Baxter’s Q-operator for the homogeneous XXX spin chain

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Abstract:
Applying the Pasquier-Gaudin procedure we construct the Baxter’s Q-operator for the homogeneous XXX model as integral operator in standard representation of $SL(2)$. The connection between Q-operator and local Hamiltonians is discussed. It is shown that operator of Lipatov’s duality symmetry arises naturally as leading term of the asymptotic expansion of Q-operator for large values of spectral parameter.
1 Introduction

The modern approach to the theory of integrable systems is given by the quantum inverse scattering method (QISM) \[4, 7\]. In the framework of QISM, eigenstates \(|\lambda_1, ..., \lambda_l\rangle\) are obtained by the algebraic Bethe ansatz (ABA) method as an excitations over the vacuum state and the spectral problem is reduced to the set of algebraic Bethe equations (BE) for the parameters \(\lambda_j\). In fact the ABA is equivalent to the construction of the eigenfunctions in a special representation as polynomials of some suitable variables.

The alternative approach is the method of Q-operator \[1\] proposed by Baxter: there exists the operator \(Q(\lambda)\) which obeys the Baxter equation. The set of the Bethe equations is equivalent to the Baxter equation for the eigenvalue \(Q(\lambda)\) of the Q-operator. This second order finite-difference equation is the simple consequence of the Baxter relation for the transfer matrix and the Q-operator \[1\].

The ABA and method of Q-operator are equivalent when eigenfunctions and therefore \(Q(\lambda)\) are polynomials. In more general "nonpolynomial" situation one could use the method of Q-operator. The Q-operator for the periodic Toda chain was constructed in the work of Pasquier and Gaudin \[2\]. The application of the Q-operator for the construction of eigenstates with arbitrary complex values of conformal wights in the case XXX spin chain was considered in the work of Korchemsky and Faddeev \[7\]. In the
present paper we construct Q-operator for the homogeneous XXX spin chain using the Pasquier-Gaudin procedure.

The presentation is organized as follows. Section 2 introduces definitions and the standard facts about Baxter equation and construction of local Hamiltonians. In Section 3 we construct the Q-operator and study some properties of the obtained Q-operator in the simplest case of homogeneous chain. In Section 4 we obtain the connection between Q-operator and local Hamiltonian. In Section 5 we consider the asymptotic expansion of Q-operator for large spectral parameter. The operator of duality symmetry introduced by L.N.Lipatov [9] appears naturally as leading term in this asymptotic. Finally, in Section 6 we summarize.

2 XXX spin chain

In this section we collect some basic facts about XXX spin chain.

2.1 R-matrix and Yang-Baxter equation

The main object is the so called R - matrix which is the solution of the Yang-Baxter equation:

\[ R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{12}(\lambda). \]  

(2.1.1)

The operator \( R_{ij}(\lambda) \) depends on some complex variable – spectral parameter \( \lambda \) and two sets of \( SL(2) \)-generators \( \vec{S}_i \) and \( \vec{S}_j \) acting in different vector spaces \( V_i \) and \( V_j \).

Fixing the representations of the spins \( s_i \) and \( s_j \) in the vector spaces \( V_i \) and \( V_j \) we obtain the following \( R \)-matrices.

- \( s = 1/2 \) in space \( V_i \) and arbitrary representation \( s \) for the \( V_j \):

\[ R_j(\lambda) = \lambda + \frac{\eta}{2} + \eta \cdot \vec{S}_j \vec{\sigma}. \]

This \( R \)-matrix is used for the construction of the Lax \( L \)-operator:

\[ L_i(\lambda) \equiv R(\lambda - \frac{\eta}{2}) = \lambda + \eta \cdot \vec{S}_i \vec{\sigma} = \begin{pmatrix} \lambda + \eta S_i^- & \eta S_i^+ \\ \eta S_i^- & \lambda - \eta S_i \end{pmatrix} \]

(2.1.2)

- The equivalent representations \( s \) in spaces \( V_i \) and \( V_j \):

\[ R_{ij}(\lambda) = P_{ij} \cdot \frac{\Gamma(J_{ij} + \eta \lambda)}{\Gamma(J_{ij} - \eta \lambda)} ; \quad J_{ij} \cdot (J_{ij} - 1) = L_{ij} \]

(2.1.3)

where \( P_{ij} \) is the permutation and \( L_{ij} \) is the "two-particle" Casimir in \( V_i \otimes V_j \). This fundamental \( R \)-matrix is the building block for the construction of the local Hamiltonians.
2.2 Baxter equation for XXX-model.

The "usual" quantum monodromy matrix $T(\lambda)$ is defined as product of the $L$-matrices in the common two-dimensional auxiliary space. $T(\lambda)$ is a $2 \times 2$ matrix with operator entries acting in the quantum space $\otimes_{i=1}^{n} V_i$.

$$T(\lambda) \equiv L_1(\lambda + c_1)L_2(\lambda + c_2)...L_n(\lambda + c_n) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$  \hspace{1cm} (2.2.1)

The quantum transfer matrix $t(\lambda)$ is obtained by taking trace of $T(\lambda)$ in the auxiliary space:

$$t(\lambda) \equiv \text{Tr} T(\lambda) = A(\lambda) + D(\lambda)$$  \hspace{1cm} (2.2.2)

Due to the Yang-Baxter equation the family of operators $t(\lambda)$ is commuting, its $\lambda$ expansion begins with power $\lambda^n$ and provides $n - 1$ commuting operators $Q_k$:

$$t(\lambda) \cdot t(\mu) = t(\mu) \cdot t(\lambda) ; \quad t(\lambda) = 2\lambda^n + \sum_{k=0}^{n-2} Q_k \lambda^k.$$  \hspace{1cm} (2.2.3)

It is possible to show that transfer matrix $t(\lambda)$ is $SL(2)$-invariant

$$[\vec{S}, t(\lambda)] = 0 ; \quad \vec{S} \equiv \sum_{k=1}^{n} \vec{S}_k.$$ 

Therefore there exists the "full" set of $n$ commuting operators: $n - 1$ operators $Q_k$ and operator $S$. Due to $SL(2)$-invariance the subspace of the eigenvectors of operator $t(\lambda)$ with eigenvalue $\tau(\lambda)$ is the $SL(2)$ - module generated by highest weight vector $\Psi$, i.e. vector space spanned by linear combinations of monomials in the $S^+$ applied to vector $\Psi$. The highest weight vector $\Psi$ is defined by the equation $S^-\Psi = 0$.

