On the Polish doughnut accretion disc via the effective potential approach

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ABSTRACT
We revisit the Polish doughnut model of accretion discs providing a comprehensive analytical description of the Polish doughnut structure. We describe a perfect fluid circularly orbiting around a Schwarzschild black hole, source of gravitational field, by the effective potential approach for the exact gravitational and centrifugal effects. This analysis leads to a detailed, analytical description of accretion disc, its toroidal surface, the thickness and the distance from the source. We determine the variation of these features with the effective potential and the fluid angular momentum. Many analytical formulas are given. In particular it turns out that the distance from the source of the inner surface of the torus increases with increasing fluid angular momentum but decreases with increasing energy function defined as the value of the effective potential for that momentum. The location of torus maximum thickness moves towards the external regions of the surface with increasing angular momentum, until it reaches a maximum and then decreases. Assuming a polytropic equation of state we investigate some specific cases.

Key words: accretion, accretion discs – black hole physics – hydrodynamics.

1 INTRODUCTION

Accretion discs are one of the most intriguing issues in high-energy astrophysics. They enter into very different contexts: from the protoplanetary discs to gamma-ray bursts (GRBs), from X-ray binaries to the active galactic nuclei (AGN). Indeed, many aspects of disc structure, its dynamics, the formation of °jets°, the exact mechanism behind accretion, its equilibrium, the confinement and stability under perturbations are still uncertain.

Analytic and semi-analytic models for accretion on to a compact object are generally stationary and axially symmetric. Thus, all physical quantities depend only on the radial distance from the centre, \( r \), and the vertical distance from the equatorial symmetry plane, \( z \).

Thick accretion disc (\( z/r \gg 1 \)) models are basically designed according to the following assumptions: first, a circular motion dynamics is prescribed. Secondly, the matter being accreted is described by a perfect fluid energy–momentum tensor. This requirement lies on the assumption that the time-scale of dynamic processes, involving pressure, centrifugal and gravitational forces, are smaller, and much smaller in the case of vertically thin discs, than thermal ones, which in turn are smaller or much smaller than the viscous time-scale. This implies that, at variance with many thin-disc models, dissipative effects like viscosity or resistivity are neglected, and accretion is a consequence of the strong gravitational field of the attractor. Indeed, this is a great advantage since angular momentum transport in the fluid is perhaps one of the most controversial aspects in thin-disc models.

In this work we focus on the Polish doughnut model. The Polish doughnut is a fully relativistic model of thick accretion disc with a toroidal shape, and is an example of opaque (large optical depth) and super-Eddington (high matter accretion rates) disc model. During evolution of dynamic processes, the functional form of angular momentum and entropy distribution depends on the initial conditions of the system and on the details of the dissipative processes. Paczynski realized that it is physically reasonable to assume ad hoc distributions. The Polish doughnut is characterized by a constant angular momentum (Abramowicz 2009a).

The development of this model was drawn up by Paczynski and his collaborators in a series of works (Fishbone& Moncrief 1976; Abramowicz, Jaroszyński & Sikora 1978; Kozłowski, Jaroszyński & Abramowicz 1978; Abramowicz, Calvani & Nobili 1980; Jaroszyński, Abramowicz & Paczynski 1980; Paczynski 1980; Paczynski & Wiita 1980; Abramowicz, Lanza & Percival 1997). Using a perfect
fluid energy–momentum tensor. Abramowicz et al. (1978) wrote the equations of the hydrodynamics for this model. Jaroszynski et al. (1980) discussed the case of a non-barotropic fluid and showed an important result concerning the pressure of the rotating fluid: for a perfect fluid matter circularly rotating around a Schwarzschild black hole, the shapes and location of the equipressure surfaces follow directly from the assumed angular momentum distribution (Jaroszynski et al. 1980). More recently, Lei et al. (2009) assumed an angular momentum distribution in the form that depends on three constant parameters, and different configurations have been studied. A significant result concerning this model, known as the von Zeipel condition, has been extensively investigated in Kozłowski et al. (1978) and Jaroszynski et al. (1980): the constant-pressure surfaces coincide with surfaces of constant density if and only if the surfaces at constant angular momentum coincide with the surfaces at constant relativistic velocity. In the static space–times the family of von Zeipel’s surfaces does not depend on the particular rotation law of the fluid, in the sense that it does not depend on nothing but the background space–time. An accurate study of the von Zeipel surfaces has been performed in Abramowicz (1971) and Chakrabarti (1990, 1991).

Paczyński realized from the study of Roche lobe in the accretion discs of the binary systems that the black hole Roche lobe overflow must induce dynamical mass loss from the disc, thus the accretion (Boyer 1965; Frank, King & Raine 2002). The accretion occurs at the point of cusp of equipotential surfaces. This process is realized by the relativistic Roche lobe overflow. This clearly is an explanation for the accretion that does not involve factors (as the dissipative ones) other than the strong gravitational field of the attractor. However, Abramowicz (1981) showed that it constitutes also an important stabilizing mechanism against the thermal and viscous instabilities locally, and against the so-called Papaloizou and Pringle instability globally (Blaes 1987).

The general relativistic effects on matter dynamics close to a Schwarzschild black hole have been modelled in an approximate pseudo-Newtonian theory by Paczyński and Wiita introducing a properly chosen non-exact gravitational potential, known as Paczyński–Wiita (PW) potential (Paczyński & Wiita 1980). This potential simulates the relativistic effects of the gravitational field acting on the fluid in the disc, in the Schwarzschild space–time. This is not the exact expression of the effective potential for gravitational and centrifugal effects, and yet the PW potential cannot properly be considered a Newtonian approximation that is valid in the limit of weak gravitational fields. The PW potential differs from the exact relativistic one by a constant in its gravitational part and it is a Newtonian way to write some of the general relativistic effects characterizing thick discs. With this approximation, the radius of a marginally bound orbit, the last stable circular orbit radius of the Schwarzschild space–time, and the form of the Keplerian angular momentum have been correctly reproduced. A step-by-step derivation of the PW potential and a detailed discussion of its main features can be found in Abramowicz (2009b). The agreement between model predictions and simulations of accretion flows has been verified and found an excellent outcome in e.g. Igumenshche & Abramowicz (2000) and Shafee et al. (2008). The equipressure surfaces for a Schwarzschild black hole have been compared with global magnetohydrodynamic numerical simulations in Fragile et al. (2007) (see also Hawley, Smarr & Wilson 1984; Hawley 1987, 1990, 1991; De Villiers & Hawley 2002). Recently, the study of the Polish doughnut model has been developed for different attractors (see Stuchlík & Slaný 2006; Stuchlík & Kovář 2008; Stuchlík, Slaný & Kovar 2009 for the Schwarzschild–de Sitter and Kerr–de Sitter space–times).

In the present work we study the Polish doughnut model in the Schwarzschild background using the approach of the general relativistic effective potential in its exact form. Gravitational and centrifugal forces carried out in the effective potential and the pressure force operate on a perfect fluid of the disc. When the latter vanishes, the hydrodynamics of the fluid describes a geodetic disc whose equations are formally resembling those of motion of a test particle orbiting in the same background. We take advantage of this formal analogy using the familiar and well-known results on the dynamics of the test particles to get a comparison between the Polish doughnut, which is supported by the pressure, and the geodetic disc. In this way we can evaluate the right weight of the pressure effects on the dynamics of fluid and the shape of the torus. Furthermore, we draw a complete and analytic description of the toroidal surface of the disc, including the analysis of its extension in space, the distance from the centre attractor, its thickness, etc., and understand how these features are modified by changing the angular momentum of the fluid and the effective potential. In particular we find that the distance from the source of the inner surface of the torus increases with increasing fluid angular momentum and decreases with increasing energy function defined as the value of the effective potential for that momentum.

In Section 2 we introduce the Polish doughnut model writing the equations of the ideal hydrodynamics for a fluid circularly orbiting in the background of the Schwarzschild space–time. We detail the fluid pressure gradients along the radial direction (Section 3) and along the polar angular direction (Section 4) to determine the regions of maximum and minimum pressure in the disc, the regions of increasing pressure and the isobar surfaces. In Section 5 we trace the profile of the toroidal disc in the Polish model by introducing and studying in detail the Boyer potential for the barotropic fluid. In Section 6 we investigate the case of polytropic equation of state. In Section 7 we analyse the proper fluid velocity, finding regions of the discs of maximum and minimum velocity. In Section 8 we discuss the relativistic angular velocity examining the properties of the von Zeipel surfaces. Conclusions are given in Section 9.

2 THE POLISH DOUGHNUT MODEL

Consider a one-species particle perfect fluid (simple fluid), where

\[ T_{ab} = (\rho + p)U_a U_b + p g_{ab} \] (1)
is the fluid energy–momentum tensor, \( \rho \) and \( p \) are the total energy density and pressure, respectively, as measured by an observer moving with the fluid, and \( g_{ab} \) is the metric tensor. The time-like flow vector field \( U \) denotes the fluid four-velocity.\(^1\)

The motion of the fluid is described by the continuity equation
\[
U^a \nabla_a \rho + (p + \rho) \nabla^a U_a = 0 ,
\]
and the Euler equation
\[
(p + \rho) U^a \nabla_a U^b + h^{ab} \nabla_b p = 0 ,
\]
where \( h_{ab} = g_{ab} + U_a U_b \) (Misner, Thorne & Wheeler 1973).

Neglecting the fluid back-reaction, we consider the fluid motion in the Schwarzschild space–time background:
\[
ds^2 = -e^{\nu(r)} dr^2 + e^{\rho(r)} (r^2 + a^2) d\theta^2 + r^2 (d\phi^2 + \sin^2 \theta d\varphi^2) ,
\]
written in standard spherical coordinates, where \( e^{\nu(r)} \equiv (1 - 2M/r) \). We define
\[
\Lambda \equiv U^t , \quad \Sigma \equiv U^r , \quad \Phi \equiv U^\theta , \quad \Theta \equiv U^\varphi ,
\]
and we introduce the set of variables \( \{ E, V, L, T \} \) by the following relations:
\[
\Lambda = \sqrt{E^2 - V^2} , \quad \Sigma = \frac{E}{\nu} , \quad \Phi = \frac{L}{r^2 \sin \theta} , \quad \Theta = \frac{T}{r^2} ,
\]
where
\[
V = \sqrt{e^{\nu(r)} \left( 1 + \frac{L^2}{r^2 \sin^2 \theta} + \frac{T^2}{r^2} \right)}
\]
is the effective potential (Misner et al. 1973). In fact, from equation (6) we obtain
\[
r^2 = (E^2 - V^2) \quad (8)
\]
(the dot represents differentiation with respect to proper time); equation (8) describes the motion inside the effective potential \( V \), defined as the energy at which the (radial) kinetic energy of the fluid element vanishes.

Equation (8) and the definitions in equations (5)–(7) are formally the same as for the test particle motion in the Schwarzschild space–time. Obviously, for the particle motion, \( U^a \) in equation (5) is the test particle four-velocity, and \( (E, L) \) in equation (6) are two constants of motion, the particle energy and angular momentum per unit of mass as seen by an observer at infinity, respectively.

Using this similarity, we can make a one-to-one comparison of the motion of the fluid under the action of the pressure forces balanced by the effective potential, with the test particle’s dynamics regulated by gravitational and centrifugal forces as described by the effective potential (Misner et al. 1973; Wald 1984; Pugliese, Quevedo & Ruffini 2011a,b,c). The comparison with the case of dust disc through the effective potential enables us to evaluate the relationship between the contribution of pressure and the gravitational and centrifugal effects to the dynamics of the system, especially in relation to the angular momentum of the fluid in rotation and the disc shape.

We consider the case of a fluid circular configuration, defined by constraints \( \Lambda = 0 \) (i.e. \( V = E \)), restricted to a fixed plane \( \sin \theta = \sigma \neq 0 \). No motion is assumed in the \( \theta \) angular direction, which means \( \Theta = 0 \). For symmetries of the problem, we always assume \( \partial_\sigma Q = 0 \) and \( \partial_\varphi Q = 0 \), where \( Q \) is a generic tensor of space–time (we can refer to this assumption as the condition of ideal hydrodynamics of equilibrium).

Within our assumptions (\( \Lambda = 0, \Theta = 0, \partial_\rho p = \partial_\varphi p = 0, \)) from Euler equation (3) we derive the expressions for the radial pressure gradient \( G_r \) and the angular pressure gradient \( G_\varphi \):
\[
G_r = \frac{\partial_r p}{\rho + p} = -\left( \frac{\nu'\rho'}{2} \Sigma^2 - r\sigma^2 \Phi^2 \right) \quad (9)
\]
and
\[
G_\varphi = \frac{\partial_\varphi p}{\rho + p} = +\sigma \sqrt{1 - \sigma^2 r^2} \Phi^2 . \quad (10)
\]

### 3 THE RADIAL PRESSURE GRADIENT \( G_r \)

The first part of the present work is dedicated to the study of fluid angular momentum; in particular, we are interested especially in the comparison between the geodetic disc case, and the case of a fluid subjected to a non-zero pressure. For this purpose we discuss the radial and angular pressure gradient in this and the following sections. This study allows us to evaluate the pressure contribution to the disc dynamics

\(^1\)The fluid four-velocity satisfies \( \nu U^\nu U_a = -1 \). We adopt the geometrical units \( c = 1 = G \) and the (–, +, +, +) signature. The radius \( r \) has unit of mass \([M]\) and the angular momentum units of \([M]^2\), the velocities \( p [U^r] = [U^t] = 1 \) and \([U^\varphi] = [M]^{-1} \). With \([U^r/U^\varphi] = [M]^{-1} \) and \([U_\varphi/U^r] = [M] \). For the sake of convenience, we always consider the dimensionless energy and effective potential \( [V] = 1 \) and an angular momentum per unit of mass \( [L]/[M] = [M] \).
along the orbital radius and the planes on which the accretion disc stretches. This analysis introduces the second part of the work in which we finally trace the profile of the disc obtained from the analysis of constant-pressure surfaces.

