Quantum simulation of Dirac fermion mode, Majorana fermion mode and Majorana-Weyl fermion mode in cavity QED lattice

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Abstract – Quantum simulation aims to simulate a quantum system using a controllable laboratory system that underlines the same mathematical model. The cavity QED lattice system is the prescribed system to simulate the relativistic quantum effect. We quantum simulate the Dirac fermion mode, the Majorana fermion mode and also the Majorana-Weyl fermion mode and a crossover between them in cavity QED lattice. We present the different analytical relations between the field operators for different mode excitations. We also present an analysis of correlation function and the effect of dissipation at the zero Majorana fermion mode of cavity QED lattice.

Introduction. – The difficulties in observing the real quantum relativistic effects have generated immense interest in quantum simulation physics. In recent years, there has been increasing interest in the simulation of relativistic quantum effects using different physical systems in which parameters tunability allows access to different physical regimes [1–11]. These difficulties of observing quantum relativistic effects stimulate us to study the quantum simulation physics of the Dirac fermion mode, the Majorana fermion mode and the Majorana-Weyl fermion mode in one-dimensional cavity QED lattice and a crossover from one mode of excitations to another using the tunability of the physical parameters of the system.

In quantum simulation, one aim is to simulate a quantum system using a controllable laboratory system that underlines the same mathematical models. Therefore it is possible to simulate a quantum system that can be neither efficiently simulated on a classical computer nor easily accessed experimentally.

The recent experimental success in engineering a strong interaction between the photons and atoms in high-quality micro-cavities opens up the possibility to use the light matter system as quantum simulators for many-body physics [9–29]. Many interesting results are coming out to understand the complicated quantum many-body system. A focus on the coupled cavities is one of the most potential candidates for an efficient quantum simulator due to the control of the micro-cavities parameters and to the success of fabrication of large-scale cavity arrays [25,26].
and Majorana fermion modes in the existing literature of cavity QED system [9–11].

The authors of ref. [3] have found the process of Dirac to Majorana fermion converter on the surface of a 3D topological insulator. A Dirac fermion injected by the voltage source is split into a pair of Majorana fermions and then finally recombined before going to drain. But in our present study there is no such split and fusion process of Dirac and Majorana fermions. In our study the system is in the Dirac fermion mode when only an atom-photon coupling or a single Rabi frequency oscillation is present in the system and the existence of Majorana fermions occurs when two atom-photon couplings and the two laser frequencies are simultaneously present in the system. For a quantum simulated Hamiltonian for the Majorana fermion mode for a specific mathematical relation between the Rabi frequencies oscillation, the atom-photon coupling strengths and the laser field detuning.

To the best of our knowledge the quantum simulation physics for the different kinds of fermionic mode in a same cavity QED system is absent in the literature.

In the present study, we simulate two model Hamiltonians through the proper tuning of Rabi frequencies and the atom-photon coupling strengths in the system which quantum simulate different modes of fermionic excitations.

The model Hamiltonian. The Hamiltonian of our present study consists of three parts:

\[ H = H_A + H_C + H_{AC}. \] (1)

The Hamiltonians are the following:

\[ H_A = \sum_{j=1}^{N} \omega_c |e_j \rangle \langle e_j | + \omega_b |b_j \rangle \langle b_j |, \]

where \( j \) is the cavity index. \( \omega_c \) and \( \omega_b \) are the energies of the state, \(|b\rangle\) and the excited state, respectively. The energy level of state \(|a\rangle\) is set as zero. \(|a\rangle\) and \(|b\rangle\) are the two stable states of an atom in the cavity and \(|c\rangle\) is the excited state of that atom in the same cavity. The following Hamiltonian describes the photons in the cavity:

\[ H_C = \omega_C \sum_{j=1}^{N} a_j^\dagger a_j + J_C \sum_{j=1}^{N} (a_j^\dagger a_{j+1} + \text{h.c.}), \]

where \( a_j^\dagger (a_j) \) is the photon creation (annihilation) operator for the photon field in the \( j \)-th cavity, \( \omega_c \) is the energy of photons and \( J_C \) is the tunneling rate of photons between neighboring cavities. The interaction between the atoms and photons and also by the driving lasers are described by

\[ H_{AC} = \sum_{j=1}^{N} \left( \frac{\Omega_b}{2} e^{-i\omega_b t} + g_a a_j \right) |e_j \rangle \langle a_j | + \text{h.c.} \] \[ + [a \leftrightarrow b]. \]

