CELLULARITY OF CYCLOTOMIC
BIRMAN–WENZL–MURAKAMI ALGEBRAS

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Dedicated to Gus Lehrer on the occasion of his 60th birthday.

Abstract. We show that cyclotomic BMW algebras are cellular algebras.

1. Introduction

In this paper, we prove that the cyclotomic Birman–Wenzl–Murakami algebras
are cellular, in the sense of Graham and Lehrer [8].

The origin of the BMW algebras was in knot theory. Shortly after the invention
of the Jones link invariant [10], Kauffman introduced a new invariant of regular
isotopy for links in $S^3$, determined by certain skein relations [11]. Birman and
Wenzl [2] and independently Murakami [16] then defined a family braid group
algebra quotients from which Kauffman’s invariant could be recovered. These
(BMW) algebras were defined by generators and relations, but were implicitly
modeled on certain algebras of tangles, whose definition was subsequently made
explicit by Morton and Traczyk [14], as follows: Let $S$ be a commutative unital
ring with invertible elements $\rho$, $q$, and $\delta_0$ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$. The
Kauffman tangle algebra $KT_{n,S}$ is the $S$–algebra of framed $(n,n)$–tangles in
the disc cross the interval, modulo Kauffman skein relations:

1. Crossing relation: $\begin{array}{c} \backslash \end{array} \backslash - \backslash \backslash = (q^{-1} - q) \left( \begin{array}{c} \backslash \end{array} \backslash - \backslash \backslash \right)\,.$

2. Untwisting relation: $\begin{array}{c} \circ \end{array} = \rho$ and $\begin{array}{c} \circ \end{array} = \rho^{-1}$

3. Free loop relation: $T \cup \begin{array}{c} \circ \end{array} = \delta_0 T$.

Morton and Traczyk [14] showed that the $n$–strand algebra $KT_{n,S}$ is free of rank
$(2n - 1)!!$ as a module over $S$, and Morton and Wassermann [15] proved that the
BMW algebras and the Kauffman tangle algebras are isomorphic.

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It is natural to “affinize” the BMW algebras to obtain BMW analogues of the affine Hecke algebras of type \( A \), see [1]. The affine Hecke algebra can be realized geometrically as the algebra of braids in the annulus cross the interval, modulo Hecke skein relations; this suggests defining the affine Kauffman tangle algebra as the algebra of framed \((n, n)\)–tangles in the annulus cross the interval, modulo Kauffman skein relations. However, Turaev [17] showed that the resulting algebra of \((0, 0)\)–tangles is a (commutative) polynomial algebra in infinitely many variables, so it makes sense to absorb this polynomial algebra into the ground ring. (The ground ring gains infinitely many parameters corresponding to the generators of the polynomial algebra.) One can also define a purely algebraic version of these algebras, by generators and relations [9], the affine BMW algebras. In [5], we showed that the two versions are isomorphic.

The affine BMW algebras have a distinguished generator \( y_1 \), which, in the geometric (Kauffman tangle) picture is represented by a braid with one strand wrapping around the hole in the annulus cross interval. Cyclotomic BMW algebras are quotients of the affine BMW algebras in which the generator \( y_1 \) satisfies a monic polynomial equation. The affine and cyclotomic BMW algebras arise naturally in connection with knot theory in the solid torus, braid representations generated by \( R \)–matrices of symplectic and orthogonal quantum groups, and the representation theory of the ordinary BMW algebras (where the affine generators become Jucys–Murphy elements). We refer the reader to [6] for further discussion and references.

In order to get a good theory for cyclotomic BMW algebras, it is necessary to impose conditions on the ground ring. An appropriate condition, known as admissibility, was introduced by Wilcox and Yu in [18]. Their condition has a simple formulation in terms of the representation theory of the 2–strand cyclotomic BMW algebra, and also translates into explicit relations on the parameters.

Let \( W_{n,S,r} \) denote the cyclotomic quotient of the \( n \)–strand affine BMW algebra, in which the affine generator \( y_1 \) satisfies a polynomial relation of degree \( r \), defined over a ring \( S \) with appropriate parameters. It has been shown in [6, 7, 19, 21] that if \( S \) is an admissible integral domain, then \( W_{n,S,r} \) is a free \( S \)–module of rank \( r^n(2n - 1)!! \), and is isomorphic to a cyclotomic version of the Kauffman tangle algebra. In this paper, we show that the techniques of [6] can be modified to yield a cellular basis of the cyclotomic BMW algebras.
The cellularity of the ordinary BMW algebras has been shown by Xi [20] and Enyang [3, 4]. It is worth pointing out that if we specialize our proof for the cyclotomic case to the ordinary BMW algebras, we end up showing that the tangle basis of [14, 15] is cellular; in fact, the proof would require only minor modifications of arguments already present in Morton–Wassermann [15].

Yu [21] has also shown that cyclotomic BMW algebras over admissible ground rings are cellular; her result is slightly more general, since she used a broader definition of admissibility. See also Remark 2.10.

2. Preliminaries

2.1. Definitions. In the following, let $S$ be a commutative unital ring containing elements $\rho, q, \text{ and } \delta_j$, $j \geq 0$, with $\rho, q, \text{ and } \delta_0$ invertible, satisfying the relation $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

**Definition 2.1.** The affine Kauffman tangle algebra $\hat{KT}_{n,S,r}$ is the $S$–algebra of framed $(n,n)$–tangles in the annulus cross the interval, modulo Kauffman skein relations, namely the crossing relation and untwisting relation, as given in the introduction, and the free loop relations: for $j \geq 0$, $T \cup \Theta_j = \rho^{-j}\delta_j T$, where $T \cup \Theta_j$ is the union of an affine tangle $T$ and a disjoint copy of the closed curve $\Theta_j$ that wraps $j$ times around the hole in the annulus cross the interval.

**Figure 2.1.** Affine $(4, 4)$–tangle diagram

Affine tangles can be represented by affine tangle diagrams. These are pieces of link diagrams in the rectangle $\mathcal{R}$, with some number of endpoints of curves on the top and bottom boundaries of $\mathcal{R}$, and a distinguished vertical segment representing the hole in the annulus cross interval. (We call this curve the flagpole.) Affine tangle diagrams are regarded as equivalent if they are regularly isotopic; see [6] for details. An affine $(n,n)$–tangle diagram is one with $n$ vertices (endpoints of curves) on the top, and $n$ vertices on the bottom edge of $\mathcal{R}$. See Figure 2.1. We label the vertices on the top edge from left to right as $1, \ldots, n$ and
those on the bottom edge from left to right as \( \bar{1}, \ldots, \bar{n} \). We order the vertices by \( 1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1} \).

**Definition 2.2.** The affine Birman–Wenzl–Murakami algebra \( \hat{W}_{n, S} \) is the algebra with generators \( y_i^{\pm 1}, g_i^{\pm 1} \) and \( e_i \) (\( 1 \leq i \leq n - 1 \)) and relations:

1. (Inverses) \( g_i g_i^{-1} = g_i^{-1} g_i = 1 \) and \( y_1 y_1^{-1} = y_1^{-1} y_1 = 1 \).
2. (Idempotent relation) \( e_i^2 = \delta_0 e_i \).
3. (Type B braid relations)
   (a) \( g_i g_{i+1} g_i = g_{i+1} g_i g_i+1 \) and \( g_i g_j = g_j g_i \) if \( |i - j| \geq 2 \).
   (b) \( y_1 y_1 y_1 g_1 = g_1 y_1 y_1 y_1 \) and \( y_1 g_j = g_j y_1 \) if \( j \geq 2 \).
4. (Commutation relations)
   (a) \( g_i e_j = e_j g_i \) and \( e_i e_j = e_j e_i \) if \( |i - j| \geq 2 \).
   (b) \( y_1 e_j = e_j y_1 \) if \( j \geq 2 \).
5. (Affine tangle relations)
   (a) \( e_i e_{i \pm 1} e_i = e_i \),
   (b) \( g_i g_{i \pm 1} e_i = e_{i \pm 1} e_i \) and \( e_i g_{i \pm 1} g_i = e_i e_{i \pm 1} \).
   (c) For \( j \geq 1 \), \( e_1 y_1 e_j = \delta_j e_1 \).
6. (Kauffman skein relation) \( g_i - g_i^{-1} = (q^{-1} - q)(e_i - 1) \).
7. (Untwisting relations) \( g_i e_i = e_i g_i = \rho^{-1} e_i \) and \( e_i g_{i \pm 1} e_i = \rho e_i \).
8. (Unwrapping relation) \( e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1 \).

Let \( X_1, G_i, E_i \) denote the following affine tangle diagrams:

\[
X_1 = \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\end{array} \quad
G_i = \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array} \quad
E_i = \begin{array}{|c|c|c|}
\hline
\hline
\hline
\hline
\hline
\end{array}
\]

**Theorem 2.3** ([5]). The affine BMW algebra \( \hat{W}_{n, S} \) is isomorphic to the affine Kauffman tangle algebra \( \hat{KT}_{n, S} \) by a map \( \varphi \) determined by \( \varphi(g_i) = G_i \), \( \varphi(e_i) = E_i \), and \( \varphi(y_1) = \rho X_1 \).

