Criteria of partial separability of multipartite qubit mixed-states

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Abstract

In this paper, we discuss the partial separability and its criteria problems of multipartite qubit mixed-states. First we strictly define what is the partial separability of a multipartite qubit system. Next we give a reduction way from N-partite qubit density matrixes to bipartite qubit density matrixes, and prove a necessary condition that a N-partite qubit mixed-state to be partially separable is its reduction to satisfy the PPT condition.

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It is known that the entanglement problem is one of the most fascinating features in modernistic quantum mechanics. Especially, it has recently been recognized that entanglement is a very important resource in quantum information processing, e.g. teleportation, quantum computation, quantum cryptography and quantum communication, etc.. In the entanglement theory, an important task is to find the criteria of separability of mixed-states. The first important result is the well-known positive partial transposition (PPT, Peres-Horodecki) criteria[1,2] for $2 \times 2$ and $2 \times 3$ systems. There are many studies about the criteria of separability for the multipartite systems, see [3-8]. About the problem of criteria of separability of multipartite quantum systems, it is not completely solved as yet.

Generally, the common so-called ‘separability’, in fact, is the ‘full-separability’. However, for multipartite systems the problems are more complex: There yet
is other concept of separability weaker than full-separability, i.e. the ‘partial
separability’, e.g. the A-BC-separability, B-AC-separability for a tripartite
qubit pure-state \( \rho_{ABC} \), etc.. Related to Bell-type inequalities and some
criteria of partial separability of multipartite systems, etc., see [9-12].

The main purpose of this paper is to discuss the partial separability and
the criteria problems of multipartite systems. First, we need to stricter define
the concept of partial separability, further we can find the simpler criteria.
Therefore, in the first part of this paper we discuss how to define strictly
the concept of the partial separability corresponding to a partition. In the
second part of this paper, we give a new way that an arbitrary N-partite
\((N \geq 3)\) qubit density matrix always can be reduced in one step through
to a bipartite qubit density matrix. In third part of this paper, we prove
an effective criterion: A necessary condition of a N-partite qubit state to
be partially separable with respect to a partition is that the corresponding
reduced bipartite qubit state in particular satisfies the PPT condition. Some
examples are given in the last part of this paper.

Suppose that \( \rho_{i_1i_2\ldots i_N} \) is a mixed-state for N-partite qubit Hilbert space
\( H = \otimes_{s=1}^{N} H_s \), of which the standard basis is \( \{ \otimes_{s=1}^{N} | i_s > \} \) \( (i_s = 0, 1) \). Let
\( \mathbb{Z}_N \) be the integer set \( \{1, 2, \ldots, N\} \). If two subsets \( (r)_P \equiv \{r_1, \ldots, r_P\} \) and
\( (s)_{N-P} \equiv \{s_1, \ldots, s_{N-P}\} \) in \( \mathbb{Z}_N \) obey
\[
1 \leq r_1 < \cdots < r_P < N, \quad 1 < s_1 < \cdots < s_{N-P} \leq N
\]
\[
(r)_P \cup (s)_{N-P} = \mathbb{Z}_N, \quad (r)_P \cap (s)_{N-P} = \emptyset
\] (1)
where \( P \) is an integer, \( 1 \leq P \leq N - 1 \), the pair \( \{(r)_P, (s)_{N-P}\} \) forms a
partition of \( \mathbb{Z}_N \), in the following we simply call it a ‘partition’, and for the
sake of stress we denote it by symbol \( (r)_P \parallel (s)_{N-P} \). corresponds to a permu-
tation \( S_{(r)_P \parallel (s)_{N-P}} \equiv \left(\begin{array}{cccc}
1, & \cdots & P, & P + 1, \cdots N \\
r_1, & \cdots & r_P, & s_1, \cdots s_{N-P}
\end{array}\right) \). By a partition
\( (r)_P \parallel (s)_{N-P} \), a new matrix \( \rho_{(r)_P \parallel (s)_{N-P}} \) from \( \rho_{i_1i_2\ldots i_N} \) can be defined now,
whose entries are determined by
\[
[\rho_{(r)_P \parallel (s)_{N-P}}]_{j_1\cdots j_N, k_1\cdots k_N} = [\rho]_{j_1r_1\cdots j_PrPs_1\cdots s_{N-P}, k_1r_1\cdots k_Pk_s_1\cdots k_s_{N-P}}
\] (2)
For instance, \( \rho_{A\parallel BCD} = \rho_{AB\parallel CD} = \rho_{ABC\parallel D} = \rho_{ABCD} \), and \( [\rho_{AC\parallel BD}]_{ijkl, rstu} = [\rho_{ABCD}]_{ijkl, rstu} \), etc.. Generally, \( \rho_{(r)_P \parallel (s)_{N-P}} \neq \)
\[\rho_{i_1 i_2 \cdots i_N}, \text{ unless } (r)_P \parallel (s)_{N-P} \text{ just maintains the natural order of } Z_N \text{ (i.e. } (r)_P = (1, \cdots, P), (s)_{N-P} = (P + 1, \cdots, N)), \text{ then } \rho_{(r)_P\parallel(s)_{N-P}} = \rho_{i_1 i_2 \cdots i_N}.\]

**Lemma.** For any partition \((r)_P \parallel (s)_{N-P},\) \(\rho_{(r)_P\parallel(s)_{N-P}}\) is still a \(N\)-partite qubit mixed-state.

