Abstract. We study a class of left-invariant pseudo-Riemannian Sasaki metrics on solvable Lie groups, which can be characterized by the property that the zero level set of the moment map relative to the action of some one-parameter subgroup \( \{ \exp tX \} \) is a normal nilpotent subgroup commuting with \( \{ \exp tX \} \), and \( X \) is not lightlike. We characterize this geometry in terms of the Sasaki reduction and its pseudo-Kähler quotient under the action generated by the Reeb vector field.

We classify pseudo-Riemannian Sasaki solvmanifolds of this type in dimension 5 and those of dimension 7 whose Kähler reduction in the above sense is abelian.

Introduction

Sasaki manifolds were introduced in [16] as an odd-dimensional counterpart to Kähler geometry; they are characterized by an almost contact metric structure \((φ, ξ, η, g)\) which is both normal and contact. Beside the analogy, they bear a strong relation to Kähler geometry in that both the cone over a Sasaki manifold and the space of leaves of the Reeb foliation carry a Kähler structure. For pseudo-Riemannian metrics, a completely analogous definition of Sasaki structure can be given, which was first considered in [17]; the relation to pseudo-Kähler geometry is the same as in the definite setting.

Arguably, the most interesting Sasaki metrics are those satisfying the Einstein condition \( \text{ric} = 2ng \), where the Einstein constant is fixed by the dimension. Both in the Riemannian and indefinite case, Einstein-Sasaki metrics are characterized by the existence of a Killing spinor (see [2]), which makes them relevant for general relativity and supersymmetry (see [9, 18]).

In this paper we focus on the homogeneous case, and particularly on invariant pseudo-Riemannian Sasaki metrics on solvmanifolds. Although we do not insist on the Einstein condition here, the prospect of applying the machinery to produce Einstein-Sasaki metrics leads us to consider standard solvmanifolds,
corresponding to semidirect products \( g \rtimes a \), where \( g \) is nilpotent, \( a \) abelian and their sum orthogonal. Indeed, all Riemannian Einstein solvmanifolds are of this type (see [12, 13]), and even in the indefinite case the standard condition has proved quite effective to produce examples (see [6, 7]). In fact, the most studied standard Lie algebras are those of Iwasawa type (or pseudo-Iwasawa, for indefinite signature), namely those for which \( \text{ad} \, X \) is symmetric for all \( X \) in \( a \).

Restricting to left-invariant pseudo-Riemannian Sasaki metrics on solvable Lie groups allows us to work at the Lie algebra level; we shall therefore refer to the structures under consideration as Sasaki structures on a Lie algebra. Our first result (Proposition 2.6) is that Sasaki Lie algebras cannot be of pseudo-Iwasawa type. This motivates us to study the more general class of standard Lie algebras, though restricting for simplicity to one-dimensional abelian factors, i.e., \( \tilde{g} = g \rtimes \text{Span} \{ e_0 \} \). In Proposition 3.3, we characterize the Sasaki condition on \( \tilde{g} \) in terms of the induced structure on \( g \). The resulting conditions on \( g \) are somewhat unwieldy.

However, the situation simplifies if we impose that \( g \) is the zero-level set of a moment map relative to the action of a one-parameter subgroup. In practice, this means that \( \phi(e_0) \) lies in the center \( z(g) \). We dub this particular class of Sasaki structures \( z \)-standard. One can then take the Sasaki reduction in the sense of contact geometry, obtaining a new Sasaki nilmanifold with additional structure, namely a derivation \( D \) commuting with \( \phi \) and satisfying a quadratic equation of the form

\[
[D^s, D^a] = hD^s - 2(D^s)^2,
\]

where \( h \) is a real constant, and \( D^s, D^a \) denote the symmetric and antisymmetric part of \( D \) (Corollary 4.3). In this setting, the Reeb field \( \xi \) is central, so one can take a further quotient and obtain a pseudo-Kähler nilmanifold in three dimensions less (Corollary 4.4); equivalently, one can interpret this quotient as a Kähler reduction of the pseudo-Kähler Lie algebra \( \tilde{g}/\text{Span} \{ \xi \} \).

This construction can be inverted: starting from a pseudo-Kähler nilmanifold with a derivation as above, one obtains a pseudo-Kähler solvmanifold in two dimensions higher, then giving a \( z \)-standard Sasaki solvmanifold by taking a circle bundle (Proposition 5.1). This procedure differs from the double extension procedure considered in [3], in that the two “extra” dimensions span a definite two-plane, rather than neutral.

We show that up to isometry, when \( D^s \) is both a derivation and diagonalizable over \( \mathbb{C} \) it can be assumed to be a projection, giving a simple explicit form to the resulting Sasaki structure (Corollary 5.6). Making use of this fact, we classify \( z \)-standard Sasaki solvmanifolds in dimension 5 (Theorem 5.7), and all those in dimension 7 whose Kähler reduction is abelian (Theorem 5.8).

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1. Pseudo-Riemannian Sasaki structures

In this section we recall some basic definitions and facts on pseudo-Riemannian Sasaki structures. For further details we refer to [5, 17].

**Definition.** An almost contact structure on a $(2n + 1)$-dimensional manifold $M$ is a triple $(\phi, \xi, \eta)$, where $\phi$ is a tensor field of type $(1,1)$, $\xi$ is a vector field, and $\eta$ is a 1-form, such that

\[ \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi^2 = -\text{Id} + \eta \otimes \xi. \]

Given a pseudo-Riemannian metric $g$ on $M$, the quadruple $(\phi, \xi, \eta, g)$ is called an almost contact metric structure if $(\phi, \xi, \eta)$ is an almost contact structure and

\[ g(\xi, \xi) = \epsilon \in \{ \pm 1 \}, \quad \eta = \epsilon \eta^\flat, \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y) \]

for any vector fields $X, Y$.

We will assume $\epsilon = 1$ in the sequel.

Note that if $(\phi, \xi, \eta, g)$ is an almost contact metric structure with $g(\xi, \xi) = \epsilon = -1$, then defining $\tilde{g} = -g$ we have that $(\phi, \xi, \eta, \tilde{g})$ is another almost contact metric structure such that $\tilde{g}(\xi, \xi) = \tilde{\epsilon} = 1$, so our assumption does not entail a loss of generality.

**Remark 1.1.** The generalized eigenspace of 0 for $\phi$ is generated by $\xi$. Therefore 0 is an eigenvalue and $\xi$ is an eigenvector, i.e., $\phi(\xi) = 0$.

**Remark 1.2.** The endomorphism $\phi$ is always skew-symmetric: indeed,

\[ g(\phi(X), Y) = -g(\phi^2 Y - \eta(Y)\xi, \phi X) = -g(X, \phi(Y)) + \eta(X)\eta(\phi(Y)) = -g(X, \phi(Y)). \]

In fact, if $\phi$ is assumed to be skew-symmetric, $g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y)$ is equivalent to $\phi^2 = -\text{Id} + \eta \otimes \xi$.

We define the fundamental 2-form associated to the almost contact metric structure $(\phi, \xi, \eta, g)$ as

\[ \Phi = g(\cdot, \phi \cdot). \]

In addition, in analogy with the Nijenhuis tensor field for complex manifolds, we define

\[ N_\phi = \phi^2 [X, Y] + [\phi X, \phi Y] - \phi(\phi X, Y) - \phi(X, \phi Y). \]

**Definition.** An almost contact metric structure $(\phi, \xi, \eta, g)$ is said to be Sasaki if $(\phi, \xi, \eta, g)$ satisfies $N_\phi + d\eta \otimes \xi = 0$ and $d\eta = 2\Phi$. 
Sasaki structures can be characterized in terms of the covariant derivative $\nabla \phi$; as usual, we indicate by $\nabla$ the Levi-Civita connection, by $R$ its curvature tensor, by $\text{ric}$ its Ricci tensor.

**Lemma 1.3** ([17, Proposition 1]). Given an almost contact metric structure $(\phi, \xi, \eta, g)$ on a manifold of dimension $2n + 1$ such that

$$\nabla_X \phi Y = g(X,Y)\xi - \eta(Y)X,$$

the following hold:

1. $\nabla_X \xi = -\phi(X)$;
2. $\xi$ is a Killing vector field;
3. $d\eta(X,Y) = 2\Phi(X,Y)$;
4. $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$;
5. $\text{ric}(\xi, X) = 2n\eta(X)$.

Arguing as in [4, Theorem 7.3.16], one obtains:

**Proposition 1.4.** Let $(\phi, \xi, \eta, g)$ be an almost contact pseudo-Riemannian metric structure on $M$. The following are equivalent:

1. $(\phi, \xi, \eta, g)$ is Sasaki;
2. the cone $(\mathbb{R}_+ \times M, J, \omega)$ is pseudo-Kähler;
3. $\nabla_X \phi Y = g(X,Y)\xi - \eta(Y)X$;
4. $\nabla_X \Phi = \eta \wedge X^\flat$.

Pseudo-Sasaki manifolds are related to pseudo-Kähler geometry in the following way. Recall that a pseudo-Kähler structure on a manifold $M$ is an almost-pseudo-Hermitian structure $(J, g, \omega)$, with the convention that $\omega = g(\cdot, J \cdot)$, such that $J$ is integrable and $\omega$ is closed; equivalently, $\omega$ is parallel with respect to the Levi-Civita connection.

Like in the Riemannian case, we have the following:

**Proposition 1.5** ([14]). Let $M$ have a pseudo-Riemannian Sasaki structure $(\phi, \xi, \eta, g)$. Then the space of leaves of the Reeb foliation has an induced pseudo-Kähler structure.

Finally, we recall that given a Sasaki structure $(\phi, \xi, \eta, g)$ and a positive constant $a$, we can define another Sasaki structure by

$$\hat{\phi} = \phi, \quad \hat{\xi} = a^{-1}\xi, \quad \hat{\eta} = a\eta, \quad \hat{g} = ag + (a^2 - a)\eta \otimes \eta.$$

Such a transformation is called a $D$-homothety. This defines an equivalence relation between Sasaki structures on a given manifold.

### 2. Sasaki Lie algebras

Throughout the paper, we consider left-invariant structures on Lie groups, which can be characterized at the Lie algebra level. Accordingly, we shall refer to pseudo-Riemannian metrics on a Lie algebra, Sasaki structures etc. to mean
objects defined at the Lie algebra level and silently extended to the Lie group by left translation.

Recall from [6] that a standard decomposition on a Lie algebra \( \mathfrak{g} \) endowed with a pseudo-Riemannian metric is an orthogonal decomposition \( \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{a} \), with \( \mathfrak{g} \) nilpotent and \( \mathfrak{a} \) abelian. A standard decomposition is pseudo-Iwasawa if \( \text{ad} X \) is symmetric for all \( X \in \mathfrak{a} \). These definitions mimic and generalize analogous definitions for Riemannian metrics (see [12]), and they have proved useful in the study of Einstein metrics ([6]).

