Distributed Neighbor Selection in Multi-agent Networks

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Abstract—Achieving consensus via nearest neighbor rules is an important prerequisite for multi-agent networks to accomplish collective tasks. A common assumption in consensus setup is that each agent interacts with all its neighbors during the process. This paper examines whether network functionality and performance can be maintained—and even enhanced—when agents interact only with a subset of their respective (available) neighbors. As shown in the paper, the answer to this inquiry is affirmative. In this direction, we show that by using the monotonicity property of the Laplacian eigenvectors, a neighbor selection rule with guaranteed performance enhancements, can be realized for consensus-type networks. For the purpose of distributed implementation, a quantitative connection between Laplacian eigenvectors and the “relative rate of change” in the state between neighboring agents is further established; this connection facilitates a distributed algorithm for each agent to identify “favorable” neighbors to interact with. Multi-agent networks with and without external influence are examined, as well as extensions to signed networks. This paper underscores the utility of Laplacian eigenvectors in the context of distributed neighbor selection, providing novel insights into distributed data-driven control of multi-agent systems.

Index Terms—Distributed neighbor selection; Laplacian eigenvectors; improving consensus convergence rate; block-cut tree; relative tempo; data-driven control.

I. INTRODUCTION

A multi-agent network is composed of a group of agents, interacting with their respective (nearest) neighboring agents, by following simple local rules; when such local rules lead to an emerging collective behavior at the network level is of great interest [1], [2], [3]. Engineered multi-agent networks are often inspired by cooperative and swarming phenomena in natural systems [4], [5], motivating the analytical frameworks for characterizing their respective collective behaviors based on nearest neighbor rule [6], [7], [8]. Achieving consensus via pairwise diffusive interactions between neighboring agents is a prototypical collective behavior of multi-agent systems [9], [7], [10], [11], [12]. In fact, consensus turns out to be a critical prerequisite in disciplines such as distributed control of networked systems [13], [14], distributed estimation over sensor networks [15], synchronization in complex networks [16], large-scale multi-agent machine learning [17], opinion dynamics [18].

The functionality and performance of a multi-agent network is dependent on the underlying network topology, realized via each agent’s interactions with its (accessible) local neighbors. Although distributed algorithms for multi-agent consensus based on the nearest neighbor rule have been extensively investigated, an important question is whether the network performance can be improved if agents only interact with a subset of their available neighbors. For instance, in leader-follower multi-robotic networks it is often assumed that each robot interacts with neighboring robots within a given sensing radius; an important observation is that the resultant network is not necessary efficient in diffusion of information from leader robots to the followers [13]. A similar situation arises in distributed optimization and estimation, where the coordination network is realized via the spatial distribution of sensors or processors [17], [19].

In the meantime, the neighbor selection is ubiquitous in both natural and artificial networks. For instance, it has been reported for example, that in flocks of starlings, birds generally interact with a subset of their nearest neighbors, rather than with all birds within a given sensing radius [20]. An analogous scenario is observed in social networks, where an individual often determines the subset of their friends to interact with on online social media; this phenomenon also occurs in real-world social interactions amongst people [21]. In the meanwhile, neighbor selection schemes are employed in peer-to-peer networks, such as BitTorrent, to save traffic costs [22]. Along the same lines, adaptive neighbor selection has been proposed to enhance the quality of predicted ratings in recommender systems [23]. The kNN imputation methods are designed to select k nearest neighbors to deal with missing data in a given dataset [24].

In multi-agent consensus problem, network topology plays a crucial role in achieving consensus and the corresponding convergence rate [25], [26], [27], [28]. A common assumption in this line of work is that each agent interacts with all its neighbors; however, there may exist interactions that degrade the overall convergence rate, i.e., the responsiveness of the multi-agent network [1], [2], [3], [29]. Our work is also inspired by an observation that the information flow between a pair of neighboring agents does not need to be bidirectional, for example, when the two agents are not equivalent according to a specific hierarchy. For instance, a rooted tree is a typical hierarchical structure for the efficient spreading of information...
from the root to other nodes; a node with a shorter distance from the root can be considered to have a higher hierarchical level. In fact, hierarchical organizations are ubiquitous in both natural and artificial large-scale networks and a lack of hierarchy could have an adverse effect on the network performance [30], [31], [32], [33], [34], [35].

A natural question thereby is whether, for a given network, the importance of agents’ neighbors (with respect to the desired performance) can be inferred from local information. This distributed neighbor selection problem is the focus of the present work. In particular, we provide a theoretical framework to reason about the neighbor selection process as well as guarantees on its performance. As we will show, the nodes (eigenvectors) of the graph Laplacian prove to be instrumental in designing this algorithm. In fact, one of these modes facilitates a systematic treatment for designing and analyzing a novel distributed neighbor selection algorithm for consensus-type networks.

The contribution of this paper is threefold. First, we will show that the neighbor selection process (effectively removing a specific subset of edges from the network) could be useful for improving the network performance. In this direction, one of our contributions involves using the eigenvector entries of the Laplacian as a criterion for neighbor selection; subsequently, we show how the corresponding reduced network maintains and even enhances the functionality of the original network. Secondly, inspired by the observation that redundant interactions amongst agents can hinder the efficiency of information propagation throughout a network, we provide theoretical guarantees on the performance enhancement of the reduced network. Third, as the Laplacian eigenvectors are global network variables, we establish a quantitative connection between the Laplacian eigenvector and the relative rate of change in work variables, we establish a quantitative connection between

II. Preliminaries

First a quick note on the notation. Let $\mathbb{R}$ and $\mathbb{Z}_+$ denote the set of real numbers and positive integers, respectively. Denote the set \{1, 2, \ldots, n\} as $\mathfrak{p}$, where $n \in \mathbb{Z}_+$. $1_n$ and $0_{n \times m}$ denote $n \times 1$ vector and $n \times m$ matrix of all ones and all zeros, respectively. Let $I_d$ denote the $d \times d$ identity matrix. The $i$th smallest eigenvalue and the corresponding normalized eigenvector of a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is signified by $\lambda_i(M)$ and $v_i(M)$, respectively. The entry located at the $i$th row and $j$th column in a matrix $M \in \mathbb{R}^{n \times n}$ is denoted by $M_{ij}$ and the $i$th entry of a vector $x$ by $[x]_i$. Let $x_{ij}$ denote $[x]_j$, for a vector $x \in \mathbb{R}^n$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is designated by $\|x\| = (x^T x)^{\frac{1}{2}}$. A vector $x \in \mathbb{R}^n$ is positive if $[x]_i > 0$ for all $i \in \mathfrak{p}$. The spectral radius of a matrix $M$ is the largest absolute value of its eigenvalues, which we denote by $\rho(M)$.

Next, we provide a few graph-theoretic constructs that will be subsequently used in the paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ denote a graph with the node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The adjacency matrix $W = (w_{ij}) \in \mathbb{R}^{n \times n}$ is such that the weight between agents $i$ and $j$ satisfies $w_{ij} \neq 0$ if and only if $(i, j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise. A graph $\mathcal{G}$ is called a signed network if there exists an edge $(i, j) \in \mathcal{E}$ such that $w_{ij} < 0$ and as unsigned otherwise. Denote the in-degree neighbor set of an agent $i \in \mathcal{V}$ as $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$.

A path from a node $i_p \in \mathcal{V}$ to a node $i_1 \in \mathcal{V}$ in a graph $\mathcal{G}$ is a concatenation of edges $\mathcal{P}(i_1, i_p) = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p)\} \subset \mathcal{E}$, where all nodes $i_1, i_2, \ldots, i_p \in \mathcal{V}$ are distinct; a node $i \in \mathcal{V}$ is reachable from a node $j \in \mathcal{V}$ if there exists a path $\mathcal{P}(i, j) \in \mathcal{G}$. A graph is connected if each pair of nodes are reachable from each other. An acyclic graph (tree) is a connected graph without cycles.

Let $S^d$ denote a star graph with $n \in \mathbb{Z}_+$ nodes. A subgraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ is a graph such that $\mathcal{V} \subset \mathcal{V}$ and $\mathcal{E} \subset \mathcal{E}$. The subgraph obtained by removing a node set $\mathcal{V}' \subset \mathcal{V}$ and all incident edges from a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is denoted by $\mathcal{G} - \mathcal{V}'$. Let $\mathcal{S} \subset \mathcal{V}$ be any subset of nodes in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then the induced subgraph $\mathcal{G}(\mathcal{S})$ is the graph whose node set is $\mathcal{S}$ and whose edge set consists of all of the edges incident to nodes in $\mathcal{S}$.

Lastly, we provide a brief synopsis of multi-agent networks. In a multi-agent network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, each agent $i \in \mathcal{V}$ has the state $x_i(t) \in \mathbb{R}^d$ (or $x_i \in \mathbb{R}^d$) at time $t$. In the sequel we will consider two distinct categories of diffusively coupled networks, namely, fully-autonomous and semi-autonomous.

In a fully autonomous network (FAN), $n \in \mathbb{Z}_+$ agents evolve their respective states through interactions characterized by an unsigned graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. In particular, each agent updates its state by adopting the diffusive interaction protocol,

$$\dot{x}_i(t) = - \sum_{i=1}^{n} w_{ij} (x_i(t) - x_j(t)), \ i \in \mathcal{V}. \quad (1)$$

1We will use “graphs” and “networks” interchangeably in this paper.
In relation with the protocol (1), we say the state of agent \( i \) is influenced by its neighbors \( j \in \mathcal{N}_i \) or, equivalently, agent \( i \) follows its neighbors \( j \in \mathcal{N}_i \). Denote the graph Laplacian of \( \mathcal{G} \) as \( L(\mathcal{G}) = (l_{ij}) \in \mathbb{R}^{n \times n} \) where \( l_{ij} = \sum_{j=1}^{n} w_{ij} \) for \( i = j \) and \( l_{ij} = -w_{ij} \) for \( i \neq j \). The collective behavior of a FAN can be characterized by the graph Laplacian as,

\[
\dot{x} = -(L(\mathcal{G}) \otimes I_d)x,
\]

where \( x = (x_1^T(t), \ldots, x_n^T(t))^T \in \mathbb{R}^{nd} \).

