The sharp $L^p$ decay of oscillatory integral operators with certain homogeneous polynomial phases in several variables

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Abstract

We obtain the $L^p$ decay of oscillatory integral operators $T_\lambda$ with certain homogeneous polynomial phase of degree $d$ in $(n + n)$-dimensions. In this paper we require that $d > 2n$. If $d/(d - n) < p < d/n$, the decay is sharp and the decay rate is related to the Newton distance. In the case of $p = d/n$ or $d/(d - n)$, we also obtain the almost sharp decay, here “almost” means the decay contains a $\log(\lambda)$ term. For otherwise, the $L^p$ decay of $T_\lambda$ is also obtained but not sharp. A counterexample also arises in this paper to show that $d/(d - n) \leq p \leq d/n$ is not necessary to guarantee the sharp decay.

Keywords: oscillatory integral operators, sharp $L^p$ decay, several variables, Newton distance

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1 Introduction

We consider the following oscillatory operator:

$$T_\lambda(f)(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x,y)} \psi(x,y)f(y) \, dy, \ n \geq 2 \tag{1.1}$$

where $x \in \mathbb{R}^n, \psi(x,y)$ is a smooth function supported in a compact neighborhood of the origin, $S(x,y) = \sum_{|\alpha| + |\beta| = d} a_{\alpha,\beta} x^\alpha y^\beta$ is a real-valued homogeneous polynomial in higher dimension with degree $d$, here $\alpha, \beta$ are multi-indices. Research on this operator centers on the decay of $L^p$ bound as the parameter $\lambda$ tends to infinity. In one dimensional case, Phong and Stein contributed a lot to this subject. In a series of their articles [7], [8], [9], [10], they developed the almost-orthogonality method to obtain the sharp $L^2$ decay of oscillatory integral operators with phase functions varying from homogeneous polynomials to real-valued analytic functions. They also clarified the relation between the decay rate and the Newton distance raised by Arnold and Varchenko in [1]. Later, the sharp $L^2$ estimate was extended to $C^\infty$ phases by Rychkov [12] and Greenblatt [2]. When $S$ is smooth and $T_\lambda$ has two-sided Whitney fold, Greenleaf and Seeger obtained the endpoint estimates for the $L^p$ decay rate of $T_\lambda$ in [4]. Yang obtained the sharp endpoint estimate in [15] with the assumption $a_{1,d-1}a_{d-1,1} \neq 0$, here $a_{\alpha,\beta}$ are coefficients of homogeneous polynomial phase function $S(x,y)$ in $\mathbb{R} \times \mathbb{R}$. Shi and Yan [13] established the sharp endpoint $L^p$ decay for arbitrary homogeneous polynomial phase functions. Later, Xiao extended this result to arbitrary analytic phases as well as presented a very specific review for this subject in [17]. Higher dimensional case even $L^2$ estimate has not been understood well. The one dimensional result of $L^2$ decay has been partially extended to $(2+1)$-dimensions by Tang [16]. The further remarkable work in higher dimension...
was obtained in [8], the authors obtained the $L^2$ estimate for the oscillatory integral operators with homogeneous polynomials satisfying various genericity assumptions.

Inspired by the method used in [13] and [12], we prove our main result by embedding $T_\lambda$ into a family of analytic operators and using complex interpolation. This method requires us to establish the $L^2 - L^2$ decay estimate as well as $H^1 - L^1$ boundedness of operators with different amplitude functions. Before we state our main theorem, some definitions should be illustrated.

**Definition** (3) If $S(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with Taylor series $\sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha y^\beta$ having no pure $x$- or $z$-term, we denote the **reduced Newton polyhedron** by

$$\mathcal{N}_0(S) = \text{convex hull} \left( \bigcup_{\alpha,\beta \neq 0} (\alpha, \beta) + \mathbb{R}_+^{\alpha + \beta} \right).$$

Then the **Newton polytope** of $S(x,y)$ (at $(0,0)$) is

$$\mathcal{N}(S) := \partial(\mathcal{N}_0(S)),$$

and the **Newton distance** $\delta(S)$ of $S$ is then

$$\delta(S) := \inf \{ \delta^{-1} > 0 : (\delta^{-1}, \ldots, \delta^{-1}) \in \mathcal{N}(S) \}.$$

These definitions correspond to the 1-dimension definitions in[9]. In our main theorem, the next definition is necessary.

**Definition** Denote the Hilbert-Schmidt norm of a matrix $A = (a_{ij})$ by

$$\|A\|_{HS} = (\text{tr}(A \cdot A^T))^{1/2} = (\sum_{i,j} |a_{ij}|^2)^{1/2}. \tag{1.2}$$

Denote by $\mathcal{O}^d(\mathbb{R}^n \times \mathbb{R}^n)$ the space of homogeneous polynomials of degree $d$ on $\mathbb{R}^n \times \mathbb{R}^n$. In fact, for oscillatory operators with homogeneous polynomial phases, we are only interested in polynomial phase functions not containing pure $x$- or $y$-terms since these leave the operator norm unchanged. Thus, we denote the space consisting of such polynomials by $\mathcal{O}^d(\mathbb{R}^n \times \mathbb{R}^n)$.

Now, we formulate our main result:

**Theorem A.** Suppose $S(x,y) \in \mathcal{O}^d(\mathbb{R}^n \times \mathbb{R}^n)$ and $d > 2n \geq 4$, if $\|S''_{xy}\|_{HS}^{1/(d-2)}$ is a norm of $\mathbb{R}^n \times \mathbb{R}^n$, then it follows

$$\|T_\lambda\|_p \lesssim \begin{cases} 
\lambda^{-\delta/2} & d/(d-n) < p < d/n, \\
\lambda^{-\delta/2} (\log(\lambda))^\delta & p = d/n \text{ or } p = d/(d-n), \\
\lambda^{-1/p'} & 1 < p < d/(d-n), \\
\lambda^{-1/p} & d/n < p < \infty.
\end{cases} \tag{1.3}$$

where $\delta$ is the Newton distance. If $d/(d-n) < p < d/n$, the decay is sharp. If $p = d/n$ or $d/(d-n)$, the decay is sharp except possibly for a $\log(\lambda)$ term. And $d/(d-n) \leq p \leq d/n$ is not necessary to guarantee the sharp decay.

