INVERSE BOUNDARY VALUE PROBLEM FOR SCHRÖDINGER EQUATION IN TWO DIMENSIONS

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ABSTRACT. We relax the regularity condition on potentials of Schrödinger equations in the uniqueness results in [2] and [15] for the inverse boundary value problem of determining a potential by Dirichlet-to-Neumann map.

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain with $\partial \Omega = \bigcup_{j=0}^{K} \Sigma_j$ where $\Sigma_j$ are smooth contours and $\Sigma_0$ is the external contour. Let $\nu = (\nu_1, \nu_2)$ be the unit outer normal to $\partial \Omega$ and let $\frac{\partial}{\partial \nu} = \nabla \cdot \nu$.

In this domain, we consider a Schrödinger equation with potential $q$: 

\begin{equation}
(\Delta + q)u = 0 \quad \text{in } \Omega.
\end{equation}

Consider the full Cauchy data

\begin{equation}
C_q = \left\{ \left( u, \frac{\partial u}{\partial \nu} \right) \bigg| \partial \Omega ; (\Delta + q)u = 0 \quad \text{in } \Omega, \ u \in H^1(\Omega) \right\}.
\end{equation}

By the inverse boundary value problem we mean an inverse problem of determining a potential in (0.1) by the full Cauchy data. Such a problem was formulated by Calderón [7]. In the two-dimensional case, we refer to Blasten [2], Brown and Uhlmann [4], Bukhgeim [5], Imanuvilov and Yamamoto [15], Nachman [17] and to Novikov [18] for the stability estimate. In [2] the author proved that the full Cauchy data uniquely determine the potential within piecewise $W^1_p(\Omega)$ with $p > 2$. The goal of this paper is to improve the regularity assumptions on the potential $q$ in the inverse boundary value problem and sharpen the results in [2] and [15].

As for the related problem of recovery of the two-dimensional conductivity, Astala and Päivärinta [1] established the uniqueness result for $L^\infty(\Omega)$ conductivities, which significantly improves the regularity assumption in [17]. If supports of Dirichlet data $f$ belong to a subboundary $\tilde{\Gamma}$ and observation of the Neumann data restricted on $\tilde{\Gamma}$, we call all such pairs of Dirichlet and Neumann data by partial Cauchy data. Under the assumption $q \in C^{4+\alpha}(\Omega)$, the uniqueness result for the partial Cauchy data was proved in Imanuvilov, Uhlmann and Yamamoto [12] for the case of arbitrary subboundary $\tilde{\Gamma}$. Guillarmou and Tzou [9] improved the assumption on potentials up to $C^{2+\alpha}(\Omega)$ with partial Cauchy data. As for other uniqueness results for general second order elliptic equation in the two dimensional case with partial Cauchy data on arbitrary subboundary, we refer to Imanuvilov, Uhlmann and Yamamoto [13], Imanuvilov and Yamamoto [14].

In dimensions $n \geq 3$, with the full Cauchy data, Sylvester and Uhlmann [20] established the uniqueness of recovery of conductivity in $C^2(\Omega)$, and later the regularity assumptions...
were relaxed up to $C^4(\overline{\Omega})$ in Päivärinta, Panchenko and Uhlmann [19] and up to $W^2_p(\Omega)$ with $p > 2n$ in Brown and Torres [3]. A recent result by Haberman and Tataru [10] establishes the uniqueness for Lipschitz continuous conductivities. For the case of partial Cauchy data, uniqueness theorems were proved under assumption that a potential of the Schrödinger equation belongs to $L^\infty(\Omega)$ (Bukhgeim and Uhlmann [6], Kenig, Sjöstrand and Uhlmann [16]).

Our main result is as follows

**Theorem 0.1.** Let $q_1, q_2 \in L^p(\Omega)$ with $p > 2$. If $C_{q_1} = C_{q_2}$ then $q_1 = q_2$.

Theorem 3.5 in [3] announces the same result as our main theorem, but the argument in lines 1-3 on p.27 clearly does not work and therefore the proof in [3] misses some details. The rest part of the paper is devoted to the proof of the theorem 0.1.

