Cuspidal integrals and subseries for $\text{SL}(3)/K_{\varepsilon}$

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Abstract

We show that for the symmetric spaces $\text{SL}(3, \mathbb{R})/\text{SO}(1, 2)_{\varepsilon}$ and $\text{SL}(3, \mathbb{C})/\text{SU}(1, 2)$ the cuspidal integrals are absolutely convergent. We further determine the behavior of the corresponding Radon transforms and relate the kernels of the Radon transforms to the different series of representations occurring in the Plancherel decomposition of these spaces. Finally we show that for the symmetric space $\text{SL}(3, \mathbb{H})/\text{Sp}(1, 2)$ the cuspidal integrals are not convergent for all Schwartz functions.
Introduction

In this article we investigate the notion of cusp forms for some symmetric spaces of split rank 2.

Harish-Chandra defined a notion of cusp forms for reductive Lie groups and he showed that the space of cusp forms coincides with the closure in the Schwartz space of the span of the discrete series of representations. This fact plays an important role in his work on the Plancherel decomposition for reductive groups. In [AFJS12] and [AFJ13] a notion of cusp forms was suggested for reductive symmetric spaces, specifically for hyperbolic spaces. This notion was adjusted in [vdBK17a] to a notion for reductive symmetric spaces of split rank 1.

The main problem one encounters when trying to define cusp forms, is convergence of the integrals involved. For a reductive symmetric space $G/H$ of split rank 1, this problem was solved in [vdBK17a] by identifying a class of parabolics subgroups $P$ of $G$, the so-called $\mathfrak{h}$-compatible parabolic subgroups, for which the integrals
\[
\int_{N_P/N_P \cap H} \phi(n) \, dn \quad (0.1)
\]
are absolutely convergent for all Schwartz functions $\phi$ on $G/H$. In [vdBK17b] it was shown that for the spaces $\text{SL}(n, \mathbb{R})/\text{GL}(n - 1, \mathbb{R})$ the condition of $\mathfrak{h}$-compatibility is necessary for the integrals (0.1) to converge for all Schwartz functions $\phi$. Cusp forms are then defined to be those Schwartz functions $\phi$ for which
\[
\int_{N_P/N_P \cap H} \phi(gn) \, dn = 0
\]
for every $\mathfrak{h}$-compatible parabolic subgroup $P$ and every $g \in G$. For reductive symmetric spaces of split rank larger than 1, the methods in [vdBK17b] cannot be used to show convergence of the integrals.

For the spaces of type $G/K_\varepsilon$, which are described in [OS80], the condition of $\mathfrak{h}$-compatibility is void. If $\sigma$ denotes the involution determining the symmetric subgroup $K_\varepsilon$, then the naive definition of cusp form involves the integrals (0.1) for all $\sigma$-parabolic subgroups $P \neq G$, i.e., all parabolic subgroups $P \neq G$ such that $\sigma(P)$ is opposite to $P$. This would require the integrals (0.1) to converge for all $\sigma$-parabolic subgroups $P$ and all Schwartz functions $\phi$.

In this article we investigate the convergence of such integrals for three reductive symmetric spaces of split rank 2 of the type described in [OS80], namely $\text{SL}(3, \mathbb{R})/\text{SO}(1, 2)_\varepsilon$, $\text{SL}(3, \mathbb{C})/\text{SU}(1, 2)$ and $\text{SL}(3, \mathbb{H})/\text{Sp}(1, 2)$. For the first two of these spaces we show that for all $\sigma$-parabolic subgroups the integrals are absolutely convergent. We further show how one can characterize the different series of representation occurring in the Plancherel decomposition of these spaces with the use of these integrals.

Using the higher-rank analogue of [vdBK17a] Section 7.2 and a careful analysis of the residues occurring in the analogue of the formula in [vdBK17a] Lemma 7.8 for the space $\text{SL}(3, \mathbb{H})/\text{Sp}(1, 2)$, Erik van den Ban showed in unpublished notes that the integrals (0.1) are not converging for all minimal $\sigma$-parabolic subgroups of $\text{SL}(3, \mathbb{H})$ and
all Schwartz functions. We give a short and easy argument showing that not even for the maximal \( \sigma \)-parabolic subgroups all of the integrals are converging. The non-convergence of the integrals for this space raises the question whether it is possible to give a useful definition for cusp forms for reductive symmetric spaces of split rank larger than 1.

The article is organized as follows. In Section 1 we describe the structure of the above mentioned symmetric spaces, their Schwartz spaces and the parabolic subgroups. For the spaces \( \text{SL}(3, \mathbb{R})/\text{SO}(1, 2)_e \) and \( \text{SL}(3, \mathbb{C})/\text{SU}(1, 2) \) we then show in Section 2 that all integrals are convergent and in Section 3 we make the connection with the Plancherel decompositions of these spaces. Finally, in Section 4 we prove that there exists a Schwartz-function on \( \text{SL}(3, \mathbb{H})/\text{Sp}(1, 2) \) such that the integrals (0.1) are divergent for some of the maximal and all of the minimal parabolic subgroups \( P \).

This paper grew out of discussions with Erik van den Ban and Henrik Schlichtkrull about explicit computations for a simple split rank 2 symmetric space. We want to thank both of them for their contribution through these discussions. In particular we want to thank Erik van den Ban for allowing us to publish our simple proof of his result on the non-convergence.

1 Structure, parabolic subgroups and Schwartz spaces

1.1 Involutions

Let \( F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \) and let \( G = \text{SL}(3, F) \). Let \( \theta \) be the usual Cartan involution

\[ \theta : g \mapsto (g^{-1})^\dagger \]

and let \( \sigma \) be the involution

\[ \sigma : g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \theta(g) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

We define \( H \) to be the connected open subgroup of \( G^\sigma \) and \( K \) to be the maximal compact subgroup \( G^\theta \). Note that

- \( H = \text{SO}(1, 2)_e \) and \( K = \text{SO}(3) \) if \( F = \mathbb{R} \);
- \( H = \text{SU}(1, 2) \) and \( K = \text{SU}(3) \) if \( F = \mathbb{C} \);
- \( H = \text{Sp}(1, 2) \) and \( K = \text{Sp}(3) \) if \( F = \mathbb{H} \).

The involutions \( \theta \) and \( \sigma \) commute. We use the same symbols for the involutions of \( \mathfrak{g} \) obtained by deriving \( \theta \) and \( \sigma \). Let \( \mathfrak{p} \) and \( \mathfrak{q} \) be the \(-1\) eigenspaces of \( \theta \) and \( \sigma \) respectively. Then

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}, \]

where \( \mathfrak{k} \) and \( \mathfrak{h} \) are the Lie algebras of \( K \) and \( H \) respectively.
1.2 \( \sigma \)-Stable maximal split abelian subalgebras

Let \( b \) be a \( \sigma \)-stable maximal split abelian subalgebra of \( g \). The split rank of \( G \) is equal to 2, while the split rank of \( H \) is equal to 1. Therefore \( \dim(b) = 2 \) and \( \dim(b \cap h) \leq 1 \). We define \( A \) to be the set of all \( \sigma \)-stable maximal split abelian subalgebra \( b \) such that \( \dim(b \cap h) = 0 \), i.e., \( b \subset q \). Note that \( H \) acts on \( A \).

**Proposition 1.1.** The action of \( H \) on \( A \) is transitive.

**Proof.** The proposition follows directly from [vdBKS14, Lemma 1.1] and [Mat79, Lemmas 4&7].

Let \( a \) be the Lie subalgebra of \( g \) consisting of all diagonal matrices. Then \( a \subset p \cap q \) and thus \( a \in A \). It follows from Proposition 1.1 that every maximal split abelian subalgebra in \( A \) is conjugate to \( a \) via an element in \( H \).

For later purposes we note here that the group \( N_{K \cap H}(a)/Z_{K \cap H}(a) \) consists of two elements: the equivalence class of the unit element and the equivalence class of

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}.
\]

(1.1)

1.3 \( H \)-conjugacy classes of minimal \( \sigma \)-parabolic subgroups

Let \( \Sigma \) be the root system of \( a \) in \( g \). For \( i \in \{1, 2, 3\} \) let \( e_i \in a^* \) be given by

\[ e_i \left( \text{diag}(x_1, x_2, x_3) \right) = x_i. \]

Then

\[ \Sigma = \{ e_i - e_j : 1 \leq i, j \leq 3, i \neq j \}. \]

We write \( g_\alpha \) for the root space of a root \( \alpha \in \Sigma \).

We define

\[
\begin{align*}
\Sigma_1 &:= \{ e_1 - e_2, e_1 - e_3, e_2 - e_3 \}, \\
\Sigma_2 &:= \{ e_2 - e_1, e_1 - e_3, e_2 - e_3 \}, \\
\Sigma_3 &:= \{ e_2 - e_1, e_3 - e_1, e_2 - e_3 \}.
\end{align*}
\]

(1.2)

Note that \( \Sigma_i \), with \( i \in \{1, 2, 3\} \), is a positive system of \( \Sigma \). We define \( P_i \) to be the minimal parabolic subgroup such that \( a \subset \text{Lie}(P_i) \) and the set of roots of \( \text{Lie}(P_i) \) in \( a \) is equal to \( \Sigma_i \).

**Proposition 1.2.** Let \( P \) be a minimal parabolic subgroup of \( G \). The following are equivalent.

(i) \( P \) is a \( \sigma \)-parabolic subgroup, i.e., \( \sigma P \) is opposite to \( P \);

(ii) \( PH \) is open in \( G \);
(iii) There exists a \( b \in A \) such that \( b \subset \text{Lie}(P) \);

(iv) \( P \) is \( H \)-conjugate to one of the parabolic subgroups \( P_1, P_2 \) or \( P_3 \).

**Proof.** (i) \( \Rightarrow \) (ii): Assume that \( \sigma P \) is opposite to \( P \). Then

\[
\text{Lie}(P) + h \supseteq \text{Lie}(P) + \frac{1 + \sigma}{2} (\text{Lie}(P)) = \text{Lie}(P) + \sigma \text{Lie}(P) = g. \tag{1.3}
\]

This proves that \( PH \) is open in \( G \).

(ii) \( \Rightarrow \) (iii): Every element in \( A \) is a subspace that is contained in \( q \), and vice versa, every maximal split abelian subspace of \( q \) is an element of \( A \). The implication now follows from [Ros79, Lemma 14].

(iii) \( \Rightarrow \) (iv): By Proposition 1.1 we may without loss of generality assume that \( a \subset \text{Lie}(P) \). Let \( k = w_0 \) if \( g_{3,2} \subset \text{Lie}(P) \) (see (1.1)); otherwise, let \( k = e \). Note that in both cases \( k \in N_{K \cap H}(a) \). Therefore \( P' := kPk^{-1} \) is a parabolic subgroup such that \( a \subset \text{Lie}(P') \). Moreover, \( g_{2,3} \subset \text{Lie}(P') \), hence \( P' \) is equal to one of the parabolic subgroups \( P_1, P_2 \) or \( P_3 \).

