A note on sum-product estimates over finite valuation rings

by

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1. Introduction. Let \( A \) be a subset of integers. We define the sum set and the product set by

\[
A + A = \{ a + b : a, b \in A \}, \quad A \cdot A = \{ a \cdot b : a, b \in A \}.
\]

In 1983, Erdős and Szemerédi \[2\] proved that there is no set \( A \) which is both highly additively structured and multiplicatively structured. More precisely, they proved that

\[
\max\{|A + A|, |A \cdot A|\} \geq |A|^{1+c}
\]

for some universal constant \( c > 0 \).

Over the last twenty years, there has been intensive progress in improving the constant \( c \) and in studying variants in different settings, for example, over finite fields and the complex numbers. We refer the interested reader to \[4, 12, 11, 13, 14\] and references therein for more details.

Let \( R \) be a finite valuation ring, i.e. a finite, local and principal ring (see \[8\] or Section 2). The first result on sum-product type problems in the context of finite valuation rings was given by Ham, Pham and Vinh \[5\]. In particular, they proved the following theorems.

**Theorem 1.1** (Ham–Pham–Vinh, \[5\]). Let \( R \) be a finite valuation ring of order \( q^r \), and \( R^* \) the set of units of \( R \). Let \( G \) be a subgroup of \( R^* \), and let \( f(x, y) = g(x)(h(x) + y) \) be defined on \( G \times R^* \), where \( g, h : G \to R^* \) are arbitrary functions. Put \( m = \max_{t \in R} |\{ x \in G : g(x)h(x) = t \}| \). For any sets \( A \subset G \) and \( B, C \subset R^* \), we have

\[
|f(A, B)||B \cdot C| \gg \min \left\{ \frac{q^r|B|}{m}, \frac{|A||B|^2|C|}{m^2 q^{2r-1}} \right\}.
\]

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Theorem 1.2 (Ham–Pham–Vinh, [5]). Let $\mathcal{R}$ be a finite valuation ring of order $q^r$, and $\mathcal{R}^*$ denote the set of units of $\mathcal{R}$. Let $G$ be a subgroup of $\mathcal{R}^*$, and let $f(x, y) = g(x)(h(x) + y)$ be defined on $G \times \mathcal{R}^*$, where $g, h : G \to \mathcal{R}^*$ are arbitrary functions. Put $m = \max_{t \in \mathcal{R}} |\{x \in G : g(x) = t\}|$. For any sets $A \subset G$ and $B, C \subset \mathcal{R}^*$, we have

$$|f(A, B)||B + C| \gg \min\left\{\frac{q^r|B|}{m}, \frac{|A||B|^2|C|}{m^2q^{2r-1}}\right\}.$$  

Here and throughout, $X \gg Y$ means that there exists a positive constant $c$ such that $X \geq cY$, and “$\ll$” is defined in a similar way. We also write $X \sim Y$ if $X \gg Y$ and $Y \gg X$. The above results are generalizations of earlier results due to Hegyvári and Hennecart [6] in the prime field setting. A prime field version for small sets can also be found in the work of Mojarrad and Pham [7].

For any $A \subset \mathcal{R}^*$, it follows from Theorem 1.2 with $g(x) = x$ and $h(x) \equiv 0$ that

$$|A \cdot A| |A + A| \gg \min\left\{q^r|A|, \frac{|A|^4}{q^{2r-1}}\right\}.$$  

Thus,

$$\max\{|A + A|, |A \cdot A|\} \gg \min\left\{q^{r/2}|A|^{1/2}, \frac{|A|^2}{q^{(2r-1)/2}}\right\}.$$  

For $A \subset \mathcal{R}$, we define $A^2 := \{x^2 : x \in A\}$. In a recent work, Yazici [15] studied another sum-product type estimate. Namely, she proved the following theorem.