Here \( g_a \) and \( g_b \) are the couplings of the cavity mode for the transition from the energy states \(|a\rangle\) and \(|b\rangle\) to the excited state. \( \Omega_a \) and \( \Omega_b \) are the Rabi frequencies of the lasers with frequencies \( \omega_a \) and \( \omega_b \), respectively.

The authors of ref. [14–16] have derived an effective spin model by considering the following physical processes: A virtual process regarding the emission and absorption of photons between the two stable states of neighboring cavity yields the resulting effective Hamiltonian as

\[ H_{xy} = \sum_{j=1}^{N} B \sigma_j^z + \sum_{j=1}^{N} \left( \frac{J_1}{2} \sigma_j^+ \sigma_{j+1}^- + \frac{J_2}{2} \sigma_j^- \sigma_{j+1}^+ + \text{h.c.} \right). \] (2)

When \( J_2 \) is real then this Hamiltonian reduces to the XY model, where \( \sigma_j^z = |b_j\rangle \langle b_j | - |a_j\rangle \langle a_j |, \sigma_j^+ = |b_j\rangle \langle a_j |, \sigma_j^- = |a_j\rangle \langle b_j |; \)

\[ H_{xy} = \sum_{i=1}^{N} B (\sigma_i^x + J_2 \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y) \] (3)

with \( J_x = (J_1 + J_2) \) and \( J_y = (J_1 - J_2) \).

We follow refs. [14, 36] to present the analytical expression for the different physical parameters of the system. \( B = \frac{\delta}{2} - \beta, \beta \) is defined in footnote 1.

\[ J_1 = \frac{\delta}{2} \omega_{ab}^2 + \omega_{ab}^2 \gamma_{ab}, J_2 = \frac{\delta}{2} \omega_{ab} \omega_b / \omega_c \gamma_{ab}, \]

where \( \gamma_{ab} = \frac{1}{N} \sum_k (a_k \omega_{ab} / \omega_c \omega_b) \), and \( \gamma_2 = \frac{1}{N} \sum_k (a_k \omega_{ab} / \omega_c \omega_b) \). \( \Delta_c = \omega_c - \omega_b, \Delta_a = \omega_c - \omega_b, \Delta_b = \omega_c - \omega_b, \Omega_a = \omega_a - \omega_b, \Omega_b = \omega_a - \omega_b, \Omega_{ab} = \omega_{ab} - \omega_c, \Omega_{cb} = \omega_{ab} - \omega_b \).

Quantum simulation for Dirac fermion mode and mathematical relation between the fields. Here we quantum simulate the Dirac fermion physics through the proper tuning of cavity QED lattice parameters. This condition can be achieved when \( J_x = J_y \). This condition implies that \( 2J_2 = 0 \). It is clear from the analytical expression of \( J_1 \) and \( J_2 \) that, to satisfy the condition, one of the atom-photon coupling strengths, \( \omega_a \) or \( \omega_b \) should be zero or one of the Rabi frequency oscillation \( \Omega_a \) or \( \Omega_b \) should be zero. The analytical expression for \( J_1 \) becomes \( J_1 = \frac{\delta}{2} |\Omega_b|^2 / \Delta_b \) or \( J_1 = \frac{\delta}{2} |\Omega_a|^2 / \Delta_a \). The other condition is \( \delta_1 = 2 \beta, \delta_2 = 2 \omega_{ab} - (\omega_a - \omega_b) \). Therefore the condition for quantum simulation relates up to the microscopic level.