To clarify the relation between Newton distance and the $L^p$ decay rate, a proposition in [8] should be mentioned.

**Proposition 1.1.** If $S(x,y) \in \mathcal{O}^d(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the rank one condition

$$\text{rank}(S''_{xy}) \geq 1, \text{ for all } (x,y) \neq (0,0),$$

then $\delta(S) = 2n/d$. 

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Obviously $S(x, y)$ satisfies the rank one condition because of the assumptions in Theorem 4, thus the Newton distance in Theorem A is actually $2n/d$.

The main tool in our proof is the interpolation of analytic families of operators which was due to Stein [15]. Here, the analytic families of operators are

$$T^+_\lambda (f)(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x, y)}\|S'_{xy}\|_{HS}\psi(x, y) f(y) \, dy, \quad z = \sigma + it \in \mathbb{C}. \quad (1.4)$$

Especially, $T^+_\lambda = T_\lambda$. Theorem A naturally follows from the interpolation between $L^2 - L^2$ decay of $T^+_\lambda$ and the $H^1 - L^1$ mapping property of $T^+_\lambda$ as well as dual arguments.

The layout of the paper is as follows. In the next section, we give the difference in our proof is that we combine the dyadic decomposition and the local oscillatory estimate

$$\|T^+_\lambda\|_2 \lesssim \begin{cases} 
C_z |\lambda|^{-1/2}, & \sigma > \sigma_2; \\
C_z |\lambda|^{-1/2} \log(\lambda), & \sigma = \sigma_2; \\
C_z |\lambda|^{-[(d-2)\sigma+n]/d}, & \sigma_1 < \sigma < \sigma_2.
\end{cases} \quad (2.5)$$

Our proof roughly follows the pattern appeared in [15] and [3] in which the authors offered a nice viewpoint of higher dimensional oscillatory integral operators. They combined the dyadic decomposition of the entire space and the local Hörmander lemma [5] on the dyadic shell to give the next lemma.

**Lemma 2.2** ([3]). For a homogeneous phase function $S(x, y)$ of degree $d$ with $S'_{xy}$ satisfying the rank one condition

$$\text{rank}(S'_{xy}) \geq 1, \text{ for all } (x, y) \neq (0, 0)$$

on $\mathbb{R}^{nx} \times \mathbb{R}^{ny} (n_X \geq n_Y \geq 2)$, there hold

$$\|T^+_\lambda\|_2 \leq \begin{cases} 
C\lambda^{-(nx+ny)/(2d)} & \text{if } d > n_X + n_Y, \\
C\lambda^{-1/2} \log \lambda & \text{if } d = n_X + n_Y, \\
C\lambda^{-1/2} & \text{if } 2 \leq d < n_X + n_Y.
\end{cases} \quad (2.6)$$

The difference in our proof is that we combine the dyadic decomposition and the local oscillatory estimate (Lemma 1.1 in [3]). Now we turn to our proof of Theorem 2.1.

**Proof.** Since the support of $\psi(x, y)$ is compact, we may assume that $\text{supp} (\psi)$ is contained in $\{(x, y) : |(x, y)| \leq 1\}$. Considering the compactness of the sphere $|(x, y)| = 1$, we can make a partition of unity over the unit sphere, and then extend it to a partition of unity on $\mathbb{R}^{2n} \setminus \{0\}$, homogeneous of degree 0. Thus, to conclude the result of (2.5), it suffices to show that for each point on the unit sphere of $\mathbb{R}^n \times \mathbb{R}^n$, an operator supported in one of its (small enough) convex conic neighborhood has the desired decay rate. Decompose the unit ball by dyadic partition of unity $\{a_k\}, \sum_{k=0}^{\infty} a_k(x, y) \equiv 1$, and

$$\text{supp} (a_k) \subseteq \{2^{-k-1} < |(x, y)| \leq 2^{-k+1}\}.$$
Set $\psi_k = \psi a_k$ and $T_{x,k}^\varphi(f)(x) = \int_{\mathbb{R}^n} e^{i \varphi(x,y)} \psi_k(x,y) f(y) \, dy$. Since the Hessian $S''_{xy}$ satisfies the rank one condition, then for each $(x_0, y_0) \in S^{2n-1}$, there exists at least a pair of indices $(i_0, j_0)$ such that $S''_{x_i y_j}(x_0, y_0) \neq 0$. Set $C_0 = \max\{|S''_{x_i y_j}(x_0, y_0)| : 1 \leq i, j \leq n\}$, thus $C_0 > 0$ for each $(x_0, y_0) \in S^{2n-1}$. Without confusion, we may assume $|S''_{x_i y_j}(x_0, y_0)| = C_0$ and there must exist a sufficiently small neighborhood $\mathcal{U}$ of $(x_0, y_0)$ on the unit sphere such that

$$C_0/2 < |S''_{x_i y_j}(x, y)| < 2C_0, \quad |S''_{x_i y_j}(x, y)| < 2C_0, \quad \forall (i, j) \neq (1, 1), \forall (x, y) \in \mathcal{U}.$$ 

Denote the conic convex hull of origin and $\mathcal{U}$ by $\mathcal{U}_c$. A finite number of such $\mathcal{U}_c$ cover the unit ball. Thus $\psi(x, y)$ can be assumed to be supported in $\mathcal{U}_c$. Obviously, on the support of $\psi_k$, we have $|S''_{x_i y_j}(x, y)| \approx 2^{-(d-2)k} C_0$.

