Improving Sample Complexity Bounds for Actor-Critic Algorithms

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Abstract

The actor-critic (AC) algorithm is a popular method to find an optimal policy in reinforcement learning. The finite-sample convergence rate for the AC and natural actor-critic (NAC) algorithms has been established recently, but under independent and identically distributed (i.i.d.) sampling and single-sample update at each iteration. In contrast, this paper characterizes the convergence rate and sample complexity of AC and NAC under Markovian sampling, with mini-batch data for each iteration, and with actor having general policy class approximation. We show that the overall sample complexity for a mini-batch AC to attain an $\epsilon$-accurate stationary point improves the best known sample complexity of AC by an order of $O\left(\frac{1}{\epsilon} \log\left(\frac{1}{\epsilon}\right)\right)$. We also show that the overall sample complexity for a mini-batch NAC to attain an $\epsilon$-accurate globally optimal point improves the known sample complexity of natural policy gradient (NPG) by $O\left(\frac{\epsilon}{\log\left(\frac{1}{\epsilon}\right)}\right)$. Our study develops several novel techniques for finite-sample analysis of RL algorithms including handling the bias error due to mini-batch Markovian sampling and exploiting the self variance reduction property to improve the convergence analysis of NAC.

1 Introduction

The goal of reinforcement learning (RL) (Sutton and Barto, 2018) is to maximize the expected total reward by taking actions according to a policy in a stochastic environment, which is modelled as a Markov decision process (MDP) (Bellman, 1957). To obtain an optimal policy, one popular method is the direct maximization of the expected total reward via gradient ascent, which is referred to as the policy gradient (PG) method (Sutton et al., 2000; Williams, 1992). In practice, PG methods often suffer from large variance caused by the Monte Carlo rollouts to acquire the value function in the policy gradient, which substantially degrades the convergence speed. To address such an issue, the actor-critic (AC) type of algorithms have been proposed (Konda and Borkar, 1999; Konda and Tsitsiklis, 2000), in which critic tracks the value function and actor updates the policy using the return of critic. The usage of critic significantly reduces the variance of the policy update and speeds up the convergence.

The first AC algorithm was proposed by (Konda and Tsitsiklis, 2000), in which actor’s updates adopt the simple stochastic policy gradient descent step. This algorithm was later extended to the natural actor-critic (NAC) algorithm in (Peters and Schaal, 2008), in which actor’s updates adopt the natural policy gradient (NPG) algorithm (Kakade, 2002). The asymptotic convergence of AC and NAC algorithms under both i.i.d. sampling and Markovian sampling have been established in (Kakade, 2002; Konda, 2002; Bhatnagar, 2010; Bhatnagar et al., 2009, 2008). The convergence rate (i.e., the finite-sample analysis) of AC and NAC has recently been studied. More specifically, (Yang et al., 2019) studied the sample complexity of AC with linear function approximation in the linear quadratic regulator (LQR) problem. (Wang et al., 2019) studied AC and NAC in a more general MDP setting, in which both actor and critic utilize overparameterized neural networks as approximation functions. (Kumar et al., 2019) studied AC with general policy class and linear function approximation for critic, but with the requirement that the true value function is in the linear
Table 1: Comparison of sample complexity of AC and NAC algorithms

| Algorithm | Reference | Sampling | Total complexity\(^1\,2\) | Accuracy\(^3\) |
|-----------|-----------|----------|-----------------------------|---------------|
| AC        | (Wang et al., 2019) | i.i.d. | \(O(\epsilon^{-4})\) | \(\epsilon + \zeta_{nn}\) |
|           | (Kumar et al., 2019) | i.i.d. | \(O(\epsilon^{-4})\) | \(\epsilon + \zeta_{approx}\) |
|           | (Qiu et al., 2019) | i.i.d. | \(O(\epsilon^{-3} \log^2(\epsilon^{-1}))\) | \(\epsilon + \zeta_{approx}\) |
|           | This paper | Markovian | \(O(\epsilon^{-2} \log(\epsilon^{-1}))\) | \(\epsilon + O(\lambda^2)\) |
| NAC       | (Wang et al., 2019) | i.i.d. | \(O(\epsilon^{-4})\) | \(\epsilon + \zeta_{nn}'\) |
|           | (Agarwal et al., 2019) | i.i.d. | \(O(\epsilon^{-4})\) | \(\epsilon + O(\sqrt{\zeta_{approx}})\) |
|           | This paper | Markovian | \(O(\epsilon^{-3} \log(\epsilon^{-1}))\) | \(\epsilon + O(\sqrt{\zeta_{approx}})\) |

\(^1\)Total complexity of AC is measured to attain an \((\epsilon + \text{error})\)-accurate stationary point \(\tilde{w}\), i.e., \(\|\nabla_w J(\tilde{w})\|_2 < \epsilon + \text{error}\). Total complexity of NAC is measured to attain an \((\epsilon + \text{error})\)-accurate global optimum \(\bar{w}\), i.e., \(J(\pi^*) - J(\bar{w}) < \epsilon + \text{error}\).

\(^2\)Total complexity does not include the dependence on the order of \(\frac{1}{1-\gamma}\), because some existing studies did not explicitly capture such dependence, and it is difficult to make a fair comparison among all exist studies with respect to the dependence on the order of \(\frac{1}{1-\gamma}\). Our complexity results do capture such dependence as specified in our theorems.

\(^3\)The parameter \(\lambda\) is a regularization constant, and can be chosen to be arbitrarily small. The errors \(0 < \zeta_{approx}, \zeta_{nn} < \infty\) and \(0 < \zeta_{approx}', \zeta_{nn}' < \infty\) are nonvanishing approximation errors determined by the expressive power of the base function class of critic and the policy class of actor, respectively. Note that (Wang et al., 2019) proves that \(\zeta_{nn}\) and \(\zeta_{nn}'\) converge to zero at a sublinear rate as the width of the neural network increases, and \(\zeta_{approx} = 0\) in (Kumar et al., 2019) is due to their assumption that the true value function is in the base function class of critic.

Although having progressed significantly, existing finite-sample analysis of AC and NAC have several limitations. First, they all assume that actor has access to the visitation distribution (or stationary distribution) to generate independent and identically distributed (i.i.d.) samples, which can hardly be satisfied in practice. Second, some studies focused only on the specific LQR setting, specific function classes for policy and value function approximation, etc. Third, existing studies focused on the case with only one single sample utilized for each actor and critic update, which may not yield stable and overall sample-efficient algorithms.

Thus, our goal in this paper is to characterize the convergence rate (i.e., sample complexity) of AC and NAC under Markovian sampling for both actor and critic, with general policy class approximation for actor, and with the more general mini-batch sampling for each actor and critic update. Our results for both AC and NAC orderwisely improve the overall sample complexity established in existing studies.
1.1 Main Contributions

We summarize our results and their comparison with the existing results in Table 1. More specifically, we characterize the convergence rate and sample complexity of AC and NAC under Markovian sampling, with actor having general policy class approximation and with mini-batch data for each iteration. We show that the overall sample complexity for a mini-batch AC to attain an $\epsilon$-accurate stationary point improves the best known sample complexity of AC (Qiu et al., 2019) by an order of $O\left(\frac{1}{\epsilon} \log\left(\frac{1}{\epsilon}\right)\right)$. We also show that the overall sample complexity for a mini-batch NAC to attain an $\epsilon$-accurate globally optimal point improves the known sample complexity of natural policy gradient (NPG) (Agarwal et al., 2019) by $O\left(\frac{1}{\epsilon} / \log\left(\frac{1}{\epsilon}\right)\right)$.

Our analysis relies on the following new technical developments. To obtain the convergence rate for critic, we develop a new technique to handle the bias error caused by mini-batch Markovian sampling in the linear stochastic approximation (SA) setting, which is different in nature from how existing studies handle single-sample bias (Bhandari et al., 2018). Our result shows that Markovian mini-batch linear SA outperforms single-sample linear SA in terms of the total computational complexity by a factor of $\log\left(\frac{1}{\epsilon}\right)$, which can be of independent interest. For AC, we develop a new technique to bound the bias error caused by the mini-batch Markovian sampling in the nonlinear SA setting for actor’s update, which is different from the bias error of linear SA in critic’s update. Thus, we improve the total sample complexity by a factor of $O\left(\frac{1}{\epsilon}\right)$ due to mini-batch Markovian sampling used in actor’s update. For NAC, we improve the analysis of the sample complexity in (Agarwal et al., 2019) by leveraging the self-reduced property of the variance error, which enables the usage of the constant stepsize to reduce actor’s iterations and the overall sample complexity.

1.2 Related Work

We include here only theoretical studies of AC and NAC that are highly related to our work. We also discuss theoretical studies on policy gradient (PG) and linear SA, which are relevant to our study over a broader scope.

AC and NAC. The first AC algorithm was proposed by (Konda and Tsitsiklis, 2000) and was later extended to NAC in (Peters and Schaal, 2008) using NPG (Kakade, 2002). The asymptotic convergence of AC and NAC algorithms under both i.i.d. sampling and Markovian sampling have been established in (Kakade, 2002; Konda, 2002; Bhatnagar, 2010; Bhatnagar et al., 2009, 2008). More recently, the convergence rate (i.e., the finite-sample rate) of AC and NAC has been studied respectively in (Wang et al., 2019; Yang et al., 2019; Kumar et al., 2019; Qiu et al., 2019) and in (Wang et al., 2019; Agarwal et al., 2019). In contrast to the above studies of AC and NAC under i.i.d. sampling and with single sample for each iteration, our study focuses on Markovian sampling and mini-batch data for each iteration. Furthermore, we assume that actor uses general policy class approximation and hence our result is also applicable to the settings with specific policy classes such as neural networks.

Policy gradient. The asymptotic convergence of PG in both the finite and infinite horizon scenarios has been established in (Williams, 1992; Baxter and Bartlett, 2001; Sutton et al., 2000; Kakade, 2002; Pirotta et al., 2015; Tadić et al., 2017; Agarwal et al., 2019). In some special RL problems such as LQR, under tabular policy, or with convex policy function approximation, PG has been shown to converge to the global optimum (Fazel et al., 2018; Malik et al., 2018; Tu and Recht, 2018; Bhandari and Russo, 2019). With general nonconcave/nonconvex function approximation, (Shen et al., 2019; Papini et al., 2018, 2017; Xu et al., 2019a, 2020a) established the convergence rate (or sample complexity) of PG and variance reduced PG for finite-horizon scenarios. For the infinite-horizon scenario and under Markovian sampling, (Karimi et al., 2019) showed that PG converges to a neighbourhood of a first-order stationary point and (Zhang et al., 2019a)
modified the algorithm so that PG is guaranteed to converge to a second-order stationary point. The convergence of more advanced PG algorithms such as TRPO/PPO has been studied in (Shani et al., 2019) for the tabular case and in (Liu et al., 2019) with the neural network function approximation. Our paper here focuses on actor-critic algorithms, where the updates of critic significantly affect the overall performance of actor that runs PG. Hence, our analysis is very different from the above studies.

**Linear SA and TD learning.** The convergence analysis of critic in AC and NAC in this paper is related to but different from the studies on on-policy TD learning, which we briefly summarize as follows. For TD learning under i.i.d. sampling (which can be modeled as linear SA with martingale noise), the asymptotic convergence has been well established in (Borkar and Meyn, 2000; Borkar, 2009), and the non-asymptotic convergence (i.e., finite-time analysis) has been provided in (Dalal et al., 2018; Kamal, 2010; Thoppe and Borkar, 2019). For TD learning under Markovian sampling (which can be modeled as linear SA with Markovian noise), the asymptotic convergence has been established in (Tsitsiklis and Van Roy, 1997; Tadić, 2001), and the non-asymptotic analysis has been provided in (Bhandari et al., 2018; Xu et al., 2020b; Srikant and Ying, 2019; Hu and Syed, 2019). Furthermore, (Zou et al., 2019) studied the SARSA algorithm under non-i.i.d. samples, which can be seen as a special case of linear SA with noise generated by dynamically changing transition kernel. In our study here, the recursion of critic follows linear SA up-algorithm under non-i.i.d. samples, which can be seen as a special case of linear SA with noise generated by dynamically changing transition kernel. In our study here, the recursion of critic follows linear SA up-

tates with mini-batch Markovian sampling, whose finite-time performance has not been studied in previous works.

## 2 Problem Formulation and Preliminaries

In this section, we introduce the background about Markov decision process (MDP), and AC and NAC algorithms. We also provide technical assumptions for our analysis.

### 2.1 Markov Decision Process

A discounted Markov decision process (MDP) is defined by a tuple \((\mathcal{S}, \mathcal{A}, P, r, \xi, \gamma)\), where \(\mathcal{S}\) and \(\mathcal{A}\) are the state and action spaces, \(P\) is the transition kernel, and \(r\) is the reward function. Specifically, at step \(t\), an agent takes an action \(a_t \in \mathcal{A}\) at state \(s_t \in \mathcal{S}\), transits into the next state \(s_{t+1} \in \mathcal{S}\) according to the transition probability \(P(s_{t+1}|s_t, a_t)\) and receives a reward \(r(s_t, a_t, s_{t+1})\). Moreover, \(\xi\) denotes the distribution of the initial state \(s_0 \in \mathcal{S}\) and \(\gamma \in (0, 1)\) denotes the discount factor. A policy \(\pi\) maps a state \(s \in \mathcal{S}\) to the actions in \(\mathcal{A}\) via a probability distribution \(\pi(\cdot|s)\).