We now suppose \( S \) (as above) has additional distinguished invertible elements \( u_1, \ldots, u_r \).

**Definition 2.4.** The cyclotomic BMW algebra \( W_{n, S, r}(u_1, \ldots, u_r) \) is the quotient of \( \hat{W}_{n, S} \) by the relation

\[
(2.1) \quad (y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0.
\]
To define the cyclotomic Kauffman tangle algebra, we begin by rewriting the relation Equation (2.1) in the form $\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) y_1^k = 0$, where $\varepsilon_j$ is the $j$–th elementary symmetric function. The corresponding relation in the affine Kauffman tangle algebra is $\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) \rho^k x_1^k = 0$.

Now we want to impose this as a local skein relation.

Definition 2.5. The cyclotomic Kauffman tangle algebra $\text{KT}_{n,S,r}(u_1, \ldots, u_r)$ is the quotient of the affine Kauffman tangle algebra $\widehat{\text{KT}}_{n,S}$ by the cyclotomic skein relation:

\[
\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) \rho^k x_1^k = 0,
\]

The sum is over affine tangle diagrams which differ only in the interior of the indicated disc and are identical outside of the disc; the interior of the disc contains an interval on the flagpole and a piece of an affine tangle diagram isotopic to $X_1^k$.

Definition 2.6. Say that $S$ is weakly admissible if $e_1$ is not a torsion element in $W_{2,S,r}$. Say that $S$ is admissible if \{ $e_1, y_1 e_1, \ldots, y_1^{r-1} e_1$ \} is linearly independent over $S$ in $W_{2,S,r}$.

These conditions can be translated into explicit conditions on the parameters of $S$; see [18, 6, 7].

Theorem 2.7 ([6, 7, 21, 19]). If $S$ is an admissible integral domain, then the assignment $e_i \mapsto E_i$, $g_i \mapsto G_i$, $y_1 \mapsto \rho X_1$ determines an isomorphism of $W_{n,S,r}$ and $\text{KT}_{n,S,r}$. Moreover these algebras are free $S$–modules of rank $r^n(2n-1)!!$.

Because of Theorems 2.3 and 2.7, we will no longer take care to distinguish between affine or cyclotomic BMW algebras and their realizations as algebras of tangles. We identify $e_i$ and $g_i$ with the corresponding affine tangle diagrams and $x_1 = \rho^{-1} y_1$ with the affine tangle diagram $X_1$. The ordinary BMW algebra $W_{n,S}$ imbeds in the affine BMW algebra $W_{n,S}$ as the subalgebra generated by the $e_i$’s and $g_i$’s.

2.2. The rank of tangle diagrams. An ordinary or affine tangle diagram $T$ with $n$ strands is said to have rank $\leq r$ if it can be written as a product $T = T_1 T_2$, where $T_1$ is an (ordinary or affine) $(r, n)$ tangle and $T_2$ is an (ordinary or affine) $(n, r)$ tangle.
2.3. The algebra involution * on BMW algebras. Each of the ordinary, affine, and cyclotomic BMW algebras admits a unique involutive algebra anti–automorphism, denoted \( a \mapsto a^* \), fixing each of the generators \( g_i, e_i \) (and \( x_1 \) in the affine or cyclotomic case). For an (ordinary or affine) tangle diagram \( T \) representing an element of one of these algebras, \( T^* \) is the diagram obtained by flipping \( T \) around a horizontal axis.

2.4. The Hecke algebra and the BMW algebra. The Hecke algebra \( H_{n,S}(q^2) \) of type \( A \) is the quotient of the group algebra \( S \mathcal{B}_n \) of the braid group, by the relations
\[
\sigma_i - \sigma_i^{-1} = (q - q^{-1}) \quad (1 \leq i \leq n - 1),
\]
where \( \sigma_i \) are the Artin braid generators. Let \( \tau_i \) denote the image of the braid generator \( \sigma_i \) in the Hecke algebra.

Given a permutation \( \pi \in \mathfrak{S}_n \), let \( \beta_\pi \) be the positive permutation braid in the braid group \( \mathcal{B}_n \) whose image in \( \mathfrak{S}_n \) is \( \pi \). A positive permutation braid is a braid in which two strands cross an most once, and all crossings are positive, that is the braid is in the monoid generated by the Artin generators \( \sigma_i \) of the braid group.

Let \( g_\pi \) be the image of \( \beta_\pi \) in \( W_{n,S} \), and \( \tau_\pi \) the image of \( \beta_\pi \) in \( H_{n,S}(q^2) \). If \( \pi \) has a reduced expression \( \pi = s_{i_1}s_{i_2} \cdots s_{i_\ell} \), then \( g_\pi = g_{i_1}g_{i_2} \cdots g_{i_\ell} \), and \( \tau_\pi = \tau_{i_1}\tau_{i_2} \cdots \tau_{i_\ell} \).

It is well known that \( \{ \tau_\pi : \pi \in \mathfrak{S}_n \} \) is a basis of the Hecke algebra \( H_{n,S}(q^2) \). The Hecke algebra has an involutive algebra anti-automorphism \( x \mapsto x^* \) determined by \( (\tau_\pi)^* = \tau_{\pi^{-1}} \).

2.5. Affine and cyclotomic Hecke algebras.

**Definition 2.8.** (See [1].) Let \( S \) be a commutative unital ring with an invertible element \( q \). The affine Hecke algebra \( \hat{H}_{n,S}(q^2) \) over \( S \) is the \( S \)-algebra with generators \( t_1, \tau_1, \ldots, \tau_{n-1} \), with relations:

1. The generators \( \tau_i \) are invertible, satisfy the braid relations, and \( \tau_i - \tau_i^{-1} = (q - q^{-1}) \).
2. The generator \( t_1 \) is invertible, \( t_1 \tau_1 t_1 = \tau_1 t_1 \tau_1 \) and \( t_1 \) commutes with \( \tau_j \) for \( j \geq 2 \).

Let \( u_1, \ldots, u_r \) be additional invertible elements in \( S \). The cyclotomic Hecke algebra \( H_{n,S,S}(q^2; u_1, \ldots, u_r) \) is the quotient of the affine Hecke algebra \( \hat{H}_{n,S}(q^2) \) by the polynomial relation \( (t_1 - u_1) \cdots (t_1 - u_r) = 0 \).

Define elements \( t_j \) (1 \( \leq j \leq n \)) in the affine or cyclotomic Hecke algebra by
\[
t_j = \tau_{j-1} \cdots \tau_1 t_1 \tau_1 \cdots \tau_{j-1}.
\]
It is well known that the ordinary Hecke algebra $H_n^S(q^2)$ imbeds in the affine Hecke algebra and that the affine Hecke algebra $\hat{H}_n^S(q^2)$ is a free $S$–module with basis the set of elements $\tau_\pi t^b$, where $\pi \in S_n$ and $t^b$ denotes a Laurent monomial in $t_1, \ldots, t_n$. Similarly, a cyclotomic Hecke algebra $H_{n,S,r}(q; u_1, \ldots, u_r)$ is a free $S$–module with basis the set of elements $\tau_\pi t^b$, where now $t^b$ is a monomial with restricted exponents $0 \leq b_i \leq r - 1$.

Let $S$ be a commutative ring with appropriate parameters $\rho, q, \delta_j$. There is an algebra homomorphism $p: \hat{W}_n^S \to \hat{H}_n^S(q^2)$ determined by $g_i \mapsto \tau_i$, $e_i \mapsto 0$, and $x_1 \mapsto t_1$. The kernel of $p$ is the ideal $I_n$ spanned by affine tangle diagrams with rank strictly less than $n$. Suppose that $S$ has additional parameters $u_1, \ldots, u_r$. Then $p$ induces a homomorphism of the cyclotomic quotients $\hat{p}: \hat{W}_{n,S,r}(u_1, \ldots, u_r) \to H_{n,S,r}(q^2; u_1, \ldots, u_r)$.

The affine and cyclotomic Hecke algebras have unique involutive algebra anti-automorphisms $^*$ fixing the generators $\tau_i$ and $t_1$. (The image of a word in the generators is the reversed word.) The quotient map $p$ respects the involutions, $p(x^*) = p(x)^*$.

We have a linear section $t: \hat{H}_n^S(q^2) \to \hat{W}_n^S$ of the map $p$ determined by $t(\tau_\pi t^b) = g_\pi x^b$. Moreover, $t(x^*) \equiv t(x)^* \mod I_n$ and $t(x)t(y) \equiv t(xy) \mod I_n$ for any $x, y \in \hat{H}_n^S(q^2)$. Analogous statements hold for the cyclotomic algebras.

2.6. **Cellular bases.** We recall the definition of *cellularity* from [8]; see also [13]. The version of the definition given here is slightly weaker than the original definition in [8]; we justify this below.

