Surface charge of horizon symmetries of a black hole with supertranslation field

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Abstract

Near-horizon symmetries are studied for static black hole solutions to Einstein equations containing supertranslation field. We consider general diffeomorphisms which preserve the gauge and near-horizon structure of the metric. Diffeomorphisms are generated by the vector fields and form a group of near-horizon symmetries. The densities of variation of the surface charge associated to asymptotic horizon symmetries of the metric are calculated in different coordinate systems connected by ”large” transformations which change supertranslation field in metric. Integrability of the variations of surface charge is considered in different coordinate systems. The case of integrable variation of the surface charge is studied in detail.

1 Introduction

The final state of gravitational collapse is a stationary metric diffeomorphic to the metric of the Kerr black hole [1, 2, 3]. General diffeomorphisms contain pure gauge transformations which are changes of coordinates and ”large” transformations which change supertranslation field in metric. Physically ”large” transformations map a physical state to another physical state with a different cloud of soft particles [4, 5, 6].

Supertranslations naturally appear in a study of symmetries of the asymptotically flat of gravity at the null infinity initiated by Bondi, van der Burg, Metzner and Sachs [7, 8]. The infinite-dimensional group of the asymptotic symmetries (BMS group) extends the Poincare group and contains a normal subgroup of supertranslations which are the angle-dependent translations of retarded time at the null infinity [9].

BMS algebra can be further enhanced to contain superrotations [10, 11, 12, 13]. Exponentiation of the infinitesimal supertranslation and superrotation generators produces finite transformations, but in distinction to supertranslations, exponentiation of the infinitesimal superrotations when acting on physical states leads to states with the energy unbounded from below [14]. One cannot introduce a physical state with a finite superrotation charge, but there exist conserved charges associated with supertranslations and superrotations [10, 11, 12, 13]. Supertranslation charges vanish except for a
charge corresponding to the mass of a state, but finite superrotation charges differ for states with different supertranslation fields.

BMS transformations are naturally formulated at the null infinity, but there is a complicated problem of extension of an asymptotically defined metric containing a supertranslation field in the bulk. In paper [14] a family of 4D vacua containing a supertranslation field was constructed in the bulk, and in paper [15] a solution-generation technique was developed and the black hole metrics diffeomorphic to the Schwarzschild metric and containing a supertranslation field were obtained.

In this paper we study the near-horizon symmetries of the black holes containing a supertranslation field. The near-horizon symmetries are the main characteristics of horizon microstates which in turn define thermodynamic properties, the entropy and evaporation of a black hole. Near-horizon symmetries were extensively investigated in a large number of papers (very incomplete list of refs. is [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39])

The near-horizon region is foliated by a set of surfaces enclosing the horizon surface and located at a distance $x$ from the horizon ($x$ is defined differently in different coordinate systems). Near-horizon symmetries are generated by transformations which preserve the horizon structure of a metric, and do not change the power of the leading in $x$ terms in the components of the metric considered at a near-horizon surface at a distance $x$ from the horizon, as $x \to 0$.

We consider the near-horizon transformations of the static black hole solutions of the Einstein equations containing a supertranslation field. Transformations are generated by the vector fields $\xi^\mu$. The metric variations $\delta g$ are elements of the tangent space to the space of metrics and are solutions to the linearized Einstein equations. On the tangent space is defined a bilinear presymplectic form. The presymplectic Lee-Wald form [20] is

$$ w^{\mu}^{\text{LW}}(\delta_1 g, \delta_2 g, g) = \delta_1 \Theta^\mu(\delta_2 g, g) - \delta_2 \Theta^\mu(\delta_1 g, g) - \Theta^\mu(g, \delta_{[1,2]} g), $$

where $\Theta^\mu$ is the boundary term in a variation of the Einstein action

$$ \delta(\sqrt{-g}R) = E(g)\delta g + \partial_\mu \Theta^\mu(\delta g, g), $$

and $E(g) = 0$ for a solution of the Einstein equations. Other forms of the presymplectic structures [22, 37] differ from the Lee-Wald form by the terms vanishing on solutions of the linearized Einstein equations. A symplectic 2-form is defined as an integral over a codimension-1 spacelike surface

$$ \Omega^{\text{LW}}(\delta_1 g, \delta_2 g, g) = \int_\Sigma w^{\mu}^{\text{LW}}(\delta_1 g, \delta_2 g, g) d^3 x_\mu. $$

The Lee-Wald presymplectic form contracted with a metric perturbation generated by a vector field $\xi^\mu$ and any metric variation $\delta g$ from the tangent space of metrics satisfies the on-shell relation

$$ w^{\mu}^{\text{LW}}(\delta g, \delta \xi g, g) \simeq dK^{\text{LW}}_\xi(\delta g, g), $$

where $K^{\text{LW}}_\xi$ is the Iyer-Wald surface charge form [21, 23] and the equality is valid up to terms vanishing on-shell. Variation of the surface charge associated with a transformation generated by a vector field $\xi^\mu$ is defined as

$$ \delta H_\xi = \oint_{\partial \Sigma} K^{\text{LW}}_\xi, $$

where integration is over a surface enclosing the horizon [20, 21, 22, 37, 23, 39].

The black hole solutions containing supertranslation field are obtained from the Schwarzschild solution (in isotropic spherical coordinates) by applying the "large" transformations containing supertranslation field. Supertranslation field is a real function on the unit sphere. The event horizon of
a metric containing supertranslation field constructed in [15] (\(\rho\)-system) is located at a surface which depends on supertranslation field. It is possible to construct another coordinate system ("\(r\)’-system), connected to the \(\rho\)-system by a "large" transformation, in which the horizon is located at the surface \(r = 2M\), where \(M\) is the mass of black hole.

We calculate the surface charge forms \(K^{\mu\nu}_\xi\) in different coordinate systems connected by "large" transformations and also within \(\rho\) and \(r\) systems in coordinate systems corresponding to different parametrizations of the unit sphere on which supertranslation field is defined.

To obtain the surface charge \(H_\xi\), variation \(\delta H_\xi\) should be integrated over the space of metrics. The unique surface charge is obtained, if the integral over the space of metrics is independent of a path of integration. We find that in the general case variation of the surface charge \(\delta H_\xi\) cannot be written as a variation of a certain functional over the space of metrics, and integration over the space of metrics does not yield a path-independent charge. We discuss a special case in which the surface charge of horizon symmetries is obtained in the closed form.

The paper is organized as follows.

In Sect.2 we review the form of the static vacuum metric containing supertranslation field in a \(\rho\)-system obtained in [15]. Next, by a "large" transformation we transform the metric to the \(r\)-system. In both \(\rho\) and \(r\)-systems we obtain the metrics in different parametrisations of the unit sphere on which is defined supertranslation field.

In Sect.3 we study diffeomorphisms preserving the near-horizon form of the metric in \(\rho\) and \(r\)-systems. We find constraints on the generators of transformations preserving the gauge and the near-horizon form of the metric.

In Sect.4 we consider supertranslations preserving the gauge and near-horizon structure of metric which are extendable in the bulk. Supertranslations form a group under the modified bracket [15]. A case of supertranslation field depending only on an angle \(\theta\) is considered in detail. It is shown that in the case of supertranslation field depending only on \(\theta\) the requirement that supertranslation preserves the gauge and the form of the metric fixes parameter of transformation through the supertranslation field \(C(\theta)\).

In Sect.5 we calculate variations of the surface charge corresponding to horizon symmetries in the \(\rho\) and \(r\)-systems. Variation of the charge is obtained by integration of the charge surface forms over the surfaces enclosing the horizon and located at a distance \(x\) from the horizon.

In the \(\rho\)-system, because of the specific form of the surface enclosing the horizon, the variation of the charge \(\delta H_\xi\) receives contributions from integrals of the surface charge densities \(K^{\mu\nu}_\xi\) with different components \((\mu,\nu) = (t,\rho), (t,\theta), (t,\varphi)\). In the \(r\)-system the only contribution is from integration of the component with \((\mu\nu) = (r,t)\) over the unit sphere.

In the charge densities we separate the leading in \(x\) terms in accordance with the power of \(x\) coming from the determinant of metric so that the resulting expression for the variation of the charge is independent on \(x\).

In Sect. 6 we discuss integrability of the variation of the surface charge in the case of supertranslation field depending only on a spherical angle \(\theta\). In the \(\rho\)-system, the variation of the charge cannot be presented as a variation of a functional over the space of metrics. In the \(r\)-system we show that the variation of the charge is integrable. Although the variations of the charge have different forms in the \(\rho\) and \(r\)-systems, performing the change of coordinates, we show that the the expressions are equal.

The last section contains a brief summary of results.
2 Static vacuum solution of the Einstein equations with supertranslation field

We begin this section with a short review of a black hole solution with a supertranslation field constructed in [15]. Next, we transform the metric to a form in which the horizon is located at the surface \( r = 2M \).

