A Critical String Theory in (3+1)+4 Dimensions and the Standard Model

J.S.Bhattacharyya
Kanchrapara College,
Kanchrapara, 743145,
India
E-mail:jtskhrbhattacharyya@gmail.com

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Abstract

We consider open strings moving in a (3+1)+4 dimensional space, assuming that there are two local $N = 1$ world-sheet supersymmetries. One is associated with the dynamics of the oriented string in (3+1) dimensional Minkowski spacetime and the other with a 4-dimensional internal space adding $SU(3)_C$ and $SU(2)_L$ quantum numbers to it. Together, the longitudinal and time components of the world-sheet fermions add the weak hypercharge to a state. At zero temperature, we consider the Fermi Sea of the world-sheet fermions as the ground state of non-interacting strings. The theory is regular in (3+1)+4 dimensions. Even though the full spectrum contains both bosons and fermions, there is no space-time supersymmetry for the lack of triality. There are three types of left-handed neutrinos. The right-handed neutrino is sterile. The vacuum energy after the tachyon induced spontaneous breaking of the $SU(2)_L$ symmetry makes the string be point objects with spin $\leq 1$ in the matter sector.
We consider open strings moving in a (3+1)+4 dimensional space. There are two \( N = 1 \) local supersymmetries. One acts on four world-sheet scalar fields and four world-sheet Majorana spinors that are coordinates and components of vectors of the (3+1) dimensional Minkowski spacetime. The other supersymmetry acts on another set of world-sheet scalars and Majorana spinors that are coordinates and components of vectors of the four-dimensional Euclidean space. Therefore, the world-sheet action is \[ S = -\frac{1}{2\pi} \int d^2 \sigma e \left[ \frac{1}{2} h^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X_\mu - i \bar{\psi}^\mu \rho^\sigma \partial_\alpha \psi_\mu \right] \]

where \( \chi_1, \chi_2 \) are the two world-sheet gravitinos corresponding to the two local supersymmetries:

\[
\delta X^{\mu,A} = \epsilon^{1,2} \bar{\psi}^{\mu,A}, \delta \bar{\psi}^{\mu,A} = -i \rho^\alpha \epsilon^{1,2} (\partial_\alpha X^{\mu,A} - \bar{\psi}^{\mu,A} \chi^{1,2}_\alpha) \]

\[
\delta e_{a}^\alpha = -2i \epsilon^{1,2} \rho^\alpha \chi^{1,2}_a, \delta \chi^{1,2}_\alpha = \nabla_\alpha \epsilon^{1,2}.
\]

Here \( \mu \) runs over the values \((0, 1, 2, 3)\) and \( A \) runs over the values \((1, 2, 3, 4)\). The local Weyl transformation that leaves the action invariant is

\[
\delta X^{\mu,A} = 0, \delta \bar{\psi}^{\mu,A} = -\frac{1}{2} \Lambda \psi^{\mu,A} \]

\[
\delta e_{a}^\alpha = \Lambda e_{a}^\alpha, \delta \chi^{1,2}_\alpha = \frac{1}{2} \Lambda \chi^{1,2}_\alpha.
\]

There is also another local fermionic symmetry given by

\[
\delta \chi^{1,2}_\alpha = i \rho_\alpha \eta^{1,2}, \delta \psi^{\mu,A} = \delta X^{\mu,A} = 0.
\]

There are altogether four local bosonic symmetries: two world-sheet reparametrizations, one local Lorentz and one local Weyl scaling. Locally they can be used to gauge the four components of the zweibein into the standard form \( e_{a}^\alpha = \delta_{a}^\alpha \) \((h_{++} = h_{--} = 0)\). Similarly, the four supersymmetries \((\epsilon^1, \epsilon^2)\) and the four superconformal symmetries \((\eta^1, \eta^2)\) can be used to set the eight components of \( \chi^1 \) and \( \chi^2 \) to zero. The Faddeev-Popov determinant for the residual reparametrization \( (\sigma \to \sigma + \xi) \) symmetry: \( \delta h_{++} = \nabla_+ \xi_+ = 0 \) and \( \delta h_{--} = \nabla_- \xi_- = 0 \) yields the pair of reparametrization ghosts and anti-ghosts \((c, b)\). Similarly the residual
supersymmetries: $\delta x_\alpha^{1,2} = \nabla_\epsilon^{1,2} = 0$ yield two pairs of superconformal ghosts and anti-ghosts $(\gamma^1, \beta^1)$ and $(\gamma^2, \beta^2)$. Thus, the gauge-fixed world-sheet action becomes

$$S = \frac{1}{\pi} \int d^2 \sigma \left[ \{ \epsilon^+ \partial_+ b_{++} + \epsilon^- \partial_+ b_{--} + \partial_+ X^\mu \partial_- X_\mu + \gamma^1 \partial_+ \gamma^{-1/2}_{1/2} \right. $$

$$+ \left. \beta_{-3/2} \partial_- \gamma^{-1/2}_{1/2} + \beta_{3/2} \partial_+ \gamma^{1/2}_{-1/2} \} + \{ \partial_+ X^A \partial_- X^A + \partial_- X^A \partial_+ X^A + \partial_+ \partial_- \psi_+^A + \partial_+ \partial_- \psi_-^A \right. $$

$$+ \left. \beta_{-3/2} \partial_- \gamma^{-1/2}_{1/2} + \beta_{3/2} \partial_+ \gamma^{1/2}_{-1/2} \} \right]$$

If we allow the time coordinate to vary from $-\infty$ to $+\infty$ and assume that the system is in thermal equilibrium at temperature $T$, then from the expression for the partition function $Z = Tr e^{-\beta H}$, we can infer that $T = 0$, because the periodic time $\frac{1}{kT}$ is infinite.

So fermions will fill all the energy levels up to the Fermi energy $E_F = \frac{1}{m} \left( \frac{N}{l} \right)^2$ (6)

where $N$ is the number of fermions and $l$ is the one dimensional spatial extension of the manifold. Hence the Fermi energy $E_F = \infty$ for massless fermions.