In this limit the Hamiltonian is reduced to

\[ H = J \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y). \] (4)

\[ \beta = \frac{1}{2} \left( \frac{44 \Omega_a^2}{4 \Delta_a} - \frac{4 \Omega_b^2}{4 \Delta_b} \right) - \frac{|\Omega_b|^2}{4 (\Delta_a - \Delta_b)} - \gamma_b s_b^2 - \gamma_1 s_a^2 + \gamma_2 s_a^4 \left( a \leftrightarrow b \right). \] (15)
After the Jordan-Wigner transformation and the Abelian bosonization study one can write the above Hamiltonian as [37,38]

\[ H_0 = \sum_{k} \frac{1}{2\pi}dk \langle k|V_F|\psi_{s,R}^\dagger(k)\psi_{s,R}(k) - \psi_{s,L}^\dagger(k)\psi_{s,L}(k) \rangle. \]  

Here we use \( e_k = kV_F \) near the Fermi points and \( \psi_{s,R}^\dagger(x) = (\psi_{s,L}^\dagger(x), \psi_{s,R}^\dagger(x)) \). We can write the above Hamiltonian in the following form of Dirac equation without any mass term:

\[ H = 2J \int dx \bar{\psi}(i\gamma_1)\partial_x \psi \]  

and \( \psi(x) = (\psi_R^\dagger(x), \psi_L^\dagger(x), \psi_R(x) = \int \frac{dk}{2\pi} \psi(k)e^{ikx} \). It is customary to introduce the Dirac matrices in the Dirac equation. The \( \gamma \) matrices are the following:

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\gamma^1 &= -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\gamma^2 &= \gamma^\mu \gamma^\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\psi_R \text{ and } \psi_L \text{ are the fermionic field for the right and left movers electron. The scalar field and its dual can be expressed as } \phi(x) = \phi_R(x) + \phi_L(x), \\
\theta(x) = -\phi_R(x) + \phi_L(x).
\end{align*}
\]

**Dirac equation for Majorana fermion mode and condition of Majorana-Weyl fermionic condition.**

Here we quantum simulate the Majorana fermion mode and the Majorana-Weyl fermion mode. Majorana thought whether it might be possible for a spin-(1/2) particle to be its own antiparticle. To get an equation alike to the Dirac equation but capable of governing a real field it is required that the \( \gamma \) matrices of that equation satisfy the Clifford algebra are purely imaginary. Here we show explicitly that the \( \gamma \) matrices of the Majorana equation are purely imaginary and the field satisfies the particle-antiparticle equivalent condition.

We also derive the condition for the excitation of the Majorana-Weyl fermion mode where the simulated mode shows the gapless excitation.

We consider \( J_1 = J_2, J_3 \) becomes \( J_1 + J_2 \) and \( J_3 = 0 \). In the micro-cavity array, the condition for \( J_1 = J_2 \) achieves when

\[ \Omega_a^2 g_a^2 \Delta_b^2 + \Omega_b^2 g_b^2 \Delta_a^2 = 2\Omega_a \Omega_b g_a g_b \Delta_a \Delta_b. \]  

The above condition implies that \( \Omega_a = \Omega_b \frac{\Delta_a}{\Delta_b} \). The only constraint is that \( \Delta_a \neq \Delta_b \), the magnetic field diverges when \( \Delta_a = \Delta_b \). At the same time, \( \Omega_a = \Omega_b \) and \( g_a = g_b \) are also not possible because this limit also leads to the condition \( \Delta_a = \Delta_b \). Suppose we consider, \( \Omega_a = \alpha_1 \Omega_b, g_a = \alpha_2 g_b, \text{ and } \Delta_a = \alpha_3 \Delta_b. \) These relations imply that \( \alpha_1^2 + \alpha_2^2 \alpha_3^2 = 2 \alpha_1 \alpha_2 \alpha_3, \alpha_1 = \alpha_2, \alpha_1 = \alpha_2, \alpha_3 \) are the numbers. These analytical relations help to implement the transverse Ising model Hamiltonian but \( \alpha_1, \alpha_2, \text{ and } \alpha_3 \) should not be equal to 1.

