On the growth factor upper bound for Aasen’s algorithm *

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Abstract

Aasen’s algorithm factorizes a symmetric indefinite matrix $A$ into $A = P^T L L^T$, where $P$ is a permutation matrix, $L$ is unit lower triangular with its first column being the first column of identity matrix, and $T$ is tridiagonal. In this note, we give a growth factor upper bound for Aasen’s algorithm which is much smaller than that given by Higham. We also show that the upper bound we given is not tight when matrix dimension is greater than or equal to 6.

Keywords: Aasen’s algorithm, growth factor, $L L^T$ factorization

AMS subject classifications. 15A23, 65F05, 65G50

1 Introduction

Aasen’s algorithm [1] factorizes a symmetric indefinite matrix $A \in \mathbb{R}^{n \times n}$ as

$$P A P^T = L L^T,$$

where $P$ is a permutation matrix, $L$ is unit lower triangular with first column $e_1$, i.e.

$$L = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ l_{n-1,2} & \ldots & 1 \\ l_{n2} & \ldots & l_{n,n-1} & 1 \end{pmatrix},$$

and

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} & t_{23} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ l_{n,n-1} & t_{n-1,n} \\ t_{n,n-1} & t_{nn} \end{pmatrix},$$

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with \( t_{ij} = t_{ji} \) \((i \neq j)\) is symmetric tridiagonal. It is well known that since it can effectively solve symmetric indefinite linear systems \( Ax = b \) when \( A \) is a dense matrix, the Aasen’s algorithm has been used in LAPACK [2].

To introduce Aasen’s algorithm, we initially ignore pivoting and assume that the first \( k \) columns of \( L \) and the first \( k - 1 \) columns of \( T \) are known. We show how to obtain the \((k + 1)\)st column of \( L \) and the \( k \)th column of \( T \) in the next. Let

\[
H = TL^T,
\]

then matrix \( H \) is upper Hessenberg, i.e.

\[
H = \begin{pmatrix}
h_{11} & h_{12} & \cdots & h_{1,n-1} & h_{1n} \\
h_{21} & h_{22} & \cdots & h_{2,n-1} & h_{2n} \\
h_{32} & h_{33} & \cdots & h_{3,n-1} & h_{3n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
h_{n,n-1} & h_{n,n-2} & \cdots & h_{nn}
\end{pmatrix},
\]

and

\[
A = LH.
\]

Then for \( 1 \leq k \leq n \),

\[
h_{ik} = t_{i,i-1}l_{k,i-1} + t_{i,i}l_{k,i} + t_{i,i+1}l_{k,i+1}, \quad i = 1, \ldots, k - 1,
\]

where \( t_{1,0} = 0, \ l_{k,0} = 0 \).

\[
h_{kk} = a_{kk} - \sum_{j=1}^{k-1} l_{kj}h_{jk},
\]

\[
v_i = a_{i,k} - \sum_{j=1}^{k} l_{ij}h_{jk}, \quad i = k + 1, \ldots, n,
\]

\[
t_{kk} = h_{kk} - t_{k,k-1}l_{k,k-1},
\]

\[
t_{k,k+1} = t_{k+1,k} = v_{k+1},
\]

\[
l_{i,k+1} = \frac{v_i}{v_{k+1}}, \quad i = k + 2, \ldots, n.
\]

Without pivoting this factorization may break down, since \( v_{k+1} \) might be zero or a very small \( v_{k+1} \) might cause numerical problems when computing \( l_{i,k+1} \). This can be solved easily by finding \( r \) such that \( |v_r| = \max\{|v_i|: i = k + 1, \ldots, n\} \), and swapping \( v_{k+1} \) and \( v_r \). (If \( v \) is all zeros, \( l_{i,k+1} \) can be set to zero.) If such a permutation is performed, it should make corresponding interchanges in \( A \) and \( L \). This pivoting strategy ensures that \( l_{ij} \leq 1 \) for \( i > j \).

For Aasen’s algorithm, Higham in [3] gave a growth factor upper bound, which is \( 4^n - 2 \), but also said whether this bound is attainable for \( n \geq 4 \) is an open problem. Cheng in [4] constructed an example in which the growth factor bound is 4 for \( n = 3 \), and reported growth factor of 7.99 for \( n = 4 \) and 14.61 for \( n = 5 \) by using direct search method.
In this note, we report that a growth factor upper bound for Aasen’s algorithm is $2^{n-1}$, which is much smaller than $4^{n-2}$ given by Higham. Moreover, we also show that the upper bound $2^{n-1}$ we given is not tight when $n \geq 6$.

