AUSLANDER-BUCHWEITZ CONTEXT AND CO-t-STRUCTURES

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Abstract. We show that the relative Auslander-Buchweitz context on a triangulated category $T$ coincides with the notion of co-t-structure on certain triangulated subcategory of $T$ (see Theorem 3.8). In the Krull-Schmidt case, we establish a bijective correspondence between co-t-structures and cosuspended, precovering subcategories (see Theorem 3.11). We also give a characterization of bounded co-t-structures in terms of relative homological algebra. The relationship between silting classes and co-t-structures is also studied. We prove that a silting class $\omega$ induces a bounded non-degenerated co-t-structure on the smallest thick triangulated subcategory of $T$ containing $\omega$. We also give a description of the bounded co-t-structures on $T$ (see Theorem 5.10). Finally, as an application to the particular case of the bounded derived category $\mathcal{D}^b(\mathcal{H})$, where $\mathcal{H}$ is an abelian hereditary category which is Hom-finite, Ext-finite and has a tilting object (see [10]), we give a bijective correspondence between finite silting generator sets $\omega = \text{add} (\omega)$ and bounded co-t-structures (see Theorem 6.7).

1. Introduction.

In [11], Hashimoto defined the “Auslander-Buchweitz context” for abelian categories, giving a new framework to homological approximation theory. The starting point of Hashimoto’s work is the theory of approximations in abelian categories developed by Auslander and Buchweitz in [1], which has been a starting point for performing relative homological algebra with respect to suitable subcategories, with applications ranging from the study of Cohen-Macaulay modules over commutative rings, to tilting theory, the theory of quasi-hereditary algebras and reductive groups, the study of homological conjectures for finite dimensional algebras, and many other topics. On the other hand, in [5], Beligiannis generalizes to exact categories the fundamental work of [1]. In particular, following Hashimoto’s ideas, he introduces the Auslander-Buchweitz context for exact categories, which are more general than abelian ones.

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In the case of mod(Λ) (the category of finitely generated modules over an artin algebra Λ), it is important to mention the work of Auslander and Reiten in [2]. They studied the notion of approximations of modules using tilting and cotilting modules, and showed that there is a bijective correspondence between the basic cotilting modules in mod(Λ), and certain precovering subcategories X of mod(Λ). The main aim in [2] is to explore the connection between various aspects of tilting theory and the theory of cotorsion pairs in mod(Λ).

As we mentioned before in [14], abelian categories used to be the proper context for the study of homological algebra. But recently, triangulated categories entered into the subject in a relevant way. In [14], an analogue of Auslander-Buchweitz approximation theory is developed.

The main aim of the present paper is to explore, in the setting described in [14], results analogous to the results of Auslander-Reiten in connection with various aspects of tilting theory and the theory of co-t-structures. To do that, we use the notions and machinery of [14], concentrating our study to the relations between Auslander-Buchweitz contexts in a triangulated category T and co-t-structures defined on T.

The notion of co-t-structure was recently introduced independently by Pauksztello [15] and Bondarko [6] (under the name “weight structures”). This notion seems to be important, and one reason for this is that they provide important information in a triangulated category T allowing the existence of nice “weight” decompositions and filtrations. Furthermore, co-t-structures provide examples of torsion theories in Krull-Schmidt triangulated categories in the sense of Iyama and Yoshino [12].

Throughout this paper, T denotes an arbitrary triangulated category. Given a class X of objects of T, the smallest triangulated (respectively, smallest thick) subcategory of T containing X is denoted by ∆T(X) (respectively, ∆T(X)).

The paper is organized as follows. In Section 1, we recall, from [14], some notions about the Auslander-Buchweitz approximation theory that will be useful in this paper.

In Section 2, we show that the notion of relative Auslander-Buchweitz context for triangulated categories T coincides with the notion of co-t-structure on ∆T(X) (see Theorem 3.3). In particular, an Auslander-Buchweitz context is the same as a bounded below co-t-structure. Moreover, we establish a bijective correspondence between the relative Auslander-Buchweitz contexts (X, Y) on T and the class of pairs (X, ω) such that X is cosuspended and ω is an X-injective weak-cogenerator in X (see Theorem 3.11).

In Section 3, we focus our attention on bounded, faithful and non-degenerate co-t-structures. A characterization of bounded co-t-structures, in terms of relative homological algebra, is also given. Furthermore, a relationship between the different types of co-t-structures is also established (see Theorem 4.20). We also provide, on one hand, a relationship between several subcategories
attached to co-t-structures; and on the other hand, some relations between relative homological dimensions. We finish the section with some results involving co-t-structures and the notion of categorical cogenerator.

In Section 4, we study the relationship between co-t-structures and silting classes. In this section, we establish a bijective correspondence between silting classes in $\mathcal{T}$ and bounded co-t-structures on the thick subcategory of $\mathcal{T}$ generated by the silting class (see Corollary 5.8). Furthermore, we give a characterization of the bounded co-t-structures on $\mathcal{T}$ (see Theorem 5.10).

In Section 5, we apply the results, obtained in Section 4, to the particular case of the bounded derived category $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H}$ is an abelian hereditary category which is Hom-finite, Ext-finite and has a tilting object. We give a bijective correspondence between finite silting generator sets $\omega = \operatorname{add}(\omega)$ and bounded co-t-structures (see Theorem 6.7). As a nice consequence, we get that any bounded co-t-structure on $\mathcal{D}^b(\mathcal{H})$ has two companions as t-structures: one on the left and the other on the right. That is, any bounded co-t-structure on $\mathcal{D}^b(\mathcal{H})$ is always left (respectively, right) adjacent to a t-structure on $\mathcal{D}^b(\mathcal{H})$ in the sense of [6].

Note that in [9], the author studies co-t-structures on triangulated categories with arbitrary coproducts (his notion of “negative subcategories” correspond to our notion of silting). In this context, he proves that any silting subcategory $\omega$ provides a co-t-structure on the smallest triangulated subcategory of $\mathcal{T}$ closed under arbitrary coproducts and containing $\omega$. Our result (Theorem 5.5), which is proved using relative homology techniques, is the analogue for thick subcategories containing $\omega$, to the Theorem 4.3.2 in [6] which was proved with different techniques.

2. Preliminaries

Throughout this paper, $\mathcal{T}$ will be a triangulated category and $[1] : \mathcal{T} \rightarrow \mathcal{T}$ its suspension functor.

In this paper, when we say that $\mathcal{C}$ is a subcategory of $\mathcal{T}$, it always means that $\mathcal{C}$ is a full subcategory which is additive and closed under isomorphisms. For a class $\mathcal{X}$ of objects of $\mathcal{T}$, we denote by $\operatorname{add}(\mathcal{X})$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$, closed under finite direct sums and direct summands.

For some classes $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{T}$, we write $\perp \mathcal{X} := \{Z \in \mathcal{T} : \operatorname{Hom}_\mathcal{T}(Z,-)|_{\mathcal{X}} = 0\}$ and $\mathcal{X}^\perp := \{Z \in \mathcal{T} : \operatorname{Hom}_\mathcal{T}(-,Z)|_{\mathcal{X}} = 0\}$. We also recall that $\mathcal{X} \ast \mathcal{Y}$ denotes the class of objects $Z \in \mathcal{T}$ for which there exists a distinguished triangle $X \rightarrow Z \rightarrow Y \rightarrow X[1]$ in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Furthermore, it is said that $\mathcal{X}$ is closed under extensions if $\mathcal{X} \ast \mathcal{Y} \subseteq \mathcal{X}$.

Recall that a class $\mathcal{X}$ of objects in $\mathcal{T}$ is said to be suspended (respectively, cosuspended) if $\mathcal{X}[1] \subseteq \mathcal{X}$ (respectively, $\mathcal{X}[-1] \subseteq \mathcal{X}$) and $\mathcal{X}$ is closed under
extensions. Observe that a suspended (respectively, cosuspended) class \( \mathcal{X} \), of objects in \( \mathcal{T} \), is a subcategory of \( \mathcal{T} \) (see [14, Lemma 2.1 (b)]).

Given a class \( \mathcal{X} \) of objects in \( \mathcal{T} \), it is said that \( \mathcal{X} \) is closed under cones if for any distinguished triangle \( A \to B \to C \to A[1] \) in \( \mathcal{T} \) with \( A, B \in \mathcal{X} \) we have that \( C \in \mathcal{X} \). Similarly, \( \mathcal{X} \) is closed under cocones if for any distinguished triangle \( A \to B \to C \to A[1] \) in \( \mathcal{T} \) with \( B, C \in \mathcal{X} \) we have that \( A \in \mathcal{X} \).

Let \( \mathcal{X} \) be a class of objects of \( \mathcal{T} \). We denote by \( \mathcal{U}_\mathcal{X} \) (respectively, \( \mathcal{X}\mathcal{U} \)) the smallest suspended (respectively, cosuspended) subcategory of \( \mathcal{T} \) containing the class \( \mathcal{X} \). Note that if \( \mathcal{X} \) is suspended (respectively, cosuspended) subcategory of \( \mathcal{T} \), then \( \mathcal{X} = \mathcal{U}_\mathcal{X} \) (respectively, \( \mathcal{X} = \mathcal{X}\mathcal{U} \)). We also recall that a subcategory \( \mathcal{U} \) of \( \mathcal{T} \), which is suspended and cosuspended, is called a triangulated subcategory of \( \mathcal{T} \). A thick subcategory of \( \mathcal{T} \) is a triangulated subcategory of \( \mathcal{T} \) which is closed under direct summands in \( \mathcal{T} \). The smallest triangulated (respectively, smallest thick) subcategory of \( \mathcal{T} \) containing \( \mathcal{X} \) is denoted by \( \Delta_\mathcal{T}(\mathcal{X}) \) (respectively, \( \Sigma_\mathcal{T}(\mathcal{X}) \)).

We recall the following well known definition (see, for example, [7] and [8]).

**Definition 2.1.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be classes of objects in a triangulated category \( \mathcal{T} \). A morphism \( f : X \to C \) in \( \mathcal{T} \) is said to be an \( \mathcal{X} \)-precover of \( C \) if \( X \in \mathcal{X} \) and \( \text{Hom}_{\mathcal{T}}(X', f) : \text{Hom}_{\mathcal{T}}(X', X) \to \text{Hom}_{\mathcal{T}}(X', C) \) is surjective \( \forall X' \in \mathcal{X} \). If any \( C \in \mathcal{Y} \) admits an \( \mathcal{X} \)-precover, then \( \mathcal{X} \) is called a precovering class in \( \mathcal{Y} \).

By dualizing the definition above, we get the notion of an \( \mathcal{X} \)-preenveloping of \( C \) and a preenveloping class in \( \mathcal{Y} \). Finally, it is said that \( \mathcal{X} \) is functorially finite in \( \mathcal{T} \) if \( \mathcal{X} \) is both precovering and preenveloping in \( \mathcal{T} \).

Now, we recall from [14], the following definitions. For a more completed discussion and properties of such notions, we suggest that the reader see [14].

**Definition 2.2.** [14] Let \( \mathcal{X} \) be a class of objects in \( \mathcal{T} \). For any natural number \( n \), we introduce inductively the class \( \varepsilon_n^\wedge(\mathcal{X}) \) as follows: \( \varepsilon_0^\wedge(\mathcal{X}) := \mathcal{X} \) and assuming defined \( \varepsilon_n^\wedge(\mathcal{X}) \), the class \( \varepsilon_{n+1}^\wedge(\mathcal{X}) \) is given by all the objects \( Z \in \mathcal{T} \) for which there exists a distinguished triangle in \( \mathcal{T} \)

\[
\begin{array}{cccc}
Z[-1] & \longrightarrow & W & \longrightarrow & X & \longrightarrow & Z
\end{array}
\]

with \( W \in \varepsilon_{n-1}^\wedge(\mathcal{X}) \) and \( X \in \mathcal{X} \).