The vacuum solution of the Einstein equations containing supertranslation field \( C(z^a) \) is
\[
    ds^2 = g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + g_{ab}dz^adz^b =
    -\frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2}dt^2 + (1 + M/2\rho_s)^4 \left[ d\rho^2 + (((\rho - E)^2 + U)\gamma_{ab} + (\rho - E)C_{ab})dz^adz^b \right],
\] where \( z^a \) are coordinates on the unit sphere. Supertranslation field \( C(z^a) \) is a real regular function on the unit sphere. Coordinates on the sphere \( z^a \) can be realized as spherical coordinates \( \theta, \varphi \) with the metric \( ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \), or as projective coordinates \( z^1 = z = \cot \frac{\theta}{2} e^{i\varphi}, z^2 = \bar{z} = \cot \frac{\theta}{2} e^{-i\varphi} \) with the metric \( ds^2 = 2\gamma_{zz}dzd\bar{z}, \quad \gamma_{zz} = 2e^{-2\psi}, \quad \psi = \ln(1 + |z|^2) \). Covariant derivatives \( D_a \) are defined with respect to the corresponding metric on the sphere. Here
\[
    \rho_s(\rho, C) = \sqrt{(\rho - C - C_{00})^2 + D_a C D^a C}.
\] \( C_{00} \) is the lowest spherical harmonic mode of \( C(z^a) \). In the following we do not write \( C_{00} \) explicitly understanding \( C \to C - C_{00} \). The horizon of metric (2.1) is located at the surface \( \rho_s = M/2 \). Here \( \rho \subset (0, +\infty) \). The tensor \( C_{ab} \) and the functions \( U \) and \( E \) are defined as
\[
    C_{ab} = -(2D_a D_b - \gamma_{ab} D^2) C,
    U = \frac{1}{8} C_{ab} C^{ba},
    E = \frac{1}{2} D^2 C + C,
\] (2.3)

The metric (2.1) in coordinates \((\rho, \theta, \varphi)\) with supertranslation field \( C(\theta, \varphi) \) was obtained from the Schwarzschild metric
\[
    ds^2 = -(\frac{1 - M/2\rho_s}{1 + M/2\rho_s})^2 dt^2 + (1 + M/2\rho_s)^4 (dx_s^2 + dy_s^2 + dz_s^2)
\]
by the diffeomorphism [15]
\[
    x_s = (\rho - C) \sin \theta \cos \varphi + \partial_\varphi C \sin \varphi / \sin \theta - \partial_\theta C \cos \theta \cos \varphi,
    y_s = (\rho - C) \sin \theta \sin \varphi - \partial_\varphi C \cos \varphi / \sin \theta - \partial_\theta C \cos \theta \sin \varphi,
    z_s = (\rho - C) \cos \theta - \partial_\theta C \sin \theta.
\] (2.4)

In coordinates \((\rho, \theta, \varphi)\) the transformed metric is
\[
    ds^2 = g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + 2g_{\theta\varphi}d\theta d\varphi + \tilde{g}_{\theta\theta}d\theta^2 + \tilde{g}_{\varphi\varphi}d\varphi^2 =
    -\frac{(\rho_s - M/2)^2}{(\rho_s + M/2)^2} dt^2 + (1 + M/2\rho_s)^4 \left[ d\rho^2 + 2(\rho - E)C_{\theta\varphi}d\theta d\varphi +
    + \left( \rho - E + \frac{1}{2} C_{\theta\theta} \right)^2 d\theta^2 + \sin^2 \theta \left( \rho - E - \frac{1}{2} C_{\theta\theta} \right)^2 d\varphi^2 \right],
\] (2.5)
where
\[ C_{\theta\theta} = -C'' + C' \cot \theta + \frac{\ddot{C}}{\sin^2 \theta}, \quad C_{\varphi \varphi} = -C_{\theta\theta} \sin^2 \theta, \quad C_{\theta \varphi} = -2(\dot{C}' - \dot{C} \cot \theta). \]

Here dot and prime are derivatives over \( \varphi \) and \( \theta \).

In variables \( \rho, z^a \) the metric with supertranslation field \( C(z, \bar{z}) \) has a form
\[
ds^2 = g_{tt} dt^2 + g_{\rho \rho} [d\rho^2 + \bar{g}_{ab} dz^a dz^b] = -\frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2} dt^2 + (1 + M/2\rho_s)^4 [d\rho^2 + 2((\rho - E)^2 + U)\gamma_{zz} dz d\bar{z} + (\rho - E)(C_{zz} dz dz + C_{\bar{z}\bar{z}} d\bar{z} d\bar{z})],
\]
where
\[ C_{zz} = -2D_z \partial_z C, \quad C_{\bar{z}\bar{z}} = -2D_{\bar{z}} \partial_{\bar{z}} C, \quad C_{zz} = 0. \]

Transformation from \((\theta, \varphi, \rho)\) to \((z, \bar{z}, \rho)\) is a pure gauge transformation.

If the supertranslation field depends only on \(|z|\), or in coordinates \((r, \theta, \varphi)\) only on \(\theta\), the metric simplifies with \(g_{\theta \varphi} = 0\) (2.5). On the other hand, if the supertranslation field depends only on \(z/\bar{z}\), or on \(\varphi\), the metric retains its general form.

Next, we transform the metric (2.6) to new variables \((r, z^a)\), where \(z^a = (z, \bar{z})\) or \((\theta, \varphi)\). The new variables are chosen so that in new variables the \(tt\) component of metric is equal to [40]
\[ g_{tt} = 1 - 2M/r \equiv V. \]

A variable \(r \geq 2M\) is defined through the variables \(\rho, \theta, \varphi\) by the relation
\[ r = \rho_s(\rho, C) \left(1 + \frac{M}{2\rho_s(\rho, C)}\right)^2. \]

Inversely, \(\rho\) is expressed through \(r\) as
\[ \rho = C + \sqrt{\frac{K^2}{4} - D_a CD_a C} = C + \frac{K}{2} \sqrt{1 - b^2}, \]
where we introduced the functions
\[ K = r - M + rV^{1/2}, \quad b_a = \frac{2D_a C}{K}, \quad b^2 = b_a b^a. \]

Differential \(d\rho(r, z^a)\) is
\[
d\rho = \rho_a dz^a + \rho_r dr \]
\[ \rho_a = \frac{K}{2\sqrt{1 - b^2}} \left(b_a \sqrt{1 - b^2} - \frac{\partial_a b^2}{2}\right), \quad \rho_r = \frac{K}{2\sqrt{1 - b^2} rV^{1/2}}. \]

Using the relations
\[ g_{tt} = \frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2} = V, \quad g_{\rho \rho} = (1 + M/2\rho_s)^4 = \frac{4r^2}{K^2}, \]
(2.11)
and introducing the transformed metric components (in variables \((r, z^a)\) the metric components are written with hats).

\[
\hat{g}_{rr} = g_{\rho\rho} \rho^2, \quad \hat{g}_{ra} = g_{\rho\rho} \rho \rho^a, \quad \hat{g}_{ab} = g_{\rho\rho} (\tilde{g}_{ab} + \rho_a \rho_b),
\]

we obtain the metric in a form

\[
ds^2 = \hat{g}_{tt} dt^2 + \hat{g}_{rr} dr^2 + 2\hat{g}_{ra} dr dz^a + \hat{g}_{ab} dz^a dz^b =
\]

\[
= -V dt^2 + \frac{4r^2}{K^2} \left[ \rho^2 dr^2 + 2\rho_r \rho_a dr dz^a + (\tilde{g}_{ab} + \rho_a \rho_b) dz^a dz^b \right].
\]

For the above expressions to be well-defined, we require that \(1 - b^2 > 0\). Because \(K\) is an increasing function of \(r\) which has its minimum at \(r = 2M\), the sufficient condition is \(1 - (2DC/M)^2 > \text{const} > 0\).

In the following we work in the units \(M = 1\).

### 3 Diffeomorphisms preserving the near-horizon form of the metric

#### 3.1 The metric in variables \(\rho, z^a\)

In this section we study diffeomorphisms which preserve the near-horizon form and the gauge of the metric (2.6) in the \(\rho\)-system. Near-horizon foliation of the space-time is done as follows. Let \(x\) be a parameter specifying the distance from a near-horizon to horizon surface (the choice of \(x\) depends on the choice of coordinates and is specified below). Horizon is located at the surface \(\rho_s = 1/2\) (\(\rho_s\) is defined in (2.2)). Near-horizon surface is defined as a surface \(\rho_s = 1/2 + x\). Assuming that at the horizon the equation \(\sqrt{(\rho - C)^2 + (D_a C) D^a C} = 1/2\) has a unique solution,

\[
\rho_H(z, \tilde{z}) = C + \sqrt{1/4 - D_a C D^a C},
\]

by continuity the equation \(\rho_s = \sqrt{(\rho - C)^2 + (D_a C) D^a C} = 1/2 + x\) in some vicinity of \(\rho_s = 1/2\) also has a unique solution

\[
\rho(x, z^a) = C + \sqrt{(1/2 + x)^2 - D_a C D^a C}.
\]

For a small \(|x| \ll 1\) we obtain

\[
\rho(x, z^a) \simeq \rho_H(z^a) + \tilde{x},
\]

where

\[
\tilde{x} = \frac{x}{2\sqrt{1/4 - (DC)^2}}.
\]

There are two branches of \(\rho: \rho = C \pm \sqrt{\rho_s^2 - (DC)^2}\). To have a smooth limit to the Schwarzschild metric, \(C \to 0\), we choose the plus sign.

In the near-horizon region the metric has a form

\[
ds^2 = (\tilde{g}_{tt} \tilde{x}^2 + O(\tilde{x}^3)) dt^2 + (\tilde{g}_{\rho\rho} + O(\tilde{x})) d\rho^2 + (\tilde{g}_{ab} + O(\tilde{x})) dz^a dz^b.
\]

Here \(\tilde{g}_{\mu\nu}\) are the \(O(\tilde{x}^0)\) parts of \(g_{\mu\nu}\).

We consider transformations generated by the vector field

\[
\xi^i = \xi^t \partial_t + \xi^\rho \partial_\rho + \xi^a \partial_a,
\]

where

\[
\xi^t = \xi^\rho = \xi^a = 0.
\]

In the close vicinity of the surface \(\rho_s \simeq 1/2\) the metric components are

\[
\hat{g}_{tt} \simeq V dt^2, \quad \hat{g}_{rr} \simeq \rho^2 dr^2, \quad \hat{g}_{ra} \simeq \rho \rho^a dr dz^a, \quad \hat{g}_{ab} \simeq (\tilde{g}_{ab} + \rho_a \rho_b) dz^a dz^b.
\]

For the above expressions to be well-defined, we require that \(1 - b^2 > 0\). Because \(K\) is an increasing function of \(r\) which has its minimum at \(r = 2M\), the sufficient condition is \(1 - (2DC/M)^2 > \text{const} > 0\).

