Multivariate Bernoulli polynomials

Genki Shibukawa*

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Abstract

We introduce a multivariate analogue of Bernoulli polynomials and give their fundamental properties: difference and differential relations, symmetry, explicit formula, inversion formula, multiplication theorem, and binomial type formula. Further, we consider a multivariate analogue of the multiple Bernoulli polynomials and give their fundamental properties.

1 Introduction

The Bernoulli numbers $B_m$ are defined by the generating function

$$\frac{u}{e^u - 1} = \sum_{m=0}^{\infty} \frac{B_m u^m}{m!}, \quad |u| < 2\pi, \quad (1.1)$$

and the Bernoulli polynomials $B_m(z)$ by means of

$$\frac{u}{e^u - 1} e^{zu} = \sum_{m=0}^{\infty} \frac{B_m(z) u^m}{m!}, \quad |u| < 2\pi. \quad (1.2)$$

Bernoulli polynomial $B_m(z)$ has the following fundamental properties (see for example [E] 1.13).

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Let us describe proofs of (1.3) - (1.10).

(1.3) It follows from the definition of Bernoulli numbers (1.1) and polynomials (1.2).

(1.4) By the generating function of Bernoulli polynomials (1.2), the index law and the definition of exponential function $e^{zu}$, we have

$$\sum_{m \geq 0} (B_m(z + 1) - B_m(z)) \frac{u^m}{m!} = \frac{u}{e^u - 1} (e^{(z+1)u} - e^{zu})$$

$$= \frac{u}{e^u - 1} (e^u e^{zu} - e^{zu})$$

$$= \frac{u}{e^u - 1} e^{zu}$$

$$= \sum_{m \geq 0} z^m \frac{u^m}{m!}$$

$$= \sum_{m \geq 0} z^m \frac{u^{m+1}}{(m+1)!} (m+1)$$

$$= \sum_{m \geq 0} m z^{m-1} \frac{u^m}{m!}.$$
\[\sum_{m \geq 0} B_m'(z) \frac{u^m}{m!} = \frac{u}{e^u - 1} \partial_z e^{zu}\]
\[= \frac{u}{e^u - 1} e^{zu}u\]
\[= \sum_{m \geq 0} B_m(z) \frac{u^m}{m!}\]
\[= \sum_{m \geq 0} B_m(z) \frac{u^{m+1}}{(m+1)!} (m+1)\]
\[= \sum_{m \geq 0} mB_{m-1}(z) \frac{u^m}{m!}.
\]

\[\sum_{m \geq 0} B_m(1-z) \frac{u^m}{m!} = \frac{u}{e^u - 1} e^{(1-z)u}\]
\[= \frac{ue^u}{e^u - 1} e^{-zu}\]
\[= \frac{-u}{e^{-u} - 1} e^{-zu}\]
\[= \sum_{m \geq 0} B_m(z) \frac{(-u)^m}{m!}\]
\[= \sum_{m \geq 0} (-1)^m B_m(z) \frac{u^m}{m!}.
\]

\[\sum_{m \geq 0} B_m(z) \frac{u^m}{m!} = \frac{u}{e^u - 1} e^{zu}\]
\[= \sum_{N \geq 0} B_N \frac{u^N}{N!} \sum_{n \geq 0} z^n \frac{u^n}{n!}\]
\[= \sum_{N \geq 0} B_N \sum_{n \geq 0} z^n \frac{u^N}{N!} \frac{u^n}{n!}\]
\[= \sum_{m \geq 0} \sum_{n=0}^m B_{m-n} z^n \binom{m}{n} \frac{u^m}{m!}.
\]
\[\sum_{m \geq 0} z^m \frac{u^m}{m!} = e^{zu}\]

\[= \frac{e^u - 1}{u} \sum_{n \geq 0} B_n(z) \frac{u^n}{n!}\]

\[= \sum_{N \geq 0} \frac{1}{N+1} \frac{u^N}{N!} \sum_{n \geq 0} B_n(z) \frac{u^n}{n!}\]

\[= \sum_{N \geq 0} \sum_{n \geq 0} \frac{1}{N+1} B_n(z) \frac{u^N u^n}{N! n!}\]

\[= \sum_{N \geq 0} \sum_{n \geq 0} \frac{1}{N+1} B_n(z) \sum_{m \geq 0} \binom{m}{n} \frac{u^m}{m!}\]

\[= \sum_{m \geq 0} \sum_{n=0}^{m} \frac{1}{m-n+1} \binom{m}{n} B_n(z) \frac{u^m}{m!}.

\[\sum_{m \geq 0}^{N-1} B_m \left( z + \frac{i}{N} \right) \frac{u^m}{m!} = \sum_{i=0}^{N-1} \frac{u}{e^u - 1} e^{(z + \frac{i}{N})u}\]

\[= \frac{u}{e^u - 1} e^{zu} \sum_{i=0}^{N-1} e^{i \frac{u}{N}}\]

\[= \frac{u}{e^u - 1} e^{zu} e^{u \frac{u}{N}} - 1\]

\[= N \frac{u}{e^u - 1} e^{Nz \frac{u}{N}}\]

\[= N \sum_{m \geq 0} B_m(Nz) \frac{1}{m!} \left( \frac{u}{N} \right)^m\]

\[= \sum_{m \geq 0} N^{1-m} B_m(Nz) \frac{u^m}{m!}.

\[\sum_{i=0}^{N-1} \frac{u}{e^u - 1} e^{(z + \frac{i}{N})u}\]
\[
\sum_{m \geq 0} B_m(z) + 1 \frac{u^m}{m!} = \frac{u}{e^u - 1} e^{(z+1)u} \\
= e^u \frac{u}{e^u - 1} e^{zu} \\
= \sum_{n \geq 0} B_n(z) e^u \frac{u^n}{n!} \\
= \sum_{n \geq 0} B_n(z) \sum_{m \geq 0} \left( \begin{array}{c} m \\ n \end{array} \right) \frac{u^m}{m!} \\
= \sum_{m \geq 0} \sum_{n \geq 0} \left( \begin{array}{c} m \\ n \end{array} \right) B_n(z) \frac{u^m}{m!}.
\]

We remark that in the above proofs we only use the following formulas which are trivial results in the one variable case.

