THE ASYMPTOTIC PROFILE OF $\chi_y$–GENERA OF HILBERT SCHEMES OF POINTS ON K3 SURFACES

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ABSTRACT. The Hodge numbers of the Hilbert schemes of points on algebraic surfaces are given by Göttsche’s formula, which expresses the generating functions of the Hodge numbers in terms of theta and eta functions. We specialize in this paper to generating functions of the $\chi_y(K3^n)$ genera of Hilbert schemes of $n$ points on K3 surfaces. We determine asymptotic values of the coefficients of the $\chi_y$-genus for $n \to \infty$ as well as their asymptotic profile.

1. INTRODUCTION AND RESULTS

The Hilbert scheme $S[n]$ of $n$ points on a complex projective surface $S$ heuristically parametrizes collections of $n$ points on the surface $S$. The geometry of such Hilbert schemes is well studied. In this paper we will be mainly considering the case when $S$ is a K3 surface. K3 surfaces are smooth, compact and simply connected surfaces with trivial canonical bundle. K3 surfaces are hyper-Kähler manifolds and exhibit a wealth of other special properties. The Hilbert schemes $K3^n$ are also hyper-Kähler manifolds, and their topological invariants determine many other interesting invariants for mathematical objects associated to K3: Gromov-Witten invariants [14, 15], stable pair invariants [19], rank $r$ sheaves of pure complex dimension 2 [22]. K3 surfaces are important in Calabi-Yau compactifications of 10-dimensional string theory to 4 and 6 dimensions. From this perspective, the Hodge numbers $h^{p,r}(S[n])$ of the cohomology give information about the number of (supersymmetric) quantum states in the lower dimensional physical theories. See for example [6].

The Hodge numbers $h^{p,r}(S[n])$ of the Hilbert schemes of $n$ points on an algebraic surface $S$ are famously given by Göttsche’s formula [10]. Göttsche’s formula expresses the generating function as an infinite product, and is in fact a simple product of Jacobi theta and eta functions. The asymptotic growth of the Euler number $\chi(S[n])$ for $n \to \infty$ can be determined since a long time using the Rademacher circle method (see for example [2, 12]) and is of interest for conformal field theory and string theory. Recently, methods are also developed to derive the asymptotic behavior of the Betti numbers which gives much more refined information about the cohomology of the Hilbert schemes [3, 5]. Closely related is the work by Hausel and Rodriguez-Villegas [13], who have determined the asymptotic profiles of Betti numbers of a class of (hyper)-Kähler manifolds. The majority of those (hyper)-Kähler manifolds in [13] appear as moduli spaces of families of mathematical objects.

In the present paper we extend the techniques developed in [3, 5], to determine the asymptotic behavior of the $\chi_y$ genus of $K3^n$. To explain the setup and results in more detail,

$^1$See for example [11] and [18] for two expository texts.
be the Hodge polynomial of a smooth complex manifold $\mathcal{M}$ with $h^{p,r}(\mathcal{M}) = \dim H^{p,r}(\mathcal{M})$. If $\mathcal{M}$ satisfies Poincaré duality, $e(\mathcal{M}; x, y)$ is a palindromic polynomial in the two variables $x$ and $y$. The polynomial specializes to several other well-known characteristic polynomials. For $x = y$, one obtains the Poincaré polynomial, i.e. the generating function of Betti numbers $b_k(\mathcal{M}) = \sum_{p+r=k} h^{p,r}(\mathcal{M})$. For $x = -1$, $e(\mathcal{M}; x, y)$ specializes to the $\chi_y(\mathcal{M})$-genus of $\mathcal{M}$: $\chi_y(\mathcal{M}) = \sum_{p,r} h^{p,r}(\mathcal{M})(-1)^p y^r = \sum_r \chi^r(\mathcal{M}) y^r$. The number $\chi^r(\mathcal{M})$ is the index of the Dolbeault complex of forms with non-holomorphic degree $r$. Finally for $x = y = -1$, $e(\mathcal{M}; x, y)$ equals the Euler number $\chi(\mathcal{M})$.

The famous formula by Göttche [10, Conjecture 3.1] expresses the generating function of Hodge polynomials of the Hilbert schemes as an infinite product formula:

$$
\sum_{n=0}^{\infty} e(\mathcal{M}; x, y) x^{-n} y^{-n} q^n = \prod_{n=1}^{\infty} \prod_{p+r=\text{odd}} (1 + x^{p-1} y^{r-1} q^n)^{h^{p,r}(S)} \prod_{p+r=\text{even}} (1 - x^{p-1} y^{r-1} q^n)^{h^{p,r}(S)}.
$$

To specialize Eq. (1.1) to a K3 surface, note that the nonvanishing Hodge numbers of a K3 surface are given by $h^{0,0}(K3) = h^{2,0}(K3) = h^{0,2}(K3) = h^{2,2}(K3) = 1$, and $h^{1,1}(K3) = 20$.

In the physical context of Calabi-Yau compactifications to 4 dimensions, the exponents of $x$ and $y$ label representations of the SU(2) rotation and U(1) R-symmetry group [6]. Specializing (1.1) by $x \to -1$, and $y \to -\zeta$, one obtains a generating function of the Laurent polynomials $\zeta^{-n} \chi_{-\zeta}(K3[n])$. The exponent of $\zeta$ then labels representations of a diagonal subgroup $\subset \text{SU}(2) \times \text{U}(1)$.

To obtain our results we use and develop techniques from analytic number theory. See for example the closely related papers [3, 4, 5, 7, 12, 16]. We expect that the techniques in this paper might in turn motivate and be relevant for questions in analytic number theory. For example, a combinatorial interpretation of the coefficients of (1.1) (and its specialization of $(x, y)$ to $(-1, -\zeta)$) in terms of colored partitions is still missing. We continue by expressing the generating function in terms of modular forms. Recall that the Jacobi theta function $\vartheta(w, \tau)$ is defined for $w \in \mathbb{C}$ and $\tau \in \mathbb{H}$:

$$
\vartheta(w; \tau) := i \zeta^{\frac{1}{2}} q^{1/8} \prod_{n=1}^{\infty} (1 - q^n) (1 - \zeta q^n) (1 - \zeta^{-1} q^{n-1}) = i \zeta^{\frac{1}{2}} q^{\frac{1}{8}} (q;q)_\infty (q\zeta;q)_\infty (\zeta^{-1}; q)_\infty,
$$

where $\zeta := e^{2\pi i w}$, $q := e^{2\pi i r}$ and $(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$ is the $q$-Pochhammer symbol. Recall also that the Dedekind $\eta$-function is defined as

$$
\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} (q; q)_\infty.
$$

Then specialization of equation (1.1) to $x = -1$ and $y = -e^{2\pi i w} = -\zeta$ gives the generating function of $\chi_{-\zeta}(K3^{[n]})$-genera:

$$
f_k(w; \tau) := \frac{g(w; \tau)^2}{\eta(\tau)^k},
$$
with \( k = 24 \) and:

\[
g(w; \tau) := i \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \eta(\tau)^3.
\]