**Proof.** We only consider the case of tripartite qubit, the general cases are completely similar (also see [11]). Notice the permutation \(S_{B\|AC},\) then we have

\[
\rho_{B\|AC} = S\rho_{ABC}S^\dagger, \quad S = \begin{bmatrix}
1 & & & \\
1 & 1 & 0 & 0 \\
& 0 & 1 & 0 \\
& & 0 & 1 \\
& & & 1
\end{bmatrix}
\]

\(S\) is an unitary matrix, therefore \(\rho_{B\|AC}\) is still a tripartite qubit mixed-state. As for the cases of \(S_{A\|BC}\) and \(S_{C\|AB},\) the lemma obviously holds. \(\Box\)

Now, we consider how to more strictly define the partial separability. Obviously, if a partition \((r)_P \parallel (s)_{N-P}\) maintains the natural order of \(Z_N\) (i.e. \((r)_P = (1, 2, \cdots, P), (s)_{N-P} = (P + 1, P + 2, \cdots, N)),\) then \(\rho_{(r)_P\parallel(s)_{N-P}} = \rho_{i_1 i_2 \cdots i_N}\) under the standard basis \(\{\otimes_{k=1}^N | i_s \rangle \},\) now the \((r)_P - (s)_{N-P}\) separability can naturally be defined as that if \(\rho_{i_1 i_2 \cdots i_N}\) can be decomposed as \(\rho_{(r)_P\parallel(s)_{N-P}} = \rho_{i_1 i_2 \cdots i_N} = \sum_{\alpha} p_{\alpha} \rho_{\alpha(r)_P \otimes \rho_{\alpha(s)_{N-P}}},\) with probabilities \(p_{\alpha,}\) where \(\rho_{\alpha(r)_P}\) and \(\rho_{\alpha(s)_{N-P}},\) respectively, are a \(P\)-partite and a \(N-P\)-partite qubit mixed-states of \(\otimes_{m=1}^P H_m\) and \(\otimes_{n=1}^{N-P} H_n\) for all \(\alpha,\) then we call \(\rho_{i_1 i_2 \cdots i_N}\) to be \((r)_P - (s)_{N-P}\)-separable. However, if the natural order of \(Z_N\) has been broken in \((r)_P \parallel (s)_{N-P}\) (i.e. \(s_1 < r_P\)), then generally \(\rho_{(r)_P\parallel(s)_{N-P}} \neq \rho_{i_1 i_2 \cdots i_N}\); the case is different from the above. For instance, we consider a normalized pure-state \(\rho_{ABCD} = |\Psi_{ABCD}\rangle \langle \Psi_{ABCD}|, |\Psi_{ABCD}\rangle \in H_A \otimes H_B \otimes H_C \otimes H_D\) of four spin-\(\frac{1}{2}\) particles \(A, B, C\) and \(D.\) Now, assume that \(|\Psi_{ABCD}\rangle\) has a special form as \(|\Psi_{ABCD}\rangle = \sum_{i,j,k,l=0,1} c_{ik} c_{jl} |i_A \rangle \otimes |j_B \rangle \otimes |k_C \rangle \otimes |l_D \rangle\), where \(c_{ik}, c_{jl} \in \mathbb{C}\). If we keep up to use the original standard basis, then we cannot directly see the partial separability, because this choice of basis is unsuitable. If we choose other nature basis \(|i_A \rangle \otimes |k_C \rangle \otimes |j_B \rangle \otimes |l_D \rangle\) (this, in
fact, means that we are using \( \rho_{AC} \otimes BD \), under which we can consider the state
\[ |\Psi_{ACBD} \rangle = |\Psi_{AC} \rangle \otimes |\Psi_{BD} \rangle, \]
where \( |\Psi_{AC} \rangle = \sum_{i,k=0,1} c_{ik} |i_A \rangle \otimes |k_C \rangle, \)
\[ |\Psi_{BD} \rangle = \sum_{j,l=0,1} c_{jl} |j_B \rangle \otimes |l_D \rangle. \]
Now, \( \rho_{AC} \otimes BD = \rho_{AC} \otimes \rho_{BD} \), where
\[ \rho_{AC} = |\Psi_{AC} \rangle < |\Psi_{AC} \rangle, \rho_{BD} = |\Psi_{BD} \rangle < |\Psi_{BD} \rangle, \]
and \( |\Psi_{ACBD} \rangle, |\Psi_{ACBD} \rangle \), in fact, are the same in physics, therefore to call \( |\Psi_{ACBD} \rangle = |\Psi_{AC} \rangle \otimes |\Psi_{BD} \rangle \),
thus we can generally define the concept of partial separability
as follows.

