ALMOST TORUS MANIFOLDS OF NON-NEGATIVE CURVATURE

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Abstract. An almost torus manifold $M$ is the odd-dimensional analog of a torus manifold. Namely, it is a $(2n+1)$-dimensional orientable manifold with an effective, isometric, almost isotropy-maximal $T^n$-action. In this paper, we show that closed, simply-connected, non-negatively curved almost torus manifolds are equivariantly diffeomorphic to a quotient by a free linear torus action on a product of spheres, thus generalizing Wiemeler's classification of simply-connected, non-negatively curved torus manifolds.

1. Introduction

The classification of compact Riemannian manifolds with positive or non-negative sectional curvature is a long-standing problem in Riemannian geometry. One successful approach to this problem has been the Grove Symmetry Program, which asks to classify such manifolds with “large” symmetries.

An important first step is to consider the case of continuous abelian symmetries, that is, of torus actions. Recently, Wiemeler classified simply-connected, non-negatively curved torus manifolds up to equivariant diffeomorphism in [23]. Recall that a torus manifold is an orientable, $2n$-dimensional manifold admitting an isotropy-maximal $T^n$-action, that is, the rank of the largest isotropy subgroup of the action equals the cohomogeneity of the action (cf. Escher and Searle [7]).

Our main result, stated below in Theorem 1, gives an equivariant diffeomorphism classification of closed, simply connected, non-negatively curved manifolds that are almost torus manifolds, that is, with an isometric, effective and almost isotropy-maximal torus action, generalizing Wiemeler's result on torus manifolds.

Theorem A. Let $T^n$ act isometrically and effectively on a closed, non-negatively curved, almost torus manifold, $M^{2n+1}$. Then $M^{2n+1}$ is equivariantly diffeomorphic to the quotient by a linear torus action on a product of spheres.

Combining this result with Theorem 1 of [7], we obtain the following corollary.

Corollary B. Let $T^k$ act isometrically, effectively and almost isotropy-maximally on a closed, non-negatively curved manifold, $M^n$ with $k \leq \lfloor 2n/3 \rfloor - 1$. Then the free rank of the $T^k$ action is equal to $2k - n + 1$. If the free rank is equal to the free dimension, then $M^n$ is equivariantly diffeomorphic to the quotient by a linear torus action on a product of spheres.

Recall that one defines a $T^k$-action on a smooth manifold, $M^n$, to be almost isotropy-maximal when there exists a point in $M$ whose isotropy group is maximal (respectively, almost maximal), namely of dimension $n - k - 1$. 

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1.1. Organization. The paper is organized as follows. In Section 2, we gather preliminary definitions and facts that we will use throughout the paper. In Section 3, we prove the analog for simply-connected, almost torus manifolds of Lemma 6.3 in [23]. In Section 4, we show how to extend the isometric $T^n$ action to a smooth $T^{n+1}$ action. In Section 5, we prove Theorem A and Corollary B.

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2. Preliminaries

In this section we will gather basic results and facts. For more details on many of the basic concepts in this paper see Escher and Searle [7].

2.1. Transformation Groups. Let $G$ be a compact Lie group acting on a smooth manifold $M$. We denote by $G_x = \{ g \in G : gx = x \}$ the isotropy group at $x \in M$ and by $G(x) = \{ gx : g \in G \} \simeq G/G_x$ the orbit of $x$. Orbits will be principal, exceptional or singular, as follows: principal orbits correspond to those orbits with the smallest possible isotropy subgroup, an orbit is called exceptional when its isotropy subgroup is a finite extension of the principal isotropy subgroup, and singular when its isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup.

The ineffective kernel of the action is the subgroup $K = \cap_{x \in M} G_x$. We say that $G$ acts effectively on $M$ if $K$ is trivial. The action is called almost effective if $K$ is finite. The action is free if every isotropy group is trivial and almost free if every isotropy group is finite. As mentioned in the Introduction, the free rank of an action is the rank of the maximal subtorus that acts almost freely. In order to further distinguish between the case when the free rank corresponds to a free action and the case when it corresponds to an almost free action, we define the free dimension to be the rank of the largest freely acting subgroup.

We will sometimes denote the fixed point set $M^G = \{ x \in M : gx = x, g \in G \}$ of the $G$-action by $\text{Fix}(M; G)$. Its dimension is defined as the maximum dimension of its connected components. One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space $M/G$, also called the cohomogeneity of the action. This dimension is clearly constrained by the dimension of the fixed point set $M^G$ of $G$ in $M$. In fact, $\dim(M/G) \geq \dim(M^G) + 1$ for any non-trivial, non-transitive action. In light of this, the fixed-point cohomogeneity of an action, denoted by $\text{cohomfix}(M; G)$, is defined by

$$\text{cohomfix}(M; G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$ 

A manifold with fixed-point cohomogeneity 0 is also called a $G$-fixed point homogeneous manifold. For product groups, we recall the definition of the following refinement of a fixed point homogeneous action (cf. [7]).

Definition 2.1 (Nested Fixed Point Homogeneous). Let $G = H \times \cdots \times H = H^l$ act isometrically and effectively on $M^n$. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ denote the connected component of largest dimension in $M^H$ and let $N_k \subseteq N_{k-1}$ denote the connected component of largest dimension
in $N_{k-1}^H$, for $k \leq 1$. We will call a manifold nested $H$-fixed point homogeneous when there exists a tower of nested $H$-fixed point sets,

$$N_j \subset N_{j-1} \subset \cdots \subset N_1 \subset M,$$

where $1 \leq j \leq \min\{l, \lfloor n/d \rfloor\}$, where $d$ denotes the codimension of $N_j$ in $M$. Moreover, $H$ acts fixed point homogeneously on $M^n$, and all induced actions of $G/H^k$ on $N_k$, for all $1 \leq k \leq l-1$, are $H$-fixed point homogeneous.

2.2. Torus Actions. In this subsection we will recall notation and facts about smooth $G$-actions on smooth $n$-manifolds, $M$, in the special case when $G$ is a torus. We first recall the definition of an isotropy-maximal torus action and an almost isotropy-maximal torus action. See [7] for more details about such actions.

**Definition 2.2 (Isotropy-Maximal Action/Almost Isotropy-Maximal Action).** Let $M^n$ be a connected manifold with an effective $T^k$-action. We call the $T^k$-action on $M^n$ isotropy-maximal, respectively, almost isotropy-maximal, if there is a point $x \in M$ such that the dimension of its isotropy group is $n-k$, respectively, $n-k-1$.

Note that the action of $T^k$ on $M$ is isotropy-maximal if and only if there exists a minimal orbit $T^k(x)$. The following lemma of [16] shows that an isotropy-maximal action on $M$ means that there is no larger torus which acts on $M$ effectively.

**Lemma 2.3.** Let $M$ be a connected manifold with an effective $T^k$-action. Let $T^l \subset T^k$ be a subtorus of $T^k$. Suppose that the action of $T^k$ restricted to $T^l$ on $M$ is isotropy-maximal. Then $T^l = T^k$.

2.3. Torus and Almost Torus Manifolds. An important subclass of manifolds admitting an effective torus action are the so-called torus manifolds. For more details on torus manifolds, we refer the reader to Hattori and Masuda [14], Buchstaber and Panov [2], and Masuda and Panov [17].

**Definition 2.4 (Torus Manifold).** A torus manifold $M$ is a $2n$-dimensional closed, connected, orientable, smooth manifold with an effective smooth action of an $n$-dimensional torus $T$ such that $M^T \neq \emptyset$.

Note that a torus manifold gives us an example of an isotropy-maximal torus action. The following definition of an almost torus manifold is then natural.

**Definition 2.5 (Almost Torus Manifold).** An almost torus manifold $M$ is a $(2n+1)$-dimensional closed, connected, orientable, smooth manifold with an effective smooth action of an $n$-dimensional torus $T^n$ such that the torus action is almost isotropy-maximal.

**Remark 2.6.** We will use the convention that in the case of a torus or almost torus manifold of non-negative curvature the torus action is isometric.

We also recall the definition of a characteristic submanifold.

**Definition 2.7.** Let $T^k$ act effectively and isometrically on a closed manifold $M^n$. Let $F$ be a connected component of $\text{Fix}(M,S^1)$ for some circle subgroup $S^1 \subset T^k$. Then $F$ is called a characteristic submanifold of $M$ if it satisfies the following properties:

1. $\text{codim}(F) = 2$ in $M$;
2. $F$ contains a $T$-fixed point.

Note that both torus and almost torus manifolds have the following properties: both are examples of $S^1$-fixed point homogeneous manifolds and moreover, the action on $T^n$ on $M^{2n}$, respectively, on $M^{2n+1}$, is an example of a nested $S^1$-fixed point homogeneous action, as in
Definition 2.9. Moreover, both torus and almost torus manifolds contain a characteristic submanifold which contains the the tower of nested fixed point sets.

An important class of $T^n$-actions on a $2n$-dimensional manifolds $M^{2n}$ is a locally standard torus action, whose definition we now recall.

Definition 2.8 (Locally Standard). A $T^n$-action on $M^{2n}$ is called locally standard if each point in $M^{2n}$ has an invariant neighborhood $U$ which is weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ invariant under the standard $T^n$-action on $\mathbb{C}^n$, that is, there exists an automorphism $\psi : T^n \rightarrow T^n$ and a diffeomorphism $f : U \rightarrow W$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T^n$ and $y \in U$.

We define an analogous concept for $T^n$ actions on a $(2n + 1)$-dimensional manifold, $M^{2n+1}$.

Definition 2.9 (Almost Locally Standard). A $T^n$-action on $M^{2n+1}$ is called almost locally standard if each point in $M^{2n+1}$ has an invariant neighborhood $U$ which is weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n \oplus \mathbb{R}$ invariant under the standard $T^n$-action on $\mathbb{C}^n$ and the trivial one on $\mathbb{R}$, that is, there exists an automorphism $\psi : T^n \rightarrow T^n$ and a diffeomorphism $f : U \rightarrow W$ such that $f(ty) = \psi(t)f(y)$ for all $t \in T^n$ and $y \in U$.

Remark 2.10. Both a locally standard and an almost locally standard torus action are such that all isotropy subgroups are connected.