(iv) \( \Rightarrow \) (i): The parabolic subgroups \( P_i \) for \( i \in \{1, 2, 3\} \) are stable under the involution \( \sigma \theta \). Therefore they are all \( \sigma \)-parabolic subgroups. Any \( H \)-conjugate of a \( \sigma \)-parabolic subgroup is again a \( \sigma \)-parabolic subgroup, hence \( P \) is a \( \sigma \)-parabolic subgroup. \( \square \)

Let \( P_\sigma \) be the set of all minimal \( \sigma \)-parabolic subgroups of \( G \). Note that \( H \) acts on \( P_\sigma \).

**Proposition 1.3.** The action of \( H \) on \( P_\sigma \) admits three orbits. Moreover,

\[
H \backslash P_\sigma = \{ [P_i] : i = 1, 2, 3 \},
\]

where \([P]\) denotes the \( H \)-conjugacy class of \( P \).

**Proof.** Let \( P \in P_\sigma \). Since \( N_{K \cap H}(a) / Z_{K \cap H}(a) \) consists of two elements, it follows from [Ros79, Corollary 17] (see also [Mat79, Proposition 1]) that there are three open \( H \)-orbits in \( G / P \). In other words, there are three \( H \)-conjugacy classes of parabolic subgroups \( P' \) such that \( P'H \) is open in \( G \). The proposition now follows from Proposition 1.2. \( \square \)

We conclude this section with a relation between \( P_1 \) and \( P_3 \). Recall \( w_0 \in N_{K \cap H}(a) \) from (1.1).

**Proposition 1.4.** \( \sigma(w_0P_1w_0^{-1}) = P_3 \).

**Proof.** Note that \( a \subset \text{Lie}(\sigma(w_0P_1w_0^{-1})) = \text{Ad}(w_0)(\sigma \text{Lie}(P_1)) \). Moreover, the set of roots of \( a \) in \( \text{Ad}(w_0)(\sigma \text{Lie}(P_1)) \) is equal to \( \Sigma_3 \). This proves the proposition. \( \square \)

### 1.4 \( H \)-conjugacy classes of maximal \( \sigma \)-parabolic subgroups

For \( i \in \{1, 2, 3, 4\} \) let \( Q_i \) be the maximal parabolic subgroup such that \( a \subset \text{Lie} Q \) and the nilradical \( n_i \) of \( \text{Lie} Q_i \) is given by

\[
\begin{align*}
n_1 &= g_{1,3} \oplus g_{2,3}, & n_2 &= g_{2,1} \oplus g_{3,1}, & n_3 &= g_{1,2} \oplus g_{1,3}, & n_4 &= g_{2,1} \oplus g_{2,3}.
\end{align*}
\]
Proposition 1.5. Let $Q$ be a maximal parabolic subgroup. The following are equivalent.

(i) $Q$ is a $\sigma$-parabolic subgroup, i.e., $\sigma Q$ is opposite to $Q$;
(ii) $QH$ is open in $G$;
(iii) $Q$ contains a minimal parabolic subgroup $P$ such that $PH$ is open;
(iv) $Q$ is $H$-conjugate to one of the parabolic subgroups $Q_1$, $Q_2$, $Q_3$ or $Q_4$.

Proof. (i) $\Rightarrow$ (ii): Assume that $\sigma Q$ is opposite to $Q$. Then (1.3) holds with $P$ replaced by $Q$ and thus it follows that $QH$ is open in $G$.
(ii) $\Rightarrow$ (iii): Assume that $QH$ is open in $G$. Every minimal parabolic subgroup has only finitely many orbits in $G/H$. Therefore there exists a minimal parabolic subgroup $P \subset Q$ such that $PH$ is open.
(iii) $\Rightarrow$ (iv): Assume that $Q$ contains a minimal parabolic subgroup $P$ such that $PH$ is open. From Proposition 1.2 it follows that $Q$ is $H$-conjugate to a maximal parabolic subgroup containing one of the minimal parabolic subgroups $P_1$, $P_2$ or $P_3$. Any maximal parabolic subgroup containing one of these three minimal parabolic subgroups is equal to $Q_1$, $Q_2$, $Q_3$ or $Q_4$.
(iv) $\Rightarrow$ (i): The parabolic subgroups $Q_i$ for $i \in \{1, 2, 3, 4\}$ are stable under the involution $\sigma$. Therefore they are all $\sigma$-parabolic subgroups. Any $H$-conjugate of a $\sigma$-parabolic subgroup is again a $\sigma$-parabolic subgroup.

Let $Q_\sigma$ be the set of all maximal $\sigma$-parabolic subgroups of $G$. Note that $H$ acts on $Q_\sigma$.

Proposition 1.6. The action of $H$ on $Q_\sigma$ admits four orbits. Moreover, $H \backslash Q_\sigma = \{ [Q_i] : i = 1, 2, 3, 4 \}$, where $[Q]$ denotes the $H$-conjugacy class of $Q$.

Proof. There are two $G$-conjugacy classes of maximal parabolic subgroups. Let $[Q]_G$ denote the $G$-conjugacy class of $Q$. Then $[Q_1]_G = [Q_2]_G \neq [Q_3]_G = [Q_4]_G$.

From [Ros79, Corollary 16 (2)] it easily seen that $Q_i$ admits two open orbits in $G/H$ for $i \in \{1, 2, 3, 4\}$. (From the proof of the corollary it follows that one may take the group $A$ in the theorem to be equal to $\exp(a)$.) In view of Proposition 1.5 it follows that there are two $H$-conjugacy classes of $\sigma$-parabolic subgroups in each $G$-conjugacy class $[Q_i]_G$, hence in total there are four $H$-conjugacy classes in $Q_\sigma$. The remaining claim follows from Proposition 1.5.

Recall $w_0 \in N_{K \cap H}(a)$ from (1.1).

Proposition 1.7. $\sigma Q_2 = Q_3$ and $\sigma (w_0 Q_1 w_0^{-1}) = Q_4$.

Proof. Since $a$ is contained in $\text{Lie}(Q_i)$ for all $i \in \{1, 2, 3, 4\}$ and both $\sigma$ and $\text{Ad}(w_0)$ stabilize $a$, it suffices to show that $\sigma n_2 = n_3$, $\sigma \text{Ad}(w_0)n_1 = n_4$.

The latter follows from a simple computation. 

6
1.5 Polar decomposition, the Schwartz space and tempered functions

In this section we discuss the polar decomposition for $G/H$ and sub-symmetric spaces of $G/H$. We further give a definition of Harish-Chandra Schwartz functions and tempered functions.

Let $L$ be a $\sigma$-stable closed subgroup of $G$. Assume that $L$ is a reductive Lie group of the Harish-Chandra class. Let $\theta_L$ be a Cartan involution of $L$ that commutes with $\sigma$ and let $K_L = L^{\theta_L}$ be the corresponding $\sigma$-stable maximal compact subgroup of $L$. Let $a_L$ be a maximal split abelian subalgebra of $I$ contained in $I \cap q$ and let $A_L = \exp(a_L)$. Finally let $H_L$ be the symmetric subgroup $L^\sigma = H \cap L$ of $L$. (Note that $L = G$, $K_L = K$, $A_L = A$ and $H_L = H$ are valid choices for the above defined subgroups.)

The space $L/H_L$ admits a polar decomposition: the map $$K_L \times A_L \to L/H_L; \quad (k, a) \mapsto ka \cdot H_L$$ is surjective. Moreover, if $a \cdot H_L \in K_L a' \cdot H_L$ with $a, a' \in A_L$, then there exists an element $k \in N_{K_L \cap H_L}(a_L)$ such that $a = ka'k^{-1}$.

Let $P_L$ be a minimal $\sigma$-parabolic subgroup of $L$. We define $\rho_{PL} \in \mathfrak{a}_L^*$ by

$$\rho_{PL}(Y) := \frac{1}{2} \text{tr} \left(\text{ad}(Y)|_{\mathfrak{a}_{P_L}}\right) \quad (Y \in \mathfrak{a}_L).$$

Let $W_L$ be the Weyl group of the root system in $\mathfrak{a}_L$.

**Definition 1.8.** A Schwartz function on $L/H_L$ is a smooth function $\phi : L/H_L \to \mathbb{C}$, such that for every $u \in \mathcal{U}(I)$ and $r \geq 0$ the seminorm

$$\mu_{a,r}^L(\phi) := \sup_{k \in K_L} \sup_{a \in A_L} \left( \sum_{w \in W_L} a^w \rho_{PL} \right) \left( 1 + \| \log(a) \|^r \right) \| (u \phi)(ka \cdot H_L) \|$$

is finite. Here the action of $\mathcal{U}(I)$ on $C^\infty(L/H_L)$ is obtained from the left-regular representation of $L$ on $C^\infty(L/H_L)$. We denote the vector space of Schwartz functions on $L/H_L$ by $\mathcal{C}(L/H_L)$ and equip $\mathcal{C}(L/H_L)$ with the topology induced by the mentioned seminorms. Our definition of Schwartz functions is equivalent to the one in [vdB92, section 17].

We further define $C^\infty_{\text{temp}}(L/H_L)$ to be the space of smooth functions on $L/H_L$ which are tempered as distributions on $L/H_L$, i.e., belong to the dual $C'(L/H_L)$ of the Schwartz space $\mathcal{C}(L/H_L)$. We equip the space $C^\infty_{\text{temp}}(L/H_L)$ with the coarsest locally convex topology such that the inclusion maps into $C^\infty(L/H_L)$ and $C'(L/H_L)$ are both continuous. Here $C^\infty(L/H_L)$ is equipped with the usual Fréchet topology and $C'(L/H_L)$ is equipped with the strong dual topology.

We finish this section with a more precise description of $\mathcal{C}(G/H)$. We define $\Phi : G \to \mathbb{R}_{>0}$ by

$$\Phi(g) = \| g \sigma(g)^{-1} \|_{HS}^2 \| \sigma(g) g^{-1} \|_{HS}^2 \quad (g \in G),$$

(1.4)
where \( \| \cdot \|_{HS} \) denotes the Hilbert-Schmidt norm on \( \text{Mat}(n, \mathbb{F}) \). We define
\[
V := \{ t \in \mathbb{R}^3 : \sum_{i=1}^3 t_i = 0 \}.
\]
For \( t \in V \) we further define
\[
a_t := \begin{pmatrix}
e^t_1 & 0 & 0 \\
0 & e^t_2 & 0 \\
0 & 0 & e^t_3
\end{pmatrix}.
\]

\textbf{Lemma 1.9.} Let \( g \in G \) and \( t \in V \). If \( g \in Ka \cdot H \), then
\[
\Phi(g) = 3 + 2 \cosh (4(t_1 - t_2)) + 2 \cosh (4(t_1 - t_3)) + 2 \cosh (4(t_2 - t_3)).
\]
In particular, \( \Phi \) is left \( K \)-invariant, right \( H \)-invariant and \( \Phi \circ \sigma = \Phi \). Moreover, \( \Phi|_A \) is invariant under the action of the Weyl group.

