Low-Temperature Thermodynamics

of $A_2^{(2)}$ and $su(3)$-invariant Spin Chains

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Abstract

We formulate the thermodynamic Bethe Ansatz (TBA) equations for the closed (periodic boundary conditions) $A_2^{(2)}$ quantum spin chain in an external magnetic field, in the (noncritical) regime where the anisotropy parameter $\eta$ is real. In the limit $\eta \to 0$, we recover the TBA equations of the antiferromagnetic $su(3)$-invariant chain in the fundamental representation. We solve these equations for low temperature and small field, and calculate the specific heat and magnetic susceptibility.

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1. Introduction and Summary

Given a quantum integrable lattice model in one space dimension, one can find the eigenvalues of the Hamiltonian in terms of solutions of the model’s Bethe Ansatz (BA) equations. (See, e.g., Ref. 1.) Unfortunately, having found the eigenvalues, one is still quite far from determining the model’s physical properties. The main reason for this is that the BA equations are in general very difficult to solve, in particular for \( N \) (the number of lattice sites) finite. Considerable simplification occurs in the thermodynamic \( (N \to \infty) \) limit. Provided one can formulate a suitable “string hypothesis” for the solutions of the BA equations, the problem is then to determine the densities \( \rho_n(\lambda) \) and \( \tilde{\rho}_n(\lambda) \) of quasi-particles and quasi-holes, respectively. In principle, this can be accomplished once one solves the so-called thermodynamic Bethe Ansatz (TBA) equations for the quantities \( \epsilon_n = T \ln (\tilde{\rho}_n/\rho_n) \). Since the TBA equations are an infinite set of coupled nonlinear integral equations, in practice one solves them perturbatively (e.g., near \( T = 0 \)).

This program\(^2\) has been successfully applied to a number of integrable lattice models. Foremost among these are integrable quantum spin chains – e.g., the spin \( 1/2 \) Heisenberg chain and its many generalizations. The large body of work\(^3\)–\(^{13}\) on the thermodynamics of quantum spin chains has had significant consequences for both quantum field theory and condensed matter physics. (For a recent introduction, see Ref. 14.)

In this paper, we focus on the closed (i.e., periodic boundary conditions) \( A_2^{(2)} \) chain, with Hamiltonian

\[
\mathcal{H} = \sum_{k=1}^{N-1} \mathcal{H}_{k,k+1} + \mathcal{H}_{N,1}, \quad \mathcal{H}_{k,k+1} = \frac{d}{du} \tilde{R}_{k,k+1}(u)\bigg|_{u=0}.
\] (1.1)

Here

\[
\tilde{R}(u) = \mathcal{P} R(u),
\] (1.2)

where \( \mathcal{P} \) is the permutation matrix, and \( R(u) \) is the \( R \)-matrix associated with the twisted
affine algebra $A_2^{(2)}$ in the fundamental representation, which depends on the so-called anisotropy parameter $\eta$. The Hilbert space is $\otimes^N V$, where $V$ is three dimensional.

This model was first explicitly constructed by Izergin and Korepin\textsuperscript{15}. The spectrum of the transfer matrix and the BA equations were first determined using the analytical BA method\textsuperscript{16}, and later using the algebraic BA method\textsuperscript{17}. The corresponding vertex model is equivalent to an $O(n)$ model on a square\textsuperscript{18} or hexagonal\textsuperscript{19} lattice. Such $O(n)$ models are relevant\textsuperscript{20} to the study of polymers.

We distinguish two regimes: $\eta$ purely imaginary and $\eta$ purely real. For zero external magnetic field, these correspond to critical and noncritical regimes, respectively. By abuse of language, we shall refer to the two regimes as “critical” and “noncritical” even for nonzero field.

For the $A_2^{(2)}$ chain in the critical regime, a general string hypothesis has not yet been formulated. A new 2-string solution was found in Ref. 21, and several new candidate 4-string solutions were found in Ref. 22. Presumably, there are new longer strings as well. In the absence of a suitable string hypothesis, the TBA equations of the critical $A_2^{(2)}$ chain cannot be formulated. An alternative approach of investigating this model, based on finite-size corrections, has been recently pursued by two groups\textsuperscript{23,24}. However, there is some disagreement between their results.

The difficulty in formulating a string hypothesis may be related to the fact that, in the critical regime, the Hamiltonian of the $A_2^{(2)}$ chain is not Hermitian. One might try to restrict the space of states to a subspace in which the Hamiltonian is Hermitian. However, since this model does not have a quantum-algebra symmetry, it is not clear how to implement this restriction.

The situation for the open $A_2^{(2)}$ chain, whose Hamiltonian is the same as (1.1) except without the final term $H_{N,1}$, may be better. As shown in Ref. 25, this model has the quantum algebra symmetry $U_q[su(2)]$, and is integrable. Indeed, the spectrum of the
transfer matrix and the BA equations have been found, using a generalization of the analytical BA method, in Ref. 26. In the critical regime, one may be able to exploit the model’s $U_q[su(2)]$ symmetry to make suitable projections on the space of states, in analogy with the $A_1^{(1)}$ case.

As a warm-up exercise before addressing these problems, we consider in this paper the closed $A_2^{(2)}$ chain in the noncritical* regime. Here the Hamiltonian is Hermitian, and there is no difficulty in formulating a string hypothesis. We probe this system at finite temperature $T$ in an external magnetic field $H$. In analogy with the $A_1^{(1)}$ case, we expect various phases in the $H - \Delta$ plane, where $\Delta \equiv \text{ch} \eta$. In particular, for $\Delta \geq 1$ there should be a range of $H \geq 0$ for which the model exhibits massless behavior.

We focus on the point $\Delta \to 1$, $H \to 0$ in the massless phase, where the model becomes $su(3)$-invariant. We solve the TBA equations in a systematic low-$T$ and small-$H$ perturbative expansion, along the lines of Johnson and McCoy. These calculations employ Wiener-Hopf techniques, which were first used in a similar context by Yang and Yang. However, here we deal with a system of integral equations which requires factorization of a matrix kernel. We then compute the free energy $F(T, H)$, and determine the specific heat and magnetic susceptibility. Using the well-known relation between low-temperature specific heat and central charge ($c$) for critical systems, we arrive at the value $c = 2$ for the $su(3)$-invariant chain. This value coincides with that obtained from finite-size corrections, and is expected from the equivalence of this model to the level-one $su(3)$ WZW model in the continuum limit. We believe that our value for the magnetic susceptibility is new.

While there are other ways of calculating the low-temperature specific heat within the general TBA approach, the method pursued here has the merit of treating this calculation in the same manner as the one for the magnetic susceptibility.

* in the sense defined above
We see no difficulty in computing thermodynamic quantities within the TBA approach at other points in the $H - \Delta$ plane. However, we are primarily interested in the massless phase, as this is where the connection with field theory is better understood.

The outline of our paper is as follows. In Section 2, we formulate the TBA equations for the closed $A_2^{(2)}$ chain in the noncritical regime, with $\eta$ real. In Section 3, we take the limit $\eta \to 0$, and arrive at the TBA equations for the antiferromagnetic $su(3)$-invariant chain in the fundamental representation. In Section 4, we solve these equations for small values of $T$ and $H$, and in Section 5 we calculate the free energy, specific heat, and magnetic susceptibility. We present further discussion of our results in Section 6.

2. TBA equations for the $A_2^{(2)}$ chain

The Hamiltonian for the closed $N$-site $A_2^{(2)}$ chain is given implicitly in terms of the $A_2^{(2)}$ $R$ matrix in Eq. (1.1). (Explicit expressions are given in Refs. 15, 16, and 25.) In an external magnetic field $H$, the corresponding energy eigenvalues are

$$E = -\sum_{k=1}^{M} \frac{\text{sh}^2 \eta}{\sin \eta(\lambda_k - \frac{i}{2}) \sin \eta(\lambda_k + \frac{i}{2})} - H (N - M),$$

(2.1)

where the real part of the complex variables $\lambda_k$ have values in the interval $[-\pi/2\eta, \pi/2\eta]$ and satisfy the Bethe Ansatz (BA) equations

$$\left[ \frac{\sin \eta(\lambda_k + \frac{i}{2})}{\sin \eta(\lambda_k - \frac{i}{2})} \right]^N = \prod_{j=1}^{M} \frac{\sin \eta(\lambda_k - \lambda_j + i) \cos \eta(\lambda_k - \lambda_j - \frac{i}{2})}{\sin \eta(\lambda_k - \lambda_j - i) \cos \eta(\lambda_k - \lambda_j + \frac{i}{2})},$$

$$k = 1, \ldots, M.$$ (2.2)

We consider the (noncritical) regime where $\eta$ is real and (without loss of generality) positive.

