A FREE BOUNDARY PROBLEM IN THERMAL INSULATION WITH A PRESCRIBED HEAT SOURCE

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Abstract. We study the thermal insulation of a bounded body \(\Omega \subset \mathbb{R}^n\), under a prescribed heat source \(f > 0\), via a bulk layer of insulating material. We consider a model of heat transfer between the insulated body and the environment determined by convection; this corresponds to Robin boundary conditions on the free boundary of the layer. We show that a minimal configuration exists and that it satisfies uniform density estimates.

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1. Introduction

Let \(\Omega \subset \mathbb{R}^n\) be an open bounded set with smooth boundary, let \(f \in L^2(\Omega)\) be a positive function and let \(\beta, C_0\) be positive constants. We consider the following energy functional

\[
F(A, v) = \int_A |\nabla v|^2 \, dL^n + \beta \int_{\partial A} v^2 \, dH^{n-1} - 2 \int_{\Omega} fv \, dL^n + C_0 L^n(A \setminus \Omega),
\]

and the variational problem

\[
\inf \left\{ F(A, v) \right\} \quad \left| \begin{array}{l}
A \supseteq \Omega \text{ open, bounded and Lipschitz} \\
v \in W^{1,2}(A), \ v \geq 0 \text{ in } A
\end{array} \right\}.
\]

This problem is related to the following thermal insulation problem: for a given heat source \(f\) distributed in a conductor \(\Omega\), find the best possible configuration of insulating material surrounding \(\Omega\). A similar problem has been studied in [2, 10] for a thin insulating layer, and in [4, 5, 7] for a prescribed temperature in \(\Omega\).

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following stationary problem, with Robin boundary condition on $\partial A$. Precisely

\[
\begin{cases}
-\Delta u_A = f & \text{in } \Omega, \\
\frac{\partial u_A^-}{\partial \nu} = \frac{\partial u_A^+}{\partial \nu} & \text{on } \partial \Omega, \\
\Delta u_A = 0 & \text{in } A \setminus \Omega, \\
\frac{\partial u_A^-}{\partial \nu} + \beta u_A = 0 & \text{on } \partial A,
\end{cases}
\]

where $u_A^-$ and $u_A^+$ denote the traces of $u_A$ on $\partial \Omega$ in $\Omega$ and in $A \setminus \Omega$ respectively. That is

\[
\int_A \nabla u_A \cdot \nabla \varphi \, d\mathcal{L}^n + \beta \int_{\partial A} u_A \varphi \, d\mathcal{H}^{n-1} = \int_\Omega f \varphi \, d\mathcal{L}^n,
\]

for all $\varphi \in W^{1,2}(A)$. The Robin boundary condition represents the case when the heat transfer with the environment is conveyed by convection.

If for any couple $(A, v)$ with $A$ an open bounded set with Lipschitz boundary containing $\Omega$ and $v \in W^{1,2}(A)$, $v \geq 0$ in $A$, we identify $v$ with $v \chi_A$, where $\chi_A$ is the characteristic function of $A$, and the set $A$ with the support of $v$, then the energy functional (1.1) becomes

\[
\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{J_v} \left( \pi^2 + v^2 \right) \, d\mathcal{H}^{n-1} - 2 \int\Omega f \, d\mathcal{L}^n + C_0 \mathcal{L}^n(\{v > 0\} \setminus \Omega),
\]

and the minimization problem (1.2) becomes

\[
\inf \left\{ \mathcal{F}(v) \bigm| v \in SBV^\perp(\mathbb{R}^n) \cap W^{1,2}(\Omega) \right\} ,
\]

where $v^\perp$ and $v$ are respectively the approximate upper and lower limits of $v$, $J_v$ is the jump set and $\nabla v$ is the absolutely continuous part of the derivative of $v$. See Section 2 for the definitions.

We state the main results of this paper in the two following theorems.

**Theorem 1.1.** Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $f \in L^2(\Omega)$, with $f > 0$ almost everywhere in $\Omega$. Assume in addition that, if $n = 2$,

\[
\|f\|^2_{L^2(\Omega)} < C_0 \lambda_\beta(B) \mathcal{L}^2(\Omega),
\]

where $B$ is a ball having the same measure of $\Omega$. Then problem (1.5) admits a solution. Moreover, if $p > n$ and $f \in L^p(\Omega)$, then there exists a positive constant $C = C(\Omega, f, p, \beta, C_0)$ such that if $u$ is a minimizer to problem (1.5) then

\[
\|u\|_\infty \leq C.
\]

**Theorem 1.2.** Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $p > n$ and let $f \in L^p(\Omega)$, with $f > 0$ almost everywhere in $\Omega$. Assume in addition that, if $n = 2$ condition (1.6) holds true. Then there exist positive constants $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0)$, $c = c(\Omega, f, p, \beta, C_0)$, $C = C(\Omega, f, p, \beta, C_0)$ such that if $u$ is a minimizer to problem (1.5) then

\[
u \geq \delta_0 \quad \mathcal{L}^n \text{-a.e. in } \{u > 0\},
\]
and the jump set $J_u$ satisfies the density estimates
\[ cr^{n-1} \leq H^{n-1}(J_u \cap B_r(x)) \leq Cr^{n-1}, \]
with $x \in \overline{J}_u$, and $0 < r < d(x, \partial \Omega)$. In particular, we have
\[ H^{n-1}(J_u \setminus J_u) = 0. \]

We refer to Section 2 for the definitions of $\lambda_\beta(B)$ in (1.6), and the distance $d(x, \partial \Omega)$ in Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.1, while Section 4 is devoted to the proof of Theorem 1.2.

We notice that the assumptions on the function $f$ do not seem to be sharp. Indeed, it is well known that (see for instance [12], Thm. 8.15), in the more regular case, the assumption $f \in L^p(\Omega)$ with $p > n/2$ ensures the boundedness of solutions to equation (1.3).

2. Notation and tools

In this section we recall some definitions and properties of the spaces $BV$, $SBV$, and $SBV^1$. We refer to [1, 3, 11] for a deep study of the properties of these functions.

In the following, given an open set $\Omega \subseteq \mathbb{R}^n$ and $1 \leq p \leq \infty$, we will denote the $L^p(\Omega)$ norm of a function $v \in L^p(\Omega)$ as $\|v\|_{L^p(\Omega)}$, in particular when $\Omega = \mathbb{R}^n$ we will simply write $\|v\|_p = \|v\|_{L^p(\mathbb{R}^n)}$.

Definition 2.1 (BV). Let $u \in L^1(\mathbb{R}^n)$. We say that $u$ is a function of bounded variation in $\mathbb{R}^n$ and we write $u \in BV(\mathbb{R}^n)$ if its distributional derivative is a Radon measure, namely
\[ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = \int_\Omega \varphi \, dD_i u \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n), \]
with $Du$ a $\mathbb{R}^n$-valued measure in $\mathbb{R}^n$. We denote with $|Du|$ the total variation of the measure $Du$. The space $BV(\mathbb{R}^n)$ is a Banach space equipped with the norm
\[ \|u\|_{BV(\mathbb{R}^n)} = \|u\|_1 + |Du|(\mathbb{R}^n). \]

Definition 2.2. Let $E \subseteq \mathbb{R}^n$ be a measurable set. We define the set of points of density 1 for $E$ as
\[ E^{(1)} = \left\{ x \in \mathbb{R}^n \mid \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 1 \right\}, \]
and the set of points of density 0 for $E$ as
\[ E^{(0)} = \left\{ x \in \mathbb{R}^n \mid \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 0 \right\}. \]

Moreover, we define the essential boundary of $E$ as
\[ \partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}). \]

Definition 2.3 (Approximate upper and lower limits). Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We define the approximate upper and lower limits of $u$, respectively, as
\[ \overline{u}(x) = \inf \left\{ t \in \mathbb{R} \mid \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u > t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\}, \]
and
\[ u(x) = \sup \left\{ t \in \mathbb{R} \left| \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u < t \})}{\mathcal{L}^n(B_r(x))} = 0 \right. \right\}. \]

We define the jump set of \( u \) as
\[ J_u = \{ x \in \mathbb{R}^n \mid u(x) < \underline{u}(x) \}. \]

We denote by \( K_u \) the closure of \( J_u \).

If \( \underline{u}(x) = \overline{u}(x) = l \), we say that \( l \) is the approximate limit of \( u \) as \( y \) tends to \( x \), and we have that, for any \( \varepsilon > 0 \),
\[ \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ |u - l| \geq \varepsilon \})}{\mathcal{L}^n(B_r(x))} = 0. \]

If \( u \in BV(\mathbb{R}^n) \), the jump set \( J_u \) is a \((n-1)\)-rectifiable set, i.e. \( J_u \subseteq \bigcup_{i \in \mathbb{N}} M_i \), up to a \( H^{n-1} \)-negligible set, with \( M_i \) a \( C^1 \)-hypersurface in \( \mathbb{R}^n \) for every \( i \). We can then define \( H^{n-1} \)-almost everywhere on \( J_u \) a normal \( \nu_u \) coinciding with the normal to the hypersurfaces \( M_i \). Furthermore, the direction of \( \nu_u(x) \) is chosen in such a way that the approximate upper and lower limits of \( u \) coincide with the approximate limit of \( u \) on the half-planes
\[ H^+_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \geq 0 \} \]
and
\[ H^-_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \leq 0 \} \]
respectively.

**Definition 2.4.** Let \( E \subseteq \mathbb{R}^n \) be a measurable set and let \( \Omega \subseteq \mathbb{R}^n \) be an open set. We define the **relative perimeter** of \( E \) inside \( \Omega \) as
\[ P(E; \Omega) = \sup \left\{ \int_E \text{div} \varphi \, d\mathcal{L}^n \mid \varphi \in C^1_c(\Omega, \mathbb{R}^n), \ |\varphi| \leq 1 \right\}. \]

If \( P(E; \mathbb{R}^n) < +\infty \) we say that \( E \) is a **set of finite perimeter**.

**Theorem 2.5** (Relative Isoperimetric Inequality). Let \( \Omega \) be an open, bounded, connected set with Lipschitz boundary. Then there exists a positive constants \( C = C(\Omega) \) such that
\[ \min \{ \mathcal{L}^n(\Omega \cap E), \mathcal{L}^n(\Omega \setminus E) \} \frac{n-1}{n} \leq CP(E; \Omega), \]
for every set \( E \) of finite perimeter.