We define the coefficients \( a_{m,k}(n) \) of \( f_k(w, \tau) \), \( k \geq 1 \), as follows

\[
f_k(w, \tau) := \sum_{m,n} a_{m,k}(n) \zeta^m q^{n-\frac{k}{24}}.
\]

We note that the coefficients \( \chi^r([K3]^{[m]}) \) are given by \((-1)^r a_{r,24}(n)\). Due to the symmetry \( w \leftrightarrow -w \) of \( f_k(w; \tau) \) it is easy to deduce that \( \chi_{-\zeta}(K3^{[m]}) \) is a palindromic polynomial of degree \( 2n \) with positive coefficients. Using this symmetry we can obtain the coefficients \( f_{m,k}(\tau) \) of \( \zeta^m \) moreover as:

\[
f_{m,k}(\tau) := \frac{2 q^{\frac{k}{24}}}{\eta(\tau)^k} \int_0^\frac{1}{2} g(w; \tau)^2 \cos(2\pi mw) dw.
\]

From \( f_{m,k}(\tau) \) the \( a_{m,k}(n) \) are consequently obtained as:

\[
a_{m,k}(n) := \frac{1}{2\pi i} \int_C \frac{f_{m,k}(\tau)}{\eta(\tau)^{n+1}} dq,
\]

where \( C \) is a circle surrounding 0 clockwise. We note that \( \sum_{m \in \mathbb{Z}} a_{m,k}(n) = p_k(n) \), with \( p_k(n) \) the number of partitions of \( n \) in \( k \) colors.

Our first result is obtained using the approach of Wright [23] as in [5].

**Theorem 1.1.** For fixed \( m \) we have, as \( n \to \infty \)

\[
a_{m,k}(n) = (2\pi)^{-\frac{3}{2}} \sum_{\ell=1}^N d_{m,k}(\ell) n^{\frac{2+2\ell+k}{4}} \left( \frac{\pi k}{6} \right)^{1+\ell+\frac{k}{2}} \left( \pi \sqrt{\frac{2kn}{3}} \right)^{\ell-\frac{k}{4}} + O\left(n^{-1} - \frac{N}{2} - \frac{k}{4} \sqrt{\frac{2kn}{3}} \right).
\]

where the \( d_{m,k}(\ell) \) are defined in equation (2.8). From this we deduce

**Corollary 1.1.** The difference \( a_{m,k}(n) - a_{r,k}(n) \), as \( n \to \infty \), is given by

\[
a_{m,k}(n) - a_{r,k}(n) = \frac{4}{15} \pi^3 (r^2 - m^2) (8n)^{\frac{2+k}{4}} \left( \frac{k}{3} \right)^{\frac{k-4}{2}} e^{\frac{\pi}{\sqrt{24}} \sqrt{\frac{2kn}{3}}} + O\left(n^{-3} - \frac{1}{2} \pi e^{\sqrt{\frac{2kn}{3}}} \right).
\]

**Remark.** Thus we see that the leading asymptotic behavior is very similar to [5 Corollary 1.2]. Note however that for large \( m \) the coefficients \( d_{m,k}(\ell) \) grow much faster in the case studied here compared to [5].

Our second main result concerns the profile of the coefficients \( a_{m,k}(n) \) for \( |m| \leq \sqrt{\pi \log n} \).

To this end we define

\[
P(m, \beta) := \frac{d^2}{dm^2} \left( m \frac{\coth \left( \frac{\beta m}{2} \right)}{2} \right) = \frac{\beta}{4} \csch^2 \left( \frac{\beta m}{2} \right) \left( \beta m \coth \left( \frac{\beta m}{2} \right) - 2 \right),
\]

with \( \csch(x) = 1/\sinh(x) \). The function \( P(m, \beta) \) satisfies \( P(0, \beta) = \frac{2}{6}, \int_{-\infty}^{\infty} P(m, \beta) dm = 1 \) and has variance \( \int_{-\infty}^{\infty} m^2 P(m, \beta) dm = \frac{2\pi^2}{3\sqrt{\pi}} \). Using a resummation argument we obtain the limiting shape of the ratio \( a_{m,k}(n)/p_k(n) \) for large \( n \). This is given by:
Theorem 1.2. For \( m \) as above we have, as \( n \to \infty \)

\[
\frac{a_{m,k}(n)}{p_k(n)} = P(m, \beta_k) \left( 1 + O\left( \beta_k^\frac{1}{4} |m|^{\frac{3}{4}} \right) \right),
\]

where \( \beta_k = \pi \sqrt{\frac{k}{6n}} \).

It is an interesting open question to which distribution the probability density function \( P(m, \beta) \) corresponds. Probability distributions occurred earlier for coefficients of inverse theta functions and for the cohomology of hyper-Kähler manifolds. For example the profile for the function \( g(w; \tau)/\eta(\tau)^k \) was conjectured by Dyson [8] (and recently proven by Bringmann and Dousse [3, Theorem 1.3]; see also [21, equation 2.13]), to be equal to

\[
P_{\log}(m, \beta) = \frac{\beta}{4} \operatorname{sech}^2\left( \frac{1}{2} \beta m \right) = \frac{\beta}{4} \cosh^{-2}\left( \frac{1}{2} \beta m \right),
\]

which coincides with the probability density function of the logistic distribution with mean 0 and variance \( \frac{\pi^2}{3} \beta^2 \). Similarly, the profiles of Betti numbers of hyper-Kähler manifolds found by Hausel and Rodriguez-Villegas allow often an interpretation as probability distributions [13]. For example, the profile of the Betti numbers of Hilbert schemes on \( \mathbb{C}^2 \) corresponds to the Gumbel distribution [13]. In a similar spirit, a Gaussian distribution is found for DT-invariants of \( \mathbb{C}^3 \) [17].