Theorem 1.3 (Yazici, [15]). Let $\mathcal{R}$ be a finite valuation ring of $q^r$ and $A$ be a subset of $\mathcal{R}$. If $|A + A||A|^2 \gg q^{3r-1}$, then

$$\max\{|A + A|, |A^2 + A^2|\} \gg q^{r/4}|A|^{3/4}.$$  

It follows from Theorem 1.3 that either the size of $A + A$ or $A^2 + A^2$ is large when $|A|$ is large enough. In this paper, we provide two improvements of this result. Our first result is stated as follows.

Theorem 1.4. Let $\mathcal{R}$ be a finite valuation ring of order $q^r$ with $q$ a power of an odd prime number. For $A \subset \mathcal{R}$ with $|A| \geq 2q^{r-1}$, we have

$$\max\{|A + A|, |A^2 + A^2|\} \gg \min\left\{q^{r/2}|A|^{1/2}, \frac{|A|^2}{q^{(2r-1)/2}}\right\}.$$  

One can check that Theorem 1.4 improves Theorem 1.3 when $|A| \gg q^{r-1/3}$. Indeed, it is clear that

$$\min\left\{q^{r/2}|A|^{1/2}, \frac{|A|^2}{q^{(2r-1)/2}}\right\} = q^{r/2}|A|^{1/2}.$$  

Thus, under the condition $|A| \gg q^{r-1/3}$, the conclusion of Theorem 1.4 is stronger than that of Theorem 1.3. We also need to compare the assumptions of these two theorems. Obviously, the assumption $|A| \gg q^{r-1/3}$ implies the condition $|A + A| |A|^2 \gg q^{3r-1}$ by using the fact that $|A + A| \geq |A|$. When $|A + A| |A|^2 \gg q^{3r-1}$ and $|A| \ll q^{-1/3}$, we have another improvement of Theorem 1.3, which is a consequence of the following theorem.

Theorem 1.5. Let $\mathcal{R}$ be a finite valuation ring of order $q^r$ with $q$ a power of an odd prime number. For $A \subset \mathcal{R}$ with $|A| \geq 2q^{r-1}$ and $|A + A| |A|^2 \gg q^{3r-1}$, we have

$$\max\{|A + A|, |A^2 + A^2|\} \gg q^{r/3}|A|^{2/3}.$$ 

It is convenient to make a brief comparison of these theorems. It follows from Theorem 1.4 that if $|A| \ll q^{r-1/3}$, then $\max\{|A + A|, |A^2 + A^2|\} \gg q^{(1-2r)/2}|A|^2$, which is better than the bound $q^{r/4}|A|^{3/4}$ of Theorem 1.3 whenever $|A| \gg q^{-2/5}$, and weaker than the threshold $q^{r/3}|A|^{2/3}$ of Theorem 1.5 whenever $|A| \ll q^{-3/8}$. Thus, in the range $q^{-3/8} \ll |A| \ll q^{-1/3}$, the bound $|A|^2 / q^{(2r-1)/2}$ is the best. If $|A + A| |A|^2 \gg q^{3r-1}$ and $2q^{r-1} \leq |A| \ll q^{-3/8}$, then the lower bound of Theorem 1.3 is the strongest. In other words, we can summarize the bounds in the following corollary.

Corollary 1.6. Let $\mathcal{R}$ be a finite valuation ring of order $q^r$ with $q$ a power of an odd prime number, and let $A \subset \mathcal{R}$.

