Abstract. We demonstrate that our gradient density estimator—corresponding to estimating the density function of the derivatives in one dimension—obtained from a finite sample set of size \( N \) using the method of stationary phase converges at the rate of \( O(1/N) \) as \( N \to \infty \). For a thrice differentiable function \( S \), the density function of its derivative \( s = S' \) is obtained via a random variable transformation of a uniformly distributed random variable defined on a closed, bounded interval \( \Omega = [0, L] \subset \mathbb{R} \) using \( s \) as the transformation function. Given \( N \) i.i.d. samples of \( S \) we prove that the integral of the scaled, discrete power spectrum of \( \phi = \exp \left( \frac{2\pi i}{L} s \right) \) increasingly approximates the integral of the density function of \( s \) over an arbitrarily small interval \( \mathcal{N}_\alpha \) at the rate of \( O(1/N) \).

In addition to its fast computability in \( O(N \log N) \), our framework for obtaining the density does not involve any parameter selection like the number of histogram bins, width of the histogram bins, width of the kernel parameter, number of mixture components etc. as required by other widely applied methods like histogramming and Parzen windows.

Key words. Keywords: Stationary phase approximation; Density estimation; Fourier transform; Convergence rate; Error bounds

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1. Introduction. Density estimation methods attempt to estimate an unobservable probability density function using observed data \([12, 13, 15]\). The observed data are treated as random samples from a large population which is assumed to be distributed according to the underlying density function. The aim of our current work is to compute the density function of the gradient—corresponding to derivative in one dimension—of a thrice differentiable function \( S \) (density function of \( S' \)) from a finite set of \( N \) samples of \( S \) using the method of stationary phase \([3, 8, 10, 17, 18]\) and bound the error between the estimated and the unknown true density as a function of \( N \). If \( s = S' \) represent the derivative of the function \( S \), the density function of \( s \) is defined via a random variable transformation of the uniformly distributed random variable \( X \) using \( s \) as the transformation function. In other words, if we define a random variable \( Y = s(X) \) where the random variable \( X \) has a uniform distribution on the interval \( \Omega = [0, L] \), the density function of \( Y \) represents the density function of \( s \). In the field of computer vision many applications arise where the density of the gradient of the image, also popularly known as the histogram of oriented gradients (HOG), directly estimated from samples of the image are employed for human and object detection \([4, 19]\). Here the image intensity plays the role of the function \( S \) and the distribution of intensity gradients or edge directions are used as the feature descriptors to characterize the object appearance and shape within an image. In the recent article \([7]\), an adaptation of the HOG descriptor called the Gradient Field HOF (GF-HOG) is used for sketch based image retrieval.

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Our current work is along the lines of our earlier efforts [5, 6] where we demonstrated using the method of stationary phase that the power spectrum of \( \phi(x) = \exp \left( \frac{iS(x)}{\tau} \right) \) increasingly approximates the density function of the gradients as the free parameter \( \tau \to 0 \). In [5] we focused on exploiting the stationary phase tool to obtain gradient densities of Euclidean distance functions in two dimensions. As the gradient norm of Euclidean distance functions is identically equal to 1 everywhere, the density of the gradients is one-dimensional and defined over the space of orientations. In [6] we generalized and established this equivalence between the power spectrum and the gradient density to arbitrary smooth functions in arbitrary finite dimensions.

The fundamental point of departure between our current work and the results proved in [5, 6] is that here we compute the gradient density from a finite, discrete samples of \( S \) rather than assuming the availability of the complete description of \( S \) on \( \Omega \). Given only \( N \) samples of \( S \) the convergence proof involving continuous Fourier transform in [5, 6] has to be substituted with its discrete counterpart. Aliasing errors [2] which are non-existent in the continuous case have to be explicitly addressed in the present discrete setting. Curious enough we find that the free parameter \( \tau \), which could be set arbitrarily close to zero in the continuous case, has to respect a lower bound proportional to \( 1/N \) in the discrete scenario and an optimum value of \( \tau \) as a function of \( N \) can be explicitly determined. Apart from establishing the equivalence between the power spectrum and the gradient density, we also estimate the rate of convergence as \( N \to \infty \) (\( \tau \to 0 \)), a result not discussed in [5, 6]. Even in one dimension we find the discrete setting to be challenging and worthy of a separate examination. The discrete, one dimensional case seems to possess most of the mathematical complexities of its higher dimensional counterpart (thought at this point we are not completely sure) and lays the foundation towards extending the convergence rate result to arbitrary finite dimensions, a task we plan to take up in the future.

2. Main Contribution. Say we have \( N \) i.i.d. samples of a thrice differentiable function \( S \) obtained at uniform intervals of \( \delta = \frac{L}{N} \) between \( [0, L] \) at locations \( y_n = (n + \frac{1}{2})\delta, 0 \leq n \leq N - 1 \) denoted by the set \( \{ S(y_n) \}_{n=0}^{N-1}. \) As before, let \( s = S' \) denote the derivative of \( s \). For all positive parameter \( \tau > 0 \), we define a function

\[
\phi^D_\tau(y_n) = \frac{1}{\sqrt{L}} \exp \left( \frac{iS(y_n)}{\tau} \right)
\]

at these \( N \) discrete locations \( y_n \) by expressing \( S \) as the phase of the wave function \( \phi^D \) and consider its discrete power spectrum at an optimum value of \( \tau \approx \frac{1}{N} \). We show that the integral of this discrete power spectrum over an arbitrary small interval \( \mathcal{N}_\alpha \) with the interval length chosen independent of \( N \) approaches the cumulative measure of the true density of \( s \) over \( \mathcal{N}_\alpha \) as \( O(1/N) \) when \( N \to \infty \). The formal mathematical statement of our result is stated in Theorem 8.1. In our current effort we affirmatively answer the following questions:

1. As the number of samples \( N \to \infty \), does the discrete power spectrum (its interval measure to be precise) increasingly approximate the true density of the derivatives?
2. If yes, can we estimate the rate of convergence as a function of \( N \)?
3. Is there a lower bound on \( \tau \) as a function of \( N \) preventing it from being set arbitrarily close to zero?
4. Is there an optimum value for \( \tau \) as a function of \( N \)?
Table 2.1 presents the list of important symbols used in the article and their interpretations.

| Symbols | Interpretation |
|---------|----------------|
| $i, \tau$ | The imaginary unit satisfying $i^2 = -1$ and a free parameter respectively. |
| $S, s, \phi, B$ | The true function, its derivative, the sinusoidal function containing $S$ in its phase, and the bound on the derivative respectively. |
| $\delta, N, \Omega, L$ | The sampling interval, number of samples, domain of $S$, and the length of domain respectively. |
| $\{y_n\}_{n=0}^{N-1}, \{u_k\}_{k=0}^{M-1}$ | Sampling and the frequency locations respectively. |
| $\{x_m\}_{m=1}^{M(u)}, \{x_t\}_{t=1}^{M(u)}$ | Interchangeable notations of the same finite set $A_u$ of cardinality $M(u)$ containing the stationary points for a given frequency value $u$. |
| $F^D_{\tau}(u_k), F^D_{\tau}(u_k)$ | Scaled discrete Fourier transform and its magnitude square respectively. |
| $f_{\tau}(u)$ | The true density of $s$ obtained via random variable transformation. |
| $\epsilon_{\tau, r, k} \in \{1, 2, 3, 4, 5\}$ | Error terms. |
| $f_{\tau}(u) = O(\tau)$ | There exist a constant $\lambda > 0$ and a bounded continuous function $\gamma(u)$ both independent of $\tau$ such that when $\tau \leq \lambda$, $|f_{\tau}(u)| \leq \tau \gamma(u)$. |

3. Nature of the true function $S$ and existence of density. The function $S$ is assumed to be continuously thrice differentiable defined on the closed interval $\Omega = [0, L]$ with length $L$ and has a non-vanishing second derivative almost everywhere on $\Omega$, i.e.

$$\mu \left( \{x \in \Omega : S''(x) = 0\} \right) = 0,$$

where $\mu$ denotes the Lebesgue measure. The assumption in (3.1) is made in order to ensure that the density function of $s$ exists almost everywhere.

Define the following sets:

$$\mathcal{B} \equiv \{x : S''(x) = s'(x) = 0\} \quad (3.2)$$

$$\mathcal{C} \equiv \{s(x) : x \in \mathcal{B}\} \cup \{s(0), s(L)\}, \quad (3.3)$$

$$\mathcal{A}_u \equiv \{x : s(x) = u\}. \quad (3.4)$$

Here, $s(0) = \lim_{x \to 0^+} s(x)$ and $s(L) = \lim_{x \to (L)^-} s(x)$. The higher derivatives of $S$ at the end points 0 and $L$ are also defined along similar lines using one-sided limits.

The main purpose of defining these one-sided limits is to exactly determine the set $\mathcal{C}$ where the density of $Y$ is not defined. Let $M(u) = |\mathcal{A}_u|$. We now state some useful lemmas whose proofs are provided in Appendix A.1, A.2 and A.3 respectively.

**Lemma 3.1.** [Finiteness Lemma] $|\mathcal{A}_u|$ is finite for every $u \notin \mathcal{C}$.

As we see from Lemma 3.1 above that for a given $u \notin \mathcal{C}$, there is only a finite collection of $x \in \Omega$ that maps to $u$ under the function $s$. The inverse map $s^{-1}(u)$ which identifies the set of $x \in \Omega$ that maps to $u$ under $s$ is ill-defined as a function as it is a one to many mapping. The objective of the following Lemma 3.2 is to define appropriate neighborhoods such that the inverse function $s^{-1}$ when restricted to those neighborhoods is well-defined.

**Lemma 3.2.** [Neighborhood Lemma] For every $u \notin \mathcal{C}$, there exist a closed neighborhood $\mathcal{N}_{a_0}(u)$ around $u$ such that $\mathcal{N}_{a_0}(u) \cap \mathcal{C}$ is empty. Furthermore, if $|\mathcal{A}_u| > 0$, 

\( \mathcal{N}_\alpha(u) \) can be chosen such that we can find a closed neighborhood \( \mathcal{N}_\alpha(x) \) around each \( x \in A_u \) satisfying the following conditions:

1. \( s(\mathcal{N}_\alpha(x)) = \mathcal{N}_\alpha(u) \).
2. \( S''(y) \neq 0, \forall y \in \mathcal{N}_\alpha(x) \).
3. The inverse function \( s^{-1}(u) : \mathcal{N}_\alpha(u) \to \mathcal{N}_\alpha(x) \) is well-defined.
4. \( S'' \) is of constant sign in \( \mathcal{N}_\alpha(x) \).
5. \( M(u) \) is constant in \( \mathcal{N}_\alpha(u) \).

