Nested Bethe Ansatz for RTT-Algebras of $sp(2n)$ Type

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Abstract—We study the highest weight representations of the RTT-algebras for the R-matrix of $sp(2n)$ type by the nested algebraic Bethe ansatz. For special representations these models were solved by Reshetikhin and by Martins and Ramos.

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INTRODUCTION

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school [1] provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT-algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors, depending on a set of complex variables. The first formulation of the Bethe vectors for the $gl(n)$-invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [2] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra. In our construction of Bethe vectors we used the new RTT-algebra $\tilde{A}$ which is defined in section 3 and is not the RTT-subalgebra of $sp(2n)$. This algebra has two RTT-subalgebras $gl(n)$ type and the study of the nested Bethe ansatz for this RTT-algebra is in progress. The simplest case for $n = 2$ was solved in [3], where the Bethe vectors and Bethe conditions were found explicitly.

Our construction of Bethe vectors is in any sense a generalization of Reshetikhin’s results [4]. Another approach to the nested Bethe ansatz for very special representations of the RTT-algebras of $sp(2n)$ type was given by Martin and Ramas [5].

In this note, due to the lack of space, we omit the proofs of many claims. Mostly, it is possible to prove them similarly to corresponding claims in [3].

1 The article is published in the original.

BASIC DEFINITIONS AND NOTATION

Let indices go through the set $\{\pm 1, \pm 2, \ldots, \pm n\}$. We will denote by $E^i_k$ the matrices that have all elements equal to zero with the exception of the element on the $i$th row and $k$th column that is equal to one. Then $I = \sum_k E^k_k$ is the unit matrix.

We will consider the R-matrix of $sp(2n)$ type, which has the shape

$$R(x, y) = \frac{1}{f(x, y)}(I \otimes I + g(x, y) \times \sum_{i, k = -n}^n E^i_k \otimes E^k_i - h(x, y) \sum_{i, k = -n}^n \varepsilon_i \varepsilon_k E^i_k \otimes E^{-i}_k),$$

where $\varepsilon_i = \text{sgn}(i)$ and

$$g(x, y) = \frac{1}{x - y}, \quad f(x, y) = 1 + g(x, y), \quad h(x, y) = \frac{1}{x - y + n + 1}.$$

This R-matrix is invertible and satisfies the Yang–Baxter equation.

The RTT-algebra of $sp(2n)$ type is an associative algebra $\mathcal{A}$ with unit, which is generated by $T^i_j(x)$, for which the monodromy operator $T(x) = \sum_{i, k = -n}^n E^i_k \otimes T^i_j(x)$ fulfills the RTT-equation

$$R_{12}(x, y)T(x)yT(y) = T(y)xT(x)R_{12}(x, y).$$

It can be obtained from the invertibility of the R-matrix that the operator $H(x) = \text{Tr}(T(x)) = \sum_{i = -n}^n T^i_i(x)$ fulfills the equation $[H(x), H(y)] = 0$ for any $x$ and $y$. 
We suppose that in the representation space \( W \) of the RTT-algebra \( \mathcal{A} \) there exists a vacuum vector \( \omega \in W \), for which \( W = \mathcal{A}\omega \) and

\[
T^i_k(x) \omega = 0 \quad \text{for} \quad i < k,
\]

\[
T^i_i(x) \omega = \lambda_i(x) \omega \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

In the vector space \( W = \mathcal{A}\omega \) we will look for eigenvectors of \( H(x) \).

**RTT-ALGEBRA \( \tilde{\mathcal{A}} \)**

In the RTT-algebra \( \mathcal{A} \) we have the RTT-subalgebras \( \mathcal{A}^{(+)} \) and \( \mathcal{A}^{(-)} \), respectively, which are generated by the elements \( T^i_k(x) \) and \( T^{-i}_{-k}(x) \), where \( i, k = 1, 2, \ldots, n \). First we will study the space \( W_0 = \mathcal{A}^{(+)} A^{(-)} \subset W = \mathcal{A}\omega \).

