ELECTRON TRANSPORT AND ELECTRON DENSITY INSIDE QUASI-ONE-DIMENSIONAL DISORDERED CONDUCTORS

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(Dated: August 1, 2019)

Abstract

We consider the problem of electron transport across a quasi-one-dimensional disordered multiply-scattering medium, and study the statistical properties of the electron density inside the system. Electrons of a given energy feed the disordered conductor from one end and the density is computed along the system and outside. The extension of the techniques that were used in the past to find information outside the sample is done in terms of the scattering properties of the two segments that form the entire conductor on each side of the observation point. The problem is of interest in various other branches of physics, as electrodynamics and elasticity.

PACS numbers: 72.10.-d, 73.23.-b, 73.63.Nm
I. INTRODUCTION

The problem of the electronic conductance in disordered mesoscopic systems has been of interest for many years (for a review, see Ref. [1] and references cited therein). The equivalence of the electronic conductance expressed in units of the quantum of conductance and the transmittance [1–3] has allowed studying the electronic transport in terms of the scattering properties of the system of interest. For very low temperatures and small chemical potential difference between the two terminals, the relevant scattering properties are those in the vicinity of the Fermi energy. That equivalence also allows many of the predictions of mesoscopic physics and localization theory to apply equally to the transport of quantum and classical waves [4–11].

In addition to studies of the conductance and transmission, which refer to quantities evaluated outside the system, the statistics of transport inside random systems has also been studied for many years [12–16]. Just as the conductance problem, the problem of electron transport inside random systems is also of interest in various other branches of physics, as it is representative of a more general wave-scattering problem in a quasi-one-dimensional (q1D) disordered system: e.g., an electromagnetic wave traveling in a disordered waveguide [17] – the interest being in the energy density inside the structure –, or an elastic wave propagating in a disordered elastic waveguide [18,19].

In this paper, we study the statistical properties of the electron density inside a quasi-one-dimensional multiply-scattering medium. The system is fed with electrons of a given energy from one end of the disordered conductor and the electron density is evaluated along the conductor and outside. For 1D systems, this was done recently in Refs. [17,20]. In Ref. [20], the expectation value $\langle W(x) \rangle$ of the intensity $W(x)$ a distance $x$ from the entrance was calculated and compared with computer simulations. In Ref. [17], emphasis was put on the statistics of the logarithm of the intensity, which shows interesting scaling properties, in a way similar to the logarithm of the conductance in the conduction problem; theoretical predictions were compared with computer simulations and microwave experiments. In the present paper, we address the more complicated problem of the electron density inside q1D systems supporting more than one propagating mode or open channel ($N \geq 1$).
The study of the statistics of electron density in the interior of random samples is performed in terms of the scattering properties of the two segments that form the entire conductor on each side of the observation point. We apply to each of the two segments, considered to be statistically independent, the maximum-entropy approach (MEA) to random-matrix theory [1] that was used in the past for the entire conductor. The MEA for the full conductor of Ref. [1] is a random-matrix theory which leads to a Fokker-Planck equation, known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [21,22], governing the “evolution” with sample length $L$ of the probability distribution (PD) $p_L(M)$ of the system transfer matrix $M$. In the so-called dense-weak-scattering limit, the resulting PD for the full system is expected to give results insensitive to microscopic details, depending only on the mean-free-path (MFP) $\ell$.

The multichannel problem is more complicated than the 1D case that was studied in Refs. [17,20]. Although we have not been able to solve the problem in its entirety, we have succeeded in finding a number of partial results, which we consider of sufficient interest to be discussed in the present publication, especially because they may encourage the development of methods towards their experimental verification.

Two comments are in order at this point:

i) In a real electrical conduction problem realized by inserting the system between two terminals (reservoirs) at different chemical potentials, the electron density inside the system would have to be calculated by adding the contribution of all incident energies at which electrons are fed by the reservoirs, with a weight given by the Fermi function of the respective reservoir. As was already mentioned, in the present paper we restrict the analysis to one energy, the more complete calculation being deferred to a later publication.

ii) The problem studied in this paper may arise in Quantum Mechanics (QM), describing electronic scattering in a disordered conductor, or, more generally, in a wave-scattering problem in a q1D disordered waveguide. In what follows, we shall refer specifically to the first type of problem and use the QM nomenclature, although the notions to be discussed below are applicable to a more general problem involving wave transport (see, e.g., Ref. [23], Sec. V-C, for an application to electromagnetic waveguides).

