Joint Numerical Range of Matrix Polynomials

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ABSTRACT

Some algebraic properties of the sharp points of the joint numerical range of a matrix polynomials are the main subject of this paper. We also consider isolated points of the joint numerical range of matrix polynomials.

Key words: joint numerical range, matrix polynomial, sharp points.

1- Introduction:

Let $A \in M_n$ be the algebra of $n \times n$ complex matrices. The classical numerical range of $A$ is the set of a complex numbers $W(A) = \{x^*Ax: x \in \mathbb{C}^n, x^*x = 1\}$ where $\mathbb{C}^n$ vector space (over $\mathbb{C}$) of complex $n$-vectors [6]. There has been many generalizations and applications of the classical numerical range, see, for example [6]. In the following, we consider a generalization of the classical numerical range. Suppose $p(\lambda) = A_0\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A\lambda + A_0$ is a matrix polynomial, where $A_0, A_1, A_2, \ldots, A_m \in M_n$ and $\lambda$ is a complex variable. Define the joint numerical range of $p(\lambda)$ as

$$JNR(p(\lambda)) = \{(x^*A_0x, x^*A_1x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\} [9].$$

This generalized joint numerical range has been discussed by [9]. On the other hand, the joint numerical range of matrix polynomials, being a continuous image of the unit sphere, is compact and connected but not necessarily convex; see Binding and Li [3]. Its convex hull is denoted by $\text{co}(JNR(p(\lambda)))$ and it plays an important role in the study of damped vibration.
systems, with a finite number of degree of freedom [7] and it is useful in various theoretical and applied subjects (see[1,2,3,4 and 5]) and their references. The aim of this paper is to give some algebraic properties of the sharp points of the joint numerical range of matrix polynomials, we also consider an isolated point of the joint numerical range of \( p(\lambda) \). The rest of this paper is organized as follows: In section 2, we present definitions and some basic results which will be used in this paper. In section 3, we prove that if \( \lambda_0 \) is a sharp point of the joint numerical range of the linear pencil \( A_1\lambda_1 - B_1, A_2\lambda_2 - B_2, \ldots, A_m\lambda_m - B_m \) then zero is a sharp point of \( JNR( A_1\lambda_1 - B_1, A_2\lambda_2 - B_2, \ldots, A_m\lambda_m - B_m ). \) and if \( \lambda_0 \) is an isolated point of \( JNR( p(\lambda) ) \), then \( p(\lambda_0) = 0 \).

2- Preliminaries:

In this section, we present some definitions and basic results on joint numerical ranges of matrix polynomials.

Definition 2.1 [8]

A point \( \lambda_0 \in NR( p(\lambda) ) \) is called a sharp point of \( NR( p(\lambda) ) \) if for a connected component \( w_s \) of \( NR( p(\lambda) ) \) there exists a disk centered at \( \lambda_0 \) and with radius \( r \) \( S(\lambda_0, r) \), \( r>0 \) and angles \( \phi_1 \) and \( \phi_2 \) with \( 0 \leq \phi_1 < \phi_2 \leq 2\pi \) such that

\[
\text{Re}(e^{i\phi_0}w_s) = \text{max}(\text{Re} z : e^{i\phi_0}z \in w_s, p(\lambda) \cap S(\lambda_0, r)),
\]

for all \( 0 \leq \phi_1, \phi_2 \).

Definition 2.2 [6]

A matrix \( A \in M_n \) is said to be unitary if \( A^*A = I \), if in addition \( A \in M_n(R) \) then \( A \) is said to be real orthogonal.

Theorem 2.3[6]

If \( A \in M_n \) the following are equivalent.

a) \( A \) is unitary.

b) \( A \) is non singular and \( A^* = A^{-1} \).

c) \( AA^* = I \).

d) \( A^* \) is unitary.

