An almost-Schur type lemma for symmetric $(2,0)$ tensors and applications

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Abstract

In our previous paper in [C], we generalized the almost-Schur lemma of De Lellis and Topping for closed manifolds with nonnegative Ricci curvature to any closed manifolds. In this paper, we generalize the above results to symmetric $(2,0)$-tensors and give the applications including $r$th mean curvatures of closed hypersurfaces in a space form and $k$ scalar curvatures for closed locally conformally flat manifolds.

1 Introduction

Recall that an $n$-dimensional Riemannian manifold $(M, g)$ is called to be Einstein if its traceless Ricci tensor $\tilde{\text{Ric}} = \text{Ric} - \frac{R}{n} g$ is identically zero. Here $\text{Ric}$ and $R$ denote Ricci curvature and scalar curvature respectively. Schur’s lemma states that the scalar curvature of an Einstein manifold of dimension $n \geq 3$ must be constant. In [dLT], De Lellis and Topping discussed the stability and rigidity of Schur’s lemma for closed manifold and proved the following almost Schur lemma, as they called.

Theorem 1.1. ([dLT]) If $(M, g)$ is a closed Riemannian manifold of dimension $n$ with nonnegative Ricci curvature, $n \geq 3$, then

$$\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2} \int_M |\text{Ric} - \frac{R}{n} g|^2,$$

and equivalently,

$$\int_M |\text{Ric} - \frac{R}{n} g|^2 \leq \frac{n^2}{(n-2)^2} \int_M |\text{Ric} - \frac{R}{n} g|^2,$$

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where $\overline{R} = \frac{\int_M R \, dv}{\text{vol}(M)}$ denotes the average of $R$ over $M$. Moreover the equality in (1.1) or (1.2) holds if and only if $M$ is Einstein.

De Lellis and Topping [dLT] also proved their estimates are sharp. First, the constants are optimal in (1.1) and (1.2) ([dLT], Section 2). Second, the curvature condition $\text{Ric} \geq 0$ cannot simply be dropped (see the examples in the proof of Prop. 2.1 and 2.2 in [dLT]). Without the condition of nonnegativity of Ricci curvature, the same type of inequalities as (1.1) and (1.2) cannot hold if the constants in these inequalities only depend on the lower bound of the Ricci curvature. In [C], we considered the case of closed manifolds without the hypothesis of nonnegativity of Ricci curvature and proved that

**Theorem 1.2.** ([C]) If $(M, g)$ is a closed Riemannian manifold of dimension $n, n \geq 3$, then

$$
\int_M (R - \overline{R})^2 \leq \frac{4n(n-1)}{(n-2)^2} \left(1 + \frac{nK}{\lambda_1}\right) \int_M |\text{Ric} - \frac{R}{n} g|^2,
$$

and equivalently,

$$
\int_M |\text{Ric} - \overline{R} g|^2 \leq \frac{n^2}{(n-2)^2} \left[1 + \frac{4(n-1)K}{n\lambda_1}\right] \int_M |\text{Ric} - \frac{R}{n} g|^2,
$$

where $\lambda_1$ denotes the first nonzero eigenvalue of Laplace operator on $(M, g)$, $K$ is nonnegative constant such that the Ricci curvature of $(M, g)$ satisfies $\text{Ric} \geq -(n-1)K$, and $\overline{R}$ denotes the average of $R = \frac{\int_M R \, dv}{\text{vol}(M)}$ over $M$.

Moreover, the equality in (1.3) or (1.4) holds if and only if $M$ is an Einstein manifold.

Observe that Theorem 1.1 is a particular case of Theorem 1.2 ($K = 0$). After the work of De Lellis and Topping, in the case of dimension $n = 3, 4$, Y. Ge and G. Wang ([GW1], [GW2]) proved that Theorem 1.1 holds under the weaker condition of nonnegative scalar curvature. However as pointed out in [dLT], this is surely not possible for $n \geq 5$ (it can be shown using constructions similar to the ones of Section 3 in [dLT]). Also, Ge, Wang and Xia [GWX] proved the case of equality in (1.1) by a different way and gave some generalization of the De Lellis-Topping’ inequalities for $k$-Einstein tensors and Lovelock curvature.
On the other hand, there is a similar phenomenon in submanifold theory. In differential geometry, a classical theorem states that a closed totally umbilical surface in the Euclidean space $$\mathbb{R}^3$$ must be a round sphere $$S^2$$ and its second fundamental form $$A$$ is a constant multiple of its metric. This theorem is also also true for hypersurfaces in $$\mathbb{R}^{n+1}$$. It is interesting to discuss the stability and rigidity of this theorem. De Lellis and Müller [dLM] obtained an optimal rigidity estimate for closed surfaces in $$\mathbb{R}^3$$. Recently, D. Perez [P] proved the following theorem for convex hypersurfaces in $$\mathbb{R}^{n+1}$$.

**Theorem 1.3.** ([P]) Let $$\Sigma$$ be a smooth, closed and connected hypersurface in $$\mathbb{R}^{n+1}$$, $$n \geq 2$$ with induced Riemannian $$g$$ and non-negative Ricci curvature, then

$$\int_{\Sigma} |A - \frac{1}{n} Hg|^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2,$$

and equivalently

$$\int_{\Sigma} (H - \overline{H})^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2,$$

where $$A$$ and $$H = \text{trace} A$$ denote the second fundamental form and the mean curvature of $$\Sigma$$ respectively, $$\overline{H} = \frac{1}{\text{Vol}_n(\Sigma)} \int_{\Sigma} H$$. In particular, the above estimate holds for smooth, closed hypersurfaces which are the boundary of a convex set in $$\mathbb{R}^{n+1}$$.

