Topography of the hot sphaleron Transitions

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Abstract

By numerical simulations in real time we provide evidence in favour of sphaleron like transitions in the hot, symmetric phase of the electroweak theory. Earlier performed observations of a change in the Chern-Simons number are supplemented with a measurement of the lowest eigenvalues of the three-dimensional staggered fermion Dirac operator and observations of the spatial extension of energy lumps associated with the transition. The observations corroborate on the interpretation of the change in Chern-Simons numbers as representing continuum physics, not lattice artifacts. By combining the various observations it is possible to follow in considerable detail the time-history of thermal fluctuations of the classical gauge-field configurations responsible for the change in the Chern-Simons number.

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1 Introduction

It is by now well known that the fermion number is not conserved in the standard electroweak theory due to the anomaly of the fermionic current and the periodic vacuum structure in the non-abelian gauge theories \[^1\]. Although the amplitude for this process is exponentially small at zero temperature a great amplification can occur at high temperature \[^2\]. The reason is that the fermion-number violating processes at zero temperature can occur only by tunneling from one classical vacuum to another, the two being connected by a large gauge transformation with winding number different from zero. At higher temperature it is possible to move between the different gauge vacua by classical thermal fluctuations. The energy barrier which separates the vacua has a height determined by the sphaleron energy \(E_{\text{sph}}\), which is the energy of the static classical solution to the electroweak theory corresponding to the lowest lying saddle point between two neighbour vacua \[^3\]. If the temperature \(T\) is so “low” that the symmetry is broken we have

\[
E_{\text{sph}} = \frac{2M_w(T)}{\alpha_w} B(\lambda/\alpha_w)
\]  

(1.1)

where \(B(\lambda/\alpha_w)\) is a factor of order one, which varies only slowly when the ratio between the Higgs coupling \(\lambda\) and \(\alpha_w = g_w^2/4\pi\) changes from zero to infinity. In (1.1) we have included a possible temperature dependence in the \(W\)-mass \(M_w\). As long as \(x \equiv E_{\text{sph}}/T \ll 1\) we can trust a one loop calculation and we get that the transition probability per volume and time for moving from one vacuum to a neighbour one is given by \[^4\] \[^5\]

\[
\Gamma = 0.007(\alpha_w T)^4 x^7 e^{-x}.
\]  

(1.2)

This calculation is strictly speaking only valid as long as the Boltzmann factor \(e^{-x}\) dominates the prefactor \(x^7\) from zero modes of the sphaleron. It is natural to expect that (1.2) extrapolates to

\[
\Gamma = \kappa(\alpha_w T)^4
\]  

(1.3)

when the symmetry is restored \((M_w(T) = 0)\) and \(x\) is formally zero \[^4\] \[^5\] \[^6\], but obviously we can get no information about the non-perturbative constant \(\kappa\) from (1.2) and analytical methods have until now failed in the symmetric phase due to infrared divergences in the high temperature expansion. The constant \(\kappa\) was determined by real time computer simulations of the \(SU(2)\) gauge-Higgs system and the result was that \(\kappa \sim O(1)\) \[^8\].

The cosmological implication of \(\kappa \sim O(1)\) is that any \(B + L\), the baryon plus lepton number, generated at \(GUT\) temperature, will be washed out before we reach
the electroweak transition. In theories where $B-L$ are strictly conserved the problem of explaining the baryon asymmetry observed in the universe is then pushed to the electroweak transition temperature.

