Non-perturbative Solution to the Quantum Interaction Problem via Schwinger’s Action Principle

C. A. M. de Melo\textsuperscript{1, 2*}, B. M. Pimentel\textsuperscript{2†}, and J.A. Ramirez\textsuperscript{3‡}

\textsuperscript{1}Instituto de Ciência e Tecnologia - Universidade Federal de Alfenas, Campus Poços de Caldas, BR 267 - Rod. Pref. José Aurélio Vilela 11.999, Km 533, 37715-400 Cidade Universitária - Poços de Caldas, MG, Brasil

\textsuperscript{2}Instituto de Física Teórica - São Paulo State University, P.O.Box 70532-2, CEP: 01156-970 - São Paulo, SP, Brasil. and

\textsuperscript{3}Instituto de Física - Universidade Federal da Bahia, Campus Universitário de Ondina, CEP 40210-340, Salvador, BA, Brazil.

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Abstract

The most realistic situations in quantum mechanics involve the interaction between two or more systems. In the most of reliable models, the form and structure of the interactions generate differential equations which are, in the most of cases, almost impossible to solve exactly. In this paper, using the Schwinger Quantum Action Principle, we found the time transformation function that solves exactly the harmonic oscillator interacting with a set of other harmonic coupled oscillators. In order to do it, we have introduced a new special set of creation and annihilation operators which leads directly to the \textit{dressed} states associated to the system, which are the real quantum states of the interacting “\textit{field-particle}” system. To obtain the closed solution, it is introduced in the same foot a set of \textit{normal mode} creation and annihilation operators of the system related to the first ones by an orthogonal transformation. We find the eigenstates, amplitude transitions and the system spectra without any approximation. At last, we show that the Schwinger Variational Principle provides the solutions in a free representation way.

* cassius@unifal-mg.br
† pimentel@ift.unesp.br
‡ johnarb@ufba.br
I. INTRODUCTION

To obtain an exact solution to the radiation-matter interaction problem results in a difficult or almost impossible task for several systems. In many cases theoretical approaches based on the intensity or asymptotic behavior of the system allow perturbative or computational treatments, giving an idea of the functional form of the solutions. Despite this, those situations are not general and are related to very specific laboratory conditions. On the other hand, systems with strong interaction escape from those approaches. In particular situations these systems can be reduced to special simple cases as the semi-classical Rabi Model (RM) and its quantum version, the so-called Jaynes-Cummings Model (JCM) \cite{1, 2}. These idealizations of the very original problem are still rich in phenomenology, admitting nonperturbative approaches to cavity confined systems: a single atom interacting with one monochromatic mode of the field in the cavity, composing a subfield of quantum optics called Cavity Quantum Electrodynamics (CQED).

There is a generalized approach to CQED called Dressed Atom which can deal with systems that can not be treated by perturbative methods and leading to analytical results, being first proposed by Cohen Tannoudji, Serge Haroche and Nicole Polonsky \cite{3–5}. In such approach, the atoms or molecules into the cavity interact with the surrounding photons forming a weakly interacting subsystem with the rest of the field; thus, the atom is “dressed” by the photons of the surrounding electromagnetic field \cite{6}. This description works since only one of the field modes inside the cavity, or a superposition of them, has photons with the necessary frequency to induce the transitions in the atoms, then we can say that those photons dress the atom and the other modes are empty and make the role of reservoir. Thus, radiation processes in cavities can be put in a convenient base allowing to deal with some cases that otherwise might not be addressed by perturbative methods.

However, one of the limitations of the Dressed State theory is given by the loss of generality when the interactions have a non-linear character; since in those cases the determination of such states loses its rigor \cite{7}. Alternatively, one can define Dressed or Renormalized Coordinates \cite{8–10}, offering an extension of the Dressed State concept which proves to be suitable to deal with systems where the interactions between the atom and modes of the field are linear. Some applications of this approach are the description of ohmic dissipation \cite{11} and the radiation problem of a dipolar charge distribution \cite{8–10, 12, 13}. It can be seen that,
regardless of the approach taken, either quantum mechanics \[3\] or path integral formalism \[12, 13\], those studies always have been accompanied by the need for representation of the system into a set of spatial coordinates.

Here we introduce a new approach to deal with systems modeled by a harmonic oscillator interacting linearly with a set of other harmonic oscillators using the Schwinger Quantum Action Principle. Instead of using Dressed Atom or Coordinates approaches, we define two sets of creation and annihilation operators for the real quantum states of each of the component of the system. The first set will be associated with measurable physical states of each individual component in the presence of interaction, \textit{i.e.}, to the harmonic oscillator and each mode of the field. The second set is associated with normal modes of the system, each of these modes expresses the state of the whole system, oscillator + field modes. Thus, we shall call the first set \textit{the creation and annihilation operators of Dressed States} of the system (DSO) and the second set \textit{the creation and annihilation operators of Normal Modes States} of the system (NMO).

The content of this work is divided into following: in the first section is given a complete description of the model and a mathematical apart about the characteristics of the interaction matrix. The second section is devoted to define the two sets of the creation and annihilation operators as well as to introduce the Normal and Dressed Hamiltonians. In the third section we briefly introduce the Schwinger Action Principle and show how to obtain the dynamical equations for the operators and the transformation function of the system. In the last sections we find the final transformation function which is calculated in the Normal Mode Operator base, and then it is transformed to the Dressed State operator base from which is derived the spectrum and generalized probability transitions. Finally, the work is ended with some observations and perspectives.

II. SCHWINGER QUANTUM ACTION PRINCIPLE

Schwinger Quantum Action Principle offers a general approach to Quantum Mechanics. This approach describes the microscopic phenomena by means of quantum measurement symbols and expressing the dynamics of operators and quantum states through a variational principle.

Since its foundation in the algebraic measurement theory \[14, 15\] this quantum formalism
emerges losing any reference to the classical analogue of the system, not being a quantization procedure. In this formulation, changes of quantum states and observables are studied in a detailed way while the correspondence principle [16] is not used \textit{a priori}. Commutation relations are derived in a totally self-consistent way [17, 18] from the analysis of the admissible variations of the operators and states of the theory. Due to its generality, Schwinger’s formulation can be used to study quantization of gauge theories without gauge fixing [19], the behaviour of classical fields on curved and torsioned backgrounds [20] and even to construct quaternionic extensions of quantum theory [21].

