Abstract
If a pure state of a qubit pair is developed over the four basis states, an equality between the four coefficients of that development, verified if and only if that state is unentangled, is already known. This paper considers an arbitrary pure state of an \(N\)-qubit system, developed over the \(2^N\) basis states. It is shown that the state is unentangled if and only if a well-chosen collection of \([2^N - (N + 1)]\) equalities between the \(2^N\) coefficients of that development is verified. The number of these equalities is large as soon as \(N \gtrsim 10\), but it is shown that this set of equalities may be classified into \((N - 1)\) subsets, which should facilitate their manipulation. This result should be useful e.g. in the contexts of blind quantum source separation and blind quantum process tomography, with an aim which should not be confused with that found when using the concept of equivalence of pure states through local unitary transformations.

Keywords
Unentanglement condition · Entanglement · \(N\)-qubit system · Pure state

1 Introduction
If both parts \(S_1\) and \(S_2\) of a bipartite quantum system \(S\) are initially prepared in a pure state, then, at a timescale allowing one to neglect any coupling between \(S\) and the rest of the universe, \(S\) may be described as being in a time-dependent pure state \(|\Psi(t)\rangle\) which obeys the Schrödinger equation. But, at that timescale, if an internal coupling transiently exists between \(S_1\) and \(S_2\), then, after it disappeared, \(|\Psi(t)\rangle\) often cannot be equated with a tensorial product \(|\Psi_1(t)\rangle \otimes |\Psi_2(t)\rangle\) describing \(S_1\) and \(S_2\), respectively: \(|\Psi(t)\rangle\) is entangled (or not separable) [3,21]. Entanglement plays a significant role in

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1 IM2NP UMR 7334, Aix-Marseille Université, CNRS, 13397 Marseille, France
2 IRAP (Institut de Recherche en Astrophysique et Planétologie), Université de Toulouse, UPS, CNRS, CNES, 14 Avenue Edouard Belin, 31400 Toulouse, France
quantum information (QI) [14] e.g. in the context of quantum computing. The idea of a Quantum Computer (QC) dates back to the 1980s [5,13], and the word qubit to name the basic cell of the quantum computer appeared in 1995 [cf. [22] and its acknowledgements]. The basic components of the future QC should be the qubit, the quantum register—a quantum device consisting of several qubits—and the quantum gate—a device aimed at controlling qubits and registers. It is presently possible to claim that quantum computing research is coming of age [16], and qubits, registers and gates on one side, quantum algorithms using abstract qubits on the other are under development. Qubits are generally supposed to have been initially prepared in a pure state, which is not always easily achieved with physical qubits, as shown e.g. for nuclear spins in [18] (cf. its page 324). Coupling between qubits induces entanglement between initially unentangled qubit pure states. In the context of QI, entanglement may be either desired, e.g. when considering a QC, because it may allow some form of parallel computing (Deutsch speaks of quantum parallelism [5]), or avoided, because e.g. coupling with the environment may cause decoherence, and then stopping a calculation. It should then be useful to be able to know whether a given pure state of a system of abstract qubits is entangled or not.

The aim of this paper is to establish a necessary and sufficient unentanglement condition (for brevity, an iff condition) for a system consisting of \( N \) distinguishable qubits which is in a pure state. This iff condition should generalize a result which we recently established for \( N = 2 \) qubits [9], in the context of a subfield of QI, based on blind source separation (BSS), which started around 1985 in a classical context and which is now a mature field. In BSS [4,8], typically, a set of users (the Writer) presents a set of simultaneous signals (input signals, called sources) at the input of a multi-user communication system (the Mixer). The sources, compelled to possess some general properties (e.g. mutual statistical independence), are then combined (mixed, in the BSS sense) in the Mixer. Another set of users (the Reader) receives the signals arriving at the Mixer output. The Writer possibly knows the sources, but the Reader does not know them, and cannot access the inputs of the Mixer. That Mixer uses one or several parameter values, unknown to the Reader, who only knows some of its general properties. The Reader’s final task is the restoration of the sources (possibly up to some so-called acceptable indeterminacies) from the signals at the Mixer output, during the “inversion phase”. An intermediate task is the determination of the unknown parameters of the Mixer, or of its inverse, made during an “adaptation phase”. Since 2007, we have been developing a quantum version of BSS, namely blind quantum source separation (BQSS). In our previous papers ([10] and references therein), we considered two distinguishable qubits numbered 1 and 2, first prepared in a pure unentangled state \( |\Psi(t_w)\rangle \), by the Writer, at time \( t_w \). The qubit pair is isolated from the rest of the world between \( t_w \) and the time \( t_r \) when the Reader can operate. At time \( t_r \), the qubit pair is therefore still in a pure state \( |\Psi(t_r)\rangle \) but, because of an undesired coupling between the qubits (viewed as the action of a Mixer), \( |\Psi(t_r)\rangle \) is generally entangled. The Reader’s task is the restoration of \( |\Psi(t_w)\rangle = |\Psi_1(t_w)\rangle \otimes |\Psi_2(t_w)\rangle \) from \( |\Psi(t_r)\rangle \) (again possibly up to some acceptable indeterminacies). In 2012, in this journal, we presented a paper devoted to methods for separating quantum sources [6]. We recently introduced blind quantum process tomography (BQPT [7,11]), the blind version of quantum process tomography (QPT [18]), which is defined in Sect. 5.
iff condition hereafter established is first aimed at extending the solution of the BQSS and BQPT problems beyond the qubit pair, but it is hoped that it could be used in other contexts e.g. that of the QC.

In Sect. 2, we first recall the iff condition for \( N = 2 \) qubits and then present already existing criteria related to pure states or statistical mixtures of quantum systems composed of several parts, in situations which are more or less different from the present one—qubits, in arbitrary number. In Sect. 3, the method to be followed for finding this iff condition is presented. This \textit{iff} condition is established in Sect. 4, through an iterative process. The results and some of their applications are discussed in Sect. 5. The possibility of using the von Neumann concept is briefly tackled in “Appendix”.

