The Goldenshluger–Lepski Method for Constrained Least-Squares Estimators over RKHSs

STEPHEN PAGE $^1$* and STEFFEN GRÜNEWÄLDER $^2$

$^1$STOR-i, Lancaster University, Lancaster, LA1 4YF, United Kingdom. E-mail: *s.page@lancaster.ac.uk

$^2$Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, United Kingdom. E-mail: s.grunewalder@lancaster.ac.uk

We study an adaptive estimation procedure called the Goldenshluger–Lepski method in the context of reproducing kernel Hilbert space (RKHS) regression. Adaptive estimation provides a way of selecting tuning parameters for statistical estimators using only the available data. This allows us to perform estimation without making strong assumptions about the estimand. In contrast to procedures such as training and validation, the Goldenshluger–Lepski method uses all of the data to produce non-adaptive estimators for a range of values of the tuning parameters. An adaptive estimator is selected by performing pairwise comparisons between these non-adaptive estimators. Applying the Goldenshluger–Lepski method is non-trivial as it requires a simultaneous high-probability bound on all of the pairwise comparisons. In the RKHS regression context, we choose our non-adaptive estimators to be clipped least-squares estimators constrained to lie in a ball in an RKHS. Applying the Goldenshluger–Lepski method in this context is made more complicated by the fact that we cannot use the $L^2$ norm for performing the pairwise comparisons as it is unknown. We use the method to address two regression problems. In the first problem the RKHS is fixed, while in the second problem we adapt over a collection of RKHSs.

Keywords: Adaptive Estimation, Goldenshluger–Lepski Method, RKHS Regression.

1. Introduction

In nonparametric statistics, it is assumed that the estimand belongs to a very large parameter space in order to avoid model misspecification. Such misspecification can lead to large approximation errors and poor estimator performance. However, it is often challenging to produce estimators which are robust against such large parameter spaces. An important tool which allows us to achieve this aim is adaptive estimation. Adaptive estimators behave as if they know the true model from a collection of models, despite being
Stephen Page and Steffen Grünewälder

a function of the data. In particular, adaptive estimators can often achieve the same optimal rates of convergence as the best estimators when the true model is known.

There are many ways of creating adaptive estimators. One way is to pass information on the true model from the data to a non-adaptive estimator through tuning parameters. For example, a Gaussian kernel estimator depends on the width parameter of the Gaussian kernel. The different width parameters define different sets of functions and represent different assumptions about the estimand.

In this paper, we study an adaptive estimation procedure called the Goldenshluger–Lepski method in the context of reproducing kernel Hilbert space (RKHS) regression. The Goldenshluger–Lepski method works by performing pairwise comparisons between non-adaptive estimators with a range of values for the tuning parameters. As far as we are aware, this is the first time that this method has been applied in the context of RKHS regression. The Goldenshluger–Lepski method, introduced in the series of papers \[6, 7, 8, 9\], is an extension of Lepski’s method. While Lepski’s method focusses on adaptation over a single parameter, the Goldenshluger–Lepski method can be used to perform adaptation over multiple parameters.

The Goldenshluger–Lepski method operates by selecting an estimator which minimises the sum of a proxy for the unknown bias and an inflated variance term. The proxy for the bias is calculated by performing pairwise comparisons between the estimator in question and all estimators which are in some sense less smooth than this estimator. A key challenge in applying the Goldenshluger–Lepski method is proving a high-probability bound on all of these pairwise comparisons simultaneously. This bound is known as a majorant.

A popular alternative to the Goldenshluger–Lepski method for constructing adaptive estimators is training and validation. Here, the data is split into a training set and a validation set. The training set is used to produce a collection of non-adaptive estimators for a range of different values for the tuning parameters and the validation set is used to select the best estimator from this collection. This selection is performed by calculating a proxy for the cost function that we wish to minimise. The estimator with the smallest value of the proxy is selected as our final estimator. One important advantage of the Goldenshluger–Lepski method in comparison to training and validation is that it uses all of the data to calculate the non-adaptive estimators. This is because it does not require data for calculating a proxy cost function. However, the Goldenshluger–Lepski method does require us to calculate a majorant, as discussed above, which is often a challenging task.

We now describe the RKHS regression problem studied in this paper in more detail. We assume that the regression function lies in an interpolation space between $L^\infty$ and an RKHS. Depending on the setting, this RKHS may be fixed or we may perform adaptation over a collection of RKHSs. The non-adaptive estimators we use in this context are clipped versions of least-squares estimators which are constrained to lie in a ball of pre-defined radius in an RKHS. These estimators are discussed in detail in \[15\]. Constraining...
an estimator to lie in a ball of predefined radius is a form of Ivanov regularisation (see [14]).

One advantage of the estimators that we consider is that there is a clear way of producing a majorant for them, especially when the RKHS is fixed. This is because we can control the estimator constrained to lie in a ball of radius \( r \) by bounding quantities of the form \( rZ \) for some random variable \( Z \) which does not depend on \( r \), such as in the proof of Lemma 3. It may be possible to use different non-adaptive estimators to address our RKHS regression problem, however this would require the calculation of a majorant for such estimators, which would generally be more difficult than the calculation of the majorant for the Ivanov-regularised estimators considered in this paper.

When the RKHS is fixed, the only tuning parameter to be selected is the radius of the ball in which the least-squares estimator is constrained to lie. Estimators for which the radius is larger are considered to be less smooth. In order to provide a majorant for the Goldenshluger–Lepski method, we must prove regression results which control these estimators for all radii simultaneously. When we perform adaptation over a collection of RKHSs, we must prove regression results which control the same estimators for all RKHSs and all ball radii in these RKHSs simultaneously. We demonstrate this approach for a collection of RKHSs with Gaussian kernels. Estimators for which both the width parameter of the Gaussian kernel is smaller and the radius of the ball in the RKHS is larger are considered to be less smooth. These results extend those of [15].

One of the main difficulties in applying the Goldenshluger–Lepski method to our RKHS regression problem is that the covariate distribution \( P \), and hence the \( L^2(P) \) norm, is unknown. This is a problem when trying to control the squared \( L^2(P) \) error of our adaptive estimator, because the Goldenshluger–Lepski method generally requires the corresponding norm to be known. This is so that the pairwise comparisons can be performed when calculating the proxy for the unknown bias of the non-adaptive estimators. In order to get around this problem, we replace the \( L^2(P) \) norm in the pairwise comparisons with its empirical counterpart, the \( L^2(P_n) \) norm. Here, \( P_n \) is the empirical distribution of the covariates. The terms added to our bound when moving our control on the squared \( L^2(P_n) \) error of our adaptive estimator to the squared \( L^2(P) \) error do not significantly increase its size.

Our main results are Theorems 11 (page 11) and 22 (page 17). These show that a fixed quantile of the squared \( L^2(P) \) error of a clipped version of the estimator produced by the Goldenshluger–Lepski method is of order \( n^{-\beta/(1+\beta)} \). Here, \( n \) is the number of data points and \( \beta \) parametrises the interpolation space between \( L^\infty \) and the RKHS containing the regression function. We use \( L^\infty \) when interpolating so that we have direct control over approximation errors in the \( L^2(P_n) \) norm. Theorem 11 addresses the case in which the RKHS is fixed and Theorem 22 addresses the case in which we perform adaptation over a collection of RKHSs with Gaussian kernels. The order \( n^{-\beta/(1+\beta)} \) for the squared \( L^2(P) \) error of the adaptive estimators matches the order of the smallest bounds obtained in [15] for the squared \( L^2(P) \) error of the non-adaptive estimators. In the sense discussed in [15], this order is the optimal power of \( n \) if we make the slightly weaker assumption...
that the regression function is an element of the interpolation space between $L^2(P)$ and the RKHS parametrised by $\beta$.

2. Literature Review

Lepski’s method was introduced in the series of papers [10, 11, 12] as a method for adaptation over a single parameter. It has since been studied in, for example, [2] and [5]. Lepski’s method selects the smoothest non-adaptive estimator from a collection, subject to a bound on a series of pairwise comparisons involving all estimators at most as smooth as the resulting estimator. The method can only adapt to one parameter because of the need for an ordering of the collection of non-adaptive estimators.

Lepski’s method has been applied to RKHS regression under the name of the balancing principle. However, as far as we are aware, Lepski’s method has not been used to target the true regression function, but instead an RKHS element which approximates the true regression function. In [3], the authors note the difficulty in using Lepski’s method to control the squared $L^2(P)$ error of an adaptive version of a support vector machine (SVM). This difficulty arises because Lepski’s method generally requires the norm we are interested in controlling to be known in order to perform the pairwise comparisons. However, $P$ is unknown in this situation.

The authors of [3] get around the problem that $P$ is unknown as follows. Lepski’s method is used to control the known squared $L^2(P_n)$ error and squared RKHS error of two different adaptive SVMs. The results of these procedures are combined to produce an adaptive SVM whose squared $L^2(P)$ error is bounded. The above alteration is also noted in [13]. Furthermore, the authors show that it is possible to greatly reduce the number of pairwise comparisons which must be performed to produce an adaptive estimator. This is done by only comparing each estimator to the estimator which is next less smooth.

The Goldenshluger–Lepski method extends Lepski’s method in order to perform adaptation over multiple parameters. The method is introduced in the series of papers [6, 7, 8, 9]. The first two papers concentrate on function estimation in the presence of white noise. The first paper considers the problem of pointwise estimation, while the second paper examines estimation in the $L^p$ norm for $p \in [1, \infty]$. The third paper produces adaptive bandwidth estimators for kernel density estimation and the fourth paper considers general methodology for selecting a linear estimator from a collection.

An example of using training and validation to perform adaptation over a Gaussian kernel parameter for an SVM can be found in [4]. The procedure produces an adaptive estimator of a bounded regression function from a range of Sobolev spaces. This estimator is analysed using union bounding, as opposed to the chaining techniques used to analyse the Goldenshluger–Lepski method in this paper.
3. Contribution

In this paper, we use the Goldenshluger–Lepski method to produce an adaptive estimator from a collection of clipped versions of least-squares estimators which are constrained to lie in a ball of predefined radius in a fixed RKHS $H$, which is separable with a bounded and measurable kernel $k$. The estimator, defined by (1), adapts over the radius of the ball. As far as we are aware, the Goldenshluger–Lepski method has not previously been applied in the context of RKHS regression. Under the assumption that the regression function comes from an interpolation space between $L^\infty$ and some RKHS $H$ from the collection, we obtain a bound on a fixed quantile of the squared $L^2(P)$ error of the same order $n^{-\beta/(1+\beta)}$ (Theorem 22 on page 17).

4. RKHSs and Their Interpolation Spaces

An RKHS $H$ on $S$ is a Hilbert space of real-valued functions on $S$ such that, for all $x \in S$, there is some $k_x \in H$ such that $h(x) = \langle h, k_x \rangle_H$ for all $h \in H$. The function $k(x_1,x_2) = \langle k_{x_1},k_{x_2} \rangle_H$ for $x_1,x_2 \in S$ is known as the kernel and is symmetric and positive-definite.

We now define interpolation spaces between a Banach space $(Z, \|\cdot\|_Z)$ and a subspace $(V, \|\cdot\|_V)$ (see [1]). The $K$-functional of $(Z, V)$ is

$$K(z,t) = \inf_{v \in V} (\|z - v\|_Z + t\|v\|_V)$$

for $z \in Z$ and $t > 0$. We define

$$\|z\|_{\beta,q} = \left( \int_0^\infty (t^{-\beta}K(z,t))^{q}t^{-1}dt \right)^{1/q} \quad \text{and} \quad \|z\|_{\beta,\infty} = \sup_{t>0} (t^{-\beta}K(z,t))$$
for \( z \in Z, \beta \in (0, 1) \) and \( 1 \leq q < \infty \). We then define the interpolation space \([Z, V]_{\beta,q}\) to be the set of \( z \in Z \) such that \( \|z\|_{\beta,q} < \infty \). The size of \([Z, V]_{\beta,q}\) decreases as \( \beta \) increases. The following result is Lemma 1 of [15], which is essentially Theorem 3.1 of [19].

**Lemma 1** Let \((Z,\|\cdot\|_Z)\) be a Banach space, \((V,\|\cdot\|_V)\) be a subspace of \(Z\) and \( z \in [Z,V]_{\beta,\infty}\). We have

\[
\inf\{\|v - z\|_Z : v \in V, \|v\|_V \leq r\} \leq \frac{\|z\|_{\beta,q}^{1/(1-\beta)}}{r^{\beta/(1-\beta)}}.
\]

From the above, we can define the interpolation spaces \([L^\infty,H]_{\beta,q}\), where \(L^\infty\) is the space of bounded measurable functions on \((S,S)\). We set \( q = \infty \) and work with the largest space of functions for a fixed \( \beta \in (0,1) \). We are then able to apply the approximation result in Lemma 1.

### 5. Problem Definition

We give a formal definition of the RKHS regression problem. For a topological space \( T \), let \( \mathcal{B}(T) \) be its Borel \( \sigma \)-algebra. Let \((S,S)\) be a measurable space and \((X_i,Y_i)\) for \( 1 \leq i \leq n \) be i.i.d. \((S \times \mathbb{R},S \otimes \mathcal{B}((0,1)))\)-valued random variables on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We assume \( X_i \sim \mathbb{P} \) and \( \mathbb{E}(Y_i^2) < \infty \), where \( \mathbb{E} \) denotes integration with respect to \( \mathbb{P} \). We have \( \mathbb{E}(Y_i|X_i) = g(X_i) \) almost surely for some function \( g \) which is measurable on \((S,S)\) (Section A3.2 of [22]). Since \( \mathbb{E}(Y_i^2) < \infty \), it follows that \( g \in L^2(\mathbb{P}) \) by Jensen’s inequality. We assume throughout that

\[
(g1) \qquad \|g\|_\infty \leq C \text{ for } C > 0.
\]

We also need to make an assumption on the behaviour of the errors of the response variables \( Y_i \) for \( 1 \leq i \leq n \). Let \( U \) and \( V \) be random variables on \((\Omega, \mathcal{F}, \mathbb{P})\). We say \( U \) is \( \sigma^2 \)-subgaussian if

\[
\mathbb{E}(\exp(tU)) \leq \exp(\sigma^2 t^2 / 2)
\]

for all \( t \in \mathbb{R} \). We say \( U \) is \( \sigma^2 \)-subgaussian given \( V \) if

\[
\mathbb{E}(\exp(tU)|V) \leq \exp(\sigma^2 t^2 / 2)
\]

almost surely for all \( t \in \mathbb{R} \). We assume

\[
(Y) \qquad Y_i - g(X_i) \text{ is } \sigma^2 \text{-subgaussian given } X_i \text{ for } 1 \leq i \leq n.
\]

### 6. Regression for a Fixed RKHS

We continue by providing simultaneous bounds on our collection of non-adaptive estimators for a fixed RKHS. Our results in this section depend on how well the regression
function $g$ can be approximated by elements of an RKHS $H$ with kernel $k$. We make the following assumptions.

$(H)$ The RKHS $H$ with kernel $k$ has the following properties:

- The RKHS $H$ is separable.
- The kernel $k$ is bounded.
- The kernel $k$ is a measurable function on $(S \times S, S \otimes S)$.

We define $\|k\|_{\text{diag}} = \sup_{x \in S} k(x, x) < \infty$. We can guarantee that $H$ is separable by, for example, assuming that $k$ is continuous and $S$ is a separable topological space (Lemma 4.33 of [20]). The fact that $H$ has a kernel $k$ which is measurable on $(S \times S, S \otimes S)$ guarantees that all functions in $H$ are measurable on $(S, S)$ (Lemma 4.24 of [20]).

Let $B_H$ be the closed unit ball of $H$ and $r > 0$. We define the estimator

$$\hat{h}_r = \arg \min_{f \in rB_H} \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - Y_i)^2$$

of the regression function $g$. We make this definition unique by demanding that $\hat{h}_r \in \text{sp}\{k_X : 1 \leq i \leq n\}$ (see Lemma 3 of [15]). We also define $\hat{h}_0 = 0$. The following combines parts of Lemmas 3 and 32 of [15].

Lemma 2 Assume $(H)$. We have that $\hat{h}_r$ is a $(H, B(H))$-valued measurable function on $(\Omega \times [0, \infty), \mathcal{F} \otimes B([0, \infty)))$, where $r$ varies in $[0, \infty)$. Furthermore, $\|\hat{h}_r - \hat{h}_s\|_H^2 \leq |r^2 - s^2|$ for $r, s \in [0, \infty)$.

Since we assume $(g1)$, that $g$ is bounded in $[-C, C]$, we can make $\hat{h}_r$ closer to $g$ by constraining it to lie in the same interval. As in [15], we define the projection $V : \mathbb{R} \to [-C, C]$ by

$$V(t) = \begin{cases} -C & \text{if } t < -C \\ t & \text{if } |t| \leq C \\ C & \text{if } t > C \end{cases}$$

for $t \in \mathbb{R}$.

We now prove a series of results which allow us to control $\hat{h}_r$ for $r \geq 0$ simultaneously, extending the results of [15] while using similar proof techniques. This is crucial in order to apply the Goldenshluger–Lepski method to these estimators. The results assign probabilities to events which occur for all $r \geq 0$ and all $h_r \in rB_H$. These events are measurable due to the separability of $[0, \infty)$ and $rB_H$, as well as the continuity in $r$ of the quantities in question, including $\hat{h}_r$ by Lemma 2. By Lemma 2 of [15], we have

$$\|\hat{h}_r - h_r\|_{L^2(P_n)}^2 \leq \frac{4}{n} \sum_{i=1}^{n} (Y_i - g(X_i))(\hat{h}_r(X_i) - h_r(X_i)) + 4\|h_r - g\|_{L^2(P_n)}^2$$
for all \( r > 0 \) and all \( h_r \in rB_H \). We can get rid of \( \hat{h}_r \) in the first term on the right-hand side by taking a supremum over \( rB_H \). After applying the reproducing kernel property and the Cauchy–Schwarz inequality, we obtain a quadratic form of subgaussians which can be controlled using Lemma 36 of [15].

**Lemma 3** Assume \((Y)\) and \((H)\). Let \( t \geq 1 \) and \( A_{1,t} \in \mathcal{F} \) be the set on which

\[
\|\hat{h}_r - h_r\|_{L^2(P_n)}^2 \leq \frac{20\|k\|_{\text{diag}}^{1/2}\sigma r^{1/2}}{n^{1/2}} + 4\|h_r - g\|_{\infty}^2
\]

simultaneously for all \( r \geq 0 \) and all \( h_r \in rB_H \). We have \( \mathbb{P}(A_{1,t}) \geq 1 - e^{-t} \).

It is useful to be able to transfer a bound on the squared \( L^2(P_n) \) error of an estimator, including the result above, to a bound on the squared \( L^2(P) \) error of the estimator. By using Talagrand’s inequality, we can obtain a high-probability bound on

\[
\sup_{r > 0} \sup_{f_1, f_2 \in rB_H} \frac{1}{r} \|Vf_1 - Vf_2\|_{L^2(P_n)}^2 - \|Vf_1 - Vf_2\|_{L^2(P)}^2
\]

by proving an expectation bound on the same quantity. By using symmetrisation (Lemma 2.3.1 of [21]) and the contraction principle for Rademacher processes (Theorem 3.2.1 of [5]), we again obtain a quadratic form of subgaussians, which in this case are Rademacher random variables.

