NEW METHODS FOR LOCAL SOLVABILITY OF QUASILINEAR
SYMMETRIC HYPERBOLIC SYSTEMS

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Abstract. In this work we establish the local solvability of quasilinear symmetric hyperbolic system using local monotonicity method and frequency truncation method. The existence of an optimal control is also proved as an application of these methods.

1. Introduction. Quasilinear symmetric and symmetrizable hyperbolic systems arise in a wide range of problems in engineering and physics. Some examples include unsteady Euler and potential equations of gas dynamics, inviscid magnetohydrodynamic (MHD) equations, shallow water equations, and Einstein field equations of general relativity to name a few (see for example [22], [13], [3]). The Cauchy problem of smooth solutions for these systems has been studied in the past using semigroup approach and fixed point arguments (see [6], [11], [8], [9], [21]). In this work, we establish the solvability of such system using two different methods, viz. local monotonicity method, which was first used in [15] to establish the solvability of stochastic Navier-Stokes equations, and a frequency truncation method ([14], [4]). The new methods we present here are motivated by applications to control theory and stochastic analysis (see for example [18], [15], [20], where such methods are used). We also formulate a simple optimal control problem and demonstrate the utility of the new methods in proving the existence of optimal control. Stochastic analysis aspects will be presented in a separate paper.

The construction of the paper is as follows. In Section 2, we describe the quasilinear symmetric hyperbolic system

\[
\frac{\partial u(t)}{\partial t} + \mathcal{A}(t,u(t))u(t) = f(t), \quad 0 \leq t \leq T,
\]

\[u(0) = u_0,\]

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where \( u = (u_1, \ldots, u_m), \ f = (f_1, \ldots, f_m), \ \mathcal{A}(t, u) = \sum_{j=1}^{n} A^j(t, x, u) \frac{\partial}{\partial x_j} \) and each \( A^j(\cdot, \cdot, \cdot) \) is an \( m \times m \) symmetric matrix with \( u_0 \in H^s(\mathbb{R}^n) \) for \( s > n/2 + 1 \). We also obtain certain conditions satisfied by the linear operator \( \mathcal{A}(t, u) \). By proving that the nonlinear term \( \mathcal{A}(t, u)u \) is locally monotone, a local in time existence and uniqueness of smooth solutions of (1) is obtained in Section 3, using a generalization of the Minty-Browder technique. By considering a truncated quasilinear symmetric hyperbolic system, the local solvability of (1) is established in Section 4 using the fact that the frequency truncated sequence of solutions is Cauchy. As an application of both of these methods, the existence of an optimal control is obtained in Section 5 for a typical control problem.

In the sequel \( \mathcal{L}(X, Y) \) denotes the space of all bounded linear operators from \( X \) to \( Y \) and \( D(A) \) denotes the domain of an operator \( A \). The main theorem of this paper is

**Theorem 1.1.** For the quasilinear symmetric hyperbolic system defined in (1):

(i) The linear operator \( \mathcal{A}(t, u) := \sum_{j=1}^{n} A^j(t, x, u) \frac{\partial}{\partial x_j} \), where \( A^j(\cdot, \cdot, \cdot) \) are \( m \times m \) symmetric matrices for \( j = 1, \ldots, n \), satisfies

\[
(\mathcal{A}(t, u)v, v)_{L^2} \geq -C\|\nabla A\|_{L^\infty} \|v\|_{L^2}^2,
\]

for \( u, v \in H^s(\mathbb{R}^n), s > n/2 + 1 \), where

\[
\|\nabla A\|_{L^\infty} = \sum_{j=1}^{n} \max_{1 \leq i \leq m} \sum_{k=1}^{m} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial}{\partial x_j} a_{ik}^j(t, x, u) \right|
\]

and \( a_{ik}^j(\cdot, \cdot, \cdot) \) is an entry of \( A^j(\cdot, \cdot, \cdot) \).

(ii) The operator \( \mathcal{B}(t, u) := J_s\mathcal{A}(t, u)J^{-s} - \mathcal{A}(t, u) \in \mathcal{L}(L^2, L^2), \) where \( J^s := (1 - \Delta)^s/2, \) satisfies

\[
\|\mathcal{B}(t, u)v\|_{L^2} \leq C\|\nabla A\|_{L^\infty}\|\nabla J^{-s}v\|_{H^{s-1}} + C(\|u\|_{L^\infty})(1 + \|u\|_{H^s})\|\nabla J^{-s}v\|_{L^\infty},
\]

for \( u, v \in H^s(\mathbb{R}^n), s > n/2 + 1 \).

(iii) Also for \( s > n/2 + 1, H^s(\mathbb{R}^n) \subset D(\mathcal{A}(t, u)), \) so that \( \mathcal{A}(t, u) \in \mathcal{L}(H^s, L^2) \) and

\[
\| (\mathcal{A}(t, u) - \mathcal{A}(t, v))w \|_{L^2} \leq C\|\nabla u A\|_{L^\infty}\|u - v\|_{L^2}\|w\|_{L^\infty},
\]

for all \( u, v, w \in H^s(\mathbb{R}^n), \) where

\[
\|\nabla u A\|_{L^\infty} = \left( \sum_{j=1}^{n} \max_{1 \leq i \leq m} \sum_{k=1}^{m} \sup_{(x, \tau) \in \mathbb{R}^n \times [0, 1]} \left| \nabla u a_{ik}^j(t, x, \tau u + (1 - \tau)v) \right|^2 \right)^{1/2}.
\]

(iv) Let \( u_0 \in H^s(\mathbb{R}^n) \) and \( f \in L^2(0, T; H^s(\mathbb{R}^n)) \) with \( s > n/2 + 1 \) be given. Then, there exists a time \( 0 < T^* < T \) and unique solution \( u(\cdot) \) of (1) such that \( u \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)). \)

Also \( u(\cdot) \) satisfies the energy estimate

\[
\sup_{0 \leq t \leq T^*} \|u(t)\|_{H^s}^2 \leq \left( \|u(0)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_0^{T^*} \|f(t)\|_{H^s}^2 dt \right) e^{(4CM + 2M')T^*}, \quad (2)
\]
for $0 < \varepsilon \leq 1$, where $M = \sup_{t \in [0,T^*]} \|\nabla A(t)\|_{L^\infty}$, $M' = \sup_{t \in [0,T^*]} C(\|u(t)\|_{L^\infty})(1 + \|\nabla u(t)\|_{L^\infty})$, and $0 < T^* < T$ is the maximal time for which the left hand side of (2) is finite.

2. Quasilinear symmetric hyperbolic system. The main ideas of this section are due to Kato ([8], [9]) and we elaborate here, since several of these results are used in subsequent sections. For the symmetric hyperbolic system (1), in order to compute the basic energy identity of Friedrichs ([13]), we use the symmetry of $A^j(\cdot, \cdot, \cdot)$ to find

$$
\left( a^{jk}_{ik} \frac{\partial v_k}{\partial x_j}, v_i \right)_{L^2} = \int_{\mathbb{R}^n} a^{jk}_{ik} \frac{\partial v_k}{\partial x_j} v_i \, dx \\
= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} (a^{jk}_{ik} v_k v_i) \, dx - \int_{\mathbb{R}^n} a^{jk}_{ik} \frac{\partial v_i}{\partial x_j} v_k \, dx - \int_{\mathbb{R}^n} \frac{\partial a^{jk}_{ik}}{\partial x_j} v_k v_i \, dx \\
= - \int_{\mathbb{R}^n} a^{jk}_{ik} \frac{\partial v_k}{\partial x_j} v_i \, dx - \int_{\mathbb{R}^n} \frac{\partial a^{jk}_{ik}}{\partial x_j} v_k v_i \, dx
$$

(3)

where $a^{jk}_{ik}(\cdot, \cdot, \cdot)$ is an entry of the matrix $A^j(\cdot, \cdot, \cdot)$. Hence from (3), we get

$$
\int_{\mathbb{R}^n} a^{jk}_{ik} \frac{\partial v_k}{\partial x_j} v_i \, dx = - \frac{1}{2} \int_{\mathbb{R}^n} \frac{\partial a^{jk}_{ik}}{\partial x_j} v_k v_i \, dx \geq - \frac{1}{2} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial a^{jk}_{ik}}{\partial x_j} \right| \int_{\mathbb{R}^n} v_k v_i \, dx
$$

and

$$
\sum_{i,k=1}^m \int_{\mathbb{R}^n} a^{jk}_{ik} \frac{\partial v_k}{\partial x_j} v_i \, dx \geq - \frac{1}{2} \sum_{i,k=1}^m \sup_{x \in \mathbb{R}^n} \left| \frac{\partial a^{jk}_{ik}}{\partial x_j} \right| \sum_{i=1}^m \int_{\mathbb{R}^n} v_i^2 \, dx.
$$

(4)

Finally by using (4), for $u, v \in H^s(\mathbb{R}^n)$, $s > n/2 + 1$, we have

$$
(\mathcal{A}(t,u)v, v)_{L^2} = \sum_{j=1}^n \left( A^j(t,x,u) \frac{\partial v}{\partial x_j}, v \right)_{L^2} \\
\geq -C \sum_{j=1}^n \left\| \frac{\partial}{\partial x_j} A^j(t,x,u) \right\|_{L^\infty} \|v\|_{L^2}^2 = -C \|\nabla A\|_{L^\infty} \|v\|_{L^2}^2,
$$

(5)

where $C$ is a constant independent of $u$ and $\|\nabla A\|_{L^\infty} = \sum_{j=1}^n \left( \left| \frac{\partial}{\partial x_j} A^j(t,x,u) \right| \right)_{L^\infty}$

with $\left( \frac{\partial}{\partial x_j} A^j(t,x,u) \right)_{L^\infty} = \max_{1 \leq i \leq m} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial a^{jk}_{ik}(t,x,u)}{\partial x_j} \right|$. Now by using the identity

$$
a^{jk}_{ik}(t,x,u_1) - a^{jk}_{ik}(t,x,u_2) = \int_0^1 \nabla_u a^{jk}_{ik}(t,x,\tau u_1 + (1-\tau)u_2) \cdot (u_1 - u_2) \, d\tau,
$$

(6)

we have

$$
\left( \frac{\partial a^{jk}_{ik}(t,x,u)}{\partial x_j}, \frac{\partial w_k}{\partial x_j} \right)_{L^2} \geq \int_{\mathbb{R}^n} \left( a^{jk}_{ik}(t,x,u) - a^{jk}_{ik}(t,x,v) \right) \frac{\partial w_k}{\partial x_j} \, dx
$$

$$
\leq \sup_{x \in \mathbb{R}^n} \left| \frac{\partial w_k}{\partial x_j} \right|^2 \int_{\mathbb{R}^n} \left| a^{jk}_{ik}(t,x,u) - a^{jk}_{ik}(t,x,v) \right|^2 \, dx
$$
\[
\leq \sup_{x \in \mathbb{R}^n} \frac{\partial w_k}{\partial x_j} \left[ \int_{\mathbb{R}^n} \int_{0}^{1} \left| \nabla u_{3i}^{j}(t, x, \tau u + (1-\tau)v) \cdot (u-v) \right|^2 d\tau dx \right]^{1/2} \\
\leq \sup_{x \in \mathbb{R}^n} \frac{\partial w_k}{\partial x_j} \left[ \int_{\mathbb{R}^n} \sup_{\tau \in [0,1]} \left| \nabla u_{3i}^{j}(t, x, \tau u + (1-\tau)v) \right|^2 \int_{0}^{1} |u-v|^2 d\tau \right]^{1/2} \\
\leq \sup_{x \in \mathbb{R}^n} \frac{\partial w_k}{\partial x_j} \sup_{x \in \mathbb{R}^n} \sup_{\tau \in [0,1]} \left| \nabla u_{3i}^{j}(t, x, \tau u + (1-\tau)v) \right|^2 |u-v|^2_{L^2},
\]

for \( u, v, w \in H^s(\mathbb{R}^n) \). Hence from (7), for \( s > n/2 + 1 \), we obtain

\[
\left\| (\mathcal{A}(t, u) - \mathcal{A}(t, v)) \right\|_{L^2} \\
\leq C \sum_{j=1}^{n} \| \nabla u A_j^{j} \|_{L^\infty} \left\| \frac{\partial w}{\partial x_j} \right\|_{L^\infty} \| u - v \|_{L^2} \\
\leq C \left( \sum_{j=1}^{n} \| \nabla u A_j^{j} \|^2_{L^\infty} \right)^{1/2} \left( \sum_{j=1}^{n} \| \frac{\partial w}{\partial x_j} \|^2_{L^\infty} \right)^{1/2} \| u - v \|_{L^2} \\
\leq C \| \nabla u A \|_{L^\infty} \| u - v \|_{L^2} \| \nabla w \|_{L^\infty} \\
\leq C \| \nabla u A \|_{L^\infty} \| u - v \|_{L^2} \| w \|_{H^s},
\]

(8)

where

\[
\| \nabla u A_j^{j} \|^2_{L^\infty} = \left( \sum_{j=1}^{n} \| \nabla u A_j^{j} \|^2_{L^\infty} \right)^{1/2}
\]

From (8), it can be seen that

\[
\| \mathcal{A}(t, u) - \mathcal{A}(t, v) \|_{L^2} \leq C \| \nabla u A \|_{L^\infty} \| u - v \|_{L^2}.
\]

(9)

We have \( J^s := (I - L)^{s/2} \) so that \( \| f \|_{H^s} = \| J^s f \|_{L^2} \), for \( f \in H^s(\mathbb{R}^n) \). Let us now recall the commutator estimates ([10]) and Moser type estimates ([21]) used in this paper.

