On quantization of $r$-matrices for Belavin-Drinfeld Triples

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Abstract

We suggest a formula for quantum universal $R$-matrices corresponding to quasitriangular classical $r$-matrices classified by Belavin and Drinfeld for all simple Lie algebras. The $R$-matrices are obtained by twisting the standard universal $R$-matrix.
1. Classical quasitriangular $r$-matrices for semisimple Lie algebras are classified by Belavin-Drinfeld triples \[1\]. The Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, \tau)\) for a simple Lie algebra \(g = g^+ \oplus h \oplus g^-\) consists of the following data: \(\Gamma_1, \Gamma_2\) are subsets of the set \(\Gamma\) of simple roots of the algebra \(g\) and \(\tau\) is a one-to-one mapping: \(\Gamma_1 \rightarrow \Gamma_2\) such that \(\tau(\alpha), \tau(\beta) \geq \alpha, \beta\) and \(\tau^k(\alpha) \neq \alpha\) for any \(\alpha, \beta \in \Gamma_1\) and any natural \(k\). The corresponding quantum \(R\)-matrices should have the form

\[ R_{12} = F_{21} \mathcal{R}_{12} F_{12}^{-1}, \tag{1} \]

where \(\mathcal{R}\) is the standard universal Drinfeld-Jimbo \(R\)-matrix for the Lie algebra \(g\). The twisting operator satisfies the cocycle equation

\[ F_{12} (\Delta \otimes \text{id}) F = F_{23} (\text{id} \otimes \Delta) F. \tag{2} \]

Therefore the problem of quantization is reduced to the problem of finding the twisting operator \(F_{12}\) for each Belavin-Drinfeld triple. In the present paper we suggest a formula for the twisting operator \(F_{12}\). We present the twisting operator in a factorized form

\[ F_{12} = F_{12}^{(N)} \cdot F_{12}^{(N-1)} \cdots F_{12}^{(2)} \cdot F_{12}^{(1)} \cdot K, \tag{3} \]

where the factors \(F^{(k)}\) are special canonical elements defined by the powers of the one-to-one map \(\tau\); the operator \(K\) belongs to \(q^h \otimes h\). A different formula for the operator \(F_{12}\) was given in \[2\]. We shall say several words about the differences at the end of the present paper.

The plan of our paper is as follows.

Our approach heavily uses the modified Cartan-Weyl basis for \(U_q(g)\). The definition of the modified simple root generators is contained in Section 2. In Section 3 we give an interpretation of Belavin-Drinfeld triples in terms of the modified basis. In Section 4 a modified Cartan-Weyl basis is introduced. The twisting operator \(F_{12}\) is constructed in Section 5. Finally, in Section 6 several examples are presented.

Everywhere below we assume the deformation parameter \(q\) to be generic (not a root of unity).

2. Modified basis for quantum universal enveloping algebras.

Consider a quantum universal enveloping algebra \(U_q(g)\) with relations
(see e.g. [3])

\[ [h_i, h_j] = 0 \, , \, [h_i, e_j] = a_{ij} e_j \, , \, [h_i, f_j] = -a_{ij} f_j \, , \]

\[ [e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}} , \] (4)

and Serre relations

\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] e_i^k e_j (e_i)^{1-a_{ij}-k} = 0 , \] (5)

\[ \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] f_i^k f_j (f_i)^{1-a_{ij}-k} = 0 , \] (6)

where

\[ \left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{[n]_q!}{[k]_q! [n-k]_q!} , \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} , \]

\( a_{ij} \) is the Cartan matrix for \( g \), \( K_i = q^{d_i h_i} \) and \( d_i \) are smallest positive integers (from the set 1, 2, 3) such that \( d_i a_{ij} = a_{ij}^{(s)} \) is symmetric matrix. The algebra \( U_q(g) \) is a Hopf algebra with the comultiplication

\[ \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i , \]

\[ \Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i , \quad \Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i . \] (7)

The antipode and the counit are

\[ S(h_i) = -h_i , \quad S(e_i) = -e_i K_i^{-1} , \quad S(f_i) = -K_i f_i , \]

\[ \epsilon(h_i) = \epsilon(e_i) = \epsilon(f_i) = 0 . \]

