Kaluza-Klein dimensional reduction and Gauss-Codazzi-Ricci equations

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Abstract

In this paper we imitate the traditional method which is used customarily in the General Relativity and some mathematical literatures to derive the Gauss-Codazzi-Ricci equations for dimensional reduction. It would be more distinct concerning geometric meaning than the vielbein method. Especially, if the lower dimensional metric is independent of reduced dimensions the counterpart of the symmetric extrinsic curvature is proportional to the antisymmetric Kaluza-Klein gauge field strength. For isometry group of internal space, the SO(n) symmetry and SU(n) symmetry are discussed. And the Kaluza-Klein instanton is also enquired. PACS:03.70;11.15

1 Introduction

Kaluza-Klein dimensional reduction is a longstanding problem which is followed with interest by theoretical physicists. It is developed from initial unification of gravitational and electromagnetic interactions to becoming the cornerstone for superstring and supergravity. (see review papers and references therein.) To depict Kaluza-Klein dimensional reduction most authors adopt the Cartan moving frame that is the vielbein method. Its form is elegant and its algorithm is rapid. But the well-known Gauss-Codazzi-Ricci equations which describe a submanifold embedded in a Riemann space are implicit. Perhaps the role played by Codazzi constraint and Ricci constraint may not clear too. Alternatively we would like to derive these equations for dimensional reduction by using the traditional method which is used customarily in the General Relativity and some mathematic literatures. The geometric meaning may be more distinct than the vielbein method. As a result we have found the substitute of so called lapse function and shifted function, that is, instead of shifted function we have Kaluza Klein gauge potential and instead of lapse function we have scalar field tensor. Especially, the symmetric extrinsic curvature tensor now is replaced by a mixed tensor which has antisymmetric part

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as well as symmetric one. When the metric of lower dimensional space is independent of reduced dimensions the tensor is proportional to the antisymmetric Kaluza-Klein gauge field strength.

The simplest Kaluza-Klein reduction is through the torus, consequently, the isometry group of internal space is $U(1)^n$. But for connecting with physics the nonAbelian group is more interest. So we try to examine SO(n) symmetry and SU(n) symmetry. Up to now there are only a few of nonlinear ansatz for truncations to the massless supermultiplets: DeWit and Nicolai demonstrated the consistency reduced on $S^7$ from 11-dimensional supergravity to the 4-dimensional SO(8) supergravity. Nastase et al. found a complete solution for the $S^4$ reduction of 11-dimensional supergravity giving rise to the 7-dimensional gauged SO(5) supergravity, and a $S^5$ reduction of IIB supergravity giving rise to the 5-dimensional gauged SO(6) supergravity was contributed by Cvetic et al. Nevertheless as indicated in ref. the consistency of these ansatz always work at the level of the equations of motion. According to usual understanding, the actions are equivalent to equations of motion. But these authors (they called above reduction the Pauli reduction) pointed out that substituting the ansatz into the higher dimensional action may not give the correct lower dimensional theory. In this paper we will see the Gauss equation denotes essentially the reduction relation of action. It may provide another avenues for further investigation. We know SU(n) group is a subgroup of SO(2n) there may be byproduct when we study the spherical reduction, SU(n) ansatz can be embedded in the SO(2n) ansatz. Besides, in present theory, the Gauss-Codazzi-Ricci equations are dependent on Kaluza-Klein gauge potential, they may be defined in different neighborhood (gauge); moreover, we have yet to study isometric group SU(n), hence except the Kaluza-Klein monopole we can also enquire the Kaluza-Klein instanton.

## 2 Tensor K and vector L, Gauss-Weingarten Formula

The standard Kaluza Klein reduction formula from D-dimensional spacetime to d-dimensional subspace is shown in the following

$$ds^2 = g_{AB} dx^A dx^B = h_{\alpha\beta} dx^\alpha dx^\beta + N_{ij}(du^i + N^i_\alpha dx^\alpha)(du^j + N^j_\beta dx^\beta)$$

in which $N_{ij} = N_{ji}$ and $N^i_\alpha$ are the generalization of lapse function and shifted function in General Relativity respectively. Physically they represent scalars and gauge fields, for Abelian theory $N^i_\alpha = A^i_\alpha$, and for nonAbelian case

$$N^i_\alpha = -\xi^P_\mu A^\mu_\alpha,$$

(2)

$\xi^P_\mu$ are Killing vectors on (D-d)-dimensional internal space satisfying

$$\xi^P_\mu \partial_\nu \xi^Q_\rho - \xi^P_\nu \partial_\mu \xi^Q_\rho = C^R_{PQ} \xi^I_R,$$

(3)
where $C_{PQ}^R$ is the structure constant of isometric group. In imitation of lapse-shifted method we can introduce the normal vectors

$$n^i_A = (N^i_A, \delta^i_j), \quad n^{Ai} = (0, N^{-1}ij), \quad n^i_A n^i_B = N^{-1}_{ij}$$

so that

$$g_{AB} = h_{AB} + N_{ij} n^i_A n^j_B, \quad h_{AB} = \begin{pmatrix} h_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}.$$  

It is easy to find the inverse metric

$$g^{AB} = h^{AB} + \begin{pmatrix} 0 & 0 \\ 0 & N^{-1}_{ij} \end{pmatrix}$$

where

$$h^{AB} = \begin{pmatrix} h^{\alpha\beta} & -N^{ij} \\ -N_{ij} & N^i_{\alpha} N^i_{\alpha} \end{pmatrix}.$$  

Obviously, we have

$$h_{AB} n^B_i = 0 = h_{AB} n^A_i.$$  

Especially, there are special components of metric $h$ which satisfy

$$h^{\beta \alpha} = \delta^{\beta \alpha}, \quad h^i = -N^i_{\alpha}, \quad h^{A}_i = h^{A} = h^i = 0.$$  

