“Squashed Entanglement” – An Additive Entanglement Measure

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In this paper, we present a new entanglement monotone for bipartite quantum states. Its definition is inspired by the so-called intrinsic information of classical cryptography and is given by the halved minimum quantum conditional mutual information over all tripartite state extensions. We derive certain properties of the new measure which we call “squashed entanglement”: it is a lower bound on entanglement of formation and an upper bound on distillable entanglement. Furthermore, it is convex, additive on tensor products, and superadditive in general.

Continuity in the state is the only property of our entanglement measure which we cannot provide a proof for. We present some evidence, however, that our quantity has this property, the strongest indication being a conjectured Fannes type inequality for the conditional von Neumann entropy. This inequality is proved in the classical case.

I. INTRODUCTION

Ever since Bennett et al. [1, 2] introduced the entanglement measures of distillable entanglement and entanglement of formation in order to measure the amount of nonclassical correlation in a bipartite quantum state, there has been an interest in an axiomatic approach to entanglement measures. One natural axiom is LOCC-monotonicity, which means that an entanglement measure should not increase under Local Operations and Classical Communication. Furthermore, every entanglement measure should vanish on the set of separable quantum states; it should be convex, additive, and a continuous function in the state. Though several entanglement measures have been proposed, it turns out to be difficult to find measures that satisfy all of the above axioms. One unresolved question is whether or not entanglement of formation is additive. This is an important question and has recently been connected to many other additivity problems in quantum information theory [24]. Other examples are distillable entanglement, which shows evidence of being neither additive nor convex [22], and relative entropy of entanglement [28], which can be proved to be nonadditive [30].

In this paper we present a functional called “squashed entanglement” which has many of these desirable properties: it is convex, additive on tensor products and superadditive in general. It is upper bounded by entanglement of formation and has recently been connected to many other additive monotones in quantum information theory [24].

The infimum is taken over all extensions of $\rho^{AB}$, i.e. over all quantum states $\rho^{AB}$ with $\rho^{AB} = Tr_{E}(\rho^{AB})$. $I(A;B|E) = S(AE) + S(BE) - S(ABE) - S(E)$ is the quantum conditional mutual information of $\rho^{AB}$. $\rho^A$ stands for the restriction of the state $\rho^{AB}$ to subsystem $A$, and $S(A) = S(\rho^A)$ is the von Neumann entropy of the underlying state, if it is clear from the context. If not, we emphasise the state in subscript, $S(A)_\rho$. Note that the dimension of $E$ is a priori unbounded.

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tions into the relationship between quantum conditional mutual information and entanglement measures, in particular entanglement of formation.

Our name for this functional comes from the idea that the right choice of a conditioning system reduces the quantum mutual information between $A$ and $B$, thus “squashing out” the non–quantum correlations. See section [11] for a similar idea in classical cryptography, which motivated the above definition.

**Example 2.** Let $\rho^{AB} = |\psi\rangle\langle\psi|^{AB}$ be a pure state. All extensions of $\rho^{AB}$ are of the form $\rho^{ABE} = \rho^{AB} \otimes \rho^{E}$; therefore

$$\frac{1}{2} I(A; B|E) = S(\rho^{A}) = E(|\psi\rangle),$$

which implies $E_{sq}(|\psi\rangle\langle\psi|) = E(|\psi\rangle)$.

**Proposition 3.** $E_{sq}$ is an entanglement monotone, i.e. it does not increase under local quantum operations and classical communication (LOCC) and it is convex.

**Proof.** According to [20] it suffices to verify that $E_{sq}$ satisfies the following two criteria:

1. For any quantum state $\rho^{AB}$ and any unlocal quantum instrument $(\mathcal{E}_k)$ — the $\mathcal{E}_k$ are completely positive maps and their sum is trace preserving [7] — performed on either subsystem,

$$E_{sq}(\rho^{AB}) \geq \sum_k p_k E_{sq}(\tilde{\rho}_k^{AB}),$$

where

$$p_k = \text{Tr}\mathcal{E}_k(\rho^{AB})$$

and $\tilde{\rho}_k^{AB} = \frac{1}{p_k} \mathcal{E}_k(\rho^{AB})$.