From the RTT-equation it is possible to show that \( T^i_k(x) w = 0 \) for any \( i, k = 1, 2, \ldots, n \) and any \( w \in W_0 \). If we denote

\[
T^{(+)k}(x) = \sum_{i=1}^{n} E^i_k \otimes T^{i}_{i_k}(x), \quad T^{(-)k}(x) = \sum_{i=1}^{n} E^{-i}_{-i_k} \otimes T^{-i}_{-i_k}(x),
\]

\[
\tilde{T}(x) = T^{(+)k}(x) + T^{(-)k}(x), \quad R^{(+)k}(x, y) = \frac{1}{f(x, y)} \times (I_+ \otimes I_+ + g(x, y) \sum_{i, k=1}^{n} E^i_k \otimes E^{-i}_{-k}),
\]

\[
R^{(-)k}(x, y) = \frac{1}{f(x, y)} (I_- \otimes I_+ + g(x, y) \sum_{i, k=1}^{n} E^{-i}_{-k} \otimes E^i_k),
\]

\[
R^{-1}(x, y) = I_- \otimes I_+ h(x, y) \sum_{i, k=1}^{n} E^i_k \otimes E^{-i}_{-k},
\]

the R-matrix

\[
\check{R}(x, y) = R^{(+)k}(x, y) + R^{(-)k}(x, y)
\]

is invertible, fulfills the Yang–Baxter equation and on the space \( W_0 \) the RTT-equations

\[
\check{R}_{1,2}(x, y) T^i_1(x) T^j_2(y) = T^j_1(y) T^i_2(x) \check{R}_{1,2}(x, y)
\]

are valid. So the R-matrix \( \check{R}(x, y) \) defines the RTT-algebra which we denote as \( \tilde{\mathcal{A}} \).

For the generators of the RTT-algebra \( \tilde{\mathcal{A}} \) we will write \( \tilde{T}^i_k(x) \) and \( \tilde{T}^{-i}_{-k}(x) \). Especially, we use this designation for the restriction of the operators \( T^i_k(x) \) and \( T^{-i}_{-k}(x) \) on the vector space \( W_0 \).

From the RTT-equation for the algebra \( \tilde{\mathcal{A}} \) it can be shown that the operators

\[
\tilde{H}^{(+)}(x) = \text{Tr}(T^{(+)}(x) x),
\]

\[
\tilde{H}^{(-)}(x) = \text{Tr}(T^{(-)}(x) x),
\]

for any \( x \) and \( y \) commute with each other.

**EIGENVECTORS AND EIGENVALUES OF THE OPERATOR \( H(x) \)**

We denote by \( \tilde{a} \) an ordered set of mutually different numbers. Let \( e_i, i = 1, 2, \ldots, n \), be the basis of the vector space \( \mathcal{V}_+ \sim \mathbb{C}^n \) and \( f^i \) its dual basis in \( \mathcal{V}_+^* \). Similarly, \( e_i \) and \( f^i \) are the basis and dual basis of the vector space \( \mathcal{V}_- \sim \mathbb{C}^n \) and \( \mathcal{V}_-^* \).

We denote \( B_i(u) = \sum_{i, k=1}^{n} e_i \otimes f^{-i}_{-k} \otimes T^{-i}_{-k}(u) \in \mathcal{V}_+ \otimes \mathcal{V}_+^* \otimes \mathcal{A} \) and define

\[
B_{1, \ldots, n}(\tilde{a}) = B_i(u_i) B_i(u_2) \ldots B_i(u_N) \in \mathcal{V}_+ \otimes \mathcal{V}_+^* \otimes \mathcal{A},
\]

\[
\mathcal{V}_+ = \mathcal{V}_+ \otimes \mathcal{V}_2 \otimes \ldots \otimes \mathcal{V}_N,
\]

\[
\mathcal{V}_+^* = \mathcal{V}_1^* \otimes \mathcal{V}_2^* \otimes \ldots \otimes \mathcal{V}_N^*.
\]

The general shape of the eigenvector will be found in the form \( B_{1, \ldots, n}(\tilde{a}), \Phi \), where

\[
\Phi = f^{(1)} \otimes e_{-1} \otimes \Phi_{y}^* \in \mathcal{V}_+^* \otimes \mathcal{V}_- \otimes W_0 = \widetilde{W}_0,
\]

