MULTIGRADED RINGS, DIAGONAL SUBALGEBRAS, AND
RATIONAL SINGULARITIES

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1. Introduction

We study the properties of F-rationality and F-regularity in multigraded rings
and their diagonal subalgebras. The main focus is on diagonal subalgebras of bi-
graded rings: these constitute an interesting class of rings since they arise naturally
as homogeneous coordinate rings of blow-ups of projective varieties.

Let $X$ be a projective variety over a field $K$, with homogeneous coordinate ring $A$.
Let $a \subset A$ be a homogeneous ideal, and $V \subset X$ the closed subvariety defined by $a$.
For $g$ an integer, we use $a_g$ to denote the $K$-vector space consisting of homogeneous
elements of $a$ of degree $g$. If $g \gg 0$, then $a_g$ defines a very ample complete linear
system on the blow-up of $X$ along $V$, and hence $K[a_g]$ is a homogeneous coordinate
ring for this blow-up. Since the ideals $a^h$ define the same subvariety $V$, the rings
$K[a^h]_g$ are homogeneous coordinate ring for the blow-up provided $g \gg h > 0$.

Suppose that $A$ is a standard $\mathbb{N}$-graded $K$-algebra, and consider the $\mathbb{N}^2$-grading
on the Rees algebra $A[at]$, where $\deg rt^j = (i, j)$ for $r \in A_i$. The connection with
diagonal subalgebras stems from the fact that if $a^h$ is generated by elements of
degree less than or equal to $g$, then

$$K[a^h]_g \cong \bigoplus_{k \geq 0} A[at]_{(gk, hk)}.$$

Using $\Delta = (g, h)\mathbb{Z}$ to denote the $(g, h)$-diagonal in $\mathbb{Z}^2$, the diagonal subalgebra
$A[at]_{\Delta} = \bigoplus_k A[at]_{(gk, hk)}$ is a homogeneous coordinate ring for the blow-up of Proj $A$
along the subvariety defined by $a$, whenever $g \gg h > 0$.

The papers [GG, GGH, GGP, TR] use diagonal subalgebras in studying blow-
ups of projective space at finite sets of points. For a polynomial ring and $a$
a homogeneous ideal, the ring theoretic properties of $K[a]$ are studied by Simis,
Trung, and Valla in [STV] by realizing $K[a]$ as a diagonal subalgebra of the Rees
algebra $A[at]$. In particular, they determine when $K[a]$ is Cohen-Macaulay for $a$
a complete intersection ideal generated by forms of equal degree, and also for $a$ the

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ideal of maximal minors of a generic matrix. Some of their results are extended by
Conca, Herzog, Trung, and Valla as in the following theorem:

**Theorem 1.1.** [CHTV, Theorem 4.6] Let $K[x_1,\ldots,x_m]$ be a polynomial ring over
a field, and let $\mathfrak{a}$ be a complete intersection ideal minimally generated by forms of
degrees $d_1,\ldots,d_r$. Fix positive integers $g$ and $h$ with $g/h > d = \max\{d_1,\ldots,d_r\}$.

Then $K[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay if and only if $g > (h-1)d - m + \sum_{j=1}^r d_j$.

When $A$ is a polynomial ring and $\mathfrak{a}$ an ideal for which $A[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay,
Lavila-Vidal [Lv1, Theorem 4.5] proved that the diagonal subalgebras $K[(\mathfrak{a}^h)_g]$ are
Cohen-Macaulay for $g \gg h > 0$, thereby settling a conjecture from [CHTV]. In
[CH] Cutkosky and Herzog obtain affirmative answers regarding the existence of
a constant $c$ such that $K[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay whenever $g \geq c h$. For more
work on the Cohen-Macaulay and Gorenstein properties of diagonal subalgebras,
see [HHR, Hy2, Lv2], and [LvZ].

As a motivating example for some of the results of this paper, consider a poly-
nomial ring $A = K[x_1,\ldots,x_m]$ and an ideal $\mathfrak{a} = (z_1,z_2)$ generated by relatively
prime forms $z_1$ and $z_2$ of degree $d$. Setting $\Delta = (d+1,1)\mathbb{Z}$, the diagonal subalgebra
$A[\mathfrak{a}^h]_\Delta$ is a homogeneous coordinate ring for the blow-up of $\text{Proj} A = \mathbb{P}^{m-1}$ along
the subvariety defined by $\mathfrak{a}$. The Rees algebra $A[\mathfrak{a}^h]_\Delta$ has a presentation

$$R = K[x_1,\ldots,x_m,y_1,y_2]/(y_2z_1 - y_1z_2),$$

where $\deg x_i = (1,0)$ and $\deg y_j = (d,1)$, and consequently $R_\Delta$ is the subalgebra of
$R$ generated by the elements $x_i, y_j$. When $K$ has characteristic zero and $z_1$ and $z_2$
are general forms of degree $d$, the results of Section 3 imply that $R_\Delta$ has rational
singularities if and only if $d \leq m$, and that it is of F-regular type if and only
if $d < m$. As a consequence, we obtain large families of rings of the form $R_\Delta$,
standard graded over a field, which have rational singularities, but which are not
of F-regular type.

It is worth pointing out that if $R$ is an $\mathbb{N}^2$-graded ring over an infinite field
$R_{(0,0)} = K$, and $\Delta = (g,h)\mathbb{Z}$ for coprime positive integers $g$ and $h$, then $R_\Delta$ is the
ring of invariants of the torus $K^*$ acting on $R$ via

$$\lambda: r \mapsto \lambda^{hi-gj} r \quad \text{where } \lambda \in K^* \text{ and } r \in R_{(i,j)}.$$ 

Consequently there exist torus actions on hypersurfaces for which the rings of in-
variants have rational singularities but are not of F-regular type.

In Section 4 we use diagonal subalgebras to construct standard graded normal
rings $R$, with isolated singularities, for which $H^2_{m}(R)_0 = 0$ and $H^2_{m}(R)_1 \neq 0$. If
$S$ is the localization of such a ring $R$ at its homogeneous maximal ideal, then,
by Danilov’s results, the divisor class group of $S$ is a finitely generated abelian
group, though $S$ does not have a discrete divisor class group. Such rings $R$ are
also of interest in view of the results of [RSS], where it is proved that the image
of $H^2_m(R)_0$ in $H^2_n(R)$ is annihilated by elements of $R^+$ of arbitrarily small positive degree; here $R^+$ denotes the absolute integral closure of $R$. A corresponding result for $H^2_m(R)_1$ is not known at this point, and the rings constructed in Section 4 constitute interesting test cases.

Section 2 summarizes some notation and conventions for multigraded rings and modules. In Section 3 we carry out an analysis of diagonal subalgebras of bigraded hypersurfaces; this uses results on rational singularities and F-regular rings proved in Sections 5 and 6 respectively.

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2. Preliminaries

In this section, we provide a brief treatment of multigraded rings and modules; see [GW1, GW2, HHR], and [HIO] for further details.

By an $\mathbb{N}^r$-graded ring we mean a ring $R = \bigoplus_{n \in \mathbb{N}^r} R_n$, which is finitely generated over the subring $R_0$. If $(R_0, m)$ is a local ring, then $R$ has a unique homogeneous maximal ideal $M = mR + R_+,$ where $R_+ = \oplus_{n \neq 0} R_n$.