We shall investigate the thermodynamics of this model, so we shall need to solve the BA equations in the $N \to \infty$ and $M \to \infty$ limit, with $M/N$ fixed. As is customary, we
adopt the “string hypothesis” which states that all the solutions \( \{\lambda_k, k = 1, \ldots, \infty\} \) are collections of \( M_n \) “strings” of “length” \( n \) of the form

\[
\lambda_{\alpha}^{(n,l)} = \lambda_{\alpha}^n + i \left( \frac{n+1}{2} - l \right),
\]

where \( l = 1, \ldots, n; \alpha = 0, 1, \ldots, M_n; n = 1, \ldots, \infty; \) and the “centers” \( \lambda_{\alpha}^n \) are real. A particular solution of the BA equations corresponds to a set of non-negative integers \( \{ M_n \} \) and the \( M_n \) real numbers \( \lambda_{\alpha}^n \) for each \( n \). Observe that the total number of \( \lambda \) variables, and BA equations, is \( M = \sum_{n=1}^{\infty} n M_n \). These string solutions are22 the same as those of the noncritical \( A_1^{(1)} \) chain4.

It will prove convenient to introduce the functions

\[
p_n(\lambda) = i \ln \left[ \frac{\sin \eta(\lambda + \frac{in}{2})}{\sin \eta(\lambda - \frac{in}{2})} \right],
\]

\[
q_n(\lambda) = i \ln \left[ \frac{\cos \eta(\lambda + \frac{in}{2})}{\cos \eta(\lambda - \frac{in}{2})} \right],
\]

and the matrices

\[
\Xi_{nm}(\lambda) = p_{n+m}(\lambda) + 2 \sum_{l=1}^{\min(n,m)-1} p_{|n-m|+2l}(\lambda) + p_{|n-m|}(\lambda) + \sum_{l=1}^{\min(n,m)} q_{2l-n-m-1}(\lambda).
\]

We now substitute (2.3) into the BA equations (2.2), and take the product of the resulting equations for \( \lambda_{\alpha}^{(n,l)} \) over the \( n \) values of \( l \). Taking the logarithm, we then obtain the following equations for the centers \( \lambda_{\alpha}^n \):

\[
h^n(\lambda_{\alpha}) = J^n_{\alpha}, \quad \alpha = 0, 1, \ldots, M_n; \quad n = 1, 2, \ldots,
\]

where

\[
h^n(\lambda) = \frac{1}{2\pi} \left\{ N p_n(\lambda) - \sum_{m=1}^{\infty} \sum_{\beta=0}^{M_m} \Xi_{nm}(\lambda - \lambda_{\beta}^m) \right\},
\]

and \( J^n_{\alpha} \) are integers or half-integers. We make the conventional assumption that \( h^n(\lambda) \) is a monotonic increasing function of \( \lambda \). Let \( J \) denote the set of allowed values of \( J^n_{\alpha} \), and
\( \tilde{J} \) its complement. If \( \lambda \) is such that \( h^{n}(\lambda) \in J \), it is said to correspond to a particle (of rapidity \( \lambda \)). If \( \lambda \) is such that \( h^{n}(\lambda) \in \tilde{J} \), it is said to correspond to a hole. Let \( \rho_{n}(\lambda) \) be the density of particles and \( \tilde{\rho}_{n}(\lambda) \) be the density of holes. Then

\[
\rho_{n}(\lambda) + \tilde{\rho}_{n}(\lambda) = \lim_{N \to \infty} \frac{1}{N} \frac{d}{d\lambda} h^{n}(\lambda).
\] (2.8)

An expression for the right hand side of this equation can be found by the substitution of \( N^{-1} \sum_{\beta} \) by \( \int d\lambda' \rho(\lambda') \) in (2.7). This leads to the equation

\[
\tilde{\rho}_{n} + \sum_{m=1}^{\infty} (A_{nm} + B_{nm}) \ast \rho_{m} = a_{n},
\] (2.9)

where

\[
A_{nm}(\lambda) = \delta_{nm}\delta(\lambda) + (1 - \delta_{nm}) a_{|n - m|}(\lambda) + a_{n + m}(\lambda)
\]

\[
+ 2 \sum_{l=1}^{\min(n,m)-1} a_{|n - m| + 2l}(\lambda),
\] (2.10)

\[
B_{nm}(\lambda) = \sum_{l=1}^{\min(n,m)} b_{2l - n - m - 1}(\lambda),
\] (2.11)

and

\[
a_{n}(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} p_{n}(\lambda) = \frac{\eta}{\pi} \frac{\sh(\eta n)}{\ch(\eta n) - \cos(2\eta \lambda)},
\] (2.12)

\[
b_{n}(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} q_{n}(\lambda) = \frac{\eta}{\pi} \frac{\sh(\eta n)}{\ch(\eta n) + \cos(2\eta \lambda)},
\] (2.13)

and \( \ast \) denotes a convolution,

\[
(f \ast g)(\lambda) = \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda' f(\lambda - \lambda') g(\lambda').
\] (2.14)

For future reference we note here that

\[
B_{nm}(\lambda) = - (s \ast A_{nm})(\lambda + \frac{\pi}{2\eta}),
\] (2.15)
where \( s(\lambda) \) is defined by

\[
s(\lambda) = \frac{\eta}{\pi} \sum_{k=-\infty}^{\infty} e^{-2i\eta k\lambda} \frac{1}{2\text{ch}(\eta k)}.
\]

We also note that \( s(\lambda) \), which can be expressed in terms of the Jacobian elliptic function \( dn \), has the property that

\[
s * a_2 = a_1 - s, \quad s * (a_{n+1} + a_{n-1}) = a_n \quad n > 1.
\]

These and other relations which we give below can be easily derived with the help of Fourier transforms, for which we use the following conventions

\[
f(\lambda) = \frac{\eta}{\pi} \sum_{k=-\infty}^{\infty} e^{-2i\eta k\lambda} \hat{f}_k, \quad \hat{f}_k = \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda e^{2i\eta k\lambda} f(\lambda).
\]

The thermodynamic limit of the energy per site is

\[
\frac{E}{N} = -2\pi \text{sh} \eta \sum_{n=1}^{\infty} \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda a_n(\lambda) \rho_n(\lambda) - H \left[ 1 - \sum_{n=1}^{\infty} n \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda \rho_n(\lambda) \right],
\]

while the entropy per site is

\[
\frac{S}{N} = \sum_{n=1}^{\infty} \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda \left[ (\rho_n + \tilde{\rho}_n) \ln (\rho_n + \tilde{\rho}_n) - \rho_n \ln \rho_n - \tilde{\rho}_n \ln \tilde{\rho}_n \right].
\]

The equilibrium value of \( \rho_n \) at temperature \( T \) is determined by extremizing the free energy per site \( F/N = (E - TS)/N \). Note that the variation of \( \tilde{\rho}_n \) is determined in terms of the set of variations \( \{\delta \rho_n\} \) by the constraint (2.9) which implies that

\[
-\frac{\delta \tilde{\rho}_n(\lambda)}{\delta \rho_m(\lambda')} = A_{nm}(\lambda - \lambda') + B_{nm}(\lambda - \lambda').
\]

Using this one finds that \( F/N \) is extremized when the functions

\[
\epsilon_n(\lambda) = T \ln \left( \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)} \right)
\]

are extremized with respect to \( \lambda' \).
satisfy the thermodynamic Bethe Ansatz (TBA) equations

\[ T \ln \left( 1 + e^{\epsilon_n/T} \right) = \sum_{m=1}^{\infty} (A_{nm} + B_{nm}) \ast T \ln \left( 1 + e^{-\epsilon_m/T} \right) - 2 \pi \frac{\text{sh} \eta}{\eta} a_n + nH . \] (2.23)

Using the TBA equations one finds that, in equilibrium,

\[ \frac{F}{N} = -T \sum_{n=1}^{\infty} \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda \ a_n(\lambda) \ln \left( 1 + e^{-\epsilon_n(\lambda)/T} \right) - H . \] (2.24)

As in other integrable models, the free energy can be re-expressed as a functional of only one of the \( \epsilon_n(\lambda) \)'s. To see this, we introduce the matrix function

\[ A_{nm}^{-1}(\lambda) = \delta(\lambda)\delta_{nm} - s(\lambda) \left( \delta_{n,m+1} + \delta_{n,m-1} \right) , \] (2.25)

where \( s(\lambda) \) was defined in (2.16). As the notation suggests, it has the property

\[ \sum_{n'=1}^{\infty} \left( A_{nn'}^{-1} * A_{n'm} \right)(\lambda) = \delta(\lambda)\delta_{nm} , \] (2.26)

which follows from (2.17). It has the further properties

\[ \sum_{m=1}^{\infty} \left( A_{nm}^{-1} * a_m \right)(\lambda) = s(\lambda)\delta_{n1} , \quad \sum_{m=1}^{\infty} A_{nm}^{-1} * m = 0 . \] (2.27)

Inserting \( A^{-1}A \) into (2.24) and using (2.27) and (2.23) with \( B_{nm} \) replaced by the right hand side of (2.15), one finds that

\[ \frac{F}{N} = -2 \pi \frac{\text{sh} \eta}{\eta} \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda \ a_1(\lambda) r(\lambda) - T \int_{-\pi/2\eta}^{\pi/2\eta} d\lambda \ r(\lambda) \ln \left( 1 + e^{\epsilon_1(\lambda)/T} \right) . \] (2.28)

Here

\[ r(\lambda) = \frac{\eta}{\pi} \sum_{k=\infty}^{\infty} e^{-2i\eta k \lambda} \frac{\hat{s}_k}{1 + (-1)^{k+1} \hat{s}_k} , \] (2.29)

where \( \hat{s}_k = 1/2 \text{ch} \eta k \) are the Fourier coefficients of \( s(\lambda) \).

