NONCOMMUTATIVE MARKED SURFACES

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To the memory of Andrei Zelevinsky
Светлой памяти Андрея Владимировича Зелевинского посвящается

ABSTRACT. The aim of the paper is to attach a noncommutative cluster-like structure to each marked surface \( \Sigma \). This is a noncommutative algebra \( A_\Sigma \) generated by “noncommutative geodesics” between marked points subject to certain triangle relations and noncommutative analogues of Ptolemy-Plücker relations. It turns out that the algebra \( A_\Sigma \) exhibits a noncommutative Laurent Phenomenon with respect to any triangulation of \( \Sigma \), which confirms its “cluster nature”. As a surprising byproduct, we obtain a new topological invariant of \( \Sigma \), which is a free or a 1-relator group easily computable in terms of any triangulation of \( \Sigma \). Another application is the proof of Laurentness and positivity of certain discrete noncommutative integrable systems.

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1. Introduction

The goal of the paper is to introduce and study noncommutative algebras attached to surfaces (with marked boundary points and punctures) and their triangulations. This provides an instance of the noncommutative cluster theory (which is the main theme of the forthcoming paper [4]).

Since each surface can be obtained by gluing edges of a polygon (actually, in many ways), the most important object of study are noncommutative polygons and their noncommutative triangulations.

In the commutative case, cluster structure (of type $A_{n-3}$) on an $n$-gon is based on the Ptolemy relations:

$$x_{ik}x_{jk} = x_{ij}x_{kl} + x_{il}x_{kj}$$

(1.1)

for all quadrilaterals $(i, j, k, \ell)$ inscribed in a circle, $1 \leq i, j, k, \ell \leq n$, so that the chords $(i, k)$ and $(j, \ell)$ are diagonals of the quadrilateral, and $x_{ij} = x_{ji}, i \neq j$ is the Euclidean length of the chord $(ij)$. The Ptolemy relations (1.1) can also be interpreted as Plücker identities for $2 \times n$ matrices.

In the noncommutative version we do not assume that $x_{ij} = x_{ji}$ and we think of $x_{ij}$ as a measurement of a directed chord from $i$ to $j$. We suggest the following noncommutative generalization of the Ptolemy identity based on the theory of noncommutative quasi-Plücker coordinates developed in [30]:

$$x_{ik}x_{jk}^{-1}x_{j\ell} = x_{i\ell} + x_{ij}x_{k\ell}^{-1}x_{jk}$$

(1.2)

for every quadrilateral $(i, j, k, \ell)$, in which $(i, k)$ and $(j, \ell)$ are the diagonals.

Note that since elements $x_{ij}$ correspond to directed arrows, the products of the form $x_{ij}x_{k\ell}^{-1}, x_{ik}x_{j\ell}^{-1}$ make sense only when $\ell = j$.

It turns out that in order to establish the noncommutative Laurent Phenomenon and thus obtain a noncommutative cluster structure on the $n$-gon, it is crucial to impose additional triangle relations (also suggested by properties of quasi-Plücker coordinates):

$$x_{ij}x_{k\ell}^{-1}x_{ki} = x_{ik}x_{j\ell}^{-1}x_{ji}$$

(1.3)

for all distinct $i, j, k$ (of course, (1.3) is redundant in the commutative case).

The triangle relations (1.3) are of fundamental importance because they allow to introduce noncommutative angles $T_{i}^{j,k} := x_{ij}^{-1}x_{jk}x_{ik}^{-1}$ in each triangle $(i, j, k)$ so that $T_{i}^{j,k} = T_{i}^{k,j}$ due to (1.3). That is, the noncommutative angle at a vertex of a triangle does not depend on the order of the remaining two vertices. The "commutative" angles were introduced by Penner in [35, Section 3] (where they were called "$h$-lengths") and each $x_{ij} = x_{ji}$ was viewed as the $\lambda$-length of the side $(i, j)$ of an ideal triangle $(i, j, k)$ (see also [22, Lemma 7.9], [20, Section 12], and [27, Section 1.2], in the latter paper the term “angle” was used, apparently, for the first time) and thus noncommutative angles together with the "noncommutative $\lambda$-lengths" $x_{ij}$ can be thought of as a totally noncommutative metric on the Lobachevsky plane. The term "angle" is justified by the following observation. The noncommutative Ptolemy relations (1.2) together with the triangle relations (1.3) are equivalent to:

$$T_{j}^{ik} = T_{j}^{ik} = T_{j}^{k}$$

(1.4)

for every quadrilateral $(i, j, k, \ell)$, in which $(i, k)$ and $(j, \ell)$ are the diagonals. In other words, the (both commutative and noncommutative) angles are additive, which justifies the name. Using additivity of noncommutative angles, we establish the first instance of the noncommutative Laurent Phenomenon for the $n$-gon with vertices $1, \ldots, n$:

$$x_{ij} = \sum_{k=1}^{j-1} x_{1,k} T_{1}^{k,k+1} x_{1,j}$$

for all $2 \leq i < j \leq n - 1$, e.g., each $x_{ij}$ is a noncommutative Laurent polynomial in $x_{1,k}, x_{k,1}, k = 2, \ldots, n - 1$ and all $x_{i,i\pm1}$. In fact, the latter elements correspond to a triangulation of the $n$-gon where each triangle has...
a vertex at 1. We generalize this to any triangulation of the \( n \)-gon in Theorem 2.10 and, as expected, the commutative “limit” of this result (with all \( x_{ij} = x_{ji} \)) specializes to the Schiffer formula (39 Theorem 1.2)).

These arguments extend verbatim if we replace a polygon with a surface \( \Sigma \) with marked points. That is, for each such \( \Sigma \) one defines a \( \mathbb{Z} \)-algebra \( A_\Sigma \) generated by \( x_{ij}^{-1} \), where \( \gamma \) runs over homotopy classes of curves on \( \Sigma \) between marked points subject to the triangle and noncommutative Ptolemy relations. The Noncommutative Laurent Phenomenon (Theorem 3.30) asserts that for a given triangulation \( \Delta \) of \( \Sigma \) each \( x_{ij} \) belongs to the subalgebra generated by all \( x_{ij}^{-1} \), \( \gamma \in \Delta \). In any case, the assignments \( \Sigma \mapsto A_\Sigma \) and \( \Sigma \mapsto T_\Sigma \) define functors from the category of surfaces with marked points to respectively the category of algebras and the category of groups (Theorem 3.16).

A surprising byproduct of our approach is that the corresponding triangle group \( T_\Delta \) (generated by all \( t_{\gamma} \), \( \gamma \in \Delta \) subject to the triangle relations) does not depend on the triangulation of \( \Sigma \), therefore, is a topological invariant of \( \Sigma \) (Theorem 3.21). Moreover, each \( T_\Delta \) is either free or a one-relator group which looks like the fundamental group of \( \Sigma \), however it is different from \( \pi_1(\Sigma) \). For instance, if \( \Sigma \) is the sphere with \( n \) punctures, then \( T_\Delta \) is a free group in 5 generators if \( n = 3 \) and it is a 1-relator torsion-free group in \( 4n - 7 \) generators if \( n \geq 4 \). It turns out that each group \( T_\Delta \) has a “universal cover” \( \overline{T}_\Sigma \) which is a group generated by \( t_{\gamma} \), as \( \gamma \) runs over all isotopy classes of directed curves on \( \Sigma \) between marked point, subject to the triangle relations (see Sections 2.5 and 3.3 for details). This group, which we refer to as big triangle group is of interest as well: if \( \Sigma \) is the \( n \)-gon, we prove (Proposition 2.7) that \( \overline{T}_\Sigma \) has a presentation with \( (n-1)(n+2)/2 \) generators and \( (n-3)^2 \) relations and expect that the multiplicative group of \( A_\Sigma \) is isomorphic to \( \overline{T}_\Sigma \).

For each marked point \( i \) on \( \Sigma \) and each triangulation \( \Delta \) we also introduce a total (noncommutative) angle \( T^i_\Delta \in A_\Sigma \) in Section 3.9 to be the sum of noncommutative angles of all adjacent triangles. Similarly to the commutative case, we establish (Theorem 3.40) that the total angles do not depend on the choice of a triangulation \( \Delta \). Thus the collection of the total angles \( \{ T^i_\Delta \} \) can be thought of as a noncommutative version of a (hyperbolic) Riemann structure on \( \Sigma \). Using them we define in Section 3.9 the algebra \( U_\Sigma \) to be the subalgebra of \( A_\Sigma \) generated by all noncommutative edges \( x_{ij} \), the inverses of the boundary edges and all noncommutative angles \( T_i \) and argue that \( U_\Sigma \) is a totally noncommutative analogue of the upper cluster algebra corresponding to \( \Sigma \) (see e.g., [2]).

As an application of our noncommutative Laurent phenomenon, taking \( \Sigma \) to be a cylinder with no punctures, one marked point on the inner boundary and \( k \) marked points on the outer boundary, we prove Laurentness of the following noncommutative recursion for each \( k \in 1 + 2\mathbb{Z}_{>0} \):

\[
\begin{cases}
U_{n-k}DU_n = C_n + U_{n-1}DU_{n+1-k} & \text{if } n \text{ is even} \\
U_nDU_{n-k} = C_n + U_{n+1-k}DU_{n-1} & \text{if } n \text{ is odd}
\end{cases}
\]

for all \( n \geq k+1 \), where \( D, \overline{D}, \) and \( C_i, i \in \mathbb{Z}_{>0} \) belong to a noncommutative ground ring so that \( C_{n+k-1} = C_{k-1} \) for \( n \in \mathbb{Z}_{>0} \).

We prove (Theorem 4.5) that for odd \( k > 0 \) this recursion has a unique solution in the group algebra \( \mathbb{Q}F_{2k+1} \) of the free group \( F_{2k+1} \) freely generated by \( D, \overline{D}, C_1, \ldots, C_{k-1}, U_1, \ldots, U_k \), more precisely, each \( U_n \) is a sum of elements of \( F_{2k+1} \). We also prove (Theorem 4.5) that the element \( H_n \) in the skew field of fractions of \( \mathbb{Q}F_{2k+1} \), \( n \geq k \), given by

\[
H_n := \begin{cases}
\overline{D}U_{n-k}U_n^{-1} + DU_{n+k-1}U_n^{-1} & \text{if } n \text{ is even} \\
U_n^{-1}U_{n+k-1}D + U_n^{-1}U_{n+k-1}D & \text{if } n \text{ is odd}
\end{cases}
\]

belongs to \( \mathbb{Z}F_{2k+1} \) and does not depend on \( n \) hence is a “noncommutative conserved quantity.”

Setting \( D = \overline{D} = C_1 = 1 \) for all \( i > 0 \), we recover the Laurentness of the noncommutative discrete dynamical system established by Di Francesco and Kedem in [32] Theorem 6.2 (conjectured by M. Kontsevich in [33], Section 3).

We finish the introduction with establishing Laurentness of the following noncommutative recursion (which specializes to the discrete integrable system recently studied by P. Di Francesco in [18], see Section 4 for details) in the skew field \( \mathcal{F} \) freely generated by \( A, \overline{A}, B, \overline{B}, U_i, V_i, U_{i+1}, i \in \mathbb{Z} \):

\[
U_{i+1,j} \overline{A}_j V_{i+1,j} = B_{i+1,j}^{-1} + U_{i+1,j+1} \overline{A}_j V_{i,j} + V_{i,j} B_j U_{i+1,j+1} = A_{i+1,j}^{-1} + V_{i+1,j} + B_j U_{i+1,j+1}, \tag{1.6}
\]

\(^1Misha\ Kapovich\ explained\ to\ us\ that\ the\ relation\ of\ this\ group\ is\ related\ to\ the\ fundamental\ group\ of\ a\ ramified\ two-fold\ cover\ of\ \Sigma.\)
Definition 2.1. Denote by \( \mathcal{A}_n \) the \( \mathbb{Q} \)-algebra generated by \( x_{ij} \) and \( x_{ij}^{-1} \), \( i, j \in [n] \), \( i \neq j \) subject to the relations:

(i) (Triangle relations) For any distinct indices \( i, j, k, \ell \in [n] \):
\[
(2.1) \quad x_{ij} x_{jk} x_{ki} = x_{ik} x_{jk} x_{ij}.
\]

(ii) (Exchange relations) For any cyclic \((i, j, k, \ell)\) in \([n] \) :
\[
(2.2) \quad x_{j\ell} = x_{jk} x_{i\ell}^{-1} x_{ik} x_{\ell j}^{-1}.
\]

We prove (Theorem 4.11) that this recursion has a (unique) solution in the group algebra \( \mathbb{Q}T_\infty \) of the free group \( T_\infty \) freely generated by \( A_i, \overline{A}_i, B_i, \overline{B}_i, U_{i,i}, V_{i,i}, U_{i,i+1}, i \in \mathbb{Z} \), more precisely, each \( U_{ij} \) and \( V_{ij} \) is a sum of elements of the group. We also prove (Theorem 4.11) that the elements \( H_{ij}^\pm \in \operatorname{Frac}(\mathbb{Q}T_\infty) \), \( i \in \mathbb{Z} \), given by
\[
(1.8) \quad H_{ij}^+ := U_{ij}^{-1} (U_{j,i-1} A_{i-1} + U_{j,i+1} A_i), \quad H_{ij}^- := V_{ij}^{-1} (V_{j,i-1} B_{i-1} + V_{j,i+1} B_i)
\]
belong to \( \mathbb{Z}T_\infty \) and do not depend on \( j \).

These examples and their treatment in Section 4 suggest the following general approach to constructing noncommutative discrete integrable systems. That is, such a system consists of a marked surface \( \Sigma \), its automorphism \( \tau : \Sigma \to \Sigma \) permuting marked points, and a triangulation \( \Delta \) so that the collection \( \mathcal{T} = \{ x_\gamma \in \mathcal{A}_\Sigma, \gamma \in \cup_{k \in \mathbb{Z}} \mathbb{T}_k(\Delta) \} \) evolves in “discrete time” \( k \in \mathbb{Z} \) and for each marked point \( p \) of \( \Sigma \), the total noncommutative angle \( T_p \) is a (noncommutative) conserved quantity. The noncommutative Laurent Phenomenon (Theorems 3.30 and 3.36) then guarantees that each \( \mathcal{T} \) belongs to the algebra isomorphic to the group algebra of \( \mathbb{T}_\Delta \).

Acknowledgments. This work was partly done during our visits to Mathematisches Forschungsinstitut Oberwolfach, Max-Planck-Institut für Mathematik, Institut des Hautes Études Scientifiques, and Centre de Recerca Matemàtica, Barcelona. We gratefully acknowledge the support of these institutions. We are very grateful to Alexander Goncharov and especially to Maxim Kontsevich for their encouragement and support. Thanks are due to George Bergman, Dolors Herbera, and Alexander Lichtman for stimulating discussions of noncommutative localizations and unique factorizations, and to Misha Kapovich, Feng Luo, and Misha Shapiro for explaining important aspects of low dimensional topology and hyperbolic geometry.

2. Noncommutative polygons

2.1. Definition and main results. For each \( n \geq 3 \) consider a cyclic order \( i \mapsto i^+ \) on \( [n] = \{1, 2, \ldots, n\} \) by
\[
i^+ = \begin{cases} i + 1 & \text{if } i < n \\ 1 & \text{if } i = n \end{cases}
\]
We also say that a sequence \( i = (i_1, \ldots, i_\ell) \) of distinct elements in \([n]\) is cyclic if a cyclic permutation \( i \mapsto (i_k, \ldots, i_\ell, i_1, \ldots, i_{k-1}) \) is strictly increasing. In particular, the sequence \((k, k+1, \ldots, n, 1, \ldots, k-1)\) is cyclic for each \( k \).

Definition 2.1. Denote by \( \mathcal{A}_n \) the \( \mathbb{Q} \)-algebra generated by \( x_{ij} \) and \( x_{ij}^{-1} \), \( i, j \in [n] \), \( i \neq j \) subject to the relations:

(i) (Triangle relations) For any distinct indices \( i, j, k, \ell \in [n] \):
\[
(2.1) \quad x_{ij} x_{jk} x_{ki} = x_{ik} x_{jk} x_{ij}.
\]

(ii) (Exchange relations) For any cyclic \((i, j, k, \ell)\) in \([n] \) :
\[
(2.2) \quad x_{j\ell} = x_{jk} x_{i\ell}^{-1} x_{ik} x_{\ell j}^{-1}.
\]
Remark 2.2. It is easy to see that the exchange relations (2.2) are equivalent to noncommutative Ptolemy relations (1.2) provided the triangle relations (2.1) hold.

At the first glance the number of relations of $\mathcal{A}_n$ greatly exceeds the number of generators, moreover, we expect that the subalgebra of $\mathcal{A}_n$, generated by all $x_{ij}$ is a free algebra in $n^2 - n$ generators.

However, we will demonstrate below that the algebra $\mathcal{A}_n$ is “rationally” generated only by $3n - 4$ free generators.

Denote by $\mathcal{F}_m$ the free group on $m$ generators, so that its group algebra $Q\mathcal{F}_m$ is the free Laurent polynomial algebra $Q[c_1^{\pm 1}, \ldots, c_m^{\pm 1}]$. Following Amitsur and Cohn (see e.g., [13] or Section 5 below) denote by $\mathcal{F}_m$ the free skew field on $m$ generators, which is the “largest” skew field of fractions of $Q\mathcal{F}_m$. The following is our first main result, in which we freely use notation of Section 5.

Theorem 2.3. For each $n \geq 2$ the algebra $\mathcal{A}_n$ contains a subalgebra $\mathcal{A}_n'$ isomorphic to the free group algebra $Q\mathcal{F}_{3n-4}$ so that $\mathcal{A}_n$ is a universal localization of $\mathcal{A}_n'$ by a certain multiplicative submonoid of $\mathcal{A}_n' \setminus \{0\}$.

We prove the theorem in Section 2.14. In fact, it will follow from a more precise assertion (Theorem 2.8). In view of universality of the localization (Lemma 5.1), Theorem 2.3 implies the following immediate corollary.

Corollary 2.4. The canonical monomorphism of algebras $\varphi': \mathcal{A}_n' \hookrightarrow \mathcal{F}_{3n-4}$ uniquely extends to a homomorphism of algebras

$$\varphi: \mathcal{A}_n \rightarrow \mathcal{F}_{3n-4}$$

In fact, we expect that $\varphi'$ is injective, so far we can deduce this from another, “innocent looking” conjectural property of the group algebras $Q\mathcal{F}_m$ (Conjecture 5.18, see also Section 2.15).

Remark 2.5. Injectivity of $\varphi'$ would imply, in particular, that $\mathcal{A}_n$ has no zero divisors, which is a rather non-trivial assertion because of the following “counter-example” which was communicated to us by George Bergman. The universal localization $Q\langle x, y \rangle [(xy)^{-1}]$ of the free algebra $Q\langle x, y \rangle$ has a zero-divisor $y(xy)^{-1}x - 1$.

Remark 2.6. Given $n' \geq n$ and an injective map $j : [n] \hookrightarrow [n']$ for some $n' > n$, clearly, the assignments $x_{ij} \mapsto x_{j(i), j(j)}$ define a homomorphism of algebras $j_* : \mathcal{A}_n \rightarrow \mathcal{A}_{n'}$. One can conjecture that each $j_*$ is injective. In fact, this would directly follow from the injectivity of each $\varphi'$.

Now we explore the “cluster” structure of $\mathcal{A}_n$. We say that a pair $(i, k)$ crosses $(j, \ell)$ if $(i, j, k, \ell)$ is cyclic. A triangulation $\Delta$ of $[n]$ is a maximal crossing-free subset of $[n] \times [n] \setminus \{(i, i) | i \in [n]\}$. Clearly, each triangulation of $[n]$ has cardinality $4n - 6$.

For each triangulation $\Delta$ of $[n]$ define:

- The subalgebra $\mathcal{A}_\Delta$ of $\mathcal{A}_n$, generated by $x_{ij}$, $i, j \in [n]$, $i \neq j$ and $x_{ij}^{-1}$, $(i, j) \in \Delta$.
- The triangle group $T_\Delta$ generated by all $t_{ij}$, $(i, j) \in \Delta$ subject to the relations:

$$t_{ij}t_{kj}^{-1}t_{ki} = t_{ik}t_{jk}^{-1}t_{ji}$$

for all $i, j, k \in [n]$ such that $(i, j), (j, k), (k, i) \in \Delta$.

Theorem 2.7. Each $T_\Delta$ is a free group in $3n - 4$ generators.

We prove Theorem 2.7 in Section 2.14. We generalize it in Theorem 3.24 to all surfaces.

Clearly, the assignments $t_{ij} \mapsto x_{ij}$, $(i, j) \in \Delta$ define a homomorphism of algebras:

$$i_\Delta : QT_\Delta \rightarrow \mathcal{A}_\Delta,$$

where $QT_\Delta$ is the group algebra of $T_\Delta$.

Remark (see, e.g., (5.1)) that for a given algebra $\mathcal{A}$ with no zero divisors and a submonoid $S \subset \mathcal{A} \setminus \{0\}$ one has a universal localization $\mathcal{A}[S^{-1}]$ of $\mathcal{A}$ by $S$.

Theorem 2.8. For each triangulation $\Delta$ of $[n]$ one has:

(a) The homomorphism $i_\Delta$ given by (2.4) is an isomorphism of algebras.

(b) $\mathcal{A}_n = \mathcal{A}_\Delta[S^{-1}]$, where $S$ is the multiplicative submonoid of $\mathcal{A}_\Delta$ generated by all $x_{ij}$.

We will prove Theorem 2.8 in Section 2.14. In fact, Theorem 2.8(a) establishes a noncommutative cluster structure on $\mathcal{A}_n$ and Theorem 2.8(b) — a noncommutative Laurent Phenomenon (see also Section 2.2).
2.2. Noncommutative Laurent Phenomenon. For each even sequence \( i = (i_1, \ldots, i_{2m}) \in [n]^{2m} \) such that adjacent indices are distinct define the monomial \( x_i \in A_n \) by:

\[
x_i := x_{i_1} x_{i_2}^{-1} x_{i_3} x_{i_4}^{-1} \cdots x_{i_{2m-1}} x_{i_{2m-2}} x_{i_{2m-1}}^{-1} x_{i_{2m-2}}^{-1} .
\]

**Definition 2.9.** For a directed chord \((i, j), i, j \in [n], i \neq j\) and a triangulation \( \Delta \) of \([n]\), we say that a sequence \( i = (i_1, \ldots, i_{2m}) \in [n]^{2m} \) is \((i, j, \Delta)\)-admissible if:

(i) \( i_1 = i, i_2 = j \) and \( (i_s, i_{s+1}) \in \Delta \) for \( s = 1, \ldots, 2m - 1 \);

(ii) each chord \((i_s, i_{s+1}), s = 1, \ldots, m - 1\) intersects \((i, j)\);

(iii) If \( p := (i_k, i_{k+1}) \cap (i, j) \neq \emptyset \) and \( q := (i_\ell, i_{\ell+1}) \cap (i, j) \neq \emptyset \) for some \( k < \ell \), then the point \( p \) of \((i, j)\) is closer in the path to \( i \) than the point \( q \).

We denote by \( \text{Adm}_{\Delta}(i, j) \) the set of all \((i, j, \Delta)\)-admissible sequences \( i \).

**Theorem 2.10.** *(Noncommutative Laurent Phenomenon)* Let \( \Delta \) be a triangulation of \([n]\). Then

\[
x_{ij} = \sum_{i \in \text{Adm}_{\Delta}(i, j)} x_i .
\]

for all \( i, j \in [n], i \neq j \).

We prove Theorem 2.10 in Section 2.3.

**Remark 2.11.** This is a noncommutative generalization of Schiffler’s formula (39).

Now we illustrate Theorem 2.10 for each *starlike* triangulation

\[
\Delta_i = \{(i, j), (j, i) \mid j \in [n] \setminus \{i\}\} \cup \{(k, k^\pm), k \in [n]\}, \quad i \in [n].
\]

**Example 2.12.** Fix \( i \in [n] \). Then for each \( k, \ell \in [n] \setminus \{i\} \) such that \((i, k, \ell)\) is cyclic, the following relation holds in \( A_n \):

\[
x_{k\ell} = \sum_s x_{ki} x_{s1}^{-1} x_{s\pm} x_{i\ell}^{-1}
\]

where summation is over all \( s = k, k^\pm, \ldots, \ell^\pm \) in cyclic order. Hence \( x_{k\ell} = i_{\Delta_i}(\sum_s t_s t_{s1}^{-1} t_{s\pm}^{-1} t_{i\ell}). \)

**Example 2.13.** (a) If \( n = 5 \) and \( \Delta = \{(1, 3), (3, 1), (1, 4), (4, 1); (i, i^\pm)\mid i \in [5]\} \), then

\[
x_{25} = x_{21} x_{41}^{-1} x_{45} + x_{23} x_{13}^{-1} x_{15} + x_{21} x_{31}^{-1} x_{34} x_{14}^{-1} x_{15} .
\]

(b) If \( n = 6 \) and \( \Delta = \{(1, 3), (3, 1), (3, 6), (6, 3), (4, 6), (6, 4); (i, i^\pm)\mid i \in [6]\} \), then

\[
x_{25} = x_{23} x_{63}^{-1} x_{65} + x_{21} x_{31}^{-1} x_{36} x_{46}^{-1} x_{45} + x_{21} x_{31}^{-1} x_{34} x_{64}^{-1} x_{65} + x_{23} x_{13}^{-1} x_{16} x_{46}^{-1} x_{45} + x_{23} x_{13}^{-1} x_{16} x_{36}^{-1} x_{34} x_{64}^{-1} x_{65} .
\]

A triangulation of a hexagon and all \((2, 5)\)-admissible sequences

In fact, we will streamline the formula for \( x_{ij} \) by introducing new coordinates \( y_{ij}^k \in A_n \) for distinct \( i, j, k \in [n] \) by:

\[
y_{ij}^k := x_{ki} x_{kj}^{-1} .
\]

We refer to \( y_{ij}^k \) as *noncommutative sectors* and denote by \( Q_n \) the subalgebra of \( A_n \) generated by all \( y_{ij}^k \) (with the convention \( y_{ii}^k = 1 \)).
Theorem 2.14. The algebra $\mathcal{Q}_n$ is generated by all $y_{ij}^k$ subject to the relations:

(i) (triangle relations):

\begin{equation}
y_{ij}^k y_{ji}^k = 1, \quad y_{ij}^k y_{jk} y_{ki} = 1
\end{equation}

for distinct $i, j, k \in [n]$ and

\begin{equation}
y_{ij}^k y_{jk} y_{ki}^k = 1
\end{equation}

for distinct $i, j, k, \ell \in [n]$.

(ii) (exchange relations) For all cyclic $(i, j, k, \ell)$ in $[n]$:

\begin{equation}
y_{ij}^k y_{ji}^k = y_{il}^k y_{ji}^k.
\end{equation}

We prove Theorem 2.14 in Section 2.10.

We start with a more economical presentation of $\mathcal{Q}_n$. Since $\mathcal{Q}_n$ is a subalgebra of $\mathcal{S}_n$ generated by all cyclic $(i, j, k)$, then

\begin{equation}
\mathcal{Q}_n \subset \mathcal{S}_n \subset \mathcal{A}_n.
\end{equation}

Example 2.16. Following Example 2.13:

(a) If $n = 5$ and $\Delta = \{(1, 3), (3, 1), (1, 4), (4, 1); (i, i^\pm) | i \in [5]\}$, then

\begin{equation}
y^2_{15} = y^4_{15} + y^2_{13} y^2_{15} + y^3_{13} y^2_{15}.
\end{equation}

(b) If $n = 6$ and $\Delta = \{(1, 3), (3, 1), (3, 6), (6, 3), (4, 6), (6, 4); (i, i^\pm) | i \in [5]\}$, then

\begin{equation}
y^2_{15} = y^3_{16} y^2_{15} + y^2_{13} y^2_{15} + y^3_{14} y^2_{15} + y^2_{13} y^2_{16} y^2_{15} + y^2_{13} y^2_{16} y^2_{15} + y^3_{13} y^2_{16} y^2_{15}.
\end{equation}

Similarly to Section 2.14, for each triangulation $\Delta$ of $[n]$ define:

- The subalgebra $\mathcal{Q}_\Delta$ of $\mathcal{Q}_n$ generated by all $y_{ij}^k$, $i, j, k \in [n]$ such that $(i, k), (j, k) \in \Delta$.
- The subgroup $U_\Delta$ of $T_\Delta$ generated by

\begin{equation}
y_{ij}^k := t_{ki}^{-1} t_{kj}
\end{equation}

for $i, j, k \in [n]$ such that $(i, k), (j, k) \in \Delta$.

Clearly, the restriction of the homomorphism $i_\Delta$ given by 2.4 to $\mathcal{Q}_\Delta \subset \mathcal{Q}T_\Delta$ is a surjective homomorphism of algebras:

\begin{equation}
i_\Delta : \mathcal{Q}U_\Delta \rightarrow \mathcal{Q}_\Delta.
\end{equation}

Since $\mathcal{Q}_\Delta$ is a subalgebra of $\mathcal{A}_\Delta$, the following is an immediate corollary of Theorem 2.8.

Corollary 2.17. For each triangulation $\Delta$ one has:

(a) The homomorphism $i_\Delta$ given by 2.4 is an isomorphism.

(b) $\mathcal{Q}_n = \mathcal{Q}_\Delta[S'^{-1}]$ for some multiplicative submonoid $S' \subset \mathcal{Q}_\Delta \setminus \{0\}$.

(c) $i_\Delta$ extends to a monomorphism of algebras $\mathcal{Q}_n \hookrightarrow \text{Frac}(\mathcal{Q}_\Delta) = \mathcal{F}_{2n-4}$.

2.3. Regular elements in noncommutative polygons. We start with a more economical presentation of $\mathcal{A}_n$. Denote by $\mathcal{U}_n$ the subalgebra of $\mathcal{A}_n$ generated by all $x_{ij}, i \neq j$ and $x_{i,i}^{-1}$. The following result is obvious.

Lemma 2.18. The algebra $\mathcal{U}_n$ satisfies the following relations

(a) (reduced triangle relations) for all $i, j \in [n], i \notin \{j^-\}$:

\begin{equation}
x_{i,j}^{-1} x_{j,j}^{-1} x_{ji} = x_{ij} x_{j,j}^{-1} x_{ji}^{-1}
\end{equation}

(b) (reduced exchange relations) for all cyclic $(i, j, k)$ in $[n]$ such that $i^- \notin \{j, k\}$:

\begin{equation}
x_{ij} x_{j,j}^{-1} x_{j,k} = x_{ik} + x_{i,j} x_{j,j}^{-1} x_{jk}, \quad x_{k,j} x_{j,j}^{-1} x_{ji} = x_{k,i} + x_{kj} x_{j,j}^{-1} x_{j,-i}.
\end{equation}
Remark 2.19. We expect that these relations are defining for the algebra $\mathcal{U}_n$.

Noncommutative Laurent phenomenon $\text{(2.10)}$ guarantees that $\mathcal{U}_n$ belongs to each subalgebra $A_{\Delta}$. The following conjecture implies, in particular, that $\mathcal{U}_n$ is a totally noncommutative analogue of the upper cluster algebra of type $A_{n-3}$.

Conjecture 2.20. For each $n \geq 2$ one has:

\begin{equation}
\mathcal{U}_n = \bigcap_{\Delta} A_{\Delta},
\end{equation}

where the intersection is over all triangulations $\Delta$ of $[n]$.

