On the possibility of traversable wormhole formation in the Galactic halo in the presence of scalar field

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Abstract In the paper we obtain traversable wormhole (TW) solutions in the Einstein gravity with a functional form of the dark matter in the galactic halo. The dark matter model is pseudo isothermal which is derived from modified gravity. For a given central density the possibility of TW is explored determining the shape functions. The null energy condition (NEC) and the weak energy conditions in the model is probed. We also study the existence of TWs considering homogeneous scalar field in addition to the dark matter halo in the galaxies. An interesting observation is that TW solutions exist even if NEC is not violated in the presence of scalar field.

1 Introduction

General Theory of Relativity (GTR) permits wormhole solutions of the Einstein field equations for a special composition of matter different from perfect fluid. Wormholes are tunnel like objects which may connect different spacetime regions within the universe, or different universes [1,2]. In recent times the study of traversable wormholes (TW) has attracted a lot of interest with the pioneering work of Ellis [3,4], Bronnikov [5] including the seminal work of Morris and Thorne [6]. The geometry of TW requires exotic matter concentrated at the wormhole throat implying violation of the energy conditions, namely null energy condition (NEC) [7]. It is speculated that the exotic matter can exist in the context of quantum field theory. The concept of dark matter, a hypothetical form of matter that makes up about 25 % of the matter composition of the Universe is predicted from observations. A number of possible candidates from particle physics and supersymmetric string theory such as axions and weakly interacting massive particles (WIMP) are considered to describe as the candidate of the dark matter though
there is no direct experimental detection of dark matter. Nevertheless, its existence are hinted in the galactic rotation curves \[8\], the galaxy clusters dynamics \[9\], and also at cosmological scales of anisotropies encoded in the cosmic microwave background measured by PLANCK \[10\].

In the literature \([11,12,13]\) considering characteristics of dark matter halos and galaxies formation of traversable wormholes are considered based on the Navarro-Frenk-White (NFW) profile \([14]\) of matter distribution. TWs may form in the outer halo of galaxies which are interesting to describe time machines. Rahaman et al. \([15]\) first proposed the possible existence of wormholes in the outer regions of the galactic halo based on the NFW density profile, then they used the Universal Rotation Curve (URC) dark matter model to obtain analogous results for the central parts of the halo \([16]\). DM has been adapted as a non-relativistic phenomenon such as NFW profile and King profile to study WH contraction \([20]\). Recently, Jusufi \textit{et al.} \([21]\) pointed out the possibility of traversable wormhole formation with a Bose-Einstein condensation dark matter halo, however, in this case approximate solutions are found. The existence of TW in the spherical stellar systems based on the fact that the dark matter halos produced in computer simulations are best described by such a profile is also investigated. Assuming a number of parameters that exists in the Einasto profile \([22,23]\) it is shown that TWs in the outer regions of spiral galaxies are possible while the inner part regions prohibits such formations. The Bose-Einstein Condensation dark matter (BEC-DM) model shows that the model is more realistic on the small scales of galaxies compared to the CDM model. For instance, the interactions between dark matter particles are very strong in the inner regions of galaxies, and thus the dark matter will no longer be cold. For the BEC-DM model, the dark matter density profile can be described by the Thomas-Fermi (TF) profile \([24]\). The possibility of formation of TWs in the dark matter halo is studied with isotropic pressure with TF profile \([25]\). The BEC-DM model predicts much less dark matter density in the central regions of galaxies compared to that found with the NFW profile. There exists another class of DM described by pseudo isothermal (PI) profile in addition to the CDM model and the BEC-DM model which is associated with the modified gravity, such as Modified Newtonian Dynamics (MOND) \([26]\). In the MOND model, the dark matter density profile is given by

\[
\rho_{DM} = \frac{\rho_0}{1 + \left(\frac{r}{R_c}\right)^2}
\]

where \(\rho_0\) is the central dark matter density and \(R_c\) is the scale radius. Using the above formalism, we adapted the DM density distribution and used to study wormhole construction for MOND. Subsequently we introduce homogeneous scalar field in addition to MOND dark matter to investigate the TW models and the matter required for their existence.

