Scale-free linear protocol design for global regulated state synchronization of discrete-time double-integrator multi-agent systems subject to actuator saturation

Zhenwei Liu  
College of Information Science and Engineering, Northeastern University  
Shenyang, China  
liuzhenwei@ise.neu.edu.cn

Ali Saberi  
School of Electrical Engineering and Computer Science, Washington State University  
Pullman, WA, USA  
saberi@wsu.edu

Anton A. Stoorvogel  
Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente  
Enschede, The Netherlands  
A.A.Stoorvogel@utwente.nl

Abstract—This paper studies global regulated state synchronization of discrete-time double-integrator multi-agent systems subject to actuator saturation by utilizing localized information exchange. We propose a scalable linear protocol that achieves global regulated state synchronization for any network with arbitrary number of agents and arbitrarily directed communication graph that has a path between each agent and exosystem which generates the reference trajectory.

Index Terms—Discrete-time; double-integrator; multi-agent systems; actuator saturation; global regulated state synchronization; scale-free

I. Introduction

The synchronization or consensus problem of multi-agent systems (MAS) has drawn a lot of investigation in recent years, because of its developing applications in a few fields, like automotive vehicle control, satellites/robots formation, sensor networks, and so forth. The work can be seen, for example in the books [1], [2], [4], [14], [15], [16], [19] and references therein.

MAS subjects to actuator saturation has been studied in both semi-global and global frameworks. Some researchers have worked on (semi) global state and output synchronizations for both continuous- and discrete-time MAS subject to actuator saturation. We can summarize the current existing literature on MAS with linear agents subject to actuator saturation as follows:

1) The semi-global synchronization has been studied in [18] for full-state coupling. For partial state coupling, we have [17], [24] which are based on the extra communication. Meanwhile, the result without the extra communication is developed in [25]. The static protocols via partial state coupling is designed in [10] for G-passive agents and G-passifiable via input feedforward agents.

2) Global synchronization for the case of full-state coupling and an undirected network has been studied by [20] for neutrally stable and double-integrator discrete-time agents. For special case, the result dealing with networks that has a directed spanning tree are provided in [5] for single integrator agents. Then, global synchronization results for partial-state coupling has been studied in [10] for both continuous- and discrete-time G-passive and G-passifiable via input feedforward agent models by a static design, which require the graph is strongly connected and detailed balanced.

Recently, we have initiated a research effort on developing scale-free protocols design for MAS, in which the agent model can be continuous- or discrete-time homogeneous and heterogeneous, and includes external disturbances, input delays, and communication delays ([6]). The scale-free framework for protocol design means that the design does not depend on the communication network, the size of network or the number of agents. In particular, for the case of linear agents subject to actuator saturation, we have designed scale-free nonlinear and linear protocols to achieve the global regulated state synchronization for both continuous- or discrete-time agents, see [13], [7], [9]. The nonlinear design is focusing on at most weakly unstable agents, i.e., eigenvalues of agents are in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems. The linear design focused only on the neutrally stable agents.

We continue the research effort on extending the scale-free linear protocol design to global regulated synchronization of discrete-time double-integrator MAS in this paper. We would like to point out that the double-integrator is a polynomially unstable system. It is also well known that the only significant class of MAS with polynomially unstable agents subject to
actuator saturation for which global synchronization can be achieved is a double-integrator MAS subject to actuator saturation. This class is important due to the practical significance. We propose a family of the linear protocols that are scalable, and any number of this family of protocols can achieve global regulated state synchronization for MAS with any communication graph with arbitrary number of agents as long as the communication graph has a path between the exosystem which generates reference trajectory and each agent.

