Degenerate Fubini-type polynomials associated with degenerate Apostol-Bernoulli and Apostol-Euler polynomials of order $\alpha$

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Abstract

In this paper, by introducing the degenerate Fubini-type polynomials, we give several relations with the help of the Faà di Bruno formula and some properties of Bell polynomials, and generating function methods. Also, we derive some new explicit formulas and recurrence relations for Fubini-type polynomials and numbers. Associating the degenerate Fubini-type polynomials newly defined here with degenerate Apostol-Bernoulli polynomials and degenerate Apostol-Euler polynomials of order $\alpha$ enables us to present additional relations for some degenerate special polynomials and numbers.

Keywords: Apostol-Bernoulli polynomials, Apostol-Euler polynomials, degenerate Bernoulli polynomials, Generalized Fubini polynomial, Stirling numbers, explicit formula, recurrence relation, generating function.

Mathematics Subject Classification 2010: 11B68, 11B37, 05A15, 05A19, 11B83, 11Y55.

1 Introduction

The higher-order Bernoulli polynomials $B^{(\alpha)}_n(x)$ and higher-order Euler polynomials $E^{(\alpha)}_n(x)$, each of degree $n$ in $x$ and in $\alpha$, are defined by means of the
generating functions \[23\]

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}
\]

and

\[
\left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!},
\]

respectively. For \(\alpha = 1\), we have the classical Bernoulli polynomials \(B_n(x)\) and Euler polynomials \(E_n(x)\), defined by means of the following generating functions:

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi)
\]

and

\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi).
\]

In particular, the rational numbers \(B_n = B_n(0)\) and integers \(E_n = 2^n E_n(1/2)\) are called classical Bernoulli numbers and Euler numbers, respectively.

The generalized Apostol-Bernoulli polynomials \(B_n^{(\alpha)}(x; \gamma)\) were defined by Luo and Srivastava by means of the generating function \[19, 21, 22\]

\[
\left( \frac{t}{\gamma e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \gamma) \frac{t^n}{n!}
\]

\((\gamma \in \mathbb{C}; \ |t| < 2\pi \text{ if } \gamma = 1; \ |t| < |\log \gamma| \text{ if } \gamma \neq 1)\),

and the generalized Apostol-Euler polynomials \(E_n^{(\alpha)}(x; \gamma)\) by means of the generating function \[20\]

\[
\left( \frac{2}{\gamma e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \gamma) \frac{t^n}{n!}
\]

\((|t| < \pi \text{ if } \gamma = 1; \ |t| < |\log(-\gamma)| \text{ if } \gamma \neq 1; \ 1^\gamma = 1)\).

Carlitz \[1\] defined degenerate Bernoulli polynomials and degenerate Euler polynomials by

\[
\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!},
\]
and
\[ \frac{2}{(1 + \lambda t)^{1/\lambda} + 1} (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} E_n(x; \lambda) \frac{t^n}{n!}. \]

For \( x = 0 \), these are called as degenerate Bernoulli and Euler numbers.

The degenerate versions of Apostol-Bernoulli polynomials and Apostol-Euler polynomials of order \( \alpha \) were introduced by [13]
\[ \left( \frac{t}{\gamma (1 + \lambda t)^{1/\lambda} - 1} \right)^\alpha (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda; \gamma) \frac{t^n}{n!}, \quad (1.1) \]
and
\[ \left( \frac{2}{\gamma (1 + \lambda t)^{1/\lambda} + 1} \right)^\alpha (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda; \gamma) \frac{t^n}{n!}, \quad (1.2) \]
respectively. Note that since \( \lim_{\lambda \to 0} (1 + \lambda t)^{1/\lambda} = e^t \), for \( \lambda \to 0 \), \( \alpha = \gamma = 1 \), the equations (1.1) and (1.2) reduce to the generating functions for classical Bernoulli and Euler polynomials, respectively.

Let us mention that the above polynomials have been discussed detailed in the literature. (See for example [6, 9, 10, 18, 20] and related references therein).

