On the dimensions of conformal repellers. Randomness and parameter dependency

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Abstract

Bowen’s formula relates the Hausdorff dimension of a conformal repeller to the zero of a ‘pressure’ function. We present an elementary, self-contained proof which bypasses measure theory and the Thermodynamic Formalism to show that Bowen’s formula holds for $C^1$ conformal repellers. We consider time-dependent conformal repellers obtained as invariant subsets for sequences of conformally expanding maps within a suitable class. We show that Bowen’s formula generalizes to such a repeller and that if the sequence is picked at random then the Hausdorff dimension of the repeller almost surely agrees with its upper and lower Box dimensions and is given by a natural generalization of Bowen’s formula. For a random uniformly hyperbolic Julia set on the Riemann sphere we show that if the family of maps and the probability law depend real-analytically on parameters then so does its almost sure Hausdorff dimension.

1 Random Julia sets and their dimensions

Let $(U,d_U)$ be an open, connected subset of the Riemann sphere avoiding at least three points and equipped with a hyperbolic metric. Let $K \subset U$ be a compact subset. We denote by $E(K,U)$ the space of unramified conformal covering maps, $f : D_f \rightarrow U$, with the requirement that the covering domain $D_f \subset K$. Denote by $Df : D_f \rightarrow \mathbb{R}_+$ the conformal derivative of $f$, see equation (2.4), and by $\|Df\| = \sup_{f^{-1}K} Df$ the maximal value of this derivative over the set $f^{-1}K$. Let $F = (f_n) \subset E(K,U)$ be a sequence of such maps. The intersection

$$J(F) = \bigcap_{n \geq 1} f_1^{-1} \circ \cdots \circ f_n^{-1}(U)$$

(1.1)

defines a uniformly hyperbolic Julia set for the sequence $F$. Let $(\Upsilon, \nu)$ be a probability space and let $\omega \in \Upsilon \rightarrow f_\omega \in E(K,U)$ be a measurable map. Suppose that the elements in the sequence $F$ are picked independently, according to the law $\nu$. Then $J(F)$ becomes a random ‘variable’. Our main objective is to establish the following

Theorem 1.1

I. Suppose that $\mathbb{E}(\log \|Df_\omega\|) < \infty$. Then the Hausdorff dimension of $J(F)$ equals almost surely its upper and lower box dimensions and is given by a generalization of Bowen’s formula.

II. Suppose in addition that: (a) The family of maps $(f_\omega)_{\omega \in \Upsilon}$ and the probability measure, $\nu$, depend uniformly real-analytically on complex parameters. (b) For any local inverse $f_\omega^{-1}$, $\log Df_\omega \circ f_\omega^{-1}$ is uniformly Lipschitz in parameters and in $\omega \in \Upsilon$. (c) The condition number, $\|Df_\omega\| \cdot \|1/Df_\omega\|$, is uniformly bounded in parameters and in $\omega \in \Upsilon$. Then the almost sure Hausdorff dimension depends real-analytically on the parameters (for more precise definitions see section 6).
Example 1.2 Let $a \in \mathbb{C}$ and $r \geq 0$ be such that $|a| + r < \frac{1}{4}$. Suppose that $c_n \in \mathbb{C}$, $n \in \mathbb{N}$ are i.i.d. random variables uniformly distributed in the closed disk $B(a, r)$ and that $N_n$, $n \in \mathbb{N}$ are i.i.d. random variables distributed according to a Poisson law of parameter $\lambda \geq 0$. We consider the sequence of maps, $F = (f_n)_{n \in \mathbb{N}}$, given by

$$f_n(z) = z^{N_n+2} + c_n.$$  \tag{1.2}

An associated `random’ Julia set may be defined through

$$J(F) = \partial \{z \in \mathbb{C} : f_n \circ \cdots \circ f_1 (z) \to \infty\} \tag{1.3}$$

As shown in section 6 the family verifies all conditions for Theorem 1.1, part I and II. The random Julia set therefore has the same almost sure Hausdorff and upper/lower box dimension $\dim(J(F)) = d(a, r, \lambda)$ which in addition depends real-analytically upon $a$, $r$ and $\lambda$. Note that the sequence of degrees, $(N_n)_{n \in \mathbb{N}}$, almost surely is unbounded when $\lambda > 0$.

Rufus Bowen, one of the founders of the Thermodynamic Formalism (henceforth abbreviated TF), saw more than twenty years ago [Bow79] a natural connection between the geometric properties of a conformal repeller and the TF for the map(s) generating this repeller. The Hausdorff dimension $\dim_H(\Lambda)$ of a smooth and compact conformal repeller $(\Lambda, f)$ is precisely the unique zero $s_{\text{crit}}$ of a ‘pressure’ function $P(s, \Lambda, f)$ having its origin in the TF. This relationship is now known as ‘Bowen’s formula’. The original proof by Bowen [Bow79] was in the context of Kleinian groups and involved a finite Markov partition and uniformly expanding conformal maps. Using TF he constructed a finite Gibbs measure of zero ‘conformal pressure’ and showed that this measure is equivalent to the $s_{\text{crit}}$-dimensional Hausdorff measure of $\Lambda$. The conclusion then follows.

Bowen’s formula apply in many other cases. For example, when dealing with expanding ‘Markov maps’ the Markov partition need not be finite and one may eventually have a neutral fixed point in the repeller [Urb96, SSU01]. One may also relax on smoothness of the maps involved. Barreira [Bar96] and also Gatzouras and Peres [GP97] were able to demonstrate that Bowen’s formula holds for classes of $C^1$ repellers. A priori, the classical TF does not apply in this setup. Gatzouras and Peres circumvene the problem by using an approximation argument and then apply the classical theory. Barreira, following the approach of Pesin [Pes88], defines the Hausdorff dimension as a Caratheodory dimension characteristic. By extending the TF itself Barreira goes closer to the core of the problem and may also consider maps somewhat beyond the $C^1$ case mentioned. The proofs are, however, fairly involved and do not generalize easily neither to a random set-up nor to a study of parameter-dependency.

In [Rue82], Ruelle showed that the Hausdorff dimension of the Julia set of a uniformly hyperbolic rational map depends real-analytically on parameters. The original approach of Ruelle was indirect, using dynamical zeta-functions, [Rue76]. Other later proofs are based on holomorphic motions, (see e.g. Zinsmeister [Zin99] and references therein). In either case it is difficult to adapt the proofs to a time-dependent and/or random set-up because the methods do not give sufficiently uniform bounds. In another context, Furstenberg and Kesten, [FK60], had shown, under a condition of log-integrability, that a random product of matrices has a unique almost sure characteristic exponent. Ruelle, in [Rue79], required in addition that the matrices contracted uniformly a positive cone and satisfied a compactness and continuity condition with respect to the underlying probablility space. He showed that under these conditions if the family of positive random matrices depends real-analytically on parameters then so does the almost sure characteristic exponent of their product. He did not, however, allow the probability law to depend on parameters. We note here that if the matrices contract uniformly a positive cone, the
topological conditions in [Rue79] may be replaced by the weaker condition of measurability + log-integrability. We also mention the more recent paper, [Rue97], of Ruelle. It is in spirit close to [Rue79] (not so obvious at first sight) but provides a more global and far more elegant point of view to the question of parameter-dependency. It has been an invaluable source of inspiration to our work.

In this article we depart from the traditional path stuck out by TF. In Part I we present a proof of Bowen’s formula, Theorem 2.1, for a $C^1$ conformal repeller which bypasses measure theory and most of the TF. Measure theory can be avoided essentially because $\Lambda$ is compact and the only element remaining from TF is a family of transfer operators which encodes geometric informations into analytic ones. Our proof is short and elementary and releases us from some of the smoothness conditions imposed by TF. An elementary proof of Bowen’s formula should be of interest in its own, at least in the author’s opinion. It generalizes, however, also to situations where a ‘standard’ approach either fails or manages only with great difficulties. We consider classes of time-dependent conformal repellers. By picking a sequence of maps within a suitable equi-conformal class one may study the associated time-dependent repeller. Under the assumption of uniform equi-expansion and equi-mixing and a technical assumption of sub-exponential ‘growth’ of the involved sequences we show, Theorem 3.7, that the Hausdorff and Box Dimensions are bounded within the unique zeros of a lower and an upper conformal pressure. Similar results were found by Barreira [Bar96, Theorem 2.1 and 3.8]. When it comes to random conformal repellers, however, the approach of Pesin and Barreira seems difficult to generalize. Kifer [Kif96] and later, also Bogenshutz and Ochs [BO99], using time-dependent TF and Martingale arguments, considered random conformal repellers for certain classes of transformations, but under the smoothness restriction imposed by TF. In Theorem 4.4, a straight-forward application of Kingmans sub-ergodic Theorem, [King68], allows us to deal with such cases without such restrictions. In addition we obtain very general formulae for the parameter-dependency of the Hausdorff dimension.

Part II is devoted to Random Julia sets on hyperbolic subsets of the Riemann sphere. Here statements and hypotheses attain much more elegant forms, cf. Theorem 1.1 and Example 1.2 above. Straight-forward Koebe estimates enables us to apply Theorem 4.4 to deduce Theorem 5.3 which in turn yields Theorem 1.1, part (I). The parameter dependency is, however, more subtle. The central ideas are then the following:

1. We introduce a ‘mirror embedding’ of our hyperbolic subset and then a related family of transfer operators and cones which a natural (real-)analytic structure.

2. We compute the pressure function as a hyperbolic fixed point of a holomorphic map acting upon the cone-family. When the family of maps depends real-analytically on parameters, then the real-analytically dependency of the dimensions, Theorem 6.22, follows from an implicit function theorem.

3. The above mentioned fixed point is hyperbolic. This implies an exponential decay of the fixed point with ‘time’ and allow us to treat a real-analytic parameter dependency with respect to the underlying probability law. This concludes the proof of Theorem 1.1.
2 Part I: $C^1$ conformal repellers and Bowen’s formula

Let $(\Lambda, d)$ be a non-empty compact metric space without isolated points and let $f : \Lambda \to \Lambda$ be a continuous surjective map. Throughout Part I we will write interchangeably $f_x$ or $f(x)$ for a map $f$ applied to a point $x$. We say that $f$ is $C^1$ conformal at $x \in \Lambda$ iff the following double limit exists:

$$Df_x = \lim_{u \neq v \to x} \frac{d(f_u, f_v)}{d(u, v)}.$$  (2.4)

The limit is called the conformal derivative of $f$ at $x$. The map $f$ is said to be $C^1$ conformal on $\Lambda$ if it is so at every point of $\Lambda$. A point $x \in \Lambda$ is said to be critical if $a_x = 0$.

The product $Df_x^{(n)} = Df_{f^{n-1}x} \cdots Df_x$ along the orbit of $x$ is the conformal derivative for the $n$th iterate of $f$. The map is said to be uniformly expanding if there are constants $C > 0$, $\beta > 1$ for which $Df_x^{(n)} \geq C \beta^n$ for all $x \in \Lambda$ and $n \in \mathbb{N}$. We say that $(\Lambda, f)$ is a $C^1$ conformal repeller if

(C1) $f$ is $C^1$ conformal on $\Lambda$.
(C2) $f$ is uniformly expanding.
(C3) $f$ is an open mapping.

For $s \in \mathbb{R}$ we define the dynamical pressure of the $s$-th power of the conformal derivative by the formula:

$$P(s, \Lambda, f) = \liminf_n \frac{1}{n} \log \sup_{y \in \Lambda} \sum_{x \in \Lambda : f^n_x = y} \left(Df_x^{(n)}\right)^{-s}.$$  (2.5)

We then have the following

Theorem 2.1 (Bowen’s formula) [Bow79, Rue82, Fal89, Bar96, GP97] Let $(\Lambda, f)$ be a $C^1$ conformal repeller. Then, the Hausdorff dimension of $\Lambda$ coincides with its upper and lower box dimensions and is given as the unique zero of the pressure function $P(s, \Lambda, f)$.

For clarity of the proof we will here impose the additional assumption of strong mixing. We have delegated to Appendix A a sketch of how to remove this restriction. We have chosen to do so because (1) the proof is really much more elegant and (2) there seems to be no natural generalisation when dealing with the time-dependent case (apart from trivialities).

More precisely, to any given $\delta > 0$ we assume that there is an integer $n_0 = n_0(\delta) < \infty$ (denoted the $\delta$-covering time for the repeller) such that for every $x \in \Lambda$:

(C4) $f^{n_0} B(x, \delta) = \Lambda$.  (2.6)

For the rest of this section, $(\Lambda, f)$ will be assumed to be a strongly mixing $C^1$ conformal repeller, thus verifying (C1)-(C4).

Recall that a countable family $\{U_n\}_{n \in \mathbb{N}}$ of open sets is a $\delta$-cover($\Lambda$) if $\text{diam } U_n < \delta$ for all $n$ and their union contains (here equals) $\Lambda$. For $s \geq 0$ we set

$$M_\delta(s, \Lambda) = \inf \left\{ \sum_n (\text{diam } U_n)^s : \{U_n\}_{n \in \mathbb{N}} \text{ is a } \delta - \text{cover}(\Lambda) \right\} \in [0, +\infty]$$
Then $M(s, \Lambda) = \lim_{\delta \to 0} M_{\delta}(s, \Lambda) \in [0, +\infty]$ exists and is called the $s$-dimensional Hausdorff measure of $\Lambda$. The Hausdorff dimension is the unique critical value $s_{\text{crit}} = \dim_H \Lambda \in [0, \infty]$ such that $M(s, \Lambda) = 0$ for $s > s_{\text{crit}}$ and $M(s, \Lambda) = \infty$ for $s < s_{\text{crit}}$. The Hausdorff measure is said to be finite if $0 < M(s_{\text{crit}}, \Lambda) < \infty$.

Alternatively we may replace the condition on the covering sets by considering finite covers by open balls $B(x, \delta)$ of fixed radii $\delta > 0$. Then the limit as $\delta \to 0$ of $M_{\delta}(s, \Lambda)$ need not exist so we replace it by taking lim sup and lim inf. We then obtain the upper, respectively the lower $s$-dimensional Box ‘measure’. The upper and lower Box Dimensions, $\dim^{B} \Lambda$ and $\dim_{B} \Lambda$, are the corresponding critical values. It is immediate that

$$0 \leq \dim_H \Lambda \leq \dim_B \Lambda \leq \dim^{B} \Lambda \leq +\infty$$

Remarks 2.2

1. Let $J(f)$ denote the Julia set of a uniformly hyperbolic rational map $f$ of the Riemann sphere. There is an open (hyperbolic) neighborhood $U$ of $J(f)$ such that $V = f^{-1}U$ is compactly contained in $U$ and such that $f$ has no critical points in $V$. Writing $d$ for the hyperbolic metric on $U$ one verifies that $(J(f), f)$ is a $C^1$ conformal repeller.

2. Let $X$ be a $C^1$ Riemannian manifold without boundaries and let $f : X \to X$ be a $C^1$ map. It is an exercise in Riemannian geometry to see that $f$ is uniformly conformal at $x \in X$ iff $f_x : T_x X \to T_{f(x)} X$ is a conformal map of tangent spaces and in that case, $Df_x = \|f_x\|$. When $\dim X < \infty$ condition (C3) follows from (C1)-(C2). We note also that being $C^1$ (the double limit in equation 2.4) rather than just differentiable is important.

2.1 Geometric bounds

We will first establish sub-exponential geometric bounds for iterates of the map $f$. In the following we say that a sequence $(b_n)_{n \in \mathbb{N}}$ of positive real numbers is sub-exponential or of sub-exponential growth if $\lim_n \sqrt[n]{b_n} = 1$. For notational convenience we will also assume that $Df_x \geq \beta > 1$ for all $x \in \Lambda$. This may always be achieved in the present set-up by considering a high enough iterate of the map $f$, possibly redefining $\beta$.

Define the divided difference,

$$f[u, v] = \begin{cases} \frac{d(f(u), f(v))}{d(u, v)} & u \neq v \in \Lambda, \\ Df_u & u = v \in \Lambda. \end{cases}$$ (2.7)

Our hypothesis on $f$ implies that $f[\cdot, \cdot]$ is continuous on the compact set $\Lambda \times \Lambda$ and not smaller than $\beta > 1$ on the diagonal of the product set. We let $\|Df\| = \sup_{u \in \Lambda} Df_u < +\infty$ denote the maximal conformal derivative on the repeller.

Choose $1 < \lambda_0 < \beta$. Uniform continuity and openness of the map $f$ show that we may find $\delta_f > 0$ and $\lambda_1 < +\infty$ such that

(C2') $\lambda_0 \leq f[u, v] \leq \lambda_1$ whenever $d(u, v) < \delta_f$.

(C3') $B(f_x, \delta_f) \subset fB(x, \delta_f)$, for all $x \in \Lambda$.

The constant $\delta_f$ gives a scale below which the map $f$ is injective, uniformly expanding and (locally) onto. In the following we will assume that values of $\delta_f > 0$, $\lambda_0 > 1$ and $\lambda_1 < +\infty$ have been found so as to satisfy conditions (C2') and (C3').
We define the distortion of $f$ at $x \in \Lambda$ and for $r > 0$ as follows:

$$
\epsilon_f(x, r) = \sup \{ \log \frac{f[u_1, u_2]}{f[u_3, u_4]} : \text{all } u_i \in B(x, \delta_f) \cap f^{-1}B(f_x, r) \}.
$$

(2.8)

This quantity tends to zero as $r \to 0^+$ uniformly in $x \in \Lambda$ (same compactness and continuity as before). Thus,

$$
\epsilon(r) = \sup_{x \in \Lambda} \epsilon_f(x, r)
$$

tends to zero as $r \to 0^+$. When $x \in \Lambda$ and the $u_i$’s are as in (2.8) then also:

$$
\left| \log \frac{f[u_1, u_2]}{Df_{u_3}} \right| \leq \epsilon(r) \quad \text{and} \quad \left| \log \frac{Df_{u_1}}{Df_{u_2}} \right| \leq \epsilon(r).
$$

(2.9)

For $n \geq \mathbb{N}$ we define the $n$-th ‘Bowen ball’ around $x \in \Lambda$,

$$
B_n(x) \equiv B_n(x, \delta_f, f) = \{ u \in \Lambda : d(f_x^k, f_u^k) < \delta_f, \ 0 \leq k \leq n \}.
$$

We say that $u$ is $n$-close to $x \in \Lambda$ if $u \in B_n(x)$. The Bowen balls act as ‘reference’ balls, getting uniformly smaller with increasing $n$. In particular, $\text{diam} B_n(x) \leq 2 \delta_f \lambda_0^{-n}$, i.e. tends to zero exponentially fast with $n$. We also see that for each $x \in \Lambda$ and $n \geq 0$ the map,

$$
f : B_{n+1}(x) \to B_n(f_x),
$$

is a uniformly expanding homeomorphism.

Expansiveness of the map $f$ means that nearby points may follow very different future trajectories. Our assumptions assure, however, that nearby points have very similar backwards histories. The following two Lemmas emphasize this point:

**Lemma 2.3 [Pairing]** For each $y, w \in \Lambda$ with $d(y, w) \leq \delta_f$ and for every $n \in \mathbb{N}$ the sets $f^{-n}\{y\}$ and $f^{-n}\{z\}$ may be paired uniquely into pairs of $n$-close points.

**Proof:** Take $x \in f^{-n}\{y\}$. The map $f^n : B_n(x) \to B_0(f^n_x) = B(y, \delta_f)$ is a homeomorphism. Thus there is a unique point $u \in f^{-n}\{z\} \cap B_n(x)$. By construction, $x \in B_n(u)$ iff $u \in B_n(x)$. Therefore $x \in f^{-n}\{y\} \cap B_n(u)$ is the unique pre-image of $y$ in the $n$-th Bowen ball around $u$ and we obtain the desired pairing. \(\Box\)

**Lemma 2.4 [Sub-exponential Distortion]** There is a sub-exponential sequence, $(c_n)_{n \in \mathbb{N}}$, such that for any two points $z, u$ which are $n$-close to $x \in \Lambda$ ($x \neq u$)

$$
\frac{1}{c_n} \leq \frac{d(f_u^n, f_x^n)}{d(u, x) \cdot Df_z^{(n)}} \leq c_n \quad \text{and} \quad \frac{1}{c_n} \leq \frac{Df_z^n}{Df_z^{(n)}} \leq c_n
$$

Proof: For all $1 \leq k \leq n$ we have that $f_u^k \in B_{n-k}(f_x^k)$. Therefore, $d(f_u^k, f_x^k) < \delta_f \lambda_0^{k-n}$ and the distortion bound (2.9) implies that

$$
\left| \log \frac{d(f_u^n, f_x^n)}{d(u, x) \cdot Df_z^{(n)}} \right| \leq \epsilon(\delta_f) + \epsilon(\delta_f \lambda_0^{-1}) + \cdots + \epsilon(\delta_f \lambda_0^{1-n}) \equiv \log c_n.
$$

Since $\lim_{r \to 0} \epsilon(r) = 0$ it follows that $\frac{1}{n} \log c_n \to 0$, whence that the sequence $(c_n)_{n \in \mathbb{N}}$ is of sub-exponential growth. This yields the first inequality and the second is proved e.g. by taking the limit $u \to x$. \(\Box\)
 Remarks 2.5 When \( K = \int_0^{\lambda_0^d} \epsilon(t)/t \, dt < +\infty \) one verifies that the distortion stays uniformly bounded, i.e. that \( c_n \leq K/(\lambda_0 - 1) < \infty \) uniformly in \( n \). This is the case, e.g. when \( \epsilon \) is Hölder continuous at zero.