**Lemma 2.1.** If \( s \geq 0 \) and \( 1 < p < \infty \), then

\[
\| J^s (fg) - f(J^s g) \|_{L^p} \leq C_p \| \nabla f \|_{L^\infty} \| J^{s-1} g \|_{L^p} + \| J^s f \|_{L^p} \| g \|_{L^\infty}.
\]

(10)

*Proof.* See Lemma XI, [10]. \( \square \)

**Lemma 2.2.** Let \( F(\cdot) \) be a smooth function of \( u \in L^\infty(\mathbb{R}^n) \cap H^{s,p}(\mathbb{R}^n) \), then for \( s > 0 \) and \( p \in (1, \infty) \), we have

\[
\| F(u) \|_{H^{s,p}} \leq C_{s,p} \| u \|_{L^\infty} \left( 1 + \| u \|_{H^{s,p}} \right).
\]

(11)

*Proof.* See Proposition 3.1.1.A., [21], Chapter 2, page 102, [23]. \( \square \)

Let us define the operator \( \mathcal{B}(t, u) \) ([8]) by

\[
\mathcal{B}(t, u)v = J^s \mathcal{A}(t, u) J^{-s} v - \mathcal{A}(t, u)v \\
= J^s \sum_{j=1}^{n} A^j(t, x, u) \frac{\partial}{\partial x_j} J^{-s} v - \sum_{j=1}^{n} A^j(t, x, u) \frac{\partial}{\partial x_j} v,
\]
\begin{align*}
&= \sum_{j=1}^{n} \left[ J^s \left( A^j(t, x, \mathbf{u}) \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right) - A^j(t, x, \mathbf{u}) J^s \left( \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right) \right], \quad (12)
\end{align*}
for \( \mathbf{u}, \mathbf{v} \in H^s(\mathbb{R}^n) \). Let us now take \( f = A^j(t, x, \mathbf{u}) \), \( g = \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \), and \( p = 2 \) in (10) to estimate the \( L^2 \)-norm of the term inside the summation in (12) as
\begin{align*}
&\left\| J^s \left( A^j(t, x, \mathbf{u}) \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right) - A^j(t, x, \mathbf{u}) J^s \left( \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right) \right\|_{L^2} \\
&\leq C \left( \left\| \nabla A^j(t, x, \mathbf{u}) \right\|_{L^\infty} \left\| J^{s-1} \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{L^2} + \left\| J^{s} A^j(t, x, \mathbf{u}) \right\|_{L^2} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{L^\infty} \right) \\
&= C \left( \left\| \nabla A^j(t, x, \mathbf{u}) \right\|_{L^\infty} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{H^{-s-1}} + \left\| A^j(t, x, \mathbf{u}) \right\|_{H^s} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{L^\infty} \right), \quad (13)
\end{align*}
where \( C \) is independent of \( \mathbf{u} \), \( \left\| \nabla A^j(t, x, \mathbf{u}) \right\|_{L^\infty} = \max_{1 \leq i \leq m} \sum_{k=1}^{m} \sup_{x \in \mathbb{R}^n} \left\| \frac{\partial}{\partial x_j} a_{ik}^j(t, x, \mathbf{u}) \right\| \), and \( \left\| A^j(t, x, \mathbf{u}) \right\|_{H^s} = \left( \sum_{i,k=1}^{m} \left\| a_{ik}^j(t, x, \mathbf{u}) \right\|_{H^s}^2 \right)^{1/2} \), which is the Frobenius norm. An application of (13) in (12) yields
\begin{align*}
&\left\| \mathcal{B}(t, \mathbf{u}) \mathbf{v} \right\|_{L^2} \\
&\leq C \sum_{j=1}^{n} \left( \left\| \nabla A^j(t, x, \mathbf{u}) \right\|_{L^\infty} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{H^{-s-1}} + \left\| A^j(t, x, \mathbf{u}) \right\|_{H^s} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{L^\infty} \right) \\
&\leq C \left( \sum_{j=1}^{n} \left\| \nabla A^j(t, x, \mathbf{u}) \right\|_{L^\infty} \right)^{1/2} \left( \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{H^{-s-1}}^2 \right)^{1/2} \\
&+ C \left( \sum_{j=1}^{n} \left\| A^j(t, x, \mathbf{u}) \right\|_{H^s} \right)^{1/2} \left( \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} J^{-s} \mathbf{v} \right\|_{L^\infty}^2 \right)^{1/2} . \quad (14)
\end{align*}
By using Moser estimates (Theorem 2.2), we have
\begin{align*}
\left\| A^j(t, x, \mathbf{u}) \right\|_{H^s} &\leq C \left( \left\| \mathbf{u} \right\|_{L^\infty} \right) \left( 1 + \left\| \mathbf{u} \right\|_{H^s} \right) . \quad (15)
\end{align*}
From (14), we get
\begin{align*}
&\left\| \mathcal{B}(t, \mathbf{u}) \mathbf{v} \right\|_{L^2} \leq C \left\| \nabla A \right\|_{L^\infty} \left\| \nabla J^{-s} \mathbf{v} \right\|_{H^{-s-1}} + C \left( \left\| \mathbf{u} \right\|_{L^\infty} \right) \left\| \nabla \mathbf{v} \right\|_{L^\infty} . \quad (16)
\end{align*}
\begin{align*}
&\leq C \left( \sum_{j=1}^{n} \left\| \nabla A^j(t, x, \mathbf{u}) \right\|_{L^\infty} \right)^{1/2} \left\| \nabla A \right\|_{L^\infty} = \left\| \nabla A \right\|_{L^\infty} .
\end{align*}
Hence for the symmetric hyperbolic system (1), we obtain the following conditions under which we prove the local solvability of (1).
\begin{itemize}
\item[(C1)] The linear operator \( \mathcal{A}(t, \mathbf{u}) := \sum_{j=1}^{n} A^j(t, x, \mathbf{u}) \frac{\partial}{\partial x_j} \), where \( A^j(\cdot, \cdot, \cdot) \)'s are \( m \times m \) symmetric matrices for \( j = 1, \cdots, n \), satisfies
\begin{align*}
\left( \mathcal{A}(t, \mathbf{u}) \mathbf{v}, \mathbf{v} \right)_{L^2} \geq -C \left\| \nabla A \right\|_{L^\infty} \left\| \mathbf{v} \right\|_{L^2}^2 , \quad (17)
\end{align*}
\end{itemize}
for \( u, v \in H^s(\mathbb{R}^n) \), \( s > n/2 + 1 \), where \( \|\nabla A\|_{L_\infty} = \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} A^j(t, x, u) \right\|_{L_\infty} \) with

\[
\left\| \frac{\partial}{\partial x_j} A^j(t, x, u) \right\|_{L_\infty} = \max_{1 \leq i \leq m} \sum_{k=1}^{m} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial}{\partial x_j} a_{ik}^j(t, x, u) \right| .
\]

For \( C\|\nabla A\|_{L_\infty} \leq \mu \), the operator \( \mathcal{A}(t, u) + \mu I \) is monotone.

(C2) There exists an operator

\[
\mathcal{B}(t, u) := \sum_{j=1}^{n} \left[ J^s \left( A^j(t, x, u) \frac{\partial}{\partial x_j} J^{-s} \right) - A^j(t, x, u) J^s \left( \frac{\partial}{\partial x_j} J^{-s} \right) \right],
\]

where \( J^s = (I - \Delta)^{s/2} \), such that \( \mathcal{B}(t, u) \in \mathcal{L}(L^2, L^2) \) with

\[
\|\mathcal{B}(t, u)v\|_{L^2} \leq C\|\nabla A\|_{L_\infty} \|\nabla J^{-s}v\|_{H^{-1}} + C((\|u\|_{L_\infty})(1 + \|u\|_{H^s}))\|\nabla J^{-s}v\|_{L_\infty},
\]

for \( u, v \in H^s(\mathbb{R}^n) \), \( s > n/2 + 1 \).

(C3) For \( s > n/2 + 1 \), we have \( H^s(\mathbb{R}^n) \subset D(\mathcal{A}(t, u)) \), so that \( \mathcal{A}(t, u) \in \mathcal{L}(H^s, L^2) \) with

\[
\|\mathcal{A}(t, u) - \mathcal{A}(t, v)\|_{\mathcal{L}(H^s, L^2)} \leq C\|\nabla A\|_{L_\infty}\|u - v\|_{L^2},
\]

for \( u, v, w \in H^s(\mathbb{R}^n) \) and

where \( \|\nabla A\|_{L_\infty} = \left( \sum_{j=1}^{n} \|\nabla A^j\|_{L_\infty}^2 \right)^{1/2} \) with

\[
\|\nabla A^j\|_{L_\infty}^2 = \max_{1 \leq i \leq m} \sum_{k=1}^{m} \sup_{(x, \tau) \in \mathbb{R}^n \times [0,1]} \left| \nabla A^j_{ik}(t, x, \tau u + (1 - \tau)v) \right|^2.
\]

3. Existence and uniqueness-local monotonicity method. In this section, we establish the unique solvability of the symmetric hyperbolic system (1) by exploiting the local monotonicity property of \( \mathcal{A}(\cdot, \cdot) \) and using the Minty-Browder type technique.

3.1. Energy estimates and local monotonicity. Let \( \{e_1, e_2, \cdots\} \) be a complete orthonormal system in \( L^2(\mathbb{R}^n) \) belonging to \( H^s(\mathbb{R}^n) \) and let \( L^2_n(\mathbb{R}^n) \) be the \( n \)-dimensional subspace of \( L^2(\mathbb{R}^n) \). Let us now consider the following system of ODE in \( L^2_n(\mathbb{R}^n) \):

\[
\frac{\partial u^n(t)}{\partial t} + \mathcal{A}(t, u^n(t))u^n(t) = f(t), \quad 0 \leq t \leq T,
\]

\[
u^n(0) = u^n_0,
\]

where \( u^n_0 \) is the orthogonal projection of \( u_0 \) into \( L^2_n(\mathbb{R}^n) \) and for simplicity we take \( f^n = f \). Since the system (22) is finite-dimensional and having locally Lipschitz coefficient, by Picard’s theorem, the system has a unique solution in some interval \([0, T]\). Let us now find the \( L^2 \) and \( H^s \) energy estimates for the system (22).
Proposition 1 (L^2-energy estimate). Let u^n(\cdot) be the unique solution of the system of ODE’s (22) with u_0 \in L^2(\mathbb{R}^n). Then, there exists a time 0 < T^* < T such that, for f \in L^2(0, T^*; L^2(\mathbb{R}^n)) and \varepsilon \leq 1, we have the following a-priori energy estimate:

\[ \|u^n(t)\|_{L^2}^2 \leq \left( \|u^n(0)\|_{L^2}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{L^2}^2 dr \right) \exp \left( \int_0^t (2C\|\nabla A(r)\|_{L^\infty} + \varepsilon) dr \right), \]

for 0 \leq t \leq T^*, and

\[ \sup_{0 \leq t \leq T^*} \|u^n(t)\|_{L^2}^2 \leq \left( \|u^n(0)\|_{L^2}^2 + \frac{1}{\varepsilon} \int_0^{T^*} \|f(t)\|_{L^2}^2 dt \right) e^{(2CM + \varepsilon)T^*}, \]

where \( M = \sup_{t \in [0, T^*]} \|\nabla A(t)\|_{L^\infty} \) and the left hand side of the inequality (24) is finite whenever M is finite.

Proof. Let us find the L^2-energy estimate starting with the energy equality

\[ \left( \frac{d}{dt} (\partial u^n(t), u^n(t)), u^n(t) \right)_{L^2} + (\partial \varphi(t, u^n)u^n(t), u^n(t))_{L^2} = (f(t), u^n(t))_{L^2}. \]  

(25)

By using (17), Cauchy-Schwartz inequality, and Young’s inequality in (25), we get

\[ \frac{1}{2} \frac{d}{dt} \|u^n(t)\|_{L^2}^2 \leq - (\partial \varphi(t, u^n)u^n(t), u^n(t))_{L^2} + \|f(t)\|_{L^2} \|u^n(t)\|_{L^2} \]

\[ \leq C\|\nabla A(t)\|_{L^\infty} \|u^n(t)\|_{L^2}^2 + \|f(t)\|_{L^2} \|u^n(t)\|_{L^2} \]

\[ \leq C\|\nabla A(t)\|_{L^\infty} \|u^n(t)\|_{L^2}^2 + \frac{1}{2\varepsilon} \|f(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|u^n(t)\|_{L^2}^2, \]

(26)

for 0 < \varepsilon \leq 1. Integrating (26) from 0 to t, we obtain

\[ \|u^n(t)\|_{L^2}^2 \leq \|u^n(0)\|_{L^2}^2 + \int_0^t (2C\|\nabla A(r)\|_{L^\infty} + \varepsilon) \|u^n(r)\|_{L^2}^2 dr + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{L^2}^2 dr. \]

(27)

An application of Gronwall’s inequality in (27) yields

\[ \|u^n(t)\|_{L^2}^2 \leq \left( \|u^n(0)\|_{L^2}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{L^2}^2 dr \right) \exp \left( \int_0^t (2C\|\nabla A(r)\|_{L^\infty} + \varepsilon) dr \right), \]

(28)

for 0 \leq t \leq T. Let us take the supremum from 0 to T in the inequality (27) to get

\[ \sup_{0 \leq t \leq T} \|u^n(t)\|_{L^2}^2 \]

\[ \leq \|u^n(0)\|_{L^2}^2 + \int_0^T (2C\|\nabla A(t)\|_{L^\infty} + \varepsilon) \|u^n(t)\|_{L^2}^2 dt + \frac{1}{\varepsilon} \int_0^T \|f(t)\|_{L^2}^2 dt. \]

(29)

Once again applying Gronwall’s inequality in (29), we find

\[ \sup_{0 \leq t \leq T} \|u^n(t)\|_{L^2}^2 \]

\[ \leq \left( \|u^n(0)\|_{L^2}^2 + \frac{1}{\varepsilon} \int_0^T \|f(t)\|_{L^2}^2 dt \right) \exp \left( \int_0^T (2C\|\nabla A(t)\|_{L^\infty} + \varepsilon) dt \right), \]

(30)
for $0 < \varepsilon \leq 1$. It is clear from the inequality (30) that the left hand side of the inequality (30) is finite whenever $\int_0^T \|\nabla A(t)\|_{L^\infty} dt < \infty$. Hence, there exists a time $0 < T^* < T$, up to which $\int_0^{T^*} \|\nabla A(t)\|_{L^\infty} dt < \infty$, so that $T^*$ is the maximal time for which the left hand side of the inequality (30) is finite. Let $M = \sup_{t \in [0, T^*]} \|\nabla A(t)\|_{L^\infty}$, then we have
\[
\sup_{0 < \varepsilon \leq T^*} \|u^n(t)\|_{H^s}^2 \leq \left(\|u^n(0)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_0^T \|f(t)\|_{H^s}^2 dt\right) e^{(2CM + \varepsilon)T^*},
\]
for $0 < \varepsilon \leq 1$, and the left hand side of the inequality (31) is finite whenever $M < \infty$. \hfill \Box

**Proposition 2** (H$^s$—energy estimate). Let $u^n(\cdot)$ be the unique solution of the system of ODE's (22) with $u_0 \in H^s(\mathbb{R}^n)$, for $s > n/2 + 1$. Then, there exists a time $0 < T^* < T$ such that, for $f \in L^2(0, T^*; H^s(\mathbb{R}^n))$ and $0 < \varepsilon \leq 1$, we have the following a-priori energy estimate:
\[
\|u^n(t)\|_{H^s}^2 \leq \left(\|u^n(0)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{H^s}^2 dr\right) \times \exp\left(\int_0^t \left(4C\|\nabla A(r)\|_{L^\infty} + \varepsilon + 2C(\|u^n(r)\|_{L^\infty} + \|\nabla u^n(r)\|_{L^\infty})\right) dr\right),
\]
for $0 \leq t \leq T^*$, and
\[
\sup_{0 \leq t \leq T^*} \|u^n(t)\|_{H^s}^2 \leq \left(\|u^n(0)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_0^{T^*} \|f(t)\|_{H^s}^2 dt\right) e^{(4CM + 2M^* + \varepsilon)T^*},
\]
where $M = \sup_{t \in [0, T^*]} \|\nabla A(t)\|_{L^\infty}$ and $M^* = \sup_{0 \leq t \leq T^*} C(\|u^n(t)\|_{L^\infty})(1 + \|\nabla u^n(t)\|_{L^\infty})$ and the left hand side of the inequality (33) is finite whenever $M, M^*$ are finite.

