Transformation devices: carpets in space and space-time

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Here we extend the theory of space-time or event cloaking into that based on the carpet or ground-plane reflective surface. Further, by recasting and generalizing a scalar acoustic wave model into a new mathematically covariant form, we also show how transformation theories for optics and acoustics can be combined into a single prescription. The single prescription, however, still respects the fundamental differences between electromagnetic and acoustic waves, which then provide us with an existence test for any desired transformation device – are the required material properties (the required constitutive parameters) physically permitted by the wave theory for which it is being designed? Whilst electromagnetism is a flexible theory permitting almost any transformation-device (T-device) design, we show that the acoustic model used here is more restricted.

I. INTRODUCTION

The concept of electromagnetic cloaking has now been with us for more than five years [1, 2], but has been recently revitalized by the introduction of the concept of space-time cloak [3, 4]. Here, in light of the many variants of spatial cloaking that now exist – ordinary cloaks, carpet cloaks [5], exterior cloaks [6], we extend the possible implementation to include space-time “carpet” cloaks. Of course, cloaking is not the only application of transformation theories, as evidenced not only by spatial illusion devices [1, 2], but also ideas for spacetime illusion devices [4] that appear to trick – but only trick – causality [1]. Other applications such as beam control [3, 5, 6] geodesic lenses [13, 14], hyperbolic materials [15, 16], and cosmological models [4, 17–19] are also possible.

Like a spatial carpet cloak, the space-time “event” carpet cloak is one sided and reflective, and is less singular than a traditional spatial cloaking implementation. Rather than hide a region of space in perpetuity, however, the “event” carpet cloak allows the region to always appear visible, even though a finite segment of its timeline is forced to take place in darkness. This dark period means that events therein are edited out of visible history. However, those missing events are covered up by clever manipulation of the light signals that communicate to observers their information about events. Alternatively, the transformation can be adapted to temporarily include extra events, rather than exclude them; thus creating a space-time “periscope”. We might even relax our preference for invisibility devices, and construct one that appears bigger on the inside than the outside – a “tardis”.

In this paper we deliberately use first order equations to model our wave mechanics [20], so that a pair of them are needed (in concert with constitutive or state equations of some kind) to generate wave behaviour. If desired, the first order equations can be substituted inside one another to give a familiar second order form, but in fact the first order formulation is both less restrictive and more general. Indeed, we will see that it is suggestive of a generalization to a Transformation Mechanics valid for any wave equations expressible in such a form, being also related to a “transformation media” concept addressing the material properties required by T-devices. Notably, here we apply the event carpet cloak and periscope transformations to both electromagnetism (EM) and a simple pressure-acoustics (p-acoustics) model, demonstrating how specific wave models allow or restrict the set of possible T-devices.

We introduce the “wave mechanics” models for EM and our p-acoustics in section II, and show how the two can be unified. This then allows us to unify transformation optics, transformation p-acoustics, and indeed any other compatible wave transformation theory into a general transformation mechanics. After this, we specialise to ground-plane T-devices – in section III we investigate those based on the (spatial) carpet cloak, and in IV we generalize them into space-time carpet T-devices. In section V we propose a scheme for implementing an EM space-time carpet cloak, before concluding in section VI.

II. WAVES AND TRANSFORMATIONS

In this paper we do not restrict ourselves to a single type of wave mechanics in which to construct ground plane T-devices. Instead, we will show that a unified procedure is possible, encompassing both electromagnetism and a simple pressure-acoustics model; and indeed any linear wave mechanics that can be described within a theory containing two second-rank field tensors and one fourth-rank constitutive tensor. We will start with the ordinary vector calculus descriptions, and then show how they can be re-expressed in the tensor form; and also describe how the tensor descriptions can be abbreviated into a more accessible matrix form. This process is of course well known for EM, but here we use an unconventional tensorization so that it more easily matches up with the p-acoustic description and other possible generalisations, as well as with straightforward matrix algebra versions. A simpler 1D introduction to this description can be seen in [4]. After the wave theory descriptions, we show how transformations intended to implement specific T-devices affect the differential equations and constitutive relations.

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A. Electromagnetism

Maxwell’s equations [27] are often written as a set of vector differential equations with two pairs of field types, and two constitutive relations. These are

\[ \partial_t D(\vec{r}, t) = \nabla \times \vec{H}(\vec{r}, t) - \vec{J}_f(\vec{r}, t) \]  
\[ \nabla \cdot \vec{B} = 0, \]  
\[ \partial_t B(\vec{r}, t) = -\nabla \times \vec{E}(\vec{r}, t), \]  
\[ \nabla \cdot \vec{E} = 0, \]  

with constitutive relations

\[ \vec{B} = \mu \vec{H} - \mu_0 \vec{E} \rightarrow \vec{H} = \eta \vec{B} + \beta \vec{E}, \]  
\[ \vec{D} = \varepsilon \vec{E} + \mu_0 \vec{H} \rightarrow \vec{D} = \varepsilon \vec{D} + \alpha \vec{F}, \]

where the permittivity matrix is \( \varepsilon' = \varepsilon - \alpha \beta \), and \( \eta \) is the inverse of the more common permeability matrix \( \mu \). Although in many circumstances the first (LH) form is preferred, for the tensor form discussed next the second (RH) form is used instead. The matrices \( \alpha \) and \( \beta \) contain information about magnetoelectric coupling, and follow \( \beta = \alpha^T \) [27].

Since we are interested primarily in freely propagating EM fields, we will assume that there are no free electric charges (i.e. \( \rho_\epsilon = 0 \)), and as is usual, we have already assumed in eqn. [24] that there are no magnetic charges.

In tensor form the differential equations have a remarkably simple form. In order to emphasise their similarities, we chose to use the usual G tensor density (i.e. \( G^{\alpha\beta} \)) but the dual of the \( \mathcal{F} \) tensor (i.e. \( \mathcal{F}^{\alpha\beta} \)), with components\( \mathcal{F}^{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon^{\gamma\delta} F_{\gamma\delta} \)), where \( \varepsilon^{\alpha\beta\gamma} \) is the antisymmetrization symbol. The equations can then be written as

\[ \partial_\alpha \mathcal{F}^{\alpha\beta} = 0, \quad \partial_\alpha G^{\alpha\beta} = J^\alpha. \]  

Using \( \mathcal{G} \) and \( \mathcal{F} \) means that we must then use a mixed form for the constitutive tensor \( \mathcal{X} \) in the constitutive relations, with

\[ G^{\alpha\beta} = \frac{1}{2} \mathcal{X}^{\alpha\beta\gamma} \mathcal{F}_{\gamma\delta} \mathcal{F}^{\delta\gamma}. \]

This use of \( \mathcal{F}^{\alpha\beta} \), \( \mathcal{G}^{\alpha\beta} \), and \( \mathcal{X}^{\alpha\beta\gamma} \), both generalizes better and maps onto the vector calculus representation more simply.