We now focus on Kilar and Simsek’s recent study [14], in which a family of Fubini-type polynomials \( a_n^{(\alpha)}(x) \) are introduced as in the following
\[ \frac{2^\alpha}{(2 - e^t)^{2\alpha}} e^{xt} = \sum_{n=0}^{\infty} a_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad \alpha \in \mathbb{N}_0 \text{ and } |t| < \log 2. \quad (1.3) \]
In particular, \( a_n^{(\alpha)}(0) = a_n^{(\alpha)} \) are called Fubini-type numbers. They gave some relationships between these polynomials and numbers, and other celebrated polynomials and numbers such as Apostol-Bernoulli numbers, the Frobenius-Euler numbers and the Stirling numbers via generating function methods and functional equations. Very recently, Srivastava and Kızılates [38] extended Fubini-type polynomials \( a_n^{(\alpha)}(x) \) to parametric kind families of the Fubini-type polynomials by considering the two special generating functions and obtained many relations concerning these and other parametric special polynomials and numbers. As emphasized therein, the Fubini-type polynomials \( a_n^{(\alpha)}(x) \) are special case of generalized Apostol-Euler polynomials \( E_n^{(\alpha)}(x; \gamma) \). More concretely, \( E_n^{(2\alpha)}(x; -1/2) = 2^{3\alpha} a_n^{(\alpha)}(x) \).
Further investigations on Fubini polynomials and numbers can be found in [12, 15, 16, 24, 37, 39], and plenty of references cited therein.

On the other hand, Qi and his colleagues have studied a number of explicit and recursive formulas, and closed forms for some significant polynomials and numbers by applying the Faà di Bruno formula (see Eq. (2.1), below), some properties of the Bell polynomials of the second kind, and a general derivative formula for a ratio of two differentiable functions. See [3–5, 8, 11, 25–31, 33, 34, 36, 40] and related references.

In this paper, we introduce degenerate version of Fubini-type polynomials as

\[
\frac{2^{\alpha}}{(2 - (1 + \lambda t)^{1/\lambda})^{2\alpha}} (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} a_n^{(\alpha)} (x; \lambda) \frac{t^n}{n!}, \quad \lambda \in \mathbb{R}. \tag{1.4}
\]

Notice that for \( x = 0 \), \( a_n^{(\alpha)} (0; \lambda) = a_n^{(\alpha)} (\lambda) \) are called degenerate Fubini-type numbers. Also, for \( \lambda \to 0 \), these reduce to Fubini-type polynomials \( a_n^{(\alpha)} (x) \) aforementioned above.

In parallel with the conclusion given in [38, Remark 4], we infer a relationship between degenerate Fubini-type polynomials and degenerate Apostol-Bernoulli polynomials of order \( \alpha \), i.e.

\[
a_n^{(\alpha)} (x; \lambda) = 2^{-\alpha} E_n^{(2\alpha)} (x; \lambda; -1/2). \tag{1.5}
\]

In this paper, we would like to use Faà di Bruno formula and some properties of Bell polynomials, and generating function methods in order to obtain some new explicit formulas, closed forms and recurrence relations for degenerate Fubini-type polynomials and numbers, and Fubini-type polynomials and numbers. Moreover, we give a relation between degenerate Fubini-type polynomials and degenerate Apostol-Bernoulli polynomials of order \( \alpha \) and deduce similar formulas for them.

## 2 Properties of second kind Bell polynomials

The Bell polynomials of the second kind \( B_{n,k} (x_1, x_2, \ldots, x_{n-k+1}) \) for \( n \geq k \geq 0 \) were defined by [2, p. 134 and 139]

\[
B_{n,k} (x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \quad l_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i l_i = n, \quad \sum_{i=1}^{n} l_i = k}}^{\infty} \frac{n!}{\prod_{i=1}^{l-k+1} l_i!} \prod_{i=1}^{l-k+1} \left( \frac{x_i \backslash l_i}{i!} \right).
\]
The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, ..., x_{n-k+1})$ by

$$
\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), ..., h^{(n-k+1)}(t)).
$$

(2.1)

For $n \geq k \geq 0$, these polynomials satisfy the following relation [2, p. 135]

$$
B_{n,k}(abx_1, ab^2x_2, ..., ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, ..., x_{n-k+1}),
$$

(2.2)

where $a$ and $b$ are any complex number. Also, for $n \geq k \geq 0$, the following formula is valid for the special case of $B_{n,k}$

$$
B_{n,k}(1, 1, ..., 1) = S(n, k),
$$

(2.3)

where $S(n, k)$ denotes the Stirling numbers of the second kind, can be generated by [2, p. 206]

$$
\frac{(e^t - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{t^n}{n!}.
$$

In [32, Remark 1], there existed the formula

$$
B_{n,k} \left( 1, 1 - \lambda, (1 - \lambda)(1 - 2\lambda), ..., \prod_{l=0}^{n-k-1} (1 - l\lambda) \right) = \frac{(-1)^k}{k!} \sum_{l=0}^{k} (-1)^l \left( \begin{array}{l} k \\ l \end{array} \right) \prod_{q=0}^{n-1} (l - q\lambda),
$$