In the following we work in the units \(M = 1\).
\( \xi^i \) are assumed to be independent of \( t \). General near-horizon transformations are required to preserve the gauge of the metric and the power \( \tilde{x}^a \) of the leading in \( \tilde{x} \) terms in the difference \( g_{\mu\nu}(\tilde{x}) - g_{\mu\nu}(0) \) up to a numerical coefficient at the leading term in \( \tilde{x} \).

The metric is written in the gauge \( g_{\rho a} = g_{\rho t} = g_{\alpha t} = 0 \). Transformations which preserve the gauge satisfy the relations

\[
L_\xi g_{\rho a} = \partial_\rho \xi^b g_{ba} + \partial_a \xi^b g_{\rho b} = 0, \tag{3.6}
\]
\[
L_\xi g_{\rho t} = \partial_\rho \xi^t g_{tt} + \partial_t \xi^b g_{\rho b} = 0, \tag{3.7}
\]
\[
L_\xi g_{\alpha t} = \partial_a \xi^t g_{tt} + \partial_t \xi^b g_{\alpha b} = 0. \tag{3.8}
\]

Conditions (3.7) and (3.8) give \( \xi^t = \text{const.} \)

At the near-horizon surface \( \rho_s = 1/2 + x \), or \( \rho = \rho_H + \tilde{x} \), and the component \( g_{tt} \) is

\[
g_{tt} \simeq -4(\rho_H - C)^2 \tilde{x}^2 + O(\tilde{x}^3). \tag{3.9}
\]

Under the action of transformation generated by vector field \( \xi^k \) the component \( g_{tt} \) is transformed as

\[
L_\xi g_{tt} = -4\frac{\rho_s - 1/2}{(\rho_s + 1/2)^3} L_\xi \rho_s, \tag{3.10}
\]

where

\[
L_\xi \rho_s = (\xi^b \partial_\rho + \xi^a D_a) \rho_s = \frac{\xi^\rho 2(\rho - C) + \xi^\rho (-2(\rho - C)D_a C + D_a(D_b C D^b C))}{2\rho_s}. \tag{3.11}
\]

To preserve the near-horizon behavior of \( g_{tt} \) (3.9), it is necessary that

\[
\xi^\rho 2(\rho - C) + \xi^\rho (-2(\rho - C)D_a C + D_a(D_b C D^b C)) = O(\tilde{x}). \tag{3.12}
\]

At the horizon, this condition gives the relation connecting \( \xi^\rho \) and \( \xi^a \)

\[
\xi^\rho (\rho_H - C) + \xi^\rho ((-\rho_H + C)D_a C + (D_a D_b C D^b C)|_{\rho = \rho_H} = 0. \tag{3.13}
\]

Because \( g_{\rho \rho} \) is a function of \( \rho_s \), condition (3.12) also ensures that \( L_\xi g_{\rho \rho} = O(\tilde{x}) \).

### 3.2 The metric in variables \( r, z^a \)

In variables \( (r, z^a) \) the horizon of the metric (2.13) is located at the surface \( r = 2 \). In the foliation of the near-horizon region through \( \rho_s = 1/2 + x \), the near-horizon surface is at \( r = 2 + \tilde{x} \), where \( \tilde{x} = 2x^2 + O(\tilde{x}^3) \). In the near-horizon region the metric (2.13) has a form

\[
ds^2 = \hat{g}_{tt} dt^2 + \hat{g}_{rr} dr^2 + 2\hat{g}_{ra} dr dz^a + \hat{g}_{ab} dz^a dz^b = \\
= (-\hat{g}_{tt} + O(\tilde{x}^2)) dt^2 + \left( \frac{\hat{g}_{rr}}{\tilde{x}} + O(\tilde{x}^{-1/2}) \right) d\tilde{x}^2 + 2\left( \frac{\hat{g}_{ra}}{\tilde{x}^{1/2}} + O(\tilde{x}^0) \right) d\tilde{x} dz^a + \\
+ (\hat{g}_{ab,0} \tilde{x}^0 + O(\tilde{x}^{1/2})) dz^a dz^b. \tag{3.14}
\]

\( \hat{g}_{\mu\nu} = O(\tilde{x}^0) \) are the coefficients at the leading in \( \tilde{x} \) terms in the metric components. The metric (3.14) is written in the gauge

\[
\hat{g}_{tt} = \hat{g}_{ta} = 0.
\]
The near-horizon transformations are generated by the vector fields

\[ \chi^k = \chi^t \partial_t + \chi^r \partial_r + \chi^a \partial_a. \]  

(3.15)

Transformations preserving the gauge conditions are

\[ L \chi \hat{g}_{rt} = \partial_r \chi^t \hat{g}_{tt} + \partial_t \chi^r \hat{g}_{rr} + \partial_r \chi^a \hat{g}_{ar} = 0 \]
\[ L \chi \hat{g}_{at} = \partial_a \chi^t \hat{g}_{tt} + \partial_t \chi^r \hat{g}_{ra} + \partial_r \chi^b \hat{g}_{ba} = 0. \]  

(3.16)

From conditions (3.16) we obtain that \( \chi^t = \text{const}. \) Transformations preserving the leading in \( \hat{x} \) behavior of the metric components are

\[ L \chi \hat{g}_{tt} = \chi^r \partial_r \hat{g}_{tt} = O(\hat{x}), \]
\[ L \chi \hat{g}_{rr} = \chi^r \partial_r \hat{g}_{rr} + \chi^a \partial_a \hat{g}_{rr} + 2 \partial_r \chi^a \hat{g}_{ar} + 2 \partial_r \chi^r \hat{g}_{rr} = O(\hat{x}^{-1}), \]
\[ L \chi \hat{g}_{ar} = \chi^r \partial_r \hat{g}_{ar} + \chi^a \partial_a \hat{g}_{ar} + \partial_a \chi^b \hat{g}_{br} + \partial_r \chi^r \hat{g}_{ar} + \partial_r \chi^a \hat{g}_{ab} = O(\hat{x}^{-1/2}), \]
\[ L \chi \hat{g}_{ab} = \chi^r \partial_r \hat{g}_{ab} + \chi^a \partial_a \hat{g}_{ab} + \partial_a \chi^c \hat{g}_{cb} + \partial_r \chi^r \hat{g}_{ra} + \partial_r \chi^b \hat{g}_{ca} = O(\hat{x}^0). \]  

(3.17)

We look for the components of the vector field \( \chi^k \) in a form of expansion in powers in \( \hat{x}^{1/2} \)

\[ \chi^k = \chi^k_0 + \hat{x}^{1/2} \chi^k_1 + \cdots. \]

From the relations (3.17) we obtain the form of generators preserving the near-horizon form of the metric

\[ \chi^r = \chi^r_1 \hat{x}, \quad \chi^a = \chi^a_0 + \hat{x}^{1/2} \chi^a_1. \]  

(3.18)

The vector fields generating the near-horizon transformations form the Lie brackets

\[ [\chi(1), \chi(2)]^k = \chi^k (12), \]  

(3.19)

where

\[ \chi^r (12), 1 = \chi^b (1, 0) \overset{\rightarrow}{\partial} b \chi^a (2, 1), \]
\[ \chi^a (12), 0 = \chi^b (1, 0) \overset{\rightarrow}{\partial} b \chi^a (2, 0), \]
\[ \chi^a (12), 1/2 = \chi^b (1, 0) \overset{\rightarrow}{\partial} b \chi^a (2, 1/2) + 1/2 \left( \chi^r (1, 1) \chi^a (2, 1/2) - (1 \rightarrow 2) \right). \]  

(3.20)

The vector field (3.15) is connected with the vector field (3.5) by a transformation

\[ \chi^r = \xi^\rho \frac{\partial r}{\partial \rho} + \xi^a \frac{\partial r}{\partial z^a} = \xi^\rho \frac{\partial r}{\partial \rho_s} \frac{\partial \rho_s}{\partial r} + \xi^a \frac{\partial r}{\partial \rho_s} \frac{\partial \rho_s}{\partial z^a}, \]
\[ \chi^a = \xi^\rho \frac{\partial z^a}{\partial \rho} + \xi^b \frac{\partial z^a}{\partial z^b} = \xi^a. \]  

(3.21)

From (2.7) and (2.8), we have

\[ \partial r/\partial \rho_s = \frac{K^2 - 1}{K^2}, \quad \partial r/\partial \rho_s = \sqrt{1 - b^2}, \quad \partial \rho_s/\partial z^a = \frac{K}{4} \left[ -2b_a \sqrt{1 - b^2} + D_a b^2 \right]. \]  

(3.22)

Using the relations (3.22), we obtain

\[ \chi^t = \xi^t, \quad \chi^r = \frac{K^2 - 1}{K^2} \left[ \xi^\rho \sqrt{1 - b^2} + \xi^a \frac{K}{4} \left( -2b_a \sqrt{1 - b^2} + D_a b^2 \right) \right], \quad \chi^a = \xi^a. \]  

(3.23)
The expression in the square brackets in $\chi^r$ is the same as in (3.13). For $|\hat{x}| \ll 1$ we have
\[
K \simeq 1 + \sqrt{2\hat{x}}, \quad b_a = 2\partial_a C(1 - \sqrt{2\hat{x}}), \quad (K^2 - 1)/K^2 = O(\hat{x}^{1/2}). \tag{3.24}
\]
At the near-horizon surface the metric component $\hat{g}_{tt}$ is
\[
\hat{g}_{tt} = -\frac{\hat{x}}{2} + O(\hat{x}^2). \tag{3.25}
\]
To have the transformed metric component $\hat{g}_{tt}$ of order $O(\hat{x})$, the vector component $\chi^r$ should be of order $O(\hat{x})$. It follows that
\[
\xi^a K^2 - 1)/K^2 = O(\hat{x}^{1/2}). \tag{3.26}
\]
Noting that $\hat{x} \sim \bar{x}^2$, we see that condition (3.26) coincides with the condition (3.13).