The quotient space of a $T^n$-manifold plays an important role in the theory. Recall that an $n$-dimensional convex polytope is called simple if the number of facets meeting at each vertex is $n$. An $n$-manifold with corners is a Hausdorff space together with a maximal atlas of local charts onto open subsets of the simplicial cone, $[0, \infty)^n \subset \mathbb{R}^n$, so that the overlap maps are homeomorphisms which preserve codimension. If, in addition, we assume that the manifold with corners is acyclic with acyclic faces, then we call it a homology polytope (see [17]). A manifold with corners is called nice if every codimension $k$ face is contained in exactly $k$ facets. Clearly, a simple convex polytope is a nice homology polytope.

The orbit space of a locally standard or almost locally standard torus action is a manifold with corners. Quasitoric manifolds have the property that their orbit space is diffeomorphic, as a manifold with corners, to a simple polytope $P^n$. Note that two simple polytopes are diffeomorphic as manifolds with corners if and only if they are combinatorially equivalent.

Even though the conditions on the torus action in case of a torus manifold are much weaker than the conditions in the case of a quasitoric manifold, torus manifolds still admit a combinatorial treatment similar to quasitoric manifolds. In particular, the orbit space of a torus manifold is a nice manifold with corners if the action is locally standard. If in addition the orbit space is acyclic with acyclic faces, the constructions below for simple polytope orbit spaces can be generalized to this case, that is, to the case of orbit spaces that are nice homology polytopes.

To each $n$-dimensional simple angle convex polytope, $P^n$, we may associate a $T^n$-manifold $Z_P$ with the orbit space $P^n$, as in [5].

Definition 2.11 (Moment Angle Manifold). For every point $q \in P^n$ denote by $G(q)$ the unique (smallest) face containing $q$ in its interior. For any simple polytope $P^n$ define the moment angle manifold

$$Z_P = (T^F \times P^n)/\sim = (T^m \times P^n)/\sim,$$

where $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in T_{G(q)}$.

Note that the equivalence relation depends only on the combinatorics of $P^n$. In fact, this is also true for the topological and smooth type of $Z_P$, that is, combinatorially equivalent.
simple polytopes yield homeomorphic, and, in fact, diffeomorphic, moment angle manifolds (see Proposition 4.3 in Panov [18] and the remark immediately following it).

The free action of $T^m$ on $T^F \times \mathbb{P}^n$ descends to an action on $\mathbb{Z}P$, with quotient $\mathbb{P}^n$. Let $\pi_Z : \mathbb{Z}P \to \mathbb{P}^n$ be the orbit map. The action of $T^m$ on $\mathbb{Z}P$ is free over the interior of $\mathbb{P}^n$, where each vertex $v \in \mathbb{P}^n$ represents the orbit $\pi_Z^{-1}(v)$ with maximal isotropy subgroup of dimension $n$.

2.4. **Torus Orbifolds.** In this subsection we gather some preliminary results about torus orbifolds. We first recall the definition of an orbifold. For more details about orbifolds and actions of tori on orbifolds, see Haefliger and Salem [13], and [10].

**Definition 2.12 (Orbifold).** An $n$-dimensional (smooth) orbifold, denoted by $O$, is a second-countable, Hausdorff topological space $|O|$, called the underlying topological space of $O$, together with an equivalence class of $n$-dimensional orbifold atlases.

In analogy with a torus manifold, we may define a torus orbifold, as follows.

**Definition 2.13 (Torus Orbifold).** A torus orbifold, $O$, is a $2n$-dimensional, closed, orientable orbifold with an effective smooth action of an $n$-dimensional torus $T$ such that $O^T \neq \emptyset$.

In [10], using results obtained for torus orbifolds, they prove the following theorem, which will be of use in the proof of Theorem A.

**Theorem 2.14.** [10] Let $M$ be an $n$-dimensional, smooth, closed, simply-connected, rationally elliptic manifold with an isotropy-maximal $T_k$-action. Then there is a product $\hat{P}$ of spheres of dimension $\geq 3$, a torus $\hat{T}$ acting linearly on $\hat{P}$, and an effective, linear action of $T_k$ on $\hat{M} = \hat{P}/\hat{T}$, such that there is a $T_k$-equivariant rational homotopy equivalence $M \simeq \hat{M}$.

2.5. **Alexandrov Geometry.** Recall that a complete, locally compact, finite dimensional length space $(X, \text{dist})$ with curvature bounded from below in the triangle comparison sense is an Alexandrov space (see, for example, Burago, Burago, and Ivanov [3]). When $M$ is a complete, connected Riemannian manifold and $G$ is a compact Lie group acting on $M$ by isometries, the orbit space $Y = M/G$ is equipped with the orbital distance metric induced from $M$, that is, the distance between $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ in $X$ is the distance between the orbits $G(p)$ and $G(q)$ as subsets of $M$. If, additionally, $M$ has sectional curvature bounded below, that is, $\sec M \geq k$, then the orbit space $X$ is an Alexandrov space with curv $X \geq k$.

The space of directions of a general Alexandrov space at a point $x$ is by definition the completion of the space of geodesic directions at $x$. In the case of orbit spaces $X = M/G$, the space of directions $\sum \mathcal{P}X$ at a point $\mathcal{P} \in X$ consists of geodesic directions and is isometric to $S^2_p/G_p$, where $S^2_p$ is the unit normal sphere to the orbit $G(p)$ at $p \in M$.

A non-empty, proper extremal set comprises points with spaces of directions which significantly differ from the unit round sphere. They can be defined as the sets which are “ideals” of the gradient flow of $\text{dist}(p, \cdot)$ for every point $p$. Examples of extremal sets are isolated points with space of directions of diameter $\leq \pi/2$, the boundary of an Alexandrov space and, in a trivial sense, the entire Alexandrov space. We refer the reader to Petrunin [19] for definitions and important results.

2.6. **Geometric results in the presence of a lower curvature bound.** We now recall some general results about $G$-manifolds with non-negative and almost non-negative curvature which we will use throughout.

We first recall the splitting theorem of Cheeger and Gromoll.
Theorem 2.15. [4] Let $M$ be a compact manifold of non-negative sectional curvature. Then $\pi_1(M)$ contains a finite normal subgroup $\psi$ such that $\pi_1((M)/\psi$ is a finite group extended by $\mathbb{Z}^k$, and $\tilde{M}$, the universal covering of $M$, splits isometrically as $\tilde{M} \times \mathbb{R}^k$, where $\tilde{M}$ is compact.

Recall that a torus manifold is an example of an $S^1$-fixed point homogeneous manifold, indeed, of a nested $S^1$-fixed point homogeneous manifold. Fixed point homogeneous manifolds of positive curvature were classified in Grove and Searle [12]. More recently, the following theorem by Spindeler, [21], gives a characterization of non-negatively curved $G$-fixed point homogeneous manifolds.

Theorem 2.16. [21] Assume that $G$ acts fixed point homogeneously on a closed, non-negatively curved Riemannian manifold $M$. Let $F$ be a fixed point component of maximal dimension. Then there exists a smooth submanifold $N$ of $M$, without boundary, such that $M$ is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of $F$ and $N$ glued together along their common boundaries;

$$M = D(F) \cup_D D(N).$$

Further, $N$ is $G$-invariant and contains all singularities of $M$ up to $F$.

A special case of the above theorem occurs when $F$ and $N$ are both fixed by the $S^1$ action in the following Double Soul Theorem.

Double Soul Theorem 2.17. [20] Let $M$ be a non-negatively curved $S^1$-fixed point homogeneous Riemannian manifold. If $\text{Fix}(M,S^1)$ contains at least two components $F$ and $N$ with maximal dimension, one of which is compact, then $F$ and $N$ are isometric and $M$ is diffeomorphic to an $S^2$-bundle over $F$ with structure group $S^1$. In other words, there is a principal $S^1$ bundle $P$ over $F$ such that $M$ is diffeomorphic to $P \times_{S^1} S^2$.

Moreover, Lemma 3.29 of [21], shows that in the case where $M$ is simply-connected, one can put an upper bound on the dimension of $N$ as follows.

Lemma 2.18. [21] Let $M$ and $N$ be as in Theorem 2.16 and assume that $M$ is simply connected and $G$ is connected. Then $N$ has codimension greater than or equal to 2 in $M$.

The following two facts from [21] when $M$ is a torus manifold of non-negative curvature, will be important for what follows.

Proposition 2.19. [21] Let $M, N$ and $F$ be as in Theorem 2.16 and assume that $M$ is a closed, simply connected torus manifold of non-negative curvature. Then $F$ is simply-connected.

Moreover, the following theorem from [6] gives us topological information about the fundamental groups of $E$, $F$, and $N$.

Theorem 2.20. Let $M^n$ be a simply-connected manifold that decomposes as the union of two disk bundles as follows:

$$M^n = D^{k_1}(N_1) \cup_D D^{k_2}(N_2).$$

If $k_1 = k_2 = 2$, then $\pi_1(N_1)$ and $\pi_1(N_2)$ are cyclic groups. Moreover,

1. If $k_i = 2$, $\pi_2(N_i) = 0$, for $i = 1, 2$ and $\pi_1(N_i)$ is infinite for some $i \in \{1, 2\}$, then $\pi_1(E) \cong \mathbb{Z}^2$.
2. If $k_i \geq 3$, for some $i \in \{1, 2\}$, then $\pi_1(E) \cong \pi_1(N_i)$.
For a non-negatively curved torus manifold, Proposition 4.5 from Wiemeler [23] shows that the quotient space, $M^{2n}/T^n = P^n$, is described as follows: $P^n$ is a nice manifold with corners all of whose faces are acyclic and $P^n$ is of the form

$$P^n = \prod_{i < r} \Sigma^{n_i} \times \prod_{i \geq r} \Delta^{n_i},$$

where $\Sigma^{n_i} = S^{2n_i}/T^{n_i}$ and $\Delta^{n_i} = S^{2n_i+1}/T^{n_i+1}$ is an $n_i$-simplex. The $T^{n_i}$-action on $S^{2n_i}$ is the suspension of the standard $T^{n_i}$-action on $\mathbb{R}^{2n_i}$ and it is easy to see that $\Sigma^{n_i}$ is the suspension of $\Delta^{n_i-1}$. Note that each $n_i$-simplex has $n_i + 1$ facets and each $\Sigma^{n_i}$ has $n_i$ facets. The number of facets of $P^n$ in this case is bounded between $n$ and $2n$.

