Unified discrete mechanics II: The space and time symmetric hyperincursive discrete Klein-Gordon equation bifurcates to the 4 incursive discrete Majorana real 4-spinors equations

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Abstract. This paper begins with the formalization of the second order hyperincursive discrete Klein-Gordon equation. The temporal second order hyperincursive discrete Klein-Gordon equation is similar to the time-symmetric hyperincursive discrete harmonic oscillator and so bifurcates into a group of 4 incursive discrete real equations of first order. In this group, two equations are the discrete time reverse of the two other equations, giving an oscillator and an anti-oscillator. Firstly, the discrete Klein-Gordon equation, with one space dimension (1D), bifurcates to 4 first order incursive discrete equations that we called the Dubois-Ord-Mann real 4-spinors equations because Ord and Mann obtained the same equations from a stochastic method. Secondly, we generalize to three spatial dimensions (3D) these discrete Dubois-Ord-Mann equations. These 4 discrete equations are then transformed to real partial differential equations which can be written under the generic form of the Dirac quantum 4-spinors equation. Thirdly, we consider a change in the order of space variables and a change of indexes of the functions of the Dubois-Ord-Mann equations. With these changes we obtain the original real 4-Spinors Majorana partial differential equations. Also we obtain the 4 incursive discrete Majorana real equations.

1. Introduction
This paper deals with our recent research on a unified discrete mechanics with the formalism of the hyperincursive discrete equation separable into incursive discrete equations.
This paper is a continuation of my two papers presented at the IXth and Xth Symposia on unified field mechanics, honoring noted French mathematical physicist Jean-Pierre Vigier.
The first paper [1] concerns the hyperincursive algorithms of classical harmonic oscillator applied to quantum harmonic oscillator separable into incursive oscillators.
The second paper [2] deals a unified discrete mechanics given by the bifurcation of the hyperincursive discrete harmonic oscillator, the hyperincursive discrete Schrödinger quantum equation, the hyperincursive discrete Klein-Gordon equation and the Dirac quantum relativist equations.
This paper is based on a series of classical papers given historically as follows.
In 1926, Oskar Klein [3] and Walter Gordon [4] presented independently what is called the Klein-Gordon equation.

In 1928, Dirac [5] introduced the relativist quantum mechanics based on this Klein-Gordon equation. His fundamental equation is based on 4-spinors, and is given by 4 first order complex partial differential equations. All the work of Dirac is well explained in his book [6].

In 1930 and 1932, Al Proca [7,8] proposed a generalization of the Dirac theory with the introduction of 4 groups of 4-spinors, and with 16 first order complex partial differential equations.

In 1937, Ettore Majorana [9] proposed a real 4-spinors Dirac equation, given by 4 first order real partial differential equations. Ettore Majorana disappears just after having written this fundamental paper. Eleano Pessa [10] presented a very interesting paper on the Majorana oscillator based of the 4 first order real partial differential equations.

In this paper, we will not introduce the theory of Klein-Gordon and Dirac, so an excellent introduction to quantum mechanics is given in the books of Albert Messiah [11].

In my second Vigier paper [2], I have demonstrated that the second order hyperincursive discrete Klein-Gordon equation bifurcates to the 4 Dirac first order equations, in one space dimension.

I obtained 4 equations, similar to the equations that G N Ord and R B Mann [11] deduced differently from a stochastic model with difference equations. A generalization is given in this paper under the name of Dubois-Ord-Mann equation.

The temporal Klein-Gordon is similar in mathematical form to the hyperincursive discrete classical harmonic oscillator.

A good introduction to incursion and hyperincursion is given in the following series of papers on the total incursive control of linear, non-linear and chaotic systems [13], on computing anticipatory systems with incursion and hyperincursion [14], on the computational derivation of quantum and relativist systems with forward-backward space-time shifts [15], on a review of incursive, hyperincursive and anticipatory systems, with the foundation of anticipation in electromagnetism [16], then, on the precision and stability analysis of Euler, Runge-Kutta and incursive algorithms for the harmonic oscillator [17], and finally, on the new concept of deterministic anticipation in natural and artificial systems [18].

I wrote a series of theoretical papers on the discrete physics with Adel Antippa on the harmonic oscillator via the discrete path approach [19], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [20], on the dual incursive system of the discrete harmonic oscillator [21], on the superposed hyperincursive system of the discrete harmonic oscillator [22], on the incursive discretization, system bifurcation, and energy conservation [23], on the hyperincursive discrete harmonic oscillator [24], on the synchronous discrete harmonic oscillator [25], on the discrete harmonic oscillator, a short compendium of formulas [26], on the time-symmetric discretization of the harmonic oscillator [27], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [28]. This discrete physics is based on the fundamental mathematical development of the hyperincursive and incursive discrete harmonic oscillator.

The main purpose of this paper is the deduction of the discrete Majorana real 4-spinors equation from our second order hyperincursive discrete Klein-Gordon equation.

This paper is essentially based on my recent papers on reversible time, retardation and anticipation in the recursive, incursive and hyperincursive Klein-Gordon quantum systems [29], on the reversible hyperincursive discrete Schrödinger quantum harmonic oscillator separable into two non-reversible incursive discrete oscillators [30], on the deduction of the first order spinor equations of the Schrödinger quantum spatial harmonic oscillator with the emergence of the fundamental energy [31], on the generalisation to three spatial dimensions of the Dubois-Ord-Mann real 4-spinors equation from the hyperincursive discrete Klein-Gordon equation [32], on the deduction of the Majorana real 4-spinors generic Dirac equation from the computable hyperincursive discrete Klein-Gordon equation [33], on the hyperincursive discrete Klein-Gordon equation for computing the Majorana real 4-spinors equation and the real 8-spinors Dirac equation [34], and finally, on the bifurcation of the
hyperincursive discrete Klein-Gordon equation to real 4-spinors Dirac equation related to the Majorana equation.

The paper is organized as follows.

The section 2 presents the Klein-Gordon partial differential equation and the second order hyperincursive discrete Klein-Gordon equation. With only the temporal component (particle at rest) the Klein-Gordon equation represents an oscillator, and the discrete Klein-Gordon equation is similar to the second order hyperincursive discrete harmonic oscillator. This hyperincursive equation is separable into a first discrete incursive oscillator depending on two functions and a second incursive oscillator depending on two other functions given by 4 first order discrete equations, where the second incursive system is the discrete time reverse of the first incursive system.

In section 3, we present the 4 incursive discrete Dubois-Ord-Mann equations in one space z dimension (1D) that are similar to the discrete harmonic oscillator, without the space term. These discrete equations correspond to 4 first order partial differential equations representing the Dubois-Ord-Mann real 4-spinors equations. We called these equations, the Dubois-Ord-Mann real 4-spinors equations, because Ord and Mann deduced the same real 4-spinors equations from a one-dimensional space stochastic model. They have not generalized their stochastic model to 3 spatial dimensions.

In section 4, then, we generalize these incursive discrete Dubois-Ord-Mann equations to three space dimensions (3D), corresponding to 4 first order partial differential equations representing the Dubois-Ord-Mann real 4-spinors equations. Let us notice that the two oscillators are entangled with the spatial dimension y. So the two incursive systems are no more independent in three space dimensions. This means that the 4 incursive discrete hyperincursive equations is a whole group of 4 equations which are not separable, except for particular conditions (particle at rest for example). This is a remarkable result, because the second order discrete hyperincursive equation bifurcates to a group of 4 discrete incursive real 4-spinors equations.

Indeed in section 5, we define a 4-spinors function and transform these 4 first order partial differential equations in the generic form of Dirac relativist quantum equation.

The discrete Dubois-Ord-Mann real 4-spinors was obtained from the deterministic space and time symmetric hyperincursive Klein-Gordon equation. It could be interesting to see how to generalize the one-dimensional real stochastic difference equations of Ord and Mann to three spatial dimensions.