**Pieri type formulas** For any nonnegative integer \( m \in \mathbb{Z} \),

\[
e^u \frac{u^m}{m!} = \sum_{n \geq 0} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{u^n}{n!}.
\]

(1.11)

Since

\[
e^u = \sum_{N \geq 0} \frac{1}{N!} u^N
\]

and comparing the terms of degree \( N + m \) in (1.11), we have

\[
\frac{u^N u^m}{N! m!} = \left( \begin{array}{c} m + N \\ m \end{array} \right) \frac{u^{m+N}}{(m+N)!}.
\]

(1.12)

In particular, the \( N = 1 \) case of (1.12) is the following :

\[
u \frac{u^m}{m!} = \left( m + 1 \right) \frac{u^{m+1}}{(m+1)!} = (m+1) \frac{u^{m+1}}{(m+1)!}.
\]

(1.13)

**Properties of \( e^{zu} \)**

\[
\partial_z e^{zu} = e^{zu} u.
\]

(1.14)

In particular, we obtain the index law of \( e^{zu} \)

\[
e^{(z+1)u} = e^{\partial_z e^{zu}} = e^{zu} e^u.
\]

(1.15)

**Other formula (trivial!)** For any nonnegative integer \( N \in \mathbb{Z} \),

\[
u^N = N! \frac{u^N}{N!}.
\]

(1.16)
On the other hand, a multivariate analogue of the formulas (1.11) - (1.16) has been studied (see Section 2), which is non-trivial results unlike the one variable case. Therefore if we give a good multivariate analogue of Bernoulli polynomials which can be applied a multivariate analogue of (1.11) - (1.16), then we drive a multivariate analogue of (1.3) - (1.10).

In this article, we introduce a multivariate analogue of Bernoulli polynomials $B_m(z)$ by Jack polynomials and others, which we call “multivariate Bernoulli polynomials”. We also provide a multivariate analogue of (1.3) - (1.10) based on a multivariate analogue of (1.11) - (1.16). Further, we consider a multivariate analogue of the multiple Bernoulli polynomials and give their fundamental properties.

The content of this article is as follows. In Section 2, we introduce a multivariate analysis which is a natural generalization of special functions for matrix arguments. In particular, we explain a multivariate analogue of (1.11) - (1.16). Section 3 is the main part of this article. In this section, we introduce multivariate Bernoulli polynomials by a generating function which is a natural multivariate analogue of (1.2), and give their fundamental properties. We also investigate a multivariate analogue of the multiple Bernoulli polynomials which is a multiple analogue of our multivariate Bernoulli polynomials in Section 4.

2 Preliminaries

Refer to [Ka], [Ko], [L], [M], [S], [VK] for the details in this section. Let $r \in \mathbb{Z}_{\geq 1}$, $d \in \mathbb{C}$ and

$$P := \{m = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\},$$

$$\delta := (r-1, r-2, \ldots, 2, 1, 0) \in P,$$

$$|z| := z_1 + \cdots + z_r,$$

$$E_k(z) := \sum_{j=1}^{r} z_j^k \frac{\partial}{\partial z_j} \quad (k \in \mathbb{Z}_{\geq 0}),$$

$$D_k(z) := \sum_{j=1}^{r} z_j^k \frac{\partial^2}{\partial z_j^2} + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^k}{z_j - z_l} \frac{\partial}{\partial z_j} \quad (k \in \mathbb{Z}_{\geq 0}).$$

For any partition $m = (m_1, \ldots, m_r) \in P$ and $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$, put

$$m_m(z) := \sum_{n \in \mathcal{S}_r, m} z^n,$$

where $\mathcal{S}_r$ is the symmetric group in $r$ letters and $z^n := z_1^{m_1} \cdots z_r^{m_r}$. We define Jack polynomials $P_m(z; \frac{d}{2})$ by the following two conditions.

1. \[ D_2(z)P_m \left( z; \frac{d}{2} \right) = P_m \left( z; \frac{d}{2} \right) \sum_{j=1}^{r} m_j (m_j - 1 - d(r-j)) \]

2. \[ P_m \left( z; \frac{d}{2} \right) = m_m(z) + \sum_{k < m} c_{mk}m_k(z). \]
Here, \( < \) is the dominance partial ordering which is defined by
\[
\mathbf{k} < \mathbf{m} \iff k_l \leq m_l, \quad i = 1, \ldots, r.
\]

Similarly, the shifted (or interpolation) Jack polynomials \( P_{\mathbf{m}}^{ip}(z; \frac{d}{2}) \) are defined by the following two conditions.
\[
(1)^{ip} P_{\mathbf{k}}^{ip}\left(\mathbf{m} + \frac{d}{2}; \frac{d}{2}\right) = 0, \quad \text{unless } \mathbf{k} \subset \mathbf{m} \in \mathcal{P}
\]
\[
(2)^{ip} P_{\mathbf{m}}^{ip}(z; \frac{d}{2}) = P_{\mathbf{m}}(z; \frac{d}{2}) + \text{(lower terms)}.
\]

Further, we put
\[
\Phi_{\mathbf{m}}^{(d)}(z) := \frac{P_{\mathbf{m}}(z; \frac{d}{2})}{P_{\mathbf{m}}(1; \frac{d}{2})} \quad \text{(normalized Jack polynomials)},
\]
\[
\Psi_{\mathbf{m}}^{(d)}(z) := \frac{P_{\mathbf{m}}^{ip}(z; \frac{d}{2})}{P_{\mathbf{m}}^{ip}(\mathbf{m} + \frac{d}{2}; \frac{d}{2})}
\]
and
\[
\left(\begin{array}{c}
z \\ \mathbf{k}
\end{array}\right)^{(d)} := \frac{P_{\mathbf{k}}^{ip}(z + \frac{d}{2}; \frac{d}{2})}{P_{\mathbf{k}}^{ip}(\mathbf{k} + \frac{d}{2}; \frac{d}{2})} \quad \text{(generalized (or Jack) binomial coefficients)},
\]
\[
0 F_0^{(d)}(z, u) := \sum_{\mathbf{m} \in \mathcal{P}} \Psi_{\mathbf{m}}^{(d)}(z) \Phi_{\mathbf{m}}^{(d)}(u) = \sum_{\mathbf{m} \in \mathcal{P}} \Phi_{\mathbf{m}}^{(d)}(z) \Psi_{\mathbf{m}}^{(d)}(u).
\]

**Special values** From [M] VI (6.14), (10.20) and [Ko] (4.8), we have
\[
P_{\mathbf{m}}\left(1; \frac{d}{2}\right) = \prod_{(i,j) \in \mathbf{m}} \frac{j - 1 + \frac{d}{2}(r - i + 1)}{m_i - j + \frac{d}{2}(m'_j - i + 1)} = \prod_{1 \leq i < j \leq r} \frac{\left(\frac{d}{2}(j - i + 1)\right)_{m_i - m_j}}{\left(\frac{d}{2}(j - i)\right)_{m_i - m_j}}.
\]  
(2.1)

Further, by [Ko] (7.4) and (7.5)
\[
P_{\mathbf{m}}^{ip}\left(\mathbf{m} + \frac{d}{2}; \frac{d}{2}\right) = \prod_{(i,j) \in \mathbf{m}} \left( m_i - j + 1 + \frac{d}{2}(m'_j - i) \right)
\]
\[
= \prod_{j=1}^r \left( \frac{d}{2}(r - j) + 1 \right) \prod_{1 \leq i < j \leq r} \frac{\left(\frac{d}{2}(j - i - 1) + 1\right)_{m_i - m_j}}{\left(\frac{d}{2}(j - i) + 1\right)_{m_i - m_j}}.
\]  
(2.2)

Although these multivariate special functions are very complicated, we write down these functions explicitly in \( r = 1, r = 2 \) and \( d = 2 \).