The quantum state engineering of cavity QED is in the state of art due to the rapid progress of technological development of this field [1]. Therefore one can achieve this limit to get the desired quantum state. One can write the final Hamiltonian as

\[ H_T = B \sum_{j=1}^{N} (\sigma_z(j) + \lambda \sigma_x(j)\sigma_z(j+1)). \]  

where \( \lambda = \frac{\mu J_1 J_3}{2 B} \). The effective Hamiltonian becomes the transverse Ising model which has been studied in the previous literature [39–41]. After the Jordan-Wigner transformation, one can recast the Hamiltonian in spinless fermionic chain,

\[ H_T = - \sum_{j} [(J_1 + J_2)(\psi_{j+1}^\dagger \psi_{j} + h.c.) - (J_1 + J_2)(\psi_{j+1}^\dagger \psi_{j} + h.c.)] - 2B \psi_j^\dagger \psi_j \]  

(if we compare this Hamiltonian with the original Kitaev’s chain model Hamiltonian [32] then \( t = J_1 + J_2, \Delta = J_1 + J_2, \mu = -2B \)).

One can also write the starting Hamiltonian as through the rotation in spin basis,

\[ H = \sum_n [\lambda \sigma_z(n) \sigma_z(n+1) + s_x(n)]. \]  

It will help us to use the order and disorder operator directly to derive the Dirac equation for the Majorana fermion.

Here our main motivation is to use some of the important results of this model Hamiltonian to discuss the relevant physics of the array of the cavity QED system. We introduce the order and disorder operator in the appendix [40,41]. These operators are defining the sites of the dual lattice, i.e., well define the operator between the nearest-neighbor site of the original lattice.

Here we define the Dirac spinor,

\[ \chi_1(n) = \sigma_z(n) \mu_z(n + 1/2) \]  

and \( \chi_2(n) = \sigma_z(n) \mu_z(n - 1/2) \). These two fields, \( \chi_1(n) \) and \( \chi_2(n) \) satisfy the following relations:

\[ \{ \chi_1(n), \chi_2(n) \} = 2\delta_{n_1,n_2}. \]

One can write down the final equation in the following form:

\[ \left( \gamma^0 \frac{\partial}{\partial t} + \gamma^3 \frac{\partial}{\partial x} + m \right) \chi = 0. \]

The detailed derivation is relegated to the appendix.

One can also write the above Majorana equation in a compact form:

\[ (i\gamma^0 \partial_{\mu} - m)\chi = 0, \]

where \( \gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \)

Therefore we prove that the spinor field satisfies the Majorana condition of spin-(1/2) particle and also the \( \gamma \) matrices are imaginary.

The condition for the massless Majorana fermion field is that \( m = 0 \), i.e., \( \lambda = 1 \). In this quantum simulation process, one can get through this massless excitation through
It is very clear from the above analytical relations that the ferromagnetic contribution decays much faster than the antiferromagnetic-type correlation function. It represents the antiferromagnetic-type correlation function. It is a ferromagnetic-type correlation and the second term represents the antiferromagnetic behavior in the asymptotic limit. The corresponding physics in the spirit of interacting light and matter, cavity QED lattice is that the excitation due to the light matter excitations is present in the alternate cavity of the lattice. This is the basic aspect of the correlation function study for the Dirac fermion mode.

We term this fermionic mode as Majorana-Weyl fermionic mode. In this study we obtain the different kinds of fermionic modes from the quantum simulation of the system, here there is no splitting of Dirac fermion mode into the Majorana fermion mode or the Majorana-Weyl fermionic mode. There are no studies in the previous literature of cavity QED where one has found a crossover from Dirac fermion modes to Majorana fermion modes [9–11].

