LINEAR RESPONSE, AND CONSEQUENCES FOR DIFFERENTIABILITY OF STATISTICAL QUANTITIES AND MULTIFRACTAL ANALYSIS

THIAGO BOMFIM AND ARMANDO CASTRO

Abstract. In this article we initially fix ourselves to smooth expanding dynamical systems. We prove the differentiability of the topological pressure, equilibrium states and their densities with respect to smooth expanding dynamical systems and any smooth potential. This is done by proving the regularity of the dominant eigenvalue of the transfer operator with respect to dynamics and potential. From that, we obtain strong consequences on the regularity of the dynamical system statistical properties, that apply in more general contexts. Indeed, we prove that the average and variance obtained from the central limit theorem vary $C^{-1}$ with respect to the $C^r$-expanding dynamics and $C^r$-potential, and also, there is a large deviations principle with its rate $C^{-1}$ with respect to the dynamics and the potential. An application for multifractal analysis is given. We also obtained some asymptotic formulas for derivatives.

1. Introduction

The thermodynamical formalism in dynamical systems was initially concerned with the existence and uniqueness of equilibrium states for a family of dynamics and potentials. Since we obtain uniqueness of equilibrium states for robust families of dynamics and potentials it is natural to wonder if the topological pressure and respective equilibrium states vary continuously with respect to the dynamic and potential under deterministic perturbations. This type of question is called statistical stability and has its importance both in mathematics and in physics, since very often, the dynamical system that describes a physical situation has a deterministic or random error.

When one obtains results of statistical stability the next natural step is to ask about highest regularity results, particularly results on the differentiability of thermodynamic objects with respect to the dynamics and the potential. In such direction, explicit formulas for the derivatives are also welcome. These differentiability results have been referred as linear response formulas (see e.g. [Rue09]). This has proved to be a hard subject not yet completely understood. In fact, linear response formulas have been obtained mostly for uniformly hyperbolic diffeomorphisms and flows in [KKPW89, Ji12, Rue97, BL07, GL06], for the SRB measure of some partially hyperbolic diffeomorphisms in [Dol04] and for one-dimensional piecewise expanding and quadratic maps in [Rue05, BS08, BS09, BS12, BBS14]. More recently, [GP17] studied the differentiable dependence of densities of SRB measures for expanding maps in the circle. The general picture is still far from complete, even in the expanding case, and for any equilibrium states.

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In article [BCV13] linear response formulas, continuity and differentiability of several thermodynamical quantities and limit theorems of a robust non-uniformly expanding setting of [VV10, CV13] were obtained. In such setting, the coexistence of expanding and contracting behavior was admitted, but the potential should be close of a constant function.

In the specific context of expanding dynamics and Hölder potentials it is known that there is one unique equilibrium state such that the statistical stability holds (see e.g. [PU10]). The linear response results in [BCV13] also hold for any smooth expanding dynamics. But such results only deal with smooth potentials close to a constant function.

In this paper, we have two main goals.

The first goal is to obtain linear response formulas for any equilibrium state $\mu_{f,\phi}$ associated to a smooth expanding dynamics $f$ and any smooth potential $\phi$. This is a kind of extension of [BCV13] for all smooth potentials, in the context of smooth expanding dynamics.

Our second goal is to derive the stability of statistical law rates related to the Central Limit Theorem and Large Deviations principle from the linear response formula. For that, we just use the linear response formula results in a abstract way, so, it is not necessary that the dynamics is uniformly expanding.

Finally, we apply the stability of statistical quantities involved in the Large Deviations principle to Multifractal Analysis. Even though we use specification property, which is peculiar to the expanding dynamics setting, we expect these last result also to apply to more general contexts of robust nonuniformly specification.

The strategy to obtain our results begins by studying the spectra of transfer operators associated to the equilibrium measures. We prove that such operators have the spectral gap property on the smooth functions space. For this result, we use the technique of cones and projective metrics. Moreover, the spectral gap property occurs uniformly for dynamics and potential that are sufficiently close. However, the dependence of the operator with respect to the dynamics is not continuous in the norm operator topology, and so, it is not possible to use the classical perturbative spectral theory. Nevertheless, the uniformity of the spectral gap enables us to employ a perturbative framework developed in [GL06], to prove that the topological pressure and the equilibrium states varies $C^{r-1}$ with respect to $C^r$ expanding dynamics and potentials. Using some ideas present in [BCV13], we obtain asymptotic formulas for the first derivative of the topological pressure. As a consequence of these regularity results, we obtain higher regularity results for certain thermodynamic quantities, limit theorems and rate of large deviations. Lastly, using [BV15] we apply our results in the study of the multifractal analysis of smooth expanding dynamics.

This article is organized as follows. In Section 2 we provide some definitions and the statement of the main results. In Section 3 we recall definitions and results on cones and projective metrics, and we also prove a uniform gap spectral property for smooth expanding dynamics and potential. Section 4 is devoted to the proof of the main results.

2. Statement of the main results

Along the text, $M$ will always denote a connected and compact Riemannian manifold, with Riemannian distance $d$. We denote by $D^r$, $r \geq 1$, the space of $C^r$ maps $f : M \to M$ such that $f$ is expanding. For an expanding map $f$ we mean that $\sup_{x \in M} \|[Df(x)]^{-1}\| < 1$.

Remark 2.1. The transfer operator $L_{f,\phi}$ acting on a function $g : M \to \mathbb{C}$ is defined as the function $L_{f,\phi}g(x) := \sum_{y \in f^{-1}(x)} e^{\phi(y)}g(y)$. 

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It follows from [Rue89] that given $f \in D^r$ and a Holder continuous potential $\phi : M \to \mathbb{R}$, there exists an unique equilibrium state $\mu_{f,\phi}$ for $f$ with respect to $\phi$. Furthermore, the support of $\mu_{f,\phi}$ is $M$. Moreover, $\mu_{f,\phi} = h_{f,\phi} \nu_{f,\phi}$, where $\mathcal{L}_{f,\phi} h_{f,\phi} = e^{P_{\text{top}}(f,\phi)} h_{f,\phi}$, $\mathcal{L}_{f,\phi}^* \nu_{f,\phi} = e^{P_{\text{top}}(f,\phi)} \nu_{f,\phi}$ and $\lambda_{f,\phi} := e^{P_{\text{top}}(f,\phi)}$ is the spectral radius of the transfer operator $\mathcal{L}_{f,\phi}$ acting on $C^0(M, \mathbb{R})$.

**Remark 2.2.** Given a integer $r \geq 1$, we will denote by $C^r(M, \mathbb{R})$ the functions space $r$–th continuously differentiable.

### 2.1. Linear response formula.

It is well known that the equilibrium state $\mu_{f,\phi}$ and the topological pressure $P_{\text{top}}(f,\phi)$ depend continuously on the expanding dynamics and are smooth with respect to the potential (see for example [PU10]). Our first main result assures that the topological pressure and equilibrium state varies smoothly with respect to the smooth expanding dynamics.

**Theorem A.** If $r \geq 2$ then the following maps are $C^{r-1}$:

i. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto P_{\text{top}}(f,\phi)$;

ii. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto h_{f,\phi} \in C^{r-1}$;

iii. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \nu_{f,\phi} \in [C^1]^*$;

iv. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \mu_{f,\phi} \in [C^1]^*$.

Moreover, $D_{f,\phi} P_{\text{top}}(f,\phi)$ acting on $(H_1, H_2)$ is given by

$$
\lambda_{f,\phi}^{-1} \sum_{j=1}^{\deg(f)} \int e^{\phi(f_j(\cdot))} D h_{f,\phi}(f_j(\cdot)) \circ [(T_{j\mid f} \circ H_1)(\cdot)] d\nu_{f,\phi} + 
$$

$$
\lambda_{f,\phi}^{-1} \sum_{j=1}^{\deg(f)} \int e^{\phi(f_j(\cdot))} h_{f,\phi}(f_j(\cdot)) D\phi(f_j(\cdot)) \circ [(T_{j\mid f} \circ H_1)(f_j(\cdot))] d\nu_f + \int H_2 d\mu_{f,\phi},
$$

where $f_j$ denote the local inverse branches of $f$ and $T_{j\mid f} \circ H_1$ is the derivative of $f \mapsto f_j$ acting on $H_1$.

The proof of the previous result follows from the gap spectral uniformity of the transfer operators acting on $C^r(M, \mathbb{C})$. This uniformity of the gap spectral follows from cones and projective metrics techniques. Note that the differentiability of the equilibrium state as a functional acting in $C^1(M, \mathbb{R})$ is stronger than a pointual linear response formula result.

**Remark 2.3.** It follows from [BCV13] a explicit formula for the first order derivative of the maximal entropy measure with respect to the smooth expanding dynamics.

The previous theorem implies the following two corollaries:

**Corollary A.** Let $r \geq 3$. The following functions are $C^{r-2}$:

i. $D^r \times \mathbb{R} \ni (f, t) \mapsto P_{\text{top}}(f, t \log |\det Df|);$

ii. $D^r \times \mathbb{R} \ni (f, t) \mapsto P_{\text{top}}(f, t \log ||Df\pm 1||);$

iii. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto h_{f,\phi}(f).$

**Corollary 2.4.** If $D^r \ni f \mapsto g_f \in C^r(M, \mathbb{R})$ is differentiable in $f_0$, then the function $D^r \ni f \mapsto \int g_f d\mu_f$ is differentiable in $f_0$. In particular, if $r \geq 3$ then the following are $C^{r-2}$

i. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \int |Df(x)||d\mu_{f,\phi}|$

ii. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \int |Df(x)^{-1}|^{-1} d\mu_{f,\phi}$

iii. $D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \int \log |\det Df(x)||d\mu_{f,\phi}|.$
2.2. Stability of the statistical laws. It is classically known that for expanding dynamics the Central Limit Theorem holds. We obtain results of stability involving the rates contained in the Central Limit Theorem:

**Theorem B.** Let \( r \geq 2 \) be and \((f, \phi) \in \mathcal{D}^r \times C^\alpha(M, \mathbb{R})\) be. If \( \psi \in C^r(M, \mathbb{R}) \) then either:

i. \( \psi = u \circ f - u + \int \psi \, d\mu_{f, \phi} \) to some \( u \in C^r(M, \mathbb{R}) \)

ii. or the sequence of measurable functions \( \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \psi \circ f^j \) converge in distribution for the normal distribution of average \( m = m_{f, \phi}(\psi) = \int \psi \, d\mu_{f, \phi} \) and variance \( \sigma^2 \) given by

\[
\sigma^2 = \sigma^2_{f, \phi}(\psi) = \int \tilde{\psi}^2 \, d\mu_{f, \phi} + 2 \sum_{n=1}^{\infty} \int \tilde{\psi}(\psi \circ f^n) \, d\mu_{f, \phi} > 0,
\]

where \( \tilde{\psi} = \psi - \int \psi \, d\mu_{f, \phi} \) is a function with zero average depending of \( f \) and \( \phi \). In other words, given a interval \( A \subset \mathbb{R} \):

\[
\mu_{f, \phi}(\{ x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \psi(f^j(x)) \in A \}) \xrightarrow{n \to +\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{u^2}{2\sigma^2}} \, du.
\]

Moreover, \( \mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \ni (f, \phi, \psi) \mapsto m_{f, \phi}(\psi) \) and \( \mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \ni (f, \phi, \psi) \mapsto \sigma^2_{f, \phi}(\psi) \) are \( C^r \) continuity.