Equation (9) can be written as

\[
\frac{\nabla \rho}{\rho + \rho} = -\frac{e^{-\nu}}{2} \left( \frac{\partial V^2}{\partial r} \right)_{L}
\]

in terms of the partial derivative of \(V_{\rho c}\) computed by keeping \(L = \text{constant}.\) Equation (11) yields

\[
\partial_r p = 0 \quad \text{for} \quad \left( \frac{\partial V^2}{\partial r} \right)_{L} = 0
\]

and

\[
\lim_{r \to \infty} G_r = 0, \quad \lim_{r \to 2M} G_r = -\infty.
\]

Assuming \(\rho > 0\) and \(p > 0\), from equation (11) it follows that pressure increases (decreases) with orbital radius \(r \) as \(V_{\rho c}\) decreases (increases), and that critical points of \(p\) (as a function of \(r\)) are the same as those of \(V_{\rho c}\) as a function of \(r\) at constant \(L\). Thus, solving equation (12) for the unknown \(L\) we find that these critical points are for

\[
L_K = \pm \sqrt{\frac{\sigma^2 r^2}{(r - 3M)}}.
\]

This function is defined in the range \(r > r_{\text{ISCO}}\), where \(r_{\text{ISCO}} = 3M\) is the last circular orbit radius for a test particle in the Schwarzschild space–time.\(^2\) Equation (12) is therefore satisfied only in the range \(r > r_{\text{ISCO}}\), for fixed \(\sigma\). For \(r = r_{\text{ISCO}}\), we have \(G_r = -1/(3M)\). The angular momentum \(L_K\) describes the isobar fluid configurations: where the fluid angular momentum is \(L = L_K\), the pressure \(p\) is constant and Euler equation (3) reduces to \(L^* \nabla (\Phi L^*) = 0\), describing the motion of a pressure-free fluid (dust). The curves \(L = L_K\) represent the critical points of pressure \(p\).

From equation (14) it follows, according to the physics of free (test) particle (and dust defined by \(p = 0\)) (Misner et al. 1973; Pugliese et al. 2011a, 2011b, 2011c), that no critical point exists in the range \([2M, r_{\text{ISCO}}]\), where \(p < 0\) (pressure always decreasing) (Abramowicz et al. 1978; Kozłowski et al. 1978; Paczyński 1980; Abramowicz 2009b; Lei et al. 2009; Kucakova, Slany & Stuchlik 2011).

The angular momentum \(L_K\) is a function of \(r^2\) for fixed \(\sigma\); \(L_K\) tends to infinity as the orbital radius approaches \(r = r_{\text{ISCO}}\), then monotonically decreases until it reaches its minimum value for \(r = r_{\text{ISCO}}\) \((L_K(r_{\text{ISCO}}) = 2\sqrt{3}\sigma^2 M)\) and \(G_r(r_{\text{ISCO}}) = -M^2/(1 - L^2/(12M^2\sigma^2))\) where \(r_{\text{ISCO}} = 6M\) is the last stable circular orbit radius for a test particle in the Schwarzschild geometry. Finally it increases for \(r > r_{\text{ISCO}}\). The angular momentum \(L_K\) is a monotonically increasing function of \(\sigma\), and for \(\sigma = 0\), it is \(L_K = 0\).

### 3.1 Radial pressure gradient \(G_r(L)\) versus angular momentum \(L\)

Fig. 1 (right-hand panel) illustrates the sign of the radial pressure gradient \(G_r\) as a function of the dimensionless angular momentum \(\mathcal{L} \equiv L/(\mathcal{M}\sigma)\) and the distance from the attractor, \(r\). As we noted in the previous section, \(G_r < 0\) (pressure decreasing) in the range \(2M < r \leq r_{\text{ISCO}}\) for every value of the angular momentum. When \(r > r_{\text{ISCO}}\), \(G_r < 0\) (pressure decreasing) for \(0 < L \leq L_K\) while \(G_r > 0\) (pressure increasing) for \(L > L_K\).

We are now interested in finding explicitly the critical points for the pressure, i.e. to find the solutions to equation (12). For a test particle within the effective potential \(V_{\rho c}\), the angular momentum \(L\) is a constant of motion. The particle motion is then described by \(V_{\rho c} = 0\) [the

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\(^2\) The angular proper velocity of the fluid with \(L = L_K\) is \(\Phi_K = L_K/r^2\sigma^2\). This function has no critical point. It is defined in \(r \in [r_{\text{ISCO}}, \infty]\), where it is a monotonically decreasing function of orbital radius \(r/M\), it increases approaching \(r = r_{\text{ISCO}}\), and goes to zero at infinity.

\(^3\) We restrict our analysis to \(\sigma \in (0, 1)\) (the function is even in \(\sigma \in [-1, 1]\)) and \(L_K \geq 0\).
The two definitions of angular constant for \( \sigma M \) (17) increases with \( L \) as the angular momentum \( L = \pm 1 \). For fixed \( \sigma \), \( l = \) is increasing for \( L = -1 \) in \([0, 10]\) and \( l = \) is decreasing for \( 2 \leq r < \) and \( L \). It is also progressively larger when approaching the equatorial plane \( \sigma = 1 \), where it is maximum.

We now face the problem of finding a relation to link \( r, \sigma \) and \( l \) to the condition of the existence of \( L \), and therefore of the velocity of the fluid \( \Phi \). For this purpose we re-define the radii in equation (15):

\[
\begin{align*}
r^+ / M & = \frac{2 \sqrt{2} \cos \left[ \frac{1}{3} \arccos \left( -\frac{\sqrt{3}}{\sqrt{3}} \right) \right]}{\sqrt{3}}, \\
r^- / M & = -\frac{2 \sqrt{2} \sin \left[ \frac{1}{3} - \frac{1}{3} \arccos \left( -\frac{\sqrt{3}}{\sqrt{3}} \right) \right]}{\sqrt{3}}.
\end{align*}
\]

Introducing the dimensionless quantity \( \ell = l/(\sigma M) > 0 \) \( \ell \equiv 1/(\sigma M) \). With this definition, \( r^+ = r^+ = r_o \), for \( \ell = 3 \sqrt{3} \), \( r^- \) approaches the horizon \( r = 2M \) as the angular momentum \( \ell \) increases (see Fig. 3, upper right-hand panel). The angular momentum \( L \) and the velocity \( \Phi \) are not defined inside the region \([r^-, r^+]\) (see Fig. 3, bottom panel and upper left-hand panel). The region \([r^- , r^+]\) increases with \( \ell \). It varies also for different \( \sigma \) : its behaviour as a function of \( \sigma \) is detailed in Appendix A.

**Figure 2.** Left-hand panel: \( L \) on the equatorial plane \( \sigma = 1 \) as a function of \( l/M \in [0, 10] \) and \( r/M \in [2, 10] \). Central panel: \( L(\sigma = 1) \) as a function of \( r/M \), for different \( l \). Asymptotically, as \( r \to \infty \), \( L \to l \). The dot–dashed curve marks \( l = l \) as a function of \( r/M \); the angular momentum \( L \) is defined for \( l < l \). \( r_o \) is a minimum of \( l \). Right-hand panel: \( L(\sigma = 1) = \) constant as a function of \( r/M \) and \( l/M \).
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Figure 3. Upper left-hand panel: \( r^+ \) and \( r^- \) as a function of \( \ell \). \( r^+ = r^- = r_{lco} = 6M \) (dashed line) when \( \ell = 3\sqrt{3/2} \) (dot-dashed line), \( r^+ = r^- = r_{lco} = 3M \) (dashed orange line) when \( \ell = 3\sqrt{3} \) (dotted line). White region corresponds to \( L' > 0 \) and the light-grey region corresponds to \( L' < 0 \). In the grey region the function \( L \) is not defined. Upper right-hand panel: \( r^+ \) (black curve) and \( r^- \) (orange curve) as a function of \( \ell = l/(\sigma M) \). \( r^+ = r^- = 3M \) (dashed line) when \( \ell = 3\sqrt{3} \) (dotted line). \( L \) is not defined in the grey region. Inset panel: \( r^+ \) (black curve) and \( r^- \) (orange curve) as a function of \( \sigma \) for different values of \( l \). Dashed line marks \( r = r_{lco} \), where \( r^+ = r^- \). Lower panel: table summarizing the intervals where \( L' > 0 \) (light grey) or \( L' < 0 \) (grey). Black boxes correspond to the interval \([r_-, r_+]\), where the function \( L \) is not defined.

Now, we investigate the critical points of the angular momentum \( L \) as a function of \( r \), solutions of \( L' = 0 \). For \( r > r_{lco} \), \( L \) is a constant with respect to the orbital radius when \( \ell = \ell_K \), where \( \ell_K/M = \sqrt{\sigma r^3/3M(r-2M)^2} \) is the Keplerian angular momentum of the fluid. The critical points of \( L \) are thus

\[
\begin{align*}
  r_+/M &= \frac{1}{3} \left( \ell^2 - 2\sqrt{\ell^2 (\ell^2 - 12)} \cos \left[ \frac{1}{3} \arccos \left( \frac{\ell^2 (54 - 18\ell^2 + \ell^4)}{(\ell^2 (\ell^2 - 12))^{3/2}} \right) \right] \right), \\
  r_-/M &= \frac{1}{3} \left( \ell^2 + 2\sqrt{\ell^2 (\ell^2 - 12)} \cos \left[ \frac{1}{3} \arccos \left( \frac{\ell^2 (54 - 18\ell^2 + \ell^4)}{(\ell^2 (\ell^2 - 12))^{3/2}} \right) \right] \right),
\end{align*}
\]

where \( r_+ = r_+ \), and \( r_+ = r_{lco} \) when \( \ell = 3\sqrt{3/2} \).

The angular momentum \( L \) is a decreasing function of \( r/M \) in all \( r > r_{lco} \) with \( \ell_K < \ell < \ell_r \), it increases with \( r \) in \( 2M < r \leq r_{lco} \) with \( 0 < \ell < \ell_r \), and in \( r > r_{lco} \) for \( 0 < \ell < \ell_K \) (see Fig. 3).

3.3 Radial pressure gradient \( G_r(l) \) versus fluid angular momentum \( l \)

The radial gradient \( G_r \) as a function of \( \ell = l/(\sigma M) \) can be written as

\[
G_r = -\frac{Mr^3 - \ell^2 M^3 (r - 2M)^2}{r(r-2M)|r^3 - \ell^2 M^3 (r - 2M)|^2}
\]

for \( \sigma \neq 0 \) (see Fig. 1). It is not defined in \( \ell = \ell_c = \sqrt{r^3/M^3 (r - 2M)} \) and \( r = r_{lco}^+ \).

We studied the sign of \( G_r \) as a function of \( L \) in Section 3.1; here we study the problem of \( G_r \) as a function of \( \ell \), the fluid constant of motion. In Appendix A, we will discuss the sign of \( G_r \) in terms of \( (l, \sigma, r) \) explicitly. According to the results found in Section 3.2, \( G_r = 0 \) (isobar surfaces) for radii \( r \) with Keplerian angular momentum \( \ell_K \). Thus, \( G_r < 0 \) in
Figure 4. Upper left-hand panel: \( r_{\pm} \) and \( r_{\pm}^+ \) as a function of \( \ell \). White regions correspond to \( p' > 0 \) and grey regions to \( p' < 0 \). Radii \( r_{\pm} \) are defined for \( \ell > 3 \sqrt{3}/2 \); the gradient \( G_r \) is not defined in \( r_{\pm} \). For \( 3 \sqrt{3}/2 < \ell \leq 3 \sqrt{3}/3 \) the radius \( r_{-} \) is a minimum of the pressure, and \( r_{+} \) is a maximum. For \( \ell > 3 \sqrt{3}/3 \) the radius \( r_{-} \) and \( r_{+} \) are both maximum of the pressure, \( r_{\pm} = r_{lsc0} \) (dashed line) when \( \ell = 3 \sqrt{3}/2 \) (dot–dashed line). Right-hand panel: \( r_{K} \) and \( K_{K} \) as a function of \( r/M \). \( r_{K} = 3 \sqrt{3}/3 \) in \( r = r_{lco} \) (dashed line), where \( G_r = -(1/3)M^{-1} \). The function is not defined in \( \ell = \ell_{K} \), while \( \ell = \ell_{K} \). In the grey regions \( p' < 0 \) and in the white regions \( p' > 0 \). The angular momentum \( \ell_{K} \) has a minimum in \( r_{lco} \) (dashed line). Inset: zoom of the region \( r/M \in [2, 3] \). Right-hand panel: \( \ell_{r} \) (black line) and \( K_{r} \) (orange line) as a function of \( r/M \). \( \ell_{r} = \ell_{K} = 3 \sqrt{3}/2 \) in \( r = r_{lco} \) (dashed line), where \( G_r = 0 \). In the grey regions \( p' < 0 \) and in the white regions \( p' > 0 \). The angular momentum \( \ell_{K} \) has a minimum in \( r_{lco} \) (dashed line). Lower panel: table summarizing the regions of increasing \((p' > 0 – \text{light-grey-shaded regions}) \) and decreasing \((p' < 0 – \text{grey-shaded regions}) \) pressure \( p \).

\[
\begin{array}{|c|c|c|c|}
\hline
 & 0 < \ell < 3 \sqrt{3}/2 & 3 \sqrt{3}/2 < \ell < 3 \sqrt{3}/3 & \ell > 3 \sqrt{3}/3 \\
\hline
2M < r < r_{lco} & p' < 0 & p' < 0 & p' < 0 \\
r_+ < r < r_{lco} & p' < 0 & p' < 0 & p' > 0 \\
r_{lco} < r < r_{+} & p' < 0 & p' < 0 & p' > 0 \\
r = r_{lco} & p' < 0 & p' < 0 & p' > 0 \\
\hline
\end{array}
\]

In terms of radii \( r_{\pm} \) and \( r_{\pm}^+ \), \( G_r > 0 \) in

\[
2M < r < r_{lco} \quad \text{for} \quad 0 \leq \ell < \ell_{r} \quad \text{and} \quad \ell > \ell_{K},
\]

\[
r_{lco} < r < r_{+} \quad \text{for} \quad 0 \leq \ell < \ell_{K} \quad \text{and} \quad \ell > \ell_{r}.
\]

These intervals are portrayed in Fig. 4 (upper panels).