\textbf{Proof.} A straightforward computation shows that
\[
\| g \sigma(g)^{-1} \|_{HS}^2 = \text{tr} \left( g \sigma(g)^{-1} (g \sigma(g)^{-1})^t \right) = \text{tr} \left( a_t^4 \right)
\]
and
\[
\| g^{-1} g \|_{HS}^2 = \text{tr} \left( \sigma(g)^{-1} (\sigma(g)^{-1})^t \right) = \text{tr} \left( a_t^{-4} \right).
\]
The equalities in (1.6) follow from (1.5). Equation (1.6) may be rewritten as
\[
\Phi(ka \cdot H) = 3 + \sum_{\alpha \in \Sigma} \cosh (4\alpha(\log a)) \quad (k \in K, a \in A).
\]
From this identity the claimed invariances are clear. \( \square \)

\textbf{Remark 1.10.} From (1.6) it follows that \( \Phi(g) \geq 9 \) for all \( g \in G \).

Let \( k = \dim_{\mathbb{R}} \mathbb{F} \). Let \( \rho_1 \) be the half the sum of the roots in \( \Sigma_1 \), see (1.2), and let \( x \in a \) be in the corresponding positive Weyl chamber. In view of Lemma 1.9
\[
3 + e^{4\rho_1 x} \leq \Phi(\exp x) \leq 3 + 3e^{4\rho_1 x}.
\]
The Weyl group invariance of \( \Phi|_A \) now implies that a smooth function \( \phi : G/H \to \mathbb{C} \) belongs to the Schwartz space \( C(G/H) \) if and only if for every \( u \in \mathcal{U}(g) \) and \( r \geq 0 \) the seminorm
\[
\mu_{u,r}(\phi) := \sup_{x \in G/H} \Phi(x)^{\frac{r}{2}} \left( \log \circ \Phi(x) \right)^r \| (u \phi)(x) \|
\]
is finite.
2 Convergence of cuspidal integrals

Throughout this section, let $F = \mathbb{R}$ or $F = \mathbb{C}$.

2.1 Main theorem

Theorem 2.1. Let $P$ be a $\sigma$-parabolic subgroup and let $P = M_P A_P N_P$ be a Langlands decomposition of $P$ so that $M_P$ and $A_P$ are $\sigma$-stable. We set $L_P := M_P A_P = P \cap \sigma(P)$.

(i) For every $\phi \in \mathcal{C}(G/H)$ and $g \in G$ the integral

$$\mathcal{R}_P \phi(g) := \int_{N_P} \phi(gn) \, dn \quad (2.1)$$

is absolutely convergent and the function $\mathcal{R}_P \phi$ thus obtained is a smooth function on $G/(L_P \cap H) N_P$.

(ii) Let $\delta_P$ be the character on $P$ given by

$$\delta_P(l) := |\text{Ad}(l)|_{\text{Lie}(P)}^{\frac{1}{2}} = |\text{Ad}(l)|_{n_P}^{\frac{1}{2}} \quad (l \in L_P).$$

Define for $\phi \in \mathcal{C}(G/H)$ the function $\mathcal{H}_P \phi \in C^\infty(L_P)$ by

$$\mathcal{H}_P \phi(l) := \delta_P(l) \mathcal{R}_P \phi(l) \quad (l \in L_P).$$

Then $\mathcal{H}_P \phi$ is right $L_P \cap H$-invariant and $\mathcal{H}_P$ defines a continuous linear map $\mathcal{C}(G/H) \to \mathcal{C}_\text{temp}(L_P/L_P \cap H)$. Moreover,

- a. if $F = \mathbb{R}$, then $\mathcal{H}_P$ defines a continuous linear map $\mathcal{C}(G/H) \to \mathcal{C}(L_P/L_P \cap H)$;
- b. if $F = \mathbb{C}$, then $\phi \mapsto \mathcal{H}_P \phi|_{M_P}$ defines a continuous linear map $\mathcal{C}(G/H) \to \mathcal{C}(M_P/M_P \cap H)$.

In the remainder of section 2 we give the proof for Theorem 2.1.

2.2 Some estimates

We define the functions

$$M : \mathbb{R} \to [9, \infty); \quad x \mapsto \max(9, x)$$

and

$$L := \log \circ M : \mathbb{R} \to [\log 9, \infty).$$

Note that $M$ and $L$ are monotonically increasing.

Lemma 2.2. Let $\kappa_1, \kappa_2 > 0$. Let further $r > 2, r_1 \geq 2$ and $r_2 \geq 0$ and assume that $r = r_1 + r_2$. Then there exists a $c > 0$ such that
\begin{align*}
(i) \quad & \int_0^\infty M(\kappa_1(s^2 + 1)^2 + \kappa_2) \kappa^\frac{1}{2} L(\kappa_1(s^2 + 1)^2 + \kappa_2)^{-r} ds \leq c \kappa_1^{-\frac{1}{2}} L(\kappa_1)^{-r_1 + 1} L(\kappa_2)^{-r_2}. \\
(ii) \quad & \int_0^\infty (s^2 + 1)^{\frac{1}{2}} L(\kappa_1(s^2 + 1)^2)^{-r} ds \leq c L(\kappa_1)^{-r_1 + 1}. \\
(iii) \quad & \int_0^\infty M(\kappa_1 s^2 + \kappa_2) \kappa^\frac{1}{2} L(\kappa_1 s^2 + \kappa_2)^{-r} ds \leq c \kappa_1^{-\frac{1}{2}} L(\kappa_2)^{-r + 2}. \\
(iv) \quad & \int_0^\infty (\kappa_1 s^2 + \kappa_2) \kappa^\frac{1}{2} L(\kappa_1 s^2 + \kappa_2)^{-r} ds \leq c \kappa_1^{-\frac{1}{2}} L(\kappa_2)^{-r + 2}.
\end{align*}

**Proof.** We start with (i) and first note that integral is smaller than or equal to \( L(\kappa_2)^{-r_2} I(\kappa_1) \), where

\[
I(\kappa_1) := \int_0^\infty M(\kappa_1(s^2 - 1)^2) \kappa^\frac{1}{2} L(\kappa_1(s^2 - 1)^2)^{-r_1} ds = \frac{1}{2} \int_{-1}^\infty (t + 1)^{\frac{1}{2}} M(\kappa_1 t^2)^{-\frac{1}{2}} L(\kappa_1 t^2)^{-r_1} dt.
\]

Note that the integral in \( I(\kappa_1) \) is absolutely convergent for every \( \kappa_1 > 0 \) and the resulting function \( I \) is continuous. We need to prove that

\[
I(\kappa_1) \leq c \kappa_1^{-\frac{1}{2}} L(\kappa_1)^{-r_1 + 1}
\]

for some \( c > 0 \). It is enough to consider small and large \( \kappa_1 \). First assume that \( \kappa_1 \leq 9 \). Then there exist \( c, c', c'' > 0 \) such that

\[
I(\kappa_1) \leq \frac{1}{2} \int_{-1}^\infty (t + 1)^{\frac{1}{2}} M(\kappa_1 t^2)^{-\frac{1}{2}} L(\kappa_1 t^2)^{-2} dt = c \int_{-1}^{\sqrt{\frac{3}{\kappa_1}}} (t + 1)^{\frac{1}{2}} dt + \kappa_1^{-\frac{1}{2}} \int_{\sqrt{\frac{3}{\kappa_1}}}^\infty (t + 1)^{\frac{1}{2}} t^{-\frac{1}{2}} \log(\kappa_1 t^2)^{-2} dt \leq c' \kappa_1^{-\frac{1}{2}} + c' \kappa_1^{-\frac{1}{2}} \int_{\log(9)}^\infty w^{-2} dw = c'' \kappa_1^{-\frac{1}{2}}.
\]

Now assume \( \kappa_1 \geq 9^3 \). We define \( \delta = \kappa_1^{-\frac{1}{4}} \). Then \( \delta \leq 9^{-1} \) and \( \kappa_1 t^2 \leq \kappa_1^\frac{1}{4} \) if and only if \( |t| \leq \delta \). Therefore, \( |t| \geq \delta \) implies \( \kappa_1 t^2 \geq \kappa_1^\frac{1}{4} \geq 9 \) and we find that there exist \( c, c' > 0 \)
such that

$$I(\kappa_1) \leq c \int_{0}^{\delta} |1 - t|^{-\frac{1}{2}} dt$$

$$+ 2\kappa_1^{-\frac{1}{4}} \log(\kappa_1^4) \int_{0}^{1} |1 - t|^{-\frac{1}{2}} t^{-\frac{1}{2}} dt$$

$$+ \kappa_1^{-\frac{1}{4}} \int_{1}^{\infty} (1 + t)^{-\frac{1}{2}} t^{-\frac{1}{2}} \log(\kappa_1 t^2)^{-r_1} dt$$

$$\leq c' \delta + c' \kappa_1^{-\frac{1}{4}} \log(\kappa_1)^{-r_1}$$

$$+ \kappa_1^{-\frac{1}{4}} \int_{1}^{\infty} t^{-1} \log(\kappa_1 t^2)^{-r_1} dt.$$

The latter is smaller than $c'' \kappa_1^{-\frac{1}{4}} L(\kappa_1)^{-r_1+1}$ for some $c'' > 0$ as

$$\int_{1}^{\infty} t^{-1} \log(\kappa_1 t^2)^{-r_1} dt = \frac{1}{2} \int_{\log(\kappa_1)}^{\infty} s^{-r_1} ds = \frac{1}{2(r_1 - 1)} \log(\kappa_1)^{-r_1+1}.$$

This proves (i).

In order to prove (ii), it suffices to consider the desired inequality only for the case with the minus signs in the integrand. Let $\delta := 3\kappa_1^{-\frac{1}{2}}$. Since $|s^2 - 1| \leq \delta$ if and only if $\kappa_1(s^2 - 1)^2 \leq 9$ we have

$$\int_{0}^{\infty} (s^2 - 1)^{-\frac{1}{2}} L(\kappa_1(s^2 - 1)^2)^{-r} ds$$

$$= \kappa_1^{\frac{1}{2}} \int_{\{s \in [0, \infty) : |s^2 - 1| \geq \delta\}} M(\kappa_1(s^2 - 1)^2)^{-\frac{1}{4}} L(\kappa_1(s^2 - 1)^2)^{-r} ds$$

$$+ c \int_{\{s \in [0, \infty) : |s^2 - 1| \leq \delta\}} (s^2 - 1)^{-\frac{1}{2}} ds \quad (2.2)$$

with $c = \log(9)^{-r}$. In view of (i) the first term on the right hand side of (2.2) is smaller than or equal to $c' L(\kappa_1)^{-r+1}$ for some $c' > 0$.

