Geodesic Rays and Kähler–Ricci Trajectories on Fano Manifolds

Tamás Darvas* and Weiyong He†

Abstract

Suppose \((X, J, \omega)\) is a Fano manifold and \(t \to r_t\) is a diverging Kähler-Ricci trajectory. We construct a bounded geodesic ray \(t \to u_t\) weakly asymptotic to \(t \to r_t\), along which Ding’s \(F\)–functional decreases. In absence of non-trivial holomorphic vector fields this proves the equivalence between geodesic stability of the \(F\)–functional and existence of Kähler–Einstein metrics. We also explore applications of our construction to Tian’s \(\alpha\)–invariant.

1 Introduction and Main Results

We consider a Fano manifold \((X, J, \omega)\). The space of smooth Kähler potentials \(\mathcal{H}\) is the set

\[
\mathcal{H} = \{u \in C^\infty(X) | \omega_u := \omega + i\partial\bar{\partial}u > 0\}.
\]

Clearly, \(\mathcal{H}\) is a Fréchet manifold as an open subset of \(C^\infty(X)\). For \(v \in \mathcal{H}\) one can identify \(T_v \mathcal{H}\) with \(C^\infty(X)\). Given \(1 \leq p < \infty\), one can introduce a Finsler-metric on \(\mathcal{H}\):

\[
\|\xi\|_{p,u} = \left(\frac{1}{\text{Vol}(X)} \int_X |u|^p \omega_u^n\right)^{\frac{1}{p}}, \quad \xi \in T_v \mathcal{H},
\]

where \(\text{Vol}(X) = \int_X \omega^n\). This is a generalization of the Mabuchi Riemannian metric initially investigated in [Ma, Se, Do], which corresponds to the case \(p = 2\). This Finsler structure (along with a more general situation involving Orlicz norms) was studied extensively in [Da4]. Below and in Section 2.1 we summarize the results that we will need the most from this work. As usual, the length of a smooth curve \([0,1] \ni t \to \alpha_t \in \mathcal{H}\) is computed by the formula:

\[
l_p(\alpha) = \int_0^1 \|\dot{\alpha}_t\|_{p,\alpha_t} \, dt.
\]

The path length distance \(d_p(u_0, u_1)\) between \(u_0, u_1 \in \mathcal{H}\) is the infimum of the length of smooth curves joining \(u_0, u_1\). In [Da4] it is proved that \(d_p(u_0, u_1) = 0\) if and only if \(u_0 = u_1\), thus \((\mathcal{H}, d_p)\) is a metric space, which is a generalization of a result of X.X. Chen in the case \(p = 2\) [C].

Let us recall some facts about geodesic segments in the Riemannian case \(p = 2\). Suppose \(S = \{0 < \text{Re } s < 1\} \subset \mathbb{C}\). Following [Se], one can compute that a smooth

\*Research supported by BSF grant 2012236.
†Research supported partially by NSF grant 1005392.
2010 Mathematics subject classification 53C55, 32W20, 32U05.
Theorem 6. By [Da4, Theorem 2] this curve is a t
The curve sequences. We define the metric For a quick review of finite energy classes we refer to [Da3, Section 2.3].

\[ u(t + ir, x) = u(t, x) \forall x \in X, t \in (0, 1), r \in \mathbb{R} \]
\[ u(0, x) = u_0(x), u(1, x) = u_1(x), x \in X. \]

Unfortunately the above problem does not have smooth solutions (see [LV, Da3]), but a unique solution in the sense of Bedford-Taylor does exist such that \( i \partial \bar{\partial} u \) has bounded coefficients (see \([C]\) with complements in \([B]\)). The most general result about regularity was proved in [BD, Brm2] (see \([H1]\) for a different approach) but regularity higher then \( C^{1,\alpha} \) is not possible by examples provided in \([DL]\).

The curve \( t \to u_t \) is called the weak geodesic joining \( u_0, u_1 \). As argued in [Da4], this same curve interacts well with all the metrics \( d_p \), i.e.

\[ d_p(u_0, u_1) = \| \partial_t u \|_{p, u_t}, \quad t \in [0, 1], p \geq 1, \]

and \( t \to u_t \) is an actual metric geodesic joining \( u_0, u_1 \) in the metric completion \((\mathcal{H}, d_p) = (\mathcal{E}^p(X, \omega), d_p)\) that we introduce now.

Given \( u \in \text{PSH}(X, \omega) \), as explained in [GZ1], one can define the non-pluripolar measure \( \omega_u^n \) that coincides with the usual Bedford-Taylor volume when \( u \) is bounded. We say that \( \omega_u^n \) has full volume \((u \in \mathcal{E}(X, \omega))\) if \( \int_X \omega_u^n = \int_X \omega^n \). Given \( v \in \mathcal{E}(X, \omega) \), we say that \( v \in \mathcal{E}^p(X, \omega) \) if

\[ E_p(v) = \int_X |v|^p \omega_v^n < \infty, \]

For a quick review of finite energy classes we refer to [Da3 Section 2.3].

Next we introduce a geodesic metric space structure on \( \mathcal{E}^p(X, \omega) \), following [Da4]. Suppose \( u_0, u_1 \in \mathcal{E}^p(X, \omega) \). Let \( \{ u_0^k \}_{k \in \mathbb{N}}, \{ u_1^k \}_{k \in \mathbb{N}} \subset \mathcal{H} \) be sequences decreasing pointwise to \( u_0 \) and \( u_1 \) respectively. By [BK, Da] it is always possible to find such approximating sequences. We define the metric \( d_p(u_0, u_1) \) as follows:

\[ d_p(u_0, u_1) = \lim_{k \to \infty} d_p(u_0^k, u_1^k). \]

As justified in [Da4 Theorem 2] the above limit exists, is well defined, and \( d_p(u_0, u_1) = 0 \) implies \( u_0 = u_1 \). Let us also define geodesics in this space. Let

\[ u_t^k : [0, 1] \to \mathcal{H}_\Delta := \text{PSH}(X, \omega) \cap \{ \Delta u \in L^\infty \} \]

be the weak geodesic joining \( u_0^k, u_1^k \). We define \( t \to u_t \) as the decreasing limit:

\[ u_t = \lim_{k \to \infty} u_t^k, \quad t \in (0, 1). \]

The curve \( t \to u_t \) is well defined and \( u_t \in \mathcal{E}^p(X, \omega), \quad t \in (0, 1), \) as follows from [Da3 Theorem 6]. By [Da4 Theorem 2] this curve is a \( d_p \)-geodesic joining \( u_0, u_1 \) and we have

\[ (\mathcal{H}, d_p) = (\mathcal{E}^p(X, \omega), d_p), \quad p \geq 1. \]
Recall that by a $\rho$-geodesic in a metric space $(M, \rho)$ we understand a curve $(a, b) \ni t \mapsto g_t \in M$ for which there exists $C > 0$ satisfying:

$$\rho(g_{t_1}, g_{t_2}) = C|t_1 - t_2|, \ t_1, t_2 \in (a, b).$$

By the definition, we have the inclusion $\mathcal{E}^p(X, \omega) \subset \mathcal{E}^{p'}(X, \omega)$, for $p' \leq p$ and also the metric $d_p$ dominates $d_{p'}$. What is more, it follows that for $u_0, u_1 \in \mathcal{E}^p(X, \omega)$, the curve defined in (9) is a geodesic with respect to both $d_p$ and $d_{p'}$ (perhaps of different length).

