TWISTOR THEORY FOR CO-CR QUATERNIONIC MANIFOLDS AND RELATED STRUCTURES

STEFANO MARCHIAFAVA AND RADU PANTILIE

Abstract

In a general and non metrical framework, we introduce the class of co-CR quaternionic manifolds, which contains the class of quaternionic manifolds, whilst in dimension three it particularizes to give the Einstein–Weyl spaces. We show that these manifolds have a rich natural Twistor Theory and, along the way, we obtain a heaven space construction for quaternionic-Kähler manifolds.

Introduction

Over any three-dimensional conformal manifold $M$, endowed with a conformal connection, there is a sphere bundle $Z$ endowed with a natural CR structure [14]. Furthermore, if $M$ is real analytic then [13] the CR structure of $Z$ is induced by a germ unique embedding of $Z$ into a three-dimensional complex manifold $\tilde{Z}$ which is the twistor space of an anti-self-dual manifold $\tilde{M}$; accordingly, $M$ is a hypersurface in $\tilde{M}$, and the latter is called the heaven space (due to [18]; cf. [14]) of $M$ (endowed with the given conformal connection).

In [17] (see Section 2), we obtained the higher dimensional versions of these constructions by introducing the notion of CR quaternionic manifold. Thus, the generic submanifolds of codimensions at most $2k − 1$, of a quaternionic manifold of dimension $4k$, are endowed with natural CR quaternionic structures. Moreover, assuming real-analyticity, any CR quaternionic manifold is obtained this way through a germ unique embedding into a quaternionic manifold [17].

Returning to the three-dimensional case, by [8], if the inclusion of $M$ into $\tilde{M}$ admits a retraction which is twistorial (that is, its fibres correspond to a (one-dimensional) holomorphic foliation on $\tilde{Z}$) then the connection used to construct the CR structure on $Z$ may be assumed to be a Weyl connection; moreover, there is a natural correspondence between such retractions and Einstein-Weyl connections on $M$. Furthermore, any Einstein–Weyl connection $\nabla$ on $M$ determines a complex surface $Z_\nabla$ and a holomorphic submersion from $\tilde{Z}$ onto it; then $Z_\nabla$ is the twistor space of $(M, \nabla)$ [8].

Furthermore, the correspondence between Einstein–Weyl spaces and their twistor...
spaces is similar to the correspondence between anti-self-dual manifolds and their twistor spaces (see, also, [16]). Also, from the point of view of Twistor Theory, the anti-self-dual manifolds are just four-dimensional quaternionic manifolds (see [9]).

This raises the obvious question: is there a natural class of manifolds, endowed with twistorial structures, which contains both the quaternionic manifolds and the three-dimensional Einstein–Weyl spaces?

In this paper, where the adopted point of view is essentially non-metrical, we answer in the affirmative to this question by introducing, in a general framework, the notion of co-CR quaternionic manifolds and we initiate the study of their twistorial properties. This notion is based on the (co-)CR quaternionic vector spaces which were introduced and classified in [17] (see Section 1, and, also, Appendix A for an alternative definition) and, up to the integrability, it is dual to the notion of CR quaternionic manifolds.

An interesting situation to consider is when a manifold may be endowed with both a CR quaternionic and a co-CR quaternionic structure which are compatible. This gives the notion of f-quaternionic manifold, which has two twistor spaces. The simplest example is provided by the three-dimensional Einstein–Weyl spaces, endowed with the twistorial structures of [14] and [8], respectively; furthermore, the above mentioned twistorial retraction admits a natural generalization to the f-quaternionic manifolds (Corollary 4.5). Also, the quaternionic manifolds may be characterised as f-quaternionic manifolds for which the two twistor spaces coincide.

Other examples of f-quaternionic manifolds are the Grassmannian $\text{Gr}_{3}^{l}((l+3, \mathbb{R})$ of oriented three-dimensional vector subspaces of $\mathbb{R}^{l+3}$ and the flag manifold $\text{Gr}_{2}^{l}(2n+2, \mathbb{C})$ of two-dimensional complex vector subspaces of $\mathbb{C}^{2n+2} (= \mathbb{H}^{n+1}$) which are isotropic with respect to the underlying complex symplectic structure of $\mathbb{C}^{2n+2}$, $(l, n \geq 1)$. The twistor spaces of their underlying co-CR quaternionic structures are the hyperquadric $Q_{l+1}$ of isotropic one-dimensional complex vector subspaces of $\mathbb{C}^{l+3}$ and $\text{Gr}_{2}^{l}(2n+2, \mathbb{C})$ itself, respectively. Also, their heaven spaces are the Wolf spaces $\text{Gr}_{2}^{l+4}(l+4, \mathbb{R})$ and $\text{Gr}_{2}(2n+2, \mathbb{C})$, respectively (see Examples 4.6 and 4.7, for details). Another natural class of f-quaternionic manifolds is described in Example 4.8.

The notion of almost f-quaternionic manifold appears, also, in a different form, in [10]. However, there it is not considered any adequate integrability condition. Also, in [5], [1] and [4] are considered, under particular dimensional assumptions and/or in a metrical framework, particular classes of almost f-quaternionic manifolds.

Let $N$ be the heaven space of a real analytic f-quaternionic manifold $M$, with $\dim N = \dim M + 1$. If the connection of the f-quaternionic structure on $M$ is induced by a torsion free connection on $M$ then the twistor space of $N$ is endowed with a natural holomorphic distribution of codimension one which is transversal to the twistor lines corresponding to the points of $N \setminus M$. Furthermore, this construction also works if, more generally, $M$ is a real analytic CR quaternionic manifold which is a $q$-umbilical hypersurface of its heaven space $N$. Then, under a non-degeneracy condition, this distribution defines a holomorphic contact structure on the twistor space of $N$. Therefore,
according to [15], it determines a quaternionic-Kähler structure on \( N \setminus M \) (cf. [5], [7]).

It is well known (see, for example, [20] and the references therein) that the three-dimensional Einstein–Weyl spaces are one of the basic ingredients in constructions of anti-self-dual (Einstein) manifolds. One of the aims of this paper is to give a first indication that the study of co-CR quaternionic manifolds will lead to a better understanding of quaternionic(-Kähler) manifolds.

1. Brief review of (co-)CR quaternionic vector spaces

The group of automorphisms of the (unital) associative algebra of quaternions \( \mathbb{H} \) is \( \text{SO}(3) \) acting trivially on \( \mathbb{R} (\subseteq \mathbb{H}) \) and canonically on \( \text{Im} \mathbb{H} \).

A linear hypercomplex structure on a (real) vector space \( E \) is a morphism of associative algebras \( \sigma : \mathbb{H} \to \text{End}(E) \). A linear quaternionic structure on \( E \) is an equivalence class of linear hypercomplex structures, where two linear hypercomplex structures \( \sigma_1, \sigma_2 : \mathbb{H} \to \text{End}(E) \) are equivalent if there exists \( a \in \text{SO}(3) \) such that \( \sigma_2 = \sigma_1 \circ a \). A hypercomplex/quaternionic vector space is a vector space endowed with a linear hypercomplex/quaternionic structure (see [2], [9]).

If \( \sigma : \mathbb{H} \to \text{End}(E) \) is a linear hypercomplex structure on a vector space \( E \) then the unit sphere \( Z \) in \( \sigma(\text{Im} \mathbb{H}) \subseteq \text{End}(E) \) is the corresponding space of admissible linear complex structures. Obviously, \( Z \) depends only of the linear quaternionic structure determined by \( \sigma \).

Let \( E \) and \( E' \) be quaternionic vector spaces and let \( Z \) and \( Z' \) be the corresponding spaces of admissible linear complex structures. A linear map \( t : E \to E' \) is quaternionic, with respect to some function \( T : Z \to Z' \), if \( t \circ J = T(J) \circ t \), for any \( J \in Z \) (see [2]). If, further, \( t \neq 0 \) then \( T \) is unique and an orientation preserving isometry (see [9]).

The basic example of a quaternionic vector space is \( \mathbb{H}^k \) endowed with the linear quaternionic structure given by its canonical (left) \( \mathbb{H} \)-module structure. Moreover, for any quaternionic vector space of dimension \( 4k \) there exists a quaternionic linear isomorphism from it onto \( \mathbb{H}^k \). The group of quaternionic linear automorphisms of \( \mathbb{H}^k \) is \( \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H}) \) acting on it by \( (\pm (a, A), x) \mapsto axA^{-1} \), for any \( \pm (a, A) \in \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H}) \) and \( x \in \mathbb{H}^k \). If we restrict this action to \( \text{GL}(k, \mathbb{H}) \) then we obtain the group of hypercomplex linear automorphisms of \( \mathbb{H}^k \).

If \( \sigma : \mathbb{H} \to \text{End}(E) \) is a linear hypercomplex structure then \( \sigma^* : \mathbb{H} \to \text{End}(E^*) \), where \( \sigma^*(q) \) is the transpose of \( \sigma(q) \), \( (q \in \mathbb{H}) \), is the dual linear hypercomplex structure. Accordingly, we define the dual of a linear quaternionic structure.