Now we turn our attention to the integral in the second term on the right-hand side of (2.2). Up to a constant it is equal to

$$J(\kappa_1) := \int_{-\min(1, \delta)}^{\delta} (t + 1)^{-\frac{1}{2}} |t|^{-\frac{1}{2}} dt.$$

Note that the integral is absolutely convergent and that the function $J$ is continuous. It suffices to prove that $J(\kappa_1) \leq c L(\kappa_1^{-1}) L(\kappa_1)^{-r+1}$ for some $c > 0$. It is enough to consider small and large $\kappa_1$. First let $\kappa_1 \leq 9$. Then there exists a $c > 0$ such that

$$J(\kappa_1) = \int_{-\delta}^{\delta} (t + 1)^{-\frac{1}{2}} |t|^{-\frac{1}{2}} dt \leq c L(\kappa_1^{-1}).$$

Next, let $\kappa_1 \gg 9$. Then there exists a $c > 0$ such that

$$J(\kappa_1) = \int_{-\delta}^{\delta} (t + 1)^{-\frac{1}{2}} |t|^{-\frac{1}{2}} dt \leq \frac{1}{2} \int_{-\delta}^{\delta} |t|^{-\frac{1}{2}} dt \leq c \kappa_1^{-\frac{1}{4}}.$$
This proves (ii).

By performing a substitution of variables \( s' = \sqrt{\kappa_1} s \) we may reduce the proof of (iii) and (iv) to the case that \( \kappa_1 = 1 \). Since \( L \) is an increasing function, 
\[
L(s^2 + \kappa_2) \leq L(\kappa_2) \geq L(s^2 + \kappa_2)^{-2}.
\]
Using that the integrand decreases as a function of \( \kappa_2 \) we find 
\[
\int_0^\infty M(s^2 + \kappa_2)^{-\frac{1}{2}} L(s^2 + \kappa_2)^{-2} ds \leq \int_0^\infty M(s^2)^{-\frac{1}{2}} L(s^2)^{-2} ds < \infty.
\]
This proves (iii). To prove (iv) it suffices to show that 
\[
\int_0^\infty (s^2 + \kappa_2)^{-\frac{1}{2}} L(s^2 + \kappa_2)^{-2} ds \leq c L(\kappa_2^{-1})
\]
for some \( c > 0 \). We may assume that \( \kappa_2 < 8 \). Now the integral on the left-hand side is smaller than or equal to 
\[
\int_0^{\sqrt{9 - \kappa_2}} (s^2 + \kappa_2)^{-\frac{1}{2}} L(s^2 + \kappa_2)^{-2} ds + \int_{\sqrt{9 - \kappa_2}}^{\infty} (s^2 + \kappa_2)^{-\frac{1}{2}} L(s^2 + \kappa_2)^{-2} ds.
\]
The second term is bounded by 
\[
\int_1^\infty (s^2)^{-\frac{1}{2}} L(s^2)^{-2} ds < \infty;
\]
the first term is equal to 
\[
c \int_0^{\sqrt{9 - \kappa_2}} (s^2 + \kappa_2)^{-\frac{1}{2}} ds = c \int_0^{\sqrt{\frac{9}{\kappa_2} - 1}} (s^2 + 1)^{-\frac{1}{2}} ds
\]
\[= c \text{arsinh}(\sqrt{\frac{9}{\kappa_2} - 1})
\]
with \( c = \log(9)^{-\frac{1}{2}} \). This proves (iv) as \( \text{arsinh}(x) \sim \log(x) \) for \( x \to \infty \). \( \Box \)

**Proposition 2.3.** Let \( r > 4 \).

(i) Assume that \( \mathbb{F} = \mathbb{R} \). There exists a \( c > 0 \) such that for every \( t \in V \)
\[
ap_t^{\rho_{Q_1}} \int_{N_{Q_1}} \Phi(a_t n)^{-\frac{1}{r}} (\log \circ \Phi(a_t n))^{-r} d n \leq c \cosh(t_1 - t_2)^{-1}(1 + \|t\|)^{-r+4}
\]
and
\[
ap_t^{\rho_{Q_3}} \int_{N_{Q_3}} \Phi(a_t n)^{-\frac{1}{r}} (\log \circ \Phi(a_t n))^{-r} d n \leq c \cosh(t_2 - t_3)^{-1}(1 + \|t\|)^{-r+4}.
\]
(ii) Assume that $\mathbb{F} = \mathbb{C}$. There exist $c, C > 0$ such that for every $t \in V$

$$a_t^{\rho_{21}} \int_{N_{Q_1}} \Phi(a_t n) \left( \frac{1}{2} \log \Phi(a_t n) \right)^r dn$$

$$\leq c \cosh(2t) - \frac{1}{2} L \left(e^{2t} \cosh(t_1 - t_2)\right)^{-r+4}$$  \hspace{1cm} (2.3)$$

$$\leq C \left(e^{-3t} \cosh(t_1 - t_2)^{-1} (1 + |t_1 - t_2|)^{-r+4}$$  \hspace{1cm} (2.4)$$

and

$$a_t^{\rho_{23}} \int_{N_{Q_3}} \Phi(a_t n) \left( \frac{1}{2} \log \Phi(a_t n) \right)^r dn$$

$$\leq c \cosh(2t - t_3) - \frac{1}{2} L \left(e^{2t} \cosh(t_2 - t_3)^{-1} (1 + |t_2 - t_3|)^{-r+4}$$  \hspace{1cm} (2.5)$$

Proof. We will prove the estimates for the parabolic subgroup $Q_1$; the proof for the estimates for $Q_3$ is similar.

Let $k = \dim_{\mathbb{R}}(\mathbb{F})$. Note that $N_{Q_1} = \{ n_{y, z} : y, z \in \mathbb{F} \}$, where

$$n_{y, z} := \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad (y, z \in \mathbb{F}).$$

Let $t \in V$. For $g = a_t n_{y, z}$ the right-hand side of (1.4) is equal to

$$I_t := e^{\frac{1}{2} + 4t - 2t_s} \int_{\mathbb{F}} \int_{\mathbb{F}} \Phi(a_t n_{y, z}) - \frac{1}{2} \left( \log \Phi(a_t n_{y, z}) \right)^r dz \ dy.$$  \hspace{1cm} (2.6)$$

Since $\Phi(x) \geq 9$ for all $x \in G$, see Remark 1.10, we have

$$\log \Phi = L \circ \Phi.$$

A straightforward computation of the right-hand side of (1.4) shows that

$$\Phi(a_t n_{y, z})$$

$$= \left( e^{4t} (1 - |z|^2)^2 + e^{4t} (1 + |y|^2)^2 + e^{4t} + 2e^{-2t_1} |y|^2 + 2e^{-2t_2} |z|^2 + 2e^{-2t_3} |y|^2 |z|^2 \right)$$

$$\times \left( e^{-4t_1} + e^{-4t_2} + e^{-4t_3} (1 + |y|^2 - |z|^2)^2 + 2e^{2t_1} |y|^2 + 2e^{2t_2} |z|^2 \right).$$

Define

$$\Phi_1(t, y, z) = \left( e^{4t} (1 + |y|^2)^2 + e^{4t} \right) \left( e^{-4t_1} + e^{-4t_2} + e^{-4t_3} (1 + |y|^2 - |z|^2)^2 \right)$$

$$\Phi_2(t, y, z) = \left( e^{4t} (1 - |z|^2)^2 + e^{4t} \right) \left( e^{-4t_1} + e^{-4t_2} + e^{-4t_3} (1 + |y|^2 - |z|^2)^2 \right)$$

We can estimate $\Phi(a_t n_{y, z})$ from below by both $\Phi_1(t, y, z)$ and $\Phi_2(t, y, z).$
Now assume that $F = \mathbb{R}$. We first use the estimate $\Phi(\alpha t n_{y,z}) \geq \Phi_1(t, y, z)$. We perform the substitution of variables $z = \sqrt{y^2 + 1} v$ and thus obtain that there exists a constant $c_1 > 0$ such that

$$I_t \leq c_1 e^{\frac{t_1 + t_2 - 2 t_s}{2}} \int_\mathbb{R} \int_\mathbb{R} M(\Phi_1(y, z))^{-\frac{1}{4}} L(\Phi_1(t, y, z))^{-r} \, dz \, dy$$

$$= c_1 e^{\frac{t_1 + t_2 - 2 t_s}{2}} \int_0^\infty \int_0^\infty (y^2 + 1)^{\frac{1}{2}}$$

$$\times M \left[ (e^{4 t_2} (y^2 + 1)^2 + e^{4 t_3}) (e^{-4 t_1} + e^{-4 t_2} + e^{-4 t_3} (y^2 + 1)^2 (v^2 - 1)^2) \right]^{-\frac{1}{4}}$$

$$\times L \left[ (e^{4 t_2} (y^2 + 1)^2 + e^{4 t_3}) (e^{-4 t_1} + e^{-4 t_2} + e^{-4 t_3} (y^2 + 1)^2 (v^2 - 1)^2) \right]^{-r} \, dv \, dy.$$

We apply Lemma 2.2(i) to the inner integral with

$$\kappa_1 = (e^{4(t_2-t_3)}(y^2 + 1)^2 + 1)(y^2 + 1)^2, \quad \kappa_2 = (e^{4t_2}(y^2 + 1)^2 + e^{4t_3})(e^{-4t_1} + e^{-4t_2}),$$

and thus we see that for every $2 \leq r_1 \leq r$ and $r_2 = r - r_1$ there exists a constant $c_2 > 0$ such that $I_t$ is smaller than or equal to

$$c_2 e^{\frac{t_1 + t_2 - 2 t_s}{2}} \int_\mathbb{R} \int_\mathbb{R} (e^{4(t_2-t_3)}(y^2 + 1)^2 + 1)^{-\frac{1}{4}} L \left( (e^{4(t_2-t_3)}(y^2 + 1)^2 + 1)(y^2 + 1)^2 \right)^{-r_1+1}$$

$$\times L \left( (e^{4t_2}(y^2 + 1)^2 + e^{4t_3})(e^{-4t_1} + e^{-4t_2}) \right)^{-r_2} \, dy.$$

(Here we have neglected a factor of $(y^2 + 1)^{-\frac{1}{4}}$ in the integrand.) We now first apply Lemma 2.2(ii) with $r_1 = r$ and $r_2 = 0$. Using that $L(x^2) \sim L(x)$ for $x \to \infty$, we thus obtain that there exists $c_3, c_4 > 0$ such that

$$I_t \leq c_3 e^{\frac{t_1 + t_2 - 2 t_s}{2}} \int_\mathbb{R} \int_\mathbb{R} (e^{4(t_2-t_3)}(y^2 + 1)^2)^{-\frac{1}{4}} L \left( e^{2(t_2-t_3)}(y^2 + 1)^2 \right)^{-r+1} \, dy$$

$$\leq c_4 e^{\frac{t_1 + t_2}{2}} L(e^{2(t_3-t_2)}) L(e^{2(t_2-t_3)})^{-r+2}.$$

Secondly we take $r_1 = 2$ and $r_2 = r - 2$ and use that

$$(e^{4t_2}(y^2 + 1)^2 + e^{4t_3})(e^{-4t_1} + e^{-4t_2}) \geq \max{(e^{4(t_3-t_1)} + e^{4(t_3-t_2)}, (y^2 + 1)^2)}.$$

