Abstract. In the context of Thurston’s geometrisation program we address the question which compact aspherical 3-manifolds admit Riemannian metrics of nonpositive curvature. We prove that a Haken manifold with, possibly empty, boundary of zero Euler characteristic admits metrics of nonpositive curvature if the boundary is non-empty or if at least one atoroidal component occurs in its canonical topological decomposition. Our arguments are based on Thurston’s Hyperbolisation Theorem. We give examples of closed graph manifolds with linear gluing graph and arbitrarily many Seifert components which do not admit metrics of nonpositive curvature.

1 Introduction

It is known since the last century that a closed surface admits particularly nice Riemannian metrics, namely metrics of constant curvature. All aspherical surfaces, i.e. all surfaces besides the sphere and the projective plane can be given a metric of nonpositive curvature.

Thurston suggested a geometrisation procedure in dimension 3. Here, constant curvature metrics are a too restricted class of model geometries, and one allows more generally complete locally homogeneous metrics. There are 8 types of 3-dimensional geometries which can occur, namely the model spaces are $S^3$, $S^2 \times R$, $R^3$, $Nil$, $Sol$, $SL(2,R)$, $H^2 \times R$ and $H^3$. The classification is due to Thurston [Th1] and a detailed exposition can be found in Scott’s
article [S]. Despite the large supply of model geometries it is far from being true that any compact 3-dimensional manifold \( M \) is geometric, that is, admits a geometric structure. One must cut \( M \) into suitable pieces. The canonical topological decomposition is obtained as follows: According to Kneser, there is a maximal decomposition of \( M \) as a finite connected sum. The summands are called prime manifolds and their homeomorphism type is determined by \( M \) if \( M \) is orientable, whereas the decomposition itself is in general not unique (Milnor). The aspherical prime manifolds can be further decomposed by cutting along incompressible embedded tori and Klein bottles, see [J-S, Jo]. The components which one obtains in this second step are Seifert fibered or atoroidal. Thurston’s Geometrisation Conjecture asserts that one can put a geometric structure of unique type on each piece of the canonical topological decomposition of \( M \). It is well-known that a closed 3-manifold is Seifert if and only if it can be given one of the geometries different from \( Sol \) and \( \mathbb{H}^3 \), see [S]. The atoroidal components conjecturally admit a hyperbolic structure and this has been proven by Thurston [Th2] for components which are Haken, i.e. contain a closed incompressible surface. In particular, all Haken 3-manifolds are geometrisable.

We are interested in those 3-manifolds \( M \) which have a chance to admit metrics of nonpositive curvature. In the context of Thurston’s geometrisation program, we address in this paper the

**Question:** Which compact aspherical 3-manifolds \( M \) admit Riemannian metrics of nonpositive sectional curvature?

The situation is well-understood for geometric 3-manifolds. If \( M \) is Haken and non-geometric, then each component in the canonical decomposition admits a geometric structure which is modelled on one of the nonpositively curved geometries \( \mathbb{R}^3, \mathbb{H}^2 \times \mathbb{R} \) and \( \mathbb{H}^3 \). The known obstructions to the existence of a nonpositively curved metric on \( M \) vanish, i.e. all solvable subgroups of \( \pi_1(M) \) are virtually abelian and centralisers virtually split. It is therefore only natural to ask whether one can put compatible nonpositively curved metrics on all components of \( M \) simultaneously. We show that, as suggested by the geometrisation program, non-geometric Haken manifolds indeed generically admit metrics of nonpositive curvature. More precisely, we prove the following existence results:

**Theorem 3.2** Suppose that \( M \) is a graph-manifold, i.e. contains only Seifert components, and has non-empty boundary. Then there exists a Riemannian
metric of nonpositive curvature on $M$.

And, relying on Thurston's Hyperbolisation Theorem [Th2]:

**Theorem 3.3** Let $M$ be a Haken manifold with, possibly empty, boundary of zero Euler characteristic. Suppose that at least one atoroidal component occurs in its canonical decomposition. Then $M$ admits a Riemannian metric of nonpositive curvature.

The Riemannian metrics can be chosen smooth and flat near the boundary. Our existence results yield new examples of closed nonpositively curved manifolds of geometric rank one with non-hyperbolic fundamental group.

