Interconnection structure preservation design for a type of port-controlled hamiltonian systems—A parametric approach

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Abstract
It is a very important issue to minimise the change of the interconnect structure matrix when stabilising a port-controlled Hamiltonian (PCH) system. Toward this problem, a parametric design method is proposed in this paper for a type of PCH systems. First, concepts of H-damping-assignable and H-damping-assigning controller are introduced, and the stability of the closed-loop system resulting from this type of controller is discussed. Secondly, a necessary and sufficient condition for the system to be H-damping-assignable is given, and, when this condition is met, a parametric general expression of all the H-damping-assigning controllers is obtained. Thirdly, the free parameters are optimised to minimize the change of the interconnect structure matrix and also the norm of the control gain matrix. The parameter optimisation is formulated as a standard quadratic programming problem, to which analytical solutions are obtained. Finally, comparative simulations are carried out for a numerical example to verify the effect and the superiority of the proposed method.

1 | INTRODUCTION

Port-controlled Hamiltonian (PCH) systems, which were firstly proposed and studied in [1], capture the dynamic behaviour of many natural phenomena, and have been found applications in many fields, such as multimachine power systems control [2–5], robotics and system [6], micro-electromechanical systems control [6], multiple underwater vehicles control [7], and wound-rotor synchronous motors control [8]. The general form of PCH systems is

\[
\begin{align*}
\dot{x} &= (J(x) - R(x) \frac{\partial H}{\partial x}(x) + g(x))u, \\
y &= g^T(x) \frac{\partial H}{\partial x}(x),
\end{align*}
\]

(1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^r \) and \( y \in \mathbb{R}^r \) are the state, the control input and the output vectors of the system, respectively; \( J(x) = -f^T(x) \) and \( R(x) = R^T(x) \) are the interconnection matrix and damping matrix, respectively; and \( H(x) \) is a continuous differentiable Hamiltonian function. Various control problems have been investigated for this type of system. For stabilisation of PCH systems, the interconnection and damping assignment passivity-based control (IDA-PBC) method is proposed in [9–11], and has inspired many scholars to continue the research in this field, such as [12–14]. Reference [15] develops this method and proposes a new constructive IDA-PBC method for PCH systems. Their treatment does not rely upon the solution of any partial differential equations. The adaptive control problem of a class of time-varying PCH systems with uncertainties and input delay is investigated in [16]. Trajectory tracking control problem is considered in [17], to which an approach based on generalised canonical transformations is proposed. For the similar problem, a practical method of neural network adaptive tracking control and a modified energy-balancing-based control are proposed in [18] and [19], respectively. Many other control issues have also been discussed in depth, such as disturbance attenuation [3], simultaneous stabilisation [20], finite-time control [21] and output synchronisation [22], etc.

As a special case of the PCH system (1), a linear PCH system is proposed in [23], which takes the form of

\[
\begin{align*}
\dot{x} &= (J - R \frac{\partial H}{\partial x}(x) + Bu, \\
y &= B^T \frac{\partial H}{\partial x}(x),
\end{align*}
\]

(2)

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where \( J = - J^T \), \( R = R^T \) and \( B \) are constant coefficient matrices, and the Hamiltonian function \( H(x) \) is taken as

\[
H(x) = \frac{1}{2} x^T Q x, \tag{3}
\]

with \( Q \in \mathbb{R}^{n \times n} \) being a symmetric constant matrix. The system to be studied in this paper also takes the form of (2), but the Hamiltonian function \( H(x) \) is no longer required to be in any special forms, such as (3). In such a case, the investigated system essentially becomes a non-linear system due to the non-linearity of \( H(x) \), different from the linear PCH system in [23] which is a linear system in essential.

An open problem is proposed in [23] about how to fully characterize the minimal change in the interconnection structure matrix \( J \) which is necessary when stabilising a PCH system. It is argued in [1, 24–26] that the interconnection structure captures the topology of many physical systems, and in many cases, we want to remain as closely as possible to the original topology of the system; thus, how to minimise the change of the interconnection structure matrix \( J \) is a very critical issue. It is pointed out in [23] that this problem is “not addressed in the current paper” and is then raised as an open problem. Therefore, motivated by this issue, a parametric approach for interconnection structure preservation design is developed in this paper. The basic idea of the proposed method is to first establish a parametric expression of the control law, and then optimise the free parameters to meet other design requirements or to achieve better performance. Two design requirements are considered in this paper: the first one is to reduce the change of the interconnection matrix \( J \) as much as possible; the second one is to minimise the norm of the control gain matrix to obtain a smaller control input. The specific contributions of this paper are summarised as follows:

1. The concepts of H-damping-assignable and H-damping-assigning controller are introduced, and the stability of the closed-loop system resulted in by this type of controller is discussed.
2. A necessary and sufficient condition for the PCH system (2) to be H-damping-assignable is given. Further, when this condition holds, a parametric general expression for all the H-damping-assigning controllers is obtained.
3. Based on the parameterisation of the control law, the problem of interconnection structure preservation and minimum control gain design is transformed into an optimisation problem for free parameters, to which analytical optimal solutions are obtained.