Distinct from fully-autonomous networks, in semi-autonomous networks (SANs), a subset of agents, referred to as leaders or informed agents, are selected to receive external control signals so as to steer the entire network towards a desired state. In this direction, consider a SAN consisting of \( n \in \mathbb{Z}_+ \) agents whose interaction structure is characterized by an unsigned graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, W) \). In a SAN, the leaders, denoted by \( \mathcal{V}_{\text{leader}} \subset \mathcal{V} \), can be directly influenced by the external input signal and the remaining agents are referred to as followers, denoted by \( \mathcal{V}_{\text{follower}} = \mathcal{V} \setminus \mathcal{V}_{\text{leader}} \). In this paper, the set of external inputs is denoted by \( \mathcal{U} = \{u_1, \ldots, u_m\} \), where \( u_l \in \mathbb{R}^d, l \in \mathcal{U} \) and \( m \in \mathbb{Z}_+ \). In this setup, it is assumed that each leader is at most influenced by one external input. In a SAN, the interaction protocol for each agent can be written as,

\[
\dot{x}_i(t) = -\sum_{j=1}^{n} w_{ij}(x_i(t) - x_j(t)) - \sum_{l=1}^{m} b_{il}(x_i(t) - u_l), \; i \in \mathcal{V},
\]

where \( b_{il} = 1 \) if and only if \( i \in \mathcal{V}_{\text{leader}} \) and \( b_{il} = 0 \) otherwise. Subsequently, the collective behavior of SAN (3) can be characterized by,

\[
\dot{x} = -(L_B(\mathcal{G}) \otimes I_d)x + (B \otimes I_d)u,
\]

where \( x = (x_1^T(t), \ldots, x_n^T(t))^T \in \mathbb{R}^{nd}, B = (b_{il}) \in \mathbb{R}^{n \times m}, \) \( u = (u_1^T, \ldots, u_m^T)^T \in \mathbb{R}^{md} \) and

\[
L_B(\mathcal{G}) = L(\mathcal{G}) + \text{diag}(B1_m).
\]

Note that the input structure can be characterized by the matrix \( B \). Since \( L_B(\mathcal{G}) \) (or \( L_B \) for brevity) is obtained from a perturbation on Laplacian matrix \( L(\mathcal{G}) \) by a diagonal matrix \( \text{diag}(B1_m) \), we refer to \( L_B(\mathcal{G}) \) as the perturbed Laplacian.

Both FAN (2) or SAN (4) are said to achieve consensus if \( \lim_{t \to \infty} ||x_i(t) - x_j(t)|| = 0 \) for all \( i, j \in \mathcal{V} \) and some norm on \( \mathbb{R}^d \).

III. A Motivational Scenario

In this section, we provide an example to motivate our contribution. Consider a SAN on a connected unsigned network \( \mathcal{G}_6 \) in Figure 1a (original network), where agent 1 is a leader with an external input \( u = 0.9 \). On one hand, the reachability (existence of a (directed) path) from the external input to each agent is a prerequisite for the agents to track this external input \( u \). This observation motivates us to ask whether the remaining edges, apart from those that can guarantee the leader-follower reachability of the network, are necessary for reaching a consensus. For instance, although the network \( \mathcal{G}'_6 \) (Figure 1c) is the minimal subgraph of \( \mathcal{G}_6 \) (in terms of the number of edges) that can guarantee the reachability from external input \( u \) to all the agents, the associated convergence performance is significantly enhanced compared with that of the original network \( \mathcal{G}_6 \) (see Figure 2). Although one can observe that network \( \mathcal{G}'_6 \) can be constructed from \( \mathcal{G}_6 \) by eliminating one of the bidirectional edges between neighboring agents in \( \mathcal{G}_6 \) (see Figure 1b), how the local information that can be employed to guide this neighbor selection process is not trivial.
Recall that the eigenvector associated with the smallest non-zero eigenvalue of perturbed Laplacian can be chosen to be positive. A relevant observation is that the entries of this eigenvector, along the directed paths from leader agents to all follower agents, are monotonically increasing (subsequently, we will show that this is not accidental). We will subsequently see that this monotonicity property plays an important role in the distributed neighbor selection process (see Figure 1b). In the sequel we first examine this observation analytically for SANs, followed by its implications for FANs.

IV. SEMI-AUTONOMOUS NETWORKS

In this section, a neighbor selection algorithm, based on the monotonicity of the eigenvector entries associated with the perturbed Laplacian, is proposed. Subsequently, the convergence rate of the multi-agent system on the reduced network, post neighbor selection process, will be examined. Furthermore, the distributed implementation of the neighbor selection process is discussed.

A. Reachability Analysis

We shall first examine the reachability property of SANs, as encoded in a Laplacian eigenvector of the underlying network. The eigen-pair \((\lambda_1(L_B), v_1(L_B))\) associated with the perturbed Laplacian \(L_B\) of a SAN, with input matrix \(B\), turns out to be an important algebraic construct revealing graph-theoretic properties of SANs. As such, we shall unveil the network reachability, encoded in the eigenvector \(v_1(L_B)\), providing useful insights for designing neighbor selection algorithm for SANs.

First, we provide some preliminary properties of \((\lambda_1(L_B), v_1(L_B))\).

**Lemma 1.** Let \(\lambda_1(L_B)\) and \(v_1(L_B)\) denote the smallest eigenvalue and the corresponding normalized eigenvector of \(L_B\) in (5), respectively. Then, \(\lambda_1(L_B) > 0\) is a simple eigenvalue of \(L_B\) and \(v_1(L_B)\) can be chosen to be positive.

**Proof.** Refer to the Appendix.

For a SAN \(\mathcal{G}\) with the input matrix \(B\), we proceed to construct a reduced subgraph of the SAN by eliminating a subset of edges between an agent and its neighboring agents, using information encoded in \(v_1(L_B)\), namely, neighbor selection. We shall refer to this class of reduced subgraphs as the following the slower neighbor (FSN) networks of \(\mathcal{G}\), since it is implied that each agent follows (or chooses to be influenced by) those neighbors whose rate of change in states are relatively slower; this statement will be made more rigorous in \\

**Definition 1 (FSN network of SANs).** Let \(\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)\) be an unsigned SAN with the input matrix \(B\). The FSN network of \(\mathcal{G}\), denoted by \(\mathcal{G} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})\), is a subgraph of \(\mathcal{G}\) such that \(\bar{\mathcal{V}} = \mathcal{V}, \bar{\mathcal{E}} \subseteq \mathcal{E}\) and \(\bar{W} = (\bar{w}_{ij}) \in \mathbb{R}^{n \times n}\), where \(\bar{w}_{ij} = w_{ij}\) if \(v_1(L_B)_{ij} > 1\) and \(\bar{w}_{ij} = 0\) when \(v_1(L_B)_{ij} \leq 1\).

According to Definition 1, the FSN network of a SAN is determined by the perturbed Laplacian, specifically by the corresponding eigenvector \(v_1(L_B)\). Note that the construction of the FSN network is essentially achieved by comparing \(v_1(L_B)_{ij}\) and 1; hence this process can be regarded as a rule of neighbor selection. We shall now proceed to reveal the leader-to-follower reachability (LF-reachability) of FSN networks.

**Theorem 1.** Let \(\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{W})\) be the FSN network of the SAN (4) on the unsigned connected network \(\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)\). Then each agent \(i \in \bar{\mathcal{V}}\) is reachable from some external input \(u_i\) in \(\bar{\mathcal{G}}\).

**Proof.** Let \(\mathcal{V}_{\text{leader}}\) and \(\mathcal{V}_{\text{follower}}\) be the leader and the follower sets of the SAN (4), respectively. According to Lemma 5, it is sufficient to show that for an arbitrary \(i \in \mathcal{V}_{\text{follower}},\) there exists a leader agent \(l \in \mathcal{V}_{\text{leader}}\) such that \(i\) is reachable from \(l\).

By contradiction, assume that there exists a subset of agents \(\{i_1, i_2, \ldots, i_s\} \subseteq \mathcal{V}_{\text{follower}}\) in the FSN network \(\bar{\mathcal{G}}\) such that \(i_k\) is not reachable from any \(l \in \mathcal{V}_{\text{leader}}\), where \(k \in \mathbb{S}\) and \(s \in \mathbb{Z}_+\).

Let \(\lambda_1\) be the smallest eigenvalue of the perturbed Laplacian matrix \(L_B(\bar{\mathcal{G}})\) with the corresponding eigenvector \(v_1\), i.e.,

\[
L_B(\bar{\mathcal{G}}) v_1 = \lambda_1 v_1. \tag{6}
\]

According to Lemma 1, one has \(\lambda_1 > 0\) and the corresponding eigenvector \(v_1\) is positive. Now consider the following two cases:

**Case 1:** There exists an isolated agent \(i' \in \{i_1, i_2, \ldots, i_s\}\) such that agent \(i'\) is not reachable from any leader agent in the FSN network \(\bar{\mathcal{G}}\). Then, according to Definition 1, one has,

\[
[v_1]_{i'} \leq [v_1]_j, \tag{7}
\]

for all \(j \in \mathcal{N}_{i'}\). Examining the \(i'\)'th row in (6) yields,

\[
\left( \sum_{j \in \mathcal{N}_{i'}} w_{i'j} \right) [v_1]_{i'} - \left( \sum_{j \in \mathcal{N}_{i'}} w_{i'j} [v_1]_j \right) = \lambda_1 [v_1]_{i'}. \tag{8}
\]

Combining (7) and (8), now yields the following inequality,

\[
\left( \sum_{j \in \mathcal{N}_{i'}} w_{i'j} \right) [v_1]_{i'} - \left( \sum_{j \in \mathcal{N}_{i'}} w_{i'j} [v_1]_j \right) \geq \lambda_1 [v_1]_{i'}. \tag{9}
\]

By canceling \([v_1]_{i'} > 0\) from both sides of the above inequality, one can obtain \(\lambda_1 \leq 0\), establishing a contradiction.

