Vortex Condensates in the Relativistic Self-Dual Maxwell-Chern-Simons-Higgs System

Dongho Chae
Department of Mathematics
Seoul National University
Seoul, 151-742, Republic of Korea
e-mail address: dhchae@math.snu.ac.kr

Namkwon Kim
Basic Science Research Institute &
Department of Mathematics
Seoul National University
Seoul, 151-742, Republic of Korea
e-mail address: nkkim@math.snu.ac.kr

Abstract

The existence of vortex condensates in the self-dual Maxwell-Chern-Simons-Higgs System on a flat torus is proved by the super-sub solution method under the assumption that the total vortex number in a given periodic domain is not too large. We also study the limiting behaviors of the solutions as the Chern-Simons coupling constant goes to some limits. In the Abelian-Higgs limit we find that our solutions strongly converges to the corresponding vortex condensates of the Abelian-Higgs system, while in the Chern-Simons limit the solutions strongly converges to the corresponding vortex condensates of Chern-Simons system.
Introduction

The Abelian-Higgs(or, Maxwell-Higgs) model was proposed by Ginzburg and Landau for phenomenological study of the superconductivity. For the critical value of a parameter in the Lagrangian we obtain a system of self-duality equations, called the Bogomol’nyi equations, describing a static configuration of vortices. This system of equations was studied rigorously by Jaffe-Taubes\[14\] for the case of whole domain of $\mathbb{R}^2$. To explain the periodic array of vortices, called vortex condensates (pioneered by Abrikosov\[1\]), Wang-Yang studied the same equations with the periodic boundary condition in $\mathbb{R}^2$. The vortices in this model has only magnetic charges. To have a theory for vortices having both the electric and the magnetic charges Hong-Kim-Pac\[7\] and Jackiw-Weinberg\[8\] proposed the Chern-Simons model(See also \[5\]).

The existence of topological multivortex solutions of the corresponding self-duality equations in $\mathbb{R}^2$ was proved by \[16\] using the variational argument similarly to \[8\], and later Spruck-Yang\[12\] constructed the solutions using a constructive iteration scheme, and investigated more detailed properties of solutions. For the Chern-Simons system with the periodic boundary condition Caffarelli-Yang constructed a solution in $\mathbb{R}^2$ under the assumption that total vortex number in the given domain is not too big, and Tarantello\[13\] refined the results of \[8\], and proved, in particular, multiplicity of solutions for some range of the Chern-Simons constant. In order to make a ”unified theory” of the Abelian-Higgs and the Chern-Simons models C. Lee, K. Lee and H. Min\[9\](See also \[10\],\[11\] and \[5\]) suggested the self-dual Maxwell-Chern-Simons-Higgs theory using so called $N = 2$ supersymmetry argument. For this theory the existence and various asymptotic properties of topological multivortex solutions were studied by the authors of this paper in \[4\]. In particular, the solutions constructed in \[4\] have the properties that in the limit of the Chern-Simons coupling constant $\kappa$ going to to zero with the electric charge $q$ kept fixed(the Abelian-Higgs limit) the solutions converge strongly to the Abelian-Higgs vortices constructed in \[14\], while in the limit $\kappa, q \to \infty$ with $2q^2/\kappa$ kept fixed the sequence of parametrized solutions becomes “weakly consistent ” to the Chern-Simons equations. Due to this weak convergence of solutions in the Chern-Simons limit the problem of rigorous justification that the self-dual Maxwell-Chern-Simons-Higgs model is really a unified theory of both of the Abelian-Higgs model and the Chern-
Simons model was not solved completely in [4].
In this paper we study the vortex condensates of the self-dual Maxwell-Chern-Simons-Higgs model in a periodic domain of $\mathbb{R}^2$. We construct a solution using the super-sub solution method under the assumption that the total number of the vortices is not too big. We also study the Abelian-Higgs limit and the Chern-Simons limit, and prove that our solutions converges strongly to the solutions of the Abelian-Higgs and the Chern-Simons vortex condensates respectively. Moreover, since our arguments leading to the strong convergence in Chern-Simons limit do not depend on our choice of periodic boundary condition, and also works for the topological solutions in $\mathbb{R}^2$ we resolve the problem of obtaining convergence in this limit for the domain $\mathbb{R}^2$ left unsolved in [4]. We thus complete the rigorous confirmation that the self-dual Maxwell-Chern-Simons-Higgs model is, in some sense, an "interpolation" of the Abelian-Higgs and the Chern-Simons models both in the periodic domain and in the whole domain of $\mathbb{R}^2$.

The organization of this paper is the following: In Section 1 we introduce the Lagrangian for the relativistic self-dual Maxwell-Chern-Simons-Higgs theory, set up a system of semilinear partial differential equations resulting from the corresponding Bogomol’nyi equations, and introduce the notion of admissible solutions following [4]. In Section 2 we construct a subsolution under an assumption of relation among the Chern-Simons coupling constant, electronic charge, and the total vortex number. In Section 3 we prove existence of an admissible solution using an iteration scheme. Finally in Section 4 we prove the strong convergence of our admissible solutions to the Abelian-Higgs and the Chern-Simons solutions in the corresponding limits respectively.

1 Preliminaries

The Lagrangian for the self-dual Maxwell-Chern-Simons-Higgs theory in [4] is

$$\mathcal{L} = (D_\mu \bar{\phi})(D_\mu \bar{\phi})^* + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - \frac{1}{2} (\bar{\partial}_\mu N)^2 - q^2 |\bar{\phi}|^2 - \frac{1}{2} (q |\bar{\phi}|^2 - \pi N - qa^2)^2$$

(1)
Here we are considering the space-time domain \( \Omega \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R} \) with the metric tensor given by
\[
g_{\mu \nu} = g^{\mu \nu} = \text{diag}(1, -1, -1),
\]
where \( \Omega \) is a two dimensional flat torus. \( A = A_\mu(x) dx^\mu \) is the gauge field, \( D_\mu = \partial_\mu - i A_\mu \) is the gauge covariant derivative, \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the curvature tensor, \( \epsilon^{\mu \nu \lambda} \) is the skew symmetric tensor with \( \epsilon^{012} = 1 \), \( \phi = \phi_1 + i \phi_2 \) is a complex valued scalar field, called the Higgs field, \( N \) is the neutral scalar field, \( \kappa > 0 \) is the Chern-Simons coupling constant, \( q > 0 \) is the charge of the electron, and finally \( A \) is the symmetry breaking scale. In (1) the second and the third terms are called the Maxwell term and the Chern-Simons term respectively.

