THE SECOND MAIN THEOREM OF HOLOMORPHIC MAPS ON SPHERICALLY SYMMETRIC KÄHLER MANIFOLDS

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Abstract. Spherically symmetric manifolds are one class of important Riemannian models in mathematics and physics which includes the most common spaces such as Euclidean spaces, balls and spheres, etc.. In this paper, we consider the Nevanlinna theory concerning value distribution of holomorphic maps from a spherically symmetric Kähler manifold into a complex projective manifold under the assumption that the dimension of sources is not less than one of targets. In our settings, a Second Main Theorem is obtained, which extends the classical Second Main Theorem of holomorphic maps defined on $\mathbb{C}^m$ and $\mathbb{B}^m_{\mathbb{C}}$. In particular, one extends the Carlson-Griffiths’ equi-distribution theory of holomorphic maps from $\mathbb{C}^m$ into complex projective manifolds. When some curvature condition is satisfied, we derive a defect relation of Nevanlinna theory.

1. Introduction

The spherically symmetric manifolds are a class of important Riemannian models in mathematics and physics having many good geometric properties (see [6, 12]), which are related close to many fields such as geometric analysis, gravity field theory and black hole theory, etc.. The familiar Euclidean spaces, upper half-spaces, balls and spheres, etc. are belonging to such class. Our main purpose of this paper is investigating Nevanlinna theory [20] concerning value distribution of holomorphic maps from a spherically symmetric Kähler manifold into complex projective manifolds under a dimension condition. Our work provides a generalization of the classical Nevanlinna theory for $\mathbb{C}^m$ and $\mathbb{B}^m_{\mathbb{C}}$ ([19, 21]), and specially a generalization of Carlson-Griffiths’ equi-distribution theory (see [4, 8, 9]).

Let us first recall the Second Main Theorem of a meromorphic function $f$ on $\mathbb{C}^m$ or $\mathbb{B}^m_{\mathbb{C}}$, where $\mathbb{B}^m_{\mathbb{C}}$ is the unit ball with standard Euclidean metric from $\mathbb{C}^m$. Given different points $a_1, \ldots, a_q \in \mathbb{C} \cup \{\infty\}$, we have the characteristic function $T_f(r)$, counting function $N_f(r, a_j)$ and proximity function $m_f(r, a_j)$ as well as simple counting function $\overline{N}_f(r, a_j)$. We refer the reader to Noguchi

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and Ru [21] for details, see also Hu [9]. The Second Main Theorem is stated as follows

(a) In the case of $\mathbb{C}^m$: for every $\delta > 0$, we have

$$(q - 2) T_f(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + O\left(\log^+ T_f(r) + \delta \log^+ r\right)$$

holds for all $r \in (0, \infty)$ outside a set $E_{\delta}$ of finite Lebesgue measure.

(b) In the case of $\mathbb{B}_C^m$: for every $\delta > 0$, we have

$$(q - 2) T_f(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + O\left(\log^+ T_f(r) + \log \frac{1}{1 - r}\right)$$

holds for all $r \in (0, 1)$ outside a set $E_{\delta}$ with $\int_{E_{\delta}} (1 - r)^{-1} dr < \infty$.

For a non-constant meromorphic function $f$, $T_f(r)$ is bounded from below by $O(\log r)$ as $r \to \infty$. Thus, the result (a) implies the Little Picard Theorem asserting that any non-constant meromorphic function can omit at most two points. Result (a) was extended to complex projective manifolds by Carlson-Griffiths-King [4, 8] (see Corollary 3.7) under the dimension condition that the dimension of target manifolds is not greater than $m$. In the case of $\mathbb{B}_C^m$, the Little Picard Theorem no longer holds, but we shall find from result (b) that $f$ can omit at most two points if $T_f(r)$ grows rapidly enough.

Recently, Atsuji investigated value distribution of meromorphic functions on a complete Kähler manifold of non-positive sectional curvature. In [2], he showed a Second Main Theorem by using the approach of Brownian motion initialized by Carne [5]. In Atsuji's Second Main Theorem, some quantities such as $\log G(r), N(r, \text{Ric})$ are involved, where $G(r)$ is the solution of certain second order ODE depending on Green functions, $N(r, \text{Ric})$ depends on the Ricci curvature of manifolds. To receive a defect relation, $G(r)$ needs to be estimated. So, Atsuji assumed some extra conditions such as Ricci curvature condition and growth condition.

In the present paper, we consider a Second Main Theorem of holomorphic maps $f : M_\sigma \to N$, where $M_\sigma$ is a spherically symmetric Kähler manifold of a pole $o$ and radius $R$ (see Section 2.2 for the definition), and here $N$ is a complex projective manifold with $\dim N \leq \dim M_\sigma$. In the investigations, we remove the restriction conditions such as completeness and non-positiveness of sectional curvature of the source manifolds. Under some growth condition and curvature condition (they seem to be necessary), a defect relation of $f$ is obtained.

Without giving an introduction to notations, we state the main results of this paper as follows
**Theorem A** (=Theorem 3.6). Let $M_\sigma$ be a spherically symmetric Kähler manifold of complex dimension $m$, with a pole $o$ and radius $R$. Let $L$ be a positive line bundle over a complex projective manifold $N$ with $\dim_{\mathbb{C}} N \leq m$, and $D \in |L|$ be of simple normal crossings. Let $f : M_\sigma \to N$ be a differentiably non-degenerate holomorphic map. Then

(a) For $R = \infty$ and every $\delta > 0$

$$T_{f,L}(r) + T_{f,K_N}(r) + T_{\mathcal{R}_{M_\sigma}}(r) \leq N_f, D(r) + O\left(\log^+ T_{f,L}(r) + \delta \log^+ \sigma(r)\right)$$

holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

(b) For $R < \infty$ and every $\delta > 0$

$$T_{f,L}(r) + T_{f,K_N}(r) + T_{\mathcal{R}_{M_\sigma}}(r) \leq N_f, D(r) + O\left(\log^+ T_{f,L}(r) + \log \frac{1}{R-r}\right)$$

holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} (R-r)^{-1} \, dr < \infty$.

Here, we interpret how the classical Second Main Theorem can be derived from ours. In the case of $M_\sigma = \mathbb{C}^m$ with standard Euclidean metric, we have $R = \infty, \sigma(r) = r$ and $\mathcal{R}_{M_\sigma} = 0$, where $\mathcal{R}_{M_\sigma}$ is the Ricci form of $M_\sigma$. By (a) of the above theorem, one can derive the Second Main Theorem of Carlson-Griffiths-King, see Corollary 3.7. Also, the classical Second Main Theorem for the case of $M_\sigma = \mathbb{B}^m_{\mathbb{C}}$ with standard Euclidean metric follows from that $R = 1, \sigma(r) = r$ and $\mathcal{R}_{\mathbb{B}^m_{\mathbb{C}}} = 0$, see Corollary 3.8. Moreover, for the meromorphic functions defined on Poincaré upper half-plane or Poincaré disc, one establishes the Second Main Theorem in the sense of the hyperbolic metric, which is proved to be equivalent to the classical one for the unit disc in the sense of the Euclidean metric, see Corollary 3.9 and its remark.

A manifold is said to be **non-parabolic** if it admits a non-constant positive superharmonic function, and **parabolic** otherwise.

**Theorem B** (=Theorem 3.12). Let $M_\sigma$ be a geodesically complete and non-compact spherically symmetric Kähler manifold of complex dimension $m$. Let $L$ be a positive line bundle over a complex projective manifold $N$ with $\dim_{\mathbb{C}} N \leq m$, and $D \in |L|$ be of simple normal crossings. Let $f : M_\sigma \to N$ be a differentiably non-degenerate holomorphic map. Assume that $M_\sigma$ is parabolic. Then for every $\delta > 0$

$$T_{f,L}(r) + T_{f,K_N}(r) + T_{\mathcal{R}_{M_\sigma}}(r) \leq N_f, D(r) + O\left(\log^+ T_{f,L}(r) + \delta \log^+ r\right)$$

holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

We consider a defect relation under some curvature condition. Let $\Theta_f(D)$ be the defect of $f$ with respect to the divisor $D$ without counting multiplicities. Then we obtain
Theorem C (=Corollary 3.15). The conditions are assumed as same as in Theorem A. In addition, assume that $M_\sigma$ has non-negative scalar curvature.

