Organization of directed multiplex networks

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We describe the complex global structure of giant components in directed multiplex networks which generalizes the well-known bow-tie structure, generic for ordinary directed networks. By definition, a directed multiplex network contains vertices of one kind and directed edges of 2 kinds. In directed multiplex networks, we distinguish a set of different giant components based on interconnectivity of their vertices, which is understood as various directed paths running entirely through edges of distinct types. If, in particular, m = 2, we define a strongly viable component as a set of vertices, in which each two vertices are interconnected by two pairs of directed paths, running through edges of each of two kinds in both directions. We show that in this case, a directed multiplex network contains, in total, 9 different giant components including the strongly viable component. In general, the total number of giant components is 3m. For uncorrelated directed multiplex networks, we obtain exactly the size and the birth point of the strongly viable component and estimate the sizes of other giant components.

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I. INTRODUCTION

The so-called bow-tie organization of giant components is a generic feature of directed networks [1,2]. This structure was first reported for the directed WWW graph, but it is actually valid for general directed networks, see Fig. 1(a). There are three different giant (i.e., containing a finite fraction of all vertices in an infinite network) components in these networks. From each of the vertices of the giant strongly connected component, S, one can reach any other its vertex by a directed path, so that each two vertices are mutually reachable. From each vertex of the giant in-component, I, one can reach the vertices of the strongly connected component by a directed path. Each vertex of the giant out-component, O, is reachable from vertices of the strongly connected component. (Note that according to this definition, the strongly connected component is included both in the giant in- and out-components.) In uncorrelated networks and, more generally, in locally tree-like networks with given degree-degree correlations, the sizes of these giant components were obtained analytically [2,3].

In recent few years, the focus of interest of the complex networks studies essentially shifted from single networks to coupled networks, networks of networks, etc., including interdependent and multiplex networks [5,11]. In the interdependent networks, each vertex in a network depends on a vertex or several vertices in other networks. As a result, removal of vertices in one network may launch a cascade of failures destroying a finite fraction of all networks [6]. Depending on the structure of these networks and the fraction of the initially removed vertices, this cascade may eliminate the networks completely or remain a finite fraction of nodes and edges undamaged [8,10]. The specific phase transition between these two situations is hybrid, which means that it combines a discontinuity and a critical singularity [11]. In the simplest representative situation, in which each vertex in interdependent networks has not more than one interdependence, the interdependent networks are actually equivalent to multiplex networks [12,13]. The multiplex networks have vertices of one kind and edges of several different kinds. Hence, multiplex networks are graphs with edges of several different colors. They can be treated as a superpositions of several graphs of distinct colors. The role of the remaining giant component of interdependent networks, in multiplex networks, plays the giant viable cluster. For each type of edges in an undirected multiplex network, each two vertices in the viable cluster are connected by at least one path following edges of that kind (i.e., there must be at least one path of each color between each two vertices) [9,10].

Previously, undirected multiplex networks were explored [10,11,21]. In the present article, we study multiplex networks, in which all edges are directed. We show that the giant components in these networks are organized in an essentially more complicated way than in ordinary directed networks. We introduce a system of different giant components based on the set of directed paths of different colors connecting the vertices in these components. Figure 1(b) demonstrates this set of giant components in directed multiplex networks with edges of two kinds. For locally tree-like networks, we find the size of strongly viable component analytically and describe its simplest structural characteristics and the hybrid phase transition associated with the emergence of this component. For directed multiplex networks with two kinds of edges, we obtain lower limit estimates for the sizes of other viable components.

This paper is organized as follows. In Section III, we introduce the set of giant viable components in directed multiplex networks, including the strongly viable compo-
II. GIANT COMPONENTS IN DIRECTED MULTIPLEX NETWORKS

Let us introduce different giant viable components in directed multiplex networks. For the sake of brevity, we define them in the particular case of multiplex networks having two kinds of edges, i.e., edges of two colors, A and B. This network can be treated as a superposition of network with edges of color A (network A) and the network with edges of color B (network B). Generalization to the case of an arbitrary number m kinds of edges is straightforward. Our definition is based on interconnectivity of different parts of these networks understood in terms of the set of different directed paths running between these parts. One can introduce directed paths of each color, i.e., the paths following directed edges of only that color.

(1) We define the strongly viable connected component, SS, in the following way. Each two vertices in this component are reachable from each other by directed paths of both colors. Clearly, SS is a subgraph of the giant strongly connected components of two networks with edges of distinct colors.