It is important to know that the Dubois-Ord-Mann real 4-spinors belong to the group of the Majorana real 4-spinors. Indeed, with two successive permutations, the 4 first order partial differential equations of the Dubois-Ord-Mann 4-spinors transform to the 4 first order partial differential equations of the Majorana real 4-spinors equation given by Pessa [10]. The first permutation consists in permuting two space variables. The second permutation consists in permuting the indexes of the functions. This was demonstrated in my recent paper [35].

But it is interesting to analyze separately the transformation of the 4 Dubois-Ord-Mann equations, firstly, in permuting the two space variables, and secondly, in permuting the indexes of the functions.

Indeed, in permuting only the two space variables, y and z, in the 4 Dubois-Ord-Mann equations, we obtain the 4 first order partial differential equations where the “Majorana Oscillator” presented in the paper of Pessa [10] is well put in evidence: the first two equations represents a first oscillator and the last two equations represents the second oscillator, that is an anti-oscillator. This is completely in agreement with my theory of the hyperincursive discrete harmonic oscillator that bifurcates to two incursive oscillators, each oscillator being the reverse of the other.

This transformation of the Dubois-Ord-Mann equations, by permutation of the space variables, y and z, was presented in a recent paper [33], where the transformed equations were called the Dubois-Majorana equations. In this paper [33], we say explicitly what follows:

“The Dubois Majorana real 4-spinors equation is in fact composed of a spatial oscillator entangled to a spatial anti-oscillator, related to the particle electron and anti-particle positron in Majorana equation. …
The Dubois-Majorana discrete real 4-spinors equation shows explicitly that the Majorana equation is in fact composed of a discrete hyperincursive spatial oscillator entangled to a spatial harmonic oscillator, related to the particle electron and anti-particle positron in Majorana equation.

We have shown that the ‘Majorana Oscillator’ in reference to the paper of Eliano Pessa [10] is explicitly explained by the hyperincursive discrete harmonic oscillator which is composed of 4 incursive discrete equations that are in fact the real 4-spinors time equation [33].

Moreover the 4 incursive discrete Dubois-Majorana equations can be used for the simulation of the ‘Majorana Oscillator’.

The incursive discrete Dubois-Majorana real 4-spinors equations, is transformed to the Majorana equations [9,10], by permutation of the indexes of the functions, as shown is our paper [33].

In section 6, we present the 4 incursive discrete Dubois-Majorana equations, and the 4 first order partial differential equations. In a recent paper [33], we have shown that these Dubois-Majorana real 4-spinors equations can be transformed to the generic form of the Dirac equation.

In section 7, indeed, in making the change in indexes of these 4 functions in the Dubois-Majorana equations, the 4-spinors real equations are transformed to the 4 real first order Majorana partial differential equations which are identical to the original Majorana equations.

In section 8, finally, we present the 4 incursive discrete Majorana equations.

2. The space and time symmetric second order hyperincursive discrete Klein-Gordon equation

In 1926, Oskar Klein [3] and Walter Gordon [4] published independently their famous equation, called the Klein-Gordon equation.

The Klein-Gordon equation with the function \( \varphi = \varphi(\mathbf{r}, t) \) in three spatial dimensions \( \mathbf{r} = (x, y, z) \) and time \( t \) is given by

\[
- \hbar^2 \partial^2 \varphi(\mathbf{r}, t) / \partial t^2 = - \hbar^2 c^2 \nabla^2 \varphi(\mathbf{r}, t) + m^2 c^4 \varphi(\mathbf{r}, t)
\]  

(1)

or, in the explicit form of the nabla operator \( \nabla \),

\[
- \hbar^2 \partial^2 \varphi / \partial t^2 = - \hbar^2 c^2 \partial_x^2 \varphi - \hbar^2 c^2 \partial_y^2 \varphi - \hbar^2 c^2 \partial_z^2 \varphi + m^2 c^4 \varphi
\]

(2)

where \( \hbar \) is the constant of Plank, \( c \) is the speed of light, and \( m \) the mass.

As we will consider the discrete Klein-Gordon equation, we make the following usual change of variables

\[
q(\mathbf{r}, t) = \varphi(\mathbf{r}, t)
\]

(3)

\[
a = \omega = mc^2 / \hbar
\]

(4)

where \( \omega \) is a frequency, so the Klein-Gordon equation (2) becomes

\[
\partial^2 q(\mathbf{r}, t) / \partial t^2 = +c^2 \partial_x^2 q(\mathbf{r}, t) / \partial x^2 + c^2 \partial_y^2 q(\mathbf{r}, t) / \partial y^2 + c^2 \partial_z^2 q(\mathbf{r}, t) / \partial z^2 - a^2 q(\mathbf{r}, t)
\]

(5)

From the Klein-Gordon equation (5), the second order hyperincursive discrete Klein-Gordon equation [32,35] is given by

\[
q(x, y, z, t + 2\Delta t) - 2q(x, y, z, t) + q(x, y, z, t - 2\Delta t) =
\]

\[
+ B^2[q(x + 2\Delta x, y, z, t) - 2q(x, y, z, t) + q(x - 2\Delta x, y, z, t)]
\]

\[
+ C^2[q(x, y + 2\Delta y, z, t) - 2q(x, y, z, t) + q(x, y - 2\Delta y, z, t)]
\]

\[
+ D^2[q(x, y, z + 2\Delta z, t) - 2q(x, y, z, t) + q(x, y, z - 2\Delta z, t)] - A^2 q(x, y, z, t)
\]

(6)

where the following parameters A, B, C, and, D,

\[
A = a (2\Delta t), \quad B = c (2\Delta t)/(2\Delta x), \quad C = c (2\Delta t)/(2\Delta y), \quad D = c (2\Delta t)/(2\Delta z)
\]

(7)

depend on the discrete interval of time \( \Delta t \), and the discrete intervals of space, \( \Delta x, \Delta y, \Delta z \), respectively.
As usually made in computer science, let us now introduce the discrete time \( t_k \), and the discrete spaces \( x_l, y_m, z_n \), as follows

\[
t_k = t_0 + k\Delta t, k = 0, 1, 2, ..., (8)
\]

where \( k \) is the integer time increment, and

\[
x_l = x_0 + l\Delta x, l = 0, 1, 2, ..., (9a)
\]
\[
y_m = y_0 + m\Delta y, m = 0, 1, 2, ..., (9b)
\]
\[
z_n = z_0 + n\Delta z, n = 0, 1, 2, ... (9c)
\]

where \( l, m, n \), are the integer space increments.

So, with these time and space increments, the second order hyperincursive discrete Klein-Gordon equation (6) becomes

\[
q(l, m, n, k + 2) − 2q(l, m, n, k) + q(l, m, n, k − 2) =
\]
\[
+ B^2[q(l + 2, m, n, k) − 2q(l, m, n, k) + q(l − 2, m, n, k)]
\]
\[
+ C^2[q(l, m + 2, n, k) − 2q(l, m, n, k) + q(l, m − 2, n, k)]
\]
\[
+ D^2[q(l, m, n + 2, k) − 2q(l, m, n, k) + q(l, m, n − 2, k)] − A^2 q(l, m, n, k) (10)
\]

This equation without spatial components (corresponding to a particle at rest) is similar to the harmonic oscillator.

For a particle at rest, the Klein-Gordon equation (5), with the function \( q(t) \) depending only on the time variable, is given by

\[
\frac{\partial^2 q(t)}{\partial t^2} = -\frac{\alpha^2}{\hbar} q(t) (11)
\]

with the frequency, \( \alpha = \omega = mc^2/\hbar \), given by the equation (4).

This equation (11) is formally similar to the equation of the harmonic oscillator for which \( q(t) \) would represent the position, \( x(t) \), and \( \partial q(t)/\partial t \) would represent the velocity, \( v(t) = \partial x(t)/\partial t \).

So, with only the temporal component, the second order hyperincursive discrete Klein-Gordon equation (10) becomes

\[
q(k + 2) − 2q(k) + q(k − 2) = −A^2 q(k) (12)
\]

that is similar to the second order hyperincursive equation of the harmonic oscillator [1].

