TWO MULTIVARIATE CENTRAL LIMIT THEOREMS

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Abstract. In this paper, explicit error bounds are derived in the approximation of rank \( k \) projections of certain \( n \)-dimensional random vectors by standard \( k \)-dimensional Gaussian random vectors. The bounds are given in terms of \( k, n \), and a basis of the \( k \)-dimensional space onto which we project. The random vectors considered are two generalizations of the case of a vector with independent, identically distributed components. In the first case, the random vector has components which are independent but need not have the same distribution. The second case deals with finite exchangeable sequences of random variables.

1. Introduction

The classical central limit theorem says that, under mild conditions, the random variable \( S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \) is approximately Gaussian, for a sequence of \( n \) independent, identically distributed random variables \( X_i \) and \( n \) large. That is, if \( X \) is a random vector of \( \mathbb{R}^n \) with i.i.d. components, then the orthogonal projection of \( X \) in the direction \( (1, 1, \ldots, 1) \) is approximately Gaussian. It is natural to ask for what other directions the asymptotic normality of individual projections of \( X \) in the direction \( (1, \ldots, 1) \) is approximately Gaussian, for a sequence of \( n \) independent, identically distributed random variables implies that for \( X \) as above,

\[
\sup_{t \in \mathbb{R}} \left| \mathbb{P}[\langle \theta, X \rangle \leq t] - \Phi(t) \right| \leq 0.8 \left( \mathbb{E}|X_i|^3 \right) \sum_{j=1}^{n} |\theta_j|^3,
\]

where \( \Phi \) denotes the standard normal distribution function and \( \theta \) is any unit vector in \( \mathbb{R}^n \). Thus \( \langle \theta, X \rangle \) is close to Gaussian as long as \( \sum_{j=1}^{n} |\theta_j|^3 \) is small. Roughly, this happens as long as there aren't a small number of coordinates of \( \theta \) controlling the value of \( \langle \theta, X \rangle \): i.e., the components of \( \theta \) are all of similar size. More generally, one could ask when higher rank projections of \( X \) are close to Gaussian; that is, not only consider the asymptotic normality of individual projections of \( X \), but also asymptotic independence of projections in different directions.

The basic result of this paper is the following quantitative bound on the distance from a rank \( k \) projection of \( X \) to a standard Gaussian random vector in a fixed dimension, where distance is measured by comparing the integrals of \( C^2 \) test functions. In the following theorem, \( C^2(\mathbb{R}^k) \) denotes the space of compactly supported, real-valued functions on \( \mathbb{R}^k \) with two continuous derivatives; \( ||\nabla g(x)||_\infty \) is the maximum length of the gradient of \( g \) and \( |g|_2 = \max_{ij} \left| \frac{\partial^2 g}{\partial x_i \partial x_j}(x) \right| \). The \( \ell_p \) norm of a vector \( \theta \in \mathbb{R}^n \) is denoted \( ||\theta||_p = (\sum_{i=1}^{n} |\theta_i|^p)^{\frac{1}{p}} \).

**Theorem 1.** Let \( X_1, \ldots, X_n \) be independent, identically distributed random variables with \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}X_1^2 = 1 \). Let \( \theta_1, \ldots, \theta_k \) be fixed vectors in \( \mathbb{R}^n \) with \( \theta_i = (\theta_{i1}, \ldots, \theta_{in}) \), such that \( \langle \theta_i, \theta_j \rangle = \delta_{ij} \). Define a random vector \( S_n \in \mathbb{R}^k \) by

\[
S_n^T = \langle \theta_i, X \rangle = \sum_{r=1}^{k} \theta_r^T X_r.
\]

Then for \( g \in C^2(\mathbb{R}^k) \),

\[
|\mathbb{E}g(S_n) - \mathbb{E}g(Z)| \leq \frac{\sqrt{k}}{2} ||\nabla g||_\infty \sqrt{\mathbb{E}X_1^4 - 1} \left( \sum_{i=1}^{k} ||\theta_i||^2_4 \right) + \frac{4}{3} k^2 |g|_2 (\mathbb{E}|X_1|^3) \left( \sum_{i=1}^{k} ||\theta_i||^3_3 \right),
\]

where \( Z \) is distributed as a standard normal random vector in \( \mathbb{R}^k \).
The following example is useful to consider. Suppose the \( \theta_i \) are such that \( |\theta_i^j| = \frac{1}{\sqrt{n}} \) for each \( j \); i.e., the \( \theta_i \) are orthogonal unit vectors in the directions of corners of the hypercube. As long as \( n \) is large and a multiple of 4, there are more such vectors than we can make use of — see [2]. Then the norm expressions in the bound above reduce to

\[
\sum_{i=1}^{k} \|\theta_i\|_2^3 = \frac{k}{\sqrt{n}} \quad \text{and} \quad \sum_{i=1}^{k} \|\theta_i\|_3^3 = \frac{k}{\sqrt{n}}.
\]

Thus for directions chosen in this way, projections of rank \( k \) are close to Gaussian as long as \( k = o\left(n^{1/6}\right) \).