**Definition.** For the partition \((r)_P || (s)_{N-P}\) of a \(N\)-partite qubit state
\( \rho_{i_1i_2\cdots i_N} \) of \( H = \bigotimes_{s=1}^{N} H_s \) is called to be \((r)_P - (s)_{N-P}\)-separable if the corresponding state \( \rho_{(r)_P || (s)_{N-P}} \) can be decomposed as
\[
\rho_{(r)_P || (s)_{N-P}} = \sum_{\alpha} p_{\alpha} \rho_{\alpha,(r)_P} \otimes \rho_{\alpha,(s)_{N-P}} \tag{4}
\]
where \( \rho_{\alpha,(r)_P} \) and \( \rho_{\alpha,(s)_{N-P}} \), respectively, are a \(P\)-partite and a \((N-P)\)-partite qubit state of \( \bigotimes_{m=1}^{P} H_{r_m} \) and \( \bigotimes_{n=1}^{N-P} H_{s_n} \) for all \( \alpha \), and \( 0 < p_{\alpha} \leq 1 \), \( \sum_{\alpha} p_{\alpha} = 1 \). If \( \rho_{i_1i_2\cdots i_N} \) is not \((r)_P - (s)_{N-P}\)-separable, then we call it \((r)_P - (s)_{N-P}\)-inseparable.

For the distinct partitions \( \rho_{i_1i_2\cdots i_N} \) can have distinct separability. Of course, if a \( \rho_{i_1i_2\cdots i_N} \) is partially inseparable for some partition, then it must be entangled. Here, in passing, we point out that how to find the general relations between the partial separability and the ordinary separability (full-separability), generally, is not a simple problem. For instance, there is such a multipartite qubit mixed-state \( \hat{\rho} \) (see the theorem 1 and its proof in \([13,14]\)), \( \hat{\rho} \) always is partially separable for all possible partitions \((r)_P || (s)_{N-P} \) \((1 \leq P \leq N - 1)\), but \( \hat{\rho} \) is entangled.

In order to find the criteria of partial separability, first we discuss how to reduce a multipartite qubit density matrix in one step through to a bipartite qubit density matrix. For a given partition \((r)_P || (s)_{N-P} \), let two sets \( (r)_P \) and \( (s)_{N-P} \), respectively, be separated anew as follows,
\[
(r')_{P'} = \{ r'_1, \cdots, r'_{P'} \}, (r'')_{P''} = \{ r''_1, \cdots, r''_{P''} \} \quad \text{(one of them can be the null set)}
\]
\[ r'_1 < r'_2 < \cdots < r'_{P'}, \quad r''_1 < r''_2 < \cdots < r''_{P''} \]
\[
(r)_P = (r')_{P'} \cup (r'')_{P''}, \quad (r')_{P'} \cap (r'')_{P''} = \emptyset \quad \text{(0} \leq P', P'' \leq P \text{ and } P' + P'' = P) \tag{5}
\]
\[(s')_{Q'} = \{s'_1, \ldots, s'_Q\}, (s'')_{Q''} = \{s''_1, \ldots, s''_{Q''}\} \) (one of them can be the null set)
\[s'_1 < s'_2 < \cdots < s'_Q, \quad s''_1 < s''_2 < \cdots < s''_{Q''}\)
\[(s)_{N-P} = (s')_{Q'} \cup (s'')_{Q''}, (s')_{Q'} \cap (s'')_{Q''} = \emptyset \quad (0 \leq Q', Q'' \leq N-P \text{ and } Q' + Q'' = N - P)\]

Now we rewrite the partition added these partitions as \([[(r')_{P'}, (r'')_{P''}]] \quad [(s')_{Q'}, (s'')_{Q''}]\].