**Lemma 3.3. [Density Lemma]** The probability density of \( Y \) on \( \mathbb{R} - C \) exists and is given by

\[
P(u) = \frac{1}{L} \sum_{m=1}^{M(u)} \frac{1}{|S''(x_m)|},
\]

where the summation is over \( A_u \) (which is the finite set of locations \( x_m \in \Omega \) where \( s(x_m) = u \) as per Lemma 3.1).

From Lemma 3.3 it is clear that the existence of the density function \( P \) at a location \( u \in \mathbb{R} \) necessitates that \( S''(x) \neq 0, \forall x \in A_u \). Since we are interested in the case where the density exists almost everywhere on \( \mathbb{R} \), we impose the constraint that the set \( B \) in (3.2), comprising of all points where \( S'' \) vanishes has a Lebesgue measure zero. It follows that \( \mu(C) = 0 \).

**4. Fourier Transform and its discrete version.** Before we define the various versions of the Fourier transforms used in this article. Though the definitions provided here might appear to be slightly different from the standard textbooks [16, 2], it is fairly straightforward to establish their equivalence. We find it to be imperative to explicitly state them in this modified form as they help to comprehend our result better.

**4.1. Discrete Fourier transform (DFT) and the scaled DFT.** The DFT of the function \( \phi_D(y_n) \) is defined as

\[
F_D^{D}(w_k) \equiv \delta N^{-1} \sum_{n=0}^{N-1} \phi_D^{D}(y_n) \exp(-i2\pi w_k y_n)
\]

where for \( 0 \leq k \leq N - 1 \),

\[
w_k = \begin{cases} 
\frac{1}{N\delta} (k - \frac{N}{2}) & \text{if } N\%2 = 0; \\
\frac{1}{N\delta} (k - \frac{N-1}{2}) & \text{if } N\%2 = 1.
\end{cases}
\]

The shifts introduced in the definition of \( w_k \) places the zero-frequency component at the center of the spectrum. The inverse DFT is given by

\[
\phi_D^{D}(y_n) = \frac{1}{N\delta} \sum_{k=0}^{N-1} F_D^{D}(w_k) \exp(i2\pi w_k y_n).
\]

For the subsequent analysis we assume that \( N \) is even. It is worth emphasizing that \( \frac{1}{N\delta} \) is the interval between the frequencies where the DFT values are defined. Traditionally, the definition of DFT and its inverse doesn’t explicitly include the sampling interval \( \delta \) as it is generally unknown and set to 1.
Let \( u_k = 2\pi\tau w_k, 0 \leq k \leq N - 1 \) denote the scaled frequencies scaled by \( 2\pi\tau \). For every value of \( \tau > 0 \), we define the scaled DFT of \( \phi^D_\tau \) and its associated discrete power spectrum as

\[
F^D_\tau(u_k) = \frac{\delta}{\sqrt{2\pi\tau}} \sum_{n=0}^{N-1} \phi^D(y_n) \exp \left(-\frac{i\tau u_k y_n}{\tau}\right) \tag{4.4}
\]

\[
P^D_\tau(u_k) = |F^D_\tau(u_k)|^2. \tag{4.5}
\]

The scale factor \( \frac{1}{\sqrt{2\pi\tau}} \) compensates for the scaling of the frequencies \( w_k \) by \( 2\pi\tau \) leading to the following lemma whose proof is given in Appendix A.4.

**Lemma 4.1.** The scaled DFT in (4.4) satisfies \( \frac{2\tau}{N\delta} \sum_{k=0}^{N-1} P^D_\tau(u_k) = 1 \).

**4.2. Scaled Fourier transform.** By defining the following constants

\[
\rho = \frac{\delta}{2} \tag{4.6}
\]

\[
\rho_1 = -\rho \tag{4.7}
\]

\[
\rho_2 = L + \rho, \tag{4.8}
\]

we construct a continuous function \( H_\delta(x) : \mathbb{R} \to [0, 1] \) as follows:

\[
H_\delta(x) = \begin{cases} 
1 & \text{if } x \in [0, L]; \\
0 & \text{if } x \leq \rho_1 \text{ or } x \geq \rho_2; \\
\frac{x - \rho_1}{\rho_2 - \rho_1} & \text{if } \rho_1 \leq x \leq 0; \\
\frac{\rho_2 - x}{\rho_2 - L} & \text{if } L \leq x \leq \rho_2.
\end{cases} \tag{4.9}
\]

Denote \( I_\delta = (\rho_1, 0) \cup (L, \rho_2) \) of length \( \delta \) where \( H_\delta \) is linearly interpolated. The linear interpolation guarantees that in \( I_\delta \), \( |H_\delta'(x)| = \frac{2}{\delta} \), a constant, and \( H_\delta''(x) = 0 \) which will prove useful in our proof. Note that though \( H_\delta \) is continuous everywhere, \( H_\delta' \) is discontinuous at the limit points of \( I_\delta \). Using one sided limits the derivatives of \( H_\delta \) can be appropriately extended to the limit points of \( I_\delta \). We then use \( H_\delta \) to define the sinusoidal function \( \phi_\tau(x) : \mathbb{R} \to \mathbb{C} \) for all \( \tau > 0 \) as

\[
\phi_\tau(x) = H_\delta(x) \frac{1}{\sqrt{L}} \exp \left(\frac{iS(x)}{\tau}\right), \tag{4.10}
\]

where we extend \( S \) beyond the precincts of \([0, L]\) such that \( s(x) = s(0), \forall x \in [\rho_1, 0] \) and \( s(x) = s(L), \forall x \in [L, \rho_2] \). As \( C \) includes \( s(0) \) and \( s(L) \), the aforementioned extension would ensure that \( \forall u \in \mathbb{R} - C, A_u \cap I_\delta = \emptyset \) as the interval \( I_\delta \) is artificially introduced and should not interfere with the computation of the density. As \( H_\delta(x) \) is identically zero outside \([\rho_1, \rho_2]\) any extension of \( S \) outside \([\rho_1, \rho_2]\) will not impact \( \phi_\tau(x) \). Imposing \( H_\delta(x) = 1 \) between \( y_0 = \frac{\delta}{2} \) and \( y_{N-1} = L - \frac{\delta}{2} \) where the \( N \) samples of \( S \) are confined assures that

\[
\phi_\tau(y_n) = \phi^D_\tau(y_n). \tag{4.11}
\]

Our subsequent analysis require that \( \phi_\tau(x) \) vanishes at the end points \( \rho_1 \) and \( \rho_2 \) and also satisfy (4.11) at the sample locations \( y_n \). \( \phi^D_\tau(0) \neq 0 \) precludes us from...
setting $\rho = 0$. The non-zero choice of $\rho$ entails the introduction of the function $H_\delta$ with properties as described in (4.9). Setting $\rho = \frac{\delta}{2}$ and imposing $H_\delta(x) = 0, \forall x \notin (\rho_1, \rho_2)$ forces $\phi_\tau((n + \frac{1}{2})\delta) = 0$ when $n \leq -1$ or $n \geq N$. This ensures that the Discrete Time Fourier Transform (DTFT) defined as

$$F^{DTFT}(w) \equiv \delta \sum_{n = -\infty}^{\infty} \phi^D(y_n) \exp(-i2\pi wy_n), \quad y_n = \left(n + \frac{1}{2}\right)\delta$$  \hspace{1cm} (4.12)

involving infinite summation coincides with the DFT expression—stated in (4.1)—that comprises of only $N$ finitely many summations at the frequencies $w_k$ i.e.,

$$F^{DTFT}(w_k) = F^D(w_k), \forall k.$$  \hspace{1cm} (4.13)

In Section 4.3 we will see that enabling this equality will help us relate the DFT with the Fourier transform through the Poisson summation formula \[16\]. The constants $\rho_1$ and $\rho_2$ defined in (4.7) and (4.8) respectively ascertains that $H_\delta(x) = 1, \forall x \in \bigcup_{k=0}^{N-1} A_{u_k}$ thereby impairing $H_\delta(x)$ from exercising any influence on the density of $s$ at the frequencies $u_k$. The definition of $H_\delta(x)$ is totally left to our discretion and can be flexed to incorporate any desirable properties.

Counterpart to the discrete versions given in (4.1) and (4.4), we define the Fourier transform (FT) and the scaled FT of $\phi_\tau(x)$ as:

$$F(w) = \int_{\rho_1}^{\rho_2} \phi_\tau(x) \exp(-i2\pi wx) \, dx,$$  \hspace{1cm} (4.14)

$$F_\tau(u) = \frac{1}{\sqrt{2\pi\tau}} \int_{\rho_1}^{\rho_2} \phi_\tau(x) \exp\left(-\frac{iu_x}{\tau}\right) \, dx$$  \hspace{1cm} (4.15)

where again by relating $u = 2\pi\tau w$ we get $F(w) = \sqrt{2\pi\tau} F_\tau(u)$ akin to (A.3).

4.3. Relating the scaled DFT and the scaled Fourier transform. The Poisson summation formula relates the DTFT with the Fourier transform ($F(w)$) where DTFT is just the periodic summation of $F(w)$ shifted by $\frac{1}{\tau}$ [10]. Using (4.13), the Poisson summation formula can be leveraged to relate the DFT and the Fourier transform at these frequencies $w_k$, specifically

$$F^D(w_k) = F^{DTFT}(w_k) = \sum_{l=-\infty}^{\infty} F\left(w_k - \frac{l}{\delta}\right).$$  \hspace{1cm} (4.16)

Defining

$$\gamma_l \equiv \frac{2\pi\tau l}{\delta}$$  \hspace{1cm} (4.17)

and using the scaled versions of the DFT and the Fourier transforms we get

$$F^D_\tau(u_k) = F^{DTFT}_\tau(u_k) = F_\tau(u_k) + \sum_{l=-\infty, l \neq 0}^{\infty} F_\tau(u_k - \gamma_l)$$  \hspace{1cm} (4.18)
where $F^{DTFT}_\tau(u)$ is the scaled DTFT given by

$$F^{DTFT}_\tau(u) \equiv \frac{F^{DTFT}(w)}{\sqrt{2\pi\tau}}$$

(4.19)

at $u = 2\pi\tau w$ and is defined for all $u$. The infinite summation $\sum_{l=-\infty, l\neq 0}^\infty F_\tau(u_k - \gamma l)$ is known as the aliasing error \cite{2}.

