Analytical and numerical studies of the cold electromagnetic LH wave equation in the mode conversion regime

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Abstract. The purpose of this study is to investigate the effect of the presence of the mode conversion (n_z < n_z critical), on the evolution of the electric field of the lower hybrid (LH) propagating wave. The propagation of the LH wave in a magnetized plasma, for both mode polarization (slow and fast), is described by a fourth order ordinary differential equation, which here has been analyzed analytically and numerically. A complete WKB solution of the 4th order equation has also been obtained showing the failure of the method near the cut-offs and the mode conversion point.

1. Introduction.
The study and the analysis of the lower hybrid waves (LH) are a broad and interesting field of research. The LH waves [1], in fact, exhibit the highest current drive efficiency at low plasma temperature thus allowing the control of the safety factor (q) profile: this is a desirable feature in ITER where it is important to maintain the burning plasma performance on a long time scale. Moreover, the LH waves are suitable for scenarios based on internal transport barrier [2]; their control on q can produce a safety factor profile with negative magnetic shear thus rising an internal barrier to the electron energy transport.

The difficulties in coupling the LH waves in severe edge condition, the concern of an array structure near the plasma and the impossibility for the LH of penetrating the core of a Q=10 ITER plasma, excluded the LH from the ITER first phase heating system. Anyway, the advantages mentioned above encourage the effort devoted to the study of the LH waves in many tokamak experiments like JET [1], Tore Supra [3], JT-60U [4], FTU [5].

Of course, the efficient transfer of electromagnetic power to the plasma is an important goal. First, the power is transferred from the radiofrequency generator to the plasma; this can be achieved if the generator impedance is made equal to the conjugate of the plasma impedance. Second, the electromagnetic wave, propagating in the plasma, has to transfer its energy to electrons or ions. In real plasmas, the non-uniformity of magnetic field and the plasma densities give rise to critical layers where dissipation may occur with a consequent loose of efficiency in energy transfer. In the region of this critical layer, the wave may be reflected (cut-off), absorbed (resonance) and converted into a companion mode (mode conversion).

LH waves, generally, have two propagating modes depending on the polarization: the slow and the fast modes characterized by an E_z/E_x and E_y/E_x wave field polarization respectively. Usually the LH antenna (waveguide grill) is able to launch the slow wave mode, with a power spectrum (power vs parallel wave-number, n_z), which depends on the structural elements of the antenna. The parallel wave
number changes when the wave is propagating into the plasma, and if the condition $n_z \leq n_{z-crit}$ is encountered, part of the slow-wave power can be transformed into the fast wave (mode conversion). Diffraction effects, localized near the mode conversion layer, can play a crucial role in spreading the power to higher components of the parallel wave-number spectrum, and consequently induce a more effective and peripheral wave absorption. An analytical and numerical approach to the problem is presented in the following section. The full electromagnetic wave equation (a fourth order ordinary differential equation for the electric field) has been analytically and numerically solved in order to verify this conjecture. A solution based on the WKB expansion [6] has also been obtained in order to show the failure of this method near the cut-offs and the mode conversion.

To simplify the study, it will be assumed that the wave propagates in a plane-stratified plasma and the wavelength $\lambda$ of the radiation is much smaller than the characteristic length $L$ of the non-homogeneities ($\lambda << L$).

2. Wave equation for an LH wave

It is possible to obtain the wave equation valid for the LH waves from the Maxwell-Vlasov system of equations

$$\nabla \times \nabla \times \vec{E}(r, \omega) - \frac{\omega^2}{c^2} \vec{E}(r, \omega) \cdot \vec{E}(r, \omega) = 0$$

(1)

where $\vec{E}(r, \omega) = I + j \frac{4\pi}{\omega} \sigma(r, \omega)$ is the cold dielectric tensor,

$$\varepsilon = \begin{pmatrix}
\varepsilon_{xx}(x) & \varepsilon_{xy}(x) & 0 \\
-\varepsilon_{xy}(x) & \varepsilon_{xx}(x) & 0 \\
0 & 0 & \varepsilon_{zz}(x)
\end{pmatrix} = \begin{pmatrix}
S(x) - jD(x) & 0 \\
0 & 0 & 0 \\
jD(x) & S(x) & 0
\end{pmatrix}$$

