Supergravities: from fields to branes

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Abstract. The quest for unification of particles and fields and for reconciliation of Quantum Mechanics and General Relativity has led us to gauge theories, string theories, supersymmetry and higher-extended objects: membranes. Our spacetime is quantum mechanical but it admits semiclassical descriptions of various “complementary” kinds that could be valid approximations in various circumstances. One of them might be supergravity in 11 dimensions the largest known interacting theory of a finite number of fields with gauged Poincaré supersymmetry. Its solitons and their dual membranes would be states in its quantum version called M-theory. We shall review the construction of its classical action by deformation of a globally supersymmetric free theory and its on-shell superspace formulation. Then we shall focus on the bosonic matter equations of the dimensional reductions on tori of dimensions 1 to 8 to exhibit their common self-duality nature. In the concluding section we shall discuss possible remnants at the quantum level and beyond the massless sector of generalised discrete U-dualities. We shall also comment on the variable dimension of spacetime descriptions and on the possibility of extending the self dual description to spacetime itself and its metric.

1. Extra-dimensions

If one considers extended objects beyond particles one must choose a number of internal dimensions \((t,s)\), ignoring the null case here, as well as the signature of target spacetime \((T,S)\). For a p-brane \((t,s) = (1,p)\). Each type \((t,s)\) object is minimally coupled to a “gauge” potential: a differential form of degree \(s+t\) and is its source. Supersymmetric versions of both the brane worldvolume and target superspace have been considered and allow a geometric description of fermionic theories with the caveat that some of the most interesting theories such as 11 dimensional supergravity are put on shell by the known such descriptions. In the context of local field theories one is limited by the spin 2 restriction namely by the observation that under rather general hypotheses one cannot allow interactions of fields of spin higher than two and it seems difficult even to deal with a finite number of spin two fields namely several gravitons. In september 1977 W. Nahm classified the possible superalgebras that are compatible with this restriction in the linearised approximation and he found assuming a unique time direction the maximal target dimension \((1,10;32)\) where the last figure is the number of odd coordinates for a Poincaré supergravity structure. He also recovered the massless spectra of 10d superstring theories ie of their low energy field theory limit. Nine months earlier Gliozzi, Scherk and Olive had identified the sector of the Ramond Neveu-Schwarz superstring that could be supersymmetric in target space. The decisive jump from the hadronic mass scale to the Planck scale for a fundamental realisation of strings

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as a microscopic theory of gravity had taken place in early 1974 see \[SS74\] and references therein.

In parallel to these developments the construction of theories invariant under rigid supersymmetry in 4 dimensions or under their gauged versions went along and the latter, which turn out to be supergravity theories, were progressively constructed as perturbative expansions starting with the \( N = 1 \) case in other words one Lorentz spinor of 4 (odd) supercharges up to two versions of \( N = 4 \) via the intermediate cases \( N = 2,3 \). The \( N = 8 \) case which has precisely the maximal number of supercharges compatible with the maximal spin 2 restriction has the same number \( 8 \times 4 = 32 \) of supercharges as the 11 dimensional theory mentioned above. It is exactly its toroidal dimensional reduction on a seven dimensional internal torus. For more historical references we refer to \[J01\]. The 7 extra-dimensions here play an ephemeral role and make the internal symmetry partly geometric as dimensional reduction is compactification followed by consistent (with the equations of motion) truncation of the theory to the zeroth Fourier components along the internal dimensions.

To conclude this introduction of extra-dimensions, let us recall that they are imposed on us by string theory or by supersymmetry, the open question is how precisely to extract in a predictive way a low energy 4d approximation. The author started working with 5 and 6 dimensions in 1975 and became really a convert after realizing that the Dirac equation lives virtually in six dimensions; the signature \((T, S)\) must be \((3, 3)\) if one insists on the existence of Majorana-Weyl spinors and of course 3 is a fascinating number. One important question is to decide what is the relevant dimension for a given problem. The choice of an appropriate number of odd (fermionic) dimensions is also important, even bosonic problems may be best analysed by imbedding them in superspace to analyse self-duality equations (or BPS conditions) in their natural setting, we shall comment on these problems in section 4.

