Superconductors are topologically ordered

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We revisit a venerable question: what is the nature of the ordering in a superconductor? We find that the answer is properly that the superconducting state exhibits topological order in the sense of Wen, \textit{i.e.} that while it lacks a local order parameter, it is sensitive to the global topology of the underlying manifold and exhibits an associated fractionalization of quantum numbers. We show that this perspective unifies a number of previous observations on superconductors and their low lying excitations and that this complex can be elegantly summarized in a purely topological action of the “BF” type and its elementary quantization. On manifolds with boundaries, the BF action correctly predicts non-chiral edge states, gapped in general, but crucial for fractionalization and establishing the ground state degeneracy. In all of this the role of the physical electromagnetic fields is central. We also observe that the BF action describes the topological order in several other physically distinct systems thus providing an example of topological universality.

I. INTRODUCTION

A. Generalities

The notions of order and disorder are fundamental to modern condensed matter physics. In their most influential form, starting with Landau and now covered in textbooks [1], they involve ordering as the breaking of a symmetry characterized by a non-zero local order parameter which is the expectation value of a (generally tensor) local operator,
\[
\psi(\vec{r}) = \langle \hat{\psi}(\vec{r}) \rangle
\]  
(1)
and disorder as the lack of such a broken symmetry,
\[
\psi(\vec{r}) = 0 .
\]  
(2)

Disordered states include classical gases and liquids, paramagnets, the Bose gas above condensation, and the Fermi liquid. The study of their instabilities to the much more numerous broken symmetry states has been an immensely fruitful endeavor, as reflected \textit{e.g.} in the variety of Fermi surface instabilities that signal the onset of order in fermion systems. Ordered states such as Neel antiferromagnets, superfluids and the forest of liquid crystal phases exhibit a rich set of interlinked properties that follow from the broken symmetry: Goldstone bosons, topological defects connected to dissipation, generalized rigidity and long range forces due to the rigidity [2]. All of these are captured elegantly in the mathematics of the sigma-model Lagrangian,
\[
\mathcal{L} = \frac{\rho_s}{2} | \nabla \theta(\vec{r}) |^2
\]  
(3)
where the field \( \theta(\vec{r}) \) contains all fluctuations of \( \psi(\vec{r}) \) with its amplitude frozen\textsuperscript{1}.

An important theme in current research in quantum condensed matter physics, specifically in the study of strongly correlated systems, is the examination of systems where this framework fails to apply. The breakdown of the framework is interesting on both sides of the dichotomy. Are there disordered states that fail to be characterized by their lack of a local order parameter, \textit{i.e.} are not adiabatically connected to the canonical disordered states? Are there ordered states that also fail to be sufficiently characterized by the order parameters they do develop? In both cases, a related but distinct question is the existence of states with fractionalized quasiparticles which must therefore necessarily fail the test of continuity. A hybrid possibility, of great interest in the context of the cuprates, is that of accessing conventional

\textsuperscript{1} Of course in a continuum description, the amplitude must go to zero on some lower dimensional manifold at the positions of the topological defects.
FIG. 1: The $\nu = \frac{1}{3}$ Laughlin liquid lacks a local order parameter, but is sensitive to the topology - on a surface of genus $g$ it exhibits $3^g$ ground states.

broken symmetry states from unconventional disordered states—in this fashion circumventing the standard limitations on the strength of the ordering as well as on the competitiveness of various instabilities\(^2\).

In this context the notion of “topological order” first articulated by Wen and collaborators in their studies of the quantum Hall states and the hypothesized chiral spin liquids, is especially important [4–7]. In these instances, the states lack local order parameters but display a weak form of order in which they are sensitive to the topology of the underlying two dimensional manifolds. Most strikingly they exhibit fractionalized quasiparticles. Further, all of these properties are encapsulated in purely topological, Chern-Simons actions that play a role analogous to the sigma model in broken symmetry states.

While topological order has been generally invoked in discussions of various exotic states, our contention in this paper is that it is, in fact, the proper characterization of the familiar superconducting state discovered by Kammerlingh Onnes. Indeed, we find that this point has been made early on, albeit without elaboration, by Wen himself [6]. In this paper we will offer a fairly complete treatment of this idea. Before turning to a more precise statement of this claim, we digress to list the set of properties a topologically ordered state can be expected to exhibit by appealing to the example of the $\nu = 1/3$ fractional quantum Hall state.

B. Topological Order in Quantum Hall States

As an instance of a topologically ordered state, the $\nu = 1/3$ fractional quantum Hall state exhibits the following relevant properties.

- It does not develop a local order parameter, \textit{i.e.} all operators constructed from finite numbers of electron operators exhibit exponentially decaying correlations. It \textit{does} develop a non-local, infinite particle, order parameter but as we shall discuss later this feature is not common to all topologically ordered systems. When the problem is exactly reformulated as that of a matter field coupled to a Chern-Simons gauge field, there is no local gauge invariant order parameter [8].

- Nevertheless, the system is sensitive to the topology of the underlying manifold. It exhibits a ground state multiplet on finite systems, separated from other states by an amount parametrically larger than the intra-multiplet splitting, whose degeneracy increases with the genus, $g$, of the manifold as $3^g$, \textit{e.g.} three states on the torus (Fig. 1).

- The state supports fractionalized quasiholes and quasielectrons with charge $\pm e/3$ which exhibit fractional braiding statistics in which they acquire a phase $e^{\pm i\pi/3}$ upon exchange among or between themselves. The existence of these quasiparticles is intimately related to the intra-multiplet splitting of the ground states. Their tunneling around non-contractible loops moves the system around in the ground state manifold and leads to the characteristic $O(e^{-L})$ ground state splitting in a generic system of finite linear dimension $L$. In the special case of the $\nu = 1/3$ state at exactly that filling in a clean system, the splitting vanishes altogether.

\(^2\) In the cuprates there is evidence that the state above $T_c$ is anomalous but also that the superconducting state is continuously connected to the BCS state. The presence of more than one order parameter in regions of their phase diagram, as shown recently in a set of experiments [3], raises the possibility that there are competing instabilities of the high temperature state. In the conventional Fermi liquid analysis at weak coupling, one generally finds that one instability dominates over all the others so the prospect of getting the competition from a non-Fermi liquid normal state is attractive.
In the clean case one can identify a topological symmetry algebra containing operators that move the system between different members of the ground state multiplet. These operators insert flux through the various non-contractible loops.

All of the above can be encoded in a long wavelength, purely topological, Chern-Simons Lagrangian,

\[ \mathcal{L} = \frac{k}{4\pi} \epsilon^{\mu\nu\sigma\tau} a_\mu \partial_\nu a_\sigma - j_\mu a^\mu \]  

with \( k = 3 \). The elementary quantization of this action defines a theory with a finite dimensional Hilbert space with the proper ground state degeneracies and its correlations in the presence of sources reproduce the quantum numbers of the quasiparticles.

The topological action further implies the existence of boundary degrees of freedom on manifolds with boundaries. In the case of the Laughlin quantum Hall states, the boundary excitations form a chiral Luttinger liquid.

C. This paper

In this paper we will show that superconductors with a gap in their single particle spectrum exhibit appropriate versions of all of the above properties: lack of a local order parameter, topological degeneracies and symmetry algebra,\(^3\) fractionalization, description by a topological field theory and edge degrees of freedom, and hence are properly described as being topologically ordered. In this discussion it will be crucial that superconductors are not mere superfluids like \(^3\)He and \(^4\)He but instead are charged superfluids with dynamic electromagnetism.

The claim of topological order for superconductors might surprise some fraction of our readers on at least two of its component pieces—that superconductors are not broken symmetry states and that they exhibit quantum number fractionalization. In fact, both ideas have been around for a while. The impossibility of finding a gauge invariant local order parameter for the state in the presence of electromagnetic gauge fields has been understood for a long time [10]. The conventional broken symmetry account, following Bardeen, Cooper and Schrieffer, holds for a neutral system whose response functions are argued to be qualitatively similar to the “screened” or “irreducible” response functions of the charged system. The point that the quasiparticles of a superconductor are electrically neutral, and hence fractionalized, was made (only!) a decade back by Rokhsar and Kivelson and again they invoked the gauge field in an essential way[11].

In the following we will be able to add to these observations an account of degeneracies on closed manifolds, a topological action, and an account of the edge states it predicts, to produce a unified portrait of topological order which can then substitute for the lack of a broken symmetry. As befits a topic with an extensive scholarly literature, we have found that much of what we have to say has precursors in the literature which we note at various points in the text. A subsidiary theme in this paper is that more than one system can exhibit the same topological structure and hence be described by the same topological field theory, and we will find it instructive to examine the correspondences. In particular we will note that the standard Ising gauge theory, the short ranged RVB state, a bilayer quantum Hall system with oppositely charged layers, and a \( U(1) \) lattice gauge theory coupled to a charge-2 scalar, will give rise to the same topological structure as the superconductor. Indeed, the last one on that list, studied in the seminal work of Fradkin and Shenker [12], illustrates our central points very elegantly.

In our discussion we will largely shy away from truly microscopic models of the superconducting state with the electronic degrees of freedom exhibited explicitly, since that level of detail is not essential for our considerations. Instead we shall study bosonic theories of the quantum Ginzburg-Landau form. In field theoretic terminology these are the relativistic abelian Higgs models governed by the Lagrangian,

\[ \mathcal{L}_{ah} = \frac{1}{2M} |iD_\mu \phi|^2 - \frac{\lambda}{4} \left( \phi^\dagger \phi \right)^2 - \frac{m^2}{2} \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu}^2 - eA_\mu j^\mu . \]  

Here \( \phi \) is a charge \(-2e\) scalar field representing the Cooper pair condensate, the covariant derivative \( iD_\mu = i\partial_\mu - 2eA_\mu \) and the field strength \( F_{\mu\nu} \) involve the physical electromagnetic field and the conserved current \( j^\mu \) with charge \( e \) is

\(^3\) We should note that the topological symmetry operators are not expected to be universal everywhere in a topologically ordered phase as shown by example in Ref. 9.
introduced to describe the gapped quasi-particles or perhaps external charges \(^4\) (We will use Greek and Roman indices to denote space-time vectors and spatial vectors respectively, and the metric \(g^{00} = 1\) and \(g^{ij} = -1\).) In 3+1 dimensions, this model is a plausible description of a gapped BCS superconductor with particle-hole symmetry but it has the topological features of interest even if the choice of a Lorentz invariant dynamics is non-generic. As an aside we note that the situation is more complicated for gapless superconductors, e.g. the d-wave cuprates, where there are gapless quasi-particles that must be incorporated in the effective low energy theory. We will return to this in the summary section.

In a final simplification, we will focus mostly on \(\mathcal{L}_{ah}\) in 2+1 dimensions. This no longer describes a physical superconductor since the electrodynamics is now that of the 2+1 dimensional Maxwell term (for instance a logarithmic potential between charges) which does not describe real electromagnetism even if the electron system is effectively two dimensional as is the case with superconducting films.\(^5\) The primary reason to examine this case nevertheless is that the analysis is simpler and more pedagogical than in 3+1 dimensions while the essential features of the two problems are the same. The chief simplification is that the topological theory for a 3+1 dimensional superconductor is a theory of particles and strings, while for 2+1 dimensions it is theory of particles only. Further, on manifolds with boundaries, the boundary theories of the 2+1 dimensional models are 1+1 dimensional, which are even easier to discuss. A secondary reason is that various theories of strong correlation in 2+1 dimensions give rise to the identical mathematics of coupled matter and gauge fields for physically neutral systems and our discussion will serve to formalize the discussion of topological order in that context as well. We should emphasize though, that while the occurrence of topological order in this class of theories is a fascinating question, especially with regard to the physics of the non-superconducting regions of the cuprate phase diagram, it has nothing to do with the topological order in the superconducting phase itself. In all such models, the real electromagnetic field would eventually be important to establish the topological order of the 3 dimensional superconducting state—a statement which should be self-explanatory at the end of the paper.

With this somewhat elaborate preamble we now turn to the technical content of the paper. In the next section, we briefly review why a superconductor cannot be characterized by a broken symmetry, i.e. why there is no gauge invariant local order parameter. In section III we discuss the nature of the excitations in a charged superconductor, and why they are fractionalized. From these we deduce the form of the topological \(BF\) action, which we then rederive from a path integral formulation of the abelian Higgs model. This action implies a ground state degeneracy which we discuss in Section IV. In Section V we digress to consider other problems that are also described by the \(BF\) theory: the lattice \(Z_2\) gauge theory, a bilayer quantum Hall system, the resonating valence bond (RVB) state and the Fradkin-Shenker problem. In Section VI we turn to the edge structure implied by the \(BF\) action in 2+1 dimensions as well as in 3+1 dimensions. The last section summarizes our main results and some technical details connected to edge actions are in an appendix.

II. NO LOCAL ORDER PARAMETER

The textbook Ginzburg-Landau description of a gapped superconductor invokes a charge \(-2e\) complex scalar field, the “superconducting order parameter”, that measures the condensation of Cooper pairs and is related to the underlying electron field by an appropriate expectation value, \(\psi(|\vec{r}|) = \langle \Psi_1(|\vec{r}|) \Psi_1(|\vec{r}|) \rangle\). This field is minimally coupled to the electromagnetic vector potential \(A^\mu\) and the dynamics of the two coupled fields are then fixed by the Ginzburg-Landau differential equations. These equations are, obviously, a fine description of superconductors with small fluctuations. At issue in the context of this paper is whether \(\psi(|\vec{r}|)\) is a local quantity.