**The \( r = 1 \) case** For non positive integer \( m \) and \( z \in \mathbb{C} \),
\[
P_{\mathbf{m}}(z; \frac{d}{2}) = z^m, \quad P_{\mathbf{m}}^{ip}(z; \frac{d}{2}) = \begin{cases} 
z(z-1) \cdots (z-m+1) & (m \neq 0) \\
1 & (m = 0) \end{cases}.
\]
Further,

\[ P_m \left( 1; \frac{d}{2} \right) = 1, \quad P_{ip}^m \left( m; \frac{d}{2} \right) = m!, \quad \Phi_{m}^{(d)}(z) = z^m, \quad \Psi_{m}^{(d)}(z) = \frac{z^m}{m!}, \]

\[ \binom{z}{(d)} \binom{k}{(d)} = \begin{cases} \frac{z(z-1) \ldots (z-k+1)}{k!} & (k \neq 0) \\ 1 & (k = 0) \end{cases}, \]

\[ 0F_0^{(d)}(z, u) = \sum_{m \geq 0} \frac{z^m}{m!} u^m = \sum_{m \geq 0} \frac{z^m u^m}{m!} = e^{zu}. \]

The \( r = 2 \) case (see [Ko] 10.3, [VK] 3.2.1) For any partition \( \mathbf{m} = (m_1, m_2) \in \mathcal{P} \) and \( \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \),

\[ P_m \left( \frac{d}{2}; z \right) = z_1^{m_1} z_2^{m_2} F_1 \left( \begin{array}{c} -m_1 + m_2, \frac{d}{2} \\ 1 - m_1 + m_2 - \frac{d}{2} \end{array} \middle| z_2 \right) \]

\[ P_{ip}^m \left( \frac{d}{2}; z \right) = (-1)^{m_1+m_2} (-z_1)_{m_2} (-z_2)_{m_1} F_2 \left( \begin{array}{c} -m_1 + m_2, \frac{d}{2}, -m_1 + 1 - \frac{d}{2} + z_1, 1 \\ 1 - m_1 + m_2 - \frac{d}{2}, -m_1 + 1 + z_2, 1 \end{array} \middle| \right). \]

Further,

\[ P_m \left( 1; \frac{d}{2} \right) = \frac{(d)_{m_1-m_2}}{(\frac{d}{2})_{m_1-m_2}}, \]

\[ P_{ip}^m \left( \mathbf{m}; \frac{d}{2} \right) = \frac{(\frac{d}{2} + 1)_{m_1} m_2! (m_1 - m_2)!}{(\frac{d}{2} + 1)_{m_1-m_2}}, \]

\[ \Phi_{m}^{(d)}(z) = \frac{(\frac{d}{2} - m_1 - m_2) z_1^{m_1} z_2^{m_2} F_1 \left( \begin{array}{c} -m_1 + m_2, \frac{d}{2}, \frac{z_2}{z_1} \\ 1 - m_1 + m_2 - \frac{d}{2} \end{array} \middle| \right)}{(d)_{m_1-m_2}}, \]

\[ \Psi_{m}^{(d)}(z) = \frac{(\frac{d}{2} + 1)_{m_1-m_2} z_1^{m_1} z_2^{m_2} F_1 \left( \begin{array}{c} -m_1 + m_2, \frac{d}{2}, \frac{z_2}{z_1} \\ 1 - m_1 + m_2 - \frac{d}{2} \end{array} \middle| \right)}{(\frac{d}{2} + 1)_{m_1} (m_1 - m_2)! m_2!}, \]

\[ \binom{z}{(d)} \binom{k}{(d)} = \frac{(\frac{d}{2} + 1)_{k_1-k_2}}{(\frac{d}{2} + 1)_{k_1} (k_1-k_2)! k_2!} (-1)^{k_1+k_2} \frac{(z_1 - \frac{d}{2})_{k_2}}{k_2} (-z_2)_{k_1}, \]

\[ 0F_0^{(d)}(z, u) = e^{z_1 u_1 + z_2 u_2} F_1 \left( \begin{array}{c} \frac{d}{2} \\ \frac{z_1}{z_2} - (z_1 - z_2)(u_1 - u_2) \end{array} \middle| \right). \]

The \( d = 2 \) case In this case, \( P_m(\mathbf{z}; 1) \) and \( P_{ip}^m(\mathbf{z}; 1) \) are Schur polynomials and shifted Schur polynomials respectively [OO].

\[ P_m(\mathbf{z}; 1) = s_m(\mathbf{z}) = \frac{\det \left( z_i^{m_j+r-j} \right)_{1 \leq i,j \leq r}}{\Delta(\mathbf{z})}, \]

\[ P_{ip}^m(\mathbf{z}; 1) = \frac{\det \left( P_{ip}^{m_j+r-j}(z_i + r - i; 1) \right)_{1 \leq i,j \leq r}}{\Delta(\mathbf{z})}. \]
where $\Delta(z) := \prod_{1 \leq i < j \leq r} (u_i - u_j)$. Further,

$$P_m(1; 1) = s_m(1) = \prod_{1 \leq i < j \leq r} \frac{(j - i)_{m_i - m_j}}{(j - i + 1)_{m_i - m_j}},$$

$$P^{ip}_m \left( m + \frac{d}{2}; 1 \right) = \prod_{j=1}^r (r - j + 1)_{m_j} \prod_{1 \leq i < j \leq r} \frac{(j - i)_{m_i - m_j}}{(j - i + 1)_{m_i - m_j}},$$

$$\Phi_m^{(2)}(z) = \prod_{1 \leq i < j \leq r} \frac{(j - i + 1)_{m_i - m_j}}{(j - i)_{m_i - m_j}} s_m(z),$$

$$\Psi_m^{(2)}(z) = \prod_{j=1}^r (r - j + 1)_{m_j} \prod_{1 \leq i < j \leq r} \frac{(j - i + 1)_{m_i - m_j}}{(j - i)_{m_i - m_j}} s_m(z),$$

$$\left( \frac{z}{k} \right)^{(2)} = \frac{1}{\Delta(z)} \det \left( \begin{array}{c} z_i + r - i \\ k_j + r - j \end{array} \right)_{1 \leq i, j \leq r}, \quad \mathcal{F}_0^{(d)}(z; u) = \frac{\det(e^{z_i u_j})_{1 \leq i, j \leq r}}{\Delta(z) \Delta(u)}.$$