4 Supertranslations extended in a bulk: symplectic transformations

In this section we consider supertranslations preserving the near-horizon form of the metric which are defined not only in a vicinity of the horizon, but extend to the bulk. Supertranslations which preserve the gauge of metric (2.1) were constructed in [15]. Supertranslation field in metric (in coordinates $\theta, \varphi$) transforms under supertranslations as
\[
\delta_T C(\theta, \varphi) = T(\theta, \varphi),
\]
where $T(\theta, \varphi)$ is an arbitrary smooth function on the unit sphere. Generator of supertranslations preserving the static gauge of the solution (2.1) in coordinates $(\rho, z^a)$ has a form
\[
\xi_T = T_{00}\partial_t - (T - T_{00})\partial_\rho + F^{ab}D_aT D_b, \tag{4.1}
\]
where
\[
F^{ab} = \frac{C^{ab} - 2\gamma^{ab}(\rho - E)}{2((\rho - E)^2 - U)}. \tag{4.2}
\]
Transformations (4.1) are defined in the bulk and form a commutative algebra under the modified bracket [15]
\[
[\xi_1, \xi_2]_{\text{mod}} = [\xi_1, \xi_2] - \delta_{T_1}\xi_2 + \delta_{T_2}\xi_1. \tag{4.3}
\]
It is explicitly verified that
\[
\xi_T^k \frac{\partial \rho_s}{\partial x^k} = \delta_T \rho_s, \tag{4.4}
\]
where
\[
\delta_T \rho_s(C) = \lim_{\varepsilon \to 0}[\rho_s(C + \varepsilon T) - \rho_s(C)]/\varepsilon, \tag{4.5}
\]
and
\[
\delta_T g_{pt} = \delta_T g_{at} = 0.
\]
General transformations (4.1) do not respect the near-horizon form of the metric (3.4) changing the component $g_{tt}$. To preserve the near-horizon form of the metric (3.4), transformation generated by (4.1) must satisfy condition (3.12).
If the supertranslation field depends only on \( \theta \), \( C = C(\theta) \), the generator of supertranslations simplifies to
\[
\xi_T = T_0 \partial_t - (T - T_0) \partial_\rho - \frac{T'}{\rho - C - C''} \partial_\theta.
\] (4.6)

The near-horizon structure of the metric is preserved provided the parameter of transformation (4.6) \( T(\theta) \) satisfies the relation
\[
-T(\rho_H - C) + T' C' = \mathcal{O}(\tilde{x}).
\] (4.7)

At the horizon, condition (4.7) is an ordinary differential equation on \( T(\theta) \) with the solution
\[
T(\theta) = a \exp \int^\theta d\theta \sqrt{1 - C'/C'}.
\] (4.8)

where \( a \) is an integration constant. Generators of supertranslations in coordinates \( (\rho, x^a) \) and \( (r, z^a) \) are connected by the transformation (3.21). In variables \( (r, z^a) \) generator of supertranslations is
\[
\chi_T = \rho \partial_t + \frac{K^2 - 1}{K^2} \left( -T \sqrt{1 - b^2} + \frac{K}{4} F^{ab} D_b T(-2g_{ab} \sqrt{1 - b^2} + D_a b^2) \right) \partial_r + F^{ab} D_b T \partial_a.
\] (4.9)

where in \( F^{ab} \) it is substituted \( \rho - C = K(1 - b^2)^{1/2}/2 \) Acting by the generator of supertranslations on the component \( \hat{g}_{tt} \), we obtain
\[
L_{\chi_T} \hat{g}_{tt} = \frac{2}{r^2} \frac{K^2 - 1}{K^2} \left( -T \sqrt{1 - b^2} + \frac{K}{4} F^{ab} D_b T(-2g_{ab} \sqrt{1 - b^2} + D_a b^2) \right).
\] (4.10)

In the near-horizon region the relations for \( K \) are (3.24). To preserve the form of \( \hat{g}_{tt} \), it is necessary that
\[
-T \sqrt{1 - b^2} + \frac{K}{4} F^{ab} D_b T(-2g_{ab} \sqrt{1 - b^2} + D_a b^2) = \mathcal{O}(\tilde{x}^{1/2}).
\] (4.11)

This imposes condition on \( T(z, \bar{z}) \)
\[
[-T \sqrt{1 - b^2} + \frac{1}{4} F^{ab} D_b T(-2g_{ab} \sqrt{1 - b^2} + D_a b^2)]_{r=0} = 0.
\] (4.12)

Eq.(4.12) for \( T \) is solved in Appendix A. In the case of supertranslated field in metric depending only on \( \theta \), relation (4.11) turns into (4.7). It is seen that in the near-horizon region in variables \( r, z^a \) the generator of supertranslations has the following structure
\[
\chi_T = \mathcal{O}(x^0) \partial_t + \mathcal{O}(x) \partial_z + \mathcal{O}(x^0) \partial_a.
\] (4.13)

5 Surface charge of asymptotic horizon symmetries

5.1 Variables \( \rho, z^a \).

In this section, we calculate the variation of the surface charge corresponding to diffeomorphisms preserving the near-horizon form of the metric. Calculations are performed both in \( \rho \) and \( r \)-systems. Variation of the surface charge associated with a symmetry generated by a vector field \( \xi^\mu \) is
\[
\delta H_{\xi}(g, h) = \frac{1}{4\pi} \int_{\partial \Sigma_{\nu}} (d^2 x)_{\mu\nu} \sqrt{-g} K^\mu_{\xi} K^{\nu}_{\xi}.
\] (5.1)
where \((d^2 x)_{\mu\nu} = (1/4) \epsilon_{\alpha\beta\mu\nu} dx^\alpha dx^\beta\). The charge density is

\[
K^\mu_\xi = \xi^\mu \nabla^\nu h - \xi^\mu \nabla_\sigma h^{\nu\sigma} + \xi_\sigma \nabla^\nu h^\nu_{\sigma} + \frac{1}{2} h \nabla^\nu \xi^\nu - h^{\mu\sigma} \nabla_\sigma \chi^\nu + \frac{\alpha}{2} h^{\mu\sigma} (\nabla^\nu \chi_\sigma + \nabla_\sigma \chi^\nu) + (\mu \leftrightarrow \nu),
\]

(5.2)

where \(\alpha = 1\) in the Barnich-Brandt form [22] and \(\alpha = 0\) in the Iyer-Wald form [21]. Here \(\partial \Sigma_\rho\) is a codimension 2 compact spacelike surface \(\rho - C - \sqrt{(1/2 + x)^2 + (DC)^2} = 0\) enclosing the horizon surface.

The metric variations are denoted as \(\delta g_{\mu\nu} \equiv h_{\mu\nu}\), the inverse variations are defined as \(h^{\mu\nu} = g^{\mu\rho} h_{\rho\lambda} g^{\lambda\nu}\), and the trace of metric variations is \(h = h_{\mu\nu} g^{\mu\nu}\).

First, we consider parametrization of the unit sphere by variables \(z, \bar{z}\). Because the metric is static, to the variation of the surface charge (5.1) contribute integrations over \((z, \bar{z}), (\rho, z)\) and \((\rho, \bar{z})\)

\[
\int \sqrt{-g} \epsilon_{\rho z} dz \wedge d\bar{z} K^{t\rho}, \quad \int \sqrt{-g} \epsilon_{t\rho z} d\rho \wedge dz K^{t\rho}, \quad \int \sqrt{-g} \epsilon_{t\bar{z} \rho} d\rho \wedge d\bar{z} K^{t\rho}.
\]

(5.3)

Because at the surface \(\Sigma_\rho\) we have \(\rho = \rho(z^a)\), we rewrite (5.3) as

\[
\delta H_\xi = \frac{1}{4\pi} \int dz \wedge d\bar{z} \sqrt{-g} \left[ K^{t\rho} - \rho_z K^{t\bar{z}} - \rho_{\bar{z}} K^{t\bar{z}} \right].
\]

(5.4)

The charge density \(K^{\rho t}_\xi\) in the Iyer-Wald form is

\[
K^{\rho t}_\xi = \xi^\rho \nabla^t h - \xi^\rho \nabla_\sigma h^{t\sigma} + \xi_\sigma \nabla^t h^\sigma - h^{\rho\sigma} \nabla_\sigma \xi^t - (\rho \leftrightarrow t).
\]

(5.5)

From (2.6) we have

\[
g(\rho, z, \bar{z}) = g_{tt} g_{\rho \rho} \tilde{g}^{(2)},
\]

where

\[
\tilde{g}^{(2)} = \tilde{g}_{zz} \tilde{g}_{z \bar{z}} - \tilde{g}_{z \bar{z}}^2 = \gamma^2 [(\rho - E)^2 - U]^2.
\]

In variables \(\rho, \theta, \varphi\) expressions (5.3)-(5.5) have the same functional form as above with the formal change \((z, \bar{z}) \to \theta, \varphi\), but \(\tilde{g}^{(2)}(\rho, \theta, \varphi)\) is

\[
\tilde{g}^{(2)} = \tilde{g}_{\theta \theta} \tilde{g}_{\varphi \varphi} - \tilde{g}^2_{\theta \varphi},
\]

and

\[
\delta H_\xi = \frac{1}{4\pi} \int d\theta \wedge d\varphi \sqrt{-g} \left[ K^{t\rho} - \rho_{\theta} K^{t\rho} - \rho_{\varphi} K^{t\rho} \right].
\]

At the surface \(\rho = \rho_H(z^a) + \tilde{x}\) we have \(\sqrt{-g} = O(\tilde{x})\), and to obtain a non-zero result for \(\delta H\), in \(K^{\rho t}_\xi\) we look for the terms of order \(O(\tilde{x}^{-1})\).

