ASYMPTOTIC ORDER OF THE GEOMETRIC MEAN ERROR FOR SOME SELF-AFFINE MEASURES

SANGUO ZHU AND SHU ZOU

Abstract. Let $E$ be a Bedford-McMullen carpet associated with a set of affine mappings $\{f_{ij}\}_{(i,j) \in G}$ and let $\mu$ be the self-affine measure associated with $\{f_{ij}\}_{(i,j) \in G}$ and a probability vector $(p_{ij})_{(i,j) \in G}$. We study the asymptotics of the geometric mean error in the quantization for $\mu$. Let $s_0$ be the Hausdorff dimension for $\mu$. Assuming a separation condition for $\{f_{ij}\}_{(i,j) \in G}$, we prove that the $n$th geometric error for $\mu$ is of the same order as $n^{-1/s_0}$.

1. Introduction

In this paper, we study the asymptotics of the geometric mean error in the quantization for the self-affine measures on Bedford-McMullen carpets. The quantization problem for probability measures has a deep background in information theory and engineering technology (cf. [9, 22]). One of the main objectives of this problem is to study the asymptotics for the errors when approximating a given probability measure with discrete probability measures that are supported on finite sets. We refer to [4, 6] for rigorous mathematical foundations of quantization theory. Besides absolutely continuous measures, the quantization problem for singular measures has attracted great attention in the past years (cf. [5, 6, 7, 8, 12, 15, 16, 18, 20]).

1.1. Quantization error and quantization coefficient. Let $\nu$ be a Borel probability measure on $\mathbb{R}^q$. For $k \in \mathbb{N}$, let $\mathcal{D}_k := \{\alpha \subset \mathbb{R}^q : 1 \leq \text{card}(\alpha) \leq k\}$. The $k$th quantization error for $\nu$ of order $r \in [0, \infty)$ is defined by

\begin{equation}
(1.1) \quad e_{k,r}(\nu) = \begin{cases} \inf_{\alpha \in \mathcal{D}_k} \left( \int d(x, \alpha)^r d\nu(x) \right)^{1/r} & r > 0 \\
\inf_{\alpha \in \mathcal{D}_k} \exp \left( \int \log d(x, \alpha) d\nu(x) \right) & r = 0 \end{cases}.
\end{equation}

By [6], we have, $e_{k,r}(\nu) \to e_{k,0}(\nu)$ as $r \to 0$, provided that $\int |x|^s d\nu(x) < \infty$ for some $s > 0$. Thus, the quantization for $\nu$ of order zero can be regarded as a limiting case of that of order $r > 0$. We call $e_{k,0}(\nu)$ the $k$th geometric mean error for $\nu$.

A set $\alpha \in \mathcal{D}_k$ is called a $k$-optimal set for $\nu$ of order $r$ if the infimum in (1.1) is attained at $\alpha$. Let $C_{k,r}(\nu)$ denote the collection of all such sets $\alpha$. By [4, Theorem 4.12], for every $r > 0$, $C_{k,r}(\mu)$ is non-empty whenever $\int |x|^r d\mu(x) < \infty$. In the following, we simply write $e_k(\nu)$ for $e_{k,0}(\nu)$ and write $C_k(\nu)$ for $C_{k,0}(\nu)$. By Theorem 2.5 of [6], we have $e_k(\nu) > e_{k+1}(\nu)$ and $C_k(\nu) \neq \emptyset$ provided that

\begin{equation}
(1.2) \quad \int_0^1 \sup_{a \in \mathbb{R}^q} \nu(B(a, s)) \frac{1}{s} ds < \infty,
\end{equation}

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where $B(a, s)$ denotes the closed ball of radius $s$ which is centered at $a$. This condition is fulfilled if there exist some constants $C, t$, such that

$$\sup_{a \in \mathbb{R}^d} \nu(B(a, \epsilon)) \leq C \epsilon^t \text{ for all } \epsilon > 0.$$  

The upper and lower quantization coefficients are natural characterizations for the asymptotic properties of the quantization errors. For $s > 0$, the $s$-dimensional upper and lower quantization coefficient for $\nu$ of order $r$ are defined by

$$Q_r^+(\nu) := \limsup_{k \to \infty} k^r \epsilon_{k,r}(\nu), \quad Q_r^-(\nu) := \liminf_{k \to \infty} k^r \epsilon_{k,r}(\nu).$$

(1.3)

The upper (lower) quantization dimension $D_r(\nu)(D_r^-(\nu))$ for $\nu$ of order $r$ is exactly the critical point at which $Q_r^+(\nu)(Q_r^-(\nu))$ “jumps” from infinity to zero (cf. [4, 6, 20]):

$$D_r(\nu) := \limsup_{k \to \infty} -\frac{\log k}{\log \epsilon_{k,r}(\nu)}, \quad D_r^-(\nu) := \liminf_{k \to \infty} -\frac{\log k}{\log \epsilon_{k,r}(\nu)}.$$

In comparison with the upper and lower quantization dimension, the upper and lower quantization coefficient provide us with more accurate information for the asymptotics of the quantization error.

In [6], Graf and Luschgy established general results on the asymptotics of the geometric mean errors for absolutely continuous distributions and self-similar measures on $\mathbb{R}^d$. One may see [5, 21, 23, 24] for more related results.

1.2. Bedford-McMullen carpets and self-affine measures. Let $m, n$ be integers with $n \geq m \geq 2$. Let $G$ be a subset of

$$\Gamma := \{0, 1, \ldots, n - 1\} \times \{0, 1, \ldots, m - 1\}$$

with $\text{card}(G) \geq 2$. We consider the following affine mappings on $\mathbb{R}^2$:

$$f_{ij} : (x, y) \mapsto \left(\frac{x + i}{n}, \frac{y + j}{m}\right), \quad (i, j) \in G.$$

There exists a unique compact set $\emptyset \neq E \subset \mathbb{R}^2$ satisfying $E = \bigcup_{(i, j) \in G} f_{ij}(E)$ [10]. The set $E$ is referred to as the self-affine set associated with $(f_{ij})_{(i, j) \in G}$. We also call $E$ a Bedford-McMullen carpet. Given a probability vector $(p_{ij})_{(i, j) \in G}$, there exists a unique Borel probability measure $\mu$ supported on $E$ such that

$$\mu = \sum_{(i, j) \in G} p_{ij} \mu \circ f_{ij}^{-1}.$$  

(1.5)

The measure $\mu$ is called the self-affine measure associated with $(f_{ij})_{(i, j) \in G}$ and $(p_{ij})_{(i, j) \in G}$. Self-affine sets and self-affine measures have been extensively studied in the past years (cf. [11, 12, 13, 14, 17, 19]).

Let $G_y := \text{Proj}_y G$ and $\vartheta := \frac{\log m}{\log n}$. We define

$$G_x(j) := \{i : (i, j) \in G\}, \quad q_j := \sum_{i \in G_x(j)} p_{ij}, \quad j \in G_y;$$

$$s_0 := -\frac{1}{\log m} \left(\vartheta \sum_{i \in G_x(j)} p_{ij} \log p_{ij} + (1 - \vartheta) \sum_{j \in G_y} q_j \log q_j\right).$$

(1.6)

By [11, 14], the Hausdorff dimension for $\mu$ is equal to $s_0$. More exactly, we have

$$\lim_{\epsilon \to 0} \frac{\log \mu(B(x, \epsilon))}{\log \epsilon} = s_0 \quad \text{for } \mu - \text{a.e. } x.$$
This, along with \cite{24}, Corollary 2.1, implies that $D(\mu) = D(\mu) = s_0$. Unfortunately, this does not provide us with accurate information for the asymptotics of the geometric mean error. In order to obtain the exact asymptotic order of the geometric mean error for $\mu$, we need to examine the finiteness and positivity of the upper and lower quantization coefficient for $\mu$ of order zero. As the main result of the present paper, we will prove

**Theorem 1.1.** Let $m \geq n \geq 3$ be fixed integers and let $\Gamma$ be as given in (1.4). Let $G$ be a subset of $\Gamma$ with $\text{card}(G) \geq 2$. Assume that

\begin{equation}
\max\{|i_1 - i_2|, |j_1 - j_2|\} \geq 2
\end{equation}

for every pair of distinct words $(i_1, j_1), (i_2, j_2) \in G$. Then for the self-affine measure $\mu$ as defined in (1.5), we have $0 < Q^{s_0}(\mu) \leq Q^{s_0}(\mu) < \infty$.

**Remark 1.2.** For $r > 0$, Kessböhmer and Zhu \cite{12} proved that $D_r(\mu) = D_r(\mu)$ and determined the exact value; they also proved the finiteness and positivity of the upper and lower quantization coefficient for $\mu$ of order $r$ in some special cases and Zhu \cite{26} proved this fact in general by associating subsets of $E$ with those of the product coding set $W := G^N \times G^N_y$ and considering an auxiliary measure which is supported on $W$.

In order to prove Theorem 1.1, we will also embed subsets of $E$ into the product coding set $W$ and consider an auxiliary product measure $\lambda$ which is supported on $W$. As is noted in \cite{26}, the images of non-overlapping rectangles (under the above-mentioned embedding) may be overlapping. This is one of the main obstacles in the way of proving the main result.