Any operator \( K \in q^{h \otimes h} \),

\[ K = q^{\sum_{ij} b_{ij} h_i \otimes h_j} , \] (8)

for arbitrary numerical matrix \( b_{ij} \), obviously satisfies the cocycle equation (2),

\[ K_{12} (\Delta \otimes id) K = K_{23} (id \otimes \Delta) K . \] (9)
Therefore one can twist the comultiplication by $K$:
\[
\tilde{\Delta}(a) := K \Delta(a) K^{-1}.
\] (10)

We change the basis in the algebra $U_q(g)$ by introducing new generators
\[
E_i = X_i e_i, \quad F_i = f_i Y_i,
\] (11)
where $X_i = \exp(\sum x_{ij} h_j)$, $Y_i = \exp(\sum y_{ij} h_j)$ and $x_{ij}$, $y_{ij}$ are some numerical matrices. We require that the comultiplication (10) for the new generators (11) has the following form:
\[
\tilde{\Delta}(E_i) = K \Delta(E_i) K^{-1} = E_i \otimes R_i^+ + 1 \otimes E_i,
\] (12)
\[
\tilde{\Delta}(F_i) = K \Delta(F_i) K^{-1} = F_i \otimes 1 + R_i^- \otimes F_i.
\]

Equations (12) relate operators $X_i$, $Y_i$ and $K$.

A comparison of (7) and (12) gives
\[
X_i = q^{-\sum_{mn} h_m b_{mn} a_{ni}} \equiv q^{-\langle b \rangle_{ai} i},
\]
\[
Y_i = q^{\sum_{mn} h_m b_{mn} a_{ni}} \equiv q^{\langle b \bar{b} \rangle_{ai}},
\]
and
\[
R_i^\pm = X_i K_i^\pm Y_i = K_i^\pm q^{-\langle b \bar{b} \rangle_{ai}} , \quad R_i^+ = K_i^2 R_i^-,
\] (13)

where $b_{mn} = b_{nm}$ is the transposed matrix.

The relations (4) and Serre relations (5), (6) for the quantum algebra $U_q(g)$ in terms of the new generators (11) take the form
\[
[E_i, F_j] = \delta_{ij} \frac{R_i^+ - R_i^-}{q^{d_i} - q^{-d_i}},
\] (14)
\[
R_i^\pm E_j = q^{\pm a_{ij}^{(s)} + A_{ij}} E_j R_i^\pm, \quad R_i^\pm F_j = q^{\mp a_{ij}^{(s)} - A_{ij}} F_j R_i^\pm,
\] (15)
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] q^{-k A_{ij}} (E_i)^k E_j (E_i)^{1-a_{ij}-k} = 0,
\] (16)
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] q^{k A_{ij}} (F_i)^k F_j (F_i)^{1-a_{ij}-k} = 0,
\] (17)
with a skewsymmetric matrix $A_{ij} = \langle \pi(b - \bar{b})a \rangle_{ij}$. 

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In the sequel we shall use $q$-commutators: $[A, B]_\mu := A B - \mu B A$. Relations (16), (17) can be conveniently rewritten in terms of $q$-commutators. For example, for $a_{ij} = 0$ the relations $[e_i, e_j] = 0 = [f_i, f_j]$ are rewritten as

$$[E_i, E_j]_{q^{A_{ij}}} = 0, \quad [F_i, F_j]_{q^{-A_{ij}}} = 0,$$

while for $a_{ij} = -1$ we have

$$[[E_i, E_j]_\mu, E_i]_\nu = 0 = [E_j, [E_i, E_j]_\mu]_\nu,$$

$$[[F_i, F_j]_\nu, F_i]_\mu = 0 = [F_j, [F_i, F_j]_\nu]_\mu,$$

where $\mu = q^{d_i + A_{ij}}, \nu = q^{d_i - A_{ij}}$.

