Robust Quantum Walk Search Without Knowing the Number of Marked Vertices

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There has been a very large body of research on searching a marked vertex on a graph based on quantum walks, and Grover’s algorithm can be regarded as a quantum walk-based search algorithm on a special graph. However, the existing quantum walk-based search algorithms suffer severely from the soufflé problem which mainly means that the success probability of finding a marked vertex could shrink dramatically even to zero when the number of search steps is greater than the right one, thus heavily reducing the robustness and practicability of the algorithm. Surprisingly, while the soufflé problem of Grover’s algorithm has attracted enough attention, how to address this problem for general quantum walk-based search algorithms is missing in the literature. Here we initiate the study of overcoming the soufflé problem for quantum walk-based search algorithms by presenting a new quantum walk-based search framework that achieves robustness without sacrificing the quantum speedup. In this framework, for any adjustable parameter $\epsilon$, the quantum algorithm can find a marked vertex on an $N$-vertex complete bipartite graph with probability at least $1 - \epsilon$, whenever the number of search steps $h$ satisfies $h \geq \ln(\frac{1}{\epsilon})\sqrt{N} + 1$. Note that the algorithm need not know the exact number of marked vertices. Consequently, we obtain quantum search algorithms with stronger robustness and practicability.

I. INTRODUCTION

Quantum walks, the analogues of classical random walks in the quantum realm, were first introduced by Aharonov, Davidovich and Zagury [1] in 1993. In the last nearly thirty years, much progress about quantum walks has been made from theory to experiments. Quantum walks have become a basic tool for designing quantum algorithms to settle a series of problems such as element distinctness [2], triangle finding [3], quantum backtracking [4], and so on [5–9]. Furthermore, they are a universal model of quantum computation [10, 11]. In the aspect of experiment study, various hardware platforms have been used to demonstrate results of quantum walks, e.g., [12–15]. There are two types of quantum walks: discrete-time quantum walks [16–20] and continuous-time quantum walks [21, 22]. In this paper, we are concerned with the discrete-time model.

A central topic in quantum walk-based algorithms is to develop efficient quantum algorithms for searching a marked vertex on a graph. This idea was initially proposed by Shenvi, Kempe, and Whaley [23] in 2003 who constructed a quantum walk search algorithm on the Boolean hypercube for finding a marked item in a dataset. Later, Ambainis, Kempe and Rivosh [24] proposed search algorithms based on quantum walks on $d$-dimensional lattices ($d \geq 2$). A major breakthrough was made in 2004: Ambainis [2] obtained the optimal query complexity of the element distinctness problem by employing quantum walk search on Johnson graphs. In 2004, Szegedy [25] studied the general theory of quantum walk search algorithms from the point of view of Markov chains. In this direction, a series of work [26–29] was put forward for searching a marked state in different Markov chains using phase estimation, interpolated quantum walks and quantum fast-forwarding.

Grover’s algorithm can be regarded as a quantum walk search algorithm on a complete graph with a self-loop on every vertex [24]. As pointed out by Brassard [30], Grover’s quantum searching technique suffers from the soufflé problem [31]. As a result, if the exact number of marked items is not known in advance, then one does not know when to stop the search iteration. Even if the number is known, the success probability of the algorithm could shrink dramatically when the number of query steps is greater than the right one, as shown in Fig. 1(a). Two strategies are often used to deal with the unknown number of solutions. One method is the exponential search algorithm [32] in which the number of iterations increases slowly but exponentially. Another strategy is to employ quantum counting [33] to estimate the number of marked items. However, they are still essentially oscillatory Grover search and fail to completely solve the soufflé problem. From a search perspective, the success probability of getting a marked item should not shrink (at least not shrink dramatically) as the number of search steps increases.

In order to overcome the soufflé problem, Grover [34] proposed a fixed-point quantum search algorithm that converges monotonically to the target, i.e., avoid overcooking by always amplifying the marked items (as shown in Fig. 1(b)). Yet, a price paid for this monotonicity is that the quadratic speedup of the original quantum search is lost. In 2014, Yoder, Low and Chuang [35] presented a new quantum search algorithm that achieves both goals—the search cannot be overcooked and also achieves optimal time scaling, a quadratic speedup over classical unordered search. This algorithm does not require that the error monotonically improves, but ensures...
that the error becomes bounded by a tunable parameter $\epsilon$, as shown in Fig. 1(c). Thus, this leads to a more robust quantum search algorithm. In addition, the fixed-point quantum search was discussed from an information perspective by Cafaro [36] and from the view of analog quantum search with suitable Hamiltonians specifying time-dependent two-level quantum systems by Cafaro and Alsing [37].

Currently, various quantum walk search algorithms also suffer from the soufflé problem. To the best of our knowledge, there has been no work considering how to avoid the soufflé problem from the perspective of quantum walk search. Actually, the problem becomes more challenging than in Grover’s quantum search. There are at least three reasons for the difficulty of addressing the soufflé problem for quantum walk search algorithms. Firstly, the search space is more complicated because of the diversity of topological structure of graphs. Secondly, more operations are involved in quantum walk search. Thirdly, it is generally difficult to get an analytical expression for the success probability and to analyze the computational complexity theoretically of a quantum walk search algorithm.

A. Our contributions

This paper considers for the first time how to address the soufflé problem confronted by quantum walk search algorithms. We initiate the first step towards this direction by designing a robust quantum walk search algorithm on complete bipartite graphs (NOT complete graphs). Note that this kind of graphs was extensively studied in quantum walk search algorithms [38–41]. The robustness feature of our algorithm ensures that for an $N$-vertex complete bipartite graph with marked vertices but without knowing the number of marked vertices, if the number of search steps $h$ satisfies $h \geq \ln(\frac{2}{\sqrt{\epsilon}})\sqrt{N} + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon$ for any given $\epsilon \in (0, 1]$ (the formal statement can be found in Theorem 1). Thus, the algorithm both avoids overcooking and keeps quadratic speedup over classical ones. Also note that our algorithm need not know the number of target vertices.

In order to obtain the above result, some nontrivial technical treatments are required. (1) First, Compared to Grover’s algorithm, the coined quantum walk search framework has two subsystems and three operations. Thus, what operations should be adjusted to create a robust version is not obvious, and we show that a model with two parameterized operations is sufficient. (2) Secondly, while one needs only consider essentially a two-dimensional state space for Grover’s algorithm, higher dimensional state spaces are involved in quantum walk search and thus it is even not easy to track the state of the quantum system. Luckily, we reveal some crucial observations (especially Lemma 4) to simplify the expression of the final state. It is worth noting that these observations are specific to quantum walks and are not seen in the robust version of Grover’s algorithm [35]. We think that these technical treatments may inspire the analysis of robust quantum walk search on other general graphs.

**Theorem 1.** Given an $N$-vertex complete bipartite graph with marked vertices but without knowing the number of marked vertices, there exists a quantum walk-based algorithm such that if the number of search steps $h$ satisfies $h \geq \ln(\frac{2}{\sqrt{\epsilon}})\max(\sqrt{N_1}, \sqrt{N_2}) + 1$, then the algorithm will output a marked vertex with probability at least $1 - \epsilon$ for any given $\epsilon \in (0, 1]$, where $N_1\ (N_2)$ is the number of the left (right) vertices in the complete bipartite graph.

The relationship among the theorems and technical lemmas obtained in this paper are depicted in Fig. 2. Theorem 1 states the main result of this paper which comes from Theorems 2 and 3 which corresponds respectively to the two cases: the marked vertices are in one side and in two sides of a complete bipartite graph. Furthermore, Lemma 2 (Lemma 3) is crucial for proving Theorem 2 (Theorem 3), with proofs given in Section IV and Appendixes A, B, C and D.

![FIG. 1. The success probability of finding a marked item as a function of steps in three search models. (a) Grover-type oscillatory search. (b) Fixed-point Search. (c) Robust search.](image)

![FIG. 2. The relationship among the theorems and technical lemmas.](image)
II. PRELIMINARIES

Graph Notation. Let $G = (V, E)$ be an undirected, unweighted graph with $N = |V|$ vertices and $m = |E|$ edges. An edge between $u$ and $v$ is denoted by $(u, v)$. For $u \in V$, $deg(u) = \{v : (u, v) \in E\}$ denotes the set of neighbors of $u$, and the degree of $u$ is denoted as $d_u = |deg(u)|$. A bipartite graph is represented as $G = (V = \{V_l \cup V_r\}, E)$, where $V_l$ ($V_r$) denotes the set of vertices in the left (right) side, with $V_l \cap V_r = \emptyset$. We use $N_l$ and $N_r$ to denote the number of left and right vertices, respectively. The number of the marked vertices in the left (right) side is $n_l$ ($n_r$). A complete bipartite graph is a bipartite graph where every vertex in the left side is connected to every vertex in the right side. For example, a complete bipartite graph in Fig. 3(a) contains 6 vertices in the left side and 4 vertices in the right side.

An $N$-vertex complete bipartite graph with the marked vertices denoted by $u$ in the left, the unmarked vertices denoted by $v$ in the left and $t$ in the right, the unmarked vertices denoted by $v$ in the left and $s$ in the right.

Coined Quantum Walk. In this model, the walker’s Hilbert space associated with an $N$-vertex graph $G = (V, E)$ is $\mathbb{H}^{N^2} = \text{span}\{|uv\rangle, u, v \in V\}$, where $u$ is the position and $v$ is the coin value representing one neighbor of $u$. The evolution operator of the coined quantum walk at each step is $U_{\text{walk}} = SC$, where the flip-flop shift operator $S$ is defined as $S|uv\rangle = |vu\rangle$, and the coin operator is $C = \sum_u |u\rangle \langle u| \otimes C_u$. The Grover diffusion coin operator $C_u$ often used is $C_u = 2|s_u\rangle \langle s_u| - I$, where $|s_u\rangle = \frac{1}{\sqrt{d_u}} \sum_{v \in deg(u)} |v\rangle$.

Given the initial state $|\Psi_0\rangle$, the walker’s state after $h$ steps is $|\Psi_h\rangle = U_{\text{walk}}^h |\Psi_0\rangle$.

Quantum Walk Search. In the quantum walk search framework, to find a marked vertex in a graph, the query oracle $Q$ is given by

$$Q|uw\rangle = \begin{cases} -|uv\rangle & \text{if } u \text{ is marked}, \\ |uw\rangle & \text{if } u \text{ is not marked}. \end{cases}$$

The evolution operator corresponding to one step of the quantum walk search is $U = SCQ$.

Given the initial state $|\Psi_0\rangle$, the walker’s state after $t$ steps is $|\Psi_t\rangle = U^t |\Psi_0\rangle$. Finally, the first register is measured and the measurement result is output.

Chebyshev polynomial. The Chebyshev polynomials of the first kind $T_n(x)$ are defined by initial values $T_0(x) = 1$, $T_1(x) = x$, and for an integer $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

The trigonometric identity $T_n(x) = \cos(n \arccos(x))$ is well known.

A result of one Quasi-Chebyshev polynomial implied in [35] is stated in the following lemma.

Lemma 1. Let $x = \cos(\theta)$ for $\theta \in [0, 2\pi]$. Let $h \geq 3$ be an odd integer. One Quasi-Chebyshev polynomial $a_h(x)$ is defined by initial values $a_0(x) = 1$, $a_1(x) = x$, and for $k = 2, \ldots, h$,

$$a_k(x) = x(1 + e^{-i(\zeta_k - \zeta_{k-1})})a_{k-1}(x) - e^{-i(\zeta_k - \zeta_{k-1})}a_{k-2}(x).$$

When $\zeta_{k+1} - \zeta_k = (-1)^{k-2}\pi - 2\arccot \left(\tan(\pi k/h)\sqrt{1 - \gamma^2}\right)$ for $k = 1, \ldots, h - 1$, where $\gamma = \frac{1}{\cos(\frac{\theta}{2}\arccos(\frac{x}{\gamma}))}$ with $\epsilon \in (0, 1]$, we have $a_h(x) = \frac{T_h(x/\gamma)}{T_h(1/\gamma)}$ with $T_h(1/\gamma) = 1/\sqrt{\epsilon}$.

