Structural Parameterizations for Equitable Coloring

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Abstract. An \( n \)-vertex graph is equitably \( k \)-colorable if there is a proper coloring of its vertices such that each color is used either \( \lfloor n/k \rfloor \) or \( \lceil n/k \rceil \) times. While classic Vertex Coloring is fixed parameter tractable under well established parameters such as pathwidth and feedback vertex set, equitable coloring is \( \mathcal{W}[1] \)-hard. We prove that Equitable Coloring is fixed parameter tractable when parameterized by distance to cluster or co-cluster graphs, improving on the FPT algorithm of Fiala et al. (2011) parameterized by vertex cover. In terms of intractability, we adapt the proof of Fellows et al. (2011) to show that Equitable Coloring is \( \mathcal{W}[1] \)-hard when simultaneously parameterized by distance to disjoint paths and number of colors. We also revisit the literature and derive other results on the parameterized complexity of the problem through minor reductions or other simple observations.

Keywords: Equitable Coloring · Parameterized Complexity · Distance to Cluster · Distance to Co-cluster · Distance to Disjoint Paths

1 Introduction

Equitable Coloring is a variant of the classical Vertex Coloring problem, where we not only want to partition an \( n \) vertex graph into \( k \) independent sets, but also that each of these sets has either \( \lfloor n/k \rfloor \) or \( \lceil n/k \rceil \) vertices. The smallest integer \( k \) for which \( G \) admits an equitable \( k \)-coloring is called the equitable chromatic number of \( G \). Lih [17] presented an extensive survey covering many of the results on Equitable Coloring developed in the last 50 years. Its focus, however, is not algorithmic, and most of the presented results are bounds on the equitable chromatic number for various graph classes.

Many complexity results for Equitable Coloring arise from a related problem, known as Bounded Coloring, as observed by Bodlaender and Fomin [2]. On Bounded Coloring, we ask that the size of the independent sets be bounded by an integer \( \ell \), which is not necessarily a function of \( k \) or \( n \). Among the positive results for Bounded Coloring, we have that the problem is solvable in polynomial time for: split graphs [7], complements of interval graphs [3], complements of bipartite graphs [8], forests [11], and trees [11]. While the algorithm for Bounded Coloring on trees was first presented by Baker and
Coffman [1], Jarvis and Zhou [10] show how to compute the minimum number of colors required to color a tree using at most $\ell$ colors using a novel characterization. For cographs, bipartite and interval graphs, there are polynomial-time algorithms when the number of colors $k$ is fixed. In terms of parameterized complexity, in [2] an $\text{XP}$ algorithm is given for \textsc{Equitable Coloring} parameterized by treewidth, while Fiala et al. show that the problem is $\text{FPT}$ parameterized by vertex cover [12]. Recently, Gomes et al. [14] proved that, when parameterized by the treewidth of the complement graph, \textsc{Equitable Coloring} is in $\text{FPT}$, and Reddy [18] did the same when the parameter is the distance to threshold graph. We discuss negative results and other minor observations in detail in Section 2.

The main contributions of this work are complexity results on \textsc{Equitable Coloring} for parameterizations that are weaker than vertex cover, in the sense that the parameters are upper bounded by the vertex cover number. In particular, we show that \textsc{Equitable Coloring} is fixed parameter tractable when parameterized by distance to cluster and by distance to co-cluster. Not only are the parameters weaker, in the first case, the algorithm is slightly faster than the one previously known for vertex cover, as it does not rely on \textsc{Integer Linear Programming}; the running time, however, is still super exponential. On the negative side, we show that the combined parameterization distance to disjoint paths and number of colors is insufficient to guarantee tractability. Along with some of the works discussed here and in Section 2, our results cover many branches of the known graph parameter hierarchy.

\textbf{Notation and Terminology.} We use standard graph theory notation and nomenclature for our parameters, following classical textbooks in the areas [4,10,9]. Define $[k] = \{1, \ldots, k\}$ and $2^S$ the powerset of $S$. A $k$-\textit{coloring} $\varphi$ of a graph $G$ is a function $\varphi : V(G) \rightarrow [k]$. Alternatively, a $k$-coloring is a $k$-partition $V(G) \sim \{\varphi_1, \ldots, \varphi_k\}$ such that $\varphi_i = \{u \in V(G) \mid \varphi(u) = i\}$. A $k$-coloring is said to be \textit{equitable} if, for every $i \in [k]$, $|n/k| \leq |\varphi_i| \leq \lceil n/k \rceil$. A $k$-coloring of $G$ is \textit{proper} if $\varphi_i$ is an independent set for every $i \in [k]$. Unless stated, all colorings are proper. A $k$-coloring is \textit{equitable} if for all $\varphi_i$, it holds that we have $\lfloor n \rfloor \leq |\varphi_i| \leq \lceil n \rceil$ for every $i \in [k]$, where $[k] = \{1, \ldots, k\}$. The \textsc{Equitable Coloring} problem asks whether or not $G$ can be equitably $k$-colored. A graph is a \textit{cluster graph} if each of its connected components is a clique; a \textit{co-cluster graph} is the complement of a cluster graph. The \textit{distance to cluster} (resp. \textit{co-cluster}) of a graph $G$, denoted by $\text{dc}(G)$ (resp. $\text{dc}(G)$), is the size of the smallest set $U$ such that $G - U$ is a cluster (resp. co-cluster) graph. Using the terminology of [4], a set $U \subseteq V(G)$ is an $\mathcal{F}$-\textit{modulator} of $G$ if the graph $G - U$ belongs to the graph class $\mathcal{F}$. When the context is clear, we omit the qualifier $\mathcal{F}$. For cluster and co-cluster graphs, one can decide if $G$ admits a modulator of size $k$ in time \textsc{FPT} on $k$ [5].
2 Literature corollaries and minor observations

