Quantum Mechanics is Either Non-Linear Or Non-Introspective

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All paradoxes can be paradoctored. –R.A. Heinlein

ABSTRACT
The measurement conundrum seems to have plagued quantum mechanics for so long that impressions of an inconsistency amongst its axioms have spawned. A demonstration that such purported inconsistency is fictitious may then be in order and is presented here. An exclusion principle of sorts emerges, stating that quantum mechanics cannot be simultaneously linear and introspective (self-observing).
1. Axioms and Assumptions

Although close to becoming a centenarian, quantum mechanics still has adolescent (although not obviously just cosmetic) problems, most notably exemplified by the conundrum known as the “quantum measurement problem”. The conundrum has been considered from very diverse points of view and phrased in many different ways, including the claim of contradiction between two of its axioms [1], hence an inherent inconsistency of quantum mechanics as a scientific theory.

The purpose of this article is to show that this particular (apparent) contradiction stems from a slight and subtle but serious misinterpretation of the axioms — a misinterpretation which however appears to be too well hidden and all too frequent to be easily dismissed as trivial. On exposing this misinterpretation, an avenue seems to open for a possible and perhaps interesting resolution of the “quantum measurement problem”. Details of this quest are however beyond our present scope.

The routine maneuver in some relevant applications is then seen to confirm the main result, stated in the title. While this will surprise no seasoned practitioner, a clear and explicit statement is to the best of knowledge of the present author nowhere to be found in print, and may therefore turn out to be welcome.

Over the years, one collection of axioms\footnote{The ‘axioms’ and ‘theorems’ of any system may always be reorganized so as to swap a ‘theorem’ with an ‘axiom’—provided the rules of deduction allow the demoted ‘axiom’ to be derived from the new circle of ‘axioms’.} has become more frequently quoted than any other. For the sake of completeness, they are [2] (with slight adaptation):

1. At any given time, $t$, the state of a physical system is defined by specifying a state-function (ket), $|\psi\rangle$, belonging to the state set $\mathcal{E}$.

2. Every quantity $A$ which can be measured (at least in principle) is ascribed an operator $A$, acting in $\mathcal{E}$; such quantities are called observables.

3. Only the eigenvalues of the operator $A$ are possible results of a single measurement of the corresponding observable $A$.

4. When the observable $A$ is measured on a system in the state $|\psi\rangle$, the probability $P(a_n)$ of obtaining the non-degenerate\footnote{Degenerate generalizations are easy and merely technical, not of principle.} eigenvalue $a_n$ of the corresponding operator $A$ is

\[
P(a_n) = \frac{|\langle u_n | \psi \rangle|^2}{\langle \psi | \psi \rangle}, \quad A |u_n\rangle = a_n |u_n\rangle, \quad (1.1)
\]

i.e., $|u_n\rangle$ is the normalized eigenstate of $A$ associated to the eigenvalue $a_n$.

5. If the measurement of the physical quantity $A$ on the system in the state $|\psi\rangle$ gives the result $a_n$, the state of the system upon the measurement is the normalized projection

\[
\frac{P_n |\psi\rangle}{\sqrt{\langle \psi | P_n |\psi\rangle}} = |u_n\rangle, \quad (1.2)
\]
of $|\psi\rangle$ onto the eigenstate associated with $a_n$.

6. The time evolution of the state vector $|\psi\rangle$ is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle,$$

(1.3)

where $H$ is the Hamiltonian operator of the system.

It is usually implicitly assumed that the state set $E$ from 1, also being the solution set of Eq. (1.3) from 6, is a vector space. Equivalently, one regards the Schrödinger equation (1.3) as being linear—just as it appears to be: Solutions of a linear equation form a vector space since the sum of any two solutions is again a solution; this then is tantamount to the ‘superposition principle’. Note that this is actually neither included, nor strictly a consequence of the above axioms. To make this implicit assumption manifest, we remove the seventh veil:

7. The Hamiltonian $H$ in axiom 6 is independent of $|\psi\rangle$.

As will hopefully become evident below, the assertion of 7 should not be regarded as another axiom, but merely as an interpretational/applicational choice; in fact, its negation (7) is equally viable and perhaps even more interesting (see below).

