A new family of analytical solutions in a four dimensional static spacetime is presented for $f(R)$-gravity. In contrast to General Relativity, we find that a non trivial black brane/string solution is supported in vacuum power law $f(R)$-gravity for appropriate values of the parameters characterizing the model and when axisymmetry is introduced in the line element. For the aforementioned solution, we perform a brief investigation over its basic thermodynamic quantities.

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1. INTRODUCTION

Modified theories of gravity have drawn the attention of the scientific community because of the geometric mechanics that they provide to describe various phenomena in nature. In this concept, new geometrodynamical degrees of freedom are introduced in the gravitational field equations in such a way so as to modify Einstein’s General Relativity (GR). These additional terms can have either theoretical or phenomenological origin [1]. Among the various proposed modified theories of gravity, $f(R)$-gravity [2] has been the main subject of study in various works over different areas of gravitational physics.

$f(R)$-gravity is a fourth-order theory, where the action integral involves a function of the spacetime scalar curvature $R$. General Relativity, with or without cosmological constant, is the special limit of $f(R)$-gravity when the theory becomes of second-order; that is, when $f$ is a linear function of the Ricci scalar. It belongs to a more general family of theories that take into account curvature terms in order to modify the Einstein-Hilbert action [3–6]. The gravitational field equations of $f(R)$-gravity are dynamically equivalent to that of O’Hanlon theory [7], where a Lagrange multiplier is introduced so as to reduce the order of the theory by increasing the number of degrees of freedom through the introduction of a scalar field $\Phi$. This scalar field is nonminimally coupled to gravity and recovers Brans-Dicke theory [10] with a zero Brans-Dicke parameter. Hence, $f(R)$-gravity is also related to families of Horndeski theories [11], which means that it is free of Ostrogradsky’s instabilities [12, 13]. For a recent discussion on the correspondence among $f(R)$-gravity and other theories through the various frames, together with relevant implications on conservation laws, see [14, 15].

As we mentioned before, the applications of $f(R)$-gravity in gravitational physics cover various subjects. As far as cosmology is concerned, the theory is used both to model the inflationary phase of the universe [16-22] and also as a dark energy candidate to describe the late-time acceleration phase [23-31]. In [32], it was found that new Kasner-like solutions exist, while some cosmological solutions in locally rotational spacetimes were derived in [33, 34]. Some effects on the Mixmaster universe can be found in [35-36]. In general, the various implications of $f(R)$-gravity - and other theories of gravitation as well - from a cosmological perspective can be seen in [37, 38]. Other studies on gravitational collapse can be encountered in [39-40], while static spherically solutions were derived in [41-44] with or without a constant Ricci scalar. Black hole solutions have been also investigated in the literature, for instance see [45-52] and references therein, as also physical phenomena like, binary black hole merge [53], anti-evaporation [54], black hole thermodynamics [55, 56] and many other.

It is well known that black string solutions can be easily constructed in Einstein’s gravity by trivially embedding black hole solutions in higher dimensions. The same is also true for certain classes of modified theories of gravitation (for example it holds for a certain type of Lovelock Lagrangians [57, 58]). In many cases, numerical solutions have been presented in the literature [59-61]. However, exact solutions are always of special interest and there exists an extended bibliography over the subject covering a great number of gravitational configurations: from three dimensional charged black strings [62], to cosmological constant solutions in an arbitrary number of dimensions [63], even in the presence

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of axionic scalar fields [64]. Some exact solutions which describe rotating black strings in $f(R)$-gravity in the presence of an electromagnetic field were derived in [65], while some asymptotic black strings solutions can be found in [66] and a cosmic string solution in four dimensions in the context of scalar tensor theory has been given in [67]. The stability of black string solutions is always an issue, since in general these geometries are unstable, see for example [68–70]. However, counterexamples of this general rule exist and stable solutions may also arise [74, 75]. In what regards other interesting gravitational solutions, a toroidal black hole has also emerged in the Einstein nonlinear sigma model [71].

In this work we start by investigating analytical solutions of power law $f(R)$-gravity in a four-dimensional static spacetime. In this context we derive the general analytical solution and try to see under which conditions interesting gravitational objects may be described by it. We find that a black brane/string solution can be distinguished for certain values of the involved parameters in the case where the metric is axisymmetric. The outline of the paper is as follows: In Section 2 we briefly discuss $f(R)$-gravity and derive the field equations for the spacetime of our consideration. In Section 3 we obtain the general solution for the induced system of equations. Section 4 includes the main results of our analysis which is the black brane/string solution of the field equations for the power-law theory $f(R) = R^k$. In Section 5 we calculate the surface gravity and thus the temperature of the system as well as the entropy on the horizon. Finally, in Section 6 we discuss our results and draw our conclusions.

2. PRELIMINARIES

The action integral of $f(R)$-gravity constitutes a modification of the Einstein-Hilbert action and is given by the following expression

$$S = \int dx^4 \sqrt{-g} f(R),$$

where $R$ is the Ricci scalar constructed from the spacetime metric $g_{\mu\nu}$. It follows from (1) that the field equations of Einstein’s GR are recovered when $f(R)$ is a linear function of $R$.