See for instance [14] for the proof of this theorem.

**Theorem 2.6.** Let \( \Omega \) be an open, bounded, connected set with Lipschitz boundary. Then there exists a constant \( C = C(\Omega) > 0 \) such that
\[ H^{n-1}(\partial^* E \cap \partial \Omega) \leq C H^{n-1}(\partial^* E \cap \Omega) \]
for every set of finite perimeter \( E \subseteq \Omega \) with \( 0 < \mathcal{L}^n(E) \leq \mathcal{L}^n(\Omega)/2 \).
We refer to Theorem 2.3 of [8] for the proof of the previous theorem, observing that if \( \Omega \) is a Lipschitz set, then it is an admissible set in the sense defined in [8] (see [16], Rem. 5.10.2).

**Theorem 2.7** (Decomposition of BV functions). Let \( u \in \text{BV}(\mathbb{R}^n) \). Then we have

\[
dDu = \nabla u \, d\mathcal{L}^n + |\pi - u| \nu_u \, d\mathcal{H}^{n-1}|_{J_u} + dD^c u,
\]

where \( \nabla u \) is the density of \( Du \) with respect to the Lebesgue measure, \( \nu_u \) is the normal to the jump set \( J_u \) and \( D^c u \) is the Cantor part of the measure \( Du \). The measure \( D^c u \) is singular with respect to the Lebesgue measure and concentrated out of \( J_u \).

**Definition 2.8.** Let \( v \in \text{BV}(\mathbb{R}^n) \), let \( \Gamma \subseteq \mathbb{R}^n \) be a \( \mathcal{H}^{n-1} \)-rectifiable set and let \( \nu(x) \) be the generalized normal to \( \Gamma \) defined for \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in \Gamma \). For \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in \Gamma \) we define the traces \( \gamma^+_\Gamma(v)(x) \) of \( v \) on \( \Gamma \) by the following Lebesgue-type limit quotient relation

\[
\lim_{r \to 0} \frac{1}{r^n} \int_{B_r^+(x)} |v(y) - \gamma^+_\Gamma(v)(x)| \, d\mathcal{L}^n(y) = 0,
\]

where

\[
B_r^+(x) = \{ y \in B_r(x) \mid \nu(x) \cdot (y - x) > 0 \},
\]

\[
B_r^-(x) = \{ y \in B_r(x) \mid \nu(x) \cdot (y - x) < 0 \}.
\]

**Remark 2.9.** Notice that, by Remark 3.79 of [1], for \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in \Gamma \), \( (\gamma^+_\Gamma(v)(x), \gamma^-_\Gamma(v)(x)) \) coincides with either \( (\pi(x), \underline{v}(x)) \) or \( (\underline{v}(x), \pi(x)) \), while, for \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in \Gamma \setminus J_u \), we have that \( \gamma^+_\Gamma(v)(x) = \gamma^-_\Gamma(v)(x) \) and they coincide with the approximate limit of \( v \) in \( x \). In particular, if \( \Gamma = J_u \), we have

\[
\gamma^+_J(v)(x) = \pi(x) \quad \gamma^-_J(v)(x) = \underline{v}(x)
\]

for \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in J_u \).

We now focus our attention on the BV functions whose Cantor parts vanish.

**Definition 2.10** (SBV). Let \( u \in \text{BV}(\mathbb{R}^n) \). We say that \( u \) is a special function of bounded variation and we write \( u \in \text{SBV}(\mathbb{R}^n) \) if \( D^c u = 0 \).

For SBV functions we have the following.

**Theorem 2.11** (Chain rule). Let \( g : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Then if \( u \in \text{SBV}(\mathbb{R}^n) \), we have

\[
\nabla g(u) = g'(u) \nabla u.
\]

Furthermore, if \( g \) is increasing,

\[
\overline{g}(u) = g(\pi), \quad \underline{g}(u) = g(u)
\]

while, if \( g \) is decreasing,

\[
\overline{g}(u) = g(u), \quad \underline{g}(u) = g(\pi).
\]

We now give the definition of the following class of functions.
Definition 2.12 (SBV$^{1/2}$). Let $u \in L^2(\mathbb{R}^n)$ be a non-negative function. We say that $u \in \text{SBV}^{1/2}(\mathbb{R}^n)$ if $u^2 \in \text{SBV}(\mathbb{R}^n)$. In addition, we define

$$J_u := J_{u^2}, \quad \pi := \sqrt{u^2}, \quad \eta := \sqrt{\pi^2},$$

$$\nabla u := \frac{1}{2u} \nabla (u^2) \chi_{\{u > 0\}}.$$

Notice that this definition extends the validity of the Chain Rule to the functions in $\text{SBV}^{1/2}(\mathbb{R}^n)$. We refer to Lemma 3.2 of [3] for the coherence of this definition.

Theorem 2.13 (Compactness in SBV$^{1/2}$). Let $u_k$ be a sequence in $\text{SBV}^{1/2}(\mathbb{R}^n)$ and let $C > 0$ be such that for every $k \in \mathbb{N}$

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, d\mathcal{L}^n + \int_{J_{u_k}} (\pi_k^2 + \eta_k^2) \, d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \, d\mathcal{L}^n < C.$$

Then there exists $u \in \text{SBV}^{1/2}(\mathbb{R}^n)$ and a subsequence $u_{k_j}$ such that

- Compactness:

$$u_{k_j} \overset{L^2_{\text{loc}}(\mathbb{R}^n)}{\longrightarrow} u$$

- Lower semicontinuity: for every open set $\Omega$ we have

$$\int_{\Omega} |\nabla u|^2 \, d\mathcal{L}^n \leq \liminf_{j \to +\infty} \int_{\Omega} |\nabla u_{k_j}|^2 \, d\mathcal{L}^n$$

and

$$\int_{J_{u} \cap \Omega} (\pi^2 + \eta^2) \, d\mathcal{H}^{n-1} \leq \liminf_{j \to +\infty} \int_{J_{u_{k_j}} \cap \Omega} \left(\pi_{k_j}^2 + \eta_{k_j}^2\right) \, d\mathcal{H}^{n-1}.$$

Definition 2.14 (Robin Eigenvalue). Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with Lipschitz boundary, let $\beta > 0$. We define $\lambda_\beta(\Omega)$ as

$$\lambda_\beta(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial \Omega} v^2 \, d\mathcal{H}^{n-1}}{\int_{\Omega} v^2 \, d\mathcal{L}^n} \ : \ v \in W^{1,2}(\Omega) \setminus \{0\} \right\}.$$  \hspace{1cm} (2.1)

Remark 2.15. Standard tools of calculus of variation ensure that the infimum in (2.1) is achieved, see for instance.

Lemma 2.16. For every $0 < r < R$, the following inequality holds

$$\lambda_\beta(B_r) \leq \left(\frac{\mathcal{L}^n(B_R)}{\mathcal{L}^n(B_r)}\right)^{\frac{2}{n}} \lambda_\beta(B_R),$$

where $B_R$ and $B_r$ are balls with radii $R$ and $r$ respectively.
Proof. Let \( \varphi \) be a minimum of (2.1) for \( \Omega = B_R \) and with \( \|\varphi\|_{2,B_R} = 1 \). We define
\[
w(x) = \varphi \left( \frac{R}{r} x \right) \quad \forall x \in B_r.
\]
Therefore,
\[
\lambda_\beta(B_r) \leq \frac{\int_{B_r} |\nabla w(x)|^2 \, d\mathcal{L}^n(x) + \int_{\partial B_r} w(x)^2 \, d\mathcal{H}^{n-1}(x)}{\int_{B_r} w(x)^2 \, d\mathcal{L}^n(x)} = \left( \frac{r}{R} \right)^{n-2} \int_{B_R} |\nabla \varphi(y)|^2 \, d\mathcal{L}^n(y) + \left( \frac{r}{R} \right)^{n-1} \int_{\partial B_R} \varphi(y)^2 \, d\mathcal{H}^{n-1}(y)
\]
\[
= \left( \frac{r}{R} \right) \lambda_\beta(B_r) = \left( \frac{\mathcal{L}^n(B_r)}{\mathcal{L}^n(B_R)} \right)^{-\frac{n}{2}} \lambda_\beta(B_R).
\]
Since \( r/R < 1 \), by minimality of \( \varphi \), we get
\[
\lambda_\beta(B_r) \leq \left( \frac{r}{R} \right)^{n-2} \lambda_\beta(B_R) = \left( \frac{\mathcal{L}^n(B_r)}{\mathcal{L}^n(B_R)} \right)^{-\frac{n}{2}} \lambda_\beta(B_R).
\]

Let \( \beta, m > 0 \), and let us denote by
\[
\Lambda_{\beta,m} = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \int_{\mathbb{R}^n} (v^2 + \pi^2) \, d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} v^2 \, d\mathcal{L}^n} \mid v \in \text{SBV}^2(\mathbb{R}^n) \setminus \{0\} \cup \mathcal{L}^n(\{v > 0\}) \leq m \right\}.
\]

Here we state a theorem, referring to Theorem 5 of [3] for the proof.

**Theorem 2.17.** Let \( B \subseteq \mathbb{R}^n \) be a ball of volume \( m \). Then
\[
\Lambda_{\beta,m} = \lambda_\beta(B).
\]

We will denote by \( d(x, \partial \Omega) \) the distance between \( x \in \mathbb{R}^n \) and the boundary \( \partial \Omega \), and for every \( \varepsilon > 0 \) we define
\[
\Omega_\varepsilon = \{ x \in \Omega \mid d(x, \partial \Omega) > \varepsilon \}.
\]

We will use the following result.

