Gravitomagnetic Effects on Collective Plasma Oscillations in Compact Stars

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Abstract

The effects of gravitomagnetic force on plasma oscillations are investigated using the kinetic theory of homogeneous electrically neutral plasma in the absence of external electric or magnetic field. The random phase assumption is employed neglecting the thermal motion of the electrons with respect to a fixed ion background. It is found that the gravitomagnetic force reduces the characteristic frequency of the plasma thus enhancing the refractive index of the medium. The estimates for the predicted effects are given for a typical white dwarf, pulsar, and neutron star.

1 Introduction

The outer crust of a compact star is formed of metal consisting almost exclusively of iron and other heavy elements. For a typical white dwarf the solid $^{56}\text{Fe}$ core is surrounded by $\text{Si}$, $\text{O}$, $\text{C}$, and $\text{He}$ ions and a gas of free electrons, whereas in neutron stars (pulsars), central densities are extremely high (up to $10^{12}\text{kg/cm}^3$) therefore the equation of state is relatively fixed. The outermost region of a neutron star is formed of a solid $^{56}\text{Fe}$ (density $\approx 10\text{kg/cm}^3$) crust forming a crystalline lattice, surrounded by degenerate electrons. The star is thus regarded as enveloped in a highly ionized degenerate solid state plasma consisting of densely packed ions and relatively free electrons.$^{[1]}$. Collectively a plasma behaves as an oscillating system having a characteristic (resonant) frequency called the plasma frequency which for a degenerate solid state plasma is well defined throughout the medium. In particular the refractive index of a plasma medium is determined by the plasma frequency, which in turn determines the transmission and reflection of radiation for the plasma.$^{[2]}$. On the other hand gravitational effects play a key role in all processes occurring in a
compact star, so that general relativistic effects are important for an adequate description of the phenomenon. Usually these effects are not directly observable but are manifest in an indirect way; for example via interaction with the magnetic field\[^3\], in the accretion of matter\[^4\], and other material and radiative processes occurring in vicinity of the star\[^5\]. Since in these situations gravitational field of the star is comparatively weak (as compared to totally collapsed objects), and magnitude of the angular velocity of the star lies well below the relativistic limit, the equations of the general theory of relativity can be expressed in an approximate form, adequate for a description of the gravitational field of the star. This approximation due to its formal analogy with classical electromagnetic theory is called the gravitoelectromagnetic (GEM) approximation. The linearized geodesic equation contains a velocity dependent gravitomagnetic \((GM)\) force whose origin lies in general relativistic frame dragging effects. It has been shown \[^6\] that due to the gravitomagnetic force a coupling between the frequency of a material oscillator, consisting of a point mass, and the angular frequency of a gravitational source establishes. The coupling is especially enhanced at the resonant frequency of the oscillator. It is therefore of theoretical as well as observational interest to study the possible effects of gravitomagnetic force on collective oscillations of a solid state plasma in compact star crustal regions where gravitational force is particularly effective.

The paper is organized as follows. In the next section we explain the GEM approximation to the general theory of relativity and develop a force law, analogous to classical electromagnetic theory, based on the geodesic equation. We then extend the formalism for a point mass oscillator, to include the collective behavior of the plasma in a GEM force field, using the kinetic theory for the plasma oscillations in section 3. The extension based on the random phase assumption leads to an analogous shift in the plasma frequency. Later in section 4 we investigate the effect of the frequency shift on refractive index of the plasma as a single radially oscillating fluid. In section 5 we summarize the main results and discuss the possibility of detection of the predicted shift. Throughout we employ the gravitational units where \(G = 1 = c\) unless mentioned otherwise.

## 2 The Gravitoelectromagnetic Approximation

In General Relativity the gravitational field of a massive object is described by the metric tensor \(g_{\alpha\beta}\). For sufficiently weak gravitating systems, such as the compact stars, a linearization the metric can be adequately assumed, where

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}
\]

and where \(\eta_{\alpha\beta} = \text{diag}(-1,-1,-1,1)\) is the Minkowski metric tensor and \(h_{\alpha\beta}\) is the perturbation to the metric such that \(h_{\alpha\beta} \ll 1\) and \(x^\alpha \equiv (x^t, x^0) = (r,t)\) are the position coordinates of test particle, with time \(t\) being the affine parameter. Further requiring motion to be slow and \(\Gamma^\alpha_{\beta\gamma}\), denoting the Christoffel symbol, the geodesic equation
\[ \frac{\alpha}{d^2 x^\alpha}{ds^2} + \Gamma_{\beta\gamma}^\beta \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0; \quad (2) \]

has the following analogue in the GEM approximation\[7,8\]

\[ \frac{d^2 r}{d t^2} = G + v \times H \quad (3) \]

where

\[ G = -\nabla \varphi, \quad H = \nabla \times 4a \quad (4) \]

and

\[ \varphi = -\iiint \frac{\rho}{r} dV, \quad a = \iiint \frac{\rho V}{r} dV \quad (5) \]

\( \rho \) being mass density and \( V \) is the volume. Remarkably the formal analogy of the above results with the classical electromagnetism extends to the gravitational field equations.

