Equations of the reaction–diffusion type with a loop algebra structure

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Abstract

A system of equations of the reaction–diffusion type is studied in the framework of both the direct and the inverse prolongation structure. We find that this system allows an incomplete prolongation Lie algebra, which is used to find the spectral problem and a whole class of nonlinear field equations containing the original ones as a special case.

1 Introduction

Mathematical models of spatial pattern formation in complex organisms are based on nonlinear interactions of at least two chemicals and on their diffusion, where autocatalysis and long–range inhibition play a crucial role [1]. Generally, these models are described by nonlinear evolution equations of the reaction–diffusion type, a class of which can be written as

\[ u_t = D_1 \Delta u + b_1 u^2 v + b_2 uv^2 + b_3 u + b_4 v + b_5, \]
\[ v_t = D_2 \Delta v + c_1 u^2 v + c_2 uv^2 + c_3 u + c_4 v + c_5, \]

where \( u = u(x, y, t) \), \( v = v(x, y, t) \), \( \Delta \) denotes the Laplace operator in two–dimensional orthonormal coordinates, \( D_1, D_2 \) are the diffusion constants, and \( b_1, c_1, \ldots \) are (constant) coefficients.

The physical meaning of \( u, v \) and the parameters \( b_1, c_1, \ldots \) depends on the biological phenomenon under consideration. For example, in the case of the activator–substrate model in its simplest form [2], one has \( b_1 = -b_3, b_2 = b_4 = \ldots \).
\( b_5 = 0, \ c_5 = -c_1, \ c_2 = c_3 = c_4 = 0, \) where \( b_1 \) and \( c_1 \) mean the cross-reaction coefficients. The functions \( u \) and \( v \) can be interpreted as the self–enhanced reactant and a substrate depleted by \( u \), respectively.

The high interest presented by the mathematical formulation of pattern generation in complex structures rises the question whether the system (1)–(2) contains integrable subcases. Due to the hardness of the subject, it is convenient to dwell upon this problem gradually, i.e. first studying a \((1+1)\)–dimensional version of Eqs. (1)–(2). To this regard, we notice that biological systems modeled by equations belonging to the class (1)–(2) are interesting also in \((1+1)\)–dimensions. One of them, which consists of Eqs. (1)–(2) where the nonlinear terms are missing, was introduced by Kondo and Asai in 1995 in relation to the study of the stripe pattern of the angelfish Pomacanthus.

In performing our programme, we remind the reader that one of the most remarkable feature of integrable nonlinear field equations is the onset of infinite dimensional Lie algebras with a loop structure. This characteristic strongly suggests that a close connection should be established between the integrability property and the infinite dimension of the Lie algebra allowed by a given nonlinear field equation. An efficacious tool to look for the existence of loop algebras for \(1+1\) dimensional nonlinear systems is based on the Estrabrook–Wahlquist (EW) prolongation theory. The application of this procedure to Eqs. (1)–(2) in \(1+1\) dimensions tells us that the system

\[
\begin{align*}
  u_t - u_{xx} + 2u^2v - 2ku &= 0, \\
  v_t + v_{xx} - 2uv^2 + 2kv &= 0,
\end{align*}
\]

where \( k \) is a constant, is endowed with a Lie algebra possessing a loop structure.

It is noteworthy that Eqs. (3)–(4) emerge also in the gauge formulation of the \(1+1\) dimensional gravity. In this context, \( u \) and \( v \) are Zweibein fields. In general, the system (3)–(4) is similar to the "fictitious" or "mirror–image" systems with negative friction, which appear into the thermo–field approach to the damped oscillator treated in [3].

Other equations which fall into the class (1)–(2) admitting nontrivial Lie algebras, although of finite dimensions, will be handled elsewhere.

Here we study systematically Eqs. (3)–(4) in both the direct and the inverse prolongation framework. The inverse prolongation method consists in starting from a given incomplete Lie algebra to find the nonlinear field equations whose prolongation structure it is.

The direct prolongation method is carried out in Sec. 2. It provides an incomplete Lie algebra (in the sense that not all of the commutators are known) which is exploited to obtain the linear eigenvalue problem associated with the system (3)–(4). A possible realization of the prolongation algebra turns out to be an infinite–dimensional Lie algebra with a loop structure of the Kac–Moody type. In Sec. 3 we deal with the inverse prolongation. We consider the incomplete Lie
algebra determined in Sec. 2 to build up the differential ideal related to Eqs. (3)–(4). In Sec. 4 an inverse prolongation procedure based on a certain realization of the Kac–Moody algebra related to Eqs. (3)–(4) is outlined. In both these inverse schemes we can generate the class of field equations whose prolongation structure the incomplete Lie algebra is. New integrable equations arise including the original ones as a special case. Finally, in Sec. 5 some comments are reported while the Appendixes A, B and C contain details of the calculations.

2 The prolongation algebra

In order to formulate the EW prolongation method for Eqs. (3)–(4) let us introduce the differential ideal defined by the set of 2–forms

\[
\alpha_1 = du \wedge dt - u_x dx \wedge dt, \quad (5)
\]

\[
\alpha_2 = dv \wedge dt - v_x dx \wedge dt, \quad (6)
\]

\[
\alpha_3 = -du \wedge dx - du_x \wedge dt + 2u(uv - k)dx \wedge dt, \quad (7)
\]

\[
\alpha_4 = -dv \wedge dx + dv_x \wedge dt - 2v(uv - k)dx \wedge dt, \quad (8)
\]

where \(\wedge\) means the wedge product. The ideal (5)–(8) turns out to be closed.