Using this description of the quotient space the following equivariant classification theorem is obtained in [23].

**Theorem 2.21.** [23] Let $M$ be a simply-connected, non-negatively curved torus manifold. Then $M$ is equivariantly diffeomorphic to a quotient of a free linear torus action of

$$Z_P = \prod_{i < r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i-1}, \quad n_i \geq 2,$$

where $Z_P$ is the moment angle complex corresponding to the polytope in Display (2.2).

In the proof of Theorem 2.21 the following lemma was important. It will also be useful for the proof of Theorem A.

**Lemma 2.22.** [23] Let $M^{2n}$ be a simply-connected torus manifold with an invariant metric of non-negative curvature. Then $M^{2n}$ is locally standard and $M^{2n}/T^n$ and all its faces are diffeomorphic (after smoothing the corners) to standard disks $D^k$. Moreover, $H^{\text{odd}}(M; \mathbb{Z}) = 0$.

We will also make use of the following theorem from [10].

**Theorem 2.23.** [10] Let $M$ be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an effective, isometric, isotropy-maximal torus action. Then $M$ is rationally elliptic.

The following corollary of Theorem 2.23 from [7] follows for a torus action via a simple adaptation of the proof in [10].

**Corollary 2.24.** Let $M$ be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an effective, isometric, almost isotropy-maximal torus action. Then $M$ is rationally elliptic.

The following Lemma from [7] gives us information about the structure of the quotient space, $P^{n-k}$ as well as a complete description of the corresponding isotropy groups.

**Lemma 2.25.** Let $T^k$ act isometrically, effectively and isotropy-maximally on $M^n$, a closed, non-negatively curved Riemannian manifold. Suppose that the quotient space $M/T$ is a disk. Then the following hold for the quotient space $P = M/T$:

1. All isotropy subgroups corresponding to boundary points on $P$ are connected; and
2. All boundary points correspond to singular orbits and all interior points correspond to principal orbits.
2.7. Lifting group actions. We now recall Theorems I.9.1 and I.9.2 of Bredon [1] which allows us to lift a group action to a covering space.

**Theorem 2.26.** [1] Let $G$ be a connected Lie group acting effectively on a connected, locally path-connected space $X$ and let $X'$ be any covering space of $X$. Then there is a covering group $G'$ of $G$ with an effective action of $G'$ on $X'$ covering the given action. Moreover, $G'$ and its action on $X'$ are unique.

The kernel of $G' \to G$ is a subgroup of the group of deck transformations of $X' \to X$. In particular, if $X' \to X$ has finitely many sheets, then so does $G' \to G$. If $G$ has a stationary point in $X$, then $G' = G$ and $\text{Fix}(X'; G')$ is the full inverse image of $\text{Fix}(X; G)$.

**Theorem 2.27.** [1] Let $G$ be a Lie group (not necessarily connected) acting on a connected and locally arcwise connected space $X$, and let $p : X' \to X$ be a covering space of $X$. Let $x_0' \in X'$ project to $x_0 \in X$ and suppose that $G$ leaves $x_0$, stationary. Then there exists a (unique) $G$-action on $X'$ leaving $x_0'$ stationary and covering the given action on $X$ if and only if the subgroup $p_*(\pi_1(X', x_0'))$ is invariant under the action of $G$ on $\pi_1(X, x_0)$.

Finally, we recall the following result of Su [22] (cf. Hattori and Yoshida [15]), which tells us that we can lift a torus action to a principal torus bundle when the base is simply-connected.

**Theorem 2.28.** [22] Let $P \to X$ be a principal $T^k$ bundle and suppose that $T^k$ acts on $X$. If $H^1(X; \mathbb{Z}) = 0$, then the $T^k$ action can be lifted to $P$.

3. Structure of almost torus manifolds

In this section, our goal is to prove the following odd-dimensional analog of Lemma 2.22.

**Theorem 3.1.** Let $M^{2n+1}$ be a closed, simply-connected, non-negatively curved almost torus manifold. Then the following hold.

1. The torus action on $M$ is almost locally standard.
2. $M/T$ is diffeomorphic (after smoothing the corners) to a standard disk $D^{n+1}$.
3. Any codimension one face of $M/T$ is diffeomorphic (after smoothing the corners) to either a standard disk $D^n$, or $S^1 \times D^{n-1}$.

**Remark 3.2.** Unlike torus manifolds, the quotient space $M/T$ may have codimension one faces that are not diffeomorphic to disks. Consider the fixed point homogeneous circle action on $S^3$, whose quotient space is $D^2$. The quotient space has only one 1-dimensional face: the boundary circle of $D^2$.

Before we begin the proof of Theorem 3.1, we establish some basic lemmas and propositions needed for its proof. Recall first that every almost torus manifold $M^{2n+1}$ contains a characteristic submanifold $F$ which corresponds to a codimension two component of the fixed point set of some circle subgroup of $T^n$, which we will denote by $\lambda(F)$, as in [23]. In particular, $M$ is a $\lambda(F)$-fixed point homogenous manifold, and by Theorem 2.16 we can decompose $M$ as follows:

$$M^{2n+1} = D(F) \cup E D(N).$$

where $E \cong \partial D(F) \cong \partial D(N)$ is the common boundary of the two disk bundles.

Moreover, since we assume $\pi_1(M^{2n+1}) = 0$, it follows by Lemma 2.18 that the codimension of $N$ is greater than or equal to 2 and via the same arguments as in the proof of Proposition 2.19 that $\pi_1(F) = 0$ provided codim$(N) > 2$. The Double Soul Theorem 2.17 implies that $\pi_1(F) = \pi_1(N) = 0$ when codim$(N) = 2$ and $N$ is also fixed by $\lambda(F)$. In all other cases, namely when codim$(N) = 2$, we know by Theorem 2.20 that the fundamental
groups of $F$ and $N$ are cyclic. We note further that whenever $\pi_1(F) = 0$, the fundamental group of $N$ will be generated by the $\lambda(F)$-orbit in $N$.

There is one more special case in which $\pi_1(F)$ is trivial, namely when $\text{codim}(N) = 2$ and $N$ is fixed by any other circle subgroup not equal to $\lambda(F)$:

**Proposition 3.3.** Let $M^{2n+1}$ be a simply-connected, non-negatively curved almost torus manifold with $n \geq 2$. Let $F$ be the characteristic submanifold fixed by $\lambda(F) \cong S^1 \subset T^n$, containing a connected component of $\text{Fix}(M,T^n)$. Suppose that $N^{2n-1}$ in the disk bundle decomposition given by Theorem 2.10 is fixed by any other $S^1$ subgroup of $T^n$. Then $\pi_1(F) = 0$.

**Proof.** Assume $\text{codim}(N) = 2$ and that $N$ is fixed by some $T^1 \subset T^n$ that is not $\lambda(F)$.

Let $T^2 := \lambda(F) \oplus T^1$. If $p_0$ is fixed by $T^n$, clearly $p_0 \in \text{Fix}(M,T^2)$. Since $\lambda(F)$ acts freely on the fiber of $E \to F$ we have $E / \lambda(F) \simeq F$. Likewise, the action of $T^1$ on the fiber of $E \to N$ is also free and $E / T^1 \simeq N$.

Consider the projections $\pi_x : E \to F$ and $\pi_y : E \to N$. From the homotopy sequences of these fibrations we obtain the following exact sequences:

$$\cdots \pi_1(\lambda(F)) \xrightarrow{\pi_{xx}} \pi_1(E) \xrightarrow{\pi_{xx}} \pi_1(F) \to 1$$

and

$$\cdots \pi_1(T^1) \xrightarrow{\pi_{xx}} \pi_1(E) \xrightarrow{\pi_{xx}} \pi_1(N) \to 1$$

where the maps $i_1$ and $i_2$ are the inclusions of the fibers over a given base point. Set $U_1 = \pi_\ast (\pi_1(\lambda(F)))$ and $U_2 = \pi_\ast (\pi_1(T^1))$. Then $\pi_1(F) \cong \pi_1(E) / U_1$ and $\pi_1(N) \cong \pi_1(E) / U_2$ and we have the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(E) & \xrightarrow{\pi_{xx}} & \pi_1(N) \\
\downarrow \pi_{xx} & & \downarrow \pi_{xx} \\
\pi_1(F) & \xrightarrow{h_1} & \pi_1(E) / U_1 U_2 \\
\end{array}$$

Here the lower map is given by $h_1 : \pi_1(F) \cong \pi_1(E) / U_1 \to \pi_1(E) / U_1 U_2$, and analogously for the map on the right.

Then since $M \simeq D(F) \cup_\partial D(N)$ with $D(F) \cap D(N) \simeq E$ connected, by Seifert-van Kampen there exist a unique homomorphism $h : \pi_1(M) \to \pi_1(E) / U_1 U_2$ making the following diagram commute:

$$\begin{array}{ccc}
\pi_1(E) & \xrightarrow{\pi_{xx}} & \pi_1(N) \\
\downarrow \pi_{xx} & & \downarrow h_2 \\
\pi_1(F) & \xrightarrow{h_1} & \pi_1(E) / U_1 U_2 \\
\end{array}$$

$$\begin{array}{ccc}
\pi_1(E) & \xrightarrow{\pi_{xx}} & \pi_1(N) \\
\downarrow \pi_{xx} & & \downarrow h \\
\pi_1(M) & \xrightarrow{h_1} & \pi_1(E) / U_1 U_2 \\
\end{array}$$

Since $h_1$ and $h_2$ are surjective, the induced map $h$ is also surjective. Since $\pi_1(M) = 0$, we have $\pi_1(E) \cong U_1 U_2$. Hence $\pi_1(E)$ is generated by the orbit $\lambda(F)(q)$ and $T^1(q)$ for a given point $q \in E$. Therefore the orbit map $\tau_q : T^2 \to E$, defined as $g \to g \cdot q$, induces a surjection $\tau_{xx} : \pi_1(T^2) \to \pi_1(E)$. Since $\text{Fix}(M,T^2)$ is non-empty, pick a point $x \in \text{Fix}(M,T^2)$ and $q_0 \in E$ such that $\pi_F(q_0) = x$. Then $\pi_F \circ \tau_{q_0}$ is the constant map, but $\pi_F \circ \tau_{q_0}$ also induces a surjection $\pi_1(T^2) \to \pi_1(F)$. Hence the induced map is the zero map and $\pi_1(F) = 0$. This completes the proof of the proposition.

