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Some examples of algebraic surfaces with canonical map of degree 20

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Dedicated to Margarida Mendes Lopes on the occasion of her sixty-fifth birthday

Abstract. In this note, we construct two minimal surfaces of general type with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K^2 = 20, 24$ such that their canonical map is of degree 20. In one of these surfaces, the canonical linear system has a non-trivial fixed part. These surfaces, to our knowledge, are the first examples of minimal surfaces of general type with canonical map of degree 20.

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1. Introduction

If $X$ is a minimal smooth complex projective surface, we denote by $\varphi_{[K_X]} : X \dasharrow \mathbb{P}^{p_g(X)-1}$ the canonical map of $X$, where $K_X$ is the canonical divisor of $X$ and $p_g(X) = \dim H^0(X, K_X)$ is the geometric genus. It is interesting to know which positive integers $d$ occur as the degree of such canonical maps for surfaces of general type. This problem is motivated by the work of A. Beauville [1]. One knows that, for surfaces of general type, the degree $d$ of the canonical map is at most 36 [9, Proposition 5.7]. While surfaces with $d = 1, 2, 3, \ldots, 8$ are easy to construct, only few surfaces with $d > 8$ have been known so far. The first example was found by U. Persson [9] in 1977; in this example, the canonical map has degree 16. Then, a surface with $d = 9$ was constructed by S. L. Tan [14] in 1992. In the last decade, some surfaces with $d = 12, 16, 24, 27, 32, 36$ were constructed by C. Rito [10–13], C. Gleissner, R. Pignatelli and C. Rito [4], Ching-Jui Lai and Sai-Kee Yeung [5], and the author [2]. In this paper, we present a way to construct surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers of the Del Pezzo surface $Y_4$ of degree 5.

Throughout this paper all surfaces are projective algebraic over the complex numbers. The linear equivalence of divisors is denoted by $\equiv$. We call a surface $X$ no non-trivial 2-torsion if the
only 2-torsion in \( \text{Pic}(X) \) is \( \mathcal{O}_X \). A character \( \chi \) of the group \( \mathbb{Z}_2^4 \) is a homomorphism from \( \mathbb{Z}_2^4 \) to \( \mathbb{C}^\times \), the multiplicative group of the non-zero complex numbers. We also use the following notations for Del Pezzo surfaces of degree 5:

**Notation 1.** We denote by \( Y_4 \) the blow-up of \( \mathbb{P}^2 \) at four points in general position \( P_1, P_2, P_3, P_4 \). Let us denote by \( l \) the pull-back of a general line in \( \mathbb{P}^2 \), by \( e_1, e_2, e_3, e_4 \) the exceptional divisors corresponding to \( P_1, P_2, P_3, P_4 \), respectively, by \( f_1, f_2, f_3, f_4 \) the strict transforms of a general line through \( P_1, P_2, P_3, P_4 \), respectively, and by \( h_{ij} \) the strict transforms of the line \( P_i P_j \), for all \( i \neq j \) in \( \{1, 2, 3, 4\} \), respectively. The anti-canonical class

\[-K_{Y_4} \equiv f_1 + f_2 + f_3 - e_4 \equiv f_1 + f_2 + f_4 - e_3 \equiv f_1 + f_3 + f_4 - e_2 \equiv f_2 + f_3 + f_4 - e_1 \]

is very ample and the linear system \( |-K_{Y_4}| \) embeds \( Y_4 \) as a smooth Del Pezzo surface of degree 5 in \( \mathbb{P}^5 \).

The construction of abelian covers was studied by R. Pardini in [7]. For details about the building data of abelian covers and their notations, we refer the reader to Section 1 and Section 2 of R. Pardini’s work ([7]). For the sake of completeness, we recall some facts on \( \mathbb{Z}_2^4 \)-covers, in a form which is convenient for our later constructions. We will denote by \( \chi_{j_1 j_2 j_3 j_4} \) the character of \( \mathbb{Z}_2^4 \) defined by

\[
\chi_{j_1 j_2 j_3 j_4}(a_1, a_2, a_3, a_4) := e^{(\pi a_1 j_1)\sqrt{-1}} e^{(\pi a_2 j_2)\sqrt{-1}} e^{(\pi a_3 j_3)\sqrt{-1}} e^{(\pi a_4 j_4)\sqrt{-1}}
\]