**Remark 2.1.** We remark normalization of various Jack polynomials. First, we list some notations of Jack polynomials and their special values at $z = 1$ (see Table 1). In this article, our notations are based on [FK]. In particular,

$$\Psi_m^{(d)}(z) = d_m \frac{1}{\left( \frac{z}{r} \right)_m} \Phi_m^{(d)}(z),$$

where

$$n := r + \frac{d}{2} r (r - 1),$$

$$(\alpha)_m := \begin{cases} \alpha (\alpha + 1) \cdots (\alpha + m - 1) & (m \in \mathbb{Z}_{>0}) \\ 1 & (m = 0) \end{cases},$$

$$(\alpha)_m := \prod_{j=1}^r \left(\alpha - \frac{d}{2} (j - 1) \right)_{m_j},$$

$$d_m := \prod_{1 \leq i < j \leq r} \frac{m_i - m_j + \frac{d}{2} (j - i)}{\frac{d}{2} (j - 1)} \frac{(\frac{d}{2} (j - i + 1))_{m_i - m_j}}{(\frac{d}{2} (j - i + 1) + 1)_{m_i - m_j}} \quad ([FK], p315).$$

From special values of Jack polynomials $P_m(z; \frac{d}{2})$ and interpolation Jack polynomials $P^{ip}_k(z + \frac{d}{2}; \frac{d}{2})$ ([2.1] and [2.2]), we have

$$d_m \frac{1}{\left( \frac{z}{r} \right)_m} = \frac{P_m(1; \frac{d}{2})}{P^{ip}_m \left( m + \frac{d}{2}; \frac{d}{2} \right)}.$$

Next, we remark the relationship between Stanley style $J_m^{(\frac{d}{2})}(z)$ and Macdonald style $P_m(z; \frac{d}{2})$

$$J_m^{(\frac{d}{2})}(z) = \left( \frac{2}{d} \right)^{|m|} \prod_{(i,j) \in m} \left( m_i - j + \frac{d}{2} (m_j' - i + 1) \right) P_m \left( z; \frac{d}{2} \right) \quad ([M] VI (10.22))$$
where a partition \( \mathbf{m} \) is identified with its diagram:

\[
\mathbf{m} = \{ s = (i, j) \mid 1 \leq i \leq r, 1 \leq j \leq m_i \}.
\]

Hence we have

\[
\Psi^{(d)}_{\mathbf{m}}(z) = \frac{P_m(z; \frac{d}{2})}{P^{\text{up}}_m(\mathbf{m} + \frac{d}{2} \delta; \frac{d}{2})} = \left( \frac{d}{2} \right)^{|\mathbf{m}|} \prod_{(i,j) \in \mathbf{m}} \left( \frac{1}{m_i - j + \frac{d}{2}(m'_j - i + 1)} \right) \frac{1}{P^{\text{up}}_m(\mathbf{m} + \frac{d}{2} \delta; \frac{d}{2})} J^{(\frac{d}{2})}_{\mathbf{m}}(z).
\]

The relationship between \( C^{(\frac{d}{2})}_{\mathbf{m}}(1) \) (Kaneko style) and \( \Psi^{(d)}_{\mathbf{m}}(1) \) (our style)

\[
C^{(\frac{d}{2})}_{\mathbf{m}}(1) = |\mathbf{m}|! \prod_{(i,j) \in \mathbf{m}} \left( j - 1 + \frac{d}{2}(r - i + 1) \right) \left( m_i - j + \frac{d}{2}(m'_j - i + 1) \right) \left( m_i - j + 1 + \frac{d}{2}(m'_j - i) \right)
\]

follows from [2.1], [2.2] and [Ka] (18). Thus, we have

\[
\Psi^{(d)}_{\mathbf{m}}(z) = \frac{1}{|\mathbf{m}|!} C^{(\frac{d}{2})}_{\mathbf{m}}(z).
\]

To summarize the above results, we obtain

\[
\Psi^{(d)}_{\mathbf{m}}(z) = d_{\mathbf{m}} \frac{1}{(r)^{\mathbf{m}}} \Phi^{(d)}_{\mathbf{m}}(z) = \frac{1}{P^{\text{up}}_m(\mathbf{m} + \frac{d}{2} \delta; \frac{d}{2})} P_m(z; \frac{d}{2}) = \frac{1}{|\mathbf{m}|!} C^{(\frac{d}{2})}_{\mathbf{m}}(z). \tag{2.3}
\]

| notation | special value at \(z = 1\) |
|----------|-----------------------------|
| Faraut-Korányi | \( \Phi^{(d)}_{\mathbf{m}}(z) \) |
| Stanley | \( \left( \frac{d}{2} \right)^{|\mathbf{m}|} \prod_{(i,j) \in \mathbf{m}} \left( j - 1 + \frac{d}{2}(r - i + 1) \right) \) (S Thm. 5.4) |
| Macdonald | \( P_m(z; \frac{d}{2}) \prod_{(i,j) \in \mathbf{m}} \frac{j - 1 + \frac{d}{2}(r - i + 1)}{m_i - j + \frac{d}{2}(m'_j - i + 1)} \) (M VI (10.20)) |
| Kaneko | \( C^{(\frac{d}{2})}_{\mathbf{m}}(z) \) |
| S | \( \Psi^{(d)}_{\mathbf{m}}(z) \prod_{(i,j) \in \mathbf{m}} \frac{m_i - j + \frac{d}{2}(m'_j - i + 1) + \frac{d}{2}(m'_j - i)}{m_i - j + \frac{d}{2}(m'_j - i) + \frac{d}{2}(m'_j - i)} \) (Ka) (18) |

Table 1: Notations and normalizations of Jack polynomials

Under the following, we provide all necessary formulas to prove our main results.

**Pieri type formulas for Jack polynomials** For any partition \( \mathbf{m} \in \mathcal{P} \),

\[
e^{\mathbf{u}} \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) = \sum_{n \in \mathcal{P}} \left( \begin{array}{c} n \\ m \end{array} \right)^{(d)} \Psi^{(d)}_{n}(\mathbf{u}) \quad (S \text{ Section 14}). \tag{2.4}
\]
Since
\[ e^{[u]} = \sum_{N \geq 0} \frac{1}{N!} |u|^N \]
and comparing the terms of degree \( N + |m| \) in (2.4), we have
\[ \frac{|u|^N}{N!} \Psi_m^{(d)}(u) = \sum_{|n| = |m| = N, \ n \in \mathcal{P}} \binom{n}{m}^{(d)} \Psi_n^{(d)}(u). \]  
(2.5)

