THEORETICAL ASPECTS OF ASTEROSEISMOLOGY: SMALL STEPS TOWARDS A GOLDEN FUTURE

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ABSTRACT

The current status of asteroseismic studies is here reviewed and the adequate techniques of analysis available today for the study of the oscillation frequencies are presented. Comments on prospects for future investigations through the possibility of getting ever more precise asteroseismic observations from ground and space are given.

Key words: asteroseismology; stellar oscillations; solar-type stars.

1. INTRODUCTION

The present conference took place at a period which marks a significant milestone in the development of stellar physics: asteroseismology, which aims to infer the structural properties of stars which display multi-mode pulsations, has entered in a new golden age. In fact, ground-based observations have reached such level of accuracy to allow investigation of several type of oscillating stars: the satellite MOST (Walker et al. 2003), the first space mission totally dedicated to the observation of stellar oscillations, has been successfully launched in June 2003 and has just started to reveal primary results on the analysis of pulsations seen outside earth’s atmosphere. In order to make full use of these observational successes, several theoretical techniques have been developed or adopted from helioseismology to probe the internal characteristics of stars other than the Sun.

Here I will review on the theoretical aspects of asteroseismology, considering methods and tools available today to manipulate observed frequencies of oscillation in order to investigate the evolutionary and structural properties of the stars. Example of application to real data set will be given for each asteroseismic diagnostic tool considered. Some recent results will be shown and comments will be given on perspectives for the future. For basic concepts on the theory of stellar oscillations I refer to classical books (e.g. Cox 1980; Unno et al. 1989); for theoretical methods and developments in asteroseismology see reviews by, e.g., Christensen-Dalsgaard (2003, 2004), Christensen-Dalsgaard and Dziembowski (2000), Dziembowski (2003b), Gough (2003), Paternò, Di Mauro and Ventura (2003), Roxburgh (2002, 2004).

2. FROM HELIO- TO ASTEROSEISMOLOGY

During the last decades, numerous observational and theoretical efforts in the study of the acoustic modes of solar oscillations, has brought to a detailed knowledge of the interior of the Sun.

The experience acquired in helioseismology on handling pulsations frequency data provides a good starting point for asteroseismic investigations. However, inferences of the interior of stars other than the Sun appear to be much more complicated and less outstanding in terms of achievable results. The large stellar distances, the point-source character of the stars, the low amplitude of the oscillations and the effect of the earth’s atmosphere on the signal, restrict the asteroseismic studies to the use of small sets of data often characterized by modes with only low harmonic degrees ($l \leq 4$). However, the main difference between helio- and asteroseismology is that global parameters of the Sun, are much better known than they are for any other star. Luminosity, effective temperature, surface composition and $v \sin i$ are obtained, within a certain error, from spectrum analysis of distant stars; age and composition are estimated approximately only for stars in clusters; masses and radii are measured only for spectroscopic binaries. In addition, all these measurements can be affected by unknown effects such as loss, accretion, or diffusion of mass. Thus, the structure of the model of the stars cannot be so well constrained such as that of the Sun.

All these problems make asteroseismic inferences quite difficult, so that the successes reached by helioseismic studies are beyond the possibility of asteroseismology. Nevertheless, several attempts have been made, during the last years, with the aim to identify oscillations in distant stars and also to model the stellar pulsational phenomena. In fact, oscillations have several advantages over all the other observables: pulsational instability has been detected in stars in all the evolutionary stages and of
different spectral type; frequencies of oscillations can be measured with high accuracy and depend in very simply way on the equilibrium structure of the model (only on two independent variables, e.g. the local adiabatic speed of sound and density); different modes are confined and probe different layers of the interior of the stars. Thus, accurate observations of the acoustic frequencies, can be used not only to study the physics of the stellar interior, but also to constrain theories of stellar evolution.

3. PULSATING STARS

The historical pulsating stars, like the classical Cepheids, W-Virginis and RR-Lyrae, characterized by a large luminosity variations, show only one or two pulsation modes, limiting the knowledge of the structural properties to the mean density or in better cases to the mass and radius. Asteroseismology is understood as the study of pulsations on stars in which many oscillation modes are excited at once. This definition is broad enough to include many types of pulsating stars. Multi-mode oscillations have been detected in stars which are spread over a significant part of the HR-diagram, reflecting different evolutionary stages from main-sequence to the white dwarf cooling sequence. Luminosity and effective temperature of the major classes of pulsating stars are shown in the HR-diagram of Fig. [1].

We can divide the stars on which it is possible to apply seismic techniques into two groups. A group of stars show oscillations with intensity amplitude in the range of millimagnitudes and includes white dwarf, δ Scuti stars, the rapidly rotating Ap stars, the β-Cephei stars, the slowly pulsating B (SPB) stars and the γ Doradus stars. Stars of these classes show modes excited by a mechanism due to a perturbation of the opacity, caused by structural conditions typical of each considered class. The second group includes the stars with tiny oscillations amplitude (around $10^{-4}$magnitudes or less) and includes the so-called solar-like stars, in which, like the Sun, oscillations are excited by turbulent convection in the outer convective envelope.