**Remark.** The modified basis for multiparametric twistings of $U_q(g)$ has been considered by T. Hodges [1].

3. Modified basis and Belavin-Drinfeld triples.

All the data from the Belavin-Drinfeld triple can be conveniently interpreted in terms of the modified basis for a suitable matrix $b_{ij}$:

**Proposition.** Let $\Gamma$ be the set of simple roots of $g$, $\Gamma_1$ and $\Gamma_2$ subsets of $\Gamma$ and $\tau$ a one-to-one mapping: $\Gamma_1 \rightarrow \Gamma_2$. Then the following equations for the matrix $b_{ij}$

$$R^+_{\alpha} = R^{-\tau(\alpha)}_i \forall \alpha \in \Gamma_1$$

where $R^\pm_i \equiv R^\pm \tau(\alpha)$, admit a solution if and only if the triple $(\Gamma_1, \Gamma_2, \tau)$ is the Belavin-Drinfeld triple.

**Proof.** Assume that a solution of equation (19) exists. We then need to prove that the mapping $\tau$ satisfies conditions:

1) for any $\alpha \in \Gamma_1$ there is a natural $k$ for which $\tau^k(\alpha) \notin \Gamma_1$, \hspace{1cm} (20)

2) for any $\alpha, \beta \in \Gamma_1$, $< \tau(\alpha), \tau(\beta)> = <\alpha, \beta>$. \hspace{1cm} (21)

The condition (20) means that $\tau$ has no cycles: $\tau^k(\alpha) \neq \alpha$ for all $\alpha \in \Gamma_1$ and $k > 0$.

Indeed, assume that $\tau$ has a cycle, $\tau^k(\alpha) = \alpha$ for some $\alpha \in \Gamma_1$ and a natural $k$. Take a minimal $k$ with this property.
Then $R^+_{\tau^k(\alpha)} = R^+_{\alpha}$ and equations (13) imply

$$R^+_{\alpha} = R^-_{\tau(\alpha)} = K^{-2}_{\tau(\alpha)} R^+_{\tau(\alpha)} = \ldots = \left( \prod_{i=1}^k K^{-2}_{\tau_i(\alpha)} \right) R^+_{\tau^k(\alpha)}$$

(22)

Therefore

$$\left( \prod_{i=1}^k K^{-2}_{\tau^i(\alpha)} \right) = 1,$$

which contradicts to the independence of generators in the Cartan subalgebra of $U_q(\mathfrak{g})$. Thus, $\tau^k(\alpha)$ never equals $\alpha$ which proves the condition (20).

To prove the condition (21), note that eq. (19) is equivalent to the following condition on the skewsymmetric matrix $A_{mn} = (\pi(b - b) a)_{mn}$

$$A_{im} + A_{m\tau(i)} + a^{(s)}_{im} + a^{(s)}_{\tau(i)m} = 0,$$

(23)

where the subscript $m$ runs over all simple roots while $i$ numerates only roots from $\Gamma_1$. Eq. (23) is obtained by commuting both sides of eq. (19) with $e_m$ (or $f_m$). Here it is important that $q$ is not a root of unity.

For indices $i, m$ corresponding to roots $\alpha_i, \alpha_m \in \Gamma_1$, eq. (23) can be rewritten in the following three equivalent forms

$$A_{i\tau(m)} + A_{\tau(m)\tau(i)} + a^{(s)}_{i\tau(m)} + a^{(s)}_{\tau(i)\tau(m)} = 0,$$

(24)

$$A_{mi} + A_{i\tau(m)} + a^{(s)}_{mi} + a^{(s)}_{\tau(m)i} = 0,$$

(25)

$$A_{m\tau(i)} + A_{\tau(i)\tau(m)} + a^{(s)}_{m\tau(i)} + a^{(s)}_{\tau(m)\tau(i)} = 0,$$

(26)

The combinations (23) + (25) and (24) + (26) of the equations are, respectively,

$$2a^{(s)}_{im} = -a^{(s)}_{\tau(m)i} - a^{(s)}_{\tau(i)m} - A_{m\tau(i)} - A_{i\tau(m)},$$

(27)

$$2a^{(s)}_{\tau(i)\tau(m)} = -a^{(s)}_{i\tau(m)} - a^{(s)}_{m\tau(i)} - A_{m\tau(i)} - A_{i\tau(m)},$$

(28)

Therefore,

$$a^{(s)}_{im} = a^{(s)}_{\tau(i)\tau(m)},$$

(29)
which is equivalent to the second condition \[ < \tau(\alpha_i), \tau(\alpha_m) > = < \alpha_i, \alpha_m > \]
for the Belavin-Drinfeld triple.