**Definition 2.9.** Let $R$ be an integral domain and $A$ a unital $R$–algebra. A cell datum for $A$ consists of an $R$–linear algebra involution $^*$ of $A$; a partially ordered set $(\Lambda, \geq)$ and for each $\lambda \in \Lambda$ a set $T(\lambda)$; and a subset $C = \{c_{\lambda}^{\lambda} : \lambda \in \Lambda \text{ and } s, t \in T(\lambda)\} \subseteq A$; with the following properties:

1. $C$ is an $R$–basis of $A$.
2. For each $\lambda \in \Lambda$, let $\tilde{A}^\lambda$ be the span of the $c_{\mu}^{\lambda}$ with $\mu > \lambda$. Given $\lambda \in \Lambda$, $s \in T(\lambda)$, and $a \in A$, there exist coefficients $r_s^\lambda(a) \in R$ such that for all $t \in T(\lambda)$:
   \[
   ac_s^\lambda t \equiv \sum_v r_s^v(a)c_v^t \mod \tilde{A}^\lambda.
   \]
3. $(c_{\lambda}^{\lambda})^* \equiv c_{\lambda}^{\lambda} \mod \tilde{A}^\lambda$ for all $\lambda \in \Lambda$ and, $s, t \in T(\lambda)$.

$A$ is said to be a cellular algebra if it has a cell datum.
For brevity, we say write that $C$ is a cellular basis of $A$.

**Remark 2.10.**

1. The original definition in [8] requires that $(c^\lambda_{s,t})^* = c^\lambda_{t,s}$ for all $\lambda, s, t$. However, one can check that the basic consequences of the definition ([8], pages 7-13) remain valid with our weaker axiom.

2. In case $2 \in R$ is invertible, one can check that our definition is equivalent to the original.

3. One can formulate a version of the “basis–free” definition of cellularity of König and Xi (see for example [12]) equivalent to our modified definition.

4. Suppose $A$ is an $R$–algebra with involution $*$, and $J$ is a $*$–closed ideal; then we have an induced algebra involution $*$ on $A/J$. Let us say that $J$ is a cellular ideal in $A$ if it satisfies the axioms of a cellular algebra (except for being unital) with cellular basis $\{c^\lambda_{s,t} : \lambda \in \Lambda_J \text{ and } s, t \in T(\lambda)\} \subseteq J$ and we have, as in point (2) of the definition of cellularity, $ac^\lambda_{s,t} \equiv \sum_v r^*_v(a)c^\lambda_{v,t} \mod \tilde{J}^\lambda$ not only for $a \in J$ but also for $a \in A$. If $J$ is a cellular ideal in $A$, and $A/J$ is cellular (with respect to the given involutions), then $A$ is cellular. With the original definition of [8], this statement would be true only if $J$ has a $*$–invariant $R$–module complement in $A$.

5. Yu [21] has also proved cellularity of the cyclotomic BMW algebras, using the original definition of cellularity of [8]; at one point, her proof requires a more delicate analysis, in order to obtain a $*$–invariant complement in $W_{n,S,r}$ of the kernel of $p : W_{n,S,r} \to H_{n,S,r}$.

### 3. Some new bases of the affine and cyclotomic BMW algebras

The basis of cyclotomic BMW algebras that we produced in [6] involved ordered monomials in the non–commuting but mutually conjugate elements

$$x'_j = g_{j-1} \cdots g_1 x_1 g_1^{-1} \cdots g_{j-1}^{-1}.$$  

To obtain this basis, we first produced a basis of the affine BMW algebra consisting of affine tangle diagrams satisfying certain topological conditions.

Here we want to produce a new finite basis of the cyclotomic BMW algebras involving monomials in the commuting, but non–conjugate, elements

$$x_j = g_{j-1} \cdots g_1 x_1 g_1 \cdots g_{j-1}.$$
At an intermediate stage of the exposition, we will also use the elements

\[ x''_j = g_{j-1}^{-1} \cdots g_1^{-1} x_1 g_1 \cdots g_{j-1}, \]

see the following figure:

\[ x_4 = \text{図}, \quad x'_4 = \text{図}, \quad x''_4 = \text{図}. \]

3.1. Flagpole descending affine tangle diagrams.

**Definition 3.1.** An *orientation* of an affine \((n, n)\)-tangle diagram is a linear ordering of the strands, a choice of an orientation of each strand, and a choice of an initial point on each closed loop.

An orientation determines a way of traversing the tangle diagram; namely, the strands are traversed successively, in the given order and orientation (the closed loops being traversed starting at the assigned initial point).

**Definition 3.2.** An oriented affine \((n, n)\)-tangle diagram is *stratified* if

1. there is a linear ordering of the strands such that if strand \(s\) precedes strand \(t\) in the order, then each crossing of \(s\) with \(t\) is an over-crossing.
2. each strand is totally descending, that is, each self-crossing of the strand is encountered first as an over-crossing as the strand is traversed according to the orientation.

We call the corresponding ordering of the strands the *stratification order*.

Note that a stratification order need not coincide with the ordering of strands determined by the orientation. *In the rest of the paper, we are going to use the following orientation and stratification order on affine tangle diagrams; when we say an affine tangle diagram is oriented or stratified, we mean with respect to this orientation and stratification order.*

**Definition 3.3.** A *verticals-second orientation* of affine tangle diagrams is one in which:

1. Non-closed strands are oriented from lower to higher numbered vertex.
2. Horizontal strands with vertices at the top of the diagram precede vertical strands, and vertical strands precede horizontal strands with vertices at the bottom of the diagram. Non-closed strands precede closed loops.
(3) Horizontal strands with vertices at the top of the diagram are ordered according to the order of their final vertices. Vertical strands and horizontal strands with vertices at the bottom of the diagram are each ordered according to the order of their initial vertices.

A *verticals–second stratification order* is one in which the order of strands agrees with that of a verticals–second orientation, except that vertical strands are ordered according to the reverse order of their initial vertices.

An affine tangle diagram without closed loops has a unique verticals–second orientation and a unique verticals–second stratification order.

A *simple winding* is a piece of an affine tangle diagram with one ordinary strand, without self-crossings, regularly isotopic to the intersection of one of the affine tangle diagrams $x_1$ or $x_1^{-1}$ with a neighborhood of the flagpole.

**Definition 3.4.** An affine tangle diagram is in *standard position* (See Figure 3.1) if:

1. It has no crossings to the left of the flagpole.
2. There is a neighborhood of the flagpole whose intersection with the tangle diagram is a union of simple windings.
3. The simple windings have no crossings and are not nested. That is, between the two crossings of a simple winding with the flagpole, there is no other crossing of a strand with the flagpole.

**Definition 3.5.** An oriented, stratified affine tangle diagram $T$ in standard position is said to be *flagpole descending* if it satisfies the following conditions:

1. $T$ is not regularly isotopic to an affine tangle diagram in standard position with fewer simple windings.
(2) The strands of \( T \) have no self–crossings.

(3) As \( T \) is traversed according to the orientation, successive crossings of ordinary strands with the flagpole descend the flagpole.

**Proposition 3.6.** The affine BMW algebra \( \hat{W}_{n,S} \) is spanned by affine tangle diagrams without closed loops that are flagpole descending and stratified.

**Proof.** This follows from [6], Proposition 2.19. \( \square \)

### 3.2. \( \mathbb{Z} \text{–Brauer} \) diagrams and liftings in the affine BMW algebras.

We recall that a Brauer diagram is a tangle diagram in the plane, in which information about over– and under–crossings is ignored. Let \( G \) be a group. A \( G \text{–Brauer} \) diagram (or \( G \text{–connector} \)) is an Brauer diagram in which each strand is endowed with an orientation and labeled by an element of the group \( G \). Two labelings are regarded as the same if the orientation of a strand is reversed and the group element associated to the strand is inverted.

Define a map \( c \) (the connector map) from oriented affine \((n, n)\)–tangle diagrams without closed loops to \( \mathbb{Z} \text{–Brauer} \) diagrams as follows. Let \( a \) be an oriented affine \((n, n)\)–tangle diagram without closed loops. If \( s \) connects two vertices \( v_1 \) to \( v_2 \), include a curve \( c(s) \) in \( c(a) \) connecting the same vertices with the same orientation, and label the oriented strand \( c(s) \) with the *winding number* of \( s \) with respect to the flagpole.¹

**Lemma 3.7 ([6], Lemma 2.21).** Two affine tangle diagrams without closed loops, with the same \( \mathbb{Z} \text{–Brauer} \) diagram, both stratified and flagpole descending, are regularly isotopic.

The symmetric group \( \mathfrak{S}_n \) can be regarded as the subset of \((n, n)\)–Brauer diagrams consisting of diagrams with only vertical strands. \( \mathfrak{S}_n \) acts on ordinary or \( \mathbb{Z} \)-labelled \((n, n)\)–Brauer diagrams on the left and on the right by the usual multiplication of diagrams, that is, by stacking diagrams.

We consider a particular family of permutations in \( \mathfrak{S}_n \). Let \( s \) be an integer, \( 0 \leq s \leq n \), with \( s \) congruent to \( n \) mod 2. Write \( f = (n - s)/2 \). Following Enyang [4], let \( D_{f,n} \) be the set of permutations \( \pi \in \mathfrak{S}_n \) satisfying:

¹The winding number \( n(s) \) is determined combinatorially as follows: traversing the strand in its orientation, list the over–crossings (⁺) and under-crossings (⁻) of the strand with the flagpole. Cancel any two successive ⁺’s or ⁻’s in the list, so the list now consists of alternating ⁺’s and ⁻’s. Then \( n(s) \) is \( \pm(1/2) \) the length of the list, ⁺ if the list begins with a ⁺, and ⁻ if the list begins with a ⁻.
Figure 3.2. Factorization of $\mathbb{Z}$–Brauer diagrams

(1) If $i, j$ are even numbers with $2 \leq i < j \leq 2f$, then $\pi(i) < \pi(j)$.
(2) If $i$ is odd with $1 \leq i \leq 2f - 1$, then $\pi(i) < \pi(i + 1)$.
(3) If $2f + 1 \leq i < j \leq n$, then $\pi(i) < \pi(j)$.