Contrasting the linearity of Eq. (1.3), the projection (collapse) in axiom 5 is discontinuous, arguably non-linear; this discrepancy is then argued to in fact imply an inconsistency of quantum mechanics and to be at the heart of the “quantum measurement problem” [1]. Whilst this latter observation remains to seem true, the claimed inconsistency turns out to be a mirage — owing in part to the implicit assumption of 7.

2. Beguilement Breakdown

Standard applications of quantum mechanics machinery are developed upon the above axioms, and with the (implicit) assertion of 7. The state vector $|\psi\rangle$ is indeed the wavefunction that describes the system under scrutiny, $S$, and so is assumed to carry all possibly knowable information about it. The differential equation (1.3) does embody the dynamical principle which determines the time-evolution (and all other characteristics) of the state vector $|\psi\rangle$ and so provides a complete description of the quantum dynamics of $S$.

However, just what is $H$? Textbooks prescribe how to determine the operator $H$, which typically looks something like $H = T + V$, where $T$ is the kinetic energy (operator) of the system and $V$ the potential energy (operator). The kinetic operator $T$ determines the evolution of the state vector $|\psi\rangle$ (and so the system $S$) in lieu of any interaction, whereas the potential operator $V$ describes the effects of all interactions affecting $S$.

The physical meaning of the assertion of 7 is that these interactions affecting $S$ are due to agents external to $S$, i.e., the Hamiltonian $H$ specifies the environment external
to the system $S$. This makes explicit the versed practitioners’ hitherto mostly implicit understanding that the Universe is being split asunder into:

1. $|\psi\rangle$, which represents the state of the (sub)system under scrutiny $S$ and carries all relevant information about it, and

2. $H$, which represents the ‘environment’ in which $S$ evolves and carries all relevant information about this ‘environment’.

This is clearly evident even in the wording of most textbook paradigms, such as ‘particle in a box’, where $|\psi\rangle$ represents the particle, $V$—the box; etc. On the technical side, the differential equation (1.3) now manifestly is linear and its solution set, $E$, therefore necessarily is a vector space — the superposition principle is applicable.

This division is very well suited for typical applications of quantum mechanics: for modeling of processes in which the scrutinized (sub)system is usually well (qualitatively and especially quantitatively) distinguished from the environment. For example, in the classic beam-splitting experiment, the electron beam is split in two, one with ‘spin-up’ and the other with ‘spin-down’. The electron beam, as described by its ket $|\psi\rangle$, is perfectly clearly distinguished from the magnets—which produce the magnetic field (environment) that interacted with the beam and caused the splitting. This magnetic field (and so the magnet producing it) is of course described by an appropriate term in the Hamiltonian $H$.

2.1. Measurement milieu

Now, the act of measurement is itself a form of interaction, patently of the system under scrutiny, $S$, with the measuring machine, $M$. Therefore, it ought to be possible to describe such an interaction by including an appropriate ‘potential’—the one which contains all the information about the measuring machine, $M$, including the time when it is set to measure.

In the ‘interaction picture’, the Schrödinger equation (1.3) reduces to

$$i\hbar \frac{d}{dt} |\psi\rangle = V_M |\psi\rangle ,$$

(2.1)

where $V_M$ is the as yet unspecified interaction potential which describes the interaction of the system under scrutiny, $S$, with measurement machine, $M$, and contains all the details about the latter.

