Casson towers and slice links

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Abstract. We prove that a Casson tower of height 4 contains a flat embedded disc bounded by the attaching circle, and we prove disc embedding results for height 2 and 3 Casson towers which are embedded into a 4-manifold, with some additional fundamental group assumptions. In the proofs we create a capped grope from a Casson tower and use a refined height raising argument to establish the existence of a symmetric grope which has two layers of caps, data which is sufficient for a topological disc to exist, with the desired boundary. As applications, we present new slice knots and links by giving direct geometric constructions of slicing discs. In particular we construct a family of slice knots which are potential counterexamples to the homotopy ribbon slice conjecture.

1. Introduction

This paper presents results on Casson towers of height 2, 3 and 4 in dimension four, and applications to the problem of slicing knots and links.

The disc embedding problem is one of the most important questions in 4-manifold topology. A. Casson introduced the influential idea of a Casson tower, which arises as the trace of repeated attempts to eliminate intersections of an immersed disc, the goal being to find a flat embedded disc [Cas86]. Briefly speaking, the height of a Casson tower is the number of stages of iterated attempts. A Casson tower $T$, itself a 4-manifold, is endowed with a framed circle $C = C(T)$ embedded in its boundary. We ask whether there exists a flat embedded disc with boundary $C$. See Definition 2.1 for details.

Casson considered a tower of infinite height, which is now called a Casson handle [Cas86]. He showed that a Casson handle is homotopy equivalent to a disc. In the original proof of the celebrated disc embedding theorem in dimension 4 [Fre82b], M. Freedman showed that a Casson handle is homeomorphic to an open 2-handle, and consequently contains a flat embedded disc with framed boundary $C(T)$. A key ingredient of the proof was Freedman’s reimbedding theorem [Fre82b, Theorem 4.4], which says that a height 6 Casson tower contains within it a height 7 tower (see [Bis94] for a detailed exposition). Iterating this, it follows that a given height 6 tower $T$ contains a Casson handle, and consequently contains a flat embedded disc with framed boundary $C(T)$. Gompf and Singh improved this disc embedding result by showing that height 5 Casson towers are sufficient for reimbedding [GSS84].

From this a natural question arises: what is the minimal height of a Casson tower required to obtain an embedded disc?

In Theorems A, B and C below we give disc embedding results for Casson towers of height 4, 3 and 2 respectively, under increasingly strong assumptions on fundamental groups. The height 2 result is particularly useful for the study of knot and link concordance, since it is often feasible to construct such a tower in $D^4$ bounding a knot or link. To compute concordance, we would like to know exactly which knots and links are slice. It seems likely that the full class of slice links is not yet known.

Work of Freedman in the 1980s and 90s [Fre82a, Fre85, Fre88, Fre93], and also later work such as [FT95b, FT05, CFT09], produced slice knots and links of great interest. Particular
focus was placed on which Whitehead doubles are slice (see Conjecture 1.1 below), since topological surgery problems in dimension four can be reduced to atomic problems [CF84] which have solutions precisely when such links are slice.

Using our height 2 Casson tower embedding theorem (Theorem C), we extend the class of known slice knots to include the new family of slice knots described in Theorem D. We construct slice discs directly, rather than using the topological surgery machine employed by many of the papers mentioned above. Our slice knots relate closely to the Topological Whitehead Double Conjecture [1] give potential counterexamples to the Homotopy Ribbon Slice Conjecture [2] and suggest a possible connection between the Homotopy Ribbon Slice Conjecture and the 4-dimensional surgery conjecture.

1.1. Casson towers of height four, three, and two

We proceed to introduce our disc embedding results for Casson towers of height four, three, and two. Let \( W \) be a 4-manifold with boundary. A framed link \( L \subset \partial W \) is slice in \( W \) if \( L \) bounds a collection of disjointly embedded flat discs in \( W \), as framed manifolds.

**Height four.** Our first main result implies that a height 4 Casson tower is in fact sufficient to obtain a flat embedded disc. In fact we give a stronger result. Briefly, define a distorted Casson tower by introducing plumbings of the top stage discs into discs of stage two or higher in a Casson tower (see Definition 4.4).

**Theorem A.** A distorted Casson tower \( T \) of height 4 contains a topologically embedded flat disc bounded by \( C(T) \) as a framed manifold.

In other words, \( C(T) \) is slice in \( T \). Since a Casson tower is vacuously a distorted Casson tower, Theorem A holds for an ordinary (non-distorted) Casson tower of height 4. This assertion seems to have been expected to be true by the experts, but to the knowledge of the authors, no proof has appeared in the literature; compare [Ray13, Footnote 1].

**Height three.** It is not known in general whether a height 3 Casson tower \( T \) contains an embedded disc with boundary \( C(T) \). Progress has been made by looking at special cases, as instigated in [CF84]. Freedman proved that the simplest Casson tower of height 3, namely the tower with a single double point at each stage, contains a disc [Fre88]. We remark that completing the analogous argument to our proof of Theorem A for a height 3 Casson towers would seem to require the surgery conjecture for non-abelian free groups. The corresponding statement to Theorem A for general height 3 Casson towers would therefore be rather interesting. The main difficulty, as so often in this subject, is to achieve \( \pi_1 \)-nullity.

Instead of looking for null homotopies internally in Casson towers, we can consider embedded Casson towers in a 4-manifold, and then try to find null homotopies inside the 4-manifold. For our height 3 result, we introduce the notion of a \( \pi_1 \)-null disc group, abbreviated to \( \pi_1 \text{ND} \) group. We postpone the precise description to Definition 3.1. (Experts will guess correctly that a \( \pi_1 \text{ND} \) group is a group for which the \( \pi_1 \)-null disc lemma holds.) A \( \pi_1 \text{ND} \) group is good in the sense that topological surgery works for it, but there might conceivably be good groups which are not \( \pi_1 \text{ND} \). The result of Freedman-Teichner and Krushkal-Quinn [FT95a, KQ00] tells us that a group of subexponential growth is \( \pi_1 \text{ND} \), as is any group obtained by extensions from \( \pi_1 \text{ND} \) groups.

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1In fact, Freedman showed that a two component link called “Whitehead 3” bounds slicing discs in the 4-ball whose complement has free fundamental group. This link is associated to the simplest Casson tower \( T \) of height 3, as explained in our Section 6.2. It turns out that each of the two slicing discs is ambiently isotopic to the standard disc in the 4-ball by [FQ00, 11.7A]. It follows that the exterior of one slicing disc is \( T \) and the other slicing disc is bounded by \( C(T) \).
A. Ray considered a framed grope bounded by \( C(T) \) in a Casson tower \( T \) \cite{Ray13}. Denote the first stage surface of this grope by \( \Sigma(T) \). This is an oriented surface embedded in \( T \) with \( \partial \Sigma(T) = C(T) \). See Figure 5 and Proposition 4.4 for more details. For a disjoint union of Casson towers \( T = \bigsqcup T_i \), denote \( C(T) = \bigsqcup C(T_i) \) and \( \Sigma(T) = \bigsqcup \Sigma(T_i) \). Denote a tubular neighbourhood of \( C(T) \) in \( \partial T \) by \( \partial_-(T) \).

**Theorem B.** Let \( W \) be a 4-manifold with boundary and suppose that \( T = \bigsqcup T_i \) is a collection of disjoint Casson towers \( T_i \) of height 3 in \( W \) such that \( \partial_-(T) \subseteq \partial W \) and the image of \( \pi_1(T_i \setminus \Sigma(T_i)) \to \pi_1(W \setminus \Sigma(T)) \) is \( \pi_1 \)-ND for each \( i \). Then the framed link \( C(T) \subseteq \partial W \) is slice in \( W \).

We remark that this result concerns links and not just knots.

**Height two.** For the height 2 case, we obtain a slicing result under a stronger simple connectivity hypothesis.

**Theorem C.** Let \( W \) be a 4-manifold with boundary and suppose \( T \) is a Casson tower of height 2 embedded in \( W \) such that the second stage \( T_{2,2} \) of \( T \) lies in a codimension zero simply connected submanifold \( \nu \subseteq W \setminus T_{1,1} \). Then the knot \( C(T) \subseteq \partial W \) is slice in \( W \).

In Theorems B and C, the slice discs are contained in a neighbourhood of a union of the tower itself and a collection of null homotopies for double point loops constructed during the proofs.

**Our proofs and gropes.** After Freedman’s original proof of the disc embedding theorem using Casson towers, the grope technology (see Definitions 2.2–2.5) has been developed in subsequent work by Quinn \cite{Qui82}, Edwards \cite{Edw84}, Freedman–Quinn \cite{FQ90}, Freedman–Teichner \cite{FT95a}, Krushkal–Quinn \cite{KQ00} and others. It turned out that gropes are effective for proving the disc embedding theorem in the non-simply connected case.

The grope technology is in fact a key ingredient of our proofs. For height four and three, our arguments hinge on Ray’s construction of a framed grope inside a Casson tower \cite{Ray13}. It enables us to connect the grope and Casson tower techniques.

In the grope setting the minimal data required for the existence of a topological disc has been quite well optimised in the decades since the original reference \cite{FQ90} was written. (The optimisation has not been enough for the surgery conjecture to be known, of course.) Up to date grope combinatorics were partially written up in \cite[Chapter 8]{FF13} by the second author and W. Politarczyk as part of the lecture notes for Freedman’s lectures for the Max Planck Institute for Mathematics semester on 4-manifolds in 2013. In the hope that they represent a useful addition to the literature, details relevant to the current paper which cannot be found in the earlier literature (e.g. \cite{FQ90}) are given below: see Grope Height Raising Lemma 3.7 and Cap Separation Lemma 3.8. In fact the proof of the latter lemma has not appeared anywhere before to the best of our knowledge. The inclusion of these details is further justified by the following corollary, which is proven by combining Ray’s grope construction with the Grope Height Raising Lemma 3.7.

**Corollary 7.1.** A Casson tower \( T \) of height 3 contains an embedded grope of height \( n \), with the same attaching circle \( C(T) \) as the Casson tower, for all \( n \).

This improves a result of Ray \cite[Theorem A (i)]{Ray13}. Further discussion of this corollary can be found in Section 7.

The proof of the height 2 result, Theorem C, requires an entirely new construction, given in Proposition 4.5 of capped gropes from Casson towers embedded in a 4-manifold, under a certain fundamental group condition. This depends on new geometric arguments and some quite delicate combinatorics. The application to slice knots discussed next utilises this construction.
1.2. Applications to slicing knots and links

We apply our results on Casson towers to present new slice knots and slice links in $S^3$. As usual, we say a knot or link in $S^3$ is slice if it is slice in $D^4$.

New slice knots. To state our results on knots, we recall that Milnor called a link $L$ in $S^3$ homotopically trivial if its components admit disjoint null-homotopies [Mil54]. That is, if there are maps $h_i: D^2 \to S^3$ such that $L = \bigsqcup h_i(S^1)$ and $h_i(D^2) \cap h_j(D^2) = \emptyset$ for $i \neq j$. We also recall that a band sum operation on a link $L$ is performed along an embedded band $D^1 \times D^1$ which joins two components of $L$ and whose interior $\text{Int}(D^1 \times D^1)$ is disjoint from $L$. Denote the untwisted Whitehead double of a link $L$ by $\text{Wh}(L)$.

**Theorem D.** Suppose $L$ is an $m$-component homotopically trivial link, and $K$ is a knot obtained from $\text{Wh}(L)$ by applying $m - 1$ band sum operations. Then $K$ is slice.

Our proof of Theorem D uses Theorem C on Casson towers of height 2. The details are discussed in Section 5.

Theorem D specialises to several interesting cases. First, taking $L$ to be a knot we see that Theorem D has, as a special case, the result of Freedman that the Whitehead double of any knot is slice. (When $L$ is a knot there are no band sums.) By applying Theorem D for $L$ a link, we obtain a large family of new slice knots. For instance, the following corollary gives a way to construct intriguing examples.

**Corollary E.** Suppose $L$ is an $m$-component homotopically trivial link, and $R$ is a ribbon knot. Consider a split union $\text{Wh}(L) \sqcup R$ in $S^3$, and choose $m$ disjoint bands which join each component of $\text{Wh}(L)$ to $R$, such that in addition the bands are disjoint from an immersed ribbon disc for $R$ in $S^3$ and are disjoint from Seifert surfaces for $\text{Wh}(L)$. Then the knot $K$ obtained from $\text{Wh}(L) \sqcup R$ by these band sum operations along the arcs is slice.

The assumption on the bands is not necessary to conclude that $K$ is slice, but we include it so that we can discuss the ribbon knot $R$ meaningfully. For example, a slice knot $K$ from Corollary E has the same Alexander polynomial as $R$; see Proposition 5.3. An explicit example is given in Figure 1; we apply band sum operations to $\text{Wh}(L) \sqcup R$, where $L$ is the 3-component link obtained from the Whitehead link by adjoining an untwisted parallel of one of the components, and $R$ is the ribbon knot $8_8$. By Corollary E the knot $K$ in Figure 1 is slice. This $K$ is a hyperbolic knot (verified by SnapPea), and consequently, is prime and non-satellite.

To the knowledge of the authors, previously known methods and results are not able to show that all of our knots are slice, except for in some special cases. Section 5.2 contains
more details on the failure of the topological surgery method to slice these knots. Another possible approach to slice a knot $K$ given by Corollary E would be to show that the link \( \text{Wh}(L) \cup R \) is slice. This is an important conjecture in topological 4-manifolds.

**Conjecture 1.1** (Topological Whitehead Double Conjecture). The Whitehead double \( \text{Wh}(L) \) of a link $L$ is freely slice if and only if $L$ is homotopically trivial.