### 2.2 Transfer operators

Let \( \mathcal{M}(\Lambda) \) denote the Banach space of bounded real valued functions on \( \Lambda \) equipped with the sup-norm. We denote by \( \chi_U \) the characteristic function of a subset \( U \subset \Lambda \) and we write \( 1 = \chi_{\Lambda} \) for the constant function \( 1(x) = 1, \forall x \in \Lambda \). For \( \phi \in \mathcal{M}(\Lambda) \) and \( s \geq 0 \) we define the positive linear transfer\(^1\) operator

\[
(L_s \phi)_y \equiv (L_{s,f} \phi)_y \equiv \sum_{x \in \Lambda; f_x = y} (Df_x)^{-s} \phi_x, \quad y \in \Lambda.
\]

Since \( \Lambda \) has a finite \( \delta_f \)-cover and \( Df \) is bounded these operators are necessarily bounded. The \( n \)’th iterate of the operator \( L_s \) is given by

\[
(L^n_s \phi)_y = \sum_{x \in \Lambda; f^n_x = y} (Df^n_x)^{-s} \phi_x.
\]

It is of importance to obtain bounds for the action of \( L_s \) upon the constant function. More precisely, for \( s \geq 0 \) we denote

\[
M_n(s) \equiv \sup_{y \in \Lambda} L^n_s 1(y) \quad \text{and} \quad m_n(s) \equiv \inf_{y \in \Lambda} L^n_s 1(y).
\]

We then define the lower, respectively the upper pressure through

\[
-\infty \leq P(s) \equiv \lim \inf_n \frac{1}{n} \log m_n(s) \quad \leq \quad P(s) \equiv \lim \sup_n \frac{1}{n} \log M_n(s) \leq +\infty.
\]

**Lemma 2.6 [Operator bounds]** For each \( s \geq 0 \), the upper and lower pressures agree and are finite. We write \( P(s) \equiv P(s) = P(s) \in \mathbb{R} \) for the common value. The function \( P(s) \) is continuous, strictly decreasing and has a unique zero, \( s_{\text{crit}} \geq 0 \).

**Proof:** Fix \( s \geq 0 \). Since the operator is positive, the sequences \( M_n = M_n(s) \) and \( m_n = m_n(s) \), \( n \in \mathbb{N} \) are sub-multiplicative and super-multiplicative, respectively. Thus,

\[
m_k m_{n-k} \leq m_n \leq M_n \leq M_k M_{n-k}, \quad \forall 0 \leq k \leq n.
\]

This implies convergence of both \( \sqrt[n]{M_n} \) and \( \sqrt[n]{m_n} \), the limit of the former sequence being the spectral radius of \( L_s \) acting upon \( \mathcal{M}(\Lambda) \). Let us sketch a standard proof for the first sequence: Fixing \( k \geq 1 \) we write \( n = pk + r \) with \( 0 \leq r < k \). Since \( k \) is fixed, \( \lim \sup_n \max_{0 \leq r < k} \sqrt[n]{M_r} = 1 \). But then \( \lim \sup_n \sqrt[n]{M_n} = \lim \sup_p \sqrt[p]{M_{pk}} \leq \sqrt[p]{M_k} \). Taking \( \lim \inf \) (with respect to \( k \)) on the right hand side we conclude that the limit exists. A similar proof works for the sequence \( (m_n)_{n \in \mathbb{N}} \). Both limits are non-zero (\( \geq m_1 > 0 \)) and finite (\( \leq M_1 < \infty \)). We need to show that the ratio \( M_n/m_n \) is of sub-exponential growth.

Consider \( w, z \in \Lambda \) with \( d(w, z) < \delta_f \) and \( n > 0 \). The Pairing Lemma shows that we may pair the pre-images \( f^{-n}\{w\} \) and \( f^{-n}\{z\} \) into pairs of \( n \)-close points, say \((w_\alpha, z_\alpha)_{\alpha \in I_n}\) over a finite

\(^1\)The ‘transfer’-terminology, inherited from statistical mechanics, refers here to the ‘transfer’ of the encoded geometric information at a small scale to a larger scale, using the dynamics of the map, \( f \).
index set \( I_n \), possibly depending on the pair \((w, z)\). Applying the second distortion bound in Lemma 2.4 to each pair yields

\[
L^n_s \mathbf{1}(z) \geq \left( \frac{1}{c_n} \right)^s L^n_s \mathbf{1}(w). \tag{2.12}
\]

Choose \( w \in \Lambda \) such that \( L^n_s \mathbf{1}(w) \geq M_n/2 \). Given an arbitrary \( y \in \Lambda \) our strong mixing assumption (C4) implies that the set \( B(w, \delta_f) \cap f^{-n_0} \{y\} \) contains at least one point. Using (2.12) we obtain

\[
L^{n+n_0}_s \mathbf{1}(y) = \sum_{z: f^0_z = y} (Df^0_z)^{-s} L^n_s \mathbf{1}(z) \geq (\|Df\|^{n_0} c_n)^{-s} \frac{M_n}{2}.
\]

Thus,

\[
m_{n+n_0} \geq (\|Df\|^{n_0} c_n)^{-s} M_n / 2 \tag{2.13}
\]

and since \( c_n \) is of sub-exponential growth then so is \( M_n/m_{n+n_0} \) and therefore also \( M_{n+n_0}/m_{n+n_0} \leq M_{n_0} M_n/m_{n+n_0} \).

The functions \( s \log \beta + P(s) \) and \( s \log \|Df\| + P(s) \) are non-increasing and non-decreasing, respectively. Also \( 0 \leq P(0) < +\infty \). It follows that \( s \mapsto P(s) \) is continuous and that \( P \) has a unique zero \( s_{\text{crit}} \geq 0 \). \( \square \)

**Remarks 2.7** Super- and sub-multiplicativity (2.11) imply the bounds\(^2\)

\[
m_n(s) \leq e^{nP(s)} \leq M_n(s), \quad n \in \mathbb{N}.
\]

Clearly, if the distortion \( c_n \) is uniformly bounded then so is the ratio, \( M_n/m_n \leq K(s) < \infty \).

To prove Theorem 2.1 it suffices to show that \( s_{\text{crit}} \leq \dim_H(\Lambda) \) and \( \dim_B(\Lambda) \leq s_{\text{crit}} \).

**2.3** \( \dim_H(\Lambda) \geq s_{\text{crit}} \)

Let \( U \subset \Lambda \) be an open non-empty subset of diameter not exceeding \( \delta_f \). We will iterate \( U \) by \( f \) until the size of \( f^k U \) becomes ‘large’ compared to \( \delta_f \). As long as \( f^k \) stays injective on \( U \) the set \( \{z \in U : f^k_z = y\} \) contains at most one element for any \( y \in \Lambda \). Therefore, for such \( k \)-values

\[
(L^k_s \chi_U)(y) \leq \sup_{z \in U} (Df^k_z)^{-s}, \quad \forall \ y \in \Lambda. \tag{2.14}
\]

Choose \( x = x(U) \in U \) and let \( k = k(U) \geq 0 \) be the largest positive integer for which \( U \subset B_k(x) \). In other words:

(a) \( d(f^l_x, f^l_u) < \delta_f \) for \( 0 \leq l \leq k \) and all \( u \in U \),

(b) \( d(f^{k+1}_x, f^{k+1}_u) \geq \delta_f \) for some \( u \in U \).

\( k(U) \) is finite because \( U \cap \Lambda \) contains at least two distinct points which are going to be separated when iterating. Because of (a) \( f^k \) is injective on \( U \) so that (2.14) applies. On the other hand, (a) and (b) implies that there is \( u \in U \) for which \( \delta_f \leq d(f^{k+1}_x, f^{k+1}_u) \leq \lambda_1(f)d(f^k_x, f^k_u) \) where \( \lambda_1(f) \)

\(^2\)Such bounds are useful in applications as they yield computable rigorous bounds for the dimensions.
was the maximal dilation of $f$ on $\delta_f$-separated points. Our sub-exponential distortion estimate shows that for any $z \in U$, 

$$\frac{\delta_f/\lambda_1(f)}{\text{diam } U} \frac{1}{Df^k_z} \leq \frac{d(f^k, f^k_y)}{d(y, x)} \frac{1}{Df^k_z} \leq c_k.$$ 

Inserting this in (2.14) and using the definition of $m_n(s)$ we see that for any $y \in \Lambda$,

$$(L_{sU}^k y) \leq (\text{diam } U)^s (\frac{\lambda_1(f)c_k}{\delta_f})^s 1 \leq (\text{diam } U)^s \left( (\frac{\lambda_1(f)c_k}{\delta_f})^s \frac{1}{m_k(s)} \right) L_{sU}^k 1. \quad (2.15)$$

Choosing now $0 < s < s_{\text{crit}}$, the sequence $m_k(s)$ tends exponentially fast to infinity [when $s_{\text{crit}} = 0$ there is nothing to show]. Since the sequence $((c_k)^s)_{k \in \mathbb{N}}$ is sub-exponential the factor in square-brackets is uniformly bounded in $k$, say by $\gamma_1(s) < \infty$ (independent of $U$). Positivity of the operator implies that for any $n \geq k(U)$

$$L_{sU}^n 1 \leq \gamma_1(s) (\text{diam } U)^s L_{sU}^n 1.$$ 

If $(U_{\alpha})_{\alpha \in \mathbb{N}}$ is an open $\delta_f$-cover of the compact set $\Lambda$ then it has a finite sub-cover, say $\Lambda \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_m}$. Taking now $n = \max\{k(U_{\alpha_1}), \ldots, k(U_{\alpha_m})\}$ we obtain

$$L_{sU}^n 1 \leq \sum_{i=1}^m L_{sU}^n U_{\alpha_i} \leq \gamma_1(s) \sum_{i=1}^m (\text{diam } U_{\alpha_i})^s L_{sU}^n 1 \leq \gamma_1(s) \sum_\alpha (\text{diam } U_\alpha)^s L_{sU}^n 1. \quad (2.16)$$

This equation shows that $\sum_\alpha (\text{diam } U_\alpha)^s$ is bounded uniformly from below by $1/\gamma_1(s) > 0$. The Hausdorff dimension of $\Lambda$ is then not smaller than $s$, whence not smaller than $s_{\text{crit}}$.

### 2.4 $\text{dim}^B \Lambda \leq s_{\text{crit}}$

Fix $0 < r < \delta_f$ and let $x \in \Lambda$. This time we wish to iterate a ball $U = B(x, r)$ until it has a ‘large’ interior and contains a ball of size $\delta_f$. This may, however, not be good enough (cf. Figure 2.4).

We also need to control the distortion. Again these two goals combine nicely when considering the sequence of Bowen balls, $B_k \equiv B_k(x), k \geq 0$. It forms a sequence of neighborhoods of $x$, shrinking to $\{x\}$. Hence, there is a smallest integer $k = k(x, r) \geq 1$ such that $B_k \subset U$. $f^k$ maps $B_k$ homeomorphically onto $B_0(f^k_x) = B(f^k_x, \delta_f)$ and positivity of $L_s$ shows that

$$L_{sU}^k U \geq L_{sU}^k B_k \geq \inf_{z \in B_k} \left( Df^k_z \right)^{-s} \chi_{B(f^k_z, \delta_f)}.$$ 

By assumption $B_{k-1} \not\subset U$ so there must be a point $y \in B_{k-1}$ with $d(y, x) \geq r$. As $y$ is $(k-1)$-close to $x$ our distortion estimate shows that for any $z \in B_k(x) \subset B_{k-1}(x)$,

$$\frac{\delta_f}{r} \frac{\|Df\|}{Df^k(z)} > \frac{d(f^{k-1}_y, f^{k-1}_z)}{d(y, x)} \frac{1}{Df^{k-1}(z)} > \frac{1}{c_{k-1}}.$$ 

Therefore,

$$L_{sU}^k U \geq r^s (\delta_f c_{k-1} \|Df\|)^{-s} \chi_{B(f^k_x, \delta_f)}.$$ 

If we iterate another $n_0 = n_0(\delta_f)$ times then $f^{n_0} B(f^k_x, \delta_f)$ covers all of $\Lambda$ due to mixing and using the definition of $M_n(s)$,

$$L_{sU}^{n_0 + n_0} U \geq r^s (\delta_f c_{k-1} \|Df\|)^{-s} 1 \geq (4r)^s \left[ \frac{(4\|Df\|^{1+n_0} \delta_f c_{k-1})^{-s}}{M_{k+n_0}(s)} \right] L_{sU}^k 1.$$
When $s > s_{\text{crit}}$, $M_{k+n_0}(s)$ tends exponentially fast to zero. As the rest is sub-exponential, the quantity in the square brackets is uniformly bounded from below by some $\gamma_2(s) > 0$. Using the positivity of the operator we see that

$$L_s^n \chi_U \geq \gamma_2(s)(4r)^s L_s^n 1,$$

whenever $n \geq k(x, r) + n_0$.

Now, let $x_1, \ldots, x_N$ be a finite maximal $2r$ separated set in $\Lambda$. Thus, the balls $\{B(x_i, 2r)\}_{i=1,\ldots,N}$ cover $\Lambda$ whereas the balls $\{B(x_i, r)\}_{i=1,\ldots,N}$ are mutually disjoint. For $n \geq \max_i k(x_i, r) + n_0$,

$$L_s^n 1 \geq \sum_i L_s^n \chi_{B(x_i, r)} \geq \gamma_2(s) N (4r)^s L_s^n 1.$$

We have deduced the bound

$$\sum_{i=1}^N (\text{diam } B(x_i, 2r))^s \leq 1/\gamma_2(s)$$

which shows that $\dim B \Lambda$ does not exceed $s$, whence not $s_{\text{crit}}$. We have proven Theorem 2.1 in the case of a strongly mixing repeller and refer to Appendix A for the extension to the general case.

**Corollary 2.8** If $\int_0^{\lambda_0} \epsilon(t)/t \, dt < +\infty$ and the repeller is strongly mixing (cf. Remark A.1) then the Hausdorff measure is finite and comprised between $1/\gamma_1(s_{\text{crit}}) > 0$ and $1/\gamma_2(s_{\text{crit}}) < +\infty$.

Proof: The hypothesis implies that for fixed $s$ the sequences $(c_n(s))_n$ and $M_n(s)/m_n(s)$ in the sub-exponential distortion and operator bounds, respectively, are both uniformly bounded in $n$ (Remarks 2.5 and 2.7). All the (finite) estimates may then be carried out at $s = s_{\text{crit}}$ and the conclusion follows. (Note that no measure theory was used to reach this conclusion).
3 Time dependent conformal repellers

Let \((K, d)\) denote a complete metric space without isolated points and let \(\Delta > 0\) be such that \(K\) is covered by a finite number, say \(N_\Delta\) balls of size \(\Delta\). To avoid certain pathologies we will also assume that \((K, d)\) is \(\Delta\)-homogeneous, i.e. that there is a constant \(0 < \delta < \Delta\) such that for any \(y \in K\),

\[
B(y, \Delta) \setminus B(y, \delta) \neq \emptyset. \tag{3.18}
\]

For example, if \(K\) is connected or consists of a finite number of connected components then \(K\) is \(\Delta\)-homogeneous.

Let \(\beta > 1\) and let \(\epsilon : [0, \Delta] \to [0, +\infty]\) be an \(\epsilon\)-function, i.e. a continuous function with \(\epsilon(0) = 0\). Consider the class \(\mathcal{E} = \mathcal{E}(\Delta, \beta, \epsilon)\) of maps \(f\) where

\[
f : \Omega_f \to K
\]

is a \(C^1\)-conformal unramified covering map of finite maximal degree, \(d_{\max}(f) = \max_{y \in K} \deg(f; y) \in \mathbb{N}\), from a non-empty (not necessarily connected) domain, \(\Omega_f \subset K\), onto \(K\), subject to the following ‘equi-uniform’ requirements: There are constants \(\delta(f) > 0\), \(\lambda_1(f) < +\infty\) and a function \(\delta_f : x \in \Omega_f \to [\delta(f), \Delta] \subset \mathbb{R}_+\) such that:

**Assumption 3.1**

\(\mathbf{(T0)}\) For all distinct \(x, x' \in f^{-1}\{y\}\) (with \(y \in K\)) the balls \(B(x, \delta_f(x))\) and \(B(x', \delta_f(x'))\) are disjoint, \((\text{local injectivity})\),

\(\mathbf{(T1)}\) For all \(x \in \Omega_f : B(f_x, \Delta) \subset f(B(x, \delta_f(x)) \cap \Omega_f)\), \((\text{openness})\),

\(\mathbf{(T2)}\) \(\beta \leq f[u, x] \leq \lambda_1(f), \forall u, x \in \Omega_f : d(u, x) < \delta_f(x)\), \((\text{dilation})\) and

\(\mathbf{(T3)}\) For all \(x \in \Omega_f : \epsilon_f(x, r) \leq \epsilon(r), \forall 0 < r \leq \Delta\), \((\text{distortion})\).

Here, \(f[\cdot, \cdot]\) is the divided difference from equation (2.7) and the distortion, a restricted version of equation (2.8), for \(x \in \Omega_f, r > 0\), is given by

\[
\epsilon_f(x, r) = \sup \left\{ \left| \log \frac{f[u_1, x]}{Df(u_2)} \right| : u_1, u_2 \in B(x, \delta_f(x)) \cap f^{-1}B(f_x, r) \right\}. \tag{3.19}
\]

We tacitly understand by writing \(f^{-1}(y)\) that we are looking at the pre-images of \(y \in K\) within \(\Omega_f\), i.e. where the map is defined. We also write \(\|Df\|\) for the supremum of the conformal derivative of \(f\) over its domain of definition, \(\Omega_f\). By (T2) and setting \(u = x\), we also see that \(\beta \leq \|Df\| \leq \lambda_1(f)\).

When \(f \in \mathcal{E}(\Delta, \beta, \epsilon)\) and \(f(x) = y \in K\) then by \(\Delta\)-homogeneity, (3.18), and property (T1), there must be \(u \in B(x, \delta_f(x))\) with \(f(u) \in B(y, \Delta) \setminus B(x, \delta)\) (\(\delta\) as in the above definition). By the definition, equation (3.19), of the distortion it follows that

\[
0 < \kappa \equiv \delta e^{-\epsilon(\Delta)} \leq \delta_f(x)Df(x), \forall x \in \Omega_f. \tag{3.20}
\]

In the following let \(\mathcal{F} = (f_k)_{k \in \mathbb{N}} \subset \mathcal{E}(\Delta, \beta, \epsilon)\) be a fixed sequence of such mappings and let us fix \(\delta_{f_k}(x)\), \(\delta_{f_k} = \inf_{x \in \Omega_{f_k}} \delta_{f_k}(x) > 0\) and \(\lambda_1(f_k)\) so as to satisfy conditions (T0)-(T3). For all \(n \geq 0\) define:

\[
\Omega_n(\mathcal{F}) = f_1^{-1} \circ \cdots \circ f_n^{-1}(K),
\]
and then
\[ \Lambda(\mathcal{F}) = \bigcap_{n \geq 1} \Omega_n(\mathcal{F}). \]

Letting \( \sigma(\mathcal{F}) = (f_{k+1})_{k \in \mathbb{N}} \) denote the shift of the sequence we set \( \Lambda_t = \Lambda(\sigma^t(\mathcal{F})), \ t \geq 0. \) \( K \) was assumed complete (though not necessarily compact) and each \( \delta(f_k) \) is strictly positive. It follows then that each \( \Lambda_t \) is closed, whence complete. Each \( \Lambda_t \) also has finite open covers of arbitrarily small diameters (obtained by pulling back a finite \( \Delta \)-cover of \( K \)), whence each \( \Lambda_t \) is compact and non-empty. Also \( f_t(\Lambda_{t-1}) = \Lambda_t \) and we have obtained a time-dependent sequence of compact conformal repellers:

\[
\Lambda_0 \xrightarrow{f_1} \Lambda_1 \xrightarrow{f_2} \Lambda_2 \longrightarrow \cdots
\]

For \( t \geq 0, k \geq 1 \) we denote by \( f_t^{(k)} = f_{t+k} \circ \cdots \circ f_{t+1} \) the \( k \)th iterated map from \( \Omega_k(\sigma^t(\mathcal{F})) \) onto \( K \) (with \( f_t^{(0)} \) being the identity map on \( K \)). We write simply \( f^{(k)} = f_0^{(k)} : \Omega_k(\mathcal{F}) \to K \) for the iterated map starting at time zero and \( Df^{(k)}(x) \) for the conformal derivative of this iterated map.

For \( x \in \Lambda_0 \) we write \( x_j = f^{(j)}(x), \ j \geq 0 \) for its forward orbit (and similarly for \( u \in \Lambda_0 \)). Using this notation we define the \( n \)th Bowen ball around \( x \):
\[
B_n(x) = \{u \in \Lambda_0 : d(x_j, u_j) < \delta f_{j+1}(x_j), 0 \leq j \leq n\}.
\]

and then also the \((n-1, \Delta)\)-Bowen ball around \( x \in \Lambda_0 \):
\[
B_{n-1, \Delta}(x) = \{u \in B_{n-1}(x) : d(x_n, u_n) < \Delta\}.
\]

Then \( f^{(n)} : B_{n-1, \Delta}(x) \to B(f^{(n)}(x), \Delta) \) is a uniformly expanding homeomorphism for all \( x \in \Lambda_0 \). When \( u \in B_{n-1, \Delta}(x) \) we say that \( u \) and \( x \) are \((n-1, \Delta)\)-close. Our hypotheses imply that being \((n-1, \Delta)\)-close is a reflexive relation (not so obvious when \( \delta_f(x) \) depends on \( x \)) as is shown in the proof of the following

**Lemma 3.2 [Pairing]** For \( n \in \mathbb{N}, y, w \in K \) with \( d(y, w) \leq \Delta \), the sets \( (f^{(n)})^{-1}\{y\} \) and \( (f^{(n)})^{-1}\{w\} \) may be paired uniquely into pairs of \((n-1, \Delta)\)-close points.