Furthermore, if \( \theta_i \) is random on the sphere, then

\[
\mathbb{E}\|\theta_i\|_2^2 \leq \sqrt{\mathbb{E} \sum_{r=1}^{n} (\theta_i^r)^4} = \sqrt{\frac{3}{n+2}}
\]

and

\[
\mathbb{E}\|\theta_i\|_3^3 = \sum_{r=1}^{n} \mathbb{E}|\theta_i^r|^3 = \frac{n \Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi^3} \Gamma\left(\frac{3}{2} + \frac{3}{2}\right)} \approx \sqrt{\frac{8}{n \pi}}.
\]

(See [3] for a straightforward approach to integrating even-degree monomials over the sphere; the proof given there extends to odd-degree monomials in the absolute values of coordinates as well.) It follows that for the \( \theta_i \) chosen at random (subject to the orthogonality condition), there are absolute constants \( c_1 \) and \( c_2 \) such that for every \( g \in C^2_c(\mathbb{R}^k) \),

\[
\mathbb{E}_g \left( |\mathbb{E}_g(S_{n,\theta}) - \mathbb{E}_g(Z)| \right) \leq \frac{1}{\sqrt{n}} \left[ c_1 k^{3/2} \|\nabla g\|_\infty \sqrt{\mathbb{E}X_1^4} - 1 + c_2 k^3 \|g\|_2 \mathbb{E}|X_1|^3 \right].
\]

This implies that a typical projection of rank \( k \) is close to Gaussian for \( k = o\left(n^{1/6}\right) \).

Theorem 1 is generalized below in two directions. In the following version, the \( X_i \) are assumed to be independent, but need not be identically distributed.

**Theorem 2.** Let \( X_1, \ldots, X_n \) be independent (not necessarily identically distributed) random variables with \( \mathbb{E}X_i = 0 \) and \( \mathbb{E}X_i^2 = 1 \) for each \( i \). Let \( \theta_1, \ldots, \theta_k \) be fixed vectors in \( \mathbb{R}^n \) with \( \theta_i = \langle \theta_i^1, \ldots, \theta_i^n \rangle \), such that \( \langle \theta_i, \theta_j \rangle = \delta_{ij} \). Define a random vector \( S_n \in \mathbb{R}^k \) by

\[
S_n^i = \langle \theta_i, X \rangle = \sum_{r=1}^{n} \theta_i^r X_r.
\]

Then for \( g \in C^2_c(\mathbb{R}^k) \),

\[
|\mathbb{E}_g(S_n) - \mathbb{E}_g(Z)| \leq \frac{\sqrt{K}}{2} \|\nabla g\|_\infty \sqrt{\max_{1 \leq i \leq n} \mathbb{E}X_i^4 - 1} \left( \sum_{i=1}^{k} \|\theta_i\|_4^4 \right) \\
+ \frac{4}{3} k^2 \|g\|_2 \left( \max_{1 \leq i \leq n} \mathbb{E}|X_i|^3 \right) \left( \sum_{i=1}^{k} \|\theta_i\|_3^3 \right),
\]

where \( Z \) is distributed as a standard normal random vector in \( \mathbb{R}^k \).

Theorem 2 can be generalized further to require the vectors \( \theta_i \) only to be linearly independent. Let \( H_g(x) \) be the Hessian matrix of \( g \) at \( x \), and let

\[
\|\|H_g\|_{op}\|_\infty = \sup_x \|H_g(x)\|_{op}
\]

where \( \|A\|_{op} \) is the operator norm of the matrix \( A \). Thus \( \|\|H_g\|_{op}\|_\infty \) is the supremum over \( x \) of the largest eigenvalue (in absolute value) of \( H_g(x) \).

**Theorem 3.** Let \( X_1, \ldots, X_n \) be independent (not necessarily identically distributed) random variables with \( \mathbb{E}X_i = 0 \) and \( \mathbb{E}X_i^2 = 1 \) for each \( i \). Let \( \theta_1, \ldots, \theta_k \) be fixed, linearly independent vectors in \( \mathbb{R}^n \) with \( \theta_i = \langle \theta_i^1, \ldots, \theta_i^n \rangle \), such that \( \|\theta_i\|_2 = 1 \) for each \( i \). Let \( c_{ij} = \langle \theta_i, \theta_j \rangle \). Define a random vector \( S_n \in \mathbb{R}^k \) by

\[
S_n^i = \langle \theta_i, X \rangle = \sum_{r=1}^{n} \theta_i^r X_r
\]
and let $\mathbf{Z}$ be a Gaussian random vector with covariance matrix $C = (c_{ij})_{i,j=1}^n$. Then for $f ∈ C^2_c(\mathbb{R}^k)$,

$$
|\mathbb{E}f(S_n) - \mathbb{E}f(\mathbf{Z})| \leq \frac{1}{2} \sqrt{k} ||\nabla f||_∞ \sqrt{\max_{1 ≤ i ≤ n} \mathbb{E}X_i^2 - 1} \left( \sum_{i=1}^k ||θ_i||^4 \right)
$$

$$
+ \frac{4}{3} \lambda k^2 \left( \mathbb{E}||f||_{op}^2 \max_{1 ≤ i ≤ n} \mathbb{E}|X_i|^3 \right) \left( \sum_{i=1}^k ||θ_i||^3 \right),
$$

(3)

where $\lambda$ is the largest eigenvalue of $C$.