□
Lemma 3.4. Let $M^{2n+1}$ be a simply-connected, non-negatively curved almost torus manifold with $n \geq 2$. Let $F$ be the characteristic submanifold fixed by $\lambda(F) \cong S^1 \subset T^n$, containing a connected component of $\text{Fix}(M; T^n)$. Let $N$ be the submanifold at maximal distance from $F$ given by the disk bundle decomposition of Theorem 2.16. We have the following possibilities for the fundamental groups of $F$ and $N$.

1. If $\text{codim}(N) > 2$, then $\pi_1(F) = 0$ and $\pi_1(E) \cong \pi_1(N)$ is cyclic.
2. If $\text{codim}(N) = 2$ and $N$ is fixed by any circle subgroup of $T^n$, then $\pi_1(F) = 0$.
3. If $\text{codim}(N) = 2$ and $N$ is not fixed by any circle subgroup of $T^n$, then $\pi_1(F)$ and $\pi_1(N)$ are both cyclic.

Note that in all cases the fundamental groups of $F$ and $N$ are cyclic groups.

The following lemma gives us important information about the structure of $F$ if it is not simply-connected, as in Part 3 of Lemma 3.4.

Lemma 3.5. Let $M^{2n+1}$ be a closed, almost torus manifold of non-negative curvature. Assume that $\pi_1(M)$ is cyclic and denote by $\tilde{M}$ the Riemannian universal cover of $M$. Then

1. If $\pi_1(M) \cong \mathbb{Z}_k$, $\tilde{M}$ is a closed, simply-connected almost torus manifold of non-negative curvature;
2. If $\pi_1(M) \cong \mathbb{Z}$, $\tilde{M} = \tilde{M} \times \mathbb{R}$, and $\tilde{M}$ is a closed, simply-connected torus manifold of non-negative curvature.

Proof. We break the proof into two cases: Case (1), where $\pi_1(M) = \mathbb{Z}_k$ and Case (2), where $\pi_1(M) = \mathbb{Z}$.

We begin with Case (1), where $\pi_1(M) = \mathbb{Z}_k$. Then the universal cover of $M$, $\tilde{M}$, by Theorem 2.15, is a closed, simply-connected, non-negatively curved $2n+1$-dimensional manifold admitting a $T^n$ action. By Theorem 2.16, since $T^n$ has non-empty fixed point set, it follows that the $T^n$ action lifts to a $T^n$ action on $\tilde{M}$, and hence $\tilde{M}$ is almost torus.

We now consider Case (2), where $\pi_1(M) = \mathbb{Z}$. In this case, by Theorem 2.15, the universal cover of $M$, $\tilde{M} = \tilde{M} \times \mathbb{R}$, and $\tilde{M}$ is a closed, simply-connected $2n$-dimensional manifold of non-negative curvature. By Theorem 2.16, since $T^n$ has non-empty fixed point set, it follows that the $T^n$ action lifts to a $T^n$ action on $\tilde{M}$. Furthermore, any non-trivial orbit of the $T^n$ action on $\tilde{M}$ must lie entirely in the $\mathbb{R}$ factor, which can be seen as follows. Any such orbit is compact and thus its image under the projection $\pi_2$ onto the second factor $\mathbb{R}$, must be compact, and hence either a point or an interval. Since $\pi_2$ is continuous and surjective, the image of the principal orbits under $\pi_2$ must be dense in $\mathbb{R}$. But the image of the principal orbits must then be the disjoint union of closed intervals which cannot be dense in $\mathbb{R}$.

In particular, this tells us that $\tilde{M}$ is invariant under the $T^n$ action and fixes the $\mathbb{R}$ factor.

Since the lifted $T^n$ action on $\tilde{M}$ is almost isotropy-maximal and trivial on the $\mathbb{R}$ factor, it follows immediately that the $T^n$ action is isotropy-maximal on $\tilde{M}$, and $\tilde{M}$ is torus.

Recall that the proof of Lemma 6.3 in [2] is by induction on the dimension of $M$ and relies on the fact that the characteristic submanifold $F$ is itself a closed, simply-connected, non-negatively curved torus manifold. The proof of Theorem 3.1 will also be by induction on the dimension of $M$. However, while the characteristic submanifold $F$ of an almost torus manifold is a closed, non-negatively curved almost torus manifold, it may not be simply-connected, so we must show separately that the induction hypothesis holds for this case.

Proof of Theorem 3.1. As mentioned earlier, the proof of the Theorem is by induction on the dimension of $M$. We therefore prove the anchor of the induction first. Let $n = 1$. Then we
have an isometric, effective action of $T^1$ on $M^3$, a closed, simply-connected, non-negatively curved almost torus manifold. The action of the circle is $S^1$-fixed point homogeneous and we know by work of Galaz-García [3] that $M^3$ is equivariantly diffeomorphic to $S^3$ with a linear $T^1$ action and decomposes as a union of disk bundles over $F = S^1 = N$, with $F$ fixed by $T^1$ and $N$ a $T^1$ orbit with trivial isotropy. By work of Bredon [11], its quotient space is $D^2$. That is, the quotient is diffeomorphic to $D^2$ and the codimension 1 face is diffeomorphic to $S^1$.

To show that the $T^1$-action is almost locally standard, we note that since $F$ is a fixed circle in $M^3$, the Slice theorem gives us directly that the action in a neighborhood of $F$ is almost locally standard. Since $N$ corresponds in this case to a $\lambda(F)$ orbit with trivial isotropy, it is clear that the action is almost locally standard in a neighborhood of $N$. This finishes the proof of the anchor of the induction.

We now assume that the result holds for any closed, simply-connected, non-negatively curved almost torus manifold of dimension $2k + 1$ with $k < n$. By Theorem 2.16 we have a decomposition

$$M = D(N) \cup E D(F),$$

where $F$ is a characteristic submanifold of $M$ and $E$ is the total space of the $S^1$-bundle associated to the normal bundle of $F$.

Since $F$ is a totally geodesic submanifold of $M$, it is non-negatively curved. Moreover, it admits a $T^{n-1}$ action that is almost isotropy-maximal. If it is simply-connected, then it satisfies the induction hypothesis and the $T^{n-1}$ action is almost locally standard, $F/T^{n-1}$ is diffeomorphic to a standard disk $D^n$, and any codimension one face of $F/T^{n-1}$ is diffeomorphic to either a standard disk $D^{n-1}$, or $S^1 \times D^{n-2}$.

In the case where $F$ is not simply-connected (and hence a non-trivial cyclic group by Lemma 3.4), we have the following proposition.

**Proposition 3.6.** Let $F^{2n-1}$ be the characteristic submanifold of $M^{2n+1}$, a closed, simply-connected, almost torus manifold. Suppose that $\pi_1(F) \neq 0$. Then, the following hold.

1. If $\pi_1(F)$ is finite, then $F/T^{n-1}$ is diffeomorphic (after smoothing the corners) to $D^n$, and the induced $T^{n-1}$ action on $F$ is almost locally standard.
2. If $\pi_1(F) \cong \mathbb{Z}$, then $F/T^{n-1}$ is diffeomorphic (after smoothing the corners) to $S^1 \times D^{n-1}$ and the induced $T^{n-1}$ action on $F$ is almost locally standard.

**Proof of Proposition 3.6.** Now, using Lemma 3.5, we see that in the case where $\pi_1(F) \cong \mathbb{Z}_k$, it follows that the universal cover of $F$, $\tilde{F}$, is a closed, simply-connected, non-negatively curved almost torus manifold and hence by the induction hypothesis the $T^{n-1}$ action on $\tilde{F}$ is almost locally standard and $\tilde{F}/T^{n-1}$ is diffeomorphic to a disk. By letting $p : \tilde{F} \rightarrow F$, $p_F : F \rightarrow F/T^{n-1}$, and $p_F : \tilde{F} \rightarrow F/T^{n-1}$ it follows that the induced map

$$p_F \circ p \circ p_F^{-1} : \tilde{F}/T^{n-1} \rightarrow F/T^{n-1}$$

is a local homeomorphism between compact Hausdorff spaces, hence it is a covering map. Since $\tilde{F}/T^{n-1}$ is diffeomorphic to a disk, it must be a regular covering map. If the covering is $m$-sheeted, then there exists a non-trivial element in the deck group acting on $m$ points of the fiber. But by Brouwer’s fixed point theorem any such transformation should have a fixed point in the disk, so it must be trivial, as all deck transformations act freely. Hence $m = 1$ and $p_F \circ p \circ p_F^{-1}$ is a diffeomorphism (after smoothing the corners) and the result follows.

Since being almost locally standard is a local condition, it immediately follows that the $T^{n-1}$ induced action on $F$ is almost locally standard.
In the case where \( \pi_1(F) \cong \mathbb{Z} \), Lemma 3.5 implies that the universal cover of \( F \) splits isometrically as \( \tilde{F} \times \mathbb{R} \), where \( \tilde{F}^{2n-2} \) is a closed, simply-connected, non-negatively curved torus manifold with the \( T^{n-1} \) action. In fact, the proof of Lemma 3.5 shows that \( T^{n-1} \) acts trivially on the \( \mathbb{R} \) factor. Using Lemma 2.22 it follows that the action of \( T^{n-1} \) on \( \tilde{F} \) is locally standard and that \( \tilde{F}/T^{n-1} \) is diffeomorphic to a disk. Thus the action of \( T^{n-1} \) on \( \tilde{F} \) is almost locally standard and \( \tilde{F}/T^{n-1} \simeq D^{n-1} \times \mathbb{R} \). Since \( p \circ p \circ p^{-1} : \tilde{F}/T^{n-1} \to F/T^{n-1} \) is a covering map and \( F \) is closed, it follows that \( F/T^{n-1} \) is diffeomorphic (after smoothing the corners) to \( D^{n-1} \times S^1 \). Moreover, the induced action of \( T^{n-1} \) on \( F \) is locally standard, and hence it follows that the induced action of \( T^{n-1} \) on \( \tilde{F} \) is also locally standard. As above, it follows that the induced \( T^{n-1} \) action on \( F \) is almost locally standard. \( \square \)

Proposition 3.6 and the induction hypothesis tell us that the \( T^{n-1} \) action on \( F \) is almost locally standard and that its quotient space is diffeomorphic to \( D^n \), except when \( \pi_1(F) \cong \mathbb{Z} \), in which case, it is diffeomorphic to \( D^{n-1} \times S^1 \). We will now show that the action of the torus on \( N \) is also locally standard and that the quotient space \( N/T \) is diffeomorphic to a disk.

