Computing curves on real rational surfaces

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Abstract

We present an algorithm for computing curves and families of curves of prescribed degree and geometric genus on real rational surfaces.

Keywords: rational surfaces, families of curves, linear series

MSC2010: 14Q10, 14D99, 68W30

1 Introduction

Suppose we are given a birational map \( \mathcal{H} : \mathbb{P}^2 \rightarrow X \subset \mathbb{P}^n \). Algorithm 2 computes all families of curves on \( X \) of degree \( \alpha \), projective dimension \( \nu - 1 \) and geometric genus \( \rho \). If \( \nu = 1 \), then each family consists of a single curve. We use the basepoint analysis method [7, Algorithm 1 and 2] to reduce the problem of computing curves in a surface to the problem of computing elements in a unimodular lattice. In order to clarify the input/output specification of Algorithm 2 we list some classical usecase examples that we can compute:

- Both 1-dimensional family of lines on a hyperboloid of one sheet. These families were discovered by [14, Christopher Wren, 1669].

1
• The 27 real and/or complex conjugated lines in a smooth cubic surface. Cayley-Salmon theorem states the existence of these lines [2, 1848].

• Four 1-dimensional families of circles on a ring torus. Two of these families are families of Villarceau circles [13, 1848].

• The 2-dimensional family of conics on the Roman surface. This surface was discovered by [12, Steiner, 1844] in Rome and is a projection of the Veronese surface (Figure 1).

• The four circles contained in the Roman surface (Figure 1).

See [5, linear.series] and [6, ns.lattice] for an implementation of algorithms used in this paper.

See Figure 1. A web of conics and the four circles on the Roman surface.

In [9, 10] algorithms are proposed to compute families of lines on $X$, in case $X$ is geometrically ruled. Such families are represented in terms of reparametrizations of $\mathcal{H}$ and it is shown that basepoints of $\mathcal{H}$ can be moved to the line at infinity. In [1] an algorithm is proposed to compute straight lines in a rational surface, using methods from differential geometry. The more general algorithm proposed in his article, has the advantage we can also compute curves that are not reached by the parametrization $\mathcal{H}$ [7, Section 4.1]. However, as the parametric degree of $\mathcal{H} : \mathbb{P}^2 \rightarrow X \subset \mathbb{P}^n$ and embedding dimension $n$ is increasing, our algorithm is less likely to terminate. The bottleneck is the basepoint analysis step [7, Remark 1].
likely that there exists rational surfaces where the methods of [1, 9, 10] are favourable, when computing straight lines contained in these surfaces.

We explain the structure of this paper by summarizing the main steps of Algorithm 2. After introducing basic notions in §2, we analyze in §3 the basepoints of the linear series corresponding to the map $\mathcal{H}$. From these basepoints we recover generators for the Neron-Severi lattice $\mathbb{N}(X)$. In §4 and §5 we compute with Algorithm 1 a set of candidate divisor classes $\mathcal{A} \subset \mathbb{N}(X)$ using input $\alpha$ and $\rho$. For each class $c \in \mathcal{A}$ we compute a linear series $|c|$ in the plane. The linear series $|c|$ is valid if it is of the requested projective dimension $\nu - 1$ and if the general curve in the linear series is irreducible. The curve $\mathcal{H}(C) \subset X$ is of given degree and geometric genus for all curves $C$ in a valid linear series $|c|$. In §6 we explain Algorithm 2 with an example. In §7 we compute circles in rational surfaces, by composing the birational map $\mathcal{H}$ with an inverse stereographic projection.

2 Preliminaries

2.1 Real varieties

We define a real variety $X$ as a complex variety together with an antiholomorphic involution $\sigma: X \to X$, called the real structure. We implicitly assume that all structures are compatible with the real structure. For example, the birational map $\mathcal{P}: \mathbb{P}^2 \dashrightarrow X$ is real unless explicitly stated otherwise. The smooth model of a surface $X$ is a birational morphism $Y \to X$ from a nonsingular surface $Y$, that does not contract exceptional curves.
2.2 Neron-Severi lattice

For computational purposes, we make the data associated the well-known Neron-Severi lattice explicit [3, page 461]. The Neron-Severi lattice $N(X)$ of a rational surface $X \subset \mathbb{P}^n$ consists of the following data:

1. A unimodular lattice defined by divisor classes on the smooth model $Y$ of $X$, modulo numerical equivalence.

2. A basis for the lattice. We say that a basis is of type 1 if the generators are $\langle e_0, e_1, \ldots, e_r \rangle$ such that the nonzero intersections are $e_0^2 = 1$ and $e_i^2 = -1$ for $1 \leq i \leq r$. We assume a basis of type 1 unless explicitly stated otherwise.