2 About existing unentanglement criteria

In our previous papers, qubits were supposed to be physically implemented as spins \( 1/2 \). We hereafter first recall the notations used for writing an arbitrary pure state \( |\Psi\rangle \) of a qubit pair. \( |\Psi\rangle \) is developed as:

\[
|\Psi\rangle = \sum_{i=1}^{4} c_i |i\rangle , \quad \text{where} \quad |1\rangle = |++\rangle , |2\rangle = |+-\rangle , |3\rangle = |-+\rangle , |4\rangle = |--\rangle
\]

where \( |++\rangle \) is an abbreviation for \( |1, +\rangle \otimes |2, +\rangle , |1, +\rangle \) being the eigenstate for the eigenvalue \( 1/2 \) in the standard basis of qubit 1. We now consider an arbitrary \( N \)-qubit system and generalize this writing: for qubit number \( k \) in an \( N \)-qubit system:

\[
s_{zk} |k, \pm\rangle = \pm \frac{1}{2} |k, \pm\rangle
\]

and for a pure state \( |\Psi\rangle \) of this \( N \)-qubit system:

\[
|\Psi\rangle = \sum_{i=1}^{2N} c_i |i\rangle
\]

where \( |1\rangle = |++ .... + + \rangle \) (all qubits in state \( |+\rangle \)), \( |2^N\rangle = |-- .... -- \rangle \) (all qubits in state \( |-- \rangle \)). Qubit 1 is in state \( |+\rangle \) in the first \( 2^{N-1} \) states, and in state \( |-- \rangle \) in the \( 2^{N-1} \) remaining states. \( |2^{N-1} + 1\rangle = |+- .... + + \rangle \) (all qubits in state \( |+\rangle \), except qubit 1, in state \( |-- \rangle \)), \( |2^N - 2\rangle = |-- .... -- + \rangle \) (all qubits in state \( |-- \), except qubit \( N - 1 \), in state \( |+\rangle \)).

We immediately prove (for \( c_1 \neq 0 \)) the following property, which we established in more detail (including the \( c_1 = 0 \) case) in [9]: a pure state \( |\Psi\rangle \) of a qubit pair is unentangled if and only if the \( c_i \) coefficients obey the equality:

\[
c_1 c_4 = c_2 c_3.
\]
If \( c_1 \neq 0 \), \(|\Psi\rangle\) can always be written as:

\[
|\Psi\rangle = c_1 \left( |++\rangle + \frac{c_2}{c_1} |+-\rangle + \frac{c_3}{c_1} |--\rangle + \frac{c_4}{c_1} |--\rangle \right).
\]

When \( c_1c_4 = c_2c_3 \), \(|\Psi\rangle\) can then be written as:

\[
|\Psi\rangle = c_1 \left( |++\rangle + \frac{c_2}{c_1} |+-\rangle + \frac{c_3}{c_1} |--\rangle + \frac{c_2c_3}{c_1^2} |--\rangle \right)
\]

\[
= c_1 \left( |+\rangle + \frac{c_3}{c_1} |-\rangle \right) \otimes \left( |+\rangle + \frac{c_2}{c_1} |-\rangle \right)
\]

which proves that when \( N = 2 \) and \( c_1c_4 = c_2c_3 \), \(|\Psi\rangle\) is unentangled.

Conversely, if \(|\Psi\rangle\) is unentangled, it may be written as:

\[
|\Psi\rangle = (a_1 |+\rangle + b_1 |--\rangle) \otimes (a_2 |+\rangle b_2 |--\rangle).
\]

Equation (9) must be consistent with Eq. (1), which imposes the following relations:

\[
c_1 = a_1a_2, \ c_2 = a_1b_2, \ c_3 = b_1a_2, \ c_4 = b_1b_2
\]

and therefore \( c_1c_4 = c_2c_3 \).

In this paper, the proposed \textit{iff} condition will take the form of a set of equalities between the \( c_i \) coefficients. In Sect. 4, it will e.g. be shown that \(|\Psi\rangle\) is unentangled, when \( N = 3 \), if and only if the following two subsets (S1, S2) of equalities, corresponding to a total of four independent equalities, are simultaneously verified:

\[
S_1 : \quad \frac{c_1}{c_2} = \frac{c_3}{c_4} = \frac{c_5}{c_6} = \frac{c_7}{c_8}
\]

\[
S_2 : \quad c_2c_8 = c_4c_6.
\]

Coming now to existing criteria, one should first mention the Schmidt and the Peres–Horodecki ones. The so-called Schmidt decomposition allows one to express an \textit{iff} condition for pure states of a bipartite system, in which the dimension of each part of the system is not restricted to 2. And an extension of the concept of entanglement to statistical mixtures is somewhat possible through the notions of separability and of entanglement witnesses [3]. The Peres–Horodecki criterion [14] is an \textit{iff} condition for the separability of the density matrix describing a state of a bipartite system, valid when the dimensions of the state spaces of \( S_1 \) and \( S_2 \) are low. In [9], it was explained why these criteria are not appropriate for the context of BQSS. They are both restricted to bipartite systems, and therefore should presently be discarded, since this paper is devoted to the general case \( N > 2 \). With these presently strong restrictions, their aim is not very different from the one in this paper.

