Jordanian $U_{h,s}gl(2)$ and its coloured realization

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Abstract

A two-parametric non-standard (Jordanian) deformation of the Lie algebra $gl(2)$ is constructed, and then, exploited to obtain a new, triangular $R$-matrix solution of the coloured Yang-Baxter equation. The corresponding coloured quantum group is presented explicitly.

To appear in Lett. Math. Phys.

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Non-standard or Jordanian deformation of the group $GL(2)$ was introduced in [1, 2]. Its two parameter generalization was given in [3] and the supersymmetric version $GL(1/1)$ was worked out in [4]. On the quantum algebra level, the non-standard deformation of the Lie algebra $sl(2)$ was first proposed by Ohn [5]. The universal $R$-matrix for $U_h sl(2)$ was written down in [6, 7, 8] and the irreducible representations were studied in [9, 10].

A unique feature governing this deformation is that the $R$-matrix which solves the quantum Yang–Baxter equation (YBE) is triangular i.e. $R_{12}R_{21} = I$. This motivates us to ask a question: Is it possible to Yang–Baxterize this solution such that now the $R$-matrix satisfies the more general spectral parameter dependent YBE? It is quite obvious that the triangularity property poses the main obstacle to carry out the usual Baxterization procedure. However, a possible way out is, to make the $R$-matrix dependent on some continuously varying colour parameters so that it obeys the coloured YBE. The colour parameters play the same role as spectral parameters and the coloured YBE and spectral YBE become equivalent.

The coloured $R$-matrix solution was introduced by Murakami [11] and various other approaches [12 - 18] have been pursued to construct them. In an interesting work [12], Burdík and Hellinger showed that coloured solutions can be obtained from deformations of non-semisimple Lie algebras like $U_q gl(2)$ which may be thought of as a splice product of $U_q sl(2)$ and $U u(1)$. The eigenvalues of the extra Casimir operator coming from the invariant subalgebra $U u(1)$ were interpreted as the colour indices. In the present letter, we adopt this approach to find a new, non-standard solution of the coloured YBE corresponding to $GL(2)$. However, in order to carry out the analysis an immediate problem arises as the non-standard deformation of $gl(2)$ is not known. Hence, our purpose becomes two-fold. The first is, to give such a quantization of $gl(2)$, and the second is, to derive coloured solution from it. So let us proceed with our first aim. We shall adhere to the convenient basis of [8].

1 The quantum algebra $U_{h,s} gl(2)$

The classical Lie algebra $gl(2)$ is defined as

$$[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_- \quad [J_+, J_-] = J_3, \quad [Z, \cdot] = 0, \quad (1.1)$$

where $Z$ is the central generator of $u(1)$.

The non-standard classical $r$-matrix associated with $gl(2)$ can be written as a combination of non-standard $r$ for $sl(2)$ and a term containing the two primitive generators $J_+$ and $Z$.

$$r = h J_3 \land J_+ + s Z \land J_+. \quad (1.2)$$

It is a solution of the classical Yang–Baxter equation and generates the Lie bialgebra with cocommutators given by $\delta(X) = [1 \otimes X + X \otimes 1, r]$:

$$\delta(J_+) = 0, \quad \delta(Z) = 0, \quad \delta(J_3) = 2h J_3 \land J_+ + 2s Z \land J_+, \quad \delta(J_-) = 2h J_- \land J_+ + s J_3 \land Z. \quad (1.3)$$

where $h$ and $s$ are the two deformation parameters.
The Jordanian quantum algebra $U_{h,s}gl(2) = U_hsl(2) \oplus Uu(1)$ can be defined as a Hopf algebra with the following structure:

Commutation relations:

$$
\begin{align*}
[J_3, J_+] &= \frac{e^{2hJ_+} - 1}{h}, & [J_3, J_-] &= -2J_- + hJ_3^2 + 2sZJ_3 + \frac{s^2}{h}Z^2, \\
[J_+, J_-] &= J_3 + \frac{s}{h}Z(1 - e^{2hJ_+}), & [Z, \cdot] &= 0.
\end{align*}
$$

(1.4)

Coproduct:

$$
\begin{align*}
\Delta(J_+) &= 1 \otimes J_+ + J_+ \otimes 1, \\
\Delta(J_3) &= 1 \otimes J_3 + J_3 \otimes e^{2hJ_+} + \frac{s}{h}Z \otimes (e^{2hJ_+} - 1), \\
\Delta(J_-) &= 1 \otimes J_- + J_- \otimes e^{2hJ_+} + s(J_3 + \frac{s}{h}Z) \otimes Z e^{2hJ_+}, \\
\Delta(Z) &= 1 \otimes Z + Z \otimes 1,
\end{align*}
$$

