BIRATIONAL SUPERRIGIDITY AND K-STABILITY OF
PROJECTIVELY NORMAL FANO MANIFOLDS OF INDEX ONE

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Abstract. We prove that every projectively normal Fano manifold in $\mathbb{P}^{n+r}$ of
index 1, codimension $r$ and dimension $n \geq 10r$ is birationally superrigid and
K-stable. This result was previously proved by Zhuang under the complete
intersection assumption.

1. Introduction

Birational superrigidity and K-stability of Fano manifolds are two important
notions with different backgrounds. The notion of birational superrigidity is mo-
tivated by the rationality problem of Fano manifolds. A Fano manifold $X$ with
the Picard number 1 is called birationally superrigid if any birational map from
$X$ to the source of another Mori fiber space is an isomorphism. It implies that $X$
is non-rational and $\text{Bir}(X) = \text{Aut}(X)$. On the other hand, the notion of K-stability
is motivated by the existence of Kähler-Einstein metric on Fano manifolds. A Fano
manifold $X$ is called K-stable if the Donaldson-Futaki invariant is positive for any
non-trivial normal test configuration. It is stronger than the K-polystability which
is equivalent to the existence of Kähler-Einstein metric [6, 7, 8, 32]. Birational
superrigidity and K-stability are unexpectedly related according to Odaka-Okada
and Stibitz-Zhuang [23, 30], and it is conjectured by Kim-Okada-Won [18] that
every birationally superrigid Fano manifold is K-stable. Both the notions are in-
tensively studied in the case of smooth Fano complete intersections of index 1:
birational superrigidity by Iskovskih-Manin, Pukhlikov, Cheltsov, de Fernex-Ein-
Mustaţă, de Fernex, Suzuki, and Zhuang [17, 24, 25, 26, 28, 29, 2, 12, 11, 10, 31, 34],
(see also the note [20] written by Kollár), and K-stability by Fujita, Stibitz-Zhuang,
and Zhuang [14, 20, 34]. Among them, Zhuang [34] proves that every smooth Fano
complete intersection of index 1 and small codimension is birationally superrigid
and K-stable.

In this paper, we replace the complete intersection assumption of the main
theorem of [34] by the projective normality:

Theorem 1.1. Every projectively normal Fano manifold in $\mathbb{P}^{n+r}$ of index 1, codi-

menion $r$, and dimension $n \geq 10r$ is birationally superrigid and K-stable.

Remark 1.2. So far we do not have any non-complete-intersection examples which
satisfy the assumption of Theorem 1.1. In fact, the Hartshorne conjecture predicts
that every smooth projective variety in $\mathbb{P}^{n+r}$ of codimension $r$ and dimension $n > 2r$
is a complete intersection, while the conjecture is widely open.

A key step involves a generalization of multiplicity bounds for cycles on smooth
complete intersections due to Pukhlikov [27, Proposition 5], Cheltsov [4, Lemma
13], and the author [31, Proposition 2.1].

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This paper is organized as follows. In Section 2, we study the ramification locus of the linear projection from a closed point. In Section 3, we prove multiplicity bounds for cycles. In Section 4, we prove a stronger version of Theorem [11, Theorem 1.1]. In Section 5, we discuss the singular case.

Throughout this paper, the base field is the field of complex numbers $\mathbb{C}$.

**Notation.** Let $X$ be a complete algebraic scheme.

- We denote by $[X]$ the fundamental cycle of $X$.
- For a closed subvariety $Z \subset X$, let $e_Z(X)$ be the Samuel multiplicity of $X$ along $Z$. We extend the definition of Samuel multiplicities to arbitrary cycles by linearity.
- For pure-dimensional cycles $\alpha, \beta$ on $X$ intersecting properly and as an irreducible component $Z$ of the intersection, we denote by $i(Z, \alpha \cdot \beta; X)$ the intersection multiplicity of $Z$ in $\alpha \cdot \beta$ whenever the intersection product $\alpha \cdot \beta$ is defined at $Z$.
- For a closed subscheme $Z \subset X$, we denote by $s(Z, X)$ be the Segre class of $Z$ in $X$.
- For a vector bundle $E$ on $X$, we denote by $c(E)$ the total Chern class of $E$, and by $c_i(E)$ the $i$-th Chern class of $E$.
- We denote by $Z_i(X)$ the group of $i$-cycles on $X$, and by $CH_i(X)$ (resp. $N_i(X)$) the group of those modulo the rational equivalence (resp. the numerical equivalence). When $X$ is smooth, we denote by $N^1(X)$ the group of $j$-cocycles on $X$ modulo the numerical equivalence.

For the definitions, we refer the reader to [15].

