QR-submanifolds and Riemannian metrics with $G_2$ holonomy

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Abstract

In this note we prove that QR-submanifolds of the hyper-Kähler manifolds under some conditions admit the $G_2$ holonomy. We give simplest examples of such QR-submanifolds namely tori.

We conjecture that all $G_2$ holonomy manifolds arise in this way.

1 Introduction

The study of $G_2$-manifolds lacks explicit examples of closed manifolds. First complete Riemannian metrics with holonomy $G_2$ are constructed by Bryant and Salamon in [1]. First compact examples are given by Joyce in [2, 3]. Later Kovalev constructs more compact examples in [4, 5]. Note that metrics constructed in [2, 3, 4, 5] are not explicit.

Lack of examples is a consequence of the fact that $G_2$-manifolds are not generally algebraic in the broad sense of the term.

In this paper we try to partially explain this fact and conjecture that $G_2$-manifolds are generally QR-submanifolds of hyper-Kähler manifolds. Roughly speaking, QR-submanifolds are real hypersurfaces of hyper-Kähler manifolds.

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2 Preliminaries

2.1 $G_2$-structure

Define a 3-form $\Omega_0$ on $\mathbb{R}^7$ by

$$\Omega_0 = x^{127} + x^{136} + x^{145} + x^{235} - x^{246} + x^{347} + x^{567}. \quad (1)$$

By $x^{ijk}$ denote the $x^i \wedge x^j \wedge x^k$. The subgroup of $GL(7, \mathbb{R})$ preserving $\Omega_0$ and orientation is called the $G_2$ group.

Let $M$ be an oriented closed 7-manifold. Suppose there exists a global 3-form $\Omega$ such that pointwise it coincides with $\Omega_0$; then $M$ is called a $G_2$-manifold or we say that $M$ carries the $G_2$-structure. It is known that the orientation and the Riemannian metric are uniquely determined by the $G_2$-structure.

2.2 Cross products

Let $M$ be a $G_2$-manifold. Suppose a multilinear alternating smooth map $P : TM \times TM \to TM$. Suppose $P$ satisfies compatibility conditions:

$$g(P(e_1, e_2), e_i) = 0, \quad i = 1, 2; \quad (2)$$

$$\|P(e_1, e_2)\|^2 = \|e_1\|^2 \|e_2\|^2 - g(e_1, e_2)^2, \quad \|e\|^2 = g(e, e). \quad (3)$$

Then $P$ is called a cross product. We also denote $P(e_1, e_2)$ by $e_1 \times e_2$.

The cross product is uniquely determined by the 3-form $\Omega$:

$$\Omega(e_1, e_2, e_3) = g(P(e_1, e_2), e_3). \quad (4)$$

Conversely, the cross product defines the metric by the following formula:

$$P(e_1, P(e_1, e_2)) = -\|e_1\|^2 e_2 + g(e_1, e_2) e_1. \quad (5)$$

Using (4), we determine the 3-form $\Omega$ from the cross product and the metric. Thus the cross product implies the $G_2$-structure and vice versa.

Recall that if cross product is parallel with respect to the metric connection, then the holonomy group of $M$ is a subgroup of $G_2$ and coincides with $G_2$ iff $\pi_1(M)$ is a finite group [2].
2.3 QR-submanifolds

Riemannian $4n$-manifold with holonomy group contained in $Sp(n)$ is called a hyper-Kähler manifold.

Suppose $M$ is a submanifold of the hyper-Kähler $\overline{M}$ such that normal bundle of $M$ is the direct sum of $\nu$ and $\nu^\perp$ and

$$J_i\nu \subset \nu, \quad J_i\nu^\perp \subset TM, \quad i = 1, 2, 3,$$

(6)

where by $J_i$ we denote the $i$th complex structure of $\overline{M}$. Then $M$ is called a QR-submanifold of $\overline{M}$.

In what follows we consider QR-submanifolds with $\dim \nu^\perp = 1$ only. We call them QR-submanifolds of the hypersurface type.

3 The main result

Theorem 1. Let $M$ be an oriented 7-manifold. If $M$ is a hypersurface type QR-submanifold of hyper-Kähler $\overline{M}$, then there exists the $G_2$-structure on $M$.

Proof. We shall construct a cross product on $M$ such that it is compatible with the induced metric.

By (6), it follows that $\xi_i = J_i\nu$ are 3 non-vanishing vector fields on $M$. This agrees with [8], where existence of two non-vanishing vector fields on arbitrary compact orientable 7-manifold was shown. Third non-vanishing vector is the cross product of the first two (see also [9]).

We may assume that $\xi_i$ are unit orthogonal with respect to the induced metric vector fields on $M$. Locally we extend $\xi_i$ to a basis. Additional vectors are denoted by $\xi_\alpha$, i.e., by Greek indices.