Since the implications of $\kappa \sim O(1)$ are important we have felt a need to corroborate on the claim in [8] that the configurations observed by computer simulations are really to be identified with continuum like configurations which change the Chern-Simon numbers by units of one. The only thing done in [8] was to measure the “naive” lattice definition of $\int_0^t \int d^3x F \tilde{F}$ as a function of time when it developed according to the classical equations of motion, starting with a hot configuration above the electroweak transition temperature $T_c$. Whenever a change in the Chern-Simons number compatible with unity and followed by some kind of plateau was observed it was classified as a “sphaleron” like transition. The rather rapid change of Chern-Simons number observed in [8] could potentially be due to lattice dislocations rather than genuine extended field configurations with a “topological” interpretation. We have in this paper tried to verify the “continuum nature” of the lattice configurations in two ways: By measuring the eigenvalues $E_n(t)$ of the three dimensional Dirac operator at time $t$ and observing that one of them dives to zero within the time period where the Chern-Simon number changes rapidly by one unit, and further by measuring the energy density as a function of time for the gauge field configurations. The approach is from this point of view similar to the work analyzing the monopole-like configurations in lattice QCD [11, 12], since these should also be thought of as three dimensional configurations.

2 The model and the simulation

The main approximation in the present work is the use of purely classical thermodynamics. It is based on the observation that the energy of the sphaleron is of order $M_w/\alpha_w$ while its extension is of order $M_w^{-1}$. For $T > T_c$ we expect similar characteristics of the sphaleron-like configurations: their energy will be of order $T$ but their extension of order $(\alpha_w T)^{-1}$. Due to the smallness of $\alpha_w$ the characteristic momenta of thermal fluctuations which form such sphaleron-like configurations are therefore much smaller than the generic quantum fluctuations of the hot plasma which are of order $T$. Hence the sphaleron-like fluctuations decouple from the quantum fluctuations and one might expect that the thermal fluctuations responsible for the change

\footnote{Of course there are no genuine sphalerons for $T > T_c$. We use the expression “sphaleron-like configuration” for any configuration where the gauge field has the same qualitative features as the sphaleron: The configuration is the one in a sequence of configurations where one of the eigenvalues of the Dirac operator is zero and it is one where there is a clear lump of energy located in a considerable region of space.}
in Chern-Simons number are well described by classical physics for temperatures above \( T_c \).

The idea is therefore to start out in a classical configuration

\[
\{ A_i^a, \phi^a; E_i^a, \pi^a \}
\]  

(2.1)
dictated by the Gibbs distribution \( \exp(-H/T) \) where \( H \) is the Hamiltonian. Since we consider the classical theory, \( H \) is just the classical Hamiltonian for the gauge-Higgs system

\[
H = \int d^3x \left[ \frac{1}{2} E_i^a E_i^a + \frac{1}{4} F_{ij}^a F_{ij}^a + |\pi|^2 + |D_i \phi|^2 + M^2 |\phi|^2 + \lambda |\phi|^4 \right]
\]  

(2.2)

and in the temporal gauge it should be supplemented by Gauss constraint in the form

\[
D_i^{ab} E_i^b = ig(\phi^\dagger \tau^a \pi - \pi^\dagger \tau^a \phi).
\]  

(2.3)

After having chosen a configuration according to Gibbs distribution we let the system evolve according to the classical equations of motion:

\[
\frac{dA_i^a}{dt} = \frac{\delta H}{\delta E_i^a} = E_i^a
\]  

(2.4)

\[
\frac{dE_i^a}{dt} = -\frac{\delta H}{\delta A_i^a}.
\]  

(2.5)

If the system (i.e. in practise the lattice) is sufficient large the temperature will be approximately constant in this micro-canonical simulation.

For details about the discretization of the above classical equations and choice of (lattice) coupling constants we refer to [8]. Here it is sufficient to say that the simulations were performed on a \( 16^3 \) lattice and that the bare values of the coupling constants for the gauge fields and the Higgs fields were choosen such that the tree-values of the theory corresponded to being in the broken phase with the Higgs mass equal the \( W \)-mass and such that one sphaleron fits onto the lattice. As noticed in [8] these constraints on the coupling constants force us for the given lattice size to work at a temperature where the symmetry is actually restored.