On the contrary, in the context of quantum communications, it is usual to speak of local unitary (LU) transformations of pure states, and when it is spoken of equivalent
states, the aim is different from the one in e.g. BQSS. If \(|\Psi(1, 2, \ldots N)\rangle\) is some pure state of a system composed of \(N\) distinguishable particles, and \(U\) a unitary operator acting on \(|\Psi(1, 2, \ldots N)\rangle\), then producing a transformed pure state \(U|\Psi(1, 2, \ldots N)\rangle\), the transformation is said to be local if \(U = U(1) \otimes U(2) \ldots \otimes U(N)\), where \(U(1), U(2), \ldots, U(N)\) are themselves unitary operators, each acting upon a single particle. The reason for the choice of the word \textit{Local} may be guessed if 1) one considers the transformation of an unentangled state \(|\Psi(1, 2, \ldots N)\rangle\), 2a) one considers two qubits, 1 in space-time zone 1, and 2 in space-time zone 2, separated by a spacelike interval, 2b) one imagines two experimenters, A(lice) and B(ob), A being able to access qubit 1 (only) and B qubit 2 (only), 2c) this situation is true for all distinct pairs of an \(N\)-qubit system. On the contrary, an instance of the effect of a non-LU transformation on an unentangled state is given in Sect. 5. Once LU transformations have been defined, two pure states are said to be equivalent if they differ by an LU transformation only [cf. e.g. the review article [26]]. When trying to define a degree of entanglement in order to classify all the possible pure states of a multipartite system, one is led to make no distinction between equivalent states. On the contrary, in the context of BQSS, this concept of equivalence through an LU transformation plays no role. If a given pure state \(|\Psi\rangle\) is written at the input of the Mixer, and a state \(|\Phi\rangle\) read at its output, the aim is not to get back some state \(|\Xi\rangle\) equivalent to \(|\Psi\rangle\), with the meaning that \(|\Xi\rangle\) and \(|\Psi\rangle\) differ by an LU transformation, but to get back state \(|\Psi\rangle\) itself, possibly up to some acceptable (in fact, far weaker than an arbitrary LU transformation) indeterminacies.

3 Towards a generalization of the relation between the \(c_i\) existing for a qubit pair

In order to extend entanglement-based BQSS methods beyond the simplest case, the qubit pair \((N = 2)\), it is highly desirable to find a set of relations, if it does exist, which, when \(N > 2\), could be substituted for the equality \(c_1c_4 = c_2c_3\). We have to find a collection of equalities between the \(c_i\) which are obeyed if and only if \(|\Psi\rangle\) is unentangled. We will consider only normalized states: \(|\Psi\rangle\langle\Psi| = 1\). Moreover, only the projector \(|\Psi\rangle\langle\Psi|\) has a physical meaning: one should not distinguish between \(|\Psi\rangle\) and \(e^{i\eta} |\Psi\rangle\) \((\eta\) is any real number). Therefore, if the complex numbers \(c_i\) are written \(c_i = \rho_i e^{i\varphi_i}\) \((\rho_i\) and \(\varphi_i\) are real numbers and \(i = 1, 2, \ldots 2N\)), then rather than the \(2^N\) phases, only e.g. the \((2^N - 1)\) phase differences \((\varphi_i - \varphi_1)\) are meaningful when defining an arbitrary pure state \(|\Psi\rangle\). Consequently, an arbitrary pure state \(|\Psi\rangle\) of the \(N\)-qubit system is defined by the value of \((2^N - 2)\) independent real numbers: \((2^N - 1)\) moduli and \((2^N - 1)\) phases. However, an unentangled pure state may be written as:

\[
|\Psi_{ue}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots |\psi_i\rangle \otimes \ldots \otimes |\psi_N\rangle
\]  

(13)

where each ordered factor describes the state of a given qubit and therefore depends upon two real numbers (a modulus, a phase). An unentangled state \(|\Psi_{ue}\rangle\) therefore depends upon \(2N\) real numbers only. Then, if \(|\Psi_{ue}\rangle\) is developed following (4), there should exist \(2[2^N - (N + 1)]\) relations between the \(2^{N+1}\) real quantities \(\{\rho_i, \varphi_i\}\) (besides those expressing that only \(|\Psi\rangle\langle\Psi|\) has a physical meaning and that \(|\Psi\rangle\) is
normalized). This result strengthens the hope that there may exist, between the $c_i$ coefficients, a set of $[2^N - (N + 1)]$ equalities which are verified if and only if $|\Psi\rangle$ is unentangled. This is presently only a hope, since normalization of $|\Psi\rangle$ and the physical meaning of $|\Psi\rangle\langle\Psi|$ lead to two constraints between real numbers, not to a constraint between the $c_i$ themselves. The present paper will not try to directly prove the existence of such relations between the $c_i$ for unentangled and only unentangled pure states, but will rather use an iterative approach in order to try and establish such general relations.

4 Finding a necessary and sufficient condition: an iterative approach

We will first examine the $N = 3$ and $N = 4$ cases in some detail, starting with $N = 3$. It is supposed that $c_i \neq 0$ for any $i$ value. The case when $c_i = 0$ for at least one $i$ value will be discussed in Sect. 5.

When $N = 3$, any pure state may be written as:

$$|\Psi\rangle = \left(\frac{c_3}{c_1}|++\rangle + \frac{c_5}{c_1}|+-\rangle + \frac{c_7}{c_1}|-+\rangle \right) \otimes c_1 |+\rangle$$
$$+ \left(\frac{c_4}{c_2}|++\rangle + \frac{c_6}{c_2}|+-\rangle + \frac{c_8}{c_2}|-+\rangle \right) \otimes c_2 |-\rangle$$

(14)

where $|+\rangle$ in the writing $c_1 |+\rangle$ and $|-\rangle$ in the writing $c_2 |-\rangle$ refer to a state of qubit no. 3. When

$$\frac{c_3}{c_1} = \frac{c_4}{c_2}, \frac{c_5}{c_1} = \frac{c_6}{c_2}, \frac{c_7}{c_1} = \frac{c_8}{c_2}$$