Without loss of generality, in the following, we assume that $\max_{i,j} |a_{ij}| = 1$ for the symmetric matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$, and no interchanges $(A \overset{\text{def}}{=} PA^T)$ are needed in Aasen’s algorithm for convenience.

This paper is organized as follows. Section 2 presents the upper bounds for the entries of the tridiagonal matrix $T$ of Aasen’s algorithm. Section 3 reports the growth factor bound of Aasen’s algorithm. We draw some conclusions in Section 4.

2 Upper bounds of entries of $T$

We first present a lemma which gives the upper bounds for the entries of the factor matrix $T$.

**Lemma 2.1.** Let symmetric matrix $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ and $\max_{i,j} |a_{ij}| = 1$ and

$$A \overset{\text{def}}{=} PA^T = LTL^T$$

in Aasen’s algorithm, $L = (l_{ij})_{i,j=1}^n$ is an unit lower triangular matrix, $L(:,1) = e_1$ (the first column of identity matrix), $|l_{ij}| \leq 1$ and $T = (t_{ij})_{i,j=1}^n$ is a symmetric triangular matrix. Then the following inequalities hold:

$$\begin{cases}
|t_{11}| \leq 1, & |t_{21}| \leq 1, & |t_{22}| \leq 1, & |l_{i2} t_{21}| \leq 1 & (3 \leq i \leq n) \\
|l_{i,j-1} t_{j-1,j} + l_{ij} t_{jj} + l_{i,j+1} t_{j+1,j}| \leq 2^{j-2} & (2 \leq j < i \leq n) \\
|t_{n,n-1} l_{n-1,n} + t_{nn}| \leq 2^{n-2} & (2d)
\end{cases}$$

**Proof.** We use mathematical induction to prove this lemma. When $n = 3$, the lemma holds true (see example in Cheng [4]). Assume when $n = k$, the lemma holds true. Then we consider $n = k + 1$. By direct calculation,

$$\begin{aligned}
a_{k+1,1} &= l_{k+1,1} t_{21} \\
a_{k+1,q} &= \sum_{j=2}^{q} l_{qj} (l_{k+1,j-1} t_{j-1,j} + l_{k+1,j} t_{jj} + l_{k+1, j+1} t_{j+1,j}), & (2 \leq q \leq k) \\
a_{k+1,k+1} &= \sum_{j=2}^{k} l_{k+1,j} (l_{k+1,j-1} t_{j-1,j} + l_{k+1,j} t_{jj} + l_{k+1, j+1} t_{j+1,j}) \\
&\hspace{1cm} + l_{k+1,k} t_{k,k+1} + t_{k+1,k+1}
\end{aligned}$$

Since $\max_{i,j} |a_{ij}| = 1$, then

$$|l_{k+1,2} t_{21}| \leq 1$$

By equation (3a), and $|l_{gh}| \leq 1$ (1 $\leq g, h \leq n$), then

$$|l_{k+1,2} t_{22} + l_{k+1,3} t_{32}| \leq 1$$

By equation (3b), and $|l_{gh}| \leq 1$ (1 $\leq g, h \leq n$), then
\[ |k_{l+1,2} t_{23} + l_{k+1,3} t_{33} + l_{k+1,4} t_{43}| \leq 2 \]
\[ |l_{k+1,3} t_{34} + l_{k+1,4} t_{44} + l_{k+1,5} t_{54}| \leq 2^2 \]
\[ \vdots \]
\[ |l_{k+1,k-2} t_{k-2,k-1} + l_{k+1,k-1} t_{k-1,k-1} + l_{k+1,k} t_{k,k-1}| \leq 2^{k-3} \]
\[ |l_{k+1,k-1} t_{k-1,k} + l_{k+1,k} t_{kk} + l_{k+1,k+1} t_{k+1,k}| \leq 2^{k-2} \]

Furthermore, from equation \((3c)\)
\[ |l_{k+1,k} t_{k,k+1} + t_{k+1,k+1}| \leq \sum_{j=2}^{k} 2^{j-2} + 1 = 2^{k-1} \] (4)

