Perturbative Analysis of Wave Interaction in Nonlinear Systems

Alex Veksler\textsuperscript{1} and Yair Zarmi\textsuperscript{1,2}
Ben-Gurion University of the Negev, Israel
\textsuperscript{1}Department of Physics, Beer-Sheva, 84105
\textsuperscript{2}Department of Solar Energy & Environmental Physics
Jacob Blaustein Institute for Desert Research, Sede-Boquer Campus, 84990

Abstract

This work proposes a new way for handling obstacles to asymptotic integrability in perturbed nonlinear PDE’s within the method of Normal Forms (NF) for the case of multi-wave solutions. Instead of including the whole obstacle in the NF, only its resonant part (if one exists) is included in the NF, and the remainder is assigned to the homological equation. This leaves the NF integrable and its solutions retain the character of the solutions of the unperturbed equation.

We exploit the freedom in the expansion to construct canonical obstacles which are confined to the interaction region of the waves. For soliton solutions (e.g., in the KdV equation), the interaction region is a finite domain around the origin; the canonical obstacles then do not generate secular terms in the homological equation. When the interaction region is infinite (or semi-infinite, e.g. in wavefront solutions of the Burgers equation), the obstacles may contain resonant terms.

The obstacles generate waves of a new type which cannot be written as functionals of the solutions of the NF. When the obstacle contributes a resonant term to the NF, this leads to a non-standard update of the wave velocity.

Keywords: nonlinear evolution equations, wave interactions, obstacles to asymptotic integrability, perturbed KdV equation, perturbed Burgers equation.

1 The problem of obstacles to asymptotic integrability

The analysis of the effect of a perturbation on wave solutions of evolution PDE’s has evolved in two different approaches. In scattering approach, a solution of the unperturbed equation is scattered off a perturbation that is turned on at \( t = 0 \). As the unperturbed solution is not a solution of the perturbed equation, its amplitude decays and its wave number, velocity and phase shift are modified. In addition, a soliton tail has been found outside the soliton sector in the case of the perturbed KdV equation. The methods used have been a combination
of the theory of inverse scattering and the multiple time expansion procedure 
[11]-[8].

The second approach deals with a different issue. Integrable nonlinear evolution equations are lowest-order approximations for the more complicated equations of the full dynamical systems (e.g., the equations of Fluid Dynamics in the cases of the Burgers & KdV equations, and Maxwell’s equations in the case of the NLS equation). To improve the approximation, one includes higher-order effects of the original physical system. In this case the perturbation is not turned on at $t = 0$. It exists for all times, going back to $t = -\infty$.

In this approach, one is searching for a zero-order solution, which has the same wave structure as the solution of the unperturbed equation, except for an update of the wave velocity by the higher-order effects. The method of Normal Forms was used for the analysis of soliton solutions of the perturbed KdV and NLS equations [12]-[14], [17], [22] and front solutions of the perturbed Burgers equation [11], [15], [20]. The method of Multiple Time Scales was used in the case of the perturbed KdV equation [18], [19], [21] and in the case of the perturbed NLS equation [23].

We focus on problems that arise in the analysis of wave solutions within the second approach. Most integrable nonlinear evolution PDE’s allow for single- as well as multi-wave solutions. The multi-wave solutions usually asymptote to well-separated single waves in the $x - t$ plane, except for interaction regions, where the multi-wave character of the solution is lost. The interaction regions may be localized (e.g., in the case of KdV-multi-soliton solutions) or semi-infinite (e.g., Burgers-multi-fronts). The main purpose of this work is to investigate the effect of a perturbation that is added to the unperturbed equation on the wave solutions of nonlinear systems.

The perturbed equations are often analyzed by the method of Normal Forms (NF) [9] - [11], briefly described in the following. Let

$$w_t = F^{(0)}[w] + \sum_{k=1} \epsilon^k F^{(k)}[w]$$

be a perturbed nonlinear evolution PDE (square brackets imply that the corresponding term is a differential polynomial in $w(x,t)$). We assume that $w(x,t)$ may be expanded in a power series in the small parameter $\epsilon$ of differential polynomials of $u(x,t)$ (NIT - Near-Identity Transformation):

$$w = \sum_{k=0} \epsilon^k u^{(k)}[u] \quad (u \equiv u^{(0)})$$

The time evolution of the zero-order term, $u(x,t)$, is assumed to be governed by the Normal Form (NF):

$$u_t = \sum_{k=0} \epsilon^k a_k S^{(k)}[u], \quad (a_0 = 1)$$
Here, $S^{(k)}$ are the resonant terms usually called the symmetries. Their time dynamics is equivalent to that of $u(x,t)$ up to first order, that is,

$$\left(u + \mu \, S^{(n)}\right)_t = F^{(0)} \left[u + \mu \, S^{(0)}\right], \quad (\mu \ll 1) \quad (4)$$