The present paper is organized as follows. In Section II, the Einstein gravity and field equations are obtained. The traversable wormhole solutions are obtained with MOND dark energy profile. In Section III, the existence of wormhole with MOND and scalar field is discussed and finally a brief discussion in Section IV.
2 Einstein Field Equation and Traversable Wormhole Solution

The Einstein field equation is given by

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \]  

(2)

where \( \mu, \nu = 0, 1, 2, 3 \) and \( T_{\mu\nu} = \text{diag}(-\rho, P_r, P_t, P_t) \) is the energy-momentum tensor, in gravitational unit \( c = 8\pi G = 1 \). The spacetime metric of a spherically symmetric traversable wormhole is given by

\[ ds^2 = -e^{2\Lambda(r)} dt^2 + \frac{dr^2}{1 - \alpha(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(3)

where \( \Lambda \) is the redshift function. For wormhole we consider \( \alpha(r) = b(r) \), where \( b(r) \) is the shape function. In order to ensure a wormhole to be traversable, there should be no event horizon. In this case \( \Lambda(r) \) should be finite which tends to zero when \( r \rightarrow \infty \). The geometry of the wormhole is determined by the shape function \( b(r) \), which at the throat of the wormhole \( r = r_0 \) should satisfy the condition \( b(r_0) = 0 \). To keep the wormhole’s throat open, the shape function \( b(r) \) should satisfy the flare out condition \( \frac{b(r)}{b(r_0)} > 0 \) and \( b'(r) < 1 \).

Using the Einstein field eqs. (2) and the metric given by eq. (3), we get

\[ \rho(r) = \frac{b'(r)}{8\pi r^2} \]  

(4)

\[ 8\pi P_r(r) = -\frac{b(r)}{r^3} + 2 \left( 1 - \frac{b(r)}{r} \right) \frac{\Lambda'(r)}{r} \]  

(5)

\[ 8\pi P_t(r) = \left[ 1 - \frac{b}{r} \right] \left[ \frac{\Lambda''(r) + \Lambda'^2 - U\Lambda' - U'}{r^2} \right] \]  

(6)

setting \( U = \frac{u''(r)}{u'(r) - u(r)} \) where \( \rho(r) \) is the energy density for dark matter, \( P_r \) and \( P_t \) are radial and transverse pressures. For a flat rotation curve for the circular stable geodesic motion in the equatorial plane of a galaxy one finds

\[ e^{2\Lambda(r)} = E_0 r^l. \]  

(7)

In the above \( l = 2(v^\phi)^2 \) where \( v^\phi \) represents the rotational velocity and \( E_0 \) is an integration constant for a large \( r \) [16][17][18]. For a typical galaxy it is found that \( v^\phi \sim 10^{-3} \) (300 km/s) within 300 kpc [19]. We assume that the density of the wormhole matter is described by the density profile of the MOND. We consider critical value \( E_0 = \frac{1}{R_c} \) where \( R_c \) is the scale radius, the spacetime metric can be written as

\[ ds^2 = -\left( \frac{r}{R_c} \right)^l dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(8)

Solving Einstein field eq. (1) using the DM density given by eq. (1), we obtain

\[ b(r) = \rho_0 R_c^3 \left[ \frac{r}{R_c} - \tan^{-1} \frac{r}{R_c} \right] + C \]  