Lastly, by the definition of the finite energy classes we have the inclusion

$$\mathcal{H}_0 = \text{PSH}(X, \omega) \cap L^\infty(X) \subset \bigcap_{p \geq 1} \mathcal{E}^p(X, \omega).$$

By the above, for $u_0, u_1 \in \mathcal{H}_0$, the curve $(0, 1) \ni t \mapsto u_t \in \mathcal{H}_0$ from (9) will be a $d_p$-geodesic joining $u_0, u_1$ for all $p \geq 1$. This observation will be crucial in future arguments.

Functionals play an important role in the investigation of special Kähler metrics. Recall that the Aubin-Mabuchi energy and Ding’s $\mathcal{F}$–functional are defined as follows:

$$AM(v) = \frac{1}{(n+1)\text{Vol}(X)} \sum_{j=0}^{n} \int_X u \omega^j \wedge (\omega + i\partial\bar{\partial}v)^{n-j},$$

$$\mathcal{F}(v) = -AM(v) - \log \int_X e^{-v+f} \omega^n,$$

where $v \in \mathcal{H}$ and $f \in C^\infty(X)$ is the Ricci potential of $\omega$, i.e. $\text{Ric} \omega = \omega + i\partial\bar{\partial}f$ normalized by

$$\int_X e^f \omega^n = 1.$$

It was proved in [Da4] that both of these functionals are continuous with respect to all metrics $d_p$, hence extend to $\mathcal{E}^p(X, \omega)$ continuously. As the map $u \mapsto \omega_u$ is translation invariant one may want normalize Kähler potentials to obtain an equivalence between metrics and potentials. This can be done by only considering potentials from the ”totally geodesic” hypersurfaces

$$\mathcal{H}_{AM} = \mathcal{H} \cap \{AM(\cdot) = 0\},$$

$$\mathcal{H}_{0, AM} = L^\infty(X) \cap \text{PSH}(X, \omega) \cap \{AM(\cdot) = 0\},$$

$$\mathcal{E}^p_{AM}(X, \omega) = \mathcal{E}^p(X, \omega) \cap \{AM(\cdot) = 0\}.$$

A smooth metric $\omega_{u_{KE}}$ is Kähler-Einstein if $\omega_{u_{KE}} = \text{Ric} \omega_{u_{KE}}$. One can study such metrics by looking at the long time asymptotics of the Hamilton’s Kähler–Ricci flow:

$$\begin{cases}
\frac{d\omega_t}{dt} = -\text{Ric} \omega_t + \omega_{r_t}, \\
r_0 = v.
\end{cases}$$

As proved by Cao [Cao], for any $v \in \mathcal{H}_{AM}$, this PDE has a smooth solution $[0, 1) \ni t \mapsto r_t \in \mathcal{H}_{AM}$. It follows from a theorem of Perelman and work of Chen-Tian, Tian-Zhu and Phong-Song-Sturm-Weinkove, that whenever a Kähler–Einstein metric cohomologous to $\omega$ exists, then $\omega_{r_t}$ converges exponentially fast to one such metric (see [CT], [TZ], [PSSW]).
We remark that our choice of normalization is different from the alternatives used in the literature (see [BEG, Chapter 6]). We choose to work with the normalization $AM(\cdot) = 0$, as this seems to be the most natural one from the point of view of Mabuchi geometry. Indeed, that Aubin-Mabuchi energy is continuous with respect to all metrics $d_p$ and is linear along the geodesic segments defined in (6). It will require some careful analysis, but as we shall see, from the point of view of long time asymptotics, this normalization is equivalent to other alternatives.

Suppose $(M, \rho)$ is a geodesic metric space and $[0, \infty) \ni t \to c_t \in M$ is a continuous curve. We say that the unit speed $\rho$–geodesic ray $[0, \infty) \ni t \to g_t \in M$ is weakly asymptotic to the curve $t \to c_t$, if there exists $t_j \to \infty$ for which there exist $\rho$–geodesic segments $[0, \rho(c_0, c_{t_j})] \ni t \to g_{t_j} \in M$ connecting $c_0$ and $c_{t_j}$ such that

$$\lim_{j \to \infty} \rho(g_{t_j}, g_t) = 0, \ t \in [0, \infty).$$

We clearly need $\lim_j \rho(c_0, c_{t_j}) = \infty$ in this last definition, hence to construct $d_p$–geodesic rays weakly asymptotic to diverging Kähler-Ricci trajectories, we first need to prove the following result, which generalizes [Da4, Theorem 6] and the main result of [Mc].

**Theorem 1.** Suppose $(X, J, \omega)$ is a Fano manifold and $p \geq 1$. There exists a Kähler–Einstein metric in $H$ if and only if every Kähler–Ricci trajectory $[0, \infty) \ni t \to r_t \in H_{AM}$ is $d_p$–bounded.

Using this theorem, the main result of [BrnBnm], the compactness theorem of [BBEGZ] and the divergence analysis of Kähler-Ricci trajectories from [R1], we can establish our main result:

**Theorem 2.** Suppose $(X, J, \omega)$ is a Fano manifold without a Kähler–Einstein metric in $H$ and $[0, \infty) \ni t \to r_t \in H_{AM}$ is a Kähler-Ricci trajectory. Then there exists a curve $[0, \infty) \ni t \to u_t \in H_{0, AM}$ which is a $d_p$–geodesic ray weakly asymptotic to $t \to r_t$ for all $p \geq 1$. In addition to this, $t \to u_t$ satisfies the following:

(i) $t \to \mathcal{F}(u_t)$ is decreasing,

(ii) the ”sup-normalized” potentials $u_t - \sup_X(u_t - u_0) \in H_0$ decrease pointwise to $u_\infty \in PSH(X, \omega)$ for which $\int_X e^{-\frac{n+1}{n}u_\infty} \omega^n = \infty$.