We say that an element $x \in A_n$ is regular if it belongs to each subalgebra $A_{\Delta}$ as $\Delta$ runs over all triangulations $\Delta$ of $[n]$. Thus, Conjecture 2.20 asserts that each regular element of $A_n$ belongs to $\mathcal{U}_n$, i.e., is a noncommutative polynomial in $x_{ij}$ and $x_{i,1,\pm}$.

2.4. Noncommutative angles. Now we take advantage of the “invariant” algebra $Q_n$ and will view the ambient algebra $A_n$ as some “Galois extension” of $Q_n$ (in fact, Proposition 2.24 below guarantees that $A_n$ is freely generated by $x_{i,1,\pm}$, $i \in [n]$ and $Q_n$).

However, we want a more symmetric and “geometric” presentation of $A_n$ over $Q_n$. The following result provides such a presentation of $A_n$, $n \geq 3$.

Proposition 2.21. The algebra $A_n$ is generated by $Q_n$ and $(T_i^{jk})_{i,j,k \in [n]}$ subject to:

(i) (triangle relations) $T_i^{jk} = T_i^{kj}$ for all distinct $i,j,k \in [n]$.

(ii) (modified exchange relations) $T_i^{jk} = T_j^{ik} + T_i^{ki}$ for any cyclic $(i,j,k,\ell)$ in $[n]$.

(iii) (consistency relations) $y_i^{jk}T_i^{jk} = y_i^{kj}T_i^{jk}$ for all distinct $i,j,k,\ell \in [n]$.

Proof. Denote by $A_n'$ the algebra whose presentation is given in the proposition. It is easy to see that the assignments $y_i^{jk} \mapsto x_{ij}^{-1}x_{kj}, T_i^{jk} \mapsto x_{ij}^{-1}x_{kj}x_{ik}^{-1}$ for distinct $i,j,k \in [n]$ define a homomorphism of algebras $A_n' \to A_n$.

On the other hand, the consistency relations imply that the element $(T_i^{jk})^{-1}y_i^{jk}$ does not depend on $k$.

The following is immediate.

Lemma 2.22. The assignments $x_{ij} \mapsto (T_i^{jk})^{-1}y_i^{jk}$ for distinct $i,j \in [n]$ define a homomorphism of algebras $f : A_n \to A_n'$.

In particular, $f(x_{ji}^{-1}x_{jk}x_{ik}^{-1}) = y_{jk}^{ij}T_j^{ik}(T_i^{jk})^{-1}y_{ik}^{ji}y_{ij}^{jk}T_i^{jk} = y_{jk}^{ij}y_{ik}^{ji}y_{ij}^{jk}T_i^{jk}$ by (2.7).

These homomorphisms are, clearly, inverse to each other and hence are isomorphisms.

The proposition is proved.

We refer to $T_i^{jk} := x_{ji}^{-1}x_{jk}x_{ik}^{-1}$ for all distinct $i,j,k \in [n]$ as noncommutative angles by a number of reasons. First, because of the triangle relations in Proposition 2.21 (so that we can attach $T_i^{jk}$ to the angle in the triangle $(i,j,k)$ at the vertex $i$) and, second, because of the modified exchange relations (ii) of Proposition 2.21 can be viewed as an “addition law” of angles in a quadrilateral. In fact, such an addition law holds in more general situation.

Corollary 2.23. For any cyclic $(i_0, i_1, \ldots, i_\ell)$ one has: $T_{i_0}^{i_1,i_0} = T_{i_0}^{i_1,i_2} + T_{i_0}^{i_2,i_3} + \cdots + T_{i_0}^{i_{\ell-1},i_\ell}$. In particular, $T_{i_0}^{i_1,n} = T_{i_0}^{i_1,i_2} + T_{i_0}^{i_1,i_3} + \cdots + T_{i_0}^{i_1,i_{n-1}}$.

Moreover, this view is supported by the following observation. For each triangulation $\Delta$ of $n$ and each $i \in [n]$ define the total angle $T^\Delta_i$ around the vertex $i$ to be the sum of all noncommutative angles in $\Delta$ at the vertex $i$. For instance, we have in Example 2.16.

\[
T_i^\Delta = T_1^{23} + T_1^{14} + T_1^{45}, \quad T_2^\Delta = T_2^{13} + T_2^{34} + T_2^{45}, \quad T_3^\Delta = T_3^{12} + T_3^{14} + T_3^{15}, \quad T_4^\Delta = T_4^{13} + T_4^{15}, \quad T_5^\Delta = T_5^{14}.
\]

Corollary 2.24. $T_i^\Delta = T_i^{1,+,\pm}$ for any triangulation $\Delta$ of $[n]$ and any $i \in [n]$. In particular, $T_i^\Delta$ does not depend on a choice of $\Delta$. 
Remark 2.25. Based on Corollary 2.24 we can view $T_i := T_i^{-i,i}$ as the total angle of the noncommutative $n$-gon at the vertex $i$. The sum of all total angles $T := T_1 + T_2 + \cdots + T_n$ also does not depend on a choice of triangulations and, in particular, can be specialized to any constant value (e.g., to $\pi \cdot (n-2)$).

Remark 2.26. The independence of $T_i$ of a choice of $\Delta$ means that $T_i$ is invariant under noncommutative mutations. We will encounter the noncommutative angles again in Section 3.

2.5. Big triangle group of noncommutative polygons. For each $n \geq 2$ let $T_n$ be a group generated by $t_{ij}, i, j \in [n], i \neq j$ subject to the triangle relations

$$t_{ij}t_{kj}^{-1}t_{ki} = t_{ik}t_{jk}^{-1}t_{ji}$$

for all distinct $i, j, k \in [n]$; and refer to this group as the big triangle group of the $n$-gon.

The following is obvious.

Lemma 2.27. For any $n \geq 3$ one has:

(a) the assignments $t_{ij} \mapsto \begin{cases} t_{ij} & \text{if } i = n \\ t_{i1} & \text{if } j = n \end{cases}$ for $i, j \in [n], i \neq j$ (with the convention $t_{11} = 1$) define an epimorphism of groups $\pi^+_n : T_n \to T_{n-1}$.

(b) The assignments $t_{ij} \mapsto t_{ij}$ for $i, j \in [n-1], i \neq j$ define an injective homomorphism of groups $T_{n-1} \hookrightarrow T_n$ which splits $\pi^+_n$.

The following result gives a presentation of $T_n$.

Proposition 2.28. For each $n \geq 3$ the group $T_n$ is generated by $t_{ij}, 1 \leq i < j \leq n$ and $t_{ii}, i = 2, \ldots, n$, subject to:

$$t_{i1}t_{j1}^{-1}t_{jk}t_{ik}^{-1}t_{ij} = t_{ik}t_{jk}^{-1}t_{ij}t_{i1}^{-1}t_{j1}$$

for all $2 \leq i < j < k \leq n$.

Proof. By a slight abuse of notation, set $T_{ij}^{jk} = t_{ji}^{-1}t_{jk}t_{ik}^{-1}$. Clearly, if $n = 3$, then $T_3$ is free in $t_{12}, t_{13}, t_{23}, t_{21}, t_{31}$. Furthermore, let $n \geq 4$. Then we can group the defining relations for $T_n$ into the following quadruples for $2 \leq i < j < k \leq n$:

$$T_{ij}^{jk} = T_{ji}^{kj}, \quad T_{jk}^{ik} = T_{kj}^{ki}$$

It is easy to see that each of these relations (2.16) is equivalent to the following quadruple of relations (here $(i', j') \in \{(i, j), (i, k), (j, k)\}$):

$$t_{i'j'} = t_{j'i'}t_{i'}^{-1}t_{j'i'}^{-1}t_{i'}, t_{i'j'}t_{j'i'}^{-1}t_{i'}^{-1}t_{j'i'} = t_{j'i'}^{-1}t_{j'i'}^{-1}t_{j'i'}^{-1}t_{j'i'}$$

Thus, eliminating the redundant generators $t_{i'j'}, t_{j'i'}$, we finish the proof of the proposition.

The following is obvious.

Lemma 2.29. For each $n$ one has:

(a) The assignments $t_{ij} \mapsto x_{ij}$ define a ring epimorphism $\pi_n : \mathbb{Z}T_n \twoheadrightarrow A_n$.

(b) For each triangulation $\Delta$ of $[n]$ the assignments $t_{ij} \mapsto t_{ij}$ define a group homomorphism $\hat{j}_\Delta : T_\Delta \to T_n$.

(c) The symmetric group $S_n$ acts on $T_n$ by automorphisms: $\sigma(t_{ij}) := t_{\sigma(i)\sigma(j)}$ for $\sigma \in S_n, i, j \in [n], i \neq j$.

Conjecture 2.30. The restriction of $\pi_n$ to $T_n$ is an isomorphism of monoids $T_n \cong A_n^\times$.

Theorem 2.31. For any triangulation $\Delta$ of $[n]$ there exists an epimorphism $\pi_\Delta : T_n \twoheadrightarrow T_\Delta$ such that

$$\hat{j}_\Delta \circ \pi_\Delta = Id_{T_\Delta}.$$

In particular, $\hat{j}_\Delta$ is an injective homomorphism $T_\Delta \hookrightarrow T_n$.

We prove Theorem 2.31 in Section 2.7.2.

The following is obvious.

Corollary 2.32. For any triangulations $\Delta, \Delta'$ of $n$ the composition $\tau_{\Delta, \Delta'} := \pi_{\Delta'} \circ \hat{j}_\Delta$ is an isomorphism $T_\Delta \to T_{\Delta'}$ such that $\tau_{\Delta, \Delta} = Id_{T_\Delta}$ and $\tau_{\Delta, \Delta'} \circ \tau_{\Delta', \Delta''} = \tau_{\Delta, \Delta''}$ for any triangulation $\Delta''$ of $[n]$. 
2.6. Representation of $A_n$ and $Q_n$ in noncommutative $2 \times n$ matrices. In what follows, we identify the free skew field generated by all $a_{1i}, a_{2i}, i \in [n]$ with $F_{2n}$ and view it as totally noncommutative rational functions on the space $Mat_{2 \times n}$ of $2 \times n$ matrices.

Following [30] and [3] (see also [31]) define $2 \times 2$-positive quasiminors by

$$Q^k_{ij} = \begin{vmatrix} a_{11} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}_{+} = \text{sgn}(i-j)(a_{1j} - a_{1i}a_{2i}^{-1}a_{2j}),$$

for $i, j \in [n]$ and positive quasi-Plücker coordinates $Q^k_{ij}$ for distinct $i, j, k \in [n]$ by:

$$Q^k_{ij} = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & a_{2i} \end{vmatrix}_{+} = \frac{1}{a_{1j}} \text{sgn}(j-i)(a_{2j} - a_{2i}a_{1i}^{-1}a_{1j})$$

(2.17)

Theorem 2.33. For each $n \geq 2$ one has:

(a) The assignments $x_{ij} \mapsto \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}_{+}, x_{ij} \mapsto \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}_{+}$ for all distinct $i, j \in [n]$ define respectively the homomorphisms of algebras

$$\varphi^+_n : A_n \to F_{2n}, \quad \varphi^-_n : A_n \to F_{2n}.$$

(b) The restrictions of $\varphi^+_n$ and $\varphi^-_n$ to $Q_n$ are equal to an algebra homomorphism $\varphi_n : Q_n \to F_{2n}$ such that $\varphi_n(y_{ij}^k) = Q^k_{ij}$ for distinct $i, j, k \in [n]$.

Proof. It follows from [30] Section 4.4 that positive quasi-Plücker coordinates satisfy (2.7), (2.8), and (2.9) This implies that the assignments

$$y_{ij} \mapsto Q^k_{ij}$$

(2.19)

for distinct $i, j \in [n]$ define a homomorphism of algebras $\varphi_n : Q_n \to F_{2n}$.

Furthermore, for any $Q$-algebras $A$ and $B$ denote by $A * B$ their free product, i.e., the universal algebra generated by $A$ and $B$ as subalgebras (with no relations between them). The most fundamental property of the free product is that any algebra homomorphisms $f_1 : A \to C$, $f_2 : B \to C$ canonically lift to an algebra homomorphism $f_1 * f_2 : A * B \to C$.

Denote by $F_m$ the free group generated by $c_i^\pm 1$, $i = 1, \ldots, m$.

By definition, the group algebra $QF_m$, is free Laurent polynomial algebra $Q < c_1^\pm 1, \ldots, c_m^\pm 1 >$.

Proposition 2.34. For each $n \geq 2$ the assignments $x_{ij} \mapsto c_i \cdot y_{i-j}^k$, $i, j \in [n], i \neq j$ define an isomorphism of algebras

$$f : A_n \cong (QF_n) * Q_n.$$

Proof. Let us prove that the homomorphism (2.20) is well-defined. We need the following obvious fact.

Lemma 2.35. Let $B$ be a $Q$-algebra and let $c_1, \ldots, c_n$ be invertible elements of $B$. Then the assignments

$$x_{ij} \mapsto c_i \cdot x_{ij}$$

(2.21)

for $i, j \in [n], i \neq j$ define a homomorphism of algebras $A_n \to B * A_n$.

By the above Lemma $B := QF_n$ generated by $c_i^\pm 1$, $i \in [n]$, the assignments (2.21) define a homomorphism of algebras

$$A_n \to (QF_n) * A_n.$$

Furthermore, the assignments $c_i \mapsto c_i \cdot x_{i-j}^{-1}$, $i \in [n]$ define an algebra homomorphism $f_1 : QF_n \to (QF_n) * A_n$ and the identity map $A_n \to A_n$ defines a homomorphism of algebras $f_2 : A_n \to (QF_n) * A_n$. This gives an algebra homomorphism $f_1 * f_2 : (QF_n) * A_n \to (QF_n) * A_n$ determined by $c_i \mapsto c_i \cdot x_{i-j}^{-1}, x_{ij} \mapsto x_{ij}$. Then the composition of the homomorphism (2.22) with $f_1 * f_2$ is a homomorphism of algebras

$$A_n \to (QF_n) * A_n.$$
given by \( x_{ij} \mapsto c_i \cdot x_{ij} \mapsto c_i \cdot x_{i,j}^{-1} \cdot x_{ij} = c_i \cdot y_{i,j}^1 \) for all \( i, j \in [n] \), \( i \neq j \). Since the image of the latter homomorphism belongs to \((\mathbb{Q}F_n) \ast \mathbb{Q}_n\), we see that the algebra homomorphism \( f : A_n \to (\mathbb{Q}F_n) \ast \mathbb{Q}_n \) given by (2.20) is well-defined.

It remains to show that \( f \) is invertible. Indeed, denote by \( f'_1 : \mathbb{Q}F_n \to A_n \) the homomorphism of algebras given by \( c_i \mapsto x_{i,i} \), \( i \in [n] \) and denote by \( f'_2 \) the natural inclusion \( \mathbb{Q}_n \hookrightarrow A_n \). This defines a homomorphism of algebras \( g = f'_1 \ast f'_2 : (\mathbb{Q}F_n) \ast \mathbb{Q}_n \to A_n \) which is determined by \( c_i \mapsto x_{i,i}, y_{ij} \mapsto y_{ij} \). This immediately implies that \((g \circ f)(x_{ij}) = g(c_i \cdot y_{i,j}^1) = x_{i,i}^{-1} \cdot y_{i,j} = x_{ij} \) for all \( i \neq j \). Therefore, \( g \circ f = 1d \). Similarly,

\[
(f \circ g)(c_i) = f(x_{i,i}^{-1}) = c_i \cdot y_{i,i}^1 = c_i \cdot 1 = c_i, \quad (f \circ g)(y_{i,j}^1) = f(y_{i,j}^1)
\]

\[
= f(x_{i,i}^{-1} \cdot x_{ij}) = f(x_{i,i}^{-1})^{-1} f(x_{ij}) = (c_i \cdot x_{i,i}^{-1})^{-1} c_i \cdot x_{ij} = x_{i,i}^{-1} \cdot x_{ij} = y_{ij}.
\]

Therefore, \( f \circ g = 1d \) as well.

The proposition is proved.

Now we can finish the proof of Theorem 2.38. Define algebra homomorphisms \( \psi_n^+, \psi_n^- : \mathbb{Q}F_n \to \mathcal{F}_2n \) by

\[
\psi_n^+(c_i) = \begin{bmatrix} a_{1i} & a_{1i,i} \\ a_{2i} & a_{2i,i} \end{bmatrix}, \quad \psi_n^-(c_i) = \begin{bmatrix} a_{1i} & a_{1i,i} \\ a_{2i} & a_{2i,i} \end{bmatrix},
\]

for \( i \in [n] \). Universality of free products gives natural homomorphisms of algebras

\[
\psi_n^+ \ast \phi_n, \psi_n^- \ast \phi_n : (\mathbb{Q}F_n) \ast \mathbb{Q}_n \to \mathcal{F}_2n,
\]

where \( \phi_n \) is given by (2.19). Composing these homomorphisms with the isomorphism (2.20), we obtain respectively algebra homomorphisms \( \varphi_n^+, \varphi_n^- : A_n \to \mathcal{F}_2n \), whose restriction to \( \mathbb{Q}_n \) equals \( \psi_n^- \).

Finally, note that

\[
\varphi_n^+(x_{ij}) = \psi_n^+ \ast \phi_n(c_i \cdot y_{i,j}^1) = \begin{bmatrix} a_{1i} & a_{1i,i} \\ a_{2i} & a_{2i,i} \end{bmatrix} \cdot Q_{i,j}^1 = \begin{bmatrix} a_{1i} & a_{1i,j} \\ a_{2i} & a_{2i,j} \end{bmatrix},
\]

\[
\varphi_n^-(x_{ij}) = \psi_n^- \ast \phi_n(c_i \cdot y_{i,j}^1) = \begin{bmatrix} a_{1i} & a_{1i,i} \\ a_{2i} & a_{2i,i} \end{bmatrix} \cdot Q_{i,j}^1 = \begin{bmatrix} a_{1i} & a_{1i,j} \\ a_{2i} & a_{2i,j} \end{bmatrix}
\]

for all distinct \( i, j \in [n] \).

Theorem 2.33 is proved.

Remark 2.36. Proposition 2.34 is a noncommutative algebraic analogue of the following assertion: if a group \( G \) acts freely on a set \( X \), then there a bijection \( X \cong G \times (X/G) \).

Remark 2.37. Unlike \( \varphi_n \) (see Theorem 2.35), the homomorphisms \( \varphi_n^+ \) and \( \varphi_n^- \) are not injective. In particular, one can show that \( x_{ij}x_{kj}^{-1} + 1 - x_{ik}x_{jk}^{-1} \in \text{Ker} \varphi_n^- \) for all distinct \( i, j, k \in [n] \).

For any groups \( G \) and \( H \) denote by \( G \ast H \) their free product. It is well-known (see, e.g., [13]) that \( \mathbb{Q}(G \ast H) = (\mathbb{Q}G) \ast (\mathbb{Q}H) \).

Proposition 2.38. For each triangulation \( \Delta \) of \( [n] \) the assignments \( t_{ij} \mapsto c_i \ast u_{i,j}^1 \) for all \( (i, j) \in \Delta \) (in the notation of (2.11)) define an isomorphism of (free) groups

\[
(2.23) \quad T_\Delta \cong \mathbb{Q}_n \ast \mathbb{Q}_\Delta.
\]

Proof. We essentially copy the proof of Proposition 2.34. Indeed, the following fact is obvious.

Lemma 2.39. Let \( G \) be any group and let \( c_1, \ldots, c_n \in G \). Then for any triangulation \( \Delta \) of \( [n] \) the assignments

\[
t_{ij} \mapsto c_i \ast t_{ij}
\]

for \( (i, j) \in \Delta \), define a homomorphism of groups \( T_\Delta \to G \ast T_\Delta \).
Clearly, the assignments \( c_i \mapsto c_i \cdot t_{i,j}^{-1} \) for \( i \in [n] \) define a group homomorphism \( F_n \to F_n \ast \mathbb{T}_\Delta \). Composing this with (2.24), taken with \( G = F_n \ast \mathbb{T}_\Delta \) and applying the multiplication homomorphism \( \mathbb{T}_\Delta \ast \mathbb{T}_\Delta \to \mathbb{T}_\Delta \), we obtain a group homomorphism: \( \mathbb{T}_\Delta \to F_n \ast \mathbb{T}_\Delta \ast \mathbb{T}_\Delta \to F_n \ast \mathbb{T}_\Delta \) given by \( t_{i,j} \mapsto c_i \cdot u_{i,j}^{-1} \) for all \( i,j \in \Delta \). Clearly, the image of this homomorphism contains all \( c_i \) and \( u_{i,j} \), \( (i,j) \in \Delta \) and is contained in \( F_n \ast \mathbb{U}_\Delta \), hence this gives a group homomorphism (2.23). It is also clear that the homomorphism \( F_n \ast \mathbb{U}_\Delta \to \mathbb{T}_\Delta \) given by \( c_i \mapsto t_{i,i}^{-1} \), \( u_{i,j}^{-1} \mapsto u_{i,j}^{-1} \) is inverse of (2.23).

The proposition is proved. \( \square \)

Taking into account that \( F_n \ast F_m \cong F_{m+n} \), we obtain an obvious corollary from Theorem 2.7.

**Corollary 2.40.** For each triangulation \( \Delta \) of \([n]\) the group \( \mathbb{U}_\Delta \) is isomorphic to \( F_{2n-4} \), the free group in \( 2n - 4 \) generators.

Furthermore, denote by \( F'_{2n-4} \) the skew sub-field of \( F_{2n} \) generated by \( \varphi_n(\mathbb{Q}_n) \), i.e., by all \( Q_{ij} \).

**Proposition 2.41.** \( F'_{2n-4} \) is isomorphic to \( F_{2n-4} \).

**Proof.** Denote:

\[
A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \end{pmatrix}, \quad B = \begin{pmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \end{pmatrix}, \quad C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]

so that \( A = [C \mid B] \).

The following lemma easily follows from Theorem 4.4.4 and Proposition 4.5.2 in [31].

**Lemma 2.42.** \( C^{-1}B = \begin{pmatrix} q_{13}^k & \cdots & q_{1n}^k \\ q_{23}^k & \cdots & q_{2n}^k \end{pmatrix} \), where \( q_{ij}^k = q_{ij}^k(A) = \text{sgn}(k-j) \text{sgn}(k-i)Q_{ij}^k \) for distinct \( i,j,k \in [n] \) are quasi-Plücker coordinates (in the notation (2.17)).

It was proved in [30] Section 4 that \( q_{ij}^k(A) = q_{ij}^k(DA) \) for all distinct \( i,j,k \in [n] \) and any invertible \( 2 \times 2 \) matrix \( D \) over \( F_{2n} \). In particular, taking \( D = C^{-1} \), we see that \( q_{ij}^k = q_{ij}^k([C \mid B]) = q_{ij}^k([I_2 \mid C^{-1}B]) \), therefore, each \( q_{ij}^k \) belongs to the skew subfield of \( F_{2n} \) generated by the matrix coefficients of \( C \) (here \( I_2 \) is the \( 2 \times 2 \) identity matrix). This proves that \( F'_{2n-4} \) is a sub-field of \( F_{2n} \) generated by the entries of \( C^{-1}B \), i.e., by all \( q_{ij}^k ; j = 3, \ldots, n \).

It remains to show that matrix coefficients of \( C^{-1}B \) (freely) generate a free subfield of \( F_{2n} \). We need the following obvious fact.

**Lemma 2.43.** Let \( F \) be a skew field, \( C \in GL_m(F) \) and \( B \in \text{Mat}_{m,n-m}(F) \) such that matrix coefficients of the partitioned matrix \( A = [C \mid B] \) generate \( F \). Then the matrix coefficients of \([C \mid C^{-1}B] \) also generate \( F \).

Now we take \( m = 2 \) and \( B,C \) as in (2.25), \( F = F_{2n} \), the free skew field freely generated by matrix coefficients of \( A = [C \mid B] \). Then \( C \in GL_2(F_{2n}) \) and \( B \in \text{Mat}_{2,n-2}(F_{2n}) \). Then, by Lemma 2.13 the matrix coefficients \( A' = [C \mid C^{-1}B] \) also generate \( F_{2n} \). Since \( A' \) is \( 2 \times n \), then Proposition 5.8 implies that the matrix coefficients of \( A' \) are free generators of \( F_{2n} \). In particular, the matrix coefficients of the \( 2 \times (n-2) \) matrix \( C^{-1}B \) are free generators of the free skew sub-field of \( F_{2n} \). That is, \( F'_{2n-4} \) is freely generated by the matrix coefficients \( q_{ij}^2 ; j = 3, \ldots, n \) of \( C^{-1}B \).

The proposition is proved. \( \square \)

**Remark 2.44.** Proposition 2.41 and its proof generalize verbatim to \( m \times n \) matrices.

**Theorem 2.45.** For each triangulation \( \Delta \) of \([n]\) the homomorphism

\[
\varphi_n \circ i'_\Delta : \mathbb{Q}_\Delta \to F'_{2n-4}
\]

is injective.

**Proof.** Taking any free generating set \( u_1, \ldots, u_{2n-4} \) of the free group \( \mathbb{U}_\Delta \cong F_{2n-4} \), we see that \( t_i := \varphi_n(i'_\Delta(u_i)) \), \( i = 1, \ldots, 2n-4 \) generate \( F'_{2n-4} \) due to the following fact.

**Lemma 2.46.** For each triangulation \( \Delta \) of \([n]\) the image \( \varphi_n(\mathbb{Q}_\Delta) \) generates the skew field \( F'_{2n-4} \).
Proof. Denote by $F_{2n-4}'$ the skew subfield of $F_{2n}$ generated by image $\varphi_n(Q_\Delta)$. Since image $Q_\Delta \subset Q_n$, we have an obvious inclusion $F_{2n-4}' \subseteq F_{2n-4}$. 

Therefore using Proposition[58] with $\ell = 2n - 4$, we see that $t_1, \ldots, t_{2n-4}$ are free generators of $F_{2n-4}'$ and hence the homomorphism (2.26) is injective.

Theorem[2.46] is proved.

2.7. Some symmetries of noncommutative polygons. First, we establish a new presentation of $A_n$ in generators $\tilde{x}_{ij}^{\pm 1} := \text{sgn}(j-i)x_{ij}^{\pm 1}$ and $\tilde{T}_{ij}^{k} = \tilde{x}_{ij}^{-1}\tilde{x}_i\tilde{x}_{jk}\tilde{x}_{ik}^{-1} = \text{sgn}(i-j)\text{sgn}(k-j)\text{sgn}(k-i)x_{ij}^{-1}x_{jk}x_{ik}^{-1}$ for distinct $i, j, k \in [n]$ (see also Section[2.4]). Also define $\tilde{y}_{ij}^k = \tilde{x}_{ki}^{-1}\tilde{x}_{kj} = \text{sgn}(i-k)\text{sgn}(j-k)y_{ij}^k$ for distinct $i, j, k \in [n]$. We need the following useful fact.

Lemma 2.47. For each $n \geq 2$ one has:

(a) The algebra $A_n$ is generated by $\tilde{x}_{ij}$ for distinct $i, j \in [n]$ subject to the relations:

\begin{equation}
\tilde{T}_{ij}^{jk} = -\tilde{T}_{ij}^{kj}
\end{equation}

for any distinct $i, j, k \in [n]$:

\begin{equation}
\tilde{T}_{ij}^{jk} + \tilde{T}_{ij}^{jk} + \tilde{T}_{ij}^{j\ell} = 0
\end{equation}

for any distinct $i, j, k, \ell \in [n]$.

(b) The algebra $Q_n$ is generated by all $\tilde{y}_{ij}^k$ subject to the relations:

\begin{equation}
\tilde{y}_{ij}^j \tilde{y}_{ji}^{-1} = 1, \quad \tilde{y}_{ij}^k \tilde{y}_{jk} \tilde{y}_{ki}^{-1} = -1
\end{equation}

for distinct $i, j, k \in [n]$,

\begin{equation}
\tilde{y}_{ij}^k \tilde{y}_{jk} \tilde{y}_{ki}^{-1} = 1, \quad \tilde{y}_{ik} \tilde{y}_{ki}^{-1} + \tilde{y}_{ij}^k \tilde{y}_{ji} = 1
\end{equation}

for distinct $i, j, k, \ell \in [n]$.

Proof. Prove (a). Denote by $A'_n$ the algebra freely generated by all $\tilde{x}_{ij}^{\pm 1}, i \neq j$. That is, $A'_n$ is the group algebra of a free group in $n^2 - n$ generators. Define $\tilde{r}_{ijk} = (\tilde{T}_{ij}^{k})^{-1}$. for all distinct $i, j, k \in [n]$. Clearly,

\[ \tilde{r}_{ijk} = \tilde{x}_{ki}^{-1}\tilde{x}_{kj}\tilde{x}_{ki}^{-1}\tilde{x}_{ik}\tilde{x}_{jk}\tilde{x}_{ik}^{-1}\tilde{x}_{ij} = -x_{k1}^{-1}x_{k2}x_{k1}^{-1}x_{k1}x_{k2}^{-1}x_{k2} = \tilde{y}_{ij}^k \tilde{y}_{jk} \tilde{y}_{ki}^{-1} = -y_{ij}^k y_{jk} y_{ki}^{-1} \]

for all distinct $i, j, k \in [n]$. Denote by $I'$ the ideal in $A'_n$ generated by all $\tilde{r}_{ijk} + 1$. Then the quotient $A'_n := A'_n/I'$ is an algebra generated by $\tilde{x}_{ij}, i, j \in [n], i \neq j$ subject to the triangle relations (2.27).

Furthermore, for any distinct $i, j, k, \ell \in [n]$ define $\tilde{r}_{ij,k,\ell} \in A'_n$ by $\tilde{r}_{ij,k,\ell} = \tilde{T}_{ij}^{k} + \tilde{T}_{ij}^{k\ell} + \tilde{T}_{ij}^{j\ell}$. Clearly, $\tilde{r}_{ij,k,\ell} = -\tilde{r}_{ij,k,\ell}$ for all $i, j, k, \ell$, i.e., $\tilde{r}_{ij,k,\ell}$ is skew-symmetric in $j, k, \ell$ because of (2.27). Note also that

\[ \tilde{r}_{ij,k,\ell} = \tilde{x}_{ji}^{-1}(x_{j1}x_{j2}^{-1}x_{j1} + x_{ji}x_{ki}^{-1}x_{ki} - x_{ji})x_{ji}^{-1} = (y_{ij}^k y_{jk} + y_{ik}^j - y_{ij}^k)\tilde{x}_{ji}^{-1} = (y_{ij}^k y_{jk} + y_{ik}^j - y_{ij}^k)\tilde{x}_{ji}^{-1} \]

for all distinct $i, j, k, \ell \in [n]$. Moreover, if $(i, j, k, \ell)$ is cyclic, i.e., $(i, k)$ crosses $(j, \ell)$, this gives:

\[ \tilde{r}_{ij,k,\ell} = \pm x_{ji}^{-1}(x_{j1}x_{j2}^{-1}x_{j1} + x_{ji}x_{ki}^{-1}x_{ki} - x_{ji})x_{ji}^{-1} \]

Therefore, if we denote by $I$ the ideal in $A'_n$ generated by all $\tilde{r}_{ij,k,\ell}$, then, obviously, $A'_n/I \cong A_n$.

This proves (a).

Part (b) also follows because the relations (2.29) and (2.30) are equivalent to (2.29), (2.31), and (2.32). The lemma is proved.

In the notation of Lemma 2.44 define the action of the symmetric group $S_n$ on the set $X = \{\tilde{x}_{ij} | i, j \in [n], i \neq j\}$ by the formula

\[ w(\tilde{x}_{ij}) = \tilde{x}_{w(i),w(j)} \]

for all $w \in S_n, i, j \in [n], i \neq j$. 


Proposition 2.48. For each $n \geq 2$ one has:

(a) The above action uniquely extends to an action of $S_n$ on $A_n'$ by algebra automorphisms.

(b) The action commutes with homomorphisms $\varphi_n^+, \varphi_n^- : A_n \to \mathbb{F}_{2n}$ given by (2.18), where the action of $S_n$ on $\mathbb{F}_{2n}$ is given by $w(a_s,i) = a_{s,w(i)}$ for $s = 1, 2$, $i \in [n]$, $w \in S_n$.

(c) The subalgebra $Q_n$ is invariant under the $S_n$-action, i.e., $w(\tilde{y}_{ij}^k) = \tilde{y}_{w(i),w(j)}^w(k)$ for all $i, j, k \in [n]$, $w \in S_n$.

Proof. Prove (a). The following fact is obvious.

Lemma 2.49. The $S_n$ action on $\tilde{X}$ uniquely extends to that on $A_n'' = Q(\tilde{X})$ by algebra automorphisms.

Thus, it suffices to prove that the $S_n$-action on $A_n''$ preserves the ideal of triangle relations (2.27) and exchange relations (2.28).

Let us prove that the ideal $\mathcal{I}$ of $A_n''$ generated by all $\tilde{r}_{ijk}$ is invariant under the $S_n$-action. Indeed, for distinct $i, j, k \in [n]$ and $w \in S_n$ one has

$$w(\tilde{r}_{ijk}) = w(\tilde{x}_{ij})w(\tilde{x}_{ki})^{-1}w(\tilde{x}_{kj})^{-1}w(\tilde{x}_{jk})w(\tilde{x}_{ik})^{-1} = \tilde{r}_{w(i),w(j),w(k)}.$$  

This proves that $S_n(\mathcal{I}) = \mathcal{I}$ hence $S_n$ acts on $A_n''$ by algebra automorphisms.

It remains to prove that the ideal of exchange relations (2.28) in $A_n''$ is invariant under the $S_n$-action. Now we show that the ideal $\mathcal{I}$ of $A_n'' = A_n'/\mathcal{I}_n$ generated by all $\tilde{r}_{i,j,k,t}$ is invariant under the $S_n$-action. Indeed,

$$w(\tilde{r}_{i,j,k,t}) = w(\tilde{T}_i^j) + w(\tilde{T}_i^k) + w(\tilde{T}_i^l) = \tilde{r}_{w(i),w(j),w(k),w(l)} + \tilde{r}_{w(i),w(l),w(k),w(j)}$$

for all distinct $i, j, k, l \in [n]$. This proves that $S_n(\mathcal{I}) = \mathcal{I}$.

Part (a) is proved.

Part (b) follows from the fact that the homomorphisms $\varphi_n^+, \varphi_n^- : A_n \to \mathbb{F}_{2n}$ from Theorem 2.35 commute with the $S_n$-action.

Part (c) is obvious.

The proposition is proved. \hfill \Box

The Lie algebra $gl_n(\mathbb{Q})$ (viewed as $Mat_{n \times n}$) naturally acts on the space $Mat_{2 \times n}$ by right multiplications, i.e., $E_{ij}(a_{s,t}) = \delta_{s,j}a_{s,i}$ for $s \in \{1, 2\}$, $i, j, t \in [n])$, where $E_{ij} \in gl_n(\mathbb{Q})$ are the matrix units.

This action uniquely extends to $\mathbb{F}_{2n}$ by the Leibniz rule: $E(fg) = E(f)g + fE(g)$, $E(h^{-1}) = -h^{-1}E(h)h^{-1}$ for any $E \in gl_n(\mathbb{Q})$, $f, g \in \mathbb{F}_{2n}$, $h \in \mathbb{F}_{2n} \setminus \{0\}$.

Proposition 2.50. For each $n \geq 2$ there exists a unique action of $gl_n(\mathbb{Q})$ on $Q_n$ by derivations such that the homomorphism $\varphi_n : Q_n \to \mathbb{F}_{2n}$ from Theorem 2.37(b) is $gl_n(\mathbb{Q})$-equivariant. The action is given by:

$$E_{i,j}(\tilde{y}_{ij}^k) = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ \tilde{y}_{i,j'}^k & \text{if } j' = j \\ -\tilde{y}_{i',j}^k & \text{if } j' = i \\ -\tilde{y}_{i',j'}^k & \text{if } j' = k \end{cases}$$

for any distinct indices $i, j, k \in [n]$.

Proof. Indeed, in view of Theorem 2.45 it suffices to prove (2.31) for $q_{ij}^k = \varphi_n(\tilde{y}_{ij}^k)$. Indeed, if we abbreviate

$$\mathcal{E}_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$$

for distinct $i, j \in [n]$, then

$$E_{i,j'}(\mathcal{E}_{ij}) = E_{i,j'}(a_{1j} - a_{1i}a_{2j}^{-1}a_{2i}) = \begin{cases} 0 & \text{if } j' \notin \{i,j\} \\ a_{1i} - a_{1i}a_{2j}^{-1}a_{2i} & \text{if } j' = j \\ -E_{i',j}(a_{1i}a_{2j}^{-1})a_{2j} & \text{if } j' = i \\ \mathcal{E}_{i',j'} & \text{if } j' = i \end{cases}$$

because $-E_{i',j}(a_{1i}a_{2j}^{-1}) = -a_{1i}a_{2j}^{-1} + a_{1i}a_{2j}^{-1}a_{2i}a_{2j}^{-1} = -\mathcal{E}_{i',j'}a_{2j}^{-1}$ and $a_{2i}a_{2j} = -\mathcal{E}_{ij}^{-1}$ for $i \neq j$. Therefore,
Corollary 2.51. The following result is a generalization of Proposition 2.21. For each $$i,j,k \in [n]$$ one has: $$E_{i,j'}(g_{ij}^k) = E_{i,j'}(x_{ki}^{-1}x_{kj}) = E_{i,j'}(x_{ki}^{-1})x_{kj} + x_{ki}^{-1}E_{i,j'}(x_{kj}) = \begin{cases} x_{ki}^{-1}E_{i,j'}(x_{kj}) & \text{if } j' = j \\ E_{i,j'}(x_{ki}^{-1})x_{kj} & \text{if } j' = i \\ E_{i,j'}(x_{ki}^{-1})x_{kj} + x_{ki}^{-1}E_{i,j'}(x_{kj}) & \text{if } j' = k \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$E_{i',k}(x_{ki}^{-1})x_{kj} + x_{ki}^{-1}E_{i',k}(x_{kj}) = -x_{ki}^{-1}(x_{ki}^{-1}x_{kj}x_{ki}^{-1})x_{kj} + x_{ki}^{-1}(x_{ki}^{-1}x_{kj}x_{ki}^{-1})x_{kj}$$

because

$$(x_{ki} - x_{kj})^{-1}x_{kj} = ((a_{1k} - a_{1i}a_{2k}^{-1}a_{2k})(a_{1k} - a_{1j}a_{2k}^{-1}a_{2k}) - (-a_{1k}^{-1}a_{2j})) = -(a_{1i}a_{2j}^{-1}a_{2j} + a_{1j}) = -x_{ij}$$

Therefore,

$$E_{i,j'}(g_{ij}^k) = \begin{cases} 0 & \text{if } j' \notin \{i, j, k\} \\ x_{ki}^{-1}x_{k,i'} & \text{if } j' = j \\ x_{ki}^{-1}x_{k,i'}x_{ki}^{-1}x_{kj} & \text{if } j' = i \\ x_{ki}^{-1}x_{k,i'}x_{ki}^{-1}x_{kj} & \text{if } j' = k \end{cases}$$

The proposition is proved. □

For $$i, j \in [n]$$ define the elements $$\tilde{y}_{ij} \in F_n$$ by:

$$\tilde{y}_{ij} = \tilde{y}_{i,-j} = \tilde{x}_{i,i}^{-1}$$

(with the convention that $$y_{ii} = 0$$). Clearly, $$\tilde{y}_{i,i} = 1$$ and $$\tilde{y}_{i,i^+} = \tilde{x}_{i,i^+}^{-1}$$.

Denote by $$\mathcal{A}_n$$ the subalgebra of $$Q_n$$ generated by all $$\tilde{y}_{ij}$$ and $$\tilde{y}_{i,i^+}^{-1}$$. The following is an immediate corollary of Proposition 2.40

Corollary 2.51. For each $$i, j, i', j' \in [n]$$ one has: $$E_{i,j'}(\tilde{y}_{ij}) = \begin{cases} 0 & \text{if } j' \notin \{i^-, i, k\} \\ \tilde{y}_{i,i'} & \text{if } j' = j \\ -\tilde{y}_{i,i'}\tilde{y}_{ij} & \text{if } j' = i^- \\ \tilde{y}_{i,i'}^{-1}\tilde{y}_{ij} & \text{if } j' = i \end{cases}$$

In particular, $$\mathcal{A}_n$$ is invariant under the $$gl_n(Q)$$-action.

Remark 2.52. Note, however, that the subalgebra $$U_n$$ of $$A_n$$ defined in Section 2.3 is not $$gl_n(Q)$$-invariant.

2.8. Extended noncommutative n-gons. In this section we define a larger algebra $$\mathcal{A}_n$$ which is an extension of $$Q_n$$ and can be viewed as a carrier of double noncommutative triangulations of the $$n$$-gon.

Definition 2.53. Let $$\mathcal{A}_n^\pm$$ be the algebra generated by $$x_{ij}^\pm$$ and $$\{x_{ij}^{-1}\}$$, $$i, j \in [n]$$, $$i \neq j$$, $$\varepsilon \in \{-, +\}$$ subject to the relations:

(i) (triangle relations) For any triple $$(i, j, k)$$ of distinct indices in $$[n]$$:

$$x_{ij}^\pm(x_{kj}^{-1})x_{ki}^\pm = x_{ik}^\pm(x_{jk}^{-1})^{-1}x_{ji}^-.$$ (2.32)

(ii) (exchange relations) For all cyclic $$(i, j, k, \ell)$$ in $$[n]$$:

$$x_{ij}^- = x_{jk}^+(x_{ik}^+)^{-1}x_{il}^- + x_{ji}^-(x_{ki}^+)^{-1}x_{kl}^+, \quad x_{ij}^+ = x_{jk}^+(x_{ik}^+)^{-1}x_{il}^- + x_{ji}^-(x_{ki}^+)^{-1}x_{kl}^+. \quad (2.33)$$

The following result is obvious.

Lemma 2.54. The assignments $$x_{ij}^\pm \mapsto x_{ij}$$ define an epimorphism of algebras $$\mathcal{A}_n \twoheadrightarrow A_n$$.
Proposition 2.55. The algebra $\tilde{A}_n$ is generated by $Q_n$ and $(T_i^{jk})_{i,j,k}^{\pm 1}$ for all distinct triples $(i,j,k)$ subject to:

(i) (triangle relations) $T_{i}^{jk} = T_{i}^{kj}$ for all distinct $i,j,k \in [n]$.

(ii) (modified exchange relations) $T_{i}^{j\ell} = T_{i}^{ij} + T_{i}^{k\ell}$ for all cyclic $(i,j,k,\ell)$ in $[n]$.

(iii) (consistency relations) $y_{ij}^i T_{i}^{jk} = y_{ij}^j T_{i}^{kj}$ for all cyclic $(i,j,k,\ell) \in [n]$.

Proof. We proceed similarly to the proof of Proposition 2.21. Denote by $\tilde{A}_n'$ the algebra whose presentation is given in the proposition. It is easy to see that the assignments $y_{ij}^i \mapsto (x_{ji}^{\pm 1} x_{ik}^{\pm 1})^1 T_i^k$. Hence, $y_{ij}^i T_{i}^{jk}(T_{i}^{ji})^{-1} y_{ij}^j T_{i}^{kj} = y_{ij}^i y_{jk}^j y_{ki}^i T_{i}^{jk}$ for all distinct $(i,j,k) \in [n]$.

On the other hand, the consistency relations imply that the kernel of $\tilde{A}_n' \to \tilde{A}_n$.

The following is immediate.

Lemma 2.56. The assignments $x_{ij}^i \mapsto (T_{i}^{jk})^{-1} y_{ij}^k$ for distinct $i,j,k \in [n]$ define a homomorphism of algebras $\tilde{f} : \tilde{A}_n \to \tilde{A}_n'$.

In particular, $\tilde{f}((x_{ji}^{\pm 1} x_{ik}^{\pm 1})^1) = y_{ij}^i T_{i}^{jk}(T_{i}^{ji})^{-1} y_{ij}^j T_{i}^{kj} = t_{ij}^k T_{i}^{jk}$ by (2.7). These homomorphisms are, clearly, inverse to each other and hence are isomorphisms. The proposition is proved.

Similarly to Section 2.21, for each triangulation $\Delta$ of $[n]$ define:

- The subalgebra $\tilde{A}_\Delta$ of $\tilde{A}_n$ generated by $x_{ik}^i$ for all distinct $k,\ell \in [n]$ and by $(x_{ij}^{\pm 1})^{-1}$ for all $(i,j) \in \Delta$.

- The group $\tilde{T}_\Delta$ generated by all $t_{ij}^k$ subject to the triangle relations (2.32).

Clearly, $\tilde{T}_\Delta$ is a free group on $5(n-2)$ generators. It is also clear that the assignments $t_{ij}^k \mapsto x_{ij}^k$, $(i,j) \in \Delta$ define a homomorphism of algebras $\tilde{f}_\Delta : \tilde{Z} T_\Delta \to \tilde{A}_\Delta$.

The following is immediate.

Proposition 2.57. (Laurent Phenomenon for extended noncommutative polygons) For each triangulation $\Delta$ of $[n]$ the homomorphism $\tilde{f}_\Delta$ is an epimorphism.

Note, however, that $\tilde{f}_\Delta$ is not an isomorphism, unlike its counterpart $f_\Delta$ given by (2.4).

Proposition 2.58. For each triangulation $\Delta$ of $[n]$ the kernel of $\tilde{f}_\Delta$ contains the elements

\[
(\partial_{i_1,i_2} - \partial_{i_2,i_1}) T_{i_1}^{i_2,i_3} + T_{i_1}^{i_2,i_3}(\partial_{i_1,i_4} - \partial_{i_4,i_1}) + (t_{i_3,i_4}^{-1} t_{i_4,i_3}^{-1} t_{i_1,i_4}^{-1} t_{i_4,i_1}^{-1} - t_{i_3,i_4}^{-1} t_{i_4,i_3}^{-1} t_{i_1,i_4}^{-1} t_{i_4,i_1}^{-1})
\]

for each 5-tuple $(i_1,i_2,i_3,i_4,i_5)$ in the cyclic order such that $(i_k,i_k) \in \Delta$ for all distinct $(k,\ell) \in [5] \times [5]$ except for $(k,\ell) = (2,4),(4,2),(2,5),(5,2)$, where we abbreviated $\partial_{ij} = (t_{ij}^{i,j})^{-1} t_{ij}^{j,i}$.

Proof. Without loss of generality, we assume that $i_k = k$ for $k = 1,2,3,4,5$. Then

\[
x_{25} = x_{21}^{-1} x_{14}^{-1} x_{45} + x_{24}^{-1} x_{15}^{-1} x_{45}^{-1}
\]

hence $x_{25} = x_{21}^{-1} x_{41}^{-1} x_{45} + x_{23}^{-1} x_{13}^{-1} x_{44}^{-1} x_{45}^{-1} x_{15}^{-1} + x_{21}^{-1} x_{31}^{-1} x_{34}^{-1} x_{44}^{-1} x_{15}^{-1}$. On the other hand,

\[
x_{25} = x_{21}^{-1} x_{31}^{-1} x_{41}^{-1} x_{45}^{-1} x_{45}^{-1} x_{15}^{-1} + x_{23}^{-1} x_{13}^{-1} x_{15}^{-1} + x_{21}^{-1} x_{31}^{-1} x_{34}^{-1} x_{44}^{-1} x_{15}^{-1}.
\]

Comparing the expressions for $x_{25}$, we obtain a relation in $\tilde{A}_\Delta$ which gives the appropriate element in the kernel of $\tilde{f}_\Delta$. The proposition is proved.

Remark 2.59. It is natural to conjecture that the kernel of $\tilde{f}_\Delta$ is generated (as a two-sided ideal in $\tilde{Z} T_\Delta$) by the elements $\tilde{f}_\Delta$.

2.9 Further generalizations and specializations.

Definition 2.60. Let $\tilde{A}_n$ be the algebra generated by all $x_{ij}^k, (x_{ij}^k)^{-1}$, where $i,j,k$ are distinct indices in $[1,n]$ subject to the relations:

(i) (triangle relations) $T_{i}^{jk} = T_{i}^{kj}$ for all distinct $i,j,k$, where $T_{i}^{jk} = (x_{ij}^{k})^{-1} x_{jk}^{p} x_{ik}^{p}$.

(ii) (exchange relations) $T_{i}^{j\ell} = T_{i}^{ijk} + T_{i}^{j\ell}$ whenever $(i,k)$ crosses $(j,\ell)$. 

The following result is obvious.

**Lemma 2.61.** (a) The assignments \( x_{ij}^k \mapsto x_{ij} \) define an epimorphism of algebras \( \tilde{A}_n \to A_n \).
(b) The assignments \( x_{ij}^k \mapsto x_{ij}^k \) define an epimorphism of algebras \( \tilde{A}_n \to \tilde{A}_n \) (as in Section 2.8).

We refer to each \( T_{ijk}^l \) as the generalized noncommutative angle and view it as a certain measure of the angle at the vertex \( i \) in the triangle \((ijk)\). For any triangulation \( \Delta \) of the \( n \)-gon and \( i \in [n] \), define the total angle \( \Theta_i^{\Delta} \) to be the sum of all noncommutative angles of all triangles of \( \Delta \) at the vertex \( i \).

**Theorem 2.62.** For any triangulations \( \Delta \) and \( \Delta' \) of the \( n \)-gon, we have \( T_{\Delta} = T_{\Delta'} \).

Furthermore, let \( A_n' \) be the algebra generated by \( x_{ij}, c_{ij}^{jk} = c_{ji}^{kj}, d_{ij}^{jk} = d_{ji}^{kj} \) and their inverses subject to the relations:

(i) (triangle relations) \( T_{i}^{jk} = T_{i}^{kj} \) for all distinct \( i,j,k \in [n] \), where

\[
T_{i}^{jk} = x_{ji}^{-1} x_{jk} x_{ik}^{-1} ;
\]

(ii) (exchange relations) \((d_{i}^{jk})^{-1}T_{i}^{jk}(c_{i}^{jk})^{-1} = (d_{i}^{kj})^{-1}T_{i}^{kj}(c_{i}^{kj})^{-1} + (d_{i}^{kl})^{-1}T_{i}^{kl}(c_{i}^{kl})^{-1} \) for any cyclic \((i,j,k,l)\) in \([n]\).

**Proposition 2.63.** The assignments \( x_{ij}^k \mapsto c_{ij}^{jk} x_{ij} d_{jk}^{ik} \) define a homomorphism of algebras:

\[(2.35) \quad \check{\varphi} : \tilde{A}_n \hookrightarrow A_n' .\]

**Proof.** Denote by \( \tilde{A}_n' \) the algebra freely generated by all \( x_{ij}^k \). Then, clearly, the assignments \( x_{ij}^k \mapsto c_{ij}^{jk} x_{ij} d_{jk}^{ik} \) define an algebra homomorphism

\[\tilde{A}_n' \to A_n' .\]

Denote \( T_{i}^{jk} := (x_{ij}^k)^{-1} x_{jk} (x_{ik}^j)^{-1} \). We need the following fact.

**Lemma 2.64.** \( \check{\varphi}'(T_{i}^{jk}) = (d_{i}^{jk})^{-1}T_{i}^{jk}(c_{i}^{jk})^{-1} \).

**Proof.** Indeed,

\[
\check{\varphi}'(T_{i}^{jk}) = \check{\varphi}'((x_{ij}^k)^{-1} x_{jk} (x_{ik}^j)^{-1}) = (c_{ij}^{jk} x_{ij} d_{ij}^{jk})^{-1} c_{ij}^{jk} x_{jk} d_{jk}^{ik} (c_{ij}^{jk} x_{ik} d_{ik}^{jk})^{-1} = (d_{ij}^{jk})^{-1} x_{ji} x_{jk} x_{ik} (c_{ij}^{jk})^{-1} = (d_{ij}^{jk})^{-1} T_{i}^{jk}(c_{ij}^{jk})^{-1} .
\]

The lemma is proved.

The lemma implies that \( \check{\varphi}'(T_{i}^{jk}) = \check{\varphi}'(T_{i}^{kj}) \) and:

\[
\check{\varphi}'(T_{i}^{ji} - T_{i}^{jk} - T_{i}^{kj}) = (d_{i}^{ji})^{-1}T_{i}^{ji}(c_{i}^{ji})^{-1} - (d_{i}^{jk})^{-1}T_{i}^{jk}(c_{i}^{jk})^{-1} - (d_{i}^{kj})^{-1}T_{i}^{kj}(c_{i}^{kj})^{-1} = 0 .
\]

This proves the proposition.

**Proposition 2.65.** For each collection of integers \( a = \{ a_{i}^{jk} = a_{i}^{kj} | i,j,k \in [n] \) are distinct \}, the assignments \( x_{ij}^k \mapsto (T_{i}^{jk} a_{i}^{jk}) x_{ij} (T_{k}^{ij})^{-a_{i}^{jk}} \) define an algebra homomorphism \( \varphi_a : \tilde{A}_n \to A_n \).

**Proof.** Clearly, \( \varphi_a = \psi \circ \check{\varphi} \), where \( \check{\varphi} \) is given by (2.35) and \( \psi : A_n' \to A_n \) is an epimorphism given by \( x_{ij} \mapsto x_{ij}, c_{ij}^{jk} \mapsto (T_{i}^{jk})^{a_{i}^{jk}}, d_{ij}^{jk} \mapsto (T_{i}^{jk})^{-a_{i}^{jk}} \).

**Remark 2.66.** Note that if \( a_{i}^{jk} = 1 \), then \( \varphi_a(x_{ij}^k) = x_{ki}^{-1} x_{kj} x_{jk} x_{ik}^{-1} x_{ij} \).
2.10. Free factorizations of $\mathcal{A}_n$ and proof of Theorem 2.14. First, we verify that the relations (2.7), (2.8), and (2.9) hold. The left hand side of the first relation (2.7) is:

$$y_{ij}^k y_{ji}^j = (x_{ki}^{-1} x_{kj})(x_{kj}^{-1} x_{ki}) = 1.$$  

Furthermore, the left hand side of the second relation (2.7) is:

$$y_{ij}^k y_{jk}^l y_{kj}^l = (x_{ki}^{-1} x_{kj})(x_{kj}^{-1} x_{jl})(x_{lj}^{-1} x_{jk}) = (x_{ki}^{-1} x_{kj}) (x_{kj}^{-1} x_{jl}) (x_{lj}^{-1} x_{jk}) = 1$$

for all distinct $i, j, k \in [n]$ by the triangle relations (2.1). Similarly, the left hand side of (2.8) is:

$$y_{ij}^j y_{jk}^k y_{ki}^i = (x_{ti}^{-1} x_{tj})(x_{tj}^{-1} x_{tk})(x_{tk}^{-1} x_{ti}) = 1$$

for all distinct quadruples $(i, j, k, \ell)$. Finally, the difference between the right and left hand sides of (2.9) is:

$$y_{ij}^j y_{jk}^k y_{ki}^i = (x_{ti}^{-1} x_{tj})(x_{tj}^{-1} x_{tk})(x_{tk}^{-1} x_{ti}) - (x_{ti}^{-1} x_{tj})(x_{tj}^{-1} x_{tk})(x_{tk}^{-1} x_{ti}) = 0$$

for all cyclic $(i, j, k, \ell)$ by the exchange relations (2.2).

Now let us show that the relations (2.7), (2.8), (2.9) are defining. Indeed, Proposition 2.34 implies an epimorphism of algebras $\mathcal{A}_n \to \mathcal{Q}_n$ given by $x_{ij} \mapsto y_{ij}^{-1}$. In other words, $\mathcal{Q}_n$ is isomorphic to the quotient of $\mathcal{A}_n$ by the ideal generated by elements $x_{i,i-1} = 1, i \in [n]$.

Therefore, we obtain the following obvious result.

Lemma 2.67. The algebra $\mathcal{Q}_n$ is generated by all $y_{ij} := y_{i,j}^{-1}$ and $y_{ij}^{-1}, i, j \in [n], i \neq j$, subject to $y_{i,i-1} = 1, i \in [i]$ and the relations (2.1), (2.2), i.e.,

$$(2.36) \quad y_{ij} y_{jk} y_{ki} = y_{ki} y_{jk} y_{ij}$$

for any distinct indices $i, j, k \in [n]$;

$$(2.37) \quad y_{j\ell} = y_{jk} y_{ki}^{-1} y_{\ell k} + y_{ji} y_{ki}^{-1} y_{\ell k}$$

for all cyclic $(l, k, j, i) \in [n]$.

Since $y_{ij}^k = y_{ij}^{-1} y_{kj}$, the relations (2.7) directly follow from (2.36) and the relations (2.9) directly follow from (2.37) (this is obvious if we “reverse engineer” the fist part of the proof and replace all $x_{ij}$ by $y_{ij}$ there).

Therefore, Theorem 2.14 is proved. \square

The following obvious corollary from the proof of Theorem 2.14 will be instrumental in Section 3.

Corollary 2.68. For each triangulation $\Delta$ of $[n]$ the $\mathcal{U}_\Delta$ is generated by $u_{ij}^k, (i, k), (j, k) \in \Delta$ subject to the relations (2.7) and (2.8), i.e., for all distinct $i, j, k, \ell \in [n]$ such that $(i, j), (j, k) \in \Delta$ one has:

$$u_{ij}^k = 1, \quad u_{ij}^k u_{ji}^j = u_{ij}^k u_{jk}^l u_{ki}^i, \quad u_{ij}^k u_{jk}^l = u_{ij}^k u_{kj}^l u_{ki}^i = 1.$$ 

2.11. Freeness of $\mathcal{T}_\Delta$ and proof of Theorem 2.27. Let $\Delta$ be a triangulation of $[n]$. Fix a directed triangulation $\Delta \subset \Delta$ so for each $(i, j) \in \Delta$ with $j \notin \{i^+, i^-\}$ exactly one out of $(i, j)$ and $(j, i)$ belongs to $\Delta$ and $\Delta$ contains all $(i, i^\pm), i \in [n]$. By definition, any such $\Delta$ has cardinality $3n - 3$.

Proposition 2.69. Given $i_0 \in [n]$. Then for any triangulation $\Delta$ and any $\Delta$ as above, the group $\mathcal{T}_\Delta$ is freely generated by $t_{ij}, (i, j) \in \Delta \setminus \{(i_0, i_0^\pm)\}$.

Proof. We proceed by induction on $n$. The assertion is obvious for $n \leq 3$. Suppose that $n \geq 4$. Then it is easy to see that there exist distinct $j_0, j_0^\pm \in [n]$ such that $(j_0, j_0^-), (j_0^+, j_0^-) \in \Delta$.

Without loss of generality we may assume that $j_0^+ = n$ and $j_0 \neq i$ (hence $j_0 \notin \{i, n-1, n, 1\}$). Then $\Delta = \Delta \setminus \{(1, n), (n, 1), (n-1, n), (n, n-1)\}$ is a triangulation of $[n-1]$ and $\Delta = \Delta \setminus \{(1, n), (n, 1), (n-1, n), (n, n-1)\}$ is the corresponding directed triangulation.

The following result is obvious.
Lemma 2.70. (a) The assignments \( t_{ij} \mapsto t_{ij} \), \((i, j) \in \hat{\Delta}, t_{n-1, n} \mapsto 1, t_{n, n-1} \mapsto 1, t_{1,n} \mapsto 1, t_{n,1} \mapsto t_{n-1,1}^{-1} t_{n,1}^{-1} \) define an epimorphism of groups \( \varphi : T_{\Delta} \to T_{\hat{\Delta}} \).

(b) \( i \circ \varphi = Id_{T_{\Delta}} \), where \( i : T_{\hat{\Delta}} \to T_{\Delta} \) is a homomorphism given by \( i(t_{ij}) = t_{ij} \) for \((i, j) \in \hat{\Delta} \).

(c) The homomorphism \( i : T_{\Delta} \to T_{\hat{\Delta}} \) is injective.

Denote by \( \Delta_0 \) the triangulation of the triangle with the vertices \( 1, n - 1, n \). Clearly, \( T_{\Delta} \) is generated by \( T_{\Delta} \) (via the embedding \( i \)) and \( T_{\Delta_0} \), more precisely,

\[
T_{\Delta} = T_{\Delta} \ast T_{\Delta_0}/((t_{n-1, n-1} \ast 1)(1 \ast t_{n-1,1}^{-1})(1 \ast t_{1, n-1}^{-1}))
\]

This and the inductive hypothesis (asserting that \( T_{\hat{\Delta}} \) is freely generated by \( t_{ij} \), \((i, j) \in \hat{\Delta} \), without loss of generality, that \((2.39)\) imply (by eliminating \( t_{n-1,1} \) and \( t_{1,n-1} \) and setting \( t_{k, \ell} := 1 \ast t_{k, \ell} \) for \((k, \ell) = (1, n), (n, 1), (1, n-1), (n-1, 1) \)) that \( T_{\Delta} \) is freely generated by all \( t_{ij}, (i, j) \in \Delta \) \{ \( (j_0, j_0^+) \). \]

The theorem is proved.

\[\square\]

2.12. Retraction of \( T_n \) onto \( T_{\Delta} \) and proof of Theorem 2.31. It suffices to construct an element \( \tau_{ij} \in T_{\Delta} \), for each pair \((i, j) \in [n] \times [n], i \neq j \) such that \( \tau_{ij} = t_{ij} \) whenever \((i, j) \in \Delta \) and for any distinct \( i, j, k \in n \) one has the triangle relation:

\[
(2.38)
\]

\[
\hat{T}_{ij}^k = T_{ij}^k
\]

where \( \hat{T}_{ij} := \tau_{ij}^{-1} \tau_{jk}^{-1} \).

We construct such \( \tau_{ij} \) by induction on \( n \). Retain notation from the proof of Theorem 2.14 and assume, without loss of generality, that \((n-1, n+1) \in \Delta \). If \( n \notin \{i, j\} \), then, by deleting the vertex \( n \) and using the natural inclusion \( T_{\Delta} \subset T_{\Delta} \) given by Lemma 2.70(c), we set \( \tau_{ij} \) to be that one which belongs to \( T_{\Delta} \). Finally, we set \( \tau_{i, n} := t_{1,n}, \tau_{n, i} := t_{n,1} \) and:

\[
\tau_{i, n-1} := \tau_{i, n-1}^{-1} \tau_{1,n-1}^{-1} \tau_{1,n}, \tau_{n, i} := \tau_{n,1} \tau_{n-1,1}^{-1} \tau_{n-1, i-1}
\]

for \( 1 < i < n \).

Now verify that so constructed elements satisfy \((2.38)\). Indeed, if \( i, j, k \in [n-1] \), we have nothing to prove because \((2.38)\) holds by the inductive hypothesis. Otherwise, it suffices to consider the case when \( k = n \) and verify:

\[
(2.39)
\]

for all \( i, j \in [n-1], i \neq j \). Indeed,

\[
\hat{T}_{ij}^{n,j} = \tau_{n, i}^{-1} \tau_{n, j}^{-1} = \tau_{n, i}^{-1} \tau_{n, j}^{-1} = \hat{T}_{ij}^{n,j}
\]

which, together with the inductive hypothesis, proves \((2.39)\).

Therefore, the assignments \( t_{ij} \mapsto \tau_{ij} \) for all \( i \neq j \) define a group epimorphism \( T_n \to T_{\Delta} \).

Theorem 2.31 is proved.

\[\square\]

2.13. Noncommutative Laurent Phenomenon and proof of Theorems 2.10 and 2.15. Clearly, Theorem 2.10 is a direct corollary of Theorem 2.15 so we will only prove the latter one. We proceed by induction on \( n \). In fact, due to the relations \((2.5)\) in the form \( \hat{y}_{ij} = y_{k,i}^j - y_{i,j}^j \) (hence \( y_{(k, i)} = y_{k,i}^j + y_{i,j}^j \)), it suffices to prove \((2.10)\) only with \( k = i^+ \) (however, we will use the inductive hypothesis without this restriction).

Indeed, if \( n \leq 3 \), the assertion is immediate. Now suppose that \( n \geq 4 \). In what follows we retain some notation of Section 2.11 that is, we fix a triangulation \( \Delta \) and suppose that \((n-1, 1) \in \Delta \) and \((j_0, j_0^+) \in \Delta \) for some \( j_0 \notin \{i, i, n-1, n \} \). If \( i \notin \{i, j\} \), then the assertion \((2.10)\) for \( \Delta \) coincides with that for \( \Delta = \Delta \setminus (1)n, (n, 1), (n-1, n), (n, n-1) \) and we have nothing to prove. Now suppose that \( n \in \{i, j\} \). Without loss of generality we may assume that \( i = n \) (the case \( j = n \) is obtained by reversing all chords in \([n]\)). Then, we will use the inductive hypothesis \((2.10)\) for \( \hat{\Delta} \) in the form:

\[
y_{i,j}^{n-1} = y_{i,1,i'} + y_{n-1,1,j}, \quad y_{i,n-1,j} = y_{(n-1, i')} \]

where the first (resp. second) summation is over all \((n-1, j, \hat{\Delta}) \) (resp. \((1, j, \hat{\Delta})\))-admissible sequences.
Using these and the relation (2.19) in the form $y_{i,j}^{n} = y_{i,j}^{n-1} + y_{i,n-1,j}^{1}$, we obtain:

$$y_{i,j}^{n} = \sum_{i'} y_{i(1,i')}^{n-1} + \sum_{i'} y_{i,n-1,j}^{1}.$$

Clearly, this gives (2.10) because each $(n, j, \Delta)$-admissible sequence is either of the form $(n, 1, i')$, where $i'$ is $(n, j, \Delta)$-admissible or of the form $(n, n-1, i'')$, where $i''$ is $(1, j, \Delta)$-admissible (and vice versa).

Theorem 2.13 is proved.

Therefore, Theorem 2.10 is proved.

2.14. Noncommutative cluster variables and proof of Theorems 2.8 and 2.3

For each triangulation $\Delta$ of $[n]$ and $(p, q) \in [n] \times [n]$, $p \neq q$ define an element $t_{pq}^{\Delta} \in QT_{\Delta}$ (in the notation of Theorem 2.10) by

$$(2.40) \quad t_{pq}^{\Delta} = \sum_{i \in \text{Adm}_{\Delta}(p,q)} t_{i},$$

where $t_{i} \in T_{\Delta}$ is given by:

$$t_{i} := t_{i_1,i_2}^{-1} t_{i_3,i_4}^{-1} t_{i_5,i_6}^{-1} \cdots t_{i_{2m-1},i_{2m-2}}^{-1} t_{i_{2m-1},i_{2m}}^{-1} \text{ for any } i \in [n]^{2m} \text{ (with the convention } t_{i i} = 1 \text{ for } i \in [n]).$$

We need the following result.

Theorem 2.71. For any triangulations $\Delta$ and $\Delta'$ of $[n]$ the assignments $t_{ij}^{\Delta'} \to t_{ij}^{\Delta}$ for $(i, j) \in [n] \times [n]$, $i \neq j$ define an isomorphism of algebras

$$(2.41) \quad \psi_{\Delta, \Delta'} : QT_{\Delta'}(S_{\Delta}^{-1}) \to QT_{\Delta}(S_{\Delta}^{-1}),$$

where $S_{\Delta}$ (resp. $S'_{\Delta}$) is a submonoid in $QT_{\Delta}$ generated by all $t_{ij}^{\Delta}$. These isomorphisms satisfy:

$$(2.42) \quad \psi_{\Delta, \Delta'} \circ \psi_{\Delta', \Delta''} = \psi_{\Delta, \Delta''}$$

for any triangulations $\Delta, \Delta', \Delta''$ of $[n]$.

Proof. First, prove the assertion for adjacent triangulations $\Delta, \Delta'$ of $\Sigma$, i.e., such that $\Delta \setminus \Delta' = \{(i, k), (k, i)\}$, $\Delta' \setminus \Delta = \{(j, \ell), (\ell, j)\}$, where $(i, j, k, \ell)$ is a cyclic quadruple.

By definition,

$$(2.43) \quad t_{ij}^{\Delta} = t_{i_1,k_1}^{-1} t_{i_2,k_2}^{-1} t_{i_3,k_3}^{-1} t_{i_4,k_4}^{-1} \cdots t_{i_{2m-1},k_{2m-1}}^{-1} t_{i_{2m},k_{2m}}^{-1} t_{i_{2m+1},k_{2m+1}}^{-1} t_{i_{2m+2},k_{2m+2}}^{-1}.$$ 

We need the following result.

Lemma 2.72. For any adjacent triangulations $\Delta, \Delta'$ of $[n]$ with $\Delta \setminus \Delta' = \{(i, k), (k, i)\}$, $\Delta' \setminus \Delta = \{(j, \ell), (\ell, j)\}$ there is a unique homomorphism of algebras $\varphi_{\Delta, \Delta'} : QT_{\Delta'} \to QT_{\Delta}((t_{ij}^{\Delta})^{-1})$ such that

$$\varphi_{\Delta, \Delta'}(t_{ij}^{\Delta'}) = \begin{cases} t_{ij}^{\Delta} & \text{if } (i', j') \neq (j, \ell) \\ t_{ij}^{\Delta'} & \text{if } (i', j') = (j, \ell) \end{cases}$$

for all $(i', j') \in \Delta'$. 

Proof. Indeed, it suffices only to prove that $\varphi_{\Delta, \Delta'}$ respects the triangle relations

$$T_{i'}^{j',k'} = T_{i'}^{k',j'}$$

for all triangles $(i', j', k')$ in $\Delta'$. Clearly, if $(i', j', k')$ belongs to $\Delta \cap \Delta'$, then we have nothing to prove. It suffices only to consider the case when $(j', k') = (j, \ell)$, i.e., we have to prove that

$$\varphi_{\Delta, \Delta'}(T_{i'}^{j'}) = \varphi_{\Delta, \Delta'}(T_{i'}^{k'})$$

for $i' \in \{i, k\}$. Taking into account that both $(i' j')$ and $(i' \ell)$ belong to $\Delta \cap \Delta'$, we have only to prove that in $QT_{\Delta}$ one has:

$$t_{ij}^{-1} \Delta_{ij}^{-1} = t_{i'j'}^{-1} \Delta_{i'j'}^{-1}.$$ 

In view of (2.43), this is equivalent to:

$$(2.44) \quad t_{ij}^{-1}(t_{i,j}^{-1} + t_{i,j}^{-1} t_{k,j})^{-1} = t_{i'j'}^{-1}(t_{i'j'}^{-1} + t_{i'j'}^{-1} t_{k,j})^{-1}.$$
If $i' = i$, then both sides of (2.44) are, clearly, equal to $T_i^{jk} + T_{\ell i}$, and if $i' = k$, then both sides of (2.44) are equal to $T_{ki}^{jk} + T_{i\ell}^{\ell k}$.

This proves that $\varphi_{\Delta,\Delta'}$ is well-defined homomorphism of algebras.

Furthermore, we prove that in the assumptions of Lemma 2.72 one has
\[(2.45) \varphi_{\Delta,\Delta'}(i_{pq}^{\Delta'}) = t_{pq}^{\Delta} \] for all $(p, q) \in [n] \times [n], p \neq q$.

Define a partial order $\vartriangleleft$ on $[n]^\bullet$ by the covering insertion relations $i \vartriangleleft i'$ if
\[(2.46) \quad i = (\ldots, i_k, i_{k+1}, i_{k+2}, \ldots), i' = (\ldots, i_k, a, i_{k+1}, i_{k+2}, \ldots) \] for any $a \in [n]$.

We need the following obvious fact.

**Lemma 2.73.** For each $i \in [n]^\bullet$ there is a unique element $[i]$ such that:
- $[i] \vartriangleleft i$.
- $[i]$ is minimal in the partial order $\vartriangleleft$.

Clearly, if $i, i' \in [n]^\bullet$ and $i \vartriangleleft i'$, then $t_i = t_{i'}$.

Furthermore, fix a distinct quadruple $P := (i, j, k, \ell)$ in $[n]$ and denote by $P$ the underlying set $\{i, j, k, \ell\}$.

For any $i = (i_1, \ldots, i_r) \in [n]^\star$, $r \geq 2$ define the *index set* $Ind_i(P) \subset [r - 1]$ by:

\[ Ind_i(P) = \{ s \in [r - 1] : \{i_s, i_{s+1}\} \in \{\{i, k\}, \{j, \ell\}\} \} \]

(with the convention that $i_k = 0$ if $k \leq 0$ and $i_k = \infty$ if $k > r$) and the *index* $ind_i(P) \in \mathbb{Z}_{\geq 0}$ by

\[ ind_i(P) = \min \{ Ind_i(P) \} \]

with the convention that $\min \emptyset := 0$.

Denote by $I_P$ the set of all sequences $i$ such that $|Ind_i(P)| = 1$.

Clearly, $I_{P'} = I_P$ for any permutation $P' = (i', j', k', \ell')$ of $P = (i, j, k, \ell)$ such that $\{i', k'\} \in \{\{i, k\}, \{j, \ell\}\}$.

**Proposition 2.74.** For each $i \in I_P$ one has $[i] \in I_P$ and $ind_{\{i\}}(P) \equiv ind_i(P) \mod 2$.

**Proof.** We need the following fact.

**Lemma 2.75.** Let $i, i' \in [n]^\bullet$ be such that $i \vartriangleleft i'$ and $i' \in I_P$. Then $i \in I_P$.

**Proof.** It suffices to prove the assertion only for $i$ and $i' = j_{\vartriangleleft}^{i_{\vartriangleleft}}(i)$ as in (2.40). Let $s' = ind_i(P)$. Since $|Ind_i(P)| = 1$, and $i'_{s' - 1} \neq i'_{s' + 1}$, $i'_{s'} \neq i'_{s' + 2}$, but $i'_{s' + 1} = i'_{s' + 3}$, then $s' \notin \{t, t + 2\}$. In particular, $\{i, a\} \notin \{\{i, k\}, \{j, \ell\}\}$. This immediately implies that $|Ind_i(P)| = 1$ and
\[(2.47) \quad Ind_i(P) = \begin{cases} \{s'\} & \text{if } s' \leq t \\ \{s' - 2\} & \text{if } s' \geq t + 3 \end{cases} \]

The lemma is proved.

Thus, for any $i \in I_P$ we see that $\{i'' \in [n]^\bullet : i'' \vartriangleleft i\} \subset I_P$, in particular, $[i] \in I_P$.

The proposition is proved. \qed

For $a, b \in [n]$ and $1 \leq s < r$ define the map $j_{\vartriangleleft}^{a_{\vartriangleleft}} : [n]^r \to [n]^{r+2}$ by $(\ldots, i_s, i_{s+1}, \ldots) \mapsto (\ldots, i_s, a, b, i_{s+1}, \ldots)$. Define a map $J_P : I_{P} \times \{-1, 1\} \to [n]^\bullet \times \{-1, 1\}$ by
\[(2.48) \quad J_P(i, \varepsilon) = (j_{\vartriangleleft}^{a_{\vartriangleleft}}(i), (\varepsilon^{-1})^{(s-1)}\chi_{\{i, j\}}(i_s)) \]

where $s := ind_i(P)$ and $\chi_{\{b, c\}}(a)$ is the characteristic function, i.e., it is 1 if $a \in \{b, c\}$ and 0 otherwise, and the pair $(i', k')$ is determined by $\{i', k'\} = P \setminus \{i_s, i_{s+1}\}$ and:

- If $s$ is odd then $i' = P \setminus \{i_s, i_{s+1}\}$, where we abbreviated $P_S := \begin{cases} \{i, j\} & \text{if } \varepsilon = -1 \\ \{k, \ell\} & \text{if } \varepsilon = 1 \end{cases}$.
- If $s$ is even then $i' = \begin{cases} \{i_{s-1}\} & \text{if } i_{s-1} \notin P \setminus \{i_s, i_{s+1}\} \\ P \setminus \{i_s, i_{s+1}, i_{s+2}\} & \text{if } i_{s+2} \notin P \setminus \{i_s, i_{s+1}\} \\ \{i, j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}$
Let \( \hat{I}_P \) be the set of all \((i, \varepsilon) \in I_P \times \{-1, 1\}\) such that
- if \( s = \text{ind}_1(P) \) is even, then \( \varepsilon = 1; \)
- if \( s = \text{ind}_1(P) \) is odd then:
  1. If \( \{i_{s-1}\} = P_{\varepsilon} \setminus \{i_s, i_{s+1}\}, \{i_{s+2}\} = P_{-\varepsilon} \setminus \{i_s, i_{s+1}\}, i_{s-2} \neq i_s, i_{s+3} \neq i_{s+1}, \) then \( i_s \in \{i, j\}. \)
  2. If \( \{i_{s-1}\} = P_{\varepsilon} \setminus \{i_s, i_{s+1}\}, \{i_{s+2}\} \neq P_{-\varepsilon} \setminus \{i_s, i_{s+1}\}, \) then \( i_{s-2} \neq i_{s+1}. \)

**Proposition 2.76.** \( J_P(\hat{I}_P) \subset \hat{I}_P, \) that is, \( J_P \) is a map \( J_P : \hat{I}_P \to \hat{I}_P. \)

**Proof.** We need the following fact.

**Lemma 2.77.** Let \( i \in I_P \) and let \( s = \text{ind}_1(P) \). Then

\[
\text{ind}_1'(i)(\text{ind}_1(P) + 1) = \text{ind}_1'(i')(\text{ind}_1(P) + 1)
\]

for any \( i', k' \in [n] \) such that \( \{i', k'\} = P \setminus \{i_s, i_{s+1}\} \).

**Proof.** Let \( s = \text{ind}_1(P) \) and \( i' := J_P(i) \). Note that \( s + 1 \in \text{ind}_1(P) \) because \( \{i_{s+1}, i_{s+2}\} \in \{i, k\}, \{j, \ell\} \).
This and the fact that \( \{i_{s'}, i_{s'+1}, i_{s'+2}, i_{s'+3}\} = P \) imply that \( s \notin \text{ind}_1(P) \) and \( s + 2 \notin \text{ind}_1(P) \). Finally, if \( s'' \leq s - 1 \) (res. \( s'' \geq s + 3 \)), then \( s'' \notin \text{ind}_1(P) \) because \( s'' \neq 2 \notin \text{ind}_1(P) \).
This proves \( \text{ind}_1'(i) = s + 1 \).

Furthermore, let \( (i, \varepsilon) \in \hat{I}_P, (i', \varepsilon') := J_P(i, \varepsilon), s := \text{ind}_1(P), s' := \text{ind}_1(P). \) By Lemma 2.77, \( s' = s + 1. \)
This, in particular, implies that \( \varepsilon' = (-1)^{(s'-1)(i, i')} \in \{1, -(1)^s\} \). If \( s \) is odd, this proves the desired inclusion \( J_P(i, \varepsilon) \in \hat{I}_P. \)
It remains to consider the case when \( s \) is even. Indeed, \( i' = J_{i', k'}(i), \) where \( i' = i_{s'}, k' = i_{s'+1} \) are given by the even case of \( \text{(2.48)}. \)
Note that

\[
P_{i'} = \begin{cases} \{i, j\} & \text{if } i_s \in \{i, j\} \\ \{k, \ell\} & \text{if } i_s \in \{k, \ell\} \end{cases}
P_{-i'} = \begin{cases} \{i, j\} & \text{if } i_{s+1} \in \{i, j\} \\ \{k, \ell\} & \text{if } i_{s+1} \in \{k, \ell\} \end{cases}
\]

hence \( \{i_s\} = \{i_{s'-1}\} = P \setminus \{i_s', i_{s'+1}\}, \{i_{s+1}\} = \{i_{s'+2}\} = P \setminus \{i_s', i_{s'+1}\}. \)
Finally, \( i_{s-2}' \neq i_s' \) and \( i_{s'}' \neq i_{s'+3} \) if and only if \( \{i_{s-1}, i_{s+2}\} \cap P = \emptyset \) hence \( \{i_s' \} = \{i, j\} \cap \{i_s, i_{s+1}\}. \)
This proves that \( J_P(i, \varepsilon) \in \hat{I}_P \) for even \( s \) as well.

The proposition is proved.

Denote by \([\hat{I}_P] \subset I_P\) the set of all \( i \in \hat{I}_P \) such that \( i = [i] \) is minimal in the partial order \( \prec \) and abbreviate \( [\hat{I}_P] := \hat{I}_P \cap ((I_P) \times \{-1, 1\}). \)

**Proposition 2.74** guarantees that the assignments \( i \mapsto [i] \) define a projection \( I_P \to [\hat{I}_P] \) (resp. \( \hat{I}_P \to [\hat{I}_P] \)).

**Proposition 2.78.** The assignments \((i, \varepsilon) \mapsto [J_P(i, \varepsilon)] \) define an involution \([J_P] : \hat{I}_P \to \hat{I}_P \).

**Proof.** Let \( (i, \varepsilon) \in [\hat{I}_P], \text{ let } s := \text{ind}_1(P), (i', \varepsilon') := [J_P(i, \varepsilon)], s' := \text{ind}_1(P). \) By definition,

\[
i' = [(\ldots, i_s, i', k', i_{s+1}, \ldots)] = \begin{cases} (\ldots, i_s, i', k', i_{s+1}, \ldots) & \text{if } i' \neq i_{s-1}, k' \neq i_{s+2} \\ (\ldots, i_{s-1}, i_{s+2}, \ldots) & \text{if } i' = i_{s-1}, k' = i_{s+2} \\ (\ldots, i_s, i', k_{s+1}, \ldots) & \text{if } i' \neq i_{s-1}, k' = i_{s+2} \\ (\ldots, i_s, k', i_{s+1}, i_{s+2}, \ldots) & \text{if } i' = i_{s-1}, k' \neq i_{s+2} \end{cases}
\]

in the notation \( \text{(2.48)}. \) In particular, \( i_s' = i', i_{s'+1} = k' \).
Note that, by Lemma 2.77 and Proposition 2.74, \( s' \equiv s + 1 \mod 2. \)

First, show that \((i', \varepsilon') \in [\hat{I}_P] \) (i.e., \([J_P]\) is well-defined). If \( s \) is odd, this is obvious. Suppose that \( s \) is even.
Then we have in each of the cases of \( \text{(2.51)} \):
- \( i' \neq i_{s-1}, k' \neq i_{s+2} \). Since \( s' = s + 1 \) and \( \{i_{s-1}', i_{s+1}', i_{s+2}'\} = P \) and \( i' \in \{i, j\}, \) clearly, \((i', \varepsilon') \in [\hat{I}_P]. \)
- \( i' = i_{s-1}, k' = i_{s+2} \). Since \( s' = s - 1 \) and \( \{i_{s-1}', i_{s+1}', i_{s+2}'\} = P = \emptyset, \) clearly, \((i', \varepsilon') \in [\hat{I}_P]. \)
- \( i' \neq i_{s-1}, k' = i_{s+2} \). Since \( s' = s + 1 \) and \( \{i_{s-1}'\} = \{i_{s-1}', i_{s+1}', i_{s+2}'\} = \{i_{s+3}\} \neq \{i_{s+1}\} = P_{-i'} \setminus \{i_{s}', i_{s'+1}'\} \) by \( \text{(2.48)} \) and \( i_{s+1}' = i_{s+2} \neq i_s = i_{s'+2}, \) clearly, \((i', \varepsilon') \in [\hat{I}_P]. \)
- \( i' = i_{s-1}, k' \neq i_{s+2} \). Since \( s' = s - 1 \) and \( \{i_{s+3}'\} = \{i_{s+3}'\} \) by \( \text{(2.50)} \), clearly, \((i', \varepsilon') \in [\hat{I}_P]. \)
Furthermore, let \((i', \varepsilon') = J_P(i', \varepsilon')\). That is,
\[ i'' = j_{P', k''}(i'), \]
where \(\varepsilon'' = (-1)^{(s' - 1)\chi(i, j)}(i'')\), \(\{i'', k''\} = \{i_s, i_{s+1}\}\) and one has (note that \(\{i'_s', i'_{s'+1}\} = \{i', k'\}\)):

- If \(s\) is even, then \(i'' = \begin{cases} \{i_s\} & \text{if } i_s \in P \setminus \{i_s, i_{s+1}\} \\
\{i, j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}\)

\begin{equation}
\tag{2.52}
\{i''\} = \begin{cases} P \setminus \{i_s, i_{s+1}\} & \text{if } i_s \in P \setminus \{i_s, i_{s+1}\} \\
\{i, j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}
\end{equation}

- If \(s\) is odd, then: \(i'' = \begin{cases} P \setminus \{i_s, i_{s+1}\} & \text{if } i_s \in P \setminus \{i_s, i_{s+1}\} \\
\{i, j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}\)

\begin{equation}
\tag{2.53}
\{i''\} = \begin{cases} P \setminus \{i_s, i_{s+1}\} & \text{if } i_s \in P \setminus \{i_s, i_{s+1}\} \\
\{i, j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}
\end{equation}

First, show that \(\varepsilon'' = \varepsilon\). Indeed, by the above, \(\varepsilon'' = (-1)^{\chi(i, j)}(i'')\). Since \(\varepsilon \in \{1, -1\}\), then the above implies that for even \(s\) one has \(\varepsilon'' = \varepsilon = 1\). If \(s\) is odd, then, by definition, \(i' \in \{i, j\}\) iff \(\varepsilon = -1\). This proves that \(\varepsilon'' = \varepsilon\) in this case as well.

Thus, it remains to prove that
\[ i \leq i''. \]

To do so, show that \(i'' = i_s\) in each case of (2.51):

- \(i' \neq i_{s-1}, k' \neq i_{s+2}, s' = s + 1, i'' = (\ldots, i_s, i', i'', k', i_{s+1}, \ldots)\), where for even \(s\) we have \(i'' = i_s\) by (2.52) and for odd \(s\) we also have \(i'' = i_s\) by (2.53) because \(i_{s'-1} = i_s\) and \(i_{s'+2} = i_{s+1}\).

- \(i' \neq i_{s-1}, k' = i_{s+2}, s' = s + 1, i'' = (\ldots, i_s, i', i'', k', i_{s+1}, \ldots)\), where for even \(s\) we have \(i'' = i_s\) by (2.52) and for odd \(s\) we have \(i'' = i_s\) by (2.53) because \(i_{s'-1} = i_s\) and \(i_{s'+2} = i_{s+1}\).

- \(i' = i_{s-1}, k' = i_{s+2}\), e.g., \(\{i_{s-1}, i_{s+2}\} = P \setminus \{i_s, i_{s+1}\}\), \(s' = s - 1, i'' = (\ldots, i_{s-1}, i'', k', i_{s+2}, \ldots)\), where for even \(s\), \(i'' = i_s\) by (2.52) and for odd \(s\) we have \(i'' = i_s\) by (2.53) because \(i_{s'-1} = i_s\) and \(i_{s'+2} = i_{s+1}\).

- \(i' = i_{s-1}, k' \neq i_{s+2}, s' = s - 1, \bar{i'} = (\ldots, i_{s-1}, i'', k', i_{s+1}, i_{s+2}, \ldots)\), where for even \(s\), \(i'' = i_s\) by (2.52) and for odd \(s\) we have \(i'' = i_s\) by (2.53) because:

\[ \begin{align*}
& i_{s-1} \in P \setminus \{i_s, i_{s+1}\}, i_{s+2} \notin P \setminus \{i_s, i_{s+1}\} & \text{hence } i_{s-2} \neq i_{s+1} \\
& i_{s+1} = i_s' \notin P \setminus \{i_s, i_{s+1}\} & \text{hence } i_{s-2} \neq i_{s+1} \\
& i_{s+2} = i_{s+1}' \notin P \setminus \{i_s, i_{s+1}\} & \text{hence } i_{s-2} \neq i_{s+1} \\
\end{align*} \]

Thus, \(i'' = i_s, k'' = i_{s+1}\) in all cases, which immediately implies (2.54) in all these cases.

This proves that \([J_P]\) is an involution on \([\bar{I}_P]\).

The proposition is proved.

Now suppose that \(P = (i, j, k, \ell)\) where \(\Delta \setminus \Delta' = \{(i, k), (k, i)\}\), \(\Delta' \setminus \Delta = \{(j, \ell), (\ell, j)\}\), as in Lemma 2.72.

In what follows, we assume that \((p, i) \cap (j, \ell) = \emptyset\) and \((p, q) \cap (i, j) \neq \emptyset\) (i.e., informally speaking, \((i, j)\) is closer to \(p\) than \((k, \ell)\)).

By Definition 2.9 of admissible sequences, if \(i \in Adm_{\Delta}(p, q) \subset [IP] \sqcup Adm_{\Delta'}(p, q)\) then \([i] = i\) is minimal, its index \(s := ind_1(P)\) is positive and unique, and \(\{i_s, i_{s+1}\} = \begin{cases} \{i, k\} & \text{if } i \in Adm_1(p, q) \\
\{j, \ell\} & \text{if } i \in Adm_1(p, q) \end{cases}\).

**Proposition 2.79.** Let \(\Delta, \Delta'\) be triangulations of \([n]\) and \(P = (i, j, k, \ell)\) as above. Then the restriction of \([J_P]\) to \((Adm_\Delta(p, q) \times \{-1, 1\}) \cap [\bar{I}_P]\) is a bijection:
\[ J_{\Delta, \Delta'} : (Adm_\Delta(p, q) \times \{-1, 1\}) \cap [\bar{I}_P] \rightarrow (Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\bar{I}_P] \]

**Proof.** We need the following obvious fact.

**Lemma 2.80.** Let \(i \in Adm_{\Delta}(p, q) \sqcup Adm_{\Delta'}(p, q)\) such that \(s := ind_1(P) > 0\). Then

- \((i)\) If \(s\) is even, then \((1, 1)\) belongs to \([\bar{I}_P]\).
- \((ii)\) If \(s\) is odd, then both \((1, 1)\) and \((1, -1)\) belong to \([\bar{I}_P]\).
Furthermore, for any triangulation $\Delta$ of $[n]$ and any $p, q \in [n]$ denote by $PreAdm_\Delta(p, q)$ the set of all $i_0 \in [n]$ such that $[i] \in Adm_\Delta(p, q)$. We need the following fact.

**Lemma 2.81.** In the assumptions of Proposition 2.79 let $i \in Adm_\Delta(p, q)$ and suppose that $s = ind_i(P) > 0$. Then:

(a) if $s$ is odd, then $J_{s', k'}(i) \in PreAdm_\Delta(p, q)$ whenever $\{i_s, i_{s+1}, i', k'\} = \{i, j, k, \ell\}$.

(b) if $s$ is even then $[J_P(i, 1)] \in Adm_\Delta(p, q) \times \{-1, 1\}$.

**Proof.** In what follows, we will write $p \leq p'$ for any points $p, p'$ in the chord $(p, q)$ such that either $p = p'$ or $p$ is closer to $p$ than $p'$.

Prove (a). Indeed, suppose that $\sum_{i} = 1$ or $\sum_{i} = q$ if $s = 2m$. Clearly, $p_+ < p_-$.

We now consider a number of cases.

**Case 1.** Suppose that $(p, q) \cap (i_s, i_{s+1}) = \emptyset$, $3 \leq s \leq 2m - 3$. By definition, $p_+ < p_0 < p_-$ Then the convexity of the $n$-gon $[n]$ implies that there exist $i''$, $k'' \in [n]$ such that $\{i'', k''\} = \{i', k'\}$ and $(p, q) \cap (i_s, i') = \emptyset$, $(p, q) \cap (i_{s+1}, k'') = \emptyset$. This and the facts that $(i'', k'') \cap (i_s, i_{s+1}) = \emptyset$ and that $i''$ does not belong to the convex hull of $p_-$, $p_+$. Thus, to prove (2.57), it suffices to show that $(p, q) \cap (i_s, i'') < (p, q) \cap (i', k') < (p, q) \cap (i, k'') \leq p_+$. In turn, this immediately implies (2.56) in this case.

**Case 2.** Suppose that $(p, q) \cap (i_s, i_{s+1}) = \emptyset$, $3 \leq s \leq 2m - 3$. By definition, $p_+ < p_0 < p_-$ Then the convexity of the $n$-gon $[n]$ implies that there exist $i''$, $k'' \in [n]$ such that $\{i'', k''\} = \{i', k'\}$ and $(p, q) \cap (i_s, i') = \emptyset$, $(p, q) \cap (i_{s+1}, k'') = \emptyset$. This and the facts that $(i'', k'') \cap (i_s, i_{s+1}) = \emptyset$ and that $i''$ does not belong to the convex hull of $p_-$, $p_+$. Thus, to prove (2.57), it suffices to show that $(p, q) \cap (i_s, i'') < (p, q) \cap (i', k') < (p, q) \cap (i, k'') \leq p_+$. In turn, this immediately implies (2.56) in this case.

**Proof.** Now we have to prove that

(2.57)

$$i' := \{i_s, i', k', i_{s+1}, \ldots\} \in Adm_\Delta(p, q),$$

where $s = ind_i(P)$ is even and $i', k'$ are as in (2.48) (note that $\{i_s, i_{s+1}\} = \{j, k\}$ and $\{i', k'\} = \{i, k\}$).

Denote $p_0 := (p, q) \cap (i_s, i_{s+1})$ and consider a number of cases.

**Case 1.** Suppose that $i_{s-1}, i_{s+2} = \{i, k\}$. Then $i' = i_{s-1}, k' = i_{s+2}$ by (2.48) and $i' \cap (i_s, i'') = \emptyset$, $(p, q) \cap (i', k') = \emptyset$.

**Case 2.** Suppose that $i_{s-1} \in \{i, k\}$, $i_{s+2} \notin \{i, k\}$ (the case $i_{s+1} \in \{i, k\}$, $i_{s+2} \notin \{i, k\}$ is identical to it). Then $k' = i_{s+2}$ by (2.48) and $i' \cap (i_s, i'') = \emptyset$. I.e., $i'$ is obtained from $i$ by simultaneously replacing $i_s$ with $i_{s-1}$ and $i_{s+1}$ with $i_{s+2}$. This immediately implies (2.57) in this case.

**Proof.** Now we have to prove that

(2.57)

$$i' := \{i_s, i', k', i_{s+1}, \ldots\} \in Adm_\Delta(p, q),$$

where $s = ind_i(P)$ is even and $i', k'$ are as in (2.48) (note that $\{i_s, i_{s+1}\} = \{j, k\}$ and $\{i', k'\} = \{i, k\}$).

