THE ISOPERIMETRIC INEQUALITY FOR A MINIMAL SUBMANIFOLD IN EUCLIDEAN SPACE

SIMON BRENDLE

Abstract. We prove an isoperimetric inequality which holds for minimal submanifolds in Euclidean space of arbitrary dimension and codimension. Our estimate is sharp if the codimension is at most 2.

1. Introduction

The isoperimetric inequality for domains in $\mathbb{R}^n$ is one of the most beautiful results in geometry. It has long been conjectured that the isoperimetric inequality still holds if we replace the domain in $\mathbb{R}^n$ by a minimal hypersurface in $\mathbb{R}^{n+1}$. In this paper, we prove this conjecture, as well as a more general inequality which holds for submanifolds of arbitrary dimension and codimension.

Theorem 1. Let $\Sigma$ be a compact $n$-dimensional submanifold of $\mathbb{R}^{n+m}$ with boundary $\partial \Sigma$, where $m \geq 2$. Then
\[
\frac{\text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, d\text{vol}}{|\partial B^n|} \geq \left( \frac{(n + m) |B^{n+m}|}{m |B^m| |B^n|} \right)^{\frac{1}{m}} \left( \frac{\text{vol}(\Sigma)}{|B^n|} \right)^{\frac{n-1}{n}},
\]
where $H$ denotes the mean curvature vector of $\Sigma$.

The standard recursion formula for the volume of the unit ball in Euclidean space gives $(n + 2) |B^{n+2}| = 2\pi |B^n|$, hence $\frac{(n+2) |B^{n+2}|}{2 |B^n|^2 |B^n|} = 1$. Thus, Theorem 1 implies a sharp isoperimetric inequality in codimension 2:

Corollary 2. Let $\Sigma$ be a compact $n$-dimensional submanifold of $\mathbb{R}^{n+2}$ with boundary $\partial \Sigma$. Then
\[
\frac{\text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, d\text{vol}}{|\partial B^n|} \geq \left( \frac{\text{vol}(\Sigma)}{|B^n|} \right)^{\frac{n-1}{n}},
\]
where $H$ denotes the mean curvature vector of $\Sigma$.

Finally, we characterize the case of equality:

Theorem 3. Let $\Sigma$ be a compact $n$-dimensional submanifold of $\mathbb{R}^{n+2}$ with boundary $\partial \Sigma$. If
\[
\frac{\text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, d\text{vol}}{|\partial B^n|} = \left( \frac{\text{vol}(\Sigma)}{|B^n|} \right)^{\frac{n-1}{n}},
\]
then $\Sigma$ is a round $n$-sphere.

This project was supported by the National Science Foundation under grant DMS-1806190 and by the Simons Foundation.
In 1921, Carleman [4] showed that every two-dimensional minimal surface which is diffeomorphic to a disk satisfies the sharp isoperimetric inequality $L^2 \geq 4\pi A$. Various authors have proved generalizations of Carleman’s theorem under weaker topological assumptions. In particular, these results apply to two-dimensional minimal surfaces with connected boundary (see [10], [14]); minimal surfaces diffeomorphic to annuli (cf. [8], [13]); and two-dimensional minimal surfaces with two boundary components (cf. [6], [11]). On the other hand, using completely different techniques, Leon Simon proved that every two-dimensional minimal surface satisfies the non-sharp isoperimetric inequality $L^2 \geq 2\pi A$. Stone [15] subsequently improved the constant in this inequality: he showed that $L^2 \geq 2\sqrt{2}\pi A$ for every two-dimensional minimal surface. We refer to [7] for a survey of these developments.

In higher dimensions, the famous Michael-Simon Sobolev inequality (cf. [12] or [1], Section 7) implies an isoperimetric inequality for minimal surfaces, albeit with a non-sharp constant. Castillon [5] gave an alternative proof of the Michael-Simon Sobolev inequality using methods from optimal transport. Finally, Almgren [2] showed that the sharp isoperimetric inequality holds in all dimensions if we assume that $\Sigma$ is area-minimizing.

Our method of proof is inspired in part by the Alexandrov-Bakelman-Pucci maximum principle (cf. [3]). An alternative way to prove Theorem 1 would be to use optimal transport; in that case, we would consider the transport map from a thin annulus in $\mathbb{R}^{n+m}$ to the submanifold $\Sigma$.

