PT-symmetry breaking in a necklace of coupled optical waveguides

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We consider parity-time (PT) symmetric arrays formed by \( N \) optical waveguides with gain and \( N \) waveguides with loss. When the gain-loss coefficient exceeds a critical value \( \gamma_c \), the PT-symmetry becomes spontaneously broken. We calculate \( \gamma_c(N) \) and prove that \( \gamma_c \rightarrow 0 \) as \( N \rightarrow \infty \). In the symmetric phase, the periodic array is shown to support 2\( N \) solitons with different frequencies and polarisations.

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I. INTRODUCTION

Since the emergence of the PT symmetry as a research avenue in quantum theory \cite{4}, the concept was embraced in several other fields, including photonics \cite{2}, plasmonics \cite{3}, Bose-Einstein condensates \cite{11} \cite{6}, and quantum optics of atomic gases \cite{7}. The PT-symmetric systems exhibit unusual phenomenology with a potential for practical utilisation. The PT optical structures, in particular, display unconventional beam refraction \cite{5} \cite{8}, Bragg scattering \cite{9}, nonreciprocal Bloch oscillations \cite{10}, loss-induced transparency \cite{11}, and conical diffraction \cite{12}. Nonlinear effects in such systems can be utilised for an efficient control of light, including all-optical low-threshold switching \cite{13} \cite{14} and unidirectional invisibility \cite{13} \cite{15}.

Experimentally, the optical PT symmetry was realised in a directional coupler consisting of two coupled waveguides with gain and loss \cite{2} \cite{11}, and in chains of such dipoles \cite{16}. The corresponding theoretical models went on to include the effects of diffraction of spatial beams and dispersion of temporal pulses, i.e., include an additional spatial or temporal dimension \cite{17}. Dispersive PT-couplers were shown to support optical solitons \cite{17} \cite{18}. Triplet, quadruplet and quintet of (nondispersive) guides were also dealt with \cite{19}.

A fundamental phenomenon observed in symmetric couplers is the spontaneous PT-symmetry breaking \cite{2} \cite{15} \cite{17} \cite{20} which occurs as the gain-loss coefficient is increased beyond a certain critical value \( \gamma_c \). This exceptional point separates the symmetric phase, where all perturbation frequencies are real, and the symmetry-broken phase, where some frequencies are complex and the corresponding modes grow exponentially. Besides demarcating the stability boundary, the exceptional point concerns us here, is how the geometry of this system and the growing multiplicity of its channels affects the symmetry-breaking point, \( \gamma_c \). We also uncover the diversity of solitons arising in such a chain.

The chain consists of \( N \) waveguides with gain and \( N \) with loss. The complex mode amplitudes, \( u_n \), satisfy

\begin{equation}
\dot{u}_n + u_n'' + 2|u_n|^2u_n + u_{n-1} + u_{n+1} = 2\Gamma_n u_n \\
(n = 1, ..., 2N),
\end{equation}

where \( \dot{u}_n \equiv \partial u_n / \partial t \) and \( u_n'' \equiv \partial^2 u_n / \partial z^2 \). In Eq. (1), \( t \) stands for time and \( z \) for the distance in the frame of reference traveling along with the pulse. The coefficient \( \Gamma_n \) equals \( \gamma > 0 \) for the waveguides with gain and \( -\gamma \) for those with loss. The active and lossy guides are either separated into two clusters or simply alternate (fig. 1). The chain forms a periodic necklace, that is, \( u_{2N+1} = u_1 \) and \( u_0 = u_{2N} \).