**Proof.** Let us take $J^s := (1 - \Delta)^{s/2}$ on the equation (22) to get
\[
\frac{\partial J^s u^n(t)}{\partial t} + J^s \mathcal{A}(t, u^n) u^n(t) = J^s f(t), \quad 0 \leq t \leq T,
\]
\[
u^n(0) = u^n_0.
\]
Let us now find the H$^s$—energy estimate by considering the energy equality
\[
\left(\frac{\partial}{\partial t} J^s u^n(t), J^s u^n(t)\right)_{L^2} + (J^s \mathcal{A}(t, u^n) u^n(t), J^s u^n(t))_{L^2} = (J^s f(t), J^s u^n(t))_{L^2}.
\]
By using Cauchy-Schwartz inequality, Young's inequality, (18), (17), and (19) in (35), we get
\[
\frac{1}{2} \frac{d}{dt} \|u^n(t)\|_{H^s}^2 \leq - (J^s \mathcal{A}(t, u^n) J^{-s} J^s u^n(t), J^s u^n(t))_{L^2} + |(J^s f(t), J^s u^n(t))_{L^2}|
\]
\[
\leq - (\mathcal{A}(t, u^n)) (J^s u^n(t), J^s u^n(t))_{L^2} + \|\mathcal{A}(t, u^n)\|_{L^\infty} \|J^s u^n(t)\|_{L^2}^2 + \|J^s f(t)\|_{L^2} \|J^s u^n(t)\|_{L^2}
\]
\[
\leq C \|\nabla A(t)\|_{L^\infty} \|J^s u^n(t)\|_{L^2}^2 + |(\mathcal{A}(t, u^n), J^s u^n(t), J^s u^n(t))_{L^2}|
\]
\[
+ \frac{1}{2\varepsilon} \|f(t)\|_{H^s}^2 + \varepsilon \|u^n(t)\|_{H^s}^2
\]
\[
\leq C \|\nabla A(t)\|_{L^\infty} \|u^n(t)\|_{H^s}^2 + \|\mathcal{A}(t, u^n)\|_{L^\infty} \|J^s u^n(t)\|_{L^2} \|J^s u^n(t)\|_{L^2} \|J^s u^n(t)\|_{L^2}.
Once again an application of Gronwall’s inequality in (39) yields
\[
\leq \left( C\|\nabla A(t)\|_{L^\infty} + \frac{\varepsilon}{2} \right) \|u^n(t)\|_{H^r}^2 + C\|\nabla A(t)\|_{L^\infty} \|\nabla u^n(t)\|_{H^{r-1}} \|u^n(t)\|_{H^r} \\
+ C (\|u^n(t)\|_{L^\infty}) (1 + \|u^n(t)\|_{H^r}) \|\nabla u^n(t)\|_{L^\infty} \|u^n(t)\|_{H^r} + \frac{1}{2\varepsilon} \|f(t)\|_{H^r}^2,
\]
for \(0 < \varepsilon \leq 1\). Integrating (36) from 0 to \(t\), we obtain
\[
\|u^n(t)\|_{H^r}^2 \leq \|u^n(0)\|_{H^r}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{H^r}^2 \, dr \\
+ \int_0^t \left( 4C\|\nabla A(r)\|_{L^\infty} + \varepsilon + 2C (\|u^n(r)\|_{L^\infty}) [1 + \|\nabla u^n(r)\|_{L^\infty}] \right) \|u^n(r)\|_{H^r}^2 \, dr.
\]
(37)

An application of Gronwall’s inequality in (37) yields
\[
\|u^n(t)\|_{H^r}^2 \leq \left( \|u^n(0)\|_{H^r}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{H^r}^2 \, dr \right) \times \\
\exp \left( \int_0^t \left( 4C\|\nabla A(r)\|_{L^\infty} + \varepsilon + 2C (\|u^n(r)\|_{L^\infty}) [1 + \|\nabla u^n(r)\|_{L^\infty}] \right) dr \right),
\]
(38)
for \(0 \leq t \leq T\). Let us take supremum from 0 to \(T\) on both sides of the inequality (38) to obtain
\[
\sup_{0 \leq t \leq T} \|u^n(t)\|_{H^r}^2 \leq \|u^n(0)\|_{H^r}^2 + \frac{1}{\varepsilon} \int_0^T \|f(t)\|_{H^r}^2 \, dt \\
+ \int_0^T \left( 4C\|\nabla A(t)\|_{L^\infty} + \varepsilon + 2C (\|u^n(t)\|_{L^\infty}) [1 + \|\nabla u^n(t)\|_{L^\infty}] \right) \|u^n(t)\|_{H^r}^2 \, dt.
\]
(39)

Once again an application of Gronwall’s inequality in (39) yields
\[
\sup_{0 \leq t \leq T} \|u^n(t)\|_{H^r}^2 \leq \left( \|u^n(0)\|_{H^r}^2 + \frac{1}{\varepsilon} \int_0^T \|f(t)\|_{H^r}^2 \, dt \right) \times \\
\exp \left( \int_0^T \left( 4C\|\nabla A(t)\|_{L^\infty} + \varepsilon + 2C (\|u^n(t)\|_{L^\infty}) [1 + \|\nabla u^n(t)\|_{L^\infty}] \right) dt \right).
\]
(40)

It is clear from the inequality (40) that the left hand side of (40) is finite whenever \(\int_0^T \|\nabla A(t)\|_{L^\infty} \, dt < \infty\) and \(\int_0^T C (\|u^n(t)\|_{L^\infty}) [1 + \|\nabla u^n(t)\|_{L^\infty}] \, dt < \infty\), since, for \(s > n/2 + 1\), we have
\[
\|u^n(t)\|_{L^\infty} \leq C\|u^n(t)\|_{H^{r-1}} \leq C\|u^n(t)\|_{H^r}, \\
\|\nabla u^n(t)\|_{L^\infty} \leq C\|\nabla u^n(t)\|_{H^{r-1}} \leq C\|u^n(t)\|_{H^r}.
\]

Hence, there exists a time \(0 < T^* < T\), such that
\[
\int_0^{T^*} \left( \|\nabla A(t)\|_{L^\infty} + C (\|u^n(t)\|_{L^\infty}) [1 + \|\nabla u^n(t)\|_{L^\infty}] \right) \, dt < \infty,
\]
and hence $T^*$ is the maximal time for which the left hand side of the inequality (40) is finite. Let us take $M = \sup_{t \in [0, T^*]} \|\nabla A(t)\|_{L^\infty}$ and $M' = \sup_{t \in [0, T^*]} C(\|u^n(t)\|_{L^\infty})(1 + \|\nabla u^n(t)\|_{L^\infty})$, then we have

$$
\sup_{0 \leq t \leq T^*} \|u^n(t)\|_{H^2} \leq \left(\|u^n(0)\|_{H^2} + \frac{1}{\varepsilon} \int_0^{T^*} \|f(t)\|_{H^2}^2 dt\right) e^{(4CM + 2M' + \varepsilon)T^*}, \tag{41}
$$

and the left hand side of the inequality (41) is finite whenever $M, M' < \infty$. \qed

Let us now prove that the nonlinear term $\mathcal{A}(t, u)u$ is locally monotone.

**Theorem 3.1** (Local Monotonicity). For any given $N > 0$, we consider the following (closed) ball:

$$
\mathcal{B}_N := \{u, v \in L^2(\mathbb{R}^n) : \|\nabla A\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|\nabla u A\|_{L^\infty} \leq N\}, \tag{42}
$$

then for any $u, v \in \mathcal{B}_N$ and each $t \in (0, T^*)$, we have

$$
(\mathcal{A}(t, u)u - \mathcal{A}(t, v)u, u - v)_{L^2} + C N \|u - v\|_{L^2}^2 \geq 0. \tag{43}
$$

Similarly, if $N(t)$ is a positive and measurable real valued function and $\mathcal{B}_N(t)$ is the following (closed) time-variable ball:

$$
\mathcal{B}_N(t) := \{u(\cdot), v(\cdot) \in L^\infty(0, T^*; L^2(\mathbb{R}^n)) : \|\nabla A(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty} \|\nabla u A(t)\|_{L^\infty} \leq N(t)\}, \tag{44}
$$

then for any $u(\cdot), v(\cdot) \in \mathcal{B}_N(t)$, and any measurable function $\rho(t)$, we have

$$
\int_0^{T^*} (\mathcal{A}(t, u)u - \mathcal{A}(t, v)u, u - v)_{L^2} e^{\rho(t)} dt + C \int_0^{T^*} N(t) \|u - v\|_{L^2}^2 e^{\rho(t)} dt \geq 0, \tag{45}
$$

where $T^*$ is the time up to which the energy estimates in Proposition 1 and Proposition 2 are finite.

**Proof.** Let us consider $(\mathcal{A}(t, u)u - \mathcal{A}(t, v)u, u - v)_{L^2}$ and use (17) to obtain

$$
(\mathcal{A}(t, u)u - \mathcal{A}(t, v)u, u - v)_{L^2} = (\mathcal{A}(t, u)(u - v), u - v)_{L^2} + (\mathcal{A}(t, u) - \mathcal{A}(t, v))v, u - v)_{L^2}
\geq -C \|\nabla A\|_{L^\infty} \|u - v\|_{L^2}^2 + ((\mathcal{A}(t, u) - \mathcal{A}(t, v))v, u - v)_{L^2}. \tag{46}
$$

Let us take the term $((\mathcal{A}(t, u) - \mathcal{A}(t, v))v, u - v)_{L^2}$ from (46) and use (8) to get

$$
|(\mathcal{A}(t, u) - \mathcal{A}(t, v))v, u - v)_{L^2}| \leq ||(\mathcal{A}(t, u) - \mathcal{A}(t, v))v||_{L^2} \|u - v\|_{L^2} \leq C \|\nabla v\|_{L^\infty} \|\nabla u A\|_{L^\infty} \|u - v\|_{L^2}^2. \tag{47}
$$

By using (47) in (46), we get

$$
(\mathcal{A}(t, u)u - \mathcal{A}(t, v)u, u - v)_{L^2} \geq -C(\|\nabla A\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|\nabla u A\|_{L^\infty}) \|u - v\|_{L^2}^2. \tag{48}
$$

Since $u, v \in \mathcal{B}_N$, from (48), we find

$$
(\mathcal{A}(t, u)u - \mathcal{A}(t, v)u, u - v)_{L^2} + C N \|u - v\|_{L^2}^2 \geq 0. \tag{49}
$$

The inequality (45) can be easily obtained from (43). \qed
Remark 1. Let us denote $F(u) = \mathcal{A}(t, u)u$. From the local monotonicity theorem (Theorem 3.1), one can deduce that $F(\cdot) + NI$ is a monotone operator in $\mathbb{B}_N \subset H^s(\mathbb{R}^n)$ and by an application of Theorem 1.3, Chapter 2 of Barbu [2], one can also prove that the operator $F(\cdot) + NI$ is in fact a maximal monotone operator in $\mathbb{B}_N$. From the local monotonicity condition (43), we know that

$$(F(u) - F(v), u - v)_{L^2} + N\|u - v\|_{L^2}^2 \geq 0, \text{ for all } u, v \in \mathbb{B}_N \subset H^s(\mathbb{R}^n).$$

Thus, we have

$$(F(u) + Nu - (F(v) + Nv), u - v)_{L^2} \geq 0, \text{ for all } u, v \in \mathbb{B}_N,$$

so that $F(\cdot) + NI$ is a monotone operator in $\mathbb{B}_N$. For the contrary, let us assume that $F(\cdot) + NI$ is not a maximal monotone operator in $\mathbb{B}_N$. Then, there exists $[x_0, y_0] \in \mathbb{B}_N \times H^{-1}(\mathbb{R}^n)$ such that $y_0 \neq F(x_0) + Nx_0$ and

$$(y_0 - (F(u) + Nu), x_0 - u)_{L^2} \geq 0, \text{ for all } u \in \mathbb{B}_N. \tag{50}$$

For any $x \in \mathbb{B}_N$, we set

$$u_\lambda = \lambda x_0 + (1 - \lambda)x, 0 \leq \lambda \leq 1,$$

and put $u = u_\lambda$ so that from (50), we get

$$(y_0 - (F(u_\lambda) + Nu_\lambda), (1 - \lambda)(x_0 - x))_{L^2} \geq 0.$$

Thus, we have

$$(y_0 - (F(u_\lambda) + Nu_\lambda), x_0 - x)_{L^2} \geq 0 \text{ for all } \lambda \in [0, 1], u \in \mathbb{B}_N. \tag{51}$$

Letting $\lambda \to 1$ in (51), one gets

$$(y_0 - (F(x_0) + Nx_0), x_0 - x)_{L^2} \geq 0 \text{ for all } x \in \mathbb{B}_N. \tag{52}$$

In (52), by taking $x_0 - x = \lambda w$, for $\lambda > 0$, dividing by $\lambda$, letting $\lambda \to 0$, and then using hemicontinuity property of $F(\cdot)$, we obtain

$$(y_0 - (F(x_0) + Nx_0), w)_{L^2} \geq 0 \text{ for all } w \in \mathbb{B}_N,$$

and hence $y_0 = F(x_0) + Nx_0$, which is a contradiction. Thus, $F(\cdot) + NI$ is a maximal monotone operator in the ball $\mathbb{B}_N \subset H^s(\mathbb{R}^n)$.