**Graphs**

A weighted graph $G$ is defined by a triple $(V,E,A)$ where $V = \{1, \ldots, N\}$ is a node set, $E$ is a set of pairs of nodes indicating connections among nodes, and $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix. Moreover, $a_{ij} = 0$ if there is no edge from node $j$ to node $i$. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A path from node $i_1$ to $i_k$ is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, \ldots, k-1$. A directed tree with root $r$ is a subgraph of the graph $G$ in which there exists a unique path from node $r$ to each node in this subgraph. A directed spanning tree is a directed tree containing all the nodes of the graph. The weighted in-degree of node $i$ is given by $d_i = \sum_{j=1}^{N} a_{ij}$. See [16]. For a weighted graph $G$, the matrix $L = \{l_{ij}\}$ with

$$l_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the Laplacian matrix associated with the graph $G$. The Laplacian matrix $L$ has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $1$. See [3].

**II. System description and problem formulation**

Consider a MAS consisting of $N$ identical discrete-time double-integrators with actuator saturation:

$$\begin{align*}
    x_i(t+1) &= Ax_i(t) + B\sigma(u_i(t)), \\
    y_i(t) &= Cx_i(t),
\end{align*}$$

where $x_i(t) \in \mathbb{R}^{2n}$, $y_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ are the state, output, and input of agent $i$, respectively, for $i = 1, \ldots, N$. And

$$A = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad C = (I \ 0).$$

Meanwhile,

$$\sigma(v) = \begin{pmatrix} \text{sat}(v_1) \\ \vdots \\ \text{sat}(v_m) \end{pmatrix},$$

where $v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^m$ and sat($v$) is standard saturation function satisfying sat($v$) = $\text{sgn}(v) \min(1, |v|)$.

The communication network provides agent $i$ with the following information,

$$\tilde{z}_i(t) = \sum_{j=1, i \neq j}^{N} a_{ij}(y_i(t) - y_j(t)),$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$. The network topology can be described by a weighted graph $G$ associated with (2), where the $a_{ij}$ are the coefficients of adjacency matrix $A$. $	ilde{z}_i(t)$ can be rewritten using associated Laplacian matrix as

$$\tilde{z}_i(t) = \sum_{j=1}^{N} l_{ij}y_j(t).$$

Now, we consider regulated state synchronization problem. We need a reference trajectory, which is generated by the following exosystem

$$x_i(t + 1) = Ax_i(t), \quad y_i(t) = Cx_i(t).$$

The objective of regulated state synchronization is

$$\lim_{t \to \infty} (x_i(t) - x_r(t)) = 0$$

for all $i = 1, \ldots, N$.

Clearly, we need some level of communication between the reference trajectory and the agents. Thus, we assume that a nonempty subset $S$ is given. The agents in this set have access to their own output relative to the reference trajectory $y_r(t) \in \mathbb{R}^p$. It means that each agent has access to the quantity

$$\psi_i(t) = \nu(y_i(t) - y_r(t)), \quad \nu_i = \begin{cases} 1, & i \in S, \\ 0, & i \notin S. \end{cases}$$

Then, the information available for each discrete-time agent can be expressed by

$$\tilde{z}_i(t) = \frac{1}{2 + D_m(t)} \sum_{j=1}^{N} a_{ij}(y_i(t) - y_j(t)) + \nu_i(y_i(t) - y_r(t)).$$

(7)

where $D_m(i)$ is the upper bound of $d_m(i) = \sum_{j=1}^{N} a_{ij}$ for $i = 1, \ldots, N$.

**Remark 1** $D_m(i)$ is still local information as explained in [9, Remark 2].

Thus, we use the node set $S$ as root set. Then, we define an expanded Laplacian matrix as

$$\tilde{L} = L + \text{diag}([\tilde{\nu}_i]) = [\tilde{l}_{ij}]_{N \times N}$$

for any graph with the Laplacian matrix $L$. Since the sum of its rows does not need to be zero, $\tilde{L}$ is not a regular Laplacian matrix associated to the graph. Furthermore, it should be emphasized that $\tilde{l}_{ij} = l_{ij}$ for $i \neq j$. Equation (7) can be rewritten as

$$\tilde{\tilde{z}}_i(t) = \frac{1}{2 + D_m(t)} \sum_{j=1}^{N} \tilde{\nu}_j(y_i(t) - y_r(t)).$$

(9)

To guarantee that each agent can achieve the required regulation, we need to make sure that there exists a path to each node starting with node from the set $S$. Therefore, we define the following set of graphs.
**Definition 1** Given a node set $S$, we denote by $G^N_S$ the set of all directed graphs with $N$ nodes containing the node set $S$, such that every node of the network graph $G \in G^N_S$ is a member of a directed tree which has its root contained in the node set $S$.

