THE C*-ALGEBRA OF A VECTOR BUNDLE

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Abstract. We prove that the Cuntz-Pimsner algebra $O_E$ of a vector bundle $E$ of rank $\geq 2$ over a compact metrizable space $X$ is determined up to an isomorphism of $C(X)$-algebras by the ideal $(1 - [E]) K^0(X)$ of the $K$-theory ring $K^0(X)$. Moreover, if $E$ and $F$ are vector bundles of rank $\geq 2$, then a unital embedding of $C(X)$-algebras $O_E \subset O_F$ exists if and only if $1 - [E]$ is divisible by $1 - [F]$ in the ring $K^0(X)$. We introduce related, but more computable $K$-theory and cohomology invariants for $O_E$ and study their completeness. As an application we classify the unital separable continuous fields with fibers isomorphic to the Cuntz algebra $O_{n+1}$ over a finite connected CW complex $X$ of dimension $d \leq 2n + 3$ provided that the cohomology of $X$ has no $m$-torsion.

1. Introduction

Let $E \in \text{Vect}(X)$ be a locally trivial complex vector bundle over a compact Hausdorff space $X$. If we endow $E$ with a hermitian metric, then the space $\Gamma(E)$ of all continuous sections of $E$ becomes a finitely generated projective Hilbert $C(X)$-module, whose isomorphism class does not depend on the choice of the metric. Since the action of $C(X)$ is central, $\Gamma(E)$ is naturally a Hilbert $C(X)$-bimodule. Let $O_E$ denote the Cuntz-Pimsner algebra associated to $\Gamma(E)$ as defined in [16]. Since $\Gamma(E)$ is projective, $O_E$ is isomorphic to the Doplicher-Roberts algebra of $\Gamma(E)$, see [7]. Let us recall that if $E$ is the Hilbert $C(X)$-module $\bigoplus_{n \geq 0} \Gamma(E)^{\otimes n}$, then $O_E$ is obtained as the quotient of the Toeplitz (or tensor) $C^*$-algebra $T_E$ generated by the multiplication operators $T_\xi : E \to E$, $T_\xi(\eta) = \xi \otimes \eta$, $\xi \in \Gamma(E)$, $\eta \in E$, by the ideal of “compact operators” $K(E)$. If $X$ is a point, then $E \cong C^n$ for some $n \geq 1$, and $O_E$ is isomorphic to the Cuntz algebra $O_n$, with the convention that $O_1 = C(T)$. In the general case, $O_E$ is a locally trivial unital $C(X)$-algebra (continuous field) whose fiber at $x$ is isomorphic to the Cuntz algebra $O_{n(x)}$, where $n(x)$ is the rank of the fiber $E_x$ of $E$, see [19 Prop. 2].

The motivation for this paper comes from an informal question of Cuntz: What are the invariants of $E$ captured by the $C(X)$-algebra $O_E$? In other words, how are $E$ and $F$ related if there is a $C(X)$-linear $*$-isomorphism $O_E \cong O_F$. We have shown in [5] that if $X$ has finite covering dimension, then all separable unital $C(X)$-algebras with fibers isomorphic to a fixed Cuntz algebra $O_n$, $n \geq 2$, are automatically locally trivial. Thus it...
is also natural to ask which of these algebras are isomorphic to Cuntz-Pimsner algebras associated to a vector bundle of constant rank $n$.

If $E$ is a line bundle, then $O_E$ is commutative with spectrum homeomorphic to the circle bundle of $E$, see [20]. One verifies that if $E, F \in Vect_1(X)$ and $X$ is path-connected, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $E \cong F$ or $E \cong \tilde{F}$, where $\tilde{F}$ is the conjugate of $F$, see Proposition 1.3. In view of this property, we shall only consider vector bundles of rank $\geq 2$. In the first part of the paper we answer the isomorphism question for $O_E$.

**Theorem 1.1.** Let $X$ be a compact metrizable space and let $E, F \in Vect(X)$ be complex vector bundles of rank $\geq 2$. Then $O_E$ embeds as a unital $C(X)$-subalgebra of $O_F$ if and only if there is $h \in K^0(X)$ such that $1 - [E] = (1 - [F])h$. Moreover, $O_E \cong O_F$ as $C(X)$-algebras if and only if there is $h$ as above of virtual rank one.

Thus the principal ideal $(1 - [E])K^0(X)$ determines $O_E$ up to isomorphism and an inclusion of principal ideals $(1 - [E])K^0(X) \subset (1 - [F])K^0(X)$ corresponds to unital embeddings $O_E \subset O_F$. In particular if $E \in Vect_{m+1}(X)$, then $O_E \cong C(X) \otimes O_{m+1}$ if and only if $[E] - 1$ is divisible by $m \geq 1$.

Let $\tilde{K}^0(X) = \ker(K^0(X) \xrightarrow{\text{rank}} H^0(X, \mathbb{Z}))$ be the subgroup of $K^0(X)$ corresponding to elements of virtual rank zero, and set $[\tilde{E}] := [E] - \text{rank}(E) \in \tilde{K}^0(X)$. We denote by $H^*(X, \mathbb{Z})$ the Čech cohomology. Using the nilpotency of $\tilde{K}^0(X)$ we derive the following:

**Theorem 1.2.** Let $X$ be a compact metrizable space of finite dimension $n$. Suppose that $\text{Tor}(K^0(X), \mathbb{Z}/m) = 0$. If $E, F \in Vect_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $([\tilde{E}] - [\tilde{F}]) \left( \sum_{k=1}^n (-1)^{k-1}m^{n-k}[\tilde{F}]^{k-1} \right)$ is divisible by $m^2$ in $\tilde{K}^0(X)$.

In view of Theorem 1.1 it is natural to seek explicit and computable invariants (e.g. characteristic classes) of a vector bundle $E$ that depend only on the principal ideal $(1 - [E])K^0(X)$ and hence which are invariants of $O_E$.

For each $m \geq 1$, consider the sequence of polynomials $p_n \in \mathbb{Z}[x]$,

$$p_n(x) = \ell(n) m^n \log \left(1 + \frac{x}{m}\right)_{[n]} = \sum_{k=1}^n (-1)^{k-1} \frac{\ell(n)}{k} m^{n-k} x^k,$$

where $\ell(n)$ denotes the least common multiple of the numbers $\{1, 2, ..., n\}$ and the index $[n]$ indicates that the formal series of the natural logarithm is truncated after its $n$th term.

**Theorem 1.3.** Let $X$ be a finite CW complex of dimension $d$ and let $E, F \in Vect_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $p_{\lfloor d/2\rfloor}([\tilde{E}]) - p_{\lfloor d/2\rfloor}([\tilde{F}])$ is divisible by $m^{\lfloor d/2\rfloor}$ in $\tilde{K}^0(X)$.

For $x \in \mathbb{R}$, we set $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$ and $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$. Theorem 1.3 extends to finite dimensional compact metrizable spaces: if $n \geq 1$ is an integer such that $\tilde{K}^0(X)^{n+1} = \{0\}$, then $p_n([\tilde{E}]) - p_n([\tilde{F}])$ is divisible by $m^n$ in $\tilde{K}^0(X)$.
whenever $O_E \cong O_F$ as $C(X)$-algebras. The same conclusion holds for infinite dimensional spaces $X$ but in that case $n$ depends on $E$ and $F$.

Concerning the completeness of the above invariant we have the following:

**Theorem 1.4.** Let $X$ be a finite CW complex of dimension $d$. Suppose that $m$ and $[d/2]!$ are relatively prime and that $\text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0$. If $E, F \in \text{Vect}_{m+1}(X)$, then $O_E \cong O_F$ as $C(X)$-algebras if and only if $p_{[d/2]}([\tilde{E}]) - p_{[d/2]}([\tilde{F}])$ is divisible by $m^{[d/2]}$ in $\tilde{K}^0(X)$.

The condition that $m$ and $[d/2]!$ are relatively prime is necessary. To show this, we take $m = 2$ and let $X$ be the complex projective space $\mathbb{C}P^2$. Then $K^0(X)$ is isomorphic to the polynomial ring $\mathbb{Z}[x]$ with $x^3 = 0$. Let $E$ and $F$ be bundles with K-theory classes $[E] = 3 + 3x$ and $[F] = 3 + x$. Then $[\tilde{E}] = 3x$ and $[\tilde{F}] = x$, so that $p_2(3x) - p_2(x) = 8(x^2 - x^3)$ is divisible by 4 and yet Theorem 1.2 shows that $O_E \not\cong O_F$, since $(\tilde{E} - \tilde{F})(2 - \tilde{F}) = 4x - 2x^2$ is not divisible by 4. The vanishing of $m$-torsion is also necessary in both Theorems 1.3 and 1.4 as it is seen by taking $m = 2$ and $X = \mathbb{R}P^2 \vee \mathbb{C}P^2$, where $\mathbb{R}P^2$ is the real projective space. Indeed, let $E, F \in \text{Vect}_3(X)$ be such that $F$ is trivial and $[\tilde{E}]|_{\mathbb{R}P^2} = z$ is the generator of $\tilde{K}^0(\mathbb{R}P^2) = \mathbb{Z}/2$ and $[\tilde{E}]|_{\mathbb{C}P^2} = 2x + 2x^2$. Then $([\tilde{E}] - [\tilde{F}])(2 - [\tilde{F}]) = (z + 2x + 2x^2)(2)^{2} = 4x + 4x^2$ is divisible by 4 and yet $O_E \not\cong C(X) \otimes O_3$ by Theorem 1.1 since $[E] = 2 + z = 2x + 2x^2$ is not divisible by 2.

Next we exhibit characteristic classes of $E$ which are invariants of $O_E$. For each $n \geq 1$ consider the polynomial $q_n \in \mathbb{Z}[x_1, \ldots, x_n]$:

\[ (2) \quad q_n = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} (-1)^{k_1 + \cdots + k_n - 1} m^{-n} (k_1 + \cdots + k_n - 1) n! \cdot \frac{1}{k_1! \cdots k_n!} \cdot x_1^{k_1} \cdots x_n^{k_n}. \]

Thus $q_1(x_1) = x_1$, $q_2(x_1, x_2) = mx_2 - x_2^2$, $q_3(x_1, x_2, x_3) = m^2 x_3 - 3mx_1 x_2 + 2x_3^2$, etc. Let $ch_n$ be the integral characteristic classes that appear in the Chern character, $ch = \sum_{n \geq 0} \frac{1}{n!} ch_n$.

