STOCHASTIC SPATIOTEMPORAL DIFFUSIVE PREDATOR-PREY SYSTEMS

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Abstract. In this paper, a spatiotemporal diffusive predator-prey system with Holling type-III is considered. By using a Lyapunov-like function, it is proved that the unique local solution of the system must be a global one if the interaction intensity is small enough. A comparison theorem is used to show that the system can be extinction or stability in mean square under some additional conditions. Finally, an unique invariant measure for the system is obtained.

1. Introduction. In the ecosystem, none of the species survives alone. The relation of two species can be described by competition, predator-prey, auspiciousness and so on. Among them, the predator-prey interaction is a significant one, which was introduced by Lotka and Volterra([10]) and has been developing rapidly in the last decades([4, 21, 5, 20, 8]). In this paper, we consider a stochastic homogeneous spatiotemporal diffusive predator-prey system with functional response. In 2016, J.F. Wang([20]) studied the dynamics of a deterministic homogeneous diffusive predator-prey system with Holling type-III, it takes the form:

\[
\begin{align*}
  u_t - d_1 \Delta u &= u(1 - u) - \frac{mu^2 v}{a^2 + u^2}, x \in D, t > 0, \\
  v_t - d_2 \Delta v &= -dv + \frac{mu^2 v}{a^2 + u^2}, x \in D, t > 0, \\
  u(x, 0) &= u_0(x), v(x, 0) = v_0(x), x \in D, \\
  \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, x \in \partial D, t > 0,
\end{align*}
\]

where \( D \subset \mathbb{R}^n (n \geq 1) \) is a bounded and smooth connected domain, \( u(x, t) \) and \( v(x, t) \) denote the densities of the predator and prey at \( t, x \in D \), respectively. \( d_1 \) and \( d_2 \) present the diffusion rates of \( u, v \), respectively. \( d \) is the mortality rate of the predator; \( m \) denotes the interaction intensity, and \( d, m, a, d_1, d_2 \) are positive constants. The initial density \( u_0, v_0 \in C(D) \) are non-negative functions. \( n \) is the

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unit outer normal, we suppose the system is a closed one so that there is no flux boundary condition.

But as we all know, in reality, random disturbance of various forms is everywhere. The disturbance which is described by White noise often plays an important role in the behaviour of the solution, even in the existence of the solutions. Almost all the statistical data showed that biological process has marked random fluctuation([6, 12, 14, 19, 22, 23]). But the deterministic systems always assume that the parameters of the models have nothing to do with the environmental disturbance. Hence the description and prediction of the deterministic systems are always less than satisfactory. Motivated by the issues, a great number of scholars introduced stochastic mathematical models instead of deterministic ones to exposite the population dynamics affected by environmental fluctuations in an ecosystem([6, 12, 14, 19, 22, 23]). Even so, as far as we know, studies about the predator-prey system with Holling type-III and the environment disturbances in an ecosystem([9, 10, 18, 19, 22, 23, 35]) are relatively rare. We introduce the space-time stochastic perturbations by White noises into the equation (1) directly, and obtain the following corresponding stochastic reaction-diffusion model:

\[
\begin{align*}
    u_t - d_1 \Delta u &= u(1-u) - \frac{mu^2v}{a^2 + u^2} + \sigma_1 u W_1(x, t), x \in D, t > 0, \\
    v_t - d_2 \Delta v &= -dv + \frac{mu^2v}{a^2 + u^2} + \sigma_2 v W_2(x, t), x \in D, t > 0, \\
    u(x, 0) &= u_0(x), v(x, 0) = v_0(x), x \in D, \\
    \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, x \in \partial D, t > 0,
\end{align*}
\]

where \(W_1(x, t), W_2(x, t)\) are independent spatially dependent Wiener fields, \(W(x, t)\) means the formal time derivative \(\frac{d}{dt} W(x, t)\) of \(W(x, t)\), and \(\sigma_1^2 > 0, \sigma_2^2 > 0\) represent the intensities of the White fields. Suppose the initial values \(u_0, v_0\) satisfy \(0 < u_0 \leq u \), \(0 < v_0 \leq v\), here \(u\) and \(v\) are both positive constants.

In the corresponding deterministic system (1), the existence and boundedness of the non-negative global solution is shown in [5] by a general result. Meanwhile, there are three nonnegative constant equilibrium solutions: \((0, 0), (1, 0)\) and \((\kappa, \kappa)\) of system (1), where

\[
\kappa^2 = \frac{a^2 d}{m - d}, \quad \kappa = \frac{(1 - \kappa)(a^2 + \kappa^2)}{m \kappa},
\]

and the positive equilibrium solution \((\kappa, \kappa)\) exists if and only if \(0 < \kappa < 1\), i.e. \(m > \max\{d, (a^2 - 1)d\}\). [20] gives a stability result regarding the equilibrium \((\kappa, \kappa)\) and \((1, 0)\) is under some conditions respectively. But for the stochastic case, because we formulate system (2) by stochastic perturbations \(\sigma_1 u W_1\) and \(\sigma_2 v W_2\) directly, there is no positive equilibrium point as a solution. Hence, the solution of (2) will not tend to a point. Things are quite different from the deterministic model, even the existence of the solution. So, we begin with discussing the existence of the global solution. Comparing with the condition for the existence of the positive equilibrium solution of (1), we find if the interaction intensity \(m\) satisfies \(m < \min\{2a^2, d - \frac{a^2}{2}\}\) and \(d > \frac{a^2}{2}\), (2) has a unique global positive mild solution. The difficulty here is if only satisfy local Lipschitz condition so the solution is a local one. Consequently, we apply a Lyapunov-like function and a stop time to overcome the troubles.

It is generally known that permanent is the most important property of a system, it means every species in this system can survive with other species together
continuously. And closely related to that is extinction. A lot of literatures talked about permanent, extinction and stability of a ecosystem, like [6, 2, 1, 15, 7]. In Section 4, by Comparison theorem, we prove that if the strength of the White noise is large, $\sigma_1 > 2d_1\lambda_2 - 2m$, the stochastic system will not be permanent. And if $\sigma_1 < 2d_1\lambda_2 - 2m$ and $\sigma_2 < 2d_2\lambda_2 - 2m$, where $-\lambda_2$ be the principle eigenvalue of $\Delta$, the system will be stability in mean square.

Finally, we prove that the global mild solution is a Markov process. Under stronger conditions $\sigma_1 < 2d_1\lambda_2 - 2 - m^2$ and $\sigma_2 < 2d_2\lambda_2 - 4m - \frac{4m+4}{2}$, of course under the conditions for the existence too, the stable system has a unique invariant measure which is a more delicate description of the $L^2$ exponential stability.

Throughout this paper, we assume $(\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P})$ is a complete probability space, $\{F_t\}_{t \geq 0}$ is a right continuous filtration and $F_0$ contains all $\mathbb{P}$-null sets.

2. Preliminary. We consider a nonlinear diffusion-reaction equation problem with White noise:

$$
\begin{align*}
\frac{\partial U}{\partial t} &= AU + f(U, x, t) + g(x, t) + \gamma(U, x, t)\mathbf{W}(x, t)(\omega), t \in [0, T], \\
\frac{\partial U}{\partial n} &= 0, \\
U(x, 0) &= U_0(x), x \in D,
\end{align*}
$$

(3)

where $D \subset \mathbb{R}^n(n \geq 1)$ is a domain, $H \triangleq L^2(D, \mathbb{R}^m)$ ($m \geq 1$), the operator

$$
A = \begin{pmatrix} 
\frac{d_1\Delta - \alpha}{m} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & \cdots & \frac{d_m\Delta - \alpha}{m}
\end{pmatrix},
$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $d_i > 0 (i = 1, \ldots, m)$ and $\alpha > 0$. We recall some basic well known results (see, e.g. [3]).