Here a link is called freely slice if there are slice discs whose complement has free fundamental group. Conjecture 1.1 was stated explicitly in [CFT09, Conjecture 1.1], but was implicit in several earlier works such as [Fro88, FL89, FT95b].

The only if direction of the conjecture implies that topological surgery does not work for the rank two free group [CT84, FL89]. Freedman confirmed the conjecture for knots and 2-component links [Fro88]. See [Kru99, Kru08, Kru14] for recent progress towards the only if direction in the 3-component case.

The best known result toward the if direction of Conjecture 1.1 is a theorem of Freedman and Teichner that if a link $L$ is homotopically trivial*, then \( \text{Wh}(L) \) is (freely) slice [FT95b], where $L$ is said to be homotopically trivial* if any link obtained from $L$ by adjoining a zero-linking parallel copy of one of the components is homotopically trivial.

Since Conjecture 1.1 remains unknown, this approach is not sufficient to slice the knots of Corollary E when one uses a link $L$ which is homotopically trivial but not homotopically trivial*. For instance, this is the case for the knot of Figure 1.

**Homotopy ribbon-slice conjecture.** Recall that the ribbon slice conjecture claims that every slice knot is a ribbon knot. More precisely, the statement depends on the category: in the smooth case, one asks whether a knot bounds a smooth slicing disc if and only if it is a ribbon knot. In the topological category, the right question is as follows. A knot $K$ in $S^3$ is said to be homotopy ribbon if there is a slicing disc $\Delta$ in $D^4$ for which the inclusion induces an epimorphism $\pi_1(S^3 \setminus K) \twoheadrightarrow \pi_1(D^4 \setminus \Delta)$.

**Conjecture 1.2** (Homotopy Ribbon Slice Conjecture). A knot is slice if and only if it is homotopy ribbon.

Our geometric method, which constructs a slicing disc directly, gives a potential counterexample to the homotopy ribbon slice conjecture. In particular, we ask an explicit question:

**Question 1.3.** Is the slice knot in Figure 1 homotopy ribbon?

We remark that no potential counterexample to the homotopy ribbon slice conjecture was known—more precisely, every previously known slice knot is either smoothly slice or homotopy ribbon, or both. For instance, many slice knots in the literature which are not smoothly slice are knots with Alexander polynomial one. These are homotopy ribbon by Freedman’s result. Slice knots obtained by the results in [FT05] are homotopy ribbon, too. Several papers in the literature (e.g. [HLR12, CHH13, CH12]) also consider slice knots produced by satellite constructions using companion knots of Alexander polynomial one. They are all homotopy ribbon. The essential reason for this is that all use (building blocks obtained by) a topological surgery construction of a slice disc exterior, with the fundamental group of a ribbon disc exterior, as in [FQ90] and [FT05]. The smoothly slice knots presented in [GST10, AT13], which are not known to be ribbon, can be seen to be homotopy ribbon by inspecting their constructions.

We remark that if the direction of Conjecture 1.1 would imply that the slice knots given by Corollary E are homotopy ribbon, and in particular that the answer to Question 1.3 is yes. It is also interesting to note that according to [CT84, Proposition 2], in order to solve 4-dimensional surgery problems, one needs the Whitehead doubled links in Conjecture 1.1 to be homotopy ribbon. We discuss more related questions at the end of Section 5.
Smooth status. The smooth status of our knots is also interesting. We think our knots are unlikely to be smoothly slice (particularly when all the clasps have the same sign); compare [Lev12]. For some special cases of Corollary E we computed the Rasmussen $s$-invariant to be nonzero, aided by a computer. Thus at least some of our examples are not smoothly slice. Recall that the Alexander polynomial satisfies $\Delta_K(t) = \Delta_R(t)$, in the notation of Corollary E. The following natural question arises.

**Question 1.4.** Is the slice knot in Figure 1 smoothly concordant to an Alexander polynomial one knot?

In particular for the knot $K$ of Figure 1 we have

$$\Delta_K(t) = \Delta_{s_8}(t) = 2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2 = (2t^2 - 2t + 1)(2t^{-2} - 2t^{-1} + 1).$$

We think the answer to Question 1.4 is likely to be no, but we do not know at present how to perform the computation of $d$-invariants which we think will be necessary to prove this. The existence of topologically but not smoothly slice knots with this property was shown in [HLR12]. We remark that their examples were constructed using satellite operations which tied in Alexander polynomial one knots. We conjecture that there are slice knots produced by Corollary E which are linearly independent from the examples in [HLR12] in the smooth knot concordance group modulo Alexander polynomial one knots.

New slice links. Using (distorted) Casson towers of height 4 and Theorem A we prove the following two results on links. To state the first, we consider the following operation, which is called ramified Whitehead doubling: for a given knot, take some number of untwisted parallel copies, and then replace each parallel copy by its untwisted Whitehead double. Either sign may be used for the clasp. We may iterate, by applying this operation again to each component produced in the previous step. If we repeat this $n$ times, where the number of parallel copies used in each iteration may change, then we say that the result is obtained by an $n$-fold ramified Whitehead doubling. Define a ramified $W_n$ link to be a link obtained from the Hopf link by replacing one component with its $n$-fold ramified Whitehead double.

**Theorem F.** Any ramified $W_n$ link is slice for $n \geq 4$.

Freedman showed that the unramified $W_3$, and consequently unramified $W_n$ for $n \geq 3$, are slice [Fres88]. Although Theorem F is for $n \geq 4$ only, the ramified $n = 4$ case gives new slice links. The case $n \geq 5$ was shown by Gompf-Singh [GS84].

As a second application of Theorem A, we prove the following:

**Theorem G.** The link in Figure 2 is slice.

In fact, more generally, we specify a class of slice links, related to distorted Casson towers of height 4, which contains the link in Figure 2. See Section 6.3 and particularly Theorem 6.2.

We also prove, using Theorem C, another slicing result, namely that links in a certain class of links, which we call “height two homotopically trivial,” have slice Whitehead doubles. This may be compared with the Topological Whitehead Double Conjecture [Ker65] which would require deleting the words “height two” from the hypothesis. See Theorem 6.1 in Section 6.1 for a discussion.

We finish the introduction with a couple of additional remarks. Our proofs of slicing results are more geometric and direct than many previous applications of the disc embedding theorem to slicing knots and links. Our method is similar in character to Kervaire and Levine’s programme [Ker65, Lev69] for high dimensional knots, in that we construct the slice discs in $D^4$, while in most previous slicing results (e.g. [Fres88, FT95b, FT05, CFT09, CFT09].
Figure 2. A slice link obtained from a distorted Casson tower of height 4. Each box labeled $-2$ designates two left-handed full twists.

they construct a slice disc exterior using 4-dimensional topological surgery machinery. The only exception known to the authors is an alternative proof of Garoufalidis-Teichner that Alexander polynomial one knots are slice [GT04]. Of course there is a reason for this: the surgery method is often remarkably effective.

Recall that while one can visualise a ribbon disc, (for example by drawing “movie pictures” of cross sections), it is nigh on impossible to visualise a slice disc for a knot or link which is topologically but not smoothly slice. Using the slicing theorems of this paper, one can at least understand a neighbourhood, in the 4-ball, used in the construction of the disc. The reader may also perhaps see it as virtuous, when constructing slice discs, to minimise the number of times Freedman’s disc embedding theorem is used; the slicing constructions of this paper only use it once per slice disc. This can be contrasted with the number of Freedman discs required to employ topological surgery and $h$-cobordism when using the homology surgery method, or even in Garoufalidis-Teichner [GT04].

Organisation of the paper. In Section 2 we give preliminary definitions of Casson towers, gropes, and related objects. In Section 3 we prove that the existence of a height 1.5 grope with a certain fundamental group condition gives rise to a flat embedded disc with the desired boundary. In Section 4 we prove our main disc embedding results for Casson towers: Theorems A, B, and C. Sections 5 and 6 present new slice knots and links respectively. Section 7 discusses the grope filtration of knots and Casson towers.

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2. Preliminary definitions

All 4-manifolds in this paper are compact and oriented. First we give the definitions of a Casson tower and a grope, which are the two main objects we will be working with.
**Definition 2.1** ([Cas86]). A Casson tower is a 4-manifold $T$ with a framed embedded circle $C = C(T)$ in the boundary, defined as below. We write $\partial T$ as a union of two codimension zero submanifolds $\partial_+$ and $\partial_-$, where $\partial_-$ is a (closed) tubular neighbourhood of $C$ in $\partial T$ and $\partial_+ = \partial T \setminus \partial_-$. We call $C$ and $\partial_-$ the attaching circle and the attaching region respectively.

A Casson tower has a height $n \in \mathbb{N}$. A Casson tower $T_1$ of height 1, which we will also call a plumbed handle, is a thickened disc $D^2 \times D^2$, with $C := \partial D^2 \times 0$, $\partial_- := \partial D^2 \times D^2$, and with some number of self-plumbings performed in the interior of $D^2 \times 0$. A self-plumbing is performed by taking two discs $D_1, D_2 \subset \text{int} D^2 \times 0$ and then identifying $D_1 \times D^2$ and $D_2 \times D^2$, viewing a 2-disc as the unit disc in $\mathbb{C}$, via $(z, w) \sim (w, \bar{z})$ to produce a positive self-intersection of $D^2 \times 0$, or $(z, w) \sim (\bar{w}, z)$ to produce a negative self-intersection. The core disc is defined to be the image of $D^2 \times 0$ in $T_1$, which is now an immersed disc. The attaching circle $C$ is framed by the restriction of the unique framing of $D^2 \times 0$ before plumbing. Equivalently, if we perform $k_+$ positive and $k_-$ negative plumbings, then the framing on $C$ is obtained by twisting the restriction to $C$ of the unique framing of the core disc in $T_1$ by $k_- - k_+$. To each double point of the core disc there is an associated double point loop on the core disc, which departs the double point into a sheet and comes back through the other sheet. By pushing it we obtain an embedded circle in $\partial_+$, which we call a double point loop on $\partial_+$. We assume double point loops are disjoint. Double point loops are framed in such a way that the plumbed handle with 2-handles attached along this framing is diffeomorphic to a 4-ball. There is a unique such framing.

The framings can be explicitly described using a standard Kirby diagram of a plumbed handle, shown in Figure 3: the circles $a_i$ are double point loops on $\partial_+$, and the framings on $C$ and $a_i$ described above are the zero-framing in Figure 3 (see [Cas86] Lemma 2 of Lecture I).

![Figure 3](image)

**Figure 3.** A standard Kirby diagram of a plumbed handle, with the attaching circle $C$ and double point loops $a_i$. Each plumbing corresponds to a dotted circle, where the sign of the plumbing determines the sign of the clasp.

For $n \geq 2$, a Casson tower of height $n$, denoted $T_n$, is constructed inductively by taking a height one Casson tower and identifying a neighbourhood of each double point loop on $\partial_+$ with the attaching region of a Casson tower $T_{n-1}$ of height $n - 1$, along the preferred framings of the double point loop and $C(T_{n-1})$. The attaching circle $C(T_n)$ and the attaching region $\partial_+$ of the new Casson tower $T_n$ are just those of the original height one tower.

The $k$th stage of the Casson tower $T_n$ is the material that was introduced by the $(n - k + 1)$th inductive step in the construction, where taking a height one tower counts as the first step. Following [Fre82b], we denote the stages of a Casson tower $T$ from $p$ through $q$ inclusive by $T_{p\rightarrow q}$.

**Definition 2.2** ([FQ90] [FT95a]). A grope of height $n$ $(n \in \mathbb{N})$ is a pair (2-complex, base circles) of a certain type described inductively below. A grope of height 1 is a disjoint
union of oriented connected surfaces each of which has connected nonempty boundary. The boundary circles are the base circles. Take a grope \( G_1 \) of height 1, and let \( \{ \alpha_1, \ldots, \alpha_{2g} \} \) be a standard symplectic basis of circles for the first homology of \( G_1 \). Then a grope of height \( n + 1 \) is formed by attaching a grope of height \( n \) with a single base circle to \( G_1 \) by identifying the base circle with \( \alpha_i \) for each \( i \). The base circles of the grope are defined to be the base circles of \( G_1 \), which we often call the boundary of the grope. The \( k \)th stage of the grope is the union of the surfaces that were introduced by the \( (n - k + 1) \)th inductive step in the construction, where taking a height one grope counts as the first step. We denote a grope of height \( n \) by \( G_n \), and denote the stages from \( p \) through \( q \) inclusive by \( G_{p}^{\infty} \).

A capped grope is constructed by attaching a disc to each of a symplectic basis of curves for the top stage surfaces. These discs are referred to as the caps. The surface stages are called the body of the grope. We also refer to the disjoint union of such 2-complexes as a capped grope. A capped grope of height \( n \), sometimes also known as a capped grope with \( n \) surface stages, will be denoted \( G_{n}^{c} \).

We note that, in this paper, a (capped) grope have a multi-component body in general. Such a grope is called a “union-of-discs-like” (capped) grope in \([FQ90]\).

As shown in Figure 4, a capped grope has a standard model embedded in \( \mathbb{R}^3 \). We use this to thicken and frame gropes so they are 4-dimensional objects.

**Figure 4.** A standard model of a grope of height 3 in \( \mathbb{R}^3 \).

**Definition 2.3.** Start with a model (capped) grope embedded in \( \mathbb{R}^3 \), and embed this in \( \mathbb{R}^4 \) via \( \mathbb{R}^3 \leftrightarrow \mathbb{R}^3 \times \mathbb{R} \cong \mathbb{R}^4 \). Take a thickening of the model in \( \mathbb{R}^4 \). We refer to this thickening as a framed (capped) grope. The attaching region \( \partial_- \) of a framed (capped) grope is the thickening of the base circles of the bottom surface stage.