The paper is organized as follows. In the next section we describe the problem to be developed in the article, and give its general mathematical formulation. In Sec. III we discuss the various regimes of interest for this class of systems: Sec. IIIA deals with q1D systems in...
the ballistic regime \((L/\ell \ll 1)\); in Sec. [III B] we study the regime of a large number of open channels, \(N \gg 1\) and for \(0 < s \equiv L/\ell \ll N\), which includes the ballistic and the diffusive regimes; Sec. [III C] deals with q1D systems in the localized regime, \(L \gg \xi = (N+1)\ell\), \(\xi\) being the localization length. In Sec. [IV] we verify the theoretical predictions of the previous sections by means of computer simulations. We give our conclusions in Sec. [V]. Various appendices, as well as a number of Supplemental Material (SM) sections, are included, in order to prove certain specific results without interrupting the main flow of the paper.

II. THE INTENSITY INSIDE A DISORDERED ELECTRONIC CONDUCTOR WITH A QUASI-1D GEOMETRY, SUPPORTING \(N\) PROPAGATING MODES

Consider the q1D scattering problem illustrated in Fig. [I] in which the scattering system is two dimensional, \(x\) and \(y\) being the longitudinal and transverse dimensions, respectively; we contemplate incidence from the left side of the sample at one given energy. Also, we shall refer specifically to the electron conduction problem and use the QM nomenclature, although our discussion is applicable to a more general problem involving wave transport [23].

![Diagram](image)

FIG. 1: The scattering problem associated with the q1D disordered conductor described in the text. The randomness of the potential in the \(y\)-direction is indicated schematically. For left incidence, the amplitudes of the incident, transmitted and reflected waves at either end of the waveguide are indicated. We are interested in the intensity a distance \(x\) from the entrance. The first and second segments of the sample contain \(n_1\) and \(n_2\) non-overlapping scattering units, represented schematically by vertical lines, and are characterized by the transfer matrices \(M_1, M_2\), respectively.
The scattering system consists of \( n_T \) non-overlapping scattering units; \( n_1 \) and \( n_2 \) of these are located on the left and on the right of the observation abscissa \( x \), respectively. Each scattering unit has a random-potential profile in the \( y \)-direction, as described more precisely at the beginning of Sec. [IV]. The whole sample has length \( L \), width \( W \), and can support \( N \) open channels.

The amplitudes of the wave incident from the left in each one of the channels form an \( N \)-dimensional vector, to be indicated as \( a^{(1)} \), whose amplitudes are written as \( a_n^{(1)} \) \((n = 1, \cdots, N)\). The incident wave function can be written as

\[
\psi_{\text{inc}}(x, y) = \sum_{n=1}^{N} a_n^{(1)} \phi_+(E_n; x) \chi_n(y) \quad (2.1a)
\]

\[
= \sum_n \sqrt{\frac{\mu}{k_n}} |a_n^{(1)}| e^{i\delta_n} e^{i k_n x} \chi_n(y) , \quad (2.1b)
\]

\[
\phi_\pm(E_n; x) = \sqrt{\mu} \frac{e^{i k_n x}}{\sqrt{k_n}} , \quad \mu = \frac{m}{2\pi \hbar^2} , \quad a_n^{(1)} = |a_n^{(1)}| e^{i \delta_n} . \quad (2.1c)
\]

Here, \( \phi_\pm(E_n; x) \) are plane waves with longitudinal momentum \( \hbar k_n \), normalized as delta functions of the energy, and

\[
\chi_n(y) = \sqrt{\frac{2}{W}} \sin \frac{n\pi y}{W} , \quad n = 1, \cdots, N , \quad (2.2)
\]

are the transverse wave functions for the \( N \) open channels.

The effect of the scattering system is to produce reflected waves on the left, described by the column vector \( r a^{(1)} \), \( r \) denoting the \( N \times N \) reflection matrix, and transmitted waves on the right, described by the column vector \( t a^{(1)} \), \( t \) being the \( N \times N \) transmission matrix.