We begin by presenting the relation between two metrics

$$h_{\alpha\beta} = h^{A}_\alpha h^{B}_\beta g_{AB}$$  

then we get by differentiating them

$$D_\gamma h_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} - N^i_{\alpha} \partial_i h_{\alpha\beta}$$

$$= (D_\gamma h^{A}_\alpha) h_{\beta B} + (D_\gamma h^{B}_\beta) h_{\alpha A} + h^{A}_\alpha h^{B}_\beta h^{C}_\gamma \partial_C g_{AB}$$

$$= h^{A}_\alpha h^{B}_\beta h^{C}_\gamma \partial_C g_{AB}$$  

in the last step we have used the equations (9). Define that

$$P^{\gamma}_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta} (D_\alpha h_{\delta\beta} + D_\beta h_{\alpha\delta} - D_\delta h_{\alpha\beta}) \equiv \Gamma^{\gamma}_{\alpha\beta} + H^{\gamma}_{\alpha\beta},$$

in which $\Gamma^{\gamma}_{\alpha\beta}$ is the Christoffel symbol in d-dimensional subspace, while

$$H^{\gamma}_{\alpha\beta} \equiv -\frac{1}{2} h^{\gamma\delta} (N^i_{\alpha} \partial_i h_{\delta\beta} + N^i_{\beta} \partial_i h_{\alpha\delta} - N^i_{\delta} \partial_i h_{\alpha\beta}).$$

Substituting eq.(11) into eq.(12) we obtain

$$P^{\gamma}_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta} h^{D}_\delta h^{A}_\alpha h^{B}_\beta (\partial_A g_{DB} + \partial_B g_{AD} - \partial_D g_{AB})$$

$$= h^{C}_{\gamma} h^{A}_\alpha h^{B}_\beta \frac{1}{2} g^{CD} (\partial_A g_{DB} + \partial_B g_{AD} - \partial_D g_{AB})$$

3
\[ h^C_C h^\gamma_C h^\gamma_C C = (h^A_C h^B_C + \Gamma^C_{AB} h^A_C h^B_C - P^D_{\alpha\beta} h^C_D) = 0, \]  
\text{(15)}

which tells us that \( \tilde{\nabla}_\alpha h^C_{\beta} \) is proportional to the normal vector fields \( n^C_i \), hence we can define tensors \( K^i_{\alpha\beta} \) by
\[ \tilde{\nabla}_\alpha h^C_{\beta} = K^i_{\alpha\beta} n^C_i, \]  
\text{(16)}

this is the Gauss formula in present case. Operator \( \tilde{\nabla}_\alpha \) we have introduced is an operator which operates on the D-dimensional index as well as d-dimensional index simultaneously.\[8\] In fact if we define the following operators
\[ \tilde{\nabla}_\beta n^C_i = D_\beta n^C_i + \Gamma^C_{BA} h^B_i n^A_i, \quad \tilde{\nabla}_\beta n^C_C = D_\beta n^C_C - \Gamma^A_{BC} h^A_{\gamma} n^B_{\gamma}; \]  
\text{(17)}

and
\[ \tilde{\nabla}_\beta u^\gamma = D_\beta u^\gamma + P^\gamma_{\beta\alpha} u^\alpha, \quad \tilde{\nabla}_\beta u^\gamma = D_\beta u^\gamma - P^\gamma_{\beta\gamma} u^\alpha. \]  
\text{(18)}

Operator \( \tilde{\nabla}_\alpha \) on \( h^C_{\beta} \) certainly agrees with eq.(15). In an earlier version\[17\] we have defined the operator \( \tilde{\nabla}_\alpha \) by using \( \partial_\alpha \) instead of \( D_\alpha \). As a result, both \( \tilde{\nabla}_\alpha g_{AB} \) and \( \tilde{\nabla}_\alpha h^C_{\beta} \) do not vanish, they belong to so called nonmetric connection. As a matter of fact in lower dimensional space the connection is just the torsion free linear connection described by Schouten\[7\]; while in higher dimensional space it is Yano’s projective connection.\[8\] Fortunately, when we redefine operator \( \tilde{\nabla}_\alpha \) as present form the metric property of connection is recovered, i.e.
\[ \tilde{\nabla}_\gamma g_{AB} = \tilde{\nabla}_\gamma h_{\alpha\beta} = 0. \]  
\text{(19)}

From eqs.(14) and (5) we obtain
\[ \tilde{\nabla}_\alpha h^C_{\beta} = D_\alpha h^C_{\beta} + \Gamma^C_{AB} h^A_{\alpha} h^B_{\beta} - P^\gamma_{\alpha\beta} h^C_{\gamma}, \]
\[ = (h^C^D - h^C^D h^D_C)(D_\alpha h^D_{\beta} + \Gamma^D_{AB} h^A_{\alpha} h^B_{\beta}) \]
\[ = N_{ij} n^D_{ij} (D_\alpha h^D_{\beta} + \Gamma^D_{AB} h^A_{\alpha} h^B_{\beta}) n^C_i. \]  
\text{(20)}

Therefore
\[ K_{\alpha\beta i} = -\frac{1}{2} [\partial_\alpha h_{\alpha\beta} + N_{ij} (D_\alpha N^j_{\beta} - D_\beta N^j_{\alpha})]. \]  
\text{(21)}

If there is nonAbelian isometric group on internal manifold, by means of eqs.(2)and(3) we get
\[ K_{\alpha\beta i} = -\frac{1}{2} (\partial_\alpha h_{\alpha\beta} - N_{ij} F^P_{\alpha\beta} \xi^j_P), \]  
\text{(22)}

in which
\[ F^P_{\alpha\beta} = \partial_\alpha A^P_{\beta} - \partial_\beta A^P_{\alpha} + C^P_{QR} A^Q_{\alpha} A^R_{\beta}. \]  
\text{(23)}
Now the geometric meaning of function $K'_{\alpha \beta}$ is quite clear: its symmetric part is a gradient of metric on submanifold which vanishes if neglect massive particles in the compactified theory. The antisymmetric part proportional to a Yang-Mills gauge field, which really can be thought of a kind of "curvature". We know in old Gauss-Codazzi-Ricci theory the corresponding K tensor is a symmetric extrinsic curvature.