From (2.28) we see that in order to compute thermodynamic quantities of our model, we need only determine \( \epsilon_1(\lambda) \). Unfortunately, the TBA equations (2.23) are coupled non-linear equations for all the \( \epsilon_n(\lambda) \). However, at low temperature these equations linearize and a determination of \( \epsilon_1(\lambda) \) becomes possible.
The type of low-temperature expansion depends crucially on the values of the anisotropy parameter \( \eta \) and the magnetic field \( H \), as shown for the \( A_1^{(1)} \) model by Johnson and McCoy\(^5\). We shall not pursue here a similar exhaustive analysis of the \( A_2^{(2)} \) model. Instead, we shall concentrate on the limits

\[
\eta \to 0, \quad H \to 0.
\]

In the limit \( \eta \to 0 \), the \( A_2^{(2)} \) model reduces to the \( su(3) \)-invariant quantum spin chain in the fundamental representation\(^{28,29}\). In the following section, we shall explain how the TBA equations of the \( su(3) \)-invariant model emerge from those of the \( A_2^{(2)} \) model. We remark that the \( su(3) \)-invariant model allows the introduction of two external (magnetic) fields because \( su(3) \) has rank two. By our limiting process we find only one combination.

3. The \( \eta \to 0 \) limit

We recall the \( A_2^{(2)} \) BA equations (2.2):

\[
\left[ \frac{\sin \eta(\lambda_k + \frac{i}{2})}{\sin \eta(\lambda_k - \frac{i}{2})} \right]^N = -\prod_{j=1}^{M} \frac{\sin \eta(\lambda_k - \lambda_j + i) \cos \eta(\lambda_k - \lambda_j - \frac{i}{2})}{\sin \eta(\lambda_k - \lambda_j - i) \cos \eta(\lambda_k - \lambda_j + \frac{i}{2})},
\]

\[
k = 1, \cdots, M.
\]

Consider a solution \( \{\lambda_1, \cdots, \lambda_M\} \). For \( \eta \to 0 \), some of the \( \lambda_k \)'s remain finite. We call these solutions \( \lambda_k^{(1)} \), \( k = 1, \cdots, M^{(1)} \). The remaining solutions become infinite; of these, we restrict ourselves (following Ref. 16) only to those solutions whose behavior as \( \eta \to 0 \) is given by \( \pm \frac{\pi}{2\eta} + \lambda_k^{(2)} \), \( k = 1, \cdots, M^{(2)} \), with \( \lambda_k^{(2)} \) finite. Rewriting (3.1) in terms of the new variables \( \lambda_k^{(1)} \) and \( \lambda_k^{(2)} \), we obtain two families of BA equations, which become in the \( \eta \to 0 \) limit

\[
\left( \frac{\lambda_k^{(1)} + \frac{i}{2}}{\lambda_k^{(1)} - \frac{i}{2}} \right)^N = -\prod_{j=1}^{M^{(1)}} \frac{\lambda_k^{(1)} - \lambda_j^{(1)} + i}{\lambda_k^{(1)} - \lambda_j^{(1)} - i} \prod_{j'=1}^{M^{(2)}} \frac{\lambda_k^{(1)} - \lambda_j^{(2)} - \frac{i}{2}}{\lambda_k^{(1)} - \lambda_j^{(2)} + \frac{i}{2}},
\]

\[
k = 1, \cdots, M^{(1)},
\]
\[ 1 = - \prod_{j=1}^{M^{(1)}} \frac{\lambda_k^{(2)} - \lambda_j^{(1)} - \frac{i}{2}}{\lambda_k^{(2)} - \lambda_j^{(1)} + \frac{i}{2}} \prod_{j'=1}^{M^{(2)}} \frac{\lambda_k^{(2)} - \lambda_{j'}^{(2)} + i}{\lambda_k^{(2)} - \lambda_{j'}^{(2)} - i} , \quad k = 1, \ldots, M^{(2)} \]  

These are precisely the BA equations for the \( su(3) \)-invariant chain\(^{29} \).

In the formulas of the previous section, we must now distinguish two classes of string centers. The string hypothesis becomes

\[ \lambda_{\alpha}^{(n,r,l)} = \lambda_{\alpha}^{(n,r)} + i \left( \frac{n+1}{2} - l \right) , \quad (3.3) \]

where \( l = 1, \ldots, n; \alpha = 0, 1, \ldots, M_n^{(r)}; r = 1, 2; n = 1, \ldots, \infty; \) and the centers \( \lambda_{\alpha}^{(n,r)} \) are real. The total number of \( \lambda \)'s of type \( r \) is given by

\[ M^{(r)} = \sum_{n=1}^{\infty} nM_n^{(r)} , \quad r = 1, 2 . \quad (3.4) \]

We correspondingly have densities of particles \( \rho_n^{(r)}(\lambda) \) and holes \( \tilde{\rho}_n^{(r)}(\lambda) \). The integral equations satisfied by these densities are

\[ \tilde{\rho}_n^{(r)} + \sum_{m=1}^{\infty} \sum_{s=1}^{2} A_{nm} * C_{rs} * \rho_m^{(s)} = a_n \delta_{r1} , \quad (3.5) \]

where \( A_{nm} \) and \( a_n \) are the corresponding \( \eta \rightarrow 0 \) limits of (2.10) and (2.12),

\[ A_{nm}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\lambda \omega} \left( \frac{1}{2} \left[ e^{-\frac{\omega}{2}|n-m|} - e^{-\frac{\omega}{2}(n+m)} \right] \right) , \quad (3.6) \]
\[ a_n(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\lambda \omega} e^{-\frac{n|\omega|}{2}} = \frac{1}{2\pi} \frac{n}{\lambda^2 + \frac{n^2}{4}} \quad (n \neq 0) , \quad (3.7) \]

\( C_{rs} \) are the components of the \( 2 \times 2 \) matrix

\[ C(\lambda) = \begin{pmatrix} \delta(\lambda) & -s(\lambda) \\ -s(\lambda) & \delta(\lambda) \end{pmatrix} , \quad (3.8) \]

where

\[ s(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\lambda \omega} \frac{1}{2 \cosh \frac{\omega}{2}} = \frac{1}{2 \cosh \pi \lambda} , \quad (3.9) \]
and $\ast$ now denotes the convolution

$$( f \ast g)(\lambda) = \int_{-\infty}^{\infty} d\lambda' f(\lambda - \lambda')g(\lambda'). \quad (3.10)$$

The TBA equations are

$$T \ln \left(1 + e^{\epsilon_n^{(r)}/T}\right) = \sum_{m=1}^{\infty} \sum_{s=1}^{2} A_{nm} \ast C_{rs} \ast T \ln \left(1 + e^{-\epsilon_m^{(s)}/T}\right) - 2\pi a_n \delta_{r1} + nH, \quad (3.11)$$

where

$$\epsilon_n^{(r)}(\lambda) = T \ln \left(\frac{\tilde{\rho}^{(r)}_n(\lambda)}{\rho^{(r)}_n(\lambda)}\right). \quad (3.12)$$

Forming the convolution of the TBA equations with $A_{nm}^{-1}$ (which is given by the expression (2.25), with $s(\lambda)$ now given by (3.9)) and with $C_{rs}^{-1}$, we obtain

$$T \ln \left(1 + e^{-\epsilon_n^{(r)}/T}\right) = \sum_{m=1}^{\infty} \sum_{s=1}^{2} A_{nm}^{-1} \ast C_{rs}^{-1} \ast T \ln \left(1 + e^{\epsilon_m^{(s)}/T}\right) + 2\pi \delta_{n1} s^{(r)}, \quad (3.13)$$

where

$$s^{(1)}(\lambda) \pm s^{(2)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\lambda\omega}}{2 \operatorname{ch} \frac{\omega}{2} \mp 1} = \frac{2}{\sqrt{3}} \frac{\operatorname{sh} ((3 \pm 1)\pi \lambda/3)}{\operatorname{sh} 2\pi \lambda}. \quad (3.14)$$