For ordered vector \( \vec{a} \) we will denote by \( \vec{\eta} \) the set of its elements and define

\[
F(x, \vec{a}) = \prod_{i=1}^{N} f(x, u_i), \quad F(\vec{\eta}, x) = \prod_{i=1}^{N} f(u_i, x),
\]

\[
\hat{R}^{(+)}_{0,1^*}(x, u) = \frac{1}{f(u, x)} (I_0 \otimes I^*_1 + g(u, x) \sum_{i, r=1}^{n} E^*_r \otimes E^i_r),
\]

\[
\hat{R}^{(-)}_{0,1^*}(x, u) = I_0 \otimes I^*_1 - k(u, x) \sum_{i, r=1}^{n} E^*_r \otimes E^i_r,
\]

\[
(R^*)^{(+)}(x, y) = \frac{1}{f(x, y)} (I^*_1 \otimes I^*_1 + g(x, y) \sum_{i, k=1}^{n} F^*_k \otimes F^i_k),
\]
Theorem. Let $\Phi \in \mathcal{W}_0$ be the eigenvectors of the operators

$$\mathcal{H}^{(+)}(x; \vec{u}) = \sum_{i=1}^{n} \hat{T}_i(x; \vec{u}), \quad \mathcal{H}^{(-)}(x; \vec{u}) = \sum_{i=1}^{n} \hat{T}^{-}_i(x; \vec{u})$$

with the eigenvalues $E^{(+)}(x; \vec{u})$ and $E^{(-)}(x; \vec{u})$. If for any $u_k \in \vec{u}$

$$F(\vec{u}, u_k) E^{(+)}(u_k; \vec{u}) = F(u_k, \vec{u}) E^{(-)}(u_k; \vec{u})$$

is valid, then $\{B_{0, \ldots, N}(\vec{u}), \Phi\}$ are the eigenvectors of the operator $H(x)$ with the eigenvalue

$$E(x; \vec{u}) = F(\vec{u}, x) E^{(+)}(x; \vec{u}) + F(x, \vec{u}) E^{(-)}(x; \vec{u}).$$

This theorem translates the original problem into the task of finding common eigenvectors $\mathcal{H}^{(+)}(x; \vec{u})$ on the space $\mathcal{W}_0$. It can be shown that the operators $\hat{T}_i(x; \vec{u})$ and $\hat{T}^{-}_i(x; \vec{u})$ reduced to this space are the generators of the RTT-algebra $\tilde{A}$ and that the vector

$$\Omega = f^1 \otimes \cdots \otimes f^i \otimes e_{-1} \otimes \cdots \otimes e_{-n} \otimes \omega \in \mathcal{W}_0$$

is the vacuum vector with the weights

$$\mu_i(x; \vec{u}) = \lambda_i(x) F(\vec{u}, x - 1), \quad \mu_{-i}(x; \vec{u}) = \lambda_{-i}(x) F(x + 1, \vec{u}),$$

$$i = 2, \ldots, n.$$

SUMMARY

In the presented paper, we deal with the construction of the Bethe vectors for the RTT-algebras of $sp(2n)$ type. We have shown that for solution of the problem it is possible to use the RTT-algebra $\tilde{A} = \tilde{A}$. This is the main result of the paper.

The next step of the construction is to find of Bethe vectors for this RTT-algebra. It can be done by means of the nested Bethe ansatz construction that uses the RTT-algebra $\tilde{A} = \tilde{A}_{n-1}$. We have already some results that we are going to publish in the next paper.

Let us mention that in the presented paper we assume the existence of the highest weight representation of the RTT-algebras of $sp(2n)$ type. Such representation must be given to obtain concrete physical models. At present, we explicitly know only two non-trivial representations of the RTT-algebra of $sp(2n)$ type. One of these representations is used in the paper [4]. Namely, it is representation for which the representation of the RTT-algebra $\tilde{A}_n$ is one-dimensional. The second representation constructed using the $R$-matrix is used in [5]. This paper used, unlike our construction, for the nested Bethe ansatz the RTT-algebra of $sp(2n - 2)$ type. We are convinced that this use of the RTT-algebra of $sp(2n - 2)$ type in the nested Bethe ansatz is only possible for a simple representation of the algebra.
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