Writing $x = (x_1, x'), y = (y_1, y')$, $\phi_k^*(x, y) = \|S''_{xy}\|_{H^1 S\psi k}(x, y)$, it follows

$$T_{x,k}^\varphi(f)(x) = \int_{\mathbb{R}^n} e^{i \varphi(x,y)} \|S''_{xy}\|_{H^1 S\psi k}(x, y) f(y) \, dy$$

Set $S_{x', y'}(x_1, y_1) = S(x_1, x', y_1, y')$, $\phi_k^{x', y'}(x_1, y_1) = \phi_k(x_1, x', y_1, y')$ as well as $f_{y'}(y_1) = f(y_1, y')$, then

$$T_{x,k}^\varphi(f)(x_1, x') = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i \varphi_{x', y'}(x_1, y_1)} \phi_k^{x', y'}(x_1, x', y_1, y') f_{y'}(y_1) \, dy_1 \, dy'$$

where $\tilde{T}_{x,k}^\varphi f_{y_1}(x_1)$ are the one dimensional oscillatory integral operators investigated in [8]. Repeating the proof of Lemma 1.1 in [8] and provided that $S_{x', y'}(x_1, y_1)$ is uniformly polynomial-like in $y_1$, we obtain

$$\|\tilde{T}_{x,k}^\varphi f_{y_1}(x_1)\|_{L^2(\mathbb{R})} \lesssim |z(z-1)| 2^{-(d-2)k} \lambda 2^{-(d-2)k} \left| |f(\cdot, y')| \right|_{L^2(\mathbb{R})}.$$ 

Combining this with the size of the support in $x'$ yields

$$\|\tilde{T}_{x,k}^\varphi\|_2 \lesssim |z(z-1)| 2^{-(d-2)k} \lambda 2^{-(d-2)k} \left| |f(\cdot, y')| \right|_{L^2(\mathbb{R})}^{1/2} 2^{-(2n-2)k/2}$$

$$= C_z 2^{-(d-2)k} \sigma 2^{(d-2)n/2} |\lambda|^{1/2}$$

$$= C_z 2^{-(d-2)k} \sigma 2^{(d-2)n} |\lambda|^{1/2}$$

(2.7)

where $C_z = |z(z-1)|$.

On the other hand, from the size estimate, it is easy to verify

$$\|\tilde{T}_{x,k}^\varphi\|_2 \lesssim 2^{-(d-2)k} \sigma 2^{-nk}.$$ 

(2.8)

The estimates in (2.7) and (2.8) are comparable if and only if

$$2^{-(d-2)k} \sigma 2^{(d-2)n/2} |\lambda|^{1/2} \sim 2^{-(d-2)k} \sigma 2^{-nk}, \text{ or } 2^k \sim |\lambda|^{1/d}$$

Thus

$$\|\tilde{T}_{x,k}^\varphi\|_2 \lesssim C_z \sum_{k=0}^{+\infty} \min\{2^{-(d-2)k} \sigma 2^{(d-2)n/2} |\lambda|^{1/2}, 2^{-(d-2)k} \sigma 2^{-nk}\}$$

$$= C_z \left[ \sum_{k=0}^{\lfloor \log_2 |\lambda| \rfloor} 2^{-(d-2)k} \sigma 2^{(d-2)n/2} |\lambda|^{1/2} + \sum_{k=\lfloor \log_2 |\lambda| \rfloor + 1}^{+\infty} 2^{-(d-2)k} \sigma 2^{-nk} \right]$$

$$= C_z \left[ \sum_{k=0}^{\lfloor \log_2 |\lambda| \rfloor} 2^{(d-2)n/2} |\lambda|^{-1/2} + \sum_{k=\lfloor \log_2 |\lambda| \rfloor + 1}^{+\infty} 2^{-(k((d-2)n+n))} \right].$$ 

(2.9)
If $\sigma > \sigma_2$, then $(d - 2n) - 2(d - 2)\sigma < 0$, the first sum in (2.10) is therefore less than $C_3 |\lambda|^{-1/2}$, the second one is less than $C_3 |\lambda|^{-1/2}$.

If $\sigma = \sigma_2$, then $(d - 2n) - 2(d - 2)\sigma = 0$, the first term is less than $C_3 |\lambda|^{-1/2} \log_2 |\lambda|$, the second term is less than $C_3 |\lambda|^{-1/2}$.

If $\sigma_1 < \sigma < \sigma_2$, then $(d - 2n) - 2(d - 2)\sigma > 0$ and $(d - 2)\sigma + n > 0$, the first sum is less than $C_3 |\lambda|^{-(d-2)\sigma+n}/d$ and so is the second sum.

Summing up the three cases above, we complete the proof of Theorem 2.1.

3. $H^1 - L^1$ mapping property of the damped oscillatory integral operators

By using the result in [11], Pan [6] establish the $H^1_E - L^1$ boundedness for oscillatory singular integral operators, where $H^1_E$ is a modified Hardy space. Later, Yang [15] and Shi [13] developed the method of Pan to get their corresponding $H^1 - L^1$ and $H^1_E - L^1$ boundedness results for the oscillatory operators with homogeneous polynomial phase function. In fact, based on these works, the next result can be obtained.

**Theorem 3.1.** Define an operator

$$T^P f(x) = \int_{\mathbb{R}^n} e^{iP(x,y)} \|S_{xy}''\|_{HS}^{\sigma_1 + it} \psi(x,y)f(y) \, dy$$

where $P(x,y) = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta$ is a higher dimensional polynomial with $c_{0, \beta} = 0$ for any $\beta$. If $\|S_{xy}''\|_{HS}^{1/(d-2)}$ is a norm of $\mathbb{R}^n \times \mathbb{R}^n$, then $T^P$ maps $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with operator norm less than $C(1 + |t|)$ in which $C$ is a constant independent of the coefficients of $P(x,y)$.

