Quasi-exactly solvable models as constrained systems

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Abstract

We discuss a universal algebraic approach to quasi-exactly solvable models which allows us to interpret them as constrained Hamiltonian systems with a finite number of physical states. Using this approach we reproduce well-known two-dimensional Lie-algebraic quasi-exactly solvable system based on Lie algebra $\mathfrak{su}(3)$.

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1 Introduction

It is well known that exactly solvable systems play very important role in quantum theory. Unfortunately number of such systems is quite limited. This considerably narrows their applications. Such a situation stimulates interest to quasi-exactly solvable systems [1, 2, 3, 4, 5, 6, 7]. In contrast to exactly solvable models in quasi-exactly solvable systems the spectral problem can be solved partially. Nevertheless such systems are very interesting. Besides modeling physical systems [8, 9, 10] they can be used as an initial point of the perturbation theory or to investigate various nonperturbative effects [11]. Furthermore, recently in the series of papers [12] (see also Refs. [13, 14]) it was revealed a connection between quasi-exactly solvable models and supersymmetric systems with nonlinear polynomial superalgebras [15].

There exist various approaches to constructing quasi-exactly solvable systems [1, 4, 2, 16]. Nevertheless all of them are not universal in the sense that they do not cover all possible quasi-exactly solvable systems. For example, the famous Lie-algebraic approach [1, 2] is used to construct quasi-exactly solvable differential operators, but does not allow, for example, to reproduce quasi-exactly solvable systems based on hidden dynamical symmetries with nonlinear algebras [7, 17, 14]. In Ref. [18] authors presented a general construction for quasi-exactly solvable differential operators, linear and nonlinear. But it cannot be directly applied to quasi-exactly solvable noncommutative systems.

Considered in this paper approach is universal because it is formulated in terms of algebraic relations and does not depend on any space of representation. Besides we show that this scheme reflects a general structure of quasi-exactly solvable systems and and their connection with constrained systems. Therefore particularly it can be applied to construct both usual and noncommutative quasi-exactly solvable systems.

The paper is organized as follows. In section 2 we formulate a universal algebraic approach to quasi-exactly solvable systems and demonstrate its connection with constrained systems. In section 3 the algebraic approach is used to reproduce a family of two-dimensional quasi-exactly solvable Hamiltonians, which in the Lie-algebraic approach are derived from a finite-dimensional representation of the Lie algebra $\mathfrak{su}(3)$. Brief discussion of results is presented in section 4.

2 Formulation of the algebraic approach

Let us consider a set of linear operators, $A_k$ with $k = 1, \ldots, n$ and $n \in \mathbb{N}$, on a Hilbert space $\mathcal{H}$ and suppose that they have the following commutation relations:

$$[A_k, A_l] = \sum_{m=1}^{n} F_{klm} A_m, \quad (1)$$

where $F_{klm}$ are, in general, some linear operators on $\mathcal{H}$. The subspace $\mathcal{F}_A = \bigcap_{k=0}^{n} \ker A_k$ is the annihilator [18] for the set of annihilating operators $A_k$. Here we imply that the set $A_k$ is complete, i.e. any linear operator $B$ on $\mathcal{H}$ with $\ker B \subset \mathcal{F}_A$ can be represented as $B = \sum_{k=1}^{n} C_k A_k$, where $C_k$ are some linear operators on $\mathcal{H}$.

Thus from (1) it follows that the annihilating operators $A_k$ generally form nonlinear
algebra. To come to quasi-exactly solvable systems we have to require

$$\dim \mathcal{F}_A < \infty,$$

i.e. $\mathcal{F}_A$ is a finite dimensional subspace of $\mathcal{H}$. If a Hermitian linear operator $H$ on $\mathcal{H}$ has commutation relations

$$[A_k, H] = \sum_{l=1}^n M_{kl} A_l,$$

where $M_{kl}$ are, in general, some linear operators on $\mathcal{H}$, then $H$ is a quasi-exactly solvable operator. Indeed, the relations (3) imply that $H$ is an invariant operator on finite dimensional space $\mathcal{F}_A$ and can be diagonalized on this space by a finite procedure.