To see that it is not, let us rephrase the question in the context of the abelian Higgs model (5). The Euler-Lagrange equations for \(\mathcal{L}_{ah}\) absent sources are of the Ginzburg-Landau form, although now extended to include a precessional dynamics at \(T = 0\). We expect the Euler-Lagrange equations to give a useful account if the fluctuations are small in

\(^4\) Note that in spite of the relativistic form we normalize the kinetic term such that \(|\phi|^2\) has the dimension of density. This will help to streamline our notation with that usually used in discussing superconductivity. In the non-relativistic limit this model becomes a time dependent Ginzburg-Landau theory. This model would exhibit the Meissner effect with a London penetration length \(\lambda_L\) coming from the gradient term. The Debye screening due to the Coulomb field would, however, only be generated because of the scalar potential, and the corresponding screening length would be given by \(\lambda_D \neq \lambda_L\). In the relativistic model both electric and magnetic screening emanate from the gradient term, and the two screening lengths are equal. Although this is not true in real systems, it simplifies our arguments and helps to highlight the conceptual points. The generalization to a real non-relativistic model is left for the reader.

\(^5\) The results presented in this paper are probably valid for thin charged superconducting films anyway. In this case we have power law rather than exponential decay of screening charges and currents, which appears sufficient to define a appropriate scaling limit and thus allow for a description in terms of a topological field theory.
the ordered phase and the fields involved develop non-zero expectation values. Naively, we would like $\phi$ to develop a nonzero expectation value but this is not possible since it transforms non-trivially under the $U(1)$ gauge symmetry,

$$\phi(\vec{r}) \rightarrow e^{i 2\pi \alpha(\vec{r})} \phi(\vec{r}) ; \quad A_\mu(\vec{r}) \rightarrow A_\mu(\vec{r}) + \partial_\mu \alpha(\vec{r}) ,$$

and Elitzur’s theorem [10] assures us that such quantities average to zero even in the “broken symmetry” phase.

The solution to the conundrum of what underlies the Ginzburg-Landau description is the non-local quantity first introduced by Dirac[13]. It is easiest to write it in operator form,

$$\phi_D(\vec{r}) = e^{i} \int d^3 r' E_{cl}(\vec{r}' - \vec{r}) \cdot \vec{A}(\vec{r}') \phi(\vec{r}) \quad (6)$$

where $E_{cl}(\vec{r})$ is the classical electric field corresponding to a point charge at the origin, i.e. $\vec{V} \cdot \vec{E}_{cl} = \delta(\vec{r})$, and $\phi$ and $\vec{A}$ are quantum field operators. A partial integration shows that the gauge transformation (6) leaves the combination $\phi_D$ invariant. The operator $\phi_D$ has a natural interpretation as the creation operator of a charged $\phi$ particle together with a coherent states of photons describing the accompanying Coulomb field which extends out to infinity. In the Coulomb gauge $\vec{V} \cdot \vec{A} = 0$, the Coulomb field is described entirely by the scalar potential, $A^0$, and $\phi_D$ reduces to $\phi$ alone. So in this gauge the Dirac order parameter appears local, as can be checked by writing $\bar{E}_{cl}(\vec{r})$ as a gradient and again integrating by parts. A superconductor is then characterized by off-diagonal long range order in $\phi_D$. Kennedy and King have given a rigorous proof of this statement using a covariant generalization of (6), and a lattice regularization, for a non-compact abelian Higgs model in two or more spatial dimensions[14].

Their proof also shows that this non-local order parameter cannot be used as one uses a local order parameter. Precisely, one finds that the temporal correlator of $\phi_D$ decays algebraically to its asymptotic value. With a local order parameter this would be a signature of Goldstone bosons. In fact, the Anderson-Higgs mechanism forbids any such bosons in the actual spectrum, which shows that a description based on $\phi_D$ does not have the character of the standard sigma model.

While we are on the subject of non-local characterizations of the superconductor, a second possibility, following ’t Hooft, is to classify phases by focussing on observables inspired by the behaviour of the gauge sector. Here the candidates are Wilson loops, and their duals, which correspond mathematically to the insertion of singular gauge transformations[16]. Physically, these dual variables ask a dimension dependent question. In 3+1 dimensions, in a superconducting phase with non-compact electrodynamics, ’t Hooft’s operator is a loop whose area law decay attests to the confinement of test magnetic monopoles by the Abrikosov flux tube that gets stretched between them. In 2+1 dimensions the ’tHooft operator $\phi_m$ acts at a point and becomes a nominally local field, $\phi_m$ creating a vortex. This yields a disorder parameter, which vanishes in the superconductor and has a finite expectation value in the normal phase of the abelian Higgs model. In words, the normal phase is identified as a condensate of vortices while the superconductor exhibits a gap to their creation.

In both of the above characterizations the restriction to non-compact gauge fields is not accidental. In a compact 3+1 dimensional gauge theory there are monopoles that obstruct the construction of $\phi_D$ and its covariant generalizations so that even a nonlocal order parameter in the spirit of Dirac is not possible[17]. The essential difficulty is that the Dirac quantization condition is not compatible with having a real valued current as in (6). It is even easier to see what goes wrong with the ’tHooft construction once dynamical monopoles are permitted. For example in 3+1 dimensions, without them, the potential energy of two static test monopoles separated by a distance $r$ in a superconductor will be linear $\sim sr$ where $s$ is the energy per unit length i.e. the tension of the Abrikosov flux line. In the presence of dynamical monopoles of mass M, the linear confinement will breakdown at a distance $r_{sc} \sim 2M/s$ where it will be energetically preferable to create a monopole-antimonopole pair from the vacuum to break up the flux line. This is the exact magnetic analog of electric screening of static electric test charges in a confining relativistic theory with massive charged particles.

While this discussion will certainly be germane when we discuss some compact gauge problems related to our main theme, readers interested solely in superconductors may suspect that they can do without it altogether. While this is true in practice, it is probably not true as a matter of principle! While Maxwell electrodynamics and indeed even the standard model have no monopoles, they do occur in most attempts at further unification, e.g. in various grand unified models, with masses expected to be in the $10^{15} - 10^{19}$ GeV range[18]. With such masses they will give rise to a “screening length” that we can estimate, for superconductors with Ginzburg-Landau parameter $\kappa = 1$ (so that the coherence length and the penetration depth are the same), as being of the simple form

$$\lambda_{mp} \sim \lambda_F \frac{Mc^2}{E_F} . \quad (7)$$

6 A similar construction, but for gauge fields alone, was given earlier by Fradkin and Susskind [15].
For a good old fashioned superconductor this yields $\lambda_{\text{monopole}} \approx 10^{10}$ km or about 70 AU which is therefore literally astronomical.\(^7\) It follows then that for samples of this size there really won’t be an order parameter which makes it all the more imperative to develop an alternative characterization of the order in the superconducting state—a task to which we now turn!

III. EXCITATIONS, FRACTIONALIZATION AND TOPOLOGICAL FIELD THEORY

Having established that a local order parameter description is not feasible for superconductors, we will now (successfully) attempt to construct a topological order description in terms of a topological field theory. We will start with the low energy excitations of the superconducting state and examine their quantum numbers and topological interactions. By encoding these in a topological actions in 2+1 and 3+1 dimensions we will inductively arrive at the desired description. Subsequently we deduce the same topological action in 2+1 dimensions from the path integral for the abelian Higgs model and close by noting that including leading irrelevant terms beyond the topological action completes the low energy description of the superconductor, much as it does for the quantum Hall effect.

A. Excitations and Fractionalization

The low energy excitations of a superconductor are the quasiparticles formed by breaking up a Cooper pair, vortices or vortex lines in 2 and 3 spatial dimensions respectively, and a set of collective modes which together form a massive photon in our relativistic setting.

To review their properties in the context of (5), we write $\phi$ in amplitude and phase variables, $\phi = \sqrt{\rho}e^{i\varphi}$ and focus deep in the ordered phase where $m^2 \ll 0$. Here we can set the amplitude equal to its classical value, $\rho = -m^2/\lambda$, and ignore its remaining massive fluctuations to rewrite (5) as

$$L_{\text{ah}} = \frac{\rho}{2M}(\partial_\mu \varphi + 2eA_\mu)^2 - \frac{1}{4}F_{\mu\nu}^2 - eA_\mu j^\mu + \ldots$$

where the dots indicate the neglected density fluctuations. If furthermore the model is defined on a topologically trivial manifold, and we disregard vortices, we may send $A_\mu \to A_\mu - \frac{1}{2}e\partial_\mu \varphi$ in a regular gauge transformation that defines unitary gauge, thus obtaining the following effective low energy Lagrangian,

$$L_{\text{eff}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{m_s^2}{2}A_\mu^2 - eA_\mu j^\mu$$

where the screening mass, $m_s$ is related to the London penetration length by $m_s^2 = \lambda L^{-2} = 4e^2\rho/M$.

The Lagrangian (9) is that of a massive abelian gauge field coupled to a conserved current. In the absence of the current it yields the gapped collective modes of the superconductor—the absence of a gapless mode is the Anderson-Higgs mechanism.

In the presence of a current, the classical equation of motion is a relativistic version of the London equation,

$$\partial_\nu F^{\nu\mu} = j^\mu - m_s^2 A^\mu = j^\mu + J^\mu_{\text{sc}},$$

where we identified $-m_s^2 A^\mu$ as the screening current in the medium. For $m_s^2 \neq 0$ (10) implies $\partial_\mu A^\mu = 0$, i.e. the screening current is conserved, and the equation of motion simplifies to,

$$(\Delta + m^2)A^\mu = j^\mu,$$

from which it follows that all classical fields and currents are exponentially screened over the length $\lambda_L$. This is a consequence of the Meissner effect and should be contrasted with the case of an ordinary metal, where only the charge and the longitudinal part of the current are screened. That the screening lengths for both components are the same is special to our Lorentz invariant setting—in general, they will be different.

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\(^7\) A naive estimate of the corresponding tunneling probability based on the Schwinger formula commonly used in QCD string phenomenology[19], gives a string life time $\sim e^{-10^{44}}$ wherein the units are evidently unimportant! A better estimate requires consideration of the instanton path which we defer to the future.
This screening has important consequences for the quantum numbers of the quasiparticles, as pointed out by Kivelson and Rokhsar [11]—they do not carry a classical charge. To see this, consider constructing a wavepacket with the quasiparticle at rest in a given frame. In that frame the scalar potential is the only non-zero component of $A^\mu$ and it decays to zero on the scale of $\lambda_\perp$. By Lorentz transforming we obtain the potentials for a quasiparticle in motion and still find that all components of $A^\mu$ are exponentially attenuated beyond $\lambda_\perp$. As no fields are generated beyond the screening length, the quasiparticle is classically neutral at long wavelengths. Again we should note that life is more complicated when the longitudinal and transverse screening lengths are different. In the extreme case of the metal, where the transverse screening length is infinite, a moving charge will give rise to a dipolar pattern of current backflow that will decay only algebraically at long distances [20]. For real superconductors this dipolar pattern will be cutoff at the scale of the London length, while the longitudinal currents and potentials will decay on the scale of the Thomas-Fermi length. In our problem the two parts are screened identically and hence there is no residue of the dipolar pattern whatsoever. This vanishing of the charge of the quasiparticles is an instance of quantum number fractionalization in that the fundamental electrons are charged. If the electrons carry spin then the quasiparticles do too and hence are spinons [11] but this is not central. For example, in a p-wave superconductor of spinless fermions there would be no change in the underlying fractionalization. Instead the proper formulation is that the quasiparticles retain a quantum, Ising charge, which we will discuss in the next subsection.\footnote{Readers familiar with the work of the Santa Barbara group [21] should note that their discussion does not involve the physical electromagnetic field and is thus physically quite different from that of [11] and ours. For us the superconducting phase is fractionalized while in [21] it is the non-superconducting phase that is fractionalized.}

This analysis has used the equation of motion (10) which deals with expectation values and has sidestepped the important question of defining operators for which the vanishing charge is a sharp observable [22]. To our knowledge, there isn’t a rigorous analysis of this question for the abelian Higgs model. There is however a more careful account of the expectation value question by Swieca [23] (for a rigorous version, see Ref. 24). Swieca proves the following: A theory in more than 3 space-time dimensions, with a mass gap and an identically conserved current, \textit{i.e.} a current satisfying $\partial_\mu F^{\mu\nu} = j^\nu$, has no charged states in the spectrum. This theorem is directly applicable to our model Lagrangian (5) if we take the total current $j^\mu = j^\mu + j_B^\mu$ in (10) as the identically conserved current. Swieca’s proof, which is based on Lorentz invariance of the current form factor, and locality of the electromagnetic field, is not obviously applicable to a non relativistic theory, and it would be interesting to establish such an extension.

Finally, we note that in writing (9) we explicitly ignored vortices and vortex loops/lines. These form the remaining low energy excitations. A vortex carrying a flux $\pi$ is also fractionalized in a sense that is sharpest for models with a lattice electrodynamics as they exhibit vortices with $2\pi$ flux as their primary excitations when decoupled from matter.

We turn now to embedding these excitations in a topological action.