**Remark 2** According to Definition 1, we know that graph set $G^N_S$ only requires the directed trees with their root in the node set $S$. In particular, $G \in G^N_S$ does not require necessarily the existence of directed spanning tree.

For any graph $G \in G^N_S$, with the associated expanded Laplacian matrix $\bar{L}$, we define

$$\bar{D} = I_N - (2I_N + D_m)^{-1}L,$$

where

$$D_m = \text{diag}(D_m(1), D_m(2), \ldots, D_m(N)).$$

It is easily verified that the matrix $\bar{D}$ has all eigenvalues in the open unit disk based on [12, Lemma 1].

Meanwhile, we introduce an additional information exchange among each agent and its neighbors. In particular, each agent $i$ ($i = 1, \ldots, N$) has access to the following additional information $\dot{\xi}_i(t)$, i.e.

$$\dot{\xi}_i(t) = \frac{1}{2 + D_m(i)} \sum_{j=1}^{N} a_{ij}(\dot{\xi}_j(t) - \xi_j(t))$$

where $\xi_j(t)$ is a variable produced internally by agent $j$ and defined in the following sections.

Then, we formulate the global regulated state synchronization problem for a MAS via linear protocols based on the additional information exchange (12).

**Problem 1** Consider a MAS described by (1), and the associated exosystem (3) and (4). Let a set of nodes $S$ be given which defines the set $G^N_S$. Let the associated network communication be given by Definition 1.

The scalable global regulated state synchronization problem with additional information exchange via linear dynamic protocol is to find a linear dynamic protocol, using only the knowledge of agent model $(A, B, C)$, of the form

$$\begin{cases}
  x_{c,i}(t+1) = A_{c,i}x_{c,i}(t) + B_{c,i}u_i(t) \\
  u_i(t) = K_{c,i}x_{c,i}(t)
\end{cases}$$

where $\dot{\xi}_i(t)$ is defined in (12) with $\xi_i(t) = H_{c,i}x_{c,i}(t)$, and $x_{c,i}(t) \in \mathbb{R}^{n_c}$, such that regulated state synchronization (5) is achieved for any $N$ and any graph $G \in G^N_S$, and for all initial conditions of the agents $x_i(0) \in \mathbb{R}^n$, the exosystem $x_{c,i}(0) \in \mathbb{R}^{n_c}$, and the protocols $x_{c,i}(t) \in \mathbb{R}^{n_c}$.

**Remark 3** Very recently, we have published a conference paper [8] which dealt with global state synchronization of discrete-time double-integrator MAS. In that paper, the scalability has some restrictions, namely it is required that the communication has a directed spanning tree and the associated root agent has an input identical to zero (and its protocol is therefore completely different than the protocols for the other agents).

### III. Scalable Protocol Design

#### A. Full-state coupling case (i.e., $C = I$)

We design the following linear dynamic protocol.

\[
\begin{cases}
  \dot{x}_i(t) = A_\Xi(t) + B\sigma(u_i(t)) + A_{\hat{\xi}}(t) - A_{\hat{\xi}}^2(t) - \frac{1}{2D_m(t)} A_\chi(t), \\
  u_i(t) = -K\dot{x}_i(t).
\end{cases}
\]

Then, we choose matrix $K = (k_1 I ~ k_2 I)$, where $k_1$ and $k_2$ satisfy the following set of inequalities:

\[
\begin{cases}
  0 < k_1 < 2, \\
  k_2 > 0, \\
  (4 + k_1 - 2k_2)(3k_1 - 2k_2) < 0.
\end{cases}
\]

