Another approach to Brownian motion

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Abstract

Motivated by the central limit theorem for weakly dependent variables, we show that the Brownian motion \( \{X(t); t \in [0,1]\} \), can be modeled as a process with independent increments, satisfying the following limiting condition

\[
\liminf_{h \downarrow 0} \mathbb{E} f(h^{-1/2}[X(s+h) - X(s)]) \geq \mathbb{E} f(X(1))
\]

almost surely for all \( 0 \leq s < 1 \), where \( \mathbb{E} f(X(1)) < \infty \) and \( f : \mathbb{R} \to \mathbb{R} \) is a symmetric, continuous, convex function with \( f(0) = 0 \), strictly increasing on \( \mathbb{R}^+ \) and satisfying the following growth condition:

\[
f(Kx) \leq K^p f(x), \text{ for a certain } p \in [1,2), \text{ all } K \geq K_0 \text{ and all } x > 0
\]

(for example, \( f(x) = x^p[A + B \ln(1 + Cx)] \), with \( x > 0 \), \( p \in [1,2), A > 0 \) and \( B, C \geq 0 \)).

Key words: Levy process, Brownian motion, processes with independent increments, central limit theorem, weakly dependent sequences.

1 Introduction

For the partial sums \( S_n = Y_1 + \cdots + Y_n \) of a centered stationary strongly mixing sequence \( \{Y_i\} \) with finite second moment, the well-known sufficient conditions for the central limit theorem are that \( \text{Var}(S_n)/n \) is slowly varying as \( n \to \infty \) and the sequence \( \{S_n^2/\sigma_n^2\} \) is uniformly integrable (\( \sigma_n^2 = \text{Var}(S_n) \)). The conditions are checkable under various mixing conditions and they lead to the central limit theorem under the normalization \( \sigma_n \) (see, Denker (1986) and Peligrad (1986) for a survey).

Dehling, Denker and Philipp (1986) proved an interesting central limit theorem using the non-traditional normalization \( \rho_n = \mathbb{E}|S_n| \). One of their results, Theorem 3, roughly states that if both sequences \( \sigma_n^2/n \) and \( \rho_n/\sqrt{n} \) are slowly varying as \( n \to \infty \), then the central limit theorem holds.

On the other hand, Braverman, Mallows and Shepp (1995) showed that if the absolute moments of partial sums of i.i.d. symmetric variables are equal to those of normal variables, then the marginals have normal distribution. This fact suggested the conjecture that probably the absolute moments alone characterize the homogeneous process with independent increments (see Bryc (2002) for a discussion on this topic and related conjectures).

Our main interest in this topic is to prove some of these conjectures and to apply them to understand the nature of the intricate normalization in Dehling, Denker and Philipp (1986).

¹ Mathematical Subject Classification (2000): 60G51, 60F05

Supported in part by a Charles Phelps Taft Memorial Fund grant and a NSA grant.
Throughout the paper, \( \{W(t); t \in [0, 1]\} \) denotes the standard Brownian motion, i.e. a Gaussian process \( \{W(t); t \in [0, 1]\} \) with independent increments, \( \mathbb{E}[W(t)] = 0 \) and \( \mathbb{E}[W(t)W(s)] = \min(t, s) \). By \( W \) we denote a standard normal variable. Also \( \mu \) denotes the Lebesgue measure, \( h \downarrow 0 \) denotes convergence over positive real numbers, \( [x] \) denotes the integer part of \( x \). For two processes with independent increments \( \{X(t); [0, 1]\} \) and \( \{Y(t); t \in [0, 1]\} \), equality \( X(t) = Y(t) \) means that their increments have the same distribution.

The process \( \{X(t); t \in [0, 1]\} \), is called homogeneous if \( X(t+s) - X(t) = X(s) \) where \( = \) means equality in distribution. Finally, the process \( \{X(t); t \in [0, 1]\} \) is called stochastically continuous if it does not have deterministic jumps, i.e. \( \mathbb{P}(|X(t+s) - X(t)| > u) \to 0 \) as \( s \to 0 \) for any \( u > 0 \) and \( t \in [0, 1] \).

Our paper is organized as follows. In Section 2 we include the representation results and their corollaries. Section 3 is dedicated to their proofs. In section 4 we give an application of the characterization results to the central limit theorem.

2 Characterization Results

As a class of potential characterizing functions, we consider non-negative functions satisfying the following conditions:

The function \( f: \mathbb{R} \to \mathbb{R} \) is symmetric, continuous, convex, strictly increasing on \( \mathbb{R}^+ \), \( f(0) = 0 \), and there exists \( p \in [1, 2) \) and \( K_0 \geq 0 \) such that \( f(Kx) \leq K^p f(x) \) for all \( K \geq K_0 \). (1)

For example, \( f(x) = x^p \), or more generally, \( f(x) = x^p[A + B \ln(1 + Cx)] \), \( x > 0 \), for a \( p \in [1, 2) \), \( A > 0 \) and \( B, C \geq 0 \) satisfies (1) for some \( p, p < p' < 2 \).

The following theorem is the main result of the paper.

**Theorem 1.** Let \( f \) be a positive function satisfying (1) and let \( \{X(t); t \in [0, 1]\} \) be a process with independent increments, \( X(0) = 0 \), and \( \mathbb{E}f(X(1)) < \infty \). Assume in addition that \( \mu \) - almost surely for \( s \in [0, 1] \),

\[
\liminf_{h \downarrow 0} \mathbb{E} f(h^{-1/2}[X(s+h) - X(s)]) \geq \mathbb{E} f(X(1)) .
\]

Then, \( \{X(t); t \in [0, 1]\} \) is a Gaussian process that admits the representation

\[
X(t) = \sigma W(t) + \mathbb{E} X(t) \quad \text{for all } t \in [0, 1].
\]

(3)

where \( \mathbb{E} X(1) = 0 \), \( \sigma = \Psi^{-1}(\mathbb{E} f(X(1))) \) and the function \( \Psi(x) = \mathbb{E} f(xW) \) is continuous and strictly increasing for \( x > 0 \).