### B. BF theories

At issue in writing down a topological action are the topological interactions among the excitations, \textit{i.e.} interactions which depend upon the topology of the field configurations (or particle worldlines) but not on the metric. A way to formalize this is by the idea of the topological scaling limit in which we examine the system at scale $R$ and keep those pieces of the correlation functions that are $O(R^0)$ as $R \to \infty$ at fixed couplings[25]. This limit is to be contrasted with the Wilsonian scaling limit in which the coupling constants are tuned so as to keep the ratio of $R$ to the correlation length $\xi$ fixed. While the latter keeps all information except at the lattice scale, the former keeps only the topological “braiding” information.

Among the quasiparticles, vortices and plasmons there is one non-trivial interaction in this limit—namely, a topological phase $\pi$ arises whenever a quasiparticle is transported around a vortex or vice-versa (Fig. 2). This can be read off from the venerable explicit solution of the Bogoliubov-de Gennes equations for a vortex [26] but more modern discussions of how it arises are enlightening [27, 28].

The presence of this interaction is why we were careful to refer to the classical neutrality of the quasiparticles in the past section. Further, this interaction has the feature that it only detects quasiparticle number modulo 2 so that quasiparticles carry an Ising charge under it thus explaining our comment to this effect in the last subsection. This sensitivity of the superconductor to particle number modulo 2 has been described as an Ising gauge invariance of the superconducting state previously [27].

This topological interaction can be readily written into a field theory. We first consider the 2+1 dimensional case where both quasiparticles and vortices are particles so we can proceed in close analogy to the bosonic Chern-Simons theory for the quantum Hall effect and attach flux and charge to the particles in such a way that the Berry phases...
FIG. 2: Topological interactions in a superconductor: quasiparticles encircling vortices ($d=2$) or threading vortex loops ($d=3$) pick up a phase $\pi$ at an arbitrary distance.

(or in the quantum Hall case, the exchange phases) appear as an Aharonov-Bohm effect. We define a unit charge quasiparticle current $j^\mu$, and a vortex current $\tilde{j}^\mu$, and couple them to electric and magnetic gauge potentials via the Lagrangian,

$$\mathcal{L}_{curr} = -a_\mu j^\mu - b_\mu \tilde{j}^\mu.$$  \hfill (12)

A simple calculation shows that in order to get a phase $\pi$ when moving a $j$ quantum around a $\tilde{j}$ quantum we need an action for the gauge potentials, which is of the “BF” type

$$\mathcal{L}_{BF} = \frac{1}{2\pi} \epsilon^{\mu\nu\sigma} b_\mu f^{(a)}_{\nu\sigma},$$  \hfill (13)

where $f^{(a)}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Putting the parts together we have the topological action,

$$\mathcal{L}_{top} = \frac{1}{\pi} \epsilon^{\mu\nu\sigma} b_\mu \partial_\nu a_\sigma - a_\mu j^\mu - b_\mu \tilde{j}^\mu.$$  \hfill (14)

The topological nature of $\mathcal{L}_{top}$ is clear from the equations of motion,

$$\tilde{j}^\mu = \frac{1}{\pi} \epsilon^{\nu\sigma} \partial_\nu a_\sigma = \frac{1}{2\pi} \epsilon^{\nu\sigma} f^{(a)}_{\nu\sigma}$$  \hfill (15)

$$j^\mu = -\frac{1}{\pi} \epsilon^{\nu\sigma} \partial_\nu b_\sigma = \frac{1}{2\pi} \epsilon^{\nu\sigma} f^{(b)}_{\nu\sigma}$$  \hfill (16)

which show that the gauge invariant field strengths are fully determined by the currents, just as in a Chern-Simons theory. These equations both have a very direct physical interpretation as we shall see later.

Two comments are in order. The first concerns the symmetry properties of the Lagrangian (14). Under the parity transformation ($x,y$) $\rightarrow$ ($-x,y$) the two potentials transform as ($a_0, a_x, a_y$) $\rightarrow$ ($a_0, -a_x, a_y$) and ($b_0, b_x, b_y$) $\rightarrow$ ($-b_0, b_x, -b_y$), while under time reversal the transformations are, ($a_0, a_x, a_y$) $\rightarrow$ ($a_0, -a_x, -a_y$) and ($b_0, b_x, b_y$) $\rightarrow$ ($-b_0, b_x, b_y$), respectively. The unusual transformation properties of the potential $b_\mu$ follows from that of the vortex current. It is easy to check that the $BF$ action is invariant under both $PT$ and $CPT$. Second, in the Lagrangian (14) both currents are integer valued. This quantization is naturally encoded by requiring that the gauge fields $a_\mu$ and $b_\mu$ be compact. In the continuum this means that they transform as

$$a_i \rightarrow a_i + \partial_i \Lambda_a$$

$$b_i \rightarrow b_i + \partial_i \Lambda_b ,$$  \hfill (17)

with gauge functions $\Lambda_{a/b} = \Lambda_{a/b} + 2\pi$. This compactness will also be evident in our rederivation of the $BF$ action from the microscopic theory in the next section.

Turning to the case of 3+1 dimensions, we have essentially the same construction, but with the difference that the vortices are now strings, and the vector potential $b$ is an antisymmetric tensor, $b_{\mu\nu}$. In form language, the action still has the structure $BF$, and written out in components it reads

$$\mathcal{L}_{BF} = \frac{1}{\pi} \epsilon^{\mu\nu\lambda} b_{\mu\nu} \partial_\sigma a_\lambda .$$  \hfill (18)
The gauge transformations of the $b$ field are given by

$$b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}$$  \hspace{1cm} (19)$$

where $\xi_{\mu}$ is a vector valued gauge parameter. The minimal coupling of the $b$ potential to the world sheet, $\Sigma$ of the strings is given by the action,

$$S_{vort} = - \int_{\Sigma} d\tau d\sigma b_{\mu\nu} = - \int_{\Sigma} \frac{d(d(x^\mu, x^\nu))}{d(\tau, \sigma)} b_{\mu\nu},$$  \hspace{1cm} (20)$$

where $(\tau, \sigma)$ are time and space like coordinates on the worldsheet. This is a direct generalization of the coupling of $a$ to the world line, $\Gamma$, of a spinon,

$$S_{sp} = - \int_{\Gamma} dx^\mu a_\mu = - \int_{\Gamma} \frac{dx^\mu}{d\tau} a_\mu.$$  \hspace{1cm} (21)$$

Combining these elements we get the topological action for the 3+1 dimensional superconductor,

$$S_{top} = \int d^4x \mathcal{L}_{BF} + S_{sp} + S_{vort}.$$  \hspace{1cm} (22)$$

The proof that this action indeed gives the correct braiding phases can be found e.g. in Ref. 30, and a discussion of this action in the context of superconductivity has appeared before in Ref. 31, more on which later.

C. The 2+1 BF theory from the abelian Higgs model

Previously we induced the $BF$ action (14) from our knowledge of the low energy excitations and their topological interactions. We now gain additional insight into its form by deriving it from the Lagrangian for the abelian Higgs model (5) by explicitly including the vortices we neglected before.

An (anti)vortex at position $\vec{r}$ is a solution of the classical equations of motion where the phase, $\varphi$ of the $\phi$ field winds $(-)2\pi$ along any closed curve encircling $\vec{r}$. The generalization to higher winding numbers and to multi-vortex configurations is obvious. For well separated points, one can also define configurations with $N_+$ vortices and $N_-$ anti-vortices, although only $N_+ - N_-$ is topologically conserved. Away from the vortex cores, the solutions again look like a pure gauge, but with the important difference that the $\partial_\tau \varphi$ term in (8) cannot be removed by a regular gauge transformation. Instead we split the phase field as $\varphi = \tilde{\varphi} + \eta$ where $\tilde{\varphi}$ is a function of the vortex positions, $\vec{y}_n$, and $\eta$ is the fluctuating quantum field. We can now perform the regular gauge transformation $A_\mu \rightarrow A_\mu - \frac{\pi}{2\epsilon} \partial_\mu \eta$. If we consider a fixed vortex configuration $\{\vec{y}_n, q_n\}$ where $q_n = \pm 1$, we can write the corresponding quantum partition function in terms of an Euclidean path integral[32],

$$Z[j^\mu, \{\vec{y}_n, q_n\}] = \int D(A_\mu) D[\phi]\ e^{- \int d^3r \mathcal{L}_E}.$$  \hspace{1cm} (23)$$

with

$$\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^2 + \frac{m_\phi^2}{2} (A_\mu - \frac{1}{e} a_\mu)^2 - e A_\mu j^\mu + \mathcal{L}_{[\phi]}.$$  \hspace{1cm} (24)$$

where we introduced the notation $a_\mu = - \frac{\pi}{4} \partial_\mu \tilde{\varphi}$ and use the metric $(+++)$. Here $\mathcal{L}_{[\phi]}$ includes both the potential terms, density derivative terms, and an explicit dependence on the vortex positions. The gauge field is now manifestly massive, and with the potential in (5), so is the density field $|\phi|$. In the effective low energy description the only effect of the density fluctuations that will be retained is the presence of a vortex current,

$$\tilde{j}^\mu(x^\mu) = \sum_n q_n \int d\tau_n \tilde{y}_n^\mu \delta^2(x^\mu - y_n^\mu(\tau)).$$  \hspace{1cm} (25)$$

where $y_n^\mu(\tau)$ parametrize the (Euclidean) world lines of the vortices. It will be convenient to parametrize the vortex current with a gauge potential $b_\mu$, as,

$$\tilde{j}^\mu(x^\mu) = \frac{i}{\pi} e^{\mu\nu\sigma} \partial_\nu a_\sigma$$  \hspace{1cm} (26)$$
The normalization is such that a unit charge $\tilde{\rho} = \delta^2(\vec{x})$ is associated with a fundamental vortex in the charge $-2e$ scalar field, i.e. $\int d^2x \tilde{\rho} = \frac{1}{2} \oint d\vec{x} \cdot \vec{a} = \frac{\mp 1}{2\pi} \oint d\eta = 1$. Ignoring the density fluctuations, and hence $L_{|\phi|}$, we can now rewrite (23) as

$$Z[j^\mu, \tilde{j}^\mu] = \int D[A]D[a] \delta[\tilde{j}^\mu - \frac{1}{\pi} e^{\mu \sigma \nu} \partial, a_\sigma] e^{-\frac{1}{\pi} \int d^4r (L_E - L_{|\phi|})} = \int D[a]D[b] e^{-\frac{1}{\pi} \int d^4r L_{\text{eff}}(a,b)},$$

where

$$e^{-\frac{1}{\pi} \int d^4r L_{\text{eff}}} = \int D[A] e^{-\frac{1}{\pi} \int d^4r L_E}$$

and

$$L_E = \frac{1}{4} F_{\mu \nu}^2 + \frac{\lambda L}{2} (A_\mu - \frac{1}{e} a_\mu)^2 - e A_\mu j^\mu - b_\mu \tilde{j}^\mu + \frac{i}{4} e^{\mu \nu \lambda} b_\mu \partial, a_\lambda.$$  

(29)

The gauge potential $b_\mu$ is a Lagrange multiplier that imposes the delta function constraint in (27). The remaining steps in deriving the low energy Lagrangian $L_{\text{eff}}$ is to shift the field $A_\mu \rightarrow A_\mu + \frac{1}{e} a_\mu$, perform the Gaussian integration over the massive $A$ field and finally doing a derivative expansion. To lowest order, and after rotating back to Minkowski space, we get, $L_{\text{eff}} = L_{\text{top}} + O(m^{-2})$, i.e. the previously derived topological action. We shall return to the higher order corrections below.

Although this derivation was for $2+1$ dimensions, essentially the same argument can be given to derive the $3+1$ dimensional action (22).

The physical significance of the potentials $a$ and $b$ is now revealed: from (24) above it is clear $a$ is nothing but the topological part of the usual vector potential $A$, i.e. the part which is a pure gauge everywhere except at the location of the point vortices as expressed by the constraint in (27).

Equation (16) expresses screening of the external current, since $db$ is just the dual form of the screening current $J \text{sc}$ in (10). Also from writing $\rho \text{sc} = -\epsilon^{ij} \partial, b_j = \partial, E \text{sc}$ it follows that $b_i = e \epsilon_{ij} E \text{sc}$, i.e. the potential $b$ is essentially the fields associated with the screening clouds induced by the external electric sources. Since the total field is zero, this still begs the question to how there can be any long range effect related to the $b$ potential. Put differently, how does a moving vortex detect a stationary charge, given that the electric field is exponentially screened? A particularly clear explanation has been given by Reznik and Aharonov, who showed that although the expectation value of the electric field is exponentially screened inside the superconductor, there is an unscreened “modular” or $Z_2$ part that give rise to the topological phase[28]. We will return to this below in the discussion of the ground state degeneracy.

In summary, there are three complementary ways to understand the topological $BF$ action for the superconductor:

1. It encodes the correct braiding phases of charges and vortices.
2. It relates the current of correctly normalized pointlike vortices in the condensate to the topological nontrivial part of the vector potential.
3. It implements local screening of external electric currents.

It should now also be clear that the topological action (14) could have been derived from any of these conditions. For instance, starting from the condition of local screening (16), the $BF$ action is obtained simply by introducing the potential $a$ as a Lagrange multiplier field.