(2.4)

which is equivalent to

$$
B_{n,k} \left( \langle \lambda \rangle_1, \langle \lambda \rangle_2, ..., \langle \lambda \rangle_{n-k+1} \right) = \frac{(-1)^k}{k!} \sum_{l=0}^{k} (-1)^l \left( \begin{array}{l} k \\ l \end{array} \right) \langle \lambda l \rangle_n,
$$

(2.5)

established in [35, Theorems 2.1 and 4.1]. In [7, Remark 7.5], the explicit formulas (2.4) and (2.5) were rearranged as

$$
B_{n,k} \left( 1, 1 - \lambda, (1 - \lambda)(1 - 2\lambda), ..., \prod_{l=0}^{n-k-1} (1 - l\lambda) \right)
$$

$$
= (-1)^k \frac{\lambda^{n-1}(n-1)!}{k!} \sum_{l=1}^{k} (-1)^l l \left( \begin{array}{l} k \\ l \end{array} \right) \left( \frac{l/\lambda - 1}{n - 1} \right)
$$

5
for $\lambda \neq 0$ and

$$B_{n,k}((\lambda)_1, \langle \lambda \rangle_2, \ldots, \langle \lambda \rangle_{n-k+1}) = (-1)^k \lambda \frac{(n-1)!}{k!} \sum_{l=1}^{k} (-1)^l \binom{k}{l} \left( \frac{\lambda l - 1}{n-1} \right).$$

(2.6)

Here, the generalized binomial coefficient $\binom{z}{w}$ is defined by

$$\binom{z}{w} = \begin{cases} \frac{\Gamma(z+1)\Gamma(w+1)}{\Gamma(z-w+1)} & \text{if } z, w, z-w \in \mathbb{C} - \{-1, -2, \ldots\}; \\ 0 & \text{if } z \in \mathbb{C} - \{-1, -2, \ldots\} \text{ and } w, z-w \in \{-1, -2, \ldots\}. \end{cases}$$

3 Results and their proofs

In this section, we give some computational formulas for degenerate Fubini-type numbers, some explicit formulas and recurrence relations for Fubini-type polynomials and numbers and consequently, degenerate Apostol-Bernoulli polynomials and degenerate Apostol-Euler polynomials of order $\alpha$. Also, we present further relations for some polynomials, considered here.

**Theorem 3.1** The degenerate Fubini-type numbers can be computed by the formula:

$$a_n^{(\alpha)}(\lambda) = (n-1)! \sum_{k=1}^{n} \frac{(-2\alpha)^k}{2^{\alpha+k}} \frac{(-1)^k}{\lambda^{k-1} k!} \sum_{l=1}^{k} (-1)^l \binom{k}{l} \left( \frac{\lambda l - 1}{n-1} \right).$$

(3.1)

where $\langle x \rangle_n$ denotes the falling factorial, defined for $x \in \mathbb{R}$ by

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1) \ldots (x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

Consequently, for the special case of the degenerate Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda; \gamma)$, the following relation holds:

$$E_n^{(2\alpha)}(0; \lambda; -1/2) = (n-1)! \sum_{k=1}^{n} \frac{(-2\alpha)^k}{2^{\alpha+k} \lambda^{k-1} k!} \sum_{l=1}^{k} (-1)^l \binom{k}{l} \left( \frac{\lambda l - 1}{n-1} \right).$$

(3.2)
Proof. If we apply $f(u) = (2 - u)^{-2\alpha}$ and $u = g(t) = (1 + \lambda t)^{1/\lambda}$ to the Faà di Bruno formula (2.1), and use (2.2) and (2.6) then, we find that

$$
\frac{d^n}{dt^n} \left( (2 - (1 + \lambda t)^{1/\lambda})^{-2\alpha} \right) 
= \sum_{k=0}^{n} \frac{d^k}{du^k} (2 - u)^{-2\alpha} B_{n,k} \left( \frac{\lambda (1 + t)^{\lambda-1}}{\lambda}, \frac{\lambda (\lambda - 1) (1 + t)^{\lambda-2}}{\lambda}, \ldots, \frac{\lambda (\lambda - 1) \ldots (\lambda - (n-k)) (1 + t)^{\lambda-(n-k+1)}}{\lambda} \right),
$$

$$
= \sum_{k=0}^{n} (-2\alpha)_k (2 - u)^{-2\alpha-k} \frac{(1 + t)^{k\lambda-n}}{\lambda^k} B_{n,k} \left( \langle \alpha \rangle_1, \langle \alpha \rangle_2, \ldots, \langle \alpha \rangle_{n-k+1} \right)
$$

$$
= \sum_{k=0}^{n} (-2\alpha)_k (2 - u)^{-2\alpha-k} \frac{(1 + t)^{k\lambda-n}}{\lambda^k} (-1)^k \lambda^{(n-1)!} \sum_{l=1}^{k} (-1)^l \binom{k}{l} \left( \frac{\lambda - 1}{\lambda - 1} \right)^{k!}.
$$