For $m = (m_1, \ldots, m_r)$ and $n = (n_1, \ldots, n_r)$ in $\mathbb{Z}^r$, we say $n > m$ (resp. $n \geq m$) if $n_i > m_i$ (resp. $n_i \geq m_i$) for each $i$.

Let $M$ be a $\mathbb{Z}^r$-graded $R$-module. For $m \in \mathbb{Z}^r$, we set $M_{\geq m} = \bigoplus_{n \geq m} M_n$, which is a $\mathbb{Z}^r$-graded submodule of $M$. One writes $M(m)$ for the $\mathbb{Z}^r$-graded $R$-module with shifted grading $[M(m)]_n = M_{m+n}$ for each $n \in \mathbb{Z}^r$.

Let $M$ and $N$ be $\mathbb{Z}^r$-graded $R$-modules. Then $\text{Hom}_R(M, N)$ is the $\mathbb{Z}^r$-graded module with $[\text{Hom}_R(M, N)]_n = \text{Hom}_R(M, N)_n$ being the abelian group consisting of degree preserving $R$-linear homomorphisms from $M$ to $N(n)$.

The functor $\text{Ext}_R^i(M, -)$ is the $i$-th derived functor of $\text{Hom}_R(M, -)$ in the category of $\mathbb{Z}^r$-graded $R$-modules. When $M$ is finitely generated, $\text{Ext}_R^i(M, N)$ and $\text{Ext}_R^i(M, N)$ agree as underlying $R$-modules. For a homogeneous ideal $a$ of $R$, the local cohomology modules of $M$ with support in $a$ are the $\mathbb{Z}^r$-graded modules $H^i_a(M) = \text{Ext}^i_R(R/a^n, M)$.

Let $\varphi: \mathbb{Z}^r \to \mathbb{Z}^s$ be a homomorphism of abelian groups satisfying $\varphi(\mathbb{N}^r) \subseteq \mathbb{N}^s$. We write $R^{\varphi}$ for the ring $R$ with the $\mathbb{N}^s$-grading where $[R^{\varphi}]_n = \bigoplus_{\varphi(m) = n} R_m$.
If $M$ is a $\mathbb{Z}^r$-graded $\mathcal{R}$-module, then $M^\varphi$ is the $\mathbb{Z}^s$-graded $\mathcal{R}^\varphi$-module with

$$[M^\varphi]_n = \bigoplus_{\varphi(m)=n} M_m.$$ 

The change of grading functor $(-)^\varphi$ is exact; by [HHR] Lemma 1.1 one has

$$H^i_{\mathfrak{m}}(M)^\varphi = H^i_{\mathfrak{m}^\varphi}(M^\varphi).$$

Consider the projections $\varphi_i: \mathbb{Z}^r \to \mathbb{Z}$ with $\varphi_i(m_1, \ldots, m_r) = m_i$, and set

$$a(\mathcal{R}^\varphi_i) = \max \{ a \in \mathbb{Z} | [H^i_{\mathfrak{m}}(\mathcal{R}^\varphi_i)]_a \neq 0 \};$$

this is the $a$-invariant of the $\mathbb{N}$-graded ring $\mathcal{R}^\varphi_i$ in the sense of Goto and Watanabe [GW1]. As in [HHR], the multigraded $a$-invariant of $\mathcal{R}$ is

$$a(\mathcal{R}) = (a(\mathcal{R}^\varphi_1), \ldots, a(\mathcal{R}^\varphi_r)).$$

Let $\mathcal{R}$ be a $\mathbb{Z}^2$-graded ring and let $g, h$ be positive integers. The subgroup $\Delta = (g, h)\mathbb{Z}$ is a diagonal in $\mathbb{Z}^2$, and the corresponding diagonal subalgebra of $\mathcal{R}$ is

$$\mathcal{R}_\Delta = \bigoplus_{k \in \mathbb{Z}} \mathcal{R}_{(gk, hk)}.$$ 

Similarly, if $M$ is a $\mathbb{Z}^2$-graded $\mathcal{R}$-module, we set

$$M_\Delta = \bigoplus_{k \in \mathbb{Z}} M_{(gk, hk)},$$

which is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\mathcal{R}_\Delta$.

**Lemma 2.1.** Let $A$ and $B$ be $\mathbb{N}$-graded normal rings, finitely generated over a field $A_0 = K = B_0$. Set $T = A \otimes_K B$. Let $g$ and $h$ be positive integers and set $\Delta = (g, h)\mathbb{Z}$. Let $a$, $b$, and $m$ denote the homogeneous maximal ideals of $A$, $B$, and $T_\Delta$ respectively. Then, for each $q \geq 0$ and $i, j, k \in \mathbb{Z}$, one has

$$H^q_{m}(T(i, j)\Delta)_k = (A_{i+gk} \otimes H^q_a(B)_{j+hk}) \oplus \bigoplus_{q_1+q_2=q+1} (H^q_a(A)_{i+gk} \otimes H^q_b(B)_{j+hk}).$$

**Proof.** Let $A^{(g)}$ and $B^{(h)}$ denote the respective Veronese subrings of $A$ and $B$. Set

$$A^{(g, i)} = \bigoplus_{k \in \mathbb{Z}} A_{i+gk} \quad \text{and} \quad B^{(h, j)} = \bigoplus_{k \in \mathbb{Z}} B_{j+hk},$$

which are graded $A^{(g)}$ and $B^{(h)}$ modules respectively. Using # for the Segre product, we have

$$T(i, j)\Delta = \bigoplus_{k \in \mathbb{Z}} A_{i+gk} \otimes_K B_{j+hk} = A^{(g, i)} # B^{(h, j)}.$$ 

The ideal $A^{(g)}_+ A$ is $a$-primary; likewise, $B^{(h)}_+ B$ is $b$-primary. The Künneth formula for local cohomology, [GW1] Theorem 4.1.5], now gives the desired result. 

**Notation 2.2.** We use bold letters to denote lists of elements, e.g., $z = z_1, \ldots, z_s$ and $\gamma = \gamma_1, \ldots, \gamma_s$. 
3. Diagonal subalgebras of bigraded hypersurfaces

We prove the following theorem about diagonal subalgebras of \( \mathbb{N}^2 \)-graded hypersurfaces. The proof uses results proved later in Sections 5 and 6.

**Theorem 3.1.** Let \( K \) be a field, let \( m, n \) be integers with \( m, n \geq 2 \), and let

\[
\mathcal{R} = K[x_1, \ldots, x_m, y_1, \ldots, y_n]/(f)
\]

be a normal \( \mathbb{N}^2 \)-graded hypersurface where \( \deg x_i = (1, 0), \deg y_j = (0, 1) \), and \( \deg f = (d, e) > (0, 0) \). For positive integers \( g \) and \( h \), set \( \Delta = (g, h)\mathbb{Z} \). Then:

1. The ring \( \mathcal{R}_\Delta \) is Cohen-Macaulay if and only if \( [(d - m)/g] < e/h \) and \( [(e - n)/h] < d/g \). In particular, if \( d < m \) and \( e < n \), then \( \mathcal{R}_\Delta \) is Cohen-Macaulay for each diagonal \( \Delta \).

2. The graded canonical module of \( \mathcal{R}_\Delta \) is \( \mathcal{R}(d - m, e - n)_\Delta \). Hence \( \mathcal{R}_\Delta \) is Gorenstein if and only if \( (d - m)/g = (e - n)/h \), and this is an integer.