Now we consider the prolongation 1–forms

\[
\omega^k = dy^k + F^k(u, u_x, v, v_x; y)dx + G^k(u, u_x, v, v_x; y)dt, \quad (9)
\]

where \(y = \{y^m\}\), \(k, m = 1, 2, \ldots, N\) \((N\) arbitrary), and \(F^k, G^k\) are, respectively, the pseudopotential and functions to be determined. By requiring that \(d\omega^k \in I(\alpha_j, \omega^k)\), \(I\) being the ideal generated by \(\alpha_j\) and \(\omega^k\), we find the constraints

\[
F^k_{u_x} - G^k_{u_x} = 0, \quad (10)
\]

\[
F^k_{v_x} = 0, \quad (11)
\]

\[
F^k_v + G^k_{v_x} = 0, \quad (12)
\]

\[
F^k_{v_x} = 0, \quad (13)
\]

\[
2u(uv - k)F^k_u - 2v(uv - k)F^k_v + G^k_u u_x + G^k_v v_x + [F, G]^k = 0, \quad (14)
\]

where \([F, G]^k = F^j G^k_{y_j} - G^j F^k_{y_j}\), \(F^k_{y_j} = \frac{\partial F^k}{\partial y_j}\), \(F^k_u = \frac{\partial F^k}{\partial u}\), and so on.

In the following, we shall omit the index \(k\), for simplicity.

The differential equations (10)–(14) yield

\[
F = a_1 uv + a_2 u + a_3 v + a_4, \quad (15)
\]

\[
G = (a_1 v + a_2)u_x - (a_1 u + a_3)v_x + [a_3, a_2]uv + [a_4, a_2]u + [a_3, a_4]v + a_5, \quad (16)
\]
and the commutation relations

\[
\begin{align*}
[a_1, a_2] &= 0, \quad [a_1, a_3] = 0, \quad [a_1, a_4] = 0, \quad [a_4, a_5] = 0, \\
[a_1, [a_3, a_2]] &= 0, \quad [a_1, [a_3, a_4]] = 0, \quad [a_1, [a_4, a_2]] = 0, \\
[a_2, [a_3, a_2]] &= 2a_2, \quad [a_2, [a_4, a_2]] = 0, \quad [a_3, [a_3, a_2]] = -2a_3, \\
[a_3, [a_3, a_4]] &= 0, \quad 2[a_4, [a_3, a_2]] + [a_1, a_5] = 0, \\
2ka_2 + [a_2, a_5] + [a_4, [a_4, a_2]] &= 0, \\
-2ka_3 + [a_3, a_5] + [a_4, [a_3, a_4]] &= 0.
\end{align*}
\]

Equations (17)–(22) define an incomplete Lie algebra. However, we can find a homomorphism between the algebra (17)–(22) and the \( sl(2, \mathbb{R}) \) algebra

\[
\begin{align*}
[X_1, X_2] &= 2X, \quad [X_2, X] = 2X_1, \quad [X, X_1] = -2X_2,
\end{align*}
\]

where \( X = [a_3, a_2] \), \( X_1 = a_2 + a_3 \) and \( X_2 = a_2 - a_3 \). This can be seen assuming that

\[
\begin{align*}
a_1 &= 0, \quad a_4 = \lambda X, \quad a_5 = -(k + 2\lambda^2)X,
\end{align*}
\]

where \( \lambda \) is a free parameter. Then, Eqs. (17)–(22) yield easily (23).

By means of (23), we can build up the spectral problem related to Eqs. (3)–(4). In doing so, we need to exploit a realization of Eqs. (23). In terms of two prolongation variables \( (y_1, y_2) \) a linear realization of (23) is given by

\[
\begin{align*}
X_1 &= y_2 \partial_{y_1} + y_1 \partial_{y_2}, \quad X_2 = y_2 \partial_{y_1} - y_1 \partial_{y_2}, \quad X = y_1 \partial_{y_1} - y_2 \partial_{y_2},
\end{align*}
\]

to which the following \( 2 \times 2 \) matrix representation

\[
\begin{align*}
X_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]

corresponds. Keeping in mind (1)–(3) and (10), with the help of (24) we obtain the spectral problem for Eqs. (3)–(4), namely

\[
\begin{align*}
y_x &= -F^T y, \quad y_t = -G^T y,
\end{align*}
\]

where \( y = (y_1, y_2)^T \),

\[
F = \begin{pmatrix} \lambda & v \\ u & -\lambda \end{pmatrix}, \quad G = \begin{pmatrix} uv - k - 2\lambda^2 & -(v_x + 2\lambda v) \\ u_x - 2\lambda u & -uv + k + 2\lambda^2 \end{pmatrix}.
\]

The compatibility condition for Eqs. (27), i.e. \( L_{1t} - L_{2x} - [L_1, L_2] = 0 \), provides just Eqs. (3)–(4).
Another possible realization of the prolongation algebra (17)–(22) is given by an infinite-dimensional Lie algebra with a loop structure, precisely an algebra of the Kac–Moody type. To this aim, let us suppose that \(a_1 = 0\). Then let us put

\[
a_2 = T_1^{(m_0)} - i T_2^{(-m_0)},
\]

\[
a_3 = T_1^{(m_0)} + i T_2^{(-m_0)},
\]

\[
a_4 = \alpha_4 T_3^{(n_4)} + \beta_4 T_3^{(-n_4)},
\]

\[
a_5 = \alpha_5 T_3^{(n_5)} + \beta_5 T_3^{(-n_5)},
\]

where \(m_0, n_4, n_5 \in \mathbb{Z}\), and the coefficients \(\alpha_j, \beta_j \in \mathbb{R}\) \((j = 4, 5)\), are to be determined in such a way that (29)–(32) satisfy the commutation relations of the (incomplete) Lie algebra (17)–(22). A straightforward calculation shows that the vector fields \(T_j^{(n)}\) \((j = 1, 2, 3; n \in \mathbb{Z})\) are elements of the Kac–Moody algebra (without central charge) governed by the commutation relations

\[
[T_j^{(n)}, T_k^{(m)}] = i \epsilon_{jkl} T_l^{(n+m)},
\]