The solution set for (14) is given in Remark 5. And the agents communicate $\dot{\xi}_i(t)$ which is chosen as $\dot{\xi}_i(t) = \dot{x}_i(t)$. Namely,

\[
\dot{\xi}_i(t) = \dot{x}_i(t) = \frac{1}{2 + D_m(i)} \sum_{j=1}^{N} a_{ij}(\dot{x}_j(t) - \dot{x}_j(t)).
\]

**Remark 4** Please note that since agent $i$ has access to $\psi_i(t)$, that implies $u_i$ is known to agent $i$.

**Remark 5** The solution set for (14) can be shown as Fig. 1, where the triangular formed by three points $(0, 0), (0, 2)$, and $(2, 3)$ is the solvable zone of $k_1, k_2$.

**Remark 6** It is interesting to point out that for discrete-time double-integrator system subject to saturation, we obtain that global stabilization by linear state feedback law is achieved if and only if the feedback gain belongs to the exactly the same set as above, see [21], [22], [23].

We have the following theorem by using the above design.

**Theorem 1** Consider a MAS described by (1) with $C = I$, and the associated exosystem (3). Let a set of nodes $S$ be given which defines the set $G^N_S$. Let the associated network communication be given by (7).

Then, the scalable global regulated state synchronization problem with additional information exchange as stated in Problem 1 is solvable. In particular, for any given $k_1$ and $k_2$ satisfying (14), the linear dynamic protocol (13) solves the global regulated state synchronization problem for any $N$ and any graph $G \in G^N_S$.

To obtain this theorem we need the following lemma.
Lemma 1 For all $u, v \in \mathbb{R}^n$, we have
\[(\sigma(v) - \sigma(u))(u - \sigma(u)) \leq 0.\] (16)

**Proof:** Note that we have:
\[(\sigma(v) - \sigma(u))(u - \sigma(u)) = \sum_{i=1}^{n} (\sigma(v_i) - \sigma(u_i))(u_i - \sigma(u_i))\] (17)
when $u = [u_1 \cdots u_n]^T$ and $v = [v_1 \cdots v_n]^T$. Next note that if $u_i \geq 1$ we have $\sigma(v_i) - \sigma(u_i) = \sigma(v_i) - 1 \leq 0$ and $u_i - \sigma(u_i) = u_i - 1 \geq 0$ and hence:
\[(\sigma(v_i) - \sigma(u_i))(u_i - \sigma(u_i)) \leq 0\] (18)

On the other hand if $u_i \leq -1$ we have $\sigma(v_i) - \sigma(u_i) = \sigma(v_i) + 1 \geq 0$ and $u_i - \sigma(u_i) = u_i + 1 \leq 0$ and (18) is still satisfied.

Finally, if $|u_i| \leq 1$ then $u_i - \sigma(u_i) = 0$ and (18) is also satisfied.

Since (18) is satisfied for all $i$ and using (17) we find (16) holds for all $u$ and $v$.

**The proof of Theorem 1:** Firstly, let $\tilde{x}_i(t) = x_i(t) - x_e(t)$, we have
\[
\begin{aligned}
\tilde{x}_i(t + 1) &= A\tilde{x}_i(t) + B\sigma(u_i(t)), \\
\chi_i(t + 1) &= A\chi_i(t) + B\sigma(u_i(t)) + \frac{1}{\sigma(D_{h_n})} \sum_{j=1}^{N} \tilde{e}_{ij} A \left[ \tilde{x}_j(t) - \chi_j(t) \right], \\
u_i(t) &= -(k_1 I \quad k_2 I) \chi_i(t),
\end{aligned}
\] (19)

where
\[
\frac{1}{\sigma(D_{h_n})} \sum_{j=1}^{N} \tilde{e}_{ij} A \chi_j(t) = A\tilde{\chi}_i(t) + \frac{u}{\sigma(D_{h_n})} A\chi_i(t).
\]