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2. THE RAMIFICATION LOCUS OF THE LINEAR PROJECTION FROM A CLOSED POINT

Let $X \subset \mathbb{P}^{n+r}$ be a non-degenerate smooth projective variety of dimension $n$ and codimension $r$. We take a closed point $p \in \mathbb{P}^{n+r}$ not contained in $X$. The choice of $p$ determines a section $s \in \Gamma(X, N_{X/\mathbb{P}^{n+r}}(-1))$. Let $\pi_p: X \to \mathbb{P}^{n+r-1}$ be the restriction of the linear projection from $p$. We define the ramification locus $R(\pi_p) \subset X$ of $\pi_p$ as the zero-scheme of the section $s$ [15, Example 14.4.15]. We consider the following condition $(\ast)$: given any closed point $p \in \mathbb{P}^{n+r}$ not contained in $X$, we have $Z \cap R(\pi_p) \neq \emptyset$ for any closed subset $Z \subset X$ of dimension $\geq r$.

**Lemma 2.1.** Assume that $X$ is a complete intersection. Then the condition $(\ast)$ is satisfied.

*Proof.* The ramification locus $R(\pi_p)$ is globally defined by $r$ hypersurfaces on $X$. The condition $(\ast)$ is obviously satisfied. $\square$

**Lemma 2.2.** Assume $n \geq 3r - 2$. Then the condition $(\ast)$ is satisfied.

*Proof.* We may assume that $r \geq 2$. We denote by $\text{Tan}(X)$ the tangent variety of $X$. We prove $\text{Tan}(X) = \mathbb{P}^{n+r}$. We denote by $\text{Sec}(X)$ the secant variety of $X$. We have $\text{Tan}(X) = \text{Sec}(X)$ as long as $n \geq r$ by the connectedness theorem of Fulton and Hansen [21, Corollary 3.4.5]. On the other hand, we have $\text{Sec}(X) = \mathbb{P}^{n+r}$ as long as $n \geq 2r$ by Zak’s theorem on linear normality [21, Corollary 3.4.26]. Therefore we have the desired equality of sets under our assumption. It follows that $R(\pi_p)$ is non-empty, and $\dim R(\pi_p) = n - r$ if $p$ is general. We check the condition $(\ast)$. 
We may assume that \( p \) is general so that \( R(\pi_p) \) is defined by a regular section. We have

\[ [R(\pi_p)] = c_r(N_{X/P^n+r}(-1)) \cap [X] \]

by \cite{15} Proposition 14.1. For the right hand side, we have

\[ c_r(N_{X/P^n+r}(-1)) \cap [X] = \sum_{i \geq 0} (-c_1(\mathcal{O}_X(1)))^i \cdot c_{r-i}(N_{X/P^n+r} \cap [X]) \]

by \cite{15} Example 3.2.2. We have

\[ c_r(N_{X/P^n+r}) \cap [X] = \sum_{i \geq 0} (\deg X \cdot c_1(\mathcal{O}_X(1)))^i \cap [X] \]

by \cite{15} Corollary 6.3. On the other hand, we have

\[ H^{2i}(X, Z) = Z \cdot (c_1(\mathcal{O}_X(1))^i \cap [X]) \]

for any \( 0 \leq i \leq r-1 \)

by the Barth-Larsen theorem \cite{21} Theorem 3.2.1 under the assumption \( n \geq 3r-2 \).

Therefore we have

\[ [R(\pi_p)] = a \cdot c_1(\mathcal{O}_X(1))^r \cap [X] \in H^{2r}(X, \mathbb{Z}) \]

for some positive integer \( a \). For any closed subset \( Z \subset X \) of dimension \( \geq r \), we have

\[ \deg(Z \cdot [R(\pi_p)]) = a \cdot \deg Z \neq 0, \]

which implies \( Z \cap R(\pi_p) \neq 0 \). The proof is done.

\[ \square \]

**Lemma 2.3.** Assume \( r = 2 \). Then the condition (⋆) is satisfied.

**Proof.** It follows from \cite{33} Chapter II, Corollary 2.5. \( \square \)

3. Multiplicity bounds for cycles

In this section, we prove the following key proposition:

**Proposition 3.1.** Let \( X \subset \mathbb{P}^N \) be a non-degenerate smooth projective variety of codimension \( r \). Assume that \( X \) satisfies the condition (⋆). Let \( \alpha \) be an effective cycle on \( X \) of codimension \( s \) such that

\[ \alpha = m \cdot c_1(\mathcal{O}_X(1))^s \cap [X] \in N^s(X). \]

Then we have \( e_x(\alpha) \leq m \) for any closed point \( x \in X \) away from a closed subset of dimension \( < r s \), where we use the convention \( \dim(\emptyset) = -1 \).

**Remark 3.2.** The complete intersection case is proved by Pukhlikov \cite{27} Proposition 5, Cheltsov \cite{4} Lemma 13, and the author \cite{31} Proposition 2.1.

