SHARP SOBOLEV TRACE INEQUALITIES FOR HIGHER ORDER DERIVATIVES

QIAOHUA YANG

Abstract. Motivated by a recent work of Ache and Chang concerning the sharp Sobolev trace inequality and Lebedev-Milin inequalities of order four on the Euclidean unit ball, we derive such inequalities on the Euclidean unit ball for higher order derivatives. By using, among other things, the scattering theory on hyperbolic spaces and the generalized Poisson kernel, we obtain the explicit formulas of extremal functions of such inequalities. Moreover, we also derive the sharp trace Sobolev inequalities on half spaces for higher order derivatives. Finally, we compute the explicit formulas of adapted metric, introduced by Case and Chang, on the Euclidean unit ball, which is of independent interest.

1. Introduction

It is well known that the Sobolev inequalities and sharp constants play an important role in problems in analysis and conformal geometry. An elementary example is the following classical Sobolev inequality on the standard sphere \((\mathbb{S}^n, g_{\mathbb{S}^n})\) for \(n \geq 3\):

\[
\left( \frac{1}{\omega_n} \int_{\mathbb{S}^n} |f|^{\frac{2n}{n-2}} \, d\sigma \right)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)\omega_n} \int_{\mathbb{S}^n} |\tilde{\nabla} f|^2 \, d\sigma + \frac{1}{\omega_n} \int_{\mathbb{S}^n} |f|^2 \, d\sigma,
\]

where \(d\sigma\) is Lebesgue measure on \(\mathbb{S}^n\), \(\omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\) is the volume of \(\mathbb{S}^n\) and \(\tilde{\nabla}\) is the sphere gradient on \((\mathbb{S}^n, g_{\mathbb{S}^n})\). Using the conformal invariance, one observes inequality (1.1) is equivalent to the sharp Sobolev inequality on \(\mathbb{R}^n\) given as follows:

\[
\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \omega_n^{\frac{2}{n-2}} \left( \int_{\mathbb{R}^n} |f|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq \int_{\mathbb{R}^n} |\nabla f|^2 \, dx,
\]

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where $\nabla f$ is the gradient of $f$ with respect to the Euclidean metric. In a limiting case, namely $n = 2$, (1.1) takes the form of Moser-Onfori inequality ([34, 36])

\[
\log \left( \frac{1}{4\pi} \int_{S^2} e^f \, d\sigma \right) \leq \frac{1}{16\pi} \int_{S^2} |\tilde{\nabla} f|^2 \, d\sigma + \frac{1}{4\pi} \int_{S^2} |f|^2 \, d\sigma.
\]

(1.3)

Inequality (1.3) has been widely used in analysis and conformal geometry, in particular, in the problem of prescribing Gaussian curvature on the sphere (see [9, 37, 38, 39]).

Another example is the Sobolev trace inequality. Denote by $B_{n+1}$ the unit ball on Euclidean space $\mathbb{R}^{n+1}$ with $S^n$ as the boundary. The Sobolev trace inequality on $B_{n+1}$ reads as follow (see [18]): for $f \in C^\infty(S^n)$ and $n \geq 2$,

\[
\frac{n-1}{2} \omega_n \left( \int_{S^n} |f|^\frac{2n}{n-1} \, d\sigma \right)^{\frac{n-1}{n}} \leq \int_{B_{n+1}} |\nabla v|^2 \, dx + \frac{n-1}{2} \int_{S^n} |f|^2 \, d\sigma,
\]

where $v$ is a smooth extension of $f$ to $B_{n+1}$. By the conformal invariance, one observes inequality (1.4) is also equivalent to the sharp Sobolev inequality on half space (see [18]):

\[
\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \omega_n \left( \int_{\mathbb{R}^n} |U(x,0)|^{\frac{2n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}_{+}^{n+1}} |\nabla U(x,y)|^2 \, dxdy,
\]

where $\mathbb{R}_{+}^{n+1} = \mathbb{R}^n \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$. In a limiting case, namely $n = 1$, (1.4) becomes the classical Lebedev-Milin inequality ([30])

\[
\log \left( \frac{1}{\pi} \int_{S^1} e^f \, d\sigma \right) \leq \frac{1}{4\pi} \int_{S^2} |\nabla v|^2 \, dx + \frac{1}{\pi} \int_{S^1} |f|^2 \, d\sigma.
\]

(1.6)

Such Sobolev trace inequalities and Lebedev-Milin inequality have also been widely used in analysis and geometry, such as Yamabe problem on manifolds with boundary (see [19]), the Bieberbach conjecture ([16]) and the compactness of isospectral planar domains (see [37, 38, 39]).

Notice that the operators involved in (1.1)-(1.6), either the Laplace operator or the conformal Laplace operator, are order two. Recently, the role played by these operators of order two has been extended to operators of higher order, such as Paneitz operator, poly-Laplacian and GJMS operator. In particular, Beckner [2] established the higher order Sobolev inequality on the standard sphere $(S^n, g_{S^n})$. We state it as follow

**Theorem 1.1** (Beckner). Let $\tilde{\Delta}$ be the Laplace-Beltrame operator on the standard sphere $(S^n, g_{S^n})$ and define

\[
\mathcal{P}_{2\gamma} = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)}, \quad B = \sqrt{-\Delta + \frac{(n - 1)^2}{4}}.
\]

(1.7)
Then for $0 < \gamma < \frac{n}{2}$,

$$
\frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\Gamma\left(\frac{n-2\gamma}{2}\right)} \frac{2^n}{\omega_n} \left( \int_{\mathbb{S}^n} |f|^{\frac{2n}{n-2\gamma}} d\sigma \right)^{\frac{n-2\gamma}{n}} \leq \int_{\mathbb{S}^n} f \mathcal{P}_2 f d\sigma.
$$

(1.8)

Equality holds only for functions of the form

$$
c|1 - \langle \zeta, \xi \rangle|^{\frac{2n-2}{2}}, \quad c \in \mathbb{R}, \quad \zeta \in \mathbb{B}^{n+1}, \quad \xi \in \mathbb{S}^n.
$$

If $\gamma = \frac{n}{2}$, then

$$
\ln \left( \frac{1}{\omega_n} \int_{\mathbb{S}^n} e^{f - \overline{f}} d\sigma \right) \leq \frac{1}{2n! \omega_n} \int_{\mathbb{S}^n} f \mathcal{P}_n f d\sigma,
$$

(1.9)

where $\overline{f} = \frac{1}{\omega_n} \int_{\mathbb{S}^n} f d\sigma$ is the integral average of $f$ on $\mathbb{S}^n$. Equality holds only for functions of the form

$$
-n \ln |1 - \langle \zeta, \xi \rangle| + c, \quad \zeta \in \mathbb{B}^{n+1}, \quad \xi \in \mathbb{S}^n, \quad c \in \mathbb{R}.
$$

By the conformal invariance, one observes inequality (1.8) is equivalent to the following sharp Sobolev inequality on $\mathbb{R}^n$ ([32], see also [15]):

**Theorem 1.2.** Let $0 < \gamma < \frac{n}{2}$. Then

$$
\frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\Gamma\left(\frac{n-2\gamma}{2}\right)} \frac{2^n}{\omega_n} \left( \int_{\mathbb{R}^n} |f|^\frac{n}{n-2\gamma} dx \right)^{\frac{n-2\gamma}{n}} \leq \int_{\mathbb{R}^n} |(-\Delta)\gamma f|^2 dx.
$$

(1.10)

Equality holds only for functions of the form

$$
c(\lambda^2 + |x - x_0|^2)^{-\frac{n-2\gamma}{2}}, \quad x \in \mathbb{R}^n,
$$

where $c \in \mathbb{R}$, $\lambda > 0$ and $x_0$ is some point in $\mathbb{R}^n$.

Recently, Ache and Chang [1] established sharp trace Sobolev inequality of order four on $\mathbb{B}^{n+1}$ for $n \geq 3$. As an application, they used this inequality to characterize the extremal metric of the main term in the log-determinant formula corresponding to the conformal Laplacian coupled with the boundary Robin operator on $\mathbb{B}^4$ (see [11, 12]). We state the results as follow.

**Theorem 1.3** (Ache and Chang). Given $f \in C^\infty(\mathbb{S}^n)$ with $n > 3$, suppose $v$ is a smooth extension of $f$ to the unit ball $\mathbb{B}^{n+1}$ which also satisfies the Neumann boundary condition

$$
\frac{\partial v}{\partial n} \bigg|_{\mathbb{S}^n} = -\frac{n-3}{2} f.
$$

(1.11)
Then we have the inequality

\[
\frac{2}{\Gamma\left(\frac{n+3}{2}\right)} \frac{\omega_n^{\frac{3}{2}}}{\Gamma\left(\frac{n-3}{2}\right)} \left( \int_{S^n} |f|^{\frac{2n}{n-3}} d\sigma \right)^{\frac{n-3}{n}} \leq \int_{B^{n+1}} |\Delta v|^2 dx + 2 \int_{S^n} |\tilde{\nabla} f|^2 d\sigma + \frac{(n+1)(n-3)}{2} \int_{S^n} |f|^2 d\sigma,
\]

where \(\Delta f\) is the Laplacian of \(f\) with respect to the Euclidean metric. Moreover, equality holds if and only if \(v\) is a biharmonic extension of a function of the form \(c|1 - \langle z_0, \xi \rangle|^\gamma\), where \(c\) is a constant, \(\xi \in S^n\), \(z_0\) is some point in \(B^{n+1}\), and \(v\) satisfies the boundary condition (1.11). When \(f = 1\), inequality (1.12) is attained by the function \(v(x) = 1 + \frac{n-3}{4}(1 - |x|^2)\).

**Theorem 1.4** (Ache and Chang). Given \(f \in C^\infty(S^3)\), suppose \(v\) is a smooth extension of \(f\) to the unit ball \(B^4\) which also satisfies the Neumann boundary condition

\[
\frac{\partial v}{\partial n} \bigg|_{S^3} = 0.
\]

Then we have the inequality

\[
\log \left( \frac{1}{2\pi} \int_{S^3} e^{3(f-T)} d\sigma \right) \leq \frac{3}{16\pi^3} \int_{B^4} |\Delta v|^2 dx + \frac{3}{8\pi^2} \int_{S^3} |\tilde{\nabla} f|^2 d\sigma.
\]

Moreover, equality holds if and only if \(v\) is a biharmonic extension of a function of the form \(-\log |1 - \langle z_0, \xi \rangle| + c\), where \(c\) is a constant, \(\xi \in S^3\), \(z_0\) is some point in \(B^4\) and \(v\) satisfies the boundary condition (1.13).

The proof of Theorem 1.3 and 1.4 relies on the use of scattering theory on hyperbolic space \((\mathbb{B}^{n+1}, g_\mathbb{B})\), where \(g_\mathbb{B} = \frac{4}{(1-|x|^2)^2} g_0\) and \(g_0 = |dx|^2\) is the Euclidean metric, and the right choice of distance function and adapted metric. We remark that the adapted metric, introduced by Case and Chang \cite{7}, is a ‘natural’ metric in the study of Sobolev inequalities (see \cite{13, 14}). The explicit formulas of adapted metric is computed by Ache and Chang \cite{1} only in the case \(\gamma = \frac{1}{2}\) and \(\gamma = \frac{3}{2}\). To the best of our knowledge, the explicit formulas of adapted metric is unknown for the rest of cases.