In \cite{26}, the author removed the possible overlappings by keeping the largest of those pairwise overlapping sets and deleting smaller ones, and then estimated the possible "loss". However, in the study of the geometric mean error, the involved integrals are usually negative and with logarithmic form, the method in \cite{26} is not applicable. Our idea here is to replace those overlappings with some other subsets of $W$ such that the sets in the final collection are pairwise disjoint. We also need to estimate the possible loss which is caused by such replacements.

The remaining part of the paper is organized as follows. In section 2, we establish an estimate for the geometric mean error for $\mu$ and reduce the asymptotics of $(e_k(\mu))_{k=1}^\infty$ to those of a number sequence $(s_k)_{k=1}^\infty$. We will need the assumption (1.7) so that the three-step procedure as depicted in \cite{13} can be applied. In section 3, we consider the coding space $W$ and determine the asymptotic order for two related number sequences $(d_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ in terms of $s_0$. In section 4, we associate subsets of $E$ with those of $W$ and remove the possible overlappings by using the above-mentioned replacements. This enables us to establish a relationship between $(s_k)_{k=1}^\infty$ and $(t_k)_{k=1}^\infty$ and complete the proof for Theorem 1.1.

2. Preliminaries

We denote by $|A|$ the diameter of a set $A \subset \mathbb{R}^2$ and $A^c$ its interior in $\mathbb{R}^2$. For $x \in \mathbb{R}$, let $[x]$ denote the largest integer not exceeding $x$. We will use the following notation in the remaining part of the paper (cf. \cite{6}):

\begin{equation}
\hat{e}_k(\nu) := \log e_k(\nu) = \inf_{\alpha \in \mathcal{D}_k} \int \log d(x, \alpha) d\nu(x), \ k \geq 1.
\end{equation}
Let \( k_0 := \vartheta^{-1} \). For every \( k < k_0 \), we define \( \Omega_k := G^k_y \); for \( k \geq k_0 \) we define

\[
\ell(k) := \lfloor k \vartheta \rfloor, \quad \Omega_k := G^{\ell(k)} \times \Omega^{k-\ell(k)}_y; \quad |\sigma| := k \text{ for } \sigma \in \Omega_k; \quad \Omega^* := \bigcup_{k=1}^{\infty} \Omega_k.
\]

For \( \omega = ((i_1, j_1), \ldots, (i_n, j_n)) \in G^h \) and \( (i, j) \in G \), we write

\[
\omega * (i, j) := ((i_1, j_1), \ldots, (i_n, j_n), (i, j)) \in G^{h+1}.
\]

In the same manner, we define \( \rho * f \) for \( \rho \in G^*_y \) and \( f \in G_y \).

Let \( \sigma = ((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k) \in \Omega^* \). We define

\[
\sigma^h := \big((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k-1\big), \quad \text{if } \ell(k) = \ell(k-1);
\]

\[
\sigma^h := \big((i_1, j_1), \ldots, (i_{\ell(k)-1}, j_{\ell(k)-1}), j_{\ell(k)}, \ldots, j_{k-1}\big), \quad \text{if } \ell(k) = \ell(k-1) + 1.
\]

We will consider the following approximate square \( F_\sigma \) (cf. 11):

\[
F_\sigma := \left[ \sum_{h=1}^{\ell(k)} \frac{i_h}{\eta^h} \sum_{h=1}^{\ell(k)} \frac{j_h}{\eta^h} + \frac{1}{\eta^{\ell(k)}} \right] \times \left[ \sum_{h=1}^{k} \frac{j_h}{\eta^h} \sum_{h=1}^{k} \frac{j_h}{\eta^h} + \frac{1}{\eta^k} \right].
\]

As is noted in [12], we have the following facts:

\[
(2.3) \quad \sqrt{2}m^{-k} \leq |F_\sigma| \leq \sqrt{\eta^2 + 1}m^{-k};
\]

\[
(2.4) \quad \mu_\sigma := \mu(F_\sigma) = \prod_{h=1}^{\ell(k)} p_{i_h, j_h} \prod_{h=\ell(k)+1}^{k} q_{j_h}.
\]

We write \( \sigma < \tau \) and call \( \tau \) a descendant of \( \sigma \), if \( \sigma, \tau \in \Omega^* \) and \( F_\tau \subset F_\sigma \). We say that \( \sigma, \tau \in \Omega^* \) are comparable if \( \sigma < \tau \) or \( \tau < \sigma \); otherwise, we call them incomparable. We define

\[
(2.5) \quad \phi_k := \text{card}(\Lambda_k); \quad \xi(k) := \min_{\sigma \in \Lambda_k} |\sigma|, \quad \xi(k) := \max_{\sigma \in \Lambda_k} |\sigma|.
\]

**Remark 2.1.** (1) We have \( E \subset \bigcup_{\sigma \in \Lambda_k} F_\sigma \) and \( \eta \leq \mu_\sigma / \mu_{\sigma^h} \leq \eta \) for every \( \sigma \in \Omega^* \).

(2) For \( \sigma, \tau \in \Lambda_k \) with \( \sigma \neq \tau \), we have \( \sigma \neq \tau \), \( \tau \neq \sigma \) and \( F_\sigma \cap F_\tau = \emptyset \).

Let \( (a_k)_{k=1}^{\infty} \) and \( (b_k)_{k=1}^{\infty} \) be number sequences. We write \( a_k \leq b_k \) if there exists some constant \( T \) such that \( a_k \leq T b_k \) for all \( k \geq 1 \). If \( a_k \leq b_k \) and \( b_k \leq a_k \), then we write \( a_k \asymp b_k \). By (2.1), one can easily see

\[
(2.6) \quad \xi(k) \leq \eta^k \leq \eta^{-1} \xi(k)^{-1}; \quad \phi_k \eta^{k+1} \leq \sum_{\sigma \in \Lambda_k} \mu_\sigma = 1 < \phi_k \eta^k.
\]

\[
(2.7) \quad \xi(k), \xi(k), \log \phi_k \asymp k; \quad \phi_k \leq \phi_{k+1} \leq \eta^{-2} \phi_k.
\]

The following lemma will allow us to focus on the sequence \( (\phi_k)_{k=1}^{\infty} \).

**Lemma 2.2.** We define \( Q^a_k(\mu) := \xi^{-1}_{a_{k}} \log \phi_k + e_{\phi_k}(\mu) \) and

\[
D^a_k(\mu) := \lim \inf_{k \to \infty} Q^a_k(\mu); \quad D^a_0(\mu) := \lim \sup_{k \to \infty} Q^a_k(\mu).
\]

Then \( D^a_k(\mu) > 0 \) iff \( D^a_k(\mu) > -\infty \); \( D^a_0(\mu) < +\infty \) iff \( D^a_0(\mu) < +\infty \).
Lemma 2.3. For \( \nu_\sigma := \mu(\cdot | F_\sigma) \circ h_\sigma \), \( K(\sigma) := \text{supp}(\nu_\sigma) \).

Let \( \alpha \in \mathbb{R}^2 \) be a finite set with \( \text{card}(\alpha) = l \), we have

\[
\int_{F_\sigma} \log d(x, \alpha) \, d\mu(x) = \mu_\sigma \int \log d(h_\sigma(x), \alpha) \, d\nu_\sigma(x) \geq \mu_\sigma (\log m^{-|\sigma|} + \hat{e}_l(\nu_\sigma)).
\]

\[(2.8)\]

Lemma 2.3. There exist constants \( C, t > 0 \) such that for all \( \sigma \in \Omega^* \) and all \( \epsilon > 0 \),

\[
\sup_{x \in \mathbb{R}^2} \nu_\sigma(B(x, \epsilon)) \leq C \epsilon^t.
\]

Proof. \( \sqcup \)

For \( \sigma \in \Omega^* \), let \( h_\sigma \) be an arbitrary similitude of similarity ratio \( m^{-|\sigma|} \). Define

\[
\nu_\sigma := \mu(\cdot | F_\sigma) \circ h_\sigma, \quad K(\sigma) := \text{supp}(\nu_\sigma).
\]

Let \( \alpha \in \mathbb{R}^2 \) be a finite set with \( \text{card}(\alpha) = l \), we have

\[
\int \log d(x, \alpha) \, d\mu(x) = \mu_\sigma \int \log d(h_\sigma(x), \alpha) \, d\nu_\sigma(x) \geq \mu_\sigma (\log m^{-|\sigma|} + \hat{e}_l(\nu_\sigma)).
\]

\[(2.9)\]

Proof. \( \sqcup \)

For \( \sigma \in \Omega^* \), let \( h_\sigma \) be an arbitrary similitude of similarity ratio \( m^{-|\sigma|} \). Define

\[
\nu_\sigma := \mu(\cdot | F_\sigma) \circ h_\sigma, \quad K(\sigma) := \text{supp}(\nu_\sigma).
\]

Proof. \( \sqcup \)

