COMPACT MINIMAL HYPERSURFACES WITH INDEX ONE IN THE COMPLEX PROJECTIVE SPACE

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Abstract. Let $\bar{M}$ be a compact minimal hypersurface in $\mathbb{C}P^{r-1}$. We prove a classification theorem of hypersurfaces of index one in $\mathbb{C}P^{r-1}$. More exactly, we prove that under the above conditions, $\bar{M}$ is a minimal Clifford hypersurface which includes the minimal geodesic spheres.

1. Introduction

Let $M$ be a minimal hypersurface of a Riemannian manifold. The Morse index is the number of negative eigenvalues of the Jacobi operator $J_M$ counting multiplicity. This notion is important since it gives us information about how unstable the hypersurface is. Therefore, this leads us to want to classify minimal hypersurfaces in known spaces according to their index. In the case of the real projective space $\mathbb{R}P^{r-1}$, do Carmo, Ritoré and Ros proved

**Theorem 1.1.**\cite{3} The only compact two-sided minimal hypersurfaces with index one in the real projective space $\mathbb{R}P^{r-1}$ are the totally geodesic spheres and the minimal Clifford hypersurfaces.

This theorem inspired us to give a generalization in the complex projective space $\mathbb{C}P^{r-1}$. The main theorem of this paper is

**Theorem 1.2.** The only compact minimal hypersurfaces with index one in the complex projective space $\mathbb{C}P^{r-1}$ are the minimal Clifford hypersurfaces. In particular they can be minimal geodesic spheres.

The classification of minimal hypersurfaces in Riemannian manifolds according to their Morse index has played an important role in the development of geometry. The theorem of Urbano\cite{11} stated that the index of a compact orientable nontotally geodesic minimal surface $\Sigma$ in $S^3$ is greater or equal to 5, and the equality holds if and only if $\Sigma$ is the Clifford torus. This last theorem was fundamental in the proof of the Willmore conjecture by Fernando Codá Marques and André Neves\cite{8}.

Theorems 1.1 and 1.2 can be used to classify minimal hypersurfaces obtained by the min-max technique applied to one-parameter sweepouts (see section 4 for more details).

**Corollary 1.3.** Let $\Sigma_i$ be the min-max hypersurface in $\mathbb{R}P^i$, $i \in \{3, 4, 5, 6, 7\}$. Then $\Sigma_i$ is a minimal Clifford hypersurface and
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2. Preliminary concepts

In this first section we will give some important definitions and basic properties given in [1] and [3]. Let \( S^N_R \) be the euclidean \( N \)-sphere of radius \( R \). The sphere \( S^1 \) acts freely on \( S^{2r-1} \subset \mathbb{R}^{2r} \) by left scalar multiplication. Then we define
\( \mathbb{C}^{r-1} := S^{2r-1}/S^1 \), i.e. the orbit space. The manifold \( \mathbb{C}^{r-1} \) is called the complex projective space of real dimension \( 2r - 2 \). Action of \( S^1 \) induces the usual Riemannian submersion

\[
\Pi : S^{2r-1} \to \mathbb{C}^{r-1},
\]

where \( \Pi(p) \) is the orbit of \( p \).

**Remark 2.1.** We will identify \( \mathbb{R}^r \) with \( \mathbb{C}^r \) in the following way:

\[(2.1) \quad c = (c_1^1, \ldots, c_r^1, c_1^2, \ldots, c_r^2) = (c_1^1 + ic_1^2, \ldots, c_r^1 + ic_r^2).\]

**Notation:** If \( c = (c_1^1 + ic_1^2, \ldots, c_r^1 + ic_r^2) \), \( d = (d_1^1 + id_1^2, \ldots, d_r^1 + id_r^2) \),

\[(2.2) \quad \langle c, d \rangle := \sum_{j=1}^r (c_j^1 + ic_j^2)(d_j^1 - id_j^2)\]

\[(2.3) \quad (x + iy) \cdot c = ((x + iy)(c_1^1 + ic_1^2), \ldots, (x + iy)(c_r^1 + ic_r^2))\]

and

\[i \cdot c = \tilde{c}\]

where the product in each component is the scalar product between complex numbers.

**Definition 2.2.** A Clifford hypersurface is the set of points \( S_{n_1}^{n_1} \times S_{n_2}^{n_2} \), such that \( n_1 \in \mathbb{Z}^+ \) and \( R_1 \in \mathbb{R}^+ \), where \( n_1 + n_2 = 2r - 2 \) and \( R_1^2 + R_2^2 = 1 \). When \( n_1 \) and \( n_2 \) are odd we will call also as Clifford hypersurface the set \( \Pi(S_{n_1}^{n_1} \times S_{n_2}^{n_2}) \).

**Remark 2.3.** The manifold \( S_{n_1}^{n_1} \times S_{n_2}^{n_2} \) is minimal if and only if \( n_1 R_2^2 = n_2 R_1^2 \).