Very recently, Ngô, Nguyen and Pham \cite{35} show that (1.12) is equivalent to the following sharp Sobolev trace inequality on half space via Möbius transform:

**Theorem 1.5.** Let \(U \in W^{2,2}(\mathbb{R}^{n+1}_+\mathbb{H})\) be satisfied the Neumann boundary condition

\[
\partial_y U(x, 0) = 0,
\]

where \(\mathbb{H}\) is the hyperbolic space.
Where $W^{2,2}(\mathbb{R}^{n+1}_+)$ is the usually Sobolev space. Then we have the sharp trace inequality

\[
\frac{2}{\Gamma(n+\frac{3}{2})} \omega_n^{\frac{n-3}{n}} \left( \int_{\mathbb{R}^n} |U(x,0)|^{\frac{2m}{n}} dx \right)^{\frac{n-3}{n}} \leq \int_{\mathbb{R}^{n+1}_+} |\Delta U(x,y)|^2 dx dy.
\]

Furthermore, equality in (1.10) holds if and only if $U$ is a biharmonic extension of a function of the form $c(1 + |x - x_0|^2)^{-(n-3)/2}$, where $c$ is a constant, $x \in \mathbb{R}^n$, $x_0$ is some fixed point in $\mathbb{R}^n$ and $U$ fulfills the boundary condition (1.15).

In the same paper, Ngô, Nguyen and Pham [35], among other results, propose a slightly different approach to prove Sobolev trace inequality of order six, while Case and Luo [8] obtained, among other results, the same sharp Sobolev trace inequalities by deeply work of on the boundary operators. However, it seems that the argument in [35, 8] would become increasingly delicate when the order of the operator is large. A clear question is “What is the situation for sharp Sobolev trace inequality of higher order?” Another question is “what is the explicit formula of extremal function of such inequations?” In this paper, we shall give the answer of both questions.

The main results of this paper are the following three theorems.

**Theorem 1.6.** Let $n > 3$ and $m \geq 1$ with $2m + 1 < n$. Given $f \in C^\infty(S^n)$, suppose $v$ is a smooth extension of $f$ to the unit ball $\mathbb{B}^{n+1}$ which also satisfies the Neumann boundary condition:

\[
\Delta^k v|_{S^n} = (-1)^k \frac{\Gamma(m+1)\Gamma(m-k+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(m-k+1)} \frac{P_{2m+1}}{P_{2m+1-2k}} f; \quad 0 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor;
\]

\[
\frac{\partial}{\partial n} \Delta^k v|_{S^n} = (-1)^{k+1} \frac{n-1-2m+2k}{2} \frac{\Gamma(m+1)\Gamma(m-k+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(m-k+1)} \frac{P_{2m+1}}{P_{2m+1-2k}} f, \quad 0 \leq k \leq \lfloor \frac{m-1}{2} \rfloor.
\]

Then we have the inequality

\[
\frac{\Gamma(m+1)\Gamma(\frac{1}{2})\Gamma(\frac{n+2m+1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(\frac{n-2m-1}{2})} \frac{P_{2m+1}}{\omega_n n} \left( \int_{S^n} |f|^{\frac{2m+1}{n-2m-1}} d\sigma \right)^{\frac{n-2m-1}{n}} \leq \int_{\mathbb{B}^{n+1}} |\nabla^{m+1} v|^2 dx + \int_{S^n} fT_m f d\sigma,
\]

where

\[
\nabla^{m+1} = \begin{cases} \Delta^{m+1}, & m = \text{odd}; \\
\nabla \Delta^{\frac{m}{2}}, & m = \text{even}, 
\end{cases}
\]
and $T_m$ is an operator of order $2m$ defined as follow: if $m$ is an odd integer, then
\[
T_m = \frac{n-1}{2} \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} \frac{\mathcal{P}_{2m+1}}{\mathcal{P}_1} + \sum_{k=1}^{\frac{m-1}{2}} (m-2k) \frac{\Gamma(m+1)^2}{\Gamma(m+\frac{1}{2})^2} \frac{\mathcal{P}_{2m+1}}{\mathcal{P}_{2m+2k-1}}.
\]

if $m$ is a even integer, then
\[
T_m = \frac{n-1}{2} \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} \frac{\mathcal{P}_{2m+1}}{\mathcal{P}_1} + \sum_{k=1}^{\frac{m-1}{2}} (m-2k) \frac{\Gamma(m+1)^2}{\Gamma(m+\frac{1}{2})^2} \frac{\mathcal{P}_{2m+1}}{\mathcal{P}_{2m+2k-1}}.
\]

Moreover, equality holds if and only if
\[
(1.19)\quad v(x) = c \int_{S^n} \frac{(1-|x|^2)^{2m+1}}{|x-\xi|^{n+2m+2}|1-\langle x_0, \xi \rangle|^{2m+1-2n}} \, d\sigma,
\]
where $c$ is a constant and $x_0$ is some point in $\mathbb{B}^{n+1}$. When $f = 1$, inequality (1.18) is attained by the function $v(x) = \sum_{k=0}^{m} \frac{(-1)^{n+1-m}(-m)_k}{2^k \Gamma(n+1)} \frac{\mathcal{P}_n}{\mathcal{P}_{n-2k}}$, where $(a)_k$ is the rising Pochhammer symbol defined in Section 2.

**Theorem 1.7.** Let $n \geq 3$ be an odd integer. Given $f \in C^\infty(S^n)$, suppose $v$ is a smooth extension of $f$ to the unit ball $\mathbb{B}^{n+1}$ which also satisfies the Neumann boundary condition
\[
\Delta^k v|_{S^n} = (-1)^k \frac{\Gamma(n+1)}{\Gamma(n-k)} \frac{\mathcal{P}_n}{\mathcal{P}_{n-2k}} f; \quad 0 \leq k \leq \left\lfloor \frac{n-1}{4} \right\rfloor;
\]
\[
(1.20)\quad \frac{\partial}{\partial n} \Delta^k v|_{S^n} = (-1)^{k+1} \frac{\Gamma(n+1)}{\Gamma(\frac{n+1}{2})} \frac{\mathcal{P}_n}{\mathcal{P}_{n-2k}} f; \quad 0 \leq k \leq \left\lfloor \frac{n-3}{4} \right\rfloor.
\]

Then we have the inequality
\[
(1.21)\quad \log \left( \frac{1}{\omega_n} \int_{S^n} e^{n(f-\mathcal{T})} \, d\sigma \right) \leq \frac{n}{2^{n+1} \pi^\frac{n+1}{2}} \frac{n}{\Gamma(\frac{n+1}{2})} \left( \int_{\mathbb{B}^{n+1}} |\nabla^{\frac{n+1}{2}} v|^2 \, dx + \int_{S^n} f T_{n-1}^f \, d\sigma \right),
\]
where the operator $T_{n-1}^{\frac{1}{2}}$ is defined in Theorem 1.6. Moreover, equality holds if and only if
\begin{equation}
(1.22) \quad v(x) = \pi^{-\frac{n}{2}} \frac{\Gamma(n)}{2^n \Gamma(n/2)} \int_{S^n} \frac{(1 - |x|^2)^n}{|x - \xi|^{2n}} (-\ln |1 - \langle x_0, \xi \rangle| + c) d\sigma
\end{equation}
where $c$ is a constant and $x_0$ is some point in $\mathbb{B}^{n+1}$.

**Remark 1.8.** We remark that one can also replace the Neumann boundary condition (1.17) or (1.20) by $\partial_n v|_{S^n} = \partial_n V_m|_{S^n}, \partial^2_n v|_{S^n} = \partial^2_n V_m|_{S^n}, \cdots$. When $k$ is small, we can compute the value through (4.2) and (4.3). For example, we have
\begin{align*}
\partial_n V_m|_{S^n} &= \frac{2m + 1 - n}{2} f, \quad \partial^2_n V_m|_{S^n} = \frac{\tilde{\Delta} f}{2m - 1} + \frac{(n - 1 - 2m)[(m - 1)n - m(2m - 1)]}{2(2m - 1)} f.
\end{align*}
However, the argument would become increasingly delicate when $k$ is large.

**Theorem 1.9.** Let $n > 2m + 1 \geq 3$ and $U(x, y) \in W^{m+1,2}(\mathbb{R}^{n+1})$ be satisfied the Neumann boundary condition
\begin{align}
(1.23) \quad & \Delta^k U_m(x, y)|_{y=0} = \frac{\Gamma(m + 1)\Gamma(m + \frac{1}{2} - k)}{\Gamma(m - k - 1)\Gamma(m + \frac{1}{2})} \Delta^k f, \quad 0 \leq k \leq \left[\frac{m}{2}\right] ; \\
& \partial_y \Delta^k U_m(x, y)|_{y=0} = 0, \quad 0 \leq k \leq \left[\frac{m - 1}{2}\right].
\end{align}
Then we have the sharp trace inequality
\begin{equation}
\frac{\Gamma(m + 1)\Gamma(\frac{1}{2}) \Gamma(\frac{n+2m+1}{2})}{\Gamma(m + \frac{1}{2}) \Gamma(\frac{n-2m-1}{2})} \omega_n^{2m+1} \left( \int_{\mathbb{R}^n} |U(x, 0)|^{\frac{2n}{n-2m-1}} dx \right)^{\frac{n-2m-1}{n}} \leq \int_{\mathbb{R}^{n+1}} \left| \nabla^{m+1} U \right|^2 dx dy.
\end{equation}
Furthermore, equality holds if and only
\begin{equation}
(1.24) \quad U(x, y) = c \int_{\mathbb{R}^n} \frac{y^{1+2m}}{|x - \xi|^2 + y^2} (\lambda^2 + |\xi - \xi_0|^2)^{-\frac{n-2m-1}{2}} d\xi,
\end{equation}
where $\lambda > 0$, $c$ is a constant and $\xi_0$ is some fixed point in $\mathbb{R}^n$.

**Remark 1.10.** We remark that, because of Lemma 5.4, one can also replace the Neumann boundary condition (1.23) by the following:
\begin{align}
(1.25) \quad & \partial^2_y U_m(x, y)|_{y=0} = \frac{\Gamma(k + \frac{1}{2})\Gamma(m - k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(m + \frac{1}{2})} \Delta^k f, \quad 0 \leq k \leq \left[\frac{m}{4}\right] ; \\
& \partial^2_y U(x, y)|_{y=0} = 0, \quad 0 \leq k \leq \left[\frac{m - 1}{4}\right].
\end{align}
Notice that the boundary condition (1.25) is different to that given by R. Yang (41).
This article is organized as follows: In Section 2, we briefly quote some properties of special functions, such as hypergeometric function and Gegenbauer polynomials, and Funk-Hecke formula for spherical harmonics which will be used in the paper. In Section 3 we first review the connection between scattering theory and conformally invariant objects on their boundaries. Next we compute the explicit formulas of the solution of Poisson equation and the adapted metrics on the model case \((\mathbb{B}^{n+1}, \mathbb{S}^n, g_{\mathbb{B}})\). In Section 4, we prove Theorem 1.6 and 1.7. The proof of Theorem 1.9 is given in Section 5.

2. Preliminaries

In this section, we quote some preliminary facts which will be needed in the sequel.

2.1. Hypergeometric function. We use the notation \(F(a, b; c; z)\) to denote

\[
F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k,
\]

where \(c \neq 0, -1, \ldots, -n, \ldots\) and \((a)_k\) is the rising Pochhammer symbol defined by

\[
(a)_0 = 0, \quad (a)_k = a(a+1) \cdots (a+k-1), \quad k \geq 1.
\]

If either \(a\) or \(b\) is a nonpositive integer, then the series terminates and the function reduces to a polynomial.

Here, we only list some of properties of hypergeometric function which will be used in the rest of paper. For more information about these functions, we refer to \([25]\), section 9.1 and \([17]\), Chapter II.