**D. The BF-Maxwell theory, Plasmons and Electrodynamic Response**

Thus far we have derived a topological action for the superconductor which includes the physics of the quasiparticles and the vortices. There are however, two significant omissions in this description. The plasmons are missing and so is the defining characteristic of the superconductor—its electrodynamic response. As neither of these are topological in nature, this is sensible. We now show that both of these omissions can be remedied by keeping the leading irrelevant (but now non-topological) terms in the action beyond the $BF$ term. These can be guessed on symmetry grounds alone but to get expressions for their coefficients we carry out the Gaussian integral over $A$ in (29) and obtain the Maxwell-BF Lagrangian, which after continuation back to Minkowski space becomes,

$$L_{\text{eff}} = \frac{1}{4} e^{\mu \sigma \nu} b_\mu \partial, a_\sigma - \frac{1}{4e^2} (f^{(a)}_{\mu \nu})^2 - \frac{1}{4} \left( \frac{e}{m_s \pi} \right)^2 (f^{(b)}_{\mu \nu})^2 - a_\mu j^\mu - b_\mu \tilde{j}^\mu.$$ 

(30)
The equations of motion for the Maxwell-BF theory read,

\[ \tilde{j}^\mu = \frac{1}{\pi} \epsilon^{\mu\nu\sigma} \partial_\nu a_\sigma + \left( \frac{e}{m_s \pi} \right)^2 \partial_\nu f_{\nu}^{\mu}_b \]

\[ j^\mu = \frac{1}{\pi} \epsilon^{\mu\nu\sigma} \partial_\nu b_\sigma + \frac{e}{2} \partial_\nu f_{\nu}^{\mu}_a. \]

In the absence of currents, and in Landau gauge (\( \partial_\mu a^\mu = \partial_\mu b^\mu = 0 \)), these can be combined to give

\[ (\Delta + m_s^2) a_\mu = 0 \]

\[ (\Delta + m_s^2) b_\mu = 0, \]

which shows that the spectrum now includes the plasmon modes. We note that an analogous argument in the quantum Hall problem leads to the Maxwell-Chern-Simons Lagrangian and thence to the gapped collective mode[33].

The reader may wonder at the resemblance of the first of Eqns. (32) to Eqn. (11) with \( j^\mu = 0 \). This is not coincidental—in going beyond the topological scaling limit we end up restoring the non-topological parts of the gauge field so that now \( a_\mu \) is \( eA^\mu \) at long wavelengths. This is also clear from (29) when we neglect derivative terms. With this insight we can now confirm that the superconductor is, in fact, a superconductor. To this end we integrate out \( b \) in the sector without quasiparticles or vortices (\( j^\mu = \tilde{j}^\mu = 0 \)) to obtain

\[ \mathcal{L}_{em} = -\frac{1}{4e^2} (\epsilon_{\mu\nu} f_{\nu}^{(a)})^2 - \frac{1}{2e^2} m_s^2 a_\mu a^\mu. \]

which upon variation gives the London equation and thus superconductivity\(^9\) Alternatively, we could have explicitly introduced a background electromagnetic field \( A^\mu \) and derived the London Lagrangian (33) directly in \( A^\mu \) by integrating out both \( a^\mu \) and \( b^\mu \).

**IV. THE GROUND STATE DEGENERACY**

We now return to the analysis of the purely topological field theory for the low energy excitations of the superconductor. Such a field theory has no bulk degrees of freedom but will possess global degrees of freedom which will lead to non-trivial ground state degeneracies on manifolds of non-trivial topology. In this section we will first derive the degeneracies predicted by the BF theory and then understand them physically in the setting of the abelian Higgs model.

As emphasized in the Introduction, one of the hallmarks of a topologically ordered state is a topology dependent ground state degeneracy, and a corresponding topological symmetry algebra. Before analyzing the superconductor it is instructive to recall how the ground state degeneracy is manifested in the simplest fractional quantum Hall setting, i.e. a Laughlin state with filling fraction \( \nu = 1/(2k + 1) \) on a torus[34]. In this case the ground state has a \( 1/\nu \) degeneracy corresponding to the number of lowest Landau level states for the center of mass, and the same degeneracy is obtained from an analysis of the topological low energy effective action, given by the Chern-Simons Lagrangian (4). Here the Wilson loops around the two cycles of the torus form a canonically conjugate pair, due to the non-zero commutator \([a_x, a_y]\). The Wilson loops measure the magnetic fluxes through the holes in the torus, so it follows that the operators connecting the different ground states correspond to magnetic flux “insertions”.

In the superconductor the ground state degeneracy is again related to the possible values of the Wilson loops—in this case for the gauge fields \( a \) and \( b \) appearing in the topological action. Here, however, there are two conjugate pairs of variables \((a_x, b_y)\) and \((b_x, a_y)\) so we expect a squaring of the ground state degeneracy as compared with the corresponding quantum Hall case. More precisely, the ground state degeneracy is in both cases determined by the possible ways to assign commuting fluxes to the “holes” in the surface.

**A. Ground state degeneracy from the BF theory**

We now formalize this argument, and show that, in the 2+1 dimensional case, the ground state degeneracy follows directly from the BF action (13) derived in the previous section. We work on the torus \((L_x, L_y)\).

\(^9\) That an Abelian Higgs model in the “London limit” can be rewritten in the dual form, was to our knowledge first explicitly pointed out by Balachandran and Teotonio-Sobrinho who in reference 31 considered the 3+1 dimensional counterparts to Eqns. (8) and (30).
In the absence of quasiparticles, the $BF$ action can be written in Hamiltonian form as,

$$S = \frac{1}{\pi} \int d^3x \{ \epsilon^{ij} \dot{a}_i b_j + a_0 ( \epsilon^{ij} \partial_i b_j ) + b_0 ( \epsilon^{ij} \partial_j a_i ) \}$$

(34)

where the Poisson brackets are encoded in the first term, the Hamiltonian is identically zero, and $a_0$ and $b_0$ are identified as Lagrange multipliers implementing the constraints,

$$\epsilon^{ij} \partial_i b_j = 0$$
$$\epsilon^{ij} \partial_j a_i = 0$$

(35)

On the torus we can solve these constraints by setting

$$a_i = \partial_i \Lambda_a + \bar{a}^i / L_i$$
$$b_i = \partial_i \Lambda_b + \bar{b}^i / L_i$$

(36)

where $\bar{a}$ and $\bar{b}$ are spatially constant, and $\Lambda_{a/b}$ are periodic functions on the torus. Upon inserting these forms in the action we find that it reduces to

$$L(\bar{a}_i, \bar{b}_i) = \frac{1}{\pi} \epsilon^{ij} \dot{a}_i \bar{b}_j$$

(37)

which identifies $\bar{a}_i$ and $\bar{b}_i$ as the physical degrees of freedom. The remaining, gauge, degrees of freedom can be eliminated by gauge fixing, e.g. by setting $\partial_i a_i = \partial_i b_i = 0$.

From (37) we obtain the canonical commutation relations,

$$[\bar{a}_x, \frac{1}{\pi} \bar{b}_y] = i ; \quad [\bar{a}_y, -\frac{1}{\pi} \bar{b}_x] = i .$$

(38)

Since these are two commuting Heisenberg algebras, it naively looks like there is a continuum of ground states corresponding to different eigenvalues of $e.g.$ $\bar{b}_x$ and $\bar{b}_y$. This is however not the case, since the gauge fields are compact on account of the quantization of quasiparticle and vortex numbers, as noted previously. Compactness implies that $\bar{a}_i \equiv \bar{a}_i + 2\pi$ and $\bar{b}_i \equiv \bar{b}_i + 2\pi$ are angular variables. It follows that we need instead to consider the operators (Wilson loops) $\mathcal{A}_i = e^{i\bar{a}_i}$ and $\mathcal{B}_i = e^{i\bar{b}_i}$ and their algebras,

$$\mathcal{A}_x \mathcal{B}_y + \mathcal{B}_y \mathcal{A}_x = 0 ; \quad \mathcal{A}_y \mathcal{B}_x + \mathcal{B}_x \mathcal{A}_y = 0 .$$

(39)

Each of these has a two dimensional representation (via two of the three Pauli matrices) whence we obtain a $2 \times 2 = 4$-fold ground state degeneracy on the torus. It also follows that $\mathcal{B}_i$ can be interpreted either as measuring the $b$-flux or inserting an $a$-flux, and vice versa for the $\mathcal{A}_i$.

**B. Ground state degeneracy in the abelian Higgs model**

The above considerations have established a fourfold ground state degeneracy on the torus (and $4^g$ on genus $g$ surfaces) but left their physical description obscure. Indeed, the argument beginning with quasiparticle and vortex braiding is somewhat indirect. To complete the analysis we now turn to a direct identification of the states in the abelian Higgs model.

The basic observation is our identification of the gauge fields in the last section. This indicates that in the basis in which $\mathcal{A}_i$ are diagonal, the states differ by the amount of magnetic flux passing through the two holes. At the outset it is important to emphasize that this is sourceless flux and better thought of as the (necessary) assignment of eigenvalues to the Wilson loops. For ground states, the flux must be an integer multiple of $\pi$, the superconducting flux quantum. In a theory where the fundamental charges are eigenvalues to the Wilson loops. For ground states, the flux is only defined modulo $2\pi$, and we get two states for each non-contractible loop. The operators $\mathcal{B}_i$ then move the system between these eigenstates. As the states are degenerate, we can just as well diagonalize the latter operators and the resulting states are characterized by even and odd values of the electric flux.

More explicitly, consider the position eigenstates $| \phi(\vec{r}), \vec{A}(\vec{r}) \rangle$ of the gauge and scalar fields in the Hamiltonian formulation of the abelian Higgs model (5). We can define the action of the operator conjugate to the $x$–Wilson loop $\mathcal{A}_x = \exp(i \oint dx \mathcal{A}_x)$ on the torus parametrized by $0 \leq x < L_x$ and $0 \leq y < L_y$ by

$$\mathcal{B}_y | \phi(\vec{r}), \vec{A}(\vec{r}) \rangle = | e^{i\alpha(\vec{r})} \phi(\vec{r}), \vec{A}(\vec{r}) + \frac{1}{2\epsilon} \nabla \phi(\vec{r}) \rangle$$

(40)
Here and in the following we really mean the equivalence class of $\alpha(\vec{r}) = \alpha(0, y) + 2\pi$.\(^{10}\) Locally, the effect of the flux insertion operator $B_y$ is just a gauge transformation; however, it changes the sign of the gauge invariant observable, the Wilson loop $A_x$. This is a global effect, caused by an improper gauge transformation, that does change the state. Analogously, we can define the conjugate pair $A_y, B_x$.

For the pure abelian Higgs model with only charge-2 matter we obtain four degenerate states on the torus corresponding to the possibilities $A_i = \pm 1$. Clearly this construction generalizes to a $4^g$ degeneracy on a closed surface of genus $g$. As the states are exactly degenerate, we can just as well choose the basis set to be eigenstates of the $B_i$ instead.

To clarify the meaning of the latter representation it is useful to give an explicit representation for the operators for the choice $\alpha(\vec{r}) = 2\pi \theta(x' - x)$,

$$A_x(y) = e^{i\int_0^{L_y} dx' A_x(x', y)}$$
$$B_y(x) = e^{i\int_0^{L_y} dy' E_x(x, y') \frac{\partial}{\partial t} \int d^2r' \alpha(\vec{r}) \bar{\rho}(\vec{r})}$$

and the corresponding pair $B_x(y)$ and $A_y(x)$; $\bar{\rho}(\vec{r})$ is the charge density operator. Both $A_x(y)$ and $B_y(x)$ are clearly gauge invariant, and have singularities along the lines at $y$ and $x$ respectively.\(^{11}\)

From the canonical equal time commutation relations, $[A^\dagger_\alpha(\vec{r}, t), E^\dagger_\beta(\vec{r}', t)] = i\hbar \delta(\vec{r} - \vec{r}')$ and $[\rho(\vec{r}, t), \phi(\vec{r}', t)] = i\hbar \delta(\vec{r}, t)\delta(\vec{r}' - \vec{r}' \pm \vec{n})$ follows the commutator algebra,

$$A_x(y)B_y(x) + B_y(x)A_x(y) = 0$$

which confirms that the operators $B_i$ create one magnetic flux quantum. We also see that $B_y(x)$ measure the total electric flux in the $\hat{x} / \hat{y}$ direction in units of $\pi/e$, so that the eigenstates defined by $B_i = \pm 1$, which are symmetric/antisymmetric linear combinations of the magnetic flux states, have the interpretation of possessing even or odd numbers of electric flux quanta in the two directions. Finally, it follows from (43) that the Wilson loop $A_x(y)$ creates one unit of electric flux in this direction[16] which completes this dual description.