**Theorem 1.5.** Let $X$ be a compact metrizable space and let $E, F \in \text{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $q_n(ch_1(E), ..., ch_n(E)) - q_n(ch_1(F), ..., ch_n(F))$ is divisible by $m^n$ in $H^{2n}(X, \mathbb{Z})$, for all $n \geq 1$.

Reducing mod $m^n$ it follows that the sequence $q_n(ch_1(E), ..., ch_n(E)) \in H^{2n}(X, \mathbb{Z}/m^n)$, $n \geq 1$, is an invariant of the $C(X)$-algebra $O_E$.

Let us denote by $O_{m+1}(X)$ the set of isomorphism classes of unital separable $C(X)$-algebras with all fibers isomorphic to $O_{m+1}$. In the second part of the paper we study the range of the map $\text{Vect}_{m+1}(X) \rightarrow O_{m+1}(X)$. This relies on the computation of the homotopy groups of $\text{Aut}(O_{m+1})$ of [5]. If $T$ is a set, we denote by $|T|$ its cardinality.

**Theorem 1.6.** Let $X$ be a finite CW complex of dimension $d$. Suppose that $m \geq [(d - 3)/2]$ and $\text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0$. Then each element of $O_{m+1}(X)$ is isomorphic to $O_E$ for some $E$ in $\text{Vect}_{m+1}(X)$. Moreover $|O_{m+1}(X)| = |\tilde{K}^0(X) \otimes \mathbb{Z}/m| = |\tilde{H}^{\text{even}}(X, \mathbb{Z}/m)|$. 
The hypotheses of Theorem 1.6 are necessary. Indeed, to see that the condition $m \geq \lceil (d - 3)/2 \rceil$ is necessary even in the absence of torsion, we note that $\mathrm{Vect}_3(S^8) = \{\ast\}$ since $\pi_7(U(3)) = 0$ by \cite{12}, whereas $\mathcal{O}_3(S^8) \cong \pi_7(\mathrm{Aut}(O_3)) \cong \mathbb{Z}/2$ by \cite{5}. To see that the condition on torsion is necessary when $m \geq \lceil (d - 3)/2 \rceil$, we note that if $X = \mathbb{R}P^2$, then $\mathrm{Vect}_3(SX) = \{\ast\}$ since $\tilde{K}^0(SX) \cong K^1(X) = \{0\}$ and $\dim(SX) = 3$, whereas $\mathcal{O}_3(SX) \cong K^1(X,\mathbb{Z}/2) \cong \mathbb{Z}/2$ by \cite{5}.

The study of the map $\mathrm{Vect}_{m+1}(X) \to \mathcal{O}_{m+1}(X)$ simplifies considerably if $X$ is a suspension as explained in Theorem 7.1 from Section 7.

In Section 2 we prove Theorems 1.1-1.2. Theorem 1.3 is proved in Section 3 and Theorem 1.4 is proved in Section 6. The proofs of Theorem 1.5 and Theorems 1.6 are given in Section 4 and respectively Section 5.

Cuntz-Pimsner algebras come with a natural $\mathbb{T}$-action and hence with a $\mathbb{Z}$-grading. The question studied by Vasselli in \cite{19} of when $O_E$ and $O_F$ are isomorphic as $\mathbb{Z}$-graded $C(X)$-algebras is not directly related to the questions addressed in this paper. I would like to thank Ezio Vasselli for making me aware of the isomorphism $O_E \cong O_F$ for line bundles, see \cite{20}.

2. When is $O_E$ isomorphic to $O_F$?

In this section we prove Theorems 1.1-1.2 and discuss the case of line bundles.

Proof. (of Theorem 1.1) We identify $K_0(C(X))$ with $K^0(X)$. Let $\iota_E$ denote the canonical unital inclusion $C(X) \to O_E$. By \cite{16}, the K-theory group $K_0(O_E)$ fits into an exact sequence

$$K_0(C(X)) \xrightarrow{1 - [E]} K_0(C(X)) \xrightarrow{\iota_E} K_0(O_E),$$

where $1 - [E]$ corresponds to the multiplication map by the element $1 - [E]$. Therefore $\ker(\iota_E)_* = (1 - [E])K^0(X)$. Suppose that $\phi : O_E \to O_F$ is a $C(X)$-linear unital $*$-homomorphism. Then $\phi \circ \iota_E = \iota_F$ and hence $\ker(\iota_E)_* \subset \ker(\iota_F)_*$. It follows that $(1 - [E])K^0(X) \subset (1 - [F])K^0(X)$ and hence $1 - [E] = (1 - [F])h$ for some $h \in K^0(X)$. If $\phi$ is an isomorphism, we deduce similarly that $1 - [F] = (1 - [E])k$ for some $k \in K^0(X)$. In that case $\text{rank}(E_x) = \text{rank}(F_x)$ for each $x \in X$ and $h$ must have constant virtual rank equal to one.

Conversely, suppose that there is $h \in K^0(X)$ such that $(1 - [E]) = (1 - [F])h$. We have $O_E \otimes O_\infty \cong O_E$ and $O_F \otimes O_\infty \cong O_F$ by \cite{3}. We are going to show the existence of a unital $C(X)$-linear embedding $O_E \subset O_F$ by producing an element $\chi \in KK_X(O_E, O_F)$ which maps $[1_{O_E}]$ to $[1_{O_F}]$ and then appeal to \cite{13}. If the virtual rank of $h$ is equal to one and hence $h$ is invertible in the ring $K^0(X)$, we show that $\chi$ is a $KK_X$-equivalence and that will imply that $O_E$ is isomorphic to $O_F$. Since the operation of suspension is an isomorphism in $KK_X$, it suffices to show that there is $\eta \in KK_X(SO_E, SO_F)$, respectively $\eta \in KK_X(SO_E, SO_F)^{-1}$, such that $\eta \circ [S\iota_E] = [S\iota_F]$. 

Let us recall that the mapping cone of a \( \alpha \) -homomorphism \( \alpha : A \to B \) is
\[
C_\alpha = \{(f, a) \in C([0, 1], B) \oplus A : f(0) = \alpha(a), f(1) = 0\}.
\]
If \( \alpha \) is a morphism of continuous \( C(X) \)-algebras, then the natural extension
\[
0 \longrightarrow SB \xrightarrow{\lambda} C_\alpha \xrightarrow{p} A \longrightarrow 0,
\]
where \( \lambda(f) = (f, 0) \) and \( p(f, a) = a \), is an extension of continuous \( C(X) \)-algebras.

Let \( KK(X) \) denote the additive category with objects separable \( C(X) \)-algebras and morphisms from \( A \) to \( B \) given by the group \( KK_X(A, B) \). Nest and Meyer have shown that \( KK(X) \) is a triangulated category \([14]\). In particular, for any diagram of separable \( C(X) \)-algebras and \( C(X) \)-linear \( \ast \)-homomorphisms
\[
\begin{array}{c}
SB \xrightarrow{\lambda} C_\alpha \xrightarrow{p} A \xrightarrow{\alpha} B \\
S\varphi \downarrow \quad \downarrow \gamma \quad \psi \quad \varphi \\
SB' \xrightarrow{\lambda'} C_{\alpha'} \xrightarrow{p} A' \xrightarrow{\alpha'} B'
\end{array}
\]
such that the right square commutes in \( KK(X) \), there is \( \gamma \in KK_X(C_\alpha, C_{\alpha'}) \) that makes to remaining squares commute. If the right square commutes up to homotopy of \( C(X) \)-linear \( \ast \)-homomorphisms, then \( \gamma \) can be chosen to be a \( C(X) \)-linear \( \ast \)-homomorphism, see \([18, \text{Prop. 2.9}]\). The general case is proved in a similar way, see \([14, \text{Appendix A}]\). Let us note that if \( \varphi \) and \( \psi \) are \( KK_X \)-equivalences, so is \( \gamma \) by the exactness of the Puppe sequence and the five-lemma.

We need another general observation. If
\[
0 \longrightarrow J \xrightarrow{j} B \xrightarrow{\pi} B/J \longrightarrow 0
\]
is an extension of separable continuous \( C(X) \)-algebras, then there is a surjective \( C(X) \)-linear \( \ast \)-homomorphism \( \mu : C_J \to SB/J \), \( \mu(f, b) = \pi \circ f \), and hence an extension of separable continuous \( C(X) \)-algebras
\[
0 \longrightarrow CJ \longrightarrow C_J \xrightarrow{\mu} SB/J \longrightarrow 0.
\]
If \( B/J \) is nuclear then it is \( C(X) \)-nuclear and since \( CJ \) is \( KK_X \)-contractible it follows that \( \mu \) must be a \( KK_X \)-equivalence, see \([2]\).

Let us recall that \( O_E \) is defined by the extension \( 0 \longrightarrow \mathcal{K}(\mathcal{E}) \xrightarrow{j_E} T_E \xrightarrow{\pi_E} 0 \).

By Theorem 4.4 and Lemma 4.7 of \([16]\) there is a commutative diagram in \( KK(X) \):
\[
\begin{array}{ccc}
\mathcal{K}(\mathcal{E}) \xrightarrow{[j_E]} T_E & & \\
\mathcal{E} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
C(X) \xrightarrow{[id]-[E]} C(X)
\end{array}
\]
where both vertical maps are $KK_X$-equivalences. Here $i_E$ is the canonical unital inclusion, $[\mathcal{E}]$ is the class in $KK_X(\mathcal{K}(\mathcal{E}), C(X))$ defined by the bimodule $\mathcal{E}$ that implements the strong Morita equivalence between $\mathcal{K}(\mathcal{E})$ and $C(X)$ and $[E]$ is the class in $KK_X(C(X), C(X))$ defined by the finitely generated $C(X)$-module $\Gamma(E)$. Note that $[id] - [E] \in KK_X(C(X), C(X))$ induces the multiplication map $K^0(X) \to K^0(X)$.

Pimsner’s statements refer to ordinary $KK$-theory but his constructions and arguments are natural and preserve the $C(X)$-structure.