Note that each surface and disc component of a model (capped) grope in \( \mathbb{R}^4 \) has a canonical framing of its normal bundle given by taking a 1-dimensional framing in \( \mathbb{R}^3 \), and extending via the trivial line bundle when we take the product with \( \mathbb{R} \).

We always regard the base circles of a framed (capped) grope as framed circles endowed with the induced framing. Similarly for the symplectic basis curves of the top stage surfaces of a framed grope. By definition, in the case of a framed capped grope, the framing for the symplectic basis curves is equal to the restriction of the unique framing of the caps.

The next definition is of proper immersions. Briefly, a properly immersed grope has embedded body, and immersed caps which are disjoint from the body.

**Definition 2.4 (FQ90).** Take a model framed capped grope as in Definition 2.3 and introduce plumbings into the model, by plumbing together caps and introducing self-plumbings in caps. A proper immersion of a capped grope into a 4-manifold \( M \) with boundary is an embedding of this plumbed model into \( M \) such that the attaching region \( \partial_- \) maps to \( \partial M \).
We remark that for a proper immersion it is allowed to plumb two caps attached to any, possibly different, body components.

We will also denote a framed grope of height $n$ by $G_n$, and a framed capped grope of height $n$ by $G^c_n$. From now on when we refer to a capped grope, we will mean a framed capped grope, often a properly immersed framed capped grope. However we will also refer to geometric operations, such as tubing and taking parallel copies, on surfaces which are part of the underlying 2-complex, hopefully without causing confusion. When there is concern about ambiguity, we will denote a (further) thickening of a (framed) capped grope by $\nu G^c_n$.

We will be interested in improving a capped grope to a one story capped tower. Briefly, a proper immersion of a one story capped tower is a capped grope with caps for the caps, that is, discs bounded by the double point loops of the caps. The first layer of caps should have self-intersections only. The second layer of caps, called the tower caps, must be disjoint from the body and the caps of the capped grope. The official definition is next.

**Definition 2.5 (FQ90).** A one-story capped tower is defined to be obtained from framed capped grope by finger moves that introduce self-plumbings into the caps and then by adjoining disjoint Whitney discs and accessory discs. We call the Whitney discs and accessory discs the tower caps. A proper immersion of a one-story capped tower into a 4-manifold $M$ is obtained by introducing plumbings (not necessarily just self-plumbings) into the tower caps and then embedding the plumbed model into $M$. Note that tower caps still miss the entire capped grope.

A reference for finger moves, Whitney discs, and accessory discs for readers not familiar with them is [FQ90, Sections 1.5 and 3.1]. In this paper, we do not need their definitions nor do we need any properties of tower caps, since we are not going to work with capped towers. Rather, once we have one we will observe (Lemma 3.5 and Theorem 3.6) that this is sufficient to activate the Freedman-Quinn machine and produce an embedded disc.

### 3. Obtaining a disc from a framed capped grope

This section proves Theorem 3.4 below, which is our main technical result. It sharpens the minimal grope data required to produce a disc, and will be used to deduce all of our various statements on Casson towers. In order to state the theorem we will introduce some terminology and background.

We will use the main theorem of [KQ00], stated as Theorem 3.2 below. The operation of asymmetric contraction on a framed capped grope of height $n$ is performed by using one cap from each dual pair of caps to surger the surface to which it is attached. The other caps are forgotten. The total asymmetric contraction of a capped grope is the disc obtained after the operation of asymmetric contraction, to create a capped grope with height lowered by one, has been performed sequentially on each new capped grope created by this operation, until there are no more caps to contract.

**Definition 3.1.** We say that a group $\pi$ is a $\pi_1$-null-disc ($\pi_1\text{ND}$) group if there exists $m \in \mathbb{N}$, such that given any 4-manifold $W$ and any properly immersed framed capped grope $G^c_n \to W$ of height at least $m$, together with a homomorphism $\rho: \pi_1(W) \to \pi$, the following holds: the total contraction of $G^c_n$ is regularly homotopic rel. boundary to an immersion of union of discs, whose double point loops have trivial image in $\pi$. We call $m$ the $\pi_1\text{ND}$-height of $\pi$.

As in [KQ00] Section 3], a group $\pi$ has subexponential growth if given any finite subset $S \subseteq \pi$ there is an integer $n$ such that the set of all products of elements of $S$ with length $n$ determine fewer than $2^n$ elements of the group $\pi$. 
Theorem 3.2 (Krushkal-Quinn [KQ00]). A subexponential growth group is $\pi_1 ND$ of $\pi_1 ND$-height 2.

We note that there is no requirement in Definition 3.1 nor in Theorem 3.2 for the body of $G^n_m$ to be connected, although in our applications it will suffice to apply the theorem to a capped grope with a connected body.

Next we show how an iterative argument, known to the experts, proves that the class of $\pi_1 ND$ groups is closed under extensions. We say that $\pi$ is obtained from $k$ extensions by $\pi_1 ND$ groups if $\pi = A_k$ and there are exact sequences $1 \to A_{i-1} \to A_i \to B_i \to 1$, for $i = 1, \ldots, k$, with $A_0 = 0$ and with $B_1, \ldots, B_k$ all $\pi_1 ND$.

In the proof of the next lemma we will use the symmetric contraction. The operation of symmetric contraction and pushing off the contraction is described in [FQ90 Section 2.3]. For readers not familiar with this notion, we remark that the key property we need is that symmetric contraction replaces a top level surface of a grope and caps attached to it with a new cap, having self-intersections in general, off which any other surfaces (such as other caps) can be pushed. The new cap becomes disjoint from the pushed off surfaces at the cost of possibly introducing new intersections between the pushed off surfaces. This can be done within an arbitrary neighbourhood of the original top level surface and its caps. In this paper, when not otherwise specified, ‘contraction’ and the verb ‘to contract’ refer by default to the symmetric contraction.

Lemma 3.3. Suppose that $\pi$ is obtained from $k$ extensions by $\pi_1 ND$ groups $B_1, \ldots, B_k$. Let $m_i$ be the $\pi_1 ND$-height of $B_i$ for $i = 1, \ldots, k$. Then $\pi$ is $\pi_1 ND$ of $\pi_1 ND$-height $m = 1 + \sum_{i=1}^k m_i$.

Proof. Let $1 \to A_{k-1} \to \pi \to B_k \to 1$ be the last extension. Contract the caps of one stage $m$ surface of $G^n_m$ at a time and push off all other caps before contracting the next stage $m$ surface. This makes a grope of height $m - 1$ with disjoint caps. Apply the $\pi_1 ND$ property of Definition 3.1 to each of the connected components of the top $m_k$ stages, plus caps, of $G^n_{m-1}$, with $W$ a neighbourhood of these stages and the caps, and compose $\rho: \pi(W) \to \pi$ with the map $\pi \to B_k$. We replace the top $m_k$ stages with caps whose double point loops map trivially to $B_k$, and so lie in $A_{k-1}$. We now have a grope of height $m - 1 - m_k = \sum_{i=1}^{k-1} m_i$. Note that the caps are still disjoint, since the regular homotopies occur in a neighbourhood of each connected component of the top $m_k$ stages plus caps. Repeating this argument for the next $k - 1$ extensions, we eventually obtain a height $m_i$ capped grope whose cap double points lie in $A_1$. Apply the $\pi_1 ND$ property once more, composing with the map $A_1 \to B_1$, to yield an immersed disc whose double point loops are trivial in $B_1$, and therefore lie in $A_0$. Since $A_0 = 0$, the proof is complete. \hfill $\Box$

We also note that the direct limit of a sequence of $\pi_1 ND$ groups with bounded $\pi_1 ND$-heights is again a $\pi_1 ND$ group.

A height $n.5$ capped grope is a height $n$ capped grope where one of each dual pair of curves on the $n$th stage surfaces has a capped surface instead of a cap attached to it. A proper immersion is defined by allowing plumbings of the caps, similarly to the height $n$ case.

Now we state the main theorem of this section. The theorem is probably known to the experts, but a detailed proof has not appeared.

Theorem 3.4 (Disc Embedding for $\pi_1 ND$ Capped Gropes). For some $n \in \frac{1}{2}\mathbb{N}$ which is at least 1.5, let $(G^n_m, \partial_\ast) \to (M, \partial M)$ be a properly immersed capped grope of height $n$ in a 4-manifold $M$ and let $\nu G^n_m$ be a further thickening of $G^n_m$. Suppose that the image of the inclusion induced map

$$\pi_1(\nu G^n_m \setminus G_{1.4}, \ast) \to \pi_1(M \setminus G_{1.4}, \ast)$$
is \( \pi_1 \text{ND} \), of \( \pi_1 \text{ND}-\text{height } m \), for all choices of basepoint \(* \) in \( \nu G^c_n \setminus G_{1-1} \). Then there are disjoint flat embedded discs in \( M \) with the same framed boundary as \( G^c_n \).

Recall that the body of a capped grope need not be connected. We also remark that in order to check that the hypothesis is true for all basepoints, it is enough to check it for some choice of basepoint in each connected component of \( \nu G^c_n \setminus G_{1-1} \).

Next we state some results from the literature which will be used during the proof of Theorem 3.4. In the following lemma, the phrase “\( G^c_n \) has transverse spheres” means that each component of the bottom stage of \( G^c_n \) has a transverse sphere which intersects \( G^c_n \) in precisely one point. Also, we say that a properly immersed capped grope \( G^c_n \) in \( M \) is \( \pi_1 \text{-null} \) if any loop in the image of \( G^c_n \) is null-homotopic in \( M \).

**Lemma 3.5** ([FQ90, Section 3.3]). Suppose \( G^c_n \to M \) is a proper immersion of a capped grope of height \( n \) at least 2, into a 4-manifold \( M \), which is \( \pi_1 \text{-null} \) and has transverse spheres. Then the embedding of the body of \( G^c_n \) extends to a proper immersion of a (union-of-discs)-like one-story capped tower with arbitrarily many surface stages.

The first step in the proof of Lemma 3.5 given in [FQ90, Section 3.3] uses grope height raising to find a capped grope with arbitrary height in a neighbourhood of the given capped grope. This capped grope is then improved to a one story capped tower. Freedman and Quinn’s statement begins with a proper immersion of a capped grope of height at least 3. In our statement we have replaced height 3 with height 2, in light of Lemma 3.7 below.

The strategy of the proof of Theorem 3.4 will be to arrange a situation where Lemma 3.5 can be applied. We will then be able to apply the following theorem of [FQ90].

**Theorem 3.6** (Freedman-Quinn). A neighbourhood of a properly immersed one story capped tower with at least four surface stages contains an embedded flat topological disc with the same framed boundary.

**Proof.** We follow the arguments of the second and third sentences of [FQ90, Proof of Theorem 5.1A]. The given data is sufficient to perform tower height raising with control. Begin with the tower height raising proposition in [FQ90, Section 3.5], and introduce control, to produce an infinite convergent tower as in [FQ90, Sections 3.6–8]. See in particular [FQ90, Proposition 3.8]. A convergent infinite tower is then shown to be homeomorphic relative to its attaching region \( \partial^- \) to an open 2-handle via “the design” and Bing shrinking [FQ90, Chapter 4]. See [FQ90, Theorem 4.1]. \( \square \)

A key step in the proof of Theorem 3.4 is to perform grope height raising.

**Lemma 3.7** (Groppe Height Raising Lemma). Let \( G^c_{1,5} \) be a height 1.5 capped grope which is properly immersed in a 4-manifold. For any \( n \in \mathbb{N} \) there exists, inside a neighbourhood of \( G^c_{1,5} \), a properly immersed height \( n \) capped grope \( G^c_n \) with the same framed boundary.

The statement that a height 1.5 capped grope can have its height raised arbitrarily is not contained in [FQ90] (the best statement given is in an exercise in their Section 2.7, and involves height 2.5). The outline of the proof of the Grope Height Raising Lemma was explained to the second author by F. Quinn and P. Teichner, in the discussion sessions associated to the MPIM lecture series of M. Freedman [FF]. For the convenience of the reader we include the details below, after stating and proving Lemma 3.8.

Consider a framed capped grope \( G^c_n \). Divide the surfaces (including the caps) above the first stage into two sides, labelled as the + and \( - \) sides, as follows. For a dual pair of curves in a symplectic basis for the first stage surface, the surface attached to one curve is labelled + and the surface attached to the dual curve is labelled --. A surface of stage 3 or higher has the same label as that of the surface to which it is attached. We therefore have + and \( - \) side height \( n-1 \) capped gropes. When beginning with a height 1.5 capped
gropes, choose labels so that we have $+$ side capped surfaces, while on the $-$ side we just have caps.

The following lemma is an important preliminary construction in the height raising process. When starting with a grope of height at least 3, this step can be avoided; use the argument of \[\text{FQ90}\]. When starting with a grope of height 1.5 or 2 however, the Cap Separation Lemma \[3.8\] below seems to be necessary.

**Lemma 3.8 (Cap Separation Lemma).** For any $n \geq 1.5$, within a neighbourhood of a height $n$ capped grope, there is a height $n$ capped grope with the same framed boundary and with the $+$ side caps disjoint from the $-$ side caps.

We believe some of the details of the proof we give to be new.

**Proof of Cap Separation Lemma** \[3.8\] We need to remove intersections between the $+$ and $-$ side caps. Let $F_-$ denote the dual capped surface to the $-$ side, which is constructed from two parallel copies of the $+$ side capped surface and an annulus which joins them together in a neighbourhood of the attaching circle for the $+$ side; see \[\text{FQ90 \ Section 2.6}\]. Note that the number of components of $F_-$ is equal to the genus of the bottom stage surface.

