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Existence and stability of solutions for a system of quadratic integral equations in Banach algebras

Abstract. The aim of this paper is to prove the existence and stability of solutions of a system of quadratic integral equations in the Banach algebra of continuous and bounded functions on unbounded rectangle. The main tool used in our considerations is the multiple fixed point theorem which is a consequence of Darbo’s fixed point theorem and the technique associated with measures of noncompactness. We also present an illustrative example.

1. Introduction

The theory of integral operators and integral equations is an important part of nonlinear analysis. It is caused by the fact that this theory is frequently applicable in other branches of mathematics and in mathematical physics, engineering, economics, biology as well in describing problems connected with real world (cf. [19, 25]). It is well-known that a lot of problems investigated in the theories of radiative transfer and neutron transport, and in the kinetic theory of gases and also several real world problems can be described with the help of various quadratic integral equations. Especially the quadratic integral equations of Chandrasekhar type can be very often encountered in several applications see [12, 18]. In the last years there appeared many papers devoted to the quadratic integral equations and several types of integral operators are investigated, we refer to the papers of Benchohra and al. [3], Banaś and al. [9, 14, 15, 16, 20, 21] and references therein.

On the other hand, the classical theory of integral operators and equations can be generalized with the help of the Stieltjes integral having kernels dependent
on one or two variables. Such an approach was presented and developed in many research papers see the results of Abbas and Benchohra [1, 2, 7], Banaś and al. [3, 15, 17].

The goal of this paper is to investigate a finite system of nonlinear quadratic integral equations of Hadamard-Volterra-Stieltjes type applying the concept of multiple fixed point of condensing operators and we will use the technique associated with measures of noncompactness. There are some papers in this direction, Aghajani and al. [4, 5], Dhage [20] and references therein.

It is worthwhile mentioning that in applications, the most useful measures of noncompactness are those defined in an axiomatic way [6, 11]. Such a direction of investigations has been initiated in the papers [10, 13], where the authors introduced the so-called condition [11] related to the operation of multiplication in an algebra and playing a crucial role in the use of the technique of measures of noncompactness in Banach algebras setting. Moreover, if we apply an approach to the measure of noncompactness concept associated with a suitable axiomatic definition, then we create the possibility to characterize solutions of investigated operator equations see [9, 12, 13, 14, 16, 21].

Consider the following system of quadratic fractional integral equations

\[
\begin{aligned}
&u_1(x, y) = \varphi(x, y) + (Gu_1)(x, y) \int_1^y \int_1^x \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
&\quad \times g(t, s, u_1(t, s), \ldots, u_n(t, s)) \frac{d_s h_1(y, s) d_t h_2(x, t)}{\Gamma(r_1) \Gamma(r_2) st} \\
&u_2(x, y) = \varphi(x, y) + (Gu_2)(x, y) \int_1^y \int_1^x \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
&\quad \times g(t, s, u_2(t, s), \ldots, u_n(t, s), u_1(t, s)) \frac{d_s h_1(y, s) d_t h_2(x, t)}{\Gamma(r_1) \Gamma(r_2) st} \\
&\vdots \\
&u_n(x, y) = \varphi(x, y) + (Gu_n)(x, y) \int_1^y \int_1^x \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
&\quad \times g(t, s, u_n(t, s), \ldots, u_{n-2}(t, s), u_{n-1}(t, s)) \frac{d_s h_1(y, s) d_t h_2(x, t)}{\Gamma(r_1) \Gamma(r_2) st},
\end{aligned}
\]

where \( x, y \in J = [1, +\infty) \times [1, b], \ r_1, r_2 > 1, \) \( \varphi: J \to \mathbb{R} \) is a continuous and bounded function, \( G: BC \to BC \) is a linear operator, \( g: J \times \mathbb{R}^n \to \mathbb{R} \) is a continuous function, \( h_1: [1, b] \times [1, b] \to \mathbb{R}, \ h_2: [1, +\infty) \times [1, +\infty) \to \mathbb{R} \) are given functions, and \( \Gamma(\cdot) \) is the Euler gamma function.

2. Preliminaries

Assume that \( (X, \|\cdot\|) \) is an infinite dimensional Banach space with zero element \( \theta \). Denote by \( B(x, r) \) the closed ball with center at \( x \) and radius \( r \). We write \( B_r \) to denote the ball \( B(\theta, r) \). If \( U \) is a subset of \( X \) then the symbols \( \overline{U}, \ \text{Conv}U \) stand for the closure and closed convex hull of \( U \), respectively. Apart from this
Existence and stability of solutions for a system of quadratic integral equations

the symbol \( \text{diam} \ U \) will denote the diameter of a set \( U \) while \( \|U\| \) denotes the norm of \( U \); that is, \( \|U\| = \sup\{\|u\| : u \in U\} \). Moreover, we denote by \( \mathcal{M}_X \) the family of all nonempty and bounded subsets of \( X \) and by \( \mathcal{N}_X \) its subfamily consisting of all relatively compact sets.