The inductive argument in Pan [6] starts with the the $L^p$ boundedness of the oscillatory singular integral operator obtained in [11]. This method requires us to consider the following operator

$$T_0(f)(x) = \int_{\mathbb{R}^n} \|S_{xy}''\|_{HS}^{\sigma_1 + it} \psi(x,y)f(y) \, dy.$$ If we set $K(x, y) = \|S_{xy}''\|_{HS}^{\sigma_1 + it} \psi(x, y)$, then the operator equals

$$T_0(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y) \, dy \quad (3.10)$$

3.1 Mapping property of $T_0$.

**Theorem 3.2.** Considering the operator $T_0$ defined in (3.10), if $\|S_{xy}''\|_{HS}^{1/(d-2)}$ is a norm of $\mathbb{R}^n \times \mathbb{R}^n$, it follows

(i) $T_0$ is of type $(p, p)$ whenever $1 < p < +\infty$;

(ii) $T_0$ is of weak type $(1, 1)$;

(iii) $T_0$ maps $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with operator norm less than $C(1 + |t|)$.

**Proof.** (i) By the assumption that $\|S_{xy}''\|_{HS}^{1/(d-2)}$ is a norm of $\mathbb{R}^n \times \mathbb{R}^n$, and the fact that the norms in finite dimensional linear normed space are equivalent, we have $\|S_{xy}''\|_{HS}^{1/(d-2)} \approx (|x|^2 + |y|^2)^{1/2} \approx |x| + |y|$. Since $K(x, y) = \|S_{xy}''\|_{HS}^{\sigma_1 + it} \psi(x, y)$ and $\psi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then

$$|K(x, y)| \lesssim \frac{1}{(|x| + |y|)^n} \approx \frac{1}{|x|^n + |y|^n}.$$
For any $f \in L^p(\mathbb{R}^n)$, $1 < p < +\infty$,

$$
\|T_0f(x)\|_p = \left( \int |T_0f(x)|^p \, dx \right)^{1/p} \leq \left( \int \int |K(x, y)||f(y)|^p \, dy \, dx \right)^{1/p} \\
\les \left( \int \int \frac{|f(y)|}{|x|^{n+1} + |y|^{n+1}} \, dy \, dx \right)^{1/p} \\
= \left( \int \int \frac{|f(|x|y)|}{1 + |y|^{n+1}} \, dy \, dx \right)^{1/p}
$$

By using the polar coordinate $x = R\theta$, $y = r\omega$, the last term equals

$$
\omega_{n-1}^{1/p} \left( \int_0^{\infty} \int_0^{\infty} \int_{S^{n-1}} |f(R\omega)|^{p} R^{n-1} \omega^{n-1} \, d\omega \, dR \right)^{1/p} \\
\leq \omega_{n-1}^{1/p} \int_0^{\infty} \left( \int_0^{\infty} \int_{S^{n-1}} |f(R\omega)|^{p} R^{n-1} \, d\omega \, dR \right)^{1/p} \frac{R^{n-1}}{1 + |r|^{n}} \, dr \\
\leq \omega_{n-1}^{1/p} \int_0^{\infty} \left( \int_0^{\infty} \int_{S^{n-1}} |f(R\omega)|^{p} R^{n-1} \omega^{n-1/p} \, d\omega \, dR \right)^{1/p} \frac{R^{n-1}}{1 + |r|^{n}} \, dr \\
= \omega_{n-1} \int_0^{\infty} \left( \int_0^{\infty} \int_{S^{n-1}} |f(R\omega)|^{p} R^{n-1} \, d\omega \, dR \right)^{1/p} \frac{R^{n-1}}{1 + |r|^{n}} \, dr \\
= \omega_{n-1} \|f\|_p \int_0^{\infty} \frac{r^{n-1/p} - r^{n-1}}{1 + |r|^{n}} \, dr.
$$

Since the integral in the last term is finite, then $\|T_0f\|_p \leq C\|f\|_p$ obviously.

(ii) For any $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, we can decompose it into $f(x) = g(x) + b(x)$ by Calderón-Zygmund decomposition. Here

$$
b = \sum b_j; \\
b_j = (f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) \, dy) \chi_{Q_j}(x)
$$

where $Q_j$ is a cube centered at $x_{Q_j}$ with side length $d_{Q_j}$. Let $Q_j^*$ denote the cube centered at $x_{Q_j}$ with side length $Md_{Q_j}$ where $M$ is a constant large enough. Thus

$$
|\{ x : |T_0f(x)| > \lambda \}| \leq |\{ x : |T_0g(x)| > \lambda/2 \}| + |\{ x : |T_0b(x)| > \lambda/2 \}| \\
\leq 2/\lambda \|g\|_1 + \sum_{j} |Q_j^*| + |\{ x \in (\cup_j Q_j^*)^c : |T_0b(x)| > \lambda/2 \}| \\
\lesssim \|g\|_1/\lambda + \|f\|_1/\lambda + |\{ x \in (\cup_j Q_j^*)^c : |T_0b(x)| > \lambda/2 \}| \\
\lesssim \|f\|_1/\lambda + |\{ x \in (\cup_j Q_j^*)^c : |T_0b(x)| > \lambda/2 \}| 
$$
From (3.10), it follows
\[\left|\{x \in (\cup_j Q_j)^c : |T_0 b(x)| > \lambda/2\}\right| \leq \frac{2}{\lambda} \int_{(\cup_j Q_j)^c} |T_0 b(x)| \, dx\]
\[= \frac{2}{\lambda} \int_{(\cup_j Q_j)^c} \int_{\mathbb{R}^n} K(x,y) b(y) \, dy \, dx\]
\[= \frac{2}{\lambda} \int_{(\cup_j Q_j)^c} \sum_j \int_{Q_j} K(x,y) b_j(y) \, dy \, dx\]
\[\leq \sum_j \frac{2}{\lambda} \int_{Q_j} \int_{Q_j} K(x,y) b_j(y) \, dy \, dx.\]

By the vanishing property of \(b_j\), we have
\[\int_{(Q_j)^c} \int_{Q_j} K(x,y) b_j(y) \, dy \, dx\]
\[= \int_{(Q_j)^c} \int_{Q_j} (K(x,y) - K(x,x_{Q_j})) b_j(y) \, dy \, dx.\]

Obviously,
\[\int_{(Q_j)^c} \int_{Q_j} (K(x,y) - K(x,x_{Q_j})) b_j(y) \, dy \, dx\]
\[\leq \sup_{y \in Q_j} \int_{(Q_j)^c} |K(x,y) - K(x,x_{Q_j})| \, dx \cdot \int_{Q_j} |b_j(y)| \, dy.\]

If we can prove that
\[\sup_{y \in Q_j} \int_{(Q_j)^c} |K(x,y) - K(x,x_{Q_j})| \, dx \leq C\] (3.11)
where \(C\) is a constant independent of \(Q_j\), on account of \(\sum_j ||b_j||_1 \leq C||f||_1\), we will conclude (ii). However, analysis of this supremum should be split into two cases as follow.

