AN ERDŐS-KO-RADO THEOREM FOR THE GROUP PSU(3, q)

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Abstract. In this paper we consider the derangement graph for the group PSU(3, q) where q is a prime power. We calculate all eigenvalues for this derangement graph and use these eigenvalues to prove that PSU(3, q) has the Erdős-Ko-Rado property and, provided that q ≠ 2, 5, another property that we call the Erdős-Ko-Rado module property.

1. Introduction

The Erdős-Ko-Rado (EKR) theorem states that the largest set of pairwise intersecting k-subsets from an n-set, when n > 2k, has size \( \binom{n-1}{k-1} \) and consists of all subsets that contain a fixed element of the n-set [7]. This is a famous result with many extensions and variations; there is a large body of work showing that versions of the EKR theorem hold for many different objects (see [9] and the references within). The focus of this work is to show that a version of the EKR theorem holds for the Projective Special Unitary group PSU(3, q) where q is a prime power.

Two permutations \( \rho \) and \( \sigma \) are said to be intersecting if \( \sigma^{-1}\rho \) has a fixed point (a permutation with no fixed points is a derangement). If \( \rho \) and \( \sigma \) are intersecting, then \( \rho \) and \( \sigma \) both map some point to a common point. Under this definition of intersection, one can ask what is the size and the structure of the largest sets of pair-wise intersecting permutations. Clearly the stabilizer of a point (and any of its cosets) are intersecting sets of permutations under this definition (indeed, these are the sets of all permutations that map some fixed \( i \) to some fixed \( j \)). We define the stabilizers of a point, and all of the cosets of the stabilizer of a point, to be the canonical intersecting sets of permutations. In [5, 10, 13] it is shown that the largest sets of intersecting permutations are exactly the canonical intersecting sets. This result can be viewed as a version of the EKR theorem for permutations, since it says that any largest set of intersecting permutations is the set of permutations that map some fixed \( i \) to some fixed \( j \).

The same question may be asked for the permutations in a given group (rather than all permutations). Define the canonical intersecting sets for a permutation group to be the stabilizers of a point and their cosets. We say that a permutation group has the EKR property if a canonical set is a largest set of pairwise interesting permutations from the group. Further, a permutation group has the strict-EKR property if any pairwise intersecting set of permutations from the group of maximum size is a canonical set. The result in each of [5, 10, 13] is that the symmetric

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group has the strict-EKR property. It has also been shown that many other groups also have the strict-EKR property [1, 3, 12, 15, 16]. It was recently shown that every 2-transitive group has the EKR property [17].

In this paper we consider the group $\text{PSU}(3, q)$. The group $\text{PSU}(3, q)$ has a two-transitive action on a set of size $q^3 + 1$, we will only consider this action. In [17] it is shown that $\text{PSU}(3, q)$ has the EKR property, here we give more details of this result and further give a result about the largest intersecting sets in $\text{PSU}(3, q)$. Namely, we prove that the characteristic vector for any maximum intersecting set in $\text{PSU}(3, q)$ is a linear combination of characteristic vectors of the canonical cliques in $\text{PSU}(3, q)$. Showing the same result for other 2-transitive groups was a key result in proving that the groups have the strict-EKR property [1, 10, 15, 16].

2. An algebraic proof of EKR theorems

One effective method to show that a group has the EKR property is to apply Hoffman’s ratio bound (also known as Delsarte’s bound) to the derangement graph of the group. This method is used in [1, 10, 15, 16, 17], and it will be the focus of this paper.

The derangement graph of a group $G$, usually denoted by $\Gamma_G$, has the elements of the group as its vertices, and two vertices are adjacent if they are not intersecting. In particular, $\rho$ and $\sigma$ from a permutation group $G$ are adjacent in $\Gamma_G$ if and only if $\sigma^{-1}\rho$ is a derangement. With this definition, a set of pairwise intersecting elements from $G$ is exactly a clique (or independent set) in $\Gamma_G$. For any group $G$, $\Gamma_G$ is a Cayley graph with the set of derangements from the group as the connection set.

It is well-known [4, 6] that it is possible to calculate the eigenvalues for a normal Cayley graph using the irreducible characters of the group. Denote the irreducible complex characters of a group $G$ by $\text{Irr}(G)$. Given $\psi \in \text{Irr}(G)$ and a subset $S$ of $G$, we write $\psi(S)$ for $\sum_{s \in S} \psi(s)$. With these definitions, we can give a formula for the eigenvalues the derangement graph for any permutation group.

**Lemma 2.1.** Let $G$ be a permutation group and $D$ the set of derangements in $G$. The spectrum of the graph $\Gamma_G$ is

$$\{\psi(D)/\psi(1) \mid \psi \in \text{Irr}(G)\}.$$

Further, if $\lambda$ is an eigenvalue of $\Gamma_G$ and $\psi_1, \ldots, \psi_s$ are the irreducible characters of $G$ such that $\lambda = \psi_i(D)/\psi_i(1)$, then the dimension of the $\lambda$ eigenspace of $\Gamma_G$ is $\sum_{i=1}^{s} \psi_i(1)^2$.

For each irreducible character of $G$ there is a corresponding eigenvalue for $\Gamma_G$, and there is $(|G| \times |G|)$-matrix $E_\psi$ with

$$[E_\psi]_{\rho, \sigma} = \psi(\sigma^{-1}\rho).$$

This matrix is a projection into the eigenspace for the eigenvalue corresponding to $\psi$. The image of this projection is a $G$-module. Throughout this paper, the group is fixed to be $\text{PSU}(3, q)$, so with an abuse of notation, we called the $\text{PSU}(3, q)$-module that is the image the projection $E_\psi$ the $\psi$-module.

As stated, the derangement graph for a group is a Cayley graph in which the connection set is the union of the conjugacy classes of derangements for the group. If $C$ is a conjugacy class in a group, then we use $\Gamma_C$ to denote the Cayley graph on
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Let $G$ be a permutation group with $d$ derangements. Assume that \( \tau \) is the minimum eigenvalue of $\Gamma_G$. Let $S$ be a coclique in $\Gamma_G$, then

\[
\frac{|S|}{|G|} \leq \left(1 - \frac{d}{\tau}\right)^{-1}.
\]

If equality is met, then $v_S - \frac{|S|}{|G|} \mathbf{1}$ is an eigenvector of $\Gamma_G$ with eigenvalue $\tau$.

It is noted in [2] that for any 2-transitive group, the character $\chi(g) = \text{fix}(g) - 1$ is an irreducible character (throughout this section we will use $\chi$ to denote this character). Then, for any group with a 2-transitive action on a set of size $n$ that has exactly $D$ derangements, the eigenvalue of the derangement graph corresponding to this irreducible character is $-|D|/(n-1)$ (see [2] for details). For many groups (for example the symmetric group, the alternating group and PGL(2, q)), this is the least eigenvalue of the derangement graph, and Hoffman’s ratio bound immediate shows that these groups have the EKR property (showing that the groups have strict-EKR property is where the work lies). Further, for both the symmetric and alternating group $\chi$ is the only character that has $-|D|/(n-1)$ as its corresponding eigenvalue. In these cases, Hoffman’s bound implies that if $S$ is any maximum coclique in $\Gamma_G$, then $v_S - \frac{|S|}{|G|} \mathbf{1}$ is in the $\chi$-module. This fact is used to characterize all maximum coclique in $\Gamma_G$ in [4] [10]. Note that span of the $\chi$-module and the all ones vector is the module corresponding to the character $\text{fix}(g)$, also known as the permutation module.

For a group $G$, define $S_{i,j}$ to be the set of all permutations in $G$ that map $i$ to $j$; if $i = j$, then $S_{i,j}$ is the stabilizer of a point, otherwise it is the coset of the stabilizer of a point. Define the length-$G$ vectors $v_{i,j}$ to be the characteristic vectors for $S_{i,j}$. The vectors $v_{i,j}$ are the characteristic vectors of the canonical cocliques. In [2] it is shown that if $G$ is 2-transitive, then

\[
E_\chi v_{i,j} = \left( -\frac{|D|}{n-1} \right) (v_{i,j} - \frac{|G|}{|S|} \mathbf{1})
\]
(this is a straight-forward calculation). Note that \( v_{i,j} - \frac{|G|}{|S|} 1 \) is orthogonal to the all ones vector, thus it is a balanced vector. From this we can conclude that the vectors \( v_{i,j} - \frac{|G|}{|S|} 1 \) are in the \( \chi \)-module. We summarize these results in the following lemma.