Here, we give some basic facts concerning measures of noncompactness, which is defined axiomatically in terms of some natural conditions.

**Definition 2.1** ([6] [11])
A mapping \( \psi : \mathcal{M}_X \to \mathbb{R}_+ \) is said to be a measure of noncompactness in \( X \) if it satisfies the following conditions:

(A1) The family \( \ker \psi = \{ A \in \mathcal{M}_X : \psi(A) = 0 \} \) is nonempty and \( \ker \psi \subset \mathcal{N}_X \);

(A2) \( A \subset B \to \psi(A) \leq \psi(B) \);

(A3) \( \psi(A) = \psi(A) \);

(A4) \( \psi(A) = \psi(\text{Conv} \ A) \);

(A5) \( \psi(\lambda A + (1 - \lambda)B) \leq \lambda \psi(A) + (1 - \lambda)\psi(B) \) for \( \lambda \in [0, 1] \);

(A6) If \( (A_n) \) is a sequence of closed sets from \( \mathcal{M}_X \) such that \( A_{n+1} \subset A_n \) \( (n = 0, 1, 2, \ldots) \) and if \( \lim_{n \to \infty} \psi(A_n) = 0 \), then the intersection set \( A_\infty = \cap_{n=0}^{\infty} A_n \) is nonempty.

The family ker \( \psi \) described in (A1) is said to be the kernel of the measure of noncompactness \( \psi \).

**Remark 2.2**
Observe that the intersection set \( A_\infty \) from (A6) is a member of the family ker \( \psi \). In fact, since \( \psi(A_\infty) \leq \psi(A_n) \) for any \( n \), we infer that \( \psi(A_\infty) = 0 \).

In the sequel we will assume that the space \( X \) has the structure of a Banach algebra. In such a case we write \( uv \) in order to denote the product of elements \( u, v \in X \). Similarly, we will write \( UV \) to denote the product of subsets \( U, V \) of \( X \); that is, \( UV = \{uv : u \in U, v \in V\} \).

**Definition 2.3** ([10] [13])
One says that the measure of noncompactness \( \mu \) defined on a Banach algebra \( X \) satisfies condition (m) if for arbitrary sets \( U, V \in \mathcal{M}_X \) the following inequality is satisfied

\[
\mu(UV) \leq \|U\| \mu(V) + \|V\| \mu(U).
\]

(m)

**Remark 2.4**
It turns out that the above defined condition (m) is very convenient in considerations connected with the use of the technique of measures of noncompactness in Banach algebras. Apart from this the majority of measures of noncompactness satisfy this condition.
We recall the well known fixed point theorem of Darbo.

**Theorem 2.5** ([5, 6, 9])

Let $\Omega$ be a nonempty, bounded, closed and convex subset of a Banach space $X$ and let $F: \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\psi(FA) \leq k \psi(A)$ for any nonempty subset $A$ of $\Omega$. Then $F$ has a fixed point in the set $\Omega$.

**Remark 2.6**

Let us denote by $\text{Fix } F$ the set of all fixed points of the operator $F$ which belongs to $\Omega$. It can be shown that the set $\text{Fix } F$ belongs to the family $\ker \psi$.

**Theorem 2.7** ([11])

Let $X_1, X_2, \ldots, X_n$ be Banach spaces and $\mu_1, \mu_2, \ldots, \mu_n$ be measures of noncompactness in $X_1, X_2, \ldots, X_n$, respectively. Assume that the function $\xi: \mathbb{R}_+^n \to \mathbb{R}_+$ is convex with $\xi(a_1, a_2, \ldots, a_n) = 0$ if and only if $a_i = 0$ for $i = 1, 2, \ldots, n$. Then

$$\mu(D) = \xi(\mu_1(D_1), \mu_2(D_2), \ldots, \mu_n(D_n))$$

defines a measure of noncompactness in the product space $X_1 \times X_2 \times \cdots \times X_n$ where $D_i$ denotes the natural projection of $D$ into $X_i$ for $i = 1, 2, \ldots, n$.

**Example 2.8** ([4])

Let $\mu$ be a measure of noncompactness on a Banach space $X$. Let $H_1(x, y) = \max(x, y)$ and $H_2(x, y) = x + y$ for any $(x, y) \in [0, \infty)^2$, then all the conditions of Theorem 2.7 are satisfied, i.e. $H_1$ and $H_2$ are convex with $H_1(x, y) = 0$ or $H_2(x, y) = 0$ if and only if $x = y = 0$. We conclude that $\bar{\mu}(D) = \max(\mu(D_1), \mu(D_2))$ and $\bar{\mu}(D) = \mu(D_1) + \mu(D_2)$ define measures of noncompactness in the space $X \times X$, where $D_i$, $i = 1, 2$ denote the natural projections of $X$.

