Near wall patch representation of wall bounded turbulence

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Recent experimental and computational studies indicate that near wall turbulent flows can be characterized by universal small scale autonomous dynamics that are modulated by large scale structures. We formulate numerical simulations of near wall turbulence in a small domain localized to the boundary, whose size scales in viscous units. To mimic the environment in which the near wall turbulence evolves, our formulation accounts for the flux of mean momentum through the upper boundary of the domain. Comparisons of the model’s two dimensional energy spectra and low order single-point statistics with the corresponding quantities computed from zero and mild favorable pressure gradient direct numerical simulations indicate it successfully captures the dynamics of the small scale near wall turbulence.

Key words:

1. Introduction

Wall bounded turbulent flows are characterized by a separation of scales between the flow in the near wall region, in which mean viscous stresses play an important role, and the flow farther away from the wall, where viscous effects are largely negligible. This separation of scales is quantified by the friction Reynolds number $Re_\tau = \delta/\delta_\nu$, where $\delta$ is the characteristic length scale of the problem at hand, such as a channel half-width, a pipe radius, or a boundary layer thickness, and $\delta_\nu = \nu/u_\tau$ is the viscous length scale, where $\nu$ is the kinematic viscosity of the fluid, $u_\tau = \sqrt{\tau_w/\rho}$, $\tau_w$ is the mean wall shear stress, and $\rho$ is the fluid density. Both the direct numerical simulation (DNS) and large eddy simulation (LES) of such wall bounded turbulent flows are expensive, as the spatial degrees of freedom required to resolve the near wall layer scale as $O(Re_\tau^{2.5})$ and $O(Re_\tau^2)$ for DNS and LES, respectively (Mizuno & Jiménez 2013). For a large class of flows of technological importance, this cost is prohibitive, even on modern high-performance computing systems.

Thanks to advances in experimental techniques and computational power, the understanding of the physics of wall bounded flows has increased greatly since the earliest investigations by Hagen (1839), Darcy (1854), and Reynolds (1895), and the later work by Millikan (1938). It is well known that there is an autonomous near wall cycle of self sustaining mechanisms (Jiménez & Moin 1991, Hamilton et al. 1995, Jeong et al.)

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Figure 1: Unfiltered, high-pass (SS), and low pass (LS) portion of the turbulent kinetic energy scaled in viscous units with $k_{cut} \delta_\nu = 0.00628$ ($\lambda_{cut}/\delta_\nu = 1000$) for a channel flow at various $Re_\tau$ (green: $Re_\tau = 550$, blue: $Re_\tau = 1000$, red: $Re_\tau = 2000$, black: $Re_\tau = 5200$). The contributions from the large scales increases with $Re_\tau$, but the contribution from the small scales is largely independent of $Re_\tau$. Figure reproduced from Lee & Moser (2019) with permission.

1997), involving low and high speed streamwise velocity streaks and coherent structures of quasi-streamwise vorticity. Jiménez & Pinelli (1999) showed that this cycle of near wall dynamics persists without any input from the turbulence farther away from the wall. Moreover, if any element of the cycle is suppressed, the near wall turbulent kinetic energy (TKE) decays, and the flow becomes laminar. However, the large-scale structures in the outer layer do modulate the turbulent fluctuations in the near wall region (Hutchins & Marusic 2007; Marusic et al. 2010; Ganapathisubramani et al. 2012), leaving their “footprint” on the autonomous cycle. This results, for instance, in an $Re_\tau$-dependent peak of the turbulent kinetic energy in the near wall region (see figure 1). Their influence increases with increasing $Re_\tau$.

Recently, Lee & Moser (2019) performed spectral analysis of channel flow DNS data for several different $Re_\tau$ (ranging from approximately 550 to 5200) to investigate the relative importance of different length scales to the production, transport, and dissipation of TKE. Their results suggest that the small scales in the near wall region behave universally. Indeed, when the spectrum of the TKE density is high-pass filtered to only include contributions from wavenumbers with magnitude larger than some $k_{cut} \delta_\nu = 0.00628$, the resulting energy is found to be independent of $Re_\tau$, as shown in figure 1. Similar results are also obtained in Samie et al. (2018) for experimental data ranging from $Re_\tau \approx 6000 – 20000$. These works, along with those above, indicate that the near wall region can be characterized by universal autonomous dynamics whose small scales are independent of $Re_\tau$. The large scale portion of the near wall turbulence, however, is the result of eddies whose size and influence on the turbulent statistics depend on $Re_\tau$.

With this understanding in mind, we endeavor to design a computational model of the universal small scales of turbulent, wall bounded flows in the case of a locally constant pressure gradient. The primary goal is to accurately represent the contribution of the small scales to the near wall turbulent statistics at a reduced cost relative to a DNS of the entire flow field. The model is formulated to simulate wall bounded turbulence only in a near wall, rectangular domain $\Omega$ localized to the boundary. The size of the rectangle scales in viscous units, so that as $Re_\tau$ increases, the domain shrinks in size relative to the size of the overall flow, but the number of computational degrees of freedom remains fixed. Assuming such a configuration can accurately model the dynamics of the near
wall, small scale features of the flow at a relatively low cost, it could be used to study the near wall statistics as a function of pressure gradient, covering, for instance, the cases of favorable, negligible and adverse pressure gradients. Additionally, the model approach could potentially be used to study more complicated physics in the near wall region, such as heat transfer, chemical reactions, turbophoresis, or surface roughness. Lastly, data generated from such simulations could inform a wall model for either a large eddy simulation or a Reynolds Averaged Navier Stokes (RANS) simulation.

The results presented in this work indicate that such a description is indeed possible. In the cases of both zero and a mild favorable pressure gradient flows, the statistics generated by the model, in particular the mean velocities, the small scale contribution to the velocity (co)variances, and the terms in the Reynolds stress budget equations all show remarkable agreement with those generated by DNS.