Throughout the paper, we use the following notations.

**Notations.** Let $i = \sqrt{-1}$, $x = (x_1, x_2), x_1, x_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\overline{z}$ denote the complex conjugate of $z \in \mathbb{C}$. We identify $x \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$ and $\xi = (\xi_1, \xi_2)$ with $\zeta = \xi_1 + i\xi_2$. We set $\partial_i = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_\tau = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. The tangential derivative on the boundary is given by $\partial_\tau = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial\Omega$. By $\mathcal{L}(X, Y)$ we denote the space of linear continuous operators from a Banach space $X$ into a Banach space $Y$. Let $B(0, \delta)$ be a ball in $\mathbb{R}^2$ of radius $\delta$ centered at $0$.

Let us introduce the operators:

$$
\partial_\tau^{-1} g = -\frac{1}{\pi} \int_\Omega \frac{g(\xi_1, \xi_2)}{\zeta - z} d\xi_1 d\xi_2, \quad \partial_z^{-1} g = -\frac{1}{\pi} \int_\Omega \frac{g(\xi_1, \xi_2)}{\xi - \overline{z}} d\xi_1 d\xi_2.
$$

Then we have

**Proposition 0.1.** A) Let $1 \leq p \leq 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_\tau^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.

B) Let $1 < p < \infty$. Then $\partial_\tau^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), W^1_p(\Omega))$.

A) is proved on p.47 in [21] and B) can be verified by using Theorem 1.32 (p.56) in [21].

Consider a holomorphic function $\Phi(x, y) = (z - (y_1 + iy_2))^2$ with $y = y_1 + iy_2$. We introduce two operators:

$$
\widehat{R}_\tau g = \frac{1}{2} e^{\tau(\Phi - \overline{\Phi})} \partial_\tau^{-1} (ge^{\tau(\Phi - \overline{\Phi})}), \quad R_\tau g = \frac{1}{2} e^{\tau(\Phi - \overline{\Phi})} \partial_z^{-1} (ge^{\tau(\Phi - \overline{\Phi})}).
$$

**Proof.** Without loss of generality, we may assume that $\Omega$ can be taken as square $(-K, K) \times (-K, K)$ for sufficiently large $K$. Indeed $\Omega \subset \subset (-K, K) \times (-K, K)$ for sufficiently large $K > 0$. We extend the potentials $q_j$, $j = 1, 2$ by zero in $(-K, K) \times (-K, K) \setminus \overline{\Omega}$. Consider the following Cauchy data:

$$
(\Delta + q)u = 0 \quad \text{in} \quad \Pi, \quad u \in H^1(\Pi)
$$

(0.3)
where $\Pi = (-K, K) \times (-K, K)$. We claim that $\tilde{C}_{q_1} = \tilde{C}_{q_2}$. Let $(u_1, \frac{\partial u_1}{\partial \nu}) \in \tilde{C}_{q_1}$ where $u_1$ is the corresponding solution to the Schrödinger equation. Consider the pair $(u_1, \frac{\partial u_1}{\partial \nu})|_{\partial \Omega}$. Since $C_{q_1} = C_{q_2}$, there exists a solution to the Schrödinger equation in the domain $\Omega$ with the potential $q_2$ such that $(u_1, \frac{\partial u_1}{\partial \nu})|_{\partial \Omega} = (u_2, \frac{\partial u_2}{\partial \nu})|_{\partial \Omega}$. Then since $q_j|_{\partial \Omega} = 0$, we extend $u_2$ in $\Pi \setminus \overline{\Omega}$ by setting $u_2 = u_1$. Then such a function $u_2$ satisfies the Schrödinger equation with the potential $q_2$ in the domain $\Pi$ and $(u_1, \frac{\partial u_1}{\partial \nu})|_{\partial \Pi} = (u_2, \frac{\partial u_2}{\partial \nu})|_{\partial \Pi}$.