The equilibrium free energy is given by

$$\frac{F}{N} = -T \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\lambda \: a_n(\lambda) \ln \left(1 + e^{-\epsilon_n^{(1)}(\lambda)/T}\right) - H. \quad (3.15)$$

With the help of (3.13), this expression can be cast in a form which depends only on $\epsilon_1^{(r)}(\lambda)$,

$$\frac{F}{N} = -2\pi \int_{-\infty}^{\infty} d\lambda \: a_1(\lambda) s^{(1)}(\lambda) - T \sum_{r=1}^{2} \int_{-\infty}^{\infty} d\lambda \: s^{(r)}(\lambda) \ln \left(1 + e^{\epsilon_1^{(r)}(\lambda)/T}\right). \quad (3.16)$$

In the following sections, we shall solve the TBA equations (3.11) for small values of $T$ and $H$, and evaluate the expression (3.16) for the free energy.
4. $T$ and $H$ expansion

We begin by rewriting the TBA equations of the $su(3)$-invariant chain (3.11) for $n = 1$ in the form

$$T \ln \left(1 + e^{\epsilon_{1}^{(r)}/T}\right) = \sum_{s=1}^{2} A_{11} \ast C_{rs} \ast T \ln \left(1 + e^{-\epsilon_{1}^{(s)}/T}\right) + \sum_{m=2}^{\infty} \sum_{s=1}^{2} A_{1m} \ast C_{rs} \ast T \ln \left(1 + e^{-\epsilon_{m}^{(s)}/T}\right) - 2\pi a_{1} \delta_{r1} + H. \quad (4.1)$$

We make the crucial assumption that $\epsilon_{m}^{(r)}(\lambda) > 0$ for $m > 1$. (The results which we obtain below do not contradict this assumption. For other examples, see Ref. 9.) This means that $\exp(-\epsilon_{m}^{(r)}/T)$ goes to zero exponentially as $T \to 0$ for $m > 1$ and can therefore be neglected. On the other hand, $\epsilon_{1}^{(r)}(\lambda)$ can have either sign. Defining $\varepsilon^{(r)}(\lambda)$ to be the $T \to 0$ limit of $\epsilon_{1}^{(r)}(\lambda)$,

$$\varepsilon^{(r)}(\lambda) = \lim_{T \to 0} \epsilon_{1}^{(r)}(\lambda), \quad (4.2)$$

we see that

$$\lim_{T \to 0} T \ln \left(1 + e^{\pm \epsilon_{1}^{(r)}/T}\right) = \pm \varepsilon^{(r)\pm}, \quad (4.3)$$

where we use the standard notation

$$\varepsilon^{-} \equiv \frac{1}{2} (\varepsilon - |\varepsilon|), \quad \varepsilon^{+} \equiv \varepsilon - \varepsilon^{-}. \quad (4.4)$$

It follows from (4.1) that

$$\varepsilon^{(r)+} = - \sum_{s=1}^{2} A_{11} \ast C_{rs} \ast \varepsilon^{(s)-} - 2\pi a_{1} \delta_{r1} + H. \quad (4.5)$$

It will be convenient for subsequent analysis to work with the following equivalent equations, which do not involve $\varepsilon^{(r)-}$:

$$\varepsilon^{(1)} = H - 2\pi s^{(1)} + h \ast \varepsilon^{(1)+} + g \ast \varepsilon^{(2)+}, \quad (4.6)$$

$$\varepsilon^{(2)} = H - 2\pi s^{(2)} + h \ast \varepsilon^{(2)+} + g \ast \varepsilon^{(1)+}, \quad (4.6)$$
where \( s^{(1)} \) and \( s^{(2)} \) are defined in (3.14), and

\[
g = -s^{(1)} + s^{(2)} \ast a_1, \quad h = -s^{(2)} + s^{(1)} \ast a_1. \tag{4.7}
\]

In order to obtain (4.6), we substitute into (4.5) the explicit expressions for \( A_{11} \) and \( C_{rs} \) as given by Eqs. (2.10) and (3.8), as well as \( \varepsilon^{(r)-} = \varepsilon^{(r)} - \varepsilon^{(r)+} \); and then we solve for \( \varepsilon^{(r)} \) in terms of \( \varepsilon^{(r)+} \) with the help of Fourier transforms.

We observe for future reference that by a similar procedure, (4.1) can be recast (after neglecting terms involving \( \epsilon^{(r)}_m \) with \( m > 1 \), but before taking the \( T \to 0 \) limit of \( \epsilon^{(r)}_1 \)) as follows:

\[
\begin{align*}
\epsilon_1^{(1)} &= H - 2\pi s^{(1)} + h \ast T \ln \left( 1 + e^{\epsilon^{(1)}_1/T} \right) + g \ast T \ln \left( 1 + e^{\epsilon^{(2)}_1/T} \right), \\
\epsilon_1^{(2)} &= H - 2\pi s^{(2)} + h \ast T \ln \left( 1 + e^{\epsilon^{(2)}_1/T} \right) + g \ast T \ln \left( 1 + e^{\epsilon^{(1)}_1/T} \right). \tag{4.8}
\end{align*}
\]

Evidently, the \( T \to 0 \) limit of these equations gives (4.6).

We first briefly consider the case \( H = 0 \), which corresponds to the ground (vacuum) state. Since \( s^{(1)}(\lambda) \) and \( s^{(2)}(\lambda) \) are positive for all \( \lambda \), the equations (4.6) have the solution

\[
\begin{align*}
\varepsilon^{(1)}(\lambda) &= -2\pi s^{(1)}(\lambda) = -\frac{2\pi}{\sqrt{3}} \frac{\text{ch}(\pi \lambda/3)}{\text{ch}(\pi \lambda)}, \\
\varepsilon^{(2)}(\lambda) &= -2\pi s^{(2)}(\lambda) = -\frac{2\pi}{\sqrt{3}} \frac{\text{sh}(\pi \lambda/3)}{\text{sh}(\pi \lambda)}. \tag{4.9}
\end{align*}
\]

From the definition (3.12) and the constraint equations (3.5), and also from the assumption that \( \epsilon^{(r)}_m > 0 \) for \( m > 1 \), it follows that the ground-state densities of particles and holes are given by

\[
\begin{align*}
\rho_1^{(1)}(\lambda) &= \frac{1}{\sqrt{3}} \frac{\text{ch}(\pi \lambda/3)}{\text{ch}(\pi \lambda)}, \quad \rho_1^{(2)}(\lambda) = \frac{1}{\sqrt{3}} \frac{\text{sh}(\pi \lambda/3)}{\text{sh}(\pi \lambda)}, \quad \rho_n^{(r)}(\lambda) = 0 \quad n > 1, \\
\tilde{\rho}_n^{(r)}(\lambda) &= 0. \tag{4.10}
\end{align*}
\]
That is, the ground state of the antiferromagnetic $su(3)$-invariant chain consists of a “condensate” of strings of length 1. Defining momenta $p^{(r)}(\lambda)$ of the quasi-particles by (see, e.g., Kirillov and Reshetikhin 13)

$$\frac{d}{d\lambda} p^{(r)}(\lambda) = \varepsilon^{(r)}(\lambda),$$

we see from (4.9) that for $\lambda \rightarrow \infty$, there is a linear dispersion relation

$$\varepsilon^{(r)} = v_s p^{(r)},$$

with the velocity of sound $v_s = 2\pi/3$.

We now turn to the case $H \neq 0$. For $H \rightarrow 0$, Eqs. (4.6) can in principle be solved by iteration, with the zeroth-order solution given by

$$\varepsilon^{(r)}_0(\lambda) = H - 2\pi s^{(r)}(\lambda).$$

Since the functions $s^{(r)}(\lambda)$ are positive and monotonically decreasing, for $H \rightarrow 0$ the functions $\varepsilon^{(r)}(\lambda)$ have a single zero. This zero is for $\lambda = O(\ln H)$. Assuming an expansion in powers of $(\ln H)^{-1}$, we conclude that

$$\varepsilon^{(r)}(\alpha^{(r)}) = 0$$

for

$$\alpha^{(r)} = -\frac{3}{2\pi} \left[ \ln \left( \frac{\sqrt{3}}{2\pi} H \right) + \ln \kappa^{(r)} + O\left( \frac{1}{\ln H} \right) \right],$$

where the constants $\kappa^{(r)}$ (which are independent of $H$) have still to be determined. We shall further assume that the functions $\varepsilon^{(r)}(\lambda)$ have no other zeros in the interval $(0, \infty)$. Hence, $\varepsilon^{(r)}(\lambda) < 0$ for $\lambda$ in the interval $(0, \alpha^{(r)})$, and $\varepsilon^{(r)}(\lambda) > 0$ for $\lambda$ in the interval $(\alpha^{(r)}, \infty)$.