**Lemma 2.3** (\([2]\)). Let \( G \) be a 2-transitive group and \( S_{i,j} \) a canonical coclique in \( G \). Then,

1. \( v_{i,j} - \frac{1}{n} 1 \) lies in the \( \chi \)-module; and
2. \( B := \{ v_{i,j} - \frac{1}{n} 1 | i,j \in \{ n-1 \} \} \) is a basis for the \( \chi \)-module of \( G \); and
3. \( \{ v_{i,j} | i,j \in \{ 1,2,\ldots,n \} \} \) is a spanning set for the permutation module.

If the characteristic vector for every maximum coclique in \( \Gamma_G \) (where \( G \) is a 2-transitive group) is in the \( \chi \)-module spanned by these vectors, then we say that group has the EKR-module property. The previous lemma implies that if \( G \) has the EKR-module property, then any maximum coclique in such a \( \Gamma_G \) is a linear combination of the characteristic vectors of the canonical cocliqu es. The symmetric and alternating group has the EKR-module property\([1,10]\), as does \( \text{PGL}(2,q) \), \( \text{PGL}(3,q) \) and the Mathieu groups \([2,15,16]\).

There are many 2-transitive groups for which the eigenvalue corresponding to the character \( \chi(g) = \text{fix}(g) - 1 \) is not the least eigenvalue of the derangement graph. In \([17]\) it is shown that these groups do indeed have the EKR property; this is done using a weighted version of Hoffman’s bound. For any graph a symmetric matrix \( A \) with rows and columns indexed by the vertices of the graph is said to be a weighted adjacency matrix of the graph if \( A_{u,v} = 0 \) whenever \( u \) and \( v \) are not adjacent vertices. The following is a weighted version of Hoffman’s ratio-bound (see \([9, \text{Section 2.4}]\) for a proof) stated only for derangement graphs.

**Lemma 2.4.** Let \( G \) be a permutation group and let \( A \) be a weighted adjacency matrix of \( \Gamma_G \). Let \( d \) be the largest eigenvalue of \( A \), and \( \tau \) the least eigenvalue of \( A \). If \( S \) is a coclique in \( \Gamma_G \), then

\[
\frac{|S|}{|G|} \leq \left( 1 - \frac{d}{\tau} \right)^{-1}.
\]

In this paper, we will calculate all the eigenvalues for \( \Gamma_{\text{PSU}(3,q)} \), we do this by calculating the eigenvalues for some graphs \( \Gamma_C \) where \( C \) is the union of some of the conjugacy classes of derangements in \( \text{PSU}(3,q) \) (these graphs are defined precisely in Sections 3 and 4). We will also calculate all the eigenvalues of the weighted adjacency matrix for derangement graph \( \Gamma_{\text{PSU}(3,q)} \) that is given in \([17]\); this weighting assigns the weights to the conjugacy classes of derangements. In particular, if \( \rho \) and \( \sigma \) are adjacent, the weight in the weighted adjacency matrix will depend only on which conjugacy class \( \rho \sigma^{-1} \) belongs to.

The ratio bound on the weighted adjacency matrix shows that \( \text{PSU}(3,q) \) has the EKR property. This was done in \([17]\), but the exact eigenvalues were not calculated. With the exact value of the eigenvalues of the adjacency matrix we can prove that \( \text{PSU}(3,q) \) has the EKR-module property. To do this we need the following new result.

**Lemma 2.5.** Assume that \( G \leq \text{Sym}(n) \) is a 2-transitive permutation group with \( d \) derangements. If
(1) there exists a weighted adjacency matrix $A$ with largest eigenvalue $k$ and least eigenvalue $\tau$ with
\[
\frac{1}{n} = (1 - \frac{k}{\tau})^{-1};
\]
and
\[
(2) \chi(g) = \text{fix}(g) - 1 \text{ is the only irreducible character of } G \text{ with eigenvalue equal to } -\frac{d}{n-1}.
\]
then $G$ has the EKR-module property.

Proof. By Lemma 2.4 and Condition 1, the size of the largest coclique is $|G|/n$.

Let $S$ be any coclique of maximum size; so $|S| = |G|/n$. The quotient graph of $G$ with respect to the partition \{\(S, G \setminus S\)\} is
\[
Q = \begin{pmatrix} 0 & d \\ \alpha & d - \alpha \end{pmatrix}
\]
where $d$ is the degree of the derangement graph for $G$ and $\alpha = \frac{d|S|}{|G|-|S|} = \frac{d}{n-1}$ (this is found by counting the edges between $S$ and $V \setminus S$ in $\Gamma_G$). The eigenvalues of $Q$ are $d$ and $-\alpha$, these are also eigenvalues of the adjacency matrix of the derangement graph on $G$ (corresponding to the trivial character and to $\chi$). The eigenvalues of a quotient graph interlace the eigenvalues of the graph, and if the interlacing is tight, the partition is equitable [11, Lemma 9.6.1]. Thus the partition \{\(S, G \setminus S\)\} is equitable. This means that each vertex in $S$ is adjacent to exactly $d$ vertices in $G \setminus S$, and each vertex in $G \setminus S$ is adjacent to exactly $\alpha$ vertices in $S$ and $d - \alpha$ vertices in $G \setminus S$.

Let $v_S$ be the characteristic vector of $S$, then since \{\(S, G \setminus V\)\} is equitable,
\[
A \left( v_S - \frac{1}{n} \right) = -\frac{d}{n-1} \left( v_S - \frac{1}{n} \right).
\]
Thus the balanced characteristic vector for any maximum coclique is a $-\left(\frac{d}{n-1}\right)$-eigenvector. Condition 2 this implies that the $-\left(\frac{d}{n-1}\right)$-eigenspace is exactly the $\chi$-module, so $v_S - \frac{1}{n}1$ is in the $\chi$-module (and that $v_S$ is in the permutation module). Since $S$ is any coclique of maximum size, the group $G$ has EKR-module property. □

The main result in this paper is that Lemma 2.5 can be applied to PSU(3, $q$). In Section 3 and Section 4 all of the eigenvalues for graphs $\Gamma_C$, where $C$ is a set of conjugacy classes of derangements in PSU(3, $q$), are calculated. Using these values, it is easy to find all the eigenvalues of the derangement graph $\Gamma_{PSU(3,q)}$ for all values of $q$, these are given in Section 5. In this section also includes a weighted adjacency matrix for PSU(3, $q$), and Hoffman’s ratio bound holds with equality for this matrix. From these results it will be clear that Lemma 2.5 holds and PSU(3, $q$) has EKR-module property. We will consider the cases where gcd(3, $q + 1$) = 1 and gcd(3, $q + 1$) = 3 separately.

3. Eigenvalues for $\Gamma_{PSU(3,q)}$ with gcd(3, $q + 1$) = 1

If 3 does not divide $q + 1$, then the size of the group PSU(3, $q$) is $(q^2 - q + 1)q^3(q + 1)^2(q - 1)$. There are two families of conjugacy classes in PSU(3, $q$) that
are derangements, these families are denoted by $C_1$ and $C_2$. The family $C_1$ consists of $(q^2 - q)/3$ conjugacy classes, and $C_2$ consists of $(q^2 - q)/6$ conjugacy classes. The rows of the character table for PSU(3, $q$) corresponding to these two families of conjugacy classes is given in the appendix. The characters are grouped in families, labelled from $\chi_1$ to $\chi_7$. The characters labelled $\chi_3$, $\chi_4$ and $\chi_5$ represent a family of characters parametrized by the variable given in the second row of the table; for the $\chi_6$ and $\chi_7$ the second row gives the number of these characters. The third row gives the dimension of the character. In this table, the $(q + 1)$-th root of unity is denoted by $e$. Many details of this table are omitted, see [18] for the complete character table of PSU(3, $q$).

The conjugacy classes of type $C_2$ are parameterized by triples from the set

$$T = \{(k, l, m) : k + l + m \equiv 0 \pmod{q + 1}, 1 \leq k < l < m \leq q + 1\}.$$  

The size of $T$ is $\frac{1}{q+1} \binom{q+1}{3} = \frac{q^2 - q}{6}$. The irreducible characters of type $\chi_5$ are also parameterized by the set $T$; for these characters we will use the triples $(u, v, w)$. The character $\chi_3$ in the table is the character $\chi(g) = \text{fix}(g) - 1$.

In this section we calculate the eigenvalues of two graphs whose union is the derangement graph of PSU(3, $q$). The first is the union of all $\Gamma_C$ where $C$ is a conjugacy class from the family $C_1$, we denote this by $\Gamma_1$. The second is the union of all $\Gamma_C$, where $C \in C_2$, this denoted by $\Gamma_2$. The eigenvalues for $\Gamma_1$ and $\Gamma_2$ are, respectively,

$$\lambda_\psi(\Gamma_1) = \sum_{C \in C_1} \lambda_\psi(C) = \sum_{C \in C_1} \frac{|C|}{\psi(1)} \psi(c), \quad \lambda_\psi(\Gamma_2) = \sum_{C \in C_2} \lambda_\psi(C) = \sum_{C \in C_2} \frac{|C|}{\psi(1)} \psi(c)$$

(where $c \in C$). From Lemma 2.1 it is clear that the $\psi$-eigenvalue of $\Gamma_{PSU(3, q)}$ is simply the sum of the $\psi$-eigenvalues of $\Gamma_1$ and $\Gamma_2$.