Note that since \( J^\alpha \) (and its vector counterpart \( \vec{J} \)) is a current density, this means that \( \mathcal{G} \) is also a density. Likewise, although the usual EM tensor \( \mathcal{F} \) is not a density, its dual \( \mathcal{F} \) is. Thus eqns. [27] must be transformed as densities. Consequently, the constitutive tensor \( \mathcal{X}^{\alpha\beta\gamma} \) is not a density.

In terms of generating wave solutions, the constitutive relations connect the two tensor differential equations together into a single system, and so allow (wave) solutions to be found (i.e. exist). This is most easily seen using the vector form, where we can use the constitutive relations to allow us to substitute one of the Maxwell curl equations into the other, giving us one of the four [23] possible electromagnetic second order wave equations.

Matrix representations of the (dual) tensor \( \mathcal{F} \) and (non-dual) tensor density \( \mathcal{G} \) can be written out in \([t,x,y,z]\) coordinates, as

\[ [\mathcal{F}^{\alpha\beta}] = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix}, \]  
\[ [\mathcal{G}^{\alpha\beta}] = \begin{bmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{bmatrix}, \]

where the antisymmetric nature of the tensors is now clearly visible. In this picture, we note also that \( \partial_\alpha \) has become the row vector \( [\partial_t, \partial_x, \partial_y, \partial_z] \), and the current density \( J^\alpha \) also becomes a row vector.

The constitutive tensor \( \mathcal{X}^{\alpha\beta\gamma} \) is rank four so that a direct matrix representation is too big to display. Nevertheless, its permittivity entries convert \( E_i \) from \( \mathcal{F}^{\alpha\beta} \) into \( D_i \) from \( \mathcal{G}^{\alpha\beta} \), thus they will have the same units as permittivity (i.e. as \( \varepsilon \)); likewise the permeability entries convert \( B_i \) from \( \mathcal{F}^{\gamma\delta} \) into \( H_i \) from \( \mathcal{G}^{\alpha\beta} \), thus they will have the same units as inverse permeability (i.e. as \( \mu^{-1} = \eta \)).

We can follow the usual method of achieving a convenient matrix-like compacted representation for \( \mathcal{F}^{\alpha\beta} \) and \( \mathcal{G}^{\alpha\beta} \), using the fact that each tensor has only 6 unique entries. For all combinations of \( \alpha \beta \) indices, we select out the relevant row and column coordinate pairs \( tx, ty, tz, yz, zx, xy \) in sequence, so we can write \( \mathcal{F}^{\alpha\beta} \) and \( \mathcal{G}^{\alpha\beta} \) as column vectors \( [\mathcal{F}^A] \) and \( [\mathcal{G}^B] \) in turn, being

\[ [\mathcal{F}^A] = \begin{bmatrix} B_x, B_y, B_z, -E_x, -E_y, -E_z \end{bmatrix}^T, \]  
\[ [\mathcal{G}^B] = \begin{bmatrix} D_x, D_y, D_z, H_x, H_y, H_z \end{bmatrix}^T. \]

As a result the rank-4 constitutive tensor can also be compacted, and so be represented by a 6x6 matrix. Again the first (upper) index \( A \) spans the matrix rows, so that

\[ \mathcal{X}_B^A = \begin{bmatrix} \alpha & -\varepsilon \\ -\eta & -\beta \end{bmatrix}, \]

so that the constitutive relation now has a matrix form,

\[ [\mathcal{G}^A] = [\mathcal{X}_B^A] [\mathcal{F}^B], \]  

which is particularly convenient for manual calculation. Normally when using this kind of representation, the all-upper index tensor \( \mathcal{X}^{\alpha\beta\gamma\delta} \) is used, but here we are using the mixed tensor form \( \mathcal{X}^{\alpha\beta\gamma} \). This means that our matrix expression looks different; notably a simple isotropic non-magnetoelectric medium, with a diagonal \( \mathcal{X}^{\alpha\beta\gamma} \) (matrix) is now block off-diagonal.

Note that the matrix \( \mathcal{X}_B^A \) only includes half of all the allowed elements of \( \mathcal{X}^{\alpha\beta\gamma} \), but the missing elements are all duplicates.
of included ones\(^1\). However, each element of \([\chi]\) nevertheless independently links any component of \([F]\) to any other of \([G]\); there is no duplication, or a priori requirement that certain constitutive parameters must be equal or otherwise closely related. As a result, each constitutive property of an electromagnetic material can (at least mathematically) be adjusted independently, meaning that any well specified T-device might be constructed.

### B. Pressure Acoustics

Here we introduce a simplified pressure acoustic (p-acoustic) theory, equivalent to one describing linear (perturbative) acoustic waves on a stationary background fluid; although it also encompasses many other types of wave which contain a scalar component. For linearized acoustic waves\(^2\) in pressure \(p\) and fluid velocity \(\vec{u}\), we have constitutive parameters \(\kappa\) (compressibility) and \(\bar{\rho}\) (mass density). As well as the traditional quantities, we also specify both a scalar \(\vec{p}\), and a momentum density \(\vec{V}\), giving a total of four quantities that in combination emphasise both the similarities and differences compared with the description of EM; this can be compared to e.g. the treatment by Sklan\(^3\). It also means that p-acoustics is also a more general model than the comparable traditional ones, which typically reduce the equations back down to \(p\) and \(\vec{u}\) or even just a second order wave equation in \(p\).

Here, however, in order to maximise the similarity with the EM notation, we will substitute the scalar \(\vec{p}\) with a number density \(P\), the fluid velocity \(\vec{u}\) with a velocity density \(\vec{v}\), and the constitutive parameters become an inverse energy \(\kappa\) and a particle mass \(\bar{\rho}\). Since the background fluid is stationary, for small amplitude waves the \((\vec{v}, \vec{V})\)\(^i\) term that would usually appear in acoustic wave equations is second order, and hence can be neglected.