Proposition 2.4 [6]

Suppose that \( Q(\lambda, t) = C_m(t)\lambda^m + C_{m-1}(t)\lambda^{m-1} + \ldots + C_0(t) \lambda + C_0 \) is a polynomial matrix in \( \lambda \) where the coefficient \( C_{j(t)} \) depend continuously on the parameter \( 0 \leq t < \varepsilon \) and \( C_{j(t)} \neq 0 \) for every \( 0 \leq t < \varepsilon \) then \( m \) roots \( \lambda_{j(t)}; \) \( j=1,2,\ldots,m \) of the equation \( Q(\lambda, t)=0 \) are continuous functions in \( t \in (0,\varepsilon) \).
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Definition 2.5 [6]
An n-by-n Hermitian matrix A is said to be positive definite if \( x^*Ax > 0 \) for all non-zero \( x \in \mathbb{C}^n \).

Definition 2.6 [6]
A matrix \( B \in M_n \) is said to be positive semi-definite if \( x^*Bx \geq 0 \) for all \( x \in \mathbb{C}^n \).

Definition 2.7 [6]
The matrix adjoint \( A^* \) of \( A \in M_n(C) \) is defined by \( A^* = A^T \) where \( A^* \) is the component-wise conjugate, and \( A^T \) is the transpose of A.

Definition 2.8 [6]
The matrix \( A \in M_n(C) \) is said to be Hermitian if \( A = A^* \), it is skew-Hermitian if \( A = -A^* \) and for any \( A \in M_n(C) \) can be written \( A = (A + A^*)/2 + (A - A^*)/2 = H(A) + S(A) \) where \( H(A) = (A + A^*)/2 \) is the Hermitian part of A, and \( S(A) = (A - A^*)/2 \) is the skew-Hermitian part of A.

Definition 2.9 [6]
Let \( p_0 \) be an element of an non-empty set A, we say that \( p_0 \) is an isolated point of A if \( \exists N_r(p_0) \) such that \( N_r(p_0) \cap A = \{ p_0 \} \).

3- Properties of Sharp points
In the following, we will restrict ourselves to the definition of sharp points. The next theorem gives a connection of these points with respect to the origin as a joint numerical range of matrix polynomials.

Theorem 3.1
Suppose that \( x_o \) is a unit vector such that \( 0 = x_o^*A_1x,...,0 = x_o^*A_nx \) belongs to the joint numerical range of \( A_1, A_2,..., A_n \in M_n \) if \( x^*A_1x,...,x^*A_nx \) have non-negative real parts for all \( x \) of the neighborhood \( S(x_o, \varepsilon) = \{ x \in \mathbb{C}^n : \| x - x_o \|_2 < \varepsilon \} \) then \( A_1 + A_2^*,...,A_n + A_n^* \) are positive semi-definite.

Proof:
Let \( \lambda_1, \lambda_2,..., \lambda_n \) be the eigenvalues of each of the Hermitian matrices \( A_1 + A_1^*,...,A_n + A_n^* \). We see that \( x^*(A_1 + A_1^*)x = y^*Dy \), \( x^*(A_n + A_n^*)x = y^*Dy \)
where \( D = \text{diag}(\lambda_1, \lambda_2,..., \lambda_n) \), \( y = u^*x \) and \( u \) is a unitary matrix. For \( y_o = u^*x_o = [y_1, y_2,..., y_n]^T \) we have \( y_o^*Dy_o = \lambda_1|y_1|^2 + + \lambda_n|y_n|^2 = 0 \) \( \cdots(*) \), since, \( \| y - y_o \|_2 = \| u^*x - x_o \|_2 = \| x - x_o \|_2, \) and \( \text{Re}(x^*A_1x) \geq 0,\cdots,\text{Re}(x^*A_nx) \geq 0 \)
for all \( x \in S(x_o, \varepsilon) \), there exists a neighborhood \( S(y_o, \varepsilon) \) such that \( y^*Dy \geq 0 \) for any \( y \in S(y_o, \varepsilon) \). Now in \( (*) \) assume that \( \lambda_k < 0 \) consider the
vector $y_k = y_o + \delta e_k = [y_1 \ldots y_k + \delta \ldots y_n]^{T}$ where $\delta \in \mathbb{C}$ such that $0 < |\delta| < \varepsilon$ and $|y_k + \delta| > |y_k|$. Then for a vector $y_\delta$ of the neighborhood $S(y_o, \varepsilon)$ we have $y_\delta^* D y_\delta = \lambda_1 |y_1|^2 + \ldots + \lambda_n |y_n|^2 = \lambda_k (|y_k + \delta|^2 + |y_k|^2) < 0$ which is contradiction. Therefore $\lambda_k \geq 0$ for $k = 1, 2, \ldots, n$ and $D \geq 0$. Hence the matrices $A_1 + A_1^*, \ldots, A_n + A_n^*$ are positive semi-definite.