Careful computation of the commutators of fermionic currents in a quantum field theory reveals that they do not always have the form anticipated from naive manipulations. Additional terms usually called the Schwinger Terms (ST) are to be expected in all current algebras [5–8]. Though apparently incompatible results were reported sometimes, the existence of two solutions for the ST different in sign only, seems to be a distinct possibility [9–23].

These STs arise due to the short distance singularities of the current - current correlation functions and can be computed in many ways as discussed in the literature. The oldest among them is the canonical method. This is what we pursued in [24] and demonstrated explicitly with massless fermions in two dimensions that if the positive energy states of fermions were filled instead of the negative energy states, the Schwinger Terms in the corresponding current algebras including the Virasoro algebra [25] would change sign. Because, the anomaly term involves only the odd powers of the Fourier indices ($m$) and replacing a vacant state with an occupied state, reverses the roles of the creation and annihilation operators of the fermions ($m \rightarrow -m$). This corresponds to anti-normal ordering of the current-current commutators. Creation operators of the fermions are brought to the right of the annihilation operators in the anti-normal ordered expression for the current-current commutators.

The infinite Fermi Sea where all positive energy states are filled up with exactly one fermion per level according to Pauli principle is a unique state too like the canonical vacuum and all states built from this would only contain de-excitations
(obtained by action of positive Fourier modes) of the fermion. The scenario is analogous to Dirac’s original hole theory [26], where the canonical vacuum was defined to be the one with all negative energy states filled up.

From the previous discussions, it appears that the Virasoro anomalies would cancel in this case, because

$$ c = 2\left[4(\epsilon\chi + c_\psi) + c_\gamma\right] + c_c $$

$$ = 2\left[4\left(1 - \frac{1}{2}\right) + 11\right] - 26 $$

$$ = 0, $$

where \( c \) is the central charge.

The solutions to the wave equations compatible with the proper boundary conditions are

$$ X^\mu = x^\mu + p^\mu \tau + \frac{1}{\alpha} e^{-i\alpha \tau} \cos \sigma $$

$$ X^A = x^A + q^A \tau + \frac{1}{\alpha} e^{-i\alpha \tau} \cos \sigma $$

$$ \psi^{\mu, A} = \sum_{n=-\infty}^{\infty} d_n^{\mu, A} e^{-in(\tau \pm \sigma)} $$

$$ c^\pm = \sum_{n=-\infty}^{\infty} c_n e^{-2in(\tau \pm \sigma)} $$

$$ b_{\pm \pm} = \sum_{n=-\infty}^{\infty} b_n e^{-2in(\tau \pm \sigma)} $$

$$ \gamma_{\pm \mp} = \sum_{n=-\infty}^{\infty} \gamma_n e^{-2in(\tau \pm \sigma)} $$

$$ \beta_{\mp \mp} = \sum_{n=-\infty}^{\infty} \beta_n e^{-2in(\tau \pm \sigma)}. $$

In the RNS model, the modes of the \( \psi \)s are half-integral in the bosonic sector and integral in the fermionic sector of the string [27,28]. We assume that the modes of the superconformal ghosts are half-integral in both the bosonic and the fermionic sectors. Integral modes in the fermionic sector would lead to infinite degeneracy of the ground state due to the presence of the zero modes of the superconformal ghosts.

The modes of the vanishing components \( T_{++} \) and \( T_{--} \) of the energy momen-
tum tensor give the Virasoro generators

\[ L^\alpha_m = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n} \alpha_n \]  
\[ L^d_m = \frac{1}{2} \sum_{-\infty}^{\infty} (n - \frac{m}{2}) d_{m-n} . d_n \]  
\[ L^c_m = \sum_{-\infty}^{\infty} (n - 2m) c_{m-n} b_n \]  
\[ L^\gamma_m = \sum_{-\infty}^{\infty} (n - \frac{3m}{2}) \gamma_{m-n} \beta_n . \]

We write the Virasoro algebra for each component as

\[ [L^{\mu,A}_m, L^{\mu,A}_{-m}] = 2m : \ L^{\mu,A}_0 : + \frac{m^3 - m}{12} \]  
\[ [L^{d,A}_m, L^{d,A}_{-m}] = 2m :: L^{d,A}_0 :: - \frac{m^3 - m}{24} \]  
\[ [L^{d,A}_m, L^{d,A}_{-m}] = 2m :: L^{d,A}_0 :: - \frac{m^3 + 2m}{24} \]  
\[ [L^c_m, L^c_{-m}] = 2m : L^c_0 :: - \frac{26m^3 - 2m}{12} \]  
\[ [L^{\gamma,1,2}_m, L^{\gamma,1,2}_{-m}] = 2m : L^{\gamma,1,2}_0 :: + \frac{11m^3 + m}{12} \].

Here :: :: is the symbol for anti-normal ordering.

One might wonder how the anomaly, which is basically a c-number central extension of an operator algebra, would have a different form with what looks like a partly upside down situation of the original Fock space (i.e. only as far as the fermions are concerned). The reason behind such a difference is the fact that the normalization coefficient of the Fermi Sea is ill-defined with respect to the canonical vacuum because of the infinite number of fermions that defines the former. One should note here that the two \( N = 1 \) world-sheet supersymmetries that are manifest in the action are not quite reflected in the usual fashion in the states. The bosonic state created by the creation operator of a boson field on the vacuum will transform into a “hole” state of the fermions obtained by the action of the destruction operator of the fermion on the Fermi vacuum. In that sense it can be termed as anti-supersymmetry.
Combining all the contributions of the individual components, we get

\[
\begin{align*}
[L_m, L_{-m} ] &= 2m(\alpha_0^d + \phi_0^d + \phi_0^c + \phi_0^\gamma) \\
&\quad + \frac{2[4(1 - \frac{1}{2} - 2) + 11 - 26]}{12}m^3 \\
&\quad - \frac{2[4(1 - \frac{1}{2} - 1) - 2]}{12}m \\
&= 2m(\alpha_0^d + \phi_0^d + \phi_0^c + \phi_0^\gamma)
\end{align*}
\]

in the bosonic sector. Here,

\[
L_0^\alpha = \sum_{\mu,A} L_0^\alpha_{\mu,A}, L_0^d = \sum_{\mu,A} L_0^{d_{\mu,A}}, L_0^\gamma = \sum_i L_0^\gamma_i.
\]

