Multidimensional Classical and Quantum Cosmology with Perfect Fluid

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ABSTRACT
A cosmological model describing the evolution of $n$ Ricci-flat spaces ($n > 1$) in the presence of 1-component perfect-fluid and minimally coupled scalar field is considered. When the pressures in all spaces are proportional to the density: $p_i = (1 - h_i)\rho$, $h_i = \text{const}$, the Einstein and Wheeler-DeWitt equations are integrated for a large variety of parameters $h_i$. Classical and quantum wormhole solutions are obtained for negative density $\rho < 0$. Some special classes of solutions, e.g. solutions with spontaneous and dynamical compactification, exponential and power-law inflations, are singled out. For $\rho > 0$ a third quantized cosmological model is considered and the Planckian spectrum of ”created universes” is obtained.

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1 Introduction

Last years the of Kaluza-Klein ideas [1, 2] (see also [3]-[6]) and superstring theory [7] greatly stimulated the interest in multidimensional cosmology (see, for example, [8]-[58] and references therein).

Classical and quantum multidimensional cosmological models were investigated in our papers starting from [28]-[31] (and for spherical symmetry case from [59, 60]). Some windows to observational effects of extra dimensions were found and analyzed such as possible variations of the effective gravitational constant, its relations with other cosmological parameters [28, 29, 37].

But the treatment of classical models may be only the necessary first step in analyzing the properties of the "early universe" and last stages of the gravitational collapse in a multidimensional approach. [The content of this paragraph is a personal opinion of the second author (V.N.M.).] In quantum multidimensional cosmology we hope to find answers to such questions as the singular state, the "creation of the Universe", the nature and value of the cosmological constant, some ideas about possible "seeds" of the observable structure of the Universe, stability of fundamental constants etc. In the third quantization scheme the problems of topological changes may be treated thoroughly. It should be noted also that the multidimensional schemes may be also used in multicomponent inflationary scenarios [66]-[69] (see for example [56]-[58]).

In this paper we consider a cosmological model describing the evolution of $n$ Ricci-flat spaces ($n > 1$) with 1-component perfect-fluid and minimally coupled scalar field as a matter source. The pressures in all space are proportional to the density of energy and the coefficients of proportionality satisfy a certain inequality (see (2.8) and (2.13)). This model is investigated in classical, quantum and third quantized cases and corresponding exact solutions are found. Some particular models of such type were considered previously by many authors (see, for example, [14, 19, 20, 21, 24, 30, 36]). In [47] some classes of exact solutions to Einstein and Wheeler-DeWitt equations with multicomponent perfect fluid and a chain of Ricci-flat internal spaces were obtained. In [49] new families of classical solutions for the model [47] (including those of Toda-like type) were considered.

The paper is organized as following. In Sec. 2 the general description of the model is performed. In Sec. 3 we integrate the Einstein equations for the model and analyze a class of exceptional (inflationary) solutions. (The solutions with $n = 2$ were considered recently in [57, 58]). The isotropization-like and Kasner-like asymptotical behaviours of the solutions are analyzed. In subsection 3.5 some special cases such as isotropic (when the pressures in all spaces are equal) and curvature-like cases are investigated. In the last case there exist solutions with so-called spontaneous and dynamical compactifications. The instanton solutions (classical wormholes) with imaginary scalar field and negative energy density are also obtained. (The interest to wormhole solutions was stimulated greatly after the papers [70, 71, 72]). We note that the exact solutions for 1-component perfect fluid (without scalar field) and a chain of Ricci-flat spaces, were obtained for the first time in [30] (see also [47]). The cosmological constant case was considered previously in [46, 48].

In Sec. 4 we consider our model at the quantum level (for pioneering papers see [73]-[78]). Here we quantize scale factors and a scalar field but treat the perfect fluid as a classical
object. Such approach is quite consistent at least in certain special situations such as $\Lambda$-term [46, 48] and curvature [40]-[43] cases.

In Sec. 4 the Wheeler-DeWitt equation for the model is solved and quantum wormhole solutions are obtained. We recall that the notion of quantum wormholes was introduced by Hawking and Page as a quantum extension of the classical wormhole paradigm (see also [83]-[85] and [41, 42, 46, 48]). They proposed to regard quantum wormholes as solutions of the Wheeler-DeWitt (WDW) equation with the following boundary conditions: (i) the wave function is exponentially damped for large ”spatial geometry”; (ii) the wave function is regular when the spatial geometry degenerates. The first condition expresses the fact that a space-time should be Euclidean at the spatial infinity. The second condition should reflect the fact that the space-time is nonsingular when a spatial geometry degenerates. (For example, the wave function should not oscillate an infinite number of times). Presented in this paper multidimensional quantum wormhole solutions may be considered as a natural extension of the corresponding solutions in [41, 42] and [46, 48] for curvature and $\Lambda$-term cases correspondingly.

In Sec. 5 a third quantized cosmology is investigated along a line as it was done in [40] and [53] for curvature and cosmological constant cases correspondingly. Here we are lead to the theory of massless conformally coupled scalar field in conformally flat generalized Milne universe [53]. In- and out-vacuums are defined and planckian spectrum for the ”created out-universes” (from in-vacuum) is obtained using standard relations [87, 88]. The temperature is shown to depend upon the equation of state. It should be noted that recently the interest to the third quantized models was stimulated by papers [89, 90] (see also [91]-[95] and references therein).

2 The model

We consider a cosmological model describing the evolution of $n$ Ricci-flat spaces in the presence of the 1-component perfect-fluid matter [47] and a homogeneous massless minimally coupled scalar field. The metric of the model

$$g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]g^{(i)},$$

(2.1)

is defined on the manifold

$$M = R \times M_1 \times \ldots \times M_n,$$

(2.2)

where the manifold $M_i$ with the metric $g^{(i)}$ is a Ricci-flat space of dimension $N_i$, $i = 1, \ldots, n$; $n \geq 2$. We take the field equations in the following form:

$$R^M_N - \frac{1}{2} \delta^M_N R = \kappa^2 T^M_N,$$

(2.3)

$$\Box \varphi = 0,$$

(2.4)

where $\kappa^2$ is the gravitational constant, $\varphi = \varphi(t)$ is scalar field, $\Box$ is the d’Alembert operator for the metric (2.1) and the energy-momentum tensor is adopted in the following form

$$T^M_N = T^M_N^{(pf)} + T^M_N^{(\phi)},$$

(2.5)
\[ (T_N^{M(pf)}) = \text{diag}(-\rho, p_1 \delta_{k_1}, \ldots, p_n \delta_{k_n}), \tag{2.6} \]
\[ T_N^{M(\phi)} = \partial^M \varphi \partial_N \varphi - \frac{1}{2} \delta_N^M (\partial \varphi)^2. \tag{2.7} \]

We put pressures of the perfect fluid in all spaces to be proportional to the density
\[ p_i(t) = (1 - \frac{u_i}{N_i}) \rho(t), \tag{2.8} \]
where \( u_i = \text{const}, \ i = 1, \ldots, n. \)

We impose also the following restriction on the vector \( u = (u_i) \in \mathbb{R}^n \)
\[ <u, u>_s = 0. \tag{2.9} \]

Here bilinear form \(<.., ..>_s: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) is defined by the relation
\[ <u, v>_s = G^{ij} u_i v_j, \tag{2.10} \]
where
\[ G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2 - D} \tag{2.11} \]
are components of the matrix inverse to the matrix of the minisuperspace metric [30, 31]
\[ G_{ij} = N_i \delta_{ij} - N_i N_j. \tag{2.12} \]

In (2.11) \( D = 1 + \sum_{i=1}^{n} N_i \) is the dimension of the manifold \( M \) (2.2).

Remark 1. This restriction (2.9) reads
\[ <u, u>_s = \gamma_0 \exp\left[ -\frac{\sum_{i=1}^{n} N_i x^i(t)}{2 - D}\right]. \tag{2.13} \]

We note that in notations of [30] \( <u, u>_s = \Delta(h)/(2 - D). \)

3 Classical solutions

The Einstein equations (2.3) imply \( \nabla_M T_N^M = 0 \) and due to (2.4) \( \nabla_M T_N^{M(pf)} = 0 \) or equivalently
\[ \dot{\rho} + \sum_{i=1}^{n} N_i \dot{x}^i (\rho + p_i) = 0. \tag{3.1} \]

From (2.8), (3.1) we get
\[ \kappa^2 \rho(t) = A \exp[-2N_i x^i(t) + u_i x^i(t)], \tag{3.2} \]
where \( A = \text{const}. \) We put \( A \neq 0 \) (the case \( A = 0 \) was considered thoroughly in [52]).

It is not difficult to verify that the field equations (2.3), (2.4) for the cosmological metric (2.1) in the harmonic time gauge
\[ \gamma_0 \equiv \sum_{i=1}^{n} N_i x^i \]
with the energy-momentum tensor from (2.5)-(2.7) and the relations (2.8), (3.2) imposed are equivalent to the Lagrange equations for the Lagrangian

\[ L = \frac{1}{2}(G_{ij}\dot{x}^i\dot{x}^j + \kappa^2\varphi^2) - \kappa^2 A \exp(u_kx^k) \]  

(3.4)

with the energy constraint

\[ E = \frac{1}{2}(G_{ij}\dot{x}^i\dot{x}^j + \kappa^2\varphi^2) + \kappa^2 A \exp(u_kx^k) = 0 \]  

(3.5)

(for \( \varphi = 0 \) see [47, 55]).