The rest of this paper is organised as follows: In the next section, the statement of the problem to be solved is presented. In Section 3, a parametric expression of the control law is established. Then, in Section 4, the free parameters are optimised to achieve the two design requirements. Finally, a numerical example is investigated in Section 5 to verify the effect of the proposed method, followed by a brief conclusion.

Symbols used in this paper are shown in Table 1.

![Table 1](image.png)

## 2 | PROBLEM FORMULATION

Consider the following PCH system:

\[
\dot{x} = (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + B(x) u, \tag{4}
\]

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^r \) are the state and the control input of the system, respectively; \( J(x) \in \mathbb{R}^{n \times n} \) is a skew-symmetric matrix which determines the interconnection structure of the system, \( R(x) \in \mathbb{R}^{n \times n} \) is a symmetric matrix describing the dissipation, \( B(x) \in \mathbb{R}^{n \times r} \) is the port matrix, and \( H(x) \in \mathbb{R} \) is a positive definite Hamiltonian function.

Before giving a statement of the problem to be discussed, let us first present the following basic result regarding the stability of the system (4).

**Lemma 1 ([27]).** Suppose that a dynamic system is described by (4) with \( x = 0 \) being an equilibrium point when \( u = 0 \). Let the Hamiltonian function \( H(x) \) be positive definite. Then, the equilibrium point \( x = 0 \) is

1. **stable** if
   \[
   R(x) \geq 0, \ x \in \Omega \subseteq \mathbb{R}^n;
   \]
2. **asymptotically stable** if
   \[
   R(x) > 0, \ x \in \Omega \subseteq \mathbb{R}^n;
   \]
3. **globally asymptotically stable** if
   \[
   R(x) > 0, \ x \in \mathbb{R}^n,
   \]
   and \( H(x) \) is radially unbounded.

The system investigated in this paper is a special case of the above system (4), which takes the form of

\[
\dot{x} = (J - R) \frac{\partial H}{\partial x}(x) + B u, \tag{5}
\]
where $J$, $R \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$ are constant coefficient matrices, with $J$ and $R$ being still skew-symmetric and symmetric, respectively; $H(x) : \mathbb{R}^n \to \mathbb{R}$ is a Hamiltonian function. Following [28], the following assumptions are introduced:

**Assumption 1.** $H(x)$ is positive definite.

**Assumption 2.** $\frac{\partial H}{\partial x}(0) = 0$.

In addition, we also impose the following assumption in this paper.

**Assumption 3.** $\text{rank}(B) = r$, $r < n$.

**Remark 1.** The above Assumption 3 is not a strong restriction. The requirement of full-column-rank $B$ is based on the fact that different control channels are usually assumed to be independent of each other. The reason for requiring $r < n$ is that any known non-linearity of the system can be completely cancelled out very easily if $r \geq n$, and in such a case control of the system is no longer a difficult problem.

The control law to be designed takes the form of

$$u = K \frac{\partial H}{\partial x}(x),$$

(6)

where $K \in \mathbb{R}^{n \times n}$ is a gain matrix to be determined. Then, the closed-loop system resulted in by (5) and (6) can be obtained by

$$\dot{x} = (J - R + BK) \frac{\partial H}{\partial x}(x)$$

$$= (J - R_d) \frac{\partial H}{\partial x}(x),$$

(7)

where

$$J_d = J + \frac{1}{2} \left( BK - K^T B^T \right),$$

(8)

$$R_d = R - \frac{1}{2} \left( BK + K^T B^T \right).$$

(9)

Inspired by the idea of damping assignment in [11], concepts of H-damping-assignable and H-damping-assigning controller are introduced.

**Definition 1.** A PCH system (5) is said to be H-damping-assignable for $R_d$ if and only if there exists a matrix $K$ such that

$$R - \frac{1}{2} \left( BK + K^T B^T \right) = R_d.$$  

(10)

In such a case, the control law (6) is called an H-damping-assigning controller for the system (5) with respect to $R_d$.

The following result can be directly obtained with the help of Lemma 1.

**Lemma 2.** Suppose that a PCH system (5) satisfies Assumptions 1–3. Let (6) be an H-damping-assigning controller for the system (5) with respect to $R_d$. Then, under the control law (6), the origin $x = 0$ is

1. a stable equilibrium point of the closed-loop system (7) if $R_d \geq 0$;
2. an asymptotically stable equilibrium point of the closed-loop system (7) if $R_d > 0$;
3. a globally asymptotically stable equilibrium point of the closed-loop system (7) if $R_d > 0$ and the Hamiltonian function $H(x)$ is radially unbounded.