For a given policy \(\pi\), we define the state value function as \(V_\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t, s_{t+1})|s_0 = s, \pi\right]\) and the state-action value function as \(Q_\pi(s, a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t, s_{t+1})|s_0 = s, a_0 = a, \pi\right]\), where \(a_t \sim \pi(\cdot|s_t)\) for all \(t \geq 0\). We also define the advantage function of the policy \(\pi\) as \(A_\pi(s, a) = Q_\pi(s, a) - V_\pi(s)\). Moreover, the state visitation measure induced by the policy \(\pi\) is defined as \(\nu_\pi(s) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = s)\) and the corresponding state-action visitation measure is defined as \(\nu_\pi(s, a) = \nu_\pi(s)\pi(a|s)\). It has been shown in (Konda, 2002) that the stationary distribution of a Markov chain with the transition kernel \(\mathbb{P}(\cdot|s, a) = \gamma \mathbb{P}(\cdot|s, a) + (1 - \gamma) \xi(\cdot)\) and the policy \(\pi\) is \(\nu_\pi(s, a)\) if the Markov chain is ergodic.

For a given policy \(\pi\), we define the expected total reward function as

\[
J(\pi) = (1 - \gamma) \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t, s_{t+1})\right] = \mathbb{E}_\xi[V_\pi(s)].
\]

The goal of reinforcement learning is to find an optimal policy \(\pi^*\) that maximizes \(J(\pi)\).
2.2 Parameterization of Policy and Value Function

In order to find the optimal policy \( \pi^* \) that maximizes \( J(\pi) \), a popular approach is to parameterize the policy and then optimize over the set of parameters.

**Parameterization of policy:** We let the policy \( \pi \) be parameterized by \( w \in \mathcal{W} \subset \mathbb{R}^d \), where the parameter space \( \mathcal{W} \) is the Euclidean space. Thus, the parameterized policy class we consider is \( \{ \pi_w : w \in \mathcal{W} \} \). Note that we allow general nonlinear parameterization of the policy \( \pi \).

Thus, the policy optimization problem is to solve the following optimization problem:

\[
\max_{w \in \mathcal{W}} J(\pi_w) := J(w),
\]

where we write \( J(\pi_w) = J(w) \) for notational simplicity. In order to solve the problem eq. (2) by gradient based approaches, the gradient \( \nabla J(w) \) is derived by (Sutton et al., 2000) as follows:

\[
\nabla J(w) = \mathbb{E}_{\nu_{\pi_w}} [A_{\pi_w}(s,a) \phi_w(s,a)],
\]

where \( \phi_w(s,a) := \nabla_w \log \pi_w(a|s) \) is called the score function, and we recall that \( A_{\pi_w}(s,a) \) denotes the advantage function.

**Parameterization of advantage function:** In the actor-critic algorithms to be introduced in Section 2.3, the advantage function \( A_{\pi_w}(s,a) \) is estimated via critic’s update using a function class \( A_{\theta}(s,a) \) parameterized by \( \theta \). Here, we approximate the advantage function \( A_{\pi_w}(s,a) \) by a linear function class with base function \( \phi(s,a) \), i.e., \( A_{\theta}(s,a) = \phi(s,a)^T \theta \). It has been shown in (Sutton et al., 2000; Konda and Borkar, 1999; Konda and Tsitsiklis, 2000) that such function approximation can replace the true advantage function \( A_{\pi_w}(s,a) \) in eq. (3) without changing the value of the gradient \( \nabla J(w) \), i.e., \( \nabla J(w) = \mathbb{E}_{\nu_{\pi_w}} [A_{\theta}(s,a) \phi_w(s,a)] \), if \( \phi(s,a) \) and \( \theta \) satisfy the following compatible conditions:

\[
\phi(s,a) = \phi_w(s,a) \quad \text{for all} \quad (s,a) \in \mathcal{S} \times \mathcal{A},
\]

and

\[
\theta \in \arg\min_{\theta \in \mathbb{R}^d} L_w(\theta) = \mathbb{E}_{\nu_{\pi_w}} [A_{\pi_w}(s,a) - \phi(s,a)^T \theta]^2.
\]

2.3 Actor-Critic and Natural Actor-Critic Algorithms

We now introduce the AC and NAC algorithms studied in this paper. As explained in Section 2.1, the transition kernel adopts \( P(\cdot|s,a) = \gamma P(\cdot|s,a) + (1 - \gamma) \xi(\cdot) \).

In this paper, we study the AC algorithm that adopts the design proposed in (Konda, 2002) (see Algorithm 1). The algorithm updates in a nested fashion. Namely, the outer loop consists of actor’s update of the parameter \( w \) to optimize the policy \( \pi_w \), and each outer-loop update is followed by an entire inner loop of critic’s \( T_c \) updates of the parameter \( \theta \) to estimate the advantage function \( A_{\theta}(s,a) \). More specifically, at step \( t \), actor updates the parameter \( w_t \) of policy \( \pi_w \) using a stochastic version of the policy gradient in eq. (3) given by

\[
w_{t+1} = w_t + \alpha_t \nabla J(w_t, \theta_t).
\]
where $\nabla \hat{J}(w_t, \theta_t) = A_{\theta_t}(s_t, a_t)\phi_{w_t}(s_t, a_t)$ and $\alpha_t > 0$ is the stepsize. Note that if $\theta_t$ is a good estimate of the minimum norm solution of eq. (4), then $\nabla \hat{J}(w_t, \theta_t)$ is a good stochastic approximation of $\nabla J(w_t)$ if we let $\phi(s, a) = \phi_{w_t}(s, a)$. This is guaranteed by critic’s update in the inner loop as we introduce below.

Critic’s update follows the mini-batch SA algorithm (see Algorithm 2), which adopts Q-Sampling$(s, a, \pi)$ (see Algorithm 3) proposed by (Zhang et al., 2019a) with an unbiased estimator of the state-action value function $Q_\pi(s, a)$ as the output. Critic’s update of the parameter $\theta$ is to find the solution for eq. (4). Clearly, the objective function $L_w(\theta)$ is quadratic, and can be minimized efficiently using the stochastic gradient descent (SGD) algorithm. We first give the full gradient as follows:

$-\nabla_\theta L_w(\theta) = \mathbb{E}_{\nu_w}[ -\phi_w(s, a)\phi_w(s, a)^\top \theta ] + \mathbb{E}_{\nu_w} [A_{\pi_w}(s, a)\phi_w(s, a)].$

To estimate the gradient $-\nabla_\theta L_w(\theta)$, note that

$$\mathbb{E}_{(s, a)\sim \nu_w} [A_{\pi_w}(s, a)\phi_w(s, a)] = \mathbb{E}_{(s, a)\sim \nu_w} [Q_{\pi_w}(s, a)\phi_w(s, a)] - \mathbb{E}_{(s, a')\sim \nu_w} [V_{\pi_w}(s)\phi_w(s, a')]$$

$$= \mathbb{E}_{(s, a)\sim \nu_w} [Q_{\pi_w}(s, a)\phi_w(s, a)] - \mathbb{E}_{(s, a')\sim \nu_w} [\mathbb{E}[Q_{\pi_w}(s, a)|s, a']\phi_w(s, a')].$$

Thus, at time $t$, with parameters $w_t$ and $\theta_t$, we can sample the MDP to estimate the gradient $-\nabla_\theta L_{w_t}(\theta_t)$ as

$$\hat{g}_i(\theta_t) = (-\phi_{w_t}(s_i, a_i)^\top \theta_t + \hat{Q}_{\pi_{w_t}}(s_i, a_i))\phi_{w_t}(s_i, a_i) - \hat{Q}_{\pi_{w_t}}(s_i, a_i)\phi_{w_t}(s_i, a_i') - \lambda \theta_t,$$

where $a_i$ and $a_i'$ are sampled independently from the policy $\pi_{w_t}(\cdot|s_i)$, and the regularization term $\lambda \theta_t$ is added here to prevent the divergence of critic’s iteration (namely, to prevent the matrix of the corresponding ordinary differential equation (ODE) of linear SA from being singular). Such a stability condition is typically required for the analysis of linear SA in the literature (Dalal et al., 2018; Bhandari et al., 2018). We further note that the base function $\phi_{w_t}$ of critic, which is also the score function of policy $\pi_{w_t}$, changes as actor updates, so that the compatible condition (Sutton et al., 2000; Konda and Borkar, 1999) can always be satisfied.

Alternatively, natural actor-critic (NAC) (see Algorithm 1) (Bhatnagar et al., 2009; Agarwal et al., 2019) utilizes natural gradient ascent (Amari, 1998; Kakade, 2002), which guarantees that the policy update is
we study here from the AC and NAC algorithms studied recently.

**Algorithm 2** Mini-batch SA($\pi_w, \beta, T_c, M$)

Initialize: critic parameter $\theta_0$

for $i = 0, \cdots, BT_c$ do
  $s_i \sim P(\cdot | s_{i-1}, a_{i-1})$
  Sample $a_i$ and $a_i'$ independently from $\pi_w(\cdot | s_i)$
end for

for $k = 0, \cdots, T_c - 1$ do
  for $i = kM, \cdots, (k+1)M - 1$ do
    $\hat{Q}(s_i, a_i) = Q$-Sampling$(s_i, a_i, \pi_w)$
    $\hat{g}_i(\theta_k) = (-\phi_w(s_i, a_i)\top\theta_k + \hat{Q}(s_i, a_i))\phi_w(s_i, a_i) - \hat{Q}(s_i, a_i)\phi_w(s_i, a_i') - \lambda\theta_k$
  end for
  $g_k(\theta_k) = \frac{1}{M} \sum_{i=kM}^{(k+1)M-1} \hat{g}_i(\theta_k)$
  $\theta_{k+1} = \theta_k + \beta g_k(\theta_k)$
end for
Output: $\theta_{T_c}$

**Algorithm 3** Q-Sampling$(s, a, \pi)$

Initialize: $\hat{Q}_\pi(s, a) = 0$, $s_0 = s$ and $a_0 = a$
$T \sim \text{Geom}(1 - \gamma^{1/2})$

for $t = 0, \cdots, T - 1$ do
  $s_{t+1} \sim P(\cdot | s_t, a_t)$
  $\hat{Q}_\pi(s, a) \leftarrow \hat{Q}_\pi(s, a) + \gamma t/2 r(s_t, a_t, s_{t+1})$
  $a_{t+1} \sim \pi(\cdot | s_{t+1})$
end for
Output: $\hat{Q}_\pi(s, a)$

invariant to the parameterization of the policy. Similarly to AC, NAC here also adopts the nested loop update with actor’s one update in the outer loop followed by an entire inner loop of critic’s update. At each step $t$, ideally actor’s update should be given by

$$w_{t+1} = w_t + \alpha_t F(w_t)\top \nabla J(w_t), \tag{5}$$

where $F(w_t)$ is the Fisher information matrix defined as $F(w_t) := \mathbb{E}_{\nu_{\pi_{w_t}}} \left[ \phi_{w_t}(s, a) \phi_{w_t}(s, a)^\top \right]$, and $F(w_t)\top$ represents the pseudoinverse of $F(w_t)$. However, it is intractable to perform the update in eq. (5) exactly as the visitation distribution $\nu_{\pi_{w_t}}$ is usually unknown. It can be checked easily that the minimum norm solution of the problem in eq. (4) satisfies $\theta_{w_t}^0 = F(w_t)\top \nabla J(w_t)$. Thus, a practical way to perform the NAC is to let critic solve eq. (4) approximately such that $\theta_t \approx \theta_{w_t}^0$ (Agarwal et al., 2019), and then the actor updates the parameter of the policy as

$$w_{t+1} = w_t + \alpha_t \theta_t.$$

**Differences from previously studied AC and NAC algorithms:** In the following, we summarize the major differences of AC and NAC in Algorithm 1 we study here from the AC and NAC algorithms studied recently in (Wang et al., 2019; Qiu et al., 2019; Kumar et al., 2019; Agarwal et al., 2019). We also comment on the new technical challenges that these differences introduce to the analysis and the impact of these different designs on the improvement of the sample complexity characterized in this paper.
Consider the critic’s parameterization for value function approximation. (Wang et al., 2019) utilizes a class of overparametrized neural network, and (Qiu et al., 2019; Kumar et al., 2019) utilize a linear function class with fixed base functions. Since the compatible conditions cannot be strictly satisfied in those cases, the parameterization of critic introduces a non-vanishing error in actor’s update. In contrast, our AC and NAC adopt the time-varying base functions for critic’s parameterization so that the compatibility conditions can be precisely satisfied at each iteration. Hence, critic does not introduce an additional error.

In Algorithm 1, both actor and critic utilize non-i.i.d. samples sequentially generated by the transition kernel of the MDP, whereas the previous studies of AC and NAC require the accessibility of i.i.d. samples from the visitation distribution. Such a difference introduces new technical challenges to bound the bias error in our analysis. Specifically, the transition kernel of the sampling MDP changes as the policy πwτ updates, and thus we need to handle the bias error caused by the time-varying Markovian sampling of data in both the updates of actor and critic.