We shall work in standard discrete-series representation of the group $SL(2)$:

$$S\Psi(x) \equiv (cx + d)^{-2s}\Psi\left(\frac{ax + b}{cx + d}\right) ; \quad S^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $SL(2)$-generators are realised as differential operators:

$$S_k = x_k \partial_k + s_k ; \quad S^- = -\partial_k ; \quad S_k^+ = x_k^2 \partial_k + 2 s_k x_k$$  \hspace{1cm} (2.2.4)

acting in the space of polynomials of the variable $x_k$. Here "spin" $s_k$ is arbitrary number. In this representation the commuting operators $Q_k$ are "local" differential operators acting in the space of polynomials of the $n$ variables $x_1, ..., x_n$ and there exists the vacuum vector $|0\rangle$:

$$B(\lambda)|0\rangle = 0 ; \quad A(\lambda)|0\rangle = \Delta_+(\lambda)|0\rangle ; \quad D(\lambda)|0\rangle = \Delta_-(\lambda)|0\rangle.$$
so that we can use the Algebraic Bethe Ansatz (ABA) method and reduce the problem of the common diagonalization of the operators $Q_k$ and $S$:

$$t(\lambda)\Psi_l = \tau(\lambda)\Psi_l ; \ S\Psi_l = (l + \sum_{k=1}^{n} s_k)\Psi_l$$

to the solution of the Bethe equation [4, 6]. The vacuum vector $|0\rangle$ is the common highest vector of the local representations of $SL(2)$:

$$|0\rangle \equiv \prod_{k=1}^{n} |0\rangle_k ; \ S^-|0\rangle_k = 0 ; \ S_k|0\rangle_k = s_k|0\rangle_k,$$

and

$$L_k(\lambda + c_k)|0\rangle_k = \left(\begin{array}{cc}
\lambda + c_k + \eta s_k & 0 \\
\cdots & \lambda + c_k - \eta s_k
\end{array}\right)|0\rangle_k$$

so that

$$\Delta_\pm(\lambda) \equiv \prod_{k=1}^{n}(\lambda + c_k \pm \eta s_k). \tag{2.2.5}$$

Let us look now the eigenvector $\Psi_l$ in the form:

$$\Psi_l \equiv |\lambda_1, ..., \lambda_l\rangle \equiv \prod_{j=1}^{l} C(\lambda_j)|0\rangle ; \ S|\lambda_1, ..., \lambda_l\rangle = (l + \sum_{k=1}^{n} s_k)|\lambda_1, ..., \lambda_l\rangle.$$

It is possible to show that vector $|\lambda_1, ..., \lambda_l\rangle$ is eigenvector of operator $t(\lambda)$ with eigenvalue:

$$\tau(\lambda) = \Delta_+(\lambda) \prod_{j=1}^{t} \frac{(\lambda - \lambda_j + \eta)}{(\lambda - \lambda_j)} + \Delta_-(\lambda) \prod_{j=1}^{t} \frac{(\lambda - \lambda_j - \eta)}{(\lambda - \lambda_j)} \tag{2.2.6}$$

on condition that the parameters $\lambda_i$ obey the Bethe equations:

$$\prod_{j=1}^{t}(\lambda_i - \lambda_j + \eta)\Delta_+(\lambda_i) = \prod_{j=1}^{t}(\lambda_i - \lambda_j - \eta)\Delta_-(\lambda_i). \tag{2.2.7}$$

It appears also that Bethe vectors $|\lambda_1, ..., \lambda_l\rangle$ are the highest weight vectors:

$$S^-|\lambda_1, ..., \lambda_l\rangle = 0.$$

In the representation (2.2.4) the highest weight vector $\Psi_l$ is represented by homogeneous, translation invariant polynomial degree $l(l = 0, 1, 2, ...)$ of $n$ variables $x_1, ..., x_n$:

$$\sum_{k=1}^{n} x_k \partial_k \Psi_l(x_1...x_n) = l\Psi_l(x_1...x_n) ; \ \sum_{k=1}^{n} \partial_k \Psi_l(x_1...x_n) = 0. \tag{2.2.8}$$
One can obtain Bethe equation from the formula for $\tau(\lambda)$ by taking residue at $\lambda = \lambda_i$ and using the fact that polynomial $\tau(\lambda)$ is regular at this point. Finally we see that equations (2.2.6,2.2.7) are equivalent to the Baxter equation for the polynomial $Q(\lambda)$:

$$\tau(\lambda)Q(\lambda) = \Delta_+(\lambda)Q(\lambda + \eta) + \Delta_-(\lambda)Q(\lambda - \eta)$$

(2.2.9)

where

$$Q(\lambda) \equiv \text{const} \cdot \prod_{j=1}^{t}(\lambda - \lambda_j).$$

(2.2.10)

### 2.3 Local Hamiltonians

Let us consider the homogeneous XXX-chain of equal spins: $c_k = 0$ and $s_k = s$ and fix the same representation $s$ in auxiliary space. In this case the quantum monodromy matrix $T_s(\lambda)$ is the product of the fundamental $R$-matrices (2.1.3):

$$T_s(\lambda) \equiv R_1(\lambda)R_2(\lambda)...R_n(\lambda)$$

The transfer matrix $t_s(\lambda)$ is obtained by taking trace of $T_s(\lambda)$ in the auxiliary space and due to the Yang-Baxter equation the families of operators $t_s(\lambda)$ and $t(\lambda)$ are commuting:

$$t_s(\lambda) \equiv \text{Tr}_s T_s(\lambda) ; t_s(\lambda)t_s(\mu) = t_s(\mu)t_s(\lambda) ; t(\lambda)t_s(\mu) = t_s(\mu)t(\lambda).$$

The $\lambda$ expansion of the log $t_s(\lambda)$ provides Hamiltonians $H_k$:

$$H_k \equiv \frac{1}{\eta} \frac{\partial^k}{\partial \lambda^k} \log t_s(\lambda) \bigg|_{\lambda=0} ; [H_k, H_l] = 0 ; [H_k, Q_l] = 0$$

(2.3.1)

where the $k$-th operator describes the interaction between $k + 1$ nearest neighbours on the chain. Due to the evident equalities (see (2.1.3)):

$$R_{ij}(0) = P_{ij} ; R_{ij}'(0) = 2\eta P_{ij} \cdot \psi(J_{ij})$$

one obtains the following expression for the first "two-particle" Hamiltonian $H_1$:

$$H_1 = \sum_{k=1}^{n} \hat{H}_{k-1,k} ; \hat{H}_{k-1,k} = \frac{1}{\eta} P_{k-1,k}R'_{k-1,k} = 2 \cdot \psi(J_{k-1,k}),$$

where $\psi(x)$ is logarithmic derivative of $\Gamma(x)$. It is convenient to work with the "shifted" Hamiltonian:

$$H = \sum_{k=1}^{n} H_{k-1,k} ; H_{k-1,k} = 2 \cdot \psi(J_{k-1,k}) - 2\psi(2s),$$

(2.3.2)

where the "shift" constant is defined by the requirement:

$$H_{k-1,k}|0\rangle = 0.$$
Let us calculate the eigenvalues of the operator $H_{k-1,k}$. Operator $H_{k-1,k}$ is $SL(2)$-invariant

$$[S_{k-1}^+ + S_k^+, H_{k-1,k}] = 0; \ [S_k - S_{k-1}, H_{k-1,k}] = 0$$

and its highest weight eigenfunctions $\Psi_l$ have the simple form in the representation (2.2.4):

$$(x_{k-1} \partial_{k-1} + x_k \partial_k) \Psi_l = l \Psi_l, \ (\partial_{k-1} + \partial_k) \Psi_l = 0 \Rightarrow \Psi_l(x_{k-1}, x_k) = (x_{k-1} - x_k)^l.$$ 

The two-particle Casimir $L_{k-1,k}$ is the second order differential operator:

$$L_{k-1,k} = -(x_{k-1} - x_k)^{2s} \partial_{k-1} \partial_k(x_{k-1} - x_k)^{2s}$$

and its eigenvalues $L_l$ and eigenvalues $J_l$ of operator $J_{k-1,k}$ can be easily calculated:

$$L_l = (2s + l)(2s + l - 1); \ J_l = 2s + l.$$ 

Finally we obtain the eigenvalues $H_l$ of the operator $H_{k-1,k}$:

$$H_l = 2\psi(2s + l) - 2\psi(2s).$$

In the representation (2.2.4) the operator $H_{k-1,k}$ can be realized as some “two-particle” integral operator acting on the variables $x_{k-1}$ and $x_k$:

$$H_{k-1,k}(x_{k-1}, x_k) = -\int_0^1 \frac{d\alpha}{\alpha^{2s-1}} \left[ \Psi(\alpha x_{k-1} + \alpha x_k) + \Psi(x_{k-1}, \alpha x_{k-1} + \alpha x_k) - 2\Psi(x_{k-1}, x_k) \right].$$