Obviously, as long as the desired damping matrix $R_d$ is selected properly and the linearly independent columns of the matrix $B$ are sufficient, the gain matrix $K$ satisfying (10) is generally not unique. Therefore, a natural idea is whether we can obtain a general expression of the matrix $K$. If we can, some special solutions which meet additional design requirements or possess better performance can be selected from all the ones.

Regarding the selection criteria for the solutions $K$, two factors are considered. The first one is so-called preservation of interconnection structure. It is pointed out in [23] that it is of great importance and necessity to fully characterise the minimal change in the interconnection structure matrix $J$, which is raised as an “open problem.” This issue is treated in this paper by minimising the following index:

$$J_p(K) = \|J - F(K)\|_F^2,$$

(11)

which characterises the difference between the topology of the closed-loop and the open-loop systems.

The second one is to minimise the norm of the control gain matrix $K$, which leads to the following index:

$$J_a(K) = \|K\|_F^2.$$

(12)

In general, minimising the controller gain will help to obtain smaller control inputs.

For the above considerations, the problem to be solved in this paper can be stated as follows.

**Problem 1.** Suppose that a PCH system (5) satisfies Assumptions 1–3. Let $R_d \in \mathbb{R}^{n \times n}$ be a given symmetric matrix.

1. Give a sufficient and necessary condition for the system (5) to be H-damping-assignable for $R_d$. Further, when this condition is met, establish a complete parametric general expression for all the H-damping-assigning controllers for the system (5) with respect to $R_d$.
2. Based on the obtained general expression, find special ones from all the H-damping-assigning controllers such that the following index is minimised

$$J(K) = \alpha_j \gamma_j(K) + \alpha_d \gamma_d(K),$$

(13)
where \( \alpha_p, \alpha_u \geq 0 \) are proper weighting factors; \( \tilde{J}_p(K) \) and \( \tilde{J}_u(K) \) are given by (11) and (12), respectively.

**Remark 2.** Regarding the selection of the weighting factors \( \alpha_p \) and \( \alpha_u \), on one hand, they are determined based on the trade-off between the importance of \( \tilde{J}_p(K) \) and \( \tilde{J}_u(K) \) according to practical needs. On the other hand, the values of \( \alpha_p \tilde{J}_p(K) \) and \( \alpha_u \tilde{J}_u(K) \) are also needed to be roughly balanced to avoid one of them being so large that it overwhelms the other one.

### 3 | PARAMETERISATION OF CONTROL LAW

#### 3.1 | Technical lemmas

Consider the following matrix equation:

\[
A^T X \pm X^T A = B,
\]

where \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{m \times m} \) are known coefficient matrices, \( X \in \mathbb{R}^{n \times m} \) is a matrix to be determined.

Denote the generalised inverse of the matrix \( A \) by \( G \), which satisfies

\[
AGA = A.
\]

Then, a pair of projection operators \( P_1 \) and \( P_2 \) are introduced by

\[
P_1 = GA, \quad P_2 = AG.
\]

The following well-known properties of \( G \) are presented in [29].

**Lemma 3 ([29]).** Let \( A \in \mathbb{C}^{n \times m} \) be a given constant matrix, and \( G \) be its generalised inverse which satisfies (15). Then

1. \( AG = I_n \) if and only if \( \text{rank}(A) = n \);
2. \( GA = I_m \) if and only if \( \text{rank}(A) = m \).

The following result presented in [30] gives a general solution to the matrix Equation (14).

**Lemma 4 ([30]).** There exists a solution to the matrix equation (14) if and only if

\[
B^T = \pm B,
\]

and

\[
(I_m - P_1^T)B(I_m - P_1) = 0,
\]

hold.

In this case, the general solution of (14) can be given by

\[
X = \frac{1}{2}G^TBP_1 + G^T B(I_m - P_1)
\]

\[
+ \left(I_m - P_2^T\right)Y + P_2^T ZP_2 A,
\]

where \( Y \in \mathbb{R}^{m \times m} \) is an arbitrary parameter matrix, and \( Z \in \mathbb{R}^{n \times n} \) is a parameter matrix satisfying the following constraint:

\[
\left(P_2^T ZP_2\right)^T = \mp P_2^T ZP_2.
\]

The following lemma proves the equivalence between two sets, which is necessary for the following discussion.

**Lemma 5.** Let \( \Omega_1 \) and \( \Omega_2 \) be two sets given by

\[\Omega_1 = \{ X \in \mathbb{R}^{n \times n} \mid X = A - A^T, \ A \in \mathbb{R}^{n \times n} \}, \]

\[\Omega_2 = \{ X \in \mathbb{R}^{n \times n} \mid X^T = -X \}.
\]

Then, we have

\[\Omega_1 = \Omega_2.\]

**Proof.** On one hand, if \( X_a \in \Omega_1 \), then there exists a \( A_a \in \mathbb{R}^{n \times n} \) such that

\[X_a = A_a - A_a^T.\]

Obviously, \( X_a^T = -X_a \) holds, thus we know \( X_a \in \Omega_2 \).