for all \( j_1, j_2, j_3, j_4, a_1, a_2, a_3, a_4 \in \mathbb{Z}_2 \). A \( \mathbb{Z}_2^4 \)-cover \( X \to Y \) can be determined by a collection of non-trivial divisors \( L_{\chi} \) labelled by characters of \( \mathbb{Z}_2^4 \) and effective divisors \( D_{\sigma} \) labelled by elements of \( \mathbb{Z}_2^4 \) of the surface \( Y \). More precisely, from [7, Theorem 2.1] we can define \( \mathbb{Z}_2^4 \)-covers as follows:

**Proposition 2.** Given \( Y \) a smooth projective surface with no non-trivial 2-torsion, let \( L_{\chi} \) be divisors of \( Y \) such that \( L_{\chi} \neq \mathcal{O}_Y \) for all non-trivial characters \( \chi \) of \( \mathbb{Z}_2^4 \) and let \( D_{\sigma} \) be effective divisors of \( Y \) for all \( \sigma \in \mathbb{Z}_2^4 \setminus \{0, 0, 0, 0\} \) such that the total branch divisor \( B := \sum_{\sigma \neq 0} D_{\sigma} \) is reduced. Then \( \{L_{\chi}, D_{\sigma}\}_{\chi, \sigma} \) is the building data of a \( \mathbb{Z}_2^4 \)-cover \( f : X \to Y \) if and only if

\[
2L_{\chi} = \sum_{\chi(\sigma) = -1} D_{\sigma}
\]

for all non-trivial characters \( \chi \) of \( \mathbb{Z}_2^4 \).

The following theorem is a result of this note:

**Theorem 3.** Let \( f : X \to Y_4 \) be a \( \mathbb{Z}_2^4 \)-cover with the building data \( \{L_{\chi}, D_{\sigma}\}_{\chi, \sigma} \) such that the following hold:

(a) Each branch component \( D_{\sigma} \) is smooth, the total branch locus \( B \) is a simple normal crossings divisor and no more than two of these divisors \( D_{\sigma} \) go through the same point;
(b) \( D_{0100} + D_{0101} + D_{0111} + D_{1000} + D_{1001} + D_{1010} + D_{1011} + D_{1100} + D_{1101} + D_{1110} + D_{1111} \in |-K_{Y_4}| \);
(c) \( h^0(K_{Y_4} + L_{\chi}) = 0 \) for all \( \chi \in \{\chi_{1000}, \chi_{0100}, \chi_{1100}\} \);
(d) The divisor \( D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \) is nef and big.

Then \( X \) is a minimal surface of general type with canonical map of degree 20 satisfying the following:

\[
p_g(X) = 3, \quad K_X^2 = 4 (D_{0001} + D_{0010} + D_{0011} - K_{Y_4})^2.
\]

Moreover, the reduced divisor supported on \( f^*(D_{0001} + D_{0010} + D_{0011}) \) is the fixed part of the canonical system \( |K_X| \).
Let us summarize the proof of Theorem 3. Assumptions (a), (b) and (d) show that the surface $X$ is a minimal surface of general type. Assumption (c) implies that the following diagram commutes (see Remark 6 for the proof):

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi|_{K_X}} & Z^4 \\
\downarrow & & \downarrow \varphi|_{K_Z} \\
Z & \xrightarrow{g} & Y_4
\end{array}
$$

In the above diagram, the intermediate surface $Z := X/\Gamma$ is the quotient surface of $X$, where $\Gamma := \langle (0, 0, 0, 1), (0, 0, 1, 0) \rangle$ is the subgroup of $Z^4$. The surface $Z$ is the bidouble cover of $Y_4$ ramified on $(D_{0100} + D_{0101} + D_{0110} + D_{0111}) + (D_{1000} + D_{1001} + D_{1010} + D_{1011}) + (D_{1100} + D_{1101} + D_{1110} + D_{1111})$.