(2) From any vertex of the in-in viable component, II, there are at least two directed paths (a path of one color and a path of the second color) to the strongly viable component SS. The in-in viable component is a subgraph of the in-components of the two one-color networks.

(3) To any vertex of the out-out viable component, OO, there are at least two directed paths of both colors from the SS component. The out-out viable component is a subgraph of the out-components of the two one-color networks.

(4) From each of the vertices of the strongly-in viable component, SI, there are at least two directed (at least one in each direction) paths of color A to SS and at least one directed path of color B to SS. The strongly-in viable component is a subgraph of the strongly connected component of network A and the in-component of network B.

(5) From each of the vertices of the in-strongly viable component, IS, there are at least two directed (at least one in each direction) paths of color B to SS and at least one directed path of color A to SS. The in-strongly viable component is a subgraph of the strongly connected component of network B and the in-component of network A.

(6) To each of the vertices of the strongly-out viable component, SO, there are at least two directed (at least one in each direction) paths of color B to SS and at least one directed path of color A from SS. The strongly-out viable component is a subgraph of the strongly connected component of network B and the out-component of network A.

(7) To each of the vertices of the out-strongly viable component, OS, there are at least two directed (at least one in each direction) paths of color A to SS and at least one directed path of color B from SS. The out-strongly viable component is a subgraph of the strongly connected component of network B and the out-component of network A.

(8) From each of the vertices of the in-out viable component, IO, there is at least one directed path of color A to SS and from SS there is at least one directed path of color B to each of vertices of IO. The in-out viable component is a subgraph of the in-component of the network A and the out-component of the network B.

(9) From each of the vertices of the out-in viable component, OI, there is at least one directed path of color B to SS and from SS there is at least one directed path of color A to each of vertices of OI. The out-in viable component is a subgraph of the out-component of the network A and the in-component of the network B.

Note that according to these definitions, SS is a subgraph of the rest giant components. Nine these components are schematically shown in Fig. 1(b). The number of the components exponentially grows with the number of colors, m, see below.

FIG. 1. (Color online) The structure of giant components in (a) ordinary directed networks and (b) in multiplex directed networks.
III. EQUATIONS FOR TREE-LIKE DIRECTED MULTIPLEX NETWORKS

The locally tree-like structure of random networks allows a simple analytical treatment based on the generating function technique. Let us consider a locally tree-like directed multiplex network with edges of $m$ colors, labeled with $i = A, B, \ldots$. For the sake of simplicity, here we only consider the directed multiplex networks, in which each of networks $i$ are uncorrelated although the numbers of different connections of a vertex may be correlated. This multiplex network is locally tree-like, that is, roughly speaking, it has no finite loops (cycles in terms of graph theory) in the infinite size limit. Introducing the notation $q_i = (q_{in,i}, q_{out,i})$ for the vector of the numbers of incoming and outgoing connections of a vertex in network $i$, we describe this multiplex network completely by the joint degree distribution $P(q_A, q_B, \ldots)$.

To find the relative size of the giant viable component, we introduce a set of probabilities which are defined in Fig. 2 as following:

$$x_i = \sum_{q_A, q_B, \ldots} \frac{q_{in,i} P(q_A, q_B, \ldots)}{(q_{in,i})} \left[ 1 - (1 - x_i) q_{out,i} \right] \prod_{j \neq i} \left[ 1 - (1 - x_j) q_{out,j} \right] \left[ 1 - (1 - y_j) q_{in,j} \right],$$

$$y_i = \sum_{q_A, q_B, \ldots} \frac{q_{out,i} P(q_A, q_B, \ldots)}{(q_{out,i})} \left[ 1 - (1 - y_i) q_{in,i} \right] \prod_{j \neq i} \left[ 1 - (1 - x_j) q_{out,j} \right] \left[ 1 - (1 - y_j) q_{in,j} \right].$$

Let us explain the right-hand terms in Eq. (1). For a vertex with $q_{out,i}$ out-going edges of kind $i$, the probability that each of these edges does not lead to the giant strongly viable connected component is $(1 - x_i)q_{out,i}$. Consequently, $\prod_{j \neq i} (1 - (1 - x_j)q_{out,j})$ is the probability that following at least one of these edges, we will reach the giant strongly viable connected component. Similarly, $\prod_{j \neq i} (1 - (1 - y_j)q_{in,j})$ is the probability that each of outgoing edges of this vertex with colors different from $i$ also will allow us to reach the giant strongly viable connected component. Also, for a vertex with $q_{in,j}$ incoming edges of kind $j$ (different from $i$), $\prod_{j \neq i} [1 - (1 - y_j)q_{in,j}]$, gives the probability that each of these in-coming edges comes from the giant strongly viable connected component. Similar arguments are valid for Eq. (2). The generating function technique allows us to represent these equations in a more compact form (see Appendix). Equations (1)–(2) for probabilities $x_i$ and $y_i$ are written for multiplex networks with edges of $m$ kinds, where $m \geq 1$. For the sake of simplicity, we consider $m = 2$, and obtain the sizes of different components.