If \( K \) has characteristic zero, and \( f \) is a generic polynomial of degree \( (d, e) \), then:

3. The ring \( \mathcal{R}_\Delta \) has rational singularities if and only if it is Cohen-Macaulay and \( d < m \) or \( e < n \).

4. The ring \( \mathcal{R}_\Delta \) is of F-regular type if and only if \( d < m \) and \( e < n \).

For \( m, n \geq 3 \) and \( \Delta = (1, 1)\mathbb{Z} \), the properties of \( \mathcal{R}_\Delta \), as determined by \( m, n, d, e \), are summarized in Figure 1.

![Figure 1. Properties of \( \mathcal{R}_\Delta \) for \( \Delta = (1, 1)\mathbb{Z} \).](image-url)
Remark 3.2. Let \( m, n \geq 2 \). A generic hypersurface of degree \((d, e) > (0, 0)\) in \( m, n \) variables is normal precisely when

\[
m > \min(2, d) \quad \text{and} \quad n > \min(2, e).
\]

Suppose that \( m = 2 = n \), and that \( f \) is nonzero. Then \( \dim R_{\Delta} = 2 \); since \( R_{\Delta} \) is generated over a field by elements of equal degree, \( R_{\Delta} \) is of F-regular type if and only if it has rational singularities; see \cite{Wa}. This is the case precisely if

\[
d = 1, \ e \leq h + 1, \quad \text{or} \quad e = 1, \ d \leq g + 1.
\]

Following a suggestion of Hara, the case \( n = 2 \) and \( e = 1 \) was used in \cite{Si} Example 7.3 to construct examples of standard graded rings with rational singularities which are not of F-regular type.

Proof of Theorem 3.1. Set \( A = K[x], B = K[y], \) and \( T = A \otimes_K B \). By Lemma 2.1 \( H^q_m(T_{\Delta}) = 0 \) for \( q \neq m + n - 1 \). The local cohomology exact sequence induced by

\[
0 \longrightarrow T(-d, -e)_\Delta \xrightarrow{f} T_{\Delta} \longrightarrow R_{\Delta} \longrightarrow 0
\]

therefore gives \( H^{q-1}_m(R_{\Delta}) = H^q_m(T(-d, -e)_{\Delta}) \) for \( q \leq m + n - 2 \), and also shows that \( H^{m+n-2}_m(R_{\Delta}) \) and \( H^{m+n-1}_m(R_{\Delta}) \) are, respectively, the kernel and cokernel of

\[
H^{m+n-1}_m(T(-d, -e)_{\Delta}) \xrightarrow{f} H^m_m(R_{\Delta})
\]

\[
[H^m_m(A(-d)) \otimes H^n_n(B(-e))]_{\Delta} \xrightarrow{f} [H^m_m(A) \otimes H^n_n(B)]_{\Delta}.
\]

The horizontal map above is surjective since its graded dual

\[
[A(d-m) \otimes B(e-n)]_{\Delta} \xleftarrow{f} [A(-m) \otimes B(-n)]_{\Delta}
\]

\[
T(d-m, e-n)_{\Delta} \xleftarrow{f} T(-m, -n)_{\Delta}
\]

is injective. In particular, \( \dim R_{\Delta} = m + n - 2 \).

It follows from the above discussion that \( R_{\Delta} \) is Cohen-Macaulay if and only if \( H^q_m(T(-d, -e)_{\Delta}) = 0 \) for each \( q \leq m + n - 2 \). By Lemma 2.1 this is the case if and only if, for each integer \( k \), one has

\[
A_{-d+gk} \otimes H^n_n(B)_{-e+hk} = 0 = H^m_m(A)_{-d+gk} \otimes B_{-e+hk}.
\]

Hence \( R_{\Delta} \) is Cohen-Macaulay if and only if there is no integer \( k \) satisfying

\[
d/g \leq k \leq (e - n)/h \quad \text{or} \quad e/h \leq k \leq (d - m)/g,
\]

which completes the proof of (1).

For (2), note that the graded canonical module of \( R_{\Delta} \) is the graded dual of \( H^{m+n-2}_m(R_{\Delta}) \), and hence that it equals

\[
\coker(T(-m, -n)_{\Delta} \xrightarrow{f} T(d-m, e-n)_{\Delta}) = R(d-m, e-n)_{\Delta}.
\]
This module is principal if and only if \( R(d - m, e - n)_\Delta = R_\Delta(a) \) for some integer \( a \), i.e., \( d - m = ga \) and \( e - n = ha \).

When \( f \) is a general polynomial of degree \((d, e)\), the ring \( R_\Delta \) has an isolated singularity. Also, \( R_\Delta \) is normal since it is a direct summand of the normal ring \( R \). By Theorem 5.1, \( R_\Delta \) has rational singularities precisely if it is Cohen-Macaulay and \( a(R_\Delta) < 0 \); this proves (3).

It remains to prove (4). If \( d < m \) and \( e < n \), then Theorem 5.2 implies that \( R \) has rational singularities. By Theorem 6.2 it follows that for almost all primes \( p \), the characteristic \( p \) models \( R_p \) of \( R \) are F-rational hypersurfaces which, therefore, are F-regular. Alternatively, \( R_p \) is a generic hypersurface of degree \((d, e) < (m, n)\), so Theorem 6.5 implies that \( R_p \) is F-regular. Since \((R_p)_\Delta\) is a direct summand of \( R_p \), it follows that \((R_p)_\Delta\) is F-regular. The rings \((R_p)_\Delta\) are characteristic \( p \) models of \( R_\Delta \), so we conclude that \( R_\Delta \) is of F-regular type.

Suppose \( R_\Delta \) has F-regular type, and let \((R_p)_\Delta\) be a characteristic \( p \) model which is F-regular. Fix an integer \( k > d/g \). Then Proposition 6.3 implies that there exists an integer \( q = p^e \) such that

\[
\text{rank}_K \left((R_p)_\Delta\right)_k \leq \text{rank}_K \left[H^m_{m+p}((\omega(q))]_k, \right.
\]

where \( \omega \) is the graded canonical module of \((R_p)_\Delta\). Using (2), we see that

\[
H^m_{m+p}((\omega(q)) = H^m_{m+p}((R_p(qd - qm, qe - qn)_\Delta)).
\]

Let \( T_p \) be a characteristic \( p \) model for \( T \) such that \( T_p/fT_p = R_p \). Multiplication by \( f \) on \( T_p \) induces a local cohomology exact sequence

\[
\cdots \longrightarrow H^m_{m+p}((T_p(qd - qm, qe - qn)_\Delta) \longrightarrow H^m_{m+p}((R_p(qd - qm, qe - qn)_\Delta)
\]

\[
\longrightarrow H^m_{m+p}((T_p(qd - qm - d, qe - qn - e)_\Delta) \longrightarrow \cdots.
\]

Since \( H^m_{m+p}((T_p(qd - qm, qe - qn)_\Delta) \) vanishes by Lemma 2.1, we conclude that

\[
\text{rank}_K \left((R_p)_\Delta\right)_k \leq \text{rank}_K \left[H^m_{m+p}((T_p(qd - qm - d, qe - qn - e)_\Delta)]_k.
\]

\[
= \text{rank}_K \left[H^m_{m+p}(A_p)_{qd - qm - d + gk} \otimes H^g_{v_q}(B_p)_{qe - qn - e + hk}
\]

Hence \( qd - qm - d + gk < 0 \); as \( d - gk < 0 \), we conclude \( d < m \). Similarly, \( e < n \). \( \square \)

We conclude this section with an example where a local cohomology module of a standard graded ring is not rigid in the sense that \( H^2_m(R)_0 = 0 \) while \( H^2_m(R)_1 \neq 0 \). Further such examples are constructed in Section 4.