**Definition 1.1** ([17]). A linear co-CR quaternionic structure on a vector space \( U \) is a pair \((E, \rho)\), where \( E \) is a quaternionic vector space and \( \rho : E \to U \) is a surjective linear map such that \( \ker \rho \cap J(\ker \rho) = \{0\} \), for any admissible linear complex structure \( J \) on \( E \).

A co-CR quaternionic vector space is a vector space endowed with a linear co-CR quaternionic structure.
Dually, a \textit{CR quaternionic vector space} is a triple \((U, E, \iota)\), where \(E\) is a quaternionic vector space and \(\iota: U \to E\) is an injective linear map such that \(\text{im } \iota + J(\text{im } \iota) = E\), for any admissible linear complex structure \(J\) on \(E\).

A map \(t: (U, E, \rho) \to (U', E', \rho')\) between co-CR quaternionic vector spaces is \textit{co-CR quaternionic linear} (with respect to some map \(T: Z \to Z'\)) if there exists a map \(\tilde{t}: E \to E'\) which is quaternionic linear (with respect to \(T\)) such that \(t \circ \rho = \rho' \circ \tilde{t}\).

By duality, we also have the notion of \textit{CR quaternionic linear map}.

Note that, if \((U, E, \iota)\) is a CR quaternionic vector space then the inclusion \(\iota: U \to E\) is CR quaternionic linear. Dually, if \((U, E, \rho)\) is a co-CR quaternionic vector space then the projection \(\rho: E \to U\) is co-CR quaternionic linear.

By working with pairs \((U, E)\), where \(E\) is a quaternionic vector space and \(U \subseteq E\) is a real vector subspace, we call \((\text{Ann } U, E^\ast)\) the \textit{dual pair} of \((U, E)\), where the annihilator \(\text{Ann } U\) is formed of those \(\alpha \in E^\ast\) such that \(\alpha|_U = 0\).

Any CR quaternionic vector space \((U, E, \iota)\) corresponds to the pair \((\text{im } \iota, E)\), whilst any co-CR quaternionic vector space \((U, E, \rho)\) corresponds to the pair \((\ker \rho, E)\). These associations define functors in the obvious way.

To any pair \((U, E)\) we associate a (coherent analytic) sheaf over \(Z\) as follows. Let \(E^{0,1}\) be the holomorphic vector bundle over \(Z\) whose fibre over any \(J \in Z\) is the \(-i\) eigenspace of \(J\). Let \(u: E^{0,1} \to Z \times (E/U)^C\) be the composition of the inclusion \(E^{0,1} \to Z \times E^C\) followed by the projection \(Z \times E^C \to Z \times (E/U)^C\).

**Definition 1.2** ([19]). \(U = U_- \oplus U_+\) is the sheaf of \((U, E)\), where \(U_- = \ker u\) and \(U_+ = \text{coker } u\).

If \((U, E)\) corresponds to a \((co-)CR\) quaternionic vector space then \(U\) is its holomorphic vector bundle, introduced in [17]. In fact, \((U, E)\) corresponds to a \(co\)-CR quaternionic vector space if and only if \(U\) is a holomorphic vector bundle and \(U = U_+\). Dually, \((U, E)\) corresponds to a CR quaternionic vector space if and only if \(U = U_-\) (note that, \(U_-\) is a holomorphic vector bundle for any pair). See [19] for more information on the functor \((U, E) \to U\).

Here are the basic examples of \((co-)CR\) quaternionic vector spaces.

**Example 1.3** (cf. [17]). 1) Let \(V_k\), \((k \geq 1)\), be the vector subspace of \(\mathbb{H}^k\) formed of all vectors of the form \((z_1, z_2, \ldots, z_k)\), where \(z_1, \ldots, z_k\) are complex numbers and \(\overline{z_k} = (-1)^k z_k\). Then \((V_k, \mathbb{H}^k)\) corresponds to a \(co\)-CR quaternionic vector space and its holomorphic vector bundle is \(O(2k)\). Hence, the dual pair is a CR quaternionic vector space and its holomorphic vector bundle is \(O(-2k)\).

2) Let \(V'_k\) = \(\{0\}\) and, for \(k \geq 1\), let \(V'_k\) be the vector subspace of \(\mathbb{H}^{2k+1}\) formed of all vectors of the form \((z_1, z_2, \ldots, z_k, \overline{z_{2k-1}} + z_{2k}, -\overline{z_{2k}})\), where \(z_1, \ldots, z_k\) are complex numbers. Then \((V'_k, \mathbb{H}^{2k+1})\) corresponds to a \(co\)-CR quaternionic vector space and its holomorphic vector bundle is \(2O(2k + 1)\). Hence, the dual pair is a CR quaternionic vector space and its holomorphic vector bundle is \(2O(-2k - 1)\).
Also, by [17], any (co-)CR quaternionic vector space is isomorphic to a product, unique up to the order of factors, in which each factor is given by Example (1.3) or (2).

**Definition 1.4.** A linear $f$-quaternionic structure on a vector space $U$ is a pair $(E, V)$, where $E$ is a quaternionic vector space such that $U, V \subseteq E$, $E = U \oplus V$ and $J(V) \subseteq U$, for any $J \in Z$.

An $f$-quaternionic vector space is a vector space endowed with a linear $f$-quaternionic structure.

Let $(U, E, V)$ be an $f$-quaternionic vector space; denote by $\iota : U \rightarrow E$ the inclusion and by $\rho : E \rightarrow U$ the projection determined by the decomposition $E = U \oplus V$.

Then $(E, \iota)$ and $(E, \rho)$ are linear CR-quaternionic and co-CR quaternionic structures, respectively, which are compatible.

The $f$-quaternionic linear maps are defined, accordingly, by using the compatible linear CR and co-CR quaternionic structures determining a linear $f$-quaternionic structure.

From any $f$-quaternionic vector space $(U, E, V)$, with $\dim E = 4k$, $\dim V = l$, there exists an $f$-quaternionic linear isomorphism onto $(\text{Im} \mathbb{H})^l \times \mathbb{H}^{4k-l}$ (this follows, for example, from the classification of (co-)CR quaternionic vector spaces [17]).

We end this section with the description of the Lie group $G$ of $f$-quaternionic linear isomorphisms of $(\text{Im} \mathbb{H})^l \times \mathbb{H}^m$. For this, let $\rho_k : \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H}) \rightarrow \text{SO}(3)$ be the Lie group morphism defined by $\rho_k(q \cdot A) = \pm q$, for any $q \cdot A \in \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H})$, $(k \geq 1)$. Denote

$$H = \{(A, A') \in (\text{Sp}(1) \cdot \text{GL}(l, \mathbb{H})) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})) \mid \rho_l(A) = \rho_m(A') \}.$$ 

Then $H$ is a closed subgroup of $\text{Sp}(1) \cdot \text{GL}(l + m, \mathbb{H})$ and $G$ is the closed subgroup of $H$ formed of those elements $(A, A') \in H$ such that $A$ preserves $\mathbb{R}^l \subseteq \mathbb{H}^l$. This follows from the fact that there are no nontrivial $f$-quaternionic linear maps from $\text{Im} \mathbb{H}$ to $\mathbb{H}$ (and from $\mathbb{H}$ to $\text{Im} \mathbb{H}$). Now, the canonical basis of $\text{Im} \mathbb{H}$ induces a linear isomorphism $(\text{Im} \mathbb{H})^l = (\mathbb{R}^l)^3$ and, therefore, an effective action $\sigma$ of $\text{GL}(l, \mathbb{R})$ on $(\text{Im} \mathbb{H})^l$. We define an effective action of $\text{GL}(l, \mathbb{R}) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H}))$ on $(\text{Im} \mathbb{H})^l \times \mathbb{H}^m$ by

$$(A, q \cdot B)(X, Y) = (q(\sigma(A)(X))q^{-1}, qYB^{-1}),$$

for any $A \in \text{GL}(l, \mathbb{R})$, $q \cdot B \in \text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$, $X \in (\text{Im} \mathbb{H})^l$ and $Y \in \mathbb{H}^m$.

**Proposition 1.5.** There exists an isomorphism of Lie groups

$$G = \text{GL}(l, \mathbb{R}) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})),$$

given by $(A, A') \mapsto (A|_{\mathbb{R}^l}, A')$, for any $(A, A') \in G$.

In particular, the group of $f$-quaternionic linear isomorphisms of $(\text{Im} \mathbb{H})^l$ is isomorphic to $\text{GL}(l, \mathbb{R}) \times \text{SO}(3)$.

Note that, the group of $f$-quaternionic linear isomorphisms of $\text{Im} \mathbb{H}$ is $\text{CO}(3)$.
2. A FEW BASIC FACTS ON CR QUATERNIONIC MANIFOLDS

In this section we recall, for the reader’s convenience, a few basic facts on CR quaternionic manifolds (we refer to [17] for further details).

A (smooth) bundle of associative algebras is a vector bundle whose typical fibre is a (finite-dimensional) associative algebra and whose structural group is the group of automorphisms of the typical fibre. Let $A$ and $B$ be bundles of associative algebras. A morphism of vector bundles $\rho : A \to B$ is called a morphism of bundles of associative algebras if $\rho$ restricted to each fibre is a morphism of associative algebras.

