Physical model of a heterogeneous medium as applied to nuclear medicine

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Abstract. In nuclear medicine, a patient is administered a radiolabeled pharmaceutical which is distributed in various organs of a patient body. The patient body with distributed emitting radiopharmaceutical is a heterogeneous medium. In this work, the problem of reconstructing the image of the human brain using single-photon emission tomography (SPET) is considered. To obtain the image, the statistical MAP-ENT algorithm is applied. To develop this algorithm, it is necessary to formulate a priori information about the object to be reconstructed. The distribution of radiopharmaceuticals in the patient’s body is a spatially inhomogeneous distribution of non-interacting radiopharmaceutical particles. This distribution is described using the Boltzmann model of a non-equilibrium isolated system. For such a system, one can write the entropy functional, which determines the a priori model. The algorithm was tested using the Hoffman mathematical model, which simulates the distribution of a radiopharmaceutical in the human brain. In numerical experiments, the properties of the MAP-ENT algorithm were studied under conditions close to clinical practice. The results have shown the prospects of the MAP-ENT algorithm.

1. Introduction
In nuclear medicine, namely single-photon emission computed tomography (SPECT) and positron emission tomography (PET), a patient is administered a radiolabeled pharmaceutical which is distributed in various organs of the patient body. The photons emitted by the radiopharmaceutical are collected by detectors located around the patient. Once the data have been collected, the reconstruction algorithm is applied to provide an image of radionuclide distribution within the patient body. Image reconstruction belongs to the class of inverse ill-posed problems of mathematical physics. The method of solution for this class of problems was developed by A.Tikhonov and was called as regularization [1]. This method was developed in deterministic form. At the moment, the Bayesian approach Maximum a Posteriori (MAP) is used for solving inverse ill-posed image reconstruction problems with stochastic data [2]. The MAP approach is based on statistical regularization which introduces a priori information in the probabilistic form. Within a framework of the Bayesian approach one needs to develop an a priori model. When solving reconstruction problems, one can propose the physical model of the object to be reconstructed and determine a priori information by using this model. We consider a cumulative model: it is assumed that the radiopharmaceutical has already accumulated in various organs and that this distribution remains constant during the imaging procedure. Radiopharmaceutical distribution in a patient body is a heterogeneous medium. In the present paper, we discuss the prior model to describe this medium in context of its applications in nuclear medicine.
2. Theory

Radiopharmaceutical distribution in a patient body is the spatially non-uniform distribution of non-interacting radiopharmaceutical particles. This distribution can be described using the Boltzmann model of non-equilibrium isolated system. We consider the distribution of particles in phase space. The entropy functional for such system can be presented as:

\[ S = -\int n(p, q) \ln n(p, q) dpdq \]  

(1)

where \( n(p, q) \) is the particle density in the phase space, \( p \) and \( q \) are pulses and coordinates of the particles. We replace the variables:

\[ dpdq = dq2\pi(2m)^{3/2}e^{1/2}d\varepsilon \]  

(2)

where \( \varepsilon \) is particle energy. Entropy functional (1) can be rewritten as:

\[ S = -\sum_{j} \left[ n(q_j, \varepsilon) \ln n(q_j, \varepsilon) 2\pi(2m)^{3/2}e^{1/2}d\varepsilon + \text{const} \right] \]  

(3)

We assume that an equilibrium energy distribution is established in each voxel \( q_j \). In the equilibrium state, particles have mean energy \( \varepsilon_0 \). Eq.(3) can be written:

\[ S = -\alpha \sum_{j} n(q_j, \varepsilon_0) \ln n(q_j, \varepsilon_0) \varepsilon_0^{1/2} \Delta \varepsilon + \text{const} \]  

(4)

\( \Delta \varepsilon \) is the energy ‘width’ where the particle spends almost all of its time. Taking into account the normalization

\[ \int \rho(\varepsilon)d\varepsilon = \rho(\varepsilon_0)\Delta \varepsilon = 1 \]  

(5)

one obtains:

\[ S = -\frac{\alpha}{\rho(\varepsilon_0)} \sum_{j} n(q_j) \ln n(q_j) + \text{const} , \]  

(6)

where \( \rho(\varepsilon_0) \) is the probability density of the energy \( \varepsilon_0 \). The mean density of emitting gamma photons \( \bar{f}_j \) is assumed to be proportional to the radiopharmaceutical concentration:

\[ \bar{f}_j \propto n(q_j) \]  

(7)

The entropy-based prior (6) is rewritten as:

\[ \ln P(\bar{f}_j) = -\beta \sum_{j} \bar{f}_j \ln \bar{f}_j \]  

(8)

The Bayesian MAP estimation (in logarithmic form) gives the solution of the reconstruction problem:

\[ \tilde{f} = \max_{f>0} \{ \ln P(\bar{f}) + \ln P(g | A\bar{f}) \} \]  

(9)
$P(\vec{f})$ is a priori probability density function and $P(g | A\vec{f})$ is the likelihood distribution of the observed data. The likelihood term $P(g | A\vec{f})$ is determined by the physics of data acquisition process. For applications in nuclear medicine, the likelihood distribution in logarithmic form is defined by the Poisson law:

$$\ln P(g | A\vec{f}) = g \ln A\vec{f} - A\vec{f} - \ln g!$$

(10)

Substitution of the entropy prior (8) and condition probability (10) in (9) gives the Maximum a Posteriori estimation with entropy-based prior (MAP-ENT):

$$\tilde{\vec{f}} = \arg \max_{\vec{f} > 0} \left\{ -\beta \sum_j f_j \ln f_j + \sum_i g_i \sum_j A_{ij} f_j - \sum_j A_{ij} f_j - \ln g_j! \right\}$$

(11)

Eq.(11) gives a solution of the reconstruction problem.

3. Numerical simulations

Numerical simulations using MAP-ENT algorithm were performed. To evaluate the method of iterative statistical regularization, the Hoffman brain phantom was used in this work. The 3D Hoffman phantom is shown in Fig.1. The image was sampled in 100*100*50 voxels. The two-dimensional (2D) projection data were generated for this phantom by using 120 views with 100*50 bins per projection. Data statistics was modeled as close as possible to the real clinical data. The method developed in [3] was used to generate realistic projection data taking into account the effect of collimator-detector response and Poisson data statistics. Fig.2a shows an example of the generated projections from one selected view.

Figure 1. The 3D Hoffman phantom. Human brain model developed for nuclear medicine applications.

Figure 2. An example of projection data, generated from the Hoffman phantom.
The MAP-ENT algorithm (11) was applied to image reconstruction using the generated projection data. The results of reconstructions are shown in Fig.3. The ‘exact’ and reconstructed images of the Hoffman brain phantom are shown in selected slices. Comparison of the ‘exact’ images of the Hoffman phantom and images reconstructed by the MAP-ENT algorithm, one can see that some fine structures are lost on the reconstructed images.

Figure 3. First row: the selected slices of the Hoffman phantom are shown; second row: images in the corresponding slices; third row: reconstructed images in the same slices.

4. Conclusion
A model of a heterogeneous environment, such as the human body, is developed for applications in nuclear medicine. Numerical studies were performed using the Hoffman phantom, imitating the structure of the human brain. The MAP-END algorithm has been applied to reconstruct this complex structure. The results obtained showed the prospects of the method and the need for its further development.

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