Permutability of Backlund Transformations for $N = 2$
Supersymmetric Sine-Gordon

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ABSTRACT

The permutability of two Backlund transformations is employed to construct a non linear superposition formula and to generate a class of solutions for the $N = 2$ super sine-Gordon model. We present explicitly the one and two soliton solutions.
1 Introduction

Backlund transformations reduce the order of the non-linear differential equations making the system sometimes effectively more tractable. Starting with a simple input solution, we may be able to solve for a more complicated one. In many cases, this may be very difficult to accomplish. A convenient and powerful way is to use the permutability theorem which provides a closed algebraic non-linear superposition formula for the solutions.

The Backlund transformation and the Permutability theorem are employed to derive a series of consistency conditions which are satisfied by soliton solutions of certain class of integrable models. Within such class, we encounter the sine-Gordon [1] and KdV [2] equations. This framework was also applied to the $N = 1$ super KdV [3] and super sinh-Gordon [4] in order to derive its soliton solutions.

The $N = 2$ super sine-Gordon model was proposed in [5] and later in [6] its algebraic structure was uncovered. Certain solutions of this model have already been constructed [7], however they were such that involve a single Grassmann parameter. In this paper we extend the non-linear superposition formulae for soliton solutions of the $N = 2$ super sine-Gordon model. These formulae are derived from the Backlund transformation proposed in [8] and the permutability condition which implies that the order of two successive Backlund transformations is irrelevant. As examples, we present explicitly the 1-and 2-soliton solutions with distinct Grassmann parameters.

Recently the Pohlmeyer reduction of $AdS_n x S_n$ superstring models have been considered [9] which in the simple case of $n = 2$ was shown [10] to be equivalent to the $N = 2$ supersymmetric sine-Gordon.

This paper is organized as follows. In Section 2 we discuss the $N = 2$ super sine-Gordon and its Backlund Transformation. In Section 3 we apply the permutability condition to derive a closed algebraic non-linear superposition formulae involving solutions of the model. Finally in Section 4 and 5 we present the 1- and 2-soliton solutions respectively. In the appendix A we present the Backlund transformation in components. In appendices B and C we give details for the derivation of the superposition formulae.

2 $N = 2$ super sine-Gordon - Backlund Transformation

Let us start by introducing the $N = 2$ superfields [5]

$$
\phi^\pm = \varphi^\pm(z^\pm, \bar{z}^\pm) + \theta^\pm \psi^\pm(z^\pm, \bar{z}^\pm) + \bar{\theta}^\pm \bar{\psi}^\pm(z^\pm, \bar{z}^\pm) + \theta^\pm \bar{\theta}^\pm F^\pm(z^\pm, \bar{z}^\pm),
$$

where

$$
z^\pm = z \pm \frac{1}{2} \theta^+ \theta^-, \quad \bar{z}^\pm = \bar{z} \pm \frac{1}{2} \bar{\theta}^+ \bar{\theta}^-.
$$

The superfield components $\phi^\pm$ can be expanded in Grassmann variables $\theta^\pm$ and $\bar{\theta}^\pm$. For instance, the component $\varphi^\pm(z^\pm, \bar{z}^\pm)$ gives rise to

$$
\varphi^\pm(z^\pm, \bar{z}^\pm) = \varphi^\pm \pm \frac{1}{2} \theta^+ \theta^- \partial_z \varphi^\pm \pm \frac{1}{2} \bar{\theta}^+ \bar{\theta}^- \partial_{\bar{z}} \varphi^\pm \pm \frac{1}{4} \theta^+ \theta^- \bar{\theta}^+ \bar{\theta}^- \partial_z \partial_{\bar{z}} \varphi^\pm.
$$
By expanding all components of $\phi^\pm$, we obtain

$$
\phi^\pm = \varphi^\pm + \theta^\pm\psi^\mp + \bar{\theta}^\pm\bar{\psi}^\mp \pm \frac{1}{2} \theta^+\theta^- \partial_z \varphi^\pm \pm \frac{1}{2} \bar{\theta}^+\bar{\theta}^- \bar{\partial}_z \varphi^\pm + \theta^\pm\bar{\theta}^\pm F^\pm \\
\pm \theta^\pm\bar{\theta}^\pm - \frac{1}{2} \partial_z \psi^\mp \mp \bar{\theta}^\pm\bar{\theta}^\pm - \frac{1}{2} \bar{\partial}_z \bar{\psi}^\mp + \frac{1}{4} \theta^+\theta^- \bar{\theta}^\pm \bar{\partial}_z \varphi^\pm.
$$

We next introduce the super derivatives

$$
D_\pm = \frac{\partial}{\partial \theta^\pm} + \frac{1}{2} \theta^\pm \partial_z, \quad \bar{D}_\pm = \frac{\partial}{\partial \bar{\theta}^\pm} + \frac{1}{2} \bar{\theta}^\pm \partial_z,
$$

satisfying the following conditions

$$
D_\pm^2 = 0, \quad \bar{D}_\pm^2 = 0, \\
\{\bar{D}_\pm, D_\pm\} = 0, \quad \{\bar{D}_\pm, D_\mp\} = 0, \\
\{D_+, D_-\} = \partial_z, \quad \{\bar{D}_+, \bar{D}_-\} = \partial_z.
$$

The equations of motion for the supersymmetric sine-Gordon model with $N = 2$ are given by [5]

$$
\bar{D}_\pm D_\pm \phi^\pm = g \sin \left( \beta \phi^\mp \right), \quad (1)
$$

where $g$ is a mass parameter and $\beta$ is the coupling constant. From now on we assume $\beta = 1$ which may be re-inserted by a convenient field reparametrization. In components, the equations of motion for the $N = 2$ super sine-Gordon reads,

$$
F^\pm = g \sin \varphi^\mp, \\
\partial_z \psi^\mp = g \cos \varphi^\mp \bar{\psi}^\mp, \\
\partial_z \bar{\psi}^\mp = -g \cos \varphi^\mp \psi^\mp, \\
\partial_z \bar{\partial}_z \varphi^\pm = -g \cos \varphi^\mp F^\mp - g \sin \varphi^\mp \psi^\mp \bar{\psi}^\mp.
$$

Moreover the the chiral, $\phi^+$ and the anti-chiral, $\phi^-$ superfields satisfy the conditions

$$
\bar{D}_\pm \phi^\mp = D_\pm \varphi^\mp = 0. \quad (2)
$$

Let us now recall the Backlund transformation for the $N = 2$ super sine-Gordon model [8]. For this purpose, consider the pair of first order differential equations

$$
D_+ \phi_1^+ = D_+ \phi_2^+ - \frac{8}{\kappa} \mathcal{F} \cos \left( \frac{\phi_1^- + \phi_2^-}{2} \right), \quad (3) \\
\bar{D}_+ \phi_1^+ = -\bar{D}_+ \phi_2^+ + \kappa \mathcal{G} \cos \left( \frac{\phi_1^- - \phi_2^-}{2} \right), \quad (4)
$$

where $\mathcal{F}$ and $\mathcal{G}$ are fermionic auxiliary superfields and $\kappa$ is an arbitrary constant. The above equation and the condition

$$
(\bar{D}_+ D_+ + D_+ \bar{D}_+) \phi_1^+ = 0,
$$

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leads to the equations of motion
\[ \bar{D}_+ D_+ \phi_2^\pm = g \sin \phi_2^\pm, \]
provided the superfields \( F \) and \( G \) satisfy
\[ \bar{D}_+ F = -\frac{\kappa g}{4} \sin \left( \frac{\phi_1^- - \phi_2^\pm}{2} \right), \quad D_+ G = -\frac{2g}{\kappa} \sin \left( \frac{\phi_1^- + \phi_2^\pm}{2} \right). \tag{5} \]

In a similar way,
\[ D_- \phi_1^- = D_- \phi_2^- + \lambda G \cos \left( \frac{\phi_1^+ + \phi_2^\pm}{2} \right), \tag{6} \]
\[ \bar{D}_- \phi_1^- = -\bar{D}_- \phi_2^- - \frac{8}{\lambda} F \cos \left( \frac{\phi_1^+ - \phi_2^\pm}{2} \right), \tag{7} \]
where \( \lambda \) is another arbitrary constant. Together with the condition
\[ (\bar{D}_- D_- + D_- \bar{D}_-) \phi_1^- = 0, \]
yields
\[ D_- D_- \phi_2^- = g \sin \phi_2^+, \]
provided \( G \) and \( F \) satisfy
\[ \bar{D}_- G = \frac{2g}{\lambda} \sin \left( \frac{\phi_1^+ - \phi_2^\pm}{2} \right), \quad D_- F = \frac{\lambda g}{4} \sin \left( \frac{\phi_1^+ + \phi_2^\pm}{2} \right). \tag{8} \]

Acting with \( D_+ \) in eqn. (3), \( \bar{D}_+ \) in (11), \( D_- \) in (6) and \( \bar{D}_- \) in (7) we find
\[ D_+ F = 0, \quad \bar{D}_+ G = 0, \quad D_- G = 0, \quad \bar{D}_- F = 0. \tag{9} \]

These last conditions allows us to rewrite the fermionic superfields into two distinct manners, i.e.,
\[ F = D_+ \Phi_1^+, \quad \bar{D}_- \Phi_2^-, \tag{10} \]
\[ G = D_- \Phi_1^- = \bar{D}_+ \Phi_2^+, \tag{11} \]
where the chiral \( \Phi_p^+ \) and anti-chiral \( \Phi_p^- \), \( p = 1, 2 \) superfields are defined as
\[ \Phi_1^+ = q_1^+(z^\pm, \bar{z}^\pm) + \theta^\pm \zeta_1^+(z^\pm, \bar{z}^\pm) + \bar{\theta}^\pm \zeta_2^+(z^\pm, \bar{z}^\pm) + \theta^\pm \bar{\theta}^\pm q_2^+(z^\pm, \bar{z}^\pm), \]
\[ \Phi_2^+ = p_1^+(z^\pm, \bar{z}^\pm) + \theta^\pm \zeta_1^+(z^\pm, \bar{z}^\pm) + \bar{\theta}^\pm \zeta_2^+(z^\pm, \bar{z}^\pm) + \theta^\pm \bar{\theta}^\pm p_2^+(z^\pm, \bar{z}^\pm). \]
The second equality in (10) implies
\[ \zeta_1^+ = \zeta_2^-, \quad q_2^+ = \partial_z p_1^-, \quad p_2^- = -\partial_z q_1^+, \quad \partial_z \zeta_2^+ = -\partial_z \zeta_1^-, \tag{12} \]
whilst the second equality in (11) implies
\[ \zeta_1^- = \zeta_2^+, \quad p_2^+ = -\partial_z q_1^-, \quad q_2^- = \partial_z p_1^+, \quad \partial_z \zeta_2^- = -\partial_z \zeta_1^+. \tag{13} \]