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2. Proof of theorem 1.1

2.1. The main term. We start by recalling the transformation properties under \( \tau \to -1/\tau \) of the Jacobi theta and the Dedekind eta function.

**Lemma 2.1.**

\[
\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \eta(\tau),
\]

\[
\vartheta\left(\frac{w}{\tau}; -\frac{1}{\tau}\right) = -i \sqrt{-i\pi} e^{\pi i w^2/\tau} \vartheta(w; \tau).
\]

To prove Theorem 1.1 we investigate the main term of \( g(w; \tau) \) by using the transformation rules of \( \eta(\tau) \) and \( \vartheta(w; \tau) \).

**Lemma 2.2.** For \( 0 < \Re(z) \ll 1, 0 \leq \Re(w) \leq 1 \) we have

\[
g\left(\frac{w}{2\pi}; \frac{iz}{2\pi}\right)^2 = \frac{\sin(\pi w)^2 \exp\left(\frac{4\pi^2 w^2}{z}\right)}{\left(\frac{\pi}{2}\right)^2 \sinh\left(\frac{2\pi^2 w^2}{z}\right)^2} \left(1 + O\left(e^{-4\pi^2 \Re(\frac{1}{2})(1-w)}\right)\right).
\]
Proof: We have
\[
g \left( \frac{w}{2\pi}; \frac{iz}{2\pi} \right)^2 = -\frac{\left( \zeta \frac{i}{2} - \zeta \frac{i}{2} \right)^2 \eta \left( \frac{i\pi}{2} \right)^6}{\vartheta \left( w; \frac{iz}{2\pi} \right)^2} = \frac{\left( \zeta \frac{i}{2} - \zeta \frac{i}{2} \right)^2}{\left( \frac{z}{2\pi} \right)^2} \vartheta \left( \frac{2\pi w - z}{2\pi} \right)^2 \eta \left( \frac{2\pi i z}{z} \right)^6,
\]
where we let \( \tau = iz/2\pi \). Then using Lemma 2.1 one obtains:
\[
g \left( \frac{w}{2\pi}; \frac{iz}{2\pi} \right)^2 = -\frac{\left( \zeta \frac{i}{2} - \zeta \frac{i}{2} \right)^2 \exp \left( \frac{4\pi^2 w^2}{z} \right)}{\left( \frac{z}{2\pi} \right)^2} \vartheta \left( \frac{2\pi w, \frac{2\pi i z}{z}}{2\pi} \right)^2 \eta \left( \frac{2\pi i z}{z} \right)^6 \Pi_{n \geq 1} \left( 1 - e^{-4\pi^2 \frac{w}{z}} \right)^4 \left( 1 - e^{-4\pi^2 \frac{w}{z}} \right)^4 \left( 1 - e^{-4\pi^2 \frac{w}{z}} \right)^4 \left( 1 - e^{-4\pi^2 \frac{w}{z}} \right)^4 \left( 1 + O \left( e^{-4\pi^2 \Re \left( \frac{i}{2} (1 - w) \right)} \right) \right),
\]
which completes the proof. \( \square \)

We continue by using Cauchy’s theorem to express the coefficients as a contour integral. We have
\[
a_{m,k}(n) = \frac{1}{2\pi i} \int_C \frac{f_{m,k}(\tau)}{q^{n+1}} dq,
\]
where \( C \) is a contour surrounding 0 counterclockwise. We choose \( e^{-\beta_k} \), with \( \beta_k := \pi \sqrt{\frac{E}{6n}} \), for the radius, and split
\[
a_{m,k}(n) = M + E,
\]
with
\[
M := \frac{1}{2\pi i} \int_{C_1} \frac{f_{m,k}(\tau)}{q^{n+1}} dq,
\]
\[
E := \frac{1}{2\pi i} \int_{C_2} \frac{f_{m,k}(\tau)}{q^{n+1}} dq,
\]
where \( C_1 \) is the arc going counterclockwise from phase \(-\beta_k\) to \( \beta_k \), and \( C_2 \) is the complement of \( C_1 \) in \( C \).

The leading term will follow from \( M \), whereas \( E \) will contribute to the error. We first consider \( M \) and split this further into
\[
M = M_1 + E_1,
\]
with
\[
(2.1) \quad M_1 := \frac{1}{2\pi i} \int_{C_1} \frac{g_{m,1}(z)}{(e^{-z}; e^{-z})_\infty q^{-(n+1)}} dq,
\]
\[
(2.2) \quad E_1 := \frac{1}{2\pi i} \int_{C_1} \frac{g_{m,2}(z)}{(e^{-z}; e^{-z})_\infty q^{-(n+1)}} dq,
\]
and \( g_{m,1}(z) \) and \( g_{m,2}(z) \) are defined by:
\[
(2.3) \quad g_{m,1}(z) := 8\pi^2 \int_0^{1/2} \frac{\sin(\pi w)^2}{\sinh \left( \frac{2\pi w}{z} \right)^2} e^{4\pi^2 w^2} \cos(2\pi mw) dw,
\]
\[
g_{m,2}(z) := 2 \int_0^{1/2} \left( g \left( \frac{w}{2\pi}; \frac{iz}{2\pi} \right) \right)^2 \frac{\sin(\pi w)^2}{\left( \frac{z}{2\pi} \right)^2 \sinh \left( \frac{2\pi w}{z} \right)^2} e^{4\pi^2 w^2} \cos(2\pi mw) dw.
\]
We insert the Taylor expansions of $\sin^2(\pi w)$, $\cos(2\pi mw)$, and $\exp\left(\frac{4\pi^2 w^2}{z}\right)$ into (2.3)

\[
\sin(\pi w)^2 = -\frac{1}{4} (e^{2\pi i w} + e^{-2\pi i w} - 2) = -\frac{1}{2} \sum_{\ell \geq 1} (-1)^\ell \frac{(2\pi w)^{2\ell}}{(2\ell)!},
\]

\[
\cos(2\pi mw) = \sum_{\ell \geq 0} (-1)^\ell \frac{(2\pi mw)^{2\ell}}{(2\ell)!},
\]

\[
\exp\left(\frac{4\pi^2 w^2}{z}\right) = \sum_{j \geq 0} \frac{(4\pi^2 w^2)^j}{j!}.
\]

leading to:

\[
g_{m,1}(z) = \sum_{\epsilon_1, \epsilon_2 \geq 1, j \geq 0, \ell \geq 0} (-1)^{\epsilon_1+\epsilon_2+j} \frac{(2\pi)^{2(\epsilon_1+\epsilon_2+j+1)}}{(2\ell_1)! (2\ell_2)!} \int_0^{\frac{z}{2\pi}} w^{2(\epsilon_1+\epsilon_2+j)} \frac{z^{j+2}}{\sinh\left(\frac{2\pi^2 w}{z}\right)} dw.
\]