**Case 2:** There exists a weak connected component \(\bar{\mathcal{G}}' = \{i_1, i_2, \ldots, i_s\} \in \{i_1, i_2, \ldots, i_s\}\), that is that agent where \(s_0 \in \mathbb{Z}_+\) and \(s_0 \leq s\). Let \(\mathcal{V}' = \{i_1, i_2, \ldots, i_s\}\). Then, one has \([v_1]_j \geq [v_1]_{i'}\) for all \(j \in \mathcal{N}_{i'}\). Again, one can conclude the contradiction \(\lambda_1 \leq 0\) by applying a similar procedure as in Case 1.

It turns out that the entries of the eigenvector \(v_1(L_B)\) of \(L_B\) in the SAN (4) is characterized by the location of leaders in the network. We provide an example to demonstrate the utility of Theorem 1.
Example 1. Consider a SAN on the network $G_8$ shown in Figure 3; each agent holds a three-dimensional state and agents 4 and 8 are leaders that are directly influenced by the homogeneous input $u = (u_1, u_2)^T$, where $u_1 = u_2 = (0.7, 0.8, 0.9)^T \in \mathbb{R}^3$. The initial state of agents is randomly selected from $[0, 1] \times [0, 1] \times [0, 1]$. Computing $v_1(L_B)$ corresponding to the perturbed Laplacian in this example yields, $v_1(L_B) = (0.46, 0.38, 0.30, 0.16, 0.46, 0.40, 0.32, 0.22)^T$.

One can observe from Figure 3 that for each agent $i \in \mathcal{V}$, there exists a directed path from $u_1$ or $u_2$ to $i$ such that the entries in $v_1(L_B)$ along this path is monotonically increasing. Therefore, the associated FSN network according to Definition 1, is as shown in Figure 4. One can observe from Figure 6 that each agent tends to track the external input directly in the FSN network (see the middle plot in Figure 6) instead of aggregating and moving together towards external input, as shown in the left plot in Figure 6.

Theorem 1 ensures that all agents in the FSN network of a SAN are influenced by the external inputs, namely, LF-reachability can be guaranteed. Therefore, a SAN (4) can exhibit either consensus or clustering over the corresponding FSN network, depending on the heterogeneity of external input; see Lemma 5 in Appendix as well as [10], [11], [41]. One can verify that the constructing FSN network using other eigenvectors than $v_1(L_B)$ do not ensure the LF-reachability according to the Definition 1.

![Fig. 3. An eight-node SAN $G_8$. The bidirectional edges between neighboring agents is identified by line with double arrows for simplicity. The entries in $v_1(L_B)$ corresponding to each agent are shown close to each node (with an accuracy of two decimal points).](image1)

![Fig. 4. The FSN network corresponding to the network $G_8$ in Figure 3.](image2)

Inspired by Theorem 1, we postulate that if one reserves the construction of FSN network for SANs (each agent now follows neighbors whose respective rates of change in state are relatively faster), the influence of external input exerted on the network can be weaken or even eliminated. One can refer to the resulting reduced network as following the faster neighbor (FFN) network. In this case, agents in the FFN network are not reachable from the external input. This can be useful when the external input, say, represents epidemics or rumors, and the network structure is rearranged in a distributed manner by each agent to attenuate the spreading process. For example, the FFN network of $G_8$ in Figure 3 is shown in Figure 5; in this case, the influence from external inputs to leaders can be eliminated since the rate of change in state of external inputs can be viewed as zero. The trajectory of SAN on FFN network is shown on the right-hand plot in Figure 6. In FFN networks, the influence structure is reversed in contrast to FSN network. Therefore, only leader agents (agents 4 and 8) are influenced by external inputs and as a result, the influence of external inputs on the follower agents have been eliminated. In this paper, we shall concentrate on FSN networks; such networks closely abstract means of enhancing the spreading process on a network.

![Fig. 5. The FFN network corresponding to the network $G_8$ in Figure 3.](image3)

B. Convergence Rate Enhancement

In order to evaluate the performance of neighbor selection based on $v_1(L_B)$, we now proceed to examine the convergence rate of SANs on the resultant FSN networks. Note that the smallest non-zero eigenvalue of the perturbed Laplacian of a SAN characterizes the convergence rate of the multi-agent system towards its steady-state, either consensus or clustering [27], [42], [43]. We provide the following result on the convergence rate of SAN on connected unsigned networks and the corresponding FSN networks.

**Theorem 2.** Let $\tilde{G} = (\mathcal{V}, \tilde{E}, \tilde{W})$ denote the FSN network of a SAN $\mathcal{G} = (\mathcal{V}, E, W)$ with the input matrix $B$. Then

$$\lambda_1(L_B(\mathcal{G})) \geq \lambda_1(L_B(\tilde{G})), $$

where the equality holds only when all agents are leaders.

**Proof.** Denote the perturbed Laplacian matrix $L_B(\mathcal{G})$ and $L_B(\tilde{G})$ as $\bar{L}_B$ and $L_B$, respectively. According to Definition 1, the FSN network $\tilde{G}$ is a directed acyclic network. Therefore, one can relabel the agents in $\tilde{G}$ such that $\bar{W}$ is a lower triangular matrix, implying that $\bar{L}_B$ is also lower triangular. Thus, the eigenvalues of $\bar{L}_B$ are exactly the entries on its diagonal.

Note that each agent in the FSN network has at least one indegree neighbor, namely, the diagonal entries, satisfying that $[\bar{L}_B]_{ii} \geq 1$ for all $i \in \mathcal{V}$, which in turn implies that $\lambda_1(L_B) \geq \lambda_1(L_B(\tilde{G}))$. Therefore, we can conclude that the convergence rate of SAN on connected unsigned networks can be improved by designing FSN networks.
Thus, respectively. According to Weyl theorem ([44, Theorem 4.3.1, p.239]), one has,
\[ \lambda_1(L_B) \leq \lambda_1(L) + \lambda_n(\text{diag}(B1_m)); \]
due to the fact \( \lambda_1(L) = 0 \) and \( \lambda_n(\text{diag}(B1_m)) = 1 \), it follows that \( \lambda_1(L_B) \leq 1 \).

One the one hand, if \( \text{diag}(B1_m) = I \), again using Weyl theorem yields,
\[ \lambda_1(L_B) \geq \lambda_1(L) + \lambda_1(I) = 1; \]
hence, \( \lambda_1(L_B) = 1 \). Now suppose that \( \text{diag}(B1_m) \neq I \). Let
\[ L_B = L + I + \Delta, \tag{9} \]
where \( \Delta \in \mathbb{R}^{n \times n} \) is a non-zero diagonal matrix whose diagonal entries are either \(-1\) or \(0\). In fact, (9) produces all possible perturbed Laplacians apart from the case that all agents are leaders. Without loss of generality, we choose \( \Delta = \text{diag}(-1, 0, \ldots, 0) \). Then, by applying Weyl theorem one more time, we have,
\[ \lambda_1(L + I + \Delta) \leq \lambda_1(L + I) + \lambda_n(\Delta) = 1. \tag{10} \]

For the matrix \( L + I \) in (10), applying Weyl theorem, now leads to \( \lambda_1(L + I) = 1 \). According to \( (L + I)(a1_n) = a1_n \), where \( a \in \mathbb{R} \), it follows that \( \text{span}(1_n) \) is an eigenspace of the matrix \( L + I \) corresponding to the eigenvalue \( \lambda_1(L + I) \).

For the matrix \( \Delta \) in (10), assume that there exists \( a_0 \in \mathbb{R} \) such that,
\[ \Delta(a_01_n) = \lambda_n(a_01_n); \]
as \( \Delta(a_01_n) = (-a_0, 0, \ldots, 0)^\top \) and \( \lambda_n(a_01_n) = (0, 0, \ldots, 0)^\top \), one has \( a_0 = 0 \).

Thus, there does not exist a common non-zero eigenvector corresponding to \( \lambda_1(L + I + \Delta) \), \( \lambda_1(L + I) \) and \( \lambda_n(\Delta) \), respectively. According to Weyl theorem, one has,
\[ \lambda_1(L + I + \Delta) < 1. \]

Thus,
\[ \lambda_1(\bar{L}_B) = \lambda_1(L_B) \]
if \( \text{diag}(B1_m) \neq I \) and
\[ \lambda_1(\tilde{L}_B) \geq \lambda_1(L_B) \]
when \( \text{diag}(B1_m) = I \).

Theorem 2 provides theoretical guarantees on the convergence rate of FSN network \( G \) as compared with the original network \( G \). Let us continue to employ Example 1 to demonstrate the convergence rate of SAN on the original network \( G \) and its related FSN network \( G \). The comparison of convergence times of SAN in Example 1 on original network and FSN network in Example 1 is demonstrated in Figures 7 and 8, respectively. In this case, the convergence rate of SAN on the original network \( G \) and FSN network \( G_{FSN} \) are \( \lambda_1(L_B(G)) = 0.1414 \) and \( \lambda_1(L_B(G_{FSN})) = 1 \), respectively.
C. Neighbor Selection in SANs

So far, we have presented a neighbor selection framework with guaranteed performance using the eigenvector of perturbed Laplacian. However, this eigenvector is a network-level quantity, hindering its direct applicability for large-scale networks. For such networks, it is desirable that decision-making relies only on local observations [45].

In this section, we establish a quantitative link between the Laplacian eigenvector and the relative rate of change in state of neighboring agents, referred to as the relative tempo. Using this approach, we connect the global property of the network to a locally measurable quantity as such, we are able to propose a fully distributed neighbor selection algorithm. In order to simplify our derivation, we first introduce the so-called selection matrix.