**Proposition 2.18.** Let \( \Omega \) be an open bounded set with \( C^{1,1} \) boundary, then there exist a constant \( C = C(\Omega) > 0 \) and a \( \varepsilon_0 = \varepsilon_0(\Omega) > 0 \) such that
\[
\mathcal{L}^n(\Omega \setminus \Omega_\varepsilon) \leq C\varepsilon \quad \forall \varepsilon < \varepsilon_0.
\]
Proof. It is well known (see for instance Theorem 17.5 of [13]) that there exist a constant $C = C(\Omega)$ and $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that

$$P(\Omega_\varepsilon) = P(\Omega) + C(\Omega) \varepsilon + O(\varepsilon^2),$$

for every $0 < \varepsilon < \varepsilon_0$. Let $r(x) = d(x, \partial \Omega)$ be the distance from the boundary of $\Omega$. By coarea formula we have

$$\mathcal{L}^n(\Omega \setminus \Omega_\varepsilon) = \int_{\{0 < r < \varepsilon\}} d\mathcal{L}^n = \int_0^\varepsilon P(\Omega_t) \, dt \leq C(\Omega) \varepsilon.$$

\[ \square \]

3. Existence of minimizers

In this section we prove Theorem 1.1: in Proposition 3.3 we prove the existence of a minimizer to problem (1.5); in Proposition 3.7 we prove the $L^\infty$ estimate for a minimizer.

In this section, we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $C^{1,1}$ boundary, that $f \in L^2(\Omega)$ is a positive function and that $\beta, C_0$ are positive constants. We consider the energy functional $\mathcal{F}$ defined in (1.4).

Lemma 3.1. Let $n \geq 2$ and assume that, if $n = 2$, condition (1.6) holds true. Then there exist two positive constants $c = c(\Omega, f, \beta, C_0)$ and $C = C(\Omega, f, \beta, C_0)$ such that if $v \in SBV^2(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, with $\mathcal{F}(v) \leq 0$ and $\Omega \subseteq \{ v > 0 \}$, then

$$\mathcal{L}^n(\{ v > 0 \}) \leq c,$$  \hspace{1cm} (3.1)

$$\|v\|_2 \leq C.$$  \hspace{1cm} (3.2)

Proof. Let $B'$ be a ball with the same measure as $\{ v > 0 \}$. By Theorem 2.17

$$0 \geq \mathcal{F}(v) \geq \lambda_\beta (B') \int_{\mathbb{R}^n} v^2 \, d\mathcal{L}^n - 2 \int_\Omega fv \, d\mathcal{L}^n + C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega).$$

By Lemma 2.16 and Hölder inequality

$$0 \geq \lambda_\beta (B) \left( \frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\{ v > 0 \})} \right)^\frac{2}{n} \|v\|_2^2 - 2\|f\|_{2,\Omega} \|v\|_2$$

$$+ C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega)$$

where $B$ is a ball with the same measure as $\Omega$. Obviously (3.3) implies that

$$\|f\|_{2,\Omega}^2 - \lambda_\beta (B) \left( \frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\{ v > 0 \})} \right)^\frac{2}{n} C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega) \geq 0.$$  \hspace{1cm} (3.3)

Let $M = \mathcal{L}^n(\{ v > 0 \})$, and notice that, since $\Omega \subseteq \{ v > 0 \}$,

$$\mathcal{L}^n(\{ v > 0 \} \setminus \Omega) = M - \mathcal{L}^n(\Omega),$$
therefore
\[ \|f\|_{2,\Omega}^2 \geq C_0 \lambda_\beta(B) \left( L^\alpha(\Omega) \right)^{\frac{2}{n}} \left( M^{1-\frac{2}{n}} - M^{-\frac{2}{n}} L^\alpha(\Omega) \right). \]

This implies (taking into account (1.6) if \( n = 2 \)) that there exists \( c = c(\Omega, f, \beta, C_0) > 0 \) such that
\[ L^\alpha(\{ v > 0 \}) < c. \]

Finally observe that by (3.3) it follows
\[ \|v\|_2 \leq C(M), \quad (3.4) \]
where
\[ C(M) = \frac{M^{\frac{2}{n}} \left( \|f\|_{2,\Omega} + \sqrt{\|f\|_{2,\Omega}^2 - C_0 \lambda_\beta(B) \left( \frac{L^\alpha(\Omega)}{M} \right)^{\frac{2}{n}} (M - L^\alpha(\Omega))} \right)}{\lambda_\beta(B) L^\alpha(\Omega)} \leq \frac{2c^{\frac{2}{n}} \|f\|_{2,\Omega}}{\lambda_\beta(B) L^\alpha(\Omega)}. \]

**Remark 3.2.** Let \( v \in \text{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \), it is always possible to choose a function \( v_0 \) such that \( v_0 = v \) in \( \mathbb{R}^n \setminus \Omega \), \( F(v_0) \leq F(v) \), and \( \Omega \subseteq \{ v_0 > 0 \} \). Indeed the function \( v_0 \in W^{1,2}(\Omega) \), weak solution to the following boundary value problem
\[ \begin{cases} -\Delta v_0 = f & \text{in } \Omega, \\ v_0 = \gamma^-_{\partial \Omega}(v) & \text{on } \partial \Omega, \end{cases} \quad (3.5) \]
satisfies
\[ \int_{\Omega} \nabla v_0 \cdot \nabla \varphi \, dL^n = \int_{\Omega} f \varphi \, dL^n \]
for every \( \varphi \in W_0^{1,2}(\Omega) \) and \( v_0 = \gamma^-_{\partial \Omega}(v) \) on \( \partial \Omega \) in the sense of the trace. Then, extending \( v_0 \) to be equal to \( v \) outside of \( \Omega \), we have that \( \Omega \subseteq \{ v_0 > 0 \} \) and \( F(v_0) \leq F(v) \).

**Proposition 3.3 (Existence).** Let \( n \geq 2 \) and, if \( n = 2 \), assume that condition (1.6) holds true. Then there exists a solution to problem (1.5).

**Proof.** Let \( \{ u_k \} \) be a minimizing sequence for problem (1.5). Without loss of generality we may always assume that, for all \( k \in \mathbb{N} \), \( F(u_k) \leq F(0) = 0 \), and, by Remark 3.2, \( \Omega \subseteq \{ u_k > 0 \} \). Therefore we have
\[ 0 \geq F(u_k) \geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \, dL^n + \beta \int_{J_{u_k}} (\frac{1}{2} u_k^2 + |u_k|^2) \, dH^{n-1} - 2 \int_{\Omega} f v \, dL^n \]
\[ \geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \, dL^n + \beta \int_{J_{u_k}} (\frac{1}{2} u_k^2 + |u_k|^2) \, dH^{n-1} - 2 \|f\|_{2,\Omega} \|u_k\|_{2,\Omega}, \]
and by (3.2),
\[
\int_{\mathbb{R}^n} |\nabla u_k|^2 \, d\mathcal{L}^n + \beta \int_{J_{u_k}} (\bar{u}_k^2 + u_k^2) \, d\mathcal{H}^{n-1} \leq C \|f\|_{2,\Omega}.
\]

Then we have that there exists a positive constant still denoted by $C$, independent on the sequence $\{u_k\}$, such that
\[
\int_{\mathbb{R}^n} |\nabla u_k|^2 \, d\mathcal{L}^n + \int_{J_{u_k}} (\bar{v}_k^2 + u_k^2) \, d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \, d\mathcal{L}^n < C. \tag{3.6}
\]

The compactness theorem in SBV$^\frac{1}{2}(\mathbb{R}^n)$ (Thm. 2.13), ensures that there exists a subsequence $\{u_{k_j}\}$ and a function $u \in$ SBV$^\frac{1}{2}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, such that $u_{k_j}$ converges to $u$ strongly in $L^1_{\text{loc}}(\mathbb{R}^n)$, weakly in $W^{1,2}(\Omega)$, almost everywhere in $\mathbb{R}^n$ and
\[
\int_{\mathbb{R}^n} |\nabla u|^2 \, d\mathcal{L}^n \leq \liminf_{j \to +\infty} \int_{\mathbb{R}^n} |\nabla u_{k_j}|^2 \, d\mathcal{L}^n,
\]
\[
\int_{J_u} (\bar{v}^2 + u^2) \, d\mathcal{H}^{n-1} \leq \liminf_{j \to +\infty} \int_{J_{u_{k_j}}} (\bar{v}_{k_j}^2 + u_{k_j}^2) \, d\mathcal{H}^{n-1}.
\]
\[
\mathcal{L}^n(\{ u > 0 \} \setminus \Omega) \leq \liminf_{j \to +\infty} \mathcal{L}^n(\{ u_{k_j} > 0 \} \setminus \Omega).
\]

Finally we have
\[
\mathcal{F}(u) \leq \liminf_{j \to +\infty} \mathcal{F}(u_{k_j}) = \inf \left\{ \mathcal{F}(v) \mid v \in \text{SBV}^\frac{1}{2}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \right\},
\]

Therefore $u$ is a minimizer to problem (1.5).

**Theorem 3.4 (Euler-Lagrange equation).** Let $u$ be a minimizer to problem (1.5), and let $v \in$ SBV$^\frac{1}{2}(\mathbb{R}^n)$ such that $J_v \subseteq J_u$. Assume that there exists $t > 0$ such that $\{ v > 0 \} \subseteq \{ u > t \}$ $\mathcal{L}^n$-a.e., and that
\[
\int_{\mathcal{J}_u \setminus \mathcal{J}_v} v^2 \, d\mathcal{H}^{n-1} < +\infty.
\]

Then
\[
\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, d\mathcal{L}^n + \beta \int_{J_u} (\bar{v}\gamma^+(v) + u\gamma^-(v)) \, d\mathcal{H}^{n-1} = \int_{\Omega} fv \, d\mathcal{L}^n, \tag{3.7}
\]

where $\gamma^\pm = \gamma^\pm_{J_u}$.