(18)

Now

\[
L_0^d = L_0^d - a_d = :L_0^d:+a_d.
\]

(19)

The normal ordering constant can be regularized with a \(\zeta\)-function \([29]\) to yield

\[
:L_0^d: = L_0^d - 2\cdot{\frac{1}{3}}
\]

in the bosonic sector.

We assume that any physical state \(|\phi\rangle\) is annihilated by all the modes of the anti-ghost \(b\). So, it has ghost number

\[
U_c^\phi |\phi\rangle = \sum_{n=2} c_{-n} b_n |\phi\rangle
\]

(21)

\[= 0.\]

It is a state that contains no ghosts, but is completely filled with anti-ghosts. Hence, from (15) we can write

\[
L_0^c |\phi\rangle = 0.
\]

(22)

The bosonization of the \((\beta, \gamma)\) system and the corresponding action are given by the expressions \([30]\)

\[
\gamma = e^{-i\phi - i\kappa} \partial \phi,
\]

(23)

\[
\beta = e^{i\phi + i\kappa}
\]

and

\[
S_{\beta, \gamma} = \int d^2\sigma \sqrt{h} \left( -\frac{1}{2\pi} \partial^\alpha \phi \partial_\alpha \phi - i\frac{4\pi}{4\pi} K_\phi R\phi + \frac{1}{2\pi} \partial^\alpha K_\kappa \partial_\alpha K_\kappa - \frac{i}{4\pi} K_\phi R\phi \right),
\]

(24)

where \(K_\phi = 2\) and \(K_\kappa = -1\). In our case, there are two \(\phi\)s \(\phi_1\) and \(\phi_2\) for two sets of superconformal ghosts.

Let \(\phi = -i\phi'\) and \(K_\phi = iK_\phi'\). Hence,

\[
S = \int d^2\sigma \sqrt{h} \left( \frac{1}{2\pi} \partial^\alpha \phi' \partial_\alpha \phi' - i\frac{4\pi}{4\pi} K_\phi' R\phi' + \frac{1}{2\pi} \partial^\alpha K_\kappa \partial_\alpha K_\kappa - \frac{i}{4\pi} K_\phi R\phi' \right).
\]

(25)
We fermionize $\varphi'$ and $\kappa$ to get two Dirac spinors

$$\psi^{(1)} = e^{i\varphi'} = \sum_{n=z+\frac{1}{2}} d_n^{(1)} e^{-in\tau}$$

and

$$\psi^{(2)} = e^{i\kappa} = \sum_{n=z} d_n^{(2)} e^{-in\tau}.$$ 

Thus, 

$$L_0^+ = \sum_{n=z+\frac{1}{2}} n\bar{d}_n^{(1)} d_n^{(1)} + \sum_{n=z} n\bar{d}_n^{(2)} d_n^{(2)}.$$ 

The reparametrization ghosts and the superconformal ghosts are superpartners. So, it is natural to expect from (21) that the ghost number

$$U^\gamma |\varphi\rangle = \sum \gamma_{-n} \beta_n |\varphi\rangle = \int d\sigma \partial \varphi' |\varphi\rangle = \int d\sigma \bar{\psi}^{(1)} \psi^{(1)} |\varphi\rangle = \sum_{-\infty}^{\infty} \bar{d}_n^{(1)} d_n^{(1)} |\varphi\rangle = \int d\sigma \partial \kappa |\varphi\rangle = \int d\sigma \bar{\psi}^{(2)} \psi^{(2)} |\varphi\rangle = \sum_{-\infty}^{\infty} \bar{d}_n^{(2)} d_n^{(2)} |\varphi\rangle = 0.$$ 

We changed the definition of the ghost fields to write

$$\gamma = e^{-i\varphi + i\kappa}$$

$$\beta = e^{i\varphi - i\kappa} \partial \kappa$$

and

$$U^\gamma = \int d\sigma \partial \kappa.$$ 

in the fifth line of equation (29). So, from (29), physical states will be annihilated by both the positive and negative modes of $\psi^{(1,2)}$. It is the absolute vacuum and not the Dirac Sea, according to Dirac’s hole theory. Neither the positive nor the
negative energy states of the particle are filled in this case. They can also be interpreted as a state completely filled with the resultant anti-particles, but containing no particles. Therefore, from (28) we can write

$$L_0^\gamma |\phi\rangle = 0.$$  \hspace{1cm} (32)

Now,

$$L_0^c =: L_0^c : + \frac{1}{12}$$  \hspace{1cm} (33)

and

$$L_0^\gamma =: L_0^\gamma : + 2 \cdot \frac{1}{24}.$$  \hspace{1cm} (34)

Substituting (20), (33) and (34) in (17) we get

$$[L_m, L_{-m}] = 2m(L_0^\alpha : + : L_0^d : + L_0^c + L_0^\gamma - \frac{1}{2})$$  \hspace{1cm} (35)

where

$$L_0 =: L_0^\alpha : + : L_0^d : + L_0^c + L_0^\gamma.$$  \hspace{1cm} (36)

From (16) we can write

$$[L_m, L_{-m}] = 2m(L_0^\alpha : + : L_0^d : + : L_0^c : + : L_0^\gamma : - \frac{1}{2})$$  \hspace{1cm} (37)

in the fermionic sector.