For scalar quantities \(P\) and \(p\), and vectors \(\vec{V}\) and \(\vec{v}\), the wave equation pair, here used to model p-acoustics, is

\[
\begin{align*}
\partial_t P(\vec{r}, t) &= -\nabla \cdot \vec{v}(\vec{r}, t) + Q_P(\vec{r}, t), \\
\partial_t \vec{V}(\vec{r}, t) &= -\nabla \cdot \vec{v}(\vec{r}, t) + \nabla \times \vec{V}(\vec{r}, t), \\
\nabla \times \vec{V} &= \Sigma, \\
\nabla \times \vec{v} &= \sigma,
\end{align*}
\]

(15.16)

where in simple cases \(P = \kappa p\) is a number density related to the pressure \(p\) by an inverse energy \(\kappa\), and the momentum density \(\vec{V} = \vec{p}v\) is related to the velocity-field density \(\vec{v}\) by a matrix of mass parameters \(\rho = [\rho_i^j]\). There are two allowed types of source, a particle number (density) source \(Q_P\), and a momentum (density) source \(\vec{Q}_V\).

The differential eqn. (2.15) is related to the conservation of particle number (and conservation of mass) in a microscopic acoustic model, and eqn. (2.16) ensures conservation of momentum. The third equation shows the p-acoustics analogue of charge, and just as for the EM case, where we are interested primarily in freely propagating acoustic fields, we set these to zero (\(\Sigma = 0, \sigma = 0\)).

The most general constitutive relations for coupling between the scalar and vector fields, can be written

\[
P = \kappa^{-1} p - \kappa^{-1}\vec{\alpha} \cdot \vec{v} \rightarrow p = \kappa P + \vec{\alpha} \cdot \vec{v}
\]

(2.18)

\[
\vec{V} = \vec{\beta} \kappa^{-1} p + \vec{\rho} \cdot \vec{v} \rightarrow \vec{V} = \vec{\beta} P + \vec{\rho} \cdot \vec{v}
\]

(2.19)

where \(\rho \cdot \vec{v} = \rho' \cdot \vec{v} + \vec{\beta}(\vec{\alpha} \cdot \vec{v})/\kappa\). Although in many circumstances the first (LH) form might be used, for the tensor form discussed next the second (RH) form is instead preferred. The vector \(\vec{\alpha}\) parameterizes some kind of a velocity \(\rightarrow\) pressure coupling, and \(\vec{\beta}\) parameterizes a pressure \(\rightarrow\) momentum density coupling.

Here the physical meanings of \(\vec{v}\), \(\vec{V}\), and \(p\), \(P\) mean that both eqns. (2.15), (2.16) transform like densities, but the constitutive relations eqns. (2.18), (2.19) do not – just as for EM.

Just as for EM, we can now embed \(P, \vec{v}\) into a tensor density, which to ensure compatible notation we will call \(\vec{F}, \vec{F}\), with contravariant components \(\alpha\beta\). We also embed the density fields \(p, \vec{V}\) into a tensor density \(G\) with components \(\alpha\beta\); and populate a constitutive tensor \(\chi\) with \(\kappa, \rho, \vec{\alpha}, \vec{\beta}\) appropriately.

The p-acoustic differential equations now appear in the same form as for EM, being

\[
\partial_\alpha \chi^{\alpha\beta} = J^\beta, \quad \partial_\beta G^{\alpha\beta} = K^\alpha,
\]

(2.20)

with source terms \(J^\alpha, K^\alpha\). The constitutive relations are

\[
G^{\alpha\beta} = \chi^{\alpha\beta} \delta^\gamma_\delta \delta^{\gamma\delta}. \quad \text{(2.21)}
\]

Here the only differences from EM are the internal structure – how the tensors are populated – and a choice to omit the factor of one half in the constitutive relations. Again, the \(G\) and \(\alpha\beta\) tensor densities and their differential equation must transform like densities, but the constitutive tensor \(\chi\) does not.

In \([t, x, y, z]\) coordinates the tensor \(\alpha\beta\) and density tensor \(G^{\alpha\beta}\) can be written out using field matrices

\[
\begin{bmatrix}
P & 0 & 0 & 0 \\
v_x & p & 0 & 0 \\
v_y & 0 & p & 0 \\
v_z & 0 & 0 & p
\end{bmatrix}, \\
\begin{bmatrix}
\alpha\beta \\
\chi^{\alpha\beta}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
v_x & p & 0 & 0 \\
v_y & 0 & p & 0 \\
v_z & 0 & 0 & p
\end{bmatrix}
\]

(2.22)

with source terms \(J^0 = Q_P\) and \(K^i = \vec{K}^i = \vec{Q}_V\). However, both \(\vec{J} = [J^i] = 0\) and \(K^0 = 0\).

Here the constitutive tensor \(\chi^{\alpha\beta}\) relates the \(P\) element from \(\alpha\beta\) to the \(p\) from \(G^{\alpha\beta}\), and is thus \(\sim \kappa\); likewise the \(v_i\) elements from \(\alpha\beta\) are related to \(V_i\) from \(G^{\alpha\beta}\) by \(\sim \rho\).

Note, however, that while this conveniently encodes the model of pressure acoustics we consider here, it is not the most general that might be formulated. The most general theory would be to consider field tensors that are symmetric, to contrast with the EM representation using antisymmetric field tensors. This would give not two sets of four amplitudes (i.e.

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\(^1\) This is why no 1/2 appears in the matrix constitutive relations. But we must still account for this doubling up when evaluating transformations later.
\(P\) and \(v_i\), with \(p\) and \(v_j\), but two sets of 10 amplitudes. However, we leave discussion of such details to a later work; including describing how pentamode acoustics \(\text{[26]}\) can be represented in this way.