Recall that a quaternionic vector bundle over a manifold $M$ is a real vector bundle $E$ over $M$ endowed with a pair $(A, \rho)$ where $A$ is a bundle of associative algebras, over $M$, with typical fibre $\mathbb{H}$ and $\rho : A \to \text{End}(E)$ is a morphism of bundles of associative algebras; we say that $(A, \rho)$ is a linear quaternionic structure on $E$ (see [6]). Standard arguments (see [9]) apply to show that a quaternionic vector bundle of (real) rank $4k$ is just a (real) vector bundle endowed with a reduction of its structural group to $\text{Sp}(1) \cdot \text{GL}(k, \mathbb{H})$.

If $(A, \rho)$ defines a linear quaternionic structure on a vector bundle $E$ then we denote $Q = \rho(\text{Im} A)$, and by $Z$ the sphere bundle of $Q$.

Recall [22] (see [9]) that, a manifold is almost quaternionic if and only if its tangent bundle is endowed with a linear quaternionic structure.

**Definition 2.1.** Let $E$ be a quaternionic vector bundle on a manifold $M$ and let $\iota : TM \to E$ be an injective morphism of vector bundles. We say that $(E, \iota)$ is an almost CR quaternionic structure on $M$ if $(E_x, \iota_x)$ is a linear CR quaternionic structure on $T_xM$, for any $x \in M$.

An almost CR quaternionic manifold is a manifold endowed with an almost CR quaternionic structure.

On any almost CR quaternionic manifold $(M, E, \iota)$ for which $E$ is endowed with a connection $\nabla$, compatible with its linear quaternionic structure, there can be defined a natural almost twistorial structure, as follows. For any $J \in Z$, let $B_J \subseteq T^*_J Z$ be the horizontal lift, with respect to $\nabla$, of $\iota^{-1}(E^J)$, where $E^J \subseteq E^C_{\pi(J)}$ is the eigenspace of $J$ corresponding to $-i$. Define $C_J = B_J \oplus (\ker d\pi)^{0,1}_J$, $(J \in Z)$. Then $C$ is an almost CR structure on $Z$ and $(Z, M, \pi, C)$ is the almost twistorial structure of $(M, E, \iota, \nabla)$.

**Definition 2.2.** An (integrable almost) CR quaternionic structure on $M$ is a triple $(E, \iota, \nabla)$, where $(E, \iota)$ is an almost CR quaternionic structure on $M$ and $\nabla$ is an almost quaternionic connection of $(M, E, \iota)$ such that the almost twistorial structure of $(M, E, \iota, \nabla)$ is integrable (that is, $C$ is integrable). Then $(M, E, \iota, \nabla)$ is a CR quaternionic manifold and the CR manifold $(Z, C)$ is its twistor space.

A main source of CR quaternionic manifolds is provided by the submanifolds of quaternionic manifolds.
Definition 2.3. Let \((M, E, \iota, \nabla)\) be a CR quaternionic manifold and let \((Z, C)\) be its twistor space. We say that \((M, E, \iota, \nabla)\) is realizable if \(M\) is an embedded submanifold of a quaternionic manifold \(N\) such that \(E = TN|_M\), as quaternionic vector bundles, and \(C = T^C \cap (T^{0,1}N)|_M\), where \(Z_N\) is the twistor space of \(N\).

Then \(N\) is the heaven space of \((M, E, \iota, \nabla)\).

By [17, Corollary 5.4], any real-analytic CR quaternionic manifold is realizable.

3. CO-CR QUATERNIONIC MANIFOLDS

An almost co-CR structure on a manifold \(M\) is a complex vector subbundle \(C\) of \(T^C M\) such that \(C + \overline{C} = T^C M\). An (integrable almost) co-CR structure is an almost co-CR structure whose space of sections is closed under the bracket.

Note that, if \(\varphi : M \to (N, J)\) is a submersion onto a complex manifold then \((d\varphi)^{-1}(T^{0,1}N)\) is a co-CR structure on \(M\); moreover, any co-CR structure is, locally, of this form.

Definition 3.1. Let \(E\) be a quaternionic vector bundle on a manifold \(M\) and let \(\rho : E \to TM\) be a surjective morphism of vector bundles. Then \((E, \rho)\) is called an almost co-CR quaternionic structure, on \(M\), if \((E_x, \rho_x)\) is a linear co-CR quaternionic structure on \(T_x M\), for any \(x \in M\). If, further, \(E\) is a hypercomplex vector bundle then \((E, \rho)\) is called an almost hyper-co-CR structure on \(M\). An almost co-CR quaternionic manifold (almost hyper-co-CR manifold) is a manifold endowed with an almost co-CR quaternionic structure (almost hyper-co-CR structure).

Any almost co-CR quaternionic (hyper-co-CR) structure \((E, \rho)\) for which \(\rho\) is an isomorphism is an almost quaternionic (hypercomplex) structure.

Example 3.2. Let \((M, c)\) be a three-dimensional conformal manifold and let \(L = (\Lambda^3 TM)^{1/3}\) be the line bundle of \(M\). Then, \(E = L \oplus TM\) is an oriented vector bundle of rank four endowed with a (linear) conformal structure such that \(L = (TM)^{\bot}\). Therefore \(E\) is a quaternionic vector bundle and \((M, E, \rho)\) is an almost co-CR quaternionic manifold, where \(\rho : E \to TM\) is the projection. Moreover, any three-dimensional almost co-CR quaternionic manifold is obtained this way.

Next, we are going to introduce a natural almost twistorial structure (see [16] for the definition of almost twistorial structures) on any almost co-CR quaternionic manifold \((M, E, \rho)\) for which \(E\) is endowed with a connection \(\nabla\) compatible with its linear quaternionic structure.

For any \(J \in Z\), let \(C_J \subseteq T^C J \cap Z \subseteq T^C J \cap (T^{0,1}Z)\) be the direct sum of \((\ker d\pi)^{0,1}_J\) and the horizontal lift, with respect to \(\nabla\), of \(\rho(E^J)\), where \(E^J\) is the eigenspace of \(J\) corresponding to \(-i\). Then \(C\) is an almost co-CR structure on \(Z\) and \((Z, M, \pi, C)\) is the almost twistorial structure of \((M, E, \rho, \nabla)\).

The following definition is motivated by [9] Remark 2.10(2).
Definition 3.3. An *(integrable almost) co-CR quaternionic manifold* is an almost co-CR quaternionic manifold \((M, E, \rho)\) endowed with a compatible connection \(\nabla\) on \(E\) such that the associated almost twistorial structure \((Z, M, \pi, C)\) is integrable (that is, \(C\) is integrable). If, further, \(E\) is a hypercomplex vector bundle and the connection induced by \(\nabla\) on \(Z\) is trivial then \((M, E, \rho, \nabla)\) is an *(integrable almost) hyper-co-CR manifold.*

Example 3.4. Let \((M, c)\) be a three-dimensional conformal manifold and let \((E, \rho)\) be the corresponding almost co-CR structure, where \(E = L \oplus TM\) with \(L\) the line bundle of \(M\). Let \(D\) be a Weyl connection on \((M, c)\) and let \(\nabla = DL \oplus D\), where \(DL\) is the connection induced by \(D\) on \(L\). It follows that \((M, E, \rho, \nabla)\) is co-CR quaternionic if and only if \((M, c, D)\) is Einstein–Weyl (that is, the trace-free symmetric part of the Ricci tensor of \(D\) is zero).

Furthermore, let \(\mu\) be a section of \(L^*\) such that the connection defined by \(D\rho(x, Y) = D_x Y + \mu X \times_c Y\) for any vector fields \(X\) and \(Y\) on \(M\), induces a flat connection on \(L^* \otimes TM\). Then \((M, E, c, \nabla)\) is a hyper-co-CR manifold, where \(\nabla = (D\rho)^L \oplus D\), with \((D\rho)^L\) the connection induced by \(D\rho\) on \(L\) (this follows from well-known results; see [10] and the references therein).

Example 3.5. Any co-CR quaternionic vector space is a co-CR quaternionic manifold, in an obvious way; moreover, the associated twistorial structure is simple and its twistor space is just its holomorphic vector bundle.

Theorem 3.6. Let \((M, E, \rho, \nabla)\) be a co-CR quaternionic manifold, \(\text{rank } E = 4k\), \(\text{rank } (\ker \rho) = l\). If the twistorial structure of \((M, E, \rho, \nabla)\) is simple then it is real analytic and its twistor space is a complex manifold of dimension \(2k - l + 1\) endowed with a locally complete family of complex projective lines \(\{Z_x\}_{x \in MC}\). Furthermore, for any \(x \in M\), the normal bundle of the corresponding twistor line \(Z_x\) is the holomorphic vector bundle of \((T_x M, E_x, \rho_x)\).