(2.3.3)

Note that these integral operators arise naturally in QCD [8]. To prove the equality (2.3.3) it is sufficient to show that eigenvalues of integral operator coincide with the eigenvalues $H_l$:

$$-2 \int_0^1 \frac{d\alpha}{\alpha^{2s-1}} [\alpha^l - 1] = 2[\psi(2s + l) - \psi(2s)].$$

The expression for the eigenvalues of the full Hamiltonian $H$ can be found by the ABA method [5]:

$$H = \frac{1}{\eta} \sum_{j=1}^l \frac{\partial}{\partial \lambda_j} \log \frac{\lambda_j + \eta s}{\lambda_j - \eta s} = \frac{1}{\eta} \sum_{j=1}^l \left[ \frac{1}{\eta s - \lambda_j} + \frac{1}{\lambda_j + \eta s} \right].$$

(2.3.4)

It is possible to rewrite this expression in terms of the $Q(\lambda)$-function (2.2.10) as follows [7]:

$$H = \frac{Q'(-\eta s)}{\eta Q(-\eta s)} - \frac{Q'(-\eta s)}{\eta Q(-\eta s)}$$

(2.3.4)

There exists an additional commuting with transfer matrix operator - shift operator $P$:

$$P\Psi(z_1, z_2 ... z_n) = \Psi(z_n, z_1 ... z_{n-1}); \ P = t_s(0).$$

(2.3.5)
Eigenvalues of the shift operator \( P \) can be found by the ABA method \[5\] also:

\[
P_l = \prod_{j=1}^{l} \frac{\lambda_j - \eta s}{\lambda_j + \eta s} = \frac{Q(\eta s)}{Q(-\eta s)}.
\]  

(2.3.6)

In the next sections we shall construct the Baxter’s Q-operator and show that Baxter equation (2.2.9) and equations (2.3.4,2.3.6) arise from the corresponding relations for the Q-operator.

3 Baxter’s Q-operator.

The Baxter’s \( Q \)-operator is the operator \( \hat{Q}(\lambda) \) with the properties \[1\]:

- \( t(\lambda)\hat{Q}(\lambda) = \Delta_+(\lambda)\hat{Q}(\lambda + \eta) + \Delta_-(\lambda)\hat{Q}(\lambda - \eta) \)
- \( \hat{Q}(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)\hat{Q}(\mu) \)
- \( t(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)t(\mu) \).

Operators \( \hat{Q}(\lambda) \) and \( t(\lambda) \) have the common set of eigenfunctions:

\[
\hat{Q}(\lambda)\Psi = Q(\lambda)\cdot\Psi; \ t(\lambda)\Psi = \tau(\lambda)\cdot\Psi
\]  

(3.0.1)

and eigenvalues of these operators obey the Baxter equation (2.2.9). Note that \( Q \)-function (2.2.10) has the natural interpretation as the eigenvalue of the Q-operator.

We construct the operator \( \hat{Q}(\lambda) \) in the standard representation of the group \( SL(2) \) in the following form:

\[
\hat{Q}(\lambda)\Psi(x) \equiv \langle RQ(\lambda; x, z)|\Psi(z) \rangle; \ RQ(\lambda; z) \equiv z^{-2s}Q(\lambda; z^{-1})
\]  

(3.0.2)

where \( R \) is the transformation of inversion. The scalar product here is the standard \( SL(2) \)-invariant scalar product for functions of the one variable:

\[
\langle \Psi(z)|\Phi(z) \rangle = \int_{|z|\leq 1} Dz \ \Psi(\bar{z})\Phi(z); \ Dz \equiv \frac{2s - 1}{\pi} \frac{d\bar{z}dz}{(1 - \bar{z}z)^{2-2s}}
\]  

(3.0.3)

and \( z \) is ”integration” or ”dumb” variable. In (3.0.2) the scalar product over all variables \( z_1...z_n \) is assumed. The \( SL(2) \)-generators \( S^\pm \) are conjugated with respect to this scalar product:

\[
\langle \Psi|S^\pm|\Phi \rangle = \langle S^\mp\Psi|\Phi \rangle; \ \langle \Psi|S|\Phi \rangle = \langle S\Psi|\Phi \rangle.
\]

Using the evident identities:

\[
R\Phi(z) = z^{-2s}\Phi(z^{-1}); \ RS^\pm\Phi(z) = -S^\mp R\Phi(z); \ RS\Phi(z) = -SR\Phi(z)
\]
we obtain the following rules for transposition:

\[
\langle RQ(\lambda; z) \mid S^\pm \Psi(z) \rangle = -\langle RS^\pm Q(\lambda; z) \mid \Psi(z) \rangle; \quad \langle Q(\lambda; z) \mid S\Psi(z) \rangle = -\langle RSQ(\lambda; z) \mid \Psi(z) \rangle
\]

(3.0.4)

In fact the construction of the Q-operator repeats the similar construction from the paper Pasquier and Gaudin [2].

The operator \( t(\lambda) \equiv \text{Tr} \, T(\lambda) \), where

\[
T(\lambda) \equiv L_1(\lambda + c_1)...L_n(\lambda + c_n),
\]

\[
L_k(\alpha_k) = \eta \cdot \left( \begin{array}{cc}
\alpha_k + x_k \partial_k + s_k & -\partial_k \\
x_k^2 \partial_k + 2s_kx_k & \alpha_k - x_k \partial_k - s_k
\end{array} \right) ; \quad \alpha_k = \frac{\lambda + c_k}{\eta}
\]

is invariant with respect to transformation of the local matrices \( L_k \) [1]:

\[
L_k \rightarrow \tilde{L}_k \equiv N_k^{-1}L_k N_{k+1}; \quad N_{n+1} \equiv N_1
\]

where \( N_k \) are the matrices with scalar elements. Simple calculation shows that matrix elements of the transformed matrix

\[
\tilde{L}_k = \begin{pmatrix}
\tilde{L}^{11}_k & \tilde{L}^{12}_k \\
\tilde{L}^{21}_k & \tilde{L}^{22}_k
\end{pmatrix}; \quad N_k = \begin{pmatrix}
0 & 1 \\
-1 & y_k
\end{pmatrix}
\]

have the form

\[
\tilde{L}^{11}_k = -(x_k - y_k)^{1+\alpha_k-s_k}\partial_k(x_k - y_k)^{s_k-\alpha_k}\\
\tilde{L}^{12}_k = -(x_k - y_k)^{1+\alpha_k-s_k}(x_k - y_{k+1})^{1-\alpha_k-s_k}\partial_k(x_k - y_k)^{s_k-\alpha_k}(x_k - y_{k+1})^{s_k+\alpha_k}\\
\tilde{L}^{21}_k = \partial_k; \quad \tilde{L}^{22}_k = (x_k - y_{k+1})^{1-\alpha_k-s_k}\partial_k(x_k - y_k)^{\alpha_k+s_k}.
\]

This expression for \( \tilde{L} \)-operator suggests to consider the function:

\[
\phi_k(\alpha_k; x_k; y_k, y_{k+1}) \equiv (x_k - y_k)^{\alpha_k-s_k}(x_k - y_{k+1})^{-\alpha_k-s_k}.
\]