In Schwinger’s formulation any infinitesimal variation of a transformation function $\langle a(t_1)|b(t_0) \rangle$ can be obtained as the matrix element of a single infinitesimal generator: the \textit{infinitesimal quantum action operator} [14, 17],

$$\delta \langle a(t_1)|b(t_0) \rangle = i \langle a(t_1)|\delta \hat{S}_{t_1,t_0}|b(t_0) \rangle = i \langle a(t_1)| \left( \hat{p} \delta \hat{q} - \hat{H} \delta t \right) \rangle_{t_0}^t |b(t_0) \rangle,$$

which is the variation \( \delta \hat{S}_{t_1,t_0} = \delta \left[ \hat{S}_{t_1,t_0} \right] \) of the quantum action operator \( \hat{S}_{t_1,t_0} = \int_{t_0}^{t_1} \hat{L}(t)dt \). The dynamical equations of Schrödinger and Heisenberg can be obtained from fixing the boundary conditions whether on the states or on the operators.

One of the essential concepts in this formalism is the fact that the dynamics do not depend on the path of the system in the physical space, being all the necessary information contained on the boundaries. This dependence is given by the infinitesimal generator \( \hat{G}(t) \) evaluated in the end time \( t_0 \) and \( t_1 \),

$$\hat{G} = \hat{p} \delta \hat{q} - \hat{H} \delta t,$$

where

$$\hat{p} \equiv \frac{\partial \hat{L}}{\partial \dot{\hat{q}}},$$

$$\hat{p} \cdot \dot{\hat{q}} - \hat{L} \equiv \hat{H}.$$

In order to integrate (11) to obtain the probability amplitude, it is necessary doing the time-ordering of the operators in the Dirac sense [16, 22] to reach \( \delta \hat{S}_{t_1,t_0} \implies \delta \hat{W}_{t_1,t_0} \) such that

$$\delta \langle a(t_1)|b(t_0) \rangle = i \langle a(t_1)|\delta \hat{S}_{t_1,t_0}|b(t_0) \rangle = i \delta \hat{W}_{t_1,t_0} \langle a(t_1)|b(t_0) \rangle$$
obtaining,

\[ \langle a(t_1)|b(t_0)\rangle = e^{i\mathcal{H}_{1-t_0}}. \quad (4) \]

Originally, the inclusion of the Lagrangian for the study of the dynamics at quantum level was made by Dirac \cite{22} arguing, among other things, in favor of its relativistic invariance. We can see that the final form of the Action Principle backs up to this idea, but Schwinger use a more convenient form of the Lagrangian in its canonical form \cite{3}, i.e., as a function of momentum and coordinates, avoiding second order differential equations for the observables.

This principle is particularly rich in analogies with classical mechanics, since the generator \( \hat{G} \), which is obtained from the variation of the action and associated with all admissible infinitesimal variations of the dynamic system, is also related to the unitary transformations which preserve both the structure and characteristics of the studied system.

### III. A USEFUL MODEL

In this work we shall use a model in which an harmonic oscillator is linearly coupled to a set of many other harmonic oscillators. Such model appears in several contexts in physics, specially when dealing with light-matter interaction.

In Ref. \cite{8} this model is used to describe a harmonic oscillator in the center of a sphere interacting with a scalar field. This model is a simplification of a radiation-matter interaction problem in which an electric dipole interacts with a electromagnetic field in the center of a spherical cavity. In this case, let \( q_0(t) \) be the position of the oscillator representing the matter content and \( \phi(r, t) \) the scalar field, the equations of motion can be defined directly as

\[ \ddot{q}_0(t) + \omega_0^2 q_0(t) = 2\pi \sqrt{g} \int_0^R d^3r \phi(r, t) \delta(r), \quad (5a) \]

\[ \frac{\partial^2 \phi(r, t)}{\partial t^2} - \nabla^2 \phi(r, t) = 2\pi \sqrt{g} q_0(t) \delta(r), \quad (5b) \]

in which \( g \) is the interaction constant between the scalar field and the harmonic oscillator, the light speed \( c = 1 \) and \( R \) is the sphere radius. The boundary conditions on the radial coordinate \( r \) demands that \( \phi(r, t) \) be nonzero only for \( 0 < r < R \). The solution for \( (5a) \) and \( (5b) \) can be expressed by performing an expansion of the scalar field as follows

\[ \phi(r, t) = \sum_{k=1}^{\infty} q_k(t) \phi_k(r), \quad (6) \]
in which, given the geometry of the system, an expansion in terms of plane waves it is not appropriate. So, instead, we use an appropriate expansion of the scalar field in terms of the resonant modes that can be originated in a spherical structure. These functions $\phi_k (r)$ are expressed in a convenient spherical base and the frequencies of these modes are given depending on the geometry. Hence, we get the following set of differential equations

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = \sum_{k=1}^{\infty} c_k q_k(t), \quad (7a)$$

$$\ddot{q}_k(t) + \omega_k^2 q_k(t) = c_k q_0(t) \quad (7b)$$

where, $\omega_k = \frac{\pi}{k}$ and $c_k = \frac{\omega_k}{k} \sqrt{2g}$. The approximations considered in this model are associated with a very specific and real situations. For instance, the object in the center of the sphere can be considered a molecule or atom whose vibrational spectrum can be approximated with a harmonic oscillator $q_0$.

Another situation which results in the same model is given by a small charge distribution compared with the wavelengths of the incident radiation. Using an appropriate gauge choice, the resulting model is a system consisting of a harmonic oscillator coupled to the modes of a radiation scalar field.

As a third example, let us consider an ionized atom or molecule in a general metallic cavity. The lagrangian operator for this system is

$$\hat{L} = \hat{L}_{\text{mat}} + \int_V d^3 r \left[ -\hat{E} (r, t) - \frac{1}{c} \frac{\partial}{\partial t} \hat{A} (r, t) - \frac{\hat{E}^2 (r, t) + \hat{H}^2 (r, t)}{2} + \frac{1}{c} \hat{j} (r, t) \cdot \hat{A} (r, t) \right]$$

where all products must be taken as symmetrized and $\hat{L}_{\text{mat}}$ describes the harmonic oscillator $q_0$ and $\hat{j} (r, t)$ it is the electric current associated with $q_0$.

The field at the interior of the cavity must resonate in normal modes which leads to a natural decomposition for the vector potential:

$$\int_V d^3 r A_\alpha (r) \cdot A_\beta (r) = \delta_{\alpha\beta}$$

$$\frac{1}{c} \hat{A} (r, t) = \sum_\alpha \hat{q}_\alpha (t) A_\alpha (r)$$

$$-\hat{E} (r, t) = \sum_\alpha \hat{p}_\alpha (t) A_\alpha (r)$$

$$\hat{j} (r, t) = \sum_\beta c_\beta \hat{q}_0 (t) A_\beta (r)$$
Using this together the radiation gauge and the boundary conditions, we find
\[
\hat{L} = \hat{L}_{\text{mat}} + \left[ \sum_{\beta} \sum_{\alpha} \hat{p}_{\beta}(t) \frac{d}{dt} \hat{q}_{\alpha}(t) \delta_{\alpha\beta} \right] + \\
- \frac{1}{2} \left( \left( \sum_{\gamma} \hat{p}_{\gamma}(t) \hat{p}_{\beta}(t) \delta_{\gamma\beta} \right) \right) + \\
- \frac{1}{2} \sum_{\beta} \sum_{\alpha} c^2 \hat{q}_{\alpha}(t) \hat{q}_{\beta}(t) \int_V d^3r \mathbf{A}_{\beta}(r) \cdot (\nabla \times (\nabla \times \mathbf{A}_{\alpha}(r))) + \\
+ \sum_{\beta} \hat{q}_{\beta}(t) c_{\beta} \hat{q}_0(t)
\]

Choosing \( \mathbf{A}_{\alpha} \) as the eigenfunctions of the double-rot operator \((\nabla \times (\nabla \times \ ))\):
\[
\nabla \times (\nabla \times \mathbf{A}_{\alpha}(r)) = \left( \frac{\omega_{\alpha}}{c} \right)^2 \mathbf{A}_{\alpha}(r)
\]
we finally arrive at the very same model.