Then by defining
\[
\tilde{x}(t) = \begin{pmatrix} \tilde{x}_1(t) \\ \vdots \\ \tilde{x}_N(t) \end{pmatrix}, \chi(t) = \begin{pmatrix} \chi_1(t) \\ \vdots \\ \chi_N(t) \end{pmatrix},
\]
\[
u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix}, \sigma(u(t)) = \begin{pmatrix} \sigma(u_1(t)) \\ \vdots \\ \sigma(u_N(t)) \end{pmatrix},
\]

one can obtain the closed-loop system in the form of
\[
\begin{aligned}
\tilde{x}(t + 1) &= (I_N \otimes A)\tilde{x}(t) + (I_N \otimes B)\sigma(u(t)) \\
\chi(t + 1) &= (I_N \otimes A)\chi(t) + (I_N \otimes B)\sigma(u(t)) + ((I_N - \bar{D}) \otimes A)(\tilde{x}(t) - \chi(t)) \\
u(t) &= -(I_N \otimes \{k_1 I \quad k_2 I\}) \chi(t).
\end{aligned}
\] (20)

Let $e(t) = \tilde{x}(t) - \chi(t)$, we have
\[
\begin{aligned}
\tilde{x}(t + 1) &= (I_N \otimes A)\tilde{x}(t) + (I_N \otimes B)\sigma(u(t)), \\
e(t + 1) &= (\bar{D} \otimes A)e(t), \\
u(t) &= -(I_N \otimes \{k_1 I \quad k_2 I\}) (\tilde{x}(t) - e(t)).
\end{aligned}
\] (21)

Then, let
\[
\tilde{x}_I(t) = (I_N \otimes \{I \quad 0\}) \tilde{x}(t) \text{ and } \tilde{x}_II(t) = (I_N \otimes \{0 \quad I\}) \tilde{x}(t),
\]
we have
\[
\begin{aligned}
\tilde{x}_I(t + 1) &= \tilde{x}_I(t) + \tilde{x}_II(t) \\
\tilde{x}_II(t + 1) &= \tilde{x}_II(t) + \sigma(u(t)) \\
e(t + 1) &= (\bar{D} \otimes A)e(t), \\
u(t) &= -k_1\tilde{x}_I(t) - k_2\tilde{x}_II(t) + (I_N \otimes \{k_1 I \quad k_2 I\}) e(t).
\end{aligned}
\] (22)

The eigenvalues of $\bar{D} \otimes A$ are of the form $\lambda_j \eta_j$, with $\lambda_j$ and $\eta_j$ eigenvalues of $\bar{D}$ and $A$, respectively. Since $|\lambda_j| < 1$ and $\eta_j \equiv 1$, we find $\bar{D} \otimes A$ is Schur stable. Therefore we find that
\[
\lim_{t \to \infty} e(t) = 0.
\] (23)

It also shows that $e_i \in \mathbb{L}_2$. Thus, we just need to prove the stability of (22). Namely, we have $\tilde{x}(t) \to 0$ as $t \to \infty$ with $e_i \in \mathbb{L}_2$, which means that the synchronization result is obtain.

Then, we consider the following weighting Lyapunov function
\[
V(t) = (1 - h)V_1(t) + hV_2(t),
\] (24)
where $h \in (0, 1)$,
\[
V_1(t) = \left[ \sigma(u(t)) - \tilde{x}_II(t) \right] \left[ \begin{pmatrix} 1 + k_2^2 & k_1 \\ k_1 & k_2 \end{pmatrix} \otimes I_{Nn} \right] \left[ \sigma(u(t)) - \tilde{x}_II(t) \right] + 2\sigma(u(t))^2(u(t) - \sigma(u(t))),
\]
\[
V_2(t) = e^T(t)P_D e(t),
\]
with $P_D > 0$ satisfying
\[
(\bar{D} \otimes A)^T P_D (\bar{D} \otimes A) - P_D \leq -2I_{2Nn}.
\] (25)