Denote $p_0 := (p, q) \cap (i_s, i_{s+1})$ and consider a number of cases.

**Case 1.** Suppose that $i_{s-1}, i_{s+2} = \{i, k\}$. Then $i' = i_{s-1}, k' = i_{s+2}$ by (2.48) and $i' \cap (i_s, i'') = \emptyset$, $(p, q) \cap (i', k') = \emptyset$.

**Case 2.** Suppose that $i_{s-1} \in \{i, k\}$, $i_{s+2} \notin \{i, k\}$ (the case $i_{s+1} \in \{i, k\}$, $i_{s+2} \notin \{i, k\}$ is identical to it). Then $k' = i_{s+2}$ by (2.48) and $i' \cap (i_s, i'') = \emptyset$. I.e., $i'$ is obtained from $i$ by simultaneously replacing $i_s$ with $i_{s-1}$ and $i_{s+1}$ with $i_{s+2}$. This immediately implies (2.57) in this case.
that \((p, q) \cap (i_{s-2}, i_{s-1}) \neq \emptyset\), then \(i'\) belongs to the interior of the convex hull of \(p, p_0, i_{s-1}, i_s\). This contradicts to that \(i'\) is a vertex of the convex \(n\)-gon \([n]\), which immediately implies \((2.57)\) in this case.

**Case 3.** Suppose that \(\{i_{s-1}, i_{s+2}\} \cap \{i, k\} = \emptyset\) Then \(i' = i, k' = k\) by \((2.48)\) and
\[
i' = (\ldots, i_s, i, k, i_{s+2}, \ldots).
\]

Thus, to prove \((2.57)\), it suffices to show that \((p, q) \cap (i_s, i) \neq \emptyset, (p, q) \cap (k, i) \neq \emptyset\). Since \((p, i) \cap (j, \ell) = \emptyset\), using the same argument as in **Case 2**, we see that if \((p, q) \cap (i_s, i) = \emptyset\), then \(i'\) belongs to the interior of the convex hull of \(p, p_0, i_{s-1}, i_s\) and if \((p, q) \cap (k, i) = \emptyset\), then \(k\) belongs to the interior of the convex hull of \(q, p_0, i_s, i_{s+1}\). This finishes the proof of \((2.57)\) in this case.

This finishes the proof of \((b)\).

Lemma \((2.81)\) is proved.

Using Lemma \((2.81)\) with \(P = (i, j, k, \ell)\) such that \((p, i) \cap (j, \ell) = \emptyset\) and \((p, q) \cap (i, j) \neq \emptyset\) and Lemma \((2.81)\) with any \(i', k'\) such that \(\{i', k'\} = \{i, k\}\), we see that
\[
[J_P][(\text{Adm}_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]] \subset (\text{Adm}_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]
\]
hence \(J_{\Delta, \Delta'}\) given by \((2.55)\) is a well-defined map
\[
(\text{Adm}_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \rightarrow (\text{Adm}_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P].
\]
Interchanging \(\Delta\) and \(\Delta'\), taking into account that \((p, j) \cap (i, k) = \emptyset\), and applying Lemma \((2.81)\) again, we see that
\[
[J_P][(\text{Adm}_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]] \subset (\text{Adm}_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P].
\]
This gives a well-defined map
\[
J_{\Delta, \Delta'} : (\text{Adm}_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \rightarrow (\text{Adm}_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P].
\]

Since \([J_P]\) is an involution by Proposition \((2.78)\), the maps \(J_{\Delta, \Delta'}\) and \(J_{\Delta', \Delta}\) are inverse of each other, hence each of them is a bijection.

The proposition is proved.

Furthermore, we need the following obvious fact.

**Lemma 2.82.** In the assumptions of Lemma \((2.72)\) let \(s \in [2m - 1]\) be odd and let \(i = (i_1, \ldots, i_{2m}) \in [n]^{2m}\), \(m \geq 1\) be such that \(i_s, i_{s+1} \neq \{j, \ell\}\) for \(r \in [2m - 1] \setminus \{s\}\).

(a) If \(\{i_s, i_{s+1}\} = \{j, \ell\}\) then \(\varphi_{\Delta, \Delta'}(t_i) = t_{j_{\Delta}^\ell(i)} + t_{k_{\Delta}^\ell(i)}\).

(b) If \(\{i_s, i_{s+1}\} = \{i, k\}\), then \(\varphi_{\Delta, \Delta'}(t_{j_{\Delta}^\ell(i)} + t_{k_{\Delta}^\ell(i)}) = t_{j_{\Delta}^\ell(i)}\).

Now we are ready to prove \((2.45)\). Indeed, \(t_{pq}^\prime = t_0 + t_- + t_+\), where
\[
t_0 = \sum_{i' \in \text{Adm}_{\Delta'}(p, q) : \text{ind}_{\Delta'}(P) = 0} t_0', \quad t_- = \sum_{i' \in \text{Adm}_{\Delta'}(p, q) : \text{ind}_{\Delta'}(P) \in 2\mathbb{Z}+1} t_0', \quad t_+ = \sum_{i' \in \text{Adm}_{\Delta'}(p, q) : \text{ind}_{\Delta'}(P) \in 2\mathbb{Z}+1} t_0'.
\]

Clearly,
\[
\varphi_{\Delta, \Delta'}(t_0) = \sum_{i \in \text{Adm}_{\Delta}(p, q) : \text{ind}_{\Delta}(P) = 0} t_i.
\]
Furthermore, combining Proposition \((2.79)\) and Lemma \((2.82)\) we obtain:
\[
\varphi_{\Delta, \Delta'}(t_-) = \sum_{i' \in \text{Adm}_{\Delta'}(p, q) : \text{ind}_{\Delta'}(P) \in 2\mathbb{Z}+1} t_{j_{\Delta}^\ell(i_{\Delta'}(i))} + t_{k_{\Delta}^\ell(i_{\Delta'}(i))} = \sum_{i \in \text{Adm}_{\Delta}(p, q) : \text{ind}_{\Delta}(P) \in 2\mathbb{Z}+1} t_i,
\]
\[
\varphi_{\Delta, \Delta'}(t_+) = \sum_{i \in \text{Adm}_{\Delta}(p, q) : \text{ind}_{\Delta}(P) \in 2\mathbb{Z}+1} \varphi_{\Delta, \Delta'}(t_{j_{\Delta}^\ell(i_{\Delta'}(i))} + t_{k_{\Delta}^\ell(i_{\Delta'}(i))}) = \sum_{i \in \text{Adm}_{\Delta}(p, q) : \text{ind}_{\Delta}(P) \in 2\mathbb{Z}+1} t_i.
\]
This finishes the proof of \((2.45)\).

Furthermore, we define a homomorphism \(\psi_{\Delta, \Delta'}\) as follows. First, composing \(\varphi_{\Delta, \Delta'}\) with the universal localization by \(S_{\Delta}\) and taking into the account that \(t_{j_{\Delta}^\ell}^2 \in S_{\Delta}\), we obtain a homomorphism of algebras:
\[
\varphi_{\Delta, \Delta'} : QT_{\Delta'} \rightarrow QT_{\Delta}([t_{j_{\Delta}^\ell}]^{-1})
\]
such that $\varphi'_{\Delta,\Delta'}(t_{ij}^{\Delta'}) = t_{ij}^{\Delta}$ for all $i,j$. Since $t_{ij}^{\Delta} \in S_{\Delta}$ is invertible in the image, $\varphi'_{\Delta,\Delta'}$ canonically extends to a homomorphisms of algebras

$$
\psi_{\Delta,\Delta'} : QT_{\Delta}[S_{\Delta}^{-1}] \to QT_{\Delta}[S_{\Delta'}^{-1}].
$$

Switching $\Delta$ and $\Delta'$ we obtain a homomorphism $\psi_{\Delta',\Delta} : QT_{\Delta}[S_{\Delta'}^{-1}] \to QT_{\Delta'}[S_{\Delta}^{-1}]$, which is, clearly, inverse of $\psi_{\Delta,\Delta'}$.

This proves Theorem 2.71 for neighboring triangulations $\Delta, \Delta'$. Now we prove Theorem 2.71 for any (non-neighboring) triangulation $\Delta, \Delta'$ of $[n]$ as follows. We say that the distance $dist(\Delta, \Delta')$ is the minimal number $d \geq 0$ such that there is a sequence of triangulations $\Delta = \Delta^{(0)}, \Delta^{(1)}, \ldots, \Delta^{(d)} = \Delta'$ of $[n]$ such that $\Delta^{(s)} \Delta^{(s+1)}$, $s \in [r - 1]$ are neighboring.

We construct appropriate $\varphi_{\Delta,\Delta'}$ by induction in $dist(\Delta, \Delta')$. If $dist(\Delta, \Delta') = 0$, then $\Delta$ and $\Delta'$ are neighboring and we have nothing to prove. Suppose that $d = dist(\Delta, \Delta') > 1$. Then there is a triangulation $\Delta''$ of $[n]$ with $dist(\Delta, \Delta'') < d$ and $dist(\Delta'', \Delta') < d - 1$. By the inductive hypothesis, there are isomorphisms

$$
\psi_{\Delta,\Delta''} : QT_{\Delta''}[S_{\Delta''}^{-1}] \to QT_{\Delta}[S_{\Delta}^{-1}], \quad \psi_{\Delta'',\Delta'} : QT_{\Delta'}[S_{\Delta'}^{-1}] \to QT_{\Delta''}[S_{\Delta''}^{-1}]
$$

such that $\psi_{\Delta,\Delta''}(t_{ij}^{\Delta''}) = t_{ij}^{\Delta}$ and $\psi_{\Delta'',\Delta'}(t_{ij}^{\Delta'}) = t_{ij}^{\Delta}$ for all $i,j$.

Define $\psi_{\Delta,\Delta'} := \psi_{\Delta,\Delta''} \circ \psi_{\Delta'',\Delta'}$. By definition, $\psi_{\Delta,\Delta'}$ is an isomorphism $QT_{\Delta}[S_{\Delta}^{-1}] \to QT_{\Delta'}[S_{\Delta'}^{-1}]$ such that $\psi_{\Delta,\Delta'}(t_{ij}^{\Delta'}) = t_{ij}^{\Delta}$ for all $i,j$. In particular, $\psi_{\Delta,\Delta'}$ does not depend on the choice of $\Delta''$. This finishes the induction.

The transitivity (2.32) also follows.

Theorem 2.71 is proved.

Furthermore, we need the following result.

**Proposition 2.83.** In the notation of Theorem 2.71 for each triangulation $\Delta$ of $[n]$ the homomorphism $i_{\Delta} : QT_{\Delta} \to A_{\Delta} \subset A_{n}$ given by (2.41) extends to an isomorphism of algebras $QT_{\Delta}[S_{\Delta}^{-1}] \to A_{n}$.

**Proof.** We need the following result.

**Lemma 2.84.** Let $\Delta$ be any triangulation of $[n]$. Then

(i) For any distinct $i,j,k \in [n]$, the elements $x_{ab} := t_{ab}^{\Delta}, \{a,b\} \subset \{i,j,k\}$ satisfy the triangle relations (2.1).

(ii) For any cyclic quadruple $(i,j,k,\ell)$ the elements $x_{ab} := t_{ab}^{\Delta}, \{a,b\} \subset \{i,j,k,\ell\}$ satisfy the exchange relations (2.2).

**Proof.** Indeed, to prove (i) note that for any distinct $i,j,k \in [n]$ there exists a triangulation $\Delta_{0}$ such that $(i,j,k)$ is a triangle in $\Delta_{0}$ therefore, the elements $t_{ab} \in T_{\Delta'}, \{a,b\} \subset \{i,j,k\}$ satisfy (2.1). Applying the isomorphism $\psi_{\Delta,\Delta_{0}}$ given by (2.44), we finish the proof of (i).

To prove (ii) note that for any cyclic $(i,j,k,\ell)$ there exists a triangulation $\Delta_{0}$ such that both triangles $(i,j,k)$ and $(j,k,\ell)$ belong to $\Delta_{0}$ (hence $(j,\ell) \notin \Delta_{0}$). By (2.43) for $\Delta_{0}$, we see that $t_{ab}^{\Delta_{0}}, \{a,b\} \subset \{i,j,k,\ell\}$ satisfy (2.2).

Thus applying the isomorphism $\psi_{\Delta,\Delta_{0}}$, we finish the proof of (ii).

The lemma is proved.

By Lemma 2.84, the assignments $x_{pq} \mapsto t_{pq}^{\Delta}$ for all distinct $p,q \in [n]$ define an epimorphism of algebras

$$
A_n \to QT_{\Delta}[S_{\Delta}^{-1}].
$$

On the other hand, by (already proved) Theorem 2.10 for each triangulation $\Delta$ of $[n]$ and any distinct $i,j \in [n]$ the element $x_{ij} \in i_{\Delta}(QT_{\Delta})$. Therefore, by the universality of localizations, $i_{\Delta}$ extends to an epimorphism of algebras $i_{\Delta} : QT[S_{\Delta}^{-1}] \to A_n$. Clearly, these two homomorphisms are mutually inverse.

This finishes the proof of Proposition 2.83.

Furthermore, denote by $S$ the submonoid of $A_{\Delta} \setminus \{0\}$ generated by all $x_{ij}$. Clearly, $S = i_{\Delta}(S_{\Delta})$ and $A_{\Delta} = i_{\Delta}(QT_{\Delta})$. Therefore, $A_{n} = A_{\Delta}[S^{-1}]$.

This proves Theorem 2.8.

Finally, Theorem 2.8 follows from Theorem 2.8 and that $A'_{\Delta} := A_{\Delta} = i_{\Delta}(QT_{\Delta})$ is the group algebra of $T_{\Delta}$, which is a free group in $3n - 4$ generators by (already proved) Theorem 2.7.
2.15. Self-similarity implies injectivity. In this section we prove the following result.

Proposition 2.85. If Conjecture 5.15 holds for \( m = 3n - 4 \), \( n \geq 4 \) and \( k = 2, \ldots, n - 2 \), then for each triangulation \( \Delta \) of \([n]\) the homomorphism of algebras

\[
A_n \to F_{3n-4}
\]

which is the canonical (by Proposition 2.83 and Lemma 7.7) extension to \( A_n \cong QT_\Delta [S_\Delta^{-1}] \) of the natural inclusion \( QT_\Delta \hookrightarrow \text{Frac}(QT_\Delta) \cong F_{3n-4} \) is also a monomorphism (hence \( A_n \) has no zero divisors).

Proof. it suffices to show that for at least one triangulation \( \Delta \) of \([n]\) the submonoid \( \hat{S}_\Delta \subset QT_\Delta \setminus \{0\} \) generated by all \( t_{ij}^\Delta \) and by \((QT_\Delta)^{\times} = Q^\times \cdot T_\Delta \) is factor-closed in the sense of Definition 5.14. Since \( T_\Delta \) is a free group by Theorem 2.7 in view of Proposition 5.15 it suffices to verify that each \( t_{ij}^\Delta \), \( (i, j) \notin \Delta \) is prime in \( QT_\Delta \) and all primes similar to \( t_{ij} \) belong to \( \hat{S}_\Delta \). Now let \( \Delta = \Delta_1 \) be the startlike triangulation as in 2.6 with \( i = 1 \).

We need the following obvious fact.

Lemma 2.86. For all \( n \geq 2 \) the group \( T_{\Delta_1} \) is freely generated by \( \tau_j := T_1^{j+1}, j = 2, \ldots, n - 1, t_{1,k}, t_{k,1}, k = 2, \ldots, n \).

Proof. Clearly, \( T_{\Delta_1} \) has a presentation \( t_{j,j+1} = t_{j,1} \tau_j t_{1,j+1}, t_{j+1,j} = t_{j+1,1} \tau_j t_{1,j} \) for \( j = 2, \ldots, n - 1 \).

This proves the lemma.

Furthermore, Corollary 2.22 implies that the monoid \( \hat{S}_{\Delta_1} \) is generated by \( T_{\Delta_1} \) and noncommutative angles

\[
T_{ij}^\Delta = \tau_i + \ldots + \tau_j
\]

for \( 2 \leq i < j \leq n \). Clearly, each \( T_{ij}^\Delta \), \( i < j - 1 \) is prime in \( QT_{\Delta_1} \). Let \( P_{ij} := Q^\times \cdot T_{\Delta_1} \cdot T_{ij}^\Delta \cdot T_{\Delta_1} \) for \( 2 \leq i < j \leq n \). By Conjecture 5.15 with \( m = 3n - 4 \), \( k = j \), that the only primes similar to \( T_{ij}^\Delta \) are elements of \( P_{ij} \). This together with Proposition 5.15 and Remark 5.14 proves that the submonoid \( Q^\times \cdot \hat{S}_{\Delta_1} \) of \( QT_{\Delta_1} \setminus \{0\} \) is factor-closed because it is generated by \( Q^\times \cdot T_{\Delta_1} \), and \( P = \bigcup_{2 \leq i < j \leq n} P_{ij} \). Therefore, Corollary 5.13 guarantees that \( QT_{\Delta_1}[S^{-1}] = QT_{\Delta_1}[Q^\times \cdot \hat{S}_{\Delta_1}] \) is a subalgebra of \( F_{3n-4} = \text{Frac}(QT_{\Delta_1}) \).

Using this and Proposition 2.83 with \( \Delta = \Delta_1 \), we finish the proof of Proposition 2.85. \( \square \)

3. Noncommutative surfaces

In this section we extend all the constructions and results of Section 2 to marked surfaces i.e., (connected compact smooth) surfaces \( \Sigma \) possibly with boundary equipped with a non-empty finite set \( I = I(\Sigma) = I_b \cup I_p \) of marked points with a subset \( I_b = I_b(\Sigma) \subset I \) of marked boundary points, the set \( I_p = I_p(\Sigma) = I \setminus I_b \) of ordinary punctures and a set \( I_s = I_s(\Sigma) \) of special punctures (which were called orbifold point of order 2 in [19], however, we will not use this terminology). We also require that each boundary component contains at least one point from \( I_b \). We denote by \( \Sigma \) the underlying topological space.

3.1. Multi-groupoid of curves on \( \Sigma \). Given points \( p_1, p_2 \in I(\Sigma) \), consider connected smooth directed curves \( C \) in \( \Sigma \setminus I_p(\Sigma) \) starting at \( p_1 \) and terminating at \( p_2 \). For a curve \( C \) denote by \( \overline{C} \) the same curve traversed from \( p_2 \) to \( p_1 \). We say that curves \( C \) and \( C' \) in \( \Sigma \) from \( p_1 \) to \( p_2 \) are equivalent if \( C \) and \( C' \) are homotopy equivalent as (connected smooth directed) curves in \( \Sigma \setminus I_p(\Sigma) \).

Pairwise non-equivalent curves from puncture 1 to puncture 2

Denote by \( \Gamma_{ij} = \Gamma_{ij}(\Sigma) \) the set of equivalence classes of curves \( C \) in \( \Sigma \) which originate at \( i \) and terminate at \( j \) then let \( \Gamma = \Gamma(\Sigma) := \bigsqcup_{i,j \in I(\Sigma)} \Gamma_{ij} \). For \( \gamma \in \Gamma_{ij} \) we denote by \( s(\gamma) \in I(\Sigma) \) (resp. \( t(\gamma) \in I(\Sigma) \)) the source \( i \) (resp. the target \( j \)).

Thus we have a natural involution \( \tau: \Gamma \overset{\sim}{\to} \Gamma (\gamma \mapsto \overline{\gamma}) \). By definition, \( \overline{\Gamma_{ij}} = \Gamma_{ji} \) for all \( i, j \in I(\Sigma) \).
For \( j \in I(\Sigma) \) denote by \( id_j \) the trivial loop at \( j \). Clearly, \( \gamma = \overline{\gamma} \) if \( \gamma \) is trivial.

It is easy to see that \( \Gamma(\Sigma) \) is finite iff \( \Sigma \) is homeomorphic to an \( n \)-gon, i.e., a disk with \( n \geq 1 \) marked points and no punctures. In that case, the assignments \( \gamma \mapsto (s(\gamma), t(\gamma)) \) define a bijection \( \Gamma \to \{ (i, j) \in [n] \mid i \neq j \} \).

We say that \( \gamma \in \Gamma(\Sigma) \) is simple if it has a non-self-intersecting representative. Denote by \( \Gamma^0(\Sigma) \) the set of all simple \( \gamma \in \Gamma(\Sigma) \).

**Definition 3.1.** We say that a pair \( (\gamma, \gamma') \) in \( \Gamma(\Sigma) \) is composable if \( t(\gamma) = s(\gamma') \) and define the composition \( \gamma'' = \gamma \circ \gamma' \) to be the pullback, under the natural projection \( \Gamma(\Sigma) \to \Gamma(\Sigma \setminus \{ I_p(\Sigma) \setminus \{ t(\gamma) \} \}) \) of the concatenation of \( \gamma \) and \( \gamma' \).

Clearly, the multi-composition \( \gamma \circ \gamma' \) is a 1-element set iff \( t(\gamma) = s(\gamma') \in I_b(\Sigma) \). Otherwise \( \gamma \circ \gamma' \) is countable.

The following is immediate.

**Lemma 3.2.** For each marked surface \( \Sigma \) the set \( \Gamma(\Sigma) \) is a multi-groupoid with the object set \( I(\Sigma) \) and the inverse given by \( \gamma^{-1} := \overline{\gamma} \).

**Remark 3.3.** A multi-category (e.g., a multi-groupoid) is a natural generalization of a category (e.g., of a groupoid) where we allow the composition of two morphisms to be a set of arrow and require the associativity \( (\gamma \circ \gamma') \circ \gamma'' = \gamma \circ (\gamma' \circ \gamma'') \), which is an equality of sets, see e.g. [16] (where the term polygroupoid was introduced).

**Remark 3.4.** If \( I_p(\Sigma) = \emptyset \), then \( \Gamma(\Sigma) \) is an ordinary groupoid (cf. [12] Section 2.2).

### 3.2. Category of surfaces and reduced curves.

**Definition 3.5.** Given a continuous map \( f : \Sigma \to \Sigma' \) with discrete fibers, we say that \( f \) is a morphism of marked surfaces \( \Sigma \to \Sigma' \) if:

- \( f^{-1}(I(\Sigma')) = I(\Sigma), f(I_s(\Sigma)) \subseteq I_s(\Sigma') \) (we abbreviate \( I' := f^{-1}(I_s(\Sigma')) \setminus I_s(\Sigma) \)).
- For each point \( p \in \Sigma \setminus I' \) there is a neighborhood \( O_p \) of \( p \) in \( \Sigma \) such that the restriction of \( f \) to \( O_p \) is injective (if \( p \in \partial \Sigma \) is a boundary point, then \( O_p \) is a “half-neighborhood”).
- For each \( p \in I' \) there is a neighborhood \( O_p \) of \( p \) in \( \Sigma \) such that the restriction of \( f \) to \( O_p \) is a two-fold cover of \( f(O_p) \) ramified at \( f(p) \).

**Theorem 3.6.** For any morphisms of marked surfaces \( f : \Sigma \to \Sigma' \) and \( f' : \Sigma' \to \Sigma'' \) the composition \( f' \circ f : \Sigma \to \Sigma'' \) is also a morphism of marked surfaces \( \Sigma \to \Sigma'' \).

We prove Theorem 3.6 in Section 3.11.

In what follows, denote by \( \text{Surf} \) the category whose objects are marked surfaces and arrows are morphisms of marked surfaces.

Note that if \( f : \Sigma \to \Sigma' \) is a morphism in \( \text{Surf} \) with \( I' = \emptyset \), then \( f \) respects (homotopy) equivalence of curves and, in particular, defines a map \( \Gamma(\Sigma) \to \Gamma(\Sigma') \). In general, this is no longer true. To fix it, we define below a stronger equivalence relation than the equivalence for curves in \( \Sigma' \).

Indeed, given \( i \in I_s(\Sigma) \), we say that a curve \( C \) in \( \Sigma \) is \( i \)-reducible if there is a self-intersection point \( p \in C \) such that the loop \( C_0 \subseteq C \) defined by \( p \) encloses exactly one special puncture \( i \); otherwise, \( C \) is \( i \)-reduced. Respectively, \( \gamma \in \Gamma(\Sigma) \) is \( i \)-reducible (resp. \( i \)-reduced) if \( \gamma \) has an \( i \)-reducible (resp. \( i \)-reduced) representative.
Denote by \([\Gamma(\Sigma)]_i\) the set of all \(i\)-reduced \(\gamma \in \Gamma(\Sigma)\), abbreviate \([\Gamma(\Sigma)] := \bigcap_{i \in I_s(\Sigma)} [\Gamma(\Sigma)]_i\) and refer to elements of \([\Gamma(\Sigma)]\) as \textit{reduced}. Clearly, \([\Gamma(\Sigma)] = \Gamma(\Sigma)\) iff \(I_s(\Sigma) = \emptyset\). It is also clear that each \(\gamma \in \Gamma^0(\Sigma)\) is reduced.

For each \(i\)-reducible \(\gamma \in \Gamma(\Sigma)\) denote by \([\gamma]_i\), the class in \(\Gamma(\Sigma)\) obtained by resolving the self-intersecting simple loop around \(i\) in (a generic representative \(C\) of) \(\gamma\) so that the resulting curve is connected (the “wrong” crossing resolution would result in creating two connected components, one of which is a circle around \(i\)).

The following is obvious.

**Lemma 3.7.** (a) The assignments \(\gamma \mapsto [\gamma]_i\) if \(\gamma\) is \(i\)-reducible define a map \(\pi_i : \Gamma(\Sigma) \rightarrow [\Gamma(\Sigma)]_i\).

(b) \(\pi_i \circ \pi_j = \pi_j \circ \pi_i\) for all \(i, j \in I_s(\Sigma)\).

(c) The assignments \(\gamma \mapsto \pi_i^N(\gamma)\) for sufficiently big \(N\) define a projection \(\pi_i^\infty : \Gamma(\Sigma) \rightarrow [\Gamma(\Sigma)]_i\).

(d) The composition \(\pi^\infty := \prod_{i \in I_s(\Sigma)} \pi_i^\infty\) is a projection \(\Gamma(\Sigma) \rightarrow [\Gamma(\Sigma)]\).

This, in particular, defines an equivalence relation on \([\Gamma(\Sigma)]\), namely for \(\gamma, \gamma' \in \Gamma(\Sigma)\) we say that any representatives \(C \in \gamma\) and \(C' \in \gamma'\) are \(I_s(\Sigma)\)-equivalent iff \(\pi^\infty(\gamma) = \pi^\infty(\gamma')\). We naturally identify \(I_s(\Sigma)\)-equivalence classes with elements of \([\Gamma(\Sigma)]\).

For each \(i \in I_s(\Sigma)\) and \(j \in I(\Sigma)\) let \(\lambda_{ij}\) denote a (unique up to \(\gamma\)) simple loop at \(j\) around \(i\) in \([\Gamma(\Sigma)]\). We refer to such loops as \textit{special}.

For \(n \geq 1, h \geq 0\) denote by \(P_n(h)\) the \(n\)-gon (i.e., a disk with \(n\) marked boundary points) with \(h\) special punctures and abbreviate \(P_n := P_n(0)\). Clearly, each special loop \(\lambda\) determines a (homeomorphic) copy of \(P_1(1)\) with the marked point set \([1]\) and the special puncture set \(\{i\}\).

**Lemma 3.8.** For any marked surface \(\Sigma\) the set \([\Gamma(\Sigma)]\) has a natural multi-groupoid structure:

\[ [\gamma] \circ [\gamma'] := [\gamma \circ \gamma'] \]

for any composable \((\gamma, \gamma')\) in the multi-groupoid \(\Gamma(\Sigma)\) with the object set \(I(\Sigma)\). Moreover,

(i) the assignments \(\gamma \mapsto [\gamma]\) define a surjective homomorphism of multi-groupoids \(\Gamma(\Sigma) \rightarrow [\Gamma(\Sigma)]\).

(ii) For each \(i \in I_s(\Sigma)\) and \(j \in I(\Sigma)\) each special loop satisfies \(\lambda_{ij} = \lambda_{ji}\).

The following result asserts functoriality of the multi-groupoid under morphisms of surfaces.

**Theorem 3.9.** Let \(f\) be any morphism of marked surfaces \(\Sigma \rightarrow \Sigma'\) and let \(\gamma \in [\Gamma(\Sigma)]\). Then

(a) for any generic representatives \(C, C' \in \gamma\), their images \(f(C)\) and \(f(C')\) are \(I_s(\Sigma')\)-equivalent.

(b) For each \(\gamma \in [\Gamma(\Sigma)]\) there exists a unique \(I_s(\Sigma')\)-equivalence class \(f(\gamma) \in [\Gamma(\Sigma')]\) such that \(f(C) \in f(\gamma)\) for any generic curve \(C \in \gamma\).
(c) $f : [\Gamma(\Sigma)] \to [\Gamma(\Sigma')]$ ($\gamma \mapsto f(\gamma)$) is a homomorphism of multi-groupoids.  
(d) The assignments $\Sigma \mapsto [\Gamma(\Sigma)]$ define a functor from $\text{Surf}$ to the category of multi-groupoids.

$L_q(\Sigma)$-equivalence of images of curves under the ramified double cover $z \mapsto z^2$ of $\mathbb{C}$

We prove Theorem 3.9 in Section 3.11.

It is well-known that marked surfaces can be glued out of polygons, i.e., for any $\Sigma$ there exists a surjective gluing morphism $f : P_n(h) \to \Sigma$ in $\text{Surf}$ with $h = |L_q(\Sigma)|$, $n \geq 1$ such that all $f(i,i^+) \in \Gamma^0(\Sigma)$ and the restriction of $f$ to the interior of $P_n(h)$ is injective. For readers’ convenience we construct such a gluing morphism $f$ in Lemma 3.47 for any triangulation of $\Sigma$.

The following fact is obvious.

Lemma 3.10. Let $\Sigma$ be a marked surface. Then $[\Gamma(\Sigma)]$ is finite if an only if $\Sigma$ is homeomorphic either a once punctured sphere or to $[n] = P_n$ or to $P_n(1)$ for some $n \geq 1$. More precisely, the assignments

$$\gamma \mapsto \begin{cases} (s(\gamma), t(\gamma),+) & \text{if the special puncture is to the right of } \gamma \\ (s(\gamma), t(\gamma),-) & \text{if the special puncture is to the left of } \gamma \end{cases}$$

define a bijection $[\Gamma(P_n(1))] \xrightarrow{\sim} \{ (i,j) \in [n] \times \{-,+,\} \}$.

3.3. Polygons in surfaces, noncommutative surfaces and functoriality. We say that a sequence $P = (\gamma_1, \ldots, \gamma_r)$ of not necessarily distinct $\gamma_i \in [\Gamma(\Sigma)]$, $i \in [r]$, is cyclic if each pair $(\gamma_i, \gamma_{i+})$, $i \in [r]$ is composable.

Definition 3.11. We say that a sequence $P = (\gamma_1, \ldots, \gamma_n)$ is an $n$-gon in $\Sigma$ if there exists a morphism $f : P_n \to \Sigma$ such that $f(i,i^+) = \gamma_i$ for $i \in [n]$. We also denote $\gamma_{ij} := f(i,j)$ for all distinct $i,j \in [n]$ (clearly, $\gamma_{ij}$ is nontrivial for all distinct $i,j \in [n]$). We will refer to such an $f$ as an accompanying to $P$ morphism.