3. A unimodular involution $\sigma_+: N(X) \rightarrow N(X)$ induced by the real structure of $X$.

4. A function $h^0: N(X) \rightarrow \mathbb{Z}_{\geq 0}$ assigning the dimension of global sections of the line bundle associated to a class.

5. Two distinguished elements $h, k \in N(X)$ corresponding to class of a hyperplane sections and the canonical class respectively.

2.3 Complete families

We assume that $X \subset \mathbb{P}^n$ is a rational surface such that linear-, algebraic- and numerical- equivalence of divisors on $X$ coincide.

A family is defined as a divisor $F \subset X \times \mathbb{P}^m$ such that the second projection $\pi_2: F \rightarrow \mathbb{P}^m$ is dominant. We say that $F$ covers $X$ if the first projection $\pi_1: F \rightarrow X$ is dominant as well. A member of $F$, corresponding to $i \in \mathbb{P}^m$, is defined as the curve $F_i := (\pi_1 \circ \pi_2^{-1})(i) \subset X$.

We can associate to a curve $C \subset X$ its class $[C] \in N(X)$ such that classes $[C]$ and $[C']$ are equal if and only if $C$ and $C'$ are members of the same family. The class $[F]$ of $F$ is defined as the class of any of its members.
We call a family $F$ complete if there exists a curve $C \subset X$ such that the set \{ $C' \subset X \mid [C'] = [C]$ \} defines exactly the set of members of $F$. A family is called irreducible if its general member is an irreducible curve. The dimension of $F$ is defined as $m$. A 0-dimensional family consists of a single curve. If $F$ is complete then $h^0([F]) = m + 1$. The arithmetic genus $p_a([F])$ of a complete irreducible family $F$ is defined as the geometric genus of its general member. The degree of $F$ is defined as the degree of any member with respect to the embedding $X \subset \mathbb{P}^n$.

### 3 Computing the Neron-Severi lattice

In this section we explain in terms of an example, how to compute the Neron-Severi lattice of a rational surface.

We consider the following birational map:

\[
\begin{align*}
\mathbb{P}^2 & \xrightarrow{H} X \subset \mathbb{P}^4 \\
(x_0 : x_1 : x_2) & \mapsto (x_1^3 + 2x_2^2x_0 - x_1x_0^2 - 2x_2x_0^2 : x_1^2x_2 : x_1x_2 - 2x_2x_0 + 2x_2x_0^2 : x_2 - 2x_2x_0 + x_2x_0^2).
\end{align*}
\]

Let $|h|$ be the linear series associated to the birational map $H$ so that $h$ is the class of hyperplane sections of $X$. With [7, Algorithm 1] we find the basepoints of $H$ in the affine chart $U_0 \subset \mathbb{P}^2$ defined by $x_0 \neq 0$:

\[
\Gamma = \{ \langle ((), p_1, 1) \rangle, \langle ((), p_2, 1) \rangle, \langle ((), p_3, 1) \rangle, \langle ((), p_4, 1) \rangle, \langle (p_5, t), p_2 \rangle \}
\]

where $p_1 = (-1, 0)$, $p_2 = (0, 0)$, $p_3 = (1, 0)$, $p_4 = (0, 1)$ and $p_5 = (2, 0)$. There are no basepoints outside $U_0$. Notice that basepoints $p_1$, $p_2$ and $p_3$ are collinear simple basepoints corresponding to $(-1 : 0 : 1)$, $(0 : 0 : 1)$ and $(1 : 0 : 1)$ in $\mathbb{P}^2$ respectively. Basepoint $p_4$ has projective coordinates
$(0 : 1 : 1)$ and basepoint $p_5$ is infinitely near to $p_4$.

The Neron-Severi lattice of $X$ has a basis of type 1:

$$N(X) = \langle e_0, e_1, e_2, e_3, e_4, e_5 \rangle,$$

The induced real structure $\sigma_* : N(X) \to N(X)$ is the identity map, since all basepoints are real. Moreover,

$$h = -k = 3e_0 - e_1 - e_2 - e_3 - e_4 - e_5,$$

since $h$ consists of cubic polynomials that pass with multiplicity one through each of the five basepoints. The smooth model $Z$ of $X$ is isomorphic to the projective plane blownup in the base locus of $\mathcal{H}$:

\[
\begin{array}{c}
\tau_1 \\
\mathbb{P}^2 \\
\tau_2
\end{array}
\]

The generator $e_0$ is the class of the pullback along $\tau_1$, of lines in the plane and $e_i$ for $1 \leq i \leq 5$ is the class of the pullback of the exceptional curve resulting from the blowup of basepoint $p_i$. The surface $Z$ is in general not a smooth model of $X$, since $\tau_2$ may contract exceptional curves. See [7, Section 4.3] for more details on computing the Neron-Severi lattice of a rational surface.