3.2. Existence and uniqueness of local solution. Let us now prove that the system (1) has a unique solution by exploiting the local monotonicity theorem (Theorem 3.1). The similar existence results for $2 - D$ Navier-Stokes equations can be found in [5].

**Theorem 3.2** (Local Existence and Uniqueness). Let $f \in L^2(0, T^*; H^s(\mathbb{R}^n))$ and $u_0 \in H^s(\mathbb{R}^n)$ with $s > n/2 + 1$, where $T^*$ is the maximal time for which the energy estimates given in Proposition 1 and Proposition 2 are finite. Then, there exists a unique solution $u \in L^\infty(0, T^*; H^s(\mathbb{R}^n))$ to the problem (1).

**Proof.** Let us prove Theorem 3.2 by using the Minty-Browder technique of local monotonicity in the following steps:

**Step (1).** Finite-dimensional Galerkin approximation of the symmetric hyperbolic system (1):
Let \( \{e_1, e_2, \cdots \} \) be a fixed complete orthonormal system in \( L^2(\mathbb{R}^n) \) belonging to \( H^s(\mathbb{R}^n) \). Let \( L^2_n(\mathbb{R}^n) := \text{span}\{e_1, e_2, \cdots, e_n\} \) be the \( n \)-dimensional subspace of \( L^2(\mathbb{R}^n) \). Let us now consider the following finite-dimensional ODE in \( L^2_n(\mathbb{R}^n) \):

\[
\frac{\partial}{\partial t}(u^n(t), v)_{L^2} + (\mathcal{A}(t, u^n)u^n, v)_{L^2} = (f(t), v)_{L^2},
\]

\[
u^n(0) = u^n_0, \tag{53}
\]

in \((0, T^*)\), for any \( v \in L^2_n(\mathbb{R}^n) \). From (53), we have

\[
\frac{\partial}{\partial t}u^n(t) + F(u^n(t)) = f(t), \text{ in } L^2_n(\mathbb{R}^n), \tag{54}
\]

\[u^n(0) = u^n_0.\]

Also (54) satisfies the energy equality

\[
||u^n(t)||_{L^2}^2 + 2 \int_0^t (F(u^n(s)) - f(s), u^n(s))_{L^2} ds = ||u^n(0)||_{L^2}^2, \tag{55}
\]

for any \( t \in (0, T^*) \).

**Step (2).** Weak convergence of the sequences \( u^n(\cdot) \) and \( F(u^n(\cdot)) \):

By using Proposition 2, we can extract subsequences \( \{u^n(\cdot)\} \) and \( \{F(u^n(\cdot))\} \) such that

\[
u^n(\cdot) \rightharpoonup u(\cdot) \text{ in } L^\infty(0, T^*; H^s(\mathbb{R}^n)), \tag{56}
\]

\[F(u^n(\cdot)) \rightharpoonup F_0(\cdot) \text{ in } L^\infty(0, T^*; H^{s-1}(\mathbb{R}^n)). \tag{57}
\]

The second convergence (57) is obtained by using the Moser estimates (11) and the algebra property of \( H^{s-1}(\mathbb{R}^n) \) as

\[
||\mathcal{A}(t, u^n)u^n(t)||_{H^{s-1}} \leq \sum_{j=1}^n \|A^j(t, x, u^n) \frac{\partial}{\partial x_j} u^n(t)\|_{H^{s-1}}
\]

\[
\leq \sum_{j=1}^n \|A^j(t, x, u^n)\|_{H^{s-1}} \|\frac{\partial u^n}{\partial x_j}\|_{H^{s-1}}
\]

\[
\leq \left( \sum_{j=1}^n \|A^j(t, x, u^n)\|_{H^{s-1}}^2 \right)^{1/2} \left( \sum_{j=1}^n \left| \frac{\partial u^n}{\partial x_j} \right|_{H^{s-1}}^2 \right)^{1/2}
\]

\[
\leq C(\|u^n\|_{L^\infty}) (1 + ||u^n||_{H^{s-1}}) \|\nabla u^n\|_{H^{s-1}}
\]

\[
\leq C(\|u^n\|_{L^\infty}) (1 + ||u^n||_{H^s}) \|u^n\|_{H^r}, \tag{58}
\]

and the right hand side of (58) is finite, since \( u^n \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \). We know that

\[
\frac{\partial}{\partial t} \left( e^{-r(t)} u^n(t) \right) = e^{-r(t)} \frac{\partial}{\partial t} u^n(t) - r'(t) e^{-r(t)} u^n(t), \tag{59}
\]

where \( r'(t) \) is the derivative of \( r(t) \).

Let us consider \( \left( \frac{\partial}{\partial t} (e^{-r(t)} u^n(t)), e^{-r(t)} u^n(t) \right) \) \( L^2 \) and use (54) and (59) to obtain

\[
\left( \frac{\partial}{\partial t} (e^{-r(t)} u^n(t)), e^{-r(t)} u^n(t) \right)_{L^2} = e^{-2r(t)} \left( \frac{\partial}{\partial t} u^n(t), u^n(t) \right)_{L^2} - e^{-2r(t)} r'(t) (u^n(t), u^n(t))_{L^2}
\]
and the energy equality

\[ e^{-2r(t)} \left( -F(u^n(t)) + f(t) - r'(t) u^n(t), u^n(t) \right)_{L^2}, \]  

(60)

Hence from (60), we have

\[ \frac{1}{2} \frac{d}{dt} e^{-2r(t)} \| u^n(t) \|_{L^2}^2 = e^{-2r(t)} \left( -F(u^n(t)) + f(t) - r'(t) u^n(t), u^n(t) \right)_{L^2}, \]  

(61)

and satisfies the energy equality

\[ e^{-2r(t)} \| u^n(t) \|_{L^2}^2 + 2 \int_0^t e^{-2r(s)} (F(u^n(s)) - f(s) + r'(s) u^n(s), u^n(s))_{L^2} ds \]
\[ = \| u^n(0) \|_{L^2}^2, \]  

(62)

for any \( t \in (0, T^*) \). Also on passing to limit in (54), the limit \( u(\cdot) \) satisfies

\[ \frac{\partial}{\partial t} u(t) + F_0(t) = f(t) \text{ in } L^2(0, T^*; L^2(\mathbb{R}^n)) \]
\[ u(0) = u_0, \]  

(63)

and the energy equality

\[ \| u(t) \|_{L^2}^2 + 2 \int_0^t (F_0(s) - f(s), u(s))_{L^2} ds = \| u(0) \|_{L^2}^2, \]  

(64)

for any \( t \in (0, T^*) \). A similar calculation of (62) yields

\[ e^{-2r(t)} \| u(t) \|_{L^2}^2 + 2 \int_0^t e^{-2r(s)} (F_0(s) - f(s) + r'(s) u(s), u(s))_{L^2} ds = \| u(0) \|_{L^2}^2, \]  

(65)

for any \( t \in (0, T^*) \). Also, it should be noted that the initial value \( u^n(0) \) converges to \( u(0) \) strongly, i.e.,

\[ \lim_{n \to \infty} \| u^n(0) - u(0) \|_{L^2} = 0. \]  

(66)

**Step (3).** Local Minty-Browder Technique:

For any \( v \in L^\infty(0, T^*; L^2_m(\mathbb{R}^n)) \) with \( m < n \), let us define

\[ r(t) = C \int_0^t \left( \| \nabla A(s) \|_{L^\infty} + \| \nabla v(s) \|_{L^\infty} \| \nabla u A(s) \|_{L^\infty} \right) ds, \]  

(67)

so that \( r'(t) = C (\| \nabla A(t) \|_{L^\infty} + \| \nabla v(t) \|_{L^\infty} \| \nabla u A(t) \|_{L^\infty}) \), a.e. From the local monotonicity theorem (Theorem 3.1), by using (48), we have

\[ \int_0^{T^*} e^{-2r(t)} \left( (F(v(t)) - F(u^n(t)), v(t) - u^n(t))_{L^2} + r'(t) (v(t) - u^n(t), v(t) - u^n(t))_{L^2} \right) dt \geq 0. \]  

(68)

In (68), we use the energy equality (62) to get

\[ \int_0^{T^*} e^{-2r(t)} \left( F(v(t)) + r'(t) v(t), v(t) - u^n(t) \right)_{L^2} dt \]
\[ \geq \int_0^{T^*} e^{-2r(t)} \left( F(u^n(t)) + r'(t) u^n(t), v(t) - u^n(t) \right)_{L^2} dt \]
\[ = \int_0^{T^*} e^{-2r(t)} \left( F(u^n(t)) + r'(t) u^n(t), v(t) \right)_{L^2} dt \]
\[ - \int_0^{T^*} e^{-2r(t)} \left( F(u^n(t)) + r'(t) u^n(t), u^n(t) \right)_{L^2} dt \]
\[
\begin{align*}
&= \int_0^{T^*} e^{-2r(t)} (F(u^n(t)) + r'(t)u^n(t), \nu(t))_{L^2} dt \\
&\quad + \frac{1}{2} \left( e^{-2r(T^*)} \|u^n(T^*)\|_{L^2}^2 - \|u^n(0)\|_{L^2}^2 \right) - \int_0^{T^*} (f(t), u^n(t))_{L^2} dt.
\end{align*}
\]

On taking liminf on both sides of (69), we obtain
\[
\begin{align*}
&\geq \int_0^{T^*} e^{-2r(t)} (F_0(t) + r'(t)u(t), \nu(t))_{L^2} dt \\
&\quad + \frac{1}{2} \liminf_{n \to \infty} \left( e^{-2r(T^*)} \|u^n(T^*)\|_{L^2}^2 - \|u^n(0)\|_{L^2}^2 \right) - \int_0^{T^*} (f(t), u(t))_{L^2} dt.
\end{align*}
\]

By using the lower semicontinuity property of the $L^2$-norm and the strong convergence of the initial data $u^n(0)$ (see 66), the second term on the right hand side of the inequality satisfies the following inequality:
\[
\liminf_{n \to \infty} \left( e^{-2r(T^*)} \|u^n(T^*)\|_{L^2}^2 - \|u^n(0)\|_{L^2}^2 \right) \geq e^{-2r(T^*)} \|u(T^*)\|_{L^2}^2 - \|u(0)\|_{L^2}^2.
\]

Hence by using (65) and (71) in (70), we find
\[
\begin{align*}
&\int_0^{T^*} e^{-2r(t)} (F(\nu(t)) + r'(t)\nu(t), \nu(t) - u(t))_{L^2} dt \\
&\quad \geq \int_0^{T^*} e^{-2r(t)} (F_0(t) + r'(t)u(t), \nu(t))_{L^2} dt \\
&\quad \quad + \frac{1}{2} \left( e^{-2r(T^*)} \|u(T^*)\|_{L^2}^2 - \|u(0)\|_{L^2}^2 \right) - \int_0^{T^*} (f(t), u(t))_{L^2} dt \\
&\quad = \int_0^{T^*} e^{-2r(t)} (F_0(t) + r'(t)u(t), \nu(t))_{L^2} dt \\
&\quad - \int_0^{T^*} e^{-2r(t)} (F_0(t) + r'(t)u(t), u(t))_{L^2} dt \\
&\quad = \int_0^{T^*} e^{-2r(t)} (F_0(t) + r'(t)u(t), \nu(t) - u(t))_{L^2} dt.
\end{align*}
\]

where in the second step, we used the energy equality (65). The estimate (72) holds for any $\nu \in L^\infty(0, T^*; L^2_m(\mathbb{R}^n))$ for any $m \in \mathbb{N}$, since the estimate (72) is independent of $m$ and $n$. It can be easily seen by a density argument that the inequality (72) remains true for any $\nu \in L^\infty(0, T^*; H^s(\mathbb{R}^n))$ for $s > n/2 + 1$. Indeed, for any $\nu \in L^\infty(0, T^*; H^s(\mathbb{R}^n))$, there exists a strongly convergent subsequence $\nu_m \in L^\infty(0, T^*; H^s(\mathbb{R}^n))$ that satisfies the inequality (72).