- The hypergeometric function \(F(a, b; c; z)\) satisfies the hypergeometric differential equation

\[
z(1-z)F'' + (c - (a + b + 1)z)F' - abF = 0.
\]

- If \(\text{Re}(c - a - b) > 0\), then \(F(a, b; c; 1)\) exists and

\[
F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\]

- Transformation formulas (1):

\[
F(a, b; c; z) = (1 - z)^{c-a-b}F(c - a, c - b; c; z).
\]

- Transformation formulas (2): if \(c - a - b\) is not an integer, then

\[
F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b; a + b - c + 1; 1 - z) + \frac{(1 - z)^{c-a-b}}{\Gamma(a)\Gamma(b)} \Gamma(c)\Gamma(a + b - c)}{\Gamma(c - a - b + 1; 1 - z).}
\]
• Differentiation formula:
\begin{equation}
\frac{d^k}{dz^k} F(a, b; c; z) = \frac{(a)_k(b)_k}{(c)_k} F(a + k, b + k; c + k; z), \quad k \geq 1.
\end{equation}

2.2. Gegenbauer polynomials. We use the notation $C^\alpha_k(x)$ to denote the Gegenbauer polynomial of degree $k$ which can be defined in terms of the generating function:
\begin{equation}
\frac{1}{(1 - 2xt + t^2)^\alpha} = \sum_{k=0}^{\infty} C^\alpha_k(x)t^k.
\end{equation}

Here, we also list some of properties of Gegenbauer polynomial and refer to \cite{40} and \cite{25}, section 8.93 for more information about this polynomial.

• Rodrigues formula:
\begin{equation}
C^\alpha_k(x) = \frac{(-1)^k \Gamma(\alpha + \frac{1}{2})\Gamma(k + 2\alpha)}{2^k k! \Gamma(2\alpha)\Gamma(\alpha + k + \frac{1}{2})} (1 - x^2)^{-\alpha + \frac{1}{2}} \frac{d^k}{dx^k} (1 - x^2)^{k + \alpha - \frac{1}{2}}.
\end{equation}

• Orthogonality and normalization: if $k \neq m$, then
\begin{equation}
\int_{-1}^{1} C^\alpha_k(x)C^\alpha_m(x)(1 - x^2)^{\alpha - \frac{1}{2}}dx = 0;
\end{equation}
if $k = m$, then
\begin{equation}
\int_{-1}^{1} [C^\alpha_k(x)]^2(1 - x^2)^{\alpha - \frac{1}{2}}dx = \frac{\pi 2^{1 - 2\alpha}\Gamma(k + 2\alpha)}{k!(k + \alpha)\Gamma(\alpha)}.
\end{equation}

• Differentiation formulas:
\begin{equation}
\frac{d^m}{dx^m} C^\alpha_k(x) = \begin{cases} 2^m \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} C^{\alpha + m}_{k-m}(x), & k - m \geq 0; \\ 0, & k - m < 0. \end{cases}
\end{equation}

Finally, we recall an integral (see \cite{25}, page 407, 3.665)
\begin{equation}
\int_0^\pi \frac{\sin^{2\mu-1} \theta}{(1 - 2t \cos \theta + t^2)^\alpha} d\theta = \frac{\Gamma(\mu)\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} F(\alpha, \alpha - \mu + \frac{1}{2}; \mu + \frac{1}{2}; t^2), \quad \text{Re} \mu > 0, |t| < 1.
\end{equation}

Using the expansion (2.11) and (2.7), we have, for $k \geq 0$,
\begin{equation}
\begin{aligned}
\int_0^\pi C^\alpha_{2k}(\cos \theta) \sin^{2\mu-1} \theta d\theta &= \frac{\Gamma(\mu)\Gamma(\frac{1}{2}) (\alpha)_k(\alpha - \mu + \frac{1}{2})_k}{\Gamma(\mu + \frac{1}{2}) (\mu + \frac{1}{2})_k} \frac{1}{k!}, \\
\int_0^\pi C^\alpha_{2k+1}(\cos \theta) \sin^{2\mu-1} \theta d\theta &= 0.
\end{aligned}
\end{equation}
2.3. **Funk-Hecke formula.** It is known that $L^2(S^n)$ can be decomposed as follow

$$L^2(S^n) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l,$$

where $\mathcal{H}_l$ is the space of spherical harmonics of degree $l$ (see [40]). For $n \geq 2$, the Funk-Hecke formula reads as follow (see e.g. [2, 24])

$$\int_{S^n} K((\xi,\eta)) Y_l(\eta) d\sigma(\eta) = \lambda_l Y_l,$$

where $K \in L^1((-1,1),(1-t^2)^{\frac{n-1}{2}} dt)$ and $Y_l \in \mathcal{H}_l$. Moreover, if $Y_l \in \mathcal{H}_l$, then

$$-\tilde{\Delta} Y_l = l(n-1+l) Y_l$$

and thus

$$BY_l = \left( l + \frac{n-1}{2} \right) Y_l, \quad P_{2\gamma} Y_l = \frac{\Gamma(l + \frac{n}{2} + \gamma)}{\Gamma(l + \frac{n}{2} - \gamma)} Y_l,$$

where $B$ and $P_{2\gamma}$ defined in (1.7).

3. **ADAPTED METRICS**

Firstly, we briefly review the definition of the fractional GJMS operator via scattering theory (see [28]). A triple $(X^{n+1}, M^n, g_+)$ is a Poincaré-Einstein manifold if

(1) $X^{n+1}$ is (diffeomorphic to) the interior of a compact manifold $\overline{X}^{n+1}$ with boundary $\partial \overline{X} = M^n$,

(2) $X^{n+1}$ is complete with $\text{Ric}(g_+) = -ng_+$, and

(3) there exists a nonnegative $\rho \in C^\infty(\overline{X})$ such that $\rho^{-1}(0) = M^n$, $d\rho \neq 0$ along $M$, and the metric $g := \rho^2 g_+$ extends to a smooth metric on $\overline{X}^{n+1}$.

A function $\rho$ satisfying (3) above is called a defining function. It is obvious that the conformal class $[h] := [g|_{TM}]$ on $M$ is well-defined for a Poincaré-Einstein manifold because $\rho$ is only determined up to multiplication by a positive smooth function on $X$.

Given a Poincaré-Einstein manifold $(X^{n+1}, M^n, g_+)$ and a representative $[h]$ on the conformal boundary, there is a uniquely defining function $\rho$ such that $g_+ = \rho^{-2}(d\rho^2 + h_\rho)$ on $M \times (0,\delta)$, where $h_\rho$ is a one-parameter family of metrics on $M$ satisfying $h_0 = h$. Given $f \in C^\infty(M)$. It has been shown (see [33, 28]) that the Poisson equation

$$-\Delta_{g_+} u - s(n-s)u = 0$$

(3.1)
has a unique solution of the form
\begin{equation}
    u = F \rho^{n-s} + H \rho^s, \quad F, H \in C^\infty(X), \quad F|_{\rho=0} = f,
\end{equation}
where $s \in \mathbb{C}$ and $s(n-s)$ do not belongs to the pure point spectrum of $-\Delta_{g_\rho}$.

The scattering operator on $M$ is defined as $S(s)f = H|_M$. If Re$(s) > \frac{n}{2}$, then the scattering operator is a meromorphic family of pseudo-differential operators. Graham and Zworski \cite{28} defined the fractional GJMS operator $P_{2\gamma}(\gamma \in (0, \frac{n}{2}) \setminus \mathbb{N})$ as follow
\begin{equation}
    P_{2\gamma}f := d_\gamma S \left( \frac{n}{2} + \gamma \right) f, \quad d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}.
\end{equation}
Here we denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. In term of $P_{2\gamma}$, the the fractional $Q$-curvature $Q_{2\gamma}$ is defined by
\begin{equation}
    Q_{2\gamma} := \frac{2}{n - 2\gamma} P_{2\gamma}(1).
\end{equation}
If $\gamma \in \mathbb{N}_0$, then $P_{2\gamma}$ is nothing but the GJMS operator on $M$ (see \cite{26}). It has been also shown by Graham and Zworski \cite{28} that the principal symbol of $P_{2\gamma}$ is is exactly the principal symbol of the fractional Laplacian $(-\Delta)^\gamma$ and satisfy an important conformal covariance property: for a conformal change of metric $\hat{h} = e^{2\tau} h$, we have
\begin{equation}
    \hat{P}_{2\gamma} f = e^{-\frac{n+2\gamma}{2}\tau} P_{2\gamma} \left( e^{\frac{n-2\gamma}{2}\tau} f \right), \quad \forall f \in C^\infty(M).
\end{equation}

Next we recall the adapted metric, introduced by Case and Chang \cite{7}, on a conformally compact Poincaré-Einstein manifold $(X^{n+1}, \partial X, g_\gamma)$. This metric is introduced for any parameter $s = \frac{n}{2} + \gamma$ with $\gamma \in (0, \frac{n}{2})$ and $s = n$ if $n$ is odd. For such an $s$, we denote by $\vartheta_s$ the solution of Poisson equation (3.1) with Dirichlet condition $f \equiv 1$. Notice that if the Yamabe constant of the boundary metric $h$ is positive, then by a result of Lee (see \cite{31}, Theorem A), we have $\vartheta_s > 0$ so that one can take $\rho_s := (\vartheta_s)^{-\frac{1}{n-s}}$ as a defining function. The metric $g_s = \rho_s^2 g_\gamma$ is called adapted metric. In the limiting case, namely $s = n$ and $n$ is an odd integer, the adapted metric, appeared in \cite{21}, is defined as $g^* = e^{2\tau}$, where
\begin{equation}
    \tau = - \frac{d}{ds} \vartheta_s|_{s=n}.
\end{equation}
We remark that $\tau$ satisfies $-\Delta_{g_\tau} \tau = n$. For more information about GJMS operator and adapted metric, we refer to \cite{4, 5, 6, 7, 8, 10, 13, 14, 21, 23, 27, 28, 29, 31, 41}.

In the rest of this section, we shall consider the model case $(\mathbb{B}^{n+1}, S^n, g_B)$, where $g_B = \frac{1}{(1-|x|^2)^2} g_0$ and $g_0 = |dx|^2$ is the Euclidean metric. The defining function is $\rho = \frac{1-|x|^2}{2}$. Firstly, we give the explicit formula of the solution of Poisson equation (3.1). The main result is the following theorem:
**Theorem 3.1.** Let $\gamma \in (0, \frac{n}{2})$, $s = \frac{n}{2} + \gamma$ and $\rho = \frac{1-|x|^2}{2}$. The solution of the following Poisson equation on the hyperbolic space $(\mathbb{B}^{n+1}, g)$

\[
\begin{cases}
-\Delta_{g_B} u - s(n-s)u = 0 \quad \text{in } \mathbb{B}^{n+1}, \\
u = F\rho^{n-s} + H\rho^s, \\
F|_{\partial \mathbb{B}^{n+1}} = f(\xi),
\end{cases}
\]

is

\[
u(x) = \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\gamma)} \int_{S^n} \left(\frac{1 - |x|^2}{2|x - \xi|^2}\right)^s f(\xi) d\sigma.
\]

Furthermore, if $f$ has an expansion in spherical harmonics, $f = \sum_{l=0}^{\infty} Y_l$, where $Y_l$ is a spherical harmonic of degree $l$, then (here we set $r = |x|$)

\[
u(x) = \rho^{n-s} \sum_{l=0}^{\infty} \varphi_l(r^2) r^l Y_l,
\]

where

\[
\varphi_l(r) = \frac{\Gamma(\gamma + \frac{1}{2}) \Gamma(\frac{n}{2} + \gamma)}{\Gamma(2\gamma) \Gamma(l + \frac{n}{2} + \gamma)} F(l + \frac{n}{2} - \gamma, 1 - \gamma, l + \frac{n+1}{2} ; r)
\]

satisfying $\varphi_1(1) = 1$.