We recall [30, 31] that the minisuperspace metric

\[ G = G_{ij}dx^i \otimes dx^i \]  

(3.6)

has the pseudo-Euclidean signature \((-+,+,\ldots,+)\), i.e. there exists a linear transformation

\[ z^a = e^a_i x^i, \]  

(3.7)

diagonalizing the minisuperspace metric (3.6), (2.12)

\[ G = \eta_{ab}dz^a \otimes dz^b = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \]  

(3.8)

where

\[ (\eta_{ab}) = (\eta^{ab}) \equiv diag(-1,+1,\ldots,+1), \]  

(3.9)

\( a, b = 0, \ldots, n - 1. \) From (3.7)-(3.8) we get

\[ \eta_{ab}e^a_i e^b_j = G_{ij} \]  

(3.10)

and as a consequence

\[ e^i_0 G_{ij} e^j_0 = \eta_{ab}, \quad e^i_a = e^a_i G^{ij} e^b_j, \]  

(3.11)

where \((e^i_a) = (e^a_i)^{-1}\). As in [47] we put

\[ e^0_i = u_i/(2q) \implies z^0 = u_i x^i/(2q), \]  

(3.12)

where here and below

\[ 2q \equiv \sqrt{-<u,u>}. \]  

(3.13)

It may be done, since \(<.,.,.>\) is a bilinear symmetrical 2-form of the signature \((-+,\ldots,+)\) and \(<u,u> \leq 0 \) [47]. An example of diagonalization (3.7) satisfying (3.12) was considered in [30, 31]. From (3.11) and (3.12) we get

\[ e^0_i = -G^{ij} e^j_0 = -u^i/(2q), \quad u^i \equiv G^{ij} u_j. \]  

(3.14)

We also denote

\[ z^n = \kappa \varphi. \]  

(3.15)
The Lagrangian (3.4) in \( z \)-variables (3.7), (3.15) (with the relation (3.12) imposed) may be rewritten as
\[
L = \frac{1}{2} \eta_{AB} \dot{z}^A \dot{z}^B - \kappa^2 A \exp(2qz^0),
\] (3.16)
where \( A, B = 0, \ldots, n \). The energy constraint (3.5) reads
\[
E = \frac{1}{2} \eta_{AB} \dot{z}^A \dot{z}^B + \kappa^2 A \exp(2qz^0) = 0.
\] (3.17)

The Lagrange equations for the Lagrangian (3.16)
\[
-\ddot{z}^0 + 2qA \exp(2qz^0) = 0,
\] (3.18)
\[
\ddot{z}^B = 0, \quad B = 1, \ldots, n,
\] (3.19)
with the energy constraint (3.17) can be easily solved. From (3.19) we have
\[
z^B = p^B t + q^B,
\] (3.20)
where \( p^B \) and \( q^B \) are constants and \( B = 1, \ldots, n \). The first integral for eq. (3.18) reads
\[
-\frac{1}{2} (\dot{z}^0)^2 + A \exp(2qz^0) + \mathcal{E} = 0.
\] (3.21)

Using (3.17), (3.20) and (3.21) we get
\[
\mathcal{E} = \frac{1}{2} \sum_{B=1}^n (p^B)^2.
\] (3.22)

We obtain the following solution of eqs. (3.18), (3.21)
\[
\exp(-2qz^0) = \begin{cases}
(A/\mathcal{E}) \sinh^2(q\sqrt{2\mathcal{E}}(t-t_0)), & \mathcal{E} > 0, \ A > 0, \\
(A/|\mathcal{E}|) \sin^2(q\sqrt{2|\mathcal{E}|}(t-t_0)), & \mathcal{E} < 0, \ A > 0, \\
2q^2 A(t-t_0)^2, & \mathcal{E} = 0, \ A > 0, \\
(|A|/\mathcal{E}) \cosh^2(q\sqrt{2\mathcal{E}}(t-t_0)), & \mathcal{E} > 0, \ A < 0,
\end{cases}
\] (3.23-26)

Here \( t_0 \) is an arbitrary constant. For real \( z^B \) (or, equivalently, for the real metric and the scalar field) we get from (3.22) \( \mathcal{E} \geq 0 \). The case \( \mathcal{E} < 0 \) may take place when a pure imaginary scalar field is considered.

### 3.1 Kasner-like parametrization for non-exceptional solutions with real scalar field

First, we consider the real case with \( \mathcal{E} > 0 \). In this case the relations (3.23) and (2.26) may be written in the following form
\[
\exp(-2qz^0) = \frac{|A|}{\mathcal{E}} f_\delta^2(q\sqrt{2\mathcal{E}}(t-t_0)),
\] (3.27)
where \( \delta \equiv A/|A| = \pm 1 \) and

\[
 f_\delta(x) \equiv \frac{1}{2}(e^x - \delta e^{-x}) = \begin{cases} 
\sinh x, & \delta = +1, \\
\cosh x, & \delta = -1. 
\end{cases}
(3.28)
\]

We introduce a new time variable by the relation

\[
\tau = T \ln \frac{\exp[q \sqrt{2E}(t-t_0)] + \sqrt{\delta}}{\exp[q \sqrt{2E}(t-t_0)] - \sqrt{\delta}} = \begin{cases} 
 T \ln \coth[\frac{1}{2}q \sqrt{2E}(t-t_0)], & \delta = +1, \\
2T \arctan \exp[-q \sqrt{2E}(t-t_0)], & \delta = -1,
\end{cases}
(3.29)
\]

where

\[
 T = T(u, A) \equiv (2q^2|A|)^{-1/2} = \left( \frac{1}{2}|A < u, u > \ast| \right)^{-1/2}.
(3.32)
\]

For \( \delta = +1 \) the variable \( \tau = \tau(t) \) is monotonically decreasing from \( +\infty \) to 0, when \( t - t_0 \) is varying from 0 to \( +\infty \). For \( \delta = -1 \) it is monotonically decreasing from \( \pi T \) to 0, when \( t - t_0 \) is varying from \( -\infty \) to \( +\infty \).

It is not difficult to verify that the following relations take place

\[
\frac{\sinh(\tau \sqrt{\delta}/T)}{\sqrt{\delta}} = \frac{1}{f_\delta(q \sqrt{2E}(t-t_0))}, \quad (3.33)
\]

\[
\frac{\tanh(\tau \sqrt{\delta}/2T)}{\sqrt{\delta}} = \exp[-q \sqrt{2E}(t-t_0)], \quad (3.34)
\]

\[
d\tau = -qT \sqrt{2E} dt / f_\delta(q \sqrt{2E}(t-t_0)). \quad (3.35)
\]

Now, we introduce the following dimensionless "Kasner-like" parameters

\[
\beta^i = -e^i_a \bar{p}^a / (q \sqrt{2E}), \quad (3.36)
\]

\[
\beta_\varphi = -p^\varphi / (q \sqrt{2E}), \quad (3.37)
\]

Here and below \( \hat{a}, \hat{b} = 1, \ldots, n - 1 \). From relations (3.7), (3.14), (3.20) and (3.6) we have

\[
x^i = -(u^i/2q^2)z^0 + e^i_\hat{a}[\hat{p}^\hat{a}(t-t_0) + \hat{q}^\hat{a}] = -(u^i/4q^2)(2qz^0) - q \sqrt{2E}(t-t_0)\beta^i + \gamma^i, \quad (3.38)
\]

where

\[
\gamma^i = e^i_\hat{a} \hat{q}^\hat{a}, \quad \hat{q}^\hat{a} = q^\hat{a} + p^\hat{a}t_0. \quad (3.39)
\]

Using (3.13), (3.27), (3.33), (3.34) and (3.38) we get the following expression for the scale factors

\[
a_i = \exp(x^i) = A_i[\sinh(r\tau/T)/r]^\sigma^i[\tanh(r\tau/2T)/r]^\beta^i, \quad (3.40)
\]

where \( r = \sqrt{\delta} \), and

\[
\sigma^i = 2u^i / < u, u > \ast, \quad A_i = (E/|A|)^{\sigma^i/2} \exp(\gamma^i), \quad (3.41)
\]

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\[ i = 1, \ldots, n. \] In analogous manner we obtain the expression for the scalar field (see (3.34), (3.37))
\[ \exp(\kappa \varphi) = \exp(z^n) = A_\varphi [\tanh(r\tau/2T)/r]^{\beta_\varphi}, \] (3.42)

\( A_\varphi > 0 \) is constant.

We define a bilinear symmetrical form \( < ., . > : R^n \times R^n \to R \) by the relation
\[ < \alpha, \beta > = G_{ij} \alpha^i \beta^j, \] (3.43)
where \( \alpha = (\alpha^i), \beta = (\beta^i) \in R^n \). Using the definitions (3.13), (3.22), (3.36), (3.37) and relations (3.11), we obtain the relations between Kasner-like parameters
\[ < \beta, \beta > + (\beta_\varphi)^2 = G_{ij} \beta^i \beta^j + (\beta_\varphi)^2 = \frac{1}{q^2} = 4/ < u, u >, \] (3.44)
and (see (3.12))
\[ u_i \beta^i = e_0^i e_\hat{a}^i P^\hat{a} = \delta_\alpha^0 P^\alpha = 0, \] (3.45)
where \( P^\hat{a} = -p^\hat{a} \sqrt{2/E} \).

Analogously to (3.45) we get \( u_i \gamma^i = 0 \) and hence (see (3.41))
\[ \prod_{i=1}^n A_i^{u_i} = E/|A|. \] (3.46)

Thus, the additional integral of motion \( E \) is a certain combination of parameters \( A_i \) and \( |A| \) depending on the equation of state (2.8).