Eqs. (3)-(9) describe the Backlund transformation for the \( N = 2 \) super sine-Gordon system. In appendix A we present these equations in components.
3 The Permutability condition

A Backlund transformation from \( \phi_0^+ \) to \( \phi_1^+ \) is described by

\[
D_+(\phi_0^+ - \phi_1^+) = -\frac{8}{\kappa_1} F^{(0,1)} \cos \left( \frac{\phi_0^- + \phi_1^-}{2} \right), \tag{14}
\]

\[
\bar{D}_+(\phi_0^+ + \phi_1^+) = \kappa_1 G^{(0,1)} \cos \left( \frac{\phi_0^- - \phi_1^-}{2} \right), \tag{15}
\]

\[
D_-(\phi_0^- - \phi_1^-) = \lambda_1 G^{(0,1)} \cos \left( \frac{\phi_0^+ + \phi_1^+}{2} \right), \tag{16}
\]

\[
\bar{D}_-(\phi_0^- + \phi_1^-) = -\frac{8}{\lambda_1} F^{(0,1)} \cos \left( \frac{\phi_0^+ - \phi_1^+}{2} \right), \tag{17}
\]

where we have introduced the superscript indices \((0, 1)\) for the auxiliary fermionic superfields denoting its dependence in \( \phi_0^\pm \) and \( \phi_1^\pm \). The latter, in turn satisfy the following condition (as in (5) and (6))

\[
\bar{D}_+ F^{(0,1)} = -\frac{\kappa_1}{4} \sin \left( \frac{\phi_0^- - \phi_1^-}{2} \right), \tag{18}
\]

\[
D_+ G^{(0,1)} = -\frac{2}{\kappa_1} \sin \left( \frac{\phi_0^+ + \phi_1^-}{2} \right), \tag{19}
\]

\[
\bar{D}_- G^{(0,1)} = \frac{2}{\lambda_1} \sin \left( \frac{\phi_0^+ - \phi_1^+}{2} \right), \tag{20}
\]

\[
D_- F^{(0,1)} = \frac{\lambda_1}{4} \sin \left( \frac{\phi_0^+ + \phi_1^+}{2} \right). \tag{21}
\]

The chiral conditions (9), i.e.,

\[
\bar{D}_- F^{(0,1)} = 0, \quad D_+ F^{(0,1)} = 0, \quad \bar{D}_+ G^{(0,1)} = 0, \quad D_- G^{(0,1)} = 0,
\]

are automatically satisfied using eqn. (10), i.e., expressing the fermionic superfields as derivatives of chiral superfields,

\[
F^{(0,1)} = D_+ \Phi_1^{+(0,1)} = \bar{D}_- \Phi_2^{-(0,1)}, \quad G^{(0,1)} = D_- \Phi_1^{-(0,1)} = \bar{D}_+ \Phi_2^{+(0,1)},
\]

where the superscript indices indicate whether the superfield \( \Phi_1^\pm \) and \( \Phi_2^\pm \) depend upon \( \phi_0^\pm \) and \( \phi_1^\pm \). Acting with super derivatives \( D_-, D_-, D_+ \) and \( D_+ \) on eqns. (14), (15), (16) and (17) respectively, we find

\[
\partial_\xi (\phi_0^+ - \phi_1^+) = -2\gamma_1 s_{0,1}^{-1} c_{0,1} + \frac{8}{\kappa_1} F^{(0,1)} D_- c_{0,1}, \tag{22}
\]

\[
\partial_\xi (\phi_0^+ + \phi_1^+) = 2g_2^2 \gamma_1 s_{0,1}^{+1} \bar{c}_{0,1} - \kappa_1 G^{(0,1)} D_- \bar{c}_{0,1}, \tag{23}
\]

\[
\partial_\xi (\phi_0^- - \phi_1^-) = -2\gamma_1 s_{0,1}^{-1} c_{0,1} + \lambda_1 G^{(0,1)} D_+ c_{0,1}, \tag{24}
\]

\[
\partial_\xi (\phi_0^- + \phi_1^-) = 2g_2^2 \gamma_1 s_{0,1}^{+1} \bar{c}_{0,1} + \frac{8}{\lambda_1} F^{(0,1)} \bar{D}_+ c_{0,1}, \tag{25}
\]

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where \( \gamma_1 = g \frac{\lambda_1}{\kappa_1} \) and

\[
\begin{align*}
    c_{j,k}^\pm &= \cos \left( \frac{\phi_j^\pm + \phi_k^\pm}{2} \right), \quad s_{j,k}^\pm = \sin \left( \frac{\phi_j^\pm + \phi_k^\pm}{2} \right), \\
    \bar{c}_{j,k}^\pm &= \cos \left( \frac{\phi_j^\pm - \phi_k^\pm}{2} \right), \quad \bar{s}_{j,k}^\pm = \sin \left( \frac{\phi_j^\pm - \phi_k^\pm}{2} \right),
\end{align*}
\] (26)

(27)

We now assume that the order of two successive Backlund transformations is irrelevant leading to the same final result. Such condition is known as the permutable theorem, i.e., \( \phi_0^\pm \xrightarrow{\gamma_1} \phi_1^\pm \xrightarrow{\gamma_2} \phi_{12}^\pm \) and in the inverse order, \( \phi_0^\pm \xrightarrow{\gamma_1} \phi_2^\pm \xrightarrow{\gamma_2} \phi_{21}^\pm \), does not change the final result, \( \phi_{12}^\pm = \phi_{21}^\pm \equiv \phi_3^\pm \).

The permutability theorem applied to the Backlund equation (14) leads to

\[
\begin{align*}
    D_+ (\phi_0^+ - \phi_1^+) &= -\frac{8}{\kappa_1} \mathcal{F}^{(0,1)} c_{0,1}^-, \\
    D_+ (\phi_1^+ - \phi_3^+) &= -\frac{8}{\kappa_2} \mathcal{F}^{(1,3)} c_{1,3}^-, \\
    D_+ (\phi_0^+ - \phi_2^+) &= -\frac{8}{\kappa_2} \mathcal{F}^{(0,2)} c_{0,2}^-, \\
    D_+ (\phi_2^+ - \phi_3^+) &= -\frac{8}{\kappa_1} \mathcal{F}^{(2,3)} c_{2,3}^-.
\end{align*}
\] (28)

Taking into account that the sum of the first two and the last two equations are the same, we obtain,

\[
\frac{1}{\kappa_1} \mathcal{F}^{(0,1)} c_{0,1}^- + \frac{1}{\kappa_2} \mathcal{F}^{(1,3)} c_{1,3}^- = \frac{1}{\kappa_2} \mathcal{F}^{(0,2)} c_{0,2}^- + \frac{1}{\kappa_1} \mathcal{F}^{(2,3)} c_{2,3}^-.
\] (29)

Similarly, from (17), we obtain

\[
\begin{align*}
    \bar{D}_- (\phi_0^- + \phi_1^-) &= -\frac{8}{\lambda_1} \mathcal{F}^{(0,1)} c_{0,1}^+, \\
    \bar{D}_- (\phi_1^- + \phi_3^-) &= -\frac{8}{\lambda_2} \mathcal{F}^{(1,3)} c_{1,3}^+, \\
    \bar{D}_- (\phi_0^- + \phi_2^-) &= -\frac{8}{\lambda_2} \mathcal{F}^{(0,2)} c_{0,2}^+, \\
    \bar{D}_- (\phi_2^- + \phi_3^-) &= -\frac{8}{\lambda_1} \mathcal{F}^{(2,3)} c_{2,3}^+.
\end{align*}
\] (30)

leading to

\[
\frac{1}{\lambda_1} \mathcal{F}^{(0,1)} c_{0,1}^+ - \frac{1}{\lambda_2} \mathcal{F}^{(1,3)} c_{1,3}^+ = \frac{1}{\lambda_2} \mathcal{F}^{(0,2)} c_{0,2}^+ - \frac{1}{\lambda_1} \mathcal{F}^{(2,3)} c_{2,3}^+.
\] (31)

We propose as solution for the non-linear superposition formula \( \phi_{12}^\pm = \phi_{21}^\pm = \phi_3^\pm \),

\[
\phi_3^\pm = \phi_0^\pm + \Gamma_\pm + \Delta_\pm,
\] (32)
where we have assumed the coefficients $\Lambda^{\pm}$ superfields and has the following form

$$\phi = \phi^+_1 - \phi^-_2,$$

$$x = \phi^+_1 - \phi^-_2,$$

$$y = \phi^-_1 - \phi^+_2,$$

$$\delta = \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2}, \quad \gamma_k = \frac{\lambda_k}{\kappa_k}.$$ 

Notice that the solution $\phi^\pm_3$ when the fermionic superfields are neglected is derived in the appendix B to be $\phi^\pm_3 = \phi^\pm_0 + \Gamma^\pm$. The term $\Delta^\pm$ comes from the contribution of the fermionic superfields and has the following form

$$\Delta^\pm = \sum_{j,k=1}^{2} \Lambda^\pm_{j,k} f_{j,k} + \Lambda^\pm_0 f_0,$$

$$f_{j,k} = \mathcal{F}^{(0,j)} \mathcal{G}^{(0,k)}, \quad f_0 = \mathcal{F}^{(0,1)} \mathcal{F}^{(0,2)} \mathcal{G}^{(0,1)} \mathcal{G}^{(0,2)},$$

where we have assumed the coefficients $\Lambda^\pm$ to be functionals of $x = (\phi^+_1 - \phi^+_2)$ and $y = (\phi^-_1 - \phi^-_2)$, i.e.,

$$\Lambda^\pm_{j,k} = \Lambda^\pm_{j,k}(x,y), \quad \Lambda^\pm_0 = \Lambda^\pm_0(x,y).$$