The selection matrix of a subset of agents \( V_s = \{i_1, \ldots, i_s\} \subset \mathcal{V} \) is defined as
\[
\phi(V_s) = (e_{i_1}, \ldots, e_{i_s})^T \in \mathbb{R}^{\times n},
\]
where \( e_{i_j} \) denotes the \( i_j \)-th column of \( I_s \) where \( j \in s \). Thereby, the state of agents in \( V_s \) can be characterized by
\[
(x_{i_1}(t)^T, \ldots, x_{i_s}(t)^T)^T = (\phi(V_s) \otimes I_s) \mathbf{x}(t).
\]

We now proceed to introduce the notion of relative tempo, characterizing the steady-state of the relative rate of change in state between two subsets of agents.

**Definition 2.** Let \( \mathcal{V}_1 \subset \mathcal{V} \) and \( \mathcal{V}_2 \subset \mathcal{V} \) be two subsets of agents in multi-agent network (4) (or 2). Then the relative tempo between agents in \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) is defined as the limiting ratio,
\[
\tau(\mathcal{V}_1, \mathcal{V}_2) = \lim_{t \to \infty} \frac{\|\phi(\mathcal{V}_1) \mathbf{l}_d \mathbf{x}_i(t)\|}{\|\phi(\mathcal{V}_2) \mathbf{l}_d \mathbf{x}_i(t)\|}.
\]

**Proof.** Refer to the Appendix

**Remark 1.** Theorem 3 provides a quantitative connection between the relative tempo (that can be constructed from local data available to each agent) and the Laplacian eigenvector of the underlying network. According to Theorem 1 and Theorem 2, such a connection enables a distributed implementation of neighbor selection for enhancing the convergence rate for the network. Furthermore, it is notable that Theorem 3 for SAN (4) is independent of initial states of agents, implying that it also holds for SANs influenced by either homogeneous or heterogeneous external inputs.

We provide an example to illustrate Theorem 3.

**Example 2.** Consider the following quantity
\[
g_{ij}(t) = \frac{\|\dot{x}_i(t)\|}{\|\dot{x}_j(t)\|}, \quad i \in \mathcal{V}, j \in \mathcal{N}_i,
\]
which satisfies
\[
\lim_{t \to \infty} g_{ij}(t) = \tau(i, j) \quad \text{(Theorem 3)}.
\]
Let us continue to examine Example 1. The trajectories of \( g_{ij}(t) \) for \( i = 7 \) and \( j \in \{3, 6, 8\} \) are shown in Figure 9. The steady-states of \( g_{ij}(t) \) are archived at around \( t = 10 \), particularly, \( g_{73}(10) = 1.057 \), \( g_{76}(10) = 0.8123 \) and \( g_{78}(10) = 1.47 \), respectively. In the meanwhile, one has \( v_1(L_B)_{73} = 1.057 \), \( v_1(L_B)_{76} = 0.8113 \) and \( v_1(L_B)_{78} = 1.4694 \). Note that such correspondences are sufficient for the construction of the associated FSN network (shown in Figure 4) using Definition 1.

**Remark 2.** Examining the convergence time of agents’ states in the original network shown in Figure 7 and that of \( g_{ij}(t) \) in Figure 9, one observes that the SAN (4) achieves a sort of "ordered" state characterized by the relative tempo, prior to the final consensus.

By Definition 1 and Theorem 3, the reduced neighbor set for each agent for constructing the FSN network can be defined as follows.

**Definition 3.** The reduced neighbor set for agent \( i \in \mathcal{V} \) to construct the associated FSN network for SAN (4) is defined as
\[
\mathcal{N}_i^{\text{FSN}} = \{j \in \mathcal{N}_i \mid v_1(L_B)_{ij} > 1\} = \{j \in \mathcal{N}_i \mid \tau(i, j) > 1\}.
\]

Intuitively, when \( \tau(i, j) > 1 \) for a pair of neighboring agents \( i, j \in \mathcal{V} \), then the state of agent \( i \) evolves towards the external input in a relatively faster rate than that of agent \( j \) (after the relative tempo reaches a steady-state). Therefore, agents in a SAN are involved in a sort of hierarchy encoded in \( v_1(L_B) \) according to Theorem 3. As such, each agent can select a specific group of neighbors to interact with for...
a given task. Typically, there are two alternatives to reduce the network structure, signifying the scenarios of “follow the slower neighbors” (FSN) and “follow the faster neighbors” (FFN), respectively. The main insight from our discussion is that \( \tau(i, j) \) can be estimated for agent \( i \in \mathcal{V} \) and \( j \in N_i \) via local measurements. Note that the corresponding data-driven reduced network exhibits LF-reachability property, as stated by Theorem 1.

V. Fully-Autonomous Networks

In this section, we proceed to investigate parallel results for FANs—the corresponding analysis turns out to be more intricate than for SANs.

A. Reachability Analysis

Recall that for the eigenvector associated with perturbed Laplacian matrix \( L_B \) in SANs (4), all elements of \( v_1(L_B) \) have the same sign. However, in the case of FANs, the entries in eigenvectors of graph Laplacian can be positive, negative or equal to zero; this is also valid for the Fiedler vector \( v_2(L) \), the eigenvector corresponding to the second smallest eigenvalue of graph Laplacian [37], [39]. This situation renders the extension of the aforementioned neighbor selection framework—from SANs to FANs—non-trivial.

In this section, we shall first examine the property of Laplacian eigenvectors related to SANs, specifically the Fiedler vector \( v_2(L) \), and then proceed to provide the neighbor selection algorithm to construct the FSN network of FANs for fast convergence. In the following discussions, we shall refer to a node corresponding to positive, negative or zero entry in \( v_2(L) \) as a positive node, negative node and zero node, respectively.

In FANs, the structural properties of the network turn out to be critical in the analysis and design of the neighbor selection algorithm. We introduce the following results related to the block decomposition of a graph [48], [39].

A cut node \( i \in \mathcal{V} \) of a graph \( G = (\mathcal{V}, \mathcal{E}, W) \) is a node such that \( G - \{i\} \) is disconnected. A block of a graph \( G \) is a maximal connected subgraph of \( G \) with no cut nodes. Consider a connected graph \( G \) with blocks \( \{B_i\} \) and cut nodes \( \{c_j\} \), where \( i, j \in \mathbb{Z}_+ \). The block-cut graph of \( G \), denoted by \( B(G) \), is defined as the graph with node set composed of blocks and cut nodes, namely, \( V(B(G)) = \{B_i\} \cup \{c_j\} \), where two nodes are adjacent in \( V(B(G)) \) if one corresponds to a block \( B_i \) and the other to a cut node such that \( c_j \in B_i \).

**Lemma 2.** [48] Let \( G = (\mathcal{V}, \mathcal{E}, W) \) be a connected graph. Then the block-cut graph of \( G \) is a tree.

Here, we provide an example to illustrate the block-cut tree associated with a connected graph.

**Example 3.** Consider a FAN \( \mathcal{G}_{12} \) shown on the left-hand plot of Figure 10. There are six blocks in \( \mathcal{G}_{12} \), that is,

\[
B_1 = \mathcal{G}\{4, 5, 6\}, \\
B_2 = \mathcal{G}\{1, 2, 3, 4\}, \\
B_3 = \mathcal{G}\{1, 11\}, \\
B_4 = \mathcal{G}\{1, 12\}, \\
B_5 = \mathcal{G}\{6, 7, 8\}, \\
B_6 = \mathcal{G}\{6, 9, 10\},
\]

and three cut nodes \( c_1 = \{1\}, c_2 = \{4\}, c_3 = \{6\} \). The block-cut tree of \( \mathcal{G}_{12} \) is shown on the right-hand plot of Figure 10.

The following result reveals the monotonicity property of the entries in \( v_2(L) \) along certain paths in block-cut tree of a graph, which will subsequently be used for constructing the FSN network associated with FANs.

**Lemma 3.** [39, Theorem 3.14] Let \( G \) be a connected graph with Laplacian matrix \( L \); let \( \lambda_2(L) \) and \( v_2(L) \) be the second smallest eigenvalue of \( L \) and the corresponding eigenvector, respectively. Then exactly one of the following two cases occurs:

**Case 1.** There is a single block \( B \) in \( G \) which contains both positive and negative nodes. Each other block has either positive nodes only, or negative nodes only, or zero nodes only. Every path \( P \) starting in \( B \) and containing just one node \( k \) in \( B \) has the property that the values at the points of cut node contained in \( P \) form either an increasing, or decreasing, or a zero sequence along this path according to whether \(|v_2(L)|_{k} > 0\), \(|v_2(L)|_{k} < 0\) or \(|v_2(L)|_{k} = 0\); in the last case all nodes in \( P \) are zero nodes.

**Case 2.** No block of \( G \) contains both positive and negative nodes. There exists a single zero cut node \( v_0 \) with non-zero neighbor nodes. Each block (with the exception of that zero cut node) has either positive nodes only, or negative nodes only, or zero nodes only. Every pure path \( P \) starting in that zero cut node has the property that the values at the points of cut node contained in \( P \) form either an increasing, or decreasing, or a zero sequence along this path and in the last case all nodes...
in $P$ are zero nodes. Every path containing both positive and negative nodes passes through that zero cut node.

As an example, the FAN $G_{12}$ in Figure 10 satisfies Case 1 in the Lemma 3. In the following discussions, we will refer to the block $B$ in Case 1 and the cut node $v_0$ in Case 2 in the Lemma 3 as core block and core node, respectively; we will refer to a block having only positive, negative, or zero nodes as a positive block, negative block, and zero block, respectively. We are now ready to present the construction of FSN networks for FANs.