Differently from the situation in dimension 2, not all closed aspherical 3-manifolds admit metrics of nonpositive curvature. Examples can already be found among geometric manifolds, namely quotients of $Nil$, $Sol$ or $SL(2,\mathbb{R})$. We show that also non-geometric Haken manifolds cannot always be equipped with a metric of nonpositive curvature. We give the following

**Example 4.2** There are closed graph-manifolds glued from arbitrarily many Seifert components which do not admit metrics of nonpositive curvature.

Note that nevertheless the fundamental groups of such manifolds are nonpositively curved in the large: We prove in [K-L2] that the fundamental group of a closed Haken 3-manifold is quasi-isometric to $Nil$, $Sol$ or the fundamental group of a closed nonpositively curved 3-manifold.

The paper is organised as follows: In section 2, we explain that any nonpositively curved metric on a Haken manifold arises from nonpositively curved metrics on its geometric components (section 2.1) and is rigid on Seifert components. The atoroidal components, on the other hand, are flexible (section 2.2). The extent of rigidity for the Seifert pieces is described in section 2.3. The rigidity is responsible for the non-existence examples (section 4.1) but leaves sufficient degrees of freedom (Proposition 2.6) to produce nonpositively curved metrics on Haken manifolds with atoroidal components or non-empty boundary (section 3).

We refer to [Ch-E] for an introduction to the geometry of nonpositive curvature and to [Ja] for concepts from 3-dimensional topology.

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2 Nonpositively curved metrics on the geometric components

2.1 Reduction to geometric components

In this section $M$ will always denote a compact Haken 3-manifold with boundary of zero Euler characteristic. We recall that a compact smooth aspherical 3-manifold is Haken if it contains a closed incompressible surface, that is, a closed smooth embedded 2-sided surface whose fundamental group is infinite and injects via the canonical inclusion homomorphism, cf. [Ja].

According to [J-S, Jo], there is a canonical topological decomposition of $M$ into compact 3-manifolds which are Seifert fibered or atoroidal. It is obtained by cutting $M$ along finitely many disjoint closed embedded incompressible tori and Klein bottles $\Sigma_i$ which we call the splitting or decomposing surfaces. We refer to the pieces of the decomposition as geometric components because they can be equipped with canonical geometric structures. If the topological decomposition of $M$ is non-trivial, the geometric components have non-empty incompressible boundary. The Seifert pieces then admit a $\mathbb{R}^3$- or $\mathbb{H}^2 \times \mathbb{R}$-structure (compare [S]) whereas the atoroidal pieces admit $\mathbb{H}^3$-structures [Th2]. A minimal topological decomposition of $M$ is unique up to isotopy.

Assume now that $M$ carries a Riemannian metric $g$ of nonpositive sectional curvature. We only allow metrics where the boundary is totally-geodesic and (hence) flat. In this geometric situation, there is an analogous geometric decomposition of $(M, g)$ along totally-geodesically embedded flat surfaces, cf. [L-S]. Since a minimal topological decomposition of $M$ is unique, it can therefore be geometrised in the presence of $g$:

**Lemma 2.1** ([L-S]) There is an isotopy of $M$ which moves the splitting surface $\Sigma := \cup_i \Sigma_i$ to a totally-geodesically embedded flat surface in $M$.

It follows that each metric of nonpositive curvature on $M$ arises modulo isotopy in the following way: Put suitable flat metrics on the decomposing
surfaces \( \Sigma \), and extend them to nonpositively curved metrics on the geometric components of the topological decomposition. It is therefore crucial to understand the following question:

**Extension Problem 2.2** Let \( X \) be a geometric component. Which flat metrics on the boundary \( \partial X \) can be extended to metrics of nonpositive curvature on \( X \)?

Atoroidal and Seifert components behave differently with respect to the extendability of metrics. As discussed below, atoroidal components are so flexible that the extension problem can be solved for all flat metrics prescribed on the boundary, whereas the rigidity of nonpositively curved Seifert manifolds puts restrictions on the solvability of the extension problem for Seifert components. The interdependence of flat metrics on the boundary tori of a nonpositively curved Seifert manifold is the source of obstructions to the existence of nonpositively curved metrics on graph-manifolds.