Lastly, we comment that for a flow in which the size of the turbulent boundary layer evolves significantly in the streamwise direction, such as in the case of a sustained adverse pressure gradient, the model formulation must be refined to account for growth. In this case an asymptotic approach similar to \[ \text{seems appropriate, especially since the assumption of a separation of scales of evolution both in space and time is stronger in the near wall region than in the entire boundary layer.} \]

The remainder of the paper is organized as follows: section 2 contains a description of the computational model and the numerical method used to integrate the equations of motion. Section 3 provides a comparison between the statistics generated by the model and the corresponding quantities from DNS for the cases of both zero and mild favorable pressure gradients. In section 4 the results are summarized, and possible applications and extensions of the model are discussed.

1.1. Notation

In the following discussion, the velocity components in the streamwise (x), wall-normal (y) and spanwise (z) directions are denoted as \( u, v, \) and \( w \), respectively, and when using index notation, these directions are labeled 1, 2, and 3, respectively. The expected value is denoted with angle brackets (as in \( \langle \cdot \rangle \)), and upper case \( U \) and \( P \) indicate the mean velocity and pressure, so that \( \langle u_i \rangle = U_i \). The velocity and pressure fluctuations are indicated with primes, e.g. \( u_i = U_i + u'_i \). Partial derivatives shortened to \( \partial_i \) signify \( \partial/\partial x_i \), differentiation in the direction \( x_i \). The mean advective derivative is \( D(\cdot)/Dt = \partial_t (\cdot) + U_j \partial_j (\cdot) \), where Einstein summation notation is implied. In general, repeated indices imply summation, with the exception of repeated Greek indices. For \( z \in \mathbb{C} \), \( z^* \) denotes the complex conjugate; for \( \vec{x} \in \mathbb{R}^n \), \( |\vec{x}| \) denotes the Euclidean norm.

The superscript ‘+’ denotes non-dimensionalisation with the kinematic viscosity \( \nu \) and the friction velocity \( u_\tau \). Lastly, symbols with a tilde \( \tilde{\cdot} \) indicate non-dimensionalisation with \( \nu \) and the parameter \( \tilde{u}_\tau \) defined below.

2. Model description

Below the issue of boundary conditions for the near wall patch is discussed before introducing the equations of motion. The exact simulation parameters used and numerical implementation are then detailed.

2.1. Motivation: computational boundaries

The goal of the computational model is to simulate the turbulent small scales in the near wall region as a function of pressure gradient only in a small, rectangular domain
Figure 2: The fluid is subject to periodic boundary conditions at the (dash-dotted) side walls, constant Dirichlet/Neumann conditions at the upper boundary $y = 2H$, and the no-slip condition at the wall $y = 0$, as described in (2.1). In addition to the constant pressure gradient assumed to be present in the near wall layer, the model includes an auxiliary pressure gradient in a “fringe region” to account for momentum transport at the computational boundary $y = 2H$. localized to the boundary. This necessarily means placing nonphysical computational boundaries in a region of chaotic, highly nonlinear dynamics. In addition to the standard no-slip condition at the lower boundary $y = 0$, the use of periodic boundary conditions at the side-walls is well established assuming the flow is statistically homogeneous in these directions. The problem of prescribing appropriate boundary conditions at the upper computational boundary, however, is nontrivial (Berselli et al. 2006; Sagaut et al. 2006). Once a mathematically well posed condition is prescribed, care must be taken to prevent the approximation from polluting the turbulent dynamics in the domain’s interior. To address this issue, the model augments the near wall computational domain with a “fringe region” in which turbulence is externally forced with an “auxiliary” forcing term. Although nonphysical, this auxiliary pressure gradient is designed to account for the mean flux of momentum and turbulent kinetic energy through the upper computational boundary. The inclusion of such a region increases the computational cost of the model, but it provides the momentum transport needed to create the “correct environment” for the evolution of turbulence in the near wall region. In this way, the fringe region mollifies the effect of the nonphysical computational boundary; similar techniques are used for designing inflow/outflow conditions in the DNS of turbulent boundary layers, for example, as well as in molecular dynamics simulations, often referred to as a “heat-bath” or “thermostat” (Berendsen et al. 1984). If one is interested in the turbulent statistics resulting from a constant pressure gradient near wall region out to a wall-normal height of $y \approx H$, the fringe region consists of a layer from $H \leq y \leq 2H$ in which the turbulence is forced by a pressure gradient larger in magnitude than the constant, background value assumed to be present in the underlying flow, as illustrated in figure 2.
2.2. Equations of motion and boundary conditions

The model is defined by the equations of motion and boundary conditions in the rectangular domain $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$:

$$
\begin{align*}
\partial_t u_i + u_j \partial_j u_i + \partial_i p' - \nu \partial_j \partial_j u_i &= f_i - \partial_i P \quad \text{in } \Omega \\
\partial_i u_i &= 0 \quad \text{in } \Omega \\
\partial_i P = dP/dx \delta_{i1} \quad \text{is constant in } \Omega \\
u \psi &\in \mathbb{R} \quad y = L_y \\
\partial_y u &= \psi \quad y = L_y \\
f_i(x, y, z, t) &= f(y) \delta_{i1} \quad \text{in } \Omega.
\end{align*}
$$

These are simply the forced incompressible Navier Stokes equations on a periodic domain in the wall-parallel directions, with the no-slip boundary condition at $y = 0$ (the wall), and no-flow through and constant viscous tangential traction $\nu \psi$ in the streamwise direction ($x$) specified at the top $y = L_y$. The forcing term $dP/dx$ models the mean pressure gradient for the real turbulent flow being modeled, while $p'$ is the model’s pressure field that evolves in time and space to maintain incompressibility. It only remains to specify $\psi$ and the forcing function $f(y)$. The former represents the viscous flux of mean momentum through the top, and the latter represents a source of streamwise momentum that makes up for the turbulent flux of mean streamwise momentum through the top that is missing, owing to the boundary conditions that imply that the Reynolds stress vanishes at the top.

