Multiple Measurement Vectors Problem: A Decoupling Property and its Applications

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Abstract—Efficient and reliable estimation in many signal processing problems encountered in applications requires adopting sparsity prior in a suitable basis on the signals and using techniques from compressed sensing (CS). In this paper, we study a CS problem known as Multiple Measurement Vectors (MMV) problem, which arises in joint estimation of multiple signal realizations when the signal samples have a common (joint) support over a fixed known dictionary. Although there is vast literature on the analysis of MMV, it is not yet fully known how the number of signal samples and their statistical correlations affects the performance of the joint estimation in MMV. Moreover, in many instances of MMV the underlying sparsifying dictionary may not be precisely known, and it is still an open problem to quantify how the dictionary mismatch may affect the estimation performance.

In this paper, we focus on \( \ell_2\)-norm regularized least squares (\( \ell_2\)-LS) as a well-known and widely-used MMV algorithm in the literature. We prove an interesting decoupling property for \( \ell_2\)-LS, where we show that it can be decomposed into two phases: i) use all the signal samples to estimate the signal covariance matrix (coupled phase), ii) plug in the resulting covariance estimate as the true covariance matrix into the Minimum Mean Squared Error (MMSE) estimator to reconstruct each signal sample individually (decoupled phase). As a consequence of this decomposition, we are able to provide further insights on the performance of \( \ell_2\)-LS for MMV. In particular, we address how the signal correlations and dictionary mismatch affects its estimation performance. We also provide numerical simulations to validate our theoretical results.

Index Terms—Compressed Sensing, Sparsifying Dictionary, Multiple Measurement Vectors, Mean Squared Error.

I. INTRODUCTION

Efficient and reliable signal estimation in a variety of fields including biology, statistics, wireless communication, etc. requires adopting sparsity in suitable signal dictionary as signal prior and using techniques and recovery algorithms from compressed sensing (CS) \[1, 2\]. To streamline the explanation of the main ideas in this paper, we start with and keep as an illustrative example throughout the paper a signal estimation problem encountered in wireless MIMO systems, although as we will generalize later, the underlying signal model and our results apply also to other similar signal processing problems.

Illustrative Example. We consider a wireless propagation model illustrated in Fig. 1 where a BS is equipped with a Uniform Linear Array (ULA) consisting of \( n \gg 1 \) antennas and serves single-antenna users. Here, we assume that there are \( T \gg 1 \) resource blocks (signal samples) and the signal from a generic user received at \( n \) BS antennas at resource block \( s \in [T] := \{1, \ldots, T\} \) is given by \[3, 4\]

\[
h(s) = \sum_{i=1}^{k} w_i(s) a(\xi_i),
\]

where \( w_i(s) \) denotes the channel gain of the \( i \)-th scatter, \( i \in [k] \), at resource block \( s \), where \( \xi_i = \sin(\theta_i) \in \Xi := [-1, 1] \) parametrizes the angle-of-arrival (AoA) \( \theta_i \in [-\pi, \pi] \) of the \( i \)-th scatterer, and where \( a(\xi) \in \mathbb{C}^n \) denotes the array response to a planar wave coming from the AoA \( \xi \) given by

\[
a(\xi) = (e^{j\pi \xi_1}, \ldots, e^{jn\pi \xi})^T.
\]

We consider a quasi-stationary scenario, as in all wireless MIMO and array processing problems, where we assume that the AoAs \( \{\xi_i : i \in [k]\} \subset \Xi \) in signal model \[1\] remain the same across all the signal samples \( \mathcal{H} := \{h(s) : s \in [T]\} \) such that they all have the same support in the AoA domain \( \Xi \), whereas the channel gains \( w_i(s) \), might vary considerably (even i.i.d.) across different scatterers \( i \in [k] \) and signal samples \( s \in [T] \).

Joint Sparsity Model. For the rest of the paper, we will use the signal model \[1\] and will consider a stochastic process \( \mathcal{H} := \{h(s) : s \in [T]\} \) whose samples have a sparse representation over a common dictionary of dimension \( n \)

\[
\mathcal{A} := \{a(\xi) : \xi \in \Xi\},
\]

where the atoms of the dictionary \( a(\xi) \in \mathbb{C}^n \) are labelled with a finite or infinite set \( \Xi \) (e.g., \( \Xi = [-1, 1] \) for the continuum set of AoAs). In particular, all signal samples share a common sparsity pattern or support \( \{\xi_i : i \in [k]\} \) over \( \Xi \) of size \( k \ll n \), where \( n \) is the signal dimension.