(3.3)

Now letting $t \to 0$, which is equivalent to $u \to 0$ on both sides of (3.3) and taking into consideration the generating function for degenerate Fubini-type numbers (for $x = 0$ in equation (1.4)) complete the proof of (3.1). From the relationship (1.5), the identity (3.2) follows readily. □

**Theorem 3.2** The Fubini-type polynomials $a_n^{(\alpha)}(x)$ possess the explicit formula

$$
a_n^{(\alpha)}(x) = 2^a \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (-2\alpha)_i (-1)^i S(k, i) x^{n-k},
$$

where $S(n, k)$ is the Stirling numbers of the second kind. Also, the Fubini-type numbers $a_n^{(\alpha)}$ can be written in the form

$$
a_n^{(\alpha)} = 2^a \sum_{i=0}^{n} (-2\alpha)_i (-1)^i S(n, i).
$$

(3.4)

Besides, the generalized Apostol-Euler numbers $E_n^{(\alpha)}(x; \gamma)$ can be expressed as

$$
E_n^{(2\alpha)}(x; -1/2) = 2^{2\alpha} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (-2\alpha)_i (-1)^i S(k, i) x^{n-k}.
$$

(3.5)
Proof. From (2.1), (2.2) and (2.3), we have

\[
\frac{d^k}{dt^k} (2 - e^t)^{-2\alpha} = \sum_{i=0}^{k} \langle -2\alpha \rangle_i (2 - e^t)^{-2\alpha - i} B_{k,i} (-e^t, -e^t, ..., -e^t)
\]

\[
= \sum_{i=0}^{k} \langle -2\alpha \rangle_i (2 - e^t)^{-2\alpha - i} (-1)^i e^{ti} B_{k,i} (1, 1, ..., 1)
\]

\[
\rightarrow \sum_{i=0}^{k} \langle -2\alpha \rangle_i (-1)^i S(k, i), \quad \text{as } t \to 0. \quad (3.6)
\]

Also, it is obvious that \( (e^{xt})^{(k)} = x^k e^{xt} \to x^k \), as \( t \to 0 \). So, by aid of the Leibnitz’s formula for the \( n \)th derivative of the product of two functions, we get

\[
\lim_{t \to 0} \frac{d^n}{dt^n} \left[ \frac{2^\alpha}{(2 - e^t)^{2\alpha} e^{xt}} \right] = 2^\alpha \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \langle -2\alpha \rangle_i (-1)^i S(k, i) x^{n-k},
\]

namely, we have \( a_n^{(\alpha)}(x) \) by (1.3). For \( x = 0 \), we immediately arrive at the identity (3.4). The equation (3.5) can be deduced from the relation between the Fubini-type polynomials \( a_n^{(\alpha)}(x) \) and generalized Apostol-Euler polynomials \( E_n^{(\alpha)}(x; \gamma) \), given by (1.5). \( \blacksquare \)

**Theorem 3.3** The Fubini-type polynomials \( a_n^{(\alpha)}(x) \) satisfy the recurrence relation

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} \langle 2\alpha \rangle_i (-1)^i S(n-k, i) a_k^{(\alpha)}(x) = 2^\alpha x^n
\]

In particular, the Fubini-type numbers \( a_n^{(\alpha)} \) provide that

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} \langle 2\alpha \rangle_i (-1)^i S(n-k, i) a_k^{(\alpha)} = 0. \quad (3.7)
\]
In analogy, the generalized Apostol-Euler polynomials $E_{n}^{(\alpha)}(x; \gamma)$ possess the recurrence relation

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} (2\alpha)_{i} (-1)^{i} S(n-k, i) E_{k}^{(2\alpha)}(x; -1/2) = 2^{4\alpha} x^{n}. \quad (3.8)$$