If additionally $(X, J)$ does not admit non–trivial holomorphic vector fields then $t \to \mathcal{F}(u_t)$ is strictly decreasing.

We note that the normalizing condition $AM(u_t) = 0$ in the above result assures that geodesic ray $t \to u_t$ is non–trivial, i.e. $u_t \neq u_0 + ct$.

This theorem provides a partial answer to a folklore conjecture, perhaps first suggested by [LNT], which says that one should be able to construct ”destabilizing” geodesic rays (strongly) asymptotic to diverging Kähler-Ricci trajectories. For a precise statement and connections with other results we refer to [R1, Conjecture 4.10].

We hope that the methods developed here will be the building blocks of future results constructing geodesic rays asymptotic to different (geometric) flow trajectories. We refer to Theorem 3.4 for a general result in this direction.
On Fano manifolds not admitting Kähler-Einstein metrics, condition (ii) above ensures the bound $\alpha(X) \leq n/(n+1)$ for Tian’s alpha invariant:

$$\alpha(X) = \sup \left\{ \alpha, \int_X e^{-\alpha(u) \sup X u} \omega^n \leq C_\alpha < +\infty, \ u \in PSH(X, \omega) \right\}.$$ 

This is a well known result of Tian \cite{T1}. The fact that the geodesic ray $t \to u_t$ is able to detect a potential $u_\infty$ satisfying $\int_X e^{-n\alpha(u) \sup X u} \omega^n = \infty$, is analogous to results of \cite{R1}, where it is shown that one can find such potential using a sequence of metrics along a diverging Kähler-Ricci trajectory as well. We refer to this paper for relations with Nadel sheaves.

It would be interesting to see if the geodesic ray produced by the above theorem is in fact unique. We prove that this ray is bounded, but it is not clear if this curve has more regularity. Finally, we believe that $t \to F(u_t)$ is strictly decreasing regardless whether $(X, J)$ admits non–trivial holomorphic vector fields or not.

Finally, we note the following theorem, which is a consequence of the above result, and in the case $p = 2$ gives the Kähler-Einstein analog of Donaldson’s conjectures on existence of constant scalar curvature metrics \cite{Da, H2}:

**Theorem 3.** Suppose $p \in \{1, 2\}$ and $(X, J, \omega)$ is a Fano manifold without non–trivial holomorphic vector–fields and $u \in \mathcal{H}$. There exists no Kähler-Einstein metric in $\mathcal{H}$ if and only if for any $u_0 \in \mathcal{H}$ there exists a $d_p$–geodesic ray $[0, \infty) \ni t \to u_t \in \mathcal{H}_{0,AM}$ with $u_0 = u$ such that the function $t \to F(u_t)$ is strictly decreasing.

**Proof.** The only if direction is a consequence of the previous theorem. Now we argue the if direction. Suppose there exists a Kähler–Einstein metric in $\mathcal{H}$. In case $p = 1$ it is enough to invoke \cite{Da4} Theorem 6]. Indeed, this result says that on a Fano manifold without non–trivial holomorphic vector–fields existence of a Kähler-Einstein metric in $\mathcal{H}$ is equivalent to the $d_1$–properness of $F$ (sublevel sets of $F$ are $d_1$–bounded). Hence the convex map $t \to F(u_t)$ is eventually strictly increasing for any $d_1$–geodesic ray $t \to u_t$.

The case $p = 2$ follows if one notices that $d_2$–geodesic rays are also $d_1$–geodesic rays. Indeed, this follows from the $CAT(0)$ property of $(\mathcal{H}, d_2) = (\mathcal{E}^2(X, \omega), d_2)$ (see \cite{Da3} Theorem 6(iii)], [CC]). Because of this, $d_2$–geodesic segments connecting different points of $(\mathcal{E}^2(X, \omega), d_2)$ are unique, hence they are always of the type described in (6), which are also $d_1$–geodesics in $(\mathcal{E}^1(X, \omega), d_1)$ (as remarked after (7)). The same statement holds for geodesic rays as well, not just segments. Now we can use \cite{Da4} Theorem 6(iii)] again to conclude the argument. \hfill $\square$

We note here that for $p = 2$ this last theorem follows from the work of Berman on K-polystability \cite{Brm}. Our approach however is purely analytical and avoids the use of the recently established equivalence between K–stability and existence of Kähler–Einstein metrics.

Although we do not pursue such generality, we remark that Theorem 1 and Theorem 2 also hold for the very general Orlicz-Finsler structures $(\mathcal{H}, d_\chi)$ studied in \cite{Da4}.

**Acknowledgements.** The first author would like to thank Yanir Rubinstein for numerous stimulating conversations related to the topic of the paper and for László Lempert for suggestions on how to improve the presentation.
2 Preliminaries

2.1 The Metric Spaces \((\mathcal{H}, d_p)\)

In hopes of characterizing convergence in \(\mathcal{E}^p(X, \omega)\) more explicitly, for \(u_0, u_1 \in \mathcal{E}^p(X, \omega)\) one introduces the following functional (see [Da4, G]):

\[
I_p(u_0, u_1) = \left( \int_X |u_0 - u_1|^p \omega_{u_0}^n \right)^{1/p} + \left( \int_X |u_0 - u_1|^p \omega_{u_1}^n \right)^{1/p}.
\]

In [Da4, Theorem 3] it is proved that there exists \(C(p) > 1\) such that

\[
\frac{1}{C} \leq I_p(u_0, u_1) \leq CI_p(u_0, u_1).
\]

This double estimate implies that there exists \(C(p) > 1\) such that

\[
\sup_X u \leq Cd_p(u, 0) + C.
\]

Also, if \(d_p(u_k, u) \to 0\) then \(u_k \to u\) a.e. and also \(\omega_{u_k}^n \to \omega_u^n\) weakly. For more details we refer to [Da4, Theorems 3-6]. We also note the following:

**Proposition 2.1.** Suppose \(\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}_0 = \text{PSH}(X, \omega) \cap L^\infty\) and \(\|u_k\|_{L^\infty} \leq D\) for some \(D > 0\). Then \(\{u_k\}_{k \in \mathbb{N}}\) is \(d_p\)-Cauchy if and only if it is \(d_1\)-Cauchy. If this condition holds then in addition the limit \(u = \lim_k u_k\) also satisfies \(\|u\|_{L^\infty} \leq D\).