To prove this, we consider the image of a \( T^n \)-orbit, \( \pi_F^{-1}(x) \), where \( x \in \text{Fix}(M; T^n) \cap F \). We have two cases:

Case 1: \( \dim(\pi_N(\pi_F^{-1}(x))) = 0 \);
Case 2: \( \dim(\pi_N(\pi_F^{-1}(x))) = 1 \).

We begin with Case 1, where \( \dim(\pi_N(\pi_F^{-1}(x))) = 0 \). We first prove the following proposition.

**Proposition 3.7.** Let \( M^{2n+1} \) be an almost torus manifold and let \( x \in \text{Fix}(M; T^n) \cap F \). Suppose that \( \dim(\pi_N(\pi_F^{-1}(x))) = 0 \). Then both \( F \) and \( N \) are simply-connected. Moreover, \( n \geq 2 \) and \( N \) is a closed, non-negatively curved almost torus manifold, fixed by a subtorus \( T^l \subset T^n \), \( 1 \leq l < n \), with \( 2l = \text{codim}(N) \) in \( M \).

The proof of this proposition is similar to the argument made for this case in [23]. We include it here for the sake of completeness.

**Proof.** Let \( \pi_N(\pi_F^{-1}(x)) = y \in N \). Since both \( \pi_F \) and \( \pi_N \) are equivariant maps, it follows that \( y \) is a \( T^n \)-fixed point. Since \( M \) is \((2n+1)\)-dimensional, it follows that the \( T^n \)-fixed point set component containing \( y \) is 1-dimensional and hence a circle, which we will denote by \( C \). Moreover, we recall that by Theorem 2.16 all singularities of the \( \lambda(F) \)-action are contained in \( F \) and in \( N \). Since \( C \) is fixed by \( \lambda(F) \) it must be contained in \( N \).

The slice representation of \( T^n \) at \( y \) is faithful, since the \( T^n \)-action on \( M \) is effective. Then, up to an automorphism of \( T^n \), the \( T^n \)-representation on \( T_y M \) is given by the standard representation on \( \mathbb{R}^{2n+1} \). There is an \( \mathbb{R} \) factor that is tangent to \( C \), on which the action is trivial. Thus we can write

\[
T_y M = \mathbb{C}^n \oplus T_y C,
\]

where we identify the normal space, \( T_y^l C \), with \( \mathbb{C}^n \), on which the \( T^n \)-representation is faithful. Because \( N \) is \( T^n \) invariant, it is clear that

\[
T_y N \cong \mathbb{C}^{n-l} \oplus T_y C
\]

for some \( l \leq n \) and the subtorus \( T^l \) that acts on the complement of \( \mathbb{C}^{n-l} \) fixes \( T_y N \). Hence, \( N \) is fixed by \( T^l \) and is of codimension \( 2l \) in \( M \), \( 1 \leq l \leq n \). Moreover, Lemma 3.3 and the Double Soul Theorem 2.17 now allows us to conclude that \( \pi_1(F) = 0 \) in this case.

It remains to show that \( \pi_1(N) = 0 \). However, since \( F \) is simply connected, it follows from the long exact sequence in homotopy of the fibration \( S^1 \xrightarrow{i} E \to F \) that \( i_* \) is onto \( \pi_1(E) \). Thus \( \pi_1(E) \) is cyclic and generated by the inclusion of the fiber.
It then follows from the long exact sequence in homotopy of the fibration $S^{2l-1} \to E \to N$, that $\pi_2$ is onto and thus $\pi_1(N)$ is generated by the inclusion of an $S^1$-orbit from $E$ to $N$, by the loop $\gamma_{x_0} : S^1 \to N$, $z \mapsto zx_0$, where $x_0 \in N$ is any base point of $N$. Since $\pi_N(\pi_F^{-1}(x)) = y$ is 0-dimensional, this implies that the map $\gamma_{x_0}$ is constant for $y = x_0$, and hence $\pi_1(N) = 0$ as desired. Finally, we note that since $N$ is simply-connected, $N$ cannot be of dimension 1.

We can now prove Parts 1 and 2 of Theorem 3.1 for Case 1.

**Proof of Parts 1 and 2 of Case 1 of Theorem 3.1** Since both $F$ and $N$ are closed, simply-connected, non-negatively curved almost torus manifolds of dimension $< 2n + 1$, we can apply the induction hypothesis to both $F$ and $N$ to show that the corresponding induced torus actions on both are almost locally standard. Moreover, the quotients of both are diffeomorphic to disks. Since both $N$ and $F$ are fixed by some subtorus of $T^n$, and the action of the subtorus on the unit normal sphere to each is of maximal symmetry rank, this then implies that the torus action on $D(F)$ and on $D(N)$ is almost locally standard and

\[
\begin{align*}
D(N)/T &\simeq N/T \times \Delta^l \simeq D^{n+1}, \\
D(F)/T &\simeq F/T \times I \simeq D^{n+1}.
\end{align*}
\]

Since $E/T = F/T$ is also diffeomorphic to a disk, it follows that $M/T$ is as well and the result holds for Case 1.

We now proceed to prove Theorem 3.1 for Case 2, where $\dim(\pi_N(\pi_F^{-1}(x))) = 1$. We will first show that there are two sub-cases to consider.

**Proposition 3.8.** Let $T^n$ act effectively, isometrically on $M^{2n+1}$, a closed, simply-connected, non-negatively curved almost torus manifold, and let $x \in \text{Fix}(M,T^n) \cap F$. Suppose that $\dim(\pi_N(\pi_F^{-1}(x))) = 1$. Then $T^n$-action fixes $N$ and either

(2.a) $\dim(N) = 2k + 2$, $0 \leq k \leq n - 2$; or
(2.b) $\dim(N) = 2k + 1$, $0 \leq k \leq n - 1$.

**Remark 3.9.** In Case 2.b, we see that when $k = n - 1$, $N$ is not fixed by any circle subgroup of $T^n$. Thus we see that in all cases except possibly for $k = n - 1$ in Case 2.b that $\pi_1(F)$ must be trivial by Part 1 of Lemma 3.4.

**Proof of Proposition 3.8** Since $\pi_F$ is an equivariant map, and $x \in F$ is a $T^n$ orbit, it follows that $\pi_F^{-1}(x) \subset E$ is a $T^n$ orbit with $T' = T^{n-1} = T^n/\lambda(F)$ isotropy.

Since the $T^n$-action on $M$ is effective, it follows from the Slice Theorem that there is an invariant neighborhood of $\pi_F^{-1}(x)$ in $E$ which is equivariantly diffeomorphic to

\[
\lambda(F) \times \mathbb{C}^{n-1} \times \mathbb{R}^2.
\]

Since $E$ has an invariant collar in $D(F)$ and $D(N)$, there is an $\mathbb{R}$-factor that is normal to $E$, and so the $\mathbb{C}^{n-1}$ is a faithful $T' = T^{n-1}$-representation and $\mathbb{R}^2$ is a trivial representation.

Since $\pi_F$ is an equivariant map, it follows that $\pi_N(\pi_F^{-1}(x))$ is a $T^n$ orbit of type $T^n/(H_0 \times T')$, where $H_0 \subset \lambda(F)$, is a finite subgroup. By a similar argument as above, we see that $\pi_N(\pi_F^{-1}(x))$ has an invariant neighborhood in $M$ diffeomorphic to

\[
\lambda(F) \times H_0 \mathbb{C}^{n-1} \times \mathbb{R}^2.
\]
where $T'$ acts effectively on $\mathbb{C}^{n-1}$ and the $H_0$-action on $\mathbb{C}^{n-1} \times \mathbb{R}^2$ commutes with the $T'$-action. Moreover, there is an $\mathbb{R}$-factor that is normal to $N$ because there was an $\mathbb{R}$-factor in Display (3.1) normal to $E$ and $\pi_N$ is an equivariant submersion. Then, as we saw before, restricting the tangent space of $N$ to the orbit $\pi_N(\pi_F^{-1}(x))$, which is diffeomorphic to $\lambda(F)/H_0$, we obtain an invariant subbundle of the restriction of $M$ to this orbit. Moreover, all invariant subbundles are of the form

$$\lambda(F) \times_{H_0} \mathbb{C}^k \times \mathbb{R}^l,$$

where $0 \leq k \leq n-1$ and $0 \leq l \leq 2$. Since one $\mathbb{R}$-factor is normal to $N$, the result follows. $\square$

We now proceed to prove Theorem 3.1 for Case 2.a, where $T^{n-1-k}$ fixes $N$ and $\text{dim}(N) = 2k + 2$, $0 \leq k \leq n-2$. We begin with the following proposition.

**Proposition 3.10.** Let $T^n$ act isometrically and effectively on $M^{2n+1}$, a closed, simply-connected almost torus manifold. Suppose that $T^{n-1-k}$ fixes $N$ and $\text{dim}(N) = 2k + 2$, $0 \leq k \leq n-2$. Then $\lambda(F)$ acts freely on $N$ and $N/\lambda(F)$ is a closed, simply-connected, non-negatively almost torus manifold of dimension $2k + 1$.

**Proof.** Suppose that $\lambda(F)$ does not act freely on $N$. Let $\Gamma$ be a subgroup of $\lambda(F)$ fixing a point $x \in N$. Then $\Gamma$ must act freely on the $S^{2n-2k-2}$ fiber of $E$ over $x \in N$, since $\lambda(F)$ acts freely on $E$. But since $\Gamma$ is a subgroup of $\lambda(F)$, it acts by orientation preserving isometries and hence must fix a point in the fiber, giving us a contradiction.