Dually, we set \( \varepsilon_n^\vee(\mathcal{X}) := \mathcal{X} \) and assuming defined \( \varepsilon_n^\vee(\mathcal{X}) \), the class \( \varepsilon_{n+1}^\vee(\mathcal{X}) \) is formed for all the objects \( Z \in \mathcal{T} \) for which there exists a distinguished triangle in \( \mathcal{T} \)

\[
\begin{array}{cccc}
Z & \longrightarrow & X & \longrightarrow & K & \longrightarrow & Z[1]
\end{array}
\]

with \( K \in \varepsilon_{n-1}^\wedge(\mathcal{X}) \) and \( X \in \mathcal{X} \). We also introduce the following classes

\[
\mathcal{X}^\wedge := \cup_{n \geq 0} \varepsilon_n^\wedge(\mathcal{X}), \quad \mathcal{X}^\vee := \cup_{n \geq 0} \varepsilon_n^\vee(\mathcal{X}) \quad \text{and} \quad \mathcal{X}^\sim := (\mathcal{X}^\wedge)^\vee.
\]

For the convenience of the reader, we include the following remark from [14].
Remark 2.3. [14, Remark 3.6 (2)] Let $(\mathcal{Y}, \omega)$ be a pair of classes of objects in $\mathcal{T}$ with $\omega \subseteq \mathcal{Y}$. If $\mathcal{Y}$ is closed under cones (respectively, cocones) then $\omega^\wedge \subseteq \mathcal{Y}$ (respectively, $\omega^\vee \subseteq \mathcal{Y}$). Indeed, assume that $\mathcal{Y}$ is closed under cones and let $M \in \omega^\wedge$. Thus $M \in \varepsilon_n^\wedge(\omega)$ for some $n \in \mathbb{N}$. If $n = 0$ then $M \in \omega \subseteq \mathcal{Y}$. Let $n > 0$, and hence there is a distinguished triangle $M[-1] \to K \to Y \to M$ in $\mathcal{T}$ with $K \in \varepsilon_n^\wedge(\omega)$ and $Y \in \mathcal{Y}$. By induction $K \in \mathcal{Y}$ and hence $M \in \mathcal{Y}$ since $\mathcal{Y}$ is closed under cones; proving that $\omega^\wedge \subseteq \mathcal{Y}$.

In what follows, to deal with the (co) resolution, relative projective and relative injective dimensions, we consider the extended natural numbers $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Here, we set the following rules: (a) $x + \infty = \infty$ for any $x \in \mathbb{N}$, and (b) $x < \infty$ for any $x \in \mathbb{N}$. Finally, we declare, by definition, that the minimum of the empty set is $\infty$. That is, $\min(\emptyset) := \infty$.

Definition 2.4. [14, Definition 2.4] Let $\mathcal{X}$ be a class of objects in $\mathcal{T}$, and let $M \in \mathcal{T}$. The $\mathcal{X}$-resolution dimension of $M$ is

$$\text{resdim}_\mathcal{X}(M) := \min \{n \in \mathbb{N} : M \in \varepsilon_n^\wedge(\mathcal{X})\}.$$ 

Dually, the $\mathcal{X}$-coresolution dimension of $M$ is

$$\text{coresdim}_\mathcal{X}(M) := \min \{n \in \mathbb{N} : M \in \varepsilon_n^\vee(\mathcal{X})\}.$$ 

The following result, and its dual version, will be used in Section 2.

Theorem 2.5. [14, Theorem 3.5] For any cosuspended subcategory $\mathcal{X}$ of $\mathcal{T}$ and any object $C \in \mathcal{T}$, the following statements hold.

(a) $\text{resdim}_\mathcal{X}(C) \leq n$ if and only if $C \in \mathcal{X}[n]$.
(b) $\mathcal{X}^\wedge = \cup_{n \geq 0} \mathcal{X}[n] = \Delta_\mathcal{T}(\mathcal{X})$.
(c) If $\mathcal{X}$ is closed under direct summands in $\mathcal{T}$, then $\mathcal{X}^\wedge = \Sigma_\mathcal{T}(\mathcal{X})$.

For the convenience of the reader, we include the dual version of [2.5]

Remark 2.6. For any suspended subcategory $\mathcal{Y}$ of $\mathcal{T}$ and any object $C \in \mathcal{T}$, the following statements hold.

(a) $\text{coresdim}_\mathcal{Y}(C) \leq n$ if and only if $C \in \mathcal{Y}[-n]$.
(b) $\mathcal{Y}^\vee = \cup_{n \geq 0} \mathcal{Y}[-n] = \Delta_\mathcal{T}(\mathcal{Y})$.
(c) If $\mathcal{Y}$ is closed under direct summands in $\mathcal{T}$, then $\mathcal{Y}^\vee = \Sigma_\mathcal{T}(\mathcal{Y})$.

We recall the notion of $\mathcal{X}$-projective (respectively, $\mathcal{X}$-injective) dimension of objects in $\mathcal{T}$.

Definition 2.7. [14] Let $\mathcal{X}$ be a class of objects in $\mathcal{T}$ and $M$ an object in $\mathcal{T}$.

(a) The $\mathcal{X}$-projective dimension of $M$ is

$$\text{pd}_\mathcal{X}(M) := \min \{n \in \mathbb{N} : \text{Hom}_\mathcal{T}(M[-i], -) |_{\mathcal{X}} = 0, \ \forall i > n\}.$$ 

(b) The $\mathcal{X}$-injective dimension of $M$ is

$$\text{id}_\mathcal{X}(M) := \min \{n \in \mathbb{N} : \text{Hom}_\mathcal{T}(-, M[i]) |_{\mathcal{X}} = 0, \ \forall i > n\}.$$
Definition 2.8. [14] Let \((\mathcal{X}, \omega)\) be a pair of classes of objects in \(\mathcal{T}\).

(a) \(\omega\) is a weak-cogenerator in \(\mathcal{X}\), if \(\omega \subseteq \mathcal{X} \subseteq \mathcal{X}[-1]*\omega\).

(b) \(\omega\) is a weak-generator in \(\mathcal{X}\), if \(\omega \subseteq \mathcal{X} \subseteq \omega*[\mathcal{X}[1]]\).

(c) \(\omega\) is \(\mathcal{X}\)-injective if \(\text{id}_{\mathcal{X}}(\omega) = 0\); and dually, \(\omega\) is \(\mathcal{X}\)-projective if \(\text{pd}_{\mathcal{X}}(\omega) = 0\).

3. Relative Auslander-Buchweitz context and co-\(t\)-structures

In this section, we give the notion of the (relative) Auslander-Buchweitz context for a triangulated category \(\mathcal{T}\), relating this notion with the concept of co-\(t\)-structure.

Definition 3.1. [6, 15] A pair \((A, B)\) of subcategories in \(\mathcal{T}\) is said to be a co-\(t\)-structure on \(\mathcal{T}\) if the following conditions hold.

(a) \(A\) and \(B\) are closed under direct summands in \(\mathcal{T}\).

(b) \(A[-1] \subseteq A\) and \(B[1] \subseteq B\).

(c) \(\text{Hom}_{\mathcal{T}}(A[-1], B) = 0\).

(d) \(\mathcal{T} = A[-1]*B\).

We will make use of the following result, stated by D. Pauksztello in [15].

Proposition 3.2. [15, Proposition 2.1] Let \((A, B)\) be a co-\(t\)-structure on \(\mathcal{T}\). Then, the following statements hold.

(a) \(A[-1]\) is a precovering class in \(\mathcal{T}\).

(b) \(B\) is a preenveloping class in \(\mathcal{T}\).

(c) \(A[-1] = \perp B\) and \(B = A[-1]*\).

(d) \(A\) and \(B\) are closed under extensions.

Lemma 3.3. Let \((A, B)\) be a co-\(t\)-structure on \(\mathcal{T}\), and \(Y\) be a class of objects in \(\mathcal{T}\). Then, the following statements hold.

(a) \(\text{id}_A(Y) \leq n\) if and only if \(Y \subseteq B[-n]\).

(b) \(\text{pd}_B(Y) \leq n\) if and only if \(Y \subseteq A[n]\).

Proof. (a) By [14] Lemma 4.2, we get the equivalence: \(\text{id}_A(Y) \leq n\) if and only if \(Y \subseteq A[-n]\). Therefore, since \((A, B)\) is a co-\(t\)-structure, it follows from [14] (c), that \(A[1][-n-1] = B[-n]\).

(b) It is dual to (a). □

The following result states that, for a co-\(t\)-structure \((A, B)\) on \(\mathcal{T}\), the class \(\omega := A \cap B\) is an \(A\)-injective weak-cogenerator in \(A\); and moreover, \(\omega\) is also a \(B\)-projective weak-generator in \(B\). Note that \(\omega = \text{add}(\omega)\).

Proposition 3.4. Let \((A, B)\) be a co-\(t\)-structure on \(\mathcal{T}\), and let \(\omega := A \cap B\). Then, the following statements hold.

(a) \(\text{id}_A(B) = 0\) and \(A \subseteq A[-1]*\omega\).

(b) \(\text{pd}_B(A) = 0\) and \(B \subseteq \omega*B[1]\).
from the preceding triangle that $A \in \mathcal{A}$. Then, by 3.1 (d), we have a distinguished triangle $C' \to C \to C'' \to C'[1]$ in $\mathcal{T}$ with $C' \in \mathcal{A}[-1]$ and $C'' \in \mathcal{B}$. Hence, by 3.2 (d), it follows that $C'' \in \mathcal{A} \cap \mathcal{B} = \omega$; proving that $\mathcal{A} \subseteq \mathcal{A}[-1] \ast \omega$.

By 3.1 for any $X \in \mathcal{T}$, there is a distinguished triangle $A \to X \to B[1] \to A[1]$ in $\mathcal{T}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Moreover, in the case $X \in \mathcal{B}$, it follows from the preceding triangle that $A \in \mathcal{A} \cap \mathcal{B} = \omega$; getting us that $\mathcal{B} \subseteq \omega \ast \mathcal{B}[1]$.

We will show the relation between the notions of cosuspended (respectively, suspended) subcategories $\mathcal{X}$, weak-cogenerator (respectively, weak-generator), $\mathcal{X}$-injective (respectively, $\mathcal{X}$-projective) and co-$t$-structures on $\overline{\mathcal{T}}(\mathcal{X})$. We only state the results for the cosuspended case and omit those for the suspended case which can be proved by similar arguments.

First, we show that any $\mathcal{X}$-injective weak-cogenerator in a cosuspended subcategory $\mathcal{X} = \text{add}(\mathcal{X})$ of $\mathcal{T}$ provides a co-$t$-structure on $\overline{\mathcal{T}}(\mathcal{X}) = \mathcal{X}^\wedge$.

**Theorem 3.5.** Let $(\mathcal{X}, \omega)$ be a pair of classes of objects in $\mathcal{T}$ which are closed under direct summands, $\mathcal{X}$ be cosuspended and $\omega$ be an $\mathcal{X}$-injective weak-cogenerator in $\mathcal{X}$. Then, the following statements hold.

(a) The pair $(\mathcal{X}^\wedge \cap \perp(\omega^\wedge))[1], \omega^\wedge)$ is a co-$t$-structure on the triangulated category $\mathcal{X}^\wedge$.

(b) $\omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1]$, $\mathcal{X} = \mathcal{X}^\wedge \cap \perp(\omega^\wedge)[1]$ and $\omega = \mathcal{X} \cap \mathcal{X}^\perp[-1]$.