**Definition 2.9** ([20])

An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F: X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

In what follows we will work in the Banach space $BC$ consisting of all real functions defined, continuous and bounded on $J$. This space is furnished with the standard norm

$$\|u\| = \sup\{\|u(x, y)\| : (x, y) \in J\}.$$

The space $BC$ has the structure of a Banach algebra, i.e. if $w, v \in BC$, we write $wv$ to denote the product of elements $u$ and $v$, and $(wv)(t, s) = w(t, s)v(t, s)$.

**Remark 2.10**

It is clear that the product space $\underbrace{BC \times \cdots \times BC}_n$ turns out to be a Banach space if equipped with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^n \|u_i\|_{BC}.$$
Existence and stability of solutions for a system of quadratic integral equations

In order to define a measure of noncompactness in the space $BC$, fix a nonempty bounded subset $Y$ of $BC$ and $T > 1$. For $u \in Y$ and $\epsilon_1, \epsilon_2 > 0$ let us denote by $\omega^T(u, \epsilon_1, \epsilon_2)$ the modulus of continuity of the function $u$ on the rectangle $[1, T] \times [1, b]$, i.e.

$$\omega^T(u, \epsilon_1, \epsilon_2) = \sup \{|u(x_2, y_2) - u(x_1, y_1)| : (x_1, y_1), (x_2, y_2) \in [1, T] \times [1, b], |x_2 - x_1| \leq \epsilon_1, |y_2 - y_1| \leq \epsilon_2\};$$

$$\omega^T(Y, \epsilon_1, \epsilon_2) = \sup \{\omega^T(u, \epsilon_1, \epsilon_2) : u \in Y\};$$

$$\omega^T_0(Y) = \lim_{\epsilon_1, \epsilon_2 \to 0} \omega^T(Y, \epsilon_1, \epsilon_2);$$

$$\omega_0(Y) = \lim_{T \to \infty} \omega^T_0(Y).$$

If $(t, s)$ is a fixed element from $J$, let us denote $Y(t, s) = \{u(t, s) : u \in Y\}$ and

$$\text{diam} Y(t, s) = \sup \|u(t, s) - v(t, s)\| : u, v \in Y.$$  

Finally, consider the function $\psi$ defined on the family $M_X$ by the formula

$$\psi(Y) = \omega_0(Y) + \lim_{s \to \infty} \sup \text{diam} Y(t, s).$$

It can be shown that the function $\psi$ is a measure of noncompactness in the space $BC$. The kernel $\ker \psi$ consists of nonempty and bounded sets $Y$ such that functions from $Y$ are locally equicontinuous on $J$ and the thickness of the bundle formed by functions from $Y$ tends to zero at infinity. This property will permit us to characterize solutions of the system considered in the next section.

**Theorem 2.11 (11, 13)**

The measure of noncompactness $\psi$ satisfies condition [m].

Let $L^1(J, \mathbb{R})$ be the Banach space of functions $u : J \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|u\|_{L^1} = \iint_J |u(x, y)|dydx.$$

**Definition 2.12**

Let $r_1, r_2 \geq 0$, $\sigma = (1, 1)$ and $r = (r_1, r_2)$. For $u \in L^1(J, \mathbb{R})$, define the Hadamard partial fractional integral of order $r$ by the expression

$$(HS I^r_{J}u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{t}\right)^{r_1-1} \left(\ln \frac{y}{s}\right)^{r_2-1} \frac{u(t, s)}{st} dsdt,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

For more details about Hadamard fractional integral see [22].

Let a nondegenerate rectangle $I = [a, b] \times [c, d] \subset \mathbb{R}^2$ be given. We consider a real function $p : I \to \mathbb{R}$ defined on $I$. For a given sub-rectangle $I = [a_1, b_1] \times [c_1, d_1] \subset I$, $a \leq a_1 \leq b_1 \leq b$, $c \leq c_1 \leq d_1 \leq d$ we set

$$m_p(I) = p(b_1, d_1) - p(b_1, c_1) - p(a_1, d_1) + p(a_1, c_1).$$
Let us define
\[ \sqrt[p]{I} = \sup \sum |m_p(T_i)|, \]
where the supremum is taken over all finite systems of nonoverlapping rectangles \( T_i \subset I \) (i.e. for the interiors \( T_i^o \) of the rectangles \( T_i \) we assume that \( T_i^o \cap T_j^o = \emptyset \), whenever \( i \neq j \)).