**Proof.** Set

\[
V(x) = \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\gamma)} \rho^{2\gamma} \int_{S^n} \left(\frac{1 - |x|^2}{2|x - \xi|^2}\right)^s f(\xi) d\sigma
= \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\gamma)} \rho^{2\gamma} \int_{S^n} \frac{f(\xi)}{1 - 2x \cdot \xi + |x|^2} \frac{n}{2} + \gamma d\sigma.
\]

By Funk-Hecke formula, if $f = \sum_{l=0}^{\infty} Y_l$, then

\[
V(x) = \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\gamma)} \rho^{2\gamma} \sum_{l=0}^{\infty} \lambda_l Y_l,
\]

where

\[
\lambda_l = (4\pi)^{\frac{n-1}{2}} \frac{l! \Gamma(\frac{n-1}{2})}{\Gamma(l + n - 1)} \int_{-1}^{1} \frac{1}{(1 - 2rt + r^2)^{\frac{n}{2} + \gamma}} C_l^{\frac{n}{2}}(t)(1 - t^2)^{\frac{n-2}{2}} dt
\]

\[
(4\pi)^{\frac{n-1}{2}} \frac{l! \Gamma(\frac{n-1}{2})}{\Gamma(l + n - 1)} \sum_{k=0}^{\infty} \int_{-1}^{1} C_k^{\frac{n}{2} + \gamma}(t) C_l^{\frac{n}{2}}(t)(1 - t^2)^{\frac{n-2}{2}} dt.
\]
Using the Rodrigues formula (2.8) and differentiation formula (2.11), we have

\[
\sum_{k=0}^{\infty} r^k \int_{-1}^{1} C_k^{n+\gamma}(t) C_{l}^{n+\gamma}(t) (1 - t^2)^{-\frac{n-2}{2}} dt
\]

\[
= (-1)^l \frac{\Gamma\left(\frac{n}{2}\right) \Gamma(l + n - 1)}{2^{l!} \Gamma(n - 1) \Gamma(l + \frac{n}{2})} \sum_{k=0}^{\infty} r^k \int_{-1}^{1} C_k^{n+\gamma}(t) \frac{d^l}{dt^l} (1 - t^2)^{l+\frac{n-2}{2}} dt
\]

(3.12)

\[
= \frac{1}{2^{l!} \Gamma(n - 1) \Gamma(l + \frac{n}{2})} \sum_{k=l}^{\infty} 2^l \frac{\Gamma\left(\frac{n}{2} + \gamma + l\right)}{\Gamma\left(l + \frac{n}{2} + \gamma\right)} r^k \int_{-1}^{1} C_k^{n+\gamma+l}(t) (1 - t^2)^{l+\frac{n-2}{2}} dt
\]

Substituting \( t = \cos \theta \), we have, by (2.12),

(3.13)

\[
\sum_{k=0}^{\infty} r^k \int_{-1}^{1} C_k^{n+\gamma+l}(t) (1 - t^2)^{l+\frac{n-2}{2}} dt = \sum_{k=0}^{\infty} r^k \int_{0}^{\pi} C_k^{n+\gamma+l}(\cos \theta) \sin^{2l+n-1} \theta d\theta
\]

\[
= \sum_{k=0}^{\infty} r^{l+2k} \int_{0}^{\pi} C_{2k}^{n+\gamma+l}(\cos \theta) \sin^{2l+n-1} \theta d\theta = \frac{\Gamma\left(l + \frac{n}{2}\right) \Gamma\left(l + n + 1\right)}{\Gamma\left(l + \frac{n+1}{2}\right)} \sum_{k=0}^{\infty} r^{l+2k} \frac{\Gamma\left(\frac{n}{2} + \gamma + l\right) k \Gamma\left(\gamma + \frac{1}{2}\right) k}{(l + \frac{n+1}{2}) k!} 1
\]

Combing (3.11) and (3.13) yields

(3.14)

\[
\lambda_l = (4\pi)^{\frac{n+1}{2}} \frac{l! \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(l + n - 1\right)} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(l + n - 1\right) \Gamma\left(\frac{n}{2} + \gamma + l\right)}{\Gamma\left(l + \frac{n}{2} + \gamma\right)} \frac{\Gamma\left(l + \frac{n+1}{2}\right)}{\Gamma\left(l + \frac{n+1}{2} + \gamma\right)} F(\gamma + \frac{1}{2}, l + \gamma, l + \frac{n+1}{2}; r^2) r^l
\]

\[
= (4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2}\right) \frac{\Gamma\left(l + n - 1\right) \Gamma\left(l + \frac{n}{2} + \gamma\right)}{\Gamma\left(l + \frac{n}{2} + \gamma\right)} F(\gamma + \frac{1}{2}, l + \gamma, l + \frac{n+1}{2}; r^2) r^l
\]

\[
= (4\pi)^{\frac{n+1}{2}} \Gamma\left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2}\right) \frac{\Gamma\left(l + n - 1\right) \Gamma\left(l + \frac{n}{2} + \gamma\right)}{\Gamma\left(l + \frac{n}{2} + \gamma\right)} F(\gamma + \frac{1}{2}, l + \gamma, l + \frac{n+1}{2}; r^2) r^l
\]

\[
= (1 - r^2)^{-2\gamma} F\left(l + \frac{n}{2} - \gamma, \frac{1}{2}, \gamma, l + \frac{n+1}{2}; 2r^2\right) r^l
\]

\[
= -2^{-2\gamma} (4\pi)^{\frac{n+1}{2}} \frac{\Gamma\left(\frac{n}{2} - 1\right) \Gamma\left(\frac{n}{2}\right) \Gamma\left(l + \frac{n}{2} + \gamma\right)}{\Gamma\left(l + \frac{n}{2} + \gamma\right)} \varphi_{\gamma}(r^2) r^l
\]
Therefore,

\[
V(x) = \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2} + \gamma\right)}{\Gamma(\gamma)} \rho^{2\gamma} \sum_{l=0}^{\infty} \lambda_l Y_l
\]

\[
= 2^{n-1-2\gamma} \frac{\Gamma(2\gamma)}{\Gamma(\gamma)\Gamma(\gamma + \frac{1}{2})} \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n-1)} \rho^{2\gamma} \sum_{l=0}^{\infty} \varphi_l(r^2) r^l Y_l
\]

\[
= \sum_{l=0}^{\infty} \varphi_l(r^2) r^l Y_l.
\]

(3.15)

To get the last equation above, we use the duplication formula

\[
\Gamma(2z) = 2^{2z-1} \frac{\Gamma(z)\Gamma(z + \frac{1}{2})}{\Gamma(\frac{1}{2})}.
\]

(3.16)

By (2.3), we have \( \varphi_l(1) = 1 \) and thus \( V(x)|_{r=1} = f(x) \).

By the uniqueness of the solution, to finish the proof, it is enough to show

\[
-\Delta_{g_\rho} \left[ \rho^{n-2\gamma} V(x) \right] - s(n-s) \rho^{s-2\gamma} V(x) = 0,
\]

(3.17)

or, equivalently,

\[
-\Delta_{g_\rho} \left[ \rho^{n-s} \varphi_l(r^2) r^l Y_l \right] - s(n-s) \rho^{n-s} \varphi_l(r^2) r^l Y_l = 0, \quad \forall l \geq 0.
\]

(3.18)

Recall the conformal laplacian on \((\mathbb{B}^{n+1}, g_\rho)\) is

\[
L_{g_\rho} = -\Delta_{g_\rho} + \frac{n-1}{4n} \text{Scal}(g_\rho),
\]

where \( \text{Scal}(g_\rho) \) is scalar curvature on \((\mathbb{B}^{n+1}, g_\rho)\). Since \((\mathbb{B}^{n+1}, g_\rho)\) has the constant sectional curvature \(-1\), we have \( \text{Scal}(g_\rho) = -n(n+1) \) and thus

\[
L_{g_\rho} = -\Delta_{g_\rho} - \frac{n^2-1}{4}.
\]

(3.19)

By the conformal covariant property of the conformal Laplacian for the change of metric, we have

\[
L_{g_\rho} f = \rho^{\frac{n+3}{2}} (-\Delta) \left( \rho^{-\frac{n-1}{2}} f \right), \quad \forall f \in C^\infty(\mathbb{B}^{n+1}),
\]

(3.20)

where \( \Delta \) is the Laplacian on Euclidean space. We have, by (3.19) and (3.20),

\[
\left( \Delta_{g_\rho} + \frac{n^2-1}{4} \right) \rho^{n-s} \varphi_l(r^2) r^l Y_l = \rho^{\frac{n+3}{2}} \Delta \left( \rho^{-\frac{n-1}{2}} \rho^{n-s} \varphi_l(r^2) r^l Y_l \right)
\]

\[
= \rho^{\frac{n+3}{2}} \Delta \left( \rho^{\frac{1}{2}-\gamma} \varphi_l(r^2) r^l Y_l \right)
\]

\[
= \rho^{\frac{n+3}{2}} \left[ \Delta \left( \rho^{\frac{1}{2}-\gamma} \varphi_l(r^2) \right) r^l Y_l + 2 \langle \nabla \left( \rho^{\frac{1}{2}-\gamma} \varphi_l(r^2) \right), \nabla r^l Y_l \rangle \right].
\]

(3.21)
To get the last equation, we use the fact $\Delta r^l Y_l = 0$ since $Y_l$ is the spherical harmonic of degree $l$. Substituting the polar coordinate formula $\Delta = \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta$ into (3.21), we have

$$\left( \Delta g_{bs} + \frac{n^2 - 1}{4} \right) \rho^{n-s} \varphi_l(r^2) r^l Y_l = \rho^{\frac{n+3}{2}} \left[ (\rho^{\frac{1}{2} - \gamma} \varphi_l(r^2))'' + \frac{n+2l}{r} \left( \rho^{\frac{1}{2} - \gamma} \varphi_l(r^2) \right)' \right] r^l Y_l.$$  

(3.22)

We compute

$$\left( \rho^{\frac{1}{2} - \gamma} \varphi_l(r^2) \right)' = 2r \rho^{\frac{1}{2} - \gamma} \varphi_l'(r^2) + (\gamma - \frac{1}{2}) r \rho^{\frac{1}{2} - \gamma} \varphi_l(r^2);$$

$$\left( \rho^{\frac{1}{2} - \gamma} \varphi_l(r^2) \right)'' = 4r^2 \rho^{\frac{1}{2} - \gamma} \varphi_l''(r^2) + 2 \rho^{\frac{1}{2} - \gamma} \left[ \rho - (1 - 2\gamma) r^2 \right] \varphi_l'(r^2) + \left[ (\gamma - \frac{1}{2}) \rho^{\frac{1}{2} - \gamma} + (\gamma^2 - \frac{1}{4}) r^2 \rho^{\frac{1}{2} - \gamma} \right] \varphi_l(r^2).$$

(3.23)

Substituting (3.23) into (3.22), we obtain

$$\left( \Delta g_{bs} + \frac{n^2 - 1}{4} \right) \rho^{n-s} \varphi_l(r^2) r^l Y_l = \rho^{\frac{n+3}{2}} \left\{ 4r^2 \rho^2 \varphi_l''(r^2) + 2 \rho \left[ l + \frac{n+1}{2} - \left( l + \frac{n+3}{2} - 2\gamma \right) r^2 \right] \varphi_l'(r^2) + \left[ (n+2l+1)(\gamma - \frac{1}{2}) \rho + (\gamma^2 - \frac{1}{4}) r^2 \right] \varphi_l(r^2) \right\} r^l Y_l.$$  

(3.24)

By (2.2), $\varphi_l(r^2)$ satisfies the hypergeometric differential equation

$$2r^2 \rho \varphi_l''(r^2) = - \left[ l + \frac{n+1}{2} - \left( l + \frac{n+3}{2} - 2\gamma \right) r^2 \right] \varphi_l'(r^2) + \left( \frac{n}{2} - \gamma \right) \left( \frac{1}{2} - \gamma \right) \varphi_l(r^2).$$

(3.25)