For the purposes of this review, however, I will concentrate mainly on solar-type stars and I will give just a summary on some theoretical results obtained by asteroseismology on some other pulsating targets at the end of this review. The principal reason for doing so is due to the excitement of the community for the recent observational successes obtained on some solar-type stars (e.g., α-Can Min and α-Cen) which have given the possibility, for the first time, of handling large sets of accurate oscillation frequencies to investigate the internal structure of the stars. Moreover, it should be pointed out, that methods and techniques developed and adapted for solar-type stars, are also valid for the other pulsating stars. A detailed and updated description of the characteristics of the main asteroseismic targets can be found in recent reviews (e.g. Dziembowski 2003b; Kurtz 2004; Paternò, Di Mauro and Ventura 2003).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{HR diagram showing several classes of pulsating stars. Courtesy of J. Christensen-Dalsgaard.}
\end{figure}

3.1. Properties of pulsations

Many of the oscillation modes which can be detected in stars are limited to only the lowest degree ones, owing to the point-like character of these sources. If the condition that $l \ll n$ is satisfied, the excited p-modes can be described in terms of the asymptotic theory (Tassoul 1980), which predicts that oscillation frequencies $\nu_{n,l}$ of acoustic modes, characterized by radial order $n$, at harmonic degree $l$ should satisfy the following approximation:

$$\nu_{n,l} = \Delta \nu \left( n + \frac{l}{2} + \alpha + \frac{1}{4} \right) + \epsilon_{n,l}, \quad (1)$$

where $\alpha$ is a function of the frequency determined by the properties of the surface layers, $\epsilon_{n,l}$ is a small correction which depends on the conditions in the stellar core. $\Delta \nu$ is the inverse of the sound travel time across the stellar diameter:

$$\Delta \nu = \left( \frac{2}{\int_0^R \frac{d^2 r}{c}} \right)^{-1}, \quad (2)$$

where $c$ is the local speed of sound at radius $r$ and $R$ is the photospheric stellar radius. The property expressed by Eq. (1) may provide almost immediate asteroseismic inferences on stellar parameters and constraints on theoretical models for a variety of solar-like stars in a wide range of evolutionary stages. For a given $l$, the acoustic spectra show a series of equally spaced peaks between p modes of same degree and adjacent $n$, whose frequency separation represents the so called large separation which is approximately equivalent to $\Delta \nu$:

$$\Delta \nu \approx \nu_{n+1,l} - \nu_{n,l} \equiv \Delta \nu_l. \quad (3)$$
The spectra are characterized by another series of peaks, whose narrow separation is $\delta \nu_l$, known as small separation:

$$\delta \nu_l \equiv \nu_{n,l} - \nu_{n-1,l+2} = (4l + 6)D_0$$  \hspace{1cm} (4)

where

$$D_0 = \frac{\Delta \nu}{4\pi^2 \nu_{n,l}} \left[ \frac{c(R)}{R} - \int_0^R \frac{dc}{dr} \frac{dr}{r} \right].$$  \hspace{1cm} (5)

Instead of the small separations one could consider the so-called 'second differences' of frequencies:

$$\delta^2 \nu_l \equiv \nu_{n+1,l} - 2\nu_{n,l} + \nu_{n-1,l}.$$  \hspace{1cm} (6)

$\Delta \nu$, and hence the general spectrum of acoustic modes, scales approximately as the square root of the mean density, that is, for fixed mass, as $R^{-3/2}$. On the other hand, the small frequency separation and the second difference, are both sensitive to the chemical composition gradient in central regions of the star and hence to its evolutionary state. Thus, the determination of both large and small frequency separation, $\Delta \nu_l$ and $\delta \nu_l$, provides measures of the mass and the age of the star (e.g. Christensen-Dalsgaard 1988). See review by Gough (2003) on the potential of frequencies separations as asteroseismic diagnostic.

Analogous properties have been derived for the g-modes. Tassoul’s theory shows that in the asymptotic regime the g-modes are nearly uniformly spaced in period, g-modes will be discussed in more details in Sec. 9 since the only types of stars in which these are commonly observed are rather special cases.

3.2. Sub-surface effects and a new seismic indicator

Seismic analysis of observed acoustic frequencies is an extremely powerful tool for the investigation of the internal structure of the stars, but the use of large and small separations can be misleading if not accurately considered. Theoretical pulsation frequencies, essential for asteroseismic investigation, are calculated on theoretical models of stars which are inevitably affected by errors. In particular, the structure of the near-surface regions of stars is quite uncertain: there are still substantial ambiguity in modelling the convective flux, considering the sources and the mechanisms of excitation and damping of the oscillations, defining an appropriate equation of state to describe the thermodynamic properties of the stellar structure, as well as in the treatment of non-adiabatic effects. Limiting the investigation to the use of small separations do not solve the problem. In fact, the small separation which is determined principally by condition in the core, also retains some sensitivity to the mean density and to the detailed properties of the stellar envelope.

To solve these problems the use of a new seismic indicator was introduced by Roxburgh and Vorontsov (2003a), which compared pulsating properties of several models with exactly the same interior structure, but different outer envelopes, obtained by keeping constant the first adiabatic exponent all along the radius or by introducing linear variation of the polytropic index in the structure of the envelope. They showed that the ratios of small to large separations,

$$r_l = \delta \nu_l / \Delta \nu_l,$$  \hspace{1cm} (7)

are independent of the structure of the outer layers and hence can be used as diagnostic of the interior of stars.

4. SOLAR-TYPE STARS

Solar-type stars are F, G and possibly K main-sequence and sub-giants stars in which oscillations are excited stochastically by vigorous near-surface convection in a broad spectrum of low amplitude p-modes, as in the Sun. There is also good evidence, both from photometry with the WIRE satellite (Buzasi et al. 2000) and ground-based velocity studies, for solar-like oscillations in G, K and semi-regular M giants. The oscillation periods are expected to be in the range from the typical 5-min (main-sequence stars), as in the Sun, up to about a few days (sub-giant and giant stars). Fig. 2 shows location in HR-diagram of the solar-type pulsating stars and of some selected targets.
4.1. Amplitudes and life-time of solar-like oscillations

One of the greatest deficiencies in modelling oscillations in stars with surface convection zones is the lack of a proper theory to describe convection and its interaction with pulsations. Several attempts have been made in recent years to address this problem. Christensen-Dalsgaard and Frandsen (1983) made the first effort to give a rough prediction of amplitudes of solar-like oscillations. They found velocity and luminosity amplitudes increasing with age and with mass along the main sequence. Simple scaling laws to estimate the amplitudes in velocity have been proposed by, e.g., Kjeldsen and Bedding (1995, 2001).