If \( p: I = [a, b] \times [c, d] \to \mathbb{R} \) and \( \gamma \in [c, d] \) (resp. \( \alpha \in [a, b] \)) is fixed, then we denote the usual variation of the function \( p(s, \gamma) \) (resp. \( p(\alpha, t) \)) in the interval \([a, b]\) (resp. \([c, d]\)) by \( b \int_a^\gamma p(\cdot, \gamma) \) (resp. \( d \int_c^\alpha p(\alpha, \cdot) \)).

**Definition 2.13** ([25])
The real function \( p: I \to \mathbb{R} \) is of bounded variation on \( I \) if \( \sqrt[p]{I} < \infty \).

**Lemma 2.14** ([25])
Let \( p: I = [a, b] \times [c, d] \to \mathbb{R} \) be such that \( \sqrt[p]{I} < \infty \) and let \( b \int_a^\gamma p(\cdot, \gamma_0) < \infty \) for some \( \gamma_0 \in [c, d] \). Then \( b \int_a^\gamma p(\cdot, \gamma) < \infty \) for all \( \gamma \in [c, d] \) and
\[ b \int_a^\gamma p(\cdot, \gamma) \leq \sqrt[p]{I} + b \int_a^\gamma p(\cdot, \gamma_0). \]

**Theorem 2.15** ([24])
If \( f: [a, b] \to \mathbb{R} \) is monotone on \([a, b]\), then \( f \) is of bounded variation on \([a, b]\) and
\[ \sqrt[p]{f} = |f(b) - f(a)|. \]

Let \( f \) and \( g \) be functions defined on the interval \([a, b]\), the Stieltjes integral of \( f \) with respect to \( g \) is designated by
\[ \int_a^b f(x)dg(x). \]

It is clear that the Riemann integral is a special case of the Stieltjes integral, obtained by setting \( g(x) = x \). The Stieltjes integral exists under several conditions, we will restrict ourselves to only one theorem in this direction.

**Theorem 2.16** ([24] [25])
The integral
\[ \int_a^b f(x)dg(x), \]
exists if the function \( f \) is continuous on \([a, b]\) and \( g \) is of finite variation on \([a, b]\), and we have
\[ \left| \int_a^b f(x)dg(x) \right| \leq \sup_{x \in [a, b]} |f(x)| \sqrt[p]{g}. \]
Lemma 2.17 \((21)\)
If the function \(f\) is continuous on \([a, b]\) and if the function \(g\) has a Riemann integrable derivative \(g'\) at every point of \([a, b]\), then
\[
\int_a^b f(x) dg(x) = \int_a^b f(x) g'(x) dx.
\]

For more properties of the Stieltjes integral see \([23, 24, 25, 26]\).

We consider the Hadamard-Stieltjes fractional integral of a function \(u: J \to \mathbb{R}\) of order \(r = (r_1, r_2)\) of the form
\[
(HS^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \frac{u(t, s)}{st} ds dh_1(y, s) dh_2(x, t),
\]
were \(h_1: [1, b] \times [1, b] \to \mathbb{R}\), \(h_2: [1, +\infty) \times [1, +\infty) \to \mathbb{R}\), and the symbols \(d_t, d_s\) indicate the integration with respect to \(t, s\) respectively.

Theorem 2.18
Let \(D\) be a nonempty, bounded and closed subset of a Banach space \(X\) and \(\mu\) an arbitrary measure of noncompactness on \(X\). If \(F: D^n \to D\) is a continuous operator and there exists a constant \(k \in [0, 1)\) such that
\[
\mu(F(Q_1 \times Q_2 \times \cdots \times Q_n)) \leq k \max(\mu(Q_1), \mu(Q_2), \ldots, \mu(Q_n))
\]
for any \(Q_1, Q_2, \ldots, Q_n\) of \(D\), then the operator \(F\) has a multiple fixed point in \(D\), i.e. there exist \(x_1^*, \ldots, x_n^* \in D\) such that
\[
\begin{cases}
F(x_1^*, x_2^*, \ldots, x_n^*) = x_1^*, \\
F(x_2^*, x_3^*, \ldots, x_n^*, x_1^*) = x_2^*, \\
\quad \vdots \\
F(x_n^*, x_1^*, \ldots, x_{n-2}^*, x_{n-1}^*) = x_n^*.
\end{cases}
\]

Proof. see \([1]\).