Proof. Let \((Z, M, \pi, C)\) be the twistorial structure of \((M, E, \rho, \nabla)\). Let \(\varphi : Z \to T\) be the submersion whose fibres are the leaves of \(C \cap \overline{C}\). Obviously, \(d\varphi(C)\) defines a complex structure on \(T\) of dimension \(2k - l + 1\). Furthermore, if for any \(x \in M\) we denote \(Z_x = \varphi(\pi^{-1}(x))\) then \(Z_x\) is a complex submanifold of \(T\) whose normal bundle is the holomorphic vector bundle of \((T_x M, E_x, \rho_x)\). The proof follows from [12] and [21, Proposition 2.5]. □
Proposition 3.7. Let \((M, E, \rho, \nabla)\) be a co-CR quaternionic manifold whose twistorial structure is simple; denote by \(\varphi : Z \to T\) the corresponding holomorphic submersion onto its twistor space. Then \((M, E, \rho, \nabla)\) is hyper-co-CR if and only if there exists a surjective holomorphic submersion \(\psi : T \to \mathbb{CP}^1\) such that the fibres of \(\psi \circ \varphi\) are integral manifolds of the connection induced by \(\nabla\) on \(Z\).

Proof. Denote by \(\mathcal{H}\) the connection induced by \(\nabla\) on \(Z\). Then \(\mathcal{H}\) is integrable if and only if \(d\varphi(\mathcal{H})\) is a holomorphic foliation on \(T\); furthermore, this foliation is simple if and only if \(E\) is hypercomplex and \(\mathcal{H}\) is the trivial connection on \(Z\).

4. \(f\)-Quaternionic Manifolds

Let \(F\) be an almost \(f\)-structure on a manifold \(M\); that is, \(F\) is a field of endomorphisms of \(TM\) such that \(F^3 + F = 0\). Denote by \(\mathcal{C}\) the eigenspace of \(F\) with respect to \(-i\) and let \(D = \mathcal{C} \oplus \ker F\). Then \(\mathcal{C}\) and \(D\) are compatible almost CR and almost co-CR structures, respectively. An (integrable almost) \(f\)-structure is an almost \(f\)-structure for which the corresponding almost CR and almost co-CR structures are integrable.

Definition 4.1. An almost \(f\)-quaternionic structure on a manifold \(M\) is a pair \((E, V)\), where \(E\) is a quaternionic vector bundle on \(M\) and \(TM\) and \(V\) are vector subbundles of \(E\) such that \(E = TM \oplus V\) and \(J(V) \subseteq TM\), for any \(J \in Z\). An almost hyper-\(f\)-structure on a manifold \(M\) is an almost \(f\)-quaternionic structure \((E, V)\) on \(M\) such that \(E\) is a hypercomplex vector bundle. An almost \(f\)-quaternionic manifold (almost hyper-\(f\)-manifold) is a manifold endowed with an almost \(f\)-quaternionic structure (almost hyper-\(f\)-structure).

With the same notations as in Definition 4.1, an almost \(f\)-quaternionic structure (almost hyper-\(f\)-structure) for which \(V\) is the zero bundle is an almost quaternionic structure (almost hypercomplex structure).

Let \(k\) and \(l\) be positive integers, \(k \geq l\), and denote by \(G_{k,l}\) the group of \(f\)-quaternionic linear isomorphisms of \((\text{Im} \mathbb{H})^l \times \mathbb{H}^{k-l}\). The next result is an immediate consequence of the description of \(G_{k,l}\) given in Section 1.

Proposition 4.2. Let \(M\) be a manifold of dimension \(4k - l\). Then any almost \(f\)-quaternionic structure \((E, V)\) on \(M\), with rank \(E = 4k\) and rank \(V = l\), corresponds to a reduction of the frame bundle of \(M\) to \(G_{k,l}\).

Furthermore, if \((P, M, G_{k,l})\) is the reduction of the frame bundle of \(M\), corresponding to \((E, V)\), then \(V\) is the vector bundle associated to \(P\) through the canonical morphism of Lie groups \(G_{k,l} \to \text{GL}(l, \mathbb{R})\).

Example 4.3. 1) A three-dimensional almost \(f\)-quaternionic manifold is just a (three-dimensional) conformal manifold.

2) Let \(N\) be an almost quaternionic manifold endowed with a Hermitian metric and let \(M\) be a hypersurface in \(N\). Then \((TN|_M, (TM)^\perp)\) is an almost \(f\)-quaternionic structure on \(M\).
Obviously, any almost \( f \)-quaternionic structures \((E, V)\) on a manifold \(M\) corresponds to a pair \((E, \iota)\) and \((E, \rho)\) of almost CR quaternionic and co-CR quaternionic structures on \(M\), where \(\iota : TM \to E\) and \(\rho : E \to TM\) are the inclusion and projection, respectively.

**Definition 4.4.** Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold. Let \((E, \iota)\) and \((E, \rho)\) be the almost CR quaternionic and co-CR quaternionic structures, respectively, corresponding to \((E, V)\). Let \(\nabla\) be a connection on \(E\) compatible with its linear quaternionic structure and let \(\tau\) and \(\tau_c\) be the almost twistorial structures of \((M, E, \iota, \nabla)\) and \((M, E, \rho, \nabla)\), respectively. We say that \((M, E, V, \nabla)\) is an \( f \)-quaternionic manifold if the almost twistorial structures \(\tau\) and \(\tau_c\) are integrable. If, further, \(E\) is hypercomplex and \(\nabla\) induces the trivial flat connection on \(Z\) then \((M, E, V, \nabla)\) is an (integrable almost) hyper-\( f \)-manifold.

Let \((M, E, V, \nabla)\) be an \( f \)-quaternionic manifold and let \(Z\) and \(Z_c\) be the twistor spaces of \(\tau\) and \(\tau_c\), respectively (we assume, for simplicity, that \(\tau_c\) is simple). Then \(Z\) is called the **CR twistor space** and \(Z_c\) is called the **twistor space** of \((M, E, V, \nabla)\).

Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold and let \(\nabla\) be a connection on \(E\) compatible with its linear quaternionic structure. Let \(C\) and \(D\) be the almost CR and almost co-CR structures on \(Z\) determined by \(\nabla\) and the underlying almost CR quaternionic and almost co-CR quaternionic structures of \((M, E, V)\), respectively. Then \(C\) and \(D\) are compatible; therefore \((M, E, V, \nabla)\) is \( f \)-quaternionic if and only if the corresponding almost \( f \)-structure on \(Z\) is integrable.

Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold, rank \(E = 4k\), rank \(V = l\), and \(D\) some compatible connection on \(M\) (equivalently, \(D\) is a linear connection on \(M\) which corresponds to a principal connection on the reduction to \(G_{k,l}\), of the frame bundle of \(M\), corresponding to \((E, V)\)). Then \(D\) induces a connection \(D^V\) on \(V\). Moreover, \(\nabla = D^V \oplus D\) is compatible with the linear quaternionic structure on \(E\).

**Corollary 4.5.** Let \((M, E, V, \nabla)\) be an \( f \)-quaternionic manifold, rank \(E = 4k\), rank \(V = l\), where \(\nabla = D^V \oplus D\) for some compatible connection \(D\) on \(M\). Denote by \(\tau\) and \(\tau_c\) the associated twistorial structures. Then, locally, the twistor space of \((M, \tau)\) is a complex manifold, of complex dimension \(2k - l + 1\), endowed with a locally complete family of complex projective lines each of which has normal bundle \(2(k - l)O(1) \oplus lO(2)\).

Furthermore, if \((M, E, V, \nabla)\) is real analytic then, locally, there exists a twistorial map from the corresponding heaven space \(N\), endowed with its twistorial structure, to \((M, \tau_c)\) which is a retraction of the inclusion \(M \subseteq N\).

**Proof.** By passing to a convex open set of \(D\), if necessary, we may suppose that \(\tau_c\) is simple. Thus, the first assertion is a consequence of Theorem 3.6. The second statement follows from the fact that there exists a holomorphic submersion from the twistor space of \(N\), endowed with its twistorial structure, to the twistor space of \((M, \tau_c)\), which maps diffeomorphically twistor lines onto twistor lines. \qed
Note that, if \( \dim M = 3 \) then Corollary 1.5 gives results of [13] and [8].

**Example 4.6.** Let \( M^{3l} = \text{Gr}_3^l(l+3, \mathbb{R}) \) be the Grassmann manifold of oriented vector subspaces of dimension 3 of \( \mathbb{R}^{l+3}, (l \geq 1) \). Alternatively, \( M^{3l} \) can be defined as the Riemannian symmetric space \( \text{SO}(l+3)/(\text{SO}(l) \times \text{SO}(3)) \). As the structural group of the frame bundle of \( M^{3l} \) is \( \text{SO}(l) \times \text{SO}(3) \), from Proposition 1.2 we obtain that \( M^{3l} \) is canonically endowed with an almost \( f \)-quaternionic structure. Moreover, if we endow \( M^{3l} \) with its Levi-Civita connection then we obtain an \( f \)-quaternionic manifold. Its twistor space is the hyperquadric \( Q_{l+1} \) of isotropic one-dimensional complex vector subspaces of \( \mathbb{C}^{l+3} \), considered as the complexification of the (real) Euclidean space of dimension \( l + 3 \). Further, the CR twistor space \( Z \) of \( M^{3l} \) can be described as the closed submanifold of \( Q_{l+1} \times M^{3l} \) formed of those pairs \((\ell, p)\) such that \( \ell \subseteq p^C \). Under the orthogonal decomposition \( \mathbb{R}^{l+4} = \mathbb{R} \oplus \mathbb{R}^{l+3} \), we can embed \( M^{3l} \) as a totally geodesic submanifold of the quaternionic manifold \( \tilde{M}^{4l} = \text{Gr}_3^l(l+4, \mathbb{R}) \) as follows: \( p \mapsto \mathbb{R} \oplus p, (p \in M^{3l}) \). Recall (see [15]) that the twistor space of \( \tilde{M}^{4l} \) is the manifold \( \tilde{Z} = \text{Gr}_2^0(l+4, \mathbb{C}) \) of isotropic complex vector subspaces of dimension 2 of \( \mathbb{C}^{l+4} \), where the projection \( \tilde{Z} \to \tilde{M} \) is given by \( q \mapsto p \), with \( q \) a self-dual subspace of \( p^C \) (in particular, \( p^C = q \oplus \bar{q} \)). Consequently, the CR twistor space \( Z \) of \( M^{3l} \) can be embedded in \( Z \) as follows: \((\ell, p) \mapsto q \), where \( q \) is the unique self-dual subspace of \((\mathbb{R} \oplus p)^C \) which intersects \( p^C \) along \( \ell \).