(a) For $R = \infty$, if $T_{f,L}(r) \geq O(\log^+ \sigma(r))$ as $r \to \infty$, then

$$\Theta_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

(b) For $R < \infty$, if $\log(R - r) = o(T_{f,L}(r))$ as $r \to R$, then

$$\Theta_f(D) \leq \left[ \frac{c_1(K_N^*)}{c_1(L)} \right].$$

2. Spherically symmetric manifolds

2.1. Laplace operators, polar coordinates and Ricci curvatures.

2.1.1. Laplace operators and polar coordinates.

Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The well-known Laplace-Beltrami operator $\Delta_M$ of $\nabla$ is defined by

$$\Delta_M = \sum_{i,j} g^{ij}(\nabla_\partial_i \nabla_\partial_j - \nabla_{\nabla_\partial_i} \partial_j),$$

where $\partial_j = \partial/\partial x_j$ and $(g^{ij})$ is the inverse of $(g_{ij})$. When acting on a function, $\Delta_M$ has the implicit formula

$$\Delta_M = \sum_{i,j} \frac{1}{\sqrt{\det(g_{st})}} \frac{\partial}{\partial x_i} \left( \sqrt{\det(g_{st})} g^{ij} \frac{\partial}{\partial x_j} \right).$$

Fix $o \in M$, one denotes by $B_o(r), S_o(r)$ the geodesic ball and geodesic sphere of radius $r$ with center at $o$ in $M$ respectively, and by $r(x)$ the Riemannian distance function of $x$ from $o$. Set $Cut^*(o) = Cut(o) \cup \{o\}$, where $Cut(o)$ is the cut locus of $o$. For $x \in M \setminus Cut^*(o)$, one can define the polar coordinates $(r, \theta)$ of $x$ with respect to the pole $o$, where $r = r(x)$ is called the polar radius and $\theta \in S^{d-1}$ is called the polar angle which provides the direction $\Gamma_\theta \in T_o M$ of the minimal geodesic connecting $o$ with $x$ at $o$, in which $d = \text{dim } M, S^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$ centered at the origin. Now write the metric $g$ of $M$ in the polar coordinate form

$$ds^2 = dr^2 + \sum_{i,j} \tilde{g}_{ij}d\theta_id\theta_j,$$

where $\theta_j$ are coordinate components of $\theta$, and $\tilde{g}_{ij}$ is the Riemannian metric on $S_o(r) \setminus Cut(o)$. This gives the Riemannian area element on $S_o(r) \setminus Cut(o)$ that

$$dA_r = \sqrt{\det(\tilde{g}_{st})}d\theta_1 \cdots d\theta_{d-1}.$$
If \( \theta_j \) are defined almost everywhere on \( S^{d-1} \), then we have

\[
\text{Area}(S_\alpha(r)) = \int_{S^{d-1}} \sqrt{\det(\tilde{g}_{st})} d\theta_1 \cdots d\theta_{d-1}
\]

In terms of polar coordinates, the Laplace-Beltrami operator is written as

\[
\Delta_M = \frac{\partial^2}{\partial r^2} + \frac{\partial \log \sqrt{\det(\tilde{g}_{st})}}{\partial r} \frac{\partial}{\partial r} + \Delta_{S_\alpha(r)},
\]

where \( \Delta_{S_\alpha(r)} \) is the induced Laplace-Beltrami operator on \( S_\alpha(r) \).

Now we turn to Hermitian manifolds. Let \((M, h)\) be a Hermitian manifold with Hermitian connection \( \tilde{\nabla} \). Note that \( M \) can be regarded as a Riemannian manifold with Riemannian metric \( g = \Re h \), thus there is also the Levi-Civita connection \( \nabla \) on \( M \). In general, \( \tilde{\nabla} \neq \nabla \) since the torsion tensor of \( \tilde{\nabla} \) may not vanish for the general Hermitian manifolds. Hence, the Laplace operator \( \tilde{\Delta}_M \) of \( \tilde{\nabla} \) does not coincide with the Laplace-Beltrami operator \( \Delta_M \) of \( \nabla \). However, the case for \( \tilde{\nabla} = \nabla \) happens when \( M \) is a Kähler manifold. Consequently,

\[
\Delta_M = \tilde{\Delta}_M = 4 \sum_{i,j} h^{ij} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}
\]

acting on a function for that \( M \) is Kählerian, where \( z_j \) are local holomorphic coordinates and \( (h^{ij}) \) is the inverse of \( (h_{ij}) \).

2.1.2. Ricci curvatures.

Let \( (M, h) \) be an \( m \)-dimensional Hermitian manifold with metric form

\[
\alpha = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j.
\]

The metric \( h \) induces a Hermitian metric \( \det(h_{ij}) \) on the anticanonical bundle \( K_M^* \). The Chern form of \( K_M^* \) associated to this metric is defined by

\[
\mathcal{R}_M := c_1(K_M^*, \det(h_{ij})) = -dd^c \log \det(h_{ij})
\]

which is usually called the Ricci form of \( M \) due to \( \mathcal{R}_M = \text{Ric}(\alpha^m) \), where

\[
d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial).
\]

Assume that \( h \) is a Kähler metric, then \( \mathcal{R}_M \) can be written as

\[
\mathcal{R}_M = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} R_{ij} dz_i \wedge d\bar{z}_j,
\]

where \( \text{RicC} = \sum_{i,j} R_{ij} dz_i \otimes d\bar{z}_j \) is the complex Ricci curvature tensor of \( h \). Regard \( M \) as a Riemannian manifold with Riemannina metric \( g = \Re h \), then
there is also a real Ricci curvature tensor written as $\text{Ric}_\mathbb{R} = \sum_{i,j} R_{ij} dx_i \otimes dx_j$ of $g$. Denote by $s_C$, $s_R$ the scalar curvatures of $h, g$ respectively, i.e.,

$$s_C = \sum_{i,j} h^{ij} R_{ij}, \quad s_R = \sum_{i,j} g^{ij} R_{ij}.$$  

Then we have

$$s_R = 2s_C = -\frac{1}{2} \Delta_M \log \det (h_{ij}).$$

2.2. Spherically symmetric manifolds.

Let $(M, g)$ be a Riemannian manifold. We say that $M$ is a manifold with a pole $o$ if $\text{Cut}(o) = \emptyset$. If, in addition, $M$ is complete or geodesically complete, then $M$ is diffeomorphic to $\mathbb{R}^d$, where $d = \dim M$. A Riemannian manifold with a pole $o$ is called a spherically symmetric manifold if the induced metric $\tilde{g}_{ij}$ on $S_o(r)$ is of the form

$$\sum_{i,j} \tilde{g}_{ij}(r, \theta) d\theta_i d\theta_j = \sigma^2(r) d\theta^2,$$

where $d\theta^2 = d\theta_1^2 + \cdots + d\theta_{d-1}^2$ is the standard Euclidean metric on $S^{d-1}$ and $\sigma$ is a positive smooth function of $r$. For convenience, one uses $M_\sigma$ to denote such manifolds.

Let a smooth positive function $\sigma$ on $(0, R)$ with $0 < R \leq \infty$, the necessary and sufficient condition (see [6]) for that such a manifold exists, is that

$$\sigma(0) = 0, \quad \sigma'(0) = 1.$$  

One calls $R$ the radius of $M_\sigma$ with respect to the pole $o$. Clearly, $R = \infty$ if $M$ is geodesically complete and non-compact. Consider a spherically symmetric manifold $M_\sigma$ of a pole $o$ and radius $R$. For $r < R$, we have

$$dA_r = \sigma^{d-1}(r) d\theta_1 \cdots d\theta_{d-1}.$$  

This means that

$$\text{Area}(S_o(r)) = \int_{S^{d-1}} \sigma^{d-1}(r) d\theta_1 \cdots d\theta_{d-1} = \omega_{d-1} \sigma^{d-1}(r),$$

$$\text{Vol}(B_o(r)) = \omega_{d-1} \int_0^r \sigma^{d-1}(t) dt,$$

where $\omega_{d-1}$ is the area of $S^{d-1}$. We also have

$$\Delta_{M_\sigma} = \frac{\partial^2}{\partial r^2} + (d-1) \frac{\sigma'}{\sigma} \frac{\partial}{\partial r} + \frac{1}{\sigma^2} \Delta_\theta,$$

where $\Delta_\theta$ is the standard Laplace-Beltrami operator on $S^{d-1}$.

Several important models

Let $M_\sigma$ be a spherically symmetric manifold of radius $R$. 