Let us assume that \(\Omega\) is a nonempty subset of the space \(BC\) and \(F\) is an operator on \(\Omega^n\) with values in \(BC^n\). Consider the following equation
\[
(u_1, \ldots, u_n)(x, y) = F(u_1, \ldots, u_n)(x, y), \quad (x, y) \in J. \quad (2)
\]

Definition 2.19
A solution \(u^* = (u_1(x, y), \ldots, u_n(x, y))\) of equation \((2)\) is said to be globally attractive if for each solution \(v^* = (v_1(x, y), \ldots, v_n(x, y))\) of \((2)\) we have
\[
\lim_{x \to \infty} \sup_{1 \leq i \leq n} (u_i(x, y) - v_i(x, y)) = 0.
\]

In the case when this limit is uniform, i.e. when for each \(\epsilon > 0\) there exists \(T > 0\) such that
\[
\sup_{1 \leq i \leq n} |u_i(x, y) - v_i(x, y)| < \epsilon, \quad x \geq T,
\]
we will say that solutions of \((2)\) are uniformly globally attractive.
Remark 2.20
This concept of stability in the case of equation \((Fu)(x, y) = u(x, y)\) can be found in [14, 21].

3. Main results

We will give an existence and stability results for the system \((1)\) applying Theorem 2.18 under the following assumptions:

\((H_1)\) The function \(g\) is continuous and there exist continuous and bounded functions \(p_i : J \to \mathbb{R}^+,\ i = 1, \ldots, n\) such that

\[
|g(x, y, u_1, \ldots, u_n) - g(x, y, v_1, \ldots, v_n)| \leq \frac{1}{n} \sum_{i=1}^{n} \frac{p_i(x, y)|u_i - v_i|}{\max_{1 \leq i \leq n}|u_i + v_i| + 1}
\]

for all \(u_i, v_i \in \mathbb{R},\ i = 1, \ldots, n\).

\((H_2)\) The operator \(G : BC \to BC\) is a bounded linear operator with spectral radius \(r_\sigma(G) < 1\).

\((H_3)\) The function \(s \mapsto h_1(y, s)\) is continuous and of bounded variation on \([1, b]\) for each fixed \(y \in [1, b]\), and the function \(t \mapsto h_2(x, t)\) is continuous and of bounded variation on \([1, +\infty)\) for each \(x \in [1, +\infty)\).

\((H_4)\) There exists a constant \(\eta > 0\) such that

\[
\sup_{x \geq 1; 1 \leq y \leq b} \left| \int_{1}^{x} \int_{1}^{y} \left( \frac{\ln x}{s} \right)^{r_1-1} \left[ 1 + |g(t, s, 0, \ldots, 0)| \right] \, d_s h_1(y, s) \, d_t h_2(x, t) \right| \leq \eta,
\]

with

\[
\rho = \frac{\eta p^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} < 1 \quad \text{and} \quad \frac{1 + \rho}{1 - \rho} \|\varphi\| < 1,
\]

where \(p^* = \sup_{1 \leq i \leq n} \|p_i\|\).

Remark 3.1

(i) In view of the assumption \((H_1)\) we infer that

\[
|g(x, y, u_1, \ldots, u_n)| \leq \frac{1}{n} \sum_{i=1}^{n} \frac{p_i(x, y)|u_i|}{\max_{1 \leq i \leq n}|u_i| + 1} + |g(x, y, 0, \ldots, 0)|
\]

for all \((x, y) \in J\) and \(u_i \in \mathbb{R}\).

(ii) The operator \(G\) is a bounded linear operator, i.e. the operator \(G\) is linear and there exists a constant \(l > 0\) such that for each \(u \in BC\),

\[
\|Gu\| \leq l\|u\|.
\]

If spectral radius of \(G\) satisfies \(r_\sigma(G) < 1\), we conclude that \(\sup_{\|u\| \neq 0} \frac{\|Gu\|}{\|u\|} < 1\).
Existence and stability of solutions for a system of quadratic integral equations

**Theorem 3.2**
Under the assumptions \([H_1], [H_4]\) the system \([1]\) has at least one solution \(u^* = (u_1^*, \ldots, u_n^*)\). Moreover, solutions of the system \([1]\) are globally attractive.

**Proof.** Consider the operator \(F: \bigcup_{n=1}^{\infty} BC \to BC\) defined by

\[
(F(u_1, \ldots, u_n))(x, y) = \varphi(x, y) + (Gu_1)(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
\times \frac{g(t, s, u_1(t, s), \ldots, u_n(t, s))}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t).
\]