2. $E_{sq}$ is convex, i.e. for all $0 \leq \lambda \leq 1$,

$$E_{sq}(\lambda \rho^{AB} + (1 - \lambda) \sigma^{AB}) \leq \lambda E_{sq}(\rho^{AB}) + (1 - \lambda) E_{sq}(\sigma^{AB}).$$

In order to prove 1, we modify the proof of theorem 11.15 in [19] for our purpose. By symmetry we may assume that the instrument $(\mathcal{E}_k)$ acts unilocally on $A$. Now, attach two ancilla systems $A'$ and $A''$ in states $|0\rangle^{A'}$ and $|0\rangle^{A''}$ to the system $ABE$ (i). To implement the quantum operation

$$\rho^{ABE} \rightarrow \tilde{\rho}^{A'BE} := \sum_k (\mathcal{E}_k \otimes \text{id}_E)(\rho^{ABE}) \otimes |k\rangle\langle k|^{A'},$$

with $(|k\rangle^{A'})_k$ being an orthonormal basis on $A'$, we perform (ii) a unitary transformation $U$ on $AA'A''$ followed by (iii) tracing out the system $A''$. Here, $i$ denotes the system $i \in \{A, B, AB\}$ after the unitary evolution $U$.

Then, for any extension of $\rho^{AB}$,

$$I(A; B|E) \overset{(i)}{=} I(AA'A''; B|E) \overset{(ii)}{=} I(\tilde{A}A'; \tilde{B}|E) \overset{(iii)}{\geq} I(\tilde{A} \tilde{A}'; \tilde{B}|E) \overset{(iv)}{\geq} \sum_k p_k I(\tilde{A}; \tilde{B}|E)_{p_k} \overset{(v)}{\geq} \sum_k 2p_k E_{sq}(\rho_{p_k}).$$

The justification of these steps is as follows: attaching auxiliary pure systems does not change the entropy of a system, step (i). The unitary evolution affects only the systems $AA'A''$ and therefore does not affect the quantum conditional mutual information in step (ii). To show that discarding quantum systems cannot increase the quantum conditional mutual information

$$I(\tilde{A} \tilde{A}'; \tilde{B}|E) \leq I(\tilde{A} \tilde{A}'; \tilde{B}|E)$$

we expand it into

$$S(\tilde{A} \tilde{A}'E) + S(BE) - S(\tilde{A} \tilde{A}'BE) - S(E) \leq S(\tilde{A} \tilde{A}'A''E) + S(BE) - S(\tilde{A} \tilde{A}'A''BE) - S(E),$$

which is equivalent to

$$S(\tilde{A} \tilde{A}'E) - S(\tilde{A} \tilde{A}'BE) \leq S(\tilde{A} \tilde{A}'A''E) - S(\tilde{A} \tilde{A}'A''BE),$$

the strong subadditivity [17]: this shows step (iii), and for step (iv) we use the **chain rule**,

$$I(XY; Z|U) = I(X; Z|U) + I(Y; Z|UY).$$

For step (v), note that the first term, $I(\tilde{A} \tilde{A}'|E)$, is non–negative and that the second term, $I(\tilde{A}; \tilde{B}|\tilde{E}A')$, is identical to the expression in the next line. Finally, we have (vi) since $\rho^{ABE}_{p_k}$ is a valid extension of $\rho_{p_k}$. As the original extension of $\rho^{AB}$ was arbitrary, the claim follows.

To prove convexity, property 2, consider any extensions $\rho^{ABE}$ and $\sigma^{ABE}$ of the states $\rho^{AB}$ and $\sigma^{AB}$, respectively. It is clear that we can assume, without loss of generality, that the extensions are defined on identical systems $E$. Combined, $\rho^{ABE}$ and $\sigma^{ABE}$ form an extension

$$\tau^{ABEE'} := \lambda \rho^{ABE} \otimes |0\rangle\langle 0|^{E'} + (1 - \lambda) \sigma^{ABE} \otimes |1\rangle\langle 1|^{E'},$$

of the state $\tau^{AB} = \lambda \rho^{AB} + (1 - \lambda) \sigma^{AB}$. The convexity of squashed entanglement then follows from the observation

$$\lambda I(A; B|E)_{\rho} + (1 - \lambda) I(A; B|E)_{\sigma} = I(A; B|EE')_{\tau} \geq 2E_{sq}(\tau^{AB}).$$

\qed
Proposition 4. $E_{sq}$ is superadditive in general, and additive on tensor products, i.e.

$$E_{sq}(\rho^{A'B'B'}) \geq E_{sq}(\rho^{AB}) + E_{sq}(\rho^{A'B'})$$

is true for every density operator $\rho^{A'B'B'}$ on $\mathcal{H}_A \otimes \mathcal{H}_A' \otimes \mathcal{H}_B \otimes \mathcal{H}_B'$, $\rho^{AB} = \text{Tr}_{A'B'}\rho^{A'B'B'}$, and $\rho^{A'B'} = \text{Tr}_{AB}\rho^{A'B'B'}$.

$$E_{sq}(\rho^{A'B'B'}) = E_{sq}(\rho^{AB}) + E_{sq}(\rho^{A'B'})$$

for $\rho^{A'B'B'} = \rho^{AB} \otimes \rho^{A'B'}$.

