THE TUTTE’S CONDITION IN TERMS OF GRAPH FACTORS

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Abstract. Let $G$ be a connected general graph of even order, with a function $f: V(G) \to \mathbb{Z}^+$. We obtain that $G$ satisfies the Tutte’s condition $o(G - S) \leq \sum_{v \in S} f(v)$ for any nonempty set $S \subset V(G)$, with respect to $f$ if and only if $G$ contains an $H$-factor for any function $H: V(G) \to 2^\mathbb{N}$ such that $H(v) \in \{J_f(v), J_f^+(v)\}$ for each $v \in V(G)$, where the set $J_f(v)$ consists of the integer $f(v)$ and all positive odd integers less than $f(v)$, and the set $J_f^+(v)$ consists of positive odd integers less than or equal to $f(v) + 1$. We also obtain a characterization for graphs of odd order satisfying the Tutte’s condition with respect to a function.

1. Introduction

This note connects Tutte’s condition with graph factors. Tutte’s theorem states that a graph $G$ has a perfect matching if and only if $o(G - S) \leq |S|$ for any set $S \subset V(G)$, where $o(G - S)$ denotes the number of odd components of the subgraph $G - S$, and $V(G)$ is the vertex set of $G$. Let $f: V(G) \to \mathbb{Z}^+$ be a function, where $\mathbb{Z}^+$ denotes the set of positive integers. The Tutte’s condition on $G$ with respect to $f$ is the condition $o(G - S) \leq f(S)$ for any nonempty set $S \subset V(G)$, where $f(S) = \sum_{v \in S} f(v)$. The Tutte’s condition with respect to the constant function $f \equiv 1$ is the condition in Tutte’s theorem.

A considerable large number of literatures on graph factors can be found in Akiyama and Kano’s book [2]. Let $H: V(G) \to 2^\mathbb{N}$ be a set-valued function. A spanning subgraph $F$ of $G$ is called an $H$-factor if $\deg_F(v) \in H(v)$. In particular, a 1-factor is exactly a perfect matching. For any vertex $x$ of $G$, denote by $G^x$ the graph obtained from $G$ by adding a new vertex $x'$ together with a new edge $xx'$.

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A graph $G$ is said to be $H$-critical if $G$ contains no $H$-factors and if the graph $G^x$ has an $H^x$-factor for every vertex $x$ of $G$, where

$$H^x(v) = \begin{cases} \{1\}, & \text{if } v = x'; \\ H(v), & \text{otherwise.} \end{cases}$$

Lovász [6] proposed the degree prescribed subgraph problem of determining the distance of a factor from a given integer set function. He [7] considered it with the restriction that the given set function $H$ is allowed, i.e., that every gap of the set $H(v)$ for each vertex $v$ is at most two. He also showed that the problem is NP-complete when the function $H$ is not allowed. Cornuéjols [3] provided a polynomial Edmonds-Johnson type alternating forest algorithm for the degree prescribed subgraph problem with $H$ allowed, which implies a Gallai-Edmonds type structure theorem.

For convenience, we denote the set of positive odd integers by $2\mathbb{N} + 1$, and

$$J_n = \begin{cases} \{1, 3, 5, \ldots, n\}, & \text{if } n \text{ is odd;} \\ \{1, 3, 5, \ldots, n - 1, n\}, & \text{if } n \text{ is even.} \end{cases}$$

Define $J_f(v) = J_{f(v)}$ for all vertices $v$.

**Theorem 1.1** (Cui and Kano [4]). A connected general graph $G$ of even order satisfies the Tutte’s condition with respect to a function $f: V(G) \to 2\mathbb{N} + 1$ if and only if $G$ contains a $J_f$-factor.

Extending the range of $f$ to be all positive integers, Egawa, Kano, and Yan [5] obtain Theorem 1.2.

**Theorem 1.2** (Egawa et al. [5]). Suppose that a connected simple graph $G$ of even order satisfies the Tutte’s condition with respect to a function $f: V(G) \to \mathbb{Z}^+$. Then $G$ contains a $J_f$-factor.

The particular case $f(v) \equiv 2n$ for some integer $n$ had been solved by Akiyama, Avis and Era [1] for $n = 1$ and by the present authors [8] for $n \geq 2$.

Without restricting the parity of order of $G$, Akiyama and Kano [2, Problem 6.14 (2)] proposed Problem 1.3. Denote by $2\mathbb{Z}^+$ the set of positive even integers.

**Problem 1.3** (Akiyama and Kano [2]). Suppose that a connected simple graph $G$ satisfies the Tutte’s condition with respect to a function $f: V(G) \to 2\mathbb{Z}^+$. Then what factor or property does $G$ have?

In the next section, we will give a characterization of graphs satisfying the Tutte’s condition with respect to a function $f$, in terms of graph factors, without any restriction on the range of $f$, and for graphs of any parity of order.
2. Main Result

In terms of graph factors, the authors [9] have characterized graphs satisfying the Tutte’s condition with respect to a function, but with the aid of either 2-colorings, or 2-edge-colorings, or 2-end-colorings; see Theorems 2.1 and 2.3. In this note, we present characterizations in terms of graph factors only; see Theorems 2.2 and 2.4.