III. ROBUST QUANTUM WALK SEARCH ON COMPLETE BIPARTITE GRAPHS

As mentioned before, the already existing quantum walk search algorithms suffer from the soufflé problem. Thus, this work devotes to addressing this problem by considering the case of searching a marked vertex in a complete bipartite graph. For that, first the coin operator $C$ and the query oracle $Q$ have to be adjusted, but the flip-flop shift operator $S$ can remain unchanged. The new evolution operator of one search step is

$$U(\alpha, \beta) = SC(\alpha)Q(\beta),$$

where the coin operator $C$ is changed to

$$C(\alpha) = \sum_u |u\rangle \langle u| \otimes \left[ (1 - e^{-i\alpha}) |s_u\rangle \langle s_u| - I \right],$$

and the query oracle $Q$ is replace by

$$Q(\beta)|uv\rangle = \begin{cases} e^{i\beta} |uv\rangle & \text{if } u \text{ is marked}, \\ |uv\rangle & \text{if } u \text{ is not marked}. \end{cases}$$

When $\alpha = \beta = \pm \pi$, this model becomes the original quantum walk search [24, 39].

The algorithm of search on a complete bipartite graph is given in Algorithm 1. In the input phase, according to the information of marked vertices and a tunable parameter $\epsilon$, the number of search steps $h$ is required to satisfy
In the second case, by Theorem 1, we have

\[
\text{Proof of Theorem 1.} \quad \text{According to the arrangement of marked vertices, two cases are discussed: they are in one side and two sides, as shown in Fig. 3(b) and Fig. 3(c), respectively.}
\]

(i) In the first case, without loss of generality, suppose that all the marked vertices lie in the left side \( V_l \). Then by Theorem 2, if \( h \geq \ln\left(\frac{\sqrt{N}}{\sqrt{N_l}}\right) + 1 \), then Algorithm 1 will output a marked vertex with probability at least \( 1 - \epsilon \).

(ii) In the second case, by Theorem 3, if \( h \geq \ln\left(\frac{\sqrt{N}}{\sqrt{N_l}}\right) + \ln\left(\sqrt{N_l} + \sqrt{N_r}\right) + 1 \), then Algorithm 1 will output a marked vertex with probability at least \( 1 - \epsilon^2 \).

Note that the parameters \( \alpha_i, \beta_i \) will be assigned with the same values in the two cases (this can be seen from the proof of Theorems 2 and 3). Thus, we need not know the arrangement of marked vertices. In addition, in the two cases, both \( \sqrt{N_l} \) and \( \max\left(\sqrt{N_l}, \sqrt{N_r}\right) \) have the same upper bound \( \max(\sqrt{N_l}, \sqrt{N_r}) \). Therefore, if \( h \geq \ln\left(\frac{\sqrt{N}}{\sqrt{N_l}}\right) + \ln\left(\sqrt{N_l} + \sqrt{N_r}\right) + 1 \), then Algorithm 1 will output a marked vertex with probability at least \( 1 - \epsilon^2 \). This completes the proof.

\[\boxed{\text{Algorithm 1 Robust quantum walk search}}\]

\textbf{Inputs:} an \( N \)-vertex complete bipartite graph with marked vertices, \( \epsilon \in (0, 1] \), and the number of search steps \( h \).

\textbf{Outputs:} a marked vertex \( v_0 \) (if \( h \) satisfies Eq. (2), it outputs a marked vertex with probability at least \( 1 - \epsilon \)).

\textbf{Procedure:}

1. Prepare the initial state \( |\Psi_0\rangle = \frac{1}{\sqrt{2N_lN_r}} \sum_u |u\rangle \otimes \sum_v \delta_{deg(u)} |v\rangle \).
2. Apply \( U(\alpha_1, \beta_1), \ldots, U(\alpha_h, \beta_h) \) in turn, where the parameters \( \alpha_i, \beta_i \) are determined by Eqs. (3) and (4).
3. Measure the first register in the computational basis. If the result is not marked, then the second register is measured. Output the measurement result.

In the first step, the initial state \( |\Psi_0\rangle = \frac{1}{\sqrt{2N_lN_r}} \sum_u |u\rangle \otimes \sum_v \delta_{deg(u)} |v\rangle \) is prepared. Then, \( U(\alpha_1, \beta_1), \ldots, U(\alpha_h, \beta_h) \) with appropriate parameters are applied to \( |\Psi_0\rangle \) in turn. The whole operator that performs \( h \) steps is \( \Gamma_h = U(\alpha_h, \beta_h) \cdots U(\alpha_1, \beta_1) \). The walker’s state after \( h \) steps is \( |\Psi_h\rangle = \Gamma_h |\Psi_0\rangle \). Finally, the two registers are measured. Note that in the previous work generally only the first register is measured, whereas here we measure the two registers. This will double the success probability for our problems as shown later. The success probability of getting a marked vertex is

\[ P_h = \sum_{u \text{ or } v \text{ is marked}} |\langle uv | \Gamma_h |\Psi_0\rangle|^2. \]

Fig. 4 illustrates the success probability of finding a marked item as a function of steps in Algorithm 1 and the one with \( \alpha = \beta = \pm \pi \) in Eq. (1). They show that Algorithm 1 is a robust search model.
Proof. According to Lemma 2, when $h$ is odd, in order to ensure $P_h \geq 1 - \epsilon$, it suffices to satisfy $|\cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon^2}}\right)\right)| \leq 1$, that is,

$$\frac{n_l}{N_l} \geq 1 - \cos^{-2}\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right). \quad (5)$$

Note that the following functions in (see for instance Ref.[42]) will be used:

$$\arccos(z) = \frac{1}{i} \ln(z + \sqrt{z^2 - 1}), \tan(i) = i \tanh(z),$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

where $\ln(\cdot)$ is the natural logarithm function, $i$ denotes the imaginary number, $x$ is a real number and $z$ is a complex number. Now we have

$$1 - \cos^{-2}\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right)$$

$$= - \tan^2\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\epsilon}}\right)\right)$$

$$= - \tan^2\left(\frac{1}{h} \ln\left(\frac{1}{\sqrt{\epsilon}} + \sqrt{\left(\frac{1}{\sqrt{\epsilon}}\right)^2 - 1}\right)\right)$$

$$= - \tan^2\left(\frac{1}{h} \ln\left(\frac{1}{\sqrt{\epsilon}} + \sqrt{\left(\frac{1}{\sqrt{\epsilon}}\right)^2 - 1}\right)\right)$$

$$= \tan^2\left(\frac{1}{h} \ln\left(\frac{1}{\sqrt{\epsilon}} + \sqrt{\left(\frac{1}{\sqrt{\epsilon}}\right)^2 - 1}\right)\right)$$

$$< \tan^2\left(\frac{1}{h} \ln\left(\frac{2}{\sqrt{\epsilon}}\right)\right)$$

$$< \left(\frac{\ln(2/\sqrt{\epsilon})}{h}\right)^2.$$
where the last inequality follows from $x \geq \tanh(x)$ for $x \geq 0$. Thus, in order to ensure the inequality (5), it suffices to set $\frac{n_l}{N_l} \geq \left( \frac{\ln(2) \sqrt{\frac{1}{h^2 \varepsilon}}}{N_l} \right)^2$, which leads to $h \geq \ln(\frac{\sqrt{2}}{\varepsilon}) \sqrt{\frac{N_l}{n_l}}$.

Similarly, when $h$ is even, in order to ensure $P_h \geq 1 - \epsilon$, it suffices to satisfy $|\cos\left(\frac{1}{h+1} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right)| \leq 1$, and $|\cos\left(\frac{1}{h+1} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right)| \leq 1$, which implies $h + 1 \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$ and $h - 1 \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$, respectively.

Thus, no matter $h$ is odd or even, $P_h \geq 1 - \epsilon$ holds for $h \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}} + 1$. This completes the proof of Theorem 2.

**Theorem 3.** In Algorithm 1, suppose that the marked vertices are in both the left and right sides. There exists a sequence of parameters $\alpha_i, \beta_i$, such that the success probability satisfies

\[ P_h = 1 - \epsilon^2 T^2_h \left(\cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right) \sqrt{1 - \frac{n_l}{N_l}}\right) \]

for odd $h$, and

\[ P_h = 1 - \epsilon^2 \left[ T^2_{h+1} \left(\cos\left(\frac{1}{h+1} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right) \sqrt{1 - \frac{n_l}{N_l}}\right) \right. \]

\[ \left. \times T^2_{h-1} \left(\cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right) \sqrt{1 - \frac{n_l}{N_l}}\right) \right] \]

for even $h$.

**Proof.** Similar to the proof of Theorem 2, by Lemma 3 one can show that

- When $h$ is odd, in order to ensure $P_h \geq 1 - \epsilon^2$, it suffices to satisfy $h \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$ and $h \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$.

- When $h$ is even, in order to ensure $P_h \geq 1 - \epsilon^2$, it suffices to satisfy $h + 1 \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$, $h - 1 \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$, and $h \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \sqrt{\frac{N_l}{n_l}}$.

Therefore, no matter $h$ is even or odd, $P_h \geq 1 - \epsilon^2$ holds for $h \geq \ln\left(\frac{2}{\sqrt{\varepsilon}}\right) \max\left(\sqrt{\frac{N_l}{n_l}}, \sqrt{\frac{N_r}{n_r}}\right) + 1$. This completes the proof of Theorem 3.

**Lemma 2.** In Algorithm 1, suppose that all the marked vertices are in the left side. There exists a sequence of parameters $\alpha_i, \beta_i$, such that the success probability satisfies

\[ P_h = 1 - \epsilon T^2_h \left(\cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right) \sqrt{1 - \frac{n_l}{N_l}}\right) \]

for odd $h$, and

\[ P_h = 1 - \epsilon^2 \left[ T^2_{h+1} \left(\cos\left(\frac{1}{h} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right) \sqrt{1 - \frac{n_l}{N_l}}\right) \right. \]

\[ \left. \times T^2_{h-1} \left(\cos\left(\frac{1}{h+1} \arccos\left(\frac{1}{\sqrt{\varepsilon}}\right)\right) \sqrt{1 - \frac{n_l}{N_l}}\right) \right] \]

for even $h$.