The first result is a statement of the eigenvalues that can be calculated directly from the character table (Table 7) given in the appendix using Equation 1.

**Lemma 3.1.** Assume that $3|q + 1$, and $\Gamma_1$ and $\Gamma_2$ are as defined above.

1. The eigenvalues of $\Gamma_1$ and $\Gamma_2$ for the trivial character are $\frac{|G|(q^2 - q)}{6(q^2 - q - 1)}$ and $\frac{|G|(q^2 - q)}{6(q^2 - q + 1)}$ (respectively).
2. For $\chi_1$ the eigenvalues of $\Gamma_1$ and $\Gamma_2$ are $\frac{|G|}{3(q^2 - q)}$ and $\frac{|G|}{3(q^2 - q + 1)}$ (respectively).
3. The eigenvalues of $\Gamma_1$ and $\Gamma_2$ for $\chi_2$ are $-\frac{|G|(q - 1)}{3q^2(q^2 - q + 1)}$ and $-\frac{|G|(q - 1)}{3q^2(q^2 - q + 1)}$ (respectively).
4. The eigenvalue of $\Gamma_1$ for each of irreducible characters of type $\chi_3, \chi_4, \chi_5, \chi_6$ is 0.
5. The eigenvalue of $\Gamma_2$ for both $\chi_6$ and $\chi_7$ is 0.

The first difficult calculation is for the eigenvalues of $\Gamma_1$ corresponding to the characters of type $\chi_3$. These characters are parameterized by a variable $u \in \{1, \ldots, q + 1\}$. To calculate the sum of the value of $\chi_3$ on all the elements in the conjugacy classes of type $C_2$, we need to determine some facts about $T$.

**Claim 3.2.** If $q$ is odd, then the following hold:

1. the element $q + 1$ occurs in exactly $(q - 1)/2$ triples in $T$;
2. any odd element from $\{1, \ldots, q\}$ occurs in exactly $(q - 1)/2$ triples in $T$;
(3) any even element from \{1, \ldots, q\} occurs in exactly \((q - 3)/2\) triples in \(T\);

Proof. If \(m = q + 1\), then for each \(k \in \{1, 2, \ldots, (q - 1)/2\}\) there is exactly one value for \(\ell \in \{k + 1, \ldots, q\}\) such that \(k + \ell = q + 1\). If \(k > (q - 1)/2\), then there are no such values of \(\ell \leq q\). Thus there are \((q - 1)/2\) triples \((k, \ell, q + 1) \in T\).

Assume \(x \in \{1, \ldots, q\}\) is odd and that \(x + y + z \equiv 0 \pmod{q + 1}\). Since \(q + 1\) is even, \(y\) and \(z\) have different parities and cannot be equal. For every odd value \(y \in \{1, \ldots, q\}\), except \(y = x\), there is a unique \(z \in \{1, \ldots, q + 1\}\) with \(x + y + z \equiv 0 \pmod{q + 1}\). Thus there are \(q - 1\) triples \(x, y, z\) with \(x + y + z \equiv 0 \pmod{q + 1}\), but this counts each odd element twice. Thus there are \((q - 1)/2\) triples in \(T\) that contain \(x\).

Finally, consider when \(x \in \{1, \ldots, q\}\) is even. In this case \(y = z = (q + 1 - x)/2 \pmod{q + 1}\) is solution to \(x + y + z \equiv 0 \pmod{q + 1}\). This triple is not in \(T\). Thus there are \((q - 1)/2 - 1 = (q - 3)/2\) triples in \(T\) that contain \(x\).

Using the same simple counting we get the parallel result for when \(q\) is even.

Claim 3.3. If \(q\) is even, then the following hold:

1. the element \(q + 1\) occurs in exactly \(q/2\) triples in \(T\);
2. any element from \(\{1, \ldots, q\}\) occurs in exactly \((q - 2)/2\) triples in \(T\);

With these two lemmas we can now find the eigenvalues of \(\Gamma_2\) for the characters of type \(\chi_3\). The characters of type \(\chi_3\) are parameterized by \(u = \{1, \ldots, q + 1\}\). For each of these characters we calculate the sum over all the different conjugacy classes of type \(C_2\) (which are parameterized by the set \(T\)), and from this we find the value of the eigenvalue.

Lemma 3.4. If \(q\) is odd, then the eigenvalue \(\Gamma_2\) for the character \(\chi_3\) parameterized by \(u = \frac{2q + 1}{2}\) is 

\[
-\frac{|G(q-1)|}{2(q+1)^2(q^2-q+1)}.
\]

Proof. Since \(u = \frac{2q + 1}{2}\), if \(i\) is even, then \(e^{3ui} = 1\); and if \(i\) is odd, \(e^{3ui} = -1\).

From Lemma 3.3, if \(0\) occurs in \((q - 1)/2\) triples, any odd number occurs \((q - 1)/2\) times in a triple of \(T\), and any even number occurs in \((q - 3)/2\) triples. Thus, in the sum

\[
\sum_{(k,l,m) \in T} e^{3uk} + e^{3ul} + e^{3um}
\]

1 will occur \(\frac{q - 1}{2} + \left(\frac{q - 1}{2}\right)\left(\frac{q - 3}{2}\right) = \left(\frac{q - 1}{2}\right)^2\) times. While \(-1\) will occur \(\left(\frac{q + 1}{2}\right)\left(\frac{q - 1}{2}\right)\) times. The total value of this sum over all \((k, l, m) \in T\) is

\[
\left(\frac{q - 1}{2}\right)^2 - \left(\frac{q + 1}{2}\right)\left(\frac{q - 1}{2}\right) = -\frac{q - 1}{2}.
\]

Thus the eigenvalue is

\[
-\frac{q - 1}{2} \cdot \frac{|G|}{(q + 1)^2 q^2 - q + 1} = -\frac{|G|(q - 1)}{2(q + 1)^2(q^2 - q + 1)}.
\]

□
Lemma 3.5. For all values of $q$, the eigenvalue of the character $\chi_3$ parameterized by any $u \neq \frac{q+1}{2}$ is\[\frac{|G|}{(q^2-q+1)(q+1)^2}.\]

Proof. First we will prove that if $u \neq \frac{q+1}{2}$, then
\[\sum_{(k,l,m) \in T} e^{3uk} + e^{3ul} + e^{3um} = 1.\]

If $q$ is even, then each $i \in \{1, \ldots, q\}$ occurs exactly $\frac{q^2-2}{2}$ times in the sets of $T$, with $q+1$ occurring $\frac{q+1}{2}$ times. Thus
\[\sum_{(k,l,m) \in T} e^{3ui} = e^{3u(q+1)} + \frac{q-2}{2} \sum_{i=1}^{q+1} e^{3ui} = 1\]
(since $\sum_{i=1}^{q+1} e^{3ui} = 0$).

If $q$ is odd, then
\[\sum_{i \text{ even}} e^{3ui} = \sum_{i=1}^{(q+1)/2} (e^2)^{3ui} = 0,\]
which implies that $\sum_{i \text{ odd}} e^{3ui} = 0$. Then from Lemma 3.2
\[\sum_{(k,l,m) \in T} e^{3uk} + e^{3ul} + e^{3um} = e^{3u(q+1)} + \sum_{i \text{ odd}} e^{3ui} + \frac{q-3}{2} \sum_{i=1}^{q+1} e^{3ui} = 1.\]

From Equation 2 the eigenvalue for this character is\[\frac{|G|}{(q^2-q+1)(q+1)} = \frac{|G|}{(q+1)^2(q^2-q+1)}.\]

This result can also be used to find the eigenvalues of $\Gamma_2$ corresponding to the characters of type $\chi_4$.

Lemma 3.6. Assume that $q$ is odd. The eigenvalue of $\Gamma_2$ for the character $\chi_4$ parameterized by $u = \frac{q+1}{2}$ is $\frac{|G|(q-1)}{2q(q+1)(q^2-q+1)}$. The eigenvalue of $\Gamma_2$ for any other character of type $\chi_4$ is equal to $-\frac{|G|}{q(q+1)^2(q^2-q+1)}$.\]

Each character of type $\chi_5$ is parameterized by triple $(u, v, w) \in T$. The value of this character on one of the conjugacy classes of type $C_2$ parameterized by $(k, \ell, m) \in T$, is $\sum_{(k,\ell,m)} e^{uk+\ell v+w m}$, where the sum is taken over all permutations of $k, \ell, m$. Define $S$ to be the set of all permutations of all the triples in $T$. Then $\sum_{(k,\ell,m) \in S} e^{uk+\ell v+w m}$ is the sum of the value of the character over every conjugacy class of type $C_2$. We next calculate the value of this sum, but first we state a well-know result (this is the generalization of the Chinese remainder theorem).