Variation with respect to the metric tensor gives the field equations

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (\nabla_\mu \nabla_{\nu} - g_{\mu\nu} \nabla^\sigma \nabla^\sigma) f_R = 0,$$

where we have assumed that we are in vacuum, i.e. there does not exist any matter source. The Ricci scalar contains second order derivatives of the coefficients of the metric $g_{\mu\nu}$, hence, relation (2) provides a fourth-order system of differential equations.

An alternative way to write the latter is by the use of Einstein’s tensor together with the definition of an energy-momentum tensor of geometric origin. In particular, we can rewrite (2) as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k_{eff} T^{eff}_{\mu\nu},$$

where $T^{eff}_{\mu\nu}$ is the effective energy momentum tensor that includes the terms which make the theory deviate from General Relativity,

$$T^{eff}_{\mu\nu} = (\nabla_\mu \nabla_{\nu} - g_{\mu\nu} \nabla^\sigma \nabla^\sigma) f_R + \frac{1}{2} (f - R f_R) g_{\mu\nu}$$

while $k_{eff} = (f_R (R))^{-1}$ is a varying gravitational constant. From the latter it follows that the theory is defined in the Jordan frame.

Apart from the limit in which $f_{,RR} (R) = 0$, where General Relativity is recovered, it can be observed that any constant Ricci curvature ($R = R_0$) solution of General Relativity also satisfies field equations (3) if the following algebraic condition relating the free parameters of the $f(R)$ function to $R_0$ [76] holds

$$2 f (R_0) - R_0 f_R (R_0) = 0.$$  

In our work we assume the following static metric with line element

$$ds^2 = -a(r)^2 dt^2 + N(r)^2 dr^2 + b(r)^2 d\phi^2 + c(r)^2 d\zeta^2$$
It is interesting to note that, when $b(r) = c(r)$ and the line element is invariant under rotations in the $\zeta - \phi$ plane, the general solution of a geometry characterized by a constant Ricci scalar $R_0 \neq 0$ is

$$ds^2 = \mp \left( r^2 - \frac{1}{r} \right) dt^2 + \frac{12}{R_0} \left( \frac{1}{r} - r^2 \right)^{-1} dr^2 + r^2 \left( d\phi^2 + d\zeta^2 \right),$$

(7)

which - as we noted earlier - is also a solution of GR in the presence of a cosmological constant. Line element (7) can be also seen to be a special case of a more general solution presented in [77] including also an electromagnetic field. Clearly, the minus branch of (7) characterizes a black brane or string (depending on the topology of the $(\phi, \zeta)$ surface) when $R_0 < 0$. For $f(R)$-gravity the constant scalar curvature is related to the parameters of the relative model through algebraic equation (5).

The existence of a solution like (7) is our motive to start investigating a more general setting given by line element (6). Thus, we begin by considering $b(r) \neq c(r)$ and turn to the general case where $R$ is not a constant. The Ricci scalar is calculated to be

$$R = -\frac{2}{N^2} \left( \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} + \frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} + \right) + 2 \frac{N'}{N^3} \left( \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right).$$

(8)

and the gravitational field equations (2) are

$$R^2 f_{,RRR} + \left[ \left( \frac{b'}{b} + \frac{c'}{c} - \frac{N'}{N} \right) f_{,RR} + f' \right] f_{,RR} - \left( \frac{a''}{a} + \frac{a'b'}{ab} + \frac{a'c'}{ac} - \frac{a'N'}{aN} \right) f_{,R} - \frac{1}{2} N^2 f = 0$$

(9)

$$R' \left( \frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} \right) f_{,RR} - \left( \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} - \frac{a'N'}{aN} - \frac{b'N'}{bN} - \frac{c'N'}{cN} \right) f_{,R} - \frac{1}{2} N^2 f = 0$$

(10)

$$R^2 f_{,RRR} + \left[ \left( \frac{a'}{a} + \frac{c'}{c} - \frac{N'}{N} \right) f_{,RR} + f' \right] f_{,RR} - \left( \frac{b'}{b} + \frac{a'b'}{ab} + \frac{b'c'}{bc} - \frac{b'N'}{bN} \right) f_{,R} - \frac{1}{2} N^2 f = 0$$

(11)

$$R^2 f_{,RRR} + \left[ \left( \frac{a'}{a} + \frac{b'}{b} - \frac{N'}{N} \right) f_{,RR} + f' \right] f_{,RR} - \left( \frac{c'}{c} + \frac{a'c'}{ac} + \frac{b'c'}{bc} - \frac{c'N'}{cN} \right) f_{,R} - \frac{1}{2} N^2 f = 0$$

(12)

An alternative way to derive the field equations (9)-(11), together with (8) for the scalar curvature, is with the use of a point-like Lagrangian in the minisuperspace approach. In particular, the action of the Euler-Lagrange vector over the Lagrangian

$$L = \frac{2}{N} \left[ f_{,R} (ca'b' + ab'c' + ba'c') + f_{,RR} (bca'R' + acb'R' + abc'R') \right] + Nabc (f - Rf_{,R})$$

(13)

with respect to the variables $\{N, a, b, c, R\}$, produces a system of equations equivalent to (8)-(12). It is important to mention that (13) is a singular Lagrangian, due to the presence of the degree of freedom $N$ for which no corresponding velocity appears. In this case $N$ plays a role similar to that of the lapse function in cosmological Lagrangians, the difference being that the evolution of the present system is in the $r$ variable.