Let $\{\mathbf{W}(x, t)\}_{0 \leq t \leq T}$ be a $\mathbb{R}$–Wiener process in $H = L^2(D, \mathbb{R}^m)$, satisfying

1. $E[\mathbf{W}(x, t)] = 0$,
2. $E[\mathbf{W}(x, t)\mathbf{W}(y, s)] = (t \wedge s)\rho(x, y)$, where the covariance function $\rho(\cdot, \cdot) : D \times D \rightarrow \mathbb{R}^1$ satisfies $\int_D \rho(x, x)dx < \infty$.

For $\forall \varphi \in H = L^2(D, \mathbb{R}^m)$, define a integral operator by

$$
(Q\varphi)(x) = \int_D \rho(x, y)\varphi(y)dy,
$$

then $TrQ = \int_D \rho(x, x)dx < \infty$.

We recall the eigenvalues and eigenvectors of the operator $-A$ in $H = L^2(D, \mathbb{R}^m)$. By the compact operator theory, the corresponding eigenvalues $\{\lambda_k\}$ and the eigenvectors $\{e_k\}$ satisfy

$$
\begin{align*}
-Ae_k &= \lambda_ke_k, k = 1, 2, \cdots, \\
\frac{\partial e_k}{\partial n} &= 0,
\end{align*}
$$

here

$$
\alpha \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \lambda_k \rightarrow \infty,
$$

and $\{e_k\} \subset H$ are the complete orthogonal basis functions of $H$. 

Define the Green function of linearized equation corresponding for the operator $A$ is

$$\Gamma(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} e_k(x) e_k(y), x, y \in D, t \in [0, T],$$

hence the equivalent integral equation of the equation (3) is

$$U(x, t) = \int_D \Gamma(x, y; t) U_0(y) dy + \int_0^t \int_D \Gamma(x, y, t - s) g(y, s) ds dy$$

$$+ \int_0^t \int_D \Gamma(x, y, t - s) f(U(y, s), y, s) ds dy$$

$$+ \int_0^t \int_D \Gamma(x, y, t - s) \gamma(U(y, s), y, s) W(y, s) ds dy.$$

(4) can be rewritten as

$$U_t = \Gamma_t U_0 + \int_0^t \Gamma_{t-s} g_s ds + \int_0^t \Gamma_{t-s} F_s(U) ds + \int_0^t \Gamma_{t-s} Y_s(U) dW_s,$$

(5) here the adjoint Green operation semigroup $\{\Gamma_t\}_{t \geq 0}$ is

$$(\Gamma_t U_0)(x) = \int_D \Gamma(x, y; t) U_0(y) dy,$$

$$U_t = U(\cdot, t), F_t(u) = f(U(\cdot, t), \cdot, y), g_t = g(\cdot, t), Y_t(U) = \gamma(U(\cdot, y), \cdot, t) \text{ and }$$

$$dW_t = W(\cdot, dt).$$

**Definition 2.1 ([3]).** $U \in L^2(\Omega \times [0, T]; H)$ is a mild solution of initial-boundary equation (3) iff

1. $\{U(\cdot, t)\}_{0 \leq t \leq T}$ is a $\mathcal{F}_t$-adapted stochastic process in $H = L^2(D, \mathbb{R}^m)$, for almost everywhere $(\omega, t) \in \Omega \times [0, T],$

$$E \int_0^T \{\|F_t(U)\|^2 + \langle QT_t(U), Y_t(U) \rangle \} dt$$

$$= E \int_0^T \int_D \{\|f(U(x, t), x, t)\|^2 + \rho(x, x) \gamma^2(U(x, t), x, t)\} dx dt$$

$$< \infty;$$

2. $\{U(\cdot, t)\}_{0 \leq t \leq T}$ satisfies the integral equation (5).

To prove there is a local solution of (3), we introduce the following three conditions:

(A1) \(\forall n > 0\), there exists a $C_n > 0$, such that if $U \in H = L^2(D, \mathbb{R}^m)$ satisfies $\|U\| \leq n$, for all $t \in [0, T]$, then

$$\|f(U, \cdot, t)\|^2 + \|\gamma(U, \cdot, t)\|^2 \leq C_n, \text{ a.s.}$$

(A2) \(\forall n > 0\), there exists a $K_n > 0$, such that $\forall U, \overline{U} \in H = L^2(D, \mathbb{R}^m)$, if $\|U\| \vee \|\overline{U}\| \leq n$, for any $t \in [0, T]$, then

$$\|f(U, \cdot, t) - f(\overline{U}, \cdot, t)\|^2 + \|\gamma(U, \cdot, t) - \gamma(\overline{U}, \cdot, t)\|^2 \leq K_n \|U - \overline{U}\|^2, \text{ a.s.}$$

(A3) $\{g(\cdot, t)\}_{0 \leq t \leq T}$ is a $H$-valued process which is predictable, and

$$E(\int_0^T \|g_t\|^2 dt)^p < \infty, p \geq 1.$$
Lemma 2.2. If the conditions (A1), (A2) and (A3) are satisfied, assume $U_0$ is a $\mathcal{F}_0$-measurable random field with $E[\|U_0\|^2] < \infty$, then the nonlinear stochastic reaction-diffusion initial-boundary problem (3) has a unique local (mild) solution $U(x,t)$, i.e., there exists an explosion time $\tau_e > 0$, such that $U(\cdot,t), 0 \leq t < \tau_e$ is a continuously adapted process in $H$ which satisfies (5).

Proof. The proof is similar to that of Theorem 6.5 in [3].

Since the Itô formula is no longer valid for a mild solution, we should introduce a strong solution approximating system to which Itô's formula can be applied.

Definition 2.3 ([3]). For $t \in [0,T]$, a strong solution of (3) is a stochastic process $U_t$, $t \in [0,T]$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, satisfying

1. $U_t \in D(A), 0 \leq t \leq T$ a.s. and $U_t$ is adapted to $\mathcal{F}_t, t \in [0,T]$;
2. $U_t$ is continuous in $t \in [0,T]$, a.s., for $\forall 0 \leq t \leq T, P(\int_0^t \|U_s\|^2 ds < \infty) = 1$,

$$U_t = U_0 + \int_0^t AU + (F_s(U_s) + g_s)ds + \int_0^t \gamma_s(U_s)dW_s,$$

holds for any $U_0 \in D(A)$ a.s., where $D(A)$ denotes the domain of $A$.

An approximating system of (3) is

$$(6) \begin{cases} dU = AU dt + R(l)f(U, x, t) + g(x,t)dt + R(l)\gamma(U, x, t)dW(x, t)(\omega), t \in [0,T], \\ \partial U/\partial n |_{\partial D} = 0, \\ U(x,0) = R(l)h(x), x \in D, \end{cases}$$

where $l \in \rho(A), \rho(A)$ is the resolvent set of $A$ and $R(l) = lR(l,A), R(l,A)$ is the resolvent of $A$.

Lemma 2.4. Assume for any $U_0 \in H$ is an given stochastic variable, $E[\|U_0\|^p] < \infty (p > 2$ is an integer). If $f(\cdot, \cdot, \cdot) + g(\cdot, \cdot, \cdot)$ in (3) satisfy not only the local Lipschitz condition (A2) but also conditions (A1), (A3), then

1. for each $l \in \rho(A)$ and all $p > 2$, there is a unique local strong solution $U(l) \in D(A)$ of (6) which is lie in $L^p(\Omega, \mathcal{F}, P; C(0,T; H))$;
2. for a stop time $\tau_M > 0$, there exists a subsequence $\{U_n\}$, such that

$$\lim_{n \to \infty} U_n = U_t, \text{ a.s.}$$

as $n \to \infty$, uniformly for $t \in [0, \tau_M]$.