It is worth noting that the annihilating operators $A_k$ can be treated as operator constraints while $\mathcal{F}_A$ can be interpreted as a subspace of “physical” states. In this context the commutation relations (3) can be represented as

$$[A_k, H] \approx 0$$

or even as $A_k H \approx 0$.

Thus, a quasi-exactly solvable system with the Hamiltonian $H$ obeying the commutation relations (4) can be interpreted as a constrained Hamiltonian system with the finite dimensional subspace of “physical” states $\mathcal{F}_A$.

For $n = 1$ the commutation relations (1) and (3) lead to the simplest superalgebra. Indeed, the relation $[A, H] \approx 0$ can be represented as

$$[A, H] = LA,$$

where $L$ is a linear operator. This relation can be rewritten in the matrix form

$$[Q, H] = 0,$$

where $Q$ and $H$ are matrix supercharge and superhamiltonian:

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H & 0 \\ 0 & L + H \end{pmatrix}.$$  

In one-dimensional case, $\mathcal{H} = \mathcal{C}^\omega(\mathbb{R}^1)$, we have

$$\{Q, Q^\dagger\} = P(H),$$

where the order of the polynomial $P(.)$ is equal to the order of the annihilating operator $A$ [12]. Thus we came to the supersymmetry with a nonlinear polynomial superalgebra. Various systems with such a nonlinear supersymmetry and their relation to quasi-exactly solvable systems were extensively studied [12].

In multi-dimensional case, $\mathcal{H} = \mathcal{C}^\omega(\mathbb{R}^d)$ with $d > 1$, the polynomial $P(.)$ has more involved structure and can depend on some other operators. For example, the two-dimensional systems with the nonlinear polynomial supersymmetry were considered in Ref. [19].

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1Here a linear operator $B \approx 0$ if $By = 0 \ \forall y \in \mathcal{F}_A$.  

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The algebraic approach (1)-(4) can be used for constructing quasi-exactly solvable Hamiltonians. Indeed, if we know the set of annihilating operators with finite-dimensional annihilator, the corresponding quasi-exactly solvable Hamiltonian can be constructed by solving the commutation relations (3).

Let $\mathcal{F}_A$ is a finite-dimensional functional space with a basis of linearly independent analytical functions. Such a basis can be used to construct the corresponding set of annihilating operators [18]. Moreover, in Ref. [18] the general form of a quasi-exactly solvable operator was derived. However, in practice such a calculation of quasi-exactly solvable Hamiltonians can be more complicated than that based on the proposed algebraic approach. Besides, the algebraic approach is more universal because its formulation is not restricted by specific realization of the Hilbert space. For example, it can be applied to noncommutative spaces.

It is worth noting that any quasi-exactly solvable system admits formulation in the algebraic form (1)-(4). Indeed, by definition in any quasi-exactly solvable system there exits a finite-dimensional subspace, say $\mathcal{F}$, which is invariant for Hamiltonian of the system. For the finite-dimensional subspace $\mathcal{F}$ it is always possible to construct a complete set operators $A_k$ annihilating this subspace, i.e. $A_k y = 0 \forall y \in \mathcal{F}$ or $A_k \approx 0$. Since the Hamiltonian is invariant operator on $\mathcal{F}$ we conclude that $A_k H \approx 0$ because $H y \in \mathcal{F} \forall y \in \mathcal{F}$. This is equivalent to (4).

Thus the general scheme of building a quasi-exactly solvable model is the following: (1) We choose a finite set of independent functions, an annihilator. (2) The corresponding complete set of annihilating operators has to be constructed. (3) We calculate the quasi-exactly solvable operator of second order using the relations (1)-(4). Existence of such an operator depends on the set of independent functions. Besides, for this operator to be a Hamiltonian it has to obey to well-known conditions [2, 6, 20].

3 Application to 2D quasi-exactly solvable systems

In Ref. [14] the algebraic approach, discussed in the last section, was applied to the annihilator $\mathcal{F}_n = \{x^k y^l : 0 \leq k \leq n, 0 \leq l \leq n\}$, where $x, y \in \mathbb{R}$ and $k, l \in \mathbb{Z}_+$, while $n \in \mathbb{N}$. The resulting Hamiltonian is equivalent to that derived from the Lie-algebraic approach with Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \otimes \mathfrak{sl}(2, \mathbb{R})$. In this section we construct the quasi-exactly solvable Hamiltonian for the following annihilator:

$$\mathcal{F}_n = \{x^k y^l : 0 \leq k + l \leq n\}. \quad (5)$$

where $x, y \in \mathbb{R}$ and $k, l \in \mathbb{Z}_+$, while $n \in \mathbb{N}$. For this annihilator the corresponding annihilating operators can be taken in the form

$$A_k = \partial_x^k \partial_y^{n-k+1} \quad (6)$$

with $k = 0, 1, \ldots, n + 1$.