The forcing function $f(y)$ is non-zero only in the fringe region $y > L_y/2$, and is constructed such that

$$
\int_0^{L_y} f(y) dy = \tau_{\text{turb}},
$$

where $\tau_{\text{turb}}$ is the turbulent flux of mean momentum through $y = L_y$ in the turbulent flow being modeled.

There are thus several parameters describing the mean momentum characteristics that must be defined to complete the specification of the computational model. However, they are related through the integrated mean momentum balance. In the real turbulent flow with a constant mean pressure gradient, the mean momentum equation integrated over $[0, L_y]$ yields (assume without loss of generality that $f = 0$):

$$
-\frac{dP}{dx}L_y + \left( \nu \frac{\partial U}{\partial y} \bigg|_{y=L_y} - \tau_{\text{turb}} \right) - \tau_w = 0
$$

The term in parentheses is $\tau_{\text{top}}$, the total momentum flux (viscous plus turbulent) through $y = L_y$. For the purpose of defining the near wall model, consider non-dimensionalizing with respect to $\tilde{u}_r = \sqrt{\tau_{\text{top}}}$ and $\nu$. In these units (signified by a tilde), $\tilde{v} = 1$ and $\tilde{\tau}_{\text{top}} = 1$, while $\tilde{\psi} = \tilde{\partial}_y \tilde{U}$, which is the fraction of $\tau_{\text{top}}$ carried by the viscous stress, is one of the parameters defining a model case. The non-dimensional pressure gradient $d\tilde{P}/d\tilde{x} = (\nu/\tilde{u}_r^3)dP/dx$ is a second model parameter. Specifying these parameters determines $\tau_w$. Dimensional analysis therefore implies that there is a two-parameter family of possible near wall turbulent flows to model.

To see how this parameterization maps to specific turbulent flows, consider two examples. First, clearly $d\tilde{P}/d\tilde{x} = 0$ for a zero-pressure-gradient turbulent boundary layer.
Assuming that the Reynolds number is high enough for $L_y$ to be in the constant stress and logarithmic layer, $\tilde{\psi} = 1/(\kappa \tilde{L}_y)$, where $\kappa$ is the von Kármán constant. Under these assumptions $\tilde{u}_r = u_r$ and hence $\tilde{L}_y = L_y^+$. For lower Reynolds numbers, $\tilde{\psi}$ will be larger, so increasing $\tilde{\psi}$ beyond this value would produce a model for a lower Reynolds number. Reducing the Reynolds number too much, however, will spoil the scale separation between the inner and outer turbulence and hence will invalidate the model.

Second, consider a high Reynolds number turbulent channel flow, with Reynolds number $Re_\tau$. In this case, it is well known that $\tau_{\text{top}}^+ = 1 - L_y^+/Re_\tau$ and $dP^+/dx^+ = -1/Re_\tau$. Furthermore, if the Reynolds number is high enough for $L_y$ to be in the log-layer, then $\psi^+ = 1/(\kappa L_y^+ y^+)$. It can then easily be shown that:

\[
\frac{d\tilde{P}}{d\tilde{x}} = - \left( \frac{Re_\tau^{1/3}}{Re_\tau - L_y^+} \right)^{3/2} \tilde{\psi} = \frac{Re_\tau}{\kappa L_y^+ (Re_\tau - L_y^+)}. \tag{2.4}
\]

For asymptotically high Reynolds number, this reduces to $d\tilde{P}/d\tilde{x} = -1/Re_\tau$ and $\tilde{\psi} = 1/\kappa L_y^+$. Consider now the mean momentum balance equation for the model system (2.1) in tilde units

\[
\frac{\partial \tilde{U}}{\partial \tilde{y}} - \langle \tilde{u}' \tilde{v}' \rangle + \int_0^{\tilde{y}} \tilde{f}(\tilde{s}) d\tilde{s} = \tilde{y} \frac{d\tilde{P}}{d\tilde{x}} + \tilde{\tau}_w, \tag{2.5}
\]

and assume the forcing function $f$ satisfies

\[
\int_{\tilde{L}_y/2}^{\tilde{L}_y} \tilde{f}(\tilde{s}) d\tilde{s} = \tilde{\tau}_{\text{turb}} = \tilde{\tau}_{\text{top}} - \tilde{\psi} = 1 - \tilde{\psi}. \tag{2.6}
\]

Then at $\tilde{y} = \tilde{L}_y$ (2.5) reduces to

\[
1 = \tilde{L}_y \frac{d\tilde{P}}{d\tilde{x}} + \tilde{\tau}_w. \tag{2.7}
\]

Hence, the model’s total shear stress augmented with the forcing function $f$ will be consistent with that of the real turbulent flow being modeled.

Figure 3 illustrates the stress balance for the favorable and zero pressure gradient cases NWP1000, NWP5200, and NWPzpg listed in table 1. Included in the figure is a stress profile resulting from simulating the equations of motion (2.1) without the extra momentum flux provided by the auxiliary pressure gradient $f$ (the cyan curve). In that case, a similar analysis to the one above shows that the statistically converged stress profile is simply a linear function with values $\tau_{\text{w}}^+ = 1$ and $\psi^+$ at $y^+ = 0$ and $y^+ = 600$, respectively. Since $\psi^+ = 1/(\kappa L_y^+ y^+)$ in the log-region, it is clear that this stress profile is in poor agreement with the theoretical profile $\tau_{\text{theory}}^+(y^+) = 1 - y^+/Re_\tau$. Moreover, this discrepancy increases with increasing $Re_\tau$, illustrating the utility of including the extra momentum flux provided by the auxiliary forcing term $f$.