Now,

$$:: L_0^d :: = : L_0^d : + 2 \cdot 2 \cdot 4 \cdot \frac{1}{24}$$  \hspace{1cm} (38)

$$= : L_0^d : + \frac{2}{3}.$$  \hspace{1cm} (38)

Substituting (33), (34) and (38) in (37) we get

$$[L_m, L_{-m}] = 2mL_0,$$  \hspace{1cm} (39)

where

$$L_0 =: L_0^\alpha : + : L_0^d : + L_0^c + L_0^\gamma$$  \hspace{1cm} (40)

again. Hence, from (35) and (39) we should write the string Hamiltonian for bosons as

$$H = L_0 - \frac{1}{2}$$  \hspace{1cm} (41)

and that for fermions as

$$H = L_0.$$  \hspace{1cm} (42)

One may ask why Virasoro algebra of the anti-commuting ghosts $c, b$ is not anti-normal ordered. The simple answer to it is that since $N = m = 0$ for the ghosts, $E_F$ is indeterminate from (6). Therefore, we cannot demand that all positive energy states be occupied at zero temperature.
If the Fourier modes of the $\psi$s are $d_n$, the vacuum state under the interchange of the creation and the annihilation operators transforms as

$$d_n |0\rangle = 0 \quad \text{for} \quad n > 0$$

(43)

to

$$d_n |0'\rangle = 0 \quad \text{for} \quad n < 0.$$  

(44)

Hence, $|0'\rangle$ is the Fermi vacuum.

It is obvious that to go from (43) to (44) we have to replace $\sigma^\alpha$ with $-\sigma^\alpha$ in (5). Under $\sigma^\alpha \rightarrow -\sigma^\alpha$, the Virasoro anomaly for fermions changes from

$$\langle 0 | [T_{++}(\sigma),T_{++}(\sigma')] | 0 \rangle = c\delta'''(\sigma)$$

(45)

to

$$\langle 0' | [T_{++}(-\sigma),T_{++}(-\sigma')] | 0' \rangle = -c\delta'''(\sigma).$$

(46)

Since there are no such states as $|0'\rangle$ for bosons, signs of Virasoro anomalies cannot change for them. This is because, the replacement $\sigma^\alpha \rightarrow -\sigma^\alpha$ does not change the sign of the kinetic energy for bosons in (5) and we can get back to its original form by replacing $X(-\sigma)$ by a new $X(\sigma)$.

The replacement $\sigma^\alpha \rightarrow -\sigma^\alpha$ interchanges the positive frequency modes with the negative frequency modes. This is not allowed in the case of bosons but allowed in the case of fermions, because if we interchange the creation and destruction operators for bosons, the commutator $[a,a^\dagger] = 1$ will change sign, leading to negative norm states. This is not so for the anti-commutator $\{d,d^\dagger\} = 1$ for fermions.

The descendant states for the definition of the Kac determinant [31] involving world-sheet fermions will be

$$|\psi\rangle = L_m L_n \cdots |h\rangle.$$  

(47)

Thus, from (16) we can write

$$\langle h | L_{-m} L_m | h \rangle = -\langle h | [L_m, L_{-m}] | h \rangle$$

(48)

$$= -\langle h | (2mL_0 - \frac{m^3 - m}{24}) | h \rangle$$

$$= -\langle h | (2mh - \frac{m^3 - m}{24}) | h \rangle,$$

where $L_0 |h\rangle = h |h\rangle$. Since $L_0$ is anti-normal ordered for world-sheet fermions, $h$ will be negative in this case from (19) and so the norm of (48) is positive [32].

Equations (35) and (39) indicate that our theory is very similar to the $D = 10, N = 1$ superstring theory in the light-cone gauge. But, if we replace $\sigma^\alpha$ with $-\sigma^\alpha$, the canonical vacuum transforms into the Fermi Sea. It has the occupation
number $\sum_{n=1}^{\infty} 1 = \zeta(0) = -\frac{1}{12}$ with respect to the canonical vacuum. So bosons become fermions and vice versa. This is reflected in the equations (17) and (37).

From the above discussions, it is clear that bosonization is possible only if the theory is regular and canonical commutation relations are unambiguous. If we bosonize the $\psi$s, we will get $c_\psi = \frac{1}{2}$ instead of $-\frac{1}{2}$ in (7). So anomalies will not cancel, and canonical commutators will be ill-defined. We can, however, bosonize them after the anomaly cancels. Various states then can be obtained from the ground state by the action of the ladder operators.

Since ghosts do not contribute to the zero-point energy, we cannot discard the longitudinal and time-like excitations without altering it. Physical states are admixtures of both, but their total contribution to the norms vanishes [33, 34].

World-sheet supersymmetry and residual reparametrization invariance are sufficient to gauge away the $+$ components of all nonzero mode oscillators of $X$ and allow us to set

$$\psi^+ = 0,$$

Therefore physical states will be annihilated by positive frequency modes of $\psi^+$ and thus, are equal mixtures of longitudinal and time like excitations of a particular frequency. We assume that $k^\mu = (k^0, 0, 0, k^3)$. Therefore,

$$|\phi\rangle = (d^0_n + d^3_n) |0\rangle$$

and

$$\langle \phi | \phi \rangle = \langle 0 | (d^0_n + d^3_n)(d^0_n + d^3_n) |0\rangle = \langle 0 | (\{d^0_n, d^0_n\} + \{d^3_n, d^3_n\}) |0\rangle = \langle 0 | (-1 + 1) |0\rangle = 0.$$

Since ghosts do not contribute, physical state conditions can be written as

$$(L_{m, \text{matter}}^1 + L_{m, \text{matter}}^2 - a \delta_m) |\phi\rangle = 0$$

for $m \geq 0$, where $a = 0$ in the fermionic sector and $\frac{1}{2}$ in the bosonic sector. We call $(\psi^\mu, X^\mu)$ collectively as $(1, \text{matter})$ and $(\psi^A, X^A)$ collectively as $(2, \text{matter})$.