Then $D_{f,n}$ is a complete set of left coset representatives of

$$(\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2) \rtimes S_f \times S_s \subseteq S_n,$$

where the $f$ copies of $\mathbb{Z}_2$ are generated by the transpositions $(2i - 1, 2i)$ for $1 \leq i \leq f$; $S_f$ permutes the $f$ blocks $[2i - 1, 2i]$ among themselves; and $S_s$ acts on the last $s$ digits $\{2f + 1, \ldots, n\}$.

An element $\pi$ of $D_{f,n}$ factors as $\pi = \pi_1 \pi_2$, where $\pi_2 \in D_{f,f}$, and $\pi_1$ is a $(2f, s)$ shuffle; i.e., $\pi$ preserves the order of $\{1, 2, \ldots, 2f\}$ and of $\{2f + 1, \ldots, 2f + s = n\}$. Moreover, $\ell(\pi) = \ell(\pi_1) + \ell(\pi_2)$.

For any $\mathbb{Z}$–Brauer diagram $D$, let $D_0$ denote the underlying ordinary Brauer diagram; that is, $D_0$ is obtained from $D$ by forgetting the integer valued labels of the strands. If $D$ is a $\mathbb{Z}$–Brauer diagram with exactly $s$ vertical strands, then $D$ has a unique factorization

$$D = \alpha d \beta^{-1},$$

where $\alpha$ and $\beta$ are elements of $D_{f,n}$, and $d$ has underlying Brauer diagram of the form $d_0 = e_1 e_2 \cdots e_{2f-1} \pi$, where $\pi$ is a permutation of $\{2f + 1, \ldots, n\}$. This factorization is illustrated in Figure 3.2.

It will be convenient to work in the affine BMW category, that is, the category whose objects are the natural numbers $0, 1, 2, \ldots$, with $\mathrm{Hom}(k, \ell)$ being the $S$–span of affine $(k, \ell)$–tangle diagrams, modulo Kauffman skein relations. Let us introduce the elements $\cup_i$ and $\cap_i$ which are the lower and upper half of $e_i$.
We collect several elementary observations. Fix integers \( n \) and \( s \) with \( 0 \leq s \leq n \) and \( s \equiv n \mod 2 \). Set \( f = (n - s)/2 \). Each of the following statements is justified by picture proofs.

**Lemma 3.8.**

1. \( e_1 e_3 \cdots e_{2f-1} = (\cap_{2f-1} \cdots \cap_3 \cap_1)(\cup_1 \cup_3 \cdots \cup_{2f-1}) \).
2. For \( k \) odd, \( 1 \leq k \leq 2f - 1 \), \( (\cup_1 \cup_3 \cdots \cup_{2f-1})x'_k = (\cup_1 \cup_3 \cdots \cup_{2f-1})x_k \).
3. For \( k \) odd, \( 1 \leq k \leq 2f - 1 \), \( x''_k(\cap_{2f-1} \cdots \cap_3 \cap_1) = x_k(\cap_{2f-1} \cdots \cap_3 \cap_1) \).
4. If \( \pi \) is a a permutation of \( \{2f + 1, 2f + 2, \ldots, n\} \), then
   \[
   (\cup_1 \cup_3 \cdots \cup_{2f-1})g_\pi = g_{\tilde{\pi}}(\cup_1 \cup_3 \cdots \cup_{2f-1}),
   \]
   where \( \tilde{\pi} \) is the permutation of \( \{1, 2, \ldots, s\} \) defined by \( \tilde{\pi}(j) = \pi(j + 2f) - 2f \).
   More generally, if \( T \) is an ordinary tangle on the strands \( \{2f + 1, 2f + 2, \ldots, n\} \), then
   \[
   (\cup_1 \cup_3 \cdots \cup_{2f-1}) \ T = \tilde{T} \ (\cup_1 \cup_3 \cdots \cup_{2f-1}),
   \]
   where \( \tilde{T} \) is the shift of \( T \) to the strands \( \{1, 2, \ldots, s\} \).
5. For \( 1 \leq k \leq s \), \( (\cup_1 \cup_3 \cdots \cup_{2f-1})x_{k+2f} = x_k(\cup_1 \cup_3 \cdots \cup_{2f-1}). \)

Now we can obtain a lifting of \( \mathbb{Z} \)-Brauer diagrams to affine tangle diagrams that are flagpole descending and stratified, using the factorization of Equation \((3.1)\). Let \( D \) be a \( \mathbb{Z} \)-Brauer diagram with exactly \( s \) vertical strands. Set \( f = (n - s)/2 \).

Consider the factorization \( D = \alpha d \beta^{-1} \), where \( \alpha, \beta \in \mathcal{D}_{f,n} \), and \( d_0 = e_1 \cdots e_{2f-1} \pi \), with \( \pi \) a permutation of \( \{2f + 1, \ldots, n\} \).

First, there is a unique (up to regular isotopy) stratified ordinary \((n, n)\)-tangle diagram \( T_{d_0} \) without closed loops or self-crossings of strands with Brauer diagram 

\[
c(T_{d_0}) = d_0 = e_1 e_3 \cdots e_{2f-1} \pi,
\]

namely

\[
T_{d_0} = e_1 e_3 \cdots e_{2f-1} g_\pi = (\cap_{2f-1} \cdots \cap_3 \cap_1)(\cup_1 \cup_3 \cdots \cup_{2f-1})g_\pi
= (\cap_{2f-1} \cdots \cap_3 \cap_1)g_{\tilde{\pi}}(\cup_1 \cup_3 \cdots \cup_{2f-1}),
\]
where $\tilde{\pi}$ is the permutation of $\{1, 2, \ldots, s\}$ defined by $\tilde{\pi}(j) = \pi(j + 2f) - 2f$.

Next, we set
\begin{equation}
T'_d = (x'_1)^{a_1} \cdots (x'_{2f-1})^{a_{2f-1}}(\cap_2 \cdots \cap_{2f-1})\alpha(\cup_1 \cup_3 \cdots \cup_{2f-1})(x'_1)^{c_1},
\end{equation}
where the exponents are determined as follows:

1. For $i$ odd, $i \leq 2f - 1$, if $d$ has a strand beginning at $i$ with label $k$, then $c_i = k$; otherwise $c_i = 0$.
2. For $i \geq 2f + 1$, if $d$ has a strand beginning at $i$ with label $k$, then $b_{i-2f} = k$; otherwise $b_{i-2f} = 0$.
3. For $i$ odd, $i \leq 2f - 1$, if $d$ has a horizontal strand ending at $\bar{i}$ with label $k$, then $a_i = k$; otherwise, $a_i = 0$.

Finally, we set
\begin{equation}
T'_D = g_\alpha T'_d (g_\beta)^*,
\end{equation}
endowed with the verticals–second orientation.

Example 3.9. For the $\mathbb{Z}$–Brauer diagram $D$ illustrated in Figure 3.2, $T'_D$ is illustrated in Figure 3.3 (where the winding number of the strands are indicated by the integers written at the left of the figure.) We have
\begin{equation*}
T'_d = (x'_1)^1(x''_3)^6(\cap_1 \cap_3)g(g_1g_2(3''_2)^5(x''_1)^7(x''_1)^-2(\cup_1 \cup_3)(x'_1)^{-1}(x'_1)^3
\end{equation*}

Lemma 3.10. $T'_D$ is flagpole descending and stratified, and has $\mathbb{Z}$–Brauer diagram equal to $D$.

Proof. Straightforward. \qed

Proposition 3.11. $U' = \{T'_D : D \text{ is a } \mathbb{Z}$–Brauer diagram$\}$ spans $\mathcal{W}_{n,S}$.

Proof. Follows from Proposition 3.6, Lemma 3.7, and Lemma 3.10. \qed

Proposition 3.12. $U' = \{T'_D : D \text{ is a } \mathbb{Z}$–Brauer diagram$\}$ is a basis of $\mathcal{W}_{n,S}$.

Proof. Essentially the same as the proof of Theorem 2.25 in [6]. \qed

Now fix an integer $r \geq 1$. Let $U'_D \in U'$ such that the integer valued labels on the strands of $D$ are restricted to lie in the interval $0 \leq k \leq r - 1$. Equivalently, the exponents of the $x'_j$, $x''_j$ appearing in $U'_D$ are restricted to be in the same interval of integers.
Proposition 3.13. The cyclotomic BMW algebra $W_{n,S,r}(u_1, \ldots, u_r)$ is spanned over $S$ by $U'_r$.

Proof. Same as the proof of Proposition 3.6 in [6]. \hfill \Box

Proposition 3.14. For any integral domain $S$ with admissible parameters, $U'_r$ is an $S$–basis of the cyclotomic BMW algebra $W_{n,S,r}(u_1, \ldots, u_r)$.