Proof. We start with superadditivity and assume that $\rho^{A'B'B'}$ on $\mathcal{H}_A \otimes \mathcal{H}_A' \otimes \mathcal{H}_B \otimes \mathcal{H}_B' \otimes \mathcal{H}_E$ is an extension of $\rho^{A'B'B'}$, i.e. $\rho^{A'B'B'} = E_{AB'} \rho^{A'B'B'} E$. Then,

$$I(A'A'; BB'|E) = I(A; BB'|E) + I(A'; BB'|EA)$$

$$= I(A; B|E) + I(A; B'|EE'B)$$

$$+ I(A'; B'|EA) + I(A'; B|EAB')$$

$$\geq I(A; B|E) + I(A'; B'|EA)$$

$$\geq 2E_{sq}(\rho^{AB}) + 2E_{sq}(\rho^{A'B'})$$

The first inequality is due to strong subadditivity of the von Neumann entropy. Note that $E$ is an extension for system $AB$ and $E$ extends system $A'B'$. Hence, the last inequality is true since squashed entanglement is defined via the infimum over all extensions of the respective states. The calculation is independent of the choice of the extension, which proves superadditivity.

A special case of the above is superadditivity on product states $\rho^{A'B'B'} := \rho^{AB} \otimes \rho^{A'B'}$. To conclude that $E_{sq}$ is indeed additive on tensor products, it therefore suffices to prove subadditivity on tensor products.

Let $\rho^{ABE}$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ be an extension of $\rho^{AB}$ and let $\rho^{A'B'E'}$ on $\mathcal{H}_A' \otimes \mathcal{H}_B' \otimes \mathcal{H}_E'$ be an extension for $\rho^{A'B'}$. It is evident that $\rho^{ABE} \otimes \rho^{A'B'E'}$ is a valid extension for $\rho^{A'B'B'} = \rho^{AB} \otimes \rho^{A'B'}$, hence

$$2E_{sq}(\rho^{A'B'B'}) \leq I(A'A'; BB'|EE')$$

$$\leq I(A; B|EE') + I(A; B'|EE'B')$$

$$+ I(A'; B'|EE'A) + I(A'; B|EAB')$$

$$= I(A; B|E) + I(A'; B'|EE'A)$$

$$= I(A; B|E) + I(A'; B'|E').$$

This inequality holds for arbitrary extensions of $\rho^{AB}$ and $\rho^{A'B'}$. We therefore conclude that $E_{sq}$ is subadditive on tensor products.

Proposition 5. $E_{sq}$ is upper bounded by entanglement of formation [12],

$$E_{sq}(\rho^{AB}) \leq E_F(\rho^{AB}).$$

Proof. Let $\{p_k, |\Psi_k\rangle\}$ be a pure state ensemble for $\rho^{AB}$:

$$\sum_k p_k |\Psi_k\rangle \langle \Psi_k|^{AB} = \rho^{AB}.$$  

The purity of the ensemble implies

$$\sum_k p_k S(A)_{\psi_k} = \frac{1}{2} \sum_k p_k I(A; B)_{\psi_k}.$$  

Consider the following extension $\rho^{ABE}$ of $\rho^{AB}$:

$$\rho^{ABE} := \sum_k p_k |\Psi_k\rangle \langle \Psi_k|^{AB} \otimes |k\rangle\langle k|^{E}.$$  

It is elementary to compute

$$\sum_k p_k S(A)_{\psi_k} = \frac{1}{2} \sum_k p_k I(A; B)_{\psi_k} = \frac{1}{2} I(A; B|E).$$  

Thus, it is clear that entanglement of formation can be regarded as an infimum over a certain class of extensions of $\rho^{AB}$. Squashed entanglement is an infimum over all extensions of $\rho^{AB}$, evaluated on the same quantity $\frac{1}{2} I(A; B|E)$ and therefore smaller or equal to entanglement of formation.