**Proof.** Since

$$\left(\frac{2^{\alpha}}{(2 - e^t)^{2\alpha}}\right) \frac{\partial}{\partial t^{n-k}} \left(\frac{2^{\alpha}}{(2 - e^t)^{2\alpha}} e^{xt}\right) = 2^{\alpha} e^{xt},$$

by keeping in mind the generating function of Fubini-type polynomials (1.3) and by proceeding as in the proof of (3.6), differentiate $n$ times with respect to $t$ on both sides to deduce that

$$\sum_{k=0}^{n} \binom{n}{k} \partial^{n-k} \left[ \frac{2^{\alpha}}{(2 - e^t)^{2\alpha}} e^{xt}\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} (2\alpha)_{i} (2 - e^t)^{2\alpha-i} (-1)^{i} e^{ti} S(n-k, i) \frac{\partial^{k}}{\partial u^{k}} \left[ \frac{2^{\alpha}}{(2 - e^t)^{2\alpha}} e^{xt}\right]$$

$$\to \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{n-k} (2\alpha)_{i} (-1)^{i} S(n-k, i) a_{k}^{(\alpha)}(x), \quad \text{as } t \to 0$$

$$= 2^{\alpha} x^{n}. \quad (3.8)$$

Setting $x = 0$ yields the equation (3.7) immediately. Formula (3.8) can be derived by the same motivation stated in the proofs of our previous theorems.

**Theorem 3.4** The following relationship holds true:

$$a_{n-2\alpha}^{(\alpha)}(x; \lambda) = \frac{B_{n}^{(2\alpha)}(x; \lambda; 1/2)}{2^{\alpha} \binom{n}{2\alpha}}, \quad (3.9)$$

where $B_{n}^{(\alpha)}(x; \lambda; \gamma)$ is the degenerate Apostol-Bernoulli polynomials of order $\alpha$, defined by (1.1).

**Proof.** If we put $\gamma = 1/2$ and replace $\alpha$ by $2\alpha$ in (1.1), we have

$$\sum_{n=0}^{\infty} B_{n}^{(2\alpha)}(x; \lambda; 1/2) \frac{t^{n}}{n!} = \left( \frac{t}{\frac{1}{2} (1 + \lambda t)^{1/\lambda} - 1} \right)^{2\alpha} (1 + \lambda t)^{x/\lambda}.$$
\[= 2^\alpha t^{2\alpha} \sum_{n=0}^{\infty} a_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} \]
\[= 2^\alpha \sum_{n=2\alpha}^{\infty} a_{n-2\alpha}(x; \lambda) \frac{t^n}{(n-2\alpha)!} \]
\[= 2^\alpha \sum_{n=2\alpha}^{\infty} \langle n \rangle_{2\alpha} a_{n-2\alpha}^{(\alpha)}(x; \lambda) \frac{t^n}{n!} , \]

which concludes the proof. \[\blacksquare\]

Let us continue to study degenerate Fubini-type polynomials \(a_n^{(\alpha)}(x; \lambda)\). Firstly, from the fact

\[(1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \frac{\langle x \rangle_n}{\lambda^n} \frac{t^n}{n!}\]
\[= \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} , \]

where \((x)_{n,\lambda} = x(x-\lambda)\ldots(x-(n-1)\lambda)\) for \(n > 0\) with \((x)_{0,\lambda} = 1\), it is easily verify that

\[\sum_{n=0}^{\infty} a_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!} = \frac{2^\alpha}{(2 - (1 + \lambda t)^{1/\lambda})} (1 + \lambda t)^{x/\lambda}\]
\[= \left( \sum_{n=0}^{\infty} a_n^{(\alpha)}(\lambda) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \right)\]
\[= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} a_{k}^{(\alpha)}(\lambda) (x)_{n-k,\lambda} \right) \frac{t^n}{n!} . \]

Comparing the coefficients \(\frac{t^n}{n!}\) gives the following theorem.

**Theorem 3.5** For \(n \geq 0\), we have

\[a_n^{(\alpha)}(x; \lambda) = \sum_{k=0}^{n} \binom{n}{k} a_{k}^{(\alpha)}(\lambda) (x)_{n-k,\lambda} . \]