Proof. Same as the proof of Theorem 5.5 in [6]. \hfill \Box

Remark 3.15. It is straightforward to generalize the content of this section to affine $(k, \ell)$–tangle diagrams. The notions of standard position, orientation, and stratification, and in particular the verticals–second orientation and stratification extend to affine $(k, \ell)$–tangle diagrams. Likewise, the notion of flagpole descending extends.

Define $(k, \ell)$–connectors to be “Brauer diagrams” with $k$ upper vertices and $\ell$ lower vertices, and likewise define $\mathbb{Z}$–weighted $(k, \ell)$–connectors. We can extend the definition of the connector map $c$ to a map from oriented affine $(k, \ell)$–tangle
diagrams without closed loops to \( \mathbb{Z} \)-weighted \((k, \ell)\)-connectors. Then the analogue of Lemma 3.7 holds.

Other results in this section can also be generalized, but we will need only a weak version of Proposition 3.11, and only for affine \((0, 2f)\)-tangle diagrams. Consider the set of affine \((0, 2f)\)-tangle diagrams of the form

\[
g_{\alpha}(x''_1)^{a_1} \cdots (x''_{2f-1})^{a_{2f-1}} (\cap_{2f-1} \cdots \cap_3 \cap_1),
\]

where \(\alpha \in D_{f,f}.\) These affine tangle diagrams are stratified and flagpole descending, and have no closed loops. Moreover, every \(\mathbb{Z}\)-weighted \((0, 2f)\)-connector has a lifting in this set. Therefore, by the analogue of Lemma 3.7, every totally descending, flagpole descending affine \((0, 2f)\)-tangle diagram without closed loops, is regularly isotopic to one of the diagrams represented in Equation (3.5).

### 3.3. New bases.

So far, we have produced bases \(\mathcal{U}'\) of the affine BMW algebras and \(\mathcal{U}_s\) of the cyclotomic BMW algebras involving (as did our previous bases in [6]) ordered monomials in the non–commuting but conjugate elements \(x'_j\) and \(x''_j\).

We will now use these bases to obtain new bases involving instead monomials in the commuting elements \(x_j\).

Consider the definition of \(T_D'\) in Equations (3.3) and (3.4). Note that

\[
(\cup_1 \cup_3 \cdots \cup_{2f-1})(x'_{2f-1})^{c_{2f-1}} \cdots (x'_3)^{c_3}(x'_1)^{c_1}
= (\cup_1 \cup_3 \cdots \cup_{2f-1})(x'_{2f-1})^{c_{2f-1}} \cdots (x'_3)^{c_3}(x'_1)^{c_1}
= (\cup_1 \cup_3 \cdots \cup_{2f-1})((x'_{2f-1})^{c_{2f-3}} \cdots (x'_3)^{c_3}(x'_1)^{c_1})(x'_{2f-1})^{c_{2f-1}},
\]

using Lemma 3.8 (2), and the commutivity of \(x_{2f-1}\) with \(\hat{W}_{2f-2,s}.\) Applying this step repeatedly, we end with

\[
(\cup_1 \cup_3 \cdots \cup_{2f-1})(x'_{2f-1})^{c_{2f-1}} \cdots (x'_3)^{c_3}(x'_1)^{c_1}
= (\cup_1 \cup_3 \cdots \cup_{2f-1})(x'_1)^{c_1}(x'_3)^{c_3} \cdots (x'_{2f-1})^{c_{2f-1}}
\]

Likewise, using Lemma 3.8 (3),

\[
(x''_1)^{a_1} \cdots (x''_{2f-1})^{a_{2f-1}} (\cap_{2f-1} \cdots \cap_3 \cap_1)
= (x''_1)^{a_1} \cdots (x''_{2f-1})^{a_{2f-1}} (\cap_{2f-1} \cdots \cap_3 \cap_1)
\]

Thus the element \(T_D'\) has the form

\[
T_D' = g_{\alpha}(x''_1)^{a_1} \cdots (x''_{2f-1})^{a_{2f-1}} (\cap_{2f-1} \cdots \cap_3 \cap_1) T
= (\cup_1 \cup_3 \cdots \cup_{2f-1})(x'_1)^{c_1}(x'_3)^{c_3} \cdots (x'_{2f-1})^{c_{2f-1}} (g_{\beta})^* T,
\]

where \(T\) is an affine tangle diagram on \(s\) strands with no horizontal strands.
We consider the cyclotomic BMW algebra $W_n = W_{n,S,r}(u_1, \ldots, u_r)$; the arguments for the affine BMW algebras are similar. If $T_D' \in \mathbb{U}_r'$, then the exponents $a_i$ and $c_i$ of $x_i$ in Equation (3.6) satisfy $0 \leq a_i, c_i \leq r - 1$.

Recall from Section 2.5 that the quotient of $W_s$ by the ideal $I_s$ spanned by affine tangle diagrams with rank strictly less than $s$ is isomorphic to the cyclotomic Hecke algebra $H_{s,S,r} = H_{s,S,r}(q^2; u_1, \ldots, u_r)$. The cyclotomic Hecke algebra $H_{s,S,r}$ is a free $S$–module with basis the set of $\tau_{\omega} t_{1}^{b_1} \cdots t_{s}^{b_s}$, with $0 \leq b_i \leq r - 1$ and $\omega \in \mathcal{S}_s$.

Therefore, the element $T \in W_s$ is congruent modulo $I_s$ to a linear combination of elements $t(\tau_{\omega} t_{1}^{b_1} \cdots t_{s}^{b_s}) = g_{\omega} x_1^{b_1} \cdots x_s^{b_s}$.

If we replace $T$ with $g_{\omega} x_1^{b_1} \cdots x_s^{b_s}$ in Equation (3.6), and then apply Lemma 3.8 (1), (4) and (5), we obtain (in place of $T_D'$) an expression

\begin{equation}
\begin{aligned}
g_\alpha x_1^{a_1} \cdots x_{2f-1}^{a_{2f-1}} (e_1 e_3 \cdots e_{2f-1} g_\omega) \\
x_1^{c_1} x_3^{c_3} \cdots x_{2f-1}^{c_{2f-1}} x_{2f+1}^{b_{2f+1}} \cdots x_n^{b_n} (g_\beta)^* \end{aligned}
\end{equation}

On the other hand, if we replace $T$ by an element of $I_s$, then we obtain (in place of $T_D'$) a linear combination of affine tangle diagrams with rank strictly less than $s$.

Given a $\mathbb{Z}$–Brauer diagram $D$, we define an element $T_D$ of the form displayed in Equation (3.7) whose associated $\mathbb{Z}$–Brauer diagram $c(U_D)$ is equal to $D$, as follows: Suppose $D$ has $2n$ vertices and $s$ vertical strands, and let $f = (n - s)/2$. Let $D$ have the factorization $D = \alpha d \beta^{-1}$, where $\alpha, \beta \in \mathcal{D}_{f,n}$, and $d_0 = c_1 \cdots e_{2f-1} \pi$, with $\pi$ a permutation of $\{2f+1, \ldots, n\}$.

Define

\begin{equation}
\begin{aligned}
T_d = x_1^{a_1} \cdots x_{2f-1}^{a_{2f-1}} (e_1 e_3 \cdots e_{2f-1} g_\pi) \\
x_1^{c_1} x_3^{c_3} \cdots x_{2f-1}^{c_{2f-1}} x_{2f+1}^{b_{2f+1}} \cdots x_n^{b_n},
\end{aligned}
\end{equation}

where the exponents are determined as follows: If $d$ has a horizontal strand beginning at $i$ with integer valued label $k$, then $c_i = k$; and $c_i = 0$ otherwise. If $d$ has a vertical strand beginning at $i$ with integer valued label $k$, then $b_i = k$; and $b_i = 0$ otherwise. If $d$ has a horizontal strand ending at $\tilde{i}$ with integer valued label $k$, then $a_i = k$; and $a_i = 0$ otherwise.

Finally, set $T_D = g_\alpha T_d (g_\beta)^*$. Then $c(U_D) = D$.

**Theorem 3.16.** Let $S$ be an admissible integral domain. Let $\mathbb{U}_r$ be the set of $T_D$ corresponding to $\mathbb{Z}$–Brauer diagrams $D$ with integer valued labels in the interval $0 \leq k \leq r - 1$. Then $\mathbb{U}_r$ is an $S$–basis of $W_{n,S,r}(u_1, \ldots, u_r)$. 
Proof. We will show that $U_r$ is spanning. Linear independence is proved as in the proof of Theorem 5.5 in [6]. It suffices to show that $U'_r$ is contained in the linear span of $U_r$.

Let $T'_D \in U'_r$, where $D$ has exactly $s$ vertical strands. We show by induction on $s$ that $T'_D$ is in the span of $U_r$. If $s = 0$, then in Equation (3.6), the tangle $T$ is missing, and $T'_D$ is already an element of $U_r$. If $s = 1$, then in Equation (3.6), the tangle $T$ is equal to a power of $x_1$, and again $T'_D \in U_r$, by Lemma 3.8 (5).