**Corollary 2.** Let \( f \) satisfies (1) and let \( \{X(t); t \in [0, 1]\} \) be a stochastic process with independent increments, \( X(0) = 0 \), satisfying the following condition:

\[
\mathbb{E} f(t^{-1/2}[X(t+s) - X(s)]) = \mathbb{E} f(W) \quad \text{for all } 0 \leq s \leq s + t \leq 1 .
\]

Then, \( \{X(t); t \in [0, 1]\} \) is a standard Brownian motion.

By taking \( f(x) = x \), the corollary gives an affirmative answer to a conjecture of Bryc and Peligrad formulated in a survey paper by Bryc (2002).

We notice that we do not impose any conditions on the sample path properties of the stochastic process \( \{X(t); t \in [0, 1]\} \). In particular, a Gaussian process satisfying (3) does not have to be a semimartingale (see for example, Jacod and Shiryaev, p.106).

For a stochastically continuous homogeneous processes, it is enough to check the limiting condition in (2) only on one subsequence, which is useful in applications.
Corollary 3. Suppose that \( \{X(t); t \geq 0\} \) is a stochastically continuous homogeneous process, with independent increments, \( X(0) = 0 \) (i.e. Levy process), \( \mathbb{E}f(X(1)) < \infty \), and assume there exists a positive sequence \( t_n \to 0 \) such that

\[
\lim \inf_{t_n \to 0} \mathbb{E}f(t_n^{-1/2}[X(t_n)]) \geq \mathbb{E}f(X(1)).
\]

(4)

Then, \( X(t) = \sigma W(t) \) for all \( t \in [0,1] \) where \( \sigma \) is defined as in Theorem 2.

In the following proposition, we show that without the stochastic continuity assumption in Corollary 3 the result is not true in general.

Proposition 4. There exists a non-Gaussian homogeneous stochastic process \( \{X(t); t \geq 0\} \) with independent increments, with \( X(0) = 0 \), such that \( 3 \) is satisfied with some positive sequence \( t_n \to 0 \).

We notice that the restriction \( p < 2 \) in 1 in Theorem 1 is necessary, in general. For example, if \( f(x) = x^p \) with \( p \geq 2 \) or more generally \( f(x) \) is a bounded twice continuously differentiable function with \( f(0) = f'(0) = 0 \), then 2 is a condition only on the variance of the increments \( X(t+u) - X(t) \) and does not imply 3.

3 Proofs

The proof is divided in a few separate lemmas, some of them are of the independent interest.

In the first lemma, we state some properties of the function \( f(x) \) satisfying Condition 1.

Lemma 5. Suppose that the function \( f \) satisfies Condition 1. Then,

(a) There exists a positive \( \alpha > 0 \) such that for all \( t \geq 2 \) and \( x > 0 \), \( f(tx) \leq t^\alpha f(x) \).

(b) In addition, there exists a positive constant \( C \) such that for all \( x, y \geq 0 \) and \( z > 1 \),

\[
f(x + y) \leq C[f(x) + f(y)], \quad f(x) \leq C(x + x^2) \quad \text{and} \quad f(z) \geq z/C.
\]

Proof. To prove the statement (a), we assume without loss of generality that \( K_0 > 2 \) in Condition 1. Then, for \( t > K_0 \), we know that \( f(tx) \leq t^\alpha f(x) \). For \( 2 \leq t \leq K_0 \),

\[
f(tx) = f(K_0(tx/K_0)) \leq K_0^p f(tx/K_0) \leq K_0^p f(x) = 2^{p \log_2(K_0)} f(x)
\]

which proves (a) with \( \alpha = p \log_2(K_0) \).

First inequality in part (b) is a simple consequence of (a) since

\[
f(x + y) \leq \frac{1}{2}(f(2x) + f(2y)) \leq 2^{\alpha-1}(f(x) + f(y))
\]

The other two assertions are simple consequences of Condition 1.

In the next lemma, we analyze some properties of expectations associated to the function \( f(x) \) satisfying condition 1.

Lemma 6. Let \( f \) satisfies Condition 1 and let \( W \) be a standard normal random variable.

(a) Let \( G(y) = \mathbb{E}f(W + y) \). Then the function \( G \) is symmetric, continuous and strictly increasing for \( y > 0 \). Also, the function \( \Psi(x) = \mathbb{E}f(xW) \) is continuous and strictly increasing for \( x > 0 \).

(b) Assume that \( Y \) is a random variable independent of \( W \) and let \( x \geq 0 \). Then, \( \mathbb{E}f(xW + Y) \geq \mathbb{E}f(xW) \), and the equality holds if and only if \( P(Y = 0) = 1 \).

(c) Assume that \( X \) and \( Y \) are independent random variables and \( \mathbb{E}f(X + Y) < \infty \). Then, \( \mathbb{E}f(X) \) and \( \mathbb{E}f(Y) < \infty \).

(d) Assume that \( X \) and \( Y \) are i.i.d. random variables with \( \mathbb{E}(X) = 0 \). Then there is a constant \( C_1 \) which depends only on \( p \) from Condition 1, such that \( \mathbb{E}f(X) \leq \mathbb{E}f(X - Y) \leq C_1 \mathbb{E}f(X). \)
Lemma 9. For any function $f$ is convex. Moreover, since \( W \) has as support all the real numbers, and \( f \) is non-constant, the function \( G \) is strictly convex. We shall also notice that, by symmetry, \( G'(0) = 0 \) and the function \( G'(x) \) is strictly positive for \( x > 0 \). The same argument works for the function \( \Psi(x) \) which proves (a). Statement (c) follows from the Fubini theorem, since an a.s. finite convex function is finite. Finally, statement (d) follows from the Jensen inequality, monotonicity of the function \( f \) on \( \mathbb{R}^+ \) and Property (b) in Lemma 5.

The following moment inequality was established by Klass and Nowicki (1997, Lemma 2.6). Although, their result was stated for \( A \leq 1 \) the adaptation is immediate by considering blocks with partial sums satisfying (5) with \( A \leq 1 \). We also formulate this lemma for an infinite number of pairs by passing to the limit.