**Remark 1.** The difference of eqs. (24) and (25) gives the following relation on the matrix \( A_{ij} \)

\[
A_{im} - A_{\tau(i)\tau(m)} = a_{mi}^{(s)} + a_{\tau(m)i}^{(s)} - a_{\tau(i)\tau(m)}^{(s)} = 0.
\]

This shows that the map \( \tau \) does not change the modified basis.

**Remark 2.** Consider two sequences of sets

\[
\Gamma_1 = \Gamma_1^{(0)} \supset \Gamma_1^{(1)} \supset \Gamma_1^{(2)} \ldots \supset \Gamma_1^{(N)} \supset \Gamma_1^{(N+1)} = \emptyset,
\]

\[
\Gamma_2 = \Gamma_2^{(0)} \supset \Gamma_2^{(1)} \supset \Gamma_2^{(2)} \ldots \supset \Gamma_2^{(N)},
\]

defined by

\[
\Gamma_1^{(k+1)} = \Gamma_1^{(k)} \cap \Gamma_2^{(k)}, \quad \Gamma_1^{(k)} \xrightarrow{\tau} \Gamma_2^{(k)}.
\]

We assume that the set \( \Gamma_1^{(N)} \) is not empty. The number \( N \) is called the degree of the triple \((\Gamma_1, \Gamma_2, \tau)\).

Introduce a set \( \overline{\Gamma}_1^{(k)} = \tau^{-k-1}(\Gamma_2^{(k)}) \in \Gamma_1 \). Then the mapping \( \tau^k: \overline{\Gamma}_1^{(k-1)} \xrightarrow{\tau^k} \overline{\Gamma}_2^{(k-1)} \neq \emptyset \) also defines a Belavin-Drinfeld triple

\[
(\overline{\Gamma}_1^{(k-1)}, \overline{\Gamma}_2^{(k-1)}, \tau^k).
\]

4. **Modified Cartan-Weyl basis and normal order of roots.**

Let \( \Delta_+ \) be the system of all positive roots of \( g \) with respect to \( \Gamma \). A construction of Cartan-Weyl basis in terms of the modified generators \( E_i \) and \( F_i \) is analogous to the usual procedure for \( U_q(g) \) (see [5]).

Recall the notion of a normal (convex) order in \( \Delta_+ \): the set \( \Delta_+ \) is ordered normally if any root \( \gamma \) which is a sum of roots \( \alpha \) and \( \beta \) is placed between \( \alpha \) and \( \beta \).

We write \( \alpha < \beta \) if the root \( \alpha \) is located to the left of the root \( \beta \). For \( \alpha < \beta \), the interval between roots \( \alpha \) and \( \beta \) is denoted by \( \{\alpha, \beta\} \).

Given a normal order in \( \Delta_+ \), the modified Cartan-Weyl basis is constructed by the following inductive procedure. The generators for the simple
roots are already defined. For a composite root $\gamma$, take a minimal interval \{\alpha, \beta\}, $\alpha < \beta$, with $\gamma = \alpha + \beta$ ("minimal" means that there is no subinterval \{\tilde{\alpha}, \tilde{\beta}\} \subset \{\alpha, \beta\} for which $\gamma = \tilde{\alpha} + \tilde{\beta}$). Assume that generators $E_\alpha, E_\beta$, $F_\alpha$ and $F_\beta$ were defined at previous steps. Then generators $E_\gamma$ and $F_\gamma$ are defined by

$$E_\gamma = [E_\alpha, E_\beta]_\mu = E_\alpha E_\beta - \mu E_\beta E_\alpha,$$  \hspace{1cm} \text{(33)}

$$F_\gamma = [F_\alpha, F_\beta]_\nu = F_\alpha F_\beta - \nu F_\beta F_\alpha.$$  \hspace{1cm} \text{(34)}

where

$$\mu = q^{-\langle\alpha, \beta\rangle + \langle\alpha, A\beta\rangle}, \hspace{1cm} \nu = q^{-\langle\alpha, \beta\rangle - \langle\alpha, A\beta\rangle}$$

and $A$ is the operator with the matrix $A_{ij}$:

$$<\alpha_i, A\alpha_j> = A_{ij}.$$  

If there are several possible minimal intervals \{\alpha, \beta\} for which $\gamma = \alpha + \beta$, the definitions \text{(33)}-\text{(34)} give proportional results.