As a result, their Lie Brackets vanish:

$$\left[F^{(0)}, S^{(n)}\right] = \sum_i \left\{ \frac{\partial F^{(0)}}{\partial u_i} \partial_x S^{(n)} - \frac{\partial S^{(n)}}{\partial u_i} \partial_x F^{(0)} \right\} = 0 \quad (5)$$

The symmetries (including $F^{(0)}$ itself) form hierarchies \[30, 31\]. It is possible to establish recursion relations among the symmetries in each hierarchy. For many equations (and all the equations our work deal with), the first symmetry is

$$S^{(1)} [u] = u_x \quad (6)$$

Substituting the NIT (2) and the NF (3) into (1) leads to a sequence of homological equations for the time evolution of $u^{(n)}$, which have to be solved order-by-order.

The motivation for assumption (3) is that a perturbative analysis that does not include the removal of resonant terms from the homological equation into the NF, usually yields secularities, that is, unbounded terms in the approximate solution. On the other hand, the NF is expected also to be integrable and to preserve the nature of the unperturbed solution. This feature is closely related to another significant one: that the main effect of adding the higher-order terms to the NF is the update of physically-valuable parameters (usually the wave velocity/the dispersion relation).

After removing the resonant terms out of the homological equations, they become:

$$\left[F^{(0)}, u^{(k)} [u]\right] + T^{(k)} [u] = 0 \quad (7)$$

where $T^{(k)} [u]$ is the contribution for all nonresonant terms of order $k$. The NIT is constructed from solutions of these equations.

However, the analysis may lead to the emergence of obstacles to integrability \[12-22\], beginning at some order in the expansion. These are terms (differential polynomials) that the perturbative expansion of the dynamic equation (Eq. 1) generates, which cannot be accounted for by the formalism. The differential polynomial structure of the obstacles is not unique and depends on the way in which the NIT is constructed.

To make the construction of the NIT possible, the usual practice has been to include these unaccounted-for terms in the NF. This makes the NF nonintegrable, hence the name “obstacles to integrability”. Including the obstacles in the NF, disturbs the wave character of its solutions. The effect of the obstacles in the case of the two-soliton solution of the normal form of the perturbed KdV equation \[12-14\] has been studied in \[22\]. The zero-order solution was
found to develop inelastic effects: appearance of a second-order radiation wave; fourth-order, time dependent, corrections to each of the wave numbers and the generation of an eighth-order soliton.

2 Approach for overcoming obstacles

2.1 The general ideas

The necessity to include the obstacles in the normal form is a consequence of the assumption, usually made in the NF expansion, that all the terms in the NIT are differential polynomials in the zero-order approximation (that is, in \( u(x,t) \)) and do not depend explicitly on the independent variables, \( t \) and \( x \). Our approach overcomes this problem by allowing the higher-order terms in the NIT to depend on these variables. To this end, we assume for the \( k \)'th order term in the NIT the following form:

\[
    u^{(k)} = u^{(k)}_d \left[ u \right] + u^{(k)}_r (x,t)
\]

In Eq. (8), \( u^{(k)}_d \left[ u \right] \) is a differential polynomial in \( u \), and \( u^{(k)}_r (x,t) \) depends explicitly on \( x \) and \( t \), and is expected to account for the obstacles. Thus, substituting the assumption (8) in the homological equation (7), we obtain:

\[
    \left[ F^{(0)} , u^{(k)}_r (x,t) \right] + R^{(k)} [u] = 0
\]

where \( R^{(k)} [u] \) stands for the obstacle of order \( k \).

Owing to the freedom inherent in the perturbative expansion, the construction of \( u^{(k)}_d \left[ u \right] \) is not unique. Unless \( u^{(k)}_d \left[ u \right] \) is chosen in an appropriate manner, the resulting obstacle may not reflect the following features of physical interest:

(i) Obstacles do not emerge in the case of single-wave solutions of the NF.