Take a parallel copy of $F_-$, contract the top stage of the parallel copy and push $F_-$ and the $-$ side caps off the contraction. Note that when we push off the contraction we may obtain new intersections of the $-$ side caps, but that is acceptable. By contracting further if necessary, we obtain a collection of spheres $F'_-$, each of which is dual to a $-$ side stage 2 surface, and the capped surface $F_-$ is modified so as to be disjoint from $F'_-$. We have that $F'_-$ is disjoint from the $-$ side apart from the transverse point, but $F'_-$ may intersect the $+$ caps, since we did not push those off the contraction. For a given $-$ cap we want one of these dual spheres to eliminated intersections with the $+$ caps; for each such intersection, push the intersection down to a $-$ side surface of stage 2 if necessary, and tube the $+$ side cap into one of the dual $F'_-$ spheres. We may obtain new intersections between $+$ side caps, since $F'_-$ may intersect $+$ side caps, and from the fact that $F'_-$ will probably not be embedded. But this is acceptable too.

We are now ready to give the proof of Lemma \[3.7\]. The proof uses arguments known to the experts; a variant appeared in \[\text{FQ90 \ Section 2.7}\]. Ours is based on that typed up by the second author and W. Politarczyk in \[\text{FF \ Section 8.3}\], which itself derived from the Freedman lectures and discussion sessions.

**Proof of Grope Height Raising Lemma** \[3.7\] By the Cap Separation Lemma \[3.8\] we may suppose that we have a properly immersed grope of height 1.5 where the $+$ caps are disjoint from the $-$ caps.

Let $F_-$ denote the dual capped surface to the $-$ side, which is constructed from two parallel copies of the $+$ side capped surface. Tube each intersection of the $-$ side caps into a parallel copy of $F_-$. This turns the $-$ side caps into capped surfaces. We have now raised the height by one on the $-$ side. We started with $(+, -)$ heights being $(1, 0)$ and now we have $(1, 1)$.

Next we raise height on the $+$ side. To achieve this we repeat the above process, with $+$ and $-$ reversed. That is, first we apply Lemma \[3.8\] to once again separate the caps on the $+$ and $-$ sides. Then we let $F_+$ denote the dual capped surface to the new $+$ side. Tube intersections of $+$ side caps into parallel copies of $F_+$. This creates a grope with $(+, -)$ height $(2, 1)$.

Now repeat as many times as desired, alternating $+$ and $-$ sides. From $(2, 1)$ we go to $(2, 3)$, then $(5, 3)$, then $(8, 5)$ and so on. The $(+, -)$ heights grow according to the Fibonacci numbers. We remark that we could instead apply the argument in \[\text{FQ90 \ Section 2.7}\]
once we get both sides of height at least 2; it raises height a little slower but creates fewer intersections along the way.

With the preliminaries at last complete, there now follows the proof of Theorem 3.4.

Proof of Theorem 3.4. Our goal is to extend our capped grope to a properly immersed one story capped tower, so that we can apply Theorem 3.6.

We may as well assume that $n = 1.5$, since the first step is to perform *g rope height raising.* (If desired one can first contract the grope until it is of height 1.5.) Recall that $m$ is the $\pi_1$-no-height of the image of $\pi_4(\nu G^c_5 \setminus G_{1-1}, *) \to \pi_1(M \setminus G_{1-1}, *)$. In each step below, we first state what to do, and then discuss how.

**Step 1.** Extend the body of $G^c_{1.5}$ to a properly immersed framed capped grope $G^c_{4+m}$ of height $4 + m$ in $\nu G^c_{1.5}$.

This is done by applying Groupe Height Raising Lemma 3.7 to raise the height of $G^c_{1.5}$ by $2.5 + m$.

**Step 2.** Extend $G_{1-(3+m)} \subset G^c_{3+m}$ to a height $(3 + m)$ properly immersed framed capped grope $G^c_{3+m}$ in a neighbourhood of $G^c_{4+m}$, such that all of the cap intersections of $G^c_{3+m}$ are self-intersections.

The argument for Step 2 is similar to an argument from the proof of Lemma 3.3: perform a sequence of contraction-push-off operations. That is, contract the top stage surfaces of $G^c_{1.5}$ one at a time inductively, pushing all other caps off the contraction before contracting the next top stage surface.

**Step 3.** Deform $G^c_{3+m}$ in such a way that each stage 2 surface $\Sigma$ of $G^c_{3+m}$ has a transverse sphere which meets $G^c_{3+m}$ at exactly one point, and that point belongs to $\Sigma$; also, caps attached to distinct top level surfaces are disjoint.

This is achieved using what is now a standard argument ([FQ90 Section 2.6]), as follows. We use the following notation: for a surface $\Sigma$ in the body of a capped grope $G^c$, denote by $G^c_5$ the capped grope consisting of the surfaces and caps on the top of $\Sigma$, including $\Sigma$ itself. Let $\Sigma$ be a stage 2 surface of $G^c_{3+m}$, and let $\Sigma'$ be the stage 2 surface dual to $\Sigma$ i.e. the attaching curves of $\Sigma$ and $\Sigma'$ are dual curves on the first stage surface $G_{1-1}$. Construct a transverse capped grope of height $m + 2$ for $\Sigma$ by taking two parallel copies of $G^c_{3+m}$, and attaching an annulus cobounded by the boundary of the parallel copies, which lies in a regular neighbourhood of $\partial \Sigma'$, and which meets $\Sigma$ at exactly one point. Note that the caps of the transverse capped grope may meet the caps of $G^c_{3+m}$, but no other caps. Contract the top stage of the dual capped grope, pushing intersections with caps of $G^c_{3+m}$ off the contraction, and then totally contract the remaining stages of the dual capped grope. This gives us a new capped grope, which we still call $G^c_{3+m}$, together with transverse spheres for the stage 2 surfaces. Each transverse sphere meets the new $G^c_{3+m}$ at a single point. The transverse spheres are not mutually disjoint, but that is permitted. Note that while the push off operation introduces intersections of caps which are not self-intersections, caps of the new $G^c_{3+m}$ which are attached to distinct top stages are still disjoint, since the top stage contraction-push-off can only introduce intersections between caps attached to the same top stage surface. This uses Step 2.

**Step 4.** Extend $G_{1-3} \subset G^c_{3+m}$ to a properly immersed framed height 3 capped grope $G^c_3$ whose caps lie in a regular neighbourhood of $G^c_{3+m}$ and whose double point loops are $\pi_1$-null in $M \setminus G_{1-1}$.

Consider a stage 4 surface $\Sigma$ of $G^c_{3+m}$. Choose a regular neighbourhood $W_\Sigma$ of $G^c_5$ in the exterior of $G_{1-3}$. Note that $G^c_5$ has height $m$. The capped gropes $G^c_5$ are mutually disjoint by the penultimate sentence of Step 3. Thus we may assume that the neighbourhoods $W_\Sigma$
are disjoint. Since $W_{\Sigma} \subset \nu G_n^c \setminus G_{1-1}$, the image of $\pi_1(W_{\Sigma}) \to \pi_1(M \setminus G_{1-1})$ lies in the image of the fundamental group of the component of $\nu G_n^c \setminus G_{1-1}$ containing $\Sigma$, which has $\pi_1$-ND-height $m$, by the hypothesis. By Definition 4.1, it follows that the total asymmetric contraction of $G_{\Sigma}^m$ is regularly homotopic rel. boundary to an immersed disc in $W_{\Sigma}$ whose double point loops are trivial in $\pi_1(M \setminus G_{1-1})$. Replacing each $G_{\Sigma}^c$ with this immersed disc, we obtain the desired properly immersed framed capped grope of height 3.

**Step 5.** Construct a one-story capped tower and then a flat embedded disc.

Now consider the union of $G_{\Sigma}^c$ taken over all stage 2 surfaces $F$ of $G_2$. This is a capped grope of height 2, which is properly immersed in $M \setminus G_{1-1}$, is $\pi_1$-null, and has transverse spheres, by Steps 3 and 4. It follows from Lemma 3.1 that the body of $\bigcup_F G_F^c$ extends to a properly immersed one story capped tower with 3 surface stages in $M \setminus G_{1-1}$. Attach this one story capped tower to $G_{1-1}$, to obtain a one story capped tower with 4 surface stages. Theorem 5.6 then yields a flat embedded disc as claimed. \(\square\)

4. Casson towers of height four, three and two

In this section we apply Theorem 3.3 on gropes to obtain results on Casson towers. We consider Casson towers of height four, three and two, in that order. As the height of the Casson tower decreases, we need stronger assumptions on fundamental groups in order to deduce the existence of embedded discs.

The following construction of Ray allows us to pass from Casson towers to gropes. Recall that the symplectic basis curves of top stage surfaces of a framed grope is framed by the induced framing, and double point loops of a plumbed handle are framed as in Definition 2.1.

**Proposition 4.1** ([Ray13, Proposition 3.1]). *A Casson tower $T$ of height $n$ contains an embedded framed grope $G_n^c$ of height $n$ with base circle equal to the attaching curve $C(T)$ as framed circles. Moreover the union of the standard symplectic basis curves on the top stage surfaces of $G_n$ is, as a framed 1-submanifold, isotopic to the union of $2^n$ parallel copies of the double point loops of the top stage of $T$, via disjointly embedded framed annuli whose interior is disjoint from $G_n$.*

We note that the first stage surface $G_{1-1}$ of the grope $G \subset T$ is denoted by $\Sigma(T)$ in the introduction.

To employ the grope technology, we need capped gropes. The following innocent observation is useful in producing capped gropes in a Casson tower. We will also present a more involved construction of a capped grope in Lemma 4.5.

**Lemma 4.2** (Capped grope in a Casson tower). *A Casson tower $T$ of height $n+1$ contains a properly immersed capped grope $G_n^c$ of height $n$, with base circle equal to $C(T)$ as framed circles. The body of $G_n^c$ is the grope $G_n$ for the subtower $T_{1-n}$ in Proposition 4.1."

**Proof.** Let $G_n$ be the framed embedded grope in $T_{1-n}$ obtained by applying Proposition 4.1 to $T_{1-n}$. We will attach caps to $G_n$, constructed from parallel copies of the core discs of the top stage plumbed handles of $T$ (together with parallels of the annuli given in Proposition 4.1). The only issue is that the caps should be framed. For this purpose, we arrange that the top stage core discs of $T$ induce the preferred framing on the double point loops of the $n$th stage plumbed handles to which they are attached. That is, each core disc of a stage $n+1$ plumbed handle should have the signed count of its self-plumbings equal to zero. We achieve this by locally introducing the requisite number of self-plumbings of appropriate sign. Now, from this and from the framing property in Proposition 4.1, it follows that we obtain framed caps. Thus we have a properly immersed framed capped grope $G_n^c$ extending $G_n$, inside $T$. \(\square\)
We also need the following lemma on a fundamental group arising from the construction in Proposition 4.1.

**Lemma 4.3.** Let $T$ be a Casson tower and let $\Sigma = \Sigma(T)$. Then
\[
\pi_1(T_{1-1} \setminus \Sigma) \cong \langle \mu, a_1, \ldots, a_k \mid [\mu, a_i], i = 1, \ldots, k \rangle
\]
where $\mu$ is a meridian to $\Sigma$ and the $a_i$ are the double point loops of $T_{1-1}$.

**Proof.** For convenience, we assume that the first stage of $T$ has one (negative) double point. We will indicate along the way how to adapt the proof for the general case.

We recall Ray’s construction from [Ray13, Proof of Proposition 3.1]. The first stage surface $\Sigma$ is shown in Figure 5, where the plumbed handle $T_{1-1}$ is described as a Kirby diagram. More precisely, $\Sigma$ is obtained by pushing the interior of the surface in Figure 5 slightly into the interior of $T_{1-1}$. The curve $a_1$ is the double point loop, which is the attaching circle for the next stage of the Casson tower $T$.

![Figure 5. Kirby diagram of a plumbed handle together with Ray’s genus one surface.](image)

Consider a collar neighbourhood $\partial T_{1-1} \times I$ of $\partial T_{1-1}$. We may assume that the height function $\partial T_{1-1} \times I \to I$ restricts to a Morse function for $\Sigma$ with 3 critical points, corresponding to two 1-handles and one 2-handle of $\Sigma$; the 1-handles are shown as dashed lines in Figure 6.

![Figure 6. A handle decomposition of the surface $\Sigma$.](image)

The handle decomposition of $\Sigma$ gives rise to a handle decomposition of the exterior of $\Sigma$ in the collar neighbourhood $\partial T_{1-1} \times I$ of $\partial T_{1-1}$:
\[
(\partial T_{1-1} \times I) \setminus \Sigma = (\partial T_{1-1} \setminus \partial \Sigma) \times I \cup (\text{two 2-handles}) \cup (\text{one 3-handle}).
\]

An $i$-handle in a handle decomposition of a surface embedded in a 4-manifold corresponds to an $(i + 1)$-handle in a handle decomposition of the exterior of the surface (for a proof, see for example [GS93, Proposition 6.2.1]). The handle attachments are shown in Figure 7, where the attaching circles and spheres are drawn with dashed lines.