As pointed out by De Lellis and Topping [dLT], Perez’s theorem holds even for the closed hypersurfaces with nonnegative Ricci curvature when the ambient space is Einstein. Indeed a slight modification of the proof of Theorem 1.3 gives

**Theorem 1.4.** Let $$(\mathbb{N}^{n+1}, \tilde{g})$$ be an Einstein manifold, $$n \geq 2$$. Let $$\Sigma$$ be a smooth, closed and connected hypersurface immersed in $$\mathbb{N}$$ with non-negative Ricci curvature, then

$$\int_{\Sigma} |A - \frac{1}{n} \overline{H}g|^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2,$$

and equivalently

$$\int_{\Sigma} (H - \overline{H})^2 \leq \frac{n}{n-1} \int_{\Sigma} |A - \frac{H}{n} g|^2,$$

where $$\overline{H} = \frac{1}{\text{Vol}_n(\Sigma)} \int_{\Sigma} H$$. 

3
Later, Zhou and the author ([CZ]) discussed the rigidity of the equalities in inequalities (1.5) and (1.6) and proved the following

**Theorem 1.5.** ([CZ]) Let \( Σ \) be a smooth, connected, oriented and closed hypersurface immersed in the Euclidean space \( \mathbb{R}^{n+1}, n \geq 2 \), with non-negative Ricci curvature. Then,

\[
\int_Σ |A - \frac{1}{n} \overline{H} g|^2 = \frac{n}{n - 1} \int_Σ |A - \frac{H}{n} g|^2, \tag{1.9}
\]

and equivalently

\[
\int_Σ (H - \overline{H})^2 = \frac{n}{n - 1} \int_Σ |A - \frac{H}{n} g|^2, \tag{1.10}
\]

holds if and only if \( Σ \) is a totally umbilical hypersurface, where \( \overline{H} = \frac{1}{\text{Vol}(Σ)} \int_Σ H \), that is, \( Σ \) is a distance sphere \( S^n \) in \( \mathbb{R}^{n+1} \).

In [CZ], the authors also studied the general case for hypersurfaces without hypothesis on convexity (that is, \( A \geq 0 \), which is equivalent to \( \text{Ric} \geq 0 \) when \( Σ \) is a closed hypersurface in \( \mathbb{R}^{n+1} \)). Precisely, the following theorem was proved.

**Theorem 1.6.** ([CZ]) Let \( (N^{n+1}, \overline{g}) \) be an Einstein manifold, \( n \geq 2 \). Let \( Σ \) be a smooth, connected, oriented and closed hypersurface immersed in \( N \) with induced metric \( g \). Then

\[
\int_Σ |A - \frac{H}{n} g|^2 \leq \frac{n}{n - 1} (1 + \frac{K}{λ_1}) \int_Σ |A - \frac{H}{n} g|^2, \tag{1.11}
\]

and equivalently

\[
\int_Σ (H - \overline{H})^2 \leq \frac{n}{n - 1} (1 + \frac{nK}{λ_1}) \int_Σ |A - \frac{H}{n} g|^2, \tag{1.12}
\]

where \( λ_1 \) is the first nonzero eigenvalue of the Laplacian operator on \( Σ \), \( K \geq 0 \) is a nonnegative constant such that the Ricci curvature of \( Σ \) satisfies \( \text{Ric} \geq -K \).

Moreover, when \( N^{n+1} \) is the Euclidean space \( \mathbb{R}^{n+1} \), the hyperbolic space \( \mathbb{H}^{n+1}(-1) \) or the closed hemisphere \( S_{n+1}^+(1) \), the equality in (1.11) or (1.12) holds if and only if \( Σ \) is a totally umbilical hypersurface, that is, \( Σ \) is a distance sphere \( S^n \) in \( N^{n+1} \).
From [dLT], [GW1], [GW2], [GWX], [C], [P] and [CZ], we observe that the inequalities mentioned above may be generalized to symmetric $\left(2,0\right)$ tensor fields. Applying such unified inequalities for symmetric $\left(2,0\right)$ tensors, we may obtain new inequalities besides the inequalities in the papers mentioned above. For this purpose, in this paper, we prove that

**Theorem 1.7.** Let $(M, g)$ be a closed Riemannian manifold of dimension $n, n \geq 2$. Let $T$ be a symmetric $\left(2,0\right)$-tensor field on $M$. If the divergence $\text{div} T$ of $T$ and the trace $B = \text{tr} T$ of $T$ satisfy $\text{div} T = c\nabla B$, where $c$ is a constant, then

$$ (nc - 1)^2 \int_M (B - \overline{B})^2 \leq n(n - 1) \left(1 + \frac{nK}{\lambda_1}\right) \int_M |T - \frac{B}{n}g|^2, \quad (1.13) $$

and equivalently,

$$ (nc - 1)^2 \int_M |T - \frac{B}{n}g|^2 \leq \left[(nc - 1)^2 + (n - 1) \left(1 + \frac{nK}{\lambda_1}\right)\right] \int_M |T - \frac{B}{n}g|^2, \quad (1.14) $$

where $\overline{B} = \frac{\int_M Bdv}{\text{Vol}(M)}$ denotes the average of $B$ over $M$, $\lambda_1$ denotes the first nonzero eigenvalue of Laplace operator on $M$ and $K$ is nonnegative constant such that the Ricci curvature of $M$ satisfies $\text{Ric} \geq -(n - 1)K$.

Further, assume the Ricci curvature $\text{Ric}$ of $M$ is positive. If $c \neq \frac{1}{n}$, the following conclusions (i), (ii) and (iii) are equivalent; if $c = \frac{1}{n}$, the following (i) and (ii) are equivalent.

(i) the equality in (1.13) or in (1.14) holds;
(ii) $T = \frac{B}{n}g$ holds on $M$;
(iii) $T = \frac{B}{n}g$ holds on $M$.

Take $K = 0$ in Theorem 1.7. We obtain corresponding inequalities with universal constants as follows,

**Theorem 1.8.** Let $(M, g)$ be a closed Riemannian manifold of dimension $n, n \geq 2$, with nonnegative Ricci curvature. Let $T$ be a symmetric $\left(2,0\right)$-tensor field on $M$. If the divergence $\text{div} T$ of $T$ and the trace $B = \text{tr} T$ of $T$ satisfy $\text{div} T = c\nabla B$, where $c$ is a constant, then

$$ (nc - 1)^2 \int_M (B - \overline{B})^2 \leq n(n - 1) \int_M |T - \frac{B}{n}g|^2, \quad (1.15) $$

5
and equivalently,
\[(nc - 1)^2 \int_M |T - \frac{\mathcal{B}}{n} g|^2 \leq [(nc - 1)^2 + 1] \int_M |T - \frac{B}{n} g|^2, \quad (1.16)\]

where \( \mathcal{B} = \frac{\int_M B dv}{\text{Vol}(M)} \) denotes the average of \( B \) over \( M \).