Let us first construct general quasi-exactly solvable operator of first order,

$$L = L_1(x, y)\partial_x + L_2(x, y)\partial_y + L_0(x, y), \quad (7)$$
where $L_i(x, y)$ are real-valued analytical functions. The commutation relations

$$[A_k, L] \approx 0,$$

where $k = 0, 1, \ldots, n+1$, lead to the following set of differential equations:

$$(l + 1)(m + 1)L_0^{(m,l)}(x, y) + (m + 1)(n - k - l + 1)L_2^{(m,l+1)}(x, y)$$

$$+ (l + 1)(k - m)L_1^{(m+1,0)}(x, y) = 0,$$  \hspace{1cm} (8)

where $-1 \leq l \leq n - k + 1$, $-1 \leq m \leq k$, $l + m \geq 1$ and $L^{(i,j)}(x, y) = \partial_x^i \partial_y^j L(x, y)$. This overdetermined system of differential equations can be reduced to the following form:

$$L_1^{(0,2)}(x, y) = 0, \quad L_1^{(3,0)}(x, y) = 0, \quad L_1^{(2,1)}(x, y) = 0,$$

$$L_2^{(0,2)}(x, y) - 2L_1^{(1,1)}(x, y) = 0, \quad L_0^{(0,1)}(x, y) + nL_1^{(1,1)}(x, y) = 0$$

plus the equations obtained by exchange $x \leftrightarrow y$. This system of differential equations has 9-parametrical solution:

$$L_1(x, y) = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_0,$$

$$L_2(x, y) = b_1 x + b_2 y + a_3 xy + a_4 y^2 + b_0,$$

$$L_0(x, y) = c_0 - n (a_3 x + a_4 y).$$

It corresponds to the set of quasi-exactly solvable differential operators

$$J = \{ x \partial_x, y \partial_x, \partial_x, \partial_y, x \partial_y, y \partial_y, x(x \partial_x + y \partial_y - n), y(x \partial_x + y \partial_y - n), \mathbb{I} \},$$  \hspace{1cm} (9)

which form a representation of the Lie algebra $u(3)$.

Now for the annihilator (5) we construct a general quasi-exactly solvable differential operator of second order,

$$H = H_{11}(x, y)\partial_x^2 + H_{12}(x, y)\partial_{xy}^2 + H_{22}(x, y)\partial_y^2 + H_1(x, y)\partial_x + H_2(x, y)\partial_y + H_0(x, y),$$

where $H_{ij}(x, y)$ and $H_i(x, y)$ are real-valued analytical functions. The commutation relations

$$[A_k, H] \approx 0,$$

where $k = 0, 1, \ldots, n+1$, lead to the following set of differential equations:

$$C_{l}^{m-k+1}C_{m}^{k}H_0^{(m,l)} + C_{l+1}^{m-k+1}C_{m}^{k}H_2^{(m,l+1)}$$

$$+ C_{l+2}^{m-k+1}C_{m}^{k}H_{22}^{(m,l+2)} + C_{l}^{m-k+1}C_{m+1}^{k}H_1^{(m+1,l)}$$

$$+ C_{l+1}^{m-k+1}C_{m+1}^{k}H_{12}^{(m+1,l+1)} + C_{l}^{m-k+1}C_{m+2}^{k}H_1^{(m+2,l)} = 0,$$

where $C_{m}^{k} = \frac{k!}{(k-m)!m!}$. This overdetermined system of differential equations can be reduced to the following equations:

$$H_{11}^{(0,3)} = 0, \quad H_{11}^{(4,1)} = 0, \quad H_{11}^{(5,0)} = 0, \quad 6H_{11}^{(4,2)} - H_{22}^{(0,4)} = 0,$$
shown that the resulting Hamiltonian is equivalent to the operator
\[ 3H_{11}^{(1,2)} - H_{12}^{(0,3)} = 0, \quad H_{1}^{(0,2)} + (n-1)H_{11}^{(1,2)} = 0, \]
\[ 3H_{1}^{(2,0)} - 6H_{22}^{(1,1)} - (n-1)\left(3H_{22}^{(1,2)} - H_{11}^{(3,0)}\right) = 0, \]
\[ 2H_{0}^{(0,1)} + n\left(2H_{1}^{(1,1)} + (n-1)H_{11}^{(2,1)}\right) = 0, \]
\[ 3H_{11}^{(2,1)} - 3H_{12}^{(1,2)} + H_{22}^{(0,3)} = 0 \]

plus the equations obtained by exchange \( x \leftrightarrow y \). This system of equations has the following 36-parametrical solution:

\[
H_{11} = \sum_{l=0}^{2} \sum_{k=0}^{4-l} a_{lk} x^l y^k, \\
H_{22} = \sum_{l=0}^{2} \sum_{k=0}^{3-l} b_{lk} x^l y^k + (a_{22} y^2 + a_{13} x y + a_{04} x^2) y^2, \\
H_{12} = \sum_{k=0}^{2} (2a_{k4-k} x y + a_{k3-k} y + b_{2-k} x_{k+1}) x^{2-k} y^k + \sum_{k=0}^{2} \sum_{l=0}^{2-k} c_{kl} x^k y^l, \\
H_1 = 2(1-n)a_{04} x^3 + ((1-n) (a_{03} - b_{12}) + f_{11}) x^2 + d_{01} x \\
+ (1-n) (a_{21} + 2x a_{22}) y^2 + (-2(n-1)a_{13} x^2 + d_{11} x + d_{10}) y + d_{00}, \\
H_2 = 2(1-n)a_{22} y^3 + ((1-n) (b_{03} - a_{12}) + d_{11}) y^2 + f_{01} y \\
+ (1-n)x^2 (2a_{04} y + b_{21}) + x (2(1-n)a_{13} y^2 + f_{11} y + f_{10}) + f_{00}, \\
H_0 = n(n-1)a_{04} x^2 - n ((n-1)b_{12} + f_{11}) x + n(n-1)a_{13} x y \\
+ n(n-1)a_{22} y^2 - n ((n-1)a_{12} + d_{11}) y + h_{00},
\]

where all the coefficients \( a_{lk}, b_{lk}, c_{kl}, d_{kl}, f_{kl}, h_{00} \) are real. By direct calculation it can be shown that the resulting Hamiltonian is equivalent to the operator

\[ H = \sum_{\alpha,\beta = 1}^{9} c_{\alpha \beta} J_{\alpha} J_{\beta}, \]

where \( c_{\alpha \beta} \in \mathbb{R} \), which corresponds to the Lie-algebraic Hamiltonian for the Lie algebra\(^2\) \( \text{su}(3) \) [2].

For the operator \( H \) to be a Hamiltonian its coefficient functions have to obey to additional constraints. The detail discussion of such constraints can be found in Refs. [2, 14].

### 4 Conclusion

In this paper we demonstrated that the algebraic approach to constructing quasi-exactly solvable systems, formulated in Ref. [14] can be reformulated in terms of constrained Hamiltonian systems. This underlines nontrivial relationship between such systems. Besides we

\(^2\)In the case of the algebra \( \text{su}(3) \) number of independent components of the matrix \( c_{\alpha \beta} \) is equal to 36.

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have shown that in the framework of this algebraic approach one can reproduce well-known
two-dimensional quasi-exactly solvable Hamiltonian corresponding to the Lie algebra \( su(3) \)
in the Lie-algebraic approach.

In contrast to the construction of quasi-exactly solvable differential operators, proposed
in Refs. [18, 2, 3, 16], considered in this paper approach is pure algebraic and not related
to specific realization of Hilbert space, where operators live. Therefore it is universal. For
example, it can be applied to constructing quasi-exactly solvable integral operators or Hamiltonians on noncommutative spaces where the other approaches do not work. We will consider
construction of such quasi-exactly solvable systems elsewhere.

Also we hope that the noted connection with constrained Hamiltonian systems will be
helpful for further development of the theory of quasi-exactly solvable systems.

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