Assume that $s > 1$ and that all that all elements of $U'_r$ with fewer than $s$ vertical strands are in the span of $U_r$. It follows from Equation (3.6), and the discussion following it, that $T'_D$ is in the span of $U_r$, modulo the ideal $I_n^{(s-1)}$ in $\hat{W}_{n,S,r}(u_1,\ldots,u_r)$ spanned by of affine tangle diagrams with rank strictly less than $s$.

Now it only remains to check that $I_n^{(s-1)}$ is spanned by elements of $U'_r$ with fewer than $s$ vertical strands. Here one only has to observe that smoothing any crossing in a tangle diagram with $k$ vertical strands produces a tangle diagram with at most $k$ vertical strands. Therefore, the algorithm from [6], Propositions 2.18 and 2.19, for writing an affine tangle diagram (with $k$ vertical strands) as a linear combination of flagpole descending affine tangle diagrams produces only affine tangle diagrams with at most $k$ vertical strands. □

For the affine case, we have the following result, with essentially the same proof:

**Theorem 3.17.** Let $S$ be any ring with appropriate parameters. $U = \{T_D : D$ is a $\mathbb{Z}$–Brauer diagram$\}$ is a basis of $\hat{W}_{n,S}$.

Let $D$ be an $\mathbb{Z}$–Brauer diagram, with $2n$ vertices and $s$ vertical strands, having factorization $D = \alpha d \beta$, and let $T_d$ be defined as in Equation (3.8) and let $T_D = g_\alpha T_d (g_\beta)^*$. We can rewrite $T_D$ as follows. Factor $\alpha$ as $\alpha = \alpha_1 \alpha_2$, with $\alpha_1$ a $(2f,s)$–shuffle, and $\alpha_2 \in D_{f,f}$, and factor $\beta$ similarly. Then

\begin{equation}
T_D = g_{\alpha_1} [g_{\alpha_2} x^a (e_1 \cdots e_{2f-1}) x^c (g_{\beta_2})^* (g_{\beta_1} x^b) (g_{\beta_1})^*],
\end{equation}

where $x^a$ is short for $x_1^{a_1} \cdots x_{2f-1}^{a_{2f-1}}$, and similarly for $x^c$, while $x^b$ denotes $x_{2f+1}^{b_{2f+1}} \cdots x_n^{b_n}$. 
4. **Cellular bases of cyclotomic BMW algebras**

4.1. **Tensor products of affine tangle diagrams.** The category of affine \((k, \ell)\)-tangle diagrams is not a tensor category in any evident fashion. Nevertheless, we can define a tensor product of affine tangle diagrams, as follows. Let \(T_1\) and \(T_2\) be affine tangle diagrams (say of size \((a, a)\) and \((b, b)\), respectively), and suppose that \(T_2\) has no closed loops. Then \(T_1 \odot T_2\) is obtained by replacing the flagpole in the affine tangle diagram \(T_2\) with the entire affine tangle diagram \(T_1\). See Figure 4.1. If we regard \(T_1\) and \(T_2\) as representing framed tangles in the annulus cross the interval \(A \times I\), then \(T_1 \odot T_2\) is obtained by inserting the entire copy of \(A \times I\) containing \(T_1\) into the hole of the copy of \(A \times I\) containing \(T_2\).

Then \(T_1 \odot T_2 \mapsto T_1 \odot T_2\) determines a linear map from \(\hat{W}_{a,S} \odot \hat{W}_{b,S}\) into \(\hat{W}_{a+b,S}\), or from \(W_{a,S,r} \odot W_{b,S,r}\) into \(W_{a+b,S,r}\). Note that \((T_1 \odot T_2)^* = T_1^* \odot T_2^*\).

These maps of affine and cyclotomic BMW algebras are not algebra homomorphisms. In fact,

\[
(1 \odot e_1)(1 \odot x_1)(1 \odot e_1) = z \odot e_1,
\]

where \(z\) is a (non–scalar) central element in \(\hat{W}_{a,S}\). Nevertheless, we have

\[
(A \odot B)(S \odot T) = AS \odot BT,
\]

if no closed loops are produced in the product \(BT\), in particular, if at least one of \(B\) and \(T\) has no horizontal strands.
4.2. Cellular bases. Using Equation (3.9) and our remarks in Section 4.1, we can rewrite the elements $T_D$ of Section 3.3 in the form

\begin{equation}
T_D = g_{\alpha_1}(g_{\alpha_2}x^a(e_1e_3\cdots e_{2f-1})x^c(g_{\beta_2})^*) \circ g_\pi x^b)(g_{\beta_1})^*
\end{equation}

Here, $D$ is a $\mathbb{Z}$–Brauer diagram with $s$ vertical strands and $f = (n - s)/2$; $\alpha_1$ and $\beta_1$ are $(n - s, s)$–shuffles; $\pi \in \mathfrak{S}_s$ and $x^b = x_1^{b_1} \cdots x_s^{b_s}$. Moreover, $\alpha_2$ and $\beta_2$ are elements of $D_{f,f}$, $x^a = x_1^{a_1}x_3^{a_3} \cdots x_{2f-1}^{a_{2f-1}}$, and similarly for $x^c$.

The affine $(2f, 2f)$–tangle diagram

$$T = g_{\alpha_2}x^a(e_1e_3\cdots e_{2f-1})x^c(g_{\beta_2})^*$$

is stratified and flagpole descending, with no vertical strands and no closed loops. Conversely, any stratified and flagpole descending affine $(2f, 2f)$–tangle diagram with no vertical strands and no closed loops is regularly isotopic to one of this form.

Note that we can factor $T$ as $T = xy^*$, where $x$ and $y$ are stratified and flagpole descending affine $(0, 2f)$–tangle diagram with no closed loops, namely

$$x = g_{\alpha_2}x^a(\cap_{2f-1}\cdots \cap_3 \cap_1)$$

and

$$y = g_{\beta_2}x^c(\cap_{2f-1}\cdots \cap_3 \cap_1).$$

By Remark 3.15, any stratified and flagpole descending affine $(0, 2f)$–tangle diagram with no closed loops is regularly isotopic to one of this form.

**Lemma 4.1.** The set of $T_D \in U$ with $s$ vertical strands equals the set of elements

$$g_\alpha(xy^* \circ g_\pi x^b))(g_\beta)^*,$$

where $x, y$ are stratified, flagpole descending affine $(0, n - s)$–tangle diagrams without closed loops or self-crossings of strands; $\alpha$ and $\beta$ are $(n - s, s)$–shuffles; $\pi \in \mathfrak{S}_s$, and $x^b = x_1^{b_1} \cdots x_s^{b_s}$.

Moreover, $T_D \in U_r$ if, and only if, the exponents $b_i$ are in the range $0 \leq b_i \leq r - 1$, and the winding numbers of $x$ and $y$ with the flagpole are in the same range.

We will show that the cyclotomic BMW algebras defined over integral, admissible rings are cellular. We fix an integral domain $S$ with admissible parameters, and write $W_{n,S,r}$ for $W_{n,S,r}(u_1, \ldots, u_r)$ and $H_{n,S,r}$ for $H_{n,S,r}(q^2; u_1, \ldots, u_r)$.

For each $s$ with $s \leq n$ and $n - s$ even, let $V^*_n$ be the span in $W_{n,S,r}$ of the set of elements $T_D \in U_r$ with $s$ vertical strands.

**Lemma 4.2.** For each $s$, let $\mathbb{B}_s$ be a basis of $H_{n,S,r}$. Let $\Sigma_s$ be the set of elements

$$g_\alpha(xy^* \circ t(b))(g_\beta)^* \in W_{n,S,r},$$
such that \( x, y \) are stratified, flagpole descending affine \((0, n - s)\)-tangle diagrams without closed loops; \( \alpha \) and \( \beta \) are \((n - s, s)\)-shuffles; and \( b \in \mathbb{B}_s \). Then \( \Sigma_s \) is a basis of \( V_n^s \).

Proof. Recall that \( \{\tau_b t^b : \pi \in \mathcal{G}_s \text{ and } 0 \leq b_i \leq r - 1\} \) is a basis of \( H_{s,S,r} \), and that \( g_b \mathbf{x}^b = t(\tau_b t^b) \). It follows from this and from Lemma 4.1 that \( V_n^s \) is the direct sum over \((\alpha, \beta, x, y)\) of

\[
V_n^s(\alpha, \beta, x, y) = \{g_\alpha(xy^* \circ t(u))(g_\beta)^* : u \in H_{s,S,r}\}
\]

and that \( u \rightarrow g_\alpha(T \circ t(u))(g_\beta)^* \) is injective. This implies the result. \( \square \)

For each \( s \) \((s \leq n \text{ and } n - s \text{ even})\), let \((C_s, \Lambda_s)\) be a cellular basis of the cyclotomic Hecke algebra \( H_{s,S,r} \). Let \( \Lambda = \{(s, \lambda) : \lambda \in \Lambda_s\} \) with partial order \((s, \lambda) \geq (t, \mu)\) if \( s < t \) or if \( s = t \) and \( \lambda \geq \mu \) in \( \Lambda_s \). For each pair \((s, \lambda) \in \Lambda\), we take \( T(s, \lambda) \) to be the set of triples \((\alpha, x, u)\), where \( \alpha \) is a \((n - s, s)\)-shuffle; \( x \) is a stratified, flagpole descending affine \((0, n - s)\)-tangle without closed loops; and \( u \in T(\lambda) \). Define

\[
c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)} = g_\alpha(xy^* \circ t(c_{u,v}^\lambda))(g_\beta)^*,
\]

and \( C \) to be the set of all \( c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)} \).