**Lemma 4** Assume \((H)\). Let \( s \geq r \geq 0 \). We have \( \mathbb{P}(A_{2,t}) \geq 1 - e^{-t} \).

To capture how well \( g \) can be approximated by elements of \( H \), we define

\[
I_\infty(g, r) = \inf \{ \|h_r - g\|_{\infty}^2 : h_r \in rB_H \}
\]

for \( r \geq 0 \). We use this measure of approximation as it is compatible with the use of the bound

\[
\|h_r - g\|_{L^2(P_n)}^2 \leq \|h_r - g\|_{\infty}^2
\]

in the proof of Lemma 3. We show that \( I_\infty(g, r) \) is continuous.

**Lemma 5** Assume \((H)\). Let \( s \geq r \geq 0 \). We have

\[
I_\infty(g, s) \leq I_\infty(g, r) \leq \left( I_\infty(g, s)^{1/2} + \|k\|_{\text{diag}}^{1/2}(s - r) \right)^2.
\]

We obtain a bound on the squared \( L^2(P) \) error of \( V\hat{h}_r \) by combining Lemmas 3 and 4.
Theorem 6 Assume \((g), (Y)\) and \((H)\). Let \(t \geq 1\) and recall the definitions of \(A_{1,t}\) and \(A_{2,t}\) from Lemmas 3 and 4. On the set \(A_{1,t} \cap A_{2,t} \in \mathcal{F}\), for which \(\mathbb{P}(A_{1,t} \cap A_{2,t}) \geq 1 - 2e^{-t}\), we have
\[
\|V \hat{h}_r - g\|_{L^2(P)} \leq \frac{2\|k\|_{\text{diag}}^{1/2} (97C + 20\sigma) rt^{1/2}}{n^{1/2}} + \frac{16\|k\|_{\text{diag}}^{1/2} Ct}{3n} + 10 I_\infty(g, r)
\]
simultaneously for all \(r \geq 0\).

7. The Goldenshluger–Lepski Method for a Fixed RKHS

We now produce bounds on our adaptive estimator for a fixed RKHS. The following result, which is a simple consequence of Lemma 3, can be used to define the majorant of the non-adaptive estimators. This motivates the definition of the adaptive estimator used in the Goldenshluger–Lepski method.

Lemma 7 Assume \((Y)\) and \((H)\). Let \(t \geq 1\) and recall the definition of \(A_{1,t}\) from Lemma 3. On the set \(A_{1,t} \in \mathcal{F}\), for which \(\mathbb{P}(A_{1,t}) \geq 1 - e^{-t}\), we have
\[
\|\hat{h}_r - \hat{h}_s\|_{L^2(P)}^2 \leq \frac{80\|k\|_{\text{diag}}^{1/2} \sigma(r + s) t^{1/2}}{n^{1/2}} + 40 I_\infty(g, r)
\]
simultaneously for all \(s \geq r \geq 0\).

Let \(R \subseteq [0, \infty)\) be closed and non-empty. The Goldenshluger–Lepski method defines an adaptive estimator using
\[
\hat{r} = \arg \min_{r \in R} \left( \sup_{s \in R, s \geq r} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(P)}^2 - \tau(r + s) \frac{t}{n^{1/2}} \right) + \frac{2(1 + \nu) \tau r}{n^{1/2}} \right)
\]
for tuning parameters \(\tau, \nu > 0\). The supremum of pairwise comparisons can be viewed as a proxy for the unknown bias, while the other term is an inflated variance term. Note that the supremum is at least the value at \(r\), so
\[
\sup_{s \in R, s \geq r} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(P)}^2 - \tau(r + s) \frac{t}{n^{1/2}} \right) + \frac{2(1 + \nu) \tau r}{n^{1/2}} \geq \frac{2\nu \tau r}{n^{1/2}}.
\]
The role of the tuning parameter \(\nu\) is simply to control this bound. The parameter \(\tau\) controls the probability with which our bound on the squared \(L^2(P)\) error of \(V \hat{h}_{\hat{r}}\) holds. We give a unique definition of \(\hat{r}\).

Lemma 8 Let \(\hat{r}\) be the infimum of all points attaining the minimum in (1). Then \(\hat{r}\) is well-defined.
It may be that \( \hat{r} \) is not a random variable on \((\Omega, \mathcal{F})\) in some cases, but we assume
\[
(\hat{r}) \quad \hat{r} \text{ is a well-defined random variable on } (\Omega, \mathcal{F})
\]
throughout. Later, we assume that \( R \) is finite, in which case \( \hat{r} \) is certainly a random variable on \((\Omega, \mathcal{F})\). If \( \hat{r} \) is a random variable on \((\Omega, \mathcal{F})\), then \( \hat{h}_r \) is a \((H, \mathcal{B}(H))\)-valued measurable function on \((\Omega, \mathcal{F})\) by Lemma 2.

By Lemma 7, the supremum in the definition of \( \hat{r} \) is at most \( 40I_\infty(g, r) \) for an appropriate value of \( t \). The definition of \( \hat{r} \) then gives us control over the squared \( L^2(P_n) \) norm of \( \hat{h}_r - \hat{h}_r \) when \( \hat{r} \leq r \). When \( \hat{r} \geq r \), we can control the squared \( L^2(P_n) \) norm of \( \hat{h}_r - \hat{h}_r \) using Lemma 7. However, we must control a term of order \( \hat{r}/n^{1/2} \) using (2) and the definition of \( \hat{r} \).

In both cases, this gives a bound on the squared \( L^2(P_n) \) norm of \( V\hat{h}_r - \hat{h}_r \). Extra terms appear when moving to a bound on the squared \( L^2(P) \) norm of \( V\hat{h}_r - \hat{h}_r \) using Lemma 4. However, these terms are very similar to the inflated variance term, and can be controlled in the same way. Applying
\[
\|V\hat{h}_r - g\|_{L^2(P)}^2 \leq 2\|V\hat{h}_r - \hat{h}_r\|_{L^2(P)}^2 + 2\|\hat{h}_r - g\|_{L^2(P)}^2
\]
gives the following result.

**Theorem 9** Assume \((Y), (H)\) and \((\hat{r})\). Let \( \tau \geq 80\|k\|_{\text{diag} \sigma}^{1/2}, \nu > 0 \) and
\[
t = \left( \frac{\tau}{80\|k\|_{\text{diag} \sigma}^{1/2}} \right)^2 \geq 1.
\]

Recall the definitions of \( A_{1,t} \) and \( A_{2,t} \) from Lemmas 3 and 4. On the set \( A_{1,t} \cap A_{2,t} \in \mathcal{F} \), for which \( P(A_{1,t} \cap A_{2,t}) \geq 1 - 2e^{-t} \), we have
\[
\|V\hat{h}_r - g\|_{L^2(P)}^2
\]
is at most
\[
\inf_{\tau \in R} \left( \max \left( \frac{2\tau r}{n^{1/2}} + \frac{1}{\nu} + \frac{97C}{80\sigma\nu} + \frac{C\tau}{2400\|k\|_{\text{diag} \sigma}^{1/2} \nu n^{1/2}} \right), \left( 40I_\infty(g, r) + \frac{2(1 + \nu)\tau r}{n^{1/2}} \right), \right.
\]
\[
\left. \frac{4(2 + \nu)\tau r}{n^{1/2}} + \frac{97C\tau r}{40\sigma n^{1/2}} + \frac{C\tau^2 r}{1200\|k\|_{\text{diag} \sigma}^{1/2} \nu^2 n} \right) + 80I_\infty(g, r) + 2\|\hat{h}_r - g\|_{L^2(P)}^2 \right).
\]

We now combine Theorems 6 and 9.

**Theorem 10** Assume \((g1), (Y), (H)\) and \((\hat{r})\). Let \( \tau \geq 80\|k\|_{\text{diag} \sigma}^{1/2}, \nu > 0 \) and
\[
t = \left( \frac{\tau}{80\|k\|_{\text{diag} \sigma}^{1/2}} \right)^2 \geq 1.
\]
Recall the definitions of $A_{1,t}$ and $A_{2,t}$ from Lemmas 3 and 4. On the set $A_{1,t} \cap A_{2,t} \in \mathcal{F}$, for which $\mathbb{P}(A_{1,t} \cap A_{2,t}) \geq 1 - 2e^{-t}$, we have

$$
\|V\hat{h}_r - g\|_{L^2(P)}^2 \leq \inf_{r \in R} \left( (1 + D_1\tau n^{-1/2})(D_2\tau r n^{-1/2} + D_3I_\infty(g,r)) \right)
$$

for constants $D_1, D_2, D_3 > 0$ not depending on $\tau, r$ or $n$.

We can obtain rates of convergence for our estimator $V\hat{h}_r$ if we make an assumption about how well $g$ can be approximated by elements of $H$. Let us assume

$$
g \in [L^\infty, H]_{\beta, \infty} \text{ with norm at most } B \text{ for } \beta \in (0, 1) \text{ and } B > 0.
$$

The assumption $(g2)$, together with Lemma 1, give

$$
I_\infty(g, r) \leq \frac{B^{2/(1 - \beta)}}{r^{2\beta/(1 - \beta)}}
$$

for $r > 0$. In order for us to apply Theorem 10 to this setting, we need to make an assumption on $R$. We assume either

$$(R1) \quad R = [0, \infty)$$

or

$$(R2) \quad R = \{bi : 0 \leq i \leq I - 1\} \cup \{an^{1/2}\} \text{ and } \rho = an^{1/2} \text{ for } a, b > 0 \text{ and } I = [an^{1/2}/b].$$

The assumption $(R1)$ is mainly of theoretical interest and would make it difficult to calculate $\hat{r}$ in practice. The estimator $\hat{r}$ can be computed under the assumption $(R2)$, since in this case $R$ is finite. We obtain a high-probability bound on a fixed quantile of the squared $L^2(P)$ error of $V\hat{h}_r$ of order $t^{1/2}n^{-\beta/(1 + \beta)}$ with probability at least $1 - e^{-t}$ when $\tau$ is an appropriate multiple of $t^{1/2}$.

**Theorem 11** Assume $(g1)$, $(g2)$, $(Y)$ and $(H)$. Let $\tau \geq 80\|k\|_{\text{diag}}^{-1/2} \sigma, \nu > 0$ and

$$
t = \left( \frac{\tau}{80\|k\|_{\text{diag}}^{-1/2} \sigma} \right)^2 \geq 1.
$$

Also assume $(R1)$ and $(\hat{r})$, or $(R2)$. Recall the definitions of $A_{1,t}$ and $A_{2,t}$ from Lemmas 3 and 4. On the set $A_{1,t} \cap A_{2,t} \in \mathcal{F}$, for which $\mathbb{P}(A_{1,t} \cap A_{2,t}) \geq 1 - 2e^{-t}$, we have

$$
\|V\hat{h}_r - g\|_{L^2(P)}^2 \leq D_1\tau n^{-\beta/(1 + \beta)} + D_2\tau^2 n^{-(1 + 3\beta)/(2(1 + \beta))}
$$

for constants $D_1, D_2 > 0$ not depending on $n$ or $\tau$.

### 8. Regression for a Collection of RKHSs

In this section, we again provide simultaneous bounds on our collection of non-adaptive estimators. Our results still depend on how well the regression function $g$ can be approxi-
mated by elements of an RKHS. However, this RKHS now comes from a collection instead of being fixed. Let $\mathcal{K}$ be a set of kernels on $S \times S$. We make the following assumptions.

(K1) The covariate set $S$ and the set of kernels $\mathcal{K}$ have the following properties:

- The covariate set $S$ is a separable topological space.
- The set of kernels $(\mathcal{K}, \|\cdot\|_{\infty})$ is separable.
- The kernel $k$ is bounded for all $k \in \mathcal{K}$.
- The kernel $k$ is continuous for all $k \in \mathcal{K}$.

Since $(\mathcal{K}, \|\cdot\|_{\infty})$ is a separable set of kernels, we have that $\mathcal{K}$ has a countable dense subset $\mathcal{K}_0$. For all $\varepsilon > 0$ and all $k \in \mathcal{K}$, there exists $k_0 \in \mathcal{K}_0$ such that

$$
\|k_0 - k\|_{\infty} = \sup_{x_1, x_2 \in S} |k_0(x_1, x_2) - k(x_1, x_2)| < \varepsilon.
$$

Let $H_k$ be the RKHS with kernel $k$ for $k \in \mathcal{K}$. Since $k$ is continuous and $S$ is a separable topological space, we have that $H_k$ is separable by Lemma 4.33 of [20]. Hence, the assumption (H) holds for $H_k$. We use the notation $\|\cdot\|_k$ and $\langle \cdot, \cdot \rangle_k$ for the norm and inner product of $H_k$.

Let $B_k$ be the closed unit ball of $H_k$ for $k \in \mathcal{K}$ and $r > 0$. We define the estimator

$$
\hat{h}_{k,r} = \arg\min_{f \in rB_k} \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - Y_i)^2
$$

of the regression function $g$. We make this definition unique by demanding that $\hat{h}_{k,r} \in \text{sp}\{kX_i : 1 \leq i \leq n\}$ (see Lemma 3 of [15]). We also define $\hat{h}_{k,0} = 0$. Since we assume (g1), that $g$ is bounded in $[-C, C]$, we can make $\hat{h}_{k,r}$ closer to $g$ by clipping it to obtain $V\hat{h}_{k,r}$.

Lemma 12 Assume (K1). We have that $\hat{h}_{k,r}$ is an $(L^\infty, \mathcal{B}(L^\infty))$-valued measurable function on $(\Omega \times \mathcal{K} \times [0, \infty), \mathcal{F} \otimes \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}([0, \infty)))$, where $k$ varies in $\mathcal{K}$ and $r$ varies in $[0, \infty)$.

Let

$$
\mathcal{L} = \{k/\|k\|_{\text{diag}} : k \in \mathcal{K}\} \cup \{0\}
$$

and

$$
D = \sup_{f_1, f_2 \in \mathcal{L}} \|f_1 - f_2\|_{\infty} \leq 2.
$$

We include 0 in the definition of $\mathcal{L}$ so that, when analysing stochastic processes over $\mathcal{L}$ using chaining, we can start all chains at 0. Note that $(\mathcal{L}, \|\cdot\|_{\infty})$ is separable since $\mathcal{L} \setminus \{0\}$ is the image of a continuous function on $(\mathcal{K}, \|\cdot\|_{\infty})$, which is itself separable. Let
$N(a,M,d)$ be the minimum size of an $a > 0$ cover of a metric space $(M,d)$, and let

$$J = \left( 162 \int_0^{D/2} \log(2N(a,L,||\cdot||_{\infty}))da + 1 \right)^{1/2}.$$ 

The next result is proved using the same method as Lemma 3. However, instead of one quadratic form of subgaussians, we obtain a supremum over $K$ of quadratic forms of subgaussians. This can be controlled by chaining using Lemma 26.

**Lemma 13** Assume $(Y)$ and (K1). Let $t \geq 1$. There exists a set $A_{3,t} \in \mathcal{F}$ with $P(A_{3,t}) \geq 1 - e^{-t}$ on which

$$\|\hat{h}_{k,r} - h_{k,r}\|_{L^2(P_n)}^2 \leq \frac{21J\|k\|_{\text{diag}}Crt^{1/2}}{n^{1/2}} + 4\|\hat{h}_{k,r} - g\|_{\infty}^2$$

simultaneously for all $k \in K$, all $r \geq 0$ and all $h_{k,r} \in rB_k$.

It is again useful to be able to transfer a bound on the squared $L^2(P_n)$ error of an estimator to a bound on the squared $L^2(P)$ error of the estimator. The result below is proved using the same method as Lemma 4, although we again obtain a supremum of quadratic forms of subgaussians which are controlled using chaining. The event in the result is measurable by Lemma 27.

**Lemma 14** Assume (K1). Let $t \geq 1$ and $A_{4,t} \in \mathcal{F}$ be the set on which

$$\sup_{f_1, f_2 \in rB_k} \left| \|Vf_1 - Vf_2\|_{L^2(P_n)}^2 - \|Vf_1 - Vf_2\|_{L^2(P)}^2 \right| \leq \frac{151J\|k\|_{\text{diag}}Crt^{1/2}}{n^{1/2}} + \frac{8\|k\|_{\text{diag}}Crt}{3n}$$

simultaneously for all $k \in K$ and all $r \geq 0$. We have $P(A_{4,t}) \geq 1 - e^{-t}$.

To capture how well $g$ can be approximated by elements of $H_k$, we define

$$I_{\infty}(g,k,r) = \inf \{ \|h_{k,r} - g\|_{\infty}^2 : h_{k,r} \in rB_k \}$$

for $k \in K$ and $r \geq 0$. We obtain a bound on the squared $L^2(P)$ error of $V\hat{h}_{k,r}$ by combining Lemmas 13 and 14.

**Theorem 15** Assume (g1), (Y) and (K1). Let $t \geq 1$ and recall the definitions of $A_{3,t}$ and $A_{4,t}$ from Lemmas 13 and 14. On the set $A_{3,t} \cap A_{4,t} \in \mathcal{F}$, for which $P(A_{3,t} \cap A_{4,t}) \geq 1 - 2e^{-t}$, we have

$$\|V\hat{h}_{k,r} - g\|_{L^2(P)}^2 \leq \frac{2J\|k\|_{\text{diag}}(151C + 21\sigma)r^{1/2}}{n^{1/2}} + \frac{16\|k\|_{\text{diag}}Crt}{3n} + 10I_{\infty}(g,k,r)$$

simultaneously for all $k \in K$ and all $r \geq 0$.  


9. The Goldenshluger–Lepski Method for a Collection of RKHSs with Gaussian Kernels

We now apply the Goldenshluger–Lepski method again in the context of RKHS regression. However, we now produce an estimator which adapts over a collection of RKHSs with Gaussian kernels. We make the following assumptions on $S$ and $K$.

(K2) The covariate set $S$ and the set of kernels $K$ have the following properties:

- The covariate set $S \subseteq \mathbb{R}^d$ for $d \geq 1$.
- The set of kernels $K = \{k_\gamma(x_1, x_2) = \gamma^{-d} \exp\left(-\|x_1 - x_2\|^2/\gamma^2\right) : \gamma \in \Gamma$ and $x_1, x_2 \in S\}$ for $\Gamma \subseteq [u, v]$ non-empty for $v \geq u > 0$.

Recalling the definitions from the previous section, we have

$$L = \{f_\gamma(x_1, x_2) = \exp\left(-\|x_1 - x_2\|^2/\gamma^2\right) : \gamma \in \Gamma$ and $x_1, x_2 \in S\} \cup \{0\}.$$

The assumption (K2) implies the assumption (K1). This is because Lemma 29 shows that $(L, \|\cdot\|_\infty)$, and hence $(K, \|\cdot\|_\infty)$, is separable. We change notation slightly. Let $H_\gamma$ be the RKHS with kernel $k_\gamma$ for $\gamma \in \Gamma$, let $\|\cdot\|_\gamma$ and $\langle \cdot, \cdot \rangle_\gamma$ be the norm and inner product of $H_\gamma$, and let $B_\gamma$ be the closed unit ball of $H_\gamma$. Furthermore, we write $\hat{h}_{\gamma, r}$ in place of $\hat{h}_{k_\gamma, r}$ and $I_\infty(g, \gamma, r)$ in place of $I_\infty(g, k_\gamma, r)$.