Here, we obtain $V_1(t)$ and $V_2(t)$ are positive due to $0 < k_1 < 2$ and $P_D > 0$, i.e., $V_1(t) > 0$ except for $(u(t), \tilde{x}_II(t)) = 0$ when
\[ V_1(t) = 0 \text{ and } V_2(t) > 0 \text{ except for } e(t) = 0 \text{ when } V_2(t) = 0. \]

Then, we have
\[
\Delta V_1(t) = V_1(t + 1) - V_1(t) \\
= -(1 - \frac{k_1}{2}) \sigma(u(t + 1)) \sigma(u(t + 1)) + 2 \sigma(u(t + 1))^2 u(t) \\
+ (1 + \frac{k_1}{2}) \sigma(u(t))^2 \sigma(u(t)) + 2 \sigma(u(t))^2 u(t) \\
+ 2 \sigma(u(t + 1))^2 (I - k_1 \sigma(u(t))^2) \\
= (1 - \frac{k_1}{2}) \sigma(u(t + 1)) \sigma(u(t + 1)) + 2 \sigma(u(t + 1))^2 u(t) \\
+ (1 + \frac{k_1}{2}) \sigma(u(t))^2 \sigma(u(t)) = 2 \sigma(u(t + 1)) - \sigma(u(t)) \\
+ 2(1 + k_1 - k_2) \sigma(u(t + 1))^2 \sigma(u(t)) \\
- (1 - \frac{k_1}{2}) \sigma(u(t + 1)) \sigma(u(t + 1)) + 2 \sigma(u(t + 1))^2 u(t) \\
+ (1 + \frac{k_1}{2}) \sigma(u(t))^2 \sigma(u(t)) \\
= 2(1 + k_1 - k_2) \sigma(u(t + 1))^2 \sigma(u(t)) \\
\]

According to condition (27), we obtain
\[
-1 + \frac{k_1}{2} + \frac{\| \Psi \|^2 (1 - h)(k_1^2 + k_2^2)}{h} + \frac{(1 + k_1 - k_2)^2}{1 - \frac{k_1}{2}} < 0. \\
\]

Let \( h \) sufficiently close to 1, the above inequality is less than zero, i.e.,
\[
\frac{\| \Psi \|^2 (1 - h)(k_1^2 + k_2^2)}{h} < \epsilon. \\
\]

It also means that \( \Phi < 0 \). Thus, we have \( \Delta V(t) < 0 \) for \( \sigma(u(t + 1)) \), \( \sigma(u(t)) \) \( \rightarrow 0 \) as \( t \rightarrow \infty \). Furthermore, when \( \Delta V(t) = 0 \), we obtain \( u(t + 1) = u(t) = 0 \) and \( e(t) = 0 \). It is easy to obtain \( \hat{x}_i(t) = \hat{x}(t) = 0 \) at \( \Delta V(t) = 0 \).

Thus, (21) is globally asymptotically stable based on LaSalle’s invariance principle, i.e., the scalable global regulated state synchronization result can be obtained via protocol (13) with condition (14).

B. Partial-state coupling case (i.e., \( C \neq 1 \))

We design a linear dynamic protocol with additional information exchanges (12) via partial-state coupling for agent \( i \in \{1, \ldots, N\} \).

\[
\begin{cases} 
\dot{x}_i(t + 1) = (A - FC)\hat{x}_i(t) + BR^i(t) \xi(t) + F^\omega_i(t) + u_i(t) \in K \chi_i(t), \\
\chi_i(t + 1) = A \chi_i(t) + B \sigma(u_i(t)) + A \xi(t) - A \hat{x}_i(t) - \sigma(u_i(t)). 
\end{cases} \\
(28)
\]