Assumption (b) shows that the canonical map of $Z$ is of degree 5 (see Remark 6 for the proof). Therefore, the canonical map of $X$ is of degree 20. As application of Theorem 3, we construct two surfaces with $d = 20$ described as follows:

**Theorem 4.** There exist minimal surfaces of general type $X$ satisfying the following

| $d$ | $K_X^2$ | $p_e(X)$ | $q(X)$ | $|K_X|$ |
|-----|---------|-----------|--------|--------|
| 20  | 20      | 3         | 0      | base point free |
| 20  | 24      | 3         | 0      | has a non-trivial fixed part |

2. $Z^4_2$-coverings

For the convenience of the reader, we leave here the relations (1) of the building data of $Z^4_2$-covers:

By [7, Theorem 3.1] if each branch component $D_0$ is smooth and the total branch locus $B$ is a simple normal crossings divisor, the surface $X$ is smooth.

Also from [7, Lemma 4.2, Proposition 4.2] we have:
By (8), the linear system Assumption (d). Since the divisor $2K_X$ is generated by $\{L_X, D_\sigma\}_{X, \sigma}$, the surface $X$ satisfies the following:

$$2K_X \equiv f^* \left(2K_Y + \sum_{\sigma \neq \emptyset} D_\sigma \right);$$

$$f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{X \neq \emptyset} L_X^{-1};$$

$$H^0(X, K_X) = H^0(Y, K_Y) \oplus \bigoplus_{X \neq \emptyset} H^0(Y, K_Y + L_X);$$

$$K_X^2 = 4 \left(2K_Y + \sum_{\sigma \neq \emptyset} D_\sigma \right)^2;$$

$$p_g(X) = p_g(Y) + \sum_{X \neq \emptyset} h^{0}(L_X + K_Y);$$

$$\chi(\mathcal{O}_X) = 16\chi(\mathcal{O}_Y) + \sum_{X \neq \emptyset} \frac{1}{2} L_X \left(L_X + K_Y \right).$$

Moreover, the canonical linear system $|K_X|$ is generated by

$$f^* \left(|K_Y + L_X|\right) + \sum_{X \neq \emptyset} R_\sigma, \ \forall \chi \in J$$

where $J := \{\chi' : |K_Y + L_{\chi'}| \neq \emptyset\}$ and $R_\sigma$ is the reduced divisor supported on $f^* (D_\sigma)$.

For the proof of the last statement of Proposition 5, we refer the reader to [4, p. 3].

3. Surfaces with $d = 20$ as $\mathbb{Z}_2^4$-covers

3.1. Proof of Theorem 3

The surface $X$ is smooth because each branch component $D_\sigma$ is smooth, the total branch locus $B$ is a normal crossings divisor and no more than two of these divisors $D_\sigma$ go through the same point. Moreover, by Proposition 5, the surface $X$ satisfies the following:

$$2K_X \equiv f^* \left(2K_Y + \sum_{\sigma \neq \emptyset} D_\sigma \right) \equiv f^* \left(D_{0001} + D_{0010} + D_{0011} - K_Y \right).$$

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [6, Section 2]). We remark that the divisor $D_{0001} + D_{0010} + D_{0011} - K_Y$ is nef and big by Assumption (d). Since the divisor $2K_X$ is the pull-back of a nef and big divisor, the canonical divisor $K_X$ is nef and big. Thus, the surface $X$ is of general type and minimal. Furthermore, from Proposition 5, the surface $X$ possesses the following invariants:

$$p_g(X) = 3, \ \ K_X^2 = 4 \left(D_{0001} + D_{0010} + D_{0011} - K_Y \right)^2.$$

We show that the canonical map $\varphi_{|K_X|}$ has degree 20. By Assumptions (b) and (c), we have

$$L_{1000} + K_Y \equiv L_{0100} + K_Y \equiv L_{1100} + K_Y \equiv \mathcal{O}_{Y_4},$$

$$h^{0}(L_X + K_Y) = 0, \ \forall \chi \in \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}.$$
where $D_{\sigma}$ are the reduced divisors supported $f^* (D_{\sigma})$, for all $\sigma$. Because the divisors $D_{0001}, D_{0010}, D_{0011}$ are common components of the three above divisors, these divisors $D_{0001}, D_{0010}, D_{0011}$ are fixed components of $|K_X|$.

