Eichler cohomology in general weights using spectral theory

Michael O. Neururer

September 3, 2014
1 Introduction

Let $\Gamma$ be a finitely generated Fuchsian group of the first kind that contains translations and $-1$. The interpretation of modular forms for $\Gamma$ as elements in certain cohomology groups was first discovered by Eichler [4]. The following theorem is due to him in the case of even weights and trivial multiplier system. The general case was proved later by Gunning [7].

**Theorem 1.1.** Let $k \in \mathbb{Z}$, $v$ a weight $k$ multiplier system for $\Gamma$ and $P_k$ the vector space of polynomials with coefficients in $\mathbb{C}$ of degree $\leq k$. Then

$$M_{k+2}(\Gamma, v) \oplus S_{k+2}(\Gamma, v) \cong H^1_{-k,v}(P_k).$$

Here $P_k$ is viewed as a $\Gamma$-module with the $|_{-k,v}$ action and $H^1_{-k,v}$ is the first cohomology group (for more details see Section 1). Theorem 1.1 has many applications in the computational theory of modular forms and the study of critical values of their $L$-functions, e.g. in algebraicity results like Manin’s period theorem [12].

The subject of this paper is a variant of Theorem 1.1 in the case of arbitrary real weight. Knopp first formulated this in 1974 [10]. To a cusp form $g$ of real weight $2 - r$ and multiplier system $\overline{v}$ he associated a cocycle with values in a space of functions $\mathcal{P}$ by

$$\phi(g)(z) : \gamma \mapsto \int_{\gamma^{-1}\infty}^{\infty} g(\tau) (\overline{\tau} - z)^{-r} d\overline{\tau}.$$ 

This induces an injective map from $S_{2-r}(\Gamma, \overline{v})$ to $H^1_{r,v}(\Gamma, \mathcal{P})$. He conjectured that this map is actually an isomorphism but he was only able to prove this for the cases $r \leq 0$ and $r \geq 2$. A partial result on this problem was obtained by Wang in 2000 [15] and it was resolved in 2013 by Knopp and Mawi [8], using Petersson’s principal part Theorem.

**Theorem 1.2.** (Knopp-Mawi)

$$S_{2-r}(\Gamma, \overline{v}) \cong H^1_{r,v}(\Gamma, \mathcal{P}).$$

A recent preprint [1] by Bruggeman, Choie and Diamantis gives a similar isomorphism for a much wider class of automorphic forms. They also provide several motivations to study cocycles of real weight. One of them is a formula of Goldfeld [6] that suggests a connection between the value $L'_f(1)$ (where
This is made more explicit in [1] where it is shown that
\[-\pi ir L'_f(1) + O_{r \to 0}(r^2) = \phi_{f_r}(S)(0),\]

where \( f_r(z) = f(z)(\eta(z)\eta(Nz))^r \) is a cusp form of weight \( 2 + r \) and \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

In this article we present a new proof of Theorem 1.2 for weights \( 2 - r > 1 \) that views the isomorphism in Knopp and Mawi’s theorem as a duality. The key construction is a pairing between \( S_{2-r}(\Gamma, \mathfrak{P}) \) and \( H^1_{r,v}(\Gamma, \mathfrak{P}) \) which we introduce in Section 2. In Section 3 we show that this pairing is perfect, which implies Theorem 1.2 for the weights we consider. The proof also implies Theorem 1.2 for the weights \( 2 - r < 0 \).

The proof proceeds as follows: Theorem 3.2 and Corollary 3.5 show that every cocycle \( \phi \) in \( Z^1_{r,v}(\Gamma, \mathfrak{P}) \) is a coboundary in \( Z^1_{r,v}(\Gamma, \mathfrak{Q}) \), where \( \mathfrak{Q} \) is a larger space of functions than \( \mathfrak{P} \). This means that there exists \( g \in \mathfrak{Q} \) such that \( \phi(\gamma) = g|_{r,v} - g \) for all \( \gamma \in \Gamma \). In the next step we assume that \( \phi \) is orthogonal to all cusp forms. Using the description of \( \phi \) as a coboundary in \( Z^1_{r,v}(\Gamma, \mathfrak{Q}) \) Proposition 3.11 shows that this orthogonality is equivalent to \( g^{r+2} \frac{d}{dz}\bar{g}(z) \) being in the image of the Maass raising operator \( K_{-r} \). Finally we apply standard results from the spectral theory of automorphic forms to show that \( \phi \) must be a coboundary in \( Z^1_{r,v}(\Gamma, \mathfrak{P}) \).

One of the advantages of the new proof is that once all the constructions are in place the theorem can be solved with standard techniques from the spectral theory of automorphic forms. The main references we use for spectral theory are the excellent articles [14] by W. Roelcke. Another advantage of the new proof is that it can easily be generalised to the case of vector-valued cusp forms. We sketch this generalisation in the last section of this article.

**Acknowledgements:** I am thankful to N. Diamantis for suggesting this topic to me and Y. Petridis for a helpful discussion on the proof of Proposition 3.11. I am particularly grateful for the countless comments, corrections and improvements that R. Bruggeman provided during the completion of this article.
1.1 Preliminaries

For any matrix \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \) we define the function \( j(\gamma, z) = cz + d \).

For another element \( \delta \) of \( \text{SL}_2(\mathbb{R}) \) one has the relation
\[
j(\gamma\delta, z) = j(\gamma, \delta z)j(\delta, z),
\]
for all \( z \) in the upper half plane. Let \( r \in \mathbb{R} \). Two useful functions when dealing with real weights, introduced by Petersson in [13], are
\[
\omega(\gamma, \delta) = \frac{1}{2\pi}(-\arg(j(\gamma\delta, z)) + \arg(j(\gamma, \delta z)) + \arg(j(\delta, z)))
\]
and
\[
\sigma_r(\gamma, \delta) = e^{2\pi ir\omega(\gamma, \delta)}.
\]
Here \( \arg \), the argument, is always chosen to lie in \((-\pi, \pi]\). The value of \( \omega(\gamma, \delta) \) is independent of \( z \) and in \{-1, 0, 1\}. From the definition it follows that
\[
\sigma_r(\gamma, \delta)j(\gamma\delta, z)^r = j(\gamma, \delta z)^rj(\delta z)^r, \quad \gamma, \delta \in \Gamma.
\]
(1)
Here \( j(\gamma, z)^r = \exp(r \cdot \log(j(\gamma, z))) \) and log is the principal branch of the complex logarithm satisfying \( \log(z) = |z| + i\arg(z) \) for all \( z \neq 0 \).

A multiplier system of weight \( r \) for \( \Gamma \) is a function \( v : \Gamma \to \mathbb{C} \) which satisfies the consistency condition
\[
v(\gamma\delta)j(\gamma\delta, z)^r = v(\gamma)v(\delta)j(\gamma, \delta z)^rj(\delta, z)^r, \quad \forall \gamma, \delta \in \Gamma,
\]
or equivalently
\[
v(\gamma\delta) = \sigma_r(\gamma, \delta)v(\gamma)v(\delta).
\]
Note that \( v \) is also a multiplier system of any weight \( r' \in \mathbb{R} \) with \( r' \equiv r \mod 2 \) and \( \overline{v} \) is a multiplier system of weight \(-r\). A multiplier system is called unitary if \( |v(\gamma)| = 1 \) for all \( \gamma \in \Gamma \). For the rest of this article we fix a unitary multiplier system \( v \) of weight \( r \).

For a function \( f \) on the upper half plane \( \mathcal{H} \) and \( \gamma \in \text{SL}_2(\mathbb{R}) \) the slash operators \( |_{r,v} \) and \( |_r \) are defined by
\[
f|_{r,v}\gamma(z) = \overline{v}(\gamma)j(\gamma, z)^{-r}f(\gamma z)
\]
and
\[
f|_r\gamma(z) = j(\gamma, z)^{-r}f(\gamma z).
\]
The consistency condition for $v$ implies that

$$f|_{r,v} \gamma \delta(z) = (f|_{r,v})|_{r,v} \delta(z), \quad \forall \gamma, \delta \in \Gamma.$$  

Let $q_0 = \infty$ and $q_1, \ldots, q_m$ be a set of representatives of the cusps of $\Gamma$. For every cusp $q$ the stabiliser subgroup $\Gamma_q$ is generated by $-1$ and one generator $\sigma_q \in \Gamma$. For $q = \infty$ it takes the form $\sigma_{\infty} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ with $\lambda > 0$. Let $f$ be holomorphic on $\mathcal{H}$ and invariant under $|_{r,v}$. The equation $f(z + \lambda) = v(\sigma_{\infty}) f(z)$ implies that $f$ has a Laurent expansion at $\infty$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_{n,0} \exp \left( \frac{2\pi i (n + \kappa_0) z}{\lambda} \right), \quad (2)$$

where $\kappa_i \in [0, 1)$ is defined for any cusp by $v(\sigma_i) = e^{2\pi i \kappa_i}$. At the other cusps the expansion is of the form

$$f(z) = (z - q_i)^r \sum_{n=N_i}^{\infty} a_{n,i} \exp \left( -2\pi i \frac{n + \kappa_i}{\lambda_i (z - q_i)} \right). \quad (3)$$

Here $\lambda_i > 0$ is given by

$$A\sigma_{q_i} A^{-1} = \begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & -q_i \end{pmatrix}$.