**Theorem 3.2**

Suppose $x_o$ is a unit vector such that $0 = x_o^* A_1 x_o = \ldots = x_o^* A_n x_o$ belongs to the joint numerical range of $A_1, A_2, \ldots, A_n \in M_n$. If $x^* A_1 x = 0, \ldots, x^* A_n x = 0$ for all $x \in S(x_o, \varepsilon)$ then $A_1 = A_2 = \ldots = A_n = 0$.

**Proof:**

For the matrices $A_1, A_2, \ldots, A_n$ we consider the Hermitian matrices,

$$H(A_1) = \frac{1}{2} (A_1 + A_1^*), H(A_2) = \frac{1}{2} (A_2 + A_2^*), \ldots, H(A_n) = \frac{1}{2} (A_n + A_n^*)$$

and

$$S(A_1) = \frac{1}{2} (A_1^* - A_1), S(A_2) = \frac{1}{2} (A_2^* - A_2), \ldots, S(A_n) = \frac{1}{2} (A_n^* - A_n)$$

then from the hypotheses that $x_0^* A_1 x_o = 0, \ldots, x_0^* A_n x_o = 0$ and $x^* A_1 x = 0, \ldots, x^* A_n x = 0$ for any $x \in S(x_o, \varepsilon)$ it is clear that $x_0^* H(A_1) x_o = 0, \ldots, x_0^* H(A_n) x_o = 0$ and $x^* H(A_1) x = 0, \ldots, x^* H(A_n) x = 0$ for each $x \in S(x_o, \varepsilon)$ and also $x_0^* S(A_1) x_o = 0, \ldots, x_0^* S(A_n) x_o = 0$ and $x^* S(A_1) x = 0, \ldots, x^* S(A_n) x = 0$ for each $x \in S(x_o, \varepsilon)$ thus by Theorem (3.1) we have $H(A_1), H(A_2), \ldots, H(A_n)$ and $S(A_1), S(A_2), \ldots, S(A_n)$ are both positive semi-definite and negative semi-definite and this implies that $H(A_1) = S(A_1) = 0, \ldots, H(A_n) = S(A_n) = 0$, then $A_1 = H(A_1) + i S(A_1) = 0, \ldots, A_n = H(A_n) + i S(A_n) = 0$.

**Theorem 3.3**

Let $A_1, A_2, \ldots, A_n \in M_n$ and $x_o$ be a unit vector such that $x_0^* A_1 x_o = x_0^* A_2 x_o = \ldots = x_0^* A_n x_o = 0$ belongs to the joint numerical range of $A_1, A_2, \ldots, A_n$, then zero is a sharp point of joint numerical range of $A_1, A_2, \ldots, A_n$ if and only if there exists $\varepsilon > 0$, $\phi_1$ and $\phi_2$ such that $\phi_1 \leq \arg(x^* A_1 x) \leq \phi_2, \phi_1 \leq \arg(x^* A_2 x) \leq \phi_2, \ldots, \phi_1 \leq \arg(x^* A_n x) \leq \phi_2$ with $\phi_2 - \phi_1 < \pi$ for all $x \in S(x_o, \varepsilon)$.