In the symmetric phase we can now measure the change in Chern-Simons number

\[
N_{cs}(t) - N_{cs}(0) = \frac{1}{32\pi^2} \int_0^t dt' \int d^3x \, F_{\mu,\nu}^a F_{\mu,\nu}^a
\]  

(2.6)

The results of a typical measurement is shown in fig.1a. One identifies two “sphaleron-like” transitions, of which the last one seems to correspond to a jump of Chern-Simons number of two units. Superimposed on these we see a band of short wavelength thermal fluctuations which carry the main part of the energy. According

\footnote{Throughout this paper we assume that we can ignore the hypercharge sector, which means that we effectively work with the Weinberg angle \( \theta_w = 0 \).}
to our arguments above the essential features of the sphaleron transitions should remain unchanged if we strip off the short wave length thermal fluctuations. We have done this for each of the time-sequence of configurations we get by solving the classical equations of motion by iterating for a given configuration the simplest relaxation equation:

$$\frac{\partial \phi}{\partial t} = -\frac{\delta H}{\delta \phi}, \frac{\partial A}{\partial t} = -\frac{\delta H}{\delta A}$$

(2.7)

This technique is well known from the study of lattice instantons and monopoles \cite{9, 10, 11, 12}. The results for the configurations of fig.1a are shown in fig.1b. Each configuration used in fig. 1a has been subjected to six cooling sweeps of the kind given in (2.7). By this process the energy stored in the gauge fields drops by almost a factor 80, in agreement with previously obtained results for instantons and monopoles. The picture of sphaleron transitions is considerable sharpened and the fact that the transition survives this cooling shows at least that the assumption of an effective decoupling of the short and long wave length thermal fluctuations when we discuss sphaleron-like transitions is internally consistent\footnote{It does of course not prove that we can actually replace the short wavelength quantum thermal fluctuations by short wavelength classical thermal fluctuations.}. The Higgs field relaxes much slower and there seems to be a large degree of decoupling between the Higgs field and the gauge field in the symmetric phase. In fact we get essentially the same Chern-Simons picture as in fig. 1b if we ignore the Higgs field and only relax the gauge field. Insensitivity with respect to the Higgs field has also been observed in the study of monopole-like configurations \cite{12}.

3 Spectral flow

Our aim is to provide further evidence that the sphaleron-like transitions have the essential characteristica of the continuum, relevant for the anomaly. One important feature, contained in the Atiyah-Patodi-Singer index theorem \cite{13} and explained in detail for instance by Christ \cite{14} is that the spectral flow of eigenvalues of the time dependent Dirac operator $H_D(A(t))$ is directly related to the change in Chern-Simons number if we consider a continuous time sequence of $SU(2)$ gauge potentials $A(t)$ starting in one gauge vacuum at $t = -\infty$ and ending up in a neighbour one at $t = +\infty$. In fact, if we consider fermions of a given chirality the change in Chern-Simons number is equal the number of times eigenvalues of $H_D(t)$ cross zero from below minus the number of times eigenvalues cross zero from above (the value zero has no special status in this context, but is conveniently chosen since the vacuum for $A_i = 0$ is identified with the filled Dirac sea).
One interesting point in the present situation is that we have at no time a true interpolation from one gauge vacuum to a neighbour one, since we are working at a finite temperature. We expect that the Atiyah-Patodi-Singer index theorem may be used in this more general context, just moving from one gauge field configuration to a neighbour one connected by a large gauge transformation, but have not attempted to provide a rigorous proof of this conjecture. In practise we try, as already mentioned, to use as smooth (i.e. cold) configurations as possible.

In the following we will be satisfied with verifying that the crossing of eigenvalues at zero is closely linked to the change of Chern-Simons number as measured in the most “naive” way, as described above. Since the concept of chirality on the lattice is non-trivial \cite{13,12} we will not at the present stage try to unravel the exact counting and assignment of chirality of the individual modes.

We now turn to the measurement of eigenvalues of the Dirac operator on the lattice. We have found it most convenient to use the staggered fermion formalism. The three-dimensional staggered fermion Dirac equation reads:

\[\sum_{i=1}^{3} \frac{1}{2} i \eta_i(n) \left[ U_i \chi(n + \hat{i}) - U_i^\dagger (n - \hat{i}) \chi(n - \hat{i}) \right] = E \chi(n) \]  

(3.1)

where \(U_i(n)\) are the \(SU(2)\) gauge field variables living on the links \(i\) located at lattice points \(n\). The formal relation to a continuum gauge connection is \(U_i(n) = \exp(iA_i(n))\). The fermion field \(\chi(n)\) is a one-component spinor and an \(SU(2)\) doublet. \(\eta(n)\) is the Kawamoto-Smit phase \((-1)^{n_1 + \cdots + n_i - 1}\). We use antisymmetric boundary conditions, in which case there are no zero modes in the free field case and it is easier to identify an eigenvalue crossing zero.