Observe that there are no terms like $\Lambda^\pm \mathcal{F}^{(0,1)} \mathcal{F}^{(0,2)}$ nor $\Lambda^\pm_0 \mathcal{G}^{(0,1)} \mathcal{G}^{(0,2)}$ due to chiral equations [2]. $\Lambda^\pm$ are determined in appendix C where,

$$\Lambda^{+}_{1,1} = \Lambda^{+}_{2,2} = -\frac{8\mu_-}{g\eta_+\eta_-} \cos\left(\frac{x}{2}\right) \sin\left(\frac{y}{2}\right),$$

$$\Lambda^{+}_{1,2} = \frac{8\mu_-}{g\eta_+\eta_-} \left(\frac{\lambda_2}{\lambda_1}\right) \sin\left(\frac{y}{2}\right),$$

$$\Lambda^{+}_{2,1} = \frac{8\mu_-}{g\eta_+\eta_-} \left(\frac{\lambda_1}{\lambda_2}\right) \sin\left(\frac{y}{2}\right),$$

$$\Lambda^{+}_0 = -\frac{32\mu_-}{(g\eta_+\eta_-)^2} \sin\left(\frac{x}{2}\right) \left[ \cos\left(\frac{y}{2}\right) \left( a + \cos x - \cos y \right) - 2\mu_+ \cos\left(\frac{x}{2}\right) \right],$$

$$\Lambda^{-}_{1,1} = \Lambda^{-}_{2,2} = \frac{8\mu_-}{g\eta_+\eta_-} \cos\left(\frac{y}{2}\right) \sin\left(\frac{x}{2}\right),$$

$$\Lambda^{-}_{1,2} = -\frac{8\mu_-}{g\eta_+\eta_-} \left(\frac{\kappa_2}{\kappa_1}\right) \sin\left(\frac{x}{2}\right),$$

$$\Lambda^{-}_{2,1} = -\frac{8\mu_-}{g\eta_+\eta_-} \left(\frac{\kappa_1}{\kappa_2}\right) \sin\left(\frac{x}{2}\right),$$

$$\Lambda^{-}_0 = -\frac{32\mu_-}{(g\eta_+\eta_-)^2} \sin\left(\frac{y}{2}\right) \left[ \cos\left(\frac{x}{2}\right) \left( a - \cos x + \cos y \right) - 2\mu_+ \cos\left(\frac{y}{2}\right) \right],$$

$$\mu^\pm = \frac{\gamma_1 \pm \gamma_2}{\gamma_1},$$

$$a = \frac{1}{2} \left( \frac{\gamma_1^2}{\gamma_2^2} + \frac{\gamma_2^2}{\gamma_1^2} \right) + 3,$$

$$\eta^\pm = \mu_+ - 2 \cos\left(\frac{x \pm y}{2}\right).$$
3.1 Solution in Components

In components the non-linear superposition formula (32) yields the following expressions,

\[ \varphi_3^+ = \varphi_0^+ + \tilde{\Gamma}_+ - \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-} \left( A_1^+ - B_1^+ + \frac{4}{g\tilde{\eta}_+\tilde{\eta}_-} C_1^+ \right), \]

\[ \psi_3^- = \psi_0^- + F_{1,2}\psi_{1,2}^- + \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-} \left( A_2^- - B_2^- + \frac{4}{g\tilde{\eta}_+\tilde{\eta}_-} C_2^- \right), \]

\[ \bar{\psi}_3^- = \bar{\psi}_0^- + F_{1,2}\bar{\psi}_{1,2}^- + \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-} \left( A_3^- - B_3^- + \frac{4}{g\tilde{\eta}_+\tilde{\eta}_-} C_3^- \right), \]

\[ \varphi_3^- = \varphi_0^- + \tilde{\Gamma}_- + \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-} \left( A_1^- - B_1^- - \frac{4}{g\tilde{\eta}_+\tilde{\eta}_-} C_1^- \right), \]

\[ \psi_3^+ = \psi_0^+ + F_{1,2}\psi_{1,2}^+ - \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-} \left( A_2^+ - B_2^+ - \frac{4}{g\tilde{\eta}_+\tilde{\eta}_-} C_2^+ \right), \]

\[ \bar{\psi}_3^+ = \bar{\psi}_0^+ + F_{1,2}\bar{\psi}_{1,2}^+ - \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-} \left( A_3^+ - B_3^+ - \frac{4}{g\tilde{\eta}_+\tilde{\eta}_-} C_3^+ \right), \]

where

\[ \tilde{\Gamma}_\pm = 2 \arctan \left[ \delta \tan \left( \frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4} \right) \right] \pm 2 \arctan \left[ \delta \tan \left( \frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4} \right) \right], \]

\[ \tilde{\eta}_\pm = \mu_+ - 2 \cos \left( \frac{\varphi_{1,2}^+ \pm \varphi_{1,2}^-}{2} \right), \]

\[ F_{1,2} = \frac{\delta}{2} \left[ \frac{\sec^2 \left( \frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4} \right)}{1 + \delta^2 \tan^2 \left( \frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4} \right)} + \frac{\sec^2 \left( \frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4} \right)}{1 + \delta^2 \tan^2 \left( \frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4} \right)} \right], \]
\[ \begin{align*}
A_1^+ &= \cos \left( \frac{\varphi_{1,2}^+}{2} \right) \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \xi_1^{+(0,1)} \xi_2^{+(0,2)} + \xi_1^{+(0,2)} \xi_2^{+(0,1)} \right), \\
A_2^+ &= -\cos \left( \frac{\varphi_{1,2}^+}{2} \right) \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \xi_1^{+(0,1)} p_2^{+(0,1)} + \xi_1^{+(0,2)} p_2^{+(0,2)} \right) \\
&\quad - \Sigma^+ \left( \frac{\xi_1^{+(0,1)} \xi_2^{+(0,1)}}{\xi_1^{+(0,2)} \xi_2^{+(0,2)}} \right) \psi_1, \\
A_3^+ &= -\cos \left( \frac{\varphi_{1,2}^+}{2} \right) \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( q_2^{+(0,1)} \xi_2^{+(0,1)} + q_2^{+(0,2)} \xi_2^{+(0,2)} \right) \\
&\quad - \Sigma^+ \left( \frac{\xi_1^{+(0,1)} \xi_2^{+(0,1)}}{\xi_1^{+(0,2)} \xi_2^{+(0,2)}} \right) \bar{\psi}_1, \\
B_1^+ &= \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\lambda_2}{\lambda_1} \xi_1^{+(0,1)} \xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2} \xi_1^{+(0,2)} \xi_2^{+(0,1)} \right), \\
B_2^+ &= -\sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\lambda_2}{\lambda_1} \xi_1^{+(0,1)} p_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2} \xi_1^{+(0,2)} p_2^{+(0,1)} \right) \\
&\quad - \Omega^+ \left( \frac{\lambda_2}{\lambda_1} \xi_1^{+(0,1)} \xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2} \xi_1^{+(0,2)} \xi_2^{+(0,1)} \right) \psi_1, \\
B_3^+ &= -\sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\lambda_2}{\lambda_1} q_2^{+(0,1)} \xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2} q_2^{+(0,2)} \xi_2^{+(0,1)} \right) \\
&\quad - \Omega^+ \left( \frac{\lambda_2}{\lambda_1} \xi_1^{+(0,1)} \xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2} \xi_1^{+(0,2)} \xi_2^{+(0,1)} \right) \bar{\psi}_1, \\
C_1^+ &= \sin \left( \frac{\varphi_{1,2}^+}{2} \right) A^+ \xi_1^{+(0,1)} \xi_2^{+(0,1)} \xi_2^{+(0,2)}, \\
C_2^+ &= \sin \left( \frac{\varphi_{1,2}^+}{2} \right) A^+ \xi_1^{+(0,1)} \xi_2^{+(0,2)} \left( \xi_2^{+(0,1)} \xi_2^{+(0,2)} + \xi_2^{+(0,1)} \xi_2^{+(0,2)} \right), \\
C_3^+ &= \sin \left( \frac{\varphi_{1,2}^+}{2} \right) A^+ \xi_2^{+(0,1)} \xi_2^{+(0,2)} \left( \xi_1^{+(0,1)} \xi_2^{+(0,2)} + \xi_1^{+(0,2)} \xi_2^{+(0,1)} \right),
\end{align*} \]
\[ A_1^- = \cos \left( \frac{\varphi_{1,2}^-}{2} \right) \sin \left( \frac{\varphi_{1,2}^+}{2} \right) \left( \zeta_1^{+(0,1)} \zeta_2^{-+(0,1)} + \zeta_1^{+(0,2)} \zeta_2^{+(0,2)} \right), \]

\[ A_2^- = -\cos \left( \frac{\varphi_{1,2}^-}{2} \right) \sin \left( \frac{\varphi_{1,2}^+}{2} \right) \left( \partial_z q_1^{+(0,1)} \zeta_2^{+(0,1)} + \partial_z q_1^{+(0,2)} \zeta_2^{+(0,2)} \right) - \Sigma^- \left( \zeta_1^{+(0,1)} \zeta_2^{+(0,1)} + \zeta_1^{+(0,2)} \zeta_2^{+(0,2)} \right) \psi_{1,2}^+, \]

\[ A_3^- = \cos \left( \frac{\varphi_{1,2}^-}{2} \right) \sin \left( \frac{\varphi_{1,2}^+}{2} \right) \left( \zeta_1^{+(0,1)} \partial_z p_1^{+(0,1)} + \zeta_1^{+(0,2)} \partial_z p_1^{+(0,2)} \right) - \Sigma^- \left( \zeta_1^{+(0,1)} \zeta_2^{+(0,1)} + \zeta_1^{+(0,2)} \zeta_2^{+(0,2)} \right) \bar{\psi}_{1,2}^+, \]