2. Proof of Theorem 1

For each point $x \in \Sigma$, we denote by $T_x\Sigma$ and $T^x\Sigma$ the tangent and normal space to $\Sigma$ at $x$, respectively. Moreover, we denote by $H$ the second fundamental form of $\Sigma$. Recall that $H$ is a symmetric bilinear form on $T_x\Sigma$ which takes values in $T^x\Sigma$. If $X$ and $Y$ are tangent vector fields on $\Sigma$, and $V$ is a normal vector field along $\Sigma$, then $\langle H(X, Y), V \rangle = -\langle D_X Y, V \rangle$, where $D$ denotes the standard connection on $\mathbb{R}^{n+m}$. The trace of the second fundamental gives the mean curvature vector, which we denote by $H$. Finally, we denote by $\eta$ the co-normal to $\partial \Sigma$ in $\Sigma$.

We now turn to the proof of Theorem 1. We first consider the case that $\Sigma$ is connected. By scaling, we may arrange that $\text{area}(\partial \Sigma) + \int_\Sigma |H| \, d\text{vol} = n \text{vol}(\Sigma)$. Since $\Sigma$ is connected, we can find a function $u : \Sigma \to \mathbb{R}$ such that $\Delta_\Sigma u = n - |H|$ in $\Sigma$ and $\langle \nabla^\Sigma u, \eta \rangle = 1$ at each point on $\partial \Sigma$. Since $|H|$ is Lipschitz continuous, the function $u$ is of class $C^{2,\alpha}$ for each $0 < \alpha < 1$ (see [9], Theorem 6.30).
We define
\[ U := \{ x : x \in \Sigma \setminus \partial \Sigma, |\nabla^{\Sigma} u| < 1 \}, \]
\[ \Omega := \{(x, y) : x \in \Sigma \setminus \partial \Sigma, y \in T_x^{\perp} \Sigma, |\nabla^{\Sigma} u(x)|^2 + |y|^2 < 1 \}, \]
\[ A := \{(x, y) \in \Omega : D_X^2 u(x) - \langle II(x), y \rangle \geq 0 \}. \]

Moreover, we define a map \( \Phi : \Omega \rightarrow \mathbb{R}^{n+m} \) by
\[ \Phi(x, y) = \nabla^{\Sigma} u(x) + y. \]

Since \( \nabla^{\Sigma} u(x) \in T_x \Sigma \) and \( y \in T_x^{\perp} \Sigma \) are orthogonal, we obtain \( |\Phi(x, y)|^2 = |\nabla^{\Sigma} u|^2 + |y|^2 < 1 \) for all \( (x, y) \in \Omega \).

**Lemma 4.** The image \( \Phi(A) \) is the open unit ball \( B^{n+m} \).

**Proof.** Clearly, \( \Phi(A) \subset \Phi(\Omega) \subset B^{n+m} \). To prove the reverse inclusion, we consider an arbitrary vector \( \xi \in \mathbb{R}^{n+m} \) such that \( |\xi| < 1 \). We define a function \( w : \Sigma \rightarrow \mathbb{R} \) by \( w(x) := u(x) - \langle x, \xi \rangle \). Using the Cauchy-Schwarz inequality, we obtain
\[ \langle \nabla^{\Sigma} w(x), \eta(x) \rangle = \langle \nabla^{\Sigma} u(x), \eta(x) \rangle - \langle \eta(x), \xi \rangle = 1 - \langle \eta(x), \xi \rangle > 0 \]
for each point \( x \in \partial \Sigma \). Consequently, the function \( w \) must attain its minimum in the interior of \( \Sigma \). Let \( \bar{x} \in \Sigma \setminus \partial \Sigma \) be a point in the interior of \( \Sigma \) such that \( w(\bar{x}) = \inf_{x \in \Sigma} w(x) \). This implies \( \xi = \nabla^{\Sigma} u(\bar{x}) + \bar{y} \) for some \( \bar{y} \in T_{\bar{x}}^{\perp} \Sigma \). Consequently, \( |\nabla^{\Sigma} u(\bar{x})|^2 + |\bar{y}|^2 = |\xi|^2 < 1 \). Moreover, we have \( D_X^2 w(\bar{x}) \geq 0 \). From this, we deduce that \( D_X^2 u(\bar{x}) - \langle II(\bar{x}), \xi \rangle \geq 0 \). Since \( \langle II(\bar{x}), \xi \rangle = \langle \nabla^{\Sigma} u(\bar{x}) + \bar{y}, \bar{y} \rangle = \langle II(\bar{x}), \bar{y} \rangle \), we conclude that \( D_X^2 u(\bar{x}) - \langle II(\bar{x}), \bar{y} \rangle \geq 0 \). Therefore, \( (\bar{x}, \bar{y}) \in A \) and \( \Phi(\bar{x}, \bar{y}) = \xi \). Thus, \( B^{n+m} \subset \Phi(A) \).