**Definition 1.1.** Let $f$ be holomorphic in $\mathcal{H}$ and invariant under $|_{r,v}$. Then $f$ is called a modular form (resp. cusp form) of weight $r$ and multiplier system $v$ for $\Gamma$ if in the Laurent expansions in (2) and (3) all $a_{n,i}$ with $n + \kappa_i < 0$ (resp. $\leq 0$) are zero. The set of modular forms is denoted by $M_r(\Gamma, v)$, the set of cusp forms by $S_r(\Gamma, v)$.

**Remark.** By the main theorem of [9] all modular forms of negative weight are 0.
1.2 Cohomology

Definition 1.2. Let $\mathcal{P}$ be the space of holomorphic functions on $\mathcal{H}$ that satisfy
\[
|f(z)| < K(|z|^A + y^{-B}), \forall z \in \mathcal{H},
\]
for positive constants $K, A$ and $B$.

A cocycle of weight $r$ and multiplier system $v$ with values in $\mathcal{P}$ is a function $\phi : \Gamma \to \mathcal{P}$ that satisfies
\[
\phi(\gamma \delta) = \phi(\gamma)|_{r,v} \delta(z) + \phi(\delta)(z), \forall \gamma, \delta \in \Gamma.
\]
We denote the space of cocycles by $Z^1_{r,v}(\Gamma, \mathcal{P})$. There is a natural map $d$ from $\mathcal{P}$ to $Z^1_{r,v}(\Gamma, \mathcal{P})$ that associates to a function $g \in \mathcal{P}$ the cocycle $dg : \gamma \mapsto g_{|_{r,v}} \gamma - g$.

A cocycle of the form $dg$ for $g \in \mathcal{P}$ is called a coboundary and the space of coboundaries is denoted by $B^1_{r,v}(\Gamma, \mathcal{P})$. The (first) Eichler cohomology group $H^1_{r,v}(\Gamma, \mathcal{P})$ is the quotient space $Z^1_{r,v}(\Gamma, \mathcal{P})/B^1_{r,v}(\Gamma, \mathcal{P})$.

A cocycle $\phi$ is called parabolic if for all cusps $q_i$ there exists a function $g_{q_i} \in \mathcal{P}$ such that
\[
\phi(\gamma) = g_{q_i}|_{r,v} \gamma - g, \forall \gamma \in \Gamma.
\]
We denote the space of parabolic cocycles by $\tilde{Z}^1_{r,v}(\Gamma, \mathcal{P})$. Since coboundaries are clearly parabolic we can form the parabolic cohomology group $\tilde{H}^1_{r,v}(\Gamma, \mathcal{P}) = \tilde{Z}^1_{r,v}(\Gamma, \mathcal{P})/B^1_{r,v}(\Gamma, \mathcal{P})$. It turns out that all cocycles are parabolic. This follows from a result that Knopp attributes to B.A. Taylor in [10].

Proposition 1.3. Let $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$ and $g \in \mathcal{P}$. Then there exists an $f \in \mathcal{P}$ with
\[
\tau f(z + 1) - f(z) = g(z), \forall z \in \mathcal{H}. \tag{4}
\]

Proof. This is Proposition 9 in [10] and a full proof is given there. We will only present the main idea here. A formal solution of (4) is given by the one-sided average
\[
f(z) = -\sum_{n=0}^{\infty} \tau^n g(z + n).
\]
However this sum does not always converge. Knopp uses the fact that $\mathcal{P}$ is closed under integration and differentiation to replace $g$ with a function $g' = g_1 + g_2$ such that the one-sided averages $f_1(z) = -\sum_{n=0}^{\infty} \tau^n g_1(z + n)$ and $f_2(z) = -\sum_{n=0}^{\infty} \tau^n g_2(z + n)$ converge and are in $\mathcal{P}$. \qed
Corollary 1.4. Let \( \epsilon \in \mathbb{C} \) with \(|\epsilon|=1\), \( s \in \mathbb{C} \setminus \{0\} \) and \( g \in \mathcal{P} \). Then there exists an \( f \in \mathcal{P} \) with
\[
\epsilon f(z+s) - f(z) = g(z), \quad \forall z \in \mathcal{H}.
\] (5)

Proof. Set \( \hat{g}(z) = g(sz) \). By Proposition 1.3 there exists \( \hat{f} \in \mathcal{P} \) that satisfies
\[
\epsilon \hat{f}(z+1) - \hat{f}(z) = \hat{g}(z), \quad \forall z \in \mathcal{H}.
\] Then \( f = \hat{f}(z/s) \) solves (5). \( \square \)

Theorem 1.5 ([10], p.627). Every cocycle in \( Z^1_{r,v}(\Gamma, \mathcal{P}) \) is parabolic, i.e.
\[
Z^1_{r,v}(\Gamma, \mathcal{P}) = \tilde{Z}^1_{r,v}(\Gamma, \mathcal{P}).
\]

Proof. Let \( \phi \in Z^1_{r,v}(\Gamma, \mathcal{P}) \). We will show that for every parabolic \( \gamma \in \Gamma \) there exists \( f \in \mathcal{P} \) such that
\[
\phi(\gamma) = f|_{r,v}\gamma - f. \quad (6)
\]
First suppose \( \gamma = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \) is a translation by \( s \neq 0 \). Then by Corollary 1.4 a function \( f \in \mathcal{P} \) with the desired property exists.

For the general case let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) fix a cusp \( q \). Then there exists an \( s \) such that
\[
A\gamma A^{-1} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = U, \text{ where } A = \begin{pmatrix} 0 & -1 \\ 1 & -q \end{pmatrix}.
\]
Replacing \( z \) by \( A^{-1}z \) in equation (6) we see that it is sufficient to show the existence of \( f \in \mathcal{P} \) with
\[
\overline{v(\gamma)} j(A^{-1}UA, A^{-1}z)^{-r} f(\gamma A^{-1}z) - f(A^{-1}z) = \phi(\gamma)(A^{-1}z). \quad (7)
\]
Setting \( \hat{f}(z) = f(A^{-1}z) \) this is equivalent to
\[
\overline{v(\gamma)} j(A^{-1}UA, A^{-1}z)^{-r} \hat{f}(z+s) - \hat{f}(z) = \phi(\gamma)(A^{-1}z). \quad (8)
\]
Equation (1) implies the two relations
\[
1 = j(AA^{-1}U, z)^{-r} = \sigma_r(A, A^{-1}U) j(A, A^{-1}Uz)^{-r} j(A^{-1}U, z)^{-r}, \quad (9)
\]
\[
j(A^{-1}UA, A^{-1}z)^{-r} = \sigma_r(A^{-1}U, A) j(A^{-1}U, z)^{-r} j(A, A^{-1}z)^{-r}. \quad (10)
\]
After multiplying equation (8) by \( j(A, A^{-1}z)^r \) and using the two relations (9) and (10) we get

\[
\epsilon F(z + s) - F(z) = j(A, A^{-1}z)^r \phi(\gamma)(A^{-1}z), \quad (11)
\]

where we set \( F(z) = j(A, A^{-1}z)^r \hat{f}(z) \) and \( \epsilon = v(\gamma)\sigma_r(A^{-1}U, A)\sigma_r(A, A^{-1}U) \). Note that \(|\epsilon| = 1\) and \( j(A, A^{-1}z)^r \phi(\gamma)(A^{-1}z) \in P \). The existence of such an \( F \in P \) again follows from Corollary 1.4. \( \square \)

2 Petersson inner product

In this section we define the pairing that is essential for our proof of Theorem 1.2.

**Definition 2.1.** Let \( r \in \mathbb{R} \) with \( 2 - r > 0 \) and \( g \) be a cusp form for the group \( \Gamma \) of weight \( 2 - r \) and unitary multiplier system \( \overline{\tau} \). Let

\[
G(z) = \int_{\infty}^{z} g(\tau)(\tau - z)^{-r} d\tau.
\]

Since \( g \) decreases exponentially towards the cusps the integral converges and \( G \) is a smooth function from \( \mathcal{H} \to \mathbb{C} \). We can define a cocycle by

\[
\phi_g^\infty : \gamma \mapsto \phi_{g, \gamma}^\infty(z) = G|_{r,v}\gamma(z) - G(z).
\]

**Proposition 2.1.** The cocycle \( \phi_g^\infty \) is in \( Z^1_{r,v}(\Gamma, P) \) and

\[
\phi_{g, \gamma}^\infty(z) = \int_{\gamma^{-1}\infty}^{\infty} \overline{g(\tau)(\tau - z)^{-r}} d\overline{\tau},
\]

for all \( \gamma \in \Gamma \).