Impressive progress has been made on hydrodynamical simulations of convection, and its interaction with pulsations, leading Houdek et al. (1999) to give estimates of the pulsation properties, including the damping rates, of main-sequence stars by using models based on mixing-length treatment of convection. According to theoretical estimations (e.g. Houdek et al. 1999, Kjeldsen and Bedding 1995) the expected luminosity variations are of about a few mmag in giant stars while in main-sequence and sub-giant stars are so small (from $10^{-3}$ up to a few $10^{-2}$ mmag) to be well below the detection limit for ground-based photometric observations. The radial velocity amplitudes are expected to be at most $50-60 \, m \, s^{-1}$ in K giants, $1-2 \, m \, s^{-1}$ in F and G sub-giants and even smaller in main-sequence solar-type stars. These amplitudes are at the edge of the photon noise limits of most of the available spectrographs. The theoretical predictions have been tested and compared with observational results, revealing a fairly good agreement for the several solar-like stars already detected. However, calculation of damping rates are still a problem. Theoretical prediction have been able to reproduce observed lifetimes of oscillations only in the case of the Sun. Observations of stars which are somewhat more evolved than the Sun, like Procyon (Barban et al. 1999), $\alpha$ Cen A (Bedding et al. 2004), $\beta$ Hyi (Bedding et al. 2002) and $\xi$ Hya (Stello et al. 2004) have shown mode lifetimes shorter than the values predicted by the theoretical models (Houdek et al. 1999). This inconsistency with observations seems to indicate that there is still some contributions to damping so far ignored in the theory.

5. STELLAR MODELLING

Reliable calculation of stellar models is the basic prerequisite for all asteroseismic investigations of stellar internal properties. The structure models are produced by employing up-to-date physical information and the most updated input parameters (composition, mass, mixing-length) in order to reproduce all the spectroscopic and photometric observations of the considered stars. Evolutionary sequences are calculated by including additional effects such as overshoot from the convective core (e.g. Di Mauro et al. 2003) during the main-sequence phase, diffusion and settling of helium and heavy elements (e.g. Vauclair 2003).

In the case of solar-type stars, detailed classical models have been produced on several well observed targets: $\alpha$ Cen (e.g. Guenther and Demarque 2000; Morel et al. 2000, Thévenin et al. 2002; Thou et al. 2003; Eggenberger et al. 2004b); $\eta$ Boo (Christensen-Dalsgaard, Bedding and Kjeldsen 1995; Guenther and Demarque 1996; Di Mauro et al. 2003, 2004; Kervella et al. 2003; Guenther 2004), Procyon A (e.g. Barban et al. 1999; Chaboyer, Demarque and Guenther 1999; Di Mauro, Christensen-Dalsgaard and Weiss 2000, Kervella et al. 2004; Provost, Martić and Berthomieu these proceedings), $\beta$ Hydri (Di Mauro, Christensen-Dalsgaard and Paterñó 2003; Fernandes and Monteiro 2003; Dravins, Lindgren and Vandenberg 1998); $\xi$ Hydrae (Teixeira et al. 2003; Christensen-Dalsgaard 2004).

Unfortunately, modelling of distant stars involves physical effects which substantially complicates the calculation. One of the most crucial problem in computing stellar structure is related to the modelling of the convective transport of energy in the interior and the associated physical effects such as the evolution of the composition in presence of convective regions (e.g. Dupret, these proceedings). See the reviews by Demarque and Robinson (2003) and D’Antona, Montalban and Mazzitelli (2004) about sophisticated models of convection.

Another trouble is represented by the fact that effects of rotation in stellar modelling are often neglected. The rotation may affect oscillation frequencies forming rather complex power spectra. In order to identify and reproduce the observed modes it is necessary to develop an appropriate treatment of rotation in the evolutionary models and in the calculation of oscillation frequencies. To date, rotational effects have been taken into account mainly using perturbation theory approaches. The Coriolis and the centrifugal forces, which in the case of moderate and fast-rotators cannot be neglected, have been taken into proper account as first-, second- and even third order perturbations. Many authors (Saio 1981; Gough and Thompson 1990; Dziembowski and Goode 1992; Soufi, Goupil and Dziembowski 1998; Karami, Christensen-Dalsgaard and Pijpers 2003) have approached the problem under various approximations, also including radial and latitudinal dependence of rotation and magnetic fields, which remove the degeneracy of the modes. However perturbation theory might be inadequate for the most rapidly rotating stars.

Calculation of the evolution and pulsations of two- or three-dimensional models, represents a solution for all these problems and turns out to be strongly recommended, but at present, very difficult to be attained (e.g. Demarque and Robinson 2003).
The dashed blue lines show evolution tracks plotted in HR-diagrams for α Cen A and B, respectively on the left and right panel. The tracks have been calculated with the Liége evolution code by A. Miglio and J. Montalban by assuming OPAL EOS. Input values of masses, age and mixing length parameters ($\alpha_{\text{MLT}}$) are indicated. The rectangles define the one-sigma error box for the observed luminosities and effective temperatures and the green lines show the one-sigma error box for the observed radius. The symbol star indicates the location of the selected models. Courtesy of A. Miglio.