This hyperincursive equation (12) is separable into a first discrete incursive oscillator depending on two functions defined by \( q_1(k), q_2(k) \), and a second incursive oscillator depending on two other functions defined by \( q_3(k), q_4(k) \), given by first order discrete equations.

So the first incursive equations are given by:

\[
q_1(k) = q_1(k − 2) − Aq_2(k − 1)
\]
\[
q_2(k + 1) = q_2(k − 1) + Aq_1(k) \quad (13a,b)
\]

and the second incursive equations are given by:

\[
q_3(k) = q_3(k − 2) + Aq_4(k)
\]
\[
q_4(k + 1) = q_4(k − 1) − Aq_3(k) \quad (13c,d)
\]

where the second incursive system is the time reverse of the first incursive system in making the time inversion \( T \)

\[
T: \Delta t \rightarrow −\Delta t \quad (14)
\]

which gives an oscillator and its anti-oscillator.
For a moving particle, the 3 discrete spaces terms in equation (10)
\[ q(l + 2, m, n, k) - 2q(l, m, n, k) + q(l - 2, m, n, k) \]
\[ q(l, m + 2, n, k) - 2q(l, m, n, k) + q(l, m - 2, n, k) \]
\[ q(l, m, n + 2, k) - 2q(l, m, n, k) + q(l, m, n - 2, k) \]
are similar to the discrete time term
\[ q(l, m, n, k + 2) - 2q(l, m, n, k) + q(l, m, n, k - 2) \].

The mathematical developments on the time-symmetric hyperincursive discrete harmonic oscillator, made in a series of papers by Adel Antippa and Daniel Dubois [19-28], are extensible to the mathematical development of the space and time symmetric hyperincursive discrete Klein-Gordon equation.

In this paper we will present the bifurcation of this equation (10) to the 4 incursive discrete Dubois-Ord-Mann real equations, to the 4 incursive discrete Dubois-Majorana real equations, and to the 4 incursive discrete Majorana real equations.

3. The 4 incursive discrete Dubois-Ord-Mann real equations in one spatial dimensions
Recently, we presented the second order hyperincursive discrete Klein-Gordon equation in one spatial dimension [2,9].

From the Klein-Gordon equation (10), the second order hyperincursive discrete Klein-Gordon equation in one spatial dimension (1D), \( z \), is given by:
\[ q(z, t + 2\Delta t) - 2q(z, t) + q(z, t - 2\Delta t) = \]
\[ + D^2[q(z + 2\Delta z, t) - 2q(z, t) + q(z - 2\Delta z, t)] - A^2q(z, t) \] (15)
where the parameters \( A \) and \( D \)
\[ A = a(2\Delta t), D = c (2\Delta t)/(2\Delta z) \] (16)
depend on the discrete intervals of time \( \Delta t \) and space \( \Delta z \) respectively.

With the discrete time \( t_k \) and the discrete space \( z_n \),
\[ t_k = t_0 + k\Delta t, k = 0,1,2, ..., \] (17)
and
\[ z_n = z_0 + n\Delta z, n = 0,1,2, ... \] (18)
the second order hyperincursive discrete Klein-Gordon equation (15) becomes
\[ q(n, k + 2) - 2q(n, k) + q(n, k - 2) = \]
\[ + D^2 [q(n + 2, k) - 2q(n, k) + q(n - 2, k)] - A^2q(n, k) \] (19)
As we have shown, this one spatial dimensional equation (19) bifurcates into 2 incursive equations for the discrete time, and into 2 incursive equations for the discrete space.

So, the function \( q \) bifurcates into four functions noted \( q_1, q_2, q_3, q_4 \).

Indeed, we obtained the following 4 discrete space time incursive equations of the 4 functions \( q_1(n, k), q_2(n, k), q_3(n, k), q_4(n, k), \) given by
\[ q_1(n, k + 1) = q_1(n, k - 1) + D[q_1(n + 1, k) - q_1(n - 1, k)] - Aq_2(n, k), \]
\[ q_2(n, k + 1) = q_2(n, k - 1) - D[q_2(n + 1, k) - q_2(n - 1, k)] + Aq_1(n, k), \]
\[ q_3(n, k + 1) = q_3(n, k - 1) + D[q_3(n + 1, k) - q_3(n - 1, k)] + Aq_4(n, k), \]
\[ q_4(n, k + 1) = q_4(n, k - 1) - D[q_4(n + 1, k) - q_4(n - 1, k)] - Aq_3(n, k). \] (20a,b,c,d)
These 4 discrete incursive Klein-Gordon equations show a reversible discrete time, and discrete retardation with forward derivative and discrete anticipation with backward derivative [15].

These two discrete incursive space time equations (20a,b,c,d) can be written as follows - First incursive system:

\[
q_1(n, k) = q_1(n, k - 2) + D[q_1(n + 1, 2k - 1) - q_1(n - 1, k - 1)] - A q_1(n, k - 1) \quad (21a)
\]
\[
q_2(n, k + 1) = q_2(n, k - 1) - D[q_2(n + 1, k) - q_2(n - 1, k)] + A q_1(n, k) \quad (21b)
\]

Second incursive system:

\[
q_3(n, k) = q_3(n, k - 2) + D[q_3(n + 1, k - 1) - q_3(n - 1, k - 1)] + A q_4(n, k - 1), \quad (22a)
\]
\[
q_4(n, k + 1) = q_4(n, k - 1) - D[q_4(n + 1, k) - q_4(n - 1, k)] - A q_3(n, k). \quad (22b)
\]

that are similar to the temporal equations (13a) and (14a,b), without the space term \(D\) (for a particle at rest).

In incursive systems, there is an order in which the equations are computed. Indeed, the equation (21a) must be computed before the equation (21b) because the equation (21b) needs the value of \(q_1(n, k)\) that is computed by the equation (21a), and also the equation (22a) must be computed before the equation (22b) because the equation (22b) needs the value of \(q_4(n, k)\) that is computed by the equation (22a).

As already noted, with the space dimension \(z\), the first incursive system is independent of the second incursive system.

We will show, in the next section, that with the extension to the three space dimensions, the first incursive system and the second incursive system will be no more separable but dependent on each other.

Let us now transform the 4 discrete equations (20,b,c,d) to partial differential equations. The discrete functions \(q_{\ell}(n, k), j = 1,2,3,4\) tend to the continuous functions \(\Psi_{\ell}(z, t)\), when the discrete space and time intervals tend to zero. At the limit,

\[
\Psi_j = \Psi_j(z, t) = \lim_{\Delta z \to 0, \Delta t \to 0} q_j(z, t), j = 1,2,3,4, \quad (23)
\]

the 4 discrete equations (20a,b,c,d) tend to the following 4 first order partial differential equations

\[
+ \partial \Psi_1/\partial t = + c \partial \Psi_1/\partial z - (mc^2/h)\Psi_2 \quad (24a)
\]
\[
+ \partial \Psi_2/\partial t = - c \partial \Psi_2/\partial z + (mc^2/h)\Psi_1 \quad (24b)
\]
\[
+ \partial \Psi_3/\partial t = + c \partial \Psi_3/\partial z + (mc^2/h)\Psi_4 \quad (24c)
\]
\[
+ \partial \Psi_4/\partial t = - c \partial \Psi_4/\partial z - (mc^2/h)\Psi_3 \quad (24d)
\]

representing the Dubois-Ord-Mann real 4-spinors equations [32].

Indeed, Ord and Mann [12] deduced 4 real equations similar to our equations (24a,b,c,d), from a one-dimensional space, \(z\), stochastic model. This is the reason why we call these equations, the Dubois-Ord-Mann real 4-spinors equations [32].

It will be shown that these equations can be put in the generic form of the Dirac 4-spinors equation.

But before, in the next section, we will extend our 1D hyperincursive system to 3 spatial dimensions.