The five contributions in \(K^{\rho t}_\xi\) are

1. \(\xi^\rho \nabla^t h - \xi^\rho \nabla_\sigma h^{t\sigma} = \xi^\rho g^{tt} \partial_t h - \xi^t g^{\rho \rho} \partial_t h,\)
2. \(- \xi^\rho \nabla_\sigma h^{ts} + \xi^t \nabla_\sigma h^\rho_\sigma,\)
3. \(\xi_\sigma \nabla^t h^{ts} - \xi_\sigma \nabla^t h^\rho_\sigma,\)
4. \(\frac{h}{2} (\nabla^\rho \xi^t - \nabla^t \xi^\rho) = \frac{h}{2} (g^{\rho \sigma} \nabla_\sigma \xi^t - g^{ts} \nabla_\sigma \xi^\rho),\)
5. \(- h^{\rho \rho} \nabla_\sigma \xi^t + h^{ts} \nabla_\sigma \xi^\rho.\)

(5.6)

Because \(h\) is independent of \(t\) and \(g^{\rho \rho} = O(\tilde{x}^0)\), the two terms in the item 1 are of order \(\tilde{x}^0\).
The first term in the item 2
\[-\xi^a \nabla_s h^{ts} = -\xi^a (\nabla_t h^{tt} + \nabla_\rho h^{\rho t} + \nabla_a h^{ta}) = -\xi^a (2\Gamma^t_\rho h^{tt} + \Gamma^t_\rho h^{\rho t} + \Gamma^t_\rho h^{tt} + \Gamma^t_\rho h^{\rho t} + \Gamma^t_\rho h^{tt} + \Gamma^t_\rho h^{\rho t} + \Gamma^t_\rho h^{tt} + \Gamma^t_\rho h^{\rho t}) = 0\]
is zero, because all \Gamma vanish. The second term in the item 2 is
\[\xi^t \nabla_s h^{\rho s} = \xi^t (\nabla_t h^{pt} + \nabla_\rho h^{\rho p} + \nabla_a h^{pa}) = \xi^t (\Gamma^t_\rho h^{tt} + \Gamma^t_\rho h^{\rho t} + O(\bar{x}^0)).\]
The first term in the item 3 is transformed as
\[\xi_s \nabla^p h^{ts} = g^{p\rho} \xi_t \nabla_\rho h^{tt} + \xi_\rho \nabla_\rho h^{tt} + \xi_a \nabla_\rho h^{ta} + \xi_s \nabla^a h^{ts} = \]
\[= g^{p\rho} [\xi_t (\partial_\rho h^{tt} + 2\Gamma^t_\rho h^{tt}) + \xi_\rho (\Gamma^t_\rho h^{\rho t} + \Gamma^t_\rho h^{tt}) + O(\bar{x}^0)] = \xi_t g^{p\rho} (\partial_\rho h^{tt} + g^{tt} g_{tt,\rho} h^{t\rho}) + O(\bar{x}^0).\]
The leading in \(\bar{x}\) terms in this expression cancel
\[\xi_t g^{p\rho} (\partial_\rho h^{tt} + g^{tt} g_{tt,\rho} h^{t\rho}) = \xi_t g^{p\rho} \left(-\frac{2\bar{h}^{tt}}{\bar{x}^3} + \frac{\bar{g}^{tt}}{\bar{x}^2} 2\bar{x} \bar{g}_{tt} \bar{h}^{tt}\right) = 0,\]
and the remaining expression is of order \(\bar{x}^0\). The second term in the item 3
\[-\xi_s g^{tt} \nabla_t h^{pa} = -g^{tt} (\xi_t \nabla_t h^{pt} + \xi_\rho \nabla_t h^{\rho t} + \xi_a \nabla_t h^{ta}) = -\xi^t (\Gamma^t_\rho h^{tt} + \Gamma^t_\rho h^{\rho t} + O(\bar{x}^0))\]
cancels the corresponding expression in item 2.
Collecting the items 4 and 5, we obtain
\[K^p_\xi = \nabla s \xi^t (\frac{h}{2} g^{ps} - h^{pa}) - \nabla s \xi^t (\frac{h}{2} g^{ts} - h^{ts}) + O(\bar{x}^0).\] (5.7)
Taking into account the form of the metric in the \(\rho\)-system, we write \(K^p_\xi\) as
\[K^p_\xi = \nabla_\rho \xi^t \left(\frac{h}{2} g^{\rho t} - h^{\rho t}\right) - \nabla_t \xi^t \left(\frac{h}{2} g^{tt} - h^{tt}\right) + O(\bar{x}^0).\] (5.8)
The leading in \(\bar{x}\) part of \(K^p_\xi\) of order \(\bar{x}^{-1}\) is
\[K^p_\xi = \Gamma^t_\rho \xi^t \left(\frac{h}{2} g^{\rho t} - h^{\rho t}\right) - \Gamma^p_\xi \xi^t \left(\frac{h}{2} g^{tt} - h^{tt}\right) = \]
\[\frac{\xi^t}{2} g_{tt,\rho} (h g^{tt} + g^{tt} h^{\rho t} + g^{tt} h^{tt}) = \frac{\xi^t}{2} g_{tt,\rho} g^{tt} h_{ab} g^{ab}.\] (5.9)
Here it was used that \(\xi^t = const\). The expression \(h_{ab} g^{ab}\) can be written in a form
\[h_{ab} g^{ab} = (h_{zz} g^{zz} + h_{zz} g^{zz} + 2h_{zz} g^{zz}) = \frac{1}{g^{(2)}} (h_{zz} g_{zz} + h_{zz} g_{zz} - 2h_{zz} g_{zz}) = \frac{\delta g^{(2)}}{g^{(2)}}.\] (5.10)
Calculating the charge density
\[K^{zt} = \xi^z \nabla^t h - \xi^z \nabla_\sigma h^{t\sigma} + \xi_\sigma \nabla^z h^{t\sigma} + \frac{1}{2} h \nabla^z \xi^t - h^{z\sigma} \nabla_\sigma \xi^t (z \leftrightarrow t),\] (5.11)
we note that contribution from the item 1 is \(O(\bar{x}^0)\), contributions from the items 2 and 3 cancel up to terms \(O(\bar{x}^0)\), and the items 4 and 5 yield
\[K^{zt} = \frac{\xi^t}{2} g_{tt,\rho} g^{tt} \left(\frac{h}{2} g^{za} - h^{za}\right) - \frac{\xi^t}{2} g_{tt,\rho} g^{zz} \left(\frac{h}{2} g^{tt} - h^{tt}\right).\] (5.12)
Expression (5.12) is transformed to a form
\[
K^{zt} = \frac{\xi}{2} g^{tt} \left[ g_{tt,z} \left( h_{\rho \rho} g^{\rho \rho} g^{zz} + h_{zz} g^{(2) -1} \right) + g_{tt,z} \left( \frac{h}{2} g^{zz} - h^{zz} \right) \right] + O(\bar{x}^0). \tag{5.13}
\]

In the same way for the charge density \(K^{zt}\) we have
\[
K^{zt} = \frac{\xi}{2} g^{tt} \left[ g_{tt,z} \left( h_{\rho \rho} g^{\rho \rho} g^{zz} + h_{zz} g^{(2) -1} \right) + g_{tt,z} \left( \frac{h}{2} g^{zz} - h^{zz} \right) \right] + O(\bar{x}^0). \tag{5.14}
\]

In variables \(t, \rho, \theta, \varphi\) we obtain the expressions of the form (5.13)-(5.14) with \(\theta, \varphi\) substituted for \(z, \bar{z}\). Although the metric components have similar functional form, the actual expressions written in variables are very different.

Let us consider the case of supertranslation field \(C(\theta)\) depending only on \(\theta\). The horizon surface is
\[
\sqrt{(\rho_H - C)^2 + C'^2} - 1/2 = 0,
\]
where \(\rho_H = \rho(\theta)\). Because \(\rho_\varphi = 0\), we have
\[
\delta H_\xi = \frac{1}{4\pi} \int d\theta \wedge \delta \varphi \sqrt{-g} [K^{\rho t} - \rho_{,\theta} K^{\theta t}]. \tag{5.15}
\]

The charge density \(K^{\rho t}\) is
\[
K^{\rho t} = \frac{\xi}{2} g^{tt} g^{\rho \rho} \frac{h_{\rho \theta} g_{\varphi \varphi} + h_{\varphi \varphi} g_{\theta \theta}}{g^{(2)}}, \tag{5.16}
\]
where \(g^{(2)} = g_{\theta \theta} g_{\varphi \varphi}\). For the density \(K^{\theta t}\) we obtain
\[
K^{\theta t} = \frac{\xi}{2} g^{tt} \left[ h_{\rho \rho} g^{\rho \rho} g^{\theta \theta} + h_{\theta \theta} g^{\theta \theta} g^{\varphi \varphi} \right]. \tag{5.17}
\]

The term with \(h_{\rho \rho}\) vanishes and does not contribute to \(\delta H_\xi\). Variation of the surface charge is
\[
\delta H_\xi = \lim_{\rho_s \to 1/2} \frac{1}{4\pi} \int d\theta \wedge d\varphi \sqrt{-g} \frac{\xi}{2} g^{tt} \left[ g_{tt,\rho} g^{\rho \rho} \left( \frac{\delta g_{\theta \theta}}{g_{\theta \theta}} + \frac{\delta g_{\varphi \varphi}}{g_{\varphi \varphi}} \right) - g_{tt,\theta} g^{\theta \theta} \frac{\delta g_{\varphi \varphi}}{g_{\varphi \varphi}} \right]. \tag{5.18}
\]

Here \(g^{\theta \theta} = -g^{\theta \theta} g^{\rho \rho}\). In the near-horizon region \(\rho_s = 1/2 + x, \ |x| \ll 1\) we have
\[
\frac{g_{tt,\rho}}{\sqrt{g_{tt}}} = 2 \frac{(\rho_s - 1/2)(\rho - C)}{(\rho_s + 1/2)} \simeq 4(\rho - C), \quad \frac{g_{tt,\theta}}{\sqrt{g_{tt}}} \simeq 4(-C')(\rho - C - C'''), \quad \frac{\rho_{,\theta}}{\sqrt{g_{\theta \theta}}} \simeq \frac{C'}{(\rho - C)}. \tag{5.19}
\]

We obtain variation of the charge as
\[
\delta H_\xi = \lim_{\rho_s \to 1/2} \frac{1}{4\pi} \int d\theta \wedge d\varphi 2\xi t \left[ (\rho - C) \delta \sqrt{g_{\theta \theta} g_{\varphi \varphi}} + \frac{C'}{(\rho - C)} \delta \sqrt{g_{\varphi \varphi}} \right], \tag{5.20}
\]
where at the horizon \(\rho_H - C = \sqrt{1/4 - C'''}\).