Since in the slow rotation approximation the deformations in mass distribution in the star due to rotation are negligible, the star can be regarded as a slowly rotating sphere of homogeneous mass density \( M \). Then the gravito-electric (GE) force acting on a unit mass in vicinity of the star is given by the Newtonian gravitational force

\[ G = -\frac{M}{r^2} \hat{r}, \quad (6) \]

and \( H \) is given by

\[ H = \frac{12}{5} MR^2 (\Omega r \frac{r}{r^5} - \frac{1}{3} \frac{\Omega}{r^3}) \quad (7) \]

where \( R \) is the radius and \( \Omega \) is the angular velocity of the gravitational source.

The second part of equation (3) has a non-Newtonian origin and is regarded to exhibit typically general relativistic effects, such as the Lense-Thirring dragging of frames. Within the GEM approximation the effect can be interpreted as ‘gravitomagnetic current’ induced in the vicinity of the gravitational source due to its rotation. It therefore plays an important role in testing Einstein’s theory of gravitation in the weak field and slow rotation approximation. Further it has been demonstrated\[9\] that the form of the GM potential does not depend on the choice of a particular frame or coordinate system used, thus making it physically significant for all material as well as radiative processes. However to measure the effects of this force on a given physical system high accuracy in experiments is required\[10\].
3 Equation of Motion for Collective Plasma Oscillations

A detailed kinetic theory of plasma oscillations in metals has been developed using the Fermi-Dirac statistics leading to the quantized theory of elementary excitations. However, the main features of collective plasma oscillations due to the effects of the gravitomagnetic field can be discussed within the classical treatment. We consider the phenomenon of radial electron oscillations confined to two dimensions, whereas the angular velocity vector of the star is inclined at an angle $\chi$ to the plane of oscillations. The components of acceleration due to gravitoelectric force $G$ to be of constant magnitude $g$ and the gravitomagnetic force on the $i$th electron has the magnitude $\dot{r}_i H \sin \chi$. The equations of motion for the radial oscillations of the $i$th electron can thus be written as

$$\ddot{r}_i = -\frac{4\pi e^2 i}{m} \sum_{j,k} \left( \frac{1}{k} \right) \exp\left[ i k (r_i - r_j) \right] - g + \dot{r}_i H \sin \chi. \quad (8)$$

where $e$ is the electronic charge and $m$ the electron mass. The prime over $k$ denotes a sum where $k = 0$ is excluded. Since acceleration due to gravitoelectric force is of a constant magnitude, its effect on the electron oscillations with respect to the fixed background of ions will be negligible. We therefore redefine $\ddot{r}_i$ as $\ddot{r}_i - g$ in the equation of motion (8). For all practical purposes electrons can be considered as point particles, therefore in an arbitrary region of unit volume the particle density is given by

$$\rho(r) = \sum_i \delta(r - r_i), \quad (9)$$

We decompose the particle density function into Fourier components as

$$\rho_k(r_i) = \int \rho(r) \exp[-ikr_i] dr = \sum_i \exp[-ikr_i], \quad (10)$$

such that

$$\rho(r) = \sum_{i,k} \exp[ik(r - r_i)] \quad (11)$$

Here it is understood that $i$ as a multiplier is the complex number denoting $\sqrt{-1}$ and not the script employed for the summation. Differentiating $\rho_k(r_i)$ with respect to time we obtain

$$\dot{\rho}_k(r_i) = -i \sum_i (k \dot{r}_i) \exp[-ikr_i], \quad (12)$$

and

$$\ddot{\rho}_k(r_i) = -i \sum_i [(k \ddot{r}_i)^2 + i k \dot{r}_i] \exp[-ikr_i]. \quad (13)$$
A substitution from (8) into (13) gives

$$\ddot{\rho}_k(r_i) = -\sum_i \{(k\dot{r}_i)^2 + ik\left(-\frac{4\pi e^2 i}{m}\sum_{i,k'} \frac{1}{k'} \rho_k \exp[i k' r_i] + \dot{r}_i \sin \chi \right) \} \exp[-ikr_i]$$

which can be simplified to give

$$\ddot{\rho}_k(r_i) = -\sum_i (k\dot{r}_i)^2 \exp[-ikr_i] - \frac{4\pi e^2}{m} \sum_{i,j,k'} \frac{k}{k'} \{ \exp[i(k' - k)r_i] \} \exp[-ik'r_j]$$

$$-i \sum_i k\dot{r}_i \sin \chi \exp[-ikr_i].$$

(14)