(9)

where \( C \) is an integration constant.
Using the boundary condition $b(r_0) = r_0$, the integration constant $C$ can be fixed for the shape function which is given by

$$b(r) = \rho_0 R_c^3 \left[ \frac{r}{R_c} - \frac{r_0}{R_c} - \left( \tan^{-1} \frac{r}{R_c} - \tan^{-1} \frac{r_0}{R_c} \right) \right] + r_0, \quad (10)$$

where $r_0$ represents the throat of the wormhole. The flaring out condition is required to be valid to keep the mouth of the wormhole open which is satisfied here as

$$b'(r) = \rho_0 R_c^2 \frac{r^2}{R_c^2} \frac{1 + \frac{r^2}{R_c^2}}{\left( 1 + \frac{r^2}{R_c^2} \right)^2} < 1 \quad (11)$$
for $\rho_0 R_c^2 \sim 1$. In Fig. (1) we plot radial variation of $b(r)$ with different $\rho_0 R_c^3$. It is found that as the central density is increased $b(r)$ is found to increases for a given radial distance. The radial variation of the flaring out condition is checked in Fig. (2) it is found that $\frac{4\pi}{R_c} < 1$ as long as $\rho_0 R_c^2 \leq 1$.

Next let us discuss the issue of energy conditions and make some regional plots to check the validity of the energy conditions. Recall that the WEC \[7\] is defined by

$$T_{\mu\nu} U^\mu U^\nu \geq 0 \text{ i.e., } \rho \geq 0, \quad \rho(r) + P_r \geq 0$$

(12)

where $U^\mu$ denotes the time like vector. This means that local energy density is positive and it gives rise to the continuity of NEC which is defined by $T_{\mu\nu} k^\mu k^\nu \geq 0$ i.e.,

$$\rho(r) + P_r \geq 0$$

(13)

where $k^\mu$ represents a null vector. The NEC is checked in Fig. (3), it is found that near the throat NEC is not obeyed but away from the throat it always satisfied which is shown for different power law index ($l$) in eq. (7). We note that for $l = 2$, NEC is always obeyed for the central density $\rho_0 < \frac{1}{R_c}$. It is evident from Fig. (4) that traversable wormholes exists with exotic matter near the throat but away from the throat exotic matter is not required in a low central density galaxy but for higher central density galaxies initially there is exotic matter then again exotic matter followed by normal matter at $r \to \infty$.

The mass profile of the galactic halo is

$$M_{DM}(r) = 4\pi \int_0^r \rho_{DM}(r') r'^2 dr'$$

(14)
which yields
\[ M_{DM}(r) = 4\pi \rho_0 R_c^2 \left[ \frac{r}{R_c} - \tan^{-1} \left( \frac{r}{R_c} \right) \right], \quad (15) \]

In Fig. (5), it is observed that dark matter increases with the increase in radial size in unit of \( R_c \) for increasing values of \( 4\pi \rho_0 R_c^2 = m_0 \). It is evident that as the mass parameter (\( m_0 \)) is increased From the eq. (15), one can find the tangential velocity \[ v^2_{tg}(r) = \frac{GM_{DM}(r)}{r} \]
which represents a constant rotational curve away from the centre of the galaxy.

