Full faithfulness for overconvergent $F$-isocrystals

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Abstract

Let $X$ be a smooth variety over a field of characteristic $p > 0$. We prove that the forgetful functor from the category of overconvergent $F$-isocrystals on $X$ to that of convergent $F$-isocrystals is fully faithful. The argument uses the quasi-unipotence theorem for overconvergent $F$-isocrystals (recently proved independently by André, Mebkhout, and the author), plus arguments of de Jong. In the process, we establish a theorem of Quillen-Suslin type (i.e., every finite projective module is free) over rings of overconvergent power series.

1 Introduction

Isocrystals, and more specifically $F$-isocrystals, were constructed to serve as $p$-adic analogues of local systems in the complex topology and lisse $l$-adic sheaves in the étale topology. Unfortunately, the category of convergent isocrystals suffers from certain pathologies, arising from the fact that the integral of a $p$-adic analytic function on a closed disc is no longer bounded near the boundary of the disc. For example, computed using convergent isocrystals, the cohomology of the affine line is infinite dimensional.

As first noted in the work of Dwork, pointed out more explicitly by Monsky and Washnitzer in the study of dagger cohomology, and systematized by Berthelot in his theory of rigid cohomology, one needs to work instead with objects satisfying an “overconvergence” condition. Adding this condition eliminates the pathologies mentioned above; in fact, Berthelot [B1] showed that the rigid cohomology with constant coefficients of an arbitrary variety over a field of characteristic $p$ is finite dimensional. (Finite dimensionality has now also been shown for rigid cohomology with coefficients in an $F$-isocrystal, which itself must be overconvergent [K4].) However, additional complications arise from the fact that it is not always clear how to “descend” certain constructions from the convergent category to the smaller overconvergent category.
The main result of this paper is a descent theorem in this vein, which resolves a conjecture formulated by Tsuzuki [T2, Conjecture 1.2.1(TF)]. Let $k$ be a field of characteristic $p > 0$, $K$ the fraction field of a complete mixed characteristic discrete valuation ring with residue field $k$, and $X$ a smooth separated $k$-scheme of finite type over $K$. Then for each integer $a > 0$, we can construct the categories $F^a$-$\text{Isoc}(X/K)$ and $F^a$-$\text{Isoc}^\dagger(X/K)$ of convergent and overconvergent $F^a$-isocrystals, respectively, on $X$; see for example Berthelot [B2]. One consequence of the construction is that there is a natural forgetful functor $j^*: F^a$-$\text{Isoc}^\dagger(X/K) \to F^a$-$\text{Isoc}(X/K)$.

**Theorem 1.1.** The forgetful functor $j^*: F^a$-$\text{Isoc}^\dagger(X/K) \to F^a$-$\text{Isoc}(X/K)$ is fully faithful.

In the case of unit-root $F$-isocrystals, this was proved for $X$ admitting a smooth compactification by Tsuzuki [T2] and for general $X$ by Etesse [E]. Tsuzuki further conjectures that the same statement holds for isocrystals without Frobenius structure, but the methods of this paper do not extend to that case.

The aforementioned results of Tsuzuki and Etesse have the effect of reducing the unit-root case of Theorem 1.1 to a local assertion about unit-root $F$-isocrystals, which had earlier been proved by Tsuzuki [T1]. We prove Theorem 1.1 by using a reduction of a similar spirit to bring the global problem down to an analogous local assertion about $F$-isocrystals.

Our proof of said local assertion closely follows de Jong’s proof of the equal characteristic analogue of Tate’s extension theorem for $p$-divisible groups. The key new ingredient is the proof of Crew’s conjecture [C] that overconvergent $F$-isocrystals are quasi-unipotent. This conjecture has recently been proved independently by André [A], Mebkhout [M] and the author [K2].

One other result of this paper may be of independent interest: we prove (Theorem 6.7) an analogue of the Quillen-Suslin theorem over any ring of overconvergent power series over a complete discrete valuation ring over field. That is, over such a ring, every finite projective module is free.

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**2 Definitions and notations**

This paper is a companion paper to [K2], and so we will adopt its notation and terminology. For the convenience of the reader, we recall the particular definitions and notations
we will need; however, we will defer to [K2, Sections 2–3] for the verification of various compatibilities.

For $k$ a field of characteristic $p > 0$, let $C(k)$ be a Cohen ring for $k$, that is, a complete discrete valuation ring with fraction field of characteristic 0, maximal ideal generated by $p$, and residue field isomorphic to $k$. (The ring $C(k)$ is unique up to noncanonical isomorphism if $k$ is not perfect; if $k$ is perfect, $C(k)$ is canonically isomorphic to the ring $W(k)$ of Witt vectors over $k$.) Let $O$ be a finite totally ramified extension of $C(k)$, let $\pi$ be a uniformizer of $O$, and fix a ring endomorphism $\sigma_0$ on $O$ lifting the absolute Frobenius $x \mapsto x^p$ on $k$. Let $q = p^f$ be a power of $p$ and put $\sigma = \sigma_0^f$. Let $\varpi$ denote the valuation on $O$ normalized so that $v_p(\varpi) = 1$, and let $\| \cdot \|$ denote the norm on $O$ given by $\|x\| = p^{-v_p(x)}$. For $x \in O$, let $\overline{x}$ denote its reduction in $k$.

Let $\Gamma$ denote the ring of bidirectional power series

$$\left\{ \sum_{i \in \mathbb{Z}} x_i u^i : x_i \in O, \lim_{i \to -\infty} v_p(x_i) = \infty \right\}.$$ 

Then $\Gamma$ is a complete discrete valuation ring, whose residue field we identify with $k((t))$ by identifying the reduction of $\sum x_i u^i$ with $\sum \varpi^i t^i$. Choose an extension of $\sigma_0$, as defined on $O$, to a ring endomorphism of $\Gamma$ lifting the $p$-th power $x \mapsto x^p$ on $K$, and mapping the subring

$$\Gamma_{\text{con}} = \left\{ \sum_{i \in \mathbb{Z}} x_i u^i : x_i \in O, \liminf_{i \to -\infty} \frac{v_p(x_i)}{-i} > 0 \right\}$$

into itself; it suffices to check that $u^{\sigma_0} \in \Gamma_{\text{con}}$. Again put $\sigma = \sigma_0^f$.

If $k$ is perfect, we can define a functor from the category of algebraic (finite or infinite) extensions of $k((t))$ to the category of complete discrete valuation rings which are unramified over $O$. Let $\Gamma^K$ denote the image of an extension $K$; if $K = k((t))$ we will omit the superscript entirely, while if $K = k((t))^{\text{perf}}$ or $K = k((t))^{\text{alg}}$, we will call the image $\Gamma^{\text{perf}}$ or $\Gamma^{\text{alg}}$, respectively. If $K$ is finite over $k((t))$, then $K$ is itself isomorphic to $k((t'))$ for some $t'$, and analogously the ring $\Gamma^K$ is abstractly isomorphic to $\Gamma$.

Each ring $\Gamma^K$ comes equipped with a canonical extension of $\sigma$. In fact, the choice of the functor depends on $\sigma_0$, at least for $K$ which are not separable over $k((t))$: for example, $\Gamma^{\text{perf}}$ is the direct limit of $\Gamma \to \Gamma \to \cdots$, where the transition maps are equal to $\sigma$. The functoriality is more limited if $k$ is not perfect, but for our purposes, it will suffice to embed $\Gamma$ into $\Gamma^K$ for $K = k^{\text{alg}}((t))$, and then construct $\Gamma^L$ functorially for extensions $L$ of $K$.