On the other hand, by Assumption (a) the three divisors $D_{0100} + D_{0101} + D_{0110} + D_{0111}, D_{1000} + D_{1010} + D_{1011}, D_{1100} + D_{1101} + D_{1110} + D_{1111}$ have no common intersection. So the linear system $|M|$ is base point free, where $M := D_{0100} + D_{0101} + D_{0110} + D_{0111}$. This together with $M^2 = 4(3l - e_1 - e_2 - e_3 - e_4)^2 = 20 > 0$ implies that the linear system $|K_X|$ is not composed with a pencil. Thus, the canonical image is $\mathbb{P}^2$, the canonical map is of degree 20, and the divisor $D_{0001} + D_{0010} + D_{0011}$ is the fixed part of $|K_X|$.

**Remark 6.** The canonical map $\varphi_{|K_X|}$ of $X$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi_{|K_Z|}$ of $Z$. Moreover, the canonical map of $Z$ is of degree 5.

In fact, by (4), we have the following decomposition:

$$H^0(X, K_X) = H^0(Y_4, K_{Y_4}) \oplus \bigoplus_{\chi \neq \chi_{0000}} H^0(Y_4, K_{Y_4} + L_{\chi}).$$

The group $\Gamma := \langle (0,0,0,1), (0,0,1,0) \rangle$ is the subgroup of $\mathbb{Z}_2^4$. Let $\Gamma^\perp$ denote the kernel of the restriction map $(\mathbb{Z}_2^4)^* \to \Gamma^*$, where $\Gamma^*$ is the character group of $\Gamma$. We have $\Gamma^\perp = \langle \chi_{1000}, \chi_{0100}, \chi_{1100} \rangle$. The subgroup $\Gamma$ acts trivially on $H^0(X, K_X)$ since $h^0(L_{\chi} + K_{Y_4}) = 0$ for all $\chi \notin \Gamma^\perp$ by Assumption (c). So the canonical map $\varphi_{|K_Z|}$ is the composition of the quotient map $X \to Z := X/\Gamma$ with the canonical map $\varphi_{|K_Z|}$ of $Z$ (see e.g. [8, Example 2.1]).

The intermediate surface $Z$ is the bidouble cover of $Y_4$ with the building data $\{D_1, D_2, D_3, L_1, L_2, L_3\}$ determined as follows:

\[
\begin{align*}
D_1 &:= D_{0100} + D_{0101} + D_{0110} + D_{0111} \equiv -K_{Y_4}, \\
D_2 &:= D_{1000} + D_{1001} + D_{1010} + D_{1011} \equiv -K_{Y_4}, \\
D_3 &:= D_{1100} + D_{1101} + D_{1110} + D_{1111} \equiv -K_{Y_4}, \\
L_1 &:= L_{1000} \equiv -K_{Y_4}, \\
L_2 &:= L_{0100} \equiv -K_{Y_4}, \\
L_3 &:= L_{1100} \equiv -K_{Y_4}.
\end{align*}
\]

Assumption (a) shows that the singularities of $Z$ are nodes and the canonical map of $Z$ is of degree $(3l - e_1 - e_2 - e_3 - e_4)^2 = 5$.

### 3.2. Constructions of the surfaces in Theorem 4

#### 3.2.1. A surface with $d = 20$, $p_g = 3$, $q = 0$, $K^2 = 20$

In this section, we construct the surface described in the first row of Theorem 4. Let $Y_4$ be a Del Pezzo surface of degree 5 (see Notation 1). We consider the following smooth divisors of $Y_4$:

\[
\begin{align*}
D_{0101} &:= h_{14} \\
D_{1001} &:= f_{11} + e_2 \\
D_{1101} &:= h_{13} \\
D_{0110} &:= f_{31} + e_1 \\
D_{1010} &:= h_{23} \\
D_{1110} &:= h_{34} \\
D_{0111} &:= h_{12} \\
D_{1011} &:= h_{24} \\
D_{1111} &:= f_{21} + e_3
\end{align*}
\]
and $D_\sigma = 0$ for the other $\sigma$, where $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$ such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

$$
L_{0001} := 2f_1 + f_2 + e_4 \\
L_{0010} := 2f_2 + f_3 + e_4 \\
L_{0100} := f_1 + f_2 + f_3 + e_4 \\
L_{1000} := f_1 + f_2 + f_3 - e_4 \\
L_{0011} := f_1 + 2f_3 - e_4 \\
L_{0101} := f_3 + f_4 \\
L_{0110} := h_{12} + h_{34} \\
L_{1001} := f_1 + f_2 \\
L_{1010} := f_1 + f_3 \\
L_{1011} := f_2 + f_4 \\
L_{1100} := f_1 + f_2 + f_3 - e_4 \\
L_{1101} := f_2 + f_3 \\
L_{1110} := f_1 + f_4 \\
L_{1111} := h_{12} + h_{34}.
$$