Proposition 3.7. Let $d = \gcd(a, b, m)$. The number of solutions to the equation $ax + by = c \pmod{m}$ is $dm$ if $d$ divides $c$, and $0$ otherwise.\]

Lemma 3.8. For $(u, v, w) \in T$, the value of
\[\sum_{(k,\ell,m) \in S} e^{uk+\ell v+w m}\]
is equal to \(-(q - 1)\) if the set \((u, v, w)\) includes \(q + 1\), and is equal to 2 otherwise.

**Proof.** Set

\[
d_u = \gcd(u, q + 1), \quad d_v = \gcd(v, q + 1), \quad d_w = \gcd(w, q + 1).
\]

Further, set \(d = \gcd(u, v, q + 1)\) (since \(u + v + w = q + 1\), this implies that \(d | w\)). Consider the number of solutions for \((k - m, \ell - m)\) in the equation

\[
uk + v\ell + wm \equiv u(k - m) + v(\ell - m) \equiv i \pmod{q + 1}
\]

where \(i \in \{0, \ldots, q\}\).

From Proposition 3.7, if \(d\) does not divide \(i\) then there are no solutions to the Equation 3 and, if \(d | i\), then there are \(d(q + 1)\) solutions. Not all of these solutions leads to a triple in \(S\), since many of these solutions may have either \(k = \ell\), \(k = m\) or \(\ell = m\). We will count the number of solutions to Equation 3 with \(k = \ell\), \(k = m\) or \(\ell = m\) and subtract these from the total number of solutions. This will give the number of triples in \(S\) for which \(uk + v\ell + wm \equiv i \pmod{q + 1}\).

Assume that \(k = m\). Equation 3 becomes

\[
v(\ell - m) \equiv i \pmod{q + 1}.
\]

If \(d_u\) divides \(i\), then there are \(d_u\) solutions for \(\ell - m\) to this equation. None these solutions corresponds to a triple in \(S\) (since \(k = m\)). Similarly, if \(d_u\) divides \(i\) there are \(d_u\) solutions in which \(\ell = m\). If \(k = \ell\), then we can rewrite Equation 3 as

\[
u(k - \ell) + w(m - \ell) \equiv w(m - \ell) \equiv i \pmod{q + 1}.
\]

If \(d_w\) divides \(i\), then this has exactly \(d_w\) solutions, otherwise there are no solutions.

We have shown that of the \(d(q + 1)\) solutions to Equation 3 there are \(d_u\) with \(k = m\), \(d_u\) with \(\ell = m\) and \(d_w\) with \(k = \ell\). None of these solutions correspond to a triple in \(S\). Next we show that these solutions are all distinct.

Assume that there is a single solution for Equation 3 with both \(k = m\) and \(k = \ell\). Then, clearly, \(k = \ell = m\) and

\[
ku + v\ell + mw = k(u + v + w) \equiv 0 \pmod{q + 1}.
\]

This, with Equation 3 implies, that \(i\) is congruent to 0 modulo \(q + 1\). So, if \(i\) not congruent to 0 no solution can satisfy any two of \(k = \ell\), \(k = m\) or \(\ell = m\). Thus we can remove these solutions from the total solutions with out removing one twice. On the other hand, if \(i\) is congruent to 0 modulo \(q + 1\), then there are solutions of the form \((k, k, k)\). Each of these solutions will be counted 3 times when we remove them, rather than the one time that is correct.

If \(i \not\equiv 0 \pmod{q + 1}\) then the number of solutions \((k, \ell, m)\) to Equation 3 with \((k, \ell, m) \in S\) is

\[
d(q + 1) - \sum_{j \in \{u, v, w\} \atop d_j | i} d_j,
\]

and if \(i \equiv 0 \pmod{q + 1}\) then the number of solutions is

\[
d(q + 1) - d_u - d_v - d_w + 2.
\]

For an integer \(i\) define \(f(i)\) to be the set of \(j \in \{u, v, w\}\) for which \(d_j\) divides \(i\). If \(d\) is a divisor of \(q + 1\), other than \(q + 1\), then \(\sum_{i \in \{1, \ldots, q+1\}} e^i = 0\). If \(d = q + 1\),
then this sum would simply be \( \sum_{i \in \{1, \ldots, q+1\}} e^i = e^{q+1} = 1. \) Thus we have that

\[
\sum_{(k, \ell, m) \in S} e^{uk + \ell v + wm} = d(q + 1) \sum_{i \in \{1, \ldots, q+1\}} e^i + 2e^0 - \left( \sum_{f(i) = \{d_u, d_v, d_w\}} e^i \right) + (d_u + d_v) \left( \sum_{f(i) = \{d_u, d_v\}} e^i \right) + (d_u + d_w) \left( \sum_{f(i) = \{d_u, d_w\}} e^i \right) + (d_v + d_w) \left( \sum_{f(i) = \{d_v, d_w\}} e^i \right).
\]

If the triple \( \{u, v, w\} \) includes \( q + 1 \), then this is equal to \( 2 - (q + 1) = -(q - 1) \), otherwise it is equal to 2.

**Lemma 3.9.** The eigenvalue of \( \Gamma_2 \) corresponding to the character of type \( \chi_5 \) which are parameterized by a triple that contains \( q + 1 \) is \( -\frac{|G|}{(q+1)^2(q^2-q+1)} \). The eigenvalue of \( \Gamma_2 \) for any other character of type \( \chi_5 \) is \( \frac{-2|G|}{(q+1)^2(q-1)(q^2-q+1)} \).

**Proof.** Consider a character of type \( \chi_5 \) that is parameterized by a triple that contains \( q + 1 \). Then the corresponding eigenvalue of \( \Gamma_2 \) is

\[
(q-1) \left( \frac{|G|}{(q+1)^2} \right) \frac{1}{(q - 1)(q^2 - q + 1)} = -\frac{|G|}{(q+1)^2(q^2-q+1)}.
\]

If \( q \) is odd, then there are \( (q-1)/2 \) triples in \( T \) that contain \( q + 1 \), while if \( q \) is even, then there are \( q/2 \) triples in \( T \) that contain \( q + 1 \). Thus this is an eigenvalue for \( (q-1)/2 \) characters when \( q \) odd, and \( q/2 \) characters if \( q \) is even.

Next consider the characters of type \( \chi_5 \) which are parameterized by a triple that does not include \( q + 1 \). By Lemma 3.8, the eigenvalue for these characters is

\[
(-2) \left( \frac{|G|}{(q+1)^2} \right) \frac{1}{(q - 1)(q^2 - q + 1)} = \frac{-2|G|}{(q+1)^2(q-1)(q^2-q+1)}.
\]

If \( q \) is odd then there are \( q(q-1)/6 - (q-1)/2 = (q-1)(q-3)/6 \) triples in \( T \) that do not contain \( q + 1 \). Further, if \( q \) is even then there are \( q(q-1)/6 - q/2 = (q^2 - 4q)/6 \) triples in \( T \) that do not contain \( q + 1 \). This gives the number of characters for which \( \frac{-2|G|}{(q+1)^2(q-1)(q^2-q+1)} \) is the eigenvalue.

**Lemma 3.10.** The eigenvalue of \( \Gamma_1 \) for the character \( \chi_7 \) is \( \frac{|G|}{(q^2-q+1)(q+1)^2(q-1)} \).

**Proof.** It can be seen from the full character table of \( \text{PSU}(3,q) \) that the sum of the \( B \) for \( \chi_7 \) over the different conjugacy classes of type \( C_1 \) is 1.

Next simply calculate

\[
\frac{|G|}{(q^2-q+1)(q+1)^2(q-1)} = \frac{|G|}{(q^2-q+1)(q+1)^2(q-1)}.
\]
The number of such pairs is $k, \ell, m$.

The conjugacy classes of derangements in $\text{PSU}(3,q)$ are also parametrized by the set $T$ of all triples $(k, \ell, m)$, with $1 \leq k < \ell \leq (q+1)/3$ and $\ell < m \leq q+1$ and $k + \ell + m = q+1$. The number of such pairs is

$$\left(\frac{q+1}{2}\right) = \frac{q^2 - q - 2}{18}.$$  

The characters of type $\chi_6$ are also parametrized by the elements of $T$. Again we use $\Gamma_i$ to denote the Cayley graph on the group $\text{PSU}(3,q)$ with connection set of all the conjugacy classes of type $C_i$, for $i = 1, 2, 3$. Again, $\Gamma_{\text{PSU}(3,q)}$ is the union of these three graphs.