IV. GIANT COMPONENTS IN MULTIPLEX NETWORKS WITH TWO KINDS OF EDGES

The probabilities $x_i$ and $y_i$ enable us to find the relative size of the giant strongly viable component exactly.
Figure 3 shows the probability that a vertex belongs to the giant strongly viable component, SS, in terms of the probabilities $x_i$ and $y_i$ for a multiplex network with two kinds of edges. Following the derivation of Eq. (II), we can obtain the relative size of this component:

$$SS = \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - x_A)^{q_{out,A}} \right] \left[ 1 - (1 - y_A)^{q_{in,A}} \right] \left[ 1 - (1 - x_B)^{q_{out,B}} \right] \left[ 1 - (1 - y_B)^{q_{in,B}} \right].$$ (3)

The finding of the sizes of the rest giant viable components is a more difficult task than for SS. The difficulty is that for these calculations, in addition to $x_A$, $y_A$, $x_B$, and $y_B$, one has to define a number of new probabilities for different classes of infinite trees. One can show that the number of these probabilities grows rapidly with $m$. In particular, for $m = 2$, we need to introduce 10 probabilities in total for each kind of edges. This is why we do not fulfill this challenging program in the present paper. Instead we estimate the relative sizes of the rest giant viable components taking into account only the probabilities $x_i$ and $y_i$. Based on the definitions of the giant viable components, we can schematically represent the probabilities that a vertex belongs to a corresponding component in terms of $x_i$ and $y_i$ as is shown in Fig. 4. For instance, for $II$, we use only probabilities $x_A$ and $x_B$, as is shown in Fig. 4(a). Because of that we actually imposed an extra constraint that vertices must have more connections to SS than it is necessary by definition. As a result we underestimate the size of the component. The analytical expression for $II$ shown in Fig. 4(a) is as follows

$$II \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - x_A)^{q_{out,A}} \right] \times \left[ 1 - (1 - x_B)^{q_{out,B}} \right].$$ (4)

Similarly, for the Out-Out component, we use the probabilities $y_A$ and $y_B$, which allows us to estimate the relative size of $OO$ as

$$OO \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - y_A)^{q_{in,A}} \right] \times \left[ 1 - (1 - y_B)^{q_{in,B}} \right],$$ (5)

see Fig. 4(b). Figs. 4(c) and (d) give the relative sizes of components $IO$ and $OI$, respectively:

$$IO \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - x_A)^{q_{out,A}} \right] \times \left[ 1 - (1 - y_B)^{q_{in,B}} \right],$$ (6)

$$OI \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - y_A)^{q_{in,A}} \right] \times \left[ 1 - (1 - x_B)^{q_{out,B}} \right].$$ (7)

According to Figs. 4(e) and (f), the relative sizes of components $IS$ and $OS$ are, respectively,

$$IS \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - x_A)^{q_{out,A}} \right] \times \left[ 1 - (1 - x_B)^{q_{out,B}} \right] \left[ 1 - (1 - y_B)^{q_{in,B}} \right],$$ (8)

$$OS \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - y_A)^{q_{in,A}} \right] \times \left[ 1 - (1 - x_B)^{q_{out,B}} \right] \left[ 1 - (1 - y_B)^{q_{in,B}} \right].$$ (9)

Finally, for relative sizes of components $SI$ and $SO$, Figs. 4(g) and (h) give, respectively,

$$SI \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - x_B)^{q_{out,B}} \right] \left[ 1 - (1 - x_A)^{q_{out,A}} \right] \times \left[ 1 - (1 - y_A)^{q_{in,A}} \right],$$ (10)

$$SO \simeq \sum_{q_A, q_B} P(q_A, q_B) \left[ 1 - (1 - y_B)^{q_{in,B}} \right] \left[ 1 - (1 - x_A)^{q_{out,A}} \right] \times \left[ 1 - (1 - y_A)^{q_{in,A}} \right].$$ (11)

Employing generating functions, one can represent Eqs. (3)–(11) in a more compact form (see Appendix).