In the particular case \( l = 1 \) we obtain the well-known fact (see [3]) that the twistor space of \( S^3 \) is \( Q_2 = (\mathbb{C}P^1 \times \mathbb{C}P^1) \). Also, the CR twistor space of \( S^3 \) can be identified with the sphere bundle of \( O(1) \oplus O(1) \). Similarly, the dual of \( M^{3l} \) is, canonically, an \( f \)-quaternionic manifold whose twistor space is an open set of \( Q_{l+1} \).

**Example 4.7.** Let \( \text{Gr}_2^0(2n+2, \mathbb{C}) \) be the complex hypersurface of the Grassmannian \( \text{Gr}_2(2n+2, \mathbb{C}) \) of two-dimensional complex vector subspaces of \( \mathbb{C}^{2n+2} (= \mathbb{H}^{n+1}) \) formed of those \( q \in \text{Gr}_2(2n+2, \mathbb{C}) \) which are isotropic with respect to the underlying complex symplectic structure \( \omega \) of \( \mathbb{C}^{2n+2} \); note that,

\[
\text{Gr}_2^0(2n+2, \mathbb{C}) = \text{Sp}(n+1)/(\text{U}(2) \times \text{Sp}(n-1)).
\]

Then \( \text{Gr}_2^0(2n+2, \mathbb{C}) \) is a real-analytic \( f \)-quaternionic manifold and its heaven space is \( \text{Gr}_2(2n+2, \mathbb{C}) \). Its twistor space is \( \text{Gr}_2^0(2n+2, \mathbb{C}) \) itself, considered as a complex manifold.

To describe the CR twistor space of \( \text{Gr}_2^0(2n+2, \mathbb{C}) \), firstly, recall that the twistor space of \( \text{Gr}_2(2n+2, \mathbb{C}) \) is the flag manifold \( F_{1,2n+1}(2n+2, \mathbb{C}) \) formed of the pairs \((\ell, p)\) with \( \ell \) and \( p \) complex vector subspaces of \( \mathbb{C}^{2n+2} \) of dimensions 1 and 2n + 1, respectively, such that \( \ell \subseteq p \).

Now, let \( Z \subseteq \text{Gr}_2^0(2n+2, \mathbb{C}) \times \text{Gr}_2^0(2n+2, \mathbb{C}) \) be formed of the pairs \((p, q)\) such that \( p \cap q \) and \( p \cap q^\perp \) are nontrivial and the latter is contained by the kernel of \( \omega|_{q^\perp} \), where the orthogonal complement is taken with respect to the underlying Hermitian metric of \( \mathbb{C}^{2n+2} \). Then the embedding \( Z \to F_{1,2n+1}(2n+2, \mathbb{C}) \), \((p, q) \mapsto (p \cap q, q^\perp + p \cap q)\) induces
a CR structure with respect to which \( Z \) is the CR twistor space of \( \text{Gr}_{2n}^{2n}(2n + 2, \mathbb{C}) \).

Note that, if \( n = 1 \) we obtain the \( f \)-quaternionic manifold of Example 4.4 with \( l = 2 \).

The next example is related to a construction of [23] (see, also, [9, Example 4.4]).

**Example 4.8.** Let \( M \) be a quaternionic manifold, \( \nabla \) a quaternionic connection on it and \( Z \) its twistor space.

Then \( Z \) is the sphere bundle of an oriented Riemannian vector bundle of rank three \( Q \). By extending the structural group of the frame bundle \((\text{SO}(Q), \mathcal{M}, \text{SO}(3, \mathbb{R}))\) of \( Q \) we obtain a principal bundle \((H, M, \mathbb{H}^*/\mathbb{Z}_2)\).

Let \( q \in S^2 (\subseteq \text{Im} \mathbb{H}) \). The morphism of Lie groups \( \mathbb{C}^* \to \mathbb{H}^*, a + bi \to a - bq \) induces an action of \( \mathbb{C}^* \) on \( H \) whose quotient space is \( Z \) (considered with its underlying smooth structure); denote by \( \psi_q : H \to Z \) the projection. Moreover, \((H, Z, \mathbb{C}^*)\) is a principal bundle on which \( \nabla \) induces a principal connection for which the \((0, 2)\) component of its curvature form is zero. Therefore the complex structures of \( Z \) and of the fibres of \( H \) induce, through this connection, a complex structure \( J_q \) on \( H \).

We, thus, obtain a hypercomplex manifold \((H, J_1, J_2, J_3)\) which is the heaven space of an \( f \)-quaternionic structure on \( \text{SO}(Q) \) (in fact, a hyper-\( f \) structure). Note that, the twistor space of \( \text{SO}(Q) \) is \( \mathbb{CP}^1 \times Z \) and the corresponding projection from \( S^2 \times \text{SO}(Q) \) onto \( \mathbb{CP}^1 \times Z \) is given by \((q, u) \mapsto (q, \psi_q(u))\), for any \((q, u) \in S^2 \times \text{SO}(Q)\).

If \( M = \mathbb{H}^k \) then the factorisation through \( \mathbb{Z}_2 \) is unnecessary and we obtain an \( f \)-quaternionic structure on \( S^{4k+3} \) with heaven space \( \mathbb{H}^{k+1} \setminus \{0\} \) and twistor space \( \mathbb{CP}^1 \times \mathbb{CP}^{2k+1} \).

Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold, with rank \( V = l \), and \((P, M, G_{k,l})\) the corresponding reduction of the frame bundle of \( M \), where \( \text{rank} E = 4k \). Then \( TM = (V \otimes Q) \oplus W \), where \( W \) is the quaternionic vector bundle associated to \( P \) through the canonical morphisms of Lie groups \( G_{k,l} \to \text{Sp}(1) \cdot \text{GL}(k - l, \mathbb{H}) \). Note that, \( W \) is the largest quaternionic vector subbundle of \( E \) contained by \( TM \).

**Theorem 4.9.** Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold and let \( D \) be a compatible torsion free connection, \( \text{rank} E = 4k \), \( \text{rank} V = l \); suppose that \((k, l) \neq (2, 2), (1, 0)\). Then \((M, E, V, \nabla)\) is \( f \)-quaternionic, where \( \nabla = D^V \oplus D \). Moreover, \( W \) is integrable and geodesic, with respect to \( D \) (equivalently, \( D_X Y \) is a section of \( W \), for any sections \( X \) and \( Y \) of \( W \)).

**Proof.** Let \( \iota : TM \to E \) be the inclusion and \( \rho : E \to TM \) the projection. It quickly follows that we may apply [17, Theorem 4.6] to obtain that \((M, E, \iota, \nabla)\) is CR quaternionic. To prove that \((M, E, \rho, \nabla)\) is co-CR quaternionic we apply [17, Theorem A.3] to \( D \). Thus, we obtain that it is sufficient to show that for any \( J \in Z \) and any \( X, Y, Z \in E^J \) we have \( R^D(\rho(X), \rho(Y))(\rho(Z)) \in \rho(E^J) \), where \( E^J \) is the eigenspace of \( J \), with respect to \(-i\), and \( R^D \) is the curvature form of \( D \); equivalently, for any \( J \in Z \) and any \( X, Y, Z \in E^J \) we have \( R^\nabla(\rho(X), \rho(Y))Z \in E^J \), where \( R^\nabla \) is the curvature form of \( \nabla \). The proof of the fact that \((M, E, V, \nabla)\) is \( f \)-quaternionic follows, similarly
to the proof of [17, Theorem 4.6]. The last statement, follows quickly from the fact that \( (\nabla_X J)(Y) \) is a section of \( W \), for any section \( J \) of \( Z \) and \( X, Y \) of \( W \). □

From the proof of Theorem 4.9 we immediately obtain the following.

**Corollary 4.10.** Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold and let \( D \) be a compatible torsion free connection, \( \text{rank} \, E \geq 8 \). Then \((M, E, \rho, \nabla)\) is co-CR quaternionic, where \( \rho : E \to TM \) is the projection and \( \nabla = D_V \oplus D \).