Proof: Fix \( f = f_n \) and let \( x \in f^{-1}\{y\} \). By (T1) \( fB(x, \delta_f(x)) \) contains \( B(f(x), \Delta) \ni w \). By (T2) there is a unique preimage \( z \in f^{-1}\{w\} \) at a distance \( d(z, x) < \delta_f(x) \leq \Delta \) to \( x \). We claim that then also \( x \in B(z, \delta_f(z)) \) (which makes the pairing unique and reflexive). If this were not so then there must be \( x_2 \in B(z, \delta_f(z)) \cap f^{-1}\{y\} \) which by (T0) must verify: \( \delta_f(x) > \delta_f(z) > d(x_2, z) > \delta_f(x_2) \). Inductively one constructs disjoint sequences \( x_1 = x, x_2, \ldots \in f^{-1}\{y\}, z_1 = z, z_2, \ldots \in f^{-1}\{w\} \) for which \( \delta_f(x_1) > \delta_f(z_1) > \delta_f(x_2) > \delta_f(z_2) > \cdots \) and this contradicts finiteness of the degree of \( f \). Returning to the sequence of mappings we obtain by recursion in \( n \) the unique pairing. \( \square \)

**Lemma 3.3 [Sub-exponential Distortion]** There is a sub-exponential sequence, \((c_n)_{n \in \mathbb{N}}\), (depending on the equi-distortion function, \( \epsilon \), but not on the actual sequence of maps) such that for any two points \( z \neq u \) which are \((n-1, \Delta)\)-close to an \( x \in \Lambda_0 \)

\[
\frac{1}{c_n} \leq \frac{d(f^{(n)}(u), f^{(n)}(x))}{d(u, x) Df^{(n)}(z)} \leq c_n \quad \text{and} \quad \frac{1}{c_n} \leq \frac{Df^{(n)}(x)}{Df^{(n)}(z)} \leq c_n
\]
Proof: As in Lemma 2.4. More precisely, we have \( \log c_n = \epsilon(\Delta) + \epsilon(\Delta/\beta) + \cdots + \epsilon(\Delta/\beta^{n-1}) \).

For \( s \geq 0, f \in \mathcal{E}(\Delta, \beta, \epsilon) \) we define as before a transfer operator \( L_{s,f} : \mathcal{M}(K) \to \mathcal{M}(K) \) by setting:

\[
(L_{s,f} \phi)(y) \equiv \sum_{x \in f^{-1}(y)} (Df(x))^{-s} \phi_x, \quad y \in K, \phi \in \mathcal{M}(K).
\]

(3.21)

We write \( L^{(n)}_s = L_{s,f_n} \circ \cdots \circ L_{s,f_1} \) for the \( n \)’th iterated operator from \( \mathcal{M}(K) \) to \( \mathcal{M}(K) \).

We denote by \( 1 = \chi_K \) the constant function which equals one on \( K \) and as in (2.10) we define (omitting the dependency on \( F \) in the notation):

\[
M_n(s) \equiv \sup_{y \in K} L^{(n)}_s 1(y) \quad \text{and} \quad m_n(s) \equiv \inf_{y \in K} L^{(n)}_s 1(y),
\]

and then the lower and upper \( s \)-conformal pressures:

\[
-\infty \leq P(s) \equiv \liminf_n \frac{1}{n} \log m_n(s) \leq \overline{P}(s) \equiv \limsup_n \frac{1}{n} \log M_n(s) \leq +\infty.
\]

In general these limits need not be equal nor finite. Explicitly, we have e.g. the formula for the lower pressure, similar to (2.5),

\[
P(s) = \liminf_n \frac{1}{n} \log \inf_{y \in K} \sum_{x \in (f^{(n)})^{-1}(y)} \left( Df^{(n)}_x \right)^{-s}.
\]

We define the following lower and upper critical exponents with values in \([0, +\infty]\):

\[
\underline{s}_{\text{crit}} = \sup\{ s \geq 0 : P(s) > 0 \} \quad \text{and} \quad \overline{s}_{\text{crit}} = \inf\{ s \geq 0 : \overline{P}(s) < 0 \}.
\]

It will be necessary to make some additional assumptions on mixing and growth rates. For our purposes the following suffices:

**Assumption 3.4**

(T4) There is \( n_0 < \infty \) such that the sequence \((f_k)_{k \in \mathbb{N}}\) is \((n_0, \Delta)\)-mixing, i.e. for any \( y \in K \), and \( t \geq 0 \),

\[
(f^{(n_0)}_t)^{-1}(y) \text{ is } \Delta\text{-dense in } (f^{(n_0)}_t)^{-1}K.
\]

(T5) The sequence \((\lambda(f_k))_{k \in \mathbb{N}}\) is sub-exponential, i.e.

\[
\lim_k \frac{1}{k} \log \lambda(f_k) = 0.
\]

**Lemma 3.5** Assuming (T0)-(T5) we have (the limits need not be finite):

\[
\overline{P}(s) = \limsup_n \frac{1}{n} m_n(s) = \limsup_n \frac{1}{n} M_n(s)
\]

\[
P(s) = \liminf_n \frac{1}{n} m_n(s) = \liminf_n \frac{1}{n} M_n(s)
\]
Proof: By a small modification (notably replacing $\delta_f$ by $\Delta$) in the last half of the proof of the Operator bounds - Lemma 2.6 - and making use of mixing (T4), we deduce similarly to (2.13) that

$$m_{n+n_0}(s) \geq (\|DF_{n+1}\| \cdots \|DF_{n+n_0}\| c_n)^{-s} M_n(s)/2.$$ 

By Lemma 3.3 the sequence $c_n$ is sub-exponential. Due to (T5) and since $n_0$ is fixed, $M_n(s)/m_{n+n_0}(s)$ is then of sub-exponential growth. Whether finite or not, the above lim inf’s and lim sup’s agree.

Lemma 3.6 Assuming (T0)-(T5) we have the following dichotomy: Either $\Lambda_0$ is a finite set or $\Lambda_0$ is a perfect set.

Proof: Suppose that $\Lambda_k$ is a singleton for some $k$ (this happens iff the degrees of the sequence of mappings is eventually one !). Then also $\Lambda_n$ is a singleton for all $n \geq k$ and $\Lambda_0$ is a finite set because all the (preceeding) maps are of finite degree. Suppose instead that no $\Lambda_k$ is reduced to a singleton and let us take $x \in \Lambda(\mathcal{F})$ as well as $n \geq n_0$. Let $z \in \Lambda_n$, $z \neq f^{(n)}(x)$. By (T1),(T2) and (T4) $z$ must have an $n$’th pre-image in $\Lambda_0$ distinct from $x$ and at a distance less than $\beta^{n_0-n}\Delta$ to $x$. Thus, $x$ is a point of accumulation of other points in $\Lambda_0$. □

We have the following (see [Bar96, Theorem 2.1 and 3.8] for similar results):

Theorem 3.7 Let $\Lambda_0$ denote the time-zero conformal repeller for a sequence of $\mathcal{E}(\Delta, \beta, \epsilon)$-maps, $(f_t)_{t \in \mathbb{N}}$, verifying conditions (T0) – (T5). Then we have the following inequalities (note that the first is actually an equality), regarding dimensions of $\Lambda_0 = \Lambda(\mathcal{F})$,

$$\underline{s}_{\text{crit}} = \dim_H \Lambda_0 \leq \dim_B \Lambda_0 \leq \dim^B \Lambda_0 \leq \overline{s}_{\text{crit}}.$$ 

If, in addition $\lim \frac{1}{n} \log m_n(\underline{s}_{\text{crit}}) = 0$ then $\underline{s}_{\text{crit}} = \overline{s}_{\text{crit}}$ and all the above dimensions agree.

Proof: When $\Lambda_0$ is a finite set it is easily seen that $\overline{\mathcal{P}}(0) = 0$ and then that $\underline{s}_{\text{crit}} = \overline{s}_{\text{crit}} = 0$ in agreement with our claim. In the following we assume that $\Lambda_0$ has no isolated points.

$(\underline{s}_{\text{crit}} \leq \dim_H \Lambda_0)$: Let $U$ be an open subset intersecting $\Lambda_0$ and of diameter not exceeding $\delta_{f_k}$. Choose $x = x(U) \in U \cap \Lambda_0$ and take again $k = k(U) \geq 0$ to be the largest integer (finite when $\Lambda_0$ is without isolated points) such that $U \subset B_k(x)$. Then there is $u \in U \setminus B_{k+1}(x) \subset B_k(x) \setminus B_{k+1}(x)$ for which we have $\delta_{f_{k+2}} \leq d(x_{k+1}, u_{k+1}) \leq \lambda_1(f_{k+1})d(x_k, u_k)$. The bound (2.15) is replaced by

$$L^{(k)}_s x_U \leq (\text{diam} U)^s \left( \frac{\lambda_1(f_{k+1})c_k}{\delta_{f_{k+2}}} \right)^s \frac{1}{m_k(s)} L^{(k)}_s 1.$$ 

By hypothesis (T5) $\lambda_1(f_k)$ is a sub-exponential sequence. $\Delta$-homogeneity, or more precisely the bound (3.20), shows that $\delta_{f_k} \geq \kappa/\lambda_1(f_k)$ is also sub-exponential. If $\underline{s}_{\text{crit}} = 0$ there is nothing to show. If $0 \leq s < \underline{s}_{\text{crit}}$ then $m_k(s)$ tends to infinity exponentially fast and the factor in the square bracket is uniformly bounded from above by a constant $\gamma_1(s) < \infty$. Now, let us take the restriction of the resulting inequality to the subset $\Lambda_k$. On the left hand side $U$ is replaced by its intersection with $\Lambda_0$ and on the right $L^{(k)}_s 1$ is replaced by $L^{(k)}_s x_{\Lambda_0}$ (since $(f^{(k)})^{-1}\Lambda_k = \Lambda_0$):

$$L^{(k)}_s x_{U \cap \Lambda_0} \leq \gamma_1(s) (\text{diam} U)^s L^{(k)}_s x_{\Lambda_0}.$$ 

We may then repeat the argument from section 2.3 to conclude that $\dim_H \Lambda_0 \geq \underline{s}_{\text{crit}}$. 

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By our initial assumption $K$ has a finite $\Delta$-cover $\{V_1, \ldots, V_{N_{\Delta}}\}$. Fix $n \geq 1$ as well as $i \in \{1, \ldots, N_{\Delta}\}$. Pick $x_i \in V_i$ and write $(f^{(n)})^{-1}\{x_i\} = \bigcup_{\alpha \in I_i} \{x_{i,\alpha}\}$ over a finite index set $I_i$. By the Pairing Lemma 3.2, we see that to each $x_{i,\alpha}$ corresponds a pre-image $V_{i,\alpha} = (f^{(n)})^{-1}V_i \cap B_{n-1,\Delta}(x_{i,\alpha})$ (the union over $\alpha$ yields a partition of $(f^{(n)})^{-1}V_i$). Whence, by sub-exponential distortion, Lemma 3.3,

$$\text{diam } V_{i,\alpha} \leq \frac{2cn\Delta}{Df^{(n)}(x_{i,\alpha})}.$$ 

Then,

$$\sum_{\alpha} (\text{diam } V_{i,\alpha})^s \leq (2cn\Delta)^s (L^n_s 1)(x_i)$$

and consequently

$$\sum_{i,\alpha} (\text{diam } V_{i,\alpha})^s \leq [N_{\Delta}(2cn\Delta)^s M_n(s)].$$

Let $s > s_{\text{crit}}$. Then $P(s) < 0$ and there is a sub-sequence $n_k$, $k \in \mathbb{N}$ for which $m_{n_k}$ and, by Lemma 3.5, also $M_{n_k}(s)$ tend exponentially fast to zero. For that sub-sequence the expression in the square-brackets is uniformly bounded in $n_k$. Since diam $V_{i,\alpha} \leq 2cn\Delta^{-n}$ which tends to zero with $n$ the family $\{V_{i,\alpha}\}_{n_k}$ exhibits covers of $\Lambda_0$ of arbitrarily small diameter. This implies that dim$_H(\Lambda)$ does not exceed $s$, whence not $s_{\text{crit}}$.

$$(\text{dim}_B \Lambda_0 \leq s_{\text{crit}}):$$ For the upper bound on the box dimensions, consider for $0 < r < \delta(f_1)$, $x \in \Lambda_0$ the ball $U = B(x, r)$ and let $k = k(x, r) \geq 1$ be the smallest integer such that $B_{k-1,\Delta}(x) \subset U$. Then there is $y \in B_{k-2,\Delta}(x) \setminus U \supset B_{k-2,\Delta}(x) \setminus B_{k-1,\Delta}(x)$. As in section 2.4 we deduce that

$$L_s^{(k)} \chi_U \geq r^s (c_{k-1}\Delta \|Df_k\|)^{-s} \chi_{B(f^{(k)}_0, \Delta)}.$$ 

Iterating another $n_0$ times we will by hypothesis (T4) cover all of $\Lambda_{k+n_0}$. Reasoning as in section 2.4 it follows that

$$L_s^{(k+n_0)} \chi_U \geq (4r)^s \left[ (4c_{k-1}\Delta \prod_{j=0}^{n_0} \|Df_{k+j}\|)^{-s} \frac{1}{M_{k+n_0}(s)} \right] L_s^{(k+n_0)} \chi_{\Lambda_0}$$

If $s > s_{\text{crit}}$ the sequence $M_k(s)$ tends to zero exponentially fast. The sub-exponential bounds in hypothesis (T5) imply that the factor in the brackets remains uniformly bounded from below. We may proceed to conclude that dim$_B \Lambda$ does not exceed $s$, whence not $s_{\text{crit}}$.

Finally, for the last assertion suppose that $\frac{1}{n} \log m_n(s_{\text{crit}}) = 0$, i.e. the limit exists and equals zero (cf. the Remark below). Then Lemma 3.5 shows that the lower and upper pressure agree and therefore $\bar{P}(s_{\text{crit}}) = \overline{P}(s_{\text{crit}}) = 0$. Now, both pressure functions are strictly decreasing (because $\beta > 1$). Therefore, $s_{\text{crit}} = \underline{s}_{\text{crit}}$ and the conclusion follows. $\square$

Remarks 3.8 A Hölder inequality (for fixed $n$) shows that $s \mapsto \frac{1}{n} \log M_n(s_{\text{crit}})$ is convex in $s$. The property of being convex is preserved when taking limsup (but in general not when taking liminf) so that $s \mapsto \overline{P}(s)$ is convex. Even when $\frac{1}{n} \log M_n(s_{\text{crit}})$ converges, however, it can happen that the limit is $+\infty$ for $s < s_{\text{crit}}$. In that case convergence of $\frac{1}{n} \log M_n(s_{\text{crit}})$ could be towards a strictly negative number and $\underline{s}_{\text{crit}}$ could turn out to be strictly smaller than $\overline{s}_{\text{crit}}$. 

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4 Random conformal maps and parameter-dependency

The distortion function, $\epsilon$, gives rise to a natural metric on $\mathcal{E} \equiv \mathcal{E}(\Delta, \beta, \epsilon)$. We assume in the following that $\epsilon$ is extended to all of $\mathbb{R}_+$ and is a strictly increasing concave function (or else replace it by an extension of its concave ‘hull’ and make it increasing). For $f, \tilde{f} \in \mathcal{E}$ we set $d_\mathcal{E}(f, \tilde{f}) = +\infty$ if there is $y \in K$ for which $\#f^{-1}y \neq \tilde{f}^{-1}y$. Note that by pairing $\#f^{-1}y$ is locally constant. When the local degrees coincide everywhere we proceed as follows: For $y \in K$, we let $\Pi_y$ denote the family of bijections, $\pi : f^{-1}y \rightarrow \tilde{f}^{-1}y$, and set

$$
\rho_{\pi,x}(f, \tilde{f}) = \epsilon \left( \frac{\beta}{\beta - 1} d(x, \pi(x)) \right) + \left| \log \frac{D\tilde{f} \circ \pi(x)}{Df(x)} \right|.
$$

The distance between $f$ and $\tilde{f}$ is then defined as

$$
d_\mathcal{E}(f, \tilde{f}) = \sup_{y \in K} \inf_{\pi \in \Pi_y} \sup_{x \in f^{-1}(y)} \rho_{\pi,x}(f, \tilde{f}). \quad (4.22)
$$

Our hypotheses on $\epsilon$ imply that $\rho_{\pi_2 \circ \pi_1, x}(f_1, f_3) \leq \rho_{\pi_1, x}(f_1, f_2) + \rho_{\pi_2, \pi_1(x)}(f_2, f_3)$ from which we deduce that $d_\mathcal{E}$ fulfills a triangular inequality. It is then checked that indeed, $d_\mathcal{E}$ defines a metric on $\mathcal{E}$.

Lemma 4.1 Let $u \leq \Delta$ and $d_\mathcal{E}(f, \tilde{f}) \leq \epsilon(u)$. Then for all $y, \tilde{y} \in K$ with $d(y, \tilde{y}) \leq u$ there exists a pairing $(x_\alpha, \tilde{x}_\alpha)$, $\alpha \in J$ (some index set) of $f^{-1}(y)$ and $\tilde{f}^{-1}(\tilde{y})$ for which $\forall \alpha \in J$,

$$
d(x_\alpha, \tilde{x}_\alpha) \leq u \quad \text{and} \quad \left| \log \frac{Df(x_\alpha)}{D\tilde{f}(\tilde{x}_\alpha)} \right| \leq 2\epsilon(u).
$$

Proof: Let $x \in K$ and choose a bijection $\pi : f^{-1}(y) \rightarrow \tilde{f}^{-1}(y)$ for which $\epsilon \left( \frac{\beta}{\beta - 1} d(x, \pi(x)) \right) \leq \epsilon(u)$ for all $x \in f^{-1}(y)$. Then $d(x, \pi(x)) \leq (1 - \frac{1}{\beta})u$, for all $x \in f^{-1}(y)$. Consider $x \in f^{-1}(y)$ and $x' = \pi(x)$. As $\tilde{f}B(x', \delta \tilde{f}(x')) \supset B(y, \Delta) \supset \tilde{y}$ there is a unique point $\tilde{x} \in B(x', \delta \tilde{f}(x'))$ for which $\tilde{y} = \tilde{f}(\tilde{x})$. As the association (for fixed $\pi$), $x \mapsto x' = \pi(x) \mapsto \tilde{x}$ is unique we have obtained a pairing. By expansion of $\tilde{f}$ we have $d(x', \tilde{x}) \leq d(y, \tilde{y})/\beta \leq u/\beta$. Therefore also, $d(x, \tilde{x}) \leq u(1 - \frac{1}{\beta}) + \frac{u}{\beta} = u$ as wanted. By definition of distortion we have $|\log D\tilde{f}(x')/D\tilde{f}(\tilde{x})| \leq \epsilon(d(y, \tilde{y})) \leq \epsilon(u)$. Also, $d_\mathcal{E}(f, \tilde{f}) \leq \epsilon(u)$ implies $|\log Df(x)/D\tilde{f}(x')| \leq \epsilon(u)$ and the last claim follows.

Given two sequences $\mathcal{F} = (f_n)_{n \in \mathbb{N}}$ and $\tilde{\mathcal{F}} = (\tilde{f}_n)_{n \in \mathbb{N}}$ in $\mathcal{E}$ we define their distance (one could here replace sup by lim-sup),

$$
d_\infty(\mathcal{F}, \tilde{\mathcal{F}}) = \sup_n d_\mathcal{E}(f_n, \tilde{f}_n). \quad (4.23)
$$

Proposition 4.2 When $d_\infty(\mathcal{F}, \tilde{\mathcal{F}}) \leq r \leq \epsilon(\Delta)$ then:

$$
|P(s, \mathcal{F}) - P(s, \tilde{\mathcal{F}})| \leq 2rs, \quad s \geq 0 \quad \text{and}
$$

$$
\left( 1 + \frac{2r}{\log \beta} \right)^{-1} \leq \frac{\mathcal{S}_{\text{crit}}(\mathcal{F})}{\mathcal{S}_{\text{crit}}(\tilde{\mathcal{F}})} \leq 1 + \frac{2r}{\log \beta}.
$$

We have the same bounds for the upper pressures, $\overline{P}$, and upper critical value, $\mathcal{S}_{\text{crit}}$. 

Proof: We perform a recursive pairing of pre-images at distances less than $u$, with $\epsilon(u) \leq r$. By Lemma 4.1 for the bounds on the derivatives we obtain

$$\frac{1}{k} \left| \log \frac{L_{s,\tilde{c}}^{(k)}(y)}{L_{s,\tilde{c}}^{(1)}(y)} \right| \leq 2rs.$$  

The first claim follows by taking a limit. For the second claim suppose that $s_{c} = s_{\text{crit}}(\mathcal{F}) < \tilde{s}_{c} = s_{\text{crit}}(\tilde{\mathcal{F}})$. Since $s \mapsto P(s,\mathcal{F}) + s \log \beta$ is non-increasing (same for $\tilde{\mathcal{F}}$) we have $(\tilde{s}_{c} - s_{c}) \log \beta \leq P(s,\tilde{\mathcal{F}}) - P(s,\mathcal{F}) \leq 2rs$ for all $\tilde{s}_{c} \geq s \geq s_{c}$. From this inequality the other bound follows.

We associate to the metric space $(\mathcal{E}, d\mathcal{E})$ its corresponding Borel $\sigma$-algebra and this allows us to construct measurable maps into $\mathcal{E}$. In the following, let $(\Omega, \mu)$ be a probability space and $\tau : \Omega \to \Omega$ a $\mu$-ergodic transformation.

**Definition 4.3** We write $\mathcal{E}_{\Omega} \equiv \mathcal{E}_{\Omega}(\Delta, \beta, \epsilon)$ for the space of measurable maps, $f : \omega \in (\Omega, \mu) \mapsto f_{\omega} \in (\mathcal{E}, d\mathcal{E})$, whose image is almost surely separable (i.e. the image of a subset of full measure contains a countable dense set). Following standard conventions we say that the map is Bochner-measurable.