**Proposition 3.3.** Let \( K \) be a field and let

\[
\mathcal{R} = K[x_1, x_2, x_3, y_1, y_2]/(f)
\]

where \( \deg x_i = (1, 0) \), \( \deg y_j = (0, 1) \), and \( \deg f = (d, e) \) for \( d \geq 4 \) and \( e \geq 1 \). Let \( g \) and \( h \) be positive integers such that \( g \leq d - 3 \) and \( h \geq e \), and set \( \Delta = (g, h)\mathbb{Z} \).

Then \( H^2_m(R_\Delta)_0 = 0 \) and \( H^2_m(R_\Delta)_1 \neq 0 \).
Proof. Using the resolution of $R$ over the polynomial ring $T$ as in the proof of Theorem 3.1, we have an exact sequence

$$H_m^2(T_\Delta) \rightarrow H_m^3(R_\Delta) \rightarrow H_m^3(T(-d, -e)_\Delta) \rightarrow H_m^4(T_\Delta).$$

Lemma 2.1 implies that $H_m^2(T_\Delta) = 0 = H_m^3(T_\Delta)$. Hence, again by Lemma 2.1,

$$H_m^2(R_\Delta)_0 = H^3(A)_{-d} \otimes B_{-e} = 0 \quad \text{and} \quad H_m^2(R_\Delta)_1 = H^3(A)_{g-d} \otimes B_{h-e} \neq 0. \quad \square$$

4. Non-rigid local cohomology modules

We construct examples of standard graded normal rings $R$ over $\mathbb{C}$, with only isolated singularities, for which $H_m^2(R)_0 = 0$ and $H_m^2(R)_1 \neq 0$. Let $S$ be the localization of such a ring $R$ at its homogeneous maximal ideal. By results of Danilov [Da1, Da2], Theorem 4.1 below, it follows that the divisor class group of $S$ is finitely generated, though $S$ does not have a discrete divisor class group, i.e., the natural map $\text{Cl}(S) \rightarrow \text{Cl}(S[[t]])$ is not bijective. Here, remember that if $A$ is a Noetherian normal domain, then so is $A[[t]]$.

**Theorem 4.1.** Let $R$ be a standard graded normal ring, which is finitely generated as an algebra over $R_0 = \mathbb{C}$. Assume, moreover, that $X = \text{Proj} R$ is smooth. Set $(S, m)$ to be the local ring of $R$ at its homogeneous maximal ideal, and $\hat{S}$ to be the $m$-adic completion of $S$. Then

1. the group $\text{Cl}(S)$ is finitely generated if and only if $H^1(X, O_X) = 0$;
2. the map $\text{Cl}(S) \rightarrow \text{Cl}(\hat{S})$ is bijective if and only if $H^1(X, O_X(i)) = 0$ for each integer $i \geq 1$; and
3. the map $\text{Cl}(S) \rightarrow \text{Cl}(S[[t]])$ is bijective if and only if $H^1(X, O_X(i)) = 0$ for each integer $i \geq 0$.

The essential point in our construction is in the following proposition:

**Theorem 4.2.** Let $A$ be a Cohen-Macaulay ring of dimension $d \geq 2$, which is a standard graded algebra over a field $K$. For $s \geq 2$, let $z_1, \ldots, z_s$ be a regular sequence in $A$, consisting of homogeneous elements of equal degree, say $k$. Consider the Rees ring $\mathcal{R} = A[z_1t, \ldots, z_st]$ with the $\mathbb{Z}^2$-grading where $\deg x = (n, 0)$ for $x \in A_n$, and $\deg z_it = (0, 1)$.

Let $\Delta = (g, h)\mathbb{Z}$ where $g, h$ are positive integers, and let $m$ denote the homogeneous maximal ideal of $\mathcal{R}_\Delta$. Then:

1. $H_m^q(\mathcal{R}_\Delta) = 0$ if $q \neq d + 1, d$; and
2. $H_m^{d+s+1}(\mathcal{R}_\Delta)_i \neq 0$ if and only if $1 \leq i \leq (a + ks - k)/g$, where $a$ is the $a$-invariant of $A$.

In particular, $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $g > a + ks - k$. 
Example 4.3. For $d \geq 3$, let $A = \mathbb{C}[x_0, \ldots, x_d]/(f)$ be a standard graded hypersurface such that $\text{Proj} A$ is smooth over $\mathbb{C}$. Take general $k$-forms $z_1, \ldots, z_{d-1} \in A$, and consider the Rees ring $R = A[z_1t, \ldots, z_{d-1}t]$. Since $(z) \subset A$ is a radical ideal, 
\[
\text{gr}((z), A) \cong A/(z)[y_1, \ldots, y_{d-1}]
\]
is a reduced ring, and therefore $R$ is a reduced ring, and therefore $\mathcal{R} = A[z_1t, \ldots, z_{d-1}t]$ is integrally closed in $A[t]$. Since $A$ is normal, so is $R$. Note that $\text{Proj} \mathcal{R}_\Delta$ is the blow-up of $\text{Proj} A$ at the subvariety defined by $(z)$, i.e., at $k^{d-1}(\deg f)$ points. It follows that $\text{Proj} \mathcal{R}_\Delta$ is smooth over $\mathbb{C}$. Hence $\mathcal{R}_\Delta$ is a standard graded $\mathbb{C}$-algebra, which is normal and has an isolated singularity.

If $\Delta = (g, h)\mathbb{Z}$ is a diagonal with $1 \leq g \leq \deg f + k(d-2) - (d+1)$ and $h \geq 1$, then Theorem 4.2 implies that
\[
H^2_m(\mathcal{R}_\Delta)_0 = 0 \quad \text{and} \quad H^2_m(\mathcal{R}_\Delta)_1 \neq 0.
\]

The rest of this section is devoted to proving Theorem 4.2. We may assume that the base field $K$ is infinite. Then one can find linear forms $x_1, \ldots, x_{d-s}$ in $A$ such that $x_1, \ldots, x_{d-s}, z_1, \ldots, z_s$ is a maximal $A$-regular sequence.

We will use the following lemma; the notation is as in Theorem 4.2.

Lemma 4.4. Let $a$ be the homogeneous maximal ideal of $A$. Set $I = (z_1, \ldots, z_s)A$. Let $r$ be a positive integer.

(1) $H^0_a(I') = 0$ if $q \neq d - s + 1, d$.

(2) Assume $d > s$. Then, $H^{d-s+1}_a(I'_i) \neq 0$ if and only if $i \leq a + ks + rk - k$.

(3) Assume $d = s$. Then, $H^{d-s+1}_a(I'_i) \neq 0$ if and only if $0 \leq i \leq a + ks + rk - k$.

Proof. Recall that $A$ and $A/I^r$ are Cohen-Macaulay rings of dimension $d$ and $d-s$, respectively. By the exact sequence
\[
0 \longrightarrow I' \longrightarrow A \longrightarrow A/I' \longrightarrow 0
\]
we obtain
\[
H^q_a(I') = \begin{cases} 
H^q_a(A) & \text{if } q = d \\
H^{d-s}_a(A/I^r) & \text{if } q = d - s + 1 \\
0 & \text{if } q \neq d - s + 1, d,
\end{cases}
\]
which proves (1).