Next, we prove two realizability results for \( f \)-quaternionic manifolds.

**Proposition 4.11.** Let \((M, E, V, \nabla)\) be an \( f \)-quaternionic manifold, \( \text{rank} \, V = 1 \), where \( \nabla = D_V \oplus D \) for some compatible connection \( D \) on \( M \). Then \((M, E, \iota, \nabla)\) is realizable, where \( \iota : TM \to E \) is the inclusion.

**Proof.** By passing to a convex open set of \( D \), if necessary, we may suppose that the twistorial structure \((Z, M, \pi, D)\) of the co-CR quaternionic manifold \((M, E, \rho)\) is simple, where \( \rho : E \to TM \) is the projection. Thus, by Theorem 3.6, we have that \((Z, M, \pi, D)\) is real analytic. It follows that \( Q^C \) is real analytic which, together with the relation \( TM = (V \otimes Q) \oplus W \), quickly gives that the twistorial structure \((Z, M, \pi, C)\) of \((M, E, \iota)\) is real analytic. By [17, Corollary 5.4] the proof is complete. □

The next result is an immediate consequence of Theorem 4.9 and Proposition 4.11.

**Corollary 4.12.** Let \((M, E, V)\) be an almost \( f \)-quaternionic manifold, with \( \text{rank} \, V = 1 \), \( \text{rank} \, E \geq 8 \), and let \( \nabla \) be a torsion free connection on \( E \) compatible with its linear quaternionic structure. Then \((M, E, \iota, \nabla)\) is realizable, where \( \iota : TM \to E \) is the inclusion.

We end this section with the following result.

**Proposition 4.13.** Let \((M, E, V, \nabla)\) be a real analytic \( f \)-quaternionic manifold, with \( \text{rank} \, V = 1 \), where \( \nabla = D_V \oplus D \) for some torsion free compatible connection \( D \) on \( M \). Let \( N \) be the heaven space of \((M, E, \iota, \nabla)\), where \( \iota : TM \to E \) is the inclusion, and denote by \( Z_N \) its twistor space. Then \( Z_N \) is endowed with a nonintegrable holomorphic distribution \( H \) of codimension one, transversal to the twistor lines corresponding to the points of \( N \setminus M \).

**Proof.** By passing to a complexification, we may assume all the objects complex analytic. Furthermore, excepting \( Z \), we shall denote by the same symbols the corresponding complexifications. As for \( Z \), this will denote the bundle of isotropic directions of \( Q \). Then any \( p \in Z \) corresponds to a vector subspace \( E_p \) of \( E \). Let \( \mathcal{F} \) be the distribution on \( Z \) such that \( \mathcal{F}_p \) is the horizontal lift, with respect to \( \nabla \), of \( \iota^{-1}(E_p) \), \( (p \in Z) \). As \((M, E, V, \nabla)\) is (complex) \( f \)-quaternionic \( \mathcal{F} \) is integrable. Moreover, locally, we may suppose that its leaf space is \( Z_N \). Let \( \mathcal{G} \) be the distribution on \( Z \) such that, at each \( p \in Z \), we have that \( \mathcal{G}_p \) is the horizontal lift of \((V_x \otimes p^\perp) \oplus W_x \), where \( x = \pi(p) \). Define
Then the complex analytic versions of Cartan’s structural equations and [11] Proposition III.2.3, straightforwardly show that $\mathcal{K}$ is projectable with respect to $\mathcal{F}$. Thus, $\mathcal{K}$ projects to a distribution $\mathcal{K}$ on $Z_N$ of codimension one. Furthermore, by using again [11] Proposition III.2.3, we obtain that $\mathcal{K}$ is nonintegrable. \hfill \Box

5. Quaternionic-Kähler manifolds as heaven spaces

A quaternionic-Kähler manifold is a quaternionic manifold endowed with a (semi-Riemannian) Hermitian metric whose Levi-Civita connection is quaternionic and whose scalar curvature is assumed nonzero.

Let $(M, E, \iota, \nabla)$ be a CR quaternionic manifold with rank $E = \dim M + 1$. Let $W$ be the largest quaternionic vector subbundle of $E$ contained by $TM$ and denote by $I$ the (Frobenius) integrability tensor of $W$. From the integrability of the almost twistorial structure of $(M, E, \iota, \nabla)$ it follows that, for any $J \in Z$, the two-form $I|_{E_J}$ takes values in $E_J/(E_J \cap W^\C)$; as this is one-dimensional the condition $I|_{E_J}$ nondegenerate has an obvious meaning.

**Definition 5.1.** A CR quaternionic manifold $(M, E, \iota, \nabla)$, with rank $E = \dim M + 1$, is nondegenerate if $I|_{E_J}$ is nondegenerate, for any $J \in Z$.

Let $M$ be a submanifold of a quaternionic manifold $N$ and $Z$ the twistor space of $N$.

Denote by $B$ the second fundamental form of $M$ with respect to some quaternionic connection $\nabla$ on $N$; that is, $B$ is the (symmetric) bilinear form on $M$, with values in $(TN|_M)/TM$, characterised by $B(X, Y) = \sigma(\nabla_X Y)$, for any vector fields $X, Y$ on $M$, where $\sigma : TN|_M \to (TN|_M)/TM$ is the projection.

**Definition 5.2.** We say that $M$ is $q$-umbilical in $N$ if for any $J \in Z|_M$ the second fundamental form of $M$ vanishes along the eigenvectors of $J$ which are tangent to $M$.

From [9] Propositions 1.8(ii) and 2.8 it quickly follows that the notion of $q$-umbilical submanifold, of a quaternionic manifold, does not depend of the quaternionic connection used to define the second fundamental form.

Note that, if $\dim N = 4$ then we retrieve the usual notion of umbilical submanifold. Also, if a quaternionic manifold is endowed with a Hermitian metric then any umbilical submanifold of it is $q$-umbilical.

The notion of $q$-umbilical submanifold of a quaternionic manifold can be easily extended to CR quaternionic manifolds. Indeed, just define the second fundamental form $B$ of $(M, E, \iota, \nabla)$ by $B(X, Y) = \frac{1}{2} \sigma(\nabla_X Y + \nabla_Y X)$, for any vector fields $X$ and $Y$ on $M$, where $\sigma : E \to E/TM$ is the projection.

**Theorem 5.3.** Let $N$ be the heaven space of a real analytic CR quaternionic manifold $(M, E, \iota, \nabla)$, with rank $E = \dim M + 1$. If $M$ is $q$-umbilical in $N$ then the twistor space $Z_N$ of $N$ is endowed with a nonintegrable holomorphic distribution $\mathcal{K}$ of codimension one, transversal to the twistor lines corresponding to the points of $N \setminus M$. Furthermore,
the following assertions are equivalent:

(i) $\mathcal{H}$ is a holomorphic contact structure on $Z_N$.

(ii) $(M, E, \iota, \nabla)$ is nondegenerate.

Proof. By passing to a complexification, we may assume all the objects complex analytic. Also, we may assume $\nabla$ torsion free. Furthermore, excepting $Z$, which will be soon described, below, we shall denote by the same symbols the corresponding complexifications.

Let $\dim N = 4k$. As the complexification of $\text{Sp}(1) \cdot \text{GL}(k, H)$ is $\text{SL}(2, \mathbb{C}) \cdot \text{GL}(2k, \mathbb{C})$, we may assume that, locally, $TN = H \otimes F$ where $H$ and $F$ are (complex analytic) vector bundles of rank 2 and $2k$, respectively. Also, $H$ is endowed with a nowhere zero section $\varepsilon$ of $\Lambda^2 H^*$ and $\nabla = \nabla^H \otimes \nabla^F$, for some connections $\nabla^H$ and $\nabla^F$ on $H$ and $F$, respectively, with $\nabla H_\varepsilon^* = 0$.

Then, by restricting to a convex neighbourhood of $\nabla$, if necessary, $Z_N$ is the leaf space of the foliation $\mathcal{F}_N$ on $PH$ which, at each $[u] \in PH$, is given by the horizontal lift, with respect to $\nabla^H$ of $[u] \otimes F_{\pi_H(u)}$, where $\pi_H : H \to N$ is the projection. Let $Z = PH |_M$ and let $\mathcal{F}$ be the foliation induced by $\mathcal{F}_N$ on $Z$. Note that, the leaf space of $\mathcal{F}$ is $Z_N$.

Let $PH + PF^*$ be the restriction to $N$ of $PH \times PF^*$. Then $([u], [\alpha]) \mapsto [u] \otimes \ker \alpha$ defines an embedding of $PH + PF^*$ into the Grassmann bundle $P$ of $(2k-1)$-dimensional vector spaces tangent to $N$. As $\nabla = \nabla^H \otimes \nabla^F$, this embedding preserves the connections induced by $\nabla^H$, $\nabla^F$ and $\nabla$ on $PH + PF^*$ and $P$. Let $\mathcal{F}_P$ be the distribution on $P$ which, at each $p \in P$, is the horizontal lift, with respect to $\nabla$, of $p \subseteq T_{\pi_P(p)}N$, where $\pi_P : P \to N$ is the projection. Then the restriction of $\mathcal{F}_P$ to $PH + PF^*$ is a distribution $\mathcal{F}'$ on $PH + PF^*$.