2. Deformation of free gauge theories

The construction of 11 dimensional supergravity action still relies on the so-called Noether method \[CJS78\]. It can be seen as a simultaneous deformation of an infinite dimensional abelian gauge algebra equivariant under some global (or rigid) Lie algebra and of one invariant of both algebras constructed out of a given set of fields, typically a set of representations on spacetime or superspace induced from Lorentz representations. This program has been called the Gupta program and we refer to \[FF79\] for early references. The name Noether comes from the fact that the germ of nonlinear deformation of the action is in fact the minimal coupling to the Noether current of the global symmetry one imposes, for instance it is the nonabelian rigid compact Lie subgroup \( G \) of Yang-Mills theory if one starts with the free action for \( \dim G \) abelian vector potentials. This construction is full of ambiguities that sometimes can be eliminated by field redefinitions and which are due for instance to the arbitrariness in the Noether current (the so-called improvement terms). It is also not guaranteed to succeed. This is a typical deformation problem and is of a cohomological nature: the obstructions are some cohomology classes, the ambiguity is a coboundary and the germ itself is a cocycle. Not surprisingly the relevant cohomology is Lie algebra cohomology, as the representations or the (nonlinear) realisations including the invariant action can be combined with the
transformation group itself to form the object one is deforming equivariantly under the rigid symmetry to be preserved.

A notorious example of obstruction is the impossibility to add a cosmological term to 11d supergravity \[ \text{BDHS} \]. It is obvious from the point of view of rigid supersymmetry as there is no corresponding de Sitter algebra as was shown by Nahm already, but starting from 11d Poincaré supergravity one can show also that there is no deformation that leads to a local theory with a cosmological term. This result may not be so surprising as the analogous theory in 4 dimensions, \( N = 2 \) supergravity, deforms only if one carefully adds a well defined gauge coupling to the vector field which has no field theoretic analogue in 11 dimensions.

2.1. Diffeomorphisms. In fact the deformation theory of the linearised diffeomorphisms to the true diffeomorphisms involves some hindsight from differential geometry. One may invoke the existence of a deformed Noether identity to extract the diffeomorphism transformation law at the first order or use a moving frame formalism and appeal to Lorentz covariance to get the wanted result. In fact one generally uses a symmetric energy momentum tensor (for instance of a matter field) to start the deformation which is not the canonical energy-momentum tensor associated to translations as it involves some rotational symmetry information as well. It ought to be possible to clarify this technical difficulty, in any case in practice one uses differential geometry to resum the diffeomorphism deformation and one concentrates on the fermionic terms and matter couplings that are the new features here. In the absence of scalars the result is polynomial and can be found in a few steps. The scalar fields appear multiplied by the gravitational coupling constant in a dimensionless combination so their non-polynomial contribution is somewhat harder to find, this was in fact our main motivation for constructing first the 11d supergravity, the \( N = 8 \) theory in 4d has actually 70 scalar fields and the 10 dimensional IIA supergravity has a scalar dilaton to be dealt with still. Both are toroidal compactifications of 11d SUGRA which does not have any scalar field.

In the general situation one starts from a degenerate, very abelian or very solvable structure and deforms it to a generic and rigid one, an infinite dimensional version of the deformation of a (pseudo-)Euclidean displacement group into a Lorentz or de Sitter group which are indeed simple and rigid. Physicists are maybe more familiar with the converse operation of contraction but this is a singular limit whereas the Gupta-Noether procedure has a formal (sometimes convergent) series expansion and belongs to the rich field of deformation theory.

2.2. Supergravities. In 11 dimensions the massless states are the onshell remnants of the metric, of a third degree gauge potential with what may seem like an abelian gauge invariance and of a spinor valued 1-form that is the gauge field of local supersymmetry. Altogether one has at the linearised level 44+84 bosonic states for each momentum and 128 fermionic degrees of freedom. The free lagrangian is the rather straightforward generalisation of the Rarita-Schwinger lagrangian for a spin 3/2 field in 4d plus the bosonic quadratic terms to be covariantised under diffeomorphisms. The rigid nonabelian supersymmetry gives a Noether current that leads to a cubic coupling, it becomes local by combining an abelian supersymmetry gauge invariance with the global Poincaré supersymmetry transformation rule. The iterative construction can now start, in fact the quadratic fermionic terms in the action are determined together with those in the transformation law of the fermions by asking for diffeomorphism and Lorentz covariance with
the help of some Clifford algebra identities involving actually only the subalgebra $Sp(32, \mathbb{R})$ that preserves the Bargmann hermitising matrix $\Gamma^0$.