Next we prove (2) and (3). Since $A/I^r$ is a standard graded Cohen-Macaulay ring of dimension $d-s$, it is enough to show that the $a$-invariant of this ring equals $a + ks + rk - k$. This is straightforward if $r = 1$, and we proceed by induction. Consider the exact sequence
\[
0 \longrightarrow I'/I'^{r+1} \longrightarrow A/I^{r+1} \longrightarrow A/I^r \longrightarrow 0.
\]
Since $z_1, \ldots, z_s$ is a regular sequence of $k$-forms, $I^r/I^{r+1}$ is isomorphic to $(A/I)(-rk)^{\binom{r+s}{r}}$. 
Thus, we have the following exact sequence:

$$0 \rightarrow H^d_a((A/I)(-rk)) \rightarrow H^d_a(A/I^{r+1}) \rightarrow H^d_a(A/I^r) \rightarrow 0.$$  

The $a$-invariant of $(A/I)(-rk)$ equals $a + ks + rk$, and that of $A/I^r$ is $a + ks + rk - k$ by the inductive hypothesis. Thus, $A/I^{r+1}$ has $a$-invariant $a + ks + rk$. □

Proof of Theorem 4.2. Let $B = K[y_1, \ldots, y_s]$ be a polynomial ring, and set

$$T = A \otimes_K B = A[y_1, \ldots, y_s].$$

Consider the $\mathbb{Z}^2$-grading on $T$ where $\deg x = (n, 0)$ for $x \in A_n$, and $\deg y_i = (0, 1)$ for each $i$. One has a surjective homomorphism of graded rings

$$T \rightarrow \mathcal{R} = A[z_1 t, \ldots, z_s t] \quad \text{where} \quad y_i \mapsto z_i t,$$

and this induces an isomorphism

$$\mathcal{R} \cong T/I_2(z_1 : : z_s).$$

The minimal free resolution of $\mathcal{R}$ over $T$ is given by the Eagon-Northcott complex

$$0 \rightarrow F^{-(s-1)} \rightarrow F^{-(s-2)} \rightarrow \cdots \rightarrow F^0 \rightarrow 0,$$

where $F^0 = T(0, 0)$, and $F^{-i}$ for $1 \leq i \leq s - 1$ is the direct sum of $(s-1)^i$ copies of

$$T(-k, -(i-1)) \oplus T(-2k, -(i-1)) \oplus \cdots \oplus T(-ik, -1).$$

Let $\mathfrak{n}$ be the homogeneous maximal ideal of $T_\Delta$. One has the spectral sequence:

$$E_2^{p,q} = H^p(H^q_\mathfrak{n}(F^*_\Delta)) \Rightarrow H^p_{\mathcal{R}}(\mathcal{R}).$$

Let $G$ be the set of $(n, m)$ such that $T(n, m)$ appears in the Eagon-Northcott complex above, i.e., the elements of $G$ are

$$(0, 0),$$

$$(-k, -1),$$

$$(-k, -2), (-2k, -1),$$

$$(-k, -3), (-2k, -2), (-3k, -1),$$

$$\vdots$$

$$(-k, -(s-1)), \ldots \quad (-s-1)k, -1).$$

Let $\mathfrak{a}$ and $\mathfrak{b}$ be the homogeneous maximal ideal of $A$ and $B$ respectively. For integers $n$ and $m$, the Küneth formula gives

$$H^q_{\mathfrak{n}}(T(n, m))$$

$$= H^q_{\mathfrak{n}}(A(n) \otimes_K B(m))$$

$$= (H^q_{\mathfrak{n}}(A(n)) \otimes B(m)) \oplus (A(n) \otimes H^q_{\mathfrak{n}}(B(m))) \oplus \bigoplus_{i+j=q+1} H^i_{\mathfrak{a}}(A(n)) \otimes H^j_{\mathfrak{b}}(B(m))$$

$$= H^q_{\mathfrak{n}}(T(n, m)) \oplus H^q_{\mathfrak{n}}(T(n, m)) \oplus \bigoplus_{i+j=q+1} H^i_{\mathfrak{a}}(A(n)) \otimes_K H^j_{\mathfrak{b}}(B(m)).$$
As $A$ and $B$ are Cohen-Macaulay of dimension $d$ and $s$ respectively, it follows that

$$H^n_q(F^\bullet) = 0 \quad \text{if } q \neq s, d, d + s - 1.$$  

In the case where $d > s$, one has

$$H^n_a(F^\bullet) = H^n_b(F^\bullet) \quad \text{and} \quad H^n_d(F^\bullet) = H^n_d(F^\bullet),$$

and if $d = s$, then

$$H^n_d(F^\bullet) = H^n_a(F^\bullet) \oplus H^n_b(F^\bullet).$$

We claim $H^n_b(F^\bullet)_\Delta = 0$. If not, there exists $(n, m) \in G$ and $\ell \in \mathbb{Z}$ such that

$$H^n_b(T(n, m))_{(g\ell, h\ell)} \neq 0.$$  

This implies that

$$H^n_b(T(n, m))_{(g\ell, h\ell)} = A(n)_{g\ell} \otimes_K H^n_b(B(m))_{h\ell} = A_{n+g\ell} \otimes_K H^n_b(B)_{m+h\ell}$$

is nonzero, so

$$n + g\ell \geq 0 \quad \text{and} \quad m + h\ell \leq -s,$$

and hence

$$-n - \frac{n}{g} \leq \ell \leq -\frac{s + m}{h}.$$  

But $(n, m) \in G$, so $n \leq 0$ and $m \geq -(s - 1)$, implying that

$$0 \leq \ell \leq -\frac{1}{h},$$

which is not possible. This proves that $H^n_b(F^\bullet)_\Delta = 0$. Thus, we have

$$H^n_d(F^\bullet)_\Delta = \begin{cases} 0 & \text{if } q \neq d, d + s - 1, \\ H^n_d(F^\bullet)_\Delta & \text{if } q = d. \end{cases}$$

It follows that

$$E^{p,q}_2 = H^p(H^n_d(F^\bullet)_\Delta) = E^{p,q}_\infty$$

for each $p$ and $q$. Therefore,

$$H^i_m(R_\Delta) = E^{i-d,d}_2 = H^{i-d}(H^n_d(F^\bullet)_\Delta) = H^{i-d}(H^n_d(F^\bullet)_\Delta) = H^i_m(R_\Delta)$$

for $d - s + 1 \leq i \leq d - 1$, and

$$H^i_m(R_\Delta) = 0 \quad \text{for } i < d - s + 1.$$  

We next study $H^i_\alpha(R)$. Since

$$R = A \oplus I(k) \oplus I^2(2k) \oplus \cdots \oplus I^r(rk) \oplus \cdots,$$

we have

$$H^i_\alpha(R) = H^i_\alpha(A) \oplus H^i_\alpha(I(k)) \oplus H^i_\alpha(I^2(2k)) \oplus \cdots \oplus H^i_\alpha(I^r(rk)) \oplus \cdots.$$  

Theorem 4.2 (1) now follow using Lemma 4.4 (1).

Assume that $d > s$. Then, by Lemma 4.4 (2), $H^{d-s+1}(I^\tau(rk)) \neq 0$ if and only if $i \leq a + ks - k$.  