Lemma 7. Let \( \{(X_k, I_{B_k}); k \geq 1\} \) be independent pairs of random variables, where \( I_B \) is an indicator variable. Assume that the function \( H : \mathbb{R} \to \mathbb{R} \) is symmetric, continuous, strictly increasing on \( \mathbb{R}^+ \), \( H(0) = 0 \) and there is a \( p > 0 \), such that \( H(Kx) \leq K^p H(x) \) for all \( K \geq 2 \), \( x > 0 \). Suppose that

\[
\sum_{i \geq 1} P(B_i) \leq A. \tag{5}
\]

Then, there exist two positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 \sum_{i \geq 1} \mathbb{E} H(X_i I_{B_i}) \leq \mathbb{E} H \left( \sum_{i \geq 1} X_i I_{B_i} \right) \leq c_2 \sum_{i \geq 1} \mathbb{E} H(X_i I_{B_i}).
\]

The next property is going to be used several times in the proofs (see for example, Rogers, (1998), namely Vitali’s argument in theorems 63 and 64)).

Property 8. Assume that \( F(x) \) is a non-decreasing function on \([0, 1]\), then

\[
\mu \{ s \in [0, 1] : \limsup_{h \to 0} h^{-1}|F(s + h) - F(s)| \geq K \} \leq (F(1) - F(0))/K.
\]

The following technical lemma is useful for handling the non-stationary case.

Lemma 9. For any function \( c(t), t \in [a, b], \)

\[
\liminf_{h \downarrow 0} \frac{|c(s + h) - c(s)|}{\sqrt{h}} = 0 \quad \mu - \text{almost surely}.
\]

Proof. Without loss of generality we can take \([a, b] = [0, 1]\) and notice first that

\[
\left\{ s \in (0, 1) : \liminf_{h \downarrow 0} \frac{|c(s + h) - c(s)|}{\sqrt{h}} > 0 \right\} = \bigcup_{n,k,m=1}^{\infty} \bigcup_{j=0}^{k-1} A_{n,k,j,m} \quad \text{where} \quad A_{n,k,j,m} = \left\{ s \in \left( j/k, (j + 1)/k \right) : |c(s)| < m, |c(s + h) - c(s)| \geq \sqrt{h}/n \quad \text{for all} \quad h \in (0, 1/k) \right\}.
\]

We say that the set \( G \subseteq [0, 1] \) and the function \( c \) satisfy Property \((G, c)\) if there exist two positive real numbers \( u \) and \( w \) such that

\[
(G, c) : \quad \text{for all} \quad s, s_1, s_2 \in G, \quad |c(s)| < w, \quad |c(s_2) - c(s_1)| \geq u \sqrt{|s_2 - s_1|}.
\]
Clearly, the set $A_{n,k,j,m}$ and the function $c$ satisfy Property ($G, c$) with $u = 1/n$ and $w = m$ whence, it is enough to show that if $G$ and $c$ satisfy Property ($G, c$), then $\mu(G) = 0$.

Let $D = c(G)$, that is the image of the set $G$. We observe that the function $c: G \to D$ is one to one and let $d = c^{-1}: D \to G$ be its inverse function. Then, the set $D$ and the function $d$ satisfy the property

$$(D, d): \quad \text{for all } u_1, u_2 \in D, \quad |d(u_2) - d(u_1)| \leq (1/u^2)|u_2 - u_1|^2, \quad D \subset (-w, w).$$

Let $N$ be a positive integer, $\delta = w/N$ and define the intervals $\Delta_i = (\delta i, \delta i + \delta], i = -N, 1-N, \ldots, N-1$. Then,

$$D \subseteq \bigcup_{i=-N}^{N-1} D \cap \Delta_i \quad \text{and so} \quad d(D) \subseteq \bigcup_{i=-N}^{N-1} d(D \cap \Delta_i)$$

which implies that the outer measure $\mu^*$ of the set $G$ is bounded by

$$\mu^*(G) = \mu^*(d(D)) \leq \sum_{i=-N}^{N-1} \mu^*[d(D \cap \Delta_i)]. \quad (6)$$

Further, we use the following ideas associated with the computation of the Hausdorff measurers. The first idea is a standard upper bound on the outer measure of a set by its diameter

$$\mu^*(A) \leq \text{diam}(A) := \sup \{|x-y| : x, y \in A\}$$

The second idea is the bound on the diameter of the image of the Lipschitz function $g: T \to W$,

$$\text{diam}(g(T)) \leq \text{diam}(T)K_{g,T}, \quad \text{where} \quad K_{g,T} = \sup \{|g(x) - g(y)|/|x-y| : x, y \in T\}.$$ \hspace{1cm}

In addition, we observe that Property $(D, d)$ implies the following simple upper bound $K_{d,D \cap \Delta_i} \leq (1/u^2)\delta$ on the Lipschitz coefficient $K_{d,D \cap \Delta_i}$ of the function $d$ on the set $D \cap \Delta_i$. These facts combined give

$$\mu^*[d(D \cap \Delta_i)] \leq \text{diam}[d(D \cap \Delta_i)] \leq \text{diam}(D \cap \Delta_i)K_{d,D \cap \Delta_i} \leq \delta^2/u^2$$

whence by \[(6)\]

$$\mu^*(G) \leq 2N \max_{i=-N,N-1} \mu^*[d(D \cap \Delta_i)] \leq 2N\delta^2/u^2 = (2w^2/u^2)/N \to 0$$

as $N \to \infty$ and so $\mu^*(G) = 0$. \hspace{1cm}

The following lemma is essential in our approach to tackle the characterization problem. We formulate it as it appears in Gikhman and Skorohod, (1975) by combining Theorem 1 on page 263 and Theorem 4 on page 270 (see also Jacod (1985)).