**IV. METHOD**

The section is devoted to the proof of Lemma 2. As shown in Fig. 3(b), vertices can be classified into three types: the marked vertices denoted by $u$ in the left, the unmarked vertices denoted by $v$ in the left and $s$ in the right. Therefore, our analysis can be simplified in a four-dimensional subspace with the orthogonal basis $\{|us\}, |su\rangle, |sv\rangle, |vs\rangle\}$ given below:

\[ |us\rangle = \frac{1}{\sqrt{n_l}} \sum_u |u\rangle \otimes \frac{1}{\sqrt{N_l - n_l}} \sum_s |s\rangle, \]

\[ |sv\rangle = \frac{1}{\sqrt{N_r}} \sum_s |s\rangle \otimes \frac{1}{\sqrt{N_l - n_l}} \sum_v |v\rangle, \]

\[ |su\rangle = \frac{1}{\sqrt{n_l}} \sum_s |s\rangle \otimes \frac{1}{\sqrt{n_l}} \sum_u |u\rangle, \]

\[ |vs\rangle = \frac{1}{\sqrt{N_l - n_l}} \sum_v |v\rangle \otimes \frac{1}{\sqrt{N_r}} \sum_s |s\rangle. \]

Note that $|\Psi_0\rangle$ can be rewritten in the above basis as $\Psi_0 = \frac{1}{\sqrt{2N_l N_r}} \left[ \sqrt{n_l N_r} |us\rangle + \sqrt{n_l N_r} |su\rangle + \sqrt{N_r (N_l - n_l)} |sv\rangle + \sqrt{N_r (N_l - n_l)} |vs\rangle \right]$. Hence, it can be expressed as a 4-dimensional vector $|\Psi_0\rangle = \frac{1}{\sqrt{2N_l N_r}} \left( \sqrt{n_l N_r} \sqrt{N_r (N_l - n_l)} \right).$
Furthermore, we have
\[
S = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad Q(\beta) = \begin{pmatrix}
ie^{i\beta} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
and
\[
C(\alpha) = \begin{pmatrix}
-e^{-i\alpha} & 0 & 0 & 0 \\
0 & C_{22} & (1-e^{-i\alpha}) \sin(\omega) & 0 \\
0 & (1-e^{-i\alpha}) \sin(\omega) & C_{33} & 0 \\
0 & 0 & 0 & -e^{-i\alpha}
\end{pmatrix}
\]
with \( \omega = \arccos(1 - \frac{2m}{N_1}) \), where \( C_{23} = \frac{(1-e^{-i\alpha})(1-\cos(\omega))}{2} - 1 \), \( C_{33} = \frac{(1-e^{-i\alpha})(1+\cos(\omega))}{2} - 1 \), and \( \cos(\omega) = 1 - \frac{2m}{N_1} \), \( \sin(\omega) = \frac{2}{\sqrt{N_1}} \sqrt{n_1 \ast (N_1 - n_1)} \).

Now some key results are given as follows: Let
\[
R(\theta) = -\begin{pmatrix}
ie^{-\frac{i\theta}{2}} & 0 & 0 & 0 \\
0 & e^{\frac{i\theta}{2}} & 0 & 0 \\
0 & 0 & e^{-\frac{i\theta}{2}} & 0 \\
0 & 0 & 0 & e^{-\frac{i\theta}{2}}
\end{pmatrix},
\]
and
\[
A(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\frac{\pi}{2}) & -ie^{i\theta} \sin(\frac{\pi}{2}) & 0 \\
0 & -ie^{-i\theta} \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
with \( \omega = \arccos(1 - \frac{2m}{N_1}) \). One can verify the following identities:
\[
C(\alpha) = e^{-\frac{i\alpha}{2}} A(\frac{\pi}{2}) R(\alpha) A(-\frac{\pi}{2}),
\]
\[
Q(\beta) S = -e^{\frac{i\beta}{2}} S R(\beta),
\]
\[
A(\alpha + \beta) = R(\beta) A(\alpha) R(-\beta),
\]
\[
R(\theta) R(-\theta) = I,
\]
\[
|\Psi_0\rangle = A(\frac{\pi}{2}) S A(\frac{\pi}{2}) |\bar{0}\rangle,
\]
where \( |\bar{0}\rangle \) denotes \((0, 0, 0, 1, 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T \). Another crucial observation is the following lemma, which will be useful later:

**Lemma 4.**
\[
SB_1 SB_2 S = B_2 SB_1,
\]
where \( B_1 = \prod_{i=0}^{m} D_i \) and \( B_2 = \prod_{i=0}^{m} D_i \) for \( D_i \in \{ A(\theta_i), R(\theta_i) \} \).

**Proof.** By calculation, the form of \( B_1 \) and \( B_2 \) is as follows:
\[
B_1 = \begin{pmatrix}
1 & 0 & B_1(22) & B_1(23) & 0 \\
0 & 0 & B_1(32) & B_1(33) & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & B_2(22) & B_2(23) & 0 \\
0 & 0 & B_2(32) & B_2(33) & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]
and
\[
B_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & B_2(22) & B_2(23) & 0 & 0 \\
0 & B_2(32) & B_2(33) & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]
where \( B_1(\cdot) \) and \( B_2(\cdot) \) are mathematical expressions of \( \theta \) and \( \omega \). Hence, we have \( SB_1 SB_2 S = B_2 SB_1 \).

Below we will prove Lemma 2 by two cases.

**Case 1: \( \eta \) is an odd integer.** First we set
\[
\alpha_k = \begin{cases}
-\beta_{h+2-k} & k = 2, 4, \ldots, h-1, \\
-\beta_{h-k} & k = 3, 5, \ldots, h.
\end{cases}
\]

Then \( |\Psi_h\rangle \) reduces to
\[
|\Psi_h\rangle \sim S \left[A(\eta_h) \ldots A(\eta_1)\right] R(\alpha_1) S R(\beta_h) [A(\zeta_h) \ldots A(\zeta_1)] |\bar{0}\rangle,
\]
which will be proven in Appendix C by using Eqs. (6)-(11). Here \( \eta_k = \eta_{h+1-k} \) and \( \zeta_k = \zeta_{h+1-k} \) for \( k = 1, 2, \ldots, h \), and
\[
\begin{align}
\eta_{k+1} - \eta_k &= \begin{cases}
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h-1, \\
-\pi + \alpha_{h-k+1} & k = 3, 5, \ldots, h,
\end{cases} \\
\zeta_{k+1} - \zeta_k &= \begin{cases}
\pi - \alpha_k & k = 2, 4, \ldots, h-1, \\
-\pi + \alpha_{h-k} & k = 3, 5, \ldots, h.
\end{cases}
\end{align}
\]

Let us have a more detailed analysis at the state evolution in Eq. (13), which can be divided into four stages as follows.

Stage \( \text{(1)} \): apply \( A(\zeta_h) \ldots A(\zeta_1) \) to the initial state. Let \( (a_0, b_0, c_0, d_0) = (0, 0, 1, 1) \) and
\[
|\mu_k\rangle = \begin{pmatrix}
a_k \\
b_k \\
c_k \\
d_k
\end{pmatrix} = A(\zeta_k) \ldots A(\zeta_1) \begin{pmatrix}
0 \\
0 \\
1 \\
1
\end{pmatrix}.
\]
for \( k = 1, 2, \ldots, h \). First note that \( A(\zeta) \) makes effect only to the 2th and 3th dimension of a 4-dimensional vector. Thus, \( a_k = 0 \) and \( d_k = 1 \) for all \( k \). Furthermore, we have

\[
|\mu_k\rangle = A(\zeta_k) |\mu_{k-1}\rangle
\]

(16)

\[
\begin{pmatrix}
0 \\
b_{k-1} \cos(\frac{\gamma}{2}) - ic_{k-1} e^{i\zeta_k} \sin(\frac{\gamma}{2}) \\
ib_{k-1} e^{-i\zeta_k} \sin(\frac{\gamma}{2}) + c_{k-1} \cos(\frac{\gamma}{2})
\end{pmatrix}
\]

, and

\[
|\mu_{k-2}\rangle = A(\zeta_{k-1})^{-1} |\mu_{k-1}\rangle
\]

(17)

\[
\begin{pmatrix}
0 \\
b_{k-1} \cos(\frac{\gamma}{2}) + ic_{k-1} e^{i\zeta_{k-1}} \sin(\frac{\gamma}{2}) \\
ib_{k-1} e^{-i\zeta_{k-1}} \sin(\frac{\gamma}{2}) + c_{k-1} \cos(\frac{\gamma}{2})
\end{pmatrix}
\]

Combined with Eqs. (16) and (17), we have

\[
c_k = -ib_{k-1} e^{-i\zeta_k} \sin(\frac{\gamma}{2}) + c_{k-1} \cos(\frac{\gamma}{2}),
\]

\[
c_{k-2} = ib_{k-1} e^{-i\zeta_{k-1}} \sin(\frac{\gamma}{2}) + c_{k-1} \cos(\frac{\gamma}{2})
\]

The recurrence formula of \( c_k(x) \) is defined by \( c_0(x) = 1, c_1(x) = x \), and for \( k = 2, \ldots, h \),

\[
c_k(x) = x(1 + e^{-i(\zeta_{k-1} - \zeta_k)}) c_{k-1}(x) - e^{-i(\zeta_{k-1} - \zeta_k)} c_{k-2}(x),
\]

with \( x = \cos(\frac{\gamma}{2}) \). By Lemma 1, when

\[
\zeta_{k+1} - \zeta_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h}) \sqrt{1 - \frac{\gamma}{2}})
\]

(18)

for \( k = 1, \ldots, h - 1 \), where \( \gamma^{-1} = \cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{2}})) \), we have

\[
c_h(x) = \frac{T_h(x)}{T_h(\frac{\pi}{2})}
\]

with \( T_h(\frac{\pi}{2}) = \frac{1}{\sqrt{2h}} \). Moreover, \( b_h(x) \) is determined by \( |b_h(x)|^2 + |c_h(x)|^2 = 1 \). Therefore, the state after \( A(\zeta_h) \ldots A(\zeta_1) \) applied to the initial state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} b_h(x) \\ c_h(x) \end{pmatrix}
\]

Stage (2): apply \( R(\alpha_1)SR(\beta_h) \) to the above state. After that, the state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_h} b_h(x) \\ c_h(x) \end{pmatrix}
\]

Stage (3): perform \( A(\eta_h) \ldots A(\eta_1) \). Let

\[
\begin{pmatrix} \tilde{a} _k \\ \tilde{b}_k \\ \tilde{c}_k \\ \tilde{d}_k \end{pmatrix} = A(\eta_k) \ldots A(\eta_1) \begin{pmatrix} e^{i\beta_h} b_h(x) \\ e^{i\beta_h} c_h(x) \\ 1 \\ c_h(x) \end{pmatrix}
\]

for \( k = 1, 2, \ldots, h \). By the property of the matrix \( A(\eta_k) \), we have \( \tilde{a}_k = e^{i\beta_h} b_h(x) \) and \( \tilde{d}_k = c_h(x) \) for all \( k \). The recurrence formula of \( \tilde{c}_k(x) \) is defined by \( \tilde{c}_0(x) = 1, \tilde{c}_1(x) = x \) and for \( k = 2, \ldots, h \),

\[
\tilde{c}_k(x) = x(1 + e^{-i(\eta_k - \eta_{k-1})}) \tilde{c}_{k-1}(x) - e^{-i(\eta_k - \eta_{k-1})} \tilde{c}_{k-2}(x),
\]

with \( x = \cos(\frac{\gamma}{2}) \). By Lemma 1, when

\[
\eta_{k+1} - \eta_k = (-1)^k \pi - 2\cot^{-1}(\tan(\frac{k\pi}{h}) \sqrt{1 - \frac{\gamma}{2}})
\]

(19)

for \( k = 1, \ldots, h - 1 \), where \( \gamma^{-1} = \cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{2}})) \), we have

\[
\tilde{c}_h(x) = \frac{T_h(x)}{T_h(\frac{\pi}{2})}
\]

with \( T_h(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \). Moreover, \( \tilde{b}_h(x) \) is determined by \( |\tilde{b}_h(x)|^2 + |\tilde{c}_h(x)|^2 = 1 \). Hence, the result state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_h} b_h(x) \\ b_h(x) \\ c_h(x) \\ \tilde{c}_h(x) \end{pmatrix}
\]

Stage (4): perform the final operation \( S \). The final state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{b}_h(x) \\ e^{i\beta_h} b_h(x) \\ e^{i\beta_h} c_h(x) \\ \tilde{c}_h(x) \end{pmatrix}
\]

Therefore, the success probability \( P_h \) is

\[
P_h = 1 - \frac{1}{2} ( |c_h(x)|^2 + |\tilde{c}_h(x)|^2 ) = 1 - \gamma T_h^2(\frac{\pi}{\gamma})
\]

\[
= 1 - \gamma T_h^2(\cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{2}})) \sqrt{1 - \frac{m_l}{N_l}}).
\]

By Eqs. (12), (14), (15), (18) and (19), \( \alpha_k, \beta_k \) can be chosen such that

\[
\alpha_k = -\beta_{h+2-k} = \pi + (\eta_{k+1} - \eta_k)
\]

\[
= 2\arccot(\tan(\frac{k\pi}{h}) \sqrt{1 - \frac{\gamma}{2}})
\]

for \( k = 2, 4, \ldots, h - 1 \), and

\[
\alpha_k = -\beta_{h-k} = \pi - (\zeta_{k+1} - \zeta_k)
\]

\[
= 2\arccot(\tan(\frac{(k-1)\pi}{h}) \sqrt{1 - \gamma^2})
\]

for \( k = 3, 5, \ldots, h \). In addition, \( \alpha_1 \) and \( \beta_h \) can be any value.

