On Two-Current Realization of KP Hierarchy

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ABSTRACT

A simple description of the KP hierarchy and its multi-hamiltonian structure is given in terms of two Bose currents. A deformation scheme connecting various W-infinity algebras and relation between two fundamental nonlinear structures are discussed. Properties of Faá di Bruno polynomials are extensively explored in this construction.

Applications of our method are given for the Conformal Affine Toda model, WZNW models and discrete KP approach to Toda lattice chain.

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1 Introduction

The integrability properties of two-dimensional conformally invariant models can be studied in terms of extended conformal algebras [1]. These $W_N$ algebras are generated by the finite dimensional higher spin objects. They can be realized in terms of the second Gelfand-Dickey Hamiltonian structure of KdV hierarchies [2] (see also [3]). The nonlinear structure of $W_N$ disappears under the large $N$ limit giving rise to the area preserving diffeomorphisms of 2-manifold generating the so-called $w_\infty$ algebra [4]. A richer linear structure appears under generalization of KdV to KP hierarchy with the first Gelfand-Dickey bracket [5]. By this procedure one obtains an infinite dimensional Lie algebra $W_\infty$ [6]. This algebra appeared in different settings in the literature, various isomorphic examples can be found e.g. in [7], [8], [9] and [10]. The importance of KP hierarchy has increased recently due to its connection to 2d-gravity coupled to matter [11].

In this paper we emphasize the algebraic structure of the KP hierarchy, in terms of two Bose currents. We show that such currents provide an unified framework for realizing several algebraic structures appearing in KP hierarchy, WZNW models, Toda theories, one-matrix models and other theories possessing W-infinity algebras. We start with $W_{1+\infty}$ and its truncations (by removing lower spin generators) all described by family of linear Watanabe-like [5] structures connected with the linear parts of Gelfand-Dickey brackets within the KP hierarchy. These are determined by

$$\Omega^{(r)}_{n,m} (u(x), h) \equiv -\sum_{k=0}^{n+r} (-1)^k h^{k-1} \binom{n+r}{k} u_{n+m+r-k} D_x^k + \sum_{k=0}^{m+r} h^{k-1} \binom{m+r}{k} D_x^k u_{n+m+r-k} (x)$$

(1.1)

The constant $h$ is a deformation parameter responsible for extending $w_\infty \rightarrow W_\infty$. The index $r$ denotes the level of truncation of the underlying $W_{1+\infty}$.

Next we turn to the nonlinear extension of $W_\infty$ algebra denoted by $\hat{W}_\infty$ [12, 13]. Within the KP hierarchy it arises from the second Gelfand-Dickey bracket and can be governed by an extra deformation parameter $c$. We can summarize the deformations involved in our construction in the following simple way:

$$w_\infty \xrightarrow{h} W_\infty \xrightarrow{c} \hat{W}_\infty$$

(1.2)

In [14] the first deformation in (1.2) was performed by changing the basis from monomials in the current $J$ to $h$ dependent Faà di Bruno polynomials [15] in $J$ and its derivatives. Within a KP hierarchy $h$ appears as one of the central elements of the second Gelfand-Dickey bracket. There are two ways of generating the nonlinear structure in $\hat{W}_\infty$. One is connected with the central element in the second Gelfand-Dickey bracket. Alternatively one can create it by change of basis, while still working with the first bracket.

The paper is organized as follows. The basics of KP formalism and the Gelfand-Dickey brackets are given in Section 2. Section 3 provides a detailed discussion of our method to realize KP-like algebras in terms of two currents ($\bar{J}, J$). Here we make an extensive use of our basis given in terms of Faà di Bruno polynomials. We find a way of producing a family of Watanabe type forms representing the different level of truncation of the original algebra. We classify this structure
as generated by rows and columns of the matrix made of Fa"{a} di Bruno like objects within first KP hierarchy bracket. Next we go to the second Gelfand-Dickey bracket, which introduces a $c$ type deformation as in of (1.2). We prove the equivalence of Drinfeld-Sokolov bracket with the nonlinear part of Gelfand-Dickey second bracket for our basis. In this way we obtain an agreement with Lenard relations for our formalism. An alternative way of producing nonlinear structure in terms of the first bracket is also discussed.

We also study multi-Hamiltonian structures directly in terms of $(\bar{J}, J)$ currents. We find a closed formula for the higher Hamiltonian brackets and for the case $c = 0$ relate it to higher Watanabe forms in (1.1).

Applications of our method are given in section 4. We make connection to WZNW models, conformal affine Toda model [16, 17, 18] and specific construction of KP hierarchy related to Toda lattice chain [19].

The properties of Fa"{a} di Bruno polynomials and some related technical identities are given in the Appendices.

1.1 Notation

Throughout this paper we adopt the following notation:

\[ \partial^k f \equiv f^{(k)} ; \quad D \equiv \frac{\partial}{\partial x} \]

\[ D = \partial/\partial x \text{ acts on all the terms appearing to the right as a derivative operator by the Leibniz rule. } \]

\[ (A)_- \text{ denotes projection on the negative powers of } D \text{ contained in the operator } A, \text{ while } (A)_+ \text{ denotes projection on the positive and zero powers of } D. \]

\[ \text{We define } \text{res}(A) \text{ as a projection on the coefficient of } D^{-1} \text{ in } A. \]

Furthermore the KP flow parameters are denoted by $t$ in the multicomponent notation $t = (t_1, t_2, \ldots)$. Also, $\Omega^{(r)}_{n,m}(u(x)) \equiv \Omega^{(r)}_{n,m}(u(x), h = 1)$. In this paper we use $h = 1$ unless stated otherwise.

2 KP Preliminaries

Consider the Lax operator:

\[ L \equiv D + \sum_{i=0}^{\infty} u_i(x,t) D^{-i-1} \]

and the flows of KP hierarchy

\[ \frac{\partial L}{\partial t_r} = [(L^r)_+ , L] \quad r = 1, 2, \ldots \]

The KP flow equation (2.2) admits a Hamiltonian structure meaning that we can rewrite (2.2) for components of $L$ as

\[ \frac{\partial u_i}{\partial t_r} = \{u_i, H_r\}_2 = \{u_i, H_{r+1}\}_1 \]

\[ \text{as generated by rows and columns of the matrix made of Fa"{a} di Bruno like objects within first} \]
where the Hamiltonians for the KP hierarchy are $H_r = \frac{1}{r} \int \text{res} L^r$ and $\{ \cdot, \cdot \}_{1,2}$ denote first and second Poisson bracket structure. These higher bracket structures are compatible with Lenard relations $\{ u_i, H_m \}_r = \{ u_i, H_{m-1} \}_{r+1}$. Let us introduce two relevant bracket structures proposed by Gelfand and Dickey \[3\]. Define $X = \sum_{i=0}^{\infty} \partial^i x_i$ and the pairing

$$\langle L | X \rangle \equiv \text{Tr} (LX) = \int \text{res} (LX) = \int \sum_{k=0}^{\infty} u_k x_k$$

(2.4)

The Gelfand-Dickey first and second bracket structures are defined as

$$\{ \langle L | X \rangle , \langle L | Y \rangle \}_{1}^{GD} \equiv \langle L | [X , Y] \rangle$$

(2.5)

$$\{ \langle L | X \rangle , \langle L | Y \rangle \}_{2}^{GD} \equiv \langle X | (LY)_-L - L(YL)_- \rangle$$

(2.6)

In components the above expression for the first bracket (2.5) becomes

$$\{ u_n(x) , u_m(y) \}_{1}^{GD} = \Omega_{n,m}^{(r=0)} (u(x)) \delta (x-y)$$

(2.7)

$$\Omega_{n,m}^{(r)} (u(x)) \equiv \sum_{k=0}^{n+r} (-1)^k \binom{n+r}{k} u_{n+r-k} D_x^k + \sum_{k=0}^{m+r} \binom{m+r}{k} D_x^k u_{m+r-k} (x) (2.8)$$

where we have introduced the Watanabe Hamiltonian structure $\Omega_{n,m}^{(0)} [5]$. The generalizations of (2.8) given by $\Omega_{n,m}^{(r)}$ with $r > 0$ will reproduce the linear part of several higher brackets. Note, that one obtains $\Omega_{n,m}^{(r)}$ from $\Omega_{n,m}^{(0)}$ by a simple shifting $u_t \to \tilde{u}_t = u_{t+r}$, meaning that if $u_n$’s satisfy (2.7) then we will obtain $\{ \tilde{u}_n(x) , \tilde{u}_m(y) \}_{1}^{GD} = \Omega_{n,m}^{(r)} (\tilde{u}(x)) \delta (x-y)$. We will summarize this relation in the following way

$$u_t \to \tilde{u}_t = u_{t+r} \quad ; \quad \Omega_{n,m}^{(r)} (\tilde{u}(x)) = \Omega_{n+r,m+r}^{(0)} (u(x))$$

(2.9)

The first Gelfand-Dickey bracket structure is isomorphic to the $W_{1+\infty}$ algebra [3, 4]. The existence of higher structures $\Omega_{n,m}^{(r)}$ is therefore related to truncation of original $W_{1+\infty}$ by removing lower spin generators.