These divisors $D_\sigma, L_\chi$ satisfy the following relations:

$$
2L_{0001} = D_{0001} + D_{0011} + D_{1001} + D_{1011} + D_{1111} = 4f_1 + 2f_2 - 2e_4 \\
2L_{0010} = D_{0010} + D_{0100} + D_{0110} + D_{1010} + D_{1110} = 4f_2 + 2f_3 - 2e_4 \\
2L_{0100} = D_{0100} + D_{0110} + D_{1001} + D_{1101} + D_{1111} = 2f_1 + 2f_3 + 2f_4 - 2e_4 \\
2L_{1000} = D_{1000} + D_{0100} + D_{0110} + D_{1010} + D_{1110} = 2f_1 + 2f_2 + 2f_4 - 2e_4 \\
2L_{0011} = D_{0011} + D_{0101} + D_{0111} + D_{1011} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{0111} = D_{0111} + D_{1011} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{1011} = D_{1011} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{1101} = D_{1101} + D_{1111} = 2h_{12} + 2h_{34} \\
2L_{1111} = D_{1111} = 2h_{12} + 2h_{34}.
$$

Thus by Proposition 2, the divisors $D_\sigma, L_\chi$ define a $\mathbb{Z}_2^4$-cover $g : X \to Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfils the hypotheses of Theorem 3. In fact, we have that

$$
D_{0010} + D_{0101} + D_{0110} + D_{1011} = h_{14} + f_{31} + e_1 + h_{12} \equiv 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1000} + D_{1001} + D_{1010} + D_{1011} = f_{11} + e_2 + h_{23} + h_{34} \equiv 3l - e_1 - e_2 - e_3 - e_4 \\
D_{1100} + D_{1101} + D_{1110} + D_{1111} = h_{13} + h_{34} + f_{21} + e_3 \equiv 3l - e_1 - e_2 - e_3 - e_4,
$$

$h^0(K_{Y_4} + L_\chi) = 0$ for all $\chi \notin \{\chi_{1000}, \chi_{0100}, \chi_{1100}\}$, and the divisor $D_{0001} + D_{0010} + D_{0011} - K_{Y_4} \equiv 3l - e_1 - e_2 - e_3 - e_4$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface $X$ is a minimal surface of general type and possesses the following invariants:

$$
k^2_X = 20, p_g(X) = 3, \chi(\mathcal{O}_X) = 4, q(X) = 0.
$$

Moreover, the canonical map $\varphi_{|K_X|}$ is of degree 20 and the linear system $|K_X|$ is base point free.

**Remark 7.** The surface $X$ has four pencils of genus 9 corresponding to the fibres $f_1, f_2, f_3, f_4$.

In the above construction, for each choice of $f_{11} \in |f_1|$, $f_{21} \in |f_2|$ and $f_{31} \in |f_3|$, we obtain a natural deformation of the surface $X$ (we refer [7, Definition 5.1] for the definition of natural deformations of an abelian cover). It is worth pointing out that a natural deformation of an abelian cover $X \to Y$ is a deformation of the map $X \to Y$ by [7, Proposition 5.1].

C. R. Mathématique — 2021, 359, n° 9, 1145-1153
**Remark 8.** The surface $X$ admits natural deformations. Moreover, all the natural deformations of $X$ are Galois.