Figure 3. The dashed blue lines show evolution tracks plotted in HR-diagrams for α Cen A and B plot- ted in a HR-diagram calculated with the evolution code of Christensen-Dalsgaard, with $M_A = 1.1095M_\odot$ and $M_B = 0.9301M_\odot$, the OPAL EOS and an age of 7.382 Gyr. The rectangles define the one-sigma error boxes around the observed luminosities and effective temperatures (asterisks). The triangles indicate the location of the selected models. Courtesy of J. Christensen-Dalsgaard.

Figure 4.

5.1. Model fitting method and application to the α Cen system

One of the most recent strategy introduced in order to find the models which better reproduce the observations is represented by an optimization procedure known as ‘model fitting’ method. Given a set of input parameters, such as mass, mixing-length parameter and composition of a star, the method consists in searching, among all the possible models, the one which best fits all the known observables. This can be obtained by performing a $\chi^2$ mini-

$$\chi^2 = \sum_i^N \left( \frac{O_{\text{mod}}^{\text{obs}} - O_{\text{mod}}^{\text{obs}}}{\sigma_i} \right)$$

where $O_{\text{obs}}$ are all the observables - luminosity, effective temperature, radius, oscillation frequencies, large and small separations, ratio of small to large separation - and $O_{\text{mod}}$ are the theoretical values calculated on the models. The observed errors of the parameters are $\sigma_i$. Extensive explanation of the method can be found in Guenther and Brown (2004) and Roxburgh (2004). Although in principle the method might represent an useful strategy, it still presents some problems: the solution depends on the initial inputs and therefore it does not explore the global space of parameters. To solve this problem, Metcalfe et al. (these proceedings) have suggested to use genetic algorithms, well developed and applied for modelling the white dwarfs, also for the case of the solar-type stars.

‘Model fitting’ is particularly valuable when applied to a set of stars in a cluster or in a binary system: the increase of parameters to be fitted, by assuming same age
and composition for all the components of the system, enhance the quality of the minimization technique. An immediate application of this optimized procedure has been performed, for example, in the case of the α Cen system by several authors (e.g., Eggenberger et al. 2004b; Guenther and Brown 2004). Here, I present the results not yet published obtained by A. Miglio and J. Montalban which implemented the Code Liègeois d’Évolution Stellaire, and by J. Christensen-Dalsgaard, whose code was implemented by T. Teixeira. Many spectroscopic measurements of both components of the α Cen system can be found in literature; for a summary see Morel et al. (2000) and Eggenberger et al. (2004b). Taking into account the numerous observational constraints listed in Table 1 including new seismological observations for α Cen A (Bedding et al. 2004) and α Cen B (Carrier and Bourbon 2003) a common solution was obtained by the two groups, and respectively shown in Figs. 3 and 4. The results indicate that a best fit with all the available observations can be obtained by assuming a mass of about $M_A = 1.11M_\odot$ for α Cen A and of about $M_B = 0.93M_\odot$ for α Cen B, identifying both components as being in the main-sequence phase. The composition was assumed such that the initial hydrogen abundance was $X = 0.7$ and the metallicity $Z = 0.03$.

6. THE POTENTIAL OF THE MIXED MODES

The properties of the solar-like oscillations are expected to change as the stellar structure evolves as a consequence of the hydrogen exhaustion in the core. According to Eq. (1), oscillation frequencies of a given harmonic degree should decrease as the star evolves and the radius increases and should be almost uniformly spaced by $\Delta \nu$ at each stage of evolution. However, while the radial modes seem to follow closely Eq. (1), the frequencies of some non-radial modes appear to increase suddenly at certain stages of evolution (Christensen-Dalsgaard, Bedding and Kjeldsen 1995; Guenther and Demarque 1996). This is due of the occurrence to the so called ‘avoided crossing’. The core contraction, as the star evolves and the radius expands, causes an increase of the local gravitational acceleration and of the gradients in hydrogen abundance, and hence of the buoyancy frequency in the deep interior of the star. As a consequence g modes with high frequencies are allowed to propagate and can interact with a p mode of similar frequency and same harmonic degree, giving rise to a mode with mixed character, which behaves as a g mode in the interior and as a p mode in the upper layers. The interaction can be explained as the coupling of two oscillators of similar frequencies. The effect of the coupling becomes much weaker for modes with higher harmonic degree, since in these cases the gravity waves are better trapped in the stellar interior and better separated from the region of propagation of acoustic waves.

6.1. Mixed modes in η Boo: main-sequence or subgiant star?

As an example of the use of mixed modes as asteroseismic diagnostic in the stars, I consider the case of η Boo. In Di Mauro et al. (2004) authors built a grid of stellar evolutionary sequences tuned to match the position of η Boo in the theoretical HR diagram, considering the inclusion of convective overshooting, and testing the use of different equations of state and of different treatments of convective transport. They concluded that present observations are consistent with two possible evolutionary scenarios for η Boo: (i) a subgiant star in post-main-sequence phase, whose oscillation spectrum contains frequencies of nonradial modes with mixed character due to avoided crossings; (ii) a less evolved star in the main-sequence phase which show p modes with no mixed character and frequencies which follow the asymptotic theory.