**Proof.** The equivalence follows from (12) and basic facts about \(L^p\) norms. The estimate \(\|u\|_{L^\infty} \leq D\) also follows, as from [Da4, Theorem 5(i)] we have \(u_k \to u\) in capacity, hence \(u_k \to u\) pointwise a.e.. \(\square\)

We recall the compactness theorem [BBEGZ, Theorem 2.17]. Before we write down the statement, let us first recall the notion of strong convergence and entropy. As introduced in [BBEGZ], we say that a sequence \(u_k \in \mathcal{H}\) converges strongly to \(u \in \mathcal{E}^1(X, \omega)\) if \(u_k \to L^1 u\) and \(AM(u_k) \to AM(u)\). The Mabuchi K-energy functional \(\mathcal{M}: \mathcal{H} \to \mathbb{R}\) is given by the following formula:

\[
\mathcal{M}(u) = RAM(u) - L(u) + H_\omega(\omega_u),
\]

where \(R = \int_X R_\omega \wedge \omega^n\) is the average scalar curvature \(R_\omega\) of \(\omega\), \(H_\omega(\omega_u) = \int_X \log(\omega_u^n/\omega^n) \omega_u^n\) is the entropy of \(\omega_u^n\) with respect to \(\omega^n\) and \(L(u)\) is the following operator:

\[
L(u) = \sum_{j=0}^{n-1} \int_X u \text{Ric } \omega \wedge \omega_u^j \wedge \omega^{n-1-j}.
\]

**Proposition 2.2.** [BBEGZ, Proposition 2.6, Theorem 2.17] Suppose \(\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H}\) is such that \(\sup_k u_k, H_\omega(\omega_{u_k}) \leq D\) for some \(D \geq 0\). Then there exists \(u \in \mathcal{E}^1(X, \omega)\) and \(k_l \to \infty\) such that \(u_{k_l} \to u\) strongly.

It follows from the results of [BBGZ, BBEGZ] that one has \(u_k \to u\) strongly if and only if \(I_1(u_k, u) \to 0\), which in turn is equivalent to \(d_1(u_k, u) \to 0\) according to (12). Putting together the last two results we can write:
Theorem 2.3. Suppose \( \{u_k\}_{k \in \mathbb{N}} \subset \mathcal{H} \) is such that \( H_p(\omega_{u_k}), \|u_k\|_{L^\infty} \leq D \) for some \( D \geq 0 \). Then there exists \( u \in \mathcal{H}_0 \) with \( \|u\|_{L^\infty} \leq D \) and \( k_i \to \infty \) such that \( d_p(u_{k_i}, u) \to 0 \) for all \( p \geq 1 \).

In our computations we will need the following bound for the \( L \) functional in the expression of the Mabuchi K-energy:

**Proposition 2.4.** For any \( p \geq 1 \) there exists \( C(p) > 1 \) such that

\[
|L(u)| \leq C_d_p(0, u), \ u \in \mathcal{H}.
\]

**Proof.** There exists \( C > 0 \) such that \( \text{Ric } \omega \leq C \omega \). We can start writing:

\[
|L(u)| \leq C \sum_{j=1}^n \int_X |u| \omega^j \wedge \omega_{u_t}^{n-j} \leq C \int_X \left| \frac{u}{2} \right| \omega_{u_t}^n
\]

\[
\leq C \left( \int_X \left| \frac{u}{2} \right|^p \omega_{u_t}^2 \right)^{1/p} \leq C d_p(0, \frac{u}{2}) \leq C d_p(0, u),
\]

where in the penultimate inequality we have used [Da4, Lemma 5.3].

Finally, we recall a result about bounded geodesics which will be very useful to us later:

**Theorem 2.5.** [Da2, Theorem 1] Given a bounded weak geodesic \([0, 1] \ni t \mapsto u_t \in \mathcal{H}_0 \) connecting \( u_0, u_1 \in \mathcal{H}_0 \), i.e. a bounded solution to (3), there exists \( M_u, m_u \in \mathbb{R} \) such that for any \( a, b \in [0, 1] \) we have

\begin{itemize}
  \item[(i)] \( \inf_X \frac{u_a - u_b}{a - b} = m_u \),
  \item[(ii)] \( \sup_X \frac{u_a - u_b}{a - b} = M_u \).
\end{itemize}

This result tells us that for a bounded weak geodesic \([0, 1] \ni t \mapsto u_t \in \mathcal{H}_0 \) the function \( t \mapsto \sup_X (u_t - u_0) \) is linear. As explained in [Da2], this implies that \( t \mapsto \tilde{u}_t = u_t - \sup_X (u_t - u_0) \in \mathcal{H} \) is a geodesic that is decreasing in \( t \) (one can see that \( \tilde{u}_t \leq 0 \)). Clearly, \( \sup_X \tilde{u}_t \) is bounded, hence the pointwise limit \( u_\infty = \lim_{t \to \infty} \tilde{u}_t \) is different from \( -\infty \). As we shall see by the end of this paper, for certain geodesic rays one can draw geometric conclusions by studying the singularity type of \( u_\infty \).

### 2.2 Diverging Kähler-Ricci Trajectories

In this short section we recall some results from [R2], where a careful analysis of diverging Kähler-Ricci trajectories has been carried out. Unfortunately, this work uses a normalization different from ours, but here we argue that the most important estimates have analogs for ”AM–normalized” trajectories as well.

It is well known that flow equation (11) can be rewritten as the scalar equation as \( \omega^n_{r_t} = e^{f - r_t + \beta(t)} \omega^n \), where \( \beta : [0, \infty) \to \mathbb{R} \) is a function chosen depending on the desired normalization condition on \( r_t \). In our investigations we will eventually use the condition \( AM(r_t) = 0 \), however most of the literature on the Kähler-Ricci flow uses a different normalization (that we will denote by \( t \to \tilde{r}_t \)) for which \( \beta(t) = 0 \) and \( \tilde{r}_0 = v + c \).
with $c$ carefully chosen (see [PSS, (2.10)]). Consequently, in this case the scalar equation becomes

$$\omega^n_{\tilde{r}_t} = e^{-\tilde{r}_t + \tilde{f}_t} \omega^n,$$

and the conversion from this normalization to the one employed by us is given by the formula

$$r_t = \tilde{r}_t - AM(\tilde{r}_t), \; t \geq 0.$$  

The following result brings together estimates for the trajectory $t \to \tilde{r}_t$ that we will need the most. Most of these are classical and well known, for the others we give a proof:  

**Proposition 2.6.** Suppose $t \to \tilde{r}_t$ is a Kähler-Ricci trajectory normalized as discussed above. For any $t \geq 0$ we have:

1. $\|\tilde{r}_t\|_{L^\infty}, \|f_\omega\|_{L^\infty} \leq C$ for some $C > 1$.
2. $-C \leq AM(\tilde{r}_t)$, in particular $-\int_X \tilde{r}_t \omega^n_{\tilde{r}_t} \leq n \int_X \tilde{r}_t \omega^n + C$ for some $C > 1$.
3. $\int_X \tilde{r}_t \omega^n_{\tilde{r}_t} \leq C, -C \leq \int_X \tilde{r}_t \omega^n$ hence also $-C \leq \sup_X \tilde{r}_t$ for some $C > 1$.
4. $-\inf_X \tilde{r}_t \leq C \sup_X \tilde{r}_t + D$ for some $C, D > 0$.
5. $-\log \left( \int_X e^{-\alpha(\tilde{r}_t - \sup_X \tilde{r}_t)} \omega^n \right) \leq (1 - \alpha)n - \alpha \sup_X \tilde{r}_t + C$ for some $C > 1$.
6. $\sup_X \tilde{r}_t - AM(\tilde{r}_t) \geq \sup_X \tilde{r}_t / C - C \geq (AM(\tilde{r}_t) - \inf_X \tilde{r}_t) / D - D$ for some $C, D > 1$.