Observe that in view of our assumptions, for any function \(u = (u_1, \ldots, u_n)\) of the space \(BC \times \cdots \times BC\) the function \(F(u_1, \ldots, u_n)\) is continuous on \(J \times \cdots \times J\). Next, let us take an arbitrary function \(u = (u_1, \ldots, u_n) \in BC \times \cdots \times BC\). Using our assumptions, for a fixed \((x, y) \in J\) we have

\[
|F(u_1, \ldots, u_n)(x, y)| \\
\leq |\varphi(x, y)| + |(Gu_1)(x, y)| \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
\times \frac{|g(t, s, u_1(t, s), \ldots, u_n(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \\
\leq |\varphi(x, y)| + \frac{|u_1(x, y)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
\times \left[ \frac{1}{n} \sum_{t=1}^{n} \max_{1 \leq i \leq n} |u_i(t, s)| + 1 \right] d_s h_1(y, s) d_t h_2(x, t) \\
\leq \|\varphi\| + \frac{p^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \|u\| \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \\
\times [1 + |g(t, s, 0, \ldots, 0)||d_s h_1(y, s) d_t h_2(x, t),
\]

hence we obtain

\[
\|F(u_1, \ldots, u_n)\| \leq \|\varphi\| + \frac{np^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \|u\|,
\]

so the function \(F(u_1, \ldots, u_n)\) is bounded on \(J \times \cdots \times J\). Then \(Fu \in BC\).

We take

\[
r = \frac{\|\varphi\|}{1 - \rho}.
\]

We deduce that the operator \(F\) transforms the set \(B_r \times \cdots \times B_r\) into the ball \(B_r\).
Further, let \( \{u_m\} = \{u_1, \cdots, u_n\} \subset B_r \times \cdots \times B_r \) such that \( \lim_{m \to \infty} u_{im} = u_i \) for \( i = 1, \ldots, n \) we get
\[
|\langle F(u_{1m}, \ldots, u_{nm}) \rangle(x, y) - \langle F(u_1, \ldots, u_n) \rangle(x, y) |
\]
\[
= \left| \varphi(x, y) + (Gu_{1m})(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \
\times \frac{|g(t, s, u_{1m}(t, s), \ldots, u_{nm}(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \\
- \varphi(x, y) + (Gu_1)(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \
\times \frac{|g(t, s, u_1(t, s), \ldots, u_n(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \right| \\
\leq (Gu_{1m})(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \
\times \frac{|g(t, s, u_{1m}(t, s), \ldots, u_{nm}(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \\
- (Gu_1)(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \
\times \frac{|g(t, s, u_1(t, s), \ldots, u_n(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \\
+ (Gu_{1m})(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \
\times \frac{|g(t, s, u_{1m}(t, s), \ldots, u_{nm}(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \\
- (Gu_1)(x, y) \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \
\times \frac{|g(t, s, u_1(t, s), \ldots, u_n(t, s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y, s) d_t h_2(x, t) \leq \
\frac{mp^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \|Gu_{1m} - Gu_1\| + \frac{p^*\|Gu_1\|}{\Gamma(r_1)\Gamma(r_2)} \sup_{1 \leq i \leq n} \|u_{im} - u_i\| \\
\times \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} d_s h_1(y, s) d_t h_2(x, t) \\
\leq \frac{mp^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \sup_{1 \leq i \leq n} \|u_{im} - u_i\| + \frac{(\ln b)^{r_2}p^*\|Gu_1\|}{\Gamma(r_1)\Gamma(r_2)} \sup_{1 \leq i \leq n} \|u_{im} - u_i\| \\
\times \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} d_s h_1(y, s) d_t h_2(x, t).
Existence and stability of solutions for a system of quadratic integral equations

Since \( \lim_{m \to \infty} \sup_{1 \leq i \leq n} \| u_{i,m} - u_i \| = 0 \), we obtain

\[
\lim_{m \to \infty} \| F(u_{1,m}, \ldots, u_{n,m}) - F(u_1, \ldots, u_n) \| = 0.
\]

Then the operator \( F \) is continuous on \( B_r \times \cdots \times B_r \).

**Remark 3.3**

By the hypothesis \((H_4)\) we notice that the quantity

\[
\sup_{(x,y) \in J} \left| \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1 - 1} \left( \ln \frac{y}{s} \right)^{r_2 - 1} \frac{d_t h_1(y,s)d_t h_2(x,t)}{\Gamma(r_1)\Gamma(r_2)st} \right|
\]

is finite.

Now, we consider the operator \( T \) defined on \( BC \times \cdots \times BC \) by

\[
T(u_1, \ldots, u_n)(x,y) = \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1 - 1} \left( \ln \frac{y}{s} \right)^{r_2 - 1} \frac{g(t,s,u_1(t,s), \ldots, u_n(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_t h_1(y,s)d_t h_2(x,t).
\]