After tensoring the C*-algebras in the above diagram by $O_\infty$ we can realize $[\mathcal{E}]^{-1}$ and $[id]-[E]$ as $KK_X$ classes of $C(X)$-linear $*$-homomorphisms $\psi_E$ and respectively $\alpha_E$. This is easily seen by using the identification $KK_X(C(X), B) \cong KK(C, B)$ for $B$ a $C(X)$-algebra and noting that $\mathcal{K}(\mathcal{E})$ contains a full projection since $\mathcal{E}$ is isomorphic to $C(X) \otimes \ell^2(\mathbb{N})$ by Kuiper’s theorem. Thus we obtain a diagram

\[
\begin{array}{ccc}
\mathcal{K}(\mathcal{E}) \otimes O_\infty & \xrightarrow{j_E \otimes id} & T_E \otimes O_\infty \\
\downarrow \psi_E & & \downarrow \psi_E \\
C(X) \otimes O_\infty & \xrightarrow{\alpha_E} & C(X) \otimes O_\infty
\end{array}
\]

that commutes in $KK(X)$. We construct a similar diagram for the bundle $F$. Let $H : C(X) \otimes O_\infty \to C(X) \otimes O_\infty$ be a $C(X)$-linear $*$-homomorphism which sends $[1_{C(X) \otimes O_\infty}]$ to $h \in K^0(X) \cong K_0(C(X) \otimes O_\infty)$. Note that $H$ is a $KK_X$-equivalence whenever $h$ is invertible in the ring $K^0(X)$. Since $1 - [E] = (1 - [F])h$ by assumption, the diagram

\[
\begin{array}{ccc}
C(X) \otimes O_\infty & \xrightarrow{\alpha_E} & C(X) \otimes O_\infty \\
\downarrow H & & \downarrow H \\
C(X) \otimes O_\infty & \xrightarrow{\alpha_E} & C(X) \otimes O_\infty
\end{array}
\]
commutes in the category $\text{KK}(X)$. The proof of the theorem is based on the following commutative diagram in $\text{KK}(X)$:

\[
\begin{array}{ccccccccc}
SO_E \otimes O_\infty & \rightarrow & C_{jE} \otimes O_\infty & \rightarrow & \mathcal{K}(\mathcal{E}) \otimes O_\infty & \rightarrow & T_E \otimes O_\infty \\
\mu_E \otimes id & \downarrow & \lambda_E & \downarrow & \psi_E & \downarrow & \alpha_E & \downarrow & i_E \otimes id \\
ST_E \otimes O_\infty & \rightarrow & C_{alphaE} & \rightarrow & C(X) \otimes O_\infty & \rightarrow & C(X) \otimes O_\infty \\
Si_E \otimes id & \downarrow & \gamma & \downarrow & H & \downarrow & \alpha_F & \downarrow & i_F \otimes id \\
SC(X) \otimes O_\infty & \rightarrow & C_{alphaF} & \rightarrow & C(X) \otimes O_\infty & \rightarrow & C(X) \otimes O_\infty \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
ST_F \otimes O_\infty & \rightarrow & C_{jF} \otimes O_\infty & \rightarrow & \mathcal{K}(\mathcal{F}) \otimes O_\infty & \rightarrow & T_F \otimes O_\infty \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
SO_F \otimes O_\infty & \rightarrow & \mathcal{K}(\mathcal{F}) \otimes O_\infty & \rightarrow & T_F \otimes O_\infty \\
\end{array}
\]

The elements $\gamma$, $\gamma_E$ and $\gamma_F$ are constructed as explained above since the category $\text{KK}(X)$ is triangulated. Moreover $\gamma_E$ and $\gamma_F$ are $KK_X$-equivalences since they are induced by the $KK_X$-equivalences $i_E, \psi_E$ and $i_F, \psi_F$. Arguing similarly, we see that $\gamma$ is a $KK_X$-equivalence whenever $h$ is invertible in the ring $K^0(X)$. The morphisms $\mu_E$ and $\mu_F$ are associated to the Toeplitz extensions $0 \rightarrow \mathcal{K}(\mathcal{E}) \rightarrow T_E \rightarrow O_E \rightarrow 0$ and respectively $0 \rightarrow \mathcal{K}(\mathcal{F}) \rightarrow T_F \rightarrow O_F \rightarrow 0$ and they are also $KK_X$-equivalences as we argued earlier in the proof. Let us note that $(\mu_E \otimes id) \circ \lambda_E \circ (Si_E \otimes id_{O_\infty}) = S_{i_E} \otimes id_{O_\infty}$ and $(\mu_F \otimes id) \circ \lambda_F \circ (Si_F \otimes id_{O_\infty}) = S_{i_F} \otimes id_{O_\infty}$ by a simple direct verification. In this way we are able to find an element

$\eta := [\mu_F \otimes id] \circ \gamma_F \circ \gamma_E^{-1} \circ [\mu_E \otimes id]^{-1} \in KK_X(SO_E \otimes O_\infty, SO_F \otimes O_\infty) \cong KK_X(SO_E, SO_F)$

such that the following diagram commutes in $\text{KK}(X)$:

\[
\begin{array}{cc}
\mathcal{K}(\mathcal{E}) & \mathcal{K}(\mathcal{F}) \\
\downarrow & \downarrow \\
SO_E & SO_F
\end{array}
\]

Unsuspending, we find $\chi \in KK_X(O_E, O_F)$ which maps $[1_{O_E}]$ to $[1_{O_F}]$, since $\chi \circ [i_E] = [i_F]$. If $h$ is invertible in $K^0(X)$, then $\chi$ is a $KK_X$-equivalence. It follows that $O_E \cong O_F$ as $C(X)$-algebras by applying Kirchberg’s isomorphism theorem [13]. If $\chi$ is just a $KK_X$-element which preserves the classes of the units, we invoke again [13] in order to lift $\chi$ to a unital $C(X)$-linear embedding $O_E \subset O_F$. □
Set $T_n(b) := \sum_{k=1}^{n}(-1)^{k-1}m^{n-k}b^{k-1}$. Then $(m + b)T_n(b) = m^n - (-b)^n$.

**Lemma 2.1.** Let $R$ be a commutative ring such that $R^{n+1} = \{0\}$ for some $n \geq 1$ and let $a, b \in R$. If there is $h \in R$ such that $a = b + mh + bh$, then $(a - b)T_n(b) = m^n h$. Conversely, if $(a - b)T_n(b) = m^n h$ for some $h \in R$, then $m^n(a - b - mh - bh) = 0$.

**Proof.** Suppose that $a - b = (m + b)h$ for some $h \in R$. Then $(a - b)T_n(b) = (m^n - (-b)^n)h = m^n h - (-b)^n h$. Since $(-b)^n h \in R^{n+1}$ must vanish it follows that $(a - b)T_n(b) = m^n h$. Conversely, suppose that $(a - b)T_n(b) = m^n h$ for some $h \in R$. Then $(a - b)T_n(b)(m + b) = m^n(m + b)h$ and hence $(a - b)(m^n - (-b)^n) = m^n(m + b)h$. But $(a - b)(-b)^n = 0$ since $R^{n+1} = \{0\}$. Therefore $m^n(a - b - mh - bh) = 0$. \qed

We are now prepared to prove Theorem 1.2.

**Proof.** Since $\dim(X) \leq n$, we can embed $X$ in $\mathbb{R}^{2n+1}$ and then find a decreasing sequence $X_i$ of polyhedra whose intersection is $X$. We have $\overline{K}^0(X_i)^{n+1} = \{0\}$ since $\dim(X_i) \leq 2n + 1$ (see the next section for further discussion). It follows that $\overline{K}^0(X)^{n+1} = \{0\}$ since $\overline{K}^0(X) \cong \varprojlim \overline{K}^0(X_i)$. Let us write $[E] - 1 = m + a$ and $[F] - 1 = m + b$ where $a = [\overline{E}]$ and $b = [\overline{F}] \in \overline{K}^0(X)$. By Theorem 2.1, $O_E \cong O_F$ if and only if $[E] - 1 = ([F] - 1)(1 + h)$ for some $h \in \overline{K}^0(X)$ and hence if and only if $a = b + mb + bh$ for some $h \in \overline{K}^0(X)$. With this observation we conclude the proof by applying Lemma 2.1. \qed

For a hermitian bundle $E$ we denote by $\overline{E}$ the conjugate bundle, by $E_0$ the set of all nonzero elements in $E$ and by $S(E)$ the unit sphere bundle of $E$.

**Proposition 2.2.** Let $E$ and $F$ be hermitian line bundles over a path-connected compact metrizable space $X$. Then $O_E \cong O_F$ as $C(X)$-algebras if and only if either $E \cong F$ or $E \cong F$.

**Proof.** Vasselli has shown that for a line bundle $E$, $O_E \cong C(S(E))$ as $C(X)$-algebras [20]. Therefore it suffices to show that there is a homeomorphism of sphere-bundles $\phi : S(E) \to S(F)$ if and only if either $E \cong F$ or $E \cong F$. The isomorphism $O_E \cong O_F$ was noted in [20]. One can argue as follows. The conjugate bundle $\overline{E}$ has the same underlying real vector bundle as $\overline{E}$ but with opposite complex structure; the identity map $E \to \overline{E}$ is conjugate linear. If we endow $\overline{E}$ with the conjugate hermitian metric it follows that the identity map is fiberwise norm-preserving and hence it identifies $S(E)$ with $S(\overline{E})$.

Conversely, suppose that there is a homeomorphism of sphere-bundles $\phi : S(E) \to S(F)$. By homogeneity we can extend $\phi$ to a fiber-preserving homeomorphism $\Phi : E \to F$ such that $\Phi(E_0) \subset F_0$. Let $p_E : E \to X$ be the projection map and let $i_E$ be the inclusion map $(E, \emptyset) \subset (E, E_0)$. Let us recall that the underlying real vector bundle $E_{\mathbb{R}}$ has a canonical preferred orientation which yields a Thom class $u_E \in H^2(E, E_0, \mathbb{Z})$, see [15, ch.9, ch.14]. Since $X$ is path connected, $\mathbb{Z} \cong H^0(X, \mathbb{Z}) \cong H^2(E, E_0, \mathbb{Z}) = Zu_E$ by the Thom isomorphism. The first Chern class $c_1(E) = \overline{E} = c_1(\overline{E}) = e(E_{\mathbb{R}}) = e(\overline{E}_{\mathbb{R}})$.
The map $\Phi$ induces a commutative diagram

$$
\begin{array}{ccc}
Z u_E & \xrightarrow{\Phi^*} & H^2(E, E_0, \mathbb{Z}) \\
\downarrow & & \downarrow \\
Z u_F & \xrightarrow{\Phi^*} & H^2(F, F_0, \mathbb{Z})
\end{array}
$$

Since $\Phi$ is a homeomorphism, $\Phi^*(u_F) = \pm u_E$ and hence $c_1(E) = \pm c_1(F)$. It follows that either $E \cong F$ or $E \cong F$. 