(a) If $R = \infty$, $\sigma(r) = r$, then $M_{\sigma} \cong \mathbb{R}^d$ (with standard Euclidean metric). The Laplace-Beltrami operator acquires the form
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{d - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\theta}. \]

(b) If $R = \infty$, $\sigma(r) = \sinh r$, then $M_{\sigma} \cong \mathbb{H}^d$, where $\mathbb{H}^d$ is the $d$-dimensional upper half-space with hyperbolic metric of sectional curvature $-1$ in $\mathbb{R}^d$. The Laplace-Beltrami operator acquires the form
\[ \Delta = \frac{\partial^2}{\partial r^2} + (d - 1) \cot r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r} \Delta_{\theta}. \]

(c) If $R = \pi$, $\sigma(r) = \sin r$, then $M_{\sigma} \cong S^d$ (endpoint with $r = \pi$ is added to $M_{\sigma}$), where $S^d$ is the $d$-dimensional unit sphere centered at 0 in $\mathbb{R}^{d+1}$. The Laplace-Beltrami operator acquires the form
\[ \Delta = \frac{\partial^2}{\partial r^2} + (d - 1) \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{\theta}. \]

2.3. Green functions for spherically symmetric manifolds.

Let $M_{\sigma}$ be a $d$-dimensional spherically symmetric manifold of a pole $o$ and radius $R$. Establish a polar coordinate system $(o, r, \theta)$ of $M_{\sigma}$. For $0 < r < R$, we shall compute the harmonic measure $d\pi_{\sigma}^r(x)$ on $S_o(r)$ with respect to $o$, as well as the Green function $g_r(o, x)$ of $\Delta_{M_{\sigma}}/2$ for $B_o(r)$ with pole at $o$ and Dirichlet boundary condition, i.e.,
\[ -\frac{1}{2} \Delta_{M_{\sigma}} g_r(o, x) = \delta_o(x) \text{ for } x \in B_o(r); \quad g_r(o, x) = 0 \text{ for } x \in S_o(r), \]
where $\delta_o$ is the Dirac function.

**Lemma 2.1.** For $0 < r < R$, we have
\[ d\pi_{\sigma}^r(x) = \frac{d\theta_1 \cdots d\theta_{d-1}}{\omega_{d-1}}, \quad g_r(o, x) = \frac{2}{\omega_{d-1}} \int_{r(x)}^r \frac{dt}{\sigma^{d-1}(t)}, \]
where $\omega_{d-1}$ is the area of the unit sphere in $\mathbb{R}^d$ with $d \geq 2$.

**Proof.** By the property of spherically symmetric manifolds, the induced area measure
\[ dS_r(x) = \sigma^{d-1}(r) d\theta_1 \cdots d\theta_{d-1} \]
on $S_o(r)$ is a rotationally invariant one with respect to $o$. On the other hand, $dS_r(x)/\text{Area}(S_o(r))$ is a probability measure on $S_o(r)$. Thus,
\[ d\pi_{\sigma}^r(x) = \frac{dS_r(x)}{\text{Area}(S_o(r))} = \frac{d\theta_1 \cdots d\theta_{d-1}}{\omega_{d-1}}. \]
Notice the relation
\[ \frac{1}{2} \frac{\partial g_r(o, x)}{\partial n} = \frac{d\pi_{\sigma}^r(x)}{dS_r(x)}. \]
where \( \partial / \partial n \) is the inward normal derivative on \( S_o(r) \). Then

\[
\frac{\partial g_r(o, x)}{\partial n} = - \frac{2}{\omega_{d-1} \sigma^{d-1}(r)}.
\]

On the other hand,

\[
\frac{1}{2} \Delta_{M_o} g_r(o, x) = \delta_o(x)
\]

for \( x \in B_o(r) \). Combine the above two equations, it is trivial to confirm that

\[
g_r(o, x) = \frac{2}{\omega_{d-1}} \int_{r(x)}^r \frac{dt}{\sigma^{d-1}(t)}.
\]

\[\square\]

In what follows, we give two examples to compute Green functions.

**Example 1.** \( M_o = \mathbb{R}^d \) (with standard Euclidean metric)

Take \( o \) as the coordinate origin of \( \mathbb{R}^d \). By \( \sigma(r) = r \) and \( r(x) = \|x\| \), one has

\[
g_r(o, x) = \frac{2}{\omega_{d-1}} \int_{\|x\|}^r \frac{dt}{t^{d-1}},
\]

which can be computed easily.

**Example 2.** \( M_o = \mathbb{H}_C \) (Poincaré upper half-plane, i.e., \( \mathbb{H}^2 \) with Poincaré metric)

Take \( o = (0, \sqrt{-1}) \). Let \( \phi : \mathbb{D} \to \mathbb{H}_C \) be the biholomorphic map as follows

\[
\phi(z) = \frac{1 - \sqrt{-1} z}{z - \sqrt{-1}}.
\]

Note that \( \phi^* h \) is the Poincaré metric on \( \mathbb{D} \), where \( h \) is the Poincaré metric on \( \mathbb{H}_C \). By \( \sigma(r) = \sinh r \) and \( \omega_1 = 2\pi \), we see that

\[
(1) \quad g_r(o, x) = \frac{2}{\pi} \int_{r(x)}^r \frac{dt}{e^t - e^{-t}} = \frac{1}{\pi} \log \frac{(e^r - 1)(e^{r(x)} + 1)}{(e^r + 1)(e^{r(x)} - 1)},
\]

where

\[
r(x) = \log \frac{1 + |\phi^{-1}(x)|}{1 - |\phi^{-1}(x)|}.
\]

3. Holomorphic maps from spherically symmetric Kähler manifolds into complex projective manifolds

3.1. Nevanlinna’s functions and First Main Theorem.
3.1.1. Nevanlinna’s functions.

We extend the notion of Nevanlinna’s functions containing characteristic function, counting function and proximity function to spherically symmetric Kähler manifolds. Let $(M_\sigma, h)$ be a spherically symmetric Kähler manifold of complex dimension $m$, with a pole $o$ and radius $R$. Then the Kähler form of $M$ is written as

$$\alpha = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$  

Fix a $r_0$ such that $0 < r_0 < R$. For any $(1,1)$-form $\eta$ on $M_\sigma$, define formally the notation

$$T_{\eta}(r) = \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{B_o(t)} \eta \wedge \alpha^{m-1}.$$  

Let $f : M_\sigma \to N$ be a holomorphic map into a complex projective manifold $N$, and $(L, h_L)$ be a positive Hermitian line bundle over $N$. Let $|L|$ be the complete linear system of all effective divisors $D_s$ with $s \in H^0(M, L)$, where $D_s$ denotes the zero divisor of a section $s$. For $r_0 < r < R$, the characteristic function of $f$ with respect to $L$ is defined by

$$T_{f,L}(r) = T_{f^*c_1(L, h_L)}(r)$$  

up to a bounded term. Since $M_\sigma$ is Kählerian, then

$$\Delta_M = 4 \sum_{i,j} h_{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$  

Thus,

$$\Delta_{M_\sigma} \log h_L = -4m \frac{f^*c_1(L, h) \wedge \alpha^{m-1}}{\alpha^m}.$$  

In terms of Green function, we have

$$T_{f,L}(r) = -\frac{1}{4} \int_{B_o(r) \setminus B_o(r_0)} g_r(o, x) \Delta_{M_\sigma} \log h_L(x) dV(x),$$  

where $dV$ is the Riemannian volume element of $M_\sigma$. Given a divisor $D \in |L|$, the counting function of $f$ with respect to $D$ is defined by

$$N_{f,D}(r) = \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{f^*D \cap B_o(t)} \alpha^{m-2}.$$  

Let $s_D$ be the canonical section of $L$, it is of zero divisor $D$. Write $s_D = \bar{s}_D e$, where $e$ is a local holomorphic frame of $L$. By Poincaré-Lelong formula [4], $N_{f,D}(r)$ has an alternate expression

$$N_{f,D}(r) = \frac{1}{4} \int_{B_o(r) \setminus B_o(r_0)} g_r(o, x) \Delta_{M_\sigma} \log |\bar{s}_D \circ f(x)|^2 dV(x).$$  

In a similar way, we define the simple counting function $\overline{N}_{f,D}(r)$ for $\text{Supp} f^*D$. 


where \( \sigma \) and formula (see [19, 21]). According to the definition of Nevanlinna’s functions on a Riemannian manifold \( \sigma \). Dynkin formula. Brownian motion, see, e.g., [1, 2, 5, 14, 15]. Dynkin formula which is viewed as a special case of the original probabilistic on a complex manifold, we need Dynkin formula (see [1, 14, 15]) which plays the Nevanlinna’s First Main Theorem. For a meromorphic function defined First Main Theorem.

Remark. When \( \sigma = \mathbb{C}^m \) with standard Euclidean metric, we have \( R = \infty \) and \( \sigma(r) = r \). Since \( d\pi^r_o(z) = d^\sigma \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1} \), the generalized definition of Nevanlinna’s functions agrees with the classical one, see [19, 21].