\((n, m \in \mathbb{Z})\), \(\epsilon_{jkl}\) being the Ricci tensor. The requirement that Eqs. (17)–(22) are fulfilled, entails \(m_0 = 0\), \(n_5 = 2n_4\), \(\alpha_4 = -\beta_4 = \sqrt{k}\) \((k > 0)\), and \(\alpha_5 = \beta_5 = -k\).

The algebra (33) admits the following realization in terms of the prolongation variables:

\[
T_1^{(m)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left[ y_2^{(m+n)} \frac{\partial}{\partial y_1^{(n)}} + y_1^{(m+n)} \frac{\partial}{\partial y_2^{(n)}} \right],
\]

\[
T_2^{(m)} = \frac{i}{2} \sum_{n \in \mathbb{Z}} \left[ y_2^{(m+n)} \frac{\partial}{\partial y_1^{(n)}} - y_1^{(m+n)} \frac{\partial}{\partial y_2^{(n)}} \right],
\]

\[
T_3^{(m)} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left[ y_1^{(m+n)} \frac{\partial}{\partial y_1^{(n)}} - y_2^{(m+n)} \frac{\partial}{\partial y_2^{(n)}} \right],
\]

where the pseudopotential \(y\) is expressed via the independent infinite-dimensional vectors \(y_1\) and \(y_2\) \((y_1^{(n)}\) and \(y_2^{(n)}\) stand for the \(n\)–component of \(y_1\) and \(y_2\), respectively).

At this point it is instructive to exploit the infinite-dimensional algebra (33) to write down again the spectral problem related to Eqs. (3)–(4). The procedure, which is outlined in Appendix A, is based on the equations \(y_x = -F, y_t = -G\) (see (9)), where \(F\) and \(G\) are given by (13)–(16) with \(a_1 = 0\), while \(a_2, a_3, a_4\) and \(a_5\) are expressed in terms of the Kac–Moody operators by (29)–(32), using the representation (34)–(36).
3 The inverse prolongation

In Sec. 2, we have applied the E–W method to find the prolongation algebra associated with Eqs. (3)–(4). In general, the prolongation of an integrable nonlinear field equation can be interpreted as a Cartan–Ehresmann connection, so that an incomplete Lie algebra of vectors field can be related to a differential ideal \( \mathfrak{g} \). On the contrary, starting from the incomplete Lie algebra (17)–(22), we can yield the differential ideal associated with Eqs. (3)–(4) specifying the form of the connection. In this way, we can generate the field equations whose prolongation structure the incomplete algebra (17)–(22) is. To this aim, let us assume that the connection

\[
\omega^k = dy^k + A_j^k \theta^j
\]  

exists, such that

\[
d\omega^k = A_j^k d\theta^j - \frac{1}{2} [A_j, A_i]^k \theta^j \wedge \theta^i \quad (\text{mod } \omega^k),
\]  

where \( \theta^j \) are 1–forms, \( A_j (j = 1, 2, \ldots, 8) \) are defined by

\[
A_1 = a_1, \quad A_2 = a_2, \quad A_3 = a_3, \quad A_4 = a_4, \quad A_5 = a_5, \\
A_6 = [a_3, a_2], \quad A_7 = [a_2, a_4], \quad A_8 = [a_3, a_4],
\]  

appearing in the incomplete Lie algebra (17)–(22), and \( \text{mod } \omega^k \) means that all the exterior products between \( \omega^k \) and 1–forms of the Grassmann algebra have not been considered. (For the reader convenience, in Appendix B we report the incomplete algebra involved in (38) where one can easily recognize the independent commutation relations).

By resorting to the commutation relations of Appendix B, Eq. (38) provides the constraints

\[
d\theta^1 = 0, \quad (40)
\]

\[
d\theta^2 + 2k\theta^2 \wedge \theta^5 - 2\theta^2 \wedge \theta^6 = 0, \quad (41)
\]

\[
d\theta^3 - 2k\theta^3 \wedge \theta^5 + 2\theta^3 \wedge \theta^6 = 0, \quad (42)
\]

\[
d\theta^4 = 0, \quad (43)
\]

\[
d\theta^5 = 0, \quad (44)
\]

\[
d\theta^6 + \theta^2 \wedge \theta^3 = 0, \quad (45)
\]

\[
d\theta^7 - \theta^2 \wedge \theta^4 + 2\theta^6 \wedge \theta^7 = 0, \quad (46)
\]

\[
d\theta^8 - \theta^3 \wedge \theta^4 - 2\theta^6 \wedge \theta^8 = 0, \quad (47)
\]

\[
2\theta^1 \wedge \theta^5 + \theta^3 \wedge \theta^7 - \theta^4 \wedge \theta^6 = 0, \quad (48)
\]