Here we present the analytical relations between the Majorana fermion operators with the order and disorder operator with the free Dirac field in the Abelian bosonization theory. Then one obtains the following sets of commutation relations:

\[
\begin{align*}
\sigma_z(x_1)\mu_z(x_2) &= \mu_z(x_2)\sigma_z(x_1)\text{sign}(x_1 - x_2), \\
\sigma_z(x_1)\chi(x_2) &= \chi(x_2)\sigma_z(x_1)\text{sign}(x_1 - x_2), \\
\mu_z(x_1)\chi(x_2) &= -\chi(x_2)\sigma_z(x_1)\text{sign}(x_1 - x_2).
\end{align*}
\]

It is very clear from the above analytical relations that \(\chi_1^\dagger = \chi_1\) and \(\chi_2^\dagger = \chi_2\). The above relation has similarity with the free Dirac field in the Abelian bosonization theory, where the Dirac field operator is a local product of two phase exponential depending on the scalar field and its dual [37,38,42], as one studies the Luttinger liquid physics in the Abelian bosonization theory.

**Correlation function study.** Here we discuss the correlation function of Dirac fermion mode, Majorana fermion mode and Majorana-Weyl fermion mode for the cavity QED system. We express our model Hamiltonians in terms of spin Hamiltonians. At first, we express our correlation function in terms of quantum spin system and finally we interpret the results from the perspective of cavity QED lattice.

The transverse component of spin-spin correlation function is:

\[
\langle \sigma_1^+(x,0)\sigma_2^-(0,0) \rangle = A \left( \frac{1}{x} \right)^{2K+1/(2K)} + B(-1)^x \left( \frac{1}{x} \right)^{1/(2K)},
\]

where \(A\) and \(B\) are the non-universal constants, i.e., these constants are system dependent. This analytical expression consists of two parts, one is for \(q = 0\) and the other for \(q = 2K\). The first term is for \(q = 0\), which represents a ferromagnetic-type correlation and the second term represents the antiferromagnetic-type correlation function. It is very clear from the above analytical expression that the ferromagnetic contribution decays much faster than the antiferromagnetic correlation. Therefore, the system shows the antiferromagnetic behavior in the asymptotic limit. The corresponding physics in the spirit of interacting light and matter, cavity QED lattice is that the excitation due to the light matter excitations is present in the alternate cavity of the lattice. This is the basic aspect of the correlation function study for the Dirac fermion mode.

Now we discuss the behavior of correlation function for the Majorana and Majorana-Weyl fermion modes based on eq. (10). We follow ref. [39], during the description of the physics of the system. It is clear from eq. (8), that \(\lambda\) is the ratio of coupling constant \((J_1 + J_2)\) and magnetic field \((B)\), i.e., the laser field detuning. When the laser field detuning is extremely large, the system is in the paramagnetic phase, i.e., in the cavity QED lattice it has no Majorana fermion excitation mode. At this point, where the spin of different sites is totally uncorrelated, one can write the correlation function as \(\langle 0|\sigma_z(i)\sigma_z(j)|0 \rangle = \delta_{ij}\).

When the laser field detuning is large but finite then we write the correlation as

\[
\langle 0|\sigma_z(i)\sigma_z(j)|0 \rangle \simeq e^{-|x_i-x_j|/\xi}.
\]

At this point, there is a fluctuation of spin flips which communicate between the sites. The spin flip is massive therefore it propagates over a short range.

On the other hand when the laser field detuning is small, \(B \ll 1\), at this point the system is in magnetically ordered state. One can write the correlation relation as

\[
\langle 0|\sigma_z(i)\sigma_z(j)|0 \rangle \simeq N_0^2,
\]

\(N_0^2 = 1\) for \(B = 0\) and \(N_0 < 1\) for \(B > (J_1 + J_2)\).

In the Majorana-Weyl fermion mode the basic behavior of the correlation is the same as that of the Majorana mode except one. In the Majorana-Weyl fermion mode, the mass is zero, therefore for large but finite \(B\), spin-flipping, i.e. the Majorana-Weyl fermionic mode propagates across the lattice. The experimental status of the cavity QED system is in the state of art, and we hope that experimentalists will measure the results of correlation function through photon emission spectra (photon correlation detection experiment).