Clearly, each $n$-gon $P = (\gamma_1, \ldots, \gamma_n)$ in $\Sigma$ is cyclic and for any $\gamma \in [\Gamma(\Sigma)]$ the pair $(\gamma, \overline{\gamma})$ is a 2-gon in $\Sigma$. It is convenient to define the interior $P^0$ of an $n$-gon $P = (\gamma_1, \ldots, \gamma_n)$ to be the image of the interior of $P_n$ under an accompanying morphism (to do so we choose generic representatives $C_i \in \gamma_i$ so that $f(i,i^+) = C_i$ for $i \in [i]$). It is also clear that $P^0$ does not depend on the choice of $f$, and different choices of $C_i \in \gamma_i$ result in homotopic to each other morphisms $f : P_n \to \Sigma$. We say that $P$ is simple if $P^0$ is homeomorphic to a disk.

We will sometimes refer to an 3-gon in $\Sigma$ respectively as a triangle and to a 4-gon – as a quadrilateral.

Definition 3.12. For a marked surface $\Sigma$ let $A_\Sigma$ be the $\mathbb{Q}$-algebra generated by all $x_\gamma$, $\gamma \in [\Gamma(\Sigma)]$ subject to

(i) $x_\gamma = 1$ if $\gamma$ is trivial.

(ii) (triangle relations) For any triangle $(\gamma_1, \gamma_2, \gamma_3)$ in $\Sigma$ one has

$$x_{\gamma_1}x_{\gamma_2}^{-1}x_{\gamma_3} = x_{\gamma_3}x_{\gamma_2}^{-1}x_{\gamma_1}.$$

Non-simple triangles in an annulus and in $P_3(1)$
(iii) (exchange relations) For any quadrilateral \((\gamma_1, \gamma_2, \gamma_3, \gamma_4)\) in \(\Sigma\):

\[
x_{\gamma_2 \gamma_4} = x_{\gamma_2} x_{\gamma_3}^{-1} x_{\gamma_4} x_{\gamma_2} + x_{\gamma_2} x_{\gamma_4} x_{\gamma_2}^{-1} x_{\gamma_4}.
\]

Likewise (similarly to Section 2.6), we define the big triangle group \(T_\Sigma\) of \(\Sigma\) to be generated by all \(t_\gamma, \gamma \in [\Gamma(\Sigma)]\) subject to:

- \(t_\gamma = 1\) if \(\gamma\) is trivial.
- (triangle relations) \(t_\gamma t_\gamma^{-1} t_\gamma = t_{\gamma_3} t_{\gamma_2}^{-1} t_{\gamma_1}\) for all triangles \((\gamma_1, \gamma_2, \gamma_3)\) in \(\Sigma\).

The following fact is obvious.

**Theorem 3.16.** For any morphism \(f : \Sigma \to \Sigma'\) in \(\text{Surf}\) the assignments \(t_\gamma \mapsto x_\gamma\) (resp. \(x_\gamma \mapsto x_{f(\gamma)}\)) define a homomorphism of groups \(f_* : A_\Sigma \to A_{\Sigma'}\) and the following diagram is commutative

\[
\begin{array}{ccc}
T_\Sigma & \longrightarrow & A_\Sigma \\
f_* \downarrow & & f_* \downarrow \\
T_{\Sigma'} & \longrightarrow & A_{\Sigma'}
\end{array}
\]

We prove Theorem 3.16 in Section 3.11.

**Definition 3.17.** For a marked surface \(\Sigma\) denote by \(\hat{\Sigma}\) the marked surface obtained from \(\Sigma\) by turning each special puncture into the ordinary one, i.e., \(\hat{\Sigma} = \Sigma, I(\hat{\Sigma}) = I(\Sigma) \cup I_s(\Sigma), I_s(\hat{\Sigma}) = \emptyset\).

Clearly, \([\Gamma(\Sigma)] \subseteq [\Gamma(\hat{\Sigma})] = \Gamma(\hat{\Sigma})\) and the complement \([\Gamma(\hat{\Sigma})] \setminus [\Gamma(\Sigma)]\) consists of classes of curves originating or terminating in formerly special punctures.

**Proposition 3.18.** The assignments \(t_\gamma \mapsto t_\gamma\) for \(\gamma \in [\Gamma(\Sigma)]\) define a homomorphism of groups

\[
T_\Sigma \to T_{\hat{\Sigma}}
\]

where \(\hat{\Sigma}\) is as in Definition 3.17.

**Remark 3.19.** It is natural to conjecture that (3.5) is injective. Note, however, that the natural identification \(Id : \Sigma \to \hat{\Sigma}\) is not a morphism in \(\text{Surf}\) since it takes \(I_s(\Sigma)\) to \(I_p(\Sigma)\), so we expect that there is no homomorphisms \(A_\Sigma \to A_{\hat{\Sigma}}\), which together with (3.5) would make the diagram (3.3) commutative, and illustrate this with the following example.

**Example 3.20.** Let \(\Sigma = P_2(1)\) with the vertex set \(I = \{1, 2\}\) and a single special puncture 0. For \(i \in I\) denote by \(\gamma_i\) the clockwise loop at \(i\) around 0 inside \(\Sigma\). For \(i, j \in I, i \neq j\) denote by \(\gamma_{ij}^+\) (resp. \(\gamma_{ij}^-\)) the boundary curve from \(i\) to \(j\) so that 0 is to the right (resp. to the left).
A quadrilateral in $P_2(1)$

We abbreviate $x_i := x_{\gamma_i}$, $x_{ij} := x_{\gamma_i}^{\pm}$ for the corresponding generators of $A_{\Sigma}$.

Then, according to Definition 3.12 $A_{\Sigma}$ has a presentation:

\[
\begin{align*}
\mathbf{x}_1 &= x_1, \quad \mathbf{x}_2 = x_2, \quad \mathbf{x}_{12} = x_{12}, \quad x_{12}^{-1} = x_{12}\mathbf{x}_{12}, \quad x_{12}^{-1} = x_{12}\mathbf{x}_{12}, \quad x_{12} = x_{12}\mathbf{x}_{12}, \quad x_{12}^{-1} = x_{12}\mathbf{x}_{12}, \\
\end{align*}
\]

Let $\tilde{\Sigma}$ be obtained from $\Sigma$ by converting all special punctures into ordinary ones (as in Definition 3.17). Therefore, curves on $\tilde{\Sigma}$ are those on $\Sigma$ plus four additional ones: directed intervals $\gamma_{0,i}$ from 0 to each $i$ and $\gamma_{i,0} := \gamma_0$. We abbreviate the generators of $A_{\tilde{\Sigma}}$ same way as in $A_{\Sigma}$ and $x_{0,i} := x_{\gamma_{0,i}}$, $x_{i,0} := x_{\gamma_{i,0}}$.

Then, according to Definition 3.12 $A_{\tilde{\Sigma}}$ has a presentation:

\[
\begin{align*}
\mathbf{x}_1 &= x_1, \quad \mathbf{x}_2 = x_2, \quad \mathbf{x}_{12} = x_{12}, \quad x_{12}^{-1} = x_{12}\mathbf{x}_{12}, \quad x_{12}^{-1} = x_{12}\mathbf{x}_{12}, \quad x_{12} = x_{12}\mathbf{x}_{12}, \quad x_{12}^{-1} = x_{12}\mathbf{x}_{12}, \\
\end{align*}
\]

In particular,

\[
\begin{align*}
x_2 &= x_{12}x_{12}^{-1} + x_{12}x_{12}^{-1} + x_{12}x_{12}^{-1} + x_{12}x_{12}^{-1}, \quad x_1 = x_{12}x_{12}^{-1}x_{21} + x_{12}x_{12}^{-1}x_{21} + x_{12}x_{12}^{-1}x_{21} + x_{12}x_{12}^{-1}x_{21}. \\
\end{align*}
\]

Therefore, there is no homomorphism $A_{\Sigma} \to A_{\tilde{\Sigma}}$ or $A_{\tilde{\Sigma}} \to A_{\Sigma}$ which would send $x_i \mapsto x_i$, $x_{ij} \mapsto x_{ij}^\pm$ (which justifies Remark 3.19).

3.4. Triangulations of marked surfaces. Let $\Sigma$ be a marked surface, given distinct $\gamma, \gamma' \in \Gamma(\Sigma)$, define their intersection number $n_{\gamma,\gamma'} \in \mathbb{Z}_{\geq 0}$ to be the number of intersection points in the interiors of their generic representatives minus the endpoints of $\gamma$ and $\gamma'$. Clearly, $n_{\gamma,\gamma'}$ is well-defined, i.e., does not depend on the choice of representatives. By definition, $n_{\gamma,\gamma'} = n_{\gamma',\gamma} = n_{\gamma',\gamma}$ for all $\gamma, \gamma'$. Note that $n_{\gamma,\gamma'} = 0$ iff $\gamma$ and $\gamma'$ do not intersect (and may have only endpoints in common).

Given a marked surface $\Sigma$, we say that a subset $\Gamma' \subset \Gamma(\Sigma)$ is non-crossing if $n_{\gamma,\gamma'} = 0$ for all distinct $\gamma, \gamma' \in \Gamma'$, i.e., one can simultaneously choose generic representatives of classes in $\Gamma'$ such that they pairwise do not intersect in $\Sigma$ and do not self-intersect (i.e., may have only endpoints in common). Furthermore, we say that $\Delta$ is a triangulation of $\Sigma$ if $\Delta$ is a maximal non-crossing subset of $\Gamma(\Sigma)$ such that $|\Sigma| = \Delta$.

Clearly, if $I_\Sigma(\Sigma) \neq \emptyset$, then any triangulation $\Delta$ of $\Sigma$ has a special loop $\lambda_{ij}$ at some $j \in I_\Sigma(\Sigma)$ around each $i \in I_\Sigma(\Sigma)$, i.e., $\lambda_{ij}$ defines a a 2-gon $(\lambda_{ij}, \lambda_{ij})$ in $\Delta$ homeomorphic to $P_1(1)$. It is customary to fix a generic representative of each $\gamma^0 \in \Delta$ so that $\Sigma$ is literally cut into triangles and $P_1(1)$’s.

It is well-known that all triangulations of $\Sigma$ are finite of same cardinality. Moreover, any triangulation $\Delta'$ can be obtained from a given triangulation $\Delta$ by a sequence of flips of diagonals in quadrilaterals in $\Delta$ (see e.g., [11] Proposition 7.10 and [19] Theorem 4.21).

Given an $r$-gon $Q = (\gamma_1, \ldots, \gamma_r)$ in $\Sigma$ and a triangulation $\Delta$ of $\Sigma$. We say that a curve $\gamma^0 \in \Delta$ is attracted to $Q$ if either $\gamma^0$ intersects $Q$ or there is a triangle $\tau = (\gamma^-, \gamma^0, \gamma^+)$ in $\Delta$ such that $\gamma^-$ intersects $Q$; denote by $\Delta_0 = \Delta_0(P, \Delta)$ the set of all $\gamma^0 \in \Delta$ attracted to $Q$.

The following is immediate.

**Theorem 3.21.** Let $\Delta$ be a triangulation of $\Sigma$. Then for each $r$-gon $Q = (\gamma_1', \ldots, \gamma_r')$ in $\Sigma$ there exists an $n$-gon $P = (\gamma_1, \ldots, \gamma_n) \in (\Delta_0(Q, \Delta))^n$ for some $n \geq r$, a triangulation $\Delta^0$ of $[n]$, and an order-preserving embedding $\iota : [r] \to [n]$ such that:

(a) $\gamma_{ij} \in \Delta_0(Q, \Delta)$ iff $(i, j) \in \Delta^0$.

(b) $\gamma_k = \gamma_{k(i), k(i+1)}$ for all $k \in [r]$ (i.e., $Q$ is a “sub-polygon” of $P$).
In fact, if $Q = (\gamma, \overline{\gamma})$, $\gamma \in [\Gamma(\Sigma)]$, we will construct a canonical polygon $P_\Delta(\gamma)$ as follows. We need the following obvious fact.

**Lemma 3.22.** Let $\Delta$ be a triangulation of $\Sigma$ and let $\gamma \in [\Gamma(\Sigma)] \setminus \Delta$. Then there exists a unique (up to relabeling) triangle $\tau_1 = (\gamma_1, \gamma_-, \gamma_+) \in \Delta^3$ such that $n_{\gamma, \gamma_0} > 0$ and the closest to $s(\gamma)$ intersection point of $\gamma$ with $\Delta$ is the intersection point of $\gamma$ and $\gamma_-$. 

The initial triangle for $\gamma$

We refer to such a triangle as initial for $\gamma$. Fix the initial triangle $\tau$ as in Lemma 3.22 and denote by $\gamma^{(1)}$ the unique (class of) curve which starts as $\gamma_-$, follows this “route” until the first intersection point of $\gamma_-$ and $\gamma$ and then “becomes” $\gamma$. Repeating this process, we obtain a new initial triangle $\tau_s = (\gamma_s, \gamma_-, \gamma_+^{(s)})$ for $\gamma^{(s)}$, $s = 1, \ldots, j - 1$, where $j \geq 2$ is unique with $\gamma^{(j)} = \gamma_j \in \Delta$. This process converges by induction in $n_{\gamma, \Delta} := \sum_{\gamma_0 \in \Delta} n_{\gamma, \gamma_0}$ because $n_{\gamma, \Delta} > n_{\gamma^{(1)}, \Delta} > \cdots > n_{\gamma^{(j)}, \Delta} = 0$. Denote $F_\Delta(\gamma) := (\gamma_1, \ldots, \gamma_j) \in \Delta^j$ and refer to this sequence as a $\Delta$-factorization of $\gamma$. By definition, $\gamma \in \gamma_1 \circ \cdots \circ \gamma_j$ in the multi-groupoid $[\Gamma(\Sigma)]$, which justifies the terminology.

Finally, we set $P_\Delta(\gamma) := (F_\Delta(\gamma), F_\Delta(\overline{\gamma}))$ and refer to it as the canonical polygon of $\gamma$ in $\Delta$ due to the following obvious result.

**Lemma 3.23.** Each $P_\Delta(\gamma) = (\gamma_1, \ldots, \gamma_n)$ is an $n$-gon in $\Delta$. 

Canonical polygon, no special punctures

Canonical polygon, no special punctures, $\gamma$ self-intersects

Canonical polygon, one special puncture
3.5. Triangle groups and their topological invariance. For each triangulation \( \Delta \) of \( \Sigma \) we define the triangle group \( T_\Delta = T_\Delta(\Sigma) \) to be generated by all \( t_\gamma \), \( \gamma \in \Delta \) subject to (same relations as in \( T_\Sigma \)):
- \( t_\gamma = 1 \) if \( \gamma \) is trivial.
- \( t_\gamma t_\delta t_\gamma^{-1} = t_\gamma t_\delta t_\gamma^{-1} \) for any triangle \( T = (\gamma_1, \gamma_2, \gamma_3) \) in \( \Delta \).

Also, for each triangulation \( \Delta \) of \( \Sigma \) denote by \( Y_\Delta \) the subgroup of \( T_\Delta \) generated by:

\[ y_{\gamma, \gamma'} := t_{\gamma'}^{-1}t_{\gamma} \]

for all \( \gamma, \gamma' \in \Delta \) such that \((\gamma, \gamma', \gamma'') \) is a triangle in \( \Delta \) for some \( \gamma'' \in \Delta \).

**Theorem 3.24.** For any two triangulations \( \Delta \) and \( \Delta' \) of a marked surface \( \Sigma \) there exists a group isomorphism:

\[ f_{\Delta, \Delta'} : T_\Delta \cong T_{\Delta'} \]

such that \( f_{\Delta, \Delta'}(Y_\Delta) = Y_{\Delta'} \).

We prove Theorem 3.24 in Section 3.11.

**Remark 3.25.** Theorem 3.24 implies that isomorphism classes of groups \( T_\Delta \) and \( Y_\Sigma \) are topological invariants of surfaces. However, by contrast with Theorem 3.16, we do not expect the assignments \( \Sigma \mapsto T_\Delta \) to be functorial.

Our next result is classification of triangle groups of marked surfaces.

**Theorem 3.26.** Let \( \Sigma \) be a marked surface with the Euler characteristic \( \chi(\Sigma) \), the set \( I = I(\Sigma) \neq \emptyset \) of marked points, the set \( I_0 \subseteq I \) of marked boundary points, and \( h = |I_0| \) special punctures. Then for any triangulation \( \Delta \) of \( \Sigma \) one has:

(a) If \( \Sigma \) has a boundary or special punctures, then \( T_\Delta \) is a free group in:
- \( |I| + 1 \) generators if \( \Sigma \) is a disk with \( |I| + |I_0| = 2 \), \( h = 0 \).
- \( 2h + 3|I| - 4 \) generators if \( \Sigma \) is a disk with \( |I| + |I_0| = 2 \), \( h > 0 \).
- \( 2h + 4|I| - (\chi(\Sigma)) - |I_0| \) generators otherwise.

(b) If \( \Sigma \) is a closed surface without special punctures, then \( T_\Delta \) is:
- Trivial if \( \Sigma \) is the sphere with \( |I| = 1 \).
- A free group in \( 3|I| - 4 \) generators if \( \Sigma \) is the sphere with \( |I| \in \{2, 3\} \).
- A free group in 2 generators if \( \Sigma \) is the real projective plane with \( |I| = 1 \).
- A 1-relator torsion free group (in the sense of Definition 5.6) in \( 4(|I| - \chi(\Sigma)) + 1 \) generators otherwise.

We prove Theorem 3.26 in Section 3.12 by choosing an appropriate triangulation of \( \Sigma \).

**Remark 3.27.** If \( \Sigma \) has \( r \) boundary components, then it is homotopy equivalent to a bunch of \( g \geq r \) circles and \( \chi(\Sigma) = 1 - g \). If \( \Sigma \) is a closed orientable (resp. non-orientable) surface, then it is homeomorphic the connected sum of \( g \) copies of the torus (resp. of the real projective plane) and \( \chi(\Sigma) = 2 - 2g \) (resp. \( \chi(\Sigma) = 2 - g \)).

**Example 3.28.** If \( \Delta \) is a triangulation of the torus, the Klein bottle, the real projective plane respectively with one, one, two (ordinary) punctures, then \( T_\Delta \) is generated by \( c_1, c_2, \bar{c}_1, \bar{c}_2, t_3 \) subject to, respectively (with some notation from the proof of Theorem 3.20 in Section 3.12):

(i) for the torus with one puncture: \( c_2 t_3 c_1 c_2^{-1} c_1 = \bar{c}_1 \bar{c}_2^{-1} c_1 t_3 \bar{c}_2 \), because \( \Delta \) is glued from a square with diagonal \((1, 3)\), where: \( S = \{1, 2\} \subset \{4\} \), \( \sigma(1) = 3 \), \( \sigma(2) = 4 \), \( \varepsilon(1) = \varepsilon(2) = + \) (equivalently, \( abcd = cbeda \) after substitution \( a = t_3, b = \bar{c}_1, c = c_2^{-1}, d = c_1, e = \bar{c}_2^{-1} \)).

(ii) for the Klein bottle with one puncture: \( c_2 t_3 c_1^{-1} c_1 = \bar{c}_1 c_2^{-1} c_1 t_3 \bar{c}_2 \), because \( \Delta \) is glued from a square with diagonal \((1, 3)\), where: \( S = \{1, 2\} \subset \{4\} \), \( \sigma(1) = 3 \), \( \sigma(2) = 4 \), \( \varepsilon(1) = + \), \( \varepsilon(2) = - \) (equivalently, \( abcde = edba \) after substitution \( a = t_3, b = \bar{c}_1, c = c_2^{-1}, d = c_1, e = c_2^{-1} \)).

(iii) for the real projective plane with two punctures: \( c_2 t_3 c_1^{-1} c_1 = \bar{c}_1 c_2^{-1} c_1 t_3 \bar{c}_2 \), because \( \Delta \) is glued from a square with diagonal \((1, 3)\), where: \( S = \{1, 2\} \subset \{4\} \), \( \sigma(1) = 3 \), \( \sigma(2) = 4 \), \( \varepsilon(1) = \varepsilon(2) = - \) (equivalently, \( abcde = edba \) after substitution \( a = t_3, b = c_1, c = c_2^{-1}, d = \bar{c}_1, e = c_2^{-1} \)).

**Example 3.29.** If \( \Delta \) is a triangulation of the sphere with \( m + 1 \) punctures, we can view it as glued from a regular \( 2m \)-gon with \( S = \{m\} \subset \{2m\} \), \( \sigma(k) = m + 1 - k \), \( \varepsilon(k) = + \) for \( k \in \{m\} \) (with some notation from the proof of Theorem 3.20 in Section 3.12). Then \( T_\Delta \) is generated by \( c_1, \ldots, c_{2m}, t_3, \ldots, t_{2m-1} \) subject to the relation \( t_3 c_4 \cdots c_{2m-2} t_{2m-1} = t_{2m-1} c_3 t_{2m-2} \cdots c_{2m-2} f_3 \).
3.6. Noncommutative Laurent Phenomenon for surfaces. The following result extends Noncommutative Laurent Phenomenon for $n$-gons (Theorem 2.10) to all marked surfaces.

**Theorem 3.30.** (Noncommutative Laurent Phenomenon for surfaces) Let $\Sigma$ be a marked surface and let $\Delta$ be a triangulation of $\Sigma$. Then for each $\gamma \in [\Gamma(\Sigma)]$ the element $x_{\gamma}$ of $A_\Sigma$ belongs to the subalgebra of $A_\Sigma$ generated by $x_{\gamma 0}^{\pm 1}$, $\gamma_0 \in \Delta$. More precisely, in the notation of Theorem 2.10, one has

$$x_{\gamma} = \sum_{i \in Adm, x_{i}(1,j)} x_i,$$

where $\Delta^0$ is the triangulation of $[n]$ assigned (as in Theorem 3.27(a)) to the canonical polygon $P_\Delta(\gamma) = (\gamma_1, \ldots, \gamma_n)$ in $\Delta$ with $\gamma = \gamma_{1,j}$, and we abbreviated

$$x_i := x_{i_1 i_2}, x_{\gamma_1 i_2}, x_{\gamma_3 i_4}, \ldots, x_{i_{2m-1} i_{2m-2}} x_{i_{2m-1} i_{2m}}$$

for any sequence $i = (i_1, \ldots, i_{2m}) \in [n]^{2m}$, $m \geq 1$.

We prove Theorem 3.30 in Section 3.11.

**Remark 3.31.** Theorem 3.30 is a noncommutative generalization of [37, Theorem 6.1].

**Example 3.32.** Let $\Sigma$ be a regular triangle with the clockwise vertex set $I = \{1, 2, 3\}$ and a special puncture 0 in the center. For $i \in I$ denote by $\lambda_i$ the the special loop at $i$ around 0. As in Example 3.27 for $i, j \in I$, $i \neq j$ denote by $\gamma_{ij}^+$ (resp. $\gamma_{ij}^-$) the curve from $i$ to $j$ so that 0 is to the right (resp. to the left) of the curve and abbreviate $x_i := x_{\lambda_i}$, $x_{ij}^+ := x_{\gamma_{ij}^+}$ for the corresponding generators of $A_\Sigma$.

Clearly, every triangulation of $\Sigma$ contains $\gamma_{12}^+, \gamma_{13}^-, \gamma_{23}^-, \gamma_{31}^+, \gamma_{13}^+, \gamma_{12}^-$. Let $\Delta$ be the triangulation of $\Sigma$ containing also $\gamma_{12}$ and $\gamma_{13}$. Then (3.6) reads:

$$x_2 = x_{21} x_{12}^{-1} x_{13} + x_{21} x_{23}^{-1} x_{12} x_{13}$$

$$x_3 = x_{31} x_{13}^{-1} x_{12} + x_{31} x_{23}^{-1} x_{13} x_{12}$$

Let $\Sigma$ be as in Definition 3.17. Therefore, simple curves on $\Sigma$ are those on $\Sigma$ plus six additional ones: directed intervals $\gamma_{0,i}$ from 0 to each $i$ and $\gamma_{i,0} := \gamma_{0,i}$. We abbreviate the generators of $A_\Sigma$ same way as in $A_\Sigma$ and $x_{0,i} := x_{\gamma_{0,i}}$, $x_{i,0} := x_{\gamma_{i,0}}$.

Let $\Delta$ be the triangulation of $\Sigma$ obtained from $\Delta$ by adding the intervals $\gamma_{0,1}$ and $\gamma_{1,0}$. Then (3.6) reads:

$$x_3 = x_{31} x_{13}^{-1} x_{12} + x_{31} x_{23}^{-1} x_{13} x_{12}$$

$$x_2 = x_{21} x_{12}^{-1} x_{13}$$

3.7. Noncommutative $(n,1)$-gon. In this section we consider the $(n,1)$-gon $\Sigma = P_n(1)$ (with the clockwise ordering of the set $[n] = I_0(P_n(1))$). We abbreviate $A_{n,1} := A_\Sigma$ and refer to it as the noncommutative $(n,1)$-gon. Clearly, $A_{n,1}$ is generated by $x_{ij}^+$ and $(x_{ij})^{-1}$, $i,j \in [n]$, where $x_{ij}^-$ is the curve corresponding to $(i,j,\pm)$ under the bijection in Lemma 3.10 where $x_{i1} = x_{23}^-$. The following is immediate.

**Lemma 3.33.** The algebra $A_{n,1}$ is generated by $(x_{ij}^+)^{\pm 1}$, $i,j \in [n]$ subject to:

(i) (triangle relations) For any distinct $i,j,k \in [n]$: $x_{ij}^+ x_{jk}^- x_{ki}^- = x_{jk}^- x_{ki}^- x_{ij}^+ = x_{ki}^- x_{ij}^+ x_{jk}^- x_{ki}^-$.

(ii) (2-gon exchange relations) For any distinct $i,j \in [n]$: $x_i = x_{ji}^- x_{ij}^+ + x_{ij}^- x_{i1}^+ x_{ij}^-$.

(iii) (4-gon exchange relations) For any cyclic $(i,j,k,\ell)$ in $[n]$ and $\varepsilon \in \{-,+,\}$:

$$x_{ji}^+ = x_{jk}^- x_{ki}^- x_{ij}^+ x_{ji}^- x_{ij}^+ x_{jk}^- x_{ki}^- x_{ij}^+$$

$$x_{ji}^- = x_{jk}^- x_{ki}^- x_{ij}^- x_{ji}^- x_{ij}^- x_{jk}^- x_{ki}^- x_{ij}^-$$

$$x_{ji}^- = x_{jk}^- x_{ki}^- x_{ij}^- x_{ji}^- x_{ij}^- x_{jk}^- x_{ki}^+ x_{ik}^+ x_{ij}^- x_{ji}^- x_{ij}^- x_{jk}^- x_{ki}^- x_{ij}^-$$
Clearly, the assignments $x^+_{ij} \mapsto x^+_{ji}$ define an involutive anti-automorphism of $\mathcal{A}_{n,1}$. One can easily show that all intersections $\gamma_{ij} \cap \gamma_{jk} \cap \gamma_{ki}$ are non-empty.

3.8. Universal localizations of noncommutative surfaces. Generalizing (2.4), for any triangulation $\Delta$ of any marked surface $\Sigma$ let $\mathcal{A}_\Sigma$ be the subalgebra of $\mathcal{A}_\Sigma$ generated by all $x_\gamma$, $\gamma \in [\Gamma(\Sigma)]$ and all $x^{-1}_\gamma$, $\gamma \in \Delta$.

Clearly, the assignments $t_\gamma \mapsto x_\gamma$, $\gamma \in \Delta$ define a homomorphisms of algebras:

$$1_\Delta : \text{Q}^{T_\Delta} \to \mathcal{A}_\Delta.$$  

The following result is a generalization of Theorem 2.8 to all marked surfaces.

Theorem 3.36. For each triangulation $\Delta$ of $\Sigma$ one has:

(a) The homomorphism $1_\Delta$ given by (3.7) is an isomorphism of algebras.

(b) $\mathcal{A}_\Sigma = \mathcal{A}_\Delta[S^{-1}]$, where $S$ is the submonoid of $\mathcal{A}_\Delta \setminus \{0\}$ generated by all $x_\gamma$, $\gamma \in [\Gamma(\Sigma)]$.

We prove Theorem 3.36 in Section 3.13.

Theorems 3.36, 3.38, and 3.39 imply the following.

Corollary 3.37. For each triangulation $\Delta$ of $\Sigma$ the homomorphism (3.7) is injective.

Theorem 3.36 implies that for each $\Sigma$ the natural homomorphism $\text{Q}^{T_\Delta} \hookrightarrow \text{Frac}(\text{Q}^{T_\Delta})$ defines a homomorphism of algebras:

$$\mathcal{A}_\Sigma \to \text{Frac}(\text{Q}^{T_\Delta}).$$

In view of Theorem 5.7, we propose the following conjecture.

Conjecture 3.38. For each $\Sigma$ the homomorphism (3.8) is injective, e.g., the submonoid $S_\Delta$ of $\text{Q}^{T_\Delta} \setminus \{0\}$ is divisible in the sense of Definition 5.4.

Remark 3.39. Conjecture 3.38 generalizes the expected injectivity of (2.4). To prove Conjecture 3.38 for non-closed surfaces (i.e., with free $T_\Delta$ according to Theorem 3.29) it would suffice to show that the monoid $S_\Delta$ is generated by $\mathbb{Q}^* \cdot T_\Delta$ and a subset of prime elements in $\text{Q}^{T_\Delta}$.

3.9. Noncommutative angles and regular elements in noncommutative surfaces. Similarly to Section 2.8 for each triangle $(\gamma_1, \gamma_2, \gamma_3)$ denote by $T_{\gamma_1, \gamma_2, \gamma_3}$ the element of $\mathcal{A}_\Sigma$ given by:

$$T_{\gamma_1, \gamma_2, \gamma_3} = x^{-1}_{\gamma_1} x_{\gamma_2} x^{-1}_{\gamma_3}$$

and refer to it as a noncommutative angle of $(\gamma_1, \gamma_2, \gamma_3)$ at $s(\gamma_1) = t(\gamma_3)$.

Given a triangulation $\Delta$ of $\Sigma$, for any $i \in I$ define the total angle $T_i^\Delta$ at $i \in I$ by:

$$T_i^\Delta := \sum T_{\gamma_1, \gamma_2, \gamma_3},$$

where the summation is over all clockwise triangles $(\gamma_1, \gamma_2, \gamma_3)$ in $\Delta$ such that $s(\gamma_1) = i$. 
Theorem 3.40. For any triangulations $\Delta$, $\Delta'$ of $\Sigma$ and $i \in I$ one has:
\[ T_i^\Delta = T_i^{\Delta'} . \]

Therefore, in what follows, we simply denote $T_i := T_i^\Delta$ for any triangulation $\Delta$ of $\Sigma$.

Furthermore, denote by $U_\Sigma$ the subalgebra of $A_\Sigma$ generated by all $x_\gamma, \gamma \in [\Gamma(\Sigma)]$, $x_0^{-1}, \gamma_0 \in \partial \Gamma(\Sigma)$ and all total angles $T_i$.

In particular, the algebra $U_n$ from \[2.3\] is naturally isomorphic to $U_{P_n}$. The following is an analogue of Lemma \[2.13\].