Now we define the matrix \(\rho[[r')_{P'}, (r'')_{P''}]]([(s')_{Q'}, (s'')_{Q''}]\) by
\[
\rho[[r')_{P'}, (r'')_{P''}]]([(s')_{Q'}, (s'')_{Q''}] = \text{the submatrix in } \rho_{i_1 \cdots i_N} \text{ consisting of all entries with form as } \rho_{x_1 x_2 \cdots x_N, y_1 y_2 \cdots y_N}
\]
which must be a 4×4 matrix, where the values of \(x_k\) and \(y_k\) \((k = 1, \ldots, N)\), respectively, are determined by
\[
\begin{align*}
x_k &= i \text{ for } k \in (r')_{P'}, \quad x_k = 1 - i \text{ for } k \in (r'')_{P''} \\
x_k &= j \text{ for } k \in (s')_{Q'}, \quad x_k = 1 - j \text{ for } k \in (s'')_{Q''} \\
y_k &= u \text{ for } k \in (r')_{P'}, \quad y_k = 1 - u \text{ for } k \in (r'')_{P''} \\
y_k &= v \text{ for } k \in (s')_{Q'}, \quad y_k = 1 - v \text{ for } k \in (s'')_{Q''}
\end{align*}
\]
where \(i, j, u, v = 0, 1\). E.g.
\[
\rho[[A,C],[B,D]] = \text{the submatrix in } \rho_{ABCD} \text{ consisting of all entries with form as } \rho_{i(i-1),j(j-1),u(u-1),v(v-1)}
\]
\[
\begin{bmatrix}
[\rho]_{00,00,1,00} & [\rho]_{00,01,01} & [\rho]_{01,01,11} & [\rho]_{01,11,11} \\
[\rho]_{01,00,11} & [\rho]_{01,01,11} & [\rho]_{01,10,11} & [\rho]_{01,11,11} \\
[\rho]_{11,10,10} & [\rho]_{11,11,10} & [\rho]_{11,11,11} & [\rho]_{11,11,11}
\end{bmatrix}
\]
\]
etc. Notice that there may be some repeated \(\rho[[r')_{P'}, (r'')_{P''}]]([(s')_{Q'}, (s'')_{Q''}]\), e.g.
\[
\rho[[r')_{P'}, (r'')_{P''}]]([(s')_{Q'}, (s'')_{Q''}]\) = \rho[[r'')_{P''}, (r')_{P'}]]([(s')_{Q'}, (s'')_{Q''}]\]
\[
\rho[[r'')_{P''}, (r')_{P'}]]([(s')_{Q'}, (s'')_{Q''}]\) = \rho[[r''(r')_{P''}, (r')_{P'}]]([(s')_{Q'}, (s'')_{Q''}]\) \quad \text{etc.}
\]
Now we define the 4×4 matrix \(\rho[[r)_{P}, (s)_{N-P}]\) by
\[
\rho[[r)_{P}, (s)_{N-P}] = \sum \text{(not repeated } \rho[[r')_{P'}, (r'')_{P''}]]([(s')_{Q'}, (s'')_{Q''}]\)).
\]
\[
\rho[[r)_{P}, (s)_{N-P}] = \sum \text{(not repeated } \rho[[r')_{P'}, (r'')_{P''}]]([(s')_{Q'}, (s'')_{Q''}]\)).
\]

\[\text{5}\]
where we take the sum for those possible \([ (r')_{p'}, (r'')_{p''} ] \parallel [(s')_{m-p'}, (s'')_{m-p''}]\) that \(\rho[(r')_{p'},(r'')_{p''}][[(s')_{m-p'},(s'')_{m-p''}]]\) are not repeated. For instance, we have (where for the set \((i_1, i_2, i_3) \equiv (A, B, C)\) we simply write \((A) \equiv (A)_1, (BC) \equiv (BC)_2,\) and \((A - BC) \equiv ((A)_1 - (BC)_2),\) etc.)

\[
\begin{align*}
\rho(A-BC) &= \rho((A),\emptyset)\parallel[(BC),\emptyset] + \rho((A),\emptyset)\parallel[(B),\emptyset] \\
\rho(B-ACD) &= \rho((B),\emptyset)\parallel[(ACD),\emptyset] + \rho((B),\emptyset)\parallel[(AC),(D)] + \rho((B),\emptyset)\parallel[(AD),(C)] + \rho((B),\emptyset)\parallel[(A),(CD)] \\
\rho(AC-BD) &= \rho((AC),\emptyset)\parallel[(ACD),\emptyset] + \rho((AC),\emptyset)\parallel[(AC),(D)] + \rho((A),(C))\parallel[(DC),(B)] + \rho((A),(C))\parallel[(A),(D)] \\
\rho(AC-BDE) &= \rho((AC),\emptyset)\parallel[(ACD),\emptyset] + \rho((AC),\emptyset)\parallel[(AC),(D)] + \rho((A),(C))\parallel[(D),(E)] + \rho((A),(C))\parallel[(AC),(D)] + \rho((A),(C))\parallel[(A),(D)] + \rho((A),(C))\parallel[(A),(B)] \\
&+ \rho((A),(C))\parallel[(B),(D)] + \rho((A),(C))\parallel[(B),(E)] \\
&+ \rho((A),(C))\parallel[(B),(D)] + \rho((A),(C))\parallel[(B),(D)] \\
&+ \rho((A),(C))\parallel[(B),(D)] + \rho((A),(C))\parallel[(B),(D)]
\end{align*}
\]

In order to vividly describe the above reduction procedures, we see the example from \(\rho_{ABCD}\) to \(\rho_{AC-BCD}\). The process \(\rho_{ABCD} \longrightarrow \rho_{AC-BCD} = \rho((AC),\emptyset)\parallel[(BD),\emptyset] + \rho((AC),\emptyset)\parallel[(B),(D)] + \rho((A),(C))\parallel[(BD),\emptyset] + \rho((A),(C))\parallel[(B),(D)]\) can be described as follows: We simply read \(\sigma_\Delta \equiv \rho((AC),\emptyset)\parallel[(BD),\emptyset],\) \(\sigma_\times \equiv \rho((AC),\emptyset)\parallel[(B),(D)],\) \(\sigma_* \equiv \rho((A),(C))\parallel[(BD),\emptyset],\) \(\sigma_\wedge \equiv \rho((A),(C))\parallel[(B),(D)]\), then \(\rho_{AC-BCD} \equiv \sigma_\Delta + \sigma_\times + \sigma_* + \sigma_\wedge\), where the entries of the submatrices \(\sigma_\Delta, \sigma_\times, \sigma_*\) and \(\sigma_\wedge\), respectively, simply are represented by ‘\(\Delta\), ‘\(\times\), ‘\(*\)’ and ‘\(\wedge\)’ (they all are some entries of \(\rho_{ABCD}\)), i.e.