5. Bound on $\tau$. As $s$ is also continuous on a compact interval $[\rho_1, \rho_2]$, the image of $s$ is also compact and hence bounded. Pick an arbitrarily small $\beta > 0$ and let

$$B \equiv \sup_{x \in [0, L]} |s(x)| + \beta$$

(5.1)

such that $|s(x)| < B, \forall x \in [0, L]$. From (4.2), note that $\max_{k=0}^{N-1} |w_k| = \frac{1}{2\pi}$, hence $\max_{k=0}^{N-1} |u_k| = \frac{\pi}{\tau}$ and from the definition of $B$ in (5.1) we have $\sup_{x \in [0, L]} |s(x)| = B - \beta$ where $\beta > 0$ is an arbitrarily small quantity. In the definition of $F^D_\tau(u_k)$ in (4.4), the frequencies $u_k$ are the histogram bins for the derivatives $s(x)$ where they are related by $u_k \leq s(y_n) < u_k + 1$. Then in order to capture all the derivatives, $\tau$ needs to be chosen such that

$$\tau \geq \frac{BL}{\pi} \left( \frac{1}{N} \right)$$

(5.2)

In the subsequent sections we will reason that the optimum value of $\tau = \frac{B\delta}{\pi}$. The linearity of the relation between the free parameter $\tau$ and the sample interval $\delta$ at this optimum value which will prove crucial in obtaining the desired $O(1/N)$ convergence of our density estimation technique. For now, we let

$$\tau = \frac{CB\delta}{\pi}$$

(5.3)

for some constant $C \geq 1$.

6. Bound on the aliasing error. We start with the following lemma whose proof is given in Appendix A.5. Let $T(x; u) = S(x) - ux$.

Lemma 6.1. [No stationary points] On the interval $[b_1, b_2] \subseteq [\rho_1, \rho_2]$ consider the integral

$$I_\tau(u) = \int_{b_1}^{b_2} H_\delta(x) \exp \left( \frac{iT(x; u)}{\tau} \right) dx$$

(6.1)

under the condition that there exist a constant $\xi > 0$ such that $|T'(x; u)| \geq \xi, \forall x \in [b_1, b_2]$ implying the absence of any stationary points. Then $I_\tau(u) = O(\tau)$.

For $u \in \left[ -\frac{\pi}{\tau}, \frac{\pi}{\tau} \right]$ we now obtain a bound on the aliasing error as a function of $\tau$. When $\tau$ satisfies (6.2), $|\gamma l| \geq 2B |l|$ and

$$|u - \gamma l| \geq |\gamma l| - |u| \geq \frac{2\pi\tau}{\delta} \left( |l| - \frac{1}{2} \right)$$

(6.2)

$$\geq 2B \left( |l| - \frac{1}{2} \right) \geq B |l|.$$  

(6.3)
Therefore,
\[ |T'(x; u - \gamma_l)| = |s(x) - (u - \gamma_l)| \]
\[ \geq |u - \gamma_l| - |s(x)| \geq B(|l| - 1) + \beta > 0 \]  
indicating that the integral in computing \( F_r(u_k - \gamma_l) \) is devoid of any stationary points. Applying Lemma 6.1 and recalling that \( H_\delta(x) = 0 \) at the end points \( \rho_1 \) and \( \rho_2 \) by construction, the expression on the right side of (A.12) vanishes. The remaining terms gives us

\[
\sqrt{2\pi L} F_r(u - \gamma_l) = \frac{2\sqrt{\pi} CB}{\pi [s(0) - (u - \gamma_l)]^2} \left[ \exp \left( \frac{iT(0; u - \gamma_l)}{\tau} \right) - \exp \left( \frac{iT(\rho_1; u - \gamma_l)}{\tau} \right) \right] 
\]
\[ + \frac{2\sqrt{\pi} CB}{\pi [s(L) - (u - \gamma_l)]^2} \left[ \exp \left( \frac{iT(L; u - \gamma_l)}{\tau} \right) - \exp \left( \frac{iT(\rho_2; u - \gamma_l)}{\tau} \right) \right] 
\]
\[ + O \left( \frac{\tau \sqrt{T}}{|B(|l| - 1) + \beta|^2} \right). \]  

Realizing that
\[ \sum_{l=-\infty \lor l \neq 0} \frac{1}{|l|^2} < \infty, \]
the infinite summation of each of the term in (6.6) and (6.7) converges individually. The total aliasing error then satisfies
\[ \sum_{l=-\infty \lor l \neq 0} F_r(u - \gamma_l) = O(\sqrt{T}). \]  

In particular,
\[ F_r(u - \gamma_l) = O \left( \frac{\sqrt{T}}{|B(|l| - 1) + \beta|^2} \right). \]  

7. Evaluation of the Fourier transform via the method of stationary phase. We now use the stationary phase approximation to obtain an expression for the Fourier transform defined in (4.15). We expand the scope of \( \tilde{u} \) beyond the finite set of \( N \) frequencies \( \tilde{u} \in \{u_k\}_{k=0}^{N-1} \) where the scaled DFT values are defined—, to any \( \tilde{u} \in \mathbb{R} - \mathcal{C} \).

If no stationary point exits in \( \Omega \) for the given \( \tilde{u} \) (\( |A_{\tilde{u}}| = 0 \)), then pursuant to Lemma 6.1 we have \( F_r(\tilde{u}) = O(\sqrt{T}) \). Otherwise, let the finite set \( A_{\tilde{u}} \) be represented by \( A_{\tilde{u}} = \{x_1, x_2, \ldots, x_{M(\tilde{u})}\} \) with \( x_m < x_{m+1} \), \( \forall m \). We break \( \{\rho_1, \rho_2\} \) into disjoint intervals such that each interval has utmost one stationary point. To this end, we choose numbers \( \{c_0, c_1, \ldots, c_{M(\tilde{u})}\} \) such that \( \rho_1 < c_0 < x_1, x_m < c_m < x_{m+1} \) and \( x_{M(\tilde{u})} < c_{M(\tilde{u})} < \rho_2 \). We set \( c_0 = 0 \) and \( c_{M(\tilde{u})} = L \) so that the open interval \( (c_0, c_{M(\tilde{u})}) \) encompasses all stationary points in \( A_{\tilde{u}} \). The choice of other constants will be discussed below. Recall that by definition \( H_\delta(x) = 1, \forall x \in [c_0, c_{M(\tilde{u})}] \). The scaled Fourier transform \( F_r(\tilde{u}) \) can be broken into:

\[ F_r(\tilde{u}) \sqrt{2\pi TL} = G_1(\tilde{u}) + G_2(\tilde{u}) + \sum_{m=1}^{M(\tilde{u})} K_m(\tilde{u}) + \tilde{K}_m(\tilde{u}) \]  

(7.1)
where

\[ G_{1,\tau}(\tilde{u}) \equiv \int_{\rho_1}^{\rho_2} H_\delta(x) \exp \left( \frac{iT(x; \tilde{u})}{\tau} \right) dx, \quad (7.2) \]

\[ G_{2,\tau}(\tilde{u}) \equiv \int_{c_M(\tilde{u})}^{c_1\rho_2} H_\delta(x) \exp \left( \frac{iT(x; \tilde{u})}{\tau} \right) dx, \quad (7.3) \]

\[ K_{m,\tau}(\tilde{u}) \equiv \int_{x_{m-1}}^{x_m} \exp \left( \frac{iT(x; \tilde{u})}{\tau} \right) dx, \quad (7.4) \]

\[ \hat{K}_{m,\tau}(\tilde{u}) \equiv \int_{c_m}^{x_m} \exp \left( \frac{iT(x; \tilde{u})}{\tau} \right) dx. \quad (7.5) \]

Evaluating \( K_{m,\tau}(\tilde{u}) \) and \( \hat{K}_{m,\tau}(\tilde{u}) \) using the method of stationary phase \([11],\) Chapter 3, Article 13 in [10], (7.1) can be expressed as

\[
F_\tau(\tilde{u}) \sqrt{2\pi \tau L} = \sum_{m=1}^{M(\tilde{u})} \exp \left( \frac{i}{\tau} [S(x_m) - \tilde{u} x_m] \right) \sqrt{\frac{2\pi \tau}{S''(x_m)}} \exp \left( \pm \frac{i\pi}{4} \right) \\
+ \epsilon_{1,\tau}(\tilde{u}) + \epsilon_{2,\tau}(\tilde{u}). \tag{7.6}
\]

Depending on whether \( S''(x_m) > 0 \) or \( < 0 \), the factor \( \pm \frac{i\pi}{4} \) in the exponent is positive or negative respectively. The error \( \epsilon_{1,\tau}(\tilde{u}) = G_{1,\tau}(\tilde{u}) + G_{2,\tau}(\tilde{u}) \) stems from computing the integral \( F_\tau(\tilde{u}) \) on \([\rho_1, c_0] \cup [c_M(\tilde{u}), \rho_2] \) which doesn’t contain any stationary points. Pursuant to Lemma 6.1, \( \epsilon_{1,\tau}(\tilde{u}) = O(\tau) \). Using the facts that \( H_\delta(x) = 1 \) at \( x \in \{\rho_1, \rho_2\} \) and \( H_\delta(x) = 1 \) at \( x \in \{c_0, c_M(\tilde{u})\} \) in (7.12), \( \epsilon_{1,\tau}(\tilde{u}) \) can be expressed as

\[
\epsilon_{1,\tau}(\tilde{u}) = -i \tau \left( \frac{\exp \left( \frac{iT(c_0; \tilde{u})}{\tau} \right)}{s(c_0) - \tilde{u}} + \frac{\exp \left( \frac{iT(c_M(\tilde{u}); \tilde{u})}{\tau} \right)}{s(c_M(\tilde{u})) - \tilde{u}} \right) \\
+ \frac{2\tau CB}{\pi |s(0) - \tilde{u}|^2} \left[ \exp \left( \frac{iT(0; \tilde{u})}{\tau} \right) - \exp \left( \frac{iT(\rho_1; \tilde{u})}{\tau} \right) \right] \\
+ \frac{2\tau CB}{\pi |s(L) - \tilde{u}|^2} \left[ \exp \left( \frac{iT(L; \tilde{u})}{\tau} \right) - \exp \left( \frac{iT(\rho_2; \tilde{u})}{\tau} \right) \right]. \tag{7.7} \]