The operators \( \tilde{L}^{ij}_k \) act on this function as follows:

\[
\tilde{L}^{11}_k \phi_k(\alpha_k) = (\alpha_k + s_k)\phi_k(\alpha_k + 1); \quad \tilde{L}^{12}_k \phi_k(\alpha_k) = 0
\]

\[
\tilde{L}^{22}_k \phi_k(\alpha_k) = (\alpha_k - s_k)\phi(\alpha_k - 1).
\]

Let us fix the dependence on the \( x \)-variables in the kernel of operator \( \hat{Q}(\lambda) \) in the form:

\[
Q(\lambda; x) \leftrightarrow \prod_{k=1}^n \phi(\alpha_k; x_k; y_k, y_{k+1}); \quad \eta\alpha_k = \lambda + c_k,
\]

where \( \{y_i\} \) - is the set of arbitrary parameters now. Then we have:

\[
t(\lambda)Q(\lambda; x; y) = \eta^n \cdot \text{Tr} \left( \prod_{k=1}^n \tilde{L}_k \phi_k(\alpha_k) = \prod_{k=1}^n \begin{pmatrix}
(\alpha_k + s_k)\phi_k(\alpha_k + 1) & 0 \\
\vdots & (\alpha_k - s_k)\phi_k(\alpha_k - 1)
\end{pmatrix}
\right).
\]
After multiplication of these triangular matrices and calculation of the trace we obtain the "right" Baxter’s relation:

\[ t(\lambda)Q(\lambda; x) = \Delta_+(\lambda)Q(\lambda + \eta; x) + \Delta_-(\lambda)Q(\lambda - \eta; x). \]

Next step we fix the dependence on the \( z \)-variables in the kernel of operator \( \hat{Q}(\lambda) \) to obtain the "left" Baxter’s relation:

\[ Q(\lambda; x, z)t(\lambda) = \Delta_+(\lambda)Q(\lambda + \eta; x, z) + \Delta_-(\lambda)Q(\lambda - \eta; x, z). \]

The rules (3.0.4) allow to move \( SL(2) \)-generators from the function \( \Psi(z) \) to the kernel of the \( Q \)-operator:

\[ \langle RQ(\lambda; x, z)|L_1...L_n\Psi(z)\rangle = \langle RL'_n...L'_1Q(\lambda; x, z)|\Psi(z)\rangle \]

where

\[ L'_k \equiv \eta \cdot \begin{pmatrix} \alpha_k - S_k & -S_k^- \\ -S_k^+ & \alpha_k + S_k \end{pmatrix} = [\sigma_2 \cdot L_k\sigma_2]^t, \]

and \( t \) means transposition. The trace of the product of \( L' \)-matrices can be calculated

\[ \text{Tr} L'_n...L'_1 = \text{Tr} [L_1...L_n]^t = \text{Tr} L_1...L_n, \]

and finally we obtain:

\[ \langle RQ(\lambda; x, z)|\text{Tr} [L_1...L_n]\Psi(z)\rangle = \langle R\text{Tr} [L_1...L_n]Q(\lambda; x, z)|\Psi(z)\rangle \]

Therefore the dependence of the kernel \( Q(\lambda; x, z) \) on the \( x \)- and \( z \)-variables have the same form.

In the sequel we shall concentrate on the case of homogeneous XXX-chain.

### 3.1 Q-operator for the homogeneous XXX-chain

In this section we consider the homogeneous XXX-chain of equal spins: \( c_i = 0 \), \( s_i = s \). The kernel:

\[ Q(\lambda; x, z) \equiv (-1)^{-2sn} \prod_{k=1}^{n} (x_k - z_k)^{-\frac{\alpha_k^+ - \lambda}{\eta}} (x_k - z_{k+1})^{-\frac{\alpha_k^+ + \lambda}{\eta}} \]

has the "true" \( x \)- and \( z \)-dependences and therefore \( Q \)-operator can be defined as follows:

\[ \hat{Q}(\lambda)\Psi(x) = \langle RQ(\lambda; x, z)|\Psi(z)\rangle = \prod_{k=1}^{n} \langle (1 - z_kx_k)^{-\frac{\alpha_k^+ - \lambda}{\eta}} (1 - z_kx_{k-1})^{-\frac{\alpha_k^+ + \lambda}{\eta}} |\Psi(z)\rangle. \]

There exists some useful integral representations for obtained \( Q \)-operator.
3.2 The $\alpha$-representation for the Q-operator

Let us consider the Q-operator:

$$\hat{Q}(\lambda)\Psi(x_1...x_n) \equiv \prod_{k=1}^{n} \langle (1 - x_{k-1}z_k)^{-\frac{\eta\lambda}{\eta}} (1 - x_k\bar{z}_k)^{-\frac{\eta\lambda}{\eta}} |\Psi(z_1...z_n) \rangle$$

and transform the $z_k$-integral using the following identity:

$$\int_{|z_k|\leq 1} Dz_k (1 - x_k\bar{z}_k)^{-a}(1 - x_{k-1}\bar{z}_k)^{-b} \Psi(z_k) =$$

$$= \frac{\Gamma(2s)}{\Gamma(a)\Gamma(b)} \cdot \int_{0}^{1} d\alpha \alpha^{a-1}(1 - \alpha)^{b-1} \Psi[\alpha x_k + (1 - \alpha)x_{k-1}] ; a + b = 2s. \quad (3.2.1)$$

To prove this identity we use the Feynman formula:

$$\frac{1}{A^a B^b} = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \cdot \int_{0}^{1} d\alpha \alpha^{a-1}(1 - \alpha)^{b-1} \frac{1}{[\alpha A + (1 - \alpha)B]^{a+b}} \quad (3.2.2)$$

and transform the product:

$$(1 - x_k\bar{z}_k)^{-a}(1 - x_{k-1}\bar{z}_k)^{-b} = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \cdot \int_{0}^{1} d\alpha \frac{\alpha^{a-1}(1 - \alpha)^{b-1}}{[1 - (\alpha x_k + (1 - \alpha)x_{k-1})\bar{z}_k]^{2s}}.$$ 

The remaining $z$-integral can be easily calculated:

$$\int_{|z_k|\leq 1} Dz_k (1 - z\bar{z})^{-2s} \Psi(z) = \Psi(x) ; x \equiv \alpha x_k + (1 - \alpha)x_{k-1}.$$ 

Finally we obtain the useful integral representation ($\alpha$-representation) for the Q-operator:

$$\hat{Q}(\lambda)\Psi(x) \equiv \prod_{k=1}^{n} \Gamma(\lambda; s) \int_{0}^{1} d\alpha_k \alpha_k^{\frac{\eta\lambda}{\eta}} \bar{\alpha}_k^{\frac{\eta\lambda}{\eta}} \Psi[...\alpha_k x_k + \bar{\alpha}_k x_{k-1}...], \quad (3.2.3)$$

where $\bar{\alpha} \equiv 1 - \alpha$ and

$$\Gamma(\lambda; s) \equiv \frac{\Gamma(2s)}{\Gamma(s + \lambda\eta^{-1})\Gamma(s - \lambda\eta^{-1})}.$$ 

Let us consider the eigenvalue problem for the Q-operator:

$$\hat{Q}(\lambda)\Psi(x) = Q(\lambda)\Psi(x),$$

where polynomial $\Psi(x)$ belongs to the space of homogeneous polynomials degree $l$ (2.2.8):

$$\Psi(x) = \sum_{p} \Psi_{p_1...p_n} x_1^{p_1}...x_n^{p_n} ; p_1 + p_2 + ... + p_n = l ; l = 0, 1, 2... \quad (3.2.4)$$
The Q-operator transforms polynomial \( \Psi(x) \) to homogeneous polynomial degree \( l \) whose coefficients are polynomials in \( \lambda \) degree \( l \). Therefore eigenvalues \( Q(\lambda) \) of the Q-operator are polynomials in \( \lambda \) degree \( l \).