Integrability of variation of the charge means that an integral of a variation over the manifold of metrics is path-independent. The expression for \(\delta H_\xi\), (5.20) is not of the form of a variation of a function over the space of metrics and cannot be integrated over the space of metrics in a path-independent way.
5.2 Variables $r, z^a$.

Let us turn to calculation of the variation of the surface charge in variables $(r, z^a)$ with the metric (2.13)

$$ds^2 = \hat{g}_{tt} dt^2 + \hat{g}_{rr} dr^2 + 2\hat{g}_{r\phi} dr dz^a + \hat{g}_{ab} dz^a dz^b.$$  

At the surface $r = 2 + \hat{x}$ enclosing the horizon surface $r = 2$, the near-horizon forms of the metric (2.13) and its inverse are

$$\hat{g}_{mn} = \begin{vmatrix} \hat{g}_{tt}\hat{x} & 0 & 0 & 0 \\ 0 & \hat{g}_{rr}/\hat{x} & \hat{g}_{rz}/\sqrt{\hat{x}} & \hat{g}_{r\bar{z}}/\sqrt{\hat{x}} \\ 0 & \hat{g}_{rz}/\sqrt{\hat{x}} & \hat{g}_{zz} & \hat{g}_{z\bar{z}} \\ 0 & \hat{g}_{r\bar{z}}/\sqrt{\hat{x}} & \hat{g}_{z\bar{z}} & \hat{g}_{\bar{z}\bar{z}} \end{vmatrix}; \quad \hat{g}^{mn} = \begin{vmatrix} \hat{g}^{tt}/\hat{x} & 0 & 0 & 0 \\ 0 & \hat{g}^{rr} & \hat{g}^{rz}\sqrt{\hat{x}} & \hat{g}^{r\bar{z}}\sqrt{\hat{x}} \\ 0 & \hat{g}^{rz}\sqrt{\hat{x}} & \hat{g}^{zz} & \hat{g}^{z\bar{z}} \\ 0 & \hat{g}^{r\bar{z}}\sqrt{\hat{x}} & \hat{g}^{z\bar{z}} & \hat{g}^{\bar{z}\bar{z}} \end{vmatrix}, \quad (5.21)$$

where $\hat{g}_{mn}$ denotes the factor of order $O(\hat{x}^0)$. Variation of the surface charge is

$$\beta H_{\chi}(\hat{g}, \hat{h}) = \frac{1}{4\pi} \int_{\Sigma_r} (d^2 x)_{rt} \sqrt{-\hat{g}} \hat{K}^{rt}_{\chi}(\delta \hat{g}, \hat{g}) \quad (5.22)$$

where

$$\hat{K}^{rt}_{\chi} = \chi^r \hat{\nabla}^t \hat{h} - \chi^t \hat{\nabla}^r \hat{h}^{ts} + \chi_a \hat{\nabla}^r \hat{h}^{at} + \frac{\hat{h}}{2} \hat{\nabla}^r \chi^t - \hat{h}^{rs} \hat{\nabla}^s \chi^t - (r \to t) \quad (5.23)$$

and $\Sigma_r$ is a surface $r = 2 + \hat{x}$. Here $\hat{g} = g_{tt}\hat{g}^{(3)}$, and $\hat{g}^{(3)}$ is determinant of the 3D part of the metric

$$\hat{g}^{(3)}(r, z^a) = \hat{g}_{rr}(\hat{g}_{zz} \hat{g}_{\bar{z}\bar{z}} - \hat{g}_{\bar{z}\bar{z}}^2) - \hat{g}_{r\bar{z}} \hat{g}_{zz} - \hat{g}_{r\bar{z}}^2 + 2\hat{g}_{r\bar{z}} \hat{g}_{zz}. \quad (5.24)$$

Using the expressions (2.8)-(2.11), determinant $\hat{g}^{(3)}$ can be written as

$$\hat{g}^{(3)} = g_{rt} \omega^2 \hat{g}^{(2)} = \frac{4r^2}{K^2} \left( \frac{K}{2\sqrt{1 - \hat{g}_{tt}^2}} \right)^2 (\hat{g}_{zz} \hat{g}_{\bar{z}\bar{z}} - \hat{g}_{\bar{z}\bar{z}}^2), \quad (5.25)$$

with $\hat{g}_{ab}$ from (2.6). Determinant of the metric $\gamma$ is of order $O(\hat{x}^0)$, and to extract a contribution nonzero at the horizon we must select in $\hat{K}^{rt}$ the terms of order $O(\hat{x}^0)$.

The five terms in (5.23) are

1. $\chi^r \hat{\nabla}^r \hat{h} - \chi^t \hat{\nabla}^t \hat{h} = \chi^r \hat{g}^{tt} \partial_t \hat{h} - \chi^t \hat{g}^{rr} \partial_r \hat{h} - \chi^t \hat{g}^{ra} \partial_a \hat{h} = O(\hat{x}^{1/2}).$
2. $\chi^r \hat{\nabla}^s \hat{h}^{ts} + \chi^t \hat{\nabla}^s \hat{h}^{rs}.$
3. $\chi_s \hat{\nabla}^r \hat{h}^{ts} - \chi_s \hat{\nabla}^t \hat{h}^{rs} = \chi_s \hat{\nabla}^r \hat{h}^{ut} - \chi_t \hat{\nabla}^r \hat{h}^{rt} - \chi_r \hat{\nabla}^t \hat{h}^{ra}.$
4. $\frac{\hat{h}}{2} (\hat{\nabla}^r \chi^t - \hat{\nabla}^t \chi^r) = \frac{\hat{h}}{2} \left[ \hat{g}^{rr} \hat{\nabla}^r \chi^t - \hat{g}^{rt} \hat{\nabla}^t \chi^t \right] + O(\hat{x}^{1/2}).$
5. $\hat{h}^r \hat{\nabla}^s \chi^t + \hat{h}^s \hat{\nabla}^s \chi^r = -\frac{1}{2} (\hat{h}^{rt} \hat{\nabla}^r \chi^t - \hat{h}^{tr} \hat{\nabla}^t \chi^r) + O(\hat{x}^{1/2}). \quad (5.26)$

Estimating two terms in the item 1, we have

$$\chi^r \hat{g}^{tt} \partial_t \hat{h} - \chi^t \hat{g}^{rr} \partial_r \hat{h} + \hat{g}^{rr} \partial_t \hat{h} = O(\hat{x}^{1/2})$$

and the item 1 does not contribute to $\hat{K}^{rt}$.

Because all the terms containing $\Gamma$ one index $t$ are zero, in the item 2 the first term vanishes

$$-\chi^r \hat{\nabla}^r \hat{h}^{ts} = -\chi^r (\hat{\nabla}^r \hat{h}^{ut} + \hat{\nabla}^r \hat{h}^{rt} + \hat{\nabla}^t \hat{h}^{ra}) = 0.$$
In the second term
\[ \chi^t \nabla_r \hat{h}^{rr} = \chi^t (\nabla_r \hat{h}^{rt} + \nabla_s \hat{h}^{rr} + \nabla_a \hat{h}^{ra}) \]
the part \( \chi^t (\nabla_r \hat{h}^{rr} + \nabla_a \hat{h}^{ra} \) is estimated as
\[ \chi^t \nabla_r \hat{h}^{rr} = \chi^t (2\hat{h}^{rr} + 2\Gamma_{rt} \hat{h}^{rt} + 2\Gamma_{ra} \hat{h}^{ra} + O(\hat{x}^{1/2})) = \]
\[ = \chi^t \left[ \hat{h}^{rr} + \left( \frac{\hat{g}^{rr} \hat{g}_{rr} - 1}{\hat{x}^2} + 2\hat{g}^{ra} \hat{g}_{ra} \left( -\frac{1}{2\hat{x}^{3/2}} \right) \right) \hat{h}^{rr} \hat{x} + O(\hat{x}^{1/2}) \right] = O(\hat{x}^{1/2}). \] (5.27)
Because of identity \( \hat{g}^{rr} \hat{g}_{rr} + \hat{g}^{ra} \hat{g}_{ar} = 1 \) the sum of the terms in round brackets in (5.27) is equal to \( -\hat{h}^{rr} \). The term \( \chi^t \nabla_a \hat{h}^{ra} \) is of order \( O(\hat{x}^{1/2}) \). In the item 2 there remains the term \( \chi^t \nabla_r \hat{h}^{rt} \).