The rows of the character table for $\text{PSU}(3,q)$ corresponding to these families of conjugacy classes is given in the appendix. In this table, the $(q+1)$-th root of unity is denoted by $e$, and the third root of unity is denoted by $\omega$. See [13] for the complete character table of $\text{PSU}(3,q)$.

The first result is a statement of the eigenvalues that can be calculated directly from the character table (Table 3 given in the appendix using Equation 1).

**Lemma 4.1.** (1) The eigenvalues for the trivial character of $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are $\frac{|G|(q^2-q-2)}{3(q^2-q+1)}$, $\frac{|G|(q^2-2)}{6(q+1)}$, and $\frac{|G|}{(q+1)^2}$ (respectively).

(2) The eigenvalues for $\chi_1$ of $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are $-\frac{|G|(q^2-q-2)}{3(q^2-q+1)}$, $\frac{|G|(q^2-2)}{3q(q-1)(q+1)}$, and $\frac{2|G|}{q(q-1)(q+1)^2}$ (respectively).
The eigenvalues for $\chi_2$ of $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are $-\frac{|G|}{q^3(q^2-q+1)}$, $-\frac{|G|}{q^3(q^2-q+1)}$, and $-\frac{|G|}{q^3(q^2-q+1)}$ (respectively).

The eigenvalue of $\Gamma_1$ for $\chi_3$, $\chi_4$, $\chi_5$, $\chi_6$ and $\chi_7$ is 0.

The eigenvalue of $\Gamma_3$ is $\frac{3|G|}{(q+1)^3(q^2-q+1)}$ for $\chi_3$ and $-\frac{3|G|}{(q+1)^3(q^2-q+1)}$ for $\chi_4$.

The eigenvalue for $\Gamma_3$ corresponding to $\chi_5$ is $\frac{3|G|}{(q+1)^3(q^2-q+1)}$ if $9|q+1$, and $\frac{3|G|}{(q+1)^3(q^2-q+1)}$ otherwise. For $\chi_7$ and $\chi_8$ the eigenvalue of $\Gamma_3$ is 0.

The next reducible character we consider is $\chi_3$. As in the case where $3|q+1$, we will find the eigenvalue corresponding to $\chi_3$ by summing the value of the character over all the conjugacy classes of type $C_2$. First we need a technical result.

**Lemma 4.2.** For $k \in \{1, 2, \ldots, (q + 1)/3 - 1\}$ the sum

$$\sum_{(k, \ell, m) \in T} e^{3ku} + e^{3kv} + e^{3kw}$$

(where $e$ is the $(q + 1)$-th root of unity) is equal to 0.

**Proof.** For any triple $(k, \ell, m) \in T$, both $k$ and $\ell$ are no more than $(q + 1)/3$ and $m$ is between $\ell$ and $q + 1$. Every distinct pair $(k, \ell)$ from $\{1, \ldots, (q + 1)/3\}$ will be in exactly one triple $(k, \ell, m)$ in the set $T$. Thus each element less than $(q + 1)/3$ will occur in exactly $(q + 1)/3 - 1 = (q - 2)/3$ triples in $T$.

Next we count the number of time an element larger than $(q + 1)/3$ (these are the elements we represent with $m$) occur in a triple in $T$. First, consider the elements $\frac{q+1}{3} + i$ where $i \in \{1, \ldots, \frac{q-2}{3}\}$. Each of these elements will be in exactly $\left\lfloor \frac{q+1}{3} \right\rfloor$ triples from $T$; these are the triples of the form

$$\left(\frac{q+1}{3} - i, \frac{q+1}{3}, \frac{q+1}{3} + i\right), \left(\frac{q+1}{3} - i + 1, \frac{q+1}{3} - 1, \frac{q+1}{3} + i\right), \ldots$$

Similarly the elements $q - i$ with $i \in \{1, \ldots, \frac{q-2}{3}\}$ will be in $\left\lfloor \frac{q+1}{3} \right\rfloor$ triples from $T$.

Consider elements $a$ and $b$ with $(q + 1)/3 < a < 2(q + 1)/3 \leq b < (q + 1)$ and

$$3a \equiv 3b \pmod{q + 1}.$$ 

These conditions imply that $a = (q + 1)/3 + i$ and $b = 2(q + 1)/3 + i = q - ((q + 1)/3 - i - 1)$. From the above comments, the element $a$ will occur $\left\lfloor \frac{q+1}{3} \right\rfloor + \left\lfloor \frac{(q+1)/3-1}{2} \right\rfloor$ times in sets in $T$ and the element $b$ will occur $\left\lfloor \frac{(q+1)/3-1}{2} \right\rfloor$ times. So there are $\left\lfloor \frac{q+1}{3} \right\rfloor + \left\lfloor \frac{(q+1)/3-1}{2} \right\rfloor = (q - 2)/6$ elements $a$ in sets in $T$ with $a \equiv 3i \pmod{q + 1}$.

If $q$ is even, each element from $\{1, \ldots, (q + 1)/3\}$ occurs in $(q - 2)/3$ triples, and for every $i$ there are in total, $(q - 2)/6$ elements $a$ in triples from $T$ with $a \equiv 3i \pmod{q + 1}$.

In this case we have that

$$\sum_{(k, \ell, m) \in T} e^{3uk} + e^{3ul} + e^{3um} = \frac{q-2}{3} \sum_{i=1}^{\frac{q+1}{3}} e^{3ui} + \frac{q-2}{6} \sum_{i=1}^{\frac{q+1}{3}} e^{3ui} = 0.$$

Next assume that $q$ is odd. If $i$ is even, then in total there are $(q - 5)/6$ elements $a$ in triples from $T$ for which $a \equiv 3i \pmod{q + 1}$. If $i$ is odd, then in total there are $(q + 1)/6$ elements $a$ in triples from $T$ for which $a \equiv 3i \pmod{q + 1}$. Thus for
for the irreducible characters of type $\chi$.

The eigenvalue of Lemma 4.5.

For any $q$ even

$$\sum_{(k,\ell,m)\in T} e^{3uk} + e^{3ui} + e^{3um} = \frac{q-2}{3} \sum_{i=1}^{q+1} e^{3ui} + \frac{q-5}{6} \sum_{i=1}^{q+1} e^{6iu} + \frac{q+1}{6} \sum_{i=1}^{q+1} e^{3u(2i-1)} = 0.$$  

With this lemma, it is easy to determine the eigenvalue for the characters $\chi_3$ and $\chi_4$.

**Lemma 4.3.** The eigenvalue of $\Gamma_2$ for any of the characters of type $\chi_3$ and $\chi_4$ is 0.

**Lemma 4.4.** Let $\omega$ be a third root of unity. Then

$$\sum_{(k,\ell,m)\in T} (\omega^{k-\ell} + \omega^{\ell-k}) = \begin{cases} \frac{2q+2}{3}, & \text{if } q+1 \equiv 0 \pmod{9}; \\ \frac{2q-4}{3}, & \text{if } q+1 \equiv 3 \pmod{9}; \\ \frac{2q-13}{3}, & \text{if } q+1 \equiv 6 \pmod{9}. \end{cases}$$

**Proof.** For any $k$ and $\ell$, the set $\{(k-\ell) \pmod{3}, (\ell-k) \pmod{3}\}$ will either be $\{1,2\}$ or $\{0,0\}$. If the value of these differences is equal to $(0,0)$, then $\omega^{k-\ell} + \omega^{\ell-k}$ equals 2, otherwise it is equal to $-1$.

Count the number of times that this set is $\{0,0\}$. Any such set has the form $(k,\ell) = (a, a+3i)$ with $a \in \{1, \ldots, \left\lfloor \frac{q+1}{3} \right\rfloor - 3i\}$ for some $i \leq \left\lfloor \frac{q+1}{3} \right\rfloor$. Thus there are

$$N_{(0,0)} := \sum_{i=1}^{\left\lfloor \frac{q+1}{3} \right\rfloor} \left( \frac{q+1}{3} - 3i \right) = \frac{q+1}{9} \sum_{i=1}^{\frac{q+1}{3}} - \frac{3}{2} \left( \left\lfloor \frac{q+1}{9} \right\rfloor - 1 \right)$$

pairs $(k,\ell)$ in which the difference is equivalent to $\{0,0\}$. The number of pairs $(k,\ell)$ in which the difference is equivalent to $\{1,2\}$ is then

$$\frac{(q+1)(q-2)}{18} - N_{(0,0)}.$$

Thus the sum of $(\omega^{k-\ell} + \omega^{\ell-k})$ over all $k$ and $\ell$ with $(k,\ell,m) \in T$ is equal to

$$-1 \left( \frac{(q+1)(q-2)}{18} - N_{(0,0)} \right) + 2N_{(0,0)} = \begin{cases} \frac{2q+2}{3}, & \text{if } q+1 \equiv 0 \pmod{9}; \\ \frac{2q-4}{3}, & \text{if } q+1 \equiv 3 \pmod{9}; \\ \frac{2q-13}{3}, & \text{if } q+1 \equiv 6 \pmod{9}. \end{cases}$$

With the previous result, it is straight-forward to calculate the eigenvalue for $\Gamma_2$ for the irreducible characters of type $\chi_5$.

