NEW PROOFS OF TWO $q$-ANALOGUES OF KOSHY’S FORMULA

EMMA YU JIN AND MARKUS E. NEBEL

Abstract. In this paper we prove a $q$-analogue of Koshy’s formula in terms of the Narayana polynomial due to Lassalle, and a $q$-analogue of Koshy’s formula in terms of $q$-hypergeometric series due to Andrews, by applying the inclusion-exclusion principle on Dyck paths and on partitions. We generalize these two $q$-analogues of Koshy’s formula for $q$-Catalan numbers to that for $q$-Ballot numbers. This work also answers an open question by Lassalle and two questions raised by Andrews in 2010. We conjecture that if $n$ is odd, then for $m \geq n \geq 1$, the polynomial $(1 + q^n)\left[\frac{m-n-1}{q}\right]$ is unimodal. If $n$ is even, for any even $j \neq 0$ and $m \geq n \geq 1$, the polynomial $(1+q^n)|j|q^{m-n-1}$ is unimodal. This implies the answer to the second problem posed by Andrews.

1. Introduction and background

Let $C_n = \frac{1}{n+1} \binom{2n}{n}$ be the $n$-th Catalan number. A recursive formula for Catalan numbers is given by Koshy as follows [8]:

\[ \sum_{r=0}^{n} (-1)^r \binom{n-r+1}{r} C_{n-r} = 0. \]

(1.1)

Since this is a hypergeometric sum identity, it can be proved by Zeilberger’s creative telescoping method [14]. Furthermore, Koshy’s formula (1.1) follows immediately from the orthogonality of a special Fibonacci polynomial due to Cigler [5,6]. However, these approaches do not provide structural insight; to this end a combinatorial proof based on the inclusion-exclusion principle is needed. Our proof of (1.1) in the context of labeled elevated Dyck paths is exactly of this kind. Here, a Dyck path of length $2n$ is a 2-dimensional lattice path from $(0,0)$ to $(2n,0)$ with allowed steps $(1,1)$ ($U$-step) and $(1,-1)$ ($D$-step) that never goes below the $x$-axis. A Dyck path $p$ is an elevated Dyck path if $\bar{p} = U \bar{p} D$ where $\bar{p}$ is a Dyck path. We use $D_n$ to denote the set of elevated Dyck paths of length $2n + 2$. Then $|D_n| = C_n$. We denote by $P$ the peak $UD$. We call $h$ an up-peak if $h = UUD$. Let $UP(p)$ (resp. $U(p)$) be the set of up-peaks (resp. $U$-steps) contained in a path $p$. We denote by $(\mathcal{M}^s, j, D_n)$ the set of elevated Dyck paths $p \in D_n$ having exactly $j$ elements from the set $\mathcal{M}(p)$ labeled by $s$. We set $a_{n,j}(\mathcal{M}) = |(\mathcal{M}^s, j, D_n)|$. For the case $\mathcal{M} = UP$, the number $a_{n,j}(UP)$ can be counted in two different ways.

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First, there is a bijection between the set \( (\mathcal{UP})^s, j, D_n \) and the set \( (\mathcal{U}^s, j, D_{n-j}) \). For a given path \( p \in D_n \) that has \( j \) up-peaks labeled by \( s \), we can get a unique path \( p' \in D_{n-j} \) that has \( j \) \( U \)-steps labeled by \( s \) if we remove the peak \( P \) from each labeled up-peak. As a result, we have

\[
(1.2) \quad a_{n,j}(\mathcal{UP}) = |(\mathcal{UP})^s, j, D_n| = |\mathcal{U}^s, j, D_{n-j}| = \binom{n-j+1}{j} C_{n-j}.
\]

Second, we set \( g_{n,m} = \{|p \in D_n : |\mathcal{UP}(p)| = m \| \}. \) For a given elevated Dyck path \( p \in D_n \) such that \( |\mathcal{UP}(p)| = m \), there are \( \binom{m}{j} \) ways to label \( j \) up-peaks by \( s \), which leads to \( a_{n,j}(\mathcal{UP}) = \sum_{m \geq j} \binom{m}{j} g_{n,m}. \) In combination with (1.2), we obtain

\[
(1.3) \quad \sum_{m \geq j} \binom{m}{j} g_{n,m} = a_{n,j}(\mathcal{UP}) = \binom{n-j+1}{j} C_{n-j}.
\]

Let \( G_n(q) \) be the ordinary generating function of \( g_{n,m} \), i.e., \( \sum_{m \geq 1} g_{n,m} q^m = G_n(q) \). Then from (1.3), we can derive

\[
(1.4) \quad G_n(q) = \sum_{j \geq 0} \frac{G_n^{(j)}}{j!} (q-1)^j = \sum_{j \geq 0} \binom{n-j+1}{j} C_{n-j} (q-1)^j,
\]

where \( G_n^{(j)}(1) = \frac{\partial^j G_n(q)}{\partial q^j} \big|_{q=1} \). By setting \( q = 0 \) on both sides of (1.4), Koshy’s formula (1.1) follows. Our proof uses the generating function approach to illustrate the inclusion-exclusion principle; see Chapter 2.3 of [12] and Chapter 4.2 of [13]. Here we will use this approach to prove (1.6) and (1.7). We can also prove the sieving identity by a direct application of the inclusion-exclusion method. To make this point clear, we will prove (2.3) in both ways in Section 2. Next we give two \( q \)-analogues of Koshy’s formula due to Lassalle and Andrews [3,10]. The Narayana number \( N_{n,k} \) and the Narayana polynomial \( N_n(q) \) are defined by

\[
(1.5) \quad N_{n,k} = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}, \quad N_n(q) = \sum_{k=1}^n N_{n,k} q^{k-1},
\]

where \( N_{n,k} \) counts the number of Dyck paths of length \( 2n \) that have \( k \) peaks \( P \). By utilizing a \( \lambda \)-identity of complete functions, Lassalle [10] proved for \( n \geq 1 \),

\[
(1.6) \quad N_n(q) = (1-q)^{n-1} + q \sum_{k=1}^{n-1} N_{n-k}(q) \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} (1-q)^{k-m-1}.
\]

Furthermore, Andrews gave another \( q \)-analogue of Koshy’s formula in terms of \( q \)-hypergeometric series. Here we adopt the standard notations of \( q \)-series, i.e.,

\[
[n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q,
\]

\[
(x; q)_n = (1-x)(1-qx) \cdots (1-q^{n-1}x), \quad [m]_q! = [n]_q! [m-n]_q! = (q; q)_m \cdot (q; q)_n (q; q)_{m-n},
\]

\[
C_n(q) = \frac{1}{[n+1]_q} \binom{2n}{n}_q.
\]
Then the before-mentioned $q$-analogue of Koshy’s formula is given by

\begin{equation}
C_n(q) = \sum_{r=1}^{n} (-1)^{r-1} T_r(n, q),
\end{equation}

and

\begin{equation}
T_r(n, q) = q^{r^2-r} \frac{(-q^{n-r+1}; q)_r}{(-q; q)_r} \cdot \left[\begin{array}{c} n-r+1 \\ r \end{array}\right]_q \cdot C_{n-r}(q).
\end{equation}

Andrews [3] raised open questions about $T_r(n, q)$, which are:

1. Is $T_r(n, q)$ a polynomial in $q$?
2. Does $T_r(n, q)$ only have nonnegative coefficients of $q$ if $n \geq 2r$?
3. Does $T_{r-1}(2r-1, -q)$ only have nonnegative coefficients?
4. What is the partition-theoretic combinatorial interpretation of $T_r(n, q)$ for $n \geq 2r$, what is that of $T_{r-1}(2r-1, -q)$, and what is the sieving process on the partitions to eliminate all the non-Catalan partitions?