Theorem 5 follows from Theorem 4 using a fairly straightforward linear algebra argument.

Theorem 4 can also be generalized in a different direction, by weakening the independence assumption. In the following version, the sequence $X_1, \ldots, X_n$ is assumed to be exchangeable, i.e., $(X_1, \ldots, X_n) ∼ (X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for any permutation $\sigma$, but $(X_1, \ldots, X_n)$ need not have independent entries. Theorem 4 is not a generalization of Theorem 2 in the strictest sense, as it has the additional technical requirement that $\sum_r θ_i^r = 0$ for each $i$. In what follows, let $|g|_1 = \max_{1 ≤ i ≤ k} \|g_i\|_∞$.

**Theorem 4.** Let $(X_1, \ldots, X_n)$ be a finite exchangeable sequence of random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_i^2 = 1$. Let $\{θ_i\}_{i=1}^k$ be an orthonormal set of vectors in $\mathbb{R}^n$, such that $\sum_{r=1}^n θ_i^r = 0$ for each $i$. Define the random vector $S_n$ in $\mathbb{R}^k$ by

$$
S_n^i = (θ_i, X) = \sum_{r=1}^n θ_i^r X_r.
$$

Then there are absolute constants $a, b, c$ such that for any $g ∈ C^2_c(\mathbb{R}^k)$,

$$
|\mathbb{E}g(S_n) - \mathbb{E}g(Z)| \leq ak|g|_1 \left( \sqrt{\mathbb{E}X_1 X_2 X_3 X_4} + \sqrt{\mathbb{E}(X_1^2 - 1)(X_2^2 - 1)} \right)
$$

$$
+ b|g|_1 \sqrt{\mathbb{E}X_1^4} \left( \sum_{i=1}^k ||θ_i||^4 \right)^2 + ck^2|g|_2 \mathbb{E}|X_1|^3 \left( \sum_{i=1}^k ||θ_i||^3 \right).
$$

(4)

In the case that the entries are independent, the first two error terms vanish; one can interpret their presence as a requirement that the dependence among the $X_i$ must be weak.

In the same way as one obtains Theorem 4 from Theorem 2, one can weaken the orthonormality requirement on the $θ_i$ of Theorem 4 to the requirement that they be linearly independent. This yields the following.

**Theorem 5.** Let $(X_1, \ldots, X_n)$ be an exchangeable sequence of random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_i^2 = 1$. Let $\{θ_i\}_{i=1}^k$ be a linearly independent set of vectors in $\mathbb{R}^n$, such that $\sum_{r=1}^n θ_i^r = 0$ for each $i$. Let $c_{ij} = ⟨θ_i, θ_j⟩$. Define the random vector $S_n$ in $\mathbb{R}^k$ by

$$
S_n^i = (θ_i, X) = \sum_{r=1}^n θ_i^r X_r,
$$

and let $\mathbf{Z}$ be a Gaussian random vector with covariance matrix $C = (c_{ij})_{i,j=1}^k$. Then there are absolute constants $a, b, c$ such that for any $g ∈ C^2_c(\mathbb{R}^k)$,

$$
|\mathbb{E}g(S_n) - \mathbb{E}g(\mathbf{Z})| \leq ak\sqrt{}\|\nabla g\|_∞ \left( \sqrt{\mathbb{E}X_1 X_2 X_3 X_4} + \sqrt{\mathbb{E}(X_1^2 - 1)(X_2^2 - 1)} \right)
$$

$$
+ b\sqrt{}\|\nabla g\|_∞ \sqrt{\mathbb{E}X_1^4} \left( \sum_{i=1}^k ||θ_i||^4 \right)^2 + ck^2\lambda \left( \|H_g\|_{op} \mathbb{E}|X_1|^3 \right) \left( \sum_{i=1}^k ||θ_i||^3 \right),
$$

(5)

where $λ$ is the largest eigenvalue of $C$. 


2. Proofs

Proof of Theorem 3 from Theorem 2. Perform the Gram-Schmidt algorithm on the set of vectors \( \{ \theta_i \} \): since the \( \theta_i \) are linearly independent, there is an invertible matrix \( B \) such that for \( \eta_i := \sum_j B_{ij}^{-1} \theta_j \), \( \langle \eta_i, \eta_j \rangle = \delta_{ij} \).