**Remark 2.4.** Let \( U_\rho(\cdot) \) be the tubular neighborhood of radius \( \rho \), \( c = \cos(\rho) \) and \( s = \sin(\rho) \). Since

\[
S_1^c \times S_3^{2r-3} = \partial U_\rho(S^1) = \partial U_\rho(\Pi^{-1}(\mathbb{C}^0)) = \Pi^{-1}(\partial U_\rho(\mathbb{C}^0)),
\]

then the hypersurface \( \Pi(S_1^c \times S_3^{2r-3}) \) corresponds to a geodesic sphere in \( \mathbb{C}^{r-1} \).

Let \( f : M \to S^{2r-1} \) be a minimal compact orientable hypersurface, \( N \) its normal vector field and \( |\sigma| \) the norm of the second fundamental form of \( f \). Then we have the equations

\[(2.4) \quad \triangle_M f + (2r - 2)f = 0 \quad \text{and} \quad \triangle_M N + |\sigma|^2 N = 0\]

**Definition 2.5.** Let \( J_M \) be the Jacobi operator of \( M \) given by \( J_M = \triangle_M + |\sigma|^2 + 2r - 2 \) and its quadratic form as

\[Q(u, u) = - \int_M uJ_M(u)dV,\]

where \( u : M \to \mathbb{R} \).

**Definition 2.6.** The Morse index of \( M \) is the number of negative eigenvalues of \( J_M \) counting multiplicities.
3. Main Theorem

**Proposition 3.1.** Let \( \tilde{M} \to \mathbb{C}P^{r-1} \) be a compact oriented minimal hypersurface in \( \mathbb{C}P^{r-1} \). Denote by \( f : M^{2r-2} \to S^{2r-1} \subset \mathbb{R}^{2r} = \mathbb{C}^{r} \) the uniquely defined compact hypersurface such that \( \Pi(f(M)) = \tilde{f}(M) \). Then \( f \) is oriented and minimal.

**Proof.** Let \( \tilde{N} \) be a unitary normal vector field on \( \tilde{M} \). Consider the unique horizontal lift \( N \) of \( \tilde{N} \) to \( M \). Then \( N \) is orthogonal to the fibers, \( \Pi \)-related to \( \tilde{N} \) (\( \Pi_{*}(N) = \tilde{N} \)) and normal to \( M \) in \( S^{2r-1} \).

Denote the second fundamental tensors by \( B : S^{2r-1} \to T^{r-1} \mathbb{R}^{2r} \). By O’Neill’s formulas [10] we have for \( u, v \)

\[
< B(u), v > = < \tilde{B}(\Pi_{*}(u)), \Pi_{*}(v) > .
\]

Consider \( \{ e_1, e_2, ..., e_{2r-3} \} \) an orthonormal basis in \( T_{\Pi(p)} \tilde{M} \). For each \( e_i \), consider its unique horizontal lift \( \tilde{e}_i \), \( i = 1, ..., 2r-3 \). Let \( e_{2r-2} \) be the unit vector tangent to the orbit of \( \Pi \), which in this case is the geodesic \( S^1 \). Therefore \( \{ e_1, ..., e_{2r-3}, e_{2r-2} \} \) is an orthonormal basis in \( T_p M \). If \( H \) and \( \tilde{H} \) are the mean curvature of \( M \) and \( \tilde{M} \) respectively, then

\[
H = \sum_{i=1}^{2r-3} < \nabla_{e_i} e_i, N > + < \nabla_{e_{2r-2}} e_{2r-2}, N > = - \sum_{i=1}^{2r-3} < e_i, \nabla_{e_i} N > + < \nabla_{e_{2r-2}} e_{2r-2}, N >
\]

\[
= \sum_{i=1}^{2r-3} < e_i, B(e_i) > + < \nabla_{e_{2r-2}} e_{2r-2}, N > = \sum_{i=1}^{2r-3} < \tilde{e}_i, \tilde{B}(\tilde{e}_i) > + < \nabla_{e_{2r-2}} e_{2r-2}, N >
\]

\[
= \tilde{H} + < \nabla_{e_{2r-2}} e_{2r-2}, N > .
\]

Since \( e_{2r-2} \) is a tangent vector to a geodesic, then \( \nabla_{e_{2r-2}} e_{2r-2} = 0 \). Therefore \( H = \tilde{H} \).

\( \square \)

**Definition 3.2.** Under the hypothesis of the last proposition, we define \( u : M \to \mathbb{R} \) to be a \( S^1 \)-equivariant function if for \( p \) and \( q \) in the same orbit we have \( u(p) = u(q) \). An eigenvalue is \( S^1 \)-equivariant if its associated eigenfunction is \( S^1 \)-equivariant.

**Remark 3.3.** The features of \( J_{\tilde{M}} \) in \( \tilde{M} \) give information about \( J_M \). If \( \tilde{u} \) is a smooth function on \( \tilde{M} \), let \( u \) be the \( S^1 \)-equivariant function on \( M \) given by \( u = \tilde{u} \circ \Pi \).