Higher Watanabe structures $\Omega_{n,m}^{(r)}$ have a nice geometric realization in terms of functions $e^{ipx} y^{n+r}$ on a cylinder $S^1 \times \mathbb{R}$ with a Lie algebra structure given by [9]

$$[f(x,y) , g(x,y)] = \sum_{k \geq 1} (-h)^{k-1} \frac{1}{k!} \left( \frac{\partial^k f \partial^k g}{\partial x^k \partial y^k} - \frac{\partial^k f \partial^k g}{\partial y^k \partial x^k} \right)$$

(2.10)

which is a commutator $[f,g] = f \circ g - g \circ f$ with respect to the product of symbols:

$$f(x,y) \circ g(x,y) = \sum_{k \geq 0} (-h)^{k-1} \frac{1}{k!} \frac{\partial^k f \partial^k g}{\partial x^k \partial y^k}$$

(2.11)

The connection becomes transparent by noticing that the mapping \[14\] $W_n^p \leftrightarrow e^{ipx} y^{n+r}$ where $W_n(x) = \sum_p W_n^p e^{ipx}$ is an isomorphism between generalizations of (2.7) with $\Omega_{n,m}^{(r)}$ (realizations of such structures will be given below) and the algebra of $e^{ipx} y^{n+r}$ under the bracket (2.10).
For the second bracket one obtains from (2.9) by direct calculation (see for instance [20]):

\[ \{u_n(x), u_m(y)\}_2^{GD} = \Omega_{n,m}^{(1)}(u(x)) \delta(x - y) + \{u_n(x), u_m(y)\}_2^{GD}\]_{\text{nonlinear}} \tag{2.12} \]

with the nonlinear part given by

\[ \{u_n(x), u_m(y)\}_2^{GD}\]_{\text{nonlinear}} = \sum_{i=0}^{m-1} \sum_{k=1}^{m-i-1} \binom{m-i-1}{k} u_i(x) D_x^k u_{m+n-i-k-1}(x) 
- \sum_{k=1}^{n} (-1)^k \binom{n}{k} u_{n+i-k}(x) D_x^k u_{m-i-1}(x) \delta(x - y) \tag{2.13} \]

So far we had only defined Gelfand-Dickey bracket structures without imposing that they reproduce the Hamiltonian structure corresponding to KP hierarchy flow equation (2.2). Taking this into account one realizes [21, 22] that a further structure is required. Following [21] we will call this structure a Drinfeld-Sokolov (DS) bracket [22] and define it as

\[ \{L, X\}, \{L, Y\}\]_{DS} = \int dx dy \left( \int_x dx' \text{res}[L, X] \right) \text{res}[L, Y] \tag{2.14} \]

\[ \{u_n(x), u_m(y)\}_{DS} = - \int dx_1 dx_2 \{u_n(x), u_0(x_1)\}_1^{GD} (-\epsilon(x_1 - x_2)) \{u_0(x_2), u_m(y)\}_1^{GD} 
- \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (-1)^{n-i} \binom{n}{i} \binom{m}{j} u_i(x) D_x^{n+m-i-j-1} u_j(x) \delta(x - y) \tag{2.15} \]

One notices from (2.13) that the DS bracket satisfies properties of derivation. Following Drinfeld-Sokolov [22, 23] we call two Poisson brackets coordinated if any linear combination of them is the bracket itself (i.e. is antisymmetric and satisfies Jacobi identity). In fact DS bracket is coordinated with both $\Omega^{(0)}$ and $\Omega^{(1)}$ linear bracket structures. The second Hamiltonian structure compatible with Lenard relations is given by [20]:

\[ \{u_n(x), u_m(y)\}_2 = \{u_n(x), u_m(y)\}_2^{GD} + \{u_n(x), u_m(y)\}_{DS} \tag{2.16} \]

3 A Bose Construction of KP Hierarchy and Faà di Bruno Polynomials

In this section we will construct KP hierarchy in terms of a pair of Bose fields $J$ and $\bar{J}$. Based on our earlier work in [14] we propose the Lax components $W_n$ as:

\[ W_n(x) \equiv (-1)^n \bar{J}(x) P_n(J(x)) \tag{3.1} \]

given in terms of Faà di Bruno polynomials:

\[ P_n(J) = (D + J)^n \cdot 1 = \exp(-\phi) \partial^n \exp(\phi) \quad \phi' = J \tag{3.2} \]
We associate to (3.1) the Lax operator given by:

\[ L = D + \frac{\bar{J}}{D + J} = D + \sum_{n=0}^{\infty} W_n D^{-1-n} \]  

(3.3)

From the definition (3.2) we find a recurrence relation

\[ \partial P_n = P_{n+1} - J P_n \]  

(3.4)

which could be used to calculate lowest order polynomials. Several other useful properties of Faà di Bruno polynomials are listed in Appendix A.

### 3.1 First KP Hierarchy Structure

In this subsection \( J \) and \( \bar{J} \) are considered as canonical variables with commutation relation

\[ \{ \bar{J}(x), J(y) \}_1 = -\delta'(x - y) \]  

(3.5)

\[ \{ \bar{J}(x), \bar{J}(y) \}_1 = \{ J(x), J(y) \}_1 = 0 \]

which leads to

\[ \{ \bar{J}(x), \exp(\pm \phi(y)) \}_1 = \mp \delta(x - y) \exp(\pm \phi(y)) \]  

(3.6)

The advantage of using the exponential representation (3.2) is that it makes relatively easy to calculate brackets between generators

\[ W_n(x) = (\frac{-1}{n}) J^n e^{-\phi(x)} \]  

(3.7)

One easily recognizes in (3.7) the first Gelfand-Dickey structure written in the Watanabe form:

\[ \{ W_n(x), W_m(y) \}_1 = \Omega^{(0)}_{nm}(W(x)) \delta(x - y) \]  

(3.8)

We now define a family of related generators

\[ W^{(k)}_n = (\frac{-1}{n+k}) (D - J)^k \bar{J} P_n(J) \quad ; \quad n, k = 0, 1, 2, \ldots \]  

(3.9)

Note, that \( W^{(0)}_n = W_n \). We now establish, with help of (3.4), the following recursive relation:

\[ W^{(k+1)}_n = -W^{(k)}_{n+1} - \partial W^{(k)}_n \quad ; \quad n, k = 0, 1, 2, \ldots \]  

(3.10)
The above relation allows the transparent interpretation in terms of the infinite matrix

\[
W = \begin{pmatrix}
W_0^{(0)} & W_0^{(1)} & W_0^{(2)} & \cdots \\
W_1^{(0)} & W_1^{(1)} & W_1^{(2)} & \cdots \\
W_2^{(0)} & W_2^{(1)} & W_2^{(2)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\tag{3.11}
\]

In such matrix notation the recurrence relation (3.10) reads as

\[\partial W = -I_+ W - WI_- \tag{3.12}\]

where we used the infinite raising and lowering matrices

\[
I_- = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}, \quad I_+ = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\tag{3.13}\]

Note that the top row of \(W\) matrix (3.11) consisting of elements

\[W_0^{(k)} = (-1)^k (D - J)^k \bar{J} \quad ; \quad k = 0, 1, 2, \ldots \tag{3.14}\]

is generated by its own Lax operator

\[L = D + \frac{1}{D - j} = D + \sum_{n=0}^{\infty} W_0^{(k)} D^{1-k} \tag{3.15}\]

It is natural to introduce in this setting the concept of adjoint operation, which maps the matrix (3.11) into its transposed (with rows and columns interchanged). In operator language we notice from (3.9) and definition (3.2) that adjoint operation is defined by swapping each \(D + J\) to the right of \(\bar{J}\) with \(D - J\) to the left of \(\bar{J}\) and vice versa. This concept will be useful when discuss below the algebra involving all generators defined in (3.9).

We now show that the generators defined in (3.9) satisfy the closed algebra and we find its form. In order to do that we introduce the quantities

\[U_{n,m}^{(r)}(x) \equiv (-1)^r \sum_{k=0}^{n} (-1)^k \binom{n}{k} W_{n+m-k}^{(r)} D_x^k \tag{3.16}\]

\[V_{n,m}^{(r)}(x) \equiv (-1)^r \sum_{k=0}^{m} \binom{m}{k} D_x^k W_{n+m-k}^{(r)} \tag{3.17}\]

In the appendix (B) we show that, these quantities satisfy the following identities

\[U_{n,m}^{(r+1)}(x) = U_{n+1,m}^{(r)}(x) + D_x U_{n,m}^{(r)}(x) \tag{3.18}\]

\[V_{n,m}^{(r)}(x) = V_{n+1,m}^{(r)}(x) + D_x V_{n,m}^{(r)}(x) \tag{3.19}\]

\[U_{n,m}^{(r+1)}(x) \delta(x - y) = [U_{n+m+1}^{(r)}(x) + D_x U_{n,m}^{(r)}(x)] \delta(x - y) \tag{3.20}\]

\[V_{n,m}^{(r+1)}(x) \delta(x - y) = [V_{n+m+1}^{(r)}(x) + D_x V_{n,m}^{(r)}(x)] \delta(x - y) \tag{3.21}\]
The bracket \((3.8)\) between the generators \(W_n(x) \equiv W_n^0(x)\) can then be written as
\[
\{W_n^0(x) , W_m^0(y)\}_1 = -[U_{n,m}^0(x) - V_{n,m}^0(x)]\delta(x - y) \tag{3.22}
\]

The bracket between any two generators \(W_n^r(x)\) can be obtained recursively from \((3.22)\) by using the identities \((3.18)-(3.21)\) and the relation \((3.10)\). For instance, using \((3.10), (3.18)\) and \((3.19)\) one gets
\[
\{W_n^1(x) , W_m^0(y)\}_1 = -\{W_{n+1}^0(x) , W_m^0(y)\}_1 - D_x \{W_n^0(x) , W_m^0(y)\}_1
\]
\[
= \left(U_{n,m}^1(x) - V_{n,m+1}^0(x)\right)\delta(x - y)
\]

Analogously, using \((3.10), (3.20)\) and \((3.21)\) one finds
\[
\{W_n^0(x) , W_m^1(y)\}_1 = \left(U_{n+1,m}^0(x) - V_{n,m+1}^1(x)\right)\delta(x - y)
\tag{3.23}
\]

By repeating the above steps we arrive at the general expression
\[
\{W_n^r(x) , W_m^s(y)\}_1 = (-1)^{r+s}\left(U_{n+s,m}^r(x) - V_{n,m+r}^s(x)\right)\delta(x - y)
\tag{3.24}
\]
completing the proof of the closure of the algebra.