### 2.2 Nonpositively curved metrics on atoroidal manifolds

Let \( X \) be an atoroidal component of \( M \). We already mentioned a deep result of Thurston [Th2], his Hyperbolisation Theorem, that the interior \( \text{int} X \) of \( X \) admits a complete metric \( g_0 \) of constant negative curvature, say \(-1\), and finite volume. The hyperbolic structure \( g_0 \) is in fact unique according to Mostow’s rigidity theorem. By modifying \( g_0 \) we can solve the extension problem:

**Proposition 2.3 (Flexibility of atoroidal components)** Any flat metric on \( \partial X \) can be extended to a smooth nonpositively curved metric on \( X \) which is flat near the boundary.

**Proof:** Let \( h \) be a prescribed flat metric on \( \partial X \). We will change the hyperbolic structure \( g_0 \) in order to extend \( h \) to all of \( X \). The ends of \( (\text{int} X, g_0) \) are hyperbolic cusps: Outside a suitable compact subset, the metric \( g_0 \) is isometric to a warped product metric

\[
e^{-2t} g_{\partial X} + dt^2
\]
on $\partial X \times \mathbb{R}^+$ where $g_{\partial X}$ is a flat metric on $\partial X$. The conformal type of $g_{\partial X}$ is determined by $g_0$. There is no relation between the metrics $g_{\partial X}$ and $h$. As a first step, we adjust the conformal type of the cusps of $g_0$ to the prescribed metric $h$. To do so it suffices to allow more general warped product metrics: after isotoping $h$ if necessary we can diagonalise $h$ with respect to $g_{\partial X}$, that is, we write $g_{\partial X}$ and $h$ as

$$g_{\partial X} = dx^2 + dy^2 \quad \text{and} \quad h = a^2 \cdot dx^2 + b^2 \cdot dy^2$$

where $dx^2$ and $dy^2$ are positive-semidefinite sub-Riemannian metrics parallel with respect to $g_{\partial X}$ with orthogonal one-dimensional kernels and $a, b$ are positive functions constant on each boundary component $\Sigma_i$. To interpolate between the conformal types of $g_{\partial X}$ and $h$, we put a metric of the form

$$e^{-2t}[\phi + (1 - \phi) \cdot a]^2 \cdot dx^2 + e^{-2t}[\phi + (1 - \phi) \cdot b]^2 \cdot dy^2 + dt^2$$

on $\partial X \times \mathbb{R}^+$ where the smooth function $\phi : \mathbb{R}^+ \to [0, 1]$ is required to be equal to 1 in a neighborhood of 0 and equal to 0 in a neighborhood of $\infty$. A curvature calculation shows that the sectional curvature of a metric of this type is negative if the component functions $e^{-t}[\phi + (1 - \phi) \cdot a]$ and $e^{-t}[\phi + (1 - \phi) \cdot b]$ are strictly monotonically decreasing and convex. This holds for all functions $\phi$ whose first and second derivatives are bounded by sufficiently small constants depending on $a$ and $b$. Hence we can find a complete negatively curved metric $g_1$ on $intX$ which is outside a suitable compact subset hyperbolic and isometric to the warped product metric

$$e^{-2t}h + dt^2$$

on $\partial X \times (T_0, \infty)$ for some $T_0 \in \mathbb{R}$.

In the second step, we replace $e^{-2t}$ by a convex and monotonically decreasing function $\psi : (T_0, \infty) \to (0, \infty)$ which coincides with $e^{-2t}$ in a neighborhood of $T_0$ and is constant in a neighborhood of $\infty$. The curvature of the resulting complete metric $g_2$ is nonpositive because $\psi$ is convex. After rescaling, $(intX, g_2)$ is outside some compact subset isometric to a Euclidean cylinder with base $(\partial X, h)$. We cut off the ends along cross sections of the cylinders and obtain a smooth nonpositively curved metric on $X$ which is flat near the boundary and extends $h$. $\square$
2.3 Nonpositively curved metrics on Seifert manifolds

We address the Extension Problem 2.2 for Seifert manifolds. Let $X$ be a Seifert fibered manifold with non-empty incompressible boundary and denote by $O$ its base orbifold. We have the exact sequence

$$1 \to \langle f \rangle \to \pi_1(X) \to \pi_1(O) \to 1$$

where the element $f$ is represented by a generic Seifert fiber. We recall that due to the cyclic normal subgroup of the fundamental group, any nonpositively curved metric $g$ on $X$ must have a special form: The universal cover decomposes as a Riemannian product