1. If $|A| \gg q^{r-1/3}$, then
   $$\max\{|A + A|, |A^2 + A^2|\} \gg q^{r/2}|A|^{1/2}.$$ 

2. If $q^{-3/8} \ll |A| \ll q^{-1/3}$, then
   $$\max\{|A + A|, |A^2 + A^2|\} \gg \frac{|A|^2}{q^{(2r-1)/2}}.$$ 

3. If $|A + A| |A|^2 \gg q^{3r-1}$ and $2q^{r-1} \leq |A| \ll q^{-3/8}$, then
   $$\max\{|A + A|, |A^2 + A^2|\} \gg q^{r/3}|A|^{2/3}.$$ 

Furthermore, we have a remark on the last statement of Corollary 1.6. If $|A + A| |A|^2 \gg q^{3r-1}$ and $|A| \ll q^{-3/8}$, then $|A + A| \gg |A|^{1/2}$. Thus $A + A$ is an expanding set. However, the lower bound $q^{r/3}|A|^{2/3}$ for $\max\{|A + A|, |A^2 + A^2|\}$ is stronger whenever $|A| \ll q^{-3/8}$. We refer the interested readers to [10] for related sum-product results in the finite ring setting.

The main difference between our method and that of Yazici is that she used the Plünnecke–Ruzsa inequality. Instead, to prove Theorems 1.3 and 1.4, we will use spectral graph theory techniques and some ideas from the work of Pham, Vinh, and De Zeeuw [10, Theorem 1.3]. Our method can be easily extended to the case of higher dimensions. In particular, we obtain
the following main results which are extensions of Theorems 1.4 and 1.5 respectively. (Here, for a positive integer \( n \), we write \( n\mathcal{A}^2 \) for the set of all elements of the form \( a_1 + \cdots + a_n \) with \( a_1, \ldots, a_n \) in \( \mathcal{A}^2 \).)

**Theorem 1.7.** Let \( \mathcal{R} \) be a finite valuation ring of order \( q^r \) with \( q \) a power of an odd prime number. For any \( \mathcal{A} \subset \mathcal{R} \) with \( |\mathcal{A}| \geq 2q^{r-1} \) and for any integer \( n > 1 \), we have

\[
\max\{|n\mathcal{A}^2|, |\mathcal{A} + \mathcal{A}|\} \gg \min\left\{ q^{r/n}|\mathcal{A}|^{(n-1)/n}, \frac{|\mathcal{A}|^{(3n-2)/n}}{q^{(n-1)(2r-1)/n}} \right\}.
\]

**Theorem 1.8.** Let \( \mathcal{R} \) be a finite valuation ring of order \( q^r \) with \( q \) a power of an odd prime number. For any \( \mathcal{A} \subset \mathcal{R} \) with \( |\mathcal{A}| \geq 2q^{r-1} \) and for any integer \( n > 1 \), we have

\[
\max\{|\mathcal{A} + \mathcal{A}|, |n\mathcal{A}^2|\} \gg q^{r+2n-1}|\mathcal{A}|^{2n-2}
\]

whenever \( |\mathcal{A} + \mathcal{A}|^{n-1}|\mathcal{A}|^n \gg q^r+(n-1)(2r-1) \).

2. The definition of finite valuation rings. We start by recalling from \([8]\) the definition of finite valuation rings. Let \( \mathcal{R} \) be a commutative ring with identity. The ring \( \mathcal{R} \) is called *local* if it has a unique maximal ideal, and \( \mathcal{R} \) is *principal* if its ideals are all principal, i.e. generated by a single element in \( \mathcal{R} \).

**Definition 2.1.** A commutative ring with identity is called a *finite valuation ring* if it finite, local and principal.

Let \( \mathcal{R} \) be a finite valuation ring. Then \( \mathcal{R} \) has a unique maximal ideal that contains all proper ideals of \( \mathcal{R} \). This implies that there exists a non-unit element \( z \) called a *uniformizer* in \( \mathcal{R} \) such that the maximal ideal is generated by \( z \). Note that the uniformizer is defined up to a unit in \( \mathcal{R} \). We also denote by \( \mathcal{R}^* \) and \( \mathcal{R}^0 \) the sets of units and of non-units in \( \mathcal{R} \), respectively.