\[ B_1^- = \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\kappa_2}{\kappa_1} \zeta_1^{+(0,1)} \zeta_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2} \zeta_1^{+(0,2)} \zeta_2^{+(0,1)} \right), \]

\[ B_2^- = -\sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\kappa_2}{\kappa_1} \partial_z q_1^{+(0,1)} \zeta_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2} \partial_z q_1^{+(0,2)} \zeta_2^{+(0,1)} \right) - \Omega^- \left( \frac{\kappa_2}{\kappa_1} \zeta_1^{+(0,1)} \zeta_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2} \zeta_1^{+(0,2)} \zeta_2^{+(0,1)} \right) \psi_{1,2}^+, \]

\[ B_3^- = \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\kappa_2}{\kappa_1} \zeta_1^{+(0,1)} \partial_z p_1^{+(0,2)} + \frac{\kappa_1}{\kappa_2} \zeta_1^{+(0,2)} \partial_z p_1^{+(0,1)} \right) - \Omega^- \left( \frac{\kappa_2}{\kappa_1} \zeta_1^{+(0,1)} \zeta_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2} \zeta_1^{+(0,2)} \zeta_2^{+(0,1)} \right) \bar{\psi}_{1,2}^+, \]

\[ C_1^- = \sin \left( \frac{\varphi_{1,2}^-}{2} \right) A^- \zeta_1^{+(0,1)} \zeta_2^{+(0,1)} \zeta_2^{+(0,2)}, \]

\[ C_2^- = -\sin \left( \frac{\varphi_{1,2}^-}{2} \right) A^- \zeta_2^{+(0,1)} \zeta_2^{+(0,2)} \left( \zeta_1^{+(0,2)} \partial_z q_1^{+(0,1)} - \zeta_1^{+(0,1)} \partial_z q_1^{+(0,2)} \right), \]

\[ C_3^- = -\sin \left( \frac{\varphi_{1,2}^-}{2} \right) A^- \zeta_1^{+(0,1)} \zeta_1^{+(0,2)} \left( \zeta_2^{+(0,2)} \partial_z p_1^{+(0,1)} - \zeta_2^{+(0,1)} \partial_z p_1^{+(0,2)} \right), \]
\[ \Sigma^+ = \cos \left( \frac{\varphi_{1,2}^+}{2} \right) \Omega^+ - \frac{1}{2} \sin \left( \frac{\varphi_{1,2}^+}{2} \right) \sin \left( \frac{\varphi_{1,2}^-}{2} \right), \]
\[ \Omega^+ = -\sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left[ \frac{1}{\eta^+} \sin \left( \frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{2} \right) + \frac{1}{\eta^-} \sin \left( \frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{2} \right) \right], \]
\[ A^+ = \cos \left( \frac{\varphi_{1,2}^-}{2} \right) \left( a + \cos \varphi_{1,2}^+ \cos \varphi_{1,2}^- \right) - 2 \mu_+ \cos \left( \frac{\varphi_{1,2}^-}{2} \right), \]
\[ \Sigma^- = \cos \left( \frac{\varphi_{1,2}^-}{2} \right) \Omega^- - \frac{1}{2} \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \sin \left( \frac{\varphi_{1,2}^+}{2} \right), \]
\[ \Omega^- = -\sin \left( \frac{\varphi_{1,2}^+}{2} \right) \left[ \frac{1}{\eta^+} \sin \left( \frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{2} \right) - \frac{1}{\eta^-} \sin \left( \frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{2} \right) \right], \]
\[ A^- = \cos \left( \frac{\varphi_{1,2}^+}{2} \right) \left( a - \cos \varphi_{1,2}^+ \cos \varphi_{1,2}^- \right) - 2 \mu_+ \cos \left( \frac{\varphi_{1,2}^+}{2} \right), \]

and denoted
\[ \varphi_{1,2}^\pm = \varphi_1^\pm - \varphi_2^\pm, \quad \psi_{1,2}^\pm = \psi_1^\pm - \psi_2^\pm, \quad \bar{\psi}_{1,2}^\pm = \bar{\psi}_1^\pm - \bar{\psi}_2^\pm. \]

From the Backlund eqns. we get (see app. A)
\[ \xi_1^{+(0,k)} = -\frac{\kappa}{8} \cos \left( \frac{\varphi_0^- + \varphi_k^-}{2} \right), \quad \xi_2^{+(0,k)} = \frac{1}{\kappa} \cos \left( \frac{\varphi_0^- + \varphi_k^-}{2} \right), \]
\[ \partial_2 q_1^{+(0,k)} = \frac{\lambda g}{4} \sin \left( \frac{\varphi_0^+ + \varphi_k^+}{2} \right), \quad \partial_2 p_2^{+(0,k)} = \frac{2g}{\kappa} \sin \left( \frac{\varphi_0^- + \varphi_k^-}{2} \right), \]
\[ q_2^{+(0,k)} = -\frac{\kappa g}{4} \sin \left( \frac{\varphi_0^- - \varphi_k^-}{2} \right), \quad \partial_2 p_1^{+(0,k)} = \frac{2g}{\lambda} \sin \left( \frac{\varphi_0^+ - \varphi_k^+}{2} \right). \]

### 4 1-Soliton Solution

Setting \( \phi_0^+ = 0 \) in the Backlund eqns. (14)-(21) we find in components,
\[ \partial_2 \xi_1^{+(0,1)} = -\frac{g^2}{\gamma_1} \cos \left( \frac{\varphi_1^+}{2} \right) \cos \left( \frac{\varphi_1^-}{2} \right) \xi_1^{+(0,1)}, \]
\[ \partial_2 \xi_1^{+(0,1)} = \gamma_1 \cos \left( \frac{\varphi_1^+}{2} \right) \cos \left( \frac{\varphi_1^-}{2} \right) \xi_1^{+(0,1)}, \]
\[ \partial_2 \xi_2^{+(0,1)} = -\frac{g^2}{\gamma_1} \cos \left( \frac{\varphi_2^+}{2} \right) \cos \left( \frac{\varphi_2^-}{2} \right) \xi_2^{+(0,1)}, \]
\[ \partial_2 \xi_2^{+(0,1)} = \gamma_1 \cos \left( \frac{\varphi_2^+}{2} \right) \cos \left( \frac{\varphi_2^-}{2} \right) \xi_2^{+(0,1)}, \]
\[ \partial_z \varphi^\pm = -\frac{2g^2}{\gamma_1} \sin \left( \frac{\varphi^\pm}{2} \right) \cos \left( \frac{\varphi^\pm}{2} \right), \]
\[ \partial_z \varphi^\pm = 2\gamma_1 \sin \left( \frac{\varphi^\pm}{2} \right) \cos \left( \frac{\varphi^\pm}{2} \right). \]

Integrating the above eqns. we get the 1-soliton solution,
\[ \psi^-_1 = \frac{8}{\kappa_1} \cos \left( \frac{\varphi_1^-}{2} \right) \xi_1^{(0,1)}, \quad \psi^+_1 = -\lambda_1 \cos \left( \frac{\varphi_1^+}{2} \right) \xi_2^{(0,1)}, \]
\[ \bar{\psi}^-_1 = \kappa_1 \cos \left( \frac{\varphi_1^-}{2} \right) \xi_2^{(0,1)}, \quad \bar{\psi}^+_1 = -\frac{8}{\lambda_1} \cos \left( \frac{\varphi_1^+}{2} \right) \xi_1^{(0,1)}, \]
\[ \varphi_1^\pm = 2 \arctan(a_1 \rho_1) \pm 2 \arctan(b_1 \rho_1), \]
\[ \zeta_1^{(0,1)} = \zeta_2^{(0,1)} = \epsilon_1 \chi_1, \]
\[ \chi_1 = \frac{\rho_1}{\sqrt{(1 + a_1^2 \rho_1^2)(1 + b_1^2 \rho_1^2)}}, \]

where \( a_1 \) and \( b_1 \) are arbitrary constants, \( \epsilon_1 \) is a grassmann parameter and
\[ \rho_1 = \exp \left( \frac{\gamma_1 z - g^2 / \gamma_1}{\gamma_1} \right). \]

The 1-soliton solution constructed in this section can be obtained from those of [7] by relating parameters since they both involve a single grassmann parameter.