4 Computing families from their classes

Suppose that $\mathcal{H} : \mathbb{P}^2 \to X$ is given by (1) and that $F \subset X \times \mathbb{P}^m$ is a complete family. In this section we explain in terms of examples how to compute from the class $[F] \in N(X)$ the following attributes as defined in §2.3 (see also [7, Section 4.4]):
• A complete linear series $L$ so that $\mathcal{H}(C) \subset X$ is a member of $F$ for all curves $C$ in $L$. By abuse of notation we shall denote $[\mathcal{H}(C)] = [F]$ in $N(X)$ by $[L]$.

• Dimension $h^0([F]) - 1$, arithmetic genus $p_a([F])$, and degree $\deg F$.

• Reducibility of $F$.

We computed in §3 the Neron-Severi lattice $N(X)$ and we know that

$$[F] = c = c_0e_0 + c_1e_1 + \ldots + c_5e_5 \in N(X),$$

for some $c_i \in \mathbb{Z}$ for $0 \leq i \leq 5$. Now there are two possibilities:

1. $c_0 > 0$ and $c_i \leq 0$ for $i > 0$. The members of the family $F$ are in this case, via $\tau_2 \circ \tau_1^*$, strict transforms of curves in $\mathbb{P}^2$ of degree $c_0$ that pass through basepoints $p_i$ with multiplicity $-c_i$.

2. $[F] = e_i$ for some $i > 0$. In this case $F$ is a 0-dimensional family whose unique member is $\tau_2(E) \subset X$, where $E \subset Z$ is an exceptional curve such that $\tau_1(E) = p_i \in \mathbb{P}^2$.

We remark that if $p_i$ is infinitely near to $p_j$ then there is exists a curve in $Z$ with class $e_i - e_j$. This (-2)-curve is contracted to an isolated singularity by $\tau_2: Z \rightarrow X$.

We will now consider several scenarios, where we determine attributes of $F$ when only its class $[F] \in N(X)$ is given.

• Suppose that $[F] = e_0 - e_4 - e_5$. Recall that [7, Algorithm 2] determines which curves in a given linear series, form a linear series with prescribed basepoints. The following linear series is defined by a monomial basis for linear polynomials in $\mathbb{R}[u,v]$: $G := (u,v,1)$. The line in $G$ that passes through basepoints $p_4$ and $p_5$ is after homogenization $L_{45} := \{ x_0 - 2x_1 + 2x_2 = 0 \}$ such that $u = \frac{x_1}{x_0}$ and $v = \frac{x_2}{x_0}$. Thus $[F] = [L_{45}]$ and the image $\mathcal{H}(L_{45}) \subset X$ is the unique member
of $F$. The linear series for $L_{45}$ consists of a single curve so that $F$ is of dimension $h^0([L_{45}]) - 1 = 0$. We verify from the arithmetic genus formula that $2p_a([L_{45}]) - 2 = [L_{45}]^2 + [L_{45}] \cdot k = -2$. The degree of the image $\mathcal{H}(L_{45}) \subset \mathbb{P}^4$ equals $h \cdot [L_{45}] = 1$, thus the unique member of $F$ is a line in $X$.

- Suppose that $[F] = e_0 - e_4$. We compute with [7, Algorithm 2] the linear subseries of lines in $G$ that pass through $p_4$. After projectivization this linear series is defined by the following tuple of linear forms $L_4 := (x_1, x_2 - x_0)$ so that $h^0([L_4]) = 2$ and $[F] = [L_4]$. Let $C_\alpha := \{ x \in \mathbb{P}^2 \mid \alpha_0 x_1 + \alpha_1 (x_2 - x_0) = 0 \}$ denote a line in the linear series of $L_4$ for all $\alpha = (\alpha_0 : \alpha_1) \in \mathbb{P}^1$. Since $h \cdot [L_4] = 2$ it follows that the image $\mathcal{H}(C_\alpha) \subset X$ is a conic for all $\alpha \in \mathbb{P}^1$. Indeed, $F$ defines a 1-dimensional complete family of conics on $X$. We consider the following parametrization of $C_\alpha$:

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\mathcal{G}} \mathbb{P}^2 \quad (\alpha_0 : \alpha_1 : t_0 : t_1) \mapsto (\alpha_0 t_0 : -\alpha_1 t_1 : \alpha_0 t_1 + \alpha_0 t_0).$$

Notice that $\mathcal{H} \circ \mathcal{G} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ is a reparametrization of $X$ such that if we fix $\alpha \in \mathbb{P}^1$ then we obtain a parametrization of the conic $\mathcal{H}(C_\alpha) \subset X$.