\[
\sigma_\Delta = \begin{bmatrix}
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta
\end{bmatrix}, \sigma_\times = \begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times
\end{bmatrix}, \sigma_* = \begin{bmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix}
\]

\[
\sigma_\wedge = \begin{bmatrix}
\wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge
\end{bmatrix}
\]

in the 16×16 matrix \(\rho_{ABCD}\), the distributions of four submatrices \(\sigma_\Delta, \sigma_\times, \sigma_*\), \(\sigma_\wedge\) are as in the following figure (\(\sigma_\times\) is just the matrix \(\rho((AC),\emptyset)\parallel[(B),(D)]\) in

\[
\begin{bmatrix}
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta \\
\Delta & \Delta & \Delta & \Delta
\end{bmatrix}
\]

\[
\begin{bmatrix}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times
\end{bmatrix}
\]

\[
\begin{bmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix}
\]

\[
\begin{bmatrix}
\wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge \\
\wedge & \wedge & \wedge & \wedge
\end{bmatrix}
\]
Eq.(8)), which determines yet four submatrixes \( \sigma_\Delta, \sigma_X, \sigma_\ast, \sigma_\wedge \)

|   | 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |
|---|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0000 | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) |
| 0001 | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| 0010 | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) |
| 0011 | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) |
| 0100 | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| 0101 | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) |
| 0110 | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) |
| 0111 | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) |
| 1000 | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) |
| 1001 | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) |
| 1010 | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| 1011 | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) | \( \wedge \) |
| 1100 | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) |
| 1101 | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) | \( \ast \) |
| 1110 | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| 1111 | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) |

(12)

Similarly, we can consider higher dimensional cases. As for the ordinary bipartite qubit state \( \rho_{AB} \), we can take \( \rho_{(A-B)} \equiv \rho_{AB} \).

Sum up, generally we can define the \( 4 \times 4 \) matrix \( \rho_{(r_P-(s)_{N-P})} \) for a given \( (r_P \parallel (s)_{N-P}) \). In addition, it is easily verified that for any partition \( (r_P \parallel (s)_{N-P}) \), \( \rho_{(s)_{N-P}-(r_P)} \) is just the transposition of \( \rho_{(u_P-(s)_{N-P})} \); therefore from viewpoint of partial separability, we don’t have to distinguish between the partitions \( (r_P \parallel (s)_{N-P}) \) and \( (s)_{N-P} \parallel (r_P) \).

**Theorem 1.** For any partition \( (r_P \parallel (s)_{N-P}) \), \( \rho_{(r_P-(s)_{N-P})} \) is a bipartite qubit mixed-state, therefore \( \rho_{(r_P-(s)_{N-P})} \), in fact, is a reduction of the \( N \)-partite qubit density matrix \( \rho_{i_{1}i_{2}...i_{N}} \).

**Proof.** The fact must proved only is that \( \rho_{(r_P-(s)_{N-P})} \) is surely a bipartite qubit density matrix. Here we only discuss in detail the cases of quadrirpartite qubit states, since the generalization is completely straightforward. In the first place, we prove that the theorem holds for a pure-state \( \rho_{ABCD} \). Suppose that \( \rho_{ABCD} = | \Psi_{ABCD} >= \Psi_{ABCD} | \) is a normalized pure-state, where \( | \Psi_{ABCD} >= \sum_{i,j,k,l=0,1} c_{ijkl} | i_{A} > \otimes | j_{B} > \otimes | k_{C} > \otimes | l_{D} > \),
\[ \sum_{i,j,k,l=0,1} |c_{ijkl}|^2 = 1. \]

Let

\[ |\Phi_\Delta\rangle = \sum_{i,j=0,1} c_{ijij} |i_x \otimes j_y \rangle, \quad |\Phi_X\rangle = \sum_{i,j=0,1} c_{ij(i-j)} |i_x \otimes j_y \rangle \]  

\[ |\Phi_*\rangle = \sum_{i,j=0,1} c_{ij(1-i-j)} |i_x \otimes j_y \rangle, \quad |\Phi_\wedge\rangle = \sum_{i,j=0,1} c_{ij(1-i)(1-j)} |i_x \otimes j_y \rangle \] (13)