**Proof.** The estimates in (i) are essentially due to Perelman [ST, TZ]. The estimates from (ii) are also well known. We recall the argument from [R2]. First we notice that

$$-\log \int_X e^{-\tilde{r}_t + f_\omega} \omega^n = -\log \int_X e^{-\tilde{r}_t} \omega^n_{\tilde{r}_t},$$

hence this quantity is uniformly bounded by (i). It is well known that $t \to F(\tilde{r}_t)$ is decreasing and now looking at the expression of $F(\tilde{r}_t)$ from (13), we conclude that there exists $C > 1$ such that $AM(\tilde{r}_t) \geq -C$. The second estimate of (ii) now follows from the well known inequality:

$$AM(\tilde{r}_t) = \frac{1}{(n+1)\text{Vol}(X)} \sum_{j=0}^n \int_X \tilde{r}_t \omega^j \wedge \omega^{n-j}_{\tilde{r}_t} \leq \frac{1}{(n+1)\text{Vol}(X)} \left( \int_X \tilde{r}_t \omega^n_{\tilde{r}_t} + n \int_X \tilde{r}_t \omega^n \right).$$

We now prove the estimate of (iii). From (13) we have $\int_X e^{\tilde{r}_t} \omega^n_{\tilde{r}_t} = \int_X e^{\tilde{r}_t + f_\omega}$. Hence the estimates of (i) yield that $\int_X e^{\tilde{r}_t} \omega^n_{\tilde{r}_t}$ is uniformly bounded. The first estimate now follows from Jensen’s inequality:

$$\frac{1}{\text{Vol}(X)} \int_X \tilde{r}_t \omega^n_{\tilde{r}_t} \leq \log \left( \frac{1}{\text{Vol}(X)} \int_X e^{\tilde{r}_t} \omega^n_{\tilde{r}_t} \right).$$

The second and third estimate of (iii) follows now from (ii). Estimate (iv) is just the Harnack estimate for the Kähler-Ricci flow. For a summary of the proof we refer to steps (i) and (iii) in the proof of [R2, Theorem 1.3], which in turn follows [T1].
We justify the estimate of \((v)\) and the roots of our argument are again from \([R2]\). To start, we notice that using equation (13) we can write

\[
-\log \left( \int_X e^{-\alpha \tilde{r} t + f t + \tilde{r} t \omega_n} \right) = -\log \left( \int_X e^{-\alpha \tilde{r} t + f t + \tilde{r} t \omega_n} \right) 
\]

\[
\leq \frac{1}{\text{Vol}(X)} \int_X (\alpha - 1) \tilde{r} t \omega_n + C \leq \frac{n(1 - \alpha)}{\text{Vol}(X)} \int_X \tilde{r} t \omega_n + C,
\]

where in the second line we have used the estimates of (i) and (ii). This finishes the proof of \((v)\).

Now we turn to the proof of the double estimate in \((vi)\). From the definition of \(AM\) and (iii) it follows that

\[
\sup_X \tilde{r} t - AM(\tilde{r} t) \geq \frac{1}{n + 1} \left( \sup_X \tilde{r} t - \frac{1}{\text{Vol}(X)} \int_X \tilde{r} t \omega_n \right) \geq \frac{1}{n + 1} \sup_X \tilde{r} t - C,
\]

and this establishes the first estimate. The second estimate follows from (iv) and the simple fact that \(\sup_X \tilde{r} t \geq AM(\tilde{r} t)\).

Finally, we phrase some of the above estimates for \(AM\–normalized\) Kähler–Ricci trajectories:

**Proposition 2.7.** Suppose \(t \to r_t\) is a Kähler–Ricci trajectory that is \(AM\–normalized\), i.e. \(AM(r_t) = 0\). Let \(t \to \tilde{r} t\) be the corresponding Kähler–Ricci trajectory normalized according to (13) that corresponds to \(t \to r_t\), i.e. \(r_t = \tilde{r} t - AM(\tilde{r} t)\). For \(t \geq 0\) the following hold:

(i) \(-\inf_X r_t \leq C \sup_X r_t + C\), for some \(C > 1\).

(ii) \(\sup_X \tilde{r} t \leq C \sup_X r_t + C \leq D \sup_X \tilde{r} t + E\), for some \(C, D, E > 1\).

(iii) For any \(p \geq 1\) we have \(\sup_X r_t / C - C \leq d_p(r_0, r_t) \leq C \sup_X r_t + C\) for some \(C > 1\).

(iv) If \(\alpha > n/(n+1)\) and \(p \geq 1\) then \(-\log \left( \int_X e^{-\alpha (r_t - \sup_X (r_t - r_0)) + f t \omega_n} \right) \leq -\varepsilon d_p(r_0, r_t) + C\) for some \(C > 1\) and \(\varepsilon > 0\).

**Proof.** The estimate in (i) follows from part (vi) of the previous proposition. This last estimate also gives the first estimate of (ii). Estimate (ii) in the previous result immediately gives the second part of (ii).

The first estimate of (iii) is just [Da4, Corollary 4]. By (12) we have that \(d_p(r_0, r_t) \leq \text{osc}_X(r_0 - r_t)\). Part (i) now implies the second estimate of (iii).

Notice that \(\alpha > n/(n+1)\) is equivalent with \((1 - \alpha)n - \alpha < 0\). The estimate of (iv) now follows after we put together parts (v) of the previous proposition with what we proved so far in this proposition.

### 3 Proof of the Main Results

First we give a proof for Theorem [11] As it turns out, the argument is about putting together the pieces developed in the preceding sections.
**Theorem 3.1.** Suppose \((X, J, \omega)\) is a Fano manifold and \(p \geq 1\). There exists a Kähler–Einstein metric in \(\mathcal{H}\) if and only if every Kähler–Ricci trajectory \([0, \infty) \ni t \to r_t \in \mathcal{H}_{AM}\) is \(d_p\)-bounded.