The scaling of the kernels is selected so that the following lemma holds. The result is immediate from Proposition 4.46 of [20] and the way that the norm of an RKHS scales with its kernel (Theorem 4.21 of [20]).

**Lemma 16** Assume (K2). Let $\gamma, \eta \in \Gamma$ with $\gamma \geq \eta$. We have $B_\gamma \subseteq B_\eta$.

By Lemma 29, the function $F : \Gamma \to L \setminus \{0\}$ by $F(\gamma) = f_\gamma$ is continuous. Hence, the function $G : \Gamma \to K$ by $G(\gamma) = k_\gamma$ is continuous. The next result then follows from Lemma 12.

**Lemma 17** Assume (K2). We have that $\hat{h}_{\gamma, r}$ is an $(L^\infty, B(L^\infty))$-valued measurable function on $(\Omega \times \Gamma \times [0, \infty), \mathcal{F} \otimes B(\Gamma) \otimes B([0, \infty)))$, where $\gamma$ varies in $\Gamma$ and $r$ varies in $[0, \infty)$.

Recall the definition of $J$ from the previous section. Lemma 30 provides us with a bound on $J$.

**Lemma 18** Assume (K2). We have

$$J \leq (81(\log(8\log(v/u) + 4) + 2) + 1)^{1/2}.$$
The following result can be used to define the majorant of the non-adaptive estimators and is a simple consequence of Lemma 13. This motivates the definition of the adaptive estimator used in the Goldenshluger–Lepski method.

**Lemma 19** Assume $(Y)$ and (K2). Let $t \geq 1$ and recall the definition of $A_{3,t}$ from Lemma 13. On the set $A_{3,t} \in \mathcal{F}$, for which $\mathbb{P}(A_{3,t}) \geq 1 - e^{-t}$, we have

$$\|\hat{h}_{\gamma,r} - \hat{h}_{\eta,s}\|^2_{L^2(P_n)} \leq \frac{84J\sigma(\gamma^{-d/2}r + \eta^{-d/2}s)t^{1/2}}{n^{1/2}} + 40I_{\infty}(g, \gamma, r)$$

simultaneously for all $\gamma, \eta \in \Gamma$ such that $\eta \leq \gamma$ and all $s \geq r \geq 0$.

Let $R \subseteq [0, \infty)$ be non-empty. The Goldenshluger–Lepski method creates an adaptive estimator by defining $(\hat{\gamma}, \hat{r})$ to be the minimiser of

$$\sup_{\gamma \in \Gamma, \eta \leq \gamma} \sup_{s \in R, s \geq r} \left( \|\hat{h}_{\gamma,r} - \hat{h}_{\eta,s}\|^2_{L^2(P_n)} - \tau(\gamma^{-d/2}r + \eta^{-d/2}s) \right) + \frac{2(1 + \nu)\tau\gamma^{-d/2}r}{n^{1/2}}$$

over $(\gamma, r) \in \Gamma \times R$ for tuning parameters $\tau, \nu > 0$. Again, the supremum of pairwise comparisons can be viewed as a proxy for the unknown bias, while the other term is an inflated variance term. Note that the supremum is at least the value at $(\gamma, r)$, which means the above is at least

$$\frac{2\nu\tau\gamma^{-d/2}r}{n^{1/2}}.$$  \hfill (5)

Again, the role of the tuning parameter $\nu$ is simply to control this bound. The parameter $\tau$ controls the probability with which our bound on the squared $L^2(P)$ error of $Vh_{\hat{\gamma},\hat{r}}$ holds. It may be that $\hat{\gamma}$ is not a well-defined random variable on $(\Omega, \mathcal{F})$ in some cases, but we assume

$(\hat{\gamma})$

$\hat{\gamma}$ is a well-defined random variable on $(\Omega, \mathcal{F})$ throughout. Later, we assume that $R$ and $\Gamma$ are finite, in which case $\hat{\gamma}$ and $\hat{r}$ are certainly well-defined random variables on $(\Omega, \mathcal{F})$. If $\hat{\gamma}$ and $\hat{r}$ are well-defined random variables on $(\Omega, \mathcal{F})$, then $h_{\hat{\gamma},\hat{r}}$ is an $(L^\infty, \mathcal{B}(L^\infty))$-valued measurable function on $(\Omega, \mathcal{F})$ by Lemma 17.

By Lemma 19, the supremum in the definition of $(\hat{\gamma}, \hat{r})$ is at most $40I_{\infty}(g, \gamma, r)$ for an appropriate value of $t$. The definition of $(\hat{\gamma}, \hat{r})$ then gives us control over the squared $L^2(P_n)$ norm of $h_{\hat{\gamma},\hat{r}} - h_{\hat{\gamma},\hat{r}}$. We can control the squared $L^2(P_n)$ norm of $h_{\hat{\gamma},\hat{r}} - h_{\gamma,r}$ using Lemma 19. In both cases, we use the boundedness of $\Gamma$ when controlling the squared $L^2(P_n)$ norm before clipping the estimators using $V$. Extra terms appear when moving from bounds on the squared $L^2(P_n)$ norm to bounds on the squared $L^2(P)$ norm using Lemma 14. We must then control terms of order $\hat{r}^{-d/2}r/n^{1/2}$ using (5) and the definition of $(\hat{\gamma}, \hat{r})$. Combining the bounds gives a bound on the squared $L^2(P)$ norm of $Vh_{\hat{\gamma},\hat{r}} - Vh_{\gamma,r}$. Applying

$$\|V\hat{h}_r - g\|^2_{L^2(P)} \leq 2\|Vh_{\hat{\gamma},\hat{r}} - Vh_{\gamma,r}\|^2_{L^2(P)} + 2\|Vh_{\gamma,r} - g\|^2_{L^2(P)}$$
Stephen Page and Steffen Grünewälder

gives the following result. Comparisons between \((\hat{r}, \hat{\gamma}), (r, \gamma)\) and \((\hat{\gamma} \land \gamma, \hat{r} \lor r)\) are demonstrated in Figure 1 for two different values of \((r, \gamma)\).

Figure 1. A demonstration of the parameter comparisons made in the proof of Theorem 20

Theorem 20 Assume \((Y)\) and \((K2)\). Let \(\tau \geq 84J\sigma, \nu > 0\) and

\[
t = \left(\frac{\tau}{84J\sigma}\right)^2 \geq 1.
\]

Recall the definitions of \(A_{3,t}\) and \(A_{4,t}\) from Lemmas 13 and 14. On the set \(A_{3,t} \cap A_{4,t} \in F\), for which \(P(A_{3,t} \cap A_{4,t}) \geq 1 - 2e^{-t}\), we have

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \leq \inf_{\gamma \in \Gamma} \inf_{r \in R} \left(320I_\infty(g, \gamma, r) + \frac{4v^{d/2}(5 + 2\nu)\tau\gamma^{-d/2}r}{u^{d/2}n^{1/2}} + \frac{302Cv^{d/2}\tau\gamma^{-d/2}r}{21u^{d/2}\sigma n^{1/2}} + \frac{4Cv^{d/2}\tau^2\gamma^{-d/2}r}{1323J^2u^{d/2}\sigma^2 n} + \frac{12v^{d/2}}{u^{d/2}\nu} + \frac{302Cv^{d/2}}{21u^{d/2}\sigma \nu} + \frac{4Cv^{d/2}\tau}{1323J^2u^{d/2}\sigma^2 \nu n^{1/2}} + 20I_\infty(g, \gamma, r) + \frac{(1 + \nu)\tau\gamma^{-d/2}r}{n^{1/2}} + 12v^{d/2} + \frac{302Cv^{d/2}}{21u^{d/2}\sigma} + \frac{4Cv^{d/2}\tau}{1323J^2u^{d/2}\sigma^2 \nu n^{1/2}} + 2\|\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2\right).
\]

We now combine Theorems 15 and 20.

Theorem 21 Assume \((g1)\), \((Y)\) and \((K2)\). Let \(\tau \geq 84J\sigma, \nu > 0\) and

\[
t = \left(\frac{\tau}{84J\sigma}\right)^2 \geq 1.
\]

Recall the definitions of \(A_{3,t}\) and \(A_{4,t}\) from Lemmas 13 and 14. On the set \(A_{3,t} \cap A_{4,t} \in F\), for which \(P(A_{3,t} \cap A_{4,t}) \geq 1 - 2e^{-t}\), we have

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \leq \inf_{\gamma \in \Gamma} \inf_{r \in R} \left((1 + D_1 \tau n^{-1/2})(D_2 \tau^2 \gamma^{-d/2}r n^{-1/2} + D_3 I_\infty(g, \gamma, r))\right)
\]
for constants \(D_1, D_2, D_3 > 0\) not depending on \(\tau, \gamma, r\) or \(n\).

We can obtain rates of convergence for our estimator \(V\hat{h}_{\gamma, r}\) if we make an assumption about how well \(g\) can be approximated by elements of \(H_\alpha\) for \(\alpha \in [u, v]\). Let us assume
\[
(g3) \quad g \in [L^\infty, H_\alpha]_{\beta, \infty} \text{ with norm at most } B \text{ for } \alpha \in [u, v], \beta \in (0, 1) \text{ and } B > 0.
\]
The assumption \((g3)\) together with Lemma 1, give
\[
I_\infty(g, \alpha, r) \leq \frac{B^2}{r^{2\beta/(1-\beta)}}
\]
for \(r > 0\). In order for us to apply Theorem 21 to this setting, we need to make assumptions on \(\Gamma\) and \(R\). We assume either \((R1)\) and
\[
(\Gamma1) \quad \Gamma = [u, v],
\]
or \((R2)\) and
\[
(\Gamma2) \quad \Gamma = \{uc^i : 0 \leq i \leq L - 1\} \cup \{v\} \text{ for } c > 1 \text{ and } L = \lceil\log(v/u)/\log(c)\rceil.
\]
The assumptions \((R1)\) and \((\Gamma1)\) are mainly of theoretical interest and would make it difficult to calculate \((\hat{\gamma}, \hat{r})\) in practice. The estimator \((\hat{\gamma}, \hat{r})\) can be computed under the assumptions \((R2)\) and \((\Gamma2)\), since in this case \(R\) and \(\Gamma\) are finite. We obtain a high-probability bound on a fixed quantile of the squared \(L^2(P)\) error of \(V\hat{h}_{r, \gamma}\) of order 
\[
t^{1/2}n^{-\beta/(1+\beta)}
\]
with probability at least \(1 - e^{-t}\) when \(\tau\) is an appropriate multiple of \(t^{1/2}\).

**Theorem 22** Assume \((g1), (g3), (Y)\) and \((K2)\). Let \(\tau \geq 84J\sigma\), \(\nu > 0\) and
\[
t = \left(\frac{\tau}{84J\sigma}\right)^2 \geq 1.
\]
Also assume \((R1), (\Gamma1), (\hat{r})\) and \((\hat{\gamma})\), or \((R2)\) and \((\Gamma2)\). Recall the definitions of \(A_{3,t}\) and \(A_{4,t}\) from Lemmas 13 and 14. On the set \(A_{3,t} \cap A_{4,t} \in \mathcal{F}\), for which \(P(A_{3,t} \cap A_{4,t}) \geq 1 - 2e^{-t}\), we have
\[
\|\hat{V} h_{\hat{\gamma}, \hat{r}} - g\|_{L^2(P)}^2 \leq D_1\tau n^{-\beta/(1+\beta)} + D_2\tau^2 n^{-(1+3\beta)/(2(1+\beta))}
\]
for constants \(D_1, D_2 > 0\) not depending on \(n\) or \(\tau\).

**10. Discussion**

In this paper, we show how the Goldenshluger–Lepski method can be applied when performing regression over an RKHS \(H\), which is separable with a bounded and measurable kernel \(k\), or a collection of such RKHSs. We produce an adaptive estimator from a collection of clipped versions of least-squares estimators which are constrained to lie in a ball of predefined radius in \(H\). Since the \(L^2(P)\) norm is unknown, we use the \(L^2(P_n)\) norm.
when calculating the pairwise comparisons for the proxy for the unknown bias of this
collection of non-adaptive estimators. When $H$ is fixed, our estimator need only adapt
to the radius of the ball in $H$. However, when $H$ comes from a collection of RKHSs with
Gaussian kernels, the estimator must also adapt to the width parameter of the kernel. As
far as we are aware, this is the first time that the Goldenshluger–Lepski method has been
applied in the context of RKHS regression. In order to apply the Goldenshluger–Lepski
method in this context, we must provide a majorant by controlling all of the non-adaptive
estimators simultaneously, extending the results of [15].

By assuming that the regression function lies in an interpolation space between $L^\infty$ and
$H$ parametrised by $\beta$, we obtain a bound on a fixed quantile of the squared $L^2(P)$ error
of our adaptive estimator of order $n^{-\beta/(1+\beta)}$. This is true for both the case in which $H$ is
fixed and the case in which $H$ comes from a collection of RKHSs with Gaussian kernels.
The order $n^{-\beta/(1+\beta)}$ for the squared $L^2(P)$ error of the adaptive estimators matches the
order of the smallest bounds obtained in [15] for the squared $L^2(P)$ error of the non-
adaptive estimators. In the sense discussed in [15], this order is the optimal power of $n$
if we make the slightly weaker assumption that the regression function is an element of
the interpolation space between $L^2(P)$ and $H$ parametrised by $\beta$.

For the case in which $H$ comes from a collection of RKHSs with Gaussian kernels, our
current results rely on the boundedness of the set $\Gamma$ of width parameters of the kernels.
This is somewhat limiting as allowing the width parameter to tend to 0 as $n$ tends to
infinity would allow us to estimate a greater collection of functions. We hope that in
the future the analysis in the proof of Theorem 20 can be extended to allow for such
flexibility.

The results in this paper warrant the investigation of whether it is possible to extend the
use of the Goldenshluger–Lepski method from the case in which $H$ comes from a collection
of RKHSs with Gaussian kernels to other cases. The analysis in this paper relies on the
fact that the closed unit ball of the RKHS generated by a Gaussian kernel increases as
the width of the kernel decreases. It may be possible to apply a similar analysis to other
situations in which $H$ belongs to a collection of RKHSs which also exhibit this nestedness
property. If the RKHSs did not exhibit this property, then a new form of analysis would
be necessary to apply the Goldenshluger–Lepski method. In particular, we would need
a new criterion for deciding on the smoothness of the non-adaptive estimators when
performing the pairwise comparisons.

Appendix A: Proof of the Regression Results for a
Fixed RKHS

We bound the distance between $\hat{h}_r$ and $h_r$ in the $L^2(P_n)$ norm for $r \geq 0$ and $h_r \in rB_H$
to prove Lemma 3.
Proof of Lemma 3 The result is trivial for $r = 0$. By Lemma 2 of [15], we have

$$\|\hat{h}_r - h_r\|_{L^2(P_n)}^2 \leq \frac{4}{n} \sum_{i=1}^{n} (Y_i - g(X_i))(\hat{h}_r(X_i) - h_r(X_i)) + 4\|h_r - g\|_{L^2(P_n)}^2$$

for all $r > 0$ and all $h_r \in rB_H$. We now bound the right-hand side. We have

$$\|h_r - g\|_{L^2(P_n)} \leq \|h_r - g\|_{L^\infty(P_n)}.$$

Furthermore,

$$\frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))(\hat{h}_r(X_i) - h_r(X_i))$$

$$\leq \sup_{f \in 2rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))f(X_i) \right|$$

$$= \sup_{f \in 2rB_H} \left| \left\langle \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))k_{X_i}, f \right\rangle \right|_H$$

$$= 2r \left\| \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))k_{X_i} \right\|_H$$

$$= 2r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} (Y_i - g(X_i))(Y_j - g(X_j))k(X_i, X_j) \right)^{1/2}$$

by the reproducing kernel property and the Cauchy–Schwarz inequality. Let $K$ be the $n \times n$ matrix with $K_{i,j} = k(X_i, X_j)$ and let $\varepsilon$ be the vector of the $Y_i - g(X_i)$. Then

$$\frac{1}{n^2} \sum_{i,j=1}^{n} (Y_i - g(X_i))(Y_j - g(X_j))k(X_i, X_j) = \varepsilon^T (n^{-2}K) \varepsilon.$$ 

Furthermore, since $k$ is a measurable function on $(S \times S, S \otimes S)$, we have that $n^{-2}K$ is an $(\mathbb{R}^{n \times n}, B(\mathbb{R}^{n \times n}))$-valued measurable matrix on $(\Omega, \mathcal{F})$ and non-negative-definite. Let $a_i$ for $1 \leq i \leq n$ be the eigenvalues of $n^{-2}K$. Then

$$\max_{1 \leq i \leq n} a_i \leq \text{tr}(n^{-2}K) \leq n^{-1}\|k\|_{\text{diag}}$$

and

$$\text{tr}((n^{-2}K)^2) = \|a\|_2^2 \leq \|a\|_1^2 \leq n^{-2}\|k\|_{\text{diag}}^2.$$

Therefore, by Lemma 36 of [15], we have

$$\varepsilon^T (n^{-2}K) \varepsilon \leq \|k\|_{\text{diag}}\sigma^2 n^{-1}(1 + 2t + 2(t^2 + t)^{1/2})$$
and
\[
\frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))(\hat{h}_r(X_i) - h_r(X_i)) \leq \frac{5\|k\|_{\text{diag}}^{1/2} \sigma_r t^{1/2}}{n^{1/2}}
\]
with probability at least \(1 - e^{-t}\). The result follows. ■

The following lemma, which is Lemma 25 of [15], is useful for proving Lemma 4.

**Lemma 23** Let \(D > 0\) and \(A \subseteq L^\infty\) be separable with \(\|f\|_\infty \leq D\) for all \(f \in A\). Let
\[
Z = \sup_{f \in A} \left| \|f\|_{L^2(P_n)}^2 - \|f\|_{L^2(P)}^2 \right|.
\]
Then, for \(t > 0\), we have
\[
Z \leq E(Z) + \left( \frac{2D^4t}{n} + \frac{4D^2 E(Z)t}{n} \right)^{1/2} + \frac{2D^2 t}{3n}
\]
with probability at least \(1 - e^{-t}\).

We bound the supremum of the difference in the \(L^2(P_n)\) norm and the \(L^2(P)\) norm over \(rB_H\) for \(r \geq 0\) to prove Lemma 4.

**Proof of Lemma 4** The result is trivial for \(r = 0\). Let
\[
Z = \sup_{r > 0} \sup_{f_1, f_2 \in rB_H} \left| \|Vf_1 - Vf_2\|_{L^2(P_n)}^2 - \|Vf_1 - Vf_2\|_{L^2(P)}^2 \right|.
\]
Furthermore, let the \(\varepsilon_i\) for \(1 \leq i \leq n\) be i.i.d. Rademacher random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), independent of the \(X_i\). Lemma 2.3.1 of [21] shows
\[
E(Z) \leq 2E \left( \sup_{r > 0} \sup_{f_1, f_2 \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1/2}Vf_1(X_i) - r^{-1/2}Vf_2(X_i))^2 \right| \right)
\]
by symmetrisation. Since
\[
|Vf_1(X_i) - Vf_2(X_i)| \leq 2C
\]
for all \(r > 0\) and all \(f_1, f_2 \in rB_H\), we find
\[
\frac{(r^{-1/2}Vf_1(X_i) - r^{-1/2}Vf_2(X_i))^2}{4C}
\]
is a contraction vanishing at 0 as a function of \(r^{-1}Vf_1(X_i) - r^{-1}Vf_2(X_i)\) for all \(1 \leq i \leq n\).