4. Generalisation to 3 spatial dimensions of the 4 incursive discrete Dubois-Ord-Mann equations

This section gives an extension of the one-dimensional hyperincursive discrete Dubois-Ord-Mann equations [32,35] to three space dimensions (3D).

With the 4 discrete functions \(q_{\ell}(l, m, n, k), j = 1,2,3,4\), the extension to three spatial dimensions of the discrete Dubois-Ord-Mann equations (20a,b,c,d) was given by the following 4 incursive discrete equations:
\[ q_1(l, m, n, k + 1) = q_1(l, m, n, k - 1) + B[q_2(l + 1, m, n, k) - q_2(l - 1, m, n, k)] \\
+ C[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] + D[q_4(l, m + 1, n, k) - q_4(l, m - 1, n, k)] - Aq_2(l, m, n, k) \]  
(25a)

\[ q_2(l, m, n, k + 1) = q_2(l, m, n, k - 1) + B[q_1(l + 1, m, n, k) - q_1(l - 1, m, n, k)] \\
- C[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] - D[q_2(l, m + 1, n, k) - q_2(l, m - 1, n, k)] + Aq_1(l, m, n, k) \]  
(25b)

\[ q_3(l, m, n, k + 1) = q_3(l, m, n, k - 1) + B[q_4(l + 1, m, n, k) - q_4(l - 1, m, n, k)] \\
- C[q_2(l, m + 1, n, k) - q_2(l, m - 1, n, k)] + D[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] + Aq_4(l, m, n, k) \]  
(25c)

\[ q_4(l, m, n, k + 1) = q_4(l, m, n, k - 1) + B[q_3(l + 1, m, n, k) - q_3(l - 1, m, n, k)] \\
+ C[q_4(l, m + 1, n, k) - q_4(l, m - 1, n, k)] - D[q_4(l, m + 1, n, k) - q_4(l, m - 1, n, k)] - Aq_3(l, m, n, k) \]  
(25d)

With the extension to the three space dimensions, the first two equations are dependent of the second two equations, which are no more separable.

The discrete Dubois-Ord-Mann real 4-spinors was obtained from the deterministic space and time symmetric hyperincursive Klein-Gordon equation. It could be interesting to see how to generalize the one-dimensional real stochastic difference equations of Ord and Mann to three spatial dimensions.

Let us now transform the 4 discrete equations (25a,b,c,d) to partial differential equations [32,35].

The discrete functions
\[ q_j(x, y, z, t) = q_j(r, t), j = 1,2,3,4 \]

tend to the continuous functions
\[ \Psi_j(x, y, z, t) = \Psi_j(r, t), j = 1,2,3,4 \]

when the discrete space and time intervals tend to zero.

At the limit,
\[ \Psi_j = \Psi_j(r, t) = \lim_{\Delta r \to 0, \Delta t \to 0} q_j(r, t), j = 1,2,3,4 \]  
(26)

the 4 discrete equations (25a,b,c,d) tend to the following 4 first order partial differential equations representing the Dubois-Ord-Mann real 4-spinors equations

\[ + \partial \Psi_1/\partial t = + c \partial \Psi_2/\partial x + c \partial \Psi_4/\partial y + c \partial \Psi_1/\partial z - (mc^2/h)\Psi_2 \]  
(27a)

\[ + \partial \Psi_2/\partial t = + c \partial \Psi_1/\partial x - c \partial \Psi_3/\partial y - c \partial \Psi_2/\partial z + (mc^2/h)\Psi_1 \]  
(27b)

\[ + \partial \Psi_3/\partial t = + c \partial \Psi_4/\partial x - c \partial \Psi_2/\partial y + c \partial \Psi_3/\partial z + (mc^2/h)\Psi_4 \]  
(27c)

\[ + \partial \Psi_4/\partial t = + c \partial \Psi_3/\partial x + c \partial \Psi_1/\partial y - c \partial \Psi_4/\partial z - (mc^2/h)\Psi_3 \]  
(27d)

Let us notice that the two oscillators given by equations (27a,b) and (27c,d) are entangled with the spatial dimension y.

So the two incursive systems are no more independent in three space dimensions.
This means that the 4 incursive discrete hyperincursive equations is a whole group of 4 equations which are not separable, except for particular conditions (particle at rest for example).

This is a remarkable result, because the second order discrete hyperincursive equation bifurcates to a group of 4 discrete incursive relativist quantum equations.

Indeed, the 4 first order partial differential equations represent a real 4-spinors in the generic form of Dirac equation.

5. The Dubois-Ord-Mann real 4-spinors equations in the generic form of the Dirac equation

This section demonstrates that the 4 first order partial differential equations (27a,b,c,d) represent a real 4-spinors in the generic form of Dirac relativist quantum equation [32,35].

In defining the 2-spinors real functions,

\[ \varphi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \varphi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}, \]

(28)

the two equations (27a,b) and the two equations (27c,d) are transformed to the following two 2-spinors real equations:

\[ + \frac{\partial \varphi_a}{\partial t} = c \left( c \alpha_1 \frac{\partial \varphi_a}{\partial x} - \sigma_2 \frac{\partial \varphi_b}{\partial y} + c \sigma_3 \frac{\partial \varphi_a}{\partial z} + \sigma_1 \varphi_a \right) \]

(29a)

\[ + \frac{\partial \varphi_b}{\partial t} = - c \left( c \alpha_1 \frac{\partial \varphi_b}{\partial x} + \sigma_2 \frac{\partial \varphi_a}{\partial y} + c \sigma_3 \frac{\partial \varphi_b}{\partial z} - \sigma_1 \varphi_b \right) \]

(29b)

where the real 2-spinors matrices \( \sigma_1, \sigma_2, \sigma_3 \), are defined by

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(30a,b,c)

and 2-Identity

\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_2. \]

(30d)

with the properties

\[ \sigma_1 \sigma_2 = \sigma_3, \sigma_2 \sigma_1 = - \sigma_3, \sigma_2 \sigma_2 = \sigma_1, \sigma_3 \sigma_3 = - \sigma_1, \sigma_3 \sigma_1 = - \sigma_2, \sigma_1 \sigma_3 = \sigma_2 \]

(31a)

These real spinor matrices are related to the Pauli spin matrices,

\[ \sigma_x = \sigma_1, \sigma_y = i \sigma_2, \sigma_z = \sigma_3 \]

(32a,b,c)

In defining the 4-spinors real function

\[ \Psi = \begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix}, \]

(33)

the two equations (29a,b) can be regrouped to the following 4-spinors real equation

\[ + \frac{\partial \Psi}{\partial t} = -c \alpha_1 \frac{\partial \Psi}{\partial x} - c \alpha_2 \frac{\partial \Psi}{\partial y} - c \alpha_3 \frac{\partial \Psi}{\partial z} + \left( mc^2 / \hbar \right) \alpha_0 \Psi \]

(34)

where the real 4-spinors matrices \( \alpha_1, \alpha_2, \alpha_3, \alpha_0 \), are given by

\[ \alpha_1 = \begin{pmatrix} \sigma_1 \\ 0 \\ 0 \\ \sigma_1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ - \sigma_2 \\ 0 \\ \sigma_2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ \sigma_3 \\ 0 \\ \sigma_3 \end{pmatrix}, \alpha_0 = \begin{pmatrix} \sigma_2 \\ 0 \\ 0 \\ - \sigma_2 \end{pmatrix}. \]

(35a,b,c,d)

with the properties

\[ \alpha_1^2 = 1_4, \alpha_2^2 = 1_4, \alpha_3^2 = 1_4, \alpha_0^2 = - 1_4 \]

(36)

where the 4-Identity is given by

\[ 1_4 = \begin{pmatrix} 1_2 \\ 0 \\ 0 \\ 1_2 \end{pmatrix} \]