The only difference in the application of the model to each situation is the spectrum \( \omega_{\alpha} \) which will depend on the particular geometry of the cavity. Naturally, it is possible to find many other physical systems described by the very same model, but these three above are enough to demonstrate its utility.

From now on let the symmetric Hamiltonian operator associated to this useful model be
\[
\hat{H} = \frac{1}{2} \left\{ \hat{p}_0^2 + \omega_0^2 \hat{q}_0^2 + \sum_{k=1}^N \left( \hat{p}_k^2 + \omega_k^2 \hat{q}_k^2 \right) \right\} - \frac{1}{2} \sum_{k=1}^N c_k (\hat{q}_0 \hat{q}_k + \hat{q}_k \hat{q}_0)
\]
in which the subindex 0 is reserved to the oscillator in the center of the sphere and the subindex \( k = \{1, ..., N\} \) is reserved to the modes of the scalar field. It is simple to show that the equations (7a) and (7b) can be derived from the Hamiltonian (8).

The Hamiltonian can be written in a matrix form
\[
\hat{H} = \frac{1}{2} \hat{p}^T \hat{p} + \frac{1}{2} \hat{q}^T \Omega^2 \hat{q}
\]
defining the vectors
\[
\hat{p}^T = (\hat{p}_0, \hat{p}_1, ..., \hat{p}_k), \\
\hat{q}^T = (\hat{q}_0, \hat{q}_1, ..., \hat{q}_k).
\]
The $\Omega^2$ is a symmetrical interaction matrix given by

$$
\Omega^2 = \begin{pmatrix}
\omega_0^2 & -c_1 & -c_2 & \ldots & -c_N \\
-c_1 & \omega_1^2 & 0 & \ldots & 0 \\
-c_2 & 0 & \omega_2^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-c_N & 0 & 0 & \ldots & \omega_N^2
\end{pmatrix}.
$$

(11)

The above matrix resumes all interactions in the system, those given by the boundary conditions of the system associated with its geometry and manifested in the frequencies of the field modes and those given by the dipolar interaction.

The matrix (11) can be diagonalized by an orthogonal transformation $T$,

$$
D^2 = T \Omega^2 T^T
$$

(12)

with $D^2 = \text{diag} \{\Omega_0^2, \Omega_1^2, \Omega_2^2, \ldots, \Omega_N^2\}$ such that it is possible to express this matrix as a product

$$
D^2 = D \cdot D.
$$

(13)

The transformation (12) and the expression (13) allow the factorization $\Omega^2 = \Omega \cdot \Omega$,

$$
\Omega \cdot \Omega = T D T^T \cdot T D T^T
$$

with $D = \text{diag} \{\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_N\}$ and

$$
\Omega = T D T^T.
$$

(14)

Actually, it is possible to find any power $\alpha$ of the $\Omega$ matrix,

$$
\Omega^\alpha = T D^\alpha T^T
$$

(15)

in which $\alpha \in \mathbb{R}$.

From now on, we assume each eigenvalue of $D\alpha$ is such that $\Omega_i^\alpha > 0$. This excludes the case of complex frequencies which could describe the penetration of each mode to the walls of the cavity and the energy losses associated with it, but help to focus on the interaction of the field with the dipole.
IV. NON-HERMITEAN OPERATORS

The quadratic structure of the Hamiltonian (9) suggests, similarly to an harmonic oscillator, that there is the possibility to define a set of creation and annihilation operators for this system, meaning that the Hamiltonian operator is factorable. Since in general such operators are not connected with the particle content, Schwinger used to call them simply non-hermitean operators. In order to define them we have to use the characteristics of the interaction matrix (11) and some concepts originated in the theory of coupled harmonic oscillators at the classical level.

Similarly to the classical case, at the quantum level the state of motion of each oscillator of the system depends on the state of motion of the other elements of the system due to the interaction. In the same way, there are states of motion of the system as a whole in which all of its constituents moves with the same frequency and phase. Those states of motion constitute the basic movements of the system, being called normal modes. There is one normal mode for each element of the whole system.

In consequence, we are talking about two sets of operators: the first, reporting the motions and excitations of each element of the system and the second, reporting the motions and excitations of the system as a whole. In the first set, each oscillator has a special behavior since its movements are affected by the presence of other elements in the system. In the second set, the group of elements of the system shows states of motion which have associated a natural frequency called resonant frequency. Each combination of motions is unique, with its own frequency, and all the combinations can be considered as a system of uncoupled oscillators.

Given the characteristics of the two sets of operators mentioned above, we shall call creation and annihilation operators of dressed states or DSO the operators describing excitations of individual system components. Similarly, the second set will be called annihilation and creation operators of the normal modes or NMO. Thus, in the present section, we shall construct the two sets of creation and annihilation operators and establish a simple relationship between them.
A. Dressed States Operators

The annihilation and creation operators that we shall define are related with the excitations of every one of the system components when the interaction is considered. Thus, these operators will create a superposition of the measurable states of each system component. The DSO and the states they create are called dressed since just as it happens with the modification of the magnetic moment of an atom by the coupling with a radio frequency field (see [6] section C) the frequencies and coupling constants of each oscillator are affected by the interaction with the other elements of the system. As a consequence, the energy states created by these operators are different with respect to those which would present the components of the system if there were no interaction.