By Theorem 3.2.1 of [5], we have
\[
E \left( \sup_{r > 0} \sup_{f_1, f_2 \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \frac{(r^{-1/2}Vf_1(X_i) - r^{-1/2}Vf_2(X_i))^2}{4C} \right| \right)< X
\]
is at most
\[2 \mathbb{E} \left( \sup_{r > 0} \sup_{f_1, f_2 \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1} V f_1(X_i) - r^{-1} V f_2(X_i)) \right| \right) \]almost surely. Therefore,
\[\mathbb{E}(Z) \leq 16C \mathbb{E} \left( \sup_{r > 0} \sup_{f_1, f_2 \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1} V f_1(X_i) - r^{-1} V f_2(X_i)) \right| \right) \leq 32C \mathbb{E} \left( \sup_{r > 0} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} V f(X_i) \right| \right) \]by the triangle inequality. Again, by Theorem 3.2.1 of [5], we have
\[\mathbb{E}(Z) \leq 64C \mathbb{E} \left( \sup_{r > 0} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} f(X_i) \right| \right) \]
since \(V\) is a contraction vanishing at 0. We have
\[\sup_{r > 0} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} f(X_i) \right| = \sup_{r > 0} \sup_{f \in rB_H} \left| \left\langle \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i k_{X_i}, r^{-1} f \right\rangle \right|_H \]
\[= \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i k_{X_i} \right\|_H \]
\[= \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j k(X_i, X_j) \right)^{1/2} \]
by the reproducing kernel property and the Cauchy–Schwarz inequality. By Jensen’s inequality, we have
\[\mathbb{E} \left( \sup_{r > 0} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} f(X_i) \right| \right) \leq \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \text{cov}(\varepsilon_i, \varepsilon_j|X) k(X_i, X_j) \right)^{1/2} \]
\[= \left( \frac{1}{n^2} \sum_{i=1}^{n} k(X_i, X_i) \right)^{1/2} \]
almost surely and again, by Jensen’s inequality, we have
\[\mathbb{E} \left( \sup_{r > 0} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} f(X_i) \right| \right) \leq \left( \frac{\|k\|_{\text{diag}}}{n} \right)^{1/2} \].
Hence, $\mathbb{E}(Z) \leq 64\|k\|_{\text{diag}}^{1/2}Cn^{-1/2}$.

Let
$$A = \left\{ r^{-1/2}Vf_1 - r^{-1/2}Vf_2 : r > 0 \text{ and } f_1, f_2 \in rB_H \right\}.$$  

We have that $(0, \infty)$, the set indexing $r$, is separable. Furthermore, $H$ is separable and so is separable in $L^\infty$ as it can be continuously embedded in $L^\infty$ due to its bounded kernel. Therefore, $rB_H \subseteq H$ is separable in $L^\infty$ for $r > 0$. Hence, we have that $A \subseteq L^\infty$ is separable. Furthermore,
$$\left\| r^{-1/2}Vf_1 - r^{-1/2}Vf_2 \right\|_{L^\infty} \leq \min \left\{ 2Cr^{-1/2}, 2\|k\|_{\text{diag}}^{1/2}C^{1/2} \right\} \leq 2\|k\|_{\text{diag}}^{1/2}C^{1/2}$$

for all $r > 0$ and all $f_1, f_2 \in rB_H$. The first term in the minimum comes from clipping using $V$, while the second term comes from the continuous embedding of $H$ in $L^\infty$ due to its bounded kernel. By Lemma 23, we have
$$Z \leq \mathbb{E}(Z) + \left( \frac{32\|k\|_{\text{diag}}^{1/2}C^2t}{n} + \frac{16\|k\|_{\text{diag}}^{1/2}C\mathbb{E}(Z)t}{n} \right)^{1/2} + \frac{8\|k\|_{\text{diag}}^{1/2}Ct}{3n}$$

with probability at least $1 - e^{-t}$. We have $\mathbb{E}(Z) \leq 64\|k\|_{\text{diag}}^{1/2}Cn^{-1/2}$ from above. The result follows.

We move the bound on the distance between $V\hat{h}_r$ and $Vh_r$ from the $L^2(P_n)$ norm to the $L^2(P)$ norm for $r \geq 0$ and $h_r \in rB_H$.

**Corollary 24** Assume $(Y)$ and $(H)$. Let $t \geq 1$ and recall the definitions of $A_{1,t}$ and $A_{2,t}$ from Lemmas 3 and 4. On the set $A_{1,t} \cap A_{2,t} \in \mathcal{F}$, for which $\mathbb{P}(A_{1,t} \cap A_{2,t}) \geq 1 - 2e^{-t}$, we have
$$\|V\hat{h}_r - Vh_r\|_{L^2(P)}^2 \leq \frac{\|k\|_{\text{diag}}^{1/2}(97C + 20\sigma)rt^{1/2}}{n^{1/2}} + \frac{8\|k\|_{\text{diag}}^{1/2}Cr}{3n} + 4\|h_r - g\|_{L^\infty}^2$$

simultaneously for all $r > 0$ and all $h_r \in rB_H$.

**Proof** By Lemma 3, we have
$$\|\hat{h}_r - h_r\|_{L^2(P_n)}^2 \leq \frac{20\|k\|_{\text{diag}}^{1/2}\sigma r^{1/2}}{n^{1/2}} + 4\|h_r - g\|_{L^\infty}^2$$

for all $r \geq 0$ and all $h_r \in rB_H$, so
$$\|V\hat{h}_r - Vh_r\|_{L^2(P_n)}^2 \leq \frac{20\|k\|_{\text{diag}}^{1/2}\sigma r^{1/2}}{n^{1/2}} + 4\|h_r - g\|_{L^\infty}^2.$$  

Since $\hat{h}_r, h_r \in rB_H$, by Lemma 4 we have
$$\|V\hat{h}_r - Vh_r\|_{L^2(P)}^2 \leq \|V\hat{h}_r - Vh_r\|_{L^2(P_n)}^2$$
Lepski for Constrained Estimators over RKHSs

\[\begin{align*}
\leq \sup_{f_1, f_2 \in B_H} & \left| \left\langle Vf_1 - Vf_2, \mathbb{E}(X_i | f_1) - \mathbb{E}(X_i | f_2) \right\rangle \right| \\
\leq & \frac{97}{n^{1/2}} \left\| k \right\|_{\text{diag}}^{1/2} \text{Crt}^{1/2} + \frac{8}{3n} \left\| k \right\|_{\text{diag}}^{1/2} \text{Crt}.
\end{align*}\]

The result follows.

We bound the changes in \(I_\infty(g,r)\) with \(r \geq 0\) to prove Lemma 5.

**Proof of Lemma 5** We have \(I_\infty(g,s) \leq I_\infty(g,r)\) since \(rB_H \subseteq sB_H\). Let \(h_s \in sB_H\). We have
\[\begin{align*}
\left\| \frac{r}{s} h_s - g \right\|_\infty & \leq \left\| \frac{r}{s} h_s - h_s \right\|_\infty + \left\| h_s - g \right\|_\infty.
\end{align*}\]

We have
\[\begin{align*}
\left\| \frac{r}{s} h_s - h_s \right\|_\infty & = \left(1 - \frac{r}{s}\right) \left\| h_s \right\|_\infty \\
& \leq (s - r) \left\| k \right\|_{\text{diag}}.
\end{align*}\]

The result follows.

We assume \((g1)\) to bound the distance between \(\hat{h}_r\) and \(g\) in the \(L^2(P)\) norm for \(r \geq 0\) and prove Theorem 6.

**Proof of Theorem 6** Note that \(Vg = g\). We have
\[\begin{align*}
\left\| V\hat{h}_r - g \right\|_{L^2(P)}^2 & \leq \left( \left\| V\hat{h}_r - Vh_r \right\|_{L^2(P)} + \left\| Vh_r - g \right\|_{L^2(P)} \right)^2 \\
& \leq 2 \left\| V\hat{h}_r - Vh_r \right\|_{L^2(P)}^2 + 2 \left\| Vh_r - g \right\|_{L^2(P)}^2 \\
& \leq 2 \left\| V\hat{h}_r - Vh_r \right\|_{L^2(P)}^2 + 2 \left\| h_r - g \right\|_{L^2(P)}^2
\end{align*}\]

for all \(r \geq 0\) and all \(h_r \in rB_H\). By Corollary 24, we have
\[\begin{align*}
\left\| V\hat{h}_r - Vh_r \right\|_{L^2(P)}^2 & \leq \frac{\left\| k \right\|_{\text{diag}}^{1/2} (97C + 20\sigma) rt^{1/2}}{n^{1/2}} + \frac{8}{3n} \left\| k \right\|_{\text{diag}}^{1/2} \text{Crt} + 4 \left\| h_r - g \right\|_\infty^2.
\end{align*}\]

Hence,
\[\begin{align*}
\left\| V\hat{h}_r - g \right\|_{L^2(P)}^2 & \leq \frac{2 \left\| k \right\|_{\text{diag}}^{1/2} (97C + 20\sigma) rt^{1/2}}{n^{1/2}} + \frac{16}{3n} \left\| k \right\|_{\text{diag}}^{1/2} \text{Crt} + 10 \left\| h_r - g \right\|_\infty^2.
\end{align*}\]

Taking an infimum over \(h_r \in rB_H\) proves the result.

**Appendix B: Proof of the Goldenshluger–Lepski Method for a Fixed RKHS**

We bound the distance between \(\hat{h}_r\) and \(\hat{h}_s\) in the \(L^2(P_n)\) norm for \(s \geq r \geq 0\) to prove Lemma 7.
Proof of Lemma 7} By Lemma 3, we have
\[
\|\hat{h}_r - \hat{h}_s\|_{L^2(\mu_n)}^2 \leq 4\|\hat{h}_r - h_r\|_{L^2(\mu_n)}^2 + 4\|h_r - g\|_{L^2(\mu_n)}^2 + 4\|g - h_s\|_{L^2(\mu_n)}^2 + 4\|h_s - \hat{h}_s\|_{L^2(\mu_n)}^2 \\
\leq \frac{80\|k\|_{\text{diag}}^{1/2}(r+s)t^{1/2}}{n^{1/2}} + 20\|h_r - g\|_{\infty}^2 + 20\|h_s - g\|_{\infty}^2.
\]
for all \( r, s \geq 0 \) and all \( h_r \in rB_H, h_s \in sB_H \). Taking an infimum over \( h_r \in rB_H \) and \( h_s \in sB_H \) gives
\[
\|\hat{h}_r - \hat{h}_s\|_{L^2(\mu_n)}^2 \leq \frac{80\|k\|_{\text{diag}}^{1/2}(r+s)t^{1/2}}{n^{1/2}} + 20I_{\infty}(g, r) + 20I_{\infty}(g, s).
\]
The result follows. \( \square \)

We prove Lemma 8.

Proof of Lemma 8} Let \( K \) be the \( n \times n \) symmetric matrix with \( K_{i,j} = k(X_i, X_j) \). By Lemma 3 of [15], we have that \( K \) is an \( (\mathbb{R}^{n\times n}, \mathcal{B}(\mathbb{R}^{n\times n})) \)-valued measurable matrix on \((\Omega, \mathcal{F})\) and that there exist an orthogonal matrix \( A \) and a diagonal matrix \( D \) which are both \((\mathbb{R}^{n\times n}, \mathcal{B}(\mathbb{R}^{n\times n}))\)-valued measurable matrices on \((\Omega, \mathcal{F})\) such that \( K = ADA^T \). Furthermore, we can demand that the diagonal entries of \( D \) are non-negative and non-increasing. Let \( m = \text{rk} K \) and
\[
\rho = \left( \sum_{i=1}^{m} D_{i,i}^{-1} (A^T Y)^2_i \right)^{1/2},
\]
which are random variables on \((\Omega, \mathcal{F})\). By Lemma 3 of [15], we have that \( \hat{h}_r \) is constant in \( r \) for \( r \geq \rho \). Hence,
\[
\inf_{r \in R} \left( \sup_{s \in R, s \geq r} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(\mu_n)}^2 - \frac{\tau(r+s)}{n^{1/2}} + \frac{2(1+\nu)\tau r}{n^{1/2}} \right) \right) = \inf_{r \in R \cap [0, \rho]} \left( \sup_{s \in R, s \geq r} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(\mu_n)}^2 - \frac{\tau(r+s)}{n^{1/2}} + \frac{2(1+\nu)\tau r}{n^{1/2}} \right) \right).
\]
By Lemma 2, we have
\[
\|\hat{h}_r - \hat{h}_s\|_{L^2(\mu_n)}^2 - \frac{\tau(r+s)}{n^{1/2}}
\]
is continuous in \( r \) for all \( s \in R \) such that \( s \geq r \). The supremum of a collection of lower semi-continuous functions is lower semi-continuous. Therefore,
\[
\sup_{s \in R, s \geq r} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(\mu_n)}^2 - \frac{\tau(r+s)}{n^{1/2}} + \frac{2(1+\nu)\tau r}{n^{1/2}} \right)
\]
is lower semi-continuous in \( r \). Hence, the infimum (7) is attained as it is the infimum of a lower semi-continuous function on a compact set. By lower semi-continuity, \( \hat{r} \) also attains the infimum and is well-defined.

We use the Goldenshluger–Lepski method to prove Theorem 9.

**Proof of Theorem 9** Since we assume \((Y)\) and \((H)\), we find that Lemma 3 holds, which implies that Lemma 7 holds. By our choice of \( t \), we have

\[
\|\hat{h}_r - \hat{h}_s\|_{L^2(P_n)}^2 \leq \frac{\tau(r + s)}{n^{1/2}} + 40I_\infty(g, r)
\]

simultaneously for all \( s, r \in R \) such that \( s \geq r \geq 0 \). Fix \( r \in R \) and suppose that \( \hat{r} \leq r \). By the definition of \( \hat{r} \) in (1) and (8), we have

\[
\|\hat{h}_r - \hat{r}\|_{L^2(P_n)}^2 = \|\hat{h}_r - \hat{h}_s\|_{L^2(P_n)}^2 - \tau(\hat{r} + r) n^{1/2} + \frac{\tau(\hat{r} + r)}{n}\n\]

\[
\leq \sup_{s \in R, s \geq \hat{r}} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(P_n)}^2 - \frac{\tau(\hat{r} + s)}{n} \right) + \frac{2\tau r}{n^{1/2}},
\]

\[
\leq \sup_{s \in R, s \geq \hat{r}} \left( \|\hat{h}_r - \hat{h}_s\|_{L^2(P_n)}^2 - \frac{\tau(r + s)}{n} \right) + \frac{2(2 + \nu)\tau r}{n^{1/2}} - \frac{2(1 + \nu)\tau^2}{n^{1/2}}.
\]

This shows

\[
\|V\hat{h}_r - V\hat{r}\|_{L^2(P_n)}^2 \leq 40I_\infty(g, r) + \frac{2(2 + \nu)\tau r}{n^{1/2}},
\]

and it follows from Lemma 4 and our choice of \( t \) that

\[
\|V\hat{h}_r - V\hat{r}\|_{L^2(P)}^2 \leq 40I_\infty(g, r) + \frac{2(2 + \nu)\tau r}{n^{1/2}} + \frac{97C\tau r}{80\sigma n^{1/2}} + \frac{C\tau^2 r}{400\|k\|_{\text{diag}}^2\sigma^2 n}.
\]

Hence,

\[
\|V\hat{h}_r - g\|_{L^2(P)}^2 \leq 2\|V\hat{h}_r - V\hat{r}\|_{L^2(P)}^2 + 2\|V\hat{r} - g\|_{L^2(P)}^2 \leq 80I_\infty(g, r) + \frac{2(2 + \nu)\tau r}{n^{1/2}} + \frac{97C\tau r}{40\sigma n^{1/2}} + \frac{C\tau^2 r}{1200\|k\|_{\text{diag}}^2\sigma^2 n} + 2\|V\hat{r} - g\|_{L^2(P)}^2.
\]

Now suppose instead that \( \hat{r} \geq r \). Since (8) holds simultaneously for all \( s, r \in R \) such that \( s \geq r \geq 0 \), we have

\[
\|\hat{r}_r - h_r\|_{L^2(P_n)}^2 \leq \frac{\tau(r + \hat{r})}{n^{1/2}} + 40I_\infty(g, r).
\]

This shows

\[
\|V\hat{r}_r - V\hat{r}\|_{L^2(P_n)}^2 \leq \frac{\tau r}{n^{1/2}} + 40I_\infty(g, r) + \frac{\tau^2}{n^{1/2}}.
\]
and it follows from Lemma \ref{lem:bound} that

\[
\|V\hat{h}_r - \hat{V}h_r\|^2_{L^2(P)} \leq \frac{\tau r}{n^{1/2}} + 40I_\infty(g, r) + \frac{\tau \hat{r}}{n^{1/2}} + \frac{97C\tau \hat{r}}{80\sigma n^{1/2}} + \frac{C\tau^2 \hat{r}}{2400\|k\|_{\text{diag}}^2\sigma^2n^{1/2}}
\]

\[
= \frac{\tau r}{n^{1/2}} + 40I_\infty(g, r) + \left(\frac{1}{2\nu} + \frac{97C}{160\sigma\nu} + \frac{C\tau}{4800\|k\|_{\text{diag}}^2\sigma^2n^{1/2}}\right)2\nu\tau r
\]

By (2), the definition of \(\hat{r}\) in (1) and (8), we have

\[
\frac{2\nu\tau \hat{r}}{n^{1/2}} \leq \sup_{s \in R, s \geq r} \left(\|\hat{h}_r - \hat{h}_s\|^2_{L^2(P, n)} - \frac{\tau(\hat{r} + s)}{n^{1/2}}\right) + \frac{2(1 + \nu)\tau \hat{r}}{n^{1/2}}
\]

\[
\leq \sup_{s \in R, s \geq r} \left(\|\hat{h}_r - \hat{h}_s\|^2_{L^2(P, n)} - \frac{\tau(r + s)}{n^{1/2}}\right) + \frac{2(1 + \nu)\tau r}{n^{1/2}}
\]

\[
\leq 40I_\infty(g, r) + \frac{2(1 + \nu)\tau r}{n^{1/2}}.
\]

Hence,

\[
\|V\hat{h}_r - g\|^2_{L^2(P)} \leq 2\|V\hat{h}_r - V\hat{h}_r\|^2_{L^2(P)} + 2\|\hat{V}h_r - g\|^2_{L^2(P)}
\]

\[
\leq \frac{2\tau r}{n^{1/2}} + 80I_\infty(g, r) + \left(\frac{1}{2\nu} + \frac{97C}{80\sigma\nu} + \frac{C\tau}{2400\|k\|_{\text{diag}}^2\sigma^2n^{1/2}}\right)\left(40I_\infty(g, r) + \frac{2(1 + \nu)\tau r}{n^{1/2}}\right)
\]

\[
+ 2\|\hat{V}h_r - g\|^2_{L^2(P)}.
\]

The result follows.