We define the partial valuations on $\Gamma^{\text{alg}}[1/p]$ as follows: for $x \in \Gamma^{\text{alg}}[1/p]$, write $x = \sum_{i=0}^\infty \varpi^i [u_i]$, where the brackets denote Teichmüller lifts, and put

$$v_n(x) = \min_{v_p(\varpi^i) \leq n} \{ v_l(u_i) \},$$
where the bar denotes reduction to \( k((t))^{\text{alg}} \) and \( v_t \) denotes the canonical extension of the valuation on \( k((t)) \), normalized so that \( v_t(t) = 1 \). For each \( r > 0 \), let \( \Gamma^K_r \) denote the subring of \( x \in \Gamma^K \) such that \( \lim_{n \to \infty} (rv_n(x) + n) = \infty \), and define the valuation

\[
    w_r(x) = \min_n \{ rv_n(x) + n \}
\]

on \( \Gamma^K_r \). Define \( \Gamma^K_{\text{con}} = \cup_{r > 0} \Gamma^K_r \). It can be shown that for \( K = k((t)) \), the ring \( \Gamma^K_{\text{con}} \) coincides with the ring \( \Gamma_{\text{con}} \) defined above. More precisely, for \( r \) sufficiently small (depending on the choice of \( \sigma_0 \)), the \( w_r \) coincide with their “naïve” counterparts, in whose definition \( v_n \) is replaced by

\[
    v_n^{\text{naïve}}(x) = \min_{j:v_p(x_j) \leq n} \{ j \},
\]

where \( x = \sum_j x_j u^j \).

We can define a Fréchet topology on \( \Gamma^K_r[,]^{[1]} \) using the norms \( w_s \) for \( 0 < s \leq r \), and take \( \Gamma^K_{\text{an},r} \) as the Fréchet completion of \( \Gamma^K_r[,]^{[1]} \). (However, this is only known to behave well when \( K \) is either finite over \( k((t)) \) or perfect.) Put \( \Gamma^K_{\text{an,con}} = \cup_{r > 0} \Gamma^K_{\text{an},r} \). For \( k = k((t)) \), the ring \( \Gamma^K_{\text{an,con}} = \Gamma_{\text{an,con}} \) coincide with the Robba ring, the ring of germs of functions analytic on some open \( p \)-adic annulus with outer radius 1. Concretely, this ring consists of series

\[
    \left\{ \sum_j x_j u^j : x_j \in \mathcal{O}[1/p], \quad \lim_{j \to -\infty} \frac{v_p(x_j)}{-j} > 0, \quad \lim_{j \to \infty} \frac{v_p(x_j)}{j} \geq 0 \right\}.
\]

It turns out that \( \Gamma^K_{\text{an,con}} \) is a Bézout ring (every finitely generated ideal is principal) if \( K \) is finite over \( k((t)) \) or \( K \) is perfect, by [K2, Theorem 3.12], but we will not explicitly need this fact.

We define a \( \sigma \)-module over an integral domain \( R \) to be a finite free \( R \)-module \( M \) equipped with an \( R \)-linear map \( F : M \otimes_{R,\sigma} R \to M \) that becomes an isomorphism over \( R[,]^{[1]} \); the tensor product notation indicates that \( R \) is viewed as an \( R \)-module via \( \sigma \). To specify \( F \), it is equivalent to specify an additive, \( \sigma \)-linear map from \( M \) to \( M \) that acts on any basis of \( M \) by a matrix invertible over \( R[,]^{[1]} \). We abuse notation and refer to this map as \( F \) as well; since we will only use the \( \sigma \)-linear map in what follows, there should not be any confusion induced by this.

For \( K = k((t)) \) and \( R \) one of \( \Gamma^K, \Gamma^K[,]^{[1]}, \Gamma^K_{\text{con}}, \Gamma^K_{\text{con},[,]^{[1]}}, \Gamma^K_{\text{an,con}} \), let \( \Omega^1_R \) be the free module over \( R \) generated by a symbol \( du \), and define the derivation \( d : R \to \Omega^1_R \) by the formula

\[
    d \left( \sum_j x_j u^j \right) = \left( \sum_j j x_j u^{j-1} \right) du.
\]

Likewise, if \( K \) is a finite extension of \( k((t)) \), we can make the same definition by writing elements as power series in terms of some \( u' \in \Gamma^K_{\text{con}} \) which lifts a uniformizer of \( K \).
We define a $(\sigma, \nabla)$-module over $R$ to be a $\sigma$-module $M$ plus a connection $\nabla : M \to M \otimes_R \Omega_R^1$ (i.e., an additive map satisfying the Leibniz rule $\nabla(cv) = c\nabla(v) + v \otimes dc$) that makes the following diagram commute:

\[\begin{array}{ccc}
M & \xrightarrow{\nabla} & M \otimes \Omega_R^1 \\
\downarrow F & & \downarrow F \otimes d\sigma \\
M & \xrightarrow{\nabla} & M \otimes \Omega_R^1
\end{array}\]

Given a $\sigma$-module or $(\sigma, \nabla)$-module $M$ and an integer $\ell$, we define the Tate twist $M(\ell)$ as the module $M$ with the action of $F$ multiplied by $q^\ell$. Note that the dual $M^* = \text{Hom}(M, R)$ does not generally have the structure of a $(\sigma, \nabla)$-module over $R$ (only over $R[[t]]$), but its Tate twist $M^*(\ell) = \text{Hom}(M, R(\ell))$ does for some $\ell$. If $v$ is an element of $M$ such that $Fv = \lambda v$ for some $\lambda \in \mathcal{O}$, we say $v$ is an eigenvector of $M$ of eigenvalue $\lambda$ and slope $v_p(\lambda)$.

There are two ways to associate a Newton polygon to a $\sigma$-module. For $R = \Gamma$ or $R = \Gamma_{\text{con}}$, the Dieudonné-Manin classification states that a $\sigma$-module over $R$ acquires a basis of eigenvectors over $\mathcal{O}$; the slopes of these eigenvectors are called the generic slopes of $M$. For $R = \Gamma_{\text{con}}$ or $R = \Gamma_{\text{an,con}}$, [K2, Theorem 3.12] states that a $\sigma$-module over $R$ acquires a basis of eigenvectors over $\Gamma_{\text{an,con}}^\text{alg}$; the slopes of these eigenvectors are called the special slopes of $M$.

We use the following refinement of the Dieudonné-Manin classification, due originally to de Jong [dJ, Proposition 5.5] and appearing also as [K2 Proposition 5.7].

**Proposition 2.1 (Descending slope filtration).** Let $M$ be a $\sigma$-module over $\Gamma_{\text{con}}$ for $k$ algebraically closed. Then there exists a finite extension $\mathcal{O}'$ of $\mathcal{O}$ such that over $\Gamma_{\text{con}}^\text{alg} \otimes_\mathcal{O} \mathcal{O}'$, $M$ admits a basis $v_1, \ldots, v_n$ such that $Fv_i = \lambda_i v_i + \sum_{j<i} c_{ij} v_j$ for some $\lambda_i \in \mathcal{O}$ and $c_{ij} \in \Gamma_{\text{con}}^\text{alg}$, with $v_p(\lambda_1), \ldots, v_p(\lambda_n)$ equal to the sequence of generic slopes of $M$ in descending order. Moreover, if $v_p(\lambda_1) = \cdots = v_p(\lambda_m)$, we can take $c_{ij} = 0$ for $i \leq m$.

We can also define $\sigma$-modules and $(\sigma, \nabla)$-modules over rings $R$ which are not discrete valuation rings. In particular, one typically makes these definitions for $R$ a (smooth) dagger algebra, a quotient of a ring $\mathcal{O}(x_1, \ldots, x_n)^\dagger$ of overconvergent power series which is smooth over $\mathcal{O}$, in which each series $\sum_I c_I x^I$ satisfies $\liminf v_p(c_I)/|I| > 0$. (Here $|I|$ denotes the sum of the indices in the index set.) One defines a $(\sigma, \nabla)$-module over $R$ as above, modulo the following changes:

1. The underlying module $M$ is allowed to be locally free, but not necessarily free.

2. The module of differentials $\Omega_R^1$ is now the $R$-algebra obtained from the free module over $\mathcal{O}(x_1, \ldots, x_n)^\dagger$ generated by $dx_1, \ldots, dx_n$ by quotienting by the necessary relations. (The result does not depend on the presentation of $R$.)
3. The connection must now satisfy an integrability condition: if $\nabla_1 : M \otimes \Omega^1_R \to M \otimes \wedge^2 \Omega^1_R$ is the natural map induced by $\nabla$, then $\nabla_1 \circ \nabla = 0$.

3 Splitting an exact sequence

We first recall [K2, Proposition 3.19(c)].

**Proposition 3.1.** For $\lambda \in O$ not a unit and $x \in \Gamma_{\text{an,con}}$, there is at most one $y \in \Gamma^K_{\text{an,con}}$ such that $\lambda y^\sigma - y = x$. Moreover, if $x \in \Gamma^K_{\text{con}}$, then so is $y$.