As the integral in \( \epsilon_{1,\tau}(\tilde{u}) \) excludes the interval \([0, L]\), the bound appearing in (A.15) has been deliberately omitted. \( \epsilon_{2,\tau}(\tilde{u}) \) represents the error from the stationary phase approximation and is \( O(\tau) \) as elucidated in Appendix B Using these error bounds we get

\[
F_\tau(\tilde{u}) = \frac{1}{\sqrt{L}} \sum_{m=1}^{M(\tilde{u})} \exp \left( \frac{i}{\tau} [S(x_m) - \tilde{u} x_m] \right) \exp \left( \pm \frac{i\pi}{4} \right) + \epsilon_{3,\tau}(\tilde{u}) \tag{7.11}
\]
where
\[ \epsilon_{3,\tau}(\tilde{u}) = \frac{\epsilon_{1,\tau}(\tilde{u}) + \epsilon_{2,\tau}(\tilde{u})}{\sqrt{2\pi\tau L}} \]  
(7.12)
\[ = \frac{2\sqrt{\pi CB}}{\pi \sqrt{2\pi L} [s(0) - \hat{u}]^2} \left[ \exp \left( \frac{iT(0; \tilde{u})}{\tau} \right) - \exp \left( \frac{iT(p_{1}; \tilde{u})}{\tau} \right) \right] \]  
(7.13)
\[ + \frac{2\sqrt{\pi CB}}{\pi \sqrt{2\pi L} [s(L) - \hat{u}]^2} \left[ \exp \left( \frac{iT(L; \tilde{u})}{\tau} \right) - \exp \left( \frac{iT(p_{2}; \tilde{u})}{\tau} \right) \right] \]  
(7.14)
\[ + O(\tau) \]  
(7.15)
\[ = O(\sqrt{\tau}). \]  
(7.16)

To understand the bound of \( O(\tau) \) for \( \epsilon_{3,\tau}(\tilde{u}) \) in (7.15) note that when we combine (7.7) and (7.8) to add the error terms \( \epsilon_{1,\tau}(\tilde{u}) \) and \( \epsilon_{2,\tau}(\tilde{u}) \), all the phase terms containing the constants \( c_{0} \) and \( c_{M(\tilde{u})} \) cancel each other. The remainder error term \( O(\tau\sqrt{\tau}) \) when divided by \( \sqrt{\tau} \) appearing in the denominator of (7.12) results in a bound of \( O(\tau) \).

The scaled power spectrum \( P_{\tau}(\tilde{u}) \equiv |F_{\tau}(\tilde{u})|^{2} \) equals
\[ P_{\tau}(\tilde{u}) = \frac{1}{L} \sum_{m=1}^{M(\tilde{u})} \frac{1}{|S'(x_{m})|} + \frac{1}{L} \sum_{m=1}^{M(\tilde{u})} \sum_{t=1; t \neq m}^{M(\tilde{u})} \chi_{m,t,\tau}(x_{m}, x_{t}, \tilde{u}) \]  
(7.17)
\[ + |\epsilon_{3,\tau}(\tilde{u})|^{2} + |\epsilon_{4,\tau}(\tilde{u})|^{2} \]  
(7.18)
where
\[ \chi_{m,t,\tau}(x_{m}, x_{t}, \tilde{u}) = \cos \left( \frac{i}{\tau} \left[ S(x_{m}) - S(x_{t}) - \tilde{u}(x_{m} - x_{t}) \right] + \theta_{m,t}(x_{m}, x_{t}) \right) \]  
(7.19)
\[ \epsilon_{4,\tau}(\tilde{u}) = \epsilon_{3,\tau}(\tilde{u}) \frac{1}{\sqrt{L}} \sum_{m=1}^{M(\tilde{u})} \exp \left( \frac{i}{\tau} \left[ S(x_{m}) - \tilde{u}x_{m} \right] \right) \exp \left( \pm \frac{i\pi}{4} \right). \]  
(7.20)

The cross terms \( \chi_{m,t,\tau}(x_{m}, x_{t}, \tilde{u}) \) germinate from having multiple spatial locations \( (x_{m}, x_{t}) \) index into the same frequency bin \( \tilde{u} \). Additionally, \( \theta_{m,t}(x_{m}, x_{t}) = 0, \frac{\pi}{2} \) or \( -\frac{\pi}{2} \) and \( \theta_{m,t}(x_{m}, x_{t}) = -\theta_{m,t}(x_{m}, x_{t}) \).

8. Rate of convergence of our density estimation method. To keep up with our analysis for any \( \tilde{u} \in R - C \) rather than confining to the set of \( N \) scaled frequencies \( \{ u_{k} \}_{k=0}^{N-1} \), we use the scaled DFT instead of the scaled DFT. Let \( P_{\tau}^{DFT}(\tilde{u}) \) represent the magnitude square of the scaled DFT. Substituting \( \tau = \frac{CB}{\pi^{2}} \), observe that the scaled frequencies lie between \( [-CB, CB] \) for all \( N \) where \( C \geq 1 \). Additionally, as \( |s(x)| < B, \forall x \in \Omega \) by definition, the true density \( P(\tilde{u}) = 0, \forall \tilde{u} \notin (-B, B) \). So we restrict ourselves to the interesting region where \( \tilde{u} \in [-CB, CB] - C \). Recollect that we explicitly avoid the set \( C \) where the density \( P(\tilde{u}) \) is not defined. The formal mathematical statement of our result can be stated as follows:

**Theorem 8.1.** For any \( \tilde{u} \in [-B, B] - C \), there exists a closed interval \( N_{\alpha}(\tilde{u}) = [\tilde{u} - \tilde{\alpha}, \tilde{u} + \tilde{\alpha}] \) with \( \tilde{\alpha} \) chosen independent of \( N \) as given by Lemma 8.2—such that when \( \tau = \frac{CB}{\pi^{2}} \), the cumulative of the difference \( P_{\tau}^{DFT}(\tilde{u}) = P(\tilde{u}) \) over \( N_{\alpha}(\tilde{u}) \) is of \( O(1/N) \) as \( N \to \infty \), i.e.,
\[ \int_{N_{\alpha}(\tilde{u})} \left[ P_{\tau}^{DFT}(\tilde{u}) - P(\tilde{u}) \right] d\tilde{u} = O \left( \frac{1}{N} \right). \]  
(8.1)
Proof. case (i) No stationary points: Plugging the $O(\sqrt{\tau})$ bound of the aliasing error and the Fourier transform in (4.18) and taking the magnitude square we get $P_{\tau}^{DFT}(\hat{u}) = O(\tau)$. As $\hat{u} \notin s([1, 1])$ and the image $s([\rho_1, \rho_2])$ is compact, there exist a neighborhood $N_{\tilde{\alpha}}(\hat{u})$ around $\hat{u}$ such that $\forall u \in N_{\tilde{\alpha}}(\hat{u})$, no stationary points exits. The selection of $\alpha$ as a function of $N$ is discussed below where we reason that the optimal choice of $\tau = (\frac{2\pi}{\hat{l}})^{\frac{1}{N}}$. By integrating $P_{\tau}^{DFT}(\hat{u})$ over $N_{\tilde{\alpha}}(\hat{u})$ the result follows.

case (ii) Existence of stationary points: Considering the magnitude square of (4.18) and plugging in (7.17) we get

$$P_{\tau}^{DFT}(\hat{u}) = P(\hat{u}) + \epsilon_{5,\tau}(\hat{u})$$

where

$$\epsilon_{5,\tau}(\hat{u}) = \frac{1}{L} \sum_{m=1}^{M(\hat{u})} \sum_{l=1}^{M(\hat{u})} \chi_{m,t,\tau}(x_m, x_t, \hat{u})$$

$$+ |\epsilon_{3,\tau}(\hat{u})|^2 + \epsilon_{4,\tau}(\hat{u}) + \epsilon_{4,\tau}(\hat{u})$$

$$+ \left| \sum_{l=-\infty, l \neq 0}^{\infty} F_{l}(\hat{u} - \gamma_l) \right|^2$$

$$+ F_{\epsilon}(\hat{u}) \left( \sum_{l=-\infty, l \neq 0}^{\infty} F_{l}(\hat{u} - \gamma_l) \right) + F_{\epsilon}(\hat{u}) \left( \sum_{l=-\infty, l \neq 0}^{\infty} F_{l}(\hat{u} - \gamma_l) \right).$$

Based on the form of the cross terms in (7.19) it is straightforward to check that $\lim_{\tau \to 0} \chi_{m,t,\tau}(x_m, x_t, \hat{u})$ doesn’t exist. Hence in order to recover the density we must integrate the power spectrum over an arbitrarily small neighborhood $N_{\tilde{\alpha}}(\hat{u})$ around $\hat{u}$ to nullify these cross terms. Lemma 3.2 endows us with one such neighborhood. Recall that from Lemma 3.2, $s^{-1}(N_{\tilde{\alpha}}(\hat{u})) = \bigcup_{m=1}^{M(\hat{u})} N_{\tilde{\alpha}}(x_m)$ where $N_{\tilde{\alpha}}(x_m)$ is the image of $s^{-1}(N_{\tilde{\alpha}}(\hat{u}))$ confined around $x_m$. We set $N_{\tilde{\alpha}}(\hat{u}) = [\tilde{u} - \tilde{\alpha}, \tilde{u} + \tilde{\alpha}]$ where we select a small enough $\tilde{\alpha}$ (independent of $\tau$) in accordance with Lemma 3.2 and also choose the remaining constants $\{c_1, c_2, \cdots, c_{M(\hat{u})-1}\}$ such that $N_{\tilde{\alpha}}(x_m) \subset [c_{m-1}, c_m], \forall m$. This would enable the definitions given in (7.22) and (7.24) concerning these constants to be extended $\forall u \in N_{\tilde{\alpha}}(\hat{u})$. Additionally, as $|s(x)| \leq B$, $\forall x \in [0, L]$ we further have $N_{\tilde{\alpha}}(\hat{u}) \subset (-B, B)$. The following two lemmas capture the desired $O(1/N)$ convergence of our density estimation. Their proofs are provided in Appendix A.6 and Appendix A.7 respectively.