### 3. THE ANALYTIC SOLUTION

The method of selecting some special form for the metric so as to determine the function $f(R)$ afterwards by construction is generally problematic and lacks physical information. So, in this work, we prefer to work in the inverse direction by first proving the integrability of the field equations; that is, the existence of an actual solution for the theory, and afterwards to try and derive a closed-form expression for it. In order to prove the integrability of the system we need to determine conservation laws that will help us reduce the order of the dynamical equations (8)-(12). Among the different ways for the determination of conservation laws the symmetry method and the singularity analysis have been extensively applied in $f(R)$-gravity in several cosmological studies [78][81].

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1 The prime “′” denotes a total derivative with respect to the variable $r$, that is $a'(r) = \frac{da(r)}{dr}$. 
As we discussed above a well-known solution admitted by the system (8)-(12) is that of General Relativity with a cosmological constant. However, that is only a particular solution and it is in general unstable, since the stability conditions depend on the form of \( f(R) \). By the term particular solution we mean that there are present fewer integration constants due to the fact that there is a difference on the physical degrees of freedom between General Relativity and \( f(R) \) (with \( f_{RR} \neq 0 \) ) gravity.

We choose to work with the symmetry method and in order to simplify the problem we redefine the lapse function

\[
N = \frac{n}{abc(Rf_R - f)},
\]

which leads to the equivalent Lagrangian that is of the form

\[
L(n, q^a, q'^a) = \frac{1}{n} G_{\alpha\beta} q^{\alpha} q'^{\beta} - n,
\]

where \( q = (a, b, c, R) \) and \( G_{\mu\nu} \) is

\[
G_{\mu\nu} = (2abc(Rf_R - f) f_R) \begin{pmatrix}
0 & c & b & \frac{abc}{f_R} \\
-1 & 0 & a & \frac{ac}{f_R} \\
b & -1 & 0 & \frac{ab}{f_R} \\
-1 & -1 & -1 & 0
\end{pmatrix}.
\]

It has been shown in [82] that linear in the momenta conserved quantities of the original constrained Lagrangian can be constructed by Killing vector fields of \( G_{\alpha\beta} \). The latter is the singular system realization of the Jacobi-Eisenhart metric used in Newtonian mechanics. The idea behind its utilization is to map solutions of a given energy to geodesic flows on Riemannian spaces [83] (for applications and examples in pseudo-Riemannian spaces see [84, 85]).

The mini-superspace is four dimensional and its metric \( G_{\mu\nu} \) is conformally flat. The simple transformation \( a = e^{(3b+4)c \sqrt{(1-2b)y+(b+1)^2} \frac{x}{2k-6}}, b = e^{\frac{4}{2k-6}[(1-2b)y+(b+1)^2]}, c = e^{\frac{2k+4}{2k-6}[(1-2b)y+(b+1)^2]}, R = e^{\frac{y}{4k-10b+4}} \),

leads \( G_{\mu\nu} \), as given by (16), to become

\[
G_{\mu\nu} = (k-1)k e^{wz^2}\text{diag}\{-1, \frac{1-k}{2k-1}, \frac{k^2-1}{2(2k-2)}, -\frac{3(k-1)^2(k+1)}{2(k-2)^2(2k-1)}\}.
\]

The minisuperspace admits six Killing fields, which in these coordinates are

\[
\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z - \partial_w, \quad \xi_4 = y\partial_x + \frac{(1-2k)x}{k-1} \partial_y, \quad \xi_5 = \frac{3}{2(2k-2)} \left( \frac{2k^2-3k+1}{2k-1} w + z\partial_x + \frac{(1-2k)^2x}{k^2-1} \partial_z - \frac{(1-2k)^2x}{k^2-1} \partial_w, \right.
\]

\[
\xi_6 = \frac{-(2k-1) \left( (6k^2-9k+3) w + 2(k-2)^2 z \right)}{2(k-2)^2} \partial_y - \frac{(1-2k)^2y}{k+1} \partial_z + \frac{(1-2k)^2y}{k+1} \partial_w.
\]

As is well known, they can be used to construct the conserved quantities of the form \( Q_i = \xi^\alpha \partial_L \frac{\partial L}{\partial q^\alpha} \). With the help of first order relations like \( \xi_1 = \text{const} \) it is easy to derive the general solution for a generic \( k \neq 1 \). Of course, as we see from (18), the cases \( k = 1/2, k = 5/4 \) and \( k = 2 \) have to be treated separately. The general solutions of these particular cases are given later in the appendix. By having excluded the aforementioned values, the final result in the original coordinates (after absorbing some unnecessary constants of integration reads):

\[
a(r) = e^{(\alpha+\gamma)r} (\cosh r) \frac{k+1}{k-1} \]

\[
N(r) = Ce^{\frac{2k+2\gamma}{k-1}r} (\cosh r) \frac{3(k+1)}{2(k+2)}(\frac{k+2}{k+1})
\]

\[
b(r) = e^{3\gamma r} (\cosh r) \frac{k-1}{k}
\]

\[
c(r) = e^{(\alpha-\gamma)r} (\cosh r) \frac{k+1}{k-1}
\]
with $C, \alpha, \beta$ and $\gamma$ being the remaining constants of integration. Of the latter three only two are actually independent, say $\alpha$ and $\gamma$, then it can be seen that \(20a-20d\) is a solution under the condition

$$
\beta = -\frac{|k - 2|}{2(k-2)(k-1)} \sqrt{\frac{\alpha^2(2k-1)(5-4k) + 2(2k-1)\gamma^2(8k^2-14k+5) - 3(2k-1)^2}{5-4k}} + \alpha(k-2)(2k-3)
$$