\[
\theta^2 \wedge \theta^5 + \theta^4 \wedge \theta^7 = 0, \quad (49)
\]

\[
\theta^2 \wedge \theta^8 + \theta^3 \wedge \theta^7 = 0, \quad (50)
\]

\[
\theta^3 \wedge \theta^5 - \theta^4 \wedge \theta^8 = 0, \quad (51)
\]

\[
\theta^5 \wedge \theta^6 = \theta^5 \wedge \theta^7 = \theta^5 \wedge \theta^8 = \theta^7 \wedge \theta^8 = 0. \quad (52)
\]
In the following, we shall consider an interesting solution of Eqs. (40)–(52). To this aim, we note that Eq. (44) is satisfied if
\[ \theta_5 = dt. \] (53)
On the other hand, Eq. (52) yields
\[ \theta_6 = \gamma_1 dt, \quad \theta_7 = \gamma_2 dt, \quad \theta_8 = \gamma_3 dt, \] (54)
where \( \gamma_j \) (\( j = 1, 2, 3 \)) are (arbitrary) 0–forms. Furthermore, Eqs. (49)–(51) imply
\[ \theta_2 + \gamma_2 \theta_4 = \eta_1 dt, \] (55)
\[ \gamma_3 \theta_2 + \gamma_2 \theta_3 = \eta_2 dt, \] (56)
\[ \theta_3 - \gamma_3 \theta_4 = \eta_3 dt, \] (57)
where \( \eta_j \) (\( j = 1, 2, 3 \)) are 0–forms.

With the help of (53) and (54), Eq. (48) gives
\[ 2\theta_1 + \gamma_2 \theta_3 - \gamma_1 \theta_4 = \eta_4 dt, \] (58)
\( \eta_4 \) being a 0–form.

Now, from Eqs. (55)–(57) we obtain
\[ \gamma_3 \eta_1 + \gamma_2 \eta_3 = \eta_2. \] (59)
Then, since
\[ d\theta^6 = d\gamma_1 \wedge dt \] (60)
(see (44)), Eq. (44) becomes
\[ d\gamma_1 \wedge dt = \eta_2 \theta^4 \wedge dt, \] (61)
where Eqs. (43) and (44) have been used.

At this point, by elaborating the constraints (41) and (42) on the basis of the preceding results (53)–(61) and taking \( \theta^4 = dx \), we arrive at the relations
\[ \gamma_{2t} + \eta_1 x - 2k \gamma_2 + 2\gamma_1 \gamma_2 = 0, \] (62)
\[ \gamma_{3t} - \eta_3 x + 2k \gamma_3 - 2\gamma_1 \gamma_3 = 0, \] (63)
\[ \gamma_{1x} - \gamma_2 \eta_3 - \gamma_3 \eta_1 = 0, \] (64)
\[ \gamma_{2x} + \eta_1 = 0, \] (65)
\[ \gamma_{3x} + \eta_3 = 0. \] (66)
Integrating Eq. (64) with respect to \( x \) and putting the function of integration equal to zero, with the help of Eqs. (65) and (66), Eqs. (62)–(63) can be easily reshaped to give
\[ \gamma_{2t} - \gamma_{2xx} - 2k \gamma_2 - 2\gamma_2^2 \gamma_3 = 0, \] (67)
\[ \gamma_{3t} + \gamma_{3xx} + 2k \gamma_3 + 2\gamma_2 \gamma_3^2 = 0. \] (68)
By identifying $\gamma_2$ and $\gamma_3$ with $-u$ and $v$, respectively, Eqs. (37)–(68) reproduce exactly the system (3)–(4). Thus, the inverse prolongation enables us to discover the integrable system (62)–(67), closely related to Eqs. (3)–(4), which can be considered as a kind of "potential form" of that system.

4 The inverse prolongation via the Kac–Moody algebra

We have shown that the incomplete Lie algebra associated with the prolongation structure of Eqs. (3)–(4) allows the realization expressed by the Kac–Moody operators (34)–(36). Here we shall exploit this realization to solve an inverse prolongation problem which may furnish, in theory, more information than the method applied in Sec. 3. To this aim, we start from the ansatz

$$\omega = dy + \left( \sum_{i=1}^{3} S_i X_i \right) dx + \left( \sum_{i=1}^{11} \psi_i(S_j, S_{jx}) X_i \right) dt,$$  \hspace{1cm} (69)

where $j = 1, 2, 3$,

$$X_1 = T_1 - iT_2, \quad X_2 = T_1 + iT_2, \quad X_3 = T_3 - T_3,$$

$$X_4 = T_3 + T_3, \quad X_5 = T_3, \quad X_6 = T_1, \quad X_7 = T_1,$$

$$X_8 = T_2, \quad X_9 = T_2, \quad X_{10} = T_3, \quad X_{11} = T_3,$$ \hspace{1cm} (70)

with $n, m \in \mathbb{Z} - \{0\}$, the operators $T_j$ being expressed by the realization (34)–(36) of the Kac–Moody algebra (33). Then, by equating the 1–form (33) to zero, we can determine the functions $\psi_j$ and, in correspondence, evolution systems of the form $S_{jt} = f (\{S_k\}, \{S_{kx}\}, \{S_{kxx}\})$, where $S_{jx} = \frac{\partial S_j}{\partial x}$, and so on.