Substituting (3.25) into (3.24), we obtain

$$\left( \Delta g_{bs} + \frac{n^2 - 1}{4} \right) \rho^{n-s} \varphi_l(r^2) r^l Y_l = \rho^{\frac{n+3}{2}} \left[ (1 + 2\gamma)(\gamma - \frac{1}{2}) \rho + (\gamma^2 - \frac{1}{4}) r^2 \right] \varphi_l(r^2) r^l Y_l = \rho^{\frac{n+3}{2}} (\gamma^2 - \frac{1}{4}) \varphi_l(r^2) r^l Y_l.$$  

This proves equality (3.18). The roof of Theorem 3.1 is thereby completed.
Before we compute the explicit formulas of adapted metric in term of $\rho$, we need the following Lemma:

**Lemma 3.2.** Let $u(x)$ be the solution of (3.6) and $f = \sum_{i=0}^{\infty} Y_i$ be the expansion in spherical harmonics. If $\gamma \in (0, \frac{n}{2}) \setminus \frac{1}{2}\mathbb{N}$, where $\frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \cdots, \frac{n}{2}, \cdots\}$, then

$$u(x) = \rho^{n-s} \sum_{l=0}^{\infty} F(l + \frac{n}{2} - \gamma, \frac{1}{2} - \gamma, 1 - 2\gamma; 2\rho)r^l Y_l +$$

(3.26)

$$\rho^s \frac{\Gamma(-\lambda)}{2^{2\lambda} \Gamma(\gamma)} \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{n}{2} + \gamma)}{\Gamma(l + \frac{n}{2} - \gamma)} F(\frac{1}{2} + \gamma, l + \frac{n}{2} + \gamma, 1 + 2\gamma; 2\rho)r^l Y_l.$$

If $\gamma = m + \frac{1}{2}$ with $m = [\gamma] < \frac{n-1}{2}$, then

(3.27)

$$u(x) = \rho^{\frac{n+1}{2} - m} \sum_{l=0}^{\infty} \left( \sum_{k=0}^{m} \frac{(l + \frac{n-1}{2} - m)k(-m)_k (2\rho)^k}{(-2m)_k k!} \right) r^l Y_l,$$

Proof. If $\gamma \in (0, \frac{n}{2}) \setminus \frac{1}{2}\mathbb{N}$, we have, by (2.4) and (2.5),

$$\varphi_l(r^2) = \frac{\Gamma(\gamma + \frac{1}{2}) \Gamma(l + \frac{n}{2} + \gamma)}{\Gamma(2\gamma) \Gamma(l + \frac{n+1}{2})} F(l + \frac{n}{2} - \gamma, \frac{1}{2} - \gamma, l + \frac{n+1}{2}; r^2)$$

$$= \frac{\Gamma(\gamma + \frac{1}{2}) \Gamma(l + \frac{n}{2} + \gamma)}{\Gamma(2\gamma) \Gamma(l + \frac{n+1}{2})} (1 - r^2)^{2\gamma} F(\frac{1}{2} + \gamma, l + \frac{n}{2} + \gamma, l + \frac{n+1}{2}; r^2)$$

$$= \frac{\Gamma(\gamma + \frac{1}{2}) \Gamma(l + \frac{n}{2} + \gamma)}{\Gamma(2\gamma) \Gamma(l + \frac{n+1}{2})} (1 - r^2)^{2\gamma}.$$

Using the duplication formula (3.16), we have

$$\varphi_l(r^2) = \rho^{2\gamma} \frac{\Gamma(-\lambda) \Gamma(l + \frac{n}{2} + \gamma)}{2^{2\lambda} \Gamma(\gamma) \Gamma(l + \frac{n}{2} - \gamma)} F(\frac{1}{2} + \gamma, l + \frac{n}{2} + \gamma, 1 + 2\gamma; 2\rho)$$

(3.28)

$$+ F(l + \frac{n}{2} - \gamma, \frac{1}{2} - \gamma, 1 - 2\gamma; 2\rho).$$
Substituting (3.28) into (3.8), we get (3.26).

If \( \gamma = m + \frac{1}{2} \), then

\[
\varphi_l(r) = \frac{\Gamma(m + 1)}{\Gamma(2m + 1)} \frac{\Gamma(l + m + \frac{n+1}{2})}{\Gamma(l + \frac{n+1}{2})} \frac{F(l + \frac{n-1}{2} - m, -m; l + \frac{n+1}{2}; r)}{\Gamma\left(l + m + \frac{n+1}{2} - m\right)}
\]

is a polynomial of degree \( m \) and thus

\[
\varphi_l(r) = \sum_{k=0}^{m} \frac{\varphi_l^{(k)}(1)}{k!} (r - 1)^k.
\]

Using (2.6) and (2.4), we have, for \( 0 \leq k \leq m \),

\[
\varphi_l^{(k)}(1) = \frac{\Gamma(m + 1)}{\Gamma(2m + 1)} \frac{\Gamma(l + m + \frac{n+1}{2})}{\Gamma(l + \frac{n+1}{2})} \frac{F(l + \frac{n-1}{2} - m + k, -m + k; l + \frac{n+1}{2} + k; 1)}{\Gamma\left(l + m + \frac{n+1}{2} - m + k\right)}
\]

\[
= (-1)^k \frac{(l + \frac{n-1}{2} - m)_{k}(\frac{n-1}{2} + m)_{k}}{(-2m)_{k}}.
\]

Substituting (3.30) into (3.29) and using (3.8), we get (3.27).

\[
\varphi_l(r) = \sum_{k=0}^{m} \frac{\varphi_l^{(k)}(1)}{k!} (r - 1)^k.
\]

Corollary 3.3. Let \( \vartheta_s \) be the solution of (3.6) when \( f = 1 \). If \( \gamma \in (0, \frac{n}{2}) \setminus \frac{1}{2} \mathbb{N} \), then

\[
\vartheta_s(r) = r^{\frac{n}{2} - \gamma} F\left(\frac{n}{2} - \gamma, \frac{1}{2} - \gamma; 1 - 2\gamma; 2\rho\right) + \rho^{\frac{n}{2} + \gamma} \frac{\Gamma(-\lambda)}{2^{n} \Gamma(\gamma) \Gamma\left(\frac{n}{2} - \gamma\right)} F\left(\frac{1}{2} + \gamma, \frac{n}{2} + \gamma; 1 + 2\gamma; 2\rho\right).
\]

If \( \gamma = m + \frac{1}{2} \) with \( m = \lfloor \gamma \rfloor < \frac{n-1}{2} \), then

\[
\vartheta_s(r) = r^{\frac{n-1}{2} - m} \sum_{k=0}^{m} \frac{(\frac{n-1}{2} - m)_{k}(\frac{n-1}{2} + m)_{k}}{(-2m)_{k}} (\frac{n-1}{2} + 2m)_{k}.
\]

Now we can compute the explicit formulas of adapted metric \( g^* \) on the model case \((B^{n+1}, S^n, g_\mathbb{B})\).

Proposition 3.4. Let \( \gamma \in (0, \frac{n}{2}) \) and \( s = \frac{n}{2} + \gamma \). On the model case \((B^{n+1}, S^n, g_\mathbb{B})\) we have
(1) if $\gamma \in (0, \frac{n}{2}) \setminus \frac{1}{2} \mathbb{N}$, then $g^* = \psi_{\gamma}^{2-\gamma} |dx|^2 = \psi_{\gamma}^{4} |dx|^2$, where

$$
\psi_{\gamma} = \frac{\Gamma(n+1-\gamma, 1/2-\gamma)}{\Gamma(n-\gamma)} F\left(\frac{n}{2} - \gamma, 1/2 - \gamma; 1 - 2\gamma; 2\rho\right) + \rho^{\gamma} \frac{1}{\Gamma(\gamma)} \frac{\Gamma(n+1-\gamma, 1/2-\gamma)}{\Gamma(n-\gamma)} F\left(\frac{n}{2} + \gamma, n+1; 1 - 2\gamma; 2\rho\right);
$$

(3.33)

(2) if $\gamma = m + \frac{1}{2} < \frac{n}{2}$ with $m = \lfloor \gamma \rfloor$, then $g^* = \psi_{m+\frac{1}{2}}^{2m+1} |dx|^2$, where

$$
\psi_{m+\frac{1}{2}} = \sum_{k=0}^{m} \frac{(n-1-2m)_{k}}{(2m)_{k}} (2\rho)^k k!;
$$

(3.34)

(3) if $\gamma = \frac{n}{2}$ and $n$ is an odd integer, then

$$
g^* = \exp \left\{ 2 \Gamma\left(\frac{n+1}{2}, \frac{1}{2} - \gamma; 1 - 2\gamma; 2\rho\right) \right\} |dx|^2.
$$

(3.35)

Proof. By the definition of $g^*$, we have $g^* = \vartheta_{\frac{n}{2}-\gamma}^{2} \varphi_{\frac{n}{2}+\gamma}$. By Corollary 3.3, we get (1) and (2). Now we prove (3.35). By the definition of $g^*$, we have $g^* = e^{2\tau \rho^{-2}} |dx|^2$, where

$$
\tau = -\frac{d}{ds} \vartheta_s |s=n = -\lim_{s \to n} \frac{\vartheta_s - \vartheta_n}{s - n} = \lim_{\gamma \to \frac{n}{2}} \frac{\vartheta_s - \vartheta_n}{\frac{n}{2} - \gamma}.
$$

(3.36)

By Corollary 3.3, $\vartheta_s = 1$. Therefore, substituting (3.31) into (3.36), we have

$$
\tau = \lim_{\gamma \to \frac{n}{2}} \frac{\rho^{\frac{n}{2} - \gamma}}{\frac{n}{2} - \gamma} \left[ F\left(\frac{n}{2} - \gamma, \frac{1}{2} - \gamma; 1 - 2\gamma; 2\rho\right) - 1 \right] + \lim_{\gamma \to \frac{n}{2}} \frac{\rho^{\frac{n}{2} - \gamma} - 1}{\frac{n}{2} - \gamma}
$$

$$
\lim_{\gamma \to \frac{n}{2}} \frac{\rho^{\frac{n}{2} + \gamma}}{\frac{n}{2} - \gamma} \frac{\Gamma(-\lambda)}{\Gamma(\gamma)} \Gamma\left(\frac{n}{2} + \gamma\right) F\left(\frac{n}{2} + \gamma, \frac{n}{2} + \gamma; 1 + 2\gamma; 2\rho\right)
$$

$$
\lim_{\gamma \to \frac{n}{2}} \frac{1}{\frac{n}{2} - \gamma} \left[ F\left(\frac{n}{2} - \gamma, \frac{1}{2} - \gamma; 1 - 2\gamma; 2\rho\right) - 1 \right] + \ln \rho
$$

$$
\rho^{\frac{n}{2}} \frac{\Gamma(-\frac{n}{2}) \Gamma(n)}{2^n \Gamma\left(\frac{n}{2}\right)} F\left(\frac{n}{2}, \frac{n}{2}; n+1; 2\rho\right).
$$