In Algorithm 1, both actor and critic utilize a mini-batch of samples for each update, whereas in the previous studies of AC and NAC, each update of actor and critic uses only one sample. Correspondingly, we develop a new technique to handle the mini-batch bias error caused by the Markovian mini-batch sampling in both the linear SA (i.e., critic’s update) and nonlinear SA (i.e., actor’s update) settings, which takes a different path from the previous analysis of single-sample updates (Bhandari et al., 2018; Zou et al., 2019; Xu et al., 2019b). Due to the mini-batch update, the sample complexity of critic’s update in AC and NAC is reduced by a factor of $O(\log(\frac{1}{\epsilon}))$, and the number of actor’s iterations in AC is reduced by a factor of $O(\frac{1}{\tau})$, both reducing the overall sample complexity.

### 2.4 Technical Assumptions

We take the following standard assumptions throughout the paper.

**Assumption 1.** For any $w, w' \in \mathbb{R}^d$ and any state-action pair $(s, a) \in S \times A$, there exist positive constants $L_\phi, C_\phi$, and $C_\pi$ such that the following hold:

1. $\|\phi_w(s, a) - \phi_{w'}(s, a)\|_2 \leq L_\phi \|w - w'\|_2$,
2. $\|\phi_w(s, a)\|_2 \leq C_\phi$,
3. $\|\pi_w(\cdot|s) - \pi_{w'}(\cdot|s)\|_{TV} \leq C_\pi \|w - w'\|_2$, where $\|\cdot\|_{TV}$ denotes the total-variation norm.

The first two items in Assumption 1 assume that the score function $\phi_w$ is smooth and bounded, which have also been adopted in previous studies (Kumar et al., 2019; Zhang et al., 2019a; Agarwal et al., 2019; Konda, 2002; Zou et al., 2019). The first two items can be satisfied by many commonly used policy classes including some canonical policies such as Boltzmann policy (Konda and Borkar, 1999) and Gaussian policy (Doya, 2000). The third item in Assumption 1 holds for any smooth policy with bounded action space or Gaussian policy. Lemma 1 in Appendix A provides such justifications.

**Assumption 2** (Ergodicity). Consider the MDP with policy $\pi_w$ and transition kernel $P(\cdot|s, a)$ or $\tilde{P}(\cdot|s, a) = \gamma P(\cdot|s, a) + (1 - \gamma)\eta(\cdot)$, where $\eta(\cdot)$ can either be $\xi(\cdot)$ or $P(\cdot|s, \hat{a})$ for any given $(s, \hat{a}) \in S \times A$. There exist constants $\kappa > 0$ and $\rho \in (0, 1)$ such that

$$\sup_{s \in S} \|P(s_t \in \cdot|s_0 = s) - \chi_{\pi_w}\|_{TV} \leq \kappa \rho^t, \quad \forall t \geq 0,$$

where $\chi_{\pi_w}$ is the stationary distribution of the corresponding MDP with transition kernel $P(\cdot|s, a)$ or $\tilde{P}(\cdot|s, a)$ under policy $\pi_w$. 


Assumption 2 has also been adopted in the previous studies (Bhandari et al., 2018; Xu et al., 2020b; Zou et al., 2019), which holds for any time-homogeneous Markov chain with finite state space or any uniformly ergodic Markov chain with general state space.

3 Main Results

In this section, we first analyze the convergence of critic’s update as a mini-batch linear SA algorithm. Based on such an analysis, we further provide the convergence rate for the AC and NAC algorithms.

3.1 Convergence Analysis of Mini-batch SA

In this section, we analyze critic’s update, which adopts the mini-batch policy evaluation algorithm described in Algorithm 2 and can be viewed more generally as a mini-batch linear SA algorithm.

It turns out that the analysis of mini-batch Markovian linear SA is very different from that of single-sample Markovian linear SA previously studied in (Bhandari et al., 2018). This is because samples in the same Markovian mini-batch are correlated with each other, which introduces bias error for each iteration itself in addition to the bias error across iterations. Existing techniques such as in (Bhandari et al., 2018) provide only ways to handle the correlation across iterations, but not the bias error for each iteration caused by a mini-batch Markovian data. Here, we develop a new analysis to handle such a bias error. Specifically, we show that such a bias error can be divided into two parts, in which the first part diminishes as the algorithm approaches to the fix point and the second part, which is caused by the correlation among samples in the same mini-batch, is averaged out as the batch size \( M \) increases. Hence, the bias error of mini-batch SA can be controlled by the mini-batch size. This is in contrast to the dependence of the bias error on the stepsize in the existing analysis of single-sample linear SA. Our result below further indicates that mini-batch linear SA orderwisely reduces the overall sample complexity of the single-sample linear SA studied before, which can be of independent interest.

To present the convergence result, for any \( w \in \mathbb{R}^d \), we define

\[
\mathcal{T}_w^\lambda = \mathbb{E}_{\nu \pi w} \left[ \phi_w(s,a)\phi_w(s,a)^\top \right] + \lambda I, \quad \bar{b}_w = \mathbb{E}_{\nu \pi w} \left[ A_{\pi w}(s,a)\phi_w(s,a) \right],
\]

and \( \theta^*_w = (\mathcal{T}_w^\lambda)^{-1}\bar{b}_w \). We further let \( C_P = C_\phi^2 + \lambda, C_b = 2r_{\max}C_\phi/(1-\gamma), \lambda_P = \min_{w \in \mathbb{R}^d} \lambda_{\min}(\mathcal{T}_w^\lambda) \), and suppose \( R_\theta \) satisfies \( \max_{w \in \mathbb{R}^d} \{ \theta^*_w \} \leq R_\theta \leq C_b/\lambda_P \). We also note that it can be shown that the expected number of samples utilized by Q-Sampling \( (s,a,\pi) \) at each iteration is less than \( N_Q = \frac{2}{1-\gamma} \) (Agarwal et al., 2019), which is useful for computing the sample complexity. Then the following theorem provides the sample complexity guarantee for mini-batch linear SA in Algorithm 2.

**Theorem 1.** Suppose Assumptions 1 and 2 hold. Consider Algorithm 2 of mini-batch linear SA with a given policy \( \pi_w \). Let the stepsize \( \beta = \frac{\lambda_P}{8C_P^2}, \) and the mini-batch size \( M \geq \max \left\{ C_P^2, \frac{2(C_\phi^2 + C_b^2+R_\theta^2)}{\epsilon} \right\} \left( \frac{2}{\lambda_P} + \frac{\lambda_P}{4C_P^2} \right) \left( \frac{192(1+(\kappa-1)\rho)}{1-\rho}\lambda_P \right) \right) \) and \( T \geq \frac{128C_P^2}{\lambda_P^2} \log \frac{2||\theta_0 - \theta^*_w||^2}{\epsilon} \), we have

\[
\mathbb{E}[||\theta_T - \theta^*_w||^2] \leq \epsilon.
\]

The expected total sample complexity is given by

\[
MTcN_Q = O \left( \frac{1}{(1-\gamma)^3} \log \left( \frac{1}{\min\{\epsilon, 1-\gamma\}} \right) \right).
\]
The proof of Theorem 1 is provided in Appendix D, where we provide a proof for the general mini-batch linear SA algorithm with Markovian update (see Theorem 4) that includes Algorithm 2 of mini-batch linear SA studied in Theorem 1 as a special case. Such a theorem is also applicable to many commonly used RL algorithms such as TD (Sutton, 1988), GTD and TDC (Maei, 2011) under mini-batch Markovian sampling.

Theorem 1 (or more generally Theorem 4) indicates that the sample complexity of mini-batch linear SA under Markovian sampling is $O(\epsilon^{-1} \log(\epsilon^{-1}))$, which outperforms single-sample linear SA (Bhandari et al., 2018) by a factor of $O(\log(\epsilon^{-1}))$. The use of mini-batch for each update is crucial for the improvement of the sample complexity, due to which the bias error is kept at the same level as the variance error (with respect to the mini-batch size), and hence does not cause order-level increase in the total sample complexity.

### 3.2 Convergence Analysis of AC

In order to analyze the AC algorithm, we first provide a property for $J(w)$ in general, in which we tighten the previous result. 

**Proposition 1.** Suppose Assumptions 1 and 2 hold. For any $w, w' \in \mathbb{R}^d$, we have

$$
\|\nabla w J(w) - \nabla w J(w')\|_2 \leq L_J \|w - w'\|_2,
$$

for all $w, w' \in \mathbb{R}^d$, where $L_J = \max(4C_\rho C_\phi + L_\phi)$ and $C_\nu = \frac{1}{2}C_\pi \left(1 + \lceil \log_\rho \kappa^{-1}\rceil \right)$. 

Proposition 1 has been given as the Lipschitz assumption in the previous studies of policy gradient and AC (Kumar et al., 2019; Qiu et al., 2019; Wang et al., 2019), whereas we provide a proof as a formal justification for it to hold. Furthermore, the same type of property has also been shown in (Zhang et al., 2019a), where the Lipschitz constant scales as $O((1 - \gamma)^{-3})$, whereas our result has a better scaling of $O((1 - \gamma)^{-1})$ and is therefore tighter.

Since the objective function $J(w)$ in eq. (2) is nonconcave in general, the convergence analysis of AC is with respect to the standard metric of $\mathbb{E} \|\nabla w J(w)\|_2$. Proposition 1 is thus crucial for such analysis. The next theorem provides the convergence guarantee of Algorithm 1.

**Theorem 2.** Consider the AC algorithm in Algorithm 1. Suppose Assumptions 1 and 2 hold, and let the stepsize $\alpha = \frac{1}{4L_J^2}$. Then we have

$$
\mathbb{E}[\|\nabla w J(w_T)\|_2^2] \leq \frac{16L_J [\mathbb{E}[J(w_T)] - J(w_0)]}{T} + 9C_\phi^4 \sum_{t=0}^{T-1} \mathbb{E}[\|\theta_t - \theta_{w_t}\|_2^2] \\
+ \frac{36R_\phi^2 C_\phi^4 [1 + (\kappa - 1)\rho]}{(1 - \rho)} \frac{1}{B} + 9C_\phi^4 C_\rho^2 \lambda^2,
$$

where $C_\rho$ (with its specific form given in Appendix B) is a positive constant depending on the policy $\pi_w$, and $\hat{T}$ is chosen uniformly from $\{1, \ldots, T\}$ as specified in Algorithm 1.

Furthermore, let $B \geq \frac{216L_J^2 C_\phi^4 (1 + (\kappa - 1)\rho)}{(1 - \rho^2)}$ and $T \geq \frac{48L_J \max(1 - \gamma)}{(1 - \gamma)\epsilon}$. Suppose the same setting of Theorem 1 holds (with $M$ and $T_c$ are defined therein) so that $\mathbb{E}[\|\theta_t - \theta^*\|_2^2] \leq \frac{\epsilon}{2\mathbb{E}^2}$ for all $0 \leq t \leq T - 1$. Then we have

$$
\mathbb{E}[\|\nabla w J(w_T)\|_2^2] \leq \epsilon + \mathcal{O}(\lambda^2),
$$

with the expected total sample complexity given by

$$(B + MT_c N_Q)T = \mathcal{O}\left(\frac{1}{(1 - \gamma)^2 \epsilon^2} \log\left(\frac{1}{\min\{\epsilon, 1 - \gamma\}}\right)\right).$$
The sample complexity of mini-batch AC in Theorem 2 not only generalizes the previous studies (Kumar et al., 2019; Qiu et al., 2019) of single-sample AC under i.i.d. sampling to Markovian sampling, but also outperforms their sample complexity orderwisely (see Table 1). To explain where the improvement comes from, the mini-batch update plays two important roles here. First, note that the previous studies adopted diminishing stepsize (Kumar et al., 2019) or stepsize inversely proportional to the iteration (e.g., $\alpha = \frac{1}{\sqrt{t}}$ (Qiu et al., 2019; Wang et al., 2019)) for actor’s update, so that the number of outer-loop (actor’s) iterations is at least $O(\varepsilon^{-2})$. Whereas in Algorithm 1, the mini-batch data used for actor’s update allows a constant stepsize for outer-loop iterations, which reduces the number of outer-loop iterations from $O(\varepsilon^{-2})$ to $O(\varepsilon^{-1})$. Second, the mini-batch data keeps the bias error in actor’s iteration at the same level of dependence on the mini-batch size $M$ as the variance error. In this way, the mini-batch size is only required to be kept at the same order-level with the number $BT_c$ of critic’s iterations, and hence the bias error does not cause order-level increase in the overall sample complexity. Thus, mini-batch improves the sample efficiency of both actor and critic’s iteration, so that the overall sample complexity of Algorithm 1 improves the existing result by a factor of $O(\varepsilon^{-1} \log(\varepsilon^{-1}))$.

The proof of Theorem 2 develops a new analysis to handle the bias error for actor’s update (which is nonlinear SA) due to Markovian sampling. This is different from the bias error for critic’s update (which is linear SA) that we handle in Theorem 1.

### 3.3 Convergence Analysis of NAC

Our analysis of NAC is inspired by the analysis of natural policy gradient (NPG) in (Agarwal et al., 2019), but we provide an improved sample complexity due to an improved analysis of actor’s iterations.