**Lemma 5.** The Jacobian determinant of \( \Phi \) is given by \( \det D \Phi(x, y) = \det(D_X^2 u(x) - \langle II(x), y \rangle) \) for \( (x, y) \in \Omega \).

**Proof.** Fix a point \( (\bar{x}, \bar{y}) \in \Omega \). Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis for the tangent space \( T_{\bar{x}} \Sigma \). Let \( \{x_1, \ldots, x_n\} \) be a system of geodesic normal coordinates around \( \bar{x} \) such that \( \frac{\partial}{\partial x_i} = e_i \) at the point \( \bar{x} \). Moreover, let \( \{\nu_1, \ldots, \nu_m\} \) be a local orthonormal frame for the normal bundle \( T^{\perp} \Sigma \). If we write \( y = \sum_{\alpha=1}^{m} y_\alpha \nu_\alpha \), then \( \{x_1, \ldots, x_n, y_1, \ldots, y_m\} \) is a system of local coordinates on the normal bundle of \( \Sigma \). We compute
\[ \left\langle \frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{y}), e_j \right\rangle = (D_X^2 u)(e_i, e_j) - \langle II(e_i, e_j), \bar{y} \rangle, \]
\[ \left\langle \frac{\partial \Phi}{\partial y_\alpha}(\bar{x}, \bar{y}), e_j \right\rangle = 0, \]
\[ \left\langle \frac{\partial \Phi}{\partial y_\alpha}(\bar{x}, \bar{y}), \nu_\beta \right\rangle = \delta_{\alpha \beta}. \]
Consequently,
\[
\det D\Phi(x, y) = \det \begin{bmatrix}
D_2^2 u(x) - \langle H(x), y \rangle & 0 \\
* & \text{id}
\end{bmatrix} = \det(D_2^2 u(x) - \langle H(x), y \rangle).
\]
This proves the assertion.

**Lemma 6.** The Jacobian determinant of \( \Phi \) satisfies \( 0 \leq \det D\Phi(x, y) \leq 1 \) for all \((x, y) \in A\).

**Proof.** Consider a point \((x, y) \in A\). Then \(D_2^2 u(x) - \langle H(x), y \rangle \geq 0\). Using the inequality \(|y| < 1\) and the identity \(\Delta_\Sigma u = n - |H|\), we obtain
\[
0 \leq \Delta_\Sigma u(x) - \langle H(x), y \rangle \leq \Delta_\Sigma u(x) + |H(x)| = n.
\]
Hence, the arithmetic-geometric mean inequality implies
\[
0 \leq \det(D_2^2 u(x) - \langle H(x), y \rangle) \leq \left(\frac{\text{tr}(D_2^2 u(x) - \langle H(x), y \rangle)}{n}\right)^n \leq 1.
\]
Using Lemma 5, we conclude that \(0 \leq \det D\Phi(x, y) \leq 1\). This completes the proof of Lemma 6.