- Suppose that $[F] = e_0 - e_1 - e_2 - e_3$. The curve in $G$, through basepoints $p_1, p_2$ and $p_3$, is after projectivization the line $L_{123} := \{ x \in \mathbb{P}^2 \mid x_2 = 0 \}$ so that $h^0([L_{123}]) = 1$. However, $h \cdot [L_{123}] = 0$ so that $L_{123}$ is contracted by $\mathcal{H}$ to an isolated singularity. In particular, $F$ does not define a family on $X$. Since $[L_4] : [L_{123}] = 1$ it follows that each member of the family of conics with class $[L_4]$, passes through this isolated singularity.

- Suppose that $[F] = e_0 - e_1 - e_5$. A necessary condition for $c \in N(X)$ being the class of a line in $X$, is that $h \cdot c = 1$. However, this condition
is not sufficient. In this case we have \([F] = (e_0 - e_1 - e_4) + (e_4 - e_5)\) so that \(h \cdot [F] = 1\), but \([F]\) cannot be the class of a line through \(p_1\) and an infinitely near point \(p_5\), since such a line also passes through \(p_4\). We can detect in \(N(X)\) that \([F]\) is not the class of a line, since \([F] \cdot (e_4 - e_5) < 0\) and \(h^0(e_4 - e_5) > 0\) so that \(e_4 - e_5\) is the class of a (-2)-curve. In particular, we find that \(F\) is not irreducible. Notice that the curve with class \(e_4 - e_5\) in the smooth model \(Z\) is contracted by \(\tau_2\) to an isolated singularity of \(X\). The line \(\mathcal{H}(L_{14}) \subset X\) with class \(e_0 - e_1 - e_4\) passes through this singular point, since \((e_0 - e_1 - e_4) \cdot (e_4 - e_5) = 1\).

- Suppose that \([F] = 2e_0 - e_1 - e_2 - e_3 - e_4 - e_5\). In this case, \([F]\) is the class of the pullback of a conic \(C \subset \mathbb{P}^2\) along \(\tau_1 : Z \to \mathbb{P}^2\). The conic \(C\) passes through \(p_1, p_2, p_3\) and \(p_4\) such that its tangent direction at \(p_4\) is determined by the infinitely near point \(p_5\). We notice that \([F] \cdot (h - e_1 - e_2 - e_3) < 0\) and therefore the line \(L_{123}\) through \(p_1, p_2\) and \(p_3\) must be a component of \(C\). Indeed, \([F] = [L_{45}] + [L_{123}]\). We compute with [7, Algorithm 2] the curves in the linear series of all conics that pass through \(p_1, \ldots, p_5\) and verify that there is a unique reducible conic with equation \(x_2(x_0 - 2x_1 + 2x_2)\). Thus \(F\) consists of a reducible conic and is therefore a reducible family.

- Suppose that \([F] = 2e_0 - e_1 - e_2 - e_3 - e_4\). In this example, \(h \cdot [F] = 2\). We compute with [7, Algorithm 2] the curves in the linear series of all conics that pass through \(p_1, \ldots, p_4\) and obtain after projectivization the reducible linear series \((x_2x_1, x_2(x_2 - x_0))\). We notice that \([F] \cdot [L_{123}] < 0\) and thus \([L_{123}]\) is indeed a fixed component of the linear series of conics passing through the simple basepoints \(p_1, p_2, p_3\) and \(p_4\). We conclude that \(F\) is a reducible family of conics.
5 Algorithms

We introduce in this section algorithms for computing complete irreducible families on rational surfaces of prescribed degree, dimension and arithmetic genus.

Algorithm 1. (find elements in Neron-Severi lattice)

- **Input:**
  1. The class of hyperplane sections $h = h_0e_0 + \ldots + h_re_r$ in a Neron-Severi lattice $N(X)$ with respect to type 1 basis $\langle e_0, \ldots, e_r \rangle$.
  2. Integers $\alpha, \beta \in \mathbb{Z}_{\geq 0}$.