In the item 3 the first term is
\[ \chi_s \nabla^s \hat{h}^{ts} = \chi_t \nabla_r \hat{h}^{pr} + \chi_r \nabla^r \hat{h}^{pr} + \chi_a \nabla^a \hat{h}^{ra} = \chi_t (\hat{g}^{rr} \nabla_r + \hat{g}^{ra} \nabla_a) \hat{h}^{tt} + \chi_r (\hat{g}^{rr} \nabla_r + \hat{g}^{ra} \nabla_a) \hat{h}^{tr} + O(\hat{x}^0). \]
The term \( \chi^t \hat{g}_{tt} \hat{g}^{ra} \nabla_a \hat{h}^{tt} \) is of order \( \hat{x}^{1/2} \). In the term
\[ \chi^t \hat{g}_{tt} \hat{g}^{rr} \nabla_a \hat{h}^{tt} = \chi^t \hat{g}_{tt} \hat{x} \hat{g}^{rr} \hat{x} \left( -\frac{\hat{h}^{tt}}{\hat{x}^2} + \hat{g}^{tt} \hat{x} \hat{g}_{tt} \hat{x} + O(\hat{x}^0) \right) \]
the leading-order parts cancel and it is also of order \( O(\hat{x}^{1/2}) \). The remaining term in the item 3, equal to \( -\chi^t \nabla_r \hat{h}^{rt} \), cancels the corresponding term in the item 2.

We obtain \( K^r t \) as
\[ K^r t = \nabla_s \chi^t \left( \frac{\hat{h}^{rs}}{2} - \hat{h}^{rs} \right) \]
\[ \nabla_s \chi^r \left( \frac{\hat{h}^{ts}}{2} - \hat{h}^{ts} \right) + O(\hat{x}^{1/2}) = \]
\[ = \chi^t \left[ \frac{\hat{h}^{rs}}{2} - \hat{h}^{rs} \right] \]
\[ \Gamma_{rt} \left( \frac{\hat{h}^{rr}}{2} - \hat{h}^{rr} \right) - \Gamma_{tt} \left( \frac{\hat{h}^{tt}}{2} - \hat{h}^{tt} \right) + O(\hat{x}^{1/2}) \]. (5.28)
Because \( g_{tt} = V(r) \), we have \( \Gamma_{rt} = \Gamma_{tt} = 0 \).

In \( \hat{K}^{rr} \) the leading terms are
\[ \hat{K}^{rr} = \chi^t \frac{\hat{g}_{tt}}{2} \hat{g}^{rr} \hat{h}^{rt} - \hat{h}^{rr} \hat{g}^{tt} - \hat{g}^{rr} \hat{h}^{tt} = \chi^t \frac{\hat{g}_{tt}}{2} \hat{g}^{tt} \hat{g}_{tt} \hat{h}^{ab} - \hat{g}^{ra} \hat{g}^{rb} \]. (5.29)
Combination in the rhs of (5.29) is presented as
\[ \hat{h}_{ab} (\hat{g}^{rr} \hat{g}^{ab} - \hat{g}^{ra} \hat{g}^{rb}) = \frac{\hat{h}_{zz} \hat{g}_{zz} + \hat{h}_{zz} \hat{g}_{zz} - 2\hat{h}_{zz} \hat{g}_{zz}}{\hat{g}^{(3)}} = \frac{\delta \hat{g}^{(2)}}{\hat{g}^{(3)}}, \] (5.30)
where
\[ \delta \hat{g}^{(2)} = \hat{g}_{zz} \hat{g}_{zz} - \hat{g}^{2}_{zz} \] Variation of the surface charge is
\[ \beta H_\chi (\hat{g}, \hat{h}) = \lim_{r \to 2} \frac{1}{4\pi} \int dz \wedge d\bar{z} \sqrt{\hat{g}^{(3)}} \chi^t g_{tt} \hat{g}^{tt} \frac{\delta \hat{g}^{(2)}}{\hat{g}^{(3)}} \] (5.31)
Substituting (5.25) and taking the limit \( r \to 2 \), we have
\[ \beta H_\chi (\hat{g}, \hat{h}) = \frac{1}{4\pi} \int d\gamma \wedge d\bar{z} \gamma_{zz} \chi^t \frac{\delta \hat{g}^{(2)}}{\sqrt{\hat{g}^{(3)}}} (1/4 - D_{a} CD^{a} C)^{1/2}. \] (5.32)
"Tilda" indicates that from the expression were extracted povers of \( \gamma_{zz} \).

The integral (5.32) is not of the form of a variation of a functional over the space of metrics. In a general case, expression (5.32) is not integrable. A special case of integrable variation of the surface charge is discussed in the next section.
6 Integrable variations of surface charges

In this section we consider an example of integrable variation of the charge. We consider the case of supertranslation field \( C(z, \bar{z}) \) in coordinate system \( r, z, \bar{z} \) depending only on \(|z|\), or in coordinates \( r, \theta, \varphi \), only on \( \theta \).

Integrability of the charge over the space of metrics means that the charge \( H_\chi(g) = \int \delta H_\chi \) is independent of a form of a path in a space of metrics.

In coordinates \( r, \theta, \varphi \) the metric (2.13) takes a form

\[
d s^2 = -Vdt^2 + \frac{dr^2}{r^2} + 2d\delta \frac{b r (\sqrt{1 - b^2} - b')}{(1 - b^2) V^{1/2}} + d\theta^2 + d\varphi^2 \sin^2 \theta (b \cot \theta - \sqrt{1 - b^2})^2,
\]

where \( b = 2C'/(\theta/K) \). The charge density \( \hat{K}^{rt}(\delta \hat{g}, \hat{g}) \) (5.29) is

\[
\hat{K}^{rt}_\chi = \frac{\chi^t}{2} \hat{g}_{tt} \hat{g}^{rt} \left[ \hat{h}_{\theta \theta} (\hat{g}^{rr} \hat{g}^{\theta \theta} - (\hat{g}^{\theta \theta})^2) + \hat{h}_{\varphi \varphi} \hat{g}^{rr} \hat{g}^{\varphi \varphi} \right].
\]

Using the relations

\[
\hat{g}^{rr} \hat{g}^{\theta \theta} - (\hat{g}^{\theta \theta})^2 = 1/\hat{g}^{(2)}(\theta, r),
\]

where \( \hat{g}^{(2)} = \hat{g}_{rr} \hat{g}_{\theta \theta} - \hat{g}_{r \theta}^2 \), and \( \hat{g}^{(2)} = \hat{g}_{\theta \theta} / V \) and noting that \( g^{rr} = V \), we write \( \hat{K}^{rt} \) as

\[
\hat{K}^{rt}_\chi = \frac{\chi^t}{2} \hat{g}_{tt} \hat{g}^{rt} \left( \frac{\hat{V}}{\hat{g}^{\theta \theta}} + \frac{\hat{V}}{\hat{g}^{\varphi \varphi}} \right) = \frac{\chi^t}{2} V \left( \frac{\delta \hat{g}_{\theta \theta}}{\hat{g}^{\theta \theta}} + \frac{\delta \hat{g}_{\varphi \varphi}}{\hat{g}^{\varphi \varphi}} \right).
\]

We obtain \( \beta H_\chi \) in a form

\[
\beta H_\chi(\hat{g}, \delta \hat{g}) = \lim_{r \to 2} \frac{1}{4\pi} \int d\theta \wedge d\varphi \sqrt{-\hat{g}_{tt} \hat{g}^{(2)}(\theta, r)} \hat{g}^{rt} \hat{K}^{rt} = \frac{1}{4\pi} \int d\theta \wedge d\varphi \sqrt{\hat{g}_{\theta \theta} \hat{g}^{\varphi \varphi}} \frac{\chi^t}{4} \frac{\delta (\hat{g}_{\theta \theta} \hat{g}_{\varphi \varphi})}{\hat{g}_{\theta \theta} \hat{g}_{\varphi \varphi}}
\]

With \( \chi^t = \text{const} \) (6.5) can be written as a variation

\[
\delta H_\chi(\hat{g}, \hat{h}) = \frac{1}{4\pi} \int d\theta \wedge d\varphi \frac{\chi^t}{2} \delta \sqrt{\hat{g}_{\theta \theta} \hat{g}_{\varphi \varphi}},
\]

and is integrable.

Let us consider calculation of \( \beta H_\chi(\hat{g}, \hat{h}) \) in parametrization of the sphere in coordinates \((z, \bar{z})\).