Case I: \(|x_{Q_j}| < 2d_{Q_j}||. In this case, \(|y - x_{Q_j}| < d_{Q_j}|| yields \(|y| < 3d_{Q_j}||. Note that each entry in \(S_{xy}''\) is a homogeneous polynomial of degree \(d - 2\), and \(|x| \approx |x - x_{Q_j}| > M d_{Q_j} ||| y |||). Thus provided that \(|\sigma_1| \leq 1/2\) we have
\[|\nabla_y K(x,y)| \leq \frac{C(1 + |t|)}{|x|^{n+1}}.\]

Therefore
\[\sup_{y \in Q_j} \int_{(Q_j)^c} |K(x,y) - K(x,x_{Q_j})| \, dx \lesssim \int_{|x - x_{Q_j}| > M d_{Q_j}} \frac{d_{Q_j}}{|x|^{n+1}} \, dx\]
\[\leq \int_{|x| > (M-2) d_{Q_j}} \frac{d_{Q_j}}{|x|^{n+1}} \, dx\]
\[\leq C.\]

Case II: \(|x_{Q_j}| \geq 2d_{Q_j}||.
In this case, since \( y \in Q_j \) then \( |y - x_{Q_j}| < d_{Q_j} \), i.e. \( \frac{1}{2}|x_{Q_j}| < |y| < \frac{3}{2}|x_{Q_j}| \). Hence

\[
\sup_{y \in Q_j} \int_{(Q_j)^c} |K(x, y) - K(x, x_{Q_j})| \, dx \\
\lesssim \int_{|x - x_{Q_j}| > M d_{Q_j}} |K(x, y) - K(x, x_{Q_j})| \, dx \\
= \int_{|x - x_{Q_j}| > M |x_{Q_j}|} \cdots \, dx + \int_{M d_{Q_j} < |x - x_{Q_j}| \leq M |x_{Q_j}|} \cdots \, dx \\
: = A + B.
\]

Observe that \( |y| \approx |x_{Q_j}| \), the estimate of \( A \) is same with Case I and we omit here. For \( B \), we have

\[
B = \int_{M d_{Q_j} < |x - x_{Q_j}| \leq M |x_{Q_j}|} |K(x, y) - K(x, x_{Q_j})| \, dx \\
\leq \int_{M d_{Q_j} < |x - x_{Q_j}| \leq M |x_{Q_j}|} |K(x, y)| + |K(x, x_{Q_j})| \, dx \\
\leq \int_{M d_{Q_j} < |x - x_{Q_j}| \leq M |x_{Q_j}|} \frac{1}{|x|^n + |y|^n} + \frac{1}{|x|^n + |x_{Q_j}|^n} \, dx \\
\leq \int_{|x - x_{Q_j}| \leq M |x_{Q_j}|} \frac{1}{|y|^n} + \frac{1}{|x_{Q_j}|^n} \, dx \\
\leq |x_{Q_j}|^n (\frac{1}{|y|^n} + \frac{1}{|x_{Q_j}|^n}) \\
\leq C.
\]

Thus, the proof of (ii) is complete.

(iii) Let \( a \) denote a \( H^1 \) atom associated with a cube \( Q \) centered at \( x_Q \) with side length \( d_Q \) and

\[
\text{supp } a \subset Q; \\
\|a\|_\infty \leq \frac{1}{|Q|}; \\
\int_Q a \, dx = 0.
\]

Our goal is to prove

\( \|T_0 a\|_1 \leq C \),

where \( C \) is independent of \( Q \). Analogous to the argument of (ii), the proof should be divided into two cases.

Case I: \( |x_Q| < 2d_Q \).

\[
\|T_0 a\|_1 = \int |T_0 a| \, dx = \int_{|x - x_Q| \leq M |d_Q|} |T_0 a| \, dx + \int_{|x - x_Q| > M |d_Q|} |T_0 a| \, dx \\
: = I_1 + I_2.
\]

From the \( L^p(1 < p < +\infty) \) boundedness of \( T_0 \) in (i), we have

\[
I_1 = \int_{|x - x_Q| \leq M |d_Q|} |T_0 a| \, dx \leq (M |d_Q|)^{n/2} \|T_0 a\|_2 \leq (M |d_Q|)^{n/2} \|a\|_2 \leq C.
\]

\( 8 \)
By employing the argument of Case I in (ii) to \( I_2 \), we obtain
\[
I_2 = \int_{(Q^\ast)^c} |T_0a| \, dx = \int_{(Q^\ast)^c} \left| \int_Q K(x,y)a(y) \, dy \right| \, dx = \int_{(Q^\ast)^c} \left| \int_Q (K(x,y) - K(x,x_Q))a(y) \, dy \right| \, dx 
\leq \sup_{y \in Q} \int_{(Q^\ast)^c} |K(x,y) - K(x,x_Q)| \, dx \cdot \int_Q |a(y)| \, dy.
\]
Thus (3.11) together with (3.13) implies \( I_2 \leq C \).