Further, assume the Ricci curvature \( \text{Ric} \) of \( M \) is positive. If \( c \neq \frac{1}{n} \), the following conclusions (i), (ii) and (iii) are equivalent; if \( c = \frac{1}{n} \), the following (i) and (ii) are equivalent.

(i) the equality in (1.15) or in (1.16) holds;
(ii) \( T = \frac{\mathcal{B}}{n} g \) holds on \( M \);
(iii) \( T = \frac{B}{n} g \) holds on \( M \).

It is a known fact that if \((M^n, g), n \geq 2\), is a connected Riemannian manifold of dimension \( n \). If \( T = \frac{B}{n} g \) and \( \text{div} T = c \nabla B \), where \( c \neq \frac{1}{n} \) is a constant, then \( B \) is constant on \( M \) and thus \( T \) is constant multiple of its metric \( g \) (see Proposition 2.1). Hence Theorems 1.7 and 1.8 discuss the stability and rigidity of this fact for closed manifolds. Especially, take \( T = \text{Ric}, A, \) etc, in Theorems 1.7 and 1.8. We obtain the corresponding inequalities mentioned before Theorems 1.7. In this paper, we will obtain two other applications as follows.

First we deal with \( r \)th mean curvatures of a closed hypersurface in a space form. Let \((N^{n+1}_a, \tilde{g})\) be an \((n + 1)\)-dimensional space form with constant sectional curvature \( a \), \( n \geq 2 \). Assume \((\Sigma, g)\) is a connected oriented closed hypersurface immersed in \((N^{n+1}_a, \tilde{g})\) with the induced metric \( g \). Associated with the second fundamental form \( A \) of \( \Sigma \), we have \( r \)th mean curvatures \( H_r \) of \( \Sigma \) and the Newton transformations \( P_r \), \( 0 \leq r \leq n \), (see their definition and related notations in Section 4). Since Reilly [R] introduced them, there have been much work in studying high order \( r \)-mean curvatures (cf. for instance, [Ro], [BC], [CR], [ALM]). It can be verified that if the Newton transformations \( P_r \) satisfy \( P_r = \frac{\text{tr} P_r}{n} g \) on \( \Sigma \), \( \Sigma \) has constant \( r \)th mean curvature and thus \( P_r \) is a constant multiple of its metric \( g \) (see Proposition 2.1 and Section 4). In this paper, we discuss the stability and rigidity of this fact.

In addition, although it is true that a closed immersed totally umbilical hypersurface \( \Sigma \) (that is, \( \Sigma \) satisfies \( P_1 = \frac{\text{tr} P_1}{n} g \) in \( \mathbb{R}^{n+1} \) must be a round sphere \( S^n \), it is unknown, to our best knowledge, if it is true that a closed immersed hypersurface \( \Sigma \) satisfying \( P_r = \frac{\text{tr} P_r}{n} g \) in \( \mathbb{R}^{n+1} \) must be a round sphere \( S^n \) for \( r \geq 2 \). When \( \Sigma \) is embedded, Ros [Ro1], [Ro2] showed that the round spheres are the only closed embedded hypersurfaces with constant \( r \)th
mean curvature in $\mathbb{R}^{n+1}$, $2 \leq r \leq n$ (recall Alexandrov theorem says that the round spheres are the only closed embedded hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature $[A]$). Hence the round spheres are the only closed embedded hypersurfaces in $\mathbb{R}^{n+1}$ with $P_r = \frac{nr}{n} g$, $2 \leq r \leq n$.

In Section 4 we prove the following

**Theorem 1.9.** Let $(N^{n+1}_a, \tilde{g})$ be a space form with constant sectional curvature $a$, $n \geq 2$. Assume $\Sigma$ is a smooth connected oriented closed hypersurface immersed in $N$ with induced metric $g$. Then for $2 \leq r \leq n$,

$$(n-r)^2 \int_{\Sigma} (s_r - \bar{s}_r)^2 \leq n(n-1)(1 + \frac{nK}{\lambda_1}) \int_{\Sigma} |P_r - \frac{(n-r)s_r}{n} g|^2, \quad (1.17)$$

and equivalently,

$$\int_{\Sigma} |P_r - \frac{(n-r)s_r}{n} g|^2 \leq n \left[ 1 + \frac{(n-1)K}{\lambda_1} \right] \int_{\Sigma} |P_r - \frac{(n-r)s_r}{n} g|^2, \quad (1.18)$$

where $s_r = \text{tr} P_r = (\frac{n}{r}) H_r$, $\bar{s}_r = \frac{\int_M s_r dv}{\text{Vol}(M)}$ denotes the average of $s_r$ over $\Sigma$, $\lambda_1$ is the first nonzero eigenvalue of the Laplacian operator on $\Sigma$, and $K \geq 0$ is a nonnegative constant such that the Ricci curvature of $\Sigma$ satisfies $\text{Ric} \geq -K$. Furthermore, it holds that

1) if the Ricci curvature $\text{Ric}$ of $\Sigma$ is positive, the following conclusions (i), (ii), and (iii) are equivalent,

(i) the equality in (1.17) or (1.18) holds;

(ii) $P_r = \frac{(n-r)s_r}{n} g$ holds on $\Sigma$;

(iii) $P_r = \frac{(n-r)\bar{s}_r}{n} g$ holds on $\Sigma$.

2) if $\Sigma$ is embedded in the Euclidean space $\mathbb{R}^{n+1}$ and the Ricci curvature $\text{Ric}$ of $\Sigma$ is positive, the equality in (1.17) or (1.18) holds if and only if $\Sigma$ is a round sphere $\mathbb{S}^{n+1}$ in $\mathbb{R}^{n+1}$.