This yields the existence of constants $c_5, c_6 > 0$ such that $I_t$ is smaller than or equal to

$$c_5 e^{\frac{t_1 + t_2 - 2 t_s}{2}} L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} \right)^{-r+4} \int_\mathbb{R} \int_\mathbb{R} (e^{4(t_2-t_3)}(y^2 + 1)^2 + 1)^{-\frac{1}{4}} L \left( (y^2 + 1)^2 \right)^{-2} \, dy$$

$$\leq c_6 e^{\frac{t_1 + t_2}{2}} L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} \right)^{-r+4}.$$

(Here we have neglected a factor of $L \left( e^{4(t_2-t_3)}((y^2 + 1)^2 + 1)(y^2 + 1)^2 \right)^{-1}$ in the integrand.) These two inequalities for $I_t$ imply the existence of a constant $c > 0$ such that for every $t \in V$ with $t_2 \geq t_1$

$$I_t \leq c e^{\frac{1}{2} - \frac{t_1}{t_2}} (1 + \| t \|)^{-r+4}.$$
We now use the estimate $\Phi(a_1 n_{y, y}) \geq \Phi_2(t, y, z)$. We perform the substitution of variables $y = \sqrt{|z^2 - 1|} v$ and thus we obtain that there exists a constant $c_1 > 0$ such that $I_t$ is smaller than or equal to

$$
c_1 e^{\frac{t_1 + t_2 - 2t_3}{2}} \int_0^\infty \int_0^\infty |z^2 - 1|^\frac{1}{2} 
\times M \left( e^{4t_1 (1 - |z|^2)^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2} + e^{-4t_3} (z^2 - 1)^2 (1 - v^2)^2) \right)^{-\frac{1}{2}} 
\times L \left( e^{4t_1 (1 - |z|^2)^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2} + e^{-4t_3} (z^2 - 1)^2 (1 - v^2)^2) \right)^{-r} dv dz.
$$

We apply Lemma 2.2 (i) to the inner integral with

$$
\kappa_1 = (e^{4(t_1 - t_3)} (1 - |z|^2)^2 + 1) (z^2 - 1)^2, \quad \kappa_2 = (e^{4t_1} (1 - |z|^2)^2 + e^{4t_3}) (e^{-4t_1} + e^{-4t_2}),
$$

and thus we see that for every $2 \leq r_1 \leq r$ and $r_2 = r - r_1$ there exists a constant $c_2 > 0$ such that $I_t$ is smaller than or equal to

$$
c_2 e^{\frac{t_1 - t_2 - 2t_3}{2}} \int_0^\infty (e^{4(t_1 - t_3)} (1 - |z|^2)^2 + 1) \left( e^{4(t_1 - t_3)} (1 - |z|^2)^2 + 1 \right)^{-r_1 + 1} 
\times L \left( e^{4t_1 (1 - |z|^2)^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2}) \right)^{-r_2} dz.
$$

Applying Lemma 2.2 (ii) to the remaining integral as above, we obtain a constant $c_3 > 0$ such that $e^{\frac{t_1 - t_2}{2}} I_t$ is smaller than or equal to

$$
c_4 \min \left( L(e^{2(t_3 - t_1)}), L(e^{2(t_1 - t_3)}), L(e^{4(t_3 - t_1)}), L(e^{4(t_3 - t_2)}) \right)^{-r+1}.
$$

It follows that there exists a $c > 0$ such that for every $t \in V$ with $t_1 \geq t_2$

$$
I_t \leq c e^{\frac{t_1 - t_2}{2} (1 + ||t||)}^{-r+4}.
$$

This proves (i).

Next, assume that $F = \mathbb{C}$. We first use the estimate $\Phi(a_1 n_{y, y}) \geq \Phi_1(t, y, z)$. After introducing polar coordinates and subsequently performing the substitution of variables $v = |z|^2 - |y|^2 - 1$, $w = |y|^2 + 1$, we obtain that there exists a constant $c_1 > 0$ such that

$$
I_t \leq e^{t_1 + t_2 - 2t_3} \int_\mathbb{R} \int_\mathbb{R} M(\Phi_1(y, z))^{-\frac{1}{2}} L(\Phi_1(t, y, z))^{-r} dz dy 
= c_1 e^{t_1 + t_2 - 2t_3} \int_0^\infty \int_{-w}^w M \left( e^{4t_2 w^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2} + e^{-4t_3} v^2) \right)^{-\frac{1}{2}} 
\times L \left( e^{4t_2 w^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2} + e^{-4t_3} v^2) \right)^{-r} dv dw 
\leq 2c_1 e^{t_1 + t_2 - 2t_3} \int_0^\infty \int_0^\infty M \left( e^{4t_2 w^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2} + e^{-4t_3} v^2) \right)^{-\frac{1}{2}} 
\times L \left( e^{4t_2 w^2 + e^{4t_3}} (e^{-4t_1} + e^{-4t_2} + e^{-4t_3} v^2) \right)^{-r} dw dv.
$$

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We apply Lemma 2.2(iii) to the inner integral with $\kappa_1 = e^{4(t_2-t_1)} + 1 + e^{4(t_2-t_3)}v^2$ and $\kappa_2 = e^{4(t_3-t_1)} + e^{4(t_3-t_2)} + v^2$ and we thus find that there exists a constant $c_2 > 0$ such that

$$I_t \leq c_2 e^{t_1-t_2-2t_3} \int_0^\infty \left( e^{4(t_2-t_1)} + 1 + e^{4(t_2-t_3)}v^2 \right)^{-\frac{1}{2}} L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} + v^2 \right)^{-r+2} dv$$

$$= c_2 e^{t_1-t_2} \int_0^\infty \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} + v^2 \right)^{-\frac{1}{2}} L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} + v^2 \right)^{-r+2} dv.$$

We may now apply Lemma 2.2(iv) to the remaining integral with $\kappa_1 = 1$ and $\kappa_2 = e^{4(t_3-t_1)} + e^{4(t_3-t_2)}$. It follows that there exists a $c > 0$ such that

$$I_t \leq ce^{t_1-t_2} L \left( (e^{4(t_3-t_1)} + e^{4(t_3-t_2)})^{-1} \right) L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} \right)^{-r+4}.$$

Using the estimate $\Phi(a_{t,y,z}) \geq \Phi_2(t, y, z)$ and a similar computation we obtain that there exists a constant $c > 0$ such that

$$I_t \leq ce^{t_2-t_1} L \left( (e^{4(t_3-t_1)} + e^{4(t_3-t_2)})^{-1} \right) L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} \right)^{-r+4}.$$

Now observe that

$$L \left( (e^{4(t_3-t_1)} + e^{4(t_3-t_2)})^{-1} \right) L \left( e^{4(t_3-t_1)} + e^{4(t_3-t_2)} \right)^{-r+4}$$

$$= L \left( (2e^{6t_3} \cosh(2t_1 - 2t_2))^{-1} \right) L \left( 2e^{6t_3} \cosh(2t_1 - 2t_2) \right)^{-r+4}$$

$$\leq L \left( e^{-6t_3} \right) L \left( 2e^{6t_3} \cosh(2t_1 - 2t_2) \right)^{-r+4}$$

$$\leq cL \left( e^{-t_3} \right) L \left( e^{3t_3} \cosh(t_1 - t_2) \right)^{-r+4}$$

for some $c > 0$. This proves the estimate (2.3). For the second inequality (2.4) we note that there exists a $c' > 0$ such that

$$L \left( e^{3t_3} \cosh(t_1 - t_2) \right) \geq c'L \left( e^{-t_3} \right)^{-1} \left( 1 + |t_1 - t_2| \right) \quad (t \in V).$$

\[\square\]

### 2.3 Proof of Theorem 2.1 for maximal $\sigma$-parabolic subgroups

In this section we give the proof for Theorem 2.1 for a maximal $\sigma$-parabolic subgroup, which we here denote $Q$. We write $L_Q$ for the $\sigma$-stable Levi subgroup of $Q$, i.e., $L_Q = \sigma(Q) \cap Q$. Let $P$ be a minimal $\sigma$-parabolic subgroup contained in $Q$ and let $Q = M_Q A_Q N_Q$ and $P = M_P A_P N_P$ be Langlands decompositions of $Q$ and $P$ respectively, such that $A_P$ and $A_Q$ are $\sigma$-stable and $A_Q \subseteq A_P$. Let $R := M_Q \cap P$. Then $R$ is a minimal $\sigma$-parabolic subgroup of $M_Q$.

We first list a number of conclusions that can be drawn from the calculations in Section 2.2.
Lemma 2.4.

1a. If $F = \mathbb{R}$, then for every $r > 4$ there exists a constant $c > 0$ such that for every $a \in A_P$

$$a^{\rho_Q} \int_{N_Q} \Phi(an)^{-\frac{k}{4}} (\log \Phi(an))^{-r} \, dn \leq c \left( \sum_{w \in W_{MQ}} a^{w \cdot PR} \right)^{-1} \left( 1 + \| \log(a) \| \right)^{-r+4}.$$

1b. If $F = \mathbb{C}$, then for every $r > 4$ there exists a constant $c > 0$ such that for every $a \in A_Q$ and $b \in A_P \cap M_Q$

$$a^{\rho_Q} \int_{N_Q} \Phi(abn)^{-\frac{k}{4}} (\log \Phi(abn))^{-r} \, dn \leq c \left( \sum_{w \in W_{MQ}} b^{w \cdot PR} \right)^{-1} \left( 1 + \| \log(b) \| \right)^{-r+4} \| \log a \|^{-3}.$$

2. For every $r > 4$ there exists a constant $C > 0$ such that for every $\phi \in C(G/H)$

$$\sup_{k \in K} \sup_{a \in A} \left| (1 + \| \log(a) \|)^{-1} a^{\rho_Q} \mathcal{R}_Q \phi(ka) \right| \leq C \mu_{1,r}(\phi),$$

3a. If $F = \mathbb{R}$, then for every $r > 0$ there exists a constant $C > 0$ such that for every $\phi \in C(G/H)$

$$\mu_{1,r}^{L_Q} (\mathcal{H}_Q \phi) \leq C \mu_{1,r+4}(\phi),$$

3b. If $F = \mathbb{C}$, then for every $r > 0$ there exists a constant $C > 0$ such that for every $\phi \in C(G/H)$

$$\sup_{a \in A_Q} \left( (1 + \| \log(a) \|)^{-r-1} \mu_{1,r}^{M_Q} (\mathcal{H}_Q \phi(a \cdot)) \right) \leq C \mu_{1,r+4}(\phi),$$

Proof. It suffices to prove the lemma for only one parabolic subgroup in each $H$-conjugacy class of maximal $\sigma$-parabolic subgroups, hence by Proposition 1.6 we may assume that $Q = Q_i$ for some $i \in \{1, 2, 3, 4\}$. Since $\Phi$ is invariant under composition with $\sigma$ and conjugation with $w_0$, it is in view of Proposition 1.7 enough to prove the theorem for $Q = Q_1$ and $Q = Q_3$. The assertions now follow from Proposition 2.3.