$I$ the unit tensor, $\sigma$ the conductivity and $E$ is the oscillating electric field of the wave. Note that in Stix’s notations the elements of the cold plasma dielectric tensor are:

$$S = 1 - \sum_s \frac{\omega^2_{ps}}{\omega^2 - \omega^2_{cs} g(\vec{r})} f(\vec{r})$$

$$D = \sum_s \frac{\omega^2_{ps}}{\omega} \frac{\omega^2_{ps}}{\omega^2 - \omega^2_{cs} g(\vec{r})}$$

$$P = 1 - \sum_s \frac{\omega^2_{ps}}{\omega^2} f(\vec{r})$$

and $f$ and $g$ are suitable space functions of density and external magnetic field, and $\omega_{ps}$, $\omega_{cs}$ are the plasma and cyclotron frequencies for the species $s$.

Several assumptions have been made to obtain the above equation from the Maxwell-Vlasov system. First of all it has been considered the field amplitude sufficiently small so that $W_{wave} \equiv |E|^2 \ll W_{thermal} \equiv kT$; this implies that the kinetic equation can be linearized. The second assumption concerns the possibility of assuming the local thermodynamic equilibrium, while the third assumption relies on the drift orbit approximation (weak space non-homogeneity). Moreover the steady state (Fourier analysis in time) and the cold plasma limit have been assumed. It’s worth noting, although of numerous simplifying assumptions, the propagation of the electromagnetic field in a non-homogenous medium is not a straightforward problem [7], [8] because the wave equation has non-constant coefficients.
Fourier transforming from space coordinate \((y,z)\) domain to the wave vector \((k_y,k_z)\) domain and assuming \(k_y = 0\) (grill-antenna radiation pattern) for slab geometry, the electric field becomes
\[
\mathbf{E}(r,\omega) = \mathbf{E}(x,\omega)e^{jk(x,k_y,k_z)}
\]
and the wave equation (1), in matrix form, reads:
\[
\begin{pmatrix}
  k_z^2 & 0 & jk_z \frac{d}{dx} \\
  0 & k_z^2 - \frac{d^2}{dx^2} & 0 \\
  jk_z \frac{d}{dx} & 0 & -\frac{d^2}{dx^2}
\end{pmatrix}
\begin{pmatrix}
  E_x \\
  E_y \\
  E_z
\end{pmatrix}
- \omega^2
\begin{pmatrix}
  \varepsilon_{xx} & \varepsilon_{xy} & 0 \\
  -\varepsilon_{xy} & \varepsilon_{xx} & 0 \\
  0 & 0 & \varepsilon_{zz}
\end{pmatrix}
\begin{pmatrix}
  E_x \\
  E_y \\
  E_z
\end{pmatrix}
= 0
\]

If \(B\) lies along the \(z\)-axis and in the hypothesis of slightly inhomogeneous plasma, equation (2) can be expanded to obtain:
\[
\begin{align*}
  \frac{d^2 E_y}{dx^2} + \frac{k_z D}{(n_z^2 - S)} \frac{dE_z}{dx} - \frac{\omega^2 (n_z^2 - S)^2 - D^2}{c^2} E_y = 0 \\
  \frac{d^2 E_z}{dx^2} + \frac{k_z D}{S} \frac{dE_y}{dx} - \frac{\omega^2 (n_z^2 - S)P}{S} E_z = 0
\end{align*}
\]
where \(n_z = k_z c / \omega\) is the refractive index and \(E_z = (-jDE_y - jk_z c^2 \omega^2 dE_z / dx) / (n_z^2 - S)\). Equation (3) will be written using normalized quantities as follow:
\[
\begin{align*}
  \frac{d^2 y(\bar{x})}{d\bar{x}^2} + \delta_0^{-1} \alpha(\bar{x}) \frac{dz(\bar{x})}{d\bar{x}} + \delta_0^{-1} \beta(\bar{x}) y(\bar{x}) &= 0 \\
  \frac{d^2 z(\bar{x})}{d\bar{x}^2} + \delta_0^{-1} \alpha(\bar{x}) \frac{dy(\bar{x})}{d\bar{x}} + \delta_0^{-1} \beta(\bar{x}) z(\bar{x}) &= 0
\end{align*}
\]
where
\[
\bar{x} = \frac{x}{a}, \quad \delta_0^{-1} = \frac{\omega a}{c},
\]
\[
y(\bar{x}) = \frac{E_y(\bar{x})}{\sqrt{E_x^2 + E_y^2 + E_z^2}}, \quad z(\bar{x}) = \frac{E_z(\bar{x})}{\sqrt{E_x^2 + E_y^2 + E_z^2}},
\]
\[
\alpha(\bar{x}) = \frac{n_z D}{n_z^2 - S}, \quad \alpha'(\bar{x}) = \frac{n_z D}{S},
\]
\[
\beta(\bar{x}) = -\frac{(n_z^2 - S)^2 - D^2}{n_z^2 - S}, \quad \beta'(\bar{x}) = -\frac{(n_z^2 - S)P}{S}
\]
\(a\) is the plasma radius.