In Section 3.2 we verified that \( L'(l_{K}) = 0 \), where \( L(l_{K}) = L_{K} \) for \( r > r_{lco} \). Here we showed that \( l_{K} \) satisfies the condition \( G_r(l_{K}) = 0 \) and therefore we can claim that \( l_{K} \) is also a critical point for pressure \( p \). This is illustrated in Fig. 4.

4 THE POLAR ANGULAR PRESSURE GRADIENT \( G_{\vartheta} \)

We now concentrate our attention on the polar angular pressure gradient \( G_{\vartheta} \). From equation (10), we have

\[
G_{\vartheta} \equiv \frac{\nabla_{\vartheta} p}{\rho + p}
\]

On the equatorial plane \( \sigma = 1, \nabla_{\vartheta} p = 0 \): the angular gradient of pressure on the equatorial plane is always zero. \( \partial_{\vartheta} p = 0 \) also for \( L = 0 \). This implies that the case \( L = 0 \) leads to a zero polar gradient of \( p \), while \( G_{\vartheta}(L = 0) = -M/[(r - 2M)r] < 0 \), i.e. pressure decreases as \( 1/r^{2} \) when approaching the horizon \( r = 2M \).

For \( L = L_{K} \), \( \partial_{\vartheta} p = 0 \) and therefore \( p \) is a function of \( \vartheta \) only. In general, for \( \sigma \neq 1 \), \( \nabla_{\vartheta} p(L = L_{K}) \neq 0 \). In fact it results in

\[
G_{\vartheta}(L = L_{K}) = \frac{\nabla_{\vartheta} p(L_{K})}{\rho + p} = \frac{M \sqrt{1 - \sigma^2}}{\sigma (r - 3M)}
\]

(25)
On the Polish doughnut accretion disc

Figure 5. Left-hand panel: $G_\sigma(L = L_K)$ as a function of $r/M \in [3, 20]$ for different values of $\sigma$. The black dashed line marks $r = r_{lsc}$. No solutions of $\partial_p \rho = 0$ and no test particle circular orbits exist for $r/M \in (2, 3]$. Right-hand panel: curves $G_\sigma(L = L_K)$ constant as a function of $r/M$ and $\sigma$.

Figure 6. Left-hand panel: $r_L^2$ as a function of $L$. Black line marks $L = 1$. In the light-grey region $\Pi_{r,\vartheta}^L < 1$ and in the white region $\Pi_{r,\vartheta}^L > 1$. Right-hand panel: curves $r_L^2$ as a function of $\ell$. Black line marks the value $\ell = 1$, dashed line marks the value $\ell \approx 3.35$, dot–dashed line marks the value $\ell = 3 \sqrt{3}$ and the dotted line marks the value $r = (1/2)(1 + \sqrt{13}) M$. In the grey region $r_{l}^L < r < r_{+}^L$, the function $\Pi_{r,\vartheta}^L$ is not defined. In the grey region, $\Pi_{r,\vartheta}^L < 1$, and in the white region, $\Pi_{r,\vartheta}^L > 1$. In $r = r_{l}^L$ (black line) it is $\Pi_{r,\vartheta}^L = 1$.

(see Fig. 5). The quantity in equation (25) is not defined in $r = r_{lco}$ and, as for $L_K$, in the region $r \in (2M, r_{lco}]$. This means that there is a fluid configuration with pressure constant along the orbital radius and a pressure variable from plane to plane extended at radial distance $r > r_{lco}$. $G_\sigma(L = L_K)$ admits critical points for $\sigma = 1$.

4.1 Pressure gradient ratio $\Pi_{r,\vartheta}$ versus angular momentum $L$

We define the pressure gradient ratio $\Pi_{r,\vartheta}(L, r, \sigma)$ as $\Pi_{r,\vartheta} = \nabla_{r,p}/\nabla_{\vartheta} \rho$. It is clearly $\Pi_{r,\vartheta} = 0$ for $L = L_K$ and it is not defined in $\sigma = 1$.

We rewrite this function as

$$\Pi_{r,\vartheta}^L \equiv [\cot \vartheta \Pi_{r,\vartheta} M] = \frac{|r^2 - ML^2(r - 3M)|}{L^2 r(r - 2M)}.$$  

This ratio is equal to 1 for

$$L_{\Pi} \equiv \sqrt{\frac{r^2}{r(r - M) - 3M^2}}, \quad r_{l}^L/M = \frac{1}{2} \left[ \frac{L^2}{L^2 - 1} + \sqrt{\frac{L^2 (13L^2 - 12)}{(L^2 - 1)^2}} \right].$$  

The ranges where $\Pi_{r,\vartheta}^L$ is larger than, smaller than or equal to 1 are summarized in Fig. 6 (left-hand panel).

$^4$ $G_\sigma(L = L_K)$ is written as an odd symmetric function of $\sigma$, but we are in an even symmetric theory in $\sigma$; as we have not explicitly given any further constraints to the unknown functions ($\rho$, $p$), we implicitly take $\sigma > 0$ and $\sqrt{1 - \sigma^2} \geq 0$ or $\vartheta \in (0, \pi/2)$. 

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4.2 Analysis of the pressure gradient ratio $\Pi_{r,\vartheta}$ versus angular momentum $l$

We consider now the ratio $\Pi_{r,\vartheta}$ as a function of $l$. We then define the function $\Pi^l_{r,\vartheta} = \Pi^l_{r,\vartheta}(r, \ell)$:

$$\Pi^l_{r,\vartheta} \equiv |\cot \vartheta M_{l,\vartheta}| = \frac{|r^3 - \ell^2 M(r - 2M)^2|}{\ell^2 r(r - 2M)^2}$$

(28)

in the range of existence of $L(l)$. Then we exclude the range $r_l^- \leq r \leq r_l^+$. The solution of $\Pi^l_{r,\vartheta} = 1$ are $r = r_l^+ > M(1 + \sqrt{3})/2$ and $\ell = \ell_{11} \in (1, \ell_{11-})$, where $\ell_{11-} = \sqrt{(41 + 11\sqrt{3})}/2 = 6.3506$ satisfies the condition $r_l^+(\ell_{11-}) = r_l^-(\ell_{11-}) \approx 2.30278 M$ (see Fig. 6, right-hand panel), with

$$\ell_{11} \equiv \sqrt{\frac{r^3}{(r + M)(r - 2M)^2}}$$

(29)

and

$$r_l^+/M = \frac{\ell^2}{\ell^2 - 1} + 2\ell^2 \sqrt\frac{\frac{1}{(\ell^2 - 1)^3}}{\cos \left(\frac{1}{3} \text{arccos}\left(\frac{1}{\ell^2}\sqrt{\frac{1}{(\ell^2 - 1)^3}}\right)\right)} \left(1 - \ell^2\right)\left(\ell^4 - 4\ell^2 - 2\right).$$

(30)

The ranges where $\Pi^l_{r,\vartheta}$ is larger than, smaller than or equal to 1 are summarized in Fig. 6 (right-hand panel).

5 THE BOYER POTENTIAL

The disc fluid configuration in the Polish doughnut model has been widely studied by many authors (see e.g. Frank et al. 2002; Abramowicz & Fragile 2011). In particular, an analytic theory of equilibrium configurations of rotating perfect fluid bodies was initially developed by Boyer (1965). The ‘Boyer’s condition’ states that the boundary of any stationary, barotropic, perfect fluid body has to be an equipotential surface $W(l, \vartheta)$ = constant. For a barotropic fluid the surfaces of constant pressure are given by the equipotential surfaces of the potential defined by the relation

$$\int_{r_{in}}^{r_{out}} \frac{dp}{\rho + p} = -(W - W_{in}),$$

(31)

where the subscript ‘in’ refers to the inner edge of the disc. It is important to note here that, in the Newtonian limit, the quantity $W$ is equal to the total potential, i.e. to the sum of the gravitational and centrifugal effects.

As mentioned in Section 3.2, all the main features of the equipotential surfaces for a generic rotation law $\Omega = \Omega(l)$ are described by the equipotential surface of the simplest configuration with uniform distribution of the angular momentum density $l$, which are very important being marginally stable (Seguin 1975). At the same time, the equipotential surfaces of the marginally stable configurations orbiting in a Schwarzschild space–time are defined by the constant $l$. It is therefore important to study the potential $W = W(l, \sigma)$ and to compare it with $W = W(L, \sigma)$.

We can classify the equipotential surfaces in three classes: closed, open and with a cusp (self-crossing surfaces, which can be either closed or open). The closed equipotential surfaces determine stationary equilibrium configurations: a fluid can fill any closed surface. The open equipotential surfaces are important to model some dynamical situations, for example the formation of jets (see e.g. Abramowicz, Karas & Lanza 1998; Stuchlík 2000; Stuchlík, Slaný & Hledík 2000; Rezzolla, Zanotti & Font 2003; Slaný & Stuchlík 2005; Stuchlík & Slaný 2006; Stuchlík & Kovář 2008; Abramowicz 2009b; Lei et al. 2009; Stuchlík et al. 2009; Kucakova et al. 2011).

The critical, self-crossing and closed equipotential surfaces are relevant in the theory of thick accretion discs since the accretion on to the black hole can occur through the cusp of the equipotential surface. According to Paczyński (Abramowicz et al. 1978; Kozłowski et al. 1978; Jaroszynski et al. 1980; Abramowicz 1981), the accretion on to the source (black hole) is driven through the vicinity of the cusp due to a little overcoming of the critical equipotential surface $W_{cusp}$ by the surface of the disc. The accretion is thus driven by a violation of the hydrostatic equilibrium, clearly ruling out viscosity as a basis for accretion (Kozłowski et al. 1978). In the Paczyński mechanism the disc surface exceeds the critical equipotential surface $W_{cusp}$ giving rise to a mechanical non-equilibrium process that allows the matter to flow into the black hole. In this accretion model the cusp of this equipotential surface corresponds to the inner edge of the disc.

We calculate now the Boyer potential for our system integrating equation (11):

$$\int_{r_{in}}^{r_{out}} \frac{dp}{\rho + p} = -\int_{r_{out}}^{r_{in}} e^{-\nu} \left(\frac{\partial V^2_{\vartheta}}{\partial r}\right) L.$$
\[ W = \ln \left[ \frac{(r - 2M)^2}{r^3 - \ell^2 M^2 (r - 2M)} \right] = \ln V_{sc}. \]  

(34)

In particular, it results in

\[ \lim_{r \to \infty} W = 0, \quad \lim_{r \to 2M} W = +\infty \]

(35)

(see also Abramowicz et al. 1980; Stuchlík et al. 2000; Rezzolla et al. 2003; Abramowicz 2005; Slaný & Stuchlík 2005; Stuchlík & Slaný 2006; Stuchlík & Kovář 2008). Clearly, \( W = 0 \) where \( V_{sc} = 1 \), and \( W \gtrsim 0 \) where \( V_{sc} \gtrsim 1 \). The maximum and minimum points of the function \( W \) are the same as the Schwarzschild effective potential, \( V_{sc} = V_{sc}(l, r) \). In particular, we are interested to study the equipotential surfaces, defined by the condition \( W = \text{constant} \), that coincide with the surfaces \( V_{sc} = K > 0 \), where \( K \equiv e^c \) is the energy of a test particle circularly orbiting around the source. In fact the surfaces of constant Boyer potential determine the shape of the torus (disc). We study these surfaces as a function of \( L = \text{constant} \) and \( l = \text{constant} \), respectively.

5.1 Analysis of the Boyer potential versus the angular momentum \( L \)

We consider the Boyer potential \( W(L, r) \) in equation (34) as a function of the angular momentum \( L \). The condition \( W = 0 \) is satisfied in

\[ \tilde{r}_W^\pm \equiv \frac{M}{4} \left( L^2 \pm \sqrt{L^2 (L^2 - 16)} \right) \]

(36)

(see Fig. 7, right-hand panel).

We note that two relevant cases occur when the angular momentum is \( L^2 = L^2_k \), where \( L^2_k \equiv \frac{2M}{1 - K^2} \);

(i) when \( 0 < K < 1, W < 0 \), in \( 2M < r < \tilde{r}_k \), where \( \tilde{r}_k \equiv \frac{2M}{1 - K^2} \); (ii) when \( K \geq 1, W \geq 0 \) in \( r > 2M \).