We assume (g1) to bound the distance between \(V\hat{h}_r\) and \(g\) in the \(L^2(P)\) norm and prove Theorem \ref{thm:main}.

Proof of Theorem \ref{thm:main} By Theorem \ref{thm:bound}, we have

\[
\|V\hat{h}_r - g\|^2_{L^2(P)} \leq \inf_{r \in R} \left((1 + D_4\tau n^{-1/2})(D_5\tau r n^{-1/2} + D_6I_\infty(g, r)) + 2\|V\hat{h}_r - g\|^2_{L^2(P)}\right)
\]

for some constants \(D_4, D_5, D_6 > 0\) not depending on \(\tau, r\) or \(n\). By Theorem \ref{thm:bound-tilde}, we have

\[
\|V\hat{h}_r - g\|^2_{L^2(P)} \leq \frac{(97C + 20\sigma)\tau r}{40\sigma n^{1/2}} + \frac{C\tau^2 r}{1200\|k\|_{\text{diag}}^2\sigma^2n} + 10I_\infty(g, r)
\]

\[
\leq D_7\tau n^{-1/2} + D_8\tau^2 r n^{-1} + 10I_\infty(g, r).
\]

for all \(r \in R\), for some constants \(D_7, D_8 > 0\) not depending on \(\tau, r\) or \(n\). This gives

\[
\|V\hat{h}_r - g\|^2_{L^2(P)} \leq \inf_{r \in R} \left((1 + D_4\tau n^{-1/2})(D_5\tau r n^{-1/2} + D_6I_\infty(g, r))\right)
\]
Hence, the result follows with

\[ D_1 = \frac{D_4D_5 + 2D_8}{D_5 + 2D_7}, \quad D_2 = D_5 + 2D_7, \quad D_3 = D_6 + 20. \]

We assume \((g2)\) to prove Theorem 11.

**Proof of Theorem 11** If we assume \((R1)\), then \(r = an^{(1-\beta)/(2(1+\beta))} \in R\) and

\[
\|Vh_\hat{r} - g\|^2_{L^2(P)} \leq (1 + D_3\tau n^{-1/2})(D_4\tau n^{-1/2} + D_5I_\infty(g,r))
\leq (1 + D_3\tau n^{-1/2}) \left( D_4\tau n^{-\beta/(1+\beta)} + \frac{D_5B^2/(1-\beta)}{a^2\beta/(1-\beta)} \right)
\]

for some constants \(D_3, D_4, D_5 > 0\) not depending on \(n\) or \(\tau\) by Theorem 10 and \((3)\). If we assume \((R2)\), then there is a unique \(r \in R\) such that

\[
an^{(1-\beta)/(2(1+\beta))} \leq r < an^{(1-\beta)/(2(1+\beta))} + b
\]

and

\[
\|Vh_\hat{r} - g\|^2_{L^2(P)} \leq (1 + D_3\tau n^{-1/2})(D_4\tau n^{-1/2} + D_5I_\infty(g,r))
\leq (1 + D_3\tau n^{-1/2}) \left( D_4\tau (an^{(1-\beta)/(2(1+\beta))} + b)n^{-1/2} + \frac{D_5B^2/(1-\beta)}{a^2\beta/(1-\beta)} \right)
\]

by Theorem 10 and \((3)\). In either case,

\[
\|Vh_\hat{r} - g\|^2_{L^2(P)} \leq D_1\tau n^{-\beta/(1+\beta)} + D_2\tau^2 n^{-(1+3\beta)/(2(1+\beta))}
\]

for some constants \(D_1, D_2 > 0\) not depending on \(n\) or \(\tau\).

**Appendix C: Proof of the Regression Results for a Collection of RKHSs**

We prove Lemma 13.

**Proof of Lemma 13** Let \(K\) be the \(n \times n\) symmetric matrix with \(K_{i,j} = k(X_i, X_j)\) for \(k \in K\). Then \(K\) is a continuous function of \(k\) and \(X\), hence it is an \((\mathbb{R}^{n \times n}, \mathcal{B}(\mathbb{R}^{n \times n}))\)-valued measurable matrix on \((\Omega \times K, \mathcal{F} \otimes \mathcal{B}(K))\), where \(k\) varies in \(K\). By Lemma 33, there
exist an orthogonal matrix $A$ and a diagonal matrix $D$ which are both $(\mathbb{R}^{n \times n}, \mathcal{B}(\mathbb{R}^{n \times n}))$-valued measurable matrices on $(\Omega \times \mathcal{K}, \mathcal{F} \otimes \mathcal{B}(\mathcal{K}))$ such that $K = ADA^T$. Since $K$ is non-negative definite, the diagonal entries of $D$ are non-negative, and we may assume that they are non-increasing. Let $m = \text{rk} K$, which is measurable on $(\Omega \times \mathcal{K}, \mathcal{F} \otimes \mathcal{B}(\mathcal{K}))$.

For $r > 0$, if
\[
    r^2 < \sum_{i=1}^{m} D^{-1}_{i,i} (A^T Y)^2_i,
\]
then define $\mu(r) > 0$ by
\[
    \sum_{i=1}^{m} \frac{D_{i,i}}{(D_{i,i} + n\mu(r))^2} (A^T Y)^2_i = r^2.
\]
Otherwise, let $\mu(r) = 0$. Let $a \in \mathbb{R}^n$ be defined by
\[
    (A^T a)^i = (D_{i,i} + n\mu(r))^{-1} (A^T Y)^i
\]
for $1 \leq i \leq m$ and $(A^T a)^i = 0$ for $m + 1 \leq i \leq n$, noting that $A^T$ has the inverse $A$ since it is an orthogonal matrix. By Lemma 3 of [15],
\[
    \hat{\Phi}_k(r) = \sum_{i=1}^{n} a_i k_{X_i}
\]
for $r > 0$ and $\hat{\Phi}_{k,0} = 0$ for $k \in \mathcal{K}$.

Since $\mu(r) > 0$ is strictly decreasing for
\[
    r^2 < \sum_{i=1}^{m} D^{-1}_{i,i} (A^T Y)^2_i
\]
and $\mu(r) = 0$ otherwise, we find
\[
    \{\mu(r) \leq \mu\} = \left\{ \sum_{i=1}^{m} \frac{D_{i,i}}{(D_{i,i} + n\mu(r))^2} (A^T Y)^2_i \leq r^2 \right\}
\]
for $\mu \in [0, \infty)$. Therefore, $\mu(r)$ is measurable on $(\Omega \times \mathcal{K} \times [0, \infty), \mathcal{F} \otimes \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}([0, \infty)))$, where $k$ varies in $\mathcal{K}$ and $r$ varies in $[0, \infty)$. Hence, the $a$ above with $\mu = \mu(r)$ for $r > 0$ is measurable on $(\Omega \times \mathcal{K} \times [0, \infty), \mathcal{F} \otimes \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}([0, \infty)))$. By Lemma 4.29 of [20], $\Phi_k : S \rightarrow H_k$ by $\Phi_k(x) = k_x$ is continuous for all $k \in \mathcal{K}$. Hence, $\Phi : \mathcal{K} \times S \rightarrow L^\infty$ by $\Phi(k,x) = k_x$ is continuous and $k_{X_i}$ for $1 \leq i \leq n$ are $(L^\infty, \mathcal{B}(L^\infty))$-valued measurable functions on $(\Omega \times \mathcal{K}, \mathcal{F} \otimes \mathcal{B}(\mathcal{K}))$. Together, these show that $\hat{\Phi}_{k, r}$ is an $(L^\infty, \mathcal{B}(L^\infty))$-valued measurable function on $(\Omega \times \mathcal{K} \times [0, \infty), \mathcal{F} \otimes \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}([0, \infty)))$, where $k$ varies in $\mathcal{K}$ and $r$ varies in $[0, \infty)$, recalling that $\hat{\Phi}_{k,0} = 0$. \hfill \blacksquare
Let $\psi_1(x) = \exp(|x|) - 1$ for $x \in \mathbb{R}$ and

$$\|Z\|_{\psi_1} = \inf\{a \in (0, \infty) : \mathbb{E}(\psi_1(Z/a)) \leq 1\}$$

for any random variable $Z$ on $(\Omega, \mathcal{F})$. Note that this infimum is attained by the monotone convergence theorem, and $\|Z\|_{\psi_1}$ increases as $|Z|$ increases pointwise. Let $L^{\psi_1}$ be the set of random variables $Z$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\|Z\|_{\psi_1} < \infty$. We have that $(L^{\psi_1}, \|\cdot\|_{\psi_1})$ is a Banach space known as an Orlicz space (see [17]).

**Lemma 25** Let $Z \in L^{\psi_1}$. We have

$$\mathbb{E}(|Z|) \leq (\log 2)\|Z\|_{\psi_1}.$$ 

Let $t \geq 0$. We have

$$|Z| \leq \|Z\|_{\psi_1}(\log 2 + t)$$

with probability at least $1 - e^{-t}$.

**Proof** We have $\mathbb{E}(\exp(|Z|/\|Z\|_{\psi_1})) \leq 2$. The first result follows from Jensen’s inequality. The second result follows from Chernoff bounding. \hfill \square

For $m \times n$ matrices $U$ and $V$, define $U \circ V$ to be the $m \times n$ matrix with

$$(U \circ V)_{i,j} = U_{i,j}V_{i,j}.$$ 

Recall that

$$\mathcal{L} = \{k/\|k\|_{\text{diag}} : k \in K\} \cup \{0\},$$

$$D = \sup_{f_1, f_2 \in \mathcal{L}} \|f_1 - f_2\|_{\infty} \leq 2,$$

$$J = \left(162 \int_0^{D/2} \log(2N(a, \mathcal{L}, \|\cdot\|_{\infty}))da + 1 \right)^{1/2}.$$ 

The following lemma is useful for proving Lemma 13.

**Lemma 26** Assume (K1). Let the $\varepsilon_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $(X_i, \varepsilon_i)$ are i.i.d. and $\varepsilon_i$ is $\sigma^2$-subgaussian given $X_i$. Let

$$W(f) = \frac{1}{n^2} \sum_{i,j=1}^n \varepsilon_i \varepsilon_j f(X_i, X_j)$$

for $f \in \mathcal{L}$. We have

$$\left\| \sup_{f \in \mathcal{L}} W(f) \right\|_{\psi_1} \leq \frac{4J^2\sigma^2}{n}.$$
Proof. Let $F$ be the $n \times n$ matrix with $F_{i,j} = f(X_i, X_j)$, where $F$ varies with $f \in \mathcal{L}$. Note that $F$ is an $(\mathbb{R}^{n \times n}, B(\mathbb{R}^{n \times n}))$-valued measurable matrix on $(\Omega, \mathcal{F})$. Then $W(f) = n^{-2}\varepsilon^T F \varepsilon$. Let $Z(f) = n^{-2}\varepsilon^T (F - I \circ F) \varepsilon$ for $f \in \mathcal{L}$. Note that $Z$ is continuous in $f$. We have
\[
\|Z(f_1) - Z(f_2)\|_{\psi_1} \leq 36\sigma^2 n^{-1}\|f_1 - f_2\|_{\infty}
\]
for $f_1, f_2 \in \mathcal{L}$ by Lemma 35. Let $d(f_1, f_2) = 36\sigma^2 n^{-1}\|f_1 - f_2\|_{\infty}$ for $f_1, f_2 \in \mathcal{L}$ and
\[
D_d = \sup_{f_1, f_2 \in \mathcal{L}} d(f_1, f_2).
\]
By Lemma 32 with $M = \mathcal{L}$ and $s_0 = 0$, we find
\[
\left\| \sup_{f \in \mathcal{L}} |Z(f)| \right\|_{\psi_1} \leq 18 \int_0^{D_d/2} \log(2N(a, \mathcal{L}, d)) \, da
\]
\[
= \frac{648\sigma^2}{n} \int_0^{D_d/2} \log(2N(a, \mathcal{L}, \|\cdot\|_{\infty})) \, da.
\]
Hence,
\[
\left\| \sup_{f \in \mathcal{L}} W(f) \right\|_{\psi_1} \leq n^{-2} \sup_{f \in \mathcal{L}} \varepsilon^T (I \circ F) \varepsilon \left\|_{\psi_1} + \frac{648\sigma^2}{n} \int_0^{D_d/2} \log(2N(a, \mathcal{L}, \|\cdot\|_{\infty})) \, da.
\]
We have
\[
n^{-2} \sup_{f \in \mathcal{L}} \varepsilon^T (I \circ F) \varepsilon \leq n^{-2} \varepsilon^T \varepsilon,
\]
noting that $F_{i,i} \in [0, 1]$ for $1 \leq i \leq n$ and $f \in \mathcal{L}$. Let $\delta_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are independent of each other and the $\varepsilon_i$, with $\delta_i \sim N(0, \sigma^2)$. Lemma 35 of [15] shows
\[
\mathbb{E} \left( \exp \left( n^{-2} t \sup_{f \in \mathcal{L}} \varepsilon^T (I \circ F) \varepsilon \right) \right) \leq \mathbb{E} \left( \exp \left( n^{-2} t \varepsilon^T \varepsilon \right) \right)
\]
\[
\leq \mathbb{E} \left( \exp \left( n^{-2} t \delta^T \delta \right) \right)
\]
\[
= \prod_{i=1}^n (1 - 2\sigma^2 n^{-2} t)^{-1/2}
\]
for $0 \leq 2\sigma^2 n^{-2} t < 1$ by computing the moment generating function of the $\delta_i^2$. We have that $(1 - x)^{-1/2} \leq \exp(x)$ for $x \in [0, 1/2]$, so
\[
\mathbb{E} \left( \exp \left( n^{-2} t \sup_{f \in \mathcal{L}} \varepsilon^T (I \circ F) \varepsilon \right) \right) \leq \prod_{i=1}^n \exp \left( 2\sigma^2 n^{-2} t \right) = \exp \left( 2\sigma^2 n^{-1} t \right)
\]
for $0 \leq 4\sigma^2 n^{-2} t \leq 1$. This bound is at most 2 and valid for
\[ t \leq \min\left(\frac{n^2}{4\sigma^2}, \frac{(\log 2)n}{2\sigma^2}\right). \]
Hence,
\[ \left\| n^{-2} \sup_{f \in \mathcal{L}} (I \circ F) \right\|_{\psi_1} \leq \max\left(\frac{4\sigma^2}{n^2}, \frac{2\sigma^2}{(\log 2)n}\right) \leq \frac{4\sigma^2}{n} \]
and
\[ \left\| \sup_{f \in \mathcal{L}} W(f) \right\|_{\psi_1} \leq \frac{648\sigma^2}{n} \int_0^{D/2} \log(2N(a, \mathcal{L}, \|\cdot\|_\infty)) da + \frac{4\sigma^2}{n}. \]
The result follows.

We bound the distance between $\hat{h}_{k,r}$ and $h_{k,r}$ in the $L^2(P_n)$ norm for $k \in \mathcal{K}$, $r \geq 0$ and $h_{k,r} \in rB_k$ to prove Lemma 13.

**Proof of Lemma 13** The result is trivial for $r = 0$. By Lemma 2 of [15], we have
\[ \|\hat{h}_{k,r} - h_{k,r}\|_{L^2(P_n)}^2 \leq \frac{4}{n} \sum_{i=1}^n (Y_i - g(X_i)) (\hat{h}_{k,r}(X_i) - h_{k,r}(X_i)) + 4\|h_{k,r} - g\|_{L^2(P_n)}^2 \]
for all $k \in \mathcal{K}$, all $r > 0$ and all $h_{k,r} \in rB_k$. We now bound the right-hand side. We have
\[ \|h_{k,r} - g\|_{L^2(P_n)}^2 \leq \|h_{k,r} - g\|_{\infty}^2. \]
Furthermore,
\[ \frac{1}{n} \sum_{i=1}^n (Y_i - g(X_i)) (\hat{h}_{k,r}(X_i) - h_{k,r}(X_i)) \]
\[ \leq \sup_{f \in 2rB_k} \left\| \frac{1}{n} \sum_{i=1}^n (Y_i - g(X_i)) f(X_i) \right\| \]
\[ = \sup_{f \in 2rB_k} \left\| \left( \frac{1}{n} \sum_{i=1}^n (Y_i - g(X_i)) k_{X_i} f \right) \right\| \]
\[ = 2r \left\| \frac{1}{n} \sum_{i=1}^n (Y_i - g(X_i)) k_{X_i} \right\| \]
\[ = 2r \left( \frac{1}{n^2} \sum_{i,j=1}^n (Y_i - g(X_i))(Y_j - g(X_j)) k(X_i, X_j) \right)^{1/2} \]
by the reproducing kernel property and the Cauchy–Schwarz inequality. Let
\[ Z = \sup_{k \in \mathcal{K}} \left( \frac{1}{\|k\|_{\text{diag}}^2} \sum_{i,j=1}^n (Y_i - g(X_i))(Y_j - g(X_j)) k(X_i, X_j) \right). \]
By Lemma 26 with \( \varepsilon_t = Y_t - g(X_t) \), we have \( \| Z \|_{\psi_1} \leq 4J^2\sigma^2n^{-1} \). By Lemma 25, we have \( Z \leq 4J^2\sigma^2(\log 2 + t)n^{-1} \) with probability at least \( 1 - e^{-t} \). The result follows. 

The following lemma is useful for proving Lemma 14.

**Lemma 27** Let

\[
A = \left\{ \|k\|_{\text{diag}}^{-1/4}(Vf_1 - Vf_2) : k \in \mathcal{K}, r > 0 \text{ and } f_1, f_2 \in rB_k \right\}.
\]

Then \( A \) is separable as a subset of \( L^\infty \).

**Proof** By Theorem 4.21 of [20], we have that

\[
\left\{ \sum_{i=1}^{m} a_i k_{s_i} : m \geq 1 \text{ and } a_i \in \mathbb{R}, s_i \in S \text{ for } 1 \leq i \leq m \right\}
\]

is dense in \( H_k \) for \( k \in \mathcal{K} \). Hence,

\[
\left\{ \sum_{i=1}^{m} a_i k_{s_i} : m \geq 1 \text{ and } a_i \in \mathbb{R}, s_i \in S \text{ for } 1 \leq i \leq m \text{ with } \sum_{i,j=1}^{m} a_i a_j k(s_i, s_j) \leq r^2 \right\}
\]

is dense in \( rB_k \subseteq H_k \) for \( k \in \mathcal{K} \) and \( r > 0 \). Since \( S \) is separable, it has a countable dense subset \( S_0 \). Let \( D_{k,r} \) be

\[
\left\{ \sum_{i=1}^{m} a_i k_{s_i} : m \geq 1 \text{ and } a_i \in \mathbb{Q}, s_i \in S_0 \text{ for } 1 \leq i \leq m \text{ with } \sum_{i,j=1}^{m} a_i a_j k(s_i, s_j) \leq r^2 \right\}
\]

for \( k \in \mathcal{K} \) and \( r > 0 \). Since the function \( \Phi_k : S \to H_k \) by \( \Phi_k(x) = k_x \) is continuous by Lemma 4.29 of [20], we have that \( D_{k,r} \) is dense in \( rB_k \subseteq H_k \) by suitable choices for \( a_i \in \mathbb{Q} \) for \( 1 \leq i \leq m \). Since \( k \) is bounded for all \( k \in \mathcal{K} \), as subsets of \( L^\infty \) we have that \( D_{k,r} \) is dense in \( rB_k \) and

\[
A = \text{cl} \left( \left\{ \|k\|_{\text{diag}}^{-1/4}(Vf_1 - Vf_2) : k \in \mathcal{K}, r > 0 \text{ and } f_1, f_2 \in D_{k,r} \right\} \right).
\]

Since \((\mathcal{K}, \|\cdot\|_{\infty})\) is separable, it has a countable dense subset \( \mathcal{K}_0 \). Hence,

\[
A = \text{cl} \left( \left\{ \|k\|_{\text{diag}}^{-1/4}(Vf_1 - Vf_2) : k \in \mathcal{K}_0, r \in (0, \infty) \cap \mathbb{Q} \text{ and } f_1, f_2 \in D_{k,r} \right\} \right)
\]

by suitable choices for \( r \in (0, \infty) \cap \mathbb{Q} \). The result follows. 