We now continue with the proof of Theorem 1. Using Lemma 4 and Lemma 6 we obtain
\[
|B^{n+m}| (1 - \rho^{n+m})
\]
\[
= \int_{\{\xi \in \mathbb{R}^{n+m} \mid \rho^2 < |\xi|^2 < 1\}} 1 \, d\xi
\]
\[
\leq \int_U \left( \int_{\{y \in T_+ \Sigma \mid \rho^2 < |\nabla_\Sigma u(x)|^2 + |y|^2 < 1\}} \det D\Phi(x, y) 1_A(x, y) \, dy \right) \, d\text{vol}(x)
\]
\[
\leq \int_U \left( \int_{\{y \in T_+ \Sigma \mid \rho^2 < |\nabla_\Sigma u(x)|^2 + |y|^2 < 1\}} 1 \, dy \right) \, d\text{vol}(x)
\]
\[
= |B^m| \int_U \left( (1 - |\nabla_\Sigma u(x)|^2)^{\frac{m}{2}} - (\rho^2 - |\nabla_\Sigma u(x)|^2)^{\frac{m}{2}} \right) \, d\text{vol}(x)
\]
for all \(0 \leq \rho < 1\). Since \(m \geq 2\), we have the elementary inequality \(b^{\frac{m}{2}} - a^{\frac{m}{2}} \leq \frac{m}{2} (b - a)\) for \(0 \leq a \leq b \leq 1\). Consequently,
\[
(1 - |\nabla_\Sigma u(x)|^2)^{\frac{m}{2}} - (\rho^2 - |\nabla_\Sigma u(x)|^2)^{\frac{m}{2}}
\]
\[
\leq \frac{m}{2} \left( (1 - |\nabla_\Sigma u(x)|^2) - (\rho^2 - |\nabla_\Sigma u(x)|^2)_+ \right) \leq \frac{m}{2} (1 - \rho^2)
\]
for all \(x \in U\) and all \(0 \leq \rho < 1\). Putting these facts, together, we obtain
\[
|B^{n+m}| (1 - \rho^{n+m}) \leq \frac{m}{2} |B^m| \, \text{vol}(U) (1 - \rho^2)
\]
for all \(0 \leq \rho < 1\). In the next step, we divide by \(1 - \rho\) and take the limit as \(\rho \to 1\). This gives
\[
(n + m) |B^{n+m}| \leq m |B^m| \, \text{vol}(U) \leq m |B^m| \, \text{vol}(\Sigma).
\]
Using the identities \( \text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, dvol = n \text{vol}(\Sigma) \) and \( |\partial B^n| = n |B^n| \), we conclude that
\[
\frac{\text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, dvol}{|\partial B^n|} = \frac{\text{vol}(\Sigma)}{|B^n|} \geq \left( \frac{(n + m) |B^{n+m}|}{m |B^m| |B^n|} \right)^{\frac{1}{m}} \left( \frac{\text{vol}(\Sigma)}{|B^n|} \right)^{\frac{m-1}{m}}.
\]

This proves Theorem 1 in the special case when \( \Sigma \) is connected.

It remains to consider the case when \( \Sigma \) is disconnected. In that case, we apply the inequality to each individual connected component of \( \Sigma \), and sum over all connected components. Since
\[
a^{\frac{m-1}{m}} + b^{\frac{m-1}{m}} > a (a + b)^{-\frac{1}{m}} + b (a + b)^{-\frac{1}{m}} = (a + b)^{\frac{m-1}{m}}
\]
for \( a, b > 0 \), we conclude that
\[
\frac{\text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, dvol}{|\partial B^n|} > \left( \frac{(n + m) |B^{n+m}|}{m |B^m| |B^n|} \right)^{\frac{1}{m}} \left( \frac{\text{vol}(\Sigma)}{|B^n|} \right)^{\frac{m-1}{m}}
\]
if \( \Sigma \) is disconnected. This completes the proof of Theorem 1.

### 3. Proof of Theorem 3

Suppose that \( \Sigma \) is a compact \( n \)-dimensional submanifold in \( \mathbb{R}^{n+2} \) satisfying
\[
\frac{\text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, dvol}{|\partial B^n|} = \left( \frac{\text{vol}(\Sigma)}{|B^n|} \right)^{\frac{m-1}{m}}.
\]
Clearly, \( \Sigma \) must be connected. By scaling, we may arrange that \( \text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, dvol = |\partial B^n| \) and \( \text{vol}(\Sigma) = |B^n| \). In particular, \( \text{area}(\partial \Sigma) + \int_{\Sigma} |H| \, dvol = n \text{vol}(\Sigma) \). Let \( u \) be the solution of the equation \( \Delta_{\Sigma} u = n - |H| \) in \( \Sigma \) with boundary condition \( \langle \nabla_{\Sigma} u, \eta \rangle = 1 \) on \( \partial \Sigma \). Let \( U, \Omega, A \), and \( \Phi : \Omega \to \mathbb{R}^{n+2} \) be defined as in Section 2.