**Lemma 4.5.** The eigenvalue of $\Gamma_2$ for $\chi_5$ is

$$\lambda_{\chi_5} = \begin{cases} \frac{6G}{(q-1)(q+1)(q^2-1)}, & \text{if } q+1 \equiv 0 \pmod{9}; \\ \frac{6(q-2)G}{(q-1)(q^2-1)(q+1)^2}, & \text{if } q+1 \equiv 3 \pmod{9}; \\ \frac{3G(2q-13)}{(q-1)(q+1)^2(q^2-1)}, & \text{if } q+1 \equiv 6 \pmod{9}. \end{cases}$$
Next we find the eigenvalues for the characters of type $\chi_6$. Each of these characters is parameterized by a triple $(u, v, w) \in T$ and each conjugacy class of type $C_2$ is parameterized by a triple $(k, \ell, m) \in T$. Again we let $S$ denote the set of all permutations of all triples from $T$. Next we prove two lemmas, from these it is straight-forward to calculate the eigenvalues of $\Gamma_2$ that correspond to the characters of type $\chi_6$.

**Lemma 4.6.** Let $(u, v, w) \in T$. If $v - u \equiv 0 \pmod{3}$, then the sum

$$\sum_{(k, \ell, m) \in S} e^{uk + v\ell + wm}$$

is equal to $-(q + 1)/3$, provided that one of $u, v$ or $w$ is divisible by $(q + 1)/3$; otherwise it is equal to 0.

**Proof.** Assume that $v - u \equiv 0 \pmod{3}$. Since $(u, v, w) \in T$ we know that 3 divides $u + v + w$, which implies that both $u - w$ and $v - w$ are divisible by 3.

We will count the number of solutions $(k, \ell, m) \in S$ to

$$uk + v\ell + wm \equiv i \pmod{q + 1} \quad (4)$$

for each $i \in \{0, \ldots, q\}$. Since $k + \ell + m \equiv 0 \pmod{q + 1}$, Equation (4) can be reduced to

$$(u - w)k + (v - w)\ell \equiv i \pmod{q + 1}.$$

Both $u - w$ and $v - w$ are divisible by 3, so Equation (4) has solutions if and only if $i$ is a multiple of 3. In this case, the equation can be further reduced to

$$\frac{(u - w)k}{3} + \frac{(v - w)\ell}{3} \equiv i/3 \pmod{q + 1/3}.$$

Set

$$d = \gcd\left(\frac{v - w}{3}, \frac{v - w}{3}, \frac{q + 1}{3}\right) = \frac{1}{3} \gcd(u, v, w, q + 1).$$

Using Proposition 3.7, Equation (4) has $d(q + 1)/3$ solutions $k, \ell \in \{1, \ldots, (q + 1)/3\}$. Further, set

$$d_u = \gcd(u, \frac{q + 1}{3}), \quad d_v = \gcd(v, \frac{q + 1}{3}), \quad d_w = \gcd(w, \frac{q + 1}{3}).$$

As in the proof of Lemma 3.8, if $d_w$ divides $i/3$, then $d_w$ of these solutions have $k = \ell$.

Counting the solutions with $k, m \leq (q + 1)/3 < \ell$ and $\ell, m \leq (q + 1)/3 < k$, the total number of solutions to Equation (4) is

$$3d\left(\frac{q + 1}{3}\right) - \sum_{j \in \{u, v, w\}} d_j \left[ \frac{i}{d_j} \right].$$

As in the proof of Lemma 3.8, this implies that

$$\sum_{(k, \ell, m) \in S} e^{uk + v\ell + wm} = -d_u \sum_{i \in \{1, \ldots, q + 1\}} \frac{e^i}{d_u \left[ \frac{i}{d_u} \right]} - d_v \sum_{i \in \{1, \ldots, q + 1\}} \frac{e^i}{d_v \left[ \frac{i}{d_v} \right]} - d_w \sum_{i \in \{1, \ldots, q + 1\}} \frac{e^i}{d_w \left[ \frac{i}{d_w} \right]}.$$

If the triple $(u, v, w)$ includes a multiple of $(q + 1)/3$, then this is equal to $-(q + 1)/3$, otherwise it is equal to 0. □
Lemma 4.7. Let \((u,v,w) \in T\). If \(v-u \not\equiv 0 \pmod{3}\), then the sum
\[
\sum_{(k,\ell,m) \in S} e^{uk+v\ell+w m}
\]
is equal to \((q-8)/3\) provided that one of \(u,v\) or \(w\) is divisible by \((q+1)/3\); otherwise it is equal to 3.

Proof. Once again, we will count the number of solutions \((k,\ell, m)\) to
\[
(5) \quad uk + v\ell + wm \equiv i \pmod{q+1}
\]
with exactly two of \(k, \ell, m\) in \(\{1, \ldots, (q+1)/3\}\) and \(k + \ell + m \equiv q+1 \pmod{q+1}\).

Set \(d = \gcd(u,v, (q+1)/3) = \gcd(u,v,w, (q+1)/3)\). Similar to the proof of Lemma 3.8, there are \(d(q+1)\) solutions to Equation \((5)\)

If \((k,\ell, m)\) is a solution to Equation \((5)\) then both
\[
(k + \frac{q+1}{3}, \ell + \frac{q+1}{3}, m + \frac{q+1}{3}), \quad (k + \frac{2(q+1)}{3}, \ell + \frac{2(q+1)}{3}, m + \frac{2(q+1)}{3})
\]
taken modulo \(q+1\) are also solutions to Equation \((5)\). We will call these the three shifted solutions. It is clear that exactly one of \(k, k+ (q+1)/3, \text{ and } k+2(q+1)/3\) is in the range \(\{1, \ldots, (q+1)/3\}\) (and obviously the same holds for \(\ell\) and \(m\)).

If \(i = 0\), then \(k = \ell = m = (q+1)/3\) is a solution, if fact it is the only solution in which all three of \(k, \ell, m\) are no larger than \((q+1)/3\). Further, \(k \equiv \ell \equiv m \equiv 0 \pmod{(q+1)/3}\) is a solution when \(i = 0\). We claim that these solutions are the only ones in which each of the three shifted solutions has exactly one element less than \((q+1)/3\). To see this, assume without loss of generality that \(k \in \{1, \ldots, (q+1)/3\}\), \(\ell \in \{(q+1)/3+1, \ldots, 2(q+1)/3\}\) and \(m \in \{2(q+1)/3+1, \ldots, q+1\}\). But this implies that

\[1 + (q+1)/3 + 1 + 2(q+1)/3 = q+4 \leq k + \ell + m,\]

so \(k + \ell + m = 2(q+1)\). This happens if and only if \(k = (q+1)/3, \ell = 2(q+1)/3\) and \(m = q+1\).

So, other than these solutions with \(i = 0\), exactly one of the three shifted solutions will have only one of \(k, \ell, m\) larger than \((q+1)/3\). Thus, of the \(d(q+1)\) solutions to Equation \((5)\) (other than then the examples above with \(i = 0\)), there are \(d(q+1)/3\) solutions \((k,\ell, m)\) to with exactly one of \(k, \ell\) or \(m\) larger than \((q+1)/3\).

Next we count the number of these solutions with \(k = \ell, k = m\) or \(\ell = m\) to determine all the solutions that are in \(S\). Similar to Lemma 3.8, we set
\[
d_u = \gcd(u, \frac{q+1}{3}), \quad d_v = \gcd(v, \frac{q+1}{3}), \quad d_w = \gcd(w, \frac{q+1}{3}).
\]

Assume that \(k = \ell\), then
\[
uk + v\ell + wm \equiv k(u + v) + w m \pmod{q+1}
\]
\[
\equiv k(q+1 - w) + (q+1 - 2k)w \pmod{q+1}
\]
\[
\equiv k(-w) - 2kw \pmod{q+1}
\]
\[
\equiv -3um \pmod{q+1}
\]

Thus Equation \((5)\) has a solution with \(k = \ell\) if and only if \(3|i\). In this case the equation becomes
\[
-um \equiv i/3 \pmod{q+1/3}.
\]
If \( d_w \) divides \( i/3 \), then there are \( d_w \) solutions to Equation 5 with \( k = \ell \). Similarly, if \( d_v \) divides \( i/3 \), there are \( d_v \) solutions with \( k = m \); and if \( d_u \) divides \( i/3 \), there are \( d_u \) solutions with \( \ell = m \).