- **Output:**
  The set of classes $c \in N(X)$ such that $h \cdot c = \alpha$, $c^2 = \beta$ and either one of the following three conditions is satisfied:
  1. $c = e_i - e_j$ for some $0 < i, j \leq r$.
  2. $c = e_i$ for some $i > 0$.
  3. $c \cdot e_i > 0$ for all $0 \leq i \leq r$.

- **Method:** We use notation $c = c_0e_0 + \ldots + c_re_r$.

  $S := \{ c \in N(X) \mid c \in \{ e_i - e_j, e_i \mid 0 < i, j \leq r \}, (h \cdot c, c^2) = (\alpha, \beta) \}$

  $f(t) := (h_0t - \alpha)^2 - (h_0^2 - h^2)(t^2 - \beta)$

  $c_0 := 1$

  while $f(c_0) \leq 0$ or $f(c_0) \leq f(c_0 - 1)$ do

  Compute all $(c_i)_{i>0} \in \mathbb{Z}_{\geq 0}^r$ such that $h_0c_0 - \alpha = \sum_{i>0} h_i c_i$ by going through integer partitions of $h_0c_0 - \alpha$.

  $S := S \cup \{ c \mid h_0c_0 - \alpha = \sum_{i>0} h_i c_i, c^2 = \beta \}$

  $c_0 := c_0 + 1$

  return $S$
Algorithm 2. (complete families on rational surfaces)

- **Input:** A birational map $\mathcal{H}: \mathbb{P}^2 \dashrightarrow X \subset \mathbb{P}^n$ and integers $\alpha, \nu, \rho \in \mathbb{Z}_{\geq 0}$.
- **Output:** Classes of complete irreducible families $F \subset X \times \mathbb{P}^{\nu-1}$ with prescribed degree $\alpha$, dimension $\nu - 1$ and geometric genus $\rho$:

$$\{ [F] \in N(X) \mid \deg F = \alpha, h^0([F]) = \nu, p_a([F]) = \rho, F \text{ is irreducible} \}.$$

- **Method:**

1. Compute $N(X)$ with type 1 basis $\langle e_0, \ldots, e_r \rangle$ by computing the base locus of $\mathcal{H}$ [7, Algorithm 1 and Section 4.3] such that $h \in N(X)$ is the class of the linear series associated to $\mathcal{H}$.

2. We call Algorithm 1 with input $h, \alpha$ and $\beta$, where $\beta$ runs from $-2$ to $\nu + \rho - 2$. Let $\mathcal{A}$ be the union of the outputs for each $\beta$. Each element in $\mathcal{A}$ is a vector with respect to the type 1 basis of $N(X)$.

3. We start with an empty set $\mathcal{B}$. With [7, Algorithm 2] we compute for each class in $\mathcal{A}$ its linear series $L$. If $h^0([L]) = \nu$ and $L$ is irreducible without unassigned basepoints in $\mathcal{H}$, then we set $\mathcal{B} := \mathcal{B} \cup \{[L]\}$.

return $\mathcal{B}$ \hspace{1cm} \triangleright

Notice that we discussed step 3 of Algorithm 2 in §4. We will see in Example 1 a scenario where the linear series has unassigned basepoints in $\mathcal{H}$.

**Theorem 1. (complete families on rational surfaces algorithm)**

*Algorithm 1 and Algorithm 2 are correct.*

**Proof.** The halting of Algorithm 1 follows from the Cauchy-Schwarz inequal-
(h_0c_0 - \alpha)^2 = \left(\sum_{i>0} h_ic_i \right)^2 \leq \left(\sum_{i>0} h_i^2 \right) \left(\sum_{i>0} c_i^2 \right) = (h_0^2 - h^2)(c_0^2 - \beta).

Thus we require that \( f(c_0) \leq 0 \). Notice that if \( f(c_0) > f(c_0 - 1) \) and \( f(c_0) > 0 \), then \( f(t) > 0 \) for all \( t \geq c_0 \). The correctness of Algorithm 1 is now straightforward.