We introduce also a ”quasi-volume” scale factor
\[ v = \prod_{i=1}^n a_i^{u_i/2} = \exp(\frac{1}{2} u_i x^i). \] (3.47)

From (3.12), (3.27), (3.33), (3.46) (see also (3.40), (3.45) we have
\[ v = v_0 \sinh(r\tau/T)/r = (E/|A|)^{1/2} (f_\delta(q\sqrt{2E}(t - t_0)))^{-1}, \] (3.48)
Here
\[ v_0 = \prod_{i=1}^n A_i^{u_i/2}. \] (3.49)

The quasi-volume scale factor oscillates for \( A < 0 \) and exponentially increases as \( \tau \to +\infty \) for \( A > 0 \).

From (3.3), (3.32), (3.35), (3.47), (3.48) we get
\[ \exp[2\gamma_0(t)]dt \otimes dt = \left( \prod_{i=1}^n a_i^{2N_i} \right) f_\delta^2(q\sqrt{2E}(t - t_0))(2q^2 E T^2)^{-1} d\tau \otimes d\tau, \]
\[ = \left( \prod_{i=1}^n a_i^{2N_i-u_i} \right) d\tau \otimes d\tau. \] (3.50)
Thus we get the following solution of the field equations (2.3) and (2.4)

\[ g = -\left(\prod_{i=1}^{n}(a_i(\tau))^{2N_i-u_i}\right)d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau)g^{(i)}, \quad (3.51) \]

\[ a_i(\tau) = A_i[\sinh(r\tau/T)/r]^{2u_i/u} \cdot \tanh(r\tau/2T)/r]^{\beta^i}, \quad (3.52) \]

\[ \exp(\kappa \varphi(\tau)) = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \quad (3.53) \]

\[ \kappa^2 \rho(\tau) = A_\prod_{i=1}^{n}(a_i(\tau))^{u_i-2N_i}, \quad (3.54) \]

\[ i = 1, \ldots, n; \text{ where } r = \sqrt{A/|A|}, T \text{ is defined in (3.32), } A_i, A_\varphi > 0 \text{ are constants and the parameters } \beta^i, \beta_\varphi \text{ satisfy the relations} \]

\[ \sum_{i=1}^{n} u_i \beta^i = 0, \quad \sum_{i,j=1}^{n} G_{ij} \beta^i \beta^j + (\beta_\varphi)^2 = -4/ < u, u >^* = 1/q^2. \quad (3.55) \]

Here \( \tau > 0 \) for \( A > 0 \) and \( 0 < \tau < \pi T \) for \( A < 0 \).

We note that the solution (3.51)-(3.55) without scalar field (\( \beta_\varphi = 0 \)) was obtained previously in [47]. For \( u_i = 2N_i \) (A-term case) the solution was considered in [48] (for \( \beta_\varphi = 0 \) see also [46]), where Euclidean wormholes were constructed.

For small values of \( \tau \) we have the following asymptotic relations

\[ a_i(\tau) \sim C_i \tau^{\beta^i}, \quad \exp(\kappa \varphi(\tau)) \sim C_\varphi \tau^{\beta_\varphi} \quad (3.56) \]

as \( \tau \to 0, \) \( i = 1, \ldots, n, \) where \( C_i, C_\varphi \) are constants and \( \bar{\beta}^i = \beta^i + \sigma^i \) are the new Kasner-like parameters, satisfying the relations

\[ u_i \bar{\beta}^i = 2, \quad G_{ij} \bar{\beta}^i \bar{\beta}^j + \beta_\varphi^2 = 0. \quad (3.57) \]

### 3.2 Exceptional solutions

Now we consider the exceptional real solutions corresponding to \( \mathcal{E} = 0 \) and \( A > 0 \) (see (3.25)). From \( \mathcal{E} = 0 \) and (3.22) we have \( p^B = 0 \) and hence

\[ z^B = q^B \quad (3.58) \]

are constant, \( B = 1, \ldots, n. \) So, \( \kappa \varphi = z^n = \text{const} \) in this case. From (3.7) and (3.12) we have

\[ x^i = -(u^i/4q^2)(2q^2 z^0) + \gamma^i, \quad \gamma^i = e_\hat{a}^i q^\hat{a}, \quad (3.59) \]

(\( \hat{a} = 1, \ldots, n - 1 \)). Using (3.13), (3.25), (3.32) and (3.59) for \( t > t_0, \) we get

\[ a_i = \exp(x^i) = [(t - t_0)/T]^{-\sigma^i} \exp(\gamma^i), \quad (3.60) \]

\( i = 1, \ldots, n. \)

Introducing the new time variable

\[ T/(t-t_0) = \exp[\pm(\tau - \tau_0)/T], \quad t > t_0, \quad (3.61) \]

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we obtain
\[ a_i(\tau) = \bar{A}_i \exp(\pm \sigma^i \tau / T), \quad (3.62) \]

where
\[ \bar{A}_i = \exp(\mp \sigma^i \tau_0 / T) \exp(\gamma^i), \quad (3.63) \]
i = 1, \ldots, n.

Analogously to (3.45) we get
\[ u_i \gamma^i = 0 \text{ and hence (see (3.63))} \]
\[ \prod_{i=1}^{n} \bar{A}^{u_i}_i = \exp(\mp 2 \tau_0 / T). \quad (3.64) \]

For the quasi-volume from (3.62) and (3.64) we get
\[ v = \prod_{i=1}^{n} a_i^{u_i/2} = \exp[\pm (\tau - \tau_0) / T]. \quad (3.65) \]

Thus, for $A > 0$ we have a family of exceptional solutions with the constant real scalar field
\[ g = -(\Pi_{i=1}^{n} (a_i(\tau) 2^{N_i} - u_i)) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^i, \quad (3.66) \]
\[ a_i(\tau) = \bar{A}_i \exp[\pm 2u^i \tau / (T < u, u >)], \quad (3.67) \]
\[ \varphi(\tau) = \text{const}, \quad (3.68) \]

and $\rho(\tau)$ is defined by (3.54). Here $\bar{A}_i > 0$ ($i = 1, \ldots, n$) are constants, and $T$ is defined in (3.32).

We note that for $A > 0$ the solution (3.67) with the sign "+" is an attractor for the solutions (3.52), i.e.
\[ a_i(\tau) \sim \bar{A}_i \exp(\sigma^i \tau / T), \quad \varphi(\tau) \sim \text{const}, \quad (3.69) \]
i = 1, \ldots, n, \text{ for } \tau \to +\infty.

**Synchronous-time parametrization.** The relations (3.67) imply
\[ \prod_{i=1}^{n} a_i^{2^{N_i} - u_i} = \bar{P}^2 \exp[\pm 2(\bar{\sigma} - 1) \tau / T], \quad (3.70) \]

where
\[ \bar{P} = \Pi_{i=1}^{n} \bar{A}_i^{N_i - \frac{1}{2} u_i}, \quad (3.71) \]
\[ \bar{\sigma} = 2N_i u^i / < u, u >_{*} = < u^{(\Lambda)}, u >_{*} / < u, u >_{*}. \quad (3.72) \]

Here and below
\[ u_i^{(\Lambda)} = 2N_i. \quad (3.73) \]

Now, we introduce the synchronous time variable $t_s$ satisfying the relation
\[ \bar{P}^2 \exp[\pm 2(\bar{\sigma} - 1) \tau / T] d\tau \otimes d\tau = dt_s \otimes dt_s. \quad (3.74) \]
First, we consider the case

\[ \bar{\sigma} \neq 1 \iff < u^{(\Lambda)} - u, u >_{\ast} \neq 0. \]  

(3.75)

Introducing \( t_s \) by the formula

\[ t_s = \frac{\bar{P}T}{|\bar{\sigma} - 1|} \exp[\pm(\bar{\sigma} - 1)\tau/T] > 0 \]  

(3.76)

we get for the scale factors

\[ a_i = a_i(t_s) = A_i t_s^{\nu^i}, \]  

(3.77)

where

\[ \nu^i = \sigma^i/(\bar{\sigma} - 1) = 2u^i/ < u^{(\Lambda)} - u, u >_{\ast} \]  

(3.78)

and

\[ A_i = \bar{A}_i |[\bar{\sigma} - 1]/(\bar{P}T)]^{\nu^i}. \]  

(3.79)

The parameters \( \nu^i \) (3.78) satisfy the following relation

\[ \nu^i(2N_i - u_i) = 2. \]  

(3.80)

The relations (3.32), (3.71), (3.79) and (3.80) imply

\[ \prod_{i=1}^{n} A_i^{u_i - 2N_i} = T^2/((\bar{\sigma} - 1)^2) = -2 < u, u >_{\ast} / (A < u^{(\Lambda)} - u, u >_{\ast}^2). \]  

(3.81)

From (3.54), (3.77), (3.80) and (3.81) we get the following formula for the density

\[ \kappa^2 \rho = \kappa^2 \rho(t_s) = \frac{-2 < u, u >_{\ast}}{< u^{(\Lambda)} - u, u >^2_{\ast} t_s^2}. \]  

(3.82)

The metric reads as

\[ g = -dt_s \otimes dt_s + \sum_{i=1}^{n} a_i^2(t_s) g^{(i)}, \]  

(3.83)

where the scale factors are defined in (3.77), \( i=1, \ldots, n \). Thus, formulas (3.54), (3.77), (3.82), (3.83) and \( \varphi = \text{const} \) describe exceptional solutions for the case (3.70). We call these solutions as the power-law inflationary solutions.