Now, we observe that

\[\sum_{n=0}^{\infty} \left( a_n^{(\alpha)}(x+1; \lambda) - a_n^{(\alpha)}(x; \lambda) \right) \frac{t^n}{n!} \]
\[
\frac{2^\alpha (1 + \lambda t)^{x/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha}} \left((1 + \lambda t)^{1/\lambda} - 1\right)
\]
\[
= \frac{2^\alpha (1 + \lambda t)^{x/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha-1}} \left(-1 + \frac{1}{2 - (1 + \lambda t)^{1/\lambda}}\right)
\]
\[
= \frac{2^\alpha (1 + \lambda t)^{x/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha}} - \sqrt{2} \frac{2^{\alpha-1/2} (1 + \lambda t)^{x/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2(\alpha-1/2)}}
\]
\[
= \sum_{n=0}^{\infty} a_n^{(\alpha)} (x; \lambda) \frac{t^n}{n!} - \sqrt{2} \sum_{n=0}^{\infty} a_n^{(\alpha-1/2)} (x; \lambda) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left(a_n^{(\alpha)} (x; \lambda) - \sqrt{2} a_n^{(\alpha-1/2)} (x; \lambda)\right) \frac{t^n}{n!}.
\]

Comparing the coefficients \( \frac{t^n}{n!} \) yields the following theorem.

**Theorem 3.6** For \( n \geq 0 \), we have

\[
a_n^{(\alpha)} (x + 1; \lambda) = 2 a_n^{(\alpha)} (x; \lambda) - \sqrt{2} a_n^{(\alpha-1/2)} (x; \lambda).
\]

Now, if we differentiate both sides of (1.4) with respect to \( t \), we write

\[
\frac{d}{dt} \left(\frac{2^\alpha (1 + \lambda t)^{x/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha}}\right)
\]
\[
= 2^\alpha \left(\frac{x (1 + \lambda t)^{(x-\lambda)/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha}} + 2\alpha \frac{(1 + \lambda t)^{(x-\lambda+1)/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha+1}}\right).
\]  \hspace{1cm} (3.10)

If we replace \( x \) by \( x_1 + x_2 + \lambda \) and evaluate the terms on both sides of (3.10), separately, then, we have

\[
\frac{d}{dt} \left(\frac{2^\alpha (1 + \lambda t)^{(x_1+x_2+\lambda)/\lambda}}{\left(2 - (1 + \lambda t)^{1/\lambda}\right)^{2\alpha}}\right) = \sum_{n=0}^{\infty} a_n^{(\alpha)} (x_1 + x_2 + \lambda; \lambda) \frac{t^n}{n!},
\]  \hspace{1cm} (3.11)
\[(x_1 + x_2 + \lambda) \frac{2^{\alpha} (1 + \lambda t)^{(x_1+x_2)/\lambda}}{2 - (1 + \lambda t)^{1/\lambda}}^{2\alpha} = (x_1 + x_2 + \lambda) \sum_{n=0}^{\infty} a_n^{(\alpha)} (x_1 + x_2; \lambda) \frac{t^n}{n!} \] 

and

\[\sqrt{2\alpha} \frac{2^{\alpha+1/2} (1 + \lambda t)^{(x_1+x_2+1)/\lambda}}{(2 - (1 + \lambda t)^{1/\lambda})^{2(\alpha+1/2)}} = \sqrt{2\alpha} \sum_{n=0}^{\infty} a_n^{(\alpha+1/2)} (x_1 + x_2 + 1; \lambda) \frac{t^n}{n!}. \] 

Substitute (3.11), (3.12) and (3.13) in (3.10) to reach the following theorem.

**Theorem 3.7** For \( n \geq 0 \), the degenerate Fubini-type polynomials \( a_n^{(\alpha)} (x; \lambda) \) satisfy the recurrence relation

\[a_{n+1}^{(\alpha)} (x_1 + x_2 + \lambda; \lambda) = (x_1 + x_2 + \lambda) a_n^{(\alpha)} (x_1 + x_2; \lambda) + \sqrt{2\alpha} a_n^{(\alpha+1/2)} (x_1 + x_2 + 1; \lambda) . \] 

Letting \( \lambda \to 0 \) and taking \( x_1 + x_2 = y \) in (3.14) allow us to derive the following formula for Fubini-type polynomials

\[a_{n+1}^{(\alpha)} (y) = ya_n^{(\alpha)} (y) + \sqrt{2\alpha} a_n^{(\alpha+1/2)} (y + 1). \]

**Remark 3.8** From the relationships (3.9) and (1.5), the counterpart identities in Theorems 3.6 and 3.7 can be presented for degenerate Apostol-Bernoulli polynomials and degenerate Apostol-Euler polynomials of order \( \alpha \).