For the correctness of Algorithm 2 we recall that \( \alpha = \deg F = h \cdot [F] \). The arithmetic genus formula states that \(-2 \leq c^2 + c \cdot k = 2p_a(c) - 2 \). It follows from Riemann-Roch theorem and Serre duality that \(-2 \leq c^2 - c \cdot k \leq 2h^0(c) - 2 \). The sum of these equations implies that

\[-2 \leq c^2 \leq h^0(c) + p_a(c) - 2.\]

Thus \( c^2 = \beta \in [-2, \nu + \rho - 2] \) and Algorithm 1 outputs potential classes of complete families. The final step of Algorithm 2 consists of verifying with [7, Algorithm 2], which of these classes form complete irreducible families. \( \square \)

6 Computing lines and conics on a surface

In §4 we computed some lines and a family of conics on \( X \) from a given birational map \( \mathcal{H}: \mathbb{P}^2 \dashrightarrow X \) as defined at (1). In this section we would like to compute all lines and families of conics of \( X \).

For computing all lines contained in \( X \), we call Algorithm 2 with input \( \mathcal{H}, \alpha = 1, \nu = 1 \) and \( \rho = 0 \). In the first step of the algorithm we compute \( N(X) \) as discussed in §3. Since the real structure \( \sigma: X \dashrightarrow X \) acts as the identity on \( N(X) \), all lines will be real. In the second step we call Algorithm 1 with
\[ \alpha = 1, \beta \in [-2, -1] \] so that the union of outputs is

\[ \mathcal{A} = \{ e_i, \ e_0 - e_i - e_j, \ 2e_0 - e_1 - \ldots - e_5 \mid 0 < i < j \leq 5 \}. \]

In the third step we verify which of these classes correspond to complete irreducible families (see §4). Thus we verify whether the general curve in the linear series of a class is irreducible. The output of Algorithm 2 is:

\[ \mathcal{B} = \{ e_1, \ e_2, \ e_3, \ e_4, \ e_0 - e_1 - e_4, \ e_0 - e_2 - e_4, \ e_0 - e_3 - e_4, \ e_0 - e_4 - e_5 \}. \]

It follows that \( X \) contains 8 lines. A line with class \( e_i \) for \( i > 0 \) is not reachable by the birational map \( \mathcal{H} \). See [7, Section 4.1] for parametrizing such unreachable curves. For the remaining classes we can compute the linear series in \( \mathbb{P}^2 \) so that the images via \( \mathcal{H} \) of curves in this linear series are lines in \( X \).

For computing all conics in \( X \) we call Algorithm 2 with \( \mathcal{H} \), \( \alpha = 2 \), \( \nu = 2 \) and \( \rho = 0 \). In this case \( \mathcal{A} = \{ e_0 - e_i, 2e_0 - e_i - e_j - e_k \mid 0 < i < j < k \leq 5 \} \) and the output of Algorithm 2 is

\[ \mathcal{B} = \{ e_0 - e_1, \ e_0 - e_2, \ e_0 - e_3, \ e_0 - e_4, \ 2e_0 - e_1 - e_2 - e_4 - e_5, \]
\[ 2e_0 - e_1 - e_3 - e_4 - e_5, \ 2e_0 - e_2 - e_3 - e_4 - e_5 \}. \]

In §4 we parametrized conics with class \( e_0 - e_4 \). The remaining families of conics can similarly be represented by a parametrization.

### 7 Computing circles on surfaces

In order to compute circles on rational surfaces we consider the Möbius sphere

\[ \mathbb{S}^n := \{ x \in \mathbb{P}^{n+1} \mid -x_0^2 + x_1^2 + \ldots + x_{n+1}^2 = 0 \}. \]
A circle in \( S^n \) is defined as a real irreducible conic. The Möbius sphere is topologically the projective closure of the one-point-compactification \( S^n \) of \( \mathbb{R}^n \), such that circles and lines in \( \mathbb{R}^n \) correspond to circles in \( S^n \). The stereographic projection with center \((1 : 0 : \ldots : 0 : 1) \in S^n\) is defined as

\[
S^n \xrightarrow{\mathcal{P}} \mathbb{P}^n
\]

\[
(x_0 : \ldots : x_{n+1}) \mapsto (x_0 - x_{n+1} : x_{n+1} : \ldots : x_n),
\]

with inverse

\[
\mathbb{P}^n \xrightarrow{\mathcal{P}^{-1}} S^n
\]

\[
(y_0 : \ldots : y_n) \mapsto (y_0^2 + \Delta : 2y_0y_1 : \ldots : 2y_0y_n : -y_0^2 + \Delta),
\]

where \( \Delta := y_1^2 + \ldots + y_n^2 \). We can recover \( \mathbb{R}^n \) from \( S^n \) as the affine chart \( \mathbb{R}^n \cong \{ x \in \mathbb{P}^n = \overline{\mathcal{P}(S^n)} \mid x_0 \neq 0 \} \). Notice that circles in \( S^n \) that pass through the center of stereographic projection correspond to lines in \( \mathbb{P}^n \).