Now we consider the case

\[ \bar{\sigma} = 1 \iff < u^{(\Lambda)} - u, u >_{\ast} = 0. \]  

(3.84)

From (3.54), (3.70) we get

\[ \kappa^2 \rho = A\bar{P}^{-2} = \text{const}. \]  

(3.85)

Introducing synchronous time \( t_s = \bar{P}\tau \), from (3.67) we get

\[ a_i(t_s) = \bar{A}_i \exp[\mp \frac{u^i}{\sqrt{- < u, u >_{\ast} T_0}} t_s], \]  

(3.86)
where

\[ T_0 = (2\kappa^2 \rho)^{-1/2}. \]  

(3.87)

The relations (3.83), (3.85)-(3.87) and \( \varphi = \text{const} \) describe the exponential-type inflation for the case (3.84).

Let us consider the synchronous time parametrization for the solutions (3.51)-(3.54). We have the following relation between the synchronous time \( t_s \) and the \( \tau \)-variable

\[ t_s = \varepsilon F(\tau), \quad \frac{dF}{d\tau} = f(\tau), \]  

(3.88)

where \( \varepsilon = \pm 1 \) and

\[ f(\tau) = \prod_{i=1}^{n}(a_i(\tau))^{N_i-\frac{1}{2}u_i} = P[\sinh(r\tau/T)/r]^{\bar{\sigma}-1}[\tanh(r\tau/2T)/r]^{\beta^iN_i}, \]  

\[ \kappa^2 \rho(\tau) = AP^{-2}[\sinh(r\tau/T)/r]^{2-2\bar{\sigma}}[\tanh(r\tau/2T)/r]^{-2\beta^iN_i}, \]  

(3.89)

with \( P = \prod_{i=1}^{n}A_i^{N_i-\frac{1}{2}u_i} \) and \( \bar{\sigma} \) is defined in (3.72). From (3.90) we have

\[ f(\tau) \sim B\tau^{p-1}, \quad \tau \to +0, \]  

(3.91)

where \( B > 0 \) is constant and

\[ p = p(\beta) = \bar{\sigma} + \beta^iN_i = (\sigma^i + \beta^i)N_i. \]  

(3.92)

Putting \( \varepsilon = \text{sign}(p) \) from (3.88) and (3.91) we get

\[ t_s \sim B_1\tau^p, \quad \tau \to +0, \]  

(3.93)

with \( B_1 = B/|p| \) (here the integration constant in (3.88) is properly fixed).

**Proposition 1.** Let \( 1/q^2 - (\beta_\varphi)^2 \geq 0 \). Then, for all \( \beta = (\beta^i) \) satisfying the relations (3.55), we have \( p(\beta) \neq 0 \) and

\[ i) \ u^iN_i < 0 \Rightarrow p(\beta) > 0, \]  

\[ ii) \ u^iN_i > 0 \Rightarrow p(\beta) < 0. \]  

(3.94)

(3.95)

The Proposition 1 is the special case of the more general Proposition 2 proved in the Appendix.

**Proposition 2.** Let two vectors \( u = (u_i), v = (v_i) \in R^n \) satisfy the inequalities:

\( < u, u >_s \equiv -4q^2 < 0 \) and \( < v, v >_s < 0 \). Then, \( u^i v_i = < u, v >_s \neq 0 \) and for all \( \beta = (\beta^i) \) satisfying the relations

\[ u_i\beta^i = 0, \quad G_{ij}\beta^i\beta^j \leq 1/q^2, \]  

(3.96)

the following relation is valid

\[ \text{sign}(u^i v_i) = -\text{sign}((\sigma^i + \beta^i)v_i), \]  

(3.97)

where \( \sigma^i = 2u^i/ < u, u >_s. \)
For the vector
\[ v_i = N_i = \frac{1}{2} u_i^{(A)} \] (3.98)
we have
\[ v^i = N^i = G^{ij} N_j = \frac{1}{2 - D} \] (3.99)
and hence
\[ < v, v >_\ast = N_i N^i = -\frac{D - 1}{D - 2} < 0. \] (3.100)
Thus the relations (3.92), (3.100) and the Proposition 2 imply the Proposition 1.

From (3.99) we get
\[ u_i N^i = \frac{1}{2 - D} \sum_{i=1}^{n} u_i, \] (3.101)
Using relation (3.93), (3.101) and Proposition 1 we obtain
\[ t_s \to +0, \quad \text{as } \tau \to +0, \] (3.102)
for
\[ A) \sum_{i=1}^{n} u_i > 0, \quad p(\beta) > 0 \] (3.103)
and
\[ t_s \to +\infty, \quad \text{as } \tau \to +0, \] (3.104)
for
\[ B) \sum_{i=1}^{n} u_i < 0, \quad p(\beta) < 0. \] (3.105)

In the limit \( \tau \to +0 \) we have \( \tau \sim (t_s/B_1)^{1/p} \) (see (3.93)) and hence (see (3.56))
\[ a_i(t_s) \sim B_i t_s^{\alpha^i}, \; \exp(\kappa \varphi(t_s)) \sim \bar{B}_\varphi t_s^{\alpha_\varphi} \] (3.106)
as \( t_s \to +0 \) in the case A) (3.103) and as \( t_s \to +\infty \) in the case B) (3.105). Here \( B_i, \bar{B}_\varphi \) are constants and
\[ \alpha^i = (\sigma^i + \beta^i)/p(\beta), \; \alpha_\varphi = \beta_\varphi/p(\beta), \] (3.107)
\( i = 1 \ldots n \). The parameters \( \alpha^i, \alpha_\varphi \) satisfy the Kasner-like relations
\[ \sum_{i=1}^{n} N_i \alpha^i = 1, \] (3.108)
\[ \sum_{i=1}^{n} N_i(\alpha^i)^2 + \alpha_\varphi^2 = 1. \] (3.109)
The first relation (3.108) is quite obvious, the second (3.109) is following from the first one, (2.12) and the relation
\[ G_{ij} \alpha^i \alpha^j + \alpha_\varphi^2 = 0, \] (3.110)
that can be readily verified using the relations (3.13), (3.41), (3.55) and (3.107).
The Kasner-like asymptotical behaviour (3.106), (3.108), (3.109) for the case A) agrees with one of the results of [55]: in the case A) the perfect fluid components with \( <u, u>_*<0 \) may be neglected near the singularity \( t_s \to +0 \) and we are lead to the Kasner-like formulas [52] (see also [39]).

Fig. 1

We note that for the case \( n=2 \) the following relation takes place:

\[
[b_2\frac{u_1}{N_1} - (1+s)\frac{u_2}{N_2}] [b_1\frac{u_2}{N_2} - (1+s)\frac{u_1}{N_1}] = (-s^2)(1+s) <u, u>_*,
\]

(3.111)

where \( b_i = 1 - \frac{1}{N_i}, i = 1, 2 \) and \( s = \sqrt{1-b_1b_2} \). This implies the relations for the light-cone lines \( <u, u>_*=0 \):

\[
l_1 : b_2(1-\xi_1) = (1+s)(1-\xi_2),
\]

(3.112)

\[
l_2 : b_1(1-\xi_2) = (1+s)(1-\xi_1),
\]

(3.113)

(see Fig.1).

### 3.3 Isotropization-like behaviour

Here we rewrite the attractor behaviour (3.69) for the non-exceptional solutions (3.51)-(3.55) with \( A > 0 \) (as \( \tau \to +\infty \)) in terms of the synchronous time variable \( t_s \). For the function (3.89) we have the following asymptotical behaviour

\[
f(\tau) \sim P\left[\frac{1}{2}\exp(\tau/T)\right]^{\sigma-1} = \bar{B} \exp[(\bar{\sigma} - 1)\tau/T],
\]

(3.114)

when \( \tau \to +\infty \) (\( \bar{B} = \text{const} \)).

First we consider the case \( \bar{\sigma} = 1 \) (see (3.84)). In this case \( f(\tau) \sim \bar{B} \) as \( \tau \to +\infty \) and hence (see (3.88))

\[
t_s = F(\tau) \sim \bar{B}\tau + C,
\]

(3.115)

as \( \tau \to +\infty \). (Due to \( u^iN_i < 0 \) and Proposition 1 \( \varepsilon = \text{sign}(p) = +1 \)). The synchronous time \( t_s \) is monotonically increasing from 0 to \( +\infty \) as \( \tau \) is varying from 0 to \( +\infty \) (see (3.93)).

From (3.54), (3.69) and (3.115) for the case \( \bar{\sigma} = 1 \) we get

\[
a_i(t_s) \sim \bar{A}_i \exp\left[-\frac{u^i}{\sqrt{-<u, u>_*} T_0} t_s\right],
\]

(3.116)

\[
\varphi(t_s) \sim \text{const},
\]

(3.117)

\[
\rho(t_s) \sim \rho_0.
\]

(3.118)

when \( t_s \to +\infty \), where \( T_0 = (2\kappa^2\rho_0)^{-1/2} \).

Now, we consider the case \( \bar{\sigma} \neq 1 \) (see (3.75)). In this case

\[
F(\tau) \sim \frac{\bar{B}T}{(\bar{\sigma} - 1)} \exp[(\bar{\sigma} - 1)\tau/T] + C,
\]

(3.119)
where $C$ is constant.

First, we consider the subcase: $\bar{\sigma} > 0$ or, equivalently, $u_iN^i < 0$ (or $\sum_{i=1}^{n} u_i > 0$, see (3.101)). We have $t_s = F(\tau)$, since $p > 0$ due to (3.103) and $\varepsilon = \text{sign}(p) = +1$. In this case $t_s$ is monotonically increasing from 0 to $T_*>0$ for $0 < \bar{\sigma} < 1$ and to $+\infty$ for $\bar{\sigma} > 1$ as $\tau$ is varying from 0 to $+\infty$ (see (3.93)). Using (3.54), (3.69) we get

$$a_i(t_s) \sim A_i(T_* - t_s)^{\nu^i},$$
$$\varphi(t_s) \sim \text{const},$$
$$\kappa^2\rho(t_s) \sim \frac{-2 < u, u >_*}{< u(A) - u, u >_*^2(T_* - t_s)^2},$$

as $t_s \to T_* - 0$, for $\bar{\sigma} < 1$. For $\bar{\sigma} > 1$ we have the asymptotic behaviour in the limit $t_s \to +\infty$ described by the relations

$$a_i(t_s) \sim A_i t_s^{\nu^i},$$
$$\varphi(t_s) \sim \text{const},$$
$$\kappa^2\rho(t_s) \sim \frac{-2 < u, u >_*}{< u(A) - u, u >_*^2 t_s^2},$$

as $t_s \to +\infty$, where $\nu^i$ is defined in (3.78).