**Definition 4 (FSN Network of FANs).** Let $G = (V, E, W)$ be an unsigned network with the block decomposition $\{B_1, \ldots , B_r\}$, where $r \in \mathbb{Z}^+$. The FSN network of $G$ is a subgraph $\bar{G} = (\bar{V}, \bar{E}, \bar{W})$ with node set $\bar{V} = V$ and edge set $\bar{E} \subseteq E$. The weight matrix $\bar{W} = (\bar{w}_{ij}) \in \mathbb{R}^{n \times n}$ satisfies for all $i \in B_p$ ($p \in \mathbb{Z}$), where $B_p$ is neither core block nor zero block, such that $\bar{w}_{ij} = w_{ij}$ if $v_2(L)_{ij} > 1$ and $\bar{w}_{ij} = 0$ if $v_2(L)_{ij} \leq 1$, where $j \in B_p \cap N_i$, and $\bar{w}_{ij} = w_{ij}$ for all remaining $(i, j) \in E$.

According to Definition 4, the construction of the FSN network is built upon the block decomposition of a graph; the edges in the core block and zero block will remain unchanged while the other edges can be changed depending on the quantity $v_2(L)_{ij}$. We now proceed to examine the reachability of the FSN network associated with FANs.

**Theorem 4.** Let $\bar{G} = (\bar{V}, \bar{E}, \bar{W})$ be the FSN network of the FAN (2) on $G = (V, E, W)$. Then, the FAN (2) achieves consensus on the associated FSN network $\bar{G}$. Moreover, the consensus value is determined by the initial values of either the agents in the core block and zero blocks or the core agent and the agents in zero blocks. 

**Proof.** Refer to the Appendix.

According to the Theorem 4, the FAN (2) can achieve consensus on the associated FSN network $\bar{G}$, however, the consensus value is generally not equal to the consensus value achieved by the original network (average of initial states of all agents). In fact, the consensus value achieved on the FSN network $\bar{G}$ is eventually the average of the initial states of agents belonging to the core block and the zero blocks or the average of the initial states of the core agent and agents belonging to zero blocks.

**Example 4.** The FSN network $\bar{G}_{12}$ corresponding to the network $G_{12}$ in Figure 10 is shown in Figure 11. The core block in $G_{12}$ is $B = G \{\{4, 5, 6\}\}$. As one can see from Figure 11, all agents except that in the core block $B$ are reachable from agents in the core block.

**B. Convergence Rate Enhancement**

We proceed to examine the convergence rate enhancement of FANs on the corresponding FSN networks. First, we provide the following result, for general FANs, that characterize the convergence rate of FAN (2) on the associated FSN network.

**Proposition 1.** Let $\bar{G} = (\bar{V}, \bar{E}, \bar{W})$ be the FSN network of a FAN $G = (V, E, W)$ characterized by (2). Let $\lambda_2(L(\bar{G}))$ and $\bar{v}_2$ be the second smallest eigenvalue of $L(\bar{G})$ and the corresponding normalized eigenvector, respectively. Then $\lambda_2(L(\bar{G}))$ is lower bounded by 

$$\lambda_2(L(\bar{G})) + \sum_{(i,j) \in \bar{E} \setminus \bar{E}} [\bar{v}_2]_i ([\bar{v}_2]_j - [\bar{v}_2]_i).$$ (13)

**Proof.** Refer to the Appendix.

One can observe from Proposition 1 that the quantitative relationship between $\lambda_2(L(\bar{G}))$ and $\lambda_2(L(G))$ is a bit vague since the second term in (13) can be negative. This motivates us to examine the special case of tree graphs in which case, the convergence rate between a FAN and the corresponding FSN network can be well-characterized.

**Theorem 5.** Let $T$ be an $n$-node tree network without zero blocks where $n \geq 4$. Let $\bar{T}$ be its FSN network characterized by (2). Let $\lambda_2(L(T))$ and $\lambda_2(L(\bar{T}))$ be the second smallest eigenvalue of $L(T)$ and $L(\bar{T})$, respectively. Then $\lambda_2(L(T)) > \lambda_2(L(\bar{T}))$.

**Proof.** Refer to the Appendix.

**Example 5.** Consider a 12-node FAN with the tree structure in Figure 12 (left) whose associated FSN network is shown in Figure 12 (right). The agents’ initial state is,

$$x(0) = (0.973, 0.649, 0.8, 0.454, 0.432, 0.825, 0.804, 0.133, 0.173, 0.391, 0.831, 0.803)^\top.$$ (14)

In this example, the core block in tree $T$ is the induced subgraph $T(\{4, 6\})$. According to Figure 13, the FAN (2) on its associated FSN network $\bar{G}_{12}$ with the initial states in (14) achieves consensus on the value $0.6396$, which is equal to the average of the initial states of the agents in the core block, namely, $\frac{1}{3} (0.4538 + 0.8253) = 0.6396$. Computing $v_2(L)$
corresponding to the Laplacian matrix $L$ of $\mathcal{T}$ in this example yields,

$$\nu_2(L) = (0.333, 0.101, 0.101, 0.079, 0.101, -0.257, -0.327, -0.327, -0.327, -0.327, 0.424, 0.424)^\top.$$ (15)

The convergence rate associated with networks $\mathcal{T}$ and $\mathcal{T}_{FSN}$ are $\lambda_2(L(\mathcal{T})) = 0.2148$ and $\lambda_2(L(\mathcal{T}_{FSN})) = 1$, respectively.

Remark 3. Notably, the convergence rate of FSN network corresponding to trees are always 1, however the convergence rate of a tree can decrease dramatically when the diameter of the tree grows. For example, that convergence rate on trees with diameter $\text{diam}(T)$ is bounded by

$$2 \left(1 - \cos \left(\frac{\pi}{\text{diam}(T) + 1}\right)\right),$$

signifying that $\lambda_2(L(T)) \to 0$ when $\text{diam}(T) \to \infty$ [49].

C. Neighbor Selection in FANs

In this part, we shall assume that the second smallest eigenvalue of the Laplacian matrix of a network is simple. Then, one has the following theorem which establishes the relationship between the relative tempo and the Fiedler vector of a graph.

\textbf{Theorem 6.} Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two subsets of agents in $\mathcal{V}$. Each agent in $\mathcal{V}$ adopts dynamics (1). If the second smallest eigenvalue of the Laplacian matrix $L$ is simple, then the relative tempo of agents in $\mathcal{V}_1$ compared to that in $\mathcal{V}_2$ is

$$\tau(\mathcal{V}_1, \mathcal{V}_2) = \frac{\|\phi(\mathcal{V}_1)\nu_2(L)\|}{\|\phi(\mathcal{V}_2)\nu_2(L)\|}.$$  

\textit{Proof.} According to Lemma 6 in Appendix, the proof follows by choosing $M = -L \otimes I_d$ in (24).

On the one hand, if the knowledge of both the core block and zero block of a FAN is available, according to the Definition 4 and Theorem 6, one can employ relative tempo to construct the FSN network instead of using the information in $\nu_2(L)$. Under the FSN network, a consensus at the value of the average of initial states associated with agents in the core block and zero block, or the average of initial states associated with agents in zero block and initial state of core node, can be reached. In the meanwhile, the FSN network can be constructed in a distributed manner, which is similar to SANs.

On the other hand, if the knowledge of both the core block and zero block of the network $\mathcal{G}$ is unavailable, a natural question to ask is whether one can determine the core block and zero block from network data. For general networks, this task can be tough. However, tree networks turn out to be tractable, since every node in a tree is a cut node. Therefore, the core block in a tree network contains at most two nodes.

Here, we discuss a class of the tree network without zero blocks and examine how to construct the corresponding FSN network only using local measurable information similar to the relative tempo. Note that for a tree network without zero blocks, either there exists a core block containing two nodes $\{u, v\}$ such that $[\nu_2]_u[\nu_2]_v < 0$, or there exists a core node $w$ such that $[\nu_2]_w = 0$. Actually, according to the proof of Lemma 6 in the Appendix, one can see that

$$\frac{e_i^\top \dot{x}_u(t)}{e_i^\top \dot{x}_v(t)} = \frac{[\nu_2]_u}{[\nu_2]_v}. \quad (16)$$

Denote

$$\tau'(u, v) = \frac{e_i^\top \dot{x}_u(t)}{e_i^\top \dot{x}_v(t)},$$

therefore, when considering the relative tempo between the two agents, one can use the quantity $\tau'(u, v)$ instead of $\tau(u, v)$.
to identify two nodes \{u, v\} in the core block for the former case, and the edge between \(u, v\) shall be maintained while the other edges shall be maintained or eliminated according to the Definition 4 by using the quantity \(\tau'(u, v)\); for the latter case, one can use the quantity \(\tau'(u, v)\) to construct the FSN network \(\tilde{G}\) directly.

According to Definition 4 and Theorem 6, the FSN network of FAN on tree networks can be defined as follows.

**Definition 5.** Let \(G = (V, \mathcal{E}, W)\) be a tree network. The reduced neighbor sets for each agent \(i \in V\) to construct the associated FSN network of FAN (2) is

\[
N_{i}^{\text{FSN}} = \{ j \in N_i \mid v_2(L)_{ij} > 1 \text{ or } v_2(L)_{ij} < 0 \} = \{ j \in N_i \mid \tau'(i, j) > 1 \text{ or } \tau'(i, j) < 0 \}.
\]

To sum up, we have established the parallel framework of neighborhood selection for FANs with a specific focus on tree networks.

**VI. EXTENSION TO SIGNED NETWORKS**

In this section, we discuss extensions of the aforementioned results to signed networks. A signed network \(G = (V, \mathcal{E}, W)\) is structurally balanced if there is a bipartition of the node set \(V\) (hence, \(V_1 \subset V\) and \(V_2 \subset V\) such that \(\mathcal{E} = V_1 \cup V_2\) and \(V_1 \cap V_2 = \emptyset\)) such that the edges within each subset is positive, but negative for edges between the two subsets [50]. An important property of a structurally balanced signed network is that there exists a quantitative connection between traditional Laplacian matrix and signed Laplacian matrix via a gauge transformation, assuming matrix form \(G = \text{diag} \{\sigma_1, \ldots, \sigma_n\}\), where \(\sigma_i \in \{1, -1\}\) and \(i \in \mathbb{R}\) [51]. We now discuss the extension of our results for structurally balanced signed networks.