There are two structural parameters associated to \( \mathcal{R} \): the cardinality of the residue field \( F = \mathcal{R}/(z) \), and the nilpotency degree of \( z \), which is the smallest positive integer \( r \) such that \( z^r = 0 \). Denote by \( q \) the cardinality of \( F \). It is known that \( q \) is a power of a prime number. Note that if \( q \) is odd then \( 2 \) is a unit in \( \mathcal{R} \), i.e. \( 2 \in \mathcal{R}^* \).

If \( \mathcal{R} \) is a finite valuation ring, and \( r \) is the nilpotency degree of \( z \), then one has a natural valuation

\[
\nu : \mathcal{R} \to \{0, 1, \ldots, r\}
\]
defined by \( \nu(0) = r \) and for \( x \neq 0 \), \( \nu(x) = k \) if \( x \in (z^k) \setminus (z^{k+1}) \). In other words, \( \nu(x) = k \) if and only if \( x = uz^k \) for some unit \( u \) in \( \mathcal{R} \). For any vector \( (x_1, \ldots, x_d) \) in \( \mathcal{R}^d \), we define

\[
\nu((x_1, \ldots, x_d)) := \min_{1 \leq i \leq d} \nu(x_i).
\]
Each abelian group \((z^k)/(z^{k+1})\) is a one-dimensional linear space over the residue field \(F = \mathcal{R}/(z)\), thus its size is \(q\). This implies that \(|(z^k)| = q^{r-k}\) for \(k = 0, 1, \ldots, r\). In particular, \(|(z)| = q^{r-1}\), \(|\mathcal{R}| = q^r\) and \(|\mathcal{R}^*| = |\mathcal{R}| - |(z)| = q^r - q^{r-1}\). We refer the readers to [8] for more details. The following are some examples of finite valuation rings

- Finite fields \(\mathbb{F}_q\) with \(q\) a power of a prime.
- Finite cyclic rings \(\mathbb{Z}_{p^r}\), where \(p\) is a prime.
- \(\mathcal{O}/(p^r)\), where \(\mathcal{O}\) is the ring of integers in a number field and \(p \in \mathcal{O}\) is a prime.
- \(\mathbb{F}_q[x]/(f^r)\), where \(f \in \mathbb{F}_q[x]\) is an irreducible polynomial.

### 3. Techniques from spectral graph theory.

We say that a bipartite graph \(G = (A \cup B, E)\) is biregular if in each of its two parts, all vertices have the same degree. If \(A\) is one of the two parts, we write \(\text{deg}(A)\) for the common degree of the vertices in \(A\). Label the eigenvalues of \(G\) so that \(|\lambda_1| \geq \cdots \geq |\lambda_n|\).

Note that in a bipartite graph, we have \(\lambda_2 = -\lambda_1\). In fact, let \(G\) be a bipartite graph. Since the adjacency matrix of \(G\) has the form \(\begin{bmatrix} 0 & S \\ S^T & 0 \end{bmatrix}\), the spectrum of \(G\) is symmetric with respect to 0, i.e. if \(\begin{bmatrix} u \\ v \end{bmatrix}\) is an eigenvector with eigenvalue \(\lambda\) then \(\begin{bmatrix} u \\ -v \end{bmatrix}\) is an eigenvector with eigenvalue \(-\lambda\). Therefore, \(\lambda_2 = -\lambda_1\).

The following is the expander mixing lemma for bipartite graphs whose detailed proof was given in [3, Lemma 4.8, p. 44].

**Lemma 3.1.** Let \(G\) be a bipartite graph with parts \(A, B\) such that all the vertices in \(A\) have degree \(a\) and all the vertices in \(B\) have degree \(b\). For any subsets \(X \subset A\) and \(Y \subset B\), let \(e(X,Y)\) denote the number of edges between \(X\) and \(Y\). Then

\[
|e(X,Y) - \frac{a}{|B|}|X||Y| | \leq \lambda_3 \sqrt{|X||Y|},
\]

where \(\lambda_3\) is the third eigenvalue of \(G\).