Lemma 3.41. The algebra $U_\Sigma$ satisfies the following relations:

(a) (reduced triangle relations) for all triangles $(\gamma_1, \gamma_2, \gamma_3)$ in $[\Gamma(\Sigma)]$ such that $\gamma_2$ is a boundary curve:
(3.11)
\[ x_{\gamma_1} x_{\gamma_2}^{-1} x_{\gamma_3} = x_{\gamma_3} x_{\gamma_2}^{-1} x_{\gamma_1} . \]

(b) (reduced exchange relations) for all quadrilaterals $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in $\Sigma$ such that $\gamma_2, \gamma_3$ are boundary curves:
(3.12)
\[ x_{\gamma_1} x_{\gamma_2} x_{\gamma_3} x_{\gamma_4} = x_{\gamma_1} + x_{\gamma_1} x_{\gamma_2}^{-1} x_{\gamma_3} x_{\gamma_4} . \]

Remark 3.42. It is natural to conjecture that the relations (3.11) and (3.12) are defining for $U_\Sigma$.

Noncommutative Laurent phenomenon \[3.30\] guarantees that $U_\Sigma$ belongs to each subalgebra $A_\Delta \subset A_\Sigma$.

The following is an analogue of Conjecture \[2.20\].

Conjecture 3.43. For each $n \geq 2$ one has:
(3.13)
\[ U_\Sigma = \bigcap_\Delta A_\Delta , \]

where the intersection is over all triangulations $\Delta$ of $\Sigma$.

We say that an element of $A_\Sigma$ is regular if it belongs to each subalgebra $A_\Delta$ as $\Delta$ runs over all triangulations of $\Sigma$. Thus, similarly to Section \[2.3\], Conjecture \[3.43\] asserts that regular elements of $A_\Sigma$ belong to $U_\Sigma$.

3.10. Noncommutative cohomology of surfaces. Given a surface $\Sigma$, for each triangle $(\gamma_1, \gamma_2, \gamma_3)$ in $\Sigma$ we define the element $\tau_{\gamma_1, \gamma_2, \gamma_3} \in A_\Sigma$ (in notation \[3.9\]) by:
\[ \tau_{\gamma_1, \gamma_2, \gamma_3} = T_{\gamma_1, \gamma_2, \gamma_3} + T_{\gamma_2, \gamma_3, \gamma_1} + T_{\gamma_3, \gamma_1, \gamma_2} . \]

That is, $\tau_{\gamma_1, \gamma_2, \gamma_3}$ is the sum of all noncommutative angles of the triangle $(\gamma_1, \gamma_2, \gamma_3)$.

Then define the algebra $H(\Sigma)$ to be the quotient of $A_\Sigma$ by the ideal generated by all $\tau_{(\gamma_1, \gamma_2, \gamma_3)} - \tau_{(\gamma_1', \gamma_2, \gamma_3')}$ as $(\gamma_1, \gamma_2, \gamma_3)$ and $(\gamma_1', \gamma_2', \gamma_3')$ run independently over all triangles of $\Sigma$. We refer to $H(\Sigma)$ as the noncommutative cohomology of $\Sigma$.

This notion is justified by the following construction.

Fix a triangulation $\Delta$ of $\Sigma$. For each loop $\theta$ in $\Sigma$ which does not pass through marked points, define the element $[\theta]_\Delta \in A_\Delta$ by:
\[ [\theta]_\Delta := \sum_\gamma \varepsilon_{\gamma_1, \gamma_2, \gamma_3}(\theta) \cdot T_{\gamma_1, \gamma_2, \gamma_3} , \]

the sum is over all clockwise triangles $(\gamma_1, \gamma_2, \gamma_3)$ in $\Delta$ such that $\theta$ intersects $\gamma_1$ and $\gamma_2$ (but not $\gamma_3$) and
\[ \varepsilon_{\gamma_1, \gamma_2, \gamma_3}(\theta) := \begin{cases} 1 & \text{if $\gamma_3$ is to the right of $\theta$} \\ -1 & \text{if $\gamma_3$ is to the left of $\theta$} \end{cases} \]

Note that if $\theta = \theta_i$ is a (small) clockwise loop around a puncture $i \in I$, then $[\theta]_\Delta = T_i^\Delta$, the total angle at $i$ (defined in \[3.10\]).

Furthermore, define $[\theta]_\Delta \in H(\Sigma)$ by
\[ [\theta]_\Delta := \pi(i_\Delta([\theta]_\Delta)) , \]

where $i_\Delta$ is the homomorphism $\mathbb{Q}\pi T_\Delta \to A_\Sigma$ given by \[3.7\] and $\pi : A_\Sigma \to H(\Sigma)$ is the canonical epimorphism.

The following immediate result is an analogue of Theorem \[3.40\].

Theorem 3.44. Given a loop on $\Sigma$ not passing through marked points, then for any triangulations $\Delta$ and $\Delta'$ of $\Sigma$ one has:
\[ [\theta]_{\Delta'} = [\theta]_{\Delta} . \]

This allows us to define a noncommutative loop $[\theta] \in H(\Sigma)$ by $[\theta] := [\theta]_{\Delta}$ for any triangulation $\Delta$ of $\Sigma$. 

3.11. Proof of Theorems 3.6, 3.9, 3.16, 3.24 and 3.30

Proof of Theorem 3.6 Clearly, the composition $f' \circ f : \Sigma \to \Sigma''$ is a continuous map with finite fibers. Also,

$$(f' \circ f)^{-1}(I(\Sigma')) = f^{-1}(f'^{-1}(I(\Sigma'))) = f^{-1}(f'(I(\Sigma'))) = f^{-1}(I(\Sigma')).$$

This verifies the first requirement of Definition 3.5 for $f' \circ f$.

Furthermore, prove that $I' \circ f = I \cup f^{-1}(I')$. Indeed,

$$I' \circ f = (f' \circ f)^{-1}(I(\Sigma')) \setminus I(\Sigma) = f^{-1}(f'^{-1}(I(\Sigma')) \setminus I(\Sigma) = I \cup f^{-1}(I')$$

since $f'^{-1}(I(\Sigma')) \setminus I(\Sigma) = I(\Sigma') \cup I', f^{-1}(I(\Sigma')) = f^{-1}(I(\Sigma')) \setminus I', f^{-1}(I') \setminus I(\Sigma) = \emptyset$, and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ for any disjoint subsets $A$ and $B$ of $\Sigma'$.

Let now $p \in \Sigma \setminus I'$. By above, this is equivalent to that $p \in \Sigma \setminus I'$. Hence there is a neighborhood $O_p$ of $p$ in $\Sigma \setminus (\cup_i O_i)$ such that the restriction of $f$ to $O_p$ is injective and a (half-)neighborhood $I(f(p))$ of $f(p)$ in $\Sigma'$ such that the restriction of $f'$ to $O_p$ is injective. In particular, $O_p := f^{-1}(O_{f(p)})$ is a neighborhood of $p$ in $\Sigma$ and the restriction of $f' \circ f$ to $O_p$ is injective. This verifies the second requirement of Definition 3.5 for $f' \circ f$.

Let now $p \in I' \setminus I$. By above, this is equivalent to that either $p \in I'$ or $f(p) \in I'$. In the first case, clearly, $f(p) \in \Sigma' \setminus I'$, therefore there is a neighborhood $O_{f(p)}$ of $f(p)$ in $\Sigma'$ such that the restriction of $f'$ to $O_{f(p)}$ is injective and $O_{f(p)} \cap O_p \setminus I'$ is a two-fold cover of the neighborhood $O_p = f(U_p)$ ramified at $f(p)$. Therefore, the restriction of $f'$ to the neighborhood $U'_p = f^{-1}(O_p \cap O_p')$ is a two-fold cover of $O_p \cap O_p'$ ramified at $f(p)$ and the restriction of $f' \circ f$ to $O_p \cap O_p'$ is injective. Thus, the restriction of $f' \circ f$ to $U'_p$ is a two-fold cover of $f(O_p \cap O_p')$ ramified at $(f' \circ f)(p)$.

In the second case, clearly, $p \in \Sigma \setminus I'$, therefore there is a neighborhood $O_p$ of $p$ in $\Sigma$ such that the restriction of $f$ to $O_p$ is injective and a neighborhood $U_p$ of $p$ in $\Sigma$ such that the restriction of $f$ to $U_p$ is a two-fold cover of the neighborhood $O_p = f(U_p)$ ramified at $f(p)$. Therefore, the restriction of $f'$ to the neighborhood $U'_p = f^{-1}(O_p \cap O'_p)$ is a two-fold cover of $O_p \cap O'_p$ ramified at $f(p)$ and the restriction of $f' \circ f$ to $O_p \cap O'_p$ is injective. Thus, the restriction of $f' \circ f$ to $O_p$ is a two-fold cover of $f(O_p \cap O'_p)$ ramified at $(f' \circ f)(p)$.

This verifies the last requirement of Definition 3.5 for $f' \circ f$.

The theorem is proved.

Proof of Theorem 3.9 Without loss of generality, it suffices to prove the first assertion in the case when $C \subset C'$ and $C' \setminus C$ is a simple loop around $i \in I'$ not enclosing any points $I(\Sigma) \cup I(\Sigma) \cup I' \setminus \{i\}$ (where we regard $C$ and $C'$ as subsets of $\Sigma$). Moreover, it suffices to take $C = \{p\}$ for some $p \in \Sigma$, $p \neq i$, so that $C'$ is a simple loop at $p$ around $i$ (e.g., $C'$ is contractible to $p$ in $\Sigma \setminus I(\Sigma)$).

By definition, there is a neighborhood $U_p$ of $p$ such that the restriction of $f$ to $U_p$ is a two-fold cover of $f(U_p)$ ramified at $f(p)$. Once again, without loss of generality, we may assume that $C'$ intersects $U_p$ and there exist exactly two distinct points $p', p'' \in C$ such that $f(p') = f(p'')$. This implies that $f(C') \subset \Sigma'$ is a (self-intersecting) loop at $f(p)$ with a single self-intersection point $f(p') = f(p'')$. If we denote by $\gamma'$ the equivalence class of $f(C')$ in $\Sigma' \setminus (I(\Sigma) \cup I(\Sigma)) \cup \{f(p)\}$, then, clearly, $[\gamma']$ is trivial.

This proves (a).

Parts (b), (c) and (d) follow.

The theorem is proved.

Proof of Theorem 3.16 We need the following fact.

Lemma 3.45. In the notation of Theorem 3.3, for any polygon $P = (\gamma_1, \ldots, \gamma_n)$ in $\Sigma$ the tuple $f(P) = (f(\gamma_1), \ldots, f(\gamma_n))$ is a polygon in $\Sigma'$.

Proof. Indeed, let $P = (\gamma_1, \ldots, \gamma_n)$ be a polygon in $\Sigma$ and let $g : P_n \to \Sigma$ be an accompanying morphism. Then $g' = f \circ g$ is a morphism $P_n \to \Sigma'$ in $\text{Surf}$ such that $g'(i, i^+) = f(\gamma_i)$ for $i \in [n]$, i.e., $f(P)$ is an $n$-gon in $\Sigma'$.
The lemma is proved.

Thus the triangle relations in $T_{\Sigma}$ are carried by $f_*$ to those in $T_{\Sigma'}$. This proves the assertion for groups. Likewise, the triangle and exchange relations in $A_{\Sigma}$ are carried by $f_*$ to those in $A_{\Sigma'}$. This proves the assertion for algebras. The commutativity of the diagram (3.4) follows.

The theorem is proved.

**Proof of Theorem 3.24** It suffices to prove the assertion only for neighboring triangulations $\Delta$ and $\Delta'$, i.e., for a quadrilateral $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in $\Delta$ such that $\Delta \setminus \Delta' = \{\gamma_{13}, \gamma_{31}\}$ and $\Delta' \setminus \Delta = \{\gamma_{24}, \gamma_{42}\}$.

The following result is obvious.

**Lemma 3.46.** In the notation as above, the assignment $t_\gamma \mapsto \begin{cases} t_{\gamma_1} t_{\gamma_2}^{-1} t_{\gamma_3} & \text{if } \gamma = \gamma_{13} \\ t_{\gamma_3} t_{\gamma_1}^{-1} t_{\gamma_2} & \text{if } \gamma = \gamma_{31} \\ t_\gamma & \text{otherwise} \end{cases}$ defines an isomorphism $\varphi_{\Delta, \Delta'} : T_{\Delta} \overset{\sim}{\rightarrow} T_{\Delta'}$.

The second assertion follows immediately because one has for $\gamma, \gamma' \in \Delta$, $\gamma' \notin \{\gamma_{13}, \gamma_{31}\}$:

$$f_{\Delta, \Delta'}(y_{\gamma, \gamma'}) = \begin{cases} t_{\gamma_1} t_{\gamma_2}^{-1} t_{\gamma_3} t_{\gamma_4} & \text{if } \gamma = \gamma_{13} \\ t_{\gamma_3} t_{\gamma_1}^{-1} t_{\gamma_2} t_{\gamma_4} & \text{if } \gamma = \gamma_{31} \\ t_\gamma & \text{otherwise} \end{cases}$$

This proves the theorem.

**Proof of Theorem 3.30** Indeed, let $f : P_n \to \Sigma$ be an accompanying map for the canonical polygon $P_{\Delta}(\gamma) = (\gamma_1, \ldots, \gamma_n)$. Then, by Theorem 3.16, the assignments $x_{ij} \mapsto x_{ij}^\gamma$ define an algebra homomorphism $f_* : A_n \to A_{\Sigma}$, where $A_n = A_{P_n}$ is the noncommutative $n$-gon as in Section 2.2. Applying $f_*$ to (2.5) with $i = 1$ yields (3.6).

The theorem is proved.

### 3.12. Noncommutative triangle groups and proof of Theorem 3.26

We need the following immediate result.

**Lemma 3.47.** For any marked surface $\Sigma$ there is $n \geq 1$, a subset $S \subset [n]$, an injective map $\sigma : S \to [n] \setminus S$, and a function $\varepsilon : S \to \{-, +\}$ such that $\Sigma$ is obtained from $P_n(h)$, $h = |I_s(\Sigma)|$ by gluing the chord $(i, i^\dagger)$ to the chord $\{\sigma(i)^+, \sigma(i)\}$ if $\varepsilon(i) = +$ and $\{\sigma(i), \sigma(i)^-\}$ if $\varepsilon(i) = -$ for all $i \in S$.

**Remark 3.48.** Clearly, for any $n \geq 2$ and any pair $(\sigma, \varepsilon)$ as in Lemma 3.47, there is a marked surface $\Sigma_{\sigma, \varepsilon}$ obtained from $P_n(h)$ by such a gluing procedure.

The following is an obvious version of Theorem 3.16.

**Lemma 3.49.** Let $f : \Sigma \to \Sigma'$ be as in Theorem 3.16 and let $\Delta$ and $\Delta'$ be triangulations of $\Sigma$ and $\Sigma'$ respectively such that $f(\Delta) \subset \Delta'$. Then the assignments $t_\gamma \mapsto t_{f(\gamma)}$ for $\gamma \in \Delta$ define homomorphism of groups $f_* : T_{\Delta} \to T_{\Delta'}$.

Combining Lemmas 3.47 and 3.49 and taking into account that under the gluing map $f : P_n(h) \to \Sigma$, the image $f(\Delta)$ of any triangulation $\Delta$ of $P_n(h)$ is a triangulation of $\Sigma = \Sigma_{\sigma, \varepsilon}$, we see that the quotient group of $T_{\Delta}$ by the relations

$$t_{i, i^-} = \begin{cases} t_{\sigma(i)^+, \sigma(i)} & \text{if } \varepsilon(i) = + \\ t_{\sigma(i), \sigma(i)^-} & \text{if } \varepsilon(i) = - \\ t_{\sigma(i)^-, \sigma(i)} & \text{if } \varepsilon(i) = + \\ t_{\sigma(i), \sigma(i)^+} & \text{if } \varepsilon(i) = - \end{cases}$$

$i \in S$, is naturally isomorphic to $T_{f(\Delta)}$ (of course, $T_{f(\Delta)} \cong T_{\Delta''}$ for any triangulation $\Delta''$ of $\Sigma$ by Theorem 3.24).

We will use this observation with the appropriately modified starlike triangulation $\Delta = \tilde{\Delta}_1$ of $P_n(h)$, where $\tilde{\Delta}_1$ is the starlike triangulation of $[n]$ as in (2.6) with $i = 1$. 
Namely, for all \( n \geq 2 \), \( \tilde{\Delta}_1 \) is obtained from \( \Delta_1 \) by adding \( h \) curves \( \gamma_{12}^{(s)} \), \( s \in [h] \) from the vertex 1 to the vertex 2 outside of \( \Delta_1 \) so that each 2-gon \( (\gamma_{12}^{(s)})^{-1} \gamma_{12}^{(s-1)} \), \( s \in [h] \) contains exactly one special puncture (here, with a slight abuse of notation, \( \gamma_{12}^{(0)} \) is the chord \((1,2)\) in \([n]\)) and a clockwise loop \( \gamma_1^{(s)} \) around each special puncture inside \((\gamma_{12}^{(s)})^{-1} \gamma_{12}^{(s-1)} \), \( s \in [h] \).

**Lemma 3.50.** Suppose that \( n \geq 2 \). Then, using same arguments as in the proof of Lemma 2.80 we see that the group \( \mathbb{T}_{\Delta_1} \) is generated by \( t_j = T_j^{1,3+} \), \( j = 3, \ldots, n-1 \), \( c_k = t_{k,k+} \), \( \tau_k = t_{k+,k} \), \( k \in [n] \), \( y_s = t_{\gamma_1^{(s)}} \), \( z_s = t_{\gamma_1^{(s)}} \), \( s \in [h] \), and \( \overline{y}_h = t_{\gamma_1^{(h-1)}} \), subject to (if \( n \geq 4 \)):

\[
\begin{align*}
(3.15) & \quad c_2 t_3 c_3 \cdots t_{n-1} c_{n-1} t_{n-1}^{-1} c_1 = \tau_1 c_n^{-1} \tau_{n-1}^{-1} t_{n-1} \tau_{n-2}^{-1} \cdots t_3 \tau_2 \\
(3.16) & \quad \overline{y}_h = \tau_1 (z_1^{-1} y_1^{-1} z_1) (z_2^{-1} y_2^{-1} z_2) \cdots (z_h^{-1} y_h^{-1} z_h) \; .
\end{align*}
\]

**Proof.** It is easy to see that \( t_{1,j} = c_1 t_2 c_2 \cdots t_{j-1} c_{j-1} \), \( t_{j,1} = \tau_1 t_{j-1} \cdots \tau_2 t_1 \) for \( j = 1, \ldots, n \). Thus, \( \mathbb{T}_{\Delta_1} \) is generated by \( t_2, \ldots, t_{n-1}, c_k, \tau_k, k = 1, \ldots, n \) subject to the relations:

\[
\mathbb{T}_n \equiv c_1 t_2 c_2 \cdots c_{n-1} c_{n-1} t_{n-1} \tau_n^{-1} \tau_{n-1}^{-1} \tau_{n-2}^{-1} \cdots \tau_2 t_1 \; .
\]

By eliminating \( t_2 \), we see that \( \mathbb{T}_{\Delta_1} \) is subject to the relation \( (3.15) \). Furthermore, the 1-gons relations in the 1-gons \( \gamma_1^{(s)} \) and triangle relations in the triangles \((\gamma_1^{(s)})^{-1} \gamma_1^{(s)} \gamma_1^{(s-1)} \) for the remaining generators \( y_s = t_{\gamma_1^{(s)}} \), \( \overline{y}_s = t_{\gamma_1^{(s-1)}} \), \( z_s = t_{\gamma_1^{(s)}} \), \( \mathbb{T}_s = t_{\gamma_1^{(s-1)}} \) for \( s \in \{0\} \cup [h] \) of \( \mathbb{T}_{\Delta_1} \) read:

\[
\begin{align*}
\overline{y}_s = z_s, \quad & \overline{y}_s = z_s^{-1} y_s - \overline{y}_s z_s^{-1} y_s, \\
\overline{y}_s = & \tau_1 (z_1^{-1} y_1^{-1} z_1) (z_2^{-1} y_2^{-1} z_2) \cdots (z_s^{-1} y_s^{-1} z_s) \; ,
\end{align*}
\]

so that the remaining generators \( z_s \) and \( y_s \), \( s \in [h] \) are free.

The lemma is proved.

Combining Lemmas 3.47 and 3.50, we see that for \( n \geq 3 \) the group \( \mathbb{T}_f(\Delta_1) \) is generated by \( t_j, j = 3, \ldots, n-1 \), \( c_k, \tau_k, k = 1, \ldots, n \), \( y_s, z_s, s \in [h] \), and \( \overline{y}_h \) subject to \( (3.15) \) and the following relations for all \( i \in S \):

\[
\begin{align*}
\sigma(i) = & \begin{cases} 
\tilde{c}_i^\prime & \text{if } \varepsilon(i) = + \\
c_i^\prime & \text{if } \varepsilon(i) = -
\end{cases}, \\
\tilde{\sigma}(i) = & \begin{cases} 
\tilde{c}_i^\prime & \text{if } \varepsilon(i) = + \\
c_i^\prime & \text{if } \varepsilon(i) = -
\end{cases}, \quad \text{where } c_i^\prime := \begin{cases} 
c_i & \text{if } i \neq \pm 1 \\
y_i & \text{if } i = 1
\end{cases}, \quad \overline{\tilde{c}}_i^\prime := \begin{cases} \overline{y}_i & \text{if } i \neq \pm 1 \\
\overline{y}_i & \text{if } i = 1
\end{cases}.
\end{align*}
\]

Thus, if \( n \geq 3 \), then the group \( \mathbb{T}_{\Delta_1} \) has \( (n-3) + 2 |S| + 2h = 3n - 3 - 2 |S| + 2h \) generators \( t_j, j = 3, \ldots, n-1 \), \( c_k, \tau_k, k \in [n] \setminus \sigma(S) \), \( y_s, z_s, s \in [h] \) and exactly one relation \( (3.15) \). Now compute the Euler characteristic of \( \Sigma \) using the triangulation \( \Delta'' \) of \( 
\Delta'' \) obtained by removing all \( h \) loops around special punctures from \( f(\Sigma) \). By definition,

\[
\chi(\Sigma) = |I| - E + T, \quad \text{where } E \text{ is the number of edges and } T \text{ is the number of triangles in } \Delta''.
\]

Clearly, \( T = n - 2 \) and \( E = (n-3) + (n-|S|) \), therefore,

\[
\chi(\Sigma) = |I| - (n-3) + (n-|S|) + n - 2 = |I| + 1 - n + |S| = |I| + 1 - \frac{n}{2} - \frac{|I_0|}{2}
\]

because \( n-2 |S| = |I_0| \). Therefore, the number of generators of \( \mathbb{T}_f(\Delta) \) is:

\[
3n - 3 - 2 |S| + 2h = 2n - 3 + |I_0| + 2h = 4(|I| + 1 - \chi(\Sigma) - \frac{|I_0|}{2}) - 3 + |I_0| + 2h = 4(|I| - \chi(\Sigma)) + 1 - |I_0| + 2h.
\]

We now consider several cases.

**Case 1.** \( n \geq 3 \) and either \( \Sigma \) has boundary, i.e., \( S \cup \sigma(S) \neq [n] \) or \( h > 0 \). The above implies that \( \mathbb{T}_f(\Delta) \) is free in \( 4(|I| - \chi(\Sigma)) - |I_0| + 2h \) generators.

**Case 2.** \( n = 2 \) (hence \( h > 0 \)). Then, clearly, \( \mathbb{T}_{\Delta_1} \) is a free group in \( 2h + 2 \) generators. Therefore:

- If \( n = 2, h \geq 2 \), then \( \mathbb{T}_f(\Delta) \) is free in \( 2h + 2 - 2 |S| \) generators, where \( |S| \in \{0,1\} \).
- If \( n = 2, h = 1 \), then \( \mathbb{T}_f(\Delta) \) is free in \( 4 - |S| \) generators, where \( |S| \in \{0,1\} \).
Case 3. $n = 1$, then $f$ is the identity map and $\Sigma$ is a disk with $|I| = |I_0| = 1$ and $h$ special punctures. If $h = 0$, then, clearly, $\mathcal{T}_\Delta$ is free in two generators $t_\gamma$ and $t_\varphi$, where $\gamma$ is the clockwise loop. Suppose that $h > 0$. Then one can choose a triangulation $\Delta$ of $\Sigma$ in such a way that, in addition to $\gamma$ it consists of a special loop $\lambda_s$, $s \in [h]$ around each special puncture and a clockwise loop $\gamma_s$ enclosing first $s$ special punctures (from the left to the right), $s = 2, \ldots, h$ (so that $\lambda_1 = \gamma_1$ and $\gamma_h = \gamma$). Then $\mathcal{T}_\Delta$ is generated by $z_s = t_{\lambda_s}$, $y_s = t_{\gamma_s}$, $\overline{y}_s = t_{\varphi_s}$, $s \in [h]$ subject to the following triangle relations in the $h - 1$ triangles $(\gamma_{s-1}, \lambda_s, \gamma_s)$, $s = 2, \ldots, h$:

$$y_{s-1}z_s^{-1} \overline{y}_s = y_s z_s^{-1} \overline{y}_{s-1}$$

for $s = 2, \ldots, h$ if $h > 2$. That is, similarly to the equations (3.16), one can solve recursively for all $\overline{y}_s$, $s = 2, \ldots, h$:

$$\overline{y}_s = (z_s y_{s-1}^{-1} y_s) \cdots (z_3 y_2^{-1} z_2 y_2) \cdot (z_2 z_1^{-1} z_2 z_1)$$

(since $y_1 = \overline{y}_1 = 1$) so that $\mathcal{T}_\Delta$ is freely generated by $z_s$, $s \in [h]$ and $y_s$, $s = 2, \ldots, h$.

This finishes the proof of Theorem 3.26(a).

Case 4. $n \geq 3$ and $\Sigma$ has no boundary, i.e., $|I_0| = 0$ hence $S \sqcup \iota(S) = [n]$ and $h = 0$. Then $n = 2|S|$ is even and $\mathcal{T}_{f(\Delta)}$ is a 1-relator torsion-free group in $4(|I| - \chi(\Sigma)) + 1$ generators $t_k$, $k = 3, \ldots, n - 1$, $c_k, \iota(c_k)$, $k \in S$. Suppose that $\Sigma$ is a sphere with $|I| \leq 3$ punctures. Then $\mathcal{T}_{f(\Delta)}$ is trivial for $|I| = 1$ because all loops are contractible, is free in 2 generators $t_\gamma$ and $t_{\gamma_1}$ if $|I| = 2$, where $\gamma$ is an arc between two punctures, and if $|I| = 3$, it is free in 5 generators, because we can take $S = \{1, 3\} \subset [4]$, $\iota(1) = 2$, $\iota(3) = 4$, $\iota(1) = \iota(3) = +$ so that $\mathcal{T}_{f(\Delta)}$ is freely generated by $c_1, \overline{c}_1, c_3, \overline{c}_3, t_1$. Otherwise, it is, clearly, non-free. This finishes the proof of Theorem 3.26(b).

The theorem is proved. □

3.13. Noncommutative curves and proof of Theorem 3.36 For each $\gamma \in [\iota(\Sigma)]$, a triangulation $\Delta$ of $\Sigma$ define the elements $t_{\gamma, \Delta} \in QT_\Delta$ same way as in Theorem 2.10

$$t_{\gamma, \Delta} := \sum_{i \in Adm_{\Delta} \mathcal{A}(1, j)} t_i,$$

(3.17)

where $\Delta^0$ is the triangulation of $[n]$ assigned (as in Theorem 3.21(a)) to $\Delta$ and the canonical polygon $P_\Delta(\gamma) = (\gamma_1, \ldots, \gamma_n)$ with $\gamma = \gamma_{1, j}$ and we abbreviated

$$t_i := t_{\gamma_1, i_2} t_{\gamma_2, i_3}^{-1} t_{\gamma_3, i_4} \cdots t_{\gamma_{m-1}, i_{2m-2}}^{-1} t_{\gamma_{2m-1}, i_{2m-1}} t_{\gamma_{2m}, i_{2m}},$$

for any sequence $i = (i_1, \ldots, i_{2m}) \in [n]^{2m}$, $m \geq 1$.

We refer each $t_{\gamma_i}$ as it as a noncommutative triangular curve.

Clearly, if $\Sigma = P_\gamma(0)$ is an $n$-gon (i.e., a disk with $I(\Sigma) = I_0(\Sigma) = [n]$) so that $\gamma = (p, q) \in [n] \times [n]$, then $t_{\gamma} = t_{\gamma, \Delta}$ as in (2.3). To finish the proof of Theorem 3.36 we need the following result.

Proposition 3.51. The assignments $x_\gamma \mapsto t_{\gamma, \Delta}$ for $\gamma \in [\iota(\Sigma)]$ define an epimorphism of algebras

$$\mathcal{A}_\Sigma \rightarrow QT_\Delta[S_\Delta^{-1}],$$

where $S_\Delta$ is the sub-monoid of $QT_\Delta$ generated by all $t_{\gamma, \Delta}$.

Proof. It suffices to show that the elements $t_{\gamma, \Delta}$ satisfy the defining relations of $\mathcal{A}_\Sigma$ from Definition 3.12.

We need the following result.

Lemma 3.52. Let $Q = (\gamma_1, \ldots, \gamma_n)$, $\Delta^0$, and $\iota : [r] \rightarrow [n]$ be as in Theorem 3.21. Then, in view of Theorem, for any accompanying morphism $f : P_n \rightarrow \Sigma$ Therefore, the assignments $(i, j) \mapsto f(i, j) = \gamma_{ij}$ restricted to $\Delta^0$ define a homomorphism of algebras $f_\Delta : QT_{\Delta^0} \rightarrow QT_\Sigma$ such that $f_\Delta(t_{\gamma_{ij}}) = t_{\gamma_{ij}}$ for $i, j \in [n]$. Since $f_\Delta(S_{\Delta^0}) \subset S_\Delta$, then passing to the universal localizations, this gives an algebra homomorphism

$$QT_{\Delta^0}[S_\Delta^{-1}] \rightarrow QT_\Delta[S_\Delta^{-1}].$$
Composing it with the isomorphism $A_n \cong QT_{\Delta}[S^{-1}_{\Delta}]$ given by Theorem 2.38 (b) and the homomorphism $A_r \to A_n$ given by $x_{k,\ell} \mapsto x_{i(k),i(\ell)}$ give the desired homomorphism (3.19). The lemma is proved. □

Using the Lemma with $r = 3, 4$, we finish the proof of the proposition. □

Since each $x_\gamma, \gamma \in [\Gamma(\Sigma)]$ is invertible in $A_\Sigma$, the universality of localization $QT_{\Delta}[S^{-1}_{\Delta}]$ implies that (3.7) extends to a homomorphism of algebras

$$QT_{\Delta}[S^{-1}_{\Delta}] \to A_\Sigma.$$ 

By the construction and Theorem 3.30 (3.7) takes each $t_\gamma^\Delta$ to $x_\Delta$ and therefore is an epimorphism $QT_{\Delta} \to A_\Delta$. In turn, (3.20) is an epimorphism as well.

Thus, we obtained two mutually inverse epimorphisms (3.19) and (3.20), which implies that they are isomorphisms of algebras. Therefore, (3.18) is an isomorphism, which proves Theorem 3.36 (b). Theorem 3.36 (a) also follows because (3.7) is a restriction to $QT_{\Delta}$ of the isomorphism (3.20) and $1_{\Delta}(QT_{\Delta}) = A_\Delta$.

Theorem 3.36 is proved. □

4. Noncommutative discrete integrable systems

4.1. An integrable system on a cylinder. Denote by $\Sigma_{1,r}$, an annulus (i.e., a cylinder) with no punctures, one marked point $p$ on the outer circle and $r$ marked points $p_1, \ldots, p_r$ on the inner circle (listed clockwise).