Case 2: \( h \) is an even integer. The proof is given in Appendix A.
V. CONCLUSION & OUTLOOK

In this paper, we investigated how to overcome the soufflé problem of quantum walk search. We presented a robust quantum walk-based algorithm for searching a marked vertex on a complete bipartite graph. The algorithm need not know any prior information about the marked vertices (e.g., the number of marked vertices), but keeps a quadratic speedup over classical search algorithms and ensures that the error is bounded by a tunable parameter $\epsilon$.

We have just initiated the first step towards robust quantum walk search. More questions are worthy of further consideration. For example, several interesting questions are listed below:

- Can the robustness feature be introduced into quantum walk search on other graphs?
- For the framework of searching a marked state in Markov chains, can we propose a robust version?
- Another interesting direction would be to explore some important problems in practical scenarios by robust quantum walk search algorithms.

We will try to address these questions in forthcoming works.

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First we set
\[
\beta_k = -\alpha_{h+1-k} \quad k = 1, 2, \ldots, h - 1.
\] (A1)

Then \( |\Psi_h \rangle \) reduces to
\[
|\Psi_h \rangle \sim R(\beta_h) [A(\phi_{h-1}) \ldots A(\phi_1)] R(\alpha_1) S [A(\psi_{h+1}) \ldots A(\psi_1)] |\vec{0} \rangle,
\] (A2)

which will be proven in Appendix C by using Eqs. (6)-(11). Here \( \phi_k = \phi_{h-k} \) for \( k = 1, 2, \ldots, h-1 \), \( \psi_k = \psi_{h+2-k} \) for \( k = 1, 2, \ldots, h+1 \), and
\[
\phi_{k+1} - \phi_k = \begin{cases} 
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k} & k = 1, 3, \ldots, h - 3,
\end{cases}
\] (A3)
\[
\psi_{k+1} - \psi_k = \begin{cases} 
\pi - \alpha_k & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k+1} & k = 1, 3, \ldots, h.
\end{cases}
\] (A4)

Similar to Eq. (13), the final state of Eq. (A2) can be obtained by the following four stages:
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} A(\psi_{h+1}) \ldots A(\psi_1) \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} b_{h+1}(x) \\ b_{h+1}(x) \\ e_{h+1}(x) \\ e_{h+1}(x) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} R(\alpha_1)S \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} b_{h+1}(x) \\ b_{h+1}(x) \\ e_{h+1}(x) \\ e_{h+1}(x) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\]

### Stage (1): apply \( A(\psi_{h+1}) \ldots A(\psi_1) \) to the initial state. Let \((a_0, b_0, c_0, d_0) = (0, 0, 1, 1)\) and
\[
\begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \end{pmatrix} = A(\psi_k) \ldots A(\psi_1) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
\]

for \( k = 1, 2, \ldots, h + 1 \). By the property of matrix \( A(\psi_i) \), we have \( a_k = 0 \) and \( d_k = 1 \) for all \( k \). The recurrence formula of \( c_k(x) \) is defined by \( c_0(x) = 1, c_1(x) = x \) and for \( k = 2, \ldots, h + 1 \),
\[
c_k(x) = x(1 + e^{-i(\psi_k - \psi_{k-1})})c_{k-1}(x) - e^{-i(\psi_k - \psi_{k-1})}c_{k-2}(x),
\]
with \( x = \cos(\frac{\pi}{2}) \). By Lemma 1, when
\[
\psi_{k+1} - \psi_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h+1})\sqrt{1 - \gamma^{-2}})
\] (A5)

with \( \gamma^{-1} = \cos(\frac{1}{h+1}\arccos(\frac{1}{x})) \) for \( k = 1, 2, \ldots, h \), we have
\[
c_{h+1}(x) = \frac{T_{h+1}(\frac{x}{\gamma})}{T_{h+1}(\frac{1}{\gamma})},
\]

with \( T_{h+1}(\frac{1}{\gamma}) = \frac{1}{\sqrt{2}} \). Moreover, \( b_{h+1}(x) \) is determined by \( |b_{h+1}(x)|^2 + |c_{h+1}(x)|^2 = 1 \). Therefore, the state after \( A(\psi_{h+1}) \ldots A(\psi_1) \) applied to the initial state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} b_{h+1}(x) \\ c_{h+1}(x) \end{pmatrix}.
\]
Stage (2): apply $R(\alpha_1)S$ to the above state. After that, the state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
    b_{h+1}(x) \\
    0 \\
    1 \\
    c_{h+1}(x)
\end{pmatrix}.
\]

Stage (3): perform $A(\phi_{h-1})...A(\phi_1)$. Let
\[
\begin{pmatrix}
    \bar{a}_k \\
    \bar{b}_k \\
    \bar{c}_k \\
    \bar{d}_k
\end{pmatrix} = A(\phi_k)...A(\phi_1) \begin{pmatrix}
    b_{h+1}(x) \\
    0 \\
    1 \\
    c_{h+1}(x)
\end{pmatrix}
\]
for $k = 1, 2, \ldots, h - 1$. By the property of matrix $A(\phi_k)$, we have $\bar{a}_k = b_{h+1}(x)$ and $\bar{d}_k = c_{h+1}(x)$ for all $k$. The recurrence formula of $\bar{c}_k(x)$ is defined by $\bar{c}_0(x) = 1, \bar{c}_1(x) = x$ and for $k = 2, \ldots, h - 1$,
\[
\bar{c}_k(x) = x(1 + e^{-i(\phi_{k-1})})\bar{c}_{k-1}(x) - e^{-i(\phi_{k-1})}\bar{c}_{k-2}(x)
\]
with $x = \cos(\frac{\pi}{2})$. By Lemma 1, when
\[
\phi_{k+1} - \phi_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h-1} )) \sqrt{1 - \gamma_2}
\]
with $\gamma_2^{-1} = \cos(\frac{1}{h-1} \arccos(\frac{\gamma_2}{h}))$ for $k = 1, 2, \ldots, h - 2$, we have
\[
\bar{c}_{h-1}(x) = \frac{T_{h-1}(\frac{x}{\gamma_2})}{T_{h-1}(\frac{1}{\gamma_2})},
\]
with $T_{h-1}(\frac{1}{\gamma_2}) = \frac{1}{\sqrt{x}}$. Moreover, $\bar{b}_{h-1}(x)$ is determined by $|\bar{b}_{h-1}(x)|^2 + |\bar{c}_{h-1}(x)|^2 = 1$. Hence, the result state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
    b_{h+1}(x) \\
    \bar{b}_{h-1}(x) \\
    \bar{c}_{h-1}(x) \\
    c_{h+1}(x)
\end{pmatrix}.
\]

Stage (4): perform the final operation $R(\beta_h)$. The final state is
\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
    b_{h+1}(x) \\
    e^{i\beta_h} \bar{b}_{h-1}(x) \\
    \bar{c}_{h-1}(x) \\
    c_{h+1}(x)
\end{pmatrix}.
\]

Therefore, the success probability $P_h$ is
\[
P_h = 1 - \frac{1}{2} |(\bar{c}_{h-1}(x)|^2 + |c_{h-1}(x)|^2)
= 1 - \frac{c}{2} \left( T_{h+1}(\frac{x}{\gamma_2}) + T_{h-1}(\frac{x}{\gamma_2}) \right)
= 1 - \frac{c}{2} \left[ T_{h+1}(\cos(\frac{1}{h+1} \arccos(\frac{1}{\sqrt{c}}))) \sqrt{1 - \frac{m_l}{N_l}} + T_{h-1}(\cos(\frac{1}{h-1} \arccos(\frac{1}{\sqrt{c}}))) \sqrt{1 - \frac{m_l}{N_l}} \right].
\]

By Eq. (A1) and Eqs. (A3)-(A6), $\alpha_k, \beta_k$ can be chosen such that
\[
\alpha_k = -\beta_{h+1-k} = \pi - (\phi_k - \phi_{k-1}) = 2\arccot(\tan(\frac{k\pi}{h+1} )) \sqrt{1 - \gamma_1^2})
\]
for $k = 2, 4, \ldots, h$, and
\[
\alpha_k = -\beta_{h+1-k} = \pi - (\psi_{k+1} - \psi_k) = 2\arccot(\tan(\frac{(k-1)\pi}{h-1} )) \sqrt{1 - \gamma_2^2})
\]
for $k = 3, 5, \ldots, h - 1$. In addition, $\alpha_1$ and $\beta_h$ can be any value.
Appendix B: Proof of Lemma 3: marked vertices in two sides

The section is devoted to the proof of Lemma 3.

Proof. Vertices in Fig. 3(c) can be divided into four types: the marked vertices denoted by \( u \) in the left and \( t \) in the right, the unmarked vertices denoted by \( v \) in the left and \( s \) in the right. Hence, our analysis can be simplified in an 8-dimensional subspace defined by the following orthogonal basis \( \{|u\rangle, |u_s\rangle, |tu\rangle, |tv\rangle, |vt\rangle, |vs\rangle, |su\rangle, |sv\rangle\} \):

\[
|ut\rangle = \frac{1}{\sqrt{n_l}} \sum_u |u\rangle \otimes \frac{1}{\sqrt{n_r}} \sum_t |t\rangle, \quad |us\rangle = \frac{1}{\sqrt{n_l}} \sum_u |u\rangle \otimes \frac{1}{\sqrt{N_r-n_r}} \sum_s |s\rangle, \\
|tu\rangle = \frac{1}{\sqrt{n_r}} \sum_t |t\rangle \otimes \frac{1}{\sqrt{n_l}} \sum_u |u\rangle, \quad |tv\rangle = \frac{1}{\sqrt{n_r}} \sum_t |t\rangle \otimes \frac{1}{\sqrt{N_l-n_l}} \sum_v |v\rangle, \\
|vt\rangle = \frac{1}{\sqrt{N_l-n_l}} \sum_v |v\rangle \otimes \frac{1}{\sqrt{n_r}} \sum_t |t\rangle, \\
|vs\rangle = \frac{1}{\sqrt{N_r-n_r}} \sum_v |v\rangle \otimes \frac{1}{\sqrt{n_l}} \sum_u |u\rangle, \\
|su\rangle = \frac{1}{\sqrt{n_l}} \sum_u |u\rangle \otimes \frac{1}{\sqrt{N_r-n_r}} \sum_s |s\rangle, \\
|sv\rangle = \frac{1}{\sqrt{N_r-n_r}} \sum_s |s\rangle \otimes \frac{1}{\sqrt{N_l-n_l}} \sum_v |v\rangle.
\]