From [L] Section 14,
\[ \left( \binom{m^i}{m}^{(d)} \right) = \left( m_i + 1 + \frac{d}{2}(r - i) \right) h_{i,j}^{(d)}(m^i), \]
where \( \epsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r \), \( m^i := m + \epsilon_i \) and
\[ h_{i,j}^{(d)}(m) := \prod_{1 \leq k \neq i \leq r} \frac{m_i - m_k - \frac{d}{2}(i - k) \pm \frac{d}{2}(j - k)}{m_i - m_k - \frac{d}{2}(i - k)} \]
In particular, the \( N = 1 \) case of (2.5) is the following :
\[ |u| \Psi_m^{(d)}(u) = \sum_{1 \leq i \leq r, \ m^i \in \mathcal{P}} \binom{m^i}{m}^{(d)} \Psi_m^{(d)}(u) = \sum_{1 \leq i \leq r, \ m^i \in \mathcal{P}} \Psi_m^{(d)}(u) \left( m_i + 1 + \frac{d}{2}(r - i) \right) h_{i,i}^{(d)}(m^i). \]  
(2.6)

**Properties of \( 0 \mathcal{F}_0^{(d)} \)** By [L] Section 14, we have
\[ E_0(z)_0 \mathcal{F}_0^{(d)} \left( ; z, u \right) = 0 \mathcal{F}_0^{(d)} \left( ; z, u \right) |u|. \]  
(2.7)
In particular, we obtain the index law of \( 0 \mathcal{F}_0^{(d)} \left( ; z, u \right) \)
\[ 0 \mathcal{F}_0^{(d)} \left( ; 1 + z, u \right) = e^{E_0(z)} 0 \mathcal{F}_0^{(d)} \left( ; z, u \right) = 0 \mathcal{F}_0^{(d)} \left( ; z, u \right) e^{[u]}. \]  
(2.8)

**Other formula**
\[ |u|^N = N! \sum_{|m| = N, \ m \in \mathcal{P}} \Psi_m^{(d)}(u) \]  
([S] Prop. 2.3 or [Ka] (17)).  
(2.9)
To summarize the above results, we obtain the following dictionary.

\[ P_m \left( 1; \frac{d}{2} \right) = 1 \Rightarrow P_m \left( 1; \frac{d}{2} \right) = \prod_{1 \leq i < j \leq r} \left( \frac{d}{2} (j - i + 1) \right)_{m_i - m_j}, \]

\[ P^p_m \left( m; \frac{d}{2} \right) = m! \Rightarrow P^p_m \left( m + \frac{d}{2} \delta; \frac{d}{2} \right) = \prod_{j=1}^{r} \left( \frac{d}{2} (r - j) + 1 \right)_{m_j}, \]

\[ \Phi^{(d)}_m(z) := z^m \Rightarrow \Phi^{(d)}_m(z) := \frac{P_m(z; \frac{d}{2})}{P_m \left( 1; \frac{d}{2} \right)}, \]

\[ \Psi^{(d)}_m(z) := \frac{z^m}{m!} \Rightarrow \Psi^{(d)}_m(z) := \frac{P_m \left( 1; \frac{d}{2} \right) \Phi^{(d)}_m(z)}{P_m \left( m + \frac{d}{2} \delta; \frac{d}{2} \right)} = \frac{P_m(z; \frac{d}{2})}{P^p_m \left( m + \frac{d}{2} \delta; \frac{d}{2} \right)}, \]

\[ \binom{m}{k} := \frac{P^p_k \left( m; \frac{d}{2} \right)}{P^p_k \left( k; \frac{d}{2} \right)} \Rightarrow \binom{m}{k} \Rightarrow \binom{m}{k} := \frac{P^p_k \left( m + \frac{d}{2} \delta; \frac{d}{2} \right)}{P^p_k \left( k + \frac{d}{2} \delta; \frac{d}{2} \right)}, \]

\[ e^{zu} = \sum_{m=0}^{\infty} \frac{1}{m!} z^m u^m \Rightarrow 0F_0^{(d)}(z, u) := \sum_{m \in \mathcal{P}} \psi^{(d)}_k(z) \psi^{(d)}_k(u) \]

\[ e^{zu} \frac{u^{m}}{m!} = \sum_{n=0}^{\infty} \binom{n}{m} \frac{u^n}{n!} \Rightarrow e^{zu} \frac{u^{m}}{m!} = \sum_{n \in \mathcal{P}} \left( \binom{n}{m} \psi^{(d)}_m(u) \right), \]

\[ \frac{u^m}{m!} = \frac{u^{m+1}}{(m+1)!} (m+1) \Rightarrow \frac{u^m}{m!} = \sum_{1 \leq i \leq r, m_i \in \mathcal{P}} \psi^{(d)}_m(u) \left( m_i + 1 + \frac{d}{2} (r - i) \right) h_{-i,i}(m^i), \]

\[ \partial_z e^{zu} = e^{zu} u \Rightarrow E_0(z)_{0F_0^{(d)}}(z, u) = 0F_0^{(d)}(z, u) |u|, \]

\[ e^{(1+z)u} = e^u e^{zu} \Rightarrow 0F_0^{(d)}(1+z, u) = e^{u} 0F_0^{(d)}(z, u), \]

\[ u^N = N! \frac{u^N}{N!} \Rightarrow \frac{|u|^N}{N!} = \sum_{|m|=N, m_i \in \mathcal{P}} \psi^{(d)}_m(u), \]
3 Multivariate Bernoulli polynomials

We define multivariate Bernoulli polynomials $B^{(d)}_m(z)$ or $B^m\left(z; \frac{d}{2}\right)$ by the following generating function.

$$\frac{u}{e^u - 1} e^{zu} = \sum_{m=0}^{\infty} B_m(z) \Psi_m(u) \quad (|u| < 2\pi)$$

$$\downarrow$$

$$\frac{|u|}{e^{|u|} - 1} \mathcal{F}_0^{(d)}(z, u) = \sum_{m \in \mathcal{P}} B^{(d)}_m(z) \Psi^{(d)}_m(u) \quad (|u_1 + \cdots + u_r| < 2\pi). \quad (3.1)$$

**Remark 3.1.** Originally, we consider the following type generating function and multivariate analogue of Bernoulli polynomials.

$$\prod_{j=1}^{r} \frac{u_j}{e^{u_j} - 1} \mathcal{F}_0^{(d)}(z, u) = \sum_{m \in \mathcal{P}} \tilde{B}^{(d)}_m(z) \Psi^{(d)}_m(u)$$

In the $d = 2$ case, this type generating function has the determinant expression

$$\prod_{j=1}^{r} \frac{u_j}{e^{u_j} - 1} \mathcal{F}_0^{(2)}(z, u) = \frac{\det \left( \frac{u_i e^{u_j}}{e^{|u|} - 1} \right)_{1 \leq i, j \leq r}}{\Delta(z) \Delta(u)}$$

and $\tilde{B}^{(2)}_m(z)$ has the Jacobi-Trudi type formula

$$\tilde{B}^{(2)}_m(z) = \frac{\det \left( B_{m+r-i}(z_j) \right)}{\Delta(z)}.$$

However, for this multivariate analogue of Bernoulli polynomials, we can not find an analogue of the formulas (1.3) - (1.10). Therefore, we investigate the above type (3.1) multivariate Bernoulli polynomials.