As with EM, the tensor constitutive representation can be compacted. Here we use the fact that the elements \(G^{xx}, G^{yy}\) and \(G^{zz}\) are all just \(p\), and so write a compact vector for \(G\) using this knowledge; similarly for \(sF\). The resulting field column vectors \(\begin{bmatrix} G^A \\ sF^B \end{bmatrix}\) are

\[
\begin{bmatrix} G^A \\ sF^B \end{bmatrix} = \begin{bmatrix} p, v_x, v_y, v_z \end{bmatrix}^T. \tag{2.23}
\]

The matrix representation of \(\chi_{\alpha \beta}^{G_{\alpha \beta}}\) has the upper index denoting rows, and the lower index denoting columns, so that the 4x4 constitutive matrix is

\[
\begin{bmatrix} \chi_{\alpha \beta}^{G_{\alpha \beta}} \end{bmatrix} = \begin{bmatrix} \kappa & \alpha^T \\ \beta & \rho \end{bmatrix}. \tag{2.25}
\]

the matrix constitutive relation is

\[
\begin{bmatrix} G^A \\ sF^B \end{bmatrix} = \chi_{\alpha \beta}^{G_{\alpha \beta}} \begin{bmatrix} G^A \\ sF^B \end{bmatrix}. \tag{2.26}
\]

\[
\begin{bmatrix} G^{ww} \\ G^a \\ G^a \end{bmatrix} = \begin{bmatrix} \kappa & \alpha^T \\ \beta_a & \rho \end{bmatrix} \begin{bmatrix} G^{ww} \\ G^a \\ G^a \end{bmatrix}. \tag{2.27}
\]

Note that we must remember that the \(G^{ww}\) element can be freely substituted by any of \(G^{xx}, G^{yy}\) or \(G^{zz}\); this acts as an implicit constraint on allowed transformations, which are only allowed to transform \(x, y, \) and \(z\) equivalently\(^2\). In a scalar wave theory such as p-acoustics, all non-isotropic wave behaviour has to derive from the constitutive parameters \(\alpha, \beta, \) or \(\rho\), and not from a transformation.

\[C. \text{ Transformations}\]

With both of our preferred sorts of waves having been expressed using the same mathematical machinery, we can now investigate the effect of coordinate transformation in either case simultaneously, leaving any more specific details to a later stage. Transformations of the coordinates are written down between the original coordinates \(x^\alpha\) and new ones \(x'^\alpha\) as

\[
T^\alpha_{\phantom{\alpha} \alpha'} = \frac{\partial x'^\alpha}{\partial x^\alpha}, \tag{2.28}
\]

so that the field density tensors \(G\) and \(F\) transform as

\[
G^{\alpha' \beta'} = \Delta^{-1} p^{\alpha' \beta'} T^\beta_{\phantom{\beta} \beta}, \tag{2.29}
\]

\[
sF^{\alpha' \beta'} = \Delta^{-1} p^{\alpha' \beta'} sF^{\beta'} T^\beta_{\phantom{\beta} \beta}. \tag{2.30}
\]

The factor \(\Delta = \text{det}(T)\) occurs because the fields are represented by tensor \(\text{distributions}\) rather than a pure tensor. Using the modulus of the determinant ensures that parity transformations changing the handedness still preserve positive volumes.

Now we can address the transformation of the constitutive tensor \(\chi\). If for EM we set \(a = 2\), and for p-Acoustics we set \(a = 1\), then

\[
G^{\alpha' \beta'} = \frac{1}{a} \chi^{\alpha' \beta'} T^\gamma_{\phantom{\gamma} \gamma}, \tag{2.31}
\]

where

\[
\chi^{\alpha' \beta'} = L^{\alpha' \gamma} \chi^{\alpha \beta} \Delta_{\gamma \gamma}', \tag{2.32}
\]

\[
L^{\alpha' \gamma} = \Delta^{-1} p^{\alpha' \beta'} T^\beta_{\phantom{\beta} \beta}' . \tag{2.33}
\]

\[
M^{\delta \gamma} = T^\delta_{\phantom{\delta} \delta} T^\gamma_{\phantom{\gamma} \gamma}', \tag{2.34}
\]

Here we have seen how coordinate transformations can be actualized by seeing how they affect the constitutive tensor. But there is also another part to the wave mechanics – the differential equations.

If our coordinate transformation is not just a deformation of our existing Cartesian coordinate system, but e.g. a change to cylindrical or polar coordinates, then the wave equations change their form. Although such radical coordinate conversions can be very useful, they are a re-parameterization of the host space-time, and not a straightforward deformation: hence the appearance of extra factors of \(r\) when converting from Cartesians to cylindrical coordinates. For brevity, we do not address this case here, and focus instead on the deforming transformations used to produce particular T-devices.

Deforming transformations (or, technically, diffeomorphisms [27]) of the coordinates (and hence of the matrix representations of the field and constitutive tensors) are simple to handle. Such transformations are easily written down between the original coordinates \(\alpha\) and new ones \(\alpha'\), and we just have \(T^\alpha_{\phantom{\alpha} \alpha'}\). When so desired, we can also compact them as we did for the EM and p-acoustic theories above, generating deformation matrices containing the appropriate products of the \(T^\alpha_{\phantom{\alpha} \alpha'}\) elements.

In general, the two field tensors might be compacted differently, so that they will have distinct compacted deformation matrices, one for \(F\), and another for \(G\). The deformation matrix equation for the vector (density) \(sF^A\) is

\[
\begin{bmatrix} sF^A \end{bmatrix} = \begin{bmatrix} M^A \end{bmatrix} \begin{bmatrix} sF^A \end{bmatrix}, \tag{2.35}
\]

whereas for the vector (density) \(G^A\) we would have

\[
\begin{bmatrix} G^A \end{bmatrix} = \begin{bmatrix} L^A \end{bmatrix} \begin{bmatrix} G^A \end{bmatrix}. \tag{2.36}
\]

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\(^2\) A more explicit version of the \(G^A\) vector might have six elements, the first three of which would represent \(G^{xx}, G^{yy}\) or \(G^{zz}\) in turn, but all being equal to \(p\); and with \(\chi_{\alpha \beta}^G\) being a 6x4 matrix, the top three rows being identical. We must take care to allow for this multiplicity when transforming \(G^A\).
This situation of distinct deformation matrices is indeed the case for p-audios (as we have seen), but for EM, we have made a point of indexing them in the same way.

Of course the matrix representation of \( \chi_{\alpha\beta}^{\text{trial}} \) (i.e. \( [\chi]_B \), or \( [\chi] \)) also transforms, as we can see by adapting the constitutive relations to the new coordinates.

\[
[L] [G] = [L] [\chi] [F] \quad (2.37)
\]
\[
[G'] = [L] [\chi] [M]^{-1} [M] [F] \quad (2.38)
\]
\[
[G'] = (L) [\chi] [M]^{-1} (M) [F] \quad (2.39)
\]
\[
= [\chi'] [F']. \quad (2.40)
\]

Now we see that if we want to physically implement the transformation \( T \) as a T-device, for any possible states of the field tensors \( F \) and \( G \), we need to change the host material so that its constitutive makeup is not simply the reference value \( \chi \), but that of the transformed constitutive makeup \( \chi' \). In what follows we will implement this for both purely spatial and the more exotic space-time cloaks in a ground-plane reference geometry.