We write $\mathcal{F}_{\omega} = (f_{\tau^{n-1}\omega})_{n\in\mathbb{N}}$ for the sequence of maps fibered at the orbit of $\omega \in \Omega$. Denote by $f_{\omega}^{(n)} = f_{\tau^{n}\omega} \circ \cdots \circ f_{\omega}$ the iterated map defined on the domain $\Omega^{n}(\mathcal{F}_{\omega}) = f_{\omega}^{-1} \circ f_{\tau(\omega)}^{-1} \circ \cdots \circ f_{\tau^{n-1}(\omega)}^{-1}(K)$. The ‘random’ Julia set, as before, is the compact, non-empty intersection

$$J(\mathcal{F}_{\omega}) = \bigcap_{n \geq 0} \Omega_{n}(\mathcal{F}_{\omega}). \quad (4.24)$$

Our assumptions imply that $(f_{1}, \ldots, f_{n}) \in \mathcal{E}^{n} \mapsto f_{1}^{-1} \circ \cdots \circ f_{n}^{-1}(K) \subset K$ is continuous when $K$ is equipped with the Hausdorff topology for its non-empty subsets. It follows that $\omega \mapsto \Omega_{n}(\mathcal{F}_{\omega})$ is measurable. Uniform contraction implies that $\Omega_{n}$ converges exponentially fast to $J(\mathcal{F}_{\omega})$ in the Hausdorff topology, whence the ‘random’ conformal repeller, $J(\mathcal{F}_{\omega})$ (a.s.) measurable for the Hausdorff $\sigma$-algebra.

Using the estimates from the previous Proposition, the function, $(f_{1}, \ldots, f_{n}) \in \mathcal{E}^{n} \mapsto M_{n}(s, (f_{1}, \ldots, f_{n}))$ is continuous. Almost sure separability of $\{f_{\omega} : \omega \in \Omega\} \subset \mathcal{E}$ implies then that $\omega \mapsto M_{n}(s, f_{\omega})$ is measurable (with the standard Borel $\sigma$-algebra on the reals). For example, if $V_{1}, V_{2}$ are open subsets of $\mathcal{E}$, the pre-image of $V_{1} \times V_{2}$ by $\omega \mapsto (f_{\omega}, f_{\tau(\omega)})$ is $f_{\omega}^{-1} \cap \tau f_{\omega}^{-1}(V_{2})$ which is measurable. The function $\mathcal{T}(s, \mathcal{F}_{\omega})$, being a lim sup of measurable functions, is then also measurable (and the same is true for $m_{n}$ and $P$). We define the distance between $f, \tilde{f} \in \mathcal{E}_{\Omega}$ to be

$$d_{\mathcal{E}, \Omega}(f, \tilde{f}) = \mu\text{-ess sup } d_{\mathcal{E}}(f_{\omega}, \tilde{f}_{\omega}) \in [0, +\infty]. \quad (4.25)$$

**Theorem 4.4** Let $\tau$ be an ergodic transformation on $(\Omega, \mu)$ and let $f = (f_{\omega})_{\omega \in \Omega} \in \mathcal{E}_{\Omega}$ be Bochner-measurable (Definition 4.3). We suppose that there is $n_{0} < \infty$ such that almost surely the sequence $\mathcal{F}_{\omega} = (f_{\tau^{n-1}\omega})_{n \in \mathbb{N}}$ is $(n_{0}, \Delta)$-mixing (Condition (T4) in Assumption 3.4).

(a) Suppose that $\mathbb{E}(\log \|DF_{\omega}\|) < +\infty$. [We say that the family is of bounded average logarithmic dilation]. Then for any $s \geq 0$ and $\mu$-almost surely, the pressure function $P(s, \mathcal{F}_{\omega})$ is independent of $\omega$. We write $P(s, f)$ for this almost sure value. The various dimensions of the random conformal repeller agree (a.s.) in value. Their common value is (a.s.) constant and given by

$$\dim \Lambda(\mathcal{F}_{\omega}) = \sup \{s \geq 0 : P(s, f) > 0\} \in [0, +\infty].$$
(b) The (a.s.) dimension is finite iff \( P(0, f) < +\infty \) (this is the case, e.g. if \( \mathbb{E} \log d_{\max}^n(f) < \infty \)) and one has the estimate,

\[
\frac{\mathbb{E} \log d_{\min}^n(f)}{\mathbb{E} \log \|Df\|} \leq \dim_{\Lambda}(\mathcal{F}_{\omega}) \leq \frac{\mathbb{E} \log d_{\max}^n(f)}{-\mathbb{E} \log \|1/Df\|}.
\]

(c) The mapping, \( f \in (\mathcal{E}_\Omega, d_{\mathcal{E}, \Omega}) \mapsto \dim_{\Lambda}(\mathcal{F}_{\omega}), \) is \( \frac{2}{\log \beta} \)-Lipschitz (at distances \( \leq \epsilon(\Delta) \)).

Proof:

Write \( \phi = \phi_{\omega} = \log \|Df_{\omega}\| \geq 0 \) and similarly \( \phi^{(n)} = \phi_{\omega}^{(n)} = \log \|Df_{\omega}^{(n)}\| \geq 0 \). Then \( \phi^{(n)} = \phi^{(k)} + \phi^{(n-k)} \circ \tau^k \), \( 0 < k < n \) and since \( \phi \) is integrable we get by Kingman’s subergodic Theorem, [King68], that the limit

\[
\lim_{n} \frac{1}{n} \phi^{(n)} \geq 0
\]

exists \( \mu \)-almost surely. As a consequence,

\[
\lim_{n} \frac{1}{n} \phi \circ \tau^n = \lim_{n} \frac{n + 1}{n} \phi^{(n+1)} - \frac{1}{n} \phi^{(n)} = 0
\]

\( \mu \)-almost surely. Thus the sequence of maximal dilations is almost surely sub-exponential (Condition (T5) of Assumption 3.4). Condition (T4) of that assumption is a.s. verified by the hypotheses stated in our Theorem. It follows by Theorem 3.7 that the Hausdorff dimension of the random repeller, \( \Lambda(\mathcal{F}_{\omega}) \) a.s. is given by \( \mathbb{E} \log m_{n}(\mathcal{F}_{\omega}) \). We wish to show that a.s. the value is constant and that a.s. \( \frac{1}{n} \log m_{n}(\mathcal{F}_{\omega}) \rightarrow 0 \) as \( n \rightarrow \infty \).

We have the following bounds for the action of the transfer operator, \( L_{s,f} \), upon a positive function, \( \phi > 0 \):

\[
\frac{d_{\min}^n(f)}{\|Df\|^s} \min \phi \leq L_{s,f} \phi \leq d_{\max}^n(f) \| \frac{1}{Df} \|^s \max \phi.
\]  

Here, \( d_{\max}^n(f) \) and \( d_{\min}^n(f) \) denotes the maximal, respectively, the minimal (local) degree of the mapping \( f \). From the lower bound we obtain in particular,

\[
\mathbb{E} \log m_{1}(s, \mathcal{F}_{\omega}) \geq \mathbb{E} \log d_{\min}^n(f) - s \mathbb{E} \log \|Df\| \geq -s \mathbb{E} \log \|Df\|.
\]

The family, \( m_{n} \), is super-multiplicative, i.e. \( m_{n}(s, \mathcal{F}_{\omega}) \geq m_{n-k}(s, \mathcal{F}_{\tau^k \omega})m_{k}(s, \mathcal{F}_{\omega}) \), for \( n, k \geq 0 \) and \( \omega \in \Omega \). Writing \( \log x = \max\{0, \log x\}, x > 0 \), we have

\[
\mathbb{E} \log \frac{1}{m_{1}(s, \mathcal{F}_{\omega})} \leq s \mathbb{E} \log \|Df\|.
\]

As the latter quantity is assumed finite, we may apply Kingman’s super-ergodic Theorem to \( m_{n} \) (i.e. the sub-ergodic Theorem to the sequence \( 1/m_{n} \)), to deduce that the limit

\[
\frac{1}{n} \lim \log m_{n}(s, \mathcal{F}_{\omega}) \in (-\infty, +\infty)
\]

exists \( \mu \)-almost surely and is a.s. constant. We write \( P(s, f) \) for this a.s. limit. From the expression for the operator and for fixed \( n \) and \( \omega \in \Omega \), the sequence, \( \|Df_{\omega}^{(n)}\|m_{n}(s, \mathcal{F}_{\omega}) \), is a non-decreasing function of \( s \). The same is then true for

\[
\frac{1}{n} \log \|Df_{\omega}^{(n)}\| + \frac{1}{n} \log m_{n}(s, \mathcal{F}_{\omega}).
\]
Apply now Kingman’s sub-ergodic, respectively super-ergodic, Theorem to these two terms to see that
\[ s \mathbb{E} \log \| Df \| + \frac{P(s,f)}{s} \in (-\infty, +\infty) \]
is a non-decreasing function of \( s \). It is seen in a similar way that
\[ s \log \beta + \frac{P(s,f)}{s} \in (-\infty, +\infty) \]
is non-increasing. These two bounds together with Theorem 3.7 imply that either (1) \( P(0,f) = \infty \), \( \frac{P(s,f)}{s} \) is infinite for all \( s \geq 0 \) and \( s_{\text{crit}} = +\infty \), or (2) \( P(0,f) < +\infty \) in which case the function \( s \mapsto \frac{P(s,f)}{s} \) is continuous, strictly decreasing and has a unique zero \( s_{\text{crit}} \). The additional condition in Theorem 3.7 is thus satisfied and \( s_{\text{crit}} \) therefore equals all of the various dimensions. The estimate, (b), for the dimensions follows from (4.26) and taking averages as above. Finally, (c) is a consequence of Proposition 4.2 and the fact that \( s_{\text{crit}} \) a.s. equals the dimensions. \( \square \)

**Example 4.5** Let \( K = \{ \phi \in \ell^2(\mathbb{N}) : \| \phi \| \leq 1 \} \) and denote by \( e_n, n \in \mathbb{N} \) the canonical basis for \( \ell^2(\mathbb{N}) \). The domains \( D_n = \text{Cl} B(\frac{2}{3} e_n, \frac{1}{3}) \), \( n \in \mathbb{N} \) maps conformally onto \( K \) by \( x \mapsto 6(x - \frac{2}{3} e_n) \). To each \( n \in \mathbb{N} \) we consider the conformal map, \( f_n, \) of degree \( n \), which maps \( D_1 \cup \ldots \cup D_n \) onto \( K \) by the above mappings. Finally let \( \nu \) be a probability measure on \( \mathbb{N} \). Picking an i.i.d. sequence of the mappings, \( f_n \), according to the distribution \( \nu \) we obtain a conformal repellor for which all dimensions almost surely agree. In this case we have equality in the estimates in Theorem 4.4 (b) so the a.s. common value for the dimensions is given by
\[ \sum_n n \nu(n) \log 6. \]

Finiteness of the dimension thus depends on \( n \) having finite average or not, cf. also [DT01, Example 2.1].

The Lipschitz continuity of the dimensions with respect to parameters is somewhat delusive because it is with respect to our particular metric on \( \mathcal{E} \). In practice, when constructing parametrized families of mappings it is really the modulus of continuity of \( Df \), i.e. the \( \epsilon \)-function in \( \mathcal{E}(K, \Delta, \epsilon) \) that comes into play:

**Example 4.6** We consider here just the case of one stationary map, \( f \in \mathcal{E} \). Let \( T_t, t \geq 0 \) be a Lipschitz motion of \( \Omega_f \). By this we mean that \( T_t^{-1} : \Omega_f \to K, t \geq 0 \), is a family of conformal mappings with \( T_0 = \text{id}, |\log DT_t^{-1}(x)| \leq t, x \in \Omega_f \) and \( \sup_{x \in K} d(x, T_t^{-1}x) \leq t \). Let \( \epsilon_{T_t}(r) \) denote the distortion function for \( T_t \) (which we may defined in the same way as for \( f \) when \( r < \Delta - t \)). A calculation then shows that for \( t \) small enough, \( \epsilon_{f \circ T_t}(r) \leq \epsilon_f(r) + \epsilon_{T_t}(r/\beta) \). One also checks that \( d_{\mathcal{E}}(f \circ T_t, f) \leq 2\epsilon_f(t) + \epsilon_{T_t}(t/\beta) + ct \). By Theorem 4.4 (c), the mapping \( t \mapsto d(t) = \dim_H \Lambda(f \circ T_t) \) for \( t \) small verifies
\[ \left| \log \frac{d(t)}{d(0)} \right| \leq \frac{2}{\log \beta} (2\epsilon_f(t) + \epsilon_{T_t}(\frac{t}{\beta}) + ct). \]

When Thermodynamic Formalism applies, in particular when a bit more smoothness is imposed, a similar result could be deduced within the framework (and restrictions) of TF. I am, however, not aware of any results published on this.
Figure 2: An example of a covering map of degree 2 and its ‘inverse’ in the universal cover. Cuts along the dotted lines become arcs in the lift. One fundamental domain is sketched in each cover.

5 Part II: Random Julia sets and parameter dependency

Let $U \subset \hat{\mathbb{C}}$ be an open non-empty connected subset of the Riemann sphere omitting at least three points. We denote by $(U, d_U)$ the set $U$ equipped with a hyperbolic metric. As normalisation we use $ds = 2|dz|/(1 - |z|^2)$ on the unit disk $\mathbb{D}$ and the hereby induced metric for the hyperbolic Riemann surface $U$ (cf. Remark 5.1 below). In particular, we have for the unit disk and $z \in \mathbb{D}$,

$$d_\mathbb{D}(0, z) = \log \frac{1 + |z|}{1 - |z|}, \quad |z| = \tanh \frac{d_\mathbb{D}(0, z)}{2}.$$ 

We write $B(u, r) \equiv B_U(u, r)$ for the hyperbolic ball of radius $r > 0$ centered at $u \in (U, d_U)$, $B_\mathbb{D}(t, r)$ for the similar hyperbolic ball in $(\mathbb{D}, d_\mathbb{D})$ and $B_\mathbb{C}(u, r) = \{z \in \mathbb{C} : |z - u| < r\}$ for a standard Euclidean ball in $\mathbb{C}$.

Recall that when $K \subset U$ is a compact subset the inclusion mapping $(\text{Int}K, d_{\text{Int}K}) \hookrightarrow (\text{Int}K, d_U)$ is a strict contraction [CG93, Theorem 4.2] by some factor $\beta = \beta(K, U) > 1$, depending on $K$ and $U$ only. We consider the family $\mathcal{E}(K, U)$ of finite degree unramified conformal covering maps

$$f : \mathcal{D}_f \to U$$

for which the domain $\mathcal{D}_f$ is a subset of the compact set $K$. We may assume without loss of generality that $K$ is the closure of its own interior. Our first goal is to show that such maps $a \text{ fortiori}$ verify conditions (T0)-(T3) from the previous section, in which the set $K$ is the same as here and the metric $d$ on $K$ is the restriction of the hyperbolic metric $d_U$ to $K$.

Let $\ell = \ell(K, U) > 0$ be the infimum length of closed non-contractible curves (sometimes called essential loops) intersecting $K$ and let $\alpha = \tanh(\ell/4)$ ($\ell = +\infty$ and $\alpha = 1$ when $U$ is simply connected). We define constants

$$\Delta = \Delta(K, U) = \log \frac{1 + \alpha/7}{1 - \alpha/7}, \quad \Delta' = \Delta'(K, U) = \log \frac{1 + \alpha/2}{1 - \alpha/2}$$

(5.27)
and for $0 \leq r < \ell/2$ the $\epsilon$-function

$$
\epsilon_\ell(r) = -6 \log \left( 1 - \frac{\tanh(r/2)}{\tanh(\ell/4)} \right).
$$

(5.28)

One has: \( \tanh \frac{\Delta}{2} = \frac{\phi}{\Delta}, \tanh \frac{\Delta'}{2} = \frac{\phi'}{\Delta'}, \Delta < \ell/14, \Delta' < \ell/4 \) and \( \epsilon(\Delta) = 6 \log 7/6 < 1 \).

**Remarks 5.1** We recall some facts about universal covering maps of Riemann surfaces: Let \( \phi : \mathbb{D} \rightarrow U \) be a universal conformal covering map of \( U \). For \( x, y \in U \) their hyperbolic distance are defined as \( d_U(x, y) = \min \{ d_\mathbb{D}(\hat{x}, \hat{y}) \} \) where the minimum is taken over lifts \( \hat{x} \in \phi^{-1}(x) \) and \( \hat{y} \in \phi^{-1}(y) \) of \( x \) and \( y \), respectively. If \( p, p' \in \phi^{-1}(y) \) are two distinct lifts of a point \( y \in K \) then \( d_\mathbb{D}(p, p') \geq \ell \). Otherwise the geodesic connecting \( p \) and \( p' \) projects to a closed non-contractible curve in \( U \) intersecting \( K \) and of length $< \ell$, contradicting our definition of $\ell$. For the same reason, the map \( \phi : B_\mathbb{D}(\ell/2) \rightarrow B(y, \ell/2) \) must be a conformal bijection which preserves distances to \( y \), i.e. if \( z \in B_\mathbb{D}(\ell/2) \) then \( d_\mathbb{D}(z, p) = d_U(\phi(z), y) \). Note, however, that \( \phi \) need not be an isometry on the full disc, since two points in \( B(y, \ell/2) \setminus K \) may have lifts closer than their lifts in \( B_\mathbb{D}(\ell/2) \).

We have the following

**Lemma 5.2 (Local Koebe Distortion)** Let \( f \in \mathcal{E}(K, U) \). Denote by \( \| Df \| \) the maximal conformal derivative of \( f \) on the set \( f^{-1}K \). We define \( \lambda_1(f) = 3 \| Df \| \). Let \( x \in D_f \cap f^{-1}K \) and set

$$
\delta_f(x) = \min \{ \log \frac{5 + \alpha/Df(x)}{5 - \alpha/Df(x)}, \Delta \}.
$$

(5.29)

Let also \( \delta_f = \min \{ \log \frac{5 + \alpha\| Df \|}{5 - \alpha\| Df \|}, \Delta \} > 0 \) be the minimum value of \( \delta_f(x) \) over the compact set \( D_f \cap f^{-1}K \). Then \( B(x, \delta_f(x)) \subset D_f \) and we have the following properties:

1. If \( x' \neq x \) is another pre-image of \( f(x) \) then \( B(x, \delta_f(x)) \) and \( B(x', \delta_f(x')) \) are disjoint.
2. \( f \) is univalent on the hyperbolic disk \( B(x, \delta_f(x)) \) and \( B(f_x, \Delta) \subset fB(x, \delta_f(x)) \).
3. \( \beta \leq f[u, x] \leq \lambda_1(f) \) for \( u \in B(x, \delta_f(x)) \).
4. If \( u, v \in B(x, \delta_f(x)) \) and \( f_u, f_v \in B(f_x, r) \) with \( 0 < r \leq \Delta \) then

$$
\log \left( \frac{d(f_x, f_u)}{d(x, u)Df(v)} \right) \leq \epsilon_\ell(r).
$$

(5.30)

Proof: Let \( C \) be a connected component of \( D_f \subset K \) and fix an \( x \in C \) for which \( y = f(x) \in K \subset U \). Pick universal conforming covering maps, \( \phi_x : \mathbb{D} \rightarrow U \) and \( \phi_y : \mathbb{D} \rightarrow U \) for which \( \phi_x(0) = x \) and \( \phi_y(0) = y \). Let \( \hat{C} = \phi_x^{-1}C \subset \mathbb{D} \) be the lift of the connected component \( C \) containing \( x \). The composed map, \( f \circ \phi_x : \hat{C} \rightarrow U \) is a conformal covering map of \( U \). Since \( \phi_y : \mathbb{D} \rightarrow U \) is a universal covering there is a unique, a fortiori conformal, map \( \psi = \psi_{x,y} : \mathbb{D} \rightarrow \hat{C} \) such that \( \psi_{x,y}(0) = 0 \) and (cf. figure 5),

$$
f \circ \phi_x \circ \psi_{x,y} \equiv \phi_y : \mathbb{D} \rightarrow U.
$$

By definition of the hyperbolic metric the conformal derivative of \( f \) at \( x \) is given by

$$
\lambda \equiv Df(x) = 1/|\psi'(0)|.
$$
More generally, if \( u = \phi_x \circ \psi(z) \in C, \ z \in \mathbb{D} \) then
\[
Df(u) = 1/D\psi(z) = \frac{1}{|\psi'(z)|} \frac{1 - |\psi(z)|^2}{1 - |z|^2}.
\]
The value does not depend on the choices of covering maps because the conformal line element
\( ds = 2|dz|/(1 - |z|^2) \) is invariant under conformal automorphisms of the unit disk (both in the source and in the image).

The map \( \psi : (\mathbb{D}, d_\mathbb{D}) \to (\hat{C}, d_{\hat{C}}) \) is non-expanding [CG93, Theorem 4.1]. As mentioned above
the inclusion (\( \hat{C}, d_{\hat{C}} \)) \( \to (\hat{C}, d_U) \) is \( \beta^{-1} \)-Lipschitz so the composed map
\( \phi : (\mathbb{D}, d_\mathbb{D}) \to (\hat{C}, d_U) \) is also \( \beta^{-1} \)-Lipschitz.