Now, we consider the subcase: $\bar{\sigma} < 0$ or, equivalently, $u_iN^i > 0$ (or $\sum_{i=1}^{n} u_i < 0$, see (3.101)) We remind that $p < 0$ due to (3.105) and $\varepsilon = \text{sign}(p) = -1$. Then, we have $t_s = -F(\tau)$ (we put $C = 0$ in (3.119)) and $t_s$ is monotonically decreasing from $+\infty$ to 0 as $\tau$ is varying from 0 to $+\infty$ (see (3.93)). In the considered subcase we have the asymptotic behaviour in the limit $t_s \to +0$ described by the relations (3.123)-(3.125).

### 3.4 Solutions with pure imaginary scalar field

Now, we consider the solutions of the field equations with the complex scalar field and the real metric. In this case $\mathcal{E}, p^1, \ldots, p^{n-1}$ are real and hence (see (3.21), (3.22)) $p^n$ is either real or pure imaginary. The case of real $p^n$ was considered above.

For the pure imaginary $p^n$ we have three subcases: a) $\mathcal{E} > 0$, b) $\mathcal{E} = 0$, c) $\mathcal{E} < 0$. In the first case $\mathcal{E} > 0$ after the reparametrization (3.29)-(3.32) we get the solutions (3.51)-(3.55) with an imaginary value of $\beta_\varphi$. The cases b) and c): $\mathcal{E} \leq 0$ take place only for $A > 0$ (see (3.24), (3.25)).

Let us consider the case $\mathcal{E} < 0$. Here, we have (see (3.36), (3.37)) the imaginary $\beta^k$:

$$\beta^k = i\tilde{\beta}^k, \quad k = 1, \ldots, n, \quad (3.126)$$

and real $\beta_\varphi$. The solution may be obtained from (3.51)-(3.55) substituting (3.126) and $\tau/T \mapsto \tau/T + i\frac{\pi}{2}$.

\[ g = -\prod_{i=1}^{n} (a_i(\tau))^{2N_i - u_i}d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau)g(i), \quad (3.127) \]
\[ a_i(\tau) = \hat{A}_i[\cosh(\tau/T)]^{\sigma^i}[f(\tau/2T)]^{\hat{\beta}_i}, \quad (3.128) \]
\[ \varphi(\tau) = c + 2i\beta_\varphi \arctan \exp(-\tau/T), \quad (3.129) \]
where \( c, \hat{A}_i \neq 0 \) are constants, \( i = 1, \ldots, n \), \( T \) is defined in (3.32), \( \sigma^i \) are defined in (3.41), \( A > 0 \) and the real parameters \( \hat{\beta}^i, \beta_\varphi \) satisfy the relations

\[
\sum_{i=1}^{n} u_i \hat{\beta}^i = 0, \quad -\sum_{i,j=1}^{n} G_{ij} \hat{\beta}^i \hat{\beta}^j + (\beta_\varphi)^2 = -4/ \langle u, u \rangle_*= 1/q^2.
\] (3.130)

Here, like in [48],

\[
f(x) \equiv [\tanh(x + i\pi/4)]^i = \exp(-2 \arctan e^{-2x})
\] (3.131)

is the smooth monotonically increasing function bounded by its asymptotics:

\[ e^{-\pi} < f(x) < 1; \ f(x) \to 1 \text{ as } x \to +\infty \text{ and } f(x) \to e^{-\pi} \text{ as } x \to -\infty. \]

The solution (3.127)-(3.130) (with \( \rho \) from (3.54)) may be also obtained from formulas (3.20), (3.24). The relation between the harmonic time and \( \tau \)-variable (3.33) for the case \( E < 0 \) is modified

\[
\cosh(\tau/T) = 1/ \sin(q \sqrt{2|E|} (t-t_0)).
\] (3.132)

For the quasi-volume scale factor we have

\[
v = \prod_{i=1}^{n} a^{u_i/2}_i = (\prod_{i=1}^{n} \hat{A}^{u_i/2}_i) \cosh(\tau/T).
\] (3.133)

The scalar field \( \varphi(t) \) varies from \( c + i\pi \beta_\varphi \) to \( c \) as \( \tau \) varies from \( -\infty \) to \( +\infty \). The solution (3.127)-(3.130) for \( \tau \in (-\infty, +\infty) \) is non-singular. Any scale factor \( a_i(\tau) \) for some \( \tau_0 \) has a minimum and

\[
a_i(\tau) \sim A_i^\pm \exp(\sigma^i|\tau|/T),
\] (3.134)

for \( \tau \to \pm\infty \).

The "Lorentzian" solutions considered above have "Euclidean" analogues for \( A < 0 \) also

\[
g = (\prod_{i=1}^{n} (a_i(\tau))^{2N_i-u_i}) d\tau \otimes d\tau + \sum_{i=1}^{n} a_i^2(\tau) g^{(i)},
\] (3.135)

\[
a_i(\tau) = \hat{A}_i [\cosh(\tau/T)]^{\sigma^i} [f(\tau/2T)]^{\hat{\beta}_i},
\] (3.136)

\[
\varphi(\tau) = c + 2i\sigma_{n+1} \arctan \exp(-\tau/T),
\] (3.137)

with the parameters \( \hat{\beta}^i, \beta_\varphi \) satisfying the relations (3.130). When all spaces \( (M_i, g^{(i)}) \) are Riemannian, this solution may be interpreted as the classical Euclidean wormhole (instanton) solution.

An interesting special case of the solution (3.135)-(3.137) occurs for \( \hat{\beta}^i = 0, i = 1, \ldots, n, \) (this corresponds to \( p^{\hat{\beta}} = 0 \))

\[
a_i(\tau) = \hat{A}_i [\cosh(\tau/T)]^{\sigma^i},
\] (3.138)

\[
\varphi(\tau) = c \pm 2i q^{-1} \arctan \exp(-\tau/T).
\] (3.139)

All scale factors (3.138) have a minimum at \( \tau = 0 \) and are symmetric with respect to the time inversion: \( \tau \mapsto -\tau \). It is necessary to stress that here, like in [48], wormhole solutions take place only in the presence of the imaginary scalar field.
3.5 Some examples

In this subsection we consider some application of the obtained above formulas.

3.5.1. Isotropic case.

First, we consider the isotropic case:

\[ u_i = hN_i \iff u = \frac{h}{2}u^{(\Lambda)}, \quad p_i = (1 - h)\rho, \]  

(3.140)

where \( h \neq 0 \) is constant. From (3.98)-(3.100) and (3.140) we get

\[ u^i = \frac{h}{2-D}, \quad < u, u >_* = -h^2 \frac{D-1}{D-2} < 0, \]  

(3.141)

and hence

\[ \sigma^i = 2u^i / < u, u >_* = \frac{2}{h(D-1)} = \sigma(h) = \sigma. \]  

(3.142)

The solution (3.51)-(3.55) reads

\[ g = -\left( \prod_{i=1}^n (a_i(\tau))^{(2-h)N_i} \right) \, d\tau \otimes d\tau + \sum_{i=1}^n a_i^2(\tau) g^{(i)}, \]  

(3.143)

\[ a_i(\tau) = A_i[\sinh(r\tau/T)/r]^{\sigma(h)}[\tanh(r\tau/2T)/r]^{\beta'}, \]  

(3.144)

\[ \exp(\kappa\varphi(\tau)) = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi}, \]  

(3.145)

\[ \kappa^2 \rho(\tau) = A \prod_{i=1}^n (a_i(\tau))^{(h-2)N_i} = A(\prod_{i=1}^n A_i^{(h-2)N_i})[\sinh(r\tau/T)/r]^{2(h-2)/h}, \]  

(3.146)

\( i = 1, \ldots, n; \) where \( r = \sqrt{A/|A|}, \)

\[ T = |h|^{-1} \left[ \frac{|A|(D-1)}{2(D-2)} \right]^{-1/2}, \]  

(3.147)

\( A_i, A_\varphi > 0 \) are constants and the parameters \( \beta^i, \beta_\varphi \) satisfy the relations

\[ \sum_{i=1}^n N_i \beta^i = 0, \quad \sum_{i=1}^n N_i (\beta^i)^2 + (\beta_\varphi)^2 = \frac{4(D-2)}{h^2(D-1)}. \]  

(3.148)

The special case of this solution with \( h = 2 \) (\( \Lambda \)-term case) was considered in [48].