**Step 1:** We review the construction of residual intersection classes due to Fulton.

**Theorem 3.3** (\cite{15}, Theorem 9.2). We consider a diagram

\[
\begin{array}{ccc}
R & \xrightarrow{a} & W \\
\downarrow b & & \downarrow j \\
D & \xrightarrow{i} & V \\
\downarrow g & & \downarrow f \\
X & \xrightarrow{i} & Y \\
\end{array}
\]

where the square is a fiber square, \( i, j, a, b \) are closed embeddings, and \( V \) is a \( k \)-dimensional variety. Assume that:

(i) \( i \) is a regular embedding of codimension \( r \);
(ii) \( ja \) embeds \( D \) as a Cartier divisor of \( V \);
(iii) \( R \) is the residual scheme to \( D \) in \( W \).
Let $N = g^* N_X Y$ and $\mathcal{O}(-D) = j^* \mathcal{O}_V(-D)$. We define the residual intersection class $\mathbb{R} \in CH_{k-r}(R)$ by the formula
\[
\mathbb{R} = \{ c(N \otimes \mathcal{O}(-D)) \cap s(R, V) \}_{k-r}.
\]
Then we have
\[
X \cdot_Y V = \{ c(N) \cap s(D, V) \}_{k-r} + \mathbb{R}
\]
in $CH_{k-r}(W)$.

**Remark 3.4.** In the setting of Theorem 3.3, the class $\mathbb{R}$ is represented by an effective cycle if $\dim \mathbb{R} = k-r$. Indeed, let $R_1, \ldots, R_t$ be the irreducible components of $\mathbb{R}$.

Then we have
\[
\mathbb{R} = s(R, V)_{k-r} = \sum_{i=1}^{t} e_i[R_i],
\]
where $e_i$ is the same as Samuel’s multiplicity $e(q)$ of the primary ideal $q$ determined by $R$ in the local ring $\mathcal{O}_{V,R_i}$, which is positive [15, Corollary 9.2.2].

We construct a residual intersection class associated to the linear projection from a closed point. Let $X \subseteq \mathbb{P}^N$ be a non-degenerate smooth projective variety of codimension $r$. We assume that $X$ satisfies the condition $(\ast)$. Let $Z \subseteq X$ be a closed subvariety with $\dim Z \geq r$. Let $p \in \mathbb{P}^N$ be a general closed point. Let $C = \bigcup_{x \in Z} \langle p, x \rangle$ be the cone of $Z$ with the vertex $p$. Let $R$ be the residual set to $Z$ in $X \cap C$. We have a diagram with a fiber square
\[
\begin{array}{ccc}
R & \rightarrow & \tilde{R} \\
\downarrow & & \downarrow \\
Z & \rightarrow & \tilde{X} \cap \tilde{C} \\
\downarrow & & \downarrow \\
X & \rightarrow & \mathbb{P}^N
\end{array}
\]

We want to define the residual intersection class $\mathbb{R} \in CH_{\dim Z+1-1}(R)$, but $Z \subset C$ is not a Cartier divisor in general.

To remedy this situation, we consider the projective bundle $\mathbb{P} = \mathbb{P}_{\mathbb{P}^N}(\mathcal{O} \oplus \mathcal{O}(-1))$ with closed subvarieties $E = \mathbb{P}_{\mathbb{P}^N}(\mathcal{O})$, $F = \mathbb{P}_{\mathbb{P}^N}(\mathcal{O}(-1))$, and $\tilde{X} = \mathbb{P}_X(\mathcal{O} \oplus \mathcal{O}(-1))$. The projective bundle $\mathbb{P}$ is the blow-up of $\mathbb{P}^{N+1}$ along a closed point $q$ with the exceptional divisor $E$; the natural projection $\mathbb{P} \rightarrow \mathbb{P}^N$ is the resolution of the linear projection $\mathbb{P}^{N+1} \rightarrow \mathbb{P}^N$ from $q$. Moreover we identify $F$ with the base $\mathbb{P}^N$, and also with its image in $\mathbb{P}^{N+1}$, a hyperplane disjoint from $q$. Let $\tilde{p} \in (p, q)$ be a closed point different from $p \in F$ and $q$. Let $\tilde{C}$ be the cone of $Z \subset F$ with the vertex $\tilde{p}$. Then $Z \subset \tilde{C}$ is a Cartier divisor. In addition, we have $q \notin \tilde{C}$ by the generality of $p$, thus we identify $\tilde{C}$ with its inverse image in $\mathbb{P}$. We let $\tilde{R}$ be the residual scheme to $Z$ in $\tilde{X} \cap \tilde{C}$. We have a diagram with a fiber square
\[
\begin{array}{ccc}
\tilde{R} & \rightarrow & \tilde{R} \\
\downarrow & & \downarrow \\
\tilde{X} \cap \tilde{C} & \rightarrow & \tilde{C} \\
\downarrow & & \downarrow \\
X & \rightarrow & \mathbb{P}^N
\end{array}
\]
Let $\tilde{R} \in CH_{\dim Z+1-r}(\tilde{R})$ be the residual intersection class. Let $f: \tilde{C} \to C$ be the restriction of the natural projection $\mathbb{P} \to \mathbb{P}^N$, which is a finite morphism. We replace $R$ by $f(\tilde{R})$ if necessary. We define $\mathbb{R}$ to be $f_\ast \tilde{R}$.