On the other hand, if \( X_b \in \Omega_2 \), then the following relation holds

\[X_b^T = -X_b.\]

Thus, the matrix \( A_b \in \mathbb{R}^{n \times n} \) satisfying

\[X_b = A_b - A_b^T,\]

can by given by

\[A_b = \frac{1}{2}X_b.\]

Therefore, we have \( X_b \in \Omega_1 \). Then, the proof is completed.

#### 3.2 | Parameterisation of control law

Through some simple deduction, (10) can be rewritten in the form of

\[
BK + K^T B^T = 2(R - R_d),
\]

(16)
which is obviously the type of the matrix equation (14) with respect to $K$.

Similarly, let $G$ be the generalised inverse of the matrix $B^T$ satisfying
\[ B^T G B^T = B^T, \] (17)
and further let
\[ P_1 = G B^T, \quad P_2 = B^T G. \] (18)

Then, with Lemma 4, the following result can be obtained immediately.

\textbf{Theorem 1.} Suppose that a PCH system (5) satisfies Assumptions 1–3, and $R_d \in \mathbb{R}^{r \times r}$ is a given symmetric matrix. Let $G$ be the generalised inverse of the matrix $B^T$ satisfying (17), and $P_1$ and $P_2$ be given by (18).

Then, the PCH system (5) is $H$-damping-assignable for $R_d$ if and only if
\[ \left(I_n - P_1^T\right) \left(R - R_d\right) \left(I_n - P_1\right) = 0, \] (19)

When the condition (19) holds, all the $H$-damping-assigning controllers for the system (5) with respect to $R_d$ can be given by (6), where
\[ K = G^T \left(R - R_d\right) P_1 + 2G^T \left(R - R_d\right) \left(I_n - P_1\right) + \left(Y - Y^T\right) B^T, \] (20)
with $Y \in \mathbb{R}^{r \times r}$ being a free parameter matrix.

\textbf{Proof.} It follows from Lemma 4 that the matrix Equation (16) has a solution if and only if
\[ 2\left(R - R_d\right)^T = 2\left(R - R_d\right), \]
and (19) holds. Since $R$ and $R_d$ are both symmetric, and considering that the matrix equation (16) is equivalent to (10), it is obvious that there exists a matrix $K$ satisfying (10) if and only if the condition (19) is satisfied.

In view of Assumption 3, it follows from Lemma 3 that
\[ P_2 = B^T \frac{1}{2} = I_n. \]

Then, according to Lemma 4, the general expression of $K$ can be given by
\[ K = G^T \left(R - R_d\right) P_1 + 2G^T \left(R - R_d\right) \left(I_n - P_1\right) + ZB^T, \]
when the condition (19) holds, with a parameter matrix $Z \in \mathbb{R}^{r \times r}$ satisfying the following constraint:
\[ Z^T = -Z. \] (21)

Introduce a new parameter matrix $Y \in \mathbb{R}^{r \times r}$ and replace $Z$ with $Y - Y^T$, then the constraint (21) will always be met for any $Y$. In addition, it is noted that this replacement does not lose any degrees of freedom according to Lemma 5. Therefore, the general expression (20) can be finally obtained with an unconstrained parameter matrix $Y \in \mathbb{R}^{r \times r}$. Then the proof is completed. \qed

\section{INTERCONNECTION STRUCTURE PRESERVING AND CONTROLLER GAIN MINIMISING DESIGN}

With the parametric general expression of $K$ obtained by Theorem 1, Problem 1.2 can be completely transformed into the following optimisation problem:

\begin{align*}
\min_{K,G,R_d,Y} & \quad J(K(G,R_d,Y)), \\
\text{s.t.} & \quad Y \in \mathbb{R}^{r \times r}; \quad G \text{ and } R_d \text{ satisfy (17) and (19)},
\end{align*} (22)

The above optimisation (22) is a non-linear programming problem, to which there is usually no analytical solutions. It can be only solved by using some numerical optimisation algorithms, such as genetic algorithms, which generally need very large computational load in practical applications, and only local sub-optimal solutions can be obtained instead of global optimal ones.

However, if the matrices $G$ and $R_d$ are predetermined, the problem (22) can be re-formulated as

\begin{align*}
\min_{Y} & \quad J(K(Y)), \\
\text{s.t.} & \quad Y \in \mathbb{R}^{r \times r},
\end{align*} (23)

In such a case, we will prove in this section that the above (23) becomes a standard quadratic programming problem, to which analytical optimal solutions can be obtained.