For the proof we use obtained \( \alpha \)-representation. Let us consider the action of Q-operator on polynomial \( \Psi(x) \):

\[
\hat{Q}(\lambda)\Psi(x) \equiv \sum_p \Psi_{p_1...p_n} \prod_{k=1}^{n} \Gamma(\lambda; s) \int_0^1 d\alpha_k \alpha_k^{\frac{n_k-\lambda}{\eta}} \frac{1}{\alpha_k^{\frac{n_k+\lambda}{\eta}}-1} [\alpha_k x_k + \bar{\alpha}_k x_{k-1}]^{p_k}.
\]

The expression for the \( \alpha_k \)-integral have the form:

\[
\Gamma(\lambda; s) \int_0^1 d\alpha_k \alpha_k^{\frac{n_k-\lambda}{\eta}} \frac{1}{\alpha_k^{\frac{n_k+\lambda}{\eta}}-1} [\alpha_k x_k + \bar{\alpha}_k x_{k-1}]^{p_k} = \sum_{m=0}^{p_k} C_{p_k,m} x_k^m x_{k-1}^{p_k-m},
\]

where coefficients \( C_{p_k,m} \)

\[
C_{p_k,m} = \frac{p_k!}{m!(p_k-m)!} \frac{\Gamma(2s)}{\Gamma(2s+p_k)} \frac{\Gamma(s-\lambda \eta^{-1}+m)}{\Gamma(s-\lambda \eta^{-1})} \frac{\Gamma(s+\lambda \eta^{-1}+p_k-m)}{\Gamma(s+\lambda \eta^{-1})}
\]

are polynomials in \( \lambda \) degree \( p_k \) because of evident equality:

\[
\frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)...(a+m-1).
\]

There are similar expressions for the remaining \( \alpha \)-integrals and we obtain that Q-operator transforms polynomial \( \Psi(x) \) to homogeneous polynomial degree \( l \) whose coefficients are polynomials in \( \lambda \) degree \( p_1 + p_2 + ... + p_n = l \).

There exists some another useful representation for the Q-operator(t-representation):

\[
\hat{Q}(\lambda)\Psi(x) \equiv \prod_{k=1}^{n} \frac{\Gamma(\lambda; s)}{(x_k - x_{k-1})^{2s-1}} \int_{x_{k-1}}^{x_k} dt_k (t_k - x_{k-1})^{\frac{n_k-\lambda}{\eta}-1} (x_k - t_k)^{\frac{n_k+\lambda}{\eta}-1} \Psi[...t_k...].
\]

This formula is obtained from the (3.2.3) by the following change of variables:

\[
t_k = \alpha_k x_k + \bar{\alpha}_k x_{k-1}.
\]

3.3 \( SL(2) \)-invariance of the Q-operator. Commutativity

We shall prove the two important properties of obtained Q-operator: \( SL(2) \)-invariance and commutativity. Let us begin from the \( SL(2) \)-invariance:

\[
S\hat{Q}(\lambda)\Psi(x) = \hat{Q}(\lambda)S\Psi(x) ;
S\Psi(x) = (cx + d)^{-2s}\Psi(Sx) ;
Sx \equiv \frac{ax + b}{cx + d}
\]

11
The simplest way is to use the representation (3.2.5). We start from $S\hat{Q}(\lambda)$:

$$S\hat{Q}(\lambda)\Psi(x) \sim (Sx_k - Sx_{k-1})^{-2s+1} \cdot \int_{Sx_{k-1}}^{Sx_k} \Psi(t) \cdot \left( t - Sx_{k-1} \right)^{\frac{a_s - \lambda}{\eta} - 1} \left( Sx_k - t \right)^{\frac{a_s + \lambda}{\eta} - 1} dt$$

and make the change of variable in $t$-integral:

$$t = S\tau = \frac{a\tau + b}{c\tau + d}; \ S\tau - Sx = \frac{\tau - x}{(c\tau + d)(cx + d)}; \ dt = \frac{d\tau}{(c\tau + d)^2}.$$ 

After this change of variable the $t$-integral is transformed to the $\tau$-integral of required form:

$$(x_k - x_{k-1})^{-2s+1} \cdot \int_{x_{k-1}}^{x_k} \Psi(S\tau) \sim \hat{Q}(\lambda)S\Psi(x).$$

It is worth to emphasize that all factors like $(cx_k + d)^{-2s}$ are cancelled in the whole product.

The second important property of Q-operator is commutativity:

$$\hat{Q}(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)\hat{Q}(\mu). \quad (3.3.1)$$

It follows that there exists a unitary operator $U$ independent on $\lambda$ which diagonalizes $\hat{Q}(\lambda)$ simultaneously for all values of $\lambda$ and therefore due to the Baxter relation operators $\hat{Q}(\lambda)$ and $t(\mu)$ commute also:

$$t(\mu)\hat{Q}(\lambda) = \hat{Q}(\lambda)t(\mu). \quad (3.3.2)$$

It is useful to visualize the Q-operator itself and the product of two Q-operators as shown in figure: the line with index $a$ between the points $x$ and $z$ represents the function $(1 - xz)^{-a}$ where $a = \frac{ns - \lambda}{\eta}$ and $b = \frac{ns - \mu}{\eta}$. The integration (3.0.3) in any four-point vertex is supposed.
Let us consider the product $\hat{Q}(\lambda)\hat{Q}(\mu)$ and the corresponding kernel:

$$\langle Q(\lambda; x) \mid Q(\mu; y) \rangle \equiv \prod_{k=1}^{n} \langle (1-x_{k-1}y_{k})^{\frac{\lambda-n_s}{n}} (1-x_{k}y_{k})^{\frac{\lambda-n_s}{n}} \mid (1-y_{k}z_{k+1})^{\frac{\mu-n_s}{n}} (1-y_{k+1}z_{k})^{\frac{\mu-n_s}{n}} \rangle.$$  

The "mechanism" of commutativity is shown in figure [2]:

![Diagram](image-url)

and is grounded on the "local" identity:

$$\langle (1-x_{k-1}y)^{-2s+a} (1-x_{k}y)^{-a} (1-yz)^{-b} (1-yz_{k+1})^{-2s+b} \rangle \cdot (1-z_{k}x_{k-1})^{a-b} = (3.3.3)$$

$$= \langle (1-x_{k-1}y)^{-2s+b} (1-x_{k}y)^{-b} (1-yz)^{-a} (1-yz_{k+1})^{-2s+a} \rangle \cdot (1-z_{k+1}x_{k})^{a-b}.$$  

The graphic representation of this identity is shown in figure (a, b are arbitrary parameters):

![Diagram](image-url)

The proof of the equality (3.3.3) can be found in Appendix.