**Theorem 3.2.** (1) Special value at $z = 0$

$$B^{(d)}_m(0) = B^m|_m, \quad (3.2)$$

(2) Difference equation

$$B^{(d)}_m(z + 1) - B^{(d)}_m(z) = \sum_{i=1}^{r} \Phi^{(d)}_{m_i}(z) \left( m_i + \frac{d}{2}(r - i) \right) h^{(d)}_{-i}(m). \quad (3.3)$$

(3) Differential equation

$$E_0(z) B^{(d)}_m(z) = \sum_{i=1}^{r} B^{(d)}_{m_i}(z) \left( m_i + \frac{d}{2}(r - i) \right) h^{(d)}_{-i}(m). \quad (3.4)$$
(4) Symmetry

\[ B_m^{(d)}(1 - z) = (-1)^{|m|} B_m^{(d)}(z) \]  \hspace{1cm} (3.5)

(5) Explicit formula

\[ B_m^{(d)}(z) = \sum_{n \subseteq m} B_{|m| - |n|}^{(d)} \binom{m}{n} \Phi_n^{(d)}(z). \]  \hspace{1cm} (3.6)

(6) Inversion formula

\[ \Phi_m^{(d)}(z) = \sum_{n \subseteq m} \frac{1}{|m| - |n| + 1} \binom{m}{n} B_n^{(d)}(z). \]  \hspace{1cm} (3.7)

(7) Multiplication formula

\[ \sum_{i=0}^{N-1} B_m^{(d)} \left( z + \frac{i}{N} \right) = N^{1-|m|} B_m^{(d)}(Nz) \]  \hspace{1cm} (3.8)

(8) Binomial formula

\[ B_m^{(d)}(z + 1) = \sum_{n \subseteq m} \binom{m}{n} B_n^{(d)}(z) \]  \hspace{1cm} (3.9)

**proof.** (1) By the definition of the multivariate Bernoulli polynomials and Bernoulli numbers, we have

\[ \sum_{m \in \mathcal{P}} B_m^{(d)}(0) \Psi_m^{(d)}(u) = |u| = \sum_{N=0}^{\infty} \frac{B_N}{N!} |u|^N. \]

On the other hand, by (2.9)

\[ \sum_{N=0}^{\infty} \frac{B_N}{N!} |u|^N = \sum_{N=0}^{\infty} B_N \sum_{|m|=N} \Psi_m^{(d)}(u) = \sum_{m \in \mathcal{P}} B_{|m|} \Psi_m^{(d)}(u). \]

(2) By (2.8) and (2.6), we have

\[ \sum_{m \in \mathcal{P}} \left( B_m^{(d)}(z + 1) - B_m^{(d)}(z) \right) \Psi_m^{(d)}(z) = \frac{|u|}{e^{|u|} - 1} \left( e^{|u|} \mathcal{F}_0^{(d)}(z + 1, u) - \mathcal{F}_0^{(d)}(z, u) \right) \]

\[ = \frac{|u|}{e^{|u|} - 1} \left( e^{|u|} \mathcal{F}_0^{(d)}(z, u) - \mathcal{F}_0^{(d)}(z, u) \right) \]

\[ = |u| \mathcal{F}_0^{(d)}(z, u) \]

\[ = \sum_{m \in \mathcal{P}} \Phi_m^{(d)}(z) |u| \Psi_m^{(d)}(u) \]

\[ = \sum_{m \in \mathcal{P}} \Phi_m^{(d)}(z) \sum_{i=1}^{r} \Psi_{m_i}^{(d)}(u) \left( m_i + 1 + \frac{d}{2}(r - i) \right) h_{r-i}^{(d)}(m^i) \]

\[ = \sum_{m \in \mathcal{P}} \sum_{i=1}^{r} \Phi_m^{(d)}(z) \left( m_i + \frac{d}{2}(r - i) \right) h_{r-i}^{(d)}(m) \Psi_m^{(d)}(u). \]
(3) By (2.7) and (2.6),
\[
\sum_{m \in \mathcal{P}} E_0(z) B_{m}^{(d)}(z) \Psi_{m}^{(d)}(u) = \frac{|u|}{e^{|u|} - 1} E_0(z) \mathcal{F}_0^{(d)}(z, u) \\
= \frac{|u|}{e^{|u|} - 1} 0 \mathcal{F}_0^{(d)}(z, u) |u| \\
= \sum_{m \in \mathcal{P}} B_{m}^{(d)}(z) |u| \Psi_{m}^{(d)}(u) \\
= \sum_{m \in \mathcal{P}} B_{m}^{(d)}(z) \sum_{i=1}^{r} \Psi_{m}^{(d)}(u) \left( m_i + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m) \\
= \sum_{m \in \mathcal{P}} \sum_{i=1}^{r} B_{m_i}^{(d)}(z) \left( m_i + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m) \Psi_{m}^{(d)}(u).
\]

(4) By (2.8),
\[
\sum_{m \in \mathcal{P}} B_{m}^{(d)}(1 - z) \Psi_{m}^{(d)}(u) = \frac{|u|}{e^{|u|} - 1} 0 \mathcal{F}_0^{(d)}(1 - z, u) \\
= \frac{|u|}{e^{|u|} - 1} 0 \mathcal{F}_0^{(d)}(z, -u) e^{|u|} \\
= \frac{|-u|}{e^{-|u|} - 1} 0 \mathcal{F}_0^{(d)}(z, -u) \\
= \sum_{m \in \mathcal{P}} B_{m}^{(d)}(z) \Psi_{m}^{(d)}(-u) \\
= \sum_{m \in \mathcal{P}} (-1)^{|m|} B_{m}^{(d)}(z) \Psi_{m}^{(d)}(u).
\]