**Example 1. (Roman surface)**

We consider the Roman surface (see Figure 1) with parametrization

\[
\mathbb{P}^2 \xrightarrow{\mathcal{R}} X \subset \mathbb{P}^3
\]

\[
(x_0 : x_1 : x_2) \mapsto (x_0^2 + x_1^2 + x_2^2 : -x_0x_1 : -x_1x_2 : x_0x_1).
\]

The inverse stereographic projection of the roman surface \( X \subset \mathbb{P}^3 \) into \( S^3 \) has parametrization

\[
\mathbb{P}^2 \xrightarrow{\mathcal{R}^{-1}} Y \subset S^3
\]

\[
(x_0 : x_1 : x_2) \mapsto \left( x_1^4 + 3x_1^2x_2^2 + x_2^4 + 3x_1^2x_0^2 + 3x_2^2x_0^2 + x_0^4 : \\
-2x_0x_1(x_1^2 + x_2^2 + x_0^2) : -2x_2x_1(x_1^2 + x_2^2 + x_0^2) : \\
2x_0x_2(x_1^2 + x_2^2 + x_0^2) : \\
-(x_1^4 + x_1^2x_2^2 + x_2^4 + 3x_1^2x_0^2 + 2x_2^2x_0^2 + x_0^4) \right).
\]
We expect a finite number of circles on the Roman surface $X \subset \mathbb{P}^3$ and thus we assume that $\nu - 1 = 0$. We call Algorithm 2 with $\mathcal{P}^{-1} \circ \mathcal{R}$ and $(\alpha, \nu, \rho) = (2, 1, 0)$.

In step 1 of Algorithm 2 we perform with [7, Algorithm 1] a basepoint analysis on the map $\mathcal{P}^{-1} \circ \mathcal{R}$ and find the following simple basepoints in $\mathbb{P}^2$ where $\eta^2 - \eta + 1 = 0$:

\[
\begin{align*}
p_1 &:= (1 : \eta - 1 : -\eta), & p_2 &:= (1 : -\eta : \eta - 1), \\
p_3 &:= (1 : -\eta + 1 : \eta), & p_4 &:= (1 : \eta : -\eta + 1), \\
p_5 &:= (1 : \eta - 1 : \eta), & p_6 &:= (1 : -\eta : -\eta + 1), \\
p_7 &:= (1 : -\eta + 1 : -\eta), & p_8 &:= (1 : \eta : \eta - 1).
\end{align*}
\]

Notice that $p_i$ is complex conjugate to $p_{i+1}$ for $i \in \{1, 3, 5, 7\}$. We denote the exceptional divisor, that results from blowing up the projective plane with center $p_i$, by $e_i$ for $1 \leq i \leq 8$. The Neron-Severi lattice of $Y \subset \mathbb{S}^3$ is generated by $N(Y) = \langle e_0, e_1, \ldots, e_8 \rangle$ with $\sigma^*(e_0) = e_0$ and $\sigma^*(e_i) = e_{i+1}$ for $1 \leq i \leq 7$. The class of hyperplane sections of $Y$ equals $h = 4e_0 - e_1 - \ldots - e_8$.

In step 2 of Algorithm 2 we call Algorithm 1 with $h \in N(Y)$ and $(\alpha, \beta) = (2, -2)$. The output is $\{ h - 2e_0 + e_i + e_j \mid 1 \leq i < j \leq 8 \}$. Next we call Algorithm 1 with $h \in N(Y)$ and $(\alpha, \beta) = (2, -1)$. In this case, the output is $\{ e_0 - e_i - e_j \mid 1 \leq i < j \leq 8 \}$.

In step 3 of Algorithm 2 we compute the linear series of $2e_0 - e_1 - \ldots - e_6$. This linear series consists of a single curve $C := \{ x \in \mathbb{P}^2 \mid x_0^2 + x_1^2 + x_2^2 = 0 \}$. However, we detect that $p_i \in C$ for $1 \leq i \leq 8$ and thus the linear series has unassigned basepoints $p_7$ and $p_8$. Therefore the class of the linear series is $2e_0 - e_1 - \ldots - e_8$, which is not the class of a circle. Since we want to compute the real circles we require that the classes are fixed under the involution. It follows that the remaining candidate classes for circles are $\{ e_0 - e_i - e_{i+1} \mid i \in \{1, 3, 5, 7\} \}$. The linear series $L_{12}$ with class $e_0 - e_1 - e_2$ is generated by the tuple $(x_0 + x_1 + x_2)$ so that the the parametrization of
\{ x \in \mathbb{P}^2 \mid x_0 + x_1 + x_2 = 0 \} \text{ composed with the map } \mathcal{P}: \mathbb{P}^2 \to X, \text{ is the parametrization of a circle in } X. \text{ Similarly, we obtain for the remaining classes the linear series } L_{34} = (x_0 - x_1 - x_2), \ L_{56} = (x_0 + x_1 - x_2) \text{ and } L_{78} = (x_0 - x_1 + x_2). \quad \triangleright

