The harmony in the Kepler and related problems

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Abstract
The technique of reduction of order developed by Nucci (J Math
Phys 37 (1996) 1772-1775) is used to produce nonlocal symmetries
additional to those reported by Krause (J Math Phys 35 (1994) 5734-
5748) in his study of the complete symmetry group of the Kepler Pro-
b lem. The technique is shown to be applicable to related problems
containing a drag term which have been used to model the motion
of low altitude satellites in the Earth’s atmosphere and further gener-
alisations. A consequence of the application of this technique is the
demonstration of the group theoretical relationship between the simple
harmonic oscillator and the Kepler and related problems.

1 Introduction
In a paper of a few years ago Krause [15] introduced a new concept into
the study of the symmetries of ordinary differential equations. He called
this a complete symmetry group and defined it by adding two properties
to the definition of a Lie symmetry group. These were that the manifol
of solutions is an homogeneous space of the group and the group is specific
to the system, ie no other system admits it. This definition required the
introduction of a new type of symmetry defined by

\[ Y = \left[ \int \xi(t,x_1,\ldots,x_N)dt \right] \partial_t + \sum_{i=1}^{N} \eta_i(t,x_1,\ldots,x_N) \partial_{x_i}. \] (1.1)

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This definition of a symmetry differs from that of a Lie point symmetry due to the presence of the integral as the coefficient function of $\partial_t$.

As an illustration of the concept of a complete symmetry group Krause used the Kepler problem and obtained three symmetries of the type of (1.1). He claimed that these three symmetries could not be obtained by means of the standard Lie point symmetry analysis. Naturally it was not long before this claim was shown by Nucci [25] to be incorrect in the case of an autonomous system. In the case of the Kepler problem, an autonomous system, one of the dependent variables can be taken to be the new independent variable and the order of the system be reduced by one. An analysis of the reduced system for Lie point symmetries leads to results different from the analysis of the original system. In particular the three additional nonlocal symmetries obtained by Krause followed from point symmetries of the reduced system.

One of the fundamental problems of mechanics is that of the Kepler problem which describes the interaction of two point particles with an inverse square law of attraction. It is well-known that this problem possesses the first integrals of the conservation of the scalar energy, the vector of angular momentum and vectors in the plane of the orbit known as Hamilton’s vector [10] and the Laplace-Runge-Lenz vector [3, 11, 1, 16, 27, 22]. The invariance Lie algebra of the first integrals under the operation of taking the Poisson Bracket is $so(4)$ (in the case of negative energy) and the Lie algebra of the five Lie point symmetries of the equation of motion $A_2 \oplus so(3)$. The algebra of the complete symmetry group has not been given. The elements of the five-dimensional algebra are

\begin{align*}
X_1 &= \partial_t \\
X_2 &= t\partial_t + \frac{2}{3}r\partial_r \\
X_3 &= x_2 \partial_{x_3} - x_3 \partial_{x_2} \\
X_4 &= x_3 \partial_{x_1} - x_1 \partial_{x_3} \\
X_5 &= x_1 \partial_{x_2} - x_2 \partial_{x_1}.
\end{align*}

(1.2)

The additional three nonlocal symmetries provided by Krause are

\begin{align*}
Y_1 &= 2 \left( \int x_1 dt \right) \partial_t + x_1 r \partial_r \\
Y_2 &= 2 \left( \int x_2 dt \right) \partial_t + x_2 r \partial_r \\
Y_1 &= 2 \left( \int x_3 dt \right) \partial_t + x_3 r \partial_r
\end{align*}

(1.3)

in which $r^2 = x_1^2 + x_2^2 + x_3^2$. 

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There have been several other systems, generalisations of the Kepler problem, which have been shown to have a similar set of conserved quantities [7, 8, 9, 20]. Just as the Laplace-Runge-Lenz vector provides a direct route to the equation of the orbit of the classical Kepler problem, the corresponding vectors of the generalised Kepler problems provide the same direct route to the equations of their orbits. The Lie point symmetry associated with the Laplace-Runge-Lenz vector of the classical Kepler problem is the rescaling symmetry, $X_2$. The generalisations of the Laplace-Runge-Lenz vector do not always have such an associated Lie point symmetry. In the case of the equation of motion

$$\ddot{r} - \left(\frac{\dot{g}}{2g} + \frac{3\dot{r}}{2r}\right)\dot{r} + \mu gr = 0,$$

(1.4)

which is a variation of the model proposed by Danby [2, 14, 23, 19] for the motion of a satellite in a low altitude orbit subject to atmospheric drag, \textit{viz}

$$\ddot{r} + \frac{\alpha\dot{r}}{r^2} + \frac{\mu r}{r^3} = 0,$$

(1.5)

and for which the generalisation of the Laplace-Runge-Lenz vector is

$$J = \dot{r} \times \hat{L} + \frac{\mu}{A^2}\dot{r},$$

(1.6)

where $L$ is the magnitude of the angular momentum, $\mathbf{L}$, and $A$ is a constant of the motion defined through

$$L = A \left(gr^3\right)^{\frac{1}{2}},$$

(1.7)

Pillay et al [26] showed that, instead of the Lie point symmetry for the classical Kepler problem, $X_2$, the nonlocal symmetry

$$G = -\frac{1}{2} \left[ \int \frac{g'r}{g} \, dt \right] \partial_t + r \partial_r,$$

(1.8)

was the corresponding associated Lie symmetry.

In this paper we intend to demonstrate the existence of far more nonlocal symmetries for the classical Kepler problem than were reported by Krause. We derive them as Lie point symmetries of a reduced system using the method of Nucci [24]. By deriving these additional nonlocal symmetries in this way we are able to say something definite about the expanded symmetry.
group. The essence of the method of Nucci is to reduce the order of the system by using the symmetry, $X_1$, which is a statement of the autonomy of the system. We recall that in a reduction of order the symmetry $Z$ which does not have the property $[X_1, Z] = \lambda X_1$ becomes an exponential nonlocal symmetry [4]. By an exponential nonlocal symmetry we mean one of the form

$$G = \exp \left[ \int f dt \right] (\tau \partial_t + \eta_i \partial_{x_i})$$

in which without a knowledge of the solution of the differential equation $f$ is not an exact derivative and $\tau$ and the $\eta_i$ are functions only of $t$, the $x_i$ and their derivatives. In this case we have nonlocal symmetries becoming local on the reduction of order. Consequently we know that the symmetries of the reduced equation quite possibly have zero Lie Bracket with $X_1$ and this will enable us to construct the algebra. Further we shall show that these considerations which are applicable to the classical Kepler problem can be extended to the generalisations such as the one in (1.4).