A. Single and Multiple Measurement Vector problem

In many signal processing problems, one needs to recover the signal \( h(s) \) as in \[1\] from a set of \( m \ll n \) linear and possibly noisy projections \( y(s) = \Psi(s) h(s) + z(s) \), where \( \Psi(s) \in \mathbb{C}^{m \times n} \) is the \( m \times n \) projection matrix applied to the sample \( s \) and where \( z(s) \) is the measurement noise. Low-dim projections \( m \ll n \) arise in signal processing problems...
is the sum of \( \ell_1 \) promoting regularizers (e.g., greedy techniques or convex optimization using joint sparsity following two classes of algorithms. The first class applies overview of the algorithms proposed for signal recovery in B. Related Works and Contribution

\[ \hat{c}(s) = \arg \min_{c \in \mathbb{C}^G} \frac{1}{2} \| x(s) - \Psi(s) A c \|^2 + \varrho \| c \|_1, \]  

where \( A = [a(\xi_1), \ldots, a(\xi_G)] \) is a finite dictionary matrix generally obtained by an approximation (finite dictionary) or a quantization (infinite dictionary) of the sparsifying dictionary \( A \) in (3) over a finite grid \( G = \{ \xi_1, \ldots, \xi_G \} \) of size \( G \), where \( c = (c_1, \ldots, c_G)^T \in \mathbb{C}^G \) denote the coefficients of the approximation of \( h(s) \) over the discrete dictionary \( A \), where \( \| c \|_1 = \sum_{i \in [G]} | c_i | \) denotes the \( \ell_1 \)-norm of \( c \), and where \( \varrho > 0 \) is a regularization parameter. Note that as is well-known in CS, the \( \ell_1 \)-norm regularization promotes the sparsity of the coefficients \( c \). As a result, in the ideal scenario where the grid \( G \) contains the true signal support \( \{ \xi_1, \ldots, \xi_K \} \subset \Xi \) (see, e.g., [1]), we expect that the coefficient vector \( \hat{c}(s) \) has non-zero values on the elements corresponding to the true atoms and is zero elsewhere. Hence, one can estimate the signal \( h(s) \) from the estimated sparse coefficients \( \hat{c}(s) \) as \( h(s) = A \hat{c}(s) \).

One can run the optimization problem (4) to estimate each signal sample \( h(s) \) from its corresponding noisy sketches \( x(s) = \Psi(s) h(s) + z(s) \) individually; this is known as the Single Measurement Vector (SMV) problem in the literature. In the presence of joint sparsity, however, it is conventional to run a joint CS estimation problem to improve the recovery performance. This is known as the Multiple Measurement Vectors (MMV) problem and is quite well studied in the literature [5-9]. A well-known algorithm and an MMV variant of \( \ell_1 \)-LS in (4) is \( \ell_2,1 \)-norm regularized least squares (\( \ell_2,1 \)-LS).

\[ \hat{C} = \arg \min_{C \in \mathbb{C}^{G \times T}} \frac{1}{2} \sum_{s \in [T]} \| x(s) - \Psi(s) A c(s) \|^2 + \varrho \sqrt{T} \| C \|_{2,1}, \]  

where \( \hat{C} = [\hat{c}(1), \ldots, \hat{c}(T)] \) is the \( G \times T \) matrix of estimated coefficients with \( \hat{c}(s) \) denoting the estimated coefficient vector of the \( s \)-th signal \( h(s) \), and where \( \| C \|_{2,1} = \sum_{i \in [G]} \| C_{i,:} \|_2 \) is the sum of \( \ell_2 \)-norm of rows of \( C \) with \( C_{i,:} \) denoting the \( i \)-th row of \( C \), and where \( \varrho > 0 \) is a regularization parameter as before. Similar to \( \ell_1 \)-LS, one can estimate the original signal samples as \( \hat{h}(s) = A \hat{c}(s), s \in [T] \). It is known that the \( \ell_2,1 \)-norm regularization promotes the row sparsity of the coefficient matrix \( C \), thus, it imposes a common (joint) sparsity pattern on the supports of the estimated signals.

**B. Related Works and Contribution**

There is a vast literature in CS on MMV and signal estimation under joint/group sparsity; we may refer to [5-9] for several earlier works and the refs. therein. A broad overview of the algorithms proposed for signal recovery in the MMV setup reveals that they generally belong to the following two classes of algorithms. The first class applies greedy techniques or convex optimization using joint sparsity promoting regularizers (e.g., \( \ell_2,1 \)-norm regularization as in \( \ell_2,1 \)-LS in (5)) to estimate the signals. The second class of algorithms use the sample covariance matrix of data and applies subspace techniques to extract the joint support. Once the support is identified, the standard least squares method can be applied to find the corresponding coefficients. Our results in this paper, broadly speaking, make a link between these two class of algorithms. More precisely, by deriving a decoupling property in the specific case of \( \ell_2,1 \)-LS belonging to the first class of algorithms, we show that \( \ell_2,1 \)-LS can be decomposed into two steps where the first step can be seen merely as the estimation of the sparse covariance of the signal samples, which highly resembles the subspace techniques applied in the second class of algorithms.

All the classical algorithms for MMV require a finite signal dictionary, thus, when the signal dictionary is infinite (as in the continuum AoA in our illustrative example), one needs to run MMV on a finite quantized version of the original dictionary. This causes a mismatch when the signal support is not covered by the quantized dictionary and may dramatically degrade the performance [10]. Recently off-grid MMV techniques have also been developed [11-13] to avoid the dictionary mismatch. Using the decoupling property for \( \ell_2,1 \)-LS, we are able to fully quantify the effect of dictionary mismatch in terms of the convex cone produced by the dictionary, which provides further insights on the effect of quantization.