(c) If $\omega'$ is an $\mathcal{X}$-injective weak-cogenerator in $\mathcal{X}$, then $\omega = \text{add}\omega'$.

**Proof.** First note that $\mathcal{X} = \mathcal{X}^\mathcal{U}$ since $\mathcal{X}$ is cosuspended.

(b) From [14 Proposition 5.9], we have that $\omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1]$ . By [14 Proposition 5.2 (b)], it follows that $\omega = \mathcal{X} \cap \mathcal{X}^\perp[-1]$ since $\mathcal{X}$ is cosuspended. Moreover, by [14 Theorem 5.10] it follows that $\mathcal{X} = \mathcal{X}^\wedge \cap \perp(\omega^\wedge)[1]$.

(a) We have that $\omega^\wedge = \mathcal{X}^\wedge \cap \mathcal{X}^\perp[-1]$ is suspended and closed under direct summands. Therefore $\mathcal{X}^\wedge \cap \perp(\omega^\wedge)[1]$ is cosuspended and closed under direct summands. So, in order to get that the given pair in (a) is a co-$t$-structure on the triangulated category $\mathcal{X}^\wedge$, it is enough to see that $\mathcal{X}^\wedge = (\mathcal{X}^\wedge \cap \perp(\omega^\wedge)) \ast \omega^\wedge$.

But this is a consequence of [14 Corollary 5.5 (b)] since $\mathcal{X}[-1] = \mathcal{X}^\wedge \cap \perp(\omega^\wedge)$.

(c) It follows from (b) and the fact that $\text{add}(\omega')$ is an $\mathcal{X}$-injective weak-cogenerator in $\mathcal{X}$. □

**Remark 3.6.** Let $\mathcal{X} = \text{add}(\mathcal{X})$ be a cosuspended subcategory of $\mathcal{T}$. Note that $\mathcal{X} \cap \mathcal{X}^\perp[-1]$ is $\mathcal{X}$-injective. Moreover, from 3.1 we get that: If there is an $\mathcal{X}$-injective weak-cogenerator $\omega = \text{add}(\omega)$ in $\mathcal{X}$ then it is unique. Consequently, there is an $\mathcal{X}$-injective weak-cogenerator $\omega = \text{add}(\omega)$ in $\mathcal{X}$ if and only if $\mathcal{X} \cap \mathcal{X}^\perp[-1]$ is a weak-cogenerator in $\mathcal{X}$.

The Auslander-Buchweitz context for abelian categories was introduced by M. Hashimoto in [11]. Inspired by that, we will introduce such a context
for a triangulated category $\mathcal{T}$. To do so, we define the notion of a relative Auslander-Buchweitz context on $\mathcal{T}$. Observe that the “relative Auslander-Buchweitz context” in triangulated categories is used for an analogue of what Hashimoto calls “weak Auslander-Buchweitz context” in abelian categories.

**Definition 3.7.** Let $(\mathcal{X}, \mathcal{Y})$ be a pair of classes of objects in $\mathcal{T}$, and let $\omega := \mathcal{X} \cap \mathcal{Y}$. The pair $(\mathcal{X}, \mathcal{Y})$ is said to be a relative Auslander-Buchweitz context on $\mathcal{T}$ if the following three conditions hold:

1. **(AB1)** $\mathcal{X}$ is cosuspended and closed under direct summands in $\mathcal{T}$.
2. **(AB2)** $\mathcal{Y}$ is suspended and closed under direct summands in $\mathcal{T}$ and $\mathcal{Y} \subseteq \mathcal{X}^\perp$.
3. **(AB3)** $\omega$ is an $\mathcal{X}$-injective weak-cogenerator in $\mathcal{X}$.

The pair $(\mathcal{X}, \mathcal{Y})$ is said to be an Auslander-Buchweitz context on $\mathcal{T}$ if $(\mathcal{X}, \mathcal{Y})$ is a relative Auslander-Buchweitz context on $\mathcal{T}$ and $\mathcal{X}^\perp = \mathcal{T}$.

**Theorem 3.8.** Let $(\mathcal{X}, \mathcal{Y})$ be a relative Auslander-Buchweitz context on $\mathcal{T}$ and $\omega := \mathcal{X} \cap \mathcal{Y}$. Then, the following statements hold.

(a) $\omega = \mathcal{X} \cap \mathcal{X}^\perp[-1]$ and $\omega^\perp = \mathcal{Y}$.

(b) $(\mathcal{X}, \mathcal{Y})$ is a co-t-structure on the triangulated category $\Delta_{\mathcal{T}}(\mathcal{X})$.

**Proof.** (a) The first equality follows from 3.5. Since $\omega \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is suspended, it follows from 3.4 (a) that $\mathcal{Y}^\perp = \omega^\perp$.

We assert that $\text{id}_{\mathcal{X}}(\mathcal{Y}) = 0$. Indeed, let $C \in \mathcal{Y} \subseteq \mathcal{X}^\perp$. Hence, by [14, Theorem 5.4], we have a distinguished triangle $Y_C \to X_C \to Y_C[1]$ in $\mathcal{T}$ with $X_C \in \mathcal{X}$ and $Y_C \in \omega^\perp \subseteq \mathcal{Y}$. Hence $X_C \in \mathcal{X} \cap \mathcal{Y} = \omega$ and so $\text{id}_{\mathcal{X}}(X_C) = 0$. On the other hand, since $\text{id}_{\mathcal{X}}(Y_C) = 0$ (see [14, Proposition 5.2 (a)]), it follows by [14, Lemma 5.7] that $\text{id}_{\mathcal{X}}(C) = 0$; proving the assertion. Finally, $\text{id}_{\mathcal{X}}(\mathcal{Y}) = 0$ and the fact that $\mathcal{X}$ is cosuspended implies by [14, Lemma 4.2] that $\mathcal{Y} \subseteq \mathcal{X}^\perp \cap \mathcal{X}^\perp[-1]$. Therefore $\mathcal{Y} \subseteq \omega^\perp$ by 3.5.

(b) Since $\omega^\perp = \mathcal{Y}$, we have that (b) follows from 3.5. $\square$

Given a class $\mathcal{X}$ of objects in $\mathcal{T}$, we recall that $\overline{\Delta}_{\mathcal{T}}(\mathcal{X})$ denotes the smallest thick subcategory of $\mathcal{T}$ containing the class $\mathcal{X}$.

**Proposition 3.9.** Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of objects in $\mathcal{T}$ such that the pair $(\mathcal{X}, \mathcal{Y})$ is a co-t-structure on the triangulated category $\overline{\Delta}_{\mathcal{T}}(\mathcal{X})$. Then, $\overline{\Delta}_{\mathcal{T}}(\mathcal{X}) = \mathcal{X}^\perp$ and $(\mathcal{X}, \mathcal{Y})$ is a relative Auslander-Buchweitz context on $\mathcal{T}$.

**Proof.** By 3.2 (d), we have that $\mathcal{X}$ is cosuspended and $\mathcal{Y}$ is suspended. In particular, from 2.8 we conclude that $\overline{\Delta}_{\mathcal{T}}(\mathcal{X}) = \mathcal{X}^\perp$. The fact that $\omega = \mathcal{X} \cap \mathcal{Y}$ is an $\mathcal{X}$-injective weak-cogenerator in $\mathcal{X}$, follows from 3.4 (a). $\square$

Now, we are in a position to state our main result in this section. In order to do that, we introduce the following classes.

**Definition 3.10.** For a given triangulated category $\mathcal{T}$, we introduce the following classes:
Theorem 3.11. Let \( T \) be a triangulated category. Then, the following statements hold.

(a) \( C_2 = C_3 \) and the correspondence \( C_1 \rightarrow C_2 \), \((\mathcal{X}, \omega) \mapsto (\mathcal{X}, \mathcal{Y} := \omega^\wedge)\)

is a bijection with inverse \( C_2 \rightarrow C_1 \) given by \((\mathcal{X}, \mathcal{Y}) \mapsto (\mathcal{X}, \omega := \mathcal{X} \cap \mathcal{Y})\).

(b) If \( T \) is an \( R \)-linear triangulated category which is Hom-finite and

Krull-Schmidt, then the correspondence \( C_4 \rightarrow C_3 \), \( \mathcal{X} \mapsto (\mathcal{X}, \mathcal{Y} := \mathcal{X}^\perp[-1] \cap \mathcal{X}^\wedge) \)

is a bijection with inverse \( C_4 \rightarrow C_3 \) given by \((\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X}\).

\[\begin{align*}
\mathcal{X}[-1] \ast (\mathcal{X}^\perp[-1] \cap \mathcal{X}^\wedge) &= (\mathcal{X} \ast (\mathcal{X}^\perp \cap \mathcal{X}^\wedge))[-1] = \mathcal{X}^\perp[-1] = \mathcal{X}^\wedge,
\end{align*}\]
giving us that \((\mathcal{X}, \mathcal{X}^\perp[-1] \cap \mathcal{X}^\wedge) \in C_3\).

Consider a pair \((\mathcal{X}, \mathcal{Y})\) \(\in C_3\). Then by \(3.9\) \(\text{and} 3.8\), it follows that \(\mathcal{Y} = \mathcal{X}^\perp[-1] \cap \mathcal{X}^\wedge\). Moreover, since the pair \((\mathcal{X}[1], \mathcal{Y}[1])\) is also a co-\(t\)-structure on \(\overline{\Delta}_T(\mathcal{X})\), we have from \(3.2\) that \(\mathcal{X} \in C_4\). Furthermore, since \(\mathcal{Y} = \mathcal{X}^\perp[-1] \cap \mathcal{X}^\wedge\), it follows that the correspondence \(\mathcal{X} \mapsto (\mathcal{X}, \mathcal{X}^\perp[-1] \cap \mathcal{X}^\wedge)\) induces a bijection, with inverse \((\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X}\), between the classes \(C_4\) and \(C_3\).
Corollary 3.12. There is a bijective correspondence \( \mathcal{X} \mapsto (\mathcal{X}, \mathcal{X}^\perp \cap \mathcal{X}^\perp[-1]) \) between cosuspended subcategories \( \mathcal{X} = \text{add} (\mathcal{X}) \) of \( \mathcal{T} \) such that \( \mathcal{X} \cap \mathcal{X}^\perp \cap \mathcal{X}^\perp[-1] \) is a weak-cogenerator in \( \mathcal{X} \), and co-t-structures \((\mathcal{X}, \mathcal{Y})\) on \( \overline{\mathcal{X}} \).

Proof. It follows from 3.11 and 3.6. 

4. Bounded, faithful and non-degenerate co-t-structures

In this section we focus our attention on bounded, faithful and non-degenerate co-t-structures. We finish the section with some results involving co-t-structures and the notion of categorical cogenerator.

Following the terminology for co-t-structures on triangulated categories given in [6], we recall the following definition.

Definition 4.1. Let \((A, B)\) be a co-t-structure on \( \mathcal{T} \). It is said that \((A, B)\) is bounded below (respectively, bounded above) if \( \bigcup_{n \in \mathbb{Z}} A[n] = \mathcal{T} \) (respectively, \( \bigcup_{n \in \mathbb{Z}} B[n] = \mathcal{T} \)). So, the pair \((A, B)\) is said to be bounded if it is bounded both below and above.

Remark 4.2. From 2.5 and 2.6, we have that a co-t-structure \((A, B)\) on \( \mathcal{T} \) is bounded below (respectively, above) if and only if \( A^\wedge = \mathcal{T} \) (respectively, \( B^\vee = \mathcal{T} \)).