Case II: \( |x_Q| \geq 2d_Q \).
\[
\|T_0a\|_1 = \int |T_0a| \, dx = \int_{|x-x_Q| \leq M|x_Q|} |T_0a| \, dx + \int_{|x-x_Q| > M|x_Q|} |T_0a| \, dx 
: = I_3 + I_4.
\]
Since \( |x_Q| \geq 2d_Q \) and \( y \in Q \), then \( \frac{1}{2}|x_Q| \leq |y| \leq \frac{3}{2}|x_Q| \), it is easy to verify
\[
I_3 = \int_{|x-x_Q| \leq M|x_Q|} |T_0a| \, dx = \int_{|x-x_Q| \leq M|x_Q|} \left| \int_Q K(x,y)a(y) \, dy \right| \, dx 
\leq \int_{|x| \leq (M+1)|x_Q|} \int_{Q} |x^n + |y|^n|a(y)| \, dy \, dx 
= \int_Q \int_{|x| \leq (M+1)|x_Q|/|y|} \frac{1}{|x^n + 1|} \, dx |a(y)| \, dy 
\leq C.
\]

Observe that
\[
I_4 \leq \int_{|x-x_Q| > M|d_Q|} |T_0a| \, dx = \int_{(Q^\ast)^c} |T_0a| \, dx 
= \int_{(Q^\ast)^c} \left| \int_Q K(x,y)a(y) \, dy \right| \, dx 
= \int_{(Q^\ast)^c} \left| \int_Q (K(x,y) - K(x,x_Q))a(y) \, dy \right| \, dx 
\leq \sup_{y \in Q} \int_{(Q^\ast)^c} |K(x,y) - K(x,x_Q)| \, dx \cdot \int_Q |a(y)| \, dy.
\]

On account of (3.11), \( I_4 \leq C \) obviously.

\[
\square
\]

### 3.2 Some useful lemmas

Before we prove Theorem 3.1 some useful lemmas should be stated.

**Lemma 3.3.** Let \( \phi(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha \) be a real-valued polynomial in \( \mathbb{R}^n \) of degree \( d \), and \( \varphi(x) \in C_0^\infty(\mathbb{R}^n) \). If \( a_\alpha \neq 0 \) for \( c_\alpha = d \) we have
\[
\left| \int_{\mathbb{R}^n} e^{i\phi(x)} \varphi(x) \, dx \right| \leq C |a_\alpha|^{-1/d} (\|\varphi\|_\infty + \|\nabla \varphi\|_1).
\]

More details about this lemma can be found in [14]. The following lemma about polynomial was due to Ricci and Stein [11].
Lemma 3.4. Let $P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ denote a polynomial in $\mathbb{R}^n$ of degree $d$. Suppose $\epsilon < 1/d$, then

$$\int_{|x| \leq 1} |P(x)|^{-\epsilon} \, dx \leq A_\epsilon \left( \sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\epsilon}.$$ 

The bound $A_\epsilon$ depends on $\epsilon$ and dimension $n$, but not on the coefficients $\{a_\alpha\}$.

3.3 Proof of Theorem 3.1

Proof. For the atoms in Hardy space, we use the same notations as the proof of (iii). To prove this theorem, we shall use induction on the degree $l$ of $y$ in $P(x,y)$.

If $l = 0$, $P(x,y)$ contains only the pure $x$-term. Then from (iii) we know

$$\|T^P a\|_1 = \|T_0 a\|_1 \leq C.$$ 

We suppose that $\|T^P a\|_1 \leq C$ holds if the degree of $P$ in $y$ is less than $l$. As the proof of (iii), we consider two cases:

Case I: $|x_Q| \leq 2d_Q$.

$$\int \left| T^P a(x) \right| \, dx = \int_{|x-x_Q| \leq Md_Q} \left| T^P a(x) \right| \, dx + \int_{|x-x_Q| > Md_Q} \left| T^P a(x) \right| \, dx$$

$$: = I_5 + I_6.$$ 

Taking absolute value in $I_5$ and recalling the argument of (i), $I_5 \leq C$ obviously.

Write $P(x,y) = \sum_{|\alpha|, |\beta| \leq l} c_{\alpha,\beta} x^\alpha y^\beta + Q(x,y)$, where $Q(x,y)$ is a polynomial with degree in $y$ less than or equal to $l - 1$. We split $I_6$ into two parts,

$$I_6 = \int_{Md_Q < |x-x_Q| < r} \left| \int e^{iP(x,y)} K(x,y)a(y) \, dy \right| \, dx +$$

$$\int_{|x-x_Q| \geq \max\{Md_Q, r\}} \left| \int e^{iP(x,y)} K(x,y)a(y) \, dy \right| \, dx$$

$$: = I_7 + I_8.$$ 

Then for $I_7$, there is

$$I_7 = \int_{Md_Q < |x-x_Q| < r} \left| T^P a \right| \, dx$$

$$= \int_{Md_Q < |x-x_Q| < r} \left| \int e^{iP(x,y)} K(x,y)a(y) \, dy \right| \, dx$$

$$= \int_{Md_Q < |x-x_Q| < r} \left| \int (e^{iP(x,y)} - e^{iQ(x,y)}) K(x,y)a(y) \, dy \right| \, dx +$$

$$\int_{Md_Q < |x-x_Q| < r} \left| \int e^{iQ(x,y)} K(x,y)a(y) \, dy \right| \, dx$$

$$: = I_9 + I_{10}.$$
For $I_9$, we have

$$I_9 = \int_{|x - x_0| < r} \left| \int \left( e^{iP(x,y)} - e^{iQ(x,y)} \right) K(x, y) a(y) \, dy \right| \, dx$$

$$\lesssim \int_{|x - x_0| < r} \int_{Q} \left| \sum_{|\alpha| \geq 1, |\beta| = l} c_{\alpha, \beta} x^\alpha y^\beta \left( \frac{1}{|x|^n + |y|^n} \right) a(y) \right| \, dy \, dx$$

$$\leq \int_{|x - x_0| < r} \int_{Q} \left| \sum_{|\alpha| \geq 1, |\beta| = l} c_{\alpha, \beta} x^\alpha y^\beta \left( \frac{1}{|x|^n} \right) a(y) \right| \, dy \, dx$$

$$\leq \int_{|x - x_0| < r} \int_{Q} \left| \sum_{|\alpha| \geq 1, |\beta| = l} |c_{\alpha, \beta}| |x|^{|\alpha| - n} |y|^{|\beta|} \left( \frac{1}{|x|^n} \right) a(y) \right| \, dy \, dx$$

$$\lesssim \int_{|x - x_0| < r} \int_{Q} \left| \sum_{|\alpha| \geq 1, |\beta| = l} |c_{\alpha, \beta}| |x|^{|\alpha| - n} |d_Q|^l |a(y)\right| \, dy \, dx$$