Before giving our results, let us introduce some notations. First, we denote an $n \times m$-dimensional permutation matrix by $\mathcal{K}_{mn}$, which is defined by
\[ \mathcal{K}_{mn} = \sum_{j=1}^{n} \left( e_j^T \otimes I_n \otimes e_j \right), \] (24)
where $e_j \in \mathbb{R}^{r \times 1}, j = 1, 2, \ldots, n$ represent vectors with the $j$-th element being 1, and the remaining elements being 0. Secondly, we define $\Gamma$, $\zeta$ and $b$ as
\[ \Gamma = \left(B \otimes I_n\right) \left(I_n - \mathcal{K}_{rr}\right), \]
\[ \zeta = \text{vec}(Y), \quad b = \text{vec}(\Theta), \] (25)
where
\[ \Theta = G^T \left(R - R_d\right) P_1 + 2G^T \left(R - R_d\right) \left(I_n - P_1\right). \] (26)
Thirdly, $M_p$, $M_u$, $\eta_p$, $\eta_u$, $\sigma_p$ and $\sigma_u$ are introduced by

$$M_p = M_{p1} - M_{p2}, \quad M_u = \Gamma^T \Gamma,$$

$$\eta_p^T = \eta_{p1}^T - \eta_{p2}^T, \quad \eta_u^T = 2b^T \Gamma,$$

$$\sigma_p = \sigma_{p1} - \sigma_{p2}, \quad \sigma_u = b^T b,$$

where $M_{pi}$, $\eta_{pi}$, $\sigma_{pi}$, $i = 1, 2$ are given by

$$M_{p1} = \Gamma^T \left( L_p \otimes (B^T B) \right) \Gamma,$$

$$M_{p2} = \Gamma^T K_u \left( B^T \otimes B \right) \Gamma,$$

$$\eta_{p1}^T = 2b^T \left( I_u \otimes (B^T B) \right) \Gamma,$$

$$\eta_{p2}^T = 2b^T K_u \left( B^T \otimes B \right) \Gamma,$$

$$\sigma_{p1} = b^T \left( I_u \otimes (B^T B) \right) b,$$

$$\sigma_{p2} = b^T K_u \left( B^T \otimes B \right) b.$$

Finally, $M$, $\eta$, $\sigma$ are defined by

$$M = \frac{1}{2} \alpha_p M_p + \alpha_u M_u,$$

$$\eta^T = \frac{1}{2} \alpha_p \eta_p^T + \alpha_u \eta_u^T,$$

$$\sigma = \frac{1}{2} \alpha_p \sigma_p + \alpha_u \sigma_u.$$

With the above preparations, the following result can be obtained.

**Theorem 2.** Suppose that $K(Y) : \mathbb{R}^{nxr} \rightarrow \mathbb{R}^{nxr}$ is a matrix function given by (20) with constant coefficients $R$, $R_d$, $P$, $G$ and $B$, and $\tilde{J}(K)$ is the index given by (13). If the matrix $M$ given by (29) is positive semi-definite, that is,

$$M \succeq 0,$$

then the optimal solution $Y_*$ to the optimisation problem (23) can be given by

$$Y_* = \text{unvec}_{x_r} \left( -\frac{1}{2} M^T \eta + (I_{2} \otimes M^T M) \xi \right),$$

where $\xi \in \mathbb{R}^{r^2}$ is a parameter vector.

**Proof.** In view of (8), (11) and (12), the index $\tilde{J}(K)$ given by (13) can be rewritten in the form of

$$\tilde{J}(K) = \alpha_p \|J - J_1\|_{F}^2 + \alpha_u \|K\|_{F}^2$$

$$= \alpha_p \left\| \frac{1}{2} (B K - K^T B^T) \right\|_{F}^2 + \alpha_u \|K\|_{F}^2$$

$$= \frac{1}{4} \alpha_p \|B K - K^T B^T\|_{F}^2 + \alpha_u \|K\|_{F}^2$$

$$= \frac{1}{4} \alpha_p J_1 + \alpha_u J_2,$$

where

$$J_1 = \|B K - K^T B^T\|_{F}^2,$$

$$J_2 = \|K\|_{F}^2.$$

It follows from (20) that the matrix $K$ can be vectorised as

$$\text{vec}(K) = \text{vec}(YB^T) - \text{vec}(Y^T B^T) + \text{vec}(\Theta)$$

$$= (B \otimes I_r) \left( \text{vec}(Y) - \text{vec}(Y^T) \right) + \text{vec}(\Theta)$$

$$= (B \otimes I_r) \left( \text{vec}(Y) - K_{p_r} \text{vec}(Y) \right) + \text{vec}(\Theta)$$

$$= \Gamma \zeta + b,$$

where $\Gamma$, $\zeta$ and $b$ are given by (25).