Corollary 4.3. There is a bijective correspondence \( \mathcal{X} \mapsto (\mathcal{X}, \mathcal{X}^\perp \cap \mathcal{X}^\perp[-1]) \) between cosuspended subcategories \( \mathcal{X} = \text{add} (\mathcal{X}) \) of \( \mathcal{T} \) such that \( \mathcal{X} \cap \mathcal{X}^\perp \cap \mathcal{X}^\perp \cap \mathcal{X}^\perp[-1] \) is a weak-cogenerator in \( \mathcal{X} \), and bounded below co-t-structures \((\mathcal{X}, \mathcal{Y})\) on \( \mathcal{T} \).

Proof. It follows from 3.12 and 4.2

Now, we prove some relationships between the relative homological dimensions attached to a co-t-structure.

Proposition 4.4. Let \((A, B)\) be a co-t-structure on \( \mathcal{T} \) and \( \omega := A \cap B \). Then

\( (a) \) \pd_B(M) = \resdim_A(M) \text{ and } \id_A(M) = \coresdim_B(M), \ \forall M \in \mathcal{T}. 

\( (b) \) \resdim_A(M) = \resdim_\omega(M), \ \forall M \in \omega^\wedge. 

\( (c) \) \coresdim_B(M) = \coresdim_\omega(M), \ \forall M \in \omega^\vee. 

Proof. By 3.2, we know that \( B = \mathcal{A}^\perp[-1] = \mathcal{A} \mathcal{U}^\perp[-1] \) and \( A = \mathcal{B}^\perp[1] = \mathcal{B}^\perp[1] \). Hence, from [14] Proposition 4.3, we get (a). Finally, (b) and (c) follows from [14] Theorem 4.4 and its dual, and the item (a).

The next result provides a relationship between several subcategories attached to co-t-structures. Furthermore, it characterizes the bounded below co-t-structures on \( \mathcal{T} \). We recall that \( \omega^\sim := (\omega^\wedge)^\vee \) for any class \( \omega \) of objects in \( \mathcal{T} \).

Theorem 4.5. Let \((A, B)\) be a co-t-structure on \( \mathcal{T} \) and \( \omega := A \cap B \). Then, the following conditions hold.
(a) \( U_\omega = \omega^\wedge = A^\wedge \cap B \) and \( U_\omega^\vee = \omega^\vee = B^\vee \cap A \).

(b) \( \Sigma_T(\omega) = \omega^\sim = \{ C \in A^\wedge : \text{id}_A(C) < \infty \} = \{ C \in B^\vee : \text{pd}_B(C) < \infty \} = A^\wedge \cap B^\vee \).

(c) The following conditions are equivalent:
   (c1) \((A, B)\) is bounded below.
   (c2) \( B \subseteq \omega^\sim \).
   (c3) \( \omega^\wedge = B \).
   (c4) \( B \subseteq A^\wedge \).

**Proof.** (a) Since \((A, B)\) is a co-t-structure on \(T\), we obtain from 3.4 that \(\omega\) is an \(A\)-injective weak-cogenerator in \(A\). Therefore, the first equalities in (a) follows from [14, Proposition 5.9], and the second ones can be proven by dualizing [14, Proposition 5.9].

(b) It follows from [14, Theorem 5.16] and its dual.

(c) (c1) \(\Rightarrow\) (c3) Let \(A^\wedge = T\) (see 4.2). Then, by 3.9, it follows that \((A, B)\) is an Auslander-Buchweitz context on \(T\). Hence \(B = \omega^\wedge\) by 3.8.

(c3) \(\Rightarrow\) (c2) Assume that \(B = \omega^\wedge\). Since \(\omega^\wedge \subseteq \omega^\sim\), we get that \(B \subseteq \omega^\sim\).

(c2) \(\Rightarrow\) (c1) Suppose that \(B \subseteq \omega^\sim\). We assert that \(T = A^\wedge\). Indeed, since \((A, B)\) is a co-t-structure on \(T\), we have that \(T = A[-1] \ast A^\perp[1] = (A \ast A^\perp)[1];\) and so \(T = A \ast A^\perp\). Thus for any \(C \in T\) there is a distinguished triangle \(Z[1] \to A \to C \to Z\) in \(T\) with \(A \in A\) and \(Z \in A^\perp\). But \(Z[1] \in A^\perp[1] = B \subseteq \omega^\sim \subseteq A^\wedge\) by (b); proving that \(C \in A^\wedge\).

(c3) \(\Leftrightarrow\) (c4) It follows from the equality \(\omega^\wedge = A^\wedge \cap B\) (see (a)). ✷

The results for bounded above co-t-structures can be stated and proved. To give an example, we give the following characterization of bounded above co-t-structures.

**Remark 4.6.** Let \((A, B)\) be a co-t-structure on \(T\) and \(\omega := A \cap B\). Then, the following conditions are equivalent:
   (a) \((A, B)\) is bounded above.
   (b) \(A \subseteq \omega^\sim\).
   (c) \(\omega^\vee = A\).
   (d) \(A \subseteq B^\vee\).

Following the terminology for t-structures on triangulated categories, and also \([6]\) and \([15]\), we give the following definitions.

**Definition 4.7.** Let \((A, B)\) be a co-t-structure on \(T\), and let \(\omega := A \cap B\). It is said that \((A, B)\) is faithful below (respectively, faithful above) if \(\sqcup_{n \in \mathbb{Z}} A[n] = \Sigma_T(\omega)\) (respectively, \(\sqcup_{n \in \mathbb{Z}} B[n] = \Sigma_T(\omega)\)). So, it is said that \((A, B)\) is faithful if it is both faithful below and above.

**Proposition 4.8.** Let \((A, B)\) be a co-t-structure on \(T\), and let \(\omega := A \cap B\). Then, the following conditions are equivalent.
   (a) \((A, B)\) is faithful below.
(b) \( A^\land = \Delta_T(\omega) \).
(c) \( A^\land \subseteq B^\lor \).
(d) \((A, B)\) is bounded above.

**Proof.** (a) \(\Leftrightarrow\) (b) It follows from 2.5.
(b) \(\Leftrightarrow\) (c) It follows from 4.5 (b).
(c) \(\Leftrightarrow\) (d) First, observe that \( A^\land \subseteq B^\lor \) is equivalent to the inclusion \( A \subseteq B^\lor \) since \( B^\lor \) is a triangulated subcategory of \( T \) (see 2.6). Therefore, by 4.6, we get the result. \(\square\)

**Corollary 4.9.** Let \((A, B)\) be a co-t-structure on \( T \). Then, \((A, B)\) is bounded if and only if it is faithful.

**Proof.** It follows from 4.8 and its dual. \(\square\)

**Theorem 4.10.** Let \((A, B)\) be a bounded co-t-structure on \( T \), and let \( \omega := A \cap B \).

Then, the following statements hold.

(a) \(\Delta_T(\omega) = (\omega^\lor)^\land = T \), \( \cup_\omega = \omega^\land = B \) and \( \omega \mathcal{M} = \omega^\lor = A \).
(b) \( \text{id}_A(C) = \text{id}_\omega(C) = \text{coresdim}_B(C) < \infty \) for all \( C \in T \).
(c) \( \text{pd}_B(C) = \text{pd}_\omega(C) = \text{resdim}_A(C) < \infty \) for all \( C \in A \).
(d) \( \text{coresdim}_B(C) = \text{coresdim}_\omega(C) < \infty \) for all \( C \in B \).

**Proof.** (a) It follows from 3.8 [4.3] and its dual.
(b) Since \( \omega^\sim = T \) (see (a)), we get from 4.8 (b) that \( \text{id}_A(C) < \infty \) for all \( C \in T \). Thus, (b) follows from 4.4 (a) and [14] Proposition 5.17 (a).
(c) Using that \((A, B)\) is a co-t-structure on \( A^\land = T \), we get from 3.4 that the pair \((A, \omega)\) satisfies the needed hypothesis in [14] Theorem 5.6; proving (c).
(d) and (e) They follow from 4.4 since \( \omega^\lor = A \) and \( \omega^\land = B \). \(\square\)

Now, we will do one application of [4.10] to the so called Rouquier’s relative dimension which was introduced in [14]. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be classes of objects in a triangulated category \( T \). Consider the subcategory \( \langle \mathcal{X} \rangle := \text{add} \left( \bigcup_{i \in \mathbb{Z}} \mathcal{X}[i] \right) \) and let \( \mathcal{X} \diamond \mathcal{Y} := \langle \mathcal{X} \ast \mathcal{Y} \rangle \). Following R. Rouquier in [16], we inductively define \( \langle \mathcal{X} \rangle_0 := 0 \) and \( \langle \mathcal{X} \rangle_n := \langle \mathcal{X} \rangle_{n-1} \diamond \langle \mathcal{X} \rangle \) for \( n \geq 1 \). So, we start with the following definition.

**Definition 4.11.** [14] Definition 6.3 Let \( T \) be a triangulated category, \( \mathcal{X} \) a class of objects in \( T \) and \( M \in T \). The \( \mathcal{X} \)-dimension of \( M \) is

\[ \dim_\mathcal{X}(M) := \min \{ n \in \mathbb{N} \text{ such that } M \in \langle \mathcal{X} \rangle_{n+1} \}. \]

For a class \( \mathcal{Y} \) of objects in \( T \), we set \( \dim_\mathcal{X}(\mathcal{Y}) := \sup \{ \dim_\mathcal{X}(Y) : Y \in \mathcal{Y} \} \).

**Corollary 4.12.** Let \((A, B)\) be a bounded co-t-structure on \( T \), and let \( \omega := A \cap B \). Then

(a) \( \max \{ \dim_A(C), \dim_B(C) \} \leq \dim_\omega(C) \) for all \( C \in T \).
Dually, by induction and using the definition of Remark 4.14. Let \( \Omega := \) and let \( \cap \) is a cogenerator in \( \mathcal{T} \). Therefore, using that Proposition 4.15.

**Proof.** Since \( \omega = \mathcal{A} \cap \mathcal{B} \), (a) follows from \([14\text{ Lemma 6.4 (b)}]\). Let \( \mathcal{X} \) be any class of objects in \( \mathcal{T} \) and \( M \in \mathcal{T} \). Since \( \mathcal{X} \subseteq \langle \mathcal{X} \rangle \), we get by \([14\text{ Proposition 6.6}]\), that \( \dim_{\mathcal{X}}(M) \leq \min \{ \resdim_{\mathcal{X}}(M), \coresdim_{\mathcal{X}}(M) \} \). Hence, the result follows from \([4.10\text{ ]}) \.

We recall the following well known notions that will be useful in what follows.

**Definition 4.13.** Let \( \omega \) be a class of objects of the triangulated category \( \mathcal{T} \), and let \( \Omega := \bigcup_{i \in \mathbb{Z}} \omega[i] \). It is said that \( \omega \) is a cogenerator in \( \mathcal{T} \), if \( \Omega^\perp = \{0\} \). Dually, \( \omega \) is a generator in \( \mathcal{T} \), if \( \Omega^\perp = \{0\} \).

**Remark 4.14.** Let \( \omega \) be a class of objects of the triangulated category \( \mathcal{T} \). So, by induction and using the definition of \( \overline{\mathcal{T}}(\omega) \), it can be seen that \( \omega \) is both a generator and a cogenerator in the triangulated category \( \overline{\mathcal{T}}(\omega) \).

**Proposition 4.15.** Let \( \mathcal{X} = \text{add}(\mathcal{X}) \) be a cosuspended subcategory of \( \mathcal{T} \) and let \( \omega \) be an \( \mathcal{X} \)-injective weak-cogenerator in \( \mathcal{X} \). Then, \( \cap_{i \in \mathbb{Z}} \mathcal{X}[i] = \{0\} \) if and only if \( \omega \) is a cogenerator in \( \overline{\mathcal{T}}(\mathcal{X}) \).