**Lemma 8.2.** Let the constant $\kappa \in \{\rho_1, 0, L, \rho_2\}$ so that $s(\kappa) \neq u, \forall u \in N_{\tilde{\alpha}}(\hat{u})$. Let $|s(\kappa) - (u - \gamma)| \geq \xi$ for some constant $\xi > 0$ where $\gamma = \frac{2\pi l}{\hat{l}} = 2BCl, l \in \mathbb{Z}$ (including $l = 0$). Define

$$\zeta_{\tau}(u) \equiv \frac{\exp \left( \frac{i\tau(s(\kappa) - u - \gamma)}{\tau} \right)}{|s(\kappa) - (u - \gamma)|^2}$$

(8.7)
for \( u \in \mathcal{N}_\tilde{\alpha}(\tilde{u}) \).

Then

\[
I_\tau = \int_{\mathcal{N}_\tilde{\alpha}(\tilde{u})} \zeta_\tau(u) \exp \left( \frac{i}{\tau} \frac{[S(x_m(u)) - u x_m(u)]}{\sqrt{|S''(x_m(u))|}} \right) du = O \left( \frac{\tau}{\xi^2} \right), \forall m.
\] (8.8)

**Lemma 8.3.** [Bound on Integrated Error Lemma] The bound on the each of the error terms in \( \epsilon_{5,\tau}(u) \) when integrated over an interval \( \mathcal{N}_\tilde{\alpha}(\tilde{u}) \) chosen independent of \( \tau \) are as summarized in Table 8.1.

**Table 8.1**

| Integrated over \( \mathcal{N}_\tilde{\alpha}(\tilde{u}) \) | Bound |
|----------------------------------------------------------|-------|
| \( \chi_{m,\tau}(x_m(u), x_t(u), u) \)                        | \( O(\tau) \) |
| \( |\epsilon_{3,\tau}(u)|^2 \)                                | \( O(\tau) \) |
| \( \epsilon_{4,\tau}(u) \)                                  | \( O(\tau) \) |
| \( \sum_{l=-\infty, l \neq 0}^{\infty} F_\tau(u - \gamma_l) \) | \( O(\tau) \) |
| \( F_\tau(u) \left( \sum_{l=-\infty, l \neq 0}^{\infty} F_\tau(u - \gamma_l) \right) \) | \( O(\tau) \) |

Based on which we could conclude that

\[
\int_{\mathcal{N}_\tilde{\alpha}(\tilde{u})} \epsilon_{5,\tau}(u) du = O(\tau).
\] (8.9)

**Optimal choice of \( \tau \) as a function of \( N \).** In Section 5 we demonstrated that if there are only \( N \) finitely many samples of \( S \) picked at intervals of \( \delta \), \( \tau \) cannot be set arbitrarily close to zero and should respect the inequality (5). Lemma 8.3 establishes that the integral of the error \( \epsilon_{5,\tau}(u) \) between the true and the estimated density over a small interval \( \mathcal{N}_\tilde{\alpha}(\tilde{u}) \) is bounded by \( \tau \) and hence we can expect the error profile to portray a decreasing trend as we tune down \( \tau \). Apropos to the aforementioned statements it is logical to conclude that the optimum value of \( \tau \) for a given \( N \) equals

\[
\tau = \frac{B\delta}{\pi} = \left( \frac{BL}{\pi} \right) \frac{1}{N}.
\] (8.10)

The inverse relation between \( \tau \) and \( N \) proves Theorem 8.1.

We also obtain the following corollary as a direct consequence of Theorem 8.1.

**Corollary 8.4.** For all \( \tilde{u} \in [-B, B] - C \) consider the closed interval \( \mathcal{N}_\alpha(\tilde{u}) = [\tilde{u} - \tilde{\alpha}, \tilde{u} + \tilde{\alpha}] \) of length \( 2\tilde{\alpha} \) satisfying Lemma 8.2. Then

\[
\lim_{\tilde{\alpha} \to 0} \lim_{N \to \infty} \int_{\mathcal{N}_\alpha(\tilde{u})} P_{DTFT}^{(\tilde{u}, \bar{u})}(u) = P(\tilde{u}).
\] (8.11)
9. Discussion. The integrals

\[ I_\tau(\tilde{u}) = \int_{\mathcal{N}_\delta(\tilde{u})} P_{\tau}^{D T F T}(u) \, du, \quad I(\tilde{u}) = \int_{\mathcal{N}_\delta(\tilde{u})} P(u) \, du \quad (9.1) \]

represent the interval measures of the density functions \( P_{\tau}^{D T F T} \) and \( P \) respectively over an interval \( \mathcal{N}_\delta(\tilde{u}) \) where the interval length can be made arbitrarily smaller but independent of \( N \). Theorem 8.1 states that given the \( N \) samples of \( \phi(x) = \exp \left( i \frac{\tilde{S}(x)}{\tau} \right) \) and when \( \tau \) is set to the optimum value of \( \frac{B}{\pi N} \), both the interval measures are almost equal with the difference between them dwindling at the fast rate of \( O(1/N) \). Recall that the scaled discrete power spectrum \( P_{\tau}^D(u_k) \) computed at the \( N \) scaled frequencies \( \{u_k\}_{k=0}^{N-1} \) spaced at increasingly smaller intervals of \( \frac{2\pi}{N} = \frac{2B}{\pi} \) are the uniform samples of the discrete time power spectrum \( P_{\tau}^{D T F T}(u) \). In Section 9.2, we show through simulations that even the approximation of the interval measure \( I_\tau(\tilde{u}) \) by its Riemann sum computed using \( P_{\tau}^D(u_k) \) converges as \( O(1/N) \) to the corresponding Riemann sum approximation of \( I(\tilde{u}) \). Though only a conjecture presently, we strongly believe that the experimentally observed \( O(1/N) \) convergence of Riemann sums can also be mathematically proven leading to the conclusion that the discrete power spectrum can potentially serve as the density estimator for the derivative of \( S \) at large values of \( N \). Once established, this will prove to be a stronger computational result strengthening our Theorem 8.1. These are fruitful topics for future research.

9.1. Advantages of our formulation.
1. One of the foremost advantage of our approach is that it recovers the gradient density function of \( S \) without explicitly computing its derivative \( s \). Since the stationary points capture gradient information and map them into the corresponding frequency bins, we can directly work with \( S \) without the need to compute its derivatives.
2. Our method is also extremely fast in terms of computational complexity. Given the \( N \) sampled values \( \tilde{S} \) the Fourier transform of \( \exp \left( i \frac{\tilde{S}(x)}{\tau} \right) \) at the optimum value of \( \tau \) can be computed in \( O(N \log N) \) and the subsequent squaring operation to obtain the power spectrum can be performed in \( O(N) \). We thus provide an \( O(N \log N) \) algorithm for approximating the density of the derivative that exhibits a fast convergence of \( O(1/N) \).
3. Our framework for obtaining the density does not involve any parameter selection like number of histogram bins, width of the histogram bins, width of the kernel parameter, number of mixture components etc. as seen in other widely applied methods like histogramming and Parzen windows [12].

9.2. Experimental justification. Below we experimentally verify the convergence of our gradient density estimator to the true density function for the function \( S(x) = \sin(\pi(x - 1)); x \in [0, 2] \). The bound \( B \) on the derivative \( s(x) = \pi \cos(\pi(x - 1)) \) equals \( B = \pi \). We would like to emphasize that unless we gain background information about the function \( S \) that is being sampled, it is generally not feasible to determine \( B \) accurately and have to be approximated (say) by setting it to the maximum absolute of the derivative computed via finite differences, i.e,

\[ B \approx \max_{n=0}^{N-2} \frac{S(g_{n+1}) - S(g_n)}{\delta}. \quad (9.2) \]
As \( S''(x) = -\pi^2 \sin(\pi(x - 1)) \) equals zero for \( x \in \{0, 1, 2\} \), the density is not defined at \( u \in \left\{ s(0), s(1), s(2) \right\} = \pm \pi \). For all other values of \( u \in (-\pi, \pi) \), the set \( A_u = \{ x_1(u), x_2(u) \} \) of cardinality \( M(u) = |A_u| = 2 \) computed as the pre-image of \( \pi \cos(\pi(x - 1)) = u \) satisfies: \( 0 < x_1(u) < 1, 1 < x_2(u) < 2 \) and \( |S''(x_1(u))| = |S''(x_2(u))| \). Using the last equality we could let \( \cos^{-1} \left( \frac{u}{\pi} \right) \) to lie in the interval \((0, \pi)\) corresponding to \( x_2(u) \) for \( u \in (-\pi, \pi) \) and compute the true density as

\[
P(u) = \frac{1}{\pi^2 \sin \left( \cos^{-1} \left( \frac{u}{\pi} \right) \right)}
\]  

(9.3)

With the above set up in place we investigate two variations of convergence results where:

1. We progressively increase the the number of samples \( N \) and for each \( N \) set \( \tau \) to its corresponding optimum value \( \frac{BL}{2N} = \frac{\delta}{N} = \delta \).
2. We fix \( N = N_0 \) and progressively decrease \( \tau \) from some high value to its optimum value \( \tau_0 = \delta_0 \).

9.2.1. Case study 1. Our Theorem 8.1 requires both the power spectrum and the true density to be integrated over a small neighborhood \( \mathcal{N}_u \) chosen independent of \( N \) to observe convergence. To this end we preselect a set of \( K = 255 \) fixed frequencies \( \{ \tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_K \} \) and consider appropriate non-overlapping neighborhoods \( \{ \mathcal{N}_\tilde{u}(\tilde{u}_k) \}_{k=1}^K \) around them in accordance with Lemma 3.2 Note that as we scale up \( N \), the number of frequency locations \( N_k \) within each neighborhood \( \mathcal{N}_\tilde{u}(\tilde{u}_k) \) where the discrete power spectrum is defined also increases and hence we could progressively approximate the integrals in Theorem 8.1 by its summation involving \( P^{\text{DTFT}}_\tau(u) \). The replacement of the integral with sums is akin to the well known Riemann summation approximation, though not exactly equivalent as the underlying function \( P^{\text{DTFT}}_\tau(u) \) whose samples we get in the form of \( P^{\text{DTFT}}_\tau(u) \) keeps varying with \( N \) (and also with \( \tau \)) as easily seen from the definition of the scaled DTFT in [4.3] that involves summation over \( N \) terms. Notwithstanding this conceptual difference and continuing to refer to it as Riemann sums, we define the error between their respective Riemann summation as

\[
\Delta_{\tau,N} = \frac{1}{K} \sum_{k=1}^{K} \frac{2\pi}{N} \left| \sum_{l=1}^{N_k} \left[ P^{D}_{\tau}(u_{l,k}) - P(u_{l,k}) \right] \right|
\]  

(9.4)

where \( u_{i,k} \) is the \( i^{th} \) frequency location within the interval \( \mathcal{N}_\tilde{u}(\tilde{u}_k) \). The spacing between the frequencies equals \( \frac{2\pi}{N} = \frac{\delta}{N} \). In Figure 9.2.1 we visualize these Riemann summations where we observe that as we increase \( N \), the interval measure of our density function (plotted right) steadily approaches the interval measure of the true density function (plotted left) corroborating our assertion that the power spectrum can increasingly, accurately serve as the gradient density estimator for large values of \( N \). In Figure 9.2.2 we plot \( \Delta_{\tau,N} \) for different values of \( \tau = \frac{2}{N} \) and find it to be linear ascertaining our Theorem 8.1.

