On an logarithmic equation by primes

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Abstract

Let \( \lfloor \cdot \rfloor \) be the floor function. In this paper we show that every sufficiently large positive integer \( N \) can be represented in the form

\[
N = \lfloor p_1 \log p_1 \rfloor + \lfloor p_2 \log p_2 \rfloor + \lfloor p_3 \log p_3 \rfloor,
\]

where \( p_1, p_2, p_3 \) are prime numbers. We also establish an asymptotic formula for the number of such representations, when \( p_1, p_2, p_3 \) do not exceed given sufficiently large positive number.

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1 Introduction and main result

A remarkable moment in analytic number theory is 1937, when Vinogradov \cite{9} proved the ternary Goldbach problem. He showed that every sufficiently large odd integer \( N \) can be represented in the form

\[
N = p_1 + p_2 + p_3,
\]

where \( p_1, p_2, p_3 \) are prime numbers.

The consequences of Vinogradov’s \cite{10} ingenious method for estimating exponential sums over primes continue to this day in analytic number theory.

In 1995 Laporta and Tolev \cite{6} investigated an analogue of the Goldbach-Vinogradov theorem. They considered the diophantine equation

\[
N = \lfloor p_1^c \rfloor + \lfloor p_2^c \rfloor + \lfloor p_3^c \rfloor,
\]

where \( p_1, p_2, p_3 \) are primes. For \( 1 < c < 17/16 \) they showed that for the sum

\[
R(N) = \sum_{N=\lfloor p_1^c \rfloor+\lfloor p_2^c \rfloor+\lfloor p_3^c \rfloor} \log p_1 \log p_2 \log p_3
\]
the asymptotic formula

\[ R(N) = \frac{\Gamma^3(1 + 1/c)}{\Gamma(3/c)} N^{3/c-1} + O\left( N^{3/c-1} \exp\left( - (\log N)^{1/3-\varepsilon} \right) \right) \]  

holds.

Subsequently the result of Laporta and Tolev was sharpened by Kumchev and Nedeva [5] to

\[ 1 < c < \frac{12}{11}, \]

by Zhai and Cao [11] to

\[ 1 < c < \frac{258}{235}, \]

by Cai [2] to

\[ 1 < c < \frac{137}{119}. \]

Overcoming all difficulties Zhang and Li [14] improved the result of Cai to

\[ 1 < c < \frac{3113}{2703} \]

and this is the best result up to now.

On the other hand recently the author [3] showed that when \( N \) is a sufficiently large positive number and \( \varepsilon > 0 \) is a small constant then the logarithmic inequality

\[ |p_1 \log p_1 + p_2 \log p_2 + p_3 \log p_3 - N| < \varepsilon, \]

has a solution in prime numbers \( p_1, p_2, p_3 \).

Motivated by these results in this paper we introduce new diophantine equation with prime numbers.

Consider the logarithmic equation

\[ N = [p_1 \log p_1] + [p_2 \log p_2] + [p_3 \log p_3], \]  

where \( N \) is a sufficiently large positive integer. Having the arguments of the aforementioned marvellous mathematicians and [3] we expect that (2) has a solutions in primes \( p_1, p_2, p_3 \). Define the sum

\[ \Gamma = \sum_{N=[p_1 \log p_1]+[p_2 \log p_2]+[p_3 \log p_3]} \log p_1 \log p_2 \log p_3. \]  

We make the first attempt and prove the following theorem.
Theorem 1. Let \( N \) is a sufficiently large positive integer. Let \( X \) is a solution of the equality
\[
X \log X = N.
\]
Then the asymptotic formula
\[
\Gamma = \frac{X^2}{1 + \log X} + O\left( X^2 \exp \left( - (\log X)^{1/3-\epsilon} \right) \right)
\]
holds.

As usual the corresponding binary problem is out of reach of the current state of analytic number theory. In other words we have the following challenge.

Conjecture 1. Let \( N \) is a sufficiently large positive integer. Then the logarithmic equation
\[
N = [p_1 \log p_1] + [p_2 \log p_2]
\]
is solvable in prime numbers \( p_1, p_2 \).

Needless to say we believe that in the near future we will see the solution of this binary logarithmic hypothesis.