In the case \( C = C(\theta) \) from the relation \( \sqrt{\hat{g}^{(3)}(r, z, \bar{z})} dz \wedge d\bar{z} = \sqrt{\hat{g}^{(3)}(r, \theta, \varphi)} d\theta \wedge d\varphi \) it follows that

\[
\hat{g}^{(3)}(r, z, \bar{z}) = \hat{g}^{(3)}(r, \theta, \varphi) (\theta_z \varphi_{\bar{z}} - \theta_{\bar{z}} \varphi_z)^2.
\]

Expressing the variation \( \delta (\hat{g}_{zz} \hat{g}_{\bar{z}z} - \hat{g}_{\bar{z}z}^2) \) through coordinates \( \theta, \varphi \), we have

\[
\delta (\hat{g}_{zz} \hat{g}_{\bar{z}z} - \hat{g}_{\bar{z}z}^2) = \delta (\hat{g}_{\theta \theta} \hat{g}_{\varphi \varphi}) (\theta_z \varphi_{\bar{z}} - \theta_{\bar{z}} \varphi_z)^2.
\]

Substituting \( \hat{g}^{(3)}(r, \theta, \varphi) = (\hat{g}_{rr} \hat{g}_{\theta \theta} - \hat{g}_{r \theta}^2) \hat{g}_{\varphi \varphi} = \hat{g}_{\theta \theta} \hat{g}_{\varphi \varphi} / V \), we obtain the variation of the surface charge in the form (6.5).
Let us show that the expression for the variation of the charge $\delta H_\chi$ in variables $r, \theta, \varphi$ is equal to the variation $\delta H_\xi$ in variables $\rho, \theta, \varphi$. Note that the expressions have different functional form, and $\delta H_\xi$ is not integrable. The charge densities in the $r$ and $\rho$ systems are connected as

$$\hat{K}^{rt} = K^{\rho t} \frac{\partial r}{\partial \rho} + K^{\theta t} \frac{\partial r}{\partial \theta}. \quad (6.6)$$

The integration measures satisfy the equality $\sqrt{-g} dr \wedge d\theta \wedge d\varphi = \sqrt{-g} (\partial r/\partial \rho) d\rho$. Using the relations

$$\frac{\partial r}{\partial \rho} \frac{\partial \rho}{\partial r} = 1, \quad \frac{\partial r}{\partial \rho} \frac{\partial \rho}{\partial \theta} + \frac{\partial r}{\partial \theta} = 0 \quad (6.7)$$

and noting that $\chi^t = \xi^t$, we obtain

$$\delta H_\chi = \int d\theta \wedge d\varphi \sqrt{-g} \hat{K}^{rt} = \int d\theta \wedge d\varphi \sqrt{-g} \left[ K^{\rho t} - \rho g K^{\theta t} \right] = \delta H_\xi. \quad (6.8)$$

Details of the derivation of the relation (6.8) are contained in Appendix B.

### 7 Summary and conclusions

In this paper we studied the near-horizon symmetries of the metric of black hole containing supertranslation field which preserve the near-horizon structure of the metric. The horizon symmetries were considered in different coordinate systems ($\rho$ and $r$-systems) connected by a "large" diffeomorphism containing supertranslation field and also in coordinate systems connected by a pure coordinate transformation which do not change supertranslation field in the metric. Foliation of the near-horizon region was defined through a smooth deformation of the horizon surface $\rho_s = 1/2$ to $\rho_s = 1/2 + x$ where $\rho_s = ((\rho - C)^2 + (DC)^2)^{1/2}$. In the $r$-system the horizon is is located at the surface $r = 2$ (in units $M = 1$), in the $\rho$-system - at the surface $\rho_H = C + (1/4 + (DC)^2)^{1/2}$. In both $\rho$ and $r$-systems we constructed diffeomorphisms which preserve the gauge and the near-horizon form of the metric in the leading order in $x$.

We discussed symplectic transformations which are extendable from the near-horizon region in the bulk. Symplectic transformations are generated by vector fields which depend on the supertranslation field in the metric $C(\theta, \varphi)$ and a field $T(\theta, \varphi)$ and act on $C(\theta, \varphi)$ as $\delta T C(\theta, \varphi) = T(\theta, \varphi)$. In a case of supertranslation field depending only on $\theta$ a condition that transformation preserves the near-horizon form of metric at the horizon is an ordinary differential equation with a solution for $T$ expressed through $C(\theta)$.

We calculated variation of the surface charge in the $\rho$ and $r$-systems and also in different parametrizations of the the unit sphere $\theta, \varphi$ and $z, \bar{z}$. In the $r$-system, variation of the charge is expressed through an integral of the charge density $\hat{K}^{\rho t}(\delta g_{mn}, \hat{g}_{mn})$ over the sphere. In the $\rho$-system, variation of the charge is obtained as a sum of three integrals over the horizon surface with charge densities $K^\rho, K^{\theta t}$ and $K^{\varphi t}$ and corresponding integrations over $d\theta d\varphi, d\rho d\varphi$ and $d\rho d\theta$. Here $\xi$ and $\chi$ are the vector fields generating asymptotic horizon transformations in the $\rho$ and $r$-systems. Surface charge densities were calculated in the leading order in $x$, and together with contributions from $\sqrt{-g}$ yield for a variation of the charge an expression independent of $x$.

In a general case the surface charges obtained by integration of variations of the charges over the space of metrics are path-dependent. In a special case of the supertranslation field in metric...
depending only on a spherical angle $\theta$, variation of the charge in the $r$-system is of the form of a variation of a functional over the space of metrics and can be integrated in a path-independent way. Integrability of the surface charge was proved for both parametrizations of the unit sphere. Although of different functional forms, the expressions for the variation of the charge are transformed one to another by the transformation from the $r$ to the $\rho$ system.

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8 Appendix A: Solution of Eq. (4.12)

In this Appendix we find a general solution for the function $T(z, \bar{z})$ in the generator of supertranslations Eq. (4.1). We solve the Eq. (4.12) which is a condition on the generator $T$ at the horizon (written in notations of Sect. 2)

$$[-T\sqrt{1-b^2} + \frac{1}{4}F^{ab}D_bT(-2b_a\sqrt{1-b^2} + D_a b^2)]_{r=2} = 0.$$  \hspace{1cm} (a1)

Eq. (a1) can be presented in a form

$$T + F^a D_a T = 0,$$  \hspace{1cm} (a2)

where

$$F^a = \frac{1}{2}F^{ac}(b_c + \partial_c\sqrt{1-b^2})|_{r=2}$$

Following the general rules of solving the differential equations with partial derivatives [41], we consider a function $W(T,z,\bar{z})$ satisfying the equation

$$T\frac{\partial W}{\partial T} + F^z \frac{\partial W}{\partial z} + F^{\bar{z}} \frac{\partial W}{\partial \bar{z}} = 0.$$  \hspace{1cm} (a3)

Eq. (a3) is solved by writing the system of ordinary differential equations

$$\frac{dT}{T} = \frac{dz}{F^z} = \frac{d\bar{z}}{F^{\bar{z}}}.$$  \hspace{1cm} (a4)

Let the independent first integrals of the Eq. (a4) be

$$\psi_1(T,z,\bar{z}) = C_1, \quad \psi_2(T,z,\bar{z}) = C_2.$$  \hspace{1cm} (a5)

The general solution of the Eq. (a3) for $W(T,z,\bar{z})$ is

$$W = f(\psi_1, \psi_2),$$  \hspace{1cm} (a6)

where $f$ is an arbitrary smooth function. The function $T(z,\bar{z})$ is implicitly determined from the equation

$$f(\psi_1, \psi_2) = 0.$$  \hspace{1cm} (a7)
9 Appendix B: Transformation $\delta H_\xi(g, h) \leftrightarrow \delta H_\chi(\hat{g}, \hat{h})$ in the case $C = C(\theta)$

In this Appendix we present details of transformations leading to the relation (6.8).

For $\rho_s = 1/2 + x$, $|x| \ll 1$, using (2.7) we have

$$\frac{\partial r}{\partial \rho} = \left(1 - \frac{1}{4\rho_s^2}\right) \frac{\partial \rho_s}{\partial \rho} = \frac{(K^2 - 1)(\rho - C)}{K^2 \rho_s} \simeq 4(K - 1)(\rho - C),$$  \hspace{1cm} (b1)

$$\frac{\partial r}{\partial \theta} = \left(1 - \frac{1}{4\rho_s^2}\right) \frac{\partial \rho_s}{\partial \theta} = \frac{(K^2 - 1)(-C')(\rho - C - C'')}{K^2 \rho_s} \simeq 4(-C'')(\rho - C - C'').$$  \hspace{1cm} (b2)

The derivative $\partial \rho/\partial r$ with $\rho$ (2.8) is obtained as

$$\frac{\partial \rho}{\partial r} = \frac{K(\partial K/\partial r)}{4\sqrt{K^2/4 - (DC)^2}} \simeq \frac{4(\rho - C)(K - 1)}{\rho - C}. \hspace{1cm} (b3)$$

At the horizon, $K = 1$, Eqs.(b1) and (b3) give

$$\frac{\partial r}{\partial \rho} \frac{\partial \rho}{\partial r} = 1. \hspace{1cm} (b4)$$

In the same way, using (b2) and (b3), we obtain the second relation (6.7)

$$\frac{\partial \rho}{\partial r} \frac{\partial r}{\partial \theta} = \frac{[4(\rho - C)(K - 1)]^{-1}(K^2 - 1)(-C')(\rho - C - C'')}{K^2 \rho_s} \simeq \frac{(-C'')(\rho - C - C'')}{\rho - C} = -\rho, \hspace{1cm} (b5)$$

Following transformations leading to (6.5), we have

$$\sqrt{-g} dr \wedge dt \wedge d\theta \wedge d\phi = \sqrt{-\hat{g}_{\theta\theta}} \hat{g}_{\phi\phi} (\partial r/\partial \rho) d\rho \wedge dt \wedge d\theta \wedge d\phi. \hspace{1cm} (b6)$$

To obtain (6.8), we use the relation

$$\hat{g}_{\theta\theta} = \frac{g_{\theta\theta}}{4(\rho - C)^2} \hspace{1cm} (b7)$$

which follows from the definition (2.12). We transform

$$\sqrt{-g} dr = \sqrt{\hat{g}_{\theta\theta}} \hat{g}_{\phi\phi} (\partial r/\partial \rho) d\rho \simeq \frac{\sqrt{\hat{g}_{\theta\theta}}\hat{g}_{\phi\phi}}{2(\rho - C)} 4(K - 1)(\rho - C) d\rho. \hspace{1cm} (b8)$$

On the other hand,

$$\sqrt{-g} d\rho \simeq \frac{(\rho_s - 1/2)}{(\rho_s + 1/2)} 4\sqrt{\hat{g}_{\theta\theta}} \hat{g}_{\phi\phi} d\rho. \hspace{1cm} (b9)$$

Because $(\rho_s - 1/2)/(\rho_s + 1/2) \simeq (K - 1)/2$, in the horizon limit $r \rightarrow 2$ expressions (b8) and (b9) are equal.

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