### 4 Conclusion

In our recent study, we have introduced and dealt with degenerate version of Fubini-type polynomials. Utilizing the Faà di Bruno formula and some properties of Bell polynomials, and generating function methods, we have derived some new explicit formulas, closed forms and recurrence relations for degenerate Fubini-type polynomials and numbers, and Fubini-type polynomials and numbers, defined by Kilar and Simsek [14]. Furthermore, by associating the degenerate Fubini-type polynomials with degenerate Apostol-Bernoulli polynomials and degenerate Apostol-Euler polynomials of order \( \alpha \), we have presented analog identities for them. As a final note, a relation involving degenerate Fubini-type polynomials and degenerate Apostol-Genocchi polynomials of order \( \alpha \), defined by [13, Eq. 2.6] can be given and further relations can be obtained.
References

[1] L. Carlitz, *Degenerate Stirling Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51–88;

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition. D. Reidel Publishing Co., Dordrecht 1974.

[3] M. C. Dağlı, *A new recursive formula arising from a determinantal expression for weighted Delannoy numbers*, Turkish J. Math **45** (2021), no. 1, 471–478; available online at https://doi.org/10.3906/mat-2009-92

[4] M. C. Dağlı, *Closed formulas and determinantal expressions for higher-order Bernoulli and Euler polynomials in terms of Stirling numbers*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 1, Art. No. 32, 8 pages; available online at https://doi.org/10.1007/s13398-020-00970-9

[5] L. Dai and H. Pan, *Closed forms for degenerate Bernoulli polynomials*, Bull. Aust. Math. Soc. **101** (2020), no. 2, 207–217.

[6] R. Dere, Y. Simsek, and H. M. Srivastava, *A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra*, J. Number Theory **133** (2013), 3245–3263; available online at https://doi.org/10.1016/j.jnt.2013.03.004

[7] B.-N. Guo, D. Lim, and F. Qi, *Maclaurin series expansions for powers of inverse (hyperbolic) sine, for powers of inverse (hyperbolic) tangent, and for incomplete gamma functions, with applications to second kind Bell polynomials and generalized logsine function*, arXiv (2021), available online at https://arxiv.org/abs/2101.10686v5

[8] B.-N. Guo and F. Qi, *An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind*, J. Anal. Number Theory **3** (2015), no. 1, 27–30.

[9] Y. He, S. Araci, and H. M. Srivastava, *Some new formulas for the products of the Apostol type polynomials*, Adv. Differ. Equ. 2016, 1–18 (2016)
[10] Y. He, S. Araci, H. M. Srivastava, and M. Acikgoz, *Some new identities for the Apostol–Bernoulli polynomials and the Apostol–Genocchi polynomials*, Appl. Math. Comput. 262 (2015), 31–41; available online at https://doi.org/10.1016/j.amc.2015.03.132

[11] S. Hu and M.-S. Kim, *Two closed forms for the Apostol–Bernoulli polynomials*, Ramanujan J. 46 (2018), no. 1, 103–117; available online at https://doi.org/10.1007/s11139-017-9907-4

[12] L. Kargın, *Some formulae for products of geometric polynomials with applications*, J. Integer Seq. 20 (2017), Article 17.4.4, 15 pp.

[13] S. Khan, T. Nahid, and M. Riyasat, *On degenerate Apostol-type polynomials and applications*, Boletín de la Sociedad Matemática Mexicana, 25 (2019), 509–528; available online at https://doi.org/10.1007/s40590-018-0220-z

[14] N. Kılar and Y. Simsek, *A new family of Fubini type numbers and polynomials associated with Apostol–Bernoulli numbers and polynomials*, J. Korean Math. Soc. 54 (2017), 1605–1621; available online at https://doi.org/10.4134/JKMS.j160597

[15] N. Kılar and Y. Simsek, *Identities and relations for Fubini type numbers and polynomials via generating functions and p–adic integral approach*, Publ. Inst. Math. (Belgr.) 106 (120) (2019), 113–123; available online at https://doi.org/10.2298/PIM1920113K

[16] D. S. Kim, T. Kim, H.-I. Kwon, and J.-W. Park, *Two variable higher-order Fubini polynomials*, J. Korean Math. Soc. 55 (2018), no. 4, 975–986; available online at https://doi.org/10.4134/JKMS.j170573

[17] T. Kim, D. S. Kim, and G.-W. Jang, *A note on degenerate Fubini polynomials*, Proc. Jangjeon Math. Soc. 20 (2017), 521–531; available online at http://dx.doi.org/10.17777/pjms2017.20.4.521

[18] D.-Q. Lu and H. M. Srivastava, *Some series identities involving the generalized Apostol type and related polynomials*, Comput. Math. Appl. 62 (2011), 3591–3602; available online at https://doi.org/10.1016/j.camwa.2011.09.010
[19] Q.-M. Luo, *On the Apostol Bernoulli polynomials*, Central European J. Math. **2** (2004), 509–515; available online at https://doi.org/10.2478/BF02475959