In fact, by [7, Definition 5.1] the natural deformations of the $\mathbb{Z}_2^4$-cover $g : X \to Y_4$ are parametrized by the direct sum of the vector spaces

$$\bigoplus_{\sigma \neq 0} H^0(Y_4, D_\sigma) \bigoplus_{\not\exists L^{0000}} H^0(Y_4, D_\sigma - L_X).$$

Moreover, all the natural deformations of $X$ are Galois if the second summand $\bigoplus_{\not\exists L^{0000}} H^0(Y_4, D_\sigma - L_X)$ is zero (see [3, Definition 3.2]). We have that

$$H^0(Y_4, D_{0110}) = H^0(Y_4, f_{31}) \cong \mathbb{C}^2$$
$$H^0(Y_4, D_{1001}) = H^0(Y_4, f_{11}) \cong \mathbb{C}^2$$
$$H^0(Y_4, D_{1111}) = H^0(Y_4, f_{21}) \cong \mathbb{C}^2$$

and $H^0(Y_4, D_\sigma) \cong \mathbb{C}$ for the other non-trivial $D_\sigma$. So the family of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, all natural deformations of $X$ are Galois since $\bigoplus_{\not\exists L^{0000}} H^0(Y_4, D_\sigma - L_X) = 0$.

### 3.2.2. A surface with $d = 20$, $p_g = 3$, $q = 0$, $K^2 = 24$

In this section, we construct the surface described in the second row of Theorem 4. We consider the following smooth divisors of a del Pezzo surface $Y_4$ of degree 5:

- $D_{0011} := e_4$
- $D_{0101} := h_{14}$
- $D_{1000} := e_2$
- $D_{1100} := h_{34}$
- $D_{0110} := f_{21}$
- $D_{1001} := h_{23}$
- $D_{1101} := h_{12} + h_{13}$
- $D_{0111} := f_{31}$
- $D_{1010} := h_{24}$
- $D_{1111} := e_3$
- $D_{1111} := e_3$

and the other $D_\sigma = 0$, where $f_1 \in |f_1|$, $f_2 \in |f_2|$ and $f_3 \in |f_3|$ such that no more than two of these divisors $D_\sigma$ go through the same point. We consider the following non-trivial divisors of $Y_4$:

- $L_{0001} := 2f_1 + f_2 - e_3$
- $L_{0010} := f_2 + l$
- $L_{0100} := f_1 + f_2 + f_3 - e_4$
- $L_{1000} := f_1 + f_2 + f_3 - e_4$
- $L_{0011} := f_1 + 2f_2 - e_3 - e_4$
- $L_{0101} := f_2 + f_3 - e_3 - e_4$
- $L_{0110} := 2f_1 + f_2 - e_3 - e_4$
- $L_{0111} := f_2 + f_3 - e_4$
- $L_{1001} := f_3 + f_4$
- $L_{1010} := f_1 + f_2 + f_3 - e_3$
- $L_{1011} := f_1 + f_4$
- $L_{1100} := f_1 + f_2 + f_3 - e_4$
- $L_{1101} := f_1 + f_2$
- $L_{1110} := l$
- $L_{1111} := f_1 + f_3$. 

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*Nguyen Bin*
These divisors $D_\sigma, L_X$ satisfy the following relations:

\[
\begin{align*}
2L_{1000} & \equiv D_{0101} + D_{0110} + D_{1010} = h_{14} + f_{21} + f_{31} \equiv 3L - e_1 - e_2 - e_3 - e_4 \\
2L_{1010} & \equiv D_{0101} + D_{0110} + D_{1010} = h_{14} + f_{21} + f_{31} \equiv 3L - e_1 - e_2 - e_3 - e_4 \\
2L_{1100} & \equiv D_{1010} + D_{1100} + D_{1110} = h_{34} + h_{12} + h_{13} + e_2 + e_3 = 3L - e_1 - e_2 - e_3 - e_4 \\
2L_{1110} & \equiv D_{1010} + D_{1100} + D_{1110} = h_{34} + h_{12} + h_{13} + e_2 + e_3 = 3L - e_1 - e_2 - e_3 - e_4 \\
2L_{1111} & \equiv D_{1010} + D_{1100} + D_{1110} + D_{1111} = h_{34} + h_{12} + h_{13} + e_2 + e_3 = 3L - e_1 - e_2 - e_3 - e_4 \\
\end{align*}
\]

Thus by Proposition 2, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^4$-cover $g : X \to Y_4$. Moreover, this $\mathbb{Z}_2^4$-cover fulfills the hypotheses of Theorem 3. In fact, we have

\[
D_{1000} + D_{0101} + D_{1010} = h_{14} + f_{21} + f_{31} = 3L - e_1 - e_2 - e_3 - e_4 \\
D_{1000} + D_{1001} + D_{1010} + D_{1011} = e_2 + h_{23} + h_{24} + f_{11} \equiv 3L - e_1 - e_2 - e_3 - e_4 \\
D_{1100} + D_{1101} + D_{1110} + D_{1111} = h_{34} + h_{12} + h_{13} + e_1 + e_3 \equiv 3L - e_1 - e_2 - e_3 - e_4.
\]