## 5 2-Soliton Solution

For the 2-soliton case we obtain from the superposition formulae \( [32] \)
\[ \varphi_3^+ = \varphi_3^{(0)} + \varphi_3^{(1)} \epsilon_1 \epsilon_2, \]
\[ \varphi_3^{(0)} = 2 \arctan \left[ \delta \tan \left( \frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4} \right) \right] + 2 \arctan \left[ \delta \tan \left( \frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4} \right) \right], \]
\[ \varphi_3^{(1)} = \frac{8\mu_-}{g \bar{\eta} + \bar{\eta}^-} \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \left( \frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{\lambda_2} \right) \chi_1 \chi_2, \]
\[ \psi^-_3 = \epsilon_1 \psi_3^{(1)} + \epsilon_2 \psi_3^{(2)}, \]
\[ \psi_3^{(1)} = \frac{8}{\kappa_1} F_{1,2} \cos \left( \frac{\varphi_1^-}{2} \right) \chi_1 \]
\[ + \frac{16}{\kappa_1 \gamma_1 \bar{\eta} + \bar{\eta}^-} \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \chi_1 \left[ \gamma_2 \sin \left( \frac{\varphi_2^-}{2} \right) - \gamma_1 \cos \left( \frac{\varphi_{1,2}^-}{2} \right) \sin \left( \frac{\varphi_1^-}{2} \right) \right], \]
\[ \psi_3^{(2)} = -\frac{8}{\kappa_2} F_{1,2} \cos \left( \frac{\varphi_2^-}{2} \right) \chi_2 \]
\[ + \frac{16}{\kappa_2 \gamma_2 \bar{\eta} + \bar{\eta}^-} \sin \left( \frac{\varphi_{1,2}^-}{2} \right) \chi_2 \left[ \gamma_1 \sin \left( \frac{\varphi_1^-}{2} \right) - \gamma_2 \cos \left( \frac{\varphi_{1,2}^-}{2} \right) \sin \left( \frac{\varphi_2^-}{2} \right) \right]. \]
\[ \tilde{\psi}_3^- = \epsilon_1 \tilde{\psi}_3^{-(1)} + \epsilon_2 \tilde{\psi}_3^{-(2)}, \]
\[ \tilde{\psi}_3^{-(1)} = \kappa_1 F_{1,2} \cos \left( \frac{\varphi_1}{2} \right) \chi_1 \]
\[ + \frac{2\kappa_1}{\gamma_2} \frac{\mu_-}{\tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \chi_1 \left[ \gamma_1 \sin \left( \frac{\varphi_2}{2} \right) - \gamma_2 \cos \left( \frac{\varphi_{1,2}}{2} \right) \sin \left( \frac{\varphi_1}{2} \right) \right], \]
\[ \tilde{\psi}_3^{-(2)} = -\kappa_2 F_{1,2} \cos \left( \frac{\varphi_2}{2} \right) \chi_2 \]
\[ + \frac{2\kappa_2}{\gamma_1} \frac{\mu_-}{\tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \chi_2 \left[ \gamma_2 \sin \left( \frac{\varphi_1}{2} \right) - \gamma_1 \cos \left( \frac{\varphi_{1,2}}{2} \right) \sin \left( \frac{\varphi_2}{2} \right) \right], \]
\[ \varphi_3^- = \varphi_3^{-(0)} + \varphi_3^{-(1)} \epsilon_1 \epsilon_2, \]
\[ \varphi_3^{-(0)} = 2 \arctan \left[ \tan \left( \frac{\varphi_{1,2} + \varphi_{1,2}}{4} \right) \right] - 2 \arctan \left[ \tan \left( \frac{\varphi_{1,2} - \varphi_{1,2}}{4} \right) \right], \]
\[ \varphi_3^{-(1)} = -\frac{8\mu_-}{g \tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \left( \frac{\kappa_2}{\kappa_1} - \frac{\kappa_1}{\kappa_2} \right) \chi_1 \chi_2, \]
\[ \psi_3^+ = \epsilon_1 \psi_3^{+(1)} + \epsilon_2 \psi_3^{+(2)}, \]
\[ \psi_3^{+(1)} = -\lambda_1 F_{1,2} \cos \left( \frac{\varphi_1}{2} \right) \chi_1 \]
\[ - \frac{2\lambda_1}{\gamma_1} \frac{\mu_-}{\tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \chi_1 \left[ \gamma_2 \sin \left( \frac{\varphi_2}{2} \right) - \gamma_1 \cos \left( \frac{\varphi_{1,2}}{2} \right) \sin \left( \frac{\varphi_1}{2} \right) \right], \]
\[ \psi_3^{+(2)} = \lambda_2 F_{1,2} \cos \left( \frac{\varphi_2}{2} \right) \chi_2 \]
\[ - \frac{2\lambda_2}{\gamma_2} \frac{\mu_-}{\tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \chi_2 \left[ \gamma_1 \sin \left( \frac{\varphi_1}{2} \right) - \gamma_2 \cos \left( \frac{\varphi_{1,2}}{2} \right) \sin \left( \frac{\varphi_2}{2} \right) \right], \]
\[ \tilde{\psi}_3^+ = \epsilon_1 \tilde{\psi}_3^{+(1)} + \epsilon_2 \tilde{\psi}_3^{+(2)}, \]
\[ \tilde{\psi}_3^{+(1)} = -\frac{8}{\lambda_1} F_{1,2} \cos \left( \frac{\varphi_1}{2} \right) \chi_1 \]
\[ - \frac{16}{\lambda_1 \gamma_2} \frac{\mu_-}{\tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \chi_1 \left[ \gamma_1 \sin \left( \frac{\varphi_2}{2} \right) - \gamma_2 \cos \left( \frac{\varphi_{1,2}}{2} \right) \sin \left( \frac{\varphi_1}{2} \right) \right], \]
\[ \tilde{\psi}_3^{+(2)} = \frac{8}{\lambda_2} F_{1,2} \cos \left( \frac{\varphi_2}{2} \right) \chi_2 \]
\[ - \frac{16}{\lambda_2 \gamma_1} \frac{\mu_-}{\tilde{\eta}_+ \tilde{\eta}_-} \sin \left( \frac{\varphi_{1,2}}{2} \right) \chi_2 \left[ \gamma_2 \sin \left( \frac{\varphi_1}{2} \right) - \gamma_1 \cos \left( \frac{\varphi_{1,2}}{2} \right) \sin \left( \frac{\varphi_2}{2} \right) \right]. \]
where
\[ \varphi_k^\pm = 2 \arctan(a_k \rho_k) \pm 2 \arctan(b_k \rho_k), \]
\[ \chi_k = \frac{\rho_k}{\sqrt{1 + a_k^2 \rho_k^2}(1 + b_k^2 \rho_k^2)}. \]

\( k = 1, 2, a_k \) and \( b_k \) are arbitrary constants, \( \epsilon_k \) is a grassmann constant and
\[ \rho_k = \exp \left( \gamma_k z - \frac{g^2}{\gamma_k} \bar{z} \right). \]

Notice that the 2-soliton solution constructed in this section generalizes those constructed in ref. [7] involving a single grassmann parameter.

Both 1- and 2-soliton solutions presented above were verified to satisfy the equations of motion.

Acknowledgments

LHY acknowledges support from Fapesp, JFG and AHZ thank CNPq for partial support.

Appendix A

In order to simplify notation let us introduce \( \varphi_\pm = \varphi_1 \pm \varphi_2, \varphi_\pm = \varphi_1^\pm \pm \varphi_2^\pm \) and similar notation for the other fields.

In components eqn. (5) becomes

- **D** \( F = -\frac{\kappa g}{4} \sin \left( \frac{\varphi_- - \varphi_2}{2} \right) \)

\[ q_2^+ = -\frac{\kappa g}{4} \sin \left( \frac{\varphi_-^2}{2} \right), \quad \partial_z \zeta_1^+ = -\frac{\kappa g}{8} \cos \left( \frac{\varphi_-^2}{2} \right) \psi_-^+, \quad \partial_z \zeta_2^+ = \frac{\kappa g}{8} \cos \left( \frac{\varphi_-^2}{2} \right) \psi_+^-, \]
\[ \partial_z \partial_z q_1^+ = \frac{\kappa g}{8} \cos \left( \frac{\varphi_-^2}{2} \right) F_-^+, + \frac{\kappa g}{16} \sin \left( \frac{\varphi_-^2}{2} \right) \psi_-^+ \psi_-^-. \]

- **D** \( G = -\frac{2\kappa g}{4} \sin \left( \frac{\varphi_1^+ + \varphi_2^+}{2} \right) \)

\[ p_2^+ = \frac{2\kappa g}{4} \sin \left( \frac{\varphi_-^2}{2} \right), \quad \partial_z \xi_1^+ = \frac{2\kappa g}{4} \cos \left( \frac{\varphi_-^2}{2} \right) \psi_+^-, \quad \partial_z \xi_2^+ = -\frac{2\kappa g}{4} \cos \left( \frac{\varphi_-^2}{2} \right) \psi_+^+, \]
\[ \partial_z \partial_z p_1^+ = -\frac{2\kappa g}{4} \cos \left( \frac{\varphi_-^2}{2} \right) F_+^-, -\frac{2\kappa g}{4} \sin \left( \frac{\varphi_-^2}{2} \right) \psi_+^+ \psi_+^-. \]
Similarly we find for (8),

- \( \bar{D}_- \mathcal{G} = \frac{2\eta}{\lambda} \sin \left( \frac{\phi_1^+ - \phi_2^+}{2} \right) \),

\[ q_2^- = \frac{2\eta}{\lambda} \sin \left( \frac{\varphi_2^+}{2} \right), \quad \partial_2 \zeta_1^- = \frac{\eta}{\lambda} \cos \left( \frac{\varphi_2^+}{2} \right) \psi_2^-, \quad \partial_2 \zeta_2^- = -\frac{\eta}{\lambda} \cos \left( \frac{\varphi_2^+}{2} \right) \psi_2^-, \]

\[ \partial_2 \partial_2 q_1^- = -\frac{\eta}{\lambda} \cos \left( \frac{\varphi_2^+}{2} \right) F_2^- - \frac{\eta}{\lambda} \sin \left( \frac{\varphi_2^+}{2} \right) \psi_2^- \psi_2^- . \]

- \( D_- \mathcal{F} = \frac{\lambda \eta}{4} \sin \left( \frac{\phi_1^+ + \phi_2^+}{2} \right) \),

\[ p_2^- = -\frac{\lambda \eta}{4} \sin \left( \frac{\varphi_2^+}{2} \right), \quad \partial_2 \xi_1^- = -\frac{\lambda \eta}{8} \cos \left( \frac{\varphi_2^+}{2} \right) \bar{\psi}_2^-, \quad \partial_2 \xi_2^- = \frac{\lambda \eta}{8} \cos \left( \frac{\varphi_2^+}{2} \right) \bar{\psi}_2^-, \]

\[ \partial_2 \partial_2 p_1^- = \frac{\lambda \eta}{8} \cos \left( \frac{\varphi_2^+}{2} \right) F_2^+ + \frac{\lambda \eta}{16} \sin \left( \frac{\varphi_2^+}{2} \right) \psi_2^- \psi_2^- . \]

From \( \mathbf{3} \) and \( \mathbf{4} \),

- \( D_+ \phi_1^+ = D_+ \phi_2^+ - \frac{8}{\kappa} \mathcal{F} \cos \left( \frac{\phi_1^+ - \phi_2^+}{2} \right) \),

\[ \psi_2^- = -\frac{8}{\kappa} \xi_1^+ \cos \left( \frac{\varphi_2^+}{2} \right), \quad F_2^+ = -\frac{8}{\kappa} \xi_2^+ \cos \left( \frac{\varphi_2^+}{2} \right), \]

\[ \partial_2 \varphi_2^+ = -\frac{4}{\kappa} \sin \left( \frac{\varphi_2^+}{2} \right) \xi_2^+ \psi_2^+ - \frac{8}{\kappa} \partial_2 q_1^+ \cos \left( \frac{\varphi_2^+}{2} \right) \].