Using equation \((3b)\) again
\[
a_{k+1,k} = \sum_{j=2}^{k} l_{kj} (l_{k+1,j-1} t_{j-1,j} + l_{k+1,j} t_{jj} + l_{k+1,j+1} t_{j+1,j})
\]
\[
= \sum_{j=2}^{k-1} l_{k+1,j} (l_{k,j-1} t_{j-1,j} + l_{kj} t_{jj} + l_{kj+1} t_{j,j+1})
\]
\[
+ l_{k+1,k} (l_{k,k-1} t_{k,k-1} + t_{kk}) + t_{k+1,k}
\]

Then
\[ |t_{k+1,k}| \leq \sum_{j=2}^{k-1} 2^{j-2} + 2^{k-2} + 1 = 2^{k-1} \]

by inequality \[(4)\]
\[ |t_{k+1,k+1}| \leq 2^{k} \]

Therefore, when \(n = k + 1\) the lemma still holds true. We thus prove the lemma. \(\square\)

3 Growth factor bound for Aasen’s algorithm

In this section, we present our main result.

Since the \(n\)th row of matrix \(A\) have the following form:

\[
a_{nj} = \sum_{j=2}^{q} l_{dj} (l_{n,j-1} t_{j-1,j} + l_{nj} t_{jj} + l_{n,j+1} t_{j+1,j}), \quad (2 \leq q \leq n-1) \] (5a)
\[
a_{nn} = \sum_{j=2}^{n-1} l_{nj} (l_{n,j-1} t_{j-1,j} + l_{nj} t_{jj} + l_{n,j+1} t_{j+1,j}) + l_{n,n-1} t_{n-1,n} + t_{nn} \] (5b)

From Lemma \(2.1\) and if \(|t_{n,n}| = 2^{n-1}\), without loss of generality, we assume that \(t_{n,n} = 2^{n-1}\), then by equation \((5b)\)

\[
l_{nj} (l_{n,j-1} t_{j-1,j} + l_{nj} t_{jj} + l_{n,j+1} t_{j+1,j}) = -2^{j-2}, \quad (2 \leq j \leq n-1) \]
\[
l_{n,n-1} t_{n-1,n} = -2^{n-2} \]
\[
t_{nn} = 2^{n-1} \] (6)
and by equation (9) and inequalities (2b) - (2d), we obtain
\[ l_{nj} = \pm 1, \quad (2 \leq j \leq n - 1). \tag{7} \]

Substituting (6) and (7) into (5a), therefore
\[ \left| \sum_{j=2}^{q} l_{nj} l_{nj} 2^{j-2} \right| \leq 1, \quad (2 \leq q \leq n - 1). \]

thus
\[ l_{nj} l_{nj} l_{nj} = -1, \quad (2 \leq j < q \leq n - 1). \tag{8} \]

furthermore,
\[ l_{ij} l_{ij} l_{ij} = -1, \quad (2 \leq j < q < i \leq n - 1). \tag{9} \]

In the next, we will derive an upper bound of \( t_{nn} \) from equation (8) and (9).

At first, we rewrite equation (9) as
\[
\begin{aligned}
& l_{nj} (l_{n,j-1} t_{j-1,j} + l_{nj} t_{jj} + l_{n,j+1} t_{j+1,j}) = -2^{j-2} + \delta_{j-2}, \quad (2 \leq j \leq n - 1) \\
& l_{n,n-1} t_{n,n-1} = -2^{n-2} + \delta_{n-2} \\
& t_{nn} = 2^{n-1} - \delta
\end{aligned}
\tag{10}
\]

where \( \delta = \sum_{j=0}^{n-2} \delta_{j} \), and \( 0 \leq \delta_{j} \leq 2^{j+1} \) (\( 0 \leq j \leq n - 2 \)). Hence the upper bound of \( t_{nn} \) is equivalent to optimization problem
\[
\begin{aligned}
\delta &= \min \sum_{j=0}^{n-2} \delta_{j} \\
\text{s.t.} \quad 0 &\leq \delta_{j} \leq 2^{j+1} \quad (0 \leq j \leq n - 2)
\end{aligned}
\tag{11}
\]

We will seek more constraints on \( \delta_{j} \) in the following. Substituting equation (8) and (10) into equation (5a), then
\[ 0 \leq \delta_{q-2} - \sum_{j=2}^{q-1} \delta_{j-2} \leq 2, \quad (3 \leq q \leq n - 1). \tag{12} \]

