Different algebras for one reality

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The most familiar formalism for the description of geometry applicable to physics comprises operations among 4-component vectors and complex real numbers; few people realize that this formalism has indeed 32 degrees of freedom and can thus be called 32-dimensional. We will revise this formalism and we will briefly show that it is best accommodated in the Clifford or geometric algebra $G_{1,3} \times \mathbb{C}$, the algebra of 4-dimensional spacetime over the complex field.

We will then explore other algebras isomorphic to that one, namely $G_{2,3}$, $G_{4,1}$ and $\mathbb{Q} \times \mathbb{Q} \times \mathbb{C}$, all of which have been used in the past by PIRT participants to formulate their respective approaches to physics. $G_{2,3}$ is the algebra of 3-space with two time dimensions, which John Carroll used implicitly in his formulation of electromagnetism in $3 + 3$ spacetime, $G_{4,1}$ was and it still is used by myself in a tentative to unify the formulation of physics and $\mathbb{Q} \times \mathbb{Q} \times \mathbb{C}$ is the choice of Peter Rowlands for his nilpotent formulation of quantum mechanics. We will show how the equations can be converted among isomorphic algebras and we also examine how the monogenic functions that I use are equivalent in many ways to Peter Rowlands nilpotent entities.

PACS numbers: 04.50.-h; 02.40.-k.

1 Introduction

We call Physics to a discipline that creates mathematical models of physical reality. In practice, we write mathematical equations whose solutions allow us to predict the outcome of experiments and observations. One physical model is just as good as the predictions it allows and the most successful models become known as physical theories.
Every model makes use of a limited set of independent variables, which can be operated among themselves; we say that the model uses an underlying algebra. The model must also give physical meaning to the independent variables and algebraic operations performed among them, so that everybody can then translate into reality the results of operations performed within the model.

In view of what was said above, one sees that an algebra is an intrinsical component of any physical model, but it happens quite often that several algebras are only apparently different and can be shown to be isomorphic to each other. When this happens, models incorporating such algebras are frequently equivalent, although the insight one has over problems addressed with two equivalent models may be entirely different. In the following sections we will discuss the algebras associated with models proposed by various authors, showing that they are in many cases isomorphic. We will also show how to convert equations between isomorphic algebras. In the case of a model proposed by John Carroll [1], considering a space with 3 spatial and 3 temporal dimensions, the associated algebra is a superalgebra of several 5-dimensional algebras, so, the isomorphisms that can be found apply only to a subalgebra of the one proposed by the author.

2 Algebras most frequently used in physics

Both Newtonian mechanics and Maxwell’s equations are models based on 4 independent variables, 3 space coordinates and 1 scalar time variable. The algebras used to operate with these variables are the algebras of real and complex numbers complemented with vector algebra, but it is easy to see that this system lacks consistency. For instance, two vectors \( \mathbf{a} \) and \( \mathbf{b} \) determine a parallelogram with area given by \( |\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{a}| \). We make use of an operation among two vectors and then define the area as a scalar quantity. It makes more sense to define a new product whose outcome is an oriented area, called outer product and denoted \( \mathbf{a} \wedge \mathbf{b} \). The outcome of the outer product is precisely the area of the parallelogram defined by the two vectors, with a sign defined by the direction of movement from one vector to the other.

Clifford algebras are based on the geometric product or simply the product of vectors, incorporating both the inner and outer products. For any two vectors it is

\[
\mathbf{a}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.
\]  

(2.1)

The geometric product is associative and so it is possible to have products of 3 vectors, leading to a grade-3 element of the type \( \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \) which, if not zero, represents an oriented volume. We can thus say that the algebra associated with the spatial part of Newtonian mechanics and Maxwell’s equations is Clifford algebra of dimension 3, also known as geometric algebra of dimension 3 and denoted \( \mathcal{G}_3 \) or \( \mathcal{G}_{3,0} \). The volume elements of this algebra, as well as the highest grade elements of any Clifford algebra, are called pseudoscalars. For an extensive treatment of geometric algebras see [2,3].
Further problems with the Newtonian and Maxwellian models reside in the fact that time is treated as scalar but it has to be differentiated from the scalar coefficients of vectors. This is solved in special relativity, because it proposes that time is to be treated as a dimension of spacetime, thus increasing the dimensionality of the associated geometric algebra to 4; the highest grade element is now a 4-dimensional hypervolume. There are two possible algebras, $\mathcal{G}_{1,3}$ and $\mathcal{G}_{3,1}$, associated with one positive and 3 negative norm frame vectors or one negative and 3 positive norm frame vectors, respectively. The former is the most common choice and it allows the formulation of most physics equations, including quantum mechanics [4]. In order to fully accommodate quantum mechanics one must, however, allow for complex coefficients, a possibility not considered in [4].