Next, we write down the Weingarten formula

$$\tilde{\nabla}_\beta n^A_i = h^B_A h^\alpha_B \tilde{\nabla}_\beta n^B_i + N_{kl} n^k_A n^l_B \tilde{\nabla}_\beta n^l_B$$

$$= - h^A_B \tilde{\nabla}_\beta h^B_A + n^A_i L_{\beta k}^A$$

$$= - h^A_B \tilde{K}_{\beta \alpha}^A + n^A_i L_{\beta \alpha}^A,$$  \hspace{1cm} (24)

in which

$$\tilde{K}_{\beta \alpha}^A = K_{\beta \alpha}^j N_j^{-1}i,$$  \hspace{1cm} (25)

or

$$\tilde{L}_{\beta ji} = N_{ij} n^A_k \tilde{\nabla}_\beta n^A_i = \frac{1}{2} N_i^{-1j} (D_\beta N_{ij} + N_{ik} \partial_j N^k_\beta - N_{jk} \partial_i N^k_\beta),$$  \hspace{1cm} (26)

which satisfies

$$\tilde{L}_{\beta ji} + \tilde{L}_{\beta ij} = D_\beta N_{ij}^{-1},$$  \hspace{1cm} (28)

and for nonAbelian case

$$\tilde{L}_{\beta ij} = \frac{1}{2} D_\beta N_{ij}^{-1} - (N_i^{-1i} \partial_i \xi P_j - N_j^{-1j} \partial_i \xi P_i) A^P_{\beta}.$$  \hspace{1cm} (29)

### 3 Gauss-Codazzi-Ricci Equations

From definition of $\tilde{\nabla}_\alpha h^C_{\beta}$ it is straightforward to calculate the antisymmetric double derivative of $h^C_{\beta}$ as follows

$$h^C_{\beta} (\tilde{\nabla}_\gamma \tilde{\nabla}_\alpha - \tilde{\nabla}_\alpha \tilde{\nabla}_\gamma) h^C_{\beta}$$

$$= h^C_{\beta} h^A_{\alpha} h^B_{\beta} h^D_{\gamma} R_{ADB}^C - S_{\alpha \gamma \beta}^D$$

$$+ h^D_{\alpha} N^{-1ij} (\partial_i h_{\alpha \gamma} + 2 K_{\alpha \gamma \eta} K_{\beta \eta \jmath}),$$  \hspace{1cm} (30)

in which $R_{ADB}^C$ is the Riemann curvature tensor of higher dimensional space and

$$S_{\alpha \gamma \beta}^D \equiv D_\alpha P^\delta_{\alpha \beta} - D_\alpha P^\delta_{\gamma \beta} + P^\delta_{\gamma \eta} P^\eta_{\alpha \beta} - P^\delta_{\alpha \eta} P^\eta_{\gamma \beta}$$

$$= R_{\alpha \gamma \beta}^D + N^i_\alpha \partial_i \Gamma^D_{\gamma \beta} - N^i_\gamma \partial_i \Gamma^D_{\alpha \beta} + D_\gamma H^D_{\alpha \beta} - D_\alpha H^D_{\gamma \beta}$$

$$+ H^D_{\gamma \eta} H^\eta_{\alpha \beta} - H^\delta_{\alpha \gamma} H^\delta_{\gamma \beta} + \Gamma^D_{\gamma \eta} H^\eta_{\alpha \beta} + \Gamma^\eta_{\alpha \beta} H^\delta_{\gamma \eta}$$
\[ -\Gamma_{\alpha\eta}^i H_{j\beta}^{\eta} - \Gamma_{\gamma\beta}^i H_{\alpha\eta}^\delta, \]  

(31)

where

\[ R_{\alpha\gamma\beta} = \partial_\eta \Gamma_{\alpha\eta}^i - \partial_\alpha \Gamma_{\gamma\eta}^i + \Gamma_{\gamma\delta}^i \Gamma_{\delta\eta}^{\alpha\beta} - \Gamma_{\alpha\delta}^i \Gamma_{\delta\eta}^{\gamma\beta} \]  

(32)

is the Riemann curvature tensor in lower dimensional space. On the other hand

\[
h_C^C (\nabla_\gamma \nabla_\eta - \nabla_\eta \nabla_\gamma) h_C^C
\]

\[
= \nabla_\eta h_C^C \nabla_\gamma h_\eta^C - \nabla_\gamma h_C^C \nabla_\eta h_\eta^C
\]

\[ = N_{ij}^{-1} h^\delta\eta (K_{\alpha\eta}^i K_{\beta\lambda}^j - K_{\gamma\lambda}^i K_{\beta\eta}^j). \]  

(33)

Equating right hand sides of (30) and (33) and contracting \( \gamma \) with \( \delta \) and \( \alpha \) with \( \beta \) we obtain the Gauss equation

\[
h^{AB} h^{CD} R_{ABCD} = h^{\alpha\beta} S_{\alpha\eta}^{\gamma} + h^{\alpha\beta} h^{\gamma\delta} N^{-1}_{ij} (K_{\alpha\delta}^i K_{\gamma\beta}^j - K_{\gamma\delta}^i K_{\alpha\beta}^j - 2 K_{\alpha\gamma}^i K_{\beta\delta}^j)
\]

\[
+ \frac{1}{2} h^{\alpha\beta} h^{\gamma\delta} N^{-1} \partial_i h_{\alpha\gamma} \partial_j h_{\beta\delta}.
\]