2 Notations

The letter \( p \) with or without subscript will always denote prime number. We denote by \( \Lambda(n) \) von Mangoldt’s function. Moreover \( e(y) = e^{2\pi i y} \). As usual \([t]\) and \( \{t\}\) denote the integer part, respectively, the fractional part of \( t \). We recall that \( t = [t] + \{t\} \) and \( \|t\| = \min(\{t\}, 1 - \{t\}) \). By \( \epsilon \) we denote an arbitrary small positive constant, not the same in all appearances. Let \( N \) be a sufficiently large positive integer. Let \( X \) is a solution of the equality
\[
X \log X = N. \tag{5}
\]
Let \( y \) be an implicit function of \( t \) defined by
\[
y \log y = t. \tag{6}
\]
The first derivative of \( y \) is
\[
y' = \frac{1}{1 + \log y}. \tag{7}
\]
Denote
\[ \tau = X^{-\frac{23}{25}}; \tag{8} \]
\[ S(\alpha) = \sum_{p \leq X} e(\alpha[p \log p]) \log p; \tag{9} \]
\[ \Theta(\alpha) = \sum_{m \leq N} \frac{1}{1 + \log y(m)} e(m\alpha); \tag{10} \]
\[ \Gamma_1 = \int_{-\tau}^{\tau} S^3(\alpha)e(-N\alpha) d\alpha; \tag{11} \]
\[ \Gamma_2 = \int_{1/2}^{\tau} S^3(\alpha)e(-N\alpha) d\alpha; \tag{12} \]
\[ \Psi_k = \int_{-1/2}^{1/2} \Theta^k(\alpha)e(-N\alpha) d\alpha, \quad k = 1, 2, 3, \ldots; \tag{13} \]
\[ \tilde{\Psi} = \int_{-\tau}^{\tau} \Theta^3(\alpha)e(-N\alpha) d\alpha. \tag{14} \]

3 Lemmas

Lemma 1. Let \( f(x) \) be a real differentiable function in the interval \([a, b]\). If \( f'(x) \) is a monotonous and satisfies \( |f'(x)| \leq \theta < 1 \). Then we have
\[ \sum_{a < n \leq b} e(f(n)) = \int_a^b e(f(x)) \, dx + \mathcal{O}(1). \]

Proof. See ([7], Lemma 4.8).

Lemma 2. Let \( x, y \in \mathbb{R} \) and \( H \geq 3 \). Then the formula
\[ e(-x\{y\}) = \sum_{|h| \leq H} c_h(x)e(hy) + \mathcal{O}\left(\min\left(1, \frac{1}{H\|y\|}\right)\right) \]
holds. Here
\[ c_h(x) = \frac{1 - e(-x)}{2\pi i(h + x)}. \]

Proof. See ([1], Lemma 12).
Lemma 3. (Van der Corput) Let $f(x)$ be a real-valued function with continuous second derivative in $[a, b]$ such that
\[ |f''(x)| \asymp \lambda, \quad (\lambda > 0) \quad \text{for} \quad x \in [a, b]. \]
Then
\[ \left| \sum_{a < n \leq b} e(f(n)) \right| \ll (b-a)\lambda^{\frac{3}{2}} + \lambda^{-\frac{1}{2}}. \]

Proof. See ([4], Ch. 1, Th. 5).

Lemma 4. For any real number $t$ and $H \geq 1$, there holds
\[ \min \left( 1, \frac{1}{H\|t\|} \right) = \sum_{h=-\infty}^{+\infty} a_h e(ht), \]
where
\[ a_h \ll \min \left( \log 2H, \frac{1}{|h|}, \frac{H}{|h|^2} \right). \]

Proof. See ([12], Lemma 2).

4 Proof of the Theorem

From (3), (9), (11) and (12) we have
\[ \Gamma = \int_{0}^{1} S^3(\alpha)e(-N\alpha) \, d\alpha = \Gamma_1 + \Gamma_2. \quad (15) \]

Estimation of $\Gamma_1$

We write
\[ \Gamma_1 = (\Gamma_1 - \tilde{\Psi}) + (\tilde{\Psi} - \Psi_3) + \Psi_3. \quad (16) \]

Bearing in mind (10) and (13) we obtain
\[ \Psi_1 = \int_{-1/2}^{1/2} \Theta(\alpha)e(-N\alpha) \, d\alpha = \frac{1}{1 + \log y(N)}. \]