Proof. Since $AR(l) = AL(l,A) = l - l^2 R(l,A)$ are bounded operators. By Lemma 2.2, (6) has a local mild solution, and Proposition 1.3.5 in [9] indicates the local mild solution is also a local strong solution. The remainder of the proof is similar to that of Proposition 1.3.6 in [9]. The difference is that we can deduce that

$$E \sup_{0 \leq t \leq T \land \tau_M} \|U_t - U_l(t)\|^p \leq \varepsilon(l, M)C(T, M)T \to 0, \text{ as } l \to \infty,$$

where $\tau_M \triangleq \inf\{t > 0 | \|U_t \wedge U_l(t)\| \geq M\} \wedge \tau_e$ is a stop time, $\tau_e$ is the explosion time mentioned in Lemma 2.2 and $\varepsilon(l, M) = o(l)$ if $0 \leq t \leq T \setminus \tau_M$. \square
3. Existence of the global positive solution. In this section, we consider the existence of the global solution for (2). The first step towards the existence is to obtain a positive local solution of (2).

Let \( \phi(x,t) = \ln u(x,t) \) and \( \psi(x,t) = \ln v(x,t) \), considering a system

\[
\begin{align*}
\phi_t &= d_1 \Delta \phi + (1 - \frac{1}{2} \sigma_1^2) - e^{\phi(x,t)} - \frac{me^{\phi(x,t) + \psi(x,t)}}{a^2 + e^{2\phi(x,t)}} + \sigma_1 W_1(x,t), x \in D, t > 0, \\
\psi_t &= d_2 \Delta \psi - (d + \frac{1}{2} \sigma_2^2) + \frac{me^{2\psi(x,t)}}{a^2 + e^{2\phi(x,t)}} + \sigma_2 W_2(x,t), x \in D, t > 0, \\
\phi(x,0) &= \ln u_0(x), \psi(x,0) = \ln v_0(x), x \in D,
\end{align*}
\]

(7)

Lemma 3.1. Assume that \( H = L^2(D, \mathbb{R}^2) \) is a Hilbert space, \( D \subset \mathbb{R}^n \) is a domain. A linear operator \( A \) from \( H \) to \( H \) is defined as \( A = \begin{pmatrix} d_1 \Delta - 1 & 0 \\ 0 & d_2 \Delta - 1 \end{pmatrix} \), where \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) is a Laplace operator. Let

\[
U(x,t) = (\phi(x,t), \psi(x,t))^T, (x,t) \in D \times [0, T],
\]

for \( \forall t \in [0, T], U(\cdot, t) \in H = L^2(D, \mathbb{R}^2) \), we rewrite (7) as the following form:

\[
\begin{align*}
\frac{\partial \tilde{U}}{\partial t} &= A \tilde{U}(x,t)dt + f(\tilde{U}(x,t), t)dt + \gamma(\tilde{U}(x,t), t)dW(x,t), x \in D, t \geq 0, \\
\frac{\partial \tilde{U}}{\partial n} |_{\partial D} &= 0, \\
\tilde{U}(x,0) &= \tilde{U}_0(x), x \in D,
\end{align*}
\]

(8)

where

\[
f(\tilde{U}(\cdot,t), t) = \begin{pmatrix} \phi(\cdot,t) + (1 - \frac{1}{2} \sigma_1^2) - e^{\phi(\cdot,t)} - \frac{me^{\phi(\cdot,t) + \psi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}} \\ \psi(\cdot,t) - (d + \frac{1}{2} \sigma_2^2) + \frac{me^{2\psi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}} \end{pmatrix},
\]

\[
\gamma(\tilde{U}(\cdot,t), t) = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix},
\]

\[
dW(x,t) = \begin{pmatrix} dW_1(x,t) \\ dW_2(x,t) \end{pmatrix}.
\]

Then (8) satisfies conditions (A1),(A2),(A3).

Proof. (1) For \( \forall n > 0, \tilde{U}(x,t) = (\phi(x,t), \psi(x,t))^T \), if \( \|\tilde{U}(\cdot,t)\|^2 = \|\phi(\cdot,t)\|^2 + \|\psi(\cdot,t)\|^2 \leq n^2 \), then we have

\[
\begin{align*}
\|f(\tilde{U}(\cdot,t), t)\|^2 &= \|\phi(\cdot,t) + (1 - \frac{1}{2} \sigma_1^2) - e^{\phi(\cdot,t)} - \frac{me^{\phi(\cdot,t) + \psi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}}\|^2 \\
&\quad + \|\psi(\cdot,t) - (d + \frac{1}{2} \sigma_2^2) + \frac{me^{2\psi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}}\|^2 \\
&\leq 3\|\phi(\cdot,t) + (1 - \frac{1}{2} \sigma_1^2)\|^2 + 3\|e^{\phi(\cdot,t)}\|^2 + 3\|\frac{me^{\phi(\cdot,t) + \psi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}}\|^2 \\
&\quad + 3\|\psi(\cdot,t)\|^2 + 3(d + \frac{1}{2} \sigma_2^2)^2 + 3\|\frac{me^{2\psi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}}\|^2 \\
&\leq 6\|\phi(\cdot,t)\|^2 + 6(1 - \frac{1}{2} \sigma_1^2)^2 + 3\|e^{\phi(\cdot,t)}\|^2 + 6m^2\|\frac{e^{2\phi(\cdot,t)}}{a^2 + e^{2\phi(\cdot,t)}}\|^2
\end{align*}
\]
\[ + 6m^2 \| \frac{e^{2\psi(t)}}{a^2 + e^{2\phi(t)}} \|^2 + 3\| \psi(t) \|^2 + 3(d + \frac{1}{2} \sigma_1^2)^2 + 3m^2 \]
\[ \leq 6n^2 + 6(1 - \frac{1}{2} \sigma_1^2)^2 + 3c^2n + 6m^2 \frac{e^{2\psi(t)}}{a^2} \|^2 + 3n^2 \]
\[ + 3(d + \frac{1}{2} \sigma_2^2)^2 + 3m^2 \]
\[ \leq 9n^2 + 9m^2 + 6 \frac{m^2}{a^2} e^{4n} + 3c^2n + 6(1 - \frac{1}{2} \sigma_1^2)^2 + 3(d + \frac{1}{2} \sigma_2^2)^2, \]

and

\[ \| \gamma(\tilde{U}(\cdot, t), t) \|^2 = \sigma_1^2 + \sigma_2^2. \]

Let \( C_n = 9n^2 + 9m^2 + 6 \frac{m^2}{a^2} e^{4n} + 3c^2n + 6(1 - \frac{1}{2} \sigma_1^2)^2 + 3(d + \frac{1}{2} \sigma_2^2)^2 + \sigma_1^2 + \sigma_2^2 \), thus

\[ \| f(\tilde{U}, \cdot, t) \|^2 + \| \gamma(\tilde{U}, \cdot, t) \|^2 \leq C_n, \text{ a.s.} \]

\((A_n1)\) is satisfied.