Recently a novel class of algorithms based on Vector Approximate Message Passing (VAMP) have been proposed for signal recovery in MMV. VAMP belongs to the Bayesian class of algorithms and requires knowing the probability distribution of the coefficients of signal samples (e.g., \( \{ w(s) : s \in [T] \} \) in [1] in our illustrative example). The performance of VAMP for i.i.d. Gaussian dictionaries can be fully specified in the large-dimensional setup by a simple State Evolution equation [14], which yields an exact characterization of the performance in terms of parameters such as signal distribution and number of MMV samples \( T \). In practice, however, signal dictionaries are structured and, in particular, far from i.i.d. Gaussian, and exact analysis of VAMP in those cases is still an open problem. The decoupling property for \( \ell_2,1 \)-LS reveals that the performance of MMV over any dictionary (even an structured one) hinges crucially on the quality of the covariance estimation phase. Although, in this paper we do not make further progress in this direction, we provide new guidelines on how one may be able to theoretically analyze the performance of MMV for such structured dictionaries.

**II. OVERVIEW OF THE RESULTS**

**A. Signal Samples as Side Information in MMV**

Let us consider the joint optimization (5). As explained before, we can estimate each signal sample \( h(s), s \in [T] \), from the optimal solution \( C \) of (5) as \( h(s) = A \hat{c}(s) \), where \( \hat{c}(s) \) denotes the \( s \)-th column of \( C \). This can be interpreted as follows: \( \ell_2,1 \)-LS yields an MMV estimator \( \hat{h}(s) = \text{MMV}(x(s); X_{:,s}) \) for each signal sample \( s \in [T] \), which depends explicitly on the corresponding sketch \( x(s) \) of signal sample \( h(s) \) as well as implicitly on all the other sketches \( X_{:,s} \), where \( X = [x(1), \ldots, x(T)] \) denotes the \( m \times T \) matrix of sketches,
and where $X_{:,s}$ denotes the $m \times (T-1)$ matrix obtained after removing the $s$-th column of $X$ corresponding to $x(s)$. One can interpret $X_{:,s}$ as some sort of side information from the other sketches, exploited by MMV in the estimation of $\hat{h}(s)$. Our goal in this paper is to better understand how this side information is used/processed by MMV.

To explain this more clearly, let us consider an alternative to MMV, a genie-aided Minimum Mean Square Error (MMSE) estimator, denoted by $\hat{h}$, that has a perfect knowledge of the covariance matrix of the samples $\Sigma_h = \mathbb{E}[h(s)h(s)^\dagger]$. In the conventional case, where the signal coefficients $\{w_i(s) : i \in [k]\}$ are independent of each other, $\Sigma_h$ is given by $\Sigma_h = \sum_{i=1}^k \gamma_i^2 a(\xi)_a(\xi_i)^\dagger$, where $\gamma_i^2 = \mathbb{E}[|w_i(s)|^2]$ denotes the strength of the $i$-th coefficient. Let us consider a scenario where the signal samples are i.i.d. and where the side information coming from the other samples is exploited by MMV and how it improves the estimation performance of an individual signal sample.

For simplicity, we focused on a Gaussian process $\mathcal{H}$; for which the MMSE estimator is linear. The results remain valid for a non-Gaussian process by replacing the MMSE by the Best Linear Unbiased Estimator (BLUE) $\hat{h}$.

III. MAIN RESULTS AND DISCUSSION

A. Main Theorems

Let $\mathcal{H} = \{h(s) : s \in [T]\}$ be the $n$-dim jointly sparse stochastic process as in signal model (1) and let $x(s) = \Psi(s)\hat{h}(s) + z(s)$ be the collection of sketches of $\mathcal{H}$ taken via $m \times n$ projection matrices $\Psi(s)$. For simplicity, we will assume that $\Psi(s)\Psi(s)^\dagger = I_m$, where $I_m$ denotes the identity matrix of order $m$. Let us define $\ell_{2,1}$-LS cost function as in (5) by

$$f(C) = \frac{1}{2} \sum_{s \in [T]} ||x(s) - \Psi(s)A\gamma||^2_2 + \vartheta \sqrt{T}||C||_{2,1}, \quad (7)$$

where $A$ denotes the $n \times G$ dictionary matrix and where $C \in [c(1), \ldots, c(T)]$ denotes the $G \times T$ matrix of coefficients of $\mathcal{H}$ over $A$.

Let us also introduce the following convex cost function

$$g(\gamma) = \sum_{s \in [T]} x(s)^\dagger (\Psi(s)A^\dagger \Psi(s)^\dagger + \vartheta I_m)^{-1} x(s) + \text{tr}(\Gamma) \quad (8)$$

where $\gamma = (\gamma_1, \ldots, \gamma_G)^\dagger \in \mathbb{R}^G_+$ is a G-dim vector consisting of positive elements, where $\Gamma := \text{diag}(\gamma)$ is a diagonal matrix with diagonal elements $\gamma_i$, and where $\text{tr}(\cdot)$ denotes the trace operator. We first prove the following theorem which links the optimal solutions of the convex cost functions in (7) and (8).