We are thus left to evaluate integrals of the shape ($j \in \mathbb{N}_0$)

(2.4)

\[
\int_0^{\frac{z}{2\pi}} \frac{w^{2j}}{\sinh\left(\frac{2\pi^2 w}{z}\right)^2} dw.
\]

We next turn the integral (2.4) into an integral up to $\infty$. The error may be bounded by (2.5)

\[
\ll \int_0^{\frac{z}{2\pi}} w^{2j} e^{-4\pi^2 w \Re\left(\frac{1}{z}\right)} dw \ll \left(\Re\left(\frac{1}{z}\right)\right)^{-2j-1} \Gamma\left(2j + 1; 2\pi^2 \Re\left(\frac{1}{z}\right)\right) \ll e^{-2\pi^2 \Re\left(\frac{1}{z}\right)},
\]

where we used the well known fact that

\[
\Gamma\left(j; x\right) \sim x^{j-1} e^{-x},
\]

as $x \to \infty$. In the new integral we make the change of variables $\frac{2\pi w}{z} = u$. The path then is given by $\text{Arg}(u) = \text{Arg}(z)$. Using the Residue Theorem we can shift the path of integration down to $\mathbb{R}$ giving

\[
\left(\frac{z}{2\pi}\right)^{2j+1} \int_0^{\infty} \frac{u^{2j}}{\sinh(\pi u)^2} du.
\]

Now define

\[
B_\ell := \int_0^{\infty} \frac{u^{2\ell}}{\sinh(\pi u)^2} du.
\]

We will need the following evaluation:

**Lemma 2.3.** We have

\[
B_\ell = \frac{(-1)^{\ell+1} B_{2\ell}}{\pi},
\]

where $B_\ell$ denotes the $\ell$th Bernoulli number.
Proof: The proof is similar in spirit to [4, Lemma 5.2]. We first extend the integral to $\mathbb{R}$. Since the poles all lie at $i\mathbb{Z}/\{0\}$, we can shift the integral away from the real axis. One obtains:

$$B_\ell = \frac{1}{2} \int_{\mathbb{R} + \frac{i}{2}} u^{2\ell} \sinh(\pi u)^2 du.$$ 

Define the function $g(u, T) := \frac{e^{2\pi T u}}{\sinh(\pi u)^2}$. Its residue is given by:

$$2\pi i \text{Res}_u = \left( \int_{\mathbb{R} + \frac{i}{2}} - \int_{\mathbb{R} + \frac{3i}{2}} \right) g(u, T) du = -4T e^{-2\pi T}.$$ 

Moreover, we have that $\int_{\mathbb{R} + \frac{3i}{2}} g(u, T) du = \int_{\mathbb{R} + \frac{i}{2}} g(u + i, T) du = e^{-2\pi T} \int_{\mathbb{R} + \frac{i}{2}} g(u, T) du$, which gives us the integral:

$$\int_{\mathbb{R} + \frac{i}{2}} g(u, T) du = \frac{4T}{1 - e^{2\pi T}}.$$ 

The generating function of the Bernoulli numbers $B_m$

$$(2.6) \quad \frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!},$$

and the expansion of the numerator of $\frac{e^{2\pi T u}}{\sinh(\pi u)^2}$ in the integral gives the desired result. □

Now combining Lemma 2.3 with the error (2.5), we have

$$(2.7) \quad g_{m,1}(z) = 2 \sum_{\ell_1 \geq 1, \ell_2 \geq 0} (-1)^j \frac{m^{2\ell_2}}{(2\ell_1)! (2\ell_2)! j!} z^{2(\ell_1 + \ell_2) + j - 1} \times \left( B_{2(\ell_1 + \ell_2 + j)} + O \left( |z|^{-2(\ell_1 + \ell_2 + j) - 1} e^{-2\pi^2 \text{Re}(\frac{1}{z})} \right) \right).$$

To evaluate the integral $M_1$ defined in (2.1), we can proceed as in [5, 23]. First we define the coefficients $d_{m,k}(\ell)$ as the Taylor coefficients of $g_{m,1}(z)$:

$$(2.8) \quad e^{-\frac{kz}{2\pi}} g_{m,1}(z) = \sum_{\ell=1}^{N} d_{m,k}(\ell) z^\ell + O(z^{N+1}).$$

The first few coefficients are given by:

$$d_{m,k}(1) = \frac{1}{6}, \quad d_{m,k}(2) = \frac{1}{30} - \frac{k}{144}, \quad d_{m,k}(3) = \frac{23}{2520} - \frac{m^2}{60} - \frac{k}{720} + \frac{k^2}{6912}.$$ 

Having obtained equation (2.8), we make two further splits:

$$M_1 = M_2 + E_2,$$

where

$$M_2 := \frac{1}{2\pi i} \int_{c_1} g_{m,1}(z) \left( \frac{z}{2\pi} \right)^{\frac{k}{2}} e^{-\frac{kz}{2\pi} + \frac{k^2z^2}{4\pi}} dq,$$

$$E_2 := \frac{1}{2\pi i} \int_{c_1} g_{m,1}(z) \left( \frac{1}{(e^{-z}; e^{-z})_{\infty}} - \left( \frac{z}{2\pi} \right)^{\frac{k}{2}} e^{-\frac{kz}{2\pi} + \frac{k^2z^2}{4\pi}} \right) dq,$$
and:

$$M_2 = M_3 + E_3,$$

where

$$M_3 := \frac{1}{2\pi i} \sum_{\ell=1}^{N} d_{m,k}(\ell) \int_{C_1} \frac{1}{q^{n+1}} \left( \frac{z}{2\pi} \right)^{\frac{\ell}{2}} e^{\frac{\ell}{4\pi} \frac{m^2}{z^2}} dq,$$

$$E_3 := \frac{1}{2\pi i} \int_{C_1} \frac{1}{q^{n+1}} \left( \frac{z}{2\pi} \right)^{\frac{\ell}{2}} e^{\frac{\ell}{4\pi} \frac{m^2}{z^2}} \left( e^{-\frac{\ell}{2\pi} g_{m,1}(z)} - \sum_{\ell=1}^{N} d_{m,k}(\ell) z^{\ell} \right) dq.$$