The solutions of the equation \( W = \ln(K) \) can be described in terms of the energies as

\[ \tilde{K}_+ \equiv \sqrt{\frac{1}{54} \left( 36 \pm \sqrt{\frac{(L^2 - 12)^2}{L^2} + L^2} \right)}, \quad \tilde{K}_- \equiv \frac{1}{\sqrt{3}} \sqrt{3 \frac{L^2 - 4}{L}}, \]

(37)

in terms of the radii as

\[ \tilde{r}_{k_+/L}^+ \equiv -\frac{2 + 2\pi \sin \left[ \frac{1}{3} \left( \pi + 2 \arccos \psi \right) \right]}{3 (K^2 - 1)}, \]

(38)

\[ \tilde{r}_{k_-/L}^- \equiv -\frac{2 + 2\pi \cos \left[ \frac{1}{3} \left( \pi + \arccos \psi \right) \right]}{3 (K^2 - 1)}, \]

(39)

\[ \tilde{r}_{k_+/L}^- \equiv -\frac{2 + 2\pi \cos \left( \frac{1}{3} \arccos \psi \right)}{3 (K^2 - 1)}, \]

(40)

(see Fig. 8), where

Figure 7. Left-hand panel: \( r_W^+ \) (orange curve) and \( r_W^- \) (black curve) as a function of \( \ell \). The black region \( r_W^- < r < r_W^+ \) is forbidden. \( V_{sc} > 1 \) in the white region and \( V_{sc} < 1 \) in the grey region. Dotted line marks \( \ell = 4 \). Right-hand panel: \( \tilde{r}_W^+ \) (orange curve) and \( \tilde{r}_W^- \) (black curve) as a function of \( L \). \( V_{sc} > 1 \) in the white region and \( V_{sc} < 1 \) in the grey region. Dotted line marks \( L = 4 \).
Figure 8. $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_3$ as a function of $L^2$ for different values of $K$ in the ranges $[0, 2\sqrt{2}/3]$ (left-hand panel), $[2\sqrt{2}/3, 1]$ (central panel) and $K > 1$ (right-hand panel). Dotted lines are the momenta $L^2_\alpha$ and $(L^2_\pm)^2$.

Figure 9. Upper left-hand panel: $\tilde{K}^+$ (black curve), $\tilde{K}^-$ (orange curve) and $\tilde{K}_\alpha$ (dashed thick curve) as a function of $L^2$. The regions of existence for radii $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_3$, solutions of $W = \ln (K)$, are marked in white, grey and light grey, respectively. Black regions are forbidden. Dot–dashed line marks $K = 2\sqrt{2}/3$.

Upper right-hand panel: $\tilde{r}_K$ as a function of $K$. Grey curve is $\tilde{r}_1$, dashed curve is $\tilde{r}_2$. In the white region it is $L^2 = L^2_\alpha$ solution of $W = \ln (K)$. No solution exists in the black region. Lower panel: table summarizing the regions of existence of the radii $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_3$. Black boxes are forbidden.

$$
\psi \equiv - \frac{8 + 9 (3K^4 - 5K^2 + 2) L^2}{(K^2 - 1)^3}, \quad \sigma \equiv (K^2 - 1) \sqrt{\frac{4 + 3 (K^2 - 1) L^2}{(K^2 - 1)^2}}; \quad (41)
$$

and in terms of the angular momenta as

$$
L^2_\alpha = \frac{4}{3(K^2 - 1)}, \quad (L^2_\pm)^2 = \frac{1}{2} \left[ \frac{27K^4 - 36K^2 + 8}{K^2 - 1} \pm \frac{K^2 (9K^2 - 8)}{(K^2 - 1)^2} \right]. \quad (42)
$$

These solutions are summarized in Fig. 9.

The critical points of the angular momentum $L^2_\pm$ are

$$
\frac{r_\pm}{M} = \frac{1}{2} \left[ \frac{K^2 (9K^2 - 8)}{(K^2 - 1)^2} \pm \frac{3K^2 - 4}{K^2 - 1} \right]. \quad (43)
$$
When $2\sqrt{3}/3 < K < 1$ there are two critical points $\tilde{r}_1^\pm$, while when $K > 1$ there is only one, $r = \tilde{r}_1^+$ (see Fig. 9, upper right-hand panel).

For $K = 1$ it is $\tilde{r}_1^\pm = r_{\text{mbo}}$, where $r_{\text{mbo}} = 4M$ is the marginally bounded orbit for a test particle in the Schwarzschild space–time, and in $K = 2\sqrt{3}/3$ it is $\tilde{r}_1^\pm = r_{\text{isc}}$.

**Closed surfaces.** The conditions for the existence of closed surfaces can be obtained by noting that, in Cartesian coordinate $(x, y)$, the closed surfaces should satisfy the condition $V_c(x = 0) = K$ with three solutions, say $y = \{y_1, y_2, y_3\}$. The closed surfaces then exist when $2\sqrt{3}/3 < K < 1$ and $(\mathcal{L}_k^0)^2 < L^2 < (\mathcal{L}_k^1)^2$ (see Fig. 9, lower panel), with $(\mathcal{L}_k^1)^2 \geq 16$. In Cartesian coordinate $(x, y)$ the surfaces $V_c = K$ are

$$x = \pm \sqrt{\frac{(2L^2 + 2y^2)^2}{(L^2 + y^2 - y^2K^2)^2}} - y^2.$$  

(44)

The maximum diameter $(x = 0)$ of the closed Boyer surface lies between the points $y = y_2$ and $y = y_3$, where

$$y_1/M \equiv \frac{1}{9} \left( \zeta - 6\theta \sin \left[ \frac{1}{3} (\pi + 2 \arccos \varepsilon) \right] \right),$$

(45)

$$y_2/M \equiv \frac{1}{9} \left( \zeta - 6\theta \cos \left[ \frac{1}{3} (\pi + \arccos \varepsilon) \right] \right),$$

(46)

$$y_3/M \equiv \frac{1}{9} \left[ \zeta + 6\theta \cos \left( \frac{1}{3} \arccos \varepsilon \right) \right],$$

(47)

with

$$\theta \equiv \sqrt{\frac{16 + 8 (3K^4 - 4K^2 + 1) (L/M)^2 + (K^2 - 1)^2 (L/M)^4}{(K^2 - 1)^2}},$$

(48)

$$\varepsilon \equiv \frac{64 + 48 (3K^4 - 4K^2 + 1) (L/M)^2 + 6 (K^2 - 1)^2 (9K^4 - 6K^2 + 2) (L/M)^4 - (K^2 - 1)^3 (L/M)^6}{(K^2 - 1)^6 \theta^3},$$

(49)

$$\zeta \equiv \frac{6 [2 + (K^2 - 1) (L/M)^2]}{(K^2 - 1)^2}$$

(50)

(see Fig. 10, left-hand panel).

Fig. 10 (right-hand panel) portrays a closed Boyer surface. We can characterize this surface introducing the following parameters:

(i) the surface maximum diameter: $\lambda \equiv y_3 - y_2$,

(ii) the distance from the source: $\delta \equiv y_2 - 2M$,

(iii) the distance from the inner surface: $\hat{\delta} \equiv y_2 - y_1$,

(iv) the distance of the inner surface from the horizon: $\hat{\delta} \equiv y_1 - 2M$,

**Figure 10.** Left-hand panel: $y_1$ (black curve), $y_2$ (orange curve) and $y_3$ (dashed curve) as a function of $L^2$ for $K = 0.96$. Right-hand panel: the closed Boyer surface at $K = 0.98$ and $L^2 = 16M^2$. $\lambda \equiv y_3 - y_2$ is the surface maximum diameter, $\delta \equiv y_2 - 2M$ is the distance from the source, $\hat{\delta} \equiv y_2 - y_1$ is the distance from the inner surface, $\hat{\delta} \equiv y_1 - 2M$ is the distance of the inner surface from the horizon, $h \equiv 2x_M$ and $X_0 \equiv y_M$. 

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for different energies. $K$ varies in the range $2 < K < L$, and decreases with $K$. Upper right-hand panel: surface maximum diameter $L$ as a function of $K$ for different values of $K$. Lower left-hand panel: maximum $x_M = x(y_M)$ as a function of $L^2$ for different energies $K$. Inset: $y_M$ as a function of $L^2$, for different values of $K$. Lower right-hand panel: distance $\delta = y_1 - 2M$ as a function of $L^2$, for different values of $K$. Inset: distance $\delta = y_2 - y_1$ as a function of $L^2$, for different values of $K$.

\begin{enumerate}
\item[(v)] the surface maximum height: $h = 2x_M.$
\item[(vi)] the quantity $X_b \equiv y_M.$
\end{enumerate}

$(x_M, y_M)$ is the critical point of the surface in equation (44). In what follows we find the constraints for the set of parameters $\{\lambda, \delta, \delta, h, X_b\}$. The point $y_1$ varies in the range $2M < y_1 < x_3, and L^2$ lies on the surface $L^2$, which is a solution of $V_s = K$ on the plane $x = 0$ for $y > 4M$ and $K \geq 1,$ and for $4M < y < y_1,$ and $K < 1$:

\begin{equation}
y_1 = \frac{2M}{[K^2 - 1]}, \quad L^2 = 2M \sqrt{\frac{K^2 y^6}{(y^2 - 4M^2)^2} + y^2 \left( \frac{K^2 y^2}{y^2 - 4M^2} - 1 \right)} \tag{51}
\end{equation}

(see Fig. 11). $\lambda$ increases with energy $K$, but decreases with the fluid angular momentum $L$. In contrast, the distance from the source $\delta = y_2 - 2M$ increases with $L$ and decreases with $K$ (see Fig. 11).

The maximum vertical distance of the closed surfaces is

\begin{equation}
y_M/M = \sqrt{\frac{3(L/M)^2 + 4\sqrt{6} (K^2 - 1) \sqrt{\frac{K^2 (L/M)^2}{(K^2 - 1)^2}} \cos \left[ \frac{1}{2} \sqrt{\frac{1}{2} (K^2 - 1) \sqrt{\frac{K^2 (L/M)^2}{(K^2 - 1)^2}}} \right]}{3 (K^2 - 1)}} \tag{52}
\end{equation}

As shown in Fig. 11, the maximum $x_M = x(y_M)$, and consequently the height $h$, increases with $K$ and decreases with $L^2$. In contrast, $y_M$ increases with both $K$ and $L^2$, until it reaches a maximum and then decreases. The distance $\delta = y_2 - y_1$ increases with $L^2$ and decreases with $K$. The distance $\delta \equiv y_1 - 2M$ increases with energy $K$ and decreases with $L^2$.

\textbf{Cusp}. The cusps, i.e. the self-crossing surfaces $W = constant$, correspond to the maxima of the potential as a function of $r$: open surfaces are maxima with energy $K > 1$, and closed surfaces are maxima with energy $K < 1$, as outlined in Fig. 12 (upper panels). It is therefore important to consider the function $V$ in the regions of closed and open Boyer surfaces. From Section 3.1 we know that the solutions of the equations $V = 0$, which correspond to maxima of the effective potential, are located on the radius $r = r^c_1$; in particular, there are closed surfaces with cusps in $r^c_1 > r_{nbo}$ with $12 \leq L^2 < 16$, and maxima located in $r^c_1 > r_{nbo}$ with $L^2 \geq 16$ are open surfaces with a cusp (see Fig. 12, upper panels).
Remark. Integrating equation (32) with $V_{sc}$ as a function of the constant of motion $L$, we obtain the following expression for the potential:

$$\tilde{W} = \frac{2L^4 M(r + 3M) + (4 + L^2) r^2 \ln(r/M - 2) - \ln(r/M)}{8r^2}. \quad (53)$$

The solutions of $\tilde{W} = c$ ensure the existence of closed, open and self-crossing surfaces.

5.2 Analysis of the Boyer potential versus the angular momentum $l$

Now we analyse the Boyer potential in terms of the fluid angular momentum $l = \text{constant}$. First, we note that the Boyer surface in equation (34) is not defined in the region $[r^-_L, r^+_L]$. We detail the study of the sign of $V_{sc}(l, r)$ in Fig. 13 (see also Fig. 7, left-hand panel). With respect to
Figure 13. Upper panel: $K_a$ (black curve), $K_b$ (orange curve), $K^+\alpha$ (dashed thick black curve) and $K^-\alpha$ (dashed thick orange curve) as a function of $\ell^2$. The coloured regions mark the existence for radii $r_{k_1}$, $r_{k_2}$ and $r_{k_3}$. The black region is forbidden. Dotted lines are $\ell^2 = 16$ and $\ell^2 = 27/2$. Lower panel: table summarizing the regions of existence for radii $r_{k_1}$, $r_{k_2}$ and $r_{k_3}$. Black boxes are forbidden.

The case of $L = \text{constant}$ (see Fig. 9 and equations 37–42), here we introduce new definitions for the energies as

$$K_a = \sqrt{\frac{\ell^2 - 36 - 2\beta_1 \sin \left(\frac{1}{3} \arcsin \alpha_1\right)}{3\ell^2 - 81}},$$

$$K_b = \sqrt{\frac{\ell^2 - 36 + 2\beta_1 \cos \left(\frac{1}{3} \arccos \alpha_1\right)}{3\ell^2 - 81}},$$

$$K^{\pm}_\alpha = \frac{1}{\sqrt{6}} \sqrt{\frac{3 \pm \sqrt{3} \sqrt{\ell^2(3\ell^2 - 16)}}{\ell^2}},$$

with

$$\alpha_1 = \left(2^3 \sqrt{3} - \left(108 \sqrt{3}\right) \ell^2 + \left(6 \sqrt{51}\right) \ell^2 - 72 \ell^3 + \ell^4\right) \ell^2 \beta_1^3,$$

$$\beta_1 = (\ell^2 - 27) \sqrt{\frac{72 + (\ell^2 - 24)^2}{(\ell^2 - 27)^2}};$$

new definitions for angular momenta as

$$\ell_a^2 = -\frac{4}{3K^2(K^2 - 1)}, \quad (\ell_a^2)^3 = \frac{1}{2} \left( \pm \sqrt{\frac{(9K^2 - 8)^3}{K^2(K^2 - 1)^3}} + \frac{27K^4 - 36K^2 + 8}{K^2(K^2 - 1)} \right);$$

and new definitions for radii as

$$r_{k_1}/M = -\frac{2 \left[1 + \sigma_k \cos \left(\frac{1}{3} \arccos \psi_k\right)\right]}{3(K^2 - 1)},$$

$$r_{k_2}/M = \frac{2 \left[-1 + \sigma_k \cos \left(\frac{1}{3} \arccos (-\psi_k)\right)\right]}{3(K^2 - 1)}.$$
where

\[ \psi_k = \frac{8 + 9K^2 (3K^2 - 5K^2 + 2) \ell^2}{\sigma_k^2}, \quad \sigma_k = (K^2 - 1) \sqrt{\frac{4 + 3K^2 (K^2 - 1) \ell^2}{(K^2 - 1)^2}}. \]  

The surfaces \( W(l, r) = \text{constant} \) exist in all the space–time with angular momentum \( \ell^2 = \ell_k^2 \) and energy \( K = K_s \), where

\[ \ell_k^2 = \frac{r^2 [2M + (K^2 - 1) r]}{M^2(r - 2M)}, \quad K_s = r \sqrt{\frac{r - 2M}{r^3 - \ell^2 M^2(r - 2M)}}. \]  

In particular, in the case of \( 0 < K < 1 \), we have \( 2M < r \leq 2M/(1 - K^2) \).