Then \( |\Psi_0\rangle \) can be rewritten in the above basis as

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2N_lN_r}} \left[ \sqrt{n_ln_r} |ut\rangle + \sqrt{n_l(N_r-n_r)} |us\rangle + \sqrt{n_rn_l} |tu\rangle + \sqrt{n_r(N_l-n_l)} |tv\rangle + \sqrt{(N_l-n_l)(N_r-n_r)} |vs\rangle + \sqrt{n_l(N_r-n_r)} |su\rangle + \sqrt{(N_l-n_l)(N_r-n_r)} |sv\rangle \right].
\]

Thus, \( |\Psi_0\rangle \) can be expressed as an 8-dimensional vector

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2N_lN_r}} \begin{pmatrix}
\sqrt{n_ln_r} \\
n_l(N_r-n_r) \\
n_rn_l \\
\sqrt{(N_l-n_l)(N_r-n_r)} \\
n_l(N_r-n_r) \\
\sqrt{n_ln_r} \\
n_r(N_l-n_l) \\
\sqrt{(N_l-n_l)(N_r-n_r)}
\end{pmatrix}.
\]

Furthermore, we have

\[
S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad Q(\beta) = \begin{pmatrix}
e^{i\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{i\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{i\beta} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{i\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{i\beta} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{i\beta} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{i\beta} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{i\beta} & 0
\end{pmatrix},
\]

and

\[
C(\alpha) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\
0 & 0 & M_2 & M_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_4 & M_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_5 & M_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_6 & M_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & M_7 & 0
\end{pmatrix}.
\]
with
\[ M_1 = \frac{(1 - e^{-i\alpha})(1 - \cos(\omega_2))}{2} - 1, \quad M_2 = \frac{(1 - e^{-i\alpha}) \sin(\omega_2)}{2}, \]
\[ M_3 = \frac{(1 - e^{-i\alpha})(1 + \cos(\omega_2))}{2} - 1, \quad M_4 = \frac{(1 - e^{-i\alpha})(1 - \cos(\omega_1))}{2} - 1, \]
\[ M_5 = \frac{(1 - e^{-i\alpha}) \sin(\omega_1)}{2}, \quad M_6 = \frac{(1 - e^{-i\alpha})(1 + \cos(\omega_1))}{2} - 1, \]

where \( \cos(\omega_1) = 1 - \frac{2n_1}{N_F}, \sin(\omega_1) = 2n_1 \sqrt{n_1(N_1 - n_1)}, \cos(\omega_2) = 1 - \frac{2n_2}{N_F}, \) and \( \sin(\omega_2) = \frac{2}{N_F} \sqrt{n_2(N_F - n_2)}. \)

Let
\[
R(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{\frac{\theta}{2}} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
and
\[
A(\theta) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix}
\cos(\frac{\theta}{2}) & -ie^{\frac{i\theta}{2}} \sin(\omega_2) \\
-ie^{-\frac{i\theta}{2}} \sin(\omega_2) & \cos(\frac{\theta}{2}) \\
0 & 0 & 1
\end{pmatrix}.
\]

We have
\[
C(\alpha) = e^{-\frac{i\alpha}{2}} A(\frac{\pi}{2}) R(\alpha) A(-\frac{\pi}{2}), \quad (B1)
\]
\[
Q(\beta) S = -e^{i\beta} SR(\beta), \quad (B2)
\]
\[
A(\alpha + \beta) = R(\beta) A(\alpha) R(-\beta), \quad (B3)
\]
\[
R(\theta) R(-\theta) = I, \quad (B4)
\]
\[
|\Psi_0\rangle = A(\frac{\pi}{2}) S A(\frac{\pi}{2}) |0\rangle, \quad (B5)
\]

where \(|0\rangle\) denotes \((0, 0, 0, 0, 0, 0, 0, 0)\). Similarly to Eq. \(11\), the following equation holds.
\[
SB_1 SB_2 S = B_2 SB_1, \quad (B6)
\]
where \(B_1 = \prod_{i=0}^{n} D_i\) and \(B_2 = \prod_{i=0}^{m} D_i\) for \(D_i \in \{A(\theta_i), R(\theta_i)\}\).

Below we will prove Lemma 3 by two cases.

\[ \square \]

1. **Case 1: \(h\) is an odd integer**

First we set
\[
\alpha_k = \begin{cases}
-\beta_{h+2-k} & k = 2, 4, \ldots, h - 1, \\
-\beta_{h-k} & k = 3, 5, \ldots, h.
\end{cases} \quad (B7)
\]

Then \(|\Psi_h\rangle\) reduces to
\[
|\Psi_h\rangle \sim S \left[ A(\eta_h) \ldots A(\eta_1) \right] R(\alpha_1) SR(\beta_h) \left[ A(\zeta_h) \ldots A(\zeta_1) \right] |0\rangle, \quad (B8)
\]
which will be proven in Appendix D by using Eqs. \(B1\)-\(B6\). Here \(\eta_k = \eta_{h+1-k}, \zeta_k = \zeta_{h+1-k}\) for \(k = 1, 2, \ldots, h\), and
\[
\eta_{k+1} - \eta_k = \begin{cases}
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 1, \\
-\pi + \alpha_{h-k+1} & k = 3, 5, \ldots, h.
\end{cases} \quad (B9)
\]
\[ \zeta_{k+1} - \zeta_k = \begin{cases} \pi - \alpha_k & k = 2, 4, \ldots, h - 1, \\ -\pi + \alpha_{h-k} & k = 3, 5, \ldots, h. \end{cases} \] (B10)

The final state of Eq. (B8) can be obtained by the following four stages:

\[
\begin{align*}
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} A(\zeta_k) \ldots A(\zeta_1) & \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e^{i\beta_h g_h(x_1)} & 0 & 0 \\ 0 & 0 & e^{i\beta_h g_h(x_2)} & 0 \\ e^{i\beta_h g_h(x_1)} & g_h(x_1) & 0 & 0 \\ e^{i\beta_h g_h(x_2)} & g_h(x_2) & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & e^{i\beta_h e_h(x_1)} & 0 \\ 0 & e^{i\beta_h e_h(x_2)} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} R(\alpha_1) \ldots R(\beta_h) \\
A(\eta_{h}) \ldots A(\eta_1) & \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_h g_h(x_1)} & e^{i\beta_h g_h(x_2)} & e^{i\beta_h e_h(x_1)} & e^{i\beta_h e_h(x_2)} \\ e^{i\beta_h g_h(x_1)} & g_h(x_1) & e^{i\beta_h e_h(x_1)} & g_h(x_1) \\ e^{i\beta_h g_h(x_1)} & f_h(x_1) & e^{i\beta_h e_h(x_1)} & f_h(x_1) \\ e^{i\beta_h g_h(x_1)} & g_h(x_1) & e^{i\beta_h e_h(x_1)} & g_h(x_1) \end{pmatrix} \\
S & \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_h g_h(x_1)} & e^{i\beta_h g_h(x_2)} & e^{i\beta_h e_h(x_1)} & e^{i\beta_h e_h(x_2)} \\ e^{i\beta_h g_h(x_1)} & f_h(x_1) & e^{i\beta_h e_h(x_1)} & f_h(x_1) \\ e^{i\beta_h g_h(x_1)} & g_h(x_1) & e^{i\beta_h e_h(x_1)} & g_h(x_1) \\ e^{i\beta_h g_h(x_1)} & g_h(x_1) & e^{i\beta_h e_h(x_1)} & g_h(x_1) \end{pmatrix}. 
\end{align*}
\]

Stage (1): apply \( A(\zeta_k) \ldots A(\zeta_1) \) to the initial state. Let

\[
(a_0, b_0, c_0, d_0, e_0, f_0, g_0, l_0) = (0, 0, 0, 0, 1, 0, 1)
\]

and

\[
\begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \\ e_k \\ f_k \\ g_k \\ l_k \end{pmatrix} = A(\zeta_k) \ldots A(\zeta_1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}
\]

for \( k = 1, 2, \ldots, h \). The elements of a 8-dimensional vector are divided into four groups: (1th, 2th), (3th, 4th), (5th, 6th), and (7th, 8th). Note that each block of matrix \( A(\zeta_i) \) acts on a corresponding group of a 8-dimensional vector. Thus, \( a_k = b_k = c_k = d_k = 0 \) for all \( k \). The recurrence formula of \( l_k(x_1) \) is defined by \( l_0(x_1) = 1, l_1(x_1) = x_1 \) and for \( k = 2, \ldots, h \)

\[
l_k(x_1) = x_1(1 + e^{-i(\zeta_k - \zeta_{k-1})})l_{k-1}(x_1) - e^{-i(\zeta_k - \zeta_{k-1})}l_{k-2}(x_1)
\]

with \( x_1 = \cos(\frac{\alpha_k}{2}) \). The recurrence formula of \( f_k(x_2) \) is defined by \( f_0(x_2) = 1, f_1(x_2) = x_2 \) and for \( k = 2, \ldots, h \)

\[
f_k(x_2) = x_2(1 + e^{-i(\zeta_k - \zeta_{k-1})})f_{k-1}(x_2) - e^{-i(\zeta_k - \zeta_{k-1})}f_{k-2}(x_2)
\]

with \( x_2 = \cos(\frac{\beta_h}{2}) \). By Lemma 1, when

\[
\zeta_{k+1} - \zeta_k = (-1)^k \pi - 2 \arccos(\tan(\frac{k\pi}{h})\sqrt{1 - \gamma^2}), \quad \text{for } k = 1, 2, \ldots, h - 1,
\]

with \( \gamma^{-1} = \cos(\frac{1}{h} \arccos(\frac{1}{\sqrt{h}})) \) for \( k = 1, 2, \ldots, h - 1 \), we have

\[
l_h(x_1) = \frac{T_h(\frac{\pi}{2})}{T_h(\frac{\pi}{2})} f_h(x_2) = \frac{T_h(\frac{\pi}{2})}{T_h(\frac{\pi}{2})},
\]

with \( T_h(\frac{1}{4}) = \frac{1}{\sqrt{h}} \). Moreover, \( e_h(x_2) \) and \( g_h(x_1) \) are determined by \( |e_h(x_2)|^2 + |f_h(x_2)|^2 = 1 \) and \( |g_h(x_1)|^2 + |l_h(x_1)|^2 = 1 \), respectively. Therefore, the state after \( A(\zeta_h) \ldots A(\zeta_1) \) applied to the initial state is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} e_h(x_2) \\ f_h(x_2) \\ g_h(x_1) \\ l_h(x_1) \end{pmatrix}.
\]
Stage (2): apply $R(\alpha_1)SR(\beta_h)$ to the above state. After that, the state is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \
 \frac{e^{i\beta_h g_0(x_1)} 0}{e^{i\beta_h c_0(x_1)} 0} \\n \frac{0 g_1(x_1)}{0 f_0(x_2)} \\n \frac{g_0(x_1) f_0(x_2)}{f_0(x_2) f_0(x_1)} \end{pmatrix}.$$ 