**Proof.** Let \( \gamma \in \Gamma \):

\[
G(\gamma z) = \int_{\infty}^{\gamma z} g(\tau)(\tau - \gamma z)^{-r} d\tau
\]

\[
= \int_{\gamma^{-1}\infty}^{\gamma^{-1}z} \overline{g(\gamma \tau)(\gamma \tau - \gamma z)^{-r}} d\gamma \tau
\]

\[
= j(\gamma, z)^r \int_{\gamma^{-1}\infty}^{\gamma^{-1}z} \overline{g(\gamma \tau)j(\gamma, \tau)^{-2+r}(\tau - z)^{-r}} d\gamma \tau.
\]

8
In the last equality we used
\[(\gamma \tau - \gamma z)^{-r} = \left(\frac{\tau - z}{j(\gamma, \tau)j(\gamma, z)}\right)^{-r} = \frac{(\tau - z)^{-r}}{j(\gamma, \tau)^{-r}j(\gamma, z)^{-r}}.\]

To prove this let
\[
\alpha = \text{arg}(\gamma \tau - \gamma z) \quad \text{and} \quad \beta = \text{arg}(\tau - z) - \text{arg}(j(\gamma, \tau)) - \text{arg}(j(\gamma, z)).
\]

We know that \(\alpha \equiv \beta \mod 2\pi\) and want to show \(\alpha = \beta\). Both \((\gamma \tau - \gamma z)\) and \((\tau - z)\) are in \(\mathcal{H}\), so their arguments are in \((0, -\pi)\). Furthermore exactly one of \(j(\gamma, \tau)\) and \(j(\gamma, z)\) will be in \(\mathcal{H}\) and one in \(\mathcal{H}^c\), so \(\pi > \beta > -2\pi\) and \(0 > \alpha > -\pi\). Together with \(\beta \equiv \alpha \mod 2\pi\) this implies \(\alpha = \beta\). Now we use the modularity of \(g\) to obtain
\[G(\gamma z) = j(\gamma, z)^r v(\gamma) \int_{\gamma^{-1} - \infty}^{\gamma^{-1} + \infty} g(\tau)(\tau - z)^{-r} d\tau, \tag{12}\]

or \(G|_{r, v, \gamma}(z) = \int_{\gamma^{-1} - \infty}^{\gamma^{-1} + \infty} \overline{g(\tau)}(\tau - z)^{-r} d\tau\). An application of Cauchy’s theorem now gives us
\[\phi_{g, \gamma}^\infty(z) = G|_{r, v, \gamma}(z) - G(z) = \left( \int_{\gamma^{-1} - \infty}^{\gamma^{-1} + \infty} - \int_{\gamma^{-1} + \infty}^{\gamma^{-1} - \infty} \right) g(\tau)(\tau - z)^{-r} d\tau = \int_{\gamma^{-1} - \infty}^{\gamma^{-1} + \infty} \overline{g(\tau)}(\tau - z)^{-r} d\tau.
\]

To see that \(\phi_{g, \gamma}^\infty\) is in \(\mathcal{P}\) first note that \((\tau - z)^{-r}\) is holomorphic in \(\mathcal{H}\) as a function of \(z\) (actually even in the slit plane \(\mathbb{C} \setminus \{\mathbb{R}_{>0} + \mathbb{R}\}\)) and the integrals in the definition of \(G\) and \(\phi_{g, \gamma}^\infty\) converge absolutely because \(g\) is a cusp form. Therefore \(\phi_{g, \gamma}^\infty(z)\) is holomorphic in \(\mathcal{H}\). To prove that \(\phi_{g, \gamma}^\infty\) is in \(\mathcal{P}\) one can use simple bounds for \(|\tau - z|^{-r}\). We sketch the procedure for the case \(-r \geq 0\) and \(\text{Im}(z) > 1\). In this case
\[|\tau - z|^{-r} \leq |\tau - z|^{-r} \leq \sum_{j=0}^{[-r]} |\tau|^{-r-j} |z|^j.
\]

One can use this to bound \(\phi_{g, \gamma}^\infty(z)\) by a polynomial in \(|z|\). The other cases are dealt with similarly.
Let $f$ be another modular form of the weight $2 - r$ and multiplier system \( \tau \). Then, since $f$ is holomorphic

\[
\frac{\partial Gf}{\partial z}(z) = g(z)(\overline{z} - z)^{-r} f(z) = (-2i)^{-r} f(z)g(z)y^{-r}.
\]

This is just a scalar times the integrand occurring in the Petersson inner product of $g$ and $f$ defined as

\[
(f, g) = \int_{\Gamma \backslash \mathcal{H}} f(z)g(z)y^{-r}dxdy.
\]

Choose a fundamental domain of $\Gamma$, $\mathcal{F}$. Then by Stoke’s theorem we have

\[
(f, g) = \frac{i}{2} \int_{\mathcal{F}} f(z)g(z)y^{-r}d\overline{z} \wedge dz = C_{2-r} \int_{\partial \mathcal{F}} f(z)G(z)dz,
\]

for $C_{2-r} = \frac{i}{2}(-2i)^r$. Now we choose a fundamental domain according to the following Proposition found in [2] on page 9.

**Proposition 2.2.** The fundamental domain $\mathcal{F}$ can be chosen such that $\partial \mathcal{F}$ consists of an even number of geodesic segments $[A_i, A_{i+1}]$ for $i = 1, \ldots, 2n$ (the indices are taken modulo $2n$) and $\alpha_i \in \Gamma$ for $i = 1, \ldots, 2n$ such that there exists an involution of $\{1, \ldots, 2n\}$, denoted by $\tau$, such that

1. $\tau$ does not have any fixed points,
2. $\alpha_i A_i = A_{\tau(i)+1}$, $\alpha_i A_{i+1} = A_{\tau(i)}$,
3. $\alpha_{\tau(i)} = \alpha_i^{-1}$,
4. $\alpha_i$ maps $[A_i, A_{i+1}]$ to $[A_{\tau(i)+1}, A_{\tau(i)}]$.

**Example 2.1.** For $\Gamma = \text{SL}_2(\mathbb{Z})$ we choose the classic fundamental domain with $A_1 = \infty$, $A_2 = e^{2\pi i/3}$, $A_3 = i$, $A_4 = A_2 + 1$. Then $\alpha_1 = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ maps $[A_1, A_2]$ to $[A_1, A_4]$ and $\alpha_2 = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ maps $[A_2, A_3]$ to $[A_4, A_3]$. So $\tau$ is the permutation that swaps 1 with 4 and 2 with 3.

---

1 $[A_i, A_{i+1}]$ denotes the geodesic in $\Gamma \backslash \mathcal{H}$ that connects $A_i$ and $A_{i+1}$ and includes $A_i$ but not $A_{i+1}$.
Remark. For general Fuchsian groups $\Gamma$ of the first kind an example of such a fundamental domain is the Ford fundamental domain (see [5])

$$\mathcal{F} = \{ z \in \mathcal{H} \mid |z| \leq \lambda/2 \text{ and } |j(\gamma, z)| > 1 \forall \gamma \in \Gamma \setminus \Gamma_\infty \},$$

where $\lambda$, the width of the cusp $\infty$, was defined in the last section. For the rest of this article we will fix this fundamental domain for $\Gamma$.

We can restate Proposition 2.2 as

$$\partial \mathcal{F} = \bigsqcup_{m=1}^{n} \left( [A_{im}, A_{im+1}[ \sqcup \alpha_{im} ] A_{im}, A_{im+1}] \right).$$

Thus the Petersson inner product of $f$ and $g$ becomes

$$C_{2-r} \sum_{m=1}^{n} \left( \int_{A_{im}}^{A_{im+1}} f(z) G(z) \, dz - \int_{\alpha_{im} A_{im}}^{\alpha_{im} A_{im+1}} f(z) G(z) \, dz \right).$$

Using the modularity of $f$ the second integral in the sum becomes

$$\int_{\alpha_{im} A_{im}}^{\alpha_{im} A_{im+1}} f(z) G(z) \, dz = \int_{A_{im}}^{A_{im+1}} f(\alpha_{im} z) G(\alpha_{im} z) d\alpha_{im} z$$

$$= \int_{A_{im}}^{A_{im+1}} f(z) G|_{r, v} \alpha_{im}(z) \, dz.$$

Finally we arrive at

$$(f, g) = C_{2-r} \sum_{m=1}^{n} \int_{A_{im}}^{A_{im+1}} f(z) \left( G(z) - G|_{r, v} \alpha_{im}(z) \right) \, dz$$

$$= -C_{2-r} \sum_{m=1}^{n} \int_{A_{im}}^{A_{im+1}} f(z) \phi_{g, \alpha_{im}}(z) \, dz.$$

Motivated by the previous calculations we define a pairing between cusp forms and cocycles:

**Definition 2.2.** Let $2 - r > 0$, $f \in S_{2-r}(\Gamma, \overline{v})$ and $\phi \in Z_{r, v}^{1}(\Gamma, \mathcal{P})$. Define

$$(f, \phi) = -C_{2-r} \sum_{m=1}^{n} \int_{A_{im}}^{A_{im+1}} f(z) \phi_{\alpha_{im}}(z) \, dz.$$  \hfill (14)

The integrals in the sum converge because $\phi_{\alpha_{im}}$ is in $\mathcal{P}$ and therefore can increase only polynomially towards the cusps, while $f$ decreases exponentially.
This pairing factors through $H^1_{r,v}(\Gamma, \mathcal{P})$ as the following argument shows. Let $\phi$ be a coboundary. This means that there exists a function $h \in \mathcal{P}$ with $\phi(\gamma) = h|_{r,v} \gamma - h$. Let $f \in S_{2-r}(\Gamma, \overline{v})$, then
\[
\int_{A_{im}}^{A_{im+1}} f(z)h|_{r,v} \alpha_{im}(z)dz = \int_{A_{im}}^{A_{im+1}} f(z)j(\alpha_{im}, z)2^{-r}v(\alpha_{im})h(\alpha_{im}z)d(\alpha_{im}z)
= \int_{A_{im}}^{A_{im+1}} f(\alpha_{im}z)h(\alpha_{im}z)d(\alpha_{im}z)
= \int_{\alpha_{im}A_i} f(z)h(z)dz.
\]
So
\[
(f, \phi) = -C_{2-r} \sum_{m=1}^{n} \int_{A_{im}}^{A_{im+1}} f(z)\phi_{\alpha_{im}}(z)dz = C_{2-r} \int_{\partial F} f(z)h(z)dz. \tag{15}
\]
The integral over the boundary is 0 because, since $f(z)h(z)$ decreases exponentially towards the cusps we can approach $\int_{\partial F} f(z)h(z)dz$ by integrals over closed paths contained in $\mathcal{H}$, which are all equal to zero, since $f(z)h(z)$ is holomorphic.