(2) For any \( n > 0 \), if \( \tilde{U}_1(x, t) = (\phi(x, t), \psi(x, t))^T \) and \( \tilde{U}_2 = (\bar{\phi}(x, t), \bar{\psi}(x, t))^T \) satisfy \( \| \tilde{U}_1(\cdot, t) \|^2 \geq \| \tilde{U}_2(\cdot, t) \|^2 \), \( \forall t \in [0, T] \) or \( (\| \phi(\cdot, t) \|^2 + \| \psi(\cdot, t) \|^2) \geq (\| \bar{\phi}(\cdot, t) \|^2 + \| \bar{\psi}(\cdot, t) \|^2) \), then

\[ \| f(\tilde{U}_1(\cdot, t), t) - f(\tilde{U}_2(\cdot, t), t) \|^2 \]
\[ = \left( \phi(\cdot, t) + (1 - \frac{1}{2} \sigma_1^2) - e^{\phi(t)} - \frac{me^{\phi(t) + \psi(t)}}{a^2 + e^{2\phi(t)}} \right) \]
\[ \psi(\cdot, t) - (d + \frac{1}{2} \sigma_2^2) + \frac{me^{2\phi(t)}}{a^2 + e^{2\phi(t)}} \]
\[ - \left( \bar{\phi}(\cdot, t) + (1 - \frac{1}{2} \sigma_1^2) - e^{\bar{\phi}(t)} - \frac{me^{\bar{\phi}(t) + \bar{\psi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \right) \]
\[ \bar{\psi}(\cdot, t) - (d + \frac{1}{2} \sigma_2^2) + \frac{me^{2\bar{\phi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \]
\[ = \left( \phi(\cdot, t) - \bar{\phi}(\cdot, t) - (e^{\phi(t)} - e^{\bar{\phi}(t)}) - \left( \frac{me^{\phi(t) + \psi(t)}}{a^2 + e^{2\phi(t)}} - \frac{me^{\bar{\phi}(t) + \bar{\psi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \right) \right) \]
\[ \psi(\cdot, t) - \bar{\psi}(\cdot, t) + \left( \frac{me^{2\phi(t)}}{a^2 + e^{2\phi(t)}} - \frac{me^{2\bar{\phi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \right), \]

then

\[ \| f(\tilde{U}_1(\cdot, t), t) - f(\tilde{U}_2(\cdot, t), t) \|^2 \]
\[ = \| \phi(\cdot, t) - \bar{\phi}(\cdot, t) - (e^{\phi(t)} - e^{\bar{\phi}(t)}) \|^2 \]
\[ + \| \psi(\cdot, t) - \bar{\psi}(\cdot, t) + \left( \frac{me^{2\phi(t)}}{a^2 + e^{2\phi(t)}} - \frac{me^{2\bar{\phi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \right) \|^2 \]
\[ \leq 3\| \phi(\cdot, t) - \bar{\phi}(\cdot, t) \|^2 + 3\| e^{\phi(t)} - e^{\bar{\phi}(t)} \|^2 + 3 \| \frac{me^{\phi(t) + \psi(t)}}{a^2 + e^{2\phi(t)}} - \frac{me^{\bar{\phi}(t) + \bar{\psi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \|^2 \]
\[ + 2\| \psi(\cdot, t) - \bar{\psi}(\cdot, t) \|^2 + 2 \| \frac{me^{2\phi(t)}}{a^2 + e^{2\phi(t)}} - \frac{me^{2\bar{\phi}(t)}}{a^2 + e^{2\bar{\phi}(t)}} \|^2 \]

(9)
$$\leq 3\|\phi(\cdot, t) - \overline{\phi}(\cdot, t)\|^2 + 3\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2 + 3m^2\|\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}} - \frac{e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\overline{\phi}(\cdot, t)}}\|^2$$

$$+ 2\|\psi(\cdot, t) - \overline{\psi}(\cdot, t)\|^2 + 2m^2\|\frac{e^{\phi(\cdot, t)} + e^{\overline{\phi}(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}}\|\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$\leq 3\|\phi(\cdot, t) - \overline{\phi}(\cdot, t)\|^2 + 3\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2 + 3m^2\|\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}} - \frac{e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\overline{\phi}(\cdot, t)}}\|^2$$

$$+ 2\|\psi(\cdot, t) - \overline{\psi}(\cdot, t)\|^2 + 2m^2\left(\frac{2\|e^{\phi(\cdot, t)}\|^2 + 2\|e^{\overline{\phi}(\cdot, t)}\|^2}{a^8}\right)\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$\leq 3\|\phi(\cdot, t) - \overline{\phi}(\cdot, t)\|^2 + 3\|\phi(\cdot, t) - \overline{\phi}(\cdot, t)\|^2 + 1 + \Theta(\phi(\cdot, t), \overline{\phi}(\cdot, t))\|\|^2$$

$$+ 3m^2\|\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}} - \frac{e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\overline{\phi}(\cdot, t)}}\|^2$$

$$+ 2m^2\left(\frac{2\|e^{\phi(\cdot, t)}\|^2 + 2\|e^{\overline{\phi}(\cdot, t)}\|^2}{a^8}\right)\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$\leq 2a^4\left(\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)} - e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}}\right)^2$$

$$+ 2\left(\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)} - e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}}\right)\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$\leq 4a^4\left(\frac{\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2 + \|e^{\psi(\cdot, t)} - e^{\overline{\psi}(\cdot, t)}\|^2}{a^2 + e^{2\phi(\cdot, t)}}\right)^2$$

$$+ 2\left(\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)} - e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}}\right)\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$\leq 4a^4\left(\frac{\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2 + \|e^{\psi(\cdot, t)} - e^{\overline{\psi}(\cdot, t)}\|^2}{a^2 + e^{2\phi(\cdot, t)}}\right)^2$$

$$+ 2\left(\frac{e^{\phi(\cdot, t) + \psi(\cdot, t)} - e^{\overline{\phi}(\cdot, t) + \overline{\psi}(\cdot, t)}}{a^2 + e^{2\phi(\cdot, t)}}\right)\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$\leq \frac{4}{a^4}\|e^{\phi(\cdot, t)}\|^2\|e^{\psi(\cdot, t)} - e^{\overline{\psi}(\cdot, t)}\|^2 + \frac{4}{a^4}\|e^{\overline{\phi}(\cdot, t)}\|^2\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$+ \frac{2}{a^4}\|e^{\phi(\cdot, t) + \psi(\cdot, t)}\|^2\|e^{\overline{\phi}(\cdot, t)} - e^{\overline{\psi}(\cdot, t)}\|^2$$

$$\leq \frac{4}{a^4}\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2 + \|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$+ \frac{2}{a^4}e^{2\phi(\cdot, t)}\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$

$$+ \frac{2}{a^4}e^{2\phi(\cdot, t)}\|e^{\phi(\cdot, t)} - e^{\overline{\phi}(\cdot, t)}\|^2$$
\[\phi \leq \frac{4}{a^2} e^{2n} \|\psi(\cdot, t) - \overline{\psi}(\cdot, t)\|^2 [1 + \Theta(\psi(\cdot, t), \overline{\psi}(\cdot, t))]^2 + \|\phi(\cdot, t) - \overline{\phi}(\cdot, t)\|^2,\]

\[\|1 + \Theta(\phi(\cdot, t), \overline{\phi}(\cdot, t))\|^2 + \frac{2e^{6n}}{a^4} \|\phi(\cdot, t) - \overline{\phi}(\cdot, t)\|^2 1 + \Theta(\phi(\cdot, t), \overline{\phi}(\cdot, t))\|^2,\]

Submit (10) into (9), we obtain

\[\|f(\tilde{U}_1(\cdot, t), t) - f(\tilde{U}_2(\cdot, t), t)\|^2 = [3 + 3(1 + \Theta(n))^2 + 8m^2 e^{2n} (1 + \Theta(n))^2 + 12m^2 e^{2n} (1 + \Theta(n))^2\]

\[+ 6 \frac{m^2 e^{6n}}{a^4} (1 + \Theta(n))^2 \|\phi - \overline{\phi}\|^2 + (2 + 12 \frac{m^2 e^{2n}}{a^4} (1 + \Theta(n))^2) \|\psi - \overline{\psi}\|^2 \leq K_n \|\phi - \overline{\phi}\|^2 + \|\psi - \overline{\psi}\|^2 \]

\[= K_n \|\tilde{U}_1 - \tilde{U}_2\|^2,\]

where \(K_n = \max\{3 + 3(1 + \Theta(n))^2 + 8m^2 e^{2n} (1 + \Theta(n))^2 + 12m^2 e^{2n} (1 + \Theta(n))^2 + 6 \frac{m^2 e^{6n}}{a^4} (1 + \Theta(n))^2, 2 + 12 \frac{m^2 e^{2n}}{a^4} (1 + \Theta(n))^2\},\)

thus

\[\|f(\tilde{U}_1(\cdot, t), t) - f(\tilde{U}_2(\cdot, t), t)\|^2 + \|\gamma(\tilde{U}_1(\cdot, t), t) - \gamma(\tilde{U}_2(\cdot, t), t)\|^2 \leq K_n \|\tilde{U}_1 - \tilde{U}_2\|^2, a.s.\]

(A_n.2) is satisfied.