**Proof.** First, by \([2.3\text{ ]}) we have that \( \overline{\mathcal{T}}(\mathcal{X}) = \mathcal{X}^\wedge = \bigcup_{n \geq 0} \mathcal{X}[n] \). We assert that \( \cap_{i \in \mathbb{Z}} \mathcal{X}[i] \subseteq \overline{\mathcal{T}}(\mathcal{X})^\perp \), where \( \Omega := \bigcup_{i \in \mathbb{Z}} \omega[i] \). Indeed, let \( M \in \cap_{i \in \mathbb{Z}} \mathcal{X}[i] \) and \( j \in \mathbb{Z} \). Hence \( M = X[j - 1] \) for some \( X \in \mathcal{X} \), and so \( \text{Hom}(M, W[j]) \simeq \text{Hom}(X, W[1]) = 0 \) for any \( W \in \omega \), proving the assertion.

Assume that \( \omega \) is a cogenerator in \( \overline{\mathcal{T}}(\mathcal{X}) \). Hence \( \Omega^\perp \cap \overline{\mathcal{T}}(\mathcal{X}) = \{0\} \) and by the assertion above, it follows that \( \cap_{i \in \mathbb{Z}} \mathcal{X}[i] = \{0\} \).

Suppose now that \( \cap_{i \in \mathbb{Z}} \mathcal{X}[i] = \{0\} \). Let \( Y \in \overline{\mathcal{T}}(\mathcal{X}) \) be non-zero. We prove the existence of an integer \( \ell \) such that \( \text{Hom}(Y, \omega[\ell]) \neq 0 \). Indeed, since \( \overline{\mathcal{T}}(\mathcal{X}) = \bigcup_{i \geq 0} \mathcal{X}[n] \), there is \( n \in \mathbb{N} \) with \( Y = X[n] \) for some \( X \in \mathcal{X} \). Furthermore, using that \( X[n] = X[n - i][i] \) and the fact that \( \mathcal{X} \) is cosuspended, it follows that \( Y \in \mathcal{X}[i] \) for any \( i \geq n \). On the other hand, since \( \cap_{j \in \mathbb{Z}} \mathcal{X}[j] = \{0\} \), we have that there is some \( j_0 < n \) such that \( Y \notin \mathcal{X}[j_0] \). We assert that \( Y \notin \mathcal{X}[i] \) for any \( i \leq j_0 \). It follows from \( \mathcal{X}[i] = \mathcal{X}[i - j_o][j_o] \subseteq \mathcal{X}[j_0] \) and \( Y \notin \mathcal{X}[j_0] \). Now, we set \( \ell := \min \{ s : j_0 < s \leq n \text{ and } Y \in \mathcal{X}[s] \} \). So we have \( Y[-\ell] \in \mathcal{X} \) and then, by using that \( \omega \) is a weak-cogenerator in \( \mathcal{X} \), there exists a distinguished triangle \( X'[-1] \to Y[-\ell] \xrightarrow{f} W \to X' \) with \( X' \in \mathcal{X} \) and \( W \in \omega \).

Hence, the morphism \( f : Y[-\ell] \to W \) is non-zero. In fact if \( f = 0 \), then \( Y[-\ell] \) would be a direct summand of \( X'[-1] \subseteq \mathcal{X}[-1] \), and so \( Y[-\ell + 1] \in \mathcal{X} \); giving a contradiction since \( Y \notin \mathcal{X}[\ell - 1] \). Thus \( \text{Hom}(Y, W[\ell]) \neq 0 \); proving the result. □
Definition 4.16. Let \((A, B)\) be a co-t-structure on \(T\). It said that the pair \((A, B)\) is **non-degenerate below** (respectively, **non-degenerate above**) if \(\cap_{i \in \mathbb{Z}} A[i] = \{0\}\) (respectively, \(\cap_{i \in \mathbb{Z}} B[i] = \{0\}\)). So, it is said that \((A, B)\) is **non-degenerate** if it is both non-degenerate below and above.

Proposition 4.17. Let \((\mathcal{X}, \mathcal{Y})\) be a bounded below co-t-structure on a triangulated category \(T\), and let \(\omega := \mathcal{X} \cap \mathcal{Y}\). Then, the following conditions are equivalent.

(a) \((\mathcal{X}, \mathcal{Y})\) is non-degenerate below.
(b) \(\omega\) is a cogenerator in \(T\).

Proof. It follows from 4.15, 3.4 (a) and 2.5 (c). \(\Box\)

Corollary 4.18. Let \((\mathcal{X}, \mathcal{Y})\) be a bounded co-t-structure on a triangulated category \(T\), and let \(\omega := \mathcal{X} \cap \mathcal{Y}\). Then, the following conditions are equivalent.

(a) \((\mathcal{X}, \mathcal{Y})\) is non-degenerate.
(b) \(\omega\) is both a generator and a cogenerator in \(T\).

Proof. It follows from 4.17 and its dual. \(\Box\)

Corollary 4.19. There is a bijective correspondence \(\mathcal{X} \mapsto (\mathcal{X}, \Xi_T(\mathcal{X}) \cap \mathcal{X}^\perp[-1])\) between cosuspended subcategories \(\mathcal{X} = \text{add}(\mathcal{X})\) of \(T\) such that \(\mathcal{X} \cap \mathcal{X}^\perp[-1]\) is both a weak-cogenerator in \(\mathcal{X}\) and a cogenerator in \(\Xi_T(\mathcal{X})\), and non-degenerate below co-t-structures \((\mathcal{X}, \mathcal{Y})\) on \(\Xi_T(\mathcal{X})\).

Proof. From 3.12, co-t-structures \((\mathcal{X}, \mathcal{Y})\) on \(\Xi_T(\mathcal{X})\) correspond bijectively to cosuspended subcategories \(\mathcal{X}\) of \(T\) such that \(\mathcal{X} \cap \mathcal{X}^\perp[-1]\) is a weak-cogenerator in \(\mathcal{X}\). Therefore, the result follows from 4.15 and 4.17. \(\Box\)

The relationship between the different types of co-t-structures is as follows.

Theorem 4.20. Let \((\mathcal{X}, \mathcal{Y})\) be a co-t-structure on a triangulated category \(T\). Then, the following statements are equivalent.

(a) \((\mathcal{X}, \mathcal{Y})\) is bounded.
(b) \((\mathcal{X}, \mathcal{Y})\) is faithful.
(c) \((\mathcal{X}, \mathcal{Y})\) is bounded and non-degenerate.
(d) \(T = \Xi_T(\mathcal{X} \cap \mathcal{Y})\).

Proof. (a) \(\Leftrightarrow\) (b) It is 4.19

(a) \(\Rightarrow\) (c) Assume that \((\mathcal{X}, \mathcal{Y})\) is bounded. Thus, by 4.2 and 4.8 (b), we get that \(T = \Xi_T(\omega)\) for \(\omega := \mathcal{X} \cap \mathcal{Y}\); and so by 4.14 we have that \(\omega\) is both a cogenerator and a generator in \(T\). Then, (c) follows from 4.18.

(c) \(\Rightarrow\) (d) It follows from 4.10 (a).

(d) \(\Rightarrow\) (a) Let \(T = \Xi_T(\mathcal{X} \cap \mathcal{Y})\). Hence we get \(T = \Xi_T(\mathcal{X}) = \Xi_T(\mathcal{Y})\).
Therefore, by 2.3 and 2.6 we get that \((\mathcal{X}, \mathcal{Y})\) is bounded. \(\Box\)
5. siltings and co-t-structures

In this section, we show that in many cases a co-t-structure can be determined by a silting set. We also study the relationship between co-t-structures, silting and relative injective classes. Following [13], we recall the notion of a silting class in triangulated categories.

**Definition 5.1.** Let $\omega$ be a class of objects in $\mathcal{T}$. It is said that $\omega$ is silting if $\text{id}_\omega(\omega) = 0$.

We denote by $\omega \mathcal{U}$ (respectively, $\mathcal{U}_\omega$) the smallest cosuspended (respectively, suspended) subcategory of $\mathcal{T}$, closed under direct summands and containing $\omega$.

**Remark 5.2.** Let $\omega$ be a class of objects in $\mathcal{T}$. We define a sequence $\{\varepsilon_i^{-}(\omega)\}_{i \geq 0}$ of classes of objects of $\mathcal{T}$ as follows. Set $\varepsilon_0^{-}(\omega) := \text{add} (\bigcup_{i \leq 0} \omega[i])$. Assume that $\varepsilon_i^{-}(\omega), \varepsilon_{i-1}^{-}(\omega), \ldots, \varepsilon_0^{-}(\omega)$ are already defined. Then, we define $\varepsilon_i^{-}(\omega)$ as the class of objects in $\mathcal{T}$, which are direct summands of objects in $\varepsilon_{i-1}^{-}(\omega) * \varepsilon_0^{-}(\omega)$. It is not hard to show that $\omega \mathcal{U} = \bigcup_{i \geq 0} \varepsilon_i^{-}(\omega)$.

**Lemma 5.3.** Let $(\mathcal{X}, \omega)$ be a pair of classes of objects in $\mathcal{T}$, such that $\omega \subseteq \mathcal{X}$. Then, the following statements hold.

(a) If $\mathcal{X}$ is cosuspended and $\mathcal{X} = \text{add}(\mathcal{X})$, then $\mathcal{X}[-1] * \omega$ is closed under direct summands.

(b) If $\omega$ is silting and closed under direct sums, then $\omega$ is closed under extensions.

**Proof.** (a) Assume that $\mathcal{X}$ is cosuspended and closed under direct summands. Let $C \in \mathcal{X}[-1] * \omega$. Then, there is a distinguished triangle $X[-1] \rightarrow C \rightarrow W \rightarrow X$ where $X \in \mathcal{X}$ and $W \in \omega$. Let $Z$ be a direct summand of $C$, hence there is distinguished triangle $Z \rightarrow C \rightarrow Z'[1]$, which splits. Using the octahedral axiom, we get distinguished triangles $\Delta_1 : Z \rightarrow C' \rightarrow V \rightarrow Z[1]$ and $\Delta_2 : Z' \rightarrow V \rightarrow X \rightarrow Z'[1]$. By the hypothesis, we have that $\mathcal{X}[-1] * \omega \subseteq \mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$; and so $C \in \mathcal{X}$, giving us that $Z$ and $Z'$ belong to $\mathcal{X}$. Thus $V \in \mathcal{X}$ (see $\Delta_2$), and hence from $\Delta_1$, we get that $Z \in \mathcal{X}[-1] * \omega$.

(b) Assume that $\omega$ is silting and closed under direct sums. Let $\Delta : W \rightarrow X \rightarrow W' \rightarrow W'[1]$ be a distinguished triangle with $W, W' \in \omega$. Using that $\text{id}_\omega(\omega) = 0$, we obtain that the triangle $\Delta$ splits; and hence $X \in \omega$ since $\omega$ is closed under direct sums. □

**Proposition 5.4.** Let $\omega$ be a silting class in $\mathcal{T}$ such that $\text{add}(\omega) = \omega$. Then $\omega$ is an $\omega \mathcal{U}$-injective weak-cogenerator in $\omega \mathcal{U}$.

**Proof.** From 5.2, we know that $\omega \mathcal{U} = \bigcup_{n \geq 0} \varepsilon_n^{-}(\omega)$. Hence, it is enough to prove, by induction on $n$, that $\varepsilon_n^{-}(\omega) \subseteq \omega \mathcal{U}[-1] * \omega$ for any $n \in \mathbb{N}$. Assume
that add \((\omega) = \omega\). In particular, we have that \( \varepsilon_0^- (\omega) = \bigoplus_{i \leq 0} \omega[i] \), where direct sums means here finite direct sums.