From equation (8), equation (10) can become
\[
\begin{aligned}
& - l_{j,j-1} t_{j-1,j} + t_{jj} - l_{j+1,j} t_{j+1,j} = -2^{j-2} + \delta_{j-2}, \quad (2 \leq j \leq n - 2) \\
& - l_{n,n-2} t_{n-2,n-1} + l_{n,n-1} t_{n-1,n-1} = 2^{n-3} + \delta_{n-3} - \delta_{n-2} \\
& l_{n,n-1} t_{n,n-1} = -2^{n-2} + \delta_{n-2} \\
& t_{nn} = 2^{n-1} - \delta
\end{aligned}
\tag{13}
\]

For \( 3 \leq i \leq n - 1 \), since the entries of matrix \( A \) have the following form:
\[
\begin{aligned}
& a_{ij} = \sum_{j=2}^{q} l_{ij} (l_{i,j-1} t_{j-1,j} + l_{ij} t_{jj} + l_{i,j+1} t_{j+1,j}), \quad (2 \leq q \leq i - 1) \tag{14a} \\
& a_{ii} = \sum_{j=2}^{i-1} l_{ij} (l_{i,j-1} t_{j-1,j} + l_{ij} t_{jj} + l_{i,j+1} t_{j+1,j}) + l_{i,i-1} t_{i-1,i} + t_{ii} \tag{14b}
\end{aligned}
\]

5
then multiply \( l_{iq} \) on both side of equation (14a) and by equation (9), for \( 2 \leq q < i \leq n - 1, \)

\[
l_{iq} a_{iq} = \sum_{j=2}^{q-1} l_{iq} l_{qj} (l_{i,j-1} t_{j-1,j} + l_{ij} t_{jj} + l_{i,j+1} t_{j+1,j})
\]

\[
= \sum_{j=2}^{q-1} (l_{j,j-1} t_{j-1,j} - t_{jj} + l_{j+1,j} t_{j+1,j}) - l_{q,q-1} t_{q-1,q} - t_{qq} + l_{iq} l_{i,q+1} t_{q+1,q}
\]

\[
= \begin{cases} 
    \sum_{j=2}^{q-1} (2^{j-2} - \delta_{j-2}) - 2^{q-2} + \delta_{q-2} & (q < i - 1) \\
    \sum_{j=2}^{q-1} (2^{j-2} - \delta_{j-2}) - 2^{q-2} + \delta_{q-2} + 2 l_{q+1,q} t_{q+1,q} & (q = i - 1) \\
    -1 + \delta_{q-2} - \sum_{j=2}^{q-1} \delta_{j-2} & (2 \leq q < i - 1 \leq n - 2) \\
    -1 + \delta_{q-2} - \sum_{j=2}^{q-1} \delta_{j-2} + 2 l_{q+1,q} t_{q+1,q} & (2 \leq q = i - 1 \leq n - 2)
\end{cases}
\]

(15)

where the third equation holds because of equation (9) and (13).

Since \( l_{iq} = \pm 1, \) and \( |a_{iq}| \leq 1, \) then \( -1 + \delta_{q-2} - \sum_{j=2}^{q-1} \delta_{j-2} \leq 1 \) produces the same condition as inequality (12). For the second equality in equation (15), we have

\[
-m_{q-2} - \sum_{j=0}^{q-3} \delta_{j} \leq l_{q+1,q} t_{q+1,q} \leq 1 - \frac{m_{q-2} - \sum_{j=0}^{q-3} \delta_{j}}{2}, \quad (2 \leq q \leq n - 2).
\]

(16)

By equations (9) and (13), then equation (14b) satisfies

\[
a_{ii} = \begin{cases} 
    4 l_{i,i-1} t_{i,i-1} + l_{i+1,i} t_{i+1,i} - \sum_{j=0}^{i-2} (2^j - \delta_j) & (3 \leq i \leq n - 2) \\
    4 l_{i,i-1} t_{i,i-1} + \sum_{j=0}^{i-2} \delta_j - \delta_{i-1} + 1 & (i = n - 1)
\end{cases}
\]

(17)