We bound the supremum of the difference in the \( L^2(P_n) \) norm and the \( L^2(P) \) norm over \( rB_k \) for \( k \in \mathcal{K} \) and \( r \geq 0 \) to prove Lemma 14.

**Proof of Lemma 14** The result is trivial for \( r = 0 \). Let

\[
Z = \sup_{k \in \mathcal{K}} \sup_{r > 0} \sup_{f_1, f_2 \in rB_k} \|k\|_{\text{diag}}^{-1/2} r^{-1} \|Vf_1 - Vf_2\|_{L^2(P_n)}^2 - \|Vf_1 - Vf_2\|_{L^2(P)}^2.
\]
We have that $Z$ is a random variable by Lemma 27. Furthermore, let the $\varepsilon_i$ for $1 \leq i \leq n$ be i.i.d. Rademacher random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of the $X_i$. Lemma 2.3.1 of [21] shows

$$
\mathbb{E}(Z) \leq 2 \mathbb{E}\left( \sup_{k \in K} \sup_{r > 0} \sup_{f_1, f_2 \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1/2} V f_1(X_i) - r^{-1/2} V f_2(X_i))^2 \right| \right)
$$

by symmetrisation. Since

$$|V f_1(X_i) - V f_2(X_i)| \leq 2C$$

for all $k \in K$, all $r > 0$ and all $f_1, f_2 \in r B_k$, we find

$$\left( \frac{r^{-1/2} V f_1(X_i) - r^{-1/2} V f_2(X_i))^2}{4C} \right)$$

is a contraction vanishing at 0 as a function of $r^{-1} V f_1(X_i) - r^{-1} V f_2(X_i)$ for all $1 \leq i \leq n$. By Theorem 3.2.1 of [5], we have

$$
\mathbb{E}\left( \sup_{k \in K} \sup_{r > 0} \sup_{f_1, f_2 \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1/2} V f_1(X_i) - r^{-1/2} V f_2(X_i))^2 \right| X \right)
$$

is at most

$$
2 \mathbb{E}\left( \sup_{k \in K} \sup_{r > 0} \sup_{f_1, f_2 \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1} V f_1(X_i) - r^{-1} V f_2(X_i)) \right| X \right)
$$

almost surely. Therefore,

$$
\mathbb{E}(Z) \leq 16C \mathbb{E}\left( \sup_{k \in K} \sup_{r > 0} \sup_{f_1, f_2 \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (r^{-1} V f_1(X_i) - r^{-1} V f_2(X_i)) \right| \right)
$$

$$
\leq 32C \mathbb{E}\left( \sup_{k \in K} \sup_{r > 0} \sup_{f \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} V f(X_i) \right| \right)
$$

by the triangle inequality. Again, by Theorem 3.2.1 of [5], we have

$$
\mathbb{E}(Z) \leq 64C \mathbb{E}\left( \sup_{k \in K} \sup_{r > 0} \sup_{f \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} f(X_i) \right| \right)
$$

since $V$ is a contraction vanishing at 0. We have

$$
\sup_{k \in K} \sup_{r > 0} \sup_{f \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i r^{-1} f(X_i) \right| \leq \sup_{k \in K} \sup_{r > 0} \sup_{f \in r B_k} \|k\|_{\text{diag}}^{-1/2} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i k X_i, r^{-1} f \right|_k
$$
\[
= \sup_{k \in \mathcal{K}} \|k\|_{\text{diag}}^{-1/2} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{i} k_{X_{i}} \right\|_{k} \\
= \sup_{k \in \mathcal{K}} \|k\|_{\text{diag}}^{-1/2} \left( \frac{1}{n^{2}} \sum_{i,j=1}^{n} \varepsilon_{i} \varepsilon_{j} k(\mathcal{X}_{i}, \mathcal{X}_{j}) \right)^{1/2}
\]

by the reproducing kernel property and the Cauchy–Schwarz inequality. By Lemma 26 with \(\sigma^{2} = 1\), Lemma 25 and Jensen’s inequality, we have \(E(Z) \leq 107JCn^{-1/2}\).

Let
\[
A = \left\{ \|k\|_{\text{diag}}^{1/4} r^{-1/2} V f_{1} - \|k\|_{\text{diag}}^{1/4} r^{-1/2} V f_{2} : k \in \mathcal{K}, r > 0 \text{ and } f_{1}, f_{2} \in rB_{k} \right\}.
\]

We have that \(A \subseteq L^{\infty}\) is separable by Lemma 27. Furthermore,
\[
\left\| \|k\|_{\text{diag}}^{1/4} r^{-1/2} V f_{1} - \|k\|_{\text{diag}}^{1/4} r^{-1/2} V f_{2} \right\|_{\infty} \leq \min \left( 2C\|k\|_{\text{diag}}^{1/4} r^{-1/2}, 2\|k\|_{\text{diag}}^{1/4} r^{-1/2} \right) \leq 2C^{1/2}
\]

for all \(k \in \mathcal{K}\), all \(r > 0\) and all \(f_{1}, f_{2} \in rB_{k}\). By Lemma 23, we have
\[
Z \leq E(Z) + \left( \frac{32C^{2}t}{n} + \frac{16C E(Z)t}{n} \right)^{1/2} + \frac{8Ct}{3n}
\]

with probability at least \(1 - e^{-t}\). We have \(E(Z) \leq 107JCn^{-1/2}\) from above. The result follows.

We move the bound on the distance between \(Vh_{k,r}\) and \(Vh_{k,r}\) from the \(L^{2}(P_{n})\) norm to the \(L^{2}(P)\) norm for \(k \in \mathcal{K}\), \(r \geq 0\) and \(h_{k,r} \in rB_{k}\).

**Corollary 28** Assume (Y) and (K1). Let \(t \geq 1\) and recall the definitions of \(A_{3,t}\) and \(A_{4,t}\) from Lemmas 13 and 14. On the set \(A_{3,t} \cap A_{4,t} \in \mathcal{F}\), for which \(P(A_{3,t} \cap A_{4,t}) \geq 1 - 2e^{-t}\), we have
\[
\|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^{2}(P)}^{2} \leq \frac{J\|k\|_{\text{diag}}^{1/2} (151C + 21\sigma)rt^{1/2}}{n^{1/2}} + \frac{8\|k\|_{\text{diag}}^{1/2} Crt}{3n} + 4\|h_{k,r} - g\|_{\infty}^{2}
\]

simultaneously for all \(k \in \mathcal{K}\), all \(r \geq 0\) and all \(h_{k,r} \in rB_{k}\).

**Proof** By Lemma 13, we have
\[
\|\hat{h}_{k,r} - h_{k,r}\|_{L^{2}(P_{n})}^{2} \leq \frac{21J\|k\|_{\text{diag}}^{1/2} \sigma rt^{1/2}}{n^{1/2}} + 4\|h_{k,r} - g\|_{\infty}^{2}
\]

for all \(k \in \mathcal{K}\), all \(r \geq 0\) and all \(h_{k,r} \in rB_{k}\), so
\[
\|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^{2}(P_{n})}^{2} \leq \frac{21J\|k\|_{\text{diag}}^{1/2} \sigma rt^{1/2}}{n^{1/2}} + 4\|h_{k,r} - g\|_{\infty}^{2}.
\]
Lepski for Constrained Estimators over RKHSs

35

Since \( \hat{h}_{k,r}, h_{k,r} \in rB_k \), by Lemma 14 we have

\[
\|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^2(P)}^2 - \|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^2(P_n)}^2 \leq \sup_{f_1, f_2 \in rB_k} \left|\|Vf_1 - Vf_2\|_{L^2(P)}^2 - \|Vf_1 - Vf_2\|_{L^2(P_n)}^2\right|
\]

\[
\leq 151J\|k\|_{\text{diag}}^{1/2} Cr t^{1/2} n^{1/2} + 8\|k\|_{\text{diag}}^{1/2} C r t
\]

The result follows.

We assume \((g1)\) to bound the distance between \(V\hat{h}_{k,r}\) and \(g\) in the \(L^2(P)\) norm for \(k \in K\) and \(r \geq 0\) and prove Theorem 15.

**Proof of Theorem 15** Note that \(Vg = g\). We have

\[
\|V\hat{h}_{k,r} - g\|_{L^2(P)}^2 \leq \left(\|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^2(P)} + \|Vh_{k,r} - g\|_{L^2(P)}\right)^2
\]

\[
\leq 2\|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^2(P)}^2 + 2\|Vh_{k,r} - g\|_{L^2(P)}^2
\]

for all \(k \in K\), all \(r \geq 0\) and all \(h_{k,r} \in rB_k\). By Corollary 28, we have

\[
\|V\hat{h}_{k,r} - Vh_{k,r}\|_{L^2(P)}^2 \leq \frac{J\|k\|_{\text{diag}}^{1/2} (151C + 21\sigma) r t^{1/2}}{n^{1/2}} + \frac{8\|k\|_{\text{diag}}^{1/2} C r t}{3n} + 4\|h_{k,r} - g\|_{\infty}^2
\]

Hence,

\[
\|V\hat{h}_{k,r} - g\|_{L^2(P)}^2 \leq \frac{2J\|k\|_{\text{diag}}^{1/2} (151C + 21\sigma) r t^{1/2}}{n^{1/2}} + \frac{16\|k\|_{\text{diag}}^{1/2} C r t}{3n} + 10\|h_{k,r} - g\|_{\infty}^2
\]

Taking an infimum over \(h_{k,r} \in rB_k\) proves the result.

**Appendix D: Proof of the Goldenshluger–Lepski Method for a Collection of RKHSs with Gaussian Kernels**

We bound the distance between \(\hat{h}_{\gamma,r}\) and \(\hat{h}_{\eta,s}\) in the \(L^2(P_n)\) norm for \(\gamma, \eta \in \Gamma\) with \(\eta \leq \gamma\) and \(s \geq r \geq 0\) to prove Lemma 19.

**Proof of Lemma 19** By Lemma 13, we have

\[
\|\hat{h}_{\gamma,r} - \hat{h}_{\eta,s}\|_{L^2(P_n)}^2 \leq 4\|\hat{h}_{\gamma,r} - h_{\gamma,r}\|_{L^2(P_n)}^2 + 4\|h_{\gamma,r} - g\|_{L^2(P_n)}^2
\]
Since we assume \( \text{Lemma 19} \) holds. By our choice of \( \gamma, \eta \), we have \( \hat{h}_{\gamma, \eta} \) for all \( \gamma, r \). Taking an infimum over 
\( h_{\gamma, r} \in rB_\gamma, h_{\eta, s} \in sB_\eta \) gives

\[
\| \hat{h}_{\gamma, r} - \hat{h}_{\eta, s} \|_{L^2(P_n)}^2 \leq \frac{84J\sigma(\gamma^{-d/2}r + \eta^{-d/2}s)^{1/2}}{n^{1/2}} + 20I_\infty(g, r) + 20I_\infty(g, s).
\]

The result follows from Lemma 16.

**Proof of Theorem 20** Since we assume \( (Y) \) and \((K2)\), which implies \((K1)\), we find that 
Lemma 13 holds, which implies that Lemma 19 holds. By our choice of \( t \), we have

\[
\| \hat{h}_{\gamma, r} - \hat{h}_{\eta, s} \|_{L^2(P_n)}^2 \leq \frac{\tau(\gamma^{-d/2}r + \eta^{-d/2}s)}{n^{1/2}} + 40I_\infty(g, r, r)
\]

simultaneously for all \( \gamma, \eta \in \Gamma \) and all \( r, s \in R \) such that \( \eta \leq \gamma \) and \( s \geq r \). Fix \( \gamma \in \Gamma \) and \( r \in R \). Then

\[
\| V\hat{h}_{\gamma, r} - V\hat{h}_{\gamma, r} \|_{L^2(P)}^2 \leq 2\| V\hat{h}_{\gamma, r} - V\hat{h}_{\gamma, \gamma, \gamma, r} \|_{L^2(P)}^2 + 2\| V\hat{h}_{\gamma, \gamma, \gamma, r} - V\hat{h}_{\gamma, r} \|_{L^2(P)}^2.
\]

We now bound the right-hand side. By \( \Gamma \subseteq [u, v] \), the definition of \( (\gamma, \hat{r}) \) in \((4) \) and \((9) \), we have

\[
\| \hat{h}_{\gamma, \hat{r}} - \hat{h}_{\gamma, \gamma, \gamma, \hat{r}} \|_{L^2(P_n)}^2 = \| \hat{h}_{\gamma, \hat{r}} - \hat{h}_{\gamma, \gamma, \gamma, \hat{r}} \|_{L^2(P_n)}^2 + \frac{\tau(\gamma^{-d/2} \hat{r} + (\gamma \wedge \gamma)^{-d/2}(\hat{r} \vee r))}{n^{1/2}}
\]

\[
\leq \sup_{\eta \in \Gamma, \gamma \leq \eta} \sup_{s \in R, s \geq \gamma} \left( \| \hat{h}_{\gamma, \hat{r}} - \hat{h}_{\eta, s} \|_{L^2(P_n)}^2 - \frac{\tau(\gamma^{-d/2} \hat{r} + (\gamma \wedge \gamma)^{-d/2}(\hat{r} \vee r))}{n^{1/2}} \right)
\]

\[
\leq \sup_{\eta \in \Gamma, \gamma \leq \eta} \sup_{s \in R, s \geq \gamma} \left( \| \hat{h}_{\gamma, \hat{r}} - \hat{h}_{\eta, s} \|_{L^2(P_n)}^2 - \frac{\tau(\gamma^{-d/2} \hat{r} + (\gamma \wedge \gamma)^{-d/2}(\hat{r} \vee r))}{n^{1/2}} \right)
\]

\[
\leq \frac{2(1 + \nu)\tau\gamma^{-d/2}r}{n^{1/2}} + \frac{2(1 + \nu)\tau\gamma^{-d/2}r}{n^{1/2}} + \frac{\nu d^2/2 \tau\gamma^{-d/2}r}{n^{1/2}} + \frac{2\nu d^2/2 \tau\gamma^{-d/2}r}{n^{1/2}}.
\]

\[
\leq 40I_\infty(g, r) + 4\| g - \hat{h}_{\eta, s} \|_{L^2(P_n)}^2 + 4\| h_{\eta, s} - \hat{h}_{\eta, s} \|_{L^2(P_n)}^2
\]
This shows

\[ \| V \hat{h}_{\gamma, r} - V \hat{h}_{\gamma, s, \eta, r} \|^2_{L_2(P_n)} \leq 40 I_{\infty}(g, \gamma, r) + \frac{u^{d/2}(3 + 2 \nu) \gamma^{-d/2} \bar{r}}{u^{d/2} n^{1/2}} + 2 \frac{\nu^d \tau_{\gamma}^{-d/2} \hat{r}^2}{u^{d/2} n^{1/2}}, \]

and it follows from Lemma 14, our choice of \( t \) and \( \Gamma \subseteq [u, v] \) that

\[
\| V \hat{h}_{\gamma, r} - V \hat{h}_{\gamma, s, \eta, r} \|^2_{L_2(P)} \leq 40 I_{\infty}(g, \gamma, r) + \frac{u^{d/2}(3 + 2 \nu) \tau_{\gamma}^{-d/2} \bar{r}}{u^{d/2} n^{1/2}} + 2 \frac{\nu^d \tau_{\gamma}^{-d/2} \hat{r}^2}{u^{d/2} n^{1/2}} \\
+ \left( \frac{151 C_r}{84 \sigma n^{1/2}} + \frac{C^2 r^2}{264 J^2 \sigma^2 n^{1/2}} \right) (\hat{\gamma} \wedge \gamma)^{-d/2} (\hat{r} \vee r) \\
\leq 40 I_{\infty}(g, \gamma, r) + \frac{u^{d/2}(3 + 2 \nu) \tau_{\gamma}^{-d/2} \bar{r}}{u^{d/2} n^{1/2}} + 2 \frac{\nu^d \tau_{\gamma}^{-d/2} \hat{r}^2}{u^{d/2} n^{1/2}} \\
+ \left( \frac{151 C_r}{84 \sigma n^{1/2}} + \frac{C^2 r^2}{264 J^2 \sigma^2 n^{1/2}} \right) \left( \frac{(u/168 u^{d/2} \sigma^2)}{2 \nu n^{1/2}} \right), \\
\frac{u^{d/2} (3 + 2 \nu) \tau_{\gamma}^{-d/2} \bar{r}}{u^{d/2} n^{1/2}} + 151 C_v u^{d/2} \tau_{\gamma}^{-d/2} \bar{r} + \frac{C y u^{d/2} \gamma^{-d/2} \bar{r}}{264 J^2 u^{d/2} \sigma^2 n^{1/2}} \\
\leq 40 I_{\infty}(g, \gamma, r) + \frac{2 \nu^d \tau_{\gamma}^{-d/2} \hat{r}^2}{u^{d/2} n^{1/2}}. \\
\]

By (5), the definition of \((\hat{\gamma}, \hat{r})\) in (4) and (9), we have

\[
\frac{2 \nu^d \gamma^{-d/2} \bar{r}}{n^{1/2}} \\
\leq \sup_{\eta \in \Gamma, \eta \leq \gamma, s \in R, s \geq \hat{r}} \left( \| \hat{h}_{\gamma, r} - \hat{h}_{\eta, r} \|^2_{L_2(P_n)} - \frac{\tau_{\gamma}^{-d/2} \hat{r}^2 \eta^{-d/2} s}{n^{1/2}} \right) + \frac{2(1 + \nu) \tau_{\gamma}^{-d/2} \hat{r}}{n^{1/2}} \\
\leq \sup_{\eta \in \Gamma, \eta \leq \gamma, s \in R, s \geq \hat{r}} \left( \| \hat{h}_{\gamma, r} - \hat{h}_{\eta, r} \|^2_{L_2(P_n)} - \frac{\tau_{\gamma}^{-d/2} \hat{r}^2 \eta^{-d/2} s}{n^{1/2}} \right) + \frac{2(1 + \nu) \tau_{\gamma}^{-d/2} \hat{r}}{n^{1/2}} \\
\leq 40 I_{\infty}(g, \gamma, r) + \frac{2(1 + \nu) \tau_{\gamma}^{-d/2} \hat{r}}{n^{1/2}}. \\
\]