It is well known that nonisomorphic Lie algebras can be isometric, meaning that the corresponding pseudo-Riemannian manifolds are isometric. The method to obtain such isometries is recalled below in Proposition 2.2. A natural question is whether one can choose a representative in an isometry class of Sasaki Lie algebras which admits a pseudo-Iwasawa decomposition. We show that this is never the case: indeed, no Sasaki Lie algebras admits a pseudo-Iwasawa decomposition. This will motivate the study of the more general standard case in the following sections.

We begin this section with an example of a standard Sasaki Lie algebra.

**Example 2.1.** Consider the 5-dimensional Lie algebra
\[
\mathfrak{g} = (0, -2e^{12} - 2e^{34}, -3e^{45}, -e^{13} + 3e^{24}, 3e^{35} - 3e^{23} - e^{14}, 2e^{12} + 2e^{34});
\]
with notation as in [15]; explicitly, we have a fixed basis \( \{e_i\} \) of \( \mathfrak{g} \) such that the dual basis \( \{e^i\} \) of \( \mathfrak{g}^* \) satisfies
\[
de 1 = 0, \quad de 2 = -2e^1 \wedge e^2 - 2e^3 \wedge e^4 \text{ and so on,}
\]
with \( d: \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^* \) denoting the Chevalley-Eilenberg operator. As observed in [8, Example 5.6], the Lie algebra \( \mathfrak{g} \) carries an Einstein-Sasaki structure given by
\[
\mathfrak{g} = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5, \\
\xi = e_5, \quad \Phi = e^{12} + e^{34}.
\]
This has a standard decomposition \( \text{Span} \{e_1\} \times \text{Span} \{e_2, e_3, e_4, e_5\} \). Notice that this metric can be obtained from the Riemannian \( \eta \)-Einstein-Sasaki metric on the Lie algebra \( \mathfrak{g}_0 \) of [1] by reversing the sign of the metric along the Reeb vector field.

Given a Lie algebra \( \mathfrak{g} \) with a metric \( g \), for any endomorphism \( f: \mathfrak{g} \to \mathfrak{g} \) we write \( f = f^s + f^a \), where \( f^s \) is symmetric and \( f^a \) is skew-symmetric relative to the metric, i.e.,
\[
f^s = \frac{1}{2}(f + f^*), \quad f^a = \frac{1}{2}(f - f^*).
\]
Consider a semidirect product \( \tilde{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{a} \), with \( \mathfrak{a} \) abelian, and fix any metric. In [10, Section 1.8] and [6, Proposition 1.19] it was shown that under certain conditions one can obtain an isometric Lie algebra by projecting on the symmetric part. These results assume that the decomposition is standard; however, the proof holds more generally, without assuming that the metric is standard and taking more general projections:
Proposition 2.2. Let $\tilde{g}$ be a pseudo-Riemannian Lie algebra (not necessarily standard) of the form $\tilde{g} = g \times a$; let $\chi : a \to \text{Der}(g)$ be a Lie algebra homomorphism such that, extending $\chi(X)$ to $\tilde{g}$ by declaring it to be zero on $a$,

$$\chi(X)^* = (\text{ad} X)^*,$$

$$[\chi(X), \text{ad} Y] = 0, \; X, Y \in a.$$  

Let $\tilde{g}^*$ be the Lie algebra $g \ltimes \chi a$. Then there is an isometry between the connected, simply connected Lie groups with Lie algebras $\tilde{g}$ and $\tilde{g}^*$, with the corresponding left-invariant metrics, whose differential at $e$ is the identity of $g \oplus a$ as a vector space.

Proof. Observe that for every $X$ in $a$, $\chi(X)$ is a derivation of $g$ that commutes with $\text{ad} a$ by (2.1), and therefore a derivation of $\tilde{g}$. For $X$ in $a$, write $\text{ad} X = A(X) + \chi(X)$, where $A(X)$ is an antisymmetric derivation of $\tilde{g}$. By construction, $A(X)$ is zero on $a$.

The rest of the proof is identical to [6, Proposition 1.19], except that one replaces $(\text{ad} X)^a$ with $A(X)$, and one cannot assume that $\exp g \exp a$ equals the whole connected, simply-connected group $\hat{G}$ with Lie algebra $\hat{g}$; however, it is clear that $\exp A(X)$ fixes the connected subgroup with Lie algebra $a$, which is what is needed.

As a consequence we have a result analogous to [6, Proposition 1.19] for nonstandard metrics:

Corollary 2.3. Let $\tilde{g}$ be a pseudo-Riemannian Lie algebra of the form $\tilde{g} = g \times a$; suppose that, for every $X$ in $a$, $(\text{ad} X)^*$ is a derivation of $\tilde{g}$ vanishing on $a$, and furthermore

$$[\text{ad} X]^*, \text{ad} Y] = 0, \; X, Y \in a.$$

Define $\chi : a \to \text{Der}(g)$ as $\chi(X) = (\text{ad} X)^*$. Let $\tilde{g}^*$ be the solvable Lie algebra $g \ltimes \chi a$. Then there is an isometry between the connected, simply connected Lie groups with Lie algebras $\tilde{g}$ and $\tilde{g}^*$, with the corresponding left-invariant metrics, whose differential at $e$ is the identity of $g \oplus a$ as a vector space.

Example 2.4. We can apply Proposition 2.2 to Example 2.1 with $a = \text{Span} \{e_5\}$, $g = \text{Span} \{e_1, e_2 - e_5, e_3, e_4\}$ to obtain an isometric Lie algebra

$$\tilde{g} = (0, -2e_{12} - 2e_{34}, -e_{13}, -e_{14}, 2e_{12} + 2e_{34}),$$

$$g = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5,$$

$$\xi = e_5, \quad \Phi = e_{12} + e_{34}.$$

This can be written as $\text{Span} \{e_2, e_3, e_4, e_5\} \times \text{Span} \{e_1\}$, with

$$\text{Span} \{e_2, e_3, e_4, e_5\} \cong (-2E^{23}, 0, 0, 2E^{23})$$

and

$$\text{ad} e_1 = 2e^2 \otimes (e_2 - e_5) + e^3 \otimes e_3 + e^4 \otimes e_4.$$

This is standard but not pseudo-Iwasawa, consistently with Proposition 2.6 below.
In the following, we will need the explicit formula for the Levi-Civita connection of a metric on a Lie algebra, namely
\begin{equation}
\nabla_w v = - \text{ad}(v)^* w - \frac{1}{2} \text{ad}(w)^* v.
\end{equation}

The formula follows immediately from the Koszul formula. In order to specialize to the standard case, we will need to fix an orthogonal basis \( \{ e_s \} \) on the abelian factor \( a \) such that \( ˜g(e_s, e_s) = \epsilon_s \).

**Lemma 2.5.** Let \( \tilde{g} \) be a Lie algebra with a standard decomposition \( \tilde{g} = g \oplus a \). Then
\[ ˜\nabla_H X = ˜\text{ad}(H)^*(X), \quad ˜\nabla_X H = - ˜\text{ad}(H)^*(X), \]
for all \( H \in a, \ X \in \tilde{g} \). In addition, if \( \{ e_i \} \) is an orthogonal basis of \( a \) and \( v, w \in g \), we have
\[ ˜\nabla_w v = - \frac{1}{2} \text{ad}(v)^* w - \frac{1}{2} \text{ad}(w)^* v + \sum_s \epsilon_s \tilde{g}(\text{ad}(e_s)^* v, w)e_s, \quad v, w \in g. \]

**Proof.** If we apply (2.2) to \( ˜\nabla \), we get
\[ ˜\nabla_H X = - ˜\text{ad}(H)^* X - \frac{1}{2} (\text{ad}H)^* X \]
\[ = - \frac{1}{2} \text{ad}(X)^* H - \frac{1}{2} \text{ad}(H)^* X = ˜\text{ad}(H)^*(X), \]
\[ ˜\nabla_X H = - ˜\text{ad}(H)^* X - \frac{1}{2} (\text{ad}X)^* H = - ˜\text{ad}(H)^*(X). \]

Now observe that \( ˜\text{ad}(v)^* w = \text{ad}(v)^* w + \sum_s \epsilon_s \tilde{g}([v, e_s], w)e_s \). Therefore,
\[ ˜\nabla_w v = - \frac{1}{2} \text{ad}(v)^* w - \frac{1}{2} \text{ad}(v)^* w - \frac{1}{2} \text{ad}(w)^* v \]
\[ = - \frac{1}{2} \text{ad}(v)w - \frac{1}{2} \text{ad}(v)^* w - \frac{1}{2} \text{ad}(w)^* v \]
\[ = - \frac{1}{2} \sum_s \epsilon_s \tilde{g}([v, e_s], w)e_s - \frac{1}{2} \sum_s \epsilon_s \tilde{g}([w, e_s], v)e_s \]
\[ = - \text{ad}(v)^* w - \frac{1}{2} \text{ad}(w)^* v \]
\[ + \frac{1}{2} \sum_s \epsilon_s (\tilde{g}(\text{ad}(e_s)^* v, w) + \tilde{g}(\text{ad}(e_s)^* v, w))e_s. \]

We can now prove the following:

**Proposition 2.6.** Let \( \tilde{g} \) be a solvable Lie algebra with a Sasaki pseudo-Riemannian metric \( g \). Then there is no pseudo-Iwasawa decomposition.

**Proof.** Assume for a contradiction that \( \tilde{g} = g \oplus a \) is a pseudo-Iwasawa decomposition. Then by Lemma 2.5 and Lemma 1.3 we have
\[ 0 = ˜\nabla_H \xi = - \phi(H), \quad H \in a. \]
This implies that $a$ is one-dimensional and spanned by $\xi$. We have
\[-\phi X = \tilde{\nabla}_X \xi = -\text{ad}(\xi)X.\]
However $\phi$ is skew-symmetric, while $\tilde{\text{ad}}(\xi)$ is symmetric, giving a contradiction.

\[\square\]

3. Sasaki structures on rank-one standard Lie algebras

In this section we consider standard decompositions of rank one, meaning that the abelian factor $a$ is one-dimensional. Accordingly, $\tilde{g}$ will be a solvable Lie algebra endowed with a standard decomposition $g \rtimes D\text{ Span}\{e_0\}$, with $D$ a derivation of $g$, and $\text{ad} e_0 = D$; we will denote by $[,]$ and $d$ the Lie bracket and exterior derivative on $g$.

Lemma 3.1. Let $g$ be a nilpotent Lie algebra with a pseudo-Riemannian metric $g$, let $D$ be a derivation, and let $\tau = \pm 1$. Then $\tilde{g} = g \rtimes D\text{ Span}\{e_0\}$ has an almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ such that
\[\tilde{g} = g + \tau e_0 \otimes e_0, \quad \tilde{\nabla}_\xi = -\phi\]
if and only if $\xi \in g$ and, writing $b = D\text{ ad}(\xi)$, for all $u, w \in g$,
\[
\begin{align*}
\phi(w) &= \frac{1}{2}(\text{ad } w)^* (\xi) + \tau g(b, w)e_0, & \phi(e_0) &= -b, \\
D(\xi) &= 0, & (\text{ad } \xi)^* &= 0, & (\text{ad } b)^*(\xi) &= 0, \\
g(w, u) &= g(\xi, w)g(\xi, u) + \tau g(b, w)g(b, u) + \frac{1}{4}g((\text{ad } w)^* \xi, (\text{ad } u)^* \xi).
\end{align*}
\]

Proof. Given $\tilde{g} = g + \tau e_0 \otimes e_0$ and $\xi \in \tilde{g}$, define $\eta = \xi^a$ and $\phi = -\tilde{\nabla}_\xi$.