Recall that $\pi_1(N)$ is generated by the $\lambda(F)$ orbit, hence the closed, non-negatively curved $(2k + 1)$-dimensional manifold, $N/\lambda(F)$, must have trivial fundamental group. Moreover, the image of $\pi_N(\pi_F^{-1}(x))$ in $N/\lambda(F)$ is a point that is invariant under the induced $T^k = (T^n/T^{n-k-1})/\lambda(F)$-action and the result holds. $\square$

The proof of the following lemma is a straightforward generalization of the analogous result for Case (ii) contained in the proof of Lemma 2.22 of [23] and is left to the reader.

**Lemma 3.11.** With the hypotheses as above,

$$E \simeq P \times_{T^{n-k-1}} S^{2n-2k-2},$$

where $P$ is the principal $T^{n-k-1}$-bundle associated to $E \to N$. Moreover, the $T^{k+1}$-action on $N$ lifts to an action on $P$. Together with the $T^{n-k-1}$-action on $S^{2n-2k-2}$, this action induces the $T^n$-action on $E$.

Now, since $N/T = (N/\lambda(F))/T/\lambda(F)$, the induction hypothesis tells us that $N/T$ is diffeomorphic to a disk, as desired. To show that $D(N)$ is almost locally standard in this case, we must generalize the proof of the analogous result for Case (ii) in Lemma 6.3 of [23]. This is done using Lemma 3.11 and we summarize the results in the following lemma.

**Lemma 3.12.** Let $M^{2n+1}$ be a closed, simply-connected, non-negatively curved almost torus manifold. Suppose that $T^{n-1-k}$ fixes $N$ and $\text{dim}(N) = 2k + 2$, $0 \leq k \leq n-2$. Then $M$ is almost locally standard in a neighborhood of $N$ and $N/T$ is diffeomorphic to a disk.

We can now prove Parts 1 and 2 of Theorem 3.1 for Case 2.a.

**Proof of Parts 1 and 2 of Case 2.a of Theorem 3.1** Since $\text{codim}(N) > 2$ in Case 2.a, Lemma 3.4 and the induction hypothesis immediately yield that $F/T$ is diffeomorphic to a disk and the induced $T^{n-1}$ action on $F$ is almost locally standard. Then, as in Case 1, we see that the $T^n$ action on $D(F)$ is almost locally standard. Lemma 3.12 implies that the $T^n$ action on $D(N)$ is almost locally standard, hence it is almost locally standard on all of $M$. 

Again, as in Case 1, we see that $D(F)/T \simeq D^{n+1}$. Moreover, $E/T \simeq F/T$ is also a disk, so to show that $M/T$ is a disk it suffices to show that $D(N)/T$ is diffeomorphic to a disk. Since

$$D^n \simeq F/T \simeq E/T \simeq (E/T^{n-k-1})/T^{k+1} \simeq (P \times_{T^{n-k-1}} \Sigma^{n-k-1})/T^{k+1},$$

and

$$(P \times_{T^{n-k-1}} \Sigma^{n-k-1})/T^{k+1} \simeq N/T^{k+1} \times \Sigma^{n-k-1} \simeq N/T^{k+1} \times D^{n-k-1}.$$

We then obtain a diffeomorphism

$$D^{n+1} \simeq N/T^{k+1} \times D^{n-k} \simeq N/T^{k+1} \times \Sigma^{n-k} \simeq D(N)/T.$$

This completes the proof of Theorem 3.1 for Case 2a. □

We now consider Case 2.b, where $T^{n-1-k}$ fixes $N^{2k+1}$, $0 \leq k \leq n-1$. We will first prove the following lemma.

**Lemma 3.13.** Let $M^{2n+1}$ be a closed, simply-connected, non-negatively curved almost torus manifold. Suppose that $T^{n-1-k}$ fixes $N$ and $\dim(N) = 2k+1$, $0 \leq k \leq n-1$. Then $N$ is a principal $\lambda(F)$-bundle over $N/\lambda(F)$, a simply connected, non-negatively curved torus manifold with the induced $T^k$ action. Moreover, $N/T^n$ and all of its faces are diffeomorphic to standard disks.

**Proof.** Let $x \in \text{Fix}(M; T^n) \cap F$. Note that the orbit of $\pi_N(\pi_F^{-1}(x))$ is of type $\lambda(F)/H_0$, where $H_0$ is a finite subgroup of $\lambda(F)$, and is invariant under the $T^k$ action and hence fixed by $T^k$. In this particular case, this implies that the action of $T^{k+1}$ on $N^{2k+1}$ is isotropy-maximal. Hence Theorem 2.23 implies that $N^{2k+1}$ is rationally $\Omega$-elliptic.

We also note that $\lambda(F)$ acts almost freely on $N^{2k+1}$ and since $\lambda(F)$ contains the fundamental group of $N$, it follows that the quotient space, $X^{2k}$, is a simply-connected, non-negatively curved, torus orbifold. This follows by noting that $N/\lambda(F) = X^{2k}$ corresponds to the soul without boundary in the proof of Theorem 2.16. Hence $X^{2k}$ is a non-negatively curved Alexandrov space.

We now claim that $X^{2k}$ is in fact a torus manifold. Note first that the quotient of $X^{2k}/T^k = N^{2k+1}/T^{k+1} = P^k$, is a simply-connected, non-negatively curved Alexandrov space with boundary. Given the structure of the $T^k$ action on $X^{2k}$, it follows by a simple adaptation of the proof of Lemma 2.25 that all points lying in the interior of $P^k$ must be regular points and thus these points correspond to principal orbits of the $T^{k+1}$ action on $N$.

Note that $P^k = X^{2k}/T^k$ has the same face poset as a product of simplices and suspended simplices by work of [10]. Note that any vertex in the boundary corresponds to a circle with $T^k$ isotropy in $N^{2k+1}$ and thus since the action of the $T^k$ is isotropy-maximal, we see that all isotropy subgroups corresponding to points on $\partial P^k$ are connected. In particular, this shows us that the $\lambda(F)$ action must be free and therefore $N/\lambda(F)$ is a simply-connected, non-negatively curved, torus manifold.

In particular, it now follows from Lemma 2.22 that $P^k = X^{2k}/T^k = N^{2k+1}/T^{k+1}$ and all its faces are diffeomorphic to disks (after smoothing corners). □

It remains to show that the $T^n$ action on a neighborhood of $N^{2k+1}$ is almost locally standard. However, this follows by using the bundle charts for $N$, which we have just seen is a principal $\lambda(F)$-bundle, and the fact that $T^{n-1-k}$ fixes $N$ and we have the following lemma.

**Lemma 3.14.** Let $M^{2n+1}$ be a closed, simply-connected, non-negatively curved almost torus manifold. Suppose that $\pi_N(\pi_F^{-1}(x))$ is of dimension 1, and $N^{2k+1}$ is fixed by $T^{n-k-1}$,
0 \leq k \leq n - 1 \text{ and that } N \text{ is the total space of a principal } S^1\text{-bundle over a locally standard } T^k\text{-manifold, } B^{2k}. \text{ Then the } T^n\text{-action on } D^{2(n-k)}(N) \text{ is almost locally standard.}

We can now prove Parts 1 and 2 of Theorem 3.1 for Case 2.b.

**Proof of Parts 1 and 2 of Case 2.b of Theorem 3.1** We will now show that the } T^n\text{ action on } M^{2n+1} \text{ is almost locally standard and that } M/T \text{ is diffeomorphic to a disk.}

We first prove that the } T^n\text{ action is almost locally standard. Since the induced action on } F \text{ is almost locally standard and since } F \text{ is fixed by } \lambda(F), \text{ it is clear that the } T^n\text{ action is almost locally standard on a neighborhood of } F. \text{ Since it is also almost locally standard on a neighborhood of } N \text{ by Lemma 3.13, we see that the action is almost locally standard on } M.

In order to show that the quotient } D(N)/T \text{ is diffeomorphic to a disk, we have two cases to consider:}

Case 2.b.i, where } k = n - 1, \text{ that is, } \dim(N) = 2n - 1;

Case 2.b.ii, where } k < n - 1, \text{ that is, } \dim(N) < 2n - 1.

However, in Case 2.b.i, by Proposition 3.6, we see that } F/T \text{ is either } D^n \text{ or } D^{n-1} \times S^1 \text{ and by Lemma 3.13, } N/T = D^{n-1}.

Then, since } F \text{ is fixed by } \lambda(F), \text{ it follows that either

\[ D(F)/T = F/T \times I \cong D^n \times I \cong D^{n+1}. \]

or

\[ D(F)/T = F/T \times I \cong D^{n-1} \times S^1 \times I. \]

Whereas, since } N \text{ is not fixed by any subtorus of } T^n, \text{ we have

\[ D(N)/T \cong N/T \times D^2 \cong D^{n+1}. \]

In the first case, since } E/T = F/T \text{ is also diffeomorphic to a disk, it is clear that } M/T \text{ is diffeomorphic to } D^{n+1}. \text{ In the second case, since } E/T = F/T \text{ is also diffeomorphic to } D^n \times S^1, \text{ it follows that the orbit space } M/T \text{ will be } D^{n-1} \times C(S^1) \cong D^{n+1}, \text{ as desired.}

In case 2.b.ii, Lemma 3.14 implies that } F \text{ is simply-connected, so by the induction hypothesis, we have that the torus action on } F \text{ is almost locally standard and } F/T \text{ is diffeomorphic to a disk. Then, since } F \text{ is fixed by } \lambda(F), \text{ it follows that

\[ D(F)/T = F/T \times I \cong D^n \times I \cong D^{n+1}. \]

Consider the } T^{n-1-k} \text{ action that fixes } N \text{ on the unit normal sphere of } N \text{ in } M. \text{ Since this action is almost isotropy-maximal, we obtain that

\[ D(N^{2k+1})/T \cong N^{2k+1}/T^{k+1} \times (\Delta^{n-k} \times I) \cong D^{n+1}. \]

Again, since } E/T = F/T \text{ is also diffeomorphic to a disk, the orbit space } M/T \text{ will be diffeomorphic to } D^{n+1}, \text{ as desired.}

\[ \square \]

It remains to prove Part 3 of Theorem 3.1. However, since the codimension one faces, } F^n \text{ all correspond to a codimension two fixed point set of a circle, they are themselves non-negatively curved almost torus manifolds. Moreover, by Lemma 3.14, the fundamental group of the corresponding codimension two fixed point set is cyclic. Hence, the result follows immediately from the induction hypothesis if the corresponding codimension two fixed point set is simply-connected. Likewise, in the case where } \pi_1(F) \text{ is non-trivial, the result follows from Proposition 3.6. This then completes the proof of Theorem 3.1.}

\[ \square \]
4. Extending the torus action by a smooth, commuting circle action

The goal of this section is to show that we can always find a smooth $T^1$ action that commutes with the isometric $T^n$ action on the closed, simply-connected, non-negatively curved almost torus manifold. This $T^1$ action then extends the $T^n$ isometric action to a smooth $T^{n+1}$ action.