(15)

it is possible to write $|\Psi\rangle$ as:

$$|\Psi\rangle = \left(\frac{c_4}{c_2}|++\rangle + \frac{c_6}{c_2}|+-\rangle + \frac{c_8}{c_2}|-+\rangle \right) \otimes (c_1 |+\rangle + c_2 |-\rangle)$$

$$\otimes (|\Psi(1,2)\rangle)$$

(16)

When, moreover, the coefficients of $|\Psi(1,2)\rangle$ obey Eq. (5), which is presently written as:

$$\frac{c_8}{c_2} = \frac{c_4}{c_2} \cdot \frac{c_6}{c_2}, \text{ i.e. } c_2 c_8 = c_4 c_6$$

(17)

then $|\Psi(1, 2)\rangle$ is an unentangled state, and $|\Psi\rangle$ is unentangled. Therefore, when $N = 3$, if subsets (11) and (12) are obeyed, $|\Psi\rangle$ is unentangled [subset (11) is equivalent to the three equations (15)].
Conversely, if $|\Psi\rangle$ is unentangled, it can be written as:

$$|\Psi(1, 2, 3)\rangle = \left(\sum_{i=1}^{4} C_i |i\rangle\right) \otimes (a_3 |+\rangle + b_3 |\rangle)$$

where the $C_i$ coefficients obey the relation $C_1 C_4 = C_2 C_3$. Identifying Eq. (18) with Eq. (4) (with $N = 3$), one gets:

$$c_1 = C_1 a_3, c_2 = C_1 b_3, c_3 = C_2 a_3, ... c_8 = C_4 b_3$$

and therefore Eqs. (11) and (12) (i.e. Conditions $S_1$ and $S_2$) are obeyed.

When $N = 3$, $|\Psi\rangle$ is therefore unentangled if and only if the two subsets of equalities (11) and (12), expressing a total of 4 (independent) equalities, are obeyed. Therefore, the above-expressed hope of finding an iff condition for unentanglement using $[2^N - (N + 1)]$ relations between the $c_i$ is satisfied when $N = 3$.

Similarly, when $N = 4$, any pure state may be written as:

$$|\Psi\rangle = \left(|+++\rangle + \frac{c_3}{c_1} |+\rangle \leq + \cdots + \frac{c_{15}}{c_1} |-\rangle \right) \otimes c_1 |+\rangle$$

$$+ \left(|+++\rangle + \frac{c_4}{c_2} |+\rangle \leq + \cdots + \frac{c_{16}}{c_2} |-\rangle \right) \otimes c_2 |\rangle$$

where now $|\rangle$ in the writing $c_1 |+\rangle$ and $|\rangle$ in the writing $c_2 |\rangle$ refer to a state of qubit no. 4. When the following set of equalities is satisfied:

$$\frac{c_3}{c_1} = \frac{c_4}{c_2}, \frac{c_5}{c_1} = \frac{c_6}{c_2}, ..., \frac{c_{15}}{c_1} = \frac{c_{16}}{c_2},$$

which can also be written as:

$$\frac{c_1}{c_2} = \frac{c_3}{c_4} = \frac{c_5}{c_6} = \cdots = \frac{c_{15}}{c_{16}},$$

$|\Psi\rangle$ may be written as:

$$|\Psi\rangle = \left(|+++\rangle + \frac{c_4}{c_2} |+\rangle \leq + \cdots + \frac{c_{16}}{c_2} |-\rangle \right) \otimes (c_1 |+\rangle + c_2 |\rangle).$$

When, moreover, the coefficients of $|\Psi(1, 2, 3)\rangle$ obey subsets (11) and (12), presently written as:

$$\frac{c_2}{c_4} = \frac{c_6}{c_8} = \frac{c_{10}}{c_{12}} = \frac{c_{14}}{c_{16}}$$

$$c_4 c_{16} = c_8 c_{12},$$

$|\Psi\rangle$ is unentangled. Therefore, when $N = 4$, if the three subsets of Eqs. (22), (24) and (25) are verified, $|\Psi\rangle$ is unentangled.
Conversely, if $|\Psi\rangle$ is unentangled, it can be written as:

$$|\Psi(1, 2, 3, 4)\rangle = \left(\sum_{i=1}^{8} C_i |i\rangle\right) \otimes (a_4 |+\rangle + b_4 |−\rangle)$$  \hspace{1cm} (26)

where the $C_i$ coefficients now obey the following sets of relations:

$$\frac{C_1}{C_2} = \frac{C_3}{C_4} = \frac{C_5}{C_6} = \frac{C_7}{C_8}$$  \hspace{1cm} (27)

$$C_2 C_8 = C_4 C_6.$$  \hspace{1cm} (28)

Now identifying Eq. (26) with Eq. (4) (with $N = 4$), one gets:

$$c_1 = C_1 a_4, \ c_2 = C_1 b_4, \ c_3 = C_2 a_4, ... \ c_{16} = C_8 b_4.$$  \hspace{1cm} (29)

These equalities were obtained through the same process as in Eq. (19), and therefore lead to the same type of equalities as in Eq. (11), but with now sixteen and not eight $c_i$ coefficients, namely:

$$\frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{15}}{c_{16}}.$$  \hspace{1cm} (30)

Moreover, from Eqs. (27), (28) and (29), one gets:

$$\frac{c_2}{c_4} = \frac{c_6}{c_8} = \frac{c_{10}}{c_{12}} = \frac{c_{14}}{c_{16}}$$  \hspace{1cm} (31)

$$c_4 c_{16} = c_8 c_{12}.$$  \hspace{1cm} (32)

Therefore, when $N = 4$, $|\Psi\rangle$ is unentangled if and only if the following three subsets of equalities are verified:

$$S_1 : \frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{15}}{c_{16}}.$$  \hspace{1cm} (33)

$$S_2 : \frac{c_2}{c_4} = \frac{c_6}{c_8} = \frac{c_{10}}{c_{12}} = \frac{c_{14}}{c_{16}}.$$  \hspace{1cm} (34)

$$S_3 : c_4 c_{16} = c_8 c_{12}.$$  \hspace{1cm} (35)

When $N = 4$, this iff condition is therefore expressed through 3 subsets of equations, expressing a total of 11 equalities, which is also again the value of $[2^N - (N + 1)]$, now for $N = 4$.