Stage (3): perform $A(\eta_h)\ldots A(\eta_1)$. Let $(\bar{a}_0, \bar{b}_0, \bar{c}_0, \bar{d}_0, \bar{e}_0, \bar{f}_0, \bar{g}_0, \bar{l}_0) = (0, 1, 0, 1, 0, 1, 0, 1)$ and

$$\begin{pmatrix} e^{i\beta_h g_0(x_1)} \bar{a}_k \
 e^{i\beta_h g_0(x_1)} \bar{b}_k \
 e^{i\beta_h c_0(x_2)} \bar{c}_k \
 e^{i\beta_h e_0(x_2)} \bar{d}_k \
 g_h(x_1) \bar{e}_k \
 g_h(x_2) \bar{f}_k \
 f_h(x_2) \bar{g}_k \
 f_h(x_2) \bar{l}_k \end{pmatrix} = A(\xi_k)\ldots A(\xi_1) \begin{pmatrix} e^{i\beta_h g_0(x_1)} 0 \
 e^{i\beta_h c_0(x_2)} 0 \
 g_h(x_1) 0 \
 f_h(x_2) 0 \end{pmatrix}$$

for $k = 1, \ldots, h$. The recurrence formula of $\tilde{d}_k(x_1)$ is defined by $\tilde{d}_0(x_1) = 1, \tilde{d}_1(x_1) = x_1$ and for $k = 2, \ldots, h$

$$\tilde{d}_k(x_1) = x(1 + e^{-i(\eta_k - \eta_{k-1})})\tilde{d}_{k-1}(x_1) - e^{-i(\eta_k - \eta_{k-1})}\tilde{d}_{k-2}(x_1),$$

with $x_1 = \cos(\frac{\pi}{h})$, and the recurrence formula of $\tilde{l}_k(x_1)$ is defined by $\tilde{l}_0(x_1) = 1, \tilde{l}_1(x_1) = x_1$ and for $k = 2, \ldots, h$

$$\tilde{l}_k(x_1) = x(1 + e^{-i(\eta_k - \eta_{k-1})})\tilde{l}_{k-1}(x_1) - e^{-i(\eta_k - \eta_{k-1})}\tilde{l}_{k-2}(x_1).$$

The recurrence formula of $\tilde{b}_k(x_2)$ is defined by $\tilde{b}_0(x_2) = 1, \tilde{b}_1(x_2) = x_2$ and for $k = 2, \ldots, h$

$$\tilde{b}_k(x_2) = x_2(1 + e^{-i(\eta_k - \eta_{k-1})})\tilde{b}_{k-1}(x_2) - e^{-i(\eta_k - \eta_{k-1})}\tilde{b}_{k-2}(x_2),$$

with $x_2 = \cos(\frac{\pi}{h})$, and the recurrence formula of $\tilde{f}_k(x_2)$ is defined by $\tilde{f}_0(x_2) = 1, \tilde{f}_1(x_2) = x_2$ and for $k = 2, \ldots, h$

$$\tilde{f}_k(x_2) = x_2(1 + e^{-i(\eta_k - \eta_{k-1})})\tilde{f}_{k-1}(x_2) - e^{-i(\eta_k - \eta_{k-1})}\tilde{f}_{k-2}(x_2).$$

By Lemma 1, when

$$\eta_{k+1} - \eta_k = (-1)^k \pi - 2\arccos\left(\tan\left(\frac{k\pi}{h}\right)\right) \sqrt{1 - \gamma^2},$$

(B12)

with $\gamma^{-1} = \cos(\frac{\pi}{h}\arccos(\frac{1}{\sqrt{e}}))$ for $k = 1, 2, \ldots, h - 1$, we have

$$\tilde{d}_h(x_1) = \tilde{l}_h(x_1) = \frac{T_h(\frac{\pi}{h})}{T_h(\frac{\pi}{h})}, \quad \tilde{b}_h(x_2) = \tilde{f}_h(x_2) = \frac{T_h(\frac{\pi}{h})}{T_h(\frac{\pi}{h})},$$

with $T_h(\frac{1}{\sqrt{e}}) = \frac{1}{\sqrt{e}}$. Moreover, $\bar{a}_h(x_2), \bar{c}_h(x_1), \bar{e}_h(x_2)$, and $\bar{g}_h(x_1)$ are determined by $|\bar{a}_h(x_2)|^2 + |\bar{b}_h(x_2)|^2 = 1, |\bar{c}_h(x_1)|^2 + |\bar{e}_h(x_2)|^2 = 1, |\bar{c}_h(x_2)|^2 + |\bar{e}_h(x_2)|^2 = 1, |\bar{g}_h(x_1)|^2 + |\bar{l}_h(x_1)|^2 = 1$, respectively. Hence, the result state is

$$\begin{pmatrix} e^{i\beta_h g_0(x_1)} \bar{a}_h(x_2) \
 e^{i\beta_h g_0(x_1)} \bar{b}_h(x_2) \
 e^{i\beta_h c_0(x_2)} \bar{c}_h(x_2) \
 e^{i\beta_h e_0(x_2)} \bar{d}_h(x_2) \
 g_h(x_1) \bar{e}_h(x_2) \
 g_h(x_2) \bar{f}_h(x_2) \
 f_h(x_2) \bar{g}_h(x_1) \
 f_h(x_2) \bar{l}_h(x_1) \end{pmatrix}.$$

Stage (4): perform the final operation $S$. The final state is

$$\begin{pmatrix} e^{i\beta_h c_0(x_2)} \bar{c}_h(x_1) \
 f_h(x_2) \bar{g}_h(x_1) \
 e^{i\beta_h g_0(x_1)} \bar{a}_h(x_2) \
 f_h(x_2) \bar{g}_h(x_1) \
 e^{i\beta_h g_0(x_1)} \bar{d}_h(x_2) \
 g_h(x_1) \bar{e}_h(x_2) \
 e^{i\beta_h c_0(x_2)} \bar{d}_h(x_2) \
 f_h(x_2) \bar{g}_h(x_1) \end{pmatrix}.$$
Hence, the success probability $P_h$ is

$$
P_h = 1 - \frac{1}{2} |f_h(x_2)\tilde{h}(x_1)|^2 - \frac{1}{2} |g_h(x_1)\tilde{f}_h(x_2)|^2 = 1 - \epsilon^2 T_h^2 \left( \frac{x_1}{\gamma} \right) T_h^2 \left( \frac{x_2}{\gamma} \right)
$$

$$
= 1 - \epsilon^2 T_h^2 \left( \cos \left( \frac{1}{h} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \left( \frac{1}{h} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \sqrt{1 - \frac{n_r}{N_r}}.
$$

By Eq. (B7) and Eqs. (B9)-(B12), $\alpha_k, \beta_k$ can be chosen such that

$$
\alpha_k = -\beta_{h+2-k} = \pi + (\eta_{k+1} - \eta_k) = 2\arccot \left( \tan \left( \frac{h\pi}{h} \right) \sqrt{1 - \gamma^2} \right)
$$

for $k = 2, 4, \ldots, h - 1$, and

$$
\alpha_k = -\beta_{h-k} = \pi - (\zeta_{k+1} - \zeta_k) = 2\arccot \left( \tan \left( \frac{(k-1)\pi}{h} \right) \sqrt{1 - \gamma^2} \right)
$$

for $k = 3, 5, \ldots, h$. In addition, $\alpha_1$ and $\beta_h$ can be any value.

2. Case 2: $h$ is an even integer

First we set

$$
\beta_k = -\alpha_{h+1-k} \quad k = 1, 2, \ldots, h - 1.
$$

Then $|\Psi_h\rangle$ reduces to

$$
|\Psi_h\rangle \sim R(\beta_h) [A(\phi_{h-1}) \cdots A(\phi_1)] R(\alpha_1) S [A(\psi_{h+1}) \cdots A(\psi_1)] |\tilde{0}\rangle,
$$

which will be proven in Appendix D by using Eqs. (B1)-(B6). Here $\phi_k = \phi_{h-k}$ for $k = 1, 2, \ldots, h - 1$, $\psi_k = \psi_{h+2-k}$ for $k = 1, 2, \ldots, h + 1$, and

$$
\phi_{k+1} - \phi_k = \begin{cases} 
\pi - \alpha_{k+1} & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k} & k = 1, 3, \ldots, h - 3,
\end{cases}
$$

$$
\psi_{k+1} - \psi_k = \begin{cases} 
\pi - \alpha_k & k = 2, 4, \ldots, h - 2, \\
-\pi + \alpha_{h-k+1} & k = 1, 3, \ldots, h.
\end{cases}
$$

The final state of Eq. (B14) is given by the four stages as follows:

\[ 
\begin{array}{c}
1 \\
\sqrt{2} \\
\end{array}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A(\psi_{h+1}) \cdots A(\psi_1) \\
1 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & e_{h+1}(x_2) \\
0 & 0 & e_{h+1}(x_2) & f_{h+1}(x_2) \\
\vdots & \vdots & \vdots & \vdots \\
0 & e_{h+1}(x_2) & 0 & f_{h+1}(x_2)
\end{pmatrix}
\begin{pmatrix}
R(\alpha_1)S \left( \frac{0}{g_{h+1}(x_1)} \right) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\psi_{h+1} \cdots \psi_1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
A(\phi_{h-1}) \cdots A(\phi_1) \\
1 \\
\end{pmatrix}
\begin{pmatrix}
0 & e_{h+1}(x_2) & 0 & f_{h+1}(x_2) \\
0 & g_{h+1}(x_1) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
e^{ih}f_{h+1}(x_2) & e^{ih}g_{h+1}(x_1) & e^{ih}l_{h+1}(x_1) & l_{h+1}(x_1)
\end{pmatrix}
\begin{pmatrix}
R(\beta_h) \left( \frac{0}{g_{h+1}(x_1)} \right) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\phi_{h+1} \cdots \phi_1 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
ge^{ih}g_{h+1}(x_1) & e^{ih}f_{h+1}(x_2) & e^{ih}l_{h+1}(x_1) & l_{h+1}(x_1)
ge^{ih}f_{h+1}(x_2) & e^{ih}g_{h+1}(x_1) & e^{ih}l_{h+1}(x_1) & l_{h+1}(x_1)
e^{ih}g_{h+1}(x_1) & e^{ih}f_{h+1}(x_2) & e^{ih}l_{h+1}(x_1) & l_{h+1}(x_1)
ge^{ih}f_{h+1}(x_2) & e^{ih}g_{h+1}(x_1) & e^{ih}l_{h+1}(x_1) & l_{h+1}(x_1)
\end{pmatrix}.
\]
Stage (1): apply $A(\psi_{k+1})...A(\psi_1)$ to the initial state. Let

$$(a_0, b_0, c_0, d_0, e_0, f_0, g_0, l_0) = (0, 0, 0, 0, 1, 0, 1)$$

and

$$\begin{pmatrix}
    a_k \\
    b_k \\
    c_k \\
    d_k \\
    e_k \\
    f_k \\
    g_k \\
    l_k
\end{pmatrix} = A(\psi_k)...A(\psi_1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

for $k = 1, 2, \ldots, h$. By the property of the matrix $A(\xi)$, we have $a_k = b_k = c_k = d_k = 0$ for all $k$. And $e_k, f_k, g_k, l_k$ can be gotten in the following. The recurrence formula of $l_k(x_1)$ is defined by $l_0(x_1) = 1, l_1(x_1) = x_1$ and for $k = 2, \ldots, h + 1$

$$l_k(x_1) = x_1 (1 + e^{-i(\psi_k - \psi_{k-1})}l_{k-1}(x_1) - e^{-i(\psi_k - \psi_{k-1})}l_{k-2}(x_1))$$

with $x_1 = \cos(\frac{k\pi}{2})$, and the recurrence formula of $f_k(x_2)$ is defined by $f_0(x_2) = 1, f_1(x_2) = x_2$ and for $k = 2, \ldots, h + 1$

$$f_k(x_2) = x_2 (1 + e^{-i(\psi_k - \psi_{k-1})})f_{k-1}(x_2) - e^{-i(\psi_k - \psi_{k-1})}f_{k-2}(x_2)$$

with $x_2 = \cos(\frac{k\pi}{2})$. By Lemma 1, when

$$\psi_{k+1} - \psi_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h+1})\sqrt{1 - \gamma_1^2})$$

with $\gamma_1^{-1} = \cos(\frac{1}{h+1}\arccos(\frac{1}{\sqrt{2}}))$ for $k = 1, 2, \ldots, h$, we have

$$l_{h+1}(x_1) = \frac{T_{h+1}(\frac{\pi}{2})}{T_{h+1}(\frac{1}{h+1})}, \quad f_{h+1}(x_2) = \frac{T_{h+1}(\frac{\pi}{2})}{T_{h+1}(\frac{1}{h+1})},$$

with $T_{h+1}(\frac{1}{h+1}) = \frac{1}{\sqrt{2}}$. Moreover, $e_{h+1}(x_2)$ and $g_{h+1}(x_1)$ are determined by $|e_{h+1}(x_2)|^2 + |f_{h+1}(x_2)|^2 = 1$ and $|g_{h+1}(x_1)|^2 + |l_{h+1}(x_1)|^2 = 1$, respectively. Therefore, the state after $A(\psi_k)...A(\psi_1)$ applied to the initial state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix}
    0 & 0 & 0 & 0 & e_{h+1}(x_2) & f_{h+1}(x_2) & g_{h+1}(x_1) & l_{h+1}(x_1)
\end{pmatrix}.$$