Write
\[\xi = v + ae_0, \quad v \in g, a \in \mathbb{R}.\]
By Lemma 2.5, we have
\[
\begin{align*}
\tilde{\nabla}_w \xi &= \tilde{\nabla}_w v + a\tilde{\nabla}_w e_0 = -\text{ad}(v)^* w - \frac{1}{2}(\text{ad } w)^* v + \tau\tilde{g}(D^a(w), v)e_0 - aD^a(w), \\
\tilde{\nabla}_{e_0} \xi &= D^a(v).
\end{align*}
\]
Since $\tilde{\phi}(X) = -\tilde{\nabla}_X \xi$, we can write
\[
\phi(w) = \text{ad}(v)^* w + \frac{1}{2}(\text{ad } w)^* v - \tau\tilde{g}(D^a(w), v)e_0 + aD^a(w),
\]
\[
\phi(e_0) = -D^a(v).
\]
This determines an almost-contact metric structure if and only if $\phi$ is skew-symmetric and
\[
\tilde{g}(X, Y) - \eta(X)\eta(Y) = \tilde{g}(\phi X, \phi Y).
\]
The skew-symmetric condition implies
\[
0 = \tilde{g}(\phi(w), e_0) + \tilde{g}(\phi(e_0), w) = -\tau^2\tilde{g}(D^a(w), v) - \tilde{g}(D^a(v), w) = -\tilde{g}(D(v), w)
\]
for all $w$ in $\mathfrak{g}$, giving $D(v) = 0$. In addition,

\[
0 = \tilde{g}(\phi(w), u) + \tilde{g}(\phi(u), w)
\]

\[
= g(\text{ad}(v)^* w, u) + g(\text{ad}(v)^* u, w) + \frac{1}{2}g((\text{ad} v)^* v, u)
\]

\[
+ \frac{1}{2}g((\text{ad} u)^* v, w) + ag(D^\alpha(w), u) + ag(D^\alpha(u), w)
\]

\[
= 2g(\text{ad}(v)^* w, u) + 2ag(D^\alpha(w), u),
\]

giving $\text{ad}(v)^* + aD^\alpha = 0$ and

\[
\phi(w) = \frac{1}{2}(\text{ad} w)^* (v) - \tau g(D^\alpha(v), w)e_0 = \frac{1}{2}(\text{ad} w)^*(v) + \tau g(D^\alpha(v), w)e_0.
\]

Evaluating (3.4) on $w, e_0$ we get

\[
-\alpha \tau g(v, w) = \tilde{g}(w, e_0) - \eta(w)\eta(e_0) = \tilde{g}(\phi(w), \phi(e_0))
\]

\[
= \tilde{g}(\frac{1}{2}(\text{ad} w)^*(v) + \tau g(D^\alpha(v), w)e_0, -D^\alpha(v))
\]

\[
= g(\frac{1}{2}(\text{ad} w)^* v + \tau g(D^\alpha(v), w)e_0, -D^\alpha(v))
\]

\[
= \frac{1}{2}g((\text{ad} w)^* v, D^\alpha(v))
\]

\[
= \frac{1}{2}g(v, [w, D^\alpha(v)]) = \frac{1}{2}g(w, (\text{ad} D^\alpha(v))^* v).
\]

This holds for all $w$ if and only if $(\text{ad} D^\alpha(v))^* v = -2\alpha \tau v$. Since $\mathfrak{g}$ is nilpotent, the operator $\text{ad} D^\alpha(v)$ and its transpose are nilpotent, so $\alpha = 0$ and $(\text{ad} D^\alpha(v))^* v = 0$. Therefore, $\xi = v$, $b = D^\alpha(v)$ and $(\text{ad} b)^* v = 0$, showing that $\phi$ takes the form (3.1) and $\xi$ satisfies (3.2). Evaluating (3.4) on $w, u$ gives

\[
g(w, u) - g(w, \xi)g(u, \xi) = \tilde{g}(\phi(w), \phi(u))
\]

\[
= g(\frac{1}{2}(\text{ad} w)^* \xi + \tau g(b, w)e_0, \frac{1}{2}(\text{ad} u)^* \xi + \tau g(b, u)e_0)
\]

\[
= \frac{1}{4}g((\text{ad} w)^* \xi, (\text{ad} u)^* (\xi)) + \tau g(b, w)g(b, u),
\]

proving (3.3).

Lastly, evaluating (3.4) on $e_0, e_0$ we get

\[
\tau = \tilde{g}(e_0, e_0) - \eta(e_0)\eta(e_0) = \tilde{g}(-b, -b) = g(b, b);
\]

however, this is a redundant condition, for $g(b, \xi) = g(D^\alpha(\xi), \xi) = 0$, so (3.3) and (3.2) imply $g(b, u) = \tau g(b, b)g(b, u)$ for all $u$, which is equivalent to $g(b, b) = \tau$.

The converse is proved in the same way. \hfill \square

Now observe that we can write

\[
g((\text{ad} w)^* (v), u) = g(v, [w, u]) = -dv^\flat(w, u) = -g((w, dv^\flat), u),
\]
so \((\text{ad} w)^* (\xi) = -(w \lrcorner d\eta)^T\). Recall that \(d\) denotes the Chevalley-Eilenberg operator on \(\mathfrak{g}\), not \(\tilde{\mathfrak{g}}\).

**Lemma 3.2.** Let \(g\) be a metric on a Lie algebra \(\mathfrak{g}\), let \(\Phi\) be a 2-form. Then
\[
\nabla_x \Phi = \frac{1}{2} \mathcal{L}_x \Phi - \frac{1}{2} (\text{ad} x)^* \Phi + \frac{1}{2} \alpha_x^\Phi,
\]
where
\[
\alpha_x^\Phi(u, w) = \Phi(\text{ad}(u)^*(x), w) - \Phi(\text{ad}(w)^*(x), u).
\]

**Proof.** Using (2.2) we have:
\[
\nabla_x \Phi(u, w) = - \Phi(\nabla_x u, w) - \Phi(u, \nabla_x w) = \frac{1}{2} (\Phi((\text{ad} x)^* u + (\text{ad} u)^x, w) - \Phi((\text{ad} x)^* w + (\text{ad} w)^x, u)) \]
\[
= - \frac{1}{2} (\text{ad} x)^* \Phi(u, w) - \frac{1}{2} \Phi(\mathcal{L}_x u, w) + \frac{1}{2} \Phi(\mathcal{L}_x w, u) + \frac{1}{2} \alpha_x^\Phi(u, w) \]
\[
= - \frac{1}{2} (\text{ad} x)^* \Phi(u, w) + \frac{1}{2} \mathcal{L}_x \Phi(u, w) + \frac{1}{2} \alpha_x^\Phi(u, w). \]
\]

**Proposition 3.3.** Let \(\mathfrak{g}\) be a nilpotent Lie algebra with a pseudo-Riemannian metric \(g\), let \(D\) be a derivation and \(\tau = \pm 1\). Then \(\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{d} \text{ Span} \{e_0\}\) has a Sasaki structure \((\varphi, \xi, \eta, \tilde{\mathfrak{g}})\) such that \(\tilde{\mathfrak{g}} = \mathfrak{g} + \tau e_0 \otimes e_0\) if and only if for some \(\xi \in \mathfrak{g}\), \(b = D a(\xi), \eta = \xi \flat, \) writing \(\alpha_x(u, w) = d\eta(\text{ad}(u)^*(x), w) - d\eta(\text{ad}(w)^*(x), u)\), the following hold for \(x, y \in \mathfrak{g}\):
\[
D(\xi) = 0, \quad (\text{ad} \xi)^* = 0, \quad (\text{ad} b)^* (\xi) = 0,
\]
\[
D^a(d\eta) = 0, \quad D^a(b) = -\tau \xi,
\]
\[
\eta \lrcorner x^b = \frac{1}{4} \alpha_x - \frac{1}{4} (\text{ad} x)^* (d\eta) + \frac{1}{4} d(\mathcal{L}_x \eta) + \tau b \lrcorner \mathfrak{s}^b + D^s(x)^b,
\]
\[
D^s(x)^a d\eta + x^a d\eta^{b} + b^a d\eta^{b} + [x, b]^{b} = 0.
\]
Then \(\phi\) is given by
\[
\phi(w) = \frac{1}{2} (\text{ad} w)^* (\xi) + \tau g(b, w)e_0, \quad \phi(e_0) = -b, \quad w \in \mathfrak{g}.
\]

**Proof.** Suppose \((\phi, \xi, \eta, \tilde{\mathfrak{g}})\) is a Sasaki structure as in the hypothesis. Since Sasaki structures satisfy \(\nabla_X \xi = -\phi(X)\), by Lemma 3.1 equations (3.1), (3.2), (3.3) hold. By Proposition 1.4, the Sasaki condition implies
\[
\eta \lrcorner X^3 = \nabla_X \Phi.
\]
We have
\[
\Phi(u, w) = \tilde{\Phi}(u, \phi(w)) = \frac{1}{2} g(u, (\text{ad} w)^*(\xi)) = -\frac{1}{2} g([u, w], \xi),
\]
Φ(e₀, w) = ˜g(e₀, φ(w)) = g(b, w).

Thus, (3.9) for \( X = e₀ \) implies

\[
0 = (\tilde{\nabla}_{e₀} Φ)(u, w) = -Φ(\tilde{\nabla}_{e₀} u, w) = -Φ(D^a(u), w) - Φ(u, D^a(w)) = \frac{1}{2} g([D^a(u), w], ξ) + \frac{1}{2} g([u, D^a(w)], ξ) = -\frac{1}{2} dη(D^a(u), w) - \frac{1}{2} dη(u, D^a(w)) = \frac{1}{2} (D^a dη)(u, w).
\]

Similarly,

\[
-τg(w, ξ) = (\tilde{\nabla}_{e₀} Φ)(e₀, w) = -Φ(e₀, \tilde{\nabla}_{e₀} w) = -Φ(e₀, D^a(w)) = -g(b, D^a(w)) = g(D^a(b), w),
\]
i.e., \( D^a(b) = -τξ \).