$$(\tilde{X}, g) \cong \mathbb{R} \times Y$$

where $Y$ is nonpositively curved, cf. [E]. The lines $\mathbb{R} \times \{y\}$ are the axes of the deck-transformation $f$, and they project down to a Seifert fibration of $X$ by closed geodesics of, apart from finitely many singular fibers, equal length. The deck-action of $\pi_1(X)$ preserves the splitting (1) and hence decomposes as the product of a representation

$$\phi : \pi_1(X) \to Isom(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$$

and a cocompact discrete action of $\pi_1(O)$ on $Y$ which induces an orbifold-metric of nonpositive curvature on $O$. Since $f$ acts on its axes by translations, the representation $\phi$ satisfies the condition

$$\phi(f) \in \mathbb{R} \setminus \{0\}.$$ (3)

We see that a nonpositively curved metric on $X$ corresponds to the choice of a nonpositively curved metric on $O$ and a representation (2) satisfying (3). It is well-known that the Seifert manifold $X$ does admit metrics of nonpositive curvature. This follows, for instance, from Theorem 5.3 in [S]; namely, the closed Seifert manifold obtained from doubling $X$ admits a metric of nonpositive curvature which yields a representation (2) for $\pi_1(X)$ so that (3) holds. We call a Seifert manifold of Euclidean respectively hyperbolic type according to whether its base orbifold $O$ admits flat or hyperbolic metrics. All nonpositively curved metrics on a Seifert manifold of Euclidean type are flat.
Suppose in the remainder of this section that $X$ has hyperbolic type. We want to determine which flat metrics $h$ on $\partial X$ can be extended to a nonpositively curved metric on $X$. One necessary condition is immediate from the above discussion: The closed geodesics in $(\partial X, h)$ which are (homotopic to) Seifert fibers must have equal lengths. This is sufficient in the non-orientable case:

**Lemma 2.4** If $X$ is non-orientable, then any flat metric on $\partial X$ so that Seifert fibers have equal length can be extended to a nonpositively curved metric on $X$.

In the orientable case, there is a second necessary condition because the values of the representation $\phi$ on the boundary tori $T_i$ are interrelated. To describe it we need some notation: The flat metric $h$ on $\partial X$ induces scalar products $\sigma_i$ on the abelian groups $\pi_1(T_i) \cong H_1(T_i, \mathbb{Z})$. The values of $\phi$ on $H_1(T_i, \mathbb{Z})$ lie in $\mathbb{R}$ and they are related to $\sigma_i$ by the formula

$$\sigma_i(f, \cdot) = \phi(f) \cdot \phi$$

on $H_1(T_i, \mathbb{Z})$.

We choose in each group $H_1(T_i, \mathbb{Z})$ an element $b_i$ which forms together with $f$ a basis compatible with the orientation of $T_i$. The compatibility condition for the flat metrics on the boundary tori is given by

**Lemma 2.5 (Rigidity of Seifert components)** If $X$ is oriented, there is a rational number $c$ which depends on the types of singular fibers and the choice of the basis elements $b_i$, so that the following is true: A flat metric $h$ on $\partial X$ can be extended to a nonpositively curved metric on $X$ if and only if the Seifert fibers of $(\partial X, h)$ have equal length and the relation

$$\sum_i \sigma_i(f, b_i) = c \cdot \|f\|^2$$

(4)

holds.

Note that the nonpositively curved metrics on $X$ can always be chosen flat near the boundary by adjusting the metric on the base-orbifold $O$.

The proofs are straight-forward. By varying the hyperbolic structure on $O$ one can realize arbitrary lengths for the boundary curves. The possible values for the representation $\phi$ on the boundary $\partial X$ can be analysed as in §
by using a canonical presentation for $\pi_1(X)$. A proof for the non-orientable case is given in [K-L1].

Observe that by varying the metric on the base orbifold $O$, exactly those modifications of the induced flat metric on $\partial X$ are possible which preserve the scalar products $\sigma_i(f, \cdot)$ with the fiber direction and, consequently, the length $\|f\|$ of the fiber. We will refer to this way of changing the flat metric on $\partial X$ as rescaling orthogonally to the Seifert fiber.

In our construction of nonpositively curved metrics on graph-manifolds in section 3 we need to extend flat metrics which are prescribed only on part of the boundary. Lemmata 2.4 and 2.5 imply that the Extension Problem 2.2 with loose ends is always solvable:

**Proposition 2.6 (Restricted flexibility of Seifert components)** Suppose that $X$ is a Seifert manifold with non-empty incompressible boundary and hyperbolic base orbifold. Let $h$ be a flat metric which is prescribed on some but not all of the boundary components of $X$ so that Seifert fibers have equal lengths. Then $h$ can be extended to a nonpositively curved metric on $X$ which is flat near the boundary.