Now, we consider the exceptional solutions for \( A > 0 \). From (3.72) and (3.141) we have

\[ \tilde{\sigma} = \sigma^i N_i = 2/h, \quad < u^{(\Lambda)} - u, u >_* = h(h-2) \frac{D-1}{D-2}. \]  

(3.149)

From (3.149) we get: \( < u^{(\Lambda)} - u, u >_* = 0 \iff h = 2 \) (\( h \neq 0 \)). The matter corresponds to cosmological constant \( \Lambda = \kappa^2 \rho > 0 \). The relations (3.86), (3.141) imply the solution [46, 48] with metric (3.83) and

\[ a_i(t_s) = \tilde{A}_i \exp[\pm \frac{t_s \sqrt{2\Lambda}}{\sqrt{(D-1)(D-2)}}]. \]  

(3.150)
For \( h \neq 2 \) (\( \Leftrightarrow < u^{(A)} - u, u >_* \neq 0 \))

\[
\nu^i = 2u^i / < u^{(A)} - u, u >_* = \frac{2}{(2-h)(D-1)} = \nu(h) = \nu.
\] (3.151)

From (3.77), (3.82), (3.141), (3.149) and (3.151) we obtain the relations for scale factors and density:

\[
a_i(t_s) = A_i t_s^{\nu(h)},
\] (3.152)

\[
\kappa^2 \rho(t_s) = \frac{2(D-2)}{(h-2)^2(D-1)t_s^2}.
\] (3.153)

For \( h < 2 \) (or \( p > -\rho \)) we have an isotropic expansion of all scale factors and for \( h > 2 \) (or \( p < -\rho \)) we have an isotropic contraction (see Fig.1).

Kasner-like behaviour. In the considered case \( \sum_{i=1}^n u_i = h(D-1) \) and hence (see (3.102)-(3.108)) the Kasner-like behaviour (3.106), (3.108), (3.109) takes place as: A) \( t_s \to +0 \), for \( h > 0 \) (or \( p < -\rho \)), and B) \( t_s \to +\infty \), for \( h < 0 \) (or \( p > \rho \)).

Isotropization-like behaviour. Using the results of subsection 3.3 and (3.149) we are lead to the following attractor behaviour:

\[
a_i(t_s) \sim A_i t_s^{\nu(h)},
\] (3.154)

\[
\kappa^2 \rho(t_s) \sim \frac{2(D-2)}{(h-2)^2(D-1)t_s^2}.
\] (3.155)

in the limits \( t_s \to +\infty \), for \( 0 < h < 2 \) (or \( -\rho < p < \rho \)) and \( t_s \to +0 \), for \( h < 0 \) (or \( p > \rho \)).

Remark 2. For the dust matter case \( h = 1 \) (\( p = 0 \)), \( \rho > 0 \), the solution (3.143)-(3.148) has the synchronous-time representation:

\[
g = -dt_s \otimes dt_s + \sum_{i=1}^n a_i^2(t_s)g^{(i)}
\]

\[
a_i(t_s) = \bar{A}_i[t_s/(t_s + T_1)]^{\beta_i/2},
\] (3.160)

\[
\exp(2\kappa\varphi(t_s)) = A_\varphi[t_s/(t_s + T_1)]^{\beta_\varphi},
\] (3.161)

\[
\kappa^2 \rho(t_s) = 2(D-2)/[(D-1)t_s(t_s + T_1)].
\] (3.162)

\( i = 1, \ldots, n \); where \( 0 < t_s < +\infty \), \( T_1 > 0 \), \( \bar{A}_i, A_\varphi > 0 \) are constants and the parameters \( \beta_i, \beta_\varphi \) satisfy the relations

\[
\sum_{i=1}^n N_i \beta_i = 0, \quad \sum_{i=1}^n N_i (\beta_i)^2 + (\beta_\varphi)^2 = 4(D-2)/(D-1).
\] (3.159)

The special case of this solution with \( \beta_\varphi = 0 \) was considered previously in [30] (for \( n = 2 \) and \( N_1 = \ldots N_n \) see [17] and [18] correspondingly.)

3.5.2 Curvature-like component.

Now we consider the perfect fluid matter with

\[
u_i = 2h(-\delta_i^1 + N_i) = hu_{i}^{(1)},
\] (3.160)
where \( h \neq 0 \) is constant and \( N_1 > 1 \). For \( h = 1 \) this component corresponds to the curvature term for the first space [31] (see below). The calculation gives

\[
u^i = -\frac{2h}{N_1} \delta_1^i, \quad <u, u>_* = -4h^2b_1 < 0,\]

(3.161)

where \( b_1 = 1 - \frac{1}{N_1} \) and

\[
<u, u^{(A)}>_* = 2u^i N_i = -4h, \quad <u, u^{(A)} - u>_* = 4h(-1 + hb_1), \quad \sigma^i = \frac{\delta_1^i}{h(N_1 - 1)}.\]

(3.162)

Using (3.160) and (3.162) we get from (3.51)-(3.55):

\[
g = -(a_1(\tau))^{2h}(\Pi_{i=1}^n(a_i(\tau))^{2N_i(1-h)})d\tau \otimes d\tau + \sum_{i=1}^n a_i^2(\tau)g^{(i)},\]

(3.163)

\[
a_1(\tau) = A_1[\sinh(r\tau/T)/r]^{\frac{1}{\beta_1(N_1-1)}}[\tanh(r\tau/2T)/r]^{\beta_1},\]

(3.164)

\[
a_i(\tau) = A_i[\tanh(r\tau/2T)/r]^{\beta_i}, \quad i > 1,
\]

(3.165)

\[
\exp(\kappa\varphi(\tau)) = A_\varphi[\tanh(r\tau/2T)/r]^{\beta_\varphi},\]

(3.166)

\[
\kappa^2\rho(\tau) = A(a_1(\tau))^{2h}(\Pi_{i=1}^n(a_i(\tau))^{2N_i(h-1)})\]

(3.167)

\[i = 1, \ldots, n; \quad \text{where } r = \sqrt{A/|A|}, \quad T = |h|^{-1}(2|A|b_1)^{-1/2}, \quad A_i, A_\varphi > 0 \]

are constants and the parameters \( \beta_1, \beta_\varphi \) satisfy the relations

\[
\beta_1 = \frac{1}{1 - N_1} \sum_{i=2}^n N_i \beta_i, \quad \left(\sum_{i=2}^n N_i \beta_i^2\right) + (N_1 - 1)[\sum_{i=2}^n N_i (\beta_i^2 + (\beta_\varphi)^2)] = N_1 h^{-2}.
\]

(3.168)

For \( h \neq h_0 \equiv b_1^{-1} = N_1/(N_1 - 1) > 1 \) we have from (3.162) \( <u, u^{(A)} - u>_* \neq 0 \) and (see (3.78))

\[
\nu^i = \delta_1^i \nu(h), \quad \nu(h) = [N_1 + h(1 - N_1)]^{-1}.
\]

(3.169)

The power-law inflationary solution for the considered case reads:

\[
g = -dt \otimes dt + A_1^2 t_\nu^2 d\nu^{(1)} + \sum_{i=2}^n A_i^2 g^{(i)},
\]

(3.170)

\[
\varphi = \text{const},
\]

(3.171)

\[
\kappa^2 \rho(t_\nu) = \frac{b_1}{2(-1 + hb_1)^2 t_\nu^2}.
\]

(3.172)

The scale factors of internal spaces in this solution are constant (we have the so-called "spontaneous compactification"). It is not difficult to show that constancy of internal scale factors leads to the equation of state (3.160).

Using the relation \( \bar{\sigma} = h_0/h \) and the analysis carried out in subsection 3.3 we obtain that the solution (3.169)-(3.172) is an attractor for non-exceptional solutions with \( \rho > 0 \) as \( t_\nu \to T_\nu - 0, \) for \( h > h_0; \) \( t_\nu \to +\infty, \) for \( 0 < h < h_0 \) and \( t_\nu \to +0, \) for \( h < 0. \) So, we obtained the solutions with the "dynamical compactification".
with a scalar field $\varphi = \varphi(t)$ and metric (2.1) defined on the manifold (2.2), where $(M_i, g^{(i)})$, $i = 2, \ldots, n$; are Ricci-flat spaces and $(M_1, g^{(1)})$ is an Einstein space of non-zero curvature, i.e. $R_{mn}g^{(i)} = \lambda_1^1g_{mn}^{(1)}$, $\lambda_1 \neq 0$. Here $n \geq 2$ and $N_i = \text{dim}M_i$. This "1-curvature model" is equivalent to a special case of the considered above model (3.160) with $h = 1$ and $A = -\frac{1}{2}\lambda_1^1N_1$. (see [39]). The solution (3.163)-(3.168) reads for this case

$$g = (a_1(\tau))^2[-d\tau \otimes d\tau + g^{(1)}] + \sum_{i=2}^n a_i^2(\tau)g^{(i)},$$

$$a_1(\tau) = A_1[\sinh(r\tau/T)/\tau]^{\frac{1}{(N_1-2)}}[\tanh(r\tau/2T)/\tau]^{\beta_1},$$

$$a_i(\tau) = A_i[\tanh(r\tau/2T)/\tau]^{\beta_i}, \quad i > 1,$$

$$\exp(\kappa\varphi(\tau)) = A_\varphi[\tanh(r\tau/2T)/\tau]^{\beta_\varphi},$$

$$\kappa^2\rho(\tau) = (a_1(\tau))^{-2}$$

$i = 1, \ldots, n$; where $r = \sqrt{-\lambda_1^1/|\lambda_1^1|}$, $T = [|\lambda_1^1|(N_1 - 1)]^{-1/2}$, $A_1, A_\varphi > 0$ are constants and the parameters $\beta_i, \beta_\varphi$ satisfy the relations

$$\beta_1 = \frac{1}{1 - N_1}\sum_{i=2}^n N_i\beta_i,$$

$$\frac{1}{1 - N_1}(\sum_{i=2}^n N_i\beta_i)^2 + \sum_{i=2}^n N_i(\beta_i)^2 + (\beta_\varphi)^2 = \frac{N_1}{(N_1 - 1)}.$$  

The power-law inflationary solution for the negative curvature case $\lambda_1 < 0$ reads:

$$g = -dt_s \otimes dt_s + A_1^2t_s^2g^{(1)} + \sum_{i=2}^n A_i^2g^{(i)},$$

$$\varphi = \text{const},$$

where $A_1^2 = |\lambda_1^1|/(N_1 - 1)$ (see (3.81), (3.172)). We are lead here to the Milne-type solution recently considered in [51].