It is easy to see that equivalence classes of curves from $p$ to $\{p_1, \ldots, p_r\}$ in $\Sigma_{1,r}$ are in a natural bijection with $\mathbb{Z}$: the $n$-th curve $\gamma_n$ goes (without self-intersections) from $p$ to $p_s$ where $s \equiv n \mod r$ and $\gamma_n$ has the winding number $q$ such that $n = rq + s$ (so that the arc is winding clockwise if $q \geq 0$ and counterclockwise if $q < 0$).

We also denote $\gamma_- \gamma_\gamma$ (resp. $\gamma_\gamma^+$) the short counterclockwise boundary arc in the inner circle from $p_i$ to the previous point $p_{i-}$ (resp. from $p_{i-}$ to $p_i$), $i \in [r]$; and by $\gamma^+$ (resp. $\gamma^+$) the clockwise (resp. counterclockwise) loop in the outer circle.

We abbreviate in the algebra $A_{\Sigma_{1,r}}$:

$$x_n := x_{\gamma_n}, \quad x_\gamma := x_{\gamma_\gamma}, \quad c_n := x_{\gamma_-_\gamma}, \quad d := x_{\gamma_-}, \quad \overline{d} := x_{\gamma_+}$$

for $n \in \mathbb{Z}$ (where we extend $\gamma_n$ periodically so that $\gamma_{n+r} = \gamma_n$ for all $n \in \mathbb{Z}$).

Since $(\gamma_n^+, \gamma_n^-, \gamma_n)$ is a 4-gon in $\Gamma(\Sigma_{1,r}) = [\Gamma(\Sigma_{1,r})]$ containing triangles $(\gamma_{n-1}, \gamma_{n-1}^-, \gamma_n^+_n)$ and $(\gamma_n, \gamma_n^+, \gamma_{n-r})$ for all $n \in \mathbb{Z}$, the following fact is immediate from Definition 3.12

**Lemma 4.1.** For each $r \geq 1$ one has in $A_{\Sigma_{1,r}}$:

(i) (triangle relations)

$$x_{n-1}c_n^{-1}x_n = x_n c_n^{-1} x_{n-1}, \quad x_n d_n^{-1} x_{n-r} = x_{n-r} d_n^{-1} x_n.$$ \hspace{1cm} (4.1)

(ii) (exchange relations) For each $n \in \mathbb{Z}$:

$$x_{n-r-1}d_n^{-1} x_n = c_n + x_{n-1} d_n^{-1} x_{n-r}, \quad x_n d_n^{-1} x_{n-r-1} = c_n + x_{n-r} d_n^{-1} x_{n-r-1}.$$ \hspace{1cm} (4.2)

Note that for each $n \in \mathbb{Z}$ the annulus $\Sigma_{1,r}$ has a triangulation

$$\Delta_n := \{ \gamma_+, \gamma_+^+; \gamma_1^{-1}, \gamma_1^-; \gamma_r^{-1}, \gamma_r^-; \gamma_m, \gamma_m; \gamma_{m+r}, \gamma_{m+r}^{-1} \}.$$  

![Triangulation Δ₁ of the cylinder Σ₁,₂](image)
Hence the group $T_r$ generated by $x_n, \overline{c}_n, n = 1, \ldots, r + 1, c_i, \overline{r}_i, i = 1, \ldots, r, d, \overline{d}$ subject to the triangle relations
\begin{equation}
\overline{x}_{r+1} = \overline{x}_1 d^{-1} x_{r+1}, \quad x_{s-1} c_s x_s = x_s c_s^{-1} x_{s-1},
\end{equation}
$s = 2, \ldots, r + 1$ (with the convention $p_{r+1} = p_r$ hence $c_{r+1} = c_r, \overline{c}_{r+1} = \overline{c}_r$ for $n \in \mathbb{Z}$) is naturally isomorphic to the triangle group $T_{\Delta_i}$. Moreover, in the notation of Section 3.3, the subalgebra $A_{\Delta_i}$ of $A_{\Sigma}$ (generated by all $x_\gamma, \gamma \in \Gamma(\Sigma_1, r)$ and all $x_{\gamma_0}, \gamma_0 \in \Delta_1$) is the group algebra $ZT_r$ by Theorem 3.30 (a).

**Proposition 4.2.** (For each $r \geq 1$ we have:
(a) Each $x_n, \overline{x}_n, n \in \mathbb{Z}$ is sum of elements of $T_r$ in $ZT_r$.
(b) The total angle $T_p \in ZT_r$ at $p$ is given by $T_p = d^{-1} x_{n-r} x_{n-1} + d^{-1} x_{n+r} x_{n+1} = \overline{x}_n^{-1} x_{n-r} d^{-1} + \overline{x}_n x_{n+r} d^{-1}$ for each $n \in \mathbb{Z}$.

**Proof.** Part (a) follows directly from Theorem 3.30 and Corollary 3.37. Part (b) follows from Proposition 4.2. Consider a triangle $\Delta_{n-r}$ and $\Delta_{n+r}$ in $\Delta_n$.

The following is an immediate corollary of Theorem 3.40.

**Lemma 4.3.** Using this and taking into account that
\begin{equation*}
T_{\gamma^+, \gamma^-} = \overline{D}^{-1} x_{n-r} x_{n-1} = \overline{x}_n^{-1} x_{n-r} d^{-1}, \quad T_{\gamma^+, \gamma^-} = \overline{x}_n^{-1} x_{n+r} d^{-1}
\end{equation*}
in the notation (3.9), we obtain (b).

The proposition is proved.

**Remark 4.4.** Using the triangulation $\Delta_n$, it is easy to see that
\begin{equation}
T_p = d^{-1} x_n x_{n-1} + d^{-1} x_{n-r} x_{n+1} + \sum_{m=n+1-r}^{n} \overline{x}_m^{-1} c_m x_m
\end{equation}
for all $n \in \mathbb{Z}$.

Clearly, by Theorem 3.40 $T_p$ does not depend on $n$.

If $r$ is even, we can refine these observations and thus recover the recursion (1.4).

Indeed, set $U_n := \begin{cases} x_n & \text{if } n \text{ is even} \\ \overline{x}_n & \text{if } n \text{ is odd} \end{cases}$, $C_n := \begin{cases} c_n & \text{if } n \text{ is even} \\ \overline{c}_n & \text{if } n \text{ is odd} \end{cases}$, and $D := d^{-1}, \overline{D} := \overline{d}^{-1}$.

By definition, $T_r$ is freely generated by $D, \overline{D}$ and $C_i, i \in [r], U_j, j \in [r + 1]$ and, by Proposition 1.2, $U_n \in QT_r$ is a sum of elements of $T_r$. This and Proposition 4.2 imply the following result.

**Theorem 4.5.** Let $r \geq 1$ be even. Then each element $U_n \in ZT_r, n \in \mathbb{Z}$ satisfies the recursion:
\begin{equation}
\begin{cases}
U_{n-r-1} DU_n = C_n + U_{n-1} DU_{n-r} & \text{if } n \text{ is even} \\
U_n DU_{n-r-1} = C_n + U_{n-r} DU_{n-1} & \text{if } n \text{ is odd}
\end{cases}
\end{equation}
(with the convention $C_{n+r} = C_r$). Furthermore, the element $H_n \in \text{Frac}(ZT_r), n \in \mathbb{Z}$, given by
\begin{equation}
H_n := \begin{cases}
DU_{n-r-1} + DU_{n+r} U_{n-1} & \text{if } n \text{ is even} \\
U_{n-1} U_{n-r} D + U_{n} U_{n+r} \overline{D} & \text{if } n \text{ is odd}
\end{cases}
\end{equation}
does not depend on $n$ and belongs to $ZT_r$.

The recursion (4.4) clearly coincides with the recursion (1.4) with $k = r + 1$ and the element $H_n$ given by (4.5) coincides with the element given by (1.5).

**Remark 4.6.** In fact, Remark 4.4 implies that the “conserved quantity” $H = H_n$ is equal (for any $n \in \mathbb{Z}$) to
\begin{equation*}
\begin{cases}
DU_{n-r} U_{n-r}^{-1} + DU_{n-r} U_{n-r}^{-1} + \sum_{m=(n+2-r)/2}^{n/2} U_{2m-1}^{-1} C_{2m} U_{2m}^{-1} + U_{2m-1}^{-1} C_{2m-1} U_{2m-2}^{-1} & \text{if } n \text{ is even} \\
U_{n-1} U_{n-r} D + U_{n-1} U_{n-r} D + \sum_{m=(n+1-r)/2}^{(n-1)/2} U_{2m-1}^{-1} C_{2m} U_{2m}^{-1} + U_{2m+1}^{-1} C_{2m+1} U_{2m+1}^{-1} & \text{if } n \text{ is odd}
\end{cases}
\end{equation*}
4.2. An integrable system on an infinite strip. In this section we establish Laurentness of another noncommutative recursion (which specializes to the discrete integrable system recently studied by P. Di Francesco in [18]). Indeed, let $\Sigma_\infty$ be a horizontal strip with marked boundary points $I = I_- \sqcup I_+$, where $I_+ = \{i, i \in \mathbb{Z}\}$ (resp. $I_- = \{-i, i \in \mathbb{Z}\}$) is the marked point set on the left (resp. on the right) boundary line. Then, clearly,

$$\Gamma(\Sigma_\infty) = \{i, j, j'\} : i, j \in \mathbb{Z}, \varepsilon, \varepsilon' \in \{-, +\}, i \neq j \text{ if } \varepsilon = \varepsilon'\}.$$

Clearly,

$$\Sigma_\infty = \bigcup_{m, + \in \mathbb{Z}, n \in \mathbb{Z}_{>0}} \Sigma_{m, -m}^n,$$

where $\Sigma_{m, -m}^n \subset \Sigma$ is the convex hull of the real intervals $[m^- + 1, m^- + n] \subset I_-, [m^+ + 1, m^+ + n] \subset I_+$. Clearly, it is an $2n$-gon embedded (as a parallelogram) into $\Sigma$, where we identify its vertex set $[2n]$ with $\{(m^- + 1), \ldots, (m^- + n)\} \cup \{(m^+ + 1), \ldots, (m^+ + n)\}$ via

$$k \mapsto \left\{\begin{array}{ll}
(m^- + k) - & \text{if } k \leq n; \\
(m^+ + 2n + 1 - k) + & \text{if } k > n.
\end{array}\right.$$

We denote by $A_{\Sigma_{m, -m}}$ a copy of $A_{2n}$ under the above identification of the vertex set $[2n]$.

Then the natural inclusions $\Sigma_{m, -m}^n \subset \Sigma_{m', -m'}^n$ for $m'^- \leq m^-, m'^+ \leq m^+, m'^- + n' \geq m^- + n, m'^+ + n' \geq m^+ + n$ are morphisms in $\mathbf{Surf}$ so they define (by Theorem 3.10) the appropriate homomorphisms of algebras $A_{\Sigma_{m, -m}^n} \rightarrow A_{\Sigma_{m', -m'}^n}$, so we denote by $A_{\Sigma}$ the direct limit $A_{\Sigma_{m, -m}^n}$ under these homomorphisms.

Clearly, the following noncommutative Ptolemy relations (in the form (2.14)) hold in $A_{\Sigma_{m, -m}^n}$:

\begin{align}
\label{eq:4.6}
x_{i+1, j, j+1, i, i}^{-1} &= x_{i+1, j, j+1, i, i}^{-1} + x_{i+1, j, j+1, i, i}^{-1}
\end{align}

\begin{align}
\label{eq:4.7}
x_{i, j, j+1, i, i}^{-1} &= x_{i, j, j+1, i, i}^{-1} + x_{i, j, j+1, i, i}^{-1}
\end{align}

for all $i, j \in \mathbb{Z}$.

Remark 4.7. It is natural to conjecture that the relations (4.6) and (4.7) are defining for $A_{\Sigma_\infty}$ and (in view of Remark 2.6) that all natural homomorphisms $A_{\Sigma_{m, -m}^n} \hookrightarrow A_{\Sigma_\infty}$ are injective.

Note that $\Sigma_\infty$ has a triangulation

$$\Delta_\infty = \{(i, -(i+1)_\pm), ((i+1)_\pm, i, i_\pm), (i, -(i+1)_\pm), (i, -(i+1)_\pm), (i, -(i+1)_\pm), (i, -(i+1)_\pm) : i \in \mathbb{Z}\}.$$

Hence the group $T_\infty$ generated by $d_{i, \pm} := x_{i, i+1, i, \pm}, \mathcal{P}_{i, \pm} := x_{i, i-1, i, \pm}, \mathcal{Y}_{i, \pm} := x_{i, -(i+1), i, \pm}, i \in \mathbb{Z}$ subject to the triangle relations

$$x_{i, \pm, j, j+1, i, i}^{-1} = y_{i, \pm, j, j+1, i, i}^{-1} = x_{i, \pm, j, j+1, i, i}^{-1}$$

for $i \in \mathbb{Z}$ is naturally isomorphic to the triangle group $T_{\Delta_\infty}$. Corollary 3.37 implies that the subalgebra $A_{\Delta_\infty}$ of $A_{\Sigma_\infty}$ (generated by all $x_\gamma, \gamma \in \Gamma(\Sigma_\infty)$) and all $x_{\gamma^0}, \gamma^0 \in \Delta_\infty$ is the group algebra $ZT_\infty$.

Proposition 4.8. In $A_{\Delta_\infty}$ we have:

(a) Each $x_{i, j, j', i}$, $i, j \in \mathbb{Z}$ is sum of elements of $T_\infty$ in $ZT_\infty$.

(b) The total angle $T_{i, j}$ at $i_\pm$ is given by

$$T_{i, j} = x_{i, j, j, i}^{-1} x_{j, i, (i+1), j}^{-1} x_{i, j, (i-1), j}^{-1} + x_{j, i, (i+1), j}^{-1} x_{i, j, (i-1), j}^{-1} = x_{i, j, j, i}^{-1} x_{j, i, (i+1), j}^{-1} x_{i, j, (i-1), j}^{-1} x_{j, i, i}^{-1},$$

for each $j \in \mathbb{Z}$. 

Lemma 4.9. \( T_{i\pm} = T((i\pm,j),((i\pm,j)(i\pm, j+1)) + T((i\pm,j),((i\pm,j)(i\pm, j)) \). \\

Using this and taking into account that \\
\[
T((i\pm,j),((i\pm,j)(i\pm, j+1)) + T((i\pm,j),((i\pm,j)(i\pm, j)) = x_{i\pm,1}^{-1} x_{i\pm,1}^{-1} + x_{i\pm,1}^{-1} x_{i\pm,1}^{-1}
\]

in the notation (4.9), we obtain (b). \\

The proposition is proved. \( \square \)

Remark 4.10. Using the triangulation \( \Delta_{\infty} \), it is easy see that \\
\[
T_{i\pm} = d_{i-1,1,y_{i-1}x_{i-1}^{-1} + \overline{d}_{i-1,y_{i-1}x_{i-1}^{-1} + \overline{d}}_{i+1,x_{i+1}y_{i+1}^{-1}} + \overline{d}_{i+1,y_{i+1}x_{i+1}^{-1} + \overline{d}}_{i-1,x_{i-1}y_{i-1}^{-1}} + x_{i-1}y_{i-1}d_{i-1}. 
\]

We can refine these observations and thus recover the recursions (1.6), (1.7). Indeed, set \\
\[
U_{ij} = x_{i-,j-}, V_{ij} = x_{i+,j-}, A_{ij} = x_{(j+1),j}^{-1} b_{(j+1),j}^{-1} + y_{(j+1),j}, B_{ij} = x_{(j+1),j}^{-1} b_{(j+1),j}^{-1} + y_{(j+1),j}.
\]

By definition, \( \mathcal{T}_{\infty} \) is freely generated by \( A_{ij}, B_{ij}, U_{ij}, V_{ij}, U_{ij+1}, i \in \mathbb{Z} \) and, by Proposition 4.8, each \( U_{ij} \) is a sum of elements of \( \mathbb{T}_R \). This and Proposition 4.8 imply the following result.

Theorem 4.11. The elements \( U_{ij}, V_{ij} \in \mathbb{Z} \mathbb{T}_\infty \), \( i, j \in \mathbb{Z} \), satisfy (1.6), (1.7). Furthermore, the elements \( H_{ij} \in \text{Frac}(\mathbb{Z} \mathbb{T}_\infty), i \in \mathbb{Z} \), given by (1.8) do not depend on \( j \) and belong to \( \mathbb{Z} \mathbb{T}_\infty \).

5. Appendix: Noncommutative localizations

Recall that for a multiplicative monoid \( S \) its linearization \( \mathbb{Z} S \) is the ring \( \mathbb{Z} S = \bigoplus_{s \in \mathbb{S}} \mathbb{Z} \cdot [s] \) with the natural extension of multiplication on \( S \). \\

Given a multiplicative submonoid \( S \) of a unital ring \( R \), define the universal localization \( R[S^{-1}] \) of \( R \) by \( S \) to be quotient of the free product \( R \ast (\mathbb{Z} S^{op}) \) by the ideal generated by all elements of the form \( s \ast [s] - 1 \), \([s] \ast [s] - 1 \) for any \( s \in S \). \\

By definition, one has a canonical ring homomorphism \\
\[
R \rightarrow R[S^{-1}].
\]

In other words, \( R[S^{-1}] \) is the unital ring \( R' \) with the universal property that one has a ring homomorphism \( R \rightarrow R' \) under which the image of each element of \( S \) in \( S \). \\

Note that (5.1) is not always injective. For each unital ring \( R \) denote by \( R^\times \) the set of all units (i.e., invertible elements) in \( R \).

The following fact is obvious.

Lemma 5.1. For any ring homomorphism \( \varphi : R \rightarrow R' \) and any submonoid \( S \subset R \setminus \{0\} \) such that \( \varphi(S) \subset (R')^\times \) there is a unique ring homomorphism \( \varphi_S : R[S^{-1}] \rightarrow R' \) such that the composition \( R \rightarrow R[S^{-1}] \rightarrow R' \) is \( \varphi \).

For each submonoid \( S \subset R \setminus \{0\} \) define its saturation \( \hat{S} \) to be the set of all \( r \in R \) such that the image of \( r \) in \( R[S^{-1}] \) is invertible. Clearly, \( \hat{S} \) is a submonoid of \( R \setminus \{0\} \) containing \( S \). We say that \( S \) is saturated if \( \hat{S} = S \). The following obvious fact justifies this definition.

Lemma 5.2. For any submonoid \( S \subset R \setminus \{0\} \) one has \( R[S^{-1}] = R[\hat{S}^{-1}] \). Moreover, \( \hat{S} \) is the largest submonoid of \( R \setminus \{0\} \) with this property.

Following Malcev and Cohn, we say that a unital ring is of class \( \mathcal{E} \) if it can be embedded into a skew-field.

Lemma 5.3. Let \( R \) be any ring of class \( \mathcal{E} \). Then for any multiplicative submonoid \( S \) of \( R \setminus \{0\} \) the canonical homomorphism (5.1) is injective.
Proof. Indeed, let \( \mathcal{F} \) be a skew field and \( \varphi : R \to \mathcal{F} \) be a monomorphism. By definition, for any submonoid \( S \) of \( R \setminus \{0\} \), \( \varphi \) factors as \( \varphi = g \circ f \), where \( f : R \to R[S^{-1}] \) and \( g : R[S^{-1}] \to \mathcal{F} \) are canonical homomorphisms. Since \( \varphi \) is a monomorphism, then \( f \) is also a monomorphism. \( \square \)

Definition 5.4. For a ring \( R \) of class \( \mathcal{E} \) we say that a submonoid \( S \) of \( R \setminus \{0\} \) is divisible if \( R[S^{-1}] \) is also of class \( \mathcal{E} \).

Following Cohn, we say that a submonoid \( S \) of \( R \setminus \{0\} \) is factor-closed if for any \( a, b \in R \setminus \{0\} \), \( ab \in S \) implies that \( a, b \in S \).

Proposition 5.5. Let \( R \) be of class \( \mathcal{E} \) and \( S \) be a divisible submonoid of \( R \setminus \{0\} \). Then the saturation \( \hat{S} \) of \( S \) is a factor-closed submonoid of \( R \setminus \{0\} \).

Proof. Since \( S \) is divisible, in particular, the canonical homomorphism \( R \to R' = R[S^{-1}] = R[\hat{S}^{-1}] \) is injective. It suffices to prove that if \( x, y \in R \) such that \( xy \in \hat{S} \), then \( x \in \hat{S} \), \( y \in \hat{S} \). Indeed, let \( z := (xy)^{-1} \) and \( t := yx - 1 \) in \( R' \). By definition, \( xyz = 1 = zyx \). This implies that \( xt = yxx - x = 1 \cdot x - x = 0 \). Since \( R' \) has no zero divisors and \( x \neq 0 \), then \( t = 0 \), i.e., \( (yz)x = 1 \). Since \( x(yz) = 1 \), we see that \( x \) is invertible in \( R' \) hence \( x \in \hat{S} \). Similarly, \( y \in \hat{S} \) as well.

The proposition is proved. \( \square \)

Below we provide a sufficient criterion for a group algebra of a group to belong to class \( \mathcal{E} \) and for divisibility of some of its submonoids.

Definition 5.6. A group \( G \) is called 1-relator torsion-free if \( G \) is isomorphic to \( F/\langle x \rangle \) where \( F \) is a finitely generated free group, \( x \in F \setminus \{1\} \) is not a proper power in \( F \), and \( \langle x \rangle \) denotes the normal subgroup of \( F \) generated by \( x \).

Results of Malcev, Newman, J. Lewin and T. Lewin (see e.g., [13, Section 8.7], [30]) imply the following.

Theorem 5.7. Let \( G \) be any finitely generated free group or any 1-relator torsion free group. Then the group algebra \( R = \mathbb{Q}G \) is of class \( \mathcal{E} \). In particular, for any submonoid \( S \subset \mathbb{Q}G \setminus \{0\} \) the canonical homomorphism \( \mathcal{E} \) is injective.

We will need the following result, which is a particular case of [40, Theorem 10.11] (here \( \mathcal{F}_\ell \) denotes a free skew field freely generated by \( \ell \) elements).

Proposition 5.8. Let \( \ell \geq 1 \) and assume that \( \ell \) elements \( t_1, \ldots, t_\ell \) of \( \mathcal{F}_\ell \) generate \( \mathcal{F}_\ell \). Then \( t_1, \ldots, t_\ell \) are free generators. In particular, the assignments \( e_i \mapsto t_i \) for \( i = 1, \ldots, \ell \) define an injective homomorphism of algebras \( \mathbb{Q}F_\ell \to \mathcal{F}_\ell \).

Following Cohn, we say that a ring \( R \) is a left (resp. right) semifir if each finitely generated left (resp. right) ideal \( J \) is isomorphic to \( R^n \) for a unique \( n = n_J \). \( R \) is called a semifir if it is both left and right semifir. We use below the standard definition of a universal \( R \)-field, see [13, Section 7.2].

Theorem 5.9. Let \( R \) be a semifir. Then:

(a) There exists a universal skew field \( \text{Frac}(R) \) containing \( R \) as a subalgebra and generated by \( R \).
(b) For any factor-closed submonoid \( S \) of \( R \setminus \{0\} \) the canonical homomorphism \( R_S \to \text{Frac}(R) \) is injective.

Proof. Recall from [13] that:

- an \( n \times n \) matrix \( A \) over a unital ring \( R \) is full if for any factorization \( A = BC \) for some \( n \times p \) matrix \( B \) and a \( p \times n \) matrix \( C \) one has \( p \geq n \);
- A homomorphism \( f : R \to R' \) is honest if the image of each full matrix is full.
- A set \( \Sigma \) of square matrices over a unital ring \( R \) is multiplicative if any upper block-triangular matrix with diagonal in \( \Sigma \) also belongs to \( \Sigma \) and \( \Sigma \) is closed under simultaneous permutation of rows and columns;
- A set \( \Sigma \) of matrices over a unital ring \( R \) is called factor-closed if \( AB \in \Sigma \) for some \( n \times n \) matrices \( A \) and \( B \) over \( R \) implies that \( A, B \in \Sigma \).
- For any set \( \Sigma \) of square matrices over a unital ring \( R \), \( R_\Sigma \) denotes the universal localization ([13, Theorem 2.1]) so that the image of each element of \( \Sigma \) under the canonical homomorphism \( R \to R_\Sigma \) is an invertible matrix (e.g., \( R_S = R[S^{-1}] \) in the notation as above);

Then Theorem 5.9(a) immediately follows from the following result.
Theorem 5.10. ([13] Section 7.5, Corollary 5.11) For each semifir \( R \) the universal localization \( \text{Frac}(R) := R_0 \), where \( \Phi \) is the set of full matrices over \( R \), is a skew field and the canonical homomorphism \( R \to \text{Frac}(R) \) is honest (hence injective).

To prove (b) we need following results from [13].

Proposition 5.11. ([13] Section 7.5, Proposition 5.7(ii)) Given unital rings \( R \) and \( R' \) and a honest homomorphism \( f : R \to R' \), then for any factor-closed multiplicative set \( \Sigma \) of square matrices over \( R \), the canonical homomorphism \( f_{:\Sigma} : R_\Sigma \to R'_\Sigma \) is injective.

For any \( S \subset R \) denote by \( \Sigma_S \) the set of all matrices over \( R \) of the form \( PMQ \) where \( P \) and \( Q \) are invertible matrices over \( R \) and \( M \) is an upper triangular matrix over \( R \) with diagonal entries in \( S \).

Lemma 5.12. ([13] Section 7.5, Lemma 10.1) Let \( R \) be a semifir. Then for any factor-closed submonoid \( S \) of \( R \setminus \{0\} \) the set \( \Sigma_S \) is factor-closed and multiplicative.

Finally, letting \( R \) be a semifir and \( R' = \text{Frac}(R) \) in Proposition 5.11 \( \Sigma = \Sigma_R \) as in Lemma 5.12 and taking into account that \( R[S^{-1}] = R_S = R_{\Sigma_R} \), we finish the proof of part (b).

Theorem 5.9 is proved.

It is well-known (see e.g., [17]) that for any finitely generated free group \( F \) its group algebra \( R = \mathbb{Q}F \) is a semifir. Therefore, Theorem 5.9 implies the following corollary.

Corollary 5.13. Let \( F \) be a finitely generated free group and \( R = \mathbb{Q}F \). Then any factor-closed submonoid \( S \) of \( R \setminus \{0\} \) is divisible, more precisely, \( R[S^{-1}] \subset \text{Frac}(R) \).

Remark 5.14. Based on Theorem 5.7, we expect that an analogue of Corollary 5.13 also holds for \( R = \mathbb{Q}G \), where \( G \) is a torsion-free 1-relator group.

Given a unital ring \( R \), following Cohn, we say that:

- Elements \( a, b \in R \) are similar if the right \( R \)-modules \( R/aR \) and \( R/bR \) are isomorphic (clearly, similarity is an equivalence relation on \( R \)).

- An element \( p \in R \setminus R^\times \) is prime if for any factorization \( p = p'p'' \) one has: either \( p' \in R^\times \) or \( p'' \in R^\times \).

- A unital ring \( R \) is a (noncommutative) unique factorization domain (UFD) if each nonzero non-unit admits a prime factorization and for any two prime factorizations of a non-unit \( x \in R \):

\[
x = p_1 \cdots p_r = q_1 \cdots q_s
\]

one has \( s = r \) and \( q_i \) is similar to \( p_{\sigma(i)} \) for \( i = 1, \ldots, r \) where \( \sigma \) is a permutation of \( \{1, \ldots, r\} \).

Proposition 5.15. Let \( R \) be a UFD and \( S \) be a submonoid of \( R \setminus \{0\} \). Then \( S \) is factor-closed iff it is generated by \( R^\times \) and a (empty or not) set \( P \) which is the union of similarity classes of prime elements in \( R \).

Proof. Denote by \( P \) the set of all primes in \( S \) and by \( S_P \) the submonoid of \( R \setminus \{0\} \) generated by \( R^\times \) and \( P \). Clearly, \( S_P \subset S \).

Suppose that \( S \) is factor-closed. Let us show that \( S = S_P \). We proceed by contradiction, i.e., suppose that there is at least one element \( x \in S \setminus S_P \). Then \( x \) is not a unit hence \( x \) has a prime factorization \( x = p_1 \cdots p_r \).

If \( r = 1 \), then \( x = p_1 \in S \) hence \( x \in S_P \) and we arrive at the contradiction. If \( r \geq 2 \), then since \( S \) is factor-closed, we have \( p_i \in S \) for \( i = 1, \ldots, r \). Hence \( x \in S_P \) and we arrive at the contradiction once again.

Suppose that \( P \) is a union of similarity classes and \( S = S_P \). Let us prove that \( S \) is factor-closed. Suppose that \( ab \in S \) for some \( a, b \in R \). Let us show that \( a, b \in S \). If either \( a \) or \( b \) is a unit, we have nothing to prove because \( R^\times \subset S \). Thus, suppose that \( a, b \in R \setminus R^\times \) and let

\[
a = p_1 \cdots p_r, \quad b = p_{r+1} \cdots p_r
\]

be respective prime factorizations with \( 1 \leq r' < r \), where \( p_1, \ldots, p_r \) are some primes in \( R \). On the other hand, since \( ab \) is a non-unit element of \( S \), it admits a prime factorization in \( S \):

\[
ab = q_1 \cdots q_s
\]

where \( q_1, \ldots, q_s \in P \). Comparing the factorizations:

\[
p_1 \cdots p_r = q_1 \cdots q_s
\]
and using the fact that \( R \) is UFD, we obtain: \( r = s \) and each \( p_i \) is similar to one of \( q_j \). Since all primes similar to each \( q_j \) belongs to \( P \), we obtain \( p_1, \ldots, p_r \in P \) hence \( a \in S \), \( b \in S \).

The proposition is proved. \( \square \)

**Remark 5.16.** The class of noncommutative UFD's is rather large: it contains group rings \( \mathbb{Q}F \), where \( F \) is any finitely generated free group (see e.g., [15 Theorem 3.4, Proposition 3.5 and Corollary]).

Note however, that similarity classes of primes may contain some “unexpected” elements. For instance, if \( R \) is the free ring in \( x, y \) then \( xy + 1 \) and \( yx + 1 \) are similar (see e.g. [15]). This motivates the following definition.

**Definition 5.17.** Given a ring \( R \), we say that an element \( a \in R \setminus \{0\} \) is **self-similar** if all elements similar to \( a \) are of the form \( uau' \), where \( u, u' \in R^\times \).

Taking into account that \( (\mathbb{Q}F)^\times = \mathbb{Q}^\times \cdot F \), for a free (or, more generally, an ordered) group \( F \) (see e.g., [35 Theorem 6.29]), we obtain the following conjectural characterization of certain self-similar primes in \( \mathbb{Q}F \).

**Conjecture 5.18.** Let \( F \) be a free group freely generated by \( t_1, \ldots, t_m \), \( m \geq 2 \). Then for \( k = 2, \ldots, m \) the element \( \tau_k := t_1 + \ldots + t_k \) is a self-similar prime, e.g., all elements of \( \mathbb{Q}F \) similar to \( \tau_k \) belong to \( \mathbb{Q}^\times \cdot F \cdot \tau_k \cdot F \).

**Remark 5.19.** This conjecture was shaped during our discussions with George Bergman, Dolores Herbera, and Alexander Lichtman. We are immensely grateful to these mathematicians.

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