For the sake of completeness, we show how one can deduce the proposition from the existence of a single nonpositively curved metric $g$ on $X$. This is done by modifying the representation $\phi : \pi_1(X) \to \text{Isom}(\mathbb{R})$. There is a simple closed loop in $O$ which separates the boundary from the singularities and orientation reversing loops. Accordingly $X$ can be obtained from gluing a Seifert manifold $Y$ and a circle bundle $Z$ over a punctured sphere along one boundary component. Denote by $a_0, \ldots, a_k$ the compatibly oriented boundary curves of a section of $Z$ so that $a_0$ corresponds to the boundary surface along which $Z$ is glued to $Y$. The fundamental group of $Z$ is presented by

$$\langle f, a_0, \ldots, a_k | a_i f a_i^{-1} = f^{\epsilon_i}, \prod a_i = 1 \rangle$$

with $\epsilon_i \in \{\pm 1\}$ and $\prod \epsilon_i = 1$. A new presentation $\phi'$ is obtained from $\phi$ as follows. We leave the presentation $\phi$ unchanged on $\pi_1(Y)$ and choose $\phi'$ on $Z$ so that $\phi'(a_i)$ is in the same component of $\text{Isom}(\mathbb{R})$ as $\phi(a_i)$ and so that the conditions

$$\phi'(f) = \phi(f), \quad \phi'(a_0) = \phi(a_0) \quad \text{and} \quad \prod_i \phi'(a_i) = \prod_i \phi(a_i)$$

are satisfied.
hold. This shows that we have the freedom to prescribe flat metrics on \( \partial X \) as claimed in Proposition 2.6.

### 3 Existence results

#### 3.1 Graph-manifolds with boundary

There is no unanimous notion of graph-manifold available in the literature. We choose the

**Definition 3.1** A graph-manifold is a Haken manifold which contains only Seifert components in its topological decomposition.

Any metric of nonpositive curvature on a graph-manifold \( M \) is rigid in the sense that it splits almost everywhere locally as a product, see section 2.3. Nevertheless, the restricted flexibility of Seifert manifolds as stated in Proposition 2.6 suffices to construct nonpositively curved metrics on \( M \) in the presence of boundary:

**Theorem 3.2** Suppose that \( M \) is a graph-manifold with non-empty boundary. Then there exists a Riemannian metric of nonpositive curvature on \( M \).

**Addendum.** Moreover, let \( \{ \gamma_i \} \) be a collection of homotopically non-trivial simple loops in the boundary, one on each component, so that none of them represents the fiber of the adjacent Seifert component. Then, given positive numbers \( l_i \), the nonpositively curved metric on \( M \) can be chosen so that the loops \( \gamma_i \) are geodesics of length \( l_i \).

We will give examples of closed graph-manifolds without metrics of nonpositive curvature in section 4.

**Proof:** The Seifert components of \( M \) have non-empty incompressible boundary and hence admit nonpositively curved metrics. We can assume that each Seifert component of Euclidean type has one end, since otherwise it would be homeomorphic to a torus or Klein bottle cross the unit interval and hence be obsolete in the topological decomposition. Furthermore, we may assume that no gluing of ends of Seifert components identifies the Seifert fibers.

Let \( n \) be the number of Seifert components of \( M \) and suppose that the claim has been proven for graph-manifolds with less than \( n \) components.
Choose a Seifert component $X$ which contributes to the boundary of $M$. It has hyperbolic type. The complement $C$ of $X$ in $M$ consists of some Seifert manifolds $X^{eu}_i$ of Euclidean type with one end and a graph-manifold $M_0$ whose Seifert components adjacent to $X$ have hyperbolic base. We pick flat metrics on the Euclidean pieces $X^{eu}_i$ and, using the induction assumption, a nonpositively curved metric on $M_0$. We can arrange this metric $g_0$ on $C$ so that all closed geodesics in the boundary which are identified with fibers in $\partial X$ via the gluing, have the same length $l$. This is achieved on $M_0$ by rescaling orthogonally to the Seifert fiber, as described in section 2.3. The metric $g_0$ determines via the gluing map a flat metric on $\partial X \cap C$ with the property that Seifert fibers have equal length $l$. It may also happen that boundary surfaces of $X$ are glued to each other. We pick flat metrics on those which are compatible with the identification and give length $l$ to the Seifert fibers as well. Our choices prescribe a flat metric on a proper part of $\partial X$ so that Seifert fibers have equal length. According to Proposition 2.4, this flat metric can be extended to a nonpositively curved metric on $X$. Combined with $g_0$, this yields a smooth nonpositively curved metric on $M$. The Addendum follows by rescaling orthogonally to the fiber and ordinary rescaling. □