### 3. K-theory invariants of $O_E$

In this section we construct a sequence $(\mu_n(E))_n$ of K-theory invariants of $O_E$. The class of the trivial bundle of rank $r$ is denoted by $r \in K^0(X)$. All the elements of the ring $K^0(X)$ are nilpotent [11].

Recall that for $E \in \operatorname{Vect}_{m+1}(X)$ we denote by $[\tilde{E}]$ the $K^0(X)$-component $[E] - (m + 1)$ of $[E]$. We introduce the following equivalence relation on $K^0(X)$: $a \sim b$, if and only if $a = b + mh + bh$ for some $h \in K^0(X)$. Rewriting $a = b + mh + bh$ as $m + a = (m + b)(1 + h)$ in $K^0(X)$, it becomes obvious that $\sim$ is an equivalence relation since $1 + h$ in invertible in the ring $K^0(X)$ with inverse $1 + \sum_{k \geq 1} (-1)^k h^k$. Moreover, if $E, F \in \operatorname{Vect}_{m+1}(X)$, then $(1 - [E])K^0(X) = (1 - [F])K^0(X)$ if and only if $[\tilde{E}] \sim [\tilde{F}]$.

Note that with our new notation, Theorem [11] shows that $O_E \cong O_F \Rightarrow [\tilde{E}] \sim [\tilde{F}]$. In other words the equivalence class of $[\tilde{E}]$ in $K^0(X)/\sim$ is an invariant of $O_E$. In order to obtain more computable invariants, for each $m \geq 1$, we use the sequence of polynomials $p_n \in \mathbb{Z}[x]$ introduced in [11]. It is immediate that

$$p_{n+1}(x) = \frac{\ell(n+1)}{\ell(n)} mp_n(x) + (-1)^n \frac{\ell(n+1)}{n+1} x^{n+1}.$$  

The first five polynomials in the sequence are:

- $p_1(x) = x$,
- $p_2(x) = 2mx - x^2$,
- $p_3(x) = 6m^2x - 3mx^2 + 2x^3$,
- $p_4(x) = 12m^3x - 6m^2x^2 + 4mx^3 - 3x^4$,
- $p_5(x) = 60m^4x - 30m^3x^2 + 20m^2x^3 - 15mx^4 + 12x^5$.

**Lemma 3.1.** For any $n \geq 1$ there are polynomials $u_n, s_{n+1} \in \mathbb{Z}[x, y]$ and $v_n \in \mathbb{Z}[x]$ such that each monomial of $s_{n+1}$ has total degree $\geq n + 1$ and

(i) $p_n(x + y) = p_n(x) + p_n(y) + xy u_n(x, y),$

(ii) $p_n(x + my + xy) = p_n(x) + mn^2 v_n(y) + s_{n+1}(x, y).$

**Proof.** It follows from the binomial formula that for any polynomial $p \in \mathbb{Z}[x]$ with $p(0) = 0$ there is a polynomial $u \in \mathbb{Z}[x, y]$ such that $p(x + y) = p(x) + p(y) + xy u(x, y)$. This
proves (i). Let us now prove (ii). Set \( V_n(x) = \sum_{k=1}^{n} (-1)^{k-1} x^k / k = \log(1 + x)_{[n]} \). Then \( p_n(x) = \ell(n)m^n V_n(x/m) \). Each monomial in \( x \) and \( y \) that appears in expansion of the series \( \sum_{k \geq n+1} (-1)^{k-1} (x + y)^{k}/k \) has total degree \( \geq n + 1 \). Therefore, the equality of formal series \( \log(1 + x + y) = \log(1 + x) + \log(1 + y) \) shows that in the reduced form of the polynomial \( r_{n+1}(x, y) := V_n(x + y + xy) - V_n(x) - V_n(y) \) all the monomials have total degree \( \geq n + 1 \). It follows that

\[
p_n(x + my + xy) = \ell(n)m^n V_n(x/m + y + x/m \cdot y)
= \ell(n)m^n V_n(x/m) + \ell(n)m^n V_n(y) + \ell(n)m^n r_{n+1}(x/m, y)
= p_n(x) + m^n \cdot v_n(y) + s_{n+1}(x, y),
\]

where \( s_{n+1}(x, y) := \ell(n)m^n r_{n+1}(x/m, y) \) and \( v_n(y) := \ell(n)V_n(y) \). Since both \( p_n \) and \( v_n \) have integer coefficients, so must have \( s_{n+1} \). \( \square \)

Let \( X \) be a finite CW complex of dimension \( d \) with skeleton decomposition

\[
\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_d = X.
\]

Consider the induced filtration of \( K^*(X) \)

\[
K^*_q(X) = \ker \left( K^*(X) \to K^*(X_{q-1}) \right) = \text{image} \left( K^*(X, X_{q-1}) \to K^*(X) \right).
\]

One has \( \{0\} = K^*_{d+1}(X) \subset K^*_d(X) \subset \cdots \subset K^*_1(X) \subset K^*_0(X) = K^*(X) \) and

\[
K^*_q(X)K^*_r(X) \subset K^*_{q+r}(X).
\]

Since the map \( K^0(X_{2d+1}) \to K^0(X_{2d}) \) is injective, it follows that \( K^0_{2d+1}(X) = K^0_{2d+2}(X) \). We will use only the even components of this filtration corresponding to \( K^0(X) \), namely

(4) \( \{0\} = K^0_{2[d/2]+2}(X) \subset K^0_{2[d/2]+1}(X) \subset \cdots \subset K^0_d(X) \subset K^0_0(X) = K^0(X) \).

Since \( \widetilde{K}^0(X) = K^0_1(X) = K^0_2(X) \) we have

(5) \( \widetilde{K}^0(X)^j \subset K^0_2(X) \).

In particular, \( \widetilde{K}^0(X)^{\lfloor d/2 \rfloor + 1} \subset K^0_{d+1}(X) \) and hence \( \widetilde{K}^0(X)^{\lfloor d/2 \rfloor + 1} = \{0\} \).

**Definition 3.2.** Let \( X \) be a finite CW complex of dimension \( d \). For each \( n \geq 1 \) we define the map

\[
\mu_n : \text{Vect}_{m+1}(X) \to \widetilde{K}^0(X) / K^0_{2d+2}(X),
\]

by \( \mu_n(E) = \pi_n([p_n(\overline{E})]) \) where \( \pi_n : K^0(X) \to \widetilde{K}^0(X) / K^0_{2d+2}(X) \) is the natural quotient map. For \( n \geq \lfloor d/2 \rfloor \), \( \mu_n(E) = p_n([\overline{E}]) \in \widetilde{K}^0(X) \) since \( K^0_{2d+2}(X) = \{0\} \).

**Theorem 3.3.** Let \( X \) be a finite CW complex and let \( E, F \in \text{Vect}_{m+1}(X) \). If \( O_E \) and \( O_F \) are isomorphic as \( C(X) \)-algebras, then \( \mu_n(E) - \mu_n(F) \) is divisible by \( m^n \), for \( n \geq 1 \).
Proof. Set \( a = \lceil \tilde{E} \rceil \) and \( b = \lceil \tilde{F} \rceil \). If \( O_E \cong O_F \), then by Theorem \[\text{1.1}\] there is \( h \in \tilde{K}^0(X) \) such that \( a = b + mh + bh \). Then, by Lemma \[\text{3.1}\](ii)

\[
p_n(a) = p_n(b + mh + bh) = p_n(b) + m^n v_n(h) + s_{n+1}(b, h),
\]

and \( s_{n+1}(b, h) \in \tilde{K}^0(X)^{n+1} \subset K^0_{2n+2}(X) \) by \[\text{(3)}\]. Thus \( \mu_n(E) - \mu_n(F) = m^n \pi_n(v_n(h)) \). \( \square \)

As a corollary, we derive Theorem \[\text{1.3}\], restated here as follows:

**Corollary 3.4.** Let \( X \) be a finite CW complex of dimension \( d \) and let \( E, F \in \text{Vect}_{m+1}(X) \). If \( O_E \cong O_F \) as \( C(X) \)-algebras, then \( p_{\lfloor d/2 \rfloor}([E]) - p_{\lfloor d/2 \rfloor}([F]) \) is divisible by \( m^{\lfloor d/2 \rfloor} \) in \( \tilde{K}^0(X) \).

**Proof.** If \( n \geq \lfloor d/2 \rfloor \), then \( K^0_{2n+2}(X) = \{0\} \) and so \( p_n(a) - p_n(b) \in m^n \tilde{K}^0(X) \). \( \square \)

**Remark 3.5.** Let us note that \( \mu_{\lfloor d/2 \rfloor}(E) \) determines \( \mu_{\lfloor d/2 \rfloor + k}(E) \) for \( k \geq 1 \). Indeed, letting \( n = \lfloor d/2 \rfloor \) it follows from \[\text{(3)}\] that

\[
\mu_{n+k}(E) = \frac{\ell(n+k)}{\ell(n)} m^k \mu_n(E),
\]

since \( \tilde{K}^0(X)^{n+k} = \{0\} \). Let us note that \( \mu_{\lfloor d/2 \rfloor}(E) \) is also related to the lower order invariants. Indeed, it follows immediately from \[\text{(3)}\] and \[\text{(5)}\] that if \( 1 \leq j \leq \lfloor d/2 \rfloor \), then

\[
\frac{\ell(j)}{\ell(j-1)} m \mu_{j-1}(E) = \pi_{j-1}(\mu_j(E)),
\]

where \( \pi_{j-1} \) stands for the quotient map \( \tilde{K}^0(X)/K^0_{2j+2}(X) \rightarrow \tilde{K}^0(X)/K^0_{2j}(X) \). From this, with \( n = \lfloor d/2 \rfloor \), we obtain

\[
\frac{\ell(n)}{\ell(n-j)} m^j \mu_{n-j}(E) = \pi_{n-j}(\mu_n(E)).
\]

Assuming that \( \text{Tor}(K^0_{2j}(X)/K^0_{2j+2}(X), \mathbb{Z}/m) = 0 \) for all \( j \geq 1 \), and that \( m \) and \( \lfloor d/2 \rfloor ! \) are relatively prime, it follows that if \( \mu_{\lfloor d/2 \rfloor}(E) - \mu_{\lfloor d/2 \rfloor}(F) \) is divisible by \( m^{\lfloor d/2 \rfloor} \), then \( \mu_j(E) - \mu_j(F) \) is divisible by \( m^j \) for all \( j \geq 1 \).