By the remark in section 4 in [1] \( J_{\tilde{M}}(u) = (J_{\tilde{M}} \tilde{u}) \circ \Pi \). It is clear that if \( u \) is a \( S^1 \)-equivariant function on \( M \), we can define a function \( \tilde{u} : \tilde{M} \to \mathbb{R} \) such that \( u = \tilde{u} \circ \Pi \). In particular

- set of eigenvalues of \( J_{\tilde{M}} \subset \) set of eigenvalues of \( J_M \).
- set of eigenvalues of \( J_{\tilde{M}} = \) set of \( S^1 \)-equivariant eigenvalues of \( J_M \).

**Lemma 3.4.** Under the assumptions of the last proposition let \( \tilde{M} \) be a hypersurface with index one, \( u : M \to \mathbb{R} \) be a smooth \( S^1 \)-equivariant function and \( \varphi \) the first eigenfunction of \( J_M \) then

1. \( \varphi \) is \( S^1 \)-equivariant.

2. Let \( u \) be a map, such that \( \int_{M} \varphi udV = 0 \) then \( Q(u, u) \geq 0 \).
Proof: (1) Let $x \in S^1$ be an arbitrary point in the sphere. Define the isometry $A_x : M \to M$ by $A_x(p) = x \cdot p$. The map $\varphi \circ A_x$ is another eigenfunction associated to the first eigenvalue $\lambda_1$ of $J_M$. The dimension of $\lambda_1$-space is one, then $k \varphi \circ A_x = \varphi$; where $k$ is a constant. Since $\varphi \geq 0$ then $k \geq 0$. On the other hand

$$k^2 \int_M (\varphi \circ A_x)^2 dV = \int_M \varphi^2 dV,$$

thus $k^2 = 1$. Therefore $k = 1$ and $\varphi \circ A_x = \varphi$, i.e. $\varphi$ is $S^1$-equivariant.

(2) Since $u$ is $S^1$-equivariant, there exists $\bar{u} : \bar{M} \to \mathbb{R}$ such that $u = \bar{u} \circ \Pi$. Then

$$2\pi \int_{\bar{M}} \bar{u} \bar{\varphi} d\bar{V} = \int_M \varphi u dV = 0$$

and

$$0 \leq Q(\bar{u}, \bar{u}) = -\int_{\bar{M}} \bar{u} J_M(\bar{u}) d\bar{V} = -\frac{1}{2\pi} \int_M u J_M(u) dV = \frac{1}{2\pi} Q(u, u). \quad \square$$

Proposition 3.5. The index of the Clifford hypersurfaces in $\mathbb{C}P^{r-1}$ is one.

Proof. Let $\Pi(S^{n_1}_{R_1} \times S^{n_2}_{R_2})$ be a typical minimal Clifford hypersurface, where $n_1 + n_2 = 2r - 2$, $R_1^2 + R_2^2 = 1$, $n_1 R_1^2 = n_2 R_2^2$ and $n_1, n_2$ odd. We look for $S^1$-equivariant negative eigenvalues of $J_M$, where $M = S^{n_1}_{R_1} \times S^{n_2}_{R_2}$ (see Remark 3.3. In the manifold $M$ we have

$$|\sigma|^2 = n_1 R_1^2 + n_2 R_2^2 = \frac{n_2 R_1^2}{R_1^2} + \frac{n_1 R_2^2}{R_2^2} = n_2 + n_1 = 2r - 2.$$ 

Since $J_M = \triangle_M + 2(2r - 2)$, we look for $S^1$-equivariant eigenvalues $\beta$ of $\triangle_M$, such that $\beta < 2(2r - 2)$. The eigenvalues of $\triangle_M$ are given by

$$\beta_{k_1, k_2} = \frac{k_1(k_1 + n_1) - 1}{R_1^2} + \frac{k_2(k_2 + n_2 - 1)}{R_2^2}$$

where $k_1, k_2 \in \mathbb{N}^0$.

Notice that $\beta_{00} = 0$ corresponds to the constant functions which are $S^1$-equivariant. When $k_1 = 1 = k_2$,

$$\beta_{11} = \frac{n_1 R_1^2}{R_1^2} + \frac{n_2 R_2^2}{R_2^2} = n_1(1 + \frac{R_2^2}{R_1^2}) + n_2(1 + \frac{R_1^2}{R_2^2}) = n_1 \frac{R_2^2}{R_1^2} + n_2 \frac{R_1^2}{R_2^2} + n_1 + n_2 = 2(2r - 2).$$

Therefore we get rid of $\beta_{k_1, k_2}$, when $k_1 \geq 1$ and $k_2 \geq 1$. Using the same argument given in [I] the eigenfunctions corresponding to $\beta_{10}$ (respectively $\beta_{01}$) are linear functions on $\mathbb{R}^{n_1+1}$ (respectively $\mathbb{R}^{n_2+1}$) which are never $S^1$-equivariant since $-id \in S^1$. It suffices to show that $\beta_{20}$ and $\beta_{02}$ are greater than $\beta_{11}$. Since $n_1 R_2^2 = n_2 R_1^2$,

$$R_2^2 R_1^2 = \frac{n_2}{n_1} \geq \frac{n_2}{n_1 + 2} \quad (3.2)$$

$$R_2^2 R_1^2 = \frac{n_2}{n_1} \leq \frac{n_2 + 2}{n_1} \quad (3.3)$$

But equations (3.2) and (3.3) are equivalent to
\[
\beta_{20} = \frac{2(n_1 + 1)}{R_1^2} \geq \frac{n_1}{R_1^2} + \frac{n_2}{R_2^2} = \beta_{11},
\]
\[
\beta_{02} = \frac{2(n_2 + 1)}{R_2^2} \geq \frac{n_1}{R_1^2} + \frac{n_2}{R_2^2} = \beta_{11}.
\]