Our next lemma generalizes this lemma, using the descending slope filtration. One can also give a direct proof within $\Gamma_{\text{an}}$; we leave this as an exercise to the interested reader.

**Lemma 3.2.** Let $M$ be a $\sigma$-module over $\Gamma_{\text{an}}$, for $K$ a finite extension of $k((t))$, whose generic slopes are all positive. Then the map $v \mapsto Fv - v$ on $M \otimes_{\Gamma^K_{\text{con}}} [1/p] \Gamma^K_{\text{an,con}}$ is injective. Moreover, if $Fv - v \in M$, then $v \in M$.

**Proof.** For both assertions, it suffices to assume $k$ is algebraically closed. We prove the second assertion first. Let $v_1, \ldots, v_n$ be the basis of $M$ over $\Gamma_{\text{alg}} \otimes O'$ given by Proposition 2.1, with $Fv_i = \lambda_i v_i + \sum_{j<i} c_{ij} v_j$. Since $v_p(\lambda_i)$ are the generic slopes of $M$, they are all positive by hypothesis. Write $v = \sum_i e_i v_i$ with $e_i \in \Gamma_{\text{alg,con}}$ and put $Fv - v = \sum_i f_i v_i$, with $f_i \in \Gamma_{\text{con}}$. Then

$$f_n = \lambda_n e_n^\sigma - e_n$$
$$f_{n-1} = \lambda_{n-1} e_{n-1}^\sigma - e_{n-1} + c_{n(n-1)} e_n^\sigma$$
$$\vdots$$
$$f_1 = \lambda_1 e_1^\sigma - e_1 + c_{21} e_2^\sigma + \cdots + c_{n1} e_n^\sigma.$$  

By Proposition 3.1, the map $x \mapsto \lambda_i x^\sigma - x$ on $\Gamma_{\text{an,con}} \otimes O'$ is injective, and if $\lambda_i x^\sigma - x \in \Gamma_{\text{con}}[1/p]$, then $x \in \Gamma_{\text{con}}[1/p]$. Applying this fact repeatedly to the above equations, we deduce successively that $e_n, e_{n-1}, \ldots, e_1$ all lie in $\Gamma_{\text{alg}}[1]$. Therefore $v$ is defined over $\Gamma_{\text{con}}[1/p] \cap \Gamma_{\text{an,con}} = \Gamma_{\text{con}}[1/p]$.

To establish the first assertion, suppose $Fv = v$; we then have the same equations as above, but with $f_i = 0$ for all $i$. By Proposition 3.1 again, we have $e_n = 0$, then $e_{n-1} = 0$, and so on. Thus $v = 0$, as desired.

We now use the previous lemma, plus the quasi-unipotence theorem, to prove the following proposition, which is analogous to [dJ, Proposition 7.1]. (In de Jong’s argument, the role of the quasi-unipotence theorem is played by Dwork’s trick.)
Proposition 3.3. Let $M$ be a $(\sigma, \nabla)$-module over $\Gamma_{\text{con}}$. Suppose there exists an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of $(\sigma, \nabla)$-modules over $\Gamma_{\text{con}}$ such that the generic slopes of $M_1$ are all greater than the generic slopes of $M_2$. Then the exact sequence splits over $\Gamma_{\text{con}}[\frac{1}{p}]$.

Proof. Let $n_1$ and $n_2$ be the ranks of $M_1$ and $M_2$, respectively. Choose a basis of $M_1$ over $\Gamma_{\text{con}}[\frac{1}{p}]$ and extend it to a basis of $M$. Then $\sigma$ acts on this basis via a block matrix of the form \(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\). To show that the exact sequence splits over a ring $R$ containing $\Gamma_{\text{con}}[\frac{1}{p}]$, it is necessary and sufficient to find a matrix $X$ over $R$ such that
\[-X + AX^\sigma D^{-1} = B,\] (1)

as then we have
\[
\begin{pmatrix} I_{n_1} & -X \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I_{n_1} \\ 0 & I_{n_2} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.
\]

We first solve (1) over a somewhat larger ring than $\Gamma_{\text{con}}[\frac{1}{p}]$, using the quasi-unipotence theorem. For any finite extension $L$ of $K = k_{\text{alg}}((t))$ and any lift $u \in \Gamma_{\text{con}}[\log u]$ of a uniformizer of $L$, define $\Gamma_{\text{an,con}}[\log u]$ as the polynomial ring over $\Gamma_{\text{an,con}}[\log u]$ in an indeterminate called “$\log u". Extend $\sigma$ from $\Gamma_{\text{an,con}}$ to $\Gamma_{\text{an,con}}[\log u]$ by setting 
\[(\log u)^\sigma = p \log u + \log(u^p/u^p),\]

using the power series expansion of $\log(1 + x)$ to define the second factor, and extend the derivation $d$ to $\log u$ by setting $d(\log u) = \frac{1}{u} \otimes du$. By the quasi-unipotence theorem (as formulated in [K2, Theorem 6.13]), there exists a finite separable extension $L$ of $K$, a finite extension $\mathcal{O}'$ of $\mathcal{O}$, and a basis $v_1, \ldots, v_{n_1}, w_1, \ldots, w_{n_2}$ of $M$ over $R_1 = \Gamma_{\text{an,con}}[\log u] \otimes \mathcal{O}'$ such that $v_1, \ldots, v_{n_1}$ form a basis of $M_1$, and
\[Fv_i = \lambda_i v_i,\]
\[\nabla v_i = 0\]
\[Fw_i = \mu_i w_i + \sum_{j=1}^{m} W_{ij} v_j\]
\[\nabla w_i = \sum_{j=1}^{n_1} Y_{ij} v_j \otimes du\]
for some $\lambda_1, \ldots, \lambda_{n_1}, \mu_1, \ldots, \mu_{n_2}$ in $\mathcal{O}$ and $W_{ij}, Y_{ij} \in R_1$.

The ring $R_1$ has the property that the map $\frac{d}{du} : R_1 \to R_1$ is surjective. (The point is that every expression $u^i(\log u)^j$ has an antiderivative, by integration by parts.) Thus we can choose $Z_{ij} \in R_1$ such that $\frac{d}{du} Z_{ij} = Y_{ij}$. Put $y_i = w_i - \sum_{j=1}^{n_1} Z_{ij} v_j$. Now $\nabla y_i = 0$.
for \( i = 1, \ldots, n \), and we may write \( Fy_i = \mu_i y_i + \sum_{j=1}^{n_1} V_{ij} v_j \) for some \( V_{ij} \in R_1 \). By the compatibility relation between \( F \) and \( \nabla \), we have \( 0 = \nabla Fy_i = \sum_{j=1}^{n_1} v_j \otimes dV_{ij} \), so \( V_{ij} \in \mathcal{O} \) for all \( i, j \). Choose \( U_{ij} \in \mathcal{O} \) such that \( \mu_i U_{ij}^\sigma - \lambda_j U_{ij} = V_{ij} \), and put \( z_i = y_i - \sum_{j=1}^{n_1} U_{ij} v_j \). Then \( Fz_i = \mu_i z_i + \sum_{j=1}^{n_1} (V_{ij} + \mu_i U_{ij} - \lambda_j U_{ij}^\sigma) = \mu_i z_i \), so \( M \) splits over \( R \) as \( M_1 \) plus the span of \( z_1, \ldots, z_{n_2} \). Thus \( (\textbf{1}) \) admits a solution \( X_1 \) over \( R_1 \).

We now descend the solution \( X_1 \) we just found to smaller and smaller rings. Before doing so, note that we can give the set of \( n_1 \times n_2 \) matrices \( X \) the structure of a \( \sigma \)-module with Frobenius given by \( X \mapsto AX^\sigma D^{-1} \). Each generic slope of this \( \sigma \)-module is a generic slope of \( A \) minus a generic slope of \( D \), and thus is positive. We thus can apply Lemma 3.2 to this \( \sigma \)-module.

Now for the descent. First put \( R_2 = \Gamma_{\text{an,con}}^L \otimes_{\mathcal{O}} \mathcal{O}' \), so that \( R_1 = R_2[\log u] \). If \( X \neq 0 \), we can write \( X = \sum_{i=0}^{M} Y_i (\log u)^i \) with \( Y_M \neq 0 \). If \( M > 0 \), we have \( -Y_M + p^m AY_M^\sigma D^{-1} = 0 \); by Lemma 3.2, this forces \( Y_M = 0 \), contradiction. Thus we must have \( M = 0 \), so \( X \) is defined over \( R_2 \).