(34)

Because of

\[
h^{AB} h^{CD} R_{ABCD} = R - 2 N^{-1} R_{ij} + N^{-1} N^{-1} R_{ijkl}
\]

\[
= R - h^{\alpha\beta} h^{\gamma\delta} N^{-1}_{ij} (2 K_{\alpha\gamma}^i K_{\beta\delta}^j - (N_{ij} \partial_j N_{\delta}^k)
\]

\[- \frac{1}{2} D_{\delta} N_{ij} (N_{ij} \partial_k h_{\alpha\beta} - N_{ij} \partial_k h_{\beta\gamma} - N_{ij} \partial_k h_{\alpha\gamma})
\]

\[- N(N_{\alpha}^l, N_{kl}), \]

(35)

\[ X(N_{\alpha}^l, N_{kl}) \equiv N^{-1}_{ij} [h^{\alpha\beta} |2 D_{\alpha} N_{ij}^\delta \partial_k N_{jk} + 2 D_{\alpha} (N_{ik} \partial_j N_{\delta}^k) - \partial_i \partial_j h_{\alpha\beta}]
\]

\[- D_{\alpha} D_{\beta} N_{ij} - D_{\alpha} N_{ij}^\delta \partial_k N_{ij} - 2 N_{kl} \partial_i N_{\delta}^k \partial_j N_{\lambda}^l]
\]

\[ + \frac{1}{2} h^{\alpha\beta} N^{-1}_{kl} [N_{im} \partial_k N_{\alpha}^m (3 N_{jm} \partial_k N_{\beta}^n)
\]

\[- N_{kn} \partial_j N_{\beta}^n - 2 N_{jm} \partial_j N_{\alpha}^m \partial_k N_{\beta}^n
\]

\[- 2 N_{im} (\partial_i N_{\alpha}^m \partial_k D_{\beta} N_{jk} - \partial_j N_{\alpha}^m \partial_k D_{\beta} N_{ik})
\]

\[ + \frac{1}{2} (3 D_{\alpha} N_{ik} D_{\beta} N_{jl} - D_{\alpha} N_{ij} D_{\beta} N_{kl}) - (2 \partial_i N_{jk})
\]

\[- \partial_i N_{ij} (2 N_{lm} D_{\alpha} N_{\alpha}^m - \partial_i h_{\alpha\beta}] +
\]

\[+ N^{-1} [\partial_i \partial_j N_{kl} - \partial_i \partial_j N_{kl}] + \frac{1}{4} N^{-1}_{mn} N^{-1}_{mn} [2 \partial_i N_{mk} (\partial_i N_{jn})
\]

\[+ \partial_j N_{nt} - 2 \partial_i N_{jt} + \partial m N_{ik} \partial n N_{jl} - \partial m N_{ij} \partial n N_{kl}
\]

\[- 4 \partial_i N_{jm} (\partial_j N_{kn} - \partial_n N_{kl}));
\]
and

\[ h^{\alpha\beta}S_{\alpha\gamma\beta} = R + V(h_{\alpha\beta}). \]  

\[ V(h_{\alpha\beta}) = N_i^\alpha \partial_i^\gamma h^{\alpha\beta} \partial_j h_{\beta\gamma} + h^{\alpha\beta}(\partial^\gamma N_i^\alpha \partial_j h_{\alpha\beta} - \partial^\gamma N_i^\alpha \partial_j h_{\beta\gamma}) + \frac{1}{2} N^\alpha \partial_i (h^{\gamma\delta} \partial_j h_{\gamma\delta}) + h^{\alpha\beta} \partial_i h_{\gamma\delta} + \frac{1}{4} h^{\alpha\beta}(2N^\gamma_i N^\delta_j) - h^{\alpha\beta} N_i^\delta N^\gamma_j (\partial_i h_{\alpha\beta} \partial_j h_{\gamma\delta} - \partial_i h_{\gamma\delta} \partial_j h_{\alpha\beta}) - N_i^\gamma \partial_i N_j^\delta \partial_j h_{\alpha\beta} + N_i^\gamma N_j^\delta \partial_i \partial_j h_{\alpha\beta} - N_i^\gamma N_j^\delta \partial_i \partial_j h_{\alpha\beta} + \frac{1}{2} h^{\alpha\beta}(2N_i^\gamma N_j^\delta \partial_j h_{\gamma\delta}) - N_i^\gamma N_j^\delta \partial_j h_{\gamma\delta} - N_i^\gamma N_j^\delta \partial_j h_{\alpha\beta} \partial_j h_{\gamma\delta}, \]  

thus

\[ R = R + h^{\alpha\beta} h^{\gamma\delta} N_i^j K_{\alpha\delta i} K_{\gamma\beta j} - K_{\gamma\delta i} K_{\alpha\beta j} + \frac{1}{2} \partial_i h_{\alpha\gamma} \partial_j h_{\beta\delta} - (N_i^k \partial_j N_k^\delta) (N_i^\gamma \partial_j h_{\alpha\beta} - N_i^\gamma \partial_j h_{\alpha\beta}) \]  

\[ + X(N_i^\alpha, N_{kl}) + V(h_{\alpha\beta}). \]  