9.2.2. Case study 2. As briefly mentioned above, the purpose of this case study is to verify that the lower bound on \( \tau = \frac{BL}{2N} \) is indeed its optimum value. To this end, we fixed \( N = N_0 = 65536 \) and computed the average summation error \( \Delta_{\tau,N_0} \) according to Eq 9.4 for varying values of \( \tau \), averaged over the preselected \( K = 255 \) fixed number of frequencies. The plot in Figure 9.3 displays the behavior of \( \Delta_{\tau,N_0} \) with \( \tau \). Note that the number of samples \( N_k \) within each neighborhood \( \mathcal{N}_\tilde{u}(\tilde{u}_k) \) does
Figure 9.1. Convergence with increasing $N$. Integral of (i) Left: True gradient density function, and (ii) Right: Estimated gradient density function.

Figure 9.2. Rate of convergence of integral error with decreasing (increasing) $\tau (N)$.

Figure 9.3. Rate of convergence of integral error with decreasing (increasing) $\tau (N)$.

not change as $N$ is held constant. However, the values of discrete power spectrum $P^{DT}(u_{l,k})$ at the frequencies $u_{l,k}$ varies with changing $\tau$. The following inference can be deduced from the profile of the graph in Figure 9.3 namely:

1. The error steadily decreases with $\tau$ as we move towards its lower bound.

2. Barring some computational inaccuracies (may be) due to finite precision representation of sinusoidal functions, the rate of decline is almost linear with $\tau (O(\tau))$ substantiating the concluding remarks of Lemma 8.3.

Appendix A. Proof of Lemmas. Below we provide the proofs for all the lemmas stated in this article.

A.1. Proof of Finiteness Lemma. Proof. We prove the result by contradiction. Observe that $A_u$ is a subset of the compact set $\Omega$. If $A_u$ is not finite, then by Theorem 2.37 in [4], $A_u$ has a limit point $x_0 \in \Omega$. Consider a sequence $\{x_n\}_{n=1}^{\infty}$, with each $x_n \in A_u$, converging to $x_0$. Since $s(x_n) = u, \forall n$, from the continuity of $s$ we get $s(x_0) = u$ and hence $x_0 \in A_u$. Since $u \in C$, $x_0 \notin \{0, L\}$. Additionally,

$$\lim_{n \to \infty} \frac{s(x_0) - s(x_n)}{x_0 - x_n} = 0 = s'(x_0) = S''(x_0)$$

implying that $x_0 \in B$ and $u \in C$ resulting in a contradiction. \[\square\]
A.2. Proof of Neighborhood Lemma. Proof. Observe that \( B \) is closed—and being a subset of \( \Omega \) is also compact—because if \( x_0 \) is a limit point of \( B \), from the continuity of \( S'' \) we have \( S''(x_0) = 0 \) and hence \( x_0 \in B \). Since \( s \) is continuous, the set \( C \) is also compact and hence \( \mathbb{R} - C \) is open. Then for \( u \notin C \), there exists an open neighborhood \( N_r(u) \) for some \( r > 0 \) around \( u \) such that \( N_r(u) \cap C = \emptyset \). By defining \( \alpha = \frac{r}{2} \), we get the required closed neighborhood \( N_\alpha \) containing \( u \).

Since \( S''(x) \neq 0, \forall x \in A_u \) and continuous, all the other points of this lemma follow directly from the inverse function theorem. As \( |A_u| \) is finite by Lemma [31], the neighborhood \( N_\alpha \) can be chosen independently of \( x \in A_u \) so that the points 1 and 3 are satisfied \( \forall x \in A_u \).

A.3. Proof of Density Lemma. Proof. Since the random variable \( X \) is assumed to have a uniform distribution on \( \Omega \), its density is given by \( f_X(x) = \frac{1}{\Omega} \) for every \( x \in \Omega \). Recall that the random variable \( Y \) is obtained via a random variable transformation from \( X \), using the function \( s \). Hence, its density function exists on \( \mathbb{R} - C \)—where we have banished the image (under \( s \)) of the measure zero set of points where \( S'' \) vanishes—and is given by Equation [3.3]. The reader may refer to [1] for a detailed explanation.

A.4. Proof of Lemma [41]. Proof. By Parseval’s theorem we have

\[
\frac{1}{N^2} \sum_{k=0}^{N-1} \left| F^D(w_k) \right|^2 = \delta \sum_{n=0}^{N-1} \left| \phi^D_n(y_n) \right|^2 = \frac{N\delta}{L} = 1. \tag{A.2}
\]

Noting that

\[
\sqrt{2\pi} \tau F^D_r(u_k) = F^D(w_k), \tag{A.3}
\]

the results follows immediately.

A.5. Proof of No-stationary-points Lemma. Proof. As \( T'(x; u) \neq 0, T(x; u) \) is strictly monotonic. Integrating by parts we get

\[
I_{\tau}(u) = -i\tau \left[ \frac{H_\delta(x) \exp \left( \frac{iT(x; u)}{\tau} \right)}{s(x) - u} \right]_{b_2}^{b_1} + i\tau \int_{b_1}^{b_2} \exp \left( \frac{iT(x; u)}{\tau} \right) [q_1(x; u) - q_2(x; u)] \, dx \tag{A.5}
\]

where

\[
q_1(x; u) = \frac{H_\delta'(x)}{s(x) - u} \quad \text{and} \quad q_2(x; u) = \frac{H_\delta(x) S''(x)}{[s(x) - u]^2}. \tag{A.6}
\]

We split the integral in [A.5] into three parts by diving at \( x = 0 \) and \( x = L \) where \( H_\delta' \) is discontinuous. As \( H_\delta'(x) = \frac{2}{\tau} \) and \( s(x) = s(0) \) between \( [b_1, 0] \), \( q_1(x; u) = \frac{2}{s(0) - u} \) and \( q_2(x, u) = 0 \) as \( S''(x) = 0 \). Recalling that \( \frac{2}{\pi} = \frac{\Omega}{\pi} \) we get

\[
\int_{b_1}^{0} \exp \left( \frac{iT(x; u)}{\tau} \right) q_1(x; u) \, dx = -i\frac{2CB}{\pi} \left[ \frac{\exp \left( i\frac{T(0; u)}{\tau} \right)}{\pi [s(0) - u]^2} - \exp \left( \frac{iT(b_1; u)}{\tau} \right) \right] \tag{A.7}
\]
On the portion \([L, b_2]\) where \(H_5'(x) = \frac{\omega}{\delta}, s(x) = s(L),\) and \(q_2(x, u) = 0\) we have

\[
\int_L^{b_2} \exp \left( \frac{iT(x; u)}{\tau} \right) q_1(x; u) \, dx = \frac{i2CB}{\pi [s(L) - u]} \left[ \exp \left( \frac{iT(b_2; u)}{\tau} \right) - \exp \left( \frac{iT(L; u)}{\tau} \right) \right]
\]  
(A.8)

We are left with the interval \([0, L]\) where \(H_5(x)\) being identically equal to 1, \(H_5' = 0\) and \(q_1(x; u)\) vanishes. Via integration by parts on the integral involving \(q_2(x; u)\) we find

\[
\int_0^L \exp \left( \frac{iT(x; u)}{\tau} \right) q_2(u, x) \, dx = -i\tau \left[ \frac{S''(x)}{\tau} \exp \left( \frac{iT(x; u)}{\tau} \right) \right]_0^L + i\tau \int_0^L \exp \left( \frac{iT(x; u)}{\tau} \right) \left[ \frac{S''(x)}{\tau} \right] dx
\]

\[
= O \left( \frac{\tau}{\xi^3} \right)
\]  
(A.11)

where we have used the premise that \(|s(x) - u| \geq \xi, \forall x \in [b_1, b_2]\. Using the results \((A.7), (A.11)\) and \((A.8)\) in \((A.5)\) we find

\[
I_{\tau}(u) = -i\tau \sum_{r=1}^2 (-1)^r \exp \left( \frac{iT(b_r; u)}{\tau} \right) \frac{H_3(b_r)}{s(b_r) - u}
\]

\[
+ \frac{2CB\tau}{\pi [s(0) - u]^2} \left[ \exp \left( \frac{iT(0; u)}{\tau} \right) - \exp \left( \frac{iT(b_1; u)}{\tau} \right) \right]
\]

\[
+ \frac{2CB\tau}{\pi [s(L) - u]^2} \left[ \exp \left( \frac{iT(L; u)}{\tau} \right) - \exp \left( \frac{iT(b_2; u)}{\tau} \right) \right]
\]

\[
+ O \left( \frac{\tau^2}{\xi^3} \right)
\]  
(A.15)

giving us a bound of \(I_{\tau}(u) = O(\tau)\).

\[\blacksquare\]

A.6. Proof of Lemma 8.2 Proof. Let \(p_m(u) = S(x_m(u)) - S(\kappa) - u(x_m(u) - \kappa) - \gamma \kappa.\) As \(s(x_m(u)) = u\) we get \(p'_m(u) = \kappa - x_m(u) \neq 0\) indicating that there are no stationary points. Defining

\[
q_m(u) = \frac{1}{[s(\kappa) - (u - \gamma)]^2 \sqrt{S''(x_m(u))(\kappa - x_m(u))}}
\]  
(A.16)

and integrating by parts we get

\[
I_{\tau} = -i\tau \left[ \exp \left( \frac{ip_m(u)}{\tau} \right) q_m(u) \right]_{\delta}^{\delta + \alpha_k} + i\tau \int_{\mathcal{N}_0(\delta)} \exp \left( \frac{ip_m(u)}{\tau} \right) q_m(u) \, du.
\]  
(A.17)
Knowing that \( \frac{d x_m(u)}{d u} = \frac{1}{S'(x_m(u))}, \) \( q'_m(u) \) can be evaluated to be

\[
q'_m(u) = \frac{2}{[s(\kappa) - (u - \gamma)]^4 |S''(x_m(u))| [\kappa - x_m(u)]^2} \\
+ \frac{1}{[s(\kappa) - (u - \gamma)]^4 |S''(x_m(u))| (S''(x_m(u))) [\kappa - x_m(u)]^2} \\
- \frac{2 [s(\kappa) - (u - \gamma)]^4 [S''(x_m(u))] \frac{3}{2} (S''(x_m(u))) [\kappa - x_m(u)]^2}
\]

We would like to emphasize the following inequality

\[
\kappa = \begin{cases} 
\rho_1; & |\kappa - x_m(u)| > |x_m(u)|; \\
\rho_2; & |\kappa - x_m(u)| > |L - x_m(u)|.
\end{cases}
\]

Furthermore, recall that \( s(\rho_1) = s(0) \) and \( s(\rho_2) = s(L) \) by construction and \( \gamma = 2BCl, l \in \mathbb{Z} \) does not depend on \( \delta \). Hence both \( q_m(u) \) and \( q'_m(u) \) in (A.19) respectively can be individually bounded independent of \( \delta \) (and also of \( \tau \)). The result then follows.