Differently from AC algorithms, due to the parameter invariant property of the NPG update, the NAC algorithm can attain the globally optimal solution in terms of the function value convergence. As shown in (Agarwal et al., 2019), NPG is guaranteed to converge to a point $\pi_{w\star}$ in the neighborhood of the global optimal $\pi^*$, which satisfies $J(\pi^*) - E[J(\pi_{w\star})] \leq \varepsilon + O(\sqrt{\zeta_{\text{approx}}})$, where $\zeta_{\text{approx}}$ represents the approximation error of the compatible function class given by

$$
\zeta'_{\text{approx}} = \max_{w \in \mathbb{R}^d, \theta \in \mathbb{R}^d} \mathbb{E}_{\nu_{w\star}}[\phi_w(s, a)^\top \theta - A_{\pi_{w\star}}(s, a)]^2.
$$

It can be shown that $\zeta'_{\text{approx}}$ is zero or small if the express power of the policy class $\pi_w$ is large, e.g., tabular policy (Agarwal et al., 2019) and overparameterized neural policy (Wang et al., 2019).

We next provide the following theorem, which not only generalizes the convergence analysis of NPG in (Agarwal et al., 2019) (Corollary 6.10) from i.i.d. sampling to Markovian sampling, but also improves the sample complexity orderwisely. We denote $D(w) = E_{\nu_{\pi\star}}[\log \pi_{\pi\star}(a|s)]$ as the KL-distance between the policy $\pi_w$ and the globally optimal policy $\pi\star$.

**Theorem 3.** Consider the NAC algorithm in Algorithm 1. Suppose Assumptions 1 and 2 hold, and let the stepsize $\alpha = \frac{\lambda_P^2}{4L_P C_P}$. Then we have

$$
J(\pi^*) - E[J(\pi_{w\star})] \\
\leq \frac{D(w_0) - E[D(w_T)]}{T(1 - \gamma)\alpha} + \frac{4r_{\max} L_P C_P}{T(1 - \gamma)\lambda_P^2} + \frac{2L_P \alpha(C_P^2 + \lambda_P^2) \sum_{t=0}^{T-1} E[\|\theta_t - \theta_{w\star}\|^2_2]}{T} \\
+ \frac{C_\phi}{1 - \gamma} \sum_{t=0}^{T-1} E[\|\theta_t - \theta_{w\star}\|^2_2] + \sqrt{\frac{1}{(1 - \gamma)^3} \nu_{\pi_{w\star}} \nu_{\pi_{w\star}} \nu_{\pi_{w\star}} \nu_{\pi_{w\star}} \nu_{\pi_{w\star}} \nu_{\pi_{w\star}} \zeta'_{\text{approx}}} + \sqrt{\frac{C_\phi C_T}{1 - \gamma} \lambda},
$$

for $\zeta_{\text{approx}} = \max_{w \in \mathbb{R}^d, \theta \in \mathbb{R}^d} \mathbb{E}_{\nu_{w\star}}[\phi_w(s, a)^\top \theta - A_{\pi_{w\star}}(s, a)]^2$.
where $C_r$ (with its form specified in Appendix B) is a positive constant depending on the policy $π_w$, and $T$ is chosen uniformly from $\{1, \cdots, T\}$ as specified in Algorithm 1.

Furthermore, let $T ≥ \max \left\{ \frac{16L_eCD(\tilde{w}_0)}{\epsilon(1-γ)λ_P}, \frac{16L_eCP_{F\max}}{\epsilon(1-γ)^2λ_P^2} \right\}$ and $T = O(\sqrt{\zeta_{\text{approx}}})$. Suppose the same setting of Theorem 1 holds (with $M$ and $T_e$ defined therein) so that

$$\mathbb{E}[\|θ_t − θ^*_w\|^2_2] ≤ \max \left\{ \frac{(1-γ)^2}{16C^2}, \frac{L_J(1-γ)C_P}{L_φλ^2(2C^2 + λ^2 + λ^2)}\right\},$$

for all $0 ≤ t ≤ T − 1$. Then we have

$$J(π^*) − \mathbb{E}[J(π_{w,T})] ≤ \epsilon + O(\sqrt{ζ_{\text{approx}}}),$$

with the expected total sample complexity given by

$$MT_eN_QT = O\left( \frac{1}{(1-γ)^2} \log\left( \frac{1}{\epsilon^3} \right) \right).$$

Differently from AC in Theorem 3, in which the convergence is guaranteed only to attain a stationary point, NAC in Theorem 3 is guaranteed to achieve a globally optimal solution in terms of the function value (i.e., the accumulative reward value), indicating that NAC converges to a neighborhood of the globally optimal policy.

The sample complexity in Theorem 3 improves Corollary 6.1 in (Agarwal et al., 2019) by a factor of $O(\frac{1}{T}/\log(\frac{1}{T}))$, which is due to our improved analysis of actor’s convergence. Specifically, we first show that the NAC update in Algorithm 1 converges to a first-order stationary point (e.g., $\nabla_wJ(w_T) → 0$) under a constant stepsize. Hence, the variance term in actor’s update, which satisfies $\mathbb{E}[\|θ_t\|^2_2] ≈ \mathbb{E}[\|θ^*_t\|^2_2] ≥ \frac{1}{λ_P^2} \mathbb{E}[\|\nabla_wJ(w_t)\|^2_2]$, converges to zero as the gradient vanishes under a constant stepsize. As a result, the constant stepsize can be used to guarantee the convergence, which yields $O(ε^{-1})$ outer-loop (i.e., actor’s) iterations rather than $O(ε^{-2})$ in (Agarwal et al., 2019) under the stepsize ($\alpha = \frac{1}{\sqrt{T}}$).

4 Conclusion

In this paper, we provide the finite-sample analysis for mini-batch AC and NAC under Markovian sampling. In particular, we show that AC converges to a first-order stationary point, and NAC converges to a neighborhood of the globally optimal solution. We develop new techniques to analyze the bias error in the linear and nonlinear SA setting under Markovian mini-batch update. We show that the overall sample complexity of mini-batch AC outperforms the existing result of the single-sample AC by a factor of $O(\frac{1}{T}/\log(\frac{1}{T}))$. We also provide an improved analysis for the NAC by exploiting the self-reduction property of the variance error, which improved the total sample complexity over the result in (Agarwal et al., 2019) by a factor of $O(\frac{1}{T}/\log(\frac{1}{T}))$. We anticipate that our techniques can be further applied to study other nonlinear SA algorithms such as Greedy-Q, nonlinear GTD (Maei, 2011), and off-policy AC algorithms (Maei, 2018; Zhang et al., 2019b).

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Appendices

A Justification of Item 3 in Assumption 1

The following lemma justifies item 3 in Assumption 1. We denote the density function of policy \( \pi_w(\cdot|s) \) as \( \frac{\pi_w(da|s)}{da} \) (if the action space \( \mathcal{A} \) is discrete, then \( \frac{\pi_w(da|s)}{da} = \pi_w(a|s) \)).

**Lemma 1.** Consider a policy \( \pi_w \) parametrized by \( w \). Consider the following two cases:

1. Density function of the policy is smooth, i.e. \( \frac{\pi_w(da|s)}{da} \) is \( L_\pi \)-Lipschitz \( (0 < L_\pi < \infty) \), and the action set is bounded, i.e. \( \int_{a \in \mathcal{A}} 1da = C_\mathcal{A} < \infty \).
2. \( \pi_w \) is the Gaussian policy, i.e., \( \pi_w(s) = \mathcal{N}(f(w), \sigma^2) \), with \( f(w) \) being \( L_f \)-Lipschitz \( (0 < L_f < \infty) \).

For both cases, we have

\[
\|\pi_w(\cdot|s) - \pi_w'(\cdot|s)\|_{TV} \leq C_\pi \|w - w'\|_2,
\]

where \( C_\pi = \frac{1}{2} \max\{L_\pi C_\mathcal{A}, \sqrt{2} L_f\} \).

**Proof.** Without loss of generality, we only consider the case when \( \mathcal{A} \) is continuous. For the first case, we have

\[
\|\pi_w(\cdot|s) - \pi_w'(\cdot|s)\|_{TV} = \frac{1}{2} \int_a \left| \frac{\pi_w(da|s)}{da} - \frac{\pi_w'(da|s)}{da} \right| da \leq \frac{1}{2} \int_a L_\pi \|w - w'\|_2 da
\]

\[
\leq \frac{1}{2} L_\pi C_\mathcal{A} \|w - w'\|_2 \leq C_\pi \|w - w'\|_2,
\]

where \( (i) \) follows from Assumption 1. For the second case, we have

\[
\|\pi_w(\cdot|s) - \pi_w'(\cdot|s)\|_{TV} \leq \frac{1}{2} \frac{KL}{D_{KL}(\pi_w(\cdot|s), \pi_w'(\cdot|s))} = \sqrt{\frac{1}{2}(f(w) - f(w'))^2}
\]

\[
= \sqrt{\frac{1}{2} L_f^2 \|w - w'\|_2^2} = \frac{\sqrt{2}}{2} L_f \|w - w'\|_2 \leq C_\pi \|w - w'\|_2.
\]

\[\square\]

B Supporting Lemmas

In this subsection, we provide supporting lemmas, which are useful to the proof of the main theorems in Section 3. The detailed proofs of these lemmas are relegated to Appendix G.

**Lemma 2.** Considering the initialization distribution \( \eta(\cdot) \) and transition kernel \( P(\cdot, a) \). Let \( \eta(\cdot) = \zeta(\cdot) \) or \( P(\cdot, a) \) for any given \( (\bar{s}, \bar{a}) \in S \times A \). Denote \( \nu_{\pi_w, \eta}(\cdot, \cdot) \) as the state-action visitation distribution of MDP with policy \( \pi_w \) and initialization distribution \( \eta(\cdot) \). Suppose Assumption 2 holds, then we have

\[
\|\nu_{\pi_w, \eta}(\cdot, \cdot) - \nu_{\pi_w, \eta}(\cdot, \cdot)\|_{TV} \leq C_\nu \|w - w'\|_2
\]

for all \( w, w' \in \mathbb{R}^d \), where \( C_\nu = C_\pi \left( 1 + \lceil \log_\rho m^{-1} \rceil + \frac{1}{1-\rho} \right) \).
Lemma 3. Suppose Assumptions 1 and 2 hold, for any \( w, w' \in \mathbb{R}^d \) and any state-action pair \((s, a) \in S \times A\). We have
\[
|Q_{\pi_w}(s, a) - Q_{\pi_{w'}}(s, a)| \leq L_Q \|w - w'\|_2,
\]
where \( L_Q = \frac{2r_{\text{max}} C_{\phi}}{1 - \gamma} \).

Lemma 4. Suppose Assumptions 1 hold, for \( w', w'' \in \mathbb{R}^d \), we have
\[
\left\| \nabla_w \mathbb{E}_{\nu_{w'}} \left[ \log \pi_{w'}(a, s) \right] - \nabla_w \mathbb{E}_{\nu_w} \left[ \log \pi_w(a, s) \right] \right\|_2 \leq L_{\phi} \|w' - w''\|_2.
\]

Lemma 5. For any \( w \in \mathbb{R}^d \), define \( \theta_w^* = (F(w) + \lambda I)^{-1} \nabla J(w) \) and \( \theta_w^+ = F(w)^{\dagger} \nabla J(w) \). We have \( \|\theta_w - \theta_w^+\|_2 \leq C_r \lambda \), where \( 0 < C_r < +\infty \) is a constant only depends on the policy class.

C Proof of Proposition 1

By definition, we have
\[
\nabla J(w) - \nabla J(w') = \int_{(s,a)} Q_{\pi_w}(s, a) \phi_w(s, a) \nu_{\pi_w}(ds, da) - \int_{(s,a)} Q_{\pi_{w'}}(s, a) \phi_{w'}(s, a) \nu_{\pi_{w'}}(ds, da)
\]
\[
= \int_{(s,a)} Q_{\pi_w}(s, a) \phi_w(s, a) \nu_{\pi_w}(ds, da) - \int_{(s,a)} Q_{\pi_w}(s, a) \phi_w(s, a) d\nu_{\pi_{w'}}(ds, da)
\]
\[
+ \int_{(s,a)} Q_{\pi_w}(s, a) \phi_w(s, a) d\nu_{\pi_{w'}}(ds, da) - \int_{(s,a)} Q_{\pi_{w'}}(s, a) \phi_{w'}(s, a) d\nu_{\pi_{w'}}(ds, da)
\]
\[
= \int_{(s,a)} Q_{\pi_w}(s, a) \phi_w(s, a) [\nu_{\pi_w}(ds, da) - \nu_{\pi_{w'}}(ds, da)]
\]
\[
+ \int_{(s,a)} [Q_{\pi_w}(s, a) \phi_w(s, a) - Q_{\pi_{w'}}(s, a) \phi_{w'}(s, a)] \nu_{\pi_{w'}}(ds, da)
\]
\[
+ \int_{(s,a)} [Q_{\pi_{w'}}(s, a) \phi_{w'}(s, a) - Q_{\pi_{w'}}(s, a) \phi_{w'}(s, a)] \nu_{\pi_{w'}}(ds, da).
\]

Thus, we have
\[
\left\| \nabla J(w) - \nabla J(w') \right\|_2 \leq \int_{(s,a)} \|Q_{\pi_w}(s, a) \phi_w(s, a)\|_2 \nu_{\pi_w}(ds, da) - \nu_{\pi_{w'}}(ds, da)
\]
\[
+ \int_{(s,a)} |Q_{\pi_w}(s, a) - Q_{\pi_{w'}}(s, a)| \|\phi_w(s, a)\|_2 \nu_{\pi_{w'}}(ds, da)
\]
\[
+ \int_{(s,a)} |Q_{\pi_{w'}}(s, a)| \|\phi_w(s, a) - \phi_{w'}(s, a)\|_2 \nu_{\pi_{w'}}(ds, da)
\]
\[
\leq \frac{r_{\text{max}} C_{\phi}}{1 - \gamma} \int_{(s,a)} \nu_{\pi_w}(ds, da) - \nu_{\pi_{w'}}(ds, da)
\]
\[
+ C_{\phi} \int_{(s,a)} |Q_{\pi_w}(s, a) - Q_{\pi_{w'}}(s, a)| \nu_{\pi_{w'}}(ds, da)
\]
\[
+ \frac{r_{\text{max}}}{1 - \gamma} \int_{(s,a)} \|\phi_w(s, a) - \phi_{w'}(s, a)\|_2 \nu_{\pi_{w'}}(ds, da)
\]
\[
\begin{align*}
\leq & \frac{2r_{\max}C_{\nu}C_{\phi}}{1-\gamma} \|w - w'\|_2 + \frac{2r_{\max}C_{\nu}C_{\phi}}{1-\gamma} \|w - w'\|_2 + r_{\max}L_{\phi} \|w - w''\|_2 \\
= & \, L_J \|w - w'\|_2,
\end{align*}
\]

where (i) follows from Lemma 2, Lemma 3 and Assumption 1.