Next, put \( R_3 = \Gamma_{\text{an,con}}^L \), so that \( R_2 = R_3 \otimes_{\mathcal{O}} \mathcal{O}' \). Choose a basis \( c_1, \ldots, c_r \) of \( \mathcal{O}' \) over \( \mathcal{O} \) with \( c_1 = 1 \). Then we can write \( X_2 \) as a linear combination of \( c_1, \ldots, c_r \) whose coefficients are matrices over \( R_1 \). The coefficient of \( c_1 \) must then also be a solution of \( (\textbf{1}) \); since \( X \) is unique by Lemma 3.2, it must coincide with its coefficient of \( c_1 \). That is, \( X \) is defined over \( R_3 \).

Next, put \( R_4 = \Gamma_{\text{con}}^L [\frac{1}{p}] \). We have \( -X + AX^\sigma D^{-1} = B \); by Lemma 3.2, since \( B \in \Gamma_{\text{con}}^L [\frac{1}{p}] = R_4 \), we must have \( X \in R_4 \).

Next, put \( R_5 = \Gamma_{\text{con}}^K [\frac{1}{p}] \). Now \( \text{Gal}(L/K) \) acts on \( \Gamma_{\text{con}}^L \) with fixed ring \( \Gamma_{\text{con}}^K \). In particular, \( \text{Gal}(L/K) \) acts on the set of solutions of \( (\textbf{1}) \). By Lemma 3.2, there is only one solution \( X \). Thus \( X \) is fixed by \( \text{Gal}(L/K) \) and so is defined over \( R_5 \).

Finally, put \( R_6 = \Gamma_{\text{con}}[\frac{1}{p}] \). We now have a \( \sigma \)-module \( Y \) over \( \Gamma_{\text{con}}[\frac{1}{p}] \) (the space of \( n_1 \times n_2 \) matrices) with all generic slopes positive, and an eigenvector \( X \) over \( \Gamma_{\text{con}}[\frac{1}{p}] \). By Proposition 5.3, \( Y \) admits a basis \( Z_1, \ldots, Z_{n_1n_2} \) on which Frobenius acts by a matrix over \( \Gamma_{\text{con}} \) with positive valuation. Write \( X = \sum c_i Z_i \), suppose there exists an integer \( m \) such that \( c_i \) is not congruent to an element of \( \Gamma_{\text{con}}[\frac{1}{p}] \) modulo \( \pi^m \) for some \( i \), and choose the smallest such \( m \). Write each \( c_i = e_i + f_i \) with \( e_i \in \Gamma_{\text{con}}[\frac{1}{p}] \) and \( f_i \not\equiv 0 \) (mod \( \pi^m \)) for some \( i \), and put \( X_1 = \sum f_i Z_i \). Then \( X_1 - AX^\sigma D^{-1} = \sum d_i Z_i \) with \( d_i \in \Gamma_{\text{con}}[\frac{1}{p}] \), but \( d_i - f_i \equiv 0 \) (mod \( \pi^m \)) because the matrix on which \( F \) acts on the \( Z_i \) has positive valuation. Then \( c_i \equiv e_i + d_i \) (mod \( \pi^m \)), contrary to the choice of \( m \). We conclude that the \( c_i \) are congruent to elements of \( \Gamma_{\text{con}}[\frac{1}{p}] \) modulo every power of \( \pi \), so belong to \( \Gamma_{\text{con}}[\frac{1}{p}] \). In other words, \( X \) is defined over \( R_6 \).

To conclude, we have shown that \( (\textbf{1}) \) admits a solution over \( \Gamma_{\text{con}}[\frac{1}{p}] \). Thus the exact sequence splits, as desired.
4 Equality of kernels

This section is nearly a carbon copy of [dJ, Section 8], to the extent that we have reproduced its title. The main changes are that we work with an arbitrary \( \sigma \) rather than a “standard” \( \sigma \), which sends some \( u \in \Gamma^K_{\mathrm{con}} \) to \( u^p \), and that we expose [dJ, Proposition 8.1] using the technical device of “generalized power series", which we hope provides a small clarification of the argument.

For \( K \) a valued field over an algebraically closed field of characteristic \( p > 0 \), the maximal immediate extension \( K^{\mathrm{imm}} \), in the sense of Kaplansky [Ka], is the maximal extension of \( K \) with value group \( \mathbb{Q} \). Kaplansky shows this field can be identified with Hahn’s field of generalized power series \( x = \sum_i x_i t^i \), where \( x_i \in k \) for each \( i \in \mathbb{Q} \), and the set \( I = I(x) \) of indices \( i \) such that \( x_i \neq 0 \) is well-ordered for each \( x \). The corresponding ring \( \Gamma^{\mathrm{imm}} \) can be described as the ring of generalized power series \( y = \sum_i y_i t^i \), where \( y_i \in \mathcal{O} \) for each \( i \in \mathbb{Q} \) and for each \( n \), the set \( I = I(n, x) \) such that \( v_p(y_i) \leq n \) is well-ordered; the partial valuation \( v_n(y) \) is equal to the smallest \( i \) such that \( v_p(y_i) \leq n \). Note that \( w \) has \( p^n \)-th roots for all \( n \) and so must be the Teichmüller lift of its reduction.

The following proposition corresponds to [dJ, Proposition 8.1].

**Proposition 4.1 (after de Jong).** For \( k \) algebraically closed, the multiplication map \( \Gamma_{\mathrm{alg}}^{\mathrm{con}} \otimes_{\Gamma_{\mathrm{con}}} \Gamma \to \Gamma_{\mathrm{alg}} \) is injective.

**Proof.** Suppose, by way of contradiction, that \( \sum_{i=1}^n f_i \otimes g_i \) is a nonzero element of \( \Gamma_{\mathrm{alg}}^{\mathrm{con}} \otimes_{\Gamma_{\mathrm{con}}} \Gamma \) such that \( \sum f_i g_i \) in \( \Gamma_{\mathrm{alg}} \), and assume \( n \) is minimal for the existence of such an element. Then the \( g_i \) are linearly independent over \( \Gamma_{\mathrm{con}} \), otherwise we could replace one of them by a linear combination of the others and thus decrease \( n \).

Embed \( L \) in \( K^{\mathrm{imm}} \), put \( \Gamma^{\mathrm{imm}} = W(K^{\mathrm{imm}}) \otimes_{W(k)} \mathcal{O} \), and use Witt vector functoriality to construct an embedding \( \Gamma_{\mathrm{alg}} \to \Gamma^{\mathrm{imm}} \). We can define the partial valuations \( v_n \), the valuations \( w_n \) and the subring \( \Gamma^{\mathrm{imm}}_{\mathrm{con}} \) using the same formulas as before; we then have \( \Gamma^{\mathrm{alg}} = \Gamma_{\mathrm{alg}} \cap \Gamma^{\mathrm{imm}}_{\mathrm{con}} \).

Let \( w \in \Gamma^{\mathrm{perf}}_{\mathrm{con}} \) be the Teichmüller lift of \( t \). We show that every element \( x \) of \( \Gamma^{\mathrm{imm}} \) can be represented uniquely by a sum \( \sum_{\alpha \in [0,1)} x_\alpha w^\alpha \) with \( x_\alpha \in \Gamma \) for all \( \alpha \). Namely, each element of \( K^{\mathrm{imm}} \) can be written uniquely as \( \sum_{\alpha \in [0,1)} c_\alpha t^\alpha \) with \( c_\alpha \in k((t)) \) for each \( \alpha \). Thus we can choose \( x_\alpha \in \Gamma \) so that \( \sum_{\alpha \in [0,1)} x_\alpha w^\alpha \equiv x \pmod{\pi} \). But by the same reasoning applied to \( (x - \sum x_\alpha w^\alpha) / \pi \), we can choose the \( x_\alpha \) so that \( \sum x_\alpha w^\alpha \equiv x \pmod{\pi^2} \), and analogously we can achieve the same congruence modulo any power of \( \pi \). The limiting values of \( x_\alpha \) give the desired decomposition. This verifies existence; for uniqueness, it suffices to note that if \( x \equiv 0 \pmod{\pi} \), by the uniqueness of the decompositions modulo \( \pi \) we have \( x_\alpha \equiv 0 \pmod{\pi^m} \) for all \( m \), by induction on \( m \).