If \( X \in \varepsilon_0^- (\omega) \), then there is a split distinguished triangle \( W' \to X \to W \to W'[1] \), where \( W' \in \bigoplus_{i < 0} \omega[i] \) and \( W \in \omega \). Hence \( X \in \omega \mathcal{U}[-1] \ast \omega \).

Let \( n > 1 \), and take \( X \in \varepsilon_n^- (\omega) \). Then, there is a distinguished triangle \( X_{n-1} \to X' \to X_0 \to X_{n-1}[1] \) with \( X_0 \in \varepsilon_0^- (\omega) \), \( X_{n-1} \in \varepsilon_{n-1}^- (\omega) \) and \( X \) is a direct summand of \( X' \). For \( X_0 \) we have an split distinguished triangle \( W' \to X_0 \to X \to W'[,1] \), where \( W' \in \bigoplus_{i < 0} \omega[i] \) and \( W \in \omega \). Therefore, by the base change argument (using the octahedral axiom), we get the following commutative and exact diagram in \( \mathcal{T} \)

\[
\begin{array}{ccc}
W[-1] & \longrightarrow & W[-1] \\
\downarrow & & \downarrow \\
X_{n-1} & \longrightarrow & Y \longrightarrow W' \longrightarrow X_{n-1}[1] \\
\downarrow & & \downarrow & & \downarrow \\
X_{n-1} & \longrightarrow & X' \longrightarrow X_0 \longrightarrow X_{n-1}[1] \\
\downarrow g & & \downarrow f & & \downarrow \\
W & \longrightarrow & W
\end{array}
\]

By induction there exist a distinguished triangle \( U[-1] \to X_{n-1} \to U \to W'' \) where \( U \in \omega \mathcal{U} \) and \( W'' \in \omega \). Since \( \text{Hom}(\bigoplus_{i < 0} \omega[i], \omega[1]) = 0 \) because \( \omega \) is silting, we have a morphism \( \alpha : W' \to U \) that can be completed to a distinguished triangle \( W' \to U \to V \to W'[1] \). By using the octahedral axiom, we get the following exact and commutative diagram in \( \mathcal{T} \)

\[
\begin{array}{ccc}
U[-1] & \longrightarrow & V[-1] \longrightarrow W' \longrightarrow U \\
\downarrow & & \downarrow & & \downarrow \\
X_{n-1} & \longrightarrow & Y \longrightarrow W' \longrightarrow X_{n-1}[1] \\
\downarrow h & & \downarrow & & \downarrow h[1] \\
W'' & \longrightarrow & W'' \longrightarrow 0 \longrightarrow W''[1] \\
\downarrow & & \downarrow & & \downarrow \\
U & \longrightarrow & V
\end{array}
\]
Corollary 5.8. Let $X_t : Y \to W' \to V$, it follows that $V[-1] \in \omega \mathcal{U}[-1]$ since $\omega \mathcal{U}[-1]$ is closed under extensions. Now, the triangle $V[-1] \to Y \to W'$ implies that $Y \in \omega \mathcal{U}[-1] \ast \omega$. Then $X' \in \omega \mathcal{U}[-1] \ast \omega \ast \omega$ since we have the triangle $W[-1] \to Y \to X' \ast \mathcal{W}$. But $\omega \ast \omega \in \omega$ (see 5.3 (b)), and so $X' \in \omega \mathcal{U}[-1] \ast \omega \ast \omega \subseteq \omega \mathcal{U}[-1] \ast \omega$. Therefore, from 5.3 (a), we conclude that $X \in \omega \mathcal{U}[-1] \ast \omega$; hence $X$ is a weak-cogenerator in $\omega \mathcal{U}$. Finally, we prove that $\mathcal{U}$ is also $\omega \mathcal{U}$-injective. Indeed, since $\text{id}_\omega(\omega) = 0$ it follows from Lemma 4.2 (a2) that $\omega \subseteq \omega \mathcal{U}^{[-1]}$; and using that $\omega \mathcal{U}^{[-1]} = \omega \mathcal{U} \mathcal{U}^{[-1]}$, we get by Lemma 4.2 (a2) that $\text{id}_{\omega \mathcal{U}}(\omega) = 0$. □

The following result is very similar to Theorem 4.3.2(II), which was proved with different techniques.

Theorem 5.5. Let $\omega$ be a silting class in $\mathcal{T}$ such that $\omega = \text{add}(\omega)$. Then, $\omega = \omega \mathcal{U} \cap \mathcal{U}_\omega$ and the pair $(\omega \mathcal{U}, \mathcal{U}_\omega)$ is a bounded co-$\mathcal{T}$-structure on $\mathcal{U}_\omega$.

Proof. Since $\Sigma_{\mathcal{T}}(\omega) = \Sigma_{\mathcal{T}}(\omega \mathcal{U})$, it follows from 5.5 that $\Sigma_{\mathcal{T}}(\omega) = \omega \mathcal{U}^{\omega}$. On the other hand, by 5.3 and 4.11 (a), we get that the pair $(\omega \mathcal{U}, \omega^{\mathcal{U}})$ is a co-$\mathcal{T}$-structure on $\mathcal{U}_\omega = \omega \mathcal{U}$ and $\omega = \omega \mathcal{U} \cap \omega^{\mathcal{U}}$. In particular, from 4.5 (a), it follows that $\mathcal{U}_\omega = \omega^{\mathcal{U}}$ and hence $\mathcal{U}_\omega = \mathcal{U}_\omega$. Therefore, the pair $(\omega \mathcal{U}, \mathcal{U}_\omega)$ is a bounded below and faithful below co-$\mathcal{T}$-structure on $\mathcal{U}_\omega$, and $\omega = \omega \mathcal{U} \cap \mathcal{U}_\omega$. So, from 4.5 we get that $(\omega \mathcal{U}, \mathcal{U}_\omega)$ is bounded on $\mathcal{U}_\omega$. Furthermore, by 4.10 (a), we obtain that $\omega \mathcal{U} = \omega \mathcal{U}$ and $\mathcal{U}_\omega = \mathcal{U}_\omega$. □

Remark 5.6. Let $\omega$ be a silting class in $\mathcal{T}$ such that $\omega = \text{add}(\omega)$. Then, by 5.3 it follows that $\omega \mathcal{U} = \omega \mathcal{U}$ and $\mathcal{U}_\omega = \mathcal{U}_\omega$.

Definition 5.7. For a given triangulated category $\mathcal{T}$, we introduce the following classes:

(a) $\mathcal{S}$ consists of all silting classes $\omega$ of $\mathcal{T}$ such that $\text{add}(\omega) = \omega$.

(b) $\mathcal{C}_b$ consists of all bounded co-$\mathcal{T}$-structures $(\mathcal{X}, \mathcal{Y})$ on $\Sigma_{\mathcal{T}}(\mathcal{X} \cap \mathcal{Y})$.

Corollary 5.8. Let $\mathcal{T}$ be a triangulated category. Then, the correspondence $\varphi : \mathcal{S} \to \mathcal{C}_b$, given by $\varphi(\omega) := (\omega \mathcal{U}, \mathcal{U}_\omega)$, is bijective.

Proof. From 5.3 it follows that $\varphi : \mathcal{S} \to \mathcal{C}_b$ is well defined and injective. Let $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}_b$, and consider $\omega := \mathcal{X} \cap \mathcal{Y}$. Since $(\mathcal{X}, \mathcal{Y})$ is a bounded co-$\mathcal{T}$-structure on $\Sigma_{\mathcal{T}}(\omega)$, we conclude by 4.10 (a) that $\varphi(\omega) = (\mathcal{X}, \mathcal{Y})$; proving that $\varphi$ is also surjective. □

Corollary 5.9. Let $\mathcal{T}$ be a triangulated category. Then, there is a bijective correspondence $(\mathcal{X}, \mathcal{Y}) \mapsto \omega := \mathcal{X} \cap \mathcal{Y}$, with inverse $\omega \mapsto (\omega \mathcal{U}, \mathcal{U}_\omega)$, between bounded co-$\mathcal{T}$-structures $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{T}$ and silting classes $\omega = \text{add}(\omega)$ such that $\mathcal{T} = \Sigma_{\mathcal{T}}(\omega)$.

Proof. It follows from 5.8 and 4.20 □

The next result characterizes when a cosuspended subcategory of $\mathcal{T}$ determines a bounded co-$\mathcal{T}$-structure on $\mathcal{T}$. 
Theorem 5.10. Let $\mathcal{T}$ be a triangulated category, and $\mathcal{X}$ be a cosuspended subcategory of $\mathcal{T}$ such that $\mathcal{X} = \text{add}(\mathcal{X})$. Then, the following statements are equivalent.

(a) There is a bounded co-t-structure $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{T}$.

(b) $\overline{\mathcal{X}}(\mathcal{X} \cap \mathcal{X}^\perp[-1]) = \mathcal{T}$.

(c) There is an $\mathcal{X}$-injective $\omega = \text{add}(\omega)$ such that $\overline{\mathcal{X}}(\omega) = \mathcal{T}$ and $\omega \subseteq \mathcal{X}$.

(d) There is a silting $\omega = \text{add}(\omega)$ such that $\overline{\mathcal{X}}(\omega) = \mathcal{T}$ and $\omega \subseteq \mathcal{X} \subseteq \omega^\perp$.

Moreover, if one of the above conditions hold, we have that $\mathcal{X} = \omega \cup \omega = \omega^\perp$, $\mathcal{Y} = \omega \cap \mathcal{Y} = \mathcal{X} \cap \mathcal{X}^\perp[-1]$.

Proof. (a) $\Rightarrow$ (d) Assume that $(\mathcal{X}, \mathcal{Y})$ is a bounded co-t-structure on $\mathcal{T}$, and let $\omega = \mathcal{X} \cap \mathcal{Y}$. Then, by 5.9 we get that $\omega = \text{add}(\omega)$ is a silting class such that $\overline{\mathcal{X}}(\omega) = \mathcal{T}$. On the other hand, since $(\mathcal{X}, \mathcal{Y})$ is bounded, it follows from 4.10 (a) that $\mathcal{X} = \omega \cup \omega = \omega^\perp$.

(d) $\Rightarrow$ (a) Suppose there is a silting class $\omega$ such that $\omega \subseteq \mathcal{X} \subseteq \omega^\perp$ and $\overline{\mathcal{X}}(\omega) = \mathcal{T}$. Hence, by 5.5, it follows that $\omega \cup \omega$ is a bounded co-t-structure on $\mathcal{T}$, and $\omega = \omega \cap \omega$. In particular, from 4.10 (a), we know that $\omega \cup \omega = \omega^\perp$. Furthermore, since $\omega \subseteq \mathcal{X}$, it follows that $\omega \cup \omega \subseteq \mathcal{X}$ and hence $\mathcal{X} = \omega \cup \omega$.

(a) $\Rightarrow$ (b) Let $(\mathcal{X}, \mathcal{Y})$ be bounded. Then, by 4.10 (b), we get that $\overline{\mathcal{X}}(\omega) = \mathcal{T}$, where $\omega := \mathcal{X} \cap \mathcal{Y} = \mathcal{X} \cap \mathcal{X}^\perp[-1]$ (see 3.2 (c)).

(b) $\Rightarrow$ (c) Let $\overline{\mathcal{X}}(\mathcal{X} \cap \mathcal{X}^\perp[-1]) = \mathcal{T}$. Since $\mathcal{X}$ is a cosuspended subcategory of $\mathcal{T}$, it follows from [14, Lemma 4.2 (a2)], that $\omega := \mathcal{X} \cap \mathcal{X}^\perp[-1]$ is $\mathcal{X}$-injective.