3.1.2. First Main Theorem.

In the classical Nevanlinna theory, Green-Jensen formula [19, 21] deduces the Nevanlinna’s First Main Theorem. For a meromorphic function defined on a complex manifold, we need Dynkin formula (see [1, 14, 15]) which plays the similar role as Green-Jensen formula. Let’s introduce a simple version of Dynkin formula which is viewed as a special case of the original probabilistic version via Brownian motion, see, e.g., [1, 2, 5, 14, 15].

Dynkin formula. Let \( u \) be a function of \( C^2 \)-class except at most a polar set of singularities on a Riemannian manifold \( M \). For \( 0 < r_0 < r \) or \( 0 \leq r_0 < r \) with \( u(o) \neq \infty \), we have

\[
\int_{S_o(r)} u(x) d\pi^r_o(x) - \int_{S_o(r_0)} u(x) d\pi^r_o(x) = \frac{1}{2} \int_{B_o(r) \setminus B_o(r_0)} g_r(o, x) \Delta_M u(x) dV(x),
\]

where \( B_o(r), S_o(r) \) are geodesic ball and geodesic sphere of radius \( r \) centered at \( o \) respectively, \( g_r(o, x) \) is the Green function of \( \Delta_M / 2 \) for \( B_o(r) \) with pole at \( o \) and Dirichlet boundary condition, and \( d\pi^r_o \) is the harmonic metric on \( S_o(r) \) with respect to \( o \). Here, \( \Delta_M u \) should be understood as distributions.

Particularly, when \( M = \mathbb{C}^m \), Dynkin formula coincides with Green-Jensen formula (see [19, 21]). According to the definition of Nevanlinna’s functions and Dynkin formula, we can easily obtain the First Main Theorem as follows

\[
F. M. T. \quad T_{f,L}(r) = m_{f,D}(r) + N_{f,D}(r) + O(1).
\]
3.2. Logarithmic Derivative Lemma.

Let \((M_\sigma, h)\) be a Hermitian model manifold of complex dimension \(m\), with a pole \(o\) and radius \(R\).

**Lemma 3.1** \([23]\). Let \(\gamma\) be an integrable function on \((0, R)\) with \(\int_0^R \gamma(r) dr = \infty\). Let \(h\) be a nondecreasing function of \(C^1\)-class on \((0, R)\). Assume that \(\lim_{r \to R} h(r) = \infty\) and \(h(r_0) > 0\) for some \(r_0 \in (0, R)\). Then for every \(\delta > 0\)

\[
h'(r) \leq h^{1+\delta}(r) \gamma(r)
\]

holds for all \(r \in (0, R)\) outside a set \(E_\delta\) with \(\int_{E_\delta} \gamma(r) dr < \infty\). In particular, when \(R = \infty\), we can take \(\gamma = 1\). Then for every \(\delta > 0\)

\[
h'(r) \leq h^{1+\delta}(r)
\]

holds for all \(r \in (0, \infty)\) outside a set \(E_\delta\) of finite Lebesgue measure.

Let \(\Gamma\) be a locally integrable function on \(M_\sigma\). Set

\[
E_{\Gamma}(r) = \int_{S_o(r)} \Gamma(x) d\pi_o(x), \quad T_{\Gamma}(r) = \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{B_o(t)} \Gamma(x) \alpha^m.
\]

We need the following so-called Calculus Lemma

**Lemma 3.2.** Let \(\gamma\) be an integrable function on \((0, R)\) with \(\int_0^R \gamma(r) dr = \infty\). Let \(\Gamma\) be a locally integrable function on \(M_\sigma\). Then for every \(\delta > 0\)

\[
E_{\Gamma}(r) \leq \frac{\pi^m}{\omega_{2m-1} m! \sigma^{(2m-1)\delta}(r) \gamma^{2+\delta}(r) T_{\Gamma}^{(1+\delta)^2}(r)}
\]

holds for all \(r \in (0, R)\) outside a set \(E_\delta\) with \(\int_{E_\delta} \gamma(r) dr < \infty\), where \(\omega_{2m-1}\) is the area of the unit sphere \(S^{2m-1}\) in \(\mathbb{R}^{2m}\).

**Proof.** Notice that

\[
\int_{B_o(r)} \Gamma(x) \alpha^m = \frac{m! \omega_{2m-1} \sigma^{2m-1}(r)}{\pi^m} \int_0^r \frac{dt}{\sigma^{2m-1}(t)} \int_{S_o(t)} \Gamma(x) d\pi_o(x),
\]

then we have

\[
\frac{d}{dr} \left( \sigma^{2m-1} dT_{\Gamma} \right) = \frac{m! \omega_{2m-1} \sigma^{2m-1}}{\pi^m} E_{\Gamma}.
\]

Using Lemma 3.1 twice (first to \(\sigma^{2m-1} T_{\Gamma}\) and then to \(T_{\Gamma}\)), then we can prove the lemma. \(\square\)

Let \(\psi\) be a meromorphic function on \(M_\sigma\). The norm of the gradient of \(\psi\) is defined by

\[
\|\nabla_{M_\sigma} \psi\|^2 = \sum_{i,j} h^{-i} \frac{\partial \psi}{\partial z_i} \frac{\partial \psi}{\partial z_j}.
\]
Identify $\psi$ with a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$. The characteristic function of $\psi$ with respect to the Fubini-Study form $\omega_{FS}$ on $\mathbb{P}^1(\mathbb{C})$ is defined by
\[
T_\psi(r) = \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{B_0(t)} f^* \omega_{FS} \wedge \alpha^{m-1}.
\]

Let $i : \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ be an inclusion, then it induces a (1,1)-form $i^* \omega_{FS}$ on $\mathbb{C}$. The Ahlfords characteristic function of $\psi$ is defined by
\[
\hat{T}_\psi(r) = \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{B_0(t)} f^* (i^* \omega_{FS}) \wedge \alpha^{m-1}.
\]
Moreover, we define the Nevanlinna characteristic function
\[
T(r, \psi) = m(r, \psi) + N(r, \psi),
\]
where
\[
m(r, \psi) = \int_{S_o(r)} \log^+ |\psi(x)| d\pi_o(x), \quad N(r, \psi) = \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{f^* \infty \cap B_0(t)} \alpha^{m-2}.
\]
It is trivial to confirm that $\hat{T}_\psi(r) \leq T_\psi(r)$ and $T(r, \psi) = \hat{T}_\psi(r) + O(1)$. Thus, we obtain
\[
T(r, \psi) \leq T_\psi(r) + O(1).
\]

On $\mathbb{P}^1(\mathbb{C})$, take a singular metric
\[
\Phi = \frac{1}{|\zeta|^2 (1 + \log^2 |\zeta|)} \sqrt{-1} \frac{d\zeta \wedge d\bar{\zeta}}{4\pi^2}.
\]
A direct computation gives that
\[
\int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 2m \pi \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m} = \frac{\|\nabla M_e \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)}.
\]
Set
\[
T_{\psi, \Phi}(r) = \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{B_0(t)} \psi^* \Phi \wedge \alpha^{m-1}.
\]

**Lemma 3.3.** We have
\[
T_{\psi, \Phi}(r) \leq T(r, \psi) + O(1).
\]

**Proof.** It yields from Fubini theorem that
\[
T_{\psi, \Phi}(r) = \int_{\mathbb{P}^1(\mathbb{C})} \Phi(\zeta) \int_{r_0}^{r} \frac{dt}{\sigma^{2m-1}(t)} \int_{\psi^* \zeta \cap B_0(t)} \alpha^{m-2} \cdot N_\psi(r, \zeta) \Phi(\zeta)
\]
\[
\leq \int_{\mathbb{P}^1(\mathbb{C})} (T(r, \psi) + O(1)) \Phi
\]
\[
= T(r, \psi) + O(1).
\]
Lemma 3.4. Let $\gamma$ be an integrable function on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $\psi \neq 0$ be a meromorphic function on $M_\sigma$. Then for every $\delta > 0$

$$\int_{S_\sigma(r)} \log^+ \frac{\|\nabla_{M_\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_\sigma^r(x)$$

$$\leq (1 + \delta)^2 \log^+ T(r, \psi) + (2 + \delta) \log^+ \gamma(r) + (2m - 1)\delta \log^+ \sigma(r) + O(1)$$

holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} \gamma(r)dr < \infty$.