To explicitly construct the ground states which all have constant density, we must include the non trivial winding modes of the $\varphi$ field,

$$A_\mu(\vec{r}, t) = \frac{1}{L} \vec{A}_\mu(t)$$
$$\varphi(\vec{r}, t) = \varphi_0(t) + \frac{2\pi}{L} \vec{n} \cdot \vec{r},$$

where $\vec{n}$ is the winding number vector. The spatially constant phase $\varphi_0$, conjugate to the total number of particles, can be absorbed by a spatially constant gauge transformation. The Hamiltonian in a fixed winding number sector is easily obtained from (8) and given by,

$$H_\vec{n} = \frac{1}{2} (\Phi_E^i)^2 + \frac{m_i^2}{2} (\vec{A}^i + \frac{\pi}{e} \vec{n}_i)^2$$

where $\Phi_E = LE^2$ is the spatially constant electric flux which is conjugate to $\vec{A}$,

$$[\Phi_E^i, \vec{A}_j] = i\delta^i_j$$

Naively there is a ground state for each winding sector, and a gap to the plasmon mode at $\hbar m_s$. Because of gauge invariance we should however identify all winding number sectors which have the same value for the Wilson loops $A_i = e^{i\pi \vec{n}_i}$. For $q = e$ there are four non-equivalent sectors corresponding to eigenvalues $\pm 1$ for the operators $A_i$. The conjugate operators $B_j = e^{i\pi \Phi_E^j}$ are precisely the “modular electric field” operators defined by Reznik and Aharonov[28], and $A_i$ and $B_j$ satisfy the algebra (39) which allows us to identify the potential $b_i$ with the modular electric field.

\(^{10}\) Here and in the following we really mean the equivalence class of $\alpha(\vec{r})$ under the addition of functions that are periodic on the torus but we will be sloppy about this without prejudice to our argument.

\(^{11}\) The nature of these singularities are, however, quite different. The singularity of $A_x(y)$ correspond to the creation of a thin line of electric flux, as discussed in the text, while the singularity in $B_y(x)$ is only a gauge artifact. This follows from the relation, $B_y(x_1)B_y^{-1}(x_2) = \exp \left\{ -\frac{\pi}{\hbar} \int d^2r' \theta(x' - x_1)\theta(x_2 - x') + \int d^2r' \frac{\partial}{\partial x} E_x(x', y) - \bar{\rho}(\vec{r}) \right\}$, and remembering that $\nabla \cdot E - \bar{\rho}$ is the generator of local gauge transformations. We see that the apparent singularity at $x$ of the operator $B_y(x)$ can be moved by a regular gauge transformation and thus has no physical significance.
FIG. 3: A vortex tunnelling process inserting a unit of magnetic flux inside the torus. In this visualization it also leaves a flux loop outside, but that is invisible to the electrons on the surface. This process connects ground states labelled by opposite values of the Wilson loop $e^{i \oint_C \vec{a} \cdot d\vec{l}} \equiv e^{i e \Phi_M}$ where $\Phi_M$ is the magnetic flux threading $C$.

Three closing comments are in order.

1. A state with definite $A_i$ necessarily has a fluctuating electric flux present which might seem problematic for a superconductor which has an infinite conductivity. This is, however, not so. The crucial point, which is not immediately obvious when one thinks about classical background electric fields, is that the matter couples to the vector potential and not the electric field and the former clearly has no effect.

2. In the dual states, while there is a definite parity of the electric flux, there still isn’t an average non-zero flux. Besides, these states are linear combinations of states that do not possess a current by the argument in (1).

3. Finally, it is worth emphasizing the importance to our analysis of the distinction that the gauge potentials $\vec{A}_i$ are not observables, but the Wilson loops $A_i^q = e^{iqA_i}$ are, where $qe$ are the charges in the system. Naively, we would be led to consider states $|\vec{n}\rangle \equiv |n_x, n_y\rangle$ with $n_x$ and $n_y$ superconducting flux quanta through the two holes. However these states are not all distinct as far as the Wilson loops go and instead form equivalence classes upon addition of $2/q$ flux quanta in either hole. We have analyzed the case of the standard superconductor where $q = 1$ and indeed that is true more generally in nature. If however, fractionally charged matter was present at a fundamental level, the ground state degeneracies would indeed be different. In such cases, consistently, the starting topological field theory would also be different since there would now be a larger set of braiding phases to encode.

C. Finite size effects and tunneling

In the last section we were a little sloppy in our discussion for pedagogical purposes. The ground states that we discussed arise in two approximations—the neglect of vortex creation/annihilation in the bulk and in the absence of any other matter, i.e. we took the quasiparticle gap to be infinity. This had the utility that ground states were now exactly degenerate for a finite system, but now we can state the more general situation.

In the general setting we must consider (i) the sensitivity of the quasiparticle field to the values of $A_i$ or equivalently processes in which two quasiparticles are created from the vacuum (the condensate) and then tunnel and recombine across a non-contractible loop and (ii) a similar process in which vortex-antivortex pair is created from the vacuum and then tunnels and recombines across a non-contractible loop. As reviewed in the introduction, such processes are responsible for motion in the ground state manifold and lead to a lifting of the topological degeneracy for finite systems.

The tunneling process that is easiest to visualize is the vortex-antivortex tunneling process shown in Fig. 3. Here a unit vortex-antivortex pair is created, they subsequently move around a cycle of the torus, and are finally annihilated. During this process they will insert a unit of magnetic flux inside the torus, thus changing the value of the corresponding $A_i$ operator. Thus this process corresponds to a tunneling between the magnetic flux states, and by itself it will mix them and lift their degeneracy by an amount $\sim e^{-L_i/\lambda_t}$ where $L_i$ is the length of the tunneling path, and $\lambda_t$ a constant of order the coherence length. Interested readers can find a quantitative computation of this process in Ref. 35, for the closely related Fradkin-Shenker system discussed below in Section V-B.

The interpretation of the quasiparticle tunneling process, shown in Fig. 4, is more subtle. Naively one might think of this as the charges pulling out an electric flux between them, but since the superconductor screens, this is not the case on average. What is true instead, is that a quasiparticle that crosses a surface changes the parity (evenness/oddness) of the fluctuating electric flux through its path. Hence this process connects the electric flux states and by itself will mix them and lift their degeneracy by an amount $\sim e^{-L_i/\xi_t}$ where $\xi_t$ is a constant of order the coherence length.

The actual finite volume ground state in the presence of both vortex and quasiparticle tunneling will be determined by a competition between the above two effects. That the topological degeneracy is recovered exponentially fast in the linear dimensions of the system is, as remarked earlier, a hallmark of topological order.
D. Ground state degeneracy in $d = 3 + 1$

Finally we present the extension of the discussion in Subsection A to $d = 3 + 1$. The action (18) can be reorganized as

$$S = \frac{1}{\pi} \int d^4 x \epsilon^{ijk} \dot{a}_i b_{jk} + a_0 (\epsilon^{ijk} \partial_j b_{jk}) + 2b_0 (\epsilon^{ijk} \partial_j a_k),$$

(47)

which identifies the four constraints in the problem. As $b_{jk}$ is antisymmetric, its independent components can be identified as $c^i = \epsilon^{ijk} b_{jk}$ and hence the constraints rewritten as

$$\partial_i c^i = 0$$
$$\epsilon^{ijk} \partial_j a_k = 0.$$  

(48)

On the 3-torus, these are solved by setting

$$c^i = \frac{(\bar{c}^i + \epsilon^{ijk} \partial_j \xi_k)}{L^{3/2}}$$
$$a_k = \frac{(\bar{a}_k + \partial_k \Lambda)}{L^{3/2}}$$

(49)

where $\xi$ and $\Lambda$ are periodic functions and we have thus separated the constant pieces of $c$ and $a$. Upon substituting these forms in (47) we obtain the analog of (37),

$$L = \frac{1}{\pi} \bar{c}^i \dot{\bar{a}}_i$$

(50)

which encodes three commuting Heisenberg algebras and thence, upon taking account of the compactness of the fields, to $2^3 = 8$ states.

V. OTHER REALIZATIONS OF THE BF THEORY

In this section we digress somewhat from the main development to examine some closely related systems. The systems are related in that they too are characterized by topological order described by the $BF$ theory—they all fail to exhibit local symmetry breaking, a pair of low energy “matter” and “gauge” excitations with the same braiding phase of $\pi$ and the attendant ground state degeneracy. In a way this is an example of universality, but in a much more limited sense than for critical point theories—for the topological scaling limit keeps much more limited information than the Wilsonian one. Our examples here are the $\mathbb{Z}_2$ lattice gauge theory, the $U(1)$ lattice gauge theory with charge-2 Higgs scalars, the short ranged resonating valence bond (RVB) state, and a particular quantum Hall bilayer.
FIG. 5: Two of the four ground states of the $Z_2$ lattice gauge theory at zero coupling, on the torus. They differ by the insertion of a $Z_2$ vortex (vison) through one of the holes of the torus—which is implemented by changing the sign on a string of bonds as shown. The pair of states thus differ in the sign of the Wilson loop $\Pi_C \sigma^x$. The remaining two states differ by vison insertion in the other hole.

A. $Z_2$ lattice gauge theory

The $Z_2$ lattice gauge theory, defined by the Hamiltonian,
\begin{equation}
H = K \sum_{P} \prod_{(ij) \in P} \sigma_{ij}^z + \Gamma \sum_{(ij)} \sigma_{ij}^x ,
\end{equation}
where the sums are over spatial plaquettes and links, has been studied extensively, and is well known to be a topological theory in the $\Gamma \to 0$ limit. In this limit all plaquettes must be unfrustrated in the ground states, $\prod_{(ij) \in P} \sigma_{ij}^z = 1$. There are four degenerate ground states on the torus of which two correspond to the configurations (really their equivalence classes under local gauge transformations) shown in Fig. 5; the remaining two are trivial extensions as discussed in the caption. Clearly all plaquettes are nonfrustrated while the Wilson loops around the cycles differ by signs in the various states. The operators that moves between the different configurations are singular gauge transformations which are the $Z_2$ counterparts of the $B$ operator introduced in (40). The conjugate, electric field states are discussed e.g. in Ref. 36. The excited states of the theory consist of Ising vortices or visons. If we now couple fundamental Ising matter sources to the gauge field,
\begin{equation}
H_m[c] = \beta \sum_{(ij)} c_i \sigma_{ij}^z c_j
\end{equation}
it is easy to see that transporting a “particle” around the vison leads to a $\pi$ phase, i.e. the $Z_2$ gauge theory has the same braiding phases [36, 37] as the $BF$ theory (14). When $K$ is finite but large and the coupling to the matter is weak, the low energy theory is still the $BF$ theory as we discuss explicitly next. For variety we will carry out the relevant treatment entirely on the lattice—it is an interesting feature of this problem that this can be done.

1. The lattice $BF$ action

As the variables in (51) are discrete, it is most convenient to work with a discretized time. To this end we begin with the classical $Z_2$ lattice gauge-matter action
\begin{equation}
S_\sigma[\sigma,c] = -K \sum_{P} \prod_{\sigma_{ij} \in P} \sigma_{ij} - \beta \sum_{(ij)} c_i \sigma_{ij} c_j ,
\end{equation}
where $\sigma_{ij}$ is an Ising variable, and the sums run over plaquettes and links on an Euclidian lattice. We now rewrite (53) in a form involving a lattice version of the $BF$ action, by using the identity,
\begin{equation}
e^{+K \prod_{\sigma_{ij}}} = f(K) \sum_{\tau = \pm 1} e^{\beta \tau} e^{i \pi (1-\tau)(1-\prod_{\sigma_{ij}})}
\end{equation}
where $2\tilde{\beta} = - \ln \tanh K$ and $f(K) = \sqrt{\frac{1}{2} \sinh(2K)}$, for each plaquette, $P$, in the partition function $Z_{Z_2} = \sum_{\sigma_{ij}} e^{-S_\sigma}$. This introduces a set of Ising variables, $\tau_{ij}$ defined on the links of the dual lattice and, and the partition function can be expressed as,
\begin{equation}
Z_{Z_2} = \sum_{\{\sigma_{ij}, \tau_{ij}, c_i\}} e^{-S_{BF}[\sigma,\tau]+\tilde{\beta} \sum_{(ij)} \tau_{ij} + \beta \sum_{(ij)} c_i \sigma_{ij} c_j} .
\end{equation}
where,

\[ S_{BF} = -\frac{i\pi}{4} \sum_{(ij)} (1 - \tau_{ij})(1 - \prod_{\gamma} \sigma_{ij}^\gamma). \]

(56)

Here \( \tau_{ij} \) denotes the plaquette on the original lattice pierced by the link \( \tau_{ij} \) on the dual one. Except for shifts, this term—which multiplies one gauge field \( \tau \) with the flux of the other \( \sigma \)—is clearly the Ising lattice analog of the continuum BF term. This piece of the action was derived by Senthil and Fisher\[37\], who also showed that by partial differentiation it can be expressed in the alternative form,

\[ S_{BF} = -\frac{i\pi}{4} \sum_{(ij)} (1 - \sigma_{ij})(1 - \prod_{\gamma} \tau_{ij}^\gamma), \]

(57)

which is manifestly invariant under the gauge transformation \( \tau_{ij} \rightarrow v_i \tau_{ij} v_j \) where the \( v_i: s \) live on the sites of the dual lattice. Because of this invariance, we can now recognize (55) as the restriction to \( v_i = 1 \) gauge of the manifestly doubly gauge invariant action

\[ Z_Z = \sum_{\{\sigma_{ij}, \tau_{ij}, c_i, v_i\}} e^{-S_{BF}[\sigma, \tau] + \beta \sum_{(ij)} v_i \tau_{ij} v_j + \beta \sum_i c_i \sigma_i c_j}, \]

(58)

If we now specialize to large \( K \) (i.e. small \( \beta \)), and small \( \beta \), we see that the BF term dominates as promised.