The original NP-complete results of Bodlaender and Jansen [3], despite being initially regarded as polynomial reductions for Bounded Coloring, are a nice source of parameterized hardness. With minimal effort, one can adapt their proofs to show that Equitable Coloring parameterized by the number of colors is $W[1]$-hard on cographs and paraNP-hard on bipartite graphs; these imply that adding the distance to these classes in the parameterization yields no additional power whatsoever. Fellows et al. [11] show that Equitable Coloring parameterized by treewidth and number of colors is $W[1]$-hard, while an XP algorithm parameterized by treewidth is given for both Equitable Coloring and Bounded Coloring by Bodlaender and Fomin [2]. In fact, the reduction shown in [11] proves that, even when simultaneously parameterized by feedback vertex set, treedepth, and number of colors, Equitable Coloring remains $W[1]$-hard. Gomes et al. [14] show that the problem parameterized by number of colors, maximum degree and treewidth is $W[1]$-hard on interval graphs. However, their intractability statement can be strengthened to number of colors and bandwidth, with no changes to the reduction. In [6], Cai proves that Vertex Coloring is $W[1]$-hard parameterized by distance to split.

We also reviewed results on parameters weaker than distance to clique, since both distance to cluster and co-cluster fall under this category. For minimum clique cover, we resort to the classic result of Garey and Johnson [13] that Partition into Triangles is NP-hard. By definition, a graph $G$ can be partitioned into vertex-disjoint triangles if and only if its complement graph can be equitably $(n/3)$-colored. The reduction given in [13] is from Exact Cover by 3-Sets, and their gadget (which we reproduce in Figure 1) has the nice property that the complement graph $\overline{G}$ has a trivial clique cover of size nine: it suffices to pick one gadget $i$ and one clique for each $a'_j$. Thus, we have that Equitable Coloring is paraNP-hard parameterized by minimum clique cover. To see that when also parameterizing by the number of colors there is an FPT algorithm, we first look at the parameterization maximum independent set $\alpha$ and number of colors $k$. First, if $\alpha < n/k$, the instance is trivially negative, so we may assume $k\alpha > n$; but, in this case, we can spend exponential time on the number of vertices and still run in FPT time. Finally, we reduce from Equitable Coloring parameterized by the number of colors $k$ to Equitable Coloring parameterized by $k$ and minimum dominating set. Note that if we take the source graph $G$ and add $\frac{n}{k}$ vertices $v_1, \ldots, v_{\frac{n}{k}}$ with $N(v_i) = V(G)$, the set $\{v_1, u\}$, with $u \in V(G)$, is a dominating set of the resulting graph $G'$; moreover, $G$ has an equitable $k$-coloring if and only if $G'$ is equitably $(k + 1)$-colorable, thus proving that Equitable Coloring parameterized by $k$ and minimum dominating set is paraNP-hard. A summary of the results discussed in this section and in the introduction is displayed in Figure 2.
Fig. 1. Exact Cover by 3-Sets to Partition into Triangles gadget of \cite{10} representing the set \( c_i = \{x_i, y_i, z_i\} \).

Fig. 2. Hasse diagram of the parameterizations of Equitable Coloring and their complexities. A single shaded box indicates that the problem is FPT; two solid boxes represent \( W[1]\)-hard even if also parameterized by the number of colors; if the inner box is dashed, the problem is paraNP-hard. If the outer box is solid and shaded, additionally using the number results in an FPT algorithm; if it is not shaded, the problem remains \( W[1]\)-hard. *’s mark our main technical contributions. Ellipses mark open cases.

3 Equitable coloring parameterized by distance to cluster

The goal of this section is to prove that Equitable Coloring can be solved in FPT time when parameterized by the distance to cluster of the input graph. As a corollary of this result, we show that unipolar graphs can be equitably \( k\)-colored in polynomial time. Throughout this section, we denote the modulator by \( U \), the connected components of \( G - U \) by \( C = \{C_1, \ldots, C_r\} \), and define \( \ell = \left\lfloor \frac{n}{k} \right\rfloor \).

The central idea of our algorithm is to guess one of the possible \( |U||E| \) colorings of the modulator and extend this guess to the clique vertices using max-flow. First, given \( U \), \( C \), and a coloring \( \varphi' \) of \( U \), we build an auxiliary graph \( H \) as follows: \( V(H) = \{s, t\} \cup A \cup W \cup V(G) \setminus U \), where \( A = \{a_1, \ldots, a_k\} \) represent the
colors we may assign to vertices, \( W = \{ w_{ij} \mid i \in [k], j \in [r] \} \) whose role is to maintain the property of the coloring, \( s \) is the source of the flow, \( t \) is the sink of the flow, and \( V(G) = \{ v_1, \ldots, v_n \} \) are the vertices of \( G \). For the arcs, we have \( E(H) = S \cup F_0 \cup F_1 \cup R \cup T \), where \( S = \{ (s, a_i) \mid i \in [k] \} \), \( F_0 = \{ (a_i, t) \mid i \in [k] \} \), \( F_1 = \{ (a_i, w_{ij}) \mid i \in [k], j \in [r] \} \), \( R = \{ (w_{ij}, v_p) \mid v_p \in C_j, N(v_p) \cap \varphi'_i = \emptyset \} \), and \( T = \{ (v_i, t) \mid v_i \in V(G) \setminus U \} \). As to the capacity of the arcs, we define \( c : E(H) \to \mathbb{N} \), with \( c(e \in S) = \ell \), \( c((a_i, t)) = |\varphi'_i \cap U| \) and \( c(e \notin S) = 1 \). Semantically, the vertices of \( A \) correspond to the \( k \) colors, while each \( w_{ij} \) ensures that cluster \( C_j \) has at most one vertex of color \( i \). Regarding the arcs, \( F_0 \) corresponds to the initial assignment of colors to the vertices of the modulator, and \( R \) encodes the adjacency between vertices of the clusters and colored vertices of the modulator. Note that the arcs \( F_0 \) and \( R \) are the only functions of the pre-coloring \( \varphi' \). An example of the constructed graph can be found in Figure 3.