However, if the measuring machine is itself of this World, and Nature really is Quantum, then there ought to exist a set of state vectors, $|\phi\rangle$, which describe the quantum states of the measuring machine. The dynamics of these $|\phi\rangle$’s then ought to be determined by another Schrödinger equation, and in the ‘interaction picture’, we have:

$$i\hbar \frac{d}{dt} |\phi\rangle = \Lambda_M |\phi\rangle ,$$

(2.2)

where the operator $\Lambda_M$ describes (among other things also) how the measuring interaction with $|\psi\rangle$ (re)acts on the measuring device.
Finally, what can one say in general of the measurement interaction operators $V_M$ and $\Lambda_M$? A moment’s reflection will satisfy the Reader that $V_M$ must depend on the state of the measuring machine, $|\phi\rangle$, and likewise that $\Lambda_M$ must depend on the state of the (sub)system under scrutiny, $|\psi\rangle$. Therefore, the two equations (2.1) and (2.2) may be written a bit more explicitly as

$$i\hbar\frac{d}{dt}|\psi\rangle = V_M(\phi)|\psi\rangle , \quad (2.3a)$$

$$i\hbar\frac{d}{dt}|\phi\rangle = \Lambda_M(\psi)|\phi\rangle . \quad (2.3b)$$

Owing to the dependence of the ‘interaction potential’ operators $V_M(\phi)$ and $\Lambda_M(\psi)$ on the state vectors $|\phi\rangle$ and $|\psi\rangle$, respectively, these two differential equations are coupled. In other words, the coupled system (2.3) is essentially self-referential.

The bottom line: the system (2.3) is non-linear.

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A remark is in order. Another description (indeed, the more standard one) of a system of two interacting parts is indeed possible where one does not proceed with a coupled system of equations (2.3). Instead, describing the combined scrutinized+measuring system, $S+M$, one introduces a product state vector $|\psi,\phi\rangle = |\psi\rangle |\phi\rangle$. This state vector would again evolve according to a third dynamical (Schrödinger) equation written very much like Eq. (1.3). In this third equation, however, the new Hamiltonian would have to be independent of $|\psi,\phi\rangle$, and would have to refer to agents external to both the measuring machine and the scrutinized (sub)system, for the superposition principle to be applicable (asserting assumption 7). More to the point, however, such a linear description is then in no way adequate for describing the measurement of the (sub)system $S$ by the machine $M$. Instead, the combined system $S+M$ may now be (meta-)measured only by agents external to $S+M$. This augmentation of the collection of involved agents then generates the quantum measurement conundrum: the infinite progression of enlarged measured+measuring systems, which may be terminated only by eventually including the (unexplained and undescribed) metaphysical mind of the observer (see also Ref. [3]). Instead, the system (2.3) appears to be more satisfactorily within the realm of quantum mechanics.

The above brief analysis presents us with two mutually exclusive options 4):

1. Quantum mechanics can be arranged to be linear (assert 7), but then cannot describe the measurement process and all the involved components.

2. Quantum mechanics can be arranged to describe the measurement process and all the involved components, but then becomes non-linear (negate 7, i.e., assert 7).

This exclusion principle seems to be built into the very setting of the axioms 1–6 together with the (usually implicit) assertion of either the assumption 7 or its negative, 7.

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4) You cannot be your cake and eat it too.
The second approach (negating assumption 7) may seem to lie beyond the standard applications of quantum mechanics, and for a good reason. Nothing in the standardly quoted (if quoted at all) axioms 1–6 describes the measuring device, so one is bound to forge an extension as was done in subsection 2.1. A guide to this end is provided as much by heuristic plausibility (see also § D) as by the historical fact that quantum mechanics already has evolved into (easily non-linear) quantum field theory. While this evolution happened for totally different reasons, it may nevertheless be helpful to reconsider old problems from this newer and perhaps more general vantage point. However, § D demonstrates that an iterative (adiabatic approximation) approach to non-linear systems very much like (2.3) is in fact standard quantum mechanics textbook material!