After a change of the integration variable $$v = z/\beta_k$$, the main term $$M_3$$ consists of contour integrals of the form:

$$I_s(\alpha) := \frac{1}{2\pi i} \int_{1+i}^{1-i} v^s e^{\alpha(v+\frac{1}{v})} \, dv, \quad \alpha > 0.$$ 

Lemma 3.1 in [5] estimates this integral in terms of the I-Bessel function:

$$I_s(\alpha) = I_{-s-1}(2\alpha) + O\left( e^{\frac{3\alpha}{2\beta_k}} \right).$$

Theorem 1.1 follows from substitution of these expressions and $$\alpha = \beta_k$$, except for the determination of the error term.

### 2.2. The error term.

We determine in this subsection the magnitude of the error terms $$E, E_1, E_2, \text{and } E_3,$$ and that they are suppressed compared to the main term. We show that the main error is due to $$E_3$$. We will use throughout that $$g(x) \ll f(x)$$ means $$g(x) = O(f(x))$$.

We start by computing bounds for the error terms coming from the different approximations near the dominant pole starting with $$E_1$$. To this end we first bound $$g_{m,2}(z)$$ near the dominant pole:

**Lemma 2.4.** We have for $$z \in C_1$$ and $$\beta_k \to 0$$:

$$g_{m,2}(z) \ll \frac{e^{-\frac{3\pi^2}{2\beta_k}}}{\beta_k^2}.$$ 

**Proof:** For $$z \in C_1$$ one can straightforwardly establish the following bounds:

$$\left| \frac{\sin(\pi w)}{1 - e^{-\frac{4\pi^2 w}{z}}} \right| \ll 1, \quad |z| \gg \beta_k, \quad \text{Re} \left( \frac{1}{z} \right) \geq \frac{1}{2\beta_k}.$$ 

Now recall that $$g_{m,2}(z)$$ is defined by

$$g_{m,2}(z) = 2 \int_{0}^{\frac{1}{2}} \left( g\left( w; \frac{iz}{2\pi} \right)^2 - \frac{\sin^2(\pi w)}{(\frac{1}{2\pi})^2 \sinh^2(\frac{2\pi w}{z})} e^{4\pi^2 w^2} \right) \cos(2\pi mw) \, dw.$$ 

Using Lemma 2.2 and the bounds above we bound $$g_{m,2}(z)$$ by

$$g_{m,2}(z) \ll \frac{1}{|z|^2} \int_{0}^{\frac{1}{2}} \left| \frac{\sin^2(\pi w)}{1 - e^{-\frac{4\pi^2 w}{z}}} \right|^2 e^{4\pi^2 \text{Re}(\frac{1}{z})(w^2-1)} \, dw \ll \frac{e^{-\frac{3\pi^2}{2\beta_k}}}{\beta_k^2},$$

where we used that $$w^2 - 1$$ has its maximum on $$[0, \frac{1}{2}]$$ at $$\frac{1}{2}$$. 

\[\square\]
With this result we find for \( E_1 \) \((2.2)\):

**Lemma 2.5.** We have for \( n \to \infty \):

\[
E_1 \ll n^{-\frac{k}{4}+\frac{1}{2}}e^{\pi \sqrt{\frac{2kn}{3}}-\frac{3}{2}\pi \sqrt{\frac{n}{k}}}.
\]

**Proof:** Using the bound of \( g_{m,2}(z) \) (see Lemma 2.4) we obtain directly:

\[
E_1 \ll \frac{e^{-\frac{3n^2}{2\beta_k}}}{\beta_k^2} \int_{c_1} \frac{q^{-(n+1)}}{(e^{-z}; e^{-z})_\infty^k} dq.
\]

The \( \eta \)-inversion formula in Lemma 2.1 is equivalent to the following transformation formula of the inverse \( q \)-Pochhammer symbol:

\[
\frac{1}{(e^{-z}; e^{-z})_\infty^k} = \sqrt{\frac{z}{2\pi}} e^{-\frac{kz^2}{24} + \frac{\pi^2}{6n}} \left( e^{-\frac{4z^2}{z}} \right)_\infty^{k}.
\]

Therefore

\[
(2.9) \quad \frac{1}{(e^{-z}; e^{-z})_\infty^k} = \left( \frac{z}{2\pi} \right)^{\frac{k}{2}} e^{-\frac{kz^2}{24} + \frac{\pi^2}{6n}} \left( 1 + O \left( e^{-\frac{4z^2}{z}} \right) \right).
\]

As a result we find for \( n \to \infty \):

\[
E_1 \ll \beta_k^{\frac{k}{2}-2} e^{-\frac{3n^2}{2\beta_k}} \int_{-\beta_k}^{\beta_k} e^{(n-\frac{k}{2})\beta_k + \frac{\pi^2}{6n} \text{Re}(\frac{1}{z})} dz.
\]

Now we investigate the exponent, which can be rewritten and bounded by

\[
\pi \sqrt{\frac{2kn}{3}} - \frac{3}{2} \pi \sqrt{\frac{6n}{k}}.
\]

This follows from the following upperbound for \( \text{Re}(\frac{1}{z}) \) on \( C_1 \):

\[
\frac{\text{Re}(z)}{|z|^2} \leq \frac{1}{\text{Re}(z)} = \frac{1}{\beta_k},
\]

and so

\[
\left( n - \frac{k}{2} \right) \beta_k + \frac{\pi^2}{6} \frac{k}{n} \text{Re} \left( \frac{1}{z} \right) < \pi \sqrt{\frac{2kn}{3}}
\]

by substituting \( \beta_k = \pi \sqrt{\frac{k}{6n}} \). Since the length of the integration path is of order \( O(\beta_k) \), we arrive at the desired result. \( \square \)

The next step is to evaluate the error \( E_2 \) coming from approximation of the \( q \)-Pochhammer symbol by its functional equation.