Corollary 6. $E_{sq}$ is upper bounded by entanglement cost:

$$E_{sq}(\rho^{AB}) \leq E_C(\rho^{AB}).$$

Proof. Entanglement cost is equal to the regularised entanglement of formation [12],

$$E_C(\rho^{AB}) = \lim_{n \to \infty} \frac{1}{n} E_F((\rho^{AB})^{\otimes n}).$$

This, together with proposition 4 and the additivity of the squashed entanglement (proposition 4) implies

$$E_C(\rho^{AB}) \leq \lim_{n \to \infty} \frac{1}{n} E_F((\rho^{AB})^{\otimes n})$$

$$\geq \lim_{n \to \infty} \frac{1}{n} E_{sq}((\rho^{AB})^{\otimes n})$$

$$= E_{sq}(\rho^{AB}).$$

Theorem 7. Squashed entanglement vanishes for every separable density matrix $\rho^{AB}$, i.e.

$$\rho^{AB} \text{ separable} \implies E_{sq}(\rho^{AB}) = 0.$$  

Conversely, if there exists a finite extension $\rho^{ABE}$ of $\rho^{AB}$ with vanishing quantum conditional mutual information, then $\rho^{AB}$ is separable, i.e.

$$I(A; B|E) = 0 \text{ and } \dim \mathcal{H}_E < \infty \implies \rho^{AB} \text{ separable.}$$
Proof. Every separable $\rho^{AB}$ can be written as a convex combination of separable pure states,

$$\rho^{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|.$$  

The quantum mutual conditional information of the extension

$$\rho^{ABE} := \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \otimes |i\rangle\langle i|$$  

, with orthonormal states (\(|i\rangle\)), is zero. Squashed entanglement thus vanishes on the set of separable states.

To proof the second assertion assume that there exists an extension $\rho^{ABE}$ of $\rho^{AB}$ with $I(A; B|E) = 0$ and $\dim \mathcal{H}_E < \infty$. Now, a recently obtained result [13] on the structure of such states $\rho^{ABE}$ applies, and as a corollary $\rho^{AB}$ is separable. \hfill $\square$

Remark 8. The minimisation in squashed entanglement ranges over extensions of $\rho^{AB}$ with a priori unbounded size. $E_{sq}(\rho) = 0$ is thus possible, even if any finite extension has strictly positive quantum conditional mutual information. Therefore, without a bound on the dimension of the extending system, the second part of theorem [7] does not suffice to conclude that $E_{sq}(\rho^{AB})$ implies separability of $\rho^{AB}$. A different approach to this question could be provided by a possibly approximate version of the main result of [13]: if there is an extension $\rho^{ABE}$ with small quantum conditional mutual information, then the $\rho^{AB}$ is close to a separable state. For further discussion on this question, see sections [13] and [14].

Note that the strict positivity of squashed entanglement for entangled states would, via corollary [10], imply strict positivity of entanglement cost for all entangled states. This is not yet proven, but conjectured as a consequence of the additivity conjecture of entanglement of formation.

Example 9. It is worth noting that in general $E_{sq}$ is strictly smaller than $E_F$ and $E_C$: consider the totally antisymmetric state $\sigma^{AB}$ in a two-qutrit system

$$\sigma^{AB} = \frac{1}{3}(|I| + |II| + |III|),$$

with

$$|I| = \frac{1}{\sqrt{2}}(|2\rangle^A|3\rangle^B - |3\rangle^A|2\rangle^B),$$

$$|II| = \frac{1}{\sqrt{2}}(|3\rangle^A|1\rangle^B - |1\rangle^A|3\rangle^B),$$

$$|III| = \frac{1}{\sqrt{2}}(|1\rangle^A|2\rangle^B - |2\rangle^A|1\rangle^B).$$

On the one hand, it is known from [31] that $E_F(\sigma^{AB}) = E_C(\sigma^{AB}) = 1$, though, on the other hand, we may consider the trivial extension,

$$E_{sq}(\sigma^{AB}) \leq \frac{1}{2} I(A; B) = \frac{1}{2} \log 3 \approx 0.792.$$  

The best known upper bound on $E_D$ for this state, the Rains bound [21], gives the only slightly smaller value $\log \frac{4}{3} \approx 0.737$. It remains open if there exist states for which squashed entanglement is smaller than the Rains bound.

**Proposition 10.** $E_{sq}$ is lower bounded by distillable entanglement $[12]$

$$E_D(\rho^{AB}) \leq E_{sq}(\rho^{AB}).$$

Proof. Consider any entanglement distillation protocol by LOCC, taking $n$ copies of the state $\rho^{AB}$ to a state $\sigma^{AB}$ such that

$$\|\sigma^{AB} - |s\rangle\langle s|^{AB}\|_1 \leq \delta,$$

with $|s\rangle$ being a maximally entangled state of Schmidt rank $s$. We may assume without loss of generality that the support of $\sigma^A$ and $\sigma^B$ is contained in the $s$-dimensional support of $\text{Tr}_E|s\rangle\langle s|$ and $\text{Tr}_A|s\rangle\langle s|$, respectively. Using propositions [4] and [3], we have