Then, (3.9) for \( X = x ∈ g \) gives

\[
g(u, ξ)g(x, w) - g(x, u)g(ξ, w) = (\tilde{\nabla}_{x} Φ)(u, w) = -Φ(\tilde{\nabla}_{x} u, w) = -Φ(\tilde{\nabla}_{x} D^a(u), w) - Φ(u, \tilde{\nabla}_{x} D^a(u)) = \frac{1}{2} g([\tilde{\nabla}_{x} D^a(u), w], ξ) - \frac{1}{2} g([u, \tilde{\nabla}_{x} D^a(u)], w) + \frac{1}{4} g([u, w] + (ad u)^*(x + (ad x)^* u, w] - [[u, w], x, ξ] + τ(b^♭ ∧ D^a(x^♭))(u, w) = \frac{1}{4} g([[ad u]^* x + (ad x)^* u, w] - [(ad u)^* x + (ad x)^* u, w] + [[u, w], x, ξ] + τ(b^♭ ∧ D^a(x^♭))(u, w) = \frac{1}{4} dη(ad(u)^* x + (ad x)^* u, w) - \frac{1}{4} dη(ad(u)^* x + (ad x)^* w, u).
\]
\[-\frac{1}{4}d\eta(x, [u, w]) + \tau(b^* \wedge D^*(x)^\flat)(u, w)\]
\[= \frac{1}{4}\alpha_x(u, w) - \frac{1}{4}(\text{ad } x)^*\eta(u, w) + \frac{1}{4}d(L_x\eta)(u, w) + \tau(b^* \wedge D^*(x)^\flat)(u, w)\]
so
\[\eta \wedge x^b = \frac{1}{4}\alpha_x - \frac{1}{4}(\text{ad } x)^*\eta + \frac{1}{4}d(L_x\eta) + \tau(b^* \wedge D^*(x)^\flat).\]

Finally,
\[0 = (\nabla_x\Phi)(c_0, w) = -\Phi(\nabla_x\nabla_x c_0, w) - \Phi(c_0, \nabla_x\nabla_x w) = \Phi(D^*(x), w) - \Phi(c_0, \nabla_x w)\]
\[= \frac{1}{2}g([w, D^*(x)], \xi) - g(b, \nabla_x w)\]
\[= \frac{1}{2}g(D^*(x), (\text{ad } w)^*\xi) + g(b, \text{ad}(w)^*\xi) + \frac{1}{2}(\text{ad } x)^*\eta(w)\]
\[= -\frac{1}{2}d\eta(w, D^*(x)) + \frac{1}{2}g(b, \text{ad}(w)(x) + (\text{ad } w)^*\xi + (\text{ad } x)^*\eta).\]

Equivalently,
\[0 = -d\eta(w, D^*(x)) + g(b, \text{ad}(w)(x) + (\text{ad } w)^*\xi + (\text{ad } x)^*\eta)\]
\[= -d\eta(w, D^*(x)) + db^*(x, w) + dx^b(b, w) + g([x, b], w)\]
\[= (D^*(x), d\eta + x_\flat db^* + b_\flat dx^b + [x, b]^*\eta)(w).\]

Conversely, define \((\phi, \xi, \eta, \tilde{g})\) as in the statement, and assume that (3.5)–(3.8) hold. Since \(\text{ad } \xi\) is antisymmetric,
\[\text{ad } \xi = -(\text{ad } \xi)^*, \quad \xi_\flat d\eta = -(\text{ad } \xi)^*(\xi)^\flat = (\text{ad } \xi)(\xi)^\flat = 0.\]

Evaluating (3.7) on \(u, \xi\), one obtains
\[g(u, \xi)g(x, \xi) - g(x, u)\]
\[= \frac{1}{4}d\eta(\text{ad}(u)^*x + (\text{ad } x)^*u, \xi) - \frac{1}{4}d\eta(\text{ad}(\xi)^*x + (\text{ad } x)^*\xi, u)\]
\[-\frac{1}{4}d\eta(\eta([u, \xi]), u) + \tau(b^* \wedge D^*(x)^\flat)(u, \xi)\]
\[= \frac{1}{4}d\eta(\eta([u, \xi]), u) - \frac{1}{4}d\eta(\eta([u, \xi]), \xi)\]
\[-\frac{1}{4}d\eta((\text{ad } x)^*\xi, u) + \tau g(b, u)g(D^*(x), \xi)\]
\[= \frac{1}{4}d\eta([\eta, \xi]) + \frac{1}{4}(\text{ad } x)^*\xi + \tau g(b, u)g(x, D^*(x)\xi)\]
\[= -\frac{1}{4}g((\text{ad } u)^*\xi, (\text{ad } x)^*\xi) - \tau g(b, u)g(x, b),\]
which is equivalent to (3.3). Since (3.5) is assumed to hold and \(\phi\) is defined so as to satisfy (3.1), Lemma 3.1 implies that \((\phi, \xi, \eta, \tilde{g})\) is an almost contact metric structure. In order to prove that it is Sasaki, one only needs to verify that (3.9) holds, which follows from the computations above. \(\Box\)
Remark 3.4. The 2-form $\alpha_x$ of Proposition 3.3 corresponds to the 2-form $\alpha^x_{\Phi}$ of Lemma 3.2 with $\Phi$ equal to $d\eta$.

Remark 3.5. Using Lemma 3.2, we see that (3.7) can be rewritten as

$$\eta \wedge x^b = \frac{1}{2} \nabla_x d\eta + \tau b^b \wedge D^*(x)^b.$$  

Using equation (2.2), we can read condition (3.8) as:

$$D_s^*(x) \eta = \nabla_x b.$$  

Remark 3.6. It is well known that on a Sasaki Lie algebra $\tilde{g}$ the center is contained in $\text{Span}\{\xi\}$; indeed, any element of the center satisfies $v \downarrow d\eta = 0$, so it is a multiple of $\xi$.

If $\tilde{g}$ has nontrivial center, then $\mathfrak{z}(\tilde{g}) = \text{Span}\{\xi\}$ and the quotient $\tilde{g} = g/\text{Span}\{\xi\}$ has an induced pseudo-Kähler structure $(\tilde{g}, J, \omega)$ by Proposition 1.5.

Remark 3.7. The equations of Proposition 3.3 simplify if we assume that the center is nontrivial, because then $\text{ad}\xi = 0$. However, the center may be trivial on a Sasaki Lie algebra, see e.g. Example 2.1. It is noteworthy that Example 2.1 is isometric to a standard Lie algebra with nontrivial center (see Example 2.4).

4. $\mathfrak{z}$-Standard Sasaki structures

In this section we study the particular case where the vector $b$ of Proposition 3.3 is central in $\tilde{g}$. More precisely, we say that a Sasaki structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on a Lie algebra $\tilde{g}$ is $\mathfrak{z}$-standard if there is a standard decomposition $\tilde{g} = g \oplus_\mathfrak{z} \text{Span}\{e_0\}$ with $b = -\phi(e_0)$ in the center of $g$ and $\tilde{g} = g + \tau e_0 \otimes e_0$, with $\tau = \pm 1$.

We will start by giving a geometric interpretation of this condition; to that end, we will need to recall a well-known construction. Let $\tilde{g}$ be a Lie algebra with a Sasaki structure $(\tilde{\xi}, \tilde{\eta}, \tilde{\phi}, \tilde{\xi})$. Let $X$ be a nonzero vector in $\tilde{g}$. The associated, left-invariant Sasaki structure on the connected, simply connected group $\tilde{G}$ with Lie algebra $\tilde{g}$ is invariant under the left action of the group $\{\exp tX\}$. The fundamental vector field $X^*$ is defined by

$$X^*_g = \frac{d}{dt}(\exp tX)g,$$

so identifying $T_g\tilde{G}$ with $\tilde{g}$ by left-translation we get

$$L_{g^{-1}}X^*_g = \frac{d}{dt}g^{-1}(\exp tX)g = \text{Ad}(g^{-1})X.$$  

The moment map $\mu: \tilde{G} \to \mathbb{R}$ is by definition

$$\mu(g) = \eta(\text{Ad}(g^{-1})X).$$

Therefore,

$$d\mu_g(L_{g^*}v) = \frac{d}{dt}|_{t=0}\mu(g \exp tv).$$
\[ \frac{d}{dt} |_{t=0} \eta(\text{Ad}(\exp -tv) \text{Ad}(g^{-1})X) = -\eta([v, \text{Ad}(g^{-1})X]). \]

Now if \( \mu(g) = 0 \), \( \text{Ad}(g^{-1})X \in \ker \eta \). This implies that \( \text{Ad}(g^{-1})X \cdot \eta \) is nonzero, i.e., there is some \( v \) such that \( \eta([v, \text{Ad}(g^{-1})X]) \neq 0 \). Thus, 0 is a regular value and \( \mu^{-1}(0) \) is a hypersurface.

Since \( X^* \) is nowhere zero, the action of \( \{\exp tX\} \) is well defined on \( \mu^{-1}(0) \). Therefore, the quotient

\[ \tilde{G}/\{\exp tX\} = \mu^{-1}(0)/\{\exp tX\} \]

is well defined (locally), and it has an induced Sasaki structure.

A \( z \)-standard Sasaki structure can be characterized as follows:

**Lemma 4.1.** Let \( \tilde{\mathfrak{g}} \) be a Lie algebra with a Sasaki structure \( (\phi, \xi, \eta, \tilde{\mathfrak{g}}) \). The following are equivalent:

(i) there is a standard decomposition \( \tilde{\mathfrak{g}} = \mathfrak{g} \rtimes D \text{Span} \{e_0\} \) with \( \phi(e_0) \) in the center of \( \mathfrak{g} \);

(ii) \( \tilde{\mathfrak{g}} \) contains a vector \( X \) with \( \tilde{\mathfrak{g}}(X, X) \neq 0 \) such that its centralizer \( z(X) \) is a nilpotent ideal of codimension one;

(iii) the simply connected Lie group \( \tilde{G} \) with Lie algebra \( \tilde{\mathfrak{g}} \) has a one-parameter subgroup \( \{\exp tX\} \) such that

- \( \tilde{g}(X, X) \neq 0 \);
- the zero set of the moment map is a normal nilpotent subgroup \( G \);

and

- \( \{\exp tX\} \) commutes with \( G \).

**Proof.** If (i) holds, observe that \( e_0 \) is not a multiple of \( \xi \) by Proposition 3.3; thus, \( X = -\phi(e_0) \) has centralizer equal to \( \mathfrak{g} \). This implies (ii).

Now assume that (ii) holds; then \( \tilde{\mathfrak{g}} \) is solvable, as it contains a codimension one nilpotent ideal. The zero level set of the moment map \( \{g \mid \eta(\text{Ad}(g^{-1})X) = 0\} \) is the connected subgroup with Lie algebra \( z(X) \), giving (iii).