**Proof:**

Suppose there exists $\varepsilon > 0$, $\phi_1$ and $\phi_2$ such that $\phi_1 \leq \arg(x^* A_1 x) \leq \phi_2$, $\phi_1 \leq \arg(x^* A_2 x) \leq \phi_2, \ldots, \phi_1 \leq \arg(x^* A_n x) \leq \phi_2$. If $w_1 = \frac{\pi}{2} - \phi_1$ and $w_2 = \frac{3\pi}{2} - \phi_2$.
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then $0 < w_2 - w_1 < \pi$, and for the matrices $e^{iw} A_1 + e^{-iw} A_1^*$, $e^{iw} A_2 + e^{-iw} A_2^*$, ..., $e^{iw} A_n - e^{-iw} A_n^*$

We have $x_0^* (e^{iw} A_1 + e^{-iw} A_1^*) x_0 = 0$, $x_0^* (e^{iw} A_2 + e^{-iw} A_2^*) x_0 = 0$, ..., $x_0^* (e^{iw} A_n + e^{-iw} A_n^*) x_0 = 0$, and $\text{Re}(x^* e^{iw} A_1 x) = \frac{1}{2} x^* (e^{iw} A_1 + e^{-iw} A_1^*) x \leq 0$.

$\text{Re}(x^* e^{iw} A_2 x) = \frac{1}{2} x^* (e^{iw} A_2 + e^{-iw} A_2^*) x \leq 0$, ...

$\text{Re}(x^* e^{iw} A_n x) = \frac{1}{2} x^* (e^{iw} A_n + e^{-iw} A_n^*) x \leq 0$,

for any $w \in [w_1, w_2]$ and for all $x \in S(x_0, e)$. Therefore by theorem (3.1) the matrices $e^{iw} A_1 + e^{-iw} A_1^*$, $e^{iw} A_2 + e^{-iw} A_2^*$, ..., $e^{iw} A_n + e^{-iw} A_n^*$ are non-negative semi definite and $\max \{ x^* (e^{iw} A_1 + e^{-iw} A_1^*) x : \|x\| = 1 \} = 0$, $\max \{ x^* (e^{iw} A_2 + e^{-iw} A_2^*) x : \|x\| = 1 \} = 0$, ..., $\max \{ x^* (e^{iw} A_n + e^{-iw} A_n^*) x : \|x\| = 1 \} = 0$.

Conversely, assume that zero is a sharp point of the joint numerical range of $A_1, A_2, ..., A_n$. Then by the definition of sharp point there exists $w_1$ and $w_2$ belongs to $[0, 2\pi]$, such that for each $w$ belongs to $[w_1, w_2]$, $\max \{ \text{Re}(z) : z \in JNR(e^{iw} (A_1, A_2, ..., A_n)) \} = 0$, where $w_1 = \frac{\pi}{2} - \varphi_1$ and $w_2 = \frac{3\pi}{2} - \varphi_2$.

so this implies that $\max \{ x^* (e^{iw} A_1 + e^{-iw} A_1^*) x : \|x\| = 1 \} = 0$, ..., $\max \{ x^* (e^{iw} A_n + e^{-iw} A_n^*) x : \|x\| = 1 \} = 0$. Because $x^* (e^{iw} A_1 + e^{-iw} A_1^*) x = \text{Re}(x^* e^{iw} A_n x)$, we obtain that $\arg(x^* A_1 x), \arg(x^* A_2 x), ..., \arg(x^* A_n x)$ belongs to $[\varphi_1, \varphi_2]$ where $\varphi_2 - \varphi_1 = \frac{3\pi}{2} - w_2 - \left( \frac{\pi}{2} - w_1 \right) < \pi$.

Theorem 3.4

Let $\lambda_0$ be a sharp point of the joint numerical range of the linear pencil $A_1 \lambda - B_1, A_2 \lambda - B_2, ..., A_n \lambda - B_n$. Then zero is a sharp point of $JNR(A_1 \lambda_0 - B_1, A_2 \lambda_0 - B_2, ..., A_n \lambda_0 - B_n)$.