On this way, following the structure of the matrices (15), the DSO operators are given by the following expressions

\[
\hat{y} = \sqrt{\frac{\Omega}{2\hbar}} (\hat{q} + i\Omega^{-1}\hat{p}), \\
\hat{y}^T = \sqrt{\frac{\Omega}{2\hbar}} (\hat{q}^T + i\hat{p}^T \Omega^{-1}) \Omega^\frac{1}{2}, \\
\hat{y}^\dagger = \sqrt{\frac{\Omega}{2\hbar}} (\hat{q} - i\Omega^{-1}\hat{p}), \\
\hat{y}^{\dagger T} = \sqrt{\frac{\Omega}{2\hbar}} (\hat{q}^T - i\hat{p}^T \Omega^{-1}) \Omega^\frac{1}{2},
\]

(16)

where the operators \(\hat{y}, \hat{y}^\dagger\) and its transposed ones, are vectors of the form

\[
\hat{y}^T = \begin{pmatrix} \hat{y}_0 & \hat{y}_1 & \ldots & \hat{y}_k \end{pmatrix}, \quad \text{and}, \quad \hat{y} = \begin{pmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_k \end{pmatrix}, \\
\hat{y}^{\dagger T} = \begin{pmatrix} \hat{y}_0^\dagger & \hat{y}_1^\dagger & \ldots & \hat{y}_k^\dagger \end{pmatrix}, \quad \text{and}, \quad \hat{y}^\dagger = \begin{pmatrix} \hat{y}_0^\dagger \\ \hat{y}_1^\dagger \\ \vdots \\ \hat{y}_k^\dagger \end{pmatrix}.
\]

As we can see, the transformations given by (16) keep the canonical commutation relations

\[
[\hat{y}, \hat{y}^\dagger] = \hat{y}^T \hat{y}^\dagger - \hat{y}^{\dagger T} \hat{y} = \frac{1}{2\hbar} \left\{ -2i\hat{q}^T \hat{p} + 2i\hat{p}^T \hat{q} \right\} = \frac{i}{\hbar} \left\{ \hat{p}^T \hat{q} - \hat{q}^T \hat{p} \right\} = 1.
\]

(17)
Therefore the transformation \((\hat{q}, \hat{p}) \rightarrow (\hat{y}, \hat{y}^\dagger)\) is canonical and the inverse relations of (16) are

\[
\begin{align*}
\hat{q} &= \sqrt{\frac{\hbar}{2}} \Omega^{\frac{1}{2}} (\hat{y} + \hat{y}^\dagger), \\
\hat{p} &= i \sqrt{\frac{\hbar}{2}} \Omega^\dagger (\hat{y}^\dagger - \hat{y}), \\
\hat{q}^T &= \sqrt{\frac{\hbar}{2}} \left( \hat{y}^T + \hat{y}^\dagger T \right) \Omega^{-\frac{1}{2}}, \\
\hat{p}^T &= i \sqrt{\frac{\hbar}{2}} \left( \hat{y}^T - \hat{y}^\dagger T \right) \Omega^\dagger. 
\end{align*}
\] (18)

Replacing (18) in the Hamiltonian (9) leads to the Dressed State Hamiltonian:

\[
\hat{H} = \frac{\hbar}{2} \left\{ \hat{y}^\dagger T \Omega \hat{y} + \hat{y}^T \Omega \hat{y}^\dagger \right\}. 
\] (19)

The solution of the problem implies the knowledge of the dynamics of each system component. However, the fact that both the states of the oscillator as the field modes are interdependent prevent us consider them isolated. This is evidenced by the fact the states created by the DSO are in a complicated entangled superposition. To solve this problem, we can make a transformation from the DSO operators to a set of operators of the normal modes of the system (NMO). This transformation can be performed using the same transformation \(T\) which was used in (16). This transformation allows to express the Hamiltonian of the system as the Hamiltonian of a set of uncoupled free harmonic oscillators which can be solved in an exact way allowing one to find the behavior of the matter oscillator and each one of the modes of the field.

**B. Normal Modes Operators**

The Hamiltonian (19) is not in a diagonal form. Besides, the matrix \(\Omega\) contains all the information about the interactions and has a structure similar to a simple harmonic oscillator when the Hamiltonian is in a matrix form, this implies that (19) can actually be put in a diagonal form using \(T\) and the normal modes operators.

The normal modes \(\hat{\xi}\) operators are defined as

\[
\begin{align*}
\hat{\xi} &= T^T \hat{y} \quad ; \quad \hat{\xi}^T = \hat{y}^T T, \\
\hat{\xi}^\dagger &= T^T \hat{y}^\dagger \quad ; \quad \hat{\xi}^{\dagger T} = \hat{y}^{\dagger T} T, 
\end{align*}
\] (20)
in which

\[ \hat{\xi}^T = \left( \hat{\xi}_0 \ \hat{\xi}_1 \ \ldots \ \hat{\xi}_k \right), \text{ and, } \hat{\xi} = \begin{pmatrix} \hat{\xi}_0 \\ \hat{\xi}_1 \\ \vdots \\ \hat{\xi}_k \end{pmatrix}, \]

\[ \hat{\xi}^{\dagger T} = \left( \hat{\xi}_0^{\dagger} \ \hat{\xi}_1^{\dagger} \ \ldots \ \hat{\xi}_k^{\dagger} \right), \text{ and, } \hat{\xi}^{\dagger} = \begin{pmatrix} \hat{\xi}_0^{\dagger} \\ \hat{\xi}_1^{\dagger} \\ \vdots \\ \hat{\xi}_k^{\dagger} \end{pmatrix}. \]

Then, the Hamiltonian (19) can be written in its diagonal form

\[ \hat{H} = \frac{\hbar}{2} \left\{ \hat{\xi}^{\dagger T} D \hat{\xi} + \hat{\xi}^T D \hat{\xi}^{\dagger} \right\} \]

\[ = \sum_{k=0}^{N} \Omega_k \left( \hat{\xi}_k^{\dagger} \hat{\xi}_k + \hat{\xi}_k \hat{\xi}_k^{\dagger} \right). \]

Taking the relations \[ [\hat{\xi}_j, \hat{\xi}_k^{\dagger}] = \delta_{jk}, \] the last expression can be written

\[ \hat{H} = \hbar \sum_{k=0}^{N} \Omega_k \left( \hat{\xi}_k^{\dagger} \hat{\xi}_k + \frac{1}{2} \right) = \hbar \sum_{k=0}^{N} \Omega_k \hat{\xi}_k^{\dagger} \hat{\xi}_k + \frac{\hbar}{2} \text{Tr} D. \quad (22) \]

The set of creation and annihilation operators of the normal modes create and destroy the collective excitations of the system. In other words, it creates and annihilates both the full field excitations and the excitations of the matter oscillator. These collective excitations have a fixed number of energy quanta. They are related to the collective behavior of the system and not with the dynamics of each of its components at the individual level. Indeed, thanks to the transformation \( T \) establishing a relationship between the operators \( (\hat{y}^{\dagger}, \hat{y}) \) and \( (\hat{\xi}^{\dagger}, \hat{\xi}) \), the dressed states of each system component can be expressed as a combination of the normal mode states of the system and vice versa.