Observe that Eqs. (4.6) are temperature-independent. Using the same approximations in the expression (3.16) for the free energy this too will be temperature-independent. To
find the leading order temperature dependence we need to compute the leading correction to the solutions \( \epsilon_1^{(r)} = \epsilon^{(r)} \) of the linearized equations (4.6). In order to obtain this correction, we make the substitution

\[
\epsilon_1^{(r)}(\lambda) = \epsilon^{(r)}(\lambda) + \eta^{(r)}(\lambda)
\] (4.16)
in (4.8) and expand to leading order in \( \eta^{(r)} \). Since \( \epsilon^{(r)} \) is a solution of the linearized equations (4.6), we shall find inhomogeneous terms in the resulting equations for \( \eta^{(r)} \).

Indeed, we have that

\[
\eta^{(1)} = h \star \left\{ T \ln \left[ 1 + e^{(\epsilon^{(1)} + \eta^{(1)})/T} \right] - \epsilon^{(1)+} \right\} + g \star \left\{ T \ln \left[ 1 + e^{(\epsilon^{(2)} + \eta^{(2)})/T} \right] - \epsilon^{(2)+} \right\},
\] (4.17)
and the similar equation for \( \eta^{(2)} \) is obtained by interchanging the superscripts (1) and (2).

Because \( \epsilon^{(r)}(\lambda) \) is an even function of \( \lambda \) with a single zero, at \( \lambda = \alpha^{(r)} \), for positive \( \lambda \), we see that (assuming \( \eta^{(r)}/T \) is small and keeping terms linear in \( \eta^{(r)} \))

\[
f \star \left\{ T \ln \left[ 1 + e^{(\epsilon^{(r)} + \eta^{(r)})/T} \right] - \epsilon^{(r)+} \right\} = \left\{ \int_{-\alpha^{(r)}}^{\alpha^{(r)}} d\lambda' f(\lambda - \lambda') T \ln \left[ 1 + e^{(\epsilon^{(r)}(\lambda') + \eta^{(r)}(\lambda'))/T} \right] - \epsilon^{(r)}(\lambda') \right\} 
\]

\[
+ \int_{-\alpha^{(r)}}^{\alpha^{(r)}} d\lambda' f(\lambda - \lambda') T \ln \left[ 1 + e^{(\epsilon^{(r)}(\lambda') + \eta^{(r)}(\lambda'))/T} \right] 
\]

\[
\approx \left\{ \int_{-\alpha^{(r)}}^{\alpha^{(r)}} d\lambda' f(\lambda - \lambda') \eta^{(r)}(\lambda') + E^{(r)}_f \right\},
\] (4.18)

where the inhomogeneous term \( E^{(r)}_f \) is given by

\[
E^{(r)}_f = f \star T \ln \left( 1 + e^{-|\epsilon^{(r)}|/T} \right).
\] (4.19)

(Here \( f \) represents either of the kernels \( h \) and \( g \) appearing in (4.17).) For \( T \to 0 \), the major contribution to the integral comes from the regions near the zeros of \( \epsilon^{(r)} \), so we expand \( \epsilon^{(r)}(\lambda) \) about \( \lambda = \alpha^{(r)} \),

\[
\epsilon^{(r)}(\lambda) = t^{(r)} (\lambda - \alpha^{(r)}) + O \left( (\lambda - \alpha^{(r)})^2 \right), \quad t^{(r)} = \frac{d\epsilon^{(r)}}{d\lambda} \bigg|_{\lambda=\alpha^{(r)}}.
\] (4.20)
One then finds that the leading $T$-dependence of $E_f^{(r)}$ is

$$E_f^{(r)}(\lambda) = \frac{2T^2}{\ell^{(r)}} \left[ f(\lambda - \alpha^{(r)}) + f(\lambda + \alpha^{(r)}) \right] \int_0^\infty du \ln (1 + e^{-u})$$

$$= \frac{\pi^2 T^2}{6\ell^{(r)}} \left[ f(\lambda - \alpha^{(r)}) + f(\lambda + \alpha^{(r)}) \right].$$  \hspace{1cm} (4.21)

Taking into account these $\eta$-independent terms in the expansion of (4.8) to $O(\eta)$, we obtain the following linear integral equation for $\eta^{(1)}$,

$$\eta^{(1)}(\lambda) = \left( \int_{-\infty}^{\lambda - \alpha^{(1)}} + \int_{\lambda + \alpha^{(1)}}^\infty \right) d\lambda' \ h(\lambda - \lambda') \eta^{(1)}(\lambda')$$

$$+ \left( \int_{-\infty}^{\lambda - \alpha^{(2)}} + \int_{\lambda + \alpha^{(2)}}^\infty \right) d\lambda' \ g(\lambda - \lambda') \eta^{(2)}(\lambda')$$

$$+ \frac{\pi^2 T^2}{6\ell^{(1)}} \left[ h(\lambda - \alpha^{(1)}) + h(\lambda + \alpha^{(1)}) \right]$$

$$+ \frac{\pi^2 T^2}{6\ell^{(2)}} \left[ g(\lambda - \alpha^{(2)}) + g(\lambda + \alpha^{(2)}) \right].$$  \hspace{1cm} (4.22)

The similar expression for $\eta^{(2)}$ is obtained by interchanging the superscripts (1) and (2) on $\eta$, $\alpha$ and $t$. These equations for $\eta^{(r)}$ and those of (4.6) for $\varepsilon^{(r)}$ complete our results for the $T$-expansion of $\epsilon_1^{(r)}$. We now turn to the expansion in powers of $\ln H$, by which from Eqs. (4.6) and (4.22) we shall generate systems of integral equations of the Wiener-Hopf type.

It will prove convenient to work with the functions

$$S^{(r)}(\lambda) = \begin{cases} e^{2\pi\alpha^{(r)}/3} \kappa^{(r)} \varepsilon^{(r)}(\lambda + \alpha^{(r)}) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases},$$  \hspace{1cm} (4.23)

and

$$T^{(r)}(\lambda) = \begin{cases} \frac{6e^{-2\pi\alpha^{(r)}/3}}{\pi^2 T^2 \kappa^{(r)}} \eta^{(r)}(\lambda + \alpha^{(r)}) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases},$$  \hspace{1cm} (4.24)

instead of the functions $\varepsilon^{(r)}(\lambda)$ and $\eta^{(r)}(\lambda)$. The factors $e^{2\pi\alpha^{(r)}/3}$ and $\left(e^{-2\pi\alpha^{(r)}/3}\right)/T^2$ in (4.23) and (4.24), respectively, are chosen such that the driving terms in the equations for $S^{(r)}(\lambda)$ and $T^{(r)}(\lambda)$ have a nonvanishing limit as $T \to 0$ and $H \to 0$. The factors of $\kappa^{(r)}$,
which were first introduced in (4.15), appear in (4.23) for a reason which will be explained below.

We return now to Eq. (4.6). We write the limits of integration explicitly, keeping in mind that \( \varepsilon^{(r)}(\lambda) > 0 \) for \(-\infty < \lambda < -\alpha^{(r)}\) and for \( \alpha^{(r)} < \lambda < \infty \); and we shift the integration variables so that they run from 0 to \( \infty \). Observe now that \( h(\lambda + 2\alpha^{(r)}) \) vanishes as \( H \to 0 \) \( (\alpha^{(r)} \to \infty) \) for finite \( \lambda \), and similarly for \( g(\lambda + \alpha^{(1)} + \alpha^{(2)}) \). The functions \( g(\lambda + \alpha^{(1)} - \alpha^{(2)}) \) and \( g(\lambda + \alpha^{(2)} - \alpha^{(1)}) \) remain finite, however, and survive in the \( H \to 0 \) limit of the equations for \( S^{(r)}(\lambda) \), which are (for \( \lambda \geq 0 \))

\[
S^{(1)}(\lambda) = \frac{2\pi}{\sqrt{3}} \left( 1 - \kappa^{(1)} e^{-2\pi \lambda/3} \right) \\
+ \int_{0}^{\infty} d\lambda' \left[ h(\lambda - \lambda') S^{(1)}(\lambda') + g(\lambda - \lambda' + \alpha^{(1)} - \alpha^{(2)}) S^{(2)}(\lambda') \right],
\]

\[
S^{(2)}(\lambda) = \frac{2\pi}{\sqrt{3}} \left( 1 - \kappa^{(2)} e^{-2\pi \lambda/3} \right) \\
+ \int_{0}^{\infty} d\lambda' \left[ g(\lambda - \lambda' + \alpha^{(2)} - \alpha^{(1)}) S^{(1)}(\lambda') + h(\lambda - \lambda') S^{(2)}(\lambda') \right].
\]