Lemma 4.3. \( (c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)})^* \equiv c^{(s,\lambda)}_{(\beta,y,v),(\alpha,x,u)} \mod \tilde{W}_{n,S,r}(s,\lambda) \).

Proof. \( (g_\alpha(xy^* \circ t(c_{u,v}^\lambda))(g_\beta)^*)^* = g_\beta(y^* x \circ t(c_{u,v}^\lambda)^*)(g_\alpha)^* \), and \( (t(c_{u,v}^\lambda))^* \equiv t(c_{v,u}^\lambda) \) modulo the span of diagrams of rank \( s \). Hence \( (c^{(s,\lambda)}_{(\alpha,x,u),(\beta,y,v)})^* \equiv c^{(s,\lambda)}_{(\beta,y,v),(\alpha,x,u)} \)

modulo the span of diagrams of rank \( s \). \( \square \)

Lemma 4.4. For any affine \((n - s, n - s)\)-tangle diagram \( A \) and affine \((s, s)\)-tangle diagram \( B \), \((A \circ B)(xy^* \circ t(c_{u,v}^\lambda))\) can be written as a linear combination of elements \( (x'y^* \circ t(c_{u',v'}^\lambda)) \), modulo \( \tilde{W}_{n,S,r}(s,\lambda) \), with coefficients independent of \( y \) and \( v \).

Proof. We have \((A \circ B)(xy^* \circ t(\tau_b \mathbf{x}^b)) = (Axy^* \circ B t(\tau_b \mathbf{x}^b))\), because \( t(\tau_b \mathbf{x}^b) \) has only vertical strands. Therefore, also \((A \circ B)(xy^* \circ t(c_{u,v}^\lambda)) = (Axy^* \circ B t(c_{u,v}^\lambda))\).

Note that \( Ax \) is an affine \((0, n - s)\)-tangle, and can be reduced using the algorithm of the proof of Propositions 2.18 and 2.19 in [6] to a linear combination of stratified, flagpole descending \((0, n - s)\)-tangles \( x' \) without closed loops. The process does not affect \( y^* \).
If $B$ has rank strictly less than $s$, then the product $(A \odot B)(xy^* \odot t(c_{u,v}^\lambda))$ is a linear combination of basis elements $T_D$ with fewer than $s$ vertical strands, so belongs to $\hat{W}_{n,S,r}$.

Otherwise, we can suppose that $B = g_\sigma x^b$. Then $B t(c_{u,v}^\lambda) = t(\tau_\sigma t^b t(c_{u,v}^\lambda)) \equiv t(\tau_\sigma t^b c_{u,v}^\lambda)$ modulo the span of basis diagrams with fewer than $s$ vertical strands. Moreover, $t(\tau_\sigma t^b c_{u,v}^\lambda)$ is a linear combination of elements $t(c_{u',v'}^\lambda)$, modulo $\hat{H}_{s,S,r}$, with coefficients independent of $v$, by the cellularity of the basis $C_s$ of $H_{s,S,r}$.

The conclusion follows from these observations. \hfill $\Box$

**Theorem 4.5.** Let $S$ be an admissible integral domain. $(C, \Lambda)$ is a cellular basis of the cyclotomic BMW algebra $W_{n,S,r}$.

**Proof.** Theorem 3.16 and Lemma 4.2 implies that $C$ is a basis of $W_{n,S,r}$, and property (3) of cellular bases holds by Lemma 4.3. It remains to verify axiom (2) for cellular bases. Thus we have to show that for $w \in W_{n,S,r}$, and for a basis element $c_{(\alpha,x,u),(\beta,y,v)}^{(s,\lambda)} = g_\alpha(xy^* \odot t(c_{u,v}^\lambda))(g_\beta)^*$, the product

\[
\tag{4.2}
w g_\alpha(xy^* \odot t(c_{u,v}^\lambda))(g_\beta)^*
\]

can be written as a linear combination of elements

\[
g_{\alpha'}(x'y^* \odot t(c_{u',v}^\lambda))(g_{\beta'})^*,
\]
modulo $\hat{W}_{n,S,r}$ (with coefficients independent of $(\beta, y, v)$).

It suffices to consider products as in Equation (4.2) with $w$ equal to $e_i$ or to $g_i$ for some $i$, or $w = x_1$. We consider first $w = e_i$ or $w = g_i$. Here there are several cases, depending on the relative position of $\alpha^{-1}(i)$ and $\alpha^{-1}(i+1)$.

Suppose that $\alpha^{-1}(i) > \alpha^{-1}(i+1)$. Then $g_\alpha = g_i g_{\alpha_1}$, where $\alpha_1^{-1}(i) < \alpha_1^{-1}(i+1)$, and $\alpha_1$ is also a $(n-s,s)$-shuffle. Thus $e_i g_\alpha = e_i g_i g_{\alpha_1} = \rho^{-1} e_i g_{\alpha_1}$. Likewise, $g_i g_\alpha = (g_i)^2 g_{\alpha_1} = g_{\alpha_1} + (q-q^{-1}) g_{\alpha_1} + (q^{-1}-q) \rho^{-1} e_i g_{\alpha_1}$. We are therefore reduced to considering the case that $\alpha^{-1}(i) < \alpha^{-1}(i+1)$.

Suppose that $\alpha^{-1}(i+1) \leq n - s$ or $n - s + 1 \leq \alpha^{-1}(i)$. Then $\alpha^{-1}(i+1) = \alpha^{-1}(i) + 1$, because $\alpha$ is a $(n-s,s)$-shuffle. Write $\chi_i$ for $g_i$ or $e_i$. We have $\chi_i g_\alpha = g_\alpha \chi_\alpha^{-1}(i)$, as one can verify with pictures. But $\chi_\alpha^{-1}(i) \in W_{n-s} \otimes W_s$, so the conclusion follows from Lemma 4.4.

It remains to examine the case that $\alpha^{-1}(i) \leq n - s$ and $\alpha^{-1}(i+1) \geq n - s + 1$. In this case, $g_i g_\alpha$ is an $(n-s,s)$-shuffle so $g_i g_\alpha(xy^* \odot t(c_{u,v}^\lambda))(g_\beta)^*$ is another basis element.
Next, we have to consider the product $e_i g_\alpha(xy^* \circ t(c_{u,v}^\lambda))(g_\beta)^*$. Define a permutation $\varrho$ by

$$
\varrho(j) = \begin{cases} 
    j & \text{if } j < \alpha^{-1}(i), \\
    j + 1 & \text{if } \alpha^{-1}(i) \leq j < n - s, \\
    \alpha^{-1}(i) & \text{if } j = n - s, \\
    \alpha^{-1}(i) + 1 & \text{if } j = n - s + 1, \\
    j - 1 & \text{if } n - s + 1 < j \leq \alpha^{-1}(i) + 1, \\
    j & \text{if } j > \alpha^{-1}(i) + 1.
\end{cases}
$$

(4.3)

Since $\varrho \in \mathfrak{S}_{n-s} \times \mathfrak{S}_r$, we have $\ell(\varrho \alpha) = \ell(\alpha) + \ell(\varrho)$ and $g_{\alpha \varrho} = g_\alpha g_\varrho$. The permutation $\alpha \varrho$ has the following properties: $\alpha \varrho(n - s) = i$; $\alpha \varrho(n - s + 1) = i + 1$; if $1 \leq a < b \leq n - s - 1$ or $n - s + 2 \leq a < b \leq n$, then $\alpha \varrho(a) < \alpha \varrho(b)$. We have

$$
e_i g_\alpha = e_i g_\alpha g_\varrho g_\varrho^{-1} = e_i g_{\alpha \varrho} g_\varrho^{-1}.
$$

(4.4)

The tangle $e_i g_{\alpha \varrho}$ is stratified and has a horizontal strand connecting the top vertices $n - s$ and $n - s + 1$. Contracting that strand, we get

$$
e_i g_{\alpha \varrho} = \cap_i g_{\sigma} \cup_{n-s},
$$

(4.5)

for a certain $(n - s - 1, s - 1)$-shuffle $\sigma$. Therefore,

$$
e_i g_{\alpha}(xy^* \circ t(c_{u,v}^\lambda))(g_\beta)^* = \cap_i g_{\sigma} \cup_{n-s} g_\varrho^{-1}(xy^* \circ t(c_{u,v}^\lambda))(g_\beta)^*.
$$

(4.6)

Moreover, $g_\varrho^{-1} \in W_{n-s} \otimes W_s \subseteq W_{n,S,r}$, so $g_\varrho^{-1}(xy^* \circ t(c_{u,v}^\lambda))(g_\beta)^*$ is congruent modulo $\bar{W}_{n,S,r}^{(\lambda)}$ to a linear combination of elements $(x'y^* \circ t(c_{u',v}^\lambda))(g_\beta)^*$, with coefficients independent of $\beta$, $y$ and $v$, by Lemma 4.4. Thus we have to consider the products

$$
\cap_i g_{\sigma} \cup_{n-s} (x'y^* \circ t(c_{u',v}^\lambda))(g_\beta)^*.
$$

(4.7)