Starting with the work by Theodor Kaluza, who proposed a 5-dimensional unification of electromagnetism with general relativity [5], some authors have used higher dimensional spaces to try and unify the equations of physics. My own work makes use of 5-dimensional spacetime and bears a strong relation to Kaluza's [6, 7]. The geometric algebra associated with 5-dimensional spacetime in this formulation is $\mathcal{G}_{4,1}$ but other authors have used the opposite signature $\mathcal{G}_{1,4}$ [8]. How different and how similar are all these approaches?

In order to answer the question we start by examining the overall dimensionality of the different algebras, starting with the algebra of physical space, $\mathcal{G}_{3,0}$. We realize that the elements of the algebra can be classified into 4 grades: scalars, vectors, areas and volumes, or better, grades 0, 1, 2 and 3. While both scalars and volumes have no associated orientation besides positive and negative, vectors and areas have 3 possible orientations, so, the total number of degrees of freedom is 8 and we say that total dimensionality is 8. In a similar way, the total dimensionality of a general geometric algebra, $\mathcal{G}_{p,q}$ is $2^{p+q}$, if only real coefficients are allowed for all grades. If complex coefficients are allowed the total dimensionality is either doubled or remains unaltered relative to the real coefficient version. Some algebras can be classified as complex algebras, because their pseudoscalar elements have negative square and commute with all other elements. In complex algebras the unit pseudoscalar doubles as the complex imaginary, so, introducing complex coefficients does not bring in any extra dimensions. In non-complex algebras the introduction of complex coefficients doubles the degrees of freedom, doubling the total dimensionality.

All geometric algebras are isomorphic to one particular matrix algebra, over one particular field that provides the coefficients. What this means is that all operations performed in a particular geometric algebra have equivalent operations in the isomorphic matrix algebra. The use of matrix algebra isomorphism is useful for classification purposes, but it is usually not recommended for performing operations since all the links with geometry are lost. Table [1] shows the matrix algebras isomorphic to the lowest order geometric algebras. The entries in the table are of the type $\mathbb{F}(n)$, which stands for algebra of $n$-dimensional matrices with coefficients in the field $\mathbb{F}$. The coefficients’ field can be real.
Table 1: Matrix representation of Clifford Algebras $\mathcal{C}\ell(p,q)$, with $p$ positive and $q$ negative norm frame vectors. The notation $\mathbb{F}(n)$ is used for the $n$-dimensional matrix algebra over the field $\mathbb{F}$ and $2\mathbb{F}(n)$ identifies the sum $\mathbb{F}(n) \oplus \mathbb{F}(n)$; $\mathbb{R}$ stands for real numbers, $\mathbb{C}$ for complex numbers and $\mathbb{Q}$ for quaternions.

| $p$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| 0   | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{Q}$ | $2\mathbb{Q}$ | $\mathbb{Q}(2)$ | $\mathbb{C}(4)$ | $\mathbb{R}(8)$ | $2\mathbb{R}(8)$ |
| 1   | $2\mathbb{R}$ | $\mathbb{R}(2)$ | $\mathbb{C}(2)$ | $\mathbb{Q}(2)$ | $2\mathbb{Q}(2)$ | $\mathbb{Q}(4)$ | $\mathbb{C}(8)$ | $\mathbb{R}(16)$ |
| 2   | $\mathbb{R}(2)$ | $2\mathbb{R}(2)$ | $\mathbb{R}(4)$ | $\mathbb{C}(4)$ | $\mathbb{Q}(4)$ | $2\mathbb{Q}(4)$ | $\mathbb{Q}(8)$ | $2\mathbb{Q}(8)$ |
| 3   | $\mathbb{C}(2)$ | $\mathbb{R}(4)$ | $2\mathbb{R}(4)$ | $\mathbb{R}(8)$ | $\mathbb{C}(8)$ | $\mathbb{Q}(8)$ | $2\mathbb{Q}(8)$ |
| 4   | $\mathbb{Q}(2)$ | $\mathbb{C}(4)$ | $\mathbb{R}(8)$ | $2\mathbb{R}(8)$ | $\mathbb{R}(16)$ | $\mathbb{C}(16)$ | $\mathbb{Q}(16)$ |
| 5   | $2\mathbb{Q}(2)$ | $\mathbb{Q}(4)$ | $\mathbb{C}(8)$ | $\mathbb{R}(16)$ | $2\mathbb{R}(16)$ | $\mathbb{R}(32)$ | $\mathbb{C}(32)$ | $\mathbb{Q}(32)$ |
| 6   | $\mathbb{Q}(4)$ | $2\mathbb{Q}(4)$ | $\mathbb{Q}(8)$ | $\mathbb{C}(16)$ | $\mathbb{R}(32)$ | $2\mathbb{R}(32)$ | $\mathbb{R}(64)$ | $\mathbb{C}(64)$ |
| 7   | $\mathbb{C}(8)$ | $\mathbb{Q}(8)$ | $2\mathbb{Q}(8)$ | $\mathbb{Q}(16)$ | $\mathbb{C}(32)$ | $\mathbb{R}(64)$ | $2\mathbb{R}(64)$ | $\mathbb{R}(128)$ |

Numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$) or quaternions ($\mathbb{Q}$). A few algebras are non-simple and are denoted $2\mathbb{F}(n)$; this means that two copies of the $\mathbb{F}(n)$ algebra are needed in the isomorphism. Looking up the table for the matrix representation of physical space algebra, $\mathcal{G}_{3,0}$, we see that we must use 2-dimensional matrices with complex coefficients. Usually we associate the frame vectors $\{\sigma_m\}$ to the Pauli matrices, as follows:

$$
\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Under the matrix isomorphism scalars are represented by the product of a real number by the identity matrix, vectors by linear combinations of matrices $\sigma_m$, areas by linear combinations of two Pauli matrix products and volumes by the product of a real number by $\sigma_1\sigma_2\sigma_3$, the notation $\hat{\sigma}_m$ being used for matrices. Since the product of the three Pauli matrices is the identity matrix multiplied by the complex imaginary, we see that the unit pseudoscalar of the algebra actually doubles as imaginary.

Minkowski spacetime is most frequently associated with $\mathcal{G}_{1,3}$ algebra, although several authors prefer the $\mathcal{G}_{3,1}$ alternative. No physical significance is attributed to the choice of signature, but one sees from Table 1 that the corresponding algebras are not isomorphic; there is probably some deep meaning in this choice that has escaped physicists so far. For the matrix representation of $\mathcal{G}_{3,1}$, the most direct route starts with Majorana gamma matrices, which have only imaginary elements, proceeding to assign the four
frame vectors from the algebra by the equation

\[ \sigma_\mu \equiv i \hat{\gamma}_\mu; \quad (2.3) \]

the notation \( \hat{\gamma}_\mu \) is used here for matrices. For the \( G_{1,3} \) algebra we should, in principle, select Pauli matrices \( \hat{\sigma}_1 \) and \( \hat{\sigma}_3 \) over the quaternion field. There is a workaround that avoids the discomfort of quaternions, which consists on allowing for 4-dimensional matrices with complex elements and restricting the matrix coefficients to real numbers. There several possible alternatives for the assignment of basis vectors to matrices, the most common being derived from Dirac-Pauli representation; this is

\[ \gamma_0 \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_m \equiv \begin{pmatrix} 0 & \hat{\sigma}_m \\ -\hat{\sigma}_m & 0 \end{pmatrix}. \quad (2.4) \]

These matrices have both real and imaginary elements, but used with real coefficients they still provide a basis representation for \( G_{1,3} \), avoiding the use of quaternions.

In 5-dimensional spacetime the representation is much easier with \( G_{4,1} \) than with \( G_{1,4} \), because the latter not only needs quaternions but it is also a non-simple algebra; we will not pay much attention to this case. With \( G_{4,1} \) we have a beautiful scenario; we can use 4-dimensional matrices with complex elements and complex coefficients. Among the various possible assignments we propose the following one, which is derived from the Dirac-Pauli representation, as we shall see below:

\[
\begin{align*}
    e_0 & \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, &
    e_1 & \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, &
    e_2 & \equiv \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \\
    e_3 & \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, &
    e_4 & \equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

We have not covered in this section the matrix representations for Carroll's \( G_{3,3} \) or Rowlands' \( Q \times Q \times \mathbb{C} \), although the former can be looked up in the table. We will consider these algebras in the next section.

### 3 Converting equations among algebras

We have seen in the previous section that there are isomorphisms between the algebras of different spaces, which means that it is feasible to translate all equations from one algebra to any of its isomorphic algebras. Although the equations can be translated, the geometric connection varies substantially and so does the insight one has over the
equations. As an example, take the Dirac equation, which appears formulated as a matrix equation in every textbook. The standard formulation does not allow any geometrical interpretation, because matrices have no connection to geometry whatsoever. The fact that Dirac equation can be translated into geometric algebra provides the necessary link to geometry and the solutions can be interpreted geometrically.