Finally, readers with an appetite for lattice manipulations can convince themselves that the braiding phases are correctly reproduced by the lattice BF action by considering the expectation values of two Wilson loops, one on the original and one on the dual lattice,

\[ \langle W_{\Gamma_1} W_{\Gamma_2} \rangle = \sum_{\{\sigma_{ij}, \tau_{ij}\}} e^{-S_{BF}[\sigma, \tau]} \prod_{(ij) \in \Gamma_1} \sigma_{ij} \prod_{(kl) \in \Gamma_2} \tau_{kl} \]

(59)

Expressing \( \sigma_{ij} = e^{i\Phi_{ij}} \), and using (57) for the action, it is easy to show that for a link present in the loop \( \Gamma_1 \), the sum over \( \sigma_{ij} = \pm 1 \) yields zero if there is not a “wrong sign” dual plaquette is attached on the dual lattice. Similarly, for a link not present in \( \Gamma_1 \), the dual plaquette must be unfrustrated. As illustrated in Fig. 6, this implies that a dual loop \( \tilde{W}_{\Gamma_1} \) will pick up a minus sign every time the curve \( \Gamma_2 \) wind around the curve \( \Gamma_1 \). Clearly the dual of this argument, i.e. binding original plaquettes to the dual links on \( \Gamma_2 \), would give the same result.

B. U(1) lattice gauge theory with charge-2 Higgs

In their influential 1979 paper on gauge-Higgs systems on the lattice, Fradkin and Shenker\[12\] analyzed a \( U(1) \) lattice gauge theory coupled to charge-2 matter and showed that it exhibited a phase where the low energy degrees of freedom reduced to those of the \( Z_2 \) gauge theory discussed above, see Fig. 7. Consequently, when the low energy theory is in its deconfined phase, the gauge-Higgs system is also described by the BF theory. This system is pretty much a truly lattice superconductor in that the gauge field also lives on a lattice. However, the compactness of the microscopic gauge field introduces features that make the characterization of its electromagnetic response problematic—there seems not to be a definition of the electrical conductivity that will distinguish the deconfined phase of interest from the confined phase. This is related to the massive character of the photon in both phases. Nevertheless, the model has other uses and has been extensively invoked in searches for spin liquids and theories of the cuprates\[36,37\] where the starting problem can often be reformulated as a \( U(1) \) theory coupled to matter but where the gauge field is now generated by the matter itself and is not related to fundamental electromagnetism. We now review the reduction of a lattice superconductor to a \( Z_2 \) gauge theory by a somewhat different method than used in the original work.

The starting point is the following lattice action,

\[ S[U, \Psi] = -\frac{K_0}{2} \sum_P \prod_{(ij) \in P} [U_{ij} + h.c.] - \frac{\gamma}{2} \sum_{(ij)} [\Psi_i U_{ij}^2 \Psi_j^\dagger + h.c.]. \]

(60)

The gauge potential, \( A_{ij} \), is defined on the links, \( U_{ij} = e^{iA_{ij}} \) and the charge 2 scalar field on sites, \( \Psi_i = e^{i\theta_i} \), and the two sums are taken over plaquettes and links of the lattice respectively. Both \( A_{ij} \) and \( \theta_i \) are angular variables defined on the interval \( [0, 2\pi] \), and in terms of these the action takes the form,

\[ S[A, \theta] = -K_0 \sum_P \cos(F_P) - \gamma \sum_{(ij)} \cos(\theta_i - \theta_j - 2A_{ij}), \]

(61)
where \( F_P = F_{ijkl} = A_{ij} + A_{jk} - A_{lk} - A_{il} \), is the lattice field strength of the plaquette \( P = (ijkl) \).

What is of relevance here is that on the \( K_0 = \infty \) line in the phase diagram, Fig. 7, the theory (60) becomes a \( Z_2 \) gauge theory. To show this, we make the following decomposition of the gauge potential,

\[
A_{ij} = \frac{1}{2} \left[ a_{ij} + \pi(1 - \sigma_{ij}) \right],
\]

(62)
corresponding to \( U_{ij} = \sigma_{ij} e^{\frac{a_{ij}}{2}} \), where \( \sigma_{ij} \) is an Ising variable, and the range of the angular variable \( a_{ij} \) is again from 0 to \( 2\pi \). We then use the following identity,

\[
\int_0^{2\pi} d\phi f(\phi) = \frac{1}{2} \sum_{\sigma = \pm 1} \int_0^{2\pi} d\chi [\frac{1}{2} \chi + \pi(1 - \sigma)]
\]

(63)
to rewrite the partition function as,

\[
Z = \prod_{(ij), k} \int_0^{2\pi} dA_{ij} d\theta_k e^{-S[A_{ij}, \Phi_k]} = \prod_{(ij), k} \int_0^{2\pi} dA_{ij} d\theta_k \frac{1}{2} \prod_{(lm)} \sum_{\sigma_{lm} = \pm 1} e^{-S'[a_{ij}, \theta_{ij}, \sigma_{lm}]}.
\]

(64)

In the \( \gamma \to \infty \) limit it is convenient to use a unitary gauge where \( \theta_i = 0 \) and the action for \( a_{ij} \) takes the form,

\[
S[a, \sigma] = -K_0 \sum_P \cos(\frac{1}{2} f_P) - \prod_{(ij) \in P} \sigma_{ij} - \gamma \sum_{(ij)} \cos(a_{ij})
\]

(65)

with \( f_P \) is the lattice field strength corresponding to \( a_{ij} \). The effective \( Z_2 \) action is now defined as,

\[
e^{-S_{\sigma}[\sigma]} = \prod_{(ij)} \int_0^{2\pi} da_{ij} e^{-S'[a, \sigma]}.
\]

(66)

and can be computed in a perturbative expansion in \( 1/\gamma \). To lowest nontrivial order we obtain,

\[
S_{\sigma}[\sigma, c] = -K \sum_{(ij) \in P} \prod_P \sigma_{ij}
\]

(67)
where $K = K_0(1 + \frac{1}{2}\gamma)$. We now add a charge $q = 1$ field $\phi_i = e^{i\theta_i}$ with the action $S[U, \phi] = (\beta_0/2) \sum_{<ij>} [\phi_i U_{ij} \phi_j + h.c.]$. Decomposing the angular variable as $\theta_i = \frac{1}{2}(\xi_i + \pi c_i)$ we have the identity, $\cos(\theta_i - \theta_j - A_{ij}) = c_i \sigma_{ij} c_j \cos \frac{1}{2}(\xi_i - \xi_j - a_{ij})$, and integrating $a_{ij}$ and $\xi_i$, gives the action

$$S_m[\beta, c] = -\beta \sum_{<ij>} c_i \sigma_{ij} c_j \quad (68)$$

which describes the coupling of an Ising matter field. Combining (67) and (68) we regain the $Z_2$ lattice action (53), and hence, by the results of the previous subsection, the BF theory as the low energy, purely topological description of the compact lattice superconductor.

C. RVB State

The short ranged RVB state of a quantum Heisenberg magnet, first proposed by Anderson [38], is a liquid of spins paired into local singlets. In the extreme short ranged case the wavefunction is made up solely of configurations $|c\rangle$ in which each spin is paired with exactly one nearest neighbor spin. A prototypical liquid wavefunction is then an equal amplitude superposition

$$|\psi\rangle = \sum_{c} |c\rangle \quad (69)$$

of such configurations. The physics of the nearest neighbor problem is captured in the quantum dimer model [39] and following the demonstration that the triangular lattice quantum dimer model supports a liquid phase [40] it has become clear that this generalizes to other non-bipartite lattices.

This liquid, RVB, phase can be readily seen to lead to a $4^g$ ground state degeneracy [41]. As shown in Fig. 8, the parity of the number of dimers crossing a non-contractible loop is invariant under a local dimer dynamics which thus yields two distinct liquid states for each such loop. In terms of our previous discussion for superconductors this is the analog of the parity of the electric flux.
FIG. 8: Topology of dimer coverings: The number of dimers crossing the non-contractible loop $C_1$ can only change by an even number under a local dimer dynamics, e.g. the resonance move shown by the dashed lines changes the number by two. Consequently, the ground states of the quantum dimer model on the torus can be labelled, in the deconfined phase, by the number of dimers modulo 2 crossing the non-contractible loops.

FIG. 9: The vison involves a string going out to infinity. A dimer configuration $c)$ is now weighted by $(-1)^{N_s(c)}$, where $N_s(c)$ is the number of dimers crossing the string.

The excitations of the RVB state are spinons and visons (vortices). A spinon is an unpaired spin while a vison involves a phase string (Fig. 9). It is not difficult to see that these gapped excitations have the familiar topological interaction with a mutual braiding phase factor of $-1$ arises. It is also an instructive exercise to see that the tunneling of spinons and vortices leads to the lifting of the ground state degeneracy. From all of this it follows then that the RVB state again has a topological description by the $BF$ action.

While the pictures drawn above pertain to two dimensions, recently it has been shown that the quantum dimer model on the FC lattice exhibits an RVB phase [42] which is then characterized by the $3 + 1$ dimensional version of the $BF$ action. Finally, we should note that in the case of the RVB, the microscopic problem is that of a strongly coupled gauge theory so a trivial reduction to the topological actions is not feasible, as it was for the weakly coupled phases of the $Z_2$ gauge theory discussed above.

D. A quantum Hall interpretation of the $BF$ theory

Finally, we observe that the $BF$ theory can be taken to describe a somewhat unusual quantum Hall system.
According to Wen and Zee[43] the general form of the topological action for an abelian quantum Hall liquid is (in an obvious form notation),

\[ \mathcal{L}_{qh} = \frac{1}{4\pi} K_{IJ} a^I d a^J + \frac{e}{2\pi} t_I a^I A d a^J - a^I j^I \]  

(70)

where \( K_{IJ} \) is a symmetric matrix, and \( t_I = \delta t_I \) a vector, both with integer entries. This action leads to a ground state degeneracy \( |\det K| \) on a surface of genus \( g \) and the true electrical charge, \( q \), of a quasiparticle with charges \( l_I \) with respect to the gauge fields \( a^I \) is given by \( q = -e t_I K_{IJ}^{-1} l_J \).

For quantum Hall systems the matrix \( K \) is taken to be positive semidominant, corresponding to the lack of time reversal invariance. The formalism can be extended, however, to time reversal invariant systems by expanding the allowed \( K \) matrices. In our case \( K^{IJ} = 2\sigma^2_{IJ} \) reproduces the \( d = 2 \) BF action. As a check, on a torus \( |\det K| = 4^4 = 4 \) as derived before.

An alternative quantum Hall representation is obtained by the transformation,

\[ a^1 \equiv a = R + L \]
\[ a^2 \equiv b = R - L \]

(71)
giving

\[ \mathcal{L}_{qh} = \frac{1}{\pi} (R d R - L d L) + j(R + L) + \tilde{j}(R - L) \]  

(72)

i.e. two decoupled \( \theta = \pi/4 \) liquids with opposite sense of time reversal breaking. Note that although the elementary quasiparticles acquire a \( e^{i\pi/4} \) phase under exchange, the original charges and vortices carry charge with respect to both layers (or, equivalently, can be thought of as composites of charges in the two layers). It is an elementary exercise to verify that the combined Berry and exchange phases come out correctly if we restrict ourselves to this sector of the expanded problem.

E. Instantons and the nature of charge

In this section we have covered a diverse set of systems that give to the \( BF \) theory in their topological scaling limit. Evidently, as we move away from that limit the differences among the systems will reassert themselves. Here we wish to comment on one of these differences, namely the nature of the charges in the various systems.

We note that the \( BF \) theory formally involves a \( U(1) \) gauge field and hence a coupling to \( U(1) \) currents. But this is misleading since in writing it we have really only encoded a finite amount of information on braiding phases—in particular these phases are insensitive to whether the quasiparticle and vortex currents are truly conserved or only conserved modulo 2. Among the systems we have considered, both currents are integer valued for the hypothetical quantum Hall system. In the ordinary superconductor the vortex number is integer valued but quasiparticle number is only defined modulo 2 since a pair of quasiparticles can always disappear into the condensate. In the \( \mathbb{Z}_2 \) gauge theory, both currents are evidently only defined modulo 2 and since the Fradkin-Shenker problem reduces to the former the same is true there.

These differences are, of course, built into the microscopic actions. Of interest here is how they can be incorporated in the \( U(1) \) description as we move beyond the topological scaling limit. The solution to this puzzle is that compact gauge fields permit finite action instantons that break the corresponding \( U(1) \) down to \( \mathbb{Z}_2 \).

For the \( a \) field the instantons are unit strength monopoles that can create or destroy two Abrikosov flux lines of strength \( 1/2e \), as illustrated in Fig. 8. The strength of the tunneling will depend on microscopic details, which determines the magnitude of the instanton action.

We now also learn how to incorporate the charge non-conserving effects of Cooper pair breaking and formation in the context of \( BF \) theory - it simply amounts to allowing monopole configurations in the dual gauge field \( b \). It is an interesting technical challenge to actually derive this prescription directly from the path integral formulation of the full abelian Higgs model.

Returning to our original question it is now clear that the inclusion of instantons is the mechanism by which the different conservation scenarios are distinguished beyond the topological scaling limit. The quantum Hall realization includes none, the ordinary superconductor includes (on reasonable scales!) only the \( b \) monopoles and the \( \mathbb{Z}_2 \) gauge theory and the Fradkin-Shenker problem require both \( a \) and \( b \) monopoles.
FIG. 10: Virtual vortex-antivortex fluctuations represented as a space-time vortex loop. Also shown are two vortices annihilating on a monopole.