We set $U_0 = 1, U_1 = \tilde{R}_r(\frac{1}{2}(\tau^{-1}q_1 - \beta_1)), U_j = \tilde{R}_r(\frac{1}{2}\tau^{-1}(q_j U_{j-1}))$ for all $j \geq 2$. The choice of constant $\beta_2$ will be made later. We construct a solution to the Schrödinger equation in the form

\[(0.4)\quad u_1 = \sum_{j=0}^{\infty} e^{rj}(-1)^j U_j.\]

First we need to show that the infinite series is convergent in $L^r(\Omega)$ with some $r > 2$.

**Proposition 0.2.** Let $u \in W^1_p(\Omega)$ for any $p > 2$. Then for any $\epsilon \in (0, 1)$ there exists a constant $C(\epsilon)$ such that

\[(0.5)\quad \|\tilde{R}_r u\|_{L^2(\Omega)} \leq C(\epsilon)\|u\|_{W^1_p(\Omega)}/r^{1-\epsilon}.\]

**Proof.** Let $\rho \in C_c^\infty(B(0, 1))$ and $\rho|_{B(0, \frac{1}{2})} = 1$. We set $\rho_r = \rho(\sqrt{r})$. Since $\tilde{R}_r u = \tilde{R}_r(\rho_r u) + \tilde{R}_r((1-\rho_r) u)$ for any positive $\epsilon$, there exists $p_0(\epsilon) > 1$ such that $\|e^{i\tau\psi} \rho_r u\|_{L^{p_0}(\Omega)} \leq C(\epsilon)\|u\|_{W^1_p(\Omega)}/r^{1-\epsilon}$. Hence applying Proposition 0.1 and the Sobolev embedding theorem, we have

\[(0.6)\quad \|\tilde{R}_r(\rho_r u)\|_{L^2(\Omega)} \leq C(\epsilon)\|u\|_{W^1_p(\Omega)}/r^{1-\epsilon}, \quad \forall \epsilon \in (0, 1).\]

Observe that

\[(0.7)\quad \int_{\Omega} \frac{1 - \rho_r}{\tau - \zeta} \frac{ue^{r(\Phi - \overline{\Phi})}}{\partial \zeta \Phi} d\zeta = \int_{\Omega} \frac{1 - \rho_r}{\tau - \zeta} \frac{ue^{r(\Phi - \overline{\Phi})}}{\partial \zeta \Phi} d\zeta = \int_{\partial \Omega} \frac{1}{2\tau \tau - \zeta} \frac{1}{\partial \zeta \Phi} d\sigma - \int_{\Omega} \frac{1}{\tau \tau - \zeta} \frac{1}{\partial \zeta \Phi} d\xi + \frac{1 - \rho_r}{\tau \partial \zeta \Phi} \frac{ue^{r(\Phi - \overline{\Phi})}}{\partial \zeta \Phi} d\xi.

Obviously, by the Sobolev embedding theorem, for any positive $\epsilon$, there exists a constant $C(\epsilon)$ such that

\[(0.8)\quad \left\|\frac{1 - \rho_r}{\tau \partial \zeta \Phi} u\right\|_{L^2(\Omega)} \leq C\|u\|_{W^1_p(\Omega)}/r^{1-\epsilon}.

For the second term on the right hand side of (0.7), we have

\[
\left| \int_{\Omega} \frac{1}{\tau \tau - \zeta} \frac{1}{\partial \zeta \Phi} d\xi \right| \leq \int_{\Omega} \left| \frac{1}{\tau \tau - \zeta} \frac{1}{\partial \zeta \Phi} \left( \frac{\partial x \partial y \partial \zeta}{\partial \zeta \Phi} \right) \right| d\xi \]

\[
+ \int_{\Omega} \left| \frac{1}{\tau \tau - \zeta} \frac{1}{\partial \zeta \Phi} \right| d\xi + \int_{\Omega} \left| \frac{1}{\tau \tau - \zeta} \frac{1}{\partial \zeta \Phi} \right| d\xi.
\]
The function \( \frac{(1-\rho)\partial u}{\partial \phi} \) is uniformly bounded in \( \tau \) in \( L^{p_1}(\Omega) \) for any \( p_1 \in (1, 2) \). Applying Proposition \( \text{[0.1]} \), we have
\[
\left\| \partial_z^{-1}\left( \frac{(1-\rho)\partial u}{\tau \partial_z \Phi} \right) \right\|_{L^2(\Omega)} \leq C\|u\|_{W^{1}_{p}(\Omega)}/\tau.
\]