(1.5)

Counit:

$$
\epsilon(X) = 0, \quad \text{for } X \in \{J_3, J_+, J_-, Z\},
$$

(1.6)

Antipode:

$$
\begin{align*}
\gamma(J_+) &= -J_+, \quad \gamma(J_3) = -J_3e^{-2hJ_+} + \frac{s}{h}Z(1 - e^{-2hJ_+}), \\
\gamma(Z) &= -Z, \quad \gamma(J_-) = -J_-e^{-2hJ_+} + s(J_3 + \frac{s}{h}Z)Ze^{-2hJ_+}.
\end{align*}
$$

(1.7)

It can be checked that $\{J_3, J_+, Z\}$ generates a Hopf subalgebra of this. Both the deformation parameters $h$ and $s$ play equal role in shaping the above non-standard deformation, unlike the two-parameter standard $(p, q)$-deformation [19] where the second parameter is relegated to the coalgebra. When $h = s$, this yields a one parameter non-standard quantization of $gl(2)$.

2 Coloured $R$-matrix $R_{h,s}^{(\lambda,\mu)}$

The universal $R$-matrix for $U_{h,s}gl(2)$ is given by

$$
\mathcal{R} = \exp\{-J_+ \otimes (hJ_3 + sZ)\} \exp\{(hJ_3 + sZ) \otimes J_+\}.
$$

(2.1)

This element solves the quantum Yang–Baxter equation

$$
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.
$$

(2.2)

and verifies the following quasi-triangular properties ($X \in U_{h,s}gl(2)$):

$$
\sigma \circ \Delta(X) = \mathcal{R}\Delta(X)\mathcal{R}^{-1}, \quad \sigma(x \otimes y) = (y \otimes x),
$$

(2.3)
In addition, $R^{-1} = R_{21}$ (i.e. $R$ is triangular), and therefore $U_{h,s}gl(2)$ is endowed with a triangular Hopf algebra structure.

Now let us denote the eigenvalue of $Z$ (Casimir like operator) by $\eta$ and the corresponding $n$-dimensional irreducible representation of the quantum algebra $[1,4]$ by $\pi^\eta_n$. Acting $\pi^\eta_n$ on the universal $R$ yields a finite dimensional coloured matrix representation. So taking the following fundamental two-dimensional matrix representation of $U_{h,s}gl(2)$

\[
\pi_\eta(J_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi_\eta(Z) = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix},
\]
\[
\pi_\eta(J_+^\eta) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi_\eta(J_-^\eta) = \begin{pmatrix} (h + \eta s)^2/2h & 0 \\ 1 & (h - \eta s)^2/2h \end{pmatrix},
\]

and substituting in ($\eta$ acts as $\lambda$ in first factor of the tensor product and $\mu$ in the second )

\[
R^{(\lambda,\mu)} = (\pi_\eta \otimes \pi_\mu)R
\]

we obtain the $4 \times 4$ matrix form of $R$

\[
R_{h,s}^{(\lambda,\mu)} = \begin{pmatrix} 1 & h + \lambda s & -(h + \mu s) & h^2 - \lambda s^2 - h s (\lambda - \mu) \\ 0 & 1 & 0 & h - \mu s \\ 0 & 0 & 1 & -(h - \lambda s) \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(2.7)

We shall call this the coloured Jordanian $R$-matrix, as it satisfies the coloured Yang-Baxter equation

\[
R_{12}^{(\lambda,\mu)} R_{13}^{(\lambda,\nu)} R_{23}^{(\mu,\nu)} = R_{23}^{(\mu,\nu)} R_{13}^{(\lambda,\nu)} R_{12}^{(\lambda,\mu)}.
\]  

(2.8)

$\lambda, \mu, \nu$ are referred to as the colour parameters, which vary continuously. Setting $\lambda = \mu = \eta$, and making the following identification

\[
h + \eta s = z', \quad h - \eta s = z
\]  

(2.9)

the coloured $R$-matrix (2.7) reduces to the two parameter $(z, z')$ $R$-matrix classified in [20], and in the limit $\lambda = \mu = 0$ to the single parameter $(z)$ Jordanian $R$-matrix [1].