**Lemma 10.** Let $X(t)$ be a stochastic process with independent increments and with $X(0) = 0$. Then, for any positive number $a$, $X(t)$ admits the representation:

$$X(t) = B(t) + \left[ c(t) + \sum_{t_k \leq t} \xi_k + \int_{x>a} xv(t, dx) + \int_{x<0} x v(t, dx) + \int_{0<|x|\leq a} x[v(t, dx) - \Pi(t, dx)] \right]$$

$$= B(t) + [c(t) + \eta(t) + T_{1,a}(t) + T_{2,a}(t) + U_a(t)]$$

$$= B(t) + c(t) + Y(t)$$

where $\eta, B, U_a, T_{1,a}, T_{2,a}$ are independent processes with independent increments. The process $B$ is the zero mean continuous component of $X$ (non–homogeneous Gaussian process) with continuous
non-decreasing variance $\sigma^2(t) = \text{Var}(B(t))$. The process $\eta$ is the deterministic time jump process, i.e. is the sum of all jumps $\xi_k$ of time $\Delta k$ occurred at deterministic times $t_k \leq t$ where the set $\{t_k\}$ is at most countable. The process $v(\Delta, A)$ counts the number of jumps of $X$ in a set $A$ in the interval of time $\Delta$ and $v(t, A) = v([0,t], A)$, where $v$ is stochastically continuous (that is $v(\{t\}, A) = 0$). Given $A \subset R - [-a, a]$ for some $a > 0$, $v(t, A)$ is a non-homogeneous Poisson process. The measure $\Pi((a, b), A) = \mathbb{E}[v((a, b), A)]$ (so $\Pi(t, A) = \mathbb{E}[v(t, A)]$) is its compensator. Moreover,

$$G(t, a) = \int_{0 < |x| \leq a} x^2 \Pi(t, dx) < \infty \quad \text{and} \quad G(t, a) \to 0 \quad \text{as} \quad a \to 0.$$ 

For future analysis of the processes that appear in the above representation it is convenient to introduce the following two notations:

Consider a stochastic process $\{Z = Z(s); s \in [0, 1]\}$. We say that the process $Z$ is $f$-negligible if

$$\mu \left\{ s \in [0, 1] : \limsup_{h \downarrow 0} \mathbb{E}f(h^{-1/2}[Z(s + h) - Z(s)]) > 0 \right\} = 0 . \quad (7)$$

Next, we consider a family of stochastic processes $\{Z_a = Z_a(s); s \in [0, 1]\}$ parameterized by $a \geq 0$. We say that the family $\{Z_a\}$ is approximately $f$-negligible if for any real $r > 0$,

$$\limsup_{a \to 0} \mu \left\{ s \in [0, 1] : \limsup_{h \downarrow 0} \mathbb{E}f(h^{-1/2}[Z_a(s + h) - Z_a(s)]) > r \right\} = 0 . \quad (8)$$

Next lemma provides some general properties about $f$-negligible processes.

**Lemma 11.** (a) If two processes $Z_1$ and $Z_2$ satisfy (7), and $a_1$ and $a_2$ are two real numbers, then the process $a_1 Z_1 + a_2 Z_2$ is also $f$-negligible.

(b) Assume that for any $a \geq 0$ the stochastic process $\{Z = Z(s); s \in [0, 1]\}$ admits the decomposition $Z = Z_a + S_a$. If for any $a$, the process $S_a$ is $f$-negligible and the family $\{Z_a\}$ is approximately $f$-negligible, then $Z$ is $f$-negligible.

(c) Suppose that the stochastic process $\{Z(s); s \in [0, 1]\}$ satisfies the inequality $\mathbb{E}f(Z(s + h) - Z(s)) \leq q(s + h) - q(s)$ where $q(s)$ is a bounded non-decreasing function. Then, the process $W$ is $f$-negligible.

(d) Consider a family of stochastic processes $\{Z_a = Z_a(s); s \in [0, 1]\}$ parameterized by $a \geq 0$, and suppose that $\mathbb{E}|Z_a(s + h) - Z_a(s)|^2 \leq q_a(s + h) - q_a(s)$ where each function $q_a(s)$ is bounded, non-decreasing and $q_a(1) \to 0$ as $a \to 0$. Then, the family $\{Z_a\}$ is approximately $f$-negligible.

**Proof.** The first and second properties are immediate consequences of the fact that $f(x + y) \leq c_f[f(x) + f(y)]$ (stated in Lemma 5) and the additivity of the Lebesgue measure.

To prove the third property, we notice that, by Condition IV and the condition imposed in this lemma,

$$\mathbb{E}f(h^{-1/2}[Z(s + h) - Z(s)]) \leq Ch^{-p/2}\mathbb{E}f(Z(s + h) - Z(s)) \leq Ch^{-p/2}(q(s + h) - q(s))$$

and, since $1 \leq p < 2$, it remains to apply Property VIII.

Finally to prove Statement (d), we let $r > 0$ and apply first Lemma 5 and then the Cauchy–Schwartz inequality to derive

$$\mu \left\{ s \in [0, 1] : \limsup_{h \downarrow 0} \mathbb{E}f(h^{-1/2}[Z_a(s + h) - Z_a(s)]) \geq r \right\} \leq \mu \left\{ s \in [0, 1] : \limsup_{h \downarrow 0} h^{-1}\mathbb{E}|Z_a(s + h) - Z_a(s)|^2 + (h^{-1}\mathbb{E}|Z_a(s + h) - Z_a(s)|^{1/2})^2 \geq (r/C) \right\} .$$
Then, we apply Property 8 along with the conditions imposed in the part (d) of this lemma in order to bound the right hand side of the above inequality by

\[ 2 \mu \left\{ s \in [0, 1) : \limsup_{h \downarrow 0} h^{-1} \mathbb{E}[Z_a(s + h) - Z_a(s)]^2 \geq A \right\} \leq q_a(1)/A \to 0 \text{ as } a \downarrow 0 \]

(where \( A = \min((r/2C)^2, (r/2C)) \)) and so the lemma follows \( \diamond \)

As one of the key steps in the proof of Theorem \( \PageIndex{1} \) we show that the jump component is \( f \)–negligible which is formulated in the following lemma.

\textbf{Lemma 12.} Assume that \( \mathbb{E}f(X(1)) < \infty \). Then, the process \( \{Y(s); s \in [0, 1]\} \) defined in Lemma \( \PageIndex{10} \) satisfies \( \PageIndex{7} \).