Now taking an arbitrary $N$ value, one may write any pure state $|\Psi\rangle$ as:

$$|\Psi\rangle = \left(|++...+\rangle + \frac{c_3}{c_1} |++...−\rangle + ... + \frac{C_2^{N-1}}{c_1} |−−...−\rangle\right) \otimes c_1 |+\rangle$$

$$+ \left(|++...+\rangle + \frac{c_4}{c_2} |++...−\rangle + ... + \frac{C_2^{N}}{c_2} |−−...−\rangle\right) \otimes c_2 |−\rangle$$  \hspace{1cm} (36)
which generalizes Eqs. (14) and (20). The reasoning which led to Eq. (11) and (33) now leads to:

\[
\frac{c_1}{c_2} = \frac{c_3}{c_4} = \ldots = \frac{c_{2N-3}}{c_{2N-2}} = \frac{c_{2N-1}}{c_{2N}}. \tag{37}
\]

This subset contains \((2^{N-1} - 1)\) independent equalities. When they are obeyed, \(|\Psi\rangle\) may be written as:

\[
|\Psi\rangle = |\Psi (1, 2, 3, \ldots N - 1)\rangle \otimes |\Psi (N)\rangle. \tag{38}
\]

This state \(|\Psi\rangle\) is unentangled if and only if, moreover, \(|\Psi (1, 2, 3, \ldots N - 1)\rangle\) itself is unentangled. If the results obtained for \(N = 3\) and \(N = 4\) may be generalized, this is obtained when \((N - 2)\) other subsets of inequalities, corresponding to a total of \([2^N - (N + 1)] - (2^{N-1} - 1)\) equalities between the \(c_i\), i.e. \((2^{N-1} - N)\) independent equalities, are satisfied. This quantity is equal to 1 if \(N = 3\) (subset \(S_2,\) Eq. (12)), and to 4 if \(N = 4\) (subsets \(S_2,\) Eq. (34), and \(S_3,\) Eq. (35)).

We now momentarily suppose that the property established for \(N = 3\) and \(N = 4\) is true for \(N - 1\) (with \(N - 1 \geq 4\)), i.e. that \(|\Psi\rangle\) is unentangled if and only if the following \((N - 2)\) subsets of equalities are simultaneously verified:

\[
S_1 : \frac{c_1}{c_2} = \frac{c_3}{c_4} = \ldots = \frac{c_{2N-3}}{c_{2N-2}} = \frac{c_{2N-1}}{c_{2N}} \tag{39}
\]

\[
S_2 : \frac{c_2}{c_4} = \frac{c_6}{c_8} = \ldots = \frac{c_{2N-2}}{c_{2N}} \tag{40}
\]

\[
S_3 : \frac{c_4}{c_8} = \frac{c_{12}}{c_{16}} = \ldots = \frac{c_{2N-4}}{c_{2N}} \tag{41}
\]

\[
S_{N-3} : \frac{c_{2N-4}}{c_{2^N}} = \frac{c_{3^2N-4}}{c_{4^2N-4}} = \frac{c_{5^2N-4}}{c_{6^2N-4}} = \frac{c_{7^2N-4}}{c_{8^2N-4}} \tag{42}
\]

\[
S_{N-2} : \frac{c_{2N-3}}{c_{2^N-3}} = \frac{c_{4^2N-3}}{c_{6^2N-3}} = \frac{c_{7^2N-3}}{c_{8^2N-3}} \tag{43}
\]

In order to help the reader anxious to see more explicitly the meaning of the dots in Eq. (42), and in Eq. (48) hereafter, the 6 subsets for \(N = 7\) are all written at the end of this section (four of them again with dots, but then with a rather obvious meaning).

We now establish that any ket \(|\Psi\rangle = \sum_{i=1}^{2^N} c_i |i\rangle\) describing a pure state of an \(N\)-qubit system is unentangled if and only if the \(N - 1\) following subsets are all obeyed:

\[
S_1 : \frac{c_1}{c_2} = \frac{c_3}{c_4} = \ldots = \frac{c_{2N-3}}{c_{2N-2}} = \frac{c_{2N-1}}{c_{2N}} \tag{45}
\]

\[
S_2 : \frac{c_2}{c_4} = \frac{c_6}{c_8} = \ldots = \frac{c_{2N-2}}{c_{2N}} \tag{46}
\]

\[
S_3 : \frac{c_4}{c_8} = \frac{c_{12}}{c_{16}} = \ldots = \frac{c_{2N-4}}{c_{2N}} \tag{47}
\]

\[
S_{N-2} : \frac{c_{2N-3}}{c_{2^N-3}} = \frac{c_{3^2N-3}}{c_{4^2N-3}} = \frac{c_{5^2N-3}}{c_{6^2N-3}} = \frac{c_{7^2N-3}}{c_{8^2N-3}} \tag{48}
\]
The approach already used for $N = 3$ and 4 is applied to an $N$-qubit system. Any pure state may be written as:

$$|\Psi\rangle = \left( |++ \ldots +\rangle + \frac{c_4}{c_2} |++ \ldots -\rangle + \ldots + \frac{c_{2N-1}}{c_1} |-- \ldots --\rangle \right) \otimes c_1 |+\rangle$$

$$+ \left( |++ \ldots +\rangle + \frac{c_4}{c_2} |++ \ldots -\rangle + \ldots + \frac{c_{2N}}{c_2} |-- \ldots --\rangle \right) \otimes c_2 |--\rangle \quad (51)$$

where now $|+\rangle$ in the writing $c_1 |+\rangle$ and $|--\rangle$ in the writing $c_2 |--\rangle$ refer to a state of qubit no. $N$. When each $c_{2k-1}/c_1$ quantity (for $k = 2, 3 \ldots N$) is equal to $c_{2k}/c_2$, this collection of relations can collectively be written as $S_1$ subset (45), and $|\Psi\rangle$ may be expressed as:

$$|\Psi\rangle = \left( |++ \ldots +\rangle + \frac{c_4}{c_2} |++ \ldots -\rangle + \ldots + \frac{c_{2N}}{c_2} |-- \ldots --\rangle \right) \otimes (c_1 |+\rangle + c_2 |--\rangle).$$

$$|\Psi(1, 2, \ldots N-1)\rangle$$

(52)