**Notation** Let \( u : M \to \mathbb{R} \) and \( V : M \to \mathbb{R}^{2r} \) be scalar and vectorial functions on \( M \) respectively, and \( V_j = \langle V, e_j \rangle \) where \( e_1, \ldots, e_{2r} \) is the canonical basis of \( \mathbb{R}^{2r} \). We define
\[
\langle \nabla u, \nabla V \rangle := \sum_{j=1}^{2r} \langle \nabla u, \nabla V_j \rangle e_j.
\]

**Theorem 3.6.** The only compact minimal hypersurfaces with index one in the complex projective space \( \mathbb{C}P^{r-1} \) are the minimal Clifford hypersurfaces. In particular they can be minimal geodesic spheres.

**Proof.** Let \( \bar{f} : \bar{M} \to \mathbb{C}P^{r-1} \) be a compact minimal hypersurface with index one in \( \mathbb{C}P^{r-1} \). Since \( \mathbb{C}P^{r-1} \) is simply connected we can assume \( \bar{f} \) orientable (see Theorem 4.7 [4]). Let \( f : M^{2r-2} \to S^{2r-1} \) as in Proposition 3.1. Consider the function \( \phi_{a,b} : M \to \mathbb{R}^{2r} \) given by
\[
\phi_{a,b} := \langle a, f \rangle \cdot f + \langle a, N \rangle \cdot N + \langle b, f \rangle \cdot N,
\]
where \( a, b \in \mathbb{R}^{2r} \). Through calculation we get
\[
\langle a, f \rangle \cdot f = \langle a, f \rangle \cdot f + \langle a, \tilde{f} \rangle \cdot \tilde{f}
\]
\[
\langle a, N \rangle \cdot N = \langle a, N \rangle \cdot N + \langle a, \tilde{N} \rangle \cdot \tilde{N}
\]
\[
\langle b, f \rangle \cdot N = \langle b, f \rangle \cdot N + \langle b, \tilde{f} \rangle \cdot \tilde{N}
\]
where \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( \mathbb{R}^{2r} \).

**Remark 3.7.** Using equations (2.2) and (2.3), \( \phi_{a,b} \) is \( S^1 \)-equivariant.

**Lemma 3.8.** The value of the Jacobi operator when applied to the function \( \phi_{a,b} \) is
\[
J_M(\phi_{a,b}) = [\langle a, f \rangle \cdot f + \langle a, \tilde{f} \rangle \cdot \tilde{f} - \langle a, N \rangle \cdot N - \langle a, \tilde{N} \rangle \cdot \tilde{N}] (|\sigma|^2 - 2r + 2) + X,
\]
where \( X : M \to \mathbb{R}^{2r} \) is a vector field tangent to \( M \).

**Proof.** Follow the same proof given in Lemma 1 [3].

**Lemma 3.9.** Given \( a, b \in \mathbb{R}^{2r} \) we have
\[
\int_M (|\sigma|^2 - 2r + 2)(\langle N, a \rangle \langle f, b \rangle + \langle \tilde{N}, a \rangle \langle \tilde{f}, b \rangle) dV = 0
\]
Proof. Using equation (2.4), the fact that $\tilde{f}$ and $\tilde{N}$ also fulfill equation (2.4) and the Divergence Theorem,
\[
\int_M (|\sigma|^2 - 2r + 2)(<N, a> < f, b > + < \tilde{N}, a > < \tilde{f}, b >) dV
= \int_M (N, a) \Delta < f, b > - < f, b > \Delta < N, a > dV
+ \int_M (\tilde{N}, a) \Delta < \tilde{f}, b > - < \tilde{f}, b > \Delta < \tilde{N}, a > dV = 0
\]
\[\square\]

Remark 3.10. Let $x \in S^1$ be an arbitrary point. Through some calculations we get
\[
< \tilde{f}, N > = 0
\]
\[
< X, \tilde{f} > = 0 = < X, \tilde{N} >
\]
\[
N(x \cdot p) = x \cdot N(p)
\]

Since $\phi_{a,b}$ are $S^1$-equivariant functions, then we can use them as test functions. From Lemma 3.8, Lemma 3.9 and Remark 3.10, we get
\[
Q(\phi_{a,b}, \phi_{a,b}) = \int_M (2r - |\sigma|^2)[< a, f >^2 + < a, \tilde{f} >^2 - < a, N >^2 - < a, \tilde{N} >^2] dV.
\]

Proposition 3.11. The map $F : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ given by
\[
F(b) = \int_M \varphi(< f, b > N + < \tilde{f}, b > \tilde{N}) dV
\]
is a linear isomorphism.