We also consider open chains. An open chain is described by Eqs. (1) without the periodicity condition:

\begin{align}
\dot{u}_1 + u_1'' + 2|u_1|^2u_1 + u_2 &= 2i\gamma u_1, \\
\dot{u}_{2N} + u_{2N}'' + 2|u_{2N}|^2u_{2N} + u_{2N-1} &= -2i\gamma u_{2N}, \\
\dot{u}_n + u_n'' + 2|u_n|^2u_n + u_{n-1} + u_{n+1} &= 2i\Gamma_n u_n \\
(n = 2, ..., 2N - 1).
\end{align}

With a suitably chosen constant matrix \( \mathcal{L} \), Eqs. (1)-(2)
can be written in a unified way:

\[ iu_n + u''_n + \sum_{m=1}^{2N} L_{nm} u_m + 2|u_n|^2 u_n = 0. \] (3)

Of particular importance for dispersive waveguides is the zero solution of Eq. (3). Let \( u_n(z, t) = 0 \) \( (n = 1, \ldots, 2N) \). This solution shall serve as a background to solitons and breathers. To classify its stability, we linearise Eq. (3) and let \( u_n = e^{i(kz - \omega t)} \), where \( k \) and \( \omega \) are assumed to be real. The combination \( \lambda = k^2 - \omega \) is then found as an eigenvalue of the linearisation matrix \( L \). The zero solution loses stability when two real eigenvalues merge and become a complex-conjugate pair.

It is fitting to note that an equivalent eigenvalue problem arises in the linearisation of the trivial solution of the symmetric array of nondispersive (\( z \)-independent) waveguides. The nondispersive array is a lattice system defined by Eq. (3) without the \( u''_n \) term. The \( N \) = 1 case corresponds to the \( PT \)-symmetric nondispersive coupler (also referred to as the dimer). The \( N = 2 \) case (the \( PT \)-quadrimer) was considered in [19].

II. OPEN ALTERNATING CHAIN

The alternating necklaces have \( \Gamma_n = (-1)^{n+1} \gamma \) for \( n = 1, \ldots, 2N \). It is convenient to consider the open chain first. In this case,

\[ L_{nm} = -2i \Gamma_n \delta_{n-m} + \delta_{n-m-1} + \delta_{n-m+1}, \] (4)

\( n, m = 1, \ldots, 2N \). Here \( \delta \) is the Kronecker delta symbol:

\[ \delta_n = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise}. \end{cases} \]

The stability eigenvalues are expressible via roots of the secular equation \( D_{2N} = 0 \), where \( \lambda = -2\alpha \) and

\[ D_{2N} = \det (L + 2\alpha I). \] (5)

The determinants with \( N = 1 \) and \( 2 \) are readily found:

\[ D_2 = 2x + 1; \]
\[ D_4 = 4x^2 + 2x - 1, \]

where \( x = 2(\gamma^2 + \alpha^2) - 1 \). Any determinant with \( N \geq 3 \) can be expanded as

\[ D_{2N} = 2x D_{2(N-1)} - D_{2(N-2)}. \] (6)

The recursion relation [6] with \( D_2 = 2x + 1 \) and \( D_4 = 4x^2 + 2x - 1 \) admits a simple solution

\[ D_{2N}(\gamma, \alpha) = U_N(x) + U_{N-1}(x), \] (7)

where \( U_N(x) \) is the Chebyshev polynomial of the second kind of \( N \)th order [21]. Using the defining property of the Chebyshev polynomials,

\[ U_N(x) = \frac{\sin((N + 1)\theta)}{\sin \theta}, \]

we evaluate the determinant \( D_{2N} \) as

\[ D_{2N} = \frac{\sin((2N + 1)\theta/2)}{\sin(\theta/2)}. \] (8)

\( D_{2N} \) has \( 2N \) simple roots

\[ \theta_n = \pm \frac{2n}{2N + 1} \pi, \quad n = 1, 2, \ldots, N. \]

On the \( (\gamma, \alpha) \)-plane, equations \( \gamma^2 + \alpha^2 = \cos^2(\theta_n/2) \) describe \( N \) concentric circles centred at the origin. As \( \gamma \) grows through \( \gamma_n = \cos(\theta_n/2) \), two opposite eigenvalues \( \lambda = -2\alpha \) converge at \( \alpha = 0 \) and become complex. The \( PT \)-symmetry breaking threshold is determined by the \( \gamma \)-intercept of the smallest circle: \( \gamma_c = \cos(\theta_N/2) \), i.e.,

\[ \gamma_c = \sin \left( \frac{\pi}{2(2N + 1)} \right). \]