- \( \bar{D}_+ \phi_1^- = -\bar{D}_+ \phi_2^- + \kappa \mathcal{G} \cos \left( \frac{\phi_1^- - \phi_2^-}{2} \right) \),

\[ \bar{\psi}_2^- = \kappa \xi_1^+ \cos \left( \frac{\varphi_2^-}{2} \right), \quad F_2^+ = \kappa \partial_2^+ \cos \left( \frac{\varphi_2^-}{2} \right), \]

\[ \partial_2 \varphi_2^+ = \frac{\kappa}{2} \sin \left( \frac{\varphi_2^-}{2} \right) \xi_2^+ \psi_2^- + \kappa \partial_2 p_1^+ \cos \left( \frac{\varphi_2^-}{2} \right) . \]

From \( \mathbf{6} \) and \( \mathbf{7} \),

- \( D_- \phi_1^- = D_- \phi_2^- + \lambda \mathcal{G} \cos \left( \frac{\phi_1^- + \phi_2^+}{2} \right) \),

\[ \psi_2^+ = \lambda \xi_1^- \cos \left( \frac{\varphi_2^-}{2} \right), \quad F_2^- = \lambda \xi_2^- \cos \left( \frac{\varphi_2^-}{2} \right), \]

\[ \partial_2 \varphi_2^- = \frac{\lambda}{2} \sin \left( \frac{\varphi_2^-}{2} \right) \xi_1^- \psi_2^- + \lambda \partial_2 q_1^- \cos \left( \frac{\varphi_2^-}{2} \right) . \]

- \( \bar{D}_- \phi_1^- = -\bar{D}_- \phi_2^- - \frac{8}{\lambda} \mathcal{F} \cos \left( \frac{\phi_1^- + \phi_2^+}{2} \right) \),

\[ \bar{\psi}_2^+ = -\frac{8}{\lambda} \xi_2^- \cos \left( \frac{\varphi_2^-}{2} \right), \quad F_2^- = -\frac{8}{\lambda} \xi_2^- \cos \left( \frac{\varphi_2^-}{2} \right), \]

\[ \partial_2 \varphi_2^- = -\frac{4}{\lambda} \sin \left( \frac{\varphi_2^-}{2} \right) \xi_2^- \psi_2^- - \frac{8}{\lambda} \partial_2 p_1^- \cos \left( \frac{\varphi_2^-}{2} \right) . \]
Appendix B

Applying the permutability theorem to eqns. (22) and (24) after neglecting the contribution proportional to fermionic superfields, we obtain the following relations

\[
\begin{align*}
\gamma_1 s_{0,1}^+c_{0,1} + \gamma_2 s_{1,3}^+c_{1,3} &= \gamma_2 s_{0,2}^+c_{0,2} + \gamma_1 s_{2,3}^+c_{2,3}, \\
\gamma_1 s_{0,1}^-c_{0,1} + \gamma_2 s_{1,3}^-c_{1,3} &= \gamma_2 s_{0,2}^-c_{0,2} + \gamma_1 s_{2,3}^-c_{2,3}.
\end{align*}
\]

Summing and subtracting the above eqns., we find

\[
\begin{align*}
\gamma_1 \left[ (s_{0,1}^+c_{0,1} \pm s_{0,1}^-c_{0,1}) - (s_{2,3}^-c_{2,3} \pm s_{2,3}^+c_{2,3}) \right] \\
+ \gamma_2 \left[ (s_{1,3}^+c_{1,3} \pm s_{1,3}^-c_{1,3}) - (s_{0,2}^-c_{0,2} \pm s_{0,2}^+c_{0,2}) \right] &= 0. \quad (35)
\end{align*}
\]

Using the identity

\[
\sin a \cos b \pm \sin b \cos a = \sin (a \pm b), \quad (36)
\]

and eqns. (26) and (27) we can rewrite (35) as

\[
\begin{align*}
\gamma_1 \left\{ \sin \left[ \left( \frac{\phi_0^+ + \phi_1^+}{2} \right) \pm \left( \frac{\phi_0^- + \phi_1^-}{2} \right) \right] - \sin \left[ \left( \frac{\phi_2^+ + \phi_3^+}{2} \right) \pm \left( \frac{\phi_2^- + \phi_3^-}{2} \right) \right] \right\} \\
\gamma_2 \left\{ \sin \left[ \left( \frac{\phi_1^+ + \phi_3^+}{2} \right) \pm \left( \frac{\phi_1^- + \phi_3^-}{2} \right) \right] - \sin \left[ \left( \frac{\phi_0^+ + \phi_2^+}{2} \right) \pm \left( \frac{\phi_0^- + \phi_2^-}{2} \right) \right] \right\} &= 0.
\end{align*}
\]

Using the fact that

\[
\sin a - \sin b = 2 \cos \left( \frac{a+b}{2} \right) \sin \left( \frac{a-b}{2} \right),
\]

yields

\[
2 \cos \left( Y^+ \pm Y^- \right) \left\{ \gamma_1 \sin \left[ (X_{1,2}^+ \pm X_{1,2}^-) - (X_{3,0}^+ \pm X_{3,0}^-) \right] \\
+ \gamma_2 \sin \left[ (X_{1,2}^+ \pm X_{1,2}^-) + (X_{3,0}^+ \pm X_{3,0}^-) \right] \right\} = 0,
\]

where we have denoted

\[
Y^\pm = \frac{\phi_0^+ + \phi_1^+ + \phi_2^+ + \phi_3^+}{4},
\]

\[
X_{j,k}^\pm = \frac{\phi_j^+ - \phi_k^+}{4}.
\]

from where it follows that

\[
(\gamma_1 + \gamma_2) \sin \left( X_{1,2}^+ \pm X_{1,2}^- \right) \cos \left( X_{3,0}^+ \pm X_{3,0}^- \right) = \\
(\gamma_1 - \gamma_2) \sin \left( X_{3,0}^+ \pm X_{3,0}^- \right) \cos \left( X_{1,2}^+ \pm X_{1,2}^- \right),
\]

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or,

\[
\tan \left( X_{3,0}^+ \pm X_{3,0}^- \right) = \left( \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \right) \tan \left( X_{1,2}^+ \pm X_{1,2}^- \right),
\]

and therefore

\[
\left( \frac{\phi_3^+ - \phi_0^+}{4} \right) \pm \left( \frac{\phi_3^- - \phi_0^-}{4} \right) = \arctan \left[ \delta \tan \left( X_{1,2}^+ \pm X_{1,2}^- \right) \right],
\]

where \( \delta = \left( \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \right) \). Adding and subtracting the above expressions we obtain

\[
\phi_3^+ = \phi_0^+ + \Gamma_\pm,
\]

with

\[
\Gamma_\pm = 2 \arctan \left[ \delta \tan \left( X_{1,2}^+ + X_{1,2}^- \right) \right] \pm 2 \arctan \left[ \delta \tan \left( X_{1,2}^+ - X_{1,2}^- \right) \right].
\]

**Appendix C**

Relations (29) and (31) can be written in matrix form,

\[
\left( \begin{array}{c}
F^{(1,3)} \\
F^{(2,3)}
\end{array} \right) = \frac{1}{Z} \left( \begin{array}{cc}
A & -B \\
C & -D
\end{array} \right) \left( \begin{array}{c}
F^{(0,1)} \\
F^{(0,2)}
\end{array} \right)
\]

(37)

where

\[
A = \kappa_2 \lambda_2 (\bar{c}_{0,1}^+ c_{2,3} + \bar{c}_{2,3}^+ c_{0,1}),
\]

\[
B = \kappa_2 \lambda_1 \bar{c}_{0,2}^+ c_{2,3} + \kappa_1 \lambda_2 \bar{c}_{2,3}^+ c_{0,2},
\]

\[
C = \kappa_2 \lambda_1 \bar{c}_{1,3}^+ c_{0,1} + \kappa_1 \lambda_2 \bar{c}_{0,1}^+ c_{1,3},
\]

\[
D = \kappa_1 \lambda_1 (\bar{c}_{0,2}^+ c_{1,3} + \bar{c}_{1,3}^+ c_{0,2}),
\]

\[
Z = \kappa_2 \lambda_1 \bar{c}_{1,3}^+ c_{2,3} - \kappa_1 \lambda_2 \bar{c}_{2,3}^+ c_{1,3}.
\]

(38)

Introduce eqn. (32) into expressions (35). Consider now the following expansions

\[
\bar{c}_{k,3}^- = c_{k,\Gamma_-} \left( 1 - \frac{\Delta_3^2}{8} \right) - \frac{\Delta_-}{2} s_{k,\Gamma_-},
\]

\[
\bar{c}_{k,3}^+ = \bar{c}_{k,\Gamma_-} \left( 1 - \frac{\Delta_3^2}{8} \right) + \frac{\Delta_+}{2} \bar{s}_{k,\Gamma_-},
\]

where we have denoted

\[
c_{k,\Gamma_-} = \cos \left( \frac{\phi^- \pm \phi_0^- + \Gamma_-}{2} \right) = \bar{c}_{k,0} \sigma_- + s_{k,0} \rho_-;
\]

\[
s_{k,\Gamma_-} = \sin \left( \frac{\phi^- \pm \phi_0^- + \Gamma_-}{2} \right) = s_{k,0} \sigma_+ + \bar{c}_{k,0} \rho_-;
\]

\[
\bar{c}_{k,\Gamma_-} = \cos \left( \frac{\phi^+ \pm \phi_0^+ - \Gamma_+}{2} \right) = \bar{c}_{k,0} \sigma_- + \bar{s}_{k,0} \rho_+;
\]

\[
\bar{s}_{k,\Gamma_-} = \sin \left( \frac{\phi^+ \pm \phi_0^+ - \Gamma_+}{2} \right) = \bar{s}_{k,0} \sigma_- - \bar{c}_{k,0} \rho_+.
\]