2.3. Simulation parameters–domain size and forcing $f$

The model makes use of the parameters summarized in table 1. As mentioned above, the simulation is scaled in wall units; a more precise statement is that the simulation is scaled in “tilde” units, that is, with $\tilde{u}_r$ and $\nu$, since these parameters can be prescribed a priori. $\tilde{u}_r$ is specified as a function of the forcing term $dP/dx$ as

\[
\tilde{u}_r^2 = 1 + L_y \frac{dP}{dx}. \tag{2.8}
\]
Figure 3: Statistical convergence for the favorable pressure gradient cases. Blue: model total stress $\tau_{\text{model}}^+(y^+) = \partial_y U^+ - \langle u'v' \rangle^+$. Red: analytic total stress $\tau_{\text{theory}}^+ = 1 + y^+ dP^+ / dx^+$. Green: Primitive $F^+(y^+)$ of auxiliary pressure gradient $f^+$. Black: $\tau_{\text{model}}^+(y^+) + F^+(y^+)$. Cyan: total stress for simulation without auxiliary pressure gradient $f^+$. Magenta: Absolute error $|\tau_{\text{model}}^+(y^+) - \tau_{\text{theory}}^+(y^+)|$ (solid lines) in the region $y^+ \in [0, 300]$ and the standard deviation of the estimated statistical error for $\tau_{\text{model}}^+$ (dashed lines).

dP/dx is set to $-(1000.512)^{-1}$, $-(5185.89)^{-1}$, and 0 for the cases NWP1000, NWP5200, and NWPzpg, respectively, and $L_y = 600$. Inserting these values into the model’s mean stress balance (2.7) implies that $u_r = 1$ will result when the flow reaches statistical equilibrium. The kinematic viscosity $\nu$ is set to unity. The size of the rectangular domain $\Omega$ is taken to be $L_x^+ = L_z^+ \approx 1500$ and $L_y^+ \approx 600$; these values are somewhat arbitrary but were arrived at through a combination of trial and error and inspiration from the results in [Lee & Moser, 2019]. Their work suggests that the contributions to the turbulent kinetic energy from modes with wavelengths $\lambda^+ < 1000$ are universal and $Re_z$-independent in a region below a wall-normal distance of approximately $y^+ = 300$. Accordingly, $L_y^+ \approx 2 \cdot 300 = 600$ is chosen to allow for a sufficiently large fringe-region to mollify the effect of the nonphysical computational boundary at $y = L_y$ (see figure 2), and both $L_x^+$ and $L_z^+$ are taken to be at least 1000.

For the domain size in the stream/spanwise directions, there generally is a balance between computational cost and the accuracy of the model, as defined by a comparison of the model’s energy spectral density with that of DNS of the entire channel. In particular, a variety of domain sizes were tested, ranging from $L_x^+ = L_z^+ = 1000$ to approximately 3500. Increasing $L_x$ and $L_z$ results in a better agreement of the model’s low-wavenumber, large-scale portion of the energy spectral density with the corresponding portion computed from DNS. The contribution of the small-scale portion of the energy density, defined by the subset of wavenumbers $[3.5]$, is found to be robust to changes in $L_x$ and $L_z$ so long as they are not taken to be too close to $L_x^+ = L_z^+ = 1000$. In particular, $L_x^+ = L_z^+ = 1500$ is found to be the smallest domain size capable of reproducing the universal small-scales illustrated in section 3.3.

Given some target turbulent flux of mean momentum $\tau_{\text{turb}}$, the auxiliary pressure gradient $f$ is defined by (2.2); it is not, however, uniquely determined. For the simulations
\[ \begin{align*}
\text{Case name} & \quad u_x & \quad \bar{u}_\tau & \quad \psi^+ & \quad L_x^+ = L_z^+ & \quad \Delta x^+ & \quad \Delta z^+ & \quad \Delta y^+ \\text{NWP1000} & 1.0006 & 0.40031 & 5.4501e-3 & 1500.9 & 600.36 & 12.51 & 5.86 & 0.002817 \\
\text{NWP5200} & 1.0033 & 0.88430 & 5.4746e-3 & 1504.9 & 601.9 & 12.54 & 5.88 & 0.002809 \\
\text{NWPzpg} & 0.98852 & 1 & 5.449e-3 & 1497.8 & 599.11 & 12.48 & 5.85 & 0.002796
\end{align*}\]

Table 1: Summary of simulation parameters: \( \psi \) is the prescribed value for the Neumann boundary condition in (2.1). \( \Delta x \) and \( \Delta z \) are in terms of Fourier modes for spectral methods: \( \Delta x = L_x/N_x \) and similarly for \( \Delta z \). For all cases, \( N_x = 120, N_z = 256 \), and the number of B-spline collocation points \( N_y = 192 \). \( \Delta y_w \) is the collocation point spacing at the wall.

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reported here, it is defined explicitly to be

\[ f(y) = \begin{cases} 
4\tau_{\text{turb}}/L_y^4 (L_y - 2y)^2 (5L_y - 4y), & y \in [L_y/2, L_y] \\
0, & y \in [0, L_y/2]
\end{cases} \]  
(2.9)

and was chosen to satisfy \( f(L_y/2) = f'(L_y/2) = f'(L_y) = 0 \).

2.4. Numerical implementation and resolution

The model (2.1) is solved numerically using the velocity-vorticity formulation due to [Kim et al. 1987](#). The equations of motion are discretized with a Fourier-Galerkin method in the stream/spanwise directions and a seventh order B-spline collocation method in the wall-normal direction ([Kwok et al. 2001](#), [Botella & Shariff 2003](#), [Lee & Moser 2015](#)). They are integrated in time with a low-storage, third order Runge-Kutta method that treats diffusive and convective terms implicitly and explicitly, respectively ([Spalart et al. 1991](#)). The numerical resolution in both space and time is chosen to be consistent with that of DNS. The number of Fourier modes, and hence the numerical resolution, used in each simulation is listed in table 1 and can be compared with, for instance, table 1 in [Lee & Moser 2015](#). In addition, the collocation point spacing in the wall-normal direction is similar to previous DNS studies; the total number of collocation points \( N_y \) is taken to be equal to the number of collocation points below \( y^+ = 600 \) in [Lee & Moser 2015](#). They are then distributed in the near wall region according to the same (shifted and rescaled) stretching function.

The model is implemented with a modified version of the Poongback DNS code ([Lee et al. 2013](#), [2014](#)); the initial condition is taken from a restart file from a DNS run that is truncated to fit in \( \Omega \) at the resolutions listed in table 1 and modified to satisfy the boundary conditions 2.1. The values for \( \psi \) are taken from DNS data.