**Lemma 2.6.** We have for \( n \to \infty \):

\[
E_2 \ll n^{-\frac{k}{4}-1}e^{\pi \sqrt{\frac{2kn}{3}}-4\pi \sqrt{\frac{n}{k}}}.
\]

**Proof:** On \( C_1 \) the following approximation is valid

\[
|z|^2 = \beta_k^2 + \text{Im}(z)^2 \leq 2\beta_k^2.
\]

The Taylor series of \( g_{m,1}(z) \) \((2.7)\) implies therefore that:

\[
g_{m,1}(z) \ll |z| \ll \beta_k.
\]
From (2.9) we know that
\[
\frac{1}{(e^{-z}; e^{-z})_{\infty}^k} - \left(\frac{z}{2\pi}\right)^k e^{-kz} = O\left(\frac{k}{z^2} e^{-\frac{kz}{2} + \frac{4\pi^2}{z}(\frac{1}{2\pi} - 1)}\right).
\]
Now we have
\[
E_2 \ll \int_{C_1} dz \ e^{n\beta_k + \frac{z^2}{6\pi k} - \frac{4\pi^2}{z^2} \frac{k}{\beta_k} + 1}.
\]
By noting that the integration path is of order \(O(\beta_k)\) and plugging in \(\beta_k\) we obtain the lemma. \(\square\)

To finish the analysis of the error terms on the major arc we calculate \(E_3\) coming from the replacement of our main term by a Taylor series.

**Lemma 2.7.** We have, as \(n \to \infty\)
\[
E_3 \ll n^{-1 - \frac{1}{4}} e^{\pi \sqrt{2kn}}.
\]

**Proof:** Using
\[
e^{-\frac{kz}{24}} g_{m,1}(z) - \sum_{\ell=1}^{N} d_{m,k}(\ell) z^{\ell} = O(z^{N+1}),
\]
and changing variables we have
\[
E_3 \ll \int_{C_1} |z|^\frac{1}{2} e^{n\beta_k + \frac{z^2}{6\pi k} - \frac{4\pi^2}{z^2} \frac{k}{\beta_k} + 1} dz.
\]
Using (2.10), \(|z| \ll \beta_k\) and that the path is of order \(O(\beta_k)\) gives the desired result. \(\square\)

To obtain an error term away from the dominant pole (also known as the minor arc) we use the following Lemma proved in [3].

**Lemma 2.8.** Assume that \(\tau = u + iv \in \mathbb{H}\) with \(Mv \leq |u| \leq \frac{1}{2}\) for \(u > 0\) and \(v \to 0\), we have that:
\[
|\langle q; q \rangle_{\infty}^{-1}| \ll \sqrt{\exp \left[\frac{1}{v} \left(\frac{\pi}{12} - \frac{1}{2\pi} \left(1 - \frac{1}{\sqrt{1 + M^2}}\right)\right)\right]}.
\]

This means that the contribution of the other roots of unity will be suppressed as we see by bounding the error term \(E\).

**Lemma 2.9.** We have, for every \(0 < \varepsilon \leq 1\), as \(n \to \infty\)
\[
E \ll n^{-\frac{k-6}{4}} e^{\pi \sqrt{2kn}} (1 - \varepsilon)\).
\]

**Proof:** We first bound \(g(w; \tau)\). To this end, we write \(g(w; \tau)\) as a sum over its poles:
\[
g(w; \tau) = 1 + (1 - \zeta) \sum_{m \geq 1} \frac{(-1)^m q^{m^2 + 1}}{1 - \zeta q^m} + (1 - \zeta^{-1}) \sum_{m \geq 1} \frac{(-1)^m q^{m^2 + 1}}{1 - \zeta^{-1} q^m}.
\]
Therefore \(g(w; \tau)\) can be bounded for \(\text{Im}(\tau) = \frac{\beta_k}{2\pi}\), \(\text{Im}(w) = 0\) and \(n \to \infty\) as follows:
\[
g(w; \tau) \ll \sum_{m \geq 1} \frac{|q|^{m^2 + 1}}{1 - |q|^m} \ll \frac{1}{1 - |q|} \sum_{m \geq 1} e^{-\beta_k m^2} \ll \beta_k^{-\frac{3}{2}} \ll n^\frac{3}{4}.
\]
Thus
\[ g(w; \tau)^2 \ll n^{3/2}. \]
We use Lemma 2.8 for the arc \( C_2 \). Recall that \( \tau = \frac{i\pi}{2n} \) and \( \Re(z) = \beta_k = \pi \sqrt{\frac{k}{6n}} \). Consequently, \( \tau = \sqrt{\frac{3}{2\pi}} \) and \( M \) in Lemma 2.8 equals 1. Using this and the bound for \( g(w; \tau) \), we directly obtain
\[ E \ll n^{3/2} \int_{C_2} \beta_k^k e^{n\beta_k + n\beta_k (1 - \frac{6}{7\pi} (1 - \frac{1}{\sqrt{2}}))} dz. \]
Using that for \( n \to \infty \), \(-\frac{6}{\pi^2} n\beta_k (1 - \frac{1}{\sqrt{2}}) < -\frac{6}{4\pi} n\beta_k \) and that the integration path is \( O(1) \) finishes the proof.

This finishes the proof of Theorem 1.1 since we have computed the main term \( M_3 \) and determined that the leading error among the error terms \( E, E_i, i = 1, 2, 3 \) is given by \( E_3 \).

3. **Proof of Theorem 1.2**

In this section we calculate the main term that contributes to the profile coming from approaching the main singularity. To conclude Theorem 1.2 we then use Wright’s circle method. Moreover we detect the error coming from terms near the dominant pole and away from the dominant pole, giving the range where the asymptotic expansion is valid.

3.1. **The main term.** To determine the profile of \( a_{m,k}(n) \) as function of \( m \) for large \( n \), we start by determining an expansion of \( g_{m,1}(z) \) which is valid for a wide range of \( m \). The range of \( m \) is \( 1 \leq |m| \leq \frac{1}{6\beta_k} \log n \). One verifies that with this range the error in Theorem 1.2 goes to zero for large \( n \). Furthermore, we set \( z = \beta_k \left( 1 + i u m^{-\frac{1}{2}} \right) \) for \(|m| \geq 1 \) with as before \( \beta_k := \pi \sqrt{\frac{k}{6n}} \).

