Deterministic walks in random media: evidence of generic scale invariance

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Deterministic walks over a random set of points in one and two dimensions ($d = 1, 2$) are considered. Points ("cities") are randomly scattered in $\mathbb{R}^d$ following a uniform distribution. A walker (a "tourist"), at each time step, goes to the nearest neighbor city that has not been visited in the past $\tau$ steps. Each initial city leads to a trajectory composed of a transient part and a final $p$-cycle attractor. The distribution of transient times, $p$-cycles and number of cities per attractor are studied. It is shown numerically that transient times (for $d = 1, 2$) follow a Poisson law with a $\tau$ dependent decay but the density of $p$-cycles follows a power law $D(p) \propto p^{-\alpha(\tau)}$ for $d = 2$. For large $\tau$, the exponent tends to $\alpha \approx 5/2$. Some analytical results are given for the $d = 1$ case. Since the power law is robust and does not depend on free parameters, this system presents "generic scale invariance". Applications to animal exploratory behavior and other local minimization problems are suggested.

The study of random walks has been very fruitful in physics and mathematics, and the theory of such stochastic processes is a well developed subject. The study of deterministic walks is also an interesting subject but presents the analytical difficulties common to the area of non-linear dynamical systems and has been less investigated [1]. Here we propose a simple and intriguing problem, a deterministic walk over a random graph with $N$ nodes that is also an example of a local ("on-line") optimization dynamics. It may be called the "local traveling salesman problem" or perhaps the "tourist problem" for short. The deterministic dynamics produces a division of the system phase space in several $O(N)$ attractor basins which trap the walker (ergodicity is broken). The problem is reminiscent of walks in rugged landscapes but the equivalent of "local minima" are cycles instead of point attractors.

The model is defined as follows: points are randomly distributed with a uniform density $\rho$ in $\mathbb{R}^d$, where $d$ is the dimensionality of the space. These points may be thought as "cities" or "safe places" and they may be viewed as vertices of a random graph. At each time step, the "tourist" follows the deterministic rule: Go to the nearest city (or place) that has not been visited in the past $\tau$ time steps. Notice that the tourist wants to minimize only the distance to the next city (a local optimization procedure), not the sum of all distances in the trajectory or some other global cost function.

Our model can be of general interest for optimization theory with local constraints, and studies of deterministic dynamical systems with quenched disorder. However, we would like to suggest some specific motivations for considering this class of problems. The local optimization procedure could be naturally related to exploratory, foraging or migratory behaviors of animals. For example, rodents present two competing drives: an exploratory drive ("curiosity") and a defensive behavior called thigmotaxis. The later refers to rodent aversion to open spaces and preference for places where its whiskers can touch touchy surfaces or objects, which provide protection [3]. It is arguable that, for biological agents (and biologically inspired robots) it could be sometimes more important to minimize the distance traveled in each movement between two safe places instead of to optimize some global cost function. In another scale, the model could describe migratory or nomadic behaviors of humans, elephants, flamingos and other animals with well developed spatial memory. Cycles could be related to stable migratory routes on environments with localized resources, for example, oceanic islands, oasis and water holes. Local procedures are also usual in optimal foraging theory [3]. The need for local optimization emerges due to short range sensorial capacities. Long distances may also imply non-additive costs: animals (and tourists) need safe places to stay during night, which puts a maximal distance that can be traveled at each time step.

Starting from a random city, the tourist performs a trajectory composed of a transient part and a final $p$-cycle attractor. In this letter we report the statistics for some relevant quantities similar to those measured in Kauffman networks [4]: a) the probability $P_t(t)$ for obtaining a transient of size $t$, defined as the number of steps before the walker enters in some attractor (irrespective to the cycle period); for large $t$, it is Poisson-like $P_t(t) \propto \exp(-t/\xi(\tau))$ with the decay time $\xi(\tau)$ growing exponentially for $d = 1$ and linearly for $d = 2$; b) the total density of attractors $D(\tau)$, which decays exponentially for $d = 1$ and as $\tau^{-1}$ for $d = 2$; c) for $d = 2$, the density of $p$-cycles $D_p(p)$ which follows a power law $D_p(p) \propto p^{-\alpha(\tau)}$ for $\tau > 0$, with $\alpha \approx 5/2$ for large $\tau$ (generic scale invariance); and d) the average number $\langle n(\tau) \rangle$ of cities present in a $p$-cycle. Also some analytical results are obtained for the $d = 1$ case.