**A.7. Proof of Bound-on-Integrated-Error Lemma.** Proof. Note that \( N_{\alpha}(\tilde{u}) \) being a closed interval includes all the limit points. As \( \{s(0), s(L)\} \in \mathcal{C} \) we have \( \{s(0), s(L)\} \cap N_{\alpha}(\tilde{u}) = \emptyset \) by selection of \( N_{\alpha}(\tilde{u}) \) as per Lemma 6.1. Hence we could find \( \xi_1, \xi_2 > 0 \) such that \( |s(0) - u| \geq \xi_1 \) and \( |s(L) - u| \geq \xi_2, \forall u \in N_{\alpha}(\tilde{u}) \). As these distances are bounded away for zero, all the bounds obtained above for \( u = \tilde{u} \) can be extended \( \forall u \in N_{\alpha}(\tilde{u}) \).

Expressing the spatial locations \( x_m \) as a function of \( u \) using the inverse function \( x_m(u) = s^{(-1)}(u) \), consider the phase of cross term defined in (7.19), namely

\[
p_{m,t}(u) = S(x_m(u)) - S(x_t(u)) - u(x_m(u) - x_t(u)) + \theta_{m,t}(x_m(u), x_t(u))
\]

where \( x_m(u) \in N_{\alpha}(x_m), x_t(u) \in N_{\alpha}(x_t) \) and \( N_{\alpha}(x_m) \cap N_{\alpha}(x_t) = \emptyset \) as \( t \neq m \). Recall that \( \theta_{m,t}(x_m(u), x_t(u)) \) depends on the sign of \( S''(x(u)) \) around \( N_{\alpha}(x_m) \) and \( N_{\alpha}(x_t) \). The constancy of the sign \( S''(x_m(u)) \) in \( N_{\alpha}(x_m), \forall m \) by Lemma 3.2 removes the variability of \( \theta_{m,t}(x_m(u), x_t(u)) \) around the same region. Checking for the stationary condition while bearing in mind that \( s(x_m(u)) = u \) we see that

\[
(p_{m,t})'_{(u)} = x_t(u) - x_m(u) \neq 0
\]

implying the absence of any stationary points. Integrating by parts once and following along the lines of the proof of Lemma 6.2 given in Appendix A.6 we get

\[
\int_{N_{\alpha}(\tilde{u})} \chi_{m,t,\tau} (x_m(u), x_t(u), u) \ du = O(\tau).
\]

Apropos to the bounds in (6.10) and \( (7.10) \) respectively, the magnitude square of both the aliasing error and \( \epsilon_{3,\tau}(u) \) is \( O(\tau), \forall u \in N_{\alpha}(\tilde{u}) \). The latter is also guaranteed by our single choice of \( \lambda \) and \( \epsilon, \forall u \in N_{\alpha}(\tilde{u}) \) elucidated in Appendix B while computing the bound for \( \epsilon_{2,\tau}(\tilde{u}) \). By extending these bounds to the integral of these terms over
\( \mathcal{N}_\alpha(\hat{u}) \) we could conclude that
\[
\int_{\mathcal{N}_\alpha(\hat{u})} |\epsilon_{3,\tau}(u)|^2 \, du = O(\tau), \quad \text{and} \quad \tag{A.24}
\]
\[
\int_{\mathcal{N}_\alpha(\hat{u})} \left[ \sum_{l=-\infty, l \neq 0}^{\infty} F_\tau(u - \gamma_l) \right]^2 \, du = O(\tau). \tag{A.25}
\]

We leverage Lemma 8.2 to bound the integral of \( \epsilon_{4,\tau}(u) \) over \( \mathcal{N}_\alpha(\hat{u}) \). Firstly, observe that the expressions in (7.13) and (7.14) are akin to the definition of \( \zeta_\tau(u) \) in Lemma 8.2. Secondly, the remaining error term in (7.15) is \( O(\tau), \forall u \in \mathcal{N}_\alpha(\hat{u}) \). Furthermore, as the sign of \( S'(x_m(u)) \) is constant in \( \mathcal{N}_\alpha(\hat{u}) \), the \( \pm \frac{\tau}{\pi} \) factor in the phase does not vary its sign. Applying Lemma 8.2 we find
\[
\int_{\mathcal{N}_\alpha(\hat{u})} \epsilon_{4,\tau}(u) \, du = O(\tau). \tag{A.27}
\]

We are left with computing
\[
\int_{\mathcal{N}_\alpha(\hat{u})} F_\tau(u) \left( \sum_{l=-\infty, l \neq 0}^{\infty} F_\tau(u - \gamma_l) \right) \, du \tag{A.28}
\]
and the integral of its conjugate. Pursuant to the Lebesgue dominated convergence theorem we can switch the infinite summation and the integral allowing us to focus independently on \( F_\tau(u - \gamma_l) \). Firstly, as \( F_\tau(u) \) is a bounded function of \( u \), the term in (6.8) when multiplied with \( F_\tau(u) \) and integrated over \( \mathcal{N}_\alpha(\hat{u}) \) produces a factor that is \( O \left( \frac{\sqrt{\tau}}{|B(\|l\| - 1) + \beta|^2} \right) \). Secondly, recall that \( F_\tau(u) \) in (7.11) is composed of two terms where the error term \( \epsilon_{3,\tau}(u) \) is \( O(\sqrt{\tau}) \). The terms in (6.6) and (6.7) being \( O \left( \frac{\sqrt{\tau}}{|B(\|l\| - 1) + \beta|^2} \right) \), \( \forall u \in \mathcal{N}_\alpha(\hat{u}) \) when multiplied with \( \epsilon_{3,\tau}(u) \) and integrated results in an expression that is \( O \left( \frac{\tau}{|B(\|l\| - 1) + \beta|^2} \right) \). To bound the integration of the product of first (main) term on the right of (7.11) with the expressions in (6.6) and (6.7), we employ Lemma 8.2 and find it to be \( O \left( \frac{\tau \sqrt{\tau}}{|B(\|l\| - 1) + \beta|^2} \right) \). Coupling these results we have
\[
\int_{\mathcal{N}_\alpha(\hat{u})} F_\tau(u) \overline{F_\tau(u - \gamma_l)} \, du = O \left( \frac{\tau}{|B(\|l\| - 1) + \beta|^2} \right). \tag{A.29}
\]

The infinite summation then leads to
\[
\int_{\mathcal{N}_\alpha(\hat{u})} F_\tau(u) \left( \sum_{l=-\infty, l \neq 0}^{\infty} F_\tau(u - \gamma_l) \right) \, du = \sum_{l=-\infty, l \neq 0}^{\infty} \int_{\mathcal{N}_\alpha(\hat{u})} F_\tau(u) \overline{F_\tau(u - \gamma_l)} \, du \\
= O(\tau). \tag{A.30}
\]
Combining (A.23), (A.24), (A.25), (A.27), and (A.30), the proof follows.
Appendix B. Expression for the error $\epsilon_{2,t}(\tilde{u})$. Let

$$ \epsilon_{2,t}(\tilde{u}) = \sum_{l=1}^{M(\tilde{u})} \epsilon_{2,t,\tau}(\tilde{u}) + \dot{\epsilon}_{2,t,\tau}(\tilde{u}) \quad \text{(B.1)} $$

where $\epsilon_{2,t,\tau}(\tilde{u})$ and $\dot{\epsilon}_{2,t,\tau}(\tilde{u})$ are the stationary phase errors incurred while evaluating $K_{t,\tau}(\tilde{u})$ and $\dot{K}_{t,\tau}(\tilde{u})$ respectively. As before, let the finite set $\{x_l^M(\tilde{u})\}$ be the location of the stationary points for the given $\tilde{u}$. The Theorem 13.1 in Chapter 3 of [10] expresses $\epsilon_{2,t,\tau}(\tilde{u})$ as

$$ \epsilon_{2,t,\tau}(\tilde{u}) = -\epsilon_{2,1,t,\tau}(\tilde{u}) + \epsilon_{2,2,t,\tau}(\tilde{u}) \quad \text{(B.2)} $$

where

$$ \epsilon_{2,1,t,\tau}(\tilde{u}) = \exp\left(\frac{iT(x_l;\tilde{u})}{\tau}\right) \int_{\eta}^{\infty} \exp\left(\frac{iv}{\tau}\right) v^{-1/2} dv \quad \text{and} \quad \epsilon_{2,2,t,\tau}(\tilde{u}) = \exp\left(\frac{iT(x_l;\tilde{u})}{\tau}\right) \int_{0}^{\eta} \exp\left(\frac{iv}{\tau}\right) Q(v;\tilde{u}) dv. \quad \text{(B.3)} $$

Here

$$ v = T(x;\tilde{u}) - T(x_l;\tilde{u}), \quad \eta = T(c_l;\tilde{u}) - T(x_l;\tilde{u}) \quad \text{and} \quad Q(v;\tilde{u}) = \frac{1}{s(x(v)) - \tilde{u}} - \frac{1}{\sqrt{2|S''(x_l)|v^2}}. \quad \text{(B.4)} $$