In the next section we review the reduction procedure of Nucci [25] and in Section 3 we apply it to the classical Kepler problem and see that there is a certain delicacy in the choice of the new independent variable. For the sake of simplicity we work in two dimensions. In Section 4 we make some observations about these symmetries, the route to further simplification and the algebra. In Section 5 we obtain the results for Danby problem [2], in Section 6 those for the generalised problem represented by (1.4), in Section 7 the symmetries for another generalisation in which the force is not only not central but is also angle dependent and in Section 8 we present our conclusions and make some pertinent observations about going to the full three dimensions.

## 2 The method of reduction of order

Consider the system of $N$ second order ordinary differential equations given by

$$\ddot{x}_i = f_i(x, \dot{x}), \quad i = 1, N$$

in which $t$ is the independent variable and $x_i, i = 1, N$ the $N$ dependent variables. These equations may be considered as equations from Newtonian mechanics, which was Krause’s approach, but there is no necessity for that to be the case. There is also no necessity for the dependent variables to represent cartesian coordinates. Indeed there is no need for the system to be
the second order. It just so happens that many of the equations which arise in practice have their origins in Newton's Second Law and so are second order equations. There is no requirement that the system be autonomous. In the case of a nonautonomous system we can apply the standard procedure of introducing a new variable \( x_{N+1} = t \) and an additional first order equation \( \dot{x}_{N+1} = 1 \) so that the system becomes formally autonomous. In our discussion we confine our attention to autonomous systems. We reduce the system (2.1) to a \( 2N \)-dimensional first order system by means of the change of variables

\[
\begin{align*}
w_1 &= x_1 \\
w_2 &= x_2 \\
& \vdots \\
w_{N-1} &= x_{N-1} \\
w_N &= x_N \\
w_{N+1} &= \dot{x}_1 \\
w_{N+2} &= \dot{x}_2 \\
& \vdots \\
w_{2N-1} &= \dot{x}_{N-1} \\
w_{2N} &= \dot{x}_N
\end{align*}
\] (2.2)

so that the system (2.1) becomes

\[
\dot{w}_i = g_i(x, w), \quad i = 1, 2N, 
\] (2.3)

where \( g_i = w_{N+i} \) for \( i = 1, N \) and \( g_i = f_i \) for \( i = N + 1, 2N \).

In the first step of the reduction of the original system (2.1) we simply follow the conventional method used to reduce a higher order system to a first order system. Any optimisation is performed in the further selection of the final variables. This selection may be motivated by the existence of a known first integral, such as angular momentum, or some specific symmetry in the original system (2.1).

We choose one of the variables \( w_i \) to be the new independent variable \( y \). For the purpose of the development here we can make the identification \( w_N = y \). By taking the quotients of the first order equations of the remaining members of the set (2.2) with (2.3N) we obtain the \((2N - 1)\)-dimensional system

\[
\frac{dw_i}{dw_N} = \frac{g_i}{g_N} = \frac{g_i}{w_{2N}}, \quad i = 1, \ldots, N - 1, N + 1, \ldots, 2N. 
\] (2.4)

We do not attempt to calculate the Lie point symmetries of the system (2.4) because the Lie point symmetries of a first order system are generalised symmetries and one has to impose some \textit{Ansatz} on the form of the symmetry. Rather we select \( n \leq N - 1 \) of the variables to be the new dependent variables and rewrite the system (2.4) as a system of \( n \) second order
equations plus $2(N - n) - 1$ first order equations. The selection of the new dependent variables is dictated by a number of considerations. The first and foremost is that we must be able to eliminate the unwanted variables from the system (2.4). After this condition has been satisfied we may look to seek variables which reflect some symmetry of the system, for example an ignorable coordinate such as the azimuthal angle in a central force problem.

After the symmetries have been calculated, they can now be translated back to symmetries of the original system as follows. Suppose that the symmetry in the original variables is given by

$$G = \tau \partial_t + \eta_i \partial_{x_i}.$$ (2.5)

The symmetry $G$ is first extended and then rewritten in terms of the new coordinates as follows

$$G^{[1]} = \tau \partial_t + \eta_i \partial_{x_i} + (\dot{\eta}_i - \dot{x}_i \dot{\tau}) \partial_{x_i}$$
$$= \tau \partial_t + \zeta_i \partial_{w_i}$$
$$= \sigma \partial_y + \xi_i \partial_{u_i},$$ (2.6)

where in the first line the summation is from 1 to $N$, in the second from 1 to $N - 1$ and $N + 1$ to $2N$ and in the third over the number of dependent variables $u_i$ (the number cannot be fixed in advance without a knowledge of the specific system); $\zeta_i = \eta_i$ for $i = 1, N - 1$ and $\zeta_i = \dot{\eta}_i - \dot{x}_i \dot{\tau}$ for $i = N + 1, 2N$; $\sigma = \eta_N$; $\xi_i = \zeta_j \partial u_i / \partial w_j$. The only way that $\tau$ appears in the symmetries of the reduced system is through its derivative with respect to time. If the non-locality in the original system occurs as a simple integral in $\tau$ of a function of the original dependent variables, $x_i$, this will be passed to the reduced system as a function of the new variables. When the point symmetries of the reduced system are computed, the form which the symmetries take in the original system can be determined from (2.6). Since $\tau$ is determined as its derivative with respect to time, the symmetry of the original system must necessarily be nonlocal unless the derivative is an exact differential. We note in passing that there is no inherent restriction on the nature of the symmetries. They could equally be contact or generalised symmetries and the same considerations would apply. The only requirement is that $\tau$ be a simple integral, not that the integrand be a point function. However, for the purposes of this paper we confine our attention to point symmetries and the integrand in $\tau$ to a point function.

In addition to the nonlocal symmetries which may be collected by this procedure the reduced system will have as point symmetries those symmetries of the original system which have the correct Lie bracket with the
symmetry $\partial_t$ which is at the basis of the reduction of order outlined above. Thus, if $G$ is a symmetry of the original system and

$$[G, \partial_t] = \lambda \partial_t, \quad (2.7)$$

$G$ will, when expressed in the appropriate coordinates, be a point symmetry of the reduced system whereas, if

$$[G, \partial_t] \neq \lambda \partial_t, \quad (2.8)$$

$G$ will not be a point symmetry of the original system but an exponential nonlocal symmetry [4]. Consequently there is the potential for a loss of symmetry in the reduction process just as there is the hope of an increase in the total number of symmetries, both point and nonlocal, known for the original system.