6.2. Mixed modes in the red giant ξ Hya

The analysis of the properties of solar-type oscillations has so far confined to stars which lie in the main-sequence or subgiant phase. However there have been reports of detections of solar-like oscillations also in giant stars, and in particular in few red giants (e.g. Buzasi et al. 2000; Merline 1999, Retter et al. 2003; Frandsen et al. 2002; Stello et al. 2004; Retter et al. 2004; Barban et al. these proceedings). Although there are difficulties in the identification of the observed modes and severe problems of theoretical interpretation, Guenther et al. (2000) and Dziembowski et al. (2001) for α Uma and Teixeira et al. (2003) and Christensen-Dalsgaard (2004) for ξ Hya succeed to draw important conclusions on the structural and pulsational properties of red giant stars. They found that observations of these stars are consistent with models in the hydrogen shell-burning phase, or more probably, in the longer lasting successive phase of core helium burning. These stars are characterized by a deep convective envelope and a small convective core. Since the density in the core is quite large, the buoyancy frequency can reach very large values in the central part, with some maxima rising from the steep changes in the molecular weight. In these conditions, g modes of high frequencies can propagate and might eventually interact with p modes giving rise to modes of mixed character. Fig. 6.
Table 2. Relevant parameters, the mass $M$, the $Age$, the luminosity $L$, the effective temperature $T_{\text{eff}}$, the surface radius $R$, the overshooting parameter $\alpha_{\text{ov}}$ and the location $r_{\text{cb}}$ of the base of the convective envelope in units of $R$, for two models of $\eta$ Boo, computed with the OPAL EOS and $Z = 0.04$. Model 1 and 2 represent evolutionary structure respectively in post-main and main-sequence phases.

| Model | $M/M_\odot$ | $Age$ (Gy) | $L/L_\odot$ | $T_{\text{eff}}$ (K) | $R/R_\odot$ | $\alpha_{\text{ov}}$ | $r_{\text{cb}}/R$ |
|-------|-------------|-------------|--------------|----------------------|--------------|----------------|----------------|
| 1     | 1.71        | 2.44        | 9.07         | 6072                 | 2.73         | 0.1            | 0.85           |
| 2     | 1.82        | 1.96        | 8.93         | 6009                 | 2.76         | 0.2            | 0.84           |

Figure 5. Echelle diagrams for Model 1 (left panel) and Model 2 (right panel) of Table 2 which are respectively in the post-main-sequence and main-sequence phases. The open symbols represent the computed frequencies. The filled symbols with error bars show observed frequencies and errors (Kjeldsen et al., 2003). Circles are used for modes with $l = 0$, triangles for $l = 1$, squares for $l = 2$, and diamonds for $l = 3$. Theoretical and observed frequencies are plotted with $\Delta \nu = 40.47 \mu$Hz. The size of the open symbols indicates the relative surface amplitude of oscillation of the modes. Crosses are employed for modes with small predicted amplitude (e.g. g modes). For details see Fig. 4 of Di Mauro et al. (2004).

shows the eigenfunctions for a pure radial mode and for a mode with $l=2$. The eigenfunction of a mixed mode is characterized by a large p-mode like displacement amplitude in the upper layers and also g mode behaviour in the core. Since the radiative damping scales as the third derivative of the displacement, it appears clear that non-radial modes are more easily damped than radial ones. As a consequence, the spectrum of the red-giant stars is quite complex, with a sequence of peaks uniformly spaced, due to radial modes, which have high amplitude and hence have more probability to be detected, and other series of peaks with low amplitude, due to the mixed modes.

7. ASTEROSEISMIC INVERSION TO INFER THE INTERNAL STRUCTURE OF STARS

The asteroseismic inversion is a powerful tool which allows to estimate the physical properties of the stars, by solving integral equations expressed in terms of the experimental data. Inversion techniques are well known and applied with success to several branches of the physics, from geophysics to the radiation theory. Applications to the helioseismic data have been studied extensively and inversion methods and techniques have been reviewed and mutually compared by several authors, leading to extraordinary results about the global properties of the Sun (see review by, e.g., Di Mauro 2003).

The experience acquired in helioseismology on inverting mode frequency data provides a good starting point for asteroseismic inversion. The several techniques developed for asteroseismic inversions support ‘linear’ or ‘non-linear’ approaches. The linear inversion is based on the linearization of the equations of stellar oscillations about a known reference model, under the assumption that perturbations are small. This results in integral equations which can be used to determine the corrections which have to be imposed on the reference model in order to obtain the observed oscillation frequencies. See the review of Basu (2003) about the applications and problems of linear inversions. The non-linear inversion based on the analytical resolution of the integral equations have recently been discussed and compared with the linear inversions by Roxburgh and Vorontsov (2003b). The choice of one or the other approach is not an easy task. The linear inversion, although it is a convenient approach to study the innermost part of the stars, represents a questionable approximation. The global parameters of the stars, such as mass, radius, luminosity and chemical composition are not well known, so that the structure of the model can-
Earlier attempts in generalizing the standard helioseismic differential methods to find the difference of structure between the observed star and a model have been applied on artificial data with encouraging results by Gough and Kosovichev (1993) and Roxburgh et al. (1998). More recently, Berthomieu et al. (2001) carried out a careful analysis of the results to be expected in inversion for stars, showing that the kernels and hence the solutions can be well concentrated only in the inner core. Patermø, Di Mauro and Ventura (2003) demonstrated that if a more realistic set of low degree modes is used, such as the accurate observations for the Sun, the inversion allows the inference of the inner core of the stars below $0.3R_{\odot}$. In addition, Basu, Christensen-Dalsgaard and Thompson (2002) demonstrated that the success of the inversion results depends strongly on the choice of variables to be inverted, and in the case of solar-type stars a preferable choice seems to be represented by the pair $u$, $Y$ (Basu 2003), where $u$ is the squared isothermal sound speed and $Y$ is the helium abundance in the convective zone. Further aspects can be also found in Thompson and Christensen-Dalsgaard (2002).