(37)
and with the following other properties

\[
\begin{align*}
\alpha_0\alpha_1 &= \left( -\sigma_2\sigma_1 \begin{array}{cc} 0 & 0 \\ 0 & \sigma_2\sigma_1 \end{array} \right), \quad \alpha_0\alpha_2 = \left( 0 \begin{array}{cc} \sigma_2^2 & 0 \\ 0 & \sigma_2^2 \end{array} \right), \quad \alpha_0\alpha_3 = \left( -\sigma_2\sigma_3 \begin{array}{cc} 0 & 0 \\ 0 & \sigma_2\sigma_3 \end{array} \right), \\
\alpha_1\alpha_2 &= \left( 0 \begin{array}{cc} -\sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{array} \right), \quad \alpha_1\alpha_3 = \left( \sigma_1\sigma_3 \begin{array}{cc} 0 & 0 \\ 0 & \sigma_1\sigma_3 \end{array} \right), \quad \alpha_2\alpha_3 = \left( 0 \begin{array}{cc} -\sigma_2\sigma_3 & 0 \\ 0 & \sigma_2\sigma_3 \end{array} \right),
\end{align*}
\]

(38)

that give the following properties

\[
\begin{align*}
\alpha_0\alpha_1 + \alpha_1\alpha_0 &= 0, \quad \alpha_0\alpha_2 + \alpha_2\alpha_0 = 0, \quad \alpha_0\alpha_3 + \alpha_3\alpha_0 = 0, \\
\alpha_1\alpha_2 + \alpha_2\alpha_1 &= 0, \quad \alpha_1\alpha_3 + \alpha_3\alpha_1 = 0, \quad \alpha_2\alpha_3 + \alpha_3\alpha_2 = 0
\end{align*}
\]

(39)

In multiplying the 4-spinors real equation (34) by \(i\hbar\), one obtains the 4-spinors equation

\[
+i\hbar\Psi/\partial t = -i\hbar (\alpha_0 \partial\Psi/\partial x - i\hbar c \alpha_2 \partial\Psi/\partial y - i\hbar c \alpha_3 \partial\Psi/\partial z + i\hbar (mc^2/\hbar) \alpha_0\Psi)
\]

(40)

and in introducing the quantum 3-momentum-vector \(\mathbf{p} = (p_x, p_y, p_z)\) given by

\[
p_x = -i\hbar \partial/\partial x, \quad p_y = -i\hbar \partial/\partial y, \quad p_z = -i\hbar \partial/\partial z,
\]

(41a,b,c)

with the energy \(E\) given by

\[
E = \hbar \partial/\partial t,
\]

(41d)

the 4-spinors equation (40) is represented in the generic form of the Dirac equation as

\[
+i\hbar\partial\Psi/\partial t = +c \alpha_x p_1\Psi + c \alpha_y p_2\Psi + c \alpha_z p_3\Psi + mc^2\beta\Psi
\]

(42)

where

\[
\alpha_x = \alpha_1, \quad \alpha_y = \alpha_2, \quad \alpha_z = \alpha_3,
\]

(43a,b,c)

and

\[
\beta = i \alpha_0
\]

(43d)

6. The 4 incursive discrete Dubois-Majorana real equations

It is important to know that the Dubois-Ord-Mann real 4-spinors belong to the group of the Majorana real 4-spinors. Indeed, with two successive permutations, the 4 first order partial differential equations of the Dubois-Ord-Mann 4-spinors transform to the 4 first order partial differential equations of the Majorana real 4-spinors equation given by Pessa [10]. The first permutation consists in permuting two space variables. The second permutation consists in permuting the indexes of the functions. This was demonstrated in my recent paper [35].

But it is interesting to analyze separately the transformation of the 4 Dubois-Ord-Mann equations, firstly, in permuting the two space variables, and secondly, in permuting the indexes of the functions. Indeed, in permuting only the two space variables, \(y\) and \(z\), in the 4 Dubois-Ord-Mann equations, we obtain the 4 first order partial differential equations where the “Majorana Oscillator” presented in the paper of Pessa [10] is well put in evidence: the first two equations represents a first oscillator and the last two equations represents the second oscillator, that is an anti-oscillator. This is completely in agreement with my theory of the hyperincursive discrete harmonic oscillator that bifurcates to two incursive oscillators, each oscillator being the reverse of the other.

This transformation of the Dubois-Ord-Mann equations, by permutation of the space variables, \(y\) and \(z\), was presented in a recent paper [33], where the transformed equations were called the Dubois-Majorana equations. In this paper [33], we say explicitly what follows:

“the Dubois Majorana real 4-spinors equation is in fact composed of a spatial oscillator entangled to a spatial anti-oscillator, related to the particle electron and anti-particle positron in Majorana equation. …
The Dubois-Majorana discrete real 4-spinors equation shows explicitly that the Majorana equation is in fact composed of a discrete hyperincursive spatial oscillator entangled to a spatial harmonic oscillator, related to the particle electron and anti-particle positron in Majorana equation.

We have shown that the “Majorana Oscillator” in reference to the paper of Eliano Pessa [10] is explicitly explained by the hyperincursive discrete harmonic oscillator which is composed of 4 incursive discrete equations that are in fact the real 4-spinors time equation” [33].

Moreover the 4 incursive discrete Dubois-Majorana equations can be used for the simulation of the “Majorana Oscillator”.

The incursive discrete Dubois-Majorana real 4-spinors equations, is transformed to the Majorana equations [9,10], by permutation of the indexes of the functions, as shown is our paper [33].

As presented recently [33,35], let us first consider the one-dimensional spatial system, based on the hyperincursive discrete harmonic oscillator, where the space variable is given by y instead of z in equation (20a,b,c,d).

The function q bifurcates into the 4 functions \(q_1(m,k), q_2(m,k), q_3(m,k), q_4(m,k)\), that depend on the 4 incursive discrete Dubois-Majorana equations, in 1D, given by

\[
q_1(m,k+1) = q_1(m,k-1) + C[q_1(m+1,k) - q_1(m-1,k)] - A q_2(m,k),
\]
\[
q_2(m,k+1) = q_2(m,k-1) - C[q_2(m+1,k) - q_2(m-1,k)] + A q_1(m,k),
\]
\[
q_3(m,k+1) = q_3(m,k-1) + C[q_3(m+1,k) - q_3(m-1,k)] + A q_4(m,k),
\]
\[
q_4(m,k+1) = q_4(m,k-1) - C[q_4(m+1,k) - q_4(m-1,k)] - A q_3(m,k),
\]

As already noted, with only one space dimension, the two first equations are independent of the two second equations. But as shown below, with only the space variable z, the 4 equations are not independent.

The generalisation of equations (44a,b,c,d) to three spatial dimensions are based on equations (25a,b,c,d), where the space variable y and z are inversed, and, so, are given by the following 4 discrete space time incursive equations of the 4 functions \(q_i(l,m,n,k), i = 1,2,3,4\),

\[
q_1(l,m,n,k+1) = q_1(l,m,n,k-1) + B[q_2(l+1,m,n,k) - q_2(l-1,m,n,k)]
+ C[q_1(l,m+1,n,k) - q_1(l,m-1,n,k)] + D[q_4(l,m,n+1,k) - q_4(l,m,n-1,k)] - A q_2(l,m,n,k),
\]

\[
q_2(l,m,n,k+1) = q_2(l,m,n,k-1) + B[q_1(l+1,m,n,k) - q_1(l-1,m,n,k)]
- C[q_2(l,m+1,n,k) - q_2(l,m-1,n,k)] - D[q_3(l,m,n+1,k) - q_3(l,m,n-1,k)] + A q_1(l,m,n,k),
\]

\[
q_3(l,m,n,k+1) = q_3(l,m,n,k-1) + B[q_4(l+1,m,n,k) - q_4(l-1,m,n,k)]
+ C[q_3(l,m+1,n,k) - q_3(l,m-1,n,k)] - D[q_2(l,m,n+1,k) - q_2(l,m,n-1,k)] + A q_4(l,m,n,k),
\]

\[
q_4(l,m,n,k+1) = q_4(l,m,n,k-1) + B[q_3(l+1,m,n,k) - q_3(l-1,m,n,k)]
- C[q_4(l,m+1,n,k) - q_4(l,m-1,n,k)] + D[q_1(l,m,n+1,k) - q_1(l,m,n-1,k)] - A q_3(l,m,n,k),
\]

that we call the incursive discrete Dubois-Majorana real equations [33].
In three spatial dimensions, the two first equations (45a,b) are no more independent of the two second equations (45c,d), due to the terms, with parameter D, of the z spatial derivation.