There exists a unique \( \sigma \in \Omega^* \) and all \( \epsilon > 0 \),

\[
\sup_{x \in \mathbb{R}^2} \nu_\sigma(B(x, \epsilon)) \leq C \epsilon^t.
\]

Proof. \( \sqcup \)

For \( \sigma \in \Omega^* \), let \( h_\sigma \) be an arbitrary similitude of similarity ratio \( m^{-|\sigma|} \). Define

\[
\nu_\sigma := \mu(\cdot | F_\sigma) \circ h_\sigma, \quad K(\sigma) := \text{supp}(\nu_\sigma).
\]

Let \( \alpha \in \mathbb{R}^2 \) be a finite set with \( \text{card}(\alpha) = l \), we have

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\int \log d(x, \alpha) \, d\mu(x) = \mu_\sigma \int \log d(h_\sigma(x), \alpha) \, d\nu_\sigma(x) \geq \mu_\sigma (\log m^{-|\sigma|} + \hat{e}_l(\nu_\sigma)).
\]

\[(2.8)\]

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\sup_{x \in \mathbb{R}^2} \nu_\sigma(B(x, \epsilon)) \leq C \epsilon^t.
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\]

Let \( \alpha \in \mathbb{R}^2 \) be a finite set with \( \text{card}(\alpha) = l \), we have

\[
\int \log d(x, \alpha) \, d\mu(x) = \mu_\sigma \int \log d(h_\sigma(x), \alpha) \, d\nu_\sigma(x) \geq \mu_\sigma (\log m^{-|\sigma|} + \hat{e}_l(\nu_\sigma)).
\]

\[(2.8)\]

\[
\sup_{x \in \mathbb{R}^2} \nu_\sigma(B(x, \epsilon)) \leq C \epsilon^t.
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\]

Let \( \alpha \in \mathbb{R}^2 \) be a finite set with \( \text{card}(\alpha) = l \), we have

\[
\int \log d(x, \alpha) \, d\mu(x) = \mu_\sigma \int \log d(h_\sigma(x), \alpha) \, d\nu_\sigma(x) \geq \mu_\sigma (\log m^{-|\sigma|} + \hat{e}_l(\nu_\sigma)).
\]

\[(2.8)\]

\[
\sup_{x \in \mathbb{R}^2} \nu_\sigma(B(x, \epsilon)) \leq C \epsilon^t.
\]

Proof. \( \sqcup \)

For \( \sigma \in \Omega^* \), let \( h_\sigma \) be an arbitrary similitude of similarity ratio \( m^{-|\sigma|} \). Define

\[
\nu_\sigma := \mu(\cdot | F_\sigma) \circ h_\sigma, \quad K(\sigma) := \text{supp}(\nu_\sigma).
\]

Let \( \alpha \in \mathbb{R}^2 \) be a finite set with \( \text{card}(\alpha) = l \), we have

\[
\int \log d(x, \alpha) \, d\mu(x) = \mu_\sigma \int \log d(h_\sigma(x), \alpha) \, d\nu_\sigma(x) \geq \mu_\sigma (\log m^{-|\sigma|} + \hat{e}_l(\nu_\sigma)).
\]

\[(2.8)\]
Proof. (1) This can be seen by (2.9) and the proof of Theorem 3.4 in [6].

(2) This is a consequence of Lemma 2.3 and [6] Lemma 5.9. One can see the proof of [25, Lemma 2.3] for the argument.

(3) For $\sigma \in \Lambda_k$, let $\beta, \gamma_\sigma$ be as defined in Remark 2.4. Let $h_0 := A_1 + A_2$ and $L_1 := D_{h_0}$. Suppose that $L_\sigma := \text{card}(\alpha_\sigma) > L_1$ for some $\sigma$. Then, by Remark 2.3 (r5), there exists a $\tau \in \Lambda_k$ such that $\alpha_\sigma = \emptyset$. Let $\tilde{\alpha}_\sigma \in C_{L_{\sigma-h_0}}(\nu_\sigma)$. We define $\beta := (\alpha \setminus \{F_\sigma(x)\}) \cup \tilde{\alpha}_\sigma \cup \beta_\sigma \cup \gamma_\tau$. Then using (2) and (2.8), one can easily deduce $\int \log d(x, \beta) d\mu(x) < \int \log d(x, \alpha) d\mu(x)$. This contradicts the optimality of $\alpha$. \hfill $\Box$

For two number sequences $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$, we write $a_k \approx b_k$ if there exists a constant $T$ such that $|a_k - b_k| \leq T$ for all (large) $k \in \mathbb{N}$. Now we are able to obtain the following estimate for $\epsilon_{\phi_k}(\mu)$:

**Proposition 2.6.** We have $\epsilon_{\phi_k}(\mu) \approx \sum_{\sigma \in \Lambda_k} \mu_\sigma \log m^{-|\sigma|}$.

**Proof.** For every $\sigma \in \Lambda_k$, let $b_\sigma$ be an arbitrary point of $F_\sigma$ and set $\beta := \{b_\sigma\}_{\sigma \in \Lambda_k}$. Then we have that $\text{card}(\beta) \leq \phi_k$ and $d(x, \beta) \leq d(x, b_\sigma)$ for all $x \in F_\sigma$. Thus,

\begin{equation}
\epsilon_{\phi_k}(\mu) \leq \sum_{\sigma \in \Lambda_k} \int_{F_\sigma} \log d(x, b_\sigma) d\mu(x) \leq \sum_{\sigma \in \Lambda_k} \mu_\sigma \log |F_\sigma|.
\end{equation}

Let $\alpha \in C_{\phi_k}(\mu)$. For $\sigma \in \Lambda_k$, let $\gamma_\sigma, \alpha_\sigma$ be as given in Remark 2.4 and (2.11). By Lemma 2.5, $\text{card}(\alpha_\sigma \cup \beta_2) \leq L_1 + A_2 =: L_2$. Further, for all $x \in F_\sigma$, we have $d(x, \alpha_\sigma) \geq d(x, \alpha_\sigma \cup \gamma_\sigma)$. This, (2.8) and Lemma 2.5 (1), yield

\begin{align*}
\epsilon_{\phi_k}(\mu) &\geq \sum_{\sigma \in \Lambda_k} \int_{F_\sigma} \log d(x, \alpha_\sigma \cup \beta_\sigma) d\mu(x) \\
&= \sum_{\sigma \in \Lambda_k} \mu_\sigma \int \log d(x, \alpha_\sigma \cup \beta_\sigma) d\nu_\sigma \circ h^{-1}(x) \\
&\geq \sum_{\sigma \in \Lambda_k} \mu_\sigma m^{-|\sigma|} + B L_2.
\end{align*}

This, (2.3) and (2.12) complete the proof of the lemma. \hfill $\Box$

**Remark 2.7.** $\epsilon_{\phi_k}(\mu)$ is closely connected with the sequence $(s_k)_{k=1}^\infty$:

\begin{equation}
s_k := \left( \sum_{\sigma \in \Lambda_k} \mu_\sigma \log m^{-|\sigma|} \right)^{-1} \sum_{\sigma \in \Lambda_k} \mu_\sigma \log \mu_\sigma.
\end{equation}

In fact, by Lemma 2.6 and (2.7), one can see that $\epsilon_{\phi_k}(\mu) \approx s_k^{-1} \log \phi_k$. Thus, the asymptotics of $\epsilon_{\phi_k}(\mu)$ reduce to those of the number sequence $(s_k)_{k=1}^\infty$.

3. Product coding space and related number sequences

3.1. Product coding space. Let $G$ and $G_y$ be endowed with discrete topology; let $G^N_y$ and $W = G^N_y \times G^N_y$ be endowed with the corresponding product topology. We denote the empty word by $\theta$. Write

\[ G^* := \bigcup_{k=1}^{\infty} G^k, \quad G^*_y := \bigcup_{k=1}^{\infty} G_y^k. \]

Let $\omega = ((i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)) \in G^k$. For $1 \leq h \leq k$, we define $|\omega| := k$, $\omega|h := ((i_1, j_1), (i_2, j_2), \ldots, (i_h, j_h))$; $p_{\omega} := p_{i_1j_1} \cdots p_{i_kj_k}$. 

For words $\omega \in G^N$ and $\tau \in G^*_1 \cup G^*_y$, we define $|\omega|, |\omega|_h, p_{\omega}$ and $|\tau|, |\tau|_h, q_{\tau}$ in the same manner. In particular,

$$|\omega| = |\tau| := \infty, p_{\omega} = q_{\tau} := 0 \text{ for } \omega \in G^N, \tau \in G^N_y; \quad p_\theta = q_\theta := 1.$$  

We write $\omega^- := \omega|_{|\omega|-1}$ if $\omega \in G^* \cup G^*_y$ and $|\omega| > 1$; otherwise, let $\omega^- := \theta$.