**Proof.** Notice that since $v \in$ SBV$^\frac{1}{2}(\mathbb{R}^n)$ with $J_v \subseteq J_u$ we have that $v \in$ SBV$^\frac{1}{2}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$. Assume $v \in$ SBV$^\frac{1}{2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. If $s \in \mathbb{R}$, recalling that $\{ v > 0 \} \subseteq \{ u > t \}$ $\mathcal{L}^n$-a.e.,
\[
u(x) + sv(x) = u(x) \geq 0 \quad \mathcal{L}^n\text{-a.e. } \forall x \in \{ u \leq t \},
\]

while, for $|s|$ small enough,
\[
u(x) + sv(x) \geq t - |s| \|v\|_\infty > 0 \quad \forall x \in \{ u > t \}.
\]
Therefore we still have

\[ u + sv \in \text{SBV}\frac{1}{2}(\mathbb{R}^n, \mathbb{R}^+). \]

Moreover by minimality of \( u \) we have, for every \( |s| \leq s_0 \)

\[
\mathcal{F}(u) \leq \mathcal{F}(u + sv) \\
= \int_{\mathbb{R}^n} |\nabla u + s\nabla v|^2 \, d\mathcal{L}^n \\
+ \int_{J_{u+sv}} \left[ (\gamma^+(u) + s\gamma^+(v))^2 + (\gamma^-(u) + s\gamma^-(v))^2 \right] \, d\mathcal{H}^{n-1} \\
- 2 \int_{\mathbb{R}^n} f(u + sv) \, d\mathcal{L}^n + C_0 \mathcal{L}^n(\{ u > 0 \}).
\]

**Claim:** The set

\[
S := \{ s \in [-s_0, s_0] \mid \mathcal{H}^{n-1}(J_u \setminus J_{u+sv}) \neq 0 \}
\]

is at most countable.

Let us define

\[
D_0 = \{ x \in J_u \mid \gamma^+(u)(x) \neq \gamma^-(u)(x) \}, \\
D_s = \{ x \in J_u \mid \gamma^+(u + sv)(x) \neq \gamma^-(u + sv)(x) \},
\]

and notice that

\[
\mathcal{H}^{n-1}(J_u \setminus D_0) = 0, \quad \mathcal{H}^{n-1}(J_{u+sv} \setminus D_s) = 0.
\]

Then we have to prove that

\[
\{ s \in [-s_0, s_0] \mid \mathcal{H}^{n-1}(D_0 \setminus D_s) \neq 0 \}
\]

is at most countable. Observe that if \( t \neq s \),

\[
(D \setminus D_t) \cap (D \setminus D_s) = \emptyset.
\]

Indeed if \( x \in D \setminus D_s \)

\[
\gamma^+(u)(x) \neq \gamma^-(u)(x), \\
\gamma^+(u) + s\gamma^+(v)(x) = \gamma^-(u) + s\gamma^-(v)(x),
\]

then

\[
\gamma^+(v)(x) \neq \gamma^-(v)(x),
\]
and so
\[ s = \frac{\gamma^-(u)(x) - \gamma^+(u)(x)}{\gamma^+(v)(x) - \gamma^-(v)(x)}. \]

If \( H^0 \) denotes the counting measure in \( \mathbb{R} \), we can write
\[
\int_{-s_0}^{s_0} H^{n-1}(D_0 \setminus D_s) \, dH^0 = H^{n-1}\left( \bigcup_{(-s_0,s_0)} D_0 \setminus D_s \right) \leq H^{n-1}(J_u) < +\infty,
\]
then the claim is proved.

We are now able to differentiate in \( s = 0 \) the function \( F(u + sv) \), and observing that \( 0 \notin S \) is a minimum for \( F(u + sv) \), we get
\[
\frac{1}{2} \delta F(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dL^n + \beta \int_{J_u} [\pi\gamma^+(v) + \omega\gamma^-(v)] \, dH^{n-1} = \int_{\Omega} fv \, dL^n = 0.
\]

If \( v \notin L^\infty(\mathbb{R}^n) \), we consider \( v_h = \min\{v, h\} \). Then
\[
\delta F(u, v_h) = 0 \quad \forall h > 0.
\]

Observe that, since \( \gamma^\pm(v_h) = \min\{\gamma^\pm(v), h\} \),
\[
\gamma^\pm(v_h) \to \gamma^\pm(v) \quad H^{n-1}\text{-a.e. in } J_u.
\]

Therefore, passing to limit for \( h \to +\infty \), by dominated convergence on the term
\[
\int_{\mathbb{R}^n} \nabla u \cdot \nabla v_h \, dL^n,
\]
and by monotone convergence on the terms
\[
\beta \int_{J_u} [\pi\gamma^+(v_h) + \omega\gamma^-(v_h)] \, dH^{n-1}, \quad \int_{\Omega} fv_h \, dL^n,
\]
we get
\[
0 = \lim_{h} \delta F(u, v_h) = \delta F(u, v).
\]

We now want to use the Euler-Lagrange equation (3.7) to prove that if \( f \) belongs to \( L^p(\Omega) \) with \( p > n \), and if \( u \) is a minimizer to problem (1.5) then \( u \) belongs to \( L^\infty(\mathbb{R}^n) \). In order to prove this we need the following

**Lemma 3.5.** Let \( m \) be a positive real number. There exists a positive constant \( C = C(m, \beta, n) \) such that, for every function \( v \in SBV^1(\mathbb{R}^n) \) with \( L^\infty(\{ v > 0 \}) \leq m \),
\[
\left( \int_{\mathbb{R}^n} v^{2-1^*} \, dL^n \right)^{\frac{1}{2}} \leq C \left[ \int_{\mathbb{R}^n} |\nabla v|^2 \, dL^n + \beta \int_{J_u} (\pi^2 + v^2) \, dH^{n-1} \right],
\]
where \( 1^* = \frac{n}{n-1} \) is the Sobolev conjugate of 1.

**Proof.** Classical Embedding of \( BV(\mathbb{R}^n) \) in \( L^{1^*}(\mathbb{R}^n) \) ensures that

\[
\left( \int_{\mathbb{R}^n} v^{2-1^*} \ d\mathcal{L}^n \right)^{1^*} \leq C(n) \| Dv^2 \| (\mathbb{R}^n)
\]

\[
= C(n) \left[ \int_{\mathbb{R}^n} 2v |\nabla v| \ d\mathcal{L}^n + \int_{J_v} (v^2 + \bar{v}^2) \ d\mathcal{H}^{n-1} \right].
\]

For every \( \varepsilon > 0 \), using Young’s and Hölder’s inequalities, we have

\[
\left( \int_{\mathbb{R}^n} v^{2-1^*} \ d\mathcal{L}^n \right)^{1^*} \leq \frac{C(n)}{\varepsilon} \int_{\mathbb{R}^n} v^2 \ d\mathcal{L}^n
\]

\[
\quad + C(n) \left[ \varepsilon \int_{\mathbb{R}^n} |\nabla v|^2 \ d\mathcal{L}^n + \int_{J_v} (v^2 + \bar{v}^2) \ d\mathcal{H}^{n-1} \right]
\]

\[
\leq \frac{C(n) m^\frac{1}{n}}{\varepsilon} \left( \int_{\mathbb{R}^n} v^{2-1^*} \ d\mathcal{L}^n \right)^{1^*}
\]

\[
\quad + C(n) \left[ \varepsilon \int_{\mathbb{R}^n} |\nabla v|^2 \ d\mathcal{L}^n + \int_{J_v} (v^2 + \bar{v}^2) \ d\mathcal{H}^{n-1} \right].
\]

Setting \( \varepsilon = 2C(n)m^\frac{1}{n} \), we can find two constants \( C(m, n), C(m, \beta, n) > 0 \) such that

\[
\left( \int_{\mathbb{R}^n} v^{2-1^*} \ d\mathcal{L}^n \right)^{1^*} \leq C(m, n) \left[ \int_{\mathbb{R}^n} |\nabla v|^2 \ d\mathcal{L}^n + \int_{J_v} (v^2 + \bar{v}^2) \ d\mathcal{H}^{n-1} \right]
\]

\[
\leq C(m, \beta, n) \left[ \int_{\mathbb{R}^n} |\nabla v|^2 \ d\mathcal{L}^n + \beta \int_{J_v} (v^2 + \bar{v}^2) \ d\mathcal{H}^{n-1} \right].
\]

\[\square\]

We refer to [15] for the following lemma.

**Lemma 3.6.** Let \( g : [0, +\infty) \to [0, +\infty) \) be a decreasing function and assume that there exist \( C, \alpha > 0 \) and \( \theta > 1 \) constants such that for every \( h > k \geq 0 \),

\[
g(h) \leq C(h - k)^{-\alpha} g(k)^\theta.
\]

Then there exists a constant \( h_0 > 0 \) such that

\[
g(h) = 0 \quad \forall h \geq h_0.
\]

In particular we have

\[
h_0 = C^\frac{1}{\alpha} g(0)^{\frac{\theta-1}{\alpha}} 2^{\theta(\theta-1)}.
\]
**Proposition 3.7** \( (L^\infty \text{ bound}) \). Let \( n \geq 2 \) and assume that, if \( n = 2 \), condition \((1.6)\) holds true. Let \( f \in L^p(\Omega) \), with \( p > n \). Then there exists a constant \( C = C(\Omega, f, p, \beta, C_0) > 0 \) such that if \( u \) is a minimizer to problem \((1.5)\), then

\[
\|u\|_{L^\infty} \leq C.
\]

**Proof.** Let \( \gamma^+ = \gamma^+_u \). For every \( \varphi, \psi \in SBV^{\frac{1}{2}}(\mathbb{R}^n) \) satisfying \( J_{\varphi}, J_{\psi} \subseteq J_u \), define

\[
a(\varphi, \psi) = \int_{\mathbb{R}^n} \nabla \varphi \cdot \nabla \psi \, d\mathcal{L}^n + \beta \int_{J_u} [\gamma^+(\varphi)\gamma^+(\psi) + \gamma^-(\varphi)\gamma^-(\psi)] \, dH^{n-1}.
\]

For every \( v \) satisfying the assumptions of Theorem 3.4, it holds that

\[
a(u, v) = \int_{\Omega} fv \, d\mathcal{L}^n.
\]

In particular, let us fix \( k \in \mathbb{R}^+ \) and define

\[
\varphi_k(x) = \begin{cases} u(x) - k & \text{if } u(x) \geq k, \\ 0 & \text{if } u(x) < k,
\end{cases}
\]

then

\[
\gamma^+(\varphi_k)(x) = \begin{cases} \pi(x) - k & \text{if } \pi(x) \geq k, \\ 0 & \text{if } \pi(x) < k,
\end{cases}
\]

and analogously for \( \gamma^-(\varphi_k) \). Furthermore, let us define

\[
\mu(k) = \mathcal{L}^n(\{ u > k \}).
\]

We want to prove that \( \mu(k) = 0 \) for sufficiently large \( k \). From Theorem 3.4, we have

\[
a(u, \varphi_k) = \int_{\Omega} f\varphi_k \, d\mathcal{L}^n, \tag{3.8}
\]

and we can observe that

\[
a(u, \varphi_k) = \int_{\{ u > k \}} |\nabla u|^2 \, d\mathcal{L}^n + \beta \int_{J_u \cap \{ u > k \}} [\pi(u - k) + u(u - k)] \, dH^{n-1} \\
\geq \int_{\{ u > k \}} |\nabla u|^2 \, d\mathcal{L}^n + \beta \int_{J_u \cap \{ u > k \}} [(u - k)^2 + (u - k)^2] \, dH^{n-1} \\
= a(\varphi_k, \varphi_k).
\]

Moreover, by minimality, \( \mathcal{F}(u) \leq \mathcal{F}(0) = 0 \) and by Remark 3.2, \( \Omega \subseteq \{ u > 0 \} \). Therefore, \((3.1)\) holds true and we can apply Lemma 3.5, having that there exists \( C = C(\Omega, f, \beta, C_0) > 0 \) such that

\[
\int_{\Omega} f\varphi_k \, d\mathcal{L}^n = a(u, \varphi_k) \geq a(\varphi_k, \varphi_k) \geq C\|\varphi_k\|_{L^1}^2. \tag{3.9}
\]
On the other hand

$$\int_{\Omega} f \varphi_k \, d\mathcal{L}^n = \int_{\Omega \cap \{ u > k \}} f(u - k) \, d\mathcal{L}^n \leq \left( \int_{\Omega \cap \{ u > k \}} f^{\frac{2}{p+1}} \, d\mathcal{L}^n \right)^{\frac{p+1}{2}} \| \varphi_k \|_{2,1^*} \tag{3.10}$$

where

$$\sigma = \frac{p(n + 1)}{2n} > 1,$$

since $p > n$. Joining (3.9) and (3.10), we have

$$\| \varphi_k \|_{2,1^*} \leq C \| f \|_{p,\Omega} \mu(k)^{\frac{n+1}{2n+1}}, \tag{3.11}$$

Let $h > k$, then

$$(h - k)^{2-1^*} \mu(h) = \int_{\{ u > h \}} (h - k)^{2-1^*} \, d\mathcal{L}^n$$

$$\leq \int_{\{ u > h \}} (u - k)^{2-1^*} \, d\mathcal{L}^n$$

$$\leq \int_{\{ u > k \}} (u - k)^{2-1^*} \, d\mathcal{L}^n = \| \varphi_k \|_{2,1^*}.$$

Using (3.11) and the previous inequality, we have

$$\mu(h) \leq C(h - k)^{-2-1^*} \mu(k)^{\frac{n+1}{(n-1)\sigma}}.$$

Since $p > n$, then $\sigma' < (n + 1)/(n - 1)$. By Lemma 3.6, we have that $\mu(h) = 0$ for all $h \geq h_0$ with $h_0 = h_0(\Omega, f, \beta, C_0) > 0$, which implies

$$\| u \|_\infty \leq h_0.$$

Proof of Theorem 1.1. The result is obtained by joining Proposition 3.3 and Proposition 3.7.

4. DENSITY ESTIMATES FOR THE JUMP SET

In this section we prove Theorem 1.2: in Proposition 4.6 we prove the lower bound for minimizers to problem (1.5); in Proposition 4.8 and Proposition 4.9 we prove the density estimates for the jump set of a minimizer to problem (1.5).

In this section, we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $C^{1,1}$ boundary, that $f \in L^p(\Omega)$, with $p > n$, is a positive function, and that $\beta, C_0$ are positive constants. We consider the energy functional $F$ defined in (1.4).

In order to show that if $u$ is a minimizer to problem (1.5) then $u$ is bounded away from 0, we will first prove that there exists a positive constant $\delta$ such that $u > \delta$ almost everywhere in $\Omega$, and then we will show that this implies the existence of a positive constant $\delta_0$ such that $u > \delta_0$ almost everywhere in the set $\{ u > 0 \}$. In the following we will denote by $U_t := \{ u < t \} \cap \Omega$. 


Remark 4.1. Let $u$ be a minimizer to (1.5), by Remark 3.2, $u$ is a solution to

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u \geq 0 & \text{on } \partial \Omega
\end{cases}
$$

\hspace{1cm}

Let $u_0 \in W^{1,2}_0(\Omega)$ be the solution to the following boundary value problem

$$
\begin{cases}
-\Delta u_0 = f & \text{in } \Omega, \\
u_0 = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{4.1}
$$

Then, by maximum principle,

$$
u \geq u_0 \quad \text{in } \Omega \subseteq \{ u > 0 \} \quad \text{and} \quad \{ u < t \} \cap \Omega = U_t \subseteq \{ u_0 < t \} \cap \Omega.
$$

Lemma 4.2. There exist two positive constants $t_0 = t_0(\Omega, f)$ and $C = C(\Omega, f)$ such that if $u$ is a minimizer to (1.5) then for every $t \in [0, t_0]$ it results

$$
\mathcal{L}^n(U_t) \leq C t.
\tag{4.2}
$$

Proof. Let $u_0$ be the solution to (4.1), fix $\varepsilon > 0$ such that the set

$$
\Omega_\varepsilon = \{ x \in \Omega \mid d(x, \partial \Omega) > \varepsilon \}
$$

is not empty. Since $u_0$ is superharmonic and non-negative in $\Omega$, by maximum principle we have that

$$
\alpha = \inf_{\Omega_\varepsilon} u_0 > 0.
$$

then $u_0$ solves

$$
\begin{cases}
-\Delta u_0 = f & \text{in } \Omega, \\
u_0 = 0 & \text{on } \partial \Omega, \\
u_0 \geq \alpha & \text{on } \partial \Omega_\varepsilon.
\end{cases}
$$

Therefore, if we consider the solution $v$ to the following boundary value problem

$$
\begin{cases}
-\Delta v = 0 & \text{in } \Omega \setminus \overline{\Omega}_\varepsilon, \\
v = 0 & \text{on } \partial \Omega, \\
v = \alpha & \text{in } \overline{\Omega}_\varepsilon,
\end{cases}
$$

we have that $u \geq u_0 \geq v$ almost everywhere in $\Omega$ and

$$
\{ u < t \} \cap \Omega = U_t \subseteq \{ u_0 < t \} \cap \Omega \subseteq \{ v < t \} \cap \Omega.
$$

Hopf Lemma implies that

there exists a constant $\tau = \tau(\Omega) > 0$ such that

$$
\frac{\partial v}{\partial \nu} < -\tau \quad \text{on } \partial \Omega.
$$
Let \( x \in \Omega \), and let \( x_0 \) be a projection of \( x \) onto the boundary \( \partial \Omega \), then

\[
|x - x_0| = d(x, \partial \Omega), \quad \frac{x - x_0}{|x - x_0|} = -\nu_\Omega(x_0),
\]

where \( \nu_\Omega \) denotes the exterior normal to \( \partial \Omega \). We can write

\[
v(x) = v(x_0) + \nabla v(x_0) \cdot (x - x_0) + o(|x - x_0|)
\]

\[
= -\frac{\partial v}{\partial \nu}(x_0)|x - x_0| + o(|x - x_0|)
\]

\[
\geq \tau |x - x_0| + o(|x - x_0|)
\]

\[
> \frac{\tau}{2} |x - x_0| = \frac{\tau}{2} d(x, \partial \Omega)
\]

(4.3)

for every \( x \) such that \( d(x, \partial \Omega) < \sigma_0 \) for a suitable \( \sigma_0 = \sigma_0(\Omega, f) > 0 \). Notice that if \( \bar{x} \in \Omega \) and \( \lim_{x \to \bar{x}} v(x) = 0 \) then necessarily \( \bar{x} \in \partial \Omega \). Therefore, there exists a \( t_0 = t_0(\Omega, f) > 0 \) such that \( v(x) < t_0 \) implies \( d(x, \partial \Omega) < \sigma_0 \). Consequently, if \( t < t_0 \), we have that

\[
\{ v < t \} \subseteq \{ d(x, \partial \Omega) < \sigma_0 \},
\]

and by (4.3), we get

\[
\mathcal{L}^n(U_t) \leq \mathcal{L}^n(\{ v < t \}) \leq \mathcal{L}^n\left( \left\{ x \in \Omega \mid d(x, \partial \Omega) \leq \frac{2}{\tau} t \right\} \right).
\]

Since \( \Omega \) is \( C^{1,1} \), by Proposition 2.18, we conclude the proof.

\[ \square \]

**Lemma 4.3.** Let \( g : [0, t_1] \to [0, +\infty) \) be an increasing, absolutely continuous function such that

\[
g(t) \leq C t^\alpha (g'(t))^\sigma \quad \forall t \in [0, t_1],
\]

(4.4)

with \( C > 0 \) and \( \alpha > \sigma > 1 \). Then there exists \( t_0 > 0 \) such that

\[
g(t) = 0 \quad \forall t \leq t_0.
\]

Precisely,

\[
t_0 = \left( \frac{C(\alpha - \sigma)}{\sigma - 1} g(t_1)^{\frac{\sigma - 1}{\sigma}} + t_1^\frac{\sigma - \alpha}{\sigma} \right)^{\frac{\alpha}{\sigma - \alpha}}.
\]

**Proof.** Assume by contradiction that \( g(t) > 0 \) for every \( t > 0 \). Inequality (4.4) implies

\[
\frac{g'}{g^\frac{\sigma}{\alpha}} \geq \frac{1}{C} t^{-\frac{\alpha}{\sigma}}.
\]
Integrating between $t_0$ and $t_1$, we have
\[ \frac{\sigma}{\sigma - 1} \left( g(t_1) \frac{\sigma - 1}{\sigma} - g(t_0) \frac{\sigma - 1}{\sigma} \right) \geq \frac{1}{C} \frac{\sigma}{\sigma - \alpha} \left( t_1^\frac{\sigma - \alpha}{\sigma} - t_0^\frac{\sigma - \alpha}{\sigma} \right). \]

Since $\alpha > \sigma > 1$, we have
\[ 0 \leq g(t_0) \frac{\sigma - 1}{\sigma} \leq \frac{\sigma - 1}{C(\alpha - \sigma)} \left( t_1^\frac{\sigma - \alpha}{\sigma} - t_0^\frac{\sigma - \alpha}{\sigma} \right) + g(t_1) \frac{\sigma - 1}{\sigma}, \]
which is a contradiction if
\[ t_0 \leq \left( \frac{C(\alpha - \sigma)}{\sigma - 1} g(t_1) \frac{\sigma - 1}{\sigma} + t_1 \frac{\sigma - \alpha}{\sigma} \right) \frac{\sigma - \alpha}{\sigma}. \]

\[ \square \]

**Remark 4.4.** Let $g$ be as in Lemma 4.3 and assume that $g(t_1) \leq K$, then $g(t) = 0$ for all $0 < t < \tilde{t}$ where
\[ \tilde{t} = \left( \frac{C(\alpha - \sigma)}{\sigma - 1} K \frac{\sigma - 1}{\sigma} + t_1 \frac{\sigma - \alpha}{\sigma} \right) \frac{\sigma - \alpha}{\sigma}. \]

We now have the tools to prove the lower bound inside $\Omega$.