\textbf{Proof.} By the property (a) of Lemma \( \PageIndex{11} \) it is enough to establish \( \PageIndex{7} \) separately for the deterministic time jump process \( \eta \) and the stochastically continuous jump process \( T_{1,a} + T_{2,a} + U_a = J \), say.

We begin by analyzing the jump process \( J \). By the properties (a) and (b) of Lemma \( \PageIndex{11} \) it is enough to show that the family \( \{U_a\} \) satisfies \( \PageIndex{8} \) and, for each \( a > 0 \), the processes \( T_{i,a} \) satisfy \( \PageIndex{7} \).

To show that the family \( \{U_a\} \) is approximately \( f \)–negligible, we notice that

\[ \mathbb{E}[(U_a(s + h) - U_a(s))^2] = \int_{0<|x|\leq a} x^2 \Pi([s, s + h], dx) = G(s + h, a) - G(s, a) \]

where, by Lemma \( \PageIndex{10} \), for each \( a > 0 \), the function \( q_a(x) = G(x, a) \) is non-decreasing and \( q_a(1) = G(1, a) \to 0 \) as \( a \to 0 \). Hence, \( \PageIndex{8} \) is an immediate consequence of property (d) of Lemma \( \PageIndex{11} \).

To finish the analysis of the stochastically continuous jump component \( J \), it is enough to show that for any \( a > 0 \) and \( i = 1, 2 \), the process \( T_{i,a} \) is \( f \)–negligible. Clearly, it is enough to treat only the stochastic process

\[ T_{1,a}(t) = \int_a^\infty x \nu(t, dx) . \]

By Property (c) of Lemma \( \PageIndex{3} \) \( \mathbb{E}f(T_{1,a}(1)) < \infty \) and by Lemma \( \PageIndex{5} \)

\[ \mathbb{E}T_{1,a}(1) = \int_a^\infty x \Pi(t, dx) < \infty \quad \text{and hence} \quad \int_a^\infty \Pi(t, dx) < \infty . \]

Using now the week convergence approximation of the Poisson process by the Bernoulli processes along with the Klass-Nowicki moment inequality from Lemma \( \PageIndex{6} \) we derive

\[ c_1 \int_a^\infty f(x) \Pi(t, dx) \leq \mathbb{E}f \left( \int_a^\infty x \nu(t, dx) \right) = \mathbb{E}f(T_{1,a}(1)) \leq c_2 \int_a^\infty f(x) \Pi(t, dx) . \]

Since \( \mathbb{E}f(T_{1,a}(1)) < \infty \), we note that

\[ q_a(t) = \int_a^\infty f(x) \Pi(t, dx) < \infty \]

and notice that for \( 0 \leq s < s + t \leq 1 \), by Condition \( \PageIndex{11} \)

\[ \mathbb{E}f \left( \int_s^\infty x \nu((s, t + s), dx) \right) \leq c_2 \int_a^\infty f(x) \Pi((s, s + h), dx) = c_2 [q_a(s + h) - q_a(s)] \]
and so, the process \( T_{1,a} \) is \( f \)-negligible by Property (c) of Lemma 11.

Now, we take care of the deterministic time jump process and notice first that, by Property (c) of Lemma 3 for any subset \( A \) of the set of points of discontinuity \( \{ t_k \} \) we have \( \mathbb{E} f \left( \sum_{k \in A} \xi_k \right) < \infty \). Furthermore, without loss of generality, we may assume that \( \mathbb{E}(\xi_k) = 0 \) and then, by Property (d) of Lemma 3 that \( \xi_k \) are symmetric. By the Kolmogorov three series theorem and symmetry

\[
\sum_k P(|\xi_k| > 1) < \infty \quad \text{and} \quad \sum_k \mathbb{E}(|\xi_k|^2 I_{(|\xi_k| \leq 1)}) < \infty .
\]

Now, for any positive \( a > 0 \), let \( Q_a \subseteq \{ t_k \} \) be a finite subset of the set of points of discontinuity such that

\[
\sum_{k : t_k \not\in Q_a} P(|\xi_k| > 1) \leq a < \infty \quad \text{and} \quad \sum_{k : t_k \not\in Q_a} \mathbb{E}(|\xi_k|^2 I_{(|\xi_k| \leq 1)}) \leq a . \tag{9}
\]

We decompose the process \( \eta \) into the form

\[
\eta(t) = \sum_{k : t_k < t, t_k \in Q_a} \xi_k + \sum_{k : t_k \leq t, t_k \not\in Q_a} \xi_k I_{(|\xi_k| > 1)} + \sum_{k : t_k \leq t, t_k \not\in Q_a} \xi_k I_{(|\xi_k| \leq 1)}
\]

\[
= I_{1,a} + I_{2,a} + I_{3,a} .
\]

The first process \( I_{1,a} \) has a finite number of jumps and obviously is \( f \)-negligible.

To analyze the second process, let \( A \) be as before, a subset of the points of discontinuity, and notice that we also have \( \mathbb{E} f \left( \sum_{k \in A} \xi_k I_{(|\xi_k| > 1)} \right) < \infty \). Next we apply the Burkholder inequality ((1973), Theorem 15.1) and we find two constants \( c_3 \) and \( c_4 \), such that for any subset \( A \subset \{ t_k \} \) we have

\[
c_3 \mathbb{E} f \left( \sum_{k \in A} \xi_k^2 I_{(|\xi_k| > 1)} \right)^{1/2} \leq \mathbb{E} f \left( \sum_{k \in A} \xi_k I_{(|\xi_k| > 1)} \right) \leq c_4 \mathbb{E} f \left( \sum_{k \in A} \xi_k^2 I_{(|\xi_k| > 1)} \right)^{1/2} . \tag{10}
\]

To estimate the quadratic term we apply the Klass–Nowicki inequality in Lemma 7 with \( X_k = \xi_k^2 \), \( B_k = (|\xi_k| > 1) \), and the function \( H(x) = f(x \sqrt{2}) \), \( x > 0 \) (which obviously satisfies the conditions of Lemma 7 with the power \( p/2 \)) and derive

\[
c_1 \sum_{k \in A} \mathbb{E} f(\xi_k I_{(|\xi_k| > 1)}) \leq \mathbb{E} f \left( \sum_{k \in A} \xi_k^2 I_{(|\xi_k| > 1)} \right)^{1/2} \leq c_2 \sum_{k \in A} \mathbb{E} f(\xi_k I_{(|\xi_k| > 1)}) . \tag{11}
\]

As a consequence, by (10) and (11)

\[
Q(t) = \sum_{k : t_k \leq t} \mathbb{E} f(\xi_k I_{(|\xi_k| > 1)}) < \infty \quad \text{and} \quad \mathbb{E} f \left( \sum_{k : s < t_k \leq s + h, t_k \not\in Q} \xi_k I_{(|\xi_k| > 1)} \right) \leq c(Q(s + h) - Q(s)) .
\]

Therefore, the process \( \{ I_{2,a}(t) \} \) satisfies the conditions of Property (c) in Lemma 11 and thus is \( f \)-negligible.