As shown in Figure 7, $\partial T_{1-1} \setminus \partial \Sigma$ is the exterior of the Whitehead link, with a boundary component (corresponding to the dotted circle) filled in with a solid torus along the zero framing. Therefore, starting with the Wirtinger presentation
\[
\langle x, y, p, q, r, s \mid y = p^{-1}ap, q = xpx^{-1}, r = sqs^{-1}, s = q^{-1}pq, r = xsx^{-1} \rangle,
\]

we obtain the fundamental group of $\Sigma(T)$.
Figure 7. A handle decomposition of $(\partial T_{1-1} \times I) \setminus \Sigma$. An arrow indicates the attaching of a handle, and the $\sim$ symbol indicates an isotopy. It is much easier to draw the attaching 2-sphere for the 3-handle after the isotopy.

of the Whitehead link, where the generators are those shown in the first diagram in Figure 5, and then adding three more relators

$$x^{-1}s^{-1}q^{-1}p^2, xy^{-1}, xqx^{-1}q^{-1}$$

which are from the Dehn filling and the 2-handle attachments respectively, we obtain a presentation of $\pi_1((\partial T_{1-1} \times I) \setminus \Sigma)$. Simplifying the presentation, we obtain:

$$\pi_1((\partial T_{1-1} \times I) \setminus \Sigma) \cong \langle x, q | [x, q] \rangle.$$  

Observe that $x = \mu$ and $q = a_1$. In the case that the clasp is a positive clasp, the relators $r = sqs^{-1}$ and $s = p^{-1}qp$ are replaced by $s = rpr^{-1}$ and $r = p^{-1}qp$. It is not too hard to check that the computation above has the same outcome with these alterations. The relator corresponding to Dehn filling also changes to $x^{-1}pxrp^{-2}$, but this relator is superfluous to simplifying the presentation in both cases.

Turn the handle decomposition of $T_{1-1}$, into a 0-handle and a 1-handle, given by the Kirby diagram in Figure 5 upside down. We see that $T_{1-1} \setminus \Sigma$ is obtained by attaching a 3-handle and a 4-handle to $(\partial T_{1-1} \times I) \setminus \Sigma$. In general, turning a handle decomposition

$$\bigcup h_0 \cup \cdots \cup \bigcup h_3$$

of a connected 4-manifold with nonempty boundary $(M, \partial M)$ upside down gives us a decomposition rel. boundary

$$\partial M \times I \cup \bigcup h_3^* \cup \cdots \cup \bigcup h_0^*,$$

where $h_i^*$ is the $(4-i)$-handle dual to the $i$-handle $h_i$. Since neither a 3-handle nor a 4-handle affect the fundamental group,

$$\pi_1(T_{1-1} \setminus \Sigma) \cong \pi_1((\partial T_{1-1} \times I) \setminus \Sigma) \cong \langle \mu, a_1 | [\mu, a_1] \rangle.$$  

For $k > 1$ double points, take $k$ copies of Figure 5 with the crossings in the clasp switched where appropriate, and connect sum the $C(T)$ curves together. This performs a boundary connect sum operation on the surfaces. Similarly, take multiple copies of the first three diagrams of Figure 5 and connect sum the $C(T)$ curves together i.e. the copies of the curve with meridians $x$ and $y$ in the first diagram of Figure 5. This composite curve represents
\( \partial \Sigma \). There is still only one 2-handle of \( \Sigma \), therefore only one 3-handle of \((\partial T_{1-1} \times I) \setminus \Sigma\). So in the ramified case the analogue of the final diagram of Figure 7 will still have just a single dashed 2-sphere. The Seifert-Van Kampen theorem applies to show that the effect of the connect sum operations is to identify the meridians labelled \( x \) in all the copies of the diagrams from Figure 7 all of these become the meridian \( \mu \). Indeed this is the only effect. By the computation above, \( \mu \) commutes with all the double point loops. We therefore have the presentation
\[
\pi_1(T_{1-1} \setminus \Sigma) \cong \langle \mu, a_1, \ldots, a_k \mid [\mu, a_i], \ i = 1, \ldots, k \rangle.
\]

\[ \square \]

### 4.1. Casson towers of height four and three

In this subsection we will prove Theorem A in the introduction, which says that a distorted Casson tower \( T \) of height 4 contains a disc bounded by \( C(T) \). We note that we do not make any assumptions about embedding a distorted Casson tower of height 4; the distorted Casson tower itself is considered as the ambient manifold.

**Definition 4.4.** A **distorted Casson tower of height** \( n \) is defined by taking a Casson tower of height \( n \), and introducing extra intersections of the cores by allowing the following operation: plumb a top stage thickened disc to another thickened disc from stage two or higher.

Note that a Casson tower of height \( n \) is a distorted Casson tower of height \( n \). As another example, a distorted tower of height \( n \) may arise if we have a height \( n - 1 \) tower \( T \) embedded in a 4-manifold \((M, \partial M)\) with \( \partial_- \subset \partial M \), and the double point loops of the top stage are null-homotopic in the complement in \( M \) of the first stage \( T_{1-1} \) of \( T \). Then a neighbourhood of the union of the height \( n - 1 \) tower and the null-homotopies of the double point loops gives rise to a distorted height \( n \) Casson tower. Some care is needed to frame the null-homotopies. Null-homotopies of different double point loops may intersect each other, and stages two or higher of the height \( n - 1 \) Casson tower.

**Proof of Theorem A**. Let \( T \) be a distorted Casson tower of height 4. First, we apply Lemma 4.2 to \( T_{1-3} \) to obtain a properly immersed capped grope \( G_2^c \) in \( T_{1-3} \), which is bounded by the framed circle \( C(T) \).

Recall that a plumbed handle is diffeomorphic to a 4-ball with 1-handles attached, and thus the fundamental group is generated by the double point loops. By an induction, the fundamental group of a Casson tower is generated by the top stage double point loops. Applying this to our case, we see that the inclusion induced map \( \pi_1(T_{2-3}) \to \pi_1(T) \) is trivial, since the 4th stage discs give null-homotopies for the double point loops of the 3rd stage plumbed handles. By Lemma 4.3 and a straightforward Seifert-Van Kampen theorem computation for \( T_{1-3} \setminus G_{1-1} = (T_{1-1} \setminus G_{1-1}) \cup T_{2-3} \), it follows that the image of \( \pi_1(T_{1-3} \setminus G_{1-1}) \) in \( \pi_1(T \setminus G_{1-1}) \) under the inclusion induced map is isomorphic to \( \mathbb{Z} \), generated by a meridian of \( G_{1-1} \). From this it also follows that the image of \( \pi_1(\nu G_2^c \setminus G_{1-1}) \) in \( \pi_1(T \setminus G_{1-1}) \) is the same \( \mathbb{Z} \). An infinite cyclic group has subexponential growth and is therefore \( \pi_1(\nu D) \) by Theorem 5.2 so the hypothesis of Theorem 5.4 is satisfied (in our case, \( G_2^c \) is connected and therefore so is \( \nu G_2^c \setminus G_{1-1} \)). Applying Theorem 5.4 we can find a flat embedded disc inside \( T \) as claimed.

\[ \square \]

In order for a Casson tower of height 3 to suffice for the existence of an embedded disc, we will need to embed the tower into a 4-manifold, with a fairly strong assumption on fundamental groups. Recall that for a Casson tower \( T \), there is a surface \( \Sigma(T) \) contained in the first stage \( T_{1-1} \), by Proposition 4.1 (see also Figure 5). Theorem 4.4 says the following:

**Let** \( W \) **be a 4-manifold with boundary and suppose that** \( T = \bigsqcup T_i \) **is a collection of disjoint Casson towers** \( T_i \) **of height 3 in** \( W \) **such that** \( \partial_- (T) \subset \partial W \) **and the image of** \( \pi_1(T_i \setminus \partial W) \) **in** \( \pi_1(W) \) **is a distorted Casson tower of height 3**. 

**Then there is a properly embedded disc bounded by** \( C(T) \) **in** \( W \) **that maps to** \( \partial_+ (T) \subset \partial W \). **The disc is homologous to 0 in** \( W \) **with respect to the tautological 2-form and it is null-homotopic in** \( W \). **Then** \( W \) **admits a 4-dimensional handle decomposition**. 

**Proof.** Let \( D \) be a properly embedded disc bounded by \( C(T) \) in \( W \) that maps to \( \partial_+ (T) \subset \partial W \). Since \( \partial_- (T) \subset \partial W \), \( D \) is homologous to 0 in \( W \) with respect to the tautological 2-form. Moreover, \( D \) is null-homotopic in \( W \) since \( W \) admits a 4-dimensional handle decomposition. 

**□**
prove it for general $n > 0$. Then the framed link $C(T) \subset \partial W$ is slice in $W$.

We note that even if $W$ is simply connected, it is quite possible that the image of this fundamental group in $\pi_1(W \setminus G_{1,-1})$ will not satisfy the $\pi_1$-ND hypothesis. For example if $T$ has more than one component, $\pi_1(W \setminus G_{1,-1})$ might contain a non-abelian free group generated by meridians to the connected components of $G_{1,-1}$.

**Proof of Theorem 4.5** Let $G^0_2$ be the height 2 capped grope in $T$ constructed as in that proof. Note that now $G^0_2$ may not be connected. A component of $\nu G^0_2$, say $V$, lies in some component $T_i$ of $T$. The inclusion induced homomorphisms on fundamental groups factor as

$$\pi_1(V \setminus G_{1,-1},*) \rightarrow \pi_1(T_i \setminus G_{1,-1},*) \rightarrow \pi_1(W \setminus G_{1,-1},*)$$

for any choice of basepoint $*$ in $V$. The hypothesis of the theorem, that the image of the second map is $\pi_1$-ND, implies that the image of this composite homomorphism is also $\pi_1$-ND. This holds because the $\pi_1$-ND property is closed under taking subgroups. Thus we can apply Theorem 3.4 to obtain the flat embedded discs that we seek. □

We remark that, as seen from the above proofs, Theorem 4.3 is indeed a consequence of Theorem 4.5. We point out that so far we only used the Disc Embedding Theorem 3.4 for a height 2 capped group, although it holds for height $\geq 1.5$.

### 4.2. Casson towers of height two

This section contains our strongest conclusions in terms of the height of Casson towers, using the strongest assumption, namely triviality, on fundamental groups. Height 2 Casson towers seem to be the most useful for slicing knots and links in $D^4$, since in practice, contrivances notwithstanding, it is often difficult to construct tall Casson towers. Applications will be given in Section 5.

Our arguments for height 2 Casson towers also involve capped gropes. For this case, we need the full power, in terms of height, of Theorem 4.3. That is, we apply the theorem to a height 1.5 capped grope. Also, we need a new construction of a capped grope from an embedded Casson tower, which is given below. The construction relies upon the properties of the embedding of the tower, and will not work without an ambient manifold.

**Proposition 4.5** (Capped grope from an embedded Casson tower). Suppose $n > 0$, $(T, \partial_\pm)$ is a Casson tower of height $n+1$ embedded in a 4-manifold $(M, \partial M)$, and the double point loops of the top stage of $T$ are null-homotopic in $M \setminus T_{1-n}$. Then there is a properly immersed capped grope $G^{c}_{n,5}$ of height $n,5$ in $M$, which extends the grope $G^{c}_{n,1}$ for the subtower $T_{1-n}$ from Proposition 4.4. In particular, the first stage surface of $G^{c}_{n,5}$ is $\Sigma(T)$, and the attaching circle of $G^{c}_{n,5}$ and $C(T)$ are equal as framed circles.

In fact, we only need the $n = 1$ case of Proposition 4.5 in this paper, but we state and prove it for general $n > 0$ for possible later use, since this does not require any additional complication.

For the proof of Proposition 4.5 we begin with a couple of lemmata. The following lemma cleans up a certain type of naturally occurring capped grope to a properly immersed height 2 framed capped grope. We need to use the notion of a twisted cap, defined as follows. Suppose $G$ is a framed grope embedded in a 4-manifold $W$. An immersed disc $D$ in $W$ bounded by a symplectic basis curve of a top stage surface $\Sigma$ of $G$ is called a twisted cap if the interior of a collar neighbourhood of $\partial D \subset D$ is disjoint from $G$, and a push-off of $\partial D$ along $\Sigma$ induces a section of the normal bundle of $D$ with relative Euler number $\pm 1$. That is, the push-off gives rise to the framing on $\partial D$ obtained by twisting the restriction of the unique framing on $D$ once (either positively or negatively).
Lemma 4.6. Suppose we are given a height 2 capped grope $G^c$ which is immersed (not properly) in a 4-manifold $M$, satisfying the following. The surface stages are disjointly embedded and framed. Each dual pair of curves on a second stage surface has two caps, one $\pm 1$ twisted cap which is disjoint from the body of the grope, and one framed cap which potentially intersects other caps and second stage surfaces. Then there is a properly immersed height 1.5 framed capped grope in a neighbourhood of $G^c$ with the same first stage surface.

Proof. Divide the second stage surfaces and caps into two sides, the + and − sides, as described just prior to Lemma 4.8. We will improve the + side first and then the − side. There are two problems to be dealt with, namely the twisted caps need to be framed and their dual caps need to be made disjoint from the body of the grope. We have the freedom to reduce height by one on the − side.

Call the $\pm 1$ twisted caps the small caps, and the other caps, which can intersect second stage grope surfaces as well as each other, the big caps.

Forget the − side big caps and apply the boundary twisting operation [FQ90] Section 1.3] to the − side small caps, so that they are framed with respect to the − side surfaces. This introduces an intersection of each +− side small cap with the − side surface to which it is attached. Then use the − side small caps to perform asymmetric surgery on the − side surfaces, changing them to immersed discs, which are the new − caps. The intersection of the small caps with the second stage surface introduced by the boundary twisting gives rise to self-intersections of the new − caps. These new − caps may also intersect other caps, but that is permitted.

Now create transverse spheres to the + side surface stages using two parallel copies of the new − side caps, and the annulus in the normal circle bundle to the attaching circle of the − cap, as we have done in several other instances in this paper (see the first paragraph of the proof of the Cap Separation Lemma 4.8 and Step 3 of the proof of Theorem 3.4). The transverse spheres we construct are immersed and may intersect + and − side caps.