In particular, we obtain consequences for the study of cohomological equation. Recall that \( \psi : M \to \mathbb{R} \) is cohomologous to a constant for a dynamic \( f : M \to M \) if there exists a constant \( c \) and a continuous function \( u : M \to \mathbb{R} \) such that the cohomological equation \( \psi = u \circ f - u + c \) holds. If \( u \in L^2(\mu) \) for some probability \( \mu \) and the previous equality holds in \( \mu \)-a.e., we say that \( \psi \) is cohomologous to a constant in \( L^2(\mu) \). When \( u \in C^r(M, \mathbb{R}) \) and the previous equality holds, we say that \( \psi \) is cohomologous to a constant in \( C^r(M, \mathbb{R}) \).

Cohomological equations appear in several problems as smoothness of invariant measures and conjugacies, mixing properties of suspended flows, rigidity of group actions, and geometric rigidity questions such as the isospectral problem. For some other important applications of cohomological equation see for instance [AK11] [W13].

**Corollary B.** Let \( r \geq 2 \). If \( \psi \in C^r(M, \mathbb{R}) \) is not cohomologous to a constant for \( f \in \mathcal{D}^r \) then the same property holds for all \( \hat{f} \) close enough to \( f \). As consequence, the sets

\[
\{ \hat{f} \in \mathcal{D}^r : \psi \text{ is cohomologous to a constant for } \hat{f} \}
\]

and

\[
\{ \hat{\psi} \in C^r(M, \mathbb{R}) : \hat{\psi} \text{ is cohomologous to a constant for } \hat{f} \}
\]

are closed.

2.3. Large deviations stability. It is well known that expanding dynamics satisfies a large deviations principle for Birkhoff’s sums. In other words, given a continuous \( \psi : M \to \mathbb{R} \) the rates

\[
\limsup_{n \to +\infty} \frac{1}{n} \log \mu_{f, \phi}(\{ x \in M : \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) - \int \psi \, d\mu_{f, \phi} \geq \varepsilon \})
\]

and

\[
\liminf_{n \to +\infty} \frac{1}{n} \log \mu_{f, \phi}(\{ x \in M : \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) - \int \psi \, d\mu_{f, \phi} \leq \varepsilon \})
\]

are well understood as functions of the error \( \varepsilon > 0 \) (for details see e.g. [DK01]).
It is known that for expanding dynamics the free energy function is well defined. More precisely, given $f \in \mathcal{D}^r, \phi, \psi \in C^r(M, \mathbb{R})$ and $t \in \mathbb{R}$, the expression

$$
\mathcal{E}_{f,\phi,\psi}(t) := \lim_{n \to \infty} \frac{1}{n} \log \int e^{tS_n \psi} \, d\mu_{f,\phi} = P_{\text{top}}(f, \phi + t\psi) - P_{\text{top}}(f, \phi)
$$

is well defined. Using that if $\psi$ is not cohomologous to a constant, then the free energy function $\mathbb{R} \ni t \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is strictly convex, we can define its Legendre transform $I_{f,\phi,\psi}$ by

$$
I_{f,\phi,\psi}(s) := \sup_{t \in \mathbb{R}} \{st - \mathcal{E}_{f,\phi,\psi}(t)\}.
$$

(see subsection 4.2 for details.)

We obtain the following stability results about the rates involving large deviations.

**Theorem C.** Suppose $r \geq 2$. Let $V$ be a compact manifold and $((f_v, \phi_v, \psi_v))_{v \in V}$ a injective and parameterized ($C^{r-1}$) family of applications in $\mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R})$. If the observable $\psi_{v_*}$ is not cohomologous to a constant, for some $v_* \in V$, then there exists an open neighborhood $U$ of $v_*$ and a open interval $I$ such that for all $v \in U$ and $[a, b] \subset I$

$$
\limsup_{n \to \infty} \frac{1}{n} \log \mu_{f_v,\phi_v} \left( x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \psi_v \circ f_v^j(x) \in [a, b] \right) \leq - \inf_{s \in [a, b]} I_{f_v,\phi_v,\psi_v}(s) \quad \text{and}
$$

$$
\liminf_{n \to \infty} \frac{1}{n} \log \mu_{f_v,\phi_v} \left( x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \psi_v \circ f_v^j(x) \in (a, b) \right) \geq - \inf_{s \in (a, b)} I_{f_v,\phi_v,\psi_v}(s).
$$

Moreover, the Legendre transform $I \times U \ni (s, v) \mapsto I_{f_v,\phi_v,\psi_v}(s)$ is $C^{r-1}$.

2.4. **Multifractal analysis.** Given a continuous observable $\psi : M \to \mathbb{R}$, a Multifractal analysis of its Birkhoff’s average means the study from the topological, dimensional or ergodic viewpoint of level sets $\{x \in M : \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) \in J\}$, where $J$ is an interval. In [BV15] these level sets are studied from the topological pressure viewpoint for dynamics exhibiting an exponential large deviations principle and a unique equilibrium state which is also a weak Gibbs measure. In particular, [BV15] Theorem B) states that if $(f, \phi) \in \mathcal{D}^r \times C^r(M, \mathbb{R})$ and $\phi \in C^r(M, \mathbb{R})$ is a observable which is not cohomologous to a constant, then

$$
R_{\mathbb{X}}_{\mu_{f,\phi,\psi,c}}(f, \phi) = P_{\text{top}}(f, \phi) - \inf_{|s - f \psi_{\mu_{f,\phi}}| \geq c} I_{f,\phi,\psi}(s),
$$

where $R_{\mathbb{X}}_{\mu_{f,\phi,\psi,c}}(f, \phi)$ is the pressure topological of the set

$$
\mathbb{X}_{\mu_{f,\phi,\psi,c}} := \{x \in M : \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) - \int \psi \, d\mu_{f,\phi} \geq c\}
$$

for $f$ with respect the $\phi$.

Recall that a system $f$ satisfies the specification property if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and any sequence of positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point $x$ in $M$ such that

$$
|d(f^{j}(x), f^{j}(x_1))| \leq \varepsilon, \quad 0 \leq j \leq n_1
$$
and 
\[ d\left(f^{j+n_1+p_1+\ldots+n_{i-1}p_{i-1}}(x), f^j(x)\right) \leq \varepsilon \]
for every \( 2 \leq i \leq k \) and \( 0 \leq j \leq n_i \). It is a classical result that expanding maps satisfy the specification property. As a consequence of Corollary C, we obtain:

**Corollary C.** Suppose \( r \geq 2 \). Let \( V \) be a compact manifold and \( \{(f_v, \phi_v, \psi_v)\}_{v \in V} \) a injective and parameterized \( (C^{r-1}) \) family of applications in \( D^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \). Then, the set \( Y := \{(v, c) \in V \times \mathbb{R}^+: c < \sup_{\eta \in M_1(f)} |f v d\mu_{f_v, \phi_v} - \int \psi_v d\eta| \} \) is an open set and

\[ Y \ni (c, v) \mapsto P_{\psi(c),\phi(c)}(f, \phi_v); \]

is \( C^{r-1} \).

Define the spectrum of a continuous function \( \psi : M \to \mathbb{R} \) as:

\[ L(\psi, f) := \{\alpha \in \mathbb{R} : \text{there exists } x \in M \text{ with } \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) = \alpha\}. \]

The set \( Y \) defined above has a close relationship with the Birkhoff’s spectrum of the observable \( \psi \). As \( f \in D^r \) has specification property we know that

\[ L(\psi, f) = \inf_{\eta \in M_1(f)} \int \psi d\eta, \sup_{\eta \in M_1(f)} \int \psi d\eta \]

(see [Tho9], Lemma 1.1). Thereby, \( Y \) is the set of \((v, c) \in V \times \mathbb{R}^+\) such that \( \int \psi v d\mu_{f_v, \phi_v} + c \) or \( \int \psi v d\mu_{f_v, \phi_v} - c \) belongs to the interior of \( L(\psi, f_v) \).

**Remark 2.5.** As we shall see the results of high regularity are consequences of spectral gap uniform of the transfer operator on smooth functions space. Thus, similar results can be obtained for a robust non-uniformly expanding setting and potential close to a constant function studied in [BCV13].

### 3. Preliminary

#### 3.1. Cones and projective metrics.**In** this section we recall basic elements of the cones and projective metrics theory, for more details see e.g. [Vi97, AM06].

Let \( E \) be a vectorial space, \( \emptyset \neq C \subset E \setminus \{0\} \) is a cone (convex) if \( \forall v_1, v_2 \in C \) and \( t > 0 \) we have \( tv_1 + v_2 \in C \). Requiring that \( C \cap (-C) = \emptyset \) we can induce a partial order on \( E \) that preserves its structure of vector space; in fact:

\[ u \preceq v \iff v - u \in C \cup \{0\} \]

The closure \( \overline{C} \) of \( C \) is defined by:

\[ w \in \overline{C} \iff \text{there exists } v \in C \text{ e } t_n \searrow 0 \text{ such that } (w + t_n v) \in C, \forall n \in \mathbb{N}. \]

We will work from here with cones such that \( \overline{C} \cap (-\overline{C}) = \{0\} \), this will allow us to define a pseudo-metric on the cone. Given \( v_1 \) and \( v_2 \in C \), define:

- \( \alpha(v_1, v_2) := \sup\{t > 0; v_2 - tv_1 \in C\} \)
- \( \beta(v_1, v_2) := \inf\{t > 0; tv_1 - v_2 \in C\} \).
Let $\Theta : C \times C \to [0, +\infty)$ be defined by:

$$\Theta(v_1, v_2) := \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)}.$$ 

$\Theta$ is known as Hilbert’s metric, and it is a pseudo-metric.

It is natural to wonder about the relationship between the projective metric and the pre-existing metrics on a normed space, in general depends of the cone that we have defined. The next proposition gives a result in this direction.