3. Dispersion relation and accessibility condition
The coefficients of equations (4) are not constant because the plasma and cyclotron frequency are functions of the magnetic field and the plasma density. These quantities can be chosen to change along \(x\) as follow, (figure 1):
\[ B_z(x) = B_0 \left(1 + \frac{\alpha}{R} x \right)^{-1} \]

\[ n(x) = N \left(1 - (Mx)^2 \right)^{\alpha/2} \]

where \( B_0 \) is the magnetic field in the center of the plasma, \( R \) is major radius and \( N \) and \( M \) are constants whose values determine the plasma density in the center of the plasma and the plasma density variation along \( x \).

For this reason the perpendicular wave number \( k_x \) (or the refractive index \( n_x \)) changes when the wave crosses the inhomogeneous plasma, while the parallel \( k_z \) remains constant and settled by the antenna periodicity. In realistic cases the magnetic field and the metric of the chosen geometry depend, generally, by two space variables (e.g. radius and poloidal angle when assuming a pseudotoroidal geometry). This means that the parallel wave number is varying too; obviously in this case the space Fourier analysis in \( z \) direction is not yet allowed, and a partial differential system of equations must be considered instead of equation (4). In order to investigate the mode conversion phenomenon, and due to the weak variation of the parallel wave number in a 2D realistic geometry [6], we can assume the invariance of the parallel wave number. The dispersion relation and the associated accessibility condition describes how \( k_x \) changes; Fourier transforming equations (4), knowing that \( d/d\bar{x} \rightarrow j\delta_0^{-1} n_x \) and \( d^2/d\bar{x}^2 \rightarrow -\delta_0^{-2} n_x^2 \) one obtains:

\[
\begin{cases}
(-n_x^2 + \beta)y(\bar{x}) + j\alpha n_x z(\bar{x}) = 0 \\
j\alpha' n_x y(\bar{x}) + (-n_x^2 + \beta') z(\bar{x}) = 0
\end{cases}
\]

The previous system can be solved if the determinant of the coefficients is equal to zero, that is:

\[ n_x^4 - (\beta + \beta' - \alpha\alpha') n_x^2 + \beta\beta' = 0 \]

Solving the fourth order equation:

\[ n_{x=\pm} = \frac{\beta + \beta' - \alpha\alpha' \pm \sqrt{(\beta + \beta' - \alpha\alpha')^2 - 4\beta\beta'}}{2} \]

which are the perpendicular refractive indexes (or wave number) for the slow wave (plus sign) and the fast wave (minus sign).

As one can’t have damped oscillations, \( n_{x=}^2 \) has to be purely propagative (\( n_{x=}^2 > 0 \)) or purely evanescent (\( n_{x=}^2 < 0 \)) and this imposes a condition on the determinant of equation (6) establishing the accessibility condition as follow:

\[ (\beta + \beta' - \alpha\alpha')^2 - 4\beta\beta' \geq 0 \]
The accessibility condition, in fact, is the condition for avoiding the coalescence of the roots of the dispersion relation. This is depicted on the right hand side of figure 2, where the mode conversion happens around the slab coordinate $x=0.25$ and $x=0.35$.