In Section [4] we completely answer (1) and (3), and show (2) for even $n$. Furthermore, in Sections 2 and 5 we prove (1.6) and (1.7) by the inclusion-exclusion method. In Sections 3 and 6 we generalize (1.6) and (1.7) for $N_n(q)$ and $C_n(q)$ to that for Ballot numbers $B_{n,r}$ and $q$-Ballot numbers $B_r(n, q)$. Finally we conjecture that if $n$ is odd, then for $n \geq n \geq 1$, the polynomial $(1 + q^n) [m \choose n-1]_q$ is unimodal. If $n$ is even, for any even $j \neq 0$ and $m \geq n \geq 1$, the polynomial $(1 + q^n)[j]_q [m \choose n-1]_q$ is unimodal. This would answer question (2) from above for odd $n$.

2. Proof of (1.6)

Before we prove (1.6), we give some main ingredients needed in the proof.

First we introduce the notions of a tower and a colored tower for an elevated Dyck path. We will choose to label the colored towers by $s$ in order to apply the inclusion-exclusion method. Recall that $D_n$ is the set of elevated Dyck paths of length $2n + 2$. For any $p \in D_n$, we will use $U^j D^j \subseteq p$ to express the fact that $p$ contains $i$ consecutive $U$-steps that are followed by $j$ consecutive $D$-steps. We call $t$ a tower of height $i$ contained in an elevated Dyck path $p$, if $t = U^j D^j \subseteq p$ and $U t D \not\subseteq p$. For any elevated Dyck path $p \in D_n$, let $T^*(p)$ be the set of towers contained in $p$. We will color the towers in $T^*(p)$ in the following way for any $p \in D_n$:

**Step 1.** Color all the towers $t \subseteq p$ that immediately follow a $U$-step.

**Step 2.** Color all the towers $t \subseteq p$ that immediately follow an uncolored tower $t' \subseteq p$ after Step 1.

We use $T(p) \subseteq T^*(p)$ to denote the set of towers contained in a path $p$ which are colored according to Steps 1 and 2. We call every $t$ in the set $T(p)$ a colored tower. We set $T^c(p) = T^*(p) - T(p)$ and let $T_2(p) \subseteq T(p)$ (resp. $T_1(p) \subseteq T(p)$) be the set of colored towers of height $\geq 2$ (resp. height $= 1$). As an example, consider the path $p = U^3 D^2 U D U D^2 \in D_4$. The colored towers in the set $T(p)$ are depicted using double lines in the rightmost elevated Dyck path of Figure 1. After Step 1, the first tower in the left-to-right order is colored. The second tower remains uncolored after Step 1, and therefore after Step 2, the third tower is colored.

Second we will use some one-to-one correspondences between two sets of labeled elevated Dyck paths. If there is a bijection between two sets $A$ and $B$, we write...
A \simeq B. Recall that \((\mathcal{M}^s, j, D_n)\) is the set of elevated Dyck paths \(p \in D_n\) having exactly \(j\) elements from the set \(\mathcal{M}(p)\) labeled by \(s\). We consider the set of elevated Dyck paths \(p \in D_n\) that have \(m\) colored towers labeled by \(s\), which is the particular case \(\mathcal{M} = \mathcal{T}\) and \(j = m\). Since each colored tower has height 1 or at least 2, we have \((\mathcal{T}^s, m, D_n) = ((\mathcal{T}_1 \cup \mathcal{T}_2)^s, m, D_n)\). Let \(\mathcal{U}^2(p)\) be the set of \(UU\)-steps of an elevated Dyck path \(p \in D_n\). We will show

**Lemma 2.1.** \((\mathcal{T}_1^s, 1, D_n) \simeq ((\mathcal{U}^2 \cup \mathcal{T}^c)^s, 1, D_{n-1}), (\mathcal{T}_2^s, 1, D_n) \simeq (\mathcal{T}^s, 1, D_{n-1})\) and \((\mathcal{T}^s, m, D_n) \simeq (\mathcal{U}^s, m, D_{n-m})\) for any \(m, 1 \leq m \leq n\).

**Proof.** We use \(U^s, U^sU^s, \) and \(t^s\) to represent a \(U\)-step, a \(UU\)-step, and a tower labeled by \(s\). For a path \(p \in D_n\) that has only one colored tower \(t\) labeled by \(s\), if the labeled and colored tower \(t\) has height 1, i.e., \(t \in \mathcal{T}_1(p)\), then \(t\) is either located between two \(U\)-steps, i.e., \(UT U \subseteq p\), or \(t\) follows an uncolored tower \(t_1\) from \(\mathcal{T}^c(p)\), i.e., \(t_1t^s \subseteq p\) where \(t_1 \in \mathcal{T}^c(p)\). Let \(g_1 : (\mathcal{T}_1^s, 1, D_n) \rightarrow ((\mathcal{U}^2 \cup \mathcal{T}^c)^s, 1, D_{n-1})\) be the map defined as follows. If \(p = \cdots UT U \cdots \in (\mathcal{T}_1^s, 1, D_n)\), then \(g_1(p) = \cdots U^sU^s \cdots \in (\mathcal{U}^2)^s, 1, D_{n-1}).\) If \(p = \cdots t_1t^s \cdots \in (\mathcal{T}_1^s, 1, D_n)\), then \(g_1(p) = \cdots t^s \cdots \in (\mathcal{T}^c)^s, 1, D_{n-1}).\) Notice that the tower \(t_1\) is also an uncolored tower of the labeled path \(g_1(p)\). This follows from the way we define the colored towers. It is obvious that the map \(g_1\) is a bijection.

For a path \(p \in D_n\) that has only one colored tower \(t\) labeled by \(s\), if the labeled and colored tower \(t\) has height at least 2, i.e., \(t \in \mathcal{T}_2(p)\), then \(t = UT_2D\) where \(t_2\) is a tower. Let \(g_2 : (\mathcal{T}_2^s, 1, D_n) \rightarrow (\mathcal{T}^s, 1, D_{n-1})\) be the map defined as follows. If \(p = \cdots t^s \cdots \cdots \cdots (UT_2D)^s \cdots \in (\mathcal{T}_2^s, 1, D_n)\), then \(g_2(p) = \cdots t^s \cdots \in (\mathcal{T}^s, 1, D_{n-1}).\) Notice that the tower \(t_2\) is also a colored tower of the labeled path \(g_2(p)\). This follows from the way we define the colored towers. It is clear that the map \(g_2\) is a bijection.

We will next prove \((\mathcal{T}^s, m, D_n) \simeq ((\mathcal{U}^2 \cup \mathcal{T} \cup \mathcal{T}^c)^s, m, D_{n-m})\) for any \(m\). The bijection for the case \(m = 1\) can be obtained by combining the bijections \(g_1\) and \(g_2\), namely, let \(f_1 : (\mathcal{T}^s, 1, D_n) \rightarrow ((\mathcal{U}^2 \cup \mathcal{T} \cup \mathcal{T}^c)^s, 1, D_{n-1})\) be the bijection defined as follows: \(f_1(p) = g_1(p)\) if \(p \in (\mathcal{T}_1^s, 1, D_n)\) and \(f_1(p) = g_2(p)\) if \(p \in (\mathcal{T}_2^s, 1, D_n)\). We can extend the bijection \(f_1\) to the general bijection \(f_m\) from the set \((\mathcal{T}^s, m, D_n)\) to the set \(((\mathcal{U}^2 \cup \mathcal{T} \cup \mathcal{T}^c)^s, m, D_{n-m})\) as follows. For any \(p \in (\mathcal{T}^s, m, D_n)\), suppose \(p\) has \(t_1, t_2, \ldots, t_m\) colored towers labeled by \(s\) from left to right. For every labeled tower \(t_i\) that has height 1 and \(UT^s_i U \subseteq p\), we label the \(U\)-steps next to \(t_i\) by \(s\) and remove \(t_i\) from \(p\), i.e., \(U^s_i U^s \subseteq f_m(p)\). For every labeled tower \(t_i\) that has height 1 and \(t t_i^s \subseteq p\), we label the tower \(t\) by \(s\) and remove \(t_i\) from \(p\), i.e., \(t^s \subseteq f_m(p)\). For every labeled tower \(t_i\) that has height at least 2, i.e., \(t_i^s = (UTD)^s \subseteq p\), we label \(t\) by \(s\) and remove a \(U\)-step and a \(D\)-step from the tower \(t_i\), i.e., \(t^s \subseteq f_m(p)\). In fact, the map \(f_m\) on the path \(p\) is equivalent to applying the bijection \(f_1\) on