### 3.1. Erdős–Rényi graphs over finite valuation rings.

Let \(d \geq 2\) be an integer. For any \(x \in \mathcal{R}^d \setminus (\mathcal{R}^0)^d\), where \(\mathcal{R}^0\) is the set of non-units in \(\mathcal{R}\), denote by \([x]\) the equivalence class of \(x\) in \(\mathcal{R}^d \setminus (\mathcal{R}^0)^d\), where \(x, y \in \mathcal{R}^d \setminus (\mathcal{R}^0)^d\) are equivalent if and only if \(x = ty\) for some \(t \in \mathcal{R}^*\). Let \(\mathcal{E}_{q,d}(\mathcal{R})\) denote the Erdős–Rényi bipartite graph \(\mathcal{E}_{q,d}(\mathcal{R}) = (A \cup B, E)\) where the vertices in each part are the points of the projective space over \(\mathcal{R}\), and two vertices \([x]\) and \([y]\) are connected if and only if \(x \cdot y = 0\). The following theorem on the spectrum of \(\mathcal{E}_{q,d}(\mathcal{R})\) was given in [8].
Theorem 3.2 (Nica, [8]). The cardinality of each part of $E_{q,d}(\mathcal{R})$ is $q^{(d-1)(r-1)}/(q-1)$ and $\deg(A)=\deg(B)=q^{(d-2)(r-1)}/(q-1)$. The third eigenvalue of $E_{q,d}(\mathcal{R})$ is at most $\sqrt{q^{(d-2)(2r-1)}}$.

4. Proof of Theorem 1.7. Since $|A| \geq 2q^{-r}$, it follows that

$$|A \cap \mathcal{R}^*| \geq |A| - |\mathcal{R}^0| = |A| - q^{-r} \geq |A|/2.$$ 

Thus we may assume that $A$ is a subset of $\mathcal{R}^*$, where $\mathcal{R}^*$ is the set of units in $\mathcal{R}$. We now prove that the size of $A^2$ is at least $\gg |A|$. Indeed, suppose $x^2 = y^2$ with $x, y \in A$. Then $(x - y)(x + y) = 0$. There are the following possibilities for the pair $(x, y)$: $x = y$, or $x = -y$, or $x - y \in (z)$ and $x + y \in (z)$. If the last case holds, then we can write $x - y = u_1z_1^k$ and $x + y = u_2z_2^k$ with $u_1, u_2 \in \mathcal{R}^*$ and some positive integers $k_1, k_2$. This leads to $2x = u_1z_1^k + u_2z_2^k \in (z)$, which gives a contradiction since both 2 and $x$ are in $\mathcal{R}^*$ and $q$ is odd. In other words, either $x = y$ or $x = -y$ and therefore $|A^2| \gg |A|$.

Define $\mathcal{D} = nA^2$. Consider the equation

$$x + (b_1 - c_1)^2 + \cdots + (b_{n-1} - c_{n-1})^2 = t,$$

where $x \in A^2$, $b_i \in A + A$, $c_i \in A$, $1 \leq i \leq n - 1$. Let $N$ be the number of solutions of this equation. We first see that $N \geq |A|^{2n-1}$. Let $U$ and $V$ be two vertex sets of the Erdős–Rényi graph $E_{q,n+1}(\mathcal{R})$ defined by

$$U := \left\{ \left( -2b_1, \ldots, -2b_{n-1}, \sum_{i=1}^{n-1} b_i^2 + x, 1 \right) : b_i \in A + A, 1 \leq i \leq n - 1, x \in A^2 \right\},$$

$$V := \left\{ \left( c_1, \ldots, c_{n-1}, 1, \sum_{i=1}^{n-1} c_i^2 - t \right) : c_i \in A, 1 \leq i \leq n - 1, t \in \mathcal{D} \right\}.$$ 