If, moreover, the $c_{2k}/c_2$ coefficients in $|\Psi(1, 2, \ldots N-1)\rangle$ obey all the equalities expressed in Eq. (39) to (44), $|\Psi(1, 2, \ldots N-1)\rangle$ is unentangled, and $|\Psi\rangle$ itself is therefore unentangled. For instance, Eq. (44) presently takes the form:

$$c_{2N-2}c_4 c_2^N = c_2^2 c_2^N c_3 c_2^N$$

(53)

which is subset (50). More generally, Eqs. (39)–(44) presently take the form of Eqs. (46)–(50), respectively. Any reader aiming at establishing these equalities should appreciate that the $c_i$ coefficients in Eq. (4) are generic quantities. For instance, $c_1$ is the coefficient for $|++ \rangle$ if $N = 3$, whereas it is the coefficient for $|++ ++ \rangle$ if $N = 5$.

Conversely, if $|\Psi(1, 2, \ldots, N)\rangle$ is unentangled, it can be written as:

$$\left( \sum_{i=1}^{2N-1} C_i |i\rangle \otimes (a_N |+\rangle + b_N |--\rangle) \right)$$

(54)

where the $C_i$ coefficients obey equalities expressed through subsets (39)–(44), with the $C_i$ instead of the $c_i$ coefficients. Then, the method used for $N = 3$ and $N = 4$ is again used for expressing each $c_i$ coefficient in expression $|\Psi\rangle = \sum_{i=1}^{2N} c_i |i\rangle$ as a function of both a $C_i$ coefficient and $a_N$ or $b_N$. This allows us to show that subsets (45)–(50) are obeyed. Therefore, if it is true that a ket $|\Psi(1, 2, \ldots N-1)\rangle$ is unentangled if and only if subsets (39)–(44) are all verified, then it is true that a ket $|\Psi(1, 2, \ldots N)\rangle$ is unentangled if and only if subsets (45)–(50) are all verified.
Considering successively \( N = 2, 3 \) and \( 4 \), it was shown above that \( |\Psi\rangle \) is unentangled if and only if a collection of equalities, structured into \((N - 1)\) subsets, is obeyed. These results and this iterative discussion from \((N - 1)\) to \( N \) finally allow us to claim that subsets (45)–(50) do collectively express an iff condition for a ket \( |\Psi\rangle \) of an \( N \)-qubit system to be unentangled.

If e.g. \( N = 7 \) (the dimension of the state space is 128 then), it is tedious but quite possible, through successive iterations, to get the explicit expressions of the six subsets of equalities expressing unentanglement. They are, respectively,

\[
S_1 : \frac{c_1}{c_2} = \frac{c_3}{c_4} = \ldots = \frac{c_{125}}{c_{126}} = \frac{c_{127}}{c_{128}} \tag{55}
\]

\[
S_2 : \frac{c_2}{c_4} = \frac{c_6}{c_8} = \ldots = \frac{c_{126}}{c_{128}} \tag{56}
\]

\[
S_3 : \frac{c_4}{c_8} = \frac{c_8}{c_{16}} = \ldots = \frac{c_{124}}{c_{128}} \tag{57}
\]

\[
S_4 : \frac{c_8}{c_{16}} = \frac{c_{24}}{c_{32}} = \ldots = \frac{c_{120}}{c_{128}} \tag{58}
\]

\[
S_5 : \frac{c_{16}}{c_{32}} = \frac{c_{48}}{c_{64}} = \frac{c_{80}}{c_{96}} = \frac{c_{112}}{c_{128}} \tag{59}
\]

\[
S_6 : \frac{c_{32}c_{128}}{c_{64}c_{96}} \tag{60}
\]

They have been written here in order to help the reader interpret the dots in Eqs. (42) and (48).

Our results for \( N = 2, 3 \) and \( 4 \) suggest that there are \([2^N - (N + 1)]\) independent equalities in Eqs. (45)–(50). We now establish this result, again using mathematical induction. We first suppose that it is true that Eqs. (39)–(44), for an \((N - 1)\)-qubit system, contain \((2^{N-1} - N)\) such equalities. Now considering an \( N \)-qubit system, we may claim that the corresponding number of such equalities is the sum of two quantities: \((2^{N-1} - 1)\), the number of equalities associated with \( S_1 \) [cf. Eq. (45)] and \((2^{N-1} - N)\) new equalities expressing that \(|\Psi(1, 2, 3, \ldots N - 1)\rangle\) in Eq. (38) is itself unentangled, which does lead to a total of \([2^N - (N + 1)]\) such equalities. This expression, which was already known to be valid for \( N = 2, 3 \) and \( 4 \), is therefore valid for any \( N \geq 2 \).

\section{Discussion}

It is possible to build other sets of equalities which are obeyed if and only if an arbitrary pure state \( |\Psi\rangle \) is unentangled. When \( N = 3 \), for instance, keeping the same approach, it is easy to replace (12) with condition \( c_1c_7 = c_3c_5 \). Then, with the same approach for \( N > 3 \), all even \( c_i \) coefficients in the subsets \( S_k \) with \( k > 1 \) are suppressed. For \( N = 6 \), e.g. the \( S_5 \) subset becomes \( c_1c_{49} = c_{17}c_{33} \). Use of even indices leads to simpler expressions, which explains the choice made in this paper.