The map \( \psi \) need not, however, be univalent on all of \( \mathbb{D} \), because a non-contractible loop
in \( C \) may be contractible in \( U \) (as is the case in figure 5).
On the other hand, the map \( \phi_y : B_\mathbb{D}(0, \ell/2) \to B(y, \ell/2) \) is a conformal bijection (Remark 5.1) so that
\[
h = \phi_x \circ \psi \circ \phi_y^{-1} : B(y, \ell/2) \to B(x, \ell/(2\beta))
\]
defines a local inverse of \( f \). In particular, we see that \( \psi \) is univalent on the disk \( B_\mathbb{D}(0, \ell/2) = B_C(0, \alpha) \).
The map, \( g : \mathbb{D} \to \mathbb{C} \), given by
\[
g(t) = \frac{\psi(t\alpha)}{\alpha\psi'(0)},
\]
is therefore univalent and normalised so that \( g(0) = 0 \) and \( g'(0) = 1 \). The Koebe distortion
Theorem [CG93, Theorem 1.6] applied to \( g \) shows that if \( |z| < \alpha \) then
\[
\frac{1}{(1 + |z|/\alpha)^2} \leq \left| \frac{\lambda \psi(z)}{z} \right| \leq \frac{1}{(1 - |z|/\alpha)^2}, \tag{5.31}
\]
\[
\frac{1 - |z|/\alpha}{(1 + |z|/\alpha)^2} \leq \left| \lambda \psi'(z) \right| \leq \frac{1 + |z|/\alpha}{(1 - |z|/\alpha)^2}.
\]

Using the first bound one verifies that
\[
\psi B_C(0, \frac{\alpha}{7}) \subset B_C(0, \frac{\alpha}{5\lambda}) \subset \psi B_C(0, \frac{\alpha}{2}). \tag{5.32}
\]
Going back to hyperbolic distances and \( U \), and noting that also \( \psi B_C(0, \alpha/7) \subset B_C(0, \alpha/7) \), we obtain
\[
hB(y, \Delta) \subset B(x, \delta_f(x)) \subset hB(y, \Delta') \subset hB(y, \ell/4).
\]
with the definition of \( \Delta, \Delta' \) and \( \delta_f(x) \) as in (5.27) and (5.29). In particular, \( B(x, \delta_f(x)) \subset C \subset D_f \).

Property (0): Let \( x' \) be another pre-image of \( y \) distinct from \( x \). Since \( B(x, \delta_f(x)) \subset C \) the
balls \( B_1 = B(x, \delta_f(x)) \) and \( B_2 = B(x', \delta_f(x')) \) are disjoint if they are in different connected
components of \( D_f \). If \( B_1 \cap B_2 \) is non-empty then we may find a shortest path, \( \gamma \subset B_1 \cup B_2 \)
connecting \( x \) and \( x' \) within \( C \). Then \( f(\gamma) \) is a closed non-contractible curve in \( U \), containing \( y \)
and of length \( < \ell/2 + \ell/2 = \ell \) which contradicts the definition of \( \ell \).

Property (1): Set \( B = B(x, \delta_f(x)) \). The first inclusion in (5.32) shows that \( fB \supset B(f(x), \Delta) \)
and since the local inverse \( h \) is well-defined and its image contains \( B \) the map \( f \) is univalent on \( B \).

Property (2), \( f[u, x] \geq \beta \): For \( v \in B(y, \ell/2) \) we have that \( d(h(v), h(y)) \leq \beta^{-1}d(y, v) \) and since
\( hB(y, \ell/2) \supset B(x, \delta_f(x)) \) we see that \( u = h(v) \in f^{-1}\{v\} \) is the point closest to \( x \in f^{-1}\{f(x)\} \).
Therefore \( d(u, x) \leq \beta^{-1}d(y, v) \) and we obtain the wanted inequality.
Property (2), \( f[u, x] \leq \lambda_1(f) \): By Schwarz’ Lemma, |\( \psi(z) \)| ≤ |\( z \)|, \( z \in \mathbb{D} \) and from the expression for the hyperbolic metric,
\[
1 \leq \frac{d_D(z, 0)}{d_D(\psi(z), 0)} \frac{|\psi(z)|}{|z|} \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]
Using the first bound in (5.31) we get for |\( z \)| < \( \alpha \),
\[
(1 - |z|/\alpha)^2 \leq \frac{f[u, x]}{Df(x)} = \frac{d_D(\psi(z), 0)D}{d_D(z, 0)} \leq \frac{(1 + |z|/\alpha)^2}{1 - |z|^2}.
\] (5.33)
In particular, for |\( z \)| ≤ \( \alpha/2 \) (corresponding to the hyperbolic radius \( \Delta' \)),
\[
f[x, u] \leq \frac{(3/2)^2}{1 - 1/4} Df(x) = 3 Df(x) \leq \lambda_1(f), \quad x \in B(x, \delta_f(x)).
\]

Property (3): The second bound in (5.31) shows that for |\( z \)|, |\( u \)| ≤ \( \hat{\tau} < \alpha \),
\[
\frac{1 - \hat{\tau}/\alpha}{(1 + \hat{\tau}/\alpha)^2} (1 - r^2) \leq \frac{Df(x)}{Df(v)} = \lambda|\psi'(u)| \frac{1 - |u|^2}{1 - |\psi(u)|^2} \leq \frac{1 + \hat{\tau}/\alpha}{(1 - \hat{\tau}/\alpha)^3}.
\]
Multiplying this and the inequality (5.33) we obtain
\[
\left| \log \frac{f[u, x]}{Df(x)} \right| \leq \log \frac{(1 + \hat{\tau}/\alpha)^3}{(1 - \hat{\tau}/\alpha)^2 (1 - \hat{\tau})^2} \leq 6 \log \frac{1}{1 - \hat{\tau}/\alpha},
\]
i.e. (5.30) with the \( \epsilon_\Delta \) function defined in (5.28). \( \square \)

This hyperbolic Koebe Lemma implies that conditions (T0)-(T3) of the previous section are verified for our class of maps, \( \mathcal{E}(K, U) \), when setting \( \Omega_f = D_f \cap f^{-1}K \) and looking at the metric space \( (K,d_U) \), the \( \epsilon \)-function \( \epsilon_\epsilon \), and finally \( \beta \), \( \Delta \) and \( \delta_f(x) \) as defined above.

**Theorem 5.3** Let \( \tau \) be an ergodic transformation on \( (\Omega, \mu) \). Let \( \mathcal{F} = (f_\omega)_{\omega \in \Omega} \in \mathcal{E}_\Omega(K, U) \) be a measurable family satisfying \( \mathbb{E}(\log \|Df_\omega\|) < +\infty \). Then \( \mu \)-almost surely the various dimensions agree and is given as the unique zero of the pressure function \( P(s) \).

Proof: We will apply Theorem 4.4. The assumption of bounded average logarithmic dilation is included in our hypothesis. We need to show that \( (n_0, \Delta) \) mixing holds for some \( n_0 \). This follows, however, directly from connectivity of \( U \) and the properties of our conformal maps. The diameter of \( K \) is finite within \( U \). Given two points \( y \) and \( z \) in \( U \) choose a path of uniformly bounded length (say less than twice the diameter of \( K \)) connecting them. By taking preimages we obtain paths of exponentially shrinking lengths. It suffices to take \( n_0 \) such that \( 2 \text{diam} K/\beta^{n_0} \leq \Delta \) and (T4) of Assumption 3.4 follows. An area estimate yields \( d(f) \text{Area}(K) = \int_{f^{-1}K} |Df|^2 d\text{Area} \leq \|Df\|^2 \text{Area}(K) \), whence
\[
d(f) \leq \|Df\|^2.
\] (5.34)
Therefore, \( \log d(f) \) is bounded on average and we may apply Theorem 4.4 to obtain the desired conclusion. For \( \phi \geq 0 \) we also have by change of variables,
\[
\int_K L_{s=2} \phi \ dA = \int_{f^{-1}K} \phi \ dA \leq \int_K \phi \ dA,
\]
which incidently shows that \( \overline{\sigma}_{\text{crit}} \leq 2 \) (as it should be !). \( \square \)
6 Mirror embedding and real-analyticity of the Hausdorff dimension

The dependence of the Hausdorff dimension on parameters may be studied through the dependence of the pressure function on those parameters. A complication arise, namely that our transfer operators do not depend analytically on the expanding map. In [Rue82], Ruelle circumvented this problem in the case of a (non-random) hyperbolic Julia set by instead looking at an associated dynamical zeta-function. Here, we shall introduce a mirror embedding which tackles the problem directly. We embed our function space into a larger space and semi-conjugate our family of transfer operators to operators with an explicit real-analytic dependency on parameters and mappings. We establish a Perron-Frobenius theorem through the contraction of cones of ‘real-analytic’ functions. The pressure function may then be calculated as the averaged action of the operator on a hyperbolic fixed point (cf. [Rue79, Rue97]) which has the wanted dependence on parameters. Finally as the pressure function cuts the horizontal axis transversally the result will follow from another implicit function Theorem.

6.1 Mirror extension and mirror embedding

Let $U$ be a hyperbolic subset of $\hat{\mathbb{C}}$ as before. We write $\overline{U} = \{ \overline{z} : z \in U \}$ for the complex conjugated domain (not the closure) and we define the mirror extension of $U$ as the product $\tilde{U} = U \times \overline{U}$. The map $j : U \to \tilde{U}$ given by $j(z) = (z, \overline{z}), \ z \in U$ is a smooth embedding of $U$ onto the mirror diagonal,

$$\text{diag}(U) = \{(z, \overline{z}) : z \in U\}.$$  

The ‘exchange-conjugation’,

$$c(u, v) = (v, \overline{u}), \ (u, v) \in U \times \overline{U}$$

defines an involution on the mirror extension leaving invariant the mirror diagonal. Let $X \subset \tilde{U}$ be an open subset. We call $X$ mirror symmetric, if $c(X) = X$. We say that $X$ is connected to the diagonal if any connected component of $X$ has a non-empty intersection with $\text{diag} \ U$. We write $A(X) = C^0(\text{Cl} \ X) \cap C^\omega(X)$ for the space of holomorphic functions on the mirror extension having a continuous extension to the boundary.

Lemma 6.1 Let $X \subset \tilde{U}$ be an open, mirror symmetric subset, connected to the diagonal and let $A = A(X)$. Then

(1) $A$ is a unital Banach algebra (in the sup-norm) with a complex involution,

$$\phi^*(u, v) = \overline{\phi(v, \overline{u})} \equiv \phi(v, u), \ (u, v) \in X, \ \phi \in A.$$  

(2) Denote by $A_\mathbb{R} = \{ \phi \in A : \phi^* = \phi \}$, the space of self-adjoint elements in $A$. Such functions are real-valued on the mirror diagonal and we have $A = A_\mathbb{R} \oplus iA_\mathbb{R}$.

(3) A function $\phi \in A$ is uniquely determined by its restriction to $(\text{diag} \ U) \cap X$.

Proof: (1) and (2) are clear. Suppose now that $\phi$ vanishes on the mirror diagonal. Because any point in $X$ is path-connected to the diagonal it suffices to show that $\phi$ vanishes on an open
neighborhood of a diagonal point \((y, \overline{y}), y \in \text{Int}K\). For \(u, v\) small enough we have a convergent power series expansion

\[
\phi(y + u, \overline{y} + v) = \sum_{k, l \geq 0} a_{k,l} u^k v^l.
\]

Setting \(u = re^{i\theta}, v = \overline{u}\) we obtain for \(r\) small enough

\[
0 = \phi(y + u, y + \overline{u}) = \sum_{m \geq 0} \sum_{k=0}^m a_{k,m-k} e^{i(2k-m)\theta},
\]

which vanishes iff \(a_{k,l} = 0\) for all \(k, l \geq 0\). \(\square\)

Consider the mirror extension, \(\hat{D} = \mathbb{D} \times \overline{\mathbb{D}} \simeq \mathbb{D}^2\), of the unit disk, \(\mathbb{D}\). We write \(d_{\mathbb{D}} = \frac{4dzd\overline{z}}{(1-z\overline{z})^2}\) for the Poincaré metric on \(\mathbb{D}\). [By abuse of notation we write \(dzd\overline{z}\) for the symmetric two tensor, \(\frac{1}{2}(dz \otimes d\overline{z} + d\overline{z} \otimes dz)\). Also note, that when \(c\) is a complex number, \(dz(c\frac{\partial}{\partial z}) = c\), but \(dz(c\frac{\partial}{\partial \overline{z}}) = \overline{c}\) (and not zero!). Below, we will use the following metric on \(\hat{D}\):

\[
d_{\hat{D}}^{(1)} = \frac{|dz_1|}{1 - z_1 \overline{z}_1} + \frac{|dz_2|}{1 - z_2 \overline{z}_2} \equiv ds_1 + ds_2, \quad (z_1, z_2) \in \hat{D}.
\]

This metric is more convenient here than the Riemannian metric, \(d_{\hat{D}}^{(2)} = \sqrt{ds_1^2 + ds_2^2}\).

**Definition 6.2** We denote by \(\text{Aut}(\mathbb{D})\) the group of holomorphic automorphisms of the disk consisting of all Möbius transformations which may be written \(R(z) = \frac{az + b}{cz + d}\), \(|a| > |b|\). To each \(R \in \text{Aut}(\mathbb{D})\) write \(\overline{R}(w) \equiv R(\overline{w}), w \in \mathbb{D}\) for the conjugated map. The pair \(\hat{R} = (R, \overline{R})\) acts isometrically on the extension, \(\hat{R}^*d_{\hat{D}} = d_{\hat{D}}\), and preserves the mirror diagonal. We denote by \(\text{Aut}(\hat{D}; \text{diag } \mathbb{D})\) the collection of such pairs and call it the group of mirror automorphisms of \(\hat{D}\). It is a subgroup of \(\text{Aut}(\mathbb{D}^2)\) which itself has a fairly simple explicit description, see e.g. [Kran00, Proposition 11.1.3].

**Proposition 6.3** The holomorphic two-form \(g_{\mathbb{D}}\) given by

\[
g_{\mathbb{D}} = \frac{4dz_1dz_2}{(1-z_1z_2)^2},
\]

is the unique symmetric holomorphic two form on \(\hat{D} = \mathbb{D} \times \overline{\mathbb{D}} = \mathbb{D}^2\) which extends the Poincaré metric on the diagonal, i.e. such that

\[
d_{\hat{D}} = j^* g_{\mathbb{D}}.
\]

A mirror automorphism preserves the holomorphic two-form, i.e. for all \(\hat{R} \in \text{Aut}(\hat{D}; \text{diag } \mathbb{D})\),

\[
\hat{R}^* g_{\mathbb{D}} = g_{\mathbb{D}}.
\]

Proof: A calculation shows that indeed we obtain an extension. By the previous Lemma, the factor \(1/(1-z_1z_2)\) is uniquely determined by its value on the diagonal. The assertion (6.36) is equivalent to the identity,

\[
R'(z_1)\overline{R}(z_2)\frac{(1-z_1z_2)^2}{(1-R(z_1)\overline{R}(z_2))^2} \equiv 1, \quad \forall(z_1, z_2) \in \hat{D},
\]
which is seen either by direct calculation or by the fact that it is indeed correct on the mirror diagonal (where it expresses the fact that $R \in \text{Aut}(\mathbb{D})$) and then by unicity of mirror holomorphic functions, Lemma 6.1(3) □

Let $\psi : \mathbb{D} \to \mathbb{D}$ be a holomorphic map without critical points. The pull-back of the Poincaré metric by $\psi$ is proportional to the Poincaré metric itself, where the factor of proportionality precisely defines the (square) of the conformal derivative,

$$\psi^*d_\psi = (D\psi)^2d_\psi, \quad D\psi > 0.$$  

It is independent of choice of conformal coordinates on $\mathbb{D}$, i.e. under conjugations by $R \in \text{Aut}(\mathbb{D})$ in either the source or in the image. We write $\psi(z) \equiv \psi(\overline{z})$, $z \in \mathbb{D}$ for the associated conjugated map. The mirror extended map, $\hat{\psi} = (\psi, \overline{\psi})$, is the unique map of $\hat{\mathbb{D}}$ for which $\hat{\psi} \circ j = j \circ \psi$. It preserves the diagonal but is, in general, not conformal on $\hat{\mathbb{D}}$ (with respect to neither $d_\mathbb{D}$ nor $d_{\mathbb{D}}^{(2)}$). It is, however, ‘conformal’ with respect to our holomorphic two-form, $g_\mathbb{D}$. More generally, if $\psi_1, \psi_2 : \mathbb{D} \to \mathbb{D}$ are two holomorphic maps, then their direct product $\Psi = (\psi_1, \psi_2)$ verifies,

$$\Psi^*g_\mathbb{D} = (D\Psi)^2g_\mathbb{D},$$

with a ‘conformal’ derivative given by the formula,

$$(D\Psi)^2 = \psi_1'(z_1)\psi_2'(z_2) \left( \frac{(1 - z_1z_2)^2}{(1 - \psi_1(z_1)\psi_2(z_2))^2} \right).$$

Let $\hat{\psi}$ be the above mirror extension of $\psi$. Then $(D\hat{\psi})^2$ is real and strictly positive on the mirror diagonal. We may then define $D\hat{\psi}$ as the unique positive square root on the mirror diagonal and extend holomorphically to all of $\hat{\mathbb{D}}$. On the mirror diagonal it coincides with the usual definition of the conformal derivative of $\psi$ on $\mathbb{D}$, i.e.

$$j^*D\hat{\psi}^2 = D\psi^2.$$  

Also, when $\Psi$ is a continuous deformation of a mirror extended map, $\Psi = \hat{\psi}$, then we may define $D\Psi$ by following the square-root along the deformation (again provided that there are no critical points).

**Lemma 6.4** Let $\Psi = (\psi_1, \psi_2)$ be a direct product map on $\hat{\mathbb{D}}$. Then for $i = 1, 2$,

$$(1 - |z_i|^2) \left| \frac{\partial}{\partial z_i} \log(D\Psi^2)(z_1, z_2) \right|$$

is conformally invariant with respect to mirror automorphisms, $\hat{R} \in \text{Aut}(\hat{\mathbb{D}}; \text{diag } \mathbb{D})$, both in the source and in the image.

**Proof:** To see this, we consider maps $\hat{R}_1, \hat{R}_2 \in \text{Aut}(\mathbb{D}; \text{diag } \mathbb{D})$ and the conjugated direct product, $\Phi = \hat{R}_2 \circ \hat{\psi} \circ \hat{R}_1$. Since $D\hat{R}_i^2 \equiv 1$, $i = 1, 2$, we have that $D\Phi^2 = D\Psi^2 \circ \hat{R}_1$. Let $(z_1, z_2) = (\hat{R}_1(u_1, u_2) = (R_1(u_1), \overline{R_1}(u_2))$. Taking the derivative with respect to $u_1$ and using $|\partial R_1/\partial u_1| = (1 - |z_1|^2)/(1 - |u_1|^2)$ we obtain

$$(1 - |z_i|^2) \left| \frac{\partial}{\partial z_i} \log(D\Psi^2)(z_1, z_2) \right| = (1 - |u_i|^2) \left| \frac{\partial}{\partial u_i} \log(D\Phi^2)(u_1, u_2) \right|$$

and thus the desired conformal invariance. □

Let $\psi : \mathbb{D} \to \mathbb{D}$ be a conformal map without critical points.
Definition 6.5 We define the injectivity radius $r = r[\psi](z) \in [0, +\infty]$ of $\psi$ at $z \in \mathbb{D}$ as the largest value such that $\psi$ is injective on a disc of hyperbolic radius $r$, centered at $z$. (In analogy with a similar notion for Riemann surfaces, see e.g. [McMull94, section 2.9]). We call $\rho = \rho[\psi](z) = \tanh \frac{r}{2}$ the Euclidean radius of injectivity. If $R \in \text{Aut}(\mathbb{D})$ maps zero to $z$, then $\psi \circ R$ is precisely injective on the Euclidean disc $B_C(0, \rho)$.

Proposition 6.6 (Mirror Koebe distortion) Let $\Psi = (\psi_1, \psi_2)$ be a direct product map on $\widehat{\mathbb{D}}$ where both maps $\psi_1$ and $\psi_2$ are conformal maps from $\mathbb{D}$ into itself and without critical points. At a given point $(z_1, z_2) \in \widehat{\mathbb{D}}$ we write $\rho_i = \rho[\psi_i](z_i)$, $i = 1, 2$ for the corresponding Euclidean radii of injectivity. We then have

$$|d \log D\Psi^2| \leq (2 + \frac{4}{\rho_1}) ds_1 + (2 + \frac{4}{\rho_2}) ds_2,$$

(6.37)

Proof: We will use conformal invariance twice. Let $(z_1, z_2) \in \widehat{\mathbb{D}}$. Fix mirror automorphisms for which $\tilde{R}_1(0, u_2) = (z_1, z_2)$ and $\tilde{R}_2(\psi_1(z_1), \psi_2(z_2)) = (w_1, 0)$. The conjugated product map, $\Phi = \tilde{R}_2 \circ \Psi \circ \tilde{R}_1 = (\phi_1, \phi_2)$, then maps $(0, u_2)$ to $(\phi_1(u_1), \phi_2(0)) = (w_1, 0)$. The conformal derivative at $(u_1, u_2)$ is then given by

$$D\Phi^2(u_1, u_2) = \phi_1'(u_1)\phi_2'(u_2)(1 - u_1u_2)^2.$$

(6.38)

Therefore,

$$(1 - |u_1|^2) \frac{\partial}{\partial u_1} \log D\Phi^2(u_1, 0)|_{u_1=0} = \frac{\phi''_1(0)}{\phi'_1(0)} - 2u_2.$$

(6.39)

Here, $\phi_1$ is univalent on the Euclidean disk of radius $\rho_1 = \rho[\psi_1](z_1)$ centered at zero. By the standard Koebe estimate, $|\phi''_1(0)/\phi'_1(0)| \leq 4/\rho_1$. Also $|u_2| \leq 1$ (in fact, $|u_2| = \tanh(d(z_1, \overline{z}_2)/2)$ for a slightly better estimate). The right hand side of (6.39) therefore does not exceed $(2 + 4/\rho_1)$. Combining with the previous Lemma on the conformal invariance of the distortion, we obtain

$$\left| \frac{\partial}{\partial z_i} \log(D\Psi^2)(z_1, z_2) \right| \leq (2 + \frac{4}{\rho_1}) \frac{1}{1 - |z_i|^2}.$$ 

Noting that $ds_1 = |dz_1|/(1 - |z_1|^2)$ and including the same estimate for the second variable we obtain the desired bound. \[\]

Let us now return to our hyperbolic space, $U \subset \widehat{\mathbb{C}}$ and the compact subset $K \subset U$. We define the constants $\Delta$ and $\Delta'$ as in (5.27). Let $K_\Delta = N_\Delta(K) \subset U$ be the $\Delta$ neighborhood of the compact set $K$. Below we will make use of constants,

$$\ell_2(K, U, \Delta) > 0 \text{ and } \alpha_2(K, U, \Delta) > 0,$$

(6.40)

defined as follows: Consider $x \in K$, $u \in K_\Delta$ and let $\gamma_{x,u}$ be a shortest geodesic between the two points. We let $\ell_{x,u} \in [0, +\infty]$ be the minimal length of non-contractible closed geodesics intersecting $\gamma_{x,u}$. Finally we let $\ell_2 = \ell_2(K, U, \Delta)$ be the infimum of all such lengths $\ell_{x,u}$. Because of $K$ and (the closure of) $K_\Delta$ being compact sets, this infimum is necessarily non-zero. We set $\alpha_2(K, U, \Delta) \equiv \tanh(\ell_2/4) \in [0, 1]$ which is then a lower bound for the Euclidean radius of injectivity of a Riemann mapping centered at a point along any of the above mentioned geodesics.