**A. Signed Semi-Autonomous Networks**

For structurally balanced signed SANs, the eigenvectors of the perturbed Laplacian can be transformed from the unsigned SANs via Gauge transformations. In this section, we provide a more complete discussion for structurally balanced signed SAN. In particular, consider the following interaction protocol on a signed SAN,

\[
\dot{x}_i(t) = -\sum_{i=1}^{n} |w_{ij}|(x_i(t) - \text{sgn}(w_{ij})x_j(t))
- \sum_{i=1}^{m} |b_{il}|(x_i(t) - \text{sgn}(b_{il})u_l), i \in V, \quad (17)
\]

where \(b_{il} \in \{1, -1\}\) if and only if \(i \in V_{\text{leader}}\) and \(b_{il} = 0\) otherwise and \(\text{sgn}(\cdot)\) denotes the sign function such that

\[
\text{sgn}(z) = \begin{cases} 
1, & z > 0; \\
-1, & z < 0; \\
0, & z = 0.
\end{cases}
\]

Denote the signed Laplacian matrix of \(G\) as \(L_s = (l_{ij}^{s}) \in \mathbb{R}^{n \times n}\), where \(l_{ij}^{s} = \sum_{j=1}^{n} |w_{ij}| \) for \(i = j\) and \(l_{ij}^{s} = -w_{ij}\) for \(i \neq j\). The collective dynamics of (17) is

\[
\dot{x} = -(L_B^{s}(G) \otimes I_d)x + (B \otimes I_d)u, \quad (18)
\]

where \(L_B^{s}(G) = L_s + \text{diag}\{B|1_m\}, \ x = (x_1^T, \ldots, x_n^T)^T \in \mathbb{R}^{dn}\), \(B = [b_{il}] \in \mathbb{R}^{n \times m}\) and \(u = (u_1^T, \ldots, u_m^T)^T \in \mathbb{R}^{dm}\).

Denote by the edge set between external input signals and the leaders and the input set as \(\mathcal{E}'\) and \(\mathcal{U} = (u_1, \ldots, u_m)\), respectively. The graph \(\tilde{G} = (\mathcal{V}, \mathcal{E}, \tilde{A})\) is directed with \(\mathcal{V} = \mathcal{V} \cup \mathcal{U}, \ \mathcal{E} = \mathcal{E} \cup \mathcal{E}'\), \(\tilde{A} = \mathcal{A} \cup B\). It is shown that the signed Laplacian matrix of the network \(\tilde{G}\) is positive semi-definite if \(\tilde{G}\) is structurally balanced [51]. We shall therefore concentrate on structurally balanced signed networks.

**Lemma 4.** Consider the signed SAN (18) on a signed network \(G = (V, \mathcal{E}, W)\). Suppose that \(G = (\mathcal{V}, E, W)\) is connected and \(G = (\mathcal{V}, E, \tilde{A})\) is structurally balanced, and let \(\lambda_1(L_B^{s}(G))\) and \(v_1(L_B^{s}(G))\) be the smallest eigenvalue of \(L_B^{s}(G)\) and the corresponding normalized eigenvector, respectively. Then, \(\lambda_1(L_B^{s}(G)) > 0\) is a simple eigenvalue of \(L_B^{s}(G)\) and \(v_1(L_B^{s}(G))\) is positive under a proper Gauge transformation.

**Proof.** The proof is a simple extension of Lemma 1 and omitted for brevity.

The FSN network for signed SANs can be defined as follows.

**Definition 6 (FSN network of signed SANs).** Let \(G = (\mathcal{V}, \mathcal{E}, W)\) be a signed SAN characterized by (18). The FSN network of \(G\), denoted by \(\tilde{G} = (\mathcal{V}, E, W)\), is a subgraph of \(G\) such that \(\mathcal{V} = \mathcal{V}, \mathcal{E} \subseteq \mathcal{E}\) and \(\tilde{W} = (\tilde{w}_{ij}) \in \mathbb{R}^{n \times n}\), where \(\tilde{w}_{ij} = w_{ij}\) if \(|v_1(L_B^{s}(G))_{ij}| > 1\) and \(\tilde{w}_{ij} = 0\) if \(|v_1(L_B^{s}(G))_{ij}| \leq 1\).

**Theorem 7.** Let \(\tilde{G} = (\mathcal{V}, E, \tilde{W})\) be the FSN network of the signed SAN \(G = (\mathcal{V}, E, W)\) characterized by (18). Suppose that \(G = (\mathcal{V}, E, W)\) is connected and \(\tilde{G} = (\mathcal{V}, E, \tilde{A})\) is structurally balanced; then all agents in \(\tilde{G}\) are reachable from the external input.

**Proof.** The proof follows from Lemma 4, proof of Theorem 1, and applying the Gauge transformation corresponding to the signed network \(\tilde{G}\).

**Theorem 8.** Let \(V_1 \subset V\) and \(V_2 \subset V\) be two subsets of agents in a connected signed SAN \(\tilde{G} = (\mathcal{V}, E, W)\) characterized by (18). If \(\tilde{G} = (\mathcal{V}, E, \tilde{A})\) is structurally balanced, then the relative tempo between agents in \(V_1\) and \(V_2\) satisfies

\[
\tau(V_1, V_2) = \frac{\|\phi(V_1)v_1(L_B^{s}(G))\|}{\|\phi(V_2)v_1(L_B^{s}(G))\|}.
\]

**Proof.** According to Lemma 6, the proof follows by choosing

\[
M = \begin{pmatrix} -L^s_B \otimes I_d & B \otimes I_d \\ 0_{md \times nd} & 0_{md \times nd} \end{pmatrix}
\]

in (24) in Appendix.

**Theorem 9.** Let \(\tilde{G} = (\mathcal{V}, E, \tilde{W})\) be the FSN network of a connected signed SAN \(G = (\mathcal{V}, E, W)\) characterized by (18). If \(\tilde{G} = (\mathcal{V}, E, \tilde{A})\) is structurally balanced, then \(\lambda_1(L_B^{s}(\tilde{G})) \geq \lambda_1(L_B^{s}(G))\).

**Proof.** The proof is a straightforward extension of Theorem 2, and omitted for brevity.
We now provide an example to illustrate the aforementioned results on signed SANs.

**Example 6.** Consider a signed SAN on the network \( G_8 \) shown in Figure 15, each agent holds a three-dimensional state and agents 4 and 8 are leaders that directly influenced by the homogeneous input \( u = (u_1, u_2)^T \), where \( u_1 = u_2 = (0.7, 0.8, 0.9)^T \in \mathbb{R}^3 \). The associated FSN network is shown in Figure 16. As one can see from Figures 17 and 18, the convergence rate of bipartite consensus is greatly improved for the FSN network.

\[ \dot{x}_i(t) = -\sum_{i=1}^{n} |w_{ij}|(x_i(t) - \text{sgn}(w_{ij})x_j(t)), i \in \mathcal{V}. \]  

\[ \dot{x} = -(L^*(G) \otimes I_d)x. \]

Denote by the unsigned network corresponding to the signed network \( \tilde{G} = (\mathcal{V}, \tilde{E}, |A|) \) with Laplacian matrix \( L \). It is shown that \( L = DL^*D \), where \( D \) is the Gauge transformation corresponding to the structurally balanced signed network \( G \) [51]. This correspondence implies that the Laplacian eigenvectors can be respectively transformed via a Gauge transformation. Therefore, the results on the unsigned FANs can be extended to the structurally balanced signed FANs through a proper Gauge transformation on the Fielder vector.

**VII. CONCLUSIONS**

This paper addresses distributed neighbor selection problem of multi-agent networks. In this direction, a theoretical framework of distributed neighbor selection for diffusively coupled multi-agent networks has been proposed. Along the way, we have highlighted the utility of Laplacian eigenvectors to further improve network performance; these eigenvectors encode hierarchical information about the network that in turn, relate to the notion of relative tempo. The latter connection is then used in a data-driven setting for the neighbor selection problem. Future works that build on the proposed framework include extensions to multi-agent time-varying networks and neighbor selection with noisy and delayed time-series data.

**APPENDIX**

The appendix contains the proofs of various results discussed in the paper.

**Proof of Lemma 1**

Proof. Note that the perturbed Laplacian matrix \( L_B \) is symmetric and diagonal dominant; as such, \( \lambda_i(L_B) \geq 0 \) for all
i ∈ \mathbb{N}_0. Assume that \( L_B \) has an eigenvalue \( \lambda_1(L_B) = 0 \) with associated eigenvector \( v_1(L_B) \in \mathbb{R}^n \). Then,
\[
L_B v_1(L_B) = L v_1(L_B) + \text{diag}(B 1_m) v_1(L_B) = 0.
\]
Multiply the above equality by \( v_1^\top(L_B) \) from left yields,
\[
v_1^\top(L_B) L v_1(L_B) + v_1^\top(L_B) \text{diag}(B 1_m) v_1(L_B) = 0.
\]
Since,
\[
v_1^\top(L_B) L v_1(L_B) \geq 0,
\]
and
\[
v_1^\top(L_B) \text{diag}(B 1_m) v_1(L_B) \geq 0,
\]
one has,
\[
v_1^\top(L_B) L v_1(L_B) = 0,
\]
and
\[
v_1^\top(L_B) \text{diag}(B 1_m) v_1(L_B) = 0.
\]
This however means that \( v_1(L_B) = 1_n \), leading to having \( v_1^\top(L_B) \text{diag}(B 1_m) v_1(L_B) > 0 \). This is a contradiction and therefore \( \lambda_1 > 0 \) for all \( i \in \mathbb{N}_0 \).