III. PERIODIC ALTERNATING CHAIN

Linearising the periodic alternating necklace [1], the corresponding secular equation is \( \tilde{D}_{2N}(\gamma, \alpha) = 0 \), where

\[ \tilde{D}_{2N} = \det (\tilde{L} + 2\alpha I). \]

The matrix \( \tilde{L} \) is given by the same expression [4], with the same \( \Gamma_n = (-1)^{n+1} \gamma \), but with \( \delta \) replaced with the cyclic Kronecker symbol \( \delta^{(2N)} \):

\[ \delta^{(2N)}_n = \begin{cases} 1, & \text{if } n \mod 2N = 0; \\ 0, & \text{otherwise}. \end{cases} \]

The determinant \( \tilde{D}_{2N} \) can be expressed via the “nonperiodic” determinants [6]:

\[ \tilde{D}_{2N} = D_{2N} - D_{2(N-1)} - 2. \]

Using [8] the determinant in question is evaluated to be

\[ \tilde{D}_{2N} = -4 \sin^2(N\theta/2), \]

with the double roots \( \theta_n = \frac{2n}{2N} \pi, n = 1, 2, \ldots, N. \)

As in the open-necklace case, equations

\[ \gamma^2 + \alpha^2 = \cos^2(\theta_n/2) \]

describe concentric circles on the \( (\gamma, \alpha) \)-plane. When \( N \) is even, the smallest circle corresponds to \( n = N/2 \) and has zero radius. When \( N \) is odd, the smallest circle corresponds to \( n = \frac{1}{2}(N - 1) \); the radius in this case is \( \sin(\pi/(2N)) \). Thus,

\[ \gamma_c = \begin{cases} 0, & N \text{ even}; \\ \sin \left( \frac{\pi}{2N} \right), & N \text{ odd}. \end{cases} \] (10)
IV. OPEN CLUSTERED NECKLACE

When the waveguides are grouped into two clusters, the gain-loss coefficient $\Gamma_n$ equals $\gamma > 0$ for $n = 1, \ldots, N$ and $-\gamma$ for $n = N + 1, \ldots, 2N$. Again, we start with the open chain, Eq. (2). The corresponding linearisation matrix $L$ is as in Eq. (4).

To find roots of the corresponding secular equation $\Delta_{2N} = 0$, where $\Delta_{2N} = \det(L + 2\alpha I)$, we expand

$$\Delta_{2N}(\gamma, \alpha) = U_N(\zeta)U_N^*(\zeta) - U_{N-1}(\zeta)U_{N-1}^*(\zeta),$$

where $U_N$ is a determinant of an $N \times N$ tridiagonal matrix

$$U_{m,n} = 2\zeta \delta_{m-n} + \delta_{|m-n|-1} \quad (m, n = 1, 2, \ldots, N),$$

and $\zeta = \alpha - i\gamma$. This determinant is nothing but the Chebyshev polynomial (of the complex argument); hence our choice of notation [21]. Defining complex $\theta$, such that $\zeta = \cos \theta$, the Chebyshev polynomial can be written as

$$U_N(\zeta) = \frac{\sin[(N + 1)\theta]}{\sin \theta}.$$

Letting $\theta = x + iy$, the secular equation reduces to

$$\sinh y \sinh[(2N + 1)y] = -\sin x \sin[(2N + 1)x].$$

Here $x^2 + y^2 \neq 0$ for $x = 0$ is not a root of $\Delta_{2N} = 0$. Note that $x$ and $y$ are the elliptic coordinates on the $(\gamma, \alpha)$ plane:

$$\alpha = \cos x \cos y, \quad \gamma = \sin x \sinh y.$$  (13)

The right-hand side of (12) is $\pi$-periodic; hence it is sufficient to consider the interval $0 \leq x \leq \pi$. The curve described by (12) consists of $N$ disconnected ovals in subintervals

$$\frac{2n-1}{2N+1} \pi \leq x \leq \frac{2n}{2N+1} \pi, \quad n = 1, \ldots, N$$

[fig.2(a)]. The transformation (13) maps these to $N$ disconnected ovals on the $(\gamma, \alpha)$ plane [shown in fig.2(a)]. Of interest to us are the points where pairs of $\alpha(\gamma)$ branches merge.