In [9], with \( N = 2 \), if at least one of the \( c_i \) coefficients is equal to 0, it was shown that condition \( c_1c_4 = c_2c_3 \) is still valid. In Sect. 4 of the present paper, it was assumed that \( c_i \neq 0 \) for any \( i \). When \( N = 3 \), if e.g. \( c_5 = 0 \), then in Eq. (14) the \( (c_5/c_1)|-+\rangle \)
term is absent and |Ψ⟩ is then unentangled only if $c_6 = 0$ [cf. the presence of the $(c_6/c_2)|−⟩$ term in Eq. (14)]. When $N > 3$, if $c_5 = 0$, then in all the subsets expressing unentanglement, the $c_6$ terms will be absent. The reason is that in Eq. (36) the (unexplicitly written) $c_5/c_1$ term of qubits 1 to $(N - 1)$ is associated with the $c_1|+⟩$ state of qubit $N$, and the corresponding state of qubits 1 to $(N - 1)$ associated with the $c_2|−⟩$ state of qubit $N$ has a $c_6/c_2$ coefficient. This reasoning may also be used if more than one $c_i$ coefficient is equal to 0.

When $N = 20$, the dimension of the state space $E_{20}$ is $2^{20}$, which is roughly $10^6$. An unentangled normalized state then depends upon 40 real numbers only. In the context of quantum information processing, it is generally considered that the wealth of the quantum behaviour originates in the existence of entanglement, but it may be important to be able to decide whether a given pure state is entangled or not, e.g. in order to achieve BQSS or BQPT, and finding an iff condition is therefore significant. The present paper has shown that, when |Ψ⟩ is unentangled, there exist $[2^N - (N + 1)]$ independent equalities between the $c_i$ coefficients, the value of which is itself roughly $10^6$ when $N = 20$. But it has also been found that these equalities may be classified into only $(N - 1)$ subsets, e.g. 19 subsets when $N = 20$. It is hoped that this classification should allow tractable operations in numerical simulations or calculations.

We now come to recent papers making use of a local unitary (LU) transformation. Ninety years after the building of modern quantum mechanics (QM), there is a vast literature devoted to its foundations (see e.g. [1,15,25]), while most physicists use QM without discussing its deep content. The following lines just aim at drawing a link between these recent papers and this literature. Paty [19] has stressed that, historically, well before the 1935 EPR paper, Einstein, at the 1927 Solvay Congress [12] (p. 256), exposed his concern about what he would later on call the incompleteness of QM. In his 1995 paper, Paty clearly and convincingly asserts that “it is only recently, indeed, that the concept of non-locality as a fundamental feature of quantum mechanics has been fully appreciated, and commentators have seldom realized that this was one of Einstein’s main points”, and that “it is in the Einstein–Podolsky–Rosen’s paper itself that non-locality is described and that emphasis is put on it”. At the same 1927 meeting, Einstein stated that the interpretation of $|Ψ|^2$ as a probability density for a single particle (rather than for an ensemble of particles) implied, for him, “a contradiction with the principle of relativity”. After 1945, a deepening of the foundations of QM partly overlapped with the development of classical and quantum information. Bell’s contributions to these foundations, from his 1966/1964 papers down to his death in 1990 [1], especially stimulated the development of both experimental tests and so-called quantum communications. In the context of quantum communications, speaking of LU transformations (cf. Sect. 2) is usual. An LU transformation $U = U(1) \otimes U(2)$ transforms a pure unentangled state $|Ψ(1) \otimes |Φ(2)⟩$ into the unentangled state $(U(1) |Ψ(1)⟩) \otimes (U(2) |Φ(2)⟩)$. On the contrary, a simple calculation shows that e.g. the non-LU transformation $U = e^{ia s_1 s_2}$ acting on the unentangled state $|+⟩$, ($a$ is some dimensional real constant) transforms it into the entangled state [cf. Eq. (5)]

$$
\frac{e^{ia/4}}{4}(|+x, +x⟩ - |−x, −x⟩) - \frac{e^{-ia/4}}{4}(|+x, −x⟩ - |−x, +x⟩)
$$

(61)
where e.g. $|+x, -x\rangle$ means $|1, +x\rangle \otimes |2, -x\rangle$, and $|i, +x\rangle$ (resp. $|i, -x\rangle$) is the eigen-ket for $s_{ix}$ for the eigenvalue $1/2$ (resp. $-1/2$), with $i = 1, 2$. LU transformations should therefore be distinguished from entanglement-inducing transformations. The latter transformations are faced e.g. in BPQT and BQSS, which are important quantum information processing problems due to their applications, including those presented hereafter.

Blind or non-blind QPT may be defined as the identification (i.e. estimation) of a given quantum process or gate, called the direct process or gate hereafter, which receives a “source state”. As discussed e.g. in [2,17,18,23,24,27], (B)QPT is a major quantum information processing tool, since it especially allows one to characterize the actual behaviour of quantum gates, which are the building blocks of the quantum computers considered in Sect. 1. The usual, i.e. non-blind, version of QPT requires one to know, hence to precisely control (i.e. prepare), the specific quantum source states used as inputs of the quantum gate to be characterized. The blind version of this tool, i.e. BQPT, then provides an attractive extension of QPT, since it allows one to use quantum source states whose values are unknown and arbitrary, except that they are requested to meet some general properties. These properties e.g. consist of unentanglement [11], which is one of the motivations for analysing unentanglement in the present paper (more details about the operation of BQPT are available e.g. in [10,11]).