Finally, suppose that (iii) holds. Since \( \mu^{-1}(0) \) is a normal nilpotent subgroup, its Lie algebra is the nilpotent ideal

\[ \mathfrak{g} = \ker X_\cdot \eta. \]

In addition, \( \mu^{-1}(0) \) contains the identity, so \( \eta(X) = 0 \). This implies that \( \mathfrak{g} \) has codimension one. By construction, \( e_0 = \phi(X) \) is orthogonal to \( \mathfrak{g} \). Since \( X \) is not lightlike, the restriction of the metric to \( \mathfrak{g} \) is definite; hence we have a standard decomposition \( \tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span} \{e_0\} \). By construction, \( \phi(e_0) = -X \), so it is central in \( \mathfrak{g} \), giving (i). \( \square \)

Given a \( z \)-standard Sasaki structure, Lemma 4.1 implies that \( \{\exp tX\} \) is central in \( G \), so the right action of \( \{\exp tX\} \) preserves the Sasaki structure and the quotient \( G/\{\exp tX\} \) is a Lie group with Lie algebra \( z(X)/\text{Span} \{X\} \), which is Sasaki by construction. Conversely, we can express \( z(X) \) as a central extension of \( X \), and then express \( \mathfrak{g} \) as a standard extension of \( z(X) \).
Example 4.2. In Example 2.4, \( \{ \exp \tau e_2 \} \) satisfies the conditions of Lemma 4.1; the three-dimensional quotient in this case is the Heisenberg algebra, with its Sasaki structure.

In the language of Proposition 3.3, we can express this as follows:

**Corollary 4.3.** Let \( g \) be a nilpotent Lie algebra with a pseudo-Riemannian metric \( g \), \( D \) a derivation and \( \tau = \pm 1 \). Assume \( \tilde{g} = g \rtimes D \) Span \( \{ e_0 \} \) has a \( \mathfrak{z} \)-standard Sasaki structure \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\). Then the following hold for \( x \in g \):

\[
\begin{align*}
D(\xi) &= 0, \quad D(b) = -2\tau \xi + hb, \quad h \in \mathbb{R}, \quad b, \xi \in \mathfrak{z}(g), \\
D^*(d\eta) &= 0, \quad D(d\eta) = 2db^\flat, \\
\eta \wedge x^b &= \frac{1}{2} \nabla_x d\eta + \tau b^\flat \wedge D^*(x)^b, \\
d\eta(D^*(x), y) &= d\eta(x, D^*(y)).
\end{align*}
\]

Furthermore, \( \phi \) is given by

\[
\phi(w) = \frac{1}{2} (ad w)^*(\xi) + \tau g(b, w)e_0, \quad \phi(e_0) = -b, \quad w \in g.
\]

In addition, \( g/\text{Span} \{ b \} \) has a Sasaki structure \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) induced by the identification \( \text{Span} \{ e_0, b \} \perp \cong g/\text{Span} \{ b \} \); at the level of the corresponding Lie groups, this amounts to taking the Sasaki reduction by the left action of the one-parameter subgroup \( \{ \exp \tau b \} \).

**Proof.** We specialize Proposition 3.3 with \( b = -\phi(e_0) \) central. Then \((ad b)^* \) and \( b, dz^x \) are zero. In particular, from (3.8), we get

\[
D^*(x) \perp d\eta + x \perp db^\flat = 0.
\]

For \( x = b \), this implies \( D^*(b) \perp d\eta = 0 \). Since \( d\eta \) is nondegenerate on \( \text{Span} \{ b, \xi \} \), this implies that \( D^*(b) \in \text{Span} \{ b, \xi \} \). Furthermore, we have

\[
g(D^*(b), \xi) = g(b, D^*(\xi)) = g(b, -b) = -\tau,
\]

so \( D^*(b) = -\tau \xi + hb \) for some real constant \( h \). Therefore,

\[
D(b) = -2\tau \xi + hb.
\]

Since \( D \) is a derivation, we have

\[
0 = D[b, x] = [D(b), x] + [b, D(x)] = -2\tau [\xi, x].
\]

Therefore \( \xi \) is in the center of \( g \).

By (3.6), \( D^*(d\eta) = 0 \), so we observe that

\[
\begin{align*}
D^*d\eta(x, y) &= Dd\eta(x, y) = -d\eta(Dx, y) - d\eta(x, Dy) \\
&= \eta([Dx, y] + [x, Dy]) = \eta(D[x, y]) \\
&= -2g(b, [x, y]) = 2db^\flat(x, y).
\end{align*}
\]
Therefore, \( D(d\eta) = 2db^\flat \) and (4.1) becomes equivalent to
\[
0 = d\eta(D^s(x), y) + \frac{1}{2}(D^s d\eta)(x, y) = \frac{1}{2}(d\eta(D^s(x), y) - d\eta(x, D^s(y))).
\]

For the last part, observe that \( g \) is the centralizer of \( b \) in \( \tilde{g} \), and apply the observation before the statement. The fact that \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) is Sasaki can be seen from \( \eta \wedge x^\flat = \frac{1}{2}\tilde{\nabla}_x d\eta \). □

We can describe the situation of Corollary 4.3 in terms of the Kähler quotient as follows:

**Corollary 4.4.** Let \( g \) be a nilpotent Lie algebra with a pseudo-Riemannian metric \( g \), \( D \) a derivation and \( \tau = \pm 1 \). Assume \( \tilde{g} = g \rtimes D \text{Span} \{e_0\} \) has a \( z \)-standard Sasaki structure \((\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\). Then \( \tilde{\eta} \) is central in \( g \) and there is \( h \in \mathbb{R} \) such that

1. \( g(\xi, \xi) = 1, g(b, b) = \tau, g(b, \xi) = 0 \);
2. the quotient \( \tilde{g} = g / \text{Span} \{b, \xi\} \) has a pseudo-Kähler structure \((\tilde{g}, J, \omega)\) with \((g, g) \rightarrow (\tilde{g}, \tilde{g})\) a Riemannian submersion, \( \omega = \frac{1}{2}d\eta \) and \( \tilde{D}(\omega) = db^\flat \);
3. relative to the splitting \( \text{Span} \{b, \xi\} \perp \text{Span} \{b\} \oplus \text{Span} \{\xi\} \), \( D \) takes the form
   \[
   D = \begin{pmatrix} \tilde{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix};
   \]
   • \([J, \tilde{D}] = 0\);
   • \( \tilde{D} \) is a derivation and \([\tilde{D}^s, \tilde{D}^s] = h\tilde{D}^s - 2(\tilde{D}^s)^2 \).

**Proof.** Define \( b = \phi(e_0) \), hence \( g(\xi, \xi) = 1 \) by definition of Sasaki and
\[
g(b, b) = gb(b, b) = \tau, g(b, \xi) = 0, \quad g(b, b) = gb(e_0, e_0) = \tau
\]
give the first condition.

Let \( \tilde{g} = g / \text{Span} \{b, \xi\} \). Then arguing as in Proposition 1.5 we see that \( \tilde{\nabla}d\eta \) is the projection of \( \nabla d\eta \); projecting the equation (3.10), we see that \( d\eta \) is \( \tilde{\nabla} \)-parallel. Furthermore, for \( x \) orthogonal to \( b, \xi \), we get by taking the interior product of (3.10) with \( \xi \) that
\[
x^\flat = \frac{1}{2}\xi^\flat \nabla_x d\eta - g(D^s(x), \xi)\tau b^\flat = \frac{1}{2}\xi^\flat \nabla_x d\eta;
\]
using Lemma 3.2, we get
\[
(4.2) \quad x^\flat = \frac{1}{4}\xi^\flat (\alpha_x - (ad x)^* d\eta + \mathcal{L}_x d\eta) = \frac{1}{4}(ad x)^* \xi^\flat d\eta.
\]
This implies that \( d\eta \) is nondegenerate. Now set
\[
J(x) = -\frac{1}{2}(x^\flat d\eta)^\sharp.
\]
Then in $\text{Span} \{ b, \xi \}^\perp$ equation (4.2) reads

$$x^\flat = -\frac{1}{4} (x, \cdot) \cdot d\eta = \frac{1}{2} J(x, \cdot) \cdot d\eta = -(J \circ J(x))^\flat = -(J^2(x))^\flat;$$

therefore, $J$ is an almost complex structure, and $(\tilde{g}, J, d\eta)$ is a pseudo-Kähler structure. In particular, we can write

$$d\eta(x, y) = 2g(x, Jy).$$

Now from Corollary 4.3 write

$$d\eta(D^a(x), y) = d\eta(x, D^a(y))$$

as

$$g(JD^a(x), y) = g(Jx, D^a(y)) = -g(x, JD^a(y));$$

i.e., $JD^a = -(JD^a)^+ = D^a J$. In addition, $D^a d\eta = 0$ can be rewritten as

$$0 = D^a d\eta(x, y) = d\eta(D^a x, y) + d\eta(x, D^a y) = 2g(D^a x, Jy) + 2g(x, JD^a y) = 2g(x, [J, D^a] y).$$

This shows that $J$ and $D$ commute.

The Lie bracket on $\tilde{g}$ and the Lie bracket on $g$ are related by

$$[x, y] = [x, y]_{\tilde{g}} - \tau db^\flat(x, y)b - d\eta(x, y)\xi;$$

$b, \xi$ are in the center for $g$. Relative to the splitting $\text{Span} \{ b, \xi \}^\perp \oplus \text{Span} \{ b \} \oplus \text{Span} \{ \xi \}$, $D$ takes the form

$$D = \begin{pmatrix} \tilde{D} & 0 & 0 \\ 0 & b & 0 \\ 0 & -2\tau & 0 \end{pmatrix}.$$

A linear map $D$ of the form (4.3) automatically satisfies $D[x, y] = [Dx, y] + [x, Dy]$ when $x$ lies in $\text{Span} \{ b, \xi \}$; therefore, $D$ is a derivation if and only if for $x, y$ in $\text{Span} \{ b, \xi \}^\perp$ one has

$$0 = D[x, y] - [Dx, y] - [x, Dy] = \tilde{D}[x, y]_{\tilde{g}} - \tau db^\flat(x, y)(hb - 2\tau \xi) - [\tilde{D}x, y]_{\tilde{g}} + \tau db^\flat(\tilde{D}x, y)b + d\eta(\tilde{D}x, y)\xi - [x, \tilde{D}y]_{\tilde{g}} + \tau db^\flat(x, \tilde{D}y)b + d\eta(x, \tilde{D}y)\xi.$$

Thus, $D$ is a derivation if and only if $\tilde{D}$ is a derivation of $\tilde{g}$ and

$$hdb^\flat(x, y) = db^\flat(\tilde{D}x, y) + db^\flat(x, \tilde{D}y);$$

$$-2db^\flat(x, y) = d\eta(\tilde{D}x, y) + d\eta(x, \tilde{D}y),$$

where the latter is again $2db^\flat = Dd\eta$.