There exist another parametrization of the solution (3.174)- (3.179) in terms of $R$-variable related to $\tau$-variable as

$$F = F(R) = 1 - (\frac{R_0}{R})^{N_1-1} = \tanh^2(\tau/2T), \quad \lambda_1 < 0,$$

$$= (\frac{R_0}{R})^{N_1-1} - 1 = \tan^2(\tau/2T), \quad \lambda_1 > 0.$$  

Here $R > R_0$ for $\lambda_1 < 0$ and $R < R_0$ for $\lambda_1 > 0$; $R_0 = A_12^{1/(N_1-1)}\sqrt{(N_1 - 1)/|\lambda_1^1|}$. In new variables the metric and the scalar field may be written as

$$g = -F^b\otimes dR \otimes dR + F^bR^2A_1^2g^{(1)} + \sum_{i=2}^n F^{\beta_i}A_i^2g^{(i)},$$

$$\exp(2\kappa\varphi(R)) = A_\varphi^2F^{\beta_\varphi},$$

$$\kappa^2\rho(R) = F^{-2}.$$
\[ A_1^2 = |\lambda^1/(N_1 - 1)|, \quad A_i, A_\varphi > 0 \] are constants and
\[ b = (1 - \sum_{i=2}^{n} N_i \beta^i)/(N_1 - 1) = (N_1 - 1)^{-1} + \beta^1, \quad (3.186) \]
and the parameters \( \beta^i(i > 1), \beta_\varphi \) satisfy the relations (3.179). The special case of the solution (3.179), (3.184)-(3.186) with \( \beta_\varphi = 0 \) (constant scalar field) was obtained earlier in [39].

Remark 3. As a special case of the presented above solution we get a scalar-vacuum analogue of spherically-symmetric Tangherlini solution [80] with \( n \) Ricci-flat internal spaces:
\[ g = -f^a dt \otimes dt + f^{b-1} dR \otimes dR + f^b R^2 d\Omega_2^2 + \sum_{i=1}^{n} f^{a_i} B_i g^{(i)}, \quad (3.187) \]
\[ \exp(2\kappa \varphi(R)) = B_\varphi f^{a_\varphi}, \quad (3.188) \]
where \( d\Omega_2^2 \) is the canonical metric on \( d \)-dimensional sphere \( S^d(d \geq 2) \), \( f = f(R) = 1 - BR^{1-d} \), \( B_\varphi, B_i > 0, B \) are constants and the parameters \( a, a_1, \ldots, a_n \) satisfy the relation
\[ b = (1 - a - \sum_{i=1}^{n} a_i N_i)/(d - 1), \quad (3.189) \]
\[ (a + \sum_{i=1}^{n} a_i N_i)^2 + (d - 1)(a_\varphi^2 + \sum_{i=1}^{n} a_\varphi^2 N_i) = d. \quad (3.190) \]
For \( a_\varphi = 0 \) see also [60, 64]. In the parametrization of the harmonic-type variable this solution was presented earlier in [62, 64].

4 Wheeler-DeWitt equation

The quantization of the zero-energy constraint (3.17) leads to the Wheeler-DeWitt (WDW) equation in the harmonic time gauge (3.3) [31, 47, 48]
\[ 2\hat{H} \Psi = \left[ \frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} - \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} + 2A \exp(2qz^0) \right] \Psi = 0. \quad (4.1) \]

We are seeking the solution of (4.1) in the form
\[ \Psi(z) = \exp(i\vec{p}z)\Phi(z^0), \quad (4.2) \]
where \( \vec{p} = (p^1, \ldots, p^n) \) is a constant vector (generally from \( \mathbb{C}^n \)), \( \vec{z} = (z^1, \ldots, z^{n-1}, z^n = \kappa \varphi) \), \( \vec{p}z \equiv \sum_{i=1}^{n} p_i z^i \). Substitution of (4.2) into (4.1) gives
\[ [-\frac{1}{2}(\frac{\partial}{\partial z^0})^2 + V_0(z^0)] \Phi = \mathcal{E} \Phi, \quad (4.3) \]
where \( \mathcal{E} = \frac{1}{2} \vec{p} \vec{p} \) and \( V_0(z^0) = -A \exp(2qz^0) \). Solving (4.3), we get
\[ \Phi(z^0) = B_i \sqrt{2q} \sqrt{-2Aq^{-1} \exp(qz^0)} \], where \( B_i = 0 \). In the parametrization of the harmonic-type variable this solution was presented earlier in [62, 64].
where $i\sqrt{2E}/q = i|\mathbf{p}|/q$, and $B = I, K$ are modified Bessel functions. We note, that

$$v = \exp(qz^0) = \prod_{i=1}^{n} a_i^{u_i/2} \quad (4.5)$$

is the ”quasi-volume” (3.47) (see (3.12)).

The general solution of Eq. (4.1) has the following form

$$\Psi(z) = \sum_{B=I,K} \int d^3\mathbf{p} \ C_B(\mathbf{p}) e^{i\mathbf{p}\mathbf{z}} B_i|\mathbf{p}|/q (\sqrt{-2Aq^{-1}} \exp(qz^0)), \quad (4.6)$$

where functions $C_B$ ($B = I, K$) belong to an appropriate class. For the $\Lambda$-term case this solution was considered in [46, 48] and for the two-component model ($n = 2$) and $\Lambda > 0$ in [94].

In the ground state we put all momenta $p^a(a = 1, \ldots, n)$ equal to zero and the ground state wave function reads

$$\Psi_0 = B_0 \left(\sqrt{-2Aq^{-1}} \exp(qz^0)\right). \quad (4.7)$$

The function $\Psi_0$ is invariant with respect to the rotation group $O(n)$.

Remark 4. Applying the arguments considered in [40, 48] one may show that the ground state wave function

$$\Psi_0^{(HH)} = I_0 \left(\frac{\sqrt{2|A|}}{q} \exp(qz^0)\right), \quad A < 0, \quad (4.8)$$

$$J_0 \left(\frac{\sqrt{2|A|}}{q} \exp(qz^0)\right), \quad A > 0, \quad (4.9)$$

satisfies the Hartle-Hawking boundary condition [96]. The special cases of this formula were considered in refs. [40] (1-curvature case) and [48] ($\Lambda$-term case).

From the equation (4.3) it follows that in the case $A < 0$ a Lorentzian region exists as well as an Euclidean one for $E > 0$. In the case $A > 0$ only the Lorentzian region occurs for $E \geq 0$ but for $E < 0$ both regions exist. The wave functions (4.2), (4.4) with $A > 0$ and $E < 0$ describe transitions between the Euclidean and Lorentzian regions, i.e. tunneling universes.

### 4.1 Quantum wormholes

Here we consider only real values of $p_i$. In this case we have $E \geq 0$.

If $A > 0$ the wave function $\Psi$ (4.2) is not exponentially damped when $v \to \infty$, i.e. the condition (i) for quantum wormholes (see the Introduction) is not satisfied. It oscillates and may be interpreted as corresponding to the classical Lorentzian solution.

For $A < 0$, the wave function (4.2) is exponentially damped for large $v$ only, when $B = K$ in (4.4). (We recall that

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \quad K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z},$$

21
for \( z \to \infty \). But in this case the function \( \Phi \) oscillates an infinite number of times, when \( v \to 0 \). So, the condition (ii) is not satisfied. The wave function describes the transition between Lorentzian and Euclidean regions.

The functions
\[
\Psi_{\vec{p}}(z) = e^{i\vec{p}\cdot \vec{z}}K_{i|\vec{p}|/q}(\sqrt{-2A}q^{-1}e^{qz^0}),
\]
may be used for constructing quantum wormhole solutions. Like in [83, 84, 46, 48] we consider superpositions of the singular solutions
\[
\hat{\Psi}_{\lambda,\vec{n}}(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \Psi_{qk\vec{n}}(z)e^{-ik\lambda},
\]
where \( \lambda \in \mathbb{R} \) and \( \vec{n} \in S^{n-1} \) is a unit vector (\( \vec{n}^2 = 1 \)). The calculation gives
\[
\hat{\Psi}_{\lambda,\vec{n}}(z) = \exp\left[-\frac{\sqrt{-2A}}{q} e^{qz^0} \cosh(\lambda - q\vec{z}\vec{n})\right].
\]
It is not difficult to verify that the formula (4.12) leads to the solutions of the WDW equation (4.1), satisfying the quantum wormholes boundary conditions.

We note that the functions
\[
\Psi_{m,\vec{n}} = H_m(x^0)H_m(x^1)\exp\left[-\frac{(x^0)^2 + (x^1)^2}{2}\right],
\]
where
\[
x^0 = (2/q)^{1/2}(-2A)^{1/4}\exp(qz^0/2)\cosh\left(\frac{1}{2}q\vec{z}\vec{n}\right),
\]
\[
x^1 = (2/q)^{1/2}(-2A)^{1/4}\exp(qz^0/2)\sinh\left(\frac{1}{2}q\vec{z}\vec{n}\right),
\]
m = 0, 1, ..., are also solutions of the WDW equation with the quantum wormhole boundary conditions. Solutions of such type were previously considered in [82, 41, 42, 46, 48]. They are called the discrete spectrum quantum wormholes (see [84]) (and may form a basis in the Hilbert space of the system [85]).