Let the cross product $P$ be given by the following formulae:

$$P(\xi_i, \xi_j) = \xi_k, \quad (ijk) \in (123),$$

(7)

$$P(\xi_i, \xi_\alpha) = J_i(\xi_\alpha),$$

(8)

$$P(\xi_\alpha, J_i(\xi_\alpha)) = \xi_i.$$  

(9)

By the definition of a hypersurface type QR-submanifold, we have that for any $\xi_\alpha$, $\xi_\beta$ there exists complex structure $J_i$ such that $J_i\xi_\alpha = \xi_\beta$. Hence formulae (7)–(9) define the cross product on all basis vectors.

Clearly, $P$ satisfies (5) and therefore $P$ is compatible with the induced metric. \qed
Let’s find out when the constructed cross product is parallel that is when holonomy is reduced to a subgroup of $G_2$.

Let $\nabla$ and $\nabla'$ be a metric connection on $\overline{M}$ and $M$ respectively.

**Claim 1.**

$$\nabla\xi_i = J_i(\nabla' n) - b(\xi_i).$$  \hspace{1cm} (10)

$$(\nabla J_i)(\xi_\alpha) = J_i \circ b(\xi_\alpha) - b \circ J_i(\xi_\alpha).$$  \hspace{1cm} (11)

**Proof.** By the Gauss formula, we have

$$\nabla\xi_i = \nabla\xi_i' + b(\xi_i),$$  \hspace{1cm} (12)

where $b(\xi_i) = b(\xi_i, \cdot)$ and $b$ is the second fundamental form.

Also, the definition of the hyper-Kähler manifold implies that

$$\nabla\xi_i = \nabla J_i(n) = (\nabla J_i)(n) + J_i(\nabla n) = J_i(\nabla n).$$  \hspace{1cm} (13)

Combining (12) and (13), we get (10).

Similarly, combining

$$\nabla(J_i\xi_\alpha) = \nabla(J_i(\xi_\alpha)) + b(J_i(\xi_\alpha)) = (\nabla J_i)(\xi_\alpha) + J_i(\nabla\xi_\alpha) + b(J_i(\xi_\alpha))$$  \hspace{1cm} (14)

and

$$\nabla(J_i\xi_\alpha) = (\nabla J_i)(\xi_\alpha) + J_i(\nabla\xi_\alpha) = J_i(\nabla\xi_\alpha),$$  \hspace{1cm} (15)

we have (11).

By definition, put

$$X_i(\xi) = J_i(\nabla\xi n) - b(J_i n, \xi), \quad Y_i(\xi, \eta) = J_i b(\xi, \eta) - b(\xi, J_i \eta).$$

**Claim 2.**

$$(\nabla P)(\xi_i, \xi_j) = X_k - X_i \times \xi_j - \xi_i \times X_j.$$  \hspace{1cm} (16)

$$(\nabla P)(\xi_i, \xi_\alpha) = Y_i(\xi_\alpha) - X_i \times \xi_\alpha.$$  \hspace{1cm} (17)

$$(\nabla P)(\xi_i, \xi_\alpha) = Y_i(\xi_\alpha) - X_i \times \xi_\alpha.$$  \hspace{1cm} (18)

**Proof.** Let’s prove (16). We differentiate (7):

$$(\nabla P)(\xi_i, \xi_j) = \nabla \xi_k - P(\nabla\xi_i, \xi_j) - P(\xi_j, \nabla\xi_i).$$  \hspace{1cm} (19)

Combining (10), (19) and (7), we obtain (16).

Similarly, if we differentiate (8) and (9), we get (17) and (18).
Recall that $\nabla P = 0$ implies that $\text{Hol}(M) \subset G_2$. If we equate with zero formulae (16)–(18), then we obtain sufficient conditions for $\nabla P = 0$. Note that (17) and (18) are equivalent.

**Theorem 2.** Suppose $M$ is an oriented 7-manifold such that $M$ is a hypersurface type QR-submanifold of the hyper-Kähler $\overline{M}$. If the following equations hold:

\[
X_k(\xi) - X_i(\xi) \times \xi_j - \xi_i \times X_j(\xi) = 0, \quad (20)
\]

\[
Y_i(\xi, \eta) - X_i(\xi) \times \eta = 0, \quad (21)
\]

for any $\xi, \eta, J\eta \in \Gamma(TM)$, $i = 1, 2, 3$; then holonomy group of $M$ is contained in $G_2$.

**Example.** Simplest examples of QR-submanifolds with holonomy contained in $G_2$ are totally geodesic hypersurfaces. These are flat tori: $T^7 \hookrightarrow T^8$ and $T^3 \times K3 \hookrightarrow T^4 \times K3$.

## 4 Conjecture

Emery Thomas proves in [8] that any $G_2$-manifold admits 3 non-vanishing unit vector fields $\xi_i$. There exists a complex structure on $\xi_i^\perp$ determined by (5). Verbitsky shows in [10] that these complex structures are integrable iff the holonomy is contained in $G_2$. Due to integrability we formulate the following

**Conjecture.** Any $G_2$ holonomy manifold is a QR-submanifold of a certain hyper-Kähler manifold satisfying the conditions of Theorem 2.

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