**Proposition 4.5.** There exists a positive constant $\delta = \delta(\Omega, f, p, \beta, C_0) > 0$ such that if $u$ is a minimizer to problem (1.5) then
\[ u \geq \delta \]
almost everywhere in $\Omega$.

**Proof.** Assume that $\Omega$ is connected and define the function
\[ u_t(x) = \begin{cases} \max\{u, t\} & \text{in } \Omega, \\ u & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \]
Recalling that $U_t = \{ u < t \} \cap \Omega$, we have
\[ J_{u_t} \setminus \partial^* U_t = J_u \setminus \partial^* U_t, \]
and on this set $u_t = u$ and $\mathring{U_t} = \mathring{U}$. 

Then we get by minimality of $u$, and using the fact that $J_{u_t} \cap \partial^* U_t \subseteq \partial \Omega$,

$$0 \geq F(u) - F(u_t)$$

$$= \int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n - 2 \int_{U_t} f(u - t) \, d\mathcal{L}^n + \beta \int_{\partial^* U_t \cap J_u} (u^2 + \overline{u}^2) \, d\mathcal{H}^{n-1} +$$

$$- \beta \int_{J_{u_t} \cap \partial^* U_t \cap J_u} \left[ t^2 + (\gamma_{\partial^* \Omega}(u))^2 \right] \, d\mathcal{H}^{n-1} - \beta \int_{(J_{u_t} \cap \partial^* U_t) \setminus J_u} (t^2 + u^2) \, d\mathcal{H}^{n-1}$$

$$\geq \int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n - 2 \beta t^2 \mathcal{H}^{n-1}(\partial^* U_t \cap \partial \Omega)$$

where we ignored all the non-negative terms except the integral of $|\nabla u|^2$, and we used that $u \leq t$ in $\partial^* U_t \setminus J_u$. By Lemma 4.2, we can choose $t$ small enough to have $\mathcal{L}^n(U_t) \leq \mathcal{L}^n(\Omega)/2$, then applying the isoperimetric inequality in Theorem 2.6 to the set $E = U_t$, we get

$$\int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n \leq 2\beta Ct^2 P(U_t; \Omega). \quad (4.5)$$

Let us define

$$p(t) = P(U_t; \Omega),$$

and consider the absolutely continuous function

$$g(t) = \int_{U_t} u |\nabla u| \, d\mathcal{L}^n = \int_0^t s p(s) \, ds.$$

By minimality of $u$ we can apply the a priori estimates (3.6) to prove the equiboundedness of $g$, i.e. there exists $K = K(\Omega, f, \beta, C_0) > 0$ such that $g(t) \leq K$ for all $t > 0$. Using the Hölder inequality and the estimate (4.5) we have

$$g(t) \leq \left( \int_{U_t} u^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}} \left( \int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}} \leq \sqrt{2\beta C t \mathcal{L}^n(U_t)^{\frac{1}{2}} (t^2 p(t))^{\frac{1}{2}}}.$$

Fix $1 > \varepsilon > 0$. Then we can write $\mathcal{L}^n(U_t) = \mathcal{L}^n(U_t)^{\varepsilon} \mathcal{L}^n(U_t)^{1-\varepsilon}$, and by Lemma 4.2 there exists a constant $C = C(\Omega, f, \beta) > 0$ such that

$$g(t) \leq C t^{2 + \frac{1-\varepsilon}{\varepsilon} \mathcal{L}^n(U_t)^{\varepsilon} p(t)^{\frac{1}{\varepsilon}}}.$$

By the relative isoperimetric inequality in Theorem 2.5, we can estimate

$$\mathcal{L}^n(U_t)^{\frac{1}{\varepsilon}} \leq C(\Omega, n) p(t)^{\frac{n}{n-1}},$$

and, noticing that $p(t) = g'(t)/t$, we get

$$g(t) \leq Ct^{\alpha}(g'(t))^\sigma,$$
where
\[ \alpha = 2 - \frac{\varepsilon}{2} \left( 1 + \frac{n}{n-1} \right), \quad \sigma = \frac{1}{2} + \frac{\varepsilon}{2} \frac{n}{n-1}. \]

In particular, if we choose
\[ \varepsilon \in \left( \frac{n-1}{n}, \frac{3n-3}{3n-1} \right), \]
we have that \( \alpha > \sigma > 1 \), and then, using Lemma 4.3 and Remark 4.4, there exists a \( \delta = \delta(\Omega, f, p, \beta, C_0) > 0 \) such that \( g(t) = 0 \) for every \( t < \delta \). Then \( L^n(\{ u < t \} \cap \Omega) = 0 \) for every \( t < \delta \), hence
\[ u \geq \delta \]
after everywhere in \( \Omega \).

When \( \Omega \) is not connected, then
\[ \Omega = \Omega_1 \cup \cdots \cup \Omega_N, \]
with \( \Omega_i \) pairwise disjoint connected open sets. Using \( u_t \) as the function \( u \) truncated inside a single \( \Omega_i \), we find constants \( \delta_i > 0 \) such that
\[ u(x) \geq \delta_i \]
after everywhere in \( \Omega_i \). Therefore choosing \( \delta = \min \{ \delta_1, \ldots, \delta_N \} \) we have \( u(x) > \delta \) almost everywhere in \( \Omega \).

Finally, following the approach in [7], we have

**Proposition 4.6 (Lower Bound).** There exists a positive constant \( \delta_0 = \delta_0(\Omega, f, p, \beta, C_0) \) such that if \( u \) is a minimizer to problem (1.5) then
\[ u \geq \delta_0 \]
after everywhere in \( \{ u > 0 \} \).

**Proof.** Let \( \delta \) be the constant in Proposition 4.5. For every \( 0 < t \leq \delta \) let us define the absolutely continuous function
\[ h(t) = \int_{\{ u \leq t \} \setminus J_u} u \left| \nabla u \right| \, dL^n = \int_0^t sP(\{ u > s \} ; \mathbb{R}^n \setminus J_u) \, ds. \]

By minimality of \( u \) we can apply the a priori estimates (3.6) to prove the equiboundedness of \( h \), i.e. there exists \( K = K(\Omega, f, \beta, C_0) > 0 \) such that \( h(t) \leq K \) for all \( t > 0 \).

We will show that \( h \) satisfies a differential inequality. For any \( 0 < t < \delta \), let us consider \( u^t = u \chi_{\{ u > t \}} \), where \( \chi_{\{ u > t \}} \) is the characteristic function of the set \( \{ u > t \} \), as a competitor for \( u \). We observe that, by Proposition 4.5,
\[ \Omega \subseteq \{ u > t \}, \text{ so we have that} \]
\[ 0 \geq F(u) - F(u') \]
\[ = \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \, d\mathcal{L}^n + \beta \int_{J_u \cap \{ u > t \}^{(0)}} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} \]
\[ + \beta \int_{J_u \cap \partial^* \{ u > t \}} u^2 \, d\mathcal{H}^{n-1} - \beta \int_{\partial^* \{ u > t \} \setminus J_u} u^2 \, d\mathcal{H}^{n-1} \]
\[ + C_0 \mathcal{L}^n (\{ 0 < u \leq t \}). \]

Rearranging the terms,
\[ \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \, d\mathcal{L}^n + \beta \int_{J_u \cap \{ u > t \}^{(0)}} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} \]
\[ + \beta \int_{J_u \cap \partial^* \{ u > t \}} u^2 \, d\mathcal{H}^{n-1} - \beta \int_{\partial^* \{ u > t \} \setminus J_u} u^2 \, d\mathcal{H}^{n-1} \]
\[ \leq \beta t^2 P \left( \left\{ u > t \right\} : \mathbb{R}^n \setminus J_u \right) = \beta t h'(t). \]

On the other hand using Hölder’s inequality, we have
\[ h(t) \leq \left( \int_{\{ u \leq t \}} |\nabla u|^2 \, d\mathcal{L}^n \right)^{\frac{2}{3}} \mathcal{L}^n (\{ 0 < u \leq t \})^{\frac{1}{3}} \left( \int_{\{ u \leq t \}} u^{2-1^*} \, d\mathcal{L}^n \right)^{\frac{1}{2}}. \] (4.7)

Classical Embedding of BV in \( L^{1^*} \) applied to \( u^2 \chi_{\{ u \leq t \}} \), ensures
\[ \left( \int_{\{ u \leq t \}} u^{2-1^*} \, d\mathcal{L}^n \right)^{\frac{1}{2^*}} \leq C(n)|D(u^2 \chi_{\{ u \leq t \}})|(\mathbb{R}^n), \]

and, using (4.6),
\[ |D(u^2 \chi_{\{ u \leq t \}})|(\mathbb{R}^n) = 2 \int_{\{ u \leq t \}} u |\nabla u| \, d\mathcal{L}^n + \beta \int_{J_u \cap \{ u > t \}^{(0)}} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} \]
\[ + \beta \int_{J_u \cap \partial^* \{ u > t \}} u^2 \, d\mathcal{H}^{n-1} + \beta \int_{\partial^* \{ u > t \} \setminus J_u} u^2 \, d\mathcal{H}^{n-1} \]
\[ \leq 2t \left( \mathcal{L}^n (\{ 0 < u \leq t \}) \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}} + 3th'(t) \]
\[ \leq \left( 2 \frac{\delta \beta}{\sqrt{C_0}} + 3 \right) t h'(t). \] (4.8)