Inside the conductor, a distance \( x \) from the left side of the sample, the wave function consists of \( N \) waves traveling to the right, whose amplitudes form the \( N \)-dimensional vector \( a \), and \( N \) waves traveling to the left, with \( N \) amplitudes forming the vector \( b \), i.e.,

\[
\psi(x, y) = \sum_{n=1}^{N} \left[ a_n \phi_+(E_n; x) + b_n \phi_-(E_n; x) \right] \chi_n(y) . \quad (2.3)
\]

For one configuration of disorder, we define the linear particle density, or intensity, at \( x \), designated as \( W(x) \), as the particle density \( |\psi(x, y)|^2 \) integrated over the lateral dimension. Using orthonormality of the transverse wave functions \( \chi_n(y) \), we have

\[
W(x) = \int_0^W |\psi(x, y)|^2 dy = \mu \sum_n \left| a_n \frac{e^{i k_n x}}{\sqrt{k_n}} + b_n \frac{e^{-i k_n x}}{\sqrt{k_n}} \right|^2 . \quad (2.4)
\]
The amplitudes $a_n, b_n$ can be calculated from the transfer matrices of the two portions of the conductor (a reminder of some general properties of transfer matrices can be found in App. A1) and the incident vector of amplitudes $a^{(1)}_n$, as explained in App. A2 (see also Refs. [17,20]). The intensity appearing in Eq. (2.4) becomes [see Eq. (A7)]

$$W(x) = \mu \sum_n \left| \sum_m \frac{e^{ik_n x}}{\sqrt{k_n}} (\alpha^{\dagger}_2 t)_{nm} - \frac{e^{-ik_n x}}{\sqrt{k_n}} (\beta^{\dagger}_2 t)_{nm} \right|^2 a^{(1)}_m^2$$

(2.5)

Here, $\alpha_2, \beta_2$ are the 11 and 12 blocks of the transfer matrix for the second portion of the wire, defined in App. A2; $t$ is the transmission matrix for the full system, defined in App. A1. If we assume random phases $\delta_m$, an average over $\delta_m$ gives [see Eq. (2.1c)]

$$\overline{W}(x) = \mu \sum_{n,m} \frac{1}{k_n} \left| e^{ik_n x} (\alpha^{\dagger}_2 t)_{nm} - e^{-ik_n x} (\beta^{\dagger}_2 t)_{nm} \right|^2 |a^{(1)}_m|^2.$$  

(2.6)

At the RHS of the sample, $x = L$, we have $\alpha_2 = I_N$ (the $N \times N$ unit matrix), $\beta_2 = 0$, so that, from Eq. (2.6), the intensity is given by

$$\overline{W}(L) = \mu \sum_{n,m} \frac{1}{k_n} |t_{nm}|^2 |a^{(1)}_m|^2.$$  

(2.7)

Notice that it is not possible to choose the incident probabilities $|a^{(1)}_m|^2$ so as to have the intensity on the RHS proportional to the transmittance $T = \sum_{nm} |t_{nm}|^2$, because $|a^{(1)}_m|^2/k_n$ cannot be made independent of $n$ and $m$.

For the squared magnitudes $|a^{(1)}_n|^2$ we make the specific choice $|a^{(1)}_n|^2 = \overline{k}/\mu$, where

$$\overline{k} = \frac{1}{N} \sum_{n=1}^N k_n$$

(2.8)

is the channel average of the longitudinal momenta. The intensity at $x$, Eq. (2.6), and at the right end, $x = L$, Eq. (2.7), averaged over the phases $\delta_n$, are then given by

$$\overline{W}(x) = \sum_{n,m} \frac{\overline{k}}{k_n} \left| e^{ik_n x} (\alpha^{\dagger}_2 t)_{nm} - e^{-ik_n x} (\beta^{\dagger}_2 t)_{nm} \right|^2,$$  

(2.9a)

$$\overline{W}(L) = \sum_{n,m} \frac{\overline{k}}{k_n} |t_{nm}|^2.$$  

(2.9b)

In contrast, the above choice for $a^{(1)}_n$ gives, for the Quantum Mechanical current through the system, the result

$$\overline{T} = \frac{\overline{k}}{\mu} \sum_{n,m} |t_{nm}|^2 = \frac{\overline{k}}{\mu} T,$$  

(2.10)
which is proportional to the transmittance $T$. Only in an equivalent-channel approximation, $k_n \equiv \overline{k}$, is the intensity at $L$, Eq. (2.9b), proportional to the transmittance.

For simplicity in writing, in the future we shall omit the bar indicating a phase average, and write $\mathcal{W}(x)$ for the quantity on the LHS of Eq. (2.9a).