The discrete functions \( q_i(x, y, z, t) = q_i(r, t), j = 1,2,3,4 \) tend to the continuous functions \( \Psi_j(x, y, z, t) = \Psi_j(r, t) \), when the discrete space and time intervals tend to zero, at the limit

\[
\Psi_j = \Psi_j(r, t) = \lim_{\Delta r \to 0, \Delta t \to 0} q_i(r, t), j = 1,2,3,4.
\] (46)

So, the 4 discrete equations (45a,b,c,d) tend to the following 4 first order partial differential equations [33] representing real 4-spinors equations

\[
+ \frac{\partial \Psi_1}{\partial t} = + c \frac{\partial \Psi_2}{\partial x} + c \frac{\partial \Psi_3}{\partial y} + c \frac{\partial \Psi_4}{\partial z} - (mc^2/h)\Psi_2
\] (47a)

\[
+ \frac{\partial \Psi_2}{\partial t} = + c \frac{\partial \Psi_1}{\partial x} - c \frac{\partial \Psi_3}{\partial y} - c \frac{\partial \Psi_4}{\partial z} + (mc^2/h)\Psi_1
\] (47b)

\[
+ \frac{\partial \Psi_3}{\partial t} = + c \frac{\partial \Psi_4}{\partial x} + c \frac{\partial \Psi_1}{\partial y} - c \frac{\partial \Psi_2}{\partial z} + (mc^2/h)\Psi_4
\] (47c)

\[
+ \frac{\partial \Psi_4}{\partial t} = + c \frac{\partial \Psi_3}{\partial x} - c \frac{\partial \Psi_4}{\partial y} + c \frac{\partial \Psi_1}{\partial z} - (mc^2/h)\Psi_3
\] (47d)

The two oscillators given by the equations (47a,b) and (47c,d) are now entangled with the spatial dimension z.

Naturally, these equations also represent real 4-spinors in the generic form of the Dirac relativist quantum equation.

In a recent paper [33], we have shown that these Dubois-Majorana real 4-spinors equations (47a,b,c,d) can be transformed to the generic form of the Dirac 4-spinors equation.

7. The Majorana equation from the Dubois-Majorana real 4-spinors

The Dubois-Majorana real 4-spinors equations (47a,b,c,d) can be rewritten in the following order

\[
+ \frac{\partial \Psi_1}{\partial t} = + c \frac{\partial \Psi_2}{\partial x} - c \frac{\partial \Psi_3}{\partial y} + c \frac{\partial \Psi_4}{\partial z} - (mc^2/h)\Psi_2
\] (48a)

\[
+ \frac{\partial \Psi_2}{\partial t} = + c \frac{\partial \Psi_3}{\partial x} - c \frac{\partial \Psi_4}{\partial y} + c \frac{\partial \Psi_1}{\partial z} + (mc^2/h)\Psi_1
\] (48b)

\[
+ \frac{\partial \Psi_3}{\partial t} = + c \frac{\partial \Psi_4}{\partial x} + c \frac{\partial \Psi_1}{\partial y} - c \frac{\partial \Psi_2}{\partial z} + (mc^2/h)\Psi_4
\] (48c)

\[
+ \frac{\partial \Psi_4}{\partial t} = + c \frac{\partial \Psi_3}{\partial x} - c \frac{\partial \Psi_1}{\partial y} + c \frac{\partial \Psi_2}{\partial z} - (mc^2/h)\Psi_3
\] (48d)

In making the change in indexes j of these 4 functions \( \Psi_j = \Psi_j(x, y, z, t), j = 1,2,3,4 \),

\[
\Psi_1 = \Psi_4, \quad \Psi_2 = \Psi_2, \quad \Psi_3 = \Psi_3, \quad \Psi_4 = \Psi_1,
\] (49)

where the functions \( \Psi_j = \Psi_j(x, y, z, t), j = 1,2,3,4 \), are defined as the Majorana functions, the 4-spinor real equations (48a,b,c,d) are transformed to the 4 real first order Majorana partial differential equations

\[
+ \frac{\partial \Psi_1}{\partial t} = + c \frac{\partial \Psi_4}{\partial x} - c \frac{\partial \Psi_1}{\partial y} + c \frac{\partial \Psi_3}{\partial z} - (mc^2/h)\Psi_4
\] (50a)

\[
+ \frac{\partial \Psi_2}{\partial t} = + c \frac{\partial \Psi_3}{\partial x} - c \frac{\partial \Psi_2}{\partial y} + c \frac{\partial \Psi_4}{\partial z} + (mc^2/h)\Psi_3
\] (50b)

\[
+ \frac{\partial \Psi_3}{\partial t} = + c \frac{\partial \Psi_2}{\partial x} + c \frac{\partial \Psi_3}{\partial y} + c \frac{\partial \Psi_1}{\partial z} - (mc^2/h)\Psi_2
\] (50c)

\[
+ \frac{\partial \Psi_4}{\partial t} = + c \frac{\partial \Psi_1}{\partial x} + c \frac{\partial \Psi_4}{\partial y} - c \frac{\partial \Psi_2}{\partial z} + (mc^2/h)\Psi_1
\] (50d)

which are identical to the original Majorana equations [9], e.g. equations (4a,b,c,d) in Pessa [10].

Let us remark that in 1937, Ettore Majorana published his last paper [9] on his Majorana Equation, before his mysterious disappearance.

The Dubois-Majorana discrete real 4-spinors equation shows explicitly that the Majorana equation is in fact composed of a discrete hyperincursive spatial oscillator entangled to a spatial harmonic anti-oscillator, related to the particle electron and anti-particle positron in Majorana equation.

The “Majorana Oscillator”, in reference to the paper of Eliano Pessa [10] is explicitly explained by
the real 4-spinors time equation
\[+\frac{\partial \Psi_1}{\partial t} = -(mc^2/h)\Psi_4 \] (51a)
\[+\frac{\partial \Psi_4}{\partial t} = + (mc^2/h)\Psi_1 \] (51b)
\[+\frac{\partial \Psi_2}{\partial t} = + (mc^2/h)\Psi_3 \] (51c)
\[+\frac{\partial \Psi_3}{\partial t} = - (mc^2/h)\Psi_2 \] (51d)
which are similar to the Dubois-Majorana temporal equations given by equations (47a,b,c,d).

8. The 4 incursive discrete Majorana real equations
Recently, we have shown that this Dubois-Majorana real 4-spinors can be simply transformed to the original Majorana equation [33,35].

Let us consider the change in indexes of the functions \(q_j = q_j(x, y, z, t), j = 1,2,3\), of the real equations in the Dubois-Majorana discrete 4-spinors equations (45a,b,c,d):
\[\bar{q}_1 = q_4, \quad \bar{q}_2 = q_2, \quad \bar{q}_3 = q_1, \quad \bar{q}_4 = q_3, \] (52)
where the functions
\[\bar{q}_j = \bar{q}_j(x, y, z, t) = \bar{q}_j(l, m, n, k), j = 1,2,3,4, \] (53)
define the discrete Majorana functions.