Finally, in order to treat the process \( \{ I_{3,a}(t) \} \) of bounded jumps, we define the finite non-decreasing function

\[
G(t) = \mathbb{E} \left( \sum_{k : t_k \leq t, t_k \not\in Q} \xi_k I_{(|\xi_k| \leq 1)} \right)^2
\]
and notice that
\[ E \left( \sum_{k:s < t_k \leq s + h, t_k \notin Q} \xi_k I_{[\xi_k \leq 1]} \right)^2 = G(s + h) - G(s). \]

Since by Relation (9), \( G(1) \leq a \to 0 \) as \( a \to 0 \) it follows that the process \( I_{3,a} \) satisfies the conditions of Property (d) in Lemma (11) and therefore the family \( \{I_{3,a}\}_{a \geq 0} \) is asymptotically \( f \)-negligible. Thus, by Property (b) in Lemma (11) the stochastic process \( \eta \) is \( f \)-negligible, which completes the proof of the lemma. \( \diamond \)

The next lemma treats the Gaussian case of Theorem 1.

**Lemma 13.** Suppose that \( \{V(t); t \in [0,1]\} \) is a stochastically continuous Gaussian process, with independent increments, \( V(0) = 0 \), and there exists \( \sigma \geq 0 \) such that
\[ \lim inf_{h \downarrow 0} \mathbb{E}f((V(t + h) - V(t))/\sqrt{h}) \geq \mathbb{E}f(\sigma W) \quad \mu - \text{almost surely}. \] (12)

Then, \( \text{Var}(V(t)) \geq \sigma^2 t \) for all \( t \in [0,1] \) and \( \text{Var}(V(1)) = \sigma^2 \) if and only if \( \text{Var}(V(t)) = \sigma^2 t \) for all \( t \in [0,1] \).

**Proof.** Denote by \( \sigma^2(t) = \text{Var}(V(t)) \), which is a non-negative, continuous, non-decreasing function. First, we notice that if \( \sigma = 0 \), then the lemma is immediate.

Since \( \sigma^2(t) \) is non-decreasing, its derivative \( (\sigma^2(t))' \) exists almost surely with respect to the Lebesgue measure \( \mu \) and to prove the lemma, it is enough to show that \( \mu \) almost surely for \( t \in [0,1] \),
\[ \sigma^2 \leq (\sigma^2(t))'. \] (13)

Denote by \( c(t) = \mathbb{E}V(t) \). Fix \( t \in (0,1) \) such that the derivative \( (\sigma^2(t))' \) exists. By Lemma 9, there exists a positive sequence \( h_\ast \downarrow 0 \) such that \( h_\ast^{-1/2}|c(t + h_\ast) - c(t)| \to 0 \). Then, since \( f(x) \) is continuous and \( |f(x)| \leq C(|x| + x^2) \), by the Lebesgue dominated convergence theorem, we obtain:
\[ \lim inf_{h_\ast, t \downarrow 0} \mathbb{E}f(h_\ast^{-1/2}(V(t + h) - V(t))) = \lim inf_{h_\ast, t \downarrow 0} \mathbb{E}f(h_\ast^{-1/2}((V(t + h) - V(t) - c(t + h_\ast) - c(t)))) = \mathbb{E}f(\sqrt{(\sigma^2(t))'W}) \]
\[ = \mathbb{E}f(\sqrt{(\sigma^2(t))'W}) \]
Thus, by the lower bound in Condition (12)
\[ \mathbb{E}f(\sigma W) = \mathbb{E}f(\sigma W) \leq \mathbb{E}f(\sqrt{(\sigma^2(t))'W}) \]
for almost all \( t \) which proves (13) by Lemma 9 Property (a).

To prove the second part of the lemma we just have to notice that
\[ 0 = \sigma^2(1) - \sigma^2 = \int_0^1 [d(\sigma^2(t)) - \sigma^2 dt] \]
whence, by (13), \( (\sigma^2(t))' = \sigma^2 \), \( \mu \)-almost surely for \( t \in [0,1] \), implying that \( \sigma^2(t) = t\sigma^2 \) for all \( t \in [0,1] \). \( \diamond \)

**Proof of Theorem 1.** We start from the representation of Lemma 10 applied to the process \( \{X(t); t \in [0,1]\} \), hence \( X(t) = B(t) + c(t) + Y(t) \) for all \( t \in [0,1] \). Since \( \mathbb{E}|X(1)| < \infty \), by Lemma 12 the discontinues component, the jump process \( Y \), satisfies \( \lim sup_{h, t \downarrow 0} \mathbb{E}|Y(h + s) - Y(s)|/\sqrt{h} = \)
0 almost surely with respect to the Lebesgue measure. Whence, by condition (2), the Gaussian component \( \{B(t) + c(t); \ t \in [0, 1]\} \) satisfies (12) with \( \sigma = \Psi^{-1}\mathbb{E}f(X(1)) \). Denote by \( \sigma^2(1) = \text{Var}(B(1)) \). From Lemma 10 and Lemma 9 we derive

\[
\mathbb{E}f(X(1)) = \mathbb{E}f(B(1) + c(1) + Y(1)) = \mathbb{E}f(\sigma(1)W + c(1) + Y(1)) \geq \mathbb{E}f(\sigma(1)W) .
\]

Moreover, by Lemma 13 we obtain \( \sigma(1) \geq \sigma \), and so, by the definition of \( \sigma \), \( \mathbb{E}f(\sigma(1)W) \geq \mathbb{E}f(X(1)) \).