Proof. By Jensen inequality

$$\int_{S_\sigma(r)} \log^+ \frac{\|\nabla_{M_\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_\sigma^r(x)$$

$$\leq \int_{S_\sigma(r)} \log \left(1 + \frac{\|\nabla_{M_\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)}\right) d\pi_\sigma^r(x)$$

$$\leq \log^+ \int_{S_\sigma(r)} \frac{\|\nabla_{M_\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_\sigma^r(x) + O(1).$$

Applying Dykin formula, Lemma 3.2 and 3.3 to get

$$\log^+ \int_{S_\sigma(r)} \frac{\|\nabla_{M_\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_\sigma^r(x)$$

$$\leq (1 + \delta)^2 \log^+ \int_{r_0}^r \frac{dt}{\sigma^{2m-1}(t)} \int_{B_\sigma(t)} \frac{\|\nabla_{M_\sigma} \psi(x)\|^2}{|\psi(x)|^2(1 + \log^2 |\psi(x)|)} d\pi_\sigma^x(x)$$

$$+(2 + \delta) \log^+ \gamma(r) + (2m - 1)\delta \log^+ \sigma(r) + O(1)$$

$$=(1 + \delta)^2 \log^+ T(r, \psi) + (2 + \delta) \log^+ \gamma(r) + (2m - 1)\delta \log^+ \sigma(r) + O(1).$$

Combining the above, the lemma is proved. □

Define

$$m \left( r, \frac{\|\nabla_{M_\sigma} \psi\|}{|\psi|} \right) = \int_{S_\sigma(r)} \log^+ \frac{\|\nabla_{M_\sigma} \psi(x)\|}{|\psi(x)|} d\pi_\sigma^r(x).$$

We have the following Logarithmic Derivative Lemma

Theorem 3.5. Let $\gamma$ be an integrable function on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $\psi \neq 0$ be a meromorphic function on $M_\sigma$. Then for every $\delta > 0$

$$m \left( r, \frac{\|\nabla_{M_\sigma} \psi\|}{|\psi|} \right) \leq \frac{2}{2} + \frac{(1 + \delta)^2}{2} \log^+ T(r, \psi) + \frac{(2 + \delta)}{2} \log^+ \gamma(r)$$

$$+(2m - 1)\delta \log^+ \sigma(r) + O(1)$$

holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} \gamma(r)dr < \infty$. 


Proof. Notice that
\[
m \left( r, \frac{\| \nabla M_\sigma \psi \|}{|\psi|} \right) \leq \frac{1}{2} \int_{S_\sigma(r)} \log^+ \frac{\| \nabla M_\sigma \psi(x) \|^2}{\psi(x)^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x) + \frac{1}{2} \int_{S_\sigma(r)} \log^+ (1 + \log^2 |\psi(x)|) d\pi_o^r(x)
\]
\[
\leq \frac{1}{2} \int_{S_\sigma(r)} \log^+ \frac{\| \nabla M_\sigma \psi(x) \|^2}{\psi(x)^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x) + \int_{S_\sigma(r)} \log \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi_o^r(x).
\]
Lemma 3.4 implies that for every \( \delta > 0 \)
\[
\frac{1}{2} \int_{S_\sigma(r)} \log^+ \frac{\| \nabla M_\sigma \psi(x) \|^2}{\psi(x)^2 (1 + \log^2 |\psi(x)|)} d\pi_o^r(x)
\]
\[
\leq \frac{(1 + \delta)^2}{2} \log^+ T(r, \psi) + \frac{(2 + \delta)}{2} \log^+ \gamma(r) + \frac{(2m - 1)\delta}{2} \log^+ \sigma(r) + O(1)
\]
holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} \gamma(r) dr < \infty \). Using Jensen inequality, it follows that
\[
\int_{S_\sigma(r)} \log \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi_o^r(x)
\]
\[
\leq \log \int_{S_\sigma(r)} \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi_o^r(x)
\]
\[
\leq \log \left( m(r, \psi) + m(r, \frac{1}{\psi}) \right) + O(1)
\]
\[
\leq \log^+ T(r, \psi) + O(1).
\]
By the above, we prove the theorem. \( \square \)

3.3. Second Main Theorem.

This subsection aims to prove the following Second Main Theorem

**Theorem 3.6.** Let \( M_\sigma \) be a spherically symmetric Kähler manifold of complex dimension \( m \), with a pole \( o \) and radius \( R \). Let \( L \) be a positive line bundle over a complex projective manifold \( N \) with \( \dim_{\mathbb{C}} N \leq m \), and \( D \in |L| \) be of simple normal crossings. Let \( f : M_\sigma \to N \) be a differentiably non-degenerate holomorphic map. Then for every \( \delta > 0 \)
\[
T_{f, L}(r) + T_{f, KN}(r) + T_{\mathfrak{n}M_\sigma}(r)
\]
\[
\leq N_{f, D}(r) + O \left( \log^+ T_{f, L}(r) + \log^+ \gamma(r) + \delta \log^+ \sigma(r) \right)
\]
holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} \gamma(r) dr < \infty \), where \( \gamma \) is an integrable function on \( (0, R) \) such that \( \int_0^R \gamma(r) dr = \infty \). We have two cases:
(a) For $R = \infty$, we take $\gamma(r) = 1$. Then for every $\delta > 0$
\[ T_{f,L}(r) + T_{f,K_N}(r) + T_{M_{\sigma}}(r) \leq N_{f,D}(r) + O\left( \log^+ T_{f,L}(r) + \delta \log^+ \sigma(r) \right) \]
holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

(b) For $R < \infty$, we take $\gamma(r) = \frac{1}{R-r}$. Then for every $\delta > 0$
\[ T_{f,L}(r) + T_{f,K_N}(r) + T_{M_{\sigma}}(r) \leq N_{f,D}(r) + O\left( \log^+ T_{f,L}(r) + \log \frac{1}{R-r} \right) \]
holds for all $r \in (0, R)$ outside a set $E_\delta$ with $\int_{E_\delta} (R-r)^{-1} \, dr < \infty$.

We first give several consequences before proving Theorem 3.6.

1. Three classical consequences

(a) $M_\sigma = \mathbb{C}^m$ (with standard Euclidean metric)

The case implies that $R = \infty$ and $\sigma(r) = r$. Then we have
\[ T_{f,L}(r) = \int_{t_0}^{r} \frac{dt}{r^{2m-1}(t)} \int_{B_{\sigma}(t)} f^* c_1(L, h_L) \wedge \alpha^{m-1}, \]
which coincides with the classical characteristic function. Since $\mathbb{C}^m$ has sectional curvature 0, then conclusion (a) in Theorem 3.6 derives immediately the classical result of Carlson-Griffiths-King (see [4, 8]) as follows

**Corollary 3.7** (Carlson-Griffiths-King). Let $L$ be a positive line bundle over a complex projective manifold $N$ with $\dim \mathbb{C}N \leq m$, and $D \in |L|$ be of simple normal crossings. Let $f : \mathbb{C}^m \to N$ be a differentiably non-degenerate holomorphic map. Then for every $\delta > 0$
\[ T_{f,L}(r) + T_{f,K_N}(r) \leq N_{f,D}(r) + O\left( \log^+ T_{f,L}(r) + \delta \log^+ r \right) \]
holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

More generalizations of Corollary 3.7 were done by Sakai [24] in terms of Kodaira dimension and by Shiffman [25] in the case of the singular divisor, see also [7, 9, 11, 16, 17, 18, 22, 26, 27].

(b) $M_\sigma = \mathbb{B}^m_\mathbb{C}$ (unit ball of complex dimension $m$ with standard Euclidean metric)

The case implies that $R = 1, \sigma(r) = r$ and $\mathcal{R}_{\mathbb{B}^m_\mathbb{C}} = 0$. We also have
\[ T_{f,L}(r) = \int_{t_0}^{r} \frac{dt}{r^{2m-1}(t)} \int_{B_{\sigma}(t)} f^* c_1(L, h_L) \wedge \alpha^{m-1}, \]
which agrees with the classical characteristic function. By Theorem 3.6 (b), it yields that
Corollary 3.8. Let $L$ be a positive line bundle over a complex projective manifold $N$ with $\dim \mathbb{C} N \leq m$, and $D \in |L|$ be of simple normal crossings. Let $f : \mathbb{B}_m^\mathbb{C} \to N$ be a differentiably non-degenerate holomorphic map, where $\mathbb{B}_m^\mathbb{C}$ is the unit ball with standard Euclidean metric. Then for every $\delta > 0$

$$T_{f,L}(r) + T_{f,K_N}(r) \leq N_{f,D}(r) + O\left( \log^+ T_{f,L}(r) + \log \frac{1}{1-r} \right)$$

holds for all $r \in (0,1)$ outside a set $E_\delta$ with $\int_{E_\delta} (1-r)^{-1} dr < \infty$.