### 7.1. Inversion for Procyon A

The Procyon binary system ($\alpha$ Cni) consists of a F5 subgiant primary and a white-dwarf secondary. Procyon A, the primary, has already attracted the attention of stellar seismologists for its proximity and brightness. Recently, Martić et al. (2004), by observing this star with the CORALIE spectrograph, have been able to identify 55 p-mode frequencies in the range $250 - 1400\mu$Hz, with harmonic degrees $l = 0 - 2$, characterized by an average large frequency separation $\Delta \nu = 53.6 \pm 0.5\mu$Hz and an average small frequency separation $\delta \nu = 5.1 \pm 0.5\mu$Hz. These results appear to be in good agreement with those obtained by Eggenberger et al. (2004a) who, from Doppler-velocity observations, have been able to identify 23 p modes in the same range of observed frequencies and with an average large separation of $\Delta \nu = 55.5\mu$Hz. The observ-
Table 4. Basic parameters, $M$, Age, $Z$, $L$, $T_{\text{eff}}$, the surface radius $R$, small and large separations for two models of Procyon A, computed with the OPAL EOS (1996). Model 3 and 4 represent evolutionary structure respectively in main-sequence and in late main-sequence phases.

| Model | $M/M_\odot$ | Age (Gy) | $Z$ | $L/L_\odot$ | $T_{\text{eff}}$ (K) | $R/R_\odot$ | $\delta
_0$ ($\mu$Hz) | $\Delta
_0$ ($\mu$Hz) |
|-------|-------------|----------|-----|-------------|----------------------|------------|----------------|----------------|
| 3     | 1.47        | 1.78     | 0.016 | 6.88        | 6501                 | 2.07       | 4.2            | 53.6           |
| 4     | 1.42        | 2.51     | 0.020 | 6.72        | 6481                 | 2.05       | 4.2            | 53.6           |

Figure 8. Echelle diagrams for Model 3 (left panel) and Model 4 (right panel) which are respectively in the main-sequence and the late-main-sequence phases. Open symbols represent the computed frequencies. The filled symbols show observed frequencies by Martić et al. (2004); blue filled symbols correspond to old identified frequencies given between brackets in Martić et al. (2004). Theoretical and observed frequencies are plotted with $\Delta
= 53.6 \mu$Hz and a reference frequency $\nu_0 = 635 \mu$Hz. Circles are used for modes with $l = 0$, triangles for $l = 1$, squares for $l = 2$, and diamonds for $l = 3$. The size of the open symbols indicates the relative surface amplitude of oscillation of the modes. Crosses are employed for modes with small predicted amplitude (e.g. g modes).

As in the case of η Boo, it appears difficult to constrain the evolution state of this star by considering only model results. The location of the star in the HR diagram identifies Procyon A as being in the late-main-sequence phase of core hydrogen burning if the mass is taken in the range $M = (1.42 - 1.46) M_\odot$. However, it is also possible that Procyon A is still in the core hydrogen-burning main-sequence phase, if models are computed by assuming a mass higher by about 5%, as shown by the evolution tracks plotted in Fig. 7.

The problem of identifying the evolutionary state of this star can be approached by studying the pulsational characteristics of the computed models. The relevant parameters are given in Table 4 for two models of Procyon A selected for the pulsation analysis, which have surface radius, large and small theoretical separations consistent with the observed values. The pulsational features are clearly illustrated by looking at the echelle diagrams for the two different models (Fig. 8). It is evident that both the two considered models show a very good agreement with the observed frequencies. The only difference which can be noticed between the two models is that for the model in the late-main-sequence phase (Model 4), the nonradial modes with $l = 1$ show considerably more scatter than the radial ones. This aspect is associated with the avoided crossings which introduce a less regular structure in the frequency spectrum. On the contrary, the modes of the models in the main-sequence phase (Model 3) show no occurrence of avoided crossing. Fig. 8 shows that the observed frequencies for $l = 1$ do not seem to indicate the presence of some irregularities similar to those seen for the computed frequencies of
the more evolved model. If this were to be confirmed by more accurate sets of observed data, Procyon A would be a star definitely identified as in the main-sequence phase.

On considering the large amount of data available, in order to obtain additional information on the internal characteristics of Procyon A, it has been thought to apply inversion techniques to the observed frequencies of this star (Martic et al. 2004). The results shown here have been obtained by applying a linear inversion procedure. If it is assumed that the equation of state of the star is known, the relative differences $\delta \omega_i/\omega_i$ between the frequencies of Procyon A and the model are related to the differences $(\delta u/u, \delta Y)$ in squared isothermal sound speed $u = p/\rho$ and helium abundance $Y$ in the convective zone, between the structure of the star and the reference model, through the following integral equation (e.g., Dziembowski, Pamyatnykh and Sienkiewicz 1990):

$$\frac{\delta \omega_i}{\omega_i} = \int_0^R K_{u,Y}^i \frac{\delta u}{u} \, dr + \int_0^R K_{Y,u}^i \delta Y \, dr + \epsilon_i , \quad (9)$$

where $K_{u,Y}^i$ and $K_{Y,u}^i$ are the kernels calculated for each mode $i = n, l$ of the set considered. Equation (9) has been solved by applying the so called SOLA method (Piipers and Thompson 1992, 1994) well used in helioseismology. Fig. 9 shows the relative differences in $u$ between Procyon A and the two selected models (Models 3 and 4 of Table 4) as functions of the fractional radius. The results indicate that the difference between Model 3 and Procyon A are extremely small, below the 5%, while Model 4 shows very large deviations indicating that linearizations cannot be applied to this model, whose properties are extremely different from those of the observed star.

From both the inversion results and analysis of the echelle diagrams, it can be concluded that present observations of oscillation frequencies of Procyon A seem to indicate that this star is in main-sequence phase. It is worth to mention that Provost et al. (these proceedings) arrived exactly to the opposite conclusion by considerations on the recent measurements of radius by Kervella et al. (2004). It is hoped that in the future more accurate observations will allow a detailed test of stellar modelling and evolution theories of Procyon A, in order to identify its present evolutionary phase.