\textbf{Note.} For the case $A_{ij} = 0$ the definition \text{(33)}-\text{(34)} of composite roots does not coincide with the definition in [5] since we use the comultiplication [6] which is different from the comultiplication in [5].

\textbf{5. Twisting operators $F_{12}$ for Belavin-Drinfeld triples.}

For a given simple Lie algebra $\mathfrak{g}$ fix a normal order in $\Delta_+$. We need the expression for the inverse of the universal $R$- matrix for the algebra $U_q(\mathfrak{g})$:

$$R^{-1} = \prod_{\beta \in \Delta_+} \exp_{q_\beta} (-\lambda a_\beta (e_\beta \otimes f_\beta)) \cdot K^{(0)},$$  \hspace{1cm} \text{(35)}

where $q_\alpha = q^{-\langle\alpha, \alpha\rangle}$, $\lambda = q - q^{-1}$ and $K^{(0)} \in q^{\mathfrak{h} \otimes \mathfrak{h}}$. The product in eq. \text{(35)} is the ordered product corresponding to the chosen normal order of roots. For precise values of the constants $a_\beta$ see [6], [5]. The function $\exp_q$ is the standard $q$-exponent,

$$\exp_q(u) = \prod_{n=0}^{\infty} (1 + (q - 1)uq^n)^{-1} = \sum_{k=0}^{\infty} \frac{u^k}{k_q!}, \hspace{1cm} k_q = \frac{q^k - 1}{q - 1}.$$  \hspace{1cm} \text{(36)}
Let \((\Gamma_1, \Gamma_2, \tau)\) be a Belavin-Drinfeld triple of degree \(N\). Define elements \(F^{(k)}\) by
\[
F^{(k)} = \prod_{\beta \in \Delta_+^{(k)}} \exp_{q^\lambda} \left( -\lambda a_\beta (E_\beta \otimes F^{(k)}_{\tau(\beta)}) \right),
\]
where in the ordered product we keep terms corresponding to only those roots \(\beta\) for which \(\tau^{(k)}(\beta)\) is defined (that is, the element \(e_\beta\) belongs to the subalgebra with generators from the subset \(\tilde{\Gamma}_1^{(k)}\) defined in (32)). This is reflected in the notation \(\beta \in \Delta_+^{(k)}\).

The expression (37) can be given a form
\[
F^{(k)} = (1 \otimes T^k) \left( K R^{-1} (K^{(0)})^{-1} K^{-1} \right),
\]
where the operator \(T\) on the elements \(F^{(k)}\) is defined by \(T(F^{(k)}_\beta) = F^{(k)}_{\tau^{(k)}(\beta)}\) wherever \(\tau^{(k)}(\beta)\) is defined; \(T(F^{(k)}_\beta) = 0\) otherwise. The operator \(K\) corresponds to the solution of eqs. (19) for the given Belavin-Drinfeld triple.

**Theorem.** For the quantum algebra \(U_q(g)\) and the Belavin-Drinfeld triple \((\Gamma_1, \Gamma_2, \tau)\) of degree \(N\) the universal twisting element \(F_{12}\) is
\[
F_{12} = F^{(N)}_{12} \cdot F^{(N-1)}_{12} \cdot \cdots \cdot F^{(2)}_{12} \cdot F^{(1)}_{12} \cdot K \equiv \tilde{F}_{12} \cdot K
\]
with the factors \(F^{(k)}_{12}\) defined in (37).

We sketch the proof shortly. It is based on explicit formulas for the coproduct of elements \(F^{(k)}_{12}\):
\[
\Delta \otimes id F^{(k)}_{12} = F^{(k)}_{23} (K^{(k)}_{23})^{-1} F^{(k)}_{13} K^{(k)}_{23},
\]
\[
(id \otimes \Delta) F^{(k)}_{12} = F^{(k)}_{12} (K^{(k)}_{12})^{-1} F^{(k)}_{13} K^{(k)}_{12},
\]
for some elements \(K^{(k)} \in q^{h \otimes h}\). The comultiplication \(\tilde{\Delta}\) is twisted as in (39).