**Lemma 3.5.** We have

$$c_1(\mathcal{O}_X(1)) \cap \mathbb{R} = c_r(N_{X/\mathbb{P}^N}(-1)) \cap [Z].$$

*Proof.* We apply Theorem 3.3 to the diagram (2). We have

$$X \cdot \mathbb{P} \cdot \tilde{C} = \left\{ c(N) \cap s(Z, \tilde{C}) \right\}_{\dim Z+1-r} + \tilde{R}$$

in $CH_{\dim Z+1-r}(\tilde{X} \cap \tilde{C})$. Since the morphism $f$ is generically one-to-one by the generality of $p$, we have $f_\ast [\tilde{C}] = [C]$. Thus we have

$$X \cdot C = f_\ast (X \cdot \mathbb{P} \cdot \tilde{C}) = \left\{ c(N) \cap f_\ast s(Z, \tilde{C}) \right\}_{\dim Z+1-r} + \mathbb{R}$$

in $CH_{\dim Z+1-r}(X \cap C)$, where the first equality follows from [15, Theorem 6.2 (a)]. Since the universal sub line bundles on $\mathbb{P}^N$, $\mathbb{P}^N_{X+1}$, and $\mathbb{P}$ all coincide on $\tilde{C}$, we use the same notation $\mathcal{O}(-1)$ when restricted. We have

$$c(N) \cap f_\ast s(Z, \tilde{C}) = c(N) \cap f_\ast (c(\mathcal{O}(1))^{-1} \cap [Z]) = \sum_{i,j \geq 0} (-c_1(\mathcal{O}(1)))^i \cdot c_j(N) \cap [Z],$$

where the first equality follows from [15, Proposition 4.1 (a)]. Then we have

$$\mathbb{R} = X \cdot C + \sum_{i \geq 1} (-1)^{i} c_1(\mathcal{O}_X(1))^{i-1} \cdot c_{r-i}(N_{X/\mathbb{P}^N}) \cap [Z].$$

Applying $c_1(\mathcal{O}_X(1)) \cap$ to the both sides, we have

$$c_1(\mathcal{O}_X(1)) \cap \mathbb{R} = X \cdot Z + \sum_{i \geq 1} (-c_1(\mathcal{O}_X(1)))^i \cdot c_{r-i}(N_{X/\mathbb{P}^N}) \cap [Z]$$

$$= c_r(N_{X/\mathbb{P}^N}(-1)) \cap [Z],$$

where the first equality follows from [15, Proposition 6.3] with

$$c_1(\mathcal{O}(1)) \cap [C] = f_\ast (c_1(\mathcal{O}(1)) \cap [\tilde{C}]) = [Z]$$

and the second equality follows from the self-intersection formula [15, Corollary 6.3]. The proof is done. □

**Lemma 3.6.** As a set, we have

$$Z \cap \tilde{R} = Z \cap R(\pi_p).$$

*Proof.* It follows from the same argument as in the proof of [27, Lemma 3]. □

By the condition $(\ast)$, we have $Z \cap R(\pi_p) \neq \emptyset$ and $\dim Z \cap R(\pi_p) = \dim Z - r$. Therefore we have $Z \cap \tilde{R} \neq \emptyset$ and $\dim Z \cap \tilde{R} = \dim Z - r$. It implies that $\dim \tilde{R} = \dim Z + 1 - r$. Therefore the class $\tilde{R}$ is represented by an effective $(\dim Z + 1 - r)$-cycle whose support is $\tilde{R}$, which implies that the class $\mathbb{R}$ is represented by an effective $(\dim Z + 1 - r)$-cycle whose support is $\tilde{R}$.

**Remark 3.7.** There is another possible definition of the residual intersection class $\mathbb{R} \in CH_{\dim Z+1-r}(R)$ for the diagram (1) using the blow-up along $Z$. We refer the reader to [15, Definition 9.2.2] for the details. We prove $\mathbb{R} = \mathbb{R}$. We apply [15, Corollary 9.2.3] to the diagram (1). We have

$$X \cdot C = \left\{ c(N) \cap s(Z, C) \right\}_{\dim Z+1-r} + \mathbb{R}.$$
The inverse image scheme $f^{-1}(Z) \subset \tilde{C}$ is the union of $Z$ and a closed subscheme supported on the inverse image $f^{-1}(Z^{ne})$ of the non-embedding locus $Z^{ne}$ on $Z$ of the linear projection from $p$. We have

$\{c(N) \cap s(Z, C)\}_{dim Z+1-r} = \{c(N) \cap f_\ast s(Z, \tilde{C})\}_{dim Z+1-r}$

by [15] Proposition 4.2] together with $dim Z^{ne} \leq dim Z - r$. Therefore we have the desired equality. The proof is done.

**Step 2:** We move cycles by multiple residual intersections.

**Lemma 3.8.** Let $X \subset \mathbb{P}^N$ be a non-degenerate smooth projective variety of codimension $r$. Assume that $X$ satisfies the condition $(\ast)$. Let $A \subset X$ be a closed subset of codimension $s$. Let $Z \subset A$ be a subvariety of dimension $rs$. Then there is a cycle $\beta$ on $X$ such that

(i) $dim \beta = s$, and $\beta$ intersects $A$ in finitely many points;

(ii) $|Z \cap Supp(\beta)| \geq deg \beta$.

**Proof.** The proof is essentially the same as in [27, Proposition 5] (see also [31, Proposition 2.1]). We only give a sketch. We take a general closed point $p \in \mathbb{P}^N$ and construct $R$ and $\mathcal{R}$ as in Step 1. We define $p_1 = p$, $R_1 = R$ and $\mathcal{R}_1 = \mathcal{R}$. We replace $R_0 = Z$ by $R_1$, and repeat the same procedure to define $p_j$, $R_j$ and $\mathcal{R}_j$ for $j = 1, \ldots, s$. We have $dim R_j = rs - (r - 1)j$ and

$$deg \mathcal{R}_j = deg \left( (c_r(N_{X/\mathbb{P}^N}(-1))^j \cap [Z]) \right).$$

In particular, we have $dim R_s = s$ and

$$deg \mathcal{R}_s = deg \left( (c_r(N_{X/\mathbb{P}^N}(-1))^s \cap [Z]) \right).$$

By careful dimension count using the ramification locus and joins of varieties as in [27, Lemma 1] (see also [31, Lemma 2.3] and [31, Lemma 2.4]) and by the condition $(\ast)$, we have

$$dim A \cap R_j = dim A \cap R_{j-1} \cap R(\pi_p) = dim A \cap R_{j-1} - r.$$

By induction, we have

$$dim A \cap R_j = rs - rj.$$

In particular, we have

$$dim A \cap R_s = 0.$$

Moreover we have

$$Z \cap R_s \supset Z \cap R_1 \cap \cdots \cap R_s \supset Z \cap R(\pi_{p_1}) \cap \cdots \cap R(\pi_{p_j})$$

and

$$|Z \cap R(\pi_{p_1}) \cap \cdots \cap R(\pi_{p_j})| = \dim \left( (c_r(N_{X/\mathbb{P}^N}(-1))^s \cap [Z]) \right).$$

Therefore $\beta = \mathcal{R}_s$ satisfies the condition. □

**Step 3:**

**Proof of Proposition 3.7.** We take $\alpha$ as in the statement. By the upper semicontinuity of Samuel multiplicity, it is enough to prove that $e_Z(\alpha) \leq m$ for any closed subvariety $Z \subset X$ of dimension $rs$. We may assume that $\alpha \neq 0$ and $Z \subset Supp(\alpha)$. Then there is an effective cycle $\beta$ on $X$ such that

(i) $dim \beta = s$, and $\beta$ intersects $\alpha$ in finitely many closed points;

(ii) $|Z \cap Supp(\beta)| \geq deg \beta.$
We have
\[
m \cdot \deg(\beta) = \alpha \cdot \beta = \sum_{x \in \text{Supp}(\alpha) \cap \text{Supp}(\beta)} i(x, \alpha \cdot \beta; X) \geq \sum_{x \in \mathbb{Z} \cap \text{Supp}(\beta)} i(x, \alpha \cdot \beta; X) \geq \sum_{x \in \mathbb{Z} \cap \text{Supp}(\beta)} e_x(\alpha) \cdot e_x(\beta) \geq e_{\mathbb{Z}}(\alpha) \cdot \deg(\beta),
\]
where the second (resp. third) inequality follows from [15, Corollary 12.4] (resp. the upper semi-continuity of Samuel multiplicity). We divide the both sides by \(\deg(\beta)\). The proof is done. \(\square\)

4. Proof of Theorem 1.1

We prove a stronger version of Theorem 1.1:

**Theorem 4.1.** Let \(X \subset \mathbb{P}^{n+r}\) be a Fano manifold of index 1, dimension \(n\) and codimension \(r\). Assume that \(X\) is \(2r\)-normal, that is, the restriction map
\[
H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(2r)) \rightarrow H^0(X, \mathcal{O}_X(2r))
\]
is surjective, and \(n \geq 10r\). Then \(X\) is birationally superrigid and K-stable.