**Effect of dissipation in Majorana zero modes in this system.** In the cavity QED system, the coupling to the environment typically leads to the spontaneous relaxation and dephasing to the atom, at the same time to photon losses from the cavity. Now we are interested in discussing the physics of the Majorana fermion mode under this dissipation, in the Majorana fermion physics the parity breaking process is the most important one. The authors of ref. [9] have already described the parity breaking dissipation channel as a single effective parity breaking channel. We have already expressed our model Hamiltonian as an optical Kitaev chain (eq. (9)), the open nature of this chain can be expressed by introducing an effective single-particle loss term in the dynamics [9].
We try to understand the single-particle losses using the Lindblad equation,
\[ \partial_t \rho = -i[H, \rho] + \Gamma \sum_{i=1}^{N} \left( b_i \rho b_i^\dagger - \frac{1}{2} \{ b_i^\dagger b_i, \rho \} \right), \]
where \( \rho \) is the density matrix of the system, \( H \) is the Hamiltonian (eq. (9)) and \( \Gamma \) is the effective single-particle decay rate associated with single cavity. \( b_i^\dagger \) \( (b_i) \) creation (annihilation) operator of dressed photon for each cavity. The authors of ref. [43] have already described the single-particle losses as the photon losses in the cavity QED lattice.

We present the effect of photon losses for the topological phase of the optical Kitaev chain where the Majorana fermion zero mode appears at the end of the chain. Here the limit of the topological phase is \( t = \Delta = (J_1 + J_2) \) and \( \delta_1 = 2\beta \) (i.e., \( B = 0 \)).

In this limit, the Hamiltonian reduces to
\[ H = \hbar t \sum_{i=1}^{N-1} c_{2i} c_{2i+1} = -\hbar t \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x. \]

We use the following relation between the fermion, Majorana fermion, and spin operators:
\[ c_{2i-1} = \psi_i + \psi_i^\dagger = \Pi_{j=1}^{i-1} (-\sigma_j^z \sigma_{j+1}^x), \]
\[ c_{2i} = -i(\psi_i - \psi_i^\dagger) = -\Pi_{j=1}^{i-1} (-\sigma_j^z \sigma_{j+1}^y). \]

Majorana zero modes corresponding to the Majorana operators \( c_1 \) and \( c_{2N} \) are localized at the ends of the chain which act as a qubit. The analytical expression for the Majorana qubit is \( \sigma_M^x = i\sigma_1 c_2 N = \Pi_{j=1}^{N} (-\sigma_j^z \sigma_{j+1}^x) \).

This analytical expression for the photonic system is different from the Kitaev’s original spinless fermionic chain model [32]. This \( \sigma_M^x \) only acts on the two end points of the system when expressed in fermionic picture but it acts in every site of the chain when we consider the photon and that finally governs the physics of cavity QED lattice. The main difference arises from the string-like operator, \( P = \Pi_{j=1}^{N} (-\sigma_j^z) \) appearing in the equation of \( \sigma_M^x \), which corresponds to the parity operator associated with the total number of fermionized photons (hard-core photon). The parity operator commutes with the optical Kitaev’s chain Hamiltonian and reduced Hamiltonian in topological limit. But it does not commute with the Lindblad operators, \( \sigma_i^- = (2\hbar c) \) [9]. Physically it means that single-particle losses break parity conservation and thus the Majorana qubit is not parity protected, i.e., the topological phase of the cavity QED system is destroyed by the decoherence but the original Majorana fermion mode exists in time scale shorter than the photon life time. The non-topological phase is the trivial phase of the system. The physics of dephasing is a non-dissipative decoherence process where no energy is exchanged with the environment, therefore the dephasing has no role in the parity-breaking process in the Majorana fermion or the Majorana-Weyl fermion physics of the system [9].

\textbf{Conclusion.} – We have presented the existence of the Dirac fermion mode, the Majorana fermion mode and also the Majorana-Weyl fermionic mode for the optical cavity array for the different relation between Rabi frequency oscillation and the atom-photon coupling strength. We have presented the crossover between the Dirac fermion modes and the Majorana fermion modes and also the condition for the appearance of the Majorana-Weyl massless mode. We have presented several analytical relations between the Majorana field, order and disorder operators. We have also studied the behavior of correlation functions and the effect of dissipation on Majorana zero mode.