The map $Z \to P$, $[u] \mapsto TM \cap ([u] \otimes F_{\pi_H(u)})$, is an embedding whose image is contained by $PH + PF^*$. Moreover, the fact that $M$ is q-umbilical in $N$ is equivalent to the fact that $\mathcal{F}$ is the restriction of $\mathcal{F}_P$ to $Z$.

If for any $([u], [\alpha]) \in PH + PF^*$ we take the preimage of $\ker(\varepsilon(u) \otimes \alpha)$ through the projection of $PH + PF^*$ we obtain a distribution of codimension one $\mathcal{G}'$ on $PH + PF^*$ which contains $\mathcal{F}'$. Furthermore, $\mathcal{G} = TZ \cap \mathcal{G}'$ is a codimension one distribution on $Z$ which contains $\mathcal{F}$.

To prove that $\mathcal{G}$ is projectable with respect to $\mathcal{F}$, firstly, observe that this is equivalent to the fact that the integrability tensor of $\mathcal{G}$ is zero when evaluated on the pairs in which one of the vectors is from $\mathcal{F}$. Thus, as $\mathcal{F}$ is integrable, $\mathcal{F} = \mathcal{F}'|_Z$ and $\mathcal{G} = TZ \cap \mathcal{G}'$, it is sufficient to prove that, at each $p \in PH + PF^*$, the integrability tensor of $\mathcal{G}'$ is zero when evaluated on the pairs formed of a vector from a basis of $\mathcal{F}'_p$ and a vector from a basis of a space complementary to $\mathcal{F}'_p$.

Let $\text{SL}(H)$ and $\text{GL}(F)$ be the frame bundles of $H$ and $F$, respectively, and let $\text{SL}(H) + \text{GL}(F)$ be the restriction to $N$ of $\text{SL}(H) \times \text{GL}(F)$. Then the kernel of the differential of the projection of $\text{SL}(H) + \text{GL}(F)$ is the trivial vector bundle over
SL(H) + GL(F) with fibre \(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2k, \mathbb{C})\). Also, note that, for any \((u, v) \in SL(H) + GL(F)\), we have that \(u \otimes v\) is a (complex-quaternionic) frame on \(N\).

Let \(G\) be the closed subgroup of \(SL(2, \mathbb{C}) \times GL(2k, \mathbb{C})\) which preserves some fixed pair \([(x_0), [\alpha_0)] \in \mathbb{CP}^1 \times P((\mathbb{C}^{2k})^*)\). Then \(PH + PF^* = (SL(H) + GL(F))/G\) and we denote \(\mathcal{F}'' = (d\mu)^{-1}(\mathcal{F}')\) and \(\mathcal{G}'' = (d\mu)^{-1}(\mathcal{G}')\), where \(\mu\) is the projection from \(SL(H) + GL(F)\) onto \(PH + PF^*\).

For any \(\xi \in \mathbb{C}^2 \otimes \mathbb{C}^{2k}\) we define a horizontal vector field \(B(\xi)\) which at any \((u, v) \in SL(H) + GL(F)\) is the horizontal lift of \((u \otimes v)(\xi)\). Then \(\mathcal{F}''\) is generated by the Lie algebra of \(G\) and all \(B(x_0 \otimes y)\) with \(\alpha_0(y) = 0\). Also, \(\mathcal{G}''\) is generated by \(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{gl}(2k, \mathbb{C})\) and all \(B(\xi)\) with \((\varepsilon_0(x_0) \otimes \alpha_0)(\xi) = 0\), where \(\varepsilon_0\) is the volume form on \(\mathbb{C}^2\).

Further, similarly to [11] Proposition III.2.3, we have \([A_1 \oplus A_2, B(x_1 \otimes x_2)] = B(A_1 x_1 \otimes x_2 + x_1 \otimes A_2 x_2)\), for any \(A_1 \in \mathfrak{sl}(2, \mathbb{C})\), \(A_2 \in \mathfrak{gl}(2k, \mathbb{C})\), \(x_1 \in \mathbb{C}^2\) and \(x_2 \in \mathbb{C}^{2k}\). Also, because \(\nabla\) is torsion free we have that, for any \(\xi, \eta \in \mathbb{C}^2 \otimes \mathbb{C}^{2k}\), the horizontal component of \([B(\xi), B(\eta)]\) is zero. These facts quickly show that, at each \((u, v) \in SL(H) + GL(F)\), the integrability tensor of \(\mathcal{G}''\) is zero when evaluated on the pairs formed of a vector from a basis of \(\mathcal{F}''_{(u, v)}\) and a vector from a basis of a space complementary to \(\mathcal{F}''_{(u, v)}\). Consequently, \(\mathcal{G}\) is projectable with respect to \(\mathcal{F}\).

Next, we shall prove that \(\mathcal{G}\) is nonintegrable. For this, firstly, observe that those \((u, v)\) in \((SL(H) + GL(F))|_M\) for which \(u \otimes v\) preserves the corresponding tangent space to \(M\) form a principal bundle, which we shall call ‘the bundle of adapted frames’, whose structural group \(K\) can be described, as follows. We may write \(\mathbb{C}^2 \otimes \mathbb{C}^{2k} = \mathfrak{gl}(2, \mathbb{C}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k-2})\) so that \(K\) is the closed subgroup of \(SL(2, \mathbb{C}) \times GL(2k, \mathbb{C})\) which preserve \(\text{Id}_{\mathbb{C}^2}\). Thus, \(K\) contains \(SL(2, \mathbb{C})\) acting on \(\mathfrak{gl}(2, \mathbb{C}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k-2})\) by \((a, (\xi, \eta)) \rightarrow (a\xi a^{-1}, \eta)\), for any \(a \in SL(2, \mathbb{C})\), \(\xi \in \mathfrak{gl}(2, \mathbb{C})\) and \(\eta \in \mathbb{C}^2 \otimes \mathbb{C}^{2k-2}\).

Note that, \(TM\) is the bundle associated to the bundle of adapted frames through the action of \(K\) on \(\mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k-2})\). Also, \(Z(\subseteq P)\) is the quotient of the bundle of adapted frames through the closed subgroup of \(K\) preserving \(\mathbb{C}\xi_0 + (\ker \xi_0 \otimes \mathbb{C}^{2k-2})\), for some fixed \(\xi_0 \in \mathfrak{sl}(2, \mathbb{C}) \setminus \{0\}\) with \(\det \xi_0 = 0\).

If we, locally, consider a principal connection on the bundle of adapted frames then we can define, similarly to above, the corresponding ‘standard horizontal vector fields’ \(B(\xi)\), for any \(\xi \in \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k-2})\), so that \(\mathcal{G}\) corresponds to the distribution generated by the Lie algebra of \(K\) and all \(B(\xi)\) with \(\xi \in \mathbb{C}^2 \otimes \mathbb{C}^{2k-2}\) or \(\xi \in \mathfrak{sl}(2, \mathbb{C})\) such that \(\xi(\ker \xi_0) \subseteq \ker \xi_0\). Thus, if we take \(\xi \in \mathfrak{sl}(2, \mathbb{C})\) with \(\xi(\ker \xi_0) \nsubseteq \ker \xi_0\) and \(A \in \mathfrak{sl}(2, \mathbb{C})\) such that \([A, \xi](\ker \xi_0) \nsubseteq \ker \xi_0\) then \(A\) and \(B(\xi)\) determine sections of \(\mathcal{G}\) whose bracket is not a section of \(\mathcal{G}\).

Finally, the equivalence of the assertions (i) and (ii) is a straightforward consequence of the fact that if we denote by \(W\) the largest complex-quaternionic subbundle of \(TN|_M\) contained by \(TM\) then \(\mathcal{F} + (d\pi)^{-1}(W) = \mathcal{G}\), where \(\pi : Z \to M\) is the projection. \(\square\)

The next result follows immediately from [15] and Theorem 5.3.
Corollary 5.4. The following assertions are equivalent, for a real analytic hypersurface $M$ embedded in a quaternionic manifold $N$:

(i) $M$ is nondegenerate and $q$-umbilical.

(ii) By passing, if necessary, to an open neighbourhood of $M$, there exists a metric $g$ on $N\setminus M$ such that $(N\setminus M, g)$ is quaternionic-Kähler and the twistor lines determined by the points of $M$ are tangent to the contact distribution, on the twistor space of $N$, corresponding to $g$.

If $\dim M = 3$ then Corollary 5.4 and [17, Corollary 5.5] give the main result of [13]. Also, the ‘quaternionic contact’ manifolds of [5] (see [7]) are nondegenerate $q$-umbilical CR quaternionic manifolds.

Appendix A. The intrinsic description of linear (co-)CR quaternionic structures

A conjugation, on a quaternionic vector space, is an involutive quaternionic automorphism (not equal to the identity); in particular, the corresponding orientation preserving isometry on the space of admissible complex structures is a symmetry in a line.

Example A.1 ([6]). Let $U^\mathbb{H} = \mathbb{H} \otimes U$ be the quaternionification of a vector space $U$ (the tensor product is taken over $\mathbb{R}$), endowed with the linear quaternionic structure induced by the multiplication to the left.