**Corollary 2.3.** The map $f \mapsto \phi_f^\infty$ from $S_{2-r}(\Gamma, \overline{v})$ to $H^1_{r,v}(\Gamma, \mathcal{P})$ is injective.

**Proof.** If $\phi_f^\infty = 0$, then $(f, \phi_f^\infty) = (f, f) = 0$ and hence $f = 0$. \qed

### 3 Duality theorem

In this section we prove that the pairing we defined in Definition 2.2, viewed as a pairing between $S_{2-r}(\Gamma, \overline{v})$ and $H^1_{r,v}(\Gamma, \mathcal{P})$ is perfect for $2-r > 1$. This is equivalent to Theorem 1.2 (when $2-r > 1$).

We already know that for every non-zero $f$ there exists a cocycle $\phi$ (e.g. $\phi = \phi_f^\infty$) such that $(f, \phi) \neq 0$, since $(f, \phi_f^\infty) = (f, f)$. To show that the pairing is perfect we therefore need to prove the following theorem.

**Theorem 3.1.** Let $2-r > 1$ and $[\phi] \in H^1_{r,v}(\Gamma, \mathcal{P})$ be represented by $\phi \in \tilde{Z}^1_{r,v}(\Gamma, \mathcal{P})$. Suppose that $(f, \phi) = 0$ for all $f \in S_{2-r}(\Gamma, \overline{v})$. Then $[\phi] = 0$. i.e. $\phi$ is a coboundary in $\tilde{Z}^1_{r,v}(\Gamma, \mathcal{P})$. 


Most constructions that follow will be valid for any real \( r \) and so, if not explicitly stated otherwise, we work in this generality. In particular we will also show Theorem \ref{thm:beta} for \( 2 - r < 0 \).

A basis of neighbourhoods of \( \infty \) is given by the sets \( H_Y(y) = \{ z \in \mathbb{H} | \text{Im}(z) > Y \} \). Let \( q \) be a cusp with \( g_q \infty = q \) for \( g_q \in \text{SL}_2(\mathbb{R}) \) such that \( g_q^{-1} \Gamma g_q \) is generated by \( T \). Then the open sets \( H_Y(q) = g_q H_Y(\infty) \) for \( Y > 0 \) form a basis of neighbourhoods of \( q \).

We define a variation of the space \( \mathcal{P} \) that will be useful in our proof. Let \( \tilde{\mathcal{Q}} \) be the space of \( C^\infty \)-functions \( f \) on \( \mathcal{H} \) such that for every cusp \( q \) of \( \Gamma \) there exists a neighbourhood \( U \subseteq \mathcal{H} \) and \( K, A, B > 0 \) such that \( f \) is holomorphic in \( U \) and
\[
|f(z)| < K(|z|^A + y^{-B}), \quad \forall z \in U.
\]

**Theorem 3.2.** Every element of \( Z^1_{r,v}(\Gamma, \mathcal{P}) \) is a coboundary in \( Z^1_{r,v}(\Gamma, \tilde{\mathcal{Q}}) \).

**Proof.** Let \( \phi \in Z^1_{r,v}(\Gamma, \mathcal{P}) \). We need to show that there exists a function \( G \in \tilde{\mathcal{Q}} \) with \( \phi(\gamma) = G|_{r,v} \gamma - G, \quad \forall \gamma \in \Gamma \). Choose \( Y \) large enough so that all the \( H_Y(q) = U_q \) are disjoint and contain no elliptic fixed points. Define \( U = \bigcup_q \text{cusp of } \Gamma \ H_Y(q) \) and \( V = \bigcup_q \text{cusp of } \Gamma \ H_{2Y}(q) \). Then \( U \) and \( V \) are \( \Gamma \)-invariant. Let \( \eta \) be a \( \Gamma \)-invariant \( C^\infty \)-function on \( \mathcal{H} \) that satisfies \( \eta(z) = 1 \) on \( V \) and \( \eta(z) = 0 \) outside \( U \) (by the smooth Urysohn lemma such a function exists). We will first try to construct a function that has \( \eta \phi \) as a coboundary. By Proposition \ref{prop:paraboliccoboundary} \( \phi \) is a parabolic cocycle so for every cusp \( q \) there exists a function \( g_q \in \mathcal{P} \) such that \( \phi(\sigma_q) = g_q|_{\sigma_q - 1} \), where \( \sigma_q \) is the generator of \( \Gamma_q \). We define \( G \) on \( U \) as follows: if \( z \in H_Y(q_i) \) for some \( i \) then \( G(z) = g_q(z) \). If \( z = \delta w \) for \( \delta \in \Gamma \) and \( w \in H_Y(q_i) \) we define
\[
G(z) = v(\delta) j(\delta, w)^r(\phi(\delta)(w) + g_q(w)).
\]

Note that this is equivalent to defining \( G|_{r,v} \delta(w) = \phi(\delta)(w) + G(w) \), so once we show that the definition of \( G(z) \) does not depend on the choice of \( \delta \), the coboundary of \( \eta G \) will be \( \eta \phi \). Suppose \( z = \delta w = \delta' w' \), for \( \delta, \delta' \in \Gamma \) and \( w, w' \in H_Y(q_i) \). We need to check that
\[
v(\delta) j(\delta, w)^r(\phi(\delta)(w) + g_q(w)) = v(\delta') j(\delta', w')^r(\phi(\delta')(w') + g_q(w')).
\]

Multiplying both sides by \( v(\delta)^{-1} j(\delta, w)^{-r} \) and using the compatibility of the multiplier system \( v \) this is equivalent to
\[
\phi(\delta)(w) + g_q(w) = [\phi(\delta') + g_q]^r(\delta'^{-1} \delta)(w).
\]
This follows from the cocycle condition on \( \phi \) and the choice of \( g_q \). Indeed, since \( w' \in \delta^{-1}H_Y(q_i) \cap H_Y(q_i) \neq \emptyset \) and since we assumed that all the \( H_Y(q) \) are disjoint, \( \delta^{-1} \delta \) must fix \( q_i \). Hence \( \delta^{-1} \delta = \pm \sigma_n^q_i \) for some \( n \geq 1 \). This implies  
\[
g_{q_i} |_{r,v}(\delta^{-1} \delta)(w) = \phi(\delta^{-1} \delta)(w) + g_{q_i}(w),
\]
and so  
\[
[\phi(\delta') + g_{q_i}] |_{r,v}(\delta^{-1} \delta)(w) = \phi(\delta)(w) - \phi(\delta^{-1} \delta)(w) + g_{q_i} |_{r,v}(\delta^{-1} \delta)(w)
\]
\[
= \phi(\delta)(w) + g_{q_i}(w).
\]
So \( \eta G \) is a well-defined function in \( \tilde{Q} \). We have thus shown that \( \eta \phi \) is a coboundary in \( Z^1_{r,v}(\Gamma, \tilde{Q}) \).

It remains to show that \( (1 - \eta) \phi \) is a coboundary. For this purpose let \( U_i, i = 1, \ldots, n \) be a finite cover of \( H \) such that every \( U_i \) is the \( \Gamma \)-orbit of an open set \( V_i \) that contains at most one elliptic point of \( \Gamma \). We denote the (finite) stabiliser of this fixed point by \( \Gamma_i \) and require furthermore that for all \( \gamma \notin \Gamma_i \) we have \( V_i \cap \gamma V_i = \emptyset \). Let \( (\epsilon_i)_{i=1}^n \) be a \( \Gamma \)-invariant \( C^\infty \)-partition of unity corresponding to this cover. We define  
\[
H_i(z) = \begin{cases} 
-(1 - \eta(z)) \epsilon_i(z) \sum_{g \in \Gamma_i} \phi(g g_i(z)^{-1})(z) & \text{if } z \in g_i(z) V_i \\
0 & \text{otherwise},
\end{cases}
\]
where \( g_i(z) \) is any element of \( \Gamma \) with \( z \in g_i(z) V_i \). This does not depend on the choice of \( g_i(z) \): If \( z \in \gamma V_i \) with \( \gamma \in \Gamma \) then we must have \( \gamma^{-1} g_i(z) \in \Gamma_i \). Thus the set \( \Gamma_i g_i(z) \) is equal to \( \Gamma_i \gamma \). So a different choice of \( g_i(z) \) just permutes the summands in the definition of \( H_i(z) \).