![Figure 1](https://www.ams.org/journal-terms-of-use)
each labeled tower $t_i$ of $p$ from left to right. It follows that $f_m$ is a bijection and therefore $(\mathcal{T}^s, m, D_n) \simeq (U^2 \cup \mathcal{T} \cup \mathcal{T}^\circ, m, D_{n-m})$ holds for any $m$. It remains to prove $U^2(p) \cup \mathcal{T}(p) \cup \mathcal{T}^\circ(p) \simeq U(p)$ for any $p \in D_n$.

In view of $\mathcal{T}^\circ(p) \cup \mathcal{T}(p) = \mathcal{T}^s(p)$ for any $p \in D_n$, we next show for any $p \in D_n$, $U^2(p) \cup \mathcal{T}^s(p) \simeq U(p)$. Each $U$-step is either part of a $UU$-step or part of a peak $P$ by considering the step that follows this $U$-step. Each peak $P$ is contained in only one tower from $\mathcal{T}^s(p)$ and each tower from $\mathcal{T}^s(p)$ has only one peak. Therefore $U(p) \simeq U^2(p) \cup \mathcal{T}^s(p)$ for any $p \in D_n$ and the proof is complete. \hfill $\square$

For a path $p$, we name the first $U$-step of a tower $t \in \mathcal{T}_2(p)$ bottom. Let $\mathcal{B}(p)$ be the set of bottoms contained in a path $p$, and let $((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n)$ be the set of elevated Dyck paths $p \in D_n$ having exactly $m$ colored towers labeled by $s$ and among these $m$ colored towers there are exactly $r$ bottoms labeled by $w$. Let $((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_n)$ be the set of elevated Dyck paths $p \in D_n$ having exactly $m$ colored towers labeled by $s$ and among these $m$ colored towers there are exactly $r$ colored towers labeled by $w$. We will prove

**Lemma 2.2.** $((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n) \simeq ((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_{n-r})$.

**Proof.** For a path $p \in ((\mathcal{T}^s, \mathcal{B}^w), (m, r), D_n)$ and for every colored tower $t \in \mathcal{T}_2(p)$ labeled by $s$ whose bottom is labeled by $w$, we remove the bottom and a $D$-step from $t$, and label the remaining colored tower by $s$ and $w$. For a path $\tilde{p} \in ((\mathcal{T}^s, \mathcal{T}^w), (m, r), D_{n-r})$ and for every colored tower $\tilde{t} \in \mathcal{T}(\tilde{p})$ labeled by $s, w$, we replace the tower $\tilde{t}^s, w$ by the tower $U^w t^s D$. This yields a bijection. \hfill $\square$

Now we are in position to prove (1.6).

**Proof.** Let $F_n(q) = q N_n(q)$. Then (1.6) is equivalent to

\[
F_n(q) = q \frac{N_n(q)}{(1-q)^n} = \frac{q}{1-q} + \sum_{k=1}^{n-1} \frac{F_{n-k}(q)}{(1-q)^{n-k}} \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} \frac{q}{(1-q)^{m+1}},
\]

where \(\frac{F_n(q)}{(1-q)^n} = \sum_{k=1}^{n} \frac{N_{n,k} q^k}{(1-q)^n}\) counts the number of elevated Dyck paths of length $2n + 2$ where each $U$-step, except the first $U$-step, has weight $\frac{1}{1-q}$ and each peak $UD$ has weight $q$. Before we proceed to the combinatorial proof, we first transform

\[
\frac{q}{(1-q)^{n+1}} \text{ according to our needs, namely,}
\]

\[
\sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} \binom{n-m}{k} \frac{q}{(1-q)^{m+1}} = \sum_{i=0}^{k-1} \sum_{m \geq i} (-1)^m \binom{k-1}{m} \binom{n-m}{k} \binom{m}{i} \left( \frac{q}{1-q} \right)^{i+1}
\]

\[
= \sum_{m=1}^{k} (-1)^{m-1} \binom{k-1}{m-1} \binom{n-k+1}{m} \left( \frac{q}{1-q} \right)^{m}.
\]
Consequently, we can express (2.1) as

\[
\frac{F_n(q)}{(1-q)^n} = \frac{q}{1-q} + \sum_{m=1}^{n-1} (-1)^{m-1} \sum_{k=m}^{n-m+1} \frac{F_{n-k}(q)}{(1-q)^n-k} \binom{n-m}{m} \binom{k-1}{m} (q \frac{1}{1-q})^m.
\]

We observe that adding a peak \( P \) right between double \( U \)-steps contributes weight \( \frac{1}{1-q} \), while adding a peak \( P \) right between \( UD \)-steps contributes weight \( \frac{1}{1-q} \). In contrast to the proof of (1.1), the reduction from (2.2) can also be derived by directly applying the inclusion-exclusion principle. We will show this identity in both ways. We set \( g_{n,k}(T) = |\{p \in D_n : |\mathcal{T}(p)| = k\}| \) and define \( g_{n,k}(T, q) \) to be the generating function of \( g_{n,k}(T) \) that weights each \( U \)-step, except the first \( U \)-step, of an elevated Dyck path \( p \in D_n \) by \( \frac{1}{1-q} \) and each peak \( UD \) of an elevated Dyck path \( p \in D_n \) by \( q \). Recall that we denote by \( (\mathcal{M}^s, m, D_n) \) the set of elevated Dyck paths \( p \in D_n \) having exactly \( m \) elements from the set \( \mathcal{M}(p) \) labeled by \( s \), and that \( a_{n,m}(\mathcal{M}) = |(\mathcal{M}^s, m, D_n)| \). Furthermore, we denote by \( ((\bigcup_{i=1}^{k} \mathcal{M}_i)^s, m, D_n) \) the set of elevated Dyck paths \( p \in D_n \) having exactly \( m \) elements from the set \( \bigcup_{i=1}^{k} \mathcal{M}_i(p) \) labeled by \( s \), and define \( a_{n,m}(\bigcup_{i=1}^{k} \mathcal{M}_i) = |((\bigcup_{i=1}^{k} \mathcal{M}_i)^s, m, D_n)| \). Then, analogous to the proof of (1.1), we consider \( \mathcal{M} = \mathcal{T} \). Let \( a_{n,m}(\mathcal{T}, q) \) be the generating function of \( a_{n,m}(\mathcal{T}) \) where each \( U \)-step, except the first \( U \)-step, of an elevated Dyck path \( p \in D_n \) is weighted by \( \frac{1}{1-q} \) and each peak \( P \) of an elevated Dyck path \( p \in D_n \) is weighted by \( q \). Then

\[
a_{n,m}(\mathcal{T}) = \sum_{k \geq m} \binom{k}{m} g_{n,k}(\mathcal{T}), \quad a_{n,m}(\mathcal{T}, q) = \sum_{k \geq m} \binom{k}{m} g_{n,k}(\mathcal{T}, q).
\]

In particular for \( m = 0 \), we have \( a_{n,0}(\mathcal{T}) = C_n \) and \( a_{n,0}(\mathcal{T}, q) = \frac{F_n(q)}{(1-q)^n} \). By introducing the generating function \( G_n(x, q) \) of \( g_{n,k}(\mathcal{T}, q) \), we obtain

\[
G_n(x, q) = \sum_{k \geq 1} g_{n,k}(\mathcal{T}, q) x^k = \sum_{m=0}^{n} \frac{G_n^{(m)}(1, q)}{m!} (x-1)^m = \sum_{m=0}^{n} a_{n,m}(\mathcal{T}, q) (x-1)^m,
\]

where \( G_n^{(m)}(1, q) = \frac{\partial^m G_n(x, q)}{\partial x^m} |_{x=1} \). By setting \( x = 0 \), we get

\[
a_{n,0}(\mathcal{T}, q) = \frac{F_n(q)}{(1-q)^n} = \sum_{m=1}^{n} (-1)^{m-1} a_{n,m}(\mathcal{T}, q).
\]

In fact (2.3) can also be derived by directly applying the inclusion-exclusion method. To be precise, we adopt the notations from [12]. Let \( A_i \) be the set of elevated Dyck paths in \( D_n \) whose \( i \)-th tower (in the left-to-right order) is a colored tower. Then \( A_1, \ldots, A_n \) are all the subsets of the set \( D_n \) and \( A_1 \cup \cdots \cup A_n = D_n \). For each subset \( T \) of \([n]\), let \( A_T = \bigcap_{i \in T} A_i \) with \( A_{\emptyset} = D_n \). Here \( A_T \) is the set of elevated Dyck paths in \( D_n \) whose \( i \)-th tower (in the left-to-right order) is a colored tower for every \( i \in T \). For \( 0 \leq m \leq n \), we set \( S_m = \sum_{|T|=m} |A_T| \) where the sum runs over all the \( m \)-subsets \( T \) of \([n]\). Therefore \( S_m \) equals the number of elevated Dyck paths in \( D_n \) that have \( m \) colored towers labeled by \( s \), which is \( a_{n,m}(T) \). So according to the
principle of inclusion-exclusion, the number $|A_1 \cap \cdots \cap A_n|$ of elevated Dyck paths in $D_n$ that have no colored towers equals

$$|A_1 \cap \cdots \cap A_n| = \sum_{m=0}^{n} (-1)^m S_m = \sum_{m=0}^{n} (-1)^m a_{n,m}(T).$$

Since every elevated Dyck path has at least one colored tower, we have $|A_1 \cap \cdots \cap A_n| = g_{n,0}(T) = 0$. Consequently, $a_{n,0}(T) = \sum_{m=1}^{n} (-1)^{m-1} a_{n,m}(T)$. In terms of the weighted elevated Dyck paths, (2.3) follows.

In order to use (2.3) to prove (2.2), we will need to give an explicit expression for $a_{n,m}(T,q)$. For any path $p \in D_n$ and any two colored towers $t_1, t_2 \in T(p)$, $t_1$ and $t_2$ are disjoint. We start by counting $a_{n,1}(T,q)$, which is the generating function for the elevated Dyck paths $p \in D_n$ having one colored tower labeled by $s$. Equivalently, $|\{T^s, 1, D_n\}| = a_{n,1}(T)$. In view of Lemma 2.1, we get $a_{n,1}(T) = a_{n-1,1}(U)$ and

$$a_{n,1}(T,q) = \frac{q}{1-q}a_{n-1,1}(U) + \frac{1}{1-q} a_{n-1,1}(T,q) = \frac{q}{1-q}a_{n-1,1}(U) + \frac{1}{1-q} a_{n-1,1}(T,q) = \sum_{k=1}^{n-1} \frac{q}{1-q} a_{n-k,1}(U,t) + a_{1,1}(T,q).$$

(2.4)

Next we will count $a_{n,m}(T,q)$ for $m \geq 2$. For a path $p$, recall that the first $U$-step of a tower $t \in T_2(p)$ is named bottom. For a given path $p \in D_n$ with $m$ colored towers labeled by $s$, let $T_{2,m}$ denote the subset of these $m$ towers that have height $\geq 2$. Since each $U$-step has weight $\frac{1}{1-q}$ and each tower in $T_{2,m}$ has one bottom, the weight on the bottoms of path $p \in D_n$ is $(\frac{q}{1-q})|T_{2,m}|$, which is equal to

$$|T_{2,m}| = \frac{|T_{2,m}|}{1-q} + \sum_{r=1}^{m} (-1)^{r-1} \left(\frac{|T_{2,m}|}{r}\right) (\frac{1}{1-q})^{r}|T_{2,m}|^{r-1}.$$

(2.5)

We will separate the weight $(\frac{q}{1-q})|T_{2,m}|$ on the bottoms of path $p$ according to (2.5). The term $(\frac{q}{1-q})|T_{2,m}|$ on the right-hand side of (2.5) corresponds to the case that each bottom is weighted by $\frac{q}{1-q}$. Accordingly, we set $f_{n,m,0}(T,q)$ to be the generating function of elevated Dyck paths $p \in D_n$ that have $m$ colored towers labeled by $s$ where each bottom has weight $\frac{q}{1-q}$, each $U$-step, other than the first $U$-step and the bottom, has weight $\frac{1}{1-q}$ and each peak has weight $q$. In the same way as for the counting of $a_{n,1}(T,q)$, the set $(T^s, m, D_n)$ is in one-to-one correspondence to the set $(U^s, m, D_{n-m})$ as shown in Lemma 2.1. Therefore,

$$a_{n,m}(T) = a_{n-m,m}(U),$$

(2.6)

$$f_{n,m,0}(T,q) = (\frac{q}{1-q})^m a_{n-m,m}(U,q).$$

The term $(\frac{q}{1-q})|T_{2,m}|^{r-1}$ on the right-hand side of (2.5) represents the number of ways to choose $r$ towers from the set $T_{2,m}$ where each bottom of these $r$ towers has weight 1 and each bottom of the remaining $(|T_{2,m}| - r)$ towers has weight $\frac{1}{1-q}$. Accordingly, for $r \geq 1$ we define $f_{n,m,r}(T)$ to be the number of elevated Dyck paths $p \in D_n$ that have $m$ colored towers labeled by $s$ and among these $m$ colored towers...
there are exactly $r$ bottoms labeled by $w$, i.e., $|\{(T^s, B^w), (m, r), D_n)\}| = f_{n, m, r}(T)$. Furthermore, let $f_{n, m, r}(T, q)$ be the generating function of $f_{n, m, r}(T)$ where each bottom labeled by $w$ has weight 1, each $U$-step, other than the bottom labeled by $w$ and the first $U$-step, has weight $\frac{1}{1-q}$ and each peak has weight $q$. From Lemma 2.2, we obtain $f_{n, m, r}(T) = |\{(T^s, T^w), (m, r), D_{n-r}\}|$ and therefore

$$f_{n, m, r}(T) = \binom{m}{r} a_{n-r, m}(T), \quad f_{n, m, r}(T, q) = \binom{m}{r} a_{n-r, m}(T, q).$$

By multiplying both sides of (2.5) by the weights on the $U$-steps (except the first $U$-step and the bottoms) and the weights on the peaks of a path $p \in D_n$, and summing over all the paths in $D_n$, we get

$$a_{n, m}(T, q) = f_{n, m, 0}(T, q) + \sum_{r=1}^{m} (-1)^{r-1} f_{n, m, r}(T, q).$$

Together with (2.6) and (2.7), we have

$$a_{n, m}(T, q) = \left(\frac{q}{1-q}\right)^m a_{n-m, m}(T, q) + \sum_{r=1}^{m} (-1)^{r-1} \binom{m}{r} a_{n-r, m}(T, q)$$

$$= \left(\frac{q}{1-q}\right)^m \left(\frac{n-m+1}{m}\right) F_{n-m}(q) \left(\frac{1}{1-q}\right)^{n-m} + \sum_{r=1}^{m} (-1)^{r-1} \binom{m}{r} a_{n-r, m}(T, q).$$

We will employ generating functions to solve (2.8). Let $A_m(x)$ be the generating function for $a_{n, m}(T, q)$ and $m \geq 2$, i.e., the $n$-th coefficient of $A_m(x)$—denoted by $[x^n]A_m(x)$—is $a_{n, m}(T, q)$. By multiplying both sides of (2.8) by $x^n$ and summing over all $n$, we obtain

$$A_m(x) = (1-x)^{-m} \sum_{n \geq m} \binom{n-m+1}{m} \left(\frac{q}{1-q}\right)^m \frac{F_{n-m}(q)}{(1-q)^{n-m}} x^n,$$

$$a_{n, m}(T, q) = [x^n]A_m(x) = \sum_{i \geq 0} \binom{n-m-i+1}{m} \left(\frac{q}{1-q}\right)^m \frac{(m+i-1)}{(m-1)} \frac{F_{n-m-i}(q)}{(1-q)^{n-m-i}}$$

$$= \sum_{k=m}^{n-k+1} \binom{n-k+1}{m} \binom{k-1}{m-1} \frac{F_{n-k}(q)}{(1-q)^{n-k}} \left(\frac{q}{1-q}\right)^m.$$

In combination of (2.3) and (2.4), the proof of (2.2) is complete. \hfill \square

3. Generalize (1.6) to Ballot numbers

The Ballot numbers $B_{n,r}$ are defined by $B_{n,r} = \frac{r+1}{2n+r+1} \binom{2n+r+1}{n}$. They count the number of paths $p$ from $(0, 0)$ to $(2n+r, r)$ with allowed $U$-step and $D$-step and each path $p$ can be decomposed as $p_1 U p_2 \cdots U p_{r+1}$ with $p_i$, $1 \leq i \leq r+1$, a Dyck path. Let $M_{n, r}(q)$ be the generating function for the path $p_1 U p_2 \cdots U p_{r+1}$ that has total length $2n$ where each peak of $p_1 U p_2 \cdots U p_{r+1}$ has weight $q$ and each $U$-step of $p_1 U p_2 \cdots p_{r+1}$ has weight $\frac{1}{1-q}$. Then we can generalize (1.6) to an equation for $M_{n, r}(q)$, i.e.,

$$M_{n, r}(q) = \frac{q(r+1)}{(1-q)^m} + \sum_{m=1}^{n-1} (-1)^{m-1} \sum_{k \geq m} q^m M_{n-k, r}(q) \frac{1-q}{(1-q)^{n-k+m}} \frac{k-1}{m-1} \frac{(n-k+1+r)}{m}.$$
The proof follows similarly to that for (1.6) and is omitted here. Next we will prove (1.7) by involution and the inclusion-exclusion method on the partitions and answer Andrews’ questions on the property of $T_r(n, q)$ given in (1.8).

4. Properties of $T_r(n, q)$

We say a polynomial $f(x) = a_n x^n + \cdots + a_0$ is reciprocal if $f(x) = x^n f(\frac{1}{x})$, i.e., $a_r = a_{n-r}$. A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be unimodal if for some $0 \leq j \leq n$ we have $a_0 \leq \cdots \leq a_j - 1 \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$. We say a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is unimodal if the sequence $a_0, a_1, \ldots, a_n$ is unimodal. If $f(x)$ and $g(x)$ are unimodal and reciprocal polynomials with non-negative coefficients, then $f(x)g(x)$ is also unimodal and reciprocal. Here we say $f(x)$ is a positive polynomial if all the coefficients of $f(x)$ are nonnegative. We say the polynomial $f(x)$ has nonnegative coefficients (resp. nonpositive coefficients) up to $x^r$ if for any $0 \leq i \leq r$, the coefficient of $x^i$ in the polynomial $f(x)$ is nonnegative (resp. nonpositive). $\omega \in \mathbb{C}$ is a primitive $k$-th root of unity if and only if $\omega^k = 1$ and for any $d \in \mathbb{Z}^+$ and $d < k$, $\omega^d \neq 1$. The $k$-th cyclotomic polynomial $\Phi_k(x) \in \mathbb{Z}[x]$ is the polynomial whose roots are the primitive $k$-th roots of unity. Let $\omega$ be any primitive $k$-th root of unity, and the polynomial $f(x)$ satisfies $f(\omega) = 0$. Then $\Phi_k(x) \mid f(x)$. The $q$-Lucas theorem is the following:

Proposition 1 ($q$-Lucas theorem). Let $m, k, d$ be positive integers, and write $m = ad + b$ and $k = rd + s$, where $0 \leq b, s \leq d - 1$. Let $\omega$ be any primitive $d$-th root of unity. Then

$$\left[ \frac{m}{k} \right]_\omega = \left( \frac{a}{r} \right) \left[ \frac{b}{s} \right]_\omega.$$  

Theorem 4.1. $T_r(n, q)$ is a polynomial in $q$. If $n$ is even, $T_r(n, q)$ is a positive polynomial. If $n$ is odd, $(1 + q)T_r(n, q)$ is a positive polynomial. In case $n = 2r - 1$, $T_r(2r - 1, -q)$ is a positive polynomial.

Proof. We can simplify $T_r(n, q)$ defined in (1.8) into

$$T_r(n, q) = q^{r^2 - r} \frac{1}{[n]_q^r} \left[ \frac{2n - 2r}{n - 1} \right]_q$$

$$= q^{r^2 - r} \left[ \frac{n}{r} \right]_q \left( 1 + q^n \right) \left[ \frac{2n - 2r - 1}{n - 2} \right]_q.$$  

Consequently from (4.1) and (4.2), we have for $r \geq 1$,

$$T_r(n, q) = ([n]_q - q[n - 1]_q)T_r(n, q)$$

$$= q^{r^2 - r} \left[ \frac{n}{r} \right]_q \left[ \frac{2n - 2r}{n - 1} \right]_q - q^{r^2 - r} \left[ \frac{n - 1}{r} \right]_q \left( q + q^{n+1} \right) \left[ \frac{2n - 2r - 1}{n - 2} \right]_q,$$

which implies $T_r(n, q)$ is a polynomial.

We will next show the polynomial $T_r(n, q)$ is positive for even $n$ and the polynomial $(1 + q)T_r(n, q)$ is positive for odd $n$. Let $d = \gcd(n, r)$; then

$$T_r(n, q) = q^{r^2 - r} \left[ \frac{n}{r} \right]_q \left[ \frac{d}{q^2} \right]_q \left[ \frac{d}{q^2} \left( 1 + q^n \right) \left( 1 - q^d \right) \right] \left[ \frac{2n - 2r}{n - 1} \right]_q.$$  

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It has been proved by Brunetti et al. that \( \left[ \frac{n}{r} \right] q^2 \frac{d_{r^2}}{[m]_q} \) is a positive polynomial based on the fact that polynomial \( [d]_q [\frac{n}{r}]_q \) is unimodal and reciprocal [1]. In the same way, let \( m = \gcd(n, 2r - 1) \); then \( \left[ \frac{m}{n} \right] q \frac{2n-2r+1}{[n-1]}_q \) is a positive polynomial. It remains to prove \( \frac{(1+q^n)(1-q)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) is a positive polynomial for even \( n \) and \( [\frac{1+q^n}{1-q^n}] [\frac{2n-2r}{n-1}]_q \) is a positive polynomial for odd \( n \). Recall that \( m = \gcd(n, 2r - 1) \). First we observe \( m \) must be odd and therefore

\[
\frac{(1 + q^n)(1 - q)}{1 - q^{2d}} \left[ \frac{2n - 2r}{n - 1} \right]_q = \frac{[m]_q [\frac{n}{r}] - q}{[2n - 2r + 1]_q} \left[ \frac{2n - 2r + 1}{n} \right]_q \frac{[n]_q^2}{[m]_q [d]_q q^2}.
\]

Here \( [n]_q \) is a polynomial since \( \gcd(m, d) = 1 \) and \( m | n, d | n \). Together with the fact that \( \left[ \frac{2n-2r+1}{n-1} \right]_q \) is a polynomial, it follows that \( \frac{(1+q^n)(1-q)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) is a polynomial. We next prove \( \frac{(1+q^n)(1-q)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) is a positive polynomial if \( n \) is even. If \( n \) is even, then \( \frac{1-q}{1-q^{2d}} [\frac{2n}{n-1}]_q \) is a polynomial since for any \( x \mid (2d) \) and \( x > 1, x \mid (2n - 2r) \) and \( x \mid (n - 1) \). If not, \( x \mid (n - 1) \), and then \( x \mid \gcd(n - 1, 2) = 1 \), contradicting the assumption. Thus by using the \( q \)-Lucas theorem, we have \( \frac{1-q}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) is a polynomial. To be precise, for any \( x \in \mathbb{Z}^+ \) such that \( x(2d) \) and \( x > 1 \), suppose \( 2n - 2r = c_1 x, n - 1 = c_2 x + t_1 \) for some \( t_1 \neq 0 \), and let \( \rho \) be any primitive \( x \)-th root of unity. Then by applying the \( q \)-Lucas theorem, we have

\[
[\frac{2n-2r}{n-1}]_q = [\frac{c_1 x}{c_2 x + t_1}]_q = [\frac{c_1}{c_2}]_q [\frac{0}{t_1}]_q = 0.
\]

This shows for any \( x \mid (2d) \) and \( x > 1 \), \( \Phi_x(q) [\frac{2n-2r}{n-1}]_q \). Since \( [2d]_q = \prod_{x \mid (2d)} \Phi_x(q) \) and any two cyclotomic polynomials are relatively prime, we can conclude that \( \prod_{x \mid (2d)} \Phi_x(q) [\frac{2n-2r}{n-1}]_q \) and therefore \( \frac{1-q}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q = \frac{1}{[2d]_q} [\frac{2n-2r}{n-1}]_q \) is a polynomial. In view of the unimodality of polynomial \( [\frac{2n-2r}{n-1}]_q \), \( (1-q)[\frac{2n-2r}{n-1}]_q \) has nonnegative coefficients up to \( q^{\left\lfloor \frac{(n-1)(n-2r+1)}{2} \right\rfloor} \). Therefore we expand \( \frac{1-q}{1-q^{2d}} \) as

\[
\frac{(1 - q)}{1 - q^{2d}} \left[ \frac{2n - 2r}{n - 1} \right]_q = (1 - q) \left[ \frac{2n - 2r}{n - 1} \right]_q + q^{2d}(1 - q) \left[ \frac{2n - 2r}{n - 1} \right]_q + \cdots,
\]

which implies that the polynomial \( \frac{(1-q)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) has nonnegative coefficients up to \( q^{\left\lfloor \frac{(n-1)(n-2r+1)}{2} \right\rfloor} \). Together with the reciprocity of \( \frac{(1-q)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \), we conclude that \( \frac{(1-q)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) has nonnegative coefficients and therefore \( T_r(n, q) \) is a positive polynomial if \( n \) is even.

In the same way, we can prove that \( \frac{(1-q^2)}{1-q^{2d}} [\frac{2n-2r}{n-1}]_q \) is a positive polynomial if \( n \) is odd. This implies \( (1+q)T_r(n, q) \) is a positive polynomial if \( n \) is odd.

By setting \( q \rightarrow -q \) and \( n \rightarrow 2r - 1 \), we can obtain

\[
T_r(2r - 1, -q) = q^{r^2 - r} \left[ \frac{2r - 1}{r} \right] \left[ \frac{1}{1 + q^{2r-1}} \right] q^{r^2 - r} [2r - 1]_q C_{r-1}(q^2),
\]

which indicates \( T_r(2r - 1, -q) \) has nonnegative coefficients. \( \square \)
5. Proof of (1.7)

We transform $T_r(n, q)$ from (4.3) into

$$\begin{align*}
T_r(n, q) &= q^{r^2} r \left[ \begin{matrix} n-1 \\ r-1 \end{matrix} \right] q^{2n-2r+1} q^{n} - q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n} \\
&\quad + q^{r^2+r} \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n} \quad \text{valid for } n \geq 2r+1, \quad \text{and}

T_r(n, q) &= q^{r^2} r \left[ \begin{matrix} n-1 \\ r-1 \end{matrix} \right] q^{2n-2r+1} q^{n} - q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n} \\
&\quad \text{valid for } 2r-1 \leq n \leq 2r.
\end{align*}$$

For simplicity, we set $a_n^{(r)} = q^{r^2+r} \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n}$, Then we can write $T_r(n, q) = a_{n-1}^{(r-1)} + a_n^{(r)} - q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n}$ for $n \geq 2r+1$ and $T_r(n, q) = a_n^{(r-1)} - q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n}$ for $2r-1 \leq n \leq 2r$. First we observe

$$\begin{align*}
\sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor -1} (-1)^{r-1} a_{n-1}^{(r-1)} + a_n^{(r)} + (-1)^{\lfloor \frac{n+1}{2} \rfloor} -1 a_n^{(\lfloor \frac{n+1}{2} \rfloor)} -1 &= a_n^{(0)} = \left[ \begin{matrix} 2n-1 \\ n \end{matrix} \right] = C_n(q) + q^{2n-1} q^{n-2},
\end{align*}$$

and it remains to prove

$$\begin{align*}
\left[ \begin{matrix} 2n-1 \\ n-2 \end{matrix} \right] q^{2n-2r+1} q^{n} - \sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{r-1} q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n} = \sum_{\ell(x)=n-2r+1, \nu \leq n-1, \ell(x)=n-2r+1, \nu \leq n-1} q^{\nu} q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2\nu}.
\end{align*}$$

We interpret (5.2) in terms of partitions. A partition $\lambda$ is defined as a finite sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ in the weakly decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. Each $\lambda_i \neq 0$ is called a part of $\lambda$. The number and the sum of parts of $\lambda$ are denoted by $\ell(\lambda)$ and $|\lambda|$, respectively. The partition-theoretic interpretation of the $q$-binomial $\left[ \begin{matrix} n+k \\ k \end{matrix} \right] q = \sum_{\lambda_1 \leq n} q^{\ell(\lambda)} q^{|\lambda|}$ as given in (1). Therefore we have

$$\begin{align*}
q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2n-2r-1} q^{n} &= \sum_{\ell(\mu) \geq n-2r+1, \mu_i \leq n-1} q^{\nu} q^{r^2} r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] q^{2\nu}.
\end{align*}$$

Given two partitions $\mu$ and $\nu$, say $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_n)$, let $\mu \cup \nu$ be the partition whose parts are $\mu_1, \mu_1, \ldots, \mu_m, \nu_1, \nu_1, \ldots, \nu_n$ in decreasing order. Let $\mu \cup \nu$ be the partition whose parts are $\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n$ in decreasing order, and let $\nu \setminus (\mu_1, \ldots, \mu_m)$ be the partition obtained from $\nu$ by removing the parts equal to $\mu_1, \ldots, \mu_m$.

For any pair $(\mu, \nu)$ such that $n-1 \geq \mu_1 \geq \cdots \geq \mu_r \geq 1$ where $\mu_i \leq n-i$ and $n-1 \geq \nu_1 \geq \cdots \geq \nu_{n-2r+1} \geq 1$, let $x$ be the smallest part in the partition $\mu \cup \nu$ with repetition. We shall construct a new pair $(\mu', \nu')$ as follows:

1. If $\mu = x$, then we choose $\mu' = (\mu_1, \ldots, \mu_{r-1})$ and $\nu' = \nu \cup (x, x)$.
2. Otherwise, $\nu_{n-2r} = \nu_{n-2r+1} = x$, and we choose $\mu' = (\mu_1, \ldots, \mu_r, x)$ and $\nu' = \nu \setminus (x, x)$.
Consequently $2|\mu| + |\nu| = 2|\mu'| + |\nu'|$ but the lengths of $\mu$ and $\mu'$ differ by 1. Indeed the map $f: (\mu, \nu) \mapsto (\mu', \nu')$ is an involution, since for any pair $(\mu', \nu')$, we can define the inverse map $g: (\mu', \nu') \mapsto (\mu, \nu)$ as follows. Let $y$ be the smallest part in the partition $\mu'' \cup \nu'$ with repetition. If $y \notin \mu'$, then we choose $\mu = \mu' \cup (y)$ and $\nu = \nu' \setminus (y, y)$. Otherwise we choose $\mu = \mu' \setminus (y)$ and $\nu = \nu' \cup (y, y)$. Clearly, $g$ is the inverse of $f$. This involution leads to (5.2).

In exactly the same way as for the proof of (1.1) and (1.6), we can prove (5.2) by counting the partitions that have exactly $k$ different parts with repetition labeled by $s$. Let $g_{n,k}(q^{n+1}[2n-1\atop n-2]_q)$ count the partitions $\lambda$ with $\lambda_1 \leq n-1$ and $\ell(\lambda) = n+1$ such that there are exactly $r$ different parts with repetition. Then the number of partitions $\lambda$ with $\lambda_1 \leq n-1$ and $\ell(\lambda) = n+1$, such that there are exactly $r$ different parts with repetition labeled by $s$, is

$$\sum_{k \geq r} \binom{k}{r} g_{n,k}(q^{n+1}[2n-1\atop n-2]_q) = q^{2+r}[n-1\atop r]_q q^{n-2r+1}[2n-2r-1\atop n-2]_q.$$  

Again, by employing the generating function of $g_{n,k}(q^{n+1}[2n-1\atop n-2]_q)$, we have

$$G_n(x, q) = \sum_{k \geq 0} g_{n,k}(q^{n+1}[2n-1\atop n-2]_q) x^k = \sum_{r \geq 0} \frac{G_n^{(r)}(1, q)}{r!} (x-1)^r$$

$$= \sum_{r \geq 0} q^{2+r}[n-1\atop r]_q q^{n-2r+1}[2n-2r-1\atop n-2]_q (x-1)^r.$$ 

By setting $x = 0$ on both sides, we get (5.2) and therefore (1.7) follows. From the proof we see that the expression of $T_r(n, q)$ given in (5.1) implies that the sieving process works on the partitions counted by $[2n-1\atop n-2]_q$.

6. Generalize (1.7) to $q$-Ballot numbers

The $q$-Ballot numbers $B_j(n, q)$ are defined by

$$B_j(n, q) = \frac{[j]_q [2n+j]_q}{[n]_q [2n+j-1]_q}.$$ 

They count the major index of lattice paths from $(0, 0)$ to $(2n+j-1, -j+1)$ with allowed $U$-step and $D$-step that never go below $y = -j+1$; see [9]. In particular, $B_1(n, q) = C_n(q)$ and $B_r(n, 1) = B_{n,r-1}$. Furthermore,

$$B_j(n, q) = \frac{[2n+j-2]_q}{[n]_q} - q^j \frac{[2n+j-2]_q}{[n]_q},$$

also counts the partitions $\lambda$ with $\lambda_1 \leq n+j-2$ and $\ell(\lambda) \leq n$ whose successive ranks are all $< j-1$; see [2]. We generalize (1.7) to an equation for $q$-Ballot numbers as follows.

**Theorem 6.1.** The $q$-Ballot numbers $B_j(n, q)$ satisfy

$$B_j(n, q) = \sum_{r=1}^{n} (-1)^{r-1} T_r^{(j)}(n, q),$$

$$T_r^{(j)}(n, q) = q^{2-r}[n\atop r]_q [2n+j-1-2r\atop n-1]_q [j]_q [n]_q.$$
where $T_r^{(j)}(n, q)$ is a polynomial for $n \geq 2r - j$. In particular $T_r^{(j)}(2r - j, -q)$ is a positive polynomial if $j \leq r$.

Proof. First we shall show $T_r^{(j)}(n, q)$ is a polynomial for $n \geq 2r - j$. By setting $d = \gcd(n, r)$ and $m = \gcd(n, 2r - j)$ we express

$$T_r^{(j)}(n, q) = q^{r - r} \left[ \frac{n}{r} \right] [d]_{q^2} [n]_{q^2} [n]_{q^2} [n]_{q^2} [n]_{q^2} [n]_{q^2} \left[ \frac{2n + j - 2r}{2n + j - 2r} \right] q \left[ \frac{2n + j - 2r}{2n + j - 2r} \right]$$

where $[n]_{q^2} [d]_{q^2}$ and $\frac{[m]_{q^2}}{\frac{[2n + j - 2r]_q}{q}}$ are positive polynomials as proved in Theorem 4.1 and it remains to show $\frac{[n]_{q^2} [n, q^2]}{[d, q^2]}$ is a polynomial. First we can simplify

$$\frac{[n]_{q^2} [n, q^2]}{[d, q^2]} = \frac{[2n]_{q^2} [n, q^2]}{[2d, q^2]}.$$  

For any $m_1 \mid m$ and $m_1 > 1$, if $m_1 \mid j$, then $\Phi_{m_1}(q) \mid [j]_q$ and therefore $\Phi_{m_1}(q) \mid \frac{[2n]_{q^2} [n, q^2]}{[d, q^2]}$. Otherwise, $m_1 \nmid j$ and therefore $m_1 \nmid d$. In fact, $m_1 \nmid (2d)$. If not, then $m_1 \mid (2r)$. Since $m_1 \mid m$, hence $m_1 \mid (2r - j)$. This leads to $m_1 \mid j$, contradicting the assumption. Let $\omega$ be any $m_1$-th primitive root of unity, i.e., $\omega^{m_1} = 1$. Then we have $[2n]_{\omega} = 0$ and $[2d]_{\omega} \neq 0$ which implies $\Phi_{m_1}(q) \mid \frac{[2n]_{q^2}}{[2d, q^2]}$. It follows that $\Phi_{m_1}(q) \mid \frac{[2n]_{q^2} [n, q]}{[2d, q^2]}$ for any $m_1 \mid m$ and $m_1 > 1$. Since any two cyclotomic polynomials are relatively prime, we have that $\frac{[2n]_{q^2} [n, q]}{[2d, q^2]}$ is a polynomial. Therefore

$$(6.3) \quad \frac{[n]_{q^2} [n, q^2]}{[d, q^2]} \left[ \frac{2n + j - 2r}{2n + j - 2r} \right] q = \left[ \frac{2n + j - 2r - 1}{2n + j - 2r - 1} \right] q$$

and $T_r^{(j)}(n, q)$ are polynomials.