8. SHARP FEATURES

Another important property of the oscillation spectra is that sharp variations localized at certain acoustic depth in the structure of pulsating stars produce a distinctive quasi-periodic signal in the frequencies of oscillation. The characteristics of such signal are related to the location and thermodynamic properties of the layer where the sharp variation occurs. Sources of sharp variations are the borders of convection zones and regions of rapid variation in the first adiabatic exponent $\Gamma_1$, such as the one that occurs in the region of second ionization of helium. Fig. 10 shows the variation of $\Gamma_1$ in the upper layers for a ZAMS model of a star of $1.2 M_\odot$ and its relative periodic signal as seen in the large separation.

A general expression (valid for low degree modes) for the signal generated is of the form (e.g. Monteiro, Christensen-Dalsgaard and Thompson 1994, 1998, 2000; Roxburgh and Vorontsov 1994):

$$\delta \omega_{n,l} \simeq A(\omega) \cos(2\omega_{n,l} \tau_d + \phi_0) , \quad (10)$$

where $A(\omega)$ is an amplitude as function of frequency $\omega$, which depends on the properties of the sharp variation; $\phi_0$ is a constant phase; $\tau_d$ is the acoustic depth of the feature. Several attempts have been tried in order to isolate the generated oscillatory components directly from the frequencies of oscillations or from linear combination of them (large separations, second differences, etc). The common approach consists in removing a smooth component from the frequencies and to fit the residual signal.
to a theoretical expression, like that of Eq. (10), which is related to the properties of the sharp feature.

This method can be applied, for example, to determine the properties of the base of the convective envelope (Monteiro, Christensen-Dalsgaard and Thompson 2000; Ballot, Turck-Chièze and García 2004) and in particular, to put limits on the extension of the convective overshoot (Monteiro, Christensen-Dalsgaard and Thompson 2002), or to investigate the border of the convective core (Mazumdar and Antia 2001; Nghiem et al. 2004). But in particular, this peculiar property of the oscillation frequencies can be used to infer the helium abundance in the stellar envelope, by studying the variation of $\Gamma_1$ in the region of second ionization of helium. Several authors have tried to develop or refine their own fitting function to infer the helium abundance (Lopes et al. 1997; Monteiro and Thompson 1998; Pérez Hernández and Christensen-Dalsgaard 1998; Miglio et al. 2003; Basu et al. 2004 and these proceedings; Houdek and Gough in these proceedings). Unfortunately this technique cannot be applied to stars with mass larger than $1.4 M_\odot$, owing to the contamination in the oscillatory signal coming from the first helium ionization zone. In fact, one of the main problems in the application of such an approach rises from the fact that signals coming from different sharp features in the interior of the star might overlap generating a complex behaviour, as it was noticed by Mazumdar and Antia (2001). A similar effect was also considered by Montgomery, Metcalfe and Winget (2003) in the case of g modes observed in white dwarfs. Very recently Miglio and Antonello (2004) have shown that in the solar-type stars the problem of distinguishing between a signal generated near the core or near the surface in the star can be easily solved by considering modes of different degree or different seismic indicators. Unfortunately this method cannot successfully be applied in the case of the white dwarfs. At the moment, L. Mantegazza et al. (private communication) is working on a new technique based on the Principal Component Analysis (Golyandina et al. 2001; Ghil et al. 2002) in order to isolate all the different oscillatory components directly from the oscillation frequencies.

It was already pointed out in Miglio et al. (2003) that a more general analysis of the effect of the bump in $\Gamma_1$ can be obtained by performing an inversion of the observed frequencies, by using the integral equation by Gough and Thompson (1991). At the moment the method has been tested only in order to find differences between a reference equilibrium model and a fictitious model which differs from the reference one only because $\Gamma_1$ has been smoothed in the region of second helium ionization. In this case, the inversion of the artificial data set of modes with $0 \leq l \leq 3$ is able to reconstruct the variations in $\Gamma_1$ for models with masses from low up to $2 M_\odot$, if the inverted set includes also low modes with low radial order ($n = 4 - 8$), whose kernels seem to have considerable weight in the upper regions.

9. THEORETICAL RESULTS ON OTHER PULSATING STARS

9.1. White dwarfs

The results obtained by applying seismic techniques to white dwarfs represent probably the biggest success of asteroseismology (e.g. Vauclair 1997; Kawaler 1998; O’Brien 2003). In fact, they represent the stars, other than the Sun, in which the largest number of oscillation frequencies has been detected. The white-dwarf stars (WDs) represent the most common end point of stellar evolution. Their physical structure is very simple: a degenerate carbon-oxygen core and an extremely thin, mostly non-degenerate, surface layer of pure H or He. Pulsations in WDs represent a short-lived phase during their evolution: the majority of them do not show pulsations except in very narrow regions of instability located along their cooling track in HR-diagram: the planetary nebulae hot nuclei (PNN), the hot DO (H deficient pre-white-dwarfs), the warm DB (WDs with He rich photospheres) and the cool DA (WDs with H rich photospheres) stages. WD variables are mainly multi-periodic pulsators in which high-order g-modes of low angular degree are excited, but the mechanism triggering the pulsations is still matter of discussion (e.g., Cox 2002).

Asteroseismology has been applied to identify the structural differences between different types of white dwarfs, and there is hope that it will permit in the future also the identification of the various origins. In fact, while theoretical models can be easily calculated without major efforts, it is still not possible to say which stellar structure might represent the precursor of each of the several types of white dwarfs.