(3.37)
To get the last equation above, we use the fact \((\frac{n}{2} - \gamma)\Gamma(\frac{n}{2} - \gamma) = \Gamma(\frac{n}{2} - \gamma + 1) \to 1\) as \(\gamma \to \frac{n}{2}\). We compute

\[
\lim_{\gamma \to \frac{n}{2}} \frac{1}{\frac{n}{2} - \gamma} \left[ F(\frac{n}{2} - \gamma; \frac{1}{2} - \gamma; 1 - 2\gamma; 2\rho) - 1 \right]
= \lim_{\gamma \to \frac{n}{2}} \frac{1}{\frac{n}{2} - \gamma} \sum_{k=1}^{\infty} \frac{\left(\frac{n}{2} - \gamma\right)k(\frac{1}{2} - \gamma)k (2\rho)^k}{(1 - 2\gamma)_k} k!
\]

\[
= \sum_{k=1}^{(n-1)/2} \frac{(k - 1)!\left(\frac{1-n}{2}\right)_k (2\rho)^k}{(1 - n)_k} k! + \frac{1}{2} \sum_{k=n}^{\infty} \frac{(k - 1)!\left(\frac{1-n}{2}\right)_{n-k} (k - \frac{n+1}{2})! (2\rho)^k}{(1 - n)_{n-1}(k - n)!} k!
\]

\[
= \sum_{k=1}^{(n-1)/2} \frac{(k - 1)!\left(\frac{1-n}{2}\right)_k (2\rho)^k}{(1 - n)_k} k! + (-1)^{n-1} \frac{2^{n-1}\Gamma(\frac{n+1}{2})}{\Gamma(n)} \rho^n \sum_{k=0}^{\infty} \frac{(k + \frac{n-1}{2})!}{(k + n)k!} (2\rho)^k.
\]

Using the duplication formula (3.16), we have

\[
(-1)^{n-1} \frac{2^{n-1}\Gamma(\frac{n+1}{2})}{\Gamma(n)} \rho^n \sum_{k=0}^{\infty} \frac{(k + \frac{n-1}{2})!}{(k + n)k!} (2\rho)^k
\]

\[
= (-1)^{n-1} \frac{2^{n-1}\Gamma(\frac{n+1}{2})^2}{\Gamma(n+1)} \rho^n F\left(\frac{n+1}{2}, n; 1 + n; 2\rho\right)
\]

\[
= (-1)^{n-1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} \rho^n F\left(\frac{n+1}{2}, n; 1 + n; 2\rho\right).
\]

Also by the duplication formula (3.16), we have

\[
\rho^n \frac{\Gamma\left(-\frac{n}{2}\right)\Gamma(n)}{2^n \Gamma\left(\frac{n}{2}\right)} F\left(\frac{n+1}{2}, n; 1 + n; 2\rho\right)
\]

\[
= \rho^n \frac{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)} F\left(\frac{n+1}{2}, n; 1 + n; 2\rho\right)
\]

\[
= (-1)^{n+1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} \rho^n F\left(\frac{n+1}{2}, n; 1 + n; 2\rho\right).
\]

To get the last equation above, we use \(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\) and the Euler’s reflection formula

\[
\Gamma\left(-\frac{n}{2}\right)\Gamma\left(1 + \frac{n}{2}\right) = \frac{\pi}{\sin \frac{n}{2}\pi} = (-1)^{n+1}\frac{n+1}{2}\pi.
\]

Substituting (3.38) into (3.37) and using (3.39)–(3.40), we get

\[
\tau = \sum_{k=1}^{(n-1)/2} \frac{(k - 1)!\left(\frac{1-n}{2}\right)_k (2\rho)^k}{(1 - n)_k} k! + \ln \rho = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)} \sum_{k=1}^{(n-1)/2} \frac{\Gamma(n-k)}{\Gamma\left(\frac{n}{2} - k\right)} (2\rho)^k + \ln \rho.
\]
The desired result follows. \( \square \)

We remark that one can also find the metric \( g^* \) in dimension \( 2m+1 \) by a “dimension continuity” in the spirit of the work of Branson (see [3]), by computing the limit

\[ e^{2\tau} \rho^{-2} = \lim_{n \to 2m+1} (\psi_{m+\frac{1}{2}})^{\frac{4}{n-2m-1}}. \]

4. **Proof of Theorem 1.6 and 1.7**

In this section, we let \( \gamma = m + \frac{1}{2} \) with \( m = \lceil \gamma \rceil \). As in the proof of Theorem 3.1, we set

\[ V_m(x) = \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2} + \gamma\right)}{\Gamma(\gamma)} \rho^{2m+1} \int_{S^n} \left( \frac{1}{|x - \xi|^2} \right)^{\frac{n+1}{2} - m} f(\xi) d\sigma \]

such that \( \rho^{\frac{n}{2} - m} V_m(x) \) is the solution of (3.6). Moreover, if \( f = \sum_{l=0}^{\infty} Y_l \), then

\[ V_m(x) = \sum_{l=0}^{\infty} \varphi_l(r^2) r^l Y_l, \]

where

\[ \varphi_l(r) = \frac{\Gamma(m + 1) \Gamma(l + \frac{n+1}{2} + m)}{\Gamma(l + \frac{n+1}{2}) \Gamma(2m+1)} F(l + \frac{n-1}{2} - m, -m, l + \frac{n+1}{2}; r). \]

**Lemma 4.1.** There holds, for \( 0 \leq k \leq m + 1 \),

\[ \Delta^k V_m(x) = 4^k \sum_{l=0}^{\infty} \frac{\Gamma(m + 1) \Gamma(l + \frac{n+1}{2} + m)}{\Gamma(l + \frac{n+1}{2}) \Gamma(2m+1)} (l + \frac{n-1}{2} - m)_k (-m)_k. \]

\[ F(l + \frac{n-1}{2} - m, -m + k, l + \frac{n+1}{2}, r^2) r^l Y_l. \]

In particular, we have

\[ \Delta^{m+1} V_m(x) = 0. \]

because of \( (-m)_{m+1} = 0 \).

**Proof.** We shall prove (4.4) by induction. It is easy to see that (4.4) is valid for \( k = 0 \). Now suppose that equation (4.4) is valid for \( k \geq 0 \). Then we have

\[ \Delta^{k+1} V_m(x) = 4^k \sum_{l=0}^{\infty} \frac{\Gamma(m + 1) \Gamma(l + \frac{n+1}{2} + m)}{\Gamma(l + \frac{n+1}{2}) \Gamma(2m+1)} (l + \frac{n-1}{2} - m)_k (-m)_k. \]

\[ F(l + \frac{n-1}{2} - m + k, -m + k, l + \frac{n+1}{2}, r^2) r^l Y_l. \]
We compute
\[
\Delta \left( F \left( l + \frac{n - 1}{2} - m + k, -m + k, l + \frac{n + 1}{2}, r^2 \right) r^l Y_l \right)
\]
\[
= \left( \partial_{rr} + \frac{n}{r} \partial_r + \frac{1}{r^2} \tilde{\Delta} \right) \sum_{j=0}^{m-k} \frac{(l + \frac{n - 1}{2} - m + k)_j (-m + k)_j 1}{(l + \frac{n + 1}{2})_j} j! r^{2j+l} Y_l
\]
\[
= \sum_{j=0}^{m-k} \frac{(l + \frac{n - 1}{2} - m + k)_j (-m + k)_j 1}{(l + \frac{n + 1}{2})_j} j! 4j \left( \frac{n - 1}{2} + j \right) r^{2j+l-2} Y_l
\]
\[
= 4 \left( l + \frac{n - 1}{2} - m + k \right) (-m + k) F \left( l + \frac{n - 1}{2} - m + k + 1, -m + k + 1, l + \frac{n + 1}{2}, r^2 \right) r^l Y_l.
\]

Substituting (4.7) into (4.6), we have
\[
\Delta^{k+1} V_m (x) = 4^{k+1} \sum_{l=0}^{\infty} \frac{\Gamma(m+1) \Gamma(l + \frac{n + 1}{2} + m)}{\Gamma(l + \frac{n - 1}{2} + 1) \Gamma(2m+1)} (l + \frac{n - 1}{2} - m)_{k+1} (-m)_{k+1} \cdot F \left( l + \frac{n - 1}{2} - m + k + 1, -m + k + 1, l + \frac{n + 1}{2}, r^2 \right) r^l Y_l.
\]

This proves the Lemma 4.1. \hfill \square

Lemma 4.2. There holds, for $0 \leq k \leq m$,
\[
\Delta^k V_m \big|_{r=1} = (-1)^k \frac{\Gamma(m+1) \Gamma(m-k + \frac{1}{2})}{\Gamma(m + \frac{1}{2}) \Gamma(m-k+1)} \mathcal{P}_{2m+1-2k} \mathcal{P}_{2m+1} f
\]
and for $0 \leq k \leq m - 1$,
\[
\partial_r \Delta^k V_m \big|_{r=1} = (-1)^{k+1} \frac{n - 1 - 2m + 2k \Gamma(m+1) \Gamma(m-k + \frac{1}{2})}{2 \Gamma(m + \frac{1}{2}) \Gamma(m-k+1)} \mathcal{P}_{2m+1-2k} \mathcal{P}_{2m+1} f.
\]

In the case $k = m$, we have
\[
\partial_r \Delta^m V_m \big|_{r=1} = (-1)^m \frac{\Gamma(m+1) \Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \left( \mathcal{P}_{2m+1} - \frac{n - 1}{2} \mathcal{P}_{2m+1} \right) f.
\]
Proof. By Lemma 4.1 and (2.3), we have, for $0 \leq k \leq m$,

$$\Delta^k V_m|_{r=1} = 4^k \sum_{l=0}^{\infty} \frac{\Gamma(m + 1) \Gamma(l + \frac{n+1}{2} + m)}{\Gamma(l + \frac{n+1}{2}) \Gamma(2m + 1)} (l + \frac{n-1}{2} - m)_k (-m)_k \cdot \frac{\Gamma(l + \frac{n+1}{2}) \Gamma(2m - 2k + 1)}{\Gamma(m + 1 - k) \Gamma(l + \frac{n+1}{2} + m - k)} Y_l$$

(4.11)

$$= 4^k \frac{\Gamma(m + 1) \Gamma(2m - 2k + 1)}{\Gamma(2m + 1) \Gamma(m - k + 1)} (-m)_k \cdot \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{n+1}{2} + m) \Gamma(l + \frac{n-1}{2} - m + k)}{\Gamma(l + \frac{n-1}{2} - m) \Gamma(l + \frac{n+1}{2} + m - k)} Y_l$$

$$= 4^k \frac{\Gamma(m + 1) \Gamma(2m - 2k + 1)}{\Gamma(2m + 1) \Gamma(m - k + 1)} (-m)_k \frac{\mathcal{P}_{2m+1}}{\mathcal{P}_{2m+1-2k}} f.$$  

To get the last equation, we use (2.13). Moreover, by duplication formula (3.16), we have

(4.12)

$$4^k \frac{\Gamma(m + 1) \Gamma(2m - 2k + 1)}{\Gamma(2m + 1) \Gamma(m - k + 1)} (-m)_k = (-1)^k \frac{\Gamma(m + 1) \Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2}) \Gamma(m - k + 1)}.$$  

Substituting (4.12) into (4.13), we get (4.8). Similarly, using (2.6) and (2.3), we have, for $0 \leq k \leq m - 1$,

$$\partial_r \Delta^k V_m|_{r=1} = 4^k \sum_{l=0}^{\infty} \frac{\Gamma(m + 1) \Gamma(l + \frac{n+1}{2} + m)}{\Gamma(l + \frac{n+1}{2}) \Gamma(2m + 1)} (l + \frac{n-1}{2} - m)_k (-m)_k \cdot \frac{(l + \frac{n-1}{2} - m + k)(-m + k)}{l + \frac{n+1}{2}} \frac{\Gamma(l + \frac{n+1}{2} + 1) \Gamma(2m - 2k)}{\Gamma(m + 1 - k) \Gamma(l + \frac{n+1}{2} + m - k)} Y_l$$

(4.13)

$$= - \left( \frac{n-1}{2} - m + k \right) \Delta^k V_m|_{r=1}.$$  

These prove (4.9). Now we prove (4.10). By Lemma 4.1

$$\Delta^m V_m(x) = 4^k \frac{\Gamma(m + 1)}{\Gamma(2m + 1)} (-m)_m \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{n+1}{2} + m)}{(l + \frac{n-1}{2}) \Gamma(l + \frac{n-1}{2} - m)} r^l Y_l.$$
Therefore, by (2.13), we have

\[ \partial_r \Delta^m V_m|_{r=1} = 4^m \frac{\Gamma(m+1)}{\Gamma(2m+1)} (-m) \sum_{l=0}^{\infty} \frac{l}{l + \frac{n-1}{2}} \frac{\Gamma(l + \frac{n+1}{2} + m)}{\Gamma(l + \frac{n-1}{2} - m)} Y_l \]

\[ = 4^m \frac{\Gamma(m+1)}{\Gamma(2m+1)} (-m) \sum_{l=0}^{\infty} \left( 1 - \frac{n-1}{2} \right) \cdot \frac{\Gamma(l + \frac{n-1}{2}) \Gamma(m+1) \Gamma(2m+1)}{\Gamma(l + \frac{n+1}{2}) \Gamma(m+\frac{1}{2})} \int_{S^n} f \frac{P_{2m+1}^2}{P_1} \frac{P_2^2}{P_1} \frac{P_1}{P_1} f \, d\sigma. \]

(4.14)

To get the last equation, we use duplication formula (3.16). The desired result follows.