Consider $J_1$ at first. Through some simple deduction, (33) can be rewritten in the form of

$$J_1 = \text{tr} \left( \left( B K - K^T B^T \right)^T \left( B K - K^T B^T \right) \right)$$

$$= \text{tr} \left( K^T B K - K^T B^T - B^T B K \right)$$

$$= -2 \text{tr} \left( K^T B^T B K \right)$$

Using the relation (35), we have

$$\text{tr} \left( K^T B^T B K \right)$$

$$= \left( \text{vec}(K) \right)^T \left( I_r \otimes (B^T B) \right) \text{vec}(K)$$

$$= \left( \Gamma \zeta + b \right)^T \left( I_r \otimes (B^T B) \right) \left( \Gamma \zeta + b \right)$$

$$= \xi^T \Gamma^T \left( I_r \otimes (B^T B) \right) \Gamma \zeta$$

$$+ \xi^T \Gamma^T \left( I_r \otimes (B^T B) \right) b$$

$$+ b^T \left( I_r \otimes (B^T B) \right) \xi + b^T \left( I_r \otimes (B^T B) \right) b$$

$$= \xi^T \Gamma^T \left( I_r \otimes (B^T B) \right) \Gamma \zeta.$$
\[ + (\Gamma^T (I_n \otimes (B^T B)) b)^T \xi \]
\[ + b^T (I_n \otimes (B^T B)) \Gamma \xi + b^T (I_n \otimes (B^T B)) b \]
\[ = \xi^T M_{p1} \xi + \eta_{p1}^T \xi + \sigma_{p1}, \quad (37) \]

and

\[ \text{tr}(BKBK) \]
\[ = (\text{vec}(K^T))^T (B^T \otimes B) \text{vec}(K) \]
\[ = (\text{vec}(K))^T \mathbf{K}_{pr} (B^T \otimes B) \text{vec}(K) \]
\[ = (\Gamma \xi + b)^T \mathbf{K}_{pr} (B^T \otimes B) (\Gamma \xi + b) \]
\[ = \xi^T \Gamma^T \mathbf{K}_{pr} (B^T \otimes B) \Gamma \xi + \xi^T \Gamma^T \mathbf{K}_{pr} (B^T \otimes B) b \]
\[ + b^T \mathbf{K}_{pr} (B^T \otimes B) \Gamma \xi + b^T \mathbf{K}_{pr} (B^T \otimes B) b \]
\[ = \xi^T \Gamma_{\xi} \mathbf{K}_{pr} (B^T \otimes B) \Gamma \xi + \xi^T \Gamma_{\xi} \mathbf{K}_{pr} (B^T \otimes B) b \]
\[ + (\Gamma^T \mathbf{K}_{pr} (B^T \otimes B) b)^T \xi \]
\[ + b^T \mathbf{K}_{pr} (B^T \otimes B) \Gamma \xi + b^T \mathbf{K}_{pr} (B^T \otimes B) b \]
\[ = \xi^T M_{p2} \xi + \eta_{p2}^T \xi + \sigma_{p2}, \quad (38) \]

where \( M_{p1}, \eta_{p1}, \sigma_{p1}, i = 1, 2 \) are given by (28). Substituting (37) and (38) into (36), yields

\[ J_1 = 2 \left( \xi^T M_{p1} \xi + \eta_{p1}^T \xi + \sigma_{p1} \right), \quad (39) \]

where \( M_{p1}, \eta_{p1}, \sigma_{p1} \) are given by (27).

Next consider the index \( J_2 \). Similarly, using the relation (35), we can transform the index (34) into

\[ J_2 = \text{tr}(K^T K) \]
\[ = (\text{vec}(K))^T \text{vec}(K) \]
\[ = (\Gamma \xi + b)^T \xi + b \]
\[ = \xi^T \Gamma_{\xi} \mathbf{K}_{pr} (B^T \otimes B) \Gamma \xi + \xi^T \Gamma_{\xi} \mathbf{K}_{pr} (B^T \otimes B) b \]
\[ + b^T \mathbf{K}_{pr} (B^T \otimes B) \Gamma \xi + b^T \mathbf{K}_{pr} (B^T \otimes B) b \]
\[ = \xi^T M_{\xi} \xi + \eta_{\xi}^T \xi + \sigma_{\xi}, \quad (40) \]

where \( M_{\xi}, \eta_{\xi}, \sigma_{\xi} \) are given by (27).

Finally, with the help of (39) and (40), the index \( J(K) \) given in (32) can be further transformed into

\[ J(K(\xi)) = \frac{1}{2} \alpha_{\xi} \left( \xi^T M_{\xi} \xi + \eta_{\xi}^T \xi + \sigma_{\xi} \right) \]
\[ + \alpha_{\eta} \xi^T M_{\eta} \xi + \eta_{\eta}^T \xi + \sigma_{\eta} \]
\[ = \xi^T M \xi + \eta^T \xi + \sigma, \quad (41) \]

where \( M, \eta, \sigma \) are given by (29).