Proof

By the equality $JNR(A_1 (\lambda + \lambda_0) - B_1, A_2 (\lambda_2 + \lambda_0) - B_2, ..., A_n (\lambda_m + \lambda_0) - B_m) = JNR(A_1 \lambda_1 - B_1, A_2 \lambda_2 - B_2, ..., A_n \lambda_m - B_m) - \lambda_0$, since $\lambda_0$ is a sharp point of $JNR(A_1 \lambda_1 - B_1, A_2 \lambda_2 - B_2, ..., A_n \lambda_m - B_m)$. This implies that zero is a sharp point
of \( JNR( A_1 \lambda_1 + ( A_1 \lambda_0 - B_1 ), A_2 \lambda_2 + ( A_2 \lambda_0 - B_2 ), \ldots, A_m \lambda_m + ( A_m \lambda_0 - B_m ) ) \), thus there exists a vector \( x_o \) such that \( x_o^* ( A_1 \lambda_0 - B_1 ) x_o = 0 \ldots, x_o^* ( A_m \lambda_0 - B_m ) x_o = 0 \) and

\[
x_o^* A_1 x_o = k_1 \neq 0 \ldots, x_o^* A_m x_o = k_m \neq 0.
\]

It is not possible to have \( x_o^* ( A_1 \lambda_0 - B_1 ) x_o = x_o^* A_1 x_o = 0 \ldots, x_o^* ( A_m \lambda_0 - B_m ) x_o = x_o^* A_m x_o = 0 \) because \( JNR( A_1 \lambda_1 + A_1 \lambda_0 - B_1, A_2 \lambda_2 + A_2 \lambda_0 - B_2, \ldots, A_m \lambda_m + A_m \lambda_0 - B_m ) = C^n \). Since zero is a sharp point of \( JNR( A_1 \lambda_1 + A_1 \lambda_0 - B_1, A_2 \lambda_2 + A_2 \lambda_0 - B_2, \ldots, A_m \lambda_m + A_m \lambda_0 - B_m ) \), there exists \( r_1, r_2, \ldots, r_m > 0 \) such that for any complex number

\[
\gamma_{x_1} = -x^* ( A_1 \lambda_0 - B_1 ) x
\]

\[
\in S(0, r_1) \cap \{ \lambda_1 < A_1 x, x + \lambda_0 < A_1 x, x > -B_1 \}
\]

\[
\gamma_{x_2} = -x^* ( A_2 \lambda_0 - B_2 ) x
\]

\[
\in S(0, r_2) \cap \{ \lambda_2 < A_2 x, x + \lambda_0 < A_2 x, x > -B_2 \}
\]

\[
\gamma_{x_m} = -x^* ( A_m \lambda_0 - B_m ) x
\]

\[
\in S(0, r_m) \cap \{ \lambda_m < A_m x, x + \lambda_0 < A_m x, x > -B_m \}
\]

We have

\[
\phi_1 \leq \arg \left( -x^* ( A_1 \lambda_0 - B_1 ) x \overline{A_1 x} \right) \leq \phi_2
\]

\[
\phi_1 \leq \arg \left( -x^* ( A_m \lambda_0 - B_m ) x \overline{A_m x} \right) \leq \phi_2
\]

with each \( \phi_2 - \phi_1 < \pi \). Moreover, by the continuity of the functions

\[
F_1( x ) = x^* A_1 x, F_2( x ) = x^* A_2 x, \ldots, F_m( x ) = x^* A_m x \text{ and } F_2( x ) = x^* ( A_2 \lambda_0 - B_2 ) x,
\]

\[
\ldots, F_m( x ) = x^* ( A_m \lambda_0 - B_m ) x \text{ for any } \epsilon > 0 \text{ there exists a neighborhood } S( x_o, \delta ) \text{ such that for any } x \in S( x_o, \delta ) \text{ } x^* A_1 x, x^* A_2 x, \ldots, x^* A_m x \text{ belong to } S( k_i, \epsilon ) \ldots, S( k_m, \epsilon )
\]

respectively and \( \gamma_{x_1}, \gamma_{x_2}, \ldots, \gamma_{x_m} \) belong to \( S(0, r_1), S(0, r_2), \ldots, S(0, r_m) \) respectively.