Given \( b > 0 \), let us consider the following scale transformation;
\[
A = b A, \quad x_\mu = \frac{x_\mu}{b}, \quad \phi = b \phi, \quad N = b N, \quad (2)
\]
then the Lagrangian (1) can be rewritten
\[
\mathcal{L} = \frac{\mathcal{L}}{b^4} = (D_\mu \phi)(D_\mu \phi)^* + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} \epsilon^{\mu \nu \lambda} A_\mu F_{\nu \lambda} - \frac{1}{2}(\partial_\mu N)^2 - q^2 N^2 |\phi|^2 - \frac{1}{2}(q|\phi|^2 - \kappa N - q\langle a \rangle)^2, \quad (3)
\]
where \( D_\mu \) and \( F_{\mu \nu} \) are defined similarly to the above, using new variables \( x_\mu \) and new gauge field \( A_\mu \), and \( \Omega \) transforms into \( \Omega \) with the area given by \( |\Omega| = b^2 |\Omega| \). From now on, we set \( b = a \) and denote \( \kappa = \frac{\kappa}{a} \). The variational equations of \( \mathcal{L} \) on \( \Omega \times \mathbb{R} \) are equivalent to those of \( \mathcal{L} \) on \( \Omega \times \mathbb{R} \);
\[
\frac{1}{2} \kappa \epsilon^{\mu \nu \lambda} F_{\nu \lambda} + \partial_\mu F_{\mu \nu} = j^\mu = i(\phi(D_\mu \phi)^* - \phi^* D_\mu \phi) \quad (4)
\]
\[
\partial_\mu \phi^* N + 2q^2 N |\phi|^2 + \kappa (q|\phi|^2 + \kappa N - q) = 0 \quad (5)
\]
\[
D_\mu (D_\mu \phi) + 2q^2 N^2 \phi + q \phi (q|\phi|^2 + \kappa N - q) = 0 \quad (6)
\]
The Gauss law (variational equation with respect to \( A_0 \)) is given by
\[
(-\Delta + 2q^2 |\phi|^2) A_0 = -\kappa F_{12}, \quad (7)
\]
while the static energy is
\[
\mathcal{E} = \int_{\Omega} \left(|D_0 \phi|^2 + |D_j \phi|^2 + \frac{1}{2} F_{j0}^2 + \frac{1}{2} F_{12}^2 + \frac{1}{2}(\partial_j N)^2 + q^2 N^2 |\phi|^2 + \frac{1}{2}(q|\phi|^2 + \kappa N - q)^2 \right) dx. \quad (8)
\]
Since the system is invariant under the following gauge transformation
\[ \phi \rightarrow e^{i\eta} \phi, \quad A \rightarrow A + \nabla \eta, \quad N \rightarrow N \]
for any smooth real valued function \( \eta \), \((\phi, A, N)\) satisfies the 't Hooft boundary conditions:\[15\];
\[
\begin{align*}
&\left. e^{i\eta(x + \tau_k)}(x + \tau_k) = e^{i\eta(x)}(x) \right. \\
&(A + \nabla \eta)(x + \tau_k) = (A + \nabla \eta)(x) \\
&N(x + \tau_k) = N(x), \quad A_0(x + \tau_k) = A_0(x)
\end{align*}
\]
(9)
Here, \( \tau_k, k = 1, 2 \) are the basis of the torus \( \Omega \). This equation leads to the following condition;
\[
\begin{align*}
&\eta_1(1, 0^-) + \eta_1(1, 0^+) + \eta_1(0, 0^+) - \eta_1(0, 1^-) + \eta_2(0^+, 1) \\
&- \eta_2(1^-, 1) + \eta_2(1^-, 0) - \eta_2(0^+, 0) + 2\pi m = 0
\end{align*}
\]
(10)
for an integer \( m \). Here, \((j, k), j, k = 0^\pm, 1^\pm \) in the arguments stands for \( j\tau_1 + k\tau_2 \). From \([8]\), \([9]\), and \([11]\), we can obtain the quantized flux-charge relation as in the pure Chern-Simons system,
\[
\Phi = \int_\Omega F_{12} = 2\pi m, \quad Q = \int_{x^0 = -2} q|\phi|^2 A_0 = \frac{\kappa}{q}\Phi.
\]
The energy \( [8] \) can be rewritten as follows by using the Bogomol’nyi type reduction:
\[
\begin{align*}
\mathcal{E} &= \int_\Omega \left\{ |(D_1 \pm iD_2) \phi|^2 + |D_0 \phi|^2 \mp iq\phi N|^2 + \frac{1}{2}(F_{j=0} \pm \partial_j N)\right. \\
&\quad + \frac{1}{2}|F_{12} \pm (q|\phi|^2 + \kappa N - q)|^2 \right\} dx \mp q \int_R F_{12} dx \\
&\quad \mp \int_\Omega (\nabla \cdot (N \nabla A_0) + i[\partial_1(\phi^*D_2\phi) - \partial_2(\phi^*D_1\phi)]) dx. \quad (11)
\end{align*}
\]
Due to the 't Hooft boundary conditions \([8]\) and \([11]\), the last term in \([11]\) vanish after applying the integration by parts and \( \int_R F_{12} dx = 2\pi m \). Thus we have a lower bound of the energy
\[
\mathcal{E} \geq 2\pi |mq|
\]
and the following Bogomol’nyi equations which saturate the lower bound;
\[
\begin{align*}
A_0 &= \mp N \quad (12) \\
(D_1 \pm iD_2) \phi &= 0 \quad (13) \\
F_{12} \pm (q|\phi|^2 + \kappa N - q) &= 0 \quad (14)
\end{align*}
\]
From now on, we choose the upper sign of the above equations and assume \( m \geq 0 \). Indeed, for the negative \( m \), we can obtain the solution \((\phi, A, N)\) by transforming simply the corresponding solution for \(-m\).