Then using $[J, D] = 0,$

$$db^\flat(x, y) = -\frac{1}{2} \tilde{D}d\eta(x, y) = \frac{1}{2} d\eta(\tilde{D}x, y) + \frac{1}{2} d\eta(x, \tilde{D}y) = g(\tilde{D}x, Jy) + g(x, J\tilde{D}y) = g(x, (D^a J + J\tilde{D}) y)$$

as

$$g(JD^a(x), y) = g(Jx, D^a(y)) = -g(x, JD^a(y)).$$
Thus
\[ 2hg(x, \tilde{\theta}) = hdb(x, y) = db(\bar{D}x, y) + db(x, \bar{D}y) = 2g(\bar{D}x, \bar{\theta}) + 2g(x, \bar{\theta}). \]

Therefore,
\[ h\tilde{\theta} = (\bar{D} - \bar{\theta})\bar{\theta} + \bar{\theta} \bar{\theta} = 2(\bar{\theta})^2 + [\bar{\theta}, \bar{\theta}]. \]

\[ \square \]

In the situation of Corollary 4.4, we will say that the pseudo-Kähler Lie algebra \( \tilde{\theta} \) is the \textit{Kähler reduction} of the \( \bar{\theta}\)-standard Sasaki structure of \( \tilde{\theta} \). Notice that \( \tilde{\theta} \) is indeed a Kähler reduction in the sense of symplectic geometry, arising from the action of \( \{\exp tb\} \) on the pseudo-Kähler nilmanifold \( \tilde{\theta}/\text{Span} \{\xi\} \).

**Example 4.5.** In Example 2.4, we have
\[ \tilde{\theta} = \text{Span} \{e_3, e_4\}, \quad \bar{D} = I, \quad b = -e_2, \quad h = 2, \quad \tau = -1, \]
\[ \omega = e^{34}, \quad db = de^2 = -2e^{34}, \quad d\eta = 2e^{34}. \]

Corollary 4.3 has a Kähler analogue, which can be viewed as a consequence of Corollary 4.4, using the fact that any pseudo-Kähler Lie algebra yields a Sasaki Lie algebra by taking a central extension. Notice that this construction only works one way in general, i.e., it is not generally true that a Sasaki Lie algebra is a central extension of a pseudo-Kähler Lie algebra. This only occurs when \( \xi \) is central, which happens to be true in the situation of Corollary 4.4.

**Proposition 4.6.** Let \( \theta \) be a nilpotent Lie algebra with a pseudo-Riemannian metric \( g \), let \( D \) be a derivation and \( \tau = \pm 1 \). Suppose that \( \tilde{\theta} = \theta \times D \text{Span} \{e_0\} \) has a pseudo-Kähler structure \((\tilde{\eta}, \tilde{\omega})\) such that \( \tilde{\theta} = g + \tau e_0 \otimes e_0 \), with \( b = -Je_0 \) in the center of \( \theta \). Then

1. the quotient \( \tilde{\theta} = \theta/\text{Span} \{b\} \) has a pseudo-Kähler structure \((\tilde{\eta}, \tilde{\omega})\) with \( \pi: (\theta, g) \rightarrow (\tilde{\theta}, \tilde{\omega}) \) a Riemannian submersion, \( \pi^* \omega = \tilde{\omega} \) and \( D(\omega) = db \);
2. relative to the splitting \( \text{Span} \{b\} \perp \text{Span} \{b\} \), \( D \) takes the form
\[ D = \begin{pmatrix} \bar{D} & 0 \\ 0 & h \end{pmatrix}; \]
3. \( [\bar{\theta}, \bar{D}] = 0 \);
4. \( \bar{D} \) is a derivation and \([\bar{\theta}, \bar{\theta}] = h\bar{\theta} - 2(\bar{\theta})^2 \).
Proof. Write $\tilde{g} = \text{Span} \{ b \}^\perp$ in $g$, and let $\omega$ be the restriction of $\tilde{\omega}$ to $\tilde{g}$. Then $\tilde{\omega} = \omega - \tau b \wedge e^0$.

Let $\mathfrak{h} = g \oplus \text{Span} \{ \xi \}$ be the central extension of $g$ by the cocycle $2\omega$, $\tilde{\mathfrak{h}}$ the quotient $\mathfrak{h}/\text{Span} \{ b \}$, and $\tilde{\mathfrak{h}}$ the semidirect product $\mathfrak{h} \rtimes_{D'} \text{Span} \{ e_0 \}$, where $D'$ is defined by

$$D'v = Dv, \quad v \in \tilde{g}, \quad D'\xi = 0, \quad D'b = Db - 2\tau \xi.$$ 

We can summarize the situation as follows

$$\tilde{\mathfrak{h}} = g \oplus \text{Span} \{ \xi \}, \quad \tilde{\mathfrak{h}} = g \oplus \text{Span} \{ b, \xi \}, \quad \tilde{\mathfrak{h}} = g \oplus \text{Span} \{ b, \xi, e_0 \}.$$ 

We can view equivalently $\tilde{\mathfrak{h}}$ as the central extension of $\tilde{g}$ by $2\tilde{\omega}$. In particular, $\tilde{\mathfrak{h}}$ has a Sasaki metric $(\tilde{\phi}, \tilde{\eta}, \tilde{\mathfrak{h}}, \tilde{\eta})$ induced by the pseudo-Kähler metric of $\tilde{g}$ (see [11]). Explicitly, $\tilde{\eta}$ is the 1-form on $\tilde{\mathfrak{h}}$ that vanishes on $\tilde{g}$, with $\tilde{\eta}(\xi) = 1$, so that $d\eta = 2\tilde{\omega}$, we have

$$\tilde{\mathfrak{h}} = \tilde{\mathfrak{g}} + \tilde{\eta} \otimes \tilde{\eta}, \quad \tilde{\phi} = \tilde{J}.$$ 

Since $b$ is central in $\mathfrak{h}$, we can apply Corollary 4.4. Then $(\tilde{g}, \tilde{J}, \tilde{\omega})$ is pseudo-Kähler, and $\tilde{D}\omega = db^\flat$,

$$D' = \begin{pmatrix} \hat{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix},$$ 

proving (1) and (2). (3) and (4) follow directly from Corollary 4.4. □

5. Construction of $z$-standard Sasaki structures

In this section we invert the reduction process of Corollary 4.4 and describe a constructive way of obtaining $z$-standard Sasaki structures. We also classify $z$-standard Sasaki structures of dimension $\leq 7$ whose Kähler reduction is abelian.

Proposition 5.1. Let $(\tilde{g}, J, \omega)$ be a pseudo-Kähler nilpotent Lie algebra. Let $\tilde{D}$ be a derivation of $\tilde{g}$, $\tau = \pm 1$, and $\tilde{g} = g \oplus \text{Span} \{ b, \xi \}$ a central extension of $g$ with a metric of the form:

$$g(x, y) = \tilde{g}(x, y), \quad g(x, b) = 0 = g(x, \xi), \quad g(\xi, \xi) = 1, \quad g(b, b) = \tau, \quad g(b, \xi) = 0,$$

where $x, y \in \tilde{g}$. Assume furthermore

- $d\xi^0 = 2\omega$, where the right-hand-side is implicitly pulled back to $g$;
- $db^0 = D\omega$, where the right-hand-side is implicitly pulled back to $g$;
- $[J, \tilde{D}] = 0$;
- $[\tilde{D}^s, \tilde{D}^s] = h\tilde{D}^s - 2(\tilde{D}^s)^2$ for some constant $h$.

Let $\tilde{g} = g \rtimes \text{Span} \{ e_0 \}$, where

$$[e_0, x] = \hat{D}x, \quad [e_0, b] = hb - 2\tau \xi, \quad [e_0, \xi] = 0.$$ 

Then $\tilde{g}$ has a $z$-standard Sasaki structure $(\phi, \eta, \xi, \tilde{g})$ given by

$$\tilde{g} = g + \tau e^0 \otimes e^0, \quad \phi(x) = J(x) + \tau g(b, x)e_0, \quad \phi(e_0) = -b, \quad x \in g.$$
Corollary 4.4. Proof. The fact that $D = \dot{D} + \tau b^\flat \otimes (hb - 2\tau \xi)$ is a derivation is proved as in Corollary 4.4.

Then we use Proposition 3.3. To prove (3.8), write
\[
\begin{align*}
\text{db}(y, x) &= D\omega(y, x) - \omega(Dy, x) - \omega(y, Dx) \\
&= -g(\dot{D}y, Jx) - g(y, J\dot{D}x) = -g(y, (\dot{D}^* J + J\dot{D})x) \\
&= -g(y, J(\dot{D} + \dot{D}^*)x) = -2\omega(y, \dot{D}^*x) = -\eta(y, \dot{D}^*x);
\end{align*}
\]
then $D^* x, db + x, db^\flat = 0$, which is equivalent to (3.8) since $b$ is central.

To prove (3.10), notice that projecting this equation to $\Lambda^2 \mathfrak{g}$ simply says that $\omega$ is parallel on $\mathfrak{g}$. The interior product with $\xi$ yields (4.2), which holds by construction. Finally, taking interior product of (3.10) with $b$ and using the fact that $D^*(b) \in \text{Span}\{b, \xi\}$, we compute
\[
0 = \frac{1}{4} b^\flat (\alpha_x - (\text{ad} x)^* d\eta + \mathcal{L}_x d\eta) + D^*(x)^b
\]
\[
= \frac{1}{4} ((\text{ad} x)^* b^\flat d\eta) + D^*(x)^b = \left(\frac{1}{2} J((\text{ad} x)^* b) + D^*(x)\right)^b.
\]
We also have $\text{ad}(x)^* b = \text{ad}(D^*(x))^* \xi = -2J(D^*(x))$. Therefore, this equation reduces to $J^2(D^*(x)) = -D^*(x)$, which is automatically satisfied.

The other hypotheses of Proposition 3.3 are trivially satisfied; therefore, $\mathfrak{g}$ has a Sasaki structure with
\[
\phi(w) = \frac{1}{2} (\text{ad} w)^* \xi + \tau g(b, w)e_0 = -w, \omega + \tau (g, b, w)e_0 = Jw + \tau (g, b, w)e_0.
\]

Remark 5.2. It is no loss of generality to assume $h \geq 0$; indeed, changing the sign of $D$, $e_0$, $b$ and $h$ gives the same Sasaki Lie algebra up to isometric isomorphism.

Remark 5.3. The hypotheses of Proposition 5.1 are preserved if one rescales both $h$ and $\dot{D}$. This yields different metrics on $\mathfrak{g}$, which are however related by a $\mathcal{D}$-homothety (in particular, they have different curvature).

Accordingly, one can assume that either $h = 0$ or $h = 2$ up to $\mathcal{D}$-homothety. The condition $h = 0$ implies that $\text{tr}(\dot{D}^*)^2 = 0$. If $\mathfrak{g}$ is Riemannian, $\dot{D}^*$ is diagonalizable, so $h = 0$ implies that $\dot{D}$ is skew-symmetric.

Remark 5.4. One can always reverse the sign of the metric $\dot{g}$ and the 2-form $\omega$ and obtain a different Sasaki metric on an isomorphic Lie algebra $\dot{\mathfrak{g}}'$; the isomorphism is realized by the mapping $b \mapsto -b'$, $\xi \mapsto -\xi'$.