3 Lie point symmetries of the reduced Kepler problem

The Lagrangian for the two-dimensional Kepler problem is

$$L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{\mu}{r}, \quad (3.1)$$

in plane polar coordinates and the two equations of motion are

$$\ddot{r} - r \dot{\theta}^2 = -\frac{\mu}{r^2}; \quad (3.2)$$

$$r \ddot{\theta} + 2 \dot{r} \dot{\theta} = 0. \quad (3.3)$$

We introduce the new variables and their time derivatives

$$\begin{align*}
w_1 &= r & \dot{w}_1 &= \dot{w}_3 \\
w_2 &= \theta & \dot{w}_2 &= \dot{w}_4 \\
w_3 &= \dot{r} & \dot{w}_3 &= w_1 w_4^2 - \frac{\mu}{w_1^2} \\
w_4 &= \dot{\theta} & \dot{w}_4 &= -\frac{2w_3 w_4}{w_1}.
\end{align*} \quad (3.4)$$

In accordance with the development in the previous section we select $w_2$ to be the new independent variable $y$. The left side of (3.4) leads to the
reduced system

\[
\frac{dw_1}{dy} = \frac{w_3}{w_4} \quad (3.5)
\]

\[
\frac{dw_3}{dy} = w_1 w_4 - \frac{\mu}{w_1^2 w_4} \quad (3.6)
\]

\[
\frac{dw_4}{dy} = -\frac{2w_3}{w_1}. \quad (3.7)
\]

From (3.5) we have \( w_3 = w_4 w_1' \), where the prime denotes differentiation with respect to the new independent variable, \( y \), and we replace (3.7) by

\[
\frac{dw_4}{dy} = -\frac{2w_4 w_1'}{w_1}. \quad (3.8)
\]

In (3.6) we replace \( w_3 \) by \( w_4 w_1' \) to obtain

\[
w_4 w_1'' - \frac{w_4 w_1'}{w_1} = w_1 w_4 - \frac{\mu}{w_1^2 w_4}. \quad (3.9)
\]

By our replacement of \( w_3 \) we have not precisely decided that the variables \( u_1 \) and \( u_2 \) are to be \( w_1 \) and \( w_4 \). If we do make this identification, we obtain a system of two equations, one of the second order and one of the first order, viz

\[
u_1'' = 2 \frac{w_1^2}{u_1} + u_1 - \frac{\mu}{w_1^2 u_2} \quad (3.10)
\]

\[
u_2' = -2 \frac{u_1' u_2}{u_1}. \quad (3.11)
\]

We observe that (3.11) is trivially integrated to give \( u_1^2 u_2 \) is a constant. (This is a consequence, naturally, of the symmetry \( \partial_\theta \) of the original system (3.2) and (3.3) which is a reflection of the fact that \( \theta \) is an ignorable coordinate.) Consequently we may just as well define our new variables to be

\[
u_1 = w_1 \quad \text{and} \quad \tilde{u}_2 = u_1^2 u_2 \quad (3.12)
\]

so that the system of equations we are to consider is

\[
u_1'' = 2 \frac{u_1^2}{u_1} + u_1 - \frac{\mu u_1^2}{u_2^2}
\]

\[
\tilde{u}_2' = 0. \quad (3.13)
\]
Hereafter we drop the tilde.

We calculate the Lie point symmetries of the system (3.13) using the well-known interactive program developed by Nucci [24] and obtain the symmetries

\begin{align*}
X_1 &= \partial_y \\
X_2 &= 2u_1 \partial u_1 + u_2 \partial u_2 \\
X_3 &= u_1 \left( \mu u_1 - u_2 \right) \partial u_1 \\
X_4 &= u_1^2 \cos y \partial u_1 \\
X_5 &= -u_1^2 \sin y \partial u_1 \\
X_6 &= \left( \mu u_1 - u_2 \right) \cos y \partial y + u_1 \left( 2\mu u_1 - u_2 \right) \sin y \partial u_1 \\
X_7 &= -\left( \mu u_1 - u_2 \right) \sin y \partial y + u_1 \left( 2\mu u_1 - u_2 \right) \cos y \partial u_1 \\
X_8 &= u_1^2 \cos 2y \partial y - u_1 \left( \mu u_1 - u_2 \right) \sin 2y \partial u_1 \\
X_9 &= u_1^2 \sin 2y \partial y + u_1 \left( \mu u_1 - u_2 \right) \cos 2y \partial u_1.
\end{align*}

(3.14)

Some of these symmetries are readily identified, but not many of them. Clearly \(X_1\) represents the rotational invariance of the system and constitutes the subalgebra, \(so(2)\), of the original system of equations. In \(X_2\) we recognise the rescaling symmetry closely associated with the Laplace-Runge-Lenz vector.

The other symmetries are not so easy to identify without making some calculation. To obtain the form of the symmetries in the original coordinates we must make use of (2.6). If we write the symmetry in the original coordinates as

\[ G = \tau \partial t + \eta \partial r + \zeta \partial \theta, \]

(3.15)

in terms of the new variables, the symmetry has the form

\[ \tilde{G} = \zeta \partial y + \eta \partial u_1 + \left[ \dot{\zeta} + \left( \frac{2\eta}{r} - \dot{\tau} \right) \dot{\theta} \right] r^2 \partial u_2, \]

(3.16)

where the coefficient functions are all expressed in terms of the original variables. The calculations are not particularly interesting and we simply list the symmetries. They are

\begin{align*}
X_1 &= \partial \theta \\
X_2 &= 3t \partial t + 2r \partial r
\end{align*}
so that we see that we lost only the time translation symmetry in the reduction of order and so gained six additional symmetries.

4 The Lie point symmetries of the reduced Kepler problem: further considerations

Further interpretation of the Lie point symmetries of the reduced generalised Kepler problem is facilitated by the redefinition of the symmetries given in (3.14). We do not alter the definitions of the first three symmetries, but we shall include them in this new listing for the sake of completeness. We now have the set of symmetries

\[
\begin{align*}
X_1 & = \partial_y \\
X_2 & = 2u_1 \partial_{u_1} + u_2 \partial_{u_2} \\
X_3 & = u_1 \left( \mu u_1 - u_2^2 \right) \partial_{u_1} \\
X_{4\pm} & = X_4 \pm iX_5 = e^{\pm iy} u_1^2 \partial_{u_1} \\
X_{5\pm} & = X_6 \mp iX_7 = e^{\pm iy} \left[ \left( \mu u_1 - u_2^2 \right) \partial_y \mp u_1 \left( 2\mu u_1 - u_2^2 \right) \partial_{u_1} \right] \\
X_{6\pm} & = X_8 \pm iX_9 = e^{\pm 2iy} \left[ u_1^2 \partial_y \pm u_1 \left( \mu u_1 - u_2^2 \right) \partial_{u_1} \right].
\end{align*}
\] (4.1)
We may search for the first integrals/invariants associated with each of these 
symmetries in the usual way. By way of concrete example we take $X_{4+}$. The 
invariants of the first extension of $X_{4+}$, viz