To prove Theorem 1.2 we continue in much the same way as in Section 1.1 using the approach of [3, 5] to perform Wright’s variant of the circle method. We recall the definition of \( a_{m,k}(n) \) (1.2):
\[
a_{m,k}(n) := \frac{1}{2\pi i} \int_{C} \frac{f_{m,k}(q)}{q^{n+1}} dq,
\]
where the contour is as in Section 2 the counterclockwise transversal of the circle \( C := \{ q \in \mathbb{C}; |q| = e^{-\beta_k} \} \). We change variables to \( z = \beta_k (1 + i u m^{-\frac{1}{2}}) \) and obtain
\[
a_{m,k}(n) = \frac{\beta_k}{2\pi m^{\frac{1}{2}}} \int_{D} f_{m,k}(e^{-z}) e^{nz} du,
\]
where \( D \) is the interval \( u \in \left[ -\frac{m^{\frac{1}{2}}}{\beta_k}, \frac{m^{\frac{1}{2}}}{\beta_k} \right] \). We split as before:
\[
a_k(m,n) = M + E,
\]
with
\[
M := \frac{\beta_k}{2\pi m^{\frac{1}{2}}} \int_{D_1} f_{m,k}(e^{-z}) e^{nz} du,
\]
\[
E := \frac{\beta_k}{2\pi m^{\frac{1}{2}}} \int_{D_2} f_{m,k}(e^{-z}) e^{nz} du,
\]
where $\mathcal{D}_1$ is the interval $u \in [-1, 1]$ and $\mathcal{D}_2$ is the complement of $\mathcal{D}_1$ in $\mathcal{D}_2$. Completely analogously to Section 2 we split $M = M_1 + E_1$ and $M_1 = M_2 + E_2$, where $M_2, E_1$ and $E_2$ are now defined as:

$$M_2 := \frac{\beta_k}{2\pi m} \int_{\mathcal{D}_1} g_{m,1}(z) \left( \frac{z}{2\pi} \right)^k e^{-\frac{kz^2}{2\pi} + nz} du,$$

$$E_1 := \frac{\beta_k}{2\pi m^3} \int_{\mathcal{D}_1} g_{m,2}(z) \left( e^{-z}; e^{-z} \right)_\infty e^{nz} du,$$

$$E_2 := \frac{\beta_k}{2\pi m^3} \int_{\mathcal{D}_1} g_{m,1}(z) \left( \frac{1}{(e^{-z}; e^{-z})_\infty} - \left( \frac{z}{2\pi} \right)^k e^{-\frac{kz^2}{2\pi} + \frac{kz^2}{6\pi}} \right) e^{nz} du.$$

In the following Lemma we give an approximation for $g_{m,1}(z)$ for $z \to 0$, which is valid for the wide range of $m$ mentioned above. We resum the sum over $\ell_2$ (the exponents of $m$) in the Taylor series for $g_{m,1}(z)$ \[2.7\]. This gives an expression in terms of hyperbolic trigonometric functions:

**Lemma 3.1.** Define

$$P(m, \beta) := \frac{\beta}{4} \left( \beta m \coth \left( \frac{\beta m}{2} \right) - 2 \right) \text{csch}^2 \left( \frac{\beta m}{2} \right).$$

Assume that $|u| \leq 1$ and $m \leq \frac{1}{6\beta_k} \log n$. Then we have as $n \to \infty$

$$g_{m,1}(z) = \left( 1 + ium^{-\frac{1}{2}} \right) P(m, \beta_k) + O\left( \beta_k m^{\frac{3}{2}} P(m, \beta_k) \right).$$

**Proof:** Recall that we determined in the previous section the Taylor series for $g_{m,1}(z)$ \[2.7\]. Using the generating function of the Bernoulli numbers \[2.6\], we can approximate this as:

$$g_{m,1}(z) = \sum_{\ell_2=0}^\infty \frac{(mz)^{2\ell_2}}{(2\ell_2)!} \left( zB_{2\ell_2+2} + O(|z|^2) \right)$$

$$= z \frac{d^2}{d(mz)^2} \left( \frac{mz}{2} \coth \left( \frac{mz}{2} \right) \right) + O(|z|^2 \cosh(mz))$$

$$= P(m, z) + O(|z|^2 \cosh(mz)).$$

We note that $z^{-1} P(m, z) =: f(mz)$ is only a function of $mz$. This function $f(x)$ is clearly smooth for $x \neq 0$, and one easily verifies that $f(x)$ is also analytic for $x = 0$:

$$f(x) = \frac{1}{6} + O(x^2).$$

We can thus make a Taylor expansion of $f(x)$ around any $x \in \mathbb{R}$. Since $|f'(x)| \leq |f(x)|$, we have:

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + O(\varepsilon^2) = f(x) + O(f(x)\varepsilon).$$

Now we apply this to $g_{m,1}(z)$ with $x = \beta_km$ and $\varepsilon = i\beta_km^{\frac{3}{2}}$:

$$g_{m,1}(z) = z f(m\beta_k) + O\left( \beta_k^{2} m^{\frac{3}{2}} (1 + m^{-\frac{1}{2}}) f(m\beta_k) \right) + O\left( \beta_k^{2} (1 + m^{-\frac{1}{2}}) \cosh(m\beta_k) \right).$$

Now we show that the first error term is larger than the second term for the full range of $m$. For that we distinguish between the cases where $\beta_km$ is bounded and where $\beta_km$ grows as $\frac{1}{6} \log(n)$. If $\beta_km$ is bounded, both $f(\beta_km)$ and $\cosh(\beta_km)$ are $O(1)$. Since in this case
\( m = O \left( n^{\frac{1}{2}} \right) \) for \( n \to \infty \), we find thus that the first error is \( O \left( n^{-\frac{2}{3}} \right) \) and the second error term is \( O(n^{-1}) \). For \( \beta_k m \to \frac{1}{6} \log(n) \) as \( n \to \infty \), we can bound the \( \cosh(\beta_k m) \) by:

\[
\cosh(\beta_k m) \ll e^{\beta_k m} \ll n^{\frac{1}{6}}.
\]

Similarly, one finds for \( f(\beta_k m) \):

\[
f(\beta_k m) \ll n^{-\frac{1}{12}} \log(n).
\]

As a result, the first error becomes \( O \left( n^{-\frac{5}{6}} \log(n)^{\frac{5}{3}} \right) \) and the second error \( O \left( n^{-\frac{2}{3}} \right) \). This concludes the proof of Lemma 3.1.