By assumption,

\[
c_{ij} = \langle \theta_i, \theta_j \rangle
= \left\langle \sum_p B_{ip} \eta_p, \sum_q B_{jq} \eta_q \right\rangle
= \sum_{p,q} B_{ip} B_{jq} \langle \eta_p, \eta_q \rangle
= \sum_p B_{ip} B_{jp}.
\]

Thus \( BB^T = C \).

Now, let \( f : \mathbb{R}^k \to \mathbb{R} \) and define \( h : \mathbb{R}^k \to \mathbb{R} \) by \( h(x) = f(Bx) \). Define \( S_n^i := \sum_{r=1}^n \eta_i^r X_r \). Then

\[
(BS_n)_i = \sum_j B_{ij} \langle \eta_j, X \rangle = \left\langle \sum_j B_{ij} \eta_j, X \right\rangle = \langle \theta_i, X \rangle = S_n^i,
\]

and so \( h(S_n) = f(BS_n) = f(S_n) \). If \( Z \) is a standard Gaussian random vector, then \( h(Z) = f(BZ) \) and \( BZ \) is a Gaussian random vector with covariance matrix \( BB^T = C \). Applying Theorem 2 for this test function \( h \) to the random vector \( S_n \) yields

\[
|Ef(S_n) - Ef(BZ)| = |Eh(S_n) - Eh(Z)|
\leq \frac{1}{2} k^{1/2} \| \nabla h \|_{\infty} \sqrt{\max_{1 \leq j \leq n} E X_j^4} - 1 \left( \sum_{i=1}^k \| \theta_i \|^3 \right)^2
+ \frac{4}{3} k^2 |h|_2 \left( \max_{1 \leq j \leq n} E |X_j|^3 \right) \left( \sum_{i=1}^k \| \theta_i \|^3 \right),
\]

so it remains to estimate \( \| \nabla h \|_{\infty} \) and \( |h|_2 \).

To estimate \( \| \nabla h \|_{\infty} \), first note that if \( B \) is viewed as an operator on \( \mathbb{R}^k \), then its operator norm is its largest singular value. That is,

\[
\|Bx\| \leq \sqrt{\lambda} \|x\|
\]

where \( \lambda \) is the largest eigenvalue of \( C \). Since \( \| \nabla h \|_{\infty} \) is the Lipschitz constant of \( h \), it follows that \( \| \nabla h \|_{\infty} \leq \sqrt{\lambda} \| \nabla f \|_{\infty} \).

To estimate \( |h|_2 \), note that

\[
\frac{\partial^2 h}{\partial x_i \partial x_p} = \sum_{r,s} B_{rp} B_{st} \frac{\partial^2 f}{\partial x_r \partial x_s}(Bx) = (B^T H_f(Bx)B)_{ip},
\]

where \( (H_f)_{ij} = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{ij} \) is the Hessian of \( f \). Now,

\[
|B^T HB_{ip}| = |\langle B^T HB_{ip}, e_p \rangle| = |\langle HBe_p, Be_p \rangle| \leq \|H\|_{op} \|B\|_{op}^2.
\]

As stated above, \( \|B\|_{op}^2 \leq \lambda \), the largest eigenvalue of \( C \); this completes the proof.

\( \square \)

Remarks:

1. To obtain a bound in Theorem 3 which doesn’t involve the operator norm of the Hessian of \( f \), one can estimate the operator norm by the Hilbert-Schmidt norm:

\[
\|H_f(Bx)\|_{op} \leq \sqrt{\sum_{r,s=1}^k \left( \frac{\partial^2 f}{\partial x_r \partial x_s}(Bx) \right)^2} \leq k |f|_2.
\]
(2) The proof of Theorem 3 from Theorem 4 is exactly the same as the proof above.

The proofs of Theorems 2 and 4 are applications of the following abstract normal approximation theorem, proved in [1].

**Theorem 6.** Let $X$ and $X'$ be two random vectors in $\mathbb{R}^k$ such that $\mathcal{L}(X) = \mathcal{L}(X')$, and let $Z \in \mathbb{R}^k$ be a standard Gaussian random vector. Suppose there is a constant $\lambda$ and random variables $E_{ij}$ such that

1. $\mathbb{E} [X_i^2 - X_j^2 \mid X] = -\lambda X_i$
2. $\mathbb{E} [(X_i^2 - X_j^2)(X_j' - X_j) \mid X] = 2\lambda \delta_{ij} + E_{ij}$.

Then if $g \in C^2(\mathbb{R}^k)$,

$$|\mathbb{E} g(X) - \mathbb{E} g(Z)| \leq \min \left\{ \frac{|g'|}{2\lambda} \sum_{i,j} \mathbb{E}|E_{ij}|, \frac{\sqrt{k}}{2\lambda} \mathbb{E} \left( \sum_{i,j} E_{ij}^2 \right)^{1/2} \right\} + \frac{k^2|g|_2}{6\lambda} \sum_i |X_i' - X_i|^3.$$  

In the contexts in which Theorem 3 is applied below, the pair of vectors $X, X'$ will in fact be constructed not only to have the same law but to be exchangeable.