Take $K = 0$ in Theorem 1.9 we prove the following

**Theorem 1.10.** Let $(N^{n+1}_a, \tilde{g})$ be a space form with constant sectional curvature $a$, $n \geq 2$. Assume $\Sigma$ is a smooth connected oriented closed hypersurface immersed in $N$ with induced metric $g$. If $\Sigma$ has nonnegative Ricci curvature, then for $2 \leq r \leq n$,

$$(n-r)^2 \int_{\Sigma} (s_r - \bar{s}_r)^2 \leq n(n-1) \int_{\Sigma} |P_r - \frac{(n-r)s_r}{n} g|^2, \quad (1.19)$$
and equivalently,
\[
\int_{\Sigma} |P_r - \left(\frac{n-r}{n}\right)s_r g|^2 \leq n \int_{\Sigma} |P_r - \left(\frac{n-r}{n}\right)s_r g|^2.
\]  
(1.20)

where \(s_r = trP_r = (\frac{n-r}{n})H_r\), \(\overline{s}_r = \int_{M} s_r dv \text{Vol}(M)\) denotes the average of \(s_r\) over \(\Sigma\). Moreover, it holds that

1) if the Ricci curvature \(\text{Ric} \) of \(\Sigma\) is positive, the following conclusions (i), (ii), and (iii) are equivalent,

   (i) the equality in (1.19) or (1.20) holds;

   (ii) \(P_r = \left(\frac{n-r}{n}\right)s_r g\) holds on \(\Sigma\);

   (iii) \(P_r = \left(\frac{n-r}{n}\right)\overline{s}_r g\) holds on \(\Sigma\).

2) if \(\Sigma\) is embedded in the Euclidean space \(\mathbb{R}^{n+1}\) and the Ricci curvature \(\text{Ric} \) of \(\Sigma\) is positive, the equality in (1.19) or (1.20) holds if and only if \(\Sigma\) is a round sphere \(S^{n+1}\) in \(\mathbb{R}^{n+1}\).

Second, we consider the \(k\)-scalar curvatures of locally conformally flat closed manifolds (see their definition in Section 5). The \(k\)-scalar curvatures have been much studied in recent years (cf, for instance, [G], [V1], [V2], etc) since they were first introduced by Viaclovsky [V1]. When \(M\) is locally conformally flat, we obtain an almost-Schur type lemma for \(k\)-scalar curvatures, \(k \geq 2\), as follows

**Theorem 1.11.** Let \((M^n, g)\) be an \(n\)-dimensional closed locally conformally flat manifold, \(n \geq 3\). Then for \(2 \leq k \leq n\), the \(k\)-scalar curvature \(\sigma_k(S_g)\) and the Newton transformation \(T_k\) associated with the Schouten tensor \(S_g\) satisfy

\[
(n-k)^2 \int_{M} (\sigma_k(S_g) - \overline{\sigma_k}(S_g))^2 \leq n(n-1)(1 + \frac{nK}{\lambda_1}) \int_{M} |T_k - \frac{(n-k)\sigma_k(S_g)}{n} g|^2,
\]  
(1.21)

and equivalently,

\[
\int_{M} |T_k - \frac{(n-k)\overline{\sigma_k}(S_g)}{n} g|^2 \leq n \left[1 + \frac{(n-1)K}{\lambda_1}\right] \int_{M} |T_k - \frac{(n-k)\sigma_k(g)}{n} g|^2,
\]  
(1.22)
where \( \overline{\sigma}_k(S_g) = \frac{\int_M \sigma_k(S_g) d\nu}{\text{Vol}(M)} \) denotes the average of \( \sigma_k(S_g) \) over \( M \), \( \lambda_1 \) is the first nonzero eigenvalue of the Laplacian operator on \( \Sigma \), \( K \geq 0 \) is a nonnegative constant such that the Ricci curvature of \( \Sigma \) satisfies \( \text{Ric} \geq -K \).

Moreover, if the Ricci curvature \( \text{Ric} \) of \( M \) is positive, the following conclusions (i), (ii), and (iii) are equivalent,

(i) the equality in (1.21) or (1.22) holds;
(ii) \( T_k = \sigma_k(S_g) g \) holds on \( M \);
(iii) \( T_k = \frac{\sigma_k(S_g)}{n} g \) holds on \( M \).

Take \( K = 0 \) in Theorem 1.11, we have the following result:

**Theorem 1.12.** Let \((M^n, g)\) be an \( n \)-dimensional locally conformally flat closed Riemannian manifold with nonnegative Ricci curvature, \( n \geq 3 \). Then for \( 2 \leq k \leq n \), the \( k \)-scalar curvature \( \sigma_k(S_g) \) and the Newton transformation \( T_k \) associated with the Schouten tensor \( S_g \) satisfy

\[
(n - k)^2 \int_M (\sigma_k(S_g) - \overline{\sigma}_k(S_g))^2 \leq n(n-1) \int_M |T_k - \frac{(n-k)\sigma_k(S_g)}{n}g|^2, \tag{1.23}
\]

and equivalently,

\[
\int_M |T_k - \frac{(n-k)\overline{\sigma}_k(S_g)}{n}g|^2 \leq n \int_M |T_k - \frac{(n-k)\sigma_k(g)}{n}g|^2. \tag{1.24}
\]

Moreover, if the Ricci curvature \( \text{Ric} \) of \( M \) is positive, (i), (ii), and (iii) are equivalent,

(i) the equality in (1.21) or (1.22) holds;
(ii) \( T_k = \frac{\sigma_k(S_g)}{n} g \) holds on \( M \);
(iii) \( T_k = \frac{\sigma_k(S_g)}{n} g \) holds on \( M \).