For every pair $\omega(1), \omega(2) \in G^* \cup G^N$, or $\omega(1), \omega(2) \in G^*_y \cup G^*_y$, we write

$$\omega(1) < \omega(2) \text{ if } |\omega(1)| \leq |\omega(2)| \text{ and } \omega(1) = \omega(2)|_{|\omega(1)|}.$$  

For $\omega = ((i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)) \in G^*$ and $\rho = (j_1, \ldots, j_k) \in G^*_y$, we write

$$[\omega] := \{\tau \in G^N : |\tau|_k = \omega\}; \quad [\rho] := \{\tau \in G^N_y : |\tau|_k = \rho\};$$  

$$[\omega \times \rho] := [\omega] \times [\rho], \quad |\omega \times \rho| := |\omega| + |\rho|.$$  

There exists a unique Borel probability measure $\lambda$ on $W$ such that

$$\lambda([\omega \times \rho]) = p_{\omega}q_{\rho}.$$  

For our purpose, we need to focus on the following sets:

$$\Phi^* := \{\omega \times \rho : \omega \in G^*, \rho \in G^*_y; \quad |\omega| = \max(|\omega \times \rho|)\}.$$  

For two words $\sigma^{(i)} = \omega^{(i)} \times \rho^{(i)} \in \Phi^*$, $i = 1, 2$, we write $\sigma^{(1)} < \sigma^{(2)}$ and call $\sigma^{(2)}$ a descendant of $\sigma^{(1)}$ if $\omega^{(1)} < \omega^{(2)}$ and $\rho^{(1)} < \rho^{(2)}$. We say that $\sigma^{(1)}, \sigma^{(2)}$ are comparable if $\sigma^{(1)} < \sigma^{(2)}$ or $\sigma^{(2)} < \sigma^{(1)}$; otherwise we call them incomparable. We write $\sigma^{(1)} = (\sigma^{(2)})^-$ if $\sigma^{(1)} < \sigma^{(2)}$ and $|\sigma^{(2)}| = |\sigma^{(1)}| + 1$.

Let $\omega = ((i_1, j_1), (i_2, j_2), \ldots, (i_{\ell(k)}, j_{\ell(k)})) \in G^*$ and $\rho = (j_{\ell(k)+1}, \ldots, j_k) \in G^*_y$. Then for $\sigma = \omega \times \rho$, we have

$$\sigma^- =: \begin{cases} 
\omega \times \rho^- & \text{if } \ell(k) = \ell(k-1) \\
\omega^- \times \rho & \text{if } \ell(k) = \ell(k-1) + 1 
\end{cases}.$$  

Remark 3.1. Let $\sigma^{(i)} = \omega^{(i)} \times \rho^{(i)} \in \Phi^*$, $i = 1, 2$. Then for the sets $[\sigma^{(i)}], i = 1, 2$, we have, either they are disjoint, or, one is a subset of the other. This can be seen as follows. If either $\omega^{(i)}, i = 1, 2$, or $\rho^{(i)}, i = 1, 2$, are incomparable, then clearly $[\sigma^{(i)}], i = 1, 2$, are disjoint; if both $\omega^{(i)}, i = 1, 2$, and $\rho^{(i)}, i = 1, 2$, are comparable, then one of $[\sigma^{(i)}], [\sigma^{(2)}]$, is contained in the other, since, by the definition of $\Phi^*$, $|\sigma^{(1)}| > |\sigma^{(2)}|$ implies that $|\rho^{(1)}| \leq |\rho^{(2)}|$.

A finite subset $\Upsilon$ of $\Phi^*$ is called a finite maximal antichain in $W$ if the words in $\Upsilon$ are pairwise incomparable and $W = \bigcup_{\tau \in \Upsilon} [\tau]$.

3.2. Two related number sequences. For the proof of the main result, we need to study the asymptotic order of two number sequences which are related to $(s_k)_{k=1}^\infty$.

The first sequence is about the words in $\Phi^*$ of the same length. We define

$$\Phi_k := \{\bar{\sigma} \in \Phi^* : |\bar{\sigma}| = k\}; \quad U_k := \sum_{\bar{\sigma} \in \Phi_k} \lambda(|\bar{\sigma}|) \log \lambda(|\bar{\sigma}|); \quad d_k := \frac{U_k}{-k \log m}.$$  

Lemma 3.2. We have $|d_k - s_0| \lesssim k^{-1}$.  

Lemma 3.3. We have

\[ d_k = \frac{1}{-k \log m} \left( \ell(k) \sum_{(i,j) \in G} p_{ij} \log p_{ij} + (k - \ell(k)) \sum_{j \in G_y} q_j \log q_j \right) \]

(3.4)

\[ = \frac{1}{\log m} \left( \frac{\ell(k)}{k} \sum_{j \in G_y} \sum_{i \in G_x(j)} p_{ij} \log p_{ij}^{-1} + \frac{k - \ell(k)}{k} \sum_{j \in G_y} q_j \log q_j^{-1} \right). \]

Note that \( p_{ij} \leq q_j \) and \( q_j = \sum_{i \in G_x(j)} p_{ij} \) for all \( j \in G_y \). We obtain

(3.5)

\[ \sum_{j \in G_y} \sum_{i \in G_x(j)} p_{ij} \log p_{ij}^{-1} \geq \sum_{j \in G_y} q_j \log q_j^{-1}. \]

Since \( \ell(k) \leq k \theta \), by the monotonicity of the function \( g \), we deduce that \( d_k \leq s_0 \). Note that \( |k \theta - \ell(k)| \leq 1 \). This, along with (3.4) and (3.5), yields

(3.6)

\[ 0 \leq s_0 - d_k \leq \frac{2}{k \log m} \sum_{(i,j) \in G} p_{ij} \log p_{ij}^{-1} \lesssim k^{-1}. \]

This completes the proof of the lemma.

Proof. For \( 0 < b \leq a \), the function \( g : x \mapsto ax + b(1 - x) \) is increasing. Note that

\[ d_k = \frac{1}{-k \log m} \left( \ell(k) \sum_{(i,j) \in G} p_{ij} \log p_{ij} + (k - \ell(k)) \sum_{j \in G_y} q_j \log q_j \right) \]

(3.4)

\[ = \frac{1}{\log m} \left( \frac{\ell(k)}{k} \sum_{j \in G_y} \sum_{i \in G_x(j)} p_{ij} \log p_{ij}^{-1} + \frac{k - \ell(k)}{k} \sum_{j \in G_y} q_j \log q_j^{-1} \right). \]

The second sequence is related to words in \( \Phi^* \) typically of different length. Let \( (\Gamma_k)_{k=1}^\infty \) be a sequence of finite maximal antichains in \( W \). We define

\[ t_k := t(\Gamma_k) := \frac{\sum_{\sigma \in \Gamma_k} \lambda(\sigma) \log \lambda(\sigma)}{\sum_{\sigma \in \Gamma_k} \lambda(\sigma) \log m^{-|\sigma|}}, \quad l_k := l(\Gamma_k) := \min |\sigma|, \quad T_k := \max |\sigma| \in \Gamma_k. \]

Lemma 3.3. We have \( |t_k - s_0| \lesssim \frac{1}{k^1}. \)

Proof. For every \( h \geq k_0 \), let \( U_h \) and \( d_h \) be as defined in (3.3). Then for \( \sigma \in \Gamma_k \),

\[ U_{|\sigma|} = \ell(|\sigma|) \sum_{(i,j) \in G} p_{ij} \log p_{ij} + (|\sigma| - \ell(|\sigma|)) \sum_{j \in G_y} q_j \log q_j. \]

For every integer \( h \geq 0 \) and \( \sigma = \omega \times \rho \in \Gamma_k \), we define

\[ \zeta(\sigma) := \lambda(|\sigma|)(\log \lambda(|\sigma|) - U_{|\sigma|}); \quad \Upsilon(\sigma, h) := \{ \rho \in \Phi_{|\sigma|+h} : \sigma \prec \rho \}. \]

If \( \ell(|\sigma|+1) = \ell(|\sigma|) \), then \( U_{|\sigma|+1} - U_{|\sigma|} = \sum_{j \in G_y} q_j \log q_j \). We deduce

\[ \sum_{\rho \in \Upsilon(\sigma, 1)} \zeta(\rho) = \sum_{j \in G_y} \lambda([\omega \times (\rho \ast j)])(\log \lambda([\omega \times (\rho \ast j)]) - U_{|\sigma|+1}) \]

\[ = \lambda(|\sigma|)(\log \lambda(|\sigma|) + \sum_{j \in G_y} q_j \log q_j) - \lambda(|\sigma|)U_{|\sigma|+1} \]

\[ = \lambda(|\sigma|)(\log \lambda(|\sigma|) - U_{|\sigma|}) = \zeta(\sigma). \]

If \( \ell(|\sigma|+1) = \ell(|\sigma|) + 1 \), then \( U_{|\sigma|+1} - U_{|\sigma|} = \sum_{(i,j) \in G} p_{ij} \log p_{ij} \). We have

\[ \sum_{\rho \in \Upsilon(\sigma, 1)} \zeta(\rho) = \sum_{(i,j) \in G} \lambda([(\omega \ast i, j)] \times \rho)(\log \lambda([(\omega \ast i, j)] \times \rho)) - U_{|\sigma|+1} \]

\[ = \lambda(|\sigma|)(\log \lambda(|\sigma|) + \sum_{(i,j) \in G} p_{ij} \log p_{ij}) - \lambda(|\sigma|)U_{|\sigma|+1} \]

\[ = \lambda(|\sigma|)(\log \lambda(|\sigma|) - U_{|\sigma|}) = \zeta(\sigma). \]
By induction, we have, \( \sum_{\rho \in \Gamma_{(\sigma,h)}} \zeta(\rho) = \zeta(\sigma) \) for all \( h \geq 1 \) and all \( \sigma \in \Gamma_k \). Hence,

\[
\sum_{\sigma \in \Gamma_k} \zeta(\sigma) = \sum_{\sigma \in \Phi_k} \zeta(\sigma) = 0.
\]