Next, one can verify the following identities
\[
\tilde{F}_{23}^{(k)} \tilde{F}_{13}^{(k+m)} = \tilde{F}_{12}^{(m)} \tilde{F}_{13}^{(k+m)} \tilde{F}_{23}^{(k)},
\]
where \(\tilde{F}^{(k)} = F^{(k)} \cdot (K^{(k)})^{-1}\).

With the help of (39) - (42) it is straightforward to check the cocycle condition (3).
**Remark 1.** Another expression for the twisting element $F$ was suggested in [2]. The expression in [2] has a factorised form as well. However, the factors $F^{(i)}$ are different; one of the differences is that each factor in [2] contains terms from $q^\mathfrak{h} \otimes \mathfrak{h}$. In our expression (39) all terms from $q^\mathfrak{h} \otimes \mathfrak{h}$ are collected; the price is the appearance of the modified basis.

**Remark 2.** The element $F$ in (39) satisfies the following analogue of the linear ABRR equation [7]:

\[
(1 \otimes T)(F_{12} R^{-1}(K^{(0)})^{-1} K^{-1}) = F_{12} K^{-1}.
\] (43)

6. Examples.

**i) $U_q(sl(3))$ case** (see [4]).
Here we have only one nontrivial Belavin-Drinfeld triple:

![Fig.1](image1.png)

This Cremmer-Gervais type triple has degree 1 and the basic relations (19) which define this triple are reduced to one equation $R_1^+ = R_2^−$. The antisymmetric matrix $A_{ij}$ is

\[
A_{ij} = \delta_{i,j+1} - \delta_{j,i+1},
\] (44)

with $1 \leq i, j \leq 2$. The corresponding universal twisting element (39) has the form

\[
F_{12} = F_{12}^{(1)} \cdot K = \exp_{q^2}(-\lambda E_1 \otimes F_2) \cdot K.
\] (45)

**ii) Cremmer-Gervais $U_q(sl(4))$ case.**
For this case the triple is given by the following diagram

![Fig.2](image2.png)
It has degree 2. The basic relations (19) which define this triple are
\[ R^+_1 = R^-_2, \quad R^+_2 = R^-_3. \]
The matrix \( A_{ij} \) is given by (44), now with \( 1 \leq i, j \leq 2 \). The corresponding universal twisting element (39) has the form
\[ F_{12} = F^{(2)}_{12} \cdot F^{(1)}_{12} \cdot K, \]
where
\[ F^{(2)}_{12} = \exp_q(-\lambda E_1 \otimes F_3), \quad F^{(1)}_{12} = \exp_q(-\lambda E_1 \otimes F_2) \\exp_q(q^{-1} \lambda [E_{12}] \otimes [F_{23}]_{q^2}) \\exp_q(-\lambda E_2 \otimes F_3). \] (48)

Here \([E_{12}] = E_1 E_2 - E_2 E_1\) and \([F_{23}]_{q^2} = F_2 F_3 - q^2 F_3 F_2\).

**Remark.** One can directly check that (45), (46) obeys the cocycle conditions (4). For (45) this check requires only the basic equation for the \( q \)-exponent,
\[ \exp_q(y) \exp_q(x) = \exp_q(x + y) \] if \( x y = q y x \). (49)

For (46) one needs two more quantum identities. The first one is the famous pentagon identity (see e.g. [8] and references therein)
\[ \exp_q(u) \exp_q(v) = \exp_q(v) \exp_q([u, v]) \exp_q(u), \] (50)

where the operators \( u \) and \( v \) satisfy the commutation (Serre) relations
\[ u [u, v] = q [u, v] u, \quad v [u, v] = q^{-1} [u, v] v. \]

The second identity is
\[ \exp_q(E) \exp_q(-R^+) \exp_q(F) = \exp_q(F) \exp_q(-R^-) \exp_q(E), \] (51)

where \( E, F \) and \( R^\pm \) generate the algebra
\[ [E, F] = (R^+ - R^-), \quad [R^+, R^-] = 0, \]
\[ R^\pm E = q^{\pm 2} E R^\pm, \quad R^\pm F = q^{\mp 2} F R^\pm. \]

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