It is well-known that the Lagrangian of Einstein gravitational equation is \( \sqrt{g}R \), hence, we have to do a conformal transformation first to assure that Gauss equation represents the Kaluza-Klein reduction of action from D-dimensional gravitation to d-dimensional gravitation plus matters (gauge fields and scalars). Since

\[ g_{AB} = \begin{pmatrix} h_{\alpha\beta} + N_i^\alpha N_i^\beta & N_i^\alpha N_i^\beta \\ N_i^\beta N_i^\alpha & N_{ij} \end{pmatrix} \]  

from the triangularization of its vielbein form it is easy to realize

\[ \det g_{AB} = \det h_{\alpha\beta} \det N_{ij}. \]  

Therefore we adopt the following conformal transformation

\[ g_{AB} \rightarrow \tilde{g}_{AB} = (\det N_{ij})^{-\frac{1}{2}} g_{AB}. \]
Comparing both expressions we obtain the two equations (17). Following definition (17) we have

\[ \sqrt{-g} \hat{R} = \sqrt{-h} \{ R + U(\det N) \} \]
\[ = \sqrt{-h} \{ R + h^{\alpha \beta} h^{\gamma \delta} N^{-1} [K_{\alpha \delta}^i K_{\gamma \beta}^j - K_{\gamma \delta}^i K_{\alpha \beta}^j ] \]
\[ + \frac{1}{2} \partial_i h_{\alpha \gamma} \partial_j h_{\beta \delta} - (N_{ik} \partial_j N_{\delta}^k - \frac{1}{2} D_{\delta} N_{ij}) (N_{\gamma}^k \partial_k h_{\alpha \beta}^i) \]
\[ + \frac{1}{2} \partial_i h_{\alpha \gamma} \partial_j h_{\beta \delta} - (N_{ik} \partial_j N_{\delta}^k + V(h_{\alpha \beta})) + \{ U(\det N) \} \}, \]

(43)

\[ U(\det N) = \frac{D - 1}{D - 2} \{ D^\alpha D_{\alpha} \ln \det N - \frac{1}{4} D^\alpha \ln \det N D_{\alpha} \ln \det N + N^{-1} \partial_i \partial_j \ln \det N \]
\[ - \frac{1}{4} \partial_i \ln \det N \partial_j \ln \det N + \frac{1}{2} h^{\alpha \beta} \partial_i h_{\alpha \beta} \partial_j \ln \det N \]
\[ - \frac{1}{2} \partial_i \partial_j \ln \det N + N^{-1} \partial_k N_{id} \partial_j N_{\gamma l} + \partial_i h_{\alpha \beta} \partial_j h_{\delta \gamma} \]
\[ - N_{\gamma l} \partial_j h_{\alpha \beta} + N_{\delta \gamma} \partial_j h_{\alpha \beta} \}

Equation (43) gives the Lagrangian reduction formula. Next, we would like to find the Codazzi equation and Ricci equation. Following definition (17) we have

\[ (\tilde{\nabla}_\gamma \tilde{\nabla}_\beta - \tilde{\nabla}_\beta \tilde{\nabla}_\gamma) n_{\alpha}^i \]
\[ = R_{\gamma \beta \delta \gamma} (h_{\beta \delta}^i h_{\gamma}^j n_{\alpha}^l - \frac{1}{2} N^{-1} D_{\gamma} N_{\beta}^i - D_{\gamma} N_{\beta}^i) \]
\[ + N_{\alpha}^i \partial_j N_{\beta}^i \}

(45)

on the one hand, and through the Weingarten formula (24) we know

\[ (\tilde{\nabla}_\gamma \tilde{\nabla}_\beta - \tilde{\nabla}_\beta \tilde{\nabla}_\gamma) n_{\alpha}^i \]
\[ = -(\tilde{\nabla}_\gamma \tilde{K}_{\beta \gamma} - \tilde{\nabla}_\beta \tilde{K}_{\gamma \alpha}^j) h_{\gamma}^j - (\tilde{K}_{\beta \gamma}^i K_{\alpha \gamma}^j - \tilde{K}_{\gamma \alpha}^i K_{\beta \gamma}^j) n_{\alpha}^i \]
\[ + (\tilde{\nabla}_\gamma L_{\beta}^i \gamma j) n_{\alpha}^j \gamma - (\tilde{K}_{\beta \gamma}^i L_{\alpha \gamma}^j \gamma - \tilde{K}_{\gamma \alpha}^i L_{\beta \gamma}^j \gamma) h_{\gamma}^j \]
\[ + (L_{\gamma} h_{\beta \gamma}^i \gamma L_{\alpha \gamma}^j \gamma - L_{\beta} h_{\gamma}^j \gamma \gamma) n_{\alpha}^i \gamma \]

(46)

on the other hand. Comparing both expressions we obtain the two equations immediately. The Codazzi equation is

\[ \tilde{\nabla}_\beta \tilde{K}_{\alpha \gamma}^i - \tilde{\nabla}_\gamma \tilde{K}_{\beta \alpha}^i + \tilde{K}_{\beta \alpha}^i L_{\gamma}^j \gamma - \tilde{K}_{\alpha \gamma}^i L_{\beta}^j \gamma \gamma \]
\[ - N^{-1} \partial_j N_{\gamma}^i \gamma (D_{\gamma} N_{\beta}^l \gamma - \partial_j N_{\gamma}^i \gamma N_{\beta}^m \gamma - N_{jm} \partial_k N_{\alpha}^m \gamma) (K_{\alpha \gamma}^l \gamma + \frac{1}{2} \partial h_{\alpha \gamma} \gamma) \]