Proof. We proceed by contradiction. Suppose there exists $b \neq 0$ such that $F(b) = 0$. Take $\phi = \phi_{0,b} = < b, f > N + < b, \tilde{f} > \tilde{N}$. From equation (3.9), $Q(\phi, \phi) = 0$. Since \[\int_M \varphi \phi dV = 0\] then $J_M(\phi) = 0$. Using equation (3.8) we have $X = 0$, i.e.
\[
X = < \nabla < b, f >, \nabla N > + < \nabla < b, \tilde{f} >, \nabla \tilde{N} > = 0.
\]

Explicit computation gives, $-X = (\nabla \bar{\tau}_p N)^T_p + (\nabla \bar{\tau}_p \tilde{N})^T_{\bar{p}}$, where $T_p$ and $T_{\bar{p}}$ denote the tangential part of $M$ at $p$ and $\bar{p}$. Then
\[
(\nabla \bar{\tau}_p N)^T_p + (\nabla \bar{\tau}_p \tilde{N})^T_{\bar{p}} = 0.
\]

Lemma 3.12. On $M$,
\[
< N(p), b > + < N(\bar{p}), b > = < N, b > + < \tilde{N}, b > = c
\]
where $c$ is a constant.

Proof. It suffices to show that for the basis $\left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{2r-2}} \right\}$ on $T_p M$ and $k = 1, \ldots, 2r - 2$
\[
\frac{\partial f}{\partial x_k}(< N, b > + < \tilde{N}, b >) = 0.
\]
By one side,
\[
\frac{\partial f}{\partial x_k}(N, b) = \nabla_{\frac{\partial f}{\partial x_k}} N, b + \nabla_{\frac{\partial f}{\partial x_k}} \hat{N}, b
\]
(3.14)
\[= \nabla_{\frac{\partial f}{\partial x_k}} N, b^T_p + \nabla_{\frac{\partial f}{\partial x_k}} \hat{N}, b^T_p\]
where we have used in the last equality \(\nabla_{\frac{\partial f}{\partial x_k}} N \in T_p M\) and \(\nabla_{\frac{\partial f}{\partial x_k}} \hat{N} \in T_{\hat{p}} M\). On the other hand, doing inner product with \(\frac{\partial f}{\partial x_k}\) in (3.11), we get
\[
(3.15) \quad < (\nabla_{b^T_p} N)^{T_p}, \frac{\partial f}{\partial x_k} > + < (\nabla_{\tilde{b}^T_p} \hat{N})^{\tilde{T}_p}, \frac{\partial f}{\partial x_k} >= 0
\]
But,
\[
(3.16) \quad < (\nabla_{b^T_p} N)^{T_p}, \frac{\partial f}{\partial x_k} >= - < A_p(b^T_p), \frac{\partial f}{\partial x_k} > = - < A_p(\frac{\partial f}{\partial x_k}), b^T_p >
\]
\[= < (\nabla_{\nabla_{\frac{\partial f}{\partial x_k}} N}^{T_p}, b^T_p >= < \nabla_{\frac{\partial f}{\partial x_k}} N, b^T_p >
\]
and
\[
(3.17) \quad < (\nabla_{\tilde{b}^T_p} \hat{N})^{\tilde{T}_p}, \frac{\partial f}{\partial x_k} >= < (\nabla_{\tilde{b}^T_p} \hat{N})^{\tilde{T}_p}, \frac{\partial f}{\partial x_k} > = < \nabla_{\frac{\partial f}{\partial x_k}} \hat{N}, b^T_p >
\]
where in (3.17), we have used the same argument used in (3.16) but in the point \(\tilde{p}\). Replacing (3.17) and (3.16) in (3.15), we get
\[
(3.18) \quad < \nabla_{\frac{\partial f}{\partial x_k}} N, b^T_p > + < \nabla_{\frac{\partial f}{\partial x_k}} \hat{N}, b^T_p >= 0.
\]
Comparing (3.18) with (3.14), it will suffice to show that \(< \nabla_{\frac{\partial f}{\partial x_k}} N, b^T_p >= 0\), where \(W \in (T_{\tilde{p}})^\perp\). Since \((T_{\tilde{p}})^\perp\) is generated by \(\{\tilde{f}, \tilde{N}\}\), its enough to show that
\[< \nabla_{\frac{\partial f}{\partial x_k}} \tilde{N}, b^T_p >= < \nabla_{\frac{\partial f}{\partial x_k}} \tilde{N}, b^T_p >= 0.
\]
Differentiating \(< \tilde{N}, \tilde{f} >= 0\) and \(< \tilde{N}, \tilde{N} >= 1\), we have
\[
0 = < d\tilde{N}, \tilde{f} > + < \tilde{N}, df > = < d\tilde{N}, \tilde{f} >= \nabla_{\frac{\partial f}{\partial x_k}} \tilde{N}
\]
\[0 = < d\tilde{N}, \tilde{N} >= \nabla_{\frac{\partial f}{\partial x_k}} \tilde{N}.
\]
\[\square\]
Using equation (3.12) in \(p\) and \(\tilde{p}\)
\[< N(p), b > + < N(\tilde{p}), b >= < N(\tilde{p}), b > - < N(p), b >= c.
\]
Summing the last equation with (3.12)
\[< N(\tilde{p}), b >= c, \text{ for all } p \in M.
\]
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Then \( < N(p), b > = 0 \). Let \( u \) be the map given by \( u = < f, b > \), therefore \( \text{Hess}(u) = < ., . > u \). If \( u = 0 \), then \( M \) is totally geodesic, i.e. an equator which is a contradiction because of the symmetry of \( M \). On the other hand, if \( u \neq 0 \) by Obata’s theorem \([9]\) \( M \) is isometric to a sphere, then using Gauss equation \( M \) is totally geodesic, which again is a contradiction.