**Closed surfaces.** Following the same procedure outlined in the previous subsection for \( V(L, r) \), we find that closed surfaces of the Boyer potential in the Cartesian coordinate \((x, y)\) are in the regions \( 2\sqrt{2}/3 < K < 1 \) and \( (\ell_k^2)^2 < \ell^2 < (\ell_k^2)^2 \), where \( \ell_k^2 > 27/2 \) (see Fig. 13). The surfaces are therefore

\[ x = \pm \left[ \frac{2M^2(K^2 - 1) + 2r^2}{K^2(P^2 - y^2) + y^2} \right]^{1/2} - y^2 \]  

(see Fig. 14, right-hand panel).

The maximum diameter \((x = 0)\) is defined by the points \( y = y_2 \) and \( y = y_3 \), where

\[ y_1/M = \sqrt{\frac{1}{9} \left[ \zeta_k - 6\theta_k \sin \left( \frac{1}{6} (\pi + 2 \arccos \epsilon_k) \right) \right]}, \]

\[ y_2/M = \sqrt{\frac{1}{9} \left[ \zeta_k - 6\theta_k \cos \left( \frac{1}{3} (\pi + 2 \arccos \epsilon_k) \right) \right]}, \]

\[ y_3/M = \sqrt{\frac{1}{9} \left[ \zeta_k + 6\theta_k \cos \left( \frac{1}{3} \arccos \epsilon_k \right) \right]} \]  

(see also equations 45–47), with

\[ \theta_k = \sqrt{\frac{16 + 8K^2(l/M)^2 + K^4(l/M)^2 - 2K^4(l/M)^2 - 12(l/M)^2 + K^6(l/M)^4}{(K^2 - 1)^4}}, \]

\[ \epsilon_k = \sqrt{\frac{64 + K^2(l/M)^2 \left( 48 + K^2 \left( 48 (3K^2 - 4) + 6 (K^2 - 1)^2 (9K^4 - 6K^2 + 2) (l/M)^2 - K^2 (K^2 - 1)^3 (l/M)^4) \right) \right)}{(K^2 - 1)^6 \ell_k^2}. \]
Figure 15. Upper left-hand panel: comparison of the closed Boyer surfaces with energy \( K = 0.98 \) and angular momentum \( l^2 = 16M^2 \) (black curves) and \( L^2 = 16M^2 \) (orange curve). It results in \( x(l^2 = 16M^2) > x(L^2 = 16M^2) \). Upper right-hand panel: comparison of the surface maximum point \( y_M/M \). Inset: comparison of the surface maximum diameter \( \lambda \). Lower left-hand panel: comparison of the distance of the inner surface from the horizon \( \bar{\delta} \equiv y_1 - 2M \). Inset: comparison of the distance from the inner surface \( \hat{\delta} \). Lower right-hand panel: comparison of the surface distance from the horizon \( \delta \). Inset: comparison of the distance from the inner surface \( \hat{\delta} \).

\[
\varsigma \equiv 6 \left[ K^4(l/M)^2 - K^2(l/M)^2 + 2 \right] / (K^2 - 1)^2
\]
(see Fig. 14, left-hand panel).

The maximum height for the surface is

\[
y_M/M \equiv \sqrt{\frac{3K^2(l/M)^2 + 4 \sqrt{6}(K^2 - 1) \sqrt{-\frac{K^4(l/M)^2}{(K^2 - 1)^2}} \cos \frac{1}{2} \arccos \left[ -\frac{1}{2} \sqrt{3} (K^2 - 1)^2 \sqrt{-\frac{K^4(l/M)^2}{(K^2 - 1)^2}} \right]}{3(K^2 - 1)}}.
\]

The surfaces for \( l = \text{constant} \) are larger than those for \( L = \text{constant} \). The two cases are compared in Fig. 15: the maximum diameter \( \lambda(l) > \lambda(L) \); the distance from the source \( \delta(l) > \delta(L) \); the distance of the inner surface from the horizon \( \bar{\delta}(l) > \bar{\delta}(L) \); its maximum height \( h(l) > h(L) \) and finally the quantity \( x_M(l) > x_M(L) \); and the distance from the inner surface \( \hat{\delta}(l) < \hat{\delta}(L) \).

**Cusps.** The closed (open) self-crossing surfaces \( W = \text{constant} \) are located on the maxima of the effective potential with energy \( K < 1 \) \((K \geq 1)\) as a function of \( r \) (see Fig. 12, lower panels). The critical points are located on \( r_c \): the maximum is \( x_c(r_c) \) when \( 3 \sqrt{3/2} < l < 3 \sqrt{3} \). The energy \( V_{sc}(r_c) \) at the maximum \( r_c \) and at \( l = 4 \) is \( V_{sc}(r_c) = 1 \), and for \( l < 4 \) it is \( V_{sc}(r_c) < 1 \).

### 6 The Polytropic Equation of State

We consider the particular case of a polytropic equation of state \( p(r) = k \rho(r)^\gamma \), where the constant \( \gamma \) is the polytropic index and \( k > 0 \) is a constant. Using this relation in equation (31) we have

\[
k \gamma \int_{r_{\text{min}}}^{r_{\text{out}}} \frac{\rho^{\gamma - 1} dr}{\rho(1 + k \rho^\gamma)} = -[W(r_{\text{out}}) - W(r_{\text{in}})].
\]
Integrating equation (72), we obtain
\[
\ln \left[ \left( \rho \left( k + \rho^{1-\gamma} \right)^{1-\gamma} \right)^{\gamma} \right] = -W, \quad \text{for} \quad \gamma \neq 1
\] (73)
and
\[
\ln \left[ \left( 1 + k \rho \right)^{1-\gamma} \right] = -W, \quad \text{for} \quad \gamma = 1
\] (74)
(isothermal case).

Solving equations (73) and (74) for \( \rho \) and using equation (34), we find, respectively,
\[
\tilde{\rho}_\gamma \equiv \left[ \frac{1}{k} \left( V_{\infty}^{1-\gamma} - 1 \right) \right]^{\frac{1-\gamma}{\gamma}}, \quad \text{for} \quad \gamma \neq 1
\] (75)
and
\[
\rho_k \equiv V_{\infty}^{1-\gamma} \frac{1}{1+k}, \quad \text{for} \quad \gamma = 1.
\] (76)

In the following we adopt the normalization \( \rho_\gamma \equiv k^{1/(\gamma-1)} \tilde{\rho}_\gamma \), which is independent of \( k \). The following limits are satisfied:
\[
\lim_{r \to \infty} \rho_\gamma = 0, \quad \lim_{r \to 2M} \rho_\gamma = \infty
\] (77)
and
\[
\lim_{r \to \infty} \rho_k = 1, \quad \lim_{r \to 2M} \rho_k = \infty.
\] (78)

We underline that for \( \gamma = 1 \), we have \( \rho_{\gamma_{\text{out}}} = k \tilde{\rho}_{\gamma_{\text{out}}} \left( V_{\infty}^{\text{out}} / V_{\infty}^{\text{in}} \right)^{1-\gamma} \), and for \( \gamma \neq 1 \), we have \( k \tilde{\rho}_{\gamma_{\text{out}}} = (k \tilde{\rho}_{\gamma_{\text{in}}})^{1-\gamma} \left( V_{\infty}^{\text{out}} / V_{\infty}^{\text{in}} \right) \). In the polytropic case, \( \rho' = p' / (k \rho^{\gamma-1}) \), thus \( \rho' = 0 \) when \( p' = 0 \) and, as \( \gamma > 0 \), the maxima (minima) of \( p \) correspond to maxima (minima) of \( \rho \).

6.1 The case \( \gamma \neq 1 \)

If the polytropic index \( \gamma \neq 1 \), the density \( \rho = \rho_\gamma \) is
\[
\rho_\gamma \equiv C^{1/(1-\gamma)},
\] (79)
with \( C \equiv \left( V_{\infty}^{\gamma-2} \right)^{1-\gamma} - 1 \) and \( (V_{\infty}^{\gamma-2}) \equiv \left( \frac{r_{\text{mbo}}^\gamma}{r_{\text{mbo}}^\gamma} - \frac{r_{\text{mbo}}^\gamma}{r_{\text{mbo}}^\gamma} \right) > 0 \).

We distinguish between two cases:

(i) \( C > 0 \) and the density \( \rho_\gamma \) is defined for all \( \gamma \);

(ii) \( C < 0 \) and the density \( \rho_\gamma \) is defined for \( \gamma = \gamma_q \equiv 1 + \frac{1}{2q} \), where \( |q| \geq 1 \) are integers (see Fig. 16, left-hand panel).

Figure 16. Left-hand panel: \( \gamma_q \) as a function of \( q \), where \( |q| \geq 1 \) are integers. For large values of \( |q| \), the index \( \gamma_q \) tends asymptotically to \( \gamma_q = 1 \). Grey region is forbidden. The maximum value of \( \gamma_q \) is \( \gamma_q = 3/2 \), the minimum is \( \gamma_q = 1/2 \). \( C < 0 \) for \( V_{\infty}^{\gamma} < 1 \) with \( \gamma_q < 1 \) in \( R_1 \) and \( V_{\infty}^{\gamma} > 1 \) with \( \gamma_q > 1 \) in \( R_2 \). Right-hand panel: \( r_{\text{in}} \) (black line), \( r_{\text{out}} \) (orange line), \( r_{\text{in}}^{\text{thick}} \) (thick black line) and \( r_{\text{out}}^{\text{thick}} \) (thick orange line) as a function of \( \ell \). Dashed line marks \( \ell = 4 \), dotted line marks \( \ell = 3 \), dot-dashed line marks \( \ell = r_{\text{mbo}} \), thick-dashed line marks \( r = r_{\text{ko}} \). The white region corresponds to the range \( R_1 \) and the grey region corresponds to \( R_2 \).
The condition $C > 0$ is satisfied in two cases: (i) where $V^2_{\text{sc}} < 1$ and $\gamma > 1$, in the ranges
\begin{equation}
R_1 \equiv \begin{cases} 
0 < \ell < 4 & \text{in } r > 2M \\
\ell \geq 4 & \text{in } 2M < r < r_{W}^+ \\
r^+ & \text{in } r > r_{W}^+
\end{cases}
\end{equation}
and (ii) where $V^2_{\text{sc}} > 1$ and $0 < \gamma < 1$, in the ranges
\begin{equation}
R_2 \equiv \begin{cases} 
4 < \ell < 3\sqrt{3} & \text{in } r_{W}^- < r < r_{W}^+ \\
\ell = 3\sqrt{3} & \text{in } r_{W}^- < r < r_{W}^+ \\
\ell > 3\sqrt{3} & \text{in } r_{W}^- < r < r_{W}^+
\end{cases}
\end{equation}
with
\begin{equation}
\frac{r_{W}^+}{M} = \frac{\ell^2 \pm \ell}{4\sqrt{\ell^2 - 16}}
\end{equation}
(see Fig. 16, right-hand panel).

We can summarize as follows: when the polytropic index $\gamma = \gamma_q$, the fluid density $\rho$ is defined for the conditions $R_1 \cup R_2$, when $\gamma \neq \gamma_q$ it is defined only for the conditions $R_1$. In the following subsections we will discuss an example of $\gamma = 5/3 \neq \gamma_q$ and the particular case $\gamma_q(q = 1) = 3/2$.

### 6.1.1 The adiabatic case: $\gamma = 5/3$

We consider now the particular case $\gamma = 5/3$. This polytropic index is adopted to describe a large variety of matter models, as the generic degenerate matter like star cores of white dwarfs (see e.g. Horedt 2004).

The density $\rho_{5/3}$ is then
\begin{equation}
\rho_{5/3} = \left(\frac{r}{r - 2M} - \frac{M^2 \ell^3}{r^5} \right)^{1/5} - 1 \right)^{3/2},
\end{equation}
defined in the range $R_1$ as in equation (80):
\begin{equation}
0 < \ell < 4 \quad \text{in } r > 2M,
\end{equation}
\begin{equation}
\ell \geq 4 \quad \text{in } 2M < r < r_{W}^+ \\
\ell \geq 4 \quad \text{in } r > r_{W}^+,
\end{equation}
where $\lim_{\ell \to 2} \rho_{5/3} = 0$ (see Fig. 17).

The critical points of $\rho_{5/3}$ can be found as solutions of $\rho'_{5/3} = 0$:
\begin{equation}
\ell = 3\sqrt{3/2} \quad \text{in } r = r_{\text{sc}}, \quad 3\sqrt{3/2} < \ell < 4 \quad \text{in } r = r_e, \quad \ell \geq 4 \quad \text{in } r = r_+,
\end{equation}
and $\rho'_{5/3} > 0$ (density increasing with the orbital radius) for
\begin{equation}
3\sqrt{3/2} < \ell \leq 4 \quad \text{in } r < r < r_e, \quad \ell > 4 \quad \text{in } r_e < r < r_+.
\end{equation}
We thus conclude that $r_-$ is a minimum and $r_+$ is a maximum of $\rho_{5/3}$ (see Fig. 18).