Remarks 6.7 In the previous section our definition of $\ell$ was somewhat simpler because a local distortion estimate sufficed. Below we need a global estimate and for this we need to control the distortion along paths (geodesics) connecting points in $K_\Delta$. 

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Figure 3: The quotients: \( \hat{\mathbb{D}} \rightarrow \hat{\mathbb{D}}/\text{diag } \Gamma \rightarrow \mathbb{D}/\Gamma \). The illustration is in the case of an annulus where \( \Gamma \) is generated by one element only. Since 4 dimensions is difficult to illustrate we have only drawn real sections.

**Proposition 6.8 (Global Mirror Distortion)** Let \( f_1, f_2 \in \mathcal{E}(K, U) \) and consider in the cover, as in the previous section, ‘inverses’ \( \psi_1 : \mathbb{D} \rightarrow \mathbb{D} \) and \( \psi_2 : \mathbb{D} \rightarrow \mathbb{D} \) of \( f_1 \) and \( f_2 \), respectively. We write \( \Psi = (\psi_1, \psi_2) : \hat{\mathbb{D}} \rightarrow \hat{\mathbb{D}} \) for the product map. For \( \xi \in \hat{K}_\Delta \) and \( \zeta \in \text{diag}(K) \), we have the bound (with the constant \( \alpha_2 \) from equation (6.40)),

\[
\left| \log \frac{D\Psi^2(\xi)}{D\Psi^2(\zeta)} \right| \leq (2 + \frac{4}{\alpha_2}) d(U(\xi, \zeta)).
\]

Proof: Let \( \gamma = (\gamma_1, \gamma_2) \) be a shortest geodesic between \( \xi \) and \( \eta \) for the metric, \( d^{(1)} \), i.e. the line element \( ds = ds_1 + ds_2 \). Then \( \gamma_1 \) and \( \gamma_2 \) are then shortest geodesics between the two coordinate projections of \( \xi \) and \( \eta \). Along \( \gamma \) we have by Proposition 6.6 the infinitesimal inequality

\[
|d \log D\Psi^2| \leq (2 + \frac{4}{\alpha_2}) ds
\]

and the result follows by integration from \( \log 1 = 0 \).

The previous distortion Lemma is, in reality, only for the extended disk. We need to establish distortion estimates for the mirror extension of \( U \). Unfortunately, it is not possible to do so globally (unless \( U \) is simply connected). We consider instead a restriction to a neighborhood of the mirror diagonal.

Let \( \Gamma \subset \text{Aut}(\mathbb{D}) \) be a surface group of \( U \) consisting of all automorphisms of \( \mathbb{D} \) that leaves invariant a given Riemann mapping \( \phi : \mathbb{D} \rightarrow U \). The mirror extended surface group, \( \text{diag } \Gamma = \{ (R, \overline{R}) : R \in \Gamma \} \) acts ‘diagonally’ upon \( \hat{\mathbb{D}} \) (it is a subgroup of \( \text{Aut}(\hat{\mathbb{D}}; \text{diag } K) \) and is normal iff \( \Gamma \) is Abelian). The quotient (see Figure 6.1),

\[ \hat{\mathbb{D}}/\text{diag } \Gamma, \]

is a complex 2-dimensional manifold. Proposition 6.3 shows that \( g \) passes down to the quotient as a holomorphic 2 form. The same is true for the distortion estimate in Proposition 6.8.

For \( r > 0 \) denote by \( \hat{K}_r \equiv N_r(\text{diag } K) \) the \( r \)-neighborhood of \( \text{diag } K \) in \( \hat{U} \). We lift \( \hat{K}_r \) to the set \( \hat{K}_{r,\Gamma} \) in \( N_r(\text{diag } \hat{\mathbb{D}}/\text{diag } \Gamma) \). We claim that if \( r < \ell/4 \) then the natural projection,

\[ \hat{K}_{r,\Gamma} \rightarrow \hat{K}_r, \]

\( \hat{\mathbb{D}}/\text{diag } \Gamma \) could be viewed as a (non-trivial) fiber-bundle over \( U \) with fiber \( \mathbb{D} \simeq \mathbb{D} \).
is a conformal isomorphism (in particular, the lift consists of one unique ‘copy’ of \( \tilde{K}_r \)). If this were not so then we could find \( z \neq z' \in \mathbb{D}/\text{diag } \Gamma \) and \( \eta, \eta' \in \text{diag } \mathbb{D}/\text{diag } \Gamma \) for which \( d(z', \eta) < r, d(z', \eta') < r \) and \( \pi(z) = \pi(z') \), i.e., projects to the same point in \( \tilde{K}_r \). But then \( d(\eta, \eta') < 2r \) and there is a non-contractible loop \( \gamma = (\gamma_1, \gamma_2) \) containing e.g. \( \eta \), intersecting \( \text{diag } K \) and of length \( \leq 4r < \ell \). Then at least one of \( \gamma_1 \) and \( \gamma_2 \) is non-contractible, of length \( < \ell \) and intersects \( K \) and this is impossible.

Our two-form, \( g \), on \( \mathbb{D}/\text{diag } \Gamma \) projects now to a unique holomorphic two-form, which we still denote \( g \), on \( \tilde{K}_{\ell/4} \). This is the unique analytic continuation of the conformal metric that we are searching for. It verifies,

\[
d_{\mathbb{U}/\text{diag } U} = j^* g
\]

Now, let \( \xi, v \in \tilde{K}_{\ell/4} \) and suppose that \( \Psi = (\psi_1, \psi_2) : \mathcal{O}(\xi) \to \mathcal{O}(v) \) is a locally defined product map between neighborhoods (in \( \tilde{K}_{\ell/4} \)) of the two points. We may then define the conformal derivative of this map through the identification

\[
\Psi^* g_{\|v} \equiv D\Psi^2(\xi) g_{\|\xi}.
\]

When \( \Psi \) preserves the diagonal, then \( D\Psi^2_{\|\text{diag } K} > 0 \) and we may define its positive square root or principal logarithm in the usual way.

### 6.2 Mirror extended transfer operators and cone contractions

Let \( f \in \mathcal{E}(K, U) \) and let \( \hat{f} = (f, \overline{f}) \) be the mirror extended map. For \( \eta \in \text{diag } (K) \), we write for its mirror-preimages

\[
P^\hat{f}_f(\eta) \equiv \hat{f}^{-1}(\eta) \cap \text{diag } K \equiv \{u_i\}_{i \in J},
\]

with \( J \) an index set. We wish to define an analytic continuation of this ensemble to points in \( \tilde{K}_\Delta \). For \( \xi \in \tilde{K}_\Delta \), pick \( \eta \in \text{diag } K \) and a path \( \gamma \) in \( \tilde{K}_\Delta \) connecting \( \eta \to \xi \). For each \( i \in J \), \( \gamma \) lifts by \( \hat{f} \) to a path \( \gamma_i \) connecting \( u_i \) to some point \( v_i \in \tilde{K}_{\Delta/\beta} \) (because of contraction of the inverse map). The collection

\[
P^\hat{f}_f(\xi) \equiv \{v_i\}_{i \in J} \subset \tilde{K}_{\Delta/\beta}
\]

yields the desired continuation. This set depends only on \( \xi \) (and \( f \), of course) but not on the choices of \( \eta \) and the path \( \gamma \). Any other choice will just give rise to a permutation of \( J \). This is true if \( \gamma \) is a shortest geodesic to the diagonal (because its length is smaller than \( \Delta < \ell/4 \)). But then it is also true for any other path as long as the path stays within \( \tilde{K}_\Delta \).

Denote by \( D\hat{f}^2(v) \), \( v \in P^\hat{f}_f(\xi) \) the holomorphic conformal derivative, (6.41), of \( \hat{f} \). When \( \eta \), whence also \( u \in P^\hat{f}_f(\eta) \), belongs to the mirror diagonal then \( D\hat{f}^2(u) > 0 \) and we define its logarithm by its principal value, \( \log D\hat{f}(u) = \frac{1}{2} \log D\hat{f}^2(u) \in \mathbb{R} \). This extends to all \( \xi \in \tilde{K}_\Delta \), \( v \in P^\hat{f}_f(\xi) \) by analytic continuation, and arguing as above, is independent of the choices made.

Recall that \( A \equiv A(\tilde{K}_\Delta) = C^0(\text{Cl } \tilde{K}_\Delta) \cap C^\omega(\tilde{K}_\Delta) \) denotes the space of holomorphic functions having a continuous extension to the boundary. We define for \( s \in \mathbb{C} \), \( \phi \in A(\tilde{K}_\Delta) \) and \( \xi \in \tilde{K}_\Delta \) the transfer operator

\[
L_{s, \hat{f}} \phi(\xi) = \sum_{v \in P^\hat{f}_f(\xi)} D\hat{f}(v)^{-s} \phi(v) \equiv \sum_{v \in P^\hat{f}_f(\xi)} e^{-s \log D\hat{f}(v)} \phi(v).
\]

For the moment let us fix a real value of \( s \geq 0 \). Then \( L_{s, \hat{f}} \) preserves \( A_\mathbb{R} \), the space of self-adjoint elements. We define for \( \sigma > 0 \) a closed proper convex cone in \( A_\mathbb{R} \),

\[
C_\sigma = \{ \phi \in A_\mathbb{R} : |\phi(\xi) - \phi(\zeta)| \leq \phi(\zeta)(e^{\sigma d(\xi, \zeta)} - 1), \xi \in \tilde{K}_\Delta, \zeta \in \text{diag } K \}.
\]
Figure 4: The cone contraction. The sliced cone $C_{\sigma', l=1}$ has an $R$-neighborhood which is contained in $C_\sigma$.

We define $\beta(\phi_1, \phi_2) = \inf \{ \lambda > 0 : \lambda \phi_1 - \phi_2 \in C_\sigma \}$ and write $d_C = \frac{1}{2} \log (\beta(\phi_1, \phi_2) \beta(\phi_2, \phi_1))$ for the corresponding projective Hilbert metric (cf. [Bir67, Liv95, Rugh02]).

Fix a point $\zeta_0 = (x_0, y_0) \in \text{diag} (K)$ and denote by $\ell \in A_R$ the (real-analytic) linear functional $\ell(\phi) = \phi(\zeta_0)$, $\phi \in A$.

We use this to introduce the sliced cone,

$$C_{\sigma, l=1} \equiv \{ \phi \in C_\sigma : \ell(\phi) = 1 \}.$$

**Lemma 6.9 (Cone contraction)** Let $s \geq 0$ and choose $\sigma \equiv \sigma(s) > 0$ large enough so that

$$\sigma' \equiv \sigma'(s) = \left(1 + \frac{2}{\alpha_2}\right) s + \frac{1}{\beta} \sigma < \sigma.$$

Then there is $\eta < 1$ such that for every $f \in E(K,U)$, the operator $L_{s, \hat{f}}$ maps $C_\sigma$ into $C_{\sigma'}$ and is an $\eta$-Lipschitz contraction for the Hilbert metric, $d_{C_\sigma}$. Furthermore,

(a) There is $k > 0$ such that $\ell(\phi) \geq k \|\phi\|$ for all $\phi \in C_\sigma$ (this corresponds to the dual cone having non-empty interior; we say that the cone is outer regular).

(b) There is $R > 0$ such that if $\phi \in C_{\sigma', l=1}$ then $B(\phi, R) \subset C_\sigma$ (C$_\sigma$ has a ‘uniformly large’ interior; we say that the cone is inner regular).

Proof: Fix $\xi = (\xi_1, \xi_2) \in \hat{K}_\Delta$, $\eta = (\eta_1, \eta_2 = \bar{\eta}_1) \in \text{diag} (K)$ and let $\hat{\gamma} = (\gamma_1, \gamma_2)$ be a shortest geodesic for the metric $ds = ds_1 + ds_2$ in $\hat{D}$. Then $\gamma_i$ is a shortest geodesic between $\xi_i$ and $\eta_i$, $i = 1, 2$. We write $d = d_{\hat{f}}(\xi, \eta) = \text{len} (\gamma_1) + \text{len} (\gamma_2)$ for the total length. By considering pre-images by $F \equiv \hat{f}$ of $\hat{\gamma}$ we obtain a pairing $(u, v)$ of the corresponding pre-images of $\xi$ and $\eta$ which verifies,

$$d_{\hat{f}}(u, v) \leq \beta^{-1} d,$$
because of contraction of the local inverse maps. By definition of \( \alpha_2 = \alpha_2(K, U, \Delta) \) and the use of Mirror Koebe distortion, Proposition 6.8, we obtain,

\[
\|L_{s,f}\phi(\xi) - L_{s,f}\phi(\eta)\| \\
\leq \sum |(DF(u))^{-s}\phi(u) - (DF(v))^{-s}\phi(v)| \\
\leq \sum |DF(u)|^{-s} |\phi(u) - \phi(v)| + |DF(u)|^{-s} - |DF(v)|^{-s} |\phi(v)| \\
\leq (e^{1 + \frac{s}{2} + \sigma d} - 1) L\phi(\eta) \\
\leq (e^{\sigma d} - 1) L\phi(\eta)
\]

where \( \sigma' = (1 + \frac{s}{2})s + \beta^{-1}\sigma \).

Thus, \( L \equiv L_{s,f} : C_{\sigma} \to C_{\sigma'} \) and we get for the projective diameter (for this standard calculation we refer to e.g. [Liv95] or [Rugh02, Appendix A]),

\[
diam_{C_{\sigma}} C_{\sigma'} \leq D = 2 \log \frac{\sigma + \sigma'}{\sigma - \sigma'} + \sigma' \text{diam} \text{K}_\Delta < \infty,
\]

where we write \(|K|\) for the diameter of \( K \) in \( U \). By Birkhoff’s Theorem (see [Bir67],[Liv95] or [Rugh02, Lemma A.4]) this implies a uniform contraction for the hyperbolic metric on \( C_{\sigma} \).

Writing \( \eta = \tanh(D/4) < 1 \) we have for \( \phi_1, \phi_2 \in C_{\sigma} \),

\[
d_C(\phi_1, \phi_2) \leq \eta d_C(\phi_1, \phi_2).
\]

Property (a) is clear from the definition of the cone which implies:

\[
|\phi(\xi)| \leq \ell(\phi) e^{\sigma \text{diam} \text{K}_\Delta}, \quad \phi \in C_{\sigma}, \xi \in \text{K}_\Delta.
\]

Set \( \kappa = \frac{2}{\tanh(\Delta/2)} \) and let \( \phi \in A(\text{K}_\Delta) \). We claim that for \( \zeta \in \text{diag}(K) \) and \( \xi \in \text{K}_\Delta \):

\[
|\phi(\xi) - \phi(\zeta)| \leq |\phi| \kappa d(\zeta, \xi). \tag{6.42}
\]

It suffices to verify this inequality in the universal cover. Consider coordinates where \( 0 \in \text{diag} \mathbb{D} \mapsto \eta \) and \( u \in \mathbb{D} \mapsto \xi \). Let \( R = \tanh(\Delta/2)/(|u_1| + |u_2|) \). If \( |t| \leq R \) then \( |u_1| + |u_2| \leq \tanh(\Delta/2) \) which implies \( d_\mathbb{D}(tu_1, 0) + d(tu_2, 0) \leq \Delta \) and then also \( d^{(1)}(tu, 0) \leq \Delta \). It follows that \( t \in B_{\mathbb{C}}(0, R) \mapsto \phi(tu) \) is analytic and bounded by \( \|\phi\| \). By the Schwarz Lemma we obtain

\[
|\phi(\xi) - \phi(\eta)| \leq \frac{2\|\phi\|}{R} \leq \frac{2\|\phi\|}{\tanh(\Delta/2)} d(\xi, \eta).
\]

Consider \( h \in C_{\sigma', \ell = 1} \) and \( \phi \in A_\mathbb{R}(\text{K}_\Delta) \). In order for \( h + \phi \) to belong to \( C_{\sigma} \) we need that

\[
\left| \frac{h_\xi + \phi_\xi}{h_\eta + \phi_\eta} - 1 \right| \leq \exp(\sigma d(\xi, \eta)) - 1
\]

is verified for all \( \xi \in \text{K}_\Delta \) and \( \eta \in \text{diag}(K) \). Setting \( d = d(\xi, \eta) \) we see that this is the case if

\[
|h(\xi) - h(\eta)| + |\phi_\xi - \phi_\eta| \leq (h(\eta) - \|\phi\|)(e^{\sigma d} - 1).
\]

Using that \( h \in C_{\sigma'}, \ell(h) = 1 \) and the above distortion estimate (6.42) for \( \phi \) we see that it suffices that for all \( d > 0 \),

\[
\|\phi\| \leq \frac{e^{\sigma d} - e^{\sigma d}}{\kappa d + e^{\sigma d} - 1} e^{-\sigma d \text{diam} K}.
\]

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Letting $d \to 0$ the right hand side tends to $(\sigma - \sigma') \exp(-\sigma' \text{diam} K)/\kappa + \sigma > 0$ and in the $d \to \infty$ limit it tends to $\exp(-\sigma' \text{diam} K) > 0$. It follows that it has a minimum $R > 0$ and we have shown property (b). \qed

Consider now a sequence $L_0, L_1, L_2, \ldots$ of operators as in the above Lemma. We write $L^{(n)} = L_n \circ L_{n-1} \circ \cdots \circ L_1$ for the $n$th iterated operator.

Lemma 6.10 There are constants $c_1, c_2 < \infty$ such that for $h, h' \in \mathbb{C}_\sigma, \ell = 1$, $\phi \in A$ and $n \geq 1$:

\[
\begin{align*}
(1) & \quad \left| \frac{L^{(n)} h}{\ell(L^{(n)} h)} - \frac{L^{(n)} h'}{\ell(L^{(n)} h')} \right| \leq c_1 \eta^n, \\
(2) & \quad \left| \frac{L^{(n)} \phi}{\ell(L^{(n)} h)} - \frac{L^{(n)} h}{\ell(L^{(n)} h)} \ell(L^{(n)} \phi) \right| \leq c_2 \eta^n |\phi|.
\end{align*}
\]

Proof: Outer regularity, i.e. Property (a) of the above Lemma, and a computation show that for $\phi_1, \phi_2 \in \mathcal{C}_\sigma, \ell = 1$, \n
\[|\phi_1 - \phi_2| \leq \frac{k + 1}{k^2} (e^{d(\phi_1, \phi_2)}/2 - 1).\]

When $\phi_1, \phi_2 \in L^{(n)} \mathcal{C}_\sigma$, $\ell(\phi_1) = \ell(\phi_2) = 1$ and $n \geq 1$ we have that $d_{\mathcal{C}_\sigma}(\phi_1, \phi_2) \leq \eta^n D$ and therefore, \n
\[|\phi_1 - \phi_2| \leq \frac{k + 1}{k^2} (e^{\eta^n D} - 1)\]

which is smaller that $c_1 \eta^n$ for a suitable choice of $c_1$. This yields the first bound.

For the second bound note that $B(h, R) \subset \mathcal{C}_\sigma$. For $\phi \in A_{\mathbb{R}}$ (small) and $h \in \mathcal{C}_\sigma$, $d_{\mathcal{C}}(h + \phi, h) \leq \frac{1}{R} |\phi| + o(|\phi|)$ and therefore \n
\[\left| \frac{L^{(n)}(h + \phi)}{\ell(L^{(n)}(h + \phi))} - \frac{L^{(n)} h}{\ell(L^{(n)}(h))} \right| \leq \frac{k + 1}{k^2} \eta^n \frac{1}{R} |\phi| + o(|\phi|)\]

By linearizing this bound (and loosing a factor of at most $\sqrt{2}$) we may extend this bound to any complex $\phi \in A$ to obtain the second inequality with $c_2 = \sqrt{\frac{2k + 1}{k^2}} R$. \qed

6.3 Analytic conformal families and mirror extensions

Let $O_{\mathbb{R}} \subset \mathbb{R}^n$ be an open set containing the origin and let $O_{\mathbb{C}} \subset \mathbb{C}^n$ be an open convex neighborhood, invariant under complex conjugation. Also, let $t \in O_{\mathbb{C}} \subset \mathbb{C}^n \to f_t \in \mathcal{E}(K, U)$ be a continuous map.