Since $|x_0| < 2d_Q$ and $\|a\|_\infty \leq \frac{1}{|Q|}$, then

$$\int_{|x - x_0| < r} \int_{Q} \left| \sum_{|\alpha| \geq 1, |\beta| = l} |c_{\alpha, \beta}| |x|^{|\alpha| - n} |d_Q|^l |a(y)\right| \, dy \, dx$$

$$\leq \int_{|x| \leq 2r} \sum_{|\alpha| \geq 1, |\beta| = l} |c_{\alpha, \beta}| |x|^{|\alpha| - n} |d_Q|^l \, dx$$

$$\lesssim |d_Q|^l \sum_{|\alpha| \geq 1, |\beta| = l} |c_{\alpha, \beta}| \|r\|^{\alpha}$$

There must exist $(\alpha_0, \beta_0)$ such that $|\alpha_0| \geq 1, |\beta_0| = l$, and

$$|d_Q|^{|\alpha_0|} |c_{\alpha_0, \beta_0}|^{1/|\alpha_0|} = \max_{|\alpha| \geq 1, |\beta| = l} |d_Q|^{|\alpha|} |c_{\alpha, \beta}|^{1/|\alpha|}.$$

Set $r^{-1} = |d_Q|^{|\alpha_0|} |c_{\alpha_0, \beta_0}|^{1/|\alpha_0|}$. Then $I_9 \leq C$ obviously. On the other hand, by inductive hypothesis, $I_{10} \leq C$. Thus we complete the argument of $I_7$. For $I_8$ we have

$$I_8 = \int_{|x - x_0| \geq \max(Md_Q, r)} \left| \int \left( e^{iP(x,y)} - e^{iQ(x,y)} \right) K(x, y) a(y) \, dy \right| \, dx$$

$$\leq \int_{|x - x_0| \geq \max(Md_Q, r)} \left| \int e^{iP(x,y)} (K(x, y) - K(x, x_Q)) a(y) \, dy \right| \, dx$$

$$+ \int_{|x - x_0| \geq \max(Md_Q, r)} \left| K(x, x_Q) \right| \left| \int e^{iP(x,y)} a(y) \, dy \right| \, dx$$

$$: = I_{11} + I_{12}.$$

From (3.11), it is easy to verify $I_{11} \leq C$. Given $|x_0| \leq 2d_Q$, we have $|K(x, x_Q)| \lesssim \frac{1}{|x|^n + |x_Q|^n} \approx \frac{1}{|x - x_0|^n}$. 

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therefore

\[ I_{12} \leq \int_{|x-x_Q| \geq r} |K(x, x_Q)| \left| \int e^{iP(x, y)} a(y) \, dy \right| \, dx \]

\[ \lesssim \int_{|x-x_Q| \geq r} \frac{1}{|x-x_Q|^n} \left| \int e^{iP(x, y)} a(y) \, dy \right| \, dx \]

\[ = \sum_{j=0}^{+\infty} \int_{R_j} \frac{1}{|x-x_Q|^n} \left| \int e^{iP(x, y)} a(y) \, dy \right| \, dx \]

\[ = \sum_{j=0}^{+\infty} \int_{R_j} \frac{1}{|x-x_Q|^n} \left| \chi_{R_j}(x) \int e^{iP(x, y)} a(y) \, dy \right| \, dx \]

\[ \lesssim \sum_{j=0}^{+\infty} \int_{R_j} \frac{1}{2^{jn}r} \left| \chi_{R_j}(x) \int e^{iP(x, y)} a(y) \, dy \right| \, dx \]

where \( R_j = \{ x \in \mathbb{R}^n : 2^j r \leq |x-x_Q| < 2^{j+1} r \} \). Set \( x = x_Q + 2^j ru, y = x_Q + d_Q v \) and \( P_j(u, v) = P(x_Q + 2^j ru, x_Q + d_Q v) \), then

\[ I_{12} \lesssim \sum_{j=0}^{+\infty} \int_{2^j r \leq |x-x_Q| < 2^{j+1} r} \frac{1}{2^{jn}r} \left| \chi_{R_j}(x) \int e^{iP(x, y)} a(y) \, dy \right| \, dx \]

\[ = \sum_{j=0}^{+\infty} \int_{|u| < 2} \left| \hat{\chi}_{R_j}(u) \int e^{iP_j(u, v)} d_Q a(x_Q + d_Q v) \, dv \right| \, du. \]

Suppose \( \varphi \in C_c^{\infty} (\mathbb{R}^n) \) and

\[ \varphi(v) \equiv 1 \quad \text{for} \quad |v| \leq 1, \quad \varphi(v) \equiv 0 \quad \text{for} \quad |v| \geq 2. \]

Define an operator \( L_j \) by

\[ L_j f(u) = \hat{\chi}_{R_j}(u) \int e^{iP_j(u, v)} \varphi(v) f(v) \, dv. \]

Then

\[ I_{12} \lesssim \sum_{j=0}^{+\infty} \int_{1 \leq |u| < 2} |L_j(b)(u)| \, du \] (3.15)

where \( b(v) = d_Q a(x_Q + d_Q v) \) is an atom associated with the unit cube centered at the origin. Set

\[ L_j(u, w) = \text{Ker}(L_j L_j^*) = \hat{\chi}_{R_j}(u) \hat{\chi}_{R_j}(w) \int e^{iP_j(u, v) - iP_j(w, v)} |\varphi(v)|^2 \, dv. \]