We shall proceed to show that \( \lambda_1(L_B) \) is simple and \( v_1(L_B) \) is positive. Denote by \( L_B = \eta I - M \), where \( M \in \mathbb{R}^{n \times n} \) is a non-negative matrix and \( \eta \) is the maximum value of the diagonal entries of \( L_B \). Then \( e^{-L_B t} = e^{M - \eta I} = e^{-\eta t} e^M \).

Note that the matrix \( M \) is non-negative, therefore, \( e^{-L_B t} \) is a non-negative matrix. In addition, since the network \( G \) is connected, \( M \) is irreducible, implying that \( e^{-L_B t} \) is a non-negative irreducible matrix. Thus, according to Perron–Frobenius theorem for irreducible non-negative matrices, the eigenvalue \( e^{-\lambda_1(L_B)} \) is simple and the corresponding eigenvector \( v_1(L_B) \) is positive.

**STEADY-STATE OF SAN ON UNSIGNED NETWORKS**

The external input \( u \) is homogeneous if \( u_i = u_j \) for all \( i \neq j \in m \) and heterogeneous otherwise. In the case of unsigned networks, the steady-state of SAN (4) is consensus (when the external input is homogeneous) or cluster consensus (when the external input is heterogeneous) [41]. Formally, the steady-state of the SAN (4) is determined by the convex hull spanned by external inputs, namely,
\[
\text{Co}(\mathcal{U}) = \left\{ \sum_{i=1}^m k_i u_i | u_i \in \mathcal{U}, k_i \geq 0, \sum_{i=1}^m k_i = 1 \right\}.
\]

Then following lemma characterizes the steady-state of the SAN (4) on unsigned networks.

**Lemma 5.** [10], [11], [41] Consider the SAN (4) on an unsigned network \( G = (\mathcal{V}, \mathcal{E}, W) \). Then, the state of all agents converge to the convex hull spanned by the external inputs for arbitrary initial conditions if and only if for each agent \( i \in \mathcal{V} \), there exists at least one external input \( u_i \in \mathcal{U} \) such that \( i \) is reachable from \( u_i \). Moreover, the steady-state of the SAN (4) admits,
\[
\lim_{t \to \infty} x(t) = (L_B^{-1} \otimes I_d)(B \otimes I_d)x_0 = (L_B^{-1}) B \otimes I_d x_u.
\]

Specifically, if \( u \) is homogeneous, then the SAN (4) achieves consensus, namely,
\[
\lim_{t \to \infty} x(t) = \left( \frac{1}{m} u^\top 1_m \right) 1_n.
\]

**PROOF OF THEOREM 3**

In order to show Theorem 3, we need the following lemma.

**Lemma 6.** Consider a multi-agent network governed by matrix ordinary differential equation
\[
\dot{x}(t) = M x(t),
\]
where \( M \in \mathbb{R}^{n \times n} \) has \( n \) linearly independent eigenvectors and \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top \). Denote the ordered eigenvalue of \( M \) as \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) with corresponding normalized eigenvectors \( \varphi_1, \varphi_2, \ldots, \varphi_n \). Let \( \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k_s} \) be the largest nonzero eigenvalue of \( M \) with the algebraic multiplicity \( s \in \mathbb{N} \). \( \phi_1 \) and \( \phi_2 \) be selection matrices of two subsets of agents \( V_1 \) and \( V_2 \), respectively. Denote \( \alpha_{qi} = \phi_q \varphi_i \in \mathbb{R}^n \), \( S = [\varphi_1, \varphi_2, \ldots, \varphi_n] \in \mathbb{R}^{n \times n} \) and \( \beta = [\beta_1, \beta_2, \ldots, \beta_n]^\top = S^{-1} x(0) \in \mathbb{R}^n \) for \( q \in \mathbb{N} \) and \( i \in \mathbb{N} \). Then
\[
\lim_{t \to \infty} \frac{\| \phi_1 x(t) \|}{\| \phi_2 x(t) \|} = \left( \sum_{i,j=1}^{\beta} \lambda_i \lambda_j \alpha_{1i}^\top \alpha_{2j} \beta_i \beta_j \right)^{1/2}
\]

**Proof.** Note that \( M = JSJ^{-1} \) where \( S = [\varphi_1, \varphi_2, \ldots, \varphi_n] \in \mathbb{R}^{n \times n} \) and \( J = \text{diag} \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \in \mathbb{R}^{n \times n} \). According to the solution to the matrix ordinary differential equation \( \dot{x}(t) = M x(t) \), the derivative of \( x(t) \) is
\[
\dot{x}(t) = M e^{J t} x(0) = S J e^{J t} S^{-1} x(0);
\]
therefore one has,
\[
\| \phi_q x(t) \|^2 = (x(t))^\top \phi_q^\top \phi_q x(t) = x(0)^\top (S^{-1})^\top e^{J t} S J e^{J t} S^{-1} x(0)
\]
\[
= \sum_{i=1}^n \lambda_i \lambda_j e^{(\alpha_{qi} + \alpha_{qj})^\top \alpha_{qi} \beta_i \beta_j}
\]

The statement of the lemma now follows from straightforward computation, that has been omitted for brevity.

We are now in the position to prove Theorem 3.

**Proof.** In Lemma 6, choose \( M = \left( \begin{array}{cc} -L_B \otimes I_d & B \otimes I_d \\ 0_{md \times nd} & 0_{md \times nd} \end{array} \right) \). Since the algebraic multiplicity of the largest nonzero eigenvalue of the matrix \( M \) is equal to 1, the proof follows from a straightforward computation.

**PROOF OF THEOREM 4**

In order to prove Theorem 4, we need the following results.

**Lemma 7.** Let \( G(S) \) be a connected induced subgraph of \( G(\mathcal{V}) \) whose node set is \( S \in \mathcal{V} \). Denote
\[
\mathcal{E}^{\text{bound}} = \{(i, j) \in \mathcal{E} | \text{i \in S and j \in \mathcal{V} \setminus S} \}
\[ \nu^\text{out-bound}_S = \{ j \in V \setminus S | \exists i \in S \text{ such that } (i, j) \in \mathcal{E}^\text{bound} \}. \]

If for all \( i \in S \) and \( j \in \nu^\text{out-bound}_S \), \([v^1_2]\) and \([v^2_2]\) have the same signs, then there exists an edge \((i, j) \in \mathcal{E}^\text{bound}\) such that \(|[v^1_2]| > |[v^2_2]|\).

**Proof.** Assume that \([v^1_2]\) and \([v^2_2]\) are positive for all \( i \in S \) and \( j \in \nu^\text{out-bound}_S \) and there does not exist an edge \((i, j) \in \mathcal{E}^\text{bound}\) such that \([v^1_2] > [v^2_2]\). Let \( \lambda_2 \) be the second smallest eigenvalue of \( L(G) \) with the corresponding eigenvector \( v^2_2 = ([v^1_2], [v^2_2], \cdots, [v^m_2]) \in \mathbb{R}^n \). Denote the agents in \( S \) as \( \{i_1, i_2, \ldots, i_p\} \); then examining all the \( i_k \)-th row in the eigen-equation \( L(G)v^2_2 = \lambda_2 v^2_2 \), yields:

\[
\left( \sum_{j \in N_{i_k}} w_{i_kj} \right) [v^1_2]_{i_k} - \sum_{j \in N_{i_k}} w_{i_kj} [v^2_2]_{j} = \lambda_2 [v^1_2]_{i_k},
\]

for all \( k \in p \). Thereby,

\[
\sum_{(i, j) \in \mathcal{E}^\text{bound}} w_{ij}([v^1_2]_{i} - [v^2_2]_{j}) = \lambda_2 \sum_{i \in S} |[v^2_2]|.
\]

Since, \([v^1_2] - [v^2_2] \leq 0, \forall (i, j) \in \mathcal{E}^\text{bound}\), one can conclude that \( \lambda_2 \leq 0 \), which is a contradiction given the fact that \( \lambda_2 > 0 \) and there exists an edge \((i, j) \in \mathcal{E}^\text{bound}\) such that \([v^1_2] > [v^2_2]\).

For the case that \([v^1_2]\) and \([v^2_2]\) are negative for all \( i \in S \) and \( j \in \nu^\text{out-bound}_S \), the proof is analogous. \( \square \)

**Lemma 8.** \([39, \text{Theorem 3.3}] \) Let \( \mathcal{G} = (V, E, W) \) be an unsigned network with associated Laplacian matrix \( L \). Let \( v_2 \) denote the eigenvector corresponding to the second smallest eigenvalue of \( L \). For any \( r_1 \geq 0 \) and \( r_2 \leq 0 \), let

\[ M(r_1) = \{ i \in V | [v^1_2]_i + r_1 \geq 0 \} \]

and

\[ M(r_2) = \{ i \in V | [v^2_2]_i + r_2 \leq 0 \}. \]

Then, the subgraph \( \mathcal{G}(r_1) (\mathcal{G}(r_2)) \) induced by \( \mathcal{G} \) on \( M(r_1) \) (\( M(r_2) \)) is connected.

We are now ready to prove Theorem 4.

**Proof.** Consider the network containing the core block \( B_0 \) in the following two cases:

**Case 1:** Let \( B_1 \) be a positive block connecting with the core block \( B_0 \) directly through the cut node \( i^* \). Firstly, we shall prove that for any node \( j \in B_1 \), there are \([v^1_2]_j \geq [v^2_2]_j\). By contradiction, assume that there exists a node \( j_0 \in B_1 \) satisfying \([v^2_2]_{j_0} < [v^2_2]_{i^*}\). Due to having \( i^* \) as a cut node, according to Lemma 8, for any node \( p \in B_0 \), one has \([v^2_2]_p > [v^2_2]_{i^*}\); this is a contradiction (with the property of \( B_0 \)). Therefore, one has \([v^2_2]_j \geq [v^2_2]_{i^*}\) for any node \( j \in B_1 \).