VI. EDGE STATES

Returning to the topological order characteristics for quantum Hall states listed in the introduction, we see that we have found analogs of all of them in superconductors save one—these are “edge states” to which we now turn. The existence of edge states, i.e. degrees of freedom localized near the boundary of a manifold with a boundary, can be deduced quite generally. To begin with, one can see qualitatively that fractionalization in the bulk implies that the missing fractional quantum numbers of the quasiparticles must migrate to the boundary and thence that the boundary must support degrees of freedom capable of absorbing these quantum numbers. This can be sharpened once one has a topological field theory in hand. While on closed manifolds the topological field theory has only global degrees of freedom, in the presence of a boundary it ceases to be purely topological and now exhibits boundary degrees of freedom.

From the quantum Hall effect we however know that the details of the boundary theory is, in general, not coded in the bulk topological action, but depends crucially on the nature of the confining potential. For instance, a polarized Laughlin state with a sharp edge will have a single chiral edge mode with a velocity given by the $\vec{E} \times \vec{B}$ drift at the edge. In a softer potential the edge can reconstruct giving pairs of counterpropagating modes which in general develop a gap.

With suitable boundary conditions, the topological field theory does define the phase space of a minimal theory needed for current conservation. In the quantum Hall case it is the electric current of the bulk quantum liquid and its associated quasiparticles. In the case of the superconductor there are two currents, described by the gaugefields $a_\mu$ and $b_\mu$ corresponding to charge and vorticity respectively. Thus, from the knowledge of the quasiparticles in the bulk one obtains a listing of the different sectors of the edge theory—which correspond to the independent ways in which quasiparticles in the bulk can influence the edge dynamics. This further allows identification of the operator spectrum at the edge. What remains is the identification of the edge Hamiltonian and while that can be constrained on symmetry grounds there remain details that only microscopics can fill in.

The choice of boundary conditions for the topological field theory is crucial - different choices give different dynamics, or even no dynamics at all. In the quantum Hall case the boundary conditions are well understood, at lest in the simplest cases, but to our knowledge there is no rigorous derivation from a microscopic approach. A brief review of the quantum Hall case is given in the Appendix. In the case of the superconductor the situation is less clear. A microscopic approach would be to study e.g. the abelian Higgs model (5) in the presence of a interface, carefully follow the steps leading to the topological $BF$ action (14) and deduce the relevant boundary conditions, which would
depend on the nature of the interface. We shall not take this route but rather, in the spirit of Section V-D, assume the kind of boundary conditions used to analyze immunocompetent quantum Hall systems. A discussion of different boundary conditions in $BF$ theories and the abelian Higgs model can be found in the work of Balachandran et. al. [31, 44].

**A. BF theory on manifold with boundary in $d=2$**

As briefly explained in the Appendix, the pertinent starting point is the Hamiltonian form (34) of the $BF$ action, which we now consider on a manifold $\Omega$ with a boundary $\partial \Omega$ parametrized by $x_1$. Under the boundary conditions $a^0|_{\partial \Omega} = b^0|_{\partial \Omega} = 0$, this action coincides with the covariant expression (14) restricted to the same domain. The constraints (35) are now solved by

$$a_i = \frac{1}{2} \partial_i \Lambda_a \quad b_i = \frac{1}{2} \partial_i \Lambda_b , \tag{73}$$

where $\Lambda_{a/b}$ take arbitrary values at the boundary. Upon inserting these forms in the action we find that it reduces to

$$S = -\frac{1}{4\pi} \int_{\partial \Omega} d^2 x \partial_0 \Lambda_a \partial_1 \Lambda_b \tag{74}$$

which shows that the only degrees of freedom live at the edge and that their phase space is that of a one dimensional boson with both chiralities present—from (74) we can read off the canonical commutation relations,

$$[\Lambda_a(x,t), -\frac{1}{4\pi} \partial_1 \Lambda_b(y,t)] = i\delta(x-y) \ . \tag{75}$$

The analysis thus far is modified when there are quasiparticles and/or vortices present in the bulk. Now the boundary line integrals of the gauge fields are non-zero but quantized, so it is necessary to allows the edge bosons to wind along the edge. Their winding numbers,

$$N_a = \frac{1}{2\pi} \int_{\partial \Omega} dx^1 \partial_1 \Lambda_a \quad N_b = \frac{1}{2\pi} \int_{\partial \Omega} dx^1 \partial_1 \Lambda_b \tag{76}$$

count the numbers of vortices and quasiparticles in the bulk respectively. Equivalently, they count the screening charges at the boundary so we can identify the edge vortex and quasiparticle densities as $\frac{1}{2\pi} \partial_1 \Lambda_a$ and $\frac{1}{2\pi} \partial_1 \Lambda_b$ respectively. In turn this identifies $\psi^{0\dagger}_a \sim e^{-i\Lambda_a/2}$ as the edge vortex creation operator and $\psi^{1\dagger}_b \sim e^{-i\Lambda_b/2}$ as the edge quasiparticle creation operator while Cooper pairs (valence bonds) and $2\pi$ vortices are created by $e^{-i\Lambda_a}$ and $e^{-i\Lambda_b}$ respectively. It is not hard to see that in a sector with $N_{a/b}$ odd $\psi^{0\dagger}_{1/a}$ picks up a factor of $-1$ upon circling the edge and hence exhibits the correct braiding. Finally, one technical point is worthy of note. The quantization conditions (76) and the set of operators identified here are not those of a compact boson of any specified radius. While this is important for a detailed understanding of the spectrum, it will not matter for the rest of our discussion.\footnote{As we were finishing this paper there appeared Ref. 45 which also notes this point as a special case in the course of a more general analysis of $BF$ theories in $2+1$ dimensions. Their point of departure, however, could not be more different!}

We turn now to the Hamiltonian, where the true nature of the currents, discussed in the last section, becomes important. If $a$ and $b$ are truly $U(1)$ fields then the edge Hamiltonian must conserve vortex and quasiparticle number and we conclude that it takes the form

$$H = \int_{\partial \Omega} dx^1 \left( \frac{\psi^{0\dagger}_a (\partial_x \Lambda_a)^2}{2} + \frac{\psi^{1\dagger}_b (\partial_x \Lambda_b)^2}{2} \right) \tag{77}$$

plus higher gradient corrections. The quadratic cross-term is ruled out by time reversal invariance. In this case the edge is gapless and exhibits Luttinger liquid behavior.
For all our remaining cases however both charges are not conserved. At a minimum Cooper pair creation/annihilation is allowed so that we must add the term

\[
H_a = \int_{\partial\Omega} d^3x \frac{g_a}{2}(e^{-i\Lambda_a} + e^{i\Lambda_a}) \equiv g_a \cos(\Lambda_a)
\]

to \(\mathcal{H}\). For the Fradkin-Shenker problem, the RVB state and the \(Z_2\) gauge theory we also need to add the dual process of vortex pair creation/annihilation,

\[
H_b = \int_{\partial\Omega} d^3x \frac{g_b}{2}(e^{-i\Lambda_b} + e^{i\Lambda_b}) \equiv g_b \cos(\Lambda_b)
\]

The resulting theory \(H + H_a + H_b\) is a dual sine-Gordon model with one of the cosines being generically the most relevant operator. It follows then, that the edge is generically gapped.

### B. BF theory on manifolds with boundary in \(d = 3\)

We now return to the action (47) and the constraints (48) but now on a manifold with a boundary. In the line with the discussion in \(d = 2\) we now write the solution to the constraints as

\[
e^i = e^{ijk} \partial_j \xi_k / L^{3/2}
\]

\[
a_k = \partial_k \Lambda / L^{3/2}
\]

where the boundary values of \(\xi_k\) and \(\Lambda\) are now unconstrained. The action now takes the form

\[
S = \frac{1}{\pi} \int_{\partial\Omega} d^3x \left( e^{ijk} \partial_j \xi_k \right)_n = \frac{1}{\pi} \int_{\partial\Omega} d^3x \left( \partial_1 \xi_2 - \partial_2 \xi_1 \right)
\]

where on the first line the subscript \(n\) indicates the normal to the bounding surface and on the second we have taken the latter to have local coordinates \((1, 2)\). Evidently this is the symplectic structure of a scalar field with \(\frac{1}{\pi} \left( \partial_1 \xi_2 - \partial_2 \xi_1 \right)\) playing the role of the conjugate momentum,

\[
[\Lambda(x, t), \frac{1}{\pi} (\partial_1 \xi_2 - \partial_2 \xi_1)(y, t)] = i\delta(x - y).
\]

Unlike in \(d = 2 + 1\) where the two edge fields enter symmetrically, we see that they have different character in \(d = 3 + 1\).

The analysis of sectors is more complex for this reason. The presence of quasiparticles in the bulk leads to the quantization

\[
N_\xi = -\frac{1}{\pi} \int_{\partial\Omega} dx^1 dx^2 (\partial_1 \xi_2 - \partial_2 \xi_1).
\]

With vortex lines first consider the situation of the infinite cylinder. Here the line integral

\[
N_a = -\frac{1}{\pi} \int_{\partial\Omega} dx^1 \partial_1 \Lambda
\]

around the circumference will count the number of vortex lines running parallel to the cylinder axis. Sectors with \(N_a \neq 0\) are manifestly locally stable but they are at infinite energies relative to the ground state. For generic bounded geometries, the situation is more complicated: vortex lines in the bulk will have to exit the surface at two points which then define vortices in the field \(\Lambda\). While one can formally define sectors of the edge theory with an arbitrary number of such vortex/anti-vortex pairs, since the bulk dynamics will force the vortex lines to move about, the actual problem can no longer be studied purely at the edge, so an edge/bulk separation is no longer possible.

Turning now to the edge “vertex” operators, we note that the quasiparticle creation operator \(\psi_\xi^\dagger \sim e^{i\Lambda}\) increases \(N_\xi\) by one. The existence of such a local operator is to be expected, e.g. in the RVB problem one can see that a spinon created in the bulk leads to the creation of a spinon at the boundary and the latter is equally a local object. For vortex lines let us restrict ourselves to the case of the cylinder. Here \(\psi_a^\dagger \sim e^{i\pi_0 \frac{x_2}{2}}\) generates shifts between different values of \(N_a\) where \(\pi_0 = \frac{1}{L} \int_{\partial\Omega} dx^2 (\partial_1 \xi_2 - \partial_2 \xi_1)\). This operator is non-local, again as one expects.

The conserving Hamiltonian is now

\[
H = \int_{\partial\Omega} dx^1 dx^2 \left( \frac{\psi_1}{2} (\nabla \Lambda)^2 + \frac{\psi_2}{2} (\nabla (\partial_1 \xi_2 - \partial_2 \xi_1))^2 \right),
\]

and the addition of quasiparticle and vortex line creation/annihilation again generically gives rise to gaps.
FIG. 11: Weakening the indicated row of plaquettes produces as set of low energy edge states (Section VI). At a critical value of these couplings a “topology changing phase transition” ensues.

C. Gapless edges and topology changing phase transitions

As we have noted above, except in the case where both quasiparticle and vortex currents are truly $U(1)$ currents, the edges will be gapped for generic values of the coupling constants. There are special values of the couplings, however, for which the edges are gapless. This gaplessness arises because in both $d = 2$ and $d = 3$ the two perturbations break the $U(1)$ symmetry down to $Z_2$ in dual ways and an Ising transition separates the two phases obtained when just one of perturbations dominates. In $d = 2$ this is well understood to happen along the line $g_a = g_b$ and the resulting theory is the familiar Majorana fermion of the critical Ising model. In $d = 3$, while an exact solution is evidently not feasible, general symmetry arguments again indicate that the critical theory is that of the Ising model.

There are two settings in which the critical Ising theory arises naturally in BF systems. First, it arises on a single edge if the microscopic model has an additional symmetry. Such a lattice model in $d = 2$ has been constructed by Wen [46] and exhibits a gapless Majorana fermion at the edge—we direct the reader there for the details. Currently we do not know of a model of a continuum superconductor that has this feature.

The second setting is that of the “topology changing phase transition” first discussed by Wen and Niu [5] in the context of the quantum Hall effect and then by Senthil and Fisher [47] in their investigation of $Z_2$ gauge theories of correlated systems—they are also responsible for the nomenclature. Here the idea is that we construct a closed manifold by sewing up a manifold with two boundaries, for specificity consider taking a cylinder and sewing it up into a torus (Fig. 11). The ground state degeneracies before and after sewing are different, so as a function of the strength of the coupling there must be a phase transition along the way. In the BF problem the disconnected edges are gapped and hence the cylinder exhibits a two-fold degeneracy from the one closed, non-contractible loop. The fully connected edges must give rise to a further two-fold degeneracy and hence we may expect an Ising transition en route. For the $Z_2$ gauge theory, this can be seen explicitly [47] by tuning the strength of a line of plaquettes. For superconductors the details are not readily worked out but the general arguments apply just as well.