Let us now take $\nu = u + \lambda w$, $\lambda > 0$, where $w \in L^\infty(0, T^*; H^s(\mathbb{R}^n))$, and substitute for $\nu$ in (72) to get
\[
\int_0^{T^*} e^{-2r(t)} (F(u(t)) + \lambda w(t)) - F_0(t) + r'(t)\lambda w(t), \lambda w(t))_{L^2} dt \geq 0.
\]

The inequality (73) becomes
\[
\int_0^{T^*} e^{-2r(t)} (\nu(t), u(t) + \lambda w(t)) - F_0(t) + r'(t)\lambda w(t), \lambda w(t))_{L^2} dt.
\]
\[ \int_0^{T^*} e^{-2r(t)} (\mathcal{A}(t, u(t) + \lambda w(t))u(t) - F_0(t) + \lambda[\mathcal{A}(t, u(t) + \lambda w(t))w(t) + r'(t)w(t)], \lambda w(t))]_{L^2} \, dt \geq 0. \]  
(74)

Let us divide the inequality (74) by \( \lambda \), use the hemicontinuity property of \( \mathcal{A}(\cdot, \cdot) \) in the second variable, and let \( \lambda \to 0 \) to obtain
\[ \int_0^{T^*} e^{-2r(t)} (\mathcal{A}(t, u(t))u(t) - F_0(t), w(t))]_{L^2} \, dt \geq 0. \]  
(75)

The term \( \int_0^{T^*} e^{-2r(t)} \lambda (\mathcal{A}(t, u(t) + \lambda w(t))w(t), w(t))]_{L^2} \, dt \) tends to 0 as \( \lambda \to 0 \), since
\[ \lim_{\lambda \to 0} \int_0^{T^*} e^{-2r(t)} (\mathcal{A}(t, u(t) + \lambda w(t))w(t), w(t))]_{L^2} \, dt = 0. \]  

The term \( \int_0^{T^*} e^{-2r(t)} \lambda (\mathcal{A}(t, u(t))w(t), w(t))]_{L^2} \, dt \) tends to 0 as \( \lambda \to 0 \), since
\[ \lim_{\lambda \to 0} \int_0^{T^*} e^{-2r(t)} \lambda (\mathcal{A}(t, u(t))w(t), w(t))]_{L^2} \, dt = 0. \]  

The estimate (76) is obtained by using the hemicontinuity property of \( \mathcal{A}(\cdot, \cdot) \) and (58), and is finite, since \( u, w \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \). Also the final term from (74) tends to 0 as \( \lambda \to 0 \), since
\[ \int_0^{T^*} e^{-2r(t)} r'(t)(w, w)]_{L^2} \, dt \]
\[ = C \int_0^{T^*} e^{-2r(t)} (\|\nabla A(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty}\|\nabla u A(t)\|_{L^\infty}) \|w\|_{H^2}^2 \, dt \]
\[ \leq C \left( M + M'' \sup_{t \in [0, T^*]} \|v(t)\|_{H^2} \right) \sup_{t \in [0, T^*]} \|w(t)\|_{H^2}^2, T^* < \infty, \]  
(77)

where \( M = \sup_{t \in [0, T^*]} \|\nabla A(t)\|_{L^\infty} \) and \( M'' = \sup_{t \in [0, T^*]} \|\nabla u A(t)\|_{L^\infty} \). The estimate (77) is finite, since \( v, w \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \) and \( M, M'' < \infty \). Hence, from (75), we finally obtain
\[ \int_0^{T^*} e^{-2r(t)} (F(u(t)) - F_0(t), w(t))]_{L^2} \, dt \geq 0, \]  
(78)
for any \( w \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \), for \( s > n/2 + 1 \).

Thus from (78), we have \( F(u(t)) = F_0(t) \) and hence \( u(\cdot) \) is a solution of the symmetric hyperbolic system (1).

**Step (4).** Uniqueness:

Let \( u_1(\cdot) \) and \( u_2(\cdot) \) be two local solutions of (1) with the same initial data \( u_0 \). Then \( (u_1 - u_2)(\cdot) \) satisfies

\[
\frac{\partial (u_1 - u_2)(t)}{\partial t} + \mathscr{A}(t, u_1(t))u_1(t) - \mathscr{A}(t, u_2(t))u_2(t) = 0, \quad 0 \leq t \leq T^*,
\]

\[
(u_1 - u_2)(0) = 0.
\]

Then it can be easily seen that \( (u_1 - u_2)(\cdot) \) satisfies the energy equality

\[
\frac{1}{2} \frac{d}{dt} \|u_1 - u_2\|_{L^2}^2 + (\mathscr{A}(t, u_1)u_1 - \mathscr{A}(t, u_2)u_2, u_1 - u_2)_{L^2} = 0.
\]

By using (17), and (8), for \( s > n/2 + 1 \), we get

\[
-(\mathscr{A}(t, u_1)u_1 - \mathscr{A}(t, u_2)u_2, u_1 - u_2)_{L^2}
= -(\mathscr{A}(t, u_1)(u_1 - u_2), u_1 - u_2)_{L^2} - ((\mathscr{A}(t, u_1) - \mathscr{A}(t, u_2))u_2, u_1 - u_2)_{L^2}
\leq C\|\nabla A(t)\|_{L^\infty} \|u_1 - u_2\|_{L^2}^2 + \|\mathscr{A}(t, u_1) - \mathscr{A}(t, u_2)\|_{L^2} \|u_1 - u_2\|_{L^2}
\leq C \|\nabla A(t)\|_{L^\infty} + \|\nabla_u A(t)(u_2(t))\|_{H^s} \|u_1 - u_2\|_{L^2}.
\]

Integrating (80) from 0 to \( T^* \) and using (81) in (80), we find

\[
\|u_1 - u_2\|_{L^2}^2 \leq 2C \int_0^{T^*} C \left( \|\nabla A(t)\|_{L^\infty} + \|\nabla A_u(t)\|_{L^\infty} \|u_2(t)\|_{H^s} \right) \|u_1 - u_2\|_{L^2}^2 dt
\leq 2C \left( M + M'' \sup_{0 \leq t \leq T^*} \|u_2(t)\|_{H^s} \right) \int_0^{T^*} \|u_1 - u_2\|_{L^2}^2 dt,
\]

where \( M = \sup_{0 \leq t \leq T^*} \|\nabla A(t)\|_{L^\infty} < \infty \) and \( M'' = \sup_{0 \leq t \leq T^*} \|\nabla_u A(t)\|_{L^\infty} < \infty \). An application of Gronwall’s inequality on (82) yield the uniqueness result, since \( u_1, u_2 \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \). By using Sobolev interpolation inequality, we have

\[
\|u_1 - u_2\|_{H^{s'}} \leq C_s \|u_1 - u_2\|_{L^2}^{1-s'/s} \|u_1 - u_2\|_{H^s}^{s'/s}
\leq C_s \|u_1 - u_2\|_{L^2}^{1-s'/s} \left( \|u_1\|_{H^{s'}}^{s'/s} + \|u_2\|_{H^{s'}}^{s'/s} \right) = 0,
\]

for \( 0 < s' < s \), since \( u_1, u_2 \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \).

**Remark 2.** If we use the Galerkin approximation and if the problem is defined on a bounded domain \( \Omega \subset \mathbb{R}^n \) or if we use the frequency truncation method (see Section 4) in the whole space \( \mathbb{R}^n \), then we can have the following simplified proof for step (3) of Theorem 3.2. This can be done by making use of the strong convergence of Galerkin approximated sequence in \( L^2(\Omega) \) due to compactness and strong convergence of frequency truncated sequence in \( L^2(\mathbb{R}^n) \) due to Proposition 5.

When the problem is defined on a bounded domain \( \Omega \) and use Galerkin approximations, in step (3) of Theorem 3.2, the following simplification can be adapted. In (55), if we take limit supremum, and use the lower semicontinuity property of the \( L^2 \)-norm, the strong convergence of the initial data \( u^0(0) \) and (56), we obtain

\[
\limsup_{n \to \infty} \int_0^t \langle F(u^n(s)), u^n(s) \rangle_{L^2} ds
\]
\[
\begin{align*}
= \limsup_{n \to \infty} \left[ \frac{1}{2} \left( \|u^n(0)\|_{L^2}^2 - \|u^n(t)\|_{L^2}^2 \right) + \int_0^t (f(s), u^n(s))_{L^2} ds \right] \\
= \frac{1}{2} \left[ \limsup_{n \to \infty} \left( \|u^n(0)\|_{L^2}^2 \right) - \liminf_{n \to \infty} \left( \|u^n(t)\|_{L^2}^2 \right) \right] + \limsup_{n \to \infty} \int_0^t (f(s), u^n(s))_{L^2} ds \\
\leq \frac{1}{2} \left( \|u(0)\|_{L^2}^2 - \|u(t)\|_{L^2}^2 \right) + \int_0^t (f(s), u(s))_{L^2} ds = \int_0^t (F_0(s), u(s))_{L^2} ds,
\end{align*}
\]
for all \( t \in (0, T^*) \). Hence, we have
\[
\limsup_{n \to \infty} \int_0^t (F(u^n(s)), u^n(s))_{L^2} ds \leq \int_0^t (F_0(s), u(s))_{L^2} ds,
\]
for all \( t \in (0, T^*) \). Since, \( F(\cdot) + NI \) is a monotone operator (in fact \( F(\cdot) + NI \) is maximal monotone in \( B_N \subset H^s(\mathbb{R}^n) \), Remark 1), by using (43), we get
\[
\int_0^{T^*} (F(u^n(t)) - F(v(t)), u^n(t) - v(t))_{L^2} dt + C \int_0^{T^*} N(t)\|u^n(t) - v(t)\|_{L^2}^2 dt \geq 0.
\]
On taking lim sup in (86), and using (85), (56) and (57), we get
\[
\int_0^{T^*} (F_0(t) - F(v(t)), u(t) - v(t))_{L^2} dt + C \limsup_{n \to \infty} \int_0^{T^*} N(t)\|u^n(t) - v(t)\|_{L^2}^2 dt \geq 0.
\]
Since \( \Omega \subset \mathbb{R}^n \) is bounded, the embedding of \( H^s(\Omega) \) in \( L^2(\Omega) \) is compact and hence we can pass to limit in the final term of (87). Thus, we have
\[
\int_0^{T^*} (F_0(t) - F(v(t)), u(t) - v(t))_{L^2} dt + C \int_0^{T^*} N(t)\|u(t) - v(t)\|_{L^2}^2 dt \geq 0.
\]
In (88), by taking \( u(t) - v(t) = \lambda w(t) \), for \( \lambda > 0 \), dividing by \( \lambda \), letting \( \lambda \to 0 \) and then by using the hemicontinuity property of \( F(\cdot) \), we get \( F_0(t) = F(u(t)) \).

**Remark 3.** From (58), it can be easily seen that
\[
\left\| \frac{\partial u^n(t)}{\partial t} \right\|_{H^{s-1}} \leq \frac{\partial f(t)}{\partial t} \left( \|u^n(t)\|_{L^\infty} + \|u^n(t)\|_{H^s} \right) \leq C(\|u^n(t)\|_{L^\infty} + \|u^n(t)\|_{H^s})\|u^n(t)\|_{H^s} + \|f(t)\|_{H^s},
\]
and hence
\[
\frac{\partial u^n(t)}{\partial t} \rightharpoonup \frac{\partial u(t)}{\partial t} \text{ in } L^2(0, T^*; H^{s-1}(\mathbb{R}^n)),
\]
since \( u^n \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \) and \( f \in L^2(0, T^*; H^{s-1}(\mathbb{R}^n)) \).

**Theorem 3.3.** Let \( f \in L^2(0, T^*; H^s(\mathbb{R}^n)) \) and \( u_0 \in H^s(\mathbb{R}^n) \) with \( s > n/2 + 1 \). Then, the unique solution \( u(\cdot) \) of (1) satisfies \( u \in C(0, T^*; H^s(\mathbb{R}^n)) \) and \( \frac{\partial u}{\partial t} \in C(0, T^*; H^{s-1}(\mathbb{R}^n)) \).

**Proof.** Using the energy estimate from Proposition 2, and (56), it can be easily seen that \( u \in C_w(0, T^*; H^s(\mathbb{R}^n)) \). Here, \( C_w \) means continuity on the interval \( (0, T^*) \) with values in the weak topology of \( H^s \), i.e., \( u \in C_w(0, T^*; H^s(\mathbb{R}^n)) \) means that for any fixed \( \phi \in H^{-s}, \langle \phi, u \rangle \) is a continuous scalar function on \( (0, T^*) \), where \( \langle \cdot, \cdot \rangle \) denote
the duality pairing of $H^{-s}$ and $H^s$. Now we show that $\|u(\cdot)\|_{H^s}$ is continuous as a function of time. A similar calculation as in Proposition 2 (see 37) yield
\[ \frac{d}{dt} \|u(t)\|_{H^s} \leq (4C\|\nabla u\|_{L^\infty} + \varepsilon + 2C (\|u(\cdot)\|_{L^\infty}) [1 + \|\nabla u(\cdot)\|_{L^\infty}]) \|u(t)\|_{H^{s+\varepsilon}} + \frac{1}{\varepsilon} \|f\|_{H^s}, \]  
for $s > n/2 + 1$ and $0 < \varepsilon \leq 1$. An application of Gronwall’s inequality show that
\[ \|u(t)\|_{H^s} \leq \left( \|u(0)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{H^s}^2 dr \right) \exp \left( \int_0^t (4C\|\nabla u(r)\|_{L^\infty} + \varepsilon + 2C (\|u(r)\|_{L^\infty}) [1 + \|\nabla u(r)\|_{L^\infty}]) dr \right) \leq \left( \|u(0)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_0^t \|f(r)\|_{H^s}^2 dr \right) e^{(4CM'+2M'+\varepsilon)t}, \]  
where $M = \sup_{\tau \in [0,T^*]} \|\nabla A(\tau)\|_{L^\infty}$ and $M' = \sup_{\tau \in [0,T^*]} C(\|u(\cdot)\|_{L^\infty})(1 + \|\nabla u(\cdot)\|_{L^\infty})$. Hence it can be seen from (91) that $\|u(\cdot)\|_{H^s}$ is continuous from the right at time $t = 0$, since $f \in L^\infty(0, T^*; H^s(\mathbb{R}^n))$. Now by applying this bound to the equation started at an arbitrary time $t \in [0, T^*]$, we have
\[ \|u(t)\|_{H^s}^2 \leq \left( \|u(\tau)\|_{H^s}^2 + \frac{1}{\varepsilon} \int_\tau^t \|f(r)\|_{H^s}^2 dr \right) e^{(4CM'+2M'+\varepsilon)(t-\tau)}. \]  
This shows that $\|u(\cdot)\|_{H^s}$ is continuous from the right at time $t = \tau$. Also the symmetric hyperbolic system given by (1) is time-reversible and so we get $\|u(\cdot)\|_{H^s}$ is continuous from the left at time $t = \tau$. Since $\tau$ is arbitrary, we find that $\|u(\cdot)\|_{H^s}$ is continuous. Since $u \in C_c(0, T^*; H^s(\mathbb{R}^n))$ and the continuity of $\|u(\cdot)\|_{H^s}$ in times implies $u \in C(0, T^*; H^s(\mathbb{R}^n))$. Now, let us prove that $u \in \text{Lip}(0, T^*; H^{s-1}(\mathbb{R}^n))$, where Lip($0, T^*; H^{s-1}(\mathbb{R}^n)$) denotes the Lipschitz continuous functions on $(0, T^*)$ with values in the norm topology of $H^{s-1}$. Let us consider
\[ \left\| \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right\|_{H^{s-1}} = \|A(t, u)u - A(t, v)v\|_{H^{s-1}} \leq \|A(t, u)(u - v)\|_{H^{s-1}} + \|A(t, u) - A(t, v)\|_{H^{s-1}} \|v\|_{H^{s-1}}, \]  
for $u, v \in C(0, T^*; H^s(\mathbb{R}^n))$. The first term in the right hand of the inequality (93) can be simplified using the algebra property of $H^{s-1}$-norm and (11) as
\[ \|A(t, u)(u - v)\|_{H^{s-1}} \leq \sum_{j=1}^n \left| A^j(t, x, u) \frac{\partial}{\partial x_j} (u - v) \right|_{H^{s-1}} \leq \sum_{j=1}^n \left| A^j(t, x, u) \right|_{H^{s-1}} \left| \frac{\partial}{\partial x_j} (u - v) \right|_{H^{s-1}} \leq \left( \sum_{j=1}^n \left| A^j(t, x, u) \right|_{H^{s-1}}^2 \right)^{1/2} \left( \sum_{j=1}^n \left| \frac{\partial}{\partial x_j} (u - v) \right|_{H^{s-1}}^2 \right)^{1/2} \leq C(\|u\|_{L^\infty}) (1 + \|u\|_{H^{s-1}}) \|\nabla (u - v)\|_{H^{s-1}} \leq C(\|u\|_{L^\infty}) (1 + \|u\|_{H^s}) \|u - v\|_{H^s}. \]  
(94)
For the second term in the right hand of the inequality (93), we use the algebra property of $H^{s-1}$–norm to find