\section*{D Proof of Theorem 1}

In this section, we first provide the proof of a more general version (given as Theorem 4) of Theorem 1 for linear SA with Markovian mini-batch updates. We then show how Theorem 4 implies Theorem 1. Throughout the paper, for two matrices \(M, N \in \mathbb{R}^{d \times d}\), we define \(\langle M, N \rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} M_{i,j} N_{i,j}\).

We consider the following linear stochastic approximation (SA) iteration with a constant stepsize:

\[
\theta_{k+1} = \theta_k + \alpha \left( \frac{1}{M} \sum_{i=kM}^{(k+1)M-1} A_{xi} \theta_k + \frac{1}{M} \sum_{i=kM}^{(k+1)M-1} b_x \right),
\]

where \(\{x_i\}_{i \geq 0}\) is a Markov chain with state space \(\mathcal{X}\). We define \(A = \mathbb{E}_{\mu}[A_x]\) and \(b = \mathbb{E}_{\mu}[b_x]\). Then the iteration eq. (6) corresponds to the following ODE:

\[
\dot{\theta} = A \theta + b.
\]

We make the following standard assumptions, which are also adopted by (Bhandari et al., 2018; Zou et al., 2019; Xu et al., 2019b, 2020b).

\textbf{Assumption 3.} For all \(x \in \mathcal{X}\), there exist constants such that the following hold

1. For all \(x\), we have \(\|A_x\|_F \leq C_A\) and \(\|b_x\|_2 \leq C_b\).

2. Matrix \(A\) is symmetric and negative definite with \(\lambda_A = 2 |\lambda_{\max}(A)| > 0\).

3. The MDP is irreducible and aperiodic, and there exist constants \(\kappa > 0\) and \(\rho \in (0, 1)\) such that

\[
\sup_{x \in \mathcal{S}} \|\mathbb{P}(x_k \in \cdot | x_0) - \mu(\cdot)\|_{TV} \leq \kappa \rho^k, \quad \forall k \geq 0,
\]

where \(\mu(\cdot)\) is the invariant distribution of the MDP.

In this case, eq. (7) has a unique fix point \(\theta^* = -A^{-1} b\) and \(\|\theta^*\|_2 \leq R_\theta\). For brevity, we use \(\hat{A}_k\) and \(\hat{b}_k\) to denote \(\frac{1}{M} \sum_{i=kM}^{(k+1)M-1} A_{xi}\) and \(\frac{1}{M} \sum_{i=kM}^{(k+1)M-1} b_{x_i}\), respectively. We also define \(g(\theta) = A \theta + b\) and \(g_k(\theta) = \hat{A}_k \theta + \hat{b}_k\). We have the following theorem on the iteration of \(\|\theta_k - \theta^*\|_2^2\).

\textbf{Theorem 4.} Suppose Assumption 3 holds. Consider the iteration eq. (6), and let \(\alpha = \frac{\lambda_A}{8C_A}\).

\[
M \geq \max \left\{ C_A^2, \frac{2(2 \lambda_2^2 R_\theta^2 + C_b^2)}{\epsilon} \right\} \left( \frac{2}{\lambda_A} + \frac{\lambda_A}{4C_A^2} \right) 192 \left[ 1 + (\kappa - 1)\rho \right] (1 - \rho) \lambda_A,
\]

and \(K \geq \frac{128C_A^2}{\lambda_A^2} \log \frac{2\|\theta_0 - \theta^*\|_2^2}{\epsilon}\). Then we have

\[
\mathbb{E}[\|\theta_K - \theta^*\|_2^2] \leq \epsilon,
\]

and the total sample complexity is given by

\[
KM = \mathcal{O} \left( \max \left\{ 1, \frac{1}{\lambda_A^2} \right\} \frac{1}{(1 - \rho)\epsilon} \log \left( \frac{1}{\epsilon} \right) \right).
\]
Proof of Theorem 4. We first proceed as follows,
\[
\|\theta_{k+1} - \theta^*\|^2_2 = \|\theta_k + \alpha g_k(\theta_k) - \theta^*\|^2_2 \\
= \|\theta_k - \theta^*\|^2_2 + 2\alpha \langle \theta_k - \theta^*, g_k(\theta_k) \rangle + \alpha^2 \|g_k(\theta_k)\|^2_2 \\
= \|\theta_k - \theta^*\|^2_2 + 2\alpha \langle \theta_k - \theta^*, g(\theta_k) \rangle + 2\alpha \langle \theta_k - \theta^*, g_k(\theta_k) - g(\theta_k) \rangle \\
+ \alpha^2 \|g_k(\theta_k) - g(\theta_k) + g(\theta_k)\|^2_2 \\
\leq (i) \|\theta_k - \theta^*\|^2_2 - \lambda_A \alpha \|\theta_k - \theta^*\|^2_2 + \frac{\lambda_A}{2} \alpha \|\theta_k - \theta^*\|^2_2 + \frac{2}{\lambda_A} \alpha \|g_k(\theta_k) - g(\theta_k)\|^2_2 \\
+ 2\alpha^2 \|g_k(\theta_k) - g(\theta_k)\|^2_2 + 2\alpha^2 \|g(\theta_k)\|^2_2 \\
\leq (ii) \left(1 - \frac{\lambda_A}{2} \alpha + 2C_A^2 \alpha^2\right) \|\theta_k - \theta^*\|^2_2 + \left(\frac{2}{\lambda_A} \alpha + 2\alpha^2\right) \|g_k(\theta_k) - g(\theta_k)\|^2_2 , \tag{8}
\]
where (i) follows from the facts that
\[
\langle \theta_k - \theta^*, g(\theta_k) \rangle = \langle \theta_k - \theta^*, A(\theta_k - \theta^*) \rangle \leq \frac{\lambda_A}{2} \|\theta_k - \theta^*\|^2_2 ,
\]
\[
\langle \theta_k - \theta^*, g_k(\theta_k) - g(\theta_k) \rangle \leq \frac{\lambda_A}{4} \|\theta_k - \theta^*\|^2_2 + \frac{1}{\lambda_A} \|g_k(\theta_k) - g(\theta_k)\|^2_2 ,
\]
and
\[
\|g_k(\theta_k) - g(\theta_k) + g(\theta_k)\|^2_2 \leq 2 \|g_k(\theta_k) - g(\theta_k)\|^2_2 + 2 \|g(\theta_k)\|^2_2 ,
\]
and (ii) follows from the fact that \(\|g(\theta_k)\|^2_2 = \|A(\theta_k - \theta^*)\|^2_2 \leq C_A \|\theta_k - \theta^*\|^2_2 \). Let \(F_k\) be the filtration of the sample \(\{x_i\}_{0 \leq i \leq k-1}\). Taking expectation on both sides of eq. (8) conditioned on \(F_k\) yields
\[
\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2_2 | F_k] \leq \left(1 - \frac{\lambda_A}{2} \alpha + 2C_A^2 \alpha^2\right) \|\theta_k - \theta^*\|^2_2 + \left(\frac{2}{\lambda_A} \alpha + 2\alpha^2\right) \mathbb{E}[\|g_k(\theta_k) - g(\theta_k)\|^2_2 | F_k] . \tag{9}
\]
Next we bound the term \(\mathbb{E}[\|g_k(\theta_k) - g(\theta_k)\|^2_2 | F_k]\) in eq. (9) as follows.
\[
\mathbb{E}[\|g_k(\theta_k) - g(\theta_k)\|^2_2 | F_k] \\
= \mathbb{E} \left[ \| (\hat{\theta}_k - A)\theta_k + \hat{b}_k - b \|^2_2 | F_k \right] \\
= \mathbb{E} \left[ \| (\hat{\theta}_k - A)(\theta_k - \theta^*) + (\hat{\theta}_k - A)\theta^* + \hat{b}_k - b \|^2_2 | F_k \right] \\
\leq 3\mathbb{E} \left[ \| (\hat{\theta}_k - A)(\theta_k - \theta^*) \|^2_2 + \| (\hat{\theta}_k - A)\theta^* \|^2_2 + \| \hat{b}_k - b \|^2_2 | F_k \right] \\
\leq 3\mathbb{E} \left[ \| \hat{\theta}_k - A \|^2_2 | F_k \right] \|\theta_k - \theta^*\|^2_2 + 3\mathbb{E} \left[ \| \hat{\theta}_k - A \|^2_2 | F_k \right] \|\theta^*\|^2_2 + 3\mathbb{E} \left[ \| \hat{b}_k - b \|^2_2 | F_k \right] . \tag{10}
\]
We next derive the upper bounds on \(\mathbb{E} \left[ \| \hat{\theta}_k - A \|^2_2 | F_k \right]\) and \(\mathbb{E} \left[ \| \hat{b}_k - b \|^2_2 | F_k \right]\), respectively.
\[
\mathbb{E} \left[ \| \hat{\theta}_k - A \|^2_2 | F_k \right] \leq \mathbb{E} \left[ \| \hat{\theta}_k - A \|^2_F | F_k \right] = \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{i=kM}^{(k+1)M-1} A_i - A \right\|^2_F | F_k \right],
\]
16
Consider the term \( \mathbb{E}[(A_{x_i} - A, A_{x_j} - A)|\mathcal{F}_k] \) with \( i \neq j \). Without loss of generality, we consider the case when \( i > j \) as follows:

\[
\mathbb{E}[(A_{x_i} - A, A_{x_j} - A)|\mathcal{F}_k] = \mathbb{E}[\mathbb{E}(A_{x_i} - A, A_{x_j} - A)|x_j]\mathcal{F}_k] = \mathbb{E}[\mathbb{E}[A_{x_i} - A, A_{x_j} - A]|\mathcal{F}_k]
\]

\[
\leq \mathbb{E}[\|E[A_{x_i}|x_j] - A\|_F \|A_{x_j} - A\|_F |\mathcal{F}_k] \leq 2C_A \mathbb{E}[\|E[A_{x_i}|x_j] - A\|_F |\mathcal{F}_k]
\]

\[
(i) \leq 4C_A^2 \kappa \rho^{i-j},
\]

where \((i)\) follows from the Assumption 3 and the fact that

\[
\|E[A_{x_i}|x_j] - A\|_F = \left\| \int_{x_i} A_{x_i} P(dx_i|x_j) - \int_{x_i} A_{x_i} \mu(dx_i) \right\|_F \leq \int_{x_i} \|A_{x_i}\|_F |P(dx_i|x_j) - \mu(dx_i)|
\]

\[
\leq C_A \int_{x_i} |P(dx_i|x_j) - \mu(dx_i)| \leq 2C_A \|P(\cdot|x_j) - \mu(\cdot)\|_{TV} \leq 2C_A \kappa \rho^{i-j}.
\]

Substituting eq. (12) into eq. (11) yields

\[
\mathbb{E}\left[\|\hat{a}_k - A\|^2_2 |\mathcal{F}_k\right] \leq \frac{1}{M^2} \left[ 4MC_A^2 + 4C_A^2 \kappa \sum_{i \neq j} \rho^{i-j} \right] \leq \frac{8C_A^2[1 + (\kappa - 1)\rho]}{(1 - \rho)M}.
\]

Following steps similar to the above, we obtain

\[
\mathbb{E}\left[\|\hat{b}_k - b\|^2_2 |\mathcal{F}_k\right] \leq \frac{8C_A^2[1 + (\kappa - 1)\rho]}{(1 - \rho)M}.
\]

Substituting eq. (13) and eq. (14) into eq. (10) yields

\[
\mathbb{E}[\|g_k(\theta_k) - g(\theta_k)\|^2_2 |\mathcal{F}_k] \leq \frac{24C_A^2[1 + (\kappa - 1)\rho]}{(1 - \rho)M} \|\theta_k - \theta^*\|^2_2 + \frac{24(C_A^2 R_\theta^2 + C_\theta^2)[1 + (\kappa - 1)\rho]}{(1 - \rho)M}.
\]