$$nE_{sq}(\rho^{AB}) = E_{sq}((\rho^{AB})^\otimes n) \geq E_{sq}(\sigma^{AB}),$$

so that it is only necessary to estimate $E_{sq}(\sigma^{AB})$ versus $E_{sq}(|s\rangle\langle s|^{AB}) = \log s$ (see example [2]). For this, let $\sigma^{ABE}$ be an arbitrary extension of $\rho^{AB}$ and consider a purification of it, $|\Psi\rangle \in \mathcal{H}_{ABEE'}$. Chain rule and monotonicity of the quantum mutual information allow us to estimate

$$I(A; B|E) = I(AE; B) - I(E; B) \geq I(A; B) - I(EE'; AB) = I(A; B) - 2S(AB).$$

Further applications of Fannes inequality [2], lemma [15] give $I(A; B) \geq 2 \log s - f(\delta) \log s$ and $2S(AB) \leq f(\delta) \log s$, with a function $f$ of $\delta$ vanishing as $\delta$ approaches 0. Hence

$$\frac{1}{2} I(A; B|E) \geq \log s - f(\delta) \log s.$$  

Since this is true for all extensions, we can put this together with eq. [2], and obtain

$$E_{sq}(\rho^{AB}) \geq \frac{1}{n} (1 - f(\delta)) \log s,$$

which, with $n \to \infty$ and $\delta \to 0$, concludes the proof, because we considered an arbitrary distillation protocol. \hfill $\square$

**Remark 11.** In the proof of proposition [10] we made use of the continuity of $E_{sq}$ in the vicinity of maximally entangled states. Similarly, $E_{sq}$ can be shown to be continuous in the vicinity of any pure state. This, together with proposition [3], the additivity on tensor products (second part of proposition [4]), and the normalisation on Bell states, suffices to prove corollary [10] and proposition [11] [15].
Corollary 12. \[ \frac{1}{2}(I(A;B) - S(AB)) \leq E_{sq}(\rho^{AB}). \]

Proof. The recently established hashing inequality provides a lower bound for the one-way distillable entanglement \( E_{\rightarrow}(\rho^{AB}) \),

\[ S(B) - S(AB) \leq E_{\rightarrow}(\rho^{AB}). \]

Interchanging the roles of \( A \) and \( B \), we have \[ \frac{1}{2}(I(A;B) - S(AB)) \leq E_{\rightarrow}(\rho^{AB}) \]

where we use the fact that one-way distillable entanglement is smaller or equal to distillable entanglement. This, together with the bound from proposition 10 implies the assertion. \[ \Box \]

III. ANALOGY TO INTRINSIC INFORMATION

Intrinsic information is a quantity that serves as a measure for the correlations between random variables in information-theoretical secret-key agreement. The intrinsic (conditional mutual) information between two discrete random variables \( X \) and \( Y \), given a third discrete random variable \( Z \), is defined as

\[ I(X;Y \mid Z) = \inf \{ I(X;Y \mid Z) : Z \text{ with } XY \rightarrow Z \rightarrow \bar{Z} \text{ a Markov chain} \}. \]

The infimum extends over all discrete channels \( Z \) to \( \bar{Z} \) that are specified by a conditional probability distribution \( P_{\bar{Z} \mid Z} \).

A first idea to utilize intrinsic information for measuring quantum correlations was mentioned in [11]. This inspired the proposal of a quantum analog to intrinsic information [4] in which the Shannon conditional mutual information plays a role similar to the quantum conditional mutual information in squashed entanglement. This proposal possesses certain good properties demanded of an entanglement measure, and it opened the discussion that has resulted in the current work.

Before we state some similarities in the properties that the intrinsic information and squashed entanglement have in common, we would like to stress their obvious relation in terms of the definitions. Let \( |\Psi\rangle^{ABC} \) be a purification of \( \rho^{AB} \) and let \( \rho^{ABE} \) be an extension of \( \rho^{AB} \) with purification \( |\Psi\rangle^{ABEE'} \). Remark that all purifications of \( \rho^{AB} \) are equivalent in the sense that there is a suitable unitary transformation on the purifying system with

\[ \mathbb{1}^{AB} \otimes U : |\Psi\rangle^{ABC} \rightarrow |\Phi\rangle^{ABEE'}. \]

Applying a partial trace operation over system \( E' \) then results in the completely positive map

\[ \Lambda : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_E), \]

\[ \text{id} \otimes \Lambda : |\Psi\rangle^{ABC} \rightarrow \rho^{ABE}. \]

Conversely, every state \( \rho^{ABE} \) constructed in this manner is an extension of \( \rho^{AB} \).