![Figure 17](https://academic.oup.com/mnras/article-abstract/428/2/952/998259 by guest on 25 July 2018)
6.1.2 The case $\gamma = 3/2$

We consider the particular case $\gamma q(q = 1) = 3/2$, which is in the extreme cases $1 < \gamma < 5/3$.

The density $\rho_{3/2} \neq 0$ is then

$$\rho_{3/2} = \left[ \left( \frac{r - 2M}{r} \right) - \frac{M^2 \ell^2}{r^2} \right]^{1/6}$$

defined in the range $R_I \cup R_{II}$ as in equations (80)–(81):

$$0 < \ell < 4 \text{ in } r > 2M,$$

$$4 \leq \ell < \frac{3\sqrt{3}}{2} \text{ in } r > 2M, \quad r \neq r^\pm_{W},$$

$$\ell \geq \frac{3\sqrt{3}}{2} \text{ in } 2M < r < r^+_{I}, \quad r > r^+_I \quad r \neq r^\pm_{W},$$

where $\lim_{r \to r^+_{I}} \rho_{3/2} = 0$ (see Fig. 19).

The critical points of $\rho_{3/2}$ can be found as solutions of $\rho_{3/2}' = 0$, or noting that $p' = (3/2)\rho' \sqrt{\rho}$. We summarize these results concluding that $r_-$ is a minimum and $r_+$ is a maximum of $\rho_{3/2}$.

6.2 The isothermal case: $\gamma = 1$

For an isothermal equation of state ($\gamma = 1$), the solution of equation (74) is $\rho = \rho_k$ (see equation 75). This function is not defined in the range $r^-_I \leq r \leq r^+_{I}$ and the fluid angular momentum is $0 < \ell < \ell_c$.

In order to describe the regions of maximum and minimum density, we study the function $\rho_k$. As $p = k\rho$, we have clearly $p' = k\rho'$. Thus $\rho' = 0$ when $p' = 0$. Moreover, as $k > 0$, the maxima (minima) of $p$ correspond to maxima (minima) of $\rho$. The existence of critical points for the isothermal case is therefore studied in Section 3.3 in terms of the critical points of $G$:

$$\ell = 3\sqrt{3} / 2 \quad r = r_{K_{I}}, \quad 3\sqrt{3} / 2 < \ell < 3\sqrt{3} \quad r = r_{K_{W}}, \quad \ell \geq 3\sqrt{3} \quad r = r_{K},$$

or also for $r > r_{K_{I}}$, $\ell = \ell_{K}$. 

Figure 18. Left-hand panel: $r^+_W$ and $r_{W}$ as a function of $\ell$. In the grey (white) region $\rho_{3/3} > 0$ ($\rho_{3/3} > 0$), and in the black region $\rho_{3/3}$ is not defined. Central panel: $\rho_{3/3}$ = constant in the plane $(x, y)$. Right-hand panel: $\rho_{3/3}$ as a function of $(x, y)$ for $l = 8M$. The black surface is $\sqrt{x^2 + y^2} = 2M$.

Figure 19. Left-hand panel: $\rho_{3/2}$ as a function of $r/M$ and $\ell$. $\rho_{3/2}$ is not defined in $[r^-_I, r^+_I]$. Central panel: $\rho_{3/3}$ = constant in the plane $(x, y)$. Right-hand panel: $\rho_{3/2}$ as a function of $(x, y)$ for $l = 4M$. The black surface is $\sqrt{x^2 + y^2} = 2M$.
The fluid proper angular velocity

This section concerns with the analysis of the fluid velocity field; in particular we are interested in assessing the orbits and the planes where the fluid proper velocity is maximum or minimum.

The fluid four-velocity along the $\phi$ angular direction is

$$\Phi = L r^2 \sigma^2 = \frac{1}{\sigma^2 r^2} \sqrt{\sigma^2 l^2 (r - 2M)^2 - \sigma^2 r^3 - l^2 (r - 2M)^2},$$

where we always consider $L > 0$ and $\Phi > 0$. We redefine $\Phi$ as a dimensionless quantity:

$$\frac{\Phi \sigma^2}{M} = \frac{M^2}{r^2} \sqrt{\frac{l^2 (r - 2M)^2}{r^3 - l^2 (r - 2M)^2}}.$$

Clearly, $\Phi \geq 0$ when $L \geq 0$, and they have the same existence conditions ($0 < l < l_c$, $2M < r < r^-_c$ and $r > r^+_c$). The behaviour of the angular velocity as a function of $\sigma$, $l$ and $r$ are portrayed in Fig. 20.

In order to derive the critical points of the proper angular velocity as a function of $r$, we consider the solutions of the equation $\Phi' = 0$. Apart from the trivial case $l = 0$ and $\Phi = 0$, the proper angular velocity is constant with respect to the radial coordinate when

$$l = l_c = \sqrt{\frac{\sigma^2 r^2 (r - 5M)}{(r - 2M)^2}}$$

for $5/2M < r < 3M$,

where $0 < l_c < 3\sqrt{3}M$. In terms of the angular momentum and radius, the orbits of $\Phi$ belong to the planes

$$\sigma = \sigma_c = \sqrt{\frac{(r - 2M)^2}{r^2 (r - 5M)}}$$

for $r_c < r < r^c$, $(\sigma = 1$ when $r = r_c$).

The convention adopted is that the matter rotates in the positive direction of the azimuthal coordinate $\phi$. 

---

**Figure 20.** Upper left-hand panel: $\Phi = a \Omega^{\phi}/M$ on the equatorial plane $\sigma = 1$ as a function of $r/M$ and $l/M$. The function is defined for $l \in [0, l_c]$. Upper right-hand panel: $\Phi = \text{constant}$ as a function of $l/M$ and $r/M$ on the equatorial plane $\sigma = 1$. The curves have minima in $r = r^-_c$. Lower left-hand panel: $\Phi = \text{constant}$ in the plane $\sigma = 1$ and $r/M$ for $l/M = 1$. The curves have maxima in $r = 5/2M$ (dashed line). Lower right-hand panel: $\Phi = \text{constant}$ in the plane $\sigma = 1$ and $r/M$ for $l/M = 5/2$. No motion is allowed for $l > l_c$ ($l = l_c$, dashed line), and $\Phi' = 0$ on $r = r_c$, $l = l_c$ and $\sigma = \sigma_c$. 

---
Figure 21. Left-hand panel: \( r_{\chi}^- = \text{constant as a function of } 0 < \sigma < 1 \) and \( 0 < l/M < 3\sqrt{3} \). Inset: \( r_{\chi}^- \) (black line) and \( r_{\chi}^+ \) (orange line) as increasing functions of \( l = l/(\sigma M) \). Right-hand panel: \( r_{\chi}^+ / M \) (orange line) and \( r_{\chi}^- / M \) (black line) as a function of \( \sigma \). Grey region corresponds to \( r \in [r_{\text{split}}, \infty] \). Dotted lines mark \( \sigma = \sigma_i \) for the selected \( l/M \).

where

\[
r_s = r_{\chi}^+ / M \equiv \frac{5}{8} \left( \frac{25}{8} + \frac{2 \ell^2}{3} - \frac{\ell^2 (\ell^2 - 36)}{6 \alpha} - \frac{\alpha}{6} \pm \frac{\sqrt{3} (\frac{125}{18} - 11 \ell^2)}{\beta} \right) \pm \frac{\beta}{8 \sqrt{3}},
\]

with

\[
\alpha = \left[ -\ell^2 \left( \ell^2 - 54 \ell^2 + 1350 - 6 \sqrt{48 \ell^4 - 2754 \ell^2 + 50625} \right) \right]^{1/3},
\]

\[
\beta = \sqrt{16\ell^2 + 75 + \frac{8\ell^2 (\ell^2 - 36)}{\alpha}} + 8 \alpha.
\]

As shown in Fig. 21, \( 2.5M < r_{\chi}^+ < 3M \) and the orbital radius \( r_s \) increases with \( \ell = l/(\sigma M) \). An alternative analysis of the proper fluid angular velocity, as a function of \((l, r, \sigma)\), is presented in Appendix B.

8 COMPARING THE FLUID RELATIVISTIC ANGULAR VELOCITY AND THE KEPLERIAN ANGULAR MOMENTUM

In this section we compare the fluid configurations with \( \partial_r p \neq 0 \) with the model \( p = \text{constant} \) where the disc is geodetic by studying the relativistic angular velocity and comparing our results with the Keplerian definitions. In Section 3.2 we have already explored the fluid behaviour with respect to \( l_K \). We summarize here the results considering the ratio

\[
\Pi_{LK} = L/L_K = \Phi/\Phi_K = \sqrt{\frac{\ell^2 M (r - 3M) (r - 2M)}{r^3 - \ell^2 M^2 (r - 2M)}},
\]

which is defined for \( 0 < \ell \leq 3\sqrt{3} \) in \( r > r_{\chi}^+ \), and for \( \ell > 3\sqrt{3} \) in \( r > r_{\chi}^- \). The regions where \( L(l) \) is larger or smaller than \( L_K = L(l_K) \) for different values of the fluid constant of motion \( l \) are portrayed in Fig. 22 (upper left-hand panel). We have \( \lim_{r \to \infty} \Pi_{LK} = \lim_{r \to r_{\chi}^-} \Pi_{LK} = 0 \). In general, \( L > L_K \) for fluid angular momentum \( l \) sufficiently large, for \( l \) sufficiently low or far from the source \( L < L_K \) (see Fig. 22, upper left-hand panel).

We are now interested in defining the critical points of the ratio \( \Pi_{LK} \). The solutions of the equation \( \Pi_{LK}' = 0 \) are, in terms of the angular momentum,

\[
\ell_{\Pi_{LK}} = \frac{r \sqrt{(10M - r) r - 18M^2}}{M (r - 2M)},
\]

and in terms of the orbital radius,

\[
r_{\Pi_{LK}} / M = \frac{1}{6} \left[ 15 + \sqrt{3} \left( \sqrt[3]{78 - 4 \ell^2} - \frac{(\ell^2 - 18)^2}{\kappa} - \kappa^{1/3} - 6 \sqrt{3} (\ell^2 - 35) \right) + \sqrt{3} \kappa \right],
\]

with

\[
\kappa = \left[ 54 (108 + 10 \ell^2 - \ell^4) + \ell^6 + 12 \sqrt{6} \sqrt{-\ell^2 (\ell^2 - 27) (\ell^4 - 54 \ell^2 + 756)} \right]^{1/3}.
\]
\[ \bar{\kappa} \equiv \sqrt{\frac{l^4 - 2l^2(18 + \kappa) + (12 + \kappa)(27 + \kappa)}{\kappa}}. \] 

These are defined for \( \eta_{\text{co}} = \eta_{\Pi_LK} < (5 + \sqrt{7}) \) and \( 0 < \eta_{\Pi_LK} < 3\sqrt{3} \). The sign of \( \Pi_{\Pi_LK} \) as a function of \( r \) and \( \ell \) is summarized in Fig. 22 (upper right-hand panel). In particular, it is manifested that \( \eta_{\Pi_LK} \) is a maximum of \( \Pi_{\Pi_LK} \). The values of this maximum, \( \Pi_{\Pi_LK}(\ell = \ell_{\Pi_LK}) \) and \( \Pi_{\Pi_LK}(r = r_{\Pi_LK}) \), are portrayed in Fig. 22 (lower right-hand panel). We note in particular that \( \Pi_{\Pi_LK} \) is an increasing function of \( \ell \); this implies that the larger the angular momentum of fluid, the larger is the ratio between \( L \) and the angular momentum of the free particle \( L_K \).

8.1 The fluid relativistic angular velocity \( \Omega \): the von Zeipel surfaces

The fluid relativistic angular velocity \( \Omega \) is

\[
\Omega = \frac{\Phi}{\Sigma} = -l \frac{g_{tt}}{\Sigma} = \frac{l(r - 2M)}{r^3 \sigma^2} = \frac{l}{r^2}. \tag{105}
\]

The function \( \Omega = \sigma \Omega(\ell) \):

\[
\Omega = \frac{\ell M(r - 2M)}{r^3} \tag{106}
\]

is defined in \( r > 2M \), and has a maximum in \( r = r_{\text{co}} \) (see Fig. 23, left-hand panel).

The surfaces known as the von Zeipel’s cylinders are defined by the following conditions: \( l = \text{constant} \) and \( \Omega = \text{constant} \) (see e.g. Abramowicz 1971; Chakrabarti 1990, 1991). In the static space–times the family of von Zeipel’s surfaces does not depend on the particular rotation law of the fluid, \( \Omega = \Omega(\ell) \), in the sense that it does not depend on nothing but the background space–time. In the case of a barotropic fluid, the von Zeipel’s theorem guarantees that the surfaces \( \Omega = \text{constant} \) coincide with the surface with constant angular momentum. More
precisely, the von Zeipel condition states that the surfaces at constant pressure coincide with the surfaces of constant density (i.e. the isobaric surfaces are also isochoric) if and only if the surfaces with the angular momentum $l = \text{constant}$ coincide with the surfaces with constant angular relativistic velocity (Abramowicz 1971; Kozłowski et al. 1978; Jaroszynski et al. 1980; Chakrabarti 1990, 1991).