Following [14, 17], we can apply \( \partial^\perp \)–Poincaré lemma to (13) to find that \( \phi \) is analytic and has \( m \) number of zeros counting multiplicities in \( \Omega \). Let \( z_j, j = 1, \ldots, k \) be the zeros of \( \phi \) with multiplicities \( m_j \) respectively. Due to the gauge invariance, we have a degree of freedom of the argument of \( \phi \) up to smooth function and we can take

\[
\phi = e^{u + i \theta}, \quad \theta = \sum_{j=1}^{k} m_j \text{Arg}(z - z_j)
\]

by applying gauge transform if necessary. Now, (7), (12), (13), and (14) reduce to the followings;

\[
\Delta u = 2q^2(e^u - 1) - 2q \kappa A_0 + 4\pi \sum_{j=1}^{k} m_j \delta(z - z_j) \quad (15)
\]

\[
\Delta A_0 = \kappa q(1 - e^u) + (\kappa^2 + 2q^2 e^u) A_0 \quad (16)
\]

Consider the equation

\[
\Delta u_0 = 4\pi \sum_{j=1}^{k} m_j \delta(z - z_j) - \frac{4\pi m}{|\Omega|}
\]

The existence of \( u_0 \) is guaranteed and \( u_0 \) is smooth except \( z = z_j \)'s and behaves like \( m_j \log(z - z_j) \) near \( z = z_j \). We note that \( u_0 \) is determined up to an additive constant. We set \( u = v + u_0 \) in (15) and (16) to have the following equations with the Dirac delta singularities removed;

\[
\Delta v = 2q^2(e^{v+u_0} - 1) - \frac{\kappa}{q} A_0 + \frac{4\pi m}{|\Omega|} \quad (17)
\]

\[
\Delta A_0 = \kappa q(1 - e^{v+u_0}) + (\kappa^2 + 2q^2 e^{v+u_0}) A_0. \quad (18)
\]

If we formally set \( \kappa = 0 \) in (17), then we obtain the equation for the Abelian-Higgs model in the periodic domain [17],

\[
\Delta v = 2q^2(e^{v+u_0} - 1) + \frac{4\pi m}{|\Omega|}. \quad (19)
\]
On the other hand, the Chern-Simons equations with the coupling constant $1/l$ is

$$
\Delta v = 4l^2 e^{v+u_0} (e^{v+u_0} - 1) + \frac{4\pi m}{|\Omega|}.
$$

We denote the solutions of (19) and (20) by $v_a$ and $v_{cs}$ respectively; they are constructed in [17] and [3] respectively. As in [4], we introduce the following definition.

**Definition 1** We call $(v, A_0) \in C^2(\Omega)$ an admissible solution of (17) and (18) if it is a solution of the equations satisfying $v + u_0 \leq 0$ and $A_0 \leq 0$.

We note that the condition of admissibility is equivalent to $|\phi|^2 \leq 1$ and $N \geq 0$ in terms of $\phi$ and $N$, which is physically natural. In the pure Chern-Simons and the Abelian Higgs model this condition follows directly from the corresponding equations, using the maximum principle. This, however, is not the case in our model.

**Proposition 1** $(v, A_0) \in H^1(\Omega)$ is admissible if and only if one of the followings hold:

(i) $v + u_0 \leq 0$

(ii) $A_0 \leq 0$

(iii) $v \leq v_a$

(iv) $\frac{2\kappa}{\kappa}(e^{v+u_0} - 1) \leq A_0$.

**Proof:** From (18), if (i) holds,

$$
\Delta A_0 \geq (\kappa^2 + 2q^2 e^{v+u_0})A_0
$$

Thus, by applying the maximum principle, we have $A_0 \leq 0$, and $(v, A_0)$ is admissible. If (ii) holds, then from (17), we have

$$
\Delta (v + u_0) \geq 2q^2 (e^{v+u_0} - 1) = 2q^2 e^t(v + u_0)
$$

for some $t \in (v + u_0, 0)$ due to the mean value theorem. Thus, (i) holds and $(v, A_0)$ is admissible. If (iii) holds, then $v + u_0 \leq 0$ since $v_a + u_0 \leq 0$ [17]. Now, if (iv) holds, from (18), we obtain

$$
\Delta A_0 = \kappa^2 \left\{ \frac{2\kappa}{\kappa}(1 - e^{v+u_0}) + A_0 \right\} + 2q^2 e^{v+u_0} A_0 \geq 2q^2 e^{v+u_0} A_0,
$$
Thus again, by the maximum principle, we have (ii). Conversely, if \((v, A_0)\) is admissible, then (i) and (ii) hold obviously. To show (iii), we note first
\[
\Delta v_a = 2q^2(e^{v_0+u_0} - 1) + \frac{4\pi m}{|\Omega|}.
\]

By the mean value theorem we have
\[
\Delta(v - v_a) \geq 2q^2(e^{v_0+u_0} - e^{v_a+u_0}) = 2q^2e^t(v - v_a).
\]

Therefore, (iii) holds by the maximum principle. Finally, by direct calculation using the fact \(A_0 \leq 0\), we obtain the estimates
\[
\Delta \left(\frac{q}{\kappa}(1 - e^{v+u_0}) + A_0\right) \leq \frac{-2q^3}{\kappa}e^{v+u_0}(1 - \frac{\kappa}{q}A_0) + \frac{q}{\kappa}(1 - \kappa q + 2q^2e^{v+u_0})A_0
\]
\[
\leq (\kappa q + 2q^2e^{v+u_0})(\frac{q}{\kappa}(1 - e^{v+u_0}) + A_0),
\]
and (iv) holds by the mean value theorem again.

\[\square\]