Thus, in the case considered the quantum wormhole solutions (with respect to quasi-volume (4.5)) exist for the matter with a negative density (3.2) (\( A < 0 \)).

5 Third quantized model.

Here we put \( A > 0 \), i.e. the density of matter is positive. We consider the case of a real \( \Psi \)-field as in [53] for simplicity. The WDW equation (4.1) corresponds to the action
\[
S = \frac{1}{2} \int d^{n+1}z \Psi \dot{\Psi}.
\]
Let us consider two bases of the solutions of the WDW equation \( \{\Psi_{in}(\vec{p}), \Psi_{in}^*(\vec{p})\}, \{\Psi_{out}(\vec{p}), \Psi_{out}^*(\vec{p})\} \)
\[
\Psi_{in}(\vec{p}) = \Psi_{in}(\vec{p}, z) = \left[2q\sinh(|\vec{p}|/q)\right]^{1/2} J_{-i|\vec{p}|/q} \left(\frac{\sqrt{2A}}{q} e^{qz^0}\right) (2\pi)^{-n/2} \exp(i\vec{p}\vec{z})
\]
\[
\Psi_{out}(\vec{p}) = \Psi_{out}(\vec{p}, z) = \frac{1}{2} \left(\frac{\pi}{q}\right)^{1/2} H_{i|\vec{p}|/q}^{(2)} \left(\frac{\sqrt{2A}}{q} e^{qz^0}\right) (2\pi)^{-n/2} \exp(i\vec{p}\vec{z})
\]
where $J_\nu$ and $H_\nu^{(2)}$ are the Bessel and Hankel functions respectively. These solutions are normalized by the following conditions

$$\left( \Psi_{\text{in}}(\vec{p}), \Psi_{\text{in}}(\vec{p}') \right) = \left( \Psi_{\text{out}}(\vec{p}), \Psi_{\text{out}}(\vec{p}') \right) = \delta \left( \vec{p} - \vec{p}' \right)$$  \hspace{1cm} (5.4)

where

$$\left( \Psi_1, \Psi_2 \right) = i \int d^n \vec{z} \left( \Psi_1^* \partial_0 \Psi_2 \right)$$  \hspace{1cm} (5.5)

is the charge form (indefinite scalar product). Here $\Psi_1 \partial \Psi_2 = \Psi_1 \partial \Psi_2 - (\partial \Psi_1) \Psi_2$. Due to asymptotic behaviour

$$\Psi_{\text{in}}(\vec{p}, z) \sim c_{\text{in}}(|\vec{p}|) \exp(i|\vec{p}|z^0), \quad v \to 0,$$
$$\Psi_{\text{out}}(\vec{p}, z) \sim \frac{c_{\text{out}}(|\vec{p}|)}{\sqrt{v}} \exp(i|\vec{p}|z - i\frac{\sqrt{2A}}{q}v), \quad v \to +\infty.$$  \hspace{1cm} (5.6) (5.7)

where $\Psi_{\text{in}}(\vec{p}, z)$ and $\Psi_{\text{out}}(\vec{p}, z)$ are negative frequency modes of “Kasner”- and “Milne”- types respectively.

The standard quantization procedure [87, 88] give us

$$\Psi(z) = \int d^n \vec{p} \left[ a_{\text{in}}^+(\vec{p}) \Psi_{\text{in}}^*(\vec{p}, z) + a_{\text{in}}(\vec{p}) \Psi_{\text{in}}(\vec{p}, z) \right]$$
$$= \int d^n \vec{p} \left[ a_{\text{out}}^+(\vec{p}) \Psi_{\text{out}}^*(\vec{p}, z) + a_{\text{out}}(\vec{p}) \Psi_{\text{out}}(\vec{p}, z) \right],$$  \hspace{1cm} (5.8)

where the non-trivial commutators read

$$[a_{\text{in}}(\vec{p}, a_{\text{in}}^+(\vec{p}')]) = [a_{\text{out}}(\vec{p}), a_{\text{out}}^+(\vec{p}')] = \delta \left( \vec{p} - \vec{p}' \right).$$  \hspace{1cm} (5.9)

In” and ”out” vacuum states satisfy the relations

$$a_{\text{in}}(\vec{p})|0, \text{in} >= a_{\text{out}}(\vec{p})|0, \text{out} >= 0.$$  \hspace{1cm} (5.10)

The modes (5.2) and (5.3) are related by the Bogoljubov transformation

$$\Psi_{\text{in}}(\vec{p}) = \alpha(|\vec{p}|) \Psi_{\text{out}}(\vec{p}) + \beta(|\vec{p}|) \Psi_{\text{out}}^*(\vec{p})$$  \hspace{1cm} (5.11)

$$\alpha(\vec{p}) = \left[ \frac{\exp(\pi |\vec{p}|/q)}{2 \sinh(\pi |\vec{p}|/q)} \right]^{1/2}, \beta(\vec{p}) = \left[ \frac{\exp(-\pi |\vec{p}|/q)}{2 \sinh(\pi |\vec{p}|/q)} \right]^{1/2}. $$  \hspace{1cm} (5.12)

The vacuums $|0, \text{in} >$ and $|0, \text{out} >$ are unitary non-equivalent. The standard calculation [87, 88] gives for a number density of “out-universes” (of ”Milne-type”) containing in the ”in-vacuum” (”Kasner-type” vacuum)

$$n(\vec{p}) = |\beta(\vec{p})|^2 = (\exp(2\pi |\vec{p}|/q) - 1)^{-1}.$$  \hspace{1cm} (5.13)

So, we obtained the Planck distribution with the temperature

$$T_{Pl} = q/2\pi = \sqrt{-<u,u>}/4\pi.$$  \hspace{1cm} (5.14)
The temperature (5.14) depends upon the vector $u = (u_i)$ (i.e. on the equation of state): $T_{Pl} = T_{Pl}(u)$. For example, we get $T_{Pl}(u^{(dust)}) = 2T_{Pl}(u^{(dust)})$. In the Zeldovich matter limit $u \to 0$ we have $T_{Pl} \to +0$.

Remark 5. In [97] a regularization of propagators (in quantum field theory) was introduced using the complex signature matrix

$$(\eta_{ab}(w)) = \text{diag}(w, 1, \ldots, 1),$$

where $w \in C \setminus (-\infty, 0]$ is the complex parameter (Wick parameter). Originally path integrals are defined (in covariant manner) for $w > 0$ (i.e. in Euclidean-like region) and then should be analytically continued to negative $w$. The Minkowsky space limit corresponds to $w = -1 + i0$ (in notations of [97] $w^{-1} = -a$). The prescription [97] is a natural realization of the Wick's rotation. In [98] the analogs of the Bogolubov-Parasjuk theorems [86] for a wide class of propagators regularized by the complex metric (5.15) were proved. This formalism may be applied for third-quantized models of the multidimensional cosmology. In this case the corresponding path integrals should be analytically continued from the interval $1 < D < 2$ ($D$ is dimension), where minisuperspace metric (2.12) is Euclidean, to $D = D_0 - i0$, $D_0 = 1 + \sum_{i=1}^a N_i$. We note also that recently J. Greensite proposed an idea of considering the space-time signature as a dynamical degree of freedom [99] (see also [100, 101]).

6 Appendix

Proof of Proposition 2. We introduce the new "diagonalized" variables

$$\beta^a = e^a_i \beta^i, \quad u_a = e^a_i u_i, \quad v_a = e^a_i v_i$$

(6.1)

where matrices $(e^a_i), (e^a_i)$ satisfy the relations (3.11)-(3.12) and (3.14). From (3.12)-(3.14) we have

$$(u_a) = (2q, \vec{0}) \quad (\sigma^a) = (\sigma^i e^a_i) = (q^{-1}, \vec{0})$$

(6.2)

and consequently (see (3.96))

$$0 = \beta^a u_i = \beta^a u_a = 2q \beta^0 \Rightarrow (\beta^a) = (0, \vec{\beta}).$$

(6.3)

From the second relation in (3.96) we get

$$G_{ij} \beta^i \beta^j = \eta_{ab} \beta^a \beta^b = \vec{\beta}^2 \leq 1/q^2.$$  

(6.4)

For the vector $(v_a) = (v_0, \vec{v})$ we have $-v_0^2 + \vec{v}^2 = < v, v > < 0$ and hence

$$|v_0| > |\vec{v}|, \quad v_0 \neq 0.$$  

(6.5)

We also obtain from (6.2) and (6.5)

$$< u, v > = -u_0 v_0 = -2qv_0 \neq 0.$$  

(6.6)
Using relations (6.2), (6.3) and (6.5) we get

\[(\sigma^i + \beta^i)v_i = (\sigma^a + \beta^a)v_a \equiv q^{-1}v_0 + \vec{\beta}\vec{v} = q^{-1}v_0(1 + \frac{q}{v_0}\vec{\beta}\vec{v}).\]  \hspace{1cm} (6.7)

Eqs. (6.4), (6.5) imply the following inequality

\[\left|\frac{q}{v_0}\vec{\beta}\vec{v}\right| \leq \frac{\vec{v}}{v_0}\frac{\vec{v}}{v_0} \leq \frac{v}{v_0} < 1.\]  \hspace{1cm} (6.8)

From (6.6)-(6.8) (and \(q > 0\)) we get the proposed identity (3.97). The proposition is proved.
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Fig. 1. The graphical representation of the allowed cone $\langle u, u \rangle < 0$ and different domains in it on the plane $\xi_i = p_i/\rho$, $i = 1, 2$, for the case $n = 2$ and $N_1 = 3$, $N_2 = 6$. The hyperbola $\bar{\sigma} = 1$ corresponds to the exponential inflation (3.86).