**Proposition 3.1.** Let $E$ be a normed space, $\| \cdot \|_i$ be semi-norms on $E$, for $i = 1, 2$, and $\preceq$ be a partial order that preserve its structure of vectorial space. Suppose that for all $v, u \in C$ holds:

$$-v \preceq u \preceq v \Rightarrow \|u\|_i \leq \|v\|_i, i = 1, 2.$$

Then; given $f, g \in C$, with $\|f\|_1 = \|g\|_1$, we have:

$$\|f - g\|_2 \leq (e^{\Theta(f, g)} - 1)\|f\|_2.$$

A fundamental result in cones theory assures that if the image of a cone $C_1$ by a linear map $L$ is bounded with respect to the projective metric of some cone $C_2$ that contains $L(C_1)$, then the map $L$ is a contraction with respect to such projective metrics:

**Theorem 3.2.** Let $E_1, E_2$ vectorial spaces, $C_i \subset E_i, i = 1, 2$, cones, $L : E_1 \to E_2$ a linear operator such that $L(C_1) \subset C_2$ and $D := \sup_{u, v \in C_1}(\Theta_2(L(u), L(v)) : u, v \in C_1)$. If $D < +\infty$ then:

$$\Theta_2(L(u), L(v)) \leq (1 - e^{-D})\Theta_1(u, v), \forall u, v \in C_1.$$

### 3.2. Uniform spectral gap

In this section we are interested in proving that given $(f, \phi) \in D^r \times C^r(M, \mathbb{R})$ the transfer operator $\mathcal{L}_{f, \phi}$ has spectral gap property as a operator acting on $C^r(M, \mathbb{C})$. Furthermore, such spectral gap can be taken uniform under small perturbations in dynamics and potentials.

We recall what we mean by the *spectral gap property*, which is stronger than the usual in the literature.

**Definition 3.3.** *(Spectral gap property.)* Given a Banach $E$ and a bounded linear operator $A : E \to E$, we will say that $A$ has the *spectral gap property* if his spectrum $\text{spec}(A)$ can be decomposed into two spectral components $\Sigma$ and $\{\lambda\}$ such that $\lambda$ is the spectral radius of $A$, is a simple positive eigenvalue and $\Sigma \cap B(0, \lambda) = \Sigma$.

Note that the *spectral gap property* for an operator $A$ is equivalent to the existence of a splitting $E = E_0 \oplus E_1$ such that:

- $E_0$ and $E_1$ are closed subspaces $A$–invariants;
- $E_1$ is a unidimensional auto space associated to spectral radius, and
- There exists $\tau \in (0, 1)$ and $k \geq 0$ such that for all $\varphi \in E_0$ we have $\|A^n\varphi\| \leq \|\varphi\|k\tau^n$, for every $n \geq 0$.

To show that $\mathcal{L}_{f, \phi}(C^r)$ has the spectral gap property in the case of smooth expanding dynamics we will use the technique of cones, in other words, we find a convex invariant cone by $\mathcal{L}_{f, \phi}$ within the space of strictly positive functions whose image by $\mathcal{L}_{f, \phi}$ has finite diameter in the projective metric associated to cone and so get convergence in the norma $C^r$ through of the convergence in the projective metric. For examples of use of the cones technique see [Li95, Vi97, LSV98, Ba00, AM06, BS09, BCV13].
For $r \geq 1$ and $\kappa > 0$ we define the cone of functions

$$\Lambda^r_\kappa := \{ \varphi \in C^r(M, \mathbb{R}) : \varphi > 0 \text{ and } \sup_{x \in M} \left\| \frac{D^s \varphi(x)}{\varphi(x)} \right\| \leq \kappa \sigma_{r,s}, \forall 1 \leq s \leq r \},$$

where $c_{r,s} = 1$, if $r \neq s$ then $c_{r,s}$ just depends on $s$ and are sufficiently small constant to occur cones invariance. Note that we omit the dependence of $c_{r,s}$ in the notation given to the cones. When $r = 1$ we have

$$\Lambda^1_\kappa = \{ \varphi \in C^1(M, \mathbb{R}) : \varphi > 0 \text{ and } \sup_{x \in M} \left\| \frac{D \varphi(x)}{\varphi(x)} \right\| \leq \kappa \}.$$

We have that $\Lambda^r_\kappa$ is a convex cone, furthermore $\Lambda^r_\kappa \cap (-\Lambda^r_\kappa) = \{0\}$.

Now we provide the cones invariance property.

The next lemma is extremely useful in the proof of invariance of the cone when $r > 1$.

**Lemma 3.4.** Suppose that for all $\kappa \geq \kappa_1$ and $\max\{i,1\} < r$ we have $L_\phi(\Lambda^i_\kappa) \subset \Lambda^i_{\lambda \kappa}$. If for $\kappa_2 > 0$ and for all $\varphi \in \Lambda^r_\kappa$, with $\kappa \geq \kappa_2$, we have $\sup_{x \in M} \left\| \frac{D^r \varphi(x)}{\varphi(x)} \right\| \leq \kappa$, then there exists $\kappa_0 > 0$ such that $L_\phi(\Lambda^i_\kappa) \subset \Lambda^i_{\lambda \kappa}$ para todo $\kappa \geq \kappa_0$.

**Proof.** Let $\kappa_0 := \max\{\kappa_2, \kappa_1 \cdot c(r, r - 1)^{1-r}\}$ be. Take $\varphi \in \Lambda^r_\kappa$, for $\kappa \geq \kappa_0$. In particular, $\sup_{x \in M} \left\| \frac{D^r \varphi(x)}{\varphi(x)} \right\| \leq \kappa c_{r,i}^{-1}$, for $i = 1, \ldots, r - 1$. As $c_{r,s}$ just depends on $s$ (unless the diagonal) we have that $\varphi \in \Lambda^r_{\kappa c_{r,i}^{-1}}$, for $i = 1, \ldots, r - 1$. Using the hypothesis, we have $L_{f, \phi} \in \Lambda^r_{\lambda \kappa c_{r,i}^{-1}}$. In particular $\sup_{x \in M} \left\| \frac{D^r L_{f, \phi} \varphi(x)}{L_{f, \phi} \varphi(x)} \right\| \leq \lambda \kappa c_{r,i}^{-1}$, for $i = 1, \ldots, r - 1$. As $\sup_{x \in M} \left\| \frac{D^r L_{f, \phi} \varphi(x)}{L_{f, \phi} \varphi(x)} \right\| \leq \kappa$, for $\kappa \geq \kappa_0$, we have $L_{f, \phi} \varphi \in \Lambda^r_{\lambda \kappa}$.

**Proposition 3.5.** Given $f \in D^r$ and $\phi \in C^r$ there exists $\kappa_0 > 0$, $c_{r,s} > 0$ and $\rho \in (0, 1)$ such that $L_{f, \phi} \Lambda^r_\kappa \subset \Lambda^r_{\rho \kappa}$, for all $\kappa \geq \kappa_0$.

**Proof.** Let $\varphi \in \Lambda^r_\kappa$ be then $L_{f, \phi} \varphi > 0$, besides, given $x \in M$ and $H \in T_x M$ with $||H|| = 1$ we have that:

$$\frac{|D L_{f, \phi} \varphi(x) \cdot H|}{L_{f, \phi} \varphi(x)} = \sum_{j=1}^{\deg(f)} e^{\phi(f_j x)} \varphi(f_j x) D \phi(f_j x) \cdot D f_j(x) \cdot H + e^{\phi(f_j x)} D \varphi(f_j x) \cdot D f_j(x) \cdot H \leq \sum_{j=1}^{\deg(f)} e^{\phi(f_j x)} \varphi(f_j x) \cdot D \phi(f_j x) \cdot D f_j(x) \cdot H \leq ||D \phi||_0 \sigma^{-1} + \sigma^{-1} \kappa,$$

hence $\sup_{x \in M} \frac{|D L_{f, \phi} \varphi(x)|}{\varphi(x)} \leq ||D \phi||_0 \sigma^{-1} + \sigma^{-1} \kappa$. Taking $\kappa_0 := \frac{2||D \phi||_0}{\sigma - 1}$ we have that

$$\sup_{x \in M} \frac{|D L_{f, \phi} \varphi(x)|}{\varphi(x)} \leq \frac{1 - \frac{\sigma^{-1}}{2} \kappa},$$

for all $\kappa \geq \kappa_0$. Thus we prove the proposition for the case $r = 1$. 

---

8
Consider now the case $r = 2$. Take $\kappa > 0$ and $\varphi \in A^2_\kappa$. Using the chain rule, we have that $D^2(L_\phi)(x)$ is a sum of the following seven terms:

\[
\begin{align*}
D^2\phi(x_j)[Df_j(x)]^2e^{\phi(x_j)}\varphi(x_j) \\
D\phi(x_j)D^2f_j(x)e^{\phi(x_j)}\varphi(x_j) \\
D\varphi(x_j)D^2f_j(x)e^{\phi(x_j)}D\phi(x_j)Df_j(x) \\
D\varphi(x_j)Df_j(x)e^{\phi(x_j)}D\phi(x_j)Df_j(x) \\
\varphi(x_j)D\phi(x_j)Df_j(x)e^{\phi(x_j)}D\phi(x_j)Df_j(x) \\
e^{\phi(x_j)}D^2\varphi(x_j)[Df_j(x)]^2 \\
e^{\phi(x_j)}D\varphi(x_j)D^2f_j(x).
\end{align*}
\]

Hence, for $x \in M$ and $H \in T_x M$, with $\|H\| = 1$, we have:

\[
\frac{|D^2L_{f,\phi\varphi}(x)\cdot H|}{L_{f,\phi\varphi}(x)} \leq \|\phi\|_2\sigma^{-1} + \|\phi\|_2 \cdot \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 \cdot \sigma^{-1} + 2c_{2,1}\kappa\sigma^{-1}\|\phi\|_2 + \|\phi\|_2^2\sigma^{-1} + \kappa\sigma^{-1} + c_{2,1}\kappa \cdot \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 \cdot \sigma^{-1} =
\]

\[
\|\phi\|_2\sigma^{-1}(1 + \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 + \|\phi\|_2) + \sigma^{-1}\kappa + \sigma^{-1}c_{2,1}\kappa \cdot \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 + 2\|\phi\|_2).
\]

So

\[
\sup_{x \in M} \left| \frac{|D^2L_{f,\phi\varphi}(x)|}{\varphi(x)} \right| \leq \|\phi\|_2\sigma^{-1}(1 + \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 + \|\phi\|_2) + \sigma^{-1}\kappa + \sigma^{-1}c_{2,1}\kappa \cdot \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 + 2\|\phi\|_2).
\]

Taking $\kappa_0 := \frac{3\|\phi\|_2(1 + \sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 + \|\phi\|_2)}{\sigma^{-1}}$ and $c_{2,1} := \frac{\sigma^{-1}}{3(\sup_{x \in M} ||Df(x)||^{-1} \cdot ||f||_2 + 2\|\phi\|_2)}$

we have

\[
\sup_{x \in M} \left| \frac{|D^2L_{f,\phi\varphi}(x)|}{\varphi(x)} \right| \leq \frac{2 + \sigma^{-1}}{3}\kappa,
\]

for all $\kappa > \kappa_0$. Therefore, using the case $r = 1$ and previous lemma, we prove the proposition for the case $r = 2$.