Further substituting eq. (15) into eq. (8) yields

\[
\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2_2 |\mathcal{F}_k] \leq \left(1 - \frac{\lambda_A}{2} \alpha + 2C_A^2 \alpha^2 + \left(\frac{2}{\lambda_A} \alpha + 2\alpha^2\right) \frac{24C_A^2[1 + (\kappa - 1)\rho]}{(1 - \rho)M}\right) \|\theta_k - \theta^*\|^2_2
\]

\[
+ \left(\frac{2}{\lambda_A} \alpha + 2\alpha^2\right) \frac{24(C_A^2 R_\theta^2 + C_\theta^2)[1 + (\kappa - 1)\rho]}{(1 - \rho)M}.
\]

Letting \( \alpha = \frac{\lambda_A}{8} \) and \( M \geq \left(\frac{2}{\lambda_A} + 2\alpha\right) \frac{192C_A^2[1 + (\kappa - 1)\rho]}{(1 - \rho)\lambda_A} \), and taking expectation over \( \mathcal{F}_t \) on both sides of the above inequality yields

\[
\mathbb{E}[\|\theta_{k+1} - \theta^*\|^2_2] \leq \left(1 - \frac{\lambda_A}{8} \alpha\right) \mathbb{E}[\|\theta_k - \theta^*\|^2_2] + \left(\frac{2}{\lambda_A} \alpha + 2\alpha^2\right) \frac{24(C_A^2 R_\theta^2 + C_\theta^2)[1 + (\kappa - 1)\rho]}{(1 - \rho)M}.
\]
Applying eq. (16) recursively from $k = 0$ to $K - 1$ yields

$$
\mathbb{E}[\|\theta_K - \theta^*\|_2^2] \\ 
\leq \left(1 - \frac{\lambda_A}{8}\right)^K \|\theta_0 - \theta^*\|_2^2 + \left(\frac{2}{\lambda_A} + 2\alpha^2\right) \frac{24(C^2_A R^2_0 + C^2_b)[1 + (\kappa - 1)\rho]}{(1 - \rho)M} \sum_{k=0}^{K-1} \left(1 - \frac{\lambda_A}{8}\right)^k \\
\leq e^{-\frac{\lambda_A}{8}K} \|\theta_0 - \theta^*\|_2^2 + \left(\frac{2}{\lambda_A} + 2\alpha\right) \frac{192(C^2_A R^2_0 + C^2_b)[1 + (\kappa - 1)\rho]}{(1 - \rho)\lambda_A M}. 
$$

(17)

Letting $K \geq \frac{128C^2_A}{\lambda_A} \log \frac{2\|\theta_0 - \theta^*\|_2^2}{\epsilon}$ and $M \geq \left(\frac{2}{\lambda_A} + \frac{\lambda_A}{4C_A}\right) \frac{384(C^2_R + C^2_b)[1 + (\kappa - 1)\rho]}{(1 - \rho)\lambda_A \epsilon}$, we have $\mathbb{E}[\|\theta_K - \theta^*\|_2^2] \leq \epsilon$.

Notice that the total sample complexity of eq. (6) is better than the results provided in (Bhandari et al., 2018) by a factor of $\log(\frac{1}{\epsilon})$. It also orderwisely matches the sample complexity of the variance-reduced TD (VRTD) provided in (Xu et al., 2020b) but has worse dependence on the conditional number $\lambda_A$.

**Proof of Theorem 1.** We show how to apply Theorem 4 to derive the sample complexity of Algorithm 2 given in Theorem 1. We define $A_x = \phi_w(s_i, a_i)\phi_w(s_i, a_i)^\top + \lambda I$, $b_x = \phi_w(s_i, a_i)\tilde{Q}_{\pi_{w}}(s_i, a_i)$ and $\theta^* = \theta^*_w$, with $R_\theta = 2C_{\phi,\text{max}}\frac{1}{\lambda A (1-\gamma)}$, and let $C_A = C^2_\phi + \lambda$, $C_b = 2r_{\text{max}}C_{\phi,\text{max}}$, and $K = T_\epsilon$. Then the results of Theorem 1 follows. \hfill \Box

### E Proof of Theorem 2

For brevity, we denote $\hat{P}_{w_t} = \frac{1}{|B_t|} \sum_{i \in B_t} P_{w_t}(s_i, a_i)$ and $v_t = \hat{P}_{w_t} \theta_t$. Following from the $L_J$-Lipschitz condition indicated in Proposition 1, we have

$$
J(w_{t+1}) \geq J(w_t) + \langle \nabla_w J(w_t), w_{t+1} - w_t \rangle - \frac{L_J}{2} \|w_{t+1} - w_t\|_2^2
$$

$$
= J(w_t) + \alpha \langle \nabla_w J(w_t), v_t - \nabla_w J(w_t) + \nabla_w J(w_t) \rangle - \frac{L_J \alpha^2}{2} \|v_t\|_2^2
$$

$$
= J(w_t) + \alpha \left\| \nabla_w J(w_t) \right\|_2^2 + \alpha \langle \nabla_w J(w_t), v_t - \nabla_w J(w_t) \rangle - \frac{L_J \alpha^2}{2} \left\| v_t - \nabla_w J(w_t) \right\|_2^2
$$

$$
\geq J(w_t) + \left(\frac{1}{2\alpha} - L_J \alpha^2\right) \left\| \nabla_w J(w_t) \right\|_2^2 - \left(\frac{1}{2\alpha} + L_J \alpha^2\right) \left\| v_t - \nabla_w J(w_t) \right\|_2^2, \tag{18}
$$

where (i) follows because

$$
\langle \nabla_w J(w_t), v_t - \nabla_w J(w_t) \rangle \geq -\frac{1}{2} \left\| \nabla_w J(w_t) \right\|_2^2 - \frac{1}{2} \left\| v_t - \nabla_w J(w_t) \right\|_2^2,
$$

and

$$
\left\| v_t - \nabla_w J(w_t) + \nabla_w J(w_t) \right\|_2^2 \leq 2 \left\| v_t - \nabla_w J(w_t) \right\|_2^2 + 2 \left\| \nabla_w J(w_t) \right\|_2^2.
$$

Taking expectation on both sides of eq. (18) and rearranging eq. (18) yield

$$
\left(\frac{1}{2\alpha} - L_J \alpha^2\right) \mathbb{E}[\left\| \nabla_w J(w_t) \right\|_2^2] \leq \mathbb{E}[J(w_{t+1})] - \mathbb{E}[J(w_t)] + \left(\frac{1}{2\alpha} + L_J \alpha^2\right) \mathbb{E}[\left\| v_t - \nabla_w J(w_t) \right\|_2^2]. \tag{19}
$$
Then we upper-bound the term $\mathbb{E}[\|v_t - \nabla w J(w_t)\|^2_2]$. By definition, we have

$$\begin{align*}
\mathbb{E}[\|v_t - \nabla w J(w_t)\|^2_2] &\leq \mathbb{E}\left[\left\|\tilde{P}_{w_t} \theta_t - \overline{P}_{w_t} \theta^*_w\right\|^2_2\right] \\
&\leq 3\mathbb{E}\left[\left\|\tilde{P}_{w_t} \theta_t - \tilde{P}_{w_t} \theta^*_w\right\|^2 + \left\|\tilde{P}_{w_t} \theta^*_w - \overline{P}_{w_t} \theta^*_w\right\|^2\right] \\
&\leq 3\mathbb{E}\left[\left\|\tilde{P}_{w_t}\right\|_2 \left\|\theta_t - \theta^*_w\right\|^2 + \left\|\tilde{P}_{w_t} - \overline{P}_{w_t}\right\|_2 \left\|\theta^*_w - \theta^*_w\right\|^2\right] \\
&\leq 3\mathbb{E}\left[\left\|\theta_t - \theta^*_w\right\|^2 + 3\mathbb{E}\left[\left\|\tilde{P}_{w_t} - \overline{P}_{w_t}\right\|^2\right] + 3C^4\mathbb{E}\left[\left\|\tilde{P}_{w_t}\right\|^2\right]\right], \quad (20)
\end{align*}$$

where $(i)$ follows from the fact that $\nabla w J(w_t) = \overline{P}_{w_t} \theta^*_w$ with $\theta^*_w = \overline{P}_{w_t} \nabla w J(w_t)$, and $(ii)$ follows from Lemma 5. To upper-bound the second term in eq. (20), we have

$$\begin{align*}
\mathbb{E}\left[\left\|\tilde{P}_{w_t} - \overline{P}_{w_t}\right\|^2\right] &\leq \mathbb{E}\left[\left\|\tilde{P}_{w_t} - \overline{P}_{w_t}\right\|^2\right] = \mathbb{E}\left[\left\|\frac{1}{B_t} \sum_{i \in B_t} P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right\|^2\right] \\
&= \frac{1}{B_t^2} \sum_{i \in B_t} \sum_{j \in B_t} \mathbb{E}\left[\left(P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)], P_{w_t}(s_j, a_j) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right)\right] \\
&\leq \frac{1}{B_t^2} \left(4B_t C^4 + \sum_{i \neq j} \mathbb{E}\left[\left(P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)], P_{w_t}(s_j, a_j) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right)\right]\right), \quad (21)
\end{align*}$$

Consider the term $\mathbb{E}\left[\left(P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)], P_{w_t}(s_j, a_j) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right)\right]$ with $i \neq j$. We denote $x_j = (s_j, a_j)$ for simplicity. Without loss of generality we consider the case $i > j$ as follows:

$$\begin{align*}
\mathbb{E}\left[\left(P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right)\right] &\leq \mathbb{E}\left[\left(P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)], P_{w_t}(s_j, a_j) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right)\right]^i \\
&\leq 2C^2\mathbb{E}\left[\left\|P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right\|_F\right] \leq 4C^4\rho^j-i, \quad (22)
\end{align*}$$

where $(i)$ follows from Assumption 2 and the fact that

$$\begin{align*}
\left\|P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right\|_F &\leq \int_{x_i} \left(P_{w_t}(s_i, a_i)p(dx_i|x_j) - \int_{x_i} P_{w_t}(s_i, a_i)\nu_{\pi w_t}(dx_i)\right) F(dx_i) \\
&\leq \int_{x_i} \left(P_{w_t}(s_i, a_i) - \mathbb{E}_{\nu_{\pi w_t}}[P_{w_t}(s, a)]\right) F(dx_i) \\
&\leq 2C^2\left\|P(-|x_i) - \nu_{\pi w_t}(\cdot)\right\|_{TV}.
\end{align*}$$
Substituting eq. (22) into eq. (21) yields

\[ \mathbb{E} \left[ \left\| \hat{P}_{w_t} - \mathcal{P}_{w_t} \right\|_2^2 \right] \leq \frac{1}{B_t^2} \left( 4C_\phi^4 B_t + 4C_\phi^4 \kappa \sum_{i \neq j} \rho^{j-i} \right) \leq \frac{1}{B_t^2} \left( 4C_\phi^4 B_t + \frac{8C_\phi^4 \kappa \rho B_t}{1 - \rho} \right) \]

Substituting eq. (23) into eq. (20) yields

\[ \mathbb{E}[\|v_t - \nabla w J(w_t)\|_2^2] \leq 3C_\phi^4 \mathbb{E} \left[ \left\| \theta_t - \theta_t^w \right\|_2^2 \right] + \frac{24R_\phi^2 C_\phi^4 [1 + (\kappa - 1)\rho]}{B_t (1 - \rho)} + 3C_\phi^4 C_v^2 \lambda^2. \]  

Substituting eq. (19) into eq. (24) yields

\[ \left( \frac{1}{2} \alpha - L_J \alpha^2 \right) \mathbb{E}[\|\nabla w J(w_t)\|_2^2] \leq \mathbb{E}[J(w_{t+1})] - \mathbb{E}[J(w_t)] + 3 \left( \frac{1}{2} \alpha + L_J \alpha^2 \right) C_\phi^4 \mathbb{E} \left[ \left\| \theta_t - \theta_t^w \right\|_2^2 \right] + 24 \left( \frac{1}{2} \alpha + L_J \alpha^2 \right) \frac{R_\phi^2 C_\phi^4 [1 + (\kappa - 1)\rho]}{B_t (1 - \rho)} + 3 \left( \frac{1}{2} \alpha + L_J \alpha^2 \right) C_\phi^4 C_v^2 \lambda^2. \]  

Letting \( \alpha = \frac{1}{4L_J} \), telescoping eq. (25) for \( t = 0 \) to \( T - 1 \), and dividing both sides by \( T/(16L_J) \) yield

\[ \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla w J(w_t)\|_2^2] \leq \frac{16L_J \mathbb{E}[J(w_T) - J(w_0)]}{T} + 9C_\phi^4 \sum_{t=0}^{T-1} \mathbb{E} \left[ \left\| \theta_t - \theta_t^w \right\|_2^2 \right] \]

\[ \quad + \frac{72R_\phi^2 C_\phi^4 [1 + (\kappa - 1)\rho]}{(1 - \rho)T} \sum_{t=0}^{T-1} \frac{1}{B_t} + 9C_\phi^4 C_v^2 \lambda^2. \]  

For all \( 0 \leq t \leq T - 1 \), let \( B_t = B \geq \frac{216L_J^2 C_\phi^4 (1 + (\kappa - 1)\rho)}{(1 - \rho) \epsilon} \) and \( \mathbb{E} \left[ \left\| \theta_t - \theta_t^w \right\|_2^2 \right] \leq \frac{\epsilon}{2TM_\phi} \) for all \( 0 \leq t \leq T - 1 \), and \( T \geq \frac{48L_J \gamma_{\max}}{(1 - \gamma) \epsilon} \), then we have

\[ \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla w J(w_t)\|_2^2] \leq \epsilon + \mathcal{O}(\lambda^2). \]