We have $|U| \sim |A + A|^{n-1}|A|$ and $|V| = |A|^{n-1} |\mathcal{D}|$. It is not hard to check that $N$ is bounded by the number of edges between $U$ and $V$ in the graph $E_{q,n+1}(\mathcal{R})$. Thus, one can apply Lemma 3.1 and Theorem 3.2 to get

$$N \ll \frac{|U||V|}{q^r} + q^{(n-1)(2r-1)/2} \sqrt{|U||V|} = \frac{|A + A|^{n-1}|A|^{n}nA^2}{q^r} + q^{(n-1)(2r-1)/2} \sqrt{|A + A|^{n-1}|A|^{n}nA^2}.$$ 

Using the fact that $N \geq |A|^{2n-1}$, we obtain

$$\max\{|nA^2|, |A + A|\} \gg \min \left\{ q^{r} |A|^{n-1}, \frac{|A|^{(3n-2)/n}}{q^{(n-1)(2r-1)/n}} \right\}.$$ 

This completes the proof of the theorem.
5. **Proof of Theorem 1.8.** Since $|A| \geq 2q^{r-1}$, as in the proof of Theorem 1.7 we may assume that $A$ is a subset of $R^*$. In this proof, we will follow the idea of [10, Theorem 1.3].

As in the proof of Theorem 1.7, we define $D = nA^2$. Let $N$ be the number of solutions of equation (1). We see that $N \geq |A|^{2n-1}$. By the Cauchy–Schwarz inequality, one has

$$N^2 \leq |D| \cdot E,$$

where $E$ is the number of tuples

$$(x, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}, y, d_1, \ldots, d_{n-1}, e_1, \ldots, e_{n-1})$$

satisfying

$$x + (b_1 - c_1)^2 + \cdots + (b_{n-1} - c_{n-1})^2 = y + (d_1 - e_1)^2 + \cdots + (d_{n-1} - e_{n-1})^2.$$

Let $U$ and $V$ be two vertex sets in the Erdős–Rényi graph $\mathcal{E}_{q,2n}(R)$ defined by

$$U := \left\{ \left( -2b_1, \ldots, -2b_{n-1}, 2d_1, \ldots, 2d_{n-1}, 1, \sum_{i=1}^{n-1} b_i^2 - \sum_{i=1}^{n-1} d_i^2 + x \right) : b_i \in A + A, d_i \in A, x \in A^2 \right\}$$

$$V := \left\{ \left( c_1, \ldots, c_{n-1}, e_1, \ldots, e_{n-1}, 1, \sum_{i=1}^{n-1} c_i^2 - \sum_{i=1}^{n-1} e_i^2 - y \right) : c_i \in A, e_i \in A + A, y \in A^2 \right\}.$$ 

We have $|U| = |V| \sim |A + A|^{n-1}|A|^n$. Moreover, $E$ is bounded by the number of edges between $U$ and $V$ in the graph $\mathcal{E}_{q,2n}(R)$. Therefore, it follows from Lemma 3.1 and Theorem 3.2 that

$$E \ll \frac{|A + A|^{2n-2}|A|^{2n}}{q^r} + q^{(n-1)(2r-1)}|A + A|^{n-1}|A|^n.$$ 

Using the facts that $N \geq |A|^{2n-1}$ and $N^2 \leq |D| \cdot E$, we derive

$$\max\{|A + A|, |nA^2|\} \gg q^{\frac{n-r}{2n-1}}|A|^{\frac{2n-2}{2n-1}}$$

whenever $|A + A|^{n-1}|A|^n \gg q^{r+(n-1)(2r-1)}$. In particular,

$$\max\{|A + A|, |nA^2|\} \gg \min\left\{ q^{\frac{r}{2n-1}}|A|^{\frac{2n-2}{2n-1}}, \frac{|A|^{(3n-2)/n}}{q^{(n-1)(2r-1)/n}} \right\},$$

which ends the proof of the theorem. 

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