For the sake of simplicity we assume that there are no correlations between edges of different colors and in- and out-degrees for each kind of edges $i$, i.e., $P(q_{in,i}, q_{out,i}) = P(q_{in,i})P(q_{out,i})$. Let us consider first the Erdős–Rényi directed multiplex networks with Poisson in-degree and out-degree distributions, $P(q_{in,i}) = \frac{q_{in,i}}{e^{q_{in,i}}}e^{-q_{in,i}}$ and $P(q_{out,i}) = \frac{q_{out,i}}{e^{q_{out,i}}}e^{-q_{out,i}}$, where $c_A$ and $c_B$ are the mean vertex degrees for edges $A$ and $B$, respectively. For the Poisson distribution, the generating function $G_i(x)$ and its first derivatives (see Appendix) are $G_i(x) = G_i^{in,i}(x) = e^{q_{in,i}x} = e^{c_A(1-x)}$.

Inspecting Eqs. (1)–(2) in the general case of $c_A \neq c_B$, we conclude that $x_i = y_i \equiv X_i$, and Eqs. (11–12) are simplified to the equation

$$X_i = (1 - e^{-c_A X_i})(1 - e^{-c_B X_i})^2.$$ (12)

The largest root of this equation plays the role of the order parameter in this problem. For the symmetric case
c_A = c_B ≡ c, the non-zero solution exists only when c exceeds the critical value c = 3.08912... .

Furthermore, equations (3)–(11) provide the following expressions for the relative sizes of the giant viable components:

\[
SS = (1 - e^{-c_A X_A})^2 (1 - e^{-c_B X_B})^2, \tag{13}
\]

\[
II = OO = IO = OI \simeq (1 - e^{-c_A X_A})(1 - e^{-c_B X_B})^2, \tag{14}
\]

\[
IS = OS \simeq X_A, \tag{15}
\]

\[
SI = SO \simeq X_B. \tag{16}
\]

For the case of \(c_A = c_B = c\), in Fig. 5 the resulting dependence of the relative size of the giant strongly viable component on \(c\) is compared with the corresponding dependence of the giant weakly viable component. The giant weakly viable component is defined as the viable component of the undirected counterpart of the directed multiplex network, in which the directedness of the edges is ignored. This component emerges at \(c = 2.4554.../2\). In Fig. 6, the lower limit estimates for the sizes of the other viable components compared with the size of the giant strongly component.

Figure 7 demonstrates that the value of the jump at a critical point for \(SS\) is maximum when the mean degrees of the two networks coincide. To obtain the critical point for the giant strongly viable component, we introduce \(g_j(X_j) \equiv X_j - (1 - e^{-c_j X_j})(1 - e^{-c_j X_j})^2\). The critical point is determined by the condition \(\det[J - I] = 0\) for the Jacobian matrix \(J\), defined as \(J_{ij} = \partial f_j/\partial x_i\), and \(I\)
is the identity matrix. Substituting the resulting values $X_A$ and $X_B$ in Eq. (13) we find the jump of the $SS$ component at the transition point. The line of critical points for the $SS$ component on the plane ($c_A, c_B$) is shown in Fig. 8.

Our approach allows us to describe the structure of the viable components. In particular, one can ask what is the probability that an edge belongs to the giant strongly viable component? By definition, as is shown in Fig. 9 if an edge belongs to the strongly viable component, then the end vertices of this edge should be connected to each other by paths of both colors running through infinity. Hence the probability that a uniformly randomly chosen edge of kind $i$ belongs to the giant strongly viable component is $P_i = x_i y_i$.

In many real systems there are more than two multiplex networks. In principle it is possible to generalize structure of viable components from two directed multiplex networks to a systems of $m$ directed multiplex networks. The main part, namely the strongly viable component of the network in which vertices are reachable from each other by directed paths of all $m$ colors, is a subgraph of the strongly connected components of all networks. One can can easily find that the total number of viable components for $m$ directed multiplex networks is $3^m$.

V. CONCLUSIONS

In this work we described the topology of directed multiplex networks. We introduced a set of giant viable components for a directed multiplex network with two kinds of edges. The definitions of these components are based on a set of directed paths between vertices which run following distinct kinds of edges. We showed how to find analytically the emergence points and the sizes of various giant viable components for directed multiplex networks with an arbitrary joint in-, out degree distribution. We found that similar to a viable component in undirected multiplex networks, hybrid transitions occur at the points of emergence of the giant viable connected components in directed multiplex networks.