Recall that by Theorem 2.16, $M$ decomposes as

$$M = D(F) \cup_E D(N),$$

where $F$ is the characteristic submanifold fixed by $\lambda(F)$, $N$ is a $T^n$-invariant submanifold at maximal distance from $F$, and $E$ is the common boundary of their respective disk bundles.

We will prove the following theorem.

**Theorem 4.1.** Let $M^{2n+1}$ be a closed, simply-connected, non-negatively curved almost torus manifold. Then we can find a unique, smooth $S^1$ action that commutes with the isometric $T^n$-almost isotropy-maximal action on $M^{2n+1}$.

The proof is by induction on $n$. There are 3 separate cases to consider, as in the proof of Theorem 3.1. As before consider the image of a $T^n$-orbit, $\pi^{-1}_F(x)$, where $x \in \text{Fix}(M; T^n) \cap F$.

Case 1: $\dim(\pi_N(\pi^{-1}_F(x))) = 0$.

Case 2: $\dim(\pi_N(\pi^{-1}_F(x))) = 1$.

Case 2.a: $N^{2k+2}$ is fixed by $T^{n-k-1} \subset T^n$, $0 \leq k \leq n - 2$.

Case 2.b: $N^{2k+1}$ is fixed by $T^{n-k-1} \subset T^n$, $0 \leq k \leq n - 1$ and when $k = n - 1$, $N$ is not fixed by any circle.

Recall that in Case 1, both $F$ and $N$ are simply-connected almost torus manifolds, with $\dim(N) \leq 2n - 1$. In Case 2, $F$ is simply-connected in all cases, except in Case 2.b, when $\dim(N) = 2n - 1$, in which case both $F$ and $N$ have (possibly non-trivial) cyclic fundamental group.

We will first show that the extension, if it exists, is unique.

**Proposition 4.2.** Let $T^k$ act smoothly and effectively on $M^n$ so that the action is almost isotropy-maximal. Then if the $T^k$ action admits a non-trivial extension to a smooth and effective isotropy-maximal $T^{k+1}$ action, then the extension is unique.

**Proof.** Suppose instead that there exist two different effective $T^{k+1}$ extensions of the $T^k$ action. Let each be denoted by $T^k \times T^1_i$, $i = 1, 2$, with $T^1_1$ and $T^1_2$ distinct smooth and effective circle actions on $M^n$.

Since principal orbits of any $G$-action are dense, we can find a point $p \in M$ such that $T^1_1(p)$ and $T^1_2(p)$ are both principal.

Now consider $T^1_1(p) \cap T^1_2(p)$. We claim that this intersection can only be the orbits themselves, in which case the two actions coincide, or it is a union of isolated points. In the latter case, we will show that the two $T^{k+1}$ actions then generate an effective $T^{k+2}$ action, which contradicts the fact that each $T^{k+1}$ action is isotropy-maximal and hence is the largest possible effective torus action on $M$, by Lemma 2.3. This will then complete the proof of the proposition.

To prove the claim, we note that as $M$ is Hausdorff, the intersection of two 1-dimensional compact sets will be closed and hence compact. Hence the orbits coincide or consist of a (finite) union of closed intervals and/or isolated points. If the intersection contains an interval, then this is equivalent to saying that the two groups intersect in a neighborhood of the identity. However, that implies that the groups are the same, as desired.

Suppose then that the orbits only intersect in a finite number of points. We will prove in this case that the two $T^{k+1}$ actions then generate an effective $T^{k+2}$ action, which will
contradict the fact that the either $T^{k+1}$ action is isotropy-maximal. Note that around every point of the intersection, $q \in T^i_1(p) \cap T^j_1(p)$, we can find an open neighborhood $U$ of $q$ and, without loss of generality, let $U \subset T^i_1(p)$, such that $U$ contains no other points of the intersection. We can assume by density of the principal orbits that no point in $T^i_1(p)$ is fixed by $T^j_2$. It follows that $T^i_1(U) \simeq I \times S^1$, and since the intersection is a finite set of points, we see that $T^i_1(T^j_1(p))$ is an $S^1$ bundle over $S^1$. Since the actions are orientable, it must be $T^2$ and the proof is complete. □

We now proceed to prove Theorem 4.1. The proof is by induction on the dimension and is separated into the cases listed above.

We begin by establishing the anchor of the induction in the following lemma.

**Lemma 4.3.** An isometric $T^1$ fixed point homogeneous action on $M^3$ of non-negative curvature may be extended to a smooth $T^2$ action.

**Proof.** The proof follows by using the construction contained in the proof of Proposition 3.6 of [11]. □

We assume that for any closed, simply-connected, non-negatively curved almost torus manifold, $M^{2k+1}$, the isometric $T^k$ action, with $k < n$, may be extended to a smooth, effective, isotropy-maximal $T^{k+1}$ action. We now proceed to prove Theorem 4.1 for each of the cases listed above.

**Proof of Case 1 of Theorem 4.1.** By the induction hypothesis, since both $F$ and $N$ are closed, simply-connected, non-negatively curved almost torus manifolds of dimension $2k+1$, with $k < n$, we may extend the induced isometric $T^{n-1}$ action on $F$ to a smooth, effective, isotropy-maximal $T^n$ action, and the induced isometric $T^{n-l}$ action, $1 \leq l \leq n-1$, on $N^{2n-2l+1}$ to a smooth, effective, isotropy-maximal $T^{n-l+1}$ action.

By Theorem 2.28, we may lift the $T^i$ action on $F$ to $E$, to obtain a smooth $T^{n+1}$ action on $E$. Since the disk bundle $E \times_{\chi(F)} D^2$ associated to the $S^1$-principal bundle $E \to F$ is diffeomorphic to $D(F)$, we can extend the action to all of $D(F)$. Note that $E$ is an $S^{2l-1}$ bundle over $N$ with structure group $T^l$ and we can write $E$ as $P \times_{T^l} S^{2l-1}$, where $P$ is a principal $T^l$ bundle over $N$. So, we may lift the smooth $T^{n-l+1}$ action on $N$ to $P$ by Theorem 2.28, $P$, and hence to the associated bundle, $E = P \times_{T^l} S^{2l-1}$. We then extend the smooth action to $D(N)$ as above. That is, $D(N)$ is diffeomorphic to $P \times_{T^l} D^{2l}$. Since the extended actions agree on their respective boundaries by Proposition 4.2, we obtain a smooth $T^{n+1}$ action on all of $M$.

**Proof of Case 2.a of Theorem 4.1.** In this case, note that $F$ is once again simply-connected, so the argument of Case 1 applies here to $D(F)$. Moreover, $N^{2k+2}$ is a principal $S^1$ bundle over a closed, simply-connected, non-negatively curved, almost torus manifold, $N^{2k+1}$. Thus, by the induction hypothesis, we may extend the isometric $T^k$ action on $N^{2k+1}$ to a smooth $T^{k+1}$ action on $N^{2k+1}$ and by Theorem 2.28, we may lift the $T^{k+1}$ action to $N^{2k+2}$ to obtain a smooth $T^{k+2}$ action on $N^{2k+2}$. We then proceed as we did in Case 1 to show that there is a smooth $T^{n+1}$ action on $D(N)$ and we are done.

**Proof of Case 2.b of Theorem 4.1.** As before we separate this case into two subcases: Case 2.b.i, where $N^{2n-1}$ is not fixed by any circle action, and Case 2.b.ii, where $N^{2k+1}$ is fixed by $T^{n-k-1} \subset T^n$, $0 \leq k < n-2$. In Case 2.b.i, we extend the action on $D(N)$ as in the proof of Proposition 3.6 of [11] to obtain a smooth $T^{n+1}$ action on $N$.
In Case 2.b.ii, we note that there is a $T^{n-k-1}$ isometric action on the unit normal $S^{2(n-k-1)+1}$ to $N$, which is an almost isotropy-maximal action on a closed, simply-connected, positively curved sphere. Hence by the induction hypothesis, we can extend the $T^{n-k-1}$ action to a smooth $T^{n-k}$ action that fixes $N$. By exponentiating we can extend the $T^{n+1}$ action to all of $D(N)$ and hence to $E$, its boundary.

Note that in Case 2.b.ii, $F$ is simply-connected, so we proceed as in Case 1. However, in Case 2.b.i, $F$ need not be simply-connected. Recall that this case only occurs when $N$ is also of codimension 2 and not fixed by any circle. It suffices to show in this case that we may extend the induced isometric $T^{n-1}$ action on $F$ to a smooth $T^n$ action.

Now, since $F$ has cyclic fundamental group, we note that if $\pi_1(F)$ is finite, then by Theorem 2.26 we may lift the $T^{n-1}$ action to $\tilde{F}$, so that the lift commutes with the deck transformations. By the induction hypothesis, we may extend the $T^{n-1}$ action on $\tilde{F}$ to a smooth $T^n$ action and this allows us to extend the $T^{n-1}$ action on $F$ to a smooth $T^n$ action. Likewise, if $\pi_1(F) \cong \mathbf{Z}$, then by Theorem 2.26 we see that the lifted action of $T^{n-1}$ on $\tilde{F} \simeq \tilde{F} \times \mathbb{R}$ is isotropy-maximal on the $\tilde{F}$ factor and trivial on the $\mathbb{R}$ factor. We can then extend the $T^{n-1}$ action on $\tilde{F}$ by an $\mathbb{R}$, obtaining a $T^{n-1} \times \mathbb{R}$ action on $\tilde{F} \times \mathbb{R}$. This action commutes with the deck transformations and hence induces a smooth $T^n$ action on $F$.