It follows that \( \sum_{\sigma \in \Gamma_k} \lambda(\sigma) \log \lambda(\sigma) = \sum_{\sigma \in \Gamma_k} \lambda(\sigma) \sum_{\rho \in \Gamma_{(\sigma,h)}} \zeta(\rho) \), Thus, we obtain

\[
\min_{\mathcal{L} \leq h \leq \mathcal{T}_k} d_h \leq t_k = \frac{\sum_{\sigma \in \Gamma_k} \lambda(\sigma) \sum_{\rho \in \Gamma_{(\sigma,h)}} \zeta(\rho)}{\sum_{\sigma \in \Gamma_k} \lambda(\sigma) \log m^{-\lambda(\sigma)}} \leq \max_{\mathcal{L} \leq h \leq \mathcal{T}_k} d_h.
\]

This and Lemma 3.2 complete the proof of the lemma.

\[\square\]

4. Proof of Theorem 1.1

Let \( \Omega^* \) be as defined in section 2. We need to associate words in \( \Omega^* \) with words in \( \Phi^* \). For \( \sigma = ((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k) \in \Omega^* \), we define

\[
\mathcal{L}(\sigma) := ((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)})) \times (j_{\ell(k)+1}, \ldots, j_k).
\]

Then \( \Phi_k := \{ \mathcal{L}(\sigma) : \sigma \in \Omega_k \} \) and \( \Phi^* = \bigcup_{k \geq 1} \Phi_k \). By (3.1),

\[
\lambda(\mathcal{L}(\sigma)) = \mu_\sigma, \quad |\mathcal{L}(\sigma)| = |\sigma|: \sigma \in \Omega^*.
\]

The difference between \( \Omega^* \) and \( \Phi^* \) lies in the fact that they have different partial orders. The partial order on \( \Omega^* \) is defined according to the geometric construction of the carpet \( E \), but this is not so for \( \Phi^* \). As a consequence, the words in \( \Omega^* \) and those in \( \Phi^* \) have descendents in different ways (cf. (2.2) and (3.2)).

**Remark 4.1.** We write \( \tilde{\Lambda}_k := \{ \mathcal{L}(\sigma) : \sigma \in \Lambda_k \} \). Let \( \sigma^{(i)} \in \Lambda_k \), \( i = 1, 2 \) be distinct words. It may happen that the sets \( \{ \mathcal{L}(\sigma^{(i)}) \} \in \tilde{\Lambda}_k \) are overlapping. This can be seen as follows. It is possible that both the following words belong to \( \Lambda_k \):

\[
\sigma^{(1)} = ((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), j_{\ell(k)+1}, \ldots, j_k);
\]

\[
\sigma^{(2)} = ((i_1, j_1), \ldots, (i_{\ell(k)}, j_{\ell(k)}), (i, j), j_{\ell(k)+1}, \ldots, j_k).
\]

We clearly have that \( \mathcal{L}(\sigma^{(1)}) \subset \mathcal{L}(\sigma^{(2)}) \) and \( [\mathcal{L}(\sigma^{(1)})] \subset [\mathcal{L}(\sigma^{(2)})] \).

In order to remove the possible overlappings in \( \tilde{\Lambda}_k \), we are going to replace \( \tilde{\Lambda}_k \) with a maximal finite antichain in \( W \). For this purpose, we need a finite sequence of integers which will be defined as follows. Let \( \xi(k) \) and \( \xi(k) \) be defined in (2.4). Then we have \( \max_{\sigma \in \tilde{\Lambda}_k} |\sigma| = \xi(k) \) and \( \min_{\sigma \in \tilde{\Lambda}_k} |\sigma| = \xi(k) \). We write

\[
M := \ell(\xi(k)) - \ell(\xi(k)) + 1.
\]

Let \( \xi(k) := \xi(k) \). We define

\[
\xi(k) = \min\{\xi(k) < h \leq \xi(k) : \ell(h) = \ell(\xi(k)) + 1\}.
\]

Assume that \( \xi(k) \) is defined. We then define

\[
\xi(k) = \min\{\xi(k) < h \leq \xi(k) : \ell(h) = \ell(\xi(k)) + 1\}.
\]

By induction, the sequence \( (\xi(k))_{k=1}^M \) is well defined.

Next, we will construct a finite maximal antichain \( \tilde{\Lambda}_M(k) \) in \( W \) by induction.

Let \( \tilde{\Lambda}_1(k) := \tilde{\Lambda}_k \) and \( F_1(k) = G_1(k) := \emptyset \). We will construct two sets \( F_2(k) \subset \tilde{\Lambda}_1(k) \cap \Phi_{\xi(k)} \) and \( G_2(k) \subset \Phi_{\xi(k)} \setminus \tilde{\Lambda}_1(k) \) and the define

\[
\tilde{\Lambda}_2(k) := (\tilde{\Lambda}_1(k) \setminus F_2(k)) \cup G_2(k).
\]
so that those words in $\tilde{\Lambda}_2(k)$ with length not exceeding $\xi_2(k)$, are pairwise incomparable. The following Lemmas 4.2, 4.4 are devoted to this goal.

**Lemma 4.2.** Assume that $\xi_2(k) > \xi_1(k) + 1$. Then the words in the following set are pairwise incomparable: $\Gamma_1(k) := \bigcup_{h=\xi_1(k)}^{\xi_2(k)-1} (\tilde{\Lambda}_1(k) \cap \Phi_h)$.

**Proof.** Let $\tilde{\sigma}, \tilde{\rho} \in \Gamma_1(k)$ with $\tilde{\sigma} \neq \tilde{\rho}, \sigma = \mathcal{L}^{-1}(\tilde{\sigma}), \rho = \mathcal{L}^{-1}(\tilde{\rho})$. By (4.2) and (4.1),

$$
\lambda([\tilde{\sigma}]) = \mu(F_\sigma) < \eta^k, \lambda([\tilde{\rho}]) = \mu(F_\rho) < \eta^k.
$$

If $|\tilde{\sigma}| = |\tilde{\rho}|$, then $\tilde{\sigma}, \tilde{\rho}$ certainly incomparable. Next, we assume that $|\tilde{\sigma}| < |\tilde{\rho}|$. Then $\xi_1(k) < |\rho| < \xi_2(k)$ and $\ell(|\rho|) = \ell(\xi_1(k)) = \ell(|\rho|) - 1$. Thus, the word $\tilde{\rho}^-$ takes the first form in (4.2); and $\tilde{\rho}^+$ take the form in (2.2). From this, we deduce that $\mathcal{L}^{-1}(\tilde{\rho}^-) = \tilde{\rho}^+$. It follows by using (4.2) that $\lambda([\tilde{\rho}^-]] = \mu(F_{\rho^+}) \geq \eta^k > \lambda([\tilde{\sigma}])$. This implies that $\tilde{\sigma} \neq \tilde{\rho}$. Since $|\tilde{\sigma}| < |\tilde{\rho}|$, we conclude that $\tilde{\sigma}, \tilde{\rho}$ are incomparable. \hfill $\square$

Next, we consider the words in $\tilde{\Lambda}_1(k) \cap \Phi_{\xi_2(k)}$.

**Lemma 4.3.** Let $\tilde{\sigma} \in \tilde{\Lambda}_1(k) \cap \Phi_{\xi_2(k)}$ with

$$
(4.3) \quad \tilde{\sigma} = ((i_1, j_1), \ldots, (i_{\ell(\xi_2(k))}, j_{\ell(\xi_2(k))}) \times (j_{\ell(\xi_2(k))} + 1, \ldots, j_{\xi_2(k)}).
$$

Assume that $\tilde{\omega} < \tilde{\sigma}$ for some $\tilde{\omega} \in \Gamma_1(k)$. Then for every $i \in G_x(j_{\ell(\xi_2(k))})$,

$$
(4.4) \quad \tilde{\sigma}(i) := ((i_1, j_1), \ldots, (i, j_{\ell(\xi_2(k))}) \times (j_{\ell(\xi_2(k))} + 1, \ldots, j_{\xi_2(k)}) \in \tilde{\Lambda}_1(k).
$$

**Proof.** Let $\tilde{\sigma} \in \tilde{\Lambda}_1(k) \cap \Phi_{\xi_2(k)}$. By (2.4), we have

$$
\mu_{\mathcal{L}^{-1}(\tilde{\sigma})} < \eta^k \leq \mu_{\mathcal{L}^{-1}(\tilde{\omega})}.
$$