□

Proof of Theorem 1.6 We firstly prove (1.18) when \( v = V_m \). By Lemma 4.2 and Green’s formula (see e.g. [20], Appendix C), we have

(4.15)

If \( m \) is an odd integer, then by Lemma 4.2 and Green’s formula, we have

(4.16)
If \( m \) is a even integer, then also by Lemma 4.2 and Green’s formula, we have
\[
\int_{B^{n+1}} \Delta V_m \Delta^m V_m dx = - \int_{S^n} |\nabla \Delta^m V_m|^2 dx - \sum_{k=1}^{m-2} (m - 2k) \frac{\Gamma(m + 1)^2}{\Gamma(m + \frac{3}{2})^2}.
\]
\[
\frac{\Gamma(k + \frac{1}{2}) \Gamma(m - k + \frac{1}{2})}{\Gamma(k + 1) \Gamma(m - k + 1)} \int_{S^n} f \frac{P_{2m+1}^2}{P_{2m+1-k} P_{2k+1}} d\sigma - \frac{n - 1 - m}{2} \left( \frac{\Gamma(m + 1) \Gamma(m + \frac{3}{2})}{\Gamma(m + \frac{1}{2}) \Gamma(\frac{m + 1}{2} + 1)} \right)^2 \int_{S^n} f \frac{P_{2m+1}^2}{P_{m+1}^2} d\sigma.
\]
(4.17)

Substituting (4.16) and (4.17) into (4.15), we have
\[
\frac{\Gamma(m + 1) \Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \int_{S^n} f P_{2m+1} f d\sigma = \int_{B^{n+1}} |\nabla^m V_m|^2 dx + \int_{S^n} f T_m d\sigma,
\]
where \( T_m \) is defined in Theorem (1.6). Therefore, by Theorem 1.1, we prove the Theorem 1.6 when \( v = V_m \) and in this case the only extremal functions is given by (1.19). For general \( v \) with the Neumann boundary condition (1.17), we claim
\[
\int_{B^{n+1}} |\nabla^{m+1} V_m|^2 dx \leq \int_{B^{n+1}} |\nabla^{m+1} v|^2 dx.
\]
(4.19)

In fact, we have
\[
0 \leq \int_{B^{n+1}} |\nabla^{m+1} (v - U_m)|^2 dx
\]
\[
= \int_{B^{n+1}} |\nabla^{m+1} v|^2 dx - \int_{B^{n+1}} |\nabla^{m+1} U_m|^2 dx
\]
\[
- 2 \int_{B^{n+1}} \nabla^{m+1} (v - U_m) \cdot \nabla^{m+1} U_m dx.
\]
Since \( v \) and \( U_m \) have the same Neumann boundary condition (1.17), we have
\[
\int_{B^{n+1}} \nabla^{m+1} (v - U_m) \cdot \nabla^{m+1} U_m dx = (-1)^{m+1} \int_{B^{n+1}} (v - U_m) \Delta^{m+1} U_m dx = 0.
\]
Therefore,
\[
\int_{B^{n+1}} |\nabla^{m+1} v|^2 dx = \int_{B^{n+1}} |\nabla^{m+1} (v - U_m)|^2 dx + \int_{B^{n+1}} |\nabla^{m+1} U_m|^2 dx
\]
\[
\geq \int_{B^{n+1}} |\nabla^{m+1} U_m|^2 dx.
\]
(4.19)

These prove the claim.
Finally, we prove the uniqueness of the extremal functions. If \( v \) is any extremal function with the Neumann boundary condition (1.17), then by (4.19) it must satisfy \( \nabla^{m+1}(v - U_m) = 0 \) and thus \( \Delta^{m+1}v = \Delta^{m+1}U_m = 0 \). By the uniqueness of the solution, we get \( v = V_m(x) \). Thus, by Theorem 1.1, the only extremal function is that given by (1.19). The proof of Theorem 1.6 is thereby completed.

**Proof of Theorem 1.7** With the same argument in the proof of Theorem 1.6, we need only consider the case \( v = V_m(x) \) with \( m = \frac{n-1}{2} \). Using (4.15)-(4.18), we get

\[
\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{S}^n} f P_n f d\sigma = \int_{\mathbb{B}^{n+1}} |\nabla^{n+1} V_m|^2 dx + \int_{\mathbb{S}^n} f T_{\frac{n+1}{2}} f d\sigma.
\]

(4.20)

Therefore, by Theorem 1.1, we have

\[
\ln \left( \frac{1}{\omega_n} \int_{\mathbb{S}^n} e^{n(f - \bar{f})} d\sigma \right) \leq \frac{n}{2(n-1)!\omega_n} \int_{\mathbb{S}^n} f P_n f d\sigma
\]

\[
\leq \frac{n}{2(n-1)!\omega_n} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \left( \int_{\mathbb{B}^{n+1}} |\nabla^{n+1} V_m|^2 dx + \int_{\mathbb{S}^n} f T_{\frac{n+1}{2}} f d\sigma \right).
\]

(4.21)

To get the last equation, we use the fact \( \omega_n = \frac{n \frac{n+1}{2} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{2^n \Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \).

With the same argument in Theorem 1.6 and using Theorem 1.1, we have get the only only extremal function is that given by (1.22). The proof of Theorem 1.7 is thereby completed.

5. **Proof of Theorem 1.9**

Since the Möbius transform \( \mathcal{M} : (\mathbb{R}^{n+1}, g_\mathbb{H}) \to (\mathbb{B}^{n+1}, g_\mathbb{B}) \), where \( g_\mathbb{H} = \frac{|dx|^2 + |dy|^2}{y^2} \), defined by

\[
\mathcal{M}(x, y) = \left( \frac{2x}{(1+y)^2 + |x|^2}, \frac{1 - |x|^2 - y^2}{(1+y)^2 + |x|^2} \right),
\]

is an isometry between the two models of hyperbolic space, by Theorem 3.1 we can solve the Poisson equation (3.1) on \((\mathbb{R}^{n+1}, g_\mathbb{H})\):
Theorem 5.1. Let $\gamma \in (0, \frac{n}{2})$ and $s = \frac{n}{2} + \gamma$. The solution of the Poisson equation on the hyperbolic space $(\mathbb{R}^{n+1}_+, g_H)$

\[
\begin{cases}
-\Delta_{g_H} u - s(n-s)u = 0 \quad \text{in } \mathbb{R}^{n+1}, \\
u = F y^{n-s} + H y^{s}, \\
F|_{\partial \mathbb{R}^{n+1}} = f(x),
\end{cases}
\]

is

\[
u(x,y) = \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + \gamma)}{\Gamma(\gamma)} \int_{\mathbb{R}^n} \left( \frac{y}{|x - \xi|^2 + y^2} \right)^s f(\xi) d\xi,
\]

where $f$ and its derivatives have fast decay at infinity (for example, in certain fractional Sobolev spaces).

Now we let $\gamma = m + \frac{1}{2}$ with $m$ an integer. In this case, we can find out the relationship between the kernel in (5.2) and the Poisson kernel. In fact, we have the following:

Lemma 5.2. There holds, for $(x, y) \in \mathbb{R}^{n+1}_+$ and $m \in \mathbb{N}$,

\[
2^{-2m} \frac{\Gamma(m + \frac{n+1}{2})}{\Gamma\left(\frac{n+1}{2}\right)} \left( \frac{y^{1+2m}}{(|x|^2 + y^2)^{\frac{n+1}{2} + m}} \right)
= 2^{2m} \frac{\Gamma(2m) \Gamma(2m-k+1) \Gamma(m-k+1) (-y)^k d^k}{k! \Gamma(m-k+1)} \left( \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \right).
\]

Proof. We shall prove it by induction. It is easy to see (5.3) is valid for $m = 0$. Suppose (5.3) is valid for $m \geq 0$. We compute

\[
yd{y}{y^{1+2m}}{(|x|^2 + y^2)^{\frac{n+1}{2} + m}} = (1 + 2m) \frac{y^{1+2m}}{(|x|^2 + y^2)^{\frac{n+1}{2} + m}} - (n + 1 + 2m) \frac{y^{3+2m}}{(|x|^2 + y^2)^{\frac{n+1}{2} + m+1}}.
\]

Therefore,

\[
2^{-2m} \frac{\Gamma(m + \frac{n+1}{2})}{\Gamma\left(\frac{n+1}{2}\right)} \left( 1 + 2m - y \frac{d}{dy} \right) \frac{y^{1+2m}}{(|x|^2 + y^2)^{\frac{n+1}{2} + m}} = \left( 1 + 2m - y \frac{d}{dy} \right) \frac{y^{1+2m}}{(|x|^2 + y^2)^{\frac{n+1}{2} + m}}
\]

\[
2^{-2m} \frac{\Gamma(m + \frac{n+1}{2})}{\Gamma\left(\frac{n+1}{2}\right)} \left( 1 + 2m - y \frac{d}{dy} \right) \sum_{k=0}^{m} \frac{2^k \Gamma(2m-k+1) \Gamma(m-k+1) (-y)^k d^k}{k! \Gamma(m-k+1)} \left( \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \right).
\]
Since

\[
(1 + 2m - y \frac{d}{dy}) \left( \sum_{k=0}^{m} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (-y)^k \frac{d^k}{dy^k} \right)
\]

\[
= \sum_{k=0}^{m} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (1 + 2m - k)(-y)^k \frac{d^k}{dy^k} +
\]

\[
\sum_{k=0}^{m} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (-y)^{k+1} \frac{d^{k+1}}{dy^{k+1}}
\]

\[
= \sum_{k=0}^{m} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (1 + 2m - k)(-y)^k \frac{d^k}{dy^k} +
\]

\[
\sum_{k=1}^{m+1} \frac{2^{k-1} \Gamma(2m - k + 2)}{(k-1)! \Gamma(m - k + 2)} (-y)^k \frac{d^k}{dy^k}
\]

\[
= \frac{1}{2} \left( \sum_{k=0}^{m+1} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (-y)^k \frac{d^k}{dy^k} \right) \frac{y}{(|x|^2 + y^2)^{n+1/2}}.
\]

We have, by (5.5) and (5.6),

\[
\left( \sum_{k=0}^{m+1} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (-y)^k \frac{d^k}{dy^k} \right) \frac{y}{(|x|^2 + y^2)^{n+1/2}} = 2^{2m+2} \Gamma(m + 1 + \frac{n+1}{2}) \frac{y^{3+2m}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{y^{3+2m}}{(|x|^2 + y^2)^{n+1/2+m+1}}
\]