The rotational velocity of a test particle within the equatorial plane is determined by \[ v^2_{tg}(r) = \frac{r}{\Lambda} \quad (17) \]
Combining eq. (17) with the expression for the test particle moving in the dark halo given by eq. (16) we get \[ r\Lambda'(r) = 4\pi \rho_0 R_c^2 \left[ 1 - \frac{R_c}{r} \tan^{-1} \left( \frac{r}{R_c} \right) \right] \quad (18) \]
On integrating
\[ A(r) = 4\pi \rho_0 R_c^2 \left[ \ln \frac{r}{R_c} \right] - R_c \tan^{-1} \left( \frac{r}{R_c} \right) + C_1 \quad (19) \]
where \( C_1 \) is an integration constant. Thus the \( g_{00} \) component of the metric tensor become
\[ e^{2\Lambda(r)} = D e^{8\pi \rho_0 R_c^2 \left[ \ln \frac{r}{\sqrt{1 + \left( \frac{r}{R_c} \right)^2}} - R_c \tan^{-1} \left( \frac{r}{R_c} \right) \right] + 2C_1}, \quad (20) \]
the new integration constant \( D \) can be absorbed by rescaling \( t \to Dt \). For finite redshift we get a constraint equation which determines \( C_1 \) as follows
\[ \lim_{r \to R_c} e^{2\Lambda(r)} = 1. \]
The limiting value
\[ \lim_{r \to R_c} e^{8\pi \rho_0 R_c^2 \left[ \ln \frac{r}{\sqrt{1 + \left( \frac{r}{R_c} \right)^2}} - R_c \tan^{-1} \left( \frac{r}{R_c} \right) \right] + 2C_1} = 1 \]
determines the constant as \[ C_1 = 2\pi R_c^2 \left( \frac{\pi}{2} + \ln 2 \right). \quad (21) \]
The wormhole metric is not asymptotically flat, therefore, we employ matching conditions by truncating the wormhole metric at radius \( R \) and connecting with the exterior Schwarzschild black hole metric as the later corresponds to vacuum
solution and asymptotically flat. Now imposing the matching condition at \( r = R \), we get
\[
8\pi \rho_0 R_0^2 \left[ \ln \left( \frac{R}{R_c} \right) \frac{R_c}{R} \left( \frac{b(R)}{R} \right) \right] + 4\pi R_0^2 \left( \frac{\rho_0}{R_c} \right) = 1 - \frac{2M}{R} \quad (22)
\]
\[
1 - \frac{b(R)}{R} = 1 - \frac{2M}{R} \quad (23)
\]
where the active mass of the galaxy (\( M \)) is determined as
\[
M = \frac{\rho_0 R_0^2}{2} \left[ \frac{R}{R_c} - \frac{r_0}{R_c} - \tan^{-1} \frac{R}{R_c} \right] + \frac{r_0}{2} \quad (24)
\]
in the gravitational unit \( \rho_0 R_0^2 \sim 1 \). The eqs. (22) and (24) implicitly provide the value of truncated radius (\( R \)) where the matching occurs. Note that \( \rho_0 \) represents galactic dark matter in the unit of \( M_{DM} \) halo/kpc\(^3\), \( r \) has units of \( M_{DM} \) halo and \( R \) and \( r_0 \) are in the units of kpc. From eq. (22), \( R_c \) is determined for density at the throat of the wormhole knowing the central dark matter density \( \rho_0 \).

3 Existence of Wormhole with MOND and scalar field

In this section we consider wormhole solutions in the presence of scalar field and MOND dark matter density profile. Therefore the energy momentum tensor consists of two terms \( T_{\mu\nu} = T_{\mu\nu}^{DM} + T_{\mu\nu}^f \). To determine the second part of stress energy tensor we consider a minimally coupled massless scalar field described by the Lagrangian:
\[
L = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (25)
\]
The equation of motion for \( \phi \) is
\[
\Box \phi = 0. \quad (26)
\]
The stress energy tensor for the scalar field is obtained from eq. (25) which is
\[
T_{\mu\nu}^f = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\sigma\eta} \phi_{,\sigma} \phi_{,\eta}. \quad (27)
\]
The conservation eq. (26) of \( \phi \) is given by
\[
\frac{\phi_{,\nu}}{\phi_{,\nu}} + \frac{1}{2} \left( \frac{1 - \frac{b(r)}{r}}{1 - \frac{b(r)}{r}} \right) + \frac{2}{r} = 0. \quad (28)
\]
On integrating we get
\[
\phi_{,\nu}^2 r^4 \left( 1 - \frac{b(r)}{r} \right) = \frac{\phi_0}{4\pi} \quad (29)
\]
where \( \phi_0 > 0 \). Now the Einstein field equations are given by
\[
\frac{b'(r)}{8\pi r^2} = \rho(r) + \frac{1}{2} \phi_{,\nu}^2 \left( 1 - \frac{b(r)}{r} \right) \quad (30)
\]
\[
- \frac{b'(r)}{8\pi r^3} + \left( 1 - \frac{b(r)}{r} \right) A'(r) = P_r(r) - \frac{1}{2} \phi_{,\nu}^2 \left( 1 - \frac{b(r)}{r} \right) \quad (31)
\]
For $\Lambda \to 0$, no horizon occurs and the radial variation of NEC is drawn in Fig. (6) for a range of values of $\phi_0$. It is found that near the throat NEC is violated but away from the throat it is obeyed which depends on $\phi_0$. The NEC is checked for a given $\phi_0 R_c^3$ and throat radius numerically which are tabulated in the Tables I-II. It is evident that as $\phi_0 R_c^3$ is increased NEC is satisfied at lower $\phi_0$ of the scalar field. In Table-III, we display the values of $\phi_0$ for a given throat radius $r_0$ with density parameter of the galaxy.