(ii) The expectation that obstacles emerge owing to interaction among waves in the multi-wave case.

Both features are realized if \( u^{(k)}_d \left[ u \right] \) is chosen to have the structure of the differential polynomial that solves the problem in the case of a single-wave solution of the NF. The choice proposed for \( u^{(k)}_d \left[ u \right] \) leads to obstacles in a "canonical" form, expressed in terms of symmetries of the unperturbed equation. The obstacles now vanish if one substitutes for \( u \) the single-wave solution of the NF. More important, as a result, they are expected to vanish away from regions of wave interaction in the multi-wave case. The reason is that away from the interaction regions, multi-wave solutions asymptote into a sum of well-separated single-wave solutions.
2.2 Construction of canonical obstacles

Starting with \(u^{(k)}[u]\) that solves the homological equation (7) for the single-wave, we find that the canonical obstacles can be written in the following form

\[
R^{(n)}[u] = \sum_{k=3}^{n+g} \gamma^n_k f_{ij}[u, \partial_x] R_{ij}[u] 
\]

where \(\gamma^n_k\) is a numerical coefficient, \(f_{ij}[u, \partial_x]\) - a differential operator of an appropriate weight and \(g\) is the gap between the index of symmetry and the order of perturbation. For example, if we define \(F^{(0)} = S^{(2)}\) (this is a widely-accepted notation), then \(g = 2\).

The obstacles of Eq. (10) vanish identically for the case of single-wave solutions of the NF. To see this, we exploit the fact that all the symmetries are proportional to one another in the case of the single-wave solutions of the NF. For “trivial” boundary condition, \(u(\xi \to -\infty) = 0\):

\[
S^{(n)} = (-1)^{n+1} v_0^{n+1} S^{(1)} \tag{12}
\]

This may be proven by induction for any hierarchy which is governed by a linear recursion relation.

The proportionality of all the symmetries leads to a simple update of the velocity of the solutions of the NF:

\[
v = \sum_{k \geq 0} \epsilon^k v_k, \quad v_k = (-1)^k a_k v_0^{k+1} \tag{13}
\]

Eq. (13) also describes the velocity update of each wave in a multi-wave solution.

2.3 Resonant contribution in obstacles?

For multi-wave solutions of the NF, the obstacles do not vanish. An important characteristic of our canonical obstacles is that they do not vanish only in the interaction regions in the \(x - t\) plane. For example, in the case of KdV solitons, the interaction region is a finite domain around the origin, whereas in the case of Burgers fronts it consists of one or more domains of finite width along semi-infinite lines. The canonical obstacles vanish exponentially fast away from the interaction regions, where the solution asymptotes to well-separated single waves. On the other hand, non-canonical obstacles are finite also outside the interaction region.

A cardinal question that now arises is whether the obstacles generate secular terms in the NIT (2). A symmetry, if contained in an obstacle, will generate a secular term through Eq. (9). We, therefore, propose to break an obstacle...
into a sum of a symmetry plus a non-resonant term. This symmetry, with its coefficient, must be included in the NF (3).

Our task is therefore to determine whether a canonical obstacle has the capacity of generating secular terms. A simple criterion for detecting this capacity is that the obstacle spreads over an infinite or a semi-infinite domain, and asymptotes to a symmetry. (This criterion is similar, although much less rigorous, than the Fredholm Alternative Theorem.)

We focus on two-wave solutions of two equations: the Korteweg-de Vries (KdV) equation with a two-soliton solution, with obstacle appearing in the second order; and the Burgers equation, with a two-front solution, where an obstacle arises already in the first order.

The interaction region, and thus the obstacles, of KdV are localized. Hence, we expect the solution of Eq. 9 for this problem to be bounded. This expectation has been verified by solving the homological equation numerically. We obtain a new bounded soliton-like wave.

On the other hand, the interaction region in the two-front solution of the Burgers equation is semi-infinite (the fronts are well separated in one half of the $x - t$ plane and merge (interacting) in the other half). The canonical obstacle asymptotically approaches the symmetry $S^{(3)}[u]$. Therefore, we expect the Burgers obstacle to generate a secular term in Eq (9). This expectation has been verified by numerically solving the equation.