where \( x \) and \( y \) are two particles. Make normalization, we obtain

\[ |\varphi_\Delta\rangle = \eta_\Delta^2 |\Phi_\Delta\rangle, \quad |\varphi_X\rangle = \eta_X^2 |\Phi_X\rangle, \quad |\varphi_*\rangle = \eta_*^2 |\Phi_*\rangle, \quad |\varphi_\wedge\rangle = \eta_\wedge^2 |\Phi_\wedge\rangle \]

and

\[ \rho_\Delta = |\varphi_\Delta\rangle \langle \varphi_\Delta|, \quad \rho_X = |\varphi_X\rangle \langle \varphi_X|, \quad \rho_* = |\varphi_*\rangle \langle \varphi_*|, \quad \rho_\wedge = |\varphi_\wedge\rangle \langle \varphi_\wedge| \]

where the normalization factors are

\[ \eta_\Delta = \sqrt{\sum_{i,j=0,1} |c_{ijij}|^2}, \quad \eta_X = \sqrt{\sum_{i,j=0,1} |c_{ij(i-j)}|^2} \]

\[ \eta_* = \sqrt{\sum_{i,j=0,1} |c_{ij(1-i-j)}|^2}, \quad \eta_\wedge = \sqrt{\sum_{i,j=0,1} |c_{ij(1-i)(1-j)}|^2} \] (14)

It can be directly verified that from Eq.(10) we have

\[ \rho_{(AC-BD)} = \eta_\Delta^2 \rho_\Delta + \eta_X^2 \rho_X + \eta_*^2 \rho_* + \eta_\wedge^2 \rho_\wedge \] (15)

where \( \rho_\Delta, \rho_X, \rho_*, \rho_\wedge \) all are bipartite qubit pure-states. It is easily seen that since \( |\Psi_{ABCD}\rangle \) is normalized, \( \eta_\Delta^2 + \eta_X^2 + \eta_*^2 + \eta_\wedge^2 = \sum_{i,j,k,l=0,1} |c_{ijkl}|^2 = 1 \).

This means that \( \rho_{(AC-BD)} \) is a bipartite qubit mixed state for this pur-state \( \rho_{ABCD} \).

Secondly, if \( \rho_{ABCD} = \sum_\alpha p_\alpha \rho_\alpha(ABCD) \) is a mixed-state, where every \( \rho_\alpha(ABCD) \) is a quasipartite qubit pure-state with probabilities \( p_\alpha \), then from Eq.(10) we have

\[ \rho_{(AC-BD)} = \sum_\alpha p_\alpha \rho_\alpha \] (14)

(\( \rho_\alpha \) is a quasipartite qubit mixed state, \( \rho_{(AC-BD)} \) is a mixed-state.

A similar way can be extended to higher dimensional case, the key is that when \( \rho_{i_1,\ldots,i_N} \) is a pure-state, then

\[ |\Psi[(r')_{p_{r'}},(r'')_{p_{r''}}][\langle s'\rangle_{m-p_{r'}},\langle s''\rangle_{m-p_{r''}}]| \langle s'\rangle_{m-p_{r'}},\langle s''\rangle_{m-p_{r''}} \rangle \ll |\Psi[(r')_{p_{r'}},(r'')_{p_{r''}}][\langle s'\rangle_{m-p_{r'}},\langle s''\rangle_{m-p_{r''}}]| > \text{id} \text{ defined by} \]

\[ |\Psi[(r')_{p_{r'}},(r'')_{p_{r''}}][\langle s'\rangle_{m-p_{r'}},\langle s''\rangle_{m-p_{r''}}]| \langle s'\rangle_{m-p_{r'}},\langle s''\rangle_{m-p_{r''}} \rangle \ll \sum_{i,j=0,1} c_{x_1x_2\ldots x_N} |x_1 \otimes \cdots \otimes x_N \rangle > \]

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\( (x_1, x_2, \cdots, x_N \text{ are determined by Eq.}(7)) \) (16)

Therefore we just have

\[
| \Psi_{i_1, \cdots, i_N} > = \sum \left( \text{not repeated} \right) | \Psi[(r')_{P_r}, (r'')_{P_r}] [(s')_{m-P_r}, (s'')_{m-P_r}] > \]

where we take the sum for those possible \([(r')_{P_r}, (r'')_{P_r}] \parallel [(s')_{m-P_r}, (s'')_{m-P_r}] \) that \( | \Psi[(r')_{P_r}, (r'')_{P_r}] [(s')_{m-P_r}, (s'')_{m-P_r}] > \) are not repeated. By using of this relation, make the similar states as in Eq.(13), and make generalization to mixes-states, we can prove that generally, a mixed-state density matrix \( \rho_{i_1 \cdots i_N} \) can be reduced through to the bipartite qubit density matrix \( \rho_{(r)_{P}-(s)_{N-P}} \).

\( \square \)

The following theorem is the main result in this paper, it is an application of PPT condition for multipartite qubit systems.