\( \square \)

From Proposition \( 3.11 \) for all \( a \in \mathbb{R}^{2r} \), there exists \( b \in \mathbb{R}^{2r} \), such that \( \phi_{a,b} \) is orthogonal to \( \varphi \). Taking an orthonormal basis \( \{a_1, ..., a_{2r}\} \) of \( \mathbb{R}^{2r} \), let \( b_i \) be the vector associated to \( a_i \) and \( \phi_i = \phi_{a_i,b_i} \); \( i = 1, ..., 2r \). Thus

\[
0 \leq \sum_{i=1}^{2r} Q(\phi_i, \phi_i)
\]

\[
= \sum_{i=1}^{2r} \int_M (2r - 2 - |\sigma|^2) < a_i, f >^2 + < a_i, \tilde{f} >^2 - < a_i, N >^2 - < a_i, \tilde{N} >^2 dV
\]

\[
= \int_M (2r - 2 - |\sigma|^2) (|f|^2 + |\tilde{f}|^2 - |N|^2 - |\tilde{N}|^2) dV = 0
\]

It implies \( Q(\phi_i, \phi_i) = 0 \), \( i = 1, ..., 2r \). Since \( \phi_i \) is orthogonal to \( \varphi \), then \( J_M(\phi_i) = 0 \). If we do inner product in equation (3.8) with \( f \) we get

\[
0 = < a_i, f > (|\sigma|^2 - 2r + 2)
\]

for all \( a_i \) in the basis, which only happens if \( |\sigma|^2 = 2r - 2 \). Using \( [6] \) and \( [2] \) \( M \) is one of the next minimal Clifford hypersurfaces

\[
S^k \sqrt{k/2r-2} \times S^{2r-2-k}/\sqrt{(2r-2-k)/2r-2}, \text{ for } k = 1, ..., |r - 1|
\]

i.e. \( M \) is a minimal Clifford hypersurface. So \( M \) is also a minimal Clifford hypersurface.

\( \square \)

4. Application

We will give some definitions and notations given in \([12]\). Let \( (M^{n+1}, g) \) be a Riemannian manifold connected closed orientable and \( H^n \) the n dimensional Hausdorff measure. When \( \Sigma^n \) is a submanifold, we use \( |\Sigma| \) to denote \( H^n(\Sigma) \).

Definition 4.1. Let \( \{\Gamma_t\}_{t \in [0,1]} \) be a family of \( H^n \) measurable closed subset of \( M \) with finite \( H^n \) measure such that

1. For each \( t \), there is a finite subset \( P_t \subset M \), such that \( \Gamma_t \) is a smooth hypersurface in \( M \setminus P_t \);
2. \( t \to H^n(\Gamma_t) \) is continuous, and \( t \to \Gamma_t \) is continuous in the Hausdorff topology;
3. \( \Gamma_t \to \Gamma_{t_0} \) smoothly in any compact \( U \subset M \setminus P_{t_0} \) as \( t \to t_0 \).

The family \( \{\Sigma_t\}_{t \in [0,1]} \) is called a sweepout of \( M \) if there exists a family of open sets \( \{\Omega_t\}_{t \in [0,1]} \), such that

1. \( \Sigma_t \subset \Omega_t \subset P_t \), for any \( t \in [0,1] \);
2. \( |\Omega_t \setminus \Omega_s| + |\Omega_s \setminus \Omega_t| \to 0 \) as \( s \to t \);
3. \( \Omega_0 = \emptyset \) and \( \Omega_1 = M \).
Definition 4.2. Given a generalized family \( \{ \Gamma_t \} \), we set
\[
L(\{ \Gamma_t \}) = \max_t H^n(\Gamma_t).
\]

Definition 4.3. Two sweepouts \( \{ \Sigma^1_t \}_{t \in [0,1]} \) and \( \{ \Sigma^2_t \}_{t \in [0,1]} \) are homotopic if there is a generalized smooth family \( \{ \Gamma_{s,t} \}_{(s,t) \in [0,1]^2} \), such that \( \Gamma_{0,t} = \Sigma^1_t \) and \( \Gamma_{1,t} = \Sigma^2_t \). A family \( \Lambda \) of sweepouts is called homotopically closed if it contains the homotopy class of each of its elements.