In particular, $T_r^{(j)}(2r - j, q)$ is a polynomial. We next prove $T_r^{(j)}(2r - j, -q)$ is a positive polynomial if $j \leq r$. If $j$ is even, then

$$T_r^{(j)}(2r - j, -q) = q^{r - r} \left[ \frac{2r - j}{2r - j} \right] q^2 \frac{1 - q^j}{1 - q^{2r - j}}$$

holds. Here $q^{r - r} \left[ \frac{2r - j}{2r - j} \right] q^2 (1 - q^j)$ has nonnegative coefficients up to $q^{r - j}$ since $\frac{1}{q^{2r - j}} \left[ \frac{2r - j}{2r - j} \right] q$ is unimodal and $j$ is even. In combination with the expansion of $\frac{1}{q^{2r - j}}$ at $q = 0$ where $j \leq r$, we conclude that $T_r^{(j)}(2r - j, -q)$ is a positive polynomial for even $j$. However, if $j$ is odd, we cannot prove the claim in the same way as before. If $j$ is odd, we have

$$(6.4) \quad T_r^{(j)}(2r - j, -q) = q^{r - r} \left[ \frac{2r - j}{2r - j} \right] q^2 \frac{1 + q^j}{1 + q^{2r - j}}.$$
Since $q^{\frac{2r-j}{r}}[2r-j]_q$ is a positive polynomial and since $j$ is odd, we find that
\((1 + q^j)q^{\frac{2r-j}{r}}[2r-j]_q^2\) is a positive polynomial. But in view of the expansion of $\frac{1}{1-q^{2r-j}}$ at $q = 0$, we get
\[T_r^{(j)}(2r - j, -q) = (1 + q^j)q^{\frac{2r-j}{r}}[2r-j]_q^2 - q^{2r-j}(1 + q^j)q^{\frac{2r-j}{r}}[2r-j]_q^2 + \cdots,\]
where the polynomial $-q^{2r-j}(1 + q^j)q^{\frac{2r-j}{r}}[2r-j]_q^2$ could make the first half of the coefficients of $(1 + q^j)q^{\frac{2r-j}{r}}[2r-j]_q^2$ become negative. Therefore, we choose to prove $T_r^{(j)}(2r - j, -q)$ in (6.4) to be a positive polynomial by considering
\[qT_r^{(j)}(2r - j, q) = q^{\frac{2r-j}{r}}[2r-j]_q^2 \frac{1 - q^j}{1 - q^{2r-j}},\]
The claim that polynomial $T_r^{(j)}(2r - j, -q)$ has nonnegative coefficients is equivalent to the claim that polynomial $qT_r^{(j)}(2r - j, q)$ has nonnegative coefficients for the odd powers of $q$, and nonpositive coefficients for the even powers of $q$. Now for $qT_r^{(j)}(2r - j, q)$ the unimodality of $q^{\frac{2r-j}{r}}[2r-j]_q$ implies that $q^{\frac{2r-j+1}{r}}[2r-j]_q(1 - q^j)$ has nonnegative coefficients for the odd powers of $q$ up to $q^{\frac{3r^2 - 2rj + j + 1}{2rj - j}}$. By expanding $\frac{1}{1-q^{2r-j}}$ as a power series at $q = 0$, we have that the odd powers of $q$ in the polynomial $qT_r^{(j)}(2r - j, q)$, come from
\[(1 + q^{4r-2j} + q^{8r-4j} + \cdots)(1 - q^2)q^{\frac{2r-j}{r}}[2r-j]_q^2,\]
from which we can conclude that the coefficients of the odd powers of $q$, up to $q^{\frac{3r^2 - 2rj + j + 1}{2rj - j}}$ in the polynomial $qT_r^{(j)}(2r - j, q)$ are nonnegative. Furthermore, we observe that the maximal degree $d_{\text{max}}$ of $q$ in the polynomial $qT_r^{(j)}(2r - j, q)$ is $3r^2 - 3r - 2rj + 2j + 1$ and the minimal degree $d_{\text{min}}$ of $q$ in the polynomial $qT_r^{(j)}(2r - j, q)$ is $r^2 - r + 1$. Let $a_i$ be the coefficient of $q^i$ in the polynomial $qT_r^{(j)}(2r - j, q)$. Then from the reciprocity of the polynomial $qT_r^{(j)}(2r - j, q)$ we get
\[a_i = a_{j+i} \text{ if } j+i = d_{\text{max}} + d_{\text{min}} = 4r^2 - 2rj - 4r + 2j + 2.\]
This implies for $i$ odd and $a_i \geq 0$, $j$ is odd and $a_j \geq 0$. Therefore we can conclude that the coefficients of all the odd powers of $q$ in the polynomial $qT_r^{(j)}(2r - j, q)$ are nonnegative. This is equivalent to say that the coefficients of all the even powers of $q$ in the polynomial $T_r^{(j)}(2r - j, q)$ are nonnegative. By expanding $\frac{1}{1-q^j}$ at $q = 0$, we have that the even powers of $q$ in the polynomial $T_r^{(j)}(2r - j, q)$ come from
\[-q^j(1 + q^{4r-2j} + q^{8r-4j} + \cdots)(1 - q^{2r-2j})q^{\frac{2r-j}{r}}[2r-j]_q^2,\]
where $-q^{2r-2j+1+j}[2r-j]_q(1 - q^{2r-2j})$ is a polynomial that has nonpositive coefficients for the even powers of $q$, up to $q^{\frac{3r^2 - 2rj - j + 1}{2rj - j}}$. Again from the reciprocity of polynomial $qT_r^{(j)}(2r - j, q)$, we conclude that the coefficients of all the even powers of $q$ in the polynomial $qT_r^{(j)}(2r - j, q)$ are nonpositive. That is to say, the coefficients of all the odd powers of $q$ in the polynomial $T_r^{(j)}(2r - j, -q)$ are nonnegative. Now it remains to prove (6.1). By similar techniques as used to prove (1.7), we
consider the partitions counted by \( q^{n+j} \left[ \begin{array}{c} 2n+j-1 \\ n-1 \end{array} \right] \) and let \( g_{n,k}(q^{n+j} \left[ \begin{array}{c} 2n+j-1 \\ n-1 \end{array} \right] q) \) count the partitions \( \lambda \) with \( \lambda_1 \leq n \) and \( \ell(\lambda) = n+j \) such that there are exactly \( k \) different parts with repetition. By following the same techniques used in the proof of (1.7) of Section 5 (see the proof of (5.4)), we can derive the identity

\[
q^{n+j} \left[ \begin{array}{c} 2n+j-1 \\ n-1 \end{array} \right] q = \sum_{r=1}^{n} (-1)^{r-1} q^{r^2+r} \left[ \begin{array}{c} n \\ r \end{array} \right] q^{n+j-2r} \left[ \begin{array}{c} 2n+j-2r-1 \\ n-1 \end{array} \right] q,
\]

which is equivalent to \((6.1)\).

**Conjecture.** Here we conjecture that if \( n \) is odd, then for \( m \geq n \geq 1 \) the polynomial \((1+q^n)\left[ \begin{array}{c} m \\ n-1 \end{array} \right]_q\) is unimodal. If \( n \) is even, then for any even \( j \neq 0 \) and \( m \geq n \geq 1 \), the polynomial \((1+q^n)\left[ \begin{array}{c} j \\ n-1 \end{array} \right]_q\) is unimodal. This, in combination with \((6.3)\), implies that \( T_r(n,q) \) is a positive polynomial for odd \( n \) and \( n \geq 2r+1 \), and \( T_{r,j}(n,q) \) given in \((6.2)\) is a positive polynomial for \( n \geq 2r-j+1 \).

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Department of Computer Science, University of Kaiserslautern, Kaiserslautern, Germany

E-mail address: jin@cs.uni-kl.de

Department of Computer Science, University of Kaiserslautern, Kaiserslautern, Germany – and – Department of Mathematics and Computer Science, University of Southern Denmark, Denmark

E-mail address: nebel@cs.uni-kl.de