### III. SPATIAL CARPET T-DEVICES

“Carpet” or ground-plane transformation devices are essentially surfaces that have been modified; in their original conception they were engineered mirrors that appeared to give simple reflections but were actually hiding some predefined properties such as an apparent expansion or shrinking of the wave-accessible space, as needed for a cloak or tardis, whilst remaining flat, but in appearance it would act like a periscope for when the observer expects to see a surface. Its diagram is not shown on fig. 1, but note that its transformation is the reverse of that for the periscope but in appearance it would act like a periscope for when the observer expects to see a carpet cloak on the plane.

We start with a planar carpet that lies flat on the \( y-z \) plane at \( x = 0 \), and decide to transform a localized region of space reaching no higher than \( x = -H \) above the plane and no further than \( y = \pm \Lambda \) sideways as shown on Fig. 1. Firstly, we can choose to restrict the space probed by incoming waves to that between the maximum height \( H \) and some lower height \( x = h \), but transform the material so that the apparent space is expanded and extends all the way from \( x = 0 \) to \( x = -H \). This T-device is the usual carpet cloak with an offset \( h \) and scaling \( R = (H-h)/H < 1 \). See Fig. 1(a).

Secondly, we might expand the actual space probed by incoming waves to that between the maximum height \( H \) and to a penetration depth below the plane of \( x = h \), but transform the material to shrink the apparent space to only that between \( x = 0 \) to \( x = -H \). This T-device is a cloaked “periscope” with a negative offset \( h < 0 \) and scaling \( R = (H-h)/H > 1 \). See Fig. 1(b), where we see that the periscope allows an observer to remain below the ground plane whilst allowing them an unrestricted view of their surroundings, just as if they were exposed above it.

Thirdly, we can define the actual space probed by incoming waves to be just the same as if there were a flat plane, but nevertheless transform the material so that the apparent space extends below the plane by \( h \). This T-device might be called a tardis, since it presents the illusion that it contains a space bigger than it is on the outside; it uses an offset \( h > 0 \) and scaling \( R = (H-h)/H < 1 \). Its diagram is shown in fig. 1(c), where the transformation is the reverse of that for the periscope, but in appearance it acts like a carpet cloak for when the observer expects to see a bump on the plane.

Lastly, we could define the space probed by incoming waves to be just the same as if there were a flat plane, but nevertheless transform the material so that the apparent space was shrunk back as if there were a bump. This T-device might be defined as an anti-tardis, but for brevity we will not discuss such a device here. Its diagram is not shown on fig. 1, but note that its transformation is the reverse of that for the carpet cloak but in appearance it would act like a periscope for when the observer expects to see a carpet cloak on the plane.

The deformation that stretches a uniform medium (unprimed coordinates) into these T-device designs (primed coordinates) is applied (only) inside the shaded regions on fig. 1, and is as follows:

\[
t' = t, \quad (3.1)
\]
\[
x' = Rx - C \frac{\Lambda - y \sgn(y)}{\Lambda} h
\]
\[
= Rx - Ch + rsy, \quad (3.2)
\]
\[
y' = y, \quad (3.3)
\]
\[
z' = z, \quad (3.4)
\]

where the ratio of actual depth \( (H-h) \) to apparent depth \( (H) \) is \( R = (H-h)/H, \) and \( r = Ch/\Lambda, \) and \( s = \sgn(y) \). Here \( C \) is set to one for the carpet cloak and periscope, but for the tardis (where \( R > 1 \)) it needs to be \( C = R \). Using a general viewpoint applicable to any of these three T-devices, we see that \( R < 1 \) and \( r < 1 \) expands the wave-accessible space, as needed for a cloak or tardis, whilst \( R > 1 \) compresses it, as needed for a periscope or anti-tardis.

The effect of this deformation from eqn. (3.2) is given by the tensor \( T \) which specifies the differential relationships between the primed and un-primed coordinates, where \( \alpha \in \{t,x,y,z\} \) and \( \alpha' \in \{t',x',y',z'\} \). If we let rows span the first (upper) index, and columns the second (lower) index, then

\[
T_{\alpha'}^\alpha = \begin{bmatrix}
\frac{\partial x'}{\partial x} & 0 & 0 & 0 \\
0 & R & rs & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad (3.5)
\]

and note that \( \det(T_{\alpha'}^\alpha) = R \). For some column 4-vector \( V_{\alpha} \), we have that

\[
V'_{\alpha'} = T_{\alpha'}^\alpha V_{\alpha}, \quad (3.6)
\]

i.e.

\[
\begin{bmatrix}
V'
V'
V'
V'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & R & rs & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
V
V
V
V
\end{bmatrix}. \quad (3.7)
\]
where the matrix $[M]$ that transforms $*F^B$ is the same. We also need the inverse of $[M]$, which is

$$
[M_A^A]^{-1} = \begin{bmatrix}
1 -rs & 0 & 0 & 0 & 0 \\
0 & R & 0 & 0 & 0 \\
0 & 0 & R & 0 & 0 \\
0 & 0 & 0 & R & 0 \\
0 & 0 & 0 & +rs & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

Starting from simple uniform background medium described only by $\varepsilon$ and $\mu$ – perhaps a vacuum with $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ – the transformed constitutive matrix defining the EM carpet T-device is then

$$
[\chi_{AB}^A] = \left[ L_A^A \right] [\chi_B^A] \left[ M_B^B \right]^{-1}.
$$

So we see a need for birefringence, as defined by the off-diagonal permittivity elements $-\varepsilon rs / R$, and the off-diagonal permeability contributions $-\varepsilon rs c^2 / R$. To get a $\mu$ matrix, we need to invert the $\eta$ sub-matrix:

$$
[\mu] = \begin{bmatrix}
1 / R & -rs / R & 0 \\
-rs / R & 1 + r^2 / R & 0 \\
0 & 0 & R
\end{bmatrix}^{-1} = \frac{1}{R} \begin{bmatrix}
R^2 + r^2 + rs & +rs & 0 \\
-rs & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

Note how the structure of the (sub)matrix $[\mu]$ matches up with that of the (sub)matrix $[\varepsilon]$, as we would expect as our transformation process demands a preservation of impedance matching.

Numerical examples indicating how the EM fields are distorted by the different carpet T-devices are shown in fig. 3.