Similarly, from (4.22), we see that the \( H \to 0 \) limit of the equations for \( T^{(r)}(\lambda) \) are (for \( \lambda \geq 0 \))

\[
T^{(1)}(\lambda) = \frac{h(\lambda)}{S^{(1)'}(0)} + \frac{g(\lambda + \alpha^{(1)} - \alpha^{(2)})}{S^{(2)'}(0)} \\
+ \int_{0}^{\infty} d\lambda' \left[ h(\lambda - \lambda') T^{(1)}(\lambda') + g(\lambda - \lambda' + \alpha^{(1)} - \alpha^{(2)}) T^{(2)}(\lambda') \right],
\]

\[
T^{(2)}(\lambda) = \frac{h(\lambda)}{S^{(2)'}(0)} + \frac{g(\lambda + \alpha^{(2)} - \alpha^{(1)})}{S^{(1)'}(0)} \\
+ \int_{0}^{\infty} d\lambda' \left[ g(\lambda - \lambda' + \alpha^{(2)} - \alpha^{(1)}) T^{(1)}(\lambda') + h(\lambda - \lambda') T^{(2)}(\lambda') \right],
\]

where

\[
S^{(r)'}(0) \equiv \left. \frac{d}{d\lambda} S^{(r)}(\lambda) \right|_{\lambda=0+} = e^{2\pi \alpha^{(r)}/3} \kappa^{(r)} t^{(r)}. \tag{4.27}
\]

These equations can be written in the standard Wiener-Hopf form

\[
S^{(r)}(\lambda) = f^{(r)}(\lambda) + b^{(r)}(\lambda) + \sum_{s=1}^{2} \int_{-\infty}^{\infty} d\lambda' K_{rs}(\lambda - \lambda') S^{(s)}(\lambda'),
\]

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\[ T^{(r)}(\lambda) = f_T^{(r)}(\lambda) + b_T^{(r)}(\lambda) + \sum_{s=1}^{2} \int_{-\infty}^{\infty} d\lambda' \ K^{rs}(\lambda - \lambda') \ T^{(s)}(\lambda'), \]
\[-\infty < \lambda < \infty, \quad (4.28)\]

where the kernels \( K^{rs}(\lambda) \) are the components of the \( 2 \times 2 \) matrix
\[
K(\lambda) = \begin{pmatrix} h(\lambda) & g(\lambda + \alpha^{(1)} - \alpha^{(2)}) \\ g(\lambda + \alpha^{(2)} - \alpha^{(1)}) & h(\lambda) \end{pmatrix}, \quad (4.29)
\]

and
\[
f_S^{(r)}(\lambda) = \begin{cases} \frac{2\pi}{\sqrt{3}} (1 - \kappa^{(r)} e^{-2\pi\lambda/3}) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}, \quad (4.30)
\]
\[
b_S^{(r)}(\lambda) = \begin{cases} 0 & \lambda > 0 \\ -\sum_{s=1}^{2} \int_{-\infty}^{\infty} d\lambda' \ K^{rs}(\lambda - \lambda') \ S^{(s)}(\lambda') & \lambda < 0 \end{cases}, \quad (4.31)
\]

and similarly
\[
f_T^{(1)}(\lambda) = \begin{cases} h(\lambda) S^{(1)'}(0) + g(\lambda + \alpha^{(1)} - \alpha^{(2)}) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}, \quad (4.32)
\]
\[
f_T^{(2)}(\lambda) = \begin{cases} h(\lambda) S^{(2)'}(0) + g(\lambda + \alpha^{(2)} - \alpha^{(1)}) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}, \quad (4.33)
\]
\[
b_T^{(r)}(\lambda) = \begin{cases} 0 & \lambda > 0 \\ -\sum_{s=1}^{2} \int_{-\infty}^{\infty} d\lambda' \ K^{rs}(\lambda - \lambda') \ T^{(s)}(\lambda') & \lambda < 0 \end{cases}. \quad (4.34)
\]

We shall solve these equations by Fourier transform. We define the Fourier coefficients of \( S^{(r)}(\lambda) \) and \( T^{(r)}(\lambda) \) by
\[
\hat{S}^{(r)}(\omega) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda\omega} S^{(r)}(\lambda), \quad \hat{T}^{(r)}(\omega) = \int_{-\infty}^{\infty} d\lambda \ e^{i\lambda\omega} T^{(r)}(\lambda). \quad (4.35)
\]

Since \( S^{(r)}(\lambda) \) and \( T^{(r)}(\lambda) \) vanish for \( \lambda < 0 \), the functions \( \hat{S}^{(r)}(\omega) \) and \( \hat{T}^{(r)}(\omega) \) are analytic in the upper-half-plane \( \text{Im} \ \omega \geq 0 \), which we denote by \( \Pi_+ \). Observe that since \( S^{(r)}(0) = 0 \) (as follows from \( \varepsilon^{(r)}(\alpha^{(r)}) = 0 \)), we have by contour integration
\[
S^{(r)}(0) = -i \lim_{|\omega| \to \infty} \omega \hat{S}^{(r)}(\omega) = 0, \quad (4.36)
\]

where the limit is taken in \( \Pi_+ \). We also note that
\[
S^{(r)'}(0) = \frac{d}{d\lambda} S^{(r)}(\lambda)\bigg|_{\lambda=0} = -\lim_{|\omega| \to \infty} \omega^2 \hat{S}^{(r)}(\omega). \quad (4.37)
\]
Care must be exercised in the derivation of this result because of the discontinuity of the derivative of $S^{(r)}(\lambda)$ at $\lambda = 0$.

The Wiener-Hopf equations for $S^{(r)}$ in Fourier space are

$$
\hat{S}^{(r)}(\omega) = \hat{f}^{(r)}(\omega) + \hat{b}^{(r)}(\omega) + \sum_{s=1}^{2} \hat{K}^{rs}(\omega) \hat{S}^{(s)}(\omega),
$$

where $\hat{K}^{rs}(\omega)$ are the components of the $2 \times 2$ matrix

$$
\hat{K}(\omega) = \begin{pmatrix}
\hat{h}(\omega) & e^{-i\omega(\alpha^{(1)} - \alpha^{(2)})}\hat{g}(\omega) \\
e^{i\omega(\alpha^{(1)} - \alpha^{(2)})}\hat{g}(\omega) & \hat{h}(\omega)
\end{pmatrix}.
$$

Observe that (for $\omega$ and $\alpha^{(r)}$ real) this matrix is Hermitian,

$$
\hat{K}(\omega)^\dagger = \hat{K}(\omega).
$$

The factors of $\kappa^{(r)}$ in the definition (4.23) of $S^{(r)}$ were chosen to arrange for this to be the case.

The Wiener-Hopf equations for $T^{(r)}$ in Fourier space are similarly found to be

$$
\hat{T}^{(r)}(\omega) = \hat{f}^{(r)}(\omega) + \hat{b}^{(r)}(\omega) + \sum_{s=1}^{2} \hat{K}^{rs}(\omega) \hat{T}^{(s)}(\omega).
$$

Since $\left(1 - \hat{K}(\omega)\right)^{-1}$ is nonsingular, Hermitian and positive-definite at $\omega = 0$, it is positive-definite for $-\infty < \omega < \infty$. Thus, Theorem 8.2 of Gohberg and Krein\textsuperscript{31} implies that the following factorization exists

$$
\left(1 - \hat{K}(\omega)\right)^{-1} = G_+^{(\omega)} G_-^{(\omega)}, \quad -\infty < \omega < \infty,
$$

where $G_+^{(\omega)}$ and $G_-^{(\omega)}$ are analytic in $\Pi_+$ with $G_+^{(\omega)} \to 1$ as $\omega \to \infty$ in $\Pi_+$, and $G_-^{(\omega)}$ and $G_-^{(\omega)}$ are analytic in $\Pi_-$ with $G_-^{(\omega)} \to 1$ as $\omega \to \infty$ in $\Pi_-$. Moreover, the fact $K(-\lambda) = K(\lambda)^T$ implies that (for $\omega$ in $\Pi_-)$

$$
G_-^{(\omega)^T} = G_+^{(-\omega)},
$$

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where the superscript $T$ denotes transpose. This lemma can be proved from the formulas developed in Ref. 31. As we shall see, explicit expressions for $G_+$ and $G_-$ are not needed to compute the free energy to the order in which we work. A similar phenomenon occurs in the Wiener-Hopf calculations of Yang and Yang$^{30}$ and Johnson and McCoy$^5$.