In Eq. (12), the sinusoid $\sin(2N + 1)x$ is modulated by a slowly changing amplitude $\sin x$. Therefore, of all $N$ ovals, the first and the last one (those with $n = 1$ and $n = N$) have the lowest maximum values of $y$. The transformation $(x, y) \rightarrow (x, \gamma)$, where $\gamma = \sin x \sinh y$, keeps the pattern horizontally periodic but elongates the central ovals still further. Therefore, the lowest value of $\gamma$ for which the merger of two real eigenvalues $\lambda = -2\alpha(\gamma)$ occurs, corresponds to the apogee of the first and the last $\gamma(x)$ ovals. Using (12), the condition $d\gamma/dx = 0$ translates into

$$\frac{\tan y}{\tan x} = \frac{\sin x \sin(2N + 1)x}{\sinh y \sinh(2N + 1)y}.$$  (14)

One can readily construct asymptotic roots of the system (12), as $N \rightarrow \infty$. Expanding $x$ and $y$ in powers of $1/2N+1$, Eqs. (12) and (14), give, respectively:

$$\xi \sinh \xi = -S \eta \sin \eta,$$

$$\xi(\sinh \xi + \xi \cosh \xi) = S \eta(\sin \eta + \eta \cos \eta),$$

where

$$x = \frac{\eta}{2N+1} + \ldots, \quad y = \frac{\xi}{2N+1} + \ldots,$$

and $S = 1$.

The system (15), (16) has an increasing sequence of roots $\eta_n, \xi_n > 0, n = 1, 2, \ldots$ [See fig.3(a)]. In particular, $\eta_1 = 5.33, \xi_1 = 1.68$. Hence we get, for each $N, x_n(\gamma_N) = \frac{\eta_n}{2N+1} + \ldots, \quad y_n(\gamma_N) = \frac{\xi_n}{2N+1} + \ldots,$

where “...” stand for corrections of order $(2N + 1)^{-2}$.

(These asymptotic expressions are accurate for $n$ such that $\eta_n$ and $\xi_n$ are much smaller than $2N + 1$.)
The curve (19) consists of \( N \leq \frac{2n - 1}{N} \pi \leq \frac{2n}{N} \pi, \quad n = 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor \).

(Here \( \lfloor \cdot \rfloor \) indicates the integer part of \( p \).) The former set will be referred to as even ovals, and the latter one as odd.

When \( N \) is large, one can find the maximum value of the function \( \gamma(x) \) in each subinterval as an expansion in powers of \( \frac{1}{N} \). The lowest maximum \( \gamma_1 \) in the odd set is given by \( \eta_1 \xi_1 N^{-2} + O(N^{-3}) \), where \( \eta_1, \xi_1 \) is the first pair of roots of the system (15)-(16) with \( S = 1 \). Substituting their numerical values, we obtain \( \gamma_1 = 8.95 N^{-2} + \ldots \).

The lowest maximum in the even set is given by \( \bar{\gamma}_1 = \bar{\eta}_1 \xi_1 N^{-2} + \ldots \), where \( \bar{\eta}_1 = 2.50, \xi_1 = 1.11 \) is the first pair of roots of the system (15)-(16) with \( S = -1 \). The maximum \( \bar{\gamma}_1 \) is lower than \( \gamma_1 \); hence it is \( \bar{\gamma}_1 \) that determines the symmetry-breaking threshold of the closed clustered necklace. Substituting for \( \bar{\eta}_1, \xi_1 \), we finally get

\[
\gamma_c = 2.77 N^{-2} + O(N^{-3}) \quad \text{as } N \to \infty. 
\]  