We now give the proof of Theorem 2.1 for maximal parabolic subgroups $Q$.

Since $N_Q H / H$ is closed in $G / H$, the Radon transform $\mathcal{R}_Q$ defines a continuous linear map

$$C^\infty_c(G/H) \to C(G/(L_Q \cap H)N_Q).$$

It follows from 1a and 1b in Lemma 2.4 that this map extends to a continuous linear map

$$\mathcal{R}_Q : C(G/H) \to C(G/(L_Q \cap H)N_Q)$$

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which for all $\phi \in C(G/H)$ and $g \in G$ is given by (2.1) with absolutely convergent integrals. Since $R_Q$ is $G$-equivariant and the left regular representation of $G$ on $C(G/H)$ is smooth, $R_Q$ in fact defines a continuous linear map $C(G/H) \to C^\infty(G/(L_Q \cap H)N_Q)$. Moreover, for every $\phi \in C(G/H)$ and $u \in U_\infty(g)$

$$u(R_Q \phi) = R_Q(u \phi).$$

Since $\delta_Q$ is a smooth and $L_Q \cap H$ invariant character on $L_Q$, $H_Q$ is a continuous linear map $C(G/H) \to C^\infty(L_Q/L_Q \cap H)$, it follows from 2 in Lemma 2.4 that $H_Q$ also defines a continuous linear map $C(G/H) \to C(L_Q/L_Q \cap H)'$ and thus $H_Q$ is in fact a continuous map to $C^\infty_{temp}(L_Q/L_Q \cap H)$.

Let $\phi \in C(G/H)$ and let $u \in U(L_Q).$ It follows from the Leibniz rule and the fact that $\delta_Q$ is a character on $L_Q$, that there exists a $v \in U(L_Q)$ such that

$$u(H_Q \phi) = u(\delta_Q(R_Q \phi)|_{L_Q}) = \delta_Q v(R_Q \phi)|_{L_Q} = \delta_Q \mu(Q(v \phi)|_{L_Q} = H_Q(v \phi).$$

If $F = \mathbb{R}$, then 3a in Lemma 2.4 implies that

$$\mu_{\kappa, r}^L(H_Q \phi) = \mu_{\kappa, r}^L(uH_Q \phi) = \mu_{\kappa, r}^L(H_Q(v \phi)) \leq C \mu_{\kappa, r}^L(d) \mu_{\kappa, r}^L(v \phi) = C \mu_{\kappa, r}^L(v \phi).$$

This proves that $H_Q$ defines a continuous linear map $C(G/H) \to C(L_Q/L_Q \cap H)$. If $F = \mathbb{C}$, the same argument, with $L_Q$ replaced by $M_Q$ show that $\phi \mapsto H_Q \phi|_{M_Q}$ defines a continuous linear map $C(G/H) \to C(M_Q/(M_Q \cap H))$.

### 2.4 Proof of Theorem 2.1 for minimal $\sigma$-parabolic subgroups

In this section we give the proof for Theorem 2.1 in case $P$ is a minimal $\sigma$-parabolic subgroup.

Let $P$ be a minimal $\sigma$-parabolic subgroup and let $Q$ be a maximal $\sigma$-parabolic subgroup containing $P$. Let $P = M_P A_P N_P$ and $Q = M_Q A_Q N_Q$ be Langlands decompositions so that $A_P$ and $A_Q$ are $\sigma$-stable and $A_Q \subseteq A_P$. Then $M_Q \cap H$ is a symmetric subgroup of $M_Q$. The group $R := P \cap M_Q$ is a minimal $\sigma$-parabolic subgroup of $M_Q$.

Note that $M_Q$ is isomorphic to the group $\{g \in GL(2, F) : |\det g| = 1\}$. We differentiate between four cases.

(a) $F = \mathbb{R}$ and $M_Q \cap H$ is compact. Then $M_Q \cap H$ is a maximal compact subgroup of $M_Q$ and hence it is isomorphic to $O(2)$.

(b) $F = \mathbb{R}$ and $M_Q \cap H$ is not compact. Then $M_Q \cap H$ is isomorphic to $O(1, 1)$.

(c) $F = \mathbb{C}$ and $M_Q \cap H$ is compact. Then $M_Q \cap H$ is a maximal compact subgroup of $M_Q$; it is isomorphic to $U(2)$.

(d) $F = \mathbb{C}$ and $M_Q \cap H$ is not compact. Then $M_Q \cap H$ is isomorphic to $U(1, 1)$. In this case

$$M_Q/M_Q \cap H \simeq PSL(2, \mathbb{C})/PSU(1, 1) \simeq SO(3, 1)_e/ SO(2, 1)_e.$$
as SO(3, 1)\(_e\)-homogeneous spaces. Note that the Schwartz space and the Radon transforms for \(M_Q/M_Q \cap H\) considered as a homogeneous space for SO(3, 1)\(_e\) coincide with the Schwartz space and the Radon transforms for \(M_Q/M_Q \cap H\) considered as a homogeneous space for \(M_Q\).

For the cases (a) and (c) it is well known and for (b) and (d) it follows from \cite{AFJ13, Theorem 5.1} that the integral
\[
\int_{N_R} \phi(n) \, dn
\]
is absolutely convergent for every continuous function \(\phi : M_Q/M_Q \cap H \to \mathbb{C}\) satisfying
\[
\sup_{k \in K \cap M_Q} \sup_{a \in AP \cap M_Q} (a^{p_R} + a^{-p_R})(1 + \| \log a \|)|\phi(ka)| < \infty.
\]
Moreover, if \(r > 1\) and \(\phi\) satisfies
\[
\sup_{k \in K \cap M_Q} \sup_{a \in AP \cap M_Q} (a^{p_R} + a^{-p_R})(1 + \| \log a \|)^r |\phi(ka)| < \infty,
\]
then the function
\[
\mathcal{R}_R \phi(g) := \int_{N_R} \phi(gn) \, dn \quad (g \in M_Q)
\]
satisfies in the cases (a), (b) and (c)
\[
\sup_{k \in K \cap M_Q} \sup_{a \in AP \cap M_Q} (1 + \| \log a \|)^{r-1} a^{p_R} \mathcal{R}_R \phi(ka) < \infty.
\]
In case (d) such strong estimates do not hold, but we still have
\[
\sup_{k \in K \cap M_Q} \sup_{a \in AP \cap M_Q} a^{p_R} \mathcal{R}_R \phi(ka) < \infty.
\]
These estimates are well known if \(M_Q \cap H\) is a maximal compact subgroup of \(M_Q\) and in the cases (b) and (d) they again follow from \cite{AFJ13, Theorem 5.1}.

In all cases the multiplication map
\[
N_R \times N_Q \to N_P; \quad (\nu, n) \to \nu n
\]
is a diffeomorphism with Jacobian equal to the constant function 1. Therefore
\[
\int_{N_P} \psi(gn) \, dn = \int_{N_R} \int_{N_Q} \psi(\nu n) \, dn \, d\nu \quad (2.7)
\]
for every \(\psi \in L^1(N_P)\).

Since \(\delta^1_{N_P} = 1\), it follows from 1a and 1b in Lemma \ref{lemma2.4} that for \(r > 5\) and all \(g \in G\) the integral
\[
\int_{N_P} \Phi(gn)^{-\frac{r}{2}} (\log \Phi(gn))^{-\frac{r}{2}} \, dn
\]
is absolutely convergent. Moreover, we have the following estimates.
(i) If $F = \mathbb{R}$, then for every $r > 5$ there exists a constant $c > 0$ such that for every $a \in A_P$

$$a^{op} \int_{N_P} \Phi(an)^{-\frac{4}{r}} (\log \Phi(an))^{-r} \, dn \leq c(1 + \|\log(a)\|)^{-r+5}.$$

(ii) If $F = \mathbb{C}$, then for every $r > 5$ there exists a constant $c > 0$ such that for every $a \in A_Q$ and $b \in A_P \cap M_Q$

$$a^{op} \int_{N_P} \Phi(abn)^{-\frac{4}{r}} (\log \Phi(abn))^{-r} \, dn \leq c\|\log a\|^3.$$

It follows from these estimates that for every $\phi \in \mathcal{C}(G/H)$ and $g \in G$ the integral

$$\mathcal{R}_P \phi(g) := \int_{N_P} \phi(gn) \, dn$$

is absolutely convergent and that the map $\mathcal{R}_P$ thus obtained is a continuous linear map from $\mathcal{C}(G/H)$ to $\mathcal{C}(G/\langle L_P \cap H \rangle N_P)$. Since $\mathcal{R}_P$ is equivariant and the left regular representation of $G$ on $\mathcal{C}(G/H)$ is smooth, $\mathcal{R}_P$ in fact defines a continuous linear map $\mathcal{C}(G/H) \to C^\infty(G/\langle L_P \cap H \rangle N_P)$.

By the estimates (i) and (ii) $\mathcal{H}_Q$ defines a continuous linear map $\mathcal{C}(G/H) \to \mathcal{C}(L_P/\langle L_P \cap H \rangle)$ and thus $\mathcal{H}_P$ is in fact a continuous map to $C^\infty_{\text{temp}}(L_P/\langle L_P \cap H \rangle)$. Finally, if $F = \mathbb{R}$, then it follows as in the proof for maximal $\sigma$-parabolic subgroups in Section 2.3 that $\mathcal{H}_P$ defines a continuous linear map $\mathcal{C}(G/H) \to \mathcal{C}(L_Q/\langle L_Q \cap H \rangle)$. The remaining assertion in case $F = \mathbb{C}$ is trivial as $M_P$ is compact. This ends the proof of Theorem 2.1.

3 Kernels

Throughout this section we assume that $F = \mathbb{R}, \mathbb{C}$.

3.1 The Plancherel decomposition and kernels of Radon transforms

For a general reductive symmetric space a description of the discrete series representations has been given by the first author [FJ80] and Matsuki and Oshima [MT84]. In our setting, it follows from the rank condition in [MT84] that the space $G/H$ does not admit discrete series. For a maximal $\sigma$-parabolic subgroup $Q$, with Langlands decomposition $Q = M_Q A_Q N_Q$ such that $M_Q$ and $A_Q$ are $\sigma$-stable, the symmetric space $M_Q/\langle M_Q \cap H \rangle$ is of rank 1 and hence admits discrete series representations if and only if $M_Q \cap H$ is not compact.