Let $(\mathfrak{g}_0, J_0, g_0, \omega_0), (\mathfrak{g}_1, J_1, g_1, \omega_1)$ be pseudo-Kähler Lie algebras, with $\mathfrak{g}_1$ abelian. Let $\rho: \mathfrak{g}_0 \rightarrow \mathfrak{g}l(\mathfrak{g}_1)$ be a representation such that
\[
(5.1) \quad \rho(X)\omega_1 = 0, \quad [J_1, \rho(X)] + [\rho(J_0 X), J_1] J_1 = 0.
\]
Then $\mathfrak{g}_0 \ltimes \mathfrak{g}_1$ has an almost Hermitian structure $(\dot{g}, \dot{J}, \omega)$, with $\dot{g} = g_0 + g_1$, $\omega = \omega_0 + \omega_1$, and $\dot{J} = (\begin{subarray}{c} J_0 & 0 \\ 0 & J_1 \end{subarray})$. It is straightforward to check that $\omega$ is closed and $J$ integrable, i.e., $\mathfrak{g}_0 \ltimes \mathfrak{g}_1$ is pseudo-Kähler. In addition, the projection $\pi_1$
on the factor \( \mathfrak{g}_1 \) is a derivation, giving a one-parameter family of derivations \( \tilde{D} = \frac{h}{2} \pi_1 \) that satisfy the hypotheses of Proposition 5.1. The resulting Sasaki extension \( \tilde{\mathfrak{g}} \) takes the form

\[
(\mathfrak{g}_0 \ltimes \mathfrak{g}_1 \oplus \text{Span } \{b, \xi\}) \rtimes \text{Span } \{e_0\}, \quad d\xi^b = 2\omega, \quad db^g = -h\omega,
\]

(5.2)

\[
[e_0, X_0] = 0, \quad [e_0, X_1] = \frac{h}{2} X_1, \quad [e_0, b] = hb - 2\tau \xi, \quad [e_0, \xi] = 0,
\]

where \( X_0 \) denotes the generic element of \( \mathfrak{g}_0 \) and \( X_1 \) the generic element of \( \mathfrak{g}_1 \).

**Proposition 5.5.** In the hypotheses of Proposition 5.1, if \( \tilde{D}^* \) is a derivation and \( [\tilde{D}^*, \tilde{D}^0] = 0 \), we can assume up to isometry that \( \tilde{\mathfrak{g}} \) is a semidirect product \( \tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes_{\rho} \mathfrak{g}_1 \), where \( \mathfrak{g}_0, \mathfrak{g}_1 \) are pseudo-Kähler with \( \mathfrak{g}_1 \) abelian, \( \tilde{D} = \frac{h}{2} \pi_1 \) and \( \tilde{\mathfrak{g}} \) takes the form (5.2).

**Proof.** Write \( \tilde{\mathfrak{g}} = \mathfrak{g} \ltimes \text{Span } \{e_0\} \), where \( \text{ad}(e_0) = \tilde{D} + hb^g \otimes (hb - 2\tau \xi) \). Then define

\[
\chi : \text{Span } \{e_0\} \to \text{Der } \mathfrak{g}, \quad \chi(e_0) = \tilde{D} + hb^g \otimes (hb - 2\tau \xi).
\]

Thus \( \chi(e_0)^* = \text{ad}(e_0)^* \) and \( [\chi(e_0), \text{ad } e_0] = 0 \). Thus, the Lie algebra \( \mathfrak{g} \ltimes_{\chi} \text{Span } \{e_0\} \) is isometric to the Lie algebra \( \tilde{\mathfrak{g}} \) constructed in Proposition 5.1. In other words, replacing \( \tilde{D} \) with \( D^* \) gives the same metric \( \tilde{g} \) up to isometry. In addition, \( D\omega = \tilde{D}\omega \), so \( dB^g \) is unchanged.

By Proposition 5.1, the minimal polynomial of \( \tilde{D} \) divides \( \rho(t) = ht - 2t^2 \). Thus \( \tilde{D} \) is diagonalizable over \( \mathbb{R} \), and takes the form

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & h^2 I
\end{pmatrix}
\]

in some basis; since \( \tilde{D} \) commutes with \( J \), its eigenspaces are \( J \)-invariant. Since it is symmetric, they are orthogonal. Since a diagonalizable derivation defines a grading, we have \( \tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes_{\rho} \mathfrak{g}_1 \), the Kähler form splits as \( \omega_0 + \omega_1 \) and

\[
J = \begin{pmatrix}
J_0 & 0 \\
0 & J_1
\end{pmatrix}.
\]

We have that \( (\mathfrak{g}_0, J_0, \omega_0) \) is Kähler, \( \mathfrak{g}_1 \) is abelian, and (5.1) holds.

**Corollary 5.6.** In the hypotheses of Proposition 5.1, if \( \tilde{D}^* \) is a derivation and it is diagonalizable over \( \mathbb{C} \), then we can assume up to isometry that \( \tilde{g} \) is a semidirect product \( \tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes_{\rho} \mathfrak{g}_1 \), where \( \mathfrak{g}_0, \mathfrak{g}_1 \) are pseudo-Kähler with \( \mathfrak{g}_1 \) abelian, \( \tilde{D} = \frac{h}{2} \pi_1 \) and \( \tilde{\mathfrak{g}} \) takes the form (5.2).

**Proof.** Denote by \( \tilde{\mathfrak{g}}^\mathbb{C} \) the complexification of \( \tilde{\mathfrak{g}} \), with the scalar product obtained by complexifying the scalar product of \( \tilde{\mathfrak{g}} \). The complexified endomorphisms \( (\tilde{D}^*)^\mathbb{C} : \tilde{\mathfrak{g}}^\mathbb{C} \to \tilde{\mathfrak{g}}^\mathbb{C} \), \( (\tilde{D}^0)^\mathbb{C} : \tilde{\mathfrak{g}}^\mathbb{C} \to \tilde{\mathfrak{g}}^\mathbb{C} \) are symmetric and antisymmetric, respectively. Furthermore, we get

(5.3)

\[
[(\tilde{D}^*)^\mathbb{C}, (\tilde{D}^0)^\mathbb{C}] = h((\tilde{D}^*)^\mathbb{C} - 2((\tilde{D}^*)^\mathbb{C})^2.
\]
By hypothesis, there exists an orthonormal basis of eigenvectors of $(\tilde{D}^s)^C$. Then $(\tilde{D}^s)^C$ is diagonal in this basis, and $(\tilde{D}^s)^C$ has zero on the diagonal. Therefore, $[(\tilde{D}^s)^C, (\tilde{D}^a)^C]$ has zero on the diagonal, so (5.3) implies that it vanishes and we can apply Proposition 5.5.

In particular, Corollary 5.6 classifies $\z$-standard Sasaki structures that reduce to an abelian Kähler Lie algebra, as positive-definiteness of the metric implies that $\tilde{D}^s$ is automatically a diagonalizable derivation in this case.

The case of indefinite signature is more flexible, as we will see below. Notice that the signature of a pseudo-Kähler metric is necessarily of the form $(2p, 2q)$.

Theorem 5.7. Let $\tilde{g}$ be a Lie algebra of dimension 5 with a $\z$-standard Sasaki structure. Then, up to isometry and $\mathcal{D}$-homothety, $\tilde{g}$ is one of

$$(0, 0, 0, -2e^{12} - 2\tau e^{35}, 0),$$

$$(0, 0, 2\tau e^{12} + 2e^{35}, -2e^{12} - 2\tau e^{35}, 0),$$

$$(e^{15}, e^{25}, 2\tau e^{12} + 2e^{35}, -2e^{12} - 2\tau e^{35}, 0),$$

and the Sasaki structure is given by

$$\tilde{g} = \pm(e^1 \otimes e^1 + e^2 \otimes e^2) + \tau e^3 \otimes e^3 + e^4 \otimes e^4 + e^5 \otimes e^5, \quad \xi = e_4, \quad \Phi = -e^{12} - \tau e^{35}.$$

Proof. The Kähler reduction $g$ is a nilpotent Lie algebra of dimension two, hence abelian. Assume first that $\tilde{g}$ has positive-definite signature. In some basis $\{e_1, e_2\}$, we have

$$\tilde{g} = e^1 \otimes e^1 + e^2 \otimes e^2, \quad \omega = -e^{12}, \quad J = e^1 \otimes e_2 - e^2 \otimes e_1.$$

Derivations that commute with $J$ lie in $\text{Span}\{I, J\}$. In particular, $\tilde{D}^s$ commutes with $\tilde{D}^a$, so Proposition 5.5 implies that up to isometry we can assume $\tilde{D} = 0$ or $\tilde{D} = \frac{1}{2} I$.

Up to $\mathcal{D}$-homothety, we can assume that either $h = 0$ or $h = 2$.

For $h = 0$, (5.2) gives

$$\tilde{g} = (0, 0, 0, -2e^{12} - 2\tau e^{35}, 0);$$

for $h = 2$, either $\tilde{D} = 0$ and

$$\tilde{g} = (0, 0, 2\tau e^{12} + 2e^{35}, -2e^{12} - 2\tau e^{35}, 0),$$

or $\tilde{D} = I$ and

$$\tilde{g} = (e^{15}, e^{25}, 2\tau e^{12} + 2e^{35}, -2e^{12} - 2\tau e^{35}, 0).$$

In either case, the metric is

$$\tilde{g} = e^1 \otimes e^1 + e^2 \otimes e^2 + \tau e^3 \otimes e^3 + e^4 \otimes e^4 + e^5 \otimes e^5.$$

Taking into consideration the negative-definite metric on $\tilde{g}$ has the effect of adding the $\pm$ signs, as per Remark 5.4. □
Notice that the third Lie algebra appearing in Theorem 5.7 is Example 2.4. We proceed to give a list of the 7-dimensional Lie algebras with a 3-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra \( \tilde{\mathfrak{g}} \) up to isometry and \( D \)-homothety. This list is given in Table 1, where we write the diagonal metric \( \tilde{g} \) as a line vector with respect to the basis \( \{ e^1, \ldots, e^7 \} \), using the convention that \( [1]_n \) is a vector of \( n \) elements, each equal to 1. For example \( [1]_4 = (1, 1, 1, 1) \) and \( (\pm [1]_4, \tau, +1, \tau) \) represents the metric
\[
\tilde{g} = \pm (e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4 + \tau e^5 \otimes e^5 + e^6 \otimes e^6 + \tau e^7 \otimes e^7).
\]