III. THE VARIOUS REGIMES

A. The ballistic regime

Here, we discuss the intensity $\mathcal{W}(x)$ defined in Eq. (2.9a), for q1D systems which are in the ballistic regime, i.e., such that

$$s \equiv \frac{L}{\ell} \ll 1 . \quad (3.1)$$

The intensity $\mathcal{W}(x)$, evaluated in the Supplemental Material (SM) [24], Sec. I, and given explicitly in Eq. SM(1.4), simplifies considerably in an equivalent-channel (EC) model, $k_n = k$, giving

$$\left( \frac{\mathcal{W}(x)}{N} \right)_{EC} = 1 - \frac{2}{N} \text{Re} \left[ e^{-2ikx} \text{Tr}(v_2^T \sqrt{\lambda_2} v_2) \right] - \frac{1}{N} \text{Tr}(\lambda_1) + \frac{1}{N} \text{Tr}(\lambda_2) - \frac{2}{N} \text{Re} \text{Tr}[(u_1 \sqrt{\lambda_1} u_1^T)(v_2^T \sqrt{\lambda_2} v_2)] + O(\lambda_3^{3/2}) . \quad (3.2)$$

Here, we write our expressions employing the polar representation summarized in App. A.3 in which the transfer matrix is parametrized in terms of the matrices $\lambda, u, v$: $\lambda$ is a diagonal $N$-dimensional matrix with non-negative elements $\lambda_n (n = 1, \cdots N)$, and $u, v$ are arbitrary $N$-dimensional unitary matrices.

The result (3.2) applies to one configuration of disorder. As shown in SM, Ref. [24], Sec. I, the ensemble average of the logarithm of the intensity can be written, upon expanding the logarithm, as

$$\left\langle \ln \left( \frac{\mathcal{W}(x)}{N} \right) \right\rangle_{s \ll 1, EC} = \frac{(N - 1)(N + 2)}{N(N + 1)} \frac{L - x}{\ell} - \frac{x}{\ell} + \left\langle O(\lambda^{3/2}) \right\rangle + \cdots . \quad (3.3)$$

In the one-channel case, $N = 1$, this result reduces to $-x/\ell$, in agreement with the analysis of Ref. [17]. Notice that, for $N > 1$, the result (3.3) acquires a dependence on $s = L/\ell$. 


We may take into account the non equivalence of channels in an approximate fashion in the following way. Eq. SM(1.4) for NECs has the structure
\[
\frac{W(x)}{N} = \frac{1}{N} \sum_n \frac{\bar{k}}{k_n} f_n = \frac{\bar{k}}{k} \frac{1}{N} \sum_n \left(1 + \frac{\bar{k} - k_n}{k_n}\right) f_n \tag{3.4a}
\]
\[
\approx \frac{\bar{k}}{k} \left(\frac{1}{N} \sum_n f_n\right) = \frac{\bar{k}}{k} \left(\frac{W(x)}{N}\right)_{EC}. \tag{3.4b}
\]

We have introduced the harmonic average \(\bar{k}\) of the \(k_n\)s, \(1/\bar{k} = 1/N \sum_n 1/k_n\), and assumed that the \(k_n\)s do not differ much from \(\bar{k}\). As a result,
\[
\langle \ln \frac{W(x)}{N} \rangle_{s \ll 1} \approx \ln \frac{\bar{k}}{k} + \langle \ln \frac{W(x)}{N} \rangle_{s \ll 1, EC}
\]
\[
\approx \ln \frac{\bar{k}}{k} + \frac{(N-1)(N+2)}{N(N+1)} \frac{L-x}{\ell} - \frac{x}{\ell} + \langle O(\lambda^{3/2}) \rangle + \cdots, \tag{3.5b}
\]
which reduces to Eq. (3.3) for ECs, as then, \(\bar{k} = \bar{k}\).

These results will be compared with computer simulations in the next section.

**B. The case of a large number of open channels. From the ballistic to the diffusive regime**

We now consider the case when the number of open channels is very large,
\[
N \gg 1, \tag{3.6a}
\]
and for
\[
0 < s = L/\ell \ll N, \tag{3.6b}
\]
conditions that cover the ballistic and diffusive regimes; we restrict the analysis to the two end points of the sample, i.e., \(x = 0, L\), for which we have explicit results.