**Proof of Theorem 2.** From the random vector $S_n$ with $S_n^i = (\theta_i, X)$, make an exchangeable pair of vectors $(S_n, S_n')$ by choosing $I \in \{1, \ldots, n\}$ at random, independent of $\{X_i\}$, and replacing $X_I$ by an independent copy $X_I'$. That is,

$$(S_n')^i = S_n^i - \theta_i^t X_I + \theta_i^t X_I'.$$

Then

$$\mathbb{E} [(S_n')^i - S_n^i \mid \{X_j\}_{j=1}^n] = -\frac{1}{n} \sum_{j=1}^n \theta_i^t X_j = -\frac{1}{n} S_n^i,$$

thus the proportionality condition of Theorem 1 holds with $\lambda = \frac{1}{n}$.

Next,

$$\mathbb{E} \left[ ((S_n')^i - S_n^i)^2 \mid \{X_j\}_{j=1}^n \right] = \mathbb{E} \left[ \left( \theta_i^t \right)^2 (X_I - X_I')^2 \mid \{X_j\}_{j=1}^n \right]$$

$$= \frac{1}{n} \sum_{\ell=1}^n \mathbb{E} \left[ \left( \theta_i^t \right)^2 (X_\ell - X_\ell')^2 \mid \{X_j\}_{j=1}^n \right]$$

$$= \frac{1}{n} \sum_{\ell=1}^n (\theta_i^t)^2 (X_\ell^2 + 1)$$

$$= \frac{2}{n} \sum_{\ell=1}^n (\theta_i^t)^2 (X_\ell^2 - 1),$$

since $\sum_{\ell}(\theta_i^t)^2 = 1$. Thus one can take $E_{ii} = \frac{1}{n} \sum_{\ell=1}^n (\theta_i^t)^2 (X_\ell^2 - 1)$. If $i \neq j$,

$$\mathbb{E} \left[ ((S_n')^i - S_n^i) (S_n')^j - S_n^j \mid \{X_j\}_{j=1}^n \right] = \mathbb{E} \left[ \theta_i^t \theta_j^t (X_I - X_I')^2 \mid \{X_j\}_{j=1}^n \right]$$

$$= \frac{1}{n} \sum_{r=1}^n \theta_i^r \theta_j^r \mathbb{E} \left[ (X_r - X_r')^2 \mid \{X_j\}_{j=1}^n \right]$$

$$= \frac{1}{n} \sum_{r=1}^n \theta_i^r \theta_j^r (X_r^2 + 1)$$

$$= \frac{1}{n} \sum_{r=1}^n \theta_i^r \theta_j^r (X_r^2 - 1),$$
where the last line follows because \( \sum_r \theta_i^r \theta_j^r = 0 \). Thus \( E_{ij} = \frac{1}{n} \sum_{r=1}^n \theta_i^r \theta_j^r (X_r^2 - 1) \) for all \( i \) and \( j \). Now,

\[
\sum_{i,j=1}^k E_{ij}^2 = \frac{1}{n^2} \sum_{i,j=1}^k \sum_{\ell,r=1}^n \theta_i^\ell \theta_j^\ell \theta_i^r \theta_j^r (X_\ell^2 - 1)(X_r^2 - 1).
\]

Making use of the facts that \( E(X_\ell^2 - 1) = 0 \) and \( X_\ell \) and \( X_r \) are independent for \( \ell \neq r \), together with the Cauchy-Schwarz inequality yields

\[
E \left[ \sum_{i,j=1}^k E_{ij}^2 \right] = \frac{1}{n} E \left[ \sum_{i,j=1}^k \sum_{\ell,r=1}^n \theta_i^\ell \theta_j^\ell \theta_i^r \theta_j^r (X_\ell^2 - 1)(X_r^2 - 1) \right]
\leq \frac{1}{n} \max_{1 \leq i \leq n} (E X_i^4 - 1) \left( \sum_{i,j=1}^k \sum_{\ell=1}^n (\theta_i^\ell)^2 (\theta_j^\ell)^2 \right).
\]

Now,

\[
\sum_{\ell=1}^n (\theta_i^\ell)^2 (\theta_j^\ell)^2 \leq \sqrt{\sum_{\ell=1}^n (\theta_i^\ell)^4} \sqrt{\sum_{\ell=1}^n (\theta_j^\ell)^4} = \|\theta_i\|_4^2 \|\theta_j\|_4^2,
\]

and so

\[
E \left[ \sum_{i,j=1}^k E_{ij}^2 \right] \leq \frac{1}{n} \left( \sum_{i=1}^k \|\theta_i\|_4^2 \right) \sqrt{\max_{1 \leq i \leq n} E X_i^4 - 1}.
\]