The groups \( \tilde{K}^0(X)/K^0_{2j}(X) \) are homotopy invariants of \( X \), and they are actually independent of the CW structure \[\text{1.1}\]. Let \( k^j(X) \) denote the reduced connective \( K \)-theory of \( X \) and let \( \beta : k^{j+2}(X) \rightarrow k^j(X) \) be the Bott operation. One can identify \( k^{2j+2}(X) \) with \( K^0(X, X_{2j}) \) in such a way that \( \beta \) corresponds to the map \( K^0(X, X_{2j}) \rightarrow K^0(X, X_{2j-2}) \). Thus the image of \( \beta^{i+1} : k^{2j+2}(X) \rightarrow k^0(X) \cong \tilde{K}^0(X) \) coincides with \( K^0_{2j}(X) \), and hence \( \mu_j(E) \) can be viewed as an element of \( k^0(X)/\beta^{i+1}k^{2j+2}(X) \).
4. Cohomology invariants of $O_E$

Let us recall that $V_n(x) = \log((1 + x)^n)$ and consider the polynomials

$$W_n(x) = \frac{n!}{\ell(n)} p_n(x) = n! m^n \log \left( 1 + \frac{x}{m} \right)_{[n]} = n! m^n V_n \left( \frac{x}{m} \right) = \sum_{r=1}^{n} (-1)^{r-1} m^{n-r} \frac{n!}{r} x^r.$$

For a polynomial $P$ in variables $x_1, \ldots, x_n$, we assign to the variable $x_k$ the weight $k$ and denote by $P(x_1, \ldots, x_n)_{(n)}$ the sum of all monomials of $P$ of total weight $n$. For example if $P(x_1, x_2, x_3) = (x_1 + \frac{x_2}{2} + \frac{x_3}{3})^2$, then $P(x_1, x_2, x_3)_{(3)} = x_1 x_2$. Consider the polynomials

$$q_n(x_1, \ldots, x_n) = W_n \left( \frac{x_1}{1!} + \cdots + \frac{x_n}{n!} \right)_{(n)} = \sum_{r=1}^{n} (-1)^{n-r} \frac{n!}{r} \left( \frac{x_1}{1!} + \cdots + \frac{x_n}{n!} \right)^r.$$

Thus

$$q_n(x_1, \ldots, x_n) = \sum_{k_1 + k_2 + \cdots + k_n = n} (-1)^{k_1 + \cdots + k_n - 1} m^{-k_1 - \cdots - k_n} \frac{n!}{1!^{k_1} \cdots n!^{k_n}} \frac{(k_1 + \cdots + k_n - 1)!}{(k_1)! (2k_2)! \cdots (nk_n)!} x_1^{k_1} \cdots x_n^{k_n}.$$

Consider also the polynomials $r_n$ obtained from $q_n$ by taking $m = 1$, i.e.

$$r_n(x_1, \ldots, x_n) = n! V_n \left( \frac{x_1}{1!} + \cdots + \frac{x_n}{n!} \right)_{(n)} = \sum_{r=1}^{n} (-1)^{n-r} \frac{n!}{r} \left( \frac{x_1}{1!} + \cdots + \frac{x_n}{n!} \right)^r.$$

**Lemma 4.1.** The polynomials $q_n(x_1, \ldots, x_n)$ and $r_n(x_1, \ldots, x_n)$ have integer coefficients.

**Proof.** We have a factorization

$$\frac{n! (k_1 + \cdots + k_n - 1)!}{1!^{k_1} \cdots n!^{k_n} k_1! \cdots k_n!} = \frac{(k_1 + 2k_2 + \cdots + nk_n)!}{(k_1)!(2k_2)! \cdots (nk_n)!} a(1, k_1) \cdots a(n, k_n)(k_1 + \cdots + k_n - 1)!,$$

where $a(j, k) = \frac{(jk)!}{j!^k} k!$. It follows that the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ is an integer since it involves a multinomial coefficient and numbers $a(j, k)$ which are easily seen to be integers by using the recurrence formula $a(j, k) = \frac{(j-1)!}{j} a(j, k-1)$ where $a(j, 1) = 1$. \hfill $\square$

Let us recall from [11] that the components of the Chern character $ch(E) = \sum_{k \geq 0} s_k(E)/k!$ involve integral stable characteristic classes $s_k$, also denoted by $ch_k$. The classes $s_k$ have two important properties. If one sets $s_0(E) = \text{rank}(E)$, then for $k \geq 0$:

$$s_k(E \oplus F) = s_k(E) + s_k(F)$$

(6) $$s_k(E \otimes F) = \sum_{i+j=k} \frac{k!}{i!j!} s_i(E)s_j(F).$$

We are now ready to prove Theorem 1.5, restated here for the convenience of the reader:
Theorem 4.2. Let $X$ be a compact metrizable space and let $E, F \in \text{Vect}_{m+1}(X)$. If $O_E \cong O_F$ as $C(X)$-algebras, then $q_n(s_1(E), ..., s_n(E)) - q_n(s_1(F), ..., s_n(F))$ is divisible by $m^n$ in $H^{2n}(X, \mathbb{Z})$, for each $n \geq 1$.

Proof. Multiplying by $n!/\ell(n)$ in Lemma 3.1 (ii), we obtain

$$W_n(y + mh + yh) - W_n(y) = m^n n! V_n(h) + \frac{n!}{\ell(n)} s_{n+1}(y, h). \tag{7}$$

Recall that $s_{n+1}$ is a polynomial with all monomials of degree at least $n + 1$. Let us make in (7) the substitutions

$$y = \frac{y_1}{1!} + \cdots + \frac{y_n}{n!}, \quad h = \frac{h_1}{1!} + \cdots + \frac{h_n}{n!},$$

where $y_k$ and $h_k$ have weight $k$. With these substitutions we have

$$W_n(y)_{(n)} = q_n(y_1, ..., y_n), \quad n! V_n(h)_{(n)} = r_n(h_1, ..., h_n),$$

whereas $\frac{n!}{\ell(n)} s_{n+1}(y, h)_{(n)} = 0$. By grouping the terms of $y + mh + yh$ of the same weight, we have

$$y + mh + yh = \frac{y_1 + mh_1}{1!} + \frac{y_2 + mh_2 + 2y_1 h_1}{2!} + \cdots + \frac{y_n + mh_n + \sum_{i+j=n} \frac{n!}{i!j!} y_i h_j}{n!},$$

and hence

$$W_n(y + mh + yh)_{(n)} = q_n(y_1 + mh_1, y_2 + mh_2 + 2y_1 h_1, ..., y_n + mh_n + \sum_{i+j=n} \frac{n!}{i!j!} y_i h_j)$$

Thus, the equation $W_n(y + mh + yh)_{(n)} - W_n(y)_{(n)} = n! V_n(h)_{(n)}$ implies that

$$q_n(y_1 + mh_1, y_2 + mh_2 + 2y_1 h_1, ..., y_n + mh_n + \sum_{i+j=n} \frac{n!}{i!j!} y_i h_j) - q_n(y_1, ..., y_n)$$

is equal to $m^n r_n(h_1, ..., h_n)$.

Suppose now $O_E \cong O_F$. Then, by Theorem 4.2 $[\tilde{E}] = [\tilde{F}] + mH + [\tilde{F}]H$ for some $H \in K^0(X)$. Using (6) it follows that for $k \geq 1$

$$s_k(E) = s_k(F) + m s_k(H) + \sum_{i+j=k} \frac{k!}{i!j!} s_i(F) s_j(H),$$

and hence

$$q_n(s_1(E), ..., s_n(E)) - q_n(s_1(F), ..., s_n(F)) = m^n r_n(s_1(H), ..., s_n(H)).$$

It follows by Theorem 4.2 that the image of $q_n(s_1(E), ..., s_n(E))$ in $H^{2n}(X, \mathbb{Z}/m^n)$ is an invariant of $O_E$ for each $n \geq 1$. The first four invariants in this sequence are:

$$\hat{s}_1(E) \in H^2(X; \mathbb{Z}/m)$$

$$m \hat{s}_2(E) - \hat{s}_1(E)^2 \in H^4(X; \mathbb{Z}/m^2)$$
Theorem 4.2 shows that we can remove the factor $s$

Proof. Let $Proposition 4.4.

Let $H$

Here we denote by $\dot{c}$ over $H$

We note that this is not really a new invariant, since it is not hard to prove that $by$ of $H$

By expressing $s_k$ in terms of Chern classes we obtain a sequence of characteristic classes of $E$ which are invariants of $O_E$. The first four classes in the sequence are:

1. $\dot{c}_1(E) \in H^2(X; \mathbb{Z}/m)$
2. $(m-1)\dot{c}_1(E)^2 - 2m\dot{c}_2(E) \in H^4(X; \mathbb{Z}/m^2)$
3. $(m^2 - 3m + 2)\dot{c}_1(E)^3 - (3m^2 - 6m)\dot{c}_1(E)\dot{c}_2(E) + 3m^2\dot{c}_3(E) \in H^6(X; \mathbb{Z}/m^3)$
4. $(m^3 - 7m^2 + 12m - 6)\dot{c}_1(E)^4 - (4m^3 - 24m^2 + 24m)\dot{c}_1(E)^2\dot{c}_2(E) + (4m^3 - 12m^2)\dot{c}_1(E)\dot{c}_3(E) + (2m^3 - 12m^2)\dot{c}_2(E)^2 - 4m^3\dot{c}_4(E) \in H^8(X; \mathbb{Z}/m^4)$

Here we denote by $\dot{c}_k(E)$ the image of the Chern class $c_k(E)$ under the coefficient map $H^{2k}(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{Z}/m^k)$.

Corollary 4.3. Let $X$ be a compact metrizable space and let $L$ and $L'$ be two line bundles over $X$. If $O_{m+L} \cong O_{m+L'}$ as $C(X)$-algebras, then $q_n(1, ..., 1)(c_1(L)^n - c_1(L')^n)$ is divisible by $m^n$ for all $n \geq 1$.

Proof. If $L$ is a line bundle, then $s_k(L) = c_1(L)^k$. Since all monomials of $q_n$ have weight $n$, $q_n(s_1(L), ..., s_n(L)) = q_n(1, ..., 1)c_1(L)^n$.

Let us note that there is a more direct way to derive cohomology invariants for $O_E$.