**Theorem 1.** Let $\hat{C} = \arg\min_{C \in \mathbb{C}^{G \times T}} f(C)$ and $\hat{\gamma} = \arg\min_{\gamma \in \mathbb{R}^G_+} g(\gamma)$ be the optimal solutions of the convex functions $f(C)$ and $g(\gamma)$ as in (7) and (8), respectively. Then,

$$\gamma_i = \frac{||C_i||_2}{\sqrt{T}}$$

for $i \in [G]$.  

**Proof:** Proof in Appendix A. □

Our second theorem makes a more direct link between the optimal solution $\gamma$ of $g(\gamma)$ in (8) and the signal estimates produced by $\ell_{2,1}$-LS via optimizing $f(C)$ in (7).

**Theorem 2.** Let $\hat{C}$ and $\hat{\gamma}$ be as in Theorem 1. Also, let $\hat{h}(s) = A\hat{C}(s)$ be the estimate of the signal sample $h(s)$ produced by $\ell_{2,1}$-LS minimization of (7), where $\hat{C}(s)$ denotes the $s$-th column of the optimal solution $\hat{C}$ of $\ell_{2,1}$-LS cost function $f(C)$.
Theorem 1, where \( \hat{h}(s) \) is given by
\[
\hat{h}(s) = A\Gamma^o A^\dagger(\Psi(s)A\Gamma^o A^\dagger + \varrho I_m)^{-1} x(s),
\]
where \( x(s) = \Psi(s)h(s) + z(s) \) denotes the noisy sketch of \( h(s) \) and \( \Gamma = \text{diag}(\gamma) \).

Proof: Proof in Appendix A.

B. MMV as a plug-in MMSE estimator

Using Theorem 1 and 2, we are now in a position to illustrate the decoupling property of \( \ell_{2,1}\)-LS algorithm for MMV as claimed in Section II (see also Fig. 2). For simplicity of explanation, we will assume a matched scenario where all the signal samples can be written as a sparse linear combination of the columns of the dictionary matrix \( A \) as \( h(s) = Aw(s) \) with some sparse vector of coefficients \( w(s) \in \mathbb{C}^G \) (see also the signal model in (1)). Assuming that the elements of \( w(s) = (w_1(s), \ldots, w_G(s))^T \) are independent with strengths \( \gamma_1 = \mathbb{E}|w_1(s)|^2 \), one can see that each signal sample \( h(s) \) and its noisy sketch \( x(s) = \Psi(s)h(s) + z(s) \) have the following covariance matrices
\[
\Sigma_h = A\Gamma^o A^\dagger, \quad \Sigma_x(s) = \Psi(s)A\Gamma^o A^\dagger\Psi(s)^\dagger + \sigma^2 I_m,
\]
where \( \Gamma^o = \text{diag}(\gamma^o) \) with \( \gamma^o = (\gamma_1^o, \ldots, \gamma_G^o)^T \), and \( \sigma^2 \) denotes the noise variance. Using Theorem 1 and 2 and also (10), we can interpret the \( \ell_{2,1}\)-LS estimator as follows:

a) Obtain and estimate \( \hat{\gamma} \) of the vector \( \gamma^o \) by optimizing the cost function \( g(\gamma) \) in (6), with a regularization \( \varrho = \sigma^2 \).

b) Use the estimate \( \hat{\gamma} \) to compute an estimate of the true covariance matrices in (10) as \( \Sigma_h = A\hat{\Gamma}^o A^\dagger, \) and \( \Sigma_x(s) = \Psi(s)A\hat{\Gamma}^o A^\dagger\Psi(s)^\dagger + \sigma^2 I_m, \) where \( \hat{\Gamma} = \text{diag}(\hat{\gamma}) \).

c) Use the estimated covariance matrices in place of the true ones, in a “plug-in” MMSE estimator as in (6), to recover each signal sample \( h(s) \) from its corresponding sketches \( x(s) \) individually, thus yielding (9) in Theorem 2.

This illustrates that \( \ell_{2,1}\)-LS can be decomposed into a coupled covariance estimation phase (steps a and b) followed by a decoupled signal estimation phase (step c) (see, e.g., Fig. 2). This underlying decoupling property also indicates that although regularization based and sample covariance (subspace) based MMV recovery algorithms overviewed in Section II-B seem quite different, they are indeed very related through the covariance estimation phase (steps a and b).