2.2. A toy model of measuring

That the system (2.3) is non-linear—and so in general not compatible with the superposition principle—should be clear from the general theory of differential equations. This is also easy to see with a simple example:

\[ i\hbar \frac{d}{dt} |\psi\rangle = a |\phi\rangle |\psi\rangle + \ldots , \]  

\[ \text{(2.4a)} \]

where the ‘interaction potentials’ \( V_M(\phi) \) and \( \Lambda_M(\psi) \) were expanded, keeping only the linear terms\(^5\). Including the omitted higher order terms (represented by the ellipses) is easily seen to only complicate matters technically, but not in principle.

With the toy model at hand, there are two radically different cases: 1. when either \( a \) or \( b \) vanishes, and 2. when both \( a, b \) are nonzero. (The case when both \( a \) and \( b \) vanish corresponds to no coupling, that is, no measurement.)

\[ b = 0: \] Now Eq. (2.4b) simply states that \( |\phi\rangle = \phi_0 \) (in the interaction picture) does not explicitly depend on time and is determined without ever asking about \( |\psi\rangle \). Thereupon, Eq. (2.4a) is indeed linear in \( |\psi\rangle \), formally solved by \( |\psi\rangle = \exp \left[ -i \hbar a \phi_0 \right] |\psi_0\rangle \).

This result epitomizes the standard practice in quantum mechanics, where \( |\psi\rangle \) evolves according to Eq. (2.4a), \( a \phi_0 \) represents the unchanging environment (potential) affecting \( |\psi\rangle \), and \( |\psi\rangle \) has no effective back-reaction onto the environment. Being time-independent, \( \phi_0 \) cannot describe a measurement process since the latter is discontinuous in time\(^6\).

This case asserts assumption 7, cannot describe the quantum dynamics of both the scrutinized (sub)system \( S \) and the measuring machine \( M \), and is linear.

\(^5\) Note that the \( |\phi\rangle \) in the product \( |\phi\rangle |\psi\rangle \) is to be reinterpreted as an operator acting on \( |\psi\rangle \), and vice versa for Eq. (2.4b). This reinterpretation is always possible and becomes trivial in any concrete representation, where the state vectors are simply wave-functions.

\(^6\) In another picture where \( \phi_0 \) is discontinuous in time so as to describe a measurement process, the evolution operator \( \exp \left[ -i \frac{\hbar}{M} a \phi_0 \right] \) and so also \( |\psi\rangle \) will be discontinuous in time. However, the moment of discontinuity must depend on \( |\psi\rangle \), as the measurement cannot happen when the (sub)system \( S \) is not in the measuring machine \( M \)—back to the general (non-linear) case below.
$a, b \neq 0$: From the system of two coupled first order differential equations (2.4), it is always possible to obtain a second order differential equation for either one of $|\psi\rangle$, $|\phi\rangle$ uncoupled from the other; this is sometimes called the integrability condition. Once this differential equation for, say, $|\psi\rangle$ alone is obtained and solved, $|\phi\rangle$ is obtained directly from Eq. (2.4a), without further integration:

$$|\phi\rangle = i\hbar \frac{d}{dt} \log (|\psi\rangle) , \quad (2.5)$$

The integrability condition for $|\psi\rangle$ is:

$$i\hbar \left[ |\psi\rangle \frac{d^2}{dt^2} |\psi\rangle - \left( \frac{d}{dt} |\psi\rangle \right)^2 - \frac{1}{a} \left( \frac{da}{dt} \right) |\psi\rangle \frac{d}{dt} |\psi\rangle \right] = b |\psi\rangle^2 \frac{d}{dt} |\psi\rangle , \quad (2.6)$$

and is manifestly non-linear and even non-homogeneous with respect to rescaling (renormalizing) the state vector $|\psi\rangle$. Therefore, the solution set of this equation, $\mathcal{E}$, is not a linear vector space and is not compatible with the superposition principle. Since Eq. (2.5) determines $|\phi\rangle$ given any $|\psi\rangle$, solving Eq. (2.6) for $|\psi\rangle$ provides a complete solution to the original system (2.4).