2 Construction of a subsolution

We say that \((w, A)\) is a subsolution(supersolution) of (17) and (18) if
\[
\Delta w \geq (\leq) 2q^2(e^{w+u_0} - 1 - \frac{\kappa}{q}A) + \frac{4\pi m}{|\Omega|} \quad (21)
\]
\[
\Delta A \geq (\leq) \kappa q(1 - e^{w+u_0}) + (\kappa^2 + 2q^2 e^{w+u_0})A \quad (22)
\]

We define an admissible subsolution(supersolution) similarly to an admissible solution. The pair \((-u_0,0)\) is obviously an admissible supersolution of (17) and (18). To construct a subsolution of (17) and (18), we first define \(f \in C^\infty(\Omega)\) by
\[
f(z) = \begin{cases} 
0, & z \in \bigcup_{j=1}^k B_{\delta}(z_j) \\
\alpha, & z \in \Omega \setminus \bigcup_{j=1}^k B_{2\delta}(z_j) \\
\frac{1}{2}\alpha(1 + \cos(\frac{2\pi |z - z_j|}{\delta})), & z \in B_{2\delta}(z_j) \setminus B_{\delta}(z_j)
\end{cases}
\]
where \(B_{\delta}(z_j) = \{ z \in \Omega \mid |z - z_j| < \delta \}\), and \(z_j\)'s are the zeros of \(\phi\) with multiplicities \(m_j\)'s respectively. Here, we have chosen \(\delta = \)
$\frac{1}{2}\min_{i \neq j}\{|z_i - z_j|, 1\}$ so that each ball $B_{2\delta}(z_j)$ is disjoint with each other, and $\alpha$ is the normalization constant to satisfy

$$\int_{\Omega} f = |\Omega|.$$  

Note that

$$1 \leq \alpha \leq \frac{|\Omega|}{|\Omega| - m \pi \delta^2} \leq 2,$$  

and

$$|\Delta f| \leq \left(\frac{\pi}{\delta}\right)^2 \alpha \quad \text{if } \delta \leq |z - z_j| \leq 2\delta \text{ for some } j$$

$$|\Delta f| = 0 \quad \text{otherwise}$$

As a candidate for a subsolution, we define $(w, A)$ as a pair of smooth solutions of the system;

$$\Delta w = \frac{4\pi m}{|\Omega|} (1 - f), \quad \max_{z \in \Omega} (w(z) + u_0(z)) = u^*$$  

$$A = \frac{q}{\kappa} (e^{w+u_0} - 1) + \frac{4\pi m}{2\kappa q |\Omega|} f,$$  

where $u^* < 0$ is a constant to be determined in Lemma 1 below. The existence of such $w$ is guaranteed by the fact that the integral over $\Omega$ of the righthand side of (24) vanishes, and that $w$ is a solution determined up to an additive constant[2].

**Lemma 1** The pair of functions, $(w, A)$, defined in (24) and (25), is a subsolution of (17) and (18) if

$$\frac{4\pi m}{|\Omega|} \leq \frac{4q^4 e^{u^*} (1 - e^{u^*}) S}{\alpha (\kappa^2 + 4q^2 e^{u^*} + (\frac{\pi}{\delta})^2)},$$

where $S = \delta^2 e^{-C(1 + |\log \delta|) \delta^2}$, $\delta$ is defined in the definition (23), and $C(\Omega^0)$ is a constant depending only on the ratio of the side lengths of $\Omega^0 = \frac{\Omega}{|\Omega|^2} = \{x \in \frac{y}{|\Omega|^2} | y \in \Omega\}$.

**Proof:** From the definitions of $w$ and $A$, (21) is satisfied obviously. To show (22) holds, we start from

$$\Delta A \geq \frac{q}{\kappa} e^{w+u_0} \Delta (w + u_0) + \frac{4\pi m}{2\kappa q |\Omega|} \Delta f$$

$$\geq -\frac{4\pi mq}{\kappa |\Omega|} e^{w+u_0} f + \frac{4\pi m}{2\kappa q |\Omega|} \Delta f \equiv LHS,$$
while
\[
\text{RHS} \equiv \kappa q (1 - e^{w+u_0}) + (\kappa^2 + 2q^2 e^{w+u_0}) A
= \frac{2q^3}{\kappa} e^{w+u_0} (e^{w+u_0} - 1) + \frac{4\pi m}{2\kappa q |\Omega|} (\kappa^2 + 2q^2 e^{w+u_0}) f
\]

Thus, the inequality \( \text{LHS} \geq \text{RHS} \) holds if
\[
\frac{4\pi m}{|\Omega|} \left(-\Delta f + (\kappa^2 + 4q^2 e^{w+u_0}) f\right) \leq 4q^4 e^{w+u_0} (1 - e^{w+u_0})
\]

This, in turn, holds for \( z \in \bigcup_{j=1}^{k} B_{\delta}(z_j) \) by the condition \( \max_{\Omega}(w + u_0) = u^* < 0 \). Thus, from our choice of \( f, (w, A) \) is a subsolution if
\[
\frac{4\pi m}{|\Omega|} \leq \frac{4q^4 T}{\alpha(\kappa^2 + 4q^2 e^{u^*} + (\frac{\pi}{\delta})^2)},
\]

where we set
\[
T = \left(1 - \max_{z \in \Omega \setminus \bigcup_{j=1}^{k} B_{\delta}(z_j)} e^{w+u_0}\right) \min_{z \in \Omega \setminus \bigcup_{j=1}^{k} B_{\delta}(z_j)} e^{w+u_0}.
\]

We are going to estimate \( T \) from now on. Due to our choice of \( w \), we have
\[
1 - \max_{z \in \Omega \setminus \bigcup_{j=1}^{k} B_{\delta}(z_j)} e^{w+u_0} \geq 1 - e^{u^*}.
\]