(c) $M_\sigma = \mathbb{H}_\mathbb{C}$ or $\mathbb{D}$ (Poincaré upper half-plane or Poincaré disc)

$\mathbb{H}_\mathbb{C}$ and $\mathbb{D}$ are two representative models in hyperbolic geometry, marking many essential differences from Euclidean geometry. It is important to study the value distribution of a meromorphic function on Poincaré models, which provides an effective tool to investigate the modular functions

$$g(\tau) = \sum_{n \geq m} c_n e^{2\pi i n \tau}$$

on $\mathbb{H}_\mathbb{C}$, where $g$ is called a modular (resp. cusp) form if $m = 0$ (resp. $m = 1$ ), see, e.g., [9, 10].

In what follows, one establishes a Second Main Theorem of meromorphic functions on Poincaré models in the sense of hyperbolic metric. When $M_\sigma = \mathbb{H}_\mathbb{C}$ or $\mathbb{D}$ with Poincaré metric, we have $R = \infty$ and $\sigma(r) = \sinh r$. Let $f$ be a meromorphic function on $\mathbb{H}_\mathbb{C}$ or $\mathbb{D}$. In this situation, we define the Ahlfords characteristic function

$$T_f(r) = \int_{r_0}^{r} \frac{dt}{\sinh t} \int_{B_o(t)} dd^c \log (1 + |f(x)|^2)$$

$$= \frac{1}{4} \int_{B_o(t) \setminus B_o(r_0)} g_r(o,x) \Delta \log (1 + |f(x)|^2) dV(x),$$

where $\Delta$ is the Laplace-Beltrami operator on $\mathbb{H}_\mathbb{C}$ or $\mathbb{D}$. Adopt the spherical distance $\| \cdot, \cdot \|$ on $\mathbb{P}^1(\mathbb{C})$. By definition, the proximity function is that

$$m_f(r,a) = \int_{S_o(r)} \frac{\log 1}{\|f(x),a\|} \pi_o^r(x)$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \log \frac{1}{\|f(r,\theta),a\|} d\theta$$

for a point $a \in \mathbb{C} \cup \{\infty\}$, where $(r,\theta)$ is the polar coordinate of $x$ with respect to the pole $o$. According to the definition (see Section 2.1.1), the counting function is defined by

$$N_f(r,a) = \int_{r_0}^{r} \frac{n_f(t,a)}{\sinh t} dt$$
for a point \( a \in \mathbb{C} \cup \{\infty\} \), where \( n_f(r, a) \) denotes the number of zeros of \( f - a \) in \( B_0(r) \) counting multiplicities. By Dynkin formula, we have the First Main Theorem

\[
T_f(r) = m_f(r, a) + N_f(r, a) + O(1).
\]

For the sake of intuition of \( T_f(r) \), we introduce the Nevanlinna characteristic function

\[
T(r, f) := m(r, f) + N(r, f),
\]

where

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r, \theta)| d\theta, \quad N(r, f) = \int_{r_0}^{r} \frac{n_f(t, \infty)}{\sinh t} dt.
\]

Note that

\[
m_f(r, \infty) = m(r, f) + O(1), \quad N_f(r, \infty) = N(r, f),
\]

hence

\[
T_f(r) = T(r, f) + O(1).
\]

Namely,

\[
T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r, \theta)| d\theta + \int_{r_0}^{r} \frac{n_f(t, \infty)}{\sinh t} dt + O(1).
\]

Let \( \omega_{FS} = dd^c \log \|w\|^2 \) be the Fubini-Study form on \( \mathbb{P}^1(\mathbb{C}) \), where \( w = [w_0 : w_1] \) is the homogeneous coordinate of \( \mathbb{P}^1(\mathbb{C}) \). Let \( a_1, \cdots, a_q \) be different points in \( \mathbb{P}^1(\mathbb{C}) \), and regard \( f = f_1/f_0 = [f_0 : f_1] \) as a holomorphic map into \( \mathbb{P}^1(\mathbb{C}) \). Theorem 3.6 (a) gives that

\[
(q - 2)T_{f, \omega_{FS}}(r) + T_{\mathbb{P}}(r) \leq \sum_{j=1}^{q} N_{f,a_j}(r) + O\left(\log^+ T_f, \omega_{FS}(r) + \delta \log^+ \sinh r\right),
\]

which implies that

\[
(q - 2)T_f(r) + T_{\mathbb{P}}(r) \leq \sum_{j=1}^{q} N_f(r, a_j) + O\left(\log^+ T_f(r) + \delta \log^+ \sinh r\right)
\]

due to

\[
T_f(r, \omega_{FS}) = \int_{r_0}^{r} \frac{dt}{\sinh t} \int_{B_0(t)} dd^c \log (|f_0(x)|^2 + |f_1(x)|^2)
\]

\[
= \int_{r_0}^{r} \frac{dt}{\sinh t} \int_{B_0(t)} dd^c \log (1 + |f(x)|^2) + \int_{r_0}^{r} \frac{dt}{\sinh t} \int_{B_0(t)} dd^c \log |f_0(x)|^2
\]

\[
= T_f(r) + N_f(r, \infty).
\]

As noted in Sections 2.2 and 2.3,

\[
\text{Area}(S_o(r)) = 2\pi \sinh r, \quad g_r(o, x) = \frac{1}{\pi} \log \frac{(e^x - 1)(e^{x(r)} + 1)}{(e^x + 1)(e^{x(r)} - 1)}.
\]
Since $s_{H^2} = s_\mathbb{D} = -1$, then a direct computation leads to

$$T_{\mathbb{H}}(r) = \int_{B_o(r) \setminus B_o(r_0)} g_r(o, x) s(x) dV(x)$$

$$= - \int_{r_0}^r dt \int_{S_o(t)} g_r(o, x) dA_t(x)$$

$$= -2 \int_{r_0}^r \log \left( \frac{e^t - 1}{e^t + 1} \right) \cdot \sinh t dt$$

$$\geq - \int_{r_0}^r \frac{(e^r - 1)(e^t + 1)}{(e^r + 1)(e^t - 1)} \cdot e^t dt$$

$$= - \log \frac{e^{2r} - 1}{e^{2r_0} - 1},$$

which implies that

$$-T_{\mathbb{H}}(r) \leq 2r + O(1).$$

Moreover,

$$\log^+ \sinh r \leq r + O(1).$$

Therefore, we conclude that

**Corollary 3.9.** Let $f$ be a nonconstant meromorphic function on $\mathbb{H}_\subset \subset$ or $\mathbb{D}$, where $\mathbb{H}_\subset \subset, \mathbb{D}$ are the Poincaré upper half-plane and Poincaré disc respectively. Let $a_1, \ldots, a_q$ be distinct points in $\mathbb{C} \cup \{\infty\}$. Then for every $\delta > 0$

$$(q - 2)T_f(r) \leq \sum_{j=1}^q N_f(r, a_j) + O\left( \log^+ T_f(r) + r \right)$$

holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

**Remark.** In fact, Corollary 3.9 is equivalent to the case where $m = 1$ and $N = P^1(\mathbb{C})$ in Corollary 3.8 i.e., the following Second Main Theorem

$$(q - 2)T_f(r) \leq \sum_{j=1}^q N_f(r, a_j) + O\left( \log^+ T_f(r) + \log \frac{1}{1 - r} \right).$$

To see this equivalence, we just need to compare the Second Main Theorem for $\mathbb{D}$ under the Poincaré metric and Euclidean metric. To avoid confusion, denote by $r, \tilde{r}$ the geodesic radius under the Poincaré metric and Euclidean metric respectively, by $r(x), \tilde{r}(x)$ the Riemannian distance functions under the Poincaré metric and Euclidean metric respectively. Similarly, we denote by $g_r(o, x), \tilde{g}_r(o, x)$ as well as $T_f(r), \tilde{T}_f(\tilde{r})$ the Green functions and characteristic functions under the metrics.
We first observe the main error terms, i.e., $O(r)$ and $O(-\log(1-\tilde{r}))$. Take $o$ as the coordinate origin of $\mathbb{D}$. By the relation
\begin{equation}
\gamma(x) = \log \frac{1 + \tilde{r}(x)}{1 - \tilde{r}(x)},
\end{equation}
we see that $r$ corresponds to
\begin{equation}
\log \frac{1 + \tilde{r}}{1 - \tilde{r}} = \log \frac{1}{1 - \tilde{r}} + O(1)
\end{equation}
due to $\tilde{r} < 1$. Thus, the two error terms are equivalent.