**Proof.** If there exists a Kähler–Einstein metric in the cohomology class of \(\omega\) then by [Da4, Theorem 6] we have that any Kähler–Ricci trajectory \(d_p\)-converges to one such metric, hence stays \(d_p\)-bounded.

For the other direction, suppose \(d_p(0, r_t)\) is bounded. By Proposition 2.7(ii)(iii), \(d_p(0, r_t)\) controls \(\sup_X \tilde{r}_t\), which in turn controls \(\|\tilde{r}_t\|_{L^\infty}\) by Proposition 2.5(iv). The regularity theory for the Kähler–Ricci flow implies now that \(t \to \tilde{r}_t\) converges exponentially fast in any \(C^k\) norm to a Kähler–Einstein metric. \qed

**Theorem 3.2.** Suppose \((X, J, \omega)\) is a Fano manifold without a Kähler–Einstein metric in \(\mathcal{H}\) and \([0, \infty) \ni t \to r_t \in \mathcal{H}_{AM}\) is a Kähler–Ricci trajectory. Then there exists a curve \([0, \infty) \ni t \to u_t \in \mathcal{H}_{0, AM}\) which is a \(d_p\)-geodesic ray weakly asymptotic to \(t \to r_t\) for all \(p \geq 1\). In addition to this, \(t \to u_t\) satisfies the following:

(i) \(t \to F(u_t)\) is decreasing,

(ii) the "sup-normalized" potentials \(u_t - \sup_X (u_t - u_0) \in \mathcal{H}_0\) decrease pointwise to \(u_\infty \in \text{PSH}(X, \omega)\) for which \(\int_X e^{-\frac{u_\infty}{n}} u_\omega^n = \infty\).

If additionally \((X, J)\) does not admit non-trivial holomorphic vector fields then \(t \to F(u_t)\) is strictly decreasing.

**Proof.** The idea of the proof is to construct a \(d_2\)-geodesic ray that satisfies all the necessary properties. At the end we will conclude that this curve is also a \(d_p\)-geodesic ray for any \(p \geq 1\).

We can assume without loss of generality that \(r_0 = 0\). As a Kähler–Einstein metric does not exist, by the previous theorem there exists \(t_0 \to \infty\) such that \(f_l = d_2(0, r_{t_0}) \to \infty\). Let \([0, f]\) \(\ni t \to u_t^l \in \mathcal{H}_\Delta\) be the rescaled weak geodesic curve of (3), joining \(r_0 = 0\) with \(r_{t_0}\). By our choice of normalization it follows that

\[
AM(u_t^l) = 0 \quad \text{and} \quad d_2(0, u_t^l) = t, \quad t \in [0, f_l].
\]

(14)

Using Proposition 2.7(i) and (iii) there exists \(C, D > 1\) such that

\[
-C d_2(0, r_{t_0}) - C \leq -D \sup_X r_{t_0} - D \leq \inf_X r_{t_0} - \sup_X r_{t_0} \leq C d_2(0, r_{t_0}) + C.
\]

Rewriting this, as \(\sup_X r_{t_0} \to \infty\), for \(l\) big enough we obtain:

\[
-C \leq -\frac{D' \sup_X u_t^l}{f_l} \leq \frac{\inf_X u_t^l}{f_l} \leq \frac{\sup_X u_t^l}{f_l} \leq C,
\]

(15)

for all \(f_l \geq 1\). As \(u_0^l = 0\) for all \(l\), using (15) and Theorem 2.5 we can conclude that

\[
-C \leq -\frac{D' \sup_X u_t^l}{t} \leq \frac{\inf_X u_t^l}{t} \leq \frac{\sup_X u_t^l}{t} \leq C, \quad t \in [0, f_l].
\]

(16)

By the results of [Brn1] and [BrnBrn] it also follows that the maps \(t \to F(u_t^l), M(u_t^l)\) are convex and non-positive. In particular, for \(t \geq 0\) we have:

\[
\frac{H_\omega(\omega_{u_t^l}) - L(u_t^l)}{t} = \frac{M(u_t^l) - M(u_0)}{t} \leq \frac{M(r_{t_0}) - M(r_0)}{f_l} \leq 0.
\]
Proposition [2.4] now implies that there exists $C > 1$ such that

$$0 \leq H_{\omega}(\omega u_t^i) \leq L(u_t^i) \leq C d_2(0, u_t^i) = Ct. \quad (17)$$

Fix now $s \geq 0$. From (16) and (17) it follows using Theorem [2.3] that there exists $l_k' \to \infty$ and $u_s \in \mathcal{H}_0$ such that $d_2(u^i_{l_k'}, u_s) \to 0$. As $AM$ is continuous with respect to $d_2$, by (14) we also have $AM(u_s) = 0$ and $d_2(0, u_s) = s$.

Building on this last observation, using a Cantor type diagonal argument, we can find sequence $l_k \to \infty$ such that for each $h \in \mathbb{Q}_+$ there exists $u_h \in \mathcal{H}_0$ satisfying $d_2(u_{l_k}', u_h) \to 0$, $AM(u_h) = 0$ and $d_2(0, u_h) = h$.

As $t \to u_t^i$ are unit speed $d_2$–geodesic segments, it follows that for any $a, b, c \in \mathbb{Q}_+$ satisfying $a < b < c$ we will also have

$$d_2(u_a, u_b) + d_2(u_b, u_c) = c - a = d_2(u_a, u_c).$$

Hence, by density we can extend $h \to u_h$ to a unit speed $d_2$–geodesic $[0, \infty) \ni t \to u_t \in \mathcal{H}_{0,AM}$ weakly asymptotic to $t \to r_t$. This $d_2$–geodesic is non–trivial, i.e. not of the form $u_t = u_0 + ct$ for some $c \in \mathbb{R}$. Indeed, this would contradict the fact $AM(u_t) = 0$ and $t \to u_t$ is unit speed with respect to $d_2$.

To show $t \to \mathcal{F}(u_t)$ is decreasing, we claim first that for any $t > 0$, $\mathcal{F}(u_0) \geq \mathcal{F}(u_t)$. First we note that $\mathcal{F}$ is continuous with respect to $d_2$. For each $l$, the map $t \to \mathcal{F}(u_t^l)$ is convex and satisfies $\mathcal{F}(u_0) \geq \mathcal{F}(u_t^l)$ hence for any $t \in [0, f_l]$ we have $\mathcal{F}(u_0) \geq \mathcal{F}(u_t^l)$. By passing to the limit, the claim is proved. As $t \to \mathcal{F}(u_t)$ is convex and $\mathcal{F}(u_0) \geq \mathcal{F}(u_t)$ for any $t \in (0, \infty)$, $\mathcal{F}$ has to be decreasing.