Hence,

\[
\| V \hat{h}_{\gamma, r} - V \hat{h}_{\gamma, s, \eta, r} \|^2_{L_2(P)} \\
\leq 40 I_{\infty}(g, \gamma, r) + \frac{u^{d/2}(3 + 2 \nu) \gamma^{-d/2} \bar{r}}{u^{d/2} n^{1/2}} + \frac{151 C_v u^{d/2} \gamma^{-d/2} \bar{r}}{84 u^{d/2} \sigma n^{1/2}} + \frac{C v u^{d/2} \gamma^{-d/2} \bar{r}}{264 J^2 u^{d/2} \sigma^2 n^{1/2}} \\
+ \left( \frac{2 u^{d/2}}{u^{d/2} \sigma^2} + \frac{151 C_v u^{d/2}}{84 u^{d/2} \sigma^2 n^{1/2}} + \frac{C v u^{d/2}}{264 J^2 u^{d/2} \sigma^2 n^{1/2}} \right) \left( \frac{20 I_{\infty}(g, \gamma, r) + \tau_{\gamma}^{-d/2} \hat{r}^2 (\hat{\gamma} \wedge \gamma)^{-d/2} (\hat{r} \vee r)}{n^{1/2}} \right). 
\]

Since (9) holds simultaneously for all \( \gamma, \eta \in \Gamma \) and all \( r, s \in R \) such that \( \eta \leq \gamma \) and \( s \geq r \), we have

\[
\| \hat{h}_{\gamma, \eta, r} - \hat{h}_{\gamma, r} \|^2_{L_2(P_n)} \leq 40 I_{\infty}(g, \gamma, r) + \frac{\tau_{\gamma}^{-d/2} \hat{r}^2 (\hat{\gamma} \wedge \gamma)^{-d/2} (\hat{r} \vee r)}{n^{1/2}}. 
\]
This shows
\[ \| V \hat{h}_{\gamma, \nu, r} - V \hat{h}_{\gamma, r} \|_{L_2(P)}^2 \leq 40 I_\infty(g, \gamma, r) + \frac{\tau (\gamma^{-d/2} r + (\hat{\gamma} \wedge \gamma)^{-d/2} (\hat{\gamma} \vee r))}{n^{1/2}}, \]
and it follows from Lemma 14, our choice of \( t \) and (10) that
\[
\begin{align*}
&\| V \hat{h}_{\gamma, \nu, r} - V \hat{h}_{\gamma, r} \|_{L_2(P)}^2 \\
&\leq 40 I_\infty(g, \gamma, r) + \frac{\tau (\gamma^{-d/2} r + (\hat{\gamma} \wedge \gamma)^{-d/2} (\hat{\gamma} \vee r))}{n^{1/2}} \\
&\quad + \left( \frac{151 C \tau}{84 \sigma n^{1/2}} + \frac{C \tau^2}{2646 J^2 \sigma^2 n} \right) ( \hat{\gamma} \wedge \gamma)^{-d/2} (\hat{\gamma} \vee r) \\
&= 40 I_\infty(g, \gamma, r) + \frac{\tau \gamma^{-d/2} r}{n^{1/2}} + \left( \frac{\tau}{n^{1/2}} + \frac{151 C \tau}{84 \sigma n^{1/2}} + \frac{C \tau^2}{2646 J^2 \sigma^2 n} \right) ( \hat{\gamma} \wedge \gamma)^{-d/2} (\hat{\gamma} \vee r) \\
&\leq 40 I_\infty(g, \gamma, r) + \frac{\tau \gamma^{-d/2} r}{n^{1/2}} \\
&\quad + \left( \frac{\tau}{n^{1/2}} + \frac{151 C \tau}{84 \sigma n^{1/2}} + \frac{C \tau^2}{2646 J^2 \sigma^2 n} \right) \left( \frac{(v/\nu)^{d/2} (\hat{\gamma}^{-d/2} \hat{r} + \gamma^{-d/2} r)}{n^{1/2}} \right) \\
&\leq 40 I_\infty(g, \gamma, r) + \frac{2 \nu v^{d/2} \nu^{-d/2} r}{n^{1/2}} + \frac{151 C v^{d/2} \nu^{-d/2} \nu^{-d/2} r}{n^{1/2}} + \frac{C v^{d/2} \nu^{-d/2} \nu^{-d/2} \nu^{-d/2} r}{n^{1/2}} \\
&\quad + \left( \frac{v^{d/2}}{u^{d/2} \nu} + \frac{151 C v^{d/2}}{108 u^{d/2} \nu} + \frac{C v^{d/2}}{5292 J u^{d/2} \sigma^2 \nu^{1/2}} \right) \left( \frac{(1 + \nu)^{d/2} \nu^{-d/2} r}{n^{1/2}} \right). \\
\end{align*}
\]
Hence,
\[
\begin{align*}
&\| V \hat{h}_{\gamma, \nu, r} - V \hat{h}_{\gamma, r} \|_{L_2(P)}^2 \\
&\leq 2 \| V \hat{h}_{\gamma, \nu, r} - V \hat{h}_{\gamma, \nu, r} \|_{L_2(P)}^2 + 2 \| V \hat{h}_{\gamma, \nu, r} - V \hat{h}_{\gamma, r} \|_{L_2(P)}^2 \\
&\leq 160 I_\infty(g, \gamma, r) + \frac{2 v^{d/2} (5 + 2 \nu) \nu^{-d/2} r}{n^{1/2}} + \frac{151 C v^{d/2} \nu^{-d/2} \nu^{-d/2} r}{n^{1/2}} + \frac{2 C v^{d/2} \nu^{-d/2} \nu^{-d/2} \nu^{-d/2} r}{n^{1/2}} \\
&\quad + \left( \frac{6 v^{d/2}}{u^{d/2} \nu} + \frac{151 C v^{d/2}}{21 u^{d/2} \nu} + \frac{2 C v^{d/2}}{1323 J v^{d/2} \sigma^2 \nu^{1/2}} \right) \left( \frac{(1 + \nu)^{d/2} \nu^{-d/2} r}{n^{1/2}} \right). \\
\end{align*}
\]
We have
\[ \| V \hat{h}_{\gamma, \nu, r} - g \|_{L_2(P)}^2 \leq 2 \| V \hat{h}_{\gamma, \nu, r} - V \hat{h}_{\gamma, r} \|_{L_2(P)}^2 + 2 \| V \hat{h}_{\gamma, r} - g \|_{L_2(P)}^2 \]
and the result follows.  

\[\blacksquare\]
We assume \((g1)\) to bound the distance between \(V\hat{h}_{\gamma, r}\) and \(g\) in the \(L^2(P)\) norm and prove Theorem 21.

**Proof of Theorem 21** By Theorem 20, we have

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \leq \inf_{\gamma \in \Gamma} \inf_{r \in R} \left( (1 + D_4\tau n^{-1/2})(D_5\tau \gamma^{-d/2} r n^{-1/2} + D_6 I_\infty(g, \gamma, r)) + 2\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \right)
\]

for some constants \(D_4, D_5, D_6 > 0\) not depending on \(\tau, \gamma, r\) or \(n\). By Theorem 15, we have

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \leq \frac{(151 C + 21 \sigma) \tau \gamma^{-d/2} r}{42 \sigma n^{1/2}} + \frac{C \tau^2 \gamma^{-d/2} r}{1323 \sigma^2 n} + 10 I_\infty(g, \gamma, r)
\]

\[
\leq D_7 \tau \gamma^{-d/2} r n^{-1/2} + D_8 \tau^2 \gamma^{-d/2} r n^{-1} + 10 I_\infty(g, \gamma, r)
\]

for all \(\gamma \in \Gamma\) and all \(r \in R\), for some constants \(D_7, D_8 > 0\) not depending on \(\tau, \gamma, r\) or \(n\). This gives

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \leq \inf_{\gamma \in \Gamma} \inf_{r \in R} \left( (1 + D_4\tau n^{-1/2})(D_5\tau \gamma^{-d/2} r n^{-1/2} + D_6 I_\infty(g, \gamma, r)) + 2D_7 \tau \gamma^{-d/2} r n^{-1/2} + 2D_8 \tau^2 \gamma^{-d/2} r n^{-1} + 20 I_\infty(g, \gamma, r) \right).
\]

Hence, the result follows with

\[
D_1 = \frac{D_4 D_5 + 2D_8}{D_5 + 2D_7}, \quad D_2 = D_5 + 2D_7, \quad D_3 = D_6 + 20.
\]

We assume \((g3)\) to prove Theorem 22.

**Proof of Theorem 22** If we assume \((R1)\) and \((\Gamma1)\), then \(\alpha \in \Gamma\) and \(r = an^{(1-\beta)/(2(1+\beta))} \in R\), so

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2 \leq (1 + D_3\tau n^{-1/2})(D_4\tau \alpha^{-d/2} r n^{-1/2} + D_5 I_\infty(g, \alpha, r))
\]

\[
\leq (1 + D_3\tau n^{-1/2}) \left( D_4\tau \alpha^{-d/2} an^{-\beta/(1+\beta)} + \frac{D_5 B^2(1-\beta)}{a^2 \beta/(1-\beta) n^2/(1+\beta)} \right)
\]

for some constants \(D_3, D_4, D_5 > 0\) not depending on \(n\) or \(\tau\) by Theorem 21 and (6). If we assume \((R2)\) and \((\Gamma2)\), then there is a unique \(\gamma \in \Gamma\) such that \(\alpha/c < \gamma \leq \alpha\) and a unique \(r \in R\) such that

\[
an^{(1-\beta)/(2(1+\beta))} \leq r < an^{(1-\beta)/(2(1+\beta))} + b.
\]

By Theorem 21, Lemma 16 and (6), we have

\[
\|V\hat{h}_{\gamma, r} - g\|_{L^2(P)}^2
\]
\[ (1 + D_3 \tau n^{-1/2})(D_4 \gamma^{-d/2} \tau n^{-1/2} + D_5 I_\infty(g, \gamma, r)) \]
\[ \leq (1 + D_3 \tau n^{-1/2}) \left( D_4 \tau e^{d/2} \alpha^{d/2} (an(1-\beta)/(2(1+\beta)) + b)n^{-1/2} + \frac{D_5 B^{2/(1-\beta)}}{a^{2\beta/(1-\beta)\gamma^{\beta/(1+\beta)}}} \right). \]

In either case,
\[ \|V \hat{h}_{\gamma,r} - g\|_{L^2(P)}^2 \leq D_1 \tau n^{-\beta/(1+\beta)} + D_2 \tau^2 n^{-(1+3\beta)/(2(1+\beta))} \]
for some constants \( D_1, D_2 > 0 \) not depending on \( n \) or \( \tau \).

**Appendix E: Covering Numbers for Gaussian Kernels**

Recall that
\[ \mathcal{L} = \{ f_\gamma(x_1, x_2) = \exp \left( -\|x_1 - x_2\|^2 / \gamma^2 \right) : \gamma \in \Gamma \text{ and } x_1, x_2 \in S \} \cup \{0\}. \]

for \( \Gamma \subseteq [u, v] \) non-empty for \( v \geq u > 0 \). We prove a continuity result about the function \( F : \Gamma \to \mathcal{L} \setminus \{0\} \) by \( F(\gamma) = f_\gamma \). We also bound the covering numbers of \( \mathcal{L} \).

**Lemma 29** Assume \((K2)\). Let \( \gamma, \eta \in \Gamma \). We have
\[ \|f_\gamma - f_\eta\|_{\infty} \leq \frac{(\gamma^2 - \eta^2)^{1/2}}{\gamma \vee \eta}. \]

For \( a \in (0, 1) \), we have \( N(a, \mathcal{L}, \|\cdot\|_{\infty}) \leq \log(v/a)^{a^2} + 2 \). For \( a \geq 1 \), we have \( N(a, \mathcal{L}, \|\cdot\|_{\infty}) = 1 \).

**Proof** Let \( \gamma \geq \eta \) and \( x_1, x_2 \in S \). We have
\[ |f_\gamma(x_1, x_2) - f_\eta(x_1, x_2)| = f_\gamma(x_1, x_2) - f_\eta(x_1, x_2) \leq \exp \left( -\|x_1 - x_2\|^2 / \gamma^2 \right) \cdot \exp \left( -\|x_1 - x_2\|^2 / \eta^2 \right). \]

This is at most \( a \in (0, 1) \) whenever \( \|x_1 - x_2\|_2 > \gamma \log(1/a)^{1/2} \). Suppose \( \|x_1 - x_2\|_2 \leq \gamma \log(1/a)^{1/2} \). We have
\[ |f_\gamma(x_1, x_2) - f_\eta(x_1, x_2)| = f_\gamma(x_1, x_2) - f_\eta(x_1, x_2) \leq \exp \left( \|x_1 - x_2\|^2 / \eta^2 \right) \left( f_\gamma(x_1, x_2) - f_\eta(x_1, x_2) \right) \leq \exp \left( \log(1/a) \left( (\gamma/\eta)^2 - 1 \right) \right) - 1. \]

This is at most \( a \) whenever
\[ \gamma \leq \left( 1 + \frac{\log(1 + a)}{\log(1/a)} \right)^{1/2} \eta. \]
Since $x/(1 + x) \leq \log(1 + x) \leq x$ for $x \geq 0$, we have

\[
\left(1 + \frac{\log(1 + a)}{\log(1/a)}\right)^{1/2} = \left(1 + \frac{\log(1 + a)}{\log(1 + (1 - a)/a)}\right)^{1/2} \\
\geq \left(1 + \frac{a/(1 + a)}{(1 - a)/a}\right)^{1/2} \\
= \left(1 + \frac{a^2}{1 - a^2}\right)^{1/2}.
\]

Hence, (11) holds whenever

\[
\gamma \leq \left(1 + \frac{a^2}{1 - a^2}\right)^{1/2} \eta,
\]

or

\[
\log(\gamma) \leq \frac{1}{2} \log \left(1 + \frac{a^2}{1 - a^2}\right) + \log(\eta).
\]

The first result follows by rearranging for $a$.

Since

\[
\log \left(1 + \frac{a^2}{1 - a^2}\right) \geq \frac{a^2/(1 - a^2)}{1 + a^2/(1 - a^2)} = a^2,
\]

(11) holds whenever $\log(\gamma) \leq a^2/2 + \log(\eta)$. Hence, for any $\gamma, \eta \in \Gamma$, we find $\|f_\gamma - f_\eta\|_\infty \leq a$ whenever $|\log(\gamma) - \log(\eta)| \leq a^2/2$. Let $b \geq 1$ and $\gamma_i \in \Gamma$ for $1 \leq i \leq b$. Recall that $\Gamma \subseteq [u, v]$. If we let

\[
\log(\gamma_i) = \log(u) + a^2(2i - 1)/2
\]

and let $b$ be such that

\[
\log(v) - (\log(u) + a^2(2b - 1)/2) \leq a^2/2,
\]

then we find the $f_{\gamma_i}$ for $1 \leq i \leq b$ form an $a$ cover of $(\mathcal{L} \setminus \{0\}, \|\cdot\|_\infty)$. Rearranging the above shows that we can choose

\[
b = \left\lceil \frac{\log(v/u)}{a^2} \right\rceil
\]

and the second result follows by adding $\{0\}$ to the cover. The third result follows from the fact that $f_\gamma(x_1, x_2) \in (0, 1]$ for all $\gamma \in \Gamma$ and all $x_1, x_2 \in S$. ■

We calculate an integral of these covering numbers.

**Lemma 30** Assume (K2). We have

\[
\int_0^{1/2} \log N(a, \mathcal{L}, \|\cdot\|_\infty)da \leq \frac{\log(2 + 4 \log(v/u))}{2} + 1.
\]
Proof We have
\[ \int_0^{1/2} \log N(a, \mathcal{L}, \|\cdot\|_\infty) \, da \leq \int_0^{1/2} \log \left( 2 + \log(v/u) a^{-2} \right) \, da \]
by Lemma 29. Changing variables to \( b = 2a \) gives
\[ \frac{1}{2} \int_0^{1} \log \left( 2 + 4 \log(v/u) b^{-2} \right) \, db \leq \frac{1}{2} \int_0^{1} \log \left( 2 + 4 \log(v/u) b^{-2} \right) \, db \]
\[ = \frac{\log(2 + 4 \log(v/u))}{2} + \int_0^{1} \log(b^{-1}) \, db. \]
Changing variables to \( s = \log(b^{-1}) \) shows
\[ \int_0^{1} \log(b^{-1}) \, db = \int_0^{\infty} s \exp(-s) \, ds = 1 \]
since the last integral is the mean of an Exponential(1) random variable. \( \blacksquare \)

Appendix F: The Orlicz Space \( L^{\psi_1} \)

Recall that \( \psi_1(x) = \exp(|x|) - 1 \) for \( x \in \mathbb{R} \),
\[ \|Z\|_{\psi_1} = \inf\{a \in (0, \infty) : \mathbb{E}(\psi_1(Z/a)) \leq 1\} \]
for any random variable \( Z \) on \( (\Omega, \mathcal{F}) \) and \( L^{\psi_1} \) is the set of random variables \( Z \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that \( \|Z\|_{\psi_1} < \infty \). We have that \( (L^{\psi_1}, \|\cdot\|_{\psi_1}) \) is a Banach space known as an Orlicz space (see \[17\]). For \( t \geq 0 \), also recall that
\[ \mathbb{E}(|Z|) \leq (\log 2)\|Z\|_{\psi_1} \text{ and } |Z| \leq \|Z\|_{\psi_1} (\log 2 + t) \]
with probability at least \( 1 - e^{-t} \) by Lemma 25. We prove a maximal inequality in \( L^{\psi_1} \) using the same method as Lemma 2.3.3 of \[5\].

Lemma 31 Let \( Z_i \in L^{\psi_1} \) for \( 1 \leq i \leq I \). Then
\[ \left\| \max_{1 \leq i \leq I} |Z_i| \right\|_{\psi_1} \leq \frac{\log(2I)}{\log(5/4)} \max_{1 \leq i \leq I} \|Z_i\|_{\psi_1}. \]

Proof Let \( M = \max_{1 \leq i \leq I} \|Z_i\|_{\psi_1} \). Also, let \( C \geq 1 \) and \( a \in (0, \infty) \). By Lemma 25, we have
\[ \mathbb{E} \left( \exp \left( \max_{1 \leq i \leq I} |Z_i|/a \right) \right) = \int_0^{\infty} \mathbb{P} \left( \max_{1 \leq i \leq I} |Z_i| > a \log t \right) \, dt \]
\[
\leq C + \int_C^\infty \mathbb{P} \left( \max_{1 \leq i \leq I} |Z_i| > a \log t \right) dt \\
\leq C + \sum_{i=1}^I \int_C^\infty \mathbb{P}(|Z_i| > a \log t) dt \\
\leq C + I \int_C^\infty 2t^{-a/M} dt.
\]

Differentiating this bound with respect to \( C \) gives \( 1 - 2IC - a/M \), so the bound is minimised by \( C = (2I)^{M/a} \). For \( a > M \), the bound becomes

\[
C + 2I - \frac{M}{a-M} C^{-(a-M)/M} = (2I)^{M/a} + \frac{M}{a-M} (2I)^{1-(a-M)/a}
\]

Let

\[
a = \frac{M \log(2I)}{\log b}
\]

for \( b > 1 \). We have

\[
\mathbb{E} \left( \exp \left( \max_{1 \leq i \leq I} |Z_i|/a \right) \right) \leq 2
\]

if \( b^{2b} \leq 4 \), the hardest case being \( I = 1 \). This holds for \( b = 5/4 \) and the result follows. 