### B. p-Acoustics carpets

For p-acoustic transformations, we need two compactified transformation matrices, with correctly arranged ordering. This is less straightforward than in EM, where rows and columns are indexed by the same list of coordinate pairs. To transform $G^{ab}$, it is compactly indexed by $ww, xt, yt, zt$, where “$w$” stands in for one of $x, y$ or $z$. Where this means that some elements may have multiple values, we will indicate this using (e.g.) $\{R, 1, 1\}$ for $x, y$, and $z$ choices respectively. We need

---

3 Remember that for EM, each element of $[L]$ not only includes the obvious contributions to $G^{\alpha\beta}$ from $G^{\alpha\beta}$, but also has a sign-flipped one from $G^{\rho\sigma}$. This accounts for the $-rs$ entry in $[L]$. 

---

**FIG. 1**: The (a) spatial carpet cloak, (b) spatial periscope, (c) spatial tardis. Continuous lines show the actual ray trajectories, dotted lines the apparent (illusory) ray trajectories. The periscope (b) needs to cover the greater distance to the back surface than it is to the apparent surface. For the carpet cloak diagram (a), we have exaggerated the transformation so that some ray path segments are stopped – but note that any realistic implemention will involve much weaker speed modulations. 

### A. EM carpets

We expressed the EM constitutive tensor using a compact $6x6$ matrix form. Therefore we also find it useful to define a compact $6x6$ deformation (transformation) matrix version of the product $L^\alpha_{\alpha} L^\beta_{\beta}$. Note that the use of the unconventional choice of (the dual) $*F^{ab}$ and (the mixed) $\chi^{ab}_{\gamma\delta}$ allows us to calculate using straightforward matrix multiplication.

The two dimensional matrix representation of the transformation matrix, as compressed for EM, is best written as an explicit transformation between the field 6-vectors to avoid error. Thus for the compressed column 6-vector $G^A$, we have

$$
\begin{bmatrix}
G^A \\
G^B \\
G^C \\
G^D \\
G^E \\
G^F
\end{bmatrix} = \left[ L_A^A \right] \left[ L_B^B \right] \left[ L_C^C \right] \left[ L_D^D \right] \left[ L_E^E \right] \left[ L_F^F \right].
$$

3 Remember that for EM, each element of $[L]$ not only includes the obvious contributions to $G^{\alpha\beta}$ from $G^{\alpha\beta}$, but also has a sign-flipped one from $G^{\rho\sigma}$. This accounts for the $-rs$ entry in $[L]$. 

$$
\begin{bmatrix}
\alpha \\
\beta \\
\gamma \\
\delta \\
\epsilon \\
\zeta
\end{bmatrix} = \begin{bmatrix}
1 & +rs & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -rs & R & 0 \\
0 & 0 & 0 & 0 & 0 & R
\end{bmatrix}.
$$

Note how the structure of the (sub)matrix $[\mu]$ matches up with that of the (sub)matrix $[\varepsilon]$, as we would expect as our transformation process demands a preservation of impedance matching.
Since \( w \) represents multiple coordinate choices, the \( \hat{w}/\hat{w}' = \hat{x}' \) element of \( \hat{L} \), i.e. \( L^{\hat{x}'}/L^{\hat{x}} = r_2^2 + r_2' \), results from a sum over all possible \( \hat{w} \hat{w}' \in \{ xx', yy', zz' \} \), in combination with \( G^{xx} = G^{zz} = p \).

Here we can immediately see that there is no way of constructing a general carpet cloak (or related T-device) with p-acoustic waves, which we can see from the fact that the transformed energy parameter \( \kappa' \) is required to have mutually incompatible values: i.e. both \((r^2 + r^2')pc^2\) and \(pc^2\) at the same time! We might try to avoid this incompatibility by restricting ourselves to only 1D x-axis propagation, so that the waves need experience only one of these \( \kappa' \) values. However, although such a 1D implementation might technically match the original conception of a carpet cloak, it is rather too simple to be of much interest: being simply a waveguide that appears longer than it really is.

### IV. SPACE-TIME CARPET T-DEVICES

The basic design used here is geometrically related to that used recently as the basis of a spatial carpet cloak for visible light [23, 24]. However, one of the two spatial coordinates is replaced by the time coordinate. This means that waves are not diverted around the event in a spatial sense, instead the leading waves (in an optical cloak, the “illumination”) are sped up, and the latter slowed down, creating a “dark” wave-free interval in which timed events are obscured. After the event, the leading part of the illumination (whether sound or light) is slowed, and the latter sped up until the initial uniform, seamless illumination has been perfectly reconstructed. In an acoustics implementation, we can think of sonar-using creatures such as bats or dolphins who rely on incoming sound waves to locate and observe events; the event cloak creates a quiet region from which no sound reflections can be heard but without (permanently) distorting any passing background noise or sonar “pings”.

As usual, we implement the cloak with the aid of what looks like a coordinate transformation. Strictly speaking, however, it is no such thing – it is a way of deforming one wave solution into another with the desired cloaking properties. In EM we would typically start (as we do here) with a reference material and solution based on a vacuum or a uniform static non-dispersive refractive index and a light reflective surface; for acoustics we would pick a stationary and featureless elastic medium or fluid with an acoustically reflective surface. If, after applying the deformation, the wave equations have the same form (i.e. are “form invariant”), then we can compare the two solutions and determine how the desired alterations to the wave mechanics (the field propagation) can be induced by complementary changes to the propagation medium.

We assume a carpet that lies flat on the \( x-z \) plane at \( x = 0 \), and decide to cloak a region \( h \) deep and \( \tau \) in duration, with the cloak’s influence extending out as far as \( H + h \) above the carpet. The deformation that stretches a uniform and static medium (un-primed coordinates) into these T-device designs (primed coordinates) is applied (only) inside the shaded re-
FIG. 3: The (a) space-time “event” carpet cloak, (b) space-time periscope, (c) space-time tardis. Continuous lines show the actual ray trajectories, dotted lines the apparent (illusory) ray trajectories. The periscope (b) needs to be embedded in a high-index host medium to allow the internal rays to cover the greater distance to the back surface than it is to the apparent surface. For the carpet cloak diagram (a), we have exaggerated the transformation so that some ray path segments are stopped – but note that any realistic implementation will involve much weaker speed modulations.

FIG. 2: The (a) space-time “event” carpet cloak, (b) space-time periscope, (c) space-time tardis. Continuous lines show the actual ray trajectories, dotted lines the apparent (illusory) ray trajectories. The periscope (b) needs to be embedded in a high-index host medium to allow the internal rays to cover the greater distance to the back surface than it is to the apparent surface. For the carpet cloak diagram (a), we have exaggerated the transformation so that some ray path segments are stopped – but note that any realistic implementation will involve much weaker speed modulations.