If \( i \not\equiv 0 \pmod{(q+1)/3} \) then the number of solutions \((k, \ell, m)\) to Equation 3 with \((k, \ell, m) \in S\) is

\[
\frac{d(q+1)}{3} - \sum_{d_j | i/3} d_j.
\]

If \( i = (q+1)/3 \) or \( 2(q+1)/3 \) then the number of solutions is

\[
\frac{d(q+1)}{3} - 1 - \sum_{d_j | i/3} d_j
\]

since we do not count the solutions with \( k \equiv \ell \equiv m \equiv 0 \pmod{q+1} \).

If \( i = 0 \) then the number of solutions is

\[
\frac{d(q+1)}{3} - d_u - d_v - d_w + 2,
\]

(since we have removed the solution with \( k = \ell = m = (q+1)/3 \) three times, rather than once we need to add 2).

For an integer \( i \) define \( f(i) \) to be the set of \( j \in \{u, v, w\} \) for which \( d_j \) divides \( i/3 \). Thus we have that

\[
\sum_{(k, \ell, m) \in S} e^{uk+vl+wm} = d(q+1)/3 \sum e^i - \left( \begin{array}{c}
(d_u + d_v + d_w) \sum_{f(i) = \{d_u, d_v, d_w\}} e^i \\
(d_u + d_v) \sum_{f(i) = \{d_u, d_v\}} e^i + (d_u + d_w) \sum_{f(i) = \{d_u, d_w\}} e^i + (d_v + d_w) \sum_{f(i) = \{d_v, d_w\}} e^i \\
+ d_u \sum_{f(i) = \{d_u\}} e^i + d_v \sum_{f(i) = \{d_v\}} e^i + d_w \sum_{f(i) = \{d_w\}} e^i + 2 \sum_{i=0}^{(q+1)/3} e^i \\
+ 2e^0 - e^{(q+1)/3} - e^{2(q+1)/3} \\
= 3 - d_u \sum_{d_u | i/3} e^i - d_v \sum_{d_v | i/3} e^i - d_w \sum_{d_w | i/3} e^i.
\right)
\]

If the triple \( \{u, v, w\} \) includes a multiple of \((q+1)/3\), then this is equal to \(3 - (q+1)/3\), otherwise it is equal to 3. \(\square\)

**Lemma 4.8.** Let \((u, v, w) \in T\). If \( v - u \equiv 0 \pmod{3} \), then the \( \chi_6 \) eigenvalue of \( \Gamma_2 \) is either 0, or

\[
\frac{|G|}{(q+1)(q-1)(q^2 - q + 1)}
\]

If \( v - u \not\equiv 0 \pmod{3} \), then the \( \chi_6 \) eigenvalue of \( \Gamma_2 \) is either

\[
\frac{|G|(q-8)}{(q+1)^2(q-1)(q^2 - q + 1)}
\]
Next we consider the character $\chi_6$, this character is indexed by the elements in $T$.

Lemma 4.9. For every triple $(u, v, w) \in T$, the $\chi_6$ eigenvalue of $\Gamma_3$ is either
\[
\begin{align*}
&\frac{6|G|}{(q-1)(q+1)^2(q^2-q+1)} \\
&\frac{3|G|}{(q-1)(q+1)^2(q^2-q+1)}.
\end{align*}
\]

Proof. If $u-v \equiv 0 \pmod{3}$ then $-3(\omega^{u-v} + \omega^{v-u}) = -6$, the $\chi_6$ eigenvalue for $\Gamma_3$ is
\[
\frac{6|G|}{(q+1)^2(q-1)(q^2-q+1)}.
\]

On the other hand, when $u-v \not\equiv 0 \pmod{3}$ it follows that $-3(\omega^{u-v} + \omega^{v-u}) = 3$. In this case the $\chi_6$ eigenvalue of $\Gamma_3$ is
\[
\frac{3|G|}{(q-1)(q^2-q+1)(q+1)^2}.
\]

The final character to consider is $\chi_8$.

Lemma 4.10. The eigenvalue of $\Gamma_1$ for $\chi_8$ eigenvalue is
\[
\frac{3|G|}{(q^2-q+1)(q+1)^2(q-1)}.
\]

Proof. By orthogonality of the characters, the sum of the values of $B$ over all the conjugacy classes of type $C_1$ is equal to 1. Thus the eigenvalue is
\[
\frac{1}{(q+1)^2(q-1)} \cdot \frac{3|G|}{(q^2-q+1)(q+1)^2(q-1)} = \frac{3|G|}{(q^2-q+1)(q+1)^2(q-1)}.
\]

We summarize the results from this section in Table 2. The character $\chi_5$ when $q+1 \equiv i \pmod{9}$ is denoted by $\chi_5^i$.

5. Eigenvalues of the Adjacency Matrices

In this section we will show that Lemma 2.5 applies to the group $\text{PSU}(3, q)$. The first step is to calculate all the eigenvalues for $\Gamma_{\text{PSU}(3,q)}$. Using the previous tables, this is straightforward for all values of $q$ and we record the results in Tables 3 and 4.

When $3 \nmid q$, the least eigenvalue of $\Gamma_{\text{PSU}(3,q)}$ is $-\frac{q|G|}{(q+1)^2(2q^2-q+1)}$ and is given by the character of type $\chi_1$. Applying Hoffman’s ratio bound gives
\[
\alpha(G_q) \leq \frac{|G|}{1 - \frac{6|G|}{2(q+1)^2(2q^2-q+1)} \frac{q|G|}{(q+1)^2(2q^2-q+1)}} = \frac{2|G|}{2 + (q-1)(q^2+q+1)} = \frac{2|G|}{(q^3+1)} = 2q^3(q+1)(q-1)
\]
But this is not the size of the largest set of intersecting permutations.
The eigenvalues for the characters $\chi_6$ in Table 2 are the only characters for which this is true.

When $3|q+1$, the least eigenvalue of $\Gamma_{PSU(3,q)}$ belongs to $\chi_1$ and is equal to $-|G|(q^3 - 3q^2 - 2)$ and, again, Hoffman’s bound does not hold with equality for the adjacency matrix. Again, the eigenvalue for $\chi_2$ is equal to the eigenvalue for the trivial character (so the degree) divided by $-q^3$, and provided that $q \neq 5$, this is the only character for which this is true.

The next step is to give a weighted adjacency matrix for $\Gamma_{PSU(3,q)}$ for which Hoffman’s ratio bound does hold with equality. This weighting was given in [17].

In the case where $3|q+1$, we will assign a weight of $a$ on the conjugacy classes in the family $C_1$, and a weight of $b$ on the conjugacy classes of type $C_2$. Let $A$ be the weighted adjacency matrix for the derangement graph for $PSU(3,q)$ with this weighting. The $(\sigma, \pi)$-entry of $A$ is $a$ if $\sigma^{-1}\pi$ is in one of the conjugacy classes of $C_1$, the entry is $b$ if $\sigma^{-1}\pi$ is in one of the conjugacy classes of $C_2$ and any other entry is equal to $0$.