Notice that the generators \(W\)'s belonging to the same column of the matrix \((3.11)\), i.e. \(r = s\), constitute a closed subalgebra. In fact it is isomorphic to a truncated Watanabe type algebra via \((2.9)\).
\[
\{W_n^r(x) , W_n^s(y)\}_1 = (-1)^r\Omega_{n,m}^r(W_n^r)\delta(x - y)
\tag{3.25}
\]
for \(n, m = 0, 1, \ldots\).

The generators appearing on a given row of the matrix \((3.11)\) also constitute a closed subalgebra. In appendix \((\text{C})\) we show that
\[
\{W_n^r(x) , W_n^s(y)\}_1 = \left(\tilde{U}_{n}^{r+s} - \tilde{V}_{n}^{s,r}W_n^r(x)\right)\delta(x - y)
\tag{3.26}
\]
where we have defined
\[
\tilde{U}_n^{r,s}(x) \equiv (-1)^n \sum_{k=0}^{r} \binom{r}{k} W_n^{r+s-k}(x) D_x^k
\tag{3.27}
\]
\[
\tilde{V}_n^{r,s}(x) \equiv (-1)^s \sum_{k=0}^{s} \binom{s}{k} D_x^k W_n^{r+s-k}(x)
\tag{3.28}
\]

The subalgebra \((3.26)\) is also a truncated Watanabe type algebra:
\[
\{W_n^r(x) , W_n^s(y)\}_1 = (-1)^n\Omega_{r,s}^n(W_n^r)\delta(x - y)
\tag{3.29}
\]
for \(r, s = 0, 1, \ldots\).
3.2 Second KP Hierarchy Structure

In this subsection we will show how to generate the second bracket structure from the representation given by (3.1). This time the algebra of $J$ and $\bar{J}$ will be defined to be

\[
\{J(x), J(y)\}_2 = J(x)\delta'(x - y) - h\delta''(x - y)
\]

\[
\{\bar{J}(x), J(y)\}_2 = 2\bar{J}(x)\delta'(x - y) + \bar{J}'(x)\delta(x - y)
\]

\[
\{J(x), J(y)\}_2 = c\delta'(x - y)
\]

where constants $h$ and $c$ can be interpreted as independent deformation parameters, see next subsection for details. Here we take $h = 1$. Recalling from (3.1) that $W_0 = \bar{J}$, $W_1 = -\bar{J}J$ and $W_2 = \bar{J}(J' + J^2)$ one can easily check that (3.30) is a unique structure for $(\bar{J}, J)$ leading to (2.16) for three lowest brackets with $c = 2$. From (3.30) we derive

\[
\left\{ \bar{J}(x), \exp(\pm \phi(y)) \right\}_2 = \mp \delta(x - y)\partial \exp(\pm \phi(y)) \pm \delta'(x - y)\exp(\pm \phi(y))
\]

repeating similar calculation as in (3.7) we get for the linear part of $\{\cdot, \cdot\}_2$ the expected result $\{W_n(x), W_m(y)\}_{2|\text{linear}} = \Omega_{nm}^{(1)}(W(x))\delta(x - y)$. To calculate the nonlinear part of the bracket we will use the exponential representation of Faà di Bruno polynomials given in (3.2). We first observe that

\[
\{\phi(x), \phi(y)\}_2 = -c\varepsilon(x - y) \quad ; \quad \phi'(x) = J(x)
\]

from which the direct calculation yields

\[
\{P_n(x), P_m(y)\}_{2|\text{nonlinear}} = -c \left[ \sum_{l=0}^{n} \sum_{p=0}^{m} \binom{n}{l} \binom{m}{p} P_{n-l}(x)P_{m-p}(y)\partial_x^l\partial_y^p \right.
\]

\[
- P_n(x) \sum_{l=0}^{m} \binom{m}{l} P_{m-l}(y)\partial_y^l - P_m(y) \sum_{l=0}^{n} \binom{n}{l} P_{n-l}(x)\partial_x^l
\]

\[
+ P_n(x)P_m(y) \varepsilon(x - y)
\]

where we wrote for brevity $P_n(J(x)) = P_n(x)$. An important point is that the pure $\varepsilon(x - y)$ terms cancel out leaving only delta functions and their derivatives in the following expression:

\[
\{P_n(x), P_n(y)\}_{2|\text{nonlinear}} = -c \left[ \sum_{l=1}^{n} \sum_{p=1}^{m} (-1)^p \binom{n}{l} \binom{m}{p} P_{n-l}(x)P_{m-p}(y)\partial_x^{l+p-1} \right] \delta(x - y)
\]

We obtain therefore the total second bracket for the generators in (3.1) as the sum of linear and nonlinear terms (after a change of variables $n - l = i$, $m - p = j$ in (3.34)):

\[
\{W_n(x), W_m(y)\}_2 = \Omega_{nm}^{(1)}(W(x))\delta(x - y)
\]

\[
- c \left[ \sum_{l=0}^{n-1} \sum_{j=0}^{m-1} (-1)^{n-i} \binom{n}{i} \binom{m}{j} W_i(x)D_x^{n-i-j-1}W_j(x) \right] \delta(x - y)
\]

for $n, m \geq 0$. 

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This result appears at first to be surprising since we recognize in (3.35) only the DS structure from (2.13) multiplied by $c$ while the nonlinear part of the second Gelfand-Dickey bracket from (2.12) appears to be missing. This raises the question whether agreement with second Hamiltonian structure of (2.16) encountered above at the lowest levels holds at the higher level for our basis (3.1). The following proposition shows that this is indeed the case.

**Proposition.** The following identity:

$$
\{W_n(x), W_m(y)\}_{2}^{GD} \bigg|_{\text{nonlinear}} = \{W_n(x), W_m(y)\}_{2}^{DS} \quad (3.36)
$$

involving the nonlinear part of the second Gelfand-Dickey bracket (2.13) and the DS structure (2.13) is valid for the generators $W_n(x) = (-1)^n J(x) P_n(J(x))$.

Let us first recall from (2.12) and (2.15) that identity (3.36) reads in components:

$$
\sum_{i=0}^{m-1} \left[ \sum_{k=1}^{m-i-1} \binom{m-i-1}{k} W_i \delta D_x^k W_{m+n-i-k-1} - \sum_{k=1}^{n} (-1)^k \binom{n}{k} W_{n+i-k} D_x^k W_{m-i-1} \right] \delta(x - y)
$$

$$
= \sum_{i=0}^{m-1} \sum_{k=0}^{m-i} \sum_{l=1}^{m-i-k} (-1)^k \binom{n}{k} \delta D_x^{k+l} W_{n+i-k} D_x^l W_{m-i-l-1} \delta(x - y) \quad (3.37)
$$

with all $W_n$’s taken at $x$. We will prove this identity by induction. First, note that both sides of (3.37) are zero for $n = 0$ and arbitrary $m$. A short calculation shows that they are also equal for $n = 1$ and arbitrary $m$:

$$
\{W_1(x), W_m(y)\}_{2}^{GD} \bigg|_{\text{nonlinear}} = \{W_1(x), W_m(y)\}_{2}^{DS} = \sum_{k=1}^{m} \binom{m}{k} W_0(x) D_x^k W_{m-k}(x) \delta(x - y) \quad (3.38)
$$

Note now that since one can factorize out $J(x)$ and $J(y)$ on both sides of (3.37) and (3.38) we can substitute there $W_i$ by $(-1)^i P_i$. Hence the proof requires showing that

$$
\{P_n(x), P_m(y)\}_{2}^{GD} \bigg|_{\text{nonlinear}} = \{P_n(x), P_m(y)\}_{2}^{DS} \quad (3.39)
$$

We make an induction assumption that (3.39) holds for some fixed $n$ and arbitrary $m$. Let us now show that this identity is also true for $n + 1$ with arbitrary $m$. Recall from (3.4) that $P_{n+1} = -\partial P_n + P_1 P_n$. The rest of the proof follows now easily from (3.38), induction assumption (3.39) and the derivation property of both brackets.

As a consequence of the above proposition we are able to rewrite relation (3.33) for $c = 2$ as

$$
\{W_n(x), W_m(y)\}_{2} = \{W_n(x), W_m(y)\}_{2}^{GD} + \{W_n(x), W_m(y)\}_{2}^{DS} \quad (3.40)
$$

where on the right hand side of (3.40) we had split nonlinear part of (3.33) equally between $\{\cdot, \cdot\}_{2}^{GD}$ and $\{\cdot, \cdot\}_{2}^{DS}$. 

9
3.3 Nonlinear Structure from the First Bracket

In this subsection we present a special realization of (1.2) in terms of the first bracket structure (3.5) by changing the basis.

Our initial basis is

$$w_n = (-1)^n \tilde{J} J^n$$

for \(n = 0, 1, 2\ldots\), which generates the area preserving diffeomorphism algebra \(w_{1+\infty}\):

$$\{w_n(x), w_m(y)\}_1 = (nw_{m+n-1}(x)D_x + mD_xw_{m+n-1}(x))\delta(x-y)$$

(3.42)

In order to introduce a deformation parameter \(h\) we replace \(\tilde{J}\) by \(\tilde{J} \equiv (h\partial - J)\tilde{J}\). It satisfies

$$\{\tilde{J}(x), \tilde{J}(y)\}_1 = 2\tilde{J}(x)\delta'(x-y) + \tilde{J}'(x)\delta(x-y)$$

(3.43)

We also have

$$\{\tilde{J}(x), J(y)\}_1 = J(x)\delta'(x-y) - h\delta''(x-y)$$

(3.44)

We then see we have generated (3.30) with \(c = 0\) out of (3.3) by changing the basis. In the basis \((\tilde{J}, J)\) we define

$$V_n \equiv (-1)^n \tilde{J} P_n(J)$$

(3.45)

satisfying \(W_\infty\) algebra with the Watanabe structure \(\Omega_{n,m}^{(1)}(V)\).