**Table 1** NEC for $\rho_0 R_c^3 = 10$ and throat radius $r_0 = 0.5$, "X" means violation and "√" means satisfied.

| $\phi_0$ | $r$  | NEC ($r \geq r_0$) |
|---------|------|---------------------|
| 0.05   | 0.310 | √                  |
| 0.08   | 0.480 | √                  |
| 0.10   | 0.422 | √                  |
| 0.20   | 0.479 | √                  |
| 0.25   | 0.50  | X                  |
| 0.26   | 0.504 | X                  |
| 0.30   | 0.518 |                     |
| 0.38   | 0.548 |                     |

**Fig. 6** Variation of NEC with $r$ in unit of $R_c$ with $4\pi R_c^3 = 1$ and different $\phi_0$.
\[ \phi_0 \leq r \geq r_0 \]

Table 2 NEC for \( \rho_0 R_c^3 = 8 \) and throat radius \( r_0 = 0.5 \), "X" means violation and "\( \sqrt{\} \)" means satisfied

| \( \phi_0 \) | \( r \) | NEC (\( r \geq r_0 \)) |
|-----------|-----|------------------|
| 0.10      | 0.339 | \( \sqrt{\} \) |
| 0.20      | 0.440 | \( \sqrt{\} \) |
| 0.30      | 0.444 | \( \sqrt{\} \) |
| 0.50      | 0.5   | \( \sqrt{\} \) |
| 0.60      | 0.520 | X                |

Table 3 Tabulation of \( \phi_0 \) with \( \rho_0 R_c^3 \) and radial distance \( r \) from which NEC is satisfied.

| \( \rho_0 R_c^3 \) | \( r \geq \) | \( \phi_0 \) |
|-------------------|--------|--------|
| 10                | 0.5    | 0.29   |
| 10                | 0.4    | 0.066  |
| 8                 | 0.5    | 0.5    |
| 6                 | 0.5    | 0.051  |
| 5                 | 0.5    | \( \sim 0 \) |

one needs a scalar field with high \( \phi_0 \) scalar field value for accommodating TW with normal matter for a given density of the galactic halo. It is also noted that for same throat radius if the density is high then one requires large value of the scalar field.

4 Discussion

In the present paper we study the possibility of traversable wormhole in the Einstein gravity with MOND dark matter profile in the presence and absence of scalar field. It is found that in the absence of scalar field one requires exotic matter at the throat of the wormhole. It is also noted that there exists two scenario (i) exotic matter at the throat and then normal matter away from the throat as NEC is obeyed and (ii) exotic matter at the throat, subsequently away from the throat NEC violates followed by the region where NEC is obeyed depending on the value of the central density. It is also found that if the central density is large the scale radius will be low for a given galactic halo mass. In the presence of homogeneous scalar field in addition to dark matter represented by MOND, we obtain a class of TW with exotic or without exotic matter depending on the values of the scalar field \( \phi_0 \) and density of the galactic halo. The later result represents a new class of TW not shown earlier. The TW in higher dimensional gravity [27] is important to probe in this context which will be taken up elsewhere.

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