Suppose that
\[ \Psi_k = \frac{1}{1 + \log y(N)}X^{k-1} + O(X^{k-2}) \quad \text{for} \quad k \geq 2. \quad (17) \]
Then
\[ \Psi_{k+1} = \sum_{m \leq N} \frac{1}{1 + \log y(m)} \left( \sum_{m_1 \leq N-m} \cdots \sum_{m_k \leq N-m} \frac{1}{1 + \log y(m_1)} \cdots \frac{1}{1 + \log y(m_k)} \right) \]
\[ = \sum_{m \leq N} \frac{1}{1 + \log y(m)} \left( \frac{1}{1 + \log y(N-m)} X^{k-1} + O(X^{k-2}) \right) \]
\[ = \sum_{m \leq N} \frac{1}{1 + \log y(m)} \cdot \frac{1}{1 + \log y(N-m)} X^{k-1} + O(X^{k-1}) \]
\[ = \frac{1}{1 + \log y(N)} X^{k} + O(X^{k-1}). \]

Consequently the supposition (17) is true.

From (5) and (6) it follows that
\[ y(N) = X. \] (18)

Bearing in mind (17) and (18) we conclude that
\[ \Psi_k = \frac{X^{k-1}}{1 + \log X} + O(X^{k-2}) \quad \text{for} \quad k \geq 2. \] (19)

Now the asymptotic formula (19) gives us
\[ \Psi_3 = \frac{X^2}{1 + \log X} + O(X). \] (20)

From (11) and (14) we get
\[ |\Gamma_1 - \tilde{\Psi}| \ll \int_{-\tau}^{\tau} |S^3(\alpha) - \Theta^3(\alpha)| \, d\alpha \]
\[ \ll \max_{|\alpha| \leq \tau} |S(\alpha) - \Theta(\alpha)| \left( \int_{-\tau}^{\tau} |S(\alpha)|^2 \, d\alpha + \int_{-1/2}^{1/2} |\Theta(\alpha)|^2 \, d\alpha \right). \] (21)

Arguing as in [3, Lemma 8] we find
\[ \int_{-\tau}^{\tau} |S(\alpha)|^2 \, d\alpha \ll X \log X. \] (22)

Square out and integrate we obtain
\[ \int_{-1/2}^{1/2} |\Theta(\alpha)|^2 \, d\alpha \ll \frac{N}{\log^2 N} \ll X. \] (23)
Now we shall estimate from above $|S(\alpha) - \Theta(\alpha)|$ for $|\alpha| \leq \tau$.
Our argument is a modification of Zhang’s and Li’s [13] argument.

From (8) and (9) we get
\[
S(\alpha) = \sum_{p \leq X} e(\alpha \log p) \log p + \mathcal{O}(\tau X) = \sum_{n \leq X} \Lambda(n)e(\alpha \log n) + \mathcal{O}(X^{1/2}) + \mathcal{O}(\tau X)
\]
\[= \sum_{n \leq X} \Lambda(n)e(\alpha \log n) + \mathcal{O}(X^{1/2}). \quad (24)\]

From $|\alpha| \leq \tau$, $y \geq 2$ and Lemma [14] we have that
\[
\sum_{1 < m \leq y} e(m\alpha) = \int_1^y e(\alpha t) \, dt + \mathcal{O}(1). \quad (25)
\]

Using (6), (7), (8), (10), (25) and partial summation we find
\[
\sum_{n \leq X} \Lambda(n)e(\alpha \log n) = \int_1^X e(\alpha \log y) \, dy \left( \sum_{n \leq y} \Lambda(n) \right)
\]
\[= \int_1^X e(\alpha \log y) \, dy + \mathcal{O}\left( X \exp \left( - (\log X)^{1/3} \right) \right)
\]
\[= \int_1^X e(\alpha t) \frac{1}{1 + \log y(t)} \, dt + \mathcal{O}\left( X \exp \left( - (\log X)^{1/3} \right) \right)
\]
\[= \int_1^N \frac{1}{1 + \log y(t)} \, dt \left( \int_1^t e(\alpha u) \, du \right) + \mathcal{O}\left( X \exp \left( - (\log X)^{1/3} \right) \right)
\]
\[= \int_1^N \frac{1}{1 + \log y(t)} \, dt \left( \sum_{1 < m \leq t} e(m\alpha) + \mathcal{O}(1) \right)
\]
\[+ \mathcal{O}\left( X \exp \left( - (\log X)^{1/3} \right) \right)
\]
\[= \sum_{m \leq N} \frac{1}{1 + \log y(m)} e(m\alpha) + \mathcal{O}\left( X \exp \left( - (\log X)^{1/3} \right) \right)
\]
\[= \Theta(\alpha) + \mathcal{O}\left( X \exp \left( - (\log X)^{1/3} \right) \right). \quad (26)
\]