Definition 6.11

1. $(f_t)_{t \in O_{\mathbb{C}}}$ is called an analytic family, if the map $\{(t, z) : t \in O_{\mathbb{C}}, z \in D_{f_t}\} \mapsto f_t(z) \in \mathbb{C}$ is analytic.

2. We say that the family $f_t$ verifies an $L$-Lipschitz condition (with $0 < L < +\infty$) if for any $z \in K_\Delta$, and any choice of local inverse $f_0^{-1}(z)$, the map $t \in O_{\mathbb{C}} \mapsto \log Df_t \circ f_t^{-1}(z) \in \mathbb{C}$ is $L$-Lipschitz.

3. We define the condition number of $f \in \mathcal{E}(K, U)$ to be \n
\[\Gamma(f) = \|Df\|_{f^{-1}K} \|1/Df\|_{f^{-1}K}.\]
It is no lack of generality to assume that the parameters are one-dimensional \((n=1)\). We may also assume that \(O_C = \mathbb{D}\), i.e. is the unit-disk and consider \(O_R = \mathbb{D} \cap \mathbb{R} = [-1, 1]\) as a real section. In the following let \(t \in \mathbb{D} \mapsto f_t \in \mathcal{E}(K, U)\) be an analytic family, verifying an \(L\)-Lipschitz condition.

**Notation 6.12** Below it is convenient to introduce

\[
\frac{d}{dt} = (1 - it) \frac{\partial}{\partial t}
\]

for the conformal derivative from \((\mathbb{D}, d_{\mathbb{D}})\) to \(\mathbb{C}\) (with the Euclidean metric). For a holomorphic map, \(h : \mathbb{D} \to \mathbb{D}\), from the disk to itself we write also

\[
\frac{D}{Dt} = \frac{1 - it}{1 - h(t)\overline{h(t)}} \frac{\partial h}{\partial t}
\]

for the conformal derivative between the disks. (Note that we do not take absolute values).

Let \(\overline{U} = U \times \overline{U}\) be the mirror extension of \(U\). Let \(\Delta > 0\) be chosen as in the previous section. We denote by \(d_{\overline{U}}\) the metric on the mirror extension of \(U\), induced by the metric on the universal cover, \(\overline{U}\). We obtain a conjugated analytic family if we set \(D_{\overline{t}} = (\overline{D_{t}}) \subset \overline{U}\) and for \(x' \in D_{\overline{t}}, \overline{f}_t(x') \equiv \overline{f_t}(x')\). Then \(\overline{f}_t(x')\) is analytic in \(t\) and \(x'\) on \(\{(t,z) : t \in \mathbb{D}, z \in D_{\overline{t}}\}\). We also define for \(t \in \mathbb{D}\) the product map \(F_t : (x,x') \in D_{\overline{t}} \times D_{\overline{t}} \mapsto (f_t(x), \overline{f_t}(x')) \in U \times \overline{U}\). Again, this map is analytic in \(x, x'\) and \(t\) on its domain of definition.

Consider \(x_0 \in D_{\overline{t}}, y_0 = f_0(x_0) \in K\) and Riemann mappings \(\phi_{x_0}\) and \(\phi_{y_0}\) defined as above. Let \(\psi_t\) be the corresponding ‘inverse’ map. Since \(\mathbb{D}\) is simply connected there is a unique holomorphic extension

\[
(t, z) \in \mathbb{D} \times \mathbb{D} \mapsto \psi_t(z) \in \mathbb{D}
\]

which analytically continues \(\psi_0\) and defines an inverse (branch) of \(f_t, t \in \mathbb{D}\). For fixed \(z \in \mathbb{D}\) the map \(t \in \mathbb{D} \mapsto \psi_t(z) \in \mathbb{D}\) is holomorphic, whence it is non-expanding and its conformal derivative can not exceed one, i.e.

\[
\left| \frac{D}{Dt} \psi_t(z) \right| \equiv \frac{1 - it}{1 - \psi_t(z) \overline{\psi_t(z)}} |\partial_t \psi_t| \leq 1
\]

The map \(\overline{\psi_t}(w) \equiv \overline{\psi_t}(\overline{w})\) defines an inverse of \(\overline{f_t}\) (in the corresponding cover). Also, \(\Psi_t(w_1, w_2) = (\psi_t(w_1), \overline{\psi_t}(w_2))\) defines an inverse of \(F_t\) in the mirror-extended cover of \(U\). Note that for \(t\) real the functions \(\psi_t\) and \(\overline{\psi_t}\) are complex conjugated and \(\Psi_t\) therefore preserves the mirror diagonal but that this is no longer true when \(t\) becomes complex.

For all \(\xi \in \hat{K}_\Delta, t \in \mathbb{D}\) we may define

\[
P_{F_t}(\xi) = \{v_i^t\}_{i \in J}
\]

as follows: Let \(\eta \in \text{diag} K\) be at a distance \(\leq \Delta\) to \(\xi\). Denote by \(\gamma\) a shortest geodesic between the two points. For every \(u_0^i \in \hat{F}_0^{-1}(\eta) \cap \text{diag} K\), we lift \(\gamma\) by \(F_0\) to a path connecting \(u_0^i\) to a point \(v_i^0 \in \hat{F}_0^{-1}(\xi)\) and we define then \(u_i^t \in F_t^{-1}(\eta)\) and also \(v_i^t \in F_t^{-1}\) by analytic continuation in \(t \in \mathbb{D}\) (by using a suitable inverse \(\Psi_t\) in the cover and projecting down). This defines the \(v_i^t\) uniquely up to permutations of \(J\) (see Figure 6.3).

If \(\xi \in \overline{U}\) then

\[
d(\Psi_t(\xi), \Psi_0(\xi)) = d(\psi_t(z_1), \psi_0(z_1)) + d(\overline{\psi_t}(z_2), \overline{\psi_0}(z_2)) \leq 2d_\mathbb{D}(t, 0).
\]
For \( t \) real we know that \( \Psi_t : \hat{K}_\Delta \to \hat{K}_{\Delta/\beta} \). When making \( t \) complex we want still to have a contraction of \( \hat{K}_\Delta \) and by the above it suffices to have \( \Delta/\beta + 2\mathbb{D}(t,0) < \Delta \) or, equivalently,

\[
|t| < \tanh \left( \frac{\Delta}{4}(1 - \frac{1}{\beta}) \right). \tag{6.44}
\]

When this condition is fulfilled we may analytically continue the transfer operator in \( t \). First note that the conformal derivative, \( DF_t(v) \) of \( F_t \), at a point \( v, \xi \in \hat{K}_\Delta \subset \hat{K}_{\ell/4} \), cf. equation (6.41). Recalling that \( DF_0(u) > 0 \) when \( u \in P_{F_0}(\eta) \), \( \eta \in \text{diag } K \) we may define \( \log DF_t(u) \in \mathbb{R} \) by its principal value and then \( \log DF_t(v) \in \mathbb{C} \) using the lift of the path \( \gamma \) and analytic continuation in \( t \).

For \( \phi \in A(\hat{K}_\Delta), \xi \in \hat{K}_\Delta, s \in \mathbb{C} \) and \( t \) verifying (6.44), we set

\[
L_{s,F_t}(\phi) = \sum_{v \in P_{F_t}(\xi)} DF_t^{-s}(v)\phi(v) = \sum_{v \in P_{F_t}(\xi)} \exp(-s \log DF_t(v))\phi(v).
\]

This uniquely defines a bounded linear operator on \( A(\hat{K}_\Delta) \).

The \( L \)-Lipschitz condition on \( f_t \) is equivalent to the assumption that \( t \in (\mathbb{D}, d_{\mathbb{D}}) \mapsto \log D\psi_t \) is \( L \)-Lipschitz. Since the maps are analytic we arrive at the equivalent condition,

\[
\left| \frac{d}{Dt} \log D\psi_t \right| \leq L. \tag{6.45}
\]

Our hypotheses ensures that the local inverse of \( F_t \), i.e. the couple \( \Psi_t = (\psi_t, \overline{\psi}_t) \), satisfies the conditions for the following Lemma to apply:

**Lemma 6.13 (Parameter distortion)** Let \( \phi_{1t}, \phi_{2t}, t \in \mathbb{D} \) be holomorphic families of conformal maps from the disk to itself, both having a conformal derivative which is \( L \)-log-Lipschitz as in (6.45). Let \( \Phi_t = (\phi_{1t}, \phi_{2t}) : \hat{\mathbb{D}} \to \hat{\mathbb{D}} \) be their direct product. Then

\[
\left| \frac{d}{Dt} \log D\Phi_t^2 \right| \leq L + 4
\]
Proof: We have (cf. the Notation 6.12),
\[ D\phi^2 = |D_u\phi|^2 = \partial_u\phi \bar{\partial}_u\phi \frac{(1 - u\bar{u})^2}{(1 - \phi \bar{\phi})^2}. \]

Taking a derivative in \( t \),
\[ \frac{d}{Dt} \log |D_u\phi|^2 = (\partial_u\phi)^{-1} \frac{d}{Dt} \partial_u\phi + 2\bar{\phi} \frac{D}{Dt}\phi. \]

In the identity,
\[ d \log D\phi^2 = 2 \text{Re} \left( \frac{d}{Dt} \log D\phi^2 \frac{dt}{1 - t^2} \right), \]
the left hand side is, by assumption, bounded in absolute value by \( L|dt|/(1 - t^2) \). But then \( |\frac{d}{Dt} \log D\phi^2| \leq \frac{L}{2} \) and also,
\[ \left| (\partial_u\phi)^{-1} \frac{d}{Dt} \partial_u\phi \right| \leq \frac{L}{2} + 2|\bar{\phi} \frac{D}{Dt}\phi| \leq \frac{L}{2} + 2. \]

Consider now the product map \( \Phi = \Phi_t \). Assume first that only \( \phi_1 \) depends on \( t \). We then use the same conjugation as in Proposition 6.6 to obtain the expression (6.38) for the conformal derivative \( D\Phi^2 \). Taking now a \( t \)-derivative we get
\[ |\frac{d}{Dt} \log D\Phi^2| = |(\partial_u_1\phi_1t)^{-1} \frac{D}{Dt}\partial_u_1\phi_1t| \leq \frac{L}{2} + 2. \]

Adding the same contribution from the \( t \)-dependence of \( \phi_2t \) we reach the desired conclusion. \[\]

**Lemma 6.14** Let \( h \in C_{\sigma'} \). Choose \( x_0 \in f_0^{-1}K \) and set \( \lambda = \|DF\|_{f^{-1}K} \). Suppose that \( d_D(t, 0) \leq \Delta(1 - \frac{1}{\beta}) \). Let \( \xi \in \hat{K}_\Delta, \eta \in \text{diag} K \) and let \( v_t \in P_{F_t}(\xi) \) and \( u_t \in P_{F_0}(\eta) \) be pairs of pre-images constructed as above. Then
\[ \left| \frac{h(v_t)e^{-s \log(D_{F_t}(v_t)/\lambda)}}{h(u_0)e^{-s_0 \log(D_{F_0}(u_0)/\lambda)}} - 1 \right| \leq e^q - 1 \]

with
\[ q = |s - s_0| \log \Gamma(f_0) + \left( 2 + |s|(2 + \frac{L}{2}) \right) d_D(t, 0) + \left( \frac{1}{\beta} + |s|(1 + \frac{2}{\alpha_2}) \right) d(\xi, \eta). \]

Proof: Since \( h \in C_{\sigma'} \) we know that
\[ \left| \frac{h(v_t)}{h(u_0)} - 1 \right| \leq e^{d(v_t, u_0)} - 1. \]

and the distance in the exponent may be bounded as follows:
\[ d(v_t, u_0) \leq d(v_t, v_0) + d(v_0, u_0) \leq 2d_D(t, 0) + \frac{1}{\beta} d(\xi, \eta). \]

Lemma 6.8 and 6.13 apply here so we also have the following inequalities,
\[ \left| \log \frac{DF_t(v_t)}{DF_0(v_0)} \right| \leq \left( 2 + \frac{L}{2} \right) d_D(t, 0) \]
and
\[ \left| \log \frac{DF_0(v_0)}{DF_0(u_0)} \right| \leq \left( 1 + \frac{2}{\alpha_2} \right) d(\xi, \eta). \]

By definition 6.11 (3) of the condition number of \( f \),
\[ \left| \log \frac{DF_0(u_0)}{\lambda} \right| \leq \log \Gamma(f). \]

The inequality,
\[ \prod e^{\alpha_i} - 1 \leq e^{\sum |a_i|} - 1, \]

is valid for any complex numbers, \( a_1, \ldots, a_n \). Now, insert the 4 estimates above to obtain the claimed inequality. \( \square \)

The following non-linear map,
\[ \pi_{s,F_t}(\phi) = \frac{L_{s,F_t} \phi}{\ell(L_{s,F_t} \phi)} \quad (6.46) \]
is well-defined when the denominator does not vanish.

**Lemma 6.15 (lemma neighborhood)** Let \( f_t \in \mathcal{E}(K,U) \), \( t \in \mathbb{D} \) verify an \( L \)-Lipschitz condition. For \( s_0 \geq 0 \) we let \( W^{s_0} \) denote the open neighborhood of \((s_0,0) \subset \mathbb{C} \times \mathbb{C}\) consisting of all \((s,t)\) that verify
\[ |s - s_0| \log \Gamma(f_0) + \left( 2 + |s|(2 + \frac{L}{2}) \right) d_\mathcal{E}(t,0) < \log \frac{4}{3}, \]
and we let
\[ \rho = \frac{1}{4} e^{-\sigma' \text{diam} \mathcal{K}_\Delta}. \]

Then for \( h \in C_{\sigma',t=1} \) and all \((s,t) \in W^{s_0}, \phi \in A(\mathcal{K}_\Delta), |\phi| < \rho \) we have
\[ 1 \leq \|\pi_{s,F_t}(h + \phi)\| \leq 5e^{2\sigma' \text{diam} \mathcal{K}_\Delta} \]
and also,
\[ \left| \frac{\ell(\lambda^s L_{s,F_t}(h + \phi))}{\ell(\lambda^{s_0} L_{s_0,F_0}(h))} - 1 \right| \leq \frac{2}{3}. \quad (6.47) \]

Proof: We first use our previous Lemma for \( \xi = \eta \). We let \( q \) and \( u_t, v_t \) be as in that Lemma. Our assumptions imply \( e^q - 1 < \frac{1}{3} \) and therefore,
\[ \left| h(u_t) e^{-s \log \frac{DF_0(u_0)}{\lambda}} - h(u_0) e^{-s_0 \log \frac{DF_0(u_0)}{\lambda}} \right| \leq \frac{1}{3} h(u_0) e^{-s_0 \log \frac{DF_0(u_0)}{\lambda}}. \]

Summing this inequality over all pairs of pre-images and then dividing by the right hand side, we obtain
\[ \left| \frac{\ell(\lambda^s L_{s,F_t} h)}{\ell(\lambda^{s_0} L_{s_0,F_0} h)} - 1 \right| \leq \frac{1}{3}. \quad (6.48) \]

In particular, \( |\lambda^s \ell(L_{s,F_t} h)| \geq \frac{2}{3} \lambda^{s_0} \ell(L_{s_0,F_0} h) \), for all \( s \) and \( t \) verifying the first condition. Using the Lemma once more, and for general \( \xi \) and \( \eta \), we see that
\[ |e^{s \log \frac{DF_0(u_0)}{\lambda}}| \leq \frac{4}{3} e^{-s_0 \log \frac{DF_0(u_0)}{\lambda}} \leq \frac{4}{3} e^{e^{-s_0 \log \frac{DF_0(u_0)}{\lambda}}} h(u_0), \]
where we have set \( \kappa = \sigma' \text{diam} \tilde{K}_\Delta \). From this we obtain

\[
\| \lambda^s L_{s,F_1} \| \leq \frac{4}{3} e^{\kappa} \lambda^{s_0} L_{s_0,F_0} h(\eta).
\]

When \( \phi \) is of norm smaller than \( \rho \) we have because \( h \in C_{0',t=1} \),

\[
\frac{\| \ell(\lambda^s L_{s,F_1} \phi) \|}{\ell(\lambda^{s_0} L_{s_0,F_0} h)} \leq \frac{4}{3} e^{\kappa} \rho = \frac{1}{3}.
\]

Finally, as \( \| h \| \leq e^{\kappa} \), we obtain the upper bound :

\[
\| \pi_{s,F_1}(h + \phi) \| = \frac{\| \lambda^s L_{s,F_1}(h + \phi) \|}{\| \lambda^s(h + \phi) \|} \leq \frac{4/3 e^{\kappa} \| h + \phi \|}{2/3 - 1/3} \leq 5e^{2\kappa}.
\]

The lower bound is clear. The bound (6.47) follows from (6.48) and (6.49).

6.4 Analytic measurable sections

Let us now return to the probability space \( (\Omega, \mu) \) and a \( \mu \)-ergodic transformation \( \tau : \Omega \to \Omega \).

We view the space \( \Omega \times A \) as a (trivial) fiber bundle over \( \Omega \) with each fiber being \( A = A(\tilde{K}_\Delta) \). We denote by \( \mathcal{A} \) the set of measurable sections of this fiber bundle and write \( \| \Phi \| \) for the \( \mu \)-essential sup of an element \( \Phi \in \mathcal{A} \). \( A \) is separable, so measurability and Bochner-measurability is here the same. Then \( \mathcal{A} \) is again a unital Banach algebra when we define the analytic operations to be performed fiber-wise. We note that measurability is preserved under such operations and also by taking uniform limits. We write \( \mathcal{A}_{\mathbb{R}} \) for the subspace of real-analytic sections. Let \( \mathcal{C}_{\sigma}(\Omega) \) denote the space of measurable cone-sections of \( \Omega \times \mathcal{C}_{\sigma} \). We write \( \mathcal{C} = \mathcal{C}_{\sigma,t=1}(\Omega) \) for the ‘sliced’ measurable cone-sections. The latter forms a bounded subset of \( \mathcal{A}_{\mathbb{R}} \).

Assumption 6.16 Let \( \mathcal{O} \subset \mathbb{R}^n \) be an open set and let \( \mathcal{O}_C \subset \mathbb{C}^n \) be a complex neighborhood of \( \mathcal{O} \). In the following we will assume that \( t \in \mathcal{O} \to F^t = (f_{t,\omega})_{\omega \in \Omega} \in \mathcal{E}_\Omega(K,U) \) is a map for which the following hold:

1. For each \( \omega \in \Omega \) the map \( t \in \mathcal{O}_C \to f_{t,\omega} \in \mathcal{E}(K,U) \) is analytic in the sense of Definition 6.11. (Note that we are implicitly assuming that for each fixed \( t \in \mathcal{O}_C \), the mapping \( \omega \in \Omega \to f_{t,\omega} \in \mathcal{E}(K,U) \) is measurable as in Definition 4.3).

2. For each \( t \in \mathbb{D} \) and \( \omega \in \Omega \), the map \( f_{t,\omega} \), \( t \in \mathbb{D} \) verifies the L-Lipschitz condition in Definition 6.11 for the same number \( 0 < L < \infty \).

3. The condition numbers \( \Gamma(f_{0,\omega}) \), \( \omega \in \Omega \) are uniformly bounded by some \( \Gamma < +\infty \).

In the following we consider an analytic family, \( t \in \mathbb{D} \mapsto F^t = (f_{t,\omega})_{\omega \in \Omega} \in \mathcal{E}_\Omega(K,U) \) verifying Assumption 6.16 above.

Let \( F_{t,\omega} = (f_{t,\omega}, \overline{f}_{t,\omega}) \) denote the holomorphic extension of \( f_{t,\omega} \) and let \( s_0 = \dim_H(J(f_{0,\cdot})) \in [0,2] \) be the (a.s.) Hausdorff dimension of the random Julia set at \( t = 0 \). We choose \( \sigma = \sigma(s_0) \) and \( \sigma' = \sigma'(s_0) \) so as to verify the Cone contraction conditions in Lemma 6.9. Let \( W^{s_0} \subset \mathbb{C}^2 \) and \( \rho > 0 \) be chosen as in Lemma 6.15 and let \( h \in \mathcal{C}_{\sigma}(\Omega) \). For \( (s,t) \in W^{s_0} \) the following ‘sliced’ cone-map,

\[
\pi_{s,t}(\Phi) \equiv \pi_{s,F_{t,\omega}}(\Phi_{t,\omega}) = \frac{L_{s,F_{t,\omega}} \Phi_{t,\omega}}{\ell(L_{s,F_{t,\omega}} \Phi_{t,\omega})},
\]
is a well-defined map $\pi_{s,t} : B(h, \rho) \rightarrow A$. By Proposition 6.15 the image is bounded in norm by $5 \exp(2\sigma' \text{diam} \hat{K}_{A})$. It takes the value of $\Phi$ at the shifted fiber $\tau_\omega$, acts with the transfer operator, normalises according to $\ell$ and assigns it to the fiber at $\omega$. Measurability of the image is a consequence of the map $(s, F_t) \mapsto L_{s, F_t}$ being continuous and $\ell$ being strictly positive on the image. The reader may note that the (non-normalised) family $(L_{s, F_t, \omega})_{\omega \in \Omega}$ need not be uniformly norm-bounded, whence need not define a bounded linear operator when acting upon sections in $A$. This is the case e.g. in our example in the introduction.

We denote by $\pi_{s_0, 0}^{(n)} : C_{\sigma'}(\Omega) \rightarrow C_{\sigma'}(\Omega)$ the $n$'th iterated map of $\pi_{s_0, 0}$ restricted to the cone-section.