$$X_{4+}^{[1]} = e^{iy} \left[ u_1^2 \partial_{u_1} + \left( 2u_1 u'_1 + iu_1^2 \right) \partial_{u'_1} \right], \quad (4.2)$$

are found from the associated Lagrange’s system

$$\frac{dy}{0} = \frac{du_1}{u_1^2} = \frac{du_2}{0} = \frac{du'_1}{2u_1 u'_1 + iu_1^2} \quad (4.3)$$

and are

$$\alpha = y, \quad \beta = u_2 \quad \text{and} \quad \gamma = \frac{u'_1 + iu_1}{u_1^2}, \quad (4.4)$$

where the first two are by inspection and the third comes from the solution of the second and fourth of $(4.3)$. The integral/invariant is a function of these three arguments and is found by demanding that the total derivative with respect to $y$ be zero when the differential equations $(3.13)$ are taken into account. We obtain the associated Lagrange’s system

$$\frac{d\alpha}{1} = \frac{d\beta}{0} = \frac{d\gamma}{-i\gamma - \frac{\mu}{\beta^2}} \quad (4.5)$$

which gives $\beta$ as one of the characteristics and

$$\omega_\pm = e^{iy} \left( \frac{u'_1 + iu_1}{u_1^2} - \frac{i\mu}{u_2^2} \right) \quad (4.6)$$

as the second characteristic. Both $\beta$ and $\omega$ are first integrals/invariants of the system $(3.13)$. Naturally we recognise the former as the angular momentum.

If we perform the same calculation with $X_{4-}$, we obtain a similar result, viz

$$\omega_- = e^{-iy} \left( \frac{u'_1 - iu_1}{u_1^2} + \frac{i\mu}{u_2^2} \right). \quad (4.7)$$

The two can be combined into one convenient expression given by

$$J_\pm = e^{\pm iy} \left( v' \mp iv \right), \quad v = \mu - \frac{u_2^2}{u_1}, \quad (4.8)$$

where we have made use of the constancy of $u_2$ to write $J_\pm = u_2^2 \omega_\pm$. For the system $(3.13)$ $J_\pm$ are two invariants and for the original system, $(3.1)$ and
the two components of the first integral known as the Laplace-Runge-Lenz vector, written in complex form.

In (4.8) we introduced a new variable \( v \). If we use this variable instead of \( u_1 \) in the system (3.13), we obtain the system

\[
\begin{align*}
v'' + v &= 0 \\
u_2' &= 0
\end{align*}
\]

so that the natural variable which arises from the invariants of \( X_{4\pm} \) is a variable which further simplifies the reduced system. This is an interesting phenomenon for we are obtaining natural variables as we progress through the process of determining the symmetries of the reduced equation. It behooves us to rewrite the symmetries in terms of this new variable. We find that

\[
\begin{align*}
X_1 &= \partial_{u_2} \\
X_2 &= \partial_y \\
X_3 &= v\partial_v \text{ mod}(u_2^2) \\
X_{4\pm} &= e^{\pm iy\partial_v} \text{ mod}(u_2^2) \\
X_{6\pm} &= e^{\pm 2iy}[\partial_y \pm iv\partial_v] \text{ mod}(u_2^2) \\
X_{8\pm} &= e^{\pm iy}[v\partial_y \pm iv^2\partial_v] \text{ mod}(u_2^2).
\end{align*}
\]

which is certainly a simpler appearance.

In the simpler form presented in (4.11) we see that the calculation of the integrals/invariants is simpler. For example, if we take \( X_{6\pm} \), the first set of characteristics comes from the solutions of the associated Lagrange’s system

\[
\frac{dy}{1} = \frac{dv}{iv} = \frac{dv'}{-iv' - 2v}.
\]

The characteristics are

\[
\alpha = ve^{\mp iy} \quad \text{and} \quad \beta = vv' \mp iv^2.
\]

The condition for the function to be an invariant (in this case not a first integral of the reduced system since the independent variable is explicitly present) is that these two characteristics satisfy the first order equation

\[
\frac{d\alpha}{\alpha} = \frac{d\beta}{\beta}.
\]
whence the invariants are given by

\[ I_{\pm} = \frac{\beta}{\alpha} = (v' \mp iv)e^{\pm iy} \]  

(4.14)

which are, of course, the two components of the Laplace-Runge-Lenz vector. We note that there is also a first integral associated with each of \( X_{6\pm} \) and this is the angular momentum which, in these coordinates, is an ignorable coordinate.

For the sake of completeness we list the first integrals/invariants associated with the symmetries listed in (4.10). They are given in the same order and with the same subscripts as the symmetries are listed.

\[
\begin{align*}
G_1 & \quad I_1 = L & \quad I_2 = J_+ J_- = 2L^2E + \mu^2 \\
G_2 & \quad I_1 = L & \quad I_2 = \frac{J_+}{J_-} \\
G_3 & \quad I_1 = L & \quad I_2 = \frac{J_+}{J_-} \\
G_{4\pm} & \quad I_1 = L & \quad I_2\pm = J_{\pm} \\
G_{5\pm} & \quad I_1 = L & \quad I_2\pm = \frac{J_+}{J_-} \\
G_{6\pm} & \quad I_1 = L & \quad I_2\pm = J_{\pm}.
\end{align*}
\]  

(4.15)

We note that in the second integral associated with \( G_1 \) we have expanded the product into the standard expression relating the square of the magnitude of the Laplace-Runge-Lenz vector with the energy and the angular momentum. For the second integral of \( G_{5\pm} \) we could have equally written \( I_2 = J_+/J_- \), but we chose to maintain the pattern of \( \pm \).