**Lemma 7.** We have \( D_{\Sigma}^2 u(x) - \langle II(x), y \rangle = g \) for all \( x \in U \) and all \( y \in T_{\Sigma} x \) satisfying \( |\nabla_{\Sigma} u(x)|^2 + |y|^2 = 1 \).

**Proof.** We argue by contradiction. Suppose that there exists a point \( \bar{x} \in U \) and a vector \( \bar{y} \in T_{\Sigma} \bar{x} \) such that \( |\nabla_{\Sigma} u(\bar{x})|^2 + |\bar{y}|^2 = 1 \) and \( D_{\Sigma}^2 u(\bar{x}) - \langle II(\bar{x}), \bar{y} \rangle \neq g \). We distinguish two cases:

**Case 1:** Suppose first that \( D_{\Sigma}^2 u(\bar{x}) - \langle II(\bar{x}), \bar{y} \rangle \) is weakly positive definite. Since \( \Delta_{\Sigma} u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle \leq \Delta_{\Sigma} u(\bar{x}) + |H(\bar{x})| = n \), the arithmetic-geometric mean inequality gives \( \det(D_{\Sigma}^2 u(\bar{x}) - \langle II(\bar{x}), \bar{y} \rangle) < 1 \). By continuity, we can find a real number \( \varepsilon \in (0,1) \) and an open neighborhood \( W \) of the point \( (\bar{x}, \bar{y}) \) such that \( \det(D_{\Sigma}^2 u(x) - \langle II(x), y \rangle) \leq 1 - \varepsilon \) for all \( (x,y) \in W \). Using Lemma 5, we obtain \( \det(D\Phi(x,y)) \leq 1 - \varepsilon \) for all \( (x, y) \in \Omega \cap W \).

**Case 2:** Suppose next that \( D_{\Sigma}^2 u(\bar{x}) - \langle II(\bar{x}), \bar{y} \rangle \) fails to be weakly positive definite. By continuity, we can find an open neighborhood \( W \) of the point \( (\bar{x}, \bar{y}) \) such that \( D_{\Sigma}^2 u(x) - \langle II(x), y \rangle \) fails to be weakly positive definite for each \( (x,y) \in W \). In particular, \( A \cap W = \emptyset \).

To summarize, we have shown that there exists an open neighborhood \( W \) of \( (\bar{x}, \bar{y}) \) such that \( \det(D\Phi(x,y)) \leq 1 - \varepsilon \) for all \( (x, y) \in A \cap W \). Using
Lemma [6] we deduce that $0 \leq \det D\Phi(x, y) 1_A(x, y) \leq 1 - \varepsilon \cdot 1_W(x, y)$ for all $(x, y) \in \Omega$. Arguing as in Section [2] we obtain

$$|B^{n+2}| (1 - \rho^{n+2})$$

$$= \int_{\{x \in \mathbb{R}^{n+2}: \rho^2 < |\xi|^2 < 1\}} 1 \, d\xi$$

$$\leq \int_U \left( \int_{\{y \in T^\perp_x \Sigma: \rho^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} \det D\Phi(x, y) 1_A(x, y) \, dy \right) \, d\nuol(x)$$

$$\leq \int_U \left( \int_{\{y \in T^\perp_x \Sigma: \rho^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} (1 - \varepsilon - 1_W(x, y)) \, dy \right) \, d\nuol(x)$$

$$= |B^2| \int_U \left( (1 - |\nabla^\Sigma u(x)|^2) - (\rho^2 - |\nabla^\Sigma u(x)|^2)_+ \right) \, d\nuol(x)$$

$$- \varepsilon \int_U \left( \int_{\{y \in T^\perp_x \Sigma: \rho^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} 1_W(x, y) \, dy \right) \, d\nuol(x)$$

$$\leq |B^2| \nuol(U) (1 - \rho^2)$$

$$- \varepsilon \int_U \left( \int_{\{y \in T^\perp_x \Sigma: \rho^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} 1_W(x, y) \, dy \right) \, d\nuol(x).$$

Dividing by $1 - \rho$ and taking the limit as $\rho \to 1$ gives

$$(n + 2) |B^{n+2}| < 2 |B^2| \nuol(U) \leq 2 |B^2| \nuol(\Sigma) = 2 |B^2| |B^n|.$$

This contradicts the fact that $(n + 2) |B^{n+2}| = 2 |B^2| |B^n|.$

**Lemma 8.** We have $D^2_{\Sigma} u(x) = g$ and $II(x) = 0$ for all $x \in U$.

**Proof.** Lemma [7] implies $D^2_{\Sigma} u(x) - \langle II(x), y \rangle = g$ for all $x \in U$ and all $y \in T^\perp_x \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Replacing $y$ by $-y$ gives $D^2_{\Sigma} u(x) + \langle II(x), y \rangle = g$ for all $x \in U$ and all $y \in T^\perp_x \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Consequently, $D^2_{\Sigma} u(x) = g$ and $\langle II(x), y \rangle = 0$ for all $x \in U$ and all $y \in T^\perp_x \Sigma$ satisfying $|\nabla^2 u(x)|^2 + |y|^2 = 1$. From this, the assertion follows.

**Lemma 9.** We have $\nuol(U) = \nuol(\Sigma)$. In particular, $U$ is a dense open subset of $\Sigma$.

**Proof.** We argue by contradiction. If $\nuol(U) < \nuol(\Sigma)$, then the arguments in Section [2] imply

$$(n + 2) |B^{n+2}| \leq 2 |B^2| \nuol(U) < 2 |B^2| \nuol(\Sigma) = 2 |B^2| |B^n|.$$

This contradicts the fact that $(n + 2) |B^{n+2}| = 2 |B^2| |B^n|.$

Combining Lemma [8] and Lemma [9] we conclude that $|\nabla^\Sigma u| \leq 1$, $D^2_{\Sigma} u = g$, and $II = 0$ at each point on $\Sigma$. Since $\Sigma$ is connected and $II = 0$ at each point on $\Sigma$, $\Sigma$ is contained in an $n$-dimensional plane $P$. Since $D^2_{\Sigma} u = g$ at each point on $\Sigma$, the function $u$ must be of the form $u(x) = \frac{1}{2} |x - p|^2 + c$ for
some point \( p \in P \) and some constant \( c \). Since \( |\nabla \Sigma u| \leq 1 \) at each point on \( \Sigma \), it follows that \( \Sigma \subset \{ x \in P : |x - p| \leq 1 \} \). Since \( \text{vol}(\Sigma) = |B^n| \), we conclude that \( \Sigma = \{ x \in P : |x - p| \leq 1 \} \). This completes the proof of Theorem 3.

4. AN IMPROVED VERSION OF THE MICHAEL–SIMON SOBOLEV INEQUALITY

In this final section, we discuss how the isoperimetric inequality in Theorem 1 implies a Sobolev inequality. This is a standard argument which we include for the convenience of the reader. Let \( \Sigma \) be a compact \( n \)-dimensional submanifold of \( \mathbb{R}^{n+m} \), where \( m \geq 2 \). Moreover, let \( f \) be a nonnegative smooth function on \( \Sigma \) which vanishes near the boundary \( \partial \Sigma \). Note that

\[
\int_{\Sigma} |\nabla \Sigma f| \, d\text{vol} = \int_0^\infty \text{area}(\{ f = s \}) \, ds
\]

by the co-area formula, and

\[
\int_{\Sigma} |H| \, f \, d\text{vol} = \int_0^\infty \left( \int_{\{f \geq s\}} |H| \, d\text{vol} \right) \, ds
\]

by Fubini’s theorem. Let \( I(t) := \int_0^t \text{vol}(\{ f \geq s \}) \frac{ds}{n+1} \). This implies \( I(t) \geq t \text{vol}(\{ f \geq t \}) \frac{m}{n+1} \) and \( I'(t) = \text{vol}(\{ f \geq t \}) \frac{m}{n+1} \). Using Fubini’s theorem, we obtain