Delorme [Del98] and independently Van den Ban and Schlichtkruhl [BS05a], [BS05b] have given a precise description of the Plancherel decomposition of a general reductive symmetric space. It follows from these descriptions that in our setting the space $L^2(G/H)$ decomposes as

$$L^2(G/H) = L^2_{P_\sigma}(G/H) \oplus L^2_{Q_\sigma}(G/H),$$

(3.1)
where $L^2_{S}(G/H)$ for $S = P_{\sigma}, Q_{\sigma}$ is a $G$-invariant closed subspace that is unitarily equivalent to a direct integral of representations that are induced from a parabolic subgroup contained in $S$. To be more precise, if $S \in \mathcal{S}$ and $= MS_{\mathfrak{S}}N_{\mathfrak{S}}$ is a Langlands decomposition of $S$ such that $A_S$ is $\sigma$-stable, then $L^2_{S}(G/H)$ is unitarily equivalent to a direct integral of representations $\text{Ind}^G_S(\xi \otimes \lambda \otimes 1)$, where $\lambda \in i\mathfrak{a}_{\mathfrak{S}}^*$ and $\xi \in \hat{M}_{\mathfrak{S}}$ is so that for some $v \in N_K(a)$

$$\text{Hom}_G\left(\xi, L^2(M_S/(M_S \cap vHv^{-1}))\right) \neq \{0\},$$

i.e., $\xi$ is equivalent to a discrete series representation for the space $M_S \cap vHv^{-1}$.

Intersecting both sides of (3.1) with $C(G/H)$ yields a decomposition

$$C(G/H) = C_{P_{\sigma}}(G/H) \oplus C_{Q_{\sigma}}(G/H),$$

where $C_{P_{\sigma}}(G/H) = C(G/H) \cap L^2_{P_{\sigma}}(G/H)$ and $C_{Q_{\sigma}}(G/H) = C(G/H) \cap L^2_{Q_{\sigma}}(G/H)$.

For a $\sigma$-parabolic subgroup $P$, we denote the kernel of $R_P$ in $C(G/H)$ by $\ker(R_P)$.

The aim of this section is to prove the following theorem.

**Theorem 3.1.**

(i) $\bigcap_{P \in \mathcal{P}_{\sigma}} \ker(R_P) = C_{Q_{\sigma}}(G/H)$.

(ii) $\bigcap_{Q \in \mathcal{Q}_{\sigma}} \ker(R_Q) = \{0\}$.

**Remark 3.2.** Let $P$ be a $\sigma$-parabolic subgroup. If $h \in H$ and $P' = ph^{-1}$, then

$$\ker(R_P) = \ker(R_{P'}).$$

Therefore,

$$\bigcap_{P \in \mathcal{P}_{\sigma}} \ker(R_P) = \bigcap_{i \in \{1,2,3\}} \ker(R_{P_i}), \quad \bigcap_{Q \in \mathcal{Q}_{\sigma}} \ker(R_Q) = \bigcap_{i \in \{1,2,3,4\}} \ker(R_{Q_i}).$$

We will prove the theorem by defining $\tau$-spherical Harish-Chandra transforms, relating them to $\tau$-spherical Fourier transforms as defined by Delorme in Section 3 of [Del98], and then using the Plancherel theorem to conclude the assertions. This program is carried out in the remainder of section 3.

### 3.2 The $\tau$-spherical Harish-Chandra transform

Let $(\tau, V_{\tau})$ be a finite dimensional representation of $K$. We write $C^\infty(G/H : \tau)$ for the space of smooth $V_{\tau}$-valued functions $\phi$ on $G/H$ that satisfy the transformation property

$$\phi(kx) = \tau(k)\phi(x) \quad (k \in K, x \in G/H).$$

We further write $C^\infty_c(G/H : \tau)$ and $C(G/H : \tau)$ for the subspaces of $C^\infty(G/H : \tau)$ consisting of compactly supported functions and Schwartz functions, respectively.
Let $W$ be the Weyl group of the root system of $\mathfrak{a}$ in $\mathfrak{g}$. Then

$$W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

For a subgroup $S$ of $G$, we define $W_S$ to be the subgroup of $W$ consisting of elements that can be realized in $N_{K\cap S}(\mathfrak{a})$.

Let $P$ be a $\sigma$-parabolic subgroup containing $A$ and let $P = M_P A_P N_P$ be a Langlands decomposition such that $A_P \subseteq A$. We write $W_P$ for a choice of a set of representatives in $N_K(\mathfrak{a})$ for the double quotient $W_P\backslash W/W_H$.

We denote by $\tau_{M_P}$ the restriction of $\tau$ to $M_P$ and define

$$C_{M_P}(\tau) := \bigoplus_{v \in W_P} C(M_P/M_P \cap vHv^{-1}, \tau_{M_P}),$$

If $\psi \in C(M_P, \tau)$, we write $\psi_v$ for the component of $\psi$ in the space $C(M_P/M_P \cap vHv^{-1}, \tau_{M_P})$. For $v \in W$ we define the parabolic subgroup $P^v$ by

$$P^v := v^{-1}Pv.$$

It follows from Theorem 2.1 that for every $\phi \in C(G/H, \tau)$, $a \in A_P$ and $m \in M_P$ the integral

$$\mathcal{H}_{P,\tau}^v \phi(a)(m) := a^{p_P} \int_{N_P^v} \phi(mavn) \, dn$$

is absolutely convergent. Moreover, for every $a \in A_P$ the function $\mathcal{H}_{P,\tau}^v \phi(a)$ belongs to $C(M_P/M_P \cap vHv^{-1}, \tau_{M_P})$, and from 3a and 3b in Lemma 2.7 it follows that the map $A_P \to C(M_P/M_P \cap vHv^{-1}, \tau_{M_P})$ thus obtained is continuous and tempered in the sense that for every continuous seminorm $\mu$ on $C(M_P/M_P \cap vHv^{-1}, \tau_{M_P})$ there exists an $r > 0$ so that

$$\sup_{a \in A_P} (1 + \|a\|)^{-r} \mu(\mathcal{H}_{P,\tau}^v \phi(a)) < \infty.$$

**Definition 3.3.** For a function $\phi \in C(G/H, \tau)$ we define its $\tau$-spherical Harish-Chandra transform $\mathcal{H}_{P,\tau} \phi$ to be the function $A_P \to C(M_P, \tau)$ given by

$$\left(\mathcal{H}_{P,\tau} \phi(a)\right)_v(m) := a^{p_P} \int_{N_P^v} \phi(mavn) \, dn \quad (v \in W_P, m \in M_P, a \in A_P).$$

### 3.3 The $\tau$-spherical Fourier transform

We continue with the assumptions and notation from the previous section. Let $\mathcal{A}_2(M_P, \tau)$ be the subspace of $C(M_P, \tau)$ consisting of all elements $\psi \in C(M_P, \tau)$ such that for every $v \in W_P$ the function $\psi_v$ is finite under the action of the algebra $D(M_P/M_P \cap vHv^{-1})$ of $M_P$-invariant differential operators on $M_P/M_P \cap vHv^{-1}$. For each $v \in W_P$ the space

$$\mathcal{A}(M_P/M_P \cap vHv^{-1}, \tau_{M_P}) := \{\psi_v : \psi \in \mathcal{A}_2(M_P, \tau)\}$$

is finite dimensional. See [Var77, Theorem 12, p. 312]. Equipped with the restriction of the inner product of $L^2(M_P/M_P \cap vHv^{-1}, V_\tau)$ this space is therefore a Hilbert space.
Since $\mathcal{A}_2(M_P, \tau)$ is the direct sum of the spaces $\mathcal{A}(M_P/M_P \cap v^{-1}Hv, \tau_{M_P})$, it is the finite direct sum of finite dimensional Hilbert spaces, and thus it is itself a finite dimensional Hilbert space.

For $\psi \in \mathcal{A}_2(M_P, \tau)$ and $\lambda \in a_{P, C}^*$ let $\psi_\lambda : G/H \to V_\tau$ be the function given by

$$
\psi_\lambda(x) := \begin{cases} 
0, & a^{-\lambda + \rho_P} \psi_v(m), \quad (x \not\in \bigcup_{v \in \mathcal{W}_P} P v H) \\
(x \in N_P \cap n v H, a \in A_P, m \in M_P, v \in \mathcal{W}_P). 
\end{cases}
$$

If $\text{Re}(\lambda - \rho_P)$ is strictly $P$-dominant we define the (unnormalized) Eisenstein integral $E(P, \psi, \lambda) : G/H \to V_\tau$ by

$$
E(P, \psi, \lambda)(x) = \int_K \tau^{-1}(kx) \psi_\lambda(kx) \, dk \quad (x \in G/H)
$$

and for other $\lambda \in a_{P, C}^*$ we define $E(P, \psi, \lambda)$ by meromorphic continuation. (See [CD98, Section 3].) The Eisenstein integral can be normalized by setting

$$
E^0(P, \psi, \lambda) := E(P, C_{P|P}(1, \lambda)^{-1}\psi, \lambda)
$$

as an identity of meromorphic functions in $\lambda$. Here $C_{P|P}(1, \lambda) \in \text{End}(\mathcal{A}_2(M_P, \tau))$ is the $c$-function determined by the asymptotic expansion in [Del98, (3.3)] for $E(P, \psi, \lambda)$ and is invertible for generic $\lambda$. The function $\lambda \mapsto C_{P|P}(1, \lambda)$ is meromorphic. In fact there exist a finite product $b = \prod_{j=1}^n (\langle \alpha_j, \cdot \rangle - c_j)$ of factors $\langle \alpha_j, \cdot \rangle - c_j$, where $\alpha_j \in \Sigma$ does not vanish on $a_P$ and $c_j \in \mathbb{C}$, with the property that $\lambda \mapsto b(\lambda)C_{P|P}(1, \lambda)$ is holomorphic on an open neighborhood of $ia_P^*$. See [CD98, Théorème 1].

For $\phi \in C_c^\infty(G/H, \tau)$ let $\mathcal{F}^0_{P, \tau} \phi(\lambda)$ be the element of $\mathcal{A}_2(M_P, \tau)$ determined by

$$
\langle \mathcal{F}^0_{P, \tau} \phi(\lambda), \psi \rangle = \int_{G/H} \langle \phi(x), E^0(P, \psi, \lambda) \rangle_x \, dx.
$$

By [CD98, Théorème 4] $\mathcal{F}^0_{P, \tau}$ defines a continuous map $\mathcal{C}(G/H : \tau) \to \mathcal{S}(ia_P^*) \otimes \mathcal{A}_2(M_P, \tau)$. The map $\mathcal{F}^0_{P, \tau}$ thus obtained is called the (normalized) $\tau$-spherical Fourier transform.

Let $\mathcal{F}_{A_P}$ be the euclidean Fourier transform on $A_P$, i.e., the transform $\mathcal{F}_{A_P} : \mathcal{S}(A_P) \to \mathcal{S}(ia_P^*)$ given by

$$
\mathcal{F}_{A_P} f(\lambda) = \int_{A_P} f(a) a^{-\lambda} \, da \quad (f \in \mathcal{S}(A_P), \lambda \in i a_P^*).
$$

The continuous extension of $\mathcal{F}_{A_P}$ to a map $\mathcal{S}'(A_P) \to \mathcal{S}'(ia_P^*)$ we also denote by $\mathcal{F}_{A_P}$.