Finally,

\[
E[(S_n')^i - S_n^i]^3 = \frac{1}{n} \sum_{j=1}^n |\theta_j^i|^3 E|X_j - X_j'|^3
\leq \frac{8}{n} \sum_{j=1}^n |\theta_j^i|^3 E|X_j|^3
\leq \frac{8\|\theta_i\|_3^3}{n} \max_{1 \leq i \leq n} E|X_i|^3,
\]

where the second line follows from the \( L_3 \) triangle inequality and the fact that \( X_j' \) has the same distribution as \( X_j \). The statement of the theorem is now an immediate consequence of Theorem 6. \( \square \)

**Proof of Theorem 6** Starting from \( S_n \), make an exchangeable pair of random vectors as follows. Choose a pair of indices \( I \neq J \) at random from \( \{1, \ldots, n\} \). Let \( \tau = (IJ) \), the permutation on \( n \) letters that transposes \( I \) and \( J \), and let

\[
X' = (X_{\tau(1)}, \ldots, X_{\tau(n)}).
\]

Then

\[
(S_n')^i = \theta_i X' = S_n^i + \theta_i^I X_I - \theta_i^J X_J + \theta_i^I X_I - \theta_i^J X_J,
\]

that is,

(7) \[
(S_n')^i - S_n^i = (\theta_i^I - \theta_i^J)(X_J - X_I).
\]
Let \( \sum' \) denote summing over distinct indices. Then

\[
\mathbb{E} \left[ (S_n^i)^2 - S_n^i \mid X \right] = \frac{1}{n(n-1)} \sum'_{r,s} (\theta^r_i X_s - \theta^r_i X_r + \theta^s_i X_r - \theta^s_i X_s)
\]

\[
= -\frac{2}{n-1} \sum_r \theta^r_i X_r
\]

\[
= -\frac{2}{n-1} S_n^i,
\]

where the second line follows as \( \sum_r \theta^r_i = 0 \). Thus the proportionality condition of Theorem 7 holds with \( \lambda = \frac{2}{n-1} \).

The next step is to compute and bound the error terms \( E_{ij} \). First, consider \( i = j \). From (7),

\[
\mathbb{E} \left[ (S_n^i)^2 - S_n^i \mid X \right] = \frac{1}{n(n-1)} \sum'_{r,s} (\theta^r_i - \theta^s_i)^2 (X_s - X_r)^2
\]

\[
= \frac{1}{n(n-1)} \sum_{r,s} (\theta^r_i - \theta^s_i)^2 (X_s - X_r)^2
\]

\[
= \frac{1}{n(n-1)} \left[ 2 \sum_{r,s} (\theta^r_i)^2 X_s^2 - 4 \sum_{r,s} (\theta^r_i)^2 X_r X_s + 2n \sum_r (\theta^r_i)^2 X_r^2
\right.

\[
- 4 \sum_{r,s} \theta^r_i \theta^s_i X_s^2 + 4 \sum_{r,s} \theta^r_i X_r \theta^s_i X_s
\left.
\right]
\]

\[
= \frac{2}{n(n-1)} \left[ \sum_r X_r^2 + n \sum_r (\theta^r_i)^2 X_s^2 - 4 \left( \sum_r (\theta^r_i)^2 X_r \right) \left( \sum_s X_s \right) + 2(S_n^i)^2 \right]
\]

\[
= \frac{4}{n-1} + \frac{2}{n(n-1)} \left[ \left( \sum_r X_r^2 - n \right) + n \left( \sum_r (\theta^r_i)^2 X_r^2 - 1 \right) - 2 \left( \sum_r (\theta^r_i)^2 X_r \right) \left( \sum_s X_s \right) + 2(S_n^i)^2 \right].
\]

The error \( E_{ii} \) can thus be taken to be

\[
E_{ii} = \frac{2}{n(n-1)} \left[ \sum_r (X_r^2 - 1) + n \sum_r (\theta^r_i)^2 (X_s^2 - 1) - 2 \left( \sum_r (\theta^r_i)^2 X_r \right) \left( \sum_s X_s \right) + 2(S_n^i)^2 \right].
\]

To bound \( \mathbb{E}[E_{ii}] \), first apply the triangle inequality and treat each of the four terms above separately. First,

\[
\mathbb{E} \left| \sum_r (X_r^2 - 1) \right| \leq \sqrt{\mathbb{E} \left( \sum_{r,s} (X_r^2 - 1)(X_s^2 - 1) \right)}
\]

\[
\leq n|\mathbb{E}(X_1^2 - 1)| + \sqrt{(n(n-1)|\mathbb{E} [(X_1^2 - 1)(X_2^2 - 1)]|}.
\]