1From [29]

$$(d^0_n + d^3_n) |\phi\rangle = 0$$

Hence,

$$d^0_n |\phi\rangle = -d^3_n |\phi\rangle$$

or

$$\langle \phi | d^0_{-n} d^3_n |\phi\rangle = \langle \phi | d^3_{-n} d^0_n |\phi\rangle$$

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We set
\[
(L_{1,\text{matter}}^1 - a\delta_m) |\phi\rangle = 0
\] (56)
Hence,
\[
L_{2,\text{matter}}^2 |\phi\rangle = 0.
\] (57)
If \(F_{m}^{1,2,\text{matter}}\) are the Fourier modes of the supercurrents \(G_{\mu}^{1,2,\text{matter}} = \psi_{\mu,A} \partial X_{\mu,A}\),
we can write
\[
F_{m}^2 = L_{2m}.
\] (58)
Hence,
\[
F_{m}^{1,2,\text{matter}} |\phi\rangle = 0,
\] (59)
where \(m\) is a positive integer for fermions and a half-integer for bosons.

From (57) and (56), we can write
\[
L_{0}^{1,\text{matter}} = a'p^2 + N_1
\] (60)
\[
= a
\]
and
\[
L_{0}^{2,\text{matter}} = a'q^2 + N_2
\] (61)
\[
= 0,
\]
where \(N_1\) and \(N_2\) are the number operators for the systems \((\psi^\mu, X^\mu)\) and \((\psi^A, X^A)\).

Therefore,
\[
q^A = 0
\] (62)
and
\[
N_2 = 0.
\] (63)

In the bosonic sector, the vertex operator for emission of the ground state
tachyon will be
\[
V_0(\tau) = e^{ip.X(\tau)},
\] (64)
because conformal dimension of \(e^{ip} = \frac{1}{2}\). Here \(p^2 = 1\) and \(\varphi = \varphi_1\).

For \(N_1 = \frac{1}{2}\), the first excited state will be a massless vector boson. The vertex operator for emission of this state will be
\[
V_v(\tau) = e^{ip.X(\tau)} |\psi(\tau)\rangle \langle \psi(\tau)|
\] (65)

For the ground state in the fermionic sector, the condition \(F_{0}^{1,\text{matter}} |\phi\rangle = 0\) yields the Dirac equation describing a massless spinor. The zero-mode condition \(F_{0}^{2,\text{matter}} |\phi\rangle = 0\) is trivial in this case.

The vertex operator for emission of this state will be
\[
V_s(\tau) = e^{ip.X(\tau)} |\psi(\tau)\rangle \langle \psi(\tau)|
\] (66)
where $\Theta_{0,1} = e^{\pm i\frac{\pi}{4}}$ are the spin operators for the pair $(\psi_\mu = 0, \psi_\mu = 1)$, where $\psi_\mu = 0 \pm i\psi_\mu = 1 = e^{\pm i\theta}$. Similarly spin operators for the pairs $(\psi_\mu = 2, \psi_\mu = 3)$, $(\psi_\mu = 1, \psi_\mu = 2)$, $(\psi_\mu = 3, \psi_\mu = 4)$ can be written as $\Theta_{1,2} = e^{\pm i\frac{\pi}{2}} \Theta_{1,2}^{\pm} = e^{\pm i\frac{\pi}{4}} \Theta_{1,2}^{\pm}$. The products $\Theta_{0,1} \Theta_{1,2}, \Theta_{1,2}^{\pm} \Theta_{3,4}^{\pm}$ and $\Theta = \Theta_{0,1} \Theta_{2,3} \Theta_{3,4}$ represent spinors of $SO(3, 1), SO(4)$ and $SO(7, 1)$.

Other vertex operators can be constructed in a similar way.

We can write the external states of a $M$-particle scattering amplitude as

$$|\phi_M\rangle = \lim_{\tau \to \pm \infty} V(\tau) |0\rangle$$

and

$$\langle \phi_1 | = \lim_{\tau \to \pm \infty} V(\tau) |0\rangle.$$  

Hence, from (64), (65) and (66), we can write

$$|\phi_M^0\rangle = \lim_{\tau \to \pm \infty} V_0(\tau) |0\rangle$$

$$= \lim_{\tau \to \pm \infty} e^{i\varphi(\tau)} e^{ip.X(\tau)} |0\rangle$$

$$= \varphi e^{ip.x} |0\rangle$$

$$|\phi_M^\varphi\rangle = \lim_{\tau \to \pm \infty} V_\varphi(\tau) |0\rangle$$

$$= \varphi e^{ip.x} |0\rangle$$

and

$$|\phi_M^\varphi\rangle = \lim_{\tau \to \pm \infty} V_\varphi(\tau) |0\rangle$$

$$= \lim_{\tau \to \pm \infty} e^{i\varphi} \Theta_{0,1} \Theta_{1,2} \Theta_{3,4}^{\pm} e^{i\varphi} |0\rangle$$

$$= \lim_{\tau \to \pm \infty} e^{i\varphi} e^{ip.2 \hat{X}_e} e^{ip.x} |0\rangle$$

$$= \varphi e^{ip.x} |0\rangle,$$

where

$$\varphi = \varphi_0 + p_\nu \tau + \text{oscillator terms.}$$

Since $\varphi$ has negative kinetic energy in (24), we put $|\varphi_0, p_\nu \rangle = -|x, p\rangle$ in the above equations.

We can put $\varphi_0 = 0$ in (69), (70) and (71) to write

$$|\phi_M^0\rangle = e^{ip.x} |0\rangle$$

$$|\phi_M^\varphi\rangle = b_{-\frac{1}{2}} \zeta e^{ip.x} |0\rangle$$

$$|\phi_M^\varphi\rangle = e^{\pm i\frac{\theta_1}{2}} e^{\pm i\frac{\theta_2}{2}} e^{\pm i\frac{\theta_3}{2}} e^{\pm i\frac{\theta_4}{2}} e^{ip.x} |0\rangle.$$
It simply states that $p_\varphi = 0$ for the external states, as mentioned before.