The total sample complexity is given by

\[ (B + MT_c N_Q)T = \mathcal{O} \left( \frac{1}{(1 - \gamma) \epsilon^2} \log \left( \frac{1}{\min\{\epsilon, 1 - \gamma\}} \right) \right). \]

**F Proof of Theorem 3**

In this section, we first show that NAC in Algorithm 1 convergences to a first-order stationary point, i.e., the norm of the gradient converges to zero. Then we present the proof of Theorem 3, in which the convergence of nested-loop NAC is characterized in terms of the function value. Following from the \( L_J \)-Lipschitz condition indicated in Proposition 1, we have

\[ J(w_{t+1}) \geq J(w_t) + \langle \nabla w J(w_t), w_{t+1} - w_t \rangle - \frac{L_J}{2} \| w_{t+1} - w_t \|_2^2 \]

\[ J(w_{t+1}) \leq J(w_t) + \frac{1}{2} \| w_{t+1} - w_t \|_2^2 \]

\[ J(w_{t+1}) \leq J(w_t) + \frac{1}{2} \| w_{t+1} - w_t \|_2^2 \]
\begin{align*}
&= J(w_t) + \alpha \langle \nabla_w J(w_t), \theta_t \rangle - \frac{L_f \alpha^2}{2} \| \theta_t \|^2 \\
&= J(w_t) + \alpha \langle \nabla_w J(w_t), \theta^*_{w_t} \rangle + \alpha \langle \nabla_w J(w_t), \theta_t - \theta^*_{w_t} \rangle - \frac{L_f \alpha^2}{2} \| \theta_t - \theta^*_{w_t} + \theta^*_{w_t} \|^2 \\
&\geq J(w_t) + \alpha \langle \nabla_w J(w_t), (P^\lambda_{w_t})^{-1} \nabla_w J(w_t) \rangle + \alpha \langle \nabla_w J(w_t), \theta_t - \theta^*_{w_t} \rangle - \frac{L_f \alpha^2}{2} \| \theta_t - \theta^*_{w_t} \|^2 \\
&- L_f \alpha^2 \| \theta^*_{w_t} \|^2 \\
&\geq J(w_t) + \alpha \left( \frac{\alpha}{C^2 + \lambda} - \frac{L_f \alpha^2}{\lambda^2} \right) \| \nabla_w J(w_t) \|^2 - \frac{\alpha}{2(C^2 + \lambda)} \| \nabla_w J(w_t) \|^2 - \left( \frac{\alpha(C^2 + \lambda)}{2} + L_f \alpha^2 \right) \| \theta_t - \theta^*_{w_t} \|^2,
\end{align*}

\text{(27)}

where \((i)\) follows from the fact that \(\frac{1}{C^2 + \lambda} \leq \left\| (P^\lambda_{w_t})^{-1} \right\|_2 \leq \frac{1}{\lambda^2} \), and

\[ \langle \nabla_w J(w_t), (P^\lambda_{w_t})^{-1} \nabla_w J(w_t) \rangle \geq \frac{1}{C^2 + \lambda} \| \nabla_w J(w_t) \|^2, \]

and

\[ \| \theta^*_{w_t} \|^2 \leq \left\| (P^\lambda_{w_t})^{-1} \nabla_w J(w) \right\| \leq \frac{1}{\lambda^2} \| \nabla_w J(w) \|^2, \]

and \((ii)\) follows from the fact

\[ \langle \nabla_w J(w_t), \theta_t - \theta^*_{w_t} \rangle \geq -\frac{1}{2(C^2 + \lambda)} \| \nabla_w J(w_t) \|^2 - \frac{C^2 + \lambda}{2} \| \theta_t - \theta^*_{w_t} \|^2. \]

Letting \(\alpha = \frac{\lambda^2}{4L_f(C^2 + \lambda)}\), taking expectation on both sides and rearranging eq. \text{(27)} yield

\[ \frac{\alpha}{4(C^2 + \lambda)} \mathbb{E}[\| \nabla_w J(w_t) \|^2] \leq \mathbb{E}[J(w_{t+1})] - \mathbb{E}[J(w_t)] + \left( \frac{\alpha(C^2 + \lambda)}{2} + L_f \alpha^2 \right) \mathbb{E} \left[ \| \theta_t - \theta^*_{w_t} \|^2 \right]. \]

\text{(28)}

Telescoping eq. \text{(28)} from \(t = 0\) to \(T - 1\) yields

\[ \frac{\alpha}{4(C^2 + \lambda)} \sum_{t=0}^{T-1} \mathbb{E}[\| \nabla_w J(w_t) \|^2] \leq \mathbb{E}[J(w_T)] - J(w_0) + \left( \frac{\alpha(C^2 + \lambda)}{2} + L_f \alpha^2 \right) \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \theta_t - \theta^*_{w_t} \|^2 \right]. \]

\text{(29)}
Dividing both sides of eq. (29) by \(\frac{\alpha T}{4(C_\phi^2 + \lambda)}\) yields

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla_w J(w_t)\|_2^2] \leq \frac{16L_f(C_\phi^2 + \lambda)^2}{\lambda^2} \mathbb{E}[J(w_T) - J(w_0)] + [2(C_\phi^2 + \lambda)^2 + \lambda^2] \sum_{t=0}^{T-1} \mathbb{E}[\|\theta_t - \theta_{w_t}^*\|_2^2]
\]

(30)

Then, given the above gradient complexity result, we proceed to prove the convergence in terms of the function value. Denote \(D(w) = D_{RL}(\pi^*(\cdot|s), \pi_w(\cdot|s)) = \mathbb{E}_{\nu^*}[\log \frac{\pi^*(a|s)}{\pi_w(a|s)}]\). We derive as follows.

\[
D(w_t) - D(w_{t+1}) = \mathbb{E}_{\nu^*}[\log(\pi_{w_{t+1}}(a|s)) - \log(\pi_{w_t}(a|s))]
\]

\[
\geq \mathbb{E}_{\nu^*}[\nabla_w \log(\pi_{w_t}(a|s))]^\top (w_{t+1} - w_t) - \frac{L_\phi}{2} \|w_{t+1} - w_t\|_2^2
\]

\[
= \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (w_{t+1} - w_t) - \frac{L_\phi}{2} \|w_{t+1} - w_t\|_2^2
\]

\[
= \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top \theta_t - \frac{L_\phi}{2} \alpha \|\theta_t\|_2^2
\]

\[
= \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top \theta^{1}_{w_t} + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_t - \theta_{w_t}^*) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_{w_t}^* - \theta_{w_t})
\]

\[
- \frac{L_\phi}{2} \alpha \|\theta_t\|_2^2
\]

\[
= \alpha \mathbb{E}_{\nu^*}[A_{\pi_{w_t}}(s,a)] + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_t - \theta_{w_t}^*) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_{w_t}^* - \theta_{w_t})
\]

\[
+ \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top \theta^{1}_{w_t} - A_{\pi_{w_t}}(s,a)
\]

\[
= \alpha (1 - \gamma) \alpha (J(\pi^*) - J(\pi_{w_t})) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_t - \theta_{w_t}^*) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_{w_t}^* - \theta_{w_t})
\]

\[
+ \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top \theta^{1}_{w_t} - A_{\pi_{w_t}}(s,a)
\]

\[
= \alpha \sqrt{\mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top \theta^{1}_{w_t} - A_{\pi_{w_t}}(s,a)}^2 - \frac{L_\phi}{2} \alpha \|\theta_t\|_2^2
\]

\[
\geq (1 - \gamma) \alpha (J(\pi^*) - J(\pi_{w_t})) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_t - \theta_{w_t}^*) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_{w_t}^* - \theta_{w_t})
\]

\[
- \alpha \sqrt{\mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top \theta^{1}_{w_t} - A_{\pi_{w_t}}(s,a)}^2 - \frac{L_\phi}{2} \alpha \|\theta_t\|_2^2
\]

\[
\geq (1 - \gamma) \alpha (J(\pi^*) - J(\pi_{w_t})) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_t - \theta_{w_t}^*) + \alpha \mathbb{E}_{\nu^*}[\phi_{w_t}(s,a)]^\top (\theta_{w_t}^* - \theta_{w_t})
\]

\[
- \alpha \sqrt{1 - \gamma} \sqrt{\mathbb{E}_{\nu_{\pi_{w_t}}} [\phi_{w_t}(s,a)]^\top \theta^{1}_{w_t} - A_{\pi_{w_t}}(s,a)}^2 - \frac{L_\phi}{2} \alpha \|\theta_t\|_2^2
\]

\[
\geq (1 - \gamma) \alpha (J(\pi^*) - J(\pi_{w_t})) - \alpha C_\phi \|\theta_t - \theta_{w_t}^*\|_2^2 - \alpha C_\phi C_\gamma \lambda
\]

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where \((i)\) follows from the \(L_\phi\)-Lipschitz condition indicated in Lemma 4, \((ii)\) follows from the fact in (Agarwal et al., 2019) that

\[
E_{\nu_{w_t}}[A_{\pi_{w_t}}(s, a)] = (1 - \gamma) \left( J(\pi^*) - J(\pi_{w_t}) \right),
\]

\((iii)\) follows from the fact that

\[
\left\| \frac{\nu_{w_t}}{\nu_{w_{t+1}}} \right\|_{\infty} \geq \frac{\nu_{w_t}}{\nu_{w_{t+1}}} \left[ \phi_{w_t}(s, a) \right]_{t}^{\dagger} \theta_{t} - A_{\pi_{w_t}}(s, a) \right\|_{2}^{2} \geq \frac{\nu_{w_{t+1}}}{\nu_{w_t}} \left[ \phi_{w_t}(s, a) \right]_{t}^{\dagger} \theta_{t} - A_{\pi_{w_t}}(s, a) \right\|_{2}^{2},
\]

\((iv)\) follows from the fact that \(\nu_{w_t} \leq (1 - \gamma)\nu_{w_{t+1}}\) (Agarwal et al., 2019; Kakade and Langford, 2002), and \((v)\) follows from Lemma 5. Recall the definition of \(\zeta_{approx}\), then we have

\[
D(w_t) - D(w_{t+1}) \geq (1 - \gamma)\alpha \left( J(\pi^*) - J(\pi_{w_t}) \right) - \alpha C_\phi \left\| \theta_t - \theta_{w_t}^* \right\|_2 - \alpha C_\phi C_{\tau}\lambda
\]

\[
= (1 - \gamma)\alpha \left( J(\pi^*) - J(\pi_{w_t}) \right) - \alpha C_\phi \left\| \theta_t - \theta_{w_t}^* \right\|_2 - \alpha C_\phi C_{\tau}\lambda
\]

\[
= (1 - \gamma)\alpha \left( J(\pi^*) - J(\pi_{w_t}) \right) - \alpha C_\phi \left\| \theta_t - \theta_{w_t}^* \right\|_2 - \alpha C_\phi C_{\tau}\lambda
\]

\[
= (1 - \gamma)\alpha \left( J(\pi^*) - J(\pi_{w_t}) \right) - \alpha C_\phi \left\| \theta_t - \theta_{w_t}^* \right\|_2 - \alpha C_\phi C_{\tau}\lambda
\]

Rearranging eq. (32) and taking expectation on both sides yield

\[
J(\pi^*) - E[J(\pi_{w_t})] \leq \frac{E[D(w_t)] - E[D(w_{t+1})]}{(1 - \gamma)\alpha} + \frac{C_\phi E\left\| \theta_t - \theta_{w_t}^* \right\|_2}{1 - \gamma} + \sqrt{\frac{1}{(1 - \gamma)^2} \left\| \frac{\nu_{w_t}}{\nu_{w_{t+1}}} \right\|_{\infty} \zeta_{approx}^{\dagger}}
\]

\[
+ \frac{L_\phi \alpha E\left\| \theta_t - \theta_{w_t}^* \right\|_2}{1 - \gamma} + \frac{L_\phi \alpha}{(1 - \gamma)\lambda_\phi^2} E\left\| \nabla_w J(w_t) \right\|_2^2 + \frac{C_\phi C_{\tau}\lambda}{1 - \gamma}.
\]

Telescoping eq. (33) from \(t = 0\) to \(T - 1\) and dividing both sides of eq. (33) by \(T\) yield

\[
J(\pi^*) - \frac{1}{T} \sum_{t=0}^{T-1} E[J(\pi_{w_t})]
\]