### 3.4 Eigenvalues of the $Q$ - operator for $n = 2$

The case $n = 2$ is the simplest one:

$$t(\lambda) = 2\lambda^2 + 2\eta^2 \vec{S}_1 \cdot \vec{S}_2 = 2\lambda^2 - 2\eta^2 s(s-1) + \eta^2 L.$$  

13
There exists only one integral of motion – two-particle Casimir: \( L \equiv (\vec{S}_1 + \vec{S}_2)^2 \). Its highest weight eigenfunctions have the form:

\[
\Psi_l(x_1, x_2) = (x_1 - x_2)^l ; \quad L\Psi_l = (l + 2s)(l + 2s - 1)\Psi_l.
\]

Due to \( SL(2) \)-invariance these functions are eigenfunctions for the \( Q \)-operator also. Let us calculate the eigenvalue \( Q_l(\lambda) \):

\[
\hat{Q}(\lambda)\Psi_l = Q_l(\lambda) \cdot \Psi_l.
\]

The simplest way is to use the \( \alpha \)-representation:

\[
\hat{Q}(\lambda)\Psi_l \equiv \Gamma^2(\lambda; s) \int_0^1 d\alpha d\beta \frac{\alpha s}{\alpha + \beta} - 1 \cdot \Psi_l[\alpha x_1 + \alpha x_2; \beta x_2 + \beta x_1],
\]

so that we obtain:

\[
Q_l(\lambda) = (-1)^l \Gamma^2(\lambda; s) \int_0^1 d\alpha d\beta \frac{\alpha s}{\alpha + \beta} - 1 \cdot \Psi_l[\alpha x_1 + \alpha x_2; \beta x_2 + \beta x_1],
\]

The eigenvalue \( Q_l(\lambda) \) was obtained in equivalent form in the paper \( [7] \) and polynomials (in \( \lambda \) \( Q_l(\lambda) \)) coincide with the Hanh orthogonal polynomials.

### 4 Q-operator for \( \lambda = \pm \eta s \) and local Hamiltonians

Let us consider the \( Q \)-operator in \( \alpha \)-representation:

\[
\hat{Q}(\lambda)\Psi(x) \equiv \prod_{k=1}^n \Gamma(\lambda; s) \int_0^1 d\alpha_k d\beta_k \frac{\alpha_k}{\alpha_k + \beta_k} - 1 \cdot \Psi_l[\alpha_k x_k + \beta_k x_k - 1],
\]

for the special value of spectral parameter \( \lambda = \eta s + \eta \epsilon \) and calculate the first two terms of the \( \epsilon \)-expansion.

We start from the \( \alpha_k \)-integral:

\[
\frac{\Gamma(2s)}{\Gamma(2s + \epsilon)\Gamma(-\epsilon)} \cdot \int_0^1 d\alpha d\epsilon \frac{\alpha^{2s+\epsilon-1}}{\alpha^{2s+\epsilon-1}} \cdot \Psi_l[\alpha x_k + \beta x_k - 1].
\]

The prefactor in this expression is proportional to \( \epsilon \) and there exists the singular \( \epsilon \)-pole term in the \( \alpha \)-integral because of singularity in the point \( \alpha = 0 \). For the calculation of the \( \epsilon \)-pole term one can put \( \alpha = 0 \) in the argument of the \( \Psi \)-function:

\[
\int_0^1 d\alpha d\alpha^{2s+\epsilon-1} \cdot \Psi_l[\alpha x_k + \beta x_k - 1] \rightarrow \frac{\Gamma(-\epsilon)\Gamma(2s + \epsilon)}{\Gamma(2s)} \cdot \Psi_l[\alpha x_k + \beta x_k - 1].
\]

In the main order of \( \epsilon \)-expansion we need the singular part of the integral only and have:

\[
\hat{Q}(\eta s)\Psi(x_1, x_2...x_k...x_n) = \Psi(x_n, x_1...x_k-1...x_n-1). \quad (4.0.1)
\]
Therefore the $Q$-operator for $\lambda = \eta s$ coincides with the “shift” operator $P$:

$$P\Psi(x_1, x_2, \ldots, x_k \ldots x_n) = \Psi(x_n, x_1, x_{k-1} \ldots x_{n-1}); \quad \dot{Q}(\eta s) = P.$$ 

In the next order of the $\epsilon$-expansion we have to extract the $\epsilon$-pole contributions from the $n - 1$ $\alpha$-integrals and the next term of the $\epsilon$-expansion from the one remaining integral. This remaining $\alpha$-integral has the form:

$$\frac{\Gamma(2s)}{\Gamma(2s + \epsilon)\Gamma(-\epsilon)} \cdot \int_0^1 d\alpha_1 \alpha_k^{-\epsilon - 1} \frac{2s + \epsilon - 1}{2} \Psi(...x_{k-2}, \alpha_k x_k + \bar{\alpha}_k x_{k-1}, x_k \ldots).$$

Note that $\epsilon$-pole contributions effectively shift all arguments of the $\Psi(x_1 \ldots x_n)$-function except for the $k$-th one. For the calculation of $\alpha$-integral it is useful to add and subtract the pole term:

$$\frac{\Gamma(2s)}{\Gamma(2s + \epsilon)\Gamma(-\epsilon)} \cdot \int_0^1 d\alpha \alpha^{-\epsilon - 1} \frac{2s + \epsilon - 1}{2} \left[ \Psi(\alpha x_k + \bar{\alpha} x_{k-1}) \pm \Psi(x_{k-1}) \right].$$

The integral with the difference is regular so we can put $\epsilon = 0$ in integrand and extract the needed contribution:

$$-\epsilon \int_0^1 d\alpha_1 \frac{2s - 1}{\bar{\alpha}} \left[ \Psi(\alpha x_k + \bar{\alpha} x_{k-1}) - \Psi(x_{k-1}) \right] + \Psi(x_{k-1}).$$

Finally we obtain the first two terms in the $\epsilon$-expansion of the $Q$-operator:

$$\dot{Q}(\eta s + \eta \epsilon) = P + \epsilon \sum_{k=1}^n H_{k-1,k}^- + O(\epsilon^2),$$

where the operator $H_{k-1,k}^-$ is defined as follows:

$$H_{k-1,k}^- \Psi(x_1 \ldots x_k \ldots x_n) = - \int_0^1 d\alpha \frac{2s - 1}{\bar{\alpha}} \left[ \Psi(...x_{k-2}, \alpha x_k + \bar{\alpha} x_{k-1}, x_k \ldots) - \Psi(...x_{k-2}, x_{k-1}, x_k \ldots) \right].$$

Note this ”two-particle” operator is not $SL(2)$-invariant.

In a similar way one can calculate the first two terms of $\epsilon$-expansion for $\lambda = -\eta s$:

$$\dot{Q}(-\eta s + \eta \epsilon) = 1 - \epsilon \sum_{k=1}^n H_{k-1,k}^+ + O(\epsilon^2),$$

where:

$$H_{k-1,k}^+ \Psi(x_1 \ldots x_k \ldots x_n) = - \int_0^1 d\alpha \frac{2s - 1}{\bar{\alpha}} \left[ \Psi(...x_{k-1}, \alpha x_k + \bar{\alpha} x_{k-1}, x_{k+1} \ldots) - \Psi(...x_{k-1}, x_k, x_{k+1} \ldots) \right].$$

Let us consider the $\epsilon$-expansion of the following combination of the $Q$-operators:

$$\dot{Q}^{-1}(\eta s)\dot{Q}(\eta s + \eta \epsilon) - \dot{Q}^{-1}(-\eta s)\dot{Q}(-\eta s + \eta \epsilon) = \epsilon \sum_{k=1}^n H_{k-1,k} + O(\epsilon^2).$$
Using the expressions for the operators $H_{k-1,k}^-$ and $H_{k-1,k}^+$ it is easy to check that operator $H_{k-1,k}^-$ acts on the variables $z_{k-1}, z_k$ only and coincides to the integral operator considered in (2.3.3). Finally we have found the following operator relations:

- $\hat{Q}(-\eta s) = 1$
- $\hat{Q}(\eta s) = P$
- $\hat{Q}^{-1}(\eta s)\hat{Q}(\eta s + \eta \epsilon) - \hat{Q}^{-1}(-\eta s)\hat{Q}(-\eta s + \eta \epsilon) = \epsilon H + O(\epsilon^2)$; $H \equiv \sum_{k=1}^n H_{k-1,k}$

Let us compare these relations to the one obtained by the ABA method (2.3.4, 2.3.6). The first relation fixes normalization of the $Q$-operator and the normalization of the eigenvalues of $Q$-operator:

$$Q(\lambda) = \prod_{j=1}^l \frac{\lambda - \lambda_j}{\eta s - \lambda_j}$$  \hspace{1cm} (4.0.2)

The second relation allows to express the eigenvalues of the "shift" $P$-operator in terms of the function $Q(\lambda)$:

$$P_i = Q(\eta s) = \prod_{j=1}^l \frac{\lambda_j - \eta s}{\lambda_j + \eta s}$$

in agreement with (2.3.6). The third relation is the operator version of the equality (2.3.4).