Analogous to the quantum group case, one can also define the coloured Jordanian braid group representation (BGR) as

\[
R^{(\lambda,\mu)} = P R^{(\lambda,\mu)}
\]  

(2.10)

where $R^{(\lambda,\mu)}$ is taken as in (2.7) and $P$ is the usual permutation matrix. This coloured BGR turns out to be a solution of the coloured braided YBE

\[
\hat{R}_{23}^{(\lambda,\mu)} \hat{R}_{12}^{(\lambda,\nu)} \hat{R}_{23}^{(\mu,\nu)} = \hat{R}_{12}^{(\mu,\nu)} \hat{R}_{23}^{(\lambda,\nu)} \hat{R}_{12}^{(\lambda,\mu)}.
\]  

(2.11)

It is worth mentioning, that $\hat{R}^{(\lambda,\mu)}$ does not satisfy the Hecke condition. Rather, its characteristic equation is of fourth order given by

\[
(\hat{R}^{(\lambda,\mu)} - 1)^3 (\hat{R}^{(\lambda,\mu)} + 1) = 0, \lambda \neq \mu
\]  

(2.12)

with only two distinct eigenvalues $+1$ and $-1$. 

(2.4)
3 Coloured quantum group $GL_{h,s}^{(\lambda,\mu)}(2)$

Let us now investigate the quantum group structure associated with the coloured Jordanian $R$-matrix. The RTT relations which normally define a quantum group are modified suitably to incorporate the coloured extension as \[3.1\]

$$R^{(\lambda,\mu)}T_{1\lambda}T_{2\mu} = T_{2\mu}T_{1\lambda}R^{(\lambda,\mu)}$$

To be compatible with the coloured $R$-matrix, now the $T$-matrix is also parametrized by the colour $\lambda$ and $\mu$ (one at a time). Since $\lambda, \mu$ are continuous variables, this implies the coloured quantum group has an infinite number of generators.

As an illustration, let us consider the two-dimensional case. Substituting $R^{(\lambda,\mu)}$ from (2.7) and $T_\lambda$ as (and likewise $T_\mu$)

$$T_\lambda = \begin{pmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{pmatrix},$$

in the coloured RTT (3.1), we obtain the following independent commutation relations for the coloured Jordanian quantum group denoted by $GL_{h,s}^{(\lambda,\mu)}(2)$:

$$[a_\lambda, c_\mu] = -(h - \mu s)c_\mu c_\lambda, \quad [a_\lambda, d_\mu] = (h + \lambda s)c_\mu a_\lambda - (h - \mu s)c_\lambda d_\mu,$$

$$[c_\lambda, d_\mu] = (h + \lambda s)c_\mu c_\lambda, \quad [b_\lambda, c_\mu] = -(h + \mu s)c_\mu a_\lambda - (h - \mu s)d_\lambda c_\mu,$$

$$[a_\lambda, b_\mu] = (h + \lambda s)a_\mu a_\lambda - (h + \lambda s)a_\lambda d_\mu + (h + \mu s)c_\lambda b_\mu - f(\lambda, \mu)c_\lambda d_\mu,$$

$$[d_\lambda, b_\mu] = (h - \lambda s)d_\lambda d_\mu - (h - \lambda s)a_\mu d_\lambda + (h - \mu s)b_\lambda c_\mu + f(\lambda, \mu)a_\mu c_\lambda,$$

$$[a_\lambda, a_\mu] = -(h + \lambda s)a_\lambda c_\mu + (h + \mu s)c_\lambda a_\mu - f(\lambda, \mu)c_\mu c_\lambda,$$

$$[b_\lambda, b_\mu] = -(h - \lambda s)a_\mu b_\lambda + (h - \mu s)b_\mu a_\lambda - (h + \lambda s)b_\lambda d_\mu + (h + \mu s)d_\lambda b_\mu + f(\lambda, \mu)(a_\mu a_\lambda - d_\lambda d_\mu),$$

$$[c_\lambda, c_\mu] = 0,$$

$$[d_\lambda, d_\mu] = -(h - \lambda s)c_\mu d_\lambda + (h - \mu s)d_\mu c_\lambda + f(\lambda, \mu)c_\mu c_\lambda.$$

Here, and in what follows

$$f(\lambda, \mu) = h^2 - \lambda \mu s^2 - hs(\lambda - \mu), \quad f(\mu, \lambda) = h^2 - \lambda \mu s^2 - hs(\mu - \lambda).$$

All other relations can be generated from the set (3.3) by simply interchanging $\lambda$ and $\mu$. Notice the presence of the last four relations like $[a_\lambda, a_\mu]$ etc. This is a peculiar characteristic of only the coloured quantum groups and does not exist for their ‘uncoloured’ counterparts since there, either $\lambda = \mu$ or both colours vanish.