Boundary twist the + side small caps to frame them with respect to the + side surfaces. This creates intersections of the + side small caps with the + side second stage surfaces. Now we just have to remove intersections of the + side caps, both big and small, with the + side surfaces. To achieve this, tube anything that intersects a + side surface into a transverse sphere. We obtain a properly immersed height 1.5 framed capped grope as required. □

Lemma 4.7. In a plumbed handle $T$ with $k$ plumbings, there is a genus $k$ framed surface $F$ with $\partial F = C(T)$ as framed circles, which has symplectic basis curves $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ satisfying the following. Each $\alpha_i$ is dual to $\beta_i$, each $\alpha_i$ bounds a $\pm 1$ twisted cap whose interior is disjoint from $F$. The union $\bigcup_i \beta_i$ is parallel to the union of the double point loops of $T$ on $\partial_s T$, via $k$ framed annuli disjointly embedded in $T$, whose interior is disjoint from $F$ and from the $\pm 1$ twisted caps.

Proof. The surface $F$ is obtained from the core disc of $T$ by a standard construction that resolves singularities by increasing the genus: replace the standard local model $(D^2 \times D^2, D^2 \times 0 \cup 0 \times D^2)$ of an intersection point by a twisted annulus in $S^3 = \partial (D^2 \times D^2) \subset D^2 \times D^2$ bounded by the Hopf link $S^1 \times 0 \cup 0 \times S^1$. To verify the framing assertions, we use a Kirby diagram argument as follows.

The Kirby diagram of a plumbed handle with $k$ plumbings in Figure 3 is isotopic to the diagram in Figure 8. Observe that $C(T)$ bounds a surface $F$ which is a band sum of $k$ untwisted annuli with $k$ twisted 1-handles attached; see Figure 3. Let $\alpha_i$ be the core circle of the $i$th twisted 1-handle, and $\beta_i$ be the core of the $i$th untwisted annulus; see Figure 3 again. The curves $\alpha_i$ and $\beta_i$ form a symplectic basis.
From Figure 8 we see that $F$ induces the zero framing on the $\beta_i$, and $\beta_i$ is parallel to the double point loop $a_i$ via a framed annulus inducing the 0-framing on both, as desired. Also, $a_i$ bounds a $\pm 1$ twisted cap whose interior lies in the interior of $D^4$ and so is disjoint from everything else. Since $F$ is a Seifert surface for $C(T), F$ induces the zero framing on $C(T)$, which is the preferred framing by Definition 2.11.

Proof of Proposition 4.3. First apply Proposition 4.1 to find a framed grope $G_n$ in $T_{1-n}$ with framed boundary $C(T)$. Consider the top stage of $T$ in $V := M - T_{1-n}$. By Lemma 4.7, in each stage $n+1$ plumbing handle of $T$, we have a framed surface bounded by its attaching circle, with $\pm 1$ twisted caps which we call a small cap, and annuli cobounded by the dual basis curve and the double point loop. By hypothesis, the double point loop is null-homotopic in $V$. Attach a null-homotopy to each annulus to obtain a cap dual to the $\pm 1$ twisted cap; we call it a big cap. This terminology was already used in the proof of Lemma 4.0. We may assume that each big cap is framed, by applying boundary twist if necessary. This gives a capped grope of height 1, which is immersed in $V$ but not properly immersed in general; the big caps may intersect other surfaces and caps.

Take $2^n$ push-offs of each of these height 1 capped grope and attach them to $G_n$ to obtain a height $n+1$ capped grope $G_{n+1}^c$. The body of $G_{n+1}^c$ and the big caps are compatibly framed, while the small caps are twisted. Now the big caps may intersect stage $n+2$ surfaces and other caps of $G_{n+1}^c$, but are disjoint from $G_{1-n}$, since the big caps lie in $V$. Note that now there are intersections between the small caps, which were introduced when we took push-offs, since the small caps are twisted. However the small caps are disjoint from the body of $G_{n+1}^c$.

Consider $G_{n-(n+1)}^c$, the top two surface stages of $G_{n+1}^c$ together with all the small and big caps. Since $G_{n-(n+1)}^c$ is a height 2 capped grope satisfying the hypotheses of Lemma 4.3, there is a properly immersed height 1.5 capped grope with the same base surfaces, which lies in a neighbourhood of $G_{n-(n+1)}^c$. Replacing $G_{n-(n+1)}^c$ in $G_{n+1}^c$ with this height 1.5 capped grope to obtain a properly immersed capped grope $G_{n,5}^c$ of height $n.5$ as desired.

We are now ready to give the proof of Theorem C from the introduction, which says: let $W$ be a 4-manifold with boundary and suppose $T$ is a Casson tower of height 2 embedded in $W$ such that the second stage $T_{2-2}$ of $T$ lies in a codimension zero simply connected submanifold $V \subseteq W - T_{1-1}$. Then the knot $C(T) \subset \partial W$ is slice in $W$.

Proof of Theorem C. Apply Proposition 4.5 to the given Casson tower $T$ of height 2, with $M := T \cup V$, to obtain a properly immersed capped grope $G_{1,5}^c$ in $T \cup V$.

Observe that $(T \cup V) \setminus G_{1-1} = V \cup (T_{1-1} \setminus G_{1-1})$, where $V$ and $(T_{1-1} \setminus G_{1-1})$ are glued along neighbourhoods of the attaching curves for $T_{2-2}$. By Lemma 4.3 and a straightforward application of the Seifert-Van Kampen theorem, it follows that $\pi_1(V \cup (T_{1-1} \setminus G_{1-1})) \cong \mathbb{Z}$. 

\[\text{Figure 8. A Kirby diagram of a plumbed handle } T \text{ with } k \text{ plumbings, with a genus } k \text{ surface } F \text{ bounded by } C(T).\]
which is \( \pi_1 \text{nd} \) by Theorem \([3,2]\). Apply Theorem \([3,4]\) to \( G_{1,5} \) in \( M := T \cup V \), to yield a flat embedded disc in \( T \cup V \) bounded by \( C(T) \).

**Remark 4.8.** In the above proof of Theorem \([C]\) we have shown that \( C(T) \) is slice in the submanifold \( T \cup V \subseteq W \).

5. Slice knots

In this section we apply the results on Casson towers of Section \([4]\) to produce a new family of slice knots in \( S^3 \).

5.1. Band sums of Whitehead doubles

In this subsection, as promised in the introduction, we use Theorem \([C]\) to give a proof of Theorem \([D]\) which we state here again for the reader’s convenience: *suppose \( L \) is an \( m \)-component homotopically trivial link, and \( K \) is a knot obtained from \( \text{Wh}(L) \) by applying \( m - 1 \) band sum operations. Then \( K \) is slice.*

We begin, in Lemma \([5.1]\), with a well-known observation on Whitehead doubles and plumbed handles. To state it we recall the definition of the (untwisted) Whitehead double of a framed link in a general 3-manifold. Let \( \text{Wh} \subseteq S^1 \times D^2 \) be the standard untwisted Whitehead knot, that is, it is obtained by taking the exterior of a component of a Whitehead link and then identifying it with \( S^1 \times D^2 \) under the zero framing. (There are two possibilities, \( \text{Wh}_+ \) and \( \text{Wh}_- \), depending on the sign of the clasp.) The zero framing on the Whitehead link induces a framing on \( \text{Wh} \subseteq S^1 \times D^2 \) which we call the zero framing. For a framed link \( L \) in a general 3-manifold \( M \), form an untwisted Whitehead double \( \text{Wh}(L) \) of \( L \), which is a framed link, by replacing a tubular neighbourhood of each component of \( L \) with \( (S^1 \times D^2, \text{Wh}) \) under the framing of \( L \). We also recall that the attaching circle and double point loops of a plumbed handle are framed as in Definition \([2.1]\).

**Lemma 5.1** (Plumbed handles for Whitehead doubles). *Suppose \( L \) is an \( m \)-component framed link in a 3-manifold \( M \). Then there exist plumbed handles \( T_i \) \((i = 1, \ldots, m)\) disjointly embedded in \( M \times [0, 1] \) such that each \( T_i \) has exactly one self-plumbing with double point loop \( \alpha_i \), \( \bigsqcup_i \text{C}(T_i) = \text{Wh}(L) \times 0 \), \( \bigsqcup_i \alpha_i = L \times 1 \) as framed links, \( T_i \cap (M \times 0) = \partial_-(T_i) \), and \( T_i \cap (M \times 1) \) is a tubular neighbourhood of \( \alpha_i \subseteq \partial_+(T_i) \).

We call the plumbed handles in Lemma \([5.1]\) the *standard plumbed handles* between \( \text{Wh}(L) \) and \( L \).

**Proof.** The best geometric way to understand our plumbed handles \( T_i \) is to construct the core discs directly: undo the clasp of the Whitehead doubling operations on \( L \) via a regular homotopy, and cap off the resulting trivial link with disjoint discs. We obtain an immersion of \( m \) discs in \( M \times [0, 1] \) bounded by \( \text{Wh}(L) \times 0 \), and then by thickening this, we obtain the plumbed handles \( T_i \). Furthermore, in this construction, by regarding the regular isotopy as a movie picture of the core discs, it can be seen that the double point loops on the core discs can be pushed to \( L \times 1 \) along embedded annuli. It follows that we may thicken the core discs in such a way that \( T_i \cap (M \times 1) \) is a tubular neighbourhood of the double point loop \( \alpha_i \subseteq \partial_+(T_i) \). The framing condition can also be verified by investigating the movie picture carefully.

The above assertions can be verified rigorously by the following alternative description. Recall that a plumbed handle \( T \) with one self-plumbing, together with the attaching circle \( C \) and the double point \( \alpha_1 \), is described by the standard Kirby diagram in Figure \([3]\) \((k = 1 \text{ for now})\), where \( C \) and \( \alpha_1 \) are zero framed by Definition \([2.1]\). In particular \( T \cong S^1 \times D^3 \). By straightening the dotted circle in the Kirby diagram, it follows that if we write \( T = S^1 \times D^2 \times I \), then we may assume \( C = \text{Wh} \times 0 \subseteq S^1 \times D^2 \times 0 \) as framed circles, and
\[ a_1 = S^1 \times 0 \times 1 \subset S^1 \times D^2 \times 1, \text{ framed by the product structure. Now, the framing of} \]
\[ L \text{ gives us an identification of a tubular neighbourhood of } L \times [0, 1] \subset M \times [0, 1] \text{ with} \]
\[ L \times D^2 \times I = \bigsqcup^n (S^1 \times D^2 \times I) = \bigsqcup^n T, m \text{ disjoint plumbed handles. By the above and by} \]
\[ the definition of Wh(L), it follows that the attaching circles of these plumbed handles form the framed link Wh(L) \times 0, \text{ and the double point loops form the framed link } L \times 1. \]

**Proof of Theorem [2].** Attach the bands used in the band sum operations for \( Wh(L) = Wh(L) \times 0 \) to the annuli \( Wh(L) \times [0, \frac{1}{2}] \subset S^3 \times [0, \frac{1}{2}] \) and push them slightly, to obtain an \( m \)-punctured disc in \( S^3 \times [0, \frac{1}{2}] \) cobounded by \( K \times 0 \) and \( Wh(L) \times \frac{1}{2} \). The zero framings on \( Wh(L) \) and \( K \) extend to a framing of the punctured disc. Thicken the punctured disc in \( S^3 \times [0, \frac{1}{2}] \) and attach the standard plumbed handles in \( S^3 \times [\frac{1}{2}, 1]\) between \( Wh(L) \) and \( L \) given by Lemma [5.4]. This constructs a single plumbed handle \( T_{1-1} \) embedded in \( S^1 \times [0, 1]\).

It has \( K \times 0 \) as the attaching circle and \( L \times 1 \) as the double points, by Lemma [5.4]. We will use \( T_{1-1} \) as the first stage of a Casson tower.

Next, view \( S^3 \times [0, 1] \) as a collar neighbourhood of the boundary of \( D^4 = S^3 \times [0, 1] \sqcup S^3 \times 1 \) (smaller \( D^4 \)). Since \( L \) is homotopically trivial, there are disjoint immersed discs in the smaller \( D^4 \), which can be thickened to plumbed handles whose attaching circles form \( L \times 1 \). We want to use these plumbed handles for the second stage. Observe the following general fact, which follows from Definition [2.4] the preferred framing of the attaching circle of a plumbed handle embedded in \( D^4 \) with \( \partial \subset S^3 \) is the zero framing in \( S^3 \). Apply this to our case: since the double point loops \( L \times 1 \) of \( T_{1-1} \) are zero framed by Lemma [5.1], it follows that we can attach these plumbed handles to \( T_{1-1} \), in the smaller \( D^4 \), to yield a height 2 Casson tower, say \( T \).

By construction, \( C(T) = K \), the second stage \( T_{2-2} \) of \( T \) lies in the smaller \( D^4 \), and the first stage \( T_{1-1} \) lies in the collar \( S^3 \times [0, 1] \) of the boundary of the bigger 4-ball \( D^4 \). Since the smaller \( D^4 \) is simply connected, we can apply Theorem [C] to obtain a flat embedded disc bounded by \( K \) as claimed.

The following is an immediate corollary. Recall that a link \( L \) in \( S^3 \) is weakly slice if it bounds a flat embedding of a punctured disc in \( D^4 \).

**Corollary 5.2.** The Whitehead double of any homotopically trivial link is weakly slice.

**Proof.** If \( L \) is an \( m \)-component homotopically trivial link, then a knot \( K \) obtained by \( m-1 \) bands sum operations on \( Wh(L) \) is slice by Theorem [B]. Attaching the bands to a slicing disc and pushing them slightly, we obtain a punctured disc bounded by \( Wh(L) \). □

As another consequence of Theorem [D], we prove Corollary [B] which says the following: suppose \( L \) is an \( m \)-component homotopically trivial link, and \( R \) is a ribbon knot. Consider a split union \( Wh(L) \sqcup R \) in \( S^3 \), and choose \( m \) disjoint bands which join each component of \( Wh(L) \) to \( R \), such that in addition the bands are disjoint from an immersed ribbon disc for \( R \) in \( S^3 \) and are disjoint from Seifert surfaces for \( Wh(L) \). Then the knot \( K \) obtained from \( Wh(L) \sqcup R \) by these band sum operations along the arcs is slice.