Now, define \( u_1 \) by
\[
u_1(z) = \begin{cases} \frac{\alpha - \ell}{\alpha} m_j \log |z - z_j|^2 & \text{if } z \in B_{2\delta}(z_j) \\ 0 & \text{otherwise} \end{cases}
\]
and set \( h = w + u_0 - u_1 \). Note that \( u_1 \leq 0 \). We obtain
\[
e^{w+u_0} = e^{u_1} e^h \geq \delta^{2m} e^h \quad \text{on } z \in \Omega \setminus \bigcup_{j=1}^{k} B_{\delta}(z_j),
\]
since \( \delta \leq \frac{1}{2} \), and \( m_j \leq m \). We have \( \max_{\Omega} h \geq \max_{\Omega}(w + u_0) = u^* \), since \( u_1 \leq 0 \). Combining these facts, we have
\[
\min_{\Omega \setminus \bigcup_{j=1}^{k} B_{\delta}(z_j)} e^{w+u_0} \geq \delta^{2m} \exp(u^* - \|h\|_{\text{osc}(\Omega)}),
\]

where we denote
\[
\|h\|_{\text{osc}(\Omega)} = \sup_{x,y \in \Omega} |h(x) - h(y)|
\]
To estimate \( \| h \|_{osc(\Omega)} \), we first calculate

\[
\Delta h = -\frac{4\pi m}{|\Omega|} f + K, \tag{27}
\]

where we set

\[
K = \begin{cases} 
\frac{4m_j \nabla f(z-z_j)}{\alpha |z-z_j|^2} + \frac{2m_j \log |z-z_j| \Delta f}{\alpha} & \text{for } \delta < |z-z_j| < 2\delta \\
0 & \text{otherwise.}
\end{cases}
\]

By direct calculation we obtain

\[
|\nabla f|_{z-z_j} \leq \frac{\alpha \pi}{2\delta^2}, \quad |\Delta f \log |z-z_j|| \leq \alpha \left(\frac{\pi}{\delta}\right)^2 |\log \delta|.
\]

Thus,

\[
\|K\|_{L^2(\Omega)} \leq \sum_{j=1}^{k} \frac{2m_j}{\alpha} \left( 2 \left\| \nabla f \right\|_{L^2(\Omega)} + \|\Delta f \log |z-z_j||_{L^2(\Omega)} \right)
\]

\[
\leq C m \frac{|\log \delta|}{\delta}, \tag{28}
\]

where \( C \) is an absolute constant. Now, using the coordinate transformation, \( x \in \Omega = |\Omega|^\frac{1}{2} y, \ y \in \frac{\Omega}{|\Omega|^\frac{1}{2}} = \Omega^0 \), (27) becomes

\[
\Delta h(y) = -4\pi m f(y) + |\Omega| K(y)
\]

on \( \Omega^0 \) with \( |\Omega^0| = 1 \). Thanks to Morrey’s imbedding inequality, we obtain

\[
\|h\|_{C^\frac{1}{2}(\Omega^0)} \leq C(\Omega^0)\|D^2 h\|_{L^2(\Omega^0)} = C(\Omega^0)\|\Delta h\|_{L^2(\Omega^0)},
\]

where \( C(\Omega^0) \) depends only on \( \Omega^0 \). Therefore,

\[
\|h\|_{osc(\Omega)} \leq C(\Omega^0)(diam(\Omega^0))^\frac{1}{2} \|\Delta h\|_{L^2(\Omega^0)}
\]

\[
\leq C(\Omega^0)4\pi m \alpha + C m \frac{|\log \delta|}{\delta}
\]

\[
\leq C(\Omega^0)(1 + \frac{|\log \delta|}{\delta}) m
\]

by the estimate (28) and the fact \( |\Omega| K(y)\|_{L^2(\Omega^0)} = |K(x)\|_{L^2(\Omega)} \).

Here, we denote various constants depending on the ratio of side-lengths of \( \Omega \) by \( C(\Omega^0) \). Therefore,

\[
T \geq \delta^{2m} e^{u^*}(1 - e^{u^*}) e^{-C(\Omega^0)(1 + \frac{|\log \delta|}{\delta}) m},
\]

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and thus the proof of the lemma is completed by (26).

3 Existence of an admissible solution

Following [4], we define a sequence \((v^i, A^i_0)\), \(i = 0, 1, 2, \ldots\) inductively by

\[
v^0 = -u_0, \quad A^0_0 = 0
\]

(29)

\[
(\Delta - d)v^i = 2q^2(e^{v^{i-1}+u_0} - 1 - \frac{\kappa}{q}A^{i-1}_0) + \frac{4\pi m}{|\Omega|} - dv^{i-1} \quad (30)
\]

\[
(\Delta - d)A^i_0 = \kappa q(1 - e^{v^i+u_0}) + (\kappa^2 + 2q^2e^{v^i+u_0})A^i_0 - dA^{i-1}_0 \quad (31)
\]

for \(d \geq 2q^2\).

**Lemma 2** Let \((w, A)\) be any admissible subsolution pair of (12) - (13), and \((v^i, A^i_0)\) be defined as in (29), (30), and (31). Then \((v^i, A^i_0)\) is monotone decreasing with \(i\) and satisfies

\[
v^i \geq w, \quad A^i_0 \geq A \quad \forall i = 0, 1, 2, \ldots \quad (32)
\]

**Proof:** We will prove only the inequality (32) by the induction argument. The monotonicity can be proved by repetition of the proof of Lemma 1 in [4]. From the admissibility condition, we obviously have \(v^0 \geq w\) and \(A^0_0 \geq A\). Assuming (32) holds for \(i\), we have, by subtracting (21) from (31),

\[
(\Delta - d)(v^{i+1} - w) \leq 2q^2(e^{v^i+u_0} - e^{w+u_0}) - 2\kappa q(A^i_0 - A) - d(v^i - w)
\]

\[
\leq 2q^2(e^{v^i+u_0} - e^{w+u_0}) - d(v^i - w)
\]

\[
\leq (2q^2e^\lambda - d)(v^i - w)
\]

for some \(\lambda \in (w + u_0, v^i + u_0)\), using the mean value theorem. From the monotonicity of \(v^i\), \(v^i + u_0 \leq 1\), which implies

\[
(\Delta - d)(v^{i+1} - w) \leq 0.
\]

Thus, applying the maximum principle, we have \(v^{i+1} \geq w\). Again by subtracting the inequality (22) from (31), we have

\[
(\Delta - d)(A^{i+1}_0 - A) \leq -\kappa q(e^{v^{i+1}+u_0} - e^{w+u_0})
\]
\begin{align*}
+ & (\kappa^2 + 2q^2 e^{v^{t+1}+u_0})(A_0^{t+1} - A) \\
+ & 2q^2(e^{v^{t+1}+u_0} - e^{w+u_0})A - d(A_0^{t} - A) \\
\leq & \ (\kappa^2 + 2q^2 e^{v^{t+1}+u_0})(A_0^{t+1} - A),
\end{align*}

using the result, \( v^{t+1} \geq w \). Thus, applying the maximum principle once more, we obtain \( A_0^{t+1} \geq A \). This completes the proof.

**Theorem 1** Given \( z_j \in \Omega \) and nonnegative integers \( m_j, j = 1, \cdots, k \) with \( \sum_j m_j = m \in \mathbb{Z}^+ \), there exists a constant \( C \) depending only on the ratio of the sidelengths of \( \Omega \) such that there exists a smooth admissible energy minimizer with zeros at \( z = z_j \)'s of multiplicity \( m_j \) if \( m \) satisfies the condition (26).

**Proof:** The existence of such minimizer is guaranteed if we have an admissible solution of (17) and (18). By Lemma 3, it suffices to prove existence of an admissible subsolution pair \((w, A)\). Now, we consider \((w, A)\) defined in (24) and (25) with \( u^* \) chosen later. By Lemma 1, \((w, A)\) is a subsolution if (26) is satisfied. \( w \leq -u_0 \) by the definition (24). Thus, the condition \( A \leq 0 \) is enough to guarantee that \((w, A)\) is admissible which reads

\[
\frac{q}{\kappa}(e^{w+u_0} - 1) + \frac{4\pi m}{2\kappa q|\Omega|} f \leq 0
\]

Thus, it is enough

\[
\frac{4\pi m}{|\Omega|} \leq 2q^2 \min_{\Omega} \frac{1 - e^{w+u_0}}{\alpha} = 2q^2 \frac{1 - e^{u^*}}{\alpha}
\]

The condition (33), in turn, follows immediately from (26).

**Remark:** The solution we have constructed under the assumptions on \( \frac{4\pi m}{|\Omega|} \) in Lemma 1 is maximal among admissible solutions. Thus it describes the most superconducting state.

We now establish some ordering properties among the admissible solutions.

**Theorem 2** Given \( z_j \) and \( m_j, j = 1, \cdots, k \) as in Theorem 1, let \((v^{\kappa_1,q_1}, A_0^{\kappa_1,q_1})\) be an admissible solution for \( \kappa = \kappa_1 \) and \( q = q_1 \), then
(i) \((v_{\kappa_1}, q_{k_2} A_0^{\kappa_1,q_1})\) is a subsolution for \(\kappa = \kappa_2 < k_1\) and \(q = q_1\).

(ii) \((v_{\kappa_1}, q_{k_2} A_0^{\kappa_1,q_1})\) is a subsolution for \(q = q_2 > q_1\) and \(\frac{q_2^2}{\kappa_1} = \frac{q_1^2}{\kappa_2}\).

(iii) \((v_{\kappa_1}, q_{k_2} A_0^{\kappa_1,q_1})\) is a subsolution for \(q = q_2 > q_1\) and \(\kappa = \kappa_2 \leq k_1\).

Proof: Note that (i) and (ii) are special cases of (iii). Thus, it suffices to prove (iii) only. From (17) we have

\[
\Delta v^{\kappa_1,q_1} - 2q_2^2(e^{v^{\kappa_1,q_1} + u_0} - 1 - \frac{\kappa_2}{q_2} A_0^{\kappa_1,q_1}) - \frac{4\pi m}{|\Omega|}
\]

\[
= 2(q_1^2 - q_2^2)(e^{v^{\kappa_1,q_1} + u_0} - 1 - \frac{\kappa_1}{q_1} A_0^{\kappa_1,q_1})
\]

\[
-2(\frac{\kappa_1}{q_1} - \frac{\kappa_2}{q_2})q_2 A_0^{\kappa_1,q_1}
\]

\[
\geq 2q_2^2 A_0^{\kappa_1,q_1}(\frac{\kappa_2}{q_2} - \frac{\kappa_1}{q_1}) \geq 0,
\]

since \(A_0 \leq 0\) and \(e^{v^{\kappa_1,q_1} + u_0} - 1 - \frac{\kappa_2}{q_2} A_0^{\kappa_1,q_1}\). Now, from (18) we obtain for \(q = q_2 \geq q_1\) and \(\kappa = \kappa_2 \leq k_1\),

\[
\Delta \frac{q_2 K_1}{q_1 K_2} A_0^{\kappa_1,q_1} - \kappa_2 q_2(1 - e^{v^{\kappa_1,q_1} + u_0}) + (\kappa_2^2 + 2q_2^2 e^{v^{\kappa_1,q_1} + u_0}) \frac{q_2 K_1}{q_1 K_2} A_0^{\kappa_1,q_1}
\]

\[
= (\frac{\kappa_2^2}{\kappa_2} - \kappa_2 q_2)(1 - e^{v^{\kappa_1,q_1} + u_0})
\]

\[
+(\kappa_2^2 - \kappa_2^2 + 2(q_1^2 - q_2^2)e^{v^{\kappa_1,q_1} + u_0}) \frac{q_2 K_1}{q_1 K_2} A_0^{\kappa_1,q_1}
\]

\[
\geq (\frac{\kappa_2^2}{\kappa_2} - \kappa_2 q_2)(1 - e^{v^{\kappa_1,q_1} + u_0} - \frac{\kappa_1}{q_1} A_0^{\kappa_1,q_1}) \geq 0
\]

This completes the proof of Theorem 2.