![Figure 3](image_url)

**Fig. 3.** (left) The input graph with \( U = \{ v_1, v_2 \} \); (right) Auxiliary graph constructed from the precoloring of \( U \). Solid arcs have unit capacity.

Now, let \( f : E(H) \to \mathbb{N} \) be the function corresponding to the max-flow from \( s \) to \( t \) obtained using any of the algorithms available in the literature. Our first observation, as given by the following lemma, is that, if no \((s, t)\)-flow saturates the outbound arcs of \( s \), then \( G \) cannot be equitably \( k \)-colored.

**Lemma 1.** If there is some \( e \in S \) with \( f(e) < \ell \), then \( G \) does not admit an equitable \( k \)-coloring that extends \( \varphi' \).

**Proof.** By the counter-positive, suppose that \( G \) admits an equitable \( k \)-coloring \( \varphi \) extending \( \varphi' \). As such, there exists some coloring \( \psi \) satisfying \( \varphi' \subset \psi \subset \varphi \) such that for all \( i \in [k] \) we have \( |\psi_i| = \ell \). To construct function \( f \), begin by setting \( f(e) = \ell \), for each \( e \in S \). If \( e = (a_i, t) \), i.e., \( e \in F_0 \), \( f(e) = |\psi_i \cap U| \). For each \( e \in T \) such that \( e = (v, t) \) and \( v \) is colored by \( \psi \), set \( f(e) = 1 \). For each \( i \in [k], j \in [r] \), if \( \psi_i \cap C_j \neq \emptyset \) then we set \( f((a_i, w_{ij})) = 1 \) and \( f((w_{ij}, v)) = 1 \), where \( v \) is the only element of \( \psi_i \cap C_j \). Arc \((w_{ij}, v)\) exists since \( v \in V(G) \setminus U \) belongs to clique \( C_j \),
has no neighbor with color $i$ in $U$, and $\psi$ is a coloring of $G$. All other arcs have $f(e) = 0$. To see that $f$ corresponds to a feasible flow of $H$ under $c$, first note that $f(e) \leq c(e)$ for every $e \in E(H)$ follows from the construction of $f$. To conclude the proof, it suffices to check that $\sum_{j \in [r]} f((a_i, w_{ij})) = \ell - |\psi \cap U|$, which must be the case, since $f((a_i, w_{ij})) = 1$ if and only if there is some $v \in C_j \cap \psi_i$. □

We may now assume that $f(e) = c(e)$ for every $e \in S$. Now, let $c'(e \notin S) = c(e)$, and $c'(e \in S) = c(e) + 1$, and run max-flow again; this time, however, we begin with the flow given by $f$, and obtain another $(s,t)$-flow represented, by $g$.

**Lemma 2.** For every $e \in S$, $g(e) \geq f(e)$.

*Proof.* Suppose that $g(e) < f(e)$ for some $e \in S$, than at some step the max-flow algorithm must have picked an augmenting path with an arc $e = (a_i, s)$, but this lead to a contradiction as the augmented path would not be simple. □

**Lemma 3.** The maximum $(s,t)$-flow $F$ given by $g$ is equal to the number of vertices of $G$ if and only if there is an equitable $k$-coloring of $G$ that extends $\varphi'$.  

*Proof.* Suppose that the maximum $(s,t)$-flow is $|V(G)|$. Reading the flow function to retrieve the coloring $\varphi$ is straightforward: for each $(w_{ij}, v_p) \in R$ with $f((w_{ij}, v_p)) = 1$, color $v_p$ with $i$. Since there is only one arc $(v, t)$ with unit capacity, there is an unique $i$ such that $v \in \varphi_i$. By Lemma $2 \ell \leq |\varphi_i|$ and, since the capacity of each outbound arc of $s$ is $\ell + 1$, $|\varphi_i| \leq \ell + 1$. That $\varphi_i$ is a proper coloring follows from the hypothesis that $\varphi'$ was a proper coloring of $G[U]$, the arcs in $R$ encode the constraints that a vertex $v \in C_j$ cannot be colored with any of the colors $\{i \mid N(v) \cap \varphi'_i \neq \emptyset\}$, and that only one vertex $v \in C_j$ satisfies $f((w_{ij}, v)) = 1$ since $c'((a_i, w_{ij})) = 1$. Finally, $\varphi$ extends $\varphi'$ since the only vertices colored by $\varphi'$ are those in $U$, and their coloring has not been modified at any point.

For the converse, take an equitable $k$-coloring $\varphi$ that extends $\varphi'$ and define the flow function as in the proof of Lemma $1$ but with $g((s, a_i)) = |\varphi_i|$. The same arguments hold, concluding the proof. □

At this point we are essentially done. Lemmas $1$ and $3$ guarantee that, if the max-flow algorithm fails to yield a large enough flow, a fixed pre-coloring of $U$ cannot be extended; moreover, the latter also implies that, if an extension is possible, max-flow correctly finds it. Now, given $U$, for each of the $O(|U|^{\ell + 1})$ possible colorings of $U$, construct $H$ and execute the above algorithm. If we are not given the modulator $U$, the same can be computed in single-exponential FPT time.