Clearly \( H_i \) is a function in \( \tilde{Q} \) and defining \( H = \sum_i H_i \) we have  
\[
H |_{r,v, \gamma}(z) = -(1 - \eta(z)) \sum_{i \text{ with } z \in U_i} \sum_{g \in \Gamma_i} \phi(g g_i(\gamma z)^{-1})(z) |_{r,v, \gamma}(z).
\]
Now we choose \( g_i(\gamma z) = \gamma g_i(z) \) to get  
\[
= -(1 - \eta(z)) \sum_{i \text{ with } z \in U_i} \sum_{g \in \Gamma_i} [\phi(g g_i(z)^{-1})(z) - \phi(\gamma)(z)]
\]
\[
= (1 - \eta(z))(\phi(\gamma)(z) + H(z)).
\]
In the definition of $\tilde{Q}$ the constants $K, a, b$ may vary from cusp to cusp. Let $Q$ be the space of functions $F$ in $\tilde{Q}$ such that there exist positive constants $K, a, b$ with

$$|F(z)| < K(|z|^A + y^{-B}), \quad \forall z \in \mathcal{H}.$$  

Note that the functions of $P$ are the holomorphic functions in $Q$.

**Proposition 3.3.** Let $F$ be in $\tilde{Q}$. If $\gamma \mapsto F|_{r,v} - F = \psi(\gamma)$ is in $Z_{r,v}^1(\Gamma, P)$ then $F$ is in $Q$.

**Proof.** This proof is similar to the proof of the main theorem of [11]. Let $M$ be the set of matrices $\gamma$ in $\Gamma$ with $\lambda/2 \leq \Re(\gamma i) < \lambda/2$. $M$ is a complete set of coset representatives of $\Gamma_\infty \setminus \Gamma$. We need a technical lemma from [10]:

**Lemma 3.4.** (Lemma 8 in [10]) There exist positive constants $K_1, A_1, B_1$ such that for all $\tau \in \mathcal{F} \cap \mathcal{H}$ and all $\gamma \in M$

$$|\psi(\gamma)(\tau)| < K_1(\Im(\gamma\tau)^{A_1} + \Im(\gamma\tau)^{-B_1}).$$

Since only finitely many cusps are in $\tilde{F}$ and since the real part of $z \in \mathcal{F}$ is bounded we can also find positive $K_2, A_2, B_2$ with

$$|F(\tau)| < K_2(\Im(\tau)^{A_2} + \Im(\tau)^{-B_2}), \quad \forall \tau \in \mathcal{F} \cap \mathcal{H}. \quad (16)$$

As in the proof of Theorem 3.2 we use the fact that $\psi$ is parabolic and hence there exists a function $g_\infty \in P$ such that $\psi(\sigma_\infty) = g_\infty|_{r,v} \sigma_\infty - g_\infty$. The equation $F|_{r,v} \sigma_\infty - F = \psi(\sigma_\infty)$ implies

$$(F - g_\infty)|_{r,v} \sigma_\infty - (F - g_\infty) = 0.$$  

$F$ is in $P$ if and only if $F - g_\infty$ is in $P$ so we can assume without loss of generality that $F$ is invariant under $\sigma_\infty$. Let $z \in \mathcal{H}$. There exists $\tau \in \mathcal{F}$ and $\gamma \in \Gamma$ such that $z = \gamma \tau$. Since $M$ is a complete set of representatives of $\Gamma_\infty \setminus \Gamma$ there is an integer $m$ and $\delta \in M$ such that $z = \sigma_\infty^m \delta \tau$. If $\delta = \Id$ then we can deduce

$$|F(z)| < K_2(\Im(\tau)^{A_2} + \Im(z)^{-B_2}),$$

from equation (16) and the fact that $F$ is $\Gamma_\infty$-invariant. Suppose $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not the identity. Then $c \neq 0$, because the only member of $M$ that fixes $\infty$
is Id. We have

\[ |F(z)| = |F(\sigma^m_{\infty}\delta\tau)| = |F(\delta\tau)| \]

(17)

\[ \leq |j(\delta, \tau)| |F(\tau)| + |\psi(\delta)(\tau)| \]

(18)

\[ < |j(\delta, \tau)| \left[ K_2(\text{Im}(\tau)^{A_2} + \text{Im}(\tau)^{-B_2}) + K_1(\text{Im}(\delta\tau)^{A_1} + \text{Im}(\delta\tau)^{-B_1}) \right]. \]

(19)

Since \( \delta \in M \) we have \( |j(\delta, \tau)| \geq 1 \) so \( y = \text{Im}(z) = \text{Im}(\tau) \). On the other hand, using \( \tau = \delta^{-1}\sigma^m_{\infty}z \) we have \( \text{Im}(\tau) = \frac{y}{|j(\delta^{-1}\sigma^m_{\infty}, z)|} \) and

\[ |j(\delta^{-1}\sigma^m_{\infty}, z)|^2 = |-cz + cm\lambda + a|^2 = c^2y^2 + (cm\lambda + a - cx)^2 \geq cy^2 > c_0y^2, \]

where \( c_0 > 0 \) depends only on \( \Gamma \). Such a \( c_0 \) exists because \( \Gamma \) is discrete. Therefore \( y \leq \text{Im}(\tau) < c_0y^{-1} \), \( \text{Im}(\tau)^{A_2} < c_0y^{-A_2} \) and \( \text{Im}(\tau)^{-B_2} \leq y^{B_2} \). Also \( |j(\delta, \tau)| = (\frac{y}{\text{Im}(\tau)})^{-r/2} \) is either \( \leq 1 \) (if \( r \leq 0 \)), or \( \leq c_0y^{-2} \) (if \( r \geq 0 \)). These inequalities inserted into (19) lead to the desired inequality of the form

\[ |F(z)| < K(|z|^A + y^{-B}), \]

for positive constants \( K, A, B \) and all \( z \in \mathcal{H} \).

\[ \square \]

**Corollary 3.5.** Every cocycle in \( Z^1_{r,v}(\Gamma, \mathcal{P}) \) is a coboundary in \( Z^1_{r,v}(\Gamma, \mathcal{Q}) \).

Let \( \phi \in Z^1_{r,v}(\Gamma, \mathcal{P}) \). By Corollary 3.5 there exists a function \( g \in \mathcal{Q} \) such that \( g|_{r,v}\gamma - g = \phi(\gamma) \) for all \( \gamma \in \Gamma \). By the same calculation as in equation (15) we have

\[ (f, \phi) = -C_{2-r} \sum_{m=1}^{n} \int_{A_{m+1}}^{A_m} f(z)(g|_{r,v}\alpha_m(z) - g(z))dz \]

\[ = C_{2-r} \int_{\partial \mathcal{F}} f(z)g(z)dz \]

\[ = C_{2-r} \int_{\mathcal{F}} \frac{\partial g}{\partial \overline{z}}dz \wedge f(z)dz. \]

Here we note again that the integrals above exist because \( g \) can only increase polynomially towards the cusps of \( \Gamma \), while \( f \) decreases exponentially.
3.1 Spectral theory of automorphic forms

To finish the proof of Theorem 3.1 we will apply spectral theory. We only give a very brief introduction here, for more details and proofs see the exposition [14] by Roelcke. In these articles Roelcke uses a variation of the slash operator which we denote by $|_{r,v}^R$

$$f|_{r,v}^R(\gamma(z)) = \left( \frac{j(\gamma, \bar{z})}{j(\gamma, z)} \right)^{r/2} \overline{v(\gamma)} f(\gamma z).$$

The connection to our slash operator is given by the following lemma:

**Lemma 3.6.** Let $f : \mathcal{H} \to \mathbb{C}$, $F(z) = y^\frac{r}{2} f(z)$ and $\gamma \in \Gamma$. Then

$$y^\frac{r}{2} (f|_{r,v}^R(\gamma(z))) = F|_{r,v}^R(\gamma(z)).$$

So a function $f$ is invariant under $|_{r,v}$ if and only if $F(z) = y^\frac{r}{2} f(z)$ is invariant under $|_{r,v}^R$.

**Definition 3.1.** Let $H_{r,v}$ be the Hilbert space of functions $f$ that are invariant under $|_{r,v}^R$ and have finite norm with respect to the scalar product

$$(f_1, f_2)^R = \int_{\mathcal{F}} f_1(z) \overline{f_2(z)} \frac{dx dy}{y^2}.$$

The weight $r$ Laplacian and the Maass raising and lowering operators are defined as

$$\Delta_r = -(z - \bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{r}{2} (z - \bar{z}) \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right),$$

$$K_r = (z - \bar{z}) \frac{\partial}{\partial z} + \frac{r}{2},$$

$$\Lambda_r = (z - \bar{z}) \frac{\partial}{\partial \bar{z}} + \frac{r}{2}.$$

Before we sum up the main properties of these operators in Proposition 3.7 we recall some definitions from operator theory.

**Definition 3.2.** Let $H$ and $H'$ be Hilbert spaces and let $T$ be a linear operator from a subspace $D$ of $H$ to $H'$. Then $T$ is called closed if:

Let $x_n$ be a sequence in $D$ that converges to $x \in H$ and suppose that $Tx_n$ converges to $y \in H'$. Then $x \in D$ and $Tx = y$. 

17
**Definition 3.3.** If $D$ is dense in $H$ then for any operator $T$ from $D$ to $H$ we can define its adjoint $T^*$ on the domain

$$\{y \in H : x \mapsto \langle Tx, y \rangle \text{ is continuous on } D\}.$$ 

Any $y$ in this set defines a linear functional on $D$ by $\phi_y : x \mapsto \langle Tx, y \rangle$. This functional can be extended to $H$ and by the Riesz representation theorem there exists $z \in H$ such that $\phi_y(x) = \langle x, z \rangle$ for all $x \in H$. We define $T^*y = z$.