This shows that the squashed entanglement equals

\[ E_{sq}(\rho^{AB}) = \inf \left\{ \frac{1}{2} I(A;B|E) : \rho^{ABE} = (\text{id} \otimes \Lambda)|\Psi\rangle^{ABC} \right\}, \]

where the infimum includes all quantum operations \( \Lambda : \mathcal{B}(\mathcal{H}_C) \rightarrow \mathcal{B}(\mathcal{H}_E) \).

In [7] it is shown that the minimisation in \( I(X;Y \mid Z) \) can be restricted to random variables \( \bar{Z} \) with a domain equal to that of \( Z \). This shows that the infimum in the definition is in effect a minimum and that the intrinsic information is a continuous function of the distribution \( P_{XYZ} \). It is interesting to note that the technique used there (and, for that matter, also in the proof that entanglement of formation is achieved as a minimum over pure state ensembles \( \rho^{AB} = \sum_k p_k |\Psi_k\rangle^{ABC} (\text{size} (\text{rank } \rho^{AB})^2) \), does not work for our problem, and so, we do not have an easy proof of the continuity of squashed entanglement. In the following section this issue will be discussed in some more detail.

In the cryptographic context in which it appears, intrinsic information serves as an upper bound for the secret-key rate \( S = S(X;Y \mid Z) \). \( S \) is the rate at which two parties, having access to repeated realisations of \( X \) and \( Y \), can distill secret correlations about which a third party, holding realisations of \( Z \), is almost ignorant. This distillation procedure includes all protocols in which the two parties communicate via a public authenticated classical channel to which the eavesdropper has access but cannot alter the transmitted messages. Clearly, one can interpret distillable entanglement as the quantum analog to the secret-key rate. On the one hand, secret quantum correlations, i.e., maximally entangled states of qubits, are distilled from a number of copies of \( \rho^{AB} \). In the classical cryptographic setting, on the other hand, one aims at distilling secret classical correlations, i.e., secret classical bits, from a number of realisations of a triple of random variables \( X, Y \) and \( Z \).

We proved in proposition 10 that squashed entanglement is an upper bound for distillable entanglement. Hence, it provides a bound in entanglement theory which is analogous to the one in information-theoretical secure key agreement, where intrinsic information bounds the secret-key rate from above.

This analogy extends further to the bound on the formation of quantum states (proposition 5 and corollary 6), where we know of a recently proven classical counterpart: namely that the intrinsic information is a lower bound on the formation cost of correlations of a triple of random variables \( X, Y \) and \( Z \) from secret correlations.

IV. THE QUESTION OF CONTINUITY

Intrinsic information, discussed in the previous section, and entanglement of formation are continuous functions.
of the probability distribution and state, respectively. This is so, because in both cases we are able to restrict the minimisation to a compact domain; in the case of intrinsic information to bounded range \( Z \) and in entanglement of formation to bounded size decompositions, noting that the functions to be minimised are continuous.

Thus, by the same general principle, we could show continuity if we had a universal bound \( d \) on the dimension of \( E \) in definition \( 1 \) in the sense that every value of \( I(A; B | E) \) obtainable by general extensions can be reproduced or beaten by an extension with a \( d \)-dimensional system \( E \). Note that if this were true, then (just as for intrinsic information and entanglement of formation) the infimum would actually be a minimum: in remark \( 7 \) we have explained that then \( E_{sq}(\rho^{AB}) = 0 \) would imply, using the result of \( 13 \), that \( \rho^{AB} \) is separable.

As it is, we cannot yet decide on this question, but we would like to present a reasonable conjecture, an inequality of the Fannes type \( 5 \) for the conditional von Neumann entropy, which we can show to imply continuity of \( E_{sq} \). Let us first revisit Fannes’ inequality in a slightly nonstandard form:

**Lemma 13.** For density operators \( \rho, \sigma \) on the same \( d \)-dimensional Hilbert space, with \( \| \rho - \sigma \|_1 \leq \epsilon \),

\[
|S(\rho) - S(\sigma)| \leq \eta(\epsilon) + \epsilon \log d,
\]

with the universal function

\[
\eta(\epsilon) = \begin{cases} \epsilon \log \epsilon, & \epsilon \leq \frac{1}{d}, \\ \frac{1}{d}, & \text{otherwise}. \end{cases}
\]

Observe that \( \eta \) is a concave function. \( \square \)