$$
\| (\mathcal{A}(t, u) - \mathcal{A}(t, v))v \|_{H^{s-1}} \\
\leq \sum_{j=1}^{n} \left\| \left( A^j(t, x, u) - A^j(t, x, v) \right) \frac{\partial v}{\partial x_j} \right\|_{H^{s-1}} \\
\leq \sum_{j=1}^{n} \| A^j(t, x, u) - A^j(t, x, v) \|_{H^{s-1}} \left\| \frac{\partial v}{\partial x_j} \right\|_{H^{s-1}} \\
\leq \left( \sum_{j=1}^{n} \| A^j(t, x, u) - A^j(t, x, v) \|_{H^{s-1}}^2 \right)^{1/2} \| v \|_{H^s}.
$$

(95)

Let us evaluate $\| A^j(t, x, u^R) - A^j(t, x, u) \|^2_{H^{s-1}}$ using the identity (6), Fubini’s theorem, and algebra property of $H^{s-1}$–norm as

$$
\| a^j_{ik}(t, x, u) - a^j_{ik}(t, x, v) \|^2_{H^{s-1}} = \int_{\mathbb{R}^n} \left| J^{s-1} \left( \int_{0}^{1} \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \cdot (u - v) d\tau \right) \right|^2 dx.
$$

$$
= \int_{\mathbb{R}^n} \left| J^{s-1} \left( \int_{0}^{1} \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \cdot (u - v) d\tau \right) \right|^2 dx.
$$

$$
\leq \int_{\mathbb{R}^n} \int_{0}^{1} \left| J^{s-1} \left( \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \cdot (u - v) \right) \right|^2 dx d\tau.
$$

$$
= \int_{0}^{1} \left\| \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \cdot (u - v) \right\|^2_{H^{s-1}} d\tau.
$$

$$
\leq \int_{0}^{1} \left\| \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \right\|^2_{H^{s-1}} \| u - v \|^2_{H^{s-1}} d\tau.
$$

$$
\leq \sup_{\tau \in [0,1]} \left\| \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \right\|^2_{H^{s-1}} \| u - v \|^2_{H^{s-1}}.
$$

(96)

From (95), it can be seen that

$$
\| (\mathcal{A}(t, u) - \mathcal{A}(t, v))v \|_{H^{s-1}} \leq C \left( \sum_{j=1}^{n} \| \nabla_u A^j \|^2_{H^{s-1}} \right)^{1/2} \| u - v \|_{H^{s-1}} \| v \|_{H^s}.
$$

(97)

where $\| \nabla_u A \|_{H^{s-1}} = \left( \sum_{j=1}^{n} \| \nabla_u A^j \|^2_{H^{s-1}} \right)^{1/2}$ with

$$
\| \nabla_u A^j \|^2_{H^{s-1}} = \sup_{\tau \in [0,1]} \sum_{i,k=1}^{m} \left\| \nabla_u a^j_{ik}(t, x, \tau u + (1 - \tau)v) \right\|^2_{H^{s-1}}.
$$
Note that $\|\nabla u A\|_{\mathcal{H}^{-1}}$ is bounded, whenever $u, v \in C(0, T^*; \mathcal{H}^s(\mathbb{R}^n))$. Combining (94) and (97), and substituting it in (93), we obtain

$$
\left\| \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right\|_{\mathcal{H}^{-1}} \leq (C(\|u\|_{L^\infty}) (1 + \|u\|_{\mathcal{H}^s}) + C\|\nabla u A\|_{\mathcal{H}^{-1}}\|v\|_{\mathcal{H}^s}) \|u - v\|_{\mathcal{H}^s}.
$$

(98)

Since $u, v \in C(0, T^*; \mathcal{H}^s(\mathbb{R}^n))$, we get $u \in \text{Lip}(0, T^*; \mathcal{H}^{s-1}(\mathbb{R}^n))$ and an application of Theorem 2.1(b), [13] yield $u \in C(0, T^*; \mathcal{H}^s(\mathbb{R}^n)) \cap C^1(0, T^*; \mathcal{H}^{s-1}(\mathbb{R}^n))$.

4. **Existence and uniqueness-frequency truncation method.** In this section, we establish the unique solvability of the symmetric hyperbolic system (1) using a frequency truncation method ([4], [14], [16], [17]). Main steps are as follows:

**Step (i).** We first consider a ball $B_R$ in the Fourier space, centered at the origin and of radius $R > 0$ to obtain the Fourier truncation $\hat{S}_R f(\xi) = 1_{B_R}(\xi)\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$. We prove that the solution $u^R(\cdot)$ of smoothed version of (1) exist and the $\mathcal{H}^s$-norm of $u^R(\cdot)$ are uniformly bounded up to time $T^*$ such that the bound is independent of $R$.

**Step (ii).** We show that the solutions $u^R(\cdot)$ of the truncated problem is Cauchy in $L^\infty(0, T; L^2(\mathbb{R}^n))$ as $R \to \infty$.

**Step (iii).** We deduce by Sobolev interpolation theorem that the truncated solution $u^R(\cdot) \to u(\cdot)$ strongly in $L^\infty(0, T^*; \mathcal{H}^s(\mathbb{R}^n))$ for $0 < s' < s$. Then we show that the exists a unique solution $u(\cdot)$ of (1) such that

(a) $u(\cdot)$ solve (1) as an equality in $L^\infty(0, T^*; \mathcal{H}^{s'-1}(\mathbb{R}^n))$,

(b) $u(\cdot)$ is a local in time solution of (1) such that $u \in C(0, T^*; \mathcal{H}^s(\mathbb{R}^n)) \cap C^1(0, T^*; \mathcal{H}^{s-1}(\mathbb{R}^n))$, for $s > n/2 + 1$.

4.1. **Truncated symmetric hyperbolic system.** Let us define the Fourier truncation $S_R$ as follows ([4]):

$$
\hat{S}_R f(\xi) = 1_{B_R}(\xi)\hat{f}(\xi),
$$

where $B_R$, a ball of radius $R$ centered at the origin and $1_{B_R}(\cdot)$ is the indicator function. For $s \geq 0$, we have ([4])

$$
\|S_R f\|_{\mathcal{H}^s(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{H}^s(\mathbb{R}^n)},
$$

(99)

$$
\|S_R f - f\|_{\mathcal{H}^s(\mathbb{R}^n)} \leq C \left( \frac{1}{R} \right)^k \|f\|_{\mathcal{H}^{s+k}(\mathbb{R}^n)},
$$

(100)

$$
\|S_R - S_R\|_{\mathcal{H}^s(\mathbb{R}^n)} \leq C \max \left\{ \left( \frac{1}{R} \right)^k , \left( \frac{1}{R^2} \right)^k \right\} \|f\|_{\mathcal{H}^{s+k}(\mathbb{R}^n)},
$$

(101)

where $C$ is a generic constant independent of $R$. Let us consider the truncated system

$$
\frac{\partial u^R(t)}{\partial t} + S_R \mathcal{A}(t, u^R)u^R(t) = S_R f(t), \ 0 \leq t \leq T, \ \ u^R(0) = S_R u_0.
$$

(102)

By taking truncated initial data $S_R u_0$, the solution $u^R(\cdot)$ of (102) lie in the space

$$
\mathcal{H}_R := \left\{ f \in L^2(\mathbb{R}^n) : \hat{f} \text{ is supported in } B_R \right\}
$$

(103)

Since $S_R \mathcal{A}(t, u^R)u^R$ is locally Lipschitz in $u^R$ (see (21)), by Picard’s theorem for infinite-dimensional ODEs (see Theorem 3.1, [14]), there exists a solution $u^R(\cdot)$.
See Proposition 1.

**Proposition 3 (L²-energy estimate).** Let \( u^R(\cdot) \) be the unique solution of the ODE (102) with \( u_0 \in L^2(\mathbb{R}^n) \). Then, there exists a time \( 0 < T^* < T \) such that, for \( f \in L^2(0, T^*; L^2(\mathbb{R}^n)) \), we have the following a-priori energy estimate for \( 0 < \varepsilon \leq 1 \):

\[
\| u^R(t) \|^2_{L^2} \leq \left( \| u_0 \|^2_{L^2} + \frac{1}{\varepsilon} \int_0^t \| f(r) \|^2_{L^2} \, dr \right) \exp \left( \int_0^t (2C\| A(r) \|_{L^\infty} + \varepsilon) \, dr \right),
\]

(104) for \( 0 \leq t \leq T^* \) and

\[
\sup_{0 \leq t \leq T^*} \| u^R(t) \|^2_{L^2} \leq \left( \| u_0 \|^2_{L^2} + \frac{1}{\varepsilon} \int_0^{T^*} \| f(t) \|^2_{L^2} \, dt \right) e^{(2CM+\varepsilon)T^*},
\]

(105) where \( M = \sup_{t \in [0, T^*]} \| \nabla A(t) \|_{L^\infty} \) and the left hand side of the inequality (105) is finite whenever \( M \) is finite. Both estimates (104) and (105) are uniformly bounded and the bounds are independent of \( R \).

**Proof.** See Proposition 1. \( \square \)

**Proposition 4 (H²-energy estimate).** Let \( u^R(\cdot) \) be the unique solution of the ODE (102) with \( u_0 \in H^s(\mathbb{R}^n) \), for \( s > n/2 + 1 \). Then, there exists a time \( 0 < T^* < T \) such that, for \( f \in L^2(0, T^*; H^s(\mathbb{R}^n)) \), we have the following a-priori energy estimate for \( 0 < \varepsilon \leq 1 \):

\[
\| u^R(t) \|^2_{H^2} \leq \left( \| u_0 \|^2_{H^2} + \frac{1}{\varepsilon} \int_0^t \| f(r) \|^2_{H^2} \, dr \right) \times
\exp \left( \int_0^t \left( 4C\| A(r) \|_{L^\infty} + \varepsilon + 2C \left( \| u^R(r) \|_{L^\infty} \right)^2 \left[ 1 + \| \nabla u^R(r) \|_{L^\infty} \right] \right) \, dr \right),
\]

(106) for \( 0 \leq t \leq T^* \) and

\[
\sup_{0 \leq t \leq T^*} \| u^R(t) \|^2_{H^2} \leq \left( \| u_0 \|^2_{H^2} + \frac{1}{\varepsilon} \int_0^{T^*} \| f(t) \|^2_{H^2} \, dt \right) e^{(4CM + 2M' + \varepsilon)T^*},
\]

(107) where \( M = \sup_{t \in [0, T^*]} \| \nabla A(t) \|_{L^\infty} \) and \( M' = \sup_{0 \leq t \leq T^*} C(\| u^R(t) \|_{L^\infty}) (1 + \| \nabla u^R(t) \|_{L^\infty}) \) and the left hand side of the inequality (107) is finite whenever \( M, M' \) are finite. Both estimates (104) and (107) are uniformly bounded and the bounds are independent of \( R \), since

\[
\| u^R(t) \|_{L^\infty} \leq C\| u^R(t) \|_{H^{s-1}} \leq C\| u^R(t) \|_{H^s};
\]

\[
\| \nabla u^R(t) \|_{L^\infty} \leq C\| \nabla u^R(t) \|_{H^{s-1}} \leq C\| u^R(t) \|_{H^s},
\]

(108) for \( s > n/2 + 1 \).

**Proof.** Let us take \( J^r := (I - \Delta)^{r/2} \) on the equation (102) to get

\[
\frac{\partial J^r u^R(t)}{\partial t} + J^r S_R \mathcal{A}(t, u^R(t)) u^R(t) = J^r S_R f(t), \quad 0 \leq t \leq T,
\]

\[
u^R(0) = S_R u_0.
\]

(109) in \( \mathcal{H}_R \) for some time interval \([0, T]\), where \( T \) depends on \( R \). The solution will exist as long as \( \| u^R \|_{H^s} \) is finite.
The operators $J^*$ and $S_R$ commute, since
\[
J^* S_R f(\xi) = (1 + |\xi|^2)^{s/2} S_R f(\xi) = (1 + |\xi|^2)^{s/2} 1_{B_R} \hat{f}(\xi)
\]
where $\hat{f}$ is the Fourier transform of $f$. The rest of the proof is same as that of Proposition 2 by using the fact that $S_R u^R = u^R$ in $\mathcal{H}_R$.

4.2. Existence and uniqueness of local solution. We will now show that $u^R(\cdot)$ is a Cauchy sequence in $L^\infty(0, T^*; L^2(\mathbb{R}^n))$.