Next,

\[
\mathbb{E} \left| \sum_r (\theta^r_i)^2 (X_s^2 - 1) \right| \leq \sqrt{\mathbb{E} \left( \sum_{r,s} (\theta^r_i)^2 (\theta^s_i)^2 (X_s^2 - 1)(X_r^2 - 1) \right)}
\]

\[
\leq \sqrt{|\mathbb{E}[X_1^2 - 1]| \sum_r (\theta^r_i)^4 + |\mathbb{E} [(X_1^2 - 1)(X_2^2 - 1)]| \sum_{r,s} (\theta^r_i)^2 (\theta^s_i)^2}
\]

\[
\leq \|\theta_i\|_4 \sqrt{|\mathbb{E}[X_1^4 - 1]| + \sqrt{|\mathbb{E} [(X_1^2 - 1)(X_2^2 - 1)]|}}.
\]
Note also that the normalization is such that \( \mathbb{E}(S_i^j)^2 = 1 \), thus only the second-last term remains to be estimated. As before, start by applying the Cauchy-Schwarz inequality:

\[
\mathbb{E} \left| \sum_{r,s} (\theta_i^r)^2 X_r X_s \right| \leq \sqrt{\mathbb{E} \sum_{k,l,r,s} (\theta_i^k)^2 (\theta_i^l)^2 X_k X_l X_r X_s}. 
\]

Now, breaking up the sum by the equality structure of the indices and using the exchangeability gives

\[
\sqrt{\mathbb{E} \sum_{k,l,r,s} (\theta_i^k)^2 (\theta_i^l)^2 X_k X_l X_r X_s}
\leq (n^2|\mathbb{E}[X_1 X_2 X_3 X_4]| + n^2\|\theta_i\|_2^2 |\mathbb{E}[X_1^2 X_2 X_3]| + n\|\theta_i\|_2^2 |\mathbb{E}[X_1^3 X_2]| + 2n\|\theta_i\|_2^2 |\mathbb{E}[X_1^3 X_2]| + 2n |\mathbb{E}[X_1^2 X_2 X_3]| + 2 |\mathbb{E}[X_1^3 X_2]|)^{\frac{1}{2}}
\]

\[
\leq n \left[ \sqrt{\mathbb{E}|X_1 X_2 X_3 X_4|} + \|\theta_i\|_2^2 \sqrt{\mathbb{E}|X_1^2 X_2 X_3|} \right] + \sqrt{n} \left[ \|\theta_i\|_2^2 \left( \sqrt{2 \mathbb{E}|X_1^3 X_2|} + \sqrt{\mathbb{E}|X_1^2 X_2|^2} \right) + \sqrt{3 \mathbb{E}|X_1^2 X_2 X_3|} \right] + \left[ \|\theta_i\|_2^2 \left( \sqrt{\mathbb{E}|X_1^4|} + \sqrt{2 \mathbb{E}|X_1^3 X_2|} + \sqrt{2 \mathbb{E}|X_1^2 X_2|^2} \right) \right] 
\]

By Hölder’s inequality and the exchangeability of the sequence,

\[
|\mathbb{E}|X_1^3 X_2| \leq \mathbb{E}|X_1^4|, \quad |\mathbb{E}|X_1^2 X_2|^2 \leq \mathbb{E}|X_1^4|, \quad \text{and} \quad |\mathbb{E}|X_1^2 X_2 X_3| \leq \mathbb{E}|X_1^4|.
\]

Also,

\[
\frac{1}{\sqrt{n}} \leq \|\theta_i\|_2^2 \leq 1
\]

since \( \|\theta_i\|_2 = 1 \). Thus there are constants \( c, c' \) such that

\[
\sqrt{\mathbb{E} \sum_{k,l,r,s} (\theta_i^k)^2 (\theta_i^l)^2 X_k X_l X_r X_s} \leq c n \sqrt{\mathbb{E}|X_1 X_2 X_3 X_4|} + c' n \|\theta_i\|_2^2 \sqrt{\mathbb{E}|X_1^4|}.
\]

All together, this shows that there are constants \( c_1, c_2, c_3 \) such that

\[
(8) \quad \mathbb{E}|E_{ii}| \leq \frac{c_1}{n} \sqrt{\mathbb{E}|X_1 X_2 X_3 X_4|} + \frac{c_2}{n} \sqrt{|\mathbb{E} (X_1^2 - 1) (X_2^2 - 1)|} + \frac{c_3}{n} \|\theta_i\|_2^2 \sqrt{\mathbb{E}|X_1^4|}.
\]

Next, consider \( E_{ij} \) for \( i \neq j \). From (7),

\[
\mathbb{E} \left[ (S_n^i)^2 - S_n^i \right] = \mathbb{E} \left[ \left( \theta_i^j - \theta_n^j \right) \left( \theta_j^i - \theta_n^i \right) (X_j - X_i)^2 \right] = \frac{1}{n(n - 1)} \sum_{k,\ell} (\theta_i^k - \theta_n^k) (\theta_j^\ell - \theta_n^\ell) (X_k - X_\ell)^2.
\]