On the other hand, for any \( p_2 \in (1, 2) \)
\[
\left\| \frac{\partial_z \rho(\sqrt{\tau z})u}{\partial \Phi} \right\|_{L^{p_2}(\Omega)} \leq C\|u\|_{C^0(\Omega)} \left\| \frac{1}{\partial_z \Phi} \right\|_{L^{p_2}(B(0, \frac{2}{\sqrt{\tau}}))} \leq C\tau^{(2-p_2)/2p_2}\|u\|_{W^{1}_{p}(\Omega)}.
\]

Thanks to this inequality, applying Proposition \( \text{[0.1]} \) again we have:
\[
\left\| \frac{1}{\tau^{\frac{1}{2}}} \partial_z^{-1}\left( \frac{\partial_z \rho(\sqrt{\tau z})u}{\partial \Phi} \right) \right\|_{L^2(\Omega)} \leq C\|u\|_{W^{1}_{p}(\Omega)}/\tau^{1-\epsilon}.
\]

For any \( p_3 > 1 \), we have
\[
\left\| \frac{(1-\rho)u}{(\partial \phi)^2} \right\|_{L^{p_3}(\Omega)} \leq C\|u\|_{C^0(\Omega)} \left\| \frac{1}{(\partial \phi)^2} \right\|_{L^{p_3}(\Omega \setminus B(0, \frac{1}{\sqrt{\tau}}))} \leq C(p_3)\|u\|_{W^{1}_{p}(\Omega)}\tau^{(2p_3-2)/2p_3}.
\]

Therefore
\[
\left\| \partial_z^{-1}\left( \frac{(1-\rho)u}{\tau(\partial \phi)^2} \right) \right\|_{L^2(\Omega)} \leq C\|u\|_{W^{1}_{p}(\Omega)}/\tau^{1-\epsilon}.
\]

From the classical representation of the Cauchy integral, we obtain
\[
\left\| \int_{\partial \Omega} \frac{(\nu_1-i\nu_2)(1-\rho)ue^{r(\phi-\zeta)}}{2\tau(\tau-\zeta)\partial \phi} d\sigma \right\|_{L^2(\Omega)} \leq C\|u\|_{W^{1}_{p}(\Omega)}/\tau.
\]

From \( \text{(0.6)} \)-\( \text{(0.12)} \) we have \( \text{(0.5)} \). 

We claim that the infinite series \( \text{(0.3)} \) is convergent in \( L^r(\Omega) \) for all sufficiently large \( \tau \). Let \( \rho \in (2, p) \). By Proposition \( \text{[0.2]} \), Proposition \( \text{[0.1]} \) and Hölder’s inequality yield the existence of a positive \( \delta(\rho) \) such that
\[
\|\tilde{R}_{\tau}u\|_{L^{\rho(\rho)}(\Omega)} \leq C\|u\|_{W^{1}_{p}(\Omega)}^{1/\delta}.
\]

Using \( \text{(0.13)} \) we have
\[
\|U_j\|_{L^{\rho(\rho)}(\Omega)} \leq \frac{C}{\tau^{\delta}} \left\| \partial_z^{-1}\left( q_1 U_{j-1} \right) \right\|_{W^{1}_{p}(\Omega)} \leq \frac{C}{2\tau^{\delta}} \left\| \partial_z^{-1}\left( \|L_{L^p(W^1_p)}\| q_1 U_{j-1} \right) \right\|_{L^p(\Omega)} \leq \frac{C}{2\tau^{\delta}} \left\| \partial_z^{-1}\left( \|L_{L^p(W^1_p)}\| q_1 \right) \right\|_{L^p(\Omega)} \|U_{j-1}\|_{L^{\rho(\rho)}(\Omega)} \leq \left( \frac{C}{2\tau^{\delta}} \right)^{j-1} \|U_1\|_{L^{\rho(\rho)}(\Omega)}.
\]