The normal modes are presented in equal number to the components of the system and constitute a set of uncoupled harmonic oscillators. Each of these modes has a natural frequency that is defined by the characteristics of the entire system. Therefore, they are essentially a linear superposition of states of motion for each system element. Also, since that the motion of each element of the system is given by a unique combination of normal modes,
we can infer that their motion will not present a single frequency. Therefore, the quantum states associated with each element of the system are not eigenstates of the Hamiltonian of the whole system.

Each normal mode represents a way in which the energy is exchanged between the elements of the system and it will depend on the number of quanta that is available between the field and the oscillator. Given that the number of excitations is fixed, the amount of those excitations taken into account will determine the size of the Hilbert space to be used. Also, since all of NMO are essentially a set of uncoupled harmonic oscillators, the quantum states associated with them can be represented as the tensor product of states associated with each of these decoupled oscillators. The states associated to DSO can be represented as a combination of the states associated to the NMO but this combination is not a simple tensor product of the states since DSO states are entangled.

V. THE LAGRANGIAN AND THE EQUATIONS OF MOTION

Given the Hamiltonian and taking into account that the system does not have constrains, the inverse Legendre transform is

\[ \hat{L} = \frac{1}{2} \dot{\hat{p}}^T \cdot \frac{d\hat{q}}{dt} + \frac{1}{2} \frac{d\hat{q}^T}{dt} \cdot \hat{p} - \hat{H}. \]

Using the transformations given in (18) we have for the Lagrangian of the system in the DSO base the following canonical form

\[
\hat{L} = i \frac{\hbar}{4} \left\{ (\dot{\hat{y}}^T - \hat{y}^T) \left( \frac{d\hat{y}}{dt} + \frac{d\hat{y}^T}{dt} \right) \right\} \\
+ i \frac{\hbar}{4} \left\{ \frac{d\hat{y}^T}{dt} + \frac{d\hat{y}^T}{dt} \right\} (\dot{\hat{y}}^T - \hat{y}^T) - \hat{H} \\
= i \frac{\hbar}{4} \left\{ \dot{\hat{y}} \frac{d\hat{y}}{dt} - \hat{y} \frac{d\hat{y}}{dt} \right\} \\
- \frac{\hbar}{2} \left\{ \dot{\hat{y}}^T \Omega \hat{y} + \hat{y}^T \Omega \hat{y}^T \right\} .
\] (23)

In the same way, with the transformation made in (20) the Lagrangian in the NMO base
\[ L = \frac{\hbar}{2} \left\{ i \hat{\xi}^T \frac{d \hat{\xi}}{dt} - i \hat{\xi}^T \frac{d \hat{\xi}^\dagger}{dt} - (\hat{\xi}^T D \hat{\xi} + \hat{\xi}^T D \hat{\xi}^\dagger) \right\} \]
\[ = \frac{\hbar}{2} \sum_{k=0}^{N} \left\{ i \hat{\xi}_k^T \frac{d \hat{\xi}_k}{dt} - i \hat{\xi}_k^T \frac{d \hat{\xi}_k^\dagger}{dt} - \Omega_k (\hat{\xi}_k^T \hat{\xi}_k + \hat{\xi}_k \hat{\xi}_k^\dagger) \right\}. \tag{24} \]

In the last expression it has been neglected the terms related with the total derivative, since they just generate the transformation from DSO base to the NMO one.

**A. Equations of Motion**

Schwinger Action Principle \[14\] can be used for the derivation of equations of motion of the quantum operators. We choose to perform a variation on the quantum Lagrangian (24) in the NMO description since in this base the matrix equations are diagonal and decoupled and the expression of any quantity emerged from here is suitable to be manipulated. After some operations we have

\[ \delta \hat{L} = \frac{\hbar}{2} \delta \hat{\xi}^T \left( i \frac{d \hat{\xi}}{dt} - D \hat{\xi} \right) - \frac{\hbar}{2} \hat{\xi}^T D \delta \hat{\xi}^\dagger - \delta \hat{\xi}^T \left( i \frac{d \hat{\xi}^\dagger}{dt} + D \hat{\xi}^\dagger \right) - \frac{\hbar}{2} \hat{\xi}^T D \delta \hat{\xi} \]
\[ + \frac{i \hbar}{2} \hat{\xi}^T \frac{d}{dt} \left( \frac{\delta \hat{\xi}}{dt} \right) - \frac{i \hbar}{2} \hat{\xi}^\dagger \frac{d}{dt} \left( \frac{\delta \hat{\xi}^\dagger}{dt} \right), \tag{25} \]

then, taking \( \delta \hat{L} = 0 \), we obtain

\[ i \frac{d \hat{\xi}}{dt} - D \hat{\xi} = 0 \quad \text{and} \quad i \frac{d \hat{\xi}^\dagger}{dt} + D \hat{\xi}^\dagger = 0, \]
\[ i \frac{d \hat{\xi}^T}{dt} - \hat{\xi}^T D = 0 \quad \text{and} \quad i \frac{d \hat{\xi}^{T\dagger}}{dt} - \hat{\xi}^{T\dagger} D = 0. \]

The solutions for the last matrix differential equations are given by

\[ \hat{\xi}(t) = \exp \left[ -iDt \right] \hat{\xi}(t_0) \tag{26} \]
\[ \text{, and } \hat{\xi}^\dagger(t) = \exp \left[ iDt \right] \hat{\xi}^\dagger(t_0), \tag{27} \]
\[ \hat{\xi}^T(t) = \hat{\xi}^T(t_0) \exp \left[ -iDt \right] \]
\[ \text{, and } \hat{\xi}^{T\dagger}(t) = \hat{\xi}^{T\dagger}(t_0) \exp \left[ iDt \right], \]

in which the vectors \( \hat{\xi}(t_0), \hat{\xi}^\dagger(t_0), \hat{\xi}^T(t_0) \) e \( \hat{\xi}^{T\dagger}(t_0) \) are defined as the set of operators with initial conditions, \( t = t_0 \).
B. Generalized Transformation Function

Following Schwinger’s Action Principle, the operator $\hat{G}(t)$ is the generator of the admissible infinitesimal transformations for the system. Its functional form can be obtained taking the solutions of the equations (24) and the Hamiltonian (22), then

$$\delta \langle \xi^\dagger, t | \xi, t_0 \rangle = \langle \xi^\dagger, t | \hat{G}(t) - \hat{G}(t_0) | \xi, t_0 \rangle$$

$$= \langle \xi^\dagger, t | \left( \hat{p}\delta\hat{q} - \hat{H}\delta t \right) \left| \right. \left. t_0 \right| \xi, t_0 \rangle.$$

Taking the relations (16) and the Hamiltonian (19) we obtain the explicit form of the generator $\hat{G}(t)$ in the DSO base:

$$\hat{G} = i\hbar \left( \hat{y}^T - \hat{y}^T \right) \left( \hat{\delta}\hat{y}\hat{\delta} \right) - \frac{\hbar}{2} \left\{ \hat{y}^T \hat{\Omega} \hat{y} + \hat{y}^T \hat{\Omega} \right\} \delta t.$$  (28)