Since \( |a_{ii}| \leq 1, \) then

\[
2^{q-1} - 2 - \sum_{j=0}^{q-2} \delta_j \leq 4 l_{q,q-1} t_{q,q-1} + l_{q+1,q} t_{q+1,q} \leq 2^{q-1} - \sum_{j=0}^{q-2} \delta_j \quad (3 \leq q \leq n - 2),
\]

(18)

and

\[
-\frac{1}{2} + \frac{\delta_{n-2} - \sum_{j=0}^{n-3} \delta_j}{4} \leq l_{n-1,n-2} t_{n-1,n-2} \leq \frac{\delta_{n-2} - \sum_{j=0}^{n-3} \delta_j}{4}.
\]

(19)

By inequalities (16) and (18), for \( 3 \leq q \leq n - 2, \)

\[
\frac{2^n - 6 - 3 \sum_{j=0}^{q-3} \delta_j - \delta_q - 2}{8} \leq l_{q,q-1} t_{q,q-1} \leq \frac{2^n - 3 \sum_{j=0}^{q-3} \delta_j - \delta_q - 2}{8}.
\]

(20)
So inequalities (16) and (20) are established simultaneously for $3 \leq q \leq n - 2$ when
\[
\begin{aligned}
&\left\{ \begin{array}{c}
\delta_{q-3} - \frac{\sum_{j=0}^{q-4} \delta_j}{2} \leq \frac{2^q - 3 \sum_{j=0}^{q-3} \delta_j - \delta_{q-2}}{8} \\
\frac{2^q - 6 - 3 \sum_{j=0}^{q-3} \delta_j - \delta_{q-2}}{8} \leq 1 - \frac{\delta_{q-3} - \sum_{j=0}^{q-4} \delta_j}{2}
\end{array} \right. \\
&\text{that is}
\end{aligned}
\]
\[
2^q - 14 \leq 7 \sum_{j=0}^{q-4} \delta_j - \delta_{q-3} + \delta_{q-2} \leq 2^q, \quad (3 \leq q \leq n - 2). \tag{21}
\]

Let $q = n - 2$ in inequality (16) and from inequality (19),
\[
\begin{aligned}
&\left\{ \begin{array}{c}
- \frac{\delta_{n-4} - \sum_{j=0}^{n-5} \delta_j}{2} \leq \frac{\delta_{n-2} - \sum_{j=0}^{n-3} \delta_j}{4} \\
- \frac{1}{2} + \frac{\delta_{n-2} - \sum_{j=0}^{n-3} \delta_j}{4} \leq 1 - \frac{\delta_{n-4} - \sum_{j=0}^{n-5} \delta_j}{2}
\end{array} \right. \\
&\text{then}
\end{aligned}
\]
\[
-6 \leq 3 \sum_{j=0}^{n-5} \delta_j - \delta_{n-4} + \delta_{n-3} - \delta_{n-2} \leq 0. \tag{22}
\]

Therefore plugging inequalities (12), (21) and (22) into optimization problem (11),
\[
\begin{aligned}
&\hat{\delta} = \min \sum_{j=0}^{n-2} \delta_j \\
&\text{s.t.} \quad 0 \leq \delta_j \leq 2^{j+1} \quad (0 \leq j \leq n - 2) \\
&0 \leq \delta_{q-2} - \sum_{j=0}^{q-3} \delta_j \leq 2 \quad (3 \leq q \leq n - 1) \\
&2^q - 14 \leq 7 \sum_{j=0}^{q-4} \delta_j - \delta_{q-3} + \delta_{q-2} \leq 2^q \quad (3 \leq q \leq n - 2) \\
&- 6 \leq 3 \sum_{j=0}^{n-5} \delta_j - \delta_{n-4} + \delta_{n-3} - \delta_{n-2} \leq 0 \quad (n \geq 4)
\end{aligned}
\]

Since $2^q - 14 > 0$ when $q \geq 4$, and observing the above optimization problem, it is easy to obtain
\[
\begin{aligned}
&\left\{ \begin{array}{c}
\delta = 0 \quad (n \leq 5) \\
\delta > 0 \quad (n \geq 6)
\end{array} \right. \tag{24}
\end{aligned}
\]

Therefore by equation (10), $t_{nn} < 2^{n-1}$ when $n \geq 6$. Using the similar argument, we also can prove that $t_{nn} > -2^{n-1}$ when $n \geq 6$. Hence, $|t_{nn}| < 2^{n-1}$ when $n \geq 6$, this motivates the following Theorem.