We perform chaining in \( L^{\psi_1} \) using the same method as Theorem 2.3.6 of [5]. Recall that \( N(a, M, d) \) is the minimum size of an \( a > 0 \) cover of a metric space \((M, d)\).

**Lemma 32** Let \( Z \) be a stochastic process on \((\Omega, \mathcal{F})\) indexed by a separable metric space \((M, d)\) on which \( Z \) is almost-surely continuous with \( \|Z(s) - Z(t)\|_{\psi_1} \leq d(s, t) \) for all \( s, t \in M \). Let \( D = \sup_{s,t \in M} d(s, t) \). Fix \( s_0 \in M \). Then

\[
\left\| \sup_{s \in M} |Z(s) - Z(s_0)| \right\|_{\psi_1} \leq \frac{4}{\log(5/4)} \int_0^{D/2} \log(2N(a, M, d)) da.
\]

**Proof** Since \((M, d)\) is separable, it has a countable dense subset \( M_0 \). We have

\[
\left\| \sup_{s \in M} |Z(s) - Z(s_0)| \right\|_{\psi_1} = \left\| \sup_{s \in M_0} |Z(s) - Z(s_0)| \right\|_{\psi_1}
\]

because \( Z \) is almost-surely continuous on \( M \). Since \( M_0 \) is countable, there exists a sequence of increasing finite subsets \( F_n \subseteq M \) for \( n \geq 1 \) whose union is \( M_0 \). We have

\[
\left\| \sup_{s \in M} |Z(s) - Z(s_0)| \right\|_{\psi_1} = \lim_{n \to \infty} \left\| \max_{s \in F_n} |Z(s) - Z(s_0)| \right\|_{\psi_1}
\]
by the monotone convergence theorem. Fix $n \geq 1$ and let $F = F_n$. Let $\delta_j = 2^{-j}D$ for $j \geq 0$. Since $F$ is finite, there exists a minimum $J \geq 0$ such that
\[ \{ t \in F : d(s, t) \leq \delta_j \} = \{ s \} \]
for all $s \in F$. Let $A_j$ for $0 \leq j \leq J - 1$ be a $\delta_j$ cover of $(M, d)$ of size $N(\delta_j, M, d)$, where we let $A_0 = \{ s_0 \}$. We define the chain $C : F \times \{ 0, \ldots, J \} \to M$ as follows. Let $C(s, J) = s$ for all $s \in F$. For $1 \leq j \leq J$, given $C(s, j)$, let $C(s, j - 1)$ be some closest point in $A_{j-1}$ to $C(s, j)$, depending on $s$ only through $C(s, j)$. We have
\[ Z(s) - Z(s_0) = \sum_{j=1}^{J} Z(C(s, j)) - Z(C(s, j - 1)) \]
for $s \in F$. Hence,
\[ \max_{s \in F} |Z(s) - Z(s_0)| \leq \sum_{j=1}^{J} \max_{s \in F} |Z(C(s, j)) - Z(C(s, j - 1))|. \]
By Lemma 31, we have
\[ \left\| \max_{s \in F} |Z(s) - Z(s_0)| \right\|_{\psi_1} \leq \sum_{j=1}^{J} \left\| \max_{s \in F} |Z(C(s, j)) - Z(C(s, j - 1))| \right\|_{\psi_1} \]
\[ \leq \sum_{j=1}^{J} \frac{\log(2N(\delta_j, M, d)\delta_{j-1})}{\log(5/4)} \]
\[ = \frac{4}{\log(5/4)} \sum_{j=1}^{J} (\delta_j - \delta_{j+1}) \log(2N(\delta_j, M, d)) \]
\[ \leq \frac{4}{\log(5/4)} \int_{\delta_{J+1}}^{\delta_1} \log(2N(a, M, d)) \, da \]
\[ \leq \frac{4}{\log(5/4)} \int_{0}^{D/2} \log(2N(a, M, d)) \, da. \]
The result follows.

**Appendix G: Subgaussian Random Variables and Symmetric Matrices**

The following result is Lemma 31 of [15], which is essentially Theorem 2.1 from [16].
Lemma 33 Let $M$ be a non-negative-definite matrix which is an $(\mathbb{R}^{n \times n}, \mathcal{B}(\mathbb{R}^{n \times n}))$-valued measurable matrix on $(\Omega, \mathcal{F})$. There exist an orthogonal matrix $A$ and a diagonal matrix $D$ which are both $(\mathbb{R}^{n \times n}, \mathcal{B}(\mathbb{R}^{n \times n}))$-valued measurable matrices on $(\Omega, \mathcal{F})$ such that $M = ADA^T$.

Recall that for $m \times n$ matrices $U$ and $V$, we define $U \circ V$ to be the $m \times n$ matrix with
\[
(U \circ V)_{i,j} = U_{i,j}V_{i,j}.
\]

The following lemma is a conditional version of Theorem 1.1 of [18], but with explicit values for the constants derived here.

Lemma 34 Let $\varepsilon_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are independent conditional on some sub-$\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$ and let
\[
E(\exp(t\varepsilon_i) | \mathcal{G}) \leq \exp(\sigma^2 t^2/2)
\]
almost surely for $t$ a random variable on $(\Omega, \mathcal{G})$. Let $M$ be an $n \times n$ symmetric matrix which is an $(\mathbb{R}^{n \times n}, \mathcal{B}(\mathbb{R}^{n \times n}))$-valued measurable matrix on $(\Omega, \mathcal{G})$. We have
\[
E(\exp(t^2 (M - I \circ M) \varepsilon) | \mathcal{G}) \leq \exp(16\sigma^4 \text{tr}(M^2) t^2)
\]
almost surely for $t$ a random variable on $(\Omega, \mathcal{G})$ such that $32\sigma^4 \text{tr}(M^2) t^2 \leq 1$.

Proof We follow the proof of Theorem 1.1 of [18]. Let
\[
Z = \varepsilon^T (M - I \circ M) \varepsilon = \sum_{i \neq j} M_{i,j} \varepsilon_i \varepsilon_j.
\]

Also, let $\phi_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are independent of each other, the $\varepsilon_i$ and $\mathcal{G}$, with $\phi_i \sim \text{Bernoulli}(1/2)$. Furthermore, let
\[
W = \sum_{i \neq j} \phi_i (1 - \phi_j) M_{i,j} \varepsilon_i \varepsilon_j.
\]

We have $Z = 4 E(W | \mathcal{G}, \varepsilon)$ almost surely, which gives
\[
\exp(tZ) \leq E(\exp(4tW) | \mathcal{G}, \varepsilon)
\]
almost surely for $t$ a random variable on $(\Omega, \mathcal{G})$ by Jensen’s inequality. Let
\[
S = \{1 \leq i \leq n : \phi_i = 1\}.
\]

We can write
\[
W = \sum_{i \in S, j \in S^C} M_{i,j} \varepsilon_i \varepsilon_j.
\]

Since the $\varepsilon_j$ are independent, we have
\[
E(\exp(tZ) | \mathcal{G}) \leq E(\exp(4tW) | \mathcal{G})
\]
almost surely. Let $\delta_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, F, P)$ which are independent of each other, the $\varepsilon_i$, the $\phi_i$ and $G$, with $\delta_i \sim \mathcal{N}(0, \sigma^2)$. Since the $\varepsilon_i$ are independent, we have

$$
\mathbb{E}(\exp(tZ)|G) \leq \mathbb{E} \left( \exp \left( 4t \sum_{j \in S^c} \sum_{i \in S} M_{i,j} \varepsilon_i \delta_j \right) \middle| G \right)
$$

almost surely. Let $F$ be the $n \times n$ matrix with $F_{i,j} = 1$ if $i = j \in S$ and 0 otherwise. Note that $F$ is an $(\mathbb{R}^{n \times n}, B(\mathbb{R}^{n \times n}))$-valued measurable matrix on $(\Omega, \sigma(\phi))$. Then

$$
\mathbb{E}(\exp(tZ)|G) \leq \mathbb{E} \left( \exp \left( 8t^2 \sigma^2 \delta^T (I - F) MFM (I - F) \delta \right) \middle| G \right)
$$

almost surely. By Lemma 33, there exist an orthogonal matrix $A$ and a diagonal matrix $D$ which are both $(\mathbb{R}^{n \times n}, B(\mathbb{R}^{n \times n}))$-valued measurable matrices on $(\Omega, \sigma(G, \phi))$ such that

$$(I - F) MFM (I - F) = ADA^T,$$

which is non-negative definite. Since $A^T \delta$ and $\delta$ have the same distribution given $G$, we have

$$
\mathbb{E}(\exp(tZ)|G) \leq \mathbb{E} \left( \exp \left( 8t^2 \sigma^2 \delta^T D \delta \right) \middle| G \right)
$$

$$
= \mathbb{E} \left( \prod_{i=1}^n \mathbb{E} \left( \exp \left( 8t^2 \sigma^2 D_{i,i} \delta_i^2 \right) \middle| G, \phi \right) \middle| G \right)
$$
Lepski for Constrained Estimators over RKHSs

47

\[
E \left( \prod_{i=1}^{n} \left( 1 - 16\sigma^4 D_{i,i} t^2 \right)^{-1/2} \bigg| \mathcal{G} \right)
\]

almost surely for \(16\sigma^4(\max_{1 \leq i \leq n} D_{i,i})t^2 < 1\) by computing the moment generating function of \(\delta_i^2\). We have that \((1-x)^{-1/2} \leq \exp(x)\) for \(x \in [0, 1/2]\), so

\[
E(\exp(tZ)|\mathcal{G}) \leq E \left( \prod_{i=1}^{n} \exp \left( 16\sigma^4 D_{i,i} t^2 \right) \bigg| \mathcal{G} \right) = E \left( \exp \left( 16\sigma^4 \text{tr}(D)t^2 \right) \bigg| \mathcal{G} \right)
\]

almost surely for \(32\sigma^4(\max_{1 \leq i \leq n} D_{i,i})t^2 \leq 1\). We have

\[
\text{tr}(D) = \text{tr}((I - F)MF(I - F)) = \sum_{i \in S} \sum_{j \in S \setminus i} M_{i,j}^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} M_{i,j}^2 = \text{tr}(M^2)
\]

and

\[
\max_{1 \leq i \leq n} D_{i,i} \leq \text{tr}(D) \leq \text{tr}(M^2).
\]

The result follows.

We move the bound on the conditional moment generating function of \(\varepsilon^T(M - I \circ M)\varepsilon\) to that of \(|\varepsilon^T(M - I \circ M)\varepsilon|\).

**Lemma 35** Let \(\varepsilon_i\) for \(1 \leq i \leq n\) be random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) which are independent conditional on some sub-\(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\) and let

\[
E(\exp(t\varepsilon_i)|\mathcal{G}) \leq \exp(\sigma^2 t^2/2)
\]

almost surely for \(t\) a random variable on \((\Omega, \mathcal{G})\). Let \(M\) be an \(n \times n\) symmetric matrix which is an \((\mathbb{R}^{n \times n}, \mathcal{B}(\mathbb{R}^{n \times n}))\)-valued measurable matrix on \((\Omega, \mathcal{G})\). We have

\[
E \left( \exp \left( t|\varepsilon^T(M - I \circ M)\varepsilon| \right) \bigg| \mathcal{G} \right) \leq \frac{1}{1 - 2^{7/2}\sigma^2 \text{tr}(M^2)^{1/2}t} e^{2^{7/2}\sigma^2 \text{tr}(M^2)^{1/2}t}
\]

almost surely for \(t \geq 0\), a random variable on \((\Omega, \mathcal{G})\), such that \(2^{7/2}\sigma^2 \text{tr}(M^2)^{1/2}t < 1\). Hence,

\[
E \left( \left| \varepsilon^T(M - I \circ M)\varepsilon \right| \frac{1}{2^{7/2}(\log 2)\sigma^2 \text{tr}(M^2)^{1/2} / \log(5/4)} \bigg| \mathcal{G} \right) \leq 2.
\]

**Proof** Let

\[
Z = \varepsilon^T(M - I \circ M)\varepsilon.
\]

By Lemma 34, we have

\[
E(\exp(tZ)|\mathcal{G}) \leq \exp \left( 16\sigma^4 \text{tr}(M^2)t^2 \right)
\]
almost surely for \( t \) a random variable on \((\Omega, \mathcal{G})\) such that \(32\sigma^4 \text{tr}(A^2)/t^2 \leq 1\). By Chernoff bounding, we have
\[
P(Z \geq z|\mathcal{G}) \leq \exp\left(-tz + 16\sigma^4 \text{tr}(M^2)/t^2\right)
\]
almost surely for \( z \geq 0 \), a random variable on \((\Omega, \mathcal{G})\), \( t \geq 0 \) and \(32\sigma^4 \text{tr}(A^2)/t^2 \leq 1\). Minimising over \( t \) gives
\[
P(Z \geq z|\mathcal{G}) \leq \exp\left(- \min\left(\frac{z^2}{32\sigma^4 \text{tr}(M^2)}, \frac{z}{2\sigma^2 \text{tr}(M^2)^{1/2}}\right)\right)
\]
amost surely. The first term in the minimum is attained by \( t = 2^{-5}\sigma^{-4} \text{tr}(M^2)^{-1/2}z \) when \( z < 2^{5/2}\sigma^2 \text{tr}(M^2)^{1/2} \), and the second term is attained by \( t = 2^{-5/2}\sigma^{-2} \text{tr}(M^2)^{-1/2} \) when \( z \geq 2^{5/2}\sigma^2 \text{tr}(M^2)^{1/2} \). In the second case, note that
\[
16\sigma^4 \text{tr}(M^2)/t^2 = \frac{1}{2} \leq \frac{z}{2\sigma^2 \text{tr}(M^2)^{1/2}}.
\]
The same result holds if we replace \( Z \) with \(-Z\) by replacing \( M \) with \(-M\). For \( C \geq 1 \) and \( t \geq 0 \), random variables on \((\Omega, \mathcal{G})\), we have
\[
\mathbb{E}(\exp(t|Z|)|\mathcal{G}) = \int_0^\infty \mathbb{P}(|Z| \geq (\log s)/t|\mathcal{G}) ds
\]
almost surely. By letting \( C \geq \exp(25/2\sigma^2 \text{tr}(M^2)^{1/2}) \), the bound becomes
\[
C + 2 \int_C^\infty s^{-2\sigma^2 \text{tr}(M^2)^{1/2}} ds.
\]
Let \( u = 2\sigma^2 \text{tr}(M^2)^{1/2}, \) a random variable on \((\Omega, \mathcal{G})\). Differentiating this bound with respect to \( C \) gives \(1 - 2C^{-u-1} \), so the bound is minimised by \( C = 2^u \). This satisfies the condition on \( C \) above as
\[
e^{2^{u}} \leq 2^{6} \leq 2^{10} \leq 2^{2^{7/2}}.
\]
For \( u < 1 \), the bound becomes
\[
C + 2 \frac{u}{1 - u} C^{-(1-u)/u} = 2^u + \frac{u}{1 - u} 2^{1-(1-u)} = \frac{1}{1 - u} 2^u.
\]
The first result follows. Let

\[ u = \frac{\log b}{\log 2} \]

for \( b > 1 \). We have

\[ \mathbb{E}(\exp(t|Z|)|G) \leq 2 \]

almost surely if \( b^2 2^b \leq 4 \). This holds for \( b = 5/4 \) and the second result follows. \( \blacksquare \)

References

[1] Jörn Bergh and Jörgen Löfström. *Interpolation Spaces. An Introduction*. Springer–Verlag, Berlin–New York, 1976.

[2] Lucien Birgé. An alternative point of view on Lepski’s method. In *State of the Art in Probability and Statistics*, volume 36 of IMS Lecture Notes Monogr. Ser., pages 113–133. Inst. Math. Statist., Beachwood, OH, 2001.

[3] Ernesto De Vito, Sergei Pereverzyev, and Lorenzo Rosasco. Adaptive kernel methods using the balancing principle. *Found. Comput. Math.*, 10(4):455–479, 2010.

[4] Mona Eberts and Ingo Steinwart. Optimal regression rates for SVMs using Gaussian kernels. *Electron. J. Stat.*, 7, 2013.

[5] Evarist Giné and Richard Nickl. *Mathematical Foundations of Infinite-Dimensional Statistical Models*. Cambridge University Press, New York, 2016.

[6] Alexander Goldenshluger and Oleg Lepski. Universal pointwise selection rule in multivariate function estimation. *Bernoulli*, 14(4):1150–1190, 2008.

[7] Alexander Goldenshluger and Oleg Lepski. Structural adaptation via Lp-norm oracle inequalities. *Probab. Theory Related Fields*, 143(1-2):41–71, 2009.

[8] Alexander Goldenshluger and Oleg Lepski. Bandwidth selection in kernel density estimation: Oracle inequalities and adaptive minimax optimality. *Ann. Statist.*, 39(3):1608–1632, 2011.

[9] Alexander Goldenshluger and Oleg Lepski. General selection rule from a family of linear estimators. *Theory Probab. Appl.*, 57(2):209–226, 2013.

[10] Oleg Lepski. Asymptotically minimax adaptive estimation. I. Upper bounds. Optimally adaptive estimates. *Theory Probab. Appl.*, 36(4), 1991.

[11] Oleg Lepski. On a problem of adaptive estimation in gaussian white noise. *Theory Probab. Appl.*, 35(3):454–466, 1991.

[12] Oleg Lepski. Asymptotically minimax adaptive estimation. II. Schemes without optimal adaptation. Adaptive estimators. *Theory Probab. Appl.*, 37(3):433–448, 1993.

[13] Shuai Lu, Peter Mathé, and Sergei V Pereverzev. Balancing principle in supervised learning for a general regularization scheme. *Appl. Comput. Harmon. Anal.*, 2018.

[14] Luca Oneto, Sandro Ridella, and Davide Anguita. Tikhonov, Ivanov and Morozov regularization for support vector machine learning. *Mach. Learn.*, 103(1):103–136, 2016.
[15] Stephen Page and Steffen Grünewälder. Ivanov-regularised least-squares estimators over large RKHSs and their interpolation spaces. arXiv preprint arXiv:1706.03678v3, 2017.

[16] Yamilet Quintana and José M. Rodríguez. Measurable diagonalization of positive definite matrices. J. Approx. Theory, 185:91–97, 2014.

[17] Malempati Madhusudana Rao and Zhong Dao Ren. Theory of Orlicz Spaces. Marcel Dekker, New York, 1991.

[18] Mark Rudelson and Roman Vershynin. Hanson–Wright inequality and sub-Gaussian concentration. Electron. Commun. Probab., 18, 2013.

[19] Steve Smale and Ding-Xuan Zhou. Estimating the approximation error in learning theory. Anal. Appl. (Singap.), 1(1):17–41, 2003.

[20] Ingo Steinwart and Andreas Christmann. Support Vector Machines. Springer–Verlag, New York, 2008.

[21] Aad W. van der Vaart and Jon A. Wellner. Weak Convergence and Empirical Processes. Springer–Verlag, New York, 1996.

[22] David Williams. Probability with Martingales. Cambridge University Press, Cambridge, 1991.