Proposition 4.4. Let $X$ be a compact metrizable space. Then $s_n(p_n([\tilde{E}]))$ is an element of $H^{2n}(X, \mathbb{Z})$ whose image in $H^{2n}(X, \mathbb{Z}/m^n)$ is an invariant of $O_E$, $n \geq 1$.

Proof. Suppose that $O_E \cong O_F$ as $C(X)$-algebras. Then, we saw in the proof of the Theorem 4.3 that $p_n([\tilde{E}]) - p_n([\tilde{F}]) = m^n c + d$ for some $c \in \tilde{K}^0(X)$ and $d \in \tilde{K}^0(X)^{n+1}$. From the multiplicative properties of the $s_n$-classes, one deduces that $s_n$ vanishes on $\tilde{K}^0(X)^{n+1}$. Therefore $s_n(p_n([\tilde{E}])) - s_n(p_n([\tilde{F}])) = m^n s_n(c)$.

We note that this is not really a new invariant, since it is not hard to prove that $s_n(p_n([\tilde{E}]) = \ell(n)q_n(s_1(E), ..., s_n(E)).$

Theorem 4.2 shows that we can remove the factor $\ell(n)$ and hence obtain a finer invariant.
5. Proof of Theorem 1.6

Recall that $O_{m+1}(X)$ denotes the set of isomorphism classes of unital separable $C(X)$-algebras with all fibers isomorphic to $O_{m+1}$. These $C(X)$-algebras are automatically locally trivial if $X$ is finite dimensional.

For a discrete abelian group $G$ and $n \geq 1$ let $K(G, n)$ be an Eilenberg-MacLane space. It is a connected CW complex $Y$ having just one nontrivial homotopy group $\pi_n(Y) \cong G$. A $K(G, n)$ space is unique up to homotopy equivalence. For a CW complex $X$, there is an isomorphism $H^n(X, G) \cong [X, K(G, n)]$.

Let us recall from [9] that if $Y$ is a connected CW complex with $\pi_1(Y)$ acting trivially on $\pi_n(Y)$ for $n \geq 1$, then $Y$ admits a Postnikov tower

$$
\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_2 \to Y_1 = K(\pi_1(Y), 1).
$$

Each space $Y_n$ carries the homotopy groups of $Y$ up to level $n$. More precisely, there exist compatible maps $Y \to Y_n$ that induce isomorphisms $\pi_i(Y) \to \pi(Y_n)$ for $i \leq n$ and $\pi_i(Y_n) = 0$ for $i > n$. Each map $Y_n \to Y_{n-1}$ is a fibration with fiber $K(\pi_n(Y), n)$. Thus $Y_n$ can be thought as a twisted product of $Y_{n-1}$ by $K(\pi_n(Y), n)$. The space $Y$ is weakly homotopy equivalent to the projective limit $\varprojlim Y_n$.

**Proposition 5.1.** Let $X$ be a finite connected CW complex and let $m \geq 1$ be an integer. Then $|O_{m+1}(X)| \leq |H^{even}(X, \mathbb{Z}/m)|$.

*Proof.* Let $Y$ be a CW complex weakly homotopy equivalent to the classifying space $B\text{Aut}(O_{m+1})$ of principal $\text{Aut}(O_{m+1})$-bundles. Then, there are bijections

$$
O_{m+1}(X) \cong [X, B\text{Aut}(O_{m+1})] \cong [X, Y].
$$

The homotopy groups of $\text{Aut}(O_{m+1})$ were computed in [4]. That calculation gives $\pi_{2k-1}(Y) = 0$ and $\pi_{2k}(Y) = \mathbb{Z}/m$, $k \geq 1$. Consequently, the Postnikov tower of $Y$ reduces to its even terms

$$
\cdots \to Y_{2k} \to Y_{2k-2} \to \cdots \to Y_2 = K(\mathbb{Z}/m, 2).
$$

The homotopy sequence of the fibration $K(\mathbb{Z}/m, 2k) \to Y_{2k} \to Y_{2k-2}$ gives for all choices of the base points an exact sequence of sets

$$
[X, K(\mathbb{Z}/m, 2k)] \to [X, Y_{2k}] \to [X, Y_{2k-2}].
$$

This shows that

$$
|[X, Y_{2k}]| \leq |[X, Y_{2k-2}]| \cdot |H^{2k}(X, \mathbb{Z}/m)|.
$$

By Whitehead’s theorem, if $n > \dim(X)/2$, then the map $Y \to Y_{2n}$ induces a bijection $[X, Y] \cong [X, Y_{2n}]$. It follows that

$$
|[X, Y]| \leq \prod_{1 \leq k \leq n} |H^{2k}(X, \mathbb{Z}/m)| = |H^{even}(X, \mathbb{Z}/m)|. \quad \Box
$$
We consider a commutative ring $R$ which admits a filtration by ideals
\[ \cdots \subset R_{k+1} \subset R_k \subset \cdots \subset R_1 = R \]
with the property that $R_q R_k \subset R_{q+k}$ and there is $n$ such that $R_{n+1} = \{0\}$. On $R$ we consider the following equivalence relation: $a \sim b$ if there is $h \in R$ such that $a = b + mh + bh$. Let us denote by $R/\sim$ the set of equivalence classes.

**Lemma 5.2.** Let $R$ be a filtered commutative ring with $R_{n+1} = \{0\}$. Suppose that $\text{Tor}(R_k/R_{k+1}, \mathbb{Z}/m) = 0$ for all $k \geq 1$. Then $|R/\sim| = |R \otimes \mathbb{Z}/m| = \prod_{k \geq 1} |R_k/R_{k+1} \otimes \mathbb{Z}/m|.$

**Proof.** Using the exact sequence for Tor, we observe first that $\text{Tor}(R_k/R_k, \mathbb{Z}/m) = 0$ for all $k \geq 1$ and hence if $h \in R$ satisfies $mh \in R_k$ for some $k$, then $h \in R_k$. Using the exact sequences
\[ 0 \to R_{k+1} \otimes \mathbb{Z}/m \to R_k \otimes \mathbb{Z}/m \to R_k/R_{k+1} \otimes \mathbb{Z}/m \to 0 \]
we see that $|R \otimes \mathbb{Z}/m| = \prod_{k \geq 1} |R_k/R_{k+1} \otimes \mathbb{Z}/m|.$ For each $k$ choose a finite subset $A_k \subset R_k$ such that the quotient map $\pi_k : R_k \to R_k/R_{k+1} \otimes \mathbb{Z}/m$ induces a bijective map $\pi_k : A_k \to R_k/R_{k+1} \otimes \mathbb{Z}/m$. Consider the map $\eta : A_1 \times A_2 \times \cdots \times A_n \to R$ defined by $\eta(a_1, \ldots, a_n) = a_1 + \cdots + a_n$. To prove the proposition it suffices to show that $\eta$ induces a bijection of $\overline{\eta} : A_1 \times A_2 \times \cdots \times A_n \to R/\sim$. First we verify that $\overline{\eta}$ is injective. Let $a_k, b_k \in A_k$, $1 \leq k \leq n$ and assume that
\[ a_1 + \cdots + a_n \sim b_1 + \cdots + b_n. \]
We must show that $a_k = b_k$ for all $k$. Set $r_k = a_k + \cdots + a_n$ and $s_k = b_k + \cdots + b_n$. Then $r_k, s_k \in R_k$. Since $a_1 + r_2 \sim b_1 + s_2$ there is $h_1 \in R_1$ such that
\[ a_1 + r_2 = b_1 + s_2 + mh_1 + b_1h_1 + s_2h_1 \]
and hence $a_1 - b_1 - mh_1 \in R_2$. Therefore $\pi_1(a_1) = \pi_1(b_1)$ and so $a_1 = b_1$. Arguing by induction, suppose that we have shown that $a_i = b_i$ for all $i \leq k-1$. Set $w = a_1 + \cdots + a_{k-1}$. By assumption $w + r_k \sim w + s_k$ and hence there is $h \in R_1$ such that
\[ w + r_k = w + s_k + mh + wh + s_kh. \]
Let us notice that if $h \in R_i$ for some $i \leq k - 1$, then equation (8) shows that $mh = (r_k - s_k) - wh - s_kh \in R_k \cup R_{i+1} = R_{i+1}$ and hence $h \in R_{i+1}$. This shows that in fact $h = h_k \in R_k$. From equation (8) we obtain
\[ a_k - b_k - mh_k = s_{k+1} - r_{k+1} + (w + s_k)h_k \in R_{k+1}. \]
This shows that $\pi_k(a_k) = \pi_k(b_k)$ and hence $a_k = b_k$.

It remains to verify that $\overline{\eta}$ is surjective. In other words for any given $x_1 \in R_1$ we must find $a_k \in A_k$, $1 \leq k \leq n$, such that $x_1 \sim a_1 + \cdots + a_n$. We do this by induction, showing that for each $k \geq 1$ there exist $a_i \in A_i$, $1 \leq i \leq k$ and $x_{k+1} \in R_{k+1}$ such that
\[ x_1 \sim a_1 + \cdots + a_k + x_{k+1}, \]
and observe that \( x_{n+1} = 0 \) since \( R_{n+1} = \{0\} \).

By the definition of \( A_1 \) there is \( a_1 \in A_1 \) such that \( \pi_1(a_1) = \pi_1(x_1) \in R_1/R_2 \otimes \mathbb{Z}/m \). Therefore there exist \( h_1 \in R_1 \) and \( y_2 \in R_2 \) such that \( a_1 = x_1 + mh_1 + y_2 \). Setting \( x_2 = x_1 h_1 - y_2 \in R_2 \) we obtain

\[
x_1 \sim x_1 + mh_1 + x_1 h_1 = a_1 + x_2.
\]

Suppose now that we found \( a_i \in A_i, 1 \leq i \leq k - 1 \) and \( x_k \in R_k \) such that

\[
x_1 \sim a_1 + \cdots + a_{k-1} + x_k.
\]

Let \( a_k \in A_k \) be such that \( \pi_k(a_k) = \pi_k(x_k) \). Then there exist \( h_k \in R_k \) and \( y_{k+1} \in R_{k+1} \) such that \( a_k = x_k + mh_k + y_{k+1} \). Thus

\[
a_1 + \cdots + a_{k-1} + x_k \sim a_1 + \cdots + a_{k-1} + x_k + mh_k + (a_1 + \cdots + a_{k-1} + x_k)h_k
\]

\[
= a_1 + \cdots + a_{k-1} + mh_k + x_k + h_k
\]

where \( x_{k+1} = (a_1 + \cdots + a_{k-1} + x_k)h_k - y_{k+1} \in R_{k+1} \). \( \square \)

**Theorem 5.3.** Let \( X \) be a finite connected CW complex of dimension \( d \) and let \( m \geq 1 \) be an integer. Suppose that \( \text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0 \) and that \( m \geq \lceil (d - 3)/2 \rceil \). Then:

(i) Any separable unital \( C(X) \)-algebra with fiber \( O_{m+1} \) is isomorphic to \( O_E \) for some \( E \in \text{Vect}_{m+1}(X) \).