Therefore, joining (4.7), (4.6), and (4.8), we have
\[ h(t) \leq C_3 (th'(t))^{1 + \frac{1}{n^*}}, \] (4.9)
where
\[ C_3 = \beta^{\frac{1}{2}} \left( \frac{\beta}{C_0} \right)^{\frac{1}{2}} C(n)^{\frac{1}{2}} \left( 2 \frac{\delta \beta}{\sqrt{C_0}} + 3 \right)^{\frac{1}{2}}. \]

By (4.9) we now want to show that there exists \( \delta_0 = \delta_0(\Omega, f, p, \beta, C_0) > 0 \) such that \( h(t) = 0 \) for every \( 0 \leq t < \delta_0 \). Indeed assume by contradiction that \( h(t) > 0 \) for every \( 0 < t \leq \delta \). We have

\[ \frac{h'(t)}{h(t)^{\frac{2n}{2n+1}}} \geq C_3^{-\frac{2n}{2n+1}} \left( \frac{2n}{2n+1} \right). \]

Integrating from \( t_0 > 0 \) to \( \delta \), we get

\[ \left( h(\delta)^{\frac{1}{2n+1}} - h(t_0)^{\frac{1}{2n+1}} \right) \geq C_4 \log \left( \frac{\delta}{t_0} \right), \]

where
\[ C_4 = C_3^{-\frac{2n}{2n+1}} \frac{2n}{2n+1}. \]

Then
\[ h(t_0)^{\frac{1}{2n+1}} \leq h(\delta)^{\frac{1}{2n+1}} + C_4 \log \left( \frac{t_0}{\delta} \right). \]

Finally, for any
\[ 0 < t_0 \leq \bar{\delta} = \delta \exp \left( -h(\delta)^{\frac{1}{2n+1}}/C_4 \right), \]
we have \( h(t_0) < 0 \), which is a contradiction. Then, setting \( \delta_0 = \delta \exp \left( -K^{\frac{1}{2n+1}}/C_4 \right) \leq \bar{\delta} \), we conclude that \( h(t) = 0 \) for any \( 0 < t < \delta_0 \), from which we have
\[ u \geq \delta_0 \]
almost everywhere in \{ \( u > 0 \) \}.

**Remark 4.7.** From Proposition 4.6, if \( u \) is a minimizer to problem (1.5), we have that
\[ \partial^* \{ u > 0 \} \subseteq J_u \subseteq K_u. \tag{4.10} \]
Indeed, on \( \partial^* \{ u > 0 \} \) we have that, by definition, \( u = 0 \) and that, since \( u \geq \delta_0 \mathcal{L}^n \text{-a.e. in } \{ u > 0 \} \), \( \bar{u} \geq \delta_0 \).

**Proposition 4.8 (Density Estimates).** There exist positive constants \( C = C(\Omega, f, p, \beta, C_0) \), \( c = c(\Omega, f, p, \beta, C_0) \) and \( \delta_1 = \delta_1(\Omega, f, p, \beta, C_0) \) such that if \( u \) is a minimizer to problem (1.5) then for every \( B_r(x) \) such that \( B_r(x) \cap \Omega = \emptyset \), we have:

(a) For every \( x \in \mathbb{R}^n \setminus \Omega \),
\[ \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C r^{n-1}; \tag{4.11} \]
(b) For every $x \in K_u$,
\[ \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq cr^n; \]  
\[ (4.12) \]

(c) The function $u$ has bounded support, namely
\[ \{ u > 0 \} \subseteq B_{1/\delta_1}. \]

Proof. This theorem is a consequence of Proposition 4.6, since we immediately have
\[ \int_{J_u \cap B_r(x)} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} \geq \delta_0^2 \mathcal{H}^{n-1}(J_u \cap B_r(x)), \]
and by minimality of $u$ we have (a)
\[
0 \geq F(u) - F(u \chi_{\mathbb{R}^n \setminus B_r(x)}) \\
\geq \int_{J_u \cap B_r(x)} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} - \int_{\partial B_r(x) \setminus J_u} u^2 \, d\mathcal{H}^{n-1} \\
\geq \int_{J_u \cap B_r(x)} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} - \int_{\partial B_r(x) \setminus \{ u>0 \}^{(1)}} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} \\
\geq \int_{J_u \cap B_r(x)} (u^2 + \pi^2) \, d\mathcal{H}^{n-1} - 2\|u\|_\infty^2 \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)}),
\]
where, in the second inequality, we have used (4.10). Thus we have
\[ \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \frac{2\|u\|_\infty^2}{\delta_0^2} \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)}) \leq Cr^{n-1}, \]
where $C = C(\Omega, f, p, \beta, C_0) > 0$.

(b) We now want to use the estimate (4.11) together with the relative isoperimetric inequality in order to get a differential inequality for the volume of $B_r(x) \cap \{ u > 0 \}^{(1)}$. Let $x \in K_u$, then for almost every $r$ we have
\[ 0 < V(r) := \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}^{(1)}) \leq k P(B_r(x) \cap \{ u > 0 \}^{(1)}) \frac{n-1}{n} \]
\[ \leq k \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)}) \frac{n-1}{n}, \]
where $k = k(\Omega, f, p, \beta, C_0) > 0$, and in the last inequality we used that (4.10) and (4.13) imply
\[ P \left( B_r(x) \cap \{ u > 0 \}^{(1)} \right) \leq \mathcal{H}^{n-1} \left( \partial B_r(x) \cap \{ u > 0 \}^{(1)} \right) + \mathcal{H}^{n-1} (J_u \cap B_r(x)) \]
\[ \leq \left( 1 + \frac{2\|u\|_\infty^2}{\delta_0^2} \right) P(B_r(x); \{ u > 0 \}^{(1)}). \]

Then we have
\[ \frac{V'(r)}{V(r)^{\frac{n-1}{n}}} \geq \frac{1}{k}, \]
which implies
\[ \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq cr^n. \]

(c) Finally, let \( x \in K_u \) such that \( d(x, \partial \Omega) \geq 1/\delta_1 \). From (4.12), noticing that \( F(u) \leq F(0) = 0 \), we have that
\[ c\delta_1^{-n} \leq \mathcal{L}^n(\{ u > 0 \} \setminus \Omega) \leq \frac{2\|u\|_{\infty}}{C_0} \int_{\Omega} f \, d\mathcal{L}^n, \]
which is a contradiction if \( \delta_1 \) is sufficiently small. Then the thesis is given by (4.10).

Finally, we have

**Proposition 4.9 (Lower Density Estimate).** There exists a positive constant \( c = c(\Omega, f, p, \beta, C_0) \) such that if \( u \) is a minimizer to problem (1.5) then

1. For any \( x \in K_u \) and \( B_r(x) \subseteq \mathbb{R}^n \setminus \Omega \),
   \[ \mathcal{H}^{n-1}(J_u \cap B_r(x)) \geq cr^{n-1}; \]

2. \( J_u \) is essentially closed, namely
   \[ \mathcal{H}^{n-1}(K_u \setminus J_u) = 0; \]

The proof of Proposition 4.9 relies on classical techniques used in [9] to prove density estimates for the jump set of almost-quasi minimizers of the Mumford-Shah functional. We refer to Theorem 5.1 of [7] and Corollary 5.4 of [7] for the details of the proof.

**Proof of Theorem 1.2.** The result is obtained by joining Proposition 4.6, Proposition 4.8, and Proposition 4.9.

**Remark 4.10.** Given the summability assumption on the function \( f \) and the lower bound given in Proposition 4.6, we have that minimizers to (1.2) are almost-quasi-minimizers of the functional \( \mathcal{G} \), defined on \( \text{SBV}^{1/2}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \) as
\[ \mathcal{G}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(J_v), \]
that is, there exists \( C(\Omega, f, p, \beta, C_0) > 0 \), \( \Lambda(\Omega, f, p, \beta, C_0) \geq \lambda \) and \( \alpha(n, p) > n - 1 \) such that, if \( B_r(x) \) is a ball of radius \( r \leq 1 \), and \( v \in \text{SBV}^{1/2}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \), with \( \{ u \neq v \} \subset B_r(x) \), then
\[ \mathcal{G}_\lambda(u; B_r(x)) \leq \mathcal{G}_\lambda(v; B_r(x)) + Cr^\alpha, \]
where
\[ \mathcal{G}_\lambda(v; B_r(x)) := \int_{B_r(x)} |\nabla v|^2 \, d\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(J_v \cap B_r(x)). \]

Indeed, let \( u \) be a minimizer to (1.2), let \( B_r(x) \) be a ball of radius \( r \leq 1 \), and let \( v \in \text{SBV}^{1/2}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \), with \( \{ u \neq v \} \subset B_r(x) \), and let
\[ w = \min \{ \max \{ v, 0 \}, \|u\|_{\infty} \}. \]
By minimality of \( u \) we have that
\[
G_\lambda(u; B_r(x)) \leq G_\lambda(v; B_r(x)) + Cr^n,
\]
where \( \lambda = \beta \delta_0^2 \) and \( \Lambda = 2 \beta \| u \|_\infty^2 \). Moreover,
\[
\int_{\Omega \cap B_r(x)} fu \, d\mathcal{L}^n \leq \| f \|_{p, \Omega} \| u \|_{\infty} \mathcal{L}^n(B_r)^{1/p'} = C(\Omega, f, p, \beta, C_0)r^\alpha,
\]
where
\[
n > \alpha = \frac{n}{p'} > n - 1.
\]
Finally, we have
\[
G_\lambda(u; B_r(x)) \leq G_\lambda(v; B_r(x)) + Cr^\alpha.
\]

Such a minimality property can be used to prove that the lower density estimate in Proposition 4.9 still holds even when \( B_r(x) \cap \Omega \) is non-empty. Indeed, the above density estimate is a consequence of the following decay lemma

**Lemma 4.11 (Decay lemma).** Let \( 1 > \gamma > n - \alpha \). There exists \( \tau_0 = \tau_0(n, \Omega, \gamma, \lambda) > 0 \) such that for every \( \tau_0 > \tau > 0 \) there exist \( r_0 = r_0(\tau, \Omega), \varepsilon_0 = \varepsilon_0(\tau, \Omega) > 0 \) such that, if \( x_0 \in \partial \Omega, r_0 > r > 0, \) and \( u \) is a almost-quasi minimizer on \( B_r = B_r(x_0) \) for the functional \( \mathcal{G} \) such that
\[
\mathcal{H}^{n-1}(J_u \cap B_r) \leq \varepsilon_0 r^{n-1},
\]
then we have that either
\[
G_\lambda(u; B_r) \leq r^{n-\gamma},
\]
or
\[
G_\lambda(u; B_{rr}) \leq r^{n-\gamma} G_\lambda(u; B_r).
\]

**Proof.** The proof of the decay lemma is similar to the one in Lemma 5.3 of [7], Section 4 of [6], Lemma 4.9 of [9]; the main difference is in the construction of the blow-up sequence of almost-quasi minimizers.