For all $\chi \in \{1, 0000, 0100, 1000\} \setminus \{0000\}$, and the divisor $D_{0001} + D_{0110} + D_{0111} - K_{Y_4} \equiv 3L - e_1 - e_2 - e_3$ is nef and big. Thus by Theorem 3 and Proposition 5, the surface $X$ is a minimal surface of general type and possesses the following invariants:

\[
K_X^2 = 24, p_g(S) = 3, \chi(f(S)) = 4, q(S) = 0.
\]

Moreover, the canonical map $\varphi_{|K_X|}$ of degree 20 and the two $(-2)$-curves coming from $\bar{e}_4$ are the fixed points of $|K_X|$. Therefore, we obtain the surface in the second row of Theorem 4.

**Remark 9.** The surface $X$ has three pencils of genus 9 corresponding the fibres $f_1, f_2, f_3$ and a pencil of genus 13 corresponding to the fibre $f_4$.

**Remark 10.** The surface $X$ admits natural deformations. Moreover, all the natural deformations of $X$ are Galois.

Similarly to Remark 8, we have that $H^0(Y_4, D_{0110}) \cong H^0(Y_4, D_{0111}) \cong H^0(Y_4, D_{1011}) \cong \mathbb{C}^2$ and $H^0(Y_4, D_\sigma) \cong \mathbb{C}$ for the other non-trivial $D_\sigma$. This implies that the family of natural deformations of $g : X \to Y_4$ is parametrized by the base space $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1$. Furthermore, all natural deformations of $X$ are Galois since $\bigoplus_{\sigma \neq 0} H^0(Y_4, D_\sigma - L_X) = 0$.

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**References**

[1] A. Beauville, “L’application canonique pour les surfaces de type général”, *Invent. Math.* 55 (1979), no. 2, p. 121-140.

[2] N. Bin, “A new example of an algebraic surface with canonical map of degree 16”, *Arch. Math.* 113 (2019), no. 4, p. 385-390.

[3] B. Fantechi, R. Pardini, “Automorphisms and moduli spaces of varieties with ample canonical class via deformations of abelian covers”, *Commun. Algebra* 25 (1997), no. 5, p. 1413-1441.
[4] C. Gleissner, R. Pignatelli, C. Rito, "New surfaces with canonical map of high degree", https://arxiv.org/abs/1807.11854, 2018.
[5] C.-J. Lai, S.-K. Yeung, "Examples of surfaces with canonical maps of maximal degree", Taiwanese J. Math. 25 (2021), no. 4, p. 699-716.
[6] M. Mendes Lopes, R. Pardini, “The geography of irregular surfaces”, in Current developments in algebraic geometry, Mathematical Sciences Research Institute Publications, vol. 59, Cambridge University Press, 2012, p. 349-378.
[7] R. Pardini, "Abelian covers of algebraic varieties", J. Reine Angew. Math. 417 (1991), p. 191-213.
[8] ———, “Canonical images of surfaces”, J. Reine Angew. Math. 417 (1991), p. 215-219.
[9] U. Persson, “Double coverings and surfaces of general type”, in Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977), Lecture Notes in Mathematics, vol. 687, Springer, 1977, p. 168-195.
[10] C. Rito, "New canonical triple covers of surfaces", Proc. Am. Math. Soc. 143 (2015), no. 11, p. 4647-4653.
[11] ———, "A surface with canonical map of degree 24", Int. J. Math. 28 (2017), no. 6, article no. 1750041 (10 pages).
[12] ———, "A surface with $q = 2$ and canonical map of degree 16", Mich. Math. J. 66 (2017), no. 1, p. 99-105.
[13] ———, "Surfaces with canonical map of maximum degree", https://arxiv.org/abs/1903.03017, 2019.
[14] S. L. Tan, "Surfaces whose canonical maps are of odd degrees", Math. Ann. 292 (1992), no. 1, p. 13-29.