Now we can state the conjecture, recalling that for a density operator \( \rho^{AB} \) on a bipartite system \( \mathcal{H}_A \otimes \mathcal{H}_B \), the conditional von Neumann entropy \( 3 \) is defined as

\[
S(A|B) := S(\rho^{AB}) - S(\rho^B).
\]

**Conjecture 14.** For density operators \( \rho, \sigma \) on the bipartite system \( \mathcal{H}_A \otimes \mathcal{H}_B \), with \( \| \rho - \sigma \|_1 \leq \epsilon \),

\[
|S(A|B)_{\rho} - S(A|B)_{\sigma}| \leq \eta(2\epsilon) + 3\epsilon \log d_A,
\]

with \( d_A = \dim \mathcal{H}_A \), or some other universal function \( f(\epsilon, d_A) \) vanishing at \( \epsilon = 0 \) on the right hand side.

Note that the essential feature of the conjectured inequality is that it only makes reference to the dimension of system \( A \). If we were to use Fannes inequality directly with the definition of the conditional von Neumann entropy, we would pick up additional terms containing the logarithm of \( d_B = \dim \mathcal{H}_B \). In the appendix we show that this conjecture is true in the classical case, or more precisely, in the more general case where the states are classical on system \( B \).

In order to show that the truth of this conjecture implies continuity of \( E_{sq} \), consider two states \( \rho^{AB} \) and \( \sigma^{AB} \) with \( \| \rho^{AB} - \sigma^{AB} \|_1 \leq \epsilon \). By well-known relations between fidelity and trace distance \( 10 \) this means that \( F(\rho^{AB}, \sigma^{AB}) \geq 1 - \epsilon \), hence \( 12 \) we can find purifications \( \left| \Psi \right>^{ABC} \) and \( \left| \Phi \right>^{ABC} \) of \( \rho^{AB} \) and \( \sigma^{AB} \) respectively, such that \( F\left( \left| \Psi \right>^{ABC}, \left| \Phi \right>^{ABC} \right) \geq 1 - \epsilon \). Using \( 10 \) once more, we get

\[
\left\| \left| \Psi \right>^{ABC} - \left| \Phi \right>^{ABC} \right\|_1 \leq 2\sqrt{\epsilon}.
\]

Now, let \( \Lambda \) be any quantum operation as in eq. \( 3 \); it creates extensions of \( \rho^{AB} \) and \( \sigma^{AB} \),

\[
\rho^{ABE} = (\text{id} \otimes \Lambda) \left| \Psi \right>^{ABC},
\]

\[
\sigma^{ABE} = (\text{id} \otimes \Lambda) \left| \Phi \right>^{ABC},
\]

with

\[
\| \rho^{ABE} - \sigma^{ABE} \|_1 \leq 2\sqrt{\epsilon}.
\]

Hence, using \( I(A; B | E) = S(A|E) + S(B|E) - S(AB|E) \), we can estimate

\[
|I(A; B | E)_{\rho} - I(A; B | E)_{\sigma}| \leq |S(A|E)_{\rho} - S(A|E)_{\sigma}| + |S(B|E)_{\rho} - S(B|E)_{\sigma}| + |S(AB)_{\rho} - S(AB)_{\sigma}| \\
\leq 3\eta(2\sqrt{\epsilon}) + 6\sqrt{\epsilon} \log(d_A d_B) \\
=: \epsilon'.
\]

Since this applies to any quantum operation \( \Lambda \) and thus to every state extension of \( \rho^{AB} \) and \( \sigma^{AB} \), respectively, we obtain

\[
|E_{sq}(\rho^{AB}) - E_{sq}(\sigma^{AB})| \leq \epsilon',
\]

with \( \epsilon' \) universally dependent on \( \epsilon \) and vanishing with \( \epsilon \to 0 \). \( \square \)

**Remark 15.** Since \( E_{sq} \) is convex it is trivially upper semicontinuous. This also follows from the fact that squashed entanglement is an infimum of continuous functions obtained by bounding the size of the dimension of system \( E \).

This observation, together with results from the general theory of convex functions, implies that squashed entanglement is continuous “almost everywhere”. Specifically, with theorem 10.1 in \( 23 \), we have:

**Proposition 16.** \( E_{sq} \) is continuous on the interior of the set of states (i.e. on the faithful states), and more generally, it is continuous when restricted to the relative interior of all faces of the state set.