**Theorem 1.** Equitable Coloring parameterized by distance to cluster can be solved in FPT time.

It is worthy to note here that there is nothing special about the capacities of the arcs in $S$; they act only as upper bounds to the number of vertices a color may be assigned to. Thus, not surprisingly, the same algorithm applies to problems
where the size of each color class is only upper bounded. This will be particularly useful in the next session. Looking at the proof of Theorem 1, the only non-polynomial step is guessing the coloring of the modulator. A straightforward corollary is that if there is a polynomial number of distinct colorings of $U$ and this family can be computed in polynomial time, we can apply the same ideas and check if an equitable $k$-coloring of the input graph exists in polynomial time. In particular, unipolar graphs (i.e where the modulator is itself a clique) satisfy the above condition. If we parameterize by distance to unipolar the problem remains $W[1]$-hard due to the hardness for split graphs. On the other hand, if we parameterize by distance to unipolar and the number of colors $k$ we have an FPT algorithm: it suffices to guess the coloring of the modulator and then note that the central clique of the resulting unipolar graph has at most $k$ vertices; as such, we can treat it as a graph with distance to cluster at most $k$ and apply our previous algorithm.

**Corollary 1.** *Equitable Coloring* can be solved in polynomial time for unipolar graphs. When parameterized by distance to unipolar, the problem remains $W[1]$-hard; if also parameterized by the number of colors, there is an FPT algorithm that solves it.

## 4 Equitable coloring parameterized by distance to co-cluster

Before proceeding to our kernelization results, we discuss an FPT algorithm when parameterized by distance to co-cluster. Interestingly, the key ingredient to our approach is the algorithm presented in Section 3, which we use to compute the transitions between states of our dynamic programming table. Much like in the previous section, we denote by $U$ the set of vertices such that $G - U$ is a co-cluster graph, and by $I = \{I_1, \ldots, I_r\}$, the independent sets of $G - U$.

The following observation follows immediately from the fact that $G - U$ is a complete $r$-partite graph; it allows us to color the sets of $I$ independently.

**Observation 1** In any $k$-coloring $\varphi$ of $G$, for every color $i$, there is at most one $j \in [r]$ such that $\varphi_i \cap I_j \neq \emptyset$.

Suppose we are already given $U$, $I$, a coloring $\psi$ of $U$, and the additional restriction that colors $P \subseteq \psi(U)$ must be used in $\ell + 1$ vertices. We index our dynamic programming table by $(S, p, q, j)$, where $S \subseteq \psi(U)$ stores which colors of the modulator still need to be extended, $p$ is the number of colors not in $P$ that must still be used $\ell + 1$ times, $q$ the colors (not in $P$) that must still be used in $\ell$ vertices, and $j \in [r]$ indicates which of the independent sets we are trying to color. Our goal is to show that $f_{\psi, P}(S, p, q, j) = 1$ if and only if there is a coloring of $G_j = G[U \cup \bigcup_{i=j}^r I_i]$ respecting the constraints given by $(S, p, q, j)$. We transition according to the following equation:

$$f_{\psi, P}(S, p, q, j) = \max_{(R, x, y) \in ext(S, p, q, j)} f_{\psi, P \setminus R}(S \setminus R, p - x, q - y, j + 1)$$

(1)
where \( \text{ext}(S, p, q, j) \) is the set of all triples \((R, x, y)\), with \( R \subseteq S \), such that each color \( i \in R \) can be extended to \( I_j \), while \( x \) and \( y \) satisfy the system:

\[
\begin{aligned}
& x(\ell + 1) + y\ell = |I_j| - \alpha_j \\
& 0 \leq x \leq p \\
& 0 \leq y \leq q
\end{aligned}
\]

where \( \alpha_j \) is the number of vertices of \( I_j \) used to extend the colors of \( R \) to \( I_j \).

Note that \(|\text{ext}(S, p, q, j)| \leq 2^{|S|}n \), so it holds that, for each fixed \( \psi \) and \( P \), our dynamic programming table can be computed in \( O^*(3^{|\psi|}) \) time if and only if we can compute \( \text{ext}(S, p, q, j) \) in \( O^*(|\text{ext}(S, p, q, j)|) \) time.

**Lemma 4.** \( \text{ext}(S, p, q, j) \) can be computed in \( O^*(|\text{ext}(S, p, q, j)|) \) time.

**Proof.** Given a triple \((R, x, y)\), with \( R \subseteq S \), satisfying the above conditions, it suffices to show that membership in \( \text{ext}(S, p, q, j) \) can be decided in polynomial time. Actually, the challenge is determining whether or not the colors in \( R \) can be extended to include \( I_j \). In order to do so, let \( U' = \{v \in U \mid \psi(v) \in R\} \), and \( G' \) be the subgraph of \( G \) induced by the vertices \( I_j \cup U' \). Observe that \( U' \) is actually a vertex cover for \( G' \); in particular, it is a modulator to cluster graph of size bounded by \( \text{dcl}(G) \). As such, we can interpret the task as follows: is there an induced subgraph of \( G' \) that be colored with \(|R|\) colors, that extends \( \psi \), the colors in \( P \cap R \) are used exactly \( \ell + 1 \) times and the others \( P \setminus R \) \( \ell \) times? This can be answered with a slight modification of the algorithm given in Section 3 after the initial max-flow is executed, i.e. if the converse of Lemma 4 holds, instead of updating the capacity of every arc leaving the source to \( \ell + 1 \) we only update those corresponding to the colors in \( P \), the same argumentation holds. \( \square \)

**Lemma 5.** \( f_{\psi, P}(S, p, q, j) = 1 \) if and only if \( \psi \) can be extended to a coloring \( \varphi \) of \( G_j \) using the colors of \( S \), with each color in \( P \) used in \( \ell + 1 \) vertices, \( p \) extra colors of size \( \ell + 1 \), and \( q \) colors of size \( \ell \).