5 The Kepler problem with drag

In the introduction we referred to the model proposed by Danby [2] for the motion of a low altitude satellite subjected to a resistive force due to the Earth’s atmosphere described by the equation of motion

\[ \ddot{\mathbf{r}} + \frac{\alpha \mathbf{r}}{r^2} + \frac{\mu \mathbf{r}}{r^3} = 0, \]  

(5.1)

where \( \alpha \) and \( \mu \) are constants. Since the direction of the angular momentum is a constant, we may analyse the problem in two dimensions using plane
polar coordinates, \((r, \theta)\). The two equations of motion are

\[
\ddot{r} - r\dot{\theta}^2 + \frac{\alpha \dot{r}}{r^2} + \frac{\mu}{r^2} = 0 \tag{5.2}
\]

\[
\ddot{\theta} + 2\dot{r}\dot{\theta} + \frac{\alpha \dot{\theta}}{r} = 0. \tag{5.3}
\]

We introduce the new variables and their time derivatives

\[
\begin{align*}
    w_1 &= r, & \dot{w}_1 &= \dot{r}, \\
    w_2 &= \theta, & \dot{w}_2 &= \dot{\theta}, \\
    w_3 &= \dot{r}, & \dot{w}_3 &= \dot{w}_1, \\
    w_4 &= \dot{\theta}, & \dot{w}_4 &= \dot{w}_2 + \frac{\alpha w_3}{w_1} - \frac{\mu}{w_1^2}.
\end{align*}
\tag{5.4}
\]

We again select \(w_2\) to be the new independent variable \(y\). The right side of (5.4) becomes

\[
\begin{align*}
    \frac{dw_1}{dy} &= \frac{w_3}{w_4}, \tag{5.5} \\
    \frac{dw_3}{dy} &= w_1 w_4 - \frac{\alpha w_3}{w_1^2 w_4} - \frac{\mu}{w_1^2 w_4} \tag{5.6} \\
    \frac{dw_4}{dy} &= -\frac{2w_3}{w_1} - \frac{\alpha w_4}{w_1^2}.
\end{align*}
\]

In this case the choice of (5.3) to eliminate \(w_3\) is obvious and we obtain the two-dimensional system

\[
\begin{align*}
    w_1' w_4 + w_1' w_4' &= w_1 w_4 - \frac{\alpha w_1'}{w_1^2} - \frac{\mu}{w_1^2 w_4} \tag{5.8} \\
    w_4' &= -\frac{2w_3'}{w_1} - \frac{\alpha}{w_1^2}.
\end{align*}
\]

In the case of (5.9) we can easily manipulate it to obtain

\[
\left( w_1^2 w_4 \right)' = \alpha \quad \Leftrightarrow \quad w_1^2 w_4 = -\alpha y + \beta, \tag{5.10}
\]

where \(\beta\) is a constant of integration, which indicates that the angular momentum is not conserved. We have a choice of defining the new variable \(u_2\) as either \(w_1^2 w_4\), which is not conserved but is a convenient variable for manipulations, or \(w_1^2 w_4 + \alpha y\), which is conserved but is not a convenient
variable for manipulation. For the present we make the former choice. In equation (5.8) we make use of (5.10) to eliminate \( w_4 \) and \( w'_4 \) to obtain
\[
\left( -\frac{1}{w_1} \right) '' (\alpha y + \beta)^2 + 2\alpha \left( -\frac{1}{w_1} \right) ' (\alpha y + \beta) + \left( -\frac{1}{w_1} \right) (\alpha y + \beta)^2 + \mu = 0. \tag{5.11}
\]
We introduce the second new variable as
\[
u_1 = -\frac{\alpha y + \beta}{w_1} \tag{5.12}
\]
so that (5.11) becomes
\[
u''_1 + u_1 = -\frac{\mu}{\alpha y + \beta}. \tag{5.13}
\]
Clearly we could regain the equation of a simple harmonic oscillator by means of the further change of variable \( v = u_1 - u_{1ps} \), where \( u_{1ps} \) is a particular solution of (5.13). Consequently the Kepler problem with drag differs from the standard Kepler problem in that the second equation of the reduced system, \( \nu_2'' = -\alpha \),
\[
\tag{5.14}
\]
has a nonzero right side. This could be eliminated by now making the second choice mentioned above. Consequently we make a final change of variables
\[
v_1 = u_1 - \int \frac{\mu \sin(y - s)ds}{\alpha s - \beta} \\
v_2 = u_2 + \alpha \tag{5.15}
\]
to obtain the same reduced system, (4.9), as we had for the Kepler problem when the introduced the sensible coordinates. Naturally we obtain the same set of symmetries as given in (4.10).

The Lie point symmetries of the reduced system translate into the symmetries
\[
\begin{align*}
\Gamma_1 &= \left( \int \frac{dt}{r^2 \dot{\theta}} \right) \partial_t \\
\Gamma_2 &= 2 \left[ \int \frac{(\alpha r + I')dt}{r(\alpha \theta - \beta)} \right] \partial_t + \left[ \frac{\alpha r + I'}{\alpha \theta - \beta} \right] \partial_r - \partial_\theta \\
\Gamma_3 &= \left[ 2t + \int \frac{I dt}{r(\alpha \theta - \beta)} \right] \partial_t + \left( r + \frac{I}{\alpha \theta - \beta} \right) \partial_r \\
\Gamma_{4\pm} &= 2 \left[ \int \frac{e^{\pm i \theta}dt}{r(\alpha \theta - \beta)} \right] \partial_t + \left[ \frac{e^{\pm i \theta}}{\alpha \theta - \beta} \right] \partial_r
\end{align*}
\]
where $I$ stands for the integral introduced in (5.15a) and $I'$ its derivative with respect to $\theta$, for the original system, (5.1).

In addition to the symmetries listed in (5.16) equation (5.1) has the point symmetry $\partial_t$ which was the symmetry used for the reduction of order. Consequently we can conclude that algebraically the Kepler problem and the Kepler problem with drag are identical.

In the above derivation we have followed a line of development in which observation and experience play major roles in reducing the system (5.4) to the simplest possible form. The need for both are considerably obviated when the interactive Lie symmetry solver devised by Nucci [24] is used. The equations to be solved suggest the appropriate variables since they are the characteristics of the partial differential equations to be solved. We illustrate this in the case of the variable related to angular momentum with the following tableau

\[
\begin{align*}
\dot{w}_2 &= y \\
\dot{w}_3 &= \frac{dw_1}{dy}w_4 \\
w_4 &= u_1, w_1 = u_2 \\
\dot{u}_1 &= -\frac{2\dot{u}_2u_1}{u_2} - \frac{\alpha}{u_2}
\end{align*}
\]

\[
\begin{align*}
\ddot{u}_2 &= \frac{(u_2^2 + 2u_2^2)u_1^2u_2 - \mu}{u_1^4u_2^2} \\
w_4 &= \frac{w_5}{w_1} \\
w_5 &= u_1, w_1 = u_2 \\
\dot{u}_1 &= -\alpha
\end{align*}
\]
\[ \ddot{u}_2 = \frac{-\mu u_2^3 + u_1^2 u_2^2 + 2 u_1^2 u_2^2}{u_1^2 u_2} \]
\[ w_5 = w_6 - \alpha y \]
\[ w_6 = u_1, w_1 = u_2 \]
\[ \dot{u}_1 = 0 \]
\[ \ddot{u}_2 = \frac{(\alpha y - u_1)^2 (u_2^2 + 2 \dot{u}_2^2) - \mu u_2^3}{(\alpha y - u_1)^2 u_2}. \]

(We have used \( w_4, w_5 \) and \( w_6 \) to successively define a new variable which is a candidate for selection as \( u_1 \). We are not introducing additional variables.)