(ii) If \( E, F \in \text{Vect}_{m+1}(X) \), then \( O_E \cong O_F \) as \( C(X) \)-algebras if and only if there is \( h \in K^0(X) \) such that \( 1 - [E] = (1 - [F])h \).

(iii) The cardinality of \( O_{m+1}(X) \) is equal to \( |\tilde{K}^0(X) \otimes \mathbb{Z}/m| = |\tilde{H}^{\text{even}}(X, \mathbb{Z}/m)| \).

**Proof.** Part (ii) is already contained in Theorem 4.1 but we state it again nevertheless since a new proof of the implication \((1 - [E])\tilde{K}^0(X) = (1 - [F])\tilde{K}^0(X) \Rightarrow O_E \cong O_F \) is given here under the assumptions from the statement of the theorem. Recall that we defined an equivalence relation on \( \tilde{K}^0(X) \) by \( a \sim b \) if and only if \( a = b + mh + bh \) for some \( h \in \tilde{K}^0(X) \). Let \( \gamma : \text{ Vect}_{m+1}(X) \to \tilde{K}^0(X) / \sim \) be the map which takes \( E \) to the equivalence class of \([E] = [E] - m - 1 \). We saw in Section 3 that \( 1 - [E] = (1 - [F])k \) for some \( k \in K^0(X) \) if and only if \([E] \sim [F] \) in \( \tilde{K}^0(X) \), i.e. \( \gamma(E) = \gamma(F) \). Let \( \omega : \text{ Vect}_{m+1}(X) \to O_{m+1}(X) \) the map which takes \( E \) to the isomorphism class of the \( C(X) \)-algebra \( O_E \). We shall construct a bijective map \( \chi \) such that the diagram

\[
\begin{array}{ccc}
\text{ Vect}_{m+1}(X) & \longrightarrow & O_{m+1}(X) \\
\gamma \downarrow & & \downarrow \chi \\
\tilde{K}^0(X) / \sim & \longrightarrow & \tilde{K}^0(X) / \sim
\end{array}
\]

is commutative. Let us note that in order to prove the parts (i), (ii) and (iii) of the theorem it suffices to verify the following four conditions.

(a) \( \gamma \) is surjective;
(b) If \( \omega(E) = \omega(F) \) then \( \gamma(E) = \gamma(F) \);
(c) \( |O_{m+1}(X)| \leq |\bar{H}^{even}(X, \mathbb{Z}/m)| \);
(d) \( |\bar{K}^0(X)/\sim| = |\bar{K}^0(X) \otimes \mathbb{Z}/m| = |\bar{H}^{even}(X, \mathbb{Z}/m)| \).

Indeed, from (a) and (b) we see that there is a well-defined surjective map \( \chi : \text{image}(\omega) \to \bar{K}^0(X)/\sim \), given by \( \chi(\omega(E)) = \gamma(E) \) and hence \( |\bar{K}^0(X)/\sim| \leq |\text{image}(\omega)| \leq |O_{m+1}(X)| \).

On the other hand from (c) and (d) we deduce that \( |O_{m+1}(X)| \leq |\bar{K}^0(X)/\sim| \). Altogether this implies that \( \omega \) is surjective and \( \chi \) is bijective.

It remains to verify the four conditions from above. If \( m \geq [(d - 3)/2] \), then the map \( \text{Vect}_{m+1}(X) \to \bar{K}^0(X), E \mapsto [E] - m - 1 \) is surjective by \[1\] Thm. 1.2 and this implies (a). Condition (b) follows from Theorem \[.1\] and condition (c) was proved in Proposition \[5.1\].

It remains to verify condition (d) using the assumption that \( \text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0 \). The first step is to use a known argument to deduce the absence of \( m \)-torsion in the \( K \)-theory of \( X \) and its skeleton filtration. We will then appeal to Lemma \[5.2\] to conclude the proof.

Let \( p \) be a prime which divides \( m \). Then \( \text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/p) = 0 \) by assumption. Let \( Z_{(p)} \) denote \( \mathbb{Z} \) localized at \( p \), i.e. the subring of \( \mathbb{Q} \) consisting of all fractions with denominator prime to \( p \). Let \( (E_r, d_r) \) be the Atiyah-Hirzebruch spectral sequence \( H^*(X, \mathbb{Z}) \Rightarrow K^*(X) \). Recall that \( E^s_2 = H^s(X, K^t(pt)) \) and \( E^s_\infty = K^{s+t}(X)/K^{s+t+1}(X) \), see \[1\]. Since \( Z_{(p)} \) is torsion free, it follows from the universal coefficient theorem that the spectral sequence \( (E_r \otimes Z_{(p)}, d_r \otimes 1) \) is convergent to \( K^*(X) \otimes Z_{(p)} \). On the other hand since all the differentials \( d_r \) are torsion operators by \[1, 2.4\] and since \( H^*(X, \mathbb{Z}) \) has no \( p \)-torsion, it follows that \( d_r \otimes 1 = 0 \) for all \( r \geq 2 \) and hence \( E_2 \otimes Z_{(p)} = E_\infty \otimes Z_{(p)} \). Therefore for all \( q \geq 0 \)

\[
H^{2q}(X, \mathbb{Z}) \otimes Z_{(p)} \cong (K^0_{2q}(X)/K^0_{2q+2}(X)) \otimes Z_{(p)}.
\]

Since \( \text{Tor}(G \otimes Z_{(p)}, \mathbb{Z}/p) \cong \text{Tor}(G, \mathbb{Z}/p) \) for all finitely generated abelian groups \( G \), it follows that \( \text{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X), \mathbb{Z}/p) = 0 \) for any prime \( p \) that divides \( m \). Therefore for all \( q \geq 0 \) we have

\[
\text{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X), \mathbb{Z}/m) = 0.
\]

This enables us to apply Lemma \[5.2\] for the ring \( R = \bar{K}^0(X) = K^0_2(X) \) filtered by the ideals \( R_q = \bar{K}^0_{2q}(X) \) to obtain that

\[
|\bar{K}^0(X)/\sim| = |\bar{K}^0(X) \otimes \mathbb{Z}/m| = \prod_{q \geq 1} |K^0_{2q}(X)/K^0_{2q+2}(X) \otimes \mathbb{Z}/m|.
\]

Since \( \text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0 \), we have \( H^{2q}(X, \mathbb{Z}/m) \cong H^{2q}(X, \mathbb{Z}) \otimes \mathbb{Z}/m \). From equation \[9\] we deduce that

\[
H^{2q}(X, \mathbb{Z}) \otimes \mathbb{Z}/m \cong (K^0_{2q}(X)/K^0_{2q+2}(X)) \otimes \mathbb{Z}/m.
\]

This completes the proof (d). \( \square \)
6. Proof of Theorem 1.4

**Lemma 6.1.** Let $R$ be a filtered commutative ring with $R_{n+1} = \{0\}$. Suppose that $\text{Tor}(R_k/R_{k+1}, \mathbb{Z}/m) = 0$ for all $k \geq 1$ and that $m$ and $n!$ are relatively prime. If $a, b \in R$ and $p_n(a) - p_n(b) \in m^nR$, then $a \sim b$.

**Proof.** We prove this by induction on $n$. Suppose first that $n = 1$. Then $p_1(a) - p_1(b) = a - b \in mR$ by assumption and so $a = b + mh$ for some $h \in R$. Since $R_2 = \{0\}$ by assumption, $bh = 0$ and so $a = b + mh + bh$, i.e. $a \sim b$. Suppose now that the statement is true for a given $n$ for all filtered rings $R$ as in the statement. Let $R$ be now a filtered ring such that $R_{n+2} = \{0\}$, $m$ and $(n+1)!$ are relatively prime and $\text{Tor}(R_k/R_{k+1}, \mathbb{Z}/m) = 0$ for all $k \geq 1$. Consider the ring $S := R/R_{n+1}$ with filtration $S_k = R_k/R_{n+1}$, $S_{n+1} = \{0\}$, and the quotient map $\pi : R \to S$. Let $a, b \in R$ satisfy $p_{n+1}(a) - p_{n+1}(b) \in m^{n+1}R$. Since $\text{Tor}(S, \mathbb{Z}/m) = 0$ and $(n+1)!$ and $m$ are relatively prime it follows that $p_n(\pi(a)) - p_n(\pi(b)) \in m^nS$. Since $S_{n+1} = \{0\}$ we obtain by the inductive hypothesis that $\pi(a) \sim \pi(b)$ in $S$ and hence $a = b + mh + bh + r_{n+1}$ for some $h \in R$ and $r_{n+1} \in R_{n+1}$. We have that $(b + mh + bh) \cdot r_{n+1} = 0$ and $r_{n+1}^2 = 0$ as these are elements of $R^{n+2} \subset R_{n+2} = \{0\}$. Therefore, by Lemma 3.1(i),

$$p_{n+1}(a) = p_{n+1}(b + mh + bh + r_{n+1}) = p_{n+1}(b + mh + bh) + p_{n+1}(r_{n+1})$$

$$= p_{n+1}(b + mh + bh) + \ell(n + 1)m^n r_{n+1}$$

On the other hand, $p_{n+1}(b + mh + bh) = p_{n+1}(b) + m^{n+1}v_{n+1}(h)$ by Lemma 3.1(ii), since $R_{n+2} = \{0\}$. Therefore

$$p_{n+1}(a) - p_{n+1}(b) = m^{n+1}v_{n+1}(h) + \ell(n + 1)m^n r_{n+1}.$$ 

Since $p_{n+1}(a) - p_{n+1}(b) \in m^{n+1}R$ by assumption, we obtain that $\ell(n + 1)m^n r_{n+1} \in m^{n+1}R$. Since $\text{Tor}(R, \mathbb{Z}/m) = 0$ we deduce that $\ell(n + 1)r_{n+1} \in mR$ and hence that $r_{n+1} = mh_{n+1}$ for some $h_{n+1} \in R$ since $m$ is relatively prime to $\ell(n + 1)$. We must have that in fact $h_{n+1} \in R_{n+1}$ since $\text{Tor}(R/R_{n+1}, \mathbb{Z}/m) = 0$ and hence $bh_{n+1} = 0$. It follows that $a \sim b$ since we can now rewrite $a = b + mh + bh + r_{n+1}$ as $a = b + m(h + h_{n+1}) + b(h + h_{n+1})$. \hfill $\square$

We are now in position to prove Theorem 1.4.