If additionally $(X, J)$ does not admit non–trivial holomorphic vector fields then $t \to \mathcal{F}(u_t)$ is strictly decreasing. Indeed, if this were not the case, then there would exist $t_0 \geq 0$ such that

$$\frac{\partial}{\partial t} \mathcal{F}(u_t) = 0, \quad t \geq t_0.$$

By Berndtsson’s convexity theorem [Brn1], this implies that $(X, J)$ admits a non–trivial holomorphic vector field, which is a contradiction.

We turn to part (ii). For $n/(n + 1) < \alpha < 1$ each curve $t \to \alpha u_t^i$ is a subgeodesic, hence it follows from [Brn1] that each map

$$t \to - \log \left( \int_X e^{-\alpha u_t^i + f_X \omega^n} \right)$$

is convex. As $u_0^i \equiv 0$, by Theorem [2.5] the function

$$t \to G_\alpha(u_t^i) = - \log \left( \int_X e^{-\alpha u_t^i + \sup_X u_t^i + f_X \omega^n} \right) = - \log \left( \int_X e^{-\alpha u_t^i + f_X \omega^n} \right) - \alpha \sup_X u_t^i$$

is also convex. By theorem [2.7iv] this implies that $G_\alpha(u_t^i) \leq -\varepsilon d_2(0, u_t^i) + C = -\varepsilon t + C$.

Similarly to $\mathcal{F}(\cdot)$, the functional $G_\alpha(\cdot)$ is also continuous with respect to $d_2$, hence by taking the limit $l_k \to \infty$ in this last estimate we obtain:

$$G_\alpha(u_t) \leq -\varepsilon t + C. \quad (18)$$

As discussed after Theorem [2.5] the decreasing limit $u_\infty = \lim_{t \to \infty} (u_t - \sup_X u_t)$ is a well defined and not identically equal to $-\infty$. Letting $t \to \infty$ in (18) we obtain that
\[ \int_X e^{-\alpha u} \omega^n = \infty. \] As \( n/(n+1) < \alpha < 1 \), the recent resolution of the openess conjecture (see [Brn2, GZh]) implies part (ii).

Finally, as \( t \to u_t \) is a bounded \( d_p \)-geodesic ray it follows by the discussion after (8) that \( t \to u_t \) is a \( d_p \) geodesic ray as well. \( \square \)

We believe \( t \to F(u_t) \) should be strictly decreasing even if \( X \) has holomorphic vector fields. We can show this when the Futaki invariant is nonzero as we elaborate below. Note that along the Kähler-Ricci trajectory \( t \to r_t \) the \( F \)-functional is strictly decreasing unless the initial metric is Kähler–Einstein. Using the identity

\[ e^{-r_t + f_{\omega_t}} \omega^n = e^{f_{\omega_t} \omega_{r_t}} \]

we can write

\[ \frac{\partial F(r_t)}{\partial t} = -\int_X f_{\omega_t} (e^{f_{\omega_t} - 1}) \omega_{r_t}^n. \]

It is natural to introduce the following quantity:

\[ \epsilon(\omega) = \inf_{u \in \mathcal{H}} \int_X f_{\omega_u} (e^{f_{\omega_u} - 1}) \omega_u^n \geq 0. \]

This quantity is clearly an invariant of \( (X, J, [\omega]) \). If \( \epsilon(\omega) > 0 \), then there exists no Kähler-Einstein metric in \( \mathcal{H} \). By Jensen’s inequality, for any \( u \in \mathcal{H} \) we have \( \int_M f_{\omega_u} \omega_u^n \leq 0 \), hence we can write

\[ \int_M f_{\omega_u} (e^{f_{\omega_u} - 1}) \omega_u^n \geq \int_M f_{\omega} e^{f_{\omega_u} - 1} \omega_u^n. \]

By [H2], the right hand side above (defined as the \( H \)-functional) is nonnegative and is uniformly bounded away from zero if the Futaki invariant is nonzero, implying in this last case the bound \( \epsilon(\omega) > 0 \). Finally, we note the following result:

**Proposition 3.3.** Suppose \( t \to r_t \) and \( t \to u_t \) are as in the previous theorem. If \( \epsilon(\omega) > 0 \), then the map \( t \to F(u_t) \) is strictly decreasing. More precisely, there exists \( C > 0 \) such that \( F(u_t) \leq F(u_0) - Ct, \ t \geq 0 \).

**Proof.** By the discussion above, we have the estimate \( F(r_t) - F(r_0) \leq -\epsilon(\omega)t \). Using the notation of the previous theorem’s proof, by the estimates of Section 2.2, there exists \( C, C' > 0 \) such that for \( l \) big enough:

\[ f_l = d_2(0, r_{t_l}) \leq C' \sup_X r_t \leq Ct_l. \]

From our observations it follows that

\[ \frac{F(u'_l) - F(u_0)}{f_l} = \frac{F(r_{t_l}) - F(r_0)}{f_l} \leq -\frac{\epsilon(\omega)}{C}. \]

By the convexity of \( F \) we can conclude that

\[ \frac{F(u'_l) - F(u_0)}{t} \leq -\frac{\epsilon(\omega)}{C}, \ t \in (0, f_l]. \]

Letting \( l \to \infty \) we obtain

\[ \frac{F(u_t) - F(u_0)}{t} \leq -\frac{\epsilon(\omega)}{C}, \ t \in (0, \infty). \]

\( \square \)
One would like to implement the ideas of this paper to construct geodesic rays asymptotic to other types of (geometric) flow trajectories. From the proof of Theorem 3.2 one can extract the following general result:

**Theorem 3.4.** Suppose \([0, \infty) \ni t \rightarrow c_t \in \mathcal{H}_{AM}\) is a curve for which there exists \(t_j \to \infty\) satisfying the following properties:

(i) There exists \(C > 1\) such that \(- \inf_X c(t_j) \leq C \sup_X c(t_j) + C, \ t \geq 0\).

(ii) We have \(\lim_{j \to \infty} \sup_X c(t_j) = +\infty\) and

\[
\limsup_{j \to \infty} \frac{\mathcal{M}(c(t_j)) - \mathcal{M}(c_0)}{\sup_X c(t_j)} < +\infty.
\]

Then there exists a curve \([0, \infty) \ni t \rightarrow u_t \in \mathcal{H}_{0,AM}\) which is a non-trivial \(d_p\)-geodesic ray weakly asymptotic to \(t \rightarrow c_t\) for all \(p \geq 1\).