6 The generalisation of the Kepler problem with drag

Equation (1.4) has, in two dimensions, the two components of the equation of motion

\[ \begin{align*}
\dot{r} &= r \dot{\theta}^2 + \frac{1}{2} \left( \frac{g'}{g} + \frac{3}{r} \right) \dot{r}^2 - \mu g r \\
\ddot{\theta} &= \frac{\dot{r} \dot{\theta}}{2r} \left( \frac{g'}{g} - \frac{1}{r} \right). 
\end{align*} \] (6.1)

We introduce the variables \( w_i, i = 1, 4 \) as above. Now the system of first order equations in these variables is

\[ \begin{align*}
\dot{w}_1 &= w_3 \\
\dot{w}_2 &= w_4 \\
\dot{w}_3 &= w_1 w_2^2 + \frac{1}{2} \left( \frac{g'}{g} + \frac{3}{r} \right) w_3^2 - \mu g w_1 \\
\dot{w}_4 &= \frac{w_3 w_4}{2w_1} \left( \frac{g'}{g} - \frac{1}{w_1} \right). 
\end{align*} \] (6.2)

As the new independent variable we take again \( y = w_2 \). The system (6.2) becomes

\[ \begin{align*}
\frac{dw_1}{dy} &= \frac{w_3}{w_4} \implies w_3 = w_4 w_1' \\
\frac{dw_3}{dy} &= w_1 w_4 + \frac{1}{2} \left( \frac{g'}{g} + \frac{3}{r} \right) w_4 w_1^2 - \mu g w_1 \frac{w_1}{w_4} \\
\frac{dw_4}{dy} &= \frac{w_3}{2w_1} \left( \frac{g'}{g} w_1 - 1 \right). 
\end{align*} \] (6.3)
When we substitute for \( w_3 \), the third of (6.3) is easily integrated to give

\[
A = \left( \frac{w_1}{g} \right)^{\frac{1}{2}} w_4, \tag{6.4}
\]

where \( A \) is an arbitrary constant of integration. The right-hand side is the same function as the characteristic of the parabolic partial differential equation produced when the system (6.3) is analysed using the code developed by Nucci [24] and this is an appropriate choice for one of the variables. We take \( u_1 = w_1 = r \) and \( u_2 = \dot{\theta}(r/g)^{1/2} \) so that with the elimination of \( w_3 \) from (6.3) we have, after a certain amount of simplification, the system of two equations

\[
\begin{align*}
  u_2 u'' - u_1 u_2 + 2 \frac{u_2 u_1^2}{u_1} - \frac{\mu u_1^2}{u_2} &= 0, \\
  u_2' &= 0. \tag{6.5}
\end{align*}
\]

We observe that (6.5) is precisely the system (3.13) and so we immediately introduce the new variable \( v = \mu - u_2^2/u_1 \) to obtain the simpler system (4.9) which has the symmetries listed in (4.10). In terms of the original variables these symmetries are

\[
\begin{align*}
  X_1 &= \partial_{\theta} \\
  X_2 &= -\left( \int \frac{g' r}{g} dt \right) \partial_t + 2r \partial_r \\
  X_3 &= \frac{1}{\mu} \left[ \int (\mu r g - r \dot{\theta}^2) \left( \frac{1}{r} - \frac{g'}{g} \right) dt \right] \partial_t + \left( \mu r g - r \dot{\theta}^2 \right) \partial_r \\
  X_{4\pm} &= \frac{1}{\mu} \left\{ \int e^{\pm i \theta} (g - rg') dt \right\} \partial_t + \left\{ e^{\pm i \theta} rg \right\} \partial_r \\
  X_{5\pm} &= \frac{1}{\mu} \left\{ \int e^{\pm i \theta} \left[ \frac{g r}{\dot{\theta}^2} \left( \mu - \frac{\dot{\theta}^2}{g} \right)^2 \left( \frac{1}{r} - \frac{g'}{g} \right) \pm 2i \left( \mu - \frac{\dot{\theta}^2}{g} \right) + \frac{2 i \dot{\theta}^2}{g r} \right] dt \right\} \partial_t \\
  &\quad + \left\{ e^{\pm i \theta} \frac{g r}{\dot{\theta}^2} \left( \mu - \frac{\dot{\theta}^2}{g} \right)^2 \right\} \partial_r + \left\{ e^{\pm i \theta} \left( \mu - \frac{\dot{\theta}^2}{g} \right) \right\} \partial_{\theta} \\
  X_{6\pm} &= \pm \frac{1}{\mu} i \left\{ \int e^{\pm 2i \theta} \left( 3 + \frac{rg' + \mu (g - rg')}{\dot{\theta}^2} \right) dt \right\} \partial_t \\
  &\quad \pm i \left\{ e^{\pm 2i \theta} \frac{\mu g}{\dot{\theta}^2} \right\} \partial_r + e^{\pm 2i \theta} \partial_{\theta}. \tag{6.6}
\end{align*}
\]
7 An example with an angle-dependent force

Sen \cite{28} obtained conserved quantities similar to those of the Kepler problem for the Hamiltonian

\[ H = \frac{1}{2} \left( \frac{p_r^2}{r^2} + \frac{p_\theta}{r^2} \right) - \frac{\mu}{r} - \frac{\alpha \sin \left( \frac{1}{2}(\theta - \beta) \right)}{r^{1/2}} \]  

(7.1)

in which the potential depends upon the azimuthal angle and \( \mu, \alpha \) and \( \beta \) are constants. Subsequently Gorringe and Leach \cite{6} showed that the equation of motion

\[ \ddot{r} + g \dot{r} + h \dot{\theta} = 0, \]  

(7.2)

where

\[ g = \frac{U''(\theta) + U(\theta)}{r^2} + 2 \frac{V'(\theta)}{r^{3/2}} \]  

and \( h = \frac{V(\theta)}{r^{3/2}} \),

(7.3)

or

\[ \ddot{r} - r \dot{\theta}^2 + g = 0 \]  

(7.4)

\[ r \ddot{\theta} + 2r \dot{\theta} + h = 0 \]  

(7.5)

in plane polar coordinates, could be solved for the orbit equation in a manner similar to that of the Kepler problem since it also possessed a Laplace-Runge-Lenz vector. The only restrictions on the functions \( U(\theta) \) and \( V(\theta) \) are that they be differentiable.