Corollary 1 Denoting the maximal admissible solution for \(\kappa, q\) by \((v^{\kappa,q}, A_0^{\kappa,q})\), we have

(i) \(v^{\kappa_1,q} \leq v^{\kappa_1,q} \leq \kappa A_0^{\kappa,q} \leq \kappa' A_0^{\kappa,q} \) for \(\kappa' < \kappa\).

(ii) \(v^{\kappa_1,q} \leq v^{\kappa_1,q} \leq q A_0^{\kappa,q} \leq q A_0^{\kappa,q} \) for \(q > q, \frac{q^2}{\kappa} = \frac{q^2}{\kappa}\).

(iii) \(v^{\kappa_1,q} \leq v^{\kappa_1,q} \leq \kappa q A_0^{\kappa,q} \leq \kappa q A_0^{\kappa,q} \) for \(\kappa' \leq \kappa\) and \(q' \geq q\).
Remark: We note that Theorem 2 and Corollary 1 hold also for $\Omega = \mathbb{R}^2$, since the argument of proof does not depend on our choice of periodic boundary condition, and the maximum principle used above also holds for topological solution in $\Omega = \mathbb{R}^2$. This observation is important in the remark at the end of the next section.

As an application of Theorem 2 we have the following:

**Theorem 3** Given $z_j$ and $m_j$, $j = 1, \cdots, k$ as in Theorem 1, there exist critical constants, $0 < \kappa_c < 1$ and $\infty > q_c > 1$ such that the following holds: An admissible solution of (17) and (18) exists if

$$\frac{4\pi m}{|\Omega|} < \min\{\kappa_c^2 q_c^4, \frac{2q^2}{q_c^2}\},$$

and does not exist if

$$\frac{4\pi m}{|\Omega|} > \min\{\kappa_c^2 q_c^4, \frac{2q^2}{q_c^2}\}. \quad (34)$$

**Proof:** Assume that there exists an admissible solution. Then, integrating (17), we have

$$0 = \int_{\Omega} \Delta v = -2q^2 \int_{\Omega} (1 - e^{u+u_0} + \frac{q}{\kappa} A_0) + 4\pi m$$

Thus, using the fact $A_0 \leq 0$, and $A_0$ is not identically zero, we have

$$\frac{4\pi m}{|\Omega|} < 2q^2. \quad (35)$$

Now integrating $\Delta(v + 2\frac{q}{\kappa} A_0)$, we have

$$0 = \frac{4q^3}{\kappa} \int_{\Omega} e^{u+u_0} A_0 + 4\pi m.$$

Then, using (iv) of the Proposition 1,

$$4\pi m \leq \frac{4q^4}{\kappa^2} \int_{\Omega} e^{u+u_0} (1 - e^{u+u_0}) < \frac{q^4}{\kappa^2} |\Omega|$$

by the inequality $t - t^2 \leq \frac{1}{4}$ for $0 \leq t \leq 1$. Thus, we have

$$\frac{4\pi m}{|\Omega|} < \frac{q^4}{\kappa^2}. \quad (36)$$
Therefore, there are constants $k_c < \infty$ and $q_c > 0$ such that the condition (34) holds. Now, by the condition (26) and Theorem 1, if $\frac{\text{Area}}{16\pi}$ is sufficiently small compared to $\min\{\frac{q_1^2}{\kappa_1^2}, \frac{2q_1}{2}\}$, then there is an admissible solution. Therefore it is enough to prove that if for a certain $\kappa_1$ and $q_1$ there exists an admissible solution, then there exists an admissible solution for all $\kappa < \kappa_1$ and $q > q_1$. By Lemma 2, it suffices to have an admissible subsolution for $\kappa < \kappa_1$ and $q > q_1$. Existence of such a subsolution is established in Theorem 2. This completes the proof of Theorem 3.

4 The Abelian-Higgs and the Chern-Simons limits

In this section we prove the strong convergence of admissible solutions both in the Abelian-Higgs limit and in the Chern-Simons limit. On the one hand, this proves rigorously that our model corresponds to an interpolation of those two models. On the other hand, combining with our existence theorem in the previous section, establishes existence of solutions of the those model equations in different methods from [17] and [3]. We recall that $v_a$ and $v_{cs}^l$ below are the Abelian-Higgs solution and the Chern-Simons solution defined by the equations (19) and (20) respectively.

**Theorem 4** Let $(v^{\kappa,q}, A^{\kappa,q}_0)$ be an admissible solution with zeros $z_j$ $m_j$, $j = 1, \cdots, k$. Then, we have

(The Abelian-Higgs limit);

$v^{\kappa,q} \rightarrow v_a$, and $A^{\kappa,q}_0 \rightarrow 0$ both in $C^\infty(\Omega)$ as $\kappa \rightarrow 0$ with $q$ kept fixed.

The Chern-Simons limit;

$v^{\kappa,q} \rightarrow v_{cs}^l$, and $\frac{q}{2} A^{\kappa,q}_0 \rightarrow e^{\nu_{cs}+u_0} - 1$ both in $H^1(\Omega)$ as $\kappa \rightarrow \infty$ with $\frac{\nu_0^2}{\kappa} = l$ kept fixed.

**Proof:** We first consider the Abelian-Higgs limit. By (i) of Corollary 1 and Proposition 1 we have

$$|A^{\kappa,q}_0| \leq \kappa |A^{q,q_0}_0| \leq \frac{\kappa q}{\kappa_0}$$
Thus $A_{0}^{\kappa,q} \to 0$ in $L^\infty(\Omega)$. Subtracting the equation (19) for $v_a$ from the equation (17) for $v^{\kappa,q}$, we obtain
\[ \Delta(v^{\kappa,q} - v_a) = 2q^2(e^{v^{\kappa,q} + u_0} - e^{v_a + u_0}) - 2\kappa q A_{0}^{\kappa,q}. \] (37)

Thus, $\|\Delta(v^{\kappa,q} - v_a)\|_{L^p(\Omega)} \leq 4q^2|\Omega|^{\frac{1}{p}}$, by Proposition 1. Using the Calderón-Zygmund inequality and the compactness of the embedding $W^{2,p}(\Omega) \hookrightarrow C^1(\Omega)$, $p > 2$, there exists a subsequence $\{v^{\kappa,q}\}$ such that $v^{\kappa,q} \to v_a$ in $C^1(\Omega)$. On the other hand, thanks to the monotonicity (i) of Corollary 1, and the admissibility, $v^{\kappa_0,q} \leq v^{\kappa,q} \leq v_a$, the original sequence $\{v^{\kappa,q}\}$ actually converges to $v_a$ in $C^1(\Omega)$. $C^\infty(\Omega)$-convergence results from (37), applying the bootstrapping argument combined with the standard elliptic regularity.