3.2 Haken manifolds with atoroidal components

By now we have collected all ingredients for our main existence result. We saw in Proposition 2.3 that atoroidal manifolds are completely flexible, and the existence theorem 3.2 for graph-manifolds with boundary allows to put nonpositively curved metrics on the complement of the atoroidal components of a Haken manifold. Combining these facts we get:

**Theorem 3.3 (Existence in presence of an atoroidal piece)** Let $M$ be a Haken manifold with, possibly empty, boundary of zero Euler characteristic. Suppose that at least one atoroidal component occurs in its canonical decomposition. Then $M$ admits a Riemannian metric of nonpositive curvature.

We obtain new examples of closed nonpositively curved manifolds which are not locally symmetric and do not admit metrics of strictly negative curvature. These manifolds have geometric rank one in the sense of Ballmann, Brin and Eberlein. The first 3-manifold examples of this kind were given by Heintze [1] and Gromov [3].
4 Closed graph-manifolds

In section 3, we proved existence of nonpositively curved metrics on Haken manifolds which have non-empty boundary or contain an atoroidal component in their topological decomposition. This leaves open the case of closed graph-manifolds which we address now.

Consider a closed non-geometric graph-manifold \( M \). As explained in section 2, a metric \( g \) of nonpositive curvature on \( M \) induces flat metrics \( g_i \) on the decomposing surfaces \( \Sigma_i \) and \( g \) locally splits as a product on each Seifert component. This rigidity interrelates the metrics \( g_i \) and serves as the source of obstructions to the existence of nonpositively curved metrics. A nonpositively curved metric exists on \( M \) if and only if we can put flat metrics on the Euclidean Seifert components and the surfaces \( \Sigma_i \) so that all orientable Seifert components with hyperbolic base satisfy the linear compatibility condition stated in Lemma 2.5. This reduces the existence question to a purely algebraic problem which is, however, in general fairly complicated. In section 4.1 below we discuss a class of graph-manifolds with linear gluing graph where the compatibility conditions can be translated into a simple combinatorial criterion for finite point configurations in hyperbolic plane. Although a majority of these manifolds admit metrics of nonpositive curvature, we will also obtain examples of non-existence.

The method in section 4.1 can be extended to the more general case when the dual graph to the canonical decomposition is a tree (joint work with Misha Kapovich). Examples of graph-manifolds with no metrics of nonpositive curvature when the dual graph contains cycles are given in [K-L1]; they arise as mapping tori of reducible surface diffeomorphisms. Recently, Buyalo and Kobelski [B-K] announced the construction of a numerical invariant for graph-manifolds which detects the existence of a nonpositively curved metric.

4.1 An example

We consider closed graph-manifolds \( M \) built from two Seifert components \( X_0 \) and \( X_{n+1} \), each with one boundary torus (denoted by \( \partial X_0 \) respectively \( \partial X_{n+1} \)), and \( n \geq 0 \) Seifert components \( X_1, \ldots, X_n \) with two boundary tori (denoted by \( \partial X_i \)). The \( X_i \) are assumed to be trivial circle bundles over compact orientable surfaces \( \Sigma_i \) of genus \( \geq 1 \) and hence they are of hyperbolic type. The graph-manifold \( M \) is obtained by performing gluing homeomor-
phisms
\[ \alpha_i : \partial_+ X_i \to \partial_- X_{i+1} \quad (i = 0, \ldots, n). \]

Denote by \( T_i := \partial_+ X_i = \partial_- X_{i+1} \) the decomposing tori of \( M \).