\textbf{Appendix.} –

Analytical relations between order and disorder operators.
\[ \mu_z^2 = 1 = \mu_x^2, \mu_z(n-1/2)\mu_z(n+1/2) = \sigma_z(n), \mu_x(n+1/2) = \sigma_x(n)\sigma_z(n+1), \mu_x(n+1/2) = \Pi_{j=1}^{n} \sigma_z(j). \]

We have presented the existence of the Dirac fermion mode, the Majorana fermion mode and also the Majorana-Weyl fermionic mode for the optical cavity array for the different relation between Rabi frequency oscillation and the atom-photon coupling strength. We have presented the crossover between the Dirac fermion modes and the Majorana fermion modes and also the condition for the appearance of the Majorana-Weyl massless mode. We have presented several analytical relations between the Majorana field, order and disorder operators. We have also studied the behavior of correlation functions and the effect of dissipation on Majorana zero mode.

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Derivation of Dirac equation for Majorana fermion field. The equation of motion for the \( \sigma_z(n) \) is the following:
\[ \frac{\partial \sigma_z(n)}{\partial \tau} = [H, \sigma_z(n)] = \sigma_z(n) \sigma_z(n). \]

The equation of motion for \( \mu_z(n+1/2) \) is the following:
\[ \frac{\partial \mu_z(n+1/2)}{\partial \tau} = \lambda \mu_z(n+1/2) \mu_z(n+1/2) = \lambda \sigma_z(n) \sigma_z(n+1/2) \mu_z(n+1/2). \]
Now we use the properties of the $\sigma$ and $\mu$ operators to derive the equation of motion for the Majorana fields $\chi_1(n)$ and $\chi_2(n)$,
\[
\frac{\partial \chi_1(n)}{\partial \tau} = \frac{\partial \sigma_z(n)}{\partial \tau} \mu_z(n + 1/2) + \sigma_z(n) \frac{\partial \mu_z(n)}{\partial \tau}, \quad (A.3)
\]
\[
\frac{\partial \chi_1(n)}{\partial \tau} = -\sigma_z(n) \mu_z(n - 1/2) \mu_z(n + 1/2) + \lambda \sigma_z(n) \sigma_z(n) \mu_z(n + 1/2), \quad (A.4)
\]
\[
\frac{\partial \chi_1(n)}{\partial \tau} = -\chi_2(n) + \lambda \chi_2(n + 1). \quad (A.5)
\]

Now the equations of motion for $\chi_2(n)$ are
\[
\frac{\partial \chi_2(n)}{\partial \tau} = \frac{\partial \sigma_z(n)}{\partial \tau} \mu_z(n - 1/2) + \sigma_z(n) \frac{\partial \mu_z(n)}{\partial \tau}, \quad (A.6)
\]
\[
\frac{\partial \chi_2(n)}{\partial \tau} = \mu_z(n - 1/2) \mu_z(n + 1/2) \sigma_z(n) \mu_z(n - 1/2) + \lambda \sigma_z(n - 1) \mu_z(n - 1/2). \quad (A.7)
\]

After a little bit of calculations and using the relation between the disorder operators (appendix), we finally arrive the equation of motion of $\chi_2(n)$ as
\[
\frac{\partial \chi_2(n)}{\partial \tau} = -\chi_1(n) + \lambda \chi_1(n - 1). \quad (A.8)
\]

These two fields, $\chi_1(n)$ and $\chi_2(n)$ satisfy the following relations, $\{\chi_1(n1), \chi_2(n2)\} = 2\delta_{n1,n2}$. One can write down the above equation in the following compact form:
\[
\left( \gamma^0 \frac{\partial}{\partial t} + \gamma^3 \frac{\partial}{\partial \tau} + m \right) \chi = 0,
\]
where $\chi^\dagger = (\chi_1, \chi_2)$ and $m = \frac{1 - \alpha}{\alpha}$, $\gamma^0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$, $\gamma^3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$.

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