(3) Compare (3) and (8) we know, in (8), \(g(x, t) \equiv 0, (A3)\) is satisfied.

By Lemma 2.2 and Lemma 3.1, (7) has a unique local solution \((\phi(x, t), \psi(x, t))\)
which is defined on \([0, \tau_x]\), and through Itô’s Formula and Lemma 2.4, \(u(x, t) = e^{\phi(x, t)}, v(x, t) = e^{\psi(x, t)}\) is the unique local positive solution of (2) on \([0, \tau_M]\).

**Theorem 3.2.** If \(m < \min\{2a^2, d - \frac{\sigma_4^2}{2}\} \text{ and } d > \frac{\sigma_4^2}{2}\), the stochastic perturbation reaction-diffusion equations (2) which depend on spatial position has a unique positive global solution \((u(x, t), v(x, t))^T\).

**Proof.** Rewrite (2) as

\[
\begin{cases}
  dU(x, t) = \overline{A}U(x, t)dt + \overline{f}(U(x, t), t)dt + \overline{\gamma}(U(x, t), t)dW(x, t), t \geq 0 \\
  \frac{\partial U}{\partial n}|_{\partial D} = 0, \\
  U(x, 0) = U_0(x),
\end{cases}
\]

where \(\overline{A} = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix}\), \(\overline{f}(\cdot, t) = \begin{pmatrix} u(1 - u) - \frac{ma^2u}{a^2 + nu} \\ -dv + \frac{mnu^2}{a^2 + nu} \end{pmatrix}\),

\(\overline{\gamma}(\cdot, t) = \begin{pmatrix} \sigma_1u & 0 \\ 0 & \sigma_2v \end{pmatrix}\) and \(dW(x, t) = \begin{pmatrix} dW_1(x, t) \\ dW_2(x, t) \end{pmatrix}\).

For a function \(V(U) \in C^2(\mathbb{R}^n; \mathbb{R})\), we define a differential operator \(L\) with (11) by

\[L(V(U)) = \langle V_U, \overline{A} + \overline{f} \rangle + \frac{1}{2} \text{trace} \langle V_{UU} \overline{\gamma}^T, \overline{\gamma} \rangle,
\]

where \(V_U = \begin{pmatrix} \frac{\partial V}{\partial U_1}, \cdots, \frac{\partial V}{\partial U_n} \end{pmatrix}, V_{UU} = \begin{pmatrix} \frac{\partial^2 V}{\partial U_i \partial U_j} \end{pmatrix}_{n \times n} \).
Since there exists a unique local positive solution \( U(x, t) = (u(x, t), v(x, t))^T \) of (11) for \( t \in [0, \tau_M] \), we need to consider the following approximation system of strong solution:

\[
\begin{aligned}
\begin{cases}
dU^n(x, t) = A^n U^n(x, t)dt + R(n)\mathcal{F}(U^n(x, t), t)dt \\
\quad + R(n)\mathcal{G}(U^n(x, t), t)d\mathbf{W}(x, t), t \geq 0 \\
\partial U^n/\partial n = 0, \\
U^n(x, 0) = R(n)U_0(x).
\end{cases}
\end{aligned}
\] (12)

From Lemma 2.4, we can see that (12) has a unique local strong solution \( U^n_t \) and \( \lim_{n \to \infty} U^n_t = U(x, t) \), a.s. uniformly for \( t \in [0, \tau_M] \), where \( U(x, t) \) is the unique local mild solution of (11). We need to proof \( \tau_M = \infty \), then the solution is global.

We choose \( \xi_0 \geq 0 \) that is sufficiently large such that \( u(x, 0), v(x, 0) \in [\frac{1}{\xi_0}, \xi_0] \). For each \( \xi \geq \xi_0 \), define

\[
\tau_\xi = \inf \{ t \in [0, \tau_M] : \min \{ u(x, t), v(x, t) \} \leq \frac{1}{\xi} \text{ or } \max \{ u(x, t), v(x, t) \} \geq \xi \}.
\]

We set inf \( \emptyset = \infty \) and then \( \tau_\xi \) is increasing as \( \xi \to \infty \). Let \( \tau_\infty = \lim_{\xi \to \infty} \tau_\xi \), where \( \tau_\infty \leq \tau_M \) a.s. We only need to prove \( \tau_\infty = \infty \), then \( \tau_M = \infty \). Otherwise, there exists a \( \delta \in (0, 1) \) and \( T > 0 \), such that

\[
\mathbb{P}[\tau_\infty \leq T] \geq \delta.
\]

Hence there exists an integer \( \xi_1 \geq \xi_0 \), for all \( \xi \geq \xi_1 \), \( \mathbb{P}[\tau_\xi \leq T] \geq \delta \).

Define a \( C^2 \) function \( V : \mathbb{R}^2_+ \to \mathbb{R}_+ \) by

\[
V(u, v) = V_1(u, v) + V_2(u, v) = (u - 1 - \log u + v - 1 - \log v) + \frac{k}{2}v^2,
\]

where \( k > 0 \), then by the definition of \( L \),

\[
L V_1 = \int_D \left[ (1 - \frac{1}{u})u(1 - u) - \frac{mu^2v}{a^2 + u^2} \right] dx + \frac{1}{2}\sigma_1^2v^2dx + \frac{1}{2}\sigma_2^2v^2dx + \int_D (1 - \frac{1}{u})d_1\Delta_1 dx + \int_D (1 - \frac{1}{v})d_2\Delta_2 dx,
\]

\[
= \int_D [(1 - u) - \frac{mu^2v}{a^2 + u^2} - (1 - u) + \frac{mu^2v}{a^2 + u^2} + \frac{1}{2}\sigma_1^2 - dv + \frac{mu^2v}{a^2 + u^2} + d
\]

\[
- \frac{mu^2v}{a^2 + u^2} + \frac{1}{2}\sigma_2^2]dx + \int_D (1 - \frac{1}{u})d_1\Delta_1 dx + \int_D (1 - \frac{1}{v})d_2\Delta_2 dx,
\]

\[
\leq \int_D [2u - u^2 - 1 + \frac{m}{a^2}u^2v - dv + d + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2]dx + \int_D (1 - \frac{1}{u})d_1\Delta_1 dx + \int_D (1 - \frac{1}{v})d_2\Delta_2 dx,
\]

\[
\leq \int_D [2u - u^2 - 1 + \frac{m}{2a^2}u^2 - dv + d + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2]dx + \int_D (1 - \frac{1}{u})d_1\Delta_1 dx + \int_D (1 - \frac{1}{v})d_2\Delta_2 dx,
\]

\[
= \int_D [(\frac{m}{2a^2} - 1)u^2 + 2u - 1 + \frac{m}{2a^2}u^2 - dv + d + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2]dx
\]
we have