If all we are interested in is the formulation of general relativity, 4-dimensional spacetime is the adequate choice, which has a total dimensionality of 16. Physics equations, however, involve the use of complex numbers, at least for quantum mechanics. The total dimensionality implied by the set of physics equations for general relativity and quantum mechanics is then 32 and our task is then to translate equations among algebras with this total dimensionality. We start with Dirac-Pauli matrices, as defined in Eq. (2.4), and we follow the usual procedure for the definition of matrix $\hat{\gamma}_5$:

$$\hat{\gamma}_5 = i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3.$$  (3.1)

The translation between Dirac algebra and the algebra of 5-dimensional spacetime, $\mathcal{G}_{4,1}$, is made directly by the following relations

$$e_\mu \equiv \hat{\gamma}_\mu \hat{\gamma}_5, \quad e_4 \equiv -\hat{\gamma}_5.$$  (3.2)

This equation can be interpreted both as a matrix or a geometric algebra equation. Indeed, if the $\gamma_\mu$ represent the frame vectors of Minkowski spacetime, the equation can be read as a geometric algebra equation and allows the transposition from Minkowski into 5-dimensional spacetime. The inverse transposition follows the rules:

$$\hat{\gamma}_\mu \equiv e_4 e_\mu, \quad \hat{\gamma}_5 \equiv -e_4.$$  (3.3)

We turn our attention now to Rowlands’ algebra, whose elements are sets of two quaternions and one complex number. For convenience we shall represent a general element of this algebra with the notation $qqc$; boldface and sanserif characters represent two independent quaternions and a normal character represents a complex number. The elements in the set can be commuted, so, the total dimensionality of the algebra is $4 \times 4 \times 2 = 32$, just as Dirac’s algebra. The basis for Rowlands’ algebra is given by the sets

$$\{1, i, j, k\},$$
$$\{1, i, j, k\},$$
$$\{i\}.$$

The first quaternion basis verifies the relations

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k;$$  (3.4)
similar relations hold for the other quaternion. In order to set up the conversion relations for \( G_{4,1} \) we start by defining 3 anticommuting elements that can be associated with the 3 physical space dimensions; for this we set
\[
e_1 \equiv i, \quad e_2 \equiv ji, \quad e_1 \equiv ki.
\] (3.5)
We note that the unit volume is now
\[
e_1 e_2 e_3 = ijk = -i.
\] (3.6)
Now we need to find an element that anticommutes with the former ones, with negative square, for \( e_0 \), and a second one, squaring to unity, for \( e_4 \). A possible choice is
\[
e_0 = j, \quad e_4 = ik.
\] (3.7)
We need to check that the unit pseudoscalar coincides with the complex imaginary, so, we do
\[
e_0 e_1 e_2 e_3 e_4 = ijik = i.
\] (3.8)
The inverse relations are very easy to establish. With the help of the above conversion relations it becomes a feasible task to convert all equations between Dirac’s, Rowlands’ and my own notations but, if physics is the same in all notations, the insight and comprehension one has over the problems at hand can gain a lot from different approaches.

The best equation to test the conversion relations is arguably the Dirac equation; this is written, in terms of matrices, as
\[
\gamma^\mu \partial_\mu \psi + im\psi = 0.
\] (3.9)
Upper indices are used here and elsewhere to denote a change of sign, with respect to the corresponding lower indices, for those elements that square to \(-1\) (\(-I\) in the matrix case). Multiplying on the left by \( \gamma^5 \) and using conversion relations from Eq. (3.3), the Dirac equation becomes
\[
e^\alpha \partial_\alpha \psi + im\psi = 0.
\] (3.10)
We now establish that \( im\psi = \partial_4 \psi \), that is, we establish that the wavefunction dependence on \( x^4 \) is harmonic and is governed by the particle’s mass. This is very similar to a compactification of coordinate \( x^4 \). The Dirac equation acquires a new form:
\[
e^\alpha \partial_\alpha \psi = \nabla \psi = 0.
\] (3.11)
The index \( \alpha \) runs from 0 to 4 and the symbol \( \nabla \) represents what is known as the vector derivative of the algebra. Any function \( \psi \) that is a solution of Eq. (3.11) is called monogenic. There are plane wave solutions for this equation, with the general form
\[
\psi = \psi_0 e^{i(p_\alpha x^\alpha + \theta)}.
\] (3.12)
The monogenic equation implies that $e^\alpha p_\alpha \psi_0 = 0$, which can only be true if $(e^\alpha p_\alpha)^2 = 0$ and $\psi_0$ includes a factor $e^\alpha p_\alpha$. We say that the vector $p = e^\alpha p_\alpha$ is a nilpotent vector. In the above cited works, Rowlands uses the nilpotency condition as first principle, but we see here how this can be derived from the monogenic condition. If one establishes monogeneity as first principle, then the nilpotency condition is implied.