An operator is called self-adjoint if it is equal to its adjoint. An operator is called essentially self-adjoint if $T \subseteq T^* = (T^*)^*$, where $T \subseteq T^*$ means that $T^*$ extends $T$.

Let

$$D_r^2 = \{f \in H_{r,v} | f \text{ twice differentiable and } -\Delta_r f \in H_{r,v}\}.$$ 

**Proposition 3.7.**

(i) $\Delta_r : D_r^2 \rightarrow H_{r,v}$ is essentially self-adjoint. It has a self-adjoint extension to a dense subset of $H_{r,v}$ that we denote by $D_r$.

(ii) The eigenfunctions of $\Delta_r$ are smooth (in fact they are even real analytic).

(iii) $K_r : D_r^2 \rightarrow H_{r+2,v}$ and $\Lambda_r : D_r^2 \rightarrow H_{r-2,v}$ can be extended to closed operators defined on $D_r$. For $f \in D_r$ and $g \in D_{2+r}$ we have

$$(K_rf,g)_R = (g, \Lambda_{2+r}f)_R.$$ 

(iv)

$$-\Delta_r = \Lambda_{r+2}K_r - \frac{r}{2}(1 + \frac{r}{2}) = K_{r-2}\Lambda_r + \frac{r}{2}(1 - \frac{r}{2}).$$ 

**Proof.** For proofs of the statements (i), (iii) and (iv) see [14]. (i) is Satz 3.2, (iii) follows from the discussion after the proof of Lemma 6.2 on page 332 and (iv) is equation (3.4) on page 305. Statement (ii) follows from the fact that $\Delta_r$ is an elliptic operator and elliptic regularity applies. For an introduction to the theory of elliptic operators see [3]. The result needed here is Corollary 8.11 in [3].

**Definition 3.4.** A cusp form of $H_{r,v}$ with eigenvalue $\lambda$ is an eigenfunction of $-\Delta_r$ with eigenvalue $\lambda$ that decreases exponentially towards the cusps of $\Gamma$. 

18
Remark. The space $S_r(\Gamma, v)$ of classical cusp forms embeds into $H_{r,v}$ by $f \mapsto y^2 f$ and the image consists exactly of the cusp forms with eigenvalue $\frac{r^2}{2}(1 - \frac{r^2}{2})$ ([14, Satz 5.2]).

The main result in [14] is a spectral decomposition of $\Delta_r$. For this purpose we introduce the Eisenstein series. Let $q$ be a cusp of $\Gamma$, $\sigma_q$ the generator of $\Gamma_q$ and $A_q \in \text{SL}_2(\mathbb{R})$ chosen such that $q = A_q^{-1}\infty$. The cusp $q$ is called singular if $v(\sigma_q) = 1$ and regular otherwise. Let $q_1, \ldots, q_m$ be a set of representatives of the singular cusps of $\Gamma$. For each of these cusps we define the Eisenstein series

$$E_{r,v}(q,z,s) = \frac{1}{2} \sum_{M \in \Gamma_q \setminus \Gamma} \sigma_r(A,M)^{-1}v(M) \left( \frac{j(AM,z)}{j(A,M,z)} \right)^{r/2} (\text{Im } AMz)^s.$$

The definition of $E_{r,v}$ depends on the choice of $A$ but a different choice of $A$ will only multiply the Eisenstein series by a constant of absolute value 1. The series above converges absolutely and uniformly for $(z,s)$ in sets of the form $K \times \{s | \text{Re } s \geq 1 + \epsilon\}$, where $K$ is a compact subset of $\mathbb{C}$. For a fixed $s$ with real part $\geq 1 + \epsilon$ one can use the absolute and uniform convergence of the series to see that $E_{r,v}(\cdot, s)$ is invariant under $|R_{r,v}$ and that

$$-\Delta_r E_{r,v}(\cdot, s) = s(1 - s)E_{r,v}(\cdot, s).$$

These series can be meromorphically continued and play an important role in the spectral decomposition of $\Delta_r$.

**Theorem 3.8.**

(i) For fixed $z \in \mathcal{H}$ the Eisenstein series $E_{r,v}^q(z, \cdot)$ can be meromorphically continued to the whole complex plane.

(ii) If, for one fixed $z$, $E_{r,v}^q(z, \cdot)$ has a pole of order $n$ at $s_0$ the function $f(z) := \lim_{s \to s_0} (s - s_0)^n E_{r,v}^q(z, s)$ is real analytic, invariant under $|R_{r,v}$ and satisfies

$$-\Delta_r f = s_0(1 - s_0)f.$$

In particular, if $E_{r,v}^q(z, \cdot)$ is holomorphic at $s_0$, then

$$-\Delta_r E_{r,v}^q(\cdot, s_0) = s_0(1 - s_0)E_{r,v}^q(\cdot, s_0).$$
Furthermore we have the following equalities:

\[ K_r E^q_{r,v}(\cdot, s_0) = \left( \frac{r}{2} + s_0 \right) E^q_{r+2,v}(\cdot, s_0), \]
\[ \Lambda_r E^q_{r,v}(\cdot, s_0) = \left( \frac{r}{2} - s_0 \right) E^q_{r+2,v}(\cdot, s_0). \]

(20)

(21)

The poles of \( E^q_{r,v}(z, \cdot) \) in the half plane defined by \( \text{Re} \ s \geq \frac{1}{2} \) are all simple and in the interval \((\frac{1}{2}, 1]\). In particular there are no poles on the line \( \text{Re} \ s = \frac{1}{2} \).

**Theorem 3.9 (Spectral expansion).** Let \( f \in \hat{D}_r \) and \( e_n \) a maximal orthonormal system of eigenfunctions of \( \Delta_r \). Then \( f \) has a spectral expansion

\[ f = \sum_n (e_n, f) e_n + \sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (\mathcal{E}^q_{r,v}(\cdot, 1/2 + i\rho), f) \mathcal{E}^q_{r,v}(z, 1/2 + i\rho) d\rho. \]

If \( f \) has compact support mod \( \Gamma \) then both parts of the spectral expansion, \( \sum (e_n, f) e_n \) and \( \sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (\mathcal{E}^q_{r,v}(\cdot, 1/2 + i\rho), f) \mathcal{E}^q_{r,v}(z, 1/2 + i\rho) d\rho \), converge absolutely and uniformly on compact subsets of \( \mathcal{H} \).

**Proof.** Both the properties of Eisenstein series and the spectral expansion are proved in the second part of [14].

We turn back to the proof of Theorem 3.1: Let \( \phi \in Z^1_{r,v}(\Gamma, \mathcal{P}) \) and \( g \in \mathcal{Q} \) such that \( \phi(\gamma) = g|_{r,v}\gamma - g \) for all \( \gamma \in \Gamma \). By applying \( \partial / \partial \bar{z} \) to the equation \( g|_{r,v}\gamma - g = \phi(\gamma) \) we see that

\[ \frac{\partial g}{\partial \bar{z}}(z) = v(\gamma) j(\gamma, z)^{-r} j(\gamma, z)^2 \frac{\partial g}{\partial \bar{z}}(\gamma z). \]

A short calculation shows that the function \( G : z \mapsto \gamma^{r/2} \frac{\partial g}{\partial \bar{z}}(z) \) is invariant under \( R_{2-r,\pi} \). Moreover \( G \) vanishes in a neighbourhood of every cusp since \( g \) is holomorphic there, so \( G \) has compact support mod \( \Gamma \) and is in \( H_{2-r,\pi} \).

To prove Theorem 3.1 we have to show that if \( \phi \) is orthogonal to \( S_{2-r}(\Gamma, \mathcal{P}) \), then \( g \in \mathcal{Q} \) can be chosen to be holomorphic. This implies that \( \phi \) is a coboundary in \( Z^1_{r,v}(\Gamma, \mathcal{P}) \).

**Lemma 3.10.** Let \( 2 - r > 0 \) and \( \phi, g \) and \( \mathcal{G} \) be as above. Then \( (f, \phi) = 0 \) for all \( f \in S_{2-r}(\Gamma, \mathcal{P}) \) if and only if \( (\bar{f}, G) = 0 \) for all cusp forms \( \bar{f} \) with eigenvalue \( \frac{r}{2}(1 - \frac{r}{2}) \).
Proof. We have the equality

$$\frac{i}{2C_{2-r}}(f, \phi) = \frac{i}{2} \int_{F} \overline{\partial g} \wedge f(z) dz = \int_{F} y^{2-r} f(z) G(z) \frac{dxdy}{y^2} = (y^{2-r} f, G)^R,$$

so \((f, \phi) = 0\) for all \(f \in S_{2-r}(\Gamma, \varpi)\) if and only if \((\tilde{f}, G)^R = 0\) for all functions of the form \(y^{2-r} f, f \in S_{2-r}(\Gamma, \varpi)\). According to the remark after Definition 3.4 these functions are exactly the cusp forms of eigenvalue \(\frac{r}{2}(1 - \frac{r}{2})\).

We can now use spectral theory to characterise functions which are orthogonal to cusp forms of eigenvalue \(\frac{r}{2}(1 - \frac{r}{2})\).