Definition 4.4. Given a homotopically closed family \( \Lambda \) of sweepouts, the width of \( M \) associated with \( \Lambda \) is defined as,
\[
W(M, \Lambda) = \inf_{\{ \Sigma_t \} \in \Lambda} L(\{ \Gamma_t \}).
\]

Theorem 4.5. \([7][12][5]\) Let \( M^{n+1} \) be connected closed orientable Riemannian manifold with positive Ricci curvature and \( 2 \leq n \leq 6 \). Then
- the min-max minimal hypersurface \( \Sigma \) is orientable of multiplicity one, which has Morse index one and \( |\Sigma| = W(M) \).
- \( W(M) = \min_{\Sigma \in S} \left\{ |\Sigma| \text{ if } \Sigma \text{ is orientable or } 2A(\Sigma) \text{ if } \Sigma \text{ is non-orientable} \right\} \), where
  \[
  S = \{ \Sigma^n \subset M^{n+1} : \Sigma \text{ is a minimal hypersurface in } M \}.
  \]

Remark 4.6. The proof of the previous theorem also applies if we get rid of the assumption that \( M \) is orientable and replace the orientability of \( \Sigma \) by two-sidedness.

Remark 4.7. There is a general version of the last theorem \([13]\), but we can not apply it here because in higher dimensions the min-max hypersurface could have singularities.

Corollary 4.8. Let \( \Sigma_i \) be the min-max hypersurface in \( \mathbb{C}P^i \), \( i \in \{2,3\} \). Then \( \Sigma_i \) is a minimal Clifford hypersurface and
\[
W(\mathbb{C}P^i) = |\Sigma_i| = \begin{cases} 
|\Pi(S^1 \sqrt{r} \times S^3 \sqrt{r})| = \frac{3\sqrt{r^2}}{8} & \text{if } i = 2 \\
|\Pi(S^3 \sqrt{r} \times S^3 \sqrt{r})| = \frac{n^3}{4} & \text{if } i = 3.
\end{cases}
\]

Proof. From Theorem 4.5 \( \Sigma_i \) has Morse index one and
\[
W(\mathbb{C}P^i) = |\Sigma_i|.
\]
Therefore from Theorem 4.2 \( \Sigma_i \) is a minimal Clifford hypersurface. The minimal Clifford hypersurfaces in \( \mathbb{C}P^{r-1} \) are \( \Pi(S^{n_1}_{R_1} \times S^{n_2}_{R_2}) \) such that
\[
n_1 + n_2 = 2r - 2, R_1 = \sqrt{\frac{n_1}{2r - 2}}, R_2 = \sqrt{\frac{n_2}{2r - 2}}, n_1, n_2 \text{ odd}
\]
and
\[
2\pi |\Pi(S^{n_1}_{R_1} \times S^{n_2}_{R_2})| = |S^{n_1}_{R_1} \times S^{n_2}_{R_2}| = \frac{4\pi^{n_1 + n_2 + 2}}{\Gamma(\frac{n_1 + 1}{2})\Gamma(\frac{n_2 + 1}{2})} R_1^{n_1} R_2^{n_2}.
\]
- In the case \( \mathbb{C}P^2 \) we only have one candidate to be the minimal Clifford hypersurface,
  \[
  \Pi(S^1 \sqrt{r} \times S^3 \sqrt{r}).
  \]
From equation (4.1)
\[ |\Pi(S^1 \times S^3)| = \frac{3\sqrt{3}\pi^2}{8}, \]
thus
\[ W(\mathbb{C}P^2) = \frac{3\sqrt{3}\pi^2}{8}. \]

- In the case $\mathbb{C}P^3$ we have two candidates to be the minimal Clifford hypersurface,
\[ \Pi(S^1 \times S^5) \] and \[ \Pi(S^3 \times S^3). \]

From equation (4.1)
\[ |\Pi(S^1 \times S^5)| = \frac{25\sqrt{5}\pi^3}{216}, \]
\[ |\Pi(S^3 \times S^3)| = \frac{\pi^3}{4} \]
thus
\[ W(\mathbb{C}P^3) = \min\left\{ \frac{25\sqrt{5}\pi^3}{216}, \frac{\pi^3}{4} \right\} = \frac{\pi^3}{4}. \]

**Remark 4.9.** In the last proof we got rid of $S^2 \times S^2$ and $S^4 \times S^2$ because they are not $S^1$-equivariant, even after a rotation.