(15)

Here the second term on the right hand side can be split into two parts one for $k' = k$ and the other for $k' \neq k$:

$$\frac{4\pi ne^2}{m} \sum_i \exp[-ikr_i] + \frac{4\pi e^2}{m} \sum_{k\neq k'} \frac{k}{k'} \sum_i \exp[-ik'r_i] \sum_j \exp[i(k' - k)r_j].$$

(16)

The phase factors, randomly distributed in the complex plane, approximately cancel each other out. Therefore the sum of such factors is small, and hence the second term being a product of such summations is negligible. Thus random phase approximation gives in equation (15)

$$\ddot{\rho}_k(r_i) = -\sum_i (k\dot{r}_i)^2 \exp[-ikr_i] - \frac{4\pi ne^2}{m} \sum_i \exp[-ikr_i]$$

$$-i \sum_i k\dot{r}_i \exp[-ikr_i] \sin \chi. $$

(17)

Further since in a metal electrons are strongly bound to the ions, their random thermal motion is negligible, therefore the contributions coming from the first term containing $k^2$ are also negligible. Using equations (10) and (12) in the second and third term of equation (17) we obtain

$$\ddot{\rho}_k(r_i) = -\frac{4\pi ne^2}{m} \rho_k + \ddot{\rho}_k H \sin \chi.$$

(18)

4 Plasma Frequency Shift and Refractive Index

To determine the resonant frequency for the Fourier component for electron density function $\rho_k(r_i)$ we assume real functional dependence on time $t$ in the form of the solution $A_k \cos \omega t$ to equation (18), where $A_k$ is the amplitude of the density fluctuation and $\omega$ is applied external frequency. The substitution leads to the following condition for resonance
\[ \omega_p^2 - \omega^2 - \omega H \sin \chi = 0. \]  \hspace{1cm} (19)\]

Since \( H \ll \omega_p \), solving (19) with \( \omega \) unknown and neglecting terms involving squares and higher powers of \( H/\omega_p \), we obtain for the plasma frequency
\[ \omega = \omega_p - \frac{H}{2} \sin \chi. \]  \hspace{1cm} (20)\]

Clearly the shift is dependent upon the rotational frequency and mass of the star, and also upon the angle of inclination \( \chi \). It follows from expression (7) that at the surface of the star \( r = R \), the gravitomagnetic force has magnitude given by
\[ H \equiv |H| \simeq \mu \sqrt{1 + 3 \cos^2 \chi}, \]  \hspace{1cm} (21)\]
where \( \mu = (4GM/5Rc^2)\Omega \), \( \Omega \) being the magnitude of the angular velocity vector. Substituting from expression (21) into (20) we obtain an expression relating plasma frequency \( \omega_p \) to the angle \( \chi \):
\[ \frac{\omega}{\omega_p} \simeq 1 - \frac{\mu \sin \chi}{2\omega_p} \sqrt{1 - 3 \cos^2 \chi}, \quad \mu \ll \omega_p. \]  \hspace{1cm} (22)\]

Here the requirements \( H \ll \omega_p \) and \( \mu \ll \omega_p \) are generally valid for typical compact stellar sources with \( \mu \) ranging from \( 0.1687 \times 10^{-3} \text{Hz} \) (for a typical white dwarf of mass \( 1M_\odot = 1.989 \times 10^{30} \text{kg} \), radius \( 7 \times 10^6 \text{m} \) and angular frequency \( 1 \text{Hz} \)) to 236.2932 Hz (for a typical neutron star of mass \( 2M_\odot \), radius \( 7.14 \times 10^6 \text{m} \), and angular frequency \( 1 \text{kHz} \)) with corresponding plasma frequency ranging approximately \( 5.65 \times 10^6 \text{Hz} \) to \( 5.65 \times 10^8 \text{Hz} \) or above. Denoting \( \tilde{\omega}_p \equiv \omega_p - H/2 \), where \( H \) is given by (21), we have in terms of plasma frequency we have the following expression for the index of refraction \( \varepsilon \) of the medium
\[ \varepsilon(\chi) = \sqrt{1 - \left( \frac{\tilde{\omega}_p}{\omega_a} \right)^2} = \sqrt{1 - \left( \frac{\omega_p}{\omega_a} - \frac{\mu \sin \chi}{2\omega_a} \sqrt{1 + 3 \cos^2 \chi} \right)^2}, \]  \hspace{1cm} (23)\]
where \( \omega_a \) is the angular frequency of waves (e.g. electromagnetic) propagating through the plasma; whose transmission or reflection, for a given compact source, is determined by the usual conditions
\[ \omega_a \leq \tilde{\omega}_p, \text{ reflection}; \]  \hspace{1cm} (24)\]
\[ \omega_a > \tilde{\omega}_p, \text{ transmission}. \]  \hspace{1cm} (25)\]