We ”extract” the symmetry $S^{(3)}[u]$ out of the obstacle with its coefficient, and transfer it into the NF (3). Thus, the NF remains solvable, but the coefficient of the first-order term in the wave-velocity update is changed. The remainder is not a canonical obstacle, that is, it is not confined to the interaction region but rather spreads over all the fronts of the zero-order solution. However, it does not contain a symmetry. The solution of the homological equation must be bounded. There is no closed form solution of the homological equation, but the numerical calculations show that this prediction is verified.

In the following section, these two examples are brought in some detail. The full results of our work will be brought in further publications. We stress that our results crucially depend on the wave nature of the solutions.

2.4 Summary and Conclusions

1. The main effect of perturbations on the interaction among waves in multiwave solutions of NL EPDE’s is the emergence of the obstacles to integrability.

2. The proper place for handling obstacles, once their resonant part has been shifted to the NF, is the homological equation. To this end, it is necessary to allow the higher-order terms in the NIT (2) to be explicit functions of the independent variables. Actually, $u^{(a)}$ will consist of two parts: the differential polynomial, $u_d^{(a)}[u]$, whose structure corresponds to the case of a single-wave solution of the NF (3) (in this case, there are no obstacles); and the function $u_f^{(a)}(x,t)$ that is supposed to account for the obstacle.
3. When the NIT is built in such a way, the obstacles (if they exist) are obtained in the canonical form \((10)\). Canonical obstacles are confined to the region of interaction among waves.

4. When the interaction region is localized (e.g., for KdV solitons), the canonical obstacles do not generate secular solutions of the homological equations \((9)\). The solution of the homological equation is then bounded, usually unavailable in closed form.

5. When the interaction region spreads over a semi-infinite range, the obstacles are expected to cause secular solutions. Then, it is necessary to identify a symmetry "hidden" inside the obstacle and remove it from the homological equation into the NF. The remainder of the obstacle remains in the homological equation and yields bounded solutions.

In both situations, the obstacles cease to be obstacles to integrability of the NF. The latter remains integrable, and its solutions (the zero-order approximation) retain the character of the unperturbed solutions. The difference between the items 4 and 5 above is that in \# 5 the wave velocity is also affected, in the order in which the obstacle exists. Focusing on two-wave solutions, it is found that the obstacle usually yields an additional wave, that may not be expressed in closed form. In general, the homological equation has to be solved numerically.

### 3 Two worked examples

#### 3.1 The perturbed KdV equation

The perturbed KdV equation is

\[
\frac{\partial w}{\partial t} = 6ww_x + w_{xxx} + \epsilon \left( 30\alpha_1 w^2 w_x + 10\alpha_2 w w_{xx} + 20\alpha_3 w_x w_{xx} + \alpha_4 w_{5x} \right) \\
+ \epsilon^2 \left( 140\beta_1 w^3 w_x + 70\beta_2 w w_{xxx} + 280\beta_3 w w_x w_{xx} + 14\beta_4 w w_{5x} \\
+ 70\beta_5 w_x^3 + 42\beta_6 w_x w_{4x} + 70\beta_7 w_{xx} w_{xxx} + \beta_8 w_{7x} \right) + O \left( \epsilon^3 \right)
\]

We assume the NIT

\[
w = u + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + O \left( \epsilon^3 \right)
\]

and the NF

\[
u_t = S^{(2)} [u] + \epsilon \alpha_4 S^{(3)} [u] + \epsilon^2 \beta_8 S^{(4)} [u] + O \left( \epsilon^3 \right)
\]

where

\[
S^{(2)} [u] = 6u u_x + u_{xxx} \\
S^{(3)} [u] = 30u^2 u_x + 10u u_{xx} + 20u_x u_{xx} + u_{5x} \\
S^{(4)} [u] = 140u^3 u_x + 70u^2 u_{xx} + 280u u_{xxx} + 14u u_{5x} \\
+ 70u_x^3 + 42u_x u_{4x} + 70u_{xx} u_{xxx} + u_{7x}
\]
A single-wave solution of the NF (16) is the well-known KdV soliton:

$$u(x, t) = \frac{2k^2}{\cosh^2 \left[ k \left( x - vt + x_0 \right) \right]}$$  \hspace{1cm} (18)

and the two-wave solution is given by the Hirota formula [26]:

$$u(x, t) = 2 \partial_x^2 \ln \left\{ 1 + g_1(x, t) + g_2(x, t) + \left( \frac{k_1 - k_2}{k_1 + k_2} \right) g_1(x, t) g_2(x, t) \right\}$$

$$\left( g_i(x, t) = \exp \left[ 2k_i(x - v_i t + x_{i0}) \right] \right)$$  \hspace{1cm} (19)