**Corollary 4.10.** Let $\Sigma_i$ be the min-max hypersurface in $\mathbb{R}P^i$, $i \in \{3, 4, 5, 6, 7\}$. Then $\Sigma_i$ is a minimal Clifford hypersurface and
\[ W(\mathbb{R}P^i) = |\Sigma_i| = \begin{cases} |\Pi(S^1 \times S^3)| = \pi^2 & \text{if } i = 3 \\ |\Pi(S^1 \times S^2)| = \frac{8\pi^2}{3\sqrt{3}} & \text{if } i = 4 \\ |\Pi(S^3 \times S^3)| = 2\pi^2 & \text{if } i = 5 \\ |\Pi(S^2 \times S^3)| = \frac{24}{25} \sqrt{\frac{3}{5}} \pi^3 & \text{if } i = 6 \\ |\Pi(S^3 \times S^3)| = \frac{\pi^4}{4} & \text{if } i = 7. \end{cases} \]

**Proof.** From Theorem 4.3 $\Sigma_i$ has index one and
\[ W(\mathbb{R}P^i) = |\Sigma_i|. \]

Therefore using Theorem 4.1 $\Sigma_i$ has to be a minimal Clifford hypersurface. The minimal Clifford hypersurfaces in $\mathbb{R}P^{r-1}$ are $\Pi(S_{R_1}^{n_1} \times S_{R_2}^{n_2})$ such that
\[ n_1 + n_2 = r - 2, \quad R_1 = \sqrt{\frac{n_1}{r-2}}, \quad R_2 = \sqrt{\frac{n_2}{r-2}}, \]
and
\[ |\Pi(S_{R_1}^{n_1} \times S_{R_2}^{n_2})| = |S_{R_1}^{n_1} \times S_{R_2}^{n_2}| = \frac{4\pi^{\frac{n_1+n_2+2}{2}} R_1^{n_1} R_2^{n_2}}{\Gamma(\frac{n_1+1}{2}) \Gamma(\frac{n_2+1}{2})}. \]

- In the case $\mathbb{R}P^3$ we only have one candidate to be the minimal Clifford hypersurface,
\[ \Pi(S^1 \times S^1). \]
From equation (4.2)
$$|\Pi(S_1^{1/4} \times S_1^{1/4})| = \pi^2$$
thus
$$W(\mathbb{R}P^3) = \pi^2.$$  

- In the case $\mathbb{R}P^4$ we only have one candidate to be the minimal Clifford hypersurface,
$$\Pi(S_1^{1/4} \times S_2^{1/4})$$
From equation (4.2)
$$|\Pi(S_1^{1/4} \times S_2^{1/4})| = \frac{8\pi^2}{3\sqrt{3}}$$
thus
$$W(\mathbb{R}P^4) = \frac{8\pi^2}{3\sqrt{3}}.$$  

- In the case $\mathbb{R}P^5$ we have two candidate to be the minimal Clifford hypersurface,
$$\Pi(S_1^{1/4} \times S_3^{1/4}) \text{ and } \Pi(S_2^{1/4} \times S_2^{1/4})$$
From equation (4.2)
$$|\Pi(S_1^{1/4} \times S_3^{1/4})| = \frac{3\sqrt{3}\pi^3}{8}$$
$$|\Pi(S_2^{1/4} \times S_2^{1/4})| = \pi^2$$
thus
$$W(\mathbb{R}P^5) = \min\left\{2\pi^2, \frac{3\sqrt{3}\pi^3}{8}\right\} = 2\pi^2.$$  

- In the case $\mathbb{R}P^6$ we have two candidate to be the minimal Clifford hypersurface,
$$\Pi(S_1^{1/4} \times S_4^{1/4}) \text{ and } \Pi(S_2^{1/4} \times S_3^{1/4})$$
From equation (4.2)
$$|\Pi(S_1^{1/4} \times S_4^{1/4})| = \frac{128\pi^3}{75\sqrt{5}}$$
$$|\Pi(S_2^{1/4} \times S_3^{1/4})| = \frac{24\sqrt{3}}{25} \sqrt{\frac{\pi^3}{5}}$$
thus
$$W(\mathbb{R}P^6) = \min\left\{\frac{128\pi^3}{75\sqrt{5}}, \frac{24\sqrt{3}}{25} \sqrt{\frac{\pi^3}{5}}\right\} = \frac{24}{25} \sqrt{\frac{3}{5}} \pi^3.$$  

- In the case $\mathbb{R}P^7$ we have three candidates to be the minimal Clifford hypersurface,
$$\Pi(S_1^{1/4} \times S_5^{1/4}), \Pi(S_2^{1/4} \times S_4^{1/4}), \Pi(S_3^{1/4} \times S_3^{1/4})$$
From equation (4.2)
\[
|\Pi(S^1 \sqrt{\pi} \times S^5 \sqrt{\pi})| = \frac{25 \sqrt{5} \pi^4}{216}
\]
\[
|\Pi(S^2 \sqrt{\pi} \times S^4 \sqrt{\pi})| = \frac{64 \pi^3}{81}
\]
\[
|\Pi(S^3 \sqrt{\pi} \times S^3 \sqrt{\pi})| = \frac{\pi^4}{4}
\]
thus
\[
W(\mathbb{R}P^7) = \min\left\{\frac{25 \sqrt{5} \pi^4}{216}, \frac{64 \pi^3}{81}, \frac{\pi^4}{4}\right\} = \frac{\pi^4}{4}.
\]

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