(21)

A first important remark that we can make by studying \(20a-20d\) is that, a typical uniform black string solution (i.e. one with $c(r) = \text{const.}$) is not possible in this setting. For the latter to happen we should require $k = 1$, which corresponds to GR and it is a value excluded from this analysis. The scalar curvature that corresponds to this solution is

$$
R = \frac{6(k-1)ke^{-2\beta(2k-1)r}}{C^2(2k-1)(4k-5)} [\cosh(r)]^{\frac{2(2-k)}{k-1}(4k-5)}.
$$

(22)

At this point we can formulate the line element in such a way so that we can identify $b(r)$ as a radial type of variable. If we, for example, choose the parameter $\beta$ to be

$$
\beta = \pm \frac{k-1}{5-4k},
$$

(23)

then $b(r)$ can become a radial variable $\tilde{r}$ in the following manner: By performing the transformation

$$
\tilde{r} = \frac{1}{2} \ln \left( \frac{\tilde{r}^{\frac{5-4k}{1-\gamma}} - 1}{\tilde{r}^{\frac{5-4k}{1-\gamma}} + 1} \right),
$$

(24)

we get $b(\tilde{r}) = 2^{\frac{1}{1-\gamma}} \tilde{r}$, with the constant value $2^{\frac{1}{1-\gamma}}$ being irrelevant since it can be absorbed in the metric with a scaling transformation. Under a transformation like \(23\), the quantities $a, b, c$ and $Ndr$ transform as scalars. After a few reparametrizations and scalings over all of the variables, we can re-write the ensuing line-element as (for simplicity instead of $\tilde{r}$ we write $r$ understanding that it is a different variable than the one used in expressions \(20a-20d\)):

$$
\begin{align*}
ds^2 &= -r^2 \left( \frac{\tilde{r}^{\frac{5-4k}{1-\gamma}} - 1}{m} \right)^{\frac{1}{1-\gamma}} \pm (\alpha + \gamma) dt^2 + r \left( \frac{\tilde{r}^{\frac{5-4k}{1-\gamma}} - 1}{m} \right)^{\frac{1}{1-\gamma}} \left( \frac{r^{4k-5}}{m} - 1 \right) ^{\frac{2k-3}{(2k-1)(4k-5)} + 2k} dr^2 \\
&\quad + r^2 d\phi^2 + r^2 \left( \frac{\tilde{r}^{\frac{5-4k}{1-\gamma}} - 1}{m} \right)^{\frac{1}{1-\gamma}} \pm (\alpha - \gamma) d\zeta^2,
\end{align*}
$$

(25)

where the constant $m$ appearing in \(25\) is associated with the initial $C$ that appears in \(20b\). Whenever a double sign appears, the upper corresponds to a solution for $k > 2$, while the lower for $k < 2$. Of course, we always have to remember that not both of $\alpha$ and $\gamma$ are free parameters, but bound through the condition $\beta = \pm \frac{k-1}{5-4k}$, with $\beta$ being given by \(21\). The properties of the resulting spacetime are related to the type of theory that we study and which is characterized by the number $k$. The scalar curvature for the metric at hand is

$$
R = \frac{6k(4k-5)}{(k-1)(2k-1)} \left( \frac{\tilde{r}^{\frac{5-4k}{1-\gamma}} + 1}{m} \right)^{\frac{2k-3}{(2k-1)(4k-5)} + 2k}.
$$

(26)

which indicates that, depending on the value of $k$, we can have curvature singularities at $r = 0, r = m^{\frac{1}{1-\gamma}}$ and at $r \to +\infty$.

As we can see from \(20\), the only way to ensure that the term $\tilde{r}^{\frac{5-4k}{1-\gamma}} - 1$ does not appear in the scalar curvature - so that it does not lead to its divergence either at $r = m^{\frac{1}{1-\gamma}}$ or at infinity - is by setting

$$
\alpha = \pm \frac{3 - 2k}{2(5-4k)}.
$$

(27)

This value inevitably results, through \(21\), in $\gamma = \pm \frac{1}{2}$ that in its turn implies $b(r) = c(r)$. Thus, we are led in this way to start considering the case where the metric admits a toroidal type of symmetry.
4. THE BLACK BRANE SOLUTION

We proceed by considering the aforementioned special case where \( b(r) = c(r) \). This is of special interest since, as we are about to see, it produces a black brane/string solution. The latter depends on how you interpret the \( \phi, \zeta \) variables and the type of topology with which you endow the two-surface that they construct. We can either consider a topology \( S^1 \times \mathbb{R} \) (cylinder), \( S^1 \times S^1 \) (torus) or \( \mathbb{R} \times \mathbb{R} \) (plane) which imply that one, both or none (respectively) of \( \phi \) and \( \zeta \) are bounded and periodic variables.