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and

\[
\begin{align*}
\sigma_\pm &= \frac{1 \pm \delta^2 \tan \left(\frac{x+u}{4}\right) \tan \left(\frac{x-u}{4}\right)}{\sqrt{1 + \delta^2 \tan^2 \left(\frac{x+u}{4}\right)} \sqrt{1 + \delta^2 \tan^2 \left(\frac{x-u}{4}\right)}}, \\
\rho_\pm &= \frac{\delta \left[\tan \left(\frac{x+u}{4}\right) \pm \tan \left(\frac{x-u}{4}\right)\right]}{\sqrt{1 + \delta^2 \tan^2 \left(\frac{x+u}{4}\right)} \sqrt{1 + \delta^2 \tan^2 \left(\frac{x-u}{4}\right)}}.
\end{align*}
\]

Next, we expand the expressions for \(A, B, C, D\) and \(Z\) in power series of \(f\) obtaining

\[
\begin{align*}
A &= A_0 + \sum_{j,k=1}^{2} A_{j,k} f_{j,k} + \mathcal{O}(f_0), \\
A_0 &= \kappa_2 \lambda_2 (c_{0,1}^+ c_{2,\Gamma^-} + c_{0,1}^- c_{2,\Gamma^+}), \\
A_{j,k} &= \frac{1}{2} \kappa_2 \lambda_2 (c_{0,1}^+ \bar{s}_{2,\Gamma^+} \Lambda_{j,k}^+ - c_{0,1}^- s_{2,\Gamma^-} \Lambda_{j,k}^-), \\
B &= B_0 + \sum_{j,k=1}^{2} B_{j,k} f_{j,k} + \mathcal{O}(f_0), \\
B_0 &= \kappa_2 \lambda_1 c_{0,2}^+ c_{2,\Gamma^-} + \kappa_1 \lambda_2 \bar{c}_{2,\Gamma^+} c_{0,2}, \\
B_{j,k} &= \frac{1}{2} (\kappa_1 \lambda_2 c_{0,2}^- \bar{s}_{2,\Gamma^+} \Lambda_{j,k}^+ - \kappa_2 \lambda_1 c_{0,2}^+ s_{2,\Gamma^-} \Lambda_{j,k}^-), \\
C &= C_0 + \sum_{j,k=1}^{2} C_{j,k} f_{j,k} + \mathcal{O}(f_0), \\
C_0 &= \kappa_1 \lambda_2 \bar{c}_{0,1}^+ c_{1,\Gamma^-} + \kappa_2 \lambda_2 \bar{c}_{0,1}^- c_{1,\Gamma^+}, \\
C_{j,k} &= \frac{1}{2} (\kappa_2 \lambda_1 \bar{c}_{0,1}^+ \bar{s}_{1,\Gamma^+} \Lambda_{j,k}^+ - \kappa_1 \lambda_2 \bar{c}_{0,1}^+ s_{1,\Gamma^-} \Lambda_{j,k}^-), \\
D &= D_0 + \sum_{j,k=1}^{2} D_{j,k} f_{j,k} + \mathcal{O}(f_0), \\
D_0 &= \kappa_1 \lambda_1 (\bar{c}_{0,2}^+ c_{1,\Gamma^-} + \bar{c}_{0,2}^- c_{1,\Gamma^+}), \\
D_{j,k} &= \frac{1}{2} \kappa_1 \lambda_1 (c_{0,2}^+ \bar{s}_{1,\Gamma^+} \Lambda_{j,k}^+ - c_{0,2}^- s_{1,\Gamma^-} \Lambda_{j,k}^-), \\
Z &= Z_0 + \sum_{j,k=1}^{2} Z_{j,k} f_{j,k} + \mathcal{O}(f_0), \\
Z_0 &= \kappa_2 \lambda_1 \bar{c}_{1,\Gamma^+} c_{2,\Gamma^-} - \kappa_1 \lambda_2 c_{1,\Gamma^-} \bar{c}_{2,\Gamma^+}, \\
Z_{j,k} &= \frac{1}{2} (\kappa_2 \lambda_1 c_{2,\Gamma^-} \bar{s}_{1,\Gamma^+} - \kappa_1 \lambda_2 c_{1,\Gamma^-} \bar{s}_{2,\Gamma^+}) \Lambda_{j,k}^+ \\
&\quad - \frac{1}{2} (\kappa_2 \lambda_1 s_{2,\Gamma^-} \bar{c}_{1,\Gamma^+} - \kappa_1 \lambda_2 s_{1,\Gamma^-} \bar{c}_{2,\Gamma^+}) \Lambda_{j,k}^-.
\end{align*}
\]
where $\mathcal{O}(f_0)$ denotes terms proportional to $f_0$. It then follows
\[
\frac{X}{Z} = \frac{X_0}{Z_0} \left[ 1 + \sum_{j,k=1}^{2} \left( \frac{X_{j,k}}{X_0} - \frac{Z_{j,k}}{Z_0} \right) f_{j,k} \right] + \mathcal{O}(f_0),
\]
where $X = \{A, B, C, D\}$.

Substituting (37), we obtain
\[
\mathcal{F}^{(1,3)} = \frac{A_0}{Z_0} \mathcal{F}^{(0,1)} - \frac{B_0}{Z_0} \mathcal{F}^{(0,2)} + \omega_1(1) \mathcal{F}^{(0,1)} f_{2,1} + \omega_2(1) \mathcal{F}^{(0,1)} f_{2,2},
\]
(39)
\[
\mathcal{F}^{(2,3)} = \frac{C_0}{Z_0} \mathcal{F}^{(0,1)} - \frac{D_0}{Z_0} \mathcal{F}^{(0,2)} + \omega_1(2) \mathcal{F}^{(0,1)} f_{2,1} + \omega_2(2) \mathcal{F}^{(0,1)} f_{2,2},
\]
(40)

where
\[
\omega_1(1) = \frac{A_0}{Z_0} \left( \frac{A_{2,1}}{A_0} - \frac{Z_{2,1}}{Z_0} \right) + \frac{B_0}{Z_0} \left( \frac{B_{1,1}}{B_0} - \frac{Z_{1,1}}{Z_0} \right),
\]
\[
\omega_2(1) = \frac{A_0}{Z_0} \left( \frac{A_{2,2}}{A_0} - \frac{Z_{2,2}}{Z_0} \right) + \frac{B_0}{Z_0} \left( \frac{B_{1,2}}{B_0} - \frac{Z_{1,2}}{Z_0} \right),
\]
\[
\omega_1(2) = \frac{C_0}{Z_0} \left( \frac{C_{2,1}}{C_0} - \frac{Z_{2,1}}{Z_0} \right) + \frac{D_0}{Z_0} \left( \frac{D_{1,1}}{D_0} - \frac{Z_{1,1}}{Z_0} \right),
\]
\[
\omega_2(2) = \frac{C_0}{Z_0} \left( \frac{C_{2,2}}{C_0} - \frac{Z_{2,2}}{Z_0} \right) + \frac{D_0}{Z_0} \left( \frac{D_{1,2}}{D_0} - \frac{Z_{1,2}}{Z_0} \right).
\]

From eqns. (28), we get
\[
D_+ (\phi_3^+ - \phi_0^+) = \frac{8}{\kappa_1} \mathcal{F}^{(0,1)} c_{0,1}^- + \frac{8}{\kappa_2} \mathcal{F}^{(1,3)} c_{1,3}^-,
\]
\[
D_+ (\phi_1^+ - \phi_2^+) = \frac{8}{\kappa_1} \mathcal{F}^{(0,1)} c_{0,1}^- - \frac{8}{\kappa_2} \mathcal{F}^{(0,2)} c_{0,2}^-.
\]

Introducing solution (32) in the first eqn. above, we find
\[
D_+ (\phi_3^+ - \phi_0^+) = D_+ (\Gamma_+ + \Delta_+) = \partial_x \Gamma_+ + D_+ (\phi_1^+ - \phi_2^+) + D_+ \Delta_+ = \frac{8}{\kappa_1} \mathcal{F}^{(0,1)} c_{0,1}^- + \frac{8}{\kappa_2} \mathcal{F}^{(1,3)} c_{1,3}^-.
\]

Using eqn. (39) in the above expression, and taking into account that $\mathcal{F}^{(0,1)}$, $\mathcal{F}^{(0,2)}$, $\mathcal{F}^{(0,1)} f_{2,1}$ and $\mathcal{F}^{(0,1)} f_{2,2}$ are independent, we arrive at the following conditions,
\[
\frac{c_{0,1}}{\kappa_1} (\partial_x \Gamma_- + 1) - \frac{c_{1,1}}{\kappa_2} \Lambda_{1,2}^+ = \frac{c_{0,2}}{\kappa_2} \Lambda_{2,1}^+ + \frac{g s_{0,2}}{4 \kappa_2} \Lambda_{1,2}^+ + \frac{g s_{0,1}}{4 \kappa_1} \Lambda_{1,1}^+ = 0,
\]
\[
\frac{c_{0,2}}{\kappa_2} \partial_x \Gamma_+ = - \frac{c_{0,1}}{\kappa_1} \frac{B_0}{Z_0} - \frac{g s_{0,1}}{4 \kappa_1} \Lambda_{2,1}^+ + \frac{g s_{0,2}}{4 \kappa_2} \Lambda_{2,2}^+ = 0,
\]
\[
\frac{c_{0,2}}{\kappa_2} \partial_x \Lambda_{1,1}^+ = \frac{c_{0,1}}{\kappa_1} \frac{\partial_x \Lambda_{2,1}^+}{4 \kappa_1} + \frac{g s_{0,2}}{4 \kappa_2} \Lambda_0^+ - \frac{c_{1,1}}{\kappa_2} \omega_1(1) + \frac{s_{1,1}}{2 \kappa_2} \left( \frac{A_0}{Z_0} \Lambda_{2,1}^- + \frac{B_0}{Z_0} \Lambda_{1,1}^- \right) = 0,
\]
\[
\frac{c_{0,2}}{\kappa_2} \partial_x \Lambda_{1,2}^+ = \frac{c_{0,1}}{\kappa_1} \frac{\partial_x \Lambda_{2,2}^+}{4 \kappa_1} + \frac{g s_{0,1}}{4 \kappa_1} \Lambda_0^+ - \frac{c_{1,1}}{\kappa_2} \omega_2(1) + \frac{s_{1,1}}{2 \kappa_2} \left( \frac{A_0}{Z_0} \Lambda_{2,2}^- + \frac{B_0}{Z_0} \Lambda_{1,2}^- \right) = 0.
\]
Moreover, the chirality condition on (32) gives
\[ D_-(\phi_3^+ - \phi_0^-) = D_-(\Gamma_+ + \Delta_+) = 0, \]
from where we obtain the following eqns.