**References**

[Brm1] R. Berman, K-polystability of Q-Fano varieties admitting Kahler-Einstein metrics, arXiv:1205.6214.

[Brm2] R. Berman, On the optimal regularity of weak geodesics in the space of metrics on a polarized manifold, arXiv:1405.6482.

[BrmBrn] R. Berman, R. Berndtsson, Convexity of the K-energy on the space of Kähler metrics, arXiv:1405.0401.

[BBGZ] R. Berman, S. Boucksom, V. Guedj, A. Zeriahi, A variational approach to complex Monge-Ampère equations, Publ. Math. Inst. Hautes Études Sci. 117 (2013), 179–245.

[BD] R. Berman, J. P. Demailly, Regularity of plurisubharmonic upper envelopes in big cohomology classes. Perspectives in analysis, geometry and topology, Progr. Math. 296, Birkhäuser/Springer, New York, 2012, 39–66.

[Brn1] B. Berndtsson, A Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem, arXiv:1103.0923.

[Brn2] B. Berndtsson, The openness conjecture for plurisubharmonic functions, arXiv:1305.5781.

[Bl] Z. Blocki, On geodesics in the space of Kähler metrics, Proceedings of the ”Conference in Geometry” dedicated to Shing-Tung Yau (Warsaw, April 2009), in ”Advances in Geometric Analysis”, ed. S. Janeczko, J. Li, D. Phong, Advanced Lectures in Mathematics 21, pp. 3-20, International Press, 2012.

[BK] Z. Blocki, S. Kołodziej, On regularization of plurisubharmonic functions on manifolds, Proceedings of the American Mathematical Society 135 (2007), 2089–2093.

[BBEGZ] S. Boucksom, R. Berman, P. Eyssidieux, V. Guedj, A. Zeriahi, Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties, arXiv:1111.7158.
[BEG] S. Boucksom, P. Eyssidieux, V. Guedj, An introduction to the Kähler-Ricci flow, Lecture Notes in Math. 2086, Springer 2013.

[CC] E. Calabi, X. X. Chen, The Space of Kähler Metrics II, J. Differential Geom. 61(2002), no. 2, 173-193.

[Cao] H. Cao, Deformation of Kaehler metrics to Kaehler-Einstein metrics on compact Kaehler manifolds, Invent. Math. 81 (1985), no. 2, 359–372.

[C] X.X. Chen, The space of Kähler metrics, J. Differential Geom. 56 (2000), no. 2, 189–234.

[CT] X.X Chen, G. Tian, Ricci flow on Kähler-Einstein surfaces, Inventiones mathematicae 2/2002; 147(3):487-544.

[Da1] T. Darvas, Morse theory and geodesics in the space of Kähler metrics, arXiv:1207.4465v3.

[Da2] T. Darvas, Weak Geodesic Rays in the Space of Kähler Metrics and the Class $\mathcal{E}(X,\omega_0)$, arXiv:1307.6822.

[Da3] T. Darvas, The Mabuchi Geometry of Finite Energy Classes, arXiv:1409.2072.

[Da4] T. Darvas, Envelopes and Geodesics in Spaces of Kähler Potentials, arXiv:1401.7318.

[DL] T. Darvas, L. Lempert, Weak geodesics in the space of Kähler metrics, Mathematical Research Letters, 19 (2012), no. 5.

[DR] T. Darvas, Y. A. Rubinstein, Kiselman’s principle, the Dirichlet problem for the Monge-Ampère equation, and rooftop obstacle problems, arXiv:1405.6548.

[De] J. P. Demailly, Regularization of closed positive currents of type (1,1) by the flow of a Chern connection, Actes du Colloque en l’honneur de P. Dolbeault (Juin 1992), édité par H. Skoda et J.M. Tépreau, Aspects of Mathematics, Vol. E 26, Vieweg, (1994) 105-126.

[Do] S. K. Donaldson - Symmetric spaces, Kähler geometry and Hamiltonian dynamics, Amer. Math. Soc. Transl. Ser. 2, vol. 196, Amer. Math. Soc., Providence RI, 1999, 13–33.

[G] V. Guedj, The metric completion of the Riemannian space of Kähler metrics, arXiv:1401.7857.

[GZh] Q. Guan, X. Zhou, Strong openness conjecture for plurisubharmonic functions, arXiv:1311.3781.

[GZ1] V. Guedj, A. Zeriahi, The weighted Monge–Ampère energy of quasiplusharmonic functions, J. Funct. Anal. 250 (2007), no. 2, 442–482.

[H1] W. He, On the space of Kähler potentials, arXiv:1208.1021.

[H2] W. He, $\mathcal{F}$-functional and geodesic stability, arXiv:1208.1020.
[LNT] G. La Nave, G. Tian, Soliton-type metrics and Kähler–Ricci flow on symplectic quotients, arXiv:0903.2413.

[LV] L. Lempert, L. Vivas, Geodesics in the space of Kähler metrics. Duke Math. J. 162 (2013), no. 7, 1369–1381.

[Ma] T. Mabuchi, Some symplectic geometry on compact Kähler manifolds I, Osaka J. Math. 24, 1987, 227–252.

[Mc] D. McFeron, The Mabuchi metric and the Kähler-Ricci flow, Proc. Amer. Math. Soc. 142 (2014), no. 3, 1005–1012.

[PSS] D.H. Phong, N. Sesum, J. Sturm, Multiplier Ideal Sheaves and the Kähler-Ricci Flow, arXiv:math/0611794.

[PSSW] D.H. Phong, J. Song, J. Sturm, B. Weinkove, The Kähler-Ricci flow and the $\bar{\partial}$–operator on vector fields, J. Differential Geom., 81 (2009), no. 7, 631–647.

[R1] Y. Rubinstein, On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow, Trans. Amer. Math. Soc. 361 (2009), 5839-5850.

[R2] Y. Rubinstein, Smooth and singular Kähler-Einstein metrics, arXiv:1404.7451.

[Se] S. Semmes, Complex Monge-Ampère and symplectic manifolds, Amer. J. Math. 114 (1992), 495–550.

[ST] N. Sesum, G. Tian, Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman), J. Inst. Math. Jussieu 7 (2008), no. 3, 575–587.

[T1] G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, Invent. Math. 89 (1987), no. 2, 225–246.

[TZ] G. Tian, X. Zhu, Convergence of Kähler-Ricci flow. J. Amer. Math. Soc. 20 (2007), no. 3, 675–699.

University of Maryland
tdarvas@math.umd.edu

University of Oregon
whe@uoregon.edu