Let

\[ \frac{\partial J(K(\xi))}{\partial \xi} = 2M\xi + \eta = 0, \quad (42) \]

then we can obtain the following least squares solution \( \xi_* \) to the above linear equation (42) with respect to \( \xi \):

\[ \xi_* = -\frac{1}{2} M^T \eta + (I_\omega - M^T M) \xi. \]

Since \( M \) is positive semi-definite, we have

\[ \frac{\partial J(K(\xi))}{\partial \xi \partial \xi^T} = 2M \geq 0. \]

Thus \( \xi_* \) is a minimum point of (41). Then, it follows from the relationship between \( \xi \) and \( Y \) described in (25) that the matrix \( Y_* \) given in (31) is the optimal solution of (23). Then the proof is completed. \( \square \)

Remark 3. It is seen from the general solution (31) that there also exists remaining degrees of freedom characterised by \( \xi \), which can be further utilised to meet other design requirements or to achieve better performance. However, these degrees of freedom will vanish if the matrix \( M \) given by (29) is invertible.

With Theorem 1 and Theorem 2, the following result is immediately obtained.

Theorem 3. Suppose that a PCH system (5) satisfies Assumptions 1–3, and \( R_d \in \mathbb{R}^{d \times n} \) is a positive definite matrix satisfying (19). Assume that \( G \) is a generalised inverse of the matrix \( B^T \) satisfying (17), and \( P_1 \) is a matrix given by (18). Let \( J(K) \) be the index defined by (13). If the matrix \( M \) given by (29) satisfy

\[ M \geq 0, \]

then the gain matrix \( K^* \) which minimises the index \( J(K) \) can be given by

\[ K^* = G^T (R - R_d) P_1 \]
\[ + 2G^T (R - R_d) (Y_* - P_1) + (Y_* - Y_*^T) B^T, \]

where \( Y_* \) is the optimum parameter obtained by (31).
5 | A NUMERICAL EXAMPLE

5.1 | Controller design

Consider a non-linear system described by (43). Choose a Hamiltonian function as

\[ H(x) = x_1^2 (\sin x_1 + 2)^2 + (\arctan x_2)^2 + \frac{1}{2} (e^{u_2})^2 + (e^{u_4} - 1)^2, \]

then the system (43) can be expressed in the form of the PCH system (5) with coefficients

\[
J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{7}{8} & 0 \\ 0 & 0 & \frac{8}{5} \end{bmatrix}, \quad R = \begin{bmatrix} -\frac{2}{5} & 0 & 0 & \frac{1}{9} \\ 0 & 2 & \frac{3}{7} & 0 \\ 0 & \frac{3}{7} & -\frac{8}{15} & \frac{2}{3} \\ \frac{1}{9} & 0 & \frac{2}{3} & 0 \end{bmatrix}.
\]

It is obvious that Assumptions 1–3 are satisfied.

The generalised inverse of the matrix \( B^T \) satisfying (17) is chosen as

\[
G = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ -\frac{3}{10} & 1 & \frac{2}{7} \\ 0 & \frac{8}{7} & 0 \\ 0 & 0 & \frac{5}{6} \end{bmatrix},
\]

and the positive definite matrix \( R_d \) is selected as

\[
R_d = \begin{bmatrix} 2.4925 & 0.2 & 0.25 & 0 \\ 0.2 & 2 & 0 & -0.5 \\ 0.25 & 0 & 2.4925 & 0 \\ 0 & -0.5 & 0 & 2.7425 \end{bmatrix}.
\]

It is verified that the condition (19) holds and the matrix \( M \) given by (29) is positive semi-definite. Thus, it follows from Theorem 3 that, when the weight coefficients are determined as

\[
\alpha_p = 65, \quad \alpha_u = 1,
\]

the optimum parameter \( Y_\ast \) and the gain matrix \( K_\ast \) can be, respectively, obtained by

\[
Y_\ast = \begin{bmatrix} 0 & 0.0397030419 & 0.110380541 \\ -0.0397030419 & 0 & -0.322358502 \\ -0.110380541 & 0.322358502 & 0 \end{bmatrix},
\]

and

\[
K_\ast = [K_{1\ast} \quad K_{2\ast}],
\]

where

\[
K_{1\ast} = \begin{bmatrix} -0.964166667 & -0.133333333 \\ -0.283116211 & 0.979591837 \\ 0.173609967 & 1.59154943 \end{bmatrix},
\]

\[
K_{2\ast} = \begin{bmatrix} -0.0840911052 & 0.0377132877 \\ -2.99142857 & 0.76889164 \\ 1.05130644 & -4.36482431 \end{bmatrix}.
\]