In doing so, we obtain

$$-y_x^{(i)} = \frac{1}{2} S_1 (\sigma_1 + i\sigma_2) y^{(i)} + \frac{1}{2} S_2 (\sigma_1 - i\sigma_2) y^{(i)} + \frac{1}{2} S_3 \sigma_3 (y^{(n+i)} - y^{(-n+i)}),$$  \hspace{1cm} (71)

$$-y_t^{(i)} = \frac{1}{2} \psi_1 (\sigma_1 + i\sigma_2) y^{(i)} + \frac{1}{2} \psi_2 (\sigma_1 - i\sigma_2) y^{(i)} +$$

$$+ \frac{1}{2} \sigma_3 \left[ \psi_3 (y^{(n+i)} - y^{(-n+i)}) + \psi_4 (y^{(n+i)} + y^{(-n+i)}) \right] +$$

$$+ \frac{1}{2} \psi_5 \sigma_3 y^{(i)} + \frac{1}{2} \psi_6 \sigma_1 y^{(n+i)} + \frac{1}{2} \psi_7 \sigma_1 y^{(-n+i)} - \frac{1}{2} \psi_8 \sigma_2 y^{(n+i)} -$$

$$- \frac{1}{2} \psi_9 \sigma_2 y^{(-n+i)} + \frac{1}{2} \psi_{10} \sigma_3 y^{(m+i)} + \frac{1}{2} \psi_{11} \sigma_3 y^{(-m+i)},$$ \hspace{1cm} (72)

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices acting on the vectors

$$y^{(i)} = \begin{pmatrix} y_1^{(i)} \\ y_2^{(i)} \end{pmatrix}.$$  \hspace{1cm} (73)
The compatibility condition $y_{x}^{(i)} = y_{t}^{(i)}$ for the system (71)-(72) yields a set of constraints involving the functions $S_{j}$ ($j = 1, 2, 3$) and $\psi_{k}$ ($k = 1, 2, \ldots, 11$). We observe that since the index $m$ is different from 0, $n$ and $-n$, two cases can occur: i) $m = 2n$ (or $m = -2n$) and ii) $m \neq 2n$, $-2n$.

**Case i**

Omitting the lengthy calculations for simplicity, for $m = 2n$ we find

$$-S_{1}\psi_{2} + S_{2}\psi_{1} - \frac{1}{2}\psi_{5x} = 0,$$

$$S_{1t} + S_{1}\psi_{5} + \frac{1}{2}S_{3}\psi_{6} - \frac{1}{2}S_{3}\psi_{7} + \frac{i}{2}S_{3}\psi_{8} - \frac{i}{2}S_{3}\psi_{9} - \psi_{1x} = 0,$$

$$S_{2t} - S_{2}\psi_{5} - \frac{1}{2}S_{3}\psi_{6} + \frac{i}{2}S_{3}\psi_{7} + \frac{i}{2}S_{3}\psi_{8} - \frac{i}{2}S_{3}\psi_{9} - \psi_{2x} = 0,$$

$$S_{1}(\psi_{3} + \psi_{4}) - S_{3}\psi_{1} - \frac{1}{2}\psi_{6x} - \frac{i}{2}\psi_{8x} = 0,$$

$$-S_{2}(\psi_{3} + \psi_{4}) + S_{3}\psi_{2} - \frac{1}{2}\psi_{6x} + \frac{i}{2}\psi_{8x} = 0,$$

$$-S_{1}\psi_{6} + iS_{1}\psi_{8} + S_{2}\psi_{6} + iS_{2}\psi_{8} + S_{3t} - (\psi_{3x} + \psi_{4x}) = 0,$$

$$-S_{1}(\psi_{3} - \psi_{4}) + S_{3}\psi_{1} - \frac{1}{2}\psi_{7x} - \frac{i}{2}\psi_{9x} = 0,$$

$$S_{2}(\psi_{3} - \psi_{4}) - S_{3}\psi_{2} - \frac{1}{2}\psi_{7x} + \frac{i}{2}\psi_{9x} = 0,$$

$$-S_{1}\psi_{7} + iS_{1}\psi_{9} + S_{2}\psi_{7} + iS_{2}\psi_{9} - S_{3t} + \psi_{3x} - \psi_{4x} = 0,$$

$$S_{1}\psi_{10} - \frac{1}{2}S_{3}\psi_{6} - \frac{i}{2}S_{3}\psi_{8} = 0,$$

$$-S_{2}\psi_{10} + \frac{1}{2}S_{3}\psi_{6} - \frac{i}{2}S_{3}\psi_{8} = 0,$$

$$\psi_{10x} = 0,$$

$$S_{1}\psi_{11} + \frac{1}{2}S_{3}\psi_{7} + \frac{i}{2}S_{3}\psi_{9} = 0,$$

$$-S_{2}\psi_{11} - \frac{1}{2}S_{3}\psi_{7} + \frac{i}{2}S_{3}\psi_{9} = 0,$$

$$\psi_{11x} = 0,$$

where $\psi_{jx} = \psi_{j, S_{k} S_{k} x} + \psi_{j, S_{k} S_{k} x} S_{k} x$. After some manipulations (see Appendix C), from Eqs. (74)-(83) we are led to the system of nonlinear evolution equations

$$U_{t} - 2aU^{2}V + 2aU - U_{x} \frac{\psi_{3}}{S_{3}} + \frac{a}{S_{3}} \left( \frac{1}{S_{3}} U_{x} \right) = 0,$$

$$V_{t} + 2aU^{2}V^{2} - 2aV - V_{x} \frac{\psi_{3}}{S_{3}} - \frac{a}{S_{3}} \left( \frac{1}{S_{3}} V_{x} \right) = 0,$$

$$S_{3t} - \psi_{3x} = 0,$$

where

$$U = \frac{S_{1}}{S_{3}}, \quad V = \frac{S_{2}}{S_{3}},$$

$a$ is an arbitrary constant, and $\psi_{3}$ is an arbitrary function depending, in general, on $\{S_{j}\}$ and $\{S_{jx}\}$.