To round out this discussion, we now review how the topology changing phase transition appears from the perspective of the BF theory. Returning to our favorite Lagrangian (34) and parametrizing the cylinder with $(x, y)$, $y$ periodic, we now write,

$$a_1 = \partial_1 \Lambda_a$$
$$a_2 = \partial_2 \Lambda_a$$

(86)

13 As an aside we note that for the $\nu = 1/2$ bosonic quantum Hall state the two fold degeneracy is reached from a phase with gapless edges and hence the transition should be expected to be of the Kosteritz-Thouless type, as shown in [5]. This will also be the case in the quantum Hall bilayer of Section V-D.
and
\[ b_1 = \partial_1 \Lambda_b, \quad b_2 = \bar{b}_2/L_2 + \partial_2 \Lambda_b \]  
(87)
where \( \Lambda_{a/b} \) are periodic in \( x_2 \) alone. These lead to the Lagrangian
\[ \pi L = \frac{\bar{b}_2}{L_2} \int dx^2 (\dot{\Lambda}_{au} - \dot{\Lambda}_{al}) + \frac{\bar{a}_2}{L_2} \int dx^2 (\dot{\Lambda}_{bu} - \dot{\Lambda}_{bl}) + \int dx^2 (\dot{\Lambda}_{au} \partial_2 \Lambda_{bu} - \dot{\Lambda}_{al} \partial_2 \Lambda_{bl}) \],
(88)
which exhibits the symplectic structure of the two bosons on the upper (\( u \)) and lower (\( l \)) edges. The addition of quasiparticle/vortex pair creation on each edge will then gap both.

Bringing the edges together will generate couplings between them by tunneling processes involving Cooper pairs and pairs of vortices. Quasiparticle tunneling will be a higher energy process while fractional vortices cannot tunnel across a gap—the same argument excludes quasiparticle tunneling in the quantum Hall effect version of this problem. Generically these processes will not be of equal strength and so the sewing of the torus will proceed in stages. For concreteness let us take the Josephson coupling to be the larger of the two. The corresponding term
\[ \cos(\Lambda_{bu} - \Lambda_{bl}) \]
will drive an Ising transition past which it will set
\[ \Lambda_b = \frac{\bar{b}_1 x_1}{L_1} + \Lambda'_b, \]
and hence reduce (88) to
\[ \pi L = \frac{\bar{b}_2}{L_2} \int dx^2 (\dot{\Lambda}_{au} - \dot{\Lambda}_{al}) + \frac{\bar{a}_2}{b_1} + \int dx^2 \partial_2 \Lambda'_b(\dot{\Lambda}_{au} - \dot{\Lambda}_{al}) \],
(89)
which is now the theory of a single boson running along the cut. The growth of the remaining coupling will freeze \( \dot{\Lambda}_{au} - \dot{\Lambda}_{al} \) and via a second Ising transition will lead to the purely topological action (74). This is the transition studied in [47].

VII. RELATED WORK

The fundamental observation at the heart of our work is the topological interaction between superconducting vortices and quasiparticles. This has a venerable history in the condensed matter literature—it is present in the solution of the Bogoliubov-de Gennes equations in the presence of a vortex where one sees a half-integer shift in angular momentum for quasiparticle states well beyond the penetration depth or coherence length [26]. Its modern formulation by Goldhaber and Kivelson [22] built on the analysis of quasiparticle fractionalization by Kivelson and Rokhsar [11] referred to in the introduction as well on the earlier work of Reznik and Aharanov [28]. In the high energy theory literature this interaction is central to the “discrete gauge theory” work starting with that of Krauss and Wilczek [27] reviewed in Ref. 48.

The particular formulation used in this paper, that of topological order, briefly appeared in Wen’s early work [6] and with a comment on the excitation spectrum that overlooked the dual role of the quasiparticles. Interestingly, Balachandran and collaborators [31] considered the problem of writing a topological field theory for the superconductor in 3+1 dimensions, as well the implications for edge structure. While our conclusions, independently reached, about the \( BF \) action in \( d = 3 + 1 \) are identical, our discussion of edge structure is quite different and \emph{prima facie} somewhat disconnected from the concerns in the earlier work. The ground state degeneracy and its lifting by tunneling are themes missing from this prior work.

Finally it is also worth repeating that the topological order discussed here for the superconducting state with electromagnetic interactions is different from the topological order discussed for states obtained by disordering an uncharged superconductor [21]. While the mathematics is similar, the physical meanings of the gauge fields are quite different.
VIII. SUMMARY AND CLOSING REMARKS

In this paper we have revisited the notion of ordering in a gapped superconductor. We find that the low energy, topological, physics of such superconductors fits conveniently into the paradigm of topological order exemplified by quantum Hall states. Mathematically, the topological $BF$ action captures this physics in all dimensions and we have used that to discuss ground state degeneracies and edge structure. Keeping the leading operators beyond the topological limit recovers the more familiar electrodynamics of the superconducting state. We have also examined physically distinct systems, such as the short ranged RVB state, which share the same topological field theory and can be considered members of a “topological universality class”. There are two obvious directions in which this analysis can be extended. First, gapless superconductors with gapless quasiparticles can be given a low energy description by the action

$$L = \frac{1}{\pi} e^{\mu\sigma} b_\mu \partial_\nu a_\sigma - b_\mu j^\mu - a_\mu j^\mu + \mathcal{L}_{qp}. \quad (90)$$

which generalizes (14) by keeping the dynamics $\mathcal{L}_{qp}$ of the gapless quasiparticle current $j^\mu$. This is no longer a purely topological action but we expect that its detailed analysis will capture the low energy physics of gapless superconductors [49]. It is also interesting to explore the connection between this formulation and the “quantum order” idea of Wen [50], who has proposed that the projective construction of interacting quantum states from mean-field states is a way to classify them. For superconductors, the mean field state can be taken to be the standard neutral BCS state tensored with the classical state in which the electromagnetic field given by the London equation $A = -\lambda J$. A projection enforcing Gauss’s law will then yield a state that presumably has the correct physics of the combined matter-field system. Such a construction can accommodate both gapped and gapless superconducting states.

Second, as in the work on the Hall effect, our abelian analysis suggests the prospect of finding “non-abelian superconductors” or “non-abelian RVB states” whose physics is captured by non-abelian generalizations of the abelian $BF$ theory. This could proceed via the non-abelian $BF$ theory discussed in the literature or (in $d = 2 + 1$) by the quantum Hall bilayer construction discussed in Section V-D. We note that the latter possibility has also been out forward by Freedman et. al. [51] from a point of departure very different from our own but with the same effect of accommodating $P$ and $T$ invariant states within the Chern-Simons class of topological field theories. We also note that Higgs phases of non-abelian Yang-Mills theories are known to exhibit topological interactions based on discrete non-abelian groups [48] and there is also a condensed matter construction of such discrete non-abelian gauge theories based on Josephson junction arrays [52]. An analogous survey of these systems from the topological order viewpoint could well prove useful.

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APPENDIX A: THE QUANTUM HALL SYSTEM WITH A BOUNDARY

Here we review the derivation of the edge action for abelian quantum Hall states specializing, for simplicity, to the Laughlin fractions $\nu = 1/k$. We begin with the topological action and include background gauge fields that allow computation of the electromagnetic response. Also including a quasiparticle current, $j$, we have the following dual Chern-Simons theory,

$$L = \frac{k}{4\pi} ad + \frac{e}{2\pi} adA - ja. \quad (A1)$$

Integrating out the $a$ field we get

$$L = -\frac{e^2}{4\pi k} AdA + \frac{e}{k} jAd - \frac{\pi}{k} j^2 j , \quad (A2)$$
where the first term gives the quantum Hall conductance, \( \sigma_H = \nu e^2/h \), the second shows that the quasiparticles have charge \( \nu e \), and the third encodes the statistical interaction making them \( \theta = \nu \pi \) anyons. For the following analysis we shall take \( j = 0 \).

On a closed surface of genus \( g \) the analysis of (A1) proceeds along the lines discussed in Section IV and yields a Hilbert state of \( k^g \) states which are the degenerate ground states of the quantum Hall fluid. We now consider how the analysis proceeds for a bounded region \( \Omega \) that has a one dimensional boundary \( \partial \Omega \)—the edge of the system.

A proper specification requires that we pick a boundary condition [7, 10], and, as discussed in the text, this should follow from the microscopic physics. In the present case, there are several ways to establish that the boundary supports a gapless chiral edge mode. We now show how this feature is reproduced by taking \( a_0 = 0 \) at the boundary.[7]

1. The edge action

With this choice, and the absence of background fields, the action corresponding to (A1) can be reorganized as,

\[
S = \frac{k}{4\pi} \int_{\Omega} d^3x [a_2 \dot{a}_1 - a_1 \dot{a}_2 + 2a_0 b] ,
\]

(A3)
to exhibit \( a_0 \) as a Lagrange multiplier field that imposes the constraint \( b = 0 \). This can be solved as

\[
a_j = -\frac{1}{k} \partial_j \chi
\]

(A4)
and on substituting this back in (A3) we find that

\[
S = -\frac{1}{4\pi k} \int_{\partial \Omega} d^2x \partial_0 \chi \partial_1 \chi
\]

(A5)
where we have chosen to parametrize the edge by the coordinate labelled 1.

We see, consequently, that for a bounded region the action depends only upon the field \( \chi \) at the boundary, i.e. the only physical degrees of freedom live at the boundary. The remaining degrees of freedom are purely gauge ones and should be eliminated by a suitable choice of gauge for the \( a \) field. Further, we see that the physical degrees of freedom are those of a chiral boson since the action (A5) specifies the canonical commutation relations of such a boson. The connection to microscopics is transparent for a circular droplet in symmetric gauge—the excitations have only momentum of only one sign. Absent an edge confining potential, these states can be thought of as degenerate ground states as indeed they appear in our choice of a theory with a vanishing Hamiltonian when \( a_0 = 0 \). For the alternate, chirality breaking, boundary condition, \( a_0 + v_0 a_1 = 0 \) the same analysis yields the nonvanishing Hamiltonian

\[
H = \frac{v}{2\pi} \int_{\partial \Omega} d^2x (\partial_1 \chi)^2
\]

Alternatively we keep the boundary condition \( a_0 = 0 \) and just add the above Hamiltonian as an allowed term in an effective edge action.

a. Including background gauge fields

If we now consider the response of the system to background (external) electromagnetic fields \( A_\mu \), we are led to the background gauge invariant action (we set \( e = 1 \)),

\[
S[a, A] = \frac{k}{4\pi} \int_{\Omega} d^3x \left[ a a + \frac{2}{k} a A \right]
\]

(A6)
which can be rewritten in the equivalent form

\[
S[a, A] = \frac{k}{4\pi} \int_{\Omega} d^3x \left[ a a + \frac{2}{k} A a \right] + \frac{1}{2\pi k} \int_{\partial \Omega} d^2x \left[ A_0 a_1 - A_1 a_0 \right]
\]

(A7)
from which it is easy to see, by functional differentiation with respect to the background field, that we have coupled the latter to the bulk current

\[
j_{\text{bulk}}^\mu = -\frac{1}{2\pi} \epsilon^{\mu \nu \lambda} \partial_\nu a_\lambda
\]

(A8)
and the edge current

\[ j_{\text{edge}}^0 = -\frac{1}{2\pi} a_1, \quad j_{\text{edge}}^1 = \frac{1}{2\pi} a_0. \]  

(A9)

We can now analyze this action with the same boundary conditions on the \(a\) fields, \textit{i.e.} \(a_0 = 0\). Then,

\[ S[a, A] = \frac{k}{4\pi} \int_\Omega d^3 x \epsilon^{ij} \dot{a}_i a_j + 2a_0 (\epsilon^{ij} \partial_i a_j + \frac{1}{k} \epsilon^{ij} \partial_i A_j) + \frac{2}{k} (\epsilon^{ij} \dot{A}_i a_j + \epsilon^{ij} a_i \partial_j A_0). \]  

(A10)

The constraint now takes the form \(\epsilon^{ij} \partial_i (a_j + \frac{1}{k} A_j) = 0\) which has the solution

\[ a_j = -\frac{1}{k} (A_j + \partial_j \chi). \]  

(A11)

To maintain background gauge invariance we require that \(\chi \to \chi - \Lambda\) when \(A_j \to A_j + \partial_j \Lambda\).

Substituting this back in (A10) we find that it reduces to

\[ S = -\frac{1}{4\pi k} \int_\Omega d^3 x r^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{1}{4\pi k} \int_{\partial \Omega} d^2 r D_0 \chi D_1 \chi + A_0 \partial_1 \chi - A_1 \partial_0 \chi \]  

(A12)

which yields both the bulk electrodynamics response captured in the Chern-Simons term and the coupling of the edge degree of freedom to the background field. In the above, \(D_{0/1} = \partial_{0/1} - a_{0/1}\). One can check directly that the above form is background gauge invariant and that the equation of continuity of current is obeyed at the boundary when the edge current is included, \textit{i.e.} the anomaly cancels. Again, for the alternate boundary condition, \(a_0 + \epsilon a_1 = 0\) the same analysis adds the gauged Hamiltonian \(H = \frac{1}{4\pi k} \int_{\partial \Omega} d^2 r (D_1 \chi)^2\).

Let us finally comment on background fields in the case of the superconductor. When electromagneticism is dynamical, a background field can only be introduced as a technical device to calculate current correlation functions. In some models of strongly correlated 2d electron systems there are electrically charged particles coupled to \textit{bona fide} 2D gauge fields. For example we could consider \textit{holons} obtained by removing the electron from a site occupied by and RVB spinon. In this case one can introduce a background electromagnetic field as in the quantum Hall case and calculate response functions. Also, a background field corresponding to \(b_\mu\) can be introduced as a technical device to calculate vortex current correlation functions.

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