By the hypothesis, $\tilde{\omega} < \tilde{\sigma}$ for some $\tilde{\omega} \in \Gamma_1(k)$. Since $\ell(|\tilde{\omega}|) = \ell(|\tilde{\omega}|) + 1$, the word $\tilde{\sigma}^-$ takes the second form in (3.2). By Remark 3.1, we obtain $\tilde{\omega} < \tilde{\sigma}^-$.

$$
(4.5) \quad \mu_{\mathcal{L}^{-1}(\tilde{\sigma}^-)} = \lambda([\tilde{\sigma}^-]) \leq \lambda([\tilde{\omega}]) = \mu_{\mathcal{L}^{-1}(\tilde{\omega})} < \eta^k.
$$

Note that $\tilde{\sigma}(i)^- = \tilde{\sigma}^-$ and $(\mathcal{L}^{-1}(\tilde{\sigma}(i)))^y = (\mathcal{L}^{-1}(\tilde{\sigma}))^y$ for every $i \in G_x(j_{\xi_2(k)}(k))$. Thus,

$$
\mu_{\mathcal{L}^{-1}(\tilde{\sigma}(i))} < \mu_{\mathcal{L}^{-1}(\tilde{\sigma}(i)^-)} = \mu_{\mathcal{L}^{-1}(\tilde{\sigma})} < \eta^k \leq \mu_{\mathcal{L}^{-1}(\tilde{\omega})} = \mu_{\mathcal{L}^{-1}(\tilde{\omega})}.
$$

By (2.4), we know that $\mathcal{L}^{-1}(\tilde{\sigma}(i)) \in \Lambda_k$ and $\tilde{\sigma}(i) \in \tilde{\Lambda}_1(k)$. \hfill $\square$

We denote the set of all the words $\tilde{\sigma}$ that fulfills the assumption in Lemma 4.3 by $\tilde{F}_2(k)$. For every $\tilde{\sigma} \in \tilde{F}_2(k)$, let us denote the set of all the words $\tilde{\sigma}(i)$ as given in (4.4) by $\mathcal{F}(\tilde{\sigma})$. Clearly, for every $i \in G_x(j_{\ell(\xi_2(k))})$, we have $\mathcal{F}(\tilde{\sigma}(i)) = \mathcal{F}(\tilde{\sigma})$. For every $\tilde{\sigma} \in \tilde{F}_2(k)$, we fix an arbitrary $\tilde{\sigma}(i)$, and denote the set of all these words $\tilde{\sigma}(i)$ by $\tilde{F}_2(k)$. Then we have $\tilde{F}_2(k) = \bigcup_{\tilde{\sigma} \in \tilde{F}_2(k)} \mathcal{F}(\tilde{\sigma})$, and $\mathcal{F}(\tilde{\sigma}) \cap \mathcal{F}(\tilde{\rho}) = \emptyset$ for every pair of distinct words $\tilde{\sigma}, \tilde{\rho} \in \tilde{F}_2(k)$.

**Lemma 4.4.** Let $\tilde{\sigma} \in \tilde{F}_2(k)$ be as given in (4.4). For every $i \in G_x(j_{\xi_2(k)})$, we define $\tilde{\sigma}(i)$ to be (by interchanging the positions of $j_{\xi_2(k)}(k)$ and $j_{\ell(\xi_2(k))}$ in (4.4)):

$$
((i_1, j_1), \ldots, (i_{\ell(\xi_2(k))}-1, j_{\ell(\xi_2(k))}-1), (i, j_{\xi_2(k)}) \times (j_{\ell(\xi_2(k))}+1, \ldots, j_{\xi_2(k)}-1, j_{\xi_2(k)}))
$$

and let $\mathcal{G}(\tilde{\sigma}) := \{\tilde{\sigma}(i) : i \in G_x(j_{\xi_2(k)})\}$. Then for every $i \in G_x(j_{\xi_2(k)})$, we have

(a1) $\tilde{\sigma}(i) \notin \tilde{\Lambda}_1(k)$; $\lambda([\tilde{\sigma}(i)]) < \eta^k$;

(a2) $\lambda([\tilde{\sigma}(i)]) \geq \eta^k$ and $(\tilde{\sigma}(i))^y \notin \tilde{\Lambda}_1(k)$;
Proof. (a1) Note that $|\tilde{\sigma}(i)| = |\tilde{\sigma}| = \xi_2(k)$. By the definition of $\xi_2(k)$, the word \((\mathcal{L}^{-1}(\tilde{\sigma}(i)))^p\) takes the following form:

\[
(i_1, j_1), \ldots, (i_{\ell(\xi_2(k)) - 1}, j_{\ell(\xi_2(k)) - 1}, j_{\xi_2(k)}, j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k) - 1}).
\]

Note that $j_{\xi_2(k)}, j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k) - 1}$ is a rearrangement of $j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k)}$. We obtain that $\mu(\mathcal{L}^{-1}(\tilde{\sigma}(i))) = \mu_{\mathcal{L}^{-1}(\tilde{\sigma})}$. Thus, (2.4) and (4.5) yield

\[
\lambda(\tilde{\sigma}(i)) = \mu_{\mathcal{L}^{-1}(\tilde{\sigma})}(\tilde{\sigma}(i)) < \mu_{\mathcal{L}^{-1}(\tilde{\sigma})} = \mu_{\mathcal{L}^{-1}(\tilde{\sigma})} \leq \mu_{\mathcal{L}^{-1}(\tilde{\sigma})} < \eta^k.
\]

By the definition \([4.3]\) of $\Lambda_k$, one gets that $\mathcal{L}^{-1}(\tilde{\sigma}(i)) \notin \Lambda_k$. Hence, $\tilde{\sigma}(i) \notin \tilde{\Lambda}_1(k)$. (a2) Since $|\tilde{\sigma}(i)| = \xi_2(k)$. By (4.3), we know that

\[
\tilde{\sigma}(i)^- = ((i_1, j_1), \ldots, (i_{\ell(\xi_2(k)) - 1}, j_{\ell(\xi_2(k)) - 1}) \times (j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k) - 1}, j_{\ell(\xi_2(k))})).
\]

On the other hand, one easily sees that $\mathcal{L}^{-1}(\tilde{\sigma}(i))$ takes the following form:

\[
((i_1, j_1), \ldots, (i_{\ell(\xi_2(k)) - 1}, j_{\ell(\xi_2(k)) - 1}), j_{\ell(\xi_2(k))}, j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k) - 1}).
\]

Since $\tilde{\sigma} \in \tilde{\Lambda}_1(k)$, we have $\mu_{\mathcal{L}^{-1}(\tilde{\sigma})} \geq \eta^k$. Note that $j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k) - 1}; j_{\ell(\xi_2(k))}$ is a rearrangement of $j_{\ell(\xi_2(k)) + 1}, \ldots, j_{\xi_2(k) - 1}$. By (2.4), we obtain

\[
\lambda(\tilde{\sigma}(i)^-) = \mu_{\mathcal{L}^{-1}(\tilde{\sigma}(i)^-)} = \mu_{\mathcal{L}^{-1}(\tilde{\sigma})} \geq \eta^k.
\]

Again, by (2.4), we have that $\tilde{\sigma}(i)^- \notin \tilde{\Lambda}_1(k)$.

With the above preparations, we now define

\[
\mathcal{G}_2(k) := \bigcup_{\tilde{\sigma} \in \mathcal{F}_2(k)} \mathcal{G}(\tilde{\sigma}), \quad \tilde{\Lambda}_2(k) := (\tilde{\Lambda}_1(k) \setminus \mathcal{F}_2(k)) \cup \mathcal{G}_2(k).
\]

Lemma 4.5. Let $\mathcal{G}_2(k)$ and $\tilde{\Lambda}_2(k)$ be as defined in (4.8). Then

- (b1) $\tilde{\Lambda}_2(k) \setminus \mathcal{G}_2(k) \subset \tilde{\Lambda}_1(k)$;
- (b2) $\lambda(\tilde{\sigma}) < \eta^k$ for all $\tilde{\sigma} \in \tilde{\Lambda}_2(k)$ and $\lambda(\tilde{\sigma}^-) \geq \eta^k$ for every $\tilde{\sigma} \in \mathcal{G}_2(k)$;
- (b3) the words in $\tilde{\Lambda}_2(k) \cap (\bigcup_{h=\xi_1(k)} \Phi_h)$ are pairwise incomparable.