For maximal gauge symmetry, we assume that $x^A$ describes a $CP^2$ to define the gauge symmetry $SU(3)_C$ over it [38,39]. Equation (62) then states that physical states are color singlet. We assume that the two coordinates $(\theta^A_0, \theta^A_0)$ we get after bosonization of the pairs $(\psi^{A=1}, \psi^{A=2})$ and $(\psi^{A=3}, \psi^{A=4})$ describe a $S^2$ to define a gauge symmetry $SU(2)_L$ on it. Right-chiral fermions are $SU(2)_L$ singlets. Therefore, we set $\theta^A_0 = \theta^A_0 = 0$ for them in (73).

To complete the Standard Model [40] an extra coordinate compactified to a circle will be needed for the definition of the $U(1)_{Y_W}$ gauge symmetry. This coordinate should merely add the weak hypercharge $Y_W$, but no new dynamics to the string.

From (64), (65) and (66), we identify $V$ with an operator $W$ of conformal dimension $\frac{1}{2}$ through the relation

$$V = e^{i\varphi}W.$$  

in the $F_2$ picture [1].

Hence,

$$W_0 = e^{ip.X}$$  

where $p^2 = -1,$

$$W_v = \psi_\nu \zeta^\nu e^{ip.X},$$

where $p^2 = 0$ and

$$W_s = \Theta e^{ip.X},$$

where $p^2 = 0$.

We define the new vertex operator in the $F_1$ picture [1,30,41] as

$$V = [F_m,W]_{\pm},$$  

From (75) and (76) we write the vertex operator for tachyon emission as

$$V_0 = [F_{m,\text{matter}},W_0]$$

$$= \psi_\nu p^\nu e^{ip.X}$$

$$= \frac{(\psi^3 \pm i\psi^0)}{\sqrt{2}} e^{ip.X}$$

$$= e^{ip.X} e^{\pm Y_W},$$

where $p^\mu = (\pm \frac{i}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})$ and $\frac{(\psi^3 \pm i\psi^0)}{\sqrt{2}} e^{\pm Y_W}.$

Assuming a mode expansion of $Y$ similar to $X$, we write

$$Y = y + Y_W \tau + \text{oscillator terms},$$  

13
to interpret \( y \) as the curled up coordinate to define the \( U(1)_Y \) gauge symmetry and the momentum \( Y_W = \pm 1 \) conjugate to it as the weak hypercharge of the state.

We absorb the longitudinal and time components of \( \psi^\mu \) into \( Y \) to write the world-sheet action as

\[
S = \int d^2 \sigma \left( \partial^\alpha X^\mu \partial^\alpha X^\mu + \partial^\alpha Y \partial^\alpha Y + \psi_\pm^i \partial_{\mp} \psi_\pm^i + \cdots \right),
\]

(79)

where \( \psi^i \)'s are the transverse components of \( \psi^\mu \). We write \( Z^a = (X^\mu, Y) \). Thus,

\[
S = \int d^2 \sigma \left( \partial^\alpha Z^a \partial^\alpha Z^a + \psi_\pm^i \partial_{\mp} \psi_\pm^i + \cdots \right).
\]

(80)

The invariance of the action under rotations of coordinate axes in the \( Z \) plane will allow us to set \( p^a = (p^\mu, Y_W) = (p^\mu, 0) \), where \( p^\mu p_\mu = 2 \), as in the case of the tachyon of the bosonic string. Hence, \( L_0 = 1 \) in the \( F_1 \) picture of the bosonic sector.

From (75) and (76), we can write the vertex operator for the massless vector boson as

\[
V_v = \left[ F^1_{m, \text{matter}}, W_v \right] = (\zeta_\mu \partial X^\mu + \psi_\mu \eta_\mu \zeta_\nu) e^{ip \cdot X} = \zeta_\mu \partial X^\mu e^{ip \cdot X}.
\]

(81)

The second term in the second line of the above equation drops out from the transversality condition \( \zeta_\mu p^\mu = 0 \).

We can set \( p^2 = -1 \) and \( Y_W^2 = 1 \) in the expression for \( W_s \) in (75) through a rotation of coordinate axes in the \( Z \) plane. Thus,

\[
W_s = e^{\pm iY} e^{ip \cdot X} \Theta.
\]

(82)

Therefore, the states \( \Theta^2_{1,2,\pm} \Theta^2_{3,4,\pm} \) should be written as

\[
\begin{bmatrix}
\nu_L \\
\epsilon_L
\end{bmatrix}
\begin{bmatrix}
p_L \\
n_L
\end{bmatrix}
\]

(83)

for the first generation of fermions (there will be three such families as we shall see later). They are conjugate isospinors with \( Y_W = \mp 1 \).

Since from (79) and (80) only \( p^2 + Y_W^2 \) has an invariant meaning, fractionally charged quarks can also be accommodated in our model by a corresponding change of mass. The \( SU(3)_C \) coupling strength should however be vanishingly small (asymptotic freedom) according to (62). Since \( q^2 = \epsilon \) in this case, where \( \epsilon \) is

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2 We can write \( \epsilon^{ijl} \psi_i \psi_j k_l e^{ik \cdot X} \) as the vertex operator for a pseudoscalar like a massless pion and \( \epsilon^{ijl} \zeta_i X_j k_l e^{ik \cdot X} \) as the vertex operator for a pseudovector with polarization \( \zeta_i \) like a massless rho meson in the \( F_1 \) picture.
a very small positive number, the mass-shell condition should be written in the limiting sense as

\[ p^2 = -\epsilon. \] (84)

According to the Kaluza-Klein theory [42, 43], the Dirac action for fermions is

\[ S_D = \int d^4x (2\pi R) \bar{\Psi} \gamma^\mu \left( \frac{p^\mu}{\sqrt{G}} - \frac{n}{R} A^\mu \right) + \gamma^5 \frac{n}{\sqrt{G} R} \Psi, \] (85)

where \( A^\mu = g^\mu_5 \) and \( G \) is the Gravitational constant. Here \( R \) is the radius of the circle defining the \( U(1) \) gauge symmetry, and \( \mu \) runs from 0 to 3.