We remark that via the transformation

$$t' = t, \quad x' = \alpha(x, t),$$

choosing the (arbitrary) function $\alpha$ in such a way that

$$\alpha_{t} = \alpha_{x} \frac{\psi_{3}}{S_{3}}.$$
the first derivatives \( U_x \) and \( V_x \) in Eqs. (89) and (90) disappear. Furthermore, assuming that
\[
\alpha_x = S_3, \quad \alpha_t = \psi_3, \quad (95)
\]
the constraint (91) is automatically satisfied and Eqs. (89) and (90) take the form
\[
\begin{align*}
U'_t - 2aU^2V + 2aU + aU_{x'x'} & = 0, \quad (96) \\
V'_t + 2aV^2U - 2aV - aV_{x'x'} & = 0. \quad (97)
\end{align*}
\]
By setting \( U = \frac{u}{\sqrt{k}}, \ V = \frac{v}{\sqrt{k}}, \ a = -k \) and rescaling \( x' \), i.e. \( x' \rightarrow \frac{1}{\sqrt{k}}x' \), the system (96)–(97) reproduces just the original equations (3)–(4).

To conclude, it is noteworthy that the inverse prolongation method based on the ansatz (69) and on the realization (34)–(36) of the Kac–Moody algebra (33), is able to predict a new system of integrable nonlinear evolution equations, precisely Eqs. (89)–(91), containing the starting equations (3)–(4) as a special case.

**Case ii**

For \( m \neq 2n \), the compatibility condition for Eqs. (71)–(72) provides the constraints
\[
\begin{align*}
-S_{3t} + \psi_{3x} - \psi_{4x} & = 0, \quad (98) \\
S_2(\psi_3 - \psi_4) - S_3\psi_2 & = 0, \quad (99) \\
-S_1(\psi_3 - \psi_4) + S_3\psi_1 & = 0, \quad (100) \\
S_{3t} - (\psi_{3x} + \psi_{4x}) & = 0, \quad (101) \\
-S_2(\psi_3 + \psi_4) + S_3\psi_2 & = 0, \quad (102) \\
S_{2t} - S_2\psi_5 - \psi_{2x} & = 0, \quad (103) \\
S_{1t} + S_1\psi_5 - \psi_{1x} & = 0, \quad (104) \\
-S_1\psi_2 + S_3\psi_1 - \frac{1}{2}\psi_{5x} & = 0. \quad (105)
\end{align*}
\]
From Equations (99) and (102) we find
\[
S_2\psi_4 = 0. \quad (106)
\]
Supposing that \( S_2 \neq 0 \), from Eqs. (99)–(101) we get
\[
\psi_2 = \frac{S_2}{S_3}\psi_3, \quad \psi_1 = \frac{S_1}{S_3}\psi_3, \quad (107)
\]
and
\[
S_{3t} - \psi_{3x} = 0. \quad (108)
\]
Equations (107) imply
\[
S_1\psi_2 - S_2\psi_1 = 0. \quad (109)
\]
Taking account of (109), from (103)

\[ \psi_{5x} = 0. \quad (110) \]

Consequently, Eqs. (104) and (103) become the decoupled equations

\[ U_t + \psi_5 U - \frac{\psi_3}{S_3} U_x = 0, \quad (111) \]
\[ V_t - \psi_5 V - \frac{\psi_3}{S_3} V_x = 0, \quad (112) \]

where \( U = \frac{S_1}{S_3}, \ V = \frac{S_2}{S_3}, \) and \( S_3 \) is linked to \( \psi_3 \) by the constraint (98). Furthermore, the function \( \psi_5 \) depends on the variable \( t \) only (see Eq. (110)). Using the change of variables (93), Eqs. (111)–(112) can be written as

\[ U_t' = -\psi_5 U, \quad V_t' = \psi_5 V, \quad (113) \]

which can be easily solved to give

\[ U(x', t') = \beta(x') e^{-\int \psi_5(t') dt'}, \quad V(x', t') = \beta(x') e^{\int \psi_5(t') dt'}, \quad (114) \]

where \( \beta(x') \) is a function of integration.

5 Concluding remarks

The analysis of the equations of the reaction–diffusion type (1)–(2), in 1 + 1 dimensions, carried out using the prolongation techniques, allows us to characterize a system, given by Eqs. (3)–(4), endowed with a loop algebra structure. This property suggests that Eqs. (3)–(4) constitute an integrable system. The direct prolongation method applied to Eqs. (1)–(2) shows that a necessary condition to have an integrable special case, is that the diffusion coefficients \( D_1 \) and \( D_2 \) are of opposite sign. (We have chosen \( D_1 = -D_2 \) for simplicity). All the subcases corresponding to the assumption \( D_1 + D_2 \neq 0 \) lead to finite–dimensional Lie algebras. This result indicates that the related systems are not integrable. On the other hand, apart from the integrable case (3)–(4), other systems belonging to the class (1)–(2) present interesting features. One of them is the Gierer–Meinhardt model in its simplest form (1)

\[ u_t = D_1 u_{xx} + b_1(u^2 v - u), \quad (115) \]
\[ v_t = D_2 v_{xx} + c_1(1 - uv^2), \quad (116) \]