C. Further insights on the decoupling property

From the decoupling property for \( \ell_{2,1}\)-LS, we can gain several insights on the qualitative performance of \( \ell_{2,1}\)-LS in the MMV setup as follows:

i) The cost function \( g(\gamma) \) in (8) has \( \text{tr}(\Gamma) = \sum_{i \in [G]} \gamma_i \), i.e., the \( \ell_1 \)-norm of \( \gamma \), as the regularizer, which is known to promote the sparsity of \( \gamma \). Thus, we expect that the optimal solution \( \hat{\gamma} \) of \( g(\gamma) \) will have only a few significantly large coefficients. Interestingly, this is consistent with the result of Theorem 1, where \( \hat{\gamma}_i = \frac{\|G_i\|}{\sqrt{f_i}} \) and the sparsity of \( \hat{\gamma} \) implies the row-sparsity of the optimal solution of \( \ell_{2,1}\)-LS in (7), which in turn follows from the \( \ell_{2,1}\)-norm regularalization.

ii) To gain a better understanding of the coupled covariance estimation phase, let us assume a case where the coefficients have a Gaussian distribution and all the signal coefficients \( \{w(s) : s \in [T]\} \) are independent from each other all having the same Gaussian distribution. Then, a straightforward computation shows that one can write the Maximum Likelihood (ML) estimate \( [15] \) of the sparse vector \( \gamma^o \) (recall that \( \gamma^o = \mathbb{E}|w_1(s)|^2 \) denotes the strength of the \( i \)-th signal coefficient) from the available sketches \( \{x(s) : s \in [T]\} \), which are also independent Gaussian vectors, as the minimization of the following cost function:
\[
l(\gamma) = \sum_{s \in [T]} x(s)^\dagger (\Psi(s)A\Gamma A^\dagger\Psi(s)^\dagger + \sigma^2 I_m)^{-1} x(s) + \log \det (\Psi(s)A\Gamma A^\dagger\Psi(s)^\dagger + \sigma^2 I_m),
\]
where \( \Gamma = \text{diag}(\gamma) \) and where \( \det(.) \) denotes the determinant. From (11), one can see the striking similarity between the cost function \( g(\gamma) \) in (8) and ML cost function \( l(\gamma) \). One can also note the fundamental difference that the \( \log \det \) regularization in (11) is replaced by the trace regularization in (5) to explicitly promote the sparsity of the vector \( \gamma \).

iii) The assumptions made for the derivation of (11) reveals that the \( \ell_{2,1}\)-norm regularization is in fact unable to capture the statistical correlation among the elements of signal coefficient, i.e., \{\( w_i(s) : i \in [G] \)\} in \( w(s) = (w_1(s), \ldots, w_G(s))^T \) (spatial correlation) and the correlation among different signal coefficients \( w(s), s \in [T] \) (temporal correlation). In view of the decoupling property derived in Theorem 1 and 2 in principle, one can improve the performance of \( \ell_{2,1}\)-LS by modifying the covariance estimation phase (e.g., the cost function) to take into account not only the joint sparsity but also the spatial/temporal signal correlations, and applying the plug-in MMSE estimator as in (9) to estimate signal samples.

iv) One can also see that \( \ell_{2,1}\)-LS exploits only the second order statistics of the signal but not higher-order ones, thus, it naturally suits Gaussian signals. For non-Gaussian signals, in contrast, one may be able to exploit the higher order signal statistics to obtain better performances (as is done in Bayesian algorithms such as VAMP discussed in Section I-B).

D. Effect of over-parametrization and dictionary mismatch

As explained in Section I-B when the signal dictionary has infinitely many elements, there may be a mismatch in \( \ell_{2,1}\)-LS optimization in (4) due to using the quantized dictionary \( A \) since the quantization grid \( G = \{\xi_1, \ldots, \xi_G\} \) may not contain the true support \( \{\xi_1, \ldots, \xi_k\} \) (see, e.g., (1)). This mismatch may also arise even with finite dictionaries when the true sparsifying dictionary is not known precisely (10).

A natural way to reduce this mismatch is to increase the dictionary size by adding additional elements (e.g., increasing the grid size \( G \) for a continuous dictionary). However, it is generally believed that enlarging the dictionary size \( G \) degrades the recovery performance in a CS problem such as (4) or (5). The underlying reasoning is that by increasing \( G \) the columns of the corresponding dictionary matrix \( A \) become highly correlated. In addition, sparse estimation under a larger dictionary requires estimating more parameters. For example,
for $\mathbf{5}$, this typically results in more spurious rows in the resulting estimate $\mathbf{C}$ and, at first sight, seems to naturally degrade the recovery performance of $\ell_{2,1}$-LS.

Using the decoupling property of $\ell_{2,1}$-LS, we can now show that although this may, in fact, be true for the recovery of $\mathbf{C}$, it does not necessarily hold when the recovery of the original signal samples $\mathcal{H} = \{h(s) : s \in [T]\}$ is of interest. To show this, we need some notation first. Let $\mathbf{A} = [a(\xi_1^G), \ldots, a(\xi_n^G)]$ be a finite (or quantized) dictionary consisting of $G$ columns. We define the cone generated by the columns of $\mathbf{A}$ by

$$K_+^A = \left\{ \sum_{i \in [G]} p_i a(\xi_i^G) a(\xi_i^G)^\dagger : p_i \geq 0 \right\}.$$  

(12)

Note that $K_+^A$ is a convex sub-cone of the cone of positive semi-definite (PSD) matrices generated by rank-1 matrices $\{a(\xi) a(\xi)^\dagger : \xi \in \{\xi_1^G, \ldots, \xi_n^G\}\}$. We call a set $\mathcal{K} \subseteq \mathbb{C}^{n \times n}$ a convex cone if any positive linear combination of the elements of $\mathcal{K}$ belongs to $\mathcal{K}$ [16].