**Theorem 2 (Criterion).** For a given partition \((r)_{P} \parallel (s)_{N-P}\), a necessary condition of a \( N \)-partite (\( N \geq 3 \)) qubit state \( \rho_{i_1 \cdots i_N} \) to be (\( r)_{P}-(s)_{N-P} \)-separable is that the reduced bipartite qubit mixed-state \( \rho_{(r)_{P}-(s)_{N-P}} \) in particular satisfies the PPT condition.

**Proof.** We only discuss in detail the case of quadripartite qubit, it can be straightforwardly generalized to the case of arbitrary \( N \)-partite qubit. In the first place, we prove that this theorem holds for a quadripartite qubit pure-state. Suppose that the pure-state \( \rho_{ABCD} \) is AC-BD-separable. This means that if we choose the natural basis \( \{| i_A > \otimes | j_C > \otimes | r_B > \otimes | s_D > \} \), then \( \rho_{ACBD} = \rho_{AC} \otimes \rho_{BD} \), where \( \rho_{AC} = | \Psi_{AC} > < \Psi_{AC} |, \quad | \Psi_{AC} > = \sum_{i,j=0,1} c_{ij} | i_A > \otimes | j_C > , \quad \sum_{i,j=0,1} | c_{ij} |^2 = 1 \), and \( \rho_{BD} = | \Psi_{BD} > < \Psi_{BD} |, \quad | \Psi_{BD} > = \sum_{r,s=0,1} d_{rs} | r_B > \otimes | s_D > , \quad \sum_{r,s=0,1} | d_{rs} |^2 = 1 \). From the above ways, it easily checked that the bipartite qubit mixed-state \( \rho_{(AC-BD)} \), in fact, can be rewritten as

\[
\rho_{(AC-BD)} = \sigma_\Delta + \sigma_\times + \sigma_\wedge + \sigma_\wedge = \sigma_{(AC)} \otimes \sigma_{(BD)} + \sigma_{(AC)} \otimes \sigma_{(BD)}
\]

\[
+ \sigma_{(AC)} \otimes \sigma_{(BD)} + \sigma_{(AC)} \otimes \sigma_{(BD)}
\]

(18)
where \( \sigma_{(AC)} = | \Phi_{(AC)} \rangle \langle \Phi_{(AC)} | \), we already write \(| \Phi_{(AC)} \rangle = \sum_{i=0,1} c_{ii} | i \rangle \). Similarly, \( \sigma_{(BD)} = | \Phi_{(BD)} \rangle \langle \Phi_{(BD)} | \), \( \rho_{i} \equiv c_{ii} \) and \(| i \rangle \otimes | i \rangle \rightarrow | i \rangle \). Therefore, for every mixed-state \( \rho_{(AC)} \), there is a decomposition as \( \rho_{(AC)} = | \Phi_{(AC)} \rangle \langle \Phi_{(AC)} | \), \( j_{x} >, f_{j} \equiv c_{ji(1-j)} \) and \(| j_{x} \rangle \otimes | (1-j)_{d} \rangle \rightarrow \rangle \), \( j_{x} > \), and similarly for \( \rho_{(BD)} \), \( \sigma_{(BD)} \), \( \Phi_{(BD)} \), etc. Now, \( \rho_{(AC-BD)} \) can be written as

\[
\rho_{(AC-BD)} = \eta_{(AC)}^{2} \eta_{(BD)}^{2} \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} \rho_{(AC)} \otimes \rho_{(BD)} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} \rho_{(AC)} \otimes \rho_{(BD)}
\]

where \( \rho_{(AC)} = (\eta_{(AC)})^{-1} | \Phi_{(AC)} \rangle \langle \Phi_{(AC)} | \), \( \eta_{(AC)} = \sqrt{\sum_{i=0,1} c_{ii}^{2}} \). Now, \( \rho_{(AC)} \) is a state of a single particle. Similarly, for \( \rho_{(AC)} \), \( \rho_{(BD)} \), \( \rho_{(AC)} \). Since

\[
\eta_{(AC)}^{2} \eta_{(BD)}^{2} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} + \eta_{(AC)}^{2} \eta_{(BD)}^{2} = \left( \eta_{(AC)}^{2} + \eta_{(AC)}^{2} \right) \left( \eta_{(BD)}^{2} + \eta_{(BD)}^{2} \right) = 1 \times 1 = 1
\]

therefore \( \rho_{(AC-BD)} \) is a separable bipartite qubit mixed-state. The PPT condition for separability of 2×2 systems is sufficient and necessary[2], thus \( \rho_{(AC-BD)} \) satisfies the PPT condition. Similarly, for other partial separability.

Secondly, we prove that this theorem holds yet for partially separable mixed-states. Suppose that \( \rho_{ABCD} \) is an AC-BD-separable mixed-state, then under the same natural basis there is a decomposition as \( \rho_{ABCD} = \sum_{\alpha} p_{\alpha} \rho_{(AC)} \otimes \rho_{(BD)} \), where \( \rho_{(AC)} \) and \( \rho_{(BD)} \) both are bipartite qubit pure-states as in the above for all \( \alpha, 0 < p_{\alpha} \leq 1 \), \( \sum_{\alpha} p_{\alpha} = 1 \). From the above reduction operation, obviously we have

\[
\rho_{(AC-BD)} = \sum_{\alpha} p_{\alpha} \left[ \rho_{(AC)} \otimes \rho_{(BD)} \right]_{(AC-BD)}
\]

According to the above mention, every \( \left[ \rho_{(AC)} \otimes \rho_{(BD)} \right]_{(AC-BD)} \) is a separable bipartite qubit mixed-state, this leads to that the convex sum \( \rho_{(AC-BD)} \).
in Eq.(21) still is a separable bipartite qubit mixed-state, and it must satisfy the PPT condition.