As above, decompose \( x \in \Gamma^{\mathrm{imm}} \) as \( \sum_{\alpha \in [0,1)} x_\alpha w^\alpha \) with \( x_\alpha \in \Gamma \); we claim that in fact \( x_\alpha \in \Gamma_{\mathrm{con}} \) for all \( \alpha \). More specifically, choose \( r, s > 0 \) such that \( w_r(u/w) = 0 \) and \( w_r(x) \geq -s \); then we claim \( rv_n(x_\alpha w^\alpha) + s + n \geq 0 \) for all \( n \). Suppose this is not the case; choose the smallest \( n \) for which \( rv_n(x_{\beta} w^{\beta}) + s + n < 0 \) for some \( \beta \). For any such \( \beta \), we must have
Let $v_p(x_\beta) = n$. Write $x_\alpha = \sum_{i \in \mathbb{Z}} x_{\alpha,i}u^i$. Let $y_\alpha$ be the sum of $x_{\alpha,i}u^i$ over all indices $i$ for which $v_p(x_{\alpha,i}) < n$, and put $z_\alpha = x_\alpha - y_\alpha$ and $z = \sum_\alpha z_\alpha w^\alpha$. Then $w_\alpha(y_\alpha) \geq 0$ for each $\alpha$, so we also have $w_\alpha(z) \geq -s$. Also, $v_p(z) \geq n$ and $rv_n(z_\beta w^\beta) + s + n < 0$ for some $\beta$. But if we put $n = v_p(\pi^N)$, we then have

$$rv_n(z_{\alpha+i} w^{\alpha+i}) + n \geq rv_0((z_{\alpha+i}/\pi^N)w^{\alpha+i}) + n$$

$$= rv_0(z/\pi^N) + n$$

$$\geq w_\tau(z) \geq -s$$

for any $\alpha \in [0,1)$ and $i \in \mathbb{Z}$. By choosing these so $\alpha + i = \beta$, we obtain a contradiction. We conclude that $w_\alpha \in \Gamma^\text{imm}$ for each $\alpha \in [0,1]$.

Now write each $f_i$ as $\sum_{\alpha \in [0,1]} f_{i,\alpha} w^\alpha$ with $f_{i,\alpha} \in \Gamma_{\con}$; then we have $\sum_{i=1}^n \sum_{\alpha \in [0,1]} f_{i,\alpha} g_i w^\alpha = 0$. By the uniqueness of this type of representation, we conclude $\sum_{i=1}^n f_{i,\alpha} g_i = 0$ for each $\alpha$. Since the $g_i$ were assumed to be linearly independent over $\Gamma_{\con}$, this implies $f_{i,\alpha} = 0$ for each $i$ and $\alpha$, and so all of the $f_i$ are zero. This contradiction completes the proof.

The following lemma corresponds to de Jong [21] Corollary 8.2, with essentially the same proof.

**Lemma 4.2 (after de Jong).** Let $M$ be a nonzero $\sigma$-module over $\Gamma_{\con} = \Gamma^K_{\con}$ with $K = k((t))$ and $k$ algebraically closed, and let $\phi : M \to \Gamma$ be a $\Gamma_{\con}$-linear injective map such that for some nonnegative integer $\ell$, $\phi(Fv) = p^\ell \phi(v)^{\sigma}$ for all $v \in M$. Then the largest generic slope of $M$ is equal to $\ell$, occurring with multiplicity 1. Moreover, $\phi^{-1}(\Gamma_{\con})$ is a $\sigma$-submodule of $M$ of dimension 1 with slope $\ell$.

**Proof.** By the previous lemma, the map $\phi : M \otimes_{\Gamma_{\con}} \Gamma_{\text{alg}} \to \Gamma \otimes_{\Gamma_{\con}} \Gamma_{\text{alg}} \to \Gamma_{\text{alg}}$ is the composition of two injections, so is injective. Let $s$ be the largest (generic) slope of $M$ and $m$ its multiplicity. Choose $\lambda$ in a finite extension of $O$ so that $\lambda$ is fixed by $\sigma$ and $v_p(\lambda) = s$. By Proposition 2.1, $M \otimes_{\Gamma_{\con}} \Gamma_{\text{alg}}$ contains $m$ linearly independent eigenvectors $v_1, \ldots, v_m$ of eigenvalue $\lambda$. Now

$$\lambda \phi(v_i) = \phi(\lambda v_i) = \phi(Fv_i) = p^\ell \phi(v_i)^{\sigma}.$$ 

This is only possible if $v_p(\lambda) = v_p(p^\ell)$, i.e., if $s = \ell$. In that case, we can take $\lambda = p^ \ell$, in which case we must have $\phi(v_i) \in \mathcal{O}_0$ for each $i$; in particular, no two of the $v_i$ can be linearly independent. This is impossible unless $m = 1$.

To complete the proof, it suffices to show that $\phi^{-1}(\Gamma_{\con})$ is nonempty. If $Fv = p^\ell v$ for $v \in M \otimes_{\Gamma_{\con}} \Gamma_{\text{alg}}$, we may choose a basis $e_1, \ldots, e_n$ of $M$, write $v = \sum c_i e_i$, let $w \in \Gamma^\text{per}$ be the Teichmüller lift of $t$, then imitate the proof of the previous lemma to write each $c_i$ as a generalized power series $\sum_{\alpha \in [0,1]} c_{i,\alpha} w^\alpha$ with $c_{i,\alpha} \in \Gamma_{\con}$. On one hand, $\phi(v) \in \mathcal{O}$. On the other hand, for each $\alpha$,

$$\phi \left( \sum_i \sum_{\alpha \in [0,1]} c_{i,\alpha} w^\alpha e_i \right) = \sum_\alpha w^\alpha \phi \left( \sum_i c_{i,\alpha} e_i \right);$$
since representations in the form \( \sum_{\alpha \in [0,1]} d_\alpha u^\alpha \) are unique, we have \( \sum_i c_i \mathfrak{e}_i \in \phi^{-1}(\mathcal{O}) \). Thus \( \phi^{-1}(\Gamma_{\mathrm{con}}) \) is nonempty, and the proof is complete. \( \square \)

5 Local full faithfulness

We now prove the local full faithfulness theorem, following [dJ, Theorem 9.1]. This theorem affirms [T2, Conjecture 2.3.2].

**Theorem 5.1.** Let \( M_1 \) and \( M_2 \) be \((\sigma, \nabla)\)-modules over \( \Gamma_{\mathrm{con}} \), and let \( f : M_1 \otimes_{\Gamma_{\mathrm{con}}} \Gamma \to M_2 \otimes_{\Gamma_{\mathrm{con}}} \Gamma \) be a morphism of \((\sigma, \nabla)\)-modules over \( \Gamma_{\mathrm{con}} \). Then there exists a morphism \( g : M_1 \to M_2 \) that induces \( f \).

**Proof.** Regard \( \nabla \) as a map from \( M \) to itself by identifying \( v \in M \) with \( v \otimes \frac{du}{u} \). For \( \ell \) sufficiently large, the Tate twist \( M_1^*(\ell) \) of the dual of \( M_1 \) is a \((\sigma, \nabla)\)-module over \( \Gamma_{\mathrm{con}} \), and there is a canonical isomorphism \( \text{Hom}(M_1, M_2)(\ell) \cong M_1^*(\ell) \otimes M_2 \). Put \( M = M_1^*(\ell) \otimes M_2 \); then \( f \) induces an additive, \( \Gamma_{\mathrm{con}} \)-linear map \( \phi : M \to \Gamma \) such that:

(a) for all \( v \in M \), \( \phi(Fv) = p^\ell \phi(v) \);

(b) for all \( v \in M \), \( \phi(\nabla v) = u \frac{du}{\sigma u} \phi(v) \).

To prove the desired result, it suffices to prove that \( \phi \) is induced from a map \( M \to \Gamma_{\mathrm{con}} \) satisfying the analogues of (a) and (b), i.e., that \( \phi(M) \subseteq \Gamma_{\mathrm{con}} \). At this point, we may assume without loss of generality that \( k \) is algebraically closed.