(5) By (2.5),
\[
\sum_{m \in \mathcal{P}} B_{m}^{(d)}(z) \Psi_{m}^{(d)}(u) = \frac{|u|}{e^{|u|} - 1} 0 \mathcal{F}_0^{(d)}(z, u) \\
= \sum_{N=0}^{\infty} B_{N} \sum_{n \in \mathcal{P}} \frac{|u|^N}{N!} \Phi_{n}^{(d)}(z) \Psi_{n}^{(d)}(u) \\
= \sum_{N=0}^{\infty} B_{N} \sum_{n \in \mathcal{P}} \frac{|u|^N}{N!} \Psi_{n}^{(d)}(u) \\
= \sum_{N=0}^{\infty} B_{N} \sum_{n \in \mathcal{P}} \Phi_{n}^{(d)}(z) \sum_{|m| - |n| = N} \binom{m}{n} \Psi_{m}^{(d)}(u) \\
= \sum_{m \in \mathcal{P}} \sum_{n \in \mathcal{P}} \Phi_{n}^{(d)}(z) \sum_{|m| - |n| = N} \binom{m}{n} \Psi_{m}^{(d)}(u).
(6) By (2.5),

\[
\sum_{m \in P} \Phi_{m}^{(d)}(z) \Psi_{m}^{(d)}(u) = 0 \mathcal{F}_{0}^{(d)}(z, u)
\]

\[
= \frac{e^{|u|} - 1}{|u|} \sum_{n \in P} B_{n}^{(d)}(z) \Psi_{n}^{(d)}(u)
\]

\[
= \sum_{N=0}^{\infty} \frac{1}{N + 1} \frac{1}{N!} |u|^{N} \sum_{n \in P} B_{n}^{(d)}(z) \Psi_{n}^{(d)}(u)
\]

\[
= \sum_{N=0}^{\infty} \frac{1}{N + 1} \sum_{n \in P} B_{n}^{(d)}(z) \frac{|u|^{N}}{N!} \Psi_{n}^{(d)}(u)
\]

\[
= \sum_{N=0}^{\infty} \frac{1}{N + 1} \sum_{n \in P} B_{n}^{(d)}(z) \sum_{|m| - |n| = N} \binom{m}{n}^{(d)} \Psi_{m}^{(d)}(u)
\]

\[
= \sum_{m \in P} \sum_{n \in m} \frac{1}{|m| - |n| + 1} B_{n}^{(d)}(z) \binom{m}{n}^{(d)} \Psi_{m}^{(d)}(u).
\]

(7) By (2.8) and the summation of a geometric series,

\[
\sum_{m \in P}^{N-1} \sum_{i=0}^{N-1} B_{m}^{(d)} \left( z + \frac{i}{N} 1 \right) \Psi_{m}^{(d)}(u) = \sum_{i=0}^{N-1} \frac{|u|}{e^{|u|} - 1} \mathcal{F}_{0}^{(d)}(z + \frac{i}{N} 1, u)
\]

\[
= \frac{|u|}{e^{|u|} - 1} \mathcal{F}_{0}^{(d)}(z, u) \sum_{i=0}^{N-1} e^{\frac{i}{N}|u|}
\]

\[
= \frac{|u|}{e^{|u|} - 1} \mathcal{F}_{0}^{(d)}(z, u) \frac{e^{|u|} - 1}{e^{\frac{|u|}{N}} - 1}
\]

\[
= N \frac{|u|}{e^{|u|} - 1} \mathcal{F}_{0}^{(d)}(Nz, \frac{u}{N})
\]

\[
= N \sum_{m \in P} B_{m}^{(d)}(Nz) \Psi_{m}^{(d)}(\frac{u}{N})
\]

\[
= \sum_{m \in P} N^{1-|m|} B_{m}^{(d)}(Nz) \Psi_{m}^{(d)}(u).
\]
By (2.8)

\[ \sum_{m \in \mathcal{P}} B^{(d)}_m (z + 1) \Psi^{(d)}_m (u) = \frac{|u|}{e^{d|u| - 1} - 1} F^{(d)}_0 (z + 1, u) \]

\[ = \frac{|u|}{e^{d|u| - 1} - 1} - e^{d|u|} \psi^{(d)} (u) \]

\[ = \sum_{n \in \mathcal{P}} B^{(d)}_n (z) e^{d|u|} \psi^{(d)} (u) \]

\[ = \sum_{n \in \mathcal{P}} B^{(d)}_n (z) \sum_{m \in \mathcal{P}} \binom{m}{n} \psi^{(d)}_m (u) \]

\[ = \sum_{m \in \mathcal{P}} \sum_{n \subset m} \binom{m}{n} B^{(d)}_n (z) \psi^{(d)}_m (u). \]

\[ \square \]

4 A multivariate analogue of the multiple Bernoulli polynomials

For \( n \)-tuple complex numbers

\[ \omega := (\omega_1, \ldots, \omega_n), \quad \omega_j \in \mathbb{C} \setminus \{0\}, \]

we define the multiple Bernoulli polynomials \( B_{n,m} (z \mid \omega) \) with a generating function

\[ e^{zu} \prod_{j=1}^{n} \frac{u - e^{\omega_j u}}{u - 1} = \sum_{m \geq 0} B_{n,m} (z \mid \omega) \psi_m (u) \quad (|\omega_j u| < 2\pi, j = 1, \ldots, n). \quad (4.1) \]

Let

\[ \hat{\omega}(j) := (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_r) \in \mathbb{C}^{r-1} \]

\[ = (\omega_1, \ldots, \omega_j, \ldots, \omega_r), \]

\[ \omega^-[j] := (\omega_1, \ldots, -\omega_j, \ldots, \omega_r) \in \mathbb{C}^r. \]

For \( B_{n,m} (z \mid \omega) \), the following formulas are well-known (see [N] (12)–(17)).

\[ B_{n,m} (cz \mid c\omega) = c^{m-n} B_{n,m} (z \mid \omega) \quad (c \in \mathbb{C}^*), \quad (4.2) \]

\[ B_{n,m} (\omega \mid -z \mid \omega) = (-1)^m B_{n,m} (z \mid \omega), \quad (4.3) \]

\[ B_{n,m} (z + \omega_j \mid \omega) - B_{n,m} (z \mid \omega) = m B_{n-1,m-1} (z \mid \hat{\omega}(j)), \quad (4.4) \]

\[ B_{n,m} (z \mid \omega^{-}[j]) = -B_{n,m} (z + \omega_j \mid \omega), \quad (4.5) \]

\[ B_{n,m} (z \mid \omega) + B_{n,m} (z \mid \omega^{-}[j]) = -m B_{n-1,m-1} (z \mid \hat{\omega}(j)), \quad (4.6) \]

\[ \frac{d}{dz} B_{n,m} (z \mid \omega) = m B_{n,m-1} (z \mid \omega). \quad (4.7) \]
We also introduce a multivariate analogue of the multiple Bernoulli polynomials by
\[
\mathcal{F}_0^{(d)}(;z,u) = \sum_{m\in \mathcal{P}} B_n^{(d)}(z; \omega) \Psi_m^{(d)}(u) \tag{4.8}
\]
and obtain a multivariate analogue of the above formulas \((4.2)\)–\((4.7)\) easily.