The surfaces $\Omega/l = s > 0$, being $s$ a constant, are defined by

$$
r_s^\pm / M \equiv -\frac{2\sqrt{\frac{M^2}{\sigma^2}} \cos \left[ \frac{1}{2} (\pi + \arccos \left[ -\frac{3\sqrt{3}}{\sqrt{\frac{M^2}{\sigma^2}}} \right]) \right]}{\sqrt{3}}, \quad r_s^+/M \equiv \frac{2\sqrt{\frac{M^2}{\sigma^2}} \cos \left[ \frac{1}{2} \arccos \left( -\frac{3\sqrt{3}}{\sqrt{\frac{M^2}{\sigma^2}}} \right) \right]}{\sqrt{3}}. \quad (107)
$$

These are a function of the product $\sigma^2s$, and they exist for $0 < \sigma^2s \leq (1/27)M^2$. In particular, $\sigma^2s = (1/27)M^2$ corresponds to $r_s^\pm = r_{lsco}$ (see Fig. 24, upper left-hand panel). In the $(x, y)$ coordinates, the $\Omega/l = s > 0$ surfaces read

$$
x_s^\pm \equiv \pm \sqrt{\left( \frac{2M^6}{M^6 - xy^2} \right)^2 - y^2} \quad (108)
$$

(see Fig. 24).

We are now interested in characterizing the angular velocity with respect to the Keplerian velocity $\Omega_K \equiv \sqrt{\frac{M}{2r^3}}$. We therefore consider the dimensionless difference $\Delta\Omega \equiv (\Omega - \Omega_K)M\sigma$:

$$
\Delta\Omega = M\sigma \frac{\ell(Mr - 2M) - \sqrt{r^3M}}{\sigma r^4}. \quad (109)
$$

$\Delta\Omega = 0$ for $r \to \infty$, $\Delta\Omega(r = 2M) = -1/2\sqrt{2}$, $\Delta\Omega(r_{lsco}) = (\ell - 3\sqrt{3})/27$ and $\Delta\Omega(r_{lsco}) = (2\ell - 3\sqrt{6})/108$ (see Fig. 23, right-hand panel). $\Delta\Omega(r = 2M)$ is negative irrespective of the value of $\ell$. The sign of $\Delta\Omega(r_{lsco})$ and $\Delta\Omega(r_{lsco})$, on the contrary, depends on the fluid angular momentum. The sign of $\Delta\Omega$ as a function of $r$ and $\ell$ is summarized in Fig. 25.

The difference $\Delta\Omega$ is maximum in $r > r_{lsco}$ when $\ell = (3/4)&\ell$. Notably, $&\ell$ is not a critical point of the angular velocity difference $\Delta\Omega$. $\Delta\Omega$ always increases in $2M < r < r_{lsco}$ for all $\ell$, and for $0 < \ell < (3/4)&\ell$ in the region $r > r_{lsco}$. Other critical points are for $\ell \geq 27/8$ in

Figure 23. Upper left-hand panel: the relativistic angular velocity $\Omega$ in units of $1/M$, for $s = 1$ as a function of $r/M$, for different values of $\ell$. Dashed line marks the Keplerian velocity $\Omega_K$. Dotted line marks $r = r_{lsco} = 3M$, a maximum of $\Omega$. For $\ell = 3\sqrt{3}/2$, it is $\Omega = \Omega_K$ in $r = r_{lsco} = 3M$; for $\ell = 3\sqrt{6}/2$, it is $\Omega = \Omega_K$ in $r = r_{lsco} = 6M$ and in $r = 6(2/2 + \sqrt{3}/2)$. Upper right-hand panel: $\Delta\Omega$ as a function of $r/M$, for different values of $\ell$. Dashed line marks the maximum of $\Delta\Omega$. Lower left-hand panel: $r_s^+$ and $r_s^-$ as a function of $\ell$. In the white (grey) regions it is $\Delta\Omega > 0$ ($\Delta\Omega < 0$). $\Delta\Omega = 0$ when $r = r_{lsco}$, and $\Delta\Omega$ is maximum in $r = r_{lsco}$ and minimum in $r = r_{lsco}^\prime$. Lower right-hand panel: $\Delta\Omega(r_{lsco})$ as a function of $\ell$. Inset: $\Delta\Omega(r_{lsco})$ as a function of $\ell$. 

On the Polish doughnut accretion disc 975
Figure 24. Upper left-hand panel: \( \frac{r^+}{M} \) as a function of \( \sigma^2 s \). Dashed lines mark the conditions \( r = r_{\text{loc}}, \sigma^2 s = (1/27)M^2 \). The white region \( (r < r_{\text{loc}}) \) corresponds to \( \Omega' / l > 0 \) and the light-grey region \( (r > r_{\text{loc}}) \) corresponds to \( \Omega' / l < 0 \). Upper right-hand panel: surfaces \( \Omega / l = s \) with \( r / M = \text{constant} \) in coordinate \( x = r \cos \vartheta, y = r \sin \vartheta \). The surface \( s = (1/27)M^2 \) is marked with a number. The grey region marks the circle \( r \leq 2M \). Arrows follow the increasing (decreasing) direction with respect to \( s \). Lower panels: surfaces \( \Omega / l = s \) in units of \( M^2 \) as a function of \( x = r \cos \vartheta, y = r \sin \vartheta \) in units of \( M \), for different values of \( s \). The grey region underlines the circle \( r \leq 2M \). Dashed circle corresponds to \( r = r_{\text{loc}} \).

Figure 25. Upper panel: \( r^\pm \) as a function of \( \ell \). Grey region corresponds to \( \Delta \Omega < 0 \) and white region to \( \Delta \Omega > 0 \). \( \Delta \Omega = 0 \) in \( r = r_{\text{loc}} \). Inset: curves \( \Delta \Omega = \text{constant} \). The regions of positive and negative \( \Delta \Omega \) are marked with + and − respectively. Lower panel: table summarizing the regions \( \Delta \Omega > 0 \) (light grey) and \( \Delta \Omega < 0 \) (grey).
and identifies the pressure, \( \ell > l \equiv \pi(\pi + \arccos \hat{\sigma}) \) is a minimum and \( r > 27 \ell \) (110) + 8 \sqrt[3]{121/81} \]
\[ L_r = L/\Delta \]
has been derived by exploiting the methodological and formal analogy with the \( r = L/\Delta \)

with
\[ \hat{\sigma} = \frac{\ell^2 (512\ell^4 - 7776\ell^2 + 19683)}{16\sqrt[3]{\ell^2 (-8\ell^2 - 81)}}^{1/2} \]
(112)

where \( r_o^- = r_o^+ = 9M \) for \( \ell = 27/8 \). \( \Delta_{\ell} \) always increases in \( 0 < \ell < 27/8 \), and for \( \ell \geq 27/8 \) in the regions (2M, \( r_o^- \)) and \( r > r_o^+ \), while it decreases in \( \ell \geq 27/8 \) in the region (\( r_o^- \), \( r_o^+ \)). The radius \( r_o^- \) is a minimum and \( r_o^+ \) a maximum point of \( \Delta_{\ell} \). The critical point values \( \Delta_{\ell}(r_o^\pm) \) are portrayed in Fig. 23 (lower panels).

We analyse the profile of the proper angular velocity, \( \Phi \), and the relativistic angular velocity, \( \Omega \), with respect to the corresponding Keplerian quantities for the case of a isothermal matter. Clearly, with respect to the general analysis, here we should take into account the specific density and pressure profiles emerging by the choice of the particular equation of state, \( p = k\rho^\gamma \).

9 SUMMARY AND CONCLUSIONS

The analysis of a stationary axisymmetric configuration of material, in equilibrium, in a Schwarzschild space–time, as emerging in the Polish doughnut framework constitutes a timely question in view of the interest in astrophysical sources, possibly resulting from super-Eddington accretion on to very compact objects, such as GRBs, AGN, binary systems and ultraluminous X-ray sources (see e.g. Fender & Belloni 2004; Soria 2007). The most characterizing features of the Polish doughnut approach is the thickness of the matter distribution across equilibrium and the existence of a region enveloping the horizon surface, where the fluid can, in principle, infall on to the black hole. The first property is of impact for a comparison with the wide spectrum of numerical simulation of a thick disc (see Font 2003; Abramowicz & Fragile 2011; Straub et al. 2012 for recent examples). The non-negligible depth of the accreting profile is typical of the regime where the gravitational effects are strong and it takes a relevant role in all those extreme phenomena associated with the gravitational collapse, characterized by a violent energy–matter release from the central compact object. The second aspect is relevant because the Polish doughnut model can account for a non-zero accretion rate of the torus even when dissipative effects are negligible. This is in contrast to the original idea by Shakura & Sunyaev (1973) that the angular momentum transport is always allowed by the shear viscosity of the accreting material. Indeed, the accreting plasma is in general quasi-ideal and the emergence of dissipative effects as those required to match the observations requires the appearance in the dynamics of a strong turbulent regime, restated as a laminar one in the presence of a significant shear viscosity. Since this picture is not yet settled down (see Balbus 2011), it is very important that an ideal hydrodynamical scheme on a Schwarzschild background like the one offered by the Polish doughnut is able to account for a material infalling on to a black hole.

In this work we revisited the Polish doughnut model of accretion discs for a perfect fluid circularly orbiting around a Schwarzschild black hole with the effective potential approach for the exact gravitational and centrifugal contributions. We take advantage of the formal analogy between the fluid when the pressure vanishes and the test particle orbiting in the same background to get a comparison between the Polish doughnut, which is supported by the pressure, and the geodetic disc. Our analysis provides a revised theoretical framework to characterize the accretion processes in presence of general relativistic effects. Indeed we formulate the Polish doughnut model in such a way that the fluid dynamics can be interpreted in terms of the fundamental stability properties of the circular orbits in Schwarzschild background.

We extensively analysed the Polish doughnut configurations for the fluid angular momentum \( l \) and the particle angular momentum \( L \), taken respectively constant throughout the entire toroidal surface. Then we propose a re-interpretation of torus physics, with respect to its shape and equilibrium dynamics in terms of parameter \( l (L) \), and in terms of the parameter \( K \), which was naturally established by introducing the effective potential for fluid motion. This new parameter \( K \) has been derived by exploiting the methodological and formal analogy with the effective potential approach to the test particles motion. Note that \( (l, K) \) fully describe the toroidal fluid configuration. This procedure made it possible to emphasize the pressure influence in the equilibrium dynamics with respect to the case of dust, the latter being treated as a set of test particles not subjected to pressure. This dual aspect, methodological and procedural, was thought to require a complete and deep analysis of the behaviour of the momenta \( l \) and \( L \) on the planes and disc orbits, of \( L \) as a function of \( l \), and of the surface characteristic parameters as a function of \( K \).

The main steps of this analysis are as follows.

Radial pressure gradient versus angular momentum \( \mathcal{L} \). We studied the pressure radial gradient as a function of \( r \) and of the angular momentum \( \mathcal{L} \). The pressure is a decreasing function of \( r < r_{\text{esc}} \) and for \( \mathcal{L} < \mathcal{L}_K \) we have \( p' < 0 \); this means that \( \mathcal{L}_K \) identifies the pressure minimum points located in \( [r_{\text{esc}}, r_{\text{isco}}] \), and pressure maximum points in \( r > r_{\text{isco}} \) (see Fig. 1, right-hand panel). However, at fixed orbit \( r \), \( \mathcal{L}_K \) is always a minimum point of \( r \), i.e. at fixed \( r \) the pressure decreases until the angular momentum reaches the values \( \mathcal{L} = \mathcal{L}_K \), and then increases with \( \mathcal{L} \) (note that \( r_{\text{isco}} \) is a minimum point of \( \mathcal{L}_K \)).

Angular momentum \( \mathcal{L} \) versus fluid angular momentum \( \ell \). We found that at fixed \( \ell \) the angular momentum \( \mathcal{L}(r, l) \) has a maximum for \( r = r_{\text{esc}} \) \( \in [r_{\text{isco}}, r_{\text{isco}}] \), and a minimum in \( r_{\text{esc}} > r_{\text{isco}} \), and \( \mathcal{L}(r) \) increases for \( r < r_{\text{esc}} \) and \( r > r_{\text{esc}} \) (Fig. 3). We identify three possibilities: \( \ell < 3\sqrt{372} \).
where the momentum $L(r)$ increases with $r$, $3/3^{3/2} < \ell < 3\sqrt{3}$, where $L$ increases with $r$ up to the maximum point $r_-$, decreases up to the minimum $r_+\hat{3} \sqrt{3}$, where $L$ increases with $r$.

Pressure gradients versus fluid angular momentum $\ell$. The fluid pressure decreases for $r < r_-$ and $r > r_+$ as well as for $\ell < 3\sqrt{3}/2$ (Fig. 4). The situation is much more complicated for fluid with higher angular momenta ($\ell > 3\sqrt{3}/2$). It appears necessary to consider fluids with $3\sqrt{3}/2 < \ell < 3\sqrt{3}$ and with $\ell > 3\sqrt{3}$ separately. In the first case we observe the presence of a ring $[r_-, r_+]$ where the fluid pressure increases with the orbital radius, $r_-$, being a pressure minimum and $r_+$, a pressure maximum. In the case $\ell > 3\sqrt{3}$ we find two rings, $[r_-, r_+]$ and $[r_1, r_3]$, where the fluid pressure is an increasing function of $r$. Fig. 4 (upper right-hand panel) describes this situation from a different point of view. For what concerns the variation of its hydrodynamic pressure with the orbits and the angular momentum, the fluid dynamics is basically split into two zones, $r < r_{Ko}$ and $r > r_{Ko}$, respectively: in the first region, the fluid pressure decreases with increasing angular momentum up to $\ell = \ell_3$ (minimum), then it increases with $\ell$ up to $\ell_k$ (maximum), finally it decreases with $\ell$. The trend is precisely the same in the region $r > r_{Ko}$, but $\ell_3$ and $\ell_k$ are now the minimum and maxima momentum, respectively, i.e. the fluid pressure decreases with $\ell < \ell_3$, grows in the range $[\ell_3, \ell_k]$ and then decreases with $\ell$. The angular momentum gradient has been studied and compared to the radial and angular gradient in Section 4.