Similarly, using the transformations (20) in the last expression, we obtain the same generator in the NMO base:

$$\hat{G} = \hbar \sum_k \left\{ \frac{i}{2} \left( \hat{\xi}_{k}^\dagger \hat{\xi}_{k}^\dagger (t_0) e^{-i\Omega_k t} - \hat{\xi}_{k}^\dagger (t_0) \delta \hat{\xi}_{k}^\dagger e^{-i\Omega_k t} \right) \right.$$  

$$- \Omega_k \left( \hat{\xi}_{k}^\dagger \hat{\xi}_{k}^\dagger + \frac{1}{2} \right) \delta t \right\}.  \quad (29)$$

Since the expressions related to (29) are decoupled, we can take the solutions (26) for a generic component $k$ and calculate the transition amplitude $\langle \xi^\dagger, t | \xi, t_0 \rangle$. In order to do that, we need to solve the variational differential equation (1), in which we substitute (26) such that $\delta \hat{W} = \hat{G}(t) - \hat{G}(t_0)$ is given by

$$\delta \hat{W} = \hbar \sum_k \left\{ \frac{i}{2} \left( \hat{\xi}_{k}^\dagger \delta \hat{\xi}_{k}^\dagger (t_0) e^{-i\Omega_k t} - \hat{\xi}_{k}^\dagger (t_0) \delta \hat{\xi}_{k}^\dagger e^{-i\Omega_k t} \right) \right.$$  

$$- \Omega_k \left( \hat{\xi}_{k}^\dagger \hat{\xi}_{k}^\dagger + \frac{1}{2} \right) \delta t \right\}.$$  

In this way, taking $\left[ \hat{\xi}_{k}^\dagger, \delta \hat{\xi}_{k} \right] = \left[ \delta \hat{\xi}_{k}^\dagger, \hat{\xi}_{k}^\dagger \right] = 0$, which are the commutation relations derived from the Schwinger’s Action Principle [14], one finds

$$\delta \hat{W} = -\hbar \sum_k \left\{ \frac{i}{2} \delta \left\{ \hat{\xi}_{k}^\dagger \delta (t_0) \right\} e^{-i\Omega_k t} - \left( \hat{\xi}_{k}^\dagger \hat{\xi}_{k}^\dagger + \frac{1}{2} \right) \Omega_k \delta t \right\}.$$  

Performing an integration of the variational equation (1), we have

$$\hat{W} = -i\hbar \sum_k \hat{\xi}_{k}^\dagger \hat{\xi}_{k} (t_0) e^{-i\Omega_k t} - \frac{\hbar}{2} \sum_k \Omega_k t$$  

$$= -i\hbar \sum_k \hat{\xi}_{k}^\dagger \hat{\xi}_{k} (t_0) e^{-i\Omega_k t} - \frac{\hbar}{2} T_r D t,$$  

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and using (4), leave a transformation function
\[
\langle \xi^\dagger, t_1 | \xi, t_0 \rangle = \exp \left\{ -\frac{i}{2} Tr D t + \sum_{k=0}^{N} \xi^\dagger_k \xi_k (t_0) e^{-i \Omega_k t} \right\}
\]
\[
= \exp \left\{ -\frac{i}{2} Tr D t \right\} \exp \left\{ \bar{\xi}^\dagger e^{-i \Omega \bar{\xi}} (t_0) \right\}. \tag{30}
\]

VI. ENERGY SPECTRUM

In the NMO base, the Hamiltonian of the system (22) is totally diagonal and equivalent to a set of \( N \) independent harmonic oscillators, each of them is actually a collective motion of the entire system. Given that the transformation \( T \) at the quantum level represents a canonical transformation, the spectrum of the system in the NMO base should be totally equivalent to the spectrum of the Hamiltonian (8). As noted above, the Hamiltonian in the NMO base is diagonal and its Hilbert space can be constructed as the product Hilbert space of \( N \) independent harmonic oscillators [23] such that
\[
\hat{H} = \sum_{k=0}^{N} \hat{H}_k = \hbar \sum_{k=0}^{N} \Omega_k \left( \hat{c}^\dagger_k \hat{c}_k + \frac{1}{2} \right), \tag{31}
\]
in which the Hilbert space of each one is given by the following tensor product
\[
\hat{H}_k = \hat{1}_0 \otimes \hat{1}_1 \otimes \cdots \otimes \hat{1}_{k-1} \otimes \hat{h}_k \otimes \hat{1}_{k+1} \otimes \cdots \otimes \hat{1}_N, \tag{32}
\]
where \( \hat{h}_k \) is the hamiltonian of one single normal mode. The identity of the \( k \)th Hilbert space is given as
\[
\hat{1}_k = \sum_{n_k} |n_k \rangle \langle n_k| \tag{33}.
\]
On this way the total identity for the system is
\[
\hat{1} = \prod_{i=0}^{N} \otimes \hat{1}_i = \prod_{i=0}^{N} \otimes \sum_{n_i} |n_i \rangle \langle n_i| = \prod_{i=0}^{N} \sum_{n_i} |n_i \rangle \langle n_i| \tag{34}.
\]
A generic element of the set of states can be given by
\[
|N \rangle = |n_0, n_1, \ldots, n_N \rangle = |n_0 \rangle \otimes |n_1 \rangle \otimes \cdots \otimes |n_N \rangle, \tag{35}
\]
and the action of the Hamiltonian over this eigenstates leaves

\[ \hat{H} |n_0, n_1, \ldots, n_N\rangle = \sum_{k=0}^{N} \hat{H}_k |n_0, n_1, \ldots, n_N\rangle \]

\[ = \sum_{k=0}^{N} |n_0\rangle \otimes \cdots \otimes \hat{H}_k |n_k\rangle \otimes \cdots \otimes |n_N\rangle \]

\[ = \sum_{k=0}^{N} E_{n_k} |n_0, n_1, \ldots, n_N\rangle. \]

It means that the energy spectrum is just a simple sum:

\[ E_{n_0n_1\ldots n_N} = \sum_{k=0}^{N} E_{n_k}. \quad (36) \]

Now we can calculate the transition amplitude for the system in the initial arbitrary state

\[ |\xi, t_0\rangle = \prod_{k=0}^{N} \otimes |\xi_k, t_0\rangle = |\xi_0, \xi_1, \ldots, \xi_N, t_0\rangle. \quad (37) \]