A sample of this solution is shown in Fig. 1. The only difference between the solutions of the unperturbed KdV equation, $w_t = 6ww_x + w_{xxx}$, and the both solutions of the NF is the update of the wave velocity:

$$v_i = -4k_i^2 - 16\epsilon\alpha_4k_i^4 - 64\epsilon^2\beta_8k_i^6 - O(\epsilon^3)$$  \hspace{1cm} (20)

There are no obstacles in the first order in the Normal Form analysis. However, an obstacle appears at the second order. Choosing the second-order term in the NIT to be

$$u^{(2)} = u_i^{(2)}(x, t) + B_1u^3 + B_2u u_{xx} + B_3u_x^2 + B_4u_{xxxx}$$

$$+ B_5u u_x q^{(1)} + B_6u_{xxx} q^{(1)} + B_7u_{xx} q^{(1)^2} + B_8u_x q^{(2)}$$

$$\left( q^{(1)} \equiv \partial_x^{-1} u, \quad q^{(2)} \equiv \partial_x^{-1} (u^2) \right)$$  \hspace{1cm} (21)

with the appropriate set of values for $\{B_k\}$ based on the form of $u^{(2)}$ in the case of the single-soliton solution, we obtain the canonical obstacle

$$R^{(2)} = \gamma_5^2 u R_{21} = \gamma_5^2 u \left( 3u^2 u_x + u u_{xxx} - u_x u_{xx} \right)$$  \hspace{1cm} (22)

($\gamma_5^2$ is built of a combination of coefficients of Eq. (14)). This obstacle is localized (see Fig. 2) and hence cannot generate a secular solution in the homological equation

$$\partial_t u_i^{(2)} = 6\partial_x \left( u u_i^{(2)} \right) + \partial_x^3 u_i^{(2)} + \gamma_5^2 u R_{21}$$  \hspace{1cm} (23)

This equation can be solved in closed form by the Green’s function method developed in the context of the Inverse Scattering approach [2, 32-34]. As our goal is only to show that the solution of Eq. (23), a numerical solution was sufficient for our purpose. It shows a new bounded soliton-like wave (Fig. 3).

### 3.2 The perturbed Burgers equation

The perturbed Burgers equation is given by

$$w_t = 2ww_x + w_{xx} + \epsilon \left( 3\alpha_1 w^2 w_x + 3\alpha_2 w w_{xx} + 3\alpha_3 w_x^2 + \alpha_4 w_{xxx} \right) + O(\epsilon^2)$$  \hspace{1cm} (24)
(we don’t write here the second-order perturbation because an obstacle emerges already in the first order). We assume, again, the NIT

\[ w = u + cu^{(1)} + O (\epsilon^2) \]  

and the NF

\[ u_1 = 2u u_x + u_{xx} + \epsilon \mu \left( 3u^2 u_x + 3u u_{xx} + 3u_x^2 + u_{xxx} \right) + O (\epsilon^2) \]  

Its single-wave solution is the shock front

\[ u(x, t) = \frac{kA \exp[k(x - vt)]}{1 + kA \exp[k(x - vt)]} \]  

and the two-front solution is a straightforward extension of Eq. (27):

\[ u(x, t) = k_1 A_1 \exp[k_1 (x - v_1 t)] + k_2 A_2 \exp[k_2 (x - v_2 t)] \]  

(its sample is shown in Fig. 4, and one can see that the fronts are interacting (i.e. merged) over a semi-infinite region). The velocity update is, again, similar for the both solutions:

\[ v_i = -k_i - \epsilon \mu k_i^2 - O (\epsilon^2) \]  

The obstacle appears in the first-order analysis. If we move all the linear term \( u_{xxx} \) from the first order into the NF, that is, take \( \mu = \alpha_4 \), and also choose \( u^{(1)} \) in the form that solves the homological equation for the single-front case plus an explicit function of \( x \) and \( t \):

\[ u^{(1)} = u_r^{(1)} (x, t) + (\alpha_1 - 2\alpha_2 - \alpha_3 + 2\alpha_4) qu_x - \frac{1}{2} (2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4) u^2 \]  

