ON IRREDUCIBLE MAPS

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The notion of irreducible map was introduced by M. Auslander and I. Reiten in [3] and plays an important role in the representation theory of artin algebras.

We recall that an artin ring $A$ is said to be an artin algebra if its center $C$ is an artin ring and $A$ is finitely generated left $A$-module. Now choose a complete set $P_1, \ldots, P_s$ of representatives of the isomorphism classes of indecomposable projectives in $\text{mod}(A)$, we will denote by $\text{pr} A$ the full subcategory of $\text{mod} A$ whose objects are $P_1, \ldots, P_s$. A map $g: X \to Y$ in $\text{mod}(A)$ is said to be irreducible if $g$ is neither a split monomorphism nor a split epimorphism and for any commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow{h} \\
Z & & 
\end{array}
$$

$f$ is a splittable monomorphism or $h$ is a splittable epimorphism.

We study irreducible maps in $\text{mod}(A)$ by using properties of the Jacobson radical of $\text{mod}(A)$. We recall that the Jacobson radical of $\text{mod}(A)$ is the subfunctor $\text{rad}$ of the two variable functor $\text{Hom}: (\text{mod}(A))^{op} \times \text{mod}(A) \to \text{Ab}$ defined by

$$
\text{rad}(X, Y) = \{f \in \text{Hom}(X, Y) \mid 1 - gf \text{ is invertible for every } g \in \text{Hom}(Y, X)\}
$$

$$
= \{f \in \text{Hom}(X, Y) \mid 1 - fh \text{ is invertible for any } h \in \text{Hom}(X, Y)\}.
$$

It is easy to prove that if $X$ and $Y$ are indecomposables, then $\text{rad}(X, Y)$ consists of all nonisomorphisms, from $X$ to $Y$.

We can prove the following result:

**Proposition 1.** Let $C$ and $D$ be indecomposables in $\text{mod}(A)$. Then

(a) A map $f: C \to X$ is irreducible iff $f \in \text{rad}(C, D)$ and $f \notin \text{rad}^2(C, D)$, where $\text{rad}^2(C, D)$ consists of all maps of the form $t_1t_2$ with $t_2 \in \text{rad}(C, X)$ and $t_1 \in \text{rad}(X, D)$.

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(b) A map

\[ g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : C \rightarrow D \oplus \cdots \oplus D \]

is irreducible iff \( g_1, \ldots, g_n, \text{rad}(C, D)/\text{rad}^2(C, D) \) are linearly independent over \( \text{End}(D)/\text{rad} \text{End}(D) \).

Using properties of \( \text{rad} \) we obtain the result stated below:

**Theorem 1.** Suppose \( f \in \text{rad}(X, Y) \). Then the following statements are equivalent

(a) \( f \) is irreducible,

(b) For any splittable monomorphism \( u : C \rightarrow X \) with \( C \) indecomposable the composed map \( fu \) is irreducible.

(c) For any splittable epimorphism \( v : Y \rightarrow D \) with \( D \) indecomposable \( vf \) is irreducible.

If the artin algebra \( \Lambda \) is infinite and of finite representation type then we have rather precise information about irreducible maps between indecomposables in \( \text{mod}(\Lambda) \). We are able to prove the following

**Theorem 2.** Suppose \( \Lambda \) is an infinite artin algebra of finite representation type and let \( X \) and \( Y \) be indecomposables in \( \text{mod}(\Lambda) \). If we denote by \( d \) and \( d' \) the dimensions of \( \text{Hom}(X, Y)/\text{rad}^2(X, Y) \) over \( \text{End}(X)/\text{rad} \text{End}(X) \) and over \( \text{End}(Y)/\text{rad} \text{End}(Y) \) respectively, then \( dd' \leq 3 \).

We also get information about the middle term of any almost split sequence in \( \text{mod}(\Lambda) \). We recall that the short exact sequence \( 0 \rightarrow A \xrightarrow{u} B \xrightarrow{v} C \rightarrow 0 \) is said to be almost split if (a) the sequence does not split. (b) For any \( h : X \rightarrow C \) nonsplitable epi there exists \( g \) with \( vg = h \). (c) For any \( h' : A \rightarrow Y \) non-splittable mono there exists \( g' : B \rightarrow Y \) with \( gu = h' \).