The Reader unsettled by the appearance of non-linear terms like $|\psi\rangle^2$ should note that in any concrete representation, the state vector $|\psi\rangle$ is replaced by the appropriate wavefunction $\psi(\cdots, t)$ and Eq. (2.6) becomes a perfectly legitimate, albeit quite complicated non-linear and non-homogeneous differential equation of second order. It remains (as always!) to ensure that the resulting solutions $\psi(\cdots, t)$ are normalizable (square-integrable).

This case negates assumption 7, does describe the quantum dynamics of both the scrutinized (sub)system $S$ and the measuring machine $M$ and is non-linear.

### 3. Complaints and Conclusions

Two complaints to the foregoing discussion and especially the above toy model come to mind immediately. First, the choice of $b \neq 0$ vs. $b = 0$ in the toy model (and $\Lambda_M(\psi) \neq 0$ vs. $\Lambda_M(\psi) = 0$ in general) seems to remain up to the Reader. Could it be that the arbitrariness of the Reader’s whim determines whether or not quantum mechanics is linear?

This complaint is misplaced, for the toy model (2.4) was precisely that—a toy model, intended merely to demonstrate the intrinsic nonlinearity. In an actual model, the measurement interaction potentials $V_M$ and $\Lambda_M$ are completely determined, depending on the details of the measurement technique and process employed; see §D.

Second, it would appear that the present result abolishes the superposition principle in quantum mechanics, in face of myriads of experiments which have in fact brought about the quantum mechanical wave-particle duality. This again is not so. While appearing as a complaint on principle, this really is a technical complaint. All experiments—and so also

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7) Should the Reader wish to champion the claim that measurement really occurs in the observer’s mind, one must then first develop a quantum theory of mind before the potentials $V_M, \Lambda_M$ can be specified in sufficient detail. Nevertheless, the present result would seem to persist.
those which imply the superposition principle—have a finite resolution. It is then easy to see that the limits on this resolution merely place a limit on the ratio of (the appropriate matrix elements of) $V_M$ and $\Lambda_M$. So, while (this version of) quantum mechanics is intrinsically non-linear, it may well be negligibly so. In this sense, the present result extends rather than abolishes the hitherto known quantum mechanics.

In addition, note that $b$ in (2.4b) and $\Lambda_M(\psi)$ in general (2.3b) are really functions of time. If chosen so that they (well-nigh) vanish except at the time of measurement when they are non-negligible, the cherished linearity of quantum mechanics is (well-nigh exactly) recovered—except during the event and at the ‘location’ of measurement.

The erudite Reader will have realized that the simple toy model (2.4) is but a special case of the ‘predator-pray’ system, and that (2.3) generalizes this considerably. These systems being non-linear, there definitely exist whole uncharted worlds of chaotic regimes and effects. Note, however, that these are radically different (and presumably wilder) than the effects studied in what is called ‘quantum chaos’.

In any case (cf. Ref. [1]), the non-linearity of the process of the “wave-function collapse” or “state vector projection” (axiom 5) is now seen to be perfectly consistent with the dynamical evolution law (1.3) (axiom 6).

4. A Diatomic Drill

Non-linearity is rather commonplace in practically all (classical and quantum) field theory, and so for this audience the present results may appear unsurprising. However, to the best of understanding of the present author, it is not widely popularized that such intrinsic non-linearity naturally extends (descends?) to quantum mechanics. The purpose of this section is to demonstrate that non-linearity is really not novel in quantum mechanics either; the outright statement of this non-linearity however is.