The eigenvalues for the characters $\chi_1$ and $\chi_2$ (respectively) of $A$ (when $3|q+1$) are

$$
\frac{1}{q(q-1)} \left( (-1)a \frac{q^2 - q}{3} \frac{|G|}{q^2 - q + 1} + 2b \frac{(q^2 - q)}{6} \frac{|G|}{(q+1)^2} \right)
$$
Table 3. Eigenvalues for the derangement graph $\text{PSU}(3, q)$ where $3 \nmid q + 1$.

| Character  | Number | Eigenvalue of $\Gamma_{\text{PSU}(3, q)}$ |
|------------|--------|------------------------------------------|
| Trivial    | 1      | $|G|(q - 1)(q^3 - 1)$                     |
| $\chi_1$   | $q - 1$| $q(q - 1)$                               |
| $\chi_2$   | 1      | $q^3$                                    |
| $\chi_3, u = \frac{q + 1}{2}$ | $q + 1$ | $\frac{|G|(q^2 - q)(q^3 + q + 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_4, u = \frac{q + 1}{2}$ | $q + 1$ | $\frac{|G|(q - 1)(q^3 + q + 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_5$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |
| $\chi_6$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |
| $\chi_7$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |
| $\chi_8$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |

Table 4. Eigenvalues for the derangement graph $\text{PSU}(3, q)$ where $3 \mid q + 1$.

| Character  | Number | Dimension | Eigenvalue of $\Gamma$ |
|------------|--------|-----------|------------------------|
| 1          | 1      | 1         | $\frac{|G|(q^3 - 3q^2 - 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_1$   | 1      | $q(q - 1)$| $\frac{|G|(q^3 - 3q^2 - 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_2$   | 1      | $q^3$     | $\frac{|G|(q^3 - 3q^2 - 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_3, u = \frac{q - 1}{2}$ | $q^2 - q + 1$ | $\frac{|G|(q^3 - 3q^2 - 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_4, u = \frac{q - 1}{2}$ | $q^2 - q + 1$ | $\frac{|G|(q^3 - 3q^2 - 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_5$   | 3      | $\frac{q^2 - 1}{3}$ | $\frac{|G|(q^3 - 3q^2 - 1)}{2(q^3 - q - 1)(q^2 + q + 1)}$ |
| $\chi_6$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |
| $\chi_7$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |
| $\chi_8$   | 3      | $\frac{(q^2 - 1)(q^3 - 1)}{3}$          |
Table 5. Table of eigenvalues of the weighted adjacency matrix for $3 \nmid q + 1$.

| Type      | Number | Dimension | eigenvalue          |
|-----------|--------|-----------|---------------------|
| Trivial   | 1      | 1         | $q^3$               |
| $\chi_1$ | 1      | $q^3$     | -1                  |
| $\chi_2$ | 1      | $q^3$     | $q(q-1)$            |
| $\chi_3$, $u = \frac{q+1}{2}$ | 1      | $q^2 - q + 1$ | -1                  |
| $\chi_4$, $u \neq \frac{q+1}{2}$ | 1      | $q^2 - q + 1$ | $\frac{1}{q}$      |
| $\chi_5$ | $\frac{(q-1)(q-3)}{6}$ | $(q-1)(q^2 - q + 1)$ | $-\frac{2}{q(q-1)}$ |
| $\chi_6$ | $(q-1)/2$ | $(q-1)(q^2 - q + 1)$ | $-\frac{4}{(q-1)^2}$ |
| $\chi_7$ | $\frac{2^2 - q - 2}{q^2 - q}$ | $(q+1)(q^2 - q + 1)$ | 0                   |

The eigenvalues of the weighted adjacency matrix $A$ can be calculated directly from Table 1.

Applying Hoffman’s ratio bound to the weighted adjacency matrix, we get in both cases that

$$\alpha(\Gamma_{PSU(3,q)}) = \frac{|PSU(3,q)|}{q^3 + 1}.$$  

Equality holds since any canonical set meets this bound. This means that the group $PSU(3,q)$ has the EKR property (this is shown in [17]). By Tables 3 and 4 provided that $q \neq 5$, only $\chi_2$ gives the eigenvalue that gives equality in the equation for Hoffman’s bound. Thus $PSU(3,q)$ satisfies all the conditions of Lemma 2.5, we conclude that every maximum coclique is in the module.

**Theorem 5.1.** For all $q \neq 5$, the group $PSU(3,q)$ has the EKR-module property.
6. Further work

We have shown that the group \( \text{PSU}(3, q) \) has the EKR-module property. This means that the characteristic vector for any maximum intersecting set of permutations from \( \text{PSU}(3, q) \) is a linear combination of the characteristic vectors of the canonical intersecting sets. It is still open if the group \( \text{PSU}(3, q) \) has the strict-EKR property, but the result in this paper gives an interesting property of the intersecting sets from \( \text{PSU}(3, q) \) which may be useful for characterizing the maximum intersecting sets.

For example, the method described in [2] proves that a group with the EKR-module property has the strict-EKR property, provided that a certain matrix has full rank. For \( q = 2 \) this matrix does not have full rank. The derangement graph for \( \text{PSU}(3, 2) \) is 8 disjoint copies of \( K_9 \), so it is clear what the maximum cocliques are and that \( \text{PSU}(3, 2) \) does not have the strict-EKR property. For \( q = 3 \), a calculation (using GAP [8]) shows that this matrix does have full rank; thus \( \text{PSU}(3, 3) \) does have the strict-EKR property. For larger values of \( q \), a more efficient method to find the rank of this smaller matrix would be needed to prove that \( \text{PSU}(3, q) \) has the strict-EKR property.

Although all 2-transitive groups have the EKR-property, it is not true that all 2-transitive groups have the strict-EKR property. It has been shown that \( \text{Sym}(n) \), \( \text{Alt}(n) \), \( \text{PGL}(2, q) \), \( \text{PGL}(3, q) \), \( \text{PSL}(2, q) \) [14] and the Mathieu groups all have the strict-EKR property by first showing that they have the EKR-module property [1] [2] [10] [15] [16] [14]. It is shown in [17] that the Suzuki groups, Ree Groups, Higman-Sims, Symplectic groups also have the EKR-module property. The result in this paper shows that every minimal almost-simple 2-transitive groups (except possibly

| Character | Number | Dimension | eigenvalue |
|-----------|--------|-----------|------------|
| \( \chi_1 \) | 1 | \( q(q - 1) \) | \( q^3 \) |
| \( \chi_2 \) | 1 | \( q^3 \) | \( -1 \) |
| \( \chi_3 \) | \( \frac{q^2 - 2}{3} \) | \( (q^2 - q + 1)^2 \) | \( \frac{6q}{q^2 - q + 4} \) |
| \( \chi_4 \) | \( \frac{q^2 - 2}{3} \) | \( q^2(q^2 - q + 1)^2 \) | \( \frac{q^2 - q + 1}{2q(q+2)^2} \) |
| \( \chi_5 \) | 3 | \( (q - 1)^2(q^2 - q + 1)^2 / 9 \) | \( \frac{(q - 1)^2(q^2 - q + 4)}{2(q+6)^2} \) |
| \( \chi_6 \) | 3 | \( (q - 1)^2(q^2 - q + 1)^2 / 9 \) | \( \frac{(q - 1)^2(q^2 - q + 4)}{2(q+6)^2} \) |
| \( \chi_7 \) | \( \frac{q^2 - g - 2}{6} \) | \( (q + 1)^2(q^2 - q + 1)^2 \) | \( 0 \) |
| \( \chi_8 \) | \( \frac{q^2 - g - 2}{18} \) | \( (q + 1)^4(q - 1)^2 \) | \( \frac{3q(2q - 1)}{(q-1)(q+1)(q^2 - q + 2)} \) |

Table 6. Table of eigenvalues of the weighted adjacency matrix for \( 3 \mid q + 1 \).
PSU(3, 5)) has the EKR-module property. Thus we conclude with the following conjecture.

**Conjecture 6.1.** Every 2-transitive group has the EKR-module property.
7. Appendix

| number | size | 1 ≤ u ≤ q + 1 | 1 ≤ u ≤ q + 1 | (u, v, w) ∈ T | (q^2 - q - 2)/2 | (q^2 - q)/6 |
|--------|------|----------------|----------------|----------------|----------------|-------------|
| C_1    | q^2 - q | q(q - 1)       | q^3            | (q - 1)(q^2 - q + 1) | (q - 1)(q^2 - q) | (q + 1)(q^2 - q) |
| C_2    | q^2 - q | q(q - 1)       | q^3            | (q - 1)(q^2 - q + 1) | (q - 1)(q^2 - q) | (q + 1)(q^2 - q) |
| C_3    | q^2 - q | q(q - 1)       | q^3            | (q - 1)(q^2 - q + 1) | (q - 1)(q^2 - q) | (q + 1)(q^2 - q) |

Table 7. Partial character Table for PSU(3, q) where 3

| number | size | 1 ≤ u ≤ q + 1 | 1 ≤ u ≤ q + 1 | (u, v, w) ∈ T | (q^2 - q - 2)/2 | (q^2 - q)/6 |
|--------|------|----------------|----------------|----------------|----------------|-------------|
| C_1    | q^2 - q | q(q - 1)       | q^3            | (q - 1)(q^2 - q + 1) | (q - 1)(q^2 - q) | (q + 1)(q^2 - q) |
| C_2    | q^2 - q | q(q - 1)       | q^3            | (q - 1)(q^2 - q + 1) | (q - 1)(q^2 - q) | (q + 1)(q^2 - q) |
| C_3    | q^2 - q | q(q - 1)       | q^3            | (q - 1)(q^2 - q + 1) | (q - 1)(q^2 - q) | (q + 1)(q^2 - q) |

Table 8. Partial character Table for PSU(3, q) where 3

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1. This value is −2 if 9/q + 1 and 1 otherwise.
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