We start from Eq. (B2), which gives the intensity at the left end of the sample and compute its expectation value. In the DMPK approach, \(\langle r_{nm} \rangle = 0\), and \(\langle |r_{nm}|^2 \rangle\) is independent of \(n, m\), so that
\[
\langle r_{nm} \rangle = 0, \tag{3.7a}
\]
\[
\langle |r_{nm}|^2 \rangle = \frac{\langle R \rangle}{N^2} = \frac{N - \langle T \rangle}{N^2} = \frac{1}{N} - \frac{\langle T \rangle}{N^2}, \tag{3.7b}
\]
in terms of the reflectance \( R = \sum_{n,m} |r_{nm}|^2 \) and the transmittance \( T = \sum_{n,m} |t_{nm}|^2 \). Since in the present regime the ensemble expectation value of the transmittance can be approximated as \( \langle T \rangle \approx \frac{N}{1+s} \),

\[
\langle T \rangle \approx \frac{N}{1+s}, \quad (3.8)
\]

Eq. (3.2) gives, on average,

\[
\frac{\langle \mathcal{W}(0) \rangle}{N} = \frac{k}{k} + \frac{1}{N} \left( \sum_{nm} \frac{k}{k_n} \right) \left( \frac{1}{N} \frac{s}{1+s} \right),
\]

\[
= \frac{k}{k} \frac{1+2s}{1+s} \approx \frac{k}{k} \times \begin{cases} 
1+s, \quad \text{for } s \ll 1, \\
2 - \frac{1}{s}, \quad \text{for } 1 \ll s \ll N.
\end{cases}
\]

so for the logarithm of this expression we obtain

\[
\ln \left( \frac{\langle \mathcal{W}(0) \rangle}{N} \right) = \ln \frac{k}{k} + \ln \frac{1+2s}{1+s} \approx \frac{k}{k} + \begin{cases} 
s, \quad \text{for } s \ll 1, \\
\ln 2 - \frac{1}{2s}, \quad \text{for } 1 \ll s \ll N.
\end{cases}
\]

At the right end of the sample, \( x = L \), Eq. (2.9b) gives

\[
\frac{\mathcal{W}(L)}{N} = \frac{1}{N} \sum_{n,m} \frac{k}{k_n} |t_{nm}|^2.
\]

(3.11)

In the DMPK approach we have

\[
\langle |t_{nm}|^2 \rangle = \frac{\langle T \rangle}{N^2},
\]

\[
\approx \frac{1}{N} \frac{1}{1+s},
\]

(3.12a)

(3.12b)

where we used the result (3.8), valid in the regime defined by Eqs. (3.6). Substituting in Eq. (3.11), we then find

\[
\frac{\langle \mathcal{W}(L) \rangle}{N} = \frac{k}{k} \frac{1}{k} \frac{1}{1+s} = \frac{k}{k} \times \begin{cases} 
1 - s + \cdots, \quad \text{for } s \ll 1, \\
\frac{1}{s} - \frac{1}{s^2} + \cdots, \quad \text{for } 1 \ll s \ll N,
\end{cases}
\]

\[
\ln \left( \frac{\langle \mathcal{W}(L) \rangle}{N} \right) = \ln \frac{k}{k} - \ln(1+s) = \ln \frac{k}{k} - \begin{cases} 
s + \cdots, \quad \text{for } s \ll 1, \\
\ln s + \cdots, \quad \text{for } 1 \ll s \ll N.
\end{cases}
\]

(3.13a)

(3.13b)

These results will be compared with computer simulations in the next section.
C. The localized regime

In this regime,

$$L \gg \xi = (N + 1)\ell,$$  \hspace{1cm} (3.14)

where $\xi$ denotes the localization length. We restrict our analysis to the left and right ends of the sample, i.e., $x = 0, L$.

In the localized regime, from Eq. (B5) we can approximate, in lowest order ($\lambda_n \gg 1$, $n = 1, \ldots, N$), the logarithm of the intensity at the left end as

$$\left( \frac{W(0)}{N} \right)_{s \gg N} \approx \frac{k}{k'} \left[ 1 - \frac{1}{N} \sum_{n=1}^{N} \tilde{k}_n \sum_{a=1}^{N} (v_{an} v_{an}^* + v_{an}^* v_{an}^*) \right] + \frac{1}{N} \sum_{n,m=1}^{N} \tilde{k}_n \left( \sum_{a=1}^{N} v_{an} v_{am}^* \right) \left( \sum_{b=1}^{N} v_{bn} v_{bm}^* \right).$$  \hspace{1cm} (3.15)

We notice that in the present approximation, $\lambda_n \gg 1$, any function of $W(0)/N$ has no $\lambda$ dependence left. The statistics of such a quantity will thus be independent of the system length $s = L/\ell$. In particular, the tendency of $\langle \ln[W(0)/N] \rangle$ to become independent of $L/\ell$ as this parameter increases is verified in the computer simulation for two open channels, $N = 2$, to be discussed later in relation with Fig. 2. This result is similar to that found in the 1D case for the average of $\ln W(x)$, which, for fixed $x$, gives $-x/\ell$ and is independent of $L/\ell$ for arbitrary values of this parameter (as long as $x < L$; see Ref. [17], Fig. 1 and related discussion).