Consider a once ionized Hydrogen molecule. Following a standard textbook [4] (§ 18.1), we first assume that the two protons are at a fixed distance $R$, and let $r_a$ and $r_b$ denote the distances of the electron from one and the other proton, respectively. The Schrödinger equation for the state vector of the electron, $|\psi\rangle$, is then

$$-\left[\frac{\hbar^2}{2m} \nabla^2 + \frac{e^2}{4\pi\epsilon_0} \left( \frac{1}{r_a} + \frac{1}{r_b} \right) \right] |\psi\rangle = E_{\text{elec.}} |\psi\rangle,$$

or

$$H_{\text{elec.}} |\psi\rangle \overset{\text{def}}{=} -\left[\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{e^2}{4\pi\epsilon_0} \left( \frac{1}{r_a} + \frac{1}{\sqrt{r_a^2 + R^2 - 2r_a R \cos \theta}} \right) \right] |\psi\rangle = E_{\text{elec.}} |\psi\rangle,$$

where $\theta$ is the angle between $\vec{R}$ and $\vec{r}_a$.

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8) Here, ‘location’ refers to a subdomain in space for coordinate representation, in momentum-space for the momentum representation, etc.
Of course, the distance $R$ between the two nuclei is not fixed, but is an observable, to be calculated as an expectation value and using the state vector $|\varphi\rangle$ that describes the state of the two protons. For the latter one, there will then exist a Schrödinger equation of the general type

$$H_{\text{nuc.}} |\varphi\rangle \overset{\text{def}}{=} \left[ -\frac{\hbar^2}{2M} \left( \vec{\nabla}^2_1 + \vec{\nabla}^2_2 \right) + \frac{2e^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|} + V_{\text{elec.}}(R) \right] |\varphi\rangle = E_{\text{nuc.}} |\varphi\rangle , \quad (4.3)$$

where $\vec{r}_i$ is the position vector of the $i^{\text{th}}$ nucleus, $\vec{\nabla}^2_i$ the Laplacian with respect to the coordinates of the $i^{\text{th}}$ nucleus.

Finally, note that

$$V_{\text{elec.}}(R) \overset{\text{def}}{=} \langle \psi | H_{\text{elec.}} | \psi \rangle , \quad R \overset{\text{def}}{=} \langle \varphi || \vec{r}_1 - \vec{r}_2 || \varphi \rangle , \quad (4.4)$$

which clearly provide a non-linear coupling between Eqs. (4.2) and (4.3). Also, one may interpret this set-up as the two nuclei, represented by $|\varphi\rangle$, ‘observing’ or ‘measuring’ (certain characteristics of) the electron, represented by $|\psi\rangle$, and the other way around.

In the linear approach, as in Ref. [4], § 18.2, one introduces a total state vector, $|\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_{\text{el}})\rangle$, for the 3-body system as a whole, and use the total Hamiltonian

$$H_{\text{tot.}} = -\frac{\hbar^2}{2M} \left( \vec{\nabla}^2_1 + \vec{\nabla}^2_2 \right) + \frac{2e^2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_1 - \vec{r}_2|}$$

$$- \frac{\hbar^2}{2m} \left( \vec{\nabla}^2_{\text{el}} \right) - \frac{2e^2}{4\pi\epsilon_0} \left( \frac{1}{|\vec{r}_1 - \vec{r}_{\text{el}}|} + \frac{1}{|\vec{r}_2 - \vec{r}_{\text{el}}|} \right)$$

which is independent of $|\Psi\rangle$. In this approach, the 3-body system is an indivisible whole and the Schrödinger equation with the Hamiltonian (4.5) is linear in $|\Psi\rangle$. Any (meta-)measurement of this 3-body system must be done by an external agent (metaobserver), with a corresponding term added to the Hamiltonian. This meta-observer itself however remains undescribed by quantum mechanics until either a non-linearly coupled system of equations akin to (2.3) is given, or $|\Psi\rangle$ is extended by another factor for this meta-observer. In the latter case, one needs a meta-meta-observer to collapse this new wavefunction...

This hopefully convinces the Reader that quantum mechanics can either be linear or describe the measurement process, but never both simultaneously.

References

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