Finally, we compare the characteristic functions, i.e., $T_f(r)$ and $\tilde{T}_f(\tilde{r})$. The similar discussions can be applied to the comparisons for counting functions and proximity functions. Under the Euclidean metric, the Green function is written as
\begin{equation}
\tilde{g}_\tilde{r}(o,x) = \frac{1}{\pi} \log \frac{\tilde{r}}{|x|} = \frac{1}{\pi} \log \frac{\tilde{r}}{\tilde{r}(x)}.
\end{equation}
which corresponds to the Green function $g_r(o,x)$ under the Poincaré metric since (1) and (4). Notice that
\[\Delta \log(1 + |f|^2) dV = \tilde{\Delta} \log(1 + |f|^2) d\tilde{V},\]
where $\Delta, \tilde{\Delta}$ denote Laplace-Beltrami operators under the Poincaré metric and Euclidean metric respectively, and $dV, d\tilde{V}$ denote volume elements under the Poincaré metric and Euclidean metric respectively. By the definition of characteristic function, we see that they are a match. Hence, the two Second Main Theorems are actually equivalent under the two metrics.

2. Some other consequences

Let $M$ be a Riemannian manifold with a point $o \in M$. We establish a polar coordinate system $(o, r, \theta)$. For any $x = (r, \theta) \in M$ such that $x \notin Cut^*(o)$, denote by $\text{Ric}_o(x)$ the Ricci curvature of $M$ at $x$ in the direction $\partial/\partial r$. Let $\omega = (\partial/\partial r, X)$ be any pair of tangent vectors from $T_xM$, where $X$ is a unit vector orthogonal to $\partial/\partial r$. Indeed, let $K_\omega(x)$ be the sectional curvature of $M$ at $x$ along the 2-section determined by $\omega$. For a $d$-dimensional spherically symmetric manifold $M_\sigma$, a direct computation yields that
\begin{equation}
\text{Ric}_o(x) = -(d - 1) \frac{\sigma''(r)}{\sigma(r)}, \quad K_\omega(x) = -\frac{\sigma''(r)}{\sigma(r)}
\end{equation}
for all $x = (r, \theta) \in M_\sigma \setminus o$.

**Lemma 3.10** (Ichihara, [12, 13]). Let $\psi$ be a smooth positive function on $(0, \infty)$ such that $\psi'(0) = 1$.

Let $M$ be a $d$-dimensional geodesically complete, non-compact manifold, and $o \in M$. Set $S(r) = \omega_d^{-1} \psi^{d-1}(r)$,
where \( \omega_{d-1} \) is the area of \( S^{d-1} \). Then

(a) If for all \( x = (r, \theta) \notin \text{Cut}^* (o) \)
\[
\text{Ric}_o(x) \geq -(d-1) \frac{\psi''(r)}{\psi(r)}, \quad \int_1^\infty \frac{dr}{S(r)} = \infty,
\]
then \( M \) is parabolic.

(b) If for all \( x = (r, \theta) \neq o \) and all \( \omega \)
\[
K_\omega(x) \geq -\frac{\psi''(r)}{\psi(r)}, \quad \int_1^\infty \frac{dr}{S(r)} < \infty,
\]
then \( M \) is non-parabolic.

**Corollary 3.11.** Let \( M_\sigma \) be a \( d \)-dimensional geodesically complete and non-compact spherically symmetric manifold. Then \( M_\sigma \) is parabolic if and only if
\[
\int_1^\infty \frac{dr}{\sigma^{d-1}(r)} = \infty.
\]

**Proof.** The conclusion follows immediately from (5) and Lemma 3.10. \( \square \)

By Corollary 3.11, \( T_{f, L}(r) \to \infty \) as \( r \to \infty \) if \( M_\sigma \) is parabolic.

**Theorem 3.12.** Let \( M_\sigma \) be a geodesically complete and non-compact spherically symmetric Kähler manifold of complex dimension \( m \). Let \( L \) be a positive line bundle over a complex projective manifold \( N \) with \( \dim \mathbb{C} N \leq m \), and \( D \in |L| \) be of simple normal crossings. Let \( f : M_\sigma \to N \) be a differentiably non-degenerate holomorphic map. Assume that \( M_\sigma \) is parabolic. Then for every \( \delta > 0 \)
\[
T_{f, L}(r) + T_{f, K_N}(r) + T_{\# M_\sigma}(r) \leq \overline{N}_{f, D}(r) + O (\log^+ r) + \delta \log^+ r
\]
holds for all \( r \in (0, \infty) \) outside a set \( E_\delta \) of finite Lebesgue measure.

**Proof.** The completeness and non-compactness imply that \( M_\sigma \) have radius \( R = \infty \). By Corollary 3.11, the parabolicity of \( M_\sigma \) implies that
\[
\int_1^\infty \frac{dr}{\sigma^{2m-1}(r)} = \infty,
\]
which leads to \( \log^+ \sigma(r) \leq O (\log^+ r) \). Apply Theorem 3.6 (a), we prove the theorem. \( \square \)

**Corollary 3.13.** Let \( M_\sigma \) be a geodesically complete and non-compact spherically symmetric Kähler manifold of complex dimension \( m \). Let \( L \) be a positive line bundle over a complex projective manifold \( N \) with \( \dim \mathbb{C} N \leq m \),
and $D \in |L|$ be of simple normal crossings. Let $f : M_\sigma \to N$ be a differen-
tiably non-degenerate holomorphic map. Assume that the Ricci curvature of $M_\sigma$ as a Riemannian manifold is non-negative. Then for every $\delta > 0$
\[ T_f,L(r) + T_f,K_N(r) \leq N_f,D(r) + O(\log^+ r + \delta \log^+ r) \]
holds for all $r \in (0, \infty)$ outside a set $E_\delta$ of finite Lebesgue measure.

**Proof.** The non-negativity of Ricci curvature implies the parabolicity of $M_\sigma$, hence the conclusion holds by using Theorem 3.12. □

**Proof of Theorem 3.6**

Proof. Write $D = \sum_{j=1}^Q D_j$ as the union of irreducible components and equip every $L_{D_j}$ with a Hermitian metric $h_j$. Then it induces a natural Hermitian
metric $h_L = h_1 \otimes \cdots \otimes h_q$ on $L$, which defines a volume form $\Omega := c_1(L,h_L)$ on $N$. Pick $s_j \in H^0(N,L_{D_j})$ with $D_j = (s_j)$ and $\|s_j\| < 1$. On $N$, one defines a singular volume form
\[ \Phi = \Omega \prod_{j=1}^Q \|s_j\|^2. \]
Set
\[ \xi^m = f^* \Phi \land \alpha^{m-n}, \quad \alpha = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^m h_{i\bar{j}} dz_i \land d\bar{z}_j. \]
Note that
\[ \alpha^m = m! \det(h_{ij}) \prod_{j=1}^m \frac{\sqrt{-1}}{2\pi} dz_j \land d\bar{z}_j. \]
A direct computation leads to
\[ dd^c \log \xi \geq f^* c_1(L,h_L) - f^* \text{Ric}(\Omega) + \mathcal{R}_{M_\sigma} - \text{Supp} f^* D \]
in the sense of currents, where $\mathcal{R}_{M_\sigma} = -dd^c \log \det(h_{ij})$. This follows that
\[ T_{dd^c \log \xi}(r) \geq T_f,L(r) + T_{f,K_N}(r) + T_{\mathcal{R}_{M_\sigma}}(r) - N_f,D(r) + O(1). \]

Next we give an upper bound of $T(r, dd^c \log \xi)$. The simple normal crossing
property of $D$ implies that there exist a finite open covering $\{U_\lambda\}$ of $N$ and
finitely many rational functions $w_{\lambda_1}, \cdots, w_{\lambda n}$ on $N$ such that $w_{\lambda_1}, \cdots, w_{\lambda n}$
are holomorphic on $U_\lambda$ for each $\lambda$ as well as
\[ dw_{\lambda_1} \land \cdots \land dw_{\lambda n}(y) \neq 0, \quad \forall y \in U_\lambda, \]
\[ D \cap U_\lambda \neq \{ w_{\lambda_1} \cdots w_{\lambda h_\lambda} = 0 \}, \quad \exists h_\lambda \leq n. \]
Indeed, we can require that $L_{D_j}|_{U_\lambda} \cong U_\lambda \times \mathbb{C}$ for $\lambda, j$. On $U_\lambda$, write
\[ \Phi = \prod_{k=1}^n \frac{\sqrt{-1}}{2\pi} dw_{\lambda k} \land d\bar{w}_{\lambda k}. \]
where \( e_\lambda \) is a positive smooth function on \( U_\lambda \). Set

\[
\Phi_\lambda = \frac{\phi_\lambda e_\lambda}{|w_{\lambda 1}|^2 \cdots |w_{\lambda n}|^2} \left( \prod_{k=1}^{n} \frac{1}{2\pi} \right) dw_{\lambda k} \wedge d\overline{w}_{\lambda k},
\]

where \( \{\phi_\lambda\} \) is a partition of unity subordinate to \( \{U_\lambda\} \). Let \( f_{\lambda k} = w_{\lambda k} \circ f \), then on \( f^{-1}(U_\lambda) \)

\[
f^*\Phi_\lambda = \frac{\phi_\lambda o f \cdot e_\lambda o f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda n}|^2} \left( \prod_{k=1}^{n} \frac{1}{2\pi} \right) \left( df_{\lambda k} \wedge d\overline{f}_{\lambda k} \right)
\]

\[
= \phi_\lambda \cdot f \cdot e_\lambda \cdot f \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq m} \left| \frac{\partial f_{\lambda 1}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda n}}{\partial z_{i_m}} \right|^2 \left| \frac{\partial f_{\lambda (h_\lambda + 1)}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda n}}{\partial z_{i_m}} \right|^2
\]

\[
= \left( \frac{1}{2\pi} \right)^n d\overline{z}_{i_1} \wedge d\overline{z}_{i_2} \wedge \cdots \wedge d\overline{z}_{i_m} \wedge d\overline{w}_{i_1} \wedge d\overline{w}_{i_2} \cdots \wedge d\overline{w}_{i_m}.
\]