\[ \frac{c_{0,1}^+}{\lambda_1} \partial_y \Gamma_+ + \frac{g s_{0,1}^+}{4 \lambda_1} \Lambda_{1,1}^+ + \frac{g s_{0,2}^+}{4 \lambda_2} \Lambda_{1,2}^+ = 0, \]
\[ \frac{c_{0,2}^+}{\lambda_2} \partial_y \Gamma_+ - \frac{g s_{0,1}^+}{4 \lambda_1} \Lambda_{2,1}^+ - \frac{g s_{0,2}^+}{4 \lambda_2} \Lambda_{2,2}^+ = 0, \]
\[ \frac{c_{0,2}^+}{\lambda_2} \partial_y \Lambda_{1,1}^+ + \frac{c_{0,1}^+}{\lambda_1} \partial_y \Lambda_{2,1}^+ + \frac{g s_{0,2}^+}{4 \lambda_2} \Lambda_0^+ = 0, \]
\[ \frac{c_{0,2}^+}{\lambda_2} \partial_y \Lambda_{1,2}^+ + \frac{c_{0,1}^+}{\lambda_1} \partial_y \Lambda_{2,2}^+ - \frac{g s_{0,1}^+}{4 \lambda_1} \Lambda_0^+ = 0. \] (42)

The two sets of eqns. namely, (41) and (42) give the following solutions,

\[ \Lambda_{1,1}^+ = \Lambda_{2,2}^+ = - \frac{8 \mu_-}{g \eta_+ \eta_-} \cos \left( \frac{x}{2} \right) \sin \left( \frac{y}{2} \right), \]
\[ \Lambda_{1,2}^+ = \frac{8 \mu_-}{g \eta_+ \eta_-} \left( \frac{\lambda_2}{\lambda_1} \right) \sin \left( \frac{y}{2} \right), \]
\[ \Lambda_{2,1}^+ = \frac{8 \mu_-}{g \eta_+ \eta_-} \left( \frac{\lambda_1}{\lambda_2} \right) \sin \left( \frac{y}{2} \right), \]
\[ \Lambda_0^+ = - \frac{32 \mu_-}{(g \eta_+ \eta_-)^2} \sin \left( \frac{x}{2} \right) \left[ \cos \left( \frac{y}{2} \right) (a + \cos x - \cos y) - 2 \mu_+ \cos \left( \frac{x}{2} \right) \right], \]

where

\[ \mu_\pm = \frac{\gamma_1 \pm \gamma_2}{\gamma_2 \gamma_1}, \]
\[ a = \frac{1}{2} \left( \frac{\gamma_1^2}{\gamma_2} + \frac{\gamma_2^2}{\gamma_1} \right) + 3, \]
\[ \eta_\pm = \mu_+ - 2 \cos \left( \frac{x \pm y}{2} \right). \] (43)

In order to determine the coefficients \( \Lambda^- \) we make use of

\[ D_-(\phi_3^- - \phi_0^-) = \frac{8}{\lambda_1} \mathcal{F}^{(0,1)} c_{0,1}^+ - \frac{8}{\lambda_2} \mathcal{F}^{(1,3)} c_{1,3}^+, \]
\[ D_-(\phi_1^- - \phi_2^-) = - \frac{8}{\lambda_1} \mathcal{F}^{(0,1)} c_{0,1}^+ + \frac{8}{\lambda_2} \mathcal{F}^{(0,2)} c_{0,2}^+, \]

which are obtained from (30). Introducing (32) in the first of these eqns. we find

\[ D_-(\phi_3^- - \phi_0^-) = D_-(\Gamma_+ + \Delta_+) = \frac{8}{\lambda_1} \mathcal{F}^{(0,1)} c_{0,1}^+ - \frac{8}{\lambda_2} \mathcal{F}^{(1,3)} c_{1,3}^+. \]
Using eqn. (39) in the above expression and taking into account that $F^{(0,1)}$, $F^{(0,2)}$, $F^{(0,1)} f_{2,1}$ and $F^{(0,1)} f_{2,2}$ are independent, we arrive at the following expressions,

\[
\frac{\bar{c}_{0,2}}{\lambda_2} \partial_y \Gamma_+ - \frac{\bar{c}_{1,1}}{\lambda_2} \left( B_0 + \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,1} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,2} \right) = 0, \\
\frac{\bar{c}_{0,1}}{\lambda_2} \partial_y \Lambda_{1,1} + \frac{\bar{c}_{0,1}}{\lambda_2} \partial_y \Lambda_{2,1} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,1} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,2} = 0, \\
\frac{\bar{c}_{0,2}}{\lambda_2} \partial_y \Lambda_{1,1} + \frac{\bar{c}_{0,1}}{\lambda_2} \partial_y \Lambda_{2,1} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,1} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,2} = 0, \\
\frac{\bar{c}_{0,2}}{\lambda_2} \partial_y \Lambda_{1,2} + \frac{\bar{c}_{0,1}}{\lambda_2} \partial_y \Lambda_{2,2} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,1} - \frac{g_s^{(2)}}{4 \lambda_2} \Lambda_{0,2} = 0.
\]

(44)

The chiral condition

\[ D_+(\phi_3 - \phi_0^*) = D_+ (\Gamma_+ + \Delta_-) = 0, \]

leads us to

\[
\frac{\bar{c}_{0,1}}{\kappa_1} \partial_y \Gamma_+ + \frac{g_s^{(2)}}{4 \kappa_1} \Lambda_{0,1} - \frac{g_s^{(2)}}{4 \kappa_1} \Lambda_{0,2} = 0, \\
\frac{\bar{c}_{0,2}}{\kappa_2} \partial_y \Gamma_+ - \frac{g_s^{(2)}}{4 \kappa_2} \Lambda_{0,1} - \frac{g_s^{(2)}}{4 \kappa_2} \Lambda_{0,2} = 0, \\
\frac{\bar{c}_{0,1}}{\kappa_1} \partial_y \Lambda_{0,1} + \frac{\bar{c}_{0,1}}{\kappa_1} \partial_y \Lambda_{0,2} + \frac{g_s^{(2)}}{4 \kappa_1} \Lambda_{0} = 0, \\
\frac{\bar{c}_{0,2}}{\kappa_2} \partial_y \Lambda_{0,1} + \frac{\bar{c}_{0,1}}{\kappa_1} \partial_y \Lambda_{0,2} - \frac{g_s^{(2)}}{4 \kappa_1} \Lambda_{0} = 0.
\]

(45)

Solving (44) and (45) for $\Lambda^-$, we find

\[
\Lambda_{1,1}^- = \frac{8 \mu_-}{g \eta_+ \eta_-} \cos \left( \frac{y}{2} \right) \sin \left( \frac{x}{2} \right), \\
\Lambda_{1,2}^- = -\frac{8 \mu_-}{g \eta_+ \eta_-} \left( k_2 \right) \sin \left( \frac{x}{2} \right), \\
\Lambda_{2,1}^- = -\frac{8 \mu_-}{g \eta_+ \eta_-} \left( k_1 \right) \sin \left( \frac{x}{2} \right), \\
\Lambda_0^- = -\frac{32 \mu_-}{(g \eta_+ \eta_-)^2} \sin \left( \frac{y}{2} \right) \left[ \cos \left( \frac{x}{2} \right) \left( a - \cos x + \cos y \right) - 2 \mu_+ \cos \left( \frac{y}{2} \right) \right],
\]

where $\mu_\pm$, $a$ and $\eta_\pm$ are given in (43).
Bibliography

[1] C. Rogers, in “Soliton Theory: a survey of results”, Ed. A. Fordy, Manchester Univ. Press. (1990)

[2] H. Wahlquist and F. Estabrook, *Phys. Rev. Lett.* **31** (1973) 1386

[3] Q.P. Liu and Y.F. Xie *Phys. Lett.* **325A** (2004) 139; Q.P. Liu and Xing-Biao Hu *J. Physics A* **38** (2005) 6371

[4] J. F. Gomes, L. H. Ymai and A. H. Zimerman, *Phys. Lett.* **373A** (2009) 1401, [arXiv:0902.2456 [math-ph]]

[5] T. Inami and Kanno, *Nucl. Phys.* **B359** (1990) 201; T. Kobayashi and T. Uematsu, *Phys. Lett.* **264B** (1991) 107

[6] “N = 2 and N = 4 supersymmetric mKdV and sinh-Gordon hierarchies” H. Aratyn, J. F. Gomes, L. H. Ymai and A. H. Zimerman
[arXiv:hep-th/0409171]

[7] H. Aratyn, J. F. Gomes, L. H. Ymai and A. H. Zimerman, *J. Physics A* **41** (2008) 312001, [arXiv:0712.0626 [nlin.SI]]

[8] J. F. Gomes, L. H. Ymai and A. H. Zimerman, JHEP **0803**, 001 (2008), [arXiv:0710.1391 [hep-th]]; J. F. Gomes, L. H. Ymai and A. H. Zimerman, J. Phys. Conf. Ser. **128**, 012004 (2008), [arXiv:0708.2407 [nlin.SI]]

[9] M. Grigoriev and A. Tseytlin, *Int. J. Mod. Phys.* **A23** (2008) 2107, [arXiv:0806.2623 [hep-th]]

[10] M. Grigoriev and A. Tseytlin, *Nucl. Phys.* **B800** (2008) 450, [arXiv:0711.0155 [hep-th]]