Since

\[ P_j(u, v) - P_j(w, v) = \sum_{|\alpha| \geq 1, |\beta| = l} c_{\alpha, \beta} [(x_Q + 2^j ru)^\alpha - (x_Q + 2^j rw)^\alpha] [x_Q + d_Q v]^\beta + \tilde{Q}(u, w, v). \]
Thus we can obtain
\[ |L_j(u, w)| \leq \tilde{x}_R(u) \tilde{x}_R(w) \sum_{|\alpha| \geq 1} c_{\alpha, \beta_0} [(x_Q + 2^j r u)^\alpha - (x_Q + 2^j r w)^\alpha] d_Q^{1/4} \]
\[ = \tilde{x}_R(u) \tilde{x}_R(w) \sum_{|\alpha| \geq 1} c_{\alpha, \beta_0} 2^j r^{|\alpha|} \left[ \left( \frac{x_Q}{2^j r} + u \right)^\alpha - \left( \frac{x_Q}{2^j r} + w \right)^\alpha \right] d_Q^{1/4} \]
\[ = \tilde{x}_R(u) \tilde{x}_R(w) \sum_{|\alpha| \geq 1} \frac{c_{\alpha, \beta_0} d_Q^j}{C_{\alpha, \beta_0} |\alpha|^{|\alpha|} d_Q^{1/4} |\alpha|^{|\alpha|}} 2^{|\alpha|} \left[ \left( \frac{x_Q}{2^j r} + u \right)^\alpha - \left( \frac{x_Q}{2^j r} + w \right)^\alpha \right] d_Q^{1/4} \]
\[ := \tilde{x}_R(u) \tilde{x}_R(w) \sum_{|\alpha| \geq 1} b_{\alpha, \beta_0} 2^{|\alpha|} \left[ \left( \frac{x_Q}{2^j r} + u \right)^\alpha - \left( \frac{x_Q}{2^j r} + w \right)^\alpha \right] d_Q^{1/4} \]

On the other hand, it is obvious that \( |L_j(u, w)| \leq C \), for a large number \( N \) we have
\[
|L_j(u, w)| \leq C \tilde{x}_R(u) \tilde{x}_R(w) \sum_{|\alpha| \geq 1} b_{\alpha, \beta_0} 2^{|\alpha|} \left[ \left( \frac{x_Q}{2^j r} + u \right)^\alpha - \left( \frac{x_Q}{2^j r} + w \right)^\alpha \right] d_Q^{1/4} \]
\[ \leq C \sum_{|\alpha| \geq 1} b_{\alpha, \beta_0} 2^{|\alpha|} \left[ \left( \frac{x_Q}{2^j r} + u \right)^\alpha - \left( \frac{x_Q}{2^j r} + w \right)^\alpha \right] d_Q^{1/4} \]

Now we come back to (3.15), by Hölder inequality
\[ \int |T^P a| \ dx = \int_{|x-x_Q| \leq 2^j |x_Q|} |T^P a| \ dx + \int_{|x-x_Q| > 2^j |x_Q|} |T^P a| \ dx \]
\[ := I_{13} + I_{14} \]

Case II: \( |x_Q| > 2d_Q \).

In this case, we decompose the integral into two parts
\[ \int |T^P a| \ dx = \int_{|x-x_Q| \leq M|x_Q|} |T^P a| \ dx + \int_{|x-x_Q| > M|x_Q|} |T^P a| \ dx \]

We shall get \( I_{13} \leq C \) from the analogue of \( I_3 \). On the other hand, \( I_{14} \) is similar to \( I_6 \), following the same pattern to deal with \( I_6 \) yields \( I_{14} \leq C \). Thus we complete our proof.
4 Optimality of decay rates and examples

The optimality of decay rates can be derived from the proof of Theorem 4.1 in [3] and we omit here. Next we give an example to demonstrate our main result.

Let $n = 2$, $d = 6$ and $S(x, y) = \frac{1}{5}(x_1^2y_1 + x_1y_1^5 + x_1x_2^3y_2 + x_1y_1^4y_2 + x_1x_2y_1 + x_2y_1^4 + x_2^5y_2 + x_2y_2^5)$, then the Hessian matrix of $S(x, y)$ is

$$S''_{xy} = \begin{pmatrix} x_1^4 + y_1^4 & x_1^4 + y_1^4 \\ x_1^4 + y_1^4 & x_2^4 + y_2^4 \end{pmatrix}$$

Hence

$$\|S''_{xy}\|^{1/(d-2)}_{HS} = \left[ (x_1^4 + y_1^4)^2 + (x_1^4 + y_1^4)^2 + (x_2^4 + y_2^4)^2 + (x_2^4 + y_2^4)^2 \right]^{1/8}.$$ 

In fact the equation above can be regarded as composition of three different simple norms. Then $\|S''_{xy}\|^{1/(d-2)}_{HS}$ is a norm in $\mathbb{R}^2 \times \mathbb{R}^2$ obviously. Thus this example satisfies the decay estimate in Theorem A.

If we let $n = 2$, $d = 6$ and $S(x, y) = \frac{1}{5}(x_1^2y_1 + x_1y_1^5 + x_2^3y_2 + x_2y_2^5)$, then the Hessian matrix of $S(x, y)$ is

$$S''_{xy} = \begin{pmatrix} x_1^4 + y_1^4 & 0 \\ 0 & x_2^4 + y_2^4 \end{pmatrix}.$$ 

This is the most simple case because the related oscillatory integral operator can be separated variables. By iterating the one-dimensional result of [13] we can show that

$$\|T_\lambda\|_p \leq C\lambda^{-1/3}$$ 

for $6/5 \leq p \leq 6$.

Thus $d/(d - n) < p < d/n$ is not necessary to guarantee the sharp decay.

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