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After contracting index $\alpha$ with $\gamma$ we get the following form

$$
\nabla_{\gamma} \tilde{K}_{\alpha j} - \nabla_{\alpha} \tilde{K}_{\beta j} + \tilde{K}_{\gamma j} L_{\beta j}^i - \tilde{K}_{\alpha j} L_{\beta j}^i
$$

$$
- \frac{1}{2} N^{-1} k_i N^{-1} \left\{ D_{\beta} \partial_k N_{\beta j} - D_{\beta} \partial_k N_{kj} + \partial_j D_{\beta} N_{kl} + (\partial_l N_{km})
- \partial_k N_{lm} \partial_j N_{m}^m + 2 \delta_j (N_{\gamma l}^m L_{\beta k m} - 2 h^{\gamma \alpha} [K_{\alpha j} L_{\beta k m} N_{\gamma j}^m] + N_{tm} \partial_j N_{\alpha}^m) + (K_{\beta \gamma l} + \partial h_{\beta \gamma l})(N_{\gamma k} L_{\alpha j m} + N_{km} \partial_j N_{\alpha}^m)
- L_{\beta k m} (\partial_j N_{lm} + \partial_l N_{jm} - \partial_m N_{jl}) + L_{\beta l m} (\partial_k N_{jm} + \partial_j N_{km} - \partial_m N_{jk}) \right\}
$$

$$
= R_{BA} h_{\beta}^B n_{\beta}^i. \quad (48)
$$

To keep consistency with the Lagrangian reduction we also perform conformal transformation (42) for Codazzi equation

$$
\tilde{R}_{BA} h_{\beta}^B n_{\alpha}^i = R_{BA} h_{\beta}^B n_{\alpha}^i + W(det N) = T_{BA} h_{\beta}^B n_{\beta}^i, \quad (49)
$$

$$
W(det N)
\equiv \frac{1}{2} N^{-1} \left\{ D_{\beta} \partial_j N_{\gamma l} + \frac{1}{2(D-2)} D_{\beta} \ln \det N \partial_j \ln \det N
- N^{-1} \left( \partial_k N_{\gamma l}^m \right) \partial_{jl} \ln \det N \right\}.
$$

In the last step we have used the higher dimensional Einstein equation where matter field tensor $T_{BA}$ is not equal to zero if D-dimensional spacetime is not pure gravity. In short eq.(49) gives out a constraint.

Finally, we write down the Ricci equation as following

$$
\nabla_{\gamma} L_{\beta j}^i - \nabla_{\beta} L_{\gamma j}^i + \tilde{K}_{\gamma j} L_{\alpha j}^i - \tilde{K}_{\alpha j} L_{\beta j}^i
$$

$$
+ L_{\gamma j}^k L_{\beta l k} - L_{\beta j}^k L_{\gamma l k} - N^{-1} j m (\partial_k N_{jk} + \partial_j N_{lk} - \partial_l N_{jk}) (K_{\gamma j m} + \frac{1}{2} \partial_{jm} h_{\beta \gamma l})
$$

$$
= R_{DBC} h_{\beta}^B h_{\gamma}^D N_{\gamma j m} n_{\beta}^i = N^{-1} i k R_{DBC} h_{\beta}^B h_{\gamma}^D. \quad (51)
$$

A simplest constraint occurs if $i=j$, i.e. the right hand side of eq.(51) vanishes.

Other constraint works when above Ricci equation is combined with Gauss equation.
4 Some examples

4.1 Isometric group SO(n) and SU(n)

Let us consider the isometric group SO(n) first. Thus, a spherical internal space $S^{n-1}(n = D - d + 1)$ is suitable for present topic, and the spherical harmonics will be a good instrument. The Killing vector can be written as

$$V^{IJ}_i = y^I \partial_i y^J - y^I \partial_i y^J.$$  (52)

By using properties of spherical harmonics

$$y^I y^I = 1, \quad \partial_i y^I \partial^i y^J + y^I y^J = \delta^{IJ}$$  (53)

and so on, we can show that

$$\partial_i V^{IJ} + \partial_j V^{IJ}_i = 0,$$  (54)

and

$$V^{IJ}_i \partial^i V^{KL} - V^{KL}_i \partial^i V^{IJ} = \frac{1}{2} (\delta^{IK} V^{JL}_j - \delta^{IL} V^{JK}_j + \delta^{IL} V^{JK}_j - \delta^{IK} V^{JL}_j),$$  (55)

which is the commutator of so(n) algebra. In this prescription

$$N^i_a = -V^i_{JJ}(u) A^{JJ}_a(x) \equiv N^i_a \partial^j y^J$$  (56)

$$N^a_j \equiv A^{JJ}_a y^J = (\mathbf{L} \cdot A^a)^{JJ} y^J$$  (57)

in which $\mathbf{L}$ is the generator of so(n).

If we introduce the scalar field tensor $T_{IJ}(x)$ which was used by authors of ref.[14] then

$$N_{ij} = \Delta^{-1} T^{-1}_{I,J} \partial_y y^I \partial_j y^J, \quad \Delta \equiv T_{IJ} y^I y^J,$$  (58)

and

$$N^{-1}_{ij} = 2 V^{IK}_i V^{JL}_j T_{K,L}.$$  (59)

Now we find

$$K_{\alpha\beta i} = -\frac{1}{2} [\partial_i h_{\alpha\beta} + \Delta^{-1} T^{-1}_{I,J} y^I \partial_i y^J (\mathbf{L} \cdot \mathbf{X})^{KJ}].$$  (60)

(in later ansatz metric $h_{\alpha\beta}$ is supposed independent of $u$) And the metric becomes

$$ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \Delta^{-1} T^{-1}_{I,J} [dy^I + (\mathbf{L} \cdot \mathbf{A}_a)^{IK} y^K dx^a][dy^J + (\mathbf{L} \cdot \mathbf{A}_b)^{JL} y^L dx^b].$$  (61)