**Proof of Corollary [B]** We first observe that a ribbon knot can be viewed as the result of band sum operations performed on a trivial link. Given a ribbon immersion \( D^2 \looparrowright S^3 \), by removing an \( \epsilon \)-neighbourhood of the singularities meeting the boundary of \( D^2 \), we obtain disjoint embedded discs, which are bounded by a trivial link. This is indeed undoing band sum operations, since each removed \( \epsilon \)-neighborhood can be replaced as a band.

Now, choose a ribbon embedding bounded by the given ribbon knot \( R \). We may assume that the feet of the bands used to produce \( K \) from \( Wh(L) \sqcup R \) are disjoint from the ribbon singularities. Undo the band sum operations, to obtain \( Wh(L) \sqcup R \) from \( K \), and then undo the band sum operations for the ribbon knot \( R \) as in the previous paragraph, to transform \( Wh(L) \sqcup R \) into a split union of \( Wh(L) \) and a trivial link. A trivial link is the Whitehead double of a trivial link, say \( L_0 \), from which it follows that our knot \( K \) is obtained by band
sum operations from the Whitehead double of the split union \( L \sqcup L_0 \). Since both \( L \) and \( L_0 \) are homotopically trivial, \( L \sqcup L_0 \) is homotopically trivial. By Theorem 1 [L] \( K \) is a slice knot. \[\square\]

5.2. Attempts to apply a surgery method

A standard surgery theoretic slicing process for a given knot \( K \) with zero-surgery manifold \( M_K \) starts with an epimorphism \( \pi_1(M_K) \to G \) onto an appropriate ribbon group \( G \); here a ribbon group is the fundamental group of the complement of a slicing disc in \( D^4 \) obtained by resolving singularities of a ribbon immersion \( D^2 \leftrightarrow S^3 \). Then one applies topological surgery over the group \( G \), to obtain a slice disc exterior whose fundamental group is \( G \).

(For implementations of this strategy for knots, see, e.g. [FQ90, FT05].) With current knowledge the surgery strategy can only be completed for certain special cases, since it is unknown whether surgery works for all ribbon groups. There are only two ribbon groups for which surgery is known to work: \( \mathbb{Z} \) and \( G_{6_1} = \langle a, t \mid t^2a^{-1} = a \rangle \), a ribbon group for the stevedore knot \( 6_1 \). They were used in [FQ90, FT05] respectively.

In most cases (including Figure 1), Corollary 2 produces a knot \( K \) admitting a natural epimorphism \( \pi_1(M_K) \to G \) onto a ribbon group \( G \) for the ribbon knot \( R \) used in the construction; see Proposition 5.3. When \( G \) is neither \( \mathbb{Z} \) nor \( G_{6_1} \), the slice disc exterior strategy cannot be carried out, at least for now, for this epimorphism, since we do not know whether \( G \) is a good group.

Proposition 5.3. Suppose \( K \) is a slice knot obtained by band sum operations on \( \text{Wh}(L) \sqcup R \) as in Corollary 1.

(1) The knots \( K \) and \( R \) have \( S \)-equivalent Seifert matrices. Consequently, \( \Delta_K(t) = \Delta_R(t) \).

(2) There is an epimorphism of \( \pi_1(M_K) \) onto \( \pi_1(M_R) \) which takes a meridian to a meridian.

Before proving Proposition 5.3 we discuss some of its consequences. First, from Proposition 5.3 [L], it follows that for any ribbon group \( G \) for \( R \), there is an epimorphism of \( \pi_1(S^3 \setminus K) \) onto \( G \), as mentioned in the introduction. The second consequence is that if \( \Delta_R(t) \) is not one and is not divisible by \( t - \frac{1}{2} \) (over \( \mathbb{Q} \)), then the topological surgery method to construct a slice disc exterior, which we discussed in the introduction, does not work for \( K \). In fact, to apply topological surgery, one needs to start with an epimorphism of \( \pi_1(S^3 \setminus K) \) onto a ribbon group \( G \) for which surgery is known to work; the only such ribbon groups are \( \mathbb{Z} \) and \( G_{6_1} := \langle a, t \mid t^2a^{-1} = a \rangle \), a ribbon group for the Stevedore knot \( 6_1 \). For the \( G = \mathbb{Z} \) case, it is known that the surgery programme slices a knot if and only if \( \Delta_K(t) = 1 \), essentially because defining a surgery problem requires a degree one normal map with target \( S^1 \times D^3 \) which restricts to a \( \mathbb{Z}[\mathbb{Z}] \) homology equivalence on the boundary. By Proposition 5.3 [L], it follows that surgery cannot be carried out if \( \Delta_R(t) \neq 1 \). Also, for \( G = G_{6_1} \), if there were an epimorphism \( \pi_1(S^3 \setminus K) \to G_{6_1} \), then it would imply that \( \Delta_K(t) \) is divisible by the “Alexander polynomial” \( \Delta_{G_{6_1}}(t) \) of \( G_{6_1} \), which is defined to be the order of the module \( (G_{6_1}^*/G_{6_1}^{'0}) \otimes \mathbb{Q} \cong H_1(G_{6_1}^*; \mathbb{Q}) \) over the PID \( \mathbb{Q}[t^{\pm 1}] \) as usual. Indeed, \( G_{6_1} \) is isomorphic to the Baumslag-Solitar group \( \langle \mathbb{Z}[t^{\pm 1}]/(t - \frac{1}{2}) \rangle \). From this it follows that if \( \Delta_K(t) \) is not divisible by \( t - \frac{1}{2} \), then there is no epimorphism of \( \pi_1(S^3 \setminus K) \) onto \( G_{6_1} \). It is conceivable that \( K \) is smoothly concordant to a knot \( J \) with \( \Delta_J = \Delta_{6_1} \), such that \( J \) can be sliced using [FT05]. In this eventuality the resulting slice disc would not be homotopy ribbon.

Next we prove Proposition 5.3.

Proof of Proposition 5.3. First we prove part 1. The standard genus 1 Seifert surface of each component of \( \text{Wh}(L) \) has Seifert matrix \([0 1\]) [11]. Since \( L \) is homotopically trivial, each
pairwise linking number of $L$ is zero. It follows that the diagonal block sum of $m$ copies of $[0,1]_m$ and a Seifert matrix of $R$ is a Seifert matrix of $K$. Consequently, $K$ has a Seifert matrix $S$-equivalent to that of $R$.

The proof of part (2) follows immediately from Lemma 5.4 below. □

**Lemma 5.4.** Suppose $L$ is an $m$-component boundary link with Seifert surface $V$, and $J$ is a knot. Suppose $K$ is a knot obtained from the split union $L ∪ J$ by $m$ band sum operations along bands which join a component of $L$ to $J$ and whose interior is disjoint from $V$. Then there is an epimorphism $π_1(M_K) → π_1(M_J)$ which takes a meridian to a meridian.

**Proof.** Let $γ_i$ be the core arc of the band joining $J$ and the $i$th component $L_i$ of $L$, for $i = 1, \ldots, m$. Using Kirby calculus it is not too hard to see (see e.g. [COT04, Proof of Theorem 4.1]) that the 3-manifold $M_K$ is obtained from $M_{L∪J}$ by zero-framed surgery along $m$ curves, say $α_i$, each of which bounds an embedded 2-disc that meets $L ∪ J$ at two transverse intersection points, contains $γ_i$, and induces the framing of $γ_i$. See Figure 9.

![Figure 9. Band sum and surgery.](image)

The standard Thom-Pontryagin construction applied to the $m$-component Seifert surface $V$ gives a map

$$S^3 \setminus ν(L) \longrightarrow \bigvee_m S^1$$

which takes $S^3 \setminus ν(L)$ to the wedge point; this induces an epimorphism $φ: π_1(M_L) → F := \text{free group of rank } m$, which takes $β_i$ to the $i$th generator $x_i ∈ F$.

Note that since $M_{L∪J} ∼ M_L # M_J$, we have that $π_1(M_{L∪J}) = π_1(M_L) * π_1(M_J)$. Furthermore, from the hypothesis that the interior of the arc $γ_i$ is disjoint from the Seifert surface $V$, it follows that $(φ * \text{id})(α_i) ∈ F * π_1(M_L)$ is of the form $x_i \cdot ζ_i$, where $ζ_i ∈ π_1(M_J)$.

Let $W$ be the cobordism between $M_K$ and $M_{L∪J}$ obtained by attaching $m$ 2-handles along the curves $α_i$ to the product $M_{L∪R} × [0, 1]$. We have an epimorphism

$$π_1(W) \cong \frac{π_1(M_{L∪J})}{⟨α_i⟩} = \frac{π_1(M_L) * π_1(M_J)}{⟨α_i⟩} \xrightarrow{φ * \text{id}} \frac{F * π_1(M_J)}{⟨x_i \cdot ζ_i⟩} \cong π_1(M_J).$$

Also, turning $W$ upside down, $W$ is obtained by attaching 2-handles to $M_K × [0, 1]$. It follows that the inclusion induces an epimorphism $π_1(M_K) → π_1(W)$. Composing this with $π_1(W)$, we obtain the desired epimorphism $π_1(M_K) → π_1(M_J)$. By construction, this takes a meridian to a meridian. □

Friedl and Teichner proposed conjectures related to necessary and sufficient conditions for being homotopy ribbon, in [FT05] Conjectures 1.6 and 1.8. It is an interesting question whether the slice knots produced by Corollary [5] satisfy their proposed conditions, that is, the Ext condition in [FT05] Conjecture 1.6 and the Poincaré duality condition in [FT05] Conjecture 1.8.
6. Slice links

In this section we present new slice links in $S^3$, using our results on Casson towers. We focus on various constructions of links involving Whitehead doubling, which are naturally related to Casson towers.

6.1. Whitehead doubles of height 2 homotopically trivial links

Let $L$ be a homotopically trivial link in the 4-ball which satisfies the following additional condition: there exists a null-homotopy i.e. a collection of maps $f_i: D^2 \to D^4$ with disjoint images with $f_i(\partial D^2) = L_i$, such that for each double point of the null-homotopy $f_i$ there is a double point loop which is null-homotopic in the complement $D^4 \setminus (\bigcup \{f_j(D^2) : j \neq i\})$. We also require that the null-homotopies belonging to double point loops of distinct $f_i(D^2)$ are disjoint. We call such links height 2 homotopically trivial. We remark that we could not find an example of a link which is height 2 homotopically trivial but not homotopically trivial$^\dagger$.

For example the Whitehead double of the Hopf link is height 2 homotopically trivial. In this case the theorem below gives another proof that the 4-fold Whitehead double of one component of the Hopf link is slice.

**Theorem 6.1.** The ant twisted Whitehead double of a height 2 homotopically trivial link is slice.

**Proof.** Let $L$ be a height 2 homotopically trivial link with $m$ components. We will construct disjoint height 2 Casson towers in $D^4$ bounded by $\text{Wh}(L)$, by an argument similar to the proof of Theorem 6.1 in Section 6.1. That is, write $D^4 = S^3 \times I \cup_{S^3 \times 1} (\text{smaller } D^4)$, invoke Lemma 6.1 to construct standard plumbed handles embedded in $S^3 \times I$ with attaching circles $\text{Wh}(L) \times 0 \subset S^3 \times 0$ and double point loops $L \times 1 \subset S^3 \times 1$, and then attach thickened null-homotopies for $L$ in the smaller $D^4$ to those standard plumbed handles. Since the null-homotopies are disjoint, we obtain $m$ disjoint height 2 Casson towers, say $\Sigma(T_1), \ldots, \Sigma(T_m)$, whose attaching circles form the zero framed link $\text{Wh}(L)$.

Note that $\pi_1((\text{smaller } D^4) \cup T_i \setminus \Sigma(T_i)) \cong \mathbb{Z}$ for each $i$, by Lemma 1.3 and a Seifert-Van Kampen argument, similarly to that used in the proof of Theorem C.

We remark that Theorem 6.1 (our height 2 Casson tower theorem) does not apply directly to $\bigsqcup T_i$, since it is only for a single Casson tower. Instead, we will construct a capped grope of height 1.5 and then apply Theorem 3.4 for the capped grope.

By the hypothesis, there are null-homotopies, in the smaller $D^4$, of the double point loops of the second stage of each $T_i$, such that the null-homotopies attached to distinct $T_i$ are disjoint. For each $i$, choose a neighbourhood $U_i$ of the union of the null-homotopies of double point loops of $T_i$ such that the sets $U_i$ are pairwise disjoint, and view $T_i$ as embedded in $T_i \cup U_i$. Arrange additionally that $U_i$ lies in the smaller $D^4$. Apply Proposition 1.5 to each $T_i$ in $T_i \cup U$, to obtain a properly immersed capped grope of height 1.5 in $T_i \cup U_i$, whose first stage is $\Sigma(T_i)$. In particular, $\Sigma(T_i)$ is bounded by the $i$th component of $\text{Wh}(L)$. Since the subsets $T_i \cup U_i$ are disjoint, the capped gropes of height 1.5 are disjoint. Let $G^c_{1.5}$ be the union of these capped gropes.