**Proof.** For the forward direction, we proceed by induction. The base case, where \( j = r + 1 \) is equal to the modulator, we can define \( f_{\psi, P}(S, p, q, r + 1) = 1 \) if and only if \( P = S = \emptyset \), and \( p = q = 0 \). Now, take an arbitrary entry of the table with \( i < r + 1 \); by Equation 4, \( f_{\psi, P}(S, p, q, j) = 1 \) if and only if there is some triple \((R, x, y)\) of \( \text{ext}(S, p, q, j) \) with \( f_{\psi, P}^\prime(S \setminus R, x, p - x, y, j + 1) = 1 \). That is, there is a coloring of \( G_{j+1} \) under the constraints imposed by \( P \setminus R \) and tuple \((S \setminus R, p - x, q - y, j + 1)\). Moreover, since \( V(G_j) \setminus V(G_{j+1}) = I_j \), no color of \( R \) is used in any vertex of \( V(G_{j+1}) \setminus U \), and by the definition of \( \text{ext}(S, p, q, j) \), it is possible to extend the coloring of \( G_{j+1} \) to include the vertices of \( I_j \) while keeping it proper and respecting the constraints imposed by \( P \).

Conversely, take a coloring \( \varphi \) of \( G_j \) satisfying the hypothesis, define \( R = \varphi(U) \cap \varphi(I_j) \), and let \( \varphi' \) be the restriction of \( \varphi \) to \( G_{j+1} \). By Observation 4, every color in \( R \subseteq S \) is used only on vertices of \( U \cup I_j \), say \( \alpha_j \) vertices of \( I_j \), and there must be integers \( 0 \leq x \leq p, 0 \leq y \leq q \) satisfying \( \ell + 1 \leq x + \ell y + \alpha_j = |I_j| \); that is, there are \( x \) colors of size \( \ell + 1 \) and \( y \) colors of size \( \ell \) used exclusively on the
remaining vertices of $I_j$. By definition, the triple $(R, x, y)$ belongs to $\text{ext}(S, p, q, j)$ and if $f_{\psi, P \setminus R}(S \setminus R, p - x, q - y, j + 1) = 1$ we are done. By induction on the number of available colors, this assertion holds since $|R| + x + y \geq 1$, otherwise $I_j = \emptyset$. \hfill \square

Finally, all that is left is to shown that the number of colorings of $U$ and the constraint set $P$ can both be computed in FPT time.

**Theorem 2.** Equitable Coloring can be solved in FPT time when parameterized by distance to co-cluster.

**Proof.** We can be guess all possible colorings $\psi$ of $|U|$ in time $O^*(|U|^{2|U|})$ and for each color of $\psi$ we need to choose between adding it to $P$ or not. So we have $O^*(|U|^{2|U|})$ cases. By Lemmas 4 and 5 each case can be solved in FPT time parameterized by $dc$, so our algorithm is FPT parameterized by $dc$. \hfill \square

It is important to note that the above algorithm does not contradict the NP-hardness of Equitable Coloring on bipartite graphs, since solving the problem on complete bipartite graphs is in P. Moreover, if $U = \emptyset$, all steps of the algorithm are performed in polynomial time, yielding the following corollary.

**Corollary 2.** Equitable Coloring of complete multipartite graphs is solvable in polynomial time.

### 5 Distance to Disjoint Paths

The last parameterization we investigate for Equitable Coloring is distance to disjoint paths, which is upper bounded by vertex cover and lower bounded by feedback vertex set. Contrary to our expectations, we show that the problem is W[1]-hard even if we also parameterize by the number of colors. To accomplish this, we make use of two intermediate problems, namely Number List Coloring and Equitable List Coloring parameterized by the number of colors. The latter is very similar to Equitable Coloring but to each vertex $v$ is assigned a list $L(v) \subseteq [k]$ of admissible colors. Number List Coloring generalizes it in the sense that now we are given a function $h : [k] \mapsto \mathbb{N}$ and color $i$ must be used exactly $h(i)$ times.

As a first step, we show that Number List Coloring parameterized by the number of colors is W[1]-hard on paths. Our proof is a modification of the one given by [11]; we show how to transform their tree gadgets into path gadgets by roughly doubling the number of colors and vertices used in the construction. The source problem is Multicolored Clique parameterized by the solution size $k$. Given an instance $(H, k)$ of Multicolored Clique, we denote the $k$ color classes by $\{V_1, \ldots, V_k\}$, the edges between $V_i$ and $V_j$ by $E(i, j)$, $|V(H)|$ by $n$ and $|E(H)|$ by $m$. We may assume that $|V_i| = N$ and $|E(i, j)| = M$ for every $i, j$; to see why this is possible, we may take $k!$ disjoint copies of $H$, each corresponding to a permutation of the color classes.
**Construction.** The first ingredients to our reduction are the colors themselves. Due to the list nature of the problem, one can easily assign semantic values to each set of colors. In our case, we separate the colors to be used in four types:

- **Selection:** The colors $S = \{\sigma(i, j) \mid (i, j) \in [k]^2, i \neq j\}$ and $S' = \{\sigma'(i, j) \mid (i, j) \in [k]^2, i \neq j\}$ are used to select which edges must belong to the clique.
- **Helper:** $Y$ and $X$ satisfy $|Y| = |X| = |S|$. These two sets of colors force the choice made at the root of the edge gadgets to be consistent across the gadget.
- **Symmetry:** The colors $E = \{\varepsilon(i, j) \mid (i, j) \in [k]^2, i < j\}$ and $E' = \{\varepsilon'(i, j) \mid (i, j) \in [k]^2, i < j\}$ guarantee that, if edge $e \in E(i, j)$ is picked from $V_i$ to $V_j$, it must also be picked from $V_j$ to $V_i$.
- **Consistency:** Colors $T = \{\tau_i(r, s) \mid i \in [k], r, s \in [k] \setminus \{i\}, r < s\}$ and $T' = \{\tau'_i(r, s) \mid i \in [k], r, s \in [k] \setminus \{i\}, r < s\}$ ensure that if the edge $uv$ is chosen between $V_i$ and $V_j$, the edge between $V_i$ and $V_j$ must also be incident to $u$.

Before detailing the gadgets themselves, we define what is, in our perception, one of the most important pieces of the proof. For each vertex $v \in V(H)$, choose an arbitrary but unique integer in the range $[n^2 + 1, n^2 + n]$ and, for each edge $e$, a unique integer in the range $[2n^2 + 1, 2n^2 + m]$. These are the up-identification numbers of vertex $v$ and edge $e$, denoted by $v_1$ and $e_1$, respectively. Now, choose a suitably huge integer $Z$, say $n^3$, and define the down-identification number for $v$ as $v_1 = Z - v_1$. These quantities play a key role on the numerical targets for the symmetry and consistency colors, tying together the choices that must be made among the various edge gadgets.

For each pair $i, j \in [k]$, with $i < j$, the input graph $G$ of **Number List Coloring** has the groups of gadgets $G(i, j)$ and $G(j, i)$, each containing $M$ edge gadgets corresponding to the edges of $E(i, j)$. We say that $G(i, j)$ is the *forward group* and that $G(j, i)$ is the *backward group*. For the description of the gadgets, we always assume $i < j$, $e \in E(i, j)$ with $u \in V_i$ and $v \in V_j$.

**Forward Edge Gadget.** The gadget $G(i, j, e)$ has a root vertex $r(i, j, e)$, with list $\{\sigma(i, j), \sigma'(i, j)\}$, and two neighbors, both with the list $\{\sigma(i, j), y(i, j)\}$, which for convenience we call $a(i, j, e)$ and $b(i, j, e)$. We equate membership of edge $e$ in the solution to **MULTICOLORED CLIQUE** to the coloring of $r(i, j, e)$ with $\sigma(i, j)$.

When discussing the vertices of the remaining vertices of the gadget, we say that a vertex is even if its distance to $r(i, j, e)$ is even, otherwise it is odd.

To $a(i, j, e)$, we append a path with $2e_1 + 2(k - 1)u_1$ vertices. First, we choose $e_1$ even vertices to assign the list $\{y(i, j), \varepsilon'(i, j)\}$. Next, for each $r$ in $j < r \leq k$, choose $u_1$ even vertices to assign the list $\{y(i, j), \tau_r'(j, r)\}$. Similarly, for each $s \neq i$ satisfying $s < j$, choose $u_1$ even vertices and assign the list $\{y(i, j), \tau_r(s, j)\}$. All the odd vertices - except $a(i, j, e)$ and $b(i, j, e)$ - are assigned the list $\{y(i, j), x(i, j)\}$. The path appended to $b(i, j, e)$ is similarly defined, except for two points: (i) the length and number of chosen vertices are proportional to $e_1$ and $u_1$; and (ii) when color $\varepsilon(i, j)$ (resp. $\tau(s, r)$) should be in the list, we add $\varepsilon'(i, j)$ (resp. $\tau_s'(s, r)$), and vice-versa. For an example of the edge gadget, please refer to Figure 4.
Backward Edge Gadget. Gadget $G(j, i, e)$ has vertices $r(j, i, e), a(j, i, e),$ and $b(j, i, e)$ defined similarly as to the forward gadget, with the root vertex having the list $\{\sigma(j, i), \sigma'(j, i)\}$, while the other two have the list $\{\sigma(j, i), y(j, i)\}$.

To $a(j, i, e), we append a path with $2e_{\downarrow} + 2(k - 1)v_{\downarrow}$ vertices. First, choose $e_{\downarrow}$ even vertices to assign the list $\{y(j, i), \varepsilon(i, j)\}$. Now, for each $r$ in $j < r \leq k$, choose $v_{\downarrow}$ even vertices to assign the list $\{y(j, i), \tau'_j(i, r)\}$. Then, for each $s \neq i$ satisfying $s < j$, choose $v_{\downarrow}$ even vertices and assign the list $\{y(j, i), \tau_j(s, i)\}$. All the odd vertices are assigned the list $\{y(j, i), x(j, i)\}$. The path appended to $b(j, i, e)$ is similarly defined, except that: (i) the length and number of chosen vertices are proportional to $e_{\uparrow}$ and $v_{\uparrow}$; and (ii) when color $\varepsilon(i, j)$ (resp. $\tau_j(s, r)$) is in the list, we replace it with $\varepsilon'(i, j)$ (resp. $\tau'_j(s, r)$), and vice-versa. Note that, for every edge gadget, either forward or backward, the number of vertices is equal to $3 + 2(e_{\uparrow} + e_{\downarrow}) + 2(k - 1)(u_{\uparrow} + u_{\downarrow}) = 3 + 2kZ$. We say that $G(i, j, e)$ is selected if $r(i, j, e)$ is colored with $\sigma(i, j)$, otherwise it is passed.