**Proposition 5.** Let $u_0 \in H^s(\mathbb{R}^n)$ and $f \in L^2(0, T^*; H^s(\mathbb{R}^n))$ for $s > n/2 + 1$, and $T^*$ is the maximal time defined in Proposition 3 and Proposition 4. Then, the family of local solutions $u^R(\cdot)$ of (102) is Cauchy (as $R \to \infty$) in $L^\infty(0, T^*; L^2(\mathbb{R}^n))$, i.e.,

\[
\lim_{R \to \infty} \sup_{t \in [0, T^*]} \|u^R - u^{R'}\|_{L^2}^2 = 0.
\]

**Proof.** If $u^R(\cdot)$ and $u^{R'}(\cdot)$ are two local solutions, then for $0 \leq t \leq T^*$, $(u^R - u^{R'})(\cdot)$ satisfy

\[
\frac{\partial (u^R - u^{R'})}{\partial t} + S_R \phi(t, u^R)u^R(t) - S_{R'} \phi(t, u^{R'})u^{R'}(t) = (S_R - S_{R'}) f(t),
\]

\[
(u^R - u^{R'})(0) = (S_R - S_{R'}) u_0.
\]

For $R' \geq R$, $(u^R - u^{R'})(\cdot)$ satisfy the energy equality

\[
\frac{1}{2} \frac{d}{dt} \|u^R - u^{R'}\|_{L^2}^2 = - \left( (S_R \phi(t, u^R)u^R(t) - S_{R'} \phi(t, u^{R'})u^{R'}(t), u^R - u^{R'}) \right)_{L^2}.
\]

The second term from the right hand side of the equality (112) can be simplified using Cauchy-Schwartz inequality, (101), and Young’s inequality as

\[
\left| \left( (S_R - S_{R'}) f(t), u^R - u^{R'} \right)_{L^2} \right| \leq \|(S_R - S_{R'}) f(t)\|_{L^2} \|u^R - u^{R'}\|_{L^2}
\]

\[
\leq \frac{C}{R^2} \|f(t)\|_{H^s} \|u^R - u^{R'}\|_{L^2} \leq \frac{C}{R^2} \|f(t)\|_{H^s} \|u^R - u^{R'}\|_{L^2}
\]

\[
\leq \frac{C}{2R^2} \|f(t)\|_{H^s}^2 + \frac{C}{2R^2} \|u^R - u^{R'}\|_{L^2}^2,
\]

for $0 < \varepsilon < 1$. The first term from the right hand side of the equality (112) can be written as

\[
\left( (S_R \phi(t, u^R)u^R(t) - S_{R'} \phi(t, u^{R'})u^{R'}(t), u^R - u^{R'}) \right)_{L^2}
\]

\[
= \left( (S_R - S_{R'}) \phi(t, u^R)u^R(t), u^R - u^{R'} \right)_{L^2}
\]

\[
+ \left( S_{R'} \left( \phi(t, u^R)u^R(t) - \phi(t, u^{R'})u^{R'}(t) \right), u^R - u^{R'} \right)_{L^2}
\]

\[
= \left( (S_R - S_{R'}) \phi(t, u^R)u^R(t), u^R - u^{R'} \right)_{L^2}
\]

\[
+ \left( S_{R'} \left[ \phi(t, u^R) - \phi(t, u^{R'}) \right] u^R(t), u^R - u^{R'} \right)_{L^2}
\]

\[
+ \left( S_{R'} \phi(t, u^{R'}) u^{R'}(t), u^R - u^{R'} \right)_{L^2}.
\]
First term from the right hand side of the equality (114) can be simplified using Cauchy-Schwartz inequality, (101), algebra property of $H^{-1}$-norm, Moser estimate (11) (with $p = 2$), and Young’s inequality as

$$\left| \left( (S_R^* - S_{R^2}) \mathcal{J}(t, u^R(t), u^R - u^{R'}) \right) \right|_{L^2}$$

$$\leq \| (S_R^* - S_{R^2}) \mathcal{J}(t, u^R(t)) \|_{L^2} \| u^R - u^{R'} \|_{L^2}$$

$$\leq \frac{C}{R^2} \| \mathcal{J}(t, u^R(t)) \|_{H^*} \| u^R - u^{R'} \|_{L^2}$$

$$\leq \frac{C}{R^2} \sum_{j=1}^n \| A^j(t, x, u^R) \frac{\partial u^R}{\partial x_j} \|_{H^{r-1}} \| u^R - u^{R'} \|_{L^2}$$

$$\leq \frac{C}{R^2} \left( \sum_{j=1}^n \| A^j(t, x, u^R) \|^2 \right)^{1/2} \left( \sum_{j=1}^n \left\| \frac{\partial u^R}{\partial x_j} \right\|^2 \right)^{1/2} \| u^R - u^{R'} \|_{L^2}$$

$$\leq \frac{C}{R^2} \left( \sum_{j=1}^n \| A^j(t, x, u^R) \|^2 \right)^{1/2} \| \nabla u^R \|_{H^{r-1}} \| u^R - u^{R'} \|_{L^2}$$

$$\leq \frac{C(\| u^R \|_{L^\infty})}{R^2} \left( 1 + \| \nabla u^R \|_{H^{r-1}} \| u^R - u^{R'} \|_{L^2} \right) \left( 1 + \| u^R - u^{R'} \|_{L^2}^2 \right). \tag{115}$$

The second term from the equality (114) can be estimated using Cauchy-Schwartz inequality, and (8) as

$$\left| \left( S_R \left[ \left( \mathcal{J}(t, u^R) - \mathcal{J}(t, u^{R'}) \right) u^R(t) \right] , u^R - u^{R'} \right) \right|_{L^2}$$

$$\leq \| \mathcal{J}(t, u^R) - \mathcal{J}(t, u^{R'}) \|_{L^2} \| u^R(t) \|_{L^2} \| u^R - u^{R'} \|_{L^2}$$

$$\leq C \| \nabla A(t) \|_{L^\infty} \| u^R \|_{H^*} \| u^R - u^{R'} \|_{L^2}^2. \tag{116}$$

The final term from the equality (114) can be simplified using (17) as

$$\left( S_{R^2} \mathcal{J}(t, u^{R'}) (u^R - u^{R'}) , u^R - u^{R'} \right)_{L^2}$$

$$= \left( \mathcal{J}(t, u^{R'}) (u^R - u^{R'}) , S_{R^2} (u^R - u^{R'}) \right)_{L^2}$$

$$= \left( \mathcal{J}(t, u^{R'}) (u^R - u^{R'}) , u^R - u^{R'} \right)_{L^2} \leq C \| \nabla A(t) \|_{L^\infty} \| u^R - u^{R'} \|_{L^2}^2. \tag{117}$$

By substituting (115), (116), and (117) in (114), we obtain

$$\left( S_R \mathcal{J}(t, u^{R'}) u^R(t) - S_{R^2} \mathcal{J}(t, u^{R'}) u^{R'}(t) , u^R - u^{R'} \right)_{L^2}$$

$$\leq C(\| u^R \|_{L^\infty}) \left( \| u^R \|_{H^*} + \| u^R \|_{H^*}^2 \right) \left( 1 + \| u^R - u^{R'} \|_{L^2}^2 \right)$$

$$+ C \left( \| \nabla A(t) \|_{L^\infty} + \| \nabla u A(t) \|_{L^\infty} \| u^R \|_{H^*} \| u^R - u^{R'} \|_{L^2} \right). \tag{118}$$
By applying (113) and (118) in (112), we find
\[ \frac{d}{dt} \| u^R - u^{R'} \|^2_{L^2} \leq \frac{C}{R^2} \| f(t) \|^2_{H^s} + \frac{C}{R^2} \| u^R - u^{R'} \|^2_{L^2} \\
+ \frac{C(\| u^R \|_{L^\infty}) (\| u^R \|_{H^s} + \| u^{R'} \|_{H^s})}{R^2} (1 + \| u^R - u^{R'} \|^2_{L^2}) \\
+ 2C \left( \| \nabla A(t) \|_{L^\infty} + \| \nabla uA^i(t) \|_{L^\infty} \| u^R \|_{H^s} \right) \| u^R - u^{R'} \|^2_{L^2}. \tag{119} \]

Let us integrate the inequality (119) in $t$, and take supremum from 0 to $T^*$, we find
\[ \sup_{0 \leq t \leq T^*} \| u^R - u^{R'} \|^2_{L^2} \]
\[ \leq \| u^R(0) - u^{R'}(0) \|^2_{L^2} + \frac{C}{R^2} \int_0^{T^*} \| f(t) \|^2_{H^s} dt + \frac{C}{R^2} \int_0^{T^*} \| u^R - u^{R'} \|^2_{L^2} dt \\
+ \frac{1}{R^2} \int_0^{T^*} C(\| u^R \|_{L^\infty}) (\| u^R \|_{H^s} + \| u^{R'} \|_{H^s}) (1 + \| u^R - u^{R'} \|^2_{L^2}) dt \\
+ 2C \int_0^{T^*} (\| \nabla A(t) \|_{L^\infty} + \| \nabla uA(t) \|_{L^\infty} \| u^R \|_{H^s}) \| u^R - u^{R'} \|^2_{L^2} dt. \tag{120} \]

By using (101), for $0 < \varepsilon < 1$, we have
\[ \| u^R(0) - u^{R'}(0) \|^2_{L^2} = \| (S_R - S_{R'}) u_0 \|^2_{L^2} \leq \frac{C}{R^2} \| u_0 \|^2_{H^s} \leq \frac{C}{R^2} \| u_0 \|^2_{H^s}. \tag{121} \]

Hence from (120), we get
\[ \sup_{0 \leq t \leq T^*} \| u^R - u^{R'} \|^2_{L^2} \]
\[ \leq \frac{C}{R^2} \left[ \| u_0 \|^2_{H^s} + \int_0^{T^*} \| f(t) \|^2_{H^s} dt \right] + \frac{C}{R^2} \int_0^{T^*} \| u^R - u^{R'} \|^2_{L^2} dt \\
+ \frac{1}{R^2} \sup_{0 \leq t \leq T^*} \left[ C(\| u^R \|_{L^\infty}) (\| u^R \|_{H^s} + \| u^{R'} \|_{H^s}) \right] \int_0^{T^*} (1 + \| u^R - u^{R'} \|^2_{L^2}) dt \\
+ 2C \sup_{0 \leq t \leq T^*} (\| \nabla A(t) \|_{L^\infty} + \| \nabla uA(t) \|_{L^\infty} \| u^R \|_{H^s}) \int_0^{T^*} \| u^R - u^{R'} \|^2_{L^2} dt. \tag{122} \]

By using Proposition 4, we have \( \sup_{0 \leq t \leq T^*} \| u^R \|_{H^s} \) and \( \sup_{0 \leq t \leq T^*} \| u^{R'} \|_{H^s} \) are uniformly bounded and the bounds are independent of $R$ and $R'$. From (122), we obtain
\[ \sup_{0 \leq t \leq T^*} \| u^R - u^{R'} \|^2_{L^2} \leq \frac{C}{R^2} \left[ \| u_0 \|^2_{H^s} + \int_0^{T^*} \| f(t) \|^2_{H^s} dt + C(K)(K + K^2)T^* \right] \\
+ \left( \frac{(C + C(K)(K + K^2))}{R^2} \right) + 2C(M + M'K) \int_0^{T^*} \| u^R - u^{R'} \|^2_{L^2} dt, \tag{123} \]

where the constants $K = \sup_{0 \leq t \leq T^*} \| u^R(t) \|_{H^s}$, $M = \sup_{0 \leq t \leq T^*} \| \nabla A(t) \|_{L^\infty}$ and $M' = \sup_{0 \leq t \leq T^*} \| \nabla uA(t) \|_{L^\infty} < \infty$ and independent of $R$ and $R'$. An application of Gronwall’s inequality on (123) yield
\[ \sup_{0 \leq t \leq T^*} \| u^R - u^{R'} \|^2_{L^2} \leq \frac{C}{R^2} \left[ \| u_0 \|^2_{H^s} + \int_0^{T^*} \| f(t) \|^2_{H^s} dt + C(K)(K + K^2)T^* \right] \times \]
We know that \( u_0 \in H^s(\mathbb{R}^n) \), \( f \in L^2(0, T^*; H^s(\mathbb{R}^n)) \), and \( M, M', K < \infty \), up to the maximal time \( T^* \). On passing \( R, R' \to \infty \), one can easily see that the right hand side of the inequality \((124)\) goes to zero and hence the sequence of solution \( u^R(\cdot) \) of \((102)\) is Cauchy (as \( R \to \infty \)) in \( L^\infty(0, T^*; L^2(\mathbb{R}^n)) \).

From Proposition 5, we get \( u^R(\cdot) \) is a Cauchy sequence in \( L^2(0, T^*; L^2(\mathbb{R}^n)) \) and it follows that \( u^R \to u \) strongly in \( L^\infty(0, T^*; L^2(\mathbb{R}^n)) \). Now we prove that the sequence \( u^R(\cdot) \) converges to \( u(\cdot) \) in \( L^\infty(0, T^*; H^s(\mathbb{R}^n)) \), for \( 0 < s' < s \) by making use of Sobolev interpolation theorem.

Proposition 6. Let \( u_0 \in H^s(\mathbb{R}^n) \) and \( f \in L^2(0, T^*; H^s(\mathbb{R}^n)) \) for \( s > n/2 + 1 \), and \( T^* \) be the maximal time defined in Proposition 3 and Proposition 4. Then, the family of local in time solution \( u^R(\cdot) \) of \((102)\) is Cauchy (as \( R \to \infty \)) in \( L^\infty(0, T^*; H^s(\mathbb{R}^n)) \), for \( 0 < s' < s \).

Proof. By using Sobolev interpolation theorem (Theorem 9.6, Remark 9.1, \[7\]), Proposition 4, and Proposition 5, for \( 0 < s' < s \), we have

\[
\sup_{0 \leq t \leq T^*} \| u^R - u^{R'} \|_{H^{s'}} \leq C_s \sup_{0 \leq t \leq T^*} \left[ \| u^R - u^{R'} \|_{L^2}^{1 - s'/s} \| u^R - u^{R'} \|_{H^{s'}}^{s'/s} \right] \leq C_s \sup_{0 \leq t \leq T^*} \| u^R - u^{R'} \|_{L^2}^{1 - s'/s} \left[ \sup_{0 \leq t \leq T^*} \| u^R \|_{H^{s'}} + \sup_{0 \leq t \leq T^*} \| u^{R'} \|_{H^{s'}} \right] \to 0 \text{ as } R, R' \to \infty,
\]

since \( u^R \to u \in L^\infty(0, T^*; L^2(\mathbb{R}^n)) \) and \( u^R, u^{R'} \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \). Hence the local in time solution \( u^R(\cdot) \) of \((102)\) is Cauchy (as \( R \to \infty \)) in \( L^\infty(0, T^*; H^s(\mathbb{R}^n)) \).