A direct inspection in Theorem 2 and (9), reveals that due to the decoupling property of $\ell_{2,1}$-LS, the dictionary matrix $\mathbf{A}$ affects the recovery performance of signal samples only through the covariance estimate $\hat{\Sigma}_h = \hat{\mathbf{A}} \hat{\mathbf{A}}^\dagger$, which belongs to the cone $K_+^A$ produced by $\mathbf{A}$ (note that $\hat{\mathbf{A}}$ is a diagonal matrix with positive diagonal elements). Moreover, assuming without loss of generality that the columns (atoms) of $\mathbf{A}$ all have the same $\ell_2$-norm $\zeta := \|A_{i,:}\|_2$, $i \in [G]$, using the fact that $\text{tr}(\mathbf{A}(\mathbf{A}^\dagger) = \zeta^2 \text{tr}(\mathbf{I})$, and by introducing the change of variable $\mathbf{K} := \mathbf{A} \mathbf{A}^\dagger$, we can reformulate the optimization of $\hat{\mathbf{A}}$ in (5) more directly in terms of the covariance matrix of the signal $\Sigma_h$ as follows:

$$\Sigma_h = \arg \min_{\mathbf{K} \in K_+^A} \sum_{s \in [T]} x(s)^\dagger (\Psi(s) \mathbf{K} \Psi(s)^\dagger + \mathbf{I}_m)^{-1} x(s)$$

$$+ \frac{1}{\zeta^2} \text{tr}(\mathbf{K}).$$

(13)

This immediately shows that the underlying dictionary can affect $\ell_{2,1}$-LS only via the covariance estimation phase in (13), which depends on the cone $K_+^A$ produced by $\mathbf{A}$. In particular, letting $\mathbf{A}$ be a larger dictionary matrix consisting of $G > G$ columns that contains all $G$ columns of the original dictionary $\mathbf{A}$, one can easily see that using the larger dictionary $\mathbf{A}$ does not degrade the performance provided that the additional elements (columns) of $\mathbf{A}$ do not widen/enlarge the convex cone $K_+^A$ considerably compared with the original cone $K_+^A$.

Using the cone produced by the dictionary, we can also obtain a better understanding of the quantization effect in scenarios where the underlying dictionary has infinitely many elements. For such a dictionary $\mathbf{A}$, we define by

$$K_+^{\mathbf{A}} := \left\{ \sum_{\xi \in \Xi'} p_\xi a(\xi) a(\xi)^\dagger : p_\xi > 0, \Xi' \subseteq \Xi, |\Xi'| < \infty \right\},$$

the cone generated by $\mathbf{A}$, where the summation is taken over all possible subsets $\Xi'$ of the labelling set $\Xi$ (see, e.g., (3)) of finite size. In view of (13), it is not difficult to see that to avoid the effect of dictionary mismatch, one needs to take a quantized dictionary $\mathbf{A} = [a(\xi_1^G), \ldots, a(\xi_n^G)]$ over a grid of sufficiently large size $G$ such that the cone $K_+^{\mathbf{A}}$ approximately exhausts the whole cone $K_+^A$ (note that $K_+^{\mathbf{A}} \subseteq K_+^A$). In particular, in special cases where the cone $K_+^{\mathbf{A}}$ has an algebraic representation, namely, it can be represented by finitely many linear constraints over the cone of PSD matrices (note that $K_+^{\mathbf{A}}$ is a subset of PSD matrices), one can fully eliminate the mismatch by estimating $\Sigma_h$ via (13) by replacing the constraint set $K_+^{\mathbf{A}}$ by $K_+^{A}$ and solving a finite-dim convex optimization problem.

IV. SIMULATION RESULTS

In this section, we illustrate the validity of the decoupling property for $\ell_{2,1}$-LS via numerical simulations.

**Basic setup.** For the simulations, we will focus on the sparse channel estimation problem illustrated in Fig.1. We assume that the signal dimension is $n = 64$ (number of BS antennas) and signal samples $\mathcal{H} = \{h(s) : s \in [T]\}$ are i.i.d. zero-mean Gaussian vectors. We consider a diffuse propagation scenario (e.g., reflection from buildings, trees, etc.) where each signal sample $h(s)$ consists of infinitely many scatterers of equal strength in the angular range $\Xi^o := [-0.1, 0.1] \subset [-1.1]$. It is straightforward to show that in this case each $h(s)$ is a Gaussian vector with a Toeplitz covariance matrix $\Sigma_h = \int a(\xi) a(\xi)^\dagger \mu(d\xi)$ where $a(\xi)$ is the array response vector as in (2) and where $\mu(d\xi)$ is a uniform distribution over the interval $\Xi^o := [-0.1, 0.1]$ and denotes the angular power spread function of the signal samples, which is not a priori known by the estimation algorithm.