Numerical Targets. Before discussing the function $h$, recall that $|E(i, j)| = M$ for every pair $i, j$ and that, for every vertex $u$ and edge $e$, the identification numbers satisfy the identity $v_{\uparrow} + v_{\downarrow} = Z$ and $e_{\uparrow} + e_{\downarrow} = Z$. We present the numerical targets of our instance - and some intuition - below.

Selection: $h(\sigma(i, j)) = 1 + 2(M - 1)$ and $h(\sigma'(i, j)) = M - 1$. Since only one edge may be chosen from $V_i$ to $V_j$, the non-selection color $\sigma'(i, j)$ must be used in $M - 1$ edges of $G(i, j)$. Thus, exactly one $G(i, j, e)$ is selected and, to achieve the target of $1 + 2(M - 1)$, for every $f \in E(i, j) \setminus \{e\}$, both $a(i, j, f)$ and $b(i, j, f)$ must also be colored with $\sigma(i, j)$.
Helper: \( h(y(i,j)) = 2 + kMZ \) and \( h(x(i,j)) = kMZ - kZ \). The goal here is that, if \( G(i,j,e) \) is selected, all the odd positions must be colored with \( y(i,j) \), otherwise even every positions must colored with it. In the latter case, the odd positions of all but one gadget of \( G(i,j) \) must be colored with \( x(i,j) \).

Symmetry: \( h(\varepsilon(i,j)) = h(\varepsilon'(i,j)) = Z \). If the previous condition holds and \( r(i,j,e) \) is colored with \( \sigma(i,j) \), then \( \varepsilon(i,j) \) appears in \( e_{i,j} \) vertices of the gadget rooted at \( r(i,j,e) \). To meet the target \( Z \), \( e_{i,j} \) vertices of another gadget must also be colored with it, as we show, the only way is if \( r(j,i,e) \) is colored with \( \sigma(j,i) \).

Consistency: \( h(\tau_i(s,r)) = h(\tau'_i(s,r)) = Z \). Similar to symmetry colors.

**Lemma 6.** If \( H \) has a \( k \)-multicolor clique, then \( G \) admits a list coloring meeting the numerical targets.

**Proof.** Let \( Q \) be a clique of \( H \) of size \( k \). Now, for each \( i,j \in [k] \), with \( i \neq j \), and \( e \in E(i,j) \), we color \( G \) as follows: if \( e \in Q \), color \( r(i,j,e) \) with \( \sigma(i,j) \), color every odd vertex with \( y(i,j) \) and every even vertex with the unique available color to it; otherwise, \( r(i,j,e) \) with \( \sigma'(i,j) \), color every even vertex with \( y(i,j) \) and all odd vertices with the unique available color. This concludes the construction of the coloring \( \varphi \) of \( G \).

As to the numerical targets, note that the colors of \( \mathcal{S} \) and \( \mathcal{S}' \) are used the appropriate number of times since only one edge of \( E(i,j) \) belongs to \( Q \). For each \( y(i,j) \in \mathcal{S} \), gadget \( G(i,j,e) \), with \( e = uv \) and \( u \in V_i \), has, at least, \( e_{i,j} + (k-1)(u_i + u_j) = kZ \), since either all odd vertices or even vertices colored with \( y(i,j) \). For the remaining two, note that the selected gadget \( G(i,j,e) \) has \( a(i,j,e) \) and \( b(i,j,e) \) also colored with \( y(i,j) \). As to the other helper color, \( x(i,j) \), we use it only in passed gadgets and, in this case, in every odd vertex (except the \( a ' \)s and \( b ' \)s); this sums up to \( \sum_{e \in E(i,j) \setminus Q} e_{i,j} + e_{i,j} + (k-1)(u_i + u_j) = kMZ - kZ \). In terms of symmetry colors, \( \varphi \) only uses \( \varepsilon(i,j) \) on the selected gadgets \( G(i,j,e) \) and \( G(j,i,e) \) (\( i < j \)), in particular, \( \varepsilon(i,j) \) is used \( e_{i,j} \) times in \( G(j,i,e) \) and \( e_{i,j} \) times on \( G(i,j,e) \), so it holds that \( |\varepsilon(\varepsilon(i,j))| = e_{i,j} + e_{i,j} = Z \). Note that the same argument applies for color \( \varepsilon'(i,j) \). Finally, for consistency colors, note that \( \tau_i(r,s) \) is used only in the selected gadgets \( G(i,r,e) \) and \( G(i,s,f) \), specifically, if \( i < r < s \), \( \tau_i(r,s) \) is used \( u_r \) times in \( G(i,r,e) \) and \( u_s \) times in \( G(i,s,f) \), since edges \( e,f \) must be incident to the same vertex of \( Q \setminus V_i \). Consequently \( \tau_i(r,s) \) is used in \( u_r + u_s = Z \) vertices. The reasoning for \( \tau'_i(r,s) \) is similar. \( \Box \)

We now proceed to the proof of the converse, i.e., if there is coloring of \( G \) that meets all the targets, then there is a \( k \)-multicolored clique of \( H \).

**Lemma 7.** In every list coloring of \( G \) satisfying \( h \), exactly one gadget of each \( G(i,j) \) is selected, each passed \( G(i,j,e) \) has all of its \( kZ \) even vertices colored with \( y(i,j) \), and the selected \( G(i,j,f) \) has all of its \( 2 + kZ \) odd vertices colored with \( y(i,j) \).