Fix any \( x_0 \in M_\sigma \), we may pick a holomorphic coordinate system \( z_1, \cdots, z_m \) near \( x_0 \) and a holomorphic coordinate system \( w_1, \cdots, w_m \) near \( f(x_0) \) so that

\[
\alpha = \left( \frac{1}{2\pi} \right)^{m} \sum_{j=1}^{m} dz_j \wedge d\overline{z}_j, \quad c_1(L, h_L)|_{f(x_0)} = \left( \frac{1}{2\pi} \right)^{n} \sum_{j=1}^{n} dw_j \wedge d\overline{w}_j.
\]

Set

\[
f^*\Phi_\lambda \wedge \alpha^{m-n} = \xi_\lambda \alpha^m.
\]

Then we have \( \xi = \sum \xi_\lambda \) and

\[
\xi_\lambda|_{x_0} = \phi_\lambda \cdot f \cdot e_\lambda \cdot f \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq m} \left| \frac{\partial f_{\lambda 1}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda n}}{\partial z_{i_m}} \right|^2 \left| \frac{\partial f_{\lambda (h_\lambda + 1)}}{\partial z_{i_1}} \right|^2 \cdots \left| \frac{\partial f_{\lambda n}}{\partial z_{i_m}} \right|^2
\]

\[
\leq \phi_\lambda \cdot f \cdot e_\lambda \cdot f \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq m} \left| \nabla_{M_\sigma} f_{\lambda 1} \right|^2 \cdots \left| \nabla_{M_\sigma} f_{\lambda n} \right|^2 \left| \nabla_{M_\sigma} f_{\lambda (h_\lambda + 1)} \right|^2 \cdots \left| \nabla_{M_\sigma} f_{\lambda n} \right|^2.
\]

Again, set

\[
(7) \quad f^*c_1(L, h_L) \wedge \alpha^{m-1} = \rho \alpha^m.
\]

Let \( f_j = w_j \circ f \) for \( 1 \leq j \leq n \), then

\[
f^*c_1(L, h_L) \wedge \alpha^{m-1}|_{x_0} = (m - 1) \sum_{j=1}^{m} \left| \nabla_{M_\sigma} f_j \right|^2 \alpha^m.
\]

That is,

\[
\rho|_{x_0} = (m - 1) \sum_{i=1}^{n} \sum_{j=1}^{m} \left| \frac{\partial f_i}{\partial z_j} \right|^2 = (m - 1) \sum_{j=1}^{n} \left| \nabla_{M_\sigma} f_j \right|^2.
\]
Combine the above, we are led to
\[
\xi_\lambda \leq \frac{\phi_\lambda \circ f \cdot \epsilon_\lambda \circ f \cdot q^{n-h_\lambda}}{(m-1)^{n-h_\lambda}} \sum_{1 \leq i_1 \neq \cdots \neq i_m \leq m} \left\| \nabla_{M_\lambda f} f_{\lambda i_1} \right\|^2 \left\| \nabla_{M_\lambda f} f_{\lambda i_m} \right\|^2 \left\| f_{\lambda i_1} \right\|^2 \left\| f_{\lambda i_m} \right\|^2 \left\| f \right\|^2
\]
on \text{on } f^{-1}(U_\lambda). \text{ Note that } \phi_\lambda \circ f \cdot \epsilon_\lambda \circ f \text{ is bounded on } M_\sigma, \text{ then it follows from } \log \xi \leq \sum_{\lambda} \log^+ \xi_\lambda + O(1) \text{ that }
\[
\log^+ \xi \leq O \left( \log^+ q + \sum_{k,\lambda} \log^+ \frac{\left\| \nabla_{M_\lambda f} f_{\lambda k} \right\|}{\left\| f_{\lambda k} \right\|} \right) + O(1)
on \text{on } M_\sigma. \text{ By Dynkin formula (see Section 3.1.2)}
\[
T_{dd^c} \log \xi(r) = \frac{1}{2} \int_{S_o(r)} \log \xi(x) d\pi_o^r(x) + O(1).
\]
By (8) and (9) with Theorem 3.5
\[
T_{dd^c} \log \xi(r) \leq O \left( \sum_{k,\lambda} \int_{S_o(r)} \log \left\| \nabla_{M_\lambda f} f_{\lambda k} \right\| d\pi_o^r(x) \right) + O(1)
\leq O \left( \sum_{k,\lambda} \log^+ \| \nabla_{M_\lambda f} f_{\lambda k} \| d\pi_o^r(x) \right) + O(1)
\leq O \left( \log^+ T_{f,L}(r) + \log^+ \int_{S_o(r)} \phi(x) d\pi_o^r(x) \right) + O(1).
\]
Lemma 3.2 and (7) imply that for every \( \delta > 0 \)
\[
\log^+ \int_{S_o(r)} \phi(x) d\pi_o^r(x) \leq (1 + \delta)^2 \log^+ T_{f,L}(r) + (2 + \delta) \log^+ \gamma(r) + (2m - 1) \delta \log^+ \sigma(r)
\]
holds for all \( r \in (0, R) \) outside a set \( E_\delta \) with \( \int_{E_\delta} \gamma(r) dr < \infty \). Thus,
\[
T_{dd^c} \log \xi(r) \leq O \left( \log^+ T_{f,L}(r) + \log^+ \gamma(r) + \delta \log^+ \sigma(r) \right) + O(1)
\]
for all \( r \in (0, R) \) outside \( E_\delta \) with \( \int_{E_\delta} \gamma(r) dr < \infty \). Combine (8) with (10), we complete the proof the theorem. \( \square \)

3.4. Defect relations.

Let \( \Theta_f(D) \) be the simple defect of \( f \) with respect to \( D \) defined by
\[
\Theta_f(D) = 1 - \lim_{r \to R} \sup_{r \to R} \frac{\overline{N}_{f,D}(r)}{T_{f,L}(r)}.
\]
For two holomorphic line bundles \( L_1, L_2 \) over \( N \), set
\[
\left\lfloor \frac{c_1(L_2)}{c_1(L_1)} \right\rfloor = \inf \left\{ t \in \mathbb{R} : t \eta_2 < t \eta_1; t \eta_1 \in c_1(L_1), t \eta_2 \in c_1(L_2) \right\}.
\]
Theorem 3.14. Assume the same conditions as in Theorem 3.6.

(a) For $R = \infty$, if $T_{f,L}(r) \geq O(\log^+ \sigma(r))$ as $r \to \infty$, then
\[
\Theta_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right] - \liminf_{r \to \infty} \frac{T_{M_\sigma}(r)}{T_{f,L}(r)}.
\]

(b) For $R < \infty$, if $\log \frac{1}{R-r} = o(T_{f,L}(r))$ as $r \to R$, then
\[
\Theta_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right] - \liminf_{r \to R} \frac{T_{M_\sigma}(r)}{T_{f,L}(r)}.
\]

Proof. The conclusions follow directly from Theorem 3.6. □

Corollary 3.15. Assume the same conditions as in Theorem 3.6. Suppose also that $M_\sigma$ has non-negative scalar curvature.

(a) For $R = \infty$, if $T_{f,L}(r) \geq O(\log^+ \sigma(r))$ as $r \to \infty$, then
\[
\Theta_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right].
\]

(b) For $R < \infty$, if $\log(R-r) = o(T_{f,L}(r))$ as $r \to R$, then
\[
\Theta_f(D) \leq \left[ \frac{c_1(K^*_N)}{c_1(L)} \right].
\]

Proof. The non-negativity of scalar curvature of $M_\sigma$ implies that $T_{M_\sigma}(r) \geq 0$, see [2]. □

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