**Proposition 3.11.** Let \(2 - r \neq 1\) and \(H\) be a smooth function in \(H_{2-r, \varpi}\) with compact support mod \(\Gamma\). Then the following are equivalent:

(i) \((\tilde{f}, H)^R = 0\) for all cusp forms \(\tilde{f}\) with eigenvalue \(\frac{r}{2}(1 - \frac{r}{2})\).

(ii) \(H = K_{-r} F + y^{2-r} E\), where \(F\) is a smooth function in \(H_{-r, \varpi}\) and \(E\) is in \(M_{2-r}(\Gamma, \varpi)\) and orthogonal to \(S_{2-r}(\Gamma, \varpi)\). If \(2 - r > 1\) this implies \(E = 0\).

**Remark.** By [9] there exist no non-zero classic modular forms of negative weight. Since, by [14, Satz 5.2], all eigenfunctions of eigenvalue \(\frac{r}{2}(1 - \frac{r}{2})\) are of the form \(y^{\frac{r}{2}} f\), where \(f\) is a classic modular form, the first condition is always satisfied in the case \(2 - r < 0\).

**Proof.** (i)\(\Rightarrow\)(ii): By [14, Satz 6.3] there is a maximal orthonormal system of eigenfunctions of \(\Delta_{2-r}\) consisting of:

1. Images of eigenfunctions of \(\Delta_{-r}\) under the Maass raising operator \(K_{-r} = (z - \overline{z}) \frac{\partial}{\partial z} - \frac{r}{2}\). We denote these by \(K_{-r} e_n\). By [14, Satz 6.3] these eigenfunctions cannot have eigenvalue \(\frac{r}{2}(1 - \frac{r}{2})\).

2. A (finite) orthonormal basis of the eigenfunctions of eigenvalue \(\frac{r}{2}(1 - \frac{r}{2})\). This set is of the form \(\{y^{\frac{r}{2}} f_1, \ldots, y^{\frac{r}{2}} f_N\}\), where the \(f_i\) form an orthonormal basis of the subspace \(N \subseteq M_{2-r}(\Gamma, \varpi)\) of modular forms that are square integrable. This subspace is equal to \(S_{2-r}(\Gamma, \varpi)\) if \(2 - r \geq 1\) but it can be larger if \(2 - r < 1\) (see [14, Satz 13.1]).
Hence the spectral expansion of $H$ is of the form

$$H = \sum_n (K_{-r}e_n, H)K_{-r}e_n + \sum_{i=1}^{N} (y^{\frac{2-r}{2}} f_i, H)y^{\frac{2-r}{2}} f_n + \sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (E_{2-r,\pi}^{q_i}(\cdot, \frac{1}{2} + i\rho), f)^R E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho) d\rho.$$  

Here we used that $\sum_n (K_{-r}e_n, H)K_{-r}e_n$ converges absolutely and uniformly on compacta to swap differentiation and summation and write it as $K_{-r}F_1 = K_{-r} (\sum_n (K_{-r}e_n, H)e_n)$.

Since $H$ is orthogonal to all cusp forms with eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$ we immediately see that $E \in N$ must be orthogonal to $S_{2-r}(\Gamma, \pi)$. If $2 - r \geq 1$ we have $N = S_{2-r}(\Gamma, \pi)$ so in that case we have $E = 0$. If $2 - r < 0$ we have $M_{2-r}(\Gamma, \pi) = \{0\}$ by [9], so in this case we also have $E = 0$.

Next we deal with $F_2$: Applying equation (20) twice we see that

$$\int_{-\infty}^{\infty} (E_{2-r,\pi}^{q_i}(\cdot, \frac{1}{2} + i\rho), H)^R E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho) d\rho,$$

$$= \int_{-\infty}^{\infty} \left( \frac{1}{2} - i\rho \right)^{-2} (K_{-r}E_{2-r,\pi}^{q_i}(\cdot, \frac{1}{2} + i\rho), H)^R K_{-r}E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho) d\rho.$$  

If $r \neq 1$ the integral

$$F_2(z) = \int_{-\infty}^{\infty} \left( \frac{1}{2} - i\rho \right)^{-2} (E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho), H)^R E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho) d\rho,$$

converges absolutely and uniformly on compacta. To see this note the integrand can be bounded above by

$$\left| \frac{1-r}{2} \right| \cdot \left| (E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho), H)^R E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho) \right|,$$

and

$$\int_{-\infty}^{\infty} (E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho), H)^R E_{2-r,\pi}^{q_i}(z, \frac{1}{2} + i\rho) d\rho.$$
converges absolutely and uniformly on compacta as it occurs in the spectral expansion of $\Lambda_{2-r} H$. So when we apply $K_{-r}$ to $F_2 = \sum_{i=1}^{m_r} \frac{1}{4\pi} F_2^i$ we can swap it with the integral and obtain

$$K_{-r} F_2 = \tilde{F}_2.$$ 

Summing up we have

$$H = K_{-r} F + E,$$

where $F = F_1 + F_2$. \hfill (22)

To see that $F$ is smooth we apply $\Lambda_{2-r}$ to equation (22):

$$\Lambda_{2-r} H = \Lambda_{2-r} K_{-r} F + \Lambda_{2-r} (y^{2-r} E) = -\Delta_{-r} F - \frac{r}{2}(1 - \frac{r}{2}) F + \Lambda_{2-r} (y^{2-r} E).$$

We see that $F$ is a solution of an elliptic differential equation and so, by elliptic regularity, $F$ is smooth.

(ii)$\Rightarrow$(i): Let $H = K_{-r} F + y^{2-r} E$ for a smooth function $F$ and let $\tilde{f}$ be a cusp form with eigenvalue $\frac{r}{2}(1 - \frac{r}{2})$. Then, since $y^{2-r} E$ is orthogonal to $\tilde{f}$,

$$(H, \tilde{f})^R = (K_{-r} F, \tilde{f})^R = (F, \Lambda_{2-r} \tilde{f})^R.$$ 

The function $f = y^{-\frac{2-r}{2}} \tilde{f}$ is in $S_{2-r}(\Gamma, \pi)$ and hence holomorphic, so we have

$$\Lambda_{2-r} \tilde{f} = \Lambda_{2-r} (y^{2-r} f) = (z - \bar{z}) \frac{\partial f}{\partial z} = 0.$$ 

Theorem 3.1 now follows from Proposition 3.11.

Proof of Theorem 3.1 and proof of Theorem 1.2 for $2 - r < 0$ and $2 - r > 1$: Let $\phi \in Z^1_{r,v}(\Gamma, \mathcal{P})$ and $g$ and $G$ be constructed as above. In the case $2 - r > 1$ suppose additionally that $(f, \phi) = 0$ for all $f \in S_{2-r}(\Gamma, \pi)$. By Proposition 3.11 and the remark after it there is a smooth $F \in H_{-r,\pi}$ with $G = K_{-r} F$, i.e.

$$y^{\frac{r+2}{2}} \frac{\partial g}{\partial z}(z) = 2iy \frac{\partial F}{\partial z} - \frac{r}{2} F = -2iy^{\frac{r}{2} + \frac{1}{2}} \frac{\partial}{\partial z}(y^{-\frac{r}{2}} F).$$

Dividing by $y^{\frac{r+2}{2}}$ and taking the complex conjugate of both sides we arrive at

$$\frac{\partial g}{\partial z}(z) = \frac{\partial}{\partial z}(2iy^{-\frac{r}{2}} F)(z).$$

23
Since $F$ is invariant under $|_{r,v}^R$, $\tilde{F}$ is invariant under $|_{r,v}^R$. By Lemma 3.6, $\tilde{F}(z) = 2iy^{-i}F$ is invariant under $|_{r,v}^R$. This invariance implies that $\tilde{g} = g - \tilde{F}$ satisfies $\tilde{g}|_{r,v,\gamma} - \tilde{g} = \phi(\gamma)$ for all $\gamma \in \Gamma$. Furthermore as a function in $H_{r,v}^\infty$, $\tilde{F}$ clearly satisfies the growth conditions for functions in $Q$ so $\tilde{g} \in Q$. Now equation 3.1 implies that $\tilde{g}$ is holomorphic, hence in $\mathcal{P}$ and so $\phi$ is indeed a coboundary in $Z_{1,r,v}(\Gamma, \mathcal{P})$.

This shows in particular that for $2-r < 0$ every cocycle in $Z_{1,r,v}^1(\Gamma, \mathcal{P})$ is a coboundary and hence $H_{1,r,v}^1(\Gamma, v) = 0$. This is the statement of 1.2, since $S_{2-r}(\Gamma, v)$ is also 0 in this case.

**Remark.** The proof fails if $0 \leq 2-r \leq 1$. We note partial results for two cases:

**Case** $0 < 2-r < 1$: We have $G = K_{-r}F + y^{2-r}E$ for $F \in H_{-r,v}^\infty$ and $E \in M_{2-r}(\Gamma, \mathcal{P}))$. Setting $\tilde{g} = g - \tilde{F}$ as above we again obtain $\tilde{g}|_{r,v,\gamma} - \tilde{g} = \phi(\gamma)$. However $\tilde{g}$ is not holomorphic in this case since

$$\frac{\partial \tilde{g}}{\partial \bar{z}}(z) = y^{-r}E(z).$$

**Case** $2-r = 1$: One can follow the proof of Proposition 3.11 to obtain

$$G = K_{-r}F + \sum_{i=1}^{m^*} \frac{1}{4\pi} \int_{-\infty}^{\infty} (E^{q_1}_{1,\pi}(\cdot, \frac{1}{2} + i\rho), f)^R E_{1,\pi}^q(z, \frac{1}{2} + i\rho) d\rho.$$  

To prove Theorem 1.2 in this case one would need to show that the second summand above is in the image of $K_{-r}^\infty$.