Two procedures can be followed to extract the solution: we can either start from the beginning, following a similar procedure like that of a previous section and consider the resulting mini-superspace using ansatz

\[
ds^2 = -a(r)^2 dt^2 + N(r)^2 dr^2 + b(r)^2 \left( d\phi^2 + d\zeta^2 \right) \tag{28}\]

for the line element, or simply arrange the parameters in the general solution (20a)-(20d) so that \( b(r) = c(r) \). Obviously, the latter happens when \( \gamma = \alpha - \beta \), which means that the resulting solution depends on the parameters \( C, \alpha \) and \( \beta \), with the latter two related through (21), when \( \gamma = \alpha - \beta \) has been substituted in it. Under these conditions it is easy to see that the general solution for the \( b(r) = c(r) \) case is

\[
a(r) = e^{(2\alpha - \beta) r} (\cosh r)^{\frac{5-k}{4}} \tag{29a}
\]

\[
N(r) = C e^{\frac{\alpha + \beta}{2} r} (\cosh r)^{\frac{(4k-7)(k-1)}{16k-4}} \tag{29b}
\]

\[
b(r) = e^{\beta r} (\cosh r)^{\frac{5-k}{4}} \tag{29c}
\]

\[
c(r) = e^{\beta r} (\cosh r)^{\frac{5-k}{4}} \tag{29d}
\]

with \( \alpha \) and \( \beta \) bound to satisfy the algebraic relation

\[
40\alpha^2 + 8\alpha^2 k(4k-9) + 4\alpha \beta (5 - 4k) + \beta^2 (4k - 5)(4k - 3) - 3(k - 2)k - 3 = 0. \tag{30}\]

Once more we can choose to study a particular solution belonging in this configuration. As in the previous section, we choose the parameter \( \beta \) to be given by (23). A value that helps us, with simple transformation like (24), to associate \( b(r) \) to a radial distance, only this time we additionally have \( c(r) = b(r) \). With the choice (23), the constraint (30) implies that

\[
\alpha = \pm \frac{1 - k}{4k - 5} \quad \text{or} \quad \alpha = \pm \frac{3 - 2k}{2(5 - 4k)}. \tag{31}\]

The first set of values leads to a solution of the form

\[
ds^2 = -r^2 dt^2 + \frac{2 \left( -2k^2 + 2k + 1 \right)}{k(k-1)(2k-1)} \left( 1 - \frac{r^{\frac{5-4k}{m}}}{m} \right)^{\frac{2k-4}{2k}} \left( \frac{r^{\frac{5-4k}{m}}}{m} - 1 \right)^{-1} dr^2 + r^2 \left( d\phi^2 + d\zeta^2 \right), \tag{32}\]

which describes a spacetime that, depending on \( k \), can have its singularity points at the origin \( r = 0 \), at infinity and at \( r = \frac{1}{\sqrt{1-k}} \). It is interesting to note that just for the value \( k = \frac{5}{4} \) the corresponding geometry has a curvature singularity only at \( r = 0 \), while \( r = \frac{1}{\sqrt{1-k}} \) is just a coordinate singularity. In all the other cases where \( r = \frac{1}{\sqrt{1-k}} \) is just a coordinate singularity, a curvature singularity at \( r \to \infty \) necessarily occurs.

The most interesting situation appears when \( \alpha = \pm \frac{3 - 2k}{2(5 - 4k)} \). With this set of values, we can write the two following line elements depending on the value of \( k \):

1. For \( 1 < k < \frac{5}{4} \)

\[
ds^2 = -r^2 \left( \frac{r^{\frac{5-4k}{m}}}{m} - 1 \right) dt^2 + \frac{r^{-4k^2 + 4k + 2}}{k(k-1)(2k-1)} \left( \frac{r^{\frac{5-4k}{m}}}{m} - 1 \right)^{-1} dr^2 + r^2 \left( d\phi^2 + d\zeta^2 \right). \tag{33}\]

\[\text{2 Again, we understand that the } r \text{ variable appearing in the following line elements is different from the one in } (28).\]

2. For $k < 1$ ($k \neq 1/2$) or $k > \frac{5}{4}$

$$ds^2 = -r^2 \left( 1 - \frac{\frac{1}{k} - 4k}{m} \right) dt^2 + r \sqrt{\frac{\frac{1}{k} + 4k + 2}{m}} \left( 1 - \frac{r^{\frac{1}{k} - 4k}}{m} \right)^{-1} dr^2 + r^2 \left( d\phi^2 + d\zeta^2 \right). \tag{34}$$

We make this distinction so that the sign of $g_{tt}$ (the coefficient of $dt^2$ in the line element) is negative in each case when $r > \frac{1}{k}$. The two line elements are associated with constant, complex scalings among the coordinates and the essential constant $m$ that appears in them. Both of them are of course solutions to the field equations [9]-[12] for an $f(R) = R^k$ theory.