which admits a closed nonabelian Lie algebra connected with the similitude group in \( \mathbb{R}^2 \) (a subgroup of the eight–parameter projective group). Such an algebra (which emerges for both the choices \( D_1 + D_2 = 0 \) and \( D_1 + D_2 \neq 0 \)), and its connection with the properties of the biological system described by Eqs. (115)–(116) will not be treated here. In this paper we have limited ourselves to
carry out a systematic analysis, within the direct and the inverse prolongation scheme, of Eqs. (3)–(4), which turn out to be a integrable particular case of the class of equations (1)–(2). The prolongation Lie algebra related to Eqs. (3)–(4) is incomplete and allows an infinite-dimensional realization of the Kac–Moody type. Moreover, the incomplete Lie algebra enables us to write the linear spectral problem associated with the system under consideration. Our results confirm that a close connection exists between the incomplete prolongation Lie algebra of the model (3)–(4) and its integrability property.

On the other hand, via the inverse prolongation approach the incomplete Lie algebra of Eqs. (3)–(4) is exploited to generate the field equations whose prolongation structure it is. The new integrable systems (62)–(63) and (89)–(91) are predicted, containing the starting equations (3)–(4) as a special case.

To conclude, two comments are in order. First, it should be important to try to extend the prolongation technique to the study of higher dimension nonlinear field equations. In this context, although up to now some works have been done at the direct prolongation level [7, 8], it seems that the inverse method has been never considered. Second, we point out that the correspondence between loop algebras and integrable equations is not unique, in the sense that the equations arising from the inverse prolongation depend on what we assume as independent variables.

**Appendix A: the spectral problem from the Kac–Moody algebra**

Let us consider the equations $y_x = -F, y_t = -G$ (see (9), (15) and (16)) with $a_1 = 0$. Combining together Eqs. (29)–(32) and Eqs. (34)–(36), we obtain

$$-y_{1,x}^{(i)} = y_2^{(i)} u + \sqrt{k} \left[ y_1^{(n_4+i)} - y_1^{(-n_4+i)} \right], \quad (117)$$

$$-y_{2,x}^{(i)} = y_1^{(i)} v - \sqrt{k} \left[ y_2^{(n_4+i)} - y_2^{(-n_4+i)} \right], \quad (118)$$

$$-y_{1,t}^{(i)} = y_2^{(i)} u_x + uv y_1^{(i)} + \sqrt{k} u \left[ -y_2^{(n_4+i)} + y_2^{(-n_4+i)} \right] - \frac{k}{2} \left[ y_1^{(2n_4+i)} + y_1^{(-2n_4+i)} \right], \quad (119)$$

$$-y_{2,t}^{(i)} = y_1^{(i)} v_x - uv y_2^{(i)} + \sqrt{k} v \left[ -y_1^{(n_4+i)} + y_1^{(-n_4+i)} \right] + \frac{k}{2} \left[ y_2^{(2n_4+i)} + y_2^{(-2n_4+i)} \right]. \quad (120)$$

Equations (117)–(120) can be cast into the matrix form

$$- \begin{pmatrix} y_1^{(i)} \\ y_2^{(i)} \end{pmatrix}_x = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \begin{pmatrix} y_1^{(i)} \\ y_2^{(i)} \end{pmatrix} + \sqrt{k} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1^{(n_4+i)} \\ y_2^{(n_4+i)} \end{pmatrix} - \begin{pmatrix} y_1^{(-n_4+i)} \\ y_2^{(-n_4+i)} \end{pmatrix}, \quad (121)$$
\[-\begin{pmatrix} y_1^{(i)} \\ y_2^{(i)} \end{pmatrix} \bigg|_t = \begin{pmatrix} uv & u_x \\ -v_x & -uv \end{pmatrix} \begin{pmatrix} y_1^{(i)} \\ y_2^{(i)} \end{pmatrix} - \]

\[-\sqrt{k} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \begin{pmatrix} y_1^{(n+i)} \\ y_2^{(n+i)} \end{pmatrix} - \begin{pmatrix} y_1^{(-n+i)} \\ y_2^{(-n+i)} \end{pmatrix} \bigg] - \]

\[-\frac{t}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1^{(2n+i)} \\ y_2^{(2n+i)} \end{pmatrix} + \begin{pmatrix} y_1^{(-2n+i)} \\ y_2^{(-2n+i)} \end{pmatrix} \bigg]. \quad (122)\]

On the other hand, by introducing the formal expansion

\[\psi(\epsilon) = \sum_{n=-\infty}^{+\infty} \epsilon^n y^{(n)} \quad (123)\]

where \(\epsilon\) is a constant,

\[\psi(\epsilon) = \begin{pmatrix} \psi_1(\epsilon) \\ \psi_2(\epsilon) \end{pmatrix}, \quad y^{(n)} = \begin{pmatrix} y_1^{(n)} \\ y_2^{(n)} \end{pmatrix} \quad (124)\]

Eqs. (121)–(122) become

\[-\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \bigg|_x = \begin{pmatrix} \lambda & u \\ v & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (125)\]

\[-\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \bigg|_t = \begin{pmatrix} uv - 2\lambda^2 - k & u_x - 2\lambda u \\ -v_x - 2\lambda v & -(uv - 2\lambda^2 - k) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (126)\]

where \(\lambda = \frac{\sqrt{k}}{2}(\epsilon^{-n_4} - \epsilon^{n_4})\) is the spectral parameter. Equations (125)–(126) reproduce just the spectral problem (27)–(28) determined starting from a finite-dimensional subalgebra of the incomplete prolongation algebra (17)–(22).