For an arbitrarily function \( u = (u_1, \ldots, u_n) \) of the space \( BC \times \cdots \times BC \) and a fixed \((x,y)\) in \( J \), using our assumptions, we get

\[
|T(u_1, \ldots, u_n)(x,y)| = \left| \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1 - 1} \left( \ln \frac{y}{s} \right)^{r_2 - 1} \frac{g(t,s,u_1(t,s), \ldots, u_n(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_t h_1(y,s)d_t h_2(x,t) \right|
\]

\[
\leq \int_1^x \int_1^y \left( \ln \frac{x}{t} \right)^{r_1 - 1} \left( \ln \frac{y}{s} \right)^{r_2 - 1} \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{p_i |u_i|(t,s)}{\max_{1 \leq i \leq n} |u_i(t,s)| + 1} \right]
\]

\[
+ |g(t,s,0, \ldots, 0)| \frac{d_t h_1(y,s)d_t h_2(x,t)}{\Gamma(r_1)\Gamma(r_2)}
\]

\[
\leq \frac{\eta (\ln b)^2 p^*}{\Gamma(r_1)\Gamma(r_2)} \sup_{1 \leq i \leq n} |u_i(t,s)|,
\]

thus

\[
\|T(u_1, \ldots, u_n)\| \leq \rho \sup_{1 \leq i \leq n} \|u_i\|.
\]

Clearly the operator \( T \) transforms the set \( B_r \times \cdots \times B_r \) into the ball \( B_r \) and

\[
F(B_r \times \cdots \times B_r) \subseteq \varphi + G(B_r) \cdot T(B_r \times \cdots \times B_r).
\]
Next, we take a nonempty \( Y \subset B_r \), using Definition 2.3 and applying Theorem 2.11 we obtain

\[
\psi(F(Y \times \cdots \times Y)) \leq \psi(G(Y) \cdot T(Y \times \cdots \times Y)) \\
\leq \|G(Y)\| \psi(T(Y \times \cdots \times Y)) + \|T(Y \times \cdots \times Y)\| \psi(G(Y)) \\
\leq \frac{\|\varphi\|}{1 - \rho} \psi(Y) + \rho \frac{\|\varphi\|}{1 - \rho} \psi(Y),
\]

hence

\[
\psi(F(Y \times \cdots \times Y)) \leq \frac{1 + \rho}{1 - \rho} \|\varphi\| \psi(Y).
\]

Finally, in view of Theorem 2.18 we deduce that \( F \) has at least one multiple fixed point in \( B_r \) which is a solution of the system (1). Moreover, taking into account the fact that the set \( \text{Fix} \, F \in \ker \psi \) and the characterization of sets belonging to \( \ker \psi \) (Remark 2.6) we conclude that all solutions of the system (1) are globally attractive in the sense of Definition 2.19.

4. Example

We consider the following system of integral equations

\[
\begin{align*}
\psi_1(x, y) &= \frac{1}{5s + t} + e^{-2y-x}u_1(x, y) \int_1^y \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
&\quad \times \left( \max_{1 \leq i \leq n} |u_i(t, s)| + 1 \right) \Gamma(r_1) \Gamma(r_2) s t d_s(\arctan(ys)) d_t \left( \frac{1}{tx} \right), \\
u_2(x, y) &= \frac{1}{5s + t} + e^{-2y-x}u_2(x, y) \int_1^y \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
&\quad \times \left( \max_{1 \leq i \leq n} |u_i(t, s)| + 1 \right) \Gamma(r_1) \Gamma(r_2) s t d_s(\arctan(ys)) d_t \left( \frac{1}{tx} \right), \\
&\vdots \\
u_n(x, y) &= \frac{1}{5s + t} + e^{-2y-x}u_n(x, y) \int_1^y \int_1^y \left( \ln \frac{x}{t} \right)^{r_1-1} \left( \ln \frac{y}{s} \right)^{r_2-1} \\
&\quad \times \left( \max_{1 \leq i \leq n} |u_i(t, s)| + 1 \right) \Gamma(r_1) \Gamma(r_2) s t d_s(\arctan(ys)) d_t \left( \frac{1}{tx} \right),
\end{align*}
\]

where \((x, y) \in J = [1, +\infty) \times [1, \pi]\), and \((r_1, r_2) = \left( \frac{3}{2}, \frac{3}{2} \right)\). Set

\[
\varphi(x, y) = \frac{1}{5s + t}, \quad \|\varphi\| = \frac{1}{6}, \quad h_1(y, t) = \arctan(ys), \quad h_2(x, s) = \frac{1}{tx},
\]

\((Gu_i)(x, y) = e^{-2y-x}u_i(x, y)\)

and

\[
g(t, s, u_1(t, s), \ldots, u_n(t, s)) = \frac{n^{-1} e^{-3st}}{\max_{1 \leq i \leq n} |u_i(t, s)| + 1}, \quad (t, s) \in J.
\]
Existence and stability of solutions for a system of quadratic integral equations

It is clear that the system (3) can be written as the system (1). Let us show that conditions \([H_1]\) hold. The function \(s \mapsto \arctan(ys)\) is continuous and nondecreasing for each fixed \(y_0 \in [1, \pi]\), so it is of bounded variation on \([1, \pi] \times [1, \pi]\), and

\[
\int_{s=1}^{s=\pi} h_2(y, s) \leq \arctan(\pi^2) - \arctan(1).
\]