In form, it looks like a D+1 dimensional metric, but with a constraint $y^I y_I = 1$. In fact, by using of these gauge fields and scalar fields the authors of ref.[14]
found a full nonlinear ansatz truncated to massless fields for 11-dimensional supergravity reduced to 7-dimensional spacetime through $S^4$ spherical reduction in which a form field ansatz $F_{(4)}$ guaranteed the consistency. But there is a Chern-Simons FFA term which makes things a little complicated. Follow closely another group of authors found a full nonlinear ansatz for 10-dimensional IIB supergravity reduced to 5 spacetime on $S^5$ which is particularly relevant for AdS/CFT correspondence. In this ansatz except the 10-dimensional gravitation we have a selfdual 5 form field. By means of this example we may use D=10 to d=5 Gauss equation to reduce the system

$$
\mathcal{L} = \sqrt{-\hat{g}} (\hat{R}_{(10)} - \frac{1}{5!} G_{ABCDEFG}^{ABCDEFG})
$$

in which ansatz $G_{(5)}$ is given in [15] (let coupling constant $g=1$)

$$
G_{\alpha\beta\gamma\delta\epsilon} = -U \epsilon_{\alpha\beta\gamma\delta\epsilon} + T_{IJ}^{-1} \epsilon_{\alpha\beta\gamma\delta\epsilon} D^0 T^{JK} y_K (L \cdot A_j)^I y_L - \frac{1}{2} T_{IJ}^{-1} \epsilon_{\alpha\beta\gamma\delta\epsilon} (L \cdot F^{\eta\zeta})^{IJ} (L \cdot A_k)^K y_M (L \cdot A_j)^L y_N - (\delta \leftrightarrow \epsilon)
$$

(63)

$$
G_{\alpha\beta\gamma i} = T_{IJ}^{-1} \epsilon_{\alpha\beta\gamma\delta\epsilon} D^0 T^{JK} y_K \partial_i y^J - T_{IK}^{-1} T_{JK}^{-1} \epsilon_{\alpha\beta\gamma\delta\epsilon} (L \cdot F^{\eta\zeta})^{IJ} (L \cdot A_k)^K y_M \partial_i y^L
$$

(64)

$$
G_{\alpha\beta\gamma} = -T_{IK}^{-1} T_{JL}^{-1} \epsilon_{\alpha\beta\gamma\delta\epsilon} (L \cdot F^{\eta\zeta})^{IJ} \partial_i y^K \partial_j y^L,
$$

(65)

where

$$
U \equiv 2 T_{IJ} T^{JK} y_K y_L - \Delta T^I_J, \quad \Delta \equiv T_{IJ} y^I y^J,
$$

(66)

$$
(L \cdot F)^{IJ} = d(L \cdot A)^{IJ} + (L \cdot A)^{IK} \wedge (L \cdot A)^J_K,
$$

(67)

$$
D_\alpha T_{IJ} = \partial_\alpha T_{IJ} + (L \cdot A_\alpha)_{IK} T^K_J + T_{IK} (L \cdot A_\alpha)_J^k.
$$

(68)

Following the ordinary logic, to substitute 10-dimensional formula(43) into (62) we ought to gain the 5-dimensional Lagrangian, i.e. the $\mathcal{L}_5$ in ref.[15]. However, gazing at eq.(43) it seems difficult to get the expectant result. In view of demonstration of consistency of known ansatz is only at the level of equations of motion, it needs more effort for checking ansatz with action. Because the calculation is complicated, so we prefer to let them for further investigation. Moreover, we need energy-momentum tensor for Codazzi constrain

$$
T_{AB} = -\frac{1}{4!} (G_{ACDEF} G_B^{CDEF} - \frac{1}{10} \hat{g}_{AB} G_{ABCDEFG}^{ABCDEFG}).
$$

(69)

Next, we examine isometric group SU(n). Since SU(n) is a subgroup of SO(2n),
we may still use the spherical harmonics to describe metric and others. Let \( \mathbf{T} \) be
the SU(n) generator in 2n-dimensional representation, and \( \mathbf{t} \) in basic representation.
To characterize Kaluza-Klein reduction for isometric group SU(n) what we have to do is to change SO(n) generator \( \mathbf{L} \) to SU(n) generator \( \mathbf{T} \). Especially we now have
\[
N^I_\alpha = (\mathbf{T} \cdot \mathbf{A})^{IJ} y_J.
\] (70)
Because the number of spherical harmonics is even, we may arrange them in pair. Let \( a = i, \cdots, n \) be the first half of \( I \), we choose that
\[
z^a = y^a + iy^{a+n},
\] (71)
and
\[
(t)^{ab} = (T)^{ab} + i(T)^{a+n \ b},
\] (72)
where
\[
(T)^{ab} = (T)^{a+n \ b+n},
\] (73)
so that
\[
N^a_\alpha = N^I_\alpha + iN^I=\alpha+n = \Re N^\alpha + i\Im N^a
\] (74)
\[
= (T \cdot \mathbf{A})^a I y_I + i(t \cdot \mathbf{A})^{a+n I} y_I = (t \cdot \mathbf{A})^{ab} z_b.
\] (75)
Two special examples are (i) SU(2)
\[
\mathbf{t} = \frac{1}{2i} \tau, \quad \tau \sim \text{Pauli matrix},
\] (76)
\[
\mathbf{T} = \frac{1}{2} \Sigma, \quad \Sigma_1 = L_{14} + L_{23}, \quad \Sigma_2 = -(L_{34} + L_{12}), \quad \Sigma_3 = -(L_{24} + L_{31});
\] (77)
(ii) SU(3)
\[
\mathbf{t} = \frac{1}{2i} \lambda, \quad \lambda \sim \text{Gell-Mann matrix},
\] (78)
\[
\mathbf{T} = \frac{1}{2} \Lambda, \quad \Lambda_1 = L_{15} + L_{24}, \quad \Lambda_2 = -(L_{12} + L_{45}), \quad \Lambda_3 = L_{14} - L_{25},
\] (79)
\[
\Lambda_4 = L_{16} + L_{34}, \quad \Lambda_5 = -(L_{13} + L_{46}), \quad \Lambda_6 = L_{26} + L_{35},
\] (80)
\[
\Lambda_7 = -(L_{23} + L_{56}), \quad \Lambda_8 = \frac{1}{\sqrt{3}}(L_{14} + L_{25} - 2L_{36}).
\] (81)
Suppose that the scalar tensor \( T_{IJ} \) keeps in real and possesses block diagonal form. The subset of "SO(2n) metric" (61) becomes
\[
g_{AB} = 
\begin{pmatrix}
    h_{\alpha \beta} + \Delta^{-1} T^{-1}_{ab} [\Re N^a N^b + \Im N^a \Im N^b] & \Delta^{-1} T^{-1}_{a \beta} N^b & \Delta^{-1} T^{-1}_{b \beta} N^a \\
    \Delta^{-1} T^{-1}_{a \alpha} N^b & \Delta^{-1} T^{-1}_{ab} N^b & \Delta^{-1} T^{-1}_{b \alpha} N^b \\
    \Delta^{-1} T^{-1}_{a \alpha} N^b & \Delta^{-1} T^{-1}_{a \beta} N^b & \Delta^{-1} T^{-1}_{ab} N^b
\end{pmatrix}
\] (82)
and \( A(\alpha, a, \beta, b, b) = 1, \cdots, D+1(= d+2n) \). Obviously, they are equivalent to
\[
ds^2 = h_{\alpha \beta} dx^\alpha dx^\beta + \Delta^{-1} [dz^a + (t \cdot \mathbf{A})^{ac} z^c]^\dagger T^{-1}_{ab} [dz^b + (t \cdot \mathbf{A})^{bd} z^d]
\] (83)
with constraint
\[ z^a \dagger z_a = 1. \]  
(84)