The scalar curvature reads

$$R = \pm \frac{6k(4k^2 - 5)r^{2(1-k)}k^{2k-3k+1}}{(k-1)(2k-1)} \tag{35}$$

with the plus corresponding to the first line element, while the minus to the second. In both situations the Kretschmann scalar $K = R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}$ is

$$K = \frac{4}{(k-2)^2(k-1)^2m^2} \left[ (56k^4 - 264k^3 + 468k^2 - 370k + 111)r^{-2(8k^2 - 16k + 9)} \left( \frac{8(k^2 - 6k + 3)}{(2k-1)(k-1)} - 2m(k-2)^2(2k-1)r^{(8k^2 + 4k + 13)} + 3m^2(8k^4 - 20k^3 + 13k^2 - 2k + 2) \right) \right]. \tag{36}$$

Thus, it can be seen that for the range of values $k < 1/2$ and $1 < k \leq 2$ the space-time has a curvature singularity at $r = 0$, while at the same time exhibits a coordinate singularity at $r = r_h = m^{\frac{1-k}{k-1}}$ (as long as $m > 0$). We have to note here that, although we had initially excluded $k = 2$ from the general solution, its consideration separately leads to the same resulting spacetimes [33] and [34], so we can include it at this point as an admissible value for $k$.

Line element [33] is appropriate for describing a (compactified) black brane or string in vacuum $f(R) = R^k$ gravity for those theories for which $1 < k < \frac{5}{4}$. On the other hand, the spacetime characterized by [34] can be used for the same purpose when $k < \frac{1}{2}$ and $\frac{5}{4} < k < 2$. In both of these cases the surface $r = r_h = m^{\frac{1-k}{k-1}}$ acts as an horizon “hiding” the singularity. Particularly for the case of the toroidal topology we have to mention the work of Vanzo [60] who was the first to consider such topological black holes. The resulting spacetime is asymptotically Ricci flat, however the curvature $R_{\kappa\lambda\mu\nu}$ does not tend to zero as $r \to \infty$. Thus, this geometry can not be smoothly deformed to any maximally symmetric space-time.

For the rest of values, that is $\frac{1}{2} < k < 1$ and $k > 2$, the curvature singularity goes form the origin to infinity. Hence, we can state that, in the theories corresponding to those values, such an object does not occur. We may now proceed with studying the basic thermodynamic quantities for the string and the compactified brane that appears in this context.

5. THERMODYNAMIC PROPERTIES

We continue by calculating basic thermodynamic quantities like the temperature and the entropy for line elements [33] and [34] for those values of $k$ for which they describe a black object. That is $1 < k < 5/4$ for the first case and $k \in (-\infty, \frac{1}{2}) \cup (\frac{5}{4}, 2]$ for the second. In both situations, we can perform a transformation $t \mapsto \tau$ of the form

$$t = \tau \pm \int \sqrt{-g_{rr}/g_{tt}} \, dr \tag{37}$$

where $g_{rr}$ corresponds to the radial variable of the metric and thus, express the latter in an ingoing Eddington - Finkelstein type of coordinate system. The minus in transformation [37] corresponds to line element [33], while the plus is for [34]. As a result, we obtain

$$ds^2 = \pm r^2 \left( 1 - \frac{\frac{1}{k} + 4k}{m} \right) dt^2 + r \sqrt{\frac{\frac{1}{k} - 4k}{m}} \left( 1 - \frac{r^{\frac{1}{k} + 4k}}{m} \right)^{-1} dt dr + r^2 \left( d\phi^2 + d\zeta^2 \right) \tag{38}$$

with the + referring to [33] and the $1 < k < \frac{5}{4}$ case, while the minus is for [34] and the rest of the values of $k$ as discussed above.
In this coordinate system we may calculate the surface gravity $\kappa$ for the given spacetimes, either from relation
\[ \kappa = \Gamma_{\mu}^i \frac{\partial}{\partial t} |_{r=r_h}, \]
where $\Gamma_{\mu}^i$ stands for the Christoffel symbols, or more formally by
\[
\nabla_{\mu}(X^\nu X_{\nu}) |_{r=r_h} = -2\kappa X_{\mu} |_{r=r_h},
\]
where $X^\mu = \frac{\partial}{\partial t}$ is the time-like Killing vector field for which a Killing horizon exists at $r = r_h$. The resulting surface gravity is given in terms of the essential constant $m$ as
\[
\kappa = \begin{cases} \frac{(4k-5)}{2(k-1)} m \frac{2k^2+2k+1}{(2k^2-1)(4k-5)}, & 1 < k < \frac{5}{4} \\ \frac{(4k-5)}{2(k-1)} m \frac{2k^2+2k+1}{(2k^2-1)(4k-5)}, & k \in (-\infty, \frac{1}{2}) \cup (\frac{5}{4}, 2] \end{cases}
\]
and allows us to easily deduce the Hawking temperature through the relation $T = \frac{\kappa}{2\pi}$. We know that the horizon entropy in $f(R)$ gravity is given in terms of the first derivative of $f$ with respect to $R$ as calculated on the horizon, i.e.
\[ S \sim \sigma f_R |_{r=r_h}, \]
where $\sigma$ is the area of the horizon. In the case that we are dealing with a string, we can use instead of $\sigma$ the area per unit length $\tilde{\sigma} = 2\pi r_h^2 = 2\pi m^{-\frac{2(4k-1)}{4k-3}}$ and talk about an entropy density $\tilde{S}$, which we may calculate to obtain
\[ \tilde{S} \sim \tilde{\sigma} f_R |_{r=r_h} \sim \begin{cases} \kappa \left( \frac{k(4k-5)}{(k-1)(2k-1)} \right)^{k-1} m^{-\frac{6(k-4)}{(2k^2-1)(4k-5)}}, & 1 < k < \frac{5}{4} \\ \kappa \left( \frac{5-4k}{2k^2-3k+1} \right)^{k-1} m^{-\frac{6(k-4)}{(2k^2-1)(4k-5)}}, & k \in (-\infty, \frac{1}{2}) \cup (\frac{5}{4}, 2] \end{cases} \]