**Theorem 4.1.** (1)
\[
B_n^{(d)}(cz; c\omega) = c^{\lvert m \rvert - n} B_n^{(d)}(z; \omega) \quad (c \in \mathbb{C}^*). \tag{4.9}
\]

(2)
\[
B_n^{(d)}(\omega^1 - z; \omega) = (-1)^{\lvert m \rvert} B_n^{(d)}(z; \omega). \tag{4.10}
\]

(3)
\[
B_n^{(d)}(z + \omega j 1; \omega) - B_n^{(d)}(z; \omega) = \sum_{i=1}^r B_{n-1,m_i}(z; \tilde{\omega}(j)) \left( m_i + \frac{d}{2} (r - i) \right) h_{i-d}^{(d)}(m). \tag{4.11}
\]

(4)
\[
B_n^{(d)}(z; \omega^{-j}) = - B_n^{(d)}(z + \omega j 1; \omega). \tag{4.12}
\]

(5)
\[
B_n^{(d)}(z; \omega) + B_n^{(d)}(z; \omega^{-j}) = - \sum_{i=1}^r B_{n-1,m_i}(z; \tilde{\omega}(j)) \left( m_i + \frac{d}{2} (r - i) \right) h_{i-d}^{(d)}(m). \tag{4.13}
\]

(6)
\[
E_0(z) B_n^{(d)}(z; \omega) = \sum_{i=1}^r B_{n,m_i}(z; \omega) \left( m_i + \frac{d}{2} (r - i) \right) h_{i-d}^{(d)}(m). \tag{4.14}
\]

**proof:** (1) From the generating function of the multiple multivariate Bernoulli polynomials and homogeneity of Jack polynomials, we have
\[
\sum_{m \in \mathcal{P}} B_n^{(d)}(cz; c\omega) \Psi_m^{(d)}(u) = \mathcal{F}_0^{(d)}(;cz,u) \prod_{i=1}^n \frac{|u|}{e^{\omega_i |u|} - 1} = c^{-n} \mathcal{F}_0^{(d)}(;z,cu) \prod_{i=1}^n \frac{|cu|}{e^{\omega_i |cu|} - 1} = c^{-n} \mathcal{F}_0^{(d)}(;z,cu) \prod_{i=1}^n \frac{|cu|}{e^{\omega_i |cu|} - 1} = c^{-n} \sum_{m \in \mathcal{P}} B_n^{(d)}(z; \omega) \Psi_m^{(d)}(cu) = \sum_{m \in \mathcal{P}} c^{\lvert m \rvert - n} B_n^{(d)}(z; \omega) \Psi_m^{(d)}(u).
\]
(2) By (2.8),

\[ \sum_{m \in P} B_{n,m}^{(d)}(\omega|1 - z | \omega)\Psi_m^{(d)}(u) = 0F_0^{(d)} \left( ; |\omega|1 - z, u \right) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1} \]

\[ = e^{\omega|u|} 0F_0^{(d)} \left( ; z, -u \right) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1} \]

\[ = 0F_0^{(d)} \left( ; z, -u \right) \prod_{i=1}^{n} \frac{|u|}{e^{-\omega_i |u|} - 1} \]

\[ = \sum_{m \in P} (-1)^{|m|} B_{n,m}^{(d)}(z | \omega)\Psi_m^{(d)}(u). \]

(3) By (2.8), (2.6)

\[ \sum_{m \in P} (B_{n,m}^{(d)}(z + \omega_j 1 | \omega) - B_{n,m}^{(d)}(z | \omega))\Psi_m^{(d)}(u) \]

\[ = \left( 0F_0^{(d)} \left( ; z + \omega_j 1, u \right) - 0F_0^{(d)} \left( ; z, u \right) \right) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1} \]

\[ = (e^{\omega_j |u|} - 1)0F_0^{(d)} \left( ; z, u \right) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1} \]

\[ = |u|0F_0^{(d)} \left( ; z, u \right) \prod_{1 \leq i \neq j \leq n} \frac{|u|}{e^{\omega_i |u|} - 1} \]

\[ = \sum_{m \in P} B_{n-1,m}^{(d)}(z | \hat{\omega}(j))|u|\Psi_m^{(d)}(u) \]

\[ = \sum_{m \in P} B_{n-1,m}^{(d)}(z | \hat{\omega}(j)) \sum_{i=1}^{r} \Psi_m^{(d)}(u) \left( m_i + 1 + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m^i) \]

\[ = \sum_{m \in P} \sum_{i=1}^{r} B_{n-1,m_i}^{(d)}(z | \hat{\omega}(j)) \left( m_i + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m)\Psi_m^{(d)}(u). \]

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(4) By (2.8)

\[
\sum_{m \in P} B_{n,m}(z \mid \omega^{-j}) \Psi_m^{(d)}(u)
= 0_F^{(d)}(z, u) \frac{|u|}{e^{-\omega_j |u|} - 1} \prod_{1 \leq j \leq n} \frac{|u|}{e^{\omega_j |u|} - 1}
= -e^{\omega_j |u|} 0_F^{(d)}(z, u) \prod_{1 \leq j \leq n} \frac{|u|}{e^{\omega_j |u|} - 1}
= -0_F^{(d)}(z + \omega_j 1, u) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1}
= \sum_{m \in P} -B_{n,m}(z + \omega_j 1 \mid \omega) \Psi_m^{(d)}(u).
\]

(5) By (4.12) and (4.11), we have

\[
B_{n,m}(z \mid \omega) + B_{n,m}(z \mid \omega^{-j})
= B_{n,m}(z \mid \omega) - B_{n,m}(z + \omega_j 1 \mid \omega)
= -\sum_{i=1}^{r} B_{n-1,m_i}(z \mid \tilde{\omega}(j)) \left( m_i + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m).
\]

(6) By (2.7) and (2.6), we have

\[
\sum_{m \in P} E_0(z) B_{n,m}(z \mid \omega) \Psi_m^{(d)}(u) = E_0(z) 0_F^{(d)}(z, u) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1}
= |u| 0_F^{(d)}(z, u) \prod_{i=1}^{n} \frac{|u|}{e^{\omega_i |u|} - 1}
= \sum_{m \in P} B_{n,m}(z \mid \omega) \Psi_m^{(d)}(u)
= \sum_{m \in P} B_{n,m}(z \mid \omega) \sum_{i=1}^{r} \Psi_m^{(d)}(u) \left( m_i + 1 + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m)
= \sum_{i=1}^{r} B_{n,m_i}(z \mid \omega) \left( m_i + \frac{d}{2} (r - i) \right) h_{-i}^{(d)}(m) \Psi_m^{(d)}(u).
\]

5 Concluding remarks

Since our multivariate Bernoulli polynomials have various properties which are regarded as a natural generalization of (1.3) - (1.10), our multivariate Bernoulli polynomials are
regarded as a good multivariate analogue of Bernoulli polynomials. Therefore we desire to
find a multivariate zeta function whose some special values are written by our multivariate
Bernoulli polynomials $B_m^{(d)}(z)$.

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Department of Mathematics, Graduate School of Science, Kobe University,
1-1, Rokkodai, Nada-ku, Kobe, 657-8501, JAPAN
E-mail: g-shibukawa@math.kobe-u.ac.jp