\( q \equiv \partial^{-1}_x u \)

then the obstacle will be get its canonical form:

\[ R^{(1)} = \gamma_3^1 R_{21} = \gamma_3^1 (S^{(2)} G^{(1)} - S^{(1)} G^{(2)}) = \gamma_3^1 (u S^{(2)} - u_x G^{(2)}) \]  

\( \gamma_3^1 = 2\alpha_1 - \alpha_2 - 2\alpha_3 + \alpha_4 \)  

This obstacle is shown in Fig. 5, and one can see that it is finite over all the interaction region. The homologals equation now reads

\[ \partial_t u_r^{(1)} = 2 \partial_x \left( u u_r^{(1)} \right) + \partial^2_x u_r^{(1)} + \gamma_3^1 R_{21} \]  

It should be remarked that of the two terms of \( R_{21} \), only \( u S^{(2)} \) may not be accounted for by the differential polyomials in the NIT. On the contrary, it
is easy to see that substituting \( u_r^{(1)} = G^{(2)} \) into the homogeneous part of the homological equation yields

\[-\partial_t G^{(2)} + 2\partial_x \left( u G^{(2)} \right) + \partial_x^2 G^{(2)} = 2u_x G^{(2)} \quad (33)\]

As expected, the numerical solution of the homological equation shows the existence of a secular term, which indicates the presence of a symmetry inside \( R_{21} \).

In order to extract this symmetry, we use the recursion relation among symmetries in the Burgers hierarchy:

\[ S^{(n+1)} = S^{(n)}_x + u S^{(n)} + u_x G^{(n)} = S^{(n)}_x + R_{n1} + 2u_x G^{(n)} \quad (34) \]

In particular, for \( n = 2 \), it reads

\[ S^{(3)} = S^{(2)}_x + R_{21} + 2u_x G^{(2)} \Rightarrow R_{21} = S^{(3)} - S^{(2)}_x - 2u_x G^{(2)} \quad (35) \]

and the homological equation (32) becomes

\[ \partial_t u_r^{(1)} = 2\partial_x \left( u u_r^{(1)} \right) + \partial_x^2 u_r^{(1)} + \gamma_3^1 \left( S^{(3)} - S^{(2)}_x - 2u_x G^{(2)} \right) \quad (36) \]

Now, we choose \( \mu \), the coefficient of \( S^{(3)} \) in the NF, to be

\[ \mu = \alpha_4 + \gamma_3^1 = 2\alpha_1 - \alpha_2 - 2\alpha_3 + 2\alpha_4 \quad (37) \]

hence correcting the update of the wave velocity (Eq. 29). Further, we add an appropriate correction to \( u_r^{(1)} \) in order to account for the term \(-2\gamma_3^1 u_x G^{(2)}\) in Eq. (36), according to Eq. (33).

Now, the equation for \( u_r^{(1)} \) becomes

\[ \partial_t u_r^{(1)} = 2\partial_x \left( u u_r^{(1)} \right) + \partial_x^2 u_r^{(1)} - \gamma_3^1 S^{(2)}_x \quad (38) \]

The term \( S^{(2)}_x \) does not asymptote to any symmetry. The numerical solution of Eq. (38) shows that a new bounded wave appears (see Fig. 6).

4 Acknowledgements

G. Burde, L. Kalyakin and Y. Kodama are acknowledged for helpful discussions.

5 Figures

Figure captions:

- Fig. 1: Two-soliton solution of the KdV NF (Eq. 19); \( k_1 = 0.3 \), \( k_2 = 0.4 \).
- Fig. 2: Canonical obstacle \( uR_{21} \) for the KdV equation (Eq. 22) for two-soliton solution; parameters as in Fig. 1.
Fig. 3: Contribution of this canonical obstacle to $u_r^{(2)}$ of Eq. (23) for zero boundary condition for $x \to -\infty$; $k_1 = 0.5$, $k_2 = 0.75$.

Fig. 4: Two-front solution of the Burgers NF (Eq. 28); $k_1 = 2$, $k_2 = -2$.

Fig. 5: Canonical obstacle $-R_{21}$ (with opposite sign) for the Burgers equation (Eq. 31) for the two-front solution; parameters as in Fig. 4.