The shift in plasma frequency in the equatorial plane (\( \chi = \pi/2 \)) is \( \omega_p - \mu/2 \) whereas in the plane orthogonal to it (\( \chi = 0 \)) the shift is \( \omega_p - \mu \). Denoting the frequency shift in the equatorial plane by \( \tilde{\omega}_{p\parallel} \) and by \( \tilde{\omega}_{p\perp} \) for the orthogonal plane, we find that the maximum difference in the frequencies transmitted through the plasma:
\[ |\tilde{\omega}_{p\perp} - \tilde{\omega}_{p\parallel}|_{\text{max}} = \frac{\mu}{2}. \]  \hspace{1cm} (26)\]
In figures (1) and (2) we give plots for the typical cases of neutron star, pulsar, and white dwarf, between the plasma frequency shift $\omega/\omega_p$ and the angle $\chi$ (using expression (22)) and between the plasma refractive index $\varepsilon$ as a function of the angle $\chi$ (using expression (23)) respectively.

5 Conclusions

In view of the above investigations we find that the ‘gravitomagnetic oscillations’ have a coupling to the plasma oscillations which become especially effective at the plasma frequency. This is so even if the plasma co-rotates with the star mainly because of the fact, that the gravitomagnetic force as well as the electron oscillations are along the radial direction, as is evident from expression (3) and (8). It is clear from figure (1) and (2) that the plasma frequency the gravitomagnetic force produces a reduction in the frequency, thus enhancing the refractive index of the plasma. Further the dependence of the shift on component of the angular frequency vector of the gravitational source to the plane of oscillations of the plasma gives a difference in the refractive index, which results in changing the allowed range of frequencies transmitted through the plasma, by a maximum amount $\mu/2$. For a typical white dwarf $\mu$ is approximately $0.1687 \times 10^{-3} \text{Hz}$, for a typical pulsar $1.6540 \text{Hz}$ and for a neutron star $236.2932 \text{Hz}$. The predicted difference in the frequency of radiation emitted by a compact star though still very small, may possibly be observed, for example in radiation spectra of various compact sources, as the observations become more refined.

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**Figure Captions:**

**Figure 1:** Plots for the shift $\omega/\omega_p$ in the plasma frequency of a compact star atmosphere as a function of the angle of inclination $\chi$ of the plane of observation to the angular frequency vector for the case of a white dwarf ($M = 1M_\odot = 1.989 \times 10^{30} kg$, $R = 7 \times 10^6 m$, $\Omega = 1Hz$, $\omega_p = 5.65 \times 10^2 Hz$), a pulsar ($M = 1.4M_\odot$, $R = 3 \times 10^4 m$, $\Omega = 30 Hz$, $\omega_p = 5.65 \times 10^4 Hz$), and a neutron star ($M = 2M_\odot$, $R = 1 \times 10^4 m$, $\Omega = 1kHz$, $\omega_p = 5.65 \times 10^6 Hz$).

**Figure 2:** Plots for the refractive index $\varepsilon$ of a compact star atmosphere as a function of the angle of inclination $\chi$ of the plane of observation to the angular frequency vector for the case of a white dwarf ($M = 1M_\odot = 1.989 \times 10^{30} kg$, $R = 7 \times 10^6 m$, $\Omega = 1Hz$, $\omega_p = 5.65 \times 10^2 Hz$), a pulsar ($M = 1.4M_\odot$, $R = 3 \times 10^4 m$, $\Omega = 30Hz$, $\omega_p = 5.65 \times 10^4 Hz$), and a neutron star ($M = 2M_\odot$, $R = 1 \times 10^4 m$, $\Omega = 1kHz$, $\omega_p = 5.65 \times 10^6 Hz$) for the propagation of an electromagnetic signal of angular frequency $10^3 Hz$, $10^5 Hz$, and $10^7 Hz$ respectively.
Figure 1:
