Non-locality can be Shared between Alice and three Bobs in unbiased settings case

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A well-known property of quantum nonlocality is monogamy. However, recent research by Silva et al. shows that multiple observers can share the nonlocality by using weak measurements [Phys. Rev. Lett. 114, 250401 (2015)]. There is an open question left in their result: whether the nonlocality of a single particle from an entangled pair can be shared among more than two observers that act sequentially and independently of each other? In this work, we analytical and numerically shows that it is possible to observe a triple violation of CHSH inequality between Alice and three Bobs when the measurements of each of the several observers at one side are unbiased with respect to the previous observers. This result overturns the conclusions and proofs of previous related work on this issue which has been widely shared in the scientific community before.

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I. INTRODUCTION

Local realism indicates the nature of the world that the measurement outcomes are pre-deterministic, and the measurement on one party of a multipartite system does not affect the other parties. However, quantum mechanics predicts that there are stronger correlations than the correlations of local hidden variables because of inherent non-locality of quantum theory [1]. The so-called Bell inequality is exploited to distinguish the differences between classical correlation and quantum correlation [3]. Subsequently, Bell-type inequalities have been studied extensively from various perspectives [4–11] and experimentally verified in many different quantum systems [12–20]. These kind of research, is not only important to deeply understand quantum theory, but also plays an crucial role in quantum information protocols, such as quantum key distribution [21], randomness generation [22–24], and entanglement certification [25]. For a background on Bell inequalities readers are referred to [2], and references therein.

Inspired by Bell’s work, Clauser, Horne, Shimony, and Holt (CHSH) derived a modified inequality [4], which provides a faithful way for experimentally testing the non-locality property in 2-qubit composite systems. However, most discussions of non-locality based on CHSH inequality focus on one pair of entangled qubits distributed to only two separated observers. Recently, a surprising result that non-locality can actually be shared among more than two observers using weak measurements, has been reported by Silva et al. [26]. In Silva’s scenario, a pair of maximally-entangled qubits are distributed to three observers Alice, Bob1 and Bob2, in which Alice accesses one qubit and the two Bobs access the other qubit. While Bob1 performs a weak measurement on his qubit and then passes it to Bob2, Alice and Bob2 perform projective measurements on their own qubits respectively. The results show that it is possible to have two Bobs simultaneously violate CHSH with Alice.

To date, a series of fruitful related theoretical researches [27–39] have been proposed by tracking this path and three experimental demonstrations have also been performed [40–42]. Especially, in Ref.[42], it shows a observation of non-locality sharing in a wide area even if Bob1’s measurement is close to a strong measurement, which is impossible in the original protocol [26].

However, there is an open question remained: whether the non-locality of a single particle from an entangled pair can be shared among more than two observers that act sequentially and independently of each other? In the original seed work [26], Silva et al. conjectured that, in the case of multiple observers, if the inputs to the various Bobs are unbiased, it is impossible to have more than a double violation of CHSH with Alice according to their numerical calculation [26]. Later, Mal et al. claimed that it has been proved analytically [28]. So far, this conclusion has been widely introduced [27–33, 35–41], and some of which even were concluded based on this conclusion. Unfortunately, we have to point out that the previous conclusion is incorrect. In this work, we analytical and numerical shows that it is possible to observe a triple violation of CHSH inequality between Alice and three Bobs when the measurements of each of the several observers at one side are unbiased with respect to the previous observers. This result overturns the conclusions and proofs of previous related work on this issue.
II. THEORETICAL FRAMEWORK

![Bell state diagram](image)

FIG. 1: Scenario of non-locality sharing of Bell state. Bell states are distributed to four observers, Alice, Bob1, Bob2, and Bob3, where three Bobs have access to the same qubit. Alice and Bob3 implement projective measurements on their qubits, whereas Bob1 and Bob2 implement weak measurements on their qubit. All these observers measured independently.

Firstly, we should briefly review the scheme allowing nonlocal sharing as illustrated in [26]. In contrast to a typical Bell test scenario for CHSH inequalities, there are four observers, Alice, Bob1, Bob2, and Bob3, performing some measurements on a 2-qubit entangled state respectively, in which three Bobs access to the same one-half of the entangled state of spin-1/2 particles. In particular, each of the observers chooses independently between two different dichotomic observables, denoted by ỹ for Alice, ỹ1, ỹ2, and ỹ3 for Bob1, Bob2, and Bob3 respectively, where x, y, ỹ1, ỹ2, ỹ3 ∈ {0, 1}. The binary outcomes of the dichotomic measurements are given by a, b1, b2, and b3, with a, b1, b2, b3 ∈ {−1, 1}. Hence, three CHSH parameters ICHSH depending on Alice-Bob1, Alice-Bob2 and Alice-Bob3 can be described as

\[
I_{\text{CHSH}}^{(1)} = \langle \tilde{\omega}_0 \otimes \tilde{\mu}_0 \rangle + \langle \tilde{\omega}_1 \otimes \tilde{\mu}_1 \rangle - \langle \tilde{\omega}_1 \otimes \tilde{\mu}_0 \rangle + \langle \tilde{\omega}_0 \otimes \tilde{\mu}_1 \rangle - \langle \tilde{\omega}_0 \otimes \tilde{\mu}_0 \rangle
\]

\[
I_{\text{CHSH}}^{(2)} = \langle \tilde{\omega}_0 \otimes \tilde{\nu}_0 \rangle + \langle \tilde{\omega}_1 \otimes \tilde{\nu}_1 \rangle - \langle \tilde{\omega}_1 \otimes \tilde{\nu}_0 \rangle + \langle \tilde{\omega}_0 \otimes \tilde{\nu}_1 \rangle + \langle \tilde{\omega}_0 \otimes \tilde{\nu}_1 \rangle - \langle \tilde{\omega}_1 \otimes \tilde{\nu}_1 \rangle
\]

\[
I_{\text{CHSH}}^{(3)} = \langle \tilde{\omega}_0 \otimes \tilde{\nu}_0 \rangle + \langle \tilde{\omega}_1 \otimes \tilde{\nu}_0 \rangle - \langle \tilde{\omega}_1 \otimes \tilde{\nu}_0 \rangle - \langle \tilde{\omega}_0 \otimes \tilde{\nu}_1 \rangle - \langle \tilde{\omega}_1 \otimes \tilde{\nu}_1 \rangle - \langle \tilde{\omega}_1 \otimes \tilde{\nu}_1 \rangle
\]

Actually, if Alice and three Bobs perform strong measurements, according to monogamy property of non-locality, it is impossible to have a simultaneous CHSH violation. In this scenario, Alice performs strong measurements on one side, three Bobs perform unsharp measurements on the other side subsystem and Bob3 perform strong measurements which depend on the precision factor G and the quality factor F as introduced in [26]. Obviously, the crucial ingredient to achieve nonlocal sharing depends on the optimal trade-off between the information gain and disturbance.

Without loss of generality, supposed that Alice and Bob each possess one half of a singlet state of spin-1/2 particles,

\[
| \psi \rangle = \frac{1}{\sqrt{2}} (| 00 \rangle + | 11 \rangle)
\]

and ρ = | ψ ⟩⟨ ψ |. Alice performs a strong measurement on her system along the corresponding operators \( \tilde{\omega}_a \) with the outcome a. After Alice’s measurement, the state in Bob’s side will be the projector in a direction opposite to Alice’s post-measurement state, here defined as \( \rho_{a|\tilde{\omega}_a} \), which is not normalized. Subsequently, Bob1 performs a weak measurement on his subsystem by the corresponding operators \( \tilde{\mu}_{y_1} \) with the quality factor \( F_1 \) and precision factor \( G_1 \) of the measurement. The reduced density matrix becomes,

\[
\rho_{a,b_1|\tilde{\omega}_a,\tilde{\mu}_{y_1}} = \frac{F_1}{2} \rho_{a|\tilde{\omega}_a} + \left( \frac{1 + b_1 G_1}{2} \right) \rho_{a|\tilde{\omega}_a} \rho_{a|\tilde{\omega}_a} + \left( \frac{1 - b_1 G_1}{2} \right) \rho_{a|\tilde{\omega}_a} \rho_{a|\tilde{\omega}_a}. \tag{3}
\]

This state is still not normalized, and \( \pi_{\tilde{\mu}_{y_1}} = \frac{I_1 \mu_{y_1}}{2} \). Similarly, we can get a non-normalized state \( \rho_{a,b_1,b_2|\tilde{\omega}_a,\tilde{\mu}_{y_1},\tilde{\mu}_{y_2}} \) after Bob2 performs a weak measurement on his subsystem by the corresponding operators \( \tilde{\nu}_{y_2} \) with the outcome \( b_2 \). Finally, Bob3 performs a strong measurement by the corresponding operators \( \tilde{\nu}_{y_3} \). The conditional probability can be given by,

\[
p(a, b_1, b_2, b_3 | \tilde{\omega}_x, \tilde{\mu}_{y_1}, \tilde{\nu}_{y_2}, \tilde{\nu}_{y_3}) = \text{Tr}[\rho_{b_3} \rho_{a,b_1,b_2|\tilde{\omega}_a,\tilde{\mu}_{y_1},\tilde{\mu}_{y_2}}]. \tag{4}
\]

Certainly, every average value defined in Eq.(1), such as \( \langle \tilde{\omega}_0 \otimes \tilde{\mu}_0 \rangle \), can be determined by,

\[
\langle \tilde{\omega}_0 \otimes \tilde{\mu}_0 \rangle = \sum_{a,b_1,b_2,b_3} (-1)^{-a b_1} p(a, b_1, b_2, b_3 | \tilde{\omega}_0, \tilde{\mu}_0, \tilde{\nu}_{y_2}, \tilde{\nu}_{y_3}). \tag{5}
\]

Hence, three CHSH parameters \( I_{\text{CHSH}} \) in Eq.(1) can be described by a linear combination of conditional probabilities respectively. Interestingly, it is easy to find that \( I_{\text{CHSH}}^{(1)} \) only depends on the measurement settings of Alice and Bob1. However, in addition to relying on the measurement of Alice and Bob2, \( I_{\text{CHSH}}^{(2)} \) also depends on the frequency with which Bob1 chose \( \mu_0 \) versus \( \mu_1 \). The reason is, even though the CHSH expression contains only conditional probabilities, the received states of Bob2 are different for different measurements of Bob1. Similarly, \( I_{\text{CHSH}}^{(3)} \) depends on the frequency with which the measurement settings chosen by both Bob1 and Bob2.

It is obvious that the former Bobs has the ability to control the mixed state that received by the later Bob in the way of performing two possible measurements with a certain proportion. Hence, there are two cases in this scenario, biased and unbiased. The biased case means the frequency with which the measurement settings chosen by each bob is not the same, and the unbiased case means the frequency with which the measurement settings chosen by each bob is equal. As is shown in [26], it is possible for more than two Bobs to violate CHSH with Alice in biased case. But for the unbiased case, this is still an open question. Previous work has tried to answer this question [26, 28]. However, we will show the opposite conclusion of the previous research on this problem below.
III. TRIPLE VIOLATION OF CHSH INEQUALITY BETWEEN ALICE AND THREE BOBS

FIG. 2: The upward view of the graphic model of \{I^{(1)}(Orange), I^{(2)}(Blue), I^{(3)}(Green), 2(Red)\}. In Red region, the three CHSH parameters (6) exceed 2 simultaneously.

In the unbiased case, three CHSH parameters \(I_{\text{CHSH}}\) depending on Alice-Bob1, Alice-Bob2 and Alice-Bob3 can be calculated as

\[
\begin{align*}
I^{(1)} &= I^{(1)\text{CHSH}} \\
I^{(2)} &= \frac{\sum \mu y_1 I^{(2)\text{CHSH}}(\mu y_1)}{2} \\
I^{(3)} &= \frac{\sum \mu y_1 \mu y_2 I^{(3)\text{CHSH}}(\mu y_1, \mu y_2)}{4}
\end{align*}
\]

where \(I^{(2)\text{CHSH}}(\mu y_1)\) and \(I^{(3)\text{CHSH}}(\mu y_1, \mu y_2)\) are reconstructed by the conditional probability \(p(a, b_1, b_2, b_3 \mid \tilde{\omega}_x, \mu y_1, \hat{\nu} y_2, \hat{\nu} y_3)\).

In order to verify the existence of triple violations, it is obvious that we should find a method to determine the optimal measurements that can achieve the maximal value of every CHSH parameter (6). Once \(\max[\min(I^{(1)}, I^{(2)}, I^{(3)})] > 2\), triple non-locality sharing will observed. Theoretically, the method of Lagrange multipliers [34] can be used to handle this problem. Unfortunately, it is too complex to obtain a simple and distinct analytical result. But we still can give a sub-optimal analytical result with special settings and the optimal result numerically.

The measurements of Alice, three Bobs, are totally independent in the entire process. So the arbitrary operator can be defined as \(\hat{O} = \tilde{O} \cdot \tilde{\sigma}\) with \(\tilde{O}(\theta, \psi) = (O^1, O^2, O^3) = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)\). Without loss of generality, the direction of the dichotomic measurements can be denoted as, \(\{(\theta_1, \psi_{11}), (\theta_2, \psi_{12})\}\) for \(\tilde{\omega}_x\), \(\{(\theta_2, \psi_{21}), (\theta_2, \psi_{22})\}\) for \(\hat{\nu} y_2\), \(\{(\theta_1, \psi_{31}), (\theta_2, \psi_{32})\}\) for \(\hat{\nu} y_2\), and \(\{(\theta_1, \psi_{41}), (\theta_2, \psi_{42})\}\) for \(\hat{\nu} y_3\). Supposed that the optimal information-disturbance trade-off condition satisfies \(F^2 + G^2 = 1\) for the weak measurements of the first two Bobs. Here we can show that, even if we choose the measurement settings consistent with the previous research [28], triple violation still exists. Alice measures in the \(\hat{X}\) or \(\hat{Z}\) direction, and the measurements performed by these Bobs is in the \(X-Z\) plane, where \(\psi_{12} = 0\) for all measurement settings. When we choose such measurement settings, \(\theta_{21} = \frac{16}{17} \pi, \theta_{22} = \frac{1}{8} \pi, \theta_{31} = \frac{17}{18} \pi, \theta_{32} = \frac{6}{7} \pi, \theta_{41} = \frac{9}{11} \pi, \theta_{42} = \frac{4}{5} \pi\), analytical forms of \(\text{Eq.(6)}\) can be obtained. In the proper range of \(G_1\) and \(G_2\), the three CHSH parameters (6) exceed 2 simultaneously, as illustrated in Fig.2. We plot a graphic model of \(\{I^{(1)}(Orange), I^{(2)}(Blue), I^{(3)}(Green), 2(Red)\}\) which depends on \(G_1\) and \(G_2\). The Fig.2 is the upward view of the graphic model. So Fig.2 shows the minimum value of four functions for different \(G_1\) and \(G_2\), \(\min[I^{(1)}, I^{(2)}, I^{(3)}, 2]\). It is clearly show that triple violation exists in an triangle range (Red). Such region is a kernel that exists in the graphic model, which is not easy to find. For example, when \(G_1 = 0.8\) and \(G_2 = 0.93\), \(I^{(1)} = 2.05\), \(I^{(2)} = 2.03\), and \(I^{(3)} = 2.02\). Although the violation is very small. But it is a tight evidence that, it is possible to observe a triple violation of CHSH inequality between Alice and three Bobs when the measurements of each of the several observers at one side are unbiased with respect to the previous observers. We have to point out that the numerical example showed in [28] is correct, but it is useless to answer the above question.

FIG. 3: The optimal value of \(\max[\min(I^{(1)}, I^{(2)}, I^{(3)})]\), when \(G_1 = 0.8\). The maximal triple violation exists when \(G_2 = 0.93\), where \(I = 2.03\).

Even it is difficult to obtain the optimal analytical result, but we can calculate the optimal result numerically. With the limits of the measurement settings mentioned above, the optimal value of \(\max[\min(I^{(1)}, I^{(2)}, I^{(3)})]\) reaches when \(I^{(1)} = I^{(2)} = I^{(3)} = I\). Triple violation exists in a larger range of proper \(G_1\) and \(G_2\). The maximal triple violation exists when \(G_1 = 0.8\) and \(G_2 = 0.93\), where \(I = 2.03\). It is shown that the optimal numerical result is not much better than the sub-optimal analytical result.

IV. THE ANALYSIS OF OPTIMAL TRIPLE VIOLATION OF CHSH INEQUALITY

In fact, to answer this open question, it is no reason to limit Alice’s measurements in the \(X\) or \(Z\) di-
Let us now consider the scenario when this limit is relaxed. We supposed that all measurement settings performed by Alice and three Bobs are in the $X-Z$ plane, where $\psi_{ij} = 0$ for all measurement settings. When we choose such measurement settings, $\theta_{11} = \frac{1}{3}\pi$, $\theta_{12} = \frac{1}{6}\pi$, $\theta_{21} = \frac{1}{3}\pi$, $\theta_{22} = \frac{1}{2}\pi$, $\theta_{31} = \frac{1}{3}\pi$, $\theta_{32} = \frac{1}{4}\pi$, $\theta_{41} = \frac{1}{4}\pi$, $\theta_{42} = \frac{1}{5}\pi$, analytical forms of Eq. (6) can be obtained. In the proper range of $G_1$ and $G_2$, the three CHSH parameters (6) exceed 2 simultaneously, as illustrated in Fig. 4. We plot a graphic model of $\{I^{(1)}(Orange), I^{(2)}(Blue), I^{(3)}(Green), 2(Red)\}$ which depends on $G_1$ and $G_2$. The Fig. 4 is the upward view of the graphic model. So Fig. 4 shows the minimum value of four functions for different $G_1$ and $G_2$, $\text{min}[I^{(1)}, I^{(2)}, I^{(3)}, 2]$. It is clearly show that triple violation exists in a triangle range (Red). Such region is a kernel that exists in the graphic model, which is not easy to find. For example, when $G_1 = 0.78$ and $G_2 = 0.92$, $I^{(1)} = 2.08$, $I^{(2)} = 2.06$, and $I^{(3)} = 2.02$. Although the violation increases not too much, but it is shown that Alice chose to measure in X or Z direction, which is not the optimal choice.

Similarly, we can calculate the optimal result numerically in this case. Actually, we have numerical evidence that the condition of $\psi_{ij} = 0$ for all measurement settings will not affect the optimal value. The optimal value of $\text{max}[\text{min}(I^{(1)}, I^{(2)}, I^{(3)})]$ reaches when $I^{(1)} = I^{(2)} = I^{(3)} = I$. Triple violation exists in a larger range of proper $G_1$ and $G_2$. The maximal triple violation exists when $G_1 = 0.78$ and $G_2 = 0.92$, where $I = 2.04$. It is shown that the optimal numerical result is not much better than the sub-optimal analytical result. The value of the CHSH violation in the sequence falls off superexponentially [26], the maximum violation that can be achieved by Bob3 with Alice, $I^{(3)}$, is only a little bigger than 2. Nevertheless, this result still enough to overturns the conclusions and proofs of previous related work on this issue which has been widely shared in the scientific community before.

FIG. 4: The upward view of the graphic model of $\{I^{(1)}(Orange), I^{(2)}(Blue), I^{(3)}(Green), 2(Red)\}$. In Red region, the three CHSH parameters (6) exceed 2 simultaneously.

![Graphic model](image)

V. CONCLUSION

We have answered an open question left in [Phys. Rev. Lett. 114, 250401 (2015)]: whether the nonlocality of a single particle from an entangled pair can be shared among more than two observers that act sequentially and independently of each other? It is clearly shown that, it is possible to observe a triple violation of CHSH inequality between Alice and three Bobs when the measurements of each of the several observers at one side are unbiased with respect to the previous observers. This result overturns the conclusions and proofs of previous related work on this issue which has been widely shared in the scientific community before.

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