2.5. Statistical convergence and sensitivity to \( \psi \)

The method of [Oliver et al. 2014](#) is used to assess the uncertainty in the statistics reported due to sampling noise. For each pressure gradient case, statistics are collected in time until the estimated statistical uncertainty in the mean stress profiles is less than a few percent. For the cases in table 1 reported here, the sampling error is less than three percent.

For each pressure gradient investigated, the sensitivity of the results to the prescribed mean viscous shear was explored by examining the response to both a doubling and halving of \( \psi \). It was found that the model statistics are robust to such changes.
3. Numerical results

The statistics reported here were computed from the three model cases NWP1000, NWP5200, and NWPzpg and are analyzed against available DNS data. The favorable pressure gradient cases are compared to the corresponding channel flow DNS performed at \( Re_\tau = 1000 \) and \( Re_\tau = 5200 \)—referred to below as LM1000 and LM5200—whose statistics are available at [http://turbulence.ices.utexas.edu](http://turbulence.ices.utexas.edu). The zero pressure gradient case is compared to the turbulent boundary layer DNS of [Borrell et al. (2013)](https://doi.org/10.1017/jfm.2013.290), [Borrell et al. (2013)](https://doi.org/10.1017/jfm.2013.290); the data used for comparison corresponds to the streamwise location at which \( Re_\tau = 2000 \), and can be found at [https://torroja.dmt.upm.es/turbdata/blayers/high_re](https://torroja.dmt.upm.es/turbdata/blayers/high_re). It is referred to below as SJM2000.

In all cases, the model correctly reproduces the mean velocities and off diagonal Reynolds shear stress \( \langle u'v' \rangle \) in the region of interest \( y^+ \in [0, 300] \). Additionally, the production, transport and dissipation terms in the Reynolds-stress budget equations are in agreement with DNS data, indicating the model is also able to capture the lifecycle of turbulent kinetic energy as it is produced by the mean, transported across scale and wall-normal distance, and ultimately dissipated by viscosity. For the favorable pressure gradient cases, the two dimensional turbulent kinetic energy (TKE) spectral densities are compared. The model correctly reproduces the large wavenumber (small-scale) portion of the spectra, along with its associated energy content. In contrast, the large-scale, low wavenumber content of energy spectral density is either missing or inaccurate owing to the model’s limited domain size in the stream/spanwise directions. In contrast to the near wall region \( y^+ \in [0, 300] \), all the statistical profiles are inaccurate for \( y^+ \gtrsim 300 \) due to the unphysical boundary conditions at \( y = L_y \), as well as the modified forcing term \( dP/dx \). These results are discussed in more depth below.

3.1. Mean velocity and shear stresses

The most basic requirement of any model of near wall turbulence is to correctly capture the equilibrium velocity profile and the associated mean wall shear stress. Figure 4 demonstrates that the model’s mean velocity profiles \( U^+ \) and log-law indicator function

\[
\beta^+(y^+) := y^+ \frac{\partial U^+}{\partial y^+}
\]  

(3.1)

profiles are in strong agreement with their DNS counterparts; the relative error in \( U^+ \) is within 0.6% for \( y^+ \in [0, 300] \). The error is similarly small for \( \beta^+ \) in the range \( y^+ \in [0, 100] \), although there is some mild disagreement in the range \( y^+ \in [100, 300] \). As expected, the profiles diverge for \( y^+ > 300 \).

In addition to the mean viscous shear stress, figure 5 shows the model’s Reynolds shear stress \( \langle u'v' \rangle \) profiles are in excellent agreement with their DNS counterparts in the region \( y^+ \in [0, 300] \). For the favorable pressure gradient cases, the error is less than 0.5%, and for the zero pressure gradient case the error is below 4%. For the former case, the total stress at \( y = L_y \) is known analytically as \( \tilde{u}_r^2 \) and imposed accordingly as \( \tilde{u}_r^2 \). In the latter, however, \( \tilde{u}_r^2 \) is still imposed via (2.8). This relation only holds in the limit \( Re_\tau \to \infty \), which can explain the relatively larger discrepancy between the \( \langle u'v' \rangle \) profiles. In both instances, recall that \( \langle u'v' \rangle \) necessarily vanishes at the upper computational boundary as a consequence of the boundary condition \( v = 0 \) in (2.1). The accuracy of the Reynolds shear stress profiles in spite of this condition demonstrates the ability of the forcing function \( f \) to successfully transport momentum to the near wall region \( y^+ \in [0, 300] \).
Figure 4: Mean velocity $U^+$ (left) and the indicator function $\beta^+ = y^+ \partial_+ U^+$ (right) versus log($y^+$). The black dashed-dotted vertical line marks $y^+ = 300$, and the cyan dashed-dotted lines plot the law-of-the-wall $U^+ = y^+$ and $U^+ = (1/\kappa) \log(y^+) + B$, where $\kappa = 0.384$ and $B = 4.27$ (Lee & Moser 2015).

Figure 5: (a) Reynolds stress $-\langle u'v' \rangle^+$ and (b) the filtered Reynolds stress $-\langle u'v' \rangle_{\text{SS}}^+$ (defined by (3.7)) as a function of log($y^+$), discussed in section 3.3. The black dashed-dotted vertical line marks $y^+ = 300$.