In order to introduce the \(c \neq 0\) into the algebra let us now define \(\mathcal{J} \equiv J - \frac{c}{2} \tilde{J}\) satisfying

$$\{\mathcal{J}(x), \mathcal{J}(y)\}_1 = c\delta'(x-y)$$

(3.46)

Further we obtain

$$\{\tilde{J}(x), \mathcal{J}(y)\}_1 = \mathcal{J}(x)\delta'(x-y) - h\delta''(x-y)$$

(3.47)

In the basis \((\tilde{J}, \mathcal{J})\) we now can define generators of the \(\hat{W}_\infty\) algebra

$$\hat{V}_n(x) = (-1)^n \mathcal{J} P_n(J)$$

(3.48)

Comparing with analysis of section 3.2 we immediately conclude that \(\hat{V}_n\) satisfies the second Hamiltonian structure (3.40).

In reference [13] there exists proposal similar in spirit to the above discussion. Despite the technical differences we show in detail how to incorporate it in our framework. Start by defining

$$\mathcal{W}_n = \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} W_{n-k+1} P_k(\tilde{J}) = \tilde{J} \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} e^{-\phi} \partial^{n-k} (\partial e^\phi) e^{-\tilde{\phi}} \partial^k e^{\tilde{\phi}}$$

$$= (-1)^n \tilde{J} e^{-\phi - \tilde{\phi}} \partial^n \left((\partial e^\phi)e^{\tilde{\phi}}\right) = (-1)^n \tilde{J} e^{-\Phi} \partial^n \left(J e^\Phi\right)$$

(3.49)

where we have introduced the current \(\tilde{J}\) satisfying \(\{\tilde{J}(x), J(y)\} = \delta'(x-y)\) with remaining brackets being zero. Also \(\tilde{\phi}\) satisfies \(\tilde{\phi}' = \tilde{J}\) and \(\Phi = \phi + \phi\). The expression (3.49) can also be rewritten as

$$\mathcal{W}_n = (-1)^n \tilde{J} e^{-\Phi} D^n e^\Phi J = (-1)^n \tilde{J} \left(D + \tilde{J} + J\right)^n J$$

(3.50)
Accordingly the Lax representation becomes:
\[
L = D + \frac{1}{D + \tilde{J} + J} J = D + \sum_{n=0}^{\infty} W_n D^{-n-1}
\] (3.51)

To calculate the algebra satisfied by $W_n$ it is useful to introduce a vertex function representation. We recall first that from equations (A.4) and (A.5) from Appendix A we have
\[
e^{\Delta_{\epsilon} \Phi(x)} = \sum_{n=0}^{\infty} P_n \left( \tilde{J} + J \right)^{\frac{\epsilon^n}{n!}}
\] (3.52)

where we have introduced for brevity a notation $\Delta_{\epsilon} \Phi(x) = \Phi(x + \epsilon) - \Phi(x)$. In this notation we can write:
\[
\tilde{J}(x)J(x + \epsilon)e^{\Delta_{\epsilon} \Phi(x)} = \sum_{n=0}^{\infty} (-1)^n W_n(x) \frac{\epsilon^n}{n!}
\] (3.53)

Hence we will be interested in calculating brackets using the generating functions from (3.53). The relevant bracket is calculated in Appendix (D) and is given by:
\[
\{ W_n(x), W_m(y) \} = \Omega^{(1)}_{nm} (W(x)) \delta(x - y)
\] (3.54)

As before the result of direct calculation of brackets resulted only in DS structure as a nonlinear part. We will now show that also in this case there is an equivalence with the Gelfand-Dickey second bracket. The proof is simplified by writing the generators $W_n$ as
\[
W_n = (-1)^n \tilde{J} R_n (\tilde{J} + J) \quad ; \quad R_n = (-1)^n \left( D + J + \tilde{J} \right)^n J
\] (3.55)

with $R_0 = J$ and $R_1 = -J' - (\tilde{J} + J)J$. Since we can always factorize $\tilde{J}(x)\tilde{J}(y)$ out in the bracket (3.54) we will really be interested in proving
\[
\{ R_n(x), R_m(y) \}^{GD}_{2} \bigg|_{\text{nonlinear}} = \{ R_n(x), R_m(y) \}^{DS}
\] (3.56)

where the detailed expressions for the left and right hand side are given in (2.13) and (2.14). One verifies easily that
\[
\{ R_0(x), R_m(y) \}^{GD}_{2} \bigg|_{\text{nonlinear}} = \{ R_0(x), R_m(y) \}^{DS} = 0
\] (3.57)

and
\[
\{ R_1(x), R_m(y) \}^{GD}_{2} \bigg|_{\text{nonlinear}} = \{ R_1(x), R_m(y) \}^{DS}
\] (3.58)

Since $R_0 = J$ and $R_1 = -R_0^2 - \tilde{J} R_0$ we obtain
\[
\{ (\tilde{J} + J)(x), R_m(y) \}^{GD}_{2} \bigg|_{\text{nonlinear}} = \{ (\tilde{J} + J)(x), R_m(y) \}^{DS}
\] (3.59)

This last relation makes now possible for us to use the recurrence relation $R_{n+1} = -R_n - (\tilde{J} + J)R_n$ to complete the induction proof.
3.4 KP Multi-Hamiltonian Structure in Terms of Currents $J, \bar{J}$

The results of subsections (3.1) and (3.2) suggest the possibility of transferring the KP flow equation (2.2) to its Hamiltonian form solely in terms of Bose currents $(J, \bar{J})$ instead of the Lax components $u_n$’s. Similar idea appeared in [23] in the setting of formalism given in [13].

In terms of the currents the equation (2.3) $\partial u_i / \partial t_r = \{ u_i, H_{r+1} \}$ for $k = 1, 2$ reads as:

$$\frac{\partial J}{\partial t_r} = \{ J, H_r \} = \{ J, H_{r+1} \} \quad (3.60)$$

$$\frac{\partial \bar{J}}{\partial t_r} = \{ \bar{J}, H_r \} = \{ \bar{J}, H_{r+1} \} \quad (3.61)$$

which rewritten in a “spinor” form becomes

$$\left( \begin{array}{c} \delta J / \delta t_r \\ \delta \bar{J} / \delta t_r \end{array} \right) = P_1 \left( \begin{array}{c} \delta H_{r+1} / \delta J \\ \delta H_{r+1} / \delta \bar{J} \end{array} \right) = P_2 \left( \begin{array}{c} \delta H_r / \delta J \\ \delta H_r / \delta \bar{J} \end{array} \right) \quad (3.62)$$

Equation (3.61) expresses compatibility relation between the first and second Hamiltonian structures. We can put it in the form of recurrence relation:

$$\left( \begin{array}{c} \delta H_{r+1} / \delta J \\ \delta H_{r+1} / \delta \bar{J} \end{array} \right) = (P_1)^{-1} P_2 \left( \begin{array}{c} \delta H_r / \delta J \\ \delta H_r / \delta \bar{J} \end{array} \right) = \left( (P_1)^{-1} P_2 \right)^{i-1} \left( \begin{array}{c} \delta H_{r+2-i} / \delta J \\ \delta H_{r+2-i} / \delta \bar{J} \end{array} \right) \quad (3.63)$$

for any $i$ such that $1 \leq i \leq r+1$. Recalling that for the general Hamiltonian matrix structure $P_i$, we have

$$\left( \begin{array}{c} \delta J / \delta t_r \\ \delta \bar{J} / \delta t_r \end{array} \right) = P_i \left( \begin{array}{c} \delta H_{r+2-i} / \delta J \\ \delta H_{r+2-i} / \delta \bar{J} \end{array} \right) \quad (3.64)$$

we find the following relation between Hamiltonian matrix structures

$$P_i = P_1 \left( (P_1)^{-1} P_2 \right)^{i-1} = P_{i-1} (P_1)^{-1} P_2 \quad i \geq 1 \quad (3.65)$$

$$P_i = P_2 \left( (P_1)^{-1} P_2 \right)^{i-2} \quad i \geq 2$$

Hence among the multi-Hamiltonian structures only $P_1$ and $P_2$ are independent. All other matrices $P_i, \ i = 3, 4, \ldots$ are related through (3.63) to $P_1$ and $P_2$. From (3.5) and (3.30) we find directly

$$P_1 = \left( \begin{array}{cc} 0 & -\partial \\ -\partial & 0 \end{array} \right), \quad P_2 = \left( \begin{array}{cc} c\partial & \partial^2 + \partial J \\ -\partial^2 + J \partial & \partial J + J \partial \end{array} \right) \quad (3.66)$$
In particular, for \( i = 3 \) we find \( P_3 = P_2 P_1^{-1} P_2 \), which gives explicitly

\[
P_3 = c \bar{P}_3 + P_3|_{c=0} = -c \begin{pmatrix}
\partial J + J \partial & \partial J + \bar{J} \partial \\
\partial J + \bar{J} \partial & 0
\end{pmatrix} - \begin{pmatrix}
0 & \partial (\partial + J)^2 \\
(-\partial + J)^2 \partial & \bar{J} (\partial + J)^2 - (-\partial + J)^2 \bar{J}
\end{pmatrix}
\] (3.67)

The higher brackets will however contain nonlocal expressions in their \( c \)-dependent parts. The simplicity of \( P_3|_{c=0} \) is maintained at the level of higher Hamiltonian structures. In fact we have:

\[
P_n|_{c=0} = (-1)^n \begin{pmatrix}
0 & \partial (\partial + J)^{n-1} \\
(-\partial + J)^{n-1} \partial & (\partial + J)^{n-1} - (-\partial + J)^{n-1} \bar{J}
\end{pmatrix}
\] (3.68)