**Lemma 4.2.** Let \(X \subset \mathbb{P}^N\) be a Fano manifold with the anti-canonical class a multiple of the hyperplane section class. Let \(Y \subset X\) be a positive-dimensional linear section. If \(X\) is \(k\)-normal, so is \(Y\).

**Proof.** It is enough to prove that, if \(Y \subset X\) is a positive-dimensional linear section, the restriction map
\[
H^0(X, \mathcal{O}_X(k)) \rightarrow H^0(Y, \mathcal{O}_Y(k))
\]
is surjective for any \(k \in \mathbb{Z}\). It is enough to prove that, if \(Y \subset X\) is a linear section, we have
\[
H^i(Y, \mathcal{O}_Y(j)) = 0 \text{ for any } 0 < i < \dim Y \text{ and } j \in \mathbb{Z}.
\]
It is enough to prove
\[
H^i(X, \mathcal{O}_X(j)) = 0 \text{ for any } 0 < i < n \text{ and } j \in \mathbb{Z}.
\]
It follows from the Kodaira vanishing theorem. The proof is done. \(\square\)

**Proof of Theorem 4.1.** The proof is essentially the same as in [34, Theorem 1.2] and [35, Lemma 3.6]. Let \(X\) as in the statement. We have \(-K_X = c_1(\mathcal{O}_X(1)) \cap [X]\) and \(r \geq 1\). We have \(\text{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]\) as long as \(n \geq r + 2\) by the Barth-Larsen theorem [21, Theorem 3.2.1]. The inequality \(n \geq r + 2\) follows from \(r \geq 1\) and the assumption \(n \geq 10r\). We may assume that \(X\) is non-degenerate. The condition (*) is satisfied by Lemma 2.2 together with the assumption \(n \geq 10r\) (or by Lemma 2.3 in the case \(r = 2\)).

We prove that \(X\) is birationally superrigid. By the Noether-Fano inequality [5, Theorem 1.26], it is enough to prove that for any movable linear system \(\mathcal{M} \subset -mK_X\), the pair \((X, \frac{1}{m}\mathcal{M})\) is canonical.

**Step 1:** We prove that
(i) there exists a closed subset of dimension \(Z \subset X\) of dimension \(\leq r - 1\) such that the pair \((X, \frac{1}{m}\mathcal{M})\) is canonical away from \(Z\);
(ii) there exists a closed subset \(Z' \subset X\) of dimension \(\leq 2r - 1\) such that the pair \((X, \frac{1}{m}\mathcal{M})\) is log canonical away from \(Z'\).
For (i), we take $D \in \mathcal{M}$. Then

$$[D] = m \cdot c_1(O_X(1)) \cap [X] \in N^1(X).$$

By Proposition 3.1 there exists a closed subset $Z$ of dimension $\leq r - 1$ such that $e_z(D) \leq m$ for any closed point $x \in X$ away from $Z$. Therefore the pair $(X, \frac{1}{m} D)$ is canonical away from $Z$ by [13, 3.14.1]. For (ii), we take $D_1, D_2 \in \mathcal{M}$ intersecting properly. Let $B = D_1 \cap D_2$. Then

$$[B] = m^2 \cdot c_1(O_X(1))^2 \cap [X] \in N^2(X).$$

By Proposition 3.1 there exists a closed subset $Z' \subset X$ such that $e_z(B) \leq m^2$ for any closed point $x \in X$ away from $Z'$. For any such $x$, let $S \subset X$ be a general linear section of dimension $2$ through $x$. Then the pair $(S, \frac{1}{m} D |_S)$ is log canonical at $x$ by [11, Theorem 0.1]. By inversion of adjunction [13, Theorem 1.1], the pair $(X, \frac{1}{m} D)$ is log canonical at $x$.

Step 2: We prove that for any closed point $x \in X$ and any general linear section $Y \subset X$ of codimension $2r - 1$ through $x$, the pair $(Y, \frac{1}{m} M |_Y)$ is Kawamata log terminal (klt) at $x$. We have $K_Y = 2(r - 1) \cdot c_1(O_Y(1)) \cap [Y]$. Let $L = O_X(2r)$. Then $L \sim Q K_Y + \frac{1}{m} M |_Y$. The pair $(Y, (1 - \epsilon) \frac{1}{m} M |_Y)$ is klt away from a finite set for all $0 < \epsilon \ll 1$, and we have

$$h^0(Y, L) \leq h^0(P^{n-r+1}, O_{P^{n-r+1}}(2r)) = \binom{n + r + 1}{2r} \leq \frac{(n - 2r + 1)^{n-2r+1}}{(n - 2r + 1)!},$$

where the first (resp. second) inequality follows from Lemma 1.2 together with the $2r$-normality of $X$ (resp. the assumption $n \geq 10r$). Then the pair $(Y, \frac{1}{m} M |_Y)$ is klt by [34, Corollary 1.8].