The rest of this paper is organized as follows. In Section 2 we prove Theorems 1.7 and 1.8. In Section 3 we recall the definitions of Newton transformation and \( r \)th symmetric function associated with a symmetric endomorphism of an \( n \)-dimensional vector space. In Section 4 we prove Theorem 1.9 by applying 1.7. In Section 5 we prove Theorem 1.11 by applying 1.7.
2 Proof of theorems on symmetric \((2, 0)\)-tensors

First we give some notations. Assume \((M, g)\) is an \(n\)-dimensional closed (that is, compact and without boundary) Riemannian manifold. Let \(\nabla\) denote the Levi-Civita connection on \((M, g)\) and also the induced connections on tensor bundles on \(M\). Let \(T\) denote a symmetric \((2, 0)\)-tensor field on \(M\). \(\text{tr}\) denotes the trace of a tensor. \(B = \text{tr}T = T^i_i = g^{ij}T_{ij}\) denotes the trace of \(T\). Here and thereafter we use Einstein summation convention. Denote by \(\overline{B} = \frac{\int_M B}{\text{vol}(M)}\) the average of \(B\) over \(M\) and \(\hat{T} = T - \frac{B}{n} g\). Denote by \(\text{div}\) the divergence of tensor field. For \(T\), \(\text{div}T = \text{tr}\nabla T\) is a \((1, 0)\)-tensor. Under the local coordinates \(\{x_i\}\) on \(M\), \(\text{div}T = g^{ij}(\nabla_i T_{jk})dx^k\), where \(\nabla_i T_{jk} = (\nabla \partial_i T)(\partial_j, \partial_k)\).

The following proposition is a well known fact, which was mentioned in the introduction.

**Proposition 2.1.** Assume \((M^n, g), n \geq 2\), is a connected Riemannian manifold of dimension \(n\). If \(T = \frac{B}{n} g\) and \(\text{div}T = c\nabla B\), where \(c \neq \frac{1}{n}\) is a constant, then \(B = \text{const}\) on \(M\) and \(T\) is constant multiple of its metric \(g\).

**Remark 2.1.** Proposition \(2.1\) can be proved directly by noting \(T = \frac{B}{n} g\) implies the identity \(\text{div}T = \frac{\nabla B}{n}\).

The argument of Theorem 1.7 is similar to the one of Theorem 1.2 (i.e. [C] Thm.1.2) and in the case of \(K = 0\), the one of Theorem 1.1 (i.e. [dLT] Thm.0.1)

**Proof of Theorem 1.7** Obviously, it suffices to prove the case \(c \neq \frac{1}{n}\). By the assumption \(\text{div}T = c\nabla B\),

\[
\text{div}\hat{T} = \text{div}T - \text{div}(\frac{B}{n} g) = \text{div}T - \frac{\nabla B}{n} = \frac{nc - \frac{1}{n}}{n}\nabla B. \tag{2.1}
\]

Let \(f\) be the unique solution of the following Poisson equation on \(M\):

\[
\Delta f = B - \overline{B}, \quad \int_M f = 0. \tag{2.2}
\]
By (2.1), (2.2) and Stokes’ formula,
\[
\int_M (B - \overline{B})^2 = \int_M (B - \overline{B}) \Delta f = - \int_M \langle \nabla B, \nabla f \rangle \\
= - \frac{n}{nc - 1} \int_M \langle \text{div} \hat{T}, \nabla f \rangle \\
= \frac{n}{nc - 1} \int_M \langle \hat{T}, \nabla^2 f \rangle \\
= \frac{n}{nc - 1} \int_M \langle \hat{T}, \nabla^2 f - \frac{1}{n} (\Delta f) g \rangle \\
\leq \frac{n}{|nc - 1|} \left( \int_M |\hat{T}|^2 \right)^{\frac{1}{2}} \left[ \int_M |\nabla^2 f - \frac{1}{n} (\Delta f) g|^2 \right]^{\frac{1}{2}} \\
= \frac{n}{|nc - 1|} \left( \int_M |\hat{T}|^2 \right)^{\frac{1}{2}} \left[ \int_M |\nabla^2 f|^2 - \frac{1}{n} \int_M (\Delta f)^2 \right]^{\frac{1}{2}}.
\tag{2.3}
\]

Recall the Bochner formula
\[
\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + \text{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla (\Delta f) \rangle,
\]
and integrate it. By the Stokes’ formula, we have
\[
\int_M |\nabla^2 f|^2 = \int_M (\Delta f)^2 - \int_M \text{Ric}(\nabla f, \nabla f). \tag{2.4}
\]

By (2.3) and (2.4),
\[
\int_M (B - \overline{B})^2 \\
\leq \frac{n}{|nc - 1|} \left( \int_M |\hat{T}|^2 \right)^{\frac{1}{2}} \left[ \frac{n - 1}{n} \int_M (\Delta f)^2 - \int_M \text{Ric}(\nabla f, \nabla f) \right]^{\frac{1}{2}}. \tag{2.5}
\]

By (2.2), \( f \equiv 0 \) if and only if \( B - \overline{B} \equiv 0 \) on \( M \). In this case, (1.13) and (1.14) obviously hold. In the following we only consider that \( f \) is not identically zero. Since the Ricci curvature has \( \text{Ric} \geq -(n - 1)K \) on \( M \),
\[
\int_M \text{Ric}(\nabla f, \nabla f) \geq -(n - 1)K \int_M |\nabla f|^2. \tag{2.6}
\]
By (2.6), (2.5) turns into
\[
\int_M (B - \overline{B})^2 \\
\leq \frac{n}{|nc - 1|} \left( \int_M |T|^2 \right)^{\frac{1}{2}} \left[ \frac{n - 1}{n} \int_M (\Delta f)^2 + (n - 1)K \int_M |\nabla f|^2 \right]^{\frac{1}{2}}. \tag{2.7}
\]