This statement is easily proven by induction. We find from (3.65) that \( P_{n+1}|_{c=0} = (P_n|_{c=0}) P_1^{-1} (P_2|_{c=0}) \). Inserting the induction assumption (3.68) into this last equation we complete the proof after a short calculation. Hence for \( c = 0 \) the corresponding \((J, \bar{J})\) algebra would take the following simple form for the \( n \)-th Hamiltonian structure \((n \geq 1)\):

\[
\{ \bar{J}(x) , J(y) \}|_{c=0} = (-1)^n (-\partial + J(x))^{n-1} \delta'(x-y)
\] (3.69)

\[
\{ \bar{J}(x) , J(y) \}|_{c=0} = (-1)^n \left( \bar{J}(x) (\partial + J(x))^{n-1} - (\partial + J(x))^{n-1} \bar{J}(x) \right) \delta(x-y)
\]

\[
\{ J(x) , J(y) \}|_{c=0} = 0
\]

It is interesting to note that the form of the bracket \( \{ \cdot, \cdot \}|_{c=0} \) in (3.69) can be reproduced by the lower bracket with modified basis. This construction goes as follows. Let \( \bar{J} \rightarrow (-\partial + J) \bar{J} = W_0^{(1)} \). Then we find

\[
\{ W_0^{(1)}(x) , J(y) \}|_{c=0} = -(-1)^n (-\partial + J(x))^{n-1} \delta'(x-y)
\] (3.70)

\[
\{ W_0^{(1)}(x) , W_0^{(1)}(y) \}|_{c=0} = -(-1)^n \left( W_0^{(1)}(x) (\partial + J(x))^{n-1} - (\partial + J(x))^{n-1} W_0^{(1)}(x) \right) \delta(x-y)
\]

Hence the form of the bracket was preserved under simultaneous change of basis and lowering the index of the bracket. Continuing this process till we reach the first bracket structure we arrive at \( \{ W_0^{(n-1)}(x) , W_0^{(n-1)}(y) \}|_{c=0} \) known from (3.23). Hence \( \{ W_n(x), W_m(y) \}|_{c=0} \) will have the same functional form as \((-1)^r \{ W_n^{(r-1)}(x) , W_m^{(r-1)}(y) \}|_{c=0} \). From here we find, recalling equation (3.23), that the algebra of \( W_n = (-1)^n \bar{J} P_n(J) \) according to the \( r \)-th order Poisson structure is given by

\[
\{ W_n(x) , W_m(y) \}|_{c=0} = \Omega_{nm}^{(r-1)} (W(x)) \delta(x-y)
\] (3.71)

In the light of above result we observe that the algebra of columns (or rows) of the matrix (3.11) is isomorphic to the higher undeformed algebras in (3.71). One also concludes that \( c \) plays a role of a deformation parameter responsible for deformation of the linear \( W_\infty \) algebra to \( \hat{W}_\infty \) algebra.
We can use the two bracket structures to construct a Lax equation as discussed in ref. [24]. Denoting \( J^1 \equiv J \) and \( J^2 \equiv \bar{J} \), we define the \( 2 \times 2 \) matrices

\[
S_i^j \equiv \left( P_1^{-1} \right)_{ik} (P_2)^{kj} \\
(U_r)_i^j \equiv \frac{\partial}{\partial J^i} \frac{\partial J^j}{\partial t_r}
\]

(3.72)

From the compatibility condition (3.61) and the fact that \( P_r \) are symplectic forms, and therefore closed, one gets that \( S \) and \( U \) constitute a Lax pair [24]

\[
\frac{dS}{dt_r} = [S, U_r]
\]

(3.73)

From (3.72) and (3.66) one easily gets

\[
S =\begin{pmatrix}
\partial - \partial^{-1} J & -\bar{J} + \partial^{-1} \bar{J} \\
-\bar{J} + \partial & \partial + J
\end{pmatrix}
\]

(3.74)

and for \( r = 1 \), \( \mathcal{H}_1 = \bar{J} \) and \( \mathcal{H}_2 = -\bar{J}J \)

\[
U_1 = \begin{pmatrix}
\partial & 0 \\
0 & \partial
\end{pmatrix}
\]

(3.75)

Using these matrices one concludes that (3.73) is compatible with the equations of motion

\[
\frac{\partial J}{\partial t_1} = J', \quad \frac{\partial \bar{J}}{\partial t_1} = \bar{J}'
\]

(3.76)

One can use (3.73) to construct conserved quantities as \( \text{Tr} \, S^n \) where \( \text{Tr} \) is some invariant trace form for these operators.

4 Applications

4.1 WZNW Type Models

The ordinary WZNW model associated to a Lie group \( G \) possesses two commuting chiral copies of the current algebra:

\[
\{ J_a(x), J_b(y) \} = f_{ab}^c J_c(x) \delta(x - y) + kg_{ab} \delta'(x - y)
\]

(4.1)

where \( f_{ab}^c \) are the structure constants of the Lie algebra \( \mathcal{G} \) of \( G \), and \( g_{ab} \) is the Killing form of \( \mathcal{G} \). The two chiral components of the energy momentum tensor are of the Sugawara form

\[
T(x) = \sum_{a,b=1}^{\text{dim}G} g^{ab} J_a(x) J_b(x)
\]

(4.2)
where $g^{ab}$ is the inverse of the Killing form. Such tensor satisfies the Virasoro algebra with vanishing central term

$$\{T(x), T(y)\} = 2T(x)\delta'(x-y) + T'(x)\delta(x-y) \quad (4.3)$$

The currents are spin one primary fields

$$\{T(x), J_n(y)\} = J_n(x)\delta'(x-y) \quad (4.4)$$

Suppose now one has a self-commuting current $\mathcal{J}$, $\{\mathcal{J}(x), \mathcal{J}(y)\} = 0$. For the non compact WZNW model this current can be, for instance, the one associated to a step operator $J(E_\alpha)$ for any root $\alpha$ of $G$. One then sees that the system $(T, \mathcal{J})$ generates an algebra isomorphic to (3.30) with $h = c = 0$, where $T$ corresponds to $\mathcal{J}$ and $\mathcal{J}$ to $J$. One can construct out of them the quantities $w_n(x) \equiv T(x)\mathcal{J}^{n-2}$ satisfying the area preserving diffeomorphism algebra, which correspond to (3.35) for $h = c = 0$, i.e.

$$\{w_n(x), w_m(y)\} = (n + m - 2)w_{n+m-2}(x)\delta'(x-y) + (m-1)(w_{n+m-2}(x))'\delta(x-y) \quad (4.5)$$

By taking now a $U(1)$ subalgebra of (4.1)

$$\{\mathcal{J}(x), \mathcal{J}(y)\} = kg\mathcal{J}\delta'(x-y) \quad (4.6)$$

one sees that the $(T, \mathcal{J})$ system now generates an algebra which is isomorphic to (3.30) for $h = 0$ and $c = kg\mathcal{J}$. The quantities $w_n$ introduced above will then generate a c-deformed (nonlinear) area preserving diffeomorphism algebra, which corresponds to (3.33) with $h = 0$, i.e.

$$\{w_n(x), w_m(y)\} = (n + m - 2)w_{n+m-2}(x)\delta'(x-y) + (m-1)(w_{n+m-2}(x))'\delta(x-y) - kg\mathcal{J}(n-2)(m-2)w_{n-1}(x)\partial_x(w_{m-1}(x)\delta(x-y)) \quad (4.7)$$

Another realization of the algebraic structure discussed in section 2 is found within the context of the two-loop Kac-Moody algebra introduced in [17] and further studied in [20]

$$[J^m_a(x), J^n_b(y)] = f_{ac}^b J^{m+n} c(x-y) + kg\partial_x\delta(x-y)\delta_{m,-n} + J^C(x)\delta(x-y)g_{ab}m\delta_{m,-n}$$

$$[J^P_a(x), J^m_b(y)] = m J^m_a(y)\delta(x-y)$$

$$[J^C(x), J^P(y)] = k\partial_x\delta(x-y)$$

$$[J^C(x), J^m_a(y)] = 0 \quad (4.8)$$

The currents $J^C$ and $J^P$ satisfy the same algebra as (3.3). Therefore the two-loop WZNW model possesses the algebraic structure discussed in subsection (3.1). In addition its Sugawara tensor together with suitably chosen spin one currents generate the symmetries discussed above for the case of ordinary WZNW models. In particular when the spin one current is $J^C$ the symmetries are those discussed in [14].

Unlike the ordinary WZNW model its two-loop version possesses the algebra (5.31) with $h \neq 0$. This is realized as follows. The Sugawara tensor can be modified, in ordinary or two-loop WZNW, by adding a derivative of a spin one current in the Cartan subalgebra. The
effect of this is to change the conformal spin of the currents and to add, for some currents, a $\delta''$ term on the r.h.s. of (4.4) (producing therefore $h \neq 0$), and also to add an anomaly term $\delta''$ on the r.h.s. of (4.3). This last anomaly is unwanted because it spoils the algebraic structure we are interested in. In the case of the two-loop WZNW model such anomaly can be avoided by choosing the current modifying the Sugawara tensor to be orthogonal to itself under the Killing form. Such current can either be $J^C$ or $J^D$. For the ordinary WZNW associated to a simple Lie group there is no such current since the Killing form restricted to the Cartan subalgebra is an Euclidean metric.

Summarizing, by modifying the Sugawara tensor of the two-loop WZNW model

$$T_{\text{2loop}}(x) = \frac{1}{2k} \left( \sum_{a,b=1}^{\dim G} \sum_{n=-\infty}^{\infty} g^{ab} J_a^n(x) J_b^{-n}(x) + 2J^D(x)J^C(x) \right)$$  \hspace{1cm} (4.9)$$

as

$$\mathcal{L}(x) = T_{\text{2loop}}(x) + \alpha \partial_x j(x)$$  \hspace{1cm} (4.10)$$

where $j$ can either be $J^C$ or $J^D$ but not a linear combination of them. The relation (4.4) for the current $K(x) = \gamma_C J^D + \gamma_D J^C$ will then be

$$\{\mathcal{L}(x), K(y)\} = K(x)\delta'(x-y) + h_j \delta''(x-y)$$  \hspace{1cm} (4.11)$$

where $h_{JC,JD} = k\alpha\gamma_{C,D}$. The relation (4.3) will not be modified since $j$ is orthogonal to itself, i.e.

$$\{\mathcal{L}(x), \mathcal{L}(y)\} = 2\mathcal{L}(x)\delta'(x-y) + \mathcal{L}'(x)\delta(x-y)$$  \hspace{1cm} (4.12)$$

we also have

$$\{K(x), K(y)\} = k(\gamma_C + \gamma_D) \delta'(x-y)$$  \hspace{1cm} (4.13)$$

Notice that $\mathcal{L}$ and $K$ satisfies the algebra (3.30) with $h = h_j$ and $c = k(\gamma_C + \gamma_D)$.