Note that the $T^n$ extended action on $F$ commutes with the $\lambda(F)$ action on $F$, since the latter is trivial. To see that the action now extends to $D(F)$, note that the $\lambda(F)$ action fixes $F$ and acts on the normal space to $F$ by rotating the fibers. As in the proof of Proposition 3.6 [11], we see that this gives us a $T^{n+1}$ action on $\nu(F)$ and hence on $D(F)$ via the exponential map.

This completes the proof of Theorem 4.1.

5. THE PROOF OF THEOREM A AND COROLLARY B

Before we prove Theorem A and Corollary B, we first prove the following proposition.

**Proposition 5.1.** The extended, smooth and effective $T^{n+1}$ action on $M^{2n+1}$, a closed, simply-connected, non-negatively curved almost torus manifold, has only connected isotropy subgroups and the quotient space $M^{2n+1}/T^{n+1}$ and all of its faces are diffeomorphic (after smoothing the corners) to disks.

We will first prove the following lemma, which is needed in the proof of Proposition 5.1.

**Lemma 5.2.** Let $M^{2n+1}$ be a closed, non-negatively curved, almost torus manifold with $\pi_1(M) \cong \mathbf{Z}$. Then the $S^1$ extension of the action to a smooth $T^{n+1}$ action is free and the quotient, $\tilde{M}^{2n} \simeq M^{2n+1}/S^1$ is a closed, simply-connected, non-negatively curved torus manifold.

**Proof.** Recall that $\tilde{M} = \tilde{M}^{2n} \times \mathbb{R}$, where $\tilde{M}^{2n}$ is a closed, simply-connected, non-negatively curved torus manifold. Moreover, the additional circle action on $M$ comes from an action of $\mathbb{R}$ on $\tilde{M}$, which acts freely on $\mathbb{R}$ and trivially on $\tilde{M}^{2n}$.

We claim that the corresponding $S^1$ action on $M^{2n+1}$ is also free, since if this $S^1$ action has non-empty fixed point set, it would to an $S^1$-action, by Theorem 2.26 contradicting the fact that it lifts to an $\mathbb{R}$-action. Moreover, if there were $\mathbb{Z}_k$ isotropy, this would imply by Theorem 2.27 that the lift of the $\mathbb{Z}_k$ isotropy subgroup would also be $\mathbb{Z}_k$, which is not a subgroup of $\mathbb{R}$, a contradiction. Finally, since $\tilde{M}$ covers $M$ and the lifted action commutes with the deck transformations, it follows that $\tilde{M}/\mathbb{R} = \tilde{M}^{2n}$ covers $M^{2n+1}/S^1$. But since both $\tilde{M}^{2n}$ and $M^{2n+1}/S^1$ are simply-connected, the result holds. □
Proof of Proposition 5.1. We prove this by induction on the dimension of $M$. The anchor of the induction is the extended $T^2$ action on $M^3 = S^3$. Note that the original isometric $T^1$ action has quotient space $M^3/T^1 = D^3$, with $T^1$ isotropy on $\partial D^3 = S^1$, and by the concavity of the distance function, $\text{dist}_{\partial D^3}(\cdot)$, any extremal sets must lie at maximal distance from the boundary. Since the action of $T^1$ on $S^3$ is almost locally standard, this implies that all isotropy groups of $T^1$ are connected. Moreover, all singularities of the action of $T^1$ on $M^3$ are contained in $F$ and $N$. $N$ is not fixed by $T^1$, since if it were, we would obtain a contradiction via the Double Soul Theorem [2.17] Hence $N$ is a principal orbit and so all points in the interior of $D^2$ correspond to principal orbits.

We now consider the quotient of $D^2$ by the additional $S^1$ action. Note that by construction, the $S^1$ action acts on $F$ and fixes $N$, acting freely on the normal $S^1$ to $N$. Moreover, the $S^1$ action on $F$ must be free, since if there is finite isotropy, this implies that the corresponding $T^1 \times Z_k$ action on the unit normal circle to $F$ is $Z_k$ ineffective and hence the $T^2$ action on $M^3$ is also $Z_k$-ineffective. Thus the $S^1$ action has no finite isotropy on $D(F)$ nor on $D(N)$. The quotient space, $D^2/S^1$ is an interval and all the interior points correspond to principal orbits of the action, with the vertices corresponding to $T^1$ and $S^1$ isotropy, respectively.

We now assume that the results hold true for any closed, simply-connected, non-negatively curved almost torus manifold, $M^{2k+1}$, with $k < n$. Recall that $M = D(F) \cup D(N)$, and we argue by cases as in the proof of Theorem 3.1 We must show here that in each case that the result holds for $D(F)/T$, $D(N)/T$ and $E/T$. Note however, that since $E/T = F/T$, it suffices to show the result holds for $F$, which follows directly from the induction hypothesis for Cases 1, 2.a and 2.b.ii.

Proof of Case 1. Here both $N$ and $F$ closed, simply-connected, non-negatively curved almost torus manifolds. So, we have by the induction hypothesis that the orbit spaces of both $F$ and $N$ are disks, all of whose faces are disks, and all isotropy subgroups are connected. Moreover, since the $S^3$ action fixing $F$ is free on the normal space to $F$, we see that the $T^{n+1}$ action on $D(F)$ has quotient space the product of $D^2/S^1 = I$ and a disk, which is a disk. All faces of the quotient of $D(F)$ are then disks, and all isotropy subgroups are connected. The result for $D(N)$ follows in the same fashion, noting that the action of the torus on the unit normal sphere to $N$ is of maximal symmetry rank and hence its quotient is a simplex. It follows that the quotient of $D(N)$ is then a product of a suspended simplex and a disk, which is diffeomorphic to a disk. All of the faces of the quotient are then diffeomorphic to disks, and all isotropy subgroups are connected. □

Proof of Case 2.a. Since $F$ is simply-connected in this case, the result follows for $D(F)$ as in Case 1.

Recall that $N^{2k+2}$ is a principal $S^1$ bundle over $\tilde{N}^{2k+1}$, a closed, simply-connected, non-negatively curved almost torus manifold. The action is extended on $N^{2k+1}$, and the induction hypothesis tells us that the result holds for $N^{2k+1}/T^{k+1} = (N^{2k+2}/S^1)/T^{k+1}$, and hence the conclusion holds for $N^{2k+2}$. We then see that the result holds for $D(N)$, since the action of the torus on the unit normal sphere to $N$ is of maximal symmetry rank. □

Proof of Case 2.b. Here we have two subcases again: Case 2.b.i, where $N$ is of codimension two and neither $F$ nor $N$ need be simply-connected, and Case 2.b.ii, where $F$ is simply-connected and $N^{2k+1}$ need not be simply-connected. We already saw in the proof of Theorem [3.1] that in both of these cases, $N^{2k+1}$ is an $S^1$-principal bundle over a non-negatively curved, simply-connected torus manifold, $M^{2k}$. Since $M^{2k}/T^k = N/T^{k+1}$, it follows by Lemma [2.22] that $N/T^{k+1}$ is a disk, all of whose faces are disks, and all isotropy
subgroups are connected. The action is extended on the normal space of $N$ and in all cases is of maximal symmetry rank on the corresponding unit normal sphere. A similar argument as in Case 1 shows that the result then holds on $D(N)$.

Now, in Case 2.b.ii, we note that since $F$ is simply-connected, we may proceed as in Case 1 to show that the result holds for $D(F)$.

It remains to show that the result holds on $D(F)$ in Case 2.b.i. Recall that $\pi_1(F)$ is cyclic in this case. If it is finite, then we may pass to its universal cover, $\tilde{F}$, which is a closed, simply-connected, non-negatively curved almost torus manifold with an extended $T^{n+1}$ action. We may then apply the induction hypothesis to $\tilde{F}$. Since, as we saw in the proof of Proposition 3.6, any smooth covering map $p : D^m \to X^m$ must be a diffeomorphism, it follows that $\tilde{F}/T^{n+1} \simeq F/T^{n+1}$, and the result holds. We can then extend the result to $D(F)$ as we did in Case 1.

Suppose now that $\pi_1(F) \cong \mathbb{Z}$. But then by Lemma 5.2, it follows that $\tilde{F}^{2n-2} = F^{2n-1}/S^1$, a closed, simply-connected, non-negatively curved torus manifold. By Lemma 2.22, $F/T^n = F^{2n-2}/T^{n+1}$ is a disk, all of whose faces are disks and all isotropy subgroups are connected. Once again, we can then extend the result to $D(F)$ as we did in Case 1, which completes the proof for this case.

This completes the proof of the proposition.

We now proceed to prove Theorem A. Observe that the $T^{n+1}$ smooth isotropy-maximal action on $M^{2n+1}$ has free rank 1 by Proposition 5.4 in [7] and because the action is isotropy-maximal. Let $T^1$ be the almost freely acting circle in $T^{n+1}$. Then $M^{2n+1}/T^1$ is a closed, simply-connected, torus orbifold. Note that if the $T^1$ action is free, then the quotient is, in fact, a torus manifold. In both cases, it is rationally elliptic, since $M^{2n+1}$ is rationally elliptic. The proof breaks into two cases: Case (1), where the $T^1$ action is free, and Case (2), where the $T^1$ action is almost free. Using Proposition 5.1, the proof then follows in essentially the same manner as the proof of Theorem A in [7] for each of these cases.

We now proceed to prove Corollary B. As mentioned above, $M$ is rationally elliptic by Corollary 2.24. The upper bound on the rank of the torus action then follows from Theorem A of [7].

Observe that since the $T^k$ action is almost isotropy-maximal, the largest possible isotropy subgroup is of dimension $n-k-1$. Since $M$ is compact, it follows by finiteness of the orbit types that the free rank is equal to $k-(n-k-1) = 2k-n+1$. If the free rank corresponds to the free dimension, then $M^n$ is a principal $T^{2k-n+1}$ bundle over an almost torus manifold, $M^{2n-2k-1}$. We can then apply Theorem 4.1 to extend the $T^{n-k-1}$ isometric action to a $T^{n-k}$ smooth action. We then apply Proposition 5.1 and proceed as in the proof of Theorem A in [7] to obtain the result.

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