**Lemma 6.17** There are constants, $c_1, c_2 < +\infty$ such that

1. For $h, h' \in C_{\sigma'}(\Omega)$ we have:
   $$|\pi_{s_0, 0}^{(n)}(h) - \pi_{s_0, 0}^{(n)}(h')| \leq c_1 \eta^n.$$

2. Taking the derivative of the $n$'th iterated map at the point $h \in C_{\sigma'}(\Omega)$ we have
   $$\|D_h \pi_{s_0, 0}^{(n)}(h)\| \leq c_2 \eta^n.$$

3. The map, $(s, t) \in W^{s_0}, \Phi \in B(h, \rho) \mapsto \pi_{s,t}(\Phi) \in A$

   is real-analytic.

Proof: (1) and (2) are reformulations of the bounds already given in Lemma 6.10 (with the constants from that lemma). A calculation shows that

$$(L_{s, F_t, \omega} \phi)^* = L_{\bar{s}, \bar{F}_t, \omega}((\phi)^*), \quad \phi \in A,$$

which implies that for $s$ and $t$ real, the operator $L_{s, F_t, \omega}$ maps $A_{\mathbb{R}}$ into $A_{\mathbb{R}}$, i.e., is real-analytic. Each $\pi_{s,t}(\phi)_{\omega}$ is analytic in $s$, $t$ and $\phi$ (for fixed $\omega$). Uniform boundedness was already shown above and a Cauchy formula (choosing $r > 0$ small enough),

$$t \mapsto \left( \int_{|t-t'|=r} \frac{\pi_{s,t}(\phi)_{\omega} \, dt'}{t - t'} \frac{dt'}{2\pi i} \right)_{\omega \in \Omega}$$

enables us to recover a power series in the $t$-variable (similarly for $s$ and $\phi$) within $A$. The map is real-analytic in the sense that it maps $(s, t) \in W^{s_0} \cap \mathbb{R}^2, \Phi \in B(h, \rho) \cap A_{\mathbb{R}}$ into $A_{\mathbb{R}}$.]

First, we consider the real case $(s, t) \in W_{\mathbb{R}}^{s_0} \equiv W^{s_0} \cap \mathbb{R}^2$. Let $h^0 \equiv 1 \in \subset A_{\mathbb{R}}$ be the unit section of our bundle and define recursively the iterates $h^{k+1} = \pi_{s_0}(h^k) \in C_{\sigma', \ell=1}(\Omega), \ k \geq 0$. Lemma 6.10 shows that $|h^{k+n} - h^k| \leq c_1 \eta(s)^k$ which tends exponentially fast to zero. The sequence thus converges uniformly in $A_{\mathbb{R}}$ towards a fixed point

$$h^* = \pi(h^*) \in C_{\sigma', \ell=1}(\Omega).$$

We are interested in the normalisation factor,

$$p_{s, t, \omega} = \ell(L_{s, F_t, \omega} h^*_{\tau_\omega})$$

at the fixed point. This function is real and strictly positive.
Lemma 6.18 We have for $s$ and $t$ real the following formula for the pressure:

$$P(s, \Lambda(F^t)) = \int \log p_{s, t, \omega} d\mu(\omega).$$

Proof: The embedding $j : K \rightarrow \text{diag} \ K \subset \mathcal{U}$ induces a pull-back $j^* : C_0^r \rightarrow C(K)$. On $C(K)$ (before the mirror embedding) we act with the operator, $L_{s, f_{t, \omega}}$, as in (3.21), and on the cone with the mirror extended operator, $L_{s, f_{t, \omega}}$. Then $Lj^* h = j^* L h$ for $h \in C_0^r$ and the cone properties show that $\ell(h) \leq \|j^* h\| \leq \ell(h) e^{\sigma \text{diam} K}$. It then follows that $\mu$-almost surely

$$P(s, \Lambda(F^t)) = \lim_{n \to \infty} \frac{1}{n} \log \|L(s, t, \omega)^{(n)}\| = \lim_{n \to \infty} \frac{1}{n} \log (\ell(s, t, \omega)^{(n)}) = \lim_{n \to \infty} \int \frac{1}{n} \log p_{s, t, \omega} d\mu(\omega).$$

The latter function is comparable to $\log \|L(s, f_{t, \omega})\|$, whence integrable, so by Birkhoff’s Theorem it converges $\mu$-almost surely towards the integral of $\log p$ as we wanted to show. \Box

Remarks 6.19 The pressure does not depend on the choice of $\ell$ (of course, it should not). If one makes another choice $\hat{\ell}$ for the normalisation this simply introduces a co-cycle that vanishes upon integration.

We will use the following version of the implicit function Theorem:

**Theorem 6.20 (Implicit Function Theorem)** Let $\pi : \mathbb{C}^2 \times \mathcal{A} \rightarrow \mathcal{A}$ be a real-analytic map defined on a neighborhood of $(x_0, \phi_0) \in \mathbb{R}^2 \times \mathcal{A}_{\mathbb{R}}$. We let $T_0 = D_\phi \pi(x_0, \phi_0)$ denote the derivative of this map with respect to $\phi$. Suppose that $\phi_0 = \pi(x_0, \phi_0) \in \mathcal{A}_{\mathbb{R}}$ and that the spectral radius of the derivative, $\rho(T_0)$, is strictly smaller than 1. Then there exists a neighborhood $U \subset \mathbb{C}^2$ of $x_0$ and a real-analytic map (unique if $U$ is small enough), $x \in U \mapsto \phi(x) \in \mathcal{A}$, for which $\phi_0 = \phi(x_0)$, $\phi(x) = \pi(x, \phi(x))$ and $\rho(D_\phi \pi(x, \phi(x))) < 1$ for all $x \in U$.

Proof: The map,

$$\Gamma(x, \phi) = (1 - T_0)^{-1}(\pi(x, \phi) - \phi_0 - T_0(\phi - \phi_0)) + \phi_0,$$

is real-analytic and verifies $\Gamma(x_0, \phi_0) = \phi_0$ and $D_\phi \Gamma(x_0, \phi_0) = 0$. We may therefore find a neighborhood $U$ of $x_0$ and a closed neighborhood $W$ of $\phi_0$ such that $\Gamma$ is a uniform contraction on the real-analytic sections, $U \rightarrow W$. The fixed point $\phi(x) = \Gamma(x, \phi(x))$, $x \in U$ is then itself a real-analytic section and has the desired properties. \Box

Lemma 6.21 The pressure function $P(s, \Lambda(F^t))$, extends to a real-analytic function $P(s, t)$, on an open neighborhood $U^{s_0} \subset \mathbb{C}^2$ of $(s_0, 0)$.

Proof: By the above implicit function Theorem there is a real-analytic map

$$(s, t) \in U^{s_0} \mapsto h^*_{s, t} \in B(h^*_{s_0, 0}, \rho) \subset \mathcal{A}.$$

defined in a neighborhood $U^{s_0} \subset W^{s_0}$ of $(s_0, 0)$. On this neighborhood we define as before, $p_{s, t, \omega} = \ell(L_{s, f_{t, \omega}} h^*_{s, t}) \in \mathbb{C}$. For fixed $\omega$ this function is clearly analytic in $(s, t) \in U^{s_0}$. Lemma 6.15 applied to our fixed point shows that when $(s, t) \in W^{s_0}$ and $\lambda_\omega = \|Df_{0, \omega}\|$ then

$$\left| \frac{\lambda_\omega p_{s, t, \omega}}{\lambda_\omega^{s_0} p_{s_0, 0, \omega}} - 1 \right| \leq \frac{2}{3}.$$
This in turn implies, $|\log(\lambda^s p_{s,t,\omega}) - \log(\lambda^s p_{s_0,0,\omega})| \leq \log 3$. Then also,

$$|\log p_{s,t,\omega}| \leq (|s| + s_0) \log \lambda + \log p_{s_0,0,\omega} + \log 3.$$ 

The right hand side is $\mu$-integrable (its integral is bounded by $(|s|+s_0)\mathbb{E}(\log \|Df_{0,\omega}\|)+P(s_0, dF^0) + \log 3$) and therefore,

$$P(s,t) = \int p_{s,t,\omega} d\mu(\omega), \quad (s,t) \in U^{s_0}$$

is well-defined and yields a real-analytic extension of the pressure. \[\]

**Theorem 6.22** Let $\tau$ be an ergodic transformation on $(\Omega, \mu)$. Let $F^t = (f_{t,\omega})_{\omega \in \Omega} \in \mathcal{E}_\Omega(K, U)$ be an analytic family verifying a uniform $L$-Lipschitz condition and with uniform bounded condition numbers, i.e. Assumption 6.16 above. Then, almost surely, the Hausdorff dimension of the random Julia set, $J(F^t_{\omega})$, (4.24) is independent of $\omega$ and depends real-analytically on $t$.

Proof: Let $t \in \mathbb{D} \cap \mathbb{R}$. We already know from Theorem 5.3 that a.s., $d(t) = \dim_H \Lambda(F^t_{\omega})$ is independent of $\omega$ and that $P(d(t), t) = 0$ whenever $(d(t), t) \in U^{s_0}, t \in \mathbb{R}$. By the previous Lemma, $P$ has a real-analytic extension and since $\frac{\partial P}{\partial d}(d(t), t) \leq \beta < 0$ for real $t$-values, we may apply another implicit function theorem to $P$ and conclude that there is an open neighborhood $V_0 \subset \mathcal{C}$ of 0 and a real-analytic function $t \in V_0 \mapsto (\hat{d}(t), t) \in U^{s_0}$ such that $P(\hat{d}(t), t) = 0$ for all $t \in V_0$. The function $\hat{d}(t)$ yields the desired real-analytic extension of the dimension. \[\]

### 6.5 Parameter dependency of the measure

Consider $\mathcal{M} \equiv \mathcal{M}(\Omega)$, the Banach space of complex measures on $\Omega$ in the variation norm. The set of probability measures, $\mathcal{P} \equiv \{\mu \in \mathcal{M} : \mu \geq 0, \mu(\Omega) = 1\}$, forms a real affine subspace of $\mathcal{M}$. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open subset.

**Definition 6.23** We say that a family of probability measures, $p_{\lambda}, \lambda \in \mathcal{O}$ is real-analytic if there is a complex neighborhood $\mathcal{O}_C \subset \mathbb{C}^n$ of $\mathcal{O}$, such that

$$\lambda \in \mathcal{O}_C \mapsto p_{\lambda} \in \mathcal{M}$$

is analytic.

**Example 6.24** The Poission law, $p_{\lambda}(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k \in \mathbb{N}$ is real-analytic in $\lambda \geq 0$. It has a complex extension to every $\lambda \in \mathbb{C}$ with a variation norm

$$\|p_{\lambda}\| = e^{\lambda} - e^{\lambda}$$

### 7 Proof of Theorem 1.1

Let $(\Omega, \mu) = (\prod_{\mathbb{N}} \mathcal{Y}, \otimes_{\mathbb{N}} \nu)$ denote the (extension of the) direct product of probability spaces and let $\tau$ be the shift on this space, i.e. $\tau(\omega) = (\omega_2, \omega_3, \ldots)$ for $\omega = (\omega_1, \omega_2, \ldots)$. With $f_{t,\omega}$ as in our Main Theorem we define $f_{t,\omega} = f_{t,\omega}$ as the random sequence of conformal maps. We suppose that each individual measure $\nu_{\lambda}$ depends analytically on a complex parameter $\lambda \in \mathbb{D}$ (setting $\lambda = t$ we obtain the statement in the Theorem). The family $F^t = (f_{t,\omega})_{\omega \in \Omega} \in \mathcal{E}_\Omega(K, U)$ verifies the conditions for Theorem 6.22 and applying this for a (real) probability measure, $\mu_{\lambda}$, $\lambda$ real,
yields part I and II of our Main Theorem, except for the real-analyticity with respect to the measure. Let $\mathcal{P}(s,t,\lambda)$ (for $\lambda$ real) denote the pressure obtained in that Theorem.

Going back to the Implicit Function Theorem, Theorem 6.20, we may find a neighborhood $U^{s_0} \subset \mathbb{C}^2$ of $(s_0,0)$ such that $D_{h\pi_{s,t}}(h_{s,t}^*)$ has spectral radius strictly smaller than one for $(s,t) \in U^{s_0}$. Possibly shrinking the neighborhood we may also find constants $C = C(s,t) < +\infty$, $\eta = \eta(s,t) < 1$ and $0 < \rho_1 \leq \rho$ such that the map $\mathfrak{p}_{s,t}^{(n)} : B(h^*, \rho_1) \to B(h^*, \rho)$ is well-defined for all $n \geq 1$ and is a $C\eta^n$-Lipschitz contraction.

For $\eta < 1$ we define, $D_\eta = \{ \lambda \in \mathbb{C}^n : \|p_\lambda\| < 1/\eta \}$, and then $D_0 = \{(s,t,\lambda) : (s,t) \in U^{s_0}, \lambda \in D_{\eta(s,t)}\}.$

Given $(s,t) \in U^{s_0},$ set $h^{(0)} = 1 \equiv \pi^{(0)} 1 \in B(h^*, \rho_1)$ and then recursively, $h^{(n)} = \pi^{(n)} 1 - \pi^{(n-1)} 1,$ $n \geq 1.$ These differences have norm smaller than $2C\eta^n.$ Also $h^{(n)}_\omega = h^{(n)}_{\omega_1,..\omega_n}$ depends only on the first $n$ iterates of $\omega$. Integrating with respect to the analytic continuation of our probability measure we see that

$$\sum_k |\ell(L_\omega h^{(k)}_{\omega_1,..\omega_k})d\mu_\lambda(\omega_1) \ldots d\mu_\lambda(\omega_k)| \leq \text{const} \sum_k (\|p_\lambda\|\eta)^k$$ \hspace{1cm} (7.50)

which is finite when $\lambda \in D_\eta.$ For $\lambda$ real,

$$\mathcal{P}(s,t,\lambda) \equiv \int \ell(L_\omega h_{\tau_\omega})d\mu_\lambda(\omega) = \sum_k \ell(L_\omega h^{(k)}_{\omega_1,..\omega_k})d\mu_\lambda(\omega_1) \ldots d\mu_\lambda(\omega_k)$$

and (7.50) shows that the right hand side extends real-analytically on the domain $(s,t,\lambda) \in D_0.$ Using transversality of this extended pressure function and once again an Implicit Function Theorem we obtain Theorem 1.1, part II, including the real-analyticity with respect to the measure. \boxit{\textbf{Remarks 7.1}} An alternative generalisation would be to pick the maps $f_{k,\omega}$ according to a Gibbs measure on a shift space over a finite alphabet. The Hausdorff dimension in this case depends real-analytically (and for the same reasons) upon the Hölder potential defining the Gibbs state. This result does not, however, cover our main example in the introduction.

\section{Proof of Example 1.2}

We define for $0 \leq \rho < 1$, the complex annulus $A_\rho = \{z \in \mathbb{C} : \rho < |z| < 1/\rho\}$ ($= \mathbb{C}^*$ for $\rho = 0$). The conditions on parameters may be written as

$$|a| + r \leq \frac{k^2}{4},$$

where $k$ is a constant $0 < k < 1$. We set $U = A_{k^2/2}$ and $K = \overline{A_{k^2/2}}$ which is a compact subset of $U$.

The maps under consideration, $f = z^{N+2} + c$, then belongs to $\mathcal{E}(K,U)$. The neighborhood $K_\Delta$ may be written as $\overline{A_\kappa}$ for some $\kappa \in [k^2/2, k/2].$ Conformal derivatives and usual derivatives are (smoothly) comparable on $f^{-1}K_\Delta$ so we are allowed to replace conformal derivatives by the standard Euclidean ones in the following. For $w = f(z) \in K_\Delta$ we have,

$$f'(z) = (N+2)z^{N+1} = (N+2)\frac{w-c}{z},$$

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which is comparable to $N$ (because both $w$ and $z$ belongs to $K$). Whence, the \textit{b.a.l.d.} condition, $\mathbb{E}(\log \|Df\|) < +\infty$, is equivalent to $\mathbb{E}(N) < +\infty$ which is clearly verified for a Poisson distribution of $N$. Also, the condition numbers $\|Df\|/\|1/Df\|$ are uniformly bounded (this is in fact true for all maps $f \in \mathcal{E}(K, U)$ for which $f^{-1}U$ is connected). If we write $f(z) = z^{N+2} + a + r\xi$, where $\xi$ is a random variable uniformly distributed in $\mathbb{D}$ then we obtain an explicit (real-) analytic parametrization of $f$ in terms of $a$ and $r$.

To see that a local inverse depends uniformly Lipschitz in parameters consider e.g.:

$$\frac{\partial f^{-1}}{\partial a} = -\frac{\partial f}{\partial a} \frac{z}{z - c},$$

which is uniformly bounded on $K$. Similarly,

$$\frac{\partial}{\partial a} \log f' \circ f^{-1} = \frac{N + 1}{z} \left( -\frac{w - c}{(N + 2)z} \right) = \frac{N + 1}{N + 2} \frac{c - w}{z^2},$$

which is again uniformly bounded, independent of the value of $N$ (but only just so!). We are in the position to apply our Main Theorem and proving the claims in Example 1.2.

\section{Removing the mixing condition}

Our mixing condition (C4) was convenient but not strictly necessary. For completeness we will show how to get rid of this condition. Our first reduction is to replace (C4) by topological transitivity. This amounts to saying that there is $n_0 = n_0(\delta)$ such that

(C4') $\bigcup_{k=0}^{n_0} f^k(B(x, \delta) \cap \Lambda) = \Lambda.$

Repeating the previous steps we see that (2.13) is replaced by the inequality

$$\max_{0 \leq k \leq n_0} m_{n+k} \geq (\lambda_1^{n_0} c_n)^{-s} M_n/2$$

from which the operator distortion bounds follow. The proof of the lower bounds for the Hausdorff dimension does not change and in the upper bounds for the Box dimension the left hand side of the inequality (2.17) is replaced by $\sum_{0 \leq j \leq n_0} L_{\gamma_1}^{j+n_0} \chi_{B(x, r)}$ which leads to the bound

$$\sum_{i=1}^{m} (\text{diam } B(x_i, 2r))^s \leq 4^s \gamma_2(s) \sum_{0 \leq j \leq n_0} \|L_j^s\|.$$
for which \( d(x_i, x_{i+1}) < \delta \) for all \( 0 \leq i \leq n \). This partitions \( \Lambda \) into \( \delta \)-connected components \( \Lambda = \Lambda_1 \cup \ldots \cup \Lambda_m \). Each \( \Lambda_i \) is \( \delta \)-separated from its complement, whence open and compact within \( \Lambda \). Thus there is a uniform bound on the number \( N_\delta \) of intermediate points needed to connect any \( x \) and \( y \) within the same component.

The partition is not Markovian. For example, for a connected hyperbolic Julia set there is only one \( \delta \)-connected component. It does, however, enjoy some Markov-like properties: If \( f \Lambda_i \cap \Lambda_j \neq \emptyset \) then \( f \Lambda_i \supset \Lambda_j \). To see this note that if \( x \in \Lambda_i \), \( y = f(x_i) \in \Lambda_j \) and \( v \in B(y, \delta) \subset \Lambda_j \) then there is (a unique) \( u \in B(x, \delta) \subset \Lambda_i \) for which \( f(u) = v \) and thus \( v \in \Lambda_i \). We may introduce a transition matrix, \( t_{ji} = 1 \) when \( f \Lambda_i \supset \Lambda_j \) and zero otherwise. A partial ordering among the partition elements \( \Lambda_i \) is then given by

\[
\Lambda_i \prec \Lambda_j \text{ iff } \exists n = n(i, j) : t^n_{ji} \geq 1
\]

and an equivalence relation

\[
\Lambda_i \sim \Lambda_j \text{ iff } \Lambda_i \prec \Lambda_j \text{ and } \Lambda_j \prec \Lambda_i.
\]

The equivalence classes provides a new partition of \( \Lambda \):

\[
\Lambda = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_k
\]

which inherits the partial ordering from before. Each equivalence class is topologically transitive and the local pressures are constant on each class. Writing \( P_i \) for the pressure on class \( i \) we have \( P_i \leq P_j \) for \( i \prec j \).

Consider now the critical \( s \)-value \( s_{\text{crit}} \) and let \( \mathcal{C}_{i_0} \) be a class which is minimal for the inherited partial ordering and such that the local pressure vanishes for every point in this class \( \mathcal{P}_x(s_{\text{crit}}, \Lambda) = 0 \), \( x \in \mathcal{C}_{i_0} \). We denote by

\[
\Lambda' = \cap_{j \geq 0} f^{-j} \mathcal{C}_{i_0}
\]

the corresponding \( f \) invariant subset of the class. This subset is topologically transitive (clear) and we claim that this set has Hausdorff and box dimensions that agree and equal \( s_{\text{crit}} \). For this it suffices to show that the pressure of that subset \( P(s_{\text{crit}}, \Lambda', f) \) vanishes.

Write for \( 1 \leq i \leq k \)

\[
N_i \phi = \chi_{\mathcal{C}_i} L_s \phi = L_s (\chi_{\mathcal{C}_i} \circ f \phi).
\]

If \( \mathcal{C}_i \prec \mathcal{C}_j \) and they are not equal then \( N_i N_j \equiv 0 \). Similarly, if \( \mathcal{C}_i \) and \( \mathcal{C}_j \) are not related the \( N_i N_j = N_j N_i \equiv 0 \). In either case we have \( (N_i + N_j)^n = N_i^n + N_j^{n-1} + \ldots N_j N_i + N_j^n \) which implies that the spectral radius of \( N_i + N_j \) is the same as the spectral radius of \( N_j \). Writing \( L_s = \sum_i N_i \) it follows that the spectral radius of \( L_{s_{\text{crit}}} \) must be the same as that of \( N_{i_0} \). But this implies precisely that \( P(s_{\text{crit}}, \Lambda', f) = 0 \).

Remarks A.1 We note that in this setting, even when distortions remain uniformly bounded the Hausdorff measure need not be finite (essentially because the powers of a matrix of spectral radius one need not be bounded when the eigenvalue one is not simple).

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