Similarly, we can prove higher dimensional cases. □

**Corollary.** If the reduced bipartite qubit mixed-state \((\rho_{i_1i_2...i_N})(r_P,S_{N-P})\) violates the PPT condition for a partition \((r_P \parallel S_{N-P})\), then \(\rho_{i_1i_2...i_N}\) is \((r_P \parallel S_{N-P})\)-inseparable and entangled.

It, in fact, is the inverse-negative proposition of Theorem 2.

**Examples.** Consider two tripartite qubit states

\[
\rho'_{ABC} = \begin{pmatrix}
0 & 1-x/4 & 1-x/4 & x/2 & -x/2 & 1-x/4 & 1-x/4 \\
1-x/4 & 0 & 1-x/4 & -x/2 & x/2 & 0 & 1-x/4 \\
x/2 & 1-x/4 & 0 & 0 & 0 & 0 & 0 \\
-1-x/2 & 0 & 0 & 1-x/4 & 0 & 0 & 0 \\
1-x/4 & 0 & 1-x/4 & 0 & 0 & 0 & 0 \\
1-x/4 & 0 & 1-x/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\rho''_{ABC} = \begin{pmatrix}
0 & 1-x/4 & 1-x/4 & x/2 & -x/2 & 1-x/4 & 1-x/4 \\
1-x/4 & 0 & 1-x/4 & -x/2 & x/2 & 0 & 1-x/4 \\
x/2 & 1-x/4 & 0 & 0 & 0 & 0 & 0 \\
-1-x/2 & 0 & 0 & 1-x/4 & 0 & 0 & 0 \\
1-x/4 & 0 & 1-x/4 & 0 & 0 & 0 & 0 \\
1-x/4 & 0 & 1-x/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

then we have

\[
(\rho'_{ABC})_{(A-BC)} = (\rho''_{ABC})_{(B-AC)} = \rho_W
\]

where \(\rho_W\) is the Werner state\([1,15]\) which consists of a singlet fraction \(x\) and a random fraction \((1-x)\),

\[
[\rho_W]_{i,j,rs} = xS_{i,j,rs} + \frac{1}{4}(1-x)\delta_{ir}\delta_{js}
\]

\[
S_{01,01} = S_{10,10} = -S_{01,10} = -S_{10,01} = \frac{1}{2}
\]

and all the other components of \(S\) vanish.
It is known[1] that when $\frac{1}{3} < x \leq 1$, $\rho_W$ violates the PPT condition, it leads to that $\rho'_{ABC}$ is A-BC-inseparable and $\rho''_{ABC}$ is B-AC-inseparable.

By using of the above theorems and corollary, in some special cases we can make a $N$-partite qubit from $2^{N-2}$ bipartite qubit states, which is partially inseparable for a given partition. As in the above, for the case of tripartite qubit we take two bipartite qubit states $\sigma_{(1)}$, $\sigma_{(2)}$ and real numbers $p_1, p_2$, $0 < p_1, p_2 \leq 1$ such that $\sigma = p_1 \sigma_{(1)} + p_2 \sigma_{(2)}$ is a bipartite qubit entangled state (then it violates the PPT condition). If we want to construct a tripartite qubit entangled state $\rho_{ABC}$ which is B-AC-inseparable, then we can take the entries of $\rho_{ABC}$ by

\[
\begin{align*}
[\rho_{ABC}]_{ijk, rst} &= p_1 [\sigma_{(1)}]_{ji, sr}, \text{ for } k = i \text{ and } t = r \\
[\rho_{ABC}]_{ijk, rst} &= p_2 [\sigma_{(2)}]_{ji, sr}, \text{ for } k = 1 - i \text{ and } t = 1 - r \\
[\rho_{ABC}]_{ijk, rst} &= 0, \text{ for the rest } (i, j, k, r, s, t = 0, 1)
\end{align*}
\]

(25)

It can be verified that $\rho_{ABC}$ is a tripartite qubit mixed-states, and is B-AC-inseparable. In fact, $\langle \rho_{ABC} \rangle_{(B-AC)} = \tau$ which violates the PPT condition. Similarly, for A-BC and C-AB. The above way can be generalized to obtain a $\rho_{p-\pi}$-inseparable state from a bipartite qubit entangled mixed-state in form as $\tau = \sum_{i=1}^{2^{N-2}} p_i \sigma_{(i)}$, where all $\sigma_{(i)}$ are some bipartite qubit states.

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