For comparison, we also consider the LMI-based control method proposed in [28], which can be readily applied to the type of the system (43). The form of the control law in [28] is exactly same as the one in this paper. Therefore, in order to make an equal comparison, the ramping matrix of the PCH system (5), namely, \( R_r = R - \frac{1}{2} (BK + K^T B^T) \), is also required to be assigned to the same \( R_d \). To achieve this goal, it follows from [28] that the following LMI with respect to \( K \) needs to be solved:

\[
\begin{cases}
\dot{x}_1 = 3u_1 - \frac{2}{9} e^{u_4} (e^{u_4} - 1) - \frac{2}{9} \arctan x_2 \frac{x_1}{x_2^2 + 1} + \frac{4}{5} x_1 (\sin x_1 + 2)^2 + \frac{4}{5} x_1^2 (\cos x_1) (\sin x_1 + 2) \\
\dot{x}_2 = 2x_1 (\sin x_1 + 2)^2 - \frac{3}{7} x_1^2 e^{2u_3} - \frac{4}{7} \arctan x_2 \frac{x_1}{x_2^2 + 1} - \frac{3}{7} x_1 e^{2u_3} + 2x_1^2 (\cos x_1) (\sin x_1 + 2) \\
\dot{x}_3 = \frac{7}{8} \frac{x_1}{x_2^2 + 1} + 8 \arctan x_2 \frac{x_1}{x_2^2 + 1} + \frac{1}{8} x_1^2 e^{2u_3} + \frac{1}{8} x_1 e^{2u_3} \\
\dot{x}_4 = \frac{1}{5} \beta u_3 - \frac{8}{3} x_3^2 e^{2u_3} - \frac{2}{9} x_1 (\sin x_1 + 2)^2 - \frac{8}{3} x_1^2 e^{2u_3} - \frac{2}{9} x_1^3 (\cos x_1) (\sin x_1 + 2)
\end{cases}
\]
\[ BK + K^T B^T < 2(R - \tilde{R}), \]

where

\[
\tilde{R} = \begin{bmatrix}
1.25 & 0.2 & 0.25 & 0 \\
0.2 & 1.75 & 0 & -0.5 \\
0.25 & 0 & 1.25 & 0 \\
0 & -0.5 & 0 & 1.5
\end{bmatrix}.
\]

Then, a solution to the above LMI can be obtained by

\[
\hat{K} = \begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix},
\]

where

\[
\hat{K}_1 = \begin{bmatrix}
-0.964169753 & -0.13333333 \ 
-0.04800004 & 0.979591837 \\
0.014862151 & 1.59154943
\end{bmatrix},
\]

and

\[
\hat{K}_2 = \begin{bmatrix} -0.153599999 & 0.07096136 \\
-2.99143915 & 1.00539249 \\
0.721950545 & -4.36483905
\end{bmatrix}.
\]

### 5.2 Numerical simulation

Comparative simulation results obtained by the proposed method and the LMI-based method are shown in Figures 1 and 2, respectively, with simulation step 0.001 s and the initial value \( x_0 = [2 \ -1 \ -0.5 \ 1]^T \). In addition, in order to further verify the effect of preserving the interconnection structure, we also calculate the index values of the two methods, and compare them in Table 2.

| Method                  | \( J(K) \) | \( \delta J_p(K) \) | \( \delta J_u(K) \) |
|-------------------------|------------|---------------------|---------------------|
| Proposed method         | 95.8392    | 61.5836             | 34.2556             |
| LMI-based method        | 108.6051   | 74.6017             | 34.0034             |

It can be seen from the comparison of Figures 1 and 2 that the two methods are roughly at the same level in terms of state response performance and control input amplitude, while it is shown in Table 1 that the proposed method is considerably better than the LMI-based method in terms of interconnection structure preservation (corresponding to the index \( \alpha_p J_p(K) \)). This result has fully reflected the superiority of the proposed method, that is, it can preserve the interconnect structure of the open-loop system as much as possible without losing too much dynamic response performance.

### 6 CONCLUSION

This paper treats the open problem raised in [23], that is, how to minimise the change of the interconnection structure matrix \( J \) when stabilising a PCH system. Regarding this issue, a parametric design approach for a type of PCH systems is proposed. The basic idea is to solve this problem in two steps. The first step is to establish a complete parameter form of the controller which guarantees the closed-loop system with desired damping matrix. The second step is to optimise the parameters to minimise the change of the interconnection structure matrix, and also the norm of the control gain matrix. In the first step,
the control law is parameterised based on the general solutions to a type of matrix equations presented in [30]. In the second step, the optimisation problem is transformed into a standard quadratic programming problem by using the obtained parametric expression of the controller, to which analytical solutions are finally obtained.

In the following works, we will try to make $G$ and $R_y$ also participate in the optimisation as design degrees of freedom. In addition, how to generalise the proposed method to the case of state-dependent $f(x)$, $R(x)$ and $B(x)$ is also one of the future research directions.

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