If one retains the equal sign and explicits the $\beta, \beta', \alpha, \alpha'$ using equation (5), the $n_z$ index can be expressed as a function of $x$ just solving the following fourth degree algebraic equation:

$$
(S - P)^2(n_z^2 - S)^2 + 2(S + P)D^2(n_z^2 - S) + (D^4 + 4PSD^3) = 0
$$

This equation can be solved analytically for $n_z$, by considering that $S \ll P$ (this condition is valid everywhere into the plasma, except the cut-off point $P = 0$ at very low density), so we have

$$
n_{z,crit} = \sqrt{S} + \frac{D}{\sqrt{P}}
$$

This condition fixes a relation between $n_z$ (fixed by the antenna periodicity) and the space variable $x$ because $S, D$ and $P$ depend on the space variable $x$ through the density and the magnetic field. The region where the accessibility condition $\left(n_z > n_{z,crit}\right)$ is fulfilled is represented in the green shadow of figure 3.
The same equation can be solved in terms of the space variable $x$, instead of $n_z$: in this case, it is possible to have an analytical expression that shows the critical layer $x_{\text{crit}}$ (where the roots of the dispersion relation have a coalescence) as function of $n_z$. To this purpose, let’s consider the following approximated expressions for $S$, $D$ and $P$:

\[
S \approx 1; \quad P \approx -\frac{\omega_{pe}^2(x)}{\omega^2}; \quad D \approx \frac{\omega_{pe}^2(x)}{\omega_0^2} = -dP(x)
\]

where $d$ has been considered as constant. The solution is given in terms of $P$ (which is an explicit function of $x$)

\[
\left(\sqrt{|P|}\right)_{\text{crit}} = \left(\frac{n_z - 1}{|d|}\right) \frac{\Omega_{ce}}{\omega}
\]

Inserting in the above equations $n_z$ and the plasma parameters (central density and magnetic field) used to obtain figure 2 (right hand side), we find $x_{\text{crit}}\approx0.25$ and $x_{\text{crit}}\approx0.35$, which corresponds to the exact values found by solving numerically equation (7).

4. Solution of the LH wave equation

The second order coupled ordinary differential equation system can be combined to obtain the following fourth order differential equations:

\[
\frac{dy(\bar{x})}{d\bar{x}}^4 + \delta_0^{-2}(\beta + \beta - \alpha \alpha') \frac{dz(\bar{x})}{d\bar{x}}^4 + \delta_0^{-4} \beta \beta' y(\bar{x}) = 0
\]

\[
\frac{dz(\bar{x})}{d\bar{x}}^4 + \delta_0^{-2}(\beta + \beta - \alpha \alpha') \frac{dy(\bar{x})}{d\bar{x}}^4 + \delta_0^{-4} \beta \beta' z(\bar{x}) = 0
\]

Just to have an idea of the behavior of the solution, it is supposed that the coefficients of equation (8) are constant (indeed they are weakly varying with space). The above differential equation can be solved analytically, obtaining:

\[
y(\bar{x}) = c_1 e^{j\beta_{\text{z}}^\prime n_z \bar{x}} + c_2 e^{-j\beta_{\text{z}}^\prime n_z \bar{x}} + c_3 e^{j\beta_{\text{z}}^\prime n_z \bar{x}} + c_4 e^{-j\beta_{\text{z}}^\prime n_z \bar{x}}
\]

where $c_1$, $c_2$, $c_3$, $c_4$ are constants to be determined with the initial conditions. Of course the same solution holds for the slow wave. It is obvious that the nature of the solution depends on the reciprocal value of the coefficients of the above equation; thus if the accessibility condition is not fulfilled the solution is evanescent, on the contrary if the accessibility condition is fulfilled the solution is oscillating ($n_z^2 > 0$) or evanescent ($n_z^2 < 0$).

The solution (9), in the oscillating case, is plotted in figure 4 for the slow and the fast wave:

![Figure 4. Transverse (fast wave) and longitudinal (slow) normalized electric field.](image)
In particular the relevant case:
$$\delta_0^{-4} (\beta + \beta' - \alpha \alpha')^2 = 4 \delta_0^{-4} \beta \beta'$$
has been considered. This relation means, as seen before, that the roots of the dispersion relation coalesce and the LH slow wave converts into the LH fast wave. In this case the above equation assumes the form:
$$\frac{d^4 y(x)}{dx^4} + 2 \delta_0^{-2} \sqrt{\beta \beta'} \frac{d^2 y(x)}{dx^2} + \delta_0^{-4} \beta \beta' y(x) = 0$$
and the solution can be written as
$$y(x) = (c_1 + c_2 x) \cos(\delta_0^{-1} n_x x) + (c_3 + c_4 x) \sin(\delta_0^{-1} n_x x) \quad \text{if} \quad n_x^2 = \sqrt{\beta \beta'}$$
$$y(x) = (c_1 + c_2 x) e^{\delta_0^{-1} n_x x} + (c_3 + c_4 x) e^{-\delta_0^{-1} n_x x} \quad \text{if} \quad n_x^2 = -\sqrt{\beta \beta'}$$