The function \(t \mapsto \frac{1}{tx} \) is continuous and decreasing for each fixed \(x_0 \in [1, +\infty)\), and

\[
\lim_{t \rightarrow +\infty} \frac{1}{tx} = 0,
\]

so it is of bounded variation on \([1, +\infty) \times [1, +\infty)\), and for each fixed \(x_0 \in [1, +\infty)\)

\[
d_t h_1(x_0, t) = -\frac{1}{x_0 t^2} dt.
\]

We have also

\[
|g(t, s, u_1, \ldots, u_n) - g(t, s, v_1, \ldots, v_n)|
\]

\[
= \frac{n^{-1} e^{-3st}}{\max_{1 \leq i \leq n} |u_i| + 1} - \frac{n^{-1} e^{-3st}}{\max_{1 \leq i \leq n} |v_i| + 1}
\]

\[
= \frac{n^{-1} e^{-3st} (\max_{1 \leq i \leq n} |u_i| - \max_{1 \leq i \leq n} |v_i|)}{(\max_{1 \leq i \leq n} |u_i| + 1)(\max_{1 \leq i \leq n} |v_i| + 1)}
\]

\[
\leq \frac{n^{-1} e^{-3st} \max_{1 \leq i \leq n} |u_i - v_i|}{\max_{1 \leq i \leq n} |u_i + v_i| + 1}
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{n} \frac{e^{-3st} |u_i - v_i|}{\max_{1 \leq i \leq n} |u_i + v_i| + 1},
\]

so for all \(i = 1, \ldots, n\) we have \(p_i(t, s) = e^{-3st}\) and \(\sup_{1 \leq i \leq n} \|p_i\| = e^{-3}\).

Obviously the operator \(G\) is linear and for each \(u\) in \(BC\), we get

\[
\sup_{\|u\| \neq 0} \frac{\|Gu\|}{\|u\|} \leq e^{-3}.
\]

Next, for a fixed \((x, y)\) in \(J\), we obtain

\[
\left| \int_{1}^{x} \int_{1}^{y} \left( \ln \frac{x}{t} \right)^{r_1 - 1} [1 + |g(t, s, 0, \ldots, 0)|] d_t h_1(y, s) d_s h_2(x, t) \right|
\]

\[
= \left| \int_{1}^{x} \int_{1}^{y} \left( \ln \frac{x}{t} \right)^{\frac{2}{3} - 1} n^{-1} e^{-3st} d_s (\arctan(ys)) d_t \left( \frac{1}{tx} \right) \right|
\]

\[
+ \left| \int_{1}^{x} \int_{1}^{y} \left( \ln \frac{x}{t} \right)^{\frac{2}{3} - 1} d_s (\arctan(ys)) d_t \left( \frac{1}{tx} \right) \right|
\]
\[
\leq \ln x \int_1^x n^{-1}e^{-3t} \left( \int_1^y d_s(\arctan(ys)) ds(\frac{1}{tx}) \right) \\
\leq \ln x [\arctan(\pi^2) - \arctan(1)] \left| \int_1^x \frac{n^{-1}e^{-3t}}{xt^2} dt \right| \\
\leq \ln x [\arctan(\pi^2) - \arctan(1)] \left| \int_1^x \frac{n^{-1}e^{-3t}}{x} dt \right| \\
\leq [\arctan(\pi^2) - \arctan(1)] \left| -\frac{\ln x}{3xe^{3x}} + \frac{\ln x}{3e^{3x}} \right|,
\]

since

\[
\lim_{x \to \infty} \left| -\frac{\ln x}{3xe^{3x}} + \frac{\ln x}{3e^{3x}} \right| = 0,
\]

then the function \( q(x) = \left| -\frac{\ln x}{3xe^{3x}} + \frac{\ln x}{3e^{3x}} \right| \) is bounded, i.e. there exists \( \alpha > 0 \) such that \( \sup_{x \geq 1} q(x) \leq \alpha \). We take

\[
\eta = \alpha [\arctan(\pi^2) - \arctan(1)].
\]

It follows that

\[
\rho = \frac{\eta p^* (\ln b)^r_2}{\Gamma(r_1) \Gamma(r_2)} = \frac{\eta (\ln \pi)^{3/2}}{e^3 \Gamma(3/2) \Gamma(3/2)} < 1,
\]

and

\[
\frac{1 + \rho} {1 - \rho} \|\varphi\| = \frac{1 + \rho} {6(1 - \rho)} < 1.
\]

Consequently from Theorem 3.2 the system (3) has at least solution in \( BC \) and solutions of the system (3) are globally attractive.

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