VI. SOLITONS

Assume the \( \mathcal{PT} \) symmetry is unbroken, \( \gamma < \gamma_c \). Let \( \phi_n \) be an eigenvector of \( \mathcal{L} \) pertaining to the (real) eigenvalue \( \lambda_n \). The transformation \( u_n = \sum \Phi_{nm} \psi_m \), with \( \Phi = \{ \phi_1, \phi_2, \ldots, \phi_{2N} \} \), casts Eq. (3) in the form

\[
i \psi_t + \psi'' + \lambda_n \psi + 2 \sum_{n,m} (\Phi^{-1})_{nm} N_n \Phi_{nm} \psi_m = 0, 
\]  

where

\[
N_n = | \sum \Phi_{nm} \psi_m |^2. 
\]  

(Physically, \( N_n = |u_n|^2 \) has the meaning of the power density in the \( n \)-th waveguide.) We will show that the vector equation (21) admits \( 2N \) independent scalar reductions.

In this paper, we confine our consideration to the case of the periodic alternating chain, Eq. (1) with \( \Gamma_n = (-1)^{n+1} \gamma \). The corresponding matrix \( \mathcal{L} \) has 2 simple and \( N - 1 \) double eigenvalues. The simple eigenvalues \( \lambda_{\pm} = \pm 2 \cos \vartheta \) have eigenvectors

\[
\phi_{\pm} = (1, \pm e^{\pm i \theta}, 1, \pm e^{\pm i \theta}, \ldots),
\]  

where \( \sin \vartheta = \gamma \). These satisfy \( |(\phi_{\pm})_j| = 1 \), for all \( j \). We now show that the rest of the eigenvectors can also be chosen to satisfy this property.

The matrix \( \mathcal{L} \) commutes with the \( Z_N \)-rotation \( \mathcal{R} \), where \( \mathcal{R}_{nm} = \delta^{(2N)}_{n-m-2} \). Therefore the basis in the invariant subspace \( S_n \) associated with the eigenvalue \( \lambda_n \) can be chosen in the form of two eigenvectors of \( \mathcal{R} \):

\[
\mathcal{R} \psi_1 = e^{2 \pi i / N} \psi_1, \quad \mathcal{R} \psi_2 = e^{-2 \pi i / N} \psi_2. 
\]  

FIG. 4. (Color online) The graphical solution of the system (15)-(19) with \( S = 1 \) (a) and \( S = -1 \) (b). The blue (solid) and red (broken) curve are described by equation (15) and (16), respectively. The black dot marks the root \( (\eta_1, \xi_1) \) in (a), and the root \( (\bar{\eta}_1, \xi_1) \) in (b).
The linearisation matrix satisfies $L \mathcal{P} = \mathcal{P} L^*$, where $\mathcal{P}$ is the inversion:

$$\mathcal{P}_{nm} = \delta^{(2N)}_{n+m-1}.$$ 

Therefore, $\mathcal{P} \tilde{\psi}_1$ and $\mathcal{P} \tilde{\psi}_2$ are also in $S_n$. Since $R \mathcal{P} = \mathcal{P} R^{-1}$, the vector $\mathcal{P} \tilde{\psi}_1$ is an eigenvector of the rotation $R$, with an eigenvalue $e^{2\pi i/N}$. That is, $\mathcal{P} \tilde{\psi}_1 = C \tilde{\psi}_1$, with some constant $C$. Since $\mathcal{P}^2 = I$, the constant $C = e^{i\chi}$, with $\chi$ real. Thus,

$$\mathcal{P} \tilde{\psi}_1 = e^{i\chi} \tilde{\psi}_1, \quad \mathcal{P} \tilde{\psi}_2 = e^{-i\chi} \tilde{\psi}_2. \quad (23)$$