The result of (82) is very similar to what is found in (85), where the charge to mass ratio of a fermion is the fundamental length \( \sqrt{G} \). In our case, it is the length \( l = \sqrt{2\alpha'} \). So, \( l = \sqrt{G} \).

From (85) we can write the gauge coupling constant as

\[ g = 2\pi R \frac{n}{R} \] (86)
\[ = 2\pi n. \]

We consider the \( M \)-point primitive amplitudes at the tree level involving bosons and fermions in the open string sector. When we calculate anomalies at zero temperature, we consider a free string. Therefore, world-sheet fermions occupy all states up to the Fermi energy, even in the bosonic sector. But interacting bosons occupy the lowest excited state (the canonical vacuum) at the absolute zero of temperature (Bose condensation). While considering interactions between bosons and fermions, we have to choose a common ground state, which is the canonical vacuum and not the Fermi Sea of the world-sheet fermions.

After replacing \( \tau \) by \(-i\tau\), we set \( \tau_1 = \infty, \tau_M = -\infty \) and \( \tau_{M-1} = 0 \) with the three real parameters of \( SL(2, R) \) to write it as

\[ g^{M-2} \int_0^\infty d\tau_2 \cdots d\tau_{M-2} \langle \phi; 1 | V(2, \tau_2) \cdots e^{H\tau_{M-2}} V(M - 2) e^{-H\tau_{M-2}} V(M - 1) | \phi; M \rangle. \] (87)

From (41) and (42) we can write

\[ H = L_0 - a. \] (88)

The value of \( a \) depends on the state on which \( H \) acts, \( a = 0 \) if it is a fermion and \( a = \frac{1}{2} \) if it is a boson.

Since \( L_0 \) commutes with an integrated vertex operator, the factor \( e^{H\tau_{M-2}} \) can be brought past successive vertices to the left until the Hamiltonian annihilates the
external state. So the equation (87) reduces to

\[ g^{M-2} \int_0^\infty d\tau_2 \cdots d\tau_{M-2} \langle \phi; 1 | V(2)e^{-H\tau_2} \cdots V(M-2)e^{-H\tau_{M-2}} | \phi; M \rangle \]

\[ = g^{M-2} \langle \phi; 1 | V(2) \frac{1}{H} \cdots V(M-2) \frac{1}{H} V(M-1) | \phi; M \rangle. \]

Some of the vertices might be of \( W \) type, the amplitude would vanish otherwise.

The string propagator \( \frac{1}{H} = \frac{1}{(L_0 - a)} = \frac{1}{L_0 - a + L_0} \) can be expanded into powers of \( \frac{L_0^2}{L_0 - a} \). The \( L_0^1 \) can then be brought past the subsequent vertices and propagators until it annhilates against the physical state at the right end of the tree. Here \( L_0^1 \) includes contributions of \( (\psi^\mu, X^\mu) \), the superconformal ghosts \( (\gamma^1, \beta^1) \) and the reparametrization ghosts \( (c, b) \) to \( L_0 \) and \( L_0^2 \) includes contributions of \( (\psi^A, X^A) \) and the superconformal ghosts \( (\gamma^2, \beta^2) \) to the same.

Thus,

\[ \frac{1}{L_0 - a} = \frac{1}{L_0^1 - a}. \] (90)

From (88) and (89), we can write the tree amplitude for \( M \) bosons as

\[ g^{M-2} \langle \phi; 1 | V(2) \frac{1}{L_0^1 - 1} V(3) \frac{1}{L_0^1 - \frac{3}{2}} \cdots V(M - 1) | \phi; M \rangle \] (91)

\[ = g^{M-2} \langle \phi'; 1 | V(2) \frac{1}{L_0 - 1} V(3) \frac{1}{L_0 - \frac{3}{2}} \cdots V(M - 1) | \phi'; M \rangle, \] (92)

where \( |\phi\rangle \) is a physical state like \( |\phi\rangle \) in the \( F_2 \) picture and \( |\phi'\rangle = L_{-\frac{1}{2}}^{1,\text{matter}} |\phi\rangle \) is the corresponding state in the \( F_1 \) picture.

The amplitude of (91) is very similar to the \( N = 1, D = 10 \) superstring theory. After the arguments cited in [1], we can show that spurious states decouple from a physical tree.

From (89), we can write the amplitude of scattering of 2 fermions and \( M - 2 \) bosons as

\[ g^{M-2} \langle \text{Fermion}; 1 | W(2) \frac{1}{L_0} V(3) \cdots \frac{1}{L_0} V(M - 1) | \text{Fermion}; M \rangle. \] (93)
From [76] we can write it as
\[
g^{M-2} \langle \text{Fermion}; 1 | W(2) \frac{F_0}{L_0} W(3) \rangle
\]
\[
\quad \cdots \frac{F_0}{L_0} W(M - 1) | \text{Fermion}; M \rangle
\]
\[
= g^{M-2} \langle \text{Fermion}; 1 | W(2) \frac{1}{F_0} W(3) \rangle
\]
\[
\quad \cdots \frac{1}{F_0} W(M - 1) | \text{Fermion}; M \rangle,
\]
where \( F_0 = F_0^1 \).

We note that \( \mathbb{CP}^2 \) does not admit spinors. But the chirality content of the total particle spectrum of an internal manifold \( \mathbb{CP}^2 \otimes M \) could well be different from that of the spinors on the \( \mathbb{CP}^2 \) only [44]. From (83) it is evident that physical states are isospinors on \( S^2 \) also. So, we calculate the Betti-Hodge numbers of \( \mathbb{CP}^2 \otimes S^2 \).