Step 3: Assume that the pair $(X, \frac{1}{m} M)$ is not canonical at $x \in Z$. Let $Y \subset X$ be a general linear section of codimension $2r - 1$ through $x$. Then the pair $(Y, \frac{1}{m} M |_Y)$ is not log canonical at $x$ by inversion of adjunction, a contradiction.

We prove that $X$ is $K$-stable. Let

$$\alpha(X) = \sup \{ t \mid (X, tD) \text{ is log canonical for any } D \in | - K_X |_Q \}$$

be the alpha invariant. By [30, Theorem 1.2], it is enough to prove that $\alpha(X) > \frac{1}{2}$. By [1, Theorem 1.5], it is enough to prove that the pair $(X, \frac{1}{2} D)$ is klt for any $D \in | - K_X |_Q$. The proof is similar using Proposition 3.1 and [34, Corollary 1.8].

The proof is done.

By analyzing the proof of Theorem 4.1 we can further strengthen the statement:

**Theorem 4.3.** Let $X \subset \mathbb{P}^{n+r}$ be a Fano manifold of index $1$, dimension $n$ and codimension $r$. Assume

$$\sum_{i=0}^{2r-1} (-1)^i \cdot \binom{2r-1}{i} \cdot h^0(O_X(2r-i)) < \frac{(n - 2r + 1)^{n-2r+1}}{(n - 2r + 1)!},$$

and $n \geq 3r - 2$. Then $X$ is birationally superrigid and $K$-stable.

**Proof.** We have

$$h^0(Y, O_Y(2r)) = \sum_{i=0}^{2r-1} (-1)^i \cdot \binom{2r-1}{i} \cdot h^0(O_X(2r-i))$$

for any linear section $Y \subset X$ of codimension $2r - 1$ by an argument similar to one in Lemma 1.2. Therefore the first assumption is equivalent to

$$h^0(Y, O_Y(2r)) < \frac{(n - 2r + 1)^{n-2r+1}}{(n - 2r + 1)!}$$

for any such $Y$. We have $n > 2r$ after all. We omit the rest of the proof. 

\[\square\]
5. The singular case

Due to Liu and Zhuang [22], the result of Zhuang [34] is generalized to the singular case (the notions of birational superrigidity and K-stability can be defined for Q-Fano varieties). Replacing the complete intersection assumption by the local complete intersection and projective normality, we prove:

**Theorem 5.1.** For integers \( \delta \geq -1 \) and \( r \geq 1 \), there exists a positive integer \( n_0(r, \delta) \) depending only on \( \delta \) and \( r \) such that, if \( X \subset \mathbb{P}^{n+r} \) is a locally complete intersection projectively normal Fano variety of index 1, codimension \( r \) and dimension \( n \geq n_0(r, \delta) \) such that

1. \( \dim \operatorname{Sing}(X) \leq \delta; \)
2. every projective tangent cone of \( X \) is a Fano complete intersection of index at least \( 4r + 2\delta + 2 \) and is smooth in dimension \( r + \delta \),

then \( X \) is birationally superrigid and K-stable.

Let \( X \subset \mathbb{P}^{n+r} \) be a non-degenerate projective variety of dimension \( n \) and codimension \( r \). For a closed point \( p \in \mathbb{P}^{n+r} \) not contained in \( X \), we define the ramification locus \( R(\pi_p) \) of the restriction \( \pi_p: X \to \mathbb{P}^{n+r} \) of the linear projection from \( p \) as the zero scheme of the section of the twisted normal sheaf \( N_{X/\mathbb{P}^{n+r}}(-1) \) associated to \( p \). We consider the following condition (**): given any closed point \( p \in \mathbb{P}^{n+r} \) not contained in \( X \), we have \( Z \cap R(\pi_p) \neq \emptyset \) for any closed subset \( Z \subset X \) of dimension \( \geq r \) disjoint from \( \operatorname{Sing}(X) \).

**Lemma 5.2.** Assume that \( X \) is a complete intersection. Then the condition (**) is satisfied.

**Proof.** The proof is the same as in Lemma 2.1. \( \square \)

**Lemma 5.3.** Assume that \( X \) is locally complete intersection and

\[ n \geq \max \{3r - 2, 2r - 1 + \delta\}, \]

where \( \delta = \dim \operatorname{Sing}(X) \). Then the condition (**) is satisfied.

**Proof.** We denote by \( \operatorname{Tan}^r(X) \) the variety of tangent stars of \( X \) (see [33, Chapter I, Definition 1.2] for the definition). We prove \( \operatorname{Tan}^r(X) = \mathbb{P}^{n+r} \). We have \( \operatorname{Tan}^r(X) = \operatorname{Sec}(X) \) as long as \( n \geq r \) by the connectedness theorem of Fulton and Hansen [33, Chapter I, Theorem 1.4]. On the other hand, we have \( \operatorname{Sec}(X) = \mathbb{P}^{n+r} \) as long as \( n \geq 2r - 1 + \delta \) by Zak’s theorem on linear normality for singular varieties [33, Chapter II, Theorem 2.1]. Therefore we have the desired equality of sets under our assumption.