We make the same reduction as in the previous cases to arrive at the two-dimensional system

\[ w_1 w_1'' w_4^2 - 2 w_1' w_2 w_4^2 - w_1 w_4^2 = w_1' h - g w_1 \]  

(7.6)

\[ w_1 w_4 w_4' + 2 w_1' w_4^2 = -h, \]  

(7.7)

where \( w_1 = r \) and \( w_4 = \dot{\theta} \) as before.

The Laplace-Runge-Lenz vector for equation (7.2) is \cite{11}

\[ \mathbf{J} = \mathbf{r} \times \mathbf{L} - U \mathbf{r} - \left[ U' + 2r^{1/2} V \right] \mathbf{\hat{\theta}}, \]  

(7.8)

where \( \mathbf{L} := \mathbf{r} \times \dot{\mathbf{r}} \) is the angular momentum. If we take the two cartesian components of \( \mathbf{J} \), viz \( J_x \) and \( J_y \), and combine them we obtain

\[ J_{\pm} = -J_x \pm iJ_y \]  

\[ = \left[ (r^3 \dot{\theta}^2 - U) \pm i \left( -r^2 r \dot{\theta} - U' - 2r^{1/2} V \right) \right] e^{\pm i \theta} \]
\[
\begin{align*}
= & \left[ (w_1^3 w_4^2 - U) \pm i \left( -w_1^2 w_4 w_3 - U' - 2w_1^{1/2} V \right) \right] e^{\pm iy} \\
= & \left[ \left( \frac{L^2}{w_1} - U \right) \pm i \left( \frac{L^2}{w_1} - U' \right) \right] e^{\pm iy}.
\end{align*}
\] (7.9)

We see that, when we write the components of the Laplace-Runge-Lenz vector in this form, we have the same structure as for the standard Kepler problem. (One could call the components the Ermanno-Bernoulli constants in honour of the original discoverers of these conserved quantities.) Immediately we have the clue to the identification of one of the new variables and we let

\[ u_1 = w_3^2 - U = \frac{L^2}{w_1} - U \] (7.10)

so that the Ermanno-Bernoulli constants for (7.2) are

\[ J_\pm = (u_1 \pm i u'_1) e^{\pm iy}. \] (7.11)

The identification of the second variable is more delicate. Equation (7.7) can be written in terms of the magnitude of the angular momentum, \( L \), as

\[ LL' = -w_1^{3/2} V(y) \] (7.12)

and, when (7.10) is taken into account, this becomes

\[ 0 = \frac{L'}{L^2} + \frac{V(y)}{(u_1 + U(y))^{3/2}}. \] (7.13)

From (7.11) we have

\[ u_1 = \frac{1}{2} \left( J_+ e^{-iy} + J_- e^{+iy} \right) = J \cos y \] (7.14)

since \( J_- = J_+^* \) and we have written \( J = |J_+| = |J_-| \). Then we can use (7.13) to define a new variable

\[ u_2 = \frac{1}{L} - \int \frac{V(y)dy}{(J \cos y + U(y))^{3/2}}. \] (7.15)

The reduced system of equations is

\[ \begin{align*}
& u_1'' + u_1 = 0 \\
& u_2' = 0
\end{align*} \] (7.16)
which is just the reduced system we obtained for the standard Kepler prob-
lem and so it has the symmetries given in (4.10). These translate to

\[ \Gamma_1 = 3 \left( \int r^2 \dot{\theta} dt \right) \partial_t + 2r^3 \dot{\theta} \partial_r \]

\[ \Gamma_2 = \left[ \int \left( \frac{2U'}{r^2 \dot{\theta}^2} + \frac{3V r^2 \dot{\theta}}{U + J \cos \theta} \right) dt \right] \partial_t + \left[ \frac{U'}{r^2 \dot{\theta}^2} + \frac{2V r^3 \dot{\theta}}{U + J \cos \theta} \right] \partial_r - \partial_{\theta} \]

\[ \Gamma_3 = 2 \left[ t - \int \frac{U dt}{r^3 \dot{\theta}^2} \right] \partial_t + \left[ r - \frac{U}{r^2 \dot{\theta}^2} \right] \partial_r \]

\[ \Gamma_{4\pm} = 2 \left( \int \frac{e^{\pm i\theta}}{r^3 \dot{\theta}^2} dt \right) \partial_t + \frac{e^{\pm i\theta}}{r^2 \dot{\theta}^2} \partial_r \]

\[ \Gamma_{5\pm} = \left\{ \int e^{\pm 2i\theta} \left[ \frac{3V r^2 \dot{\theta}}{U + J \cos \theta} + \frac{2(U' \mp iU)}{r^3 \dot{\theta}^2} \right] dt \right\} \partial_t - e^{\pm 2i\theta} \left[ \frac{2V r^3 \dot{\theta}}{U + J \cos \theta} \pm \frac{U' \mp iU}{r^2 \dot{\theta}} \right] \partial_r + e^{\pm 2i\theta} \partial_{\theta} \]

\[ \Gamma_{6\pm} = \left\{ \int e^{\pm i\theta} \left[ 2U' \left( 1 - \frac{U}{r^3 \dot{\theta}^2} \right) - 3 \left( r^2 \dot{\theta} \pm ir^3 \dot{\theta}^2 - 2r^{1/2}V \mp iU \right) \right] dt \right\} \partial_t - e^{\pm i\theta} \left[ \frac{2V r^2 \dot{\theta}}{U + J \cos \theta} \left( U' \mp i \left( r^3 \dot{\theta}^2 - U \right) \right) \left( 1 - \frac{U}{r^3 \dot{\theta}^2} \right) \right] \partial_r + e^{\pm i\theta} \left[ r^3 \dot{\theta}^2 - U \right] \partial_{\theta} \]

(7.17)

for the original system, (7.2). In addition there is the symmetry, \( \partial_t \), which

was used for the reduction of order.

Again we see the very close connection between the structure of the
Ermanno-Bernoulli constants and the appropriate variables for the reduction
of order.

8 Conclusions and observations

For the Kepler problem in three dimensions we obtain the same symmetries
as in (3.14) with the addition of

\[ X_{10} = \partial_{u_3} \]

(8.1)

where \( u_3 \) is the azimuthal angle, \( \phi \). Consequently our analysis in the lower
dimensional configurational space is justified by the result that the addi-
tional dimension simply adds another ignorable coordinate to the original

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system and so a trivial first order ordinary differential equation to the reduced system.