We define an abstract free abelian group \( A \) of rank 2 by identifying the homology groups \( H_1(T_i, \mathbb{Z}) \) with each other in a canonical way: For \( i = 1, \ldots, n \), we define the elements \( x_{i-1} \in H_1(T_{i-1}, \mathbb{Z}) \) and \( x_i \in H_1(T_i, \mathbb{Z}) \) to be equivalent iff their images under the inclusion monomorphisms
\[ H_1(\partial_\pm X_i, \mathbb{Z}) \hookrightarrow H_1(X_i, \mathbb{Z}) \]
coincide. The Seifert fibers of the components \( X_i \) yield distinguished elements \( f_0, \ldots, f_{n+1} \in A \). In addition there are elements \( b_0, b_{n+1} \in A \) which correspond to the well-defined horizontal directions in the boundary tori of the one-ended components; they can be defined as generators of the kernels of the natural homomorphisms
\[ H_1(\partial_+ X_0, \mathbb{Z}) \to H_1(X_0, \mathbb{Z}) \]
and
\[ H_1(\partial_- X_{n+1}, \mathbb{Z}) \to H_1(X_{n+1}, \mathbb{Z}). \]

Note that there are no well-defined horizontal directions in the boundary tori of components with more than one end. The collection of vectors \( b_0, b_{n+1}, f_0, \ldots, f_{n+1} \) encodes the gluing maps and hence the topology of the closed graph-manifold \( M \).

To put flat metrics on the tori \( T_i \) is equivalent to choosing scalar products, i.e. positive-definite symmetric bilinear forms \( \sigma_i \) on the group \( A \). We recall that the space \( \mathcal{H} \) of projective equivalence classes of scalar products on \( A \) carries a natural metric which is isometric to hyperbolic plane. A geodesic \( c \) in \( \mathcal{H} \) corresponds to a splitting \( V = L_1 \oplus L_2 \) of the 2-dimensional vector space \( V := A \otimes_{\mathbb{Z}} \mathbb{R} \) into 1-dimensional subspaces; namely \( c \) consists of all (classes of) scalar products with respect to which the splitting is orthogonal. The ideal boundary points in the geometric compactification of \( \mathcal{H} \) correspond to projective equivalence classes of positive-semidefinite forms or, in other words, to 1-dimensional subspaces of \( V \). The geometric boundary \( \partial_{geo} \mathcal{H} \) can hence be canonically identified with the projective line \( \mathbb{P}V \). In particular, each element \( a \in A \) can be interpreted as a point \([a] \in \partial_{geo} \mathcal{H} \).
As in Lemma 2.5 we obtain the following necessary and sufficient compatability conditions for the scalar products \( \sigma_0, \ldots, \sigma_n \) on \( A \) so that the corresponding flat metrics on the decomposing tori \( T_i \) can be extended to a nonpositively curved metric on \( M \): For the one-ended components we have

\[
\sigma_0(f_0, b_0) = 0 \quad \text{and} \quad \sigma_n(f_{n+1}, b_{n+1}) = 0,
\]

because the horizontal boundary curves representing \( b_0, b_{n+1} \) are homologically trivial in \( X_0 \) respectively \( X_{n+1} \). For the components with two ends we get

\[
\sigma_{i-1}(f_i, \cdot) = \sigma_i(f_i, \cdot) \quad \text{for } i = 1, \ldots, n.
\]

This translates into geometric conditions relating the ideal points \([b_0], [b_{n+1}], [f_0], \ldots, [f_{n+1}] \in \partial_{geo} \mathcal{H}\) and the points \([\sigma_0], \ldots, [\sigma_n] \in \mathcal{H}\):

1. \([\sigma_0]\) lies on the geodesic with ideal endpoints \([b_0]\) and \([f_0]\).  \([\sigma_n]\) lies on the geodesic with endpoints \([b_{n+1}]\) and \([f_{n+1}]\).

2. \([\sigma_{i-1}]\) and \([\sigma_i]\) lie on a geodesic asymptotic to \([f_i]\) \((i = 1, \ldots, n)\).

Note that this is a condition on the conformal types of the metrics on the tori \( T_i \). But since the gluing graph does not contain cycles, a collection of compatible conformal structures gives rise to a collection of compatible flat metrics. Our discussion yields the following

**Criterion.** There exists a metric of nonpositive curvature on \( M \) if and only if there is a configuration of points \([\sigma_0], \ldots, [\sigma_n] \in \mathcal{H}\) which satisfies the above conditions 1 and 2.