Now, we consider the Chern-Simons limit. By (ii) of Corollary 1, we have
\[ 0 \geq A_{0}^{\kappa,q} \geq \frac{q_0}{q} A_{0}^{\kappa_0,q_0}, \]
thus $A_{0}^{\kappa,q} \to 0$ in $L^\infty(\Omega)$. Since $v^{\kappa,q}$ is monotone increasing as $\kappa \to \infty$ and $v^{\kappa,q} \leq -u_0$, there exists a pointwise limit, $v^{l}_{cs} \in L^2(\Omega)$ and
\[ \lim_{\kappa,q \to \infty} \int |v^{\kappa,q} - v^{l}_{cs}|^2 = 0 \]
by the Lebesgue dominated convergence theorem. Applying the Lebesgue dominated convergence theorem again, we have $e^{v^{\kappa,q} + u_0} - 1 \to e^{v^{l}_{cs} + u_0} - 1$ in $L^p(\Omega)$ for any $1 \leq p < \infty$. Now, integrating the (17), we have
\[ 2q^2 \int_{\Omega} (e^{v^{\kappa,q} + u_0} - 1 - \frac{\kappa}{q} A_{0}^{\kappa,q}) + 4\pi m = 0 \]
Thus, by the admissibility, $\|e^{v^{\kappa,q} + u_0} - 1 - \frac{\kappa}{q} A_{0}^{\kappa,q}\|_{L^1(\Omega)} = \frac{4\pi m}{2q^2} \to 0$ as $q \to \infty$. Therefore, $\frac{\kappa}{q} A_{0}^{\kappa,q} = \frac{\kappa}{q} A_{0}^{\kappa,q} \to e^{v^{l}_{cs} + u_0} - 1$ in $L^p(\Omega)$ for any $1 \leq p < \infty$. Subtracting (18) after multiplication by $\frac{2q}{\kappa}$ from (17), we have
\[ \Delta(v^{\kappa,q} + \frac{2q}{\kappa} A_{0}^{\kappa,q}) = \frac{4q^3}{\kappa} e^{v^{\kappa,q} + u_0} A_{0}^{\kappa,q} + \frac{4\pi m}{|\Omega|}. \] (38)

Multiplying (18) by a periodic test function $\phi \in C^2(\Omega)$, and taking the limit of the equation, we have
\[ LHS = \lim_{\kappa,q \to \infty} \int (v^{\kappa,q} + \frac{2q}{\kappa} A_{0}^{\kappa,q}) \Delta \phi = \int v^{l}_{cs} \Delta \phi \]
\[
\lim_{\kappa, q \to \infty} \int \frac{4q^3}{\kappa} e^{v^{\kappa,q}+u_0} A_0^{\kappa,q} \phi = 4l^2 \lim_{\kappa, q \to \infty} \left( \int e^{v^{\kappa,q}+u_0} (1 - e^{v^{\kappa,q}+u_0}) \phi \right)
+ \int e^{v^{\kappa,q}+u_0} (1 - e^{v^{\kappa,q}+u_0} + \frac{\kappa}{q} A_0^{\kappa,q}) \phi \right) = 4l^2 \int e^{v^{l}_{cs}+u_0} (1 - e^{v^{l}_{cs}+u_0}) \phi.
\]

Thus \( v^{l}_{cs} \) is a weak solution of the Chern-Simons equation. Using the standard elliptic regularity theory, we obtain \( v^{l}_{cs} \in C^\infty(\Omega) \). Thus, \( v^{l}_{cs} \) is indeed a classical solution of the Chern-Simons equation. To show the convergence in \( H^1(\Omega) \), we first subtract the corresponding Chern-Simons equation from (38) and integrating it after multiplying by \( v^{\kappa,q} - v^{l}_{cs} \) to get

\[
\int |\nabla (v^{\kappa,q} - v^{l}_{cs})|^2 = - \int \frac{4q^3}{\kappa} e^{v^{\kappa,q}+u_0} A_0^{\kappa,q} (v^{\kappa,q} - v^{l}_{cs})
+ \int \Delta (v^{\kappa,q} - v^{l}_{cs}) \frac{q}{\kappa} A_0^{\kappa,q}
\leq C \int |v^{\kappa,q} - v^{l}_{cs}| + C \frac{q}{\kappa} \to 0,
\]

where we used

\[
\left| \frac{4q^3}{\kappa} e^{v^{\kappa,q}+u_0} A_0^{\kappa,q} \right| \leq 4l^2 \left| \frac{\kappa}{q} A_0^{\kappa,q} \right| \leq 4l^2,
\]

and

\[
\left| \Delta (v^{\kappa,q} - v^{l}_{cs}) \right| \leq \left( |\Delta v^{\kappa,q}| + |\Delta v^{l}_{cs}| \right) \leq 16\pi m.
\]

This completes the proof of Theorem 4.

\[ \square \]

**Remark:** As noted in the remark after Corollary 1, since the Theorem 2 and Corollary also hold for \( \Omega = \mathbb{R}^2 \), and the above proof, using the monotonicity properties of Corollary 1 crucially, also work for \( \Omega = \mathbb{R}^2 \) with trivial modifications, we can deduce that the sequence of admissible solutions of our system converges in \( H^1(\mathbb{R}^2) \) to the Chern-Simons solutions in the Chern-Simons limit.
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