Fig. 6: Bounded contribution of the obstacle $S_x^{(2)}$ to $u_r^{(1)}$ of Eq. (38); parameters as in Fig. 4.
6 References

References

[1] Kaup, D.J., SIAM J. Appl. Math., 31, 121 (1976).
[2] Kaup, D.J. & Newell A.S., Proc. Roy. Soc. London A361, 413 (1978).
[3] Karpman, V.I. & Maslov E.M., Sov. Phys. JETP, 46, 281 (1977).
[4] Newell, A.C., The Inverse Scattering Transform, pp. 177-242 in Solitons, ed. by R.K. Bullough & P.J. Caudrey (Springer-Verlag, 1980).
[5] Kalyakin, L.A., Uspekhi Mat. Nauk, 44, 5-34 (1989), (in Russian); Russian Math. Surveys, 44, 3-42 (1989), (in English).
[6] Kalyakin, L.A., Teor. i Mat. Phiz., 92, 62-76 (1992), (in Russian); Theor. Math. Phys., 92, 736-747 (1992), (in English).
[7] Kalyakin, L.A., Physica 87D, 193-200 (1995).
[8] Kalyakin, L.A. & Lazarev V.A., Teor. i Mat. Fiz., 112, 92-102 (1997) (in Russian).
[9] Kodama, Y. & Taniuti, T., J. Phys. Soc. Jpn., 45, 298 (1978).
[10] Kodama, Y., J. Phys. Soc. Jpn., 45, 311 (1978).
[11] Kodama, Y. & Taniuti, T., J. Phys. Soc. Jpn., 47, 1706 (1979).
[12] Kodama, Y., Phys. Lett. 112A, 193 (1985).
[13] Kodama, Y., Physica 16D, 14 (1985).
[14] Kodama, Y., Normal Form and Solitons, pp. 319-340 in Topics in Soliton Theory and Exactly Solvable Nonlinear Equation, ed. by M.J. Ablowitz et al. (World Scientific, Singapore, 1987).

[15] Fokas, T. & Luo, L., Contemp. Math, 200, 85 (1996).

[16] Focas, A.S., Grimshaw, R. H. J. & Pelinovsky, D. E., J. Math. Phys., 37, 3415 (1996).

[17] Kodama, Y. and Mikhailov, A.V., Obstacles to asymptotic integrability, pp. 173-204 in Algebraic Aspects of Integrable Systems, eds., A.S. Fokas and I.M. Gelfand (Birkhauser, Boston, 1997).

[18] Kraenkel, R.A., Manna, M.A., Merle, V., Montero, J.C., and Pereira, J.G., Phys. Rev. 54E, 2976 (1996).

[19] Kraenkel, R.A., Phys. Rev. E57, 4775 (1998).

[20] Kraenkel, R.A., Pereira, J.G. & de Rey Neto, E.C., Phys. Rev. E58, 2526 (1998).

[21] Kraenkel, R.A., Senthilvelan, M. and Zenchuk, A.I., J. Math. Phys., 41, 3160 (2000).

[22] Hiraoka, Y. & Kodama, Y., Lecture notes, Euro Summer School 2001, The Isaac Newton Institute, Cambridge, August 15-25 (2002).

[23] Degasperis, A., Manakov. S.V., Santini, P.M., Physica 100D, 187-211 (1997).

[24] Miura, R.M., Gardner, C.S. & Kruskal, M.D., J. Math. Phys., 9, 1204 (1968).

[25] Fuchssteiner, B. and Fokas, A.S., Physica 4D, 47 (1981).

[26] Hirota, R., Phys. Rev. Lett., 27, 1192-1194 (1971).

[27] Bona, J.L., Pritchard, W.G. and Scott, L.R., Phys. Fluids, 23, 438 (1980).

[28] Kivshar, Y.S. and Malomed, B.A., Phys. Lett. 115A, 377 (1986).

[29] Kodama, Y., Phys. Lett. 123A, 276 (1987).

[30] Ablowitz, M.J. & Clarkson, P.A., Solitons, Nonlinear Evolution Equations and Inverse Scattering (Cambridge University Press, 1991).

[31] Tasso, H., J. Phys. A, 29, 7779-7784 (1996).

[32] Keener J.P. & McLaughin, Phys. Rev. 16A, 777 (1977).

[33] Sachs R.L., SIAM J. Math. Anal., 14, 674 (1983).

[34] Sachs R.L., SIAM J. Math. Anal., 15, 468 (1984).