Let \( u_k \) be a sequence of almost-quasi minimizer on \( B_{r_k} \) contradicting the lemma, with \( \lim_k r_k = 0 \). To reach a contradiction one usually constructs a sequence of functions \( \tilde{u}_k \) on the unit ball, related to the sequence \( u_k \), that converges to an harmonic function \( v \). In order to prove that \( v \) is harmonic we construct a sequence of admissible test functions \( \psi_k \) on \( B_{r_k} \) and use the minimality property of \( u_k \). If \( d(x_0, \Omega) > 0 \), then the test function are only required to be in \( \text{SBV}(B_{r_k}) \), while, if \( x_0 \in \partial \Omega \) the additional constraint \( \psi_k \in \text{SBV}(B_{r_k}) \cap W^{1,2}(\Omega \cap B_{r_k}) \) should be treated with more carefulness.

Without loss of generality let \( x_0 = 0 \) and let \( E_k = r_k^{2-n} G_\lambda(u_k; B_{r_k}) \), and define
\[
v_k(x) = \frac{1}{E_k^{1/2}} u_k(r_k x).
\]
For any $k$, we extend $u_k \in W^{1,2}(\Omega \cap B_{r_k})$ to $Lu_k \in W^{1,2}(B_{r_k})$, which is a function such that $u_k - Lu_k \equiv 0$ in $\Omega$. Let us define, with a slight abuse of notation,

$$Lv_k(x) = \frac{1}{E^{1/2}_k} Lu_k(r_k x), \quad w_k = v_k - Lv_k,$$

so that, by construction, and by properties of the blow-up,

$$\lim inf_k L^n(\{w_k = 0\}) \geq \lim inf_k r_k^{-n} L^n(\Omega \cap B_{r_k}) > 0. \quad (4.14)$$

This is the key property: by Poincaré inequality in SBV, there exist $\tilde{w}_k$ truncated functions, such that

$$\lim_k L^n(\{w_k \neq \tilde{w}_k\}) = 0, \quad (4.15)$$

and, up to subtracting medians, $w_k$ converge in $L^2$ to some Sobolev function. By (4.14) and (4.15), and considering that $\tilde{w}_k$ is a truncation of $w_k$, then for big enough $k$, up to $L^n$-negligible sets,

$$\{w_k = 0\} \subseteq \{w_k = \tilde{w}_k\}.$$

This means that if we define $\tilde{v}_k = \tilde{w}_k + Lv_k$, then the scaled back functions

$$\tilde{u}_k(x) := E^{1/2}_k \tilde{v}_k \left( \frac{x}{r_k} \right)$$

respect the property $\tilde{u}_k \equiv u_k$ in $\Omega \cap B_{r_k}$. Moreover, it is possible to choose an extension $L$ (see Lem. 4.12) such that, combining the Poincaré inequality in SBV and the Poincaré inequality in $W^{1,2}$, then there exist constants $c_k$ such that $\tilde{v}_k - c_k$ converge in $L^2$ to a function $v \in W^{1,2}(B_1)$. This ensures that, if we take $\rho \leq \rho'$ small enough, $\eta$ cut-off functions between $B_{\rho}$ and $B_{\rho'}$, and $\varphi \in W^{1,2}(B_1)$, then the test functions $\psi_k = E^{1/2}_k \varphi_k(x/r_k)$, with

$$\varphi_k = \left( \eta(\varphi + c_k) + (1 - \eta) \tilde{v}_k \right) \chi_{B_{\rho'}} + v \chi_{B_1 \setminus B_{\rho'}},$$

are admissible test functions for any $\varphi \in W^{1,2}(B_1)$, leading to similar computations that can be found in the aforementioned papers. \hfill \Box

**Lemma 4.12.** Let $\Omega$ an open set with Lipschitz boundary, and let $x_0 \in \partial \Omega$. There exist positive constants $\rho_0 = \rho_0(\Omega, x_0)$, $C = C(\Omega, x_0)$, $\delta = \delta(\Omega, x_0) > 1$, and an extension operator

$$L : W^{1,2}(\Omega) \rightarrow W^{1,2}(B_{\rho_0}(x_0))$$

such that, for any $u \in W^{1,2}(\Omega)$, and for any $r < \rho_0$, we have that $Lu \equiv u$ in $\Omega \cap B_{\rho_0}(x_0)$ and

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, d\mathcal{L}^n \leq C \int_{\Omega \cap B_{r+}(x_0)} |\nabla u|^2 \, d\mathcal{L}^n. \quad (4.16)$$

**Proof.** We can assume without loss of generality that $x_0 = 0$, and, if $s$ is small enough, we have that, up to rotations,

$$\Omega \cap B_s = \{ (x', x_n) \in B_s \mid \gamma(x') < x_n \},$$
for a suitable Lipschitz function $\gamma$, with $\gamma(0) = 0$. We denote by $\Phi$ the diffeomorphism that flattens the boundary $\partial \Omega$, namely

$$\Phi(x', x_n) = (x', x_n - \gamma(x')),$$

$$\Phi^{-1}(y', y_n) = (y', y_n + \gamma(y')).$$

Let $M = \|\nabla \gamma\|_\infty$, we claim that for any $r < (1 + M)^{-2}s$ we have

$$\Phi(B_r) \subset B_{(1+M)r} \subset \Phi(B_{(1+M)^2r}). \quad (4.17)$$

Indeed, let $x \in B_r$, then

$$|\Phi(x)|^2 \leq |x|^2 + 2|x_n\gamma(x)| + |\gamma(x)|^2,$$

so that, we have

$$|\gamma(x)| \leq |x||\nabla \gamma\|_\infty,$$

and then

$$|\Phi(x)| \leq (1 + M)r.$$

In a similar way, we have that for any $x \in B_{(1+M)r}$,

$$|\Phi^{-1}(x)| \leq (1 + M)^2r,$$

thus the claim is proved.

Let us take a ball $B_t$ such that $\Phi^{-1}(B_t) \subset B_s$, which we can find thanks to $(4.17)$, and let us reflect the function $v(x) = u(\Phi(x))$ as follows: for any $x \in B_t$, we define

$$Lv(x) = \begin{cases} v(x) & \text{if } x_n < 0, \\ -3v(x', -x_n) + 4v(x', -\frac{x_n}{2}) & \text{if } x_n > 0, \end{cases}$$

which is still a Sobolev function in $B_t$. Moreover, we have

$$\int_{B_t} |\nabla Lv|^2 \, d\mathcal{L}^n \leq C \int_{B_t \cap \{x_n < 0\}} |\nabla v|^2 \, d\mathcal{L}^n,$$

where $C$ is independent of $\Omega$. We put $Lu(x) = Lv(\Phi^{-1}(x))$, and by change of variables, we get

$$\int_{\Phi^{-1}(B_t)} |\nabla Lu|^2 \, d\mathcal{L}^n \leq C(\Omega) \int_{\Omega \cap \Phi^{-1}(B_t)} |\nabla u|^2 \, d\mathcal{L}^n. \quad (4.18)$$

Finally, taking $\rho_0 = (1 + M)^{-2}s$, and $t_0 = (1 + M)^{-1}s$, we have $B_{\rho_0} \subset \Phi^{-1}(B_{t_0})$. Therefore, denoting by $\delta = (1 + M)^2$, by $(4.17)$ and $(4.18)$, we get, for $r < \rho_0$,

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, d\mathcal{L}^n \leq \int_{\Phi^{-1}(B_{\sqrt{s}})} |\nabla Lu|^2 \, d\mathcal{L}^n \leq C \int_{\Omega \cap \Phi^{-1}(B_{\sqrt{s}})} |\nabla u|^2 \, d\mathcal{L}^n \leq C \int_{\Omega \cap B_{\rho_0}} |\nabla u|^2 \, d\mathcal{L}^n. \Box$$
Remark 4.13. Notice that if $\Omega$ is bounded, the constants in Lemma 4.12 can be chosen independent of the point $x_0$.

Remark 4.14. Let $u$ be a minimizer to (1.5) and let $A = \{ u > 0 \} \setminus K_u$, then the boundary of $A$ is equal to $K_u$: in first place, assume by contradiction that there exists an $x \in (\partial A) \setminus K_u$, then $u$ is superharmonic in a small ball centered in $x$ with radius $r$. Therefore, being

$$\{ u > 0 \} \cap B_r(x) \neq \emptyset,$$

it is necessary that $u > 0$ in the entire ball, and then $x \notin \partial A$, which is a contradiction. In other words,

$$\partial A \subseteq K_u$$

By the same argument we also have that $A$ is open, and moreover $J_u \subseteq \partial A$, then

$$K_u \subseteq \partial A.$$

In particular, the pair $(A, u)$ is a minimizer for the functional

$$F(E, v) = \int_E |\nabla v|^2 \, d\mathcal{L}^n - 2 \int_{\Omega} f v \, d\mathcal{L}^n + \int_{\partial E} (v^2 + \bar{v}^2) \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(E \setminus \Omega)$$

over all pairs $(E, v)$ with $E$ open set of finite perimeter containing $\Omega$ and $v \in W^{1,2}(E)$.

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