Finally for slightly inhomogeneous plasma, the length of the non-homogeneities is greater than the wavelength of the radiation involved ($\lambda << L$). Thus a WKB approximation (Wentzel, Kramers, Brillouin) can be applied to the system (8). Making the following ansatz:
$$y(x) = y_0(x) e^{i \delta_0^{-1} S_0(x)}$$
$$z(x) = z_0(x) e^{i \delta_0^{-1} S_0(x)}$$
Substituting in equation (8) and retaining only the leading terms in the $\delta_0^{-1}$ expansion, one obtains:
$$y(x) = w_1 \left( n_{x+}^2 - \frac{\gamma}{2} \right)^{-1/2} n_{x+}^{-1/2} \exp \left( j \delta_0^{-1} f n_{x+} dx \right) + w_2 \left( n_{x-}^2 - \frac{\gamma}{2} \right)^{-1/2} n_{x-}^{-1/2} \exp \left( -j \delta_0^{-1} f n_{x-} dx \right)$$
$$+ w_3 \left( n_{x+}^2 - \frac{\gamma}{2} \right)^{-1/2} n_{x-}^{-1/2} \exp \left( j \delta_0^{-1} f n_{x-} dx \right) + w_4 \left( n_{x-}^2 - \frac{\gamma}{2} \right)^{-1/2} n_{x+}^{-1/2} \exp \left( -j \delta_0^{-1} f n_{x+} dx \right)$$

where $w_1$, $w_2$, $w_3$, $w_4$ are constants to be determined with the initial conditions and $\gamma = \beta + \beta' - \alpha \alpha'$. The same solutions hold for $z$.

It’s worth noting the WKB approximation fails when the accessibility condition is not fulfilled ($\lambda = n_{x+} c \sin$ means that $(\beta + \beta - \alpha \alpha')^2 - 4 \beta \beta' = 0$) and in the cut offs ($n_x \to 0$) where the condition of a slightly varying medium is not more satisfied ($\lambda << \left( \frac{dn_x}{n_x dx} \right)^{-1} = L$).

5. Numerical solution of the system
The ordinary differential equation system, described by equation (8), has also been solved by means of a numerical scheme based on the finite difference method. In this case the complication arises when considering the coupling terms $\alpha(\vec{x})$ and $\alpha'(\vec{x})$; the resulting matrix to be inverted loses its tridiagonal peculiarity and becomes a banded matrix of seven diagonals. A routine of the IMSL mathematical library has been used to invert this matrix; this routine solves a system of linear algebraic equations, which have a real banded coefficient matrix. First it uses a routine to compute an “LU” factorization of the coefficient matrix and to estimate the condition number of the matrix: large condition number means the matrix is poorly invertible and the accuracy of solution of a linear system may be bad.

The results are shown in figure 5. The electric field for both polarization $E_y$ and $E_z$ is plotted vs the slab variable $x$ when both wave are propagative, $n_{x+}=2$ (figure 5-left hand side), and when the mode conversion is present in the plasma, $n_{x+}=1.7$ (figure 5 right hand side).

It is worth noting that the numerical solution, when the accessibility condition is satisfied (propagative slow and fast modes), is very similar to the analytical solution obtained for constant coefficients (equation (9)).
Figure 5. Transverse and longitudinal normalized electric field for (left hand side) propagative modes and (right hand side) when a confluence between fast and slow modes is present.

6. Conclusions
An analysis of the fourth order ordinary differential equation, describing the propagation of the LH wave for both polarizations (slow and fast), has been performed when a confluence point is located inside the plasma (mode conversion layer).

The equation has been studied analytically and numerically and the wave oscillating electric field has been determined before and after the mode conversion.

A complete analytical expression for the electric field has also been obtained when treating the wave equations by means of the WKB approximation, showing the failure of the solutions near the cut-offs and mode conversion points. This problem has been examined thoroughly in order to establish the behavior of the propagating slow wave in presence of the mode conversion, the role played by diffraction in the off-axis absorption of the wave and the consequent loose of efficiency in energy transfer.

7. References
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