Stage (2): apply $R(\alpha_1)S$ to the above state. After that, the state is

$$\frac{1}{\sqrt{2}} \begin{pmatrix}
    0 & 0 & 0 & 0 & e_{h+1}(x_2) & f_{h+1}(x_2) & g_{h+1}(x_1) & l_{h+1}(x_1)
\end{pmatrix}.$$

Stage (3): perform $A(\phi_{h-1})...A(\phi_1)$. Let $(a_0, b_0, c_0, d_0, e_0, f_0, g_0, l_0) = (0, 1, 0, 1, 0, 1, 0, 1)$ and

$$\frac{1}{\sqrt{2}} \begin{pmatrix}
    g_{h+1}(x_1) & a_k & g_{h+1}(x_1) & b_k & e_{h+1}(x_2) & l_{h+1}(x_1) & e_k & f_k & l_{h+1}(x_2)
\end{pmatrix} = A(\phi_k)...A(\phi_1) \frac{1}{\sqrt{2}} \begin{pmatrix}
    0 & g_{h+1}(x_1) & 0 & e_{h+1}(x_2) & 0 & l_{h+1}(x_1) & 0 & f_{h+1}(x_2)
\end{pmatrix}$$

for $k = 1, \ldots, h - 1$. The recurrence formula of $d_k(x_1)$ is defined by $d_0(x_1) = 1, d_1(x_1) = x_1$ and for $k = 2, \ldots, h - 1$

$$d_k(x_1) = x_1 (1 + e^{-i(\phi_k - \phi_{k-1})}d_{k-1}(x_1) - e^{-i(\phi_k - \phi_{k-1})}d_{k-2}(x_1)$$

for $k = 1, \ldots, h - 1$.
with \( x_1 = \cos(\frac{\theta}{\gamma}) \), and the recurrence formula of \( \bar{l}_k(x_1) \) is defined by \( \bar{l}_0(x_1) = 1, \bar{l}_1(x_1) = x_1 \) and for \( k = 2, \ldots, h - 1 \)
\[
\bar{l}_k(x_1) = x_1(1 + e^{-i(\phi_k - \phi_{k-1})})\bar{l}_{k-1}(x_1) - e^{-i(\phi_k - \phi_{k-1})}\bar{l}_{k-2}(x_1).
\]
The recurrence formula of \( \bar{b}_k(x) \) is defined by \( \bar{b}_0(x_2) = 1, \bar{b}_1(x_2) = x_2 \) and for \( k = 2, \ldots, h - 1 \)
\[
\bar{b}_k(x_2) = x_2(1 + e^{-i(\phi_k - \phi_{k-1})})\bar{b}_{k-1}(x_2) - e^{-i(\phi_k - \phi_{k-1})}\bar{b}_{k-2}(x_2)
\]
with \( x_2 = \cos(\frac{\theta}{\gamma}) \), and the recurrence formula of \( \bar{f}_k(x_2) \) is defined by \( \bar{f}_0(x_2) = 1, \bar{f}_1(x_2) = x_2 \) and for \( k = 2, \ldots, h - 1 \)
\[
\bar{f}_k(x_2) = x_2(1 + e^{-i(\phi_k - \phi_{k-1})})\bar{f}_{k-1}(x_2) - e^{-i(\phi_k - \phi_{k-1})}\bar{f}_{k-2}(x_2).
\]

By Lemma 1, when
\[
\phi_{k+1} - \phi_k = (-1)^k \pi - 2\arccot(\tan(\frac{k\pi}{h-1})\sqrt{1 - \gamma_2^2}) \quad (B18)
\]
with \( \gamma_2^{-1} = \cos(\frac{1}{h-1} \arccos(\frac{1}{\gamma})) \) for \( k = 1, 2, \ldots, h - 2 \), we have
\[
\bar{d}_{h-1}(x_1) = \bar{l}_{h-1}(x_1) = \frac{T_{h-1}(\frac{\gamma_1}{\gamma_2})}{T_{h-1}(\frac{\gamma_1}{\gamma_2})}, \quad \bar{b}_{h-1}(x_2) = \bar{f}_{h-1}(x_2) = \frac{T_{h-1}(\frac{\gamma_2}{\gamma_1})}{T_{h-1}(\frac{\gamma_2}{\gamma_1})},
\]
with \( T_{h-1}(\frac{1}{\gamma_2}) = \frac{1}{\gamma_2} \). Moreover, \( \bar{a}_h(x_2), \bar{c}_h(x_1), \bar{c}_h(x_2), \) and \( \bar{g}_h(x_1) \) are determined by |\( \bar{a}_h(x_2) \)\|^2 + |\( \bar{b}_h(x_2) \)\|^2 = 1, |\( \bar{c}_h(x_1) \)\|^2 + |\( \bar{d}_h(x_1) \)\|^2 = 1, |\( \bar{c}_h(x_2) \)\|^2 + |\( \bar{f}_h(x_2) \)\|^2 = 1, and |\( \bar{g}_h(x_1) \)\|^2 + |\( \bar{f}_h(x_1) \)\|^2 = 1, respectively. Hence, the result state is
\[
\begin{pmatrix}
\frac{1}{\sqrt{2}}
g_{h-1}(x_2)
g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\end{pmatrix}.
\]

Stage ④: perform the final operation \( R(\beta_h) \). The final state is
\[
\begin{pmatrix}
\frac{1}{\sqrt{2}}
\left( e^{i\beta_h}g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\right)
\left( e^{i\beta_h}g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\right)
\left( e^{i\beta_h}g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\right)
\left( e^{i\beta_h}g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\right)
\left( e^{i\beta_h}g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\right)
\left( e^{i\beta_h}g_{h+1}(x_1)\bar{a}_{h-1}(x_2)
g_{h+1}(x_1)\bar{b}_{h-1}(x_2)
\right)
\end{pmatrix}.
\]

Therefore, the success probability \( P_h \) is
\[
P_h = 1 - \frac{1}{2}|c_{h+1}(x_1)d_{h-1}(x_2)|^2 - \frac{1}{2}|c_{h+1}(x_2)d_{h-1}(x_1)|^2
\]
\[
= 1 - \frac{\epsilon^2}{2} \left( T_{h+1}^2(\frac{\gamma_1}{\gamma_2}) \right) T_{h-1}^2(\frac{\gamma_1}{\gamma_2}) + \frac{T_{h+1}^2(\frac{\gamma_2}{\gamma_1}) T_{h-1}^2(\frac{\gamma_1}{\gamma_2})}{2}
\]
\[
= 1 - \frac{\epsilon^2}{2} \left[ T_{h+1}^2 \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \left( 1 - \frac{n_f}{N_f} \right) + \right.
\]
\[
\left. T_{h-1}^2 \left( \cos \left( \frac{1}{h-1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \left( 1 - \frac{n_f}{N_f} \right) \right] + \left. T_{h+1}^2 \left( \cos \left( \frac{1}{h+1} \arccos \left( \frac{1}{\sqrt{\epsilon}} \right) \right) \right) \left( 1 - \frac{n_f}{N_f} \right) \right]\]
By Eq. (B13) and Eqs. (B15)-(B18), \( \alpha_k, \beta_k \) can be chosen such that
\[
\alpha_k = - \beta_{h+1-k} = \pi - (\phi_k - \phi_{k-1}) = 2\arccot \left( \tan \left( \frac{k\pi}{h+1} \right) \right) \sqrt{1 - \gamma^2_k}
\]
for \( k = 2, 4, \ldots, h \), and
\[
\alpha_k = - \beta_{h+1-k} = \pi - (\psi_{k+1} - \psi_k) = 2\arccot \left( \tan \left( \frac{(k-1)\pi}{h-1} \right) \right) \sqrt{1 - \gamma^2_k}
\]
for \( k = 3, 5, \ldots, h-1 \). In addition, \( \alpha_1 \) and \( \beta_h \) can be any value.

Appendix C: Proof of Equations (13) and (A2)

In this appendix, we give the detailed proof of Equations (13) and (A2).
Recall that
\[
|\Psi_h\rangle = U(\alpha_h, \beta_h) \ldots U(\alpha_1, \beta_1) |\Psi_0\rangle = SC(\alpha_h)Q(\beta_h) \ldots SC(\alpha_1)Q(\beta_1) |\Psi_0\rangle.
\]
Using \( SS = I \), we have
\[
|\Psi_h\rangle = SC(\alpha_h)Q(\beta_h) \ldots SC(\alpha_1)Q(\beta_1) SS |\Psi_0\rangle.
\]
Using Eq. (6) and Eq. (7), we have
\[
|\Psi_h\rangle \sim SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2}) \ldots SR(\beta_h)A(\frac{\pi}{2})R(\alpha_{h-1})A(-\frac{\pi}{2}) S |\Psi_0\rangle.
\]
The number of \( S \) is \( h+2 \). Two cases need to be considered.

Case 1: \( h \) is odd. By Eq. (11), we have,
\[
|\Psi_h\rangle \sim SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2}) \ldots SR(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) \left\{ \begin{array}{l}
SR(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) SR(\beta_{h+3})A(\frac{\pi}{2})R(\alpha_{h+2})A(-\frac{\pi}{2}) S
\end{array} \right\}
\]
\[
R(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) \ldots SR(\beta_2)A(\frac{\pi}{2})R(\alpha_1)A(-\frac{\pi}{2}) S |\Psi_0\rangle
\]
\[
=SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2}) \ldots SR(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) \left\{ \begin{array}{l}
R(\beta_{h+1})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) SR(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+2})A(-\frac{\pi}{2})
\end{array} \right\}
\]
\[
R(\beta_{h+1})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) \ldots SR(\beta_2)A(\frac{\pi}{2})R(\alpha_1)A(-\frac{\pi}{2}) S |\Psi_0\rangle
\]
\[
=SA(\frac{\pi}{2})R(\alpha_h)A(-\frac{\pi}{2}) \ldots SR(\beta_h)A(\frac{\pi}{2})R(\alpha_{h-1})A(-\frac{\pi}{2}) \ldots R(\alpha_2)A(-\frac{\pi}{2}) S |\Psi_0\rangle.
\]
The process is as follows. We first select the formula
\[
\left\{ \begin{array}{l}
SR(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) SR(\beta_{h+3})A(\frac{\pi}{2})R(\alpha_{h+2})A(-\frac{\pi}{2}) S
\end{array} \right\}
\]
containing the middle \( S \), and then this formula reduces to
\[
\left\{ \begin{array}{l}
R(\beta_{h+1})A(\frac{\pi}{2})R(\alpha_{h+1})A(-\frac{\pi}{2}) SR(\beta_{h+2})A(\frac{\pi}{2})R(\alpha_{h+2})A(-\frac{\pi}{2})
\end{array} \right\}
\]
Here, two cases need to be discussed.