Using the factorization (4.42), the Wiener-Hopf equation (4.38) for $\hat{S}^{(r)}$ can be rewritten (in $2 \times 2$ matrix notation) as

$$G_+^{-1} \hat{S} = G_- \left( \hat{f}_S + \hat{b}_S \right). \quad (4.44)$$

Observe that the left hand side is analytic and bounded in $\Pi_+$, whereas $G_- \hat{b}_S$ is analytic and bounded in $\Pi_-$. The term $G_- \hat{f}_S$ has a decomposition as

$$G_- \hat{f}_S = P_- \left( G_- \hat{f}_S \right) + P_+ \left( G_- \hat{f}_S \right), \quad (4.45)$$

where $P_\pm \left( G_- \hat{f}_S \right)$ is analytic in $\Pi_\pm$. This decomposition is uniquely specified by the requirement $P_\pm \left( G_- \hat{f}_S \right) \to 0$ for $\omega \to \infty$ in $\Pi_\pm$. Taking the $P_+$ projection of (4.44), we have that

$$\hat{S} = G_+ \ P_+ \left( G_- \hat{f}_S \right). \quad (4.46)$$

From (4.30) we compute that

$$\hat{f}_S^{(r)}(\omega) = \frac{2\pi i}{\sqrt{3}} \left( \frac{1}{\omega + i\epsilon} - \frac{\kappa^{(r)}}{\omega + 2\pi i/3} \right), \quad (4.47)$$

where one is to take $\epsilon \to 0$ at the end. The decomposition (4.45) of $G_- \hat{f}_S$ is then found by subtracting the residues of $\hat{f}_S$, i.e.,

$$G_-^{-1}(\omega) \hat{f}_S(\omega) = \frac{2\pi i}{\sqrt{3}} \left\{ \frac{1}{\omega + i\epsilon} \left( \frac{1}{G_-(\omega) - G_-(\omega-\epsilon)} \right) \left( \frac{1}{1} \right) - \frac{1}{\omega + 2\pi i/3} \left( G_-(\omega) - G_-(\omega - 2\pi i/3) \right) \left( \frac{\kappa^{(1)}}{\kappa^{(2)}} \right) \right\}$$

$$+ \frac{2\pi i}{\sqrt{3}} \left\{ \frac{1}{\omega + i\epsilon} \frac{\kappa^{(r)}}{G_-(\omega)} \left( \frac{1}{1} \right) - \frac{1}{\omega + 2\pi i/3} \frac{G_-(\omega - 2\pi i/3)}{\kappa^{(1)}} \left( \frac{\kappa^{(1)}}{\kappa^{(2)}} \right) \right\}. \quad (4.48)$$

Hence,

$$\hat{S}(\omega) = \frac{2\pi i}{\sqrt{3}} \frac{1}{\omega + i\epsilon} G_+(\omega) G_-(0) \left( \frac{1}{1} \right) - \frac{2\pi i}{\sqrt{3}} \frac{1}{\omega + 2\pi i/3} G_+(\omega) G_-(\omega - 2\pi i/3) \left( \frac{\kappa^{(1)}}{\kappa^{(2)}} \right). \quad (4.49)$$

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The boundary condition (4.36) can now be seen to be equivalent to the condition

\[ G_-( -2\pi i/3 ) \begin{pmatrix} \kappa_1^{(1)} \\ \kappa_2^{(2)} \end{pmatrix} = G_-( 0 ) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (4.50) \]

which can be solved for the parameters \( \kappa^{(r)} \). Using this result we conclude that \( \hat{S}(\omega) \) is given by

\[ \hat{S}(\omega) = \frac{2\pi i}{\sqrt{3}} \left( \frac{1}{\omega + \imath \epsilon} - \frac{1}{\omega + 2\pi i/3} \right) G_+^{(\omega)} G_-( 0 ) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (4.51) \]

From Eq. (4.37) and the fact that \( G_+^{(\omega)} \rightarrow 1 \) as \( |\omega| \rightarrow \infty \) in \( \Pi_+ \), we find that

\[ S^{(r)'}(0) = \frac{4\pi^2}{3\sqrt{3}} G_-( 0 ) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (4.52) \]

We turn now to the equation (4.41) for \( \hat{T}(\omega) \). Proceeding as before, we use the factorization (4.42) to arrive at the formal solution

\[ \hat{T} = G_+ \, P_+ \left( G_- \hat{f}_T \right) \quad (4.53) \]

(cf. Eq. (4.46)). The explicit calculation of \( \hat{f}^{(r)}_T \) is difficult, but can be avoided by the following trick. Consider the functions \( f^{(r)}(\lambda) \) defined as

\[ f^{(1)}(\lambda) = \frac{h(\lambda)}{S^{(1)'}(0)} + \frac{g(\lambda + \alpha^{(1)} - \alpha^{(2)})}{S^{(2)'}(0)}, \]

\[ f^{(2)}(\lambda) = \frac{h(\lambda)}{S^{(2)'}(0)} + \frac{g(\lambda + \alpha^{(2)} - \alpha^{(1)})}{S^{(1)'}(0)}, \quad (4.54) \]

where \( \lambda \) ranges over the entire real line. From Eqs. (4.32) and (4.33), it is evident that

\[ \hat{f}^{(r)}_T(\lambda) = f^{(r)}_+(\lambda), \quad (4.55) \]

where

\[ f_+(\lambda) \equiv \begin{cases} f(\lambda) & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}, \quad f_-(\lambda) \equiv \begin{cases} 0 & \lambda > 0 \\ f(\lambda) & \lambda < 0 \end{cases}, \quad f = f_+ + f_- \quad (4.56) \]
The Fourier transform of $f^{(r)}(\lambda)$ is readily computed, and can be expressed in terms of the kernel $\hat{K}$,

$$\hat{f}(\omega) = \hat{K}(\omega) \left( \frac{1}{S^{(1)'}(0)} \frac{1}{S^{(2)'}(0)} \right).$$  \hspace{1cm} (4.57)

From the factorization (4.42), it follows that

$$\hat{f}_+ + \hat{f}_- = (1 - G_-^{-1}G_+^{-1}) \left( \frac{1}{S^{(1)'}(0)} \frac{1}{S^{(2)'}(0)} \right).$$  \hspace{1cm} (4.58)

After multiplying both sides of this equation by $G_-$, we see that the $P_+$ projection of $G_- \hat{f}_+$ is given by

$$P_+ \left( G_- \hat{f}_+ \right) = \alpha - G_+^{-1} \left( \frac{1}{S^{(1)'}(0)} \frac{1}{S^{(2)'}(0)} \right),$$  \hspace{1cm} (4.59)

where $\alpha$ is a constant. Requiring the right hand side to vanish for $|\omega| \to \infty$ in $\Pi_+$ determines this constant to be

$$\alpha = \left( \frac{1}{S^{(1)'}(0)} \frac{1}{S^{(2)'}(0)} \right).$$  \hspace{1cm} (4.60)

We conclude from (4.53) that $\hat{T}(\omega)$ is given by

$$\hat{T}(\omega) = (G_+(\omega) - 1) \left( \frac{1}{S^{(1)'}(0)} \frac{1}{S^{(2)'}(0)} \right),$$  \hspace{1cm} (4.61)

where $S^{(r)'}(0)$ is given by (4.52).

To summarize this section: we have made the expansion (4.16) of $\epsilon_1^{(r)}(\lambda)$, and we have changed in Eqs. (4.23), (4.24) from the variables $\epsilon^{(r)}(\lambda), \eta^{(r)}(\lambda)$ to the variables $S^{(r)}(\lambda), T^{(r)}(\lambda)$, respectively. Using Wiener-Hopf methods, we have determined in Eqs. (4.51), (4.61) the corresponding Fourier transforms $\hat{S}^{(r)}(\omega), \hat{T}^{(r)}(\omega)$ in the limits $T \to 0$ and $H \to 0$. These expressions involve $G_+(\omega)$ and $G_-^{(r)}(\omega)$, which appear in the factorization (4.42). In the next section, we shall use these results to calculate the free energy.
5. The free energy

Substituting the expansion (4.16) of $\epsilon_i^{(r)}$ into the expression (3.16) for the free energy (keeping in mind the discussion immediately following (4.16)), we obtain

$$\frac{F}{N} = e_0 - \frac{\pi^2 T^2}{3} \sum_{r=1}^{2} s^{(r)}(\alpha^{(r)}) \alpha^{(r)} - 2 \sum_{r=1}^{2} \int_{\alpha^{(r)}}^{\infty} d\lambda \, s^{(r)}(\lambda) \left[ \varepsilon^{(r)}(\lambda) + \eta^{(r)}(\lambda) \right],$$  \hspace{1cm} (5.1)

where the ground state energy per site $e_0$ is given by

$$e_0 = -2\pi \int_{-\infty}^{\infty} d\lambda \, a_1(\lambda) s^{(1)}(\lambda).$$  \hspace{1cm} (5.2)