**Case 2:** Let \( B_1 \) be a positive block such that all its neighbor blocks are positive. Denote by \( \mathcal{N}_i^B = \{ B_{i_1}, B_{i_2}, \ldots, B_{i_q} \} \) as the neighbor blocks of \( B_i \) with the corresponding cut nodes \( \{i_1, i_2, \ldots, i_q\} \). Let

\[ i^* = \argmin_{k \in \mathcal{Q}} [v^2_2]_{i_k}, \]

which connects the blocks \( B_i \) and \( B_{i^*} \).

Then we shall show that for any node \( j \in B_i \), \([v^1_2]_j \geq [v^2_2]_{i^*}\). By contradiction, assume that there exists \( j_0 \in B_i \) satisfying \([v^2_2]_{j_0} < [v^2_2]_{i^*}\); then according to Lemma 8, for any node \( p \in B_{i^*} \), one has \([v^1_2]_p > [v^1_2]_{i^*}\). Otherwise, assume that there exists a node \( p_0 \in B_{i^*} \) such that \([v^1_2]_{p_0} < [v^1_2]_{i^*}\), and choose

\[ r = \min([-v^2_2]_{j_0}, [v^2_2]_{p_0}). \]

Note that \( i^* \) is a cut node; then the subgraph \( \mathcal{G}(r) \) induced by \( \mathcal{G} \) on \( M(r) \) is disconnected. Therefore, one has \([v^1_2]_p > [v^1_2]_{i^*}\) for any node \( p \in B_{i^*} \). In addition, for any \( k \in q_i \) and \( k \neq i^* \), one has \([v^1_2]_q > [v^1_2]_{i_k}\) for any node \( q \in B_{i_k} \). This is due to having \( i_k \) as a cut node and \([v^1_2]_{i_k} > [v^1_2]_{i^*}\). However, in view of Lemma 7, this is a contradiction. Therefore, for any node \( j \in B_i \), one has \([v^1_2]_j > [v^1_2]_{i^*}\).

Based on the above two scenarios, in the following, let \( B_1 \) be an arbitrary positive block and denote by \( \mathcal{N}_i^B = \{ B_{i_1}, B_{i_2}, \ldots, B_{i_q} \} \) as the neighbor blocks of \( B_i \) with the corresponding cut nodes \( \{i_1, i_2, \ldots, i_q\} \). Let

\[ i^* = \argmin_{k \in \mathcal{Q}} [v^2_2]_{i_k}, \]

connecting blocks \( B_i \) and \( B_{i^*} \). We shall prove that any node \( j \in B_i \) can be reached by the cut node \( i^* \) on the network \( \mathcal{G} \). Let \( \mathcal{G}(\{i_1, \ldots, i_q\}) \) be weakly connected, not reachable from the cut node \( i^* \) on network \( \mathcal{G} \), where \( s_0 \in \mathbb{Z}_+ \). Denote \( \mathcal{E}^\text{bound} = \{(i, j) | i \in \{1,2, \cdots, i_q\}, j \in V \setminus \{i_1, \cdots, i_q\}\} \); then according to Definition 4, for any edge \((i, j) \in \mathcal{E}^\text{bound}\), one has \([v^1_2]_i \leq [v^1_2]_j\), which is a contradiction in view of Lemma 7. Hence, any node \( j \in B_i \), can be reached from the cut node \( i^* \) on \( \mathcal{G} \).

For the case that \( B_i \) is a negative block, the proof process is similar to the above two cases. For the case that the network only contains the core node \( r_0 \), the proof is similar to the above two cases. Consequently, the consensus is guaranteed using Lemma 5.

Next, we prove that the convergence value of the network \( \mathcal{G} \) is equal to the average initial states of the agents in the core block and zero blocks for Case 1 in Lemma 3 or the agents in zero blocks and the core agent for Case 2 in Lemma 3. We consider the general case with \( m \) agents in the core block and zero blocks for Case 1 in Lemma 3 or the agents in zero blocks and the core agent for Case 2 in Lemma 3. Denote by \( L = \{1, 2, \ldots, m\} \) and \( F = \{m + 1, m + 2, \ldots, n\} \), where \( m \in \mathbb{Z}_+ \) and \( m \leq n \). Then we can represent the Laplacian matrix of \( \mathcal{G} \) as,

\[ L = \begin{bmatrix} L_{11} & 0_{m \times (n-m)} \\ L_{21} & L_{22} \end{bmatrix}, \]

where \( L_{11} \in \mathbb{R}^{m\times m} \), \( L_{22} \in \mathbb{R}^{(n-m)\times (n-m)} \), \( L_{21} \in \mathbb{R}^{(n-m)\times m} \), and \( L_{22} \) is nonsingular. Due to \( L_{11} = 0 \), then one has \( L_{21}1_{m} + L_{22}1_{n-m} = 0 \). Therefore, \( L_{22}^{-1} L_{21}1_{m} = \)}
Denote by $\mathbf{x}_L = (\mathbf{x}^T_1(t), \ldots, \mathbf{x}^T_m(t))^T \in \mathbb{R}^{md}$ and $\mathbf{x}_F = (\mathbf{x}^T_{m+1}(t), \ldots, \mathbf{x}^T_m(t))^T \in \mathbb{R}^{(m-n)d}$. Thereby,

$$\dot{x}_L = -(L_{11} \otimes I_d)x_L,$$

$$\dot{x}_F = -(L_{22} \otimes I_d)x_F - (L_{21} \otimes I_d)x_L.$$

Letting,

$$X = \mathbf{x}_F + (L_{22}^{-1} 21 \otimes I_d)x_L,$$

one has,

$$\lim_{t \to \infty} X = 0.$$ Therefore, $\dot{x}_F = -(L_{22}^{-1} 21 \otimes I_d)x_L.$

By applying Rayleigh theorem [44, Theorem 4.2.2, p.235],

Moreover, according to the expansion,

$$\begin{align*}
&L(\mathcal{T}) = \begin{bmatrix} 0 & 0 \\ L_{\mathcal{V} \setminus \{u,v\}} \end{bmatrix}, \\
&L_{\{u,v\}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},
\end{align*}$$

denotes the Laplacian matrix associated with the core block, and $L_{\mathcal{V} \setminus \{u,v\}}$ is a lower triangular matrix with all diagonal elements equal to one. Therefore, the second smallest eigenvalue of $L(\mathcal{T})$ is equal to one. Again, in view of the Lemma 9, we conclude that $\lambda_2(L(\mathcal{T})) > \lambda_2(L(\mathcal{T})).$}

**Proof of Theorem 5**

**Lemma 9.** [49] Let $\mathcal{T}$ be a non-star tree network with $n \geq 4$ nodes ($\mathcal{T} \neq S_n$), and $\lambda_2(L(\mathcal{T}))$ be the second smallest eigenvalue of $L(\mathcal{T})$. Then $\lambda_2(L(\mathcal{T})) < 0.59$.

We are now to prove the Theorem 5.

**Proof.** Note that every node is a cut node in a tree graph. Then according to Lemma 3, there are two cases. For the case that there exists one core node in $\mathcal{T}$, the Laplacian matrix $L(\mathcal{T})$ of the FSN network $\mathcal{T}$ can be decomposed as,

$$L(\mathcal{T}) = \begin{bmatrix} 0 & 0 \\ L_{\mathcal{V} \setminus \{u,v\}} \end{bmatrix},$$

where $L_{\mathcal{V} \setminus \{u,v\}}$ is a lower triangular matrix, with all diagonal elements equal to one. Therefore, the second smallest eigenvalue of $L(\mathcal{T})$ is equal to one. Hence, using Lemma 9, it follows that $\lambda_2(L(\mathcal{T})) > \lambda_2(L(\mathcal{T})).$

Now we proceed to consider the case that there exists one core block with two nodes; denote these nodes by $u$ and $v$. Then, the Laplacian matrix $L(\mathcal{T})$ of the FSN network $\mathcal{T}$ can be decomposed as,

$$L(\mathcal{T}) = \begin{bmatrix} 0 & 0 \\ L_{\{u,v\}} \end{bmatrix},$$

where

$$L_{\{u,v\}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

denotes the Laplacian matrix associated with the core block, and $L_{\mathcal{V} \setminus \{u,v\}}$ is a lower triangular matrix with all diagonal elements equal to one. Therefore, the second smallest eigenvalue of $L(\mathcal{T})$ is equal to one. Again, in view of the Lemma 9, we conclude that $\lambda_2(L(\mathcal{T})) > \lambda_2(L(\mathcal{T})).$}

**References**

[1] M. Meshah and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*, Princeton University Press, 2010.

[2] J. Qin, Q. Ma, Y. Shi, and L. Wang, “Recent advances in consensus of multi-agent systems: A brief survey,” *IEEE Transactions on Industrial Electronics*, vol. 64, no. 6, pp. 4972–4983, 2016.

[3] Y. Cao, W. Yu, W. Ren, and G. Chen, “An overview of recent progress in the study of distributed multi-agent coordination,” *IEEE Transactions on Industrial Informatics*, vol. 9, no. 1, pp. 427–438, 2013.

[4] T. Vicsek and A. Zafeiris, “Collective motion,” *Physics Reports*, vol. 517, no. 3, pp. 71–140, 2012.

[5] G. Beni, “Swarm intelligence,” *Complex Social and Behavioral Systems: Game Theory and Agent-Based Models*, pp. 791–818, 2020.

[6] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, and O. Shochet, “Novel type of phase transition in a system of self-driven particles,” *Physical Review Letters*, vol. 75, no. 6, p. 1226, 1995.

[7] A. Jadbabaie, J. Lin, and A. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[8] R. Olfati-Saber, “Flocking for multi-agent dynamic systems: Algorithms and theory,” *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 401–420, 2006.

[9] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[10] Z. Lin, B. Francis, and M. Maggiore, “Necessary and sufficient graphical conditions for formation control of unicycles,” *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 121–127, 2005.

[11] W. Ren, R. W. Beard et al., “Consensus seeking in multiagent systems under dynamically changing interaction topologies,” *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.