The general case is analogous computation of higher order derivatives of $L_{f,\phi\varphi}$, through the chain rule, and the use of the previous lemma.

The next corollary tells us that we can take a constant invariant cone on sufficiently small neighborhoods of the dynamic and potential. It will be fundamental in the uniformity of the spectral gap.

**Corollary 3.6.** Given $f \in D^r$ and $\phi \in C^r(M, \mathbb{R})$ there exists neighborhoods $\mathcal{F}^r$ of $f$ and $\mathcal{W}^r$ of $\phi$, as well as constants $\kappa > 0, c_{r,s} > 0$ and $\rho \in (0, 1)$ such that: if $(\hat{f}, \hat{\phi}) \in \mathcal{F}^r \times \mathcal{W}^r$ then $L_{\hat{f},\hat{\phi}}\Lambda^r_\kappa \subset \Lambda^r_\rho$ and $L_{\hat{f},\hat{\phi}}\Lambda^r_\kappa,\delta \subset \Lambda^r_\rho,\delta$.

**Proof.** It follows directly from estimates made in the previous proposition. 

The next proposition, among other things, will show that the image for cones studied by transfer operator has finite diameter.
Proposition 3.7. Given $0 < \rho < 1$, the cone $\Lambda^r_{\rho k}$ has finite diameter in relation the projective metric induced by cone $\Lambda^r_k$.

Proof. Just prove that $\theta_{\kappa}(\varphi, 1)$ is uniformly bounded for all $\varphi \in \Lambda^r_{\rho k}$, with $\sup_{\varphi} \frac{\varphi}{\inf_{\varphi}} = 1$. Take then $\varphi \in \Lambda^r_{\rho k}$.

Claim 1: $\beta_{\kappa}(\varphi, 1) \leq \frac{1}{\inf_{\varphi}(1 - \rho)}$.

In fact; we will prove that for $t_0 := \frac{1}{\inf_{\varphi}(1 - \rho)}$ will occur $t_0\varphi - 1 \in \Lambda^r_k$. As $\rho < 1$ we have $t_0\varphi - 1 > 0$, and given $x \in M, H \in T_xM$, com $||H|| = 1$,

$$\frac{|D^s(t_0\varphi - 1)(x) \cdot H|}{t_0\varphi(x) - 1} = \frac{t_0|D^s\varphi(x) \cdot H|}{t_0\varphi(x) - 1} \leq \frac{t_0\varphi(x)}{t_0\varphi(x) - 1} \cdot \rho \kappa c^{r-s} \leq \kappa c^{r-s}.$$  

Claim 2: $\alpha_{\kappa}(\varphi, 1) \geq \frac{1}{\sup_{\varphi}(1 + \rho)}$.

In fact; we will prove that for $t_1 := \frac{1}{\sup_{\varphi}(1 + \rho)}$ will occur $1 - t_1\varphi \in \Lambda^r_k$. We have that $1 - t_1\varphi > 0$ and given $x \in M, H \in T_xM$, com $||H|| = 1$,

$$\frac{|D^s(1 - t_1\varphi)(x) \cdot H|}{1 - t_1\varphi(x)} \leq \frac{t_1|D^s\varphi(x) \cdot H|}{1 - t_1\varphi(x)} \leq \frac{t_1\varphi(x)}{1 - t_1\varphi(x)} \cdot \varphi(x)\rho \kappa c^{r-s} \leq \kappa c^{r-s}.$$  

Follows from the Claim 1 and 2 that

$$\theta(\varphi, 1) \leq \log \frac{\sup_{\varphi}(1 + \rho)}{\inf_{\varphi}(1 - \rho)} = \log \frac{\sup_{\varphi}}{\inf_{\varphi}} + \log \frac{1 + \rho}{1 - \rho} \leq \sup_{x \in M} \{\|D\varphi(x)\|\} \text{diam}(M) + \log \frac{1 + \rho}{1 - \rho} \leq \rho \kappa c \text{diam}(M) + \log \frac{1 + \rho}{1 - \rho}. \square$$

Now we will prove the spectral gap property in $C^r(M, \mathbb{C})$. Before of the proof, remember that follows from the Hahn-Banach’s geometric forms that there exists a probability $\nu_{f, \phi}$ such that $\mathcal{L}_{f, \phi}^*\nu_{f, \phi} = \lambda_{f, \phi}\nu_{f, \phi}$, where $\lambda_{f, \phi}$ is the spectral radius of $\mathcal{L}_{f, \phi}$ acting on $C^0$. Fix then $\nu_{f, \phi}$ as a probability with this property. We shall see that $\nu_{f, \phi}$ is unique. We denote $\frac{\mathcal{L}_{f, \phi}}{\lambda_{f, \phi}}$ by $\tilde{\mathcal{L}}_{f, \phi}$.

Theorem 3.8. If $f \in D^r$ and $\phi \in C^r$ then $\mathcal{L}_{f, \phi}(\cdot|\cdot)$ has the spectral gap property.

Proof. Take $\kappa_0, c_{r,s}$ and $\rho$ as in the proposition 3.5. Let $\varphi, \psi \in \Lambda^r_{\kappa_0}$ be and $\theta_+$ be the projective metric metric associated to cone of the positive functions. By theorem 3.2 for $n, k \geq 1$ we have:

$$\Theta_+ (\tilde{\mathcal{L}}_{f, \phi}^{n+k}(\varphi), \tilde{\mathcal{L}}_{f, \phi}^n(\psi)) \leq \Theta_{\kappa_0}(\tilde{\mathcal{L}}_{f, \phi}^{n+k}(\varphi), \tilde{\mathcal{L}}_{f, \phi}^n(\psi)) \leq \Delta \tau^{n-1},$$  

(3.1) where $\Delta$ is the $\theta_{\kappa_0}$-diameter of the cone $\Lambda^r_{\rho k_0}$ and $\tau := 1 - e^{-\Delta} \in (0, 1)$. Note that $(\varphi_n := \tilde{\mathcal{L}}_{f, \phi}^n(\varphi))_{n \geq 1}$ is Cauchy in relation the $\Theta_+$, we already know that $\Theta_+$ is complete, so there exists
In fact; since that $\tilde{L}^n_{f,\phi}(\varphi) \xrightarrow{n \to \infty} h_\varphi$ and $\int h_\varphi d\nu_{f,\phi} = \int \varphi d\nu_{f,\phi}$. As $\int \tilde{L}^n_{f,\phi}(\varphi) d\nu_{f,\phi} = \int \varphi d\nu_{f,\phi}$ we can apply the proposition 3.1 in the supreme norm and in the semi-norm of the integral. Thus $\tilde{L}^n_{f,\phi}(\varphi) \xrightarrow{C_0} h_\varphi$ and thereby $L_{f,\phi} h_{f,\phi} = \lambda_{f,\phi} h_{f,\phi}$.

Claim 1: If $\varphi \in \Lambda^r_{\kappa_0}$ then $||\varphi_{n+k} - \varphi_n||_r \leq \Delta r^{-1}(\kappa_0(e^\Delta + 2) + e^\Delta)||h_{\varphi}||_\infty$.

In fact; applying the proposition 3.1 in the supreme norm sup and $\nu_{f,\phi}$, besides the estimate 3.1, we have

$$||\varphi_{n+k} - \varphi_n||_\infty \leq (\Theta^+(\varphi_{n+k}, \varphi_n) - 1) \cdot ||\varphi_{n+k}||_\infty$$

$$\leq \Theta^+(\varphi_{n+k}, \varphi_n) e^\Delta ||\varphi_{n+k}||_\infty$$

$$\leq \Delta r^{-1} e^\Delta ||\varphi_{n+k}||_\infty.$$  \hspace{1cm} (3.2)

Note also that, as $\int \varphi_n d\nu_{f,\phi} = \int \varphi_{n+k} d\nu_{f,\phi}$ and $\varphi_n, \varphi_{n+k}$ are strictly positives continuous functions then

$$\beta_{\kappa_0} (\varphi_n, \varphi_{n+k}) \geq 1 \geq \alpha_{\kappa_0} (\varphi_n, \varphi_{n+k}),$$

hence

$$\Theta_{\kappa_0} (\varphi_n, \varphi_{n+k}) \leq \Delta r^{-1} \Rightarrow \frac{\beta_{\kappa_0} (\varphi_n, \varphi_{n+k})}{\alpha_{\kappa_0} (\varphi_n, \varphi_{n+k})} \leq e^{\Delta r^{-1}} \Rightarrow |1 - \alpha_{\kappa_0} (\varphi_n, \varphi_{n+k})| \leq \Delta r^{-1}.$$  \hspace{1cm} (3.3)

Thus,

$$||D^s (\varphi - \varphi_{n+k})||_\infty \leq ||D^s (\varphi - \alpha_{\kappa_0} (\varphi_n, \varphi_{n+k}) \cdot \varphi_{n+k})||_\infty + |\alpha_{\kappa_0} (\varphi_n, \varphi_{n+k}) - 1| \cdot ||D^s (\varphi_{n+k})||_\infty \leq$$

$$c_{r,s}^{-s} \kappa_0 ||\varphi_n - \alpha_{\kappa_0} (\varphi_n, \varphi_{n+k}) \cdot \varphi_{n+k}||_\infty + |\alpha_{\kappa_0} (\varphi_n, \varphi_{n+k}) - 1| \cdot ||D^s (\varphi_{n+k})||_\infty \leq$$

$$c_{r,s}^{-s} \kappa_0 \left[ ||\varphi_n - \varphi_{n+k}||_\infty + |1 - \alpha_{\kappa_0} (\varphi_n, \varphi_{n+k})| \cdot ||\varphi_{n+k}||_0 \right] + c_{r,s}^{-s} \kappa_0 |\alpha_{\kappa_0} (\varphi_n, \varphi_{n+k}) - 1| \cdot ||\varphi_{n+k}||_\infty \leq$$

$$c_{r,s}^{-s} \kappa_0 \left[ ||\varphi_n - \varphi_{n+k}||_\infty + ||\varphi_{n+k}||_\infty 2 \Delta r^{-1} \right] \leq$$

$$c_{r,s}^{-s} \kappa_0 \Delta r^{-1} e^\Delta ||\varphi_{n+k}||_\infty + ||\varphi_{n+k}||_\infty 2 \Delta r^{-1}.$$  \hspace{1cm} (3.4)

As $\varphi_n$ converge uniformly, using the estimates (3.2) and (3.3) we have that $\varphi_n$ is a Cauchy sequence in $C^r$. Furthermore, doing $k \rightarrow +\infty$, we have that

$$||h_{\varphi} - \varphi_n||_r \leq \Delta r^{-1} (\kappa_0(e^\Delta + 2) + e^\Delta)||h_{\varphi}||_\infty.$$  \hspace{1cm} (3.4)

Follows of the Claim 1 that $\varphi_n$ converge for $h_\varphi$ in the $C^r$ norm and $h_\varphi \in \Lambda^r_{\kappa_0}$.