Therefore, we can deduce that

Integrate \((14)\) from 0 to \(\tau_\xi \wedge T\),

\[
\begin{align*}
&\int_0^{\tau_\xi \wedge T} dV(U^n_t) \\
&= \int_0^{\tau_\xi \wedge T} (V_t U^n_t, \overline{A} + R(n)\overline{f}(U^n_t)) dt + \int_0^{\tau_\xi \wedge T} (V_t U^n_t, R(n)\overline{\gamma}(U^n_t) dW(x, t)) dt \\
&\quad + \int_0^{\tau_\xi \wedge T} \frac{1}{2} \text{trace} V_t U^n_t (R(n)\overline{\gamma})^T R(n)\overline{\gamma} dt \\
&= V(U^n_{\tau_\xi \wedge T}) - V(U^n_0).
\end{align*}
\]

Therefore, we can deduce that

\[
\begin{align*}
\text{EV}(U^n_{\tau_\xi \wedge T}) &= V(U^n_0) + \mathbb{E} \int_0^{\tau_\xi \wedge T} LV(U^n_t) dt + \int_0^{\tau_\xi \wedge T} \mathbb{E}(V_t U^n_t, (R(n) - I)\overline{f}(U^n_t)) dt
\end{align*}
\]
Assume that would not be true, which contradicts the Lemma 4.1. The condition that we need is that the uniformly Lipschitz condition was replaced by local Lipschitz condition. Indeed, if that would not be the case, there must exist \( (t, x, v) \in D \) such that \( u(x, v) = 0 \), \( v(x, v) \) equals to \( x \) or \( \frac{1}{\xi} \), that is

\[
V(u(x, \tau_\xi), v(x, \tau_\xi)) \\
\geq (\xi - 1 - \log \xi) \wedge (\frac{1}{\xi} - 1 - \log \frac{1}{\xi}) \wedge (\xi - 1 - \log \xi + \frac{k}{2\xi^2}) \\
\wedge (\frac{1}{\xi} - 1 - \log \frac{1}{\xi} + \frac{k}{2\xi^2}),
\]

hence

\[
V(u_0, v_0) + C \mu(D) T \\
\geq \mathbb{E}[1_{\Omega_c} V(u(x, \tau_\xi), v(x, \tau_\xi))] \\
\geq \varepsilon (\xi - 1 - \log \xi) \wedge (\frac{1}{\xi} - 1 - \log \frac{1}{\xi}) \wedge (\xi - 1 - \log \xi + \frac{k}{2\xi^2}) \\
\wedge (\frac{1}{\xi} - 1 - \log \frac{1}{\xi} + \frac{k}{2\xi^2}).
\]

Let \( \xi \to \infty, \infty > V(u_0, v_0) + C \mu(D) T \geq \infty \), so we must have \( T_\infty = \infty \), a.s. \qed

4. Extinction and stability in mean square. Consider the following two stochastic equations \((k = 1, 2)\),

\[
\frac{\partial}{\partial t} u^{(k)}(x, t) = A u^{(k)}(x, t) + F^{(k)}(\omega, x, t, v^{(k)}(x, t)) + \Upsilon(\omega, x, t, v^{(k)}(x, t)) \tilde{W}(x, t)
\]

where \( t \geq 0, x \in D \) with initial value \( u^{(k)}(\omega, x, t, v^{(k)}(x, t)) \in \Omega \times D \to \mathbb{R}^1 \) and Neumann boundary conditions, \( A = \nu \Delta, \nu > 0 \).

**Lemma 4.1** ([9]). Assume that \( \theta^{(k)} \) are uniformly bounded and let \( F^{(k)} \) and \( G \) be uniformly Lipschitz continuous and uniformly bounded, \( k = 1, 2 \). If

\[
u^{(1)} \geq \nu^{(2)}, \text{a.s.}
\]

and

\[
F^{(1)}(\omega, x, t, v) \geq F^{(2)}(\omega, x, t, v), \text{a.s.}
\]

where \((x, t) \in D \times [0, \infty), v \in \mathbb{R}^1, \) then \( v^{(1)} \geq v^{(2)} \) holds almost surely.

In fact, Lemma 4.1 remains true if the uniformly Lipschitz condition was replaced by local Lipschitz condition. Indeed, if that would not be the case, there must exist \( R, T > 0 \), such that

\[
P( \sup_{x \in D, 0 \leq t \leq T} v^{(2)}(x, t) - v^{(1)}(x, t) > 0) \geq \sup_{x \in D, 0 \leq t \leq T} \| v^{(1)}(x, t) \| \vee \| v^{(2)}(x, t) \| \leq R > 0.
\]

Then, for \( F^{(k)}, \Upsilon \) on \( \Omega \times D \times [0, T] \times [-R, R] \), the conclusion in Lemma 4.1 would not be true, which contradicts the Lemma 4.1.
Theorem 4.2. Suppose that the initial function \( u_0, v_0 \) of (2) satisfying \( u_0, v_0 \in C(\mathcal{D}) \), and \( \sigma_1^2 > 2, \sigma_2^2 > 2m \), we have
\[
\lim_{t \to \infty} \|u(\cdot, t)\|_{C(\mathcal{D})} = 0, a.s.
\]
\[
\lim_{t \to \infty} \|v(\cdot, t)\|_{C(\mathcal{D})} = 0, a.s.
\]
Proof. Let \( B = B' = C(\mathcal{D}), H = L^2(D, \mathbb{R}^2) \). Consider \((u_1(t), v_1(t))_{t \geq 0} \in B' \times H\) such that
\[
\begin{aligned}
u_{1t} &= d_1\Delta u_1(x, t) + u_1(x, t) + \sigma_1 u_1(x, t)\dot{\bar{W}}_1(x, t), x \in D, t > 0, \\
v_{1t} &= d_2\Delta v_1(x, t) + mv_1(x, t) + \sigma_2 v_1(x, t)\dot{\bar{W}}_2(x, t), x \in D, t > 0, \\
u_1(x, 0) &= u_0(x), v_1(x, 0) = v_0(x), x \in D.
\end{aligned}
\]
(15)
By Comparison Theorem Lemma 4.1, we obtain
\[
0 \leq u(x, t) \leq u_1(x, t),
\]
\[
0 \leq v(x, t) \leq v_1(x, t).
\]
For the mild solution \( u_1(x, t) \) of (15),
\[
u_1(x, t) = (\Gamma_u(t)u_0)(x)\exp\left((1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_1\right),
\]
where \( x \in D \), \( \Gamma_u \) is the semigroup generated by \( d_1\Delta \) in \( C(\mathcal{D}) \). Since \( \Gamma_u \) is a contraction semigroup and
\[
\exp\left((1 - \frac{\sigma_1^2}{2}) + \sigma_1 \frac{W_1}{t}\right) \to 0, t \to \infty \ a.s.,
\]
in addition, \( \frac{W_1}{t} \) \( \to 0 \) and \( \sigma_1^2 > 2 \), we have \( \|u_1(x, t)\|_{C(\mathcal{D})} \to 0 \) as \( t \to \infty \). Therefore,
\[
\|u(x, t)\|_{C(\mathcal{D})} \to 0 \text{ as } t \to \infty.
\]
Similarly, when \( \sigma_2^2 > 2m \), \( \|v(x, t)\|_{C(\mathcal{D})} \to 0 \) as \( t \to \infty \).

The interpretation of the above result is: if the intensities \( \sigma_1^2, \sigma_2^2 \) of White noise are large, for example, \( \sigma_1^2 > 2, \sigma_2^2 > 2m \), the prey and the predator will die out.