5  Asymptotic expansion of $Q$-operator for $\lambda \to \infty$

L.N. Lipatov [9] has found some beautiful symmetry of the XXX model: duality transformation. In this section we show that operator $S$ of duality arises naturally as leading term in asymptotic of $Q$-operator for large $\lambda$.

To start with let us define some transformation which is analogous to the Fourier transformation from the coordinate representation to the momentum representation.

5.1 Momentum representation

Let us define the transformation $T$ from the function $\bar{\Psi}(x)$ in "momentum" representation to the used so far function $\Psi(x)$ in "coordinate" representation:

$$\Psi(x) = T[\bar{\Psi}(x)] ; \Psi(x_1...x_n) \equiv \bar{\Psi}(\partial_{a_1}...\partial_{a_n}) \left. \prod_{k=1}^n \frac{1}{[1 - a_k x_k]^{2s}} \right|_{a=0}.$$  

This transformation maps polynomials to polynomials and can be represented as composition of Laplace transformation and inversion:

$$T[\bar{\Psi}(x)] = \frac{1}{\Gamma(2s)} R \int_0^\infty dt e^{-tx} t^{2s-1} \bar{\Psi}(t).$$

16
Using the well known properties of Laplace transformation and eqs. (3.0.4) it is easy to derive the expression for the $SL(2)$-generators in "momentum" representation:

\[
T \left[ x \Psi(x) \right] = [x^2 \partial + 2sx] \Psi(x) ;
T \left[ (x\partial^2 + 2s\partial) \Psi(x) \right] = \partial \Psi(x)
\]

\[
T \left[ (x\partial + s) \Psi(x) \right] = (x\partial + s) \Psi(x).
\]

To obtain the rules for the transformation of commuting operators $Q_k$ (2.2.3) from one representation to the another we start from the very beginning and consider transformation of the $L$-operator. The $L$-operator in coordinate representation is the $T$-transformation from the $L'$-operator in "momentum" representation:

\[
L = \left( \begin{array}{cc}
\lambda + \eta [x \partial + s] & -\eta \\
\eta [x^2 \partial + 2s\partial] & \lambda - \eta [x \partial + s]
\end{array} \right) ;
L' = \left( \begin{array}{cc}
\lambda + \eta [x \partial + s] & -\eta [x \partial^2 + 2s\partial] \\
\eta x & \lambda - \eta [x \partial + s]
\end{array} \right) = \sigma_2 \bar{L} \sigma_2,
\]

where

\[
\bar{L} = \left( \begin{array}{cc}
\lambda - \eta [x \partial + s] & -\eta x \\
\eta [x \partial^2 + 2s\partial] & \lambda + \eta [x \partial + s]
\end{array} \right).
\]

The $\sigma_2$-matrices are cancelled for the transfer matrix and we can work directly with $\bar{L}$.

Finally we have the formal rules for the transformation from one representation to the another one:

\[
\eta \to -\eta ;
x \to \partial,
\]

and operators $x$ and $\partial$ have to be "normal ordered": all $\partial$ stay on the right from the $x$.

### 5.2 Asymptotic expansion for large $\lambda$

Let us calculate the asymptotic of the $Q$-operator:

\[
\hat{Q}(\lambda) \Psi(x) = \prod_{k=1}^{n} \Gamma(\lambda; s) \int_0^1 \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \!
\[= \Gamma^{-1}(\lambda; s)\Psi[\partial_{a_k}] \frac{1}{[1 - a_k x_k]^{s-\lambda \eta^{-1}}[1 - a_k x_{k-1}]^{s+\lambda \eta^{-1}}},\]

where the Feynman formula (3.2.2) is used in "opposite" direction.

For the calculation of the asymptotic it is useful to rescale variables \(a_i\) and use the standard expansion for logarithm:

\[a_i \rightarrow \eta a_i \frac{\lambda}{\lambda}; \quad \left(1 - \frac{ax}{\lambda}\right)^{\lambda-s} = \exp\left\{\frac{-ax}{\lambda} + \frac{2sa x - a^2 x^2}{2\lambda} + \ldots\right\}.\]

Let us consider the contribution with \(a_k\):

\[\bar{\Psi}\left(\partial_{a_k}\right) \frac{1}{[1 - a_k x_k]^{s-\lambda \eta^{-1}}[1 - a_k x_{k-1}]^{s+\lambda \eta^{-1}}} \bigg|_{a_k=0} = \bar{\Psi}(\lambda \eta^{-1} \partial) \exp\left\{a(x_{k-1} - x_k) + \eta(x_{k-1} + x_k) (2sa + (x_{k-1} - x_k)a^2) + \ldots\right\} \bigg|_{a=0} = \bar{\Psi}(\lambda \eta^{-1} z) + \frac{\eta(x_{k-1} + x_k)}{2\lambda} (x \partial^2 + 2s \partial) \bar{\Psi}(\lambda \eta^{-1} x) \bigg|_{x=x_{k-1}-x_k} + O(\lambda^{-2})\]

Polynomial \(\Psi(x)\) is homogeneous so that:

\[\bar{\Psi}(\lambda \eta^{-1} x) = (\lambda \eta^{-1})^l \bar{\Psi}(x)\]

and finally we obtain the first two terms of asymptotic expansion of the Q-operator for large \(\lambda\):

\[\hat{Q}(\lambda) = \sum_{k=0}^{l} \hat{Q}_k \cdot (\lambda \eta^{-1})^{l-k}; \quad \hat{Q}_0 \Psi(x) = \bar{\Psi}(x_n - x_1, x_1 - x_2, \ldots, x_{n-1} - x_n) \quad (5.2.2)\]

\[\hat{Q}_1 \Psi(x) = \frac{1}{2} \sum_{k=1}^{n} (x_k + x_{k-1})(z_k \partial_{z_k}^2 + 2s \partial_{z_k}) \bar{\Psi}(z) \bigg|_{z_k=x_{k-1}-x_k}.\]

It seems that all operators \(\hat{Q}_k\) are local differential operators in momentum representation.