\[
\int_{\Sigma} f \frac{n}{n-1} \, d\text{vol} = \frac{n}{n-1} \int_0^\infty t \frac{1}{n+1} \text{vol}(\{ f \geq t \}) \, dt
\]

\[
\leq \frac{n}{n-1} \int_0^\infty I(t) \frac{1}{n+1} I'(t) \, dt
\]

\[
= \left( \int_0^\infty \text{vol}(\{ f \geq s \}) \frac{n-1}{n} \, ds \right) \frac{m}{n+1}.
\]

On the other hand, applying Theorem 1 to the super-level sets of \( f \) gives

\[
\frac{\text{area}(\{ f = s \}) + \int_{\{f \geq s\}} |H| \, d\text{vol}}{|\partial B^n|} \geq \left( \frac{(n+m) \, |B^{n+m}|}{m \, |B^n| \, |B^m|} \right)^{\frac{1}{n}} \left( \frac{\text{vol}(\{ f \geq s \})}{|B^n|} \right)^{\frac{n-1}{n}}
\]

whenever \( s \in (0, \infty) \) is a regular value of \( f \). Integrating over \( s \in (0, \infty) \), we conclude that

\[
\frac{\int_{\Sigma} (|\nabla \Sigma f| + |H| \, f) \, d\text{vol}}{|\partial B^n|} \geq \left( \frac{(n+m) \, |B^{n+m}|}{m \, |B^n| \, |B^m|} \right)^{\frac{1}{n}} \left( \frac{\int_{\Sigma} f \frac{n}{n-1} \, d\text{vol}}{|B^n|} \right)^{\frac{n-1}{n}}
\]

This inequality is sharp for \( m = 2 \).

REFERENCES

[1] W. Allard, On the first variation of a varifold, Ann. of Math. 95, 417–491 (1972)
[2] F.J. Almgren, Jr., Optimal isoperimetric inequalities, Indiana Univ. Math. J. 35, 451–547 (1986)
[3] X. Cabré, Elliptic PDEs in probability and geometry. Symmetry and regularity of solutions, Discrete Cont. Dyn. Systems A 20, 425–457 (2008)
[4] T. Carleman, Zur Theorie der Minimalflächen, Math. Z. 9, 154–160 (1921)
[5] P. Castillon, *Submanifolds, isoperimetric inequalities and optimal transportation*, J. Funct. Anal. 259, 79–103 (2010)

[6] J. Choe, *The isoperimetric inequality for a minimal surface with radially connected boundary*, Ann. Scuola Norm. Sup. Pisa 17, 583–593 (1990)

[7] J. Choe, *Isoperimetric inequalities of minimal submanifolds*, Global Theory of Minimal Surfaces, Clay Math. Proc. vol. 2, pp. 325–369, Amer. Math. Soc., Providence RI, 2005

[8] J. Feinberg, *The isoperimetric inequality for doubly-connected minimal surfaces in $\mathbb{R}^n$*, J. d’Anal. Math. 32, 249–278 (1977)

[9] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 2001

[10] C.C. Hsiung, *Isoperimetric inequalities for two-dimensional Riemannian manifolds with boundary*, Ann. of Math. 73, 213–220 (1961)

[11] P. Li, R. Schoen, and S.T. Yau, *On the isoperimetric inequality for minimal surfaces*, Ann. Scuola Norm. Sup. Pisa 11, 237–244 (1984)

[12] J.H. Michael and L.M. Simon, *Sobolev and mean value inequalities on generalized submanifolds of $\mathbb{R}^n$*, Comm. Pure Appl. Math. 26, 316–379 (1973)

[13] R. Osserman and M. Schiffer, *Doubly-connected minimal surfaces*, Arch. Rational Mech. Anal. 58, 285–307 (1975)

[14] W.T. Reid, *The isoperimetric inequality and associated boundary problems*, J. Math. Mech. 8, 897–906 (1959)

[15] A. Stone, *On the isoperimetric inequality on a minimal surface*, Calc. Var. PDE 17, 309–391 (2003)

Department of Mathematics, Columbia University, New York NY 10027