Focus for a moment on the product $\cap_i g_{\sigma} \cup_{n-s} (x' \odot 1)$, and write $x'$ in the form

$$
g_{\alpha_2} x_{a_1}^{a_1} x_{a_3}^{a_3} \cdots x_{a_{n-s}}^{a_{n-s}} \cap_i \cap_{n-s-1} \cap_{n-s-2} \cap_{n-s-3} \cap_1,
$$

with $\alpha_2 \in \mathcal{D}_{f,f}$ $(f = (n - s)/2)$. Figure 4.2 provides a guide to the computations. Write $a = a_{n-s-1}$. We have

$$
\cap_i g_{\sigma} \cup_{n-s} (x' \odot 1)
$$

(4.8)

$$
= \cap_i g_{\sigma} \cup_{n-s} (g_{\alpha_2} x_{a_1}^{a_1} x_{a_3}^{a_3} \cdots x_{a_{n-s}}^{a_{n-s}})(\cap_{n-s-1} \cap_{n-s-2} \cap_{n-s-3} \cap_1)
$$

$$
= \cap_i g_{\sigma} (g_{\alpha_2} x_{a_1}^{a_1} x_{a_3}^{a_3} \cdots x_{a_{n-s}}^{a_{n-s}}) \cup_{n-s} \cup_{n-s-1} \cap_{n-s-3} \cap_{n-s-2} \cap_1.
$$
Note that
\[
x_{n-s-1}^a \cup_{n-s} x_{n-s-1} = \rho^{-a} \cup_{n-s} x_{n-s}^a \cap_{n-s-1} x_{n-s}^a = \cup_{n-s} \cap_{n-s-1} x_{n-s}^a + 1,
\]
by [5], Lemma 6.8 and Remark 6.9. Thus,
\[
\cap_i g_\sigma \cup_{n-s}(x' \circ 1) = \cap_i g_\sigma \cup_{n-s}(g_{\alpha_2 x_{1}^{a_2} x_{3}^{a_3} \cdots x_{n-s-3}^{a_{n-s-3}}}(\cap_{n-s-1} \cdots \cap_1) x_{n-s}^a + 1
\]
(4.9)
where $x''$ is another stratified, flagpole descending affine $(0, n-s)$–tangle diagram (without closed loops) with the property that the strand incident with the vertex $n-s$ has winding number 0 with the flagpole. See the second stage in Figure 4.2. (In the figure, a “bead” on the $j$–th strand is supposed to indicate a power of $x_j$; a bead labelled by $a$ indicates $x_j^a$.)

Since this affine tangle diagram is stratified, the strand incident with the top vertex 1 can be pulled straight, and the horizontal strand connecting the bottom vertices $\tilde{i}$ and $\tilde{i} + 1$ can be pulled up. The result is an affine tangle diagram with the factorization $g_\pi(x'' \circ x_1^a)$, where $x''$ is a stratified, flagpole descending affine
(0, n−s)–tangle diagram and gc is a positive permutation braid; this is illustrated in the final stage of Figure 4.2.

Consequently, we have:

\[ g_\pi(x'y^* \odot t(c_{u,v}^\lambda))(g_\beta)^* \]

By Lemma 4.4, this is congruent mod \( \hat{W}_{n,S,r}^{(s,\lambda)} \) to a linear combination of terms

\[ g_\pi(x'y^* \odot t(c_{u,v}^\lambda))(g_\beta)^* , \]

with coefficients independent of \( y, \beta, \) and \( v. \)

Finally, the permutation \( \pi \) can be factored as \( \pi = \pi_1 \pi_2, \) where \( \pi_1 \) is an \((n−s,s)–shuffle, \pi_2 \in \mathfrak{S}_{n−s} \times \mathfrak{S}_s \) and \( \ell(\pi_1) = \ell(\pi_1) + \ell(\pi_2). \) Consequently, \( g_\pi = g_{\pi_1}g_{\pi_2}, \) where \( g_{\pi_2} \in W_{n−s} \otimes W_s. \) We can now apply Lemma 4.4 again to rewrite the product of Equation (4.11) as a linear combination of elements \( g_{\pi_1}(x'y^* \odot t(c_{u,v}^\lambda))(g_\beta)^* , \) modulo \( \hat{W}_{n,S,r}^{(s,\lambda)} \), with coefficients independent of \( \beta, y, \) and \( v. \) This completes the proof of the case: \( w = e_i, \alpha^{-1}(i) < n−s \) and \( n−s+1 \leq \alpha^{-1}(i+1). \)

It remains to consider the product

\[ x_1 g_\alpha(x'y^* \odot t(c_{u,v}^\lambda))(g_\beta)^* . \]

Since \( \alpha \) is an \((n−s,s)–shuffle, \) either \( \alpha^{-1}(1) = 1 \) or \( \alpha^{-1}(1) = n−s + 1. \) In case \( \alpha^{-1}(1) = 1, \) we have \( x_1g_\alpha = g_\alpha x_1, \) and the result follows by applying Lemma 4.4. If \( \alpha^{-1}(1) = n−s + 1, \) then we can write \( g_\alpha \) as \( g_\alpha = g_{\alpha_2}(g_1g_2 \cdots g_{n−s−1}), \) where \( g_{\alpha_2} \) is a word in \( g_j, \) \( j \geq 2. \) In this case, \( x_1g_\alpha = g_{\alpha_2}(g_1g_2^{-1} \cdots g_{n−s−1})x_{n−s+1}. \)

Now the result follows by first applying Lemma 4.4, and then expanding each \( g_j^{-1} \) in terms of \( g_j \) and \( e_j \) and appealing to the previous part of the proof. \( \square \)

**References**

1. Susumu Ariki, *Representations of quantum algebras and combinatorics of Young tableaux*, University Lecture Series, vol. 26, American Mathematical Society, Providence, RI, 2002, Translated from the 2000 Japanese edition and revised by the author. MR MR1911030 (2004b:17022)
2. Joan S. Birman and Hans Wenzl, *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. **313** (1989), no. 1, 249–273. MR 90g:57004
3. John Enyang, *Cellular bases for the Brauer and Birman-Murakami-Wenzl algebras*, J. Algebra **281** (2004), no. 2, 413–449. MR MR2098377 (2005f:20008)
4. *Specht modules and semisimplicity criteria for Brauer and Birman-Murakami-Wenzl algebras*, J. Algebraic Combin. **26** (2007), no. 3, 291–341. MR 2348099
5. Frederick M. Goodman and Holly Hanschild, *Affine Birman–Wenzl–Murakami algebras and tangles in the solid torus*, Fund. Math. **190** (2006), 77–137. MR MR2232856
6. Frederick M. Goodman and Holly Hauschild Mosley, *Cyclotomic Birman-Wenzl-Murakami algebras I: Freeness and realization as tangle algebras*, preprint (2006, revised 2007), arXiv:math/0612064.

7. , *Cyclotomic Birman-Wenzl-Murakami algebras II: Admissibility relations and freeness*, preprint (2006, revised 2007), arXiv:math/0612065.

8. J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. 123 (1996), no. 1, 1–34. MR MR1376244 (97h:20016)

9. Reinhard Häring-Oldenburg, *Cyclotomic Birman-Murakami-Wenzl algebras*, J. Pure Appl. Algebra 161 (2001), no. 1-2, 113–144. MR MR1834081 (2002c:20055)

10. Vaughan F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103–111. MR MR766964 (86e:57006)

11. Louis H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. 318 (1990), no. 2, 417–471. MR MR958895 (90g:57007)

12. Steffen König and Changchang Xi, *Cellular algebras: inflations and Morita equivalences*, J. London Math. Soc. (2) 60 (1999), no. 3, 700–722. MR MR1753809 (2001a:16020)

13. Andrew Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, University Lecture Series, vol. 15, American Mathematical Society, Providence, RI, 1999. MR MR1711316 (2001a:20006)

14. Hugh Morton and Paweł Traczyk, *Knots and algebras*, Contribuciones Matematicas en homenaje al profesor D. Antonio Plans Sanz de Bremond (E. Martín-Peindador and A. Rodez Usan, eds.), University of Zaragoza, Zaragoza, 1990, pp. 201–220.

15. Hugh Morton and Antony Wassermann, *A basis for the Birman-Wenzl algebra*, Unpublished manuscript (1989, revised 2000), 1–29.

16. Jun Murakami, *The Kauffman polynomial of links and representation theory*, Osaka J. Math. 24 (1987), no. 4, 745–758. MR MR927059 (89c:57007)

17. Vladimir Turaev, *The Conway and Kauffman modules of a solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988), no. Issled. Topol. 6, 79–89, 190. MR MR964255 (90g:57012)

18. Stewart Wilcox and Shona Yu, *The cyclotomic BMW algebra associated with the two string type B braid group*, preprint (2006), arXiv:math/0611518.

19. , in preparation (2008).

20. Changchang Xi, *On the quasi-heredity of Birman-Wenzl algebras*, Adv. Math. 154 (2000), no. 2, 280–298. MR MR1784677 (2001g:20008)

21. Shona Yu, *The cyclotomic Birman–Murakami–Wenzl algebras*, Ph.D. Thesis, University of Sydney (2007).