The tardis gives the onlooker the temporary impression as if they briefly put their head out above it; it is specified under allowing them an unrestricted view of their surroundings, just to remain always below the ground plane whilst temporarily allowing them an unrestricted view of their surroundings, just as if they briefly put their head out above it; it is specified using $h < 0$. Thus, for example, a sensor can be briefly exposed to take readings whilst usually remaining in shelter below the plane. The tardis gives the onlooker the temporary impression of a space bigger than it actually is; this requires that $C = R$. Although in this carpet implementation the space-time tardis is the same as a space-time carpet cloak for a dimple, in a non-carpet radial implementation the effect is decidedly more startling!

Using a general viewpoint applicable to any of these T-devices, we see that $H$ is the apparent depth detectable by an observer, positive-valued $h$ apparently expands the accessible space, and negative $h$ apparently shrinks it.

To ensure that the deformed solution includes no waves travelling faster than the maximum wave speed $c$, we also specify that the aspect ratio $(H - h)/ct$ is less than one. As seen on fig. 3(a), this deformation pushes a small triangular region of the wave illumination away from the carpet plane. Despite this, as the waves approach the carpet plane from the (below) left, the deviations from ordinary straight-line propagation that occur within the blue-outlined triangle are compensated for, so that when waves emerge travelling towards the (top) left, they appear to have reflected normally from the carpet plane. Note that the cloaking device reverses the speed changes at $t = 0$: on the figure, at early times $t < 0$, forward waves are slowed or stopped and backward waves are unaffected, but at late times $t > 0$, it is the backward waves that are slowed or stopped, with forward waves travelling normally. For the practical situation, i.e. cloaks where $(H - h) > h$, waves are not stopped but slowed.

The effect of this deformation is given by the transformation matrix $T'_{\alpha}^\alpha$, which specifies the differential relationships between the deformed (primed) and reference (unprimed) coordinates, where $\alpha \in \{t',x',y',z'\}$ and $\alpha' \in \{r,x,y,z\}$. If we let rows span the first (upper) index, and columns the second (lower) index, then

$$T'_{\alpha}^\alpha = \left[\frac{\partial x'^\alpha}{\partial x^\alpha}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ rs & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (4.5)$$

and note that $\det(T'_{\alpha}^\alpha) = R$. For some column 4-vector $V^\alpha$, we have that

$$\begin{bmatrix} V^{t'} \\ V^{x'} \\ V^{y'} \\ V^{z'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ rs & R & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^t \\ V^x \\ V^y \\ V^z \end{bmatrix}. \quad (4.7)$$

When we apply the deformation implied by the coordinate transformation, our field tensors assume the required properties, which here might generate a space-time carpet cloak. Of course, our deformation also affects the constitutive tensor, whose deformed properties correspond to what we have to implement to actually make the space-time carpet cloak.

A. EM event carpets

We expressed the EM constitutive tensor using a compact 6x6 matrix form. Therefore we also find it useful to define a compact 6x6 deformation (transformation) matrix version of the product $L'_{\alpha}^\alpha L^\beta_{\beta}$. Note that the use of the unconventional
choice of (the dual) $F^{\alpha\beta}$ and (the mixed) $\chi^{\alpha\beta\gamma}$ allows us to calculate using straightforward matrix multiplication.

The two dimensional matrix representation of the transformation matrix, as compressed for EM, is best written as an explicit transformation between the field 6-vectors to avoid error. Thus for the compressed column 6-vector $G^A$, we have\(^5\)

\[
\begin{bmatrix}
G^A' \\
G^B' \\
G^C' \\
G^D' \\
G^E' \\
G^F'
\end{bmatrix} = \begin{bmatrix}
L_{AA} & L_{AB} \\
L_{BA} & L_{BB} \\
L_{CA} & L_{CB} \\
L_{DA} & L_{DB} \\
L_{EA} & L_{EB} \\
L_{FA} & L_{FB}
\end{bmatrix} \begin{bmatrix}
G^A \\
G^B \\
G^C \\
G^D \\
G^E \\
G^F
\end{bmatrix}
\]

Starting from simple background medium described only by $\varepsilon$ and $\mu$, the transformed constitutive matrix defining the EM spacetime carpet T-device is then

\[
\begin{bmatrix}
G^A' \\
G^B' \\
G^C' \\
G^D' \\
G^E' \\
G^F'
\end{bmatrix} = \begin{bmatrix}
L_{AA}' & L_{AB}' \\
L_{BA}' & L_{BB}' \\
L_{CA}' & L_{CB}' \\
L_{DA}' & L_{DB}' \\
L_{EA}' & L_{EB}' \\
L_{FA}' & L_{FB}'
\end{bmatrix} \begin{bmatrix}
G^A \\
G^B \\
G^C \\
G^D \\
G^E \\
G^F
\end{bmatrix}
\]

\[
M_{BB}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & R & 0 & 0 & 0 & 0 \\
0 & 0 & R & 0 & 0 & 0 \\
0 & 0 & 0 & R & 0 & 0 \\
0 & 0 & +rs & 0 & 1 & 0 \\
0 & -rs & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Here we see the slowing/speeding behaviour of an ordinary space-time cloak \(\mathcal{L}_{BB}^{A}\), but with a direction-sensitivity caused by magneto-electric effects $\pm rs\varepsilon/R$ in the material; this means that ingoing light at a point can/will travel faster or slower than outgoing light, as demanded by the design.

\[\text{V. IMPLEMENTATION}\]

Unlike the standard space-time cloak \(\mathcal{L}_{BB}^{A}\) and related T-devices \(\mathcal{L}_{BB}^{A}\), these carpet T-devices cannot easily be simplified by choosing a preferred direction. We can see from fig.

\[\text{B. p-Acoustics event carpets}\]