In our analytical calculations, we considered uncorrelated and locally tree-like directed multiplex networks. We mostly focused on directed multiplex networks with two kinds of edges. In this case, the total number of giant viable components is 9. In general, for the multiplex networks with $m$ kinds of edges, there are $3^m$ distinct viable components. For uncorrelated directed multiplex networks, we found exactly the size of the central, strongly viable component. We also estimated from below the sizes of the rest giant viable components. An exact calculation of the sizes and other characteristics of these components is a challenging task for a future work.

Our study has revealed an essentially more rich global organization of directed multiplex networks compared to single directed networks. We suggest that the knowledge of the details of this complex structure will lead to a better understanding of processes taking place in these networks and of their function.

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Appendix A: Generating function technique for directed multiplex networks

The generating function $G(x, y)$ of a given in-, out-degree distribution $P_{in\ or\ out}(x,y)$ for a directed network is

$$G(x, y) \equiv \sum_{q_{in\ or\ out}} P(q_{in\ or\ out}) x^{q_{in\ or\ out}} y^{q_{in\ or\ out}}. \quad (A1)$$

Using the function $G(x, y)$, one can obtain the generating functions $G_{in}^1(x)$ and $G_{out}^1(y)$ of the joint in-, out-degree distribution of the end vertex of a randomly chosen incoming or out-going edge, respectively:

$$G_{in}^1(x) = \frac{\partial G(x, y)|_{y=1}}{\partial x |_{y=1}}, \quad (A2)$$

$$G_{out}^1(y) = \frac{\partial G(x, y)|_{x=1}}{\partial y |_{x=1}}. \quad (A3)$$

If we assume that there is no correlation between the degrees $q_A, q_B, \ldots$ of a vertex, so that $P(q_A, q_B) = P(q_A)P(q_B) \ldots$, then using these definitions, Eqs. (A1)-(A3) can be represented as

$$x_i = \left[1 - G_{in}^{1,i}(1 - x_i)\right] \prod_{j \neq i} \left[1 - G_j^j(1 - x_j, 1)ight] - G_j(1, 1 - y_j) + G_j(1 - x_j, 1 - y_j), \quad (A4)$$

$$y_i = \left[1 - G_{out}^{1,i}(1 - y_i)\right] \prod_{j \neq i} \left[1 - G_j^j(1 - x_j, 1)\right] - G_j(1, 1 - y_j) + G_j(1 - x_j, 1 - y_j), \quad (A5)$$

where the index $i$ refers to types of edges $A, B, \ldots$.

The relative sizes of the giant viable components can be written in terms of generating functions as follows:

$$II = \left[1 - G^A(1 - x_A, 1)\right] \left[1 - G^B(1 - x_B, 1)\right], \quad (A6)$$

$$OO = \left[1 - G^A(1, 1 - y_A)\right] \left[1 - G^B(1, 1 - y_B)\right], \quad (A7)$$

$$IO = \left[1 - G^A(1 - x_A, 1)\right] \left[1 - G^B(1, 1 - y_B)\right], \quad (A8)$$

$$OI = \left[1 - G^A(1, 1 - y_A)\right] \left[1 - G^B(1 - x_B, 1)\right], \quad (A9)$$

$$IS = \left[1 - G^A(1 - x_A, 1)\right] \left[1 - G^B(1 - x_B, 1) - G^B(1, 1 - y_B)\right], \quad (A10)$$

$$SI = \left[1 - G^A(1 - x_A, 1) - G^A(1, 1 - y_A)\right] \left[1 - G^B(1 - x_B, 1)\right], \quad (A11)$$

$$OS = \left[1 - G^A(1, 1 - y_A)\right] \left[1 - G^B(1 - x_B, 1) - G^B(1, 1 - y_B)\right], \quad (A12)$$

$$SO = \left[1 - G^A(1 - x_A, 1) - G^A(1, 1 - y_A)\right] \left[1 - G^B(1, 1 - y_B)\right], \quad (A13)$$

$$SS = \left[1 - G^A(1 - x_A, 1) - G^A(1, 1 - y_A)\right] \left[1 - G^B(1 - x_B, 1) - G^B(1, 1 - y_B) + G^B(1 - x_B, 1 - y_B)\right]. \quad (A14)$$

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