**Theorem 3.** Assume \( \Lambda \) is an infinite artin algebra of finite representation type. Let

\[ 0 \rightarrow A \rightarrow n_1B_1 \oplus n_2B_2 \oplus \cdots \oplus n_sB_s \rightarrow A' \rightarrow 0 \]

be an almost split sequence in \( \text{mod}(\Lambda) \) with \( B_i \) indecomposable, \( B_i \nless B_j \) if \( i \neq j \) and \( n_iB_i \) means the direct sum of \( n_i \) copies of \( B_i \).

(a) \( n_i \leq 3 \) for every \( i = 1, \ldots, s \).

(b) If for some \( i \) \( n_i \geq 2 \), then \( n_j = 1 \) if \( j \neq i \).

(c) If \( \Lambda \) is a finite dimensional algebra over an algebraically closed field \( k \), then \( n_i = 1 \) for any \( i \).
The idea of the proof is the following:

We can assume \( \Lambda \) indecomposable, denote by \( C \) the center of \( \Lambda \). Then \( K = C/\text{rad} C \) is a field. Now consider \( X \) and \( Y \) in \( \text{mod}(\Lambda) \). We define the set \( I(X, Y) = \{ f \in \text{rad}(X, Y)/\text{rad}^2(X, Y) \mid f \text{ is an irreducible map} \} \). Now we put \( K_X^* = \text{units of End}(X)/\text{rad End}(X) \), and the same for \( K_Y \). In some cases \( I(X, Y) \) is an affine \( K \)-variety and the \( K \)-algebraic group \( K_X^* \times K_Y^* \) acts on \( I(X, Y) \). Then using properties of irreducible maps [4] and the Gabriel-Tits argument [5] we get our theorem.

Following M. Auslander a skelletally small preadditive category \( C \) is said to be prevariety if: (a) Any object in \( C \) can be decomposed as finite sum of indecomposable objects in \( C \). (b) Any idempotent in \( C \) splits. (c) \( \text{End}(M) \) for any indecomposable object of \( C \) is a local ring. We recall that the Auslander graph \( \Gamma(C) \) of \( C \) is defined as follows:

Choose a complete set of representatives \( M_i, i \in I, \) of all the isomorphism classes of indecomposable objects in \( C \). Then the vertices of \( \Gamma(C) \) are the elements of \( I \). We put an arrow from \( i \) to \( j \) iff there exists an irreducible map \( f: M_i \rightarrow M_j \).

Now we define the Auslander species of \( C \) by attaching to each point \( i \in \Gamma(C) \) the division ring \( K_i = \text{End}(M_i)/\text{rad End}(M_i) \), and to each arrow \( i \rightarrow j \) in \( \Gamma(C) \) the \( K_i - K_j \) bimodule \( M_{ij} = \text{rad}(M_i, M_j)/\text{rad}^2(M_i, M_j) \).

We note that when \( \Lambda \) is an artin algebra and \( C = \text{pr}(\Lambda) \) then the Auslander species of \( C \) is just the Dlab-Ringel species of \( \Lambda \) (see [6]). As in [7] we can associate to the Auslander species of \( \text{mod}(\Lambda) \) a tensor category \( T_{\Lambda} \).

We define \( \text{rad}^i(X, Y) \) in similar way as \( \text{rad}^2(X, Y) \) was defined. We put \( \text{rad}^\infty(X, Y) = \bigcap_{i \geq 1} \text{rad}^i(X, Y) \). Here \( \text{rad}^\infty(X, Y) \) is an ideal in \( \text{mod}(\Lambda) \). Then as in [7] we have the following:

**Proposition 2.** If \( \Lambda \) is either an hereditary artin algebra of finite representation type or a finite dimensional algebra over an algebraically closed field \( k \), then there exists a full functor \( G: T \rightarrow \text{mod}(\Lambda)/\text{rad}^\infty \) such that for any indecomposable module \( M \) in \( \text{mod}(\Lambda) \) there exists \( M' \) in \( T_{\Lambda} \) with \( G(M) \cong M' \).

Observe that if \( \Lambda \) is of finite representation type then \( \text{rad}^\infty(X, Y) = 0 \) for any \( X \) and \( Y \) in \( \text{mod}(\Lambda) \).

Using properties of hereditary artin algebras proved in [2] we can describe \( \text{Ker} \ G \) in terms of almost split sequences.

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