Theorem 4.3. Let \(-\lambda_\Delta\) be the principle eigenvalue of \( \Delta \), if \( \sigma_1^2 < 2d_1\lambda_\Delta - 2, \sigma_2^2 < 2d_2\lambda_\Delta - 2m \), then for arbitrary \( u_0 \in H, v_0 \in H \),
\[
E\|u(x, t)\|^2 \leq \|u_0\|^2 e^{-\nu_u t},
\]
\[
E\|v(x, t)\|^2 \leq \|v_0\|^2 e^{-\nu_v t},
\]
for some constants \( \nu_u > 0, \nu_v > 0 \).
Proof. Let \( V(u) = \|u\|^2 \). For a single stochastic equation
\[
\begin{aligned}
u_{1t} &= d_1\Delta u_1(x, t) + u_1(x, t) + \sigma_1 u_1(x, t)\dot{\bar{W}}_1(x, t), x \in D, t > 0, \\
u_1(x, 0) &= u_0(x),
\end{aligned}
\]
(16)
we need to consider the approximation system of strong solution
\[
\begin{aligned}
u_{1nt} &= d_1\Delta u_1^n(x, t) + R(n)u_1^n(x, t) + R(n)\sigma_1 u_1^n(x, t)\dot{\bar{W}}_1(x, t), x \in D, t > 0, \\
u_1^n(x, 0) &= R(n)u_0^n(x),
\end{aligned}
\]
(17)
where \( 0 \leq n_0 \leq n \in \rho(\Delta) \) for some \( n_0 \in \rho(\Delta) \), \( \rho(\Delta) \) is the resolvent set of \( \Delta \), and \( R(n) = nR(n, \Delta), R(n, \Delta) \) is the resolvent of \( \Delta \). We can easily deduce that
\[
LV(u_1) = 2(u_1, d_1\Delta u_1)_H + \langle u, u \rangle_H + \sigma_1^2(u, u)_H \leq (-2d_1\lambda_\Delta + 2 + \sigma_1^2)\|u_1\|^2,
\]
where $-\lambda_\Delta$ is the principal eigenvalue of $\Delta$. Choose a positive constant $\mu_u$ such that $\mu_u < 2d_1\lambda_\Delta - 2 - \sigma_1^2$, applying Itô’s Formula to $e^{\mu_u t}V(u_1^n)$,

$$e^{\mu_u t}V(u_1^n(x, t)) - V(u_1^n(x, 0))$$

$$= \mu_u \int_0^t e^{\mu_u s}||u_1^n||^2 ds + \int_0^t 2e^{\mu_u s}||u_1^n||^2 ds + R(n)u_1^n H ds$$

$$+ \int_0^t 2e^{\mu_u s}(u_1^n, R(n)\sigma_1 u_1^n dW_1(x, s)) H$$

$$+ \int_0^t e^{\mu_u s}\text{trace}(R(n)\sigma_1 u_1^n)(R(n)\sigma_1 u_1^n)^T ds.$$

Therefore, by taking expectation we obtain

$$e^{\mu_u t}EV(u_1^n(x, t)) \leq V(u_1^n(x, 0)) + (\mu_u - 2d_1\lambda_\Delta + 2 + \sigma_1^2) \int_0^t e^{\mu_u s}E||u_1^n||^2 ds$$

$$+ 2\int_0^t e^{\mu_u s}E((u_1^n, (R(n) - I)u_1^n) H ds$$

$$+ e^{\mu_u s}\text{trace}[(R(n)\sigma_1 u_1^n)(R(n)\sigma_1 u_1^n)^T - (\sigma_1 u_1^n)(\sigma_1 u_1^n)^T) ds].$$

Using Lemma 2.4, there exists a subsequence of $\{n\} \in \rho(\Delta)$, still denoting it by $\{n\}$, such that $u_1^n \to u_1$ a.s. for $T \to \infty$. Let $n \to \infty$ in (18), for arbitrary $t > 0$,

$$e^{\mu_u t}EV(u_1(x, t)) \leq V(u_1(x, 0)) + (\mu_u - 2d_1\lambda_\Delta + 2 + \sigma_1^2) \int_0^t e^{\mu_u s}E||u_1(x, s)||^2 ds$$

Hence

$$E||u_1(x, t)||^2 \leq ||u_0||^2 e^{-\mu_u t}, u_0 \in H, t \geq 0.$$

By the proof in Theorem 4.2, $0 < u(x, t) \leq u_1(x, t)$, we have

$$E||u(x, t)||^2 \leq ||u_0||^2 e^{-\mu_u t}, u_0 \in H, t \geq 0.$$

Similarly, if we choose $0 < \mu_v < 2d_2\lambda_\Delta - 2m - \sigma_2^2$,

$$E||v(x, t)||^2 \leq ||v_0||^2 e^{-\mu_v t}, u_0 \in H, t \geq 0.$$

4.1. Markov property, uniqueness of invariant measure and ergodic. In this section, we prove the Markov property for the solution of (11) and follow the methods in [18] and [17] to seek for an unique invariant measure $\mu$ for (11).

For a $H$-valued random variable $X$, and a probability measure $P$ on $\Omega$, then by $\mathcal{L}(X)$ we denote the law of $X$:

$$\mathcal{L}(X)(\Lambda) = P(\omega : X(\omega) \in \Lambda), \Lambda \subset H.$$

Let $U(t, s, u)$ be the unique mild solution of

$$\begin{cases}
    dU = \overline{A}U dt + \overline{f}(U, t) dt + \overline{g}(U, t) d\overline{W}(x, t), t \geq s \\
    \frac{\partial U}{\partial n}|_{\partial D} = 0, \\
    U(s) = u,
\end{cases}$$

(19)
Following [17], let \( P_{s,t} \) and \( P(s, x, t; \Pi) \), \( t \geq 0, u \in H, \Pi \in \mathcal{B}(H) \) be the corresponding transition semigroup and transition function to \( U(t, s, u) \), here \( \mathcal{B}(H) \) is the smallest \( \sigma \)-field containing all closed (or open) subsets of \( H \). Thus
\[
P_{s,t}(x) = E[\varphi(U(t, s, u))), \varphi \in B_b(H), u \in H
\]
and
\[
P(s, x; t, \Pi) = P(s, t) \chi_{\Pi}(x) = \mathcal{L}(U(t, s, u))(\Pi), t \geq 0, u \in H, \Pi \in \mathcal{B}(H)
\]
where \( \chi_{\Pi} \) is the characteristic function of the set \( \Pi \) and \( B_b(H) \) is the Banach space of all real bounded Borel functions, endowed with the sup norm.

First, we have the Markov property of \( U(t, s, U_0), t \geq s \).