From the form of $\tilde{S}$ we can see that the latter is positive only if $0 < k < \frac{1}{2}$ or when $k$ is a negative even integer. Thus, we can distinguish a class of $f(R) = R^k$ theories - corresponding to these range of values for $k$ - in which only solution (33) results in a well defined and positive entropy.

In the case where we consider a toroidal topology, so that the solution represents a compactified black brane, none of the previous basic arguments changes, only now we need to use, instead of $\tilde{\sigma}$, the area $\sigma = 4\pi r_h^2$, which implies that the resulting entropy $S$ is different from $\tilde{S}$ up to a positive multiplicative constant.

Of course, for a complete thermodynamic description one should also take into account the mass/energy of the system. However, this would be a highly non-trivial calculation, since the space-time under consideration is not asymptotically maximally symmetric.

6. DISCUSSION

The main scope of this work is to find closed-form solutions for a four dimensional static spacetimes in $f(R)$-gravity. We derived the general solution for power-law $f(R)$-gravity with $f(R) = R^k$. In contrast to General Relativity where black string solutions are allowed only in the presence of the cosmological constant or matter, we saw that in $f(R)$-gravity, specifically in the $f(R) = R^k$ theory, black string solutions are supported by the field equations for some ranges of the power $k$ even in vacuum.

It is important to mention that solutions (33) and (34) reduce to (7) with constant non-zero Ricci scalar in the specific case when $k = 2$. In the generic case of course, when $k \neq 2$, we can see from (33) that (33) in its full generality is not a constant Ricci scalar solution. However, (33) and (34) are still only a special cases resulting from the most general expressions (29a) - (29b) with an appropriate fixing of the integration constants.

We now continue with the discussion of the effective gravitational potential for the spacetimes (33) and (34). The Hamiltonian of the geodesic equations for the corresponding line element is
\[
\pm r^2 \left( \frac{r - 4k}{m} - 1 \right) \left( \frac{dr}{ds} \right)^2 \pm r^2 \left( \frac{r - 4k^2 + 4k + 2}{r - 4k - 2} \right) \left( \frac{dr}{ds} \right)^2 + r^2 \left( \left( \frac{d\phi}{ds} \right)^2 + \left( \frac{d\zeta}{ds} \right)^2 \right) = -\mu^2
\]
where $\mu^2 = 1$ for a timelike particle and $\mu^2 = 0$, for photons. Also, wherever we have a double sign, the upper one corresponds to (33), while the lower to (34).

The static axisymmetric spacetime (6) in general admits a four dimensional Killing vector $\partial_t$ and the $E^2$ Lie algebra in the plane $\phi - \zeta$. The corresponding conservation laws are
\[
\pm r^2 \left( \frac{r - 4k}{m} - 1 \right) \left( \frac{dr}{ds} \right) = E_0, \quad r^2 \left( \frac{d\phi}{ds} \right) = I_\phi, \quad r^2 \left( \frac{d\zeta}{ds} \right) = I_\zeta,
\]
and
\[ r^2 \left( \zeta \frac{d\phi}{ds} - \phi \frac{d\zeta}{ds} \right) = I_\phi \zeta. \] (45)

For simplicity let us consider from now on \( m = 1 \), so that the coordinate singularity rests at the radius \( r = 1 \). With the use of the conservation laws the Hamiltonian (43) becomes
\[ \left( \frac{dr}{ds} \right)^2 = r^{-2 \frac{4k^2 - 3k + 2}{(k - 1)(2k - 1)}} \left[ (E_0^2 \pm I_0^2 \pm \mu^2 r^2) r^{\frac{4k}{k - 1}} + r^{\frac{3k}{k - 1}} (I_0^2 + \mu^2 r^2) \right] \] (46)
where \( I_0^2 = I_\phi^2 + I_\zeta^2 \). The latter expression corresponds to the Hamiltonian of classical particle of energy \( E \) with effective potential
\[ V_{\text{eff}}(r) = - \left( r^{-2 \frac{4k^2 - 3k + 2}{(k - 1)(2k - 1)}} \left[ (E_0^2 \pm I_0^2 \pm \mu^2 r^2) r^{\frac{4k}{k - 1}} + (I_0^2 + \mu^2 r^2) r^{\frac{3k}{k - 1}} \right] + E \right). \] (47)

Once more the upper signs correspond to the \( 1 < k < \frac{2}{3} \) case, while the lower are for \( k \in (-\infty, \frac{1}{2}) \cup (\frac{2}{3}, 2] \). An important observation is that at the coordinate singularity, \( r = 1 \), the effective potential becomes
\[ V_{\text{eff}}(r \to 1) = - (E_0^2 + E). \]