**Appendix B: the incomplete Lie algebra in terms of \(A_j\)**

From Eqs. (39) and using systematically the Jacobi identity, the incomplete Lie algebra (17)–(22) takes the form

\[[A_1, A_2] = [A_1, A_3] = [A_1, A_4] = [A_4, A_5] = [A_1, A_6] = [A_1, A_7] = [A_1, A_8] = [A_2, A_7] = [A_3, A_4] = [A_3, A_5] = 2[A_6, A_4] = 0 \quad (127)\]

\[[A_2, A_3] = -A_6, \quad [A_2, A_4] = A_7, \quad [A_3, A_4] = A_8, \quad [A_2, A_6] = 2A_2, \quad [A_3, A_6] = -2A_3 \quad (128)\]

\[[A_1, A_5] = 2[A_6, A_4] \quad (129)\]
\[ [A_2, A_3] = [A_4, A_7] - 2kA_2, \quad \text{(130)} \]
\[ [A_3, A_5] = -[A_4, A_8] + 2kA_3, \quad \text{(131)} \]
\[ [A_7, A_3] = [A_8, A_2] + [A_4, A_6], \quad \text{(132)} \]
\[ [A_6, A_7] = -2A_7, \quad [A_6, A_8] = 2A_8. \quad \text{(133)} \]

**Appendix C: derivation of Eqs. (89)–(90)**

In order to scrutinize Eqs. (74)–(88), let us consider first Eqs. (85) and (88), from which
\[ \psi_{10} = a, \quad \psi_{11} = b \quad \text{(134)} \]
where \( a, b \) are supposed to be constant.

Then, combining together Eqs. (86) and (88) we have
\[ \psi_9 = -ib \frac{S_3}{S_3} (S_2 - S_1), \quad \text{(135)} \]
\[ \psi_7 = -b \frac{S_3}{S_3} (S_1 + S_2), \quad \text{(136)} \]
with the help of (134).

Now, from Eqs. (83)–(84) we obtain (see (134))
\[ \psi_8 = \frac{ia}{S_3} (S_2 - S_1), \quad \text{(137)} \]
\[ \psi_6 = \frac{a}{S_3} (S_1 + S_2). \quad \text{(138)} \]

By virtue of (135)–(138), Eqs. (81)–(84) can be elaborated to give
\[ (a - b) \left( \frac{S_1 S_2}{S_3^2} \right)_x = 0. \quad \text{(139)} \]

From Eqs. (74)–(88), it can be shown that the only solution to Eq. (139) is \( a = b \). Furthermore, Eqs. (133)–(138) imply
\[ \psi_6 = -\psi_7, \quad \psi_8 = -\psi_9. \quad \text{(140)} \]

From Eqs. (77)–(78) and (80)–(81) we find
\[ 2(S_1 + S_2) \psi_4 - i(\psi_8 + \psi_9)_x = 0. \quad \text{(141)} \]
This constraint tells us that (see (140))
\[ \psi_4 = 0, \quad \text{(142)} \]
under the assumption $S_1 + S_2 \neq 0$. On the other hand, we deduce

\begin{align*}
\psi_6 + i\psi_8 &= 2a\frac{S_8}{S_3}, \\
\psi_6 - i\psi_8 &= 2a\frac{S_8}{S_3}, \\
\psi_7 + i\psi_9 &= -2a\frac{S_1}{S_3}, \\
\psi_7 - i\psi_9 &= -2a\frac{S_2}{S_3},
\end{align*}

(143)–(146)

from (133)–(138). Thus, Eq. (79) provides

\[ S_3 t - \psi_3 x = 0. \]  

(147)

At this stage, inserting the expressions (143)–(144) into Eqs. (77)–(78) we get

\[ \psi_3 = \frac{S_3}{S_1} \psi_1 + \frac{a}{S_1} \left( \frac{S_1}{S_3} \right)_x = \frac{S_3}{S_2} \psi_2 - \frac{a}{S_2} \left( \frac{S_2}{S_3} \right)_x. \]  

(148)

Equations (75)–(76) become (see (143)–(146))

\begin{align*}
S_1 t + S_1 \psi_5 + 2aS_1 - \psi_1 x &= 0, \\
S_2 t - S_2 \psi_5 - 2aS_2 - \psi_2 x &= 0.
\end{align*}

(149)–(150)

Coming back to Eq. (74) and exploiting Eq. (148), we have

\[ \psi_5 = -2a \frac{S_1 S_2}{S_3}, \]  

(151)

up to an arbitrary function of time. Finally, substitution from (151) into Eqs. (149)–(150) gives

\begin{align*}
S_1 t - 2a \frac{S_1 S_2}{S_3} + 2aS_1 - \psi_1 x &= 0, \\
S_2 t + 2a \frac{S_1 S_2}{S_3} - 2aS_2 - \psi_2 x &= 0.
\end{align*}

(152)–(153)

Putting in these equations $U = \frac{S_1}{S_3}$, $V = \frac{S_2}{S_3}$, and

\begin{align*}
\psi_1 &= \frac{S_4}{S_3} \psi_3 - \frac{a}{S_3} \left( \frac{S_4}{S_3} \right)_x, \\
\psi_2 &= \frac{S_4}{S_3} \psi_3 + \frac{a}{S_3} \left( \frac{S_4}{S_3} \right)_x,
\end{align*}

(154)–(155)

(see (148)), we arrive at the system (89)–(90).

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