These completes the proof of Lemma 5.2. □

In the rest paper, we let \( \Delta_x = \sum_{i=1}^{n} \partial_{x_i x_i} \) and \( \Delta = \Delta_x + \partial_{yy} \). Since the Poisson kernel \( e^{-y\sqrt{-\Delta_x}} \) on \( \mathbb{R}^{n+1}_+ \) is given by (see e.g. [40])

\[
e^{-y\sqrt{-\Delta_x}} = \frac{\pi^{-\frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}},
\]

we have, by Theorem 5.1 and Lemma 5.2, that the solution of (5.1) is

\[
u(x, y) = y^{\frac{n+1}{2} - m - 2m} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)} \left( \sum_{k=0}^{m} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} (-y)^k \frac{d^k}{dy^k} \right) e^{-y\sqrt{-\Delta_x}} f
\]

\[
= y^{\frac{n+1}{2} - m - 2m} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m + \frac{1}{2}\right)} \sum_{k=0}^{m} \frac{2^k \Gamma(2m - k + 1)}{k! \Gamma(m - k + 1)} y^k (-\Delta_x)^k e^{-y\sqrt{-\Delta_x}} f.
\]
Lemma 5.3. Let $U_m(x, y) = u(x, y)y^{-\frac{m+1}{2}+m}$, where $u(x, y)$ is given by (5.8). Then for $0 \leq k \leq m+1$, we have

$$\Delta^k U_m(x, y) = (-1)^k \frac{2^{2k-2m}\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m-k+1)\Gamma(m+\frac{1}{2})}. \quad (5.9)$$

$$\sum_{j=0}^{m-k} \frac{2^j \Gamma(2m-2k-j+1)}{j! \Gamma(m-k-j+1)} y^j (-\Delta_x)^{k+\frac{1}{2}} e^{-y\sqrt{-\Delta_x}} f.$$ 

In particular, $\Delta^{m+1} u(x, y) = 0$ because of $(-m)_{m+1} = 0$. Moreover,

$$\Delta^k U_m(x, y)|_{y=0} = \frac{\Gamma(m+1)\Gamma(m+\frac{1}{2}-k)}{\Gamma(m-k+1)\Gamma(m+\frac{1}{2})} \Delta_x^k f, \quad 0 \leq k \leq m; \quad (5.10)$$

$$\partial_y \Delta^k U_m(x, y)|_{y=0} = 0, \quad 0 \leq k \leq m-1;$$

$$\partial_y \Delta^m U_m(x, y)|_{y=0} = (-1)^m \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} (-\Delta_x)^{m+\frac{1}{2}} f.$$

Proof. We prove (5.9) by induction. Obviously, (5.9) is valid for $k = 0$. Suppose (5.9) is valid for $k$. Then

$$\Delta^{k+1} U_m(x, y) = (-1)^k \frac{2^{2k-2m}\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m-k+1)\Gamma(m+\frac{1}{2})} \Delta^k U_m(x, y)$$

$$\sum_{j=0}^{m-k} \frac{2^j \Gamma(2m-2k-j+1)}{j! \Gamma(m-k-j+1)} \Delta (y^j (-\Delta_x)^{k+\frac{1}{2}} e^{-y\sqrt{-\Delta_x}} f)$$

$$=(-1)^k \frac{2^{2k-2m}\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m-k+1)\Gamma(m+\frac{1}{2})} \sum_{j=0}^{m-k} \frac{2^j \Gamma(2m-2k-j+1)}{j! \Gamma(m-k-j+1)}$$

$$\left(j(j-1)y^{j-2}(-\Delta_x)^{\frac{j}{2}} - 2jy^j(-\Delta_x)^{\frac{j+1}{2}}\right) (-\Delta_x)^{k+\frac{1}{2}} e^{-y\sqrt{-\Delta_x}} f$$

$$=(-1)^k \frac{2^{2k-2m}\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m-k+1)\Gamma(m+\frac{1}{2})} \left(\sum_{j=2}^{m-k} \frac{2^j \Gamma(2m-2k-j+1)}{(j-2)! \Gamma(m-k-j+1)} (-\Delta_x)^{\frac{j}{2}}

- 2 \sum_{j=1}^{m-k} \frac{2^j \Gamma(2m-2k-j+1)}{(j-1)! \Gamma(m-k-j+1)} (-\Delta_x)^{\frac{j+1}{2}}\right) (-\Delta_x)^{k+\frac{1}{2}} e^{-y\sqrt{-\Delta_x}} f.$$
A simple calculation shows

\[
\sum_{j=0}^{m-k-2} \frac{2^j j!}{j+2} \frac{\Gamma(2m-2k-j-1)}{\Gamma(m-k-j-1)} (-\Delta_x)^{j+\frac{3}{2}} - 2 \sum_{j=0}^{m-k-1} \frac{2^{j+2} j!}{j+1} \frac{\Gamma(2m-2k-j)}{\Gamma(m-k-j)} (-\Delta_x)^{j+\frac{3}{2}} = 4(k-m) \sum_{j=0}^{m-k-1} \frac{2^j j! \Gamma(2m-2k-j-1)}{\Gamma(m-k-j)} y^j (-\Delta_x)^{j+\frac{3}{2}} e^{-y\sqrt{-\Delta_x}} f.
\]

and thus we have

\[
\Delta^{k+1} U_m(x, y) = (-1)^{k+1} \frac{2^{k+2-2m} \Gamma(m+1) \Gamma(\frac{1}{2})}{\Gamma(m-k) \Gamma(m+\frac{1}{2})} \sum_{j=0}^{m-k-1} \frac{2^j j! \Gamma(2m-2k-j-1)}{\Gamma(m-k-j)} y^j (-\Delta_x)^{j+\frac{3}{2}} e^{-y\sqrt{-\Delta_x}} f.
\]

These prove (5.9).

By (5.9) and (3.16), we have

\[
\Delta^k U_m(x, y) |_{y=0} = (-1)^k \frac{2^{k-2m} \Gamma(m+1) \Gamma(\frac{1}{2})}{\Gamma(m-k+1) \Gamma(m+\frac{1}{2})} \Gamma(2m-2k+1) (-\Delta_x)^k f
\]

\[
= \frac{\Gamma(m+1) \Gamma(m+\frac{1}{2} - k)}{\Gamma(m-k+1) \Gamma(m+\frac{1}{2})} \Delta_x^k f.
\]

Similarly, \( \partial_y \Delta^k U_m(x, y) |_{y=0} = 0 \) when \( 0 \leq k \leq m - 1 \) and

\[
\partial_y \Delta^m U_m(x, y) |_{y=0} = (-1)^m \frac{\Gamma(m+1) \Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} (-\Delta_x)^{m+\frac{1}{2}} f.
\]

These complete the proof of Lemma 5.3. \( \square \)

We can also compute the Neumann boundary condition \( \partial_y^k U_m |_{y=0} \) for \( 0 \leq k \leq 2m \). In fact, we have the following:

**Lemma 5.4.** There holds, for \( 0 \leq k \leq m \),

\[
(5.11) \quad \partial_y^k U_m(x, y) |_{y=0} = \frac{\Gamma(k+\frac{1}{2}) \Gamma(m-k+\frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(m+\frac{1}{2})} \Delta_x^k f
\]

and for \( 0 \leq k \leq m - 1 \),

\[
(5.12) \quad \partial_y^{2k+1} U_m(x, y) |_{y=0} = 0.
\]
Proof. Firstly, we prove (5.11) by induction. It is easy to see (5.11) is valid for \( k = 0 \). Suppose that (5.11) is valid for \( k \), i.e.,

\[
\lim_{y \to 0^+} \partial^2_y \left( \frac{\Gamma(m+1)}{\Gamma(2m+1)} \sum_{j=0}^{m-1} \frac{2^j \Gamma(2m-2k-j+1)}{j! \Gamma(m-j+1)} \right) y^j (-\Delta_x)^{\frac{k+1}{2}} e^{-y\sqrt{-\Delta_x}} f 
\]

(5.13)

Replacing \( m \) by \( m-1 \) in (5.13), we have

\[
\lim_{y \to 0^+} \partial^2_y \left( \frac{\Gamma(m)}{\Gamma(2m-1)} \sum_{j=0}^{m-1} \frac{2^j \Gamma(2m-2k-j+1)}{j! \Gamma(m-k-j+1)} \right) y^j (-\Delta_x)^{\frac{k}{2}} e^{-y\sqrt{-\Delta_x}} f 
\]

(5.14)

By (5.9), we have

\[
\partial^{2k+2}_y U_m(x, y) = \partial^2_y \left[ \Delta U_m(x, y) - \Delta_x U_m(x, y) \right] 
\]

\[
= \partial^2_y \left[ - \frac{m}{m - \frac{1}{2}} \frac{\Gamma(m)}{\Gamma(2m-1)} \sum_{j=0}^{m-1} \frac{2^j \Gamma(2m-2k-j+1)}{j! \Gamma(m-k-j+1)} \right] y^j (-\Delta_x)^{\frac{k}{2}} e^{-y\sqrt{-\Delta_x}} f 
\]

(5.15)

Combining (5.13)-(5.15) yields

\[
\lim_{y \to 0^+} \partial^{2k+2}_y U_m(x, y) = \frac{m}{m - \frac{1}{2}} \frac{\Gamma(k + \frac{1}{2}) \Gamma(m - k - \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(m - \frac{1}{2})} \Delta_x^{k+1} f - \frac{\Gamma(k + \frac{1}{2}) \Gamma(m - k + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(m + \frac{1}{2})} \Delta_x^{k+1} f 
\]

\[
= \frac{\Gamma(k + 1 + \frac{1}{2}) \Gamma(m - k - \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(m + \frac{1}{2})} \Delta_x^{k+1} f. 
\]

These prove (5.11). The proof of (5.12) is completely analogous to that of (5.11) and we omit. \( \square \)
Proof of Theorem 1.9: With the same argument in the proof of Theorem 1.6, we need only consider the case \( u = U_m(x, y) \). By Lemma 5.3 and Green’s formula, we have

\[
0 = \int_{\mathbb{R}^{n+1}} U_m \Delta^{m+1} U_m \, dx \, dy
\]

\[
= \int_{\mathbb{R}^{n+1}} \Delta U_m \Delta^m U_m \, dx \, dy + \int_{\mathbb{R}^n} U_m \partial_y \Delta^m U_m \, dx - \int_{\mathbb{R}^n} \partial_y U_m \Delta^m U_m \, dx
\]

\[
= \int_{\mathbb{R}^{n+1}} \Delta U_m \Delta^m U_m \, dx \, dy + (-1)^m \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \int_{\mathbb{R}^n} U_m(x, 0)(-\Delta_x)^{m+\frac{1}{2}} U_m(x, 0) \, dx
\]

\[
= (-1)^{m-1} \int_{\mathbb{R}^{n+1}} |\nabla^{m+1} U_m|^2 \, dx \, dy + (-1)^m \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \int_{\mathbb{R}^n} U_m(x, 0)(-\Delta_x)^{m+\frac{1}{2}} U_m(x, 0) \, dx.
\]

Therefore, by Theorem 1.2, we have

\[
\int_{\mathbb{R}^{n+1}} |\nabla^{m+1} U_m(x, y)|^2 \, dx \, dy = \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \int_{\mathbb{R}^n} U_m(x, 0)(-\Delta_x)^{m+\frac{1}{2}} U_m(x, 0) \, dx
\]

\[
\geq \frac{\Gamma(m+1)\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \frac{\Gamma(\frac{n+2m+1}{2})}{\Gamma(\frac{n-2m-1}{2})} \omega_n^{\frac{2m+1}{n}} \left( \int_{\mathbb{R}^n} |U_m(x, 0)|^2 \, dx \right)^{\frac{n-2m-1}{n}}.
\]

With the same argument in Theorem 1.6 and using Theorem 1.2, we have get the only only extremal function is that given by (1.24). These complete the proof of Theorem 1.9.

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School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, People’s Republic of China
E-mail address: qhyang.math@gmail.com; qhyang.math@whu.edu.cn