Therefore there exists \( \tau_0 \) such that for all \( \tau > \tau_0 \)
\[
\|U_j\|_{L^{\rho(\rho)}(\Omega)} \leq \frac{1}{2^j} \|U_1\|_{L^{\rho(\rho)}(\Omega)} \quad \forall j \geq 2.
\]

Then the convergence of the series is proved.
Since
\[
(\Delta + q_1)(U_j e^{r\Phi}) = 4\partial_x\partial_z(e^{r\Phi}\tilde{R}_\tau(\frac{1}{2}\partial_z^{-1}(q_1 U_{j-1}))) + q_1 U_j e^{r\Phi}
\]
\[
= 2\partial_x(e^{r\Phi}\frac{1}{2}\partial_z^{-1}(q_1 U_{j-1})) + q_1 U_j e^{r\Phi} = q_1 U_{j-1} e^{r\Phi} + q_1 U_j e^{r\Phi},
\]
the infinite series (0.4) represents the solution to the Schrödinger equation. By Proposition 0.2 we have
\[
\left\|\sum_{j=2}^{\infty}(-1)^j U_j\right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty.
\]
Similarly we construct the complex geometric optics solution for the Schrödinger equation with the potential \( q_2 \)
\[
v = \sum_{j=0}^{\infty} e^{-r\Phi}(-1)^j V_j, \quad V_0 = 1, V_1 = \mathcal{R}_{-\tau}(\partial_z^{-1}q_2 - \beta_2), \quad V_j = \mathcal{R}_{-\tau}(\partial_z^{-1}(q_2 V_{j-1})),
\]
where constant \( \beta_2 \) will be fixed later.

By Proposition 0.2 the following asymptotic formula holds true:
\[
\left\|\sum_{j=2}^{\infty}(-1)^j V_j\right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty.
\]
Since the Cauchy data (0.2) for potentials \( q_1 \) and \( q_2 \) are equal, there exists a solution \( u_2 \) to the Schrödinger equation with the potential \( q_2 \) such that \( u_1 = u_2 \) on \( \partial\Omega \) and \( \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \) on \( \partial\Omega \). Setting \( u = u_1 - u_2 \) we obtain
\[
(\Delta + q_2)u = (q_2 - q_1)u_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0.
\]
Denote \( q = q_1 - q_2 \). Taking the scalar product of equation (0.18) and the function \( v \) we have:
\[
\int_\Omega qu_1vdx = 0.
\]
By (0.4), (0.15), (0.17) and (0.16), we have
\[
0 = \int_\Omega qu_1vdx = \int_\Omega qe^{r(\Phi - \Phi)}(1 - U_1 - V_1)dx + o\left(\frac{1}{\tau}\right) = \int_\Omega qe^{r(\Phi - \Phi)}dx + \frac{1}{4}\int_\Omega (\partial_x^{-1}q(\partial_z^{-1}q_1 - \beta_1) + \partial_x^{-1}q(\partial_z^{-1}q_2 - \beta_2))e^{r(\Phi - \Phi)}dx + o\left(\frac{1}{\tau}\right) \text{ as } \tau \to +\infty.
\]
Let \( \{q_{j,\epsilon}\}_{\epsilon \in (0,1)} \subset C^\infty_0(\Omega) \) be a sequences of functions such that
\[
q_{j,\epsilon} \to q_j \quad \text{in } L^p(\Omega) \quad \text{as } \epsilon \to +0, \quad \forall j \in \{1, 2\}.
\]
We set \( q_{\epsilon} = q_{1,\epsilon} - q_{2,\epsilon} \),
\[
g = \frac{1}{4}(\partial_x^{-1}q(\partial_z^{-1}q_1 - \beta_1) + \partial_x^{-1}q(\partial_z^{-1}q_2 - \beta_2)), \quad g_{\epsilon} = \frac{1}{4}(\partial_x^{-1}q_{\epsilon}(\partial_z^{-1}q_{1,\epsilon} - \beta_1) + \partial_x^{-1}q_{\epsilon}(\partial_z^{-1}q_{2,\epsilon} - \beta_2)).
\]
By Proposition 0.1 we see that
\[
g_{\epsilon} \to g \quad \text{in } C^0(\Omega) \quad \text{as } \epsilon \to +0.
\]
We remind the following classical result of Hörmander [11]. Consider the oscillatory integral operator:

\[ T_\tau f(x) = \int_\Omega e^{-\tau \psi(x,y)} a(x,y) f(y) dy, \]

where \( \psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) and \( a \in C^\infty_0(\mathbb{R}^2 \times \mathbb{R}^2) \). We introduce the following matrix

\[ H_\psi(x,y) = (\partial^2_{x,y} \psi). \]

**Theorem 0.2.** ([11]) Suppose that \( \det H_\psi \neq 0 \) on \( \text{supp} a \). Then there exists a constant \( \hat{C} > 0 \) such that

\[ \| T_\tau \|_{L^2 \to L^2} \leq \frac{\hat{C}}{2\tau}. \]

We set \( \psi(x, y) = 2(x_1 - y_1)(x_2 - y_2) \). Then

\[ H_\psi(x, y) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \]

and \( \det H_\psi(x, y) = -4 \). Then the condition in Theorem 0.2 holds true.

We set \( a(x, y) = \chi(x)\chi(y) \) where \( \chi \in C^\infty(\mathbb{R}^n) \) and \( \chi|_{\Omega} \equiv 1 \). Then, by Theorem 0.2, there exists a constant \( C \) independent of \( \tau \) such that

\[
\| T_\tau \|_{L^2 \to L^2} + \| T_{-\tau} \|_{L^2 \to L^2} \leq \frac{C}{\tau}.
\]

Setting \( f = (q - q_\epsilon)\chi_\Omega \), we have

\[
\| T_\tau (q - q_\epsilon) \|_{L^2(\Omega)} + \| T_{-\tau} (g - g_\epsilon) \|_{L^2(\Omega)} \leq C(\epsilon)/\tau, \quad C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to +0.
\]

Hence \( \text{mes}\{x \in \Omega : |(T_\tau (q - q_\epsilon))(x)| \geq C(\epsilon)\} \leq 1/\tau^2 \). By the stationary phase argument (e.g., [8]), we have

\[
\int_\Omega q_\epsilon e^{\tau (\Phi(z,y) - \overline{\Phi}(z,y))} dx = \frac{2\pi q_\epsilon(y)}{\tau} + C(y, \tau), \lim_{\tau \to +\infty} \sup_{y \in \Omega} \frac{|C(y, \tau)|}{\tau} = 0 \quad \text{as} \quad \tau \to +\infty.
\]

Suppose that the function \( q \) is not identically equal to zero. Then there exists a positive number \( \alpha \) such that \( \text{mes}\{x \in \Omega : |q(x)| \geq \alpha\} = \delta > 0 \). For any \( \epsilon \in (0, \frac{1}{2}) \) there exists \( \tau_0 \) such that

\[
\text{mes}\{x \in \Omega : |(T_\tau (q - q_\epsilon))(x)| \leq C(\epsilon)\} \geq \text{mes}(\Omega) - \delta/9, \quad \forall \tau \geq \tau_0.
\]

By (0.21) and Egorov’s theorem, there exists a set \( \mathcal{O} \subset \{x \in \Omega : |q(x)| \geq \alpha\} \) such that

\[
\text{mes} \mathcal{O} = \delta/5 \quad \text{and} \quad \sup_{x \in \mathcal{O}} |(q - q_\epsilon)(x)| + \sup_{x \in \mathcal{O}} |(g - g_\epsilon)(x)| \to 0 \quad \text{as} \quad \epsilon \to +0.
\]