We normalise $\tilde{\psi}_1, \tilde{\psi}_2$ so that $(\tilde{\psi}_1)_1 = (\tilde{\psi}_2)_1 = 1$. Eq.(22) tells us that

$$(\tilde{\psi}_1)_{1+2\ell} = e^{-2\pi i \ell/N}, \quad (\tilde{\psi}_2)_{1+2\ell} = e^{2\pi i \ell/N}, \quad \ell = 1, 2, \ldots.$$ 

On the other hand, Eq.(23) gives

$$(\tilde{\psi}_1)_{2N-2\ell} = e^{-i\chi+2\pi i \ell/N}, \quad (\tilde{\psi}_2)_{2N-2\ell} = e^{i\chi-2\pi i \ell/N}.$$ 

Thus all eigenvectors of $\mathcal{L}$ have unimodular components:

$$|\Phi_{nm}| = |(\Phi_m)_n| = 1; \quad n, m = 1, 2, \ldots, 2N. \quad (24)$$

Returning to (21), the scalar reduction is defined by letting $\psi_m = \psi_{d_m-M}$, with some fixed $M$. In view of [24], this gives $N_n = |\psi|^2$ for all $n$. All components of (21) become identically zero, except the one with $\ell = M$, which becomes

$$i \dot{\psi} + \psi'' + \lambda M \psi + 2|\psi|^2 \psi = 0. \quad (25)$$

Each nonlinear Schrödinger equation (25), with $M = 1, \ldots, 2N$, supports a soliton

$$\psi = e^{i\Omega a} \text{sech}(az),$$

with the frequency $\Omega = a^2 + \lambda M$. Thus the original $\mathcal{PT}$-symmetric system [1] has $2N$ coexisting soliton solutions, different in their frequencies and polarisations.

We should emphasise the difference between these vector solitons and (spatial) solitons in a $\mathcal{PT}$-symmetric optical lattice [22]. While the solitons in the waveguide necklace [1] are localised as functions of $z$, the lattice solitons [24] are localised as functions of $n$ (i.e., in the transverse direction). The $n$ dependence determines the power density distribution over the $2N$ channels; this distribution is uniform in the case of the vector solitons of Eq.(1).

**VII. CONCLUDING REMARKS**

In conclusion, we have determined the symmetry breaking points for four different geometries of the necklace. Generically, there is a finite interval of the gain-loss coefficient where the $\mathcal{PT}$ symmetry is unbroken. The only exception is the periodic chain of $2N$ alternating waveguides with even $N$. Here $\gamma_c = 0$, i.e., the symmetry is spontaneously broken for an arbitrarily small $\gamma$.

The alternating arrays admit an explicit solution; the transition points are given by [9] for the open chains and by [10] for the periodic necklaces. In both cases $\gamma_c \sim \frac{1}{N}$ for large $N$. In the clustered geometry, the threshold values are expressible via roots of a simple transcendental equation — Eq.(12) in the case of the open chain, and Eq.(19) in the periodic situation. Eqs.(17) and (20) yield the corresponding asymptotic results. Here, the $\mathcal{PT}$ symmetry breaks quicker: $\gamma_c \sim \frac{1}{2N} \sim N \to \infty$.

It is interesting to note that a similar $\gamma_c \sim \frac{1}{2N}$ law was detected in a disordered $\mathcal{PT}$-symmetric chain with the clustered arrangement of gain and loss, in the limit of large localisation lengths of the eigenmodes [22].

Our $\gamma_c(N)$ values remain valid for the arrays of nondispersive $\mathcal{PT}$-symmetric couplers [Eqs.(1), (2)] without the $u''_n$ term. In particular, our conclusion that the limit of the sequence $\gamma_c(N)$ as $N \to \infty$ exists and equals 0, is in agreement with the symmetry-breaking threshold for the infinite alternating chain [24].

Finally, we have demonstrated that the alternating periodic necklace supports $2N$ coexisting soliton solutions. These vector solitons are characterised by the uniform distribution of the power density over their $2N$ components and are different in their frequencies and polarisations. It is natural to expect that the other three $\mathcal{PT}$-symmetric waveguide arrangements (open-alternating, open- and periodic-clustered) will also exhibit $2N$ different solitons each.

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