Since the first nonzero eigenvalue \(\lambda_1\) of Laplace operator on \(M\) satisfies
\[
\lambda_1 = \inf \{ \frac{\int_M |\nabla \varphi|^2}{\int_M \varphi^2} : \varphi \in C^\infty(M) \text{ is not identically zero and } \int_M \varphi = 0 \},
\]

\[
\int_M |\nabla f|^2 = - \int_M f \Delta f = - \int_M f(B - \overline{B}) \\
\leq \left( \frac{\int_M f^2}{\lambda_1} \right)^{\frac{1}{2}} \left[ \int_M (B - \overline{B})^2 \right]^{\frac{1}{2}} \\
\leq \left( \frac{\int_M |\nabla f|^2}{\lambda_1} \right)^{\frac{1}{2}} \left[ \int_M (B - \overline{B})^2 \right]^{\frac{1}{2}}.
\]

Then
\[
\int_M |\nabla f|^2 \leq \frac{1}{\lambda_1} \int_M (B - \overline{B})^2. \tag{2.8}
\]

Substitute (2.8) into (2.7) and note that \(K \geq 0\). We have
\[
\int_M (B - \overline{B})^2 \\
\leq \frac{n}{|nc - 1|} \left( \int_M |T|^2 \right)^{\frac{1}{2}} \left[ \frac{n - 1}{n} \int_M (B - \overline{B})^2 + \left( \frac{(n - 1)K}{\lambda_1} \right) \int_M (B - \overline{B})^2 \right]^{\frac{1}{2}} \\
= \frac{n^\frac{1}{2}(n - 1)^{\frac{1}{2}}}{|nc - 1|} \left( 1 + \frac{nK}{\lambda_1} \right)^{\frac{1}{2}} \left[ \int_M |T|^2 \right]^{\frac{1}{2}} \left[ \int_M (B - \overline{B})^2 \right]^{\frac{1}{2}} \tag{2.9}
\]

(2.9) implies that
\[
\int_M (B - \overline{B})^2 \leq \frac{n(n - 1)}{(nc - 1)^2} \left( 1 + \frac{nK}{\lambda_1} \right) \int_M |T|^2. \tag{2.10}
\]
Thus we have inequality (1.13):

\[(nc - 1)^2 \int_M (B - B)^2 \leq n(n - 1)(1 + \frac{nK}{\lambda_1}) \int_M |T - \frac{B}{n} g|^2.
\]

By the identity \(|T - \frac{B}{n} g|^2 = |T - \frac{B}{n} g|^2 + \frac{1}{n}(B - B)^2\), we have inequality (1.14):

\[(nc - 1)^2 \int_M |T - \frac{B}{n} g|^2 \leq \left[(nc - 1)^2 + (n - 1)(1 + \frac{nK}{\lambda_1})\right] \int_M |T - \frac{B}{n} g|^2.
\]

Now with the assumption of positivity of Ricci curvature Ric of \(M\), we may prove the case of equalities in (1.13) and (1.14). Obviously, if \(T = \frac{B}{n} g\) holds on \(M\), the equalities in (1.13) and (1.14) hold. On the other hand, suppose the equality in (1.13) (or equivalently (1.14)) holds. If \(c = \frac{1}{n}\), it is obvious that \(T = \frac{B}{n} g\) on \(M\). If \(c \neq \frac{1}{n}\), we may take \(K = 0\). By the proof of (1.13), the equality in (1.13) holds if and only if

1) \(\text{Ric}(\nabla f, \nabla f) = 0\) on \(M\) and

2) \(T - \frac{B}{n} g\) and \(\nabla^2 f - \frac{1}{n}(\Delta f) g\) are linearly dependent.

Note \(\text{Ric} > 0\) and 1). It must holds that \(\nabla f \equiv 0\) on \(M\). Then \(f \equiv 0\). Thus \(B = B\) on \(M\). By (1.13), we obtain that \(T = \frac{B}{n} g\) on \(M\). Hence conclusions (i) and (ii) are equivalent. Obviously (iii) implies (ii). When \(c \neq \frac{1}{n}\), if (ii) holds, by the above argument, (ii) implies \(B = B\) on \(M\). Thus (iii) also holds.

\[\square\]

We have a corollary of Theorem 1.7 as follows:

**Corollary 2.1.** Let \((M^n, g), n \geq 2\), be a closed Riemannian manifold of dimension \(n\). Let \(T\) be a symmetric \((2, 0)\)-tensor field on \(M\). If the divergence \(\text{div} T\) of \(T\) and the trace \(B = \text{tr} T\) of \(T\) satisfy \(\text{div} T = c\nabla B\), where \(c \neq \frac{1}{n}\) is a constant, then

\[\int_M (B - B)^2 \leq C_{(Kd^2)} \int_M |T - \frac{B}{n} g|^2, \tag{2.11}\]

and

\[\int_M |T - \frac{B}{n} g|^2 \leq \overline{C}_{(Kd^2)} \int_M |T - \frac{B}{n} g|^2; \tag{2.12}\]

where \(K\) is a positive constant. such that the Ricci curvature of \(M\) satisfies \(\text{Ric} \geq -(n - 1)K\), \(d\) denotes the diameter of \(M\) and \(C_{(Kd^2)}\) and \(\overline{C}_{(Kd^2)}\) are constants only depending on \(Kd^2\).
Proof of Corollary 2.1. When Ric $\geq -(n-1)K$, where constant $K > 0$, Li and Yau \cite{LY} proved that the first nonzero eigenvalue $\lambda_1$ has the lower bound:

$$\lambda_1 \geq \alpha = \frac{1}{(n-1)d^2 \exp[1 + \sqrt{1 + 4(n-1)^2Kd^2}]},$$

where $d$ denotes the diameter of $M$. So

$$\frac{K}{\lambda_1} \geq \frac{K}{\alpha} = \frac{(n-1)Kd^2 \exp[1 + \sqrt{1 + 4(n-1)^2Kd^2}]}{1 + n(n-1)Kd^2 \exp[1 + \sqrt{1 + 4(n-1)^2Kd^2}]}. $$

By Theorem 1.7 we obtain inequality (2.11) with the constant

$$C_{(Kd^2)} = \frac{4n(n-1)}{(n-2)^2} \left(1 + n(n-1)Kd^2 \exp[1 + \sqrt{1 + 4(n-1)^2Kd^2}]\right).$$

Inequality (2.11) implies inequality (2.12). □

Remark 2.2. There are other lower estimates $\alpha$ of $\lambda_1$ using the diameter $d$ and negative lower bound $-(n-1)K$ of Ricci curvature (for example, see \cite{KMYZ}). Hence we may have other values of constant $C_{(Kd^2)}$ and $\overline{C}_{(Kd^2)}$.