We claim $H^n_b(F^\bullet)_\Delta = 0$. If not, there exists $(n, m) \in G$ and $\ell \in \mathbb{Z}$ such that

$$H^n_b(T(n, m))_{(g\ell, h\ell)} \neq 0.$$  

This implies that

$$H^n_b(T(n, m))_{(g\ell, h\ell)} = A(n)_{g\ell} \otimes_K H^n_b(B(m))_{h\ell} = A_{n+g\ell} \otimes_K H^n_b(B)_{m+h\ell}$$

is nonzero, so

$$n + g\ell \geq 0 \quad \text{and} \quad m + h\ell \leq -s,$$

and hence

$$-n - \frac{n}{g} \leq \ell \leq -\frac{s + m}{h}.$$  

But $(n, m) \in G$, so $n \leq 0$ and $m \geq -(s - 1)$, implying that

$$0 \leq \ell \leq -\frac{1}{h},$$

which is not possible. This proves that $H^n_b(F^\bullet)_\Delta = 0$. Thus, we have

$$H^n_d(F^\bullet)_\Delta = \begin{cases} 0 & \text{if } q \neq d, d + s - 1, \\ H^n_d(F^\bullet)_\Delta & \text{if } q = d. \end{cases}$$

It follows that

$$E^{p,q}_2 = H^p(H^n_d(F^\bullet)_\Delta) = E^{p,q}_\infty$$

for each $p$ and $q$. Therefore,

$$H^i_m(R_\Delta) = E^{i-d,d}_2 = H^{i-d}(H^n_d(F^\bullet)_\Delta) = H^{i-d}(H^n_d(F^\bullet)_\Delta) = H^i_m(R_\Delta)$$

for $d - s + 1 \leq i \leq d - 1$, and

$$H^i_m(R_\Delta) = 0 \quad \text{for } i < d - s + 1.$$  

We next study $H^i_\alpha(R)$. Since

$$R = A \oplus I(k) \oplus I^2(2k) \oplus \cdots \oplus I^r(rk) \oplus \cdots,$$

we have

$$H^i_\alpha(R) = H^i_\alpha(A) \oplus H^i_\alpha(I(k)) \oplus H^i_\alpha(I^2(2k)) \oplus \cdots \oplus H^i_\alpha(I^r(rk)) \oplus \cdots.$$  

Theorem 4.2 (1) now follow using Lemma 4.4 (1).

Assume that $d > s$. Then, by Lemma 4.4 (2), $H^{d-s+1}(I^\tau(rk)) \neq 0$ if and only if $i \leq a + ks - k$.  

Assume that \( d = s \). Then, by Lemma 4.4 (3), \( H^{d-s+1}_a(I^r(rk))_i \neq 0 \) if and only if \( -rk \leq i \leq a + ks - k \).

In each case, \( H^{d-s+1}_a(R)(gi, hi)_i \neq 0 \) if and only if \( 1 \leq i \leq a + ks - k \).

5. Rational singularities

Let \( R \) be a normal domain, essentially of finite type over a field of characteristic zero, and consider a desingularization \( f: Z \to \text{Spec} R \), i.e., a proper birational morphism with \( Z \) a nonsingular variety. One says \( R \) has rational singularities if \( R^i \omega_Z = 0 \) for each \( i \geq 1 \); this does not depend on the choice of the desingularization \( f \). For \( \mathbb{N} \)-graded rings, one has the following criterion due to Flenner [Fl] and Watanabe [Wa1].

**Theorem 5.1.** Let \( R \) be a normal \( \mathbb{N} \)-graded ring which is finitely generated over a field \( R_0 \) of characteristic zero. Then \( R \) has rational singularities if and only if it is Cohen-Macaulay, \( a(R) < 0 \), and the localization \( R_p \) has rational singularities for each \( p \in \text{Spec} R \setminus \{ R_+ \} \).

When \( R \) has an isolated singularity, the above theorem gives an effective criterion for determining if \( R \) has rational singularities. However, a multigraded hypersurface typically does not have an isolated singularity, and the following variation turns out to be useful:

**Theorem 5.2.** Let \( R \) be a normal \( \mathbb{N}^r \)-graded ring such that \( R_0 \) is a local ring essentially of finite type over a field of characteristic zero, and \( R \) is generated over \( R_0 \) by elements

\[
x_{11}, x_{12}, \ldots, x_{1t_1}, \quad x_{21}, x_{22}, \ldots, x_{2t_2}, \ldots, \quad x_{r1}, x_{r2}, \ldots, x_{rt_r},
\]

where \( \deg x_{ij} \) is a positive integer multiple of the \( i \)-th unit vector \( e_i \in \mathbb{N}^r \). Then \( R \) has rational singularities if and only if

1. \( R \) is Cohen-Macaulay,
2. \( R_p \) has rational singularities for each \( p \) belonging to \( \text{Spec} R \setminus (V(x_{11}, x_{12}, \ldots, x_{1t_1}) \cup \ldots \cup V(x_{r1}, x_{r2}, \ldots, x_{rt_r})) \), and
3. \( a(R) < 0 \), i.e., \( a(R^{\mathbb{N}^r}) < 0 \) for each coordinate projection \( \varphi_i: \mathbb{N}^r \to \mathbb{N} \).

Before proceeding with the proof, we record some preliminary results.

**Remark 5.3.** Let \( R \) be an \( \mathbb{N} \)-graded ring. We use \( R^\mathbb{N} \) to denote the Rees algebra with respect to the filtration \( F_n = R_{\geq n} \), i.e.,

\[
R^\mathbb{N} = F_0 \oplus F_1 T \oplus F_2 T^2 \oplus \cdots.
\]
When considering $\text{Proj } R^2$, we use the $\mathbb{N}$-grading on $R^2$ where $[R^2]_n = F_n T^n$. The inclusion $R = [R^2]_0 \hookrightarrow R^2$ gives a map

$$\text{Proj } R^2 \xrightarrow{f} \text{Spec } R.$$ 

Also, the inclusions $R_n \hookrightarrow F_n$ give rise to an injective homomorphism of graded rings $R \hookrightarrow R^2$, which induces a surjection

$$\text{Proj } R^2 \twoheadrightarrow \text{Proj } R.$$ 

**Lemma 5.4.** Let $R$ be an $\mathbb{N}$-graded ring which is finitely generated over $R_0$, and assume that $R_0$ is essentially of finite type over a field of characteristic zero.

If $R_p$ has rational singularities for all primes $p \in \text{Spec } R \setminus V(R_+)$, then $\text{Proj } R^2$ has rational singularities.

**Proof.** Note that $\text{Proj } R^2$ is covered by affine open sets $D_+(r T^n)$ for integers $n \geq 1$ and homogeneous elements $r \in R_{\geq n}$. Consequently, it suffices to check that $[R^2_{(r T^n)}]_0$ has rational singularities. Next, note that

$$[R^2_{(r T^n)}]_0 = R + \frac{1}{r}[R]_{\geq n} + \frac{1}{r^2}[R]_{\geq 2n} + \cdots.$$ 

In the case $\deg r > n$, the ring above is simply $R_r$, which has rational singularities by the hypothesis of the lemma. If $\deg r = n$, then

$$[R^2_{(r T^n)}]_0 = [R_r]_{\geq 0}.$$ 

The $\mathbb{Z}$-graded ring $R_r$ has rational singularities and so, by [Wa1, Lemma 2.5], the ring $[R_r]_{\geq 0}$ has rational singularities as well. \hfill \square

**Lemma 5.5.** [Hy2, Lemma 2.3] Let $R$ be an $\mathbb{N}$-graded ring which is finitely generated over a local ring $(R_0, m)$. Suppose $[H^i_{m+r_+}(R)]_{\geq 0} = 0$ for all $i \geq 0$. Then, for all ideals $a$ of $R_0$, one has

$$[H^i_{a+r_+}(R)]_{\geq 0} = 0 \quad \text{for all } i \geq 0.$$ 

We are now in a position to prove the following theorem, which is a variation of [Fl, Satz 3.1], [Wa1, Theorem 2.2], and [Hy1, Theorem 1.5].