Claim 2: $\ker(L_\varphi - \lambda_{f,\phi} I) \cap C^r(M, \mathbb{C})$ has dimension one.

In fact; since that $\lambda_{f,\phi} \in \mathbb{R}$ is enough prove that $\ker(L_\varphi - \lambda_{f,\phi} I) \cap C^r(M, \mathbb{R})$ has dimension one. Let $h_{f,\phi} := \lim_{n \to +\infty} \tilde{L}^n_{f,\phi}(1)$ be and $u \in \ker(L_{f,\phi}^r(M, \mathbb{R}) \cap \Lambda^r_{\kappa_0}$ be. By (3.1) there exists $t_1 > 0$ such that $t_1 u = h$. Thereby, by Claim 1, for all $\varphi \in \Lambda^r_{\kappa_0}$ we have that $\tilde{L}^n_{f,\phi}(\varphi) \xrightarrow{C^r} \int \varphi d\nu_{f,\phi} \cdot h_{f,\phi}$. Given $v \in \ker(L_{f,\phi}^r(C^r(M, \mathbb{R})$, there exists a constant $B > 0$ such that $v + B$ is a element of $\Lambda^r_{\kappa_0}$; hence $v = \lim \tilde{L}^n_{f,\phi}(v + B) - \lim \tilde{L}^n_{f,\phi}(B) = \int v d\nu_{f,\phi} \cdot h_{f,\phi}$.  \hspace{1cm} (3.4)
Thus, ker(\(L_{f, \phi}|_{C^r} - \lambda f, \phi I\)) = \{thf, \phi : t \in \mathbb{R}\}.

Let \(E_1 := \ker(L_{f, \phi}|_{C^r} - \lambda f, \phi I)\) be and \(E_0 := \{\varphi \in C^r(M, \mathbb{C}) : \int \varphi d\nu f, \phi = 0\}\) be. If \(h_{f, \phi}\) it is defined as the previous claim then \(\int h_{f, \phi} d\nu f, \phi = 1\) and \(E_1 = \{t \cdot h_{f, \phi} : t \in \mathbb{C}\}\). Note that \(E_0, E_1\) are \(L_{f, \phi}|_{C^r}\)-invariant closed subspaces and \(C^r(M, \mathbb{C}) = E_1 \oplus E_0\).

**Claim 3:** \(\text{spec}(\hat{L}_{f, \phi}|_{E_0}) \subset B(0, \lambda_1), \) where \(0 < \lambda_1 < 1\).

In fact, endow \(E_0\) of the \(C^r\) norm and take \(\varphi \in E_0 \cap C^r(M, \mathbb{R})\), with \(||\varphi||_r = 1\). So \(\varphi + 1 + \frac{1}{\kappa_0} \in \Lambda^r\). By previous discussions, we know that \(\hat{L}_{f, \phi}^n(\varphi + 1 + \frac{1}{\kappa_0}) \overset{C^r}{\longrightarrow} \int (\varphi + 1 + \frac{1}{\kappa_0}) d\nu f, \phi \cdot h_{f, \phi} = (1 + \frac{1}{\kappa_0}) \cdot h_{f, \phi}.\) Thus:

\[
||\hat{L}_{f, \phi}^n(\varphi)||_r = ||\hat{L}_{f, \phi}^n(\varphi + 1 + \frac{1}{\kappa_0}) - \hat{L}_{f, \phi}^n(1 + \frac{1}{\kappa_0})||_r \leq ||\hat{L}_{f, \phi}^n(\varphi + 1 + \frac{1}{\kappa_0}) - (1 + \frac{1}{\kappa_0}) \cdot h_{f, \phi}||_r + ||\hat{L}_{f, \phi}^n(1 + \frac{1}{\kappa_0}) - (1 + \frac{1}{\kappa_0}) \cdot h_{f, \phi}||_r \leq \Delta^{n-1}(\kappa_0(e^\Delta + 2) + e^\Delta)(3 + \frac{2}{\kappa_0})||h_{f, \phi}||_\infty.
\]

Anallogous, for \(\varphi \in E_0\) complex function with \(||\varphi||_r = 1\) we have \(||\hat{L}_{f, \phi}^n(\varphi)||_r \leq \Delta^{n-1}(\kappa_0(e^\Delta + 2) + e^\Delta)(3 + \frac{2}{\kappa_0})||h_{f, \phi}||_\infty.\) Hence \(\hat{L}_{f, \phi}|_{E_0}\) is a contraction in the \(C^r\) norm for \(n\) big enough and so \(\text{spec}(\hat{L}_{f, \phi}|_{E_0}) \subset B(0, \lambda_1), \) where \(0 < \lambda_1 < 1\).

Then follows from the previous claims the spectral gap property. \(\square\)

**Corollary 3.9.** Given \(f \in D^r\) and \(\phi \in C^r\) there exists a unique \(\nu \in \ker(L_{f, \phi}|_{C^r} - \lambda f, \phi I)\) such that \(\nu(1) = 1\).

**Proof.** Let \(\nu \in \ker(L_{f, \phi}|_{C^r} - \lambda f, \phi I)\) be such that \(\nu(1) = 1\). Given \(\varphi \in C^r(M, \mathbb{R})\), we know from the previous theorem that \(\hat{L}_{f, \phi}\varphi \to \int \varphi d\nu f, \phi \cdot h_{f, \phi},\) hence

\[
\nu(\varphi) = \lim \nu(\hat{L}_{f, \phi}\varphi) = \nu(\int \varphi d\nu f, \phi \cdot h_{f, \phi}) = \int \varphi d\nu f, \phi \nu(h_{f, \phi}) = \int \varphi d\nu f, \phi \lim \nu(\hat{L}_{f, \phi}^n 1) = \int \varphi d\nu f, \phi.
\]

Thus prove the uniqueness. \(\square\)

The next corollary assures us uniformity in the spectral gap. Let’s remember that \(D^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \longmapsto \frac{d\nu f, \phi}{d\nu f, \phi}\) is continuous, endowing the image with the supreme norm (see e.g. [PU10]).

**Corollary 3.10.** Given \(f \in D^r\) and \(\phi \in C^r(M, \mathbb{R})\) there exists neighborhoods \(\mathcal{F}^r\) of \(f\) and \(\mathcal{W}^r\) of \(\phi\), beyond of constants \(k \geq 0\) and \(\tau \in (0, 1)\) such that: if \((\hat{f}, \hat{\phi}) \in \mathcal{F}^r \times \mathcal{W}^r\) then given \(\varphi \in C^r(M, \mathbb{R})\) we have

\[
||\hat{L}_{\hat{f}, \hat{\phi}}^n - \int \varphi d\nu \hat{f}, \hat{\phi} \cdot h_{\hat{f}, \hat{\phi}}||_r \leq k\tau^n ||\varphi||_r.
\]

for all \(n \in \mathbb{N}\).

**Proof.** It follows directly from estimates of the Claim 3, items (i) and (ii) of the previous theorem, as well as the corollary 3.6. \(\square\)
4. Proof of the main results

4.1. Linear response formula. This section is devoted to the proof of our Linear response formula result (Theorem 4).

We already know that holds statistical stability for expanding dynamics:

**Proposition 4.1.** If \( r \geq 1 \), then the following maps are continuous:

\[
\begin{align*}
& (i) \quad \mathcal{D}^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto P_{\text{top}}(f, \phi); \\
& (ii) \quad \mathcal{D}^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \frac{d\mu_{f,\phi}}{df}, \text{ endowing the image with the } C^0\text{-norm}; \\
& (iii) \quad \mathcal{D}^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \nu_{f,\phi}, \text{ endowing the image with the weak* topology}; \\
& (iv) \quad \mathcal{D}^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto \mu_{f,\phi}, \text{ endowing the image with the weak* topology}.
\end{align*}
\]

**Proof.** See e.g. [PU10]. \( \square \)

Now we will discuss the linear response formula results. We want to obtain differentiability results of \( P_{\text{top}}(f, \phi), \frac{d\mu_{f,\phi}}{df} \) and \( \mu_{f,\phi} \) with respect to \( f \) and \( \phi \). Note that these elements can be obtained by the spectral projection of \( \mathcal{L}_{f,\phi} \) on the eigenspace associated to the dominant eigenvalue. Thereby, we will discuss about the differentiability of the Spectral Theory point of view (for details about Spectral Theory see by example [K95]).

Let \( A : E \to E \) be a bounded operator with the spectral gap property, let \( \gamma \) a closed curve \( C^1 \) such that the bounded connected component determined by it contains the dominant eigenvalue of \( A \) and the region which is exterior to the curve contains other spectral component of \( A \). By semi continuity of the spectral components, there is a \( \delta > 0 \) such that if \( ||A - \hat{A}|| < \delta \) then \( \gamma \) separates two spectral components of \( \hat{A} \). We denote the spectral component of \( \hat{A} \) which is limited by the connected component by \( \Lambda_{\hat{A}} \). If we denote by \( P_{\hat{A}} \) the spectral projection associated to the spectral component \( \Lambda_{\hat{A}} \) of \( \hat{A} \), by the holomorphic functional calculus we have that \( P_{\hat{A}} = \frac{1}{2\pi i} \int_{\gamma} (zI - \hat{A})^{-1}dz \). As the curve \( \gamma \) is fixed, the differentiability of \( P_{\hat{A}} \) with respect to \( \hat{A} \), is the same of \( (zI - \hat{A})^{-1} \) with respect to \( \hat{A} \). It is known by the spectral theory that, in fact, \( (zI - \hat{A})^{-1} \) is analytic with respect to \( \hat{A} \) (we still have uniform convergence of Taylor series when we vary \( z \) along of the curve \( \gamma \)) and therefore \( P_{\hat{A}} \) is analytic with respect to \( \hat{A} \).

Let us now return to the case of transfer operator.