If $q \in S^2$ then the association $q' \otimes u \mapsto -qq'q \otimes u$, for any $q' \in \mathbb{H}$ and $u \in U$, defines a conjugation on $U^\mathbb{H}$.

In fact, more can be proved.

Proposition A.2. Any pair of distinct commuting conjugations $\tau_1$ and $\tau_2$ on a quaternionic vector space $E$ determine a quaternionic linear isomorphism $E = U^\mathbb{H}$, for some vector space $U$, so that $\tau_1$ and $\tau_2$ are defined, as in Example A.1, by two orthogonal imaginary unit quaternions.

Proof. Let $T_1, T_2 : Z \to Z$ be the orientation preserving isometries corresponding to $\tau_1, \tau_2$, respectively, where $Z$ is the space of admissible linear complex structures on $E$.

As $T_1$ and $T_2$ are commuting symmetries in lines $\ell_1$ and $\ell_2$, respectively, it follows that either $\ell_1 = \ell_2$ or $\ell_1 \perp \ell_2$. In the former case, we would have $T_1 T_2 = \text{Id}_Z$ which, together with the fact that $\tau_1$ and $\tau_2$ are commuting involutions, implies $\tau_1 = \tau_2$, a contradiction. Thus, if $\ell_1$ and $\ell_2$ are generated by $I$ and $J$, respectively, then $IJ = -IJ$; denote $K = IJ$.

Now, $E = U^+ \oplus U^-$, where $U^\pm = \ker(\tau_1 \mp \text{Id}_E)$. Furthermore, as $\tau_1 \tau_2 = \tau_2 \tau_1$, we have $U^+ = V^+ \oplus V^-$ and $U^- = W^+ \oplus W^-$, where $V^\pm = \ker(\tau_2|_{U^\pm} \mp \text{Id}_{U^\pm})$ and $W^\pm = \ker(\tau_2|_{U^\mp} \mp \text{Id}_{U^\mp})$.

A straightforward argument shows that $IV^+ = V^-$, $JV^+ = W^+$ and $KV^+ = W^-$. 
Thus, if we denote $U = V^+$ then $E = U \oplus IU \oplus JU \oplus KU$ and the association $q \otimes u \mapsto q_0u + q_1Ju + q_2Ju + q_3Ku$, for any $q = q_0 + q_1i + q_2j + q_3k \in \mathbb{H}$ and $u \in U$, defines a quaternionic linear isomorphism from $U^\mathbb{H}$ onto $E$ which is as required. \qed

The quaternionification of a linear map is defined in the obvious way. Then a quaternionic linear map between the quaternionifications of two vector spaces is the quaternionification of a linear map if and only if it intertwines two distinct commuting conjugation.

Let $U$ be a vector space and let $\Lambda$ be the space of conjugations on $U^\mathbb{H}$.

The next proposition reformulates a result of [6].

Proposition A.3. There exist natural correspondences between the following:

(i) Linear quaternionic structures on $U$;
(ii) Quaternionic vector subspaces $B \subseteq U^\mathbb{H}$ such that $U^\mathbb{H} = B \oplus \sum_{\tau \in \Lambda} \tau(B)$;
(iii) Quaternionic vector subspaces $C \subseteq U^\mathbb{H}$ such that $U^\mathbb{H} = C \oplus \bigcap_{\tau \in \Lambda} \tau(C)$.

Furthermore, the correspondences are such that $C = \sum_{\tau \in \Lambda} \tau(B)$ and $B = \bigcap_{\tau \in \Lambda} \tau(C)$.

We can now give the intrinsic description of linear CR quaternionic structures.

Proposition A.4. There exists a natural correspondence between the following:

(i) Linear CR quaternionic structures on $U$;
(ii) Quaternionic vector subspaces $C \subseteq U^\mathbb{H}$ such that
   (ii1) $C \cap \bigcap_{\tau \in \Lambda} \tau(C) = 0$,
   (ii2) $C + \sigma(C) = U^\mathbb{H}$, for any $\sigma \in \Lambda$.

Proof. If $(E, \iota)$ is a linear CR quaternionic structure on $U$ then $C = (\iota^\mathbb{H})^{-1}(C_E)$ satisfies assertion (ii), where $C_E$ is the quaternionic vector subspace of $E^\mathbb{H}$ given by assertion (iii) of Proposition A.3.

Conversely, if $C$ is as in (ii) then on defining $E = U^\mathbb{H}/C$ and $\iota$ to be the composition of the inclusion of $U$ into $U^\mathbb{H}$ followed by the projection from the latter onto $E$ we obtain the corresponding linear CR quaternionic structure. \qed

Finally, by duality, we also have.

Proposition A.5. There exists a natural correspondence between the following:

(i) Linear co-CR quaternionic structures on $U$;
(ii) Quaternionic vector subspaces $B \subseteq U^\mathbb{H}$ such that
   (ii1) $U^\mathbb{H} = B + \sum_{\tau \in \Lambda} \tau(B)$,
   (ii2) $B \cap \sigma(B) = 0$, for any $\sigma \in \Lambda$.

References

[1] D. V. Alekseevsky, Y. Kamishima, Pseudo-conformal quaternionic CR structure on $(4n + 3)$-dimensional manifolds, Ann. Mat. Pura Appl. (4), 187 (2008) 487–529.
[2] D. V. Alekseevsky, S. Marchiafava, Quaternionic structures on a manifold and subordinated structures, Ann. Mat. Pura Appl., 171 (1996) 205–273.
[3] P. Baird, J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.), no. 29, Oxford Univ. Press, Oxford, 2003.

[4] A. Bejancu, H. R. Farran, On totally umbilical QR-submanifolds of quaternion Kaehlerian manifolds, *Bull. Austral. Math. Soc.*, 62 (2000) 95–103.

[5] O. Biquard, Métriques d’Einstein asymptotiquement symétriques, *Astérisque*, 265 (2000).

[6] E. Bonan, Sur les G-structures de type quaternionien, *Cahiers Topologie Géom. Différentielle*, 9 (1967) 389–461.

[7] D. Duchemin, Quaternionic contact structures in dimension 7, *Ann. Inst. Fourier (Grenoble)*, 56 (2006) 851–885.

[8] N. J. Hitchin, Complex manifolds and Einstein’s equations, *Twistor geometry and nonlinear systems (Primorsko, 1980)*, 73–99, Lecture Notes in Math., 970, Springer, Berlin, 1982.

[9] S. Ianu¸s, S. Marchiafava, L. Ornea, R. Pantilie, Twistorial maps between quaternionic manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9 (2010) 47–67.

[10] T. Kashiwada, F. Martin Cabrera, M. M. Tripathi, Non-existence of certain 3-structures, *Rocky Mountain J. Math.*, 35 (2005) 1953–1979.

[11] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, I, II, Wiley Classics Library (reprint of the 1963, 1969 original), Wiley-Interscience Publ., Wiley, New-York, 1996.

[12] K. Kodaira, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, *Ann. of Math.*, 75 (1962) 146–162.

[13] C. R. LeBrun, $\mathcal{K}$-space with a cosmological constant, *Proc. Roy. Soc. London Ser. A*, 380 (1982) 171–185.

[14] C. R. LeBrun, Twistor CR manifolds and three-dimensional conformal geometry, *Trans. Amer. Math. Soc.*, 284 (1984) 601–616.

[15] C. R. LeBrun, Quaternionic-Kähler manifolds and conformal geometry, *Math. Ann.*, 284 (1989) 357–376.

[16] E. Loubeau, R. Pantilie, Harmonic morphisms between Weyl spaces and twistorial maps II, *Ann. Inst. Fourier (Grenoble)*, 60 (2010) 433–453.

[17] S. Marchiafava, L. Ornea, R. Pantilie, Twistor Theory for CR quaternionic manifolds and related structures, *Monatsh. Math.*, (in press).

[18] E. T. Newman, Heaven and its properties, *General Relativity and Gravitation*, 7 (1976) 107–111.

[19] R. Pantilie, The classification of the real vector subspaces of a quaternionic vector space, Preprint, IMAR, Bucharest, 2011.

[20] R. Pantilie, J. C. Wood, Twistorial harmonic morphisms with one-dimensional fibres on self-dual four-manifolds, *Q. J. Math.*, 57 (2006) 105–132.

[21] H. Rossi, LeBrun’s nonrealizability theorem in higher dimensions, *Duke Math. J.*, 52 (1985) 457–474.

[22] S. Salamon, Differential geometry of quaternionic manifolds, *Ann. Sci. École Norm. Sup. (4)*, 19 (1986) 31–55.

[23] A. Swann, HyperKähler and quaternionic Kähler geometry, *Math. Ann.*, 289 (1991) 421–450.

E-mail address: marchiaf@mat.uniroma1.it, radu.pantilie@imar.ro

S. Marchiafava, Dipartimento di Matematica, Istituto “Guido Castelnuovo”, Università degli Studi di Roma “La Sapienza”, Piazzale Aldo Moro, 2 - I 00185 Roma - Italia

R. Pantilie, Institutul de Matematică “Simion Stoilow” al Academiei Române, C.P. 1-764, 014700, București, România