Then, we choose matrix \( K = \begin{pmatrix} k_1 \ & \ k_2 \end{pmatrix} \), where \( k_1, k_2 \) satisfy condition (14). In this protocol, the agents communicate
\[
\begin{pmatrix} 
\xi(t) \\
\xi(t)
\end{pmatrix} = \begin{pmatrix} 
\chi(t) \\
\chi(t)
\end{pmatrix}, \\
\xi(t) = \begin{pmatrix} 
\xi(t) \\
\xi(t)
\end{pmatrix},
\]

i.e., each agent has access to additional information
\[
\begin{pmatrix} 
\hat{x}_i(t) \\
\hat{x}_i(t)
\end{pmatrix} = \begin{pmatrix} 
\hat{x}_i(t) \\
\hat{x}_i(t)
\end{pmatrix},
\]

where
\[
\begin{pmatrix} 
\hat{x}_i(t) \\
\hat{x}_i(t)
\end{pmatrix} = \begin{pmatrix} 
\hat{x}_i(t) \\
\hat{x}_i(t)
\end{pmatrix},
\]

and \( \hat{x}_i(t) \) is defined in (7).

We propose the following theorem.

**Theorem 2** Consider a MAS described by (1), and the associated exosystem (3) and (4). Let a set of nodes \( S \) be given which defines the set \( \mathbb{N}_S \). Let the associated network communication be given by (7).

Then, the scalable global regulated state synchronization problem with additional information exchange as stated in
Problem 1 is solvable. In particular, for any given $k_1$ and $k_2$ satisfying (14), the linear dynamic protocol (28) solves the global regulated state synchronization problem for any $N$ and any graph $G \in \mathcal{G}^{N}$. 

The proof of Theorem 2: Similar to Theorem 1, we have the matrix expression of closed-loop system by defining $\tilde{x}(t) = x_i(t) - x_j(t)$,

$$
\tilde{x}(t) = \begin{pmatrix}
\tilde{x}_1(t) \\
\vdots \\
\tilde{x}_N(t)
\end{pmatrix}, \quad \chi(t) = \begin{pmatrix}
\chi_1(t) \\
\vdots \\
\chi_N(t)
\end{pmatrix}, \\
\theta(t) = \begin{pmatrix}
\theta_1(t) \\
\vdots \\
\theta_N(t)
\end{pmatrix},
$$

\begin{align*}
u(t) &= \begin{pmatrix}
u_1(t) \\
\vdots \\
\nu_N(t)
\end{pmatrix}, \\
\sigma(u(t)) &= \begin{pmatrix}
\sigma(u_1(t)) \\
\vdots \\
\sigma(u_N(t))
\end{pmatrix}, \\
\tilde{x}(t) &= \begin{pmatrix}1_{N} \otimes \{0 \quad 0\} \end{pmatrix} \tilde{x}(t), \quad \tilde{x}_{\ell}(t) = \begin{pmatrix}1_{N} \otimes \{0 \quad 1\} \end{pmatrix} \tilde{e}(t), \\
\ell(t) &= \begin{pmatrix}(\tilde{D} \otimes A)\tilde{e}(t) + \tilde{e}(t)
\end{pmatrix}, \\
\ell(t) &= \begin{pmatrix}1_{N} \otimes \{A \otimes FC\} \end{pmatrix} \tilde{e}(t), \\
u(t) &= -\begin{pmatrix}1_{N} \otimes \{k_1 I \quad k_2 I\} \end{pmatrix} \chi(t).
\end{align*}

Since the eigenvalues of $A - FC$ and $\tilde{D} \otimes A$ are in open unit disk, we just need to prove the stability of $\tilde{x}_i(t)$ and $\tilde{x}_j(t)$.

Similar to the proof of Theorem 1, we can obtain the scalable global regulated state synchronization result. ■

IV. Numerical examples

Due to the limitation of space, we move the Numerical Examples part to the ArXiv version of this paper. See [11].

V. Conclusion

In this paper, we have developed the scalable global regulated state synchronization for discrete-time double-integrator MAS subject to actuator saturation. We proposed a family of parameterized linear protocols with two design parameters. Meanwhile, these two parameters can be arbitrarily chosen from the set shown in Fig. 1. These scalable protocols are designed solely based on agent models without utilizing any network information or the size of network and are universal.

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