From (24) and (26) it follows that
\[
\max_{|\alpha| \leq \tau} |S(\alpha) - \Theta(\alpha)| \ll X \exp \left( - (\log X)^{1/3} \right). \quad (27)
\]
Taking into account (21), (22), (23) and (27) we conclude
\[ \Gamma_1 - \tilde{\Psi} \ll X^2 \exp \left( - (\log X)^{1/3-\varepsilon} \right). \] (28)

Using (8), (13), (14) and working as in ([8], Lemma 2.8) we deduce
\[ |\Psi_3 - \tilde{\Psi}| \ll \int_{\tau \leq |\alpha| \leq 1/2} |\Theta(\alpha)|^3 \, d\alpha \ll \int_{\tau}^{1/2} \frac{d\alpha}{\alpha^3} \ll X^{\frac{25}{26}}. \] (29)

Summarizing (16), (20), (28) and (29) we obtain
\[ \Gamma_1 = \frac{X^2}{1 + \log X} + O \left( X^2 \exp \left( - (\log X)^{1/3-\varepsilon} \right) \right). \] (30)

**Estimation of \( \Gamma_2 \)**

From (12) we get
\[ \Gamma_2 \ll \max_{\tau \leq |\alpha| \leq 1-\tau} |S(\alpha)| \int_0^1 |S(\alpha)|^2 \, d\alpha \ll X (\log X) \max_{\tau \leq |\alpha| \leq 1-\tau} |S(\alpha)|. \] (31)

By (9) and Lemma 2 with \( x = \alpha, \, y = n \log n \) and
\[ H = X^{\frac{1}{26}} \] (32)

it follows
\[ S(\alpha) = \sum_{n \leq X} \Lambda(n)e(\alpha n \log n)e(-\alpha \{ n \log n \}) + O(X^{1/2}) \]
\[ = \sum_{|h| \leq H} c_h(\alpha) \sum_{n \leq X} \Lambda(n)e((h + \alpha)n \log n) \]
\[ + O \left( (\log X) \sum_{n \leq X} \min \left( 1, \frac{1}{H\|n \log n\|} \right) \right). \]

Therefore
\[ \max_{\tau \leq |\alpha| \leq 1-\tau} |S(\alpha)| \ll (S_1 + S_2) \log X, \] (33)

where
\[ S_1 = \max_{\tau \leq |\alpha| \leq H+1} \left| \sum_{n \leq X} \Lambda(n)e(\alpha n \log n) \right|, \] (34)
\[ S_2 = \sum_{n \leq X} \min \left( 1, \frac{1}{H\|n \log n\|} \right). \] (35)
Bearing in mind (8), (32) and (34), according to ([3], Lemma 9) we conclude

\[ S_1 \ll X^{24/25} \log^3 X. \] (36)

By (32), (35), Lemma 3, Lemma 4 and \( Y \leq X/2 \) we obtain

\[ S_2 \ll (\log X) \sum_{Y < n \leq 2Y} \min \left( 1, \frac{1}{H \|n \log n\|} \right) \]
\[ \leq (\log X) \sum_{h = -\infty}^{+\infty} |a_h| \sum_{Y < n \leq 2Y} e(\log n) \]
\[ \ll (\log X) \left( \frac{Y \log 2H}{H} + \frac{Y^{1/2} \log 2H}{H} \sum_{h \leq H} h^{1/2} + Y^{1/2} H \sum_{h > H} h^{-3/2} \right) \]
\[ \ll XH^{-1} \log^2 X \]
\[ \ll X^{24/25} \log^2 X. \] (37)

From (31), (33), (36) and (37) we find

\[ \Gamma_2 \ll X^{49/25} \log^5 X. \] (38)

The end of the proof

Bearing in mind (15), (30) and (38) we establish the asymptotic formula (4).

The Theorem is proved.

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