### 4.2 Toda Theories and $W_\infty$ Algebra

Let us now say some words about the symmetries one gets when WZNW models are reduced. When the ordinary non compact WZNW is reduced to the Conformal Toda models (CT) \[23\] the Sugawara tensor is modified in order to preserve the conformal symmetry. This modification introduces an anomaly in the Virasoro algebra which spoils the structures leading to the area preserving diffeomorphism algebra (4.3) and its $c$-deformed version (4.7). After reduction, the quantities $w_n = T J^{n-2}$ given above, will probably be non local, and will not satisfy a simple algebraic structure. It would be interesting to explore what sort of structure the area preserving symmetry of the ordinary WZNW leads to in the CT models after reduction.

In the case of the reduction of the two-loop WZNW model to the Conformal Affine Toda models (CAT) \[17\] some of the symmetries discussed above survive. This is true mainly because the $J^C$ current is untouched by the reduction procedure, and so is a local current for the CAT model. In order to preserve the conformal symmetry the Sugawara tensor (4.9) is modified as \[17\]

$$L(x) = T_{\text{2loop}}(x) + \partial_x \left( 2J^\delta(x) + hJ^D(x) \right)$$  \hspace{1cm} (4.14)$$

as

$$\{L(x), K(y)\} = k(\gamma_C + \gamma_D) \delta'(x-y)$$  \hspace{1cm} (4.13)$$

Notice that $L$ and $K$ satisfies the algebra (3.30) with $h = h_j$ and $c = k(\gamma_C + \gamma_D)$.
where $J_\delta = k \text{Tr} \left( \hat{g}^{-1} \partial_+ \hat{g} \cdot H^0 \right)$ with $\hat{g} = \frac{1}{2} \sum_{\alpha > 0} \alpha / \alpha^2$ and $h$ is the Coxeter number of the underlying semisimple Lie algebra. Such tensor satisfies the Virasoro algebra with an anomaly

$$- 4 \left( \hat{\delta} \right)^2 k \delta''(x - y)$$

(4.15)

An important point is that the $J^C$ current can be used to modify further the tensor (4.14) in order to cancel such anomaly. We then define

$$U(x) \equiv L(x) + \frac{2\hat{\delta}^2}{h} \partial_x J^C(x)$$

(4.16)

which is still conserved and satisfies

$$\{ U(x), U(y) \} = 2U(x)\delta'(x - y) + U'(x)\delta(x - y)$$

(4.17)

In addition

$$\{ U(x), J^C(y) \} = J^C(x)\delta'(x - y) + hk\delta''(x - y)$$

(4.18)

$$\{ J^C(x), J^C(y) \} = 0$$

(4.19)

The algebra (4.17)-(4.19) gives rise to the $W_\infty$ algebra like under the second bracket (3.30), with $c = 0$.

Therefore the CAT model not only possesses an area preserving symmetry but also a richer structure as discussed in [14]. Let us remark that the CAT model does not possess an algebra of the type (3.5). The currents $J^C$ and $J^D$ of the two-loop WZNW satisfy the algebra (3.5). However after the reduction the current $J^D$ becomes non local in the CAT model field variables. The algebra of $J^C$ and $J^D$ is then non local and certainly not isomorphic to (3.5). Another point is that the algebra (3.5) does not possess highest weight unitary representations [26]. Therefore such symmetry would jeopardize the hope of the CAT model being unitary at the quantum level.

### 4.3 KP Hierarchy and One-Matrix Model

Here we establish a connection between our realization of Section 3 with the construction of the KP hierarchy associated with the Discrete Linear System:

$$Q\Psi = \lambda \Psi$$

(4.20)

$$\frac{\partial \Psi}{\partial t_r} = Q^r_+ \Psi$$

(4.21)

proposed recently by Bonora and Xiong in [19]. In (4.20) and (4.21)

$$\Psi \equiv \begin{pmatrix} \vdots \\ \Psi_{n-1} \\ \Psi_n \\ \Psi_{n+1} \\ \vdots \end{pmatrix}$$

$$Q = I_+ + \sum_{i=0}^{\infty} a_i I_-$$

(4.22)
with the raising/lowering matrices $I_{\pm}$ from eq. (3.13).

For completeness we summarize the major steps taken in [19].

The compatibility condition of equations (4.20) and (4.21) gives rise to the discrete KP hierarchy i.e.,

$$\frac{\partial Q}{\partial t_r} = [Q^r_+, Q]$$  (4.23)

Taking $r = 1$ in (4.21) one gets (with $\partial \equiv \partial/\partial t_1$)

$$\partial \Psi_n = \Psi_{n+1} + a_0(n) \Psi_n$$  (4.24)

which can be rewritten as

$$\Psi_{n+1} = (\partial - a_0(n)) \Psi_n \rightarrow \Psi_n = \hat{B}_n \Psi_{n+1}, \quad \hat{B}_n \equiv \frac{1}{\partial - a_0(n)}$$  (4.25)

In this notation we can describe the action of $Q$ on $\Psi$ as

$$(Q\Psi)_n = \Psi_{n+1} + \sum_{i=0}^{\infty} a_i \hat{B}_{n-i} \hat{B}_{n-i+1} \cdots \hat{B}_{n-1} \Psi_n$$

$$= \partial \Psi_n + \sum_{i=1}^{\infty} a_i \hat{B}_{n-i} \hat{B}_{n-i+1} \cdots \hat{B}_{n-1} \Psi_n$$  (4.26)

which allows rewriting $Q$ in the space of $\Psi$’s as a differential operator $\hat{Q}$ labeled by $n$

$$\hat{Q}_n \Psi_n = (Q\Psi)_n \rightarrow \hat{Q}_n \equiv \partial + \sum_{i=1}^{\infty} a_i \hat{B}_{n-i} \hat{B}_{n-i+1} \cdots \hat{B}_{n-1}$$  (4.27)

Rewriting the Discrete KP hierarchy (4.23) in terms of components $a_i(n)$ we get for the first flow:

$$\frac{\partial a_i(n)}{\partial t_1} = a_{i+1}(n + 1) - a_{i+1}(n) + a_i(n) (a_0(n) - a_0(n - i))$$  (4.28)

The form of this equation is such that the choice $a_i(n) = 0, \quad \forall i \geq 2$ at $t_1 = 0$ will be preserved. In addition, we denote

$$a_0(n) = S_n, \quad a_1(n) = R_n$$  (4.29)

The flow equation (4.28) gives therefore

$$\frac{\partial S_n}{\partial t_1} = R_{n+1} - R_n$$  (4.30)

$$\frac{\partial R_n}{\partial t_1} = R_n (S_n - S_{n-1})$$  (4.31)

for $n \in \mathbb{Z}$. One recognizes in from (4.22) and (4.29) the Jacobi matrix relevant for one-matrix models [27]. In order to connect with the one-dimensional Toda model we insert

$$S_n = -\partial \varphi_n/\partial t_1$$ in the flow equation (4.31). This results in

$$R_n = \exp (\varphi_n - \varphi_{n-1})$$  (4.32)
Now (4.30) takes the form of one-dimensional Toda equation

\[ \frac{\partial^2 \varphi_n}{\partial t^2} = \exp (\varphi_n - \varphi_{n-1}) - \exp (\varphi_{n+1} - \varphi_n) \] (4.33)

The corresponding truncated \( \hat{Q}_n \) operator becomes

\[ \hat{Q}_n = \partial + R_n \hat{B}_{n-1} = \partial + R_n \frac{1}{\partial - S_{n-1}} = \partial + R_n \frac{1}{\partial - S_n + R'_n/R_n} \] (4.34)

where in the last identity use was made of the flow equation (4.30) in order to express \( \hat{Q}_n \) in terms of the fields taken at the same site \( n \). We can rearrange \( \hat{Q}_n \) in a way which we will allow us to make a contact with our \((\bar{J}, J)\) approach to KP hierarchy. Consider

\[ \partial \left( R_n \frac{1}{\partial - S_n + R'_n/R_n} \right) = R'_n \frac{1}{\partial - S_n + R'_n/R_n} + R_n \partial \frac{1}{\partial - S_n + R'_n/R_n} \] (4.35)

adding and subtracting \((-R_n S_n \frac{1}{\partial - S_n + R'_n/R_n})\) to the right hand side of the last identity we arrive after some algebra at

\[ R_n \hat{B}_{n-1} = \hat{B}_n R_n \] (4.36)

Hence the \( \hat{Q}_n \) operator can be written as

\[ \hat{Q}_n = \partial + \hat{B}_n R_n = \partial + \frac{1}{\partial - S_n} R_n \] (4.37)

in which we recognize the Lax operator introduced in (3.13) when we identify \( S_n = J \) and \( R_n = \bar{J} \).

Another method to express \( \hat{Q}_n \) by variables from the same \( n \)-sector would be to use (4.30) to write \( \hat{Q}_n = \partial + (R_{n-1} + S'_{n-1}) \hat{B}_{n-1} \). Here we make the identification \( S_{n-1} = -J \) and \( R_{n-1} = \bar{J} \). We obtain in this way the Lax operator \( D + (\bar{J} - J') (D + J)^{-1} \) with coefficients of the form \((-1)^n (\bar{J} - J') P_n(J)\). In fact the change \( J \to \bar{J} - J' \) preserves the algebra (3.3). Also the form of (3.30) is preserved under this transformation provided \( c = -2h \) (but with \( h \to -h \)).