**Theorem 2.1** (Lu and Wang [9]). A connected general graph $G$ of even order satisfies the Tutte’s condition with respect to a function $f: V(G) \to \mathbb{Z}^+$ if and only if $G$ contains an $H$-factor for any coloring $g: V(G) \to \{B, R\}$, where

$$H(v) = \begin{cases} J_f(v), & \text{if } g(v) = R; \\ 2\mathbb{N} + 1, & \text{if } g(v) = B. \end{cases}$$

For any function $f: V(G) \to \mathbb{Z}^+$, let $J_f^+(v)$ be the set of positive odd integers that are less than or equal to $f(v) + 1$. In other words,

$$J_f^+(v) = \{m \in 2\mathbb{N} + 1: m \leq f(v) + 1\} = \begin{cases} J_f(v), & \text{if } f(v) \text{ is odd}; \\ J_f(v)+1, & \text{if } f(v) \text{ is even}. \end{cases}$$

Define a set

$$\mathcal{H}_f = \{H: V(G) \to 2\mathbb{N} \mid H(v) \in \{J_f(v), J_f^+(v)\} \text{ for each } v \in V(G)\}.$$

**Theorem 2.2.** A connected general graph $G$ of even order satisfies the Tutte’s condition with respect to a function $f: V(G) \to \mathbb{Z}^+$ if and only if $G$ contains an $H$-factor for any $H \in \mathcal{H}_f$.

**Proof.** Let $G$ be a connected general graph of even order, with a function $f: V(G) \to \mathbb{Z}^+$. We shall show the necessity and sufficiency respectively.

**Necessity.** Let $H \in \mathcal{H}_f$. Consider the function $f': V(G) \to \mathbb{Z}^+$ defined by

$$f'(v) = \max_{x \in H(v)} x = \begin{cases} f(v) + 1, & \text{if } H(v) = J_f^+(v) \text{ and } f(v) \text{ is even}; \\ f(v), & \text{otherwise}. \end{cases}$$

From the premise, we infer immediately

$$o(G - S) \leq f(S) \leq f'(S) \quad \text{for any set } S \subset V(G).$$

Applying Theorem 2.1 with the coloring $g$ such that $g^{-1}(R) = V(G)$, one obtains that $G$ contains an $J_{f'}$-factor, i.e., an $H$-factor.

**Sufficiency.** Let $S \subset V(G)$. Consider the function $H \in \mathcal{H}_f$ defined by

$$H(v) = \begin{cases} J_f(v), & \text{if } v \in S; \\ J_f^+(v), & \text{otherwise}. \end{cases}$$
From premise, the graph $G$ has an $H$-factor, say, $F$. Let $C$ be any odd component of the subgraph $G - S$. Then for each $v \in C$, we have $H(v) = J_f(v)$ and thus the degree $d_F(v)$ is odd. By parity argument, we have $E_F(V(C), S) \neq \emptyset$. Therefore, one may deduce that

$$o(G - S) \leq \sum_C |E_F(V(C), S)| \leq f(S).$$

This completes the proof. □

We remark that Theorem 2.2 reduces to Theorem 1.1 if $f(V(G)) \subseteq 2\mathbb{N} + 1$. In fact, when $f(V(G)) \subseteq 2\mathbb{N} + 1$, we obtain $J_f = J_f^+$ and

$$H_f = \{H : V(G) \to 2\mathbb{N} | H(v) = J_f(v) \text{ for each } v \in V(G)\} = \{J_f\}.$$

**Theorem 2.3** (Lu and Wang [9]). Let $G$ be a connected general graph. Then $G$ satisfies the Tutte’s condition with respect to a function $f : V(G) \to \mathbb{Z}^+$ if and only if for any coloring $g : V(G) \to \{B, R\}$, the graph $G$ either contains an $H$-factor or is $H$-critical, where

$$H(v) = \begin{cases} J_f(v), & \text{if } g(v) = R; \\ 2\mathbb{N} + 1, & \text{if } g(v) = B. \end{cases}$$

By a proof similar to that of Theorem 2.2, one may obtain the following result.

**Theorem 2.4.** Let $G$ be a connected general graph of odd order. Then $G$ satisfies the Tutte’s condition with a function $f : V(G) \to \mathbb{Z}^+$ if and only if the graph $G$ either contains an $H$-factor or is $H$-critical, for any $H \in H_f$.

**Proof.** Omitted. □

Combining Theorems 2.2 and 2.4 gives an answer to Problem 1.3.

**References**

[1] J. Akiyama, D. Avis and H. Era, On a $\{1, 2\}$-factor of a graph, TRU Math. 16(2) (1980), 97–102.

[2] J. Akiyama and M. Kano, Factors and Factorizations of Graphs — Proof Techniques in Factor Theory, Springer-Verlag Berlin Heidelberg, 2011.

[3] G. Cornuojol, General factors of graphs, J. Combin. Theory Ser. B 45 (1988), 185–198.

[4] Y. Cui and M. Kano, Some results on odd factors of graphs, J. Graph Theory 12 (1988), 327–333.

[5] Y. Egawa, M. Kano, and Z. Yan, $(1, f)$-Factors of graphs with odd property, Graphs Combin. 32 (2016), 103–110.

[6] L. Lovász, The factorization of graphs, Combinatorial Structures and their Applications, In: Proc. Calgary Internat. Conf., Calgary, Alta., 1969, pp. 243–246 (1970).
[7] L. Lovász, The factorization of graphs. II, Acta Math. Hungar. 23 (1972), 223–246.
[8] H. Lu and D.G.L. Wang, On Cui-Kano's characterization problem on graph factors, J. Graph Theory 74(3) (2013), 335–343.
[9] H. Lu and D.G.L. Wang, A Tutte-type characterization for graph factors, SIAM J. Discrete Math., 31 (2017), 1149-1159.
[10] T. Niessen, A characterization of graphs having all $(g, f)$-factors, J. Combin. Theory Ser. B 72 (1998), 152–156.
[11] A. Sebő, General antifactors of graphs, J. Combin. Theory Ser. B 58 (1993), 173–184.
[12] J. Szabó, Good characterizations for some degree constrained subgraphs, J. Combin. Theory Ser. B 99(2) (2009), 436–446.

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