In fig. 1c we have shown the time evolution of the lowest positive eigenvalue for the gauge field configurations which were responsible for the change in Chern-Simons number shown in fig. 1b. These are the configurations which are cooled and thereby have lost most of the short range thermal fluctuations. However, even in the case where the full thermal fluctuations are present we see essentially the same picture. We have chosen to show in fig. 1c the eigenvalue from fig. 1b only because it is easier to identify the Chern-Simons number on this figure.

It is seen that the first diving of an eigenvalue to zero coincides with the first sphaleron transition of fig. 1b. We have not shown the higher eigenvalues of the Dirac equation, but they do not get below 0.15. They show however a similar (relative) diving as the lowest mode, and they should, since the choice of zero as the point to count the crossing of eigenvalues is arbitrary, as mentioned above.

The situation is more complicated for the next change of Chern-Simon number seen in fig. 1b since it consists of two successive jumps. Accordingly we see indeed
in fig. 1c the diving of two eigenvalues. A closer look at fig. 1b reveals that the Chern-Simons number seems to stop for some time at the value 1/2, precisely the value of the sphaleron, which we know has a zero mode \[16, 17\]. Notice that this plateau at 1/2 is present both before and after cooling and from this point of view should be taken seriously as a continuum configuration. Its presence is very clearly reflected in the behaviour of the lowest eigenvalue. As long as the configuration stays at a value of \( N_{cs}(t) \approx 1/2 \) the eigenvalue is close to zero as we have shown in fig. 2. An obvious interpretation of this behaviour could be that the system spends some time in a sphaleron like configuration without being able to make up its mind into which valley (gauge vacuum) to fall. If that is the case the word “sphaleron” is indeed appropriately chosen for this configuration.

### 4 Energy lumps

In order to understand better the nature of the gauge field configurations responsible for the change in the Chern-Simons numbers and the spectral flow of eigenvalues in fig. 1 we have recorded the actual energy distribution of the gauge fields. It was impossible to see any clear picture when all thermal fluctuations were included. Only for the cooled configurations did a clear picture emerge. (Again this is consistent with the experience from instantons and monopoles). The average energy of the gauge fields after cooling is around 0.039 and we have chosen to show the energy concentrations of the gauge field at two times: before the first sphaleron-like transition and in the middle of the transition, where the eigenvalue of the Dirac operator is close to zero (0.002). In both cases we show a sequence of 3D pictures where regions of space occupied by cubes have an energy density above a certain threshold (fig. 3). The three thresholds chosen are 0.06, 0.07 and 0.08. Fig. 3a-3c illustrate a typical situation before the rapid change in Chern-Simons number while fig. 3d-3f show the spatial distribution in the middle of the transition. We see a marked difference in the concentration of gauge-field energy for the two situations. In the case of the zero eigenvalue one can talk about a genuine extended object while the other situation reflects typical fluctuations in the energy density which will always be present if we pick a random configuration. We see that the sphaleron-like configurations in no way can be considered as nice symmetric configuration, but this is not really to be expected. As a dual representation we show in fig. 4 the energy density for the extended object present in the case where the eigenvalue is zero (fig. 3d-3f) in a plane where the concentration in the core of the energy lump is high.
5 Discussion