Let us first consider the simplest case when \( M \) is obtained from gluing two Seifert pieces with one end:

**Example 4.1** If \( n = 0 \), then \( M \) admits a metric of nonpositive curvature if and only if the gluing map \( \alpha_0 : \partial_+ X_0 \to \partial_- X_1 \) preserves the canonical bases of the first homology groups, i.e. if \( \alpha_{0,*} : H_1(\partial_+ X_0, \mathbb{Z}) \to H_1(\partial_- X_1, \mathbb{Z}) \) satisfies:

\[
\alpha_{0,*}\{\pm b_0, \pm f_0\} = \{\pm b_1, \pm f_1\}
\]

**Proof:** If a nonpositively curved metric exists on \( M \), then the splittings \( A = \langle b_0 \rangle \oplus \langle f_0 \rangle \) and \( A = \langle b_1 \rangle \oplus \langle f_1 \rangle \) must be orthogonal with respect to \( \sigma_0 \).
The sets \( \{ \pm b_0, \pm f_0 \} \) and \( \{ \pm b_1, \pm f_1 \} \) contain the shortest lattice vectors with respect to \( \sigma_0 \) and therefore coincide. \( \square \)

One can as well produce examples with arbitrary number of Seifert components: Let \( C_0 \) be the geodesic in \( \mathcal{H} \) with endpoints \( [f_0] \) and \( [b_0] \). For \( i = 1, \ldots, n \) define \( C_i \) to be the union of all geodesics in \( \mathcal{H} \) which are asymptotic to \( [f_i] \) and intersect \( C_{i-1} \). This is an increasing sequence of convex subsets of \( \mathcal{H} \), and \( C_i \) consists of all conformal structures on \( T_i \) which can be induced by a nonpositively curved metric on \( X_0 \cup \ldots \cup X_i \). There exists a nonpositively curved metric on \( M \) if and only if the geodesic connecting \( [f_{n+1}] \) and \( [b_{n+1}] \) intersects \( C_n \). The contrary can easily be arranged, choose for instance:

\[
\begin{align*}
  f_i &:= f_0 + i \cdot b_0 \quad \text{for } i = 1, \ldots, n+1 \\
  b_{n+1} &:= f_0 + (n+2) \cdot b_0
\end{align*}
\]

Then \( b_0, f_0, \ldots, f_{n+1}, b_{n+1} \) lie in this order on the circle \( \partial_{geo} \mathcal{H} \) and no metric of nonpositive curvature exists on the corresponding graph-manifold \( M \). This yields:

**Example 4.2** There exist closed graph-manifolds with arbitrarily many Seifert components which do not admit metrics of nonpositive curvature.

**References**

[B-K] S. Buyalo and V. Kobelski, *Geometrisation of graph-manifolds II: isometric states*, in preparation.

[Ch-E] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North Holland 1975.

[E] P. Eberlein, *A canonical form for compact nonpositively curved manifolds whose fundamental groups have nontrivial center*, Math. Ann. 260, (1982), vol. 1, 23-29.

[G] M. Gromov, *Manifolds of negative curvature*, J. Diff. Geom. 13 (1978), 223-230.

[H] E. Heintze, *Mannigfaltigkeiten negativer Krümmung*, Habilitationsschrift, Universität Bonn 1976.
[Ja] W. Jaco, *Lectures on three-manifold topology*, Amer. Math. Soc. 43 (1980).

[J-S] W. Jaco and P. Shalen, *Seifert fibred spaces in 3-manifolds*, Mem. Amer. Math. Soc. 220 (1979).

[Jo] K. Johannson, *Homotopy equivalences of 3-manifolds with boundary*, Springer LNM 761 (1979).

[K-L1] M. Kapovich and B. Leeb, *Actions of discrete groups on nonpositively curved spaces*, preprint 1994.

[K-L2] M. Kapovich and B. Leeb, *On quasi-isometries of graph-manifold groups*, preprint 1994.

[L] B. Leeb, *3-manifolds with(out) metrics of nonpositive curvature*, PhD Thesis, University of Maryland, 1992.

[L-S] B. Leeb and P. Scott, *A geometric characteristic splitting in all dimensions*, in preparation.

[S] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), 401-487.

[Th1] W. Thurston, *The geometry and topology of 3-manifolds*, lecture notes, Princeton University.

[Th2] W. Thurston, *Hyperbolic structures on 3-manifolds, I*, Ann. of Math. 124 (1986), 203-246.