Then there exists \( \epsilon_0 > 0 \) such that

\[
\sup_{x \in \mathcal{O}} |(q - q_\epsilon)(x)| + \sup_{x \in \mathcal{O}} |(g - g_\epsilon)(x)| \leq \frac{\alpha}{10}, \quad \forall \epsilon > \epsilon_0.
\]

Increasing \( \epsilon_0 \) if this is necessary, we may assume that

\[
C(\epsilon) \leq \alpha/10, \quad \forall \epsilon > \epsilon_0.
\]
Now let us fix $\epsilon$ and $\tau_0$ such that $(0.26)$ and $(0.29)$ hold true. It follows from $(0.27)$ and $(0.26)$ that $\text{mes}(O \cap \{ x \in \Omega : |(T_\tau(q - q_\epsilon))(x)| \leq C(\epsilon) \}) \geq \delta/10$ for all sufficiently large $\tau$. Hence there exists a sequence $\tau_k \to +\infty$ such that we can choose a sequence $y(\tau_k) \in O \cap \{ x \in \Omega : |(T_\tau(q - q_\epsilon))(x)| \leq C(\epsilon) \}$ satisfying $y(\tau_k) \to \hat{y}$.

By the stationary phase argument and the fact that $\Omega$ is square, setting $\beta_1 = \partial_x^{-1}q_{1,\epsilon}(\hat{y})$ and $\beta_2 = \partial_x^{-1}q_{2,\epsilon}(\hat{y})$, we have

$$
(0.30) \quad \int_\Omega (\partial_x^{-1}q_1(\partial_x^{-1}q_{1,\epsilon} - \partial_x^{-1}q_{1,\epsilon}(\hat{y}))) + \partial_x^{-1}q_2(\partial_x^{-1}q_{2,\epsilon} - \partial_x^{-1}q_{2,\epsilon}(\hat{y})))e^{\tau(\Phi(z,y(\tau_j)) - \Phi(z,y(\tau_j)))} \, dx = o\left(\frac{1}{\tau_j}\right) \quad \text{as } \tau_j \to +\infty.
$$

By $(0.20)$ and $(0.30)$, we have

$$
0 = \left| \int_\Omega q e^{\tau(\Phi(z,y(\tau_j)) - \Phi(z,y(\tau_j)))} \, dx + \int_\Omega g e^{\tau(\Phi(z,y(\tau_j)) - \Phi(z,y(\tau_j)))} \, dx + o\left(\frac{1}{\tau_k}\right) \right|
$$

$$
- \left| \int_\Omega (q - g) e^{\tau(\Phi(z,y(\tau_j)) - \Phi(z,y(\tau_j)))} \, dx \right|
$$

$$
- \left| \int_\Omega (g - q) e^{\tau(\Phi(z,y(\tau_j)) - \Phi(z,y(\tau_j)))} \, dx \right|
$$

$$
\geq \frac{2\pi|q_\epsilon(y(\tau_k))|}{\tau} - |(T_\tau(q - q_\epsilon))(y(\tau_k))| - |(T_{-\tau}(g - q_\epsilon))(y(\tau_k))| - o\left(\frac{1}{\tau_k}\right) \quad \text{as } \tau_k \to +\infty.
$$

By $(0.29)$ and $(0.28)$, we obtain

$$
0 \geq \frac{2\pi|q_\epsilon(y)|}{\tau_k} - \frac{\alpha}{10\tau_k} - o\left(\frac{1}{\tau_k}\right) \geq \frac{9\alpha}{10\tau_k} - \frac{\alpha}{10\tau_k} - o\left(\frac{1}{\tau_k}\right) \quad \text{as } \tau_k \to +\infty.
$$

Then for sufficiently large $\tau_k$ we arrive at the contradiction. □

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