Using the relations (33) and the orthogonality relation \( \langle n_i|m_j\rangle = \delta_{m,m;i,j} \), we have

\[ |\xi, t\rangle = e^{-i\hat{H}t/\hbar} |\xi, t_0\rangle \]

\[ = \exp \left\{ -i \frac{t}{\hbar} \sum_{k=0}^{N} \hat{H}_k \right\} |\xi_0, \xi_1, \ldots, \xi_N, t_0\rangle \]

\[ = \exp \left\{ -i \frac{t}{\hbar} \sum_{k=0}^{N} E_k \right\} \prod_{k=0}^{N} \langle n_k |\xi_k, t_0\rangle. \]

Therefore,

\[ \langle \xi^\dagger, t_1 |\xi, t_0\rangle = \prod_{j=0}^{N} \left( \sum_{n_j} e^{-i E_{n_j} t / \hbar} \langle n_j |\xi_j, t_0\rangle \right)^2. \quad (38) \]

On the other hand, due to the series expansion of the transformation function (30),

\[ \langle \xi^\dagger, t_1 |\xi, t_0\rangle = \prod_{k=0}^{N} \sum_{n=0}^{\infty} \frac{\xi_k^{\dagger n} \xi_n(t_0)}{n!} e^{-i \Omega_k (n + \frac{1}{2}) t}. \quad (39) \]

Comparing (38) and (39),

\[ \prod_{k=0}^{N} \left( \sum_{n=0}^{\infty} e^{-i E_{n_k} t / \hbar} \langle n_k |\xi_k, t_0\rangle \right)^2 = \prod_{k=0}^{N} \sum_{n=0}^{\infty} \frac{\xi_k^{\dagger n} \xi_n(t_0)}{n!} e^{-i \Omega_k (n + \frac{1}{2}) t}. \]

we obtain the eigenfunctions

\[ \langle \xi_k^\dagger, t_0 |n_k\rangle = \frac{\xi_k^{\dagger n}}{\sqrt{n!}} \text{ and } \langle n_k |\xi_k, t_0\rangle = \frac{\xi_k}{\sqrt{n!}}. \]
and the exact form of the spectrum of the system

\[ E_{n_0 n_1 ... n_N} = \sum_{k=0}^{N} E_{n_k} = \sum_{k=0}^{N} \hbar \Omega_k \left( n_k + \frac{1}{2} \right) = \sum_{k=0}^{N} \hbar \Omega_k n_k + \frac{\hbar}{2} Tr D. \]

From the last expression for the spectrum, we have the vacuum energy of the system

\[ E_{0,0,0,0...0} = \frac{\hbar}{2} Tr D = \frac{\hbar}{2} Tr \Omega. \]

VII. TRANSITION PROBABILITIES

The expression for the transition amplitude (30) is given in the NMO base of the system. However, this information is shown in a useless form if we want to study the behavior of an individual element of the system since the states associated with NMO are a superposition of DSO.

The transition amplitude in the NMO base can be understood in the following way. Let us take \(|N\rangle\) and \(|M\rangle\) two arbitrary states in the normal base, then

\[
\langle M | e^{-i \hat{H} \frac{\hbar}{2} t} | N \rangle = \prod_{k=0}^{N} e^{-i \hbar \Omega_k (n_k + \frac{1}{2}) t} \delta_{m_k, n_k} = e^{-i \hbar \sum_{s=0}^{N} \Omega_s (n_s + \frac{1}{2}) t} \prod_{k=0}^{N} \delta_{m_k, n_k}.
\]

This expression is the probability amplitude that the system in the state \(|N\rangle\) at the initial time do a transition to the state \(|M\rangle\) at the time \(t\). However, to extract useful physical information as, for instance, the behavior of the oscillator in the center of the cavity or the dynamics of a specific field mode, we need to evaluate the transition amplitude between the two corresponding dressed states of the system. This information is achieved in the non-diagonalized basis. For this reason we reformulate the generalized transition amplitude (39) to the DSO representation using the transformation (20):

\[
\xi_k^\dagger = \sum_{j=0}^{N} [T]_{jk} y_j^\dagger, \tag{40}
\]

in which \([T]_{jk}\) is a element of the \(T\) matrix and the \(\xi_k^\dagger\) is the eigenvalue of the operator \(\hat{\xi}_k^\dagger\) acting over the \(|\xi_k\rangle\), associated to the the \(k\)-th normal mode. Thus, using (40), the
expression (39) is given by

\[ \langle y^\dagger, t | y, t_0 \rangle = \exp \left\{ -\frac{i}{2} Tr D t \right\} \prod_{r=0}^{N} \prod_{s=0}^{N} \exp \left\{ y_s^\dagger J_{rs} (t) y_r (t_0) \right\}, \]  

(41)
in which it is defined

\[ \sum_{k=0}^{N} [T]_{sk} [T]_{kr} e^{-i\Omega_k t} = J_{rs} (t). \]  

(42)
The expression (41) is the generalized transition amplitude for the system in the Dressed State representation. After some manipulations, we can extract information about the transition between any two dressed modes both of the field or the oscillator in the center of the cavity:

\[ \langle y^\dagger_s, t | y_r, t_0 \rangle = \exp \left\{ -\frac{i}{2N} Tr D t \right\} \exp \left\{ y_s^\dagger J_{rs} (t) y_r (t_0) \right\}. \]  

(43)
The above expression can also be decomposed in the number base for the dressed states, using

\[ |\langle y^\dagger_k, t_0 | n_k \rangle = \frac{y_k^{\dagger n_k}}{\sqrt{n_k!}} \quad \text{and} \quad |\langle n_k | y_k, t_0 \rangle = \frac{y_k^{n_k}}{\sqrt{n_k!}}, \]  

(44)
in which the overline means that the number state is associated with the dressed base. Then, the transition (43) between any two states in the DSO base is given by

\[ \langle y^\dagger_s, t | y_r, t_0 \rangle = \exp \left\{ -\frac{i}{2N} Tr D t \right\} \sum_{n=0}^{\infty} \frac{y_s^\dagger n (t) \sqrt{n}}{\sqrt{n!}} J_{rs}^n (t) \frac{y_n (t_0)}{\sqrt{n!}}, \]  

(45)
in which the subscripts \( r \) and \( s \) are related with the individual components of the system. From the expressions (44) and (45) we obtain the following transition amplitude

\[ \langle n_s, t | n_r \rangle = \exp \left\{ -\frac{i}{2N} Tr D t \right\} J_{rs}^n (t), \]

\[ J_{rs}^n (t) = \sum_{0 \leq l_m \leq n} \frac{n!}{\prod_{n=0}^{l_j} l_0! l_1! \ldots l_m!} \prod_{p=0}^{m} \left( [T]_{sk} [T]_{kr} e^{-i\Omega_k t} \right)^{l_p}. \]

For the probabilities, we can show that

\[ \sum_{r=0}^{N} |J_{rs} (t)|^2 = 1, \]

knowing that \( T \) is orthogonal. Writing this expression for \( n = 0 \) corresponding to the oscillator and in the center of the cavity and \( k = \{1, \ldots, N\} \) for the modes of the field, one finds

\[ |J_{00} (t)|^2 + \sum_{k=0}^{N} |J_{0k} (t)|^2 = 1, \]

\[ |J_{k_0} (t)|^2 + \sum_{k_2=0}^{N} |J_{k_1 k_2} (t)|^2 = 1. \]
Such expressions are related with the probabilities to have a transition between any two elements of the system with $n$ quanta. Then, the atom or any field mode initially in the $n$-th state can change to any another state in a later time $^{8\ 12}$.