We aim for a double expansion of $F/N$ to quadratic order in both $T$ and $H$. To this end, we rewrite (5.1) in terms of $S^{(r)}(\lambda)$ and $T^{(r)}(\lambda)$. Consider first the $T^2$ term. For $H \to 0$, we can make the approximation

$$s^{(r)}(\alpha^{(r)}) = \frac{1}{\sqrt{3}} e^{-2\pi \alpha^{(r)} / 3},$$  \hspace{1cm} (5.3)

and hence this term becomes

$$-\frac{\pi^2 T^2}{3\sqrt{3}} \sum_{r=1}^{2} \kappa^{(r)}.$$  \hspace{1cm} (5.4)

We next manipulate the integral in the last term as follows:

$$\int_{\alpha^{(r)}}^{\infty} d\lambda \, s^{(r)}(\lambda) \left[ \varepsilon^{(r)}(\lambda) + \eta^{(r)}(\lambda) \right]$$

$$= \int_{0}^{\infty} d\lambda \, s^{(r)}(\lambda + \alpha^{(r)}) \left[ \frac{e^{-2\pi \alpha^{(r)} / 3}}{\kappa^{(r)}} S^{(r)}(\lambda) + \frac{\pi^2 T^2 \kappa^{(r)}}{6e^{-2\pi \alpha^{(r)} / 3}} T^{(r)}(\lambda) \right].$$  \hspace{1cm} (5.5)

Taking the $\alpha^{(r)} \to \infty$ limit of this expression, we arrive at the following expression for $F/N$,

$$\frac{F}{N} = e_0 - \frac{\sqrt{3} H^2}{2\pi^2} \sum_{r=1}^{2} \kappa^{(r)} \int_{0}^{\infty} d\lambda \, e^{-2\pi \lambda / 3} S^{(r)}(\lambda)$$

$$- \frac{\pi^2 T^2}{3\sqrt{3}} \sum_{r=1}^{2} \kappa^{(r)} \left[ \int_{0}^{\infty} d\lambda \, e^{-2\pi \lambda / 3} T^{(r)}(\lambda) + \frac{1}{S^{(r)}(0)} \right].$$  \hspace{1cm} (5.6)
Writing the functions $S^{(r)}(\lambda)$ and $T^{(r)}(\lambda)$ as the Fourier transforms of $\hat{S}(\omega)$ and $\hat{T}(\omega)$, respectively, and then performing both the $\lambda$ and $\omega$ integrals, we obtain

$$
\frac{F}{N} = e_0 - \frac{\sqrt{3}H^2}{2\pi^2} A - \frac{\pi^2 T^2}{3\sqrt{3}} B,
$$

(5.7)

with

$$
A = \sum_{r=1}^{2} \kappa^{(r)} \hat{S}^{(r)}(2\pi i/3) = \kappa^T \hat{S}(2\pi i/3),
$$

$$
B = \sum_{r=1}^{2} \kappa^{(r)} \left[ \hat{T}^{(r)}(2\pi i/3) + \frac{1}{S^{(r)'}(0)} \right] = \kappa^T \left[ \hat{T}(2\pi i/3) + \left( \frac{1}{S^{(1)'}(0)} \right) \frac{1}{S^{(2)'}(0)} \right].
$$

(5.8)

where we again switch to a matrix notation.

Since both $A$ and $B$ involve $\kappa^T$, we begin by observing from (4.50) that

$$
\kappa = \begin{pmatrix} \kappa^{(1)} \\ \kappa^{(2)} \end{pmatrix} = G_- (-2\pi i/3)^{-1} G_-(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

(5.9)

Taking the transpose of this equation, and using the property (4.43), we obtain

$$
\kappa^T = \begin{pmatrix} \kappa^{(1)} & \kappa^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} G_+(0) G_+(2\pi i/3)^{-1}.
$$

(5.10)

We now consider $A$. Evaluating $\hat{S}(2\pi i/3)$ using the expression (4.51), we obtain

$$
A = \frac{\sqrt{3}}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} G_+(0) G_-(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

(5.11)

Recalling the factorization equation (4.42) and the explicit expression (4.39) for $\hat{K}(\omega)$, we conclude that

$$
A = \frac{\sqrt{3}}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \left( 1 - \hat{K}(0) \right)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{3}.
$$

(5.12)

There remains to compute $B$. Evaluating $\hat{T}(2\pi i/3)$ using the expression (4.61), we obtain (after a crucial cancelation)

$$
B = \begin{pmatrix} 1 & 1 \end{pmatrix} G_+(0) \begin{pmatrix} 1/S^{(1)'}(0) \\ 1/S^{(2)'}(0) \end{pmatrix}.
$$

(5.13)
Recall Eq. (4.52),
\[
\begin{pmatrix}
S^{(1)'}(0) \\
S^{(2)'}(0)
\end{pmatrix} = \frac{4\pi^2}{3\sqrt{3}} G^-(0) \begin{pmatrix}
1 \\
1
\end{pmatrix}.
\] (5.14)

Taking the transpose of this equation and again using the property (4.43), we see that
\[
B = \frac{3\sqrt{3}}{4\pi^2} \begin{pmatrix}
S^{(1)'}(0) \\
S^{(2)'}(0)
\end{pmatrix} \begin{pmatrix}
1/S^{(1)'}(0) \\
1/S^{(2)'}(0)
\end{pmatrix} = \frac{3\sqrt{3}}{2\pi^2}.
\] (5.15)

The expression for the free energy per site is therefore
\[
\frac{F}{N} = e_0 - \frac{3}{2\pi^2} H^2 - \frac{1}{2} T^2.
\] (5.16)

As foreseen above, this result was obtained without using explicit expressions for $G_+$ and $G_-$. It follows that the magnetic susceptibility and specific heat, to lowest order, are given by
\[
\chi = -\frac{\partial^2}{\partial H^2} \left( \frac{F}{N} \right) \bigg|_T = \frac{3}{\pi^2},
\]
\[
C_H = -T \frac{\partial^2}{\partial T^2} \left( \frac{F}{N} \right) \bigg|_H = T.
\] (5.17)

For a critical chain, the free energy per site is given by\textsuperscript{32,33}
\[
\frac{F}{N} = e_0 - \frac{\pi c}{6v_s} T^2 + \cdots,
\] (5.18)

where $c$ is the central charge and $v_s$ is the velocity of sound. For our model, $v_s = 2\pi/3$ (see Eq. (4.12)), and hence $c = 2$. 

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6. Discussion

For an integrable model, the algebraic calculations leading to the energy eigenvalues in terms of solutions of the BA equations are quite elegant and precise. The same cannot be said for the corresponding thermodynamic calculations, at least in their present formulation. This is both surprising and disappointing. For instance, one expects that there should be an analogue of the Sugawara construction for integrable models, by which one could determine the low-temperature specific heat (central charge) by purely algebraic means. The fact that an explicit Wiener-Hopf factorization is not needed to compute such properties also suggests that an alternative approach should be possible. Indeed, very recently, progress has been made\textsuperscript{35} towards an algebraic formulation of the thermodynamics of integrable models.

Despite its shortcomings, the approach which we have followed here to investigate the low-temperature thermodynamics of the closed $su(3)$-invariant chain in the fundamental representation is nevertheless practical. It reproduces the known result for the central charge, and evidently, it can be implemented for $su(n)$.

We recall\textsuperscript{33} that, in the continuum limit, integrable spin-$s$ $su(2)$-invariant chains\textsuperscript{12} are described by level $k = 2s$ $su(2)$ WZW models. Moreover, Affleck has found\textsuperscript{36} a simple relation between the magnetic susceptibility $\chi$ and the level, namely $\chi v_s = k/2\pi$. Presumably, there is a generalization of this relation to the $su(n)$ case, for which there are $n - 1$ (rank of $su(n)$) magnetic susceptibilities $\chi_i$, $i = 1, \cdots, n - 1$. We have determined for $su(3)$ a particular linear combination of magnetic susceptibilities, which is dictated by the special imbedding $so(3) \subset su(3)$ which characterizes the $A_2^{(2)}$ chain. Our result may be relevant to the $su(n)$-generalization of Affleck’s relation.

Having treated the noncritical regime of the $A_2^{(2)}$ chain, the task now is to investigate the critical regime. As noted in the Introduction, this may be more feasible for the open
chain, which has $U_q[su(2)]$ symmetry. We hope to report on this problem in the future.

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