BQSS may be seen as a quantum information processing problem where one aims at handling the altered quantum state available at the output of a direct quantum process/gate which typically involves undesired coupling between its qubits, this process and its input being initially unknown. This BQSS problem is e.g. handled by (1) first identifying that direct gate with BQPT, thus using only the output of that gate and unentanglement or other properties, (2) then deriving a quantum gate that performs the inverse transform of that of the direct gate, and (3) then feeding that “inverse gate” with the altered states available at the output of the direct gate during final operation, so as to restore the corresponding source, i.e. non-altered, states. One may anticipate that this approach will be useful e.g. in situations where data are stored in a register of qubits of a quantum computer, for subsequent use. Due to non-idealities of the physical implementation of that register, the qubits which form it may be coupled (e.g. when qubits are implemented as the spins of electrons which are close to one another). As time goes on, the register state will therefore evolve in a complicated way due to qubit coupling, thus making the final value of that register state not directly usable in the target application of the quantum computer. BQSS may then be used to restore the initially stored register state, before providing it to the part of the quantum computer which uses these non-altered data to perform the target task of that computer.

6 Conclusion

In the 2009 review article devoted to entanglement [14], the Horodecki team noticed that “it appears that this new resource is complex and difficult to detect”. Experimental

1 Other approaches perform BQSS directly, i.e. without first resorting to BQPT.
and theoretical aspects were both involved. If one focuses this comment on the idea that establishing whether a pure state is entangled or not is a cumbersome task, the following remarks may be made. In the present paper, devoted to an arbitrary number, $N$, of distinguishable qubits, it has been shown that if a pure state of that $N$-qubit system is developed over the $2^N$ basis states of the generalized standard basis (or of some arbitrary well-defined basis) as $|\Psi\rangle = \sum_i c_i |i\rangle$, one is then led to introduce $(N - 1)$ subsets of equalities, which are verified if and only if $|\Psi\rangle$ is unentangled. It should however be realized that if $N = 20$, these 19 subsets together collect $[2^N - (N + 1)]$ equalities, which is here roughly equal to $2^{20}$, i.e. approximately $10^6$. While the complexity of the problem is reflected in the fact that the number of equalities roughly grows as $2^N$, it is hoped that the necessary and sufficient condition established in this paper, which introduced a systematic ordering within these equalities, through a classification into $(N - 1)$ subsets, may in practice help in the manipulation of the entanglement concept.

A The von Neumann entropy and the establishment of the iff condition

The entropy concept, which did not appear in this paper yet, is briefly considered here. The von Neumann entropy of a quantum system in a pure or mixed state described by a density operator $\rho$ is the trace $S = -Tr(\rho \ln \rho)$. This concept cannot directly be used in an attempt to find an iff condition for the unentanglement of a pure state $|\Psi\rangle$ of an $N$-qubit system, since its von Neumann entropy is zero for both unentangled and entangled pure states. But this $N$-qubit system can be viewed as a bipartite system $\Sigma$, composed of parts $\Sigma_A$ and $\Sigma_B$, and if $\Sigma$ is described by $\rho$, one may first introduce reduced density operators $\rho_A = Tr_B \rho$ and $\rho_B = Tr_A \rho$ (see e.g. [20]).

From now on, we focus on the situation when $\rho = |\Psi\rangle\langle\Psi|$. Both $\Sigma_A$ and $\Sigma_B$ possess orthonormal basis states $|\psi_i^A\rangle$ and $|\chi_i^B\rangle$ (Schmidt decomposition), where the sum of the squares of the real non-negative so-called Schmidt coefficients $\lambda_i$ is equal to 1 (see e.g. [18]). Moreover, $\rho_A$ and $\rho_B$ have the same eigenvalues, equal to $\lambda_i^2$ [18]. One introduces the entropies for $\Sigma_A$ and $\Sigma_B$, respectively, $S_A = -Tr_A(\rho_A \ln \rho_A)$ and $S_B = -Tr_B(\rho_B \ln \rho_B)$, and, as a result of both the Schmidt decomposition and the just mentioned property of the eigenvalues of $\rho_A$ and $\rho_B$, $S_A = S_B = -\sum_i \lambda_i^2 \ln \lambda_i^2$. Then $S = 0$, while $S_A = S_B \geq 0$, and $|\Psi\rangle$ is unentangled if and only if $S_A = S_B$ is equal to zero. A means of establishing an iff condition through the reduced entropy concept therefore does in principle exist. But the fact that the reduced entropy of a bipartite system is related to the Schmidt decomposition immediately suggests that, if this concept is used as a tool for establishing an iff condition for the $c_i$ introduced in this paper, the difficulty will be at least as great as the one found with the Schmidt criterion, already discussed in Sect. 2.

Let us first examine the two-qubit case: $A$ is qubit 1 and $B$ qubit 2. Then, keeping our previous notations, $|\Psi\rangle = \sum_{i=1}^{4} c_i |i\rangle$, one has first to express the condition $S_A = 0$ as a function of the $c_i$ coefficients, but this means: (1) calculating the expression of $\rho_A$, (2) calculating its eigenvalues, (3) calculating $S_A$ and solving the equation $S_A = 0$. 

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The reader may verify that a tedious calculation leads to our well-known result:

\[(c_1 c_4 - c_2 c_3) = 0.\]  

(62)

The next simplest situation is \(N = 3\), and one may first introduce \(\rho_3\), the reduced entropy for qubit no. 3, and focus on the corresponding reduced entropy \(S_3 = -Tr(\rho_3 \ln \rho_3)\), which is zero iff \(|\Psi\rangle\) is unentangled. This necessitates first to calculate all the elements of the reduced density matrix \(\rho_3\), each one a complicated sum involving our \(c_i\) coefficients, and secondly to find an analytical expression for the eigenvalues of \(\rho_3\). But, once this is done, one knows that if and only if one and only one eigenvalue is nonzero, and therefore equal to one, then the state is unentangled. Considering the reduced entropy \(S_3\), i.e. manipulating sums of quantities involving logarithms, is therefore unnecessary.

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