Let \( N \subseteq M \) be the kernel of \( \phi \) on \( M \); then \( N \) is clearly closed under \( \sigma \) and \( \nabla \) and saturated, so we may form the quotient \((\sigma, \nabla)\)-module \( M/N \), and the induced map \( \psi : M/N \to \Gamma \) is injective. By Lemma 4.2, the largest slope of \( M/N \) is \( \ell \) occurring with multiplicity 1, and \( P = \psi^{-1}(\Gamma_{\mathrm{con}}) \) is a \((\sigma, \nabla)\)-module of \( M/N \) of dimension 1 with slope \( \ell \). We now show that \( P \) is also closed under \( \nabla \). If \( v = \psi^{-1}(1) \), then \( Fv = p^\ell v \) implies \( \frac{du^\sigma/du}{u^\sigma/u} F(\nabla v) = p^\ell \nabla v \), and so

\[
\frac{du^\sigma/du}{u^\sigma/u} p^\ell \psi(\nabla v)^\sigma = p^\ell \psi(\nabla v).
\]

However, \( \frac{du^\sigma}{du} \) is divisible by \( \pi \) because \( u^\sigma \equiv u^p \pmod{\pi} \), so the two sides of the above equation have different \( p \)-adic valuation unless \( \psi(\nabla v) = 0 \). Since \( \psi \) is injective, we conclude \( \nabla v = 0 \), so \( P \) is a \((\sigma, \nabla)\)-submodule of \( M/N \).

Since the slope \( \ell \) of \( P \) is greater than all of the other slopes of \( M/N \), we have an exact sequence

\[0 \to P \to M/N \to (M/N)/P \to 0\]
of \((\sigma, \nabla)\)-modules satisfying the conditions of Proposition 3.3, we conclude that \(M/N\) splits as a direct sum \(P \oplus Q\) of \((\sigma, \nabla)\)-modules. If \(Q\) is nonzero, we may apply Lemma 4.2 once again to it, to conclude that its largest slope is \(\ell\), but this contradicts the fact that all slopes of \(Q\) are smaller than \(\ell\). Thus \(Q\) must be the zero module and \(\phi(M) = \psi(M/N) = \psi(P) = \Gamma_{con}\), proving the desired result.

6 Rigid analytic Quillen-Suslin

In the next section, we will reduce the global full faithfulness theorem to a computation involving a finite projective module over the ring \(K\langle x_1, \ldots, x_n \rangle^\dagger\) of overconvergent power series in \(n\) variables over \(K\). One might expect, in analogy to the Quillen-Suslin theorem, that such a module must necessarily be free; since it is not too difficult to show that this is actually the case, we include a proof here following (and ultimately reducing to) the proof of the Quillen-Suslin theorem given by Lang [La].

For an algebra \(A\) complete with respect to a nonarchimedean absolute value \(|\cdot|\) and \(\rho = (\rho_1, \ldots, \rho_n)\) an \(n\)-tuple of positive reals, we define \(A\langle t_1, \ldots, t_n \rangle_\rho\) as the ring of formal power series which converge on the closed polydisc of radius \(\rho\); that is,

\[
A\langle t_1, \ldots, t_n \rangle_\rho = \left\{ \sum_I c_I t^I : c_I \in A, \lim_{I \to \infty} |c_I|\rho^I = 0 \right\}.
\]

Here \(I = (i_1, \ldots, i_n)\) runs over tuples of nonnegative integers, \(t^I = t_1^{i_1} \cdots t_n^{i_n}\), \(\rho^I = \rho_1^{i_1} \cdots \rho_n^{i_n}\), and \(\sum I = i_1 + \cdots + i_n\). The ring \(A\langle t_1, \ldots, t_n \rangle_\rho\) is complete for the nonarchimedean absolute value

\[
\left| \sum_I c_I t^I \right| = \max_I \{|c_I|\rho^I\}.
\]

For \(n = 1\), we define the leading term \(L(f)\) of a nonzero element \(f = \sum_{i=1}^\infty c_i t^i\) of \(A\langle t \rangle_\rho\) as the monomial \(c_j t^j\) for \(j\) the largest integer such that \(|c_j|\rho^j = \max_i \{|c_i|\rho^i\}\); we refer to \(c_j\) as the leading coefficient of \(f\). We define the degree of \(f\), denoted \(\deg(f)\), as the degree of its leading term. Note that \(|L(f g) - L(f)L(g)| < |L(f g)|\); in particular, \(\deg(f g) = \deg(f) + \deg(g)\).

We use two key lemmas to reduce from power series to polynomials. The first is a form of Weierstrass preparation.

Lemma 6.1 (Weierstrass preparation). Let \(\mathfrak{o}\) be a ring which is complete with respect to a nonarchimedean absolute value \(|\cdot|\), and put \(A = \mathfrak{o}\langle t \rangle_\rho\) for some \(\rho\). Suppose the leading coefficient of \(f \in A\) is a unit in \(\mathfrak{o}\). Then there exists a unit \(u\) in \(A\) such that \(u^{-1} f = \sum_{i=0}^j b_i t^i\) with \(b_j\) a unit and and \(|b_i|\rho^i \leq |b_j|\rho^j\) for \(i < j\).

Proof. Put \(f = \sum_{i \geq 0} f_i t^i\), \(n = \deg(f)\) and \(g = \sum_{i \leq n} f_i t^i\). Let \(B\) be the ring of Laurent series \(\sum_{i \in \mathbb{Z}} c_i t^i\) with \(c_i \in \mathfrak{o}\) such that \(|c_i|\rho^i\) remains bounded as \(i \to -\infty\) and goes to infinity.
as \( i \to +\infty \). Then as in [K2 Lemma 6.3] (or [K1 Lemma 4.1.1]), \( g^{-1}f \) factors in \( B \) as \( uv \) with \( u = \sum_{i \geq 0} u_i t^i \in A \), \( v = 1 + \sum_{i < 0} v_i t^i \), \( u_0 \) a unit, and \( |v_i| \rho^j \leq 1 \) for all \( i \). (Namely, one constructs a sequence of approximate factorizations which converge under \( | \cdot | \). The necessary initial condition is that the leading term of \( g^{-1}f \) be \textit{strictly} larger than all other terms, which it is.) In the equation \( u^{-1}f = gv \), the left side belongs to \( A \) while the right side has no powers of \( t \) beyond \( t^n \). Thus \( u \) has the desired property.

The second ingredient in our reduction is an argument commonly seen in the proof of Noether normalization over finite fields (see for instance [Lal Theorem VIII.2.1]). We state it two ways, once for \( \mathfrak{o} \) a field, once for \( \mathfrak{o} \) not a field.

**Lemma 6.2.** Let \( \mathfrak{o} \) be a complete discrete valuation field. Let \( \rho = (\rho_1, \ldots, \rho_n) \) be a tuple of positive reals and put \( \rho' = (\rho_1, \ldots, \rho_{n-1}) \). Suppose \( u \in \mathfrak{o} \) is a unit with \( |u| \rho_n^m = 1 \). Put \( B = \mathfrak{o}\langle t_1, \ldots, t_{n-1}\rangle_{\rho'} \) and \( A = \mathfrak{o}\langle t_1, \ldots, t_n\rangle_{\rho} = B\langle t_n\rangle_{\rho_n} \). Let \( T_j : A \to A \) be the continuous \( \mathfrak{o}\)-algebra isomorphism with

\[
T_j(t_n) = t_n, \quad T_j(t_i) = t_i + (ut_n^m)^{j-i} \quad (i = 1, \ldots, n-1).
\]

Given \( a \in A \), for all sufficiently large \( j \) (depending on \( a \)), the leading coefficient (in \( t_n \)) of \( T_j(a) \) as an element of \( (\mathfrak{o}\langle t_1, \ldots, t_{n-1}\rangle_{\rho'})\langle t_n\rangle_{\rho_n} \) is a unit in \( \mathfrak{o} \).