The discrete time step is simply a label: it does not
measure the actual physical time spent when the walker travels between the points. This independence makes irrelevant the density $\rho$ of cities because only relative distances are important (which city is the nearest) and not absolute distances, contrasting to standard random walks where the mean length step defines an intrinsic length scale. The only parameter is the memory window $\tau$. Self-avoidance is limited to this window and trajectories can intersect outside this range. If $\tau = 0$ (no memory) the tourist goes simply to the nearest city until it finds two cities that are reciprocally nearest neighbors, entering in a 2-cycle. In this simple case, attractors may be identified with geometrical (cluster) properties. For $\tau = N - 1$ the trajectory is totally self-avoiding and one has a kind of TSP nearest-neighbor algorithm \( \mathbb{B} \). The interesting cases are the intermediate ones. For example, if $\tau = 1$, the last visited city cannot be revisited, and only $p$-cycles with $p \geq 3$ can exist. For generic $\tau$, the relation $p \geq \tau + 2$ holds.

In the numerical experiments, $N$ points are randomly scattered following a uniform distribution in the interval $[0, 1]^d$. Each point has $d$ spatial coordinates $(x^1, x^2, \ldots, x^d)$. The cities receive arbitrarily labeled as $i = 1, \ldots, N$ and one constructs the Euclidean distance matrix $\mathbf{D}$ (for example, $D_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ in $d = 2$). The dynamics can be performed over the entries of this matrix instead of on real space. Starting from some city, one gets a transient trajectory until the walker enters some periodic attractor and is trapped. The number of steps before the walker enters the cycle defines the transient length $t$. The period $p$ and the number of different cities $n$ that pertain to the attractor are also determined. The same city can be visited more than once, thus $n \leq p$.

A finite size study showed that the behavior of the system is smooth as a function of $1/N$, so we have used $N = 3000$ as a reasonable number for our simulations. Since each city is used as a starting point, a landscape with $N$ cities produces $N$ different transients. The statistics over $N_R$ realizations of sets of cities (“landscapes”) are collected. Unless stated otherwise, the results have been obtained by using $N_R = 100$.

A natural question is if there is some critical $\tau$ that produces a phase transition, for example the emergence of an untrapped (percolating) transient state. The distribution of transient times does not suggest this possibility because it is Poisson, $P_\tau(t) \sim \exp(-t/\xi(\tau))$. This is shown for $d = 1$ in Fig. 1 and for $d = 2$ in Fig. 2. The exponential decay may be understood as follows. One finds numerically that the total number of attractors $N_A(\tau)$ is proportional to $N$, that is, the total density of attractors $\mathcal{D}(\tau) = N_A(\tau)/N$ is constant and depends only on $\tau$. These attractors are scattered in phase space in a random uniform manner. Like points scattered randomly in space, one expects that the distribution of distances between attractors follows a Poisson law when these distances are larger than the attractor size $p$. By supposing that transient times are proportional to these distances, a Poisson law follows also for the transient times.

For $d = 1$, the characteristic times grow as $\xi(\tau) \propto \exp(\gamma \tau)$ (inset Fig. 1) and the total density decays as $\mathcal{D}(\tau) \propto \exp(-\gamma' \tau)$ (Fig. 3a). For $d = 2$, one observes the linear dependence $\xi(\tau) \propto \tau^d, \delta = 1.0$ (inset Fig. 2); the total density decays as a power law $\mathcal{D}(\tau) \propto \tau^{-\delta'}$ (Fig. 3b). The average transient is proportional to the average distance between attractors, (which are inversely proportional to the attractor density). This means that $\mathcal{D}(\tau) \cdot \xi(\tau)$ should be constant, that is, $\gamma = \gamma'$ and $\delta = \delta'$. This is indeed the case (see Fig. 3c), but for $d = 1$, although the exponential terms cancel, a linear residue remains. A better expression for the $1D$ decay time is $\xi(\tau) = c \tau \exp(\gamma \tau)$. We conjecture that the linear prefactor arises from the transient time spent in the cities of the attractor before it stabilizes (this time is larger for $d = 1$ systems).