The Boyer surfaces and polytropic equation of state. In Section 5 we drew a complete and analytic description of the toroidal surface of the disc. A key role of this analysis was played by the effective potential approach: the toroidal disc can be described once one gives the effective potential and the fluid angular momentum. Our results concerning the disc shape and structure can be summarized as follows.

(i) The distance from the source of the torus inner surface ($\delta = y_2 - 2M$) increases with increasing angular momentum of the fluid but decreases with increasing energy function defined as the value of the effective potential for that momentum.

(ii) The surface maximum height (torus thickness $- h$) increases with energy and decreases with angular momentum: the torus becomes thinner for high angular momenta, and thicker for high energies.

(iii) The location of maximum thickness of the torus moves towards the external regions with increasing angular momentum and energy, until it reaches a maximum an then decreases.

(iv) The surface maximum diameter ($\lambda$) increases with energy, but decreases with the fluid angular momentum.

(v) The distance of the torus inner surface from the structure inner surface ($\delta = y_2 - y_1$) increases with angular momentum and decreases with energy.

(vi) The distance of the structure inner surface from the horizon ($\delta = y_1 - 2M$) increases with energy and decreases with the fluid angular momentum.

The accreting fluids with a polytropic equation of state, divided into two classes identified by their polytropic index, have been studied in Section 6.

The fluid angular velocity. In Section 7 we have analysed the fluid proper angular velocity $\Phi$ and have compared the proper velocity with the Keplerian one $\Phi_K$ in Section 8: the velocity $\Phi < \Phi_K$ in $r > r_1$, and for all $r$ when $\ell < 3\sqrt{3}/2$ (see Fig. 22). As for the analysis of the angular momentum $L(r, l)$, a distinction is made between fluids with $\ell \in [3\sqrt{3}/2, 3\sqrt{3}]$, characterized by a ring $[r_-, r_+]$ where the proper velocity is higher than the Keplerian one, and fluids with $\ell > 3\sqrt{3}$, where the proper fluid velocity is greater than the Keplerian one in the inner regions, $r < r_\infty$. Finally, we have investigated the regions and momenta where the difference $\Phi - \Phi_K$ is maximum (see Fig. 22). We concluded the study of velocity fields by analysing the fluid relativistic velocity $\Omega$ and the von Zeipel surfaces in Section 8. The velocity $\Omega$ has a maximum at $r_{Ko}$ and increases with the increase of $\ell$. We also studied the differences $\Delta \Omega (\Omega - \Omega_K)\sigma$ between the relativistic fluid velocity and the Keplerian one $\Omega_K$ (see Fig. 23). At fixed angular momentum $\ell > 3\sqrt{3}/2$, this difference has a maximum and a minimum, while for $\ell < 3\sqrt{3}/2$ it is always decreasing. The fluid velocity $\Omega$ is lower than the Keplerian one $\Omega_K$ for $r < r_-$ and $r > r_+$, and for any $r$ when $\ell < 3\sqrt{3}/2$, instead for fluid with angular momentum greater than $\ell = 3\sqrt{3}/2$ there is a ring $[r_-, r_+]$ where the fluid velocity $\Omega$ is larger than the Keplerian one (see Fig. 25).

The fundamental merit of the present work is that the analysis of the Polish doughnut features maintain the explicit presence of the effective potential in all the basic expressions describing the matter distribution. This in fact reveals a direct link between our study with the behaviour of free test particles moving the gravitational field of the central object and following circular orbit (the dust, pressure-free limit of the present analysis). What is significant here is the possibility to compare configuration of the considered fluid, as described by certain values of the parameters $K$ and $l$, with the behaviour of the test particle system, characterized by the same values of the corresponding parameters $K$ and $L$. These latter quantities have a precise meaning for the particle (energy and angular momentum as viewed at infinite distance), while the corresponding parameters for the fluid must, on this level, regarded as a classification criterion for the torus morphology. Thus retaining the effective potential in the Polish doughnut treatment allows us to identify the Boyer surfaces in terms of parameters that have a precise meaning for a different, but closely related, context, the pressure-free fluid, i.e. the fundamental features of the Schwarzschild space–time. If this study cannot yet directly offer a paradigm for the comparison with the observations, nonetheless, it makes a concrete step in this direction.

We are now able to constrain the morphology of the equilibrium configuration by fixing the two parameters $K$ and $l$, i.e. by specifying their value, we can define the basic features of the torus shape and of the velocity field in different space regions (this is synthetically sketched in the 10 final remarks, achieved by our systematic analysis), and the comparison with the corresponding profile of the particle motion, where the physical comprehension is settled down, offers a valuable tool to interpret what the observations trace out. Our model will be completed by including other relevant ingredients, such as dissipative effects and the magnetic field, and this is probably the natural development of the conceptual paradigm we have fixed here.
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APPENDIX A: THE RADIAL GRADIENT \( G_r \) AS A FUNCTION OF \( \sigma \)

In Sections 3.1 and 3.3 we detailed the study of the radial gradient \( G_r \) as a function of the angular momentum \( L \) and of the fluid angular momentum \( l \) with a re-parametrization that is independent of the equatorial plane \( \sigma \). However, an explicit study of \( G_r \) as a function of \( \sigma \) is
useful for two main reasons: first, to have a direct comparison when the re-parametrization is not possible (as for the polar gradient \( \mathbf{G}_\sigma \)). Secondly, the angular dependence is necessary to build a three-dimensional characterization of the torus.

In Section 3 we have shown that the critical points of the pressure are defined by condition (14). In terms of the polar coordinates, it reads

\[
\sigma^1_K = \frac{L}{M} \sqrt{\frac{M(r - 3M)}{r^2}}. \tag{A1}
\]

The condition of existence of \( \sigma^1_K \) are \( r > r_{lco} \) for \( 0 < l < 2\sqrt{3}M \), and \( r_{lco} < r \leq r^+_L (\sigma = 1) \) and \( r \geq r^+_L (\sigma = 1) \) for \( L > 2\sqrt{3}M \), where \( r^+_L (\sigma = 1) \) are the circular orbit radii for a test particle in Schwarzschild space–time defined by equation (15), evaluated at \( \sigma = 1 \). The point \( r_{lco} \) is a maximum for \( \sigma^1_L \); the condition \( r^+_L = r^+_L = r_{lco} \) is fulfilled when \( \sigma^1_K = \sigma^1_K^* = \sqrt{L^2/2\sqrt{3}M} \). When \( L = 2\sqrt{3}M \), we have \( \sigma^1_K^* = 1 \).

The sign of \( G_r \) is summarized in Fig. A1.

We characterize now the angular momentum \( L \) as an explicit function of \( l \) and \( \sigma \). We know from Section 3.2 that \( L \) is not defined in the interval \([r^-_L, r^+_L]\), where \( r^-_L \) are introduced in equation (19). In terms of \( r \) this region corresponds to \( 0 \leq a \leq a_L \), where \( a_L = \sqrt{\frac{2(r - 3M)}{r^2}} \). \( a_L \) is defined for \( r > 2M \) when \( 0 < l \leq 3\sqrt{3}M \), and for \( 2M < r < r^-_L (\sigma = 1) \) and \( r > r^-_L (\sigma = 1) \) when \( l > 3\sqrt{3}M, r = r_{lco} \) is a maximum for \( \sigma^1_L \), where \( \sigma^1_L = \sigma^1_L = \sqrt{\frac{L^2}{2\sqrt{3}M}} \). When \( l = 3\sqrt{3}M \), it is \( \sigma^1_L = 1 \).

The critical points of \( L \) are determined by the condition

\[
\sigma^1_K = \sqrt{\frac{L^2(r - 2M)^2}{M^2}}. \tag{A2}
\]

(see equation 20). \( \sigma^1_K \) is defined for \( r > r_{lco} \) when \( 0 < l \leq 3\sqrt{3}/2M \), and for \( r_{lco} < r < r^-_L (\sigma = 1) \) and \( r \geq r^-_L (\sigma = 1) \) when \( 3\sqrt{3}/2M < l < 3\sqrt{3}/2M \), and also for \( r \geq r^-_L (\sigma = 1) \) for \( l \geq 3\sqrt{3}/2M \), where \( r_{lco} (\sigma = 1) \) are the radii defined by equation (21), evaluated at \( \sigma = 1 \). \( \sigma^1_K \) is maximum for \( r = r_{lco} \), where \( \sigma^1_K = \sigma^1_K^* = \sqrt{2\sigma^1_L} \). These results are summarized in Table A1.

Finally the study of \( G_r \) as a function of \((r, l, \sigma)\) is illustrated in Fig. A2.
Figure A2. Conditions for $G_r > 0$ (Left-hand panel) and $G_r < 0$ (right-hand panel). Dotted lines mark $l = 3\sqrt{3}M$, dot-dashed lines $l = 3\sqrt{3}M$, dashed lines $r = r_{\text{rco}}$ and dotted gray lines $r = r_{\text{rco}}$. Black regions are forbidden. Insets: zoom of the region $l/M \in [6, 11]$ and $r/M \in [2, r_{\text{rco}}]$, here $r_\pm = r_\pm (\sigma = 1)$ and $r_\pm^+ = r_\pm^+$. In details: conditions for $p' > 0$: $\sigma \in (\sigma_1, \sigma_2^\pm)$ when $l \in (0, 3\sqrt{3}M)$ and $r > r_{\text{rco}}$, and when $l/M \in (3\sqrt{3}/2, 3\sqrt{3})$ and $r \in (r_{\text{rco}}, r_-)$ and $r > r_\pm$, and when $l > 3\sqrt{3}M$ and $r > r_\pm$: $\sigma \in (\sigma_2^-, 1)$ when $l/M \in (3\sqrt{3}/2, 3\sqrt{3})$ and $r \in (r_-, r_\pm)$, and when $l > 3\sqrt{3}M$ and $r \in (r_\pm^-, r_\pm)$; $\sigma \in (\sigma_1^+, 1)$ when $l > 3\sqrt{3}M$ and $r \in (2M, r_{\text{rco}})$, and when $l > 3\sqrt{3}M$ and $r \in (2M, r_\pm^-)$. Right panel: conditions for $G_r > 0$: $\sigma \in (0, \sigma_1)$ and $\sigma \in (\sigma_2^+, 1)$ when $l \in (0, 3\sqrt{3}/2M)$ and $r > r_{\text{rco}}$, and when $l/M \in (3\sqrt{3}/2, 3\sqrt{3})$ and $r \in (r_{\text{rco}}, r_-)$ and $r > r_\pm$, and when $l > 3\sqrt{3}M$ and $r > r_\pm$: $\sigma \in (0, \sigma_1)$ when $l/M \in (3\sqrt{3}/2, 3\sqrt{3})$ and $r \in (r_-, r_\pm)$, and when $l > 3\sqrt{3}M$ and $r \in (r_\pm^-, r_-)$; $\sigma \in (0, \sigma_1^+)$. 

**APPENDIX B: ANALYSIS OF THE PROPER VELOCITY PROFILE**

We redefine the radius in equation (97) in terms of $\sigma$ and $l$:

$$r_\pm / M = \frac{1}{4} \left( \frac{5}{2} \alpha - \frac{\beta}{\sqrt{\sigma}} \right),$$

where

$$\tilde{\beta} = \sqrt{\frac{264\sqrt{3}(l/M)^2\alpha - 4(l/M)^4 - 4(l/M)^3\alpha + \alpha^2}{\alpha^2\sigma^2\beta}}$$

$$\alpha = \left[ -4(l/M)^2 \left( 1350\sigma^4 + (l/M)^4 - 6\sigma^2 \left[ 4(l/M)^2 + \sqrt{48(l/M)^2 - 2754\sigma^2(l/M)^2 + 50625\sigma^2} \right] \right) \right]^{1/3},$$

$$\beta = \sqrt{\frac{8(l/M)^2 + \alpha^2 + \alpha^2 \left( 75\alpha - 288(l/M)^2 \right)}{\alpha^2\sigma^2}}.$$

For $\sigma \neq 0$ (for $\sigma = 0$, we have $r_\pm = 2M$).

The fluid proper angular velocity increases with $r/M (\Phi' > 0)$ when

$0 < l < 3\sqrt{3}M$ for $0 < \sigma \leq \sigma_c$ and $2M < r < r_\pm^-$,

$\sigma_c < \sigma < 1$ and $2M < r < r_\pm^-$;

$l > 3\sqrt{3}M$ for $0 < \sigma \leq 1$ and $2M < r < r_\pm^-$.

$\Phi' < 0$ when

$0 < l < 3\sqrt{3}M$ for $0 < \sigma < \sigma_c$ and $r > r_\pm^-$.

$\sigma = \sigma_c$ and $r > r_\pm^-$. 

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$\Phi' < 0$ and $\Phi' > 0$. Solid line marks $\sigma^*_l$ as a function of $l/M$, dashed line marks $l/M = 3\sqrt{3}$.

\[ \sigma < 1 \quad \text{and} \quad r > r^+_x ; \quad (B10) \]

\[ l = 3\sqrt{3}M \quad \text{for} \quad 0 < \sigma < 1 \quad \text{and} \quad r > r^+_l ; \quad (B11) \]

\[ \sigma = 1 \quad \text{and} \quad r > r^-_l ; \quad (B12) \]

\[ l > 3\sqrt{3}M \quad \text{for} \quad 0 < \sigma \leq 1 \quad \text{and} \quad r > r^+_l \quad (B13) \]

(see Fig. B1). It follows that the angular velocity increases in $2M < r < r_x < 3M$ until it reaches the point $r = r_x$, which is a maximum, where $2.5M < r_x < 3M$ with angular momentum $l_x$ on the equatorial plane $\sigma^*_x$, then it decreases with radius (see Fig. B1).