### 3.2. Energy spectral density

For two points $(x, y, z), (x', y', z') \in \mathbb{R} \times [0, 2\delta] \times \mathbb{R}$ in an infinitely long channel, define the separation distances $r_x = x - x'$ and $r_z = z - z'$. For a turbulent flow that is statistically homogeneous in the stream and spanwise directions, the two point correlation tensor

$$R_{ij}(r_x, y, r_z) := \langle u'_i(x + r_x, y, z + r_z) u'_j(x, y, z) \rangle$$

(3.2)

is a function only of $r_x$, $y$, and $r_z$. Taking the Fourier transform of (3.2) in the variables $r_x$ and $r_z$ defines the energy spectral density $E_{ij}(k_x, y, k_z)$, which encodes the average
contribution to the turbulent kinetic energy from different length scales as a function of the wall-normal variable $y$. The Reynolds stress tensor can be recovered by taking the limit $(r_x, r_z) \to (0, 0)$ in (3.2), or by integrating the energy spectral density over all wavespace

$$
\langle u'_i u'_j \rangle(y) = \int \int E_{ij}(k_x, y, k_z) dk_x dk_z. \tag{3.3}
$$

For a wall bounded flow in a full size domain, the low-wavenumber contributions to the Reynolds stress represent the mean influences of the large scale structures on the near wall dynamics. As is well known [Hutchins & Marusic 2007; Marusic et al. 2010; Lee & Moser 2017; Samie et al. 2018; Lee & Moser 2019] these low-wavenumber features of the near wall flow depend on $Re_\tau$. In contrast, there exists strong evidence that the high-wavenumber (small-scale) contributions to the Reynolds stress profiles are universal and independent of $Re_\tau$ (Samie et al. 2018; Lee & Moser 2019). By design, the model’s domain size does not allow for the accurate representation of the very large-scale structures known to exist in the near wall region, and thus it is not reasonable to expect its Reynolds stress profiles to agree with those computed from DNS (with the exception of the shear stress $\langle u'v' \rangle$, as discussed in section 3.1). Instead, it is more reasonable to expect the model to correctly capture the dynamics of the universal small scales elucidated by Samie et al. (2018) and Lee & Moser (2019) associated with the autonomous cycle of Jiménez & Pinelli (1999).

To determine whether or not this is the case, the model’s spectra $E_{ij}$ are compared to their DNS counterparts. The spectra are visualized in so-called log-polar coordinates [Lee & Moser 2019], in which the wavenumber magnitude $k = \sqrt{k_x^2 + k_z^2}$ is represented on a logarithmic scale. For fixed wall-normal location, the log-polar coordinates are defined as

$$
k_x^\# := \frac{k_x}{k} \log_{10} \left( \frac{k}{k_{\text{ref}}} \right) \tag{3.4}
$$

$$
k_z^\# := \frac{k_z}{k} \log_{10} \left( \frac{k}{k_{\text{ref}}} \right)
$$

where $k_{\text{ref}}$ is an arbitrary reference wavenumber that must be smaller than the smallest nonzero wavenumber included in the spectrum, taken here to be $k_{\text{ref}}^+ = 1/50000$. Two advantages of these coordinates are that lines of constant $k_z/k_x$ have slopes of $k_z/k_x$, and lines of constant magnitude $k$ map to circles. In this way, the orientation and alignments of the Fourier modes are easily interpreted; see Lee & Moser (2019) for a more detailed discussion of these coordinates.

The two dimensional spectral densities of the streamwise and wall-normal velocity variances are shown in figures 6 and 7 respectively. The spectra are visualized at the wall-normal locations $y^+ = 15$, $y^+ = 100$ and $y^+ = 300$ for the favorable pressure gradient simulations NWP1000, NWP5200, LM1000, and LM5200.

In each of the cases, the streamwise spectra $E_{11}$ consists primarily of energy concentrated along the $k_z^\#$ axis, with Fourier modes for which $k_z^+ / k_x^+ \gtrsim 10$ (Lee & Moser 2019). These correspond to structures that are strongly elongated in the streamwise $x$-direction, such as the well-known near wall low and high speed streaks. The channel flow data LM1000 and LM5200 (columns two and four in figure 6) show that this energy exhibits two distinct features.

The first is an “inner energy site” (Samie et al. 2018), a triangular shaped region in the near wall layer $y^+ \approx 15$ distributed primarily between wavelengths $\lambda^+ = 100$ and $\lambda^+ = 1000$ that can be attributed to the autonomous near wall dynamics described by
Figure 6: Two-dimensional spectra of the streamwise velocity variance \( \langle u' u' \rangle^+ \) in log-polar coordinates, as defined by equation (3.4). \( \lambda^+ = 10 \) on the outermost dotted circle and increases by a factor of 10 for each dotted circle moving inward.

Hamilton et al. (1995); Jeong et al. (1997); Jiménez & Pinelli (1999), and others. The model \( E_{11} \) spectra, shown in columns one and three in figure 6, qualitatively reproduce the inner energy site, indicating that it properly captures the dynamics of the near wall, small scale energetic motions.

The second feature is a concentration of energy at relatively low wavenumbers (in the range \( 1000 < \lambda^+ < 10000 \)) along the \( k^\#_z \) axis at each of the wall-normal locations \( y^+ = 15 \), \( y^+ = 100 \), and \( y^+ = 300 \). These are due to the very large scale motions (VLSMs) imposed from the outer layer flow described by Hutchins & Marusic (2007) and Marusic et al. (2010). These VLSMs contribute energy in the near wall region around \( y^+ = 15 \), and farther away from the wall they are responsible for the majority of the turbulent kinetic energy. As both \( y^+ \) and \( Re_\tau \) increase, the energy becomes more concentrated and is found at larger wavelengths, consistent with the attached eddy hypothesis of Townsend (1976).

In addition, the VLSMs modulate the near wall cycle through nonlinear interactions, creating large scale variations in the local wall shear stress that result in local variations in the dominant (most-energetic) wavelength (Lee & Moser 2019). Consequently, the spectral peak of the inner energy site for the DNS data is reduced and “smeared out” as a function of \( Re_\tau \); the \( Re_\tau = 5200 \) peak is about ten percent lower than the \( Re_\tau = 1000 \) peak.

By design, the model can only support wavenumbers associated to wavelengths less than or equal to \( L_+^x = L_+^z \approx 1500 \), meaning the VLSMs present in real wall bounded flow are not represented by the model. As a result, there is no energy associated with such large scale structures; the concentration of energy at low wavenumbers along the
Figure 7: Two-dimensional spectra of the wall-normal velocity variance $\langle v'v' \rangle^+$ in log-polar coordinates, as defined by equation (3.4). $\lambda^+ = 10$ on the outermost dotted circle and increases by a factor of 10 for each dotted circle moving inward.