**Proof.** By Theorem 1.3.1 it suffices to show that $[\widetilde{E}] \sim [\widetilde{F}]$ whenever $p_{[d/2]}([\widetilde{E}]) - p_{[d/2]}([\widetilde{F}])$ is divisible by $m_{[d/2]}$. We have seen in the proof of Theorem 5.3 that if $\text{Tor}(H^*(X, \mathbb{Z}), \mathbb{Z}/m) = 0$ then $\text{Tor}(K^0_{2q}(X)/K^0_{2q+2}(X), \mathbb{Z}/m) = 0$. Therefore the desired implication follows from Lemma 6.1 applied to the ring $K^0(X)$ filtered as in (4) with $n = [d/2]$. \hfill $\square$
7. SUSPENSIONS

In this final part of the paper we study \( \mathcal{O}_{m+1}(SX) \) and the image of the map \( \text{Vect}(SX) \to \mathcal{O}_{m+1}(SX) \). We shall use the universal coefficient exact sequence

\[
0 \to K^1(X) \otimes \mathbb{Z}/m \overset{\beta}{\to} K^1(X, \mathbb{Z}/m) \overset{\beta}{\to} \text{Tor}(K^0(X), \mathbb{Z}/m) \to 0,
\]

where \( \beta \) is the Bockstein operation and \( \rho \) is induced by the coefficient map \( \rho \).

**Theorem 7.1.** Let \( X \) be a compact metrizable space and let \( m \geq 1 \).

(i) There is a bijection \( \gamma : \mathcal{O}_{m+1}(SX) \to K_1(C(X) \otimes \mathcal{O}_{m+1}) \cong K^1(X, \mathbb{Z}/m) \).

(ii) If \( E, F \in \text{Vect}_{m+1}(SX) \), then \( O_E \cong O_F \) as \( C(SX) \)-algebras if and only if \( [E] - [F] \in mK^0(SX) \).

(iii) If \( A \in \mathcal{O}_{m+1}(SX) \) and \( \beta(\gamma(A)) \neq 0 \), then \( A \) is not isomorphic to \( O_E \) for any \( E \in \text{Vect}_{m+1}(SX) \).

**Proof.** Part (i) is proved in [5]. We revisit the argument from [5] as it is needed for the proof of the other two parts. Let \( v_1, ..., v_{m+1} \) be the canonical generators of \( O_{m+1} \).

There is a natural map \( \gamma_0 : \text{Aut}(O_{m+1}) \to U(O_{m+1}) \) which maps an automorphism \( \varphi \) to the unitary \( \sum_{j=1}^{m+1} \varphi(v_j)v_j^* \). We showed in [5] Thm. 7.4 that \( \gamma_0 \) induces a bijection of homotopy classes \( [X, \text{Aut}(O_{m+1})] \to [X, U(O_{m+1})] \). By [17] there is a *-isomorphism \( \nu : O_{m+1} \to M_{m+1}(O_{m+1}) \). We have bijections \( \mathcal{O}_{m+1}(SX) \cong [SX, \text{BAut}(O_{m+1})] \cong [X, \text{Aut}(O_{m+1})] \) and

\[
[X, \text{Aut}(O_{m+1})] \overset{\gamma_0}{\longrightarrow} [X, U(O_{m+1})] \overset{\nu}{\longrightarrow} [X, U(M_{m+1}(O_{m+1}))] \cong K_1(C(X) \otimes \mathcal{O}_{m+1}) .
\]

The composition of these maps defines the bijection \( \gamma \) from (i). We are now prepared to prove (ii) and (iii). Consider the monomorphism of groups \( \alpha : U(m + 1) \to \text{Aut}(O_{m+1}) \) introduced in [8]. If \( u \in U(m + 1) \) has components \( u_{ij} \), then \( \alpha_u(v_j) = \sum_{i=1}^{m+1} u_{ij}v_i \). The map \( \alpha \) induces a map \( BU(m + 1) \to B\text{Aut}(O_{m+1}) \) which in its turn induces the natural map \( \alpha_* : \text{Vect}_{m+1}(Y) \to \mathcal{O}_{m+1}(Y) \) that we are studying. Let \( \eta \) be the composition of the maps from the diagram

\[
U(m + 1) \xrightarrow{\alpha} \text{Aut}(O_{m+1}) \xrightarrow{\gamma_0} U(O_{m+1}) \xrightarrow{\nu} U(M_{m+1}(O_{m+1})).
\]

An easy calculation shows that \( \eta(u) = \sum_{i,j=1}^{m+1} u_{ij} \nu(v_i v_j^*) \), where \( u_{ij} \) are the components of the unitary \( u \). Let us observe that \( \eta \) is induced by a unital *-homomorphism \( \tilde{\eta} : M_{m+1}^1(\mathbb{C}) \to M_{m+1}(O_{m+1}) \). It follows that there is a unitary \( w \in M_{m+1}(O_{m+1}) \) such that \( w\tilde{\eta}(a)w^* = a \otimes 1_{O_{m+1}} \), for all \( a \in M_{m+1}(\mathbb{C}) \). This implies that \( \eta \) will induce the coefficient map \( \rho : K^1(X) \to K^1(X, \mathbb{Z}/m) \).
We a commutative diagram

\[
\begin{align*}
\text{Vect}_{m+1}(SX) & \longrightarrow [SX, BU(m+1)] \longrightarrow [X, U(m+1)] \longrightarrow [X, U(M_{m+1} \mathbb{C})] \\
\alpha_* & \downarrow \quad \alpha_* \downarrow \quad \eta_* \\
\mathcal{O}_{m+1}(SX) & \longrightarrow [SX, B\text{Aut}(O_{m+1})] \longrightarrow [X, \text{Aut}(O_{m+1})] \longrightarrow [X, U(M_{m+1}(O_{m+1}))]
\end{align*}
\]

and hence a commutative diagram

\[
\begin{align*}
\text{Vect}_{m+1}(SX) & \xrightarrow{\alpha_*} \mathcal{O}_{m+1}(SX) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
K^1(X) & \xrightarrow{x_m} K^1(X) \xrightarrow{\rho} K^1(X, \mathbb{Z}/m) \xrightarrow{\beta} \text{Tor}(K^0(X), \mathbb{Z}/m) \rightarrow 0
\end{align*}
\]

Now both (ii) and (iii) follow from the commutativity of the diagram above and the exactness of its bottom row. \qed

\begin{thebibliography}{15}
\bibitem{atiyah1} M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In \textit{Proc. Sympos. Pure Math., Vol. III}, pages 7–38. American Mathematical Society, Providence, R.I., 1961.
\bibitem{bauval} A. Bauval. \textit{RKK(X)-nuklearité (d’après G. Skandalis).} \textit{K-Theory}, 13(1):23–40, 1998.
\bibitem{blanchard} É. Blanchard and E. Kirchberg. Non-simple purely infinite C*-algebras: the Hausdorff case. \textit{J. Funct. Anal.}, 207:461–513, 2004.
\bibitem{dadarlat} M. Dadarlat. The homotopy groups of the automorphism group of Kirchberg algebras. \textit{J. Noncommut. Geom.}, 1(1):113–139, 2007.
\bibitem{dadarlat2} M. Dadarlat. Continuous fields of C*-algebras over finite dimensional spaces. \textit{Adv. Math.}, 222(5):1850–1881, 2009.
\bibitem{dadarlat3} M. Dadarlat and G. A. Elliott. One-parameter continuous fields of Kirchberg algebras. \textit{Comm. Math. Phys.}, 274(3):795–819, 2007.
\bibitem{doplicher} Doplicher, Sergio; Pinzari, Claudia; Zuccante, Rita \textit{The C*-algebra of a Hilbert bimodule}. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 no. 2, 263–281, 1998.
\bibitem{enomoto} M. Enomoto, H. Takehana, and Y. Watatani, \textit{Automorphisms on Cuntz algebras}. Math. Japon. 24 , no. 2, 231–234, 1979.
\bibitem{hatcher} A. Hatcher. \textit{Algebraic topology}. Cambridge University Press, Cambridge, 2002.
\bibitem{husemoller} D. Husemoller. \textit{Fibre Bundles}. Number 20 in Graduate Texts in Mathematics. Springer Verlag, New York, 3rd. edition, 1966, 1994.
\bibitem{kervaire} M. Karoubi. \textit{K-theory}. Number 226 in Grundlehren der mathematischen Wissenschaften. Springer Verlag, 1978.
\bibitem{ker} M. A. Kervaire. Some nonstable homotopy groups of Lie groups. \textit{Illinois J. Math.}, 4:161–169, 1960.
\bibitem{kirchberg} E. Kirchberg. Das nicht-kommutative Michael-auswahlprinzip und die klassifikation nicht-einfacher algebren. In \textit{C*-algebras}, pages 92–141, Berlin, 2000. Springer. (Münster, 1999).
\bibitem{meyer} R. Meyer and R. Nest. The Baum-Connes conjecture via localisation of categories. \textit{Topology}, 45(2):209–259.
\bibitem{milnor} J. W. Milnor and J. D. Stasheff. Characteristic classes. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.
\end{thebibliography}
[16] M. Pimsner. A class of $C^*$-algebras generalizing both Cuntz–Krieger algebras and crossed products by Z. In D. Voiculescu, editor, Fields Inst. Commun., volume 12, pages 189–212, 1997.

[17] M. Rørdam. Classification of nuclear, simple $C^*$-algebras, volume 126 of Encyclopaedia Math. Sci. Springer, Berlin, 2002.

[18] C. Schochet. Topological methods for $C^*$-algebras. III. Axiomatic homology. Pacific J. Math., 114(2):399–445, 1984.

[19] E. Vasselli. The $C^*$-algebra of a vector bundle of fields of Cuntz algebras. J. Funct. Anal., 222(2):491–502, 2005.

[20] E. Vasselli. Continuous fields of C*-algebras arising from extensions of tensor C*-categories J. Funct. Anal. 199(1), 122–152, April 2003.

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