**Remark 1. (circles versus lines)**

Suppose that \( \mathcal{H}: \mathbb{P}^2 \to X \subset \mathbb{P}^4 \) is as defined at (1). In order to compute circles in \( X \) we compute with Algorithm 2 complete irreducible families of conics in the surface with parametrization \( \mathcal{P}^{-1} \circ \mathcal{H}: \mathbb{P}^2 \to Y \subset S^4 \). We expect at least 8 circles in \( Y \), which are the image of \( \mathcal{P}^{-1} \) of the 8 real lines in \( X \) we computed in §6. The circles in \( Y \), that do not pass through the center \((1 : 0 : 0 : 0 : 0 : 1)\) of the stereographic projection \( \mathcal{P}: S^4 \to \mathbb{P}^4 \), correspond to circles in \( X \). \quad \triangleright

**Remark 2. (alternative approach)**

In case [7, Algorithm 1] does not terminate there is an alternative method to compute circles. Suppose that \( \mathcal{H}: \mathbb{P}^2 \to X \subset \mathbb{P}^4 \) is as defined at (1). We consider the finite set \( B := \mathcal{H}^{-1}(X \cap A) \subset \mathbb{P}^2 \) where \( A := \{ x \in \mathbb{P}^4 \mid x_0 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \} \) is the *Euclidean absolute*. The real irreducible conics in \( \mathbb{P}^4 \) that meet \( A \) in two complex conjugate points correspond to circles. Thus if \( C \subset X \) is a real irreducible conic and \( \# \mathcal{H}^{-1}(C) \cap B = 2 \) then \( C \) is a circle. For example, we compute with [7, Algorithm 2] the curves in the linear series \( L_4 \) of §4 that pass through two given complex conjugate points of \( B \). \quad \triangleright

## 8 Acknowledgements

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Bibliography

[1] J.G. Alcázar and J. Caravantes. On the computation of the straight lines contained in a rational surface. *arXiv:1603.03959*, 2018.

[2] A. Cayley. On the triple tangent planes of surfaces of the third order. *Cambridge and Dublin Math. J.*, 4:118138, 1848.

[3] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley-Interscience, New York, 1978.

[4] S. Holzer and O. Labs. *surfex 0.90*. Technical report, University of Mainz, University of Saarbrücken, 2008. [www.surfex.AlgebraicSurface.net](http://www.surfex.AlgebraicSurface.net).

[5] N. Lubbes. Sage library for basepoint analysis of linear series, 2017. [http://github.com/niels-lubbes/linear_series](http://github.com/niels-lubbes/linear_series).

[6] N. Lubbes. Sage library for computations in Neron-Severi lattice, 2017. [http://github.com/niels-lubbes/ns_lattice](http://github.com/niels-lubbes/ns_lattice).

[7] N. Lubbes. Computing basepoints of linear series in the plane. *arXiv:1805.03452*, 2018.

[8] *Povray*. Persistence of Vision Pty. Ltd., 2004. [http://www povray.org/download/](http://www.povray.org/download/).

[9] J.R. Sendra, D. Sevilla, and C. Villarino. Covering rational ruled surfaces. *Math. Comp.*, 86(308):2861–2875, 2017.

[10] L.Y. Shen and S. Pérez-Díaz. Characterization of rational ruled surfaces. *J. Symbolic Comput.*, 63:21–45, 2014.

[11] W.A. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2012. [http://www.sagemath.org](http://www.sagemath.org).
[12] J. Steiner. Jacob steiner’s gesammelte werke. 1881.

[13] Y. Villarceau. Theoreme sur le tore. *Nouvelles annales de mathématiques*, 7:345–347, 1848. http://eudml.org/doc/95880.

[14] C. Wren. Generatio corporis cylindroidis hyperbolicis, elaborandis lenti bus hyperbolicis accommodati. 4:961–962, 1669.

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