3 Newton transformations and the $r$th elementary symmetric function

Let $\sigma_r : \mathbb{R}^r \rightarrow \mathbb{R}$ denote the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_r(x_{i_1}, \ldots, x_{i_r}) = \sum_{i_1 < \ldots < i_r} x_{i_1} \ldots x_{i_r}, 1 \leq r \leq n.$$

Let $V$ be an $n$-dimensional vector space and $A : V \rightarrow V$ be a symmetric linear transformation. If $\eta_1, \ldots, \eta_n$ are the eigenvalues of $A$ corresponding the orthonormal eigenvectors $\{e_i\}, i = 1, \ldots, n$ respectively, define the $r$th symmetric functions $\sigma_r(A)$ associated with $A$ by

$$\sigma_0(A) = 1,$$

$$\sigma_r(A) = \sigma_r(\eta_1, \ldots, \eta_k), 1 \leq r \leq n. \quad (3.1)$$

For convenience of the notation, we simply denote $\sigma_r(A)$ by $\sigma_r$ if there is no confusion. The Newton transformations $P_r : V \rightarrow V$, associated with $A, 0 \leq r \leq n$, are defined by

$$P_0 = I,$$

$$P_r = \frac{1}{\sigma_r(A)} \left[\sigma_r(A) P_r - \sigma_{r-1}(A) P_{r-1}\right],$$

where $I$ is the identity transformation on $V$. By Theorem 1.7 we obtain inequality (2.11) with the constant $C_{(Kd^2)} = 4n(n-1)\left(1 + n(n-1)Kd^2 \exp[1 + \sqrt{1 + 4(n-1)^2Kd^2}]\right)$. Inequality (2.11) implies inequality (2.12). □

Remark 2.2. There are other lower estimates $\alpha$ of $\lambda_1$ using the diameter $d$ and negative lower bound $-(n-1)K$ of Ricci curvature (for example, see \cite{KMYZ}). Hence we may have other values of constant $C_{(Kd^2)}$ and $\overline{C}_{(Kd^2)}$.
\[ P_r = \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j = \sigma_r I - \sigma_{r-1} A + ... + (-1)^r A^r, r = 1, \ldots, n. \]

By definition, \( P_r = \sigma_r I - AP_{r-1}, P_n = 0. \) It was proved in [R] that \( P_r \) has the following basic properties:

(i) \[ P_r(e_i) = \frac{\partial \sigma_{r+1}}{\partial \eta_i} e_i; \]
(ii) \[ \text{tr}(P_r) = (n-r)\sigma_r; \]
(iii) \[ \text{tr}(AP_r) = (r+1)\sigma_{r+1}. \]

Obviously, each \( P_r \) corresponds a symmetric \((2,0)\)-tensor on \( V \), still denoted by \( P_r \).

4 High order mean curvatures of hypersurfaces in space forms

Assume \((N, \tilde{g})\) is an \((n+1)\)-dimensional Riemannian manifold, \( n \geq 2 \). Suppose \((\Sigma, g)\) is a smooth connected oriented closed hypersurface immersed in \((N, \tilde{g})\) with induced metric \( g \). Let \( \nu \) denote the outward unit normal to \( \Sigma \) and \( A = (h_{ij}) \) denote the second fundamental form \( A : T_p \Sigma \otimes_s T_p \Sigma \to \mathbb{R} \), defined by \( A(X,Y) = -\langle \tilde{\nabla}_X Y, \nu \rangle \), where \( X,Y \in T_p \Sigma, p \in \Sigma \), \( \tilde{\nabla} \) denote the Levi-Civita connection of \((N, \tilde{g})\). \( A \) determines an equivalent \((1,1)\)-tensor, called the shape operator \( A \) of \( \Sigma \): \( T_p \Sigma \to T_p \Sigma \), given by \( AX = \tilde{\nabla}_X \nu \). \( \Sigma \) is called totally umbilical if \( A \) is multiple of its metric \( g \) at every point of \( \Sigma \), that is, \( A = \frac{\text{tr} A}{n} g \) on \( \Sigma \). Now we recall the definition of \( r \)th mean curvatures of a hypersurface, which was introduced by Reilly [R], (cf. [Ro]).

Let \( \eta_i, i = 1, \ldots, n \) denote the principle curvatures of \( \Sigma \) at \( p \), which are the eigenvalues of \( A \) at \( p \) corresponding the orthonormal eigenvectors \( \{ e_i \}, i = 1, \ldots, n \) respectively. By Section 3 we have the \( r \)th symmetric functions \( \sigma_r(A) \) associated with \( A \), denoted by \( s_r = \sigma_r(A) \) and the Newton transformations \( P_r \) associated with \( A \) at \( p \), \( 0 \leq r \leq n \). We have

**Definition 4.1.** The \( r \)th mean curvature \( H_r \) of \( \Sigma \) at \( p \) is defined by \( s_r = \binom{n}{r} H_r, 0 \leq r \leq n. \)

For instance, \( H_1 = \frac{\text{tr} A}{n} = \frac{H}{n} \) (in this paper, we also call \( H = \text{tr} A \) the mean curvature of \( \Sigma \), conformal to the previous related papers [P], [CZ], etc). \( H_n \)
is the Gauss-Kronecker curvature. When the ambient space $N$ is a space form $N_{n+1}^a$ with constant sectional curvature $a$,

$$
\text{Ric} = (n-1)aI + HA - A^2.
$$

$$
R = \text{trRic} = n(n-1)c + H^2 - |A|^2 = n(n-1)a + 2s_2.
$$

Hence $H_2$ is, modulo a constant, the scalar curvature of $\Sigma$.