This Lemma leads us to the last split \( M_2 = M_3 + E_3 \) with:

\[
M_3 := \frac{1}{2\pi m^{\frac{1}{2}}} \int_{D_1} z \, P(m, \beta_k) \left( \frac{z}{2\pi} \right)^{\frac{1}{2}} \, e^{-\frac{kz^2}{24} + \frac{k\pi}{6} + nz} \, du,
\]

\[
E_3 := \frac{\beta_k}{2\pi m^{\frac{1}{2}}} \int_{D_1} \left( g_{m,1}(z) - \frac{z}{\beta_k} \, P(m, \beta_k) \right) \left( \frac{z}{2\pi} \right)^{\frac{1}{2}} \, e^{-\frac{kz^2}{24} + \frac{k\pi}{6} + nz} \, du.
\]

In order to determine the main term, we first define the following function

\[
J_s(\alpha) := \frac{1}{2\pi i} \int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^s \, e^{\alpha(v + \frac{1}{3})} dv, \quad \alpha > 0,
\]

and recall that Ref. [3, Lemma 4.2] shows that these integrals may be related to \( I_B \)-Bessel functions (analogously to \( I_s(\alpha) \) in Section 2):

**Lemma 3.2.** As \( n \to \infty \)

\[
J_s(\alpha) = I_{s-1}(2\alpha) + O \left( \exp \left( \alpha \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right).
\]

With this lemma we prove the following proposition for \( M_3 \):

**Proposition 3.3.** We have

\[
M_3 = P(m, \beta_k) \, p_k(n) \left( 1 + O \left( n^{-\frac{1}{2}} \right) \right).
\]

**Proof:** The change of variables \( v = 1 + iu m^{-\frac{1}{3}} \) gives:

\[
M_3 = \frac{\beta_k^{k+1}}{(2\pi)^{\frac{1}{2}}} P(m, \beta_k) \frac{1}{2\pi} \int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^{\frac{k}{2} + 1} e^{\pi v(n - \frac{k}{6})} \sqrt{\frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{6}} \, dv.
\]

We approximate the integral over \( v \) for \( n \to \infty \) by

\[
\frac{1}{2\pi i} \int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^{\frac{k}{2} + 1} e^{\pi v \sqrt{\frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{6}}} \left( 1 + O \left( n^{-\frac{1}{2}} v \right) \right) \, dv.
\]

Now using the definition of \( J_s(\alpha) \) this equals:

\[
J_{\frac{k}{2} + 1} \left( \pi \sqrt{\frac{kn}{6}} \right) + O \left( n^{-\frac{1}{2}} \, J_{\frac{k}{2} + 2} \left( \pi \sqrt{\frac{kn}{6}} \right) \right).
\]
Using Lemma 3.2 and the asymptotic expansion of the Bessel function [1],

$$I_s(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^\frac{3}{4}}\right),$$

equation \ref{eqn} is further approximated by:

$$\frac{e^{\pi \sqrt{\frac{2kn}{3}}} + O\left(\frac{e^{\pi \sqrt{\frac{2kn}{3}}}}{m^\frac{1}{2}}\right) + O\left(\exp\left(\pi \sqrt{\frac{kn}{6}} \left(1 + \frac{1}{1 + m^{-\frac{2}{3}}}\right)\right)\right)}{\pi \sqrt{2\left(\frac{3}{2}kn\right)^\frac{3}{4}}},$$

One easily sees that the first error term is the largest one and so we have for the leading term $M_3$:

$$M_3 = \frac{\beta_k^{k+1}}{(2\pi)^\frac{3}{2}} P(m, \beta_k) \frac{e^{\pi \sqrt{\frac{2kn}{3}}}}{\pi \sqrt{2\left(\frac{3}{2}kn\right)^\frac{3}{4}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Now using the following well-known formula [12, 20] for $p_k(n)$, for $n \to \infty$:

$$p_k(n) = 2 \left(\frac{k}{3}\right)^{k+1} (8n)^{-k+3} e^{\pi \sqrt{\frac{2kn}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$
we finish the proof.

3.2. The error term. In this subsection, we discuss the error terms $E_1, E_2, E_3$ near the dominant pole. We also determine the error $E$ to Theorem 1.2 away from the dominant pole, which is due to the minor arc $D_2$.

To start with $E_1$, one can easily verify that the bound $g_{m,2}(z) \ll \frac{1}{\beta_k^2} e^{-\frac{4n^2}{2^{3/2}}} \beta_k$ obtained in Lemma 2.4 for $z \in C_1$ continues to hold for $z \in D_1$ and $m \geq 1$. Similarly, the proof of Lemma 2.5 mostly goes through, except that the length of $D_1$ is of order $\beta_k m^{-\frac{1}{3}}$. As a result we have now:

$$E_1 \ll n^{-\frac{k}{2}} m^{-\frac{1}{3}} e^{\pi \sqrt{\frac{2kn}{3}}} \frac{1}{\beta_k^{k+1}} \sqrt{n^3}.$$

To establish the bound for $E_2$, we note that for $z \in D_1$ and $m \geq 1$, $g_{m,1}(z) \ll P(m, \beta)$. We can follow again roughly the proof of Lemma 2.6 using now $g_{m,1}(z) \ll P(m, \beta)$ and that the length of $D_1$ is of order $\beta_k m^{-\frac{1}{3}}$. Then one obtains:

$$E_2 \ll n^{-\frac{k}{2}} m^{-\frac{1}{3}} P(m, \beta_k) e^{\pi \sqrt{\frac{2kn}{3}}} \frac{1}{\beta_k^{k+1}} \sqrt{n^3}.$$

Similarly, for $E_3$ we find with Lemma 3.1 and the length of $D_1$, one obtains:

$$E_3 \ll n^{-\frac{k+6}{2}} m^\frac{1}{3} e^{\pi \sqrt{\frac{2kn}{3}}}.$$

Finally, one can also verify that the proof for $E$ in Section 2 is now applicable with $M = m^{-\frac{1}{2}}$. This leads to:

$$E \ll n^{-\frac{k+6}{2}} e^{\pi \sqrt{\frac{2kn}{3}}} \left(1 - \frac{1}{m^2}\right).$$

Therefore the dominant pole is indeed the one for $z = O(n^{-\frac{1}{2}})$. Comparing now all error terms we see that $E_3$ is again the dominating error, which concludes the proof of Theorem 1.2.
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