This fact together with Relation (14) imply that \( \text{Var}(B) = \sigma^2 \). Moreover, by the second part of Lemma 13, we obtain that \( \sigma^2(t) = \sigma^2 t \) for all \( t \in [0, 1] \) and, by Lemma 6 \( P(c(1) + Y(1) = 0) = 1 \), implying that \( Y(1) \) is degenerate. Since the process \( Y(t) \) has independent increments if follows that all increments are degenerate, which establishes (3). Moreover, \( \mathbb{E}X(1) = 0 \) because \( c(1) + Y(1) = 0 \) almost surely.\( \diamond \)

**Remark and proof of Corollary 3** As it follows from the proof of Theorem 11 Condition (2) can be slightly weakened to consider subsequences \( h_n \to 0 \) such that the centering function \( c(t) \) satisfies \( h_n^{-1/2}(c(t + h_n) - c(t)) \to 0 \). In particular, for homogeneous stochastically continuous processes, the centering sequence is defined by the continuous solution of the Cauchy equation

\[
c(x + y) = c(x) + c(y)
\]

implying that \( c(t) = qt, \ t \in [0, 1] \). Thus, the representation (3) in Corollary 3 is then immediate. Finally, \( \mathbb{E}X(1) = q = 0 \), which completes the proof of the corollary.\( \diamond \)

**Proof of Proposition 4** First, we choose a positive sequence \( t_n \downarrow 0 \) such that the set \( T = \{t_n; \ n \geq 0\} \) is independent with respect to the rational field. Then, by using Zorn lemma, we construct the Hamel basis \( B \subset R \) such that \( T \subset B \). In order to construct the function \( k \) that satisfies the Cauchy equation (15), we define it first on the set \( B \) by

\[
k(b) = 0, \quad \text{if} \ b \notin T \quad \text{and} \quad k(t_i) = f^{-1}(1)\sqrt{t_i}, \ i = 1, \ldots
\]

Then, the solution to (15) is given by setting \( c(\Sigma r_i b_i) = \Sigma r_i c(b_i) \), (see for example Hardy, Littlewood and Polyá (1952)). Now, let \( \{Y(t); t \geq 0\} \) be a homogeneous Poisson process with rate 1 and \( b > 0 \) be such that \( \mathbb{E}f(bY(1)) = 1 \). Define

\[
X(t) = bY(t) + k(t) - tk(1), \ t \geq 0 .
\]

Then, \( \{X(t); t \geq 0\} \) is a homogeneous stochastic process with independent increments, with \( X(0) = 0 \). Notice, that \( X(1) = bY(1) \) and \( \mathbb{E}f(bY(t_n))/\sqrt{t_n} \to 0 \) by Lemma 12 whence, by construction, we derive

\[
\mathbb{E}f(X(1)) = \mathbb{E}f(bY(1)) = 1 = \lim_{t_n \to 0} \mathbb{E}f(t_n^{-1/2}|X(t_n)|) = \lim_{t_n \to 0} f(k(t_n)/\sqrt{t_n}) = f(f^{-1}(1)) = 1
\]

completing the proof of this proposition.\( \diamond \)

### 4 Application to the Central limit theorem

This section was motivated by Theorem 3 in Dehling, Denker and Philipp (1986). We give several applications of the characterization results from Section 1 to extend and develop their result in several directions.
The $L_p$ characterization of the Gaussian processes obtained in this paper allows to avoid the traditional techniques based on the characteristic functions in order to prove the CLT. Moreover, besides a certain dependence condition, the additional conditions are imposed to the moments of order $p \in [1,2)$ only. Corollary 16 is applied to derive the following central limit theorem. Let $W$ have a standard normal distribution and let $\|x\|_p = (\mathbb{E}|x|^p)^{1/p}$.

**Theorem 14.** Suppose that $\{X_k; k = 1, 2, \ldots\}$ is a strictly stationary sequence and $p$ a fixed real, $p \in [1,2)$. Assume $\mathbb{E}|X_0|^p < \infty$ and let $S_n = X_1 + \ldots + X_n$, $n = 1,2,\ldots$, $S_0 = 0$. Define the normalizing sequence $\rho_n = \|S_n\|_p/\|W\|_p$, and assume that

(i) For any positive integer $k$ and real number $x$,

$$
\lim_{n \to \infty} \left| \mathbb{E} \exp(ixS_n/\rho_n) - (\mathbb{E} \exp(ixS_{n/k}/\rho_n))^k \right| = 0 \tag{16}
$$

(ii) $\rho_n \to \infty$ and there exists a positive integer $K > 1$ such that $\rho_{Kn}/\rho_n \to \sqrt{K}$ as $n \to \infty$.

(iii) $\{|S_n|/\rho_n|^p; n = 1,2,\ldots\}$ is an uniformly integrable family.

Then, $S_n/\rho_n \to^D N(0,1)$.

**Corollary 15.** Let $\{X_n; n \geq 0\}$ be a strictly stationary sequence of integrable random variables as in Theorem 14 satisfying the condition (16). Let $p$ be a fixed real number $p \in [1,2)$ and assume there is a sequence of constants $b_n = \sqrt{n}h(n)$, where $h(n)$ is a function slowly varying at $\infty$ such that the family $\{|S_n|^p/b_n; n \geq 1\}$ is uniformly integrable. Then, $\lim\|S_n\|_p/b_n = c$ if and only if $S_n/b_n$ converges in distribution to $N(0,\|W\|_p^2 \cdot c^2)$.

If the second moments are finite then we immediately derive from the above corollary:

**Corollary 16.** Let $\{X_n; n \geq 0\}$ be a strictly stationary sequence of square integrable random variables satisfying the condition (16) and assume that $\sigma_n = \text{stdev}(S_n) = \sqrt{n}h(n)$, where $h(n)$ is a function slowly varying at $\infty$. Let $p$ be a fixed real number $p \in [1,2)$. Then, $\lim_{n \to \infty} \|S_n\|_p/\sigma_n = c$ if and only if $S_n/\sigma_n$ converges in distribution to $N(0,\|W\|_p^2 \cdot c^2)$.