For p-acoustic transformations, we need two compactified transformation matrices, with correctly arranged ordering. Again, and unlike EM, $G^{\alpha\beta}$ is compactly indexed by $aa, xt, yt, zt$, where “a” stands in for one of $x, y$ or $z$; and any multiple valued elements are indicated using a set-like notation. We need $[L^A_{AA}], \{G^A\}$, which is

\[
\begin{bmatrix}
G^A \\
G^B' \\
G^C' \\
G^D' \\
G^E' \\
G^F'
\end{bmatrix} = \begin{bmatrix}
L_{AA} & L_{AB} \\
L_{BA} & L_{BB} \\
L_{CA} & L_{CB} \\
L_{DA} & L_{DB} \\
L_{EA} & L_{EB} \\
L_{FA} & L_{FB}
\end{bmatrix} \begin{bmatrix}
G^A \\
G^B \\
G^C \\
G^D \\
G^E \\
G^F
\end{bmatrix}
\]

\[
G^B_α'^t = \frac{1}{R} \begin{bmatrix}
R^2 & 1 & 1 & \{rsR, 0, 0\} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
G^B_β'^t = \begin{bmatrix}
G^B_{AA'} & \{\{R^2, 1, 1\}, \{rsR, 0, 0\}\} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

To transform $F^{\alpha\beta}$, compactly indexed by $tt, xt, yt, zt$, we need $[M_{BB}]$

\[
\begin{bmatrix}
F^{tt} \\
F^{xt} \\
F^{yt} \\
F^{zt} \\
\end{bmatrix} = \begin{bmatrix}
M_{BB} & M_{BF} \\
M_{FB} & M_{FF}
\end{bmatrix} \begin{bmatrix}
F^{tt} \\
F^{xt} \\
F^{yt} \\
F^{zt} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
G^B_{AA'} & \{G^B_{AA'}\} & \{G^B_{AA'}\} & \{G^B_{AA'}\} \\
\end{bmatrix}
\]

\[
M_{BB}^{-1} = \begin{bmatrix}
R & 0 & 0 & 0 \\
-rs & 1 & 0 & 0 \\
0 & 0 & R & 0 \\
0 & 0 & 0 & R
\end{bmatrix}
\]

\[
\begin{bmatrix}
L_{AA} & L_{AB} \\
L_{BA} & L_{BB} \\
L_{CA} & L_{CB} \\
L_{DA} & L_{DB} \\
\end{bmatrix} \begin{bmatrix}
G^A \\
G^B \\
G^C \\
G^D \\
\end{bmatrix} = \begin{bmatrix}
\{R^2 - r_s^2, 1, 1\}c^2 & \{rs, 0, 0\} & 0 & 0 \\
\end{bmatrix}
\]

\[
G^B_{AA'} = \begin{bmatrix}
\{R^2 - r_s^2, 1, 1\}c^2 & \{rs, 0, 0\} & 0 & 0 \\
\end{bmatrix}
\]

\[
\{G^B_{AA'}\} = \begin{bmatrix}
\{R^2 - r_s^2, 1, 1\}c^2 & \{rs, 0, 0\} & 0 & 0 \\
\end{bmatrix}
\]

Again we can see that there is no way of constructing a general space-time cloak (or related T-device) with p-acoustic waves, which we can see from the fact that the modulus $\kappa'$ and $\alpha'_t$ cross-coupling are both required to have two incompatible values. In contrast to the spatial carpet cloak, a one dimensional (in x) space-time carpet cloak can make sense; just as the one dimensional space-time EM cloak makes sense \(\mathcal{L}_{BB}^{A}\). However, the necessarily bi-directional nature of the design means that we would require a controllable coupling $\alpha'_t = rs\varphi$ enabling the velocity field $v_t = \pm F_{tt}$ to affect the pressure $p = G^{tt}$, as well as a $\beta'_t = -rs\varphi$ enabling the density $P = \pm F_{tt}$ to affect the momentum density $V_t = G^{tt}$.
that the cloaking zone needs to support rays travelling both forward and backward, and the forward and backward rays need to have different phase velocities. This can be achieved (at least conceptually) by making a cloak from a moving medium, and would require material speeds comparable to the wave speed. Given the problems of sourcing and sinking the medium at the boundaries of the cloak, and in the necessary careful timing, this is unlikely to be straightforward. It would, however, be considerably easier in water waves than for EM.

To implement an electromagnetic space-time carpet, we might restrict ourselves to a polarization sensitive device, and link polarization to propagation direction to mimic a direction-dependent velocity by means of a polarization-dependent velocity. In this case, we can use a kind of novel birefringence to achieve the speed contrast between forward and backward waves, as long as we keep to a 2D spatial geometry (e.g. the xy plane); this is shown on fig. 4. In some respects, the concept is similar to a more exotic implementation of the calcite crystal carpet cloak [28-29]. A polarization switching mirror surface [33] – the carpet – enables the concept by switching a (e.g.) vertically ($z$) polarized inward waves into a horizontally polarized outward wave. Incoming light has (e.g.) its electric field polarized along $z$, while the outgoing fields can be independently controlled by $\mu_x$ acting on $H_z$. Angle independence is ensured by making $\varepsilon_x = \varepsilon_y$, and $\mu_x = \mu_y$.

then has its magnetic field polarized along $z$, thus we can independently dynamically control those as required by means of the $\mu_z$ permeability. An angle-independence is thus achieved, preserving our 2D spacetime carpet T-device, since in-plane field components $E_x,E_y,B_x,B_y$ experience unchanged material properties. Whilst this will certainly be challenging to construct, one important simplification is that the dynamic $\varepsilon_z$ and $\mu_z$ properties required are matched in strength to each other, which will hopefully simplify the necessary technology.

In pressure acoustics, the scalar nature of the wave’s pressure component prohibits us from accessing a similar trick, and so there is no alternative but to consider how we might achieve the unconventional coupling where the velocity field would be able to cause changes to the pressure. Once again, although the general nature of the mathematical formalism has enabled us to unify the transformation aspects of our chosen T-device, the specific physics of a given type of wave nevertheless restricts what can be done in principle, as well as in practise.

VI. SUMMARY

We have derived the material parameters needed for new kind of event cloak – the space-time carpet cloak – and shown what material properties are required to implement it. By unifying the mathematical treatment of EM and a scalar wave theory such as pressure acoustics, we have also shown the way towards a means of unifying transformation theories for different wave types. The unified transformation mechanics theory was applied to a range of spatial T-devices based on the ordinary carpet cloak, e.g. the periscope and the tardis. As noted in the original event cloak paper [3], a covariant wave notation greatly facilitates the design of space-time T-devices [4], which is further emphasised here, with a scalar wave theory being matched up with the usual covariant formulation of EM, and the subsequent ease of comparison and contrast between the spatial carpet T-devices from section III with the space-time carpet T-devices from section IV. We also proposed (in section V) a basic scheme laying out a possible route to selecting the metamaterial properties needed to implement an electromagnetic spacetime carpet cloak.

Acknowledgments

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