VIII. CONCLUSIONS AND OUTLOOK

In the case of interacting systems modeled by $(7a)$ and $(7b)$, we show that it is possible to define unambiguously a transformation in which some analogies with a classical coupled systems can be considered. Here, the system, as a whole can be studied as a set of coupled harmonic oscillators each of them having a different coupling constant. With the proposed approach, the normal modes operators create simple decoupled states associated with the whole system. On the other hand, the dressed states operators create quantum states that each individual oscillator can show in the presence of interaction with the rest of the system.

In the Dressed State formalism introduced in $^{3\ 5}$, the states of each element of the system are constructed by hand, expressing them as a special combination of the states of the whole system, such construction leads to entangled states. Besides that, in this work, we have introduced those states by mean of the Dressed State Operators (DSO) which create those Dressed States naturally. In our approach those states are not constructed by hand, it is the structure of the DSO which leads to the entangled states, as it can be easily checked solving the problem for only one mode of the field.

The quantum state of each part of the system, the oscillator or each field mode, is defined with a set of parameters which is modified by the presence of the interaction. For instance, the frequency is redefined as a function of the other parameters of the system

$$\omega \rightarrow \Omega (\omega_0, \{\omega_j\}, \{c_k\} ; j,k \in \{1,2,\ldots,N\} ),$$

if they are compared with the free interaction parameters, $c_i = 0$ in $^{8}$.

The orthogonality of the transformation $T$ preserves the geometrical structure of the operators, being at the quantum level a canonical transformation which preserves the canonical structure of the theory in both, normal and dressed representations. Thus, the transformation functions in the normal and dressed representations have the same information. In the dressed state representation, the transformation function allows us to know all the amplitude transitions between the states of each one of the elements in the system since the dressed
states are not eigenstates of the system.

The transformations \[16\), \[18\) and \[20\) show that in some cases the Rotating Wave Approximation (RWA) is unnecessary. This is because those transformations lead to a system which can be solved in an exact way, what means that the RWA approximation is not necessary if the correct definition of the transformation is used.

The Dressed State Approach, developed in this work, can be applied to the study of problems in which the dissipation and the interaction of a system with a bath or reservoir can be also included. This approach can even be suitable to deal with non-linear interacting system via a perturbative treatment. Some of these points will be explored elsewhere.

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[1] E. T. Jaynes, *Microwave Laboratory Report No. 502*, Stanford University, (1958).
[2] E. T. Jaynes and F. W. Cummings, *Proc. I.E.E.E.* 51, 89 (1963)
[3] C. Cohen-Tannoudji, 1994 “Atoms in Electromagnetic Fields” (Singapore: World Scientific).
   C. Cohen-Tannoudji, “Optical pumping and interactions of atoms with the electromagnetic field”, in “Cargese lectures in physics”, Vol.2, ed. by Levy M. (Gordon and Breach, New-York, 1968), p.347.
[4] N. Polonsky, 1964 Doctoral thesis Ecole Normale Supérieure, Paris. N. Polonsky, C. Cohen-Tanoudji, *J. Physique* 26, 409 (1965).
[5] S. Haroche, 1964 Doctoral thesis Ecole Normale Supérieure, Paris. C. Cohen Tanoudji, S. Haroche, *C.R. Acad. Sci.* 262, 37 (1966).
[6] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, *Atom-Photon Interactions, Basic Processes and Applications* (Wiley-VCH, Weinheim 2004); C. Cohen-Tannoudji, S. Haroche,”Le concept d’atome "habillé" par des photons; Quelques exemples d’application in Polarisation, Matiere et Rayonnement”, p. 191, Livre jubilaire en (Presses Universitaires de France, 1969).
[7] A. P. C. Malbouisson *Time evolution of confined quantum systems - a non perturbative approach* - JHEP Proceedings-Workshop on Integrable theories, solitons and duality (2002).

[8] N. P. Andion, A. P. C. Malbouisson, A. Mattos Neto, *Jour. Phys A* 34, 3735-3749 (2001).

[9] G. Flores-Hidalgo, A. P. C. Malbouisson, *Phys. Rev A* 66, 042118 (2002).

[10] G. Flores-Hidalgo, A. P. C. Malbouisson, Y. W. Milla, *Phys. Rev A* 65, 0634144 (2002).

[11] G. Flores-Hidalgo, A. P. C. Malbouisson, Y. W. Milla, *Phys. Lett. A* 311, 82 (2003).

[12] R. Casana, G. Flores-Hidalgo and B. M. Pimentel, *Physica A* 374 (2007) 600-610.

[13] G. Flores-Hidalgo, Y. W. Milla, *Phys. Rev A* 38, 7527 (2005).

[14] J. S. Schwinger, "Quantum Kinematics and Dynamics", (Westview Press, New York 1991); J. S. Schwinger, Quantum Mechanics "Symbolism of Atomic Measurements", (Springer, Tokio 2001)

[15] C.A.M. de Melo, B.M. Pimentel and J.A. Ramirez, *Rev. Bras. Ens. Fís.* 33, n. 3, 3306 (2011)

[16] P.A.M. Dirac, *Princípios de Mecânica Cuántica* (Ediciones Ariel - Madrid 1967).

[17] C.A.M. de Melo, B.M. Pimentel and J.A. Ramirez, *Rev. Bras. de Ens. Fís.* 35, n. 4, 4302 (2013)

[18] T. W. B. Kibble, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 249, No. 1258 (Jan. 13, 1959), pp. 441-444.

[19] C.A.M. de Melo, B.M. Pimentel and P.J. Pompeia, *Il Nuovo Cimento della Società Italiana di Fisica B* 121, p. 193-200, (2006).

[20] R. Casana, C.A.M. de Melo and B.M. Pimentel, *Braz. J. of Phys.* 35, 1151-1154, (2005).

[21] C.A.M. de Melo and B.M. Pimentel, *Adv. App. Cliff. Alg.* 20, p. 745-763, (2010).

[22] P.A.M. Dirac, *Physikalische Zeitschrift der Sowjetunion*, Band 3, Heft 1 (1933), pp. 64-72. A translated version of this paper can be found in, *Selected Papers on Electrodynamics*, editada por Julian S. Schwinger (Dover Publications, New York - 1958), pp. 312.

[23] F.A.Berezin, “The Method of Second Quantization”, (Academic Press, New York, 1966)