A property of natural interest is the density $D_p(\rho)$ of $p$-cycles, estimated as the number of different $p$-cycles divided by $N$, in the limit of very large systems. Evaluating this quantity requires careful enumeration because, when starting from all the possible initial states, one must not count the same attractor twice. Notice that $\mathcal{D}(\tau) = \sum_p D_p(\rho)$. For $d = 1$, $D_p(\rho)$ is certainly non-Poisson, although the evidence for a power law is weak (Fig. 4a). For $d = 2$, one observes clear power laws $D_p(\rho) \propto p^{-\alpha(\tau)}$ (Fig. 4b). The exponent $\alpha(\tau)$ stabilizes around $\alpha = 5/2$ for large $\tau$ (inset Fig. 4b). Since there is no fine tuning of any explicit parameter in our system (such as $\tau$), the scale invariance is “generic” \( \mathbb{B} \), that is, intrinsic to the problem. This is the most surprising result of our study. We conjecture that this scale invariance is related to two factors: a) a uniform distribution of points has a single length scale, $\lambda = \rho^{1/d}$ but this length is irrelevant to the dynamics since only relative distances are considered when making a move; and b) the window $\tau$ defines a minimal length $p_{\min} = \tau + 2$ but not a maximal one. A clear explanation of this power law is still lacking.

We stress that this problem is not related only to geometrical properties since the cycles are appear only due to the introduced dynamics. Naively, one could think that a $p$-cycle is a geometrical object, for example a cluster where the distances between the points are smaller than any distance outside the cluster. This indeed is a sufficient but not necessary condition to obtain a $p$-attractor. For example, for $d = 2$ (Fig. 5a), a walker with memory $\tau = 1$ starts from city $A$ and finds the 4-cycle $ABCD$. Although city $E$ is close to the cluster (since $BE < AB$), it is never visited because $BC < BE$ and $CD < CE$. However, if the tourist starts from city $C$, one gets a 3-cycle that includes city $E!$. This degeneracy and superposition of attractors can be understood observing that Fig. 5 shows trajectories in configuration space, not in phase space. In phase space, points correspond to $\tau + 1$-uples $(X_1, \ldots, X_{\tau+1})$ where $X_i$ is the position $(x, y)$ of the tourist at time $t$ and trajectories
never intersect. Only for \( \tau = 0 \) the configuration space is equivalent to the phase space.

Finally, we present the average number of cities \( \langle n_\tau(p) \rangle \) pertaining to cycles of period \( p \) (Fig. 6). For \( d = 1 \) there is almost no dispersion in the number of cities per attractor. A \( p \) -cycle has \( n(p) \) cities. We also found that, for \( n > 2(\tau + 2) \), the following relation holds for even cycles:

\[
n_\tau(p) = p/2 + \tau + 1.
\]

To see how this relation emerges, notice that for each \( \tau \) there is a minimum cycle of period \( p_\tau = \tau + 2 \), which we call a base block (Fig. 5b). A base block is composed of \( n_\tau = \tau + 2 \) cities. The next cycles follow specific constructions (Fig. 5c). But when \( n \) is large, geometrical constraints impose that the most common \( p \)-cycles are made of two base blocks (one in each attractor extremity) joined by \( n_I \) intermediate cities (see Fig. 5d). An attractor with \( n \) cities thus have \( n_I = n - 2n_\tau \) intermediate points. These intermediate cities contribute to the total period with \( p_I = 2n_I + 2 \) steps (since for \( n_I = 0 \), the joining of the base blocks contributes with two steps). Thus, the total period is \( p = 2 \times p_\tau + p_I = 2(n - \tau - 1) \), which leads to Eq. (1). This relation holds for cycles with \( n \geq 2n_\tau = 2(\tau + 2) \), because only these cycles can incorporate two independent base blocks. For \( \tau = 1 \), this is the unique conceivable manner of constructing cycles, meaning that odd cycles are prohibited (and also \( p = 6 \) cycles, see Fig. 4a). For \( \tau > 1 \), it is possible to construct odd cycles by using internal loops (an example with \( \tau = 2 \) is given in Fig. 5e).

In \( d = 2 \), the attractors are polygons with different forms and shapes so that this strict relation between periods and cities does not hold, although \( \langle n \rangle \) also scales linearly with \( p \) (not shown). For \( \tau = 1 \), one finds that odd cycles are less probable than even cycles (Fig. 4b), which is reminiscent of the \( d = 1 \) behavior. Indeed, this occurs because elongated odd attractors in two-dimensional space are prohibited by the same geometrical constraints present in the one-dimensional case.