**Lemma 3.4.** Let $\phi \in \mathcal{C}(G/H, \tau)$. Then

$$
\langle \mathcal{F}^0_{P, \tau} \phi(\lambda), \psi \rangle = \mathcal{F}_{A_P} \left( \langle \mathcal{H}_{P, \tau} \phi(\cdot), C_{P|P}(1, \lambda)^{-1}\psi \rangle(\lambda) \right) (\lambda \in i a_P^*, \psi \in \mathcal{A}_2(M_P, \tau)).
$$

as an identity of tempered distributions on $ia_P^*$. 

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Proof. Let

$$\Gamma_P := \sum_{\alpha \in \Sigma(g,a) \subseteq \mathfrak{a}_P} \mathbb{R}_{\geq 0} H_\alpha,$$

where $H_\alpha \in \mathfrak{a}^*$ is the element so that

$$\alpha(Y) = \langle Y, H_\alpha \rangle \quad (Y \in \mathfrak{a}).$$

Let $B \subseteq \mathfrak{a}$ be so that $\text{supp}(\phi) \subseteq K \exp(B)H$. It follows from [Kui13, Corollary 4.2] that $\text{supp}(\mathcal{H}_{P,\tau} \phi) \subseteq \exp((B + \Gamma_P) \cap a_P)$. Since $\mathcal{H}_{P,\tau} \phi$ is a tempered $C(M_P/M_P \cap vHv^{-1}, \tau_{M_P})$-valued function, it follows that for every $t \in (0, 1]$ the function

$$a \mapsto a^{-t_P} \mathcal{H}_{P,\tau} \phi(a)$$

belongs to $\mathcal{S}(A_P)$ and the map

$$[0, 1] \to \mathcal{S}'(A_P); \quad t \mapsto (\cdot)^{-t_P} \mathcal{H}_{P,\tau} \phi$$

is continuous. The proof is analogous to the proof of [Kui13, Lemma 5.7]. For every $\psi \in \mathcal{A}_2(M_P, \tau), \lambda \in i\mathfrak{a}_P^\vee$ and $t > 0$

$$\mathcal{F}_{A_P} \left( \langle \mathcal{H}_{P,\tau} \phi(\cdot), \psi \rangle \right)(t\rho_P + \lambda)
= \int_{A_P} a^{-\lambda + (1-t)\rho_P} \int_{N_P} \sum_{v \in W_P} \int_{M_P/M_P \cap vHv^{-1}} \langle \phi(manvH), \psi_v(m) \rangle_{\tau} dm \; dn \; da
= \sum_{v \in W_P} \int_{M_P/M_P \cap vHv^{-1}} \int_{A_P} \int_{N_P} \langle \phi(manvH), \psi_{t_P \rho + \lambda}(manvH) \rangle_{\tau} dm \; dn \; da
= \int_{G/H} \langle \phi(x), \psi_{t_P \rho + \lambda}(x) \rangle_{\tau} dx.$$

Since the measure on $G/H$ is invariant, the last integral is equal to

$$\int_K \int_{G/H} \langle \phi(kx), \psi_{t_P \rho + \lambda}(kx) \rangle_{\tau} dx \; dk
= \int_{G/H} \int_K \langle \phi(x), \tau(k^{-1}) \psi_{t_P \rho + \lambda}(kx) \rangle_{\tau} dx \; dk
= \int_{G/H} \langle \phi(x), E(P, \psi, t\rho_P + \lambda) \rangle_{\tau} dx.$$

After replacing $\psi$ by $C_{P\mid P}(1, t\rho_P + \lambda)^{-1} \psi$, we have thus obtained the identity

$$\langle \mathcal{F}_{P,\tau} \phi(t\rho_P + \lambda), \psi \rangle = \mathcal{F}_{A_P} \left( \langle \mathcal{H}_{P,\tau} \phi(\cdot), C_{P\mid P}(1, t\rho_P + \lambda)^{-1} \psi \rangle \right)(t\rho_P + \lambda).$$

The assertion in the lemma now follows by taking the limit for $t \downarrow 0$ on both sides of the equation. \qed
3.4 Proof of Theorem 3.1

For a $K$-invariant subspace $V$ of $L^2(G/H)$ and a finite subset $\vartheta$ of $\widehat{K}$, we denote by $V_\vartheta$ the subspace of $V$ of all $K$-finite vectors with isotypes contained in $\vartheta$. By continuity and equivariance of the Radon transforms, it suffices to prove that for all finite subset $\vartheta$ of $\hat{K}$

\[
\bigcap_{P \in \mathcal{P}_\sigma} \ker(\mathcal{R}_P)_\vartheta = C_{Q_\sigma}(G/H)_\vartheta, \quad \bigcap_{Q \in \mathcal{Q}_\sigma} \ker(\mathcal{R}_Q)_\vartheta = \{0\}.
\]

Let $C(K)_\vartheta$ be the space of $K$-finite continuous functions on $K$, whose right $K$-types belong to $\vartheta$ and let $\tau$ denote the right regular representation of $K$ on $V_\tau := C(K)_\vartheta$. Then the canonical map

\[
\varsigma : C(G/H)_\vartheta \to C(G/H : \tau)
\]

given by

\[
\varsigma \phi(x)(k) = \phi(kx) \quad (\phi \in C(G/H)_\vartheta, k \in K, x \in G/H)
\]
is a linear isomorphism. Let $\phi \in C(G/H)_\vartheta$ and let $P$ be a $\sigma$-parabolic subgroup. Then $\mathcal{R}_P \phi = 0$ for every $v \in \mathcal{W}_P$ if and only if $\mathcal{H}_{P,\tau}(\varsigma \phi) = 0$. We note that $\varsigma(C_P(G/H)_\vartheta)$ equals the space $C(G/H, \tau)^{(P)}$ defined in Théorème 2 in [Del98]. It follows from Lemma 3.4 and the fact that the $c$-function $C_{P|P}(1, \lambda)$ is invertible for generic $\lambda$ that

\[
\ker(\mathcal{H}_{P,\tau}) \subseteq \ker(\mathcal{F}^0_{P,\tau})
\]

with equality if $P$ is a minimal $\sigma$-parabolic subgroup.

Note that $\mathcal{F}^0_{G,\tau} = \{0\}$ since $G/H$ does not admit discrete series representation. It now follows from [Del98 Théorème 2] that if $P \in \mathcal{P}_\sigma$ and $Q \in \mathcal{Q}_\sigma$, then

\[
\ker(\mathcal{H}_{P,\tau}) = C(G/H, \tau)^{(Q)}, \quad \ker(\mathcal{H}_{Q,\tau}) \subseteq C(G/H, \tau)^{(P)}.
\]

Moreover, the identity (2.7) implies that $\ker(\mathcal{H}_{Q,\tau}) \subseteq \ker(\mathcal{H}_{P,\tau})$. The proof for Theorem 3.1 is now concluded by noting that

\[
C(G/H, \tau)^{(Q)} \cap C(G/H, \tau)^{(P)} = \{0\},
\]

so that $\ker(\mathcal{H}_{Q,\tau}) = \{0\}$.

4 Divergence of cuspidal integrals for $\text{SL}(3, \mathbb{H})/\text{Sp}(1, 2)$

In this section, let $\mathbb{F}$ be equal to $\mathbb{H}$. For $y, z \in \mathbb{H}$, let $n_{y,z}$ be given by (2.5) and let $\Phi$ be given by (1.4).

Lemma 4.1. Let $\epsilon < \frac{1}{6}$. Then the integral

\[
\int_{\mathbb{H}} \int_{\mathbb{H}} \Phi(n_{y,z})^{-1-\epsilon} \, dy \, dz
\]
is divergent.
Proof. It follows from (2.6) that \( \Phi(n_{y,z}) \) is equal to
\[
(1 - |z|^2)^2 + (1 + |y|^2)^2 + 1 + 2|y|^2 + 2|z|^2 + 2|y|^2|z|^2.
\]
Let \( R > 1 \). There exists a constant \( c > 0 \) such that for every \( y, z \in \mathbb{H} \) satisfying \( 1 < |y| < |z| < |y| + 1 \) we have
\[
(1 - |z|^2)^2 + (1 + |y|^2)^2 + 1 + 2|y|^2 + 2|z|^2 + 2|y|^2|z|^2 \leq c|y|^4
\]
and
\[
2 + (1 + |y|^2 - |z|^2)^2 + 2|y|^2 + 2|z|^2 \leq c|y|^2.
\]
Therefore there exists a constant \( C > 0 \) such that
\[
\int_{\mathbb{H}} \int_{\mathbb{H}} \Phi(n_{y,z})^{-1 - \epsilon} \, dy \, dz \geq C \int_{R}^{\infty} r^{6r^{-6-\epsilon}} \, dr.
\]
The latter integral is divergent for \( \epsilon < \frac{1}{6} \).

**Proposition 4.2.** There exists a positive function \( \phi \in \mathcal{C}(G/H) \) such that for every \( Q \in Q_\sigma \) that is \( H \)-conjugate to \( Q_1 \) or \( Q_4 \) and for every \( g \in G \) the integral
\[
\int_{N_Q} \phi(g^n) \, dn
\]
is divergent.

**Proof.** Let \( \Xi \) be Harish-Chandra’s bi-\( K \)-invariant spherical function \( \phi_0 \) on \( G \). Define \( \Theta : G/H \to \mathbb{R}_+ \) by
\[
\Theta(x) = \sqrt{\Xi(g\sigma(g)^{-1})} \quad (x \in G/H).
\]
Let \( 0 < \epsilon < \frac{1}{6} \) and let \( \phi = \Theta^{-1 - \epsilon} \). It follows from Propositions 17.2 and 17.3 in [vdB92] that \( \phi \in \mathcal{C}(G/H) \) and that for every \( g \in G \) there exists a \( c > 0 \) such that
\[
\phi(gx) \geq c\Phi(x)^{-1 - \epsilon} \quad (x \in G/H).
\]
The proposition now follows from the Proposition [17.7] and Lemma 4.1.

**Corollary 4.3.** Let \( \phi \in \mathcal{C}(G/H) \) be as in Proposition 4.2. For every \( P \in \mathcal{P}_\sigma \) and every \( g \in G \) the integral
\[
\int_{N_P} \phi(g^n) \, dn
\]
is divergent.

**Proof.** Note that \( n_{Q_1} \subset p_1 \). Let \( g \in G \). By Tonelli’s theorem
\[
\int_{N_{P_1}} \phi(g^n) \, dn = \int_{N_{P_1}/N_{Q_1}} \int_{N_{Q_1}} \phi(gnn_1) \, dn_1 \, dn.
\]
It follows from Proposition 4.2 and Fubini’s theorem that the right-hand side is divergent. Similarly we have \( n_{Q_4} \subset p_2 \cap p_3 \) and thus we see that the integrals for \( P = P_2 \) and \( P = P_3 \) are divergent as well. The assertion now follows from Proposition [4.3].

26
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