For a closed point \( p \in \mathbb{P}^{n+r} \) not contained in \( X \), we denote by \( R'(\pi_p) \) be the J-ramification locus of the restriction of the linear projection from \( p \) (see [33, Chapter II, Section 1] for the definition and property of unramified morphisms in the sense of Johnson, or J-unramified morphisms). Then \( R'(\pi_p) \neq \emptyset \), and \( \dim R'(\pi_p) \leq n - r \) for general \( p \). We prove that \( R(\pi_p) \neq \emptyset \), and \( \dim R(\pi_p) = n - r \) for general \( p \). By definition of \( R(\pi_p) \) and \( R'(\pi_p) \), we have

\[ R(\pi_p) \supseteq R'(\pi_p), \quad R(\pi_p) \cap X^{\text{sm}} = R'(\pi_p) \cap X^{\text{sm}}. \]

Thus the complement in \( R(\pi_p) \) of \( R'(\pi_p) \) is supported on \( \operatorname{Sing}(X) \), while we have \( \dim \operatorname{Sing}(X) = \delta \leq n - r \) by the assumption. Now it is enough to observe that \( R(\pi_p) \) is locally defined by \( r \) equations in \( X \) and we have \( \dim R(\pi_p) \geq n - r \). It follows that \( R(\pi_p) \) is defined by a regular section for general \( p \). We have

\[ [R(\pi_p)] = c_0(N_{X/\mathbb{P}^{n+r}}(-1)) \cap [X] \]

by [15, Proposition 14.1].
To check the condition (⋆⋆), it is enough to prove that
\[ c_r(N_{X/P^n+r}(-1)) \cap [Z] \neq 0 \]
for any closed subvariety \( Z \subset X \) of dimension \( r \). By the Barth-Larsen theorem for locally complete intersection varieties [21], Corollary 3.5.13, we have
\[ H_{2i}(X, Z) = H_{2i}(P^{n+r}, Z) \]
for any \( 0 \leq i \leq r - 1 \).
Therefore the homology class of a subvariety of \( X \) of dimension \( r - 1 \) is uniquely determined by its degree. Combined with the self-intersection formula [15], Corollary 6.3, it follows that the function
\[ Z_r(X) \to \mathbb{Z}, \, Z \mapsto c_r(N_{X/P^n+r}(-1)) \cap [Z] \]
is a \( \cdot \) deg \( Z \), where \( a \) is a constant not depending on \( Z \). Taking a general linear section of dimension \( r \), we have \( a \neq 0 \). The proof is done.

Lemma 5.4. Assume \( r = 2 \). Then the condition (⋆⋆) is satisfied.
Proof. It follows from [33, Chapter II, Corollary 2.5]. □

Proposition 5.5. Let \( X \subset P^n \) be a non-degenerate projective variety of codimension \( r \). Assume that \( X \) satisfies the condition (⋆⋆). Let \( \alpha \) be an effective cycle on \( X \) such that
\[ \alpha = m \cdot c_1(O_X(1))^s \cap [X] \in N_{N-r-s}(X). \]
Then we have \( c_Z(\alpha) \leq m \) for any closed point \( x \in X \) away from a closed subset of dimension \( \leq rs+\delta \), where \( \delta = \dim \operatorname{Sing}(X) \) and we use the convention \( \dim(\emptyset) = -1 \).

Remark 5.6. The complete intersection case is proved by Pukhlikov [27, Proposition 5] and the author [31, Proposition 2.1].

Proof. We take \( \alpha \) as in the statement. It is enough to prove that \( c_Z(\alpha) \leq m \) for any closed subvariety \( Z \subset X \) of dimension \( rs \) disjoint from \( \operatorname{Sing}(X) \). We move \( Z \) by multiple residual intersections so that the residual intersections avoid \( \operatorname{Sing}(X) \). The rest of the proof is similar. □

Proof of Theorem 5.7. The proof is essentially the same as in [22, Theorem 1.3]. We use the Barth-Larsen theorem for locally complete intersection varieties [21, Corollary 3.5.13] combined with the universal coefficient theorem to compute the Picard group. The condition (⋆⋆) is satisfied by Lemma 5.3 when \( n \) is large enough for fixed \( \delta \) and \( r \) (or by Lemma 5.3 in the case \( r = 2 \)). Then Proposition 5.5 is used to bound the dimension of the non-canonical, non-klt or non-lc locus of certain pairs on \( X \) as Proposition 5.1 in the proof of Theorem 1.1. We refer the reader to [22] for the details. □

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