Higher order terms come from the requirement of supercovariance. We should stress that the derivation of an off shell action without auxiliary fields (those fields that would be needed to have a true representation of supersymmetry) is delicate because the algebra does not close off shell. The reason is clear, one has eliminated the (still unknown) auxiliary fields by using their equations of motion. For bosonic auxiliary fields the latter are exchanged with the fermionic equations of motion by supersymmetry. Nevertheless the fermionic equations must transform without derivatives of the supersymmetry parameters in other words involve supercovariant derivatives ie be “supercovariant”. This requirement, it turns out, controls all the quartic terms one needs to get an invariant action. We refer to the original paper for standard factors of 2 to get supercovariant equations out of a nonsupercovariant action. The final check of invariance follows from a rather formidable Pauli-Fierz identity somewhat simplified by restriction to the symplectic 32x32 matrices. We refer to a recent extension to supergravity in first order formalism for more formulas, one result of that paper is that beyond 4d the first order formalism (with independent Lorentz connection) is not as useful as in 4 dimensions $^{[JS99]}$. This suggests that the superspace formalism should be more involved as well and the restriction to a Lorentz connection should be relaxed, there are already indications that an abelian gauge group should be added but probably the structure is more subtle.

The superspace formulation of 11d SUGRA was discovered in 1980 by two different groups but it was simplified significantly recently $^{[H97]}$, let us refer also to $^{[CGNN00]}$.

3. Universal instantons

There is a famous connection between self-duality equations and existence of unbroken supersymmetry. Let us recall that the so-called BPS condition started life as a solvable limit of dyon solutions where the similarity between the adjoint Higgs field and a spacelike extra component of the Yang-Mill potential becomes exact, there was no fermion in the picture. It is the stationary version of the famous instantonic self-duality equation of pure Euclidean Yang-Mills theory. Subsequently and case by case a suitable supersymmetric extension of each theory admitting “self-dual” solutions was always constructed in which a Killing spinor ie a covariantly constant spinor can be interpreted as an unbroken supersymmetry of the bosonic background which implies the saturation of the Bogomol’ny bound. The first analysis of this phenomenon was given in $^{[OW78]}$ in the case of rigid supersymmetry. So a bosonic self-duality equation becomes the condition of preservation of some supersymmetry and stability can be reinterpreted as the property of the supersymmetry algebra that some bosonic generators are sums of squares of fermionic ones. In the bosonic case the converse of dimensional reduction has been first coined group disintegration and then oxidation, we are advocating now a superoxidation mechanism.

In a way the next sections address the opposite problem we are going to show that all the bosonic matter equations of toroidally compactified 11d SUGRA can be rewritten as self-duality equations of a generalised but universal type once one doubles the field content. It is a standard procedure in the analysis of differential
systems to introduce auxiliary variables to render the system first order. The non-trivial observation maybe is now that our rather intricate systems are always defined by a finite dimensional superalgebra (ie $\mathbb{Z}_2$-graded Lie algebra) and have a universal form. The occurrence of fermionic symmetries is surprising for bosonic equations but can be understood from the odd character of odd degree gauge potentials like the three form of 11d SUGRA [CJLP98].

We shall call self-duality equation any equation relating some curvatures $\mathcal{F}$ and of the form
\[
\mathcal{F} = \ast S \mathcal{F},
\]
where $S$ is an operator of square plus or minus one that compensates for the same property of the Hodge duality, but more fundamentally $S$ exchanges the generators of the superalgebra associated to gauge potentials and those associated to their duals.

### 3.1. Middle degree

The prototype examples are of course the 4d Maxwell equations written in terms of electric and magnetic potentials with dual field strengths. Similarly in 2d the principal sigma model or more generally the symmetric space sigma models can be rewritten in the above form, at least for the propagating degrees of freedom. We recall that the typical structure is that of a coset space $KG\backslash G$ where $KG$ is the maximal compact subgroup of $G$. There are two descriptions, first the gauge fixed one where one chooses a representative of each coset but the better one restores the $KG$ gauge invariance and allows the symmetry under $G$ to become manifest. In the latter case however the self-duality (in 2d at this stage) involves the components of the field strength orthogonal to $KG$ only
\[
\mathcal{F} = (dg.g^{-1})^\perp.
\]
In fact a harmonic scalar function and its conjugate form a first order self dual pair and one can restore the $SL(2,\mathbb{R})$ invariance subgroup of the 2d conformal group by the same trick.