**Theorem 3.1.** Let symmetric indefinite matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, then the growth factor bound of Aasen’s algorithm is $2^{n-1}$, but that is not tight when $n \geq 6$. 


We construct examples for \( n = 4 \) and \( n = 5 \) to show the upper bound is attainable. In the process of solving optimization problem (23), the upper bound is attainable for \( n = 4 \) and \( n = 5 \) only when the following conditions are satisfied:

\[
\begin{align*}
&\ell_{n-1, n-2} = 0 \\
&l_n, n-1 = \pm 1
\end{align*}
\]  

(25)

We show the following examples with \( 0 \leq \delta \leq 2 \):

\[
A = \begin{pmatrix}
1 & 1 & -1 & 1 \\
1 & \frac{\delta}{2} - 1 & 1 & \delta - 1 \\
-1 & 1 & 1 & -1 \\
1 & \delta - 1 & -1 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & -1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-1 & \frac{\delta}{2} - 1 \\
\delta & 2 + \frac{\delta}{2} \\
-4 & 8 - 2\delta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\]

and

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & \frac{\delta}{2} + \frac{\delta}{2} & 1 - \frac{\delta}{2} & 1 - \delta \\
1 & 1 - \frac{\delta}{2} & 1 - \delta & 2\delta - 1 \\
1 & \delta - 1 & 1 & 1 & -1 \\
-1 & 1 - \delta & 2\delta - 1 & -1 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 - \frac{\delta}{2} & -1 + \frac{5\delta}{2} \\
1 - \delta & 4 - \delta - 8 + 4\delta \\
-8 + 4\delta & 16 - 12\delta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & -1 \\
1 & 1 & 1 \\
1 & 1
\end{pmatrix}
\]

When \( \delta \to 0^+ \) in these two matrices \( A \), the upper bound is attainable, and condition (25) are also satisfied. For the following \( n = 6 \) example,

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & -1 \\
1 & \delta/2 - 3/4 & -1/2 & \delta - 1 & \delta - 1 & 1 - \delta \\
1 & -1/2 & -1 & -1 & 1 & -1 \\
1 & \delta - 1 & -1 & 5\delta - 3 & 1 & 2\delta - 1 \\
1 & \delta - 1 & 1 & 1 & 1 & -1 \\
-1 & 1 - \delta & -1 & 2\delta - 1 & -1 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & -1 \\
0 & 1 \\
0 & -1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 - \frac{3}{4} + \frac{\delta}{2} & \frac{1}{4} - \frac{1}{2}\delta \\
\frac{1}{4} - \frac{1}{2}\delta & -\frac{3}{4} + \frac{\delta}{2} \\
-3 + 3\delta & -1 \\
\delta & 8 - 3\delta - 16 + 8\delta \\
-16 + 8\delta & 32 - 20\delta
\end{pmatrix}
\]
Since $|a_{ij}| \leq 1$, then $\frac{2}{5} \leq \delta \leq \frac{4}{5}$, and the growth factor is $24 < 2^5$ when $n = 6$.

**Remark 3.1.** When using Aasen’s algorithm [1] or partitioned version of Aasen’s algorithm [5] on these above two matrices, we will find that the growth factor bound is less than $2^{n-1}$. That’s because $L(i+2 : n, i+1)$ is set to zero when $t_{i+1,i} = 0$ in these algorithms. If we modify $L(i+2 : n, i+1) = 1$ in Aasen’s algorithm and partitioned algorithm when $t_{i+1,i} = 0$, then we can obtain the bound $2^{n-1}$. These two matrices $A$ are approximately singular when $\delta \to 0^+$, but we can obtain the same growth factor if we embed these two matrices $A$ in a well-conditioned matrix [6] of twice the dimension:

$$
\begin{pmatrix}
A \\
(1-\epsilon)I \\
0
\end{pmatrix}
$$

where $0 < \epsilon < 1$.

4 Conclusion

Since it is very effective to solve symmetric indefinite linear systems, variants of Aasen’s algorithm have been used in LAPACK. We estimate the growth factor upper bound of Aasen’s algorithm by direct computation and obtain its value is $2^{n-1}$, which is much smaller than that given by Higham. We also prove the bound we given is not tight when matrix size is greater than 6 and construct some matrix examples to verify our theoretic analysis for $n = 4, 5, 6$.

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