Proof. (b1) This is an immediate consequence of (4.8). (b1) By (4.6) and (4.7), we have $\lambda(\tilde{\sigma}) < \eta^k \leq \lambda(\tilde{\sigma}^-)$ for every $\tilde{\sigma} \in \mathcal{G}_2(k)$. For every $\tilde{\sigma} \in \tilde{\Lambda}_2(k) \setminus \mathcal{G}_2(k) \subset \tilde{\Lambda}_1(k)$, we certainly have $\lambda(\tilde{\sigma}^-) < \eta^k$. Thus, (b2) holds. (b2) Let $\tilde{\sigma} \in \Gamma_1(k)$ (see Lemma 4.2) and $\tilde{\sigma} \in \mathcal{G}_2(k)$. Then $|\tilde{\sigma}^-| < |\tilde{\sigma}|$. By (2.4) and (b1), we have $\lambda(\tilde{\sigma}) < \eta^k \leq \lambda(\tilde{\sigma}^-)$, which implies that $\tilde{\sigma}$ and $\tilde{\sigma}$ are incomparable. By the definition of $\mathcal{F}_2(k)$, for every $\tilde{\sigma} \in (\tilde{\Lambda}_1(k) \cap \Phi_{\tilde{\xi}_1(k)}) \setminus \mathcal{F}_2(k)$ and $\tilde{\sigma} \in \Gamma_1(k)$, we have $\tilde{\sigma}, \tilde{\sigma}$ are incomparable; in addition, such a word $\tilde{\sigma}$ is certainly incomparable with every word in $\mathcal{G}_2(k)$ since they are different words and are of the same length. Combining the above analysis and Lemma 4.2, we obtain (b3).
Next, we define $\mathcal{F}_{l+1}(k), \mathcal{G}_{l+1}(k)$ and $\tilde{\Lambda}_{l+1}(k)$ such that (c1)-(c3) hold for $h = l + 1$.

Claim 1: If $\xi_{l+1}(k) > \xi_{l}(k)+1$, then the words in $\Gamma_l(k) := \bigcup_{h=\xi_{l}(k)}^{\xi_{l+1}(k)-1} (\tilde{\Lambda}_l(k) \cap \Phi_h)$ are pairwise incomparable. This can be seen as follows. By (c1), we have

$$
\mathcal{D}_l := \tilde{\Lambda}_l(k) \cap \bigcup_{h=\xi_{l}(k)+1}^{\xi_{l+1}(k)-1} \Phi_h = \tilde{\Lambda}_l(k) \cap \bigcup_{h=\xi_{l}(k)+1}^{\xi_{l+1}(k)-1} \Phi_h \subset \tilde{\Lambda}_l(k).
$$

For every pair of distinct words $\tilde{\rho}, \tilde{\tau} \in \mathcal{D}_l$, if $|\tilde{\tau}| = |\tilde{\rho}|$, then they are certainly incomparable; otherwise, we may assume that $|\tilde{\tau}| > |\tilde{\rho}|$. Note that $\ell(|\tilde{\tau}|) = \ell(|\tilde{\rho}|) - 1 = \xi_l(k)$. We have $\mathcal{L}^{-1}(\tilde{\tau}^{-}) = (\mathcal{L}^{-1}(\tilde{\tau}))^\circ$. Hence,

$$
\lambda(|\tilde{\tau}^-|) = \mu(\mathcal{L}^{-1}(\tilde{\tau}))^\circ \geq \eta^k > \lambda(|\tilde{\rho}|).
$$

This implies that $\tilde{\rho}, \tilde{\tau}$ are incomparable. By (c2), we know that (4.9) also holds for every $\tilde{\tau} \in \mathcal{D}_l$ and $\rho \in \tilde{\Lambda}_l(k) \cap \bigcup_{h=\xi_{l}(k)}^{\xi_{l+1}(k)-1} \Phi_h$. Since $|\tilde{\tau}| \geq \xi_l(k) + 1 > |\tilde{\rho}|$, we obtain that $\tilde{\rho}, \tilde{\tau}$ are incomparable. By the above analysis and (c3), the claim follows.

We denote by $\mathcal{F}_{l+1}(\tilde{\sigma})$ the set of all the words $\tilde{\sigma} \in \tilde{\Lambda}_l(k) \cap \Phi_{\xi_{l+1}(k)}$ such that $\tilde{\omega} < \tilde{\sigma}$ for some $\tilde{\omega} \in \Gamma_l(k)$. For $\tilde{\sigma} \in \mathcal{F}_{l+1}(k)$, let $\mathcal{F}(\tilde{\sigma})$ and $\mathcal{G}(\tilde{\sigma})$ be defined in the same way as we did for $\tilde{\sigma} \in \mathcal{F}_2(k)$ and let $\tilde{\mathcal{F}}_{l+1}(k)$ be defined accordingly. We define

$$
\mathcal{G}_{l+1}(k) := \bigcup_{\tilde{\sigma} \in \mathcal{F}_{l+1}(k)} \mathcal{G}(\tilde{\sigma}); \quad \tilde{\Lambda}_{l+1}(k) := (\tilde{\Lambda}_l(k) \setminus \mathcal{F}_{l+1}(k)) \cup \mathcal{G}_{l+1}(k).
$$

Then we have $\tilde{\Lambda}_{l+1}(k) \setminus \mathcal{G}_{l+1}(k) \subset \tilde{\Lambda}_l(k)$. Hence, by (c2), $\lambda(|\tilde{\tau}|) < \eta^k$ for every $\tilde{\tau} \in \tilde{\Lambda}_{l+1}(k) \setminus \mathcal{G}_{l+1}(k)$. Further, By (c2) and the argument in Lemma 4.4 for $\tilde{\tau} \in \mathcal{G}_{l+1}(k)$ and $\tilde{\sigma} \in \Gamma_l(k)$ (cf. (4.10) and (4.17)), we have

$$
\lambda(|\tilde{\tau}|) < \eta^k \leq \lambda(|\tilde{\tau}^-|); \quad \lambda(|\tilde{\tau}|) \leq \lambda(|\tilde{\omega}|) < \eta^k.
$$

Hence, we obtain that $\tilde{\sigma} \neq \tilde{\tau}$. Since $|\tilde{\tau}| > |\tilde{\sigma}|$, we obtain that $\tilde{\sigma}, \tilde{\tau}$ are incomparable. As in Lemma 4.5, by the definition of $\mathcal{G}_{l+1}(k)$ one can see that, for every pair $\tilde{\tau} \in (\tilde{\Lambda}_{l+1}(k) \cap \Phi_{\xi_{l+1}(k)}) \setminus \mathcal{G}_{l+1}(k), \tilde{\rho} \in \Gamma_l(k) \cup \mathcal{G}_{l+1}(k)$

$\tilde{\tau}, \tilde{\rho}$ are incomparable. Combining the above analysis with (c3), we obtain that the words in $\tilde{\Lambda}_{l+1}(k) \cap \bigcup_{h=\xi_{l}(k)+1}^{\xi_{l+1}(k)-1} \Phi_h$ are pairwise incomparable. Thus, (c1)-(c3) hold with $l + 1$ in place of $l$.

By induction, we obtain sets $\mathcal{F}_M(k), \mathcal{G}_M(k)$ and $\tilde{\Lambda}_M(k)$ and (c1)-(c3) are fulfilled for $l = M$. One can see that $\tilde{\Lambda}_M(k) = (\tilde{\Lambda}_1(k) \setminus \bigcup_{l=1}^{M} \mathcal{F}_l(k)) \cup (\bigcup_{l=1}^{M} \mathcal{G}_l(k))$.

Lemma 4.6. $\tilde{\Lambda}_M(k)$ is a finite maximal antichain in $W$.

Proof. By the construction of $\tilde{\Lambda}_M(k)$, we know that, the words in $\tilde{\Lambda}_M(k)$ with length not exceeding $\xi_M(k)$, are pairwise incomparable. Now, if $\xi_M(k) < \xi(k)$, then we have $\ell(h) = \ell(\xi_M(k))$ for all $\xi_M(k) + 1 \leq h \leq \xi(k)$ and

$$
\tilde{\Lambda}_M(k) \cap \bigcup_{h=\xi_M(k)+1}^{\xi(k)} \Phi_h \subset \tilde{\Lambda}_1(k).
$$

Using the same argument as that in the proof for Claim 1, one can see that the words in $\tilde{\Lambda}_M(k)$ are pairwise incomparable.

Note that $\sum_{\tilde{\tau} \in \tilde{\Lambda}_M(k)} \lambda(|\tilde{\tau}|) = \sum_{\tilde{\tau} \in \tilde{\Lambda}_1(k)} \lambda(|\tilde{\tau}|) = 1$. Suppose that $\tilde{\rho} \notin \bigcup_{\tilde{\tau} \in \tilde{\Lambda}_M(k)} |\tilde{\tau}|$ for some $\tilde{\rho} \in W$. Then by Remark 3.1 there exists some $\tilde{\zeta} \in \Phi^*$ with $|\tilde{\zeta}| > |\xi(k)|$. 


such that \( \tilde{\mathcal{A}} \cap \tilde{\mathcal{F}} = \emptyset \) for all \( \tilde{\tau} \in \tilde{\mathcal{A}}_M(k) \). This implies that \( \sum_{\tilde{\tau} \in \tilde{\mathcal{A}}_M(k)} \lambda(\tilde{\tau}) < 1 \), which is a contradiction. Thus, we obtain \( W = \bigcup_{\tilde{\tau} \in \tilde{\mathcal{A}}_M(k)} \tilde{\tau} \) and \( \tilde{\mathcal{A}}_M(k) \) is a finite maximal antichain. \( \square \)

Using the following lemma, we establish an estimate for the difference that is caused by the replacements of \( F(\tilde{\sigma}) \) with \( G(\tilde{\sigma}) \) for \( \tilde{\sigma} \in F_h(k) \) and \( 2 \leq h \leq M \).