The main example that led to our discovery of the general structure is the case of the 28 vector potentials of 4d $N=8$ SUGRA that cannot form a representation of the duality symmetry group $G = E_7(7)$ unless one combines them with the 28 (Hodge) duals. The scalar fields in that theory obey the equations of the sigma model $KG\backslash G$ again and the self-duality equation for the vectors reads in that case
\[
g.\mathcal{F} = \ast S_g \mathcal{F},
\]
where $g$ stands for the 56 dimensional matrix representation of $G$ and $S$ has to be an invariant operator for $KG = SU(8)$ [CJ79]. This structure has been extended to the compactifications of 11d SUGRA on a 3-torus and on a 5-torus in [CJLP97] for the field strengths of degree half that of the spacetime volume form.

### 3.2. Self-duality for all forms

From there it was natural to try an extension to all fields, and we succeeded for all bosonic forms leaving aside for the time being the graviton and the fermions. We expect the latter to transform only under the compact subgroup $KG$ and under the Lorentz (spin) group. We are now going to exhibit a vast generalisation of $G$ or at least of its Borel subgroup. This is quite typical of broken symmetries in polynomial situations in which the components of some group element $g$ appear also polynomially in its inverse $g^{-1}$ which occurs also
as we have seen in the equations. The way to permit this is of course nilpotence and this is why the coset spaces appear usually in their Iwasawa parametrisation. One must restore the local $K G$ invariance to have simple formulas for the fermionic couplings and for the full action of $G$. We refer to [CJLP98] for the compactified cases but we shall illustrate our general structure in the 11 dimensional case; the 4-form field strength has a dual that has a non abelian piece. A compact way to encode the equation of motion and the Bianchi identity is to define a supergroup element and its field strength or curvature by

$$\mathcal{E} = e^{\text{exp}(A_3 T) e^{\text{exp}(A'_6 T')}}$$

(3.4)

$$\mathcal{F} = d\mathcal{E} \mathcal{E}^{-1}.$$  

(3.5)

This is a generalised sigma model structure, one pair of generators for each form and its dual, a theory is then specified by the choice of a supergroup law. The action of the involution $S$ is simply the exchange of $T$ and $T'$. 11d SUGRA is defined by the superalgebra

$$\{T, T\}_+ = T'.$$  

(3.6)

4. Conclusion

4.1. Discrete symmetries: arithmetic groups. It is now increasingly plausible that the internal symmetries of the massless sectors of the various compactifications of string theories are continuous versions of discrete groups of symmetries of the full theories. This was argued by A. Sen, P. Townsend, C. Hull, E. Witten and more recently by M. Green and his collaborators. Assuming these duality symmetries does allow us to control nonperturbative effects and sometimes even to resum perturbation series when the space of “modular forms” is small enough. Striking dualities relate different parametrisations, for instance the size of the periodic eleventh dimension of maximal Supergravity emerges as the IIA string coupling constant in 10 dimensions and the Planck lengths are related as usual.

4.2. Doubled spacetime. The most striking property of our self duality results in my opinion is that gravitation that is somehow spectator in 11 dimensions enters the game and fuses completely with matter forms as one descends to 3 spacetime dimensions by toroidal compactification. I have been emphasizing this repeatedly and I consider this as a challenge for my lifetime. I proposed a provocative picture at the conference where the main missing link was precisely a kind of “Doubled spacetime” that would implement the selfduality on spacetime itself and on its geometry. I reproduce a diplomatic version of the picture in fig. 1.

As a conclusion I may just mention that since the conference the Coxeter groups of hyperbolic Kac-Moody algebras exhibited in [J82] have been discovered to control chaos and not only symmetries of the homogeneous reduction of supergravities to one dimension of time. As for the main problem namely the extension of our self-dual formalism to the gravitational sector, little has been established beyond the linear level yet. It is important to keep in mind that the full diffeomorphism groups should appear and not only their linear subgroups which have already led to a rather precise systematics for the extension and overextension of Dynkin diagrams. It is not the place here to list speculative papers or incomplete analyses of facts that indeed make me believe there is a big surprise waiting for us around the corner.
Figure 1. The 2000 version of a theoretical cathedral

| Theory      | M−−−−− |
|-------------|--------|
| D−Brane     | D−space? |
| Anomalies   | Sustrings |
| Wick        | Strings |
| Einstein    | Relativities |
| Poincaré    | Lorentz |
| Newton      | Gravity |
| GEOMETRY/ | QUANTA |

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