**Theorem 5.6.** Let $R$ be an $\mathbb{N}$-graded normal ring which is finitely generated over $R_0$, and assume that $R_0$ is a local ring essentially of finite type over a field of characteristic zero. Then $R$ has rational singularities if and only if

1. $R$ is Cohen-Macaulay,
2. $R_p$ has rational singularities for all $p \in \text{Spec } R \setminus V(R_+)$, and
3. $a(R) < 0$.

**Proof.** It is straightforward to see that conditions (1)–(3) hold when $R$ has rational singularities, and we focus on the converse. Consider the morphism

$$Y = \text{Proj } R^2 \xrightarrow{f} \text{Spec } R.$$
Let $g: Z \to Y$ be a desingularization of $Y$; the composition

$$Z \xrightarrow{g} Y \xrightarrow{f} \text{Spec} R$$

is then a desingularization of $\text{Spec} R$. Note that $Y = \text{Proj} R^\sharp$ has rational singularities by Lemma 5.3 so

$$g_* \mathcal{O}_Z = \mathcal{O}_Y \quad \text{and} \quad R^q g_* \mathcal{O}_Z = 0 \quad \text{for all} \quad q \geq 1.$$ 

Consequently the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q g_* \mathcal{O}_Z) \Rightarrow H^{p+q}(Z, \mathcal{O}_Z)$$

degenerates, and we get $H^p(Z, \mathcal{O}_Z) = H^p(Y, \mathcal{O}_Y)$ for all $p \geq 1$. Since $\text{Spec} R$ is affine, we also have $R^p(g \circ f)_* \mathcal{O}_Z = H^p(Z, \mathcal{O}_Z)$. To prove that $R$ has rational singularities, it now suffices to show that $H^p(Y, \mathcal{O}_Y) = 0$ for all $p \geq 1$. Consider the map $\pi: Y \to X = \text{Proj} R$. We have

$$H^p(Y, \mathcal{O}_Y) = H^p(X, \pi_* \mathcal{O}_X) = \bigoplus_{n \geq 0} H^p(X, \mathcal{O}_X(n)) = [H^p_{R^+}(R)]_{\geq 0}.$$ 

By condition (1), we have $[H^p_{m+R^+}(R)]_{\geq 0} = 0$ for all $p \geq 0$, and so Lemma 5.5 implies that $[H^p_{R^+}(R)]_{\geq 0} = 0$ for all $p \geq 0$ as desired. $\square$

**Proof of theorem 5.2.** If $R$ has rational singularities, it is easily seen that conditions (1)–(3) must hold. For the converse, we proceed by induction on $r$. The case $r = 1$ is Theorem 5.6 established above, so assume $r \geq 2$. It suffices to show that $R_{\mathfrak{M}}$ has rational singularities where $\mathfrak{M}$ is the homogeneous maximal ideal of $R$. Set

$$m = \mathfrak{M} \cap [R^{r-1}]_0,$$

and consider the $\mathbb{N}$-graded ring $S$ obtained by inverting the multiplicative set $[R^{r-1}]_0 \setminus m$ in $R^{r-1}$. Since $R_{\mathfrak{M}}$ is a localization of $S$, it suffices to show that $S$ has rational singularities. Note that $a(S) = a(R^{r-1})$, which is a negative integer by (1). Using Theorem 5.6 it is therefore enough to show that $R_{\mathfrak{P}}$ has rational singularities for all $\mathfrak{P} \in \text{Spec} R \setminus V(x_{r_1}, x_{r_2}, \ldots, x_{r_t})$. Fix such a prime $\mathfrak{P}$, and let $\psi: Z^r \to Z^{r-1}$ be the projection to the first $r-1$ coordinates. Note that $R^\psi$ is the ring $R$ regraded such that $\deg x_{r_j} = 0$, and the degrees of $x_{ij}$ for $i < r$ are unchanged. Set

$$p = \mathfrak{P} \cap [R^{r-1}]_0,$$

and let $T$ be the ring obtained by inverting the multiplicative set $[R^{r-1}]_0 \setminus p$ in $R^{r-1}$. It suffices to show that $T$ has rational singularities. Note that $T$ is an $\mathbb{N}^{r-1}$-graded ring defined over a local ring $(T_0, \mathfrak{p})$, and that it has homogeneous maximal ideal $\mathfrak{p} + bT$ where

$$b = (R^{\psi})_+ = (x_{ij} \mid i < r)R.$$
Using the inductive hypothesis, it remains to verify that $a(T) < 0$. By condition (1), for all integers $1 \leq j \leq r - 1$, we have

$$[H^i_{\frak{m}}(R)^{\varphi_j}]_{i \geq 0} = 0 \quad \text{for all } i \geq 0,$$

and using Lemma 5.5 it follows that

$$[H^i_{p+b}(R)^{\varphi_j}]_{i \geq 0} = 0 \quad \text{for all } i \geq 0.$$

Consequently $a(T^{\varphi_j}) < 0$ for $1 \leq j \leq r - 1$, which completes the proof.  \hfill \Box

6. F-regularity

For the theory of tight closure, we refer to the papers [HH1, HH2] and [HH3]. We summarize results about F-rational and F-regular rings:

**Theorem 6.1.** The following hold for rings of prime characteristic.

1. Regular rings are F-regular.
2. Direct summands of F-regular rings are F-regular.
3. F-rational rings are normal; an F-rational ring which is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.
4. F-rational Gorenstein rings are F-regular.
5. Let $R$ be an $\mathbb{N}$-graded ring which is finitely generated over a field $R_0$. If $R$ is weakly F-regular, then it is F-regular.

**Proof.** For (1) and (2) see [HH1] Theorem 4.6 and [HH1] Proposition 4.12 respectively; (3) is part of [HH2] Theorem 4.2, and for (4) see [HH2] Corollary 4.7, Lastly, (5) is [LS] Corollary 4.4.

The characteristic zero aspects of tight closure are developed in [HH4]. Let $K$ be a field of characteristic zero. A finitely generated $K$-algebra $R = K[x_1, \ldots, x_m]/\frak{a}$ is of F-regular type if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$, and a finitely generated free $A$-algebra

$$R_A = A[x_1, \ldots, x_m]/\frak{a}_A,$$

such that $R \cong R_A \otimes_A K$ and, for all maximal ideals $\mu$ in a Zariski dense subset of $\text{Spec } A$, the fiber rings $R_A \otimes_A A/\mu$ are F-regular rings of characteristic $p > 0$. Similarly, $R$ is of F-rational type if for a dense subset of $\mu$, the fiber rings $R_A \otimes_A A/\mu$ are F-rational. Combining results from [Ha, HW, MS, Sm] one has:

**Theorem 6.2.** Let $R$ be a ring which is finitely generated over a field of characteristic zero. Then $R$ has rational singularities if and only if it is of F-rational type. If $R$ is $\mathbb{Q}$-Gorenstein, then it has log terminal singularities if and only if it is of F-regular type.
Proposition 6.3. Let $K$ be a field of characteristic $p > 0$, and $R$ an $\mathbb{N}$-graded normal ring which is finitely generated over $R_0 = K$. Let $\omega$ denote the graded canonical module of $R$, and set $d = \dim R$.