The operator \(S\) of duality transformation is defined in the following way:

\[S \Psi(x_1 \ldots x_n) \equiv \bar{\Psi}(x_n - x_1, x_1 - x_2, \ldots, x_{n-1} - x_n).\]

This operator coincides with the leading term \(\hat{Q}_0\) of asymptotic expansion and therefore operator \(S\) commutes with all integrals of motion \(Q_k\) (2.2.3). Its common eigenfunction have to be eigenfunction of the \(S\)-operator:

\[S \Psi(x_1 \ldots x_n) = S_l \cdot \Psi(x_1 \ldots x_n) \Leftrightarrow \Psi(x_n - x_1, x_1 - x_2, \ldots, x_{n-1} - x_n) = S_l \cdot \Psi(x_1 \ldots x_n)\]

Note that subspace of the common eigenvectors of \(Q_k\) with some eigenvalues \(q_k\) is the \(SL(2)\) - module generated by highest weight vector \(\Psi\). The highest weight vector \(\Psi\) is
defined by the equation $S^-\Psi = 0$ and can be constructed by the ABA method (2.2.8). From the expression for the eigenvalue of the Q-operator (4.0.2) one can derives the expression for the $S_l$:

$$Q(\lambda) = \prod_{j=1}^{l} \frac{\lambda - \lambda_j}{-\eta s - \lambda_j} \rightarrow \frac{(-\lambda)^l}{\prod_{j=1}^{l}(\eta s + \lambda_j)} ; \hat{Q}(\lambda) \rightarrow (\lambda\eta^{-1})^l \cdot S.$$ 

Therefore the eigenvalue of the duality operator for the highest weight vector of $SL(2)$-module has the form:

$$S_l = \frac{(-\eta)^l}{\prod_{j=1}^{l}(\eta s + \lambda_j)}.$$ 

It is easy to see that all other vectors from $SL(2)$-module form the "zero" subspace:

$$\Phi(x_1...x_n) = S^+\Psi(x_1...x_n) \rightarrow S\Phi(x_1...x_n) = S(x_1 + ... + x_n)\Psi(x_1...x_n) = 0.$$ 

6 Conclusions.

We have constructed the Baxter’s Q-operator for the homogeneous XXX spin chain and have checked the consistence of obtained results with the corresponding formulae obtained in the framework of the ABA-method. We have found the connection between Q-operator and operator of duality symmetry.

The considered construction can be applied to the inhomogeneous XXX-model but we are not able to obtain the useful and compact representation for the Q-operator in this case.

There exists some more universal approach to the construction of the quantum Q-operator. As V.B.Kuznetsov and E.K.Sklyanin informed me, they have recently obtained similar results [11] using the approach of [10] based on correspondence between the quantum Q-operator and classical Bäcklund transformation.

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Appendix

In this Appendix we prove the identity:

$$\langle (1 - x_{k-1}y)^{-2s+a}(1 - x_ky)^{-a}|(1 - yz_k)^{-b}(1 - yz_{k+1})^{-2s+b}\cdot (1 - z_kx_{k-1})^{a-b} =$$
This transformation can be obtained as follows:

\[ ((1 - x_{k-1}y) - 2s + b(1 - x_ky)^{-b} | (1 - yz_k)^{-a} (1 - yz_{k+1})^{-2s + a} \cdot (1 - z_{k+1}x_k)^{a-b} \]  \hspace{1cm} (B.1)

First of all note that the equality (3.2.1) can be rewritten in integral form:

\[ \langle (1 - x_{k-1}z)^{-a} (1 - x_{k-1}z)^{-b} | \Psi(z) \rangle = \frac{\Gamma(2s)}{\Gamma(a)\Gamma(b)} \frac{1}{(x_k - x_{k-1})^{2s-1}} \int_{x_{k-1}}^{x_k} dt (t - x_{k-1})^{b-1} (x_k - t)^{a-1} \Psi(t) \]

and therefore the identity (3.1) is equivalent to the following identity for integrals:

\[ \frac{\Gamma(2s)}{\Gamma(2s - a)\Gamma(a)} \frac{(1 - x_{k-1}z_k)^{a-b}}{(x_k - x_{k-1})^{2s-1}} \int_{x_{k-1}}^{x_k} \frac{dt (t - x_{k-1})^{a-1} (x_k - t)^{2s-a-1}}{(1 - tz_k)^b (1 - tz_{k+1})^{2s-b}} = \]

\[ = \frac{\Gamma(2s)}{\Gamma(2s - b)\Gamma(b)} \frac{(1 - x_kz_{k+1})^{a-b}}{(x_k - x_{k-1})^{2s-1}} \int_{x_{k-1}}^{x_k} \frac{d\tau (\tau - x_{k-1})^{b-1} (x_k - \tau)^{2s-b-1}}{(1 - \tau z_k)^a (1 - \tau z_{k+1})^{2s-a}} \]  \hspace{1cm} (B.2)

Let us start from the t-integral. There exists the bilinear transformation with the properties:

\[ z = Sx = \frac{Ax - C}{Cx + D} \; ; \; Sx_k = z_k \; , \; Sx_{k-1} = z_{k+1} \]

This transformation can be obtained as follows:

\[ \frac{z - z_k}{z - z_{k+1}} = R \frac{x - x_k}{x - x_{k-1}} \Rightarrow z = \frac{x (x_k - x_{k-1}R) + z_k x_{k-1}R - x_kz_{k+1}}{x(1 - R) + z_kR - z_{k+1}} = \frac{Ax - C}{Cx + D} \]

and therefore:

\[ A = (x_k - x_{k-1}R) \; ; \; D = z_kR - z_{k+1} \; ; \; C = 1 - R \; ; \; R = \frac{1 - x_kz_{k+1}}{1 - z_kx_{k-1}} \]

It is worth to emphasize the additional properties:

\[ Sz_k = x_k \; , \; Sz_{k+1} = x_{k-1} \; ; \; \frac{Cz_{k+1} + D}{Cz_k + D} = R \]

Let us make the same bilinear transformation in t-integral:

\[ t = S\tau = \frac{A\tau - C}{C\tau + D} \; , \; 1 - tz_k = \frac{Cz_k + D}{C\tau + D}(1 - \tau x_k) \; ; \; x_k = S\tau_k \; , \; x_{k-1} = Sz_{k+1} \]

Then we obtain:

\[ \frac{1}{(x_k - x_{k-1})^{2s-1}} \int_{x_{k-1}}^{x_k} \frac{dt (t - x_{k-1})^{a-1} (x_k - t)^{2s-a-1}}{(1 - tz_k)^b (1 - tz_{k+1})^{2s-b}} = \]

\[ = \frac{(Cz_{k+1} + D)^{a-b}}{(Cz_k + D)^{a-b}} \frac{1}{(z_k - z_{k+1})^{2s-1}} \int_{z_{k+1}}^{z_k} \frac{d\tau (\tau - z_{k+1})^{a-1} (z_k - \tau)^{2s-a-1}}{(1 - \tau x_k)^b (1 - \tau x_{k-1})^{2s-b}} \]
On the next step we transform the $\tau$-integral using the $\alpha$-representation and the Feynman formula:

$$\frac{1}{(z_k - z_{k+1})^{2s-1}} \cdot \int_{z_{k+1}}^{z_k} d\tau \frac{(\tau - z_{k+1})^{a-1}(z_k - \tau)^{2s-a-1}}{(1 - \tau x_k)^b(1 - \tau x_{k-1})^{2s-b}} = \int_0^1 d\alpha \frac{\alpha^{a-1}\bar{\alpha}^{2s-a-1}}{\Gamma(2s)} \cdot \int_0^1 d\beta \frac{\beta^{b-1}\bar{\beta}^{2s-b-1}}{\Gamma(2s-b)} \cdot \int_0^1 d\alpha \frac{\alpha^{a-1}\bar{\alpha}^{2s-a-1}}{\Gamma(a)\Gamma(2s-a)} \cdot \frac{1}{\Gamma(b)\Gamma(2s-b)} \cdot \int_{x_{k-1}}^{x_k} d\tau \frac{(\tau - x_{k-1})^{b-1}(x_k - \tau)^{2s-b-1}}{(1 - \tau z_k)^a(1 - \tau z_{k+1})^{2s-a}}.$$

Collect all together we obtain the equality (B.2).

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