In its matrix version, Dirac’s equation accepts column matrix solutions, which are called Dirac spinors. In order to find the geometric equivalent of these we define 4 orthogonal idempotent elements by the relations

\[
\begin{align*}
f_1 &= \frac{1}{4} (1 + e_3)(1 + ie_1e_2), \\
f_2 &= \frac{1}{4} (1 - e_3)(1 + ie_1e_2), \\
f_3 &= \frac{1}{4} (1 - e_3)(1 - ie_1e_2), \\
f_4 &= \frac{1}{4} (1 + e_3)(1 - ie_1e_2).
\end{align*}
\]

These elements are called idempotents because their powers are always equal to the element itself. They are orthogonal because the product of any two different idempotents returns zero; They also add to unity. We can then split the original monogenic function into four components as in

\[
\psi = \sum_{i=1}^{4} \psi_i f_i = \sum_{i=1}^{4} \psi_i.
\]

Each of the terms $\psi_i$ is still a monogenic function and it is the geometric version of a Dirac spinor. Rowlands’ nilpotents have 4 components and they are also another form of spinors.

Now, the case of Carroll’s $G_{3,3}$ algebra does not readily fall into the algebras we have discussed above because, being 6-dimensional, it has a total dimensionality of 64, doubling the dimensionality of those algebras. However, Carroll argues that there is one special time dimension, which corresponds to ordinary time, and two orthogonal time dimensions, which must be treated differently. Carroll’s proposed wavefunction is the solution of the second order equation

\[
- (\partial_{s1}^2 + \partial_{s2}^2 + \partial_{s3}^2)\psi + m^2 \psi + \partial_{t3}\psi = 0; \quad (3.15)
\]

where

\[
m^2 \psi = (\partial_{t1}^2 + \partial_{t2}^2)\psi. \quad (3.16)
\]

For the purpose of this equation we can define a combined time coordinate, using $t1$ and $t2$, by

\[
tc = \frac{1}{2} (t1 + t2). \quad (3.17)
\]
Equation (3.16) is then a $G_{2,3}$ algebra equation and we see from Table 1 that this algebra is isomorphic to $G_{4,1}$. In order to convert between the two algebras we define the vectors for $G_{2,3}$ by

$$
e_{sm} = i e_m,$$
$$
e_{t3} = i e_0,$$
$$
e_{tc} = e_4.$$  \hspace{1cm} (3.18)

With this conversion it is easy to verify that Eq. (3.16) is indeed a second order version of Eq. (3.11). We don’t discuss here other implications of Carroll’s 6-dimensional approach, the purpose of this discussion being only to show that there is an implied 5-dimensional algebra isomorphic to the other ones presented above.

4 Conclusion

Many authors resort to different algebras for the exposition of their own approaches to fundamental physical equations, such as Maxwell’s equations, Dirac’s equation and Einstein’s equations. Quite frequently authors propose their own versions of those equations, highlighting the virtues of their approaches. The task of comparing results is difficult because the form of both equations and their solutions is dependent on the particular algebra that the author has chosen. We have shown that the algebra used by Dirac has an overall dimensionality of 32, the same as several 5-dimensional algebras proposed by different authors. The tensor algebra that most people use for general relativity is indeed a 16-dimensional sub-algebra of the Dirac algebra, so, it does not need to be addressed specifically.

Particular examples of algebras isomorphic to the Dirac algebra are those used by Rowlands [9, 10] and the author himself [6, 7]. We have shown how to convert between those two algebras and the Dirac algebra. A slightly different case occurs with the algebra used by Carroll [1], because this has an overall dimensionality of 64. Here we have shown that some of the proposed equations can be set in an algebra isomorphic to the previous ones and we presented the means for converting equations between Carroll’s algebra and the remaining ones.

The choice of a particular algebra is irrelevant from the point of view of the mathematical validity of equations, but it may make a significant difference to the perception and comprehension of the physics behind the equations. Quite often, no single choice of an algebra offers the definitive approach to an equation. Looking at a particular problem from different angles usually broadens our perspective over that problem, so, it makes sense to have equivalent equations written in varied algebras. However, we need to be able to convert among algebras in order to unify the various approaches.
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