From (47) we can define four different powers which lead the behaviour of \( V_{\text{eff}}(r) \). Those are given by
\[ P_1 = \frac{2(k - 2)}{(k - 1)(2k - 1)} , \quad P_2 = - \frac{8k^2 - 16k + 9}{(k - 1)(2k - 1)}, \]
\[ P_3 = - \frac{4k^2 - 10k + 7}{(k - 1)(2k - 1)} \quad \text{and} \quad P_4 = \frac{2(2k^2 - 2k - 1)}{(k - 1)(2k - 1)}. \]
In Fig. 1 the evolution of the powers \( P_1 - 4 \) is presented where it can be seen for which values of \( k \) the effective potential \( V_{\text{eff}} \) becomes a constant far from the singularity. Furthermore, the qualitative evolution of the effective potential \( V_{\text{eff}} \) outside the horizon can be seen, for three values of \( k \), in Figs. 2 and 3.

Before we conclude, it is important to mention that another special solution can be recovered from the general expressions (29a)-(29b). It is a power-law solution which is similar to that of the cosmological Bianchi type I axisymmetric spacetime in \( f(R) = R^k \), that leads to a Kasner-like universe [32]. Therefore, in a similar way more general solutions may be constructed also in a cosmological context.

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Appendix A: Solutions when \( k = 2, k = 5/4 \) and \( k = 1/2 \)

For the sake of completeness we just state here the solutions that emerge for the cases which we needed to exclude throughout the initial calculations, that is \( k = 2, k = 5/4 \) and \( k = 1/2 \). We state those solutions for the general anisotropic line element (6). The case \( c(r) = b(r) \) can be recovered with a simple fixing of parameters. By working in a similar manner as in the generic \( k \) case, we obtain the following results:

- When \( k = 2 \) the spacetime is characterized again by the functions (20a)-(20c), only now the algebraic condition among the constants is not (21) but rather
\[ \beta = \frac{1}{2} \left( \pm \sqrt{-9\alpha^2 - 6\gamma^2 + 2} - \alpha \right). \] (A1)

The \( b(r) = c(r) \) case can be recovered in the same manner as for the generic \( k \) solution, by setting \( \gamma = \alpha - \beta \) in the corresponding expressions.
FIG. 1: Evolution of the powers $P_1$ to $P_4$ which lead the behavior of the effective potential (47). When all the powers are negative the effective potential far from the singularity becomes constant.

FIG. 2: Qualitative evolution of the effective potential (47) for timelike test particle, $\mu^2 = 1$, $k = -0.5$ (Left Fig.) and $k = 1.1$ (Right Fig.). Solid line is for $I_0 = 1$, dashed line is for $I_0 = 2.5$ and dot-dot line is for $I_0 = 3$. For simplicity we considered $E = 0$.

- For $k = 5/4$ the line element that satisfies the field equations reads

$$ds^2 = -e^{\sigma + \eta} dt^2 + e^{\rho + \epsilon} dr^2 + e^\frac{1}{2}(\beta r + \epsilon) d\phi^2 + e^\frac{1}{2}(\nu - r^2(2\alpha + \beta - 3\nu + 1)) d\zeta^2$$

(A2)

together with the following algebraic condition among the constants appearing in it

$$4\alpha^2 + 2\alpha(\beta - 3\nu + 1) + \beta^2 - 3\beta\nu + \beta + 3\nu^2 - 4\nu + 1 = 0.$$  

(A3)
FIG. 3: Qualitative evolution of the effective potential \( V(47) \) for timelike test particle, \( \mu^2 = 1, \ k = -0.5 \) (Left Fig.) and \( k = 1.1 \) (Right Fig.). Solid line is for \( I_0^2 = 1 \), dashed line is for \( I_0^2 = 2.5 \) and dot-dot line is for \( I_0^2 = 3 \). For simplicity we considered \( \mathcal{E} = 0 \).

Thus, the geometry has two essential constants. The case \( b(r) = c(r) \) is recovered when \( \alpha = \frac{1}{2}(-2\beta + 3\nu - 1) \). One can easily express this solution in a gauge where the coefficient \( b(r) \) of \( d\phi^2 \) can be considered as a radial distance by performing a transformation \( r \to \tilde{r} \) that involves the Lambert \( W \) function,

\[
r = \frac{4 \ln(\tilde{r}) - W(\frac{\tilde{r}^{2/3}}{\beta})}{\beta},
\]

On the other hand, for \( k = 1/2 \) we get the solution

\[
ds^2 = -e^{\alpha r} dt^2 + e^{\nu r} dr^2 + e^{\beta \phi^2} d\phi^2 + e^{\gamma r} d\zeta^2,
\]

where the constants appearing in the solution need to satisfy

\[
\alpha^2 + (\gamma - \nu)(\alpha + \beta + \gamma) + \alpha\beta + \beta^2 = 0.
\]

Clearly, the \( b(r) = c(r) \) solution corresponds to taking \( \gamma = \beta \). By performing the transformation \( r = \frac{\ln(\tilde{r})}{\beta} \), which corresponds to \( b(\tilde{r}) = \tilde{r} \), it can be seen that this spacetime may exhibit a singularity at the origin \( \tilde{r} = 0 \) or at infinity, depending on the values of the free parameters.

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