Asymptotic theory, which gives a clear prediction of period spacing for these stars, has helped to determine the total mass, the rotation rate, the magnetic field strength and even the mass of the outer layers of some selected targets. In particular, since the rate of evolution of white dwarfs depends primarily on the thickness of their surface layers, it might be possible to measure the galactic age through the luminosity function (e.g. Wood and Oswalt, 1998). However, the most interesting characteristics studied by asteroseismology are the structure and composition of the C/O cores of white dwarfs and the evidence for crystallization of the cores in the most massive stars (DAVs). As shown by Montgomery and Winget (1999), crystallization leaves a typical signature on the period spectrum with an amplitude which increases as the fraction of crystallized matter increases. Recently, two recent debates have been opened on the possibility to measure the fraction of crystallized matter (see Metcalfe, Montgomery and Kanaan 2004; Brassard and Fontaine these proceedings) and the $^{12}C(\alpha, \gamma)^{16}O$ reaction rates in white dwarfs (see Metcalfe, Salaris and Winget 2002; Fontaine and Brassard, 2002) by studying their pulsational properties.

Some WD variables have, or are expected to have, spec-
tra rich enough in modes to provide useful constraints on their rotational rates. Attempts to derive the internal rotational profiles by using the observed splitting and adopting forward calculations and inversion techniques have been presented in several works (e.g. Kawaler, Sekii and Gough 1999, Vauclair et al. 2002).

Finally, another asteroseismic achievement is represented by the prediction that, during the pre-white-dwarf stage, the dominant energetic process is the neutrino cooling. Since evolutionary changes affect the pulsation periods, the detection of secular period changes in the hottest WD pulsators allow to measure their cooling rates, providing experimental constraints on the theories of neutrino emission in dense plasmas and thermal neutrino production rates (O’Brien and Kawaler 2000).

9.2. δ Scuti stars

δ-Scuti stars are typically population I, A and F main-sequence or slightly post main-sequence objects of about $2 - 2.5 M_\odot$. They are located in the lower part of the classical Cepheid instability strip where the $\kappa$-mechanism is expected to drive low-order radial and non-radial p-modes, modes with a mixed p- and g-mode character, and possibly g-modes. Most of the δ-Scuti stars are moderate or rapid rotators with surface velocities up to $100 - 200 \text{ km s}^{-1}$.

The identification of the oscillation modes is a very complex task for these stars, since the asymptotic theory does not apply to the excited modes (low-order p-modes). However, the main problem depends on the scarcity of observed modes, among the hundreds of excited pulsations predicted by theory. Rapid rotation and possible differential rotation produce a rather complex power spectrum, making more difficult the identification of modes without additional information.

Thus, at the present, asteroseismology is able to put only some constraints on the internal structure of δ Scuti stars. Many interesting and comprehensive reviews on asteroseismology of δ-Scuti stars can be found in the recent literature (e.g., Breger and Montgomery 2000; Goupil and Talon 2002; Paparo, these proceedings).

9.3. Rapidly oscillating Ap (roAp) stars

The rapidly oscillating Ap (roAp) variables, a subgroup of the chemically peculiar A-type stars, are H core-burning stars of mass $\approx 2 M_\odot$, characterized by strong dipole magnetic fields of the order of a few kG, which are the most intensive fields in main-sequence stars. Their position in the H-R diagram overlaps the δ-Scuti star instability strip (see Fig. 1), but the two groups of variables differ significantly in the behaviour of the excited modes. The roAp stars pulsate in high-order ($n \geq 20$), low-degree p-modes. Most of the roAp stars pulsate in almost pure dipole ($l = 1$) modes. The reason why stars with similar luminosity and effective temperature may show so different pulsation characteristics is not completely understood, even though it is clear that the strong magnetic fields could play an important role in selecting the excited modes.

The parallax measurements have provided a primary opportunity to check independently the predictions of roAp stars seismology. The results pointed out the tendency for seismic predicted values to be systematically slightly smaller than the Hypparcos ones (Matthews et al. 1999).

The observed pulsation properties of roAp stars were originally explained in terms of the so-called oblique pulsator model (Kurtz 1982), in which pulsation and magnetic axes are mutually aligned, but tilted with respect to the rotation axis. The oblique pulsator model has been improved in order to better reproduce the observed power spectra of roAp stars (e.g. Dziembowski and Goode 1996) and more recently revisited by Bigot and Dziembowski (2002) which found out that the rotational effects seem to be so strong to result in a pulsation axis inclined...
with respect to both the rotation and the magnetic axes. However, the new theory still needs to be tested.

The excitation mechanism for the roAp stars is still an unresolved problem, although extensively debated over the years. The $\kappa$-mechanism in the He II ionization region which drives $\delta$-Scuti star oscillations has been demonstrated to be inadequate for exciting high-frequency modes in roAp stars, even though the two groups of variables share the same location in the H-R diagram. A great deal of alternative models were then proposed, each of them studying various effects which could affect the $\kappa$-mechanism giving rise to mode instability, such as a reduced convective efficiency due to the action of the magnetic field (Balmforth et al. 2001), settling of He (Dolez and Gough 1982), presence of a stellar wind (Dolez, Gough and Vauclair 1988) or a chromosphere (Gautschy, Saio, Harzenmoser 1998).

10. CONCLUSION

Asteroseismology constitutes a unique approach to the direct investigation of stellar structure and evolution which can significantly improve our knowledge of astrophysics, giving information on the structure, dynamics and evolutionary stage of the stars. Asteroseismology is still far from the great results of helioseismology, but it is clear that its success will depend on the amount and quality of data collected by the ground-based observatories, the significant improvement in the accuracy expected from the future space missions, and the progresses in the development of the application to the stars of techniques for the data inversion and study of the effects of sharp features on the oscillation frequencies.

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