**Remark 4.2.** By [Fr72], given a compact and connected smooth Riemannian manifold \( M \), the space of applications \( C^r(M,M) \) can be seen as manifold modeled by Banach space \( C^r(M,\mathbb{R}^{2\dim(M)}) \). Furthermore, given \( f \in C^r(M,M) \) its tangent space can be identified by sections \( \Gamma_f := \{ \gamma \in \mathcal{C}^r(M, TM) : \gamma(x) \in T_{f(x)}M, \forall x \in M \} \).

Thus, the differentiability of the spectral projection would imply the differentiability \( \lambda_{f,\phi} \) and \( \mu_{f,\phi} \) with respect the \( f \) and \( \phi \) if the transfer operators depended smoothly with respect to dynamics and the potential. However, in general, it is only true that in general the transfer operator depends analytically on the potential (see [PU10]). So, we have

**Proposition 4.3.** Fixed \( f \in \mathcal{D}^r \), the following applications are analytical:

\[
\begin{align*}
& \text{i. } C^r(M, \mathbb{R}) \ni \phi \mapsto \lambda_{f,\phi}; \\
& \text{ii. } C^r(M, \mathbb{R}) \ni \phi \mapsto h_{f,\phi} \in C^r(M, \mathbb{R}); \\
& \text{iii. } C^r(M, \mathbb{R}) \ni \phi \mapsto \nu_{f,\phi} \in [C^r(M, \mathbb{R})]^*; \\
& \text{iv. } C^r(M, \mathbb{R}) \ni \phi \mapsto \mu_{f,\phi} \in [C^r(M, \mathbb{R})]^*.
\end{align*}
\]
On the other hand, the transfer operator does not vary continuously with respect to the dynamics, even in the expanding case, as an operator acting on $C^r(M, \mathbb{R})$ (see [CV13, Example 4.14]). Nevertheless, in [BCV13], It is proved that the transfer operator is smooth with respect to smooth dynamics in a weaker sense, namely, requiring less regularity of its counter-domain. More accurately,

**Proposition 4.4.** Let $r \geq 1, 1 \leq k \leq r$. Let $\text{Diff}_{\text{loc}}^r(M,M)$ be the local $C^r$-diffeomorphisms space on a compact and connected Riemannian manifold $M$ and let $\phi \in C^r(M, \mathbb{R})$ be some fixed potential. Then the map

$$\text{Diff}_{\text{loc}}^r(M) \ni f \mapsto L(C^r(M, \mathbb{R}), C^{r-k}(M, \mathbb{R}))$$

is $C^k$.

**Remark 4.5.** It follows from the estimates made in the proof of the previous proposition that, in small neighborhoods of the local diffeomorphism, the derivative of $f \mapsto L_{f,\phi}$ are uniformly bounded.

Therefore, we can not guarantee that $\lambda_{f,\phi}$ and $\mu_{f,\phi}$ are smooth with respect to dynamics $f$ using the classical spectral theory, because it would only guarantee such differentiability if we had the differentiability of the transfer operator as operator acting on the same space.

For the reader convenience, we recall here the context of [GL06, Theorem 8.1]. Let $r \geq 2$ be, $\mathcal{B}^0 \supset \cdots \supset \mathcal{B}^r$ be Banach spaces, $I$ be a Banach manifold and $\{A_t\}_{t \in I}$ be a bounded linear operators family acting on the Banach spaces $\mathcal{B}^i$ such that $I \ni t \mapsto A_t \in \mathcal{L}(\mathcal{B}^0, \mathcal{B}^0)$ is continuous, where $\mathcal{L}(\mathcal{B}^i, \mathcal{B}^{i-j})$ is the bounded linear operators space of $\mathcal{B}^i$ on $\mathcal{B}^{i-j}$. Furthermore, assume that

$$\exists M > 0, \forall t \in I, \forall g \in \mathcal{B}^0, ||A_t^m g||_{\mathcal{B}^0} \leq CM^n ||g||_{\mathcal{B}^0}$$

and

$$\exists \alpha < M, \forall t \in I, ||A_t^m g||_{\mathcal{B}^1} \leq C\alpha^n ||g||_{\mathcal{B}^1} + CM^n ||g||_{\mathcal{B}^0}. \quad (4.2)$$

Assume also that for $j = 1, \ldots, r-1$, there exists the $j$-th derivative of the map $I \ni t \mapsto A_t \in \mathcal{L}(\mathcal{B}^r, \mathcal{B}^{r-j})$. Denoting by $Q_j$ the $j$-th derivative that acts from $I$ to $\mathcal{L}(\mathcal{B}^r, \mathcal{B}^{r-j})$, assume that for all $i \in [j,r]$ we have that $Q_i$ is bounded as a map of $I$ in $\mathcal{L}(\mathcal{B}^i, \mathcal{B}^{i-j})$. In these terms, it follows from [GL06, Theorem 8.1] that:

**Theorem 4.6.** For $\varrho > \alpha$ and $\delta > 0$, denote by $\mathcal{V}_{\varrho,\delta}$ the set of complex numbers $z$ such that $|z| \geq \varrho$ and, for all $1 \leq k \leq r$, assume that the distance of $z$ to spectrum of $A_t|_{\mathcal{B}^k}$ is bigger or equal than $\delta$. Then, the map $I \times \mathcal{V}_{\varrho,\delta} \ni (t,z) \mapsto (z - A_t)^{-1} \in \mathcal{L}(\mathcal{B}^r, \mathcal{B}^0)$ is $C^{r-1}$.

**Remark 4.7.** It follows also from [GL06, Theorem 8.1] that if $T_i(t,z)$ denotes the $i$-th derivative of $I \times \mathcal{V}_{\varrho,\delta} \ni (t,z) \mapsto (z - A_t)^{-1} \in \mathcal{L}(\mathcal{B}^r, \mathcal{B}^0)$ in $(t, z)$ then fixed $t \in I$ we have that

$$\frac{|T_i(t+h_1,z+h_2) - T_i(t,z) - T_{i+1}(t,z) \cdot (h_1, h_2)|}{||h_1, h_2||}$$

converges to 0, when $(h_1, h_2)$ converges to 0, uniformly in $z \in \mathcal{V}_{\varrho,\delta}$.

Using this theorem and Theorem 3.10 about uniform spectral gap property, we can prove the Theorem [A].

Before the proof of the theorem we need of a strong stability result, namely:

**Proposition 4.8.** The mapping $\mathcal{D}^r \ni f \mapsto h_f = h_{f,\phi} \in C^{r-1}(M, \mathbb{R})$ is continuous.
Proof. Fix $f_0 \in \mathcal{D}^r$. By Theorem 3.10 we have

$$||\tilde{L}_{f,\phi}^n 1 - h_{f,\phi}||_0 \leq k\tau^n, \forall f \in \mathcal{D}^r.$$ 

Take $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$||\tilde{L}_{f,\phi}^n(1) - h_{f,\phi}||_0 < \frac{\epsilon}{3}, \forall f \in \mathcal{D}^r.$$ 

By proposition 4.1 $\mathcal{D}^r \ni f \mapsto \lambda_{f,\phi}$ is continuous. Using the Proposition 4.3 there exists a neighborhood $V$ of $f_0$ so that: if $f \in V$ we have that

$$||\tilde{L}_{f,\phi}^n(1) - \tilde{L}_{f_0,\phi}^n(1)||_{r-1} < \frac{\epsilon}{3}.$$ 

Therefore, for $f \in V$:

$$||h_{f,\phi} - h_{f_0,\phi}||_0 \leq ||h_{f,\phi} - \tilde{L}_{f,\phi}^n(1)||_{r-1} + ||\tilde{L}_{f,\phi}^n(1) - \tilde{L}_{f_0,\phi}^n(1)||_{r-1} + ||\tilde{L}_{f_0,\phi}^n(1) - h_{f_0,\phi}||_0 < \epsilon.$$

This prove that $\mathcal{D}^r \ni f \mapsto h_{f,\phi} \in C^r-1(M, \mathbb{R})$ is continuous in $f_0$. 

\[ \square \]

Proof of the Theorem A. Fix $f_0 \in \mathcal{D}^r$. Note initially that $\mathcal{L}_{f,\phi}(M, \mathbb{R})$ and $\mathcal{L}_{f,\phi}(C^r)$ have the same spectral radius, which is equal to $e^{\varphi_\top f}(f_0)$. By the previous proposition there exists a neighborhood $U$ of $f_0$ so that $\sup_{f \in U} ||\mathcal{L}_{f,\phi}||_0 < +\infty$. By statistical stability (Proposition 4.3) $\lambda_{f,\phi}$ is continuous. So, we can assume without loss of generality that $\sup_{f \in U} \lambda_{f,\phi} = +\infty$. Furthermore, there exists a closed $C^1$-curve $\gamma$ such that the bounded connected component determined by $\gamma$ contains the spectral radius of $\mathcal{L}_{f,\phi}(C^r)$ for any $f \in U$, and the unbounded connected component contains the rest of the spectrum of $f$, for any $f \in U$. Let $P_f$ be the spectral projection of $\mathcal{L}_{f,\phi}(C^r)$ associated to its spectral radius. We already know that

$$P_f = \frac{1}{2\pi} \int \varphi(zI - \mathcal{L}_{f,\phi})^{-1}dz$$

for all $f \in U$. We will use the previous theorem to prove that

$$U \times \gamma \ni (zI - \mathcal{L}_{f,\phi})^{-1} \in L(C^r, C^{r-1})$$

is $C^{r-1}$ and that the rests of the $C^{r-1}$-differentiability go to 0 uniformly in relation a $z \in \gamma$, when we fix $f \in U$. For this is enough choose correctly the elements contained in Theorem 4.6. In the previous theorem take $\mathcal{B} = C^0(M, \mathbb{R}), \mathcal{B}^1 = C^{r-1}(M, \mathbb{R}), \mathcal{B}^2 = C^r(M, \mathbb{R}), \mathcal{I} = U, t = f$, and so $A_t = A_f = \mathcal{L}_{f,\phi}$. By Proposition 4.3 in order to apply Theorem 4.6. It is enough to observe that the hypotheses (4.1) and (4.2) hold. By Theorem 3.10

$$||\tilde{L}_{f,\phi}g - \int g d\nu_{f,\phi} \cdot h_{f,\phi}||_0 \leq k\tau^n, \forall g \in C^{r-1}(M, \mathbb{R}), \forall f \in U.$$ 

Take $M := sup_{f \in U} ||\mathcal{L}_{f,\phi}||_1 < +\infty$, $\alpha := sup_{f \in U} \lambda_{f,\phi} \cdot \tau$ and $C := max\{k, sup_{f \in U} ||h_{f,\phi}||_0\}$. Therefore,

$$||\mathcal{L}_{f,\phi}^n g||_0 \leq M^n ||g||_0, \forall g \in C^0(M, \mathbb{R}) \text{ and } \forall f \in U,$$

and

$$||\mathcal{L}_{f,\phi}^n g||_0 \leq k(\lambda_{f,\phi} \tau)^n + \lambda_{f,\phi}^n ||g||_0 \cdot ||h_{f,\phi}||_0 \cdot sup_{f \in U} ||\mathcal{L}_{f,\phi}||_0^n \cdot ||g||_0.$$ 

Thus, we can apply the previous theorem and the Remark 4.7 concluding that $U \times \gamma \ni (zI - \mathcal{L}_{f,\phi})^{-1} \in L(C^r, C^{r-1})$ is $C^{r-1}$ and that the rests of the $C^{r-1}$-differentiability go to 0 uniformly in relation a $z \in \gamma$, when we fix $f \in U$. So, $U \ni f \mapsto P_f \in L(C^r, C^{r-1})$ is $C^{r-1}$. As $\lambda_{f,\phi} = \frac{\varphi_\top f (P_f)}{P_f}$, $h_{f,\phi} = P_f$ and $\nu_{f,\phi}(g) = \frac{P_g}{P_f}, \forall g \in C^0(M, \mathbb{R})$, we prove the first part of the theorem.