As in [11], for small values of $v$, $x - x_l$ can be expanded—as a function of $v$—in asymptotic series of the form

$$ x - x_l \sim \sum_{l=0}^{\infty} d_lv^{(l+1)/2} \quad \text{(B.5)} $$

where the coefficients $d_l$ may be obtained by following the standard procedures for reverting the series. In particular, $d_0 = \frac{2}{|S''(x_l)|}$. The other constants $d_1, d_2, \cdots$ are a function of second and higher derivatives of $T$ around the stationary point $x_l$. Hence they depend only on the nature of the function $S$ around $x_l$ and not directly on the frequency $\tilde{u}$. The indirect dependency on $\tilde{u}$ is only through its corresponding stationary point $x_l$ as elucidated below. Differentiating with respect to $v$ we get

$$ \frac{1}{s(x(v)) - \tilde{u}} = \frac{dx}{dv} \sim \sum_{l=0}^{\infty} \left(\frac{l+1}{2}\right) d_lv^{(l-1)/2}. \quad \text{(B.6)} $$
Letting \( q_l = \frac{i\pi d_l}{2} \), \( Q(v; \tilde{u}) \) can be seen to admit a series expansion,

\[
Q(v; \tilde{u}) \sim a_1 + a_2 v^{\frac{1}{2}} + a_3 v + a_4 v^{\frac{3}{2}} + \cdots ,
\]

as \( v \to 0^+ \). It is also shown in [11] that

\[
\int_0^n |Q'(v; \tilde{u})| \, dv < \infty.
\]

Computing [B.4] by integration by parts and noticing that \( \lim_{v \to 0^+} Q(v; \tilde{u}) = a_1 \) we get

\[
\epsilon_{2,2,t,\tau}(\tilde{u}) = -i\tau \exp \left( \frac{iT(c_i; \tilde{u})}{\tau} \right) \left[ \frac{1}{s(c_i) - \tilde{u}} - \frac{1}{\sqrt{2|S''(x_i)|} [T(c_i; \tilde{u}) - T(x_i; \tilde{u})]^{\frac{1}{2}}} \right]
\]

\[
+ a_1 i\tau \exp \left( \frac{iT(x_i; \tilde{u})}{\tau} \right)
\]

\[
+ i\tau \exp \left( \frac{iT(x_i; \tilde{u})}{\tau} \right) \int_0^n \exp \left( \frac{iv}{\tau} \right) Q'(v; \tilde{u}) \, dv.
\]

The finiteness of [B.12] assures that [B.15] is bounded. Our next task is to capture this bound as a function of \( \tau \).

Based on the series expression for \( Q(v; \tilde{u}) \) in [B.11] we see that \( Q'(v; \tilde{u}) = O \left( v^{\frac{1}{2}} \right) \) and \( Q''(v; \tilde{u}) = O \left( v^{\frac{1}{2}} \right) \) independent of \( \tau \) as \( v \to 0^+ \). Then there exist constants \( \lambda > 0 \) and \( \epsilon > 0 \)—independent of \( \tau \)—such that \( |Q'(v; \tilde{u})| \leq \epsilon v^{\frac{1}{2}} \) and \( |Q''(v; \tilde{u})| \leq \epsilon v^{\frac{3}{2}} \) when \( v \leq \lambda \). In the subsequent paragraph we would like to add an important technical note on the choice of \( \lambda \) and \( \epsilon \). The reader may choose to skip the next paragraph without loss of continuity but bear in mind to refer to it when we discuss the proof of Lemma 8.3 in Appendix A.7.

As mentioned above, the constants \( d_1, d_2, \cdots \) in [B.9] depend only on the property of \( S \) around \( x_i \) and not directly on \( \tilde{u} \). However, as \( \tilde{u} \) is varied (say) over a small compact interval \( N_\theta(\tilde{u}) \) (which we soon require in Lemma 8.3), the corresponding stationary point \( x_i(\tilde{u}) \), now explicitly expressed as the function of \( \tilde{u} \), moves around in the compact interval \( s^{(i-1)}(N_\theta(\tilde{u})) = N_\theta(x_i) \) influencing the constants in [B.9] and creating an indirect dependency of them on \( \tilde{u} \). It can be verified from [9] that the constant \( d_2 \) (and thereby \( a_2 \)) which decides the aforementioned growth rate of \( Q'(v; \tilde{u}) \) and \( Q''(v; \tilde{u}) \) as \( v \to 0^+ \) varies \( \propto \frac{1}{|S''(x_i(\tilde{u}))|^{\frac{1}{2}}} \) with \( S''(x_i(\tilde{u})) \) being the only derivative of \( S \) appearing in the denominator. As we proceed, we will be soon see that our choice of neighborhood \( N_\theta(\tilde{u}) \) will be pursuant to Lemma 3.2 where in \( N_\theta(x_i), S''(x_i(\tilde{u})) \neq 0 \) and bounded away from zero. This in turn enable us to choose a single value for each the constants \( \lambda \) and \( \epsilon \) for all \( \tilde{u} \in N_\theta(\tilde{u}) \).

Since we are interested in \( N \to \infty \) or equivalently \( \tau \to 0 \), let \( \tau \) be such that \( \tau \leq \lambda \). Our subsequent steps closely follow Theorem 12.3 in Chapter 3 of [10]. Lack of a strong constraint—\( Q'(v; \tilde{u}) = O \left( v^{\frac{1}{2}} \right) \) and \( Q''(v; \tilde{u}) = O \left( v^{\frac{3}{2}} \right) \)—precludes us from directly applying Theorem 12.3 to prove a stronger assertion. However, the weaker constraints on \( Q' \) and \( Q'' \) (\( O \) instead of \( o \)) leads to an equivalently weak but a sufficiently desirable result.
Diving the integral \((B.15)\) at \(v = \tau\) we get
\[
\left| \int_0^\tau \exp \left( \frac{iv}{\tau} \right) Q'(v; \tilde{u}) \, dv \right| \leq \epsilon \int_0^\tau \frac{dv}{v} \, dv = 2\epsilon \sqrt{\tau}. \tag{B.16}
\]

Using integration by parts we find
\[
\eta \hat{\lambda} \exp \left( \frac{iv}{\tau} \right) Q'(v; \tilde{u}) \, dv = \frac{i\eta}{\tau} \left[ \exp \left( \frac{in}{\tau} \right) Q'(n; \tilde{u}) - \exp \left( \frac{i\tau}{\tau} \right) Q' \left( \tau; \tilde{u} \right) \right] \tag{B.18}
\]
\[
- \frac{i}{\tau} \int_0^\tau \exp \left( \frac{iv}{\tau} \right) Q''(v; \tilde{u}) \, dv \tag{B.19}
\]
\[
- \frac{i}{\tau} \int_0^\tau \exp \left( \frac{iv}{\tau} \right) Q''(v; \tilde{u}) \, dv. \tag{B.20}
\]

Recalling that \(Q''(v; \hat{u}) \leq \epsilon \tau^{-3}\) when \(v \leq \lambda\) we further have
\[
\left| \frac{i}{\tau} \int_0^\tau \exp \left( \frac{iv}{\tau} \right) Q''(v; \tilde{u}) \, dv \right| \leq \tau \int_0^\tau \epsilon v^{-2} \, dv = 2\epsilon \sqrt{\tau} \tag{B.21}
\]
and as \(\lambda\) is independent of \(\tau\) and \(Q''(v; \hat{u})\) is bounded away from zero for \(v \in [\lambda, \eta]\) we get
\[
\left| \frac{i}{\tau} \int_0^\tau \exp \left( \frac{iv}{\tau} \right) Q''(v; \tilde{u}) \, dv \right| \leq \tau \int_0^\tau |Q''(v; \tilde{u})| \, dv = O(\tau). \tag{B.22}
\]

Using the bound \(|Q'(\tau; \tilde{u})| \leq \epsilon \tau^{-\frac{1}{2}}\) in \((B.18)\) and combining \((B.16), (B.18), (B.21)\) and \((B.22)\) we arrive at
\[
i \tau \int_0^\eta \exp \left( \frac{iv}{\tau} \right) Q'(v; \tilde{u}) \, dv = O(\tau \sqrt{\tau}). \tag{B.23}
\]
as \(\tau \to 0\) \((N \to \infty)\).

Plugging \((B.23)\) in \((B.13)\) and subtracting \((B.8)\) gives us
\[
\epsilon_{2,1,\tau}(\tilde{u}) = i\tau a_1 \exp \left( \frac{iT(x_1; \tilde{u})}{\tau} \right) - i\tau \frac{\exp \left( \frac{iT(c_1; \tilde{u})}{\tau} \right)}{s(c_1) - \tilde{u}} + O(\tau^2) + O(\tau \sqrt{\tau}). \tag{B.24}
\]

We would like to add the following important remark about the first term on the right side of \((B.24)\). The computation of the error \(\hat{\epsilon}_{2,1,\tau}(\tilde{u})\) along similar lines on the interval \([c_{l-1}, x_l]\) will produce the exact expression but with a negative sign. These
two terms cancel with each other leaving no expression in $\epsilon_{2,\tau}(u)$ containing $T(x_t; \tilde{u})$ in the phase. The total stationary phase error at the critical point $x_t$ equals

$$
\epsilon_{2,t,\tau} (\tilde{u}) + \tilde{\epsilon}_{2,t,\tau} (\tilde{u}) = \frac{i\tau}{s(c_{t-1}) - \tilde{u}} \exp \left( \frac{iT(c_{t-1}; \tilde{u})}{\tau} \right) - \frac{i\tau}{s(c_t) - \tilde{u}} \exp \left( \frac{iT(c_t; \tilde{u})}{\tau} \right)
$$

(B.26)

$$+ O(\tau \sqrt{\tau}).
$$

(B.27)

Being a telescopic series, the adjacent terms cancel each other when summed resulting in

$$
\epsilon_{2,\tau}(u) = \frac{i\tau}{s(c_0) - \tilde{u}} \exp \left( \frac{iT(c_0; \tilde{u})}{\tau} \right) - \frac{i\tau}{s(c_M(\tilde{u})) - \tilde{u}} \exp \left( \frac{iT(c_M(\tilde{u}); \tilde{u})}{\tau} \right)
$$

(B.28)

$$+ O(\tau \sqrt{\tau}).
$$

(B.29)

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