## 4 Vector-valued modular forms

In this section we generalise Theorem 1.2 to vector-valued cusp forms. Let $\rho : \Gamma \to U(n)$ be a unitary representation of $\Gamma$ on $\mathbb{C}^n$ and $v$ a unitary multiplier system of weight $r$. Let $F$ be a function from $\mathcal{H}$ to $\mathbb{C}^n$. The slash operator $|_{\rho,v}^R$ is defined by

$$F|_{r,v,\rho,\gamma}(z) = j(\gamma, z)^{-r}v(\gamma)\rho(\gamma)^{-1}F(\gamma z).$$

**Definition 4.1.** A function $f : \mathcal{H} \to \mathbb{C}^n$ is a modular form for $\Gamma$ of weight $r$, representation $\rho$ and multiplier system $v$ if the following conditions are satisfied:
(i) $f$ is holomorphic on $\mathcal{H}$.

(ii) $f(z) = f|_{r,v,\rho} \gamma(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$.

(iii) If $q$ is a cusp of $\Gamma$ and $A\infty = q$ then for any $\epsilon > 0$

$$j(A, z)^{-r} f(Az) \text{ is bounded for } y \geq \epsilon.$$  

If $f$ satisfies additionally the condition

(iii') If $q$ is a cusp of $\Gamma$ and $A\infty = q$ then there exists $\epsilon > 0$

$$j(A, z)^{-r} f(Az) = O_{y \to \infty}(e^{-\epsilon y}),$$

it is cusp form. The set of modular forms (resp. cusp forms) of this kind is denoted by $M_{r}(\Gamma, v, \rho)$ (resp. $S_{r}(\Gamma, v, \rho)$).

Let $\mathcal{P}^n$ be the set of vector valued functions $f(z) = (f_1(z), \ldots, f_n(z))$ such that all $f_i$ are in $\mathcal{P}$. The slash operator $|_{r,v,\rho}$ defines a $\Gamma$-action on $\mathcal{P}^n$ and so we can define the cohomology groups $H^{1}_{r,v,\rho}(\Gamma, \mathcal{P}^n)$ and $\tilde{H}^{1}_{r,v,\rho}(\Gamma, \mathcal{P}^n)$. Just as in the 1-dimensional case they turn out to be the same. The proof of this fact relies on a generalisation of Corollary 1.4:

**Proposition 4.1.** Let $U \in U(n)$, $s \neq 0$ and $g \in \mathcal{P}^n$. Then there exists $f \in \mathcal{P}^n$ such that

$$U^* f(z + s) - f(z) = g(z), \quad \forall z \in \mathcal{H}. \quad (23)$$

**Proof.** Since $U$ is diagonalisable there exists a $V \in U(n)$ and a diagonal $D \in U(n)$ with

$$U = V^* D V.$$

Multiplying equation (23) with $V$

$$D^* V f(z + s) - V f(z) = V g(z). \quad (24)$$

Let $\epsilon_1, \ldots, \epsilon_n$ be the diagonal entries of $D$ and $G = V g = (G_1, \ldots, G_n) \in \mathcal{P}^n$. We can use Corollary 1.4 to find solutions $F_i \in \mathcal{P}$ for

$$\bar{\epsilon}_i F_i(z + s) - F_i(z) = G_i(z).$$

Then $f = V^{-1}(F_1, \ldots, F_n)$ is in $\mathcal{P}^n$ and satisfies (24).

This can be used to show

**Theorem 4.2.** Every cocycle in $Z^{1}_{v,\rho}(\Gamma, \mathcal{P}^n)$ is parabolic.
4.1 Petersson inner product

Let $2 - r > 0$ and $f, g$ be in $S_{2-r}(\Gamma, \mathcal{P}, \rho^{-1})$. The Petersson inner product of $f$ and $g$ is defined by

$$(f, g) = \int_{\mathcal{F}} \langle f(z), g(z) \rangle y^{-r} dxdy,$$

where $< (a_i), (b_i) > = \sum_{i=1}^{n} a_i \overline{b_i}$ is the usual scalar product on $\mathbb{C}^n$. We will repeat the constructions of Section 2.

**Lemma 4.3.** Let $g$ be in $S_{2-r}(\Gamma, \mathcal{P}, \rho^{-1})$, then

$$\phi_{g}^{\infty}(z) : \gamma \mapsto \phi_{g, \gamma}^{\infty}(z) = \int_{\gamma^{-1}\infty}^{\infty} g(\tau)(\tau - z)^{-r} d\tau,$$

is a cocycle in $Z_{r, \rho}^{1}(\Gamma, \mathcal{P}^n)$.

Again we can use Stoke’s theorem to show

$$(f, g) = -C_{2-r} \sum_{m=1}^{n} \int_{A_{im}+1}^{A_{im}} \langle f(z), \phi_{g, \alpha_{im}}^{\infty}(z) \rangle dz.$$

Using this we define a pairing between $S_{2-r}(\Gamma, \mathcal{P}, \rho^{-1})$ and $H_{r,v,\rho}^{1}(\Gamma, \mathcal{P}^n)$ as follows. Let $f \in S_{2-r}(\Gamma, \rho^{-1}, \mathcal{P})$ and $[\phi] \in H_{r,v}^{1}(\Gamma, \mathcal{P}^n)$ represented by $\phi$. Then

$$(f, [\phi]) = (f, \phi) = -C_{2-r} \sum_{m=1}^{n} \int_{A_{im}}^{A_{im}+1} \langle f(z), \phi_{\alpha_{im}}(z) \rangle dz,$$

is well-defined (independent of the representative $\phi$) and furthermore we have the theorem, analogous to Theorem 1.2.

**Theorem 4.4.** Let $\nu$ and $\rho$ be as above and $2 - r > 1$. The pairing defined above is perfect, so the map $f \mapsto \phi_{f}^{\infty}$ induces an isomorphism

$$S_{2-r}(\Gamma, \mathcal{P}, \rho^{-1}) \cong H_{r,v,\rho}^{1}(\Gamma, \mathcal{P}^n).$$

If $2 - r < 0$ we have

$$S_{2-r}(\Gamma, \mathcal{P}, \rho^{-1}) \cong H_{r,v,\rho}^{1}(\Gamma, \mathcal{P}^n) \cong \{0\}.$$

**Proof.** All the constructions of Section 3 work in the vector-valued case. In particular every statement we cited from [14] is already formulated in the vector-valued case. The fact that every vector-valued modular form of negative weight is 0 is also stated in [14] as a consequence of Satz 5.3 and generalises the main theorem of [9].
References

[1] R. Bruggeman, Y. Choie, and N. Diamantis. Holomorphic automorphic forms and cohomology. 2014.

[2] Henri Cohen. Haberland’s formula and numerical computation of Petersson scalar products. In ANTS X, volume 1 of The Open Book Series, pages 249–270, San Diego, United States, 2013. Mathematical Sciences Publisher.

[3] Gilbarg D. and Trudinger N.S. Elliptic Partial Differential Equations of Second Order. Springer Verlag, 2001.

[4] M. Eichler. Eine Verallgemeinerung der Abelschen Integrale. Math. Z., 67:267–298, 1957.

[5] L. R. Ford. The fundamental region for a Fuchsian group. Bulletin of the American Mathematical Society, 31(9-10):531–539, 11 1925.

[6] D. Goldfeld. Special values of derivatives of L-functions. In Conference Proceedings, volume 15. Canadian Mathematical Society, 1995.

[7] R. C. Gunning. The Eichler cohomology groups and automorphic forms. Trans. Amer. Math. Soc., 100:44–62, 1961.

[8] Marvin Knopp and Henok Mawi. Eichler cohomology theorem for automorphic forms of small weights. Proc. Amer. Math. Soc., 138(2):395–404, 2010.

[9] Marvin I. Knopp. Notes on automorphic functions: An entire automorphic form of positive dimension is zero. J. Res. Nat. Bur. Standards Sect. B, 71B:167–169, 1967.

[10] Marvin I. Knopp. Some new results on the Eichler cohomology of automorphic forms. Bull. Amer. Math. Soc., 80:607–632, 1974.

[11] Marvin I. Knopp. On the growth of entire automorphic integrals. Results Math., 8(2):146–152, 1985.

[12] Y. Manin. Periods of parabolic forms and p-adic Hecke series. Mathematics of the USSR-Sbornik, 1973.
[13] Hans Petersson. Zur analytischen Theorie der Grenzkreisgruppen. *Mathematische Annalen*, 115(1):518–572, 1938.

[14] Walter Roelcke. Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II. *Math. Ann. 167 (1966), 292–337; ibid.*, 168:261–324, 1966.

[15] Xueli Wang. A conjecture on the Eichler cohomology of automorphic forms. *Sci. China Ser. A*, 43(7):734–742, 2000.