Expanding this expression and making use of the facts that \( \langle \theta_i, \theta_j \rangle = 0 \) and \( \sum_r \theta_i^r = \sum_r \theta_j^r = 0 \) gives that the right-hand side is equal to

\[
\frac{2}{n(n - 1)} \left[ n \sum_k \theta_i^k \theta_j^k X_k^2 - 2 \left( \sum_k \theta_i^k \theta_j^k X_k \right) \left( \sum_\ell X_\ell \right) + 2 S_n^i S_n^j \right] =: E_{ij}.
\]
As in the case of $E_{ii}$, to estimate $\mathbb{E}|E_{ij}|$, apply the triangle inequality to the expression above and estimate each term separately. First,

$$
\mathbb{E}\left| \sum_r \theta_i^r \theta_j^r X_r X_s \right| \leq \sqrt{\mathbb{E}\sum_r (\theta_i^r)^2 (\theta_j^r)^2 \sum_s (\theta_i^s)^2 (\theta_j^s)^2}
$$

$$
= \sqrt{\langle \mathbb{E}X_1^4 - \mathbb{E}X_i^2 \mathbb{E}X_j^2 \rangle \sum_r (\theta_i^r)^2 (\theta_j^r)^2}
$$

$$
\leq \|\theta_i\|_4 \|\theta_j\|_4 \sqrt{\mathbb{E}X_1^4},
$$

where the second line follows from exchangeability and the fact that $\langle \theta_i, \theta_j \rangle = 0$. By the normalization, $\|E_{ii} \mathbb{E}S_i^2\| \leq 1$, and it remains to estimate the middle term. As before, this is done by applying the Cauchy-Schwarz inequality, breaking up the sum by equality structure of the indices, and using exchangeability and the orthonormality conditions on the $\theta_i$ to simplify the result. This process yields

$$
\mathbb{E}\left| \sum_{r,s} \theta_i^r \theta_j^s X_r X_s \right| \leq \sqrt{\sum_r (\theta_i^r)^2 (\theta_j^r)^2 \left[ n^2 \mathbb{E}X_1^4 + n^2 \|\mathbb{E}X_i^2 \mathbb{E}X_j^2 \| + 2n \mathbb{E}X_i^2 \mathbb{E}X_j^2 + n \mathbb{E}X_i^2 \mathbb{E}X_j^2 \right]}
$$

$$
\leq \|\theta_i\|_4 \|\theta_j\|_4 c n \sqrt{\mathbb{E}X_1^4},
$$

for some constant $c$. It follows that there is another constant $a$ such that for all $i \neq j$,

$$
\mathbb{E}|E_{ij}| \leq \frac{a}{n} \|\theta_i\|_4 \|\theta_j\|_4 \sqrt{\mathbb{E}X_1^4}.
$$

It now follows that

$$
\frac{1}{n} \sum_{i,j=1}^k \mathbb{E}|E_{ij}| \leq c k \sqrt{\mathbb{E}X_1^4 X_2 X_3 X_4} + c' k \sqrt{\left( \mathbb{E}(X_1^2 - 1)(X_2^2 - 1) \right)} + c'' \sqrt{\mathbb{E}X_1^4 \sum_{i} \|\theta_i\|_4^2}.
$$

To complete the application of Theorem 3, it remains to estimate $\mathbb{E}|(S_i^1)^j - S_i^n|^3$. By (7),

$$
\mathbb{E}|(S_i^1)^j - S_i^n|^3 = \mathbb{E}|(\theta_i^j - \theta_i^n)(X_j - X_1)|^3
$$

$$
= \frac{1}{n(n-1)} \sum_{r,s} \mathbb{E}|(\theta_i^r - \theta_i^s)(X_s - X_r)|^3
$$

$$
\leq \frac{8 \mathbb{E}|X_1|^3}{n(n-1)} \sum_{r,s} |\theta_i^r - \theta_i^s|^3
$$

$$
\leq \frac{8 \mathbb{E}|X_1|^3}{n(n-1)} \sum_{r,s} (|\theta_i^r| + |\theta_i^s|)^3
$$

$$
= \frac{8 \mathbb{E}|X_1|^3}{n(n-1)} \left[ 2n \|\theta_i\|_3^3 + 6 \|\theta_i\|_1 \right].
$$

Here the third line follows from the $L_3$ triangle inequality and exchangeability, and the last line by expanding the cube and using the normalization condition on $\theta_i$. Note that, by Hölder’s inequality and the fact that $\|\theta_i\|_1 = 1$ for each $i$,

$$
\|\theta_i\|_1 \leq \sqrt{n}
$$

and

$$
\|\theta_i\|_3^3 \geq \frac{1}{\sqrt{n}},
$$

thus the second term above can be absorbed into the first with a change in constant. It follows that

$$
\frac{1}{n} \sum_i \mathbb{E}|(S_i^1)^j - S_i^n|^3 \leq c \mathbb{E}|X_1|^3 \sum_{i=1}^k \|\theta_i\|_3^3.
$$
This accounts for the remaining error term in Theorem 3.

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