[20] Q.-M. Luo, *Apostol–Euler polynomials of higher order and Gaussian hypergeometric functions*, Taiwanese J. Math. **10** (2006), 917–925; available online at https://doi.org/10.11650/twjm/1500403883

[21] Q.-M. Luo and H. M. Srivastava, *Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials*, J. Math. Anal. Appl. **308** (2005), 290–302; available online at https://doi.org/10.1016/j.jmaa.2005.01.020

[22] Q.-M. Luo and H. M. Srivastava, *Some relationships between the Apostol Bernoulli and Apostol Euler polynomials*, Comput. Math. Appl. **51** (2006), 631–642; available online at https://doi.org/10.1016/j.camwa.2005.04.018

[23] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer-Verlag, Berlin, 1924.

[24] F. Qi, *Determinantal expressions and recurrence relations for Fubini and Eulerian polynomials*, J. Interdiscip. Math. **22** (2019), 317–335; Available online at https://doi.org/10.1080/09720502.2019.1624063

[25] F. Qi, *A determinantal expression and a recursive relation of the Delannoy numbers*, Acta Univ. Sapientiae Math. **13** (2021), no. 1, in press; arXiv prprint (2020), available online at https://arxiv.org/abs/2003.12572

[26] F. Qi, *An explicit formula for the Bell numbers in terms of the Lah and Stirling numbers*, Mediterr. J. Math. **13** (2016), no. 5, 2795–2800; available online at https://doi.org/10.1007/s00009-015-0655-7

[27] F. Qi, *Derivatives of tangent function and tangent numbers*, Appl. Math. Comput. **268** (2015), 844–858; available online at http://dx.doi.org/10.1016/j.amc.2015.06.123

[28] F. Qi, V. Cernanova, X.-T. Shi, and B.-N. Guo, *Some properties of central Delannoy numbers*, J. Comput. Appl. Math. **328** (2018), 101–115; available online at https://doi.org/10.1016/j.cam.2017.07.013
[29] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory, **159** (2016), 89–100; Available online at https://doi.org/10.1016/j.jnt.2015.07.021

[30] F. Qi and B.-N. Guo, *Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials*, Mediterr. J. Math. 14 (2017), no. 3, Article 140, 14 pages; available online at https://doi.org/10.1007/s00009-017-0939-1

[31] F. Qi and B.-N. Guo, *Some Determinantal expressions and recurrence relations of the Bernoulli polynomials*, Mathematics, **4** (2016), no. 4, 1–11.

[32] F. Qi and B.-N. Guo, *Viewing some ordinary di erential equations from the angle of derivative polynomials*, Iran. J. Math. Sci. Inform. 16 (2021), no. 1, in press.

[33] F. Qi, D. Lim, and B.-N. Guo, *Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113** (2019), 1–9; available online at https://doi.org/10.1007/s13398-017-0427-2

[34] F. Qi, D.-W. Niu, and B.-N. Guo, *Some identities for a sequence of unnamed polynomials connected with the Bell polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 2, 557–567; available online at https://doi.org/10.1007/s13398-018-0494-z

[35] F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, *Closed formulas and identities for the Bell polynomials and falling factorials*, Contrib. Discrete Math. **15**(1) (2020), 163–174; available online at https://doi.org/10.11575/cdm.v15i1.68111

[36] F. Qi and M.-M. Zheng, *Explicit expressions for a family of the Bell polynomials and applications*, Appl. Math. Comput. **258** (2015), 597–607; available online at https://doi.org/10.1016/j.amc.2015.02.027

[37] S. K. Sharma, W. A. Khan, and C. S. Ryoo, *A parametric kind of the degenerate Fubini numbers and polynomials*, Mathematics **8**(3) (2020), Article no: 405; available online at https://doi.org/10.3390/math8030405
[38] H. M. Srivastava and C. Kızılataş, *A parametric kind of the Fubini-type polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 113 (2019), 3253–3267; available online at https://doi.org/10.1007/s13398-019-00687-4

[39] D.-D. Su and Y. He, *Some Identities for the two variable Fubini polynomials*, Mathematics 7(2) (2019), Article no: 115; available online at https://doi.org/10.3390/math7020115

[40] C.-F. Wei and F. Qi, *Several closed expressions for the Euler numbers*, J. Inequal. Appl. 2015, 219 (2015).