$k^\#_z$ axis (corresponding to wavelengths $\lambda^+ \gtrsim 1000$) present in the DNS spectra is not present in that of the model. This is true both in the near wall region, and farther away from the wall at $y^+ = 100$ and $y^+ = 300$.

Furthermore, the model does not capture the nonlinear modulations of the autonomous cycle by the VLSMs. For instance, even though the model supports wavenumber content along the $\lambda^+ = 1000$ band, its spectra is not simply a spectral truncation of the DNS spectra. Additionally, the peak of the inner energy site is nearly identical for the two model cases, differing by only a few percent. These differences between spectra of the model and DNS highlight the important role the VLSMs play in the turbulent near wall layer.

The $E_{22}$ spectra are largest in the wavenumber regions for which the $E_{11}$ spectra are peaked, as discussed in [Lee & Moser (2019)]. Additionally, the distribution of energy generally becomes more isotropic with increasing wall-normal distance $y^+$. Figure 7 shows the DNS energy density $E_{22}$ is primarily, but not exclusively, located at the small scales, i.e. at wavenumbers located outside the $\lambda^+ = 1000$ band. Because the model adequately resolves such structures, its energy density $E_{22}$ is overall in good agreement with the DNS spectra, especially in the near wall region. Farther away from the wall the agreement is not quite as good since the DNS spectra are peaked at lower wavenumbers. Accordingly, the model’s unfiltered energy profiles $\langle v'v' \rangle$ shown in figure 8 indeed show excellent agreement with the corresponding DNS profiles in region $y^+ \in [0, 300]$; they are nearly identical for $y^+ \lesssim 50$ and only display slight discrepancies for $y^+ \in [50, 300]$. 
Lastly, the model’s energy density $E_{33}$ (not shown) compares to the DNS spectra in a similar manner to the $E_{11}$ spectra. It clearly approximates the small scales in the near wall region well, but it fails to capture the modulation by the large scale structures at each wall-normal location.

### 3.3. Universality of small scale motions

To better quantify the universal nature of the small scale motions and assess the model’s ability to reproduce them, the energy spectral density is high-pass filtered and then integrated in order to measure the energy content associated with the small scales. Let $K$ denote the set of wavenumbers supported by a simulation, and let $k_{\text{cut}} = 2\pi/\lambda_{\text{cut}}$ with $\lambda_{\text{cut}}^+ = 1000$. Define $K_{\text{SS}}$ to be the subset of $K$ with the property that $(k_x, k_z) \in K_{\text{SS}}$ if

$$\min\{|k_x|, |k_z|\} > k_{\text{cut}}.$$  \hfill (3.5)

The $K_{\text{SS}}$ sets are meant to contain large wavenumbers associated with the universal small scales. Here $k_{\text{cut}}$ is chosen from observing the two-dimensional spectra; the inner energy site associated to the autonomous cycle is contained outside the $\lambda^+ = 1000$ band.

The Reynolds stresses are then given as

$$\langle u'_i u'_j \rangle (y) = \sum_{(k_x, k_z) \in K} E_{ij}(k_x, y, k_z),$$  \hfill (3.6)

and the small scales energy can be quantified as

$$\langle u'_i u'_j \rangle_{\text{SS}} (y) = \sum_{(k_x, k_z) \in K_{\text{SS}}} E_{ij}(k_x, y, k_z).$$  \hfill (3.7)

The velocity covariance $\langle u'v' \rangle$ and variances $\langle u'_\alpha u'_\alpha \rangle$, $\alpha = 1, 2, 3$ and their high-pass fil-
tered counterparts are shown in figures 5 and 8, respectively. As previously mentioned, the model’s unfiltered \( \langle u'v' \rangle \) and \( \langle v'v' \rangle \) profiles both agree quite well with the corresponding DNS profiles. The model’s streamwise and spanwise velocity variances, however, display nontrival discrepancies with the DNS profiles, as expected from the observed differences in the two dimensional spectra. In contrast, the model’s high-pass filtered profiles all show excellent agreement with the high-pass filtered DNS quantities. In all cases the agreement is exceptionally strong in the region \( y^+ \in [0, 100] \), although there are some relatively mild discrepancies for \( y^+ \in [100, 300] \). Moreover, it is clear that the high-pass filtered quantities are nearly \( Re_\tau \) independent; the collapse of the \( \langle u'u' \rangle_{SS} \) profiles is particularly convincing. Although two-dimensional spectra data is not available for the zero-pressure gradient DNS case SJM2000, the \( \langle u'u' \rangle_{SS} \) profiles computed from the case NWPzpg are included for completeness; they display the same universal behavior as the favorable pressure gradient flows. These observations lend support to the conclusion that the small scales in the near wall region are universal, and that the difference in the Reynolds stress profiles as a function of \( Re_\tau \) is due to the increasing influence of the VLSMs. Previous results of this type obtained in both Lee & Moser (2019) and Samie et al. (2018) involve high-pass filtering the entire turbulent flow field, in which there are nonlinear interactions between wavenumbers across all the scales of motion. It is particularly remarkable, however, that the model reproduces the universal behavior of the small scales without the dynamic modulation of the near wall autonomous cycle by the large scale structures.