**Lemma 4.7.** There exists a constant \( C_1 \) such that, for every \( \tilde{\sigma} \in F_h(k) \),

\[
\left| \sum_{\tilde{\omega} \in F(\tilde{\sigma})} \lambda(\tilde{\omega}) \log \lambda(\tilde{\omega}) - \sum_{\tilde{\tau} \in G(\tilde{\sigma})} \lambda(\tilde{\tau}) \log \lambda(\tilde{\tau}) \right| \leq C_1 \sum_{\tilde{\omega} \in F(\tilde{\sigma})} \lambda(\tilde{\omega}).
\]

**Proof.** Let \( \tilde{\sigma} \in F_h(k) \) be as given in (4.3). We write

\[
\sigma^\dagger := ((i_1, j_1), \ldots, (i_{\ell_{(\xi_2(k))}}, j_{\ell_{(\xi_2(k))}}), (i_{\ell_{(\xi_2(k))}+1}, \ldots, j_{\xi_2(k)-1}).
\]

By the definition of \( F(\tilde{\sigma}) \) and that of the measure \( \lambda \), we have

\[
\sum_{\tilde{\omega} \in F(\tilde{\sigma})} \lambda(\tilde{\omega}) \log \lambda(\tilde{\omega}) = \sum_{\tilde{\tau} \in G(\tilde{\sigma}), q_{j_{\ell_{(\xi_2(k))}}} \in F_{j_{\ell_{(\xi_2(k))}}}} \lambda(\tilde{\tau}) \log \lambda(\tilde{\tau})
\]

\[
= q_{j_{\ell_{(\xi_2(k))}}} q_{j_{\xi_2(k)}} (\lambda(\sigma^\dagger)) \log(\lambda(\sigma^\dagger)) + \lambda(\sigma^\dagger) \log q_{j_{\xi_2(k)}}.
\]

(4.10)

In an analogous manner, we have

\[
\sum_{\tilde{\tau} \in G(\tilde{\sigma})} \lambda(\tilde{\tau}) \log \lambda(\tilde{\tau}) = q_{j_{\ell_{(\xi_2(k))}}, q_{j_{\xi_2(k)}}} (\lambda(\sigma^\dagger)) \log(\lambda(\sigma^\dagger))
\]

\[
+ \lambda(\sigma^\dagger) \log q_{j_{\xi_2(k)}}.
\]

(4.11)

Further, one can observe that

\[
\lambda(F(\tilde{\sigma})) := \sum_{\tilde{\omega} \in F(\tilde{\sigma})} \lambda(\tilde{\omega}) = q_{j_{\ell_{(\xi_2(k))}}, q_{j_{\xi_2(k)}}, \lambda(\sigma^\dagger)} \geq q^{\kappa \lambda(\sigma^\dagger)}.
\]

Hence, \( \lambda(\sigma^\dagger) \leq \frac{\lambda(F(\tilde{\sigma}))}{q^{\kappa \lambda(\tilde{\sigma})}} \). Set \( C_0 := -\frac{\lambda(F(\tilde{\sigma}))}{q^{\kappa \lambda(\tilde{\sigma})}} \). By (4.10)-(4.12), we obtain

\[
\left| \sum_{\tilde{\omega} \in F(\tilde{\sigma})} \lambda(\tilde{\omega}) \log \lambda(\tilde{\omega}) - \sum_{\tilde{\tau} \in G(\tilde{\sigma})} \lambda(\tilde{\tau}) \log \lambda(\tilde{\tau}) \right| \leq q^{-2} C_0 \lambda(F(\tilde{\sigma})).
\]

The lemma follows by defining \( C_1 := q^{-2} C_0 \). \( \square \)

We are now able to determine the asymptotic order for \( (s_k)_{k=1}^\infty \). We have

**Lemma 4.8.** Let \( s_k \) be as defined in (2.13). Then we have \( |s_k - s_0| \lesssim k^{-1} \).
Proof. By the construction of \(\tilde{\Lambda}_M(k)\), we have
\[
(4.13) \quad \tilde{\Lambda}_M(k) = \left( \tilde{\Lambda}_k \setminus \bigcup_{h=1}^M \bigcup_{\bar{\sigma} \in \mathcal{F}_h(k)} \mathcal{F}(\bar{\sigma}) \right) \cup \left( \bigcup_{h=1}^M \bigcup_{\bar{\sigma} \in \mathcal{F}_h(k)} G(\bar{\sigma}) \right).
\]
By the definition of \(G(\bar{s})\), we have \(|\bar{\tau}| = |\bar{s}|\) for all \(\bar{\tau} \in G(\bar{s})\) and \(\bar{\sigma} \in \mathcal{F}(\bar{s})\). Further, \(\lambda([\bar{s}]) = \sum_{\bar{\sigma} \in \Lambda} \lambda([\bar{\sigma}])\).

Combining (4.13) and (4.14), we obtain
\[
(4.14) \quad \sum_{\bar{\tau} \in \Lambda} \lambda([\bar{\tau}]) = \sum_{\bar{\sigma} \in \Lambda} \lambda([\bar{\sigma}]).
\]

Combining (4.13) and (4.14), we have
\[
(4.15) \quad \sum_{\sigma \in \tilde{\Lambda}_M(k)} \lambda([\bar{\tau}]) \log m^{-|\bar{\tau}|} = \sum_{\bar{\sigma} \in \Lambda} \lambda([\bar{\sigma}]) \log m^{-|\bar{\sigma}|}.
\]

Also, by (4.13), we have
\[
\sum_{\bar{\tau} \in \tilde{\Lambda}_M(k)} \lambda([\bar{\tau}]) \log \lambda([\bar{\tau}]) = \sum_{\bar{\tau} \in \Lambda} \lambda([\bar{\tau}]) \log \lambda([\bar{\tau}])
+ \sum_{h=1}^M \sum_{\bar{\sigma} \in \mathcal{F}_h(k)} \sum_{\bar{\tau} \in G(\bar{\sigma})} \lambda([\bar{\tau}]) \log \lambda([\bar{\tau}]).
\]

Using this and Lemma 4.1.3, we deduce
\[
\Delta_k := \left| \sum_{\bar{\tau} \in \tilde{\Lambda}_M(k)} \lambda([\bar{\tau}]) \log \lambda([\bar{\tau}]) - \sum_{\bar{\sigma} \in \Lambda} \lambda([\bar{\sigma}]) \log \lambda([\bar{\sigma}]) \right|
\]
\[
= \left| \sum_{h=1}^M \sum_{\bar{\sigma} \in \mathcal{F}_h(k)} \sum_{\bar{\tau} \in G(\bar{s})} \lambda([\bar{\tau}]) \log \lambda([\bar{\sigma}]) - \sum_{h=1}^M \sum_{\bar{\sigma} \in \mathcal{F}_h(k)} \sum_{\bar{\tau} \in G(\bar{s})} \lambda([\bar{\tau}]) \log \lambda([\bar{\sigma}]) \right|
\]
\[
\leq C_1 \sum_{h=1}^M \sum_{\bar{\sigma} \in \mathcal{F}_h(k)} \lambda(F(\bar{\sigma})) \leq C_1.
\]

Note that \(l(\tilde{\Lambda}_M(k)) = \xi(k)\). Thus, by (4.13), (4.14) and (2.14), we obtain
\[
|s_k - t(\tilde{\Lambda}_M(k))| = \left| \sum_{\sigma \in \Lambda} \mu_\sigma \log \mu_\sigma \right| - \left| \sum_{\sigma \in \tilde{\Lambda}_M(k)} \mu_\sigma \log m^{-|\sigma|} \right| \leq \left| \sum_{\sigma \in \tilde{\Lambda}_M(k)} \lambda([\bar{\sigma}]) \log m^{-|\bar{\sigma}|} \right| \leq \frac{1}{\xi(k)} \leq \frac{1}{k}
\]

Further, by Lemmas 2.2.4, 5.3 and 4.4, we deduce
\[
(4.16) \quad |t(\tilde{\Lambda}_M(k)) - s_0| \lesssim k^{-1}.
\]

Combining (4.16) and (4.17), we conclude that
\[
|s_k - s_0| \lesssim |s_k - t(\tilde{\Lambda}_M(k))| + |t(\tilde{\Lambda}_M(k)) - s_0| \lesssim k^{-1}.
\]

This completes the proof of the lemma. \(\square\)
Proof of Theorem 1.1 By (2.6), we have

\[ Q_k^{s_0}(\mu) \approx s_0^{-1} \log \phi_k + s_0^{-1} \sum_{\sigma \in \Lambda_k} \mu_\sigma \log \mu_\sigma. \]

From this, (2.7), Proposition 2.7 and Lemmas 4.8, we deduce

\[ |Q_k^{s_0}(\mu)| \approx |(s_0^{-1} - s_k^{-1}) \log \phi_k| \gg 1. \]

Thus, by Lemma 2.2 we conclude that \( 0 < Q_k^{s_0}(\mu) \leq \Omega_k^{s_0}(\mu) < \infty. \)

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