**Proof.** Write \( a = \sum_i a_it_i \). For \( j \) sufficiently large, the leading term in \( T_j(a) \) will have degree  
\[m(i_1j^{n-1} + \cdots + i_{n-1}j^{n-2} + i_n),\]
where \( I = (i_1, \ldots, i_n) \) is the last tuple in lexicographic order that minimizes \( |a_I| \rho' \). Moreover, the coefficient will be a power of \( u \) plus smaller terms, which is a unit. \( \square \)

**Lemma 6.3.** Let \( \mathfrak{o} \) be a complete discrete valuation ring. Let \( \rho = (\rho_1, \ldots, \rho_n) \) be a tuple of real numbers greater than 1, and put \( \rho' = (\rho_1, \ldots, \rho_{n-1}) \). Put \( B = \mathfrak{o}\langle t_1, \ldots, t_{n-1}\rangle_{\rho'} \) and \( A = \mathfrak{o}\langle t_1, \ldots, t_n\rangle_{\rho} = B\langle t_n\rangle_{\rho_n} \). Let \( T_j : A \to A \) be the continuous \( \mathfrak{o}\)-algebra isomorphism with

\[
T_j(t_n) = t_n, \quad T_j(t_i) = t_i + t_n^{ mj^{n}-i} \quad (i = 1, \ldots, n-1).
\]

Given \( a \in A \), for all sufficiently large \( j \) (depending on \( a \)) and all sufficiently small \( \lambda > 0 \) (depending on \( a \) and \( j \)), the leading coefficient (in \( t_n \)) of \( T_j(a) \) as an element of \( (\mathfrak{o}\langle t_1, \ldots, t_{n-1}\rangle_{\lambda\rho'})\langle t_n\rangle_{\rho_n} \) is a unit in \( \mathfrak{o} \).

**Proof.** This time, modulo \( \mathfrak{m} \), the leading term of \( T_j(a) \) (as a polynomial in \( t_n \)) has unit coefficient; the same will be true of the leading term of \( T_j(a) \) within \( (\mathfrak{o}\langle t_1, \ldots, t_{n-1}\rangle_{\lambda\rho'})\langle t_n\rangle_{\rho_n} \) provided that \( \lambda \) is sufficiently small. \( \square \)

As a consequence of the dichotomy between Lemmas 6.2 and 6.3, the results we are about to prove have two forms: one over a complete discrete valuation field, in which \( \rho \) does not change during the proof; and another over a complete discrete valuation ring, in which
the conclusion holds after replacing \( \rho \) by \( \rho^\lambda \) for some \( \lambda \) with \( 0 < \lambda \leq 1 \). For simplicity, we will state only the field versions explicitly until we reach Theorem 6.7.

An \( n \)-tuple \((f_1, \ldots, f_n)\) of elements of a ring \( R \) is unimodular if its elements generate the unit ideal of \( R \). Identifying \( n \)-tuples with column vectors, we say that two tuples \( f \) and \( g \) are equivalent, notated \( f \sim g \), if there exists an invertible \( n \times n \) matrix \( M \) over \( R \) such that \( Mf = g \); this is clearly an equivalence relation. In this terminology, our analogue of the main theorem of Quillen and Suslin is the following.

**Proposition 6.4.** Let \( K \) be a field complete with respect to a (nontrivial) nonarchimedean absolute value and let \( \rho = (\rho_1, \ldots, \rho_n) \) be a tuple such that for \( i = 1, \ldots, n \), some power of \( \rho_i \) is the norm of an element of \( K \). Let \( f \) be a unimodular tuple over \( A = K\langle t_1, \ldots, t_n\rangle_\rho \). Then \( f \sim (1, 0, \ldots, 0) \).

We set some common notation for this proof and the next: for \( \rho = (\rho_1, \ldots, \rho_n) \), put \( \rho' = (\rho_1, \ldots, \rho_{n-1}) \) and \( B = K\langle t_1, \ldots, t_{n-1}\rangle_{\rho'} \). Also define \( T_j \) as in Lemma 6.2 (or Lemma 6.3 in the non-field case).

**Proof.** We prove the theorem by induction on \( n \). If \( n = 0 \), there is nothing to prove. Suppose \( n > 0 \); by Lemmas 6.1 and 6.2, for \( j \) sufficiently large, each element of \( T_j(f) \) is a unit in \( A \) times a polynomial over \( B \) whose leading coefficient is a unit. By [La, Theorem XXI.3.4], \( T_j(f) \) is equivalent over \( A \) to a unimodular tuple over \( B \). By the induction hypothesis, the latter is equivalent to \((1, 0, \ldots, 0)\). Hence \( T_j(f) \sim (1, 0, \ldots, 0) \); applying \( T_j^{-1} \) to the resulting matrix yields \( f \sim (1, 0, \ldots, 0) \), as desired.

To apply this result to projective modules, we need an analogue of an older result of Serre, which amounts to the computation of \( K_0 \) of a polynomial ring. (The hypothesis on \( \rho \) ensures that \( K\langle t_1, \ldots, t_n\rangle_\rho \) is noetherian; see for instance [vdP].)

**Proposition 6.5.** Let \( K \) be a field complete with respect to a (nontrivial) nonarchimedean absolute value and let \( \rho = (\rho_1, \ldots, \rho_n) \) be a tuple such that for \( i = 1, \ldots, n \), some power of \( \rho_i \) is the norm of an element of \( K \). Then every finite module over \( A = K\langle t_1, \ldots, t_n\rangle_\rho \) has a finite free resolution.

**Proof.** We proceed by induction on \( n \); again, there is nothing to prove if \( n = 0 \). Note that by [La, Theorem XXI.2.7], if

\[
0 \to M_1 \to M \to M_2 \to 0
\]

is a short exact sequence of modules over a ring \( R \) and any two of \( M, M_1, M_2 \) have finite free resolutions, then so does the third. Thus it suffices to show that every ideal \( I \) of \( A \) has a finite free resolution.

Let \( f_1, \ldots, f_m \) be generators of \( I \). By Lemmas 6.1 and 6.2, for \( j \) sufficiently large, \( T_j(f_i) \) can be written as a unit of \( A \) times a polynomial in \( t_n \) over \( B \). Thus as an \( A \)-module, \( I \) is
isomorphic to $J \otimes_{B[t]} A$ for some ideal $J$ of $B[t]$. By [La, Theorem XXI.2.8] and the induction hypothesis that every finite module over $B$ has a finite free resolution, we deduce that $J$ has a finite free resolution over $B[t]$. Hence $I$ has a finite free resolution over $A$, completing the induction.

Putting everything together, we get a result that implies the main theorem of this section. Note that the complex analogue of Proposition 6.6 is also true; this follows from a theorem of Lin [Li].

**Proposition 6.6.** Let $K$ be a field complete with respect to a (nontrivial) nonarchimedean absolute value and let $\rho = (\rho_1, \ldots, \rho_n)$ be a tuple such that some power of each $\rho_i$ is the norm of an element of $K$. Then every finite projective module over $K\langle t_1, \ldots, t_n \rangle$ is free.

**Proof.** By Proposition 6.5, every finite projective module $M$ over $K\langle t_1, \ldots, t_n \rangle$ has a finite free resolution; by [La, Theorem XXI.2.1], $M$ is stably free (the direct sum of some finite free module with $M$ is finite free). By [La, Theorem XXI.3.6], every stably free module over a ring with the unimodular extension property is free. By Proposition 6.4, $K\langle t_1, \ldots, t_n \rangle$ has the unimodular extension property. Thus every finite projective module over $K\langle t_1, \ldots, t_n \rangle$ is stably free and hence free, as desired.

**Theorem 6.7.** Every finite projective module over $K\langle t_1, \ldots, t_n \rangle$ or $\mathcal{O}\langle t_1, \ldots, t_n \rangle$ is free.

**Proof.** The ring $K\langle t_1, \ldots, t_n \rangle$ is the direct limit of the rings $K\langle t_1, \ldots, t_n \rangle$ over all tuples $\rho$ with $\rho_i > 1$ for all $i$, so every finite projective over $K\langle t_1, \ldots, t_n \rangle$ is the base extension of a finite projective over $K\langle t_1, \ldots, t_n \rangle$ for some $\rho$. The result now follows from Proposition 6.6. For $\mathcal{O}\langle t_1, \ldots, t_n \rangle$, the same argument holds up to replacing $\rho$ by $\rho^\lambda$ for $0 < \lambda < 1$ unspecified (and replacing calls to Lemma 6.2 with Lemma 6.3), but the conclusion in the dagger algebra is unaffected.

### 7 Local to global

We now proceed from the local full faithfulness theorem to a global statement, by using a geometric lemma to “push forward” the problem from a general variety to an affine space, where the reduction to the local theorem is straightforward.