One of the known properties of $P_r$ is the following

**Lemma 4.1.** (cf \[R\], or \[Ro\], or \[ALM\]) When the ambient space is a space form $N_{n+1}^a$, $\text{div} P_r = 0$, $0 \leq r \leq n$.

Now we prove Theorem 1.9.

**Proof of Theorem 1.9.**

By Section 3, $\text{tr} P_r = (n-r)s_r$. Denote by $\bar{s}_r = \frac{\int \Sigma s_r}{\text{Vol}(\Sigma)}$. By Lemma 4.1, $\text{div} P_r = 0$. Take $T = P_r$ and $B = (n-r)s_r$ in Theorem 1.7. We have

$$(n-r)^2 \int \Sigma (s_r - \bar{s}_r)^2 \leq n(n-1)(1 + \frac{nK}{\lambda_1}) \int \Sigma |P_r - \frac{(n-r)s_r}{n}g|^2,$$

and equivalently,

$$
\int \Sigma |P_r - \frac{(n-r)s_r}{n}g|^2 \leq n \left(1 + \frac{(n-1)K}{\lambda_1}\right) \int \Sigma |P_r - \frac{(n-r)s_r}{n}g|^2,
$$

which are (1.17) and (1.18) respectively.

Now we prove conclusions 1) and 2) in Theorem 1.9. If the Ricci curvature of $\Sigma$ is positive, by Theorem 1.7, conclusion 1) holds and $s_r = \bar{s}_r$ is constant on $\Sigma$. If $\Sigma$ is also embedded in $\mathbb{R}^{n+1}$, by the Ros’ theorem \[Ro2\] that a closed embedded hypersurface in $\mathbb{R}^{n+1}$ with constant $r$th mean curvature must be a distance sphere $\mathbb{S}^{n+1}$, $2 \leq r \leq n$, we obtain conclusion 2).

$\square$

**Remark 4.1.** If $r = 1$, $P_1 = s_1I - A = H I - A$. $P_1$ is equivalent to the symmetric $(2,0)$-tensor $P_1 = H g - A$. So (1.17) turns to

$$
\int \Sigma (H - \bar{H})^2 \leq \frac{n}{n-1} \left(1 + \frac{nK}{\lambda_1}\right) \int \Sigma |H g - A - \frac{(n-1)H}{n}g|^2
$$

$$
= \frac{n}{n-1} \left(1 + \frac{nK}{\lambda_1}\right) \int \Sigma |A - \frac{H}{n}g|^2.
$$

(4.1)
In particular, if $K = 0$,

$$\int_\Sigma (H - \overline{H})^2 \leq \frac{n}{n-1} \int_\Sigma \vert A - \frac{H}{n} g \vert^2. \tag{4.2}$$

(4.1) and (4.2) are (1.12) and (1.8) respectively, which were proved in [CZ] and [P] respectively if $\Sigma$ is a closed hypersurface immersed in an Einstein manifold. This is because $\text{div} P_1 = 0$ even if the ambient space is Einstein.

When $r = 2$, we have

$$P_2 = \frac{(n-2)s_2}{n} g = \frac{R}{n} I - \text{Ric},$$

by direct computation,

As a symmetric $(2,0)$-tensor, $P_2 = \frac{R}{n} g - \text{Ric}$. Hence (1.17) turns to

$$\int_\Sigma (s_2 - \overline{s_2})^2 \leq \frac{n(n-1)}{(n-2)^2} (1 + \frac{nK}{\lambda_1}) \int_\Sigma \vert P_2 - \frac{(n-2)s_2}{n} g \vert^2,$$

which is

$$\int_\Sigma \vert R - \overline{R} \vert^2 \leq \frac{4n(n-1)}{(n-2)^2} (1 + \frac{nK}{\lambda_1}) \int_\Sigma \vert \text{Ric} - \frac{R}{n} g \vert^2. \tag{4.3}$$

(4.3) was proved in [C] and in the case of $K = 0$, was proved in [dLT].

If $r = n$, (1.17) is trivial.

5 $k$-scalar curvature of locally conformal flat manifolds.

We first recall the definition of the $k$-scalar curvatures of a Riemannian manifold, introduced by Viaclovsky in [V1]. If $(M^n, g)$ be an $n$-dimensional Riemannian manifold, $n \geq 3$, the Schouten tensor of $M$ is

$$S_g = \frac{1}{n-2} \left( \text{Ric} - \frac{1}{2(n-1)} Rg \right).$$

By definition, $S_g : TM \to TM$ is a symmetric $(1,1)$-tensor field. By Section 3 we have the symmetric $k$th function $\sigma_k(S_g)$ and the Newton transformations $T_k(S_g) = T_k$ associated with $S_g$, $1 \leq k \leq n$. We call $\sigma_k(S_g)$ the $k$-scalar curvatures of $M$.

It was proved that
Lemma 5.1. ([V1]) If \((M, g)\) is locally conformally flat, then for \(1 \leq k \leq n\), \(\text{div}T_k(S_g) = 0\).

Because of Lemma 5.1, we can apply Theorem 1.7 to \(T_k(S_g)\) to obtain Theorem 1.11.

Remark 5.1. When \(k = 1\), \(\sigma_1(S_g) = \text{tr}S_g = \frac{R}{2(n-1)}, T_1 = \sigma_1(S_g)I - S_g\). As a symmetric \((2,0)\)-tensor, \(T_1 = -\frac{1}{n-2}(\text{Ric} - \frac{Rg}{2})\). Hence (1.21) turns to (1.3)

\[
\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2} \left(1 + \frac{nK}{\lambda_1}\right) \int_M |\text{Ric} - \frac{R}{n}g|^2,
\]

and in particular, if \(K = 0\), (1.21) turns to (1.1)

\[
\int_M (R - \bar{R})^2 \leq \frac{4n(n-1)}{(n-2)^2} \int_M |\text{Ric} - \frac{R}{n}g|^2.
\]

(1.3) and (1.1) were proved in [C] and [dLT] respectively without the hypothesis that \(M\) is locally conformally flat. The reason is that \(\text{div}T_1 = 0\) (the contracted second Bianchi identity) holds on any Riemannian manifold.

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