With these changes in indexes, the equations (45a,b,c,d) transform to the 4 discrete incursive equations of the functions \(\bar{q}_j, j = 1,2,3,4,\)
\[\bar{q}_1(l, m, n, k + 1) = \bar{q}_1(l, m, n, k - 1) + B[q_4(l + 1, m, n, k) - q_4(l - 1, m, n, k)] - C[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] + D[q_2(l, m, n + 1, k) - q_2(l, m, n - 1, k)] - A[q_1(l, m, n, k)] \]
\[\bar{q}_2(l, m, n, k + 1) = \bar{q}_2(l, m, n, k - 1) + B[q_3(l + 1, m, n, k) - q_3(l - 1, m, n, k)] - C[q_2(l, m + 1, n, k) - q_2(l, m - 1, n, k)] + D[q_1(l, m, n + 1, k) - q_1(l, m, n - 1, k)] + A[l, m, n, k] \]
\[\bar{q}_3(l, m, n, k + 1) = \bar{q}_3(l, m, n, k - 1) + B[q_2(l + 1, m, n, k) - q_2(l - 1, m, n, k)] + C[q_1(l, m + 1, n, k) - q_1(l, m - 1, n, k)] - D[q_3(l, m, n + 1, k) - q_3(l, m, n - 1, k)] - A[l, m, n, k] \]
\[\bar{q}_4(l, m, n, k + 1) = \bar{q}_4(l, m, n, k - 1) + B[q_3(l + 1, m, n, k) - q_3(l - 1, m, n, k)] + C[q_2(l, m + 1, n, k) - q_2(l, m - 1, n, k)] - D[q_4(l, m, n + 1, k) - q_4(l, m, n - 1, k)] + A[l, m, n, k] \] (54a,b,c,d)
with
\[\bar{A} = A = a(2\Delta t), \quad \bar{B} = B = c\Delta t/\Delta x, \quad \bar{C} = C = c\Delta t/\Delta y, \quad \bar{D} = D = c\Delta t/\Delta z \] (55a,b,c,d)
where \(\Delta t\) and \(\Delta x, \Delta y, \Delta z\) are the discrete intervals of time and space respectively.

9. Conclusion
In this paper, we consider the deduction of the discrete Majorana real 4-spinors equation from our hyperincursive discrete Klein-Gordon equation.
From the Klein-Gordon partial differential equation, we presented the space and time symmetric second order hyperincursive discrete Klein-Gordon equation. With only the temporal component, the Klein-Gordon equation represents an oscillator, and the discrete Klein-Gordon equation is similar to the second order hyperincursive discrete harmonic oscillator. This hyperincursive equation is separable into a first discrete incursive oscillator depending on two functions and a second incursive oscillator depending on two other functions given by 4 first order discrete equations, where the second incursive system is the discrete time reverse of the first incursive system.

In one space z dimension (1D), we presented the 4 incursive discrete Dubois-Ord-Mann equations that are similar to the temporal equations without the space term (for a particle at rest). These discrete equations correspond to 4 first order partial differential equations representing the Dubois-Ord-Mann real 4-spinors equations. We called these equations, the Dubois-Ord-Mann real 4-spinors equations, because Ord and Mann deduced the same real 4-spinors equations from a one-dimensional space stochastic model.

Then we generalized these incursive discrete Dubois-Ord-Mann equations to three space dimensions (3D), corresponding to 4 first order partial differential equations representing the Dubois-Ord-Mann real 4-spinors equations. Let us notice that the two oscillators are entangled with the spatial dimension y. So the two incursive systems are no more independent in three space dimensions. This means that the 4 incursive discrete hyperincursive equations is a whole group of 4 equations which are not separable, except for particular conditions (particle at rest for example). This is a remarkable result, because the second order discrete hyperincursive equation bifurcates to a group of 4 discrete incursive relativist quantum equations. Indeed, the 4 first order partial differential equations represent a real 4-spinors in the generic form of Dirac relativist quantum equation.

Let us remark that the discrete Dubois-Ord-Mann real 4-spinors was obtained from the deterministic space and time symmetric hyperincursive Klein-Gordon equation. It could be interesting to see how to generalize the one-dimensional real stochastic difference equations of Ord and Mann to three spatial dimensions.

It is important to know that the Dubois-Ord-Mann real 4-spinors belong to the group of the Majorana real 4-spinors. Indeed, with two successive permutations, the 4 first order partial differential equations of the Dubois-Ord-Mann 4-spinors transform to the 4 first order partial differential equations of the Majorana real 4-spinors equation.

We analyzed separately the transformation of the 4 Dubois-Ord-Mann equations.

In permuting only the two space variables in the 4 Dubois-Ord-Mann equations, we obtained the Dubois-Majorana equations.

With the 4 incursive discrete Dubois-Majorana equations, the “Majorana Oscillator” presented in the paper of Pessa is well put in evidence: the first two discrete equations represent a first oscillator and the last two discrete equations represent the second oscillator that is an anti-oscillator. This is completely in agreement with my theory of the hyperincursive discrete harmonic oscillator that bifurcates to two incursive oscillators, each oscillator being the reverse of the other. The particle electron and the anti-particle positron in Majorana equation are related to this oscillator and anti-oscillator.

Moreover the 4 incursive discrete Dubois-Majorana equations can be used for the simulation of the “Majorana Oscillator.”

Finally, the incursive discrete Dubois-Majorana real 4-spinors equations are transformed, by permutation of the indexes of the functions, to the 4 incursive discrete Majorana real 4-spinors equations.

Let us notice that, in a recent paper [34] we have demonstrated that these 4 discrete incursive Majorana equations bifurcate to the original Dirac equation, with an original method.

References
[1] Dubois Daniel M 2016 Hyperincursive algorithms of classical harmonic oscillator applied to quantum harmonic oscillator separable into incursive oscillators Unified Field Mechanics,
Natural Science Beyond the Veil of Spacetime: Proc. of the IXth Symp Honoring Noted French Mathematical Physicist Jean-Pierre Vigier 16-19 November 2014 Baltimore USA
Eds R L Amoroso, L H Kauffman et al Singapore World Scientific 55-65

[2] Dubois Daniel M 2018 Unified discrete mechanics: Bifurcation of hyperincursive discrete harmonic oscillator, Schrödinger’s quantum oscillator, Klein-Gordon’s equation and Dirac’s quantum relativist equations Unified Field Mechanics II, Formulations and Empirical Tests: Proc. of the Xth Symp Honoring Noted French Mathematical Physicist Jean-Pierre Vigier 25-28 July 2016 Porto Novo Italy Eds R L Amoroso, L H Kauffman et al Singapore World Scientific 158-177

[3] Klein Oskar 1926 Quantentheorie und fünfdimensionale relativitätstheorie Zeitschrift für Physik 37 895

[4] Gordon Walter 1926 Der comptoneffekt nach Schrödingerschen theorie Zeitschrift für Physik 40 117

[5] Dirac P A M 1928 The quantum theory of the electron Proc of the Royal Society A Mathematical, Physical and Engineering Sciences 117 778-610-624

[6] Dirac P A M 1964 Lectures on Quantum Mechanics New York Academic Press

[7] Proca Al 1930 Sur l’équation de Dirac J. Phys. Radium 1 7 235-248

[8] Proca Al 1932 Quelques observations concernant un article, sur l’équation de Dirac J. Phys. Radium 3 4 172-184

[9] Majorana Ettore 1937 Teoria simmetrica dell’elettrone e del positrone Il Nuovo Cim 14 171

[10] Pessa Eliano 2006 The Majorana oscillator Electronic Journal of Theoretical Physics EJTP 3 10 285-292

[11] Messiah Albert 1965 Mecanique Quantique Tomes 1 & 2 Paris Dunod

[12] Ord G N and Mann R B 2003 Entwined paths, difference equations, and the Dirac equation Phys Rev A 67 022105

[13] Dubois Daniel M 1995 Total incursive control of linear, non-linear and chaotic systems Advances in Computer Cybernetics ed G E Lasker Canada: The International Institute for Advanced Studies in Systems Research and Cybernetics II 167-171 ISBN 0921836236