Continuity near pure states (see remark \( 17 \)) thus implies continuity of \( E_{sq} \) on the set of all rank-2 density operators. \( \square \)
V. CONCLUSION

In this paper we have presented a new measure of entanglement, which by its very definition allows for rather simple proofs of monotonicity under LOCC, convexity, additivity for tensor products and superadditivity in general, all by application of the strong subadditivity property of quantum entropy. We showed the functional, which we call “squashed entanglement”, to be lower bounded by the distillable entanglement and upper bounded by the entanglement cost. Thus, it has most of the “good” properties demanded by the axiomatic approaches [14, 20, 29] without suffering from the disadvantages of other superadditive entanglement monotones. The one proposed in [8], for example, diverges on the set of pure states.

The one desirable property from the wish list of axiomatic entanglement theory that we could not yet prove is continuity. We have shown, however, that squashed entanglement is continuous near pure states and in the relative interior of the faces of state space. Continuity in general would follow from a conjectured Fannes type inequality for the conditional von Neumann entropy. The proof of this conjecture thus remains the great challenge of the present work. It might well be of wider applicability in quantum information theory and certainly deserves further study.

Another question to be asked is whether or not there exist states that are nonseparable but, nonetheless, have zero squashed entanglement. We expect this not to be the case: if not by means of proving that the infimum in squashed entanglement is achieved, then by means of an approximate version of the result of [13]. The relation to entanglement measures other than entanglement is achieved, then by means of proving that the infimum in squashed entanglement is achieved, then by means of an approximate version of the result of [13]. The relation to entanglement measures other than entanglement of formation, entanglement cost and distillable entanglement remains open in general. If $E_{sq} = 0$ would imply separability, however, it would follow that for the class of PPT states, squashed entanglement is larger than entanglement measures based on the partial transpose operation, like relative entropy of entanglement, the logarithmic negativity and the Rains bound.

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APPENDIX A: THE CLASSICAL CASE OF THE CONDITIONAL FANNES INEQUALITY

In this appendix we prove the conjecture [13] for states

$$\rho^{AB} = \sum_k p_k \rho_k^A \otimes |k\rangle |k\rangle^B, \quad (A1)$$

$$\sigma^{AB} = \sum_k q_k \sigma_k^A \otimes |k\rangle |k\rangle^B, \quad (A2)$$

with an orthogonal basis $(|k\rangle)_k$ and of $\mathcal{H}_B$, probability distributions $(p_k)$ and $(q_k)$, and states $\rho_k^A$ and $\sigma_k^A$ on $A$. Note that this includes the case of a pair of classical random variables. In this case, the states $\rho_k^A$ and $\sigma_k^A$ are all diagonal in the same basis $(|j\rangle)_j$ of $\mathcal{H}_A$ and thus $\rho^{AB}$ and $\sigma^{AB}$ describe joint probability distributions on a cartesian product.

The key to the proof is that for states of the form $[A1]$

$$S(A|B)_\rho = \sum_k p_k S(\rho_k^A),$$

and similarly for the states given in eq. $[A2]$.

First of all, the assumption implies that

$$\epsilon \geq \|\rho^B - \sigma^B\|_1 = \sum_k |p_k - q_k|.$$

Hence, we can successively estimate,

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq \sum_k |p_k S(\rho_k^A) - q_k S(\sigma_k^A)|$$

$$\leq \sum_k p_k |S(\rho_k^A) - S(\sigma_k^A)| + \sum_k |p_k - q_k| S(\sigma_k^A)$$

$$\leq \sum_k p_k (\eta(\epsilon_k) + \epsilon_k \log d_A) + \epsilon \log d_A$$

$$\leq \eta(2\epsilon) + 3\epsilon \log d_A,$$

using the triangle inequality twice in the first and second lines, then using $S(\sigma_k^A) \leq \log d_A$, applying the Fannes inequality, lemma [13] in the third (with $\epsilon_k := \|\rho_k^A - \sigma_k^A\|_1$), and finally making use of the concavity of its upper bound. To complete this step, we have to show

$$\sum_k q_k \epsilon_k \leq 2\epsilon,$$

which is done as follows:

$$\epsilon \geq \|\rho^{AB} - \sigma^{AB}\|_1 = \sum_k \|p_k \rho_k^A - q_k \sigma_k^A\|_1$$

$$\geq \sum_k \left(\|p_k \rho_k^A - q_k \sigma_k^A\|_1 - \|p_k \sigma_k^A - q_k \sigma_k^A\|_1\right)$$

$$\geq \sum_k p_k \epsilon_k - \epsilon,$$

where in the second line we have used the triangle inequality. \qed
Note that in the case of pure states the conjecture is directly implied by Fannes inequality, lemma 13, since $S(AB) = 0$ and $S(A) = S(B)$. Clearly, a proof of the general case cannot proceed along these lines as do not have the possibility to present the conditional von Neumann entropy as an average of entropies on $A$.

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