We have shown that the configurations responsible for the change in Chern-Simons number on the lattice indeed seem to share the characteristics of true continuum configurations with the same properties. They have considerable spatial extension and eigenvalues of the Dirac equation will dive to zero at some point during the change of the Chern-Simons number. For the sake of clarity we have in this paper concentrated on a particular simulation and two sphaleron like transitions, but we have performed a whole sequence of such simulations and in the process of analyzing the results we have in fact seen an even more detailed relationship between the three quantities: the Chern-Simons number, the energy lumps and the eigenvalues. One type of behaviour is the following: when performing some cooling steps we still find some fluctuations in Chern-Simons numbers which can not be classified as jumps of order one and where two eigenvalues will dive towards zero, somewhat displaced. A possible interpretation is that we have two sphaleron like configurations, the appearance slightly displaced in time, with values of $N_{cs}(t)$ of opposite sign. Another type of behaviour which we have observed is one where an eigenvalue seemingly crosses zero and comes back again or which moves close to zero and then return. At the same time the Chern-Simons number will move close to approximately 1/2 and return to zero again. It is again tempting to view this as a gauge field configuration which moves to a sphaleron-like configuration and then return back to the same vacuum-like configuration from which it originated. In this way it seems possible to map out in detail the movement of gauge field configuration relevant for the change of Chern-Simons number and to get a detailed knowledge of the mechanism by which thermal fluctuations are able to co-operate and create sphaleron-like configurations.

Many things could be improved: One could obviously gain a lot if it was possible to use significantly larger lattices\footnote{Unfortunately we do not have the computer facilities needed for such calculations.}. It would then be possible to go to smaller temperatures and smoother configurations, and maybe even to address the same questions in the region where the symmetry is broken, but where the classical transitions from one vacuum to another are not yet suppressed.

Another important improvement if one wants to dig into a more detailed investigation of the flow of eigenvalues is the chirality of the eigenmodes. A suitable definition has been given in \cite{15, 12}.

A final very important question which remains to be addressed is the role of the Higgs field. This role is still obscure to us. As already reported long ago \cite{18} it seems that the Higgs field in the symmetric phase partly decouples from the gauge field. Since we effectively are in the symmetric phase the Higgs field will fluctuate...
and have zeroes, and when we iterate a few times the relaxation equations it still seems somewhat decoupled from the gauge field. At smaller temperatures one would expect a strong coupling of the phases of the Higgs field and the gauge field. Since the real fermions in the electroweak theory couple to the Higgs field it is indeed important to understand the role of this field. In the continuum the situation is only clear in the broken phase. If a field configuration in the broken phase (gauge fields and Higgs fields) interpolates in time between two non-equivalent vacua the Higgs field has (by purely topological reasons) to develop a zero in between the vacuum configurations where the magnitude of the Higgs field is constant. This zero of the Higgs field is precisely what is present in the sphaleron configuration and it is this zero which allows the Dirac operator, coupled both to gauge- and Higgs fields of the sphaleron, to have a normalizable energy eigenmode with eigenvalue zero, representing precisely the crossing of zero of the energy levels.\[16, 17\]. We do not know of any detailed investigation of the role of the Higgs field in the symmetric phase.

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Figure Caption

Fig.1 Measurements of the Chern-Simons as a function of time. Fig. 1a and shows time evolution with thermal fluctuations included, while fig. 1b shows the same time evolution with the thermal fluctuations partly stripped off. Fig. 1c shows the lowest eigenvalue for the configurations which are responsible for the Chern-Simons numbers shown in fig. 1b.

Fig.2 Fig. 2a shows in more detail the behaviour of the Chern-Simons number in the neighbourhood of the second sphaleron transition recorded in fig. 1b. Fig. 2b shows details of the lowest eigenvalue corresponding to fig. 2a.

Fig.3 Fig. 3a-3f are 3D pictures of the regions in space where the energy concentration is larger than 0.06, 0.07 and 0.08 for two configurations before and in the middle of the first sphaleron transition shown in fig. 1b. The first three figures refer to a configuration before the transition and for which there are no small eigenvalues, while the last three figures refer to a configuration in the middle of the transition. This configuration has an eigenvalue of the Dirac operator which is approximately zero.

Fig.4 An energy level plot in the $x - y$-plan of fig. 1c at $z$-height where the energy concentration in the core is high.