Hence with fields \( R_k \) and \( S_k \) in \( \hat{Q}_n \) appearing in the same \( n \) or \( n-1 \) sector we obtain the Lax representation for KP hierarchy identical to the systems associated with the first row or first column of our matrix (3.11).

### A Faá di Bruno Polynomials. Definitions and Identities

In 1855 Faá di Bruno considered the following problem [15]. Let variables \( x \) and \( y \) be related through a smooth function \( \phi \): \( x = \phi(y) \). Let \( f \) be another smooth function of \( x \). Find closed expression for the \( n \)-th derivative of \( f \) with respect to \( y \) or in another words find how does \( \partial^n f \) behave under change of variables \( x \to y \). The formula which Faá di Bruno found to
provide an answer to the above problem involves a nice piece of combinatorics and is given in terms of Bell polynomials $B_{n,k}$ by:

$$\partial^n_y f = \sum_{k=1}^{n} \partial^k_x f \ B_{n,k} \left( \phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(n-k+1)} \right) \quad (A.1)$$

$$B_{n,k} \left( x_1, x_2, \ldots, x_l \right) \equiv \sum_{i_1+i_2+\ldots+i_l \geq 0} \frac{n!}{i_1!i_2!\ldots i_l!} \left( \frac{x_1}{1!} \right)^{i_1} \left( \frac{x_2}{2!} \right)^{i_2} \ldots \left( \frac{x_l}{l!} \right)^{i_l} \quad (A.2)$$

where $\phi^{(i)} = \partial^i \phi$ and summation is over integers $i_1, i_2, \ldots, i_l \geq 0$ satisfying

$$i_1 + 2i_2 + 3i_3 + \ldots + li_l = n$$

$$i_1 + i_2 + i_3 + \ldots + i_l = k$$

where clearly $l_{\text{max}} = n - k + 1$.

In order to make connection to definition of Faà di Bruno polynomials used in this paper let us take the special function $f(x) = \exp(x)$ and multiply both sides of (A.1) by $f^{-1}$ with the result

$$P_n(J) \equiv e^{-\phi} \partial^n e^\phi = \sum_{k=1}^{n} B_{n,k}(J) \quad (A.3)$$

$$= \sum_{i_1+2i_2+\ldots+n} \frac{n!}{i_1!i_2!\ldots i_l!} \left( \frac{J}{1!} \right)^{i_1} \left( \frac{J'}{2!} \right)^{i_2} \ldots \left( \frac{J^{(l-1)}}{l!} \right)^{i_l} \quad J \equiv \phi'$$

where $B_{n,k}(J) \equiv B_{n,k} (J, J', J'', \ldots)$. As a corollary of (A.3) we can find a generating function for Faà di Bruno polynomials. Consider namely:

$$e^{\phi(x+\epsilon) - \phi(x)} = \exp \left\{ \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} J^{(k-1)} \right\} \quad (A.4)$$

On the other hand applying Taylor expansion to $e^{\phi(x+\epsilon)}$ we arrive at an expression for the generating function:

$$\exp \left\{ \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} J^{(k-1)} / k! \right\} = \sum_{k=0}^{\infty} P_k(J) \epsilon^k / k! \quad (A.5)$$

From equation (A.5) one can easily prove an identity

$$P_n \left( f + g \right) = \sum_{k=0}^{n} \binom{n}{k} P_k(f) P_{n-k}(g) \quad ; \quad n = 0, 1, \ldots \quad (A.6)$$

valid for arbitrary functions $f$ and $g$. An alternative expression for Faà di Bruno polynomials is: $P_n(J) = (D + J)^n \cdot 1$ leading to a recurrence relation $\partial P_n = P_{n+1} - JP_n$. Few lowest order polynomials are listed here for convenience:

$$P_0 = 1 \quad ; \quad P_1 = J \quad ; \quad P_2 = J^2 + J' \quad ; \quad P_3 = J^3 + 3JJ' + J'' \quad \text{etc.} \quad (A.7)$$
The exponential representation in (A.3) turns out to be useful in proving several identities involving Faà di Bruno polynomials. First we recall the following basic formulas:

\[ \partial^n f = \sum_{\alpha=0}^{n} (-1)^\alpha \binom{n}{\alpha} D^{n-\alpha} f D\alpha \] (A.8)

\[ D^n f = \sum_{\alpha=0}^{n} \binom{n}{\alpha} \partial^{n-\alpha} f D^\alpha \] (A.9)

\[ f D^n = \sum_{\alpha=0}^{n} (-1)^\alpha \binom{n}{\alpha} D^n (\partial^\alpha f) \] (A.10)

\[ D^{-n} f = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{(n+\alpha-1)!}{\alpha!(n-1)!} \partial^\alpha f D^{-n-\alpha} \] (A.11)

where \( \partial^k f \equiv f^{(k)} \) and \( D = \partial/\partial x \) acts to the right as a derivative operator.

We derive now a number of useful identities (most of them are listed in [3]). Using (A.9) we find:

\[ (D + J)^n = e^{-\phi} D^n e^\phi = \sum_{l=0}^{n} \binom{n}{l} e^{-\phi} \partial^{n-l} e^\phi D^l = \sum_{l=0}^{n} \binom{n}{l} P_{n-l}(J) D^l \] (A.12)

Using (A.8) we get:

\[ P_n(J) = e^{-\phi} \partial^n e^\phi = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e^{-\phi} D^{n-l} e^\phi D^l = \sum_{l=0}^{n} (-1)^l \binom{n}{l} (D + J)^{n-l} D^l \] (A.13)

while using (A.10) we get:

\[ P_n(J) = e^{-\phi} \partial^n e^\phi = \sum_{l=0}^{n} (-1)^l \binom{n}{l} D^n \left( \partial^l e^{-\phi} \right) e^\phi = \sum_{l=0}^{n} (-1)^l \binom{n}{l} D^{n-l} \left( (D - J)^l \right) \] (A.14)

From (A.9) we also find:

\[ (D - J)^n = e^\phi D^n e^{-\phi} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} D^l \left( \partial^{n-l} e^\phi \right) e^{-\phi} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} D^l P_{n-l}(J) \] (A.15)

Similarly we get:

\[ P_n(J) (D - J)^m = \partial^m e^\phi D^m e^{-\phi} \]

\[ = \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} D^l e^{-\phi} \partial^{m-n} e^\phi = \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} D^l P_{m+n-l}(J) \] (A.16)

from which we get

\[ P_n(J) \sum_{k=0}^{m} (-1)^k \binom{m}{k} D^k P_{m-k}(J) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} D^k P_{n+m-k}(J) \] (A.17)
Finally let us also list for completeness

\[ D^n = D^n(e^\phi e^{-\phi}) = \sum_{l=0}^{n} \binom{n}{l} e^{-\phi} \left( \partial^{n-l} e^\phi \right) e^\phi D^l e^{-\phi} \]

\[ = \sum_{l=0}^{n} \binom{n}{l} P_l(J) (D - J)^{n-l} \quad (A.18) \]

and

\[ D^n = \left( e^{-\phi} e^\phi \right) D^n = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e^{-\phi} D^{n-l} e^\phi \partial^l e^\phi \]

\[ = \sum_{l=0}^{n} (-1)^l \binom{n}{l} (D + J)^{n-l} P_l(J) \quad (A.19) \]

Note also from (A.11) that

\[ e^{-\phi} D^{-1} e^\phi = \sum_{l=0}^{\infty} (-1)^l e^{-\phi} \partial^l e^\phi D^{l-1} = \sum_{l=0}^{\infty} (-1)^l P_l(J) D^{l-1} \quad (A.20) \]

and hence \((D + J)^{-1} = e^{-\phi} D^{-1} e^\phi\).

### B Appendix B

Here we give a proof of the identities (3.18)-(3.21). From (3.16) we have

\[ U_{n+1,m}(x) + D_x U_{n,m}(x) = (-1)^r \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} W_{n+m-k+1}^{(r)} D_x^k \]

\[ + (-1)^r \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_x W_{n+m-k}^{(r)} D_x^k + (-1)^r \sum_{k=0}^{n} (-1)^k \binom{n}{k} W_{n+m-k}^{(r)} D_x^{k+1} \]

\[ \equiv I + II + III \quad (B.1) \]

Making the shift \(k+1 \rightarrow k\) in III and adding it to I one gets

\[ (-1)^r (I + III) = W_{n+m+1}^{(r)} + \sum_{k=1}^{n+1} (-1)^k \left( \binom{n+1}{k} - \binom{n}{k-1} \right) W_{n+m-k+1}^{(r)} D_x^k \]

\[ \equiv W_{n+m+1}^{(r+1)} \quad (B.2) \]

The term \(k = n + 1\) does not contribute to the sum. Using then the identity

\[ \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad (B.3) \]

one notices that, as a consequence of (3.11), the sum \(I + II + III\) is equal to \(U_{n,m}^{(r+1)}(x)\). So (3.18) is proven.
From (3.17)
\[ V_{n+1,m}^{(r)}(x) + D_x V_{n,m}^{(r)}(x) = (-1)^r \sum_{k=0}^{m} \binom{m}{k} D_x^k W_{n+m-k+1}^{(r)} \]
\[ + (-1)^r \sum_{k=0}^{m} \binom{m}{k} D_x^{k+1} W_{n+m-k}^{(r)} \equiv I + II \]
(B.4)

Make the shift \( k + 1 \rightarrow k \) in \( II \). Combine \( I \), without the term \( k = 0 \), with \( II \), without the term \( k = m + 1 \), and use identity (B.3). One then gets that \( I + II \) is equal to \( V_{n,m+1}^{(r)}(x) \). So, (3.19) is proven.