For $X$ a smooth $k$-scheme of finite type and $\mathcal{E}$ an overconvergent $F$-isocrystal, let $H^p_F(X, \mathcal{E})$ denote the set of elements of $\mathcal{E}$ which are killed by $\nabla$ and fixed by $F$. Then the full faithfulness of $j^* : F^\alpha\text{-Isoc}^\dagger(X/K) \to F^\alpha\text{-Isoc}(X/K)$ for $X$ follows from the fact that the rank of $H^p_F(\mathcal{E}, X)$ is the same whether computed in the overconvergent or convergent category. The argument is the same as in Theorem 5.1 that, given overconvergent $F$-isocrystals $\mathcal{E}_1$ and $\mathcal{E}_2$ on $X$, $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ is again an overconvergent $F$-isocrystal, and the morphisms from $\mathcal{E}_1$ to $\mathcal{E}_2$ correspond to elements of $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ fixed by Frobenius and killed by $\nabla$, in either the convergent or overconvergent category.
We now focus on showing that \( H^0_{\mathcal{F}}(X, \mathcal{E}) \) is the same in the convergent and overconvergent categories. The definition of \( H^0_{\mathcal{F}}(X, \mathcal{E}) \) is local on \( X \) in both cases, and by the following lemma, which follows from the main result of \( \text{[K5]} \), we may cover \( X \) with open affine subsets which are finite étale covers of affine spaces. (The case of \( k \) infinite and perfect is covered by \( \text{[K3]} \); one could reduce to this case with a bit of extra work.)

**Lemma 7.1.** Let \( X \) be a separated \( k \)-scheme of finite type of pure dimension \( n \) and let \( x \) be a smooth geometric point of \( X \). Then there exists an open dense subset \( U \) of \( X \), containing \( x \) and defined over \( k \), such that \( U \) admits a finite étale map over \( k \) to affine \( n \)-space.

Thus it suffices to consider the case \( X \) equal to such an open dense subset \( U \). Let \( f : X \to \mathbb{A}^n \) be a finite étale morphism and \( \mathcal{E} \) an overconvergent \( F \)-isocrystal on \( X \). There is a pushforward construction in the overconvergent and convergent categories \([\text{12}, \text{Proposition 5.1.2}]\) such that \( H^0(X, \mathcal{E}) = H^0(\mathbb{A}^n, f_*\mathcal{E}) \). Thus it suffices to consider isocrystals on affine space itself.

Take \( f = a \), so that \( q = p^a \). Let \( R = \mathcal{O}(x_1, \ldots, x_n)^\dagger \) be the ring of overconvergent power series in \( n \) variables over \( \mathcal{O} \), and let \( \sigma \) be the Frobenius lift on \( R \) sending \( \sum c_i x_i^\ell \) to \( \sum c_i^\ell x_i^{p\ell} \). Then the data of an overconvergent \( F \)-isocrystal \( \mathcal{E} \) on \( \mathbb{A}^n \) is simply that of a \((\sigma, \nabla)\)-module \( M \) on \( R[\frac{1}{p}] \); we can find an integer \( \ell \) such that the Tate twist \( M(\ell) \) can be defined over \( R \). Then \( M(\ell) \) is free over \( R \) by Theorem \( \text{[6.7]} \). Let \( R^\wedge \) denote the \( \pi \)-adic completion of \( R \); then the elements of \( H^0_{\mathcal{F}}(\mathbb{A}^n, \mathcal{E}) \) in the convergent and overconvergent categories correspond to the elements \( v \in M(\ell) \otimes_R R^\wedge \) and \( M(\ell) \), respectively, such that \( Fv = p^\ell v \) and \( \nabla v = 0 \).

Note that a power series in \( x_1, \ldots, x_n \) over \( \mathcal{O} \) lies in \( R \) if and only if it is overconvergent in each variable separately. In other words, if \( S_i \) is the valuation subring of the fraction field of the ring of null power series in all of the \( x \)'s other than \( x_i \), and \( R_i = S_i[x]^\dagger \), then \( \cap_i R_i = R \) (the intersection taking place in the completed fraction field of \( R \)).

We use the above observation to “take apart” \( M \). Namely, by definition, the module \( \Omega^1_R \) is freely generated over \( R \) by \( dx_1, \ldots, dx_n \). Let \( \Omega^1_{R/S_i} \) be the free module generated over \( R_i \) by \( dx_i \). If we put \( k = S_i \), then \( M(\ell) \otimes_R R_i \) has a natural structure as \((\sigma, \nabla_i)\)-module over \( R_i \), with \( \nabla_i \) given by composing the given map \( M(\ell) \to M(\ell) \otimes \Omega^1_R = \oplus_i M(\ell) \otimes R_i dx_i \) with the projection onto the \( i \)-th factor and the map \( R dx_i \to R_i dx_i \).

Identify \( R_i \) with a subring of \( \Gamma_{\text{con}}^S \) so that \( x_i^{-1} \) reduces to a uniformizer of the residue field. If \( v \in M(\ell) \otimes_R R^\wedge \) satisfies \( Fv = p^\ell v \) and \( \nabla v = 0 \), then Theorem \( \text{[5.1]} \) applied to \( M(\ell) \otimes_R R_i \) implies that \( v \in M(\ell) \otimes_R R_i \) for each \( i \). Since \( M(\ell) \) is free over \( R \) and \( \cap_i R_i = R \), we deduce \( v \in M(\ell) \). We thus conclude that \( H^0(\mathbb{A}^n, \mathcal{E}) \) has the same rank in the convergent and overconvergent categories; as noted above, this suffices to prove Theorem \( \text{[11]} \).
References

[A] Y. André, Filtrations de type Hasse-Arf et monodromie $p$-adique, *Invent. Math.* 148 (2002), 285–317.

[B1] P. Berthelot, Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$, in Introductions aux cohomologies $p$-adiques (Luminy, 1984), *Mém. Soc. Math. France* 23 (1986), 7–32.

[B2] P. Berthelot, Finitude et pureté cohomologique en cohomologie rigide (with an appendix in English by A.J. de Jong), *Invent. Math.* 128 (1997), 329–377.

[C] R. Crew, Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve, *Ann. Scient. Éc. Norm. Sup.* 31 (1998), 717–763.

[dJ] A.J. de Jong, Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic, *Invent. Math.* 134 (1998), 301–333.

[E] J.-Y. Etesse, Descente étale des $F$-isocristaux surconvergents et rationalité des fonctions $L$ de schémas abéliens, *Ann. Scient. Éc. Norm. Sup.* 35 (2002), 575–603.

[Ka] I. Kaplansky, Maximal fields with valuations, *Duke Math. J.* 9 (1942), 303–321.

[K1] K.S. Kedlaya, Descent theorems for overconvergent $F$-crystals, Ph.D. thesis, Massachusetts Institute of Technology, 2000.

[K2] K.S. Kedlaya, A $p$-adic local monodromy theorem, preprint, arXiv: math.AG/0110124 to appear in *Annals of Math.*

[K3] K.S. Kedlaya, Étale covers of affine spaces in positive characteristic, *C.R. Acad. Sci. Paris Ser. I* 335 (2002), 921–926.

[K4] K.S. Kedlaya, Finiteness of rigid cohomology with coefficients, preprint, arXiv: math.AG/0208027.

[K5] K.S. Kedlaya, More étale covers of affine spaces in positive characteristic, preprint, arXiv: math.AG/0303382.

[La] S. Lang, *Algebra*, third edition, Addison-Wesley, 1993.

[Li] V. Ja. Lin, Holomorphic fiberings and multivalued functions of elements of a Banach algebra (Russian), *Funk. Anal. i Priložen.* 7 (1973), 43–51.

[M] Z. Mebkhout, analogue $p$-adique du Théorème de Turrittin et le Théorème de la monodromie $p$-adique, *Invent. Math.* 148 (2002), 319–351.
[T1] N. Tsuzuki, The overconvergence of morphisms of etale $\phi-\nabla$-spaces on a local field, *Comp. Math.* **103** (1996), 227–239.

[T2] N. Tsuzuki, Morphisms of $F$-isocrystals and the finite monodromy theorem for unit-root $F$-isocrystals, *Duke Math. J.* **111** (2002), 385–418.

[vdP] M. van der Put, Non-archimedean function algebras, *Indag. Math.* **33** (1971), 60–77.