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\[
\begin{align*}
\leq & \frac{D(w_0) - E[D(w_T)]}{T(1-\gamma)^{\alpha}} + \frac{C_{\phi} \sum_{t=0}^{T-1} E[\|\theta_t - \theta_{w_t}^*\|^2]}{T(1-\gamma)} + \frac{L_{\phi} \alpha \sum_{t=0}^{T-1} E[\|\theta_t - \theta_{w_t}^*\|^2]}{T(1-\gamma)} \\
& + \frac{L_{\phi} \alpha \sum_{t=0}^{T-1} E[\|\nabla_{w_t} J(w_t)\|^2]}{T(1-\gamma) \lambda_P^2} + \sqrt{\frac{1}{(1-\gamma)^3} \left\| \frac{\nu_{\pi_w}^*}{\nu_{\pi_w}} \right\|_{\infty}^2} \sqrt{\zeta_{\text{approx}}} + \frac{C_{\phi} C_{r} \lambda}{1-\gamma},
\end{align*}
\]

where \((i)\) follows from eq. (30). If we let \(E[\|\theta_t - \theta_{w_t}^*\|^2] \leq \max \left\{ \frac{(1-\gamma)^2}{16C_{\phi}^2} \epsilon^2, \frac{L_{J}(1-\gamma)(C_{\phi}^2 + \lambda)}{L_{\phi} \lambda_P^2 [2(C_{\phi}^2 + \lambda)^2 + \lambda_P^2 + \lambda_P^2]} \right\} \) for all \(0 \leq t \leq T-1\), and \(T \geq \max \left\{ \frac{16L_{J}(C_{\phi}^2 + \lambda)^2 D(w_0)}{\epsilon (1-\gamma) \lambda_P^2}, \frac{16L_{J}(C_{\phi}^2 + \lambda)^2 r_{\text{max}}}{\epsilon (1-\gamma)^2 \lambda_P^2} \right\}\), then we have

\[
J(\pi^*) - \frac{1}{T} \sum_{t=0}^{T-1} E[J(\pi_w)] \leq \epsilon + \frac{1}{(1-\gamma)^3} \left\| \frac{\nu_{\pi_w}^*}{\nu_{\pi_w}} \right\|_{\infty} \sqrt{\zeta_{\text{approx}}} + \frac{C_{\phi} C_{r} \lambda}{1-\gamma}.
\]

The total sample complexity is given by \(TMT_{\epsilon} N_Q = O\left(\frac{1}{(1-\gamma)^2} \epsilon^2 \log \frac{1}{\min_{\epsilon, (1-\gamma)}}\right)\).

\[\text{G} \quad \text{Proof of Supporting Lemmas in Appendix B}\]

**Proof of Lemma 2.** The proof of this lemma is similar to the proof of Lemma 6 in (Zou et al., 2019) with the following difference. (Zou et al., 2019) considers the case with the finite action space, we extend their result to the case with possible infinite action space. Define the transition kernel \(\tilde{P}(\cdot|s,a) = \gamma P(\cdot|s,a) + (1-\gamma) I(\cdot)\). Denote \(P_{\pi_w,I}(\cdot)\) as the state visitation distribution of the MDP with policy \(\pi_w\) and initialization distribution \(I(\cdot)\), and it satisfies that \(\nu_{\pi_w,I}(s,a) = P_{\pi_w,I}(s) \pi_w(a|s)\). (Konda, 2002) showed that the stationary distribution of the MDP with transition kernel \(\tilde{P}(\cdot|s,a)\) and policy \(\pi_w\) is \(P_{\pi_w,I}(\cdot)\). Following from Theorem 3.1 in (Mitrophanov, 2005), we obtain

\[
\|P_{\pi_w,I}(\cdot) - P_{\pi_w',I}(\cdot)\|_{TV} \leq \left( \|\log_P m^{-1}\| + \frac{1}{1-\rho} \right) \|K_w - K_{w'}\|,
\]

where \(K_w, K_{w'}\) are state to state transition kernel of MDP with policy \(\pi_w, \pi_{w'}\) respectively and \(\|\cdot\|\) is the operator norm of a transition kernel: \(\|P\| := \sup_{\|q\|_{TV} = 1} \|qP\|_{TV}\). Note here we define the total variation of a distribution \(q(s)\) as \(\|q\|_{TV} = \int_s q(ds)\). Then we obtain

\[
\|K_w - K_{w'}\| = \sup_{\|q\|_{TV} = 1} \left\| \int_s q(ds) (K_w - K_{w'})(s, \cdot) \right\|_{TV}
\]

\[
= \frac{1}{2} \sup_{\|q\|_{TV} = 1} \int_s q(ds) \left| \int_s q(ds) (K_w(s, ds') - K_{w'}(s, ds')) \right|
\]

\[
\leq \frac{1}{2} \sup_{\|q\|_{TV} = 1} \int_s q(ds) \left| K_w(s, ds') - K_{w'}(s, ds') \right|
\]

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\[
\frac{1}{2} \sup_{||\nu||:TV=1} \int_s \int_q (\nu(ds) - \nu(da)) \left| \int_a \tilde{P}(ds', s, a)(\pi'(da | s) - \pi(da | s)) \right| \\
\leq \frac{1}{2} \sup_{||\nu||:TV=1} \int_s \int_q (\nu(ds) - \nu(da)) \left| \int_a \tilde{P}(ds', s, a) \right| \\
= \sup_{||\nu||:TV=1} \int_s \int_q (\nu(ds) || \pi'(\cdot | s) - \pi(\cdot | s)||_{TV}) \\
\leq C_\pi || w' - w ||_2, \quad (35)
\]

where \((i)\) follows from Assumption 1. Substituting eq. (35) into eq. (34) yields

\[
|| P_{\pi_w, I}(\cdot) - P_{\pi_{w'}, I}(\cdot) ||_{TV} \leq C_\pi \left( || \log m^{-1} || + \frac{1}{1 - \rho} \right) || w' - w ||_2, \quad (36)
\]

Then we bound \(|| \nu_{\pi_w, I}(\cdot, \cdot) - \nu_{\pi_{w'}, I}(\cdot, \cdot) ||_{TV} \) as follows.

\[
|| \nu_{\pi_w, I}(\cdot, \cdot) - \nu_{\pi_{w'}, I}(\cdot, \cdot) ||_{TV} = \sup_{||\nu||:TV=1} \int_s \int_q (\nu(ds) - \nu(da)) \\
= \frac{1}{2} \sup_{||\nu||:TV=1} \int_s \int_q |P_{\pi_w, I}(ds)\pi_w(da) - P_{\pi_{w'}, I}(ds)\pi_{w'}(da)| \\
= \frac{1}{2} \sup_{||\nu||:TV=1} \int_s \int_q |P_{\pi_w, I}(ds)\pi_w(da) - P_{\pi_{w'}, I}(ds)\pi_{w'}(da)| + \frac{1}{2} \sup_{||\nu||:TV=1} \int_s \int_q |P_{\pi_{w'}, I}(ds)\pi_{w'}(da) - P_{\pi_{w'}, I}(ds)\pi_{w'}(da)| \\
\leq C_\pi || w - w' ||_2 + \frac{1}{2} \int_s |P_{\pi_{w'}, I}(ds) - P_{\pi_{w'}, I}(ds')| \\
= C_\pi || w - w' ||_2 + C_\pi \left( || \log m^{-1} || + \frac{1}{1 - \rho} \right) || w' - w ||_2 \\
= C_\pi || w' - w ||_2,
\]

where \((i)\) follows from Lemma 1.

**Proof of Lemma 3.** By definition, we have \(Q_{\pi_w}(s, a) = \frac{1}{1 - \gamma} \int \gamma \mathbb{P}(s_t = \hat{s}, a_t = \hat{a} | s_0 = s, a_0 = a, \pi_w) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = \hat{s}, a_t = \hat{a} | s_0 = s, a_0 = a, \pi_w) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = \hat{s}, a_t = \hat{a} | s_0 = s, a_0 = a, \pi_w) \) is the state-action visitation distribution of the MDP with policy \(\pi_w\) and initialization distribution \(P(\cdot | s_0 = s, a_0 = a)\). Thus, \(Q_{\pi_w}(s, a) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}(s_t = \hat{s}, a_t = \hat{a} | s_0 = s, a_0 = a)\). We denote \(Q_{\pi_w}(s)\) as the state stationary distribution for such a MDP. It then follows that

\[
|Q_{\pi_w}(s, a) - Q_{\pi_{w'}}(s, a)| \\
= \frac{1}{1 - \gamma} \int \gamma \mathbb{P}(s_t = \hat{s}, a_t = \hat{a} | s_0 = s, a_0 = a, \pi_w) - \int \gamma \mathbb{P}(s_t = \hat{s}, a_t = \hat{a} | s_0 = s, a_0 = a, \pi_{w'}) \\
\]

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where (i) follows from Lemma 2.

\begin{proof}[Proof of Lemma 4] By definition, for $w', w'' \in \mathbb{R}^d$, we obtain
\begin{align*}
\|\nabla_w\mathbb{E}_{\mu_w} \left[ \log \pi_{w'}(a, s) \right] - \nabla_w\mathbb{E}_{\mu_w} \left[ \log \pi_{w''}(a, s) \right] \|_2 &= \left\| \int_{(s,a)} \phi_{w'}(s, a) \nu_{\pi'}(ds, da) - \int_{(s,a)} \phi_{w''}(s, a) \nu_{\pi''}(ds, da) \right\|_2 \\
&\leq \int_{(s,a)} \| \phi_{w'}(s, a) - \phi_{w''}(s, a) \|_2 \nu_{\pi'}(ds, da) \\
&\leq \int_{(s,a)} \| \phi_{w'}(s, a) - \phi_{w''}(s, a) \|_2 \nu_{\pi'}(ds, da) \leq \int_{(s,a)} L_\phi \| w' - w'' \|_2 \nu_{\pi'}(ds, da) = L_\phi \| w' - w'' \|_2 ,
\end{align*}
where (i) follows from Assumption 1.
\end{proof}

\begin{proof}[Proof of Lemma 5] By definition, $F(w) \in \mathbb{R}^{d \times d}$ is a symmetric matrix. Thus, if $\text{rank}(F(w)) = k \leq d$, then there exist matrix $\Gamma_w \in \mathbb{R}^{d \times d}$ and $A_w \in \mathbb{R}^{d \times d}$ such that $F(w) = A_w^\top \Gamma_w A_w$, where $\Gamma_w = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots, 0]$ and $A_w^\top = [\psi_1, \psi_2, \ldots, \psi_k, \psi_{k+1}, \psi_{k+2}, \ldots, \psi_d]$ is an orthogonal matrix with $\{\psi_1, \psi_2, \ldots, \psi_k\}$ spans over the column space $\text{Col}(F(w))$ and $\{\psi_{k+1}, \psi_{k+2}, \ldots, \psi_d\} \perp \text{Col}(F(w))$. Without loss of generality, we assume that for all $w$, the linear matrix equation $F(w)x = \nabla J(w)$ has at least one solution $x_w^* \in \mathbb{R}^d$. Then we have
\begin{align*}
\theta_w^* &= (F(w) + \lambda I)^{-1} \nabla J(w) \\
&= (A_w^\top \Gamma_w A_w + \lambda I)^{-1} \nabla J(w) \\
&= A_w^\top \Gamma_w A_w + \lambda I)^{-1} A_w \nabla J(w) \\
&= \text{diag} \left[ \frac{1}{\lambda_1 + \lambda}, \ldots, \frac{1}{\lambda_k + \lambda}, \ldots, \frac{1}{\lambda} \right] A_w \nabla J(w) \\
&\equiv (i) \text{diag} \left[ \frac{1}{\lambda_1 + \lambda}, \ldots, \frac{1}{\lambda_k + \lambda}, \ldots, \frac{1}{\lambda} \right] [\psi_1^\top \nabla J(w), \ldots, \psi_k^\top \nabla J(w), 0, \ldots, 0]^\top \\
&= A_w^\top \left[ \frac{1}{\lambda_1 + \lambda} \psi_1^\top \nabla J(w), \ldots, \frac{1}{\lambda_k + \lambda} \psi_k^\top \nabla J(w), 0, \ldots, 0 \right]^\top ,
\end{align*}
where (i) follows from the fact that $\nabla J(w) \in \text{Col}(F(w))$ and $\{\psi_{k+1}, \psi_{k+2}, \ldots, \psi_d\} \perp \text{Col}(F(w))$. Similarly, we also have
\begin{align*}
\theta_w^1 &= (F(w)^\dagger \nabla J(w) \\
&= (A_w^\top \Gamma_w A_w)^\dagger \nabla J(w) \\
&= A_w^\top (\Gamma_w)^\dagger A_w \nabla J(w) \\
&= \text{diag} \left[ \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}, 0, \ldots, 0 \right] A_w \nabla J(w) \\
&\equiv (i) \text{diag} \left[ \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}, 0, \ldots, 0 \right] [\psi_1^\top \nabla J(w), \ldots, \psi_k^\top \nabla J(w), 0, \ldots, 0]^\top
\end{align*}

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\[
\theta^* - \theta^+ \approx \Lambda^\top w \left[ \sum_{i=1}^{k} \left( \frac{1}{\lambda_i} \psi_i^\top \nabla J(w) \right) \right].
\]

Thus we have
\[
\theta^* - \theta^+ = \Lambda^\top w \left[ \sum_{i=1}^{k} \left( \frac{1}{\lambda_i} \psi_i^\top \nabla J(w) \right) \right].
\]

We can further obtain
\[
\left\| \theta^* - \theta^+ \right\| \leq \frac{\lambda}{\lambda_{\min}} \left\| A_w \right\| \left\| \nabla J(w) \right\| \leq C \lambda.
\]

where in (i) we define \( \lambda_{\min} = \min_{w \in \mathbb{R}^d} \min_{0 \leq i \leq k_w} \lambda_{w,i} \), with \( \lambda_{w,i} \) being the i-th element in \( \Gamma_w \) and \( k_w \) being the rank of the matrix \( F(w) \).

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