The proof of (3.20) is very similar to that for (3.19). Just notice that \( D_y \) will act directly on \( \delta(x-y) \) and then \( D_x^k D_y \delta(x-y) = D_x^{k+1} \delta(x-y) \).

From (3.19) we have
\[ V_{n+1,m}^{(r)}(x) \delta(x-y) + (\partial_x V_{n,m}^{(r)}(x)) \delta(x-y) - D_y V_{n,m}^{(r)}(x) \delta(x-y) = V_{n,m+1}^{(r)}(x) \delta(x-y) \]
(B.5)

Using (3.10) one has
\[ V_{n+1,m}^{(r)}(x) \delta(x-y) + (\partial_x V_{n,m}^{(r)}(x)) \delta(x-y) = (-1)^r \sum_{k=0}^{m} \binom{m}{k} D_x^k (W_{n+m-k+1}^{(r)}(x)) \]
\[ + \partial_x W_{n+m-k}^{(r)}(x) \delta(x-y) = V_{n,m}^{(r+1)}(x) \]
Substituting this in (B.5) one gets (3.21).

**C Appendix C**

We now prove the relation (3.26). By applying the identity (3.10) successively one gets
\[ W_m^{(r)}(x) = (-1)^{m-n} \sum_{k=0}^{m-n} \binom{m-n}{k} \partial_x^{m-n-k} W_n^{(r+k)}(x) \]
(C.1)
for \( m \geq n \). Then from (3.10) one gets
\[ U_{n+s,n}^{(r)}(x) = (-1)^{n+r+s} \sum_{k=0}^{n+s} \sum_{l=0}^{n+s-k} \binom{n+s}{k} \binom{n+s-k}{l} (\partial_x^{n+s-k-l} W_n^{(r+l)}(x)) D_x^k \]
(C.2)
Now using the identity
\[ \binom{m}{k} \binom{m-k}{l} = \binom{m}{l} \binom{m-l}{k} \]
(C.3)
and the fact that
\[ \sum_{i=0}^{m} \sum_{j=0}^{m-i} = \sum_{j=0}^{m} \sum_{i=0}^{m-j} \]
(C.4)
one gets
\[ U_{n+s,n}^{(r)}(x) = (-1)^{n+r+s} \sum_{l=0}^{n+r+s} \sum_{k=0}^{n+s-l} \binom{n+s}{l} \binom{n+s-l}{k} \left( \frac{\partial_x^{n+s-k-l} W_n^{(r+l)}}{l!} \right) \left( \frac{\partial_x^{n+r-k-l} W_n^{(s+l)}}{l!} \right) D_x^k \]

where we have used (A.9) and made the shift \( n + s - l \rightarrow l \).

From (3.17) and (C.1) we have
\[ \Phi(x) = (3.17) (C.1) \]

Using (C.5) and (C.10) one can easily get (3.26) from (3.24).

\[ \sum_{k=0}^{n+r-s} \sum_{l=0}^{n+r-s-k} (-1)^k \binom{n+r}{k} \binom{n+r-k}{l} \left( \frac{\partial_x^{n+r-k-l} W_n^{(s+l)}}{l!} \right) \left( \frac{\partial_x^{n+r-k-l} W_n^{(s+l)}}{l!} \right) D_x^k \]

where in the last equality we used (A.9). Now using (C.4) for the double sum in \( k \) and \( l \) and also
\[ \sum_{k=0}^{n+x-s} \sum_{l=0}^{n+x-s-k} (-1)^k \binom{n+r}{k} \binom{n+r-k}{l} \left( \frac{\partial_x^{n+r-k-l} W_n^{(s+l)}}{l!} \right) \left( \frac{\partial_x^{n+r-k-l} W_n^{(s+l)}}{l!} \right) D_x^k \]

one gets
\[ \Psi_{n,n}^{(s)}(x) = (-1)^{n+r+s} \sum_{l=0}^{n+r+s} \sum_{m=0}^{n+r-l-n+r-l} \binom{n+r}{l} \binom{n+r-l}{m} \left( \frac{\partial_x^{n+r-m-l} W_n^{(s+l)}}{l!} \right) \left( \frac{\partial_x^{n+r-m-l} W_n^{(s+l)}}{l!} \right) D_x^k \]

Making the shift \( k - m \rightarrow k \) in the sum in \( k \) and using
\[ \sum_{i=0}^{m} (-1)^i \binom{m}{i} = \delta_{m,0} \]

one gets
\[ \Psi_{n,n}^{(s)}(x) = (-1)^{n+r+s} \sum_{l=0}^{n+r+s} (-1)^l \binom{n+r}{l} W_n^{(n+r-s-l)} D_x^l \]

where we have made the shift \( n + r - l \rightarrow l \).

Using (C.5) and (C.10) one can easily get (3.26) from (3.24).

\[ \Delta \text{ Algebra for } \mathcal{W}_n \text{ Generators} \]

Here we will be interested in calculating brackets using the generating functions from (3.33). The relevant bracket is given by:
\[ \{ J(x) J(x + \epsilon) e^{\Delta \phi(x)}, J(y) J(y + \eta) e^{\Delta \phi(y)} \} = e^{\Delta \phi(y)} e^{\Delta \phi(x)} \times \]
Similarly for second and third terms on the r.h.s. of (D.3) we find
\[g\) for these two terms:
\begin{align*}
\eta \\
\text{comparing with (3.34) is reproducing the DS structure so for the time being we do not need}
\end{align*}

The last term of (D.1) will provide the nonlinear part:
\[\times \left[ \tilde{J}(y) J(x + \epsilon) \delta'(x - y - \eta) - \tilde{J}(x) J(y + \eta) \delta'(y - x - \epsilon) \\
- \tilde{J}(y) J(y + \eta) J(x + \epsilon) \delta(x - y - \eta) + \tilde{J}(x) J(y + \eta) J(x + \epsilon) \delta(y - x - \epsilon) \\
+ \tilde{J}(x) \tilde{J}(y) J(y + \eta) \delta(x + \epsilon - y) - \tilde{J}(x) \tilde{J}(y) J(x + \epsilon) \delta(y + \eta - x) \\
- 2 \tilde{J}(x) \tilde{J}(y) J(x + \epsilon) J(y + \eta) \times \left( \varepsilon(x + \epsilon - y - \eta) - \varepsilon(x + \epsilon - y) \times \\
- \varepsilon(x - y - \eta) + \varepsilon(x - y) \right) \right] \]

The last term of (D.1) will provide the nonlinear part:
\[-2 \tilde{J}(x) \tilde{J}(y) \sum_{n,m,l,p} (e^{-\Phi(x)} \partial^n J(x) e^{\Phi(x)} \frac{e^{n-l}}{(n-l)!}) \times \]
\[\left( \partial^l \partial_y (x-y) \frac{\eta^p}{l! p!} - \partial^l \partial_x (x-y) \frac{\eta^p}{l! p!} + \varepsilon(x-y) \right) \]
\[= -2 \tilde{J}(x) \tilde{J}(y) \sum_{n,m,l,p} (n+l) \left( \frac{n}{l} \right) \left( \frac{m}{p} \right) (-1)^{p} (e^{-\Phi(x)} \partial^n J(x) e^{\Phi(x)}) \times \]
\[(e^{-\Phi(y)} \partial^m J(y) e^{\Phi(y)}) \partial_x^{l+p} (x-y) \frac{\eta^m}{n! m!} \]
\[(D.2) \]

After working with delta function on the right hand side of (D.1) one finds
\[\left\{ \tilde{J}(x) J(x + \epsilon) e^{\Delta_x \Phi(x)} , \tilde{J}(y) J(y + \eta) e^{\Delta_y \Phi(y)} \right\} \]
\[= \tilde{J}(y) J(y + \eta + \epsilon) e^{\Delta_y \Phi(y)} \delta'(x - y - \eta) + \tilde{J}(x) J(x + \eta + \epsilon) e^{\Delta_x \Phi(x)} \delta'(x + \epsilon - y) \]
\[+ \delta(x + \epsilon - y) \frac{\partial}{\partial(\epsilon + \eta)} \tilde{J}(x) J(x + \eta + \epsilon) e^{\Delta_x \Phi(x)} \]
\[\left[ \tilde{J}(y) J(y + \eta + \epsilon) e^{\Delta_y \Phi(y)} + \text{last term} \right] \]

by the last term we mean the last term of equation (D.1), which as shown in (D.2) after comparing with (3.34) is reproducing the DS structure so for the time being we do not need to worry about it.

We concentrate first on first and fourth terms on the r.h.s. of (D.3). Expanding in \(\epsilon\) and \(\eta\) gives for these two terms:
\[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{m+1} \frac{\epsilon^n \eta^m}{n! m!} (-1)^{m+n} \binom{m+1}{s} W_{m+n-s-1}(y) \delta^{(s)}(x-y) \]
\[(D.4) \]

Similarly for second and third terms on the r.h.s. of (D.3) we find
\[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{n+1} \frac{\epsilon^n \eta^m}{n! m!} (-1)^{m+n-s-1} \binom{n+1}{s} W_{m+n-s-1}(x) \delta^{(s)}(x-y) \]
\[(D.5) \]
Adding to this the last term we finally obtain the following bracket

\[ \{ \mathcal{W}_n(x), \mathcal{W}_m(y) \} = \sum_{s=0}^{m+1} \binom{m+1}{s} D_x^s \mathcal{W}_{m+n-s+1}(x) \delta(x-y) \]  

(D.6)

\[ - \sum_{s=0}^{n+1} (-1)^s \binom{n+1}{s} \mathcal{W}_{m+n-s-1}(x) D_x^s \delta(x-y) \]

\[ - 2 \left( \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (-1)^{n-i} \binom{n}{i} \binom{m}{j} \mathcal{W}_i(x) D_x^{n+m-i-j-1} \mathcal{W}_j(x) \right) \delta(x-y) \]

where we copied the results for the last term from previous calculations. The first two terms reproduce \( \Omega_{nm}^{(1)}(\mathcal{W}(x)) \).

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