For \( S^2 \) or \( CP^1 \) they are given by the matrix
\[
b^{S^2}_{pq} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
and those for \( CP^2 \) are given by the matrix
\[
b^{CP^2}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Therefore, for \( CP^2 \otimes S^2 \) they are given by the matrix
\[
b^{CP^2 \otimes S^2}_{pq} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

This, after proper addition of cells becomes
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}.
\]

The matrix can be identified with the Hodge diamond of a six dimensional Calabi-Yau Manifold [45–48]. So, the number of families will be
\[
\frac{X}{2} = b_{11} - b_{21}
\]
\[
= 3.
\]
Only open strings with opposite charges at the same mass level can join ends to form closed strings. The charges of the massless fermions in the R-sector cannot match those of the massless gauge bosons in the NS-sector to cancel each other, because unlike in the $D = 10, N = 1$ theory, they are not gauginos. Therefore, NS-R states of the closed string like gravitinos will be absent. This is consistent with the fact that there is no supersymmetry in the open string spectrum. The presence of the ground state tachyon only in the NS sector, is also inconsistent with the presence of the NS-R states.

At lower energies, string theory is consistent with quantum field theory [49]. We can independently verify it from the $\alpha' \to 0$ limit of the primitive amplitudes. Since the gauge group in our model of string theory is $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, we should identify the ground state tachyon of the bosonic sector of the open string with the Higgs boson of the Standard Model.

If we consider finite temperature field theory below a critical temperature, the universe stays in a false vacuum at the zero of the Higgs field in a supercooled state before rolling down to a true vacuum as dictated by field theoretic considerations. It breaks the SU(2) symmetry spontaneously [50–57].

To study the entire process in the context of string theory, we consider the action of the nonlinear sigma model [58] that we regularize by dimensional regularization to yield

\[
S = - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma \sqrt{h} h^{\alpha \beta} \partial_\alpha X \partial_\beta X \quad (96)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma e^{\epsilon \phi} \partial X \partial X \quad (97)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma e^{\epsilon \phi} g_{\mu \nu} \partial X^\mu \partial X^\nu \quad (98)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma e^{\epsilon \phi} (\eta_{\mu \nu} - R_{\mu \rho \nu \sigma} (X_0) x^\rho x^\sigma) \quad (99)
\]

\[
\partial X^\mu \partial X^\nu \quad (100)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma [(\partial X \partial X - \frac{1}{2\alpha'} R_{\mu \nu}) (1 + \epsilon \phi)] \quad (101)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma [(\partial X \partial X - \frac{1}{2\alpha'} \lambda \eta_{\mu \nu}) (1 + \epsilon \phi)] \quad (102)
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{4\pi \alpha'} \int d^{2(1+)} \sigma [(1 - \frac{\alpha' \lambda}{2\epsilon}) \partial X \partial X - \frac{\alpha' \lambda}{2\epsilon} \phi \partial X \partial X].
\]

Reparametrization invariance of the world-sheet action allows us to set $h_{\alpha \beta} = \eta_{\alpha \beta} e^{\phi}$ in (97). We used Riemann normal coordinates to write $g_{\mu \nu} = g_{\mu \nu} (X) = \eta_{\mu \nu} - R_{\mu \rho \nu \sigma} (X_0) x^\rho x^\sigma$ in (99) ($X = X_0 + x$ are locally inertial coordinates at $X_0$) and put the logarithmically divergent contraction $\lim_{\sigma \to \sigma'} < x^\mu (\sigma) x^\nu (\sigma') > =
\[ \frac{\alpha'^2}{2\epsilon} \] We used the lambda-vacuum solution to the Einstein field equation [59] to write \[ R_{\mu\nu} = \lambda \eta_{\mu\nu} \] in (101), where \( \lambda \) is proportional to the vacuum energy (latent heat of the supercooled state) per unit volume.

The \( \phi \) dependent term in (102) will change the zero-point energy by an amount proportional to \( \lambda \phi_0 \). Therefore, the mass of states will change. It breaks the conformal invariance of the world-sheet action (102). Hence, one can define only the subalgebra \( OSp(1|2) \) of the super-Virasoro algebra consistently. Therefore, there will only be spin \( \leq 1 \) states in the matter sector. String theory is no longer applicable, because higher spin states are absent from the spectrum. Basic interactions will solely be described by particle physics, as the Regge slope \( \alpha'' = \lim_{\epsilon \to 0} \frac{\alpha'}{1 - \frac{\alpha'}{2\epsilon}} \) vanishes in the limit \( \epsilon \to 0 \) for finite \( \lambda \).

The quartic Higgs self-coupling \( \lambda_H \) we get from the low energy limit of the Veneziano amplitude [60] is \( \lambda_H \approx 1 \). Therefore, the vacuum expectation value of the Higgs field

\[ <\phi_H> = \sqrt{\frac{\mu^2}{\lambda_H}} \approx 1, \]  

where \( \mu \) is the mass of the Higgs. Hence, the Dirac mass of fermions derived through the Yukawa coupling \( \bar{\psi}\psi\phi_H \) is also of the order of unity.

The sterile right-handed neutrino should exist [61], because it is not projected out by any projection operator and can only interact via gravity. It derives a Majorana mass [62] from its coupling \( (\nu_R)^c\nu_R + \nu_R(\nu_R)^c\phi_G \) to the tachyon of the gravity sector, through Spontaneous Symmetry Breaking. Therefore, our model corresponds to an extension of the Standard Model.

Since the four-tachyon amplitude \( A_{4t} \approx \sin \frac{\pi t}{8} \) in the closed string sector [63] vanishes in the low energy limit \( t \to 0 \), the Majorana mass of a right-handed neutrino should be very large from (103).

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