3.4. Production

The production of turbulent kinetic energy in a wall bounded flow is primarily due to the large mean velocity gradient in the wall normal direction \( \partial U/\partial y \). In a flow that is homogeneous in the stream/spanwise directions with \( V = W = 0 \), the only \( \langle u'_\alpha u'_\alpha \rangle \) term with a nonzero production is \( \langle u'u' \rangle \); it is given by

\[
\mathcal{P}_{11} = -2 \frac{\partial U}{\partial y} \langle u'v' \rangle.
\]  

(3.8)

The two dimensional spectra of \( \mathcal{P}_{11} \) is accordingly defined as

\[
E_{11}^P(k_x, y, k_z) := -2 \frac{\partial U}{\partial y} (y) E_{12}(k_x, y, k_z).
\]  

(3.9)

The spectral analysis of channel flow data in Lee & Moser (2019) demonstrated that in contrast to the near wall energy spectra \( E_{11} \), the near wall production spectra \( E_{11}^P \) contains only a high wavenumber peak (see columns two and four in figure 9). It follows that the large scale motions in the near wall region, and hence the energy that they contain, are due to energy transport (either in \( y \) or in scale), rather than local production. This observation suggests that the model should be able to correctly capture the near wall energy production, even though the VLSMs are not present. The production spectra shown in figure 9 show that this is indeed true. At both \( y^+ = 15 \) and \( y^+ = 30 \), the NWP1000 and NWP5200 spectra are qualitatively similar to that of DNS, including the regions of negative production occurring over a range of scales around \( \lambda^+ = 100 \). Farther away from the wall, the large scale structures increasingly influence the energy production, and their influence increases with \( Re_\tau \). At \( y^+ = 300 \), the large scale influences dominate the DNS production spectra, and the model is not able to reproduce such low wavenumber features.

The one-dimensional, premultiplied production profiles are shown in figure 10, and they are consistent with the aforementioned observations regarding the two-dimensional
Figure 9: Two-dimensional, premultiplied spectra $y^+ (E_{11}^p)^+$ in log-polar coordinates, as defined by equation (3.4). $A^+ = 10$ on the outermost dotted circle and increases by a factor of 10 for each dotted circle moving inward.

spectra. The DNS profiles are approximately $Re_{\tau}$-independent for $y^+ \lesssim 70$, and the corresponding model profiles show strong agreement for $y^+ \lesssim 100$. However, they begin to show relatively modest discrepancies for $y^+ \gtrsim 200$.

3.5. Dissipation, pressure strain and transport

After being produced by the mean, turbulent kinetic energy is redistributed across scales and velocity components, transported both towards and away from the wall, and ultimately dissipated by viscosity. The relative strength, or importance, of these processes as a function of wall-normal distance can be measured by the terms in the Reynolds stress budget equation (Pope 2000). Exhaustive analyses of the behavior of these terms for wall bounded flows can be found in Hoyas & Jiménez (2008); Richter (2015); Mizuno (2015, 2016); Aulery et al. (2016); Lee & Moser (2019), and other references therein. A general conclusion to be drawn from these works is that, similar to the production and velocity variances, the small scale contributions to the terms in the budget equation are universal in the near wall region, and differences in the profiles as a function of $Re_{\tau}$ can
be attributed to modulations by large scale motions. As a consequence, the terms in the budget equation produced by the model are seen to compare favorably with those produced by DNS in the region $y^+ \in [0, 300]$. The model’s premultiplied dissipation

$$\epsilon_{\alpha\alpha} = -2\nu \left< \frac{\partial u'_\alpha}{\partial x_k} \frac{\partial u'_\alpha}{\partial x_k} \right>$$

profiles shown in Figure 11, for example display strong agreement with the corresponding DNS profiles in the range $y^+ \in [0, 100]$. They show modest departures from the DNS profiles for $y^+ \in [100, 300]$ and completely diverge for $y^+ > 300$, as expected. The pressure strain and viscous, pressure, and turbulent transport terms (not displayed), all feature similar or better agreement than the dissipation profiles.

4. Conclusions

The reduced order computational model of wall bounded turbulence proposed here consists of a rectangular patch $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$ whose size scales in viscous units. Similar to the work of Jiménez & Pinelli (1999), the model seeks to isolate wall
turbulence from the influence of very large scale motions present both farther away from and down to the wall. The model is cheap relative to a full DNS because it does not include wavenumbers associated with length scales larger than its domain size in the wall parallel directions, and it only simulates turbulence out to a wall normal distance of $L_y^+ \approx 600$. The model was inspired by the multiscale formulation proposed by ? in which local “quasi-DNS” are coupled to a large eddy simulation. By assuming knowledge of the mean shear stress at the location $y = L_y$, the model accounts for the flux of mean momentum through the upper domain boundary in order to provide the correct “environment” for the near wall turbulence to evolve.

The primary modeling objective is to generate accurate near wall statistical quantities as a function of pressure gradient. For both negligible and mild favorable pressure gradients, the mean velocities and their derivatives, the small scale contribution to the turbulent kinetic energy, and the production, dissipation and transport terms in the Reynolds stress budget equation are found to be in close agreement with the statistics from available DNS. For boundary layers with an adverse pressure gradient, there will be significant evolution of the mean quantities in the streamwise direction. In this case, a more refined model formulation will be required to account for the effect of boundary layer growth on the near wall fluctuations, for instance based on asymptotic expansions as in ?.

One of the motivations for developing the near wall model is to generate data to inform wall models in the setting of LES and RANS computations relatively inexpensively. Although not reported here, the model is found to be useful in reproducing the correct first order wall-law that results from an asymptotic expansion of the spatially filtered Navier-Stokes equations in a small scale parameter $\epsilon$—taken to be the ratio of the viscous length scale $\nu/u_\tau$ to the numerical resolution, or filtering width $\Delta$, of a large eddy simulation in the wall-parallel directions, as inspired by lubrication theory. Future work will involve using the model to generate data necessary to implement the higher order, time-dependent wall law that results from the asymptotic expansion. In principle, however, the model can be used to calibrate a wide range of wall models resulting from alternative definitions of a large eddy simulation.

Lastly, although not pursued in this study, the reduced order dynamical system designed here could be used to economically perform numerical experiments on wall turbulence, as was done for example by [Jiménez & Pinelli (1999)] who tested dynamical theories of time-dependent near wall processes by carefully modifying the equations of motion to suppress various candidate processes. The approach pursued here could also be used to conduct relatively inexpensive computational studies of the interaction between near wall turbulence and more complicated physical processes such as heat transfer, chemical reactions, turbophoresis, or roughness effects. Potential application areas include turbomachinery, atmospheric modeling, combustion modeling, and oceanography.

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