Following O’Brein (1987) we say that a strictly stationary sequence $\{X_k; k = 1,2,\ldots\}$ is $r$-strongly-mixing, if

$$
\alpha_r(n) = \sup \left\{ \frac{1}{r} \left( \sum_{k=0}^{r-1} P(A \cap B_k) - P(A)P(B) \right) \to 0 \text{ as } n \to \infty \right\}
$$

where the supremum is taken over all positive integers $m$; $A \in \mathcal{F}_0^m$, $B \in \mathcal{F}_{m+n}^\infty$, and $B_k$ is a shift of $B$ for $k$ steps (if $B = \{X_1, X_2, \ldots \} \in E$ for some Borel $E$, then $B_k = \{X_{k+1}, X_{k+2}, \ldots \} \in E$).

It follows from Jakubowski (1993), Proposition 5.3 that $r$-strongly mixing sequences satisfy the weak asymptotically independence condition (16). O’Brein (1987) pointed out that instantaneous functions of a stationary Harris chain with period $d > 1$ are $d$-strongly mixing and thus, by Jakubowski (1993), they satisfy (16). However, they are not mixing in a classical ergodic sense. Also, strongly mixing condition implies $r$-strong mixing. In particular, Theorem 3 in Dehling, Denker and Philipp (1986) follows from Corollary 16 applied with $p = 1$.

The regularity condition (ii) in Theorem 14 is not easy to check. However, using arguments similar to Jakubowski (1993) it follow that conditions (i), (iii) and the central limit theorem $S_n/\rho_n \to^D N(0,1)$ imply (ii). Moreover, one can argue as in Dehling, Denker and Philipp (1986) that the regularity condition can be checked empirically, using for example the bootstrap procedure. As it is pointed out
in Peligrad (1998), the limit theorems for bootstrapped estimators of dependent sequences require less restrictive conditions than the corresponding limit theorems for the original sequences.

**Proof of Theorem 14** First, we derive a useful consequence of condition (ii). We notice that, for any non-negative integer \( j \), \( ||S_i+j||/||S_i|| - 1 \leq ||S_i+j - S_i||/||S_i|| \leq j/||X_i||/||S_i|| \to 0 \). Next, let \( n = K' m + j \) where \( j \in \{0, 1, \ldots, K' - 1\} \). Then, \( \rho_n \to 0 \) as \( n \to \infty \). Now we consider the normalized triangular array \( S_j^{(n)} = S_j/\rho_n \) and observe that

\[
|S_{[nK-r]}^{(n)}|^p = |S_{[nK-r]}|^p/\rho_n = |S_{[nK-r]}|/\rho_n \to 0
\]

as \( n \to \infty \). Now consider the normalized triangular array \( S_j^{(n)} = S_j/\rho_n \) and observe that

\[
|S_{[nK-r]}^{(n)}|^p = |S_{[nK-r]}|^p/\rho_n = |S_{[nK-r]}|/\rho_n \to 0
\]

and so, the sequence \( \{|S_{[nK-r]}^{(n)}|^p; n = 1, 2, \ldots\} \) is uniformly integrable by Condition (iii) of Theorem 14 and Relation (17).

In order to prove this theorem it is enough to show that for any subsequence \( (n') \subseteq (n) \) there exists another subsequence \( (n'') \subseteq (n') \) such that \( S_{[n'']}^{(n'')} \to^D N(0, 1) \). By the Helly diagonalization technique we construct a subsequence \( (n'') \subseteq (n') \) such that \( S_{[n'']}^{(n'')} \to^D X^{(k)} \) for each positive integer \( k \in N' \). Now, \( S_{[n'']}^{(n'')} \to^D X^{(1)} \), and by Condition (i), \( X^{(1)} \) is infinitely divisible (a similar result was established in Proposition 3.1. in Samur (1984)). To prove it, fix the integer \( k \). By (i), we notice that \( X_{n''1}^{(k)} + \ldots + X_{n''k}^{(k)} \to^D X^{(1)} \), where \( \{X_{n''i}^{(k)}; i = 1, \ldots, k\} \) are \( k \) independent copies of \( S_{[n'']}^{(n'')} \). On the other hand, it is easy to see that \( X_{n''1}^{(k)} + \ldots + X_{n''k}^{(k)} \to^D X_1^{(k)} + \ldots + X_k^{(k)} \) where \( X_1^{(k)}, \ldots, X_k^{(k)} \) are independent copies of \( X^{(k)} \). By the uniqueness of the limit we obtain \( X_1^{(k)} + \ldots + X_k^{(k)} \to^D X^{(1)} \), for any \( k \geq 1 \). Therefore, without loss of generality we can take \( X^{(k)} = X(1/k) \), for \( k \in N' \), where \( \{X(t); t \geq 0\} \) is a separable homogeneous stochastic process with right continuous sample path, with independent increments and with \( X(0) = 0 \). Moreover \( \{X(t); t \geq 0\} \) can be assumed stochastically continuous.

Notice, \( ||S_{[nK-r]}^{(n)}||_p = ||W||_p\rho_n/\rho_n \to K^{-r/2}||W||_p \). Since the sequence \( \{|S_{[nK-r]}^{(n)}|^p; n = 1, 2, \ldots\} \) is uniformly integrable, we then derive that \( \liminf_{t \to 0} ||X(t_k)||_p/\sqrt{t_k} \geq \mathbb{E}(|X(1)|)^p \), where \( t_k = 1/k \), for \( k \in N' \) and it remains to apply Corollary 3 which completes the proof of the theorem.

**Acknowledgment.** The authors would like to thank Ildar Ibragimov for pointing out relevant references. We would also like to thank Wlodek Bryc for useful discussions on the absolute moments of sums of random variables and on some problems solved in this paper. We are indebted to the referee whose remarks and suggestions were very inspiring and significantly improved the presentation of this paper.

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