Another analytical result for the \( d = 1 \) case can be obtained. Consider points \( x_i \) randomly scattered along the real line, defining segments of size \( s_i = x_i - x_{i-1} \). Without loss of generality, we assume that \( \rho = 1 \), which means that \( \langle s \rangle = 1 \). It is easy to see that the distribution of interval sizes \( P(s) \) follows a Poisson distribution \( P(s) = \Theta(s) \exp(-s) \), where \( \Theta(s) \) is the Heaviside step function. For \( \tau = 0 \), there exist only 2-cycles attractors, which correspond to pairs of reciprocal nearest neighbors. The probability \( P_2 \) for this configuration is equal to the probability that \( s_{i-1} > s_i \) and \( s_{i+1} > s_i \). Since \( s_{i-1} \) and \( s_{i+1} \) are drawn independently, one gets:

\[
P_2 = \int_0^\infty ds_i P(s_i) P(s_{i-1} > s_i | s_i) P(s_{i+1} > s_i | s_i) = \int_0^\infty ds_i e^{-s_i} \left( \int_0^\infty ds e^s \right)^2 = 1/3,
\]

that is, on average, one third of the sequences of four points leads to reciprocal nearest neighbors and so to 2-cycles. Since the number of sequences of four points is, in the large \( N \) limit, equal to the number of points, one obtains \( D_0(2) = 1/3 \). This has been fully confirmed by our numerical simulations (see "Methods").

The model may be generalized by introducing a stochastic component (a “temperature” \( T = 1/\beta \)). For example, the probability for the tourist to travel from its present city \( j \) to some city \( i \) may be a function of the distance, say, \( P(j \rightarrow i) \propto \exp(-\beta D_{ij}/\lambda) \), where \( \lambda = \rho^{1/d} \) normalizes the distances. In this case we expect, for \( T = \beta^{-1} \ll 1 \), a punctuated-equilibrium behavior with sporadic transitions between attractor basins. It is an open question to determine if there is a critical temperature \( T_c \) where full ergodicity is recovered.

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FIG. 1. Distribution of transient times $P_{\tau}(t)$ for $d = 1$. From right to left: $\tau = 5, 4, 3, 2$ and 0 (squares near the $y$ axis). Inset: Decay time $\xi(\tau)$.

FIG. 2. Distribution $P_{\tau}(t)$ of transient times for $d = 2$. From right to left: $\tau = 10, 5, 3, 2, 1$ and 0. Inset: Decay time $\xi(\tau)$.

FIG. 3. Total density of attractors $D(\tau)$: a) $d = 1$; b) $d = 2$; c) $D(\tau) \cdot \xi(\tau)$ for $d = 1$ (squares) and $d = 2$ (circles), error bars smaller than symbol size.

FIG. 4. Examples of attractor densities $D_{\tau}(p)$: a) $d = 1$, $\tau = 1$ (squares) and $\tau = 6$ (filled circles); b) $d = 2$, $\tau = 1$ (squares) and 10 (filled circles), $N = 3000$ and $N_R = 700$. Inset: exponent $\alpha(\tau)$. For $\tau = 0$ only 2-cycles exist: $D_0(2) = 0.333 \pm 0.001$, for $d = 1$ and $D_0(2) = 0.31 \pm 0.01$ for $d = 2$.

FIG. 5. a) Example of superposition of attractors for $d = 2$ and $\tau = 1$: starting from $A$ one obtains the 4-cycle $ABCD$, but starting on $C$ one gets the 3-cycle $CBE$; b) the 3-cycle base block for $\tau = 1$; c) the 4-cycle for $\tau = 1$; d) next permissible cycle for $\tau = 1$: a 8-cycle made of two base blocks; e) example of odd ($p = 13$) cycle for $\tau = 2$ which is possible because of an internal loop.

FIG. 6. Number of cities per attractor $\langle n(p) \rangle$. a) $d = 1$, $\tau = 1$ (filled circles) and $\tau = 5$ (circles), theoretical curves (solid) $n(p) = p/2 + \tau + 1$. 

Fig. 1 - Lima - PRL
Fig. 2 - Lima - PRL
Fig. 3 - Lima - PRL
Fig. 4 - Lima - PRL

(a) $d = 1$

(b) $d = 2$
Fig. 5 - Lima - PRL
Fig. 6 - Lima - PRL