[14] Dubois Daniel M 1998 Computing anticipatory systems with incursion and hyperincursion Computing Anticipatory Systems: Conf. Proc. of CASYS - First Int Conf 11-15 August 1997 Liege Belgium ed D M Dubois Woodbury New York American Institute of Physics AIP CP 437 3-29

[15] Dubois Daniel M 1999 Computational derivation of quantum and relativist systems with forward-backward space-time shifts Computing Anticipatory Systems: Conf Proc of CASYS’98 Second Int Conf 10-14 August 1998 Liege Belgium Ed D M Dubois Woodbury, New York American Institute of Physics AIP CP 465 435-456

[16] Dubois Daniel M 2000 Review of incursive, hyperincursive and anticipatory systems – foundation of anticipation in electromagnetism Computing Anticipatory Systems: Conf. Proc. of CASYS’99 Third Int. Conf. (9-14 August 1999 Liege Belgium Ed D M Dubois Melville, New York: American Institute of Physics AIP CP 517 3-30

[17] Dubois Daniel M and Kalisz Eugenia 2004 Precision and stability analysis of Euler, Runge-Kutta and incursive algorithms for the harmonic oscillator International Journal of Computing Anticipatory Systems 14 21-36 ISSN 1373-5411 ISBN 2-930396-00-8

[18] Dubois Daniel M 2014 The New Concept of Deterministic Anticipation in Natural and Artificial Systems International Journal of Computing Anticipatory Systems Vol 26 pp 3-15 ISSN 1373-5411 ISBN 2-930396-15-6

[19] Antippa Adel F and Dubois Daniel M 2002 The harmonic oscillator via the discrete path approach International Journal of Computing Anticipatory Systems Volume 11 141–153 ISSN 1373-5411

[20] Antippa Adel F and Dubois Daniel M 2004 Anticipation, orbital stability, and energy conservation in discrete harmonic oscillators Computing Anticipatory Systems: Conf. Proc.
of CASYS’03–Sixth Int. Conf. (11–16 August 2003 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 718 3–44

[21] Antippa Adel F and Dubois Daniel M 2006 The dual incursive system of the discrete harmonic oscillator Computing Anticipatory Systems: Conf. Proc. of CASYS’05–Seventh Int. Conf. (8–13 August 2005 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 839 11–64

[22] Antippa Adel F and Dubois Daniel M 2006 The Superposed hyperincursive system of the discrete harmonic oscillator Computing Anticipatory Systems: Conf. Proc. of CASYS’05–Seventh Int. Conf. (8–13 August 2005 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 839 65–126

[23] Antippa Adel F and Dubois Daniel M 2007 Incursive discretization, system bifurcation, and energy conservation Journal of Mathematical Physics 48 1 012701

[24] Antippa Adel F and Dubois Daniel M 2008 Hyperincursive discrete harmonic oscillator Journal of Mathematical Physics 49 3 (032701)

[25] Antippa Adel F and Dubois Daniel M 2008 Synchronous Discrete harmonic oscillator Computing Anticipatory Systems: Conf. Proc. of CASYS’07–Eighth Int. Conf. (6–11 August 2007 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 1051 82–99

[26] Antippa Adel F and Dubois Daniel M 2010 Discrete Harmonic Oscillator: A Short Compendium of Formulas Computing Anticipatory Systems: Conf. Proc. of CASYS’09–Ninth Int. Conf. (3–8 August 2009 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 1303 111–120

[27] Antippa Adel F and Dubois Daniel M 2010 Time-Symmetric Discretization of The Harmonic Oscillator Computing Anticipatory Systems: Conf. Proc. of CASYS’09–Ninth Int. Conf. (3–8 August 2009 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 1303 121–125

[28] Antippa Adel F and Dubois Daniel M 2010 Discrete Harmonic Oscillator: Evolution of Notation and Cumulative Erratum Computing Anticipatory Systems: Conf. Proc. of CASYS’09–Ninth Int. Conf. (3–8 August 2009 Liege Belgium) ed D M Dubois (Melville, New York: American Institute of Physics) AIP CP 1303 126–130

[29] Dubois Daniel M 2017 Reversible Time, Retardation and Anticipation in the Recursive, Incursive and Hyperincursive Klein-Gordon Quantum Systems Proc. of the Symp. on Reversible Time, Retardation and Anticipation in Quantum Physics, Biology and Cybernetics: held as part of the 29th Int. Conf. on Systems Research, Informatics and Cybernetics (July 31–August 4 2017 Baden-Baden Germany) Ed D M Dubois and G E Lasker (Canada: The International Institute for Advanced Studies in Systems Research and Cybernetics) I 19-23 ISBN 978-1-897546-42-0

[30] Dubois Daniel M 2017 Reversible Hyperincursive Discrete Schrödinger Quantum Harmonic Oscillator Separable Into Two Non-Reversible Incursive Discrete Oscillators Proc. of the Symp. on Reversible Time, Retardation and Anticipation in Quantum Physics, Biology and Cybernetics: held as part of the 29th Int. Conf. on Systems Research, Informatics and Cybernetics (July 31–August 4 2017 Baden-Baden Germany) Ed D M Dubois and G E Lasker (Canada: International Institute for Advanced Studies in Systems Research and Cybernetics) I 31-36 ISBN 978-1-897546-42-0

[31] Dubois Daniel M 2017 Deduction of the First Order Spinor Equations of the Schrödinger Quantum Spatial Harmonic Oscillator with the Emergence of the Fundamental Energy Proc. of the Symp. on Reversible Time, Retardation and Anticipation in Quantum Physics, Biology and Cybernetics: held as part of the 29th Int. Conf. on Systems Research, Informatics and Cybernetics (July 31–August 4 2017 Baden-Baden Germany) Eds D M Dubois and G E Lasker (Canada: International Institute for Advanced Studies in Systems Research and Cybernetics) I 77-82 ISBN 978-1-897546-42-0
[32] Dubois Daniel M. 2018 Generalisation to Three Spatial Dimensions of the Dubois-Ord-Mann Real 4-Spinors Equation from the Hyperincursive Discrete Klein-Gordon Equation Proc. of the Symp. on Anticipative Models in Physics, Relativistic Quantum Physics, Biology and Informatics: held as part of the 30th Int. Conf. on Systems Research, Informatics and Cybernetics (July 30-August 3 2018 Baden-Baden Germany) Ed D M Dubois and G E Lasker (Canada: International Institute for Advanced Studies in Systems Research and Cybernetics) I 27-32 ISBN 978-1-897546-80-2

[33] Dubois Daniel M 2018 Deduction of the Majorana Real 4-Spinors Generic Dirac Equation from the Computable Hyperincursive Discrete Klein-Gordon Equation Proc. of the Symp. on Anticipative Models in Physics, Relativistic Quantum Physics, Biology and Informatics: held as part of the 30th Int. Conf. on Systems Research, Informatics and Cybernetics (July 30-August 3 2018 Baden-Baden Germany) Ed D M Dubois and G E Lasker (Canada: International Institute for Advanced Studies in Systems Research and Cybernetics) I 33-38 ISBN 978-1-897546-80-2

[34] Dubois Daniel M 2018 The Hyperincursive Discrete Klein-Gordon Equation for Computing the Majorana Real 4-Spinors Equation and the Real 8-spinors Dirac Equation Proc. of the Symp. on Anticipative Models in Physics, Relativistic Quantum Physics, Biology and Informatics: held as part of the 30th Int. Conf. on Systems Research, Informatics and Cybernetics (July 30-August 3 2018 Baden-Baden Germany) Ed D M Dubois and G E Lasker (Canada: International Institute for Advanced Studies in Systems Research and Cybernetics) I 73-78 ISBN 978-1-897546-80-2

[35] Dubois Daniel M 2018 Bifurcation of the Hyperincursive Discrete Klein-Gordon Equation to Real 4-Spinors Dirac Equation Related to the Majorana Equation Acta Systemica International Journal of the International Institute for Advanced Studies in Systems Research and Cybernetics (IIAS) XVIII 2 23-28 ISSN 1813-4769