thus by equation \( \arg( x^* A_1 x ) + \arg( \gamma_{x_1} ) = \arg( x^* ( A_1 \lambda_0 - B_1 ) x ) \), \( \arg( x^* A_2 x ) + \arg( \gamma_{x_2} ) = \arg( x^* ( A_2 \lambda_0 - B_2 ) x ) \), \ldots, \( \arg( x^* A_m x ) + \arg( \gamma_{x_m} ) = \arg( x^* ( A_m \lambda_0 - B_m ) x ) \), we have that each of \( \arg( x^* ( A_1 \lambda_0 - B_1 ) x ) \), \( \arg( x^* ( A_2 \lambda_0 - B_2 ) x ) \), \ldots, \( \arg( x^* ( A_m \lambda_0 - B_m ) x ) \) belongs to the \( \{ \theta_1, \theta_2 \} \) for any \( x \in S( x_o, \delta ) \) and for suitable \( \theta_1, \theta_2 \) with \( \theta_2 - \theta_1 < \pi \). And
then by Theorem (3.3) it is clear that zero is a sharp point of the 
\( JNR(A_n \lambda - B_1, A_2 \lambda - B_2, \ldots, A_m \lambda - B_m) \).

Degenerate cases of sharp points are the isolated points, and we have 
the following statement:

**Theorem 3.5**

Let \( p(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \ldots + A_1 \lambda + A_0 \) be an \( nxn \) matrix polynomial such that zero does not belong to the joint numerical range 
\( JNR(A_m) \) of the leading coefficient matrix. If \( \lambda_o \) is an isolated point of the 
joint numerical range \( JNR(p(\lambda_o)) \) then \( p(\lambda_o) = 0 \).

**Proof:**

Assume \( \lambda_o = 0 \) then there exists a unit vector such that 
\( <A, x, y> = 0 \), \( j = 0, 1, 2, \ldots, m \). This means that \( <A, x, y> = 0 \), now if 
\( <A, y, y> = 0 \) for any unit vector \( y \) then the matrix \( A_n \) satisfies \( A_o = 0 \). We assume that there exists a unit vector \( y \) with 
\( <A, y, y> > 0 \); \( j = 0, 1, 2, \ldots, m \) i.e. 
\( <A_o, y, y> > 0 \). We consider a polynomial 
\( Q(\lambda, t) = <A_m (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y), (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y) > \lambda^m + \ldots + <A_1 (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y), 
(\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y) >. \) This satisfies 
\( <A_m (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y), (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y) > > 0 \), because zero does not 
belong to the joint numerical range of the leading coefficient \( A_m \). Now it is 
sufficient to prove that there exists a sequence \( (t_n) \) for which \( t_n \) tends to 
zero as \( n \) tends to infinity and \( Q(\lambda_n, t_n) = 0 \), at first we fix \( 0 < t < 1 \) the condition 
\( Q(0, t) = <A_n (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y), (\cos \frac{\lambda t}{2} x + \sin \frac{\lambda t}{2} y) >= 0 \), is equal to 
\( <A_n (x + \tan(\frac{\lambda t}{2}) y, x + \tan(\frac{\lambda t}{2}) y) >= 0 \) on the other hand we consider the function 
\( <A_n (x + sy), x + sy >= s \), where we have \( <A_n, y, y> = 0 \) by the choice of the unit 
vector \( y \). \( <A_n, x, y> + <A_n, y, x> = 0 \) then we have \( <A_n (x + sy), x + sy >= 0 \) 
for sufficiently small \( s > 0 \) and if \( <A_n, y, y> + <A_n, y, x> = 0 \) then we have 
\( <A_n (x + sy), x + sy >= s^2 <A_n, y, x> = 0 \). Thus we conclude that \( Q(0, t) = 0 \) for a 
sufficiently small \( t > 0 \) so that \( <A_n (x + \tan(\frac{\lambda t}{2}) y, x + \tan(\frac{\lambda t}{2}) y) >= 0 \). Hence the 
equation \( Q(\lambda, t) = 0 \) in \( \lambda \) has \( m \) roots by the fundamental theorem of algebra. 
The roots of the algebraic equation depends continuously on the coefficient, 
hence \( P(0) = A_0 = 0 \).
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