From Proposition 6, it follows that \( u^R \to u \) strongly in \( L^\infty(0, T^*; H^s(\mathbb{R}^n)) \). Now we prove that the nonlinear term \( S_R \mathcal{A}(t, u^R) \to \mathcal{A}(t, u) \) strongly in \( L^\infty(0, T^*; H^{s'-1}(\mathbb{R}^n)) \).

Proposition 7. For \( s' > n/2 + 1 \) and \( 0 < s' < s \), the nonlinear term \( S_R \mathcal{A}(t, u^R) \) converges to \( \mathcal{A}(t, u) \) strongly in \( L^\infty(0, T^*; H^{s'-1}(\mathbb{R}^n)) \) as \( R \to \infty \).

Proof. Let us estimate \( \| S_R \mathcal{A}(t, u^R) u^R - \mathcal{A}(t, u) u \|_{H^{s'-1}} \) as

\[
\| S_R \mathcal{A}(t, u^R) u^R - \mathcal{A}(t, u) u \|_{H^{s'-1}} \leq \| S_R \mathcal{A}(t, u^R) u^R - \mathcal{A}(t, u^R) u \|_{H^{s'-1}} + \| S_R \mathcal{A}(t, u) u - \mathcal{A}(t, u) u \|_{H^{s'-1}}. \tag{126}
\]

For the first term from the right hand side of the inequality \((126)\), we use \((99), (93)\) and \((98)\) to get

\[
\| S_R \mathcal{A}(t, u^R) u^R - \mathcal{A}(t, u^R) u \|_{H^{s'-1}} \leq \| \mathcal{A}(t, u^R) (u^R - u) \|_{H^{s'-1}} + \| (\mathcal{A}(t, u^R) - \mathcal{A}(t, u)) u \|_{H^{s'-1}} \leq (C(\| u^R \|_{L^\infty}) (1 + \| u^R \|_{H^{s'}}) + C \| \nabla u A \|_{H^{s'-1}} \| u \|_{H^{s'}}) \| u^R - u \|_{H^{s'}}. \tag{127}
\]
For $0 < \varepsilon < 1$, the second term from the right hand side of the inequality (126) can be estimated using (100), the algebra property of $H^{s-\varepsilon-1}$ norm, and (11) as
\[
\|S_R \mathcal{A}(t,u)\|_{H^{s-1}} \leq \frac{C}{R^\varepsilon} \|\mathcal{A}(t,u)\|_{H^{s-1+\varepsilon}} \leq \frac{C}{R^\varepsilon} \sum_{j=1}^n \left\|A^j(t,x,u) \frac{\partial u}{\partial x_j}\right\|_{H^{s-1+\varepsilon}}^{1/2} \left(\sum_{j=1}^n \left\|\frac{\partial u}{\partial x_j}\right\|_{H^{s-1+\varepsilon}}^2\right)^{1/2} 
\leq \frac{C_\varepsilon}{R^\varepsilon} \left(1 + \|u\|_{H^{s-1+\varepsilon}}\right) \|\nabla u\|_{H^{s-1+\varepsilon}} 
\leq \frac{C_\varepsilon}{R^\varepsilon} \left(1 + \|u(t)\|_{H^s}\right) \|u(t)\|_{H^s},
\] (128)

A substitution of (127) and (128) in (126) yield
\[
\|S_R \mathcal{A}(t,u^R) - \mathcal{A}(t,u)\|_{H^{s-1}} \leq \left(\frac{C(\|u^R(t)\|_{L^\infty})}{R^\varepsilon}\right) (1 + \|u^R(t)\|_{H^s}) + \|\nabla u A(t)\|_{H^{s-1}} \|u(t)\|_{H^s} \|u^R - u\|_{H^s} 
+ \frac{C(\|u(t)\|_{L^\infty})}{R^\varepsilon} (1 + \|u(t)\|_{H^s}) \|u(t)\|_{H^s}.
\] (129)

The right hand side of the inequality (129) tend to zero as $R \to \infty$, since $u^R \to u$ in $L^\infty(0,T^*;H^s(\mathbb{R}^n))$ and $u,u^R \in L^\infty(0,T^*;H^s(\mathbb{R}^n))$.

\[\square\]

**Remark 4.** Since $L^\infty(0,T^*;H^s(\mathbb{R}^n)) \subset L^2(0,T^*;H^s(\mathbb{R}^n))$, from Proposition 5 and Proposition 6, we have

\[
\mathcal{A}(t,u^R) \to \mathcal{A}(t,u) \text{ in } L^2(0,T^*;H^s(\mathbb{R}^n)) \text{ and } S_R \mathcal{A}(t,u^R) \to \mathcal{A}(t,u) \text{ in } L^2(0,T^*;H^{s-1}(\mathbb{R}^n)).
\]
(130)

Also, for $0 < \varepsilon < 1$, we have
\[
\|S_R u_0 - u_0\|_{H^s} \to 0 \text{ as } R \to \infty.
\] (132)

**Remark 5.** For the convergence of the derivative $\frac{\partial u^R(t)}{\partial t}$, we use (99), and (58) to obtain
\[
\left\|\frac{\partial u^R}{\partial t}\right\|_{H^{s-1}} \leq C(\|u^R(t)\|_{L^\infty}) (1 + \|u^R(t)\|_{H^s}) \|u^R(t)\|_{H^s} + \|f(t)\|_{H^s},
\] (133)

and
\[
\left\|\frac{\partial u^R}{\partial t} - \frac{\partial u}{\partial t}\right\|_{H^{s-1}} \leq \|S_R \mathcal{A}(t,u^R) - \mathcal{A}(t,u)\|_{H^s} \|u^R(t)\|_{H^s} + \|\mathcal{A}(t,u)\|_{H^s} 
\to 0 \text{ in } L^2(0,T^*;H^{s-1}(\mathbb{R}^n)),
\] (134)

since we know that $S_R \mathcal{A}(t,u^R) \to \mathcal{A}(t,u) \text{ in } L^\infty(0,T^*;H^{s-1}(\mathbb{R}^n))$ and $\mathcal{A}(t,u) \to f$ in $L^2(0,T^*;H^{s-1}(\mathbb{R}^n))$. 
Theorem 4.1 (Local Existence and Uniqueness). Let \( f \in L^2(0, T^*; H^s(\mathbb{R}^n)) \) and \( u_0 \in H^s(\mathbb{R}^n) \) with \( s > n/2 + 1 \) and let \( T^* \) be the maximal time for which the energy estimates in Proposition 3 and Proposition 4 are finite. Then, there exists a unique solution \( u \in L^\infty(0, T^*; H^s(\mathbb{R}^n)) \) to the problem (1).

Proof. Existence: By using Proposition 5, Proposition 6, Proposition 7, Remark 4 and Remark 5, we have \( u(\cdot) \) solves (1) as an equality in \( L^\infty(0, T; H^{s'}(\mathbb{R}^n)) \), for \( s' < s \) and \( s' > n/2 + 1 \). The uniform bounds in Proposition 4 and Banach-Alaoglu theorem allows us to extract a subsequence \( u^{(m)} \) such that

\[
u^{(m)} \rightharpoonup u \text{ in } L^\infty(0, T^*; H^s(\mathbb{R}^n)).
\]

Hence the limit \( u \) satisfies \( u \in L^\infty(0, T^*; H^s(\mathbb{R}^n)). \)

Uniqueness: Uniqueness results for the solution of symmetric hyperbolic system (1) is given in Theorem 3.2. \( \square \)

Remark 6. We can also show that \( u \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)) \) with the same arguments as in Theorem 3.3.

5. Existence of optimal controls. In this section, we consider a control problem for the symmetric hyperbolic system, where the control appear as a “body force”, and prove the existence of an optimal control. The similar ideas for establishing the existence of an optimal control for the Navier-Stokes equations can be found in [19]. The controlled symmetric hyperbolic system is given by

\[
\begin{align*}
\frac{\partial u(t)}{\partial t} + \mathcal{A}(t, u) u(t) &= \mathcal{K} U(t), &0 \leq t \leq T^*, \\
u(0) &= u_0,
\end{align*}
\]

(135)

where \( \mathcal{K} \) is a bounded linear operator from \( L^2(\mathbb{R}^n) \) to \( H^s(\mathbb{R}^n) \) and \( U(\cdot) \) is the control. We formulate the control problem of finding \( U \) to minimize the cost functional

\[
J(U) = \frac{1}{2} \int_0^{T^*} \| u(t) - u_d(t) \|_{L^2}^2 \, dt + \frac{1}{2} \int_0^{T^*} \| U(t) \|_{L^2}^2 \, dt \to \inf,
\]

(136)

where \( u_d(t) \) is a desired solution.

Theorem 5.1. Let \( u_0 \in H^s(\mathbb{R}^n) \), for \( s > n/2 + 1 \), be given. Then, there exists an optimal pair

\[
(u, \bar{U}) \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)) \times L^2(0, T^*; H^s(\mathbb{R}^n)),
\]

satisfying the controlled symmetric hyperbolic system (135) such that

\[
J(\bar{U}) = \min \left\{ J(U) : U \in L^2(0, T^*; L^2(\mathbb{R}^n)), (u, U) \text{ satisfies (135)} \right\}
\]

(137)

with \( u \in C(0, T^*; H^s(\mathbb{R}^n)) \) and \( \frac{\partial u}{\partial t} \in C(0, T^*; H^{s-1}(\mathbb{R}^n)) \).

Proof. By using Theorem 3.2, and Theorem 3.3, one can easily prove that for each \( U \in L^2(0, T^*; L^2(\mathbb{R}^n)) \), there exists a unique solution \( u \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)) \) such that \( J(U) < +\infty \).

(138)

Noting that \( J(U) \geq 0 \), and

\[
J(U) \to +\infty \text{ as } \| U(t) \|_{L^2(0, T^*; L^2(\mathbb{R}^n))} \to \infty,
\]

(139)
we have for any $R > 0$,
\[ J(U) \leq \frac{1}{2} R^2 \] implies that $\|U(t)\|_{L^2(0,T^*;L^2(\mathbb{R}^n))} \leq R$. \hspace{1cm} (140)
Hence, the a-priori estimate (32) implies
\[ \sup_{0 \leq t \leq T^*} \|u(t)\|_{H^s} \leq C(R). \] \hspace{1cm} (141)
By the definition of infimum, there exists a minimizing sequence
\[ (u^n, U^n) \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)) \times L^2(0, T^*; L^2(\mathbb{R}^n)), \] satisfying the symmetric hyperbolic system (135) such that
\[ \inf J(U) = \lim_{n \to \infty} J(U^n), \quad \|U^n(t)\|_{L^2(0,T^*;L^2(\mathbb{R}^n))} \leq R, \quad \text{and} \quad \sup_{0 \leq t \leq T^*} \|u^n(t)\|_{H^s} \leq C(R). \] \hspace{1cm} (142)
We can extract a subsequence $(u^{n'}, U^{n'})$ such that
\[ (u^{n'}, U^{n'}) \xrightarrow{w} (\tilde{u}, \tilde{U}) \text{ in } L^\infty(0, T^*; H^s(\mathbb{R}^n)) \times L^2(0, T^*; L^2(\mathbb{R}^n)), \] and by Theorem 3.3, \hspace{1cm} (143)
\[ (\tilde{u}, \tilde{U}) \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)) \times L^2(0, T^*; L^2(\mathbb{R}^n)), \] \hspace{1cm} (144)
and $(\tilde{u}, \tilde{U})$ satisfies (135). Since the integrand $J(U)$ is convex, by Proposition 1.3.5, [1], $J(\cdot)$ is weakly lowersemicontinuous. That is, for
\[ (u^n, U^n) \to (\tilde{u}, \tilde{U}) \text{ in } L^2_w(0, T^*; L^2(\mathbb{R}^n)) \times L^2_w(0, T^*; L^2(\mathbb{R}^n)), \] \hspace{1cm} (145)
where $L^2_w(0, T^*; L^2(\mathbb{R}^n))$ denotes the space $L^2(0, T^*; L^2(\mathbb{R}^n))$ endowed with the weak topology, we have
\[ J(\tilde{u}) \leq \liminf_{n \to \infty} J(u^n). \] \hspace{1cm} (146)
Thus for the minimizing subsequence chosen above, we have
\[ J(\tilde{U}) \leq \liminf_{n' \to \infty} J(U^{n'}) = \lim_{n' \to \infty} J(U^{n'}) = \inf J(U), \] \hspace{1cm} (147)
and
\[ J(\tilde{U}) = \min \left\{ J(U) : U \in L^2(0,T^*;L^2(\mathbb{R}^n)), (u,U) \text{ satisfies (135)} \right\} \] \hspace{1cm} (148)
with $u \in C(0,T^*;H^s(\mathbb{R}^n))$ and $\frac{\partial u}{\partial t} \in C(0,T^*;H^{s-1}(\mathbb{R}^n))$.

\[ \square \]

Remark 7. We can also prove the theorem by extracting a subsequence (frequency truncation) $(u^{Rn}, U^{Rn})$ such that (Theorem 4.1)
\[ (u^{Rn}, U^{Rn}) \xrightarrow{w} (\tilde{u}, \tilde{U}) \text{ in } L^\infty(0, T^*; H^s(\mathbb{R}^n)) \times L^2(0, T^*; L^2(\mathbb{R}^n)), \] \hspace{1cm} (149)
and satisfies (135) as an equality in $L^2(0, T^*; H^{s-1}(\mathbb{R}^n))$. Hence, an application of Theorem 3.3 gives
\[ (\tilde{u}, \tilde{U}) \in C(0, T^*; H^s(\mathbb{R}^n)) \cap C^1(0, T^*; H^{s-1}(\mathbb{R}^n)) \times L^2(0, T^*; L^2(\mathbb{R}^n)). \] \hspace{1cm} (150)
Then, the existence of an optimal control can be obtained by following same steps as in the proof of Theorem 5.1.
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