(c) $\Rightarrow$ (a) Assume the hypothesis in (c). In particular, $\omega$ is silting since $\text{id}_\mathcal{X}(\omega) = 0$. Thus, from 5.9, it follows that $\omega \cup \omega$ is a bounded co-t-structure on $\mathcal{T}$ and also that $\omega = \omega \cap \omega$. Furthermore $\omega \cup \omega \subseteq \mathcal{X}$ since $\omega \subseteq \mathcal{X}$. On the other hand, since $\text{pd}_\omega(\mathcal{X}) = \text{id}_\mathcal{X}(\omega) = 0$, it follows from [14, Lemma 4.2 (a1)], that $\mathcal{X} \subseteq \omega \cup \omega = \omega \cup \omega$ (see 3.2 (c)); and hence $\mathcal{X} = \omega \cup \omega$. \qed

6. co-t-structures on $\mathbf{D}^b(\mathcal{H})$

Throughout this section, $k$ denotes an algebraically closed field and $\mathcal{H}$ an abelian hereditary $k$-category which is Hom-finite, Ext-finite and has a tilting object. We will consider the bounded derived category $\mathbf{D}^b(\mathcal{H})$ which is triangulated and has been intensively studied (see, for example, [9] and [10]).

In this section, we give a description of the bounded co-t-structures on $\mathcal{T} := \mathbf{D}^b(\mathcal{H})$. In this case, the obtained results take a more complete form that in the preceding section.

In what follows, we need the following useful lemma. For details, we refer the reader to [3].
Lemma 6.1. \[3\] Let \( \omega \) be a set in the triangulated category \( \text{D}^b(\mathcal{H}) \). Then, the following statements hold.

(a) \( \omega \) is a generator in \( \text{D}^b(\mathcal{H}) \) if and only if it is a cogenerator in \( \text{D}^b(\mathcal{H}) \).
(b) Let \( \omega \) be a silting class in \( \text{D}^b(\mathcal{H}) \). Then, \( \omega \) is a generator in \( \text{D}^b(\mathcal{H}) \) if and only if \( \Delta_{\text{D}^b(\mathcal{H})}(\omega) = \text{D}^b(\mathcal{H}) \).

Proof. (a) It follows from \[3\] Lemma 2.1 since \( \text{D}^b(\mathcal{H}) \) has a Serre duality.
(b) \( \Rightarrow \) By \[3\] Corollary 3.2 (b)], we know that, for every complex \( X \in \mathcal{T} \), there is a distinguished triangle \( W \to X \to L \to W[1] \) such that \( W \in \Delta_{\text{D}^b(\mathcal{H})}(\omega) \) and \( L \in \Delta_{\text{D}^b(\mathcal{H})}(\omega)^\perp \). If \( \omega \) is a generator then \( L = 0 \) and so \( X \simeq W \in \Delta_{\mathcal{T}}(\omega) \), proving that \( \Delta_{\text{D}^b(\mathcal{H})}(\omega) = \text{D}^b(\mathcal{H}) \). \( \square \)

Proposition 6.2. Let \((\mathcal{X}, \mathcal{Y})\) be a co-t-structure on \( \text{D}^b(\mathcal{H}) \). Then, the following statements are equivalent.

(a) \((\mathcal{X}, \mathcal{Y})\) is bounded.
(b) \((\mathcal{X}, \mathcal{Y})\) is non-degenerate below and bounded below.
(c) \((\mathcal{X}, \mathcal{Y})\) is non-degenerate above and bounded above.

Proof. (a) \( \Rightarrow \) (b) It follows from \[4\] Proposition 4.20.
(b) \( \Rightarrow \) (a) Assume the hypothesis in (b). Then, by \[4\] Proposition 4.17, we get that \( \omega \) is a cogenerator in \( \text{D}^b(\mathcal{H}) \). Hence, from 6.1 and 4.20, we conclude that \((\mathcal{X}, \mathcal{Y})\) is bounded.

Finally, the equivalence between (c) and (a), can be proven in a similar way we did for (a) and (b). \( \square \)

The following result gives a characterization, in terms of generators and cogenerators, of the bounded co-t-structures on \( \text{D}^b(\mathcal{H}) \).

Theorem 6.3. Let \( \mathcal{X} \) be a cosuspended subcategory of \( \text{D}^b(\mathcal{H}) \) such that \( \mathcal{X} = \text{add}(\mathcal{X}) \). Then, the following statements are equivalent.

(a) There is a bounded co-t-structure \((\mathcal{X}, \mathcal{Y})\) on \( \text{D}^b(\mathcal{H}) \).
(b) \( \mathcal{X} \cap \mathcal{X}^\perp[-1] \) is a generator set in \( \text{D}^b(\mathcal{H}) \).
(c) There is an \( \mathcal{X} \)-injective set \( \omega = \text{add}(\omega) \), which is a cogenerator in \( \text{D}^b(\mathcal{H}) \) and \( \omega \subseteq \mathcal{X} \).

Proof. Since \( \mathcal{X} \) is cosuspended, it follows by \[13\] Lemma 4.2 (a2)] that \( \mathcal{X} \cap \mathcal{X}^\perp[-1] \) is \( \mathcal{X} \)-injective; and hence, it is silting. Therefore, the result follows from \[5\] Proposition 6.10 and \[6\]. \( \square \)

Corollary 6.4. Let \( \mathcal{X} \) be a cosuspended subcategory of \( \text{D}^b(\mathcal{H}) \) such that \( \mathcal{X} = \text{add}(\mathcal{X}) \). If \( \mathcal{X} \cap \mathcal{X}^\perp[-1] \) is a generator set in \( \mathcal{T} \), then \( \mathcal{X}^\wedge = \mathcal{T} \) and \( \mathcal{X} \) is a precovering class in \( \mathcal{T} \).

Proof. It follows from \[6\] Proposition 6.3 (a) and \[3\] Proposition 3.11 (b). \( \square \)
Corollary 6.5. Let \((\mathcal{X}, \omega)\) be a pair of classes of objects of \(D^b(\mathcal{H})\), which are closed under direct summands, and let \(\mathcal{X}\) be cosuspended. Then, the following conditions are equivalent.

(a) \(\omega\) is an \(\mathcal{X}\)-injective weak-cogenerator in \(\mathcal{X}\); \(\bigcap_{i \in \mathbb{Z}} \mathcal{X}[i] = \{0\}\) and \(\mathcal{X}^\perp = D^b(\mathcal{H})\).
(b) \(\omega \subseteq \mathcal{X} \subseteq \mathcal{X}^\perp \) and \(\omega = \text{add} (\omega)\) is a silting cogenerator set in \(D^b(\mathcal{H})\).
(c) \(\omega = \text{add} (\omega) \subseteq \mathcal{X}\) and \(\omega\) is an \(\mathcal{X}\)-injective cogenerator set in \(D^b(\mathcal{H})\).
(d) \(\omega = \mathcal{X} \cap \mathcal{X}^\perp [-1]\) and \(\omega\) is a generator set in \(D^b(\mathcal{H})\).

Moreover, if one of the above conditions hold, we have that \(\mathcal{X} = \omega \mathcal{U} = \omega^\perp\).

Proof. (a) \(\Rightarrow\) (b) By \ref{thm:6.4} (a) and \ref{thm:2.4}, there is a bounded below co-t-structure \((\mathcal{X}, \mathcal{Y})\) on \(D^b(\mathcal{H})\). Furthermore, by \ref{thm:4.13}, we get that \(\omega\) is a cogenerator set in \(D^b(\mathcal{H})\); and so, by \ref{thm:6.1} and \ref{thm:4.20}, it follows that \((\mathcal{X}, \mathcal{Y})\) is bounded. Hence (b) follows from \ref{thm:6.10} (d).

(b) \(\Rightarrow\) (a) By \ref{thm:6.1} it follows that \(\Sigma_{D^b(\mathcal{H})}(\omega) = D^b(\mathcal{H})\). Therefore, the condition (d) in \ref{thm:5.10} holds. Hence (a) follows by \ref{thm:3.11} (a) and \ref{thm:4.15}.

(b) \(\Leftrightarrow\) (c) It follows from \ref{thm:6.1}, \ref{thm:5.10} and \ref{thm:6.3}.

(a) \(\Leftrightarrow\) (d) It follows from \ref{thm:6.3}, \ref{thm:3.4}, \ref{thm:6.1} and \ref{thm:4.15}.

Let \(\omega\) be a class of objects of \(D^b(\mathcal{H})\). We say that \(\omega\) is of finite type if there exist a finite number of pairwise non isomorphic indecomposable objects \(W_1, W_2, \cdots, W_n\) in \(D^b(\mathcal{H})\) satisfying that \(\text{add}(\omega) = \text{add}(\{W_1, W_2, \cdots, W_n\})\).

In such a case, we set \(\text{ind}(\omega) := \{W_1, W_2, \cdots, W_n\}\) and \(\text{rk}(\omega) := n\). We also denote by \(\text{rk}_{K_0}(\mathcal{H})\) the rank of the Grothendieck group associated with \(\mathcal{H}\).

Lemma 6.6. \cite{3} The following statements holds.

(a) If \(\omega\) is a silting set in \(D^b(\mathcal{H})\), then \(\text{rk}(\omega) \leq \text{rk}_{K_0}(\mathcal{H})\).
(b) Let \(\mathcal{Y} = \text{add}(\mathcal{Y})\) be a suspended and precovering subcategory of \(D^b(\mathcal{H})\), and let \(\omega := \mathcal{Y} \cap \mathcal{Y}^\perp[1]\). Then, \(\text{rk}(\omega) = \text{rk}_{K_0}(\mathcal{H})\) if and only if \(\omega\) is a generator in \(D^b(\mathcal{H})\). Furthermore, if this is the case, then \(\mathcal{Y} = \mathcal{U}_\omega = \mathcal{U}_{\omega}\).

Proof. (a) By \ref{thm:5.4}, we know that \(\omega\) is \(\omega\mathcal{U}\)-injective. Therefore, the item (a) is just the dual of \cite{3} Theorem 2.3 (b)]).

(b) This is \cite{3} Corollary 4.4. Observe that the equality \(\mathcal{U}_\omega = \mathcal{U}_{\omega}\) follows from \ref{thm:5.6}.