We mainly investigate the effect of dictionary mismatch as one of the direct consequences of the decoupling property proved for $\ell_{2,1}$-LS (see, Section III-D). Note that we have intentionally decided to consider a continuous angular power spread function $\mu(d\xi)$ for simulations since, in contrast with discrete AoAs illustrated in Fig.1 and signal model (1), for such a continuous distribution the underlying dictionary atoms in the sparse representation of signal are not trivially known.

We consider three different sparsifying dictionaries: the original continuum dictionary $\mathbf{A} = \{a(\xi) : \xi \in [0, 1]\}$ as in (5), and discrete dictionaries $\mathbf{A}_n = \{a(\xi^G) : i \in [n]\}$ and $\mathbf{A}_{2n} = \{a(\xi^G) : i \in [2n]\}$ obtained by quantizing $\mathbf{A}$ over a uniform grid of $\Xi$ of size $n$ and $2n$, respectively. One can check that for the dictionary $\mathbf{A}$ consisting of complex exponential atoms as in (2), the dictionary matrices corresponding to $\mathbf{A}_n$ and $\mathbf{A}_{2n}$ would be the standard Fourier matrix and the oversampled Fourier matrix with an oversampling factor 2.

We consider a window consisting of $T = 100$ signal samples and assume that each projection operator $\Psi(s), s \in [T]$ is a random all-zero matrix with only one 1 in each row and samples $m = 0.5 n$ (50% sampling) of the components of $h(s)$ (antenna selection). We also assume that the locations of the sampled elements are random in each $\Psi(s)$ and vary i.i.d. for different $s \in [T]$.

To obtain the simulation results for $\mathbf{A}_n$ and $\mathbf{A}_{2n}$, we directly solve $\ell_{2,1}$-LS in (5) with a regularization parameter $\sigma = \sigma^2$ to estimate $\hat{\mathbf{C}}$ and, thus, $\hat{h}(s)$ for $s \in [T]$. For the continuum dictionary $\mathbf{A}$, however, (5) is not directly solvable since, roughly speaking, it requires using a dictionary matrix with infinitely many columns and estimating a coefficient matrix $\mathbf{C}$ with infinitely many rows. Instead, we apply the decoupling...
property of $\ell_2,1$-LS and use an indirect approach, namely, we solve (13) to estimate the signal covariance matrix and then apply the plug-in MMSE estimator (6) to estimate the signal samples. To do this, we need to specify the cone generated by the dictionary $A$ in (13). Fortunately, for the dictionary of complex exponentials (array responses in (2)) considered here, it is well-known that the cone generated by the dictionary coincides with the cone of $n \times n$ PSD Hermitian Toeplitz matrices. Since this cone can be parameterized by $n$ complex numbers, the covariance estimation phase in (13) is a finite-dim optimization problem and can be exactly solved.

We run the simulations for 100 i.i.d. realizations of the signal samples to estimate the Normalized Mean Square Error (NMSE) of the $\ell_2,1$-LS MMV estimator. We compare the NMSE of the MMV estimator with that of the MMSE estimator, where we compute the latter by calculating the Toeplitz covariance matrix $\Sigma_h$ of the signal samples and applying (6).

**Simulation results.** Fig. 3 illustrates the simulation results. It is seen that MMV with the dictionary $A_m$ is highly mismatched and because of this mismatch its NMSE gets saturated at high Signal-to-Noise Ratios (SNRs). Interestingly, by enlarging the dictionary to $A_{2n}$, the NMSE improves dramatically. It is important to note that $\ell_2,1$-LS over the larger dictionary $A_{2n}$ requires estimating twice more number of parameters than that over $A_m$, but it still yields a better estimation performance.

Another interesting observation one can make from Fig. 3 is that by moving form $A_{2n}$ to yet a larger (in fact infinite-dim) dictionary, the MMV performance keeps improving but only slightly. This can be qualitatively justified by the fact that the cone of $n \times n$ PSD Toeplitz matrices is well approximated by the cone generated by $A_{2n}$. Roughly speaking, this can be justified by Szegő theorem [17], which states that the space of $n \times n$ PSD Toeplitz matrices is approximately diagonalizable (under an appropriate metric) with the DFT matrix of dimension $n$ in the asymptotic regime as $n \to \infty$.

V. CONCLUSIONS AND FURTHER REMARKS

In this paper, we showed a decoupling property for $\ell_2,1$-LS algorithm used for joint signal estimation in MMV setups. Specifically, we proved that $\ell_2,1$-LS is equivalent to the following two-step procedure: i) use the whole sketches obtained from all signal samples to estimate their covariance matrix (coupled phase), ii) plug in the resulting covariance estimate as the true covariance matrix into the MMSE estimator to reconstruct each signal sample from its corresponding sketch individually (decoupled phase). Using the decoupling property, we could provide several insights on the performance of the $\ell_2,1$-LS as an MMV estimator. For example, we explained that, due to its linear form as the plug-in MMSE estimator, $\ell_2,1$-LS is unable to take advantage of the higher-order signal statistics and is far from optimal for sparse non-Gaussian signal processes. Moreover, using this property, we were able to quantify the effect of dictionary mismatch on the performance of $\ell_2,1$-LS when the underlying sparsifying dictionary of the process is not precisely known. For this specific case, we provided numerical simulations to illustrate the validity of the decoupling property.
This optimization can be reparameterized with \( \Gamma = \frac{DD^\dagger}{\sqrt{T}} = \text{diag}(\frac{|d_1|^2}{\sqrt{T}}, \ldots, \frac{|d_d|^2}{\sqrt{T}}) \in \mathbb{D}_+ \), where \( \mathbb{D}_+ \) denotes the space of all \( G \times G \) diagonal matrices with positive diagonal elements. With this reparameterization, we obtain