In above IIB supergravity ansatz it seems that an SU(3) isometric group ansatz can be embeded in it.

### 4.2 Kaluza-Klein monopole and instanton

Because that the Gauss-Codazzi-Ricci equations depend on gauge potential \( A \), we have pointed that these equations may be set up in distinct neighborhoods.\[17\]

For 11-dimensional Kaluza-Klein monopole the metric is denoted as
\[ ds_{11}^2 = e^{-\phi} ds_{10}^2 + e^{2\phi}(dx^{10} + A^\pm)^2, \]  
(85)

and
\[ ds_{10}^2 = e^\phi dx^\mu dx_\mu + e^{-\frac{7}{6}} ds_3^2, \quad \mu = 0, \ldots, 6 \]  
(86)

\[ ds_3^2 = dy_i dy_i = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  
(87)

in which \( \phi \) is a dilaton and gauge fields will write in the Wu-Yang gauge\[19\]
\[ A^\pm = \frac{Q_m}{2r(y_3 \pm r)}(y_1 dy_2 - y_2 dy_1) = \frac{1}{2} Q_m(\pm - \cos \theta) d\phi, \]  
(88)

\[ F_2 = \frac{1}{\sqrt{-h}} \frac{Q_m}{r^2} \epsilon ijk y_i dy_j dy_k = dA^\pm. \]  
(89)

These construct a monopole bundle over base space \( S^2 \) with fiber U(1).

As for instanton we have to look for a fiber bundle over base space \( S^4 \) with fiber SU(2). Starting from eq.\((75)\) we set \( t = \frac{1}{2} \tau \) then we need to take a \( S^4 \) part out of the d-dimensional metric. It would be better to choose the polar coordinates of 5-dimensional de Sitter space with radius \( a/2 \)\[20\]
\[ ds_4^2 = [dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)]/(1 + \frac{r^2}{a^2})^2 \]  
(90)

in which
\[ \sigma_i = \frac{1}{r^2}(x_i dx_0 - x_0 dx_i + \epsilon_{ijk} x_j dx_k), \quad r^2 = x^\mu x_\mu \quad \mu = 0, 1, 2, 3. \]  
(91)

The gauge potential of BPST instanton will be
\[ \frac{1}{2} \tau \cdot A^{(+)}_\mu = \frac{r^2}{r^2 + a^2} i \sigma \cdot \tau = -\frac{\sigma_{\mu\nu} x^\nu}{r^2 + a^2}, \]  
(92)

in "north" hemisphere of \( S^4 \); and
\[ \frac{1}{2} \tau \cdot A^{(-)}_\mu = \frac{a^2}{r^2 + a^2} i \sigma \cdot \tau = \frac{a^2 \sigma_{\mu\nu} x^\nu}{r^2(r^2 + a^2)}, \]  
(93)
in "south" hemisphere of $S^4$, where
\[
\sigma_{ij} = \bar{\sigma}_{ij} = \frac{1}{2} \epsilon_{ijk} \tau_k, \quad \sigma_{0i} = -\bar{\sigma}_{0i} = \frac{1}{2} \tau_i, \quad \sigma_{\mu\nu} = -\sigma_{\nu\mu}, \quad (94)
\]
The field strengths are
\[
\mathcal{F}^{(+)} = \frac{2 i a^2 \tau_k}{(r^2 + a^2)^2} (dr \wedge r \sigma_k + \frac{1}{2} r^2 \epsilon_{kij} \sigma_i \wedge \sigma_j), \quad (95)
\]
\[
\mathcal{F}^{(-)} = h \mathcal{F}^{(+)} h^{-1}, \quad h = \frac{t - ix \cdot \tau}{r}. \quad (96)
\]
Of course, we can also use the t’Hooft or Jackiw-Nohl-Rebbi multiple instanton solution or other instanton solution.

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