Theorem 6.7. There are bijective correspondences

\[(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{Y}, \quad \mathcal{Y} \mapsto \omega := \mathcal{Y} \cap \mathcal{Y}^\perp[1]\] and \(\omega \mapsto (\omega \mathcal{U}, \mathcal{U}_\omega)\)

between the following classes:

(a) Bounded co-t-structures \((\mathcal{X}, \mathcal{Y})\) on \(D^b(\mathcal{H})\).
(b) Suspended and precovering subcategories \(\mathcal{Y} = \text{add}(\mathcal{Y})\) of \(D^b(\mathcal{H})\) such that \(\text{rk}(\mathcal{Y} \cap \mathcal{Y}^\perp[1]) = \text{rk}_{K_0}(\mathcal{H})\).
(c) Silting sets \(\omega = \text{add}(\omega)\) in \(D^b(\mathcal{H})\) such that \(\text{rk}(\omega) = \text{rk}_{K_0}(\mathcal{H})\).
Proof. By [3] Corollary 4.5 and [6.6] (b), we have that the correspondence \( \mathcal{Y} \mapsto \mathcal{Y} \cap \perp \mathcal{Y}[1] \) between the classes of items (b) and (c) is bijective with inverse \( \omega \mapsto \mathcal{U} \). We prove now that the correspondence \( \mathcal{X} \mapsto \mathcal{X} \cap \mathcal{Y} \) between the classes of items (a) and (c) is bijective with inverse \( \omega \mapsto (\omega, \mathcal{U}, \mathcal{U}_\omega) \). Indeed, let \( \mathcal{X} \) be a pair belonging to item (a). By [5.10] and [6.3] it follows that \( \mathcal{X} \cap \mathcal{Y} = \mathcal{Y} = \mathcal{U}_\omega \mathcal{Y} \). Hence, by applying [3] Corollary 3.2 (b), we get that \( \mathcal{Y} \) is a bounded and precovering subcategory of \( D^b(\mathcal{H}) \). Therefore, from [6.6] we get that \( \mathcal{X} \cap \mathcal{Y} \) belongs to the item (c). Furthermore, from [5.11] we conclude that \( \beta(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}, \mathcal{Y}) \). Let \( \omega \) be a class belonging to the item (c). In particular, by [5.5] we have that \( \beta(\omega) = (\omega, \mathcal{U}, \mathcal{U}_\omega) \) is a bounded non-degenerate co-t-structure on \( D^b(\mathcal{H}) \) and \( \omega = \mathcal{U} \cap \mathcal{U}_\omega = \alpha \beta(\omega) \). But, using the bijective correspondence between the classes of items (b) and (c), we get that \( \mathcal{U}_\omega \) is a bounded and precovering subcategory of \( D^b(\mathcal{H}) \). Therefore, from [6.6] we obtain that \( \omega \) is a generator in \( D^b(\mathcal{H}) \); and so \( \sum(\mathcal{U}_\omega) = \sum(D^b(\mathcal{H})) \) (see [6.1]), proving that \( (\omega, \mathcal{U}, \mathcal{U}_\omega) \) is a bounded co-t-structure on \( D^b(\mathcal{H}) \). That is, \( \beta(\omega) \) belongs to the item (a). ❑

Remark 6.8. The item (b) in [6.7] is equivalent to the following one:

(b') Suspended subcategories \( \mathcal{Y} = \text{add}(\mathcal{Y}) \) of \( D^b(\mathcal{H}) \) such that \( \text{rk}(\mathcal{Y} \cap \perp \mathcal{Y}[1]) = \text{rk}K_0(\mathcal{H}) \). Moreover, if (b') holds, then we have that \( \omega := \mathcal{Y} \cap \perp \mathcal{Y}[1] \) is a generator set in \( D^b(\mathcal{H}) \) and \( \mathcal{Y} = \mathcal{U}_\omega \).

Proof. Let \( \mathcal{Y} = \text{add}(\mathcal{Y}) \) be a suspended subcategory of \( D^b(\mathcal{H}) \), and let \( \omega := \mathcal{Y} \cap \perp \mathcal{Y}[1] \) be such that \( \text{rk}(\omega) = \text{rk}K_0(\mathcal{H}) \). Then, from [6.7] (a), we have that \( (\omega, \mathcal{U}, \mathcal{U}_\omega) \) is a bounded co-t-structure on \( D^b(\mathcal{H}) \). Thus \( \sum(D^b(\mathcal{H}))(\omega) = D^b(\mathcal{H}) \) and \( \omega \) is a generator set in \( D^b(\mathcal{H}) \) (see [5.10]). In particular \( (\sum(D^b(\mathcal{H}))(\omega)) = \{0\} \); and therefore, from [3] Theorem 4.2 (b), we conclude that \( \mathcal{Y} = \mathcal{U}_\omega \). Finally, using [3] Corollary 3.2, we get that \( \mathcal{Y} \) is precovering in \( D^b(\mathcal{H}) \). ❑

Corollary 6.9. There are bijective correspondences

\[
(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X}, \quad \mathcal{X} \mapsto \omega := \mathcal{X} \cap \mathcal{X}^\perp[-1] \quad \text{and} \quad \omega \mapsto (\omega, \mathcal{U}, \mathcal{U}_\omega)
\]

between the following classes:

(a) Bounded co-t-structures \( (\mathcal{X}, \mathcal{Y}) \) on \( D^b(\mathcal{H}) \).
(b) Cosuspended and preenveloping subcategories \( \mathcal{X} = \text{add}(\mathcal{X}) \) of \( D^b(\mathcal{H}) \) such that \( \text{rk}(\mathcal{X} \cap \mathcal{X}^\perp[-1]) = \text{rk}K_0(\mathcal{H}) \).
(c) Silting sets \( \omega = \text{add}(\omega) \) in \( D^b(\mathcal{H}) \) such that \( \text{rk}(\omega) = \text{rk}K_0(\mathcal{H}) \).
(d) Cosuspended subcategories \( \mathcal{X} = \text{add}(\mathcal{X}) \) of \( D^b(\mathcal{H}) \) such that \( \text{rk}(\mathcal{X} \cap \mathcal{X}^\perp[-1]) = \text{rk}K_0(\mathcal{H}) \).

Proof. Let \( \mathcal{T} := D^b(\mathcal{H}) \). In order to prove the result, using [6.7] 6.8 and the duality principle for triangulated categories, it is enough to prove the following statement: if \( (\mathcal{X}, \mathcal{Y}) \) is a bounded co-t-structure on \( \mathcal{T} \), then \( (\mathcal{Y}^\perp, \mathcal{X}^\perp) \) is so on...
the opposite triangulated category $\mathcal{T}^{op}$. Observe, firstly, that this statement is true since the boundedness property is a self-dual notion; and secondly, $\mathcal{T}^{op} \cong \mathsf{D}^b(\mathcal{H}^{op})$ where $\mathcal{H}^{op}$ is also an abelian hereditary $k$-category which is Hom-finite, Ext-finite and has a tilting object (see [10 Proposition 1.9]). □

As a nice consequence, from 6.7 (b) and 6.9 (b), is the following corollary, saying that any bounded co-t-structure on $\mathsf{D}^b(\mathcal{H})$ has two companions as t-structures: one on the left and the other on the right. For the convenience of the reader, we recall the definition of t-structure.

**Definition 6.10.** [4] A pair $(\mathcal{A}, \mathcal{B})$ of subcategories in $\mathcal{T}$ is said to be a t-structure on $\mathcal{T}$ if the following conditions hold.

(a) $\mathcal{A}[1] \subseteq \mathcal{A}$ and $\mathcal{B}[1] \subseteq \mathcal{B}$.

(b) $\text{Hom}_\mathcal{T}(\mathcal{A}, \mathcal{B}[1]) = 0$.

(c) $\mathcal{T} = \mathcal{A} \ast \mathcal{B}[1]$.

**Corollary 6.11.** Let $(\mathcal{X}, \mathcal{Y})$ be a bounded co-t-structure on $\mathsf{D}^b(\mathcal{H})$. Then, the pairs $(\mathcal{X}[-1], \mathcal{X})$ and $(\mathcal{Y}, \mathcal{Y}^\perp[1])$ are both t-structures on $\mathsf{D}^b(\mathcal{H})$.

**Proof.** From 6.7 (b) and [13 Proposition 1.3], it follows that $\mathcal{Y}$ is an aisle in $\mathsf{D}^b(\mathcal{H})$. Thus $(\mathcal{Y}, \mathcal{Y}^\perp[1])$ is a t-structure on $\mathsf{D}^b(\mathcal{H})$. Furthermore, by 6.9 (b) and the dual of [13 Proposition 1.3], we get that $\mathcal{X}$ is a co-aisle in $\mathsf{D}^b(\mathcal{H})$, and so, $(\mathcal{X}[-1], \mathcal{X})$ is a t-structure on $\mathsf{D}^b(\mathcal{H})$. □

**Remark 6.12.** The previous result, says that a bounded co-t-structure on $\mathsf{D}^b(\mathcal{H})$ is always left (respectively, right) adjacent to a t-structure on $\mathsf{D}^b(\mathcal{H})$ in the sense of [6].

**Corollary 6.13.** Let $(\mathcal{X}, \mathcal{Y})$ be a bounded co-t-structure on $\mathsf{D}^b(\mathcal{H})$. Then $\mathcal{X}$ and $\mathcal{Y}$ are functorially finite in $\mathsf{D}^b(\mathcal{H})$.

**Proof.** It follows from 6.7, 6.9 and 6.12. □

**Corollary 6.14.** Let $\omega = \text{add} (\omega)$ be a silting generator set in $\mathsf{D}^b(\mathcal{H})$. Then $\omega \mathcal{U}$ and $\mathcal{U} \omega$ are functorially finite in $\mathsf{D}^b(\mathcal{H})$, $\omega \mathcal{U}^\perp = \mathsf{D}^b(\mathcal{H}) = \mathcal{U} \omega^\perp$ and $\text{rk} (\omega) = \text{rk} K_0 (\mathcal{H})$.

**Proof.** From 6.1 (b) and 6.6 we know that $(\omega \mathcal{U}, \mathcal{U} \omega)$ is a bounded co-t-structure on $\mathsf{D}^b(\mathcal{H})$. Hence the result follows from 6.13 and 6.7 (c). □

**Corollary 6.15.** Let $\omega$ be a silting set in $\mathsf{D}^b(\mathcal{H})$. Then, $\omega$ is a generator in $\mathsf{D}^b(\mathcal{H})$ if and only if $\text{rk} (\omega) = \text{rk} K_0 (\mathcal{H})$.

**Proof.** Consider $\omega' := \text{add} (\omega)$. Observe that $\omega := \text{add} (\omega')$ and $\omega'$ is also a silting set in $\mathsf{D}^b(\mathcal{H})$.

($\Rightarrow$) Suppose that $\omega$ is a generator in $\mathsf{D}^b(\mathcal{H})$; and hence $\omega'$ is so. Then by 6.14 we get $\text{rk} (\omega) = \text{rk} K_0 (\mathcal{H})$ since $\text{rk} (\omega) = \text{rk} (\omega')$.

($\Leftarrow$) Assume now that $\text{rk} (\omega) = \text{rk} K_0 (\mathcal{H})$. Thus $\text{rk} (\omega') = \text{rk} K_0 (\mathcal{H})$ and so from 6.7 it follows that $(\omega \mathcal{U}, \mathcal{U} \omega)$ is a bounded co-t-structure on $\mathsf{D}^b(\mathcal{H})$. 

Therefore by \(4.8\) and \(4.9\) we get that 
\[
\Delta_{D^b(H)}(\omega) = \Delta_{D^b(H)}(\omega') = D^b(H).
\]
Hence \(\omega\) is a generator in \(D^b(H)\) (see \(6.1\)).

**Corollary 6.16.** Let \(\mathcal{Y} = \text{add}(\mathcal{Y})\) be a suspended subcategory of \(D^b(H)\) and let \(\omega := \mathcal{Y} \cap \mathcal{Y}[1]\). If \(\text{rk}(\omega) = \text{rk} K_0(H)\), then \(\mathcal{Y}\) is functorially finite in \(D^b(H)\), \(\mathcal{Y} = \mathcal{U}_\omega\) and \(\mathcal{Y}^\perp = D^b(H)\).

**Proof.** Let \(\text{rk}(\omega) = \text{rk} K_0(H)\). Then, by \(6.8\) it follows that \(\omega\) is a generator set in \(D^b(H)\) and \(\mathcal{Y} = \mathcal{U}_\omega\). So the result now follows from \(6.14\). □

**Corollary 6.17.** Let \(\mathcal{X} = \text{add}(\mathcal{X})\) be a cosuspended subcategory of \(D^b(H)\) and let \(\omega := \mathcal{X} \cap \mathcal{X}^[-1]\). If \(\text{rk}(\omega) = \text{rk} K_0(H)\), then \(\mathcal{X}\) is functorially finite in \(D^b(H)\), \(\mathcal{X} = \mathcal{M} \) and \(\mathcal{X}^\wedge = D^b(H)\).

**Proof.** It follows from \(6.16\) and the discussion given in the proof of \(6.9\). □

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