If $*$ is a basepoint in the $i$th component of $\nu G^c_{1.5}$, then the inclusion induced map

$$\pi_1(\nu G^c_{1.5} \setminus G_{1.5}, *) \to \pi_1(D^4 \setminus G_{1.5}, *)$$

factors through $\pi_1((\text{smaller } D^4) \cup (T_i \setminus \Sigma(T_i))) \cong \mathbb{Z}$, and thus has $\pi_1$ ND image (in fact the image is $\mathbb{Z}$ too). It follows that we may apply Theorem 3.4 to the whole capped grope $G^c_{1.5}$ to find disjoint slice discs bounded by $\text{Wh}(L)$. $\square$
6.2. Kirby diagrams for distorted Casson towers

In this subsection we discuss Kirby diagrams for arbitrary distorted Casson towers, prior to their application in our construction of another family of new slice links, presented in subsection 6.3.

First we recall that a plumbing operation between two 2-handles in a Kirby diagram gives us a new 1-handle and a clasp (whose sign is equal to the sign of the plumbing) between the attaching circles of the 2-handles, as shown in Figure 10. As a reference, see for instance [GS99, Example 6.1.3].

\[ \begin{array}{c}
\overset{h}{\text{h}} & \overset{h'}{\text{h'}} \\
\end{array} \quad \rightarrow \quad \overset{10}{\text{10}} \quad \overset{h}{\text{h}} \quad \overset{h'}{\text{h'}} \]

Figure 10. Plumbing of 2-handles in a Kirby diagram.

Start with a standard Kirby diagram of a Casson tower. For example, see Figure 11, which is a Casson tower of height 4. (It is a good exercise, for those not familiar with this diagram, to build it using the above plumbing operation; or see [Cas86], [Fre82], Section 2, [FQ99], Chapter 12, [GS99], Example 6.1.3.) So far all the plumbing operations are self-plumbings. As our temporary convention, a circle without a dot or a label designates a zero-framed attaching circle of a 2-handle.

\[ \begin{array}{c}
\overset{C(T)}{\text{C(T)}} \end{array} \]

Figure 11. A Casson tower of height 4. Unlabelled circles without a dot are zero-framed.

A useful fact is that (the core of) each 2-handle is a codimension zero subset of (the core of) a plumbed handle of stage two or higher. It follows that by applying the above plumbing operation for 2-handles, we can plumb a stage 4 disc to another disc of stage two or higher. For instance, by plumbing a stage 4 disc in Figure 11 to a stage 2 disc and by plumbing another stage 4 disc to a stage 3 disc, we obtain a distorted Casson tower described by the Kirby diagram in Figure 12.
Figure 12. A Kirby diagram of a distorted Casson tower of height 4.

Elimination of all 2-handles. In case of a (non-distorted) Casson tower, it is well known that one can eliminate 1- and 2-handles in pairs to obtain a Kirby diagram without 2-handles. A standard procedure, which is a “top-to-bottom” elimination, is as follows. Start with a diagram such as Figure 11 (for which this procedure will be “right-to-left”). Slide each rightmost 1-handle, which is associated to a self-intersection of the top stage disc, under the adjacent 1-handle on its left. Then eliminate the 1-handle under which it slid, together with the linking 2-handle. Iterate this, to obtain a Kirby diagram with \( k \) 1-handles, where \( k \) is the number of self-intersections of the final stage discs. In fact \( C(T) \) remains as an unknotted circle, and the 1-handles form a ramified iterated Whitehead double of a meridian of \( C(T) \).

For a distorted Casson tower diagram, the plumbings performed between 2-handles as in Figure 10 may prevent the elimination of a 1-handle and a linking 2-handle in the above procedure. Instead, in order to obtain a Kirby diagram of a distorted Casson tower without 2-handles, we will perform the “bottom-to-top” elimination discussed below, which is “left-to-right” in case of Figure 12.

For this elimination the (well known) modification shown in Figure 13 is useful; \( R \) denotes an arbitrary tangle diagram, and the hatched band designates parallel strands. The box with label \(-2\) designates two negative full twists. (If we had a negative clasp in the first picture, we would have \(+2\) instead.) The first step in Figure 13 is an isotopy which straightens the 1-handle, and the second step is handle sliding and cancellation.

Figure 13. A modification of a Kirby diagram.
Now start with a Kirby diagram drawn as in Figure 12. First apply the move in Figure 13 to the leftmost part to eliminate the leftmost 1-handle, and a 2-handle linking it. By this, $C(T)$ becomes the Whitehead double of the attaching circle of the eliminated 2-handle; see Figure 14. Repeatedly apply the move in Figure 13 to eliminate the next leftmost 1-handles and 2-handles in pairs. Eventually we obtain a Kirby diagram with $k$ 1-handles and no 2-handles, where $k$ is the number of intersections of the top stage discs and stage $\geq 2$ discs. For instance, if we start with Figure 12, we have $k = 5$ since there are 3 self-intersections of stage 4 discs, and 2 “distorting” intersections between stage 4 and lower stage discs. See Figures 15 and 16.

A consequence of this is that a distorted Casson handle is diffeomorphic to the boundary connected sum of $k$ copies of $S^1 \times D^3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{A distorted Casson tower diagram with a 1-handle and a 2-handle eliminated.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15.png}
\caption{Further elimination of handles in a distorted Casson tower diagram.}
\end{figure}
6.3. Distorted 4-fold iterated ramified Whitehead doubles

In this subsection we give a new family of slice links. The main ingredients are Theorem A on distorted Casson towers of height 4 and the Kirby diagrams we obtained in Section 6.2. First we begin with a general construction of links, without requiring a distorted Casson tower. Then we will relate such a link to a distorted Casson tower, by connecting combinatorial choices involved in the construction of the link to intersection data of the corresponding distorted Casson tower.

Construction of links. We start with the split union of arbitrary number of Hopf links. Choose one component from each Hopf link, and denote the union of the chosen components by \( L_1 \). Denote the union of the other components by \( L_2 \). In what follows, Whitehead doubles and parallels are always untwisted, and taken in a tubular neighborhood which is thin enough to be disjoint from anything we have considered previously. Also, a band sum is always assumed to be between components of split sublinks along a “straight” band; more precisely, whenever two components \( J \) and \( J' \) of a link are joined by a band, there is a separating 2-sphere \( S \) in \( S^3 \) disjoint from the link, and the band passes through \( S \) exactly once and is disjoint from anything we have considered previously. This determines the result of the band sum uniquely up to isotopy. Now the construction is described below.

1. Replace each component of \( L_2 \) with \( \text{Wh}(L_2) \). Perform some band sum operations to combine distinct components of \( L_1 \) and call the result \( L'_1 \). Remember a meridian of each component of \( L'_1 \) for later use, without adding it to the link.

2. Replace \( L'_1 \) with \( \text{Wh}(L'_1) \), perform some band sum operations to combine distinct components of \( \text{Wh}(L'_1) \), and call the result \( L''_1 \). The sublink \( \text{Wh}(L_2) \) is left unchanged. Remember a meridian of \( L''_1 \).

3. Perform (2) once again for \( L''_1 \) in place of \( L'_1 \) and call the result \( L'''_1 \). Remember a meridian of each component of \( L'''_1 \) for later use.
(4) Perform (2) once again for $L''_1$ in place of $L'_1$. This time we perform band sum operations on $\text{Wh}(L''_2)$ until we obtain a knot, say $J$.

(5) Perform the following operation some number of times: choose a remembered meridian of a component of $L'_1$ and a remembered meridian of a component of either $L'_1$, $L''_1$ or $L'''_1$. Band sum them, add a meridional circle of the band to our link, and modify $J$ by performing $\pm 1$ surgery on the banded together meridians, then $\mp 1$ surgery on each of the meridians individually. This introduces a clasp between strands enclosed by the meridians. If the same meridian is chosen more than once during the iteration, use a parallel copy.

The final outcome is the union of $\text{Wh}(L_2)$ and $J$ modified in Step (5). Remembered meridians are not included.

Maybe our construction is best understood by an example: see Figure 17.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{slice_link_construction.png}
\caption{A construction of a slice link. Each box designates $-2$ full twists. The meridians $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_4$ are those of $L'_1$, $L''_1$, $L'''_1$, and $L''''_1$, respectively, and Step (5) is performed for the pairs $(\mu_1, \mu_2)$ and $(\mu_3, \mu_4)$ to obtain the last link as the final outcome.}
\end{figure}

\textbf{Theorem 6.2.} Any link constructed as above is slice.
By Theorem 6.2, the last link in Figure 17 is slice. As another example which is simpler, the link in Figure 2 is slice. Thus Theorem C is a consequence of Theorem 6.2. Indeed, the link in Figure 2 is obtained by applying the above construction to a distorted Casson tower of height 4 which has one plumbed handle with one self-plumbing at each stage and has one distorting intersection between the stage 4 and stage 2 discs.

Proof of Theorem 6.2. We claim that a link $L$ obtained by the above construction is the union of the curve $C(T)$ and the dotted circles representing 1-handles in a Kirby diagram of a height 4 distorted Casson tower $T$ without 2-handles. For instance, observe that Figure 16 and the final picture in Figure 17 are identical. In fact, our construction of $L$ corresponds to a top-to-bottom construction of a distorted Casson tower, as follows. For each component of $\text{Wh}(L_2)$ in Step 1, take a disc with a single local kink, which we call a pre-stage-4 disc. Whenever we perform band sum of components of $L_1$ in Step 2, take a boundary connected sum of the associated pre-stage-4 discs. The resulting discs with (multi-)kinks are the stage 4 discs of our tower $T$. In Step 3, whenever we take a Whitehead double of a component, take a disc with a single local kink, which we call a pre-stage-3 disc, and attach the associated stage 4 disc to the pre-stage-3 disc along the double point loop. Again, whenever we perform a band sum, take the boundary connected sum of the pre-stage-3 discs. The result is stage 3 discs with the stage 4 discs attached. Continue in the same way for steps 4 and 5 to produce stages 2 and 1. We arrive at a non-distorted Casson tower of height 4. Finally, for each triple of ±1 surgeries occurring in step 5, plumb a stage 4 disc to a stage 4, 3 or 2 disc, where the choice of the meridians determines which discs to plumb. The 2-handle elimination procedure described in Section 6.2 applies to the standard Kirby diagram of the resulting distorted Casson tower, from which the claim follows. The lemma stated below now completes the proof.

Lemma 6.3. Let $L$ be the union of the curve $C(T)$ and the dotted circles representing 1-handles in a Kirby diagram of a height 4 distorted Casson tower $T$ without 2-handles. Then, as a link in $S^3$, $L$ is slice.

Proof. The Kirby diagram without 2-handles determines an embedding of $T$ into the 4-ball, to wit, $T$ is the exterior of the standard slicing discs $\Delta_i$ bounded by the dotted circles (which form a trivial link). By Theorem A, the curve $C(T)$ bounds a flat disc $\Delta$ in $T$. As $\Delta$ is disjoint from the discs $\Delta_i$, $L$ is slice.

The above lemma also applies to give another family of slice links: recall that Theorem F in the introduction states that any ramified $\text{Wh}_n$ link is slice for $n \geq 4$.

Proof of Theorem F. It suffices to show that any ramified $\text{Wh}_4$ link is slice. Recall that a ramified $\text{Wh}_4$ link $L$ is the union of $C(T)$, and the dotted circles in a Kirby diagram obtained by a top-to-bottom elimination of 2-handles applied to the standard Kirby diagram of a (non-distorted) Casson tower of height 4; here $C(T)$ remains as an unknotted circle and the other components form a 4th iterated ramified Whitehead double of a meridian of $C(T)$. By Lemma 6.3, $L$ is slice.

7. The grope filtration of knots and Casson towers of height 3

In this section we make the observation that we can use the improved initial hypothesis in the Grope Height Raising Lemma to slightly extend results from [Ray13] on the grope filtration of the knot concordance group. The grope filtration first appeared in the literature in [CT07], although it was already implicit in [COT03]. By definition a knot in $S^3$ lies in the $n$th term $G^{(n)}$ of the filtration if it bounds an embedded framed grope $G_n$ in $D^4$. Ray shows that a knot in $S^3$ which bounds a Casson tower of height 3 is $(n)$-solvable for all $n$. This follows from the corollary below, by [COT03] Theorem 8.11]. She also shows...
that a knot which bounds a Casson tower of height $n$ bounds a grope of height $n$ \cite[Theorem A (i)]{Ray13}. In fact, a height 3 Casson tower is enough to obtain this conclusion for all $n$.

**Corollary 7.1.** A Casson tower $T_3$ of height 3 contains an embedded framed grope $G_n$ of height $n$, with the same attaching circle as the Casson tower, for all $n$.

**Proof.** Apply Proposition \ref{prop:framed_grope} to construct a properly immersed framed capped grope of height 2 inside $T_3$, as described in the beginning of the proof of Theorem \ref{thm:Casson_tower}. Apply grope height raising, as in Lemma \ref{lem:grope_height_raising} to obtain a properly immersed framed capped grope of height $n$, and then ignore the caps. \hfill \Box

It is interesting to contrast Corollary \ref{cor:Casson_tower} with Theorem B. As mentioned above, Ray showed in \cite{Ray13} that a link in $S^3$ which bounds a Casson tower of height 3 in $D^4$ is $(n)$-solvable, in the sense of \cite{COT03}, for all $n$. Corollary \ref{cor:Casson_tower} shows that the link also lies in the intersection of the grope filtration. A link which bounds a grope of height $n + 2$ is $(n)$-solvable \cite[Theorem 8.11]{COT03}, but by \cite[Corollary 6.8]{Ott12} the converse does not hold, so for finite terms of the filtrations, having a height $n + 2$ embedded grope is stronger than having an $(n)$-solution. As observed in \cite{Ray13}, we can deduce the existence of an $(n)$-solution from the existence of an immersed grope of height $n + 2$ with the bottom two stages embedded. It is not known whether the infinite intersections of the filtrations coincide.

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