The coproduct and counit for the coloured algebra are given by \[3.4\]

$$\Delta(T_\lambda) = T_\lambda \otimes T_\lambda, \quad \epsilon(T_\lambda) = 1.$$

Contrary to the algebra, the coalgebra is a function of only one colour parameter.

The quantum determinant corresponding to $T_\lambda$ is

$$D_\lambda = a_\lambda d_\lambda - b_\lambda c_\lambda - (h + \lambda s)a_\mu c_\lambda.$$
The commutation relations of $D$ two parameter the coloured YBE for $R$. It is shown that this quantum algebra is triangular (Jordanian). Starting from the colour in this work, we have constructed a new two-parametric deformation of the Lie algebra $gl(2)$ of non-commuting nature, i.e. 

$$\begin{align*}
D & = a_\lambda d_\lambda - c_\lambda b_\lambda + (h - \lambda s)c_\lambda d_\lambda 
\end{align*} \quad (3.5)$$

It can be checked that $D$ is still 'group like'

$$D(T_\lambda T_\lambda') = D(T_\lambda).D(T_\lambda'), \quad \text{if} \quad [(T_\lambda)_{ij}, (T_\lambda')_{kl}] = 0, \quad (3.6)$$

The commutation relations of $D$ with the elements of $T_\lambda$ are:

$$\begin{align*}
[D_\lambda, a_\mu] &= (h - \lambda s)D_\lambda c_\mu - \{(h + \mu s)a_\mu d_\lambda - (h + \lambda s)c_\mu b_\lambda + f(\mu, \lambda)c_\mu d_\lambda\}c_\lambda \\
&\quad + \{(h + \lambda s)a_\lambda c_\mu - (h + \mu s)c_\lambda a_\mu + f(\lambda, \mu)c_\lambda a_\mu\}c_\lambda \\
&\quad + \{(h + \mu s)c_\lambda a_\lambda d_\mu - (h + \lambda s)c_\mu b_\lambda + f(\lambda, \mu)c_\mu d_\lambda\}c_\lambda \\
[D_\lambda, b_\mu] &= (h + \lambda s)a_\mu D_\lambda + (h - \lambda s)D_\lambda d_\mu + s(\mu - \lambda)(h + \lambda s)c_\lambda a_\mu c_\lambda \\
&\quad - \{(h - \lambda s)a_\mu d_\lambda - (h - \mu s)b_\mu c_\lambda + f(\lambda, \mu)a_\mu c_\lambda\}c_\lambda \\
&\quad - \{(h + \lambda s)(h - \lambda s)c_\lambda b_\lambda + (h - \mu s)^2 c_\lambda d_\mu\}c_\lambda. 
\end{align*} \quad (3.7)$$

Clearly the determinant is not central unless the colour parameters vanish. These relations also exhibit an interesting feature that the determinants associated with different colours are of non-commuting nature, i.e. $[D_\lambda, D_\mu] \neq 0$, unlike the coloured $q$-deformed $GL(2)$ where they behave as commuting operators [21].

If $D_\lambda$ is non-singular for every $\lambda$, then the above structure can be extended to a Hopf algebra by suitably defining left and right inverses as

$$\gamma(T_\lambda) = D_\lambda^{-1} \begin{pmatrix}
d_\lambda - (h - \lambda s)c_\lambda & -b_\lambda - (h - \lambda s)(d_\lambda - a_\lambda) + (h - \lambda s)^2 c_\lambda \\
-c_\lambda & a_\lambda + (h - \lambda s)c_\lambda
\end{pmatrix} \quad (3.8)$$

The contents of this section generalize the results of [3]. The entire Hopf algebra structure of $GL_{z,z'}(2)$ can be easily recovered in the monochromatic limit $\lambda = \mu$ and letting $(z, z')$ in [23] go to $(-z, -z')$.

4 Conclusions

In this work, we have constructed a new two-parametric deformation of the Lie algebra $gl(2)$. It is shown that this quantum algebra is triangular (Jordanian). Starting from the colour representation of its universal $R$-matrix, we have obtained a new, non-standard solution of the coloured YBE for $GL(2)$. Using this coloured solution we have also explored the quantum group structure associated with it. We have thus obtained a coloured Jordanian quantum group $GL_{h, s}^{(\lambda, \mu)}(2)$ (infinite dimensional) incorporating the one parameter $GL_z(2)$ and the two parameter $GL_{z, z'}(2)$ Jordanian quantum groups as its finite-dimensional subalgebras. It
would be interesting to find out if the Jordanian $R$-matrix and its super extension given in [4] can be regarded as different manifestations of a more general coloured solution.

Acknowledgements

I would like to thank Prof. Edward Letzter for warm hospitality at Texas A & M University.

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