**Theorem 4.4.** Under the same condition as in Theorem 3.2, for arbitrary \( \varphi \in B_b(H) \) and \( t > l > s \), we have
\[
E[\varphi(U(t, s, U_0))|\mathcal{F}_l] = E[\varphi(U(l, y))]|_{y = U(t, s, U_0)} \tag{20}
\]

**Proof.** Without any loss of generality, we assume \( \varphi \in C_b(H) \), where \( C_b(H) \) denote a Banach space of all bounded continuous real valued function defined on \( H \). By Lemma 1.1 in [17], there exists a sequence of \( H \)-valued \( \mathcal{F}_l \)-measurable simple functions \( X_m : \Omega \to H, X_m = \sum_{k=1}^{N} \tau_k \chi_{\{\tau_k \leq X_m \leq \tau_{k+1} \}}, n \in \mathbb{N}, \) where \( \tau_1, \tau_2, \cdots, \tau_m \) are positive distinct and \( \Omega = \bigcup_{k=1}^{N} \{X_m = \tau_k\} \), such that
\[
|X_m(\omega) - U(l, s, U_0)(\omega)| \downarrow 0, n \to \infty, \text{ for all } \omega \in \Omega.
\]

Since the unique of mild solution implies \( U(t, s, U_0) = U(t, l, U(l, s, U_0)) \), for any \( B \in \mathcal{F}_l \), we have
\[
\int_B \varphi(U(t, s, U_0))dP = E[\chi_B \varphi(U(t, l, U(l, s, U_0)))]
\]
\[
= \lim_{m \to \infty} E[\chi_B \varphi(U(t, l, X_m(\omega)))]
\]
\[
= \lim_{m \to \infty} \sum_{k=1}^{N} E[\chi_B \chi_{\{X_m = \tau_k\}} \varphi(U(t, l, \tau_k))]\]
\[
= \lim_{m \to \infty} \sum_{k=1}^{N} E[\chi_B \chi_{\{X_m = \tau_k\}} E[\varphi(U(t, l, \tau_k))]]
\]
\[
= \lim_{m \to \infty} E[\chi_B \sum_{k=1}^{N} \chi_{\{X_m = \tau_k\}} E[\varphi(U(t, l, \tau_k))]]
\]
\[
= \lim_{m \to \infty} E[\chi_B E[\varphi(U(t, l, y))]_{y = X_m}]
\]
\[
= \int_B E[\varphi(U(t, l, y))]|_{y = U(t, s, U_0)}dP.
\]
i.e.
\[
E[\varphi(U(t, s, U_0))|\mathcal{F}_l] = E[\varphi(U(t, l, y))]|_{y = U(t, s, U_0)}
\]

Let \( M(H) \) be the space of all bounded measures on \((H, \mathcal{B}(H))\). For any \( \varphi \in B_b(H) \) and any \( \mu \in M(H) \), we denote \( \langle \varphi, \mu \rangle = \int_H \varphi(x)\mu(dx) \). The dual semigroup \( P^* \) is defined as
\[
\langle \varphi, P^*_{s,t}\mu \rangle = \langle P_{s,t}\varphi, \mu \rangle, \forall \varphi \in B_b(H), \mu \in M(H).
\]
A measure $\mu$ is said to be invariant measure for (11) if $P_{s,t}^*\mu = \mu$, $\forall t > 0$.

By Proposition 11.2 in [17], if for some initial condition $U_0$, $\mathcal{L}(U(t,0,U_0)) \to \mu$ weakly as $t \to +\infty$, then $\mu$ is an invariant measure for (11). In addition, Proposition 11.2 in [17] means, for the uniqueness of the invariant measure of (11), it is sufficient to show $P_{s,t}^*\delta_{U_0} \to \mu$ weakly as $t \to \infty$.

**Theorem 4.5.** Assume the parameters in (11) satisfy $m < \min\{2a^2, d - \frac{\sigma^2}{a^2}\}$, $d > \frac{\sigma^2}{2}$, $\sigma^2 < 2d_1\lambda_\Delta - 2 - m^2$ and $\sigma^2 < 2d_2\lambda_\Delta - 4m - \frac{4m+1}{a^4}$, then there exists exactly one invariant measure $\mu$ for (11).

**Proof.** By a series of simple but cumbersome calculations, for (11), we have
\[
2(\bar{\mathcal{A}}(U_1 - U_2) + \bar{\mathcal{F}}(U_1) - \bar{\mathcal{F}}(U_2), U_1 - U_2) + trace(\bar{\sigma}(U_1) - \bar{\sigma}(U_2))Q(\bar{\sigma}(U_1) - \bar{\sigma}(U_2))^T
\leq (-2d_1\lambda_\Delta + 2 + m^2 + \sigma^2)||u_1 - u_2||^2 + (-2d_2\lambda_\Delta + 4m + \frac{4m+1}{a^4} + \sigma^2)||v_1 - v_2||^2
\leq - \varrho ||U_1 - U_2||^2 + \frac{4m}{a^4}||v_2||^2
\]
where $U_1 = (u_1,v_1)$, $U_2 = (u_2,v_2)$, $\varrho$ is a positive constant.

Also, by Theorem 4.3, there is a positive constant $C_1$, such that
\[
E\|U(.t)\|^2 \leq C_1\|U_0\|^2.
\]
where $C_1$ is a positive constant.

In this theorem, $U(t,s,U_0)$ will denote the unique mild solution of
\[
\begin{cases}
    dU = \bar{\mathcal{A}}U dt + \bar{\mathcal{F}}(U,t)dt + \bar{\sigma}(U,t)dW, \ t \geq s \\
    \partial U |_{\partial D} = 0 \\
    U(s) = U_0(x),
\end{cases}
\]
the corresponding approximation system of strong solution is
\[
\begin{cases}
    dU^n = \bar{\mathcal{A}}U^n dt + R(n)(\bar{\mathcal{F}}(U^n,t)dt + \bar{\sigma}(U^n,t)dW, \ t \geq s \\
    \partial U |_{\partial D} = 0 \\
    U^n(s) = R(n)U_0.
\end{cases}
\]

Theorem 3.2 implies the strong solution $U^n_t$ of (23) satisfies $\lim_{n \to \infty} U^n(t,s,U_0) = U(t,s,U_0)$, a.s. uniformly for $t \in [0, +\infty)$.

For $s_2 > s_1 > 0$, let $Z(t) = U(t,-s_1,U_0) - U(t,-s_2,U_0)$ and $Z^n(t) = U^n(t,-s_1,U_0) - U^n(t,-s_2,U_0), \ t \geq -s_1$, then $Z^n(t)$ satisfies
\[
\begin{cases}
    dZ^n = \bar{\mathcal{A}}Z^n dt + R(n)(\bar{\mathcal{F}}(U^n(t,-s_1,U_0)) - \bar{\mathcal{F}}(U^n(t,-s_2,U_0))dt \\
    + R(n)(\bar{\sigma}(U^n(t,-s_1,U_0)) - \bar{\sigma}(U^n(t,-s_2,U_0)))dW, \ t \geq -s_1 \\
    \partial Z^n |_{\partial D} = 0, \\
    Z^n(-s_1) = R(n)(U_0 - U^n(-s_1,-s_2,U_0)).
\end{cases}
\]

Similar to the method in Theorem 4.3, for $\|Z^n\|^2$, by Ito’s Formula, Lemma 2.4 and Theorem 4.3, taking account into the assumption that $0 < v_0 \leq \theta_v$, we have
\[
\frac{d}{dt}E[\|Z(t)\|^2] = E[2(AZ(t) + \bar{\mathcal{F}}(U(t,-s_1,U_0)))]
\]
Obviously, the sequence of random variables \( \{U(0, s_1, U_0)\}_{s_1 \geq 0} \) is a Cauchy sequence in \( L^2(D, H) \) as \( s_1 \to +\infty \). Thus \( \{U(0, s_1, U_0)\}_{s_1 \geq 0} \) is convergent to a random variable \( \eta \in L^2(D, H) \), which is independent of \( U_0 \). The law of \( \eta \) is the invariant measure. In fact, \[
\mathcal{L}(U(t, 0, U_0)) = \mathcal{L}(U(0, -t, U_0)) \to \mathcal{L}(\eta) \overset{\Delta}{=} \mu, \text{ weakly as } t \to +\infty.
\]
which is equivalent to \[
P_t^* \delta_{U_0} \to \mu \text{ weakly as } t \to \infty.
\]

**Remark 1.** If \( \mu \) is the unique invariant measure for \( P_{s,t} \), then it is ergodic.

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