| n. | \( \tilde{\mathfrak{g}} \) | Metric \( \tilde{g} \) |
|----|----------------|------------------|
| 1. | 0, 0, 0, 0, 0, \(-2\tau e^{12} - 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_4, \tau, +1, \tau)\) |
| 2. | 0, 0, 0, 0, 2\tau e^{12} + 2\tau e^{34} + 2\tau e^{57}, -2\tau e^{12} - 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_4, \tau, +1, \tau)\) |
| 3. | 0, 0, e^{37}, e^{17}, 2\tau e^{12} + 2\tau e^{34} + 2\tau e^{57}, -2\tau e^{12} - 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_4, \tau, +1, \tau)\) |
| 4. | e^{17}, e^{27}, e^{37}, e^{47}, 2\tau e^{12} + 2\tau e^{34} + 2\tau e^{57}, -2\tau e^{12} - 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_4, \tau, +1, \tau)\) |
| 5. | 0, 0, 0, 0, 0, \(-2\tau e^{12} + 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |
| 6. | 0, 0, 0, 0, 2\tau e^{12} - 2\tau e^{34} + 2\tau e^{57}, -2\tau e^{12} + 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |
| 7. | 0, 0, e^{17}, e^{27}, e^{37}, e^{47}, 2\tau e^{12} + 2\tau e^{34} + 2\tau e^{57}, -2\tau e^{12} + 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |
| 8. | e^{17}, e^{27}, e^{37}, e^{47}, 2\tau e^{12} + 2\tau e^{34} + 2\tau e^{57}, -2\tau e^{12} + 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |
| 9. | \(\frac{1}{2}e^{17} + 2\lambda e^{27} - \frac{1}{2}e^{37} - \lambda e^{17}, -2\lambda e^{17} + \frac{1}{2}e^{27} + \lambda e^{37} - \frac{1}{2}e^{47}, -\tau e^{12} + \tau e^{14} - \tau e^{34}, -2\tau e^{12} + 2\tau e^{34}, -2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |
| 10. | \(\frac{1}{2}e^{17} + 2\lambda e^{27} - \frac{1}{2}e^{37} - \lambda e^{17}, -2\lambda e^{17} + \frac{1}{2}e^{27} + \lambda e^{37} - \frac{1}{2}e^{47}, -\tau e^{12} + \tau e^{14} - \tau e^{34}, -2\tau e^{12} + 2\tau e^{34}, -2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |
| 11. | \(\frac{1}{2}e^{17} + 2\lambda e^{27} + \frac{1}{2}e^{37} - \lambda e^{17}, -2\lambda e^{17} + \frac{1}{2}e^{27} + \lambda e^{37} + \frac{1}{2}e^{47}, -\tau e^{12} + 3\tau e^{14} - \tau e^{34} + 2\tau e^{57}, -2\tau e^{12} + 2\tau e^{34} - 2\tau e^{57}, 0\) | \((\pm [1]_2, \tau, +1, \tau)\) |

**Theorem 5.8.** Let \( \tilde{\mathfrak{g}} \) be a Lie algebra of dimension 7 with a 3-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra \( \mathfrak{g} \). Then, up to isometry and \( D \)-homothety, the metric Lie algebra \( (\tilde{\mathfrak{g}}, \tilde{g}) \) is one of the Lie algebras appearing in Table 1 and the Sasaki structure is given by

\[
\xi = (e^6)^9 = e_6, \quad \eta = e^6, \quad 2\Phi = d\eta = de^6
\]

with respect to the basis \( \{ e^1, \ldots, e^7 \} \) of Table 1.

**Proof.** We first consider the case where \( \tilde{\mathfrak{g}} \) is positive definite, applying Corollary 5.6 and proceeding as in the proof of Theorem 5.7.
If $h = 0$, we get

$$\begin{align*}
& (0, 0, 0, 0, 0, -2e^{12} - 2e^{34} - 2\tau e^{57}, 0);
& \text{for } h = 2, \text{ we have the three possibilities } \hat{D} = 0, \hat{D} = e^3 \otimes e_3 + e^4 \otimes e_4, \hat{D} = I, \text{ corresponding to }
& (0, 0, 0, 2\tau e^{12} + 2\tau e^{34} + 2e^{57}, -2e^{12} - 2e^{34} - 2\tau e^{57}, 0),
& (0, 0, e^{37}, e^{47}, 2\tau e^{12} + 2\tau e^{34} + 2e^{57}, -2e^{12} - 2e^{34} - 2\tau e^{57}, 0),
& (e^{17}, e^{27}, e^{37}, e^{47}, 2\tau e^{12} + 2\tau e^{34} + 2e^{57}, -2e^{12} - 2e^{34} - 2\tau e^{57}, 0).
\end{align*}$$

The negative definite case gives rise to the same Lie algebras, with the restriction of the metric to $\hat{g}$ of opposite sign.

In the neutral case, we can assume

$$\hat{g} = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4,$$

$$\omega = -e^{12} + e^{34},$$

$$J = e^1 \otimes e_2 - e^2 \otimes e_1 + e^3 \otimes e_4 - e^4 \otimes e_3.$$ If $\hat{D}^s$ is diagonalizable, Corollary 5.6 applies and computations as above yield

$$\begin{align*}
& (0, 0, 0, 0, 0, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0),
& (0, 0, 0, 0, 2\tau e^{12} - 2\tau e^{34} + 2e^{57}, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0),
& (0, 0, e^{37}, e^{47}, 2\tau e^{12} - 2\tau e^{34} + 2e^{57}, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0),
& (e^{17}, e^{27}, e^{37}, e^{47}, 2\tau e^{12} - 2\tau e^{34} + 2e^{57}, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0).
\end{align*}$$

If $\hat{D}^s$ is not diagonalizable, we can exploit the $U(1,1)$ symmetry preserving the pseudo-Kähler structure of $\hat{g}$. Indeed, a symmetric derivation commuting with $J$ is effectively an element of $\mathfrak{su}(1,1)$, with $U(1,1)$ acting on it by the adjoint action. Write $\hat{D}^s = tI + \hat{D}_0^s$, where $\hat{D}_0^s$ is traceless. Then $\hat{D}_0^s$ can therefore be viewed as an element of $\mathfrak{su}(1,1)$. Now $SU(1,1)$ is isomorphic to $SL(2,\mathbb{R})$ via the Cayley isomorphism

$$(5.4) \quad \text{SL}(2,\mathbb{R}) \ni g \mapsto CgC^{-1} \in SU(1,1),$$

where $C = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$. The action of $SL(2,\mathbb{R})$ on its Lie algebra is conjugation, so any nondiagonalizable element of $\mathfrak{su}(2,\mathbb{R})$ is in the $SL(2,\mathbb{R})$-orbit of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Reading this in $\mathfrak{su}(1,1)$ via (5.4) and multiplying by $-i$, we see that $\hat{D}_0^s$ corresponds to the complex matrix $\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$; writing it as a real matrix, we obtain

$$\hat{D}^s = \begin{pmatrix} (t + \frac{1}{2})I & -\frac{1}{2}I \\ \frac{1}{2}I & (t - \frac{1}{2})I \end{pmatrix},$$
A derivation $\hat{D}$ that satisfies $[\hat{D}, J] = 0$ and is not diagonalizable takes the form
\[
\hat{D} = \begin{pmatrix}
x & \lambda_2 & \lambda_5 - 1 & -\lambda_6 \\
-\lambda_2 & x & \lambda_6 & \lambda_5 - 1 \\
\lambda_5 & \lambda_6 & x - 1 & \lambda_8 \\
-\lambda_6 & \lambda_5 & -\lambda_8 & x - 1
\end{pmatrix}.
\]

Now, thanks to Proposition 2.2, we can consider any derivation $\hat{D}'$ such that $\hat{D}' \circ \lambda = \lambda \circ \hat{D}'$. This yields $y = x$, $\mu_5 = \lambda_5$, $\mu_6 = \lambda_6$ and $\mu_2 - \mu_8 = \lambda_2 - \lambda_8$, hence we can consider $\hat{D}$ to be
\[
\hat{D} = \begin{pmatrix}
x & \lambda_2 & \lambda_5 - 1 & -\lambda_6 \\
-\lambda_2 & x & \lambda_6 & \lambda_5 - 1 \\
\lambda_5 & \lambda_6 & x - 1 & 0 \\
-\lambda_6 & \lambda_5 & 0 & x - 1
\end{pmatrix}.
\]

Again we distinguish two cases depending on $h$. If $h = 0$, then equation $[\hat{D}' \circ \lambda, \hat{D}' \circ \lambda] = h \hat{D}' \circ \lambda^2$ yields
\[
\hat{D} = \begin{pmatrix}
\frac{1}{2} & 2\lambda & -\frac{1}{2} & -\lambda \\
-2\lambda & \frac{1}{2} & \lambda & -\frac{1}{2} \\
\frac{1}{2} & \lambda & -\frac{1}{2} & 0 \\
-\lambda & \frac{1}{2} & 0 & -\frac{1}{2}
\end{pmatrix}.
\]

Hence we set $d\xi^b = -2e^{12} + 2e^{34}$, $db^a = -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}$, and the first Lie algebra extension is
\[
g = (0, 0, 0, 0, -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34}),
\]
with metric
\[
(5.5) \quad g = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + \tau b^a \otimes b^b + \xi^b \otimes \xi^b.
\]
The Sasaki extension $\tilde{g} = g \otimes \text{Span} \{e_0\}$ is determined by
\[
d\xi^b = -2e^{12} + 2e^{34}, \quad db^a = -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34},
\]
\[
[e_0, x] = \hat{D}x, \quad [e_0, \xi] = 0, \quad [e_0, b] = -2\tau \xi;
\]
hence the Lie algebra is
\[
\tilde{g} = \begin{pmatrix}
\frac{1}{2}e^{17} + 2\lambda e^{27} - \frac{1}{2}e^{37} - \lambda e^{47}, -2\lambda e^{17} + \frac{1}{2}e^{27} + \lambda e^{37} - \frac{1}{2}e^{47} \\
\frac{1}{2}e^{17} + \lambda e^{27} - \frac{1}{2}e^{37} - \lambda e^{47}, -2\lambda e^{17} + \frac{1}{2}e^{27} - \frac{1}{2}e^{47} \\
-\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0
\end{pmatrix}.
\]
If $h = 2$, then equation $[\hat{D}^s, \hat{D}^a] = h\hat{D}^s - 2(\hat{D}^s)^2$ yields two distinct solutions for $\hat{D}$:

$$\hat{D}_1 = \left( \begin{array}{cccc}
\frac{1}{2} & 2\lambda & -\frac{3}{2} & -\lambda \\
-2\lambda & \frac{1}{2} & \frac{3}{2} & -\lambda \\
-\frac{1}{2} & \lambda & \frac{1}{2} & 0 \\
-\lambda & \frac{1}{2} & \frac{1}{2} & 0
\end{array} \right) \quad \text{or} \quad \hat{D}_2 = \left( \begin{array}{cccc}
\frac{3}{2} & 2\lambda & \frac{1}{2} & -\lambda \\
-2\lambda & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{3}{2} & \lambda & \frac{1}{2} & 0 \\
-\lambda & \frac{3}{2} & \frac{1}{2} & 0
\end{array} \right).$$

For $\hat{D}_1$ we get $db^s = -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}$, hence $g = (0, 0, 0, 0, -\tau e^{12} + \tau e^{14} - \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34})$;

for $\hat{D}_2$ we get $db^s = -3\tau e^{12} + 3\tau e^{14} - \tau e^{23} + \tau e^{34}$ and $g = (0, 0, 0, 0, -3\tau e^{12} + 3\tau e^{14} - \tau e^{23} + \tau e^{34}, -2e^{12} + 2e^{34})$.

In both cases, the metric is given by (5.5). The resulting Lie algebras $\tilde{g}$ correspond to $n.10$ and $n.11$ in Table 1. □

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