Now we prove the second part of the theorem. We want to calculate an asymptotic formula for $D_{f,\phi}P_{\text{top}}(f, \phi)$. As it is already known that $\partial_{\phi}P_{\text{top}}(f, \phi) = \mu_{f,\phi}$ (see for example
Lemma 4.9. (Local Differentiability of inverse branches) Let \( r \geq 1, 0 \leq k \leq r \) and \( f : M \to M \) be a \( C^r \)-local diffeomorphism on a compact connected manifold \( M \). Let \( B = B(x, \delta) \subset M \) some ball such that the inverse branches \( f_1, \ldots, f_s : B \to M \) are well defined diffeomorphisms onto their images. Then \( C^r(M, M) \ni f \mapsto (f_1, \ldots, f_s) \in C^{r-k} \) is a \( C^k \) map.

We denote the derivative of \( f \mapsto f_j \) by \( T_{j,f} \).

Lemma 4.10. Let \( f \in \text{Diff}^r_{\text{loc}} \) and \( \phi \in C^r(M, \mathbb{R}) \) be. Given \( H \in \Gamma^r_f, g_1, g_2 \in C^r(M, \mathbb{R}) \) and \( t \in \mathbb{R} \) it holds

\[
\begin{align*}
    i) \quad & D_f(L^n_{f,\phi}(g))_{f_0} \cdot H = \sum_{i=1}^n L_i^n_{f,\phi}(D_f L^n_{f,\phi}(L_i^{n-i}(g))_{f_0} \cdot H); \\
    ii) \quad & \text{there exists} \ c_{f,\phi} > 0 \ \text{so that} \ \|D_f L^n_{f,\phi}(g))_{f_0} \cdot H\|_0 \leq c_{f,\phi} \|g\|_1 \|H\|_1 \ \text{and} \ c_{f,\phi} \ \text{can be taken} \\
    iii) \quad & D_f L^n_{f,\phi}(g_1 + tg_2)_{f_0} \cdot H = D_f L^n_{f,\phi}(g_1)_{f_0} \cdot H + t D_f L^n_{f,\phi}(g_2)_{f_0} \cdot H.
\end{align*}
\]

We remark that \( A \circ H \) will denote an operator \( A \) acting on vector \( H \). Fix \( f_0 \in \mathcal{D}^r \). By uniform spectral gap (Theorem \[3.10\]) and the continuity of \( \mathcal{D}^r \ni f \mapsto h \), there exists \( K > 0 \) and neighborhood \( W \) of \( f_0 \) so that \( K^{-1} \leq \int \mathcal{L}^1_f dv_f \leq K \) for all \( f \in W \) and \( n \in \mathbb{N} \). In particular, \( \lim_{n \to \infty} \frac{1}{n} \log \int L^n_{f,1} dv_{f_0} = \log \lambda_f \) uniformly with respect to \( f \in W \). Consider the sequence of applications \( P_n : W \to \mathbb{R} \) given by \( P_n(f) = \frac{1}{n} \log \int L^n_{f,1} dv_{f_0} \), that converges uniformly to \( \log \lambda_f \). We will see that the \( D_f P_n \) converges uniformly. By chain rule:

\[
\begin{align*}
    D_f P_n f \circ H &= \int \frac{D_f L^n_{f,1}(f) \circ H dv_{f_0}}{n \cdot \int L^n_{f,1}(dv_{f_0})} = \frac{\int \sum_{i=1}^n L_i^n_{f,1}(D_f L^n_{f,1}(L_i^{n-i}(g)) \circ (f \circ H) dv_{f_0})}{n \cdot \int L^n_{f,1}(dv_{f_0})} = \\
    &= \frac{B_n(f, \phi) \circ H}{\lambda_f \int L^n_{f,1}(dv_{f_0})} + \frac{C_n(f, \phi) \circ H}{\lambda_f \int L^n_{f,1}(dv_{f_0})},
\end{align*}
\]

where

\[
    B_n(f, \phi) \circ H = \frac{1}{n} \int \sum_{i=1}^n L_i^n_{f,1}( \sum_{j=1}^{\deg(f)} e^{\phi(f_j)}(c)) \cdot D_f L_i^{n-i}(1) \circ \left( [T_{j,f} H] \right) dv_{f_0}
\]

and

\[
    C_n(f, \phi) \circ H = \frac{1}{n} \int \sum_{i=1}^n L_i^n_{f,1}( \sum_{j=1}^{\deg(f)} e^{\phi(f_j)}(c)) \cdot \tilde{L}_i^{n-i}(1) \circ \left( D_f \phi_{f_j}(c) \cdot [T_{j,f} H] \right) dv_{f_0}.
\]

Mutatis mutandis the proof contained in [BCV13 Lemma 4.13], taking into account the uniform spectral gap, we obtain:

Claim 1: \( B_n(f, \phi) \circ H \) is uniformly convergent on \( \hat{f} \) and \( \hat{H} \in \Gamma_f^r \), with \( \|H\|_r \leq 1 \), to the expression \( \int \sum_{j=1}^{\deg(f)} e^{\phi(f_j)(c)} \cdot D_h f_i \phi_{f_j}(c) \circ \left( [T_{j,f} H_1] \right) dv_f \cdot \int h_f dv_{f_0} \).
Claim 2: $C_n(f) \cdot H$ is uniformly convergent on $\hat{f}$ and $H \in \Gamma^r$, with $\|H\|_r \leq 1$, to the expression

$$\int \sum_{j=1}^{\deg(f)} e^{\phi(f)(j)} \cdot h_{f,\phi}(\hat{f}_j(j)) \cdot D\phi_{|f_j(j)} \circ [(T_{j|f} \circ H)(\hat{f}_j(j))]) d\nu_{f} \cdot \int h_{f,\phi} d\nu_{f_0}.$$ 

As $D_f\pi_f \circ (H) = \frac{B_n(f,\phi) \circ H}{\lambda_f \int L_f^n(1) d\nu_{f_0}} + \frac{C_n(f,\phi) \circ H}{\lambda_f \int L_f^n(1) d\nu_{f_0}}$, using the two claims and that $\int \tilde{L}_f^n(1) d\nu_{f_0}$ converges for $\int h_{f,\phi} d\nu_{f_0}$ uniformly in relation to $\hat{f}$ we obtain that $D_f\pi_f \circ H$ is uniformly convergent on $\hat{f}$ and $H \in \Gamma^r$, such that $\|H\|_r = 1$, to the sum

$$\lambda_f^{-1} \int \sum_{j=1}^{\deg(f)} e^{\phi(f)(j)} \cdot Dh_{f,\phi}(\hat{f}_j(j)) \circ [(T_{j|f} \circ H_1)(\cdot)] d\nu_{f}$$

$$+ \lambda_f^{-1} \int \sum_{j=1}^{\deg(f)} e^{\phi(f)(j)} \cdot h_{f,\phi}(\hat{f}_j(j)) \cdot D\phi_{|f_j(j)} \circ [(T_{j|f} \circ H)(\hat{f}_j(j))]) d\nu_{f}.$$ 

This completes the proof of the theorem. □

4.2. Large deviations stability. This section is devoted to the proof of Corollary $\Box$

Furthermore, we know that if $\psi$ is cohomologous to a constant then $\mathbb{R} \ni t \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is affine; however, if $\psi$ not is cohomologous to a constant then $\mathbb{R} \ni t \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is real analytic and strictly convex. As a consequence of Theorem $\Box$ we have:

**Proposition 4.11.** Let $r \geq 2$. Then the following applications are $C^{r-1}$:

1. $\mathbb{R} \times \mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \ni (t, f, \phi, \psi) \mapsto \mathcal{E}_{f,\phi,\psi}(t)$,
2. $\mathbb{R} \times \mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \ni (t, f, \phi, \psi) \mapsto \mathcal{E}_{f,\phi,\psi}(t)$.

Assume that $\psi$ is not cohomologous to a constant and that $\int \psi d\mu_{f,\phi} = 0$. Using that $\mathbb{R} \ni t \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is strictly convex it is well defined its Legendre transform $I_{f,\phi,\psi}$ by

$$I_{f,\phi,\psi}(s) := \sup_{t \in \mathbb{R}} \{ st - \mathcal{E}_{f,\phi,\psi}(t) \}.$$ 

This function is non-negative and strictly convex since $\mathcal{E}_{f,\phi,\psi}$ is also strictly convex and $I_{f,\phi,\psi}(s) = 0$ if and only if $s = \int \psi d\mu_{f,\phi}$. We can define the Legendre transform to $\psi$ which is not cohomologous to a constant by $I_{f,\phi,\psi}(t) := I_{f,\phi,\psi} - \int \psi d\mu_{f,\phi} (t - \int \psi d\mu_{f,\phi})$. Using the differentiability of the free energy function it is not hard to check the variational property

$$I_{f,\phi,\psi}(\mathcal{E}'_{f,\phi,\psi}(t)) = t \mathcal{E}'_{f,\phi,\psi}(t) - \mathcal{E}_{f,\phi,\psi}(t).$$ 

Furthermore, $s \mapsto I_{f,\phi,\psi}(s)$ is strictly convex.

Using the differentiability of the free energy function, one can get results from large deviations by Gartner-Ellis’s theorem, with rate function given by the Legendre transform (see e.g. [CRL98]). Our contribution here is to prove the regularity of the large deviations rate function with respect to the expanding dynamics.

**Proof of the Theorem $\Box$** Let $\psi_{\nu}$ be a observable not cohomologous to a constant. As $\psi_{\nu}$ is cohomologous to a constant if only $\mathcal{E}_{f,\phi,\psi}(t)$ is affine and by Theorem $\Box$ we have that there
exists a neighborhood \( U \) of \( v_* \) so that \( \psi_v \) is not cohomologous to a constant for all \( v \in U \). Thus, we prove the first part of the corollary.