**Case 1.1:** When \( h \mod 4 = 1 \), \( \beta_i = \alpha_{h+2-i} \) for \( i = 2, 4, \ldots, h-1 \), and \( \beta_i = \alpha_{h-i} \) for \( i = 1, 3, \ldots, h-2 \), using Eqs. (8) and (9), we have

\[
|\Psi_h\rangle \sim S[A(\frac{\pi}{2})A(\frac{\pi}{2})A(\frac{\pi}{2})A(\frac{\pi}{2})\ldots] \langle \alpha_1 | A(\frac{\pi}{2})\ldots A(\frac{\pi}{2})R(\alpha_{h-1})A(\frac{\pi}{2})R(\alpha_h)|0\rangle
\]

Thus, we can get Eq. (13) by rewriting Eq. (C1) and Eq. (C2) as follows.

\[
|\Psi_h\rangle \sim S[A(\eta_k)\ldots A(\eta_1)] R(\alpha_1) SR(\beta_h) [A(\zeta_k)\ldots A(\zeta_1)] |0\rangle,
\]

where \( k = \eta_{h+1-k} \) and \( \zeta_k = \zeta_{h+1-k} \) for \( k = 1, 2, \ldots, h \), and

\[
\eta_{k+1} - \eta_k = \begin{cases} \pi - \alpha_{k+1} & k = 2, 4, \ldots, h-1, \\ -\pi + \alpha_{h-k+1} & k = 3, 5, \ldots, h, \end{cases}
\]

\[
\zeta_{k+1} - \zeta_k = \begin{cases} \pi - \alpha_k & k = 2, 4, \ldots, h-1, \\ -\pi + \alpha_{h-k} & k = 3, 5, \ldots, h. \end{cases}
\]

Here, when \( h \mod 4 = 1 \),

\[
\eta_{(h+1)/2} = \frac{\pi}{2} + \frac{\alpha_{(h+5)/2} + \alpha_{(h+9)/2} + \ldots + \alpha_{h-2} + \alpha_h}{2} - \frac{\alpha_3 + \alpha_5 + \ldots + \alpha_{(h-3)/2} + \alpha_{(h+1)/2}}{2},
\]
\[ \zeta_{(h+1)/2} = \frac{\pi}{2} + (\alpha_{(h+3)/2} + \alpha_{(h+7)/2} + \ldots + \alpha_{h-3} + \alpha_{h-1}) \\
- (\alpha_2 + \alpha_4 + \ldots + \alpha_{(h-5)/2} + \alpha_{(h-1)/2}), \]

and when \( h \) mod 4 = 3,

\[ \eta_{(h+1)/2} = -\frac{\pi}{2} + (\alpha_{(h+3)/2} + \alpha_{(h+7)/2} + \ldots + \alpha_{h-2} + \alpha_h) \\
- (\alpha_3 + \alpha_5 + \ldots + \alpha_{(h-5)/2} + \alpha_{(h-1)/2}), \]

\[ \zeta_{(h+1)/2} = -\frac{\pi}{2} + (\alpha_{(h+1)/2} + \alpha_{(h+5)/2} + \ldots + \alpha_{h-3} + \alpha_{h-1}) \\
- (\alpha_2 + \alpha_4 + \ldots + \alpha_{(h-7)/2} + \alpha_{(h-3)/2}). \]

**Case 2:** \( h \) is even. Using Eq. (11), there is

\[
|\Psi_h\rangle \sim S \left[ A \left( \frac{\pi}{2} \right) R(\alpha_h) A(-\frac{\pi}{2}) \ldots S \left[ R(\beta_{\frac{h}{2}+1}) A \left( \frac{\pi}{2} \right) \right] R(\alpha_{\frac{h}{2}}) A \left( \frac{\pi}{2} \right) \right] |\Psi_0\rangle
\]

Using Eq. (10) and \( A(-\frac{\pi}{2}) A(\frac{\pi}{2}) = I \), we have

\[
|\Psi_h\rangle \sim S \left[ A \left( \frac{\pi}{2} \right) R(\alpha_h) A(-\frac{\pi}{2}) \ldots S R(\beta_{\frac{h}{2}+1}) A \left( \frac{\pi}{2} \right) \ldots R(\beta_{\frac{h}{2}} A \left( \frac{\pi}{2} \right) \ldots R(\beta_1 A \left( \frac{\pi}{2} \right) \right)] |\Psi_0\rangle
\]

Here, two cases need to be discussed.

**Case 2.1:** When \( h \) mod 4 = 0, and \( \beta_i = \alpha_{h+1-i} \) for \( i = 1, 2, \ldots, h-1 \), using Eqs. (8) and (9), we have

\[
|\Psi_h\rangle \sim R(\beta_h) A \left( \frac{\pi}{2} \right) A(-\frac{\pi}{2} + \alpha_{h+1}) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_{h+1} \right) A(\frac{\pi}{2} - \alpha_3 + \alpha_{h+1} + \alpha_{h-3}) \ldots A \left( \frac{\pi}{2} - \alpha_3 - \alpha_5 \ldots - \alpha_{k_1} + \alpha_{k_1+2} \ldots + \alpha_{h-1} \right) \\
A \left( \frac{\pi}{2} - \alpha_3 + \alpha_{h+1} \right) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_{h-3} \right) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_{h-1} \right) A(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3}) \ldots \\
A \left( \frac{\pi}{2} - \alpha_3 + \alpha_{h-3} \right) A \left( \frac{\pi}{2} - \alpha_3 + \alpha_{h-1} \right) A(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3}) \ldots \\
A \left( \frac{\pi}{2} - \alpha_2 - \alpha_4 \ldots - \alpha_{k_2} + \alpha_{k_2+2} \ldots + \alpha_{h} \right) \\
A \left( \frac{\pi}{2} - \alpha_2 + \alpha_h + \alpha_{h-2} \right) A \left( \frac{\pi}{2} - \alpha_2 + \alpha_h + \alpha_{h-2} \right) A \left( \frac{\pi}{2} + \alpha_2 \right) A \left( \frac{\pi}{2} + \alpha_2 \right) |\vec{0}\rangle.
\]

where \( k_1 = h/2 - 1, k_2 = h/2 \).
Case 2.2: When \( h \mod 4 = 2 \), and \( \beta_i = \alpha_{h+1-i} \) for \( i = 1, 2, \ldots, h-1 \), using Eqs. (8) and (9), we have

\[
|\Psi_h\rangle \sim R(\beta_h)A(\frac{\pi}{2})A(-\frac{\pi}{2} + \alpha_{h-1})A\left(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1}\right)A\left(-\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3}\right)
\]

\[
\ldots A\left(\frac{\pi}{2} - \alpha_3 - \alpha_5 \cdots - \alpha_{k+1} + \alpha_{k+2} \cdots + \alpha_{h-1}\right)
\]

\[
A\left(-\frac{\pi}{2} - \alpha_3 + \alpha_{h-1} + \alpha_{h-3}\right)A\left(\frac{\pi}{2} - \alpha_3 + \alpha_{h-1}\right)A\left(-\frac{\pi}{2} + \alpha_{h-1}\right)A\left(\frac{\pi}{2}\right)R(\alpha_1)S
\]

\[
A\left(-\frac{\pi}{2} + \alpha_h\right)A\left(\frac{\pi}{2} - \alpha_2 + \alpha_h\right)A\left(-\frac{\pi}{2} - \alpha_2 + \alpha_h + \alpha_{h-2}\right)
\]

\[
\ldots A\left(-\frac{\pi}{2} - \alpha_2 - \alpha_4 \cdots - \alpha_{k+2} + \alpha_{k+3} \cdots + \alpha_{h}\right)
\]

\[
A\left(-\frac{\pi}{2} - \alpha_2 + \alpha_h + \alpha_{h-2}\right)A\left(\frac{\pi}{2} - \alpha_2 + \alpha_h\right)A\left(-\frac{\pi}{2} + \alpha_h\right)A\left(\frac{\pi}{2}\right)|0\rangle,
\]

where \( k_1 = h/2, k_2 = h/2 - 1 \).

Using Eqs. (C3) and (C4) we can get Eq. (A2),

\[
|\Psi_h\rangle \sim R(\beta_h)\left[A(\phi_{h-1})\ldots A(\phi_1)\right]R(\alpha_1)S\left[A(\psi_{h+1})\ldots A(\psi_1)\right]|0\rangle,
\]

where \( \phi_k = \phi_{h-k} \) for \( k = 1, 2, \ldots, h-1 \), \( \psi_k = \psi_{h+2-k} \) for \( k = 1, 2, \ldots, h+1 \), and

\[
\phi_{k+1} - \phi_k = \begin{cases} 
\pi - \alpha_{k+1} & \text{for } k = 2, 4, \ldots, h-2, \\
-\pi + \alpha_{h-k} & \text{for } k = 1, 3, \ldots, h-3,
\end{cases}
\]

\[
\psi_{k+1} - \psi_k = \begin{cases} 
\pi - \alpha_k & \text{for } k = 2, 4, \ldots, h-2, \\
-\pi + \alpha_{h-k+1} & \text{for } k = 1, 3, \ldots, h.
\end{cases}
\]

Here, when \( h \mod 4 = 0 \), there are

\[
\phi_{h/2} = -\frac{\pi}{2} + (\alpha_{h/2+1} + \alpha_{h/2+3} + \ldots + \alpha_{h-3} + \alpha_{h-1}) - (\alpha_3 + \alpha_5 + \ldots + \alpha_{h/2-3} + \alpha_{h/2-1}),
\]

\[
\psi_{h/2+1} = \frac{\pi}{2} + (\alpha_{h/2+2} + \alpha_{h/2+4} + \ldots + \alpha_{h-2} + \alpha_h) - (\alpha_2 + \alpha_4 + \ldots + \alpha_{h/2-2} + \alpha_{h/2}),
\]

and when \( h \mod 4 = 2 \), there are

\[
\phi_{h/2} = \frac{\pi}{2} + (\alpha_{h/2+2} + \alpha_{h/2+4} + \ldots + \alpha_{h-3} + \alpha_{h-1}) - (\alpha_3 + \alpha_5 + \ldots + \alpha_{h/2-2} + \alpha_{h/2}),
\]

\[
\psi_{h/2+1} = -\frac{\pi}{2} + (\alpha_{h/2+1} + \alpha_{h/2+3} + \ldots + \alpha_{h-2} + \alpha_h) - (\alpha_2 + \alpha_4 + \ldots + \alpha_{h/2-3} + \alpha_{h/2-1}).
\]

Appendix D: Proof of Equations (B8) and (B14)

The process of proving Eq. (13) ((A2)) uses Eqs. (6)-(11) in Appendix C. There are the same mathematical equations in two cases (see Eqs. (6)-(11) and Eqs. (B1)-(B6)), although their subspace dimensions in the analysis are different. Therefore, the proof of Eq. (B8) (Eq. (B14)) is the same as Eq. (13) ((A2)).