**Proof.** For the first statement, if no gadget was selected, \( \sigma'(i,j) \) would be used \( M > M - 1 \); if more than one is selected, \( \sigma'(i,j) \) does not meet the target. If \( G(i,j,f) \) is selected than \( a(i,j,f) \) and \( b(i,j,f) \) are colored with \( y(i,j) \). After
removing these vertices, we are left with two even paths of which at most half of its vertices are colored with the same color. As such there are at most $2 + f_i + (k-1)u + f + (k-1) = 2 + kZ$ vertices colored with $y(i, j)$ in this gadget; moreover this bound is achieved if and only if the odd vertices are colored with $y(i, j)$. For each passed $G(i, j, e)$, $a(i, j, e)$ and $b(i, j, e)$ are colored with $\sigma(i, j)$, otherwise the numerical target of $\sigma(i, j)$ cannot be met. After the removal of $a, b, r(i, j, e)$ we are again left with two even paths. Thus, at most $e_i + (k-1)u + e + (k-1) = kZ$ vertices have color $y(i, j)$ in this gadget. To see that $y(i, j)$ must be used only for even vertices of $G(i, j, e)$ to meet this bound, note that, if this is not done, then $x(i, j)$ will never meet its bound, since exactly $kM - kZ$ vertices remain (after the coloring of $G(i, j, f)$) that can be colored with $x(i, j)$, which is precisely its target.

\textbf{Lemma 8.} In every list coloring $\varphi$ of $G$, if $G(i, j, e)$ is selected, so is $G(j, i, e)$.

\textit{Proof.} Suppose $i < j$. By Lemma 7 we know that for a selected gadget every odd vertex is colored with $y(i, j)$, so each even vertex is colored with a non-helper color. Note that color $c'(i, j)$ is used $e_1$ times in gadget $G(i, j, e)$. Now, we need to select one backward gadget of $G(j, i)$; suppose we select gadget $G(j, i, f)$, $f \neq e$. Again by Lemma 7 the number of occurrences of $c'(i, j)$ is $f_1$ times in $G(j, i, f)$, but we have that $e_1 + f_1 \neq Z$, a contradiction that $\varphi$ satisfies $h$.

\textbf{Lemma 9.} In every list coloring $\varphi$ of $G$, if $G(i, j, e)$ is selected and $e = uv$, then, for every $s \neq i$, the edge $f$ of $H$ corresponding to the selected gadget $G(i, s, f)$ must be incident to $u$.

\textit{Proof.} We divide our proof in two cases. First, suppose $i < j$ and $s < j$. By Lemma 7, $\tau(s, j)$ is used in $u$ vertices of $G(i, j, e)$. Now, note that the only possible gadgets $G(i, s, f)$ that can be chosen such that the endpoint $w$ of $f$ in $V_i$ satisfies $u + w = Z$ must have $w = u$, since $\tau(s, j)$ is used $w_1$ times in gadget $G(i, s, f)$. The case $j < s$ is similar, but we replace $\tau$ with $\tau'$.

Theorem 3 follows without much effort from the previous lemmas. For the first corollary, we add isolated vertices with lists of size one to the input of \textsc{Number List Coloring} while keeping it a disjoint union of paths so as to make all colors have the same numerical target. For the second, we add a clique of size $r$, the number of colors of the instance of \textsc{Equitable List Coloring}, and label them using the integers $[r]$; afterwards, for each vertex $u$ of the input graph that does not have color $i$ in its list, we add an edge between the $i$-th vertex of the clique and $u$.

\textbf{Theorem 3.} \textsc{Number List Coloring} on paths parameterized by the number of colors that appear on the lists is W[1]-hard.

\textbf{Corollary 3.} \textsc{Equitable List Coloring} on paths parameterized by the number of colors that appear on the lists is W[1]-hard.

\textbf{Corollary 4.} \textsc{Equitable Coloring} parameterized by the number of colors and distance to disjoint paths is W[1]-hard.
6 Final Remarks

In this work we presented an extensive study of multiple parameterizations for the Equitable Coloring problem, obtaining both tractability and intractability results. Specifically, we proved that it is fixed parameter tractable when parameterized by distance to cluster and distance to co-cluster, and as corollaries that there is an FPT algorithm when parameterized by distance to unipolar and number of colors. Meanwhile, the problem remains $\mathcal{W}[1]$-hard when simultaneously parameterized by distance to disjoint paths and number of colors. We also revisited previous works in the literature and restated them in terms of parameterized complexity. This review settled the complexity for some parameterizations weaker than distance to clique and show that our results are, in a sense, optimal: searching for parameters weaker than distance to (co-)cluster will most likely not yield FPT algorithms. Vertex Coloring is notoriously hard to find polynomial kernels for, as shown by Jansen and Kratsch [15]; in fact, most of the parameterizations under which classical coloring admits a polynomial kernel do not make Equitable Coloring tractable. These results point to a quite bleak future for kernelization algorithms for Equitable Coloring. However, Reddy [18] does present a polynomial kernel parameterized by distance to threshold and number of colors, so perhaps not all hope is lost. All things considered, we believe that future work should be directed to the study of the parameterizations max leaf number or feedback edge set, as these are the two main cases we leave open. Another interesting direction would be improvements on the running times of known tractable cases, in particular we are interested in determining whether it is possible to avoid a super exponential dependency on parameters such as vertex cover and distance to clique, i.e., if we can avoid going through all possible colorings of the modulators.

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