Now we provide the second part of the corollary. Using the variational property of the Legendre transform and that \( E''_{f,\phi,\psi}(t) > 0 \), we have that for all

\[
\int_{\mathbb{R}} \psi_v d\mu_{\phi_v + t\psi_v} \in (\inf_{t \in \mathbb{R}} \int \psi_v d\mu_{\phi_v + t\psi_v}, \inf_{t \in \mathbb{R}} \int \psi_v d\mu_{\phi_v + t\psi_v})
\]

there exists a unique \( t = t(s, v) \) such that \( s = E'_{f,\phi,\psi}(t) \) and

\[
I_{f,\phi,\psi}(s) = s \cdot t(s, v) - E_{f,\phi,\psi}(t(s, v)).
\]

By taking a smaller neighborhood \( U \) of \( v_* \) if necessary, there exists an open interval \( J \) with the equation (4.3) holds for \( s \in J \) and \( v \in U \). Consider now the skew-product

\[
F : V \times J \to V \times \mathbb{R}
\]

\[
(v, t) \mapsto (v, E'_{f,\phi,\psi}(t)).
\]

It is injective because it is strictly decreasing along of fibers. As \( V \times J \) is a compact metric space, then \( F \) is a homeomorphism on the image \( F(V \times J) \). In fact it is a diffeomorphism \( C^{r-1} \). Thus, applying the function implicit theorem we have that for all \( (v, s) \in F(V \times J) \) there exists a unique \( t = t(v, s) \), depending \( C^{r-1} \) with respect to \( (v, s) \), so such \( F(v, t(v, s)) = (v, s) \) and \( s = E'_{f,\phi,\psi}(t) \). Follows then of the equation (4.3) that \( (s, v) \mapsto I_{f,\phi,\psi}(s) \) is \( C^{r-1} \). □

4.3. Stability of the statistical laws. This section is devoted to the a proof of Theorem K and Corollary B.

Before we prove the Theorem K recall part of the Livsicht’s theorem that ensures regularity in the solution of a cohomological equation when we have a hyperbolic dynamics:

**Proposition 4.12.** Let \( f \in \mathcal{D}^r \) be and \( \psi \in C^r(M, \mathbb{R}) \) be. If \( S_n \psi(p) = 0 \) for all \( p \) such that \( f^n(p) = p \) then there exists \( u \in C^r(M, \mathbb{R}) \) such that \( \psi = u \circ f - u \).

**Proof.** See for example [K05]. □

**Proof of the Theorem K.** We prove first that \( \psi \) is cohomologous to a constant in \( L^2(\mu_{f,\phi}) \) if, only if, \( \psi \) is cohomologous to a constant in \( C^r(M, \mathbb{R}) \). Thus the first part of the corollary follows of the already know Limit Central Theorem for expanding dynamics (see e.g. []). Let then \( \psi \) cohomologous to a constant in \( L^2(\mu_{f,\phi}) \). Note that

\[
E''_{f,\phi,\psi}(0) = \lim_{n \to +\infty} \frac{1}{n} \left( \int (S_n \psi)^2 d\mu_{f,\phi} - \left( \int S_n \psi d\mu_{f,\phi} \right)^2 \right)
= \lim_{n \to +\infty} \frac{1}{n} \int (\tilde{\psi})^2 d\mu_{f,\phi}
= \int \tilde{\psi}^2 d\mu_{f,\phi} + 2 \lim_{n \to +\infty} \int \sum_{j=1}^{n-1} \left( 1 - \frac{1}{n} \right) \tilde{\psi} \circ f^j \tilde{\psi} d\mu_{f,\phi}
= \int \tilde{\psi}^2 d\mu_{f,\phi} + 2 \sum_{j=1}^{\infty} \int \tilde{\psi} \circ f^j \tilde{\psi} d\mu_{f,\phi}
= \frac{\sigma_{f,\phi}(\psi)^2}{\sigma_{f,\phi}(\psi)}.
\]

By the dichotomy of the Limit Central Theorem, we have that \( E''_{f,\phi,\psi}(0) = 0 \) and thus \( E_{f,\phi,\psi} \) not is strictly convex. So \( \psi \) is cohomologous to a constant. Hence \( S_n \psi(x) = n \int \psi d\mu_{f,\phi} \) for all periodic point \( x \) of period \( n \). Thus, using the previous proposition and conclude that \( \psi = u \circ f - u + \int \psi d\mu_{f,\phi} \) for any function \( u \in C^r(M, \mathbb{R}) \). So we proved the first part of the theorem.

Now prove the second part of the theorem. The regularity of \( \mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \ni (f, \phi, \psi) \mapsto m_{f,\phi}(\psi) \) it is a direct application of the Theorem A. Finally, fixe (\( f_0, \phi_0 \)) \( \in \mathcal{D}^r \times C^r(M, \mathbb{R}) \times C^r(M, \mathbb{R}) \).
For each \((f, \phi) \in \mathcal{D}^r \times C^r(M, \mathbb{R})\) define \(E_{0,f,\phi} := \ker \nu_{f,\phi} \cap C^r(M, \mathbb{R})\), and \(T_{f,\phi}g = (g - \int gd\mu_{f,\phi}) \cdot h_{f,\phi}\) for all \(g \in C^r(M, \mathbb{R})\). So \(T_{f,\phi}\) it is a bounded operator and restrict the \(E_{0,f,\phi_0}\) is a linear isomorphism on \(E_{0,f,\phi}\), with \(T_{f,\phi}^{-1}g = \frac{g}{h_{f,\phi}} - \int \frac{\phi}{h_{f,\phi}}d\nu_{f,\phi_0}\). Given \(\psi \in C^r(M, \mathbb{R})\) we already know that \(\sigma_{f,\phi}^2(\psi) = \int \psi^2 d\mu_{f,\phi} + 2 \sum_{n=1}^{\infty} \int \psi(f \circ F^n) d\mu_{f,\phi}\), where \(\tilde{\psi} = \psi - \int \psi d\mu_{f,\phi}\). By simple computation have

\[
\int \tilde{\psi}(\psi \circ F^n) d\mu_{f,\phi} = \int \tilde{\psi}L^n_{f,\phi}(\psi \cdot h_{f,\phi}) d\nu_{f,\phi} = \int \tilde{\psi}T_{f,\phi}^{-1}L^n_{f,\phi}T_{f,\phi}T_{f,\phi}^{-1}(\psi \cdot h_{f,\phi}) d\mu_{f,\phi}.
\]

Thus

\[
\sigma_{f,\phi}^2(\psi) = - \int \tilde{\psi}^2 d\mu_{f,\phi} + 2 \int \tilde{\psi}(I - T_{f,\phi}^{-1}L^n_{f,\phi}T_{f,\phi}T_{f,\phi}^{-1}(\psi \cdot h_{f,\phi}) d\mu_{f,\phi}.
\]

Using the same ideas contained in proof of the Theorem A one can apply the Theorem B concluding that \(\mathcal{D}^r \times C^r(M, \mathbb{R}) \ni (f, \phi) \mapsto (I - T_{f,\phi}^{-1}L^n_{f,\phi}T_{f,\phi}T_{f,\phi}^{-1})\) is \(C^r\). This completes the proof of the theorem.

4.4. Multifractal analysis. This section is devoted to proof of the Corollary C.

Proof of the Corollary C. From what we have already commented in section 2.4, the corollary is a direct consequence of Theorem C since we provide \(\eta := \{v, c \in V \times \mathbb{R}^+ : c < \sup_{\eta \in \mathcal{M}_1(f_v)} |\int \psi_v d\mu_{f_v,\phi_v} - \int \psi_v d\eta|\} \) is an open set. As we already know that \(v \mapsto \mu_{f_v,\phi_v}\) is continuous, to prove that \(Y\) is open it is enough to prove that: fixed \(v_*, \in V\) and \(\eta \in \mathcal{M}_1(f_{v_*})\), given \(\epsilon > 0\) there exists a neighborhood \(V\) of \(v_*\) such that if \(v \in V\) then there exists \(\eta_v \in \mathcal{M}_1(f_v)\) with \(|\int \psi_v d\eta_v - \int \psi_v d\eta| \leq \epsilon\).

Let us prove this fact. Fix \(v_* \in V\). Given \(\epsilon > 0\), by specification property of expanding dynamics, there exists \(N(\epsilon)\) so that any finite many pieces orbit of \(f_{v_*}\) can be shadowed by a point \(x \in M\) using \(f_{v_*}\), less than \(N(\epsilon)\) iterates between the end of the a orbit and the next orbit. In general \(N(\epsilon)\) depends on the dynamics. However, as our dynamics are expanding, as long as we choose a small neighborhood \(V_1\) of \(v_*\), we can assume that \(N(\epsilon)\) is the same for all dynamic \(f_v\) with \(v \in V_1\). By Birkhoff’s ergodic theorem, there exists a \(x \in M\) and \(n > \frac{N(\epsilon)}{N(\epsilon)}\) such that \(\frac{1}{n} \sum_{i=0}^{n-1} \psi_{v_*}(f_{v_*}(x)) - \int \psi_{v_*} d\eta| < \frac{\epsilon}{2}\). As the parameterization is continuous there exists a neighborhood \(\tilde{V} \subset V_1\) of \(v_*\) so that \(d(f_{v_*}^j(y), f_v^j(y)) < \frac{\epsilon}{2\|v\|_1}\), for all \(j \in \{1, \ldots, n - 1\}, y \in M\) and \(v \in \tilde{V}\). Thus, for each \(v \in \tilde{V}\) and \(l \in \mathbb{N}\) there exists a \(y_{v,l} \in M\) such that when iterate \(y_{v,l}\) by \(f_v\) we shadow the \(l\) times the orbit piece \(x, \ldots, f_v^{n-l}(x)\), with a mistake in each jump of \(N(\epsilon)\). Taking a point of accumulation of probabilities \(\left(\frac{1}{(\eta + N(\epsilon))^{-1}} \sum_{l=0}^{(n+N(\epsilon))^{-1}} \delta_{f_{v_*}^l(y_{v,l})}\right)_{l \in \mathbb{N}}\) we construct a probability \(f\)-invariant \(\eta_v\) with \(|\int \psi_v d\eta_v - \int \psi_v d\eta| \leq \epsilon\) and therefore we conclude that \(Y\) is open. This completes the proof of the corollary.
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Thiago Bomfim. Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.

E-mail address: tbnunes@ufba.br

URL: https://sites.google.com/site/homepageofthiagobomfim/

Armando Castro. Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.

E-mail address: armando@impa.br