Suppose $R$ is F-regular. Then, for each integer $k$, there exists $q = p^e$ such that

$$\operatorname{rank}_K R_k \leq \operatorname{rank}_K [H^d_m(\omega^{(q)})]_k.$$  

Proof. If $d \leq 1$, then $R$ is regular and the assertion is elementary. Assume $d \geq 2$. Let $\xi \in [H^d_m(\omega)]_0$ be an element which generates the socle of $H^d_m(\omega)$. Since the map $\omega^{[q]} \mapsto \omega^{(q)}$ is an isomorphism in codimension one, $F^c(\xi)$ may be viewed as an element of $H^d_m(\omega^{(q)})$ as in [Wa2].

Fix an integer $k$. For each $e \in \mathbb{N}$, set $V_e$ to be the kernel of the vector space homomorphism

$$R_k \longrightarrow [H^d_m(\omega^{(p^e)})]_k,$$  

where $c \mapsto cF^e(\xi)$.

If $cF^{e+1}(\xi) = 0$, then $F^c(cF^{e+1}(\xi)) = c^p F^{e+1}(\xi) = 0$; since $R$ is F-pure, it follows that $cF^e(\xi) = 0$. Consequently the vector spaces $V_e$ form a descending sequence

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots.$$  

The hypothesis that $R$ is F-regular implies $\bigcap_e V_e = 0$. Since each $V_e$ has finite rank, $V_e = 0$ for $e \gg 0$. Hence the homomorphism (6.3.1) is injective for $e \gg 0$. \hfill $\square$

We next record tight closure properties of general $\mathbb{N}$-graded hypersurfaces. The results for F-purity are essentially worked out in [HR].

Theorem 6.4. Let $A = K[x_1, \ldots, x_m]$ be a polynomial ring over a field $K$ of positive characteristic. Let $d$ be a nonnegative integer, and set $M = (\binom{d+m-1}{d} - 1)$. Consider the affine space $\mathbb{A}^M_K$ parameterizing the degree $d$ forms in $A$ in which $x_1^d$ occurs with coefficient 1.

Let $U$ be the subset of $\mathbb{A}^M_K$ corresponding to the forms $f$ for which $A/fA$ F-pure. Then $U$ is a Zariski open set, and it is nonempty if and only if $d \leq m$.

Let $V$ be the set corresponding to forms $f$ for which $A/fA$ is F-regular. Then $V$ contains a nonempty Zariski open set if $d < m$, and is empty otherwise.

Proof. The set $U$ is Zariski open by [HR page 156] and it is empty if $d > m$ by [HR Proposition 5.18]. If $d \leq m$, the square-free monomial $x_1 \cdots x_d$ defines an F-pure hypersurface $A/(x_1 \cdots x_d)$. A linear change of variables yields the polynomial

$$f = x_1(x_1 + x_2) \cdots (x_1 + x_d)$$  

in which $x_1^d$ occurs with coefficient 1. Hence $U$ is nonempty for $d \leq m$.

If $d \geq m$, then $A/fA$ has $a$-invariant $d - m \geq 0$ so $A/fA$ is not F-regular. Suppose $d < m$. Consider the set $W \subseteq \mathbb{A}^M_K$ parameterizing the forms $f$ for which $A/fA$ is F-pure and $(A/fA)_{\mathfrak{m}_1}$ is regular; $W$ is a nonempty open subset of $\mathbb{A}^M_K$. Let $f$ correspond to a point of $W$. The element $\mathfrak{m}_1 \in A/fA$ has a power which
is a test element; since $A/fA$ is F-pure, it follows that $\mathfrak{p}_1$ is a test element. Note
that $\mathfrak{p}_2, \ldots, \mathfrak{p}_m$ is a homogeneous system of parameters for $A/fA$ and that $\mathfrak{p}_1^{d-1}$
generates the socle modulo $(\mathfrak{p}_2, \ldots, \mathfrak{p}_m)$. Hence the ring $A/fA$ is F-regular
if and only if there exists a power $q$ of the prime characteristic $p$ such that
\[
q x_1^{(d-1)q+1} \notin (x_2^q, \ldots, x_m^q, f)A.
\]
The set of such $f$ corresponds to an open subset of $W$; it remains to verify that
this subset is nonempty. For this, consider
\[
f = x_1^d + x_2 \cdots x_{d+1},
\]
which corresponds to a point of $W$, and note that $A/fA$ is F-regular since
\[
x_1^{(d-1)p+1} \notin (x_2^p, \ldots, x_m^p, f)A.
\]

These ideas carry over to multi-graded hypersurfaces; we restrict below to the
bigraded case. The set of forms in $K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ of degree $(d, e)$ in
which $x_1^d y_1^e$ occurs with coefficient 1 is parametrized by the affine space $\mathbb{A}_K^N$
where $N = \binom{d+m-1}{d-1}'$.

**Theorem 6.5.** Let $B = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial
ring over a field $K$ of positive characteristic. Consider the $\mathbb{N}^2$-grading on $B$ with
deg $x_i = (1, 0)$ and deg $y_j = (0, 1)$. Let $d, e$ be nonnegative integers, and consider
the affine space $\mathbb{A}_K^N$ parameterizing forms of degree $(d, e)$ in which $x_1^d y_1^e$
occurs with coefficient 1.

Let $U$ be the subset of $\mathbb{A}_K^N$ corresponding to forms $f$ for which $B/fB$ is F-pure.
Then $U$ is a Zariski open set, and it is nonempty if and only if $d \leq m$ and $e \leq n$.

Let $V$ be the set corresponding to forms $f$ for which $B/fB$ is F-regular. Then $V$
contains a nonempty Zariski open set if $d < m$ and $e < n$, and is empty otherwise.

**Proof.** The argument for F-purity is similar to the proof of Theorem 6.4, if $d \leq m$
and $e \leq n$, then the polynomial $x_1 \cdots x_d y_1 \cdots y_e$ defines an F-pure hypersurface.

If $B/fB$ is F-regular, then $a(B/fB) < 0$ implies $d < m$ and $e < n$. Conversely,
if $d < m$ and $e < n$, then there is a nonempty open set $W$ corresponding to forms $f$ for
which the hypersurface $B/fB$ is F-pure and $(B/fB)_{\mathfrak{p}_1, \mathfrak{p}_2}$ is regular. In
this case, $x_1 y_1 \in B/fB$ is a test element. The socle modulo the parameter ideal $(x_1 - y_1, x_2, \ldots, x_m, y_2, \ldots, y_n)B/fB$ is generated by $x_1^{d+e-1}$, so $B/fB$ is F-regular
if and only if there exists a power $q = p^e$ such that
\[
x_1^{(d+e-1)q+1} \notin (x_1^q - y_1^q, x_2^q, \ldots, x_m^q, y_2^q, \ldots, y_n^q, f)B.
\]
The subset of $W$ corresponding to such $f$ is open; it remains to verify that it is
nonempty. For this, use $f = x_1^q y_1^q + x_2 \cdots x_{d+1} y_2 \cdots y_{e+1}$.
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