In this paper we have examined the process of reduction of order introduced by Nucci [25] to derive the additional nonlocal symmetries required for the complete specification of the Kepler problem in the context not only of the Kepler problem but also in some generalisations which have appeared in the literature and which possess certain characteristics in common with the Kepler problem. In particular we have found that the possession of a conserved vector similar to that of the Laplace-Runge-Lenz vector, whether or not the magnitude of the angular momentum is conserved, leads in all cases to a reduced system consisting of the simple harmonic oscillator and a trivial first order ordinary differential equation. In the reduced system the Lie point symmetries can be written in a fairly simple fashion. When translated to the original system, they are not so simple in appearance. However, one can determine the algebraic properties of the several systems studied from those of the reduced system provided one adds the Lie point symmetry used in the reduction of order, \( \partial_t \). The Lie algebra of Lie point symmetries of the reduced system is \( A_1 \oplus sl(3, \mathbb{R}) \) and consequently the original systems each have the algebra \( 2A_1 \oplus sl(3, \mathbb{R}) \) since the symmetry used in the reduction of order has a zero Lie bracket with the other symmetries.

Considering the results obtained in this paper we can envisage a reversal of the procedure. Instead of taking a system which has a vector of the type of a Laplace-Runge-Lenz vector we could simply commence with the reduced system and introduce some transformation of the two “reduced” variables and an Ansatz on the relationship defining the variable we have been denoting by \( w_3 \). One could expect to obtain many “lame ducks”! However, there is one aspect which has the potential for some application. Many of the systems for which Laplace-Runge-Lenz vectors have been obtained do not have a known Hamiltonian representation. By the procedures of transformations treated in this paper one could seek to commence with the Hamiltonian of the Kepler problem and find Hamiltonians for the other systems. In the case of the system (7.3) the existence of a Hamiltonian has been shown only in the restricted case treated by Sen. The attractions for the applications in quantum mechanics are obvious.

In the reduced system the components of the Laplace-Runge-Lenz vector, the Ermanno-Bernoulli constants, are simply the two linearly independent first order invariants of the simple harmonic oscillator. In fact in the reduced system we have a separation in the new variables of the Ermanno-Bernoulli constants in the second order equation and the conservation of a generalised
angular momentum in the first order equation. We recall that for higher dimensional oscillators there exist the conserved components of the Jauch-Hill-Fradkin tensor \( [13, 5] \) which play an important role in the description of the orbit and their time-dependent counterparts which give the actual trajectory of the particle \( [17, 18, 21] \). Naturally these tensors have no role to play in the type of problem considered in this paper. However, it is intriguing to ponder the identity that the corresponding problem of Kepler type would have. (For a recent contribution to this more general problem see \( [14] \).)

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References

[1] Bernoulli J (1710) Extrait de la Réponse de M Bernoulli à M Herman, datée de Basle le 7 Octobre, 1710 Hist Acad Roy Sci: Mém Math Phys 521-533

[2] Danby J M B (1962) Fundamentals of Celestial Mechanics (Macmillan, New York)

[3] Ermanno G J (1710) Metodo d’investigare l’Orbite de’ Pianeti, nell’ipotesi che le forze centrali o pure le gravità degli stessi Pianeti sono in ragione reciproca de’ quadrati delle distanze, che i medesimi tengono dal Centro, a cui si dirigono le forze stesse G Lett Ital 2 447-467

[4] Géronimi C, Feix M R & Leach P G L (1997) Exponential nonlocal symmetries and nonnormal reduction of order Preprint: MAPMO,
Département mathématiques, Université d’Orléans, Orléans la Source, 45067 Orléans Cedex 2, France

[5] Fradkin D M (1965) Three-dimensional isotropic harmonic oscillator and SU$_3$ Amer J Phys 33 207-211

[6] Gorringe V M & Leach P G L (1987) Conserved vectors for the autonomous system $\ddot{r} + g(r, \theta)\dot{r} + h(r, \theta)\dot{\theta} = 0$ Physica 27D 243-248.

[7] Gorringe V M & Leach P G L (1988) Hamiltonlike vectors for a class of Kepler problems with drag Celest Mech 41 125-130

[8] Gorringe V M & Leach P G L (1989) Conserved vectors and orbit equations for autonomous systems governed by the equation of motion $\ddot{r} + f\dot{r} + gr = 0$ Amer J Phys 57 432-435

[9] Gorringe V M & Leach P G L (1993) Kepler’s third law and the oscillator’s isochronism Amer J Phys 61 991-995

[10] Hamilton W R (1847) On the application of the method of quaternions to some dynamical questions Proc Roy Irish Acad 3 344-353, Appendix III, xxxvi-l

[11] Hermann J (1710) Extrait d’une lettre de M Herman à M Bernoulli, datée de Padoue le 12 Juillet, 1710 Hist Acad Roy Sci: Mém Math Phys 519-521

[12] Iwai Toshihiro & Sunako Takehiko (2000) Dynamical and symmetry groups of the SU(2) Kepler problem J Geom Phys 33 326-355

[13] Jauch J M & Hill E L (1940) On the problem of degeneracy in quantum mechanics Phys Rev 57 641-645

[14] Jezewski D J & Mittleman D (1983) Integrals of motion for the classical two-body problem with drag Int J Non-Linear Mech 18 119-124

[15] Krause J (1994) On the complete symmetry group of the classical Kepler system J Math Phys 35 5734-5748

[16] Laplace P S (1798) Traité de Mécanique Céleste (Villars, Paris)

[17] Leach P G L (1978) Quadratic Hamiltonians, quadratic invariants and the symmetry group SU($n$) J Math Phys 19 446-451
[18] Leach P G L (1980) Quadratic Hamiltonians: the four classes of invariants, their interrelations and symmetries *J Math Phys* **21** 32-37

[19] Leach P G L (1987) The first integrals and orbit equation for the Kepler problem with drag *J Phys A: Math Gen* **20** 1997-2004

[20] Leach P G L & Gorringe V M (1987) Variations on Newton’s Keplerian theme *S Afr J Sci* **83** 550-555

[21] Lemmer R L (1996) *The Paradigms of Mechanics* (dissertation, Department of Mathematics and Applied Mathematics, University of Natal, Durban, South Africa)

[22] Lenz W (1924) Über den Berwegungsverlauf und die Quantenzustände der gestörten Keplerbewegung *Z Phys* **24** 197-207

[23] Mittleman D & Jezewski D J (1982) An analytic solution to the classical two-body problem with drag *Celest Mech* **28** 401-413

[24] Nucci M C (1996) Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: manual and floppy disk *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. III: New Trends*, N H Ibragimov ed (Boca Raton: CRC Press) 415-481.

[25] Nucci M C (1996) The complete Kepler group can be derived by Lie group analysis *J Math Phys* **37** 1772-1775

[26] Pillay T & Leach P G L (1999) Generalised Laplace-Runge-Lenz vectors and nonlocal symmetries *S Afr J Sci* **95** 403-407

[27] Runge C (1923) *Vector Analysis* (Methuen, London)

[28] Sen T (1987) A class of integrable potentials *J Math Phys* **28** 2841-2850