\[
\hat{\Gamma} = \arg\min_{\Gamma \in \mathbb{D}_+} \sum_{t=1}^{T} x(s)^\dagger \left( \Psi(s)^\dagger A_\Gamma A^\dagger \Psi(s)^\dagger + \varrho I_m \right)^{-1} x(s) + \text{tr}(\Gamma),
\]

(19)

It is immediately seen that the optimal solution \( \hat{\Gamma} \) (a diagonal matrix with positive diagonal elements \( \hat{\gamma} \in \mathbb{R}_+^G \)) coincides with the optimal solution of the cost function \( g(\gamma) \) (with \( \Gamma = \text{diag}(\gamma) \)) in (8). Moreover, using the reparameterization \( \Gamma = \text{tr}(DD^\dagger) \) in (18), the fact that the optimal solution \( \hat{C} \) of \( f(C) \) in (7) is equivalently given by \( D^*V^* \) in terms of the optimal solution \( (V^*, D^*) \) of (7), and also the identity (16), we have

\[
\hat{\gamma}_i = \frac{d_i^2}{\sqrt{T}} = \frac{\|\hat{c}_i\|_2}{\sqrt{T}},
\]

(20)

where \( \hat{\gamma}_i \) denotes the \( i \)-th component of \( \hat{\gamma} \). This shows the underlying connection between the optimal solution of \( f(C) \) in (7) and that of \( g(\gamma) \) in (8) as stated in the theorem. This completes the proof.

**B. Proof of Theorem 2**

Let \( \hat{C} = \arg\min_{C \in \mathbb{C}^G \times T} f(C) \) be the optimal solution of (7) and let \( s \in [T] \) be a fixed number. From the optimality of \( \hat{C} \), it results that \( \hat{c}(s) := \hat{c}_{i,s} \) (the \( s \)-th column of \( \hat{C} \)) is the optimal solution of the function \( f_s : \mathbb{C}^G \rightarrow \mathbb{R} \) defined by

\[
f_s(c(s)) := \frac{1}{2} \left\| \Psi(s)^\dagger A^\dagger \hat{c}(s) - x(s) \right\|_2^2 + \varrho \sqrt{T} \left\| \hat{C}_{\sim s}, c(s) \right\|_2^2,
\]

(21)

where \( \hat{C}_{\sim s} \) is the \( G \times (T-1) \) matrix obtained by removing the \( s \)-th column of \( \hat{C} \) and where \( [\hat{C}_{\sim s}, c(s)] \) denotes the \( G \times T \) matrix obtained by adjoining \( c(s) \) as the \( s \)-th column to \( \hat{C}_{\sim s} \). Note that if \( \hat{C}_{i,s} = 0 \), namely, the \( i \)-th row of the optimal solution \( \hat{C} \) is an all-zero vector, then the optimal solution \( \hat{c}(s) = (\hat{c}_1(s), \ldots, \hat{c}_G(s))^\dagger \) of (21) should satisfy \( \hat{c}_i(s) = 0 \). So we focus only on the nonzero rows of \( \hat{C} \). Let \( i \in [G] \) be such a row. Then, taking the derivative of the cost function \( f_s(c(s)) \) in (21) with respect to \( c_i(s) \), we obtain

\[
A^\dagger_i \Psi(s)^\dagger (\Psi(s)^\dagger A^\dagger \hat{c}(s) - x(s)) + \varrho \sqrt{T} \left\| \hat{c}_i(s) \right\|_2 = 0.
\]

(22)

By introducing the diagonal matrix \( L \) with the diagonal elements \( L_{i,i} = \frac{\varrho \sqrt{T}}{\left\| \hat{c}_i \right\|_2} \), we can write (22) for the non-zero elements more compactly as

\[
A^\dagger \Psi(s)^\dagger (\Psi(s)^\dagger A \hat{c}(s) - x(s)) + L \hat{c}(s) = 0.
\]

(23)

Note that if the \( i \)-th row of \( \hat{C} \) is zero, then \( L_{i,i} = \infty \), which forces the \( i \)-th component of \( \hat{c}(s) \) to be zero. Thus, with some abuse of notation, (23) also holds for the zero elements. Thus,

\[
\hat{c}(s) = (A^\dagger \Psi(s)^\dagger (\Psi(s)^\dagger A + L)^{-1} A^\dagger \Psi(s)^\dagger) x(s).\]

This completes the proof.

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