As an illustration of the approximation we have used, the reflection coefficient $R_{mn} = |r_{mn}|^2$ of Eq. (B4c) and the total reflection coefficient $R_n = \sum_m R_{mn}$ would be given, in the same approximation, by

$$R_{mn} \approx \sum_{a,b=1}^{N} v_{an} v_{am}^* (v_{bn} v_{bm}^*)$$  \hspace{1cm} (3.16a)

$$R_n = \sum_{m=1}^{N} R_{mn} \approx \sum_{a,b=1}^{N} v_{an} v_{bn}^* \sum_{m=1}^{N} v_{am} v_{bm}^* = \sum_{a,b=1}^{N} v_{an} v_{bn}^* \delta_{ab} = \sum_{a=1}^{N} v_{an} v_{sn}^* = 1,$$  \hspace{1cm} (3.16b)

meaning that a wave incident in channel $n$ is fully reflected into all $N$ backward channels, with no transmission left. This is the crudest approximation to localization.

If we adopt an equivalent-channel (EC) approximation, Eq. (3.15) reduces to

$$\left( \frac{W(0)}{N} \right)_{s \gg N, EC} = 2 - \frac{1}{N} \sum_{a,n=1}^{N} (v_{an} v_{an}^* + v_{an}^* v_{an}^*) ,$$  \hspace{1cm} (3.17)
The expectation value of the logarithm of (3.17) is given by

\[
\left\langle \ln \frac{W(0)}{N} \right\rangle_{s \gg N,EC} = \left\langle \ln \left[ 1 - \frac{1}{2N} \sum_{a,n=1}^{N} (v_{an}v_{an} + v_{an}^* v_{an}^*) \right] \right\rangle_0 + \ln 2 ,
\]

where the index 0 indicates an average over the matrices \( v \), distributed according to the invariant measure of the group \( U(N) \) \([1]\). This is evaluated in SM, Ref. \([24]\), Sec. 2, for \( N = 2 \). From Eq. (2.6b)SM, we have

\[
\left\langle \ln \frac{W(0)}{2} \right\rangle_{s \gg 2,EC} = -0.122351 + \ln 2 = 0.570796.
\]

At the right end of the sample, \( x = L \), the intensity is given by Eq. (2.9b). Only in an equivalent-channel (EC) approximation it coincides with the transmittance, i.e.,

\[
W(L) = \sum_{m,n=1}^{N} |t_{mn}|^2 = T ,
\]

which, in the polar representation, is

\[
\frac{W(L)}{N} = \frac{1}{N} \sum_{a=1}^{N} \frac{1}{1 + \lambda_a} .
\]

In the localized regime, (3.20) gives

\[
\left\langle \ln \frac{W(L)}{N} \right\rangle_{s \gg N,EC} = \left\langle \ln T \right\rangle_{s \gg N,EC} - \ln N \approx -2 \frac{L}{(N+1)\ell} - \ln N .
\]

These results will be compared with computer simulations in the next section.

**IV. COMPUTER SIMULATIONS**

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To check the theoretical results of the previous sections, we present a number of computer simulations of random q1D systems supporting \( N \) propagating modes, in which the disordered potential is a random function of position. We use the model employed in Ref. \([25]\), in which the scattering units consist of thin potential slices, idealized as equidistant delta potentials \( (d \text{ being their separation}) \), perpendicular to the longitudinal direction of the conductor, the variation of the potential in the transverse direction being random. The sets
FIG. 2: Computer simulations for \( N = 2 \) open channels and some theoretical results for the quantities shown in the various panels. The values of the parameters used in the simulations are given in the top part of the figure. a) Numerical results for \( \langle \ln(W(0)/N) \rangle \), as a function of \( s = L/\ell \). The inset shows the ballistic regime in more detail. The last, right-most point, marked \( B \) in the figure, has to be compared with the first, left-most point in Fig. 3b. b) Numerical results for \( \langle \ln(W(L)/N) \rangle \), as a function of \( s = L/\ell \). The last, right-most point, marked \( A \) in the figure, has to be compared with the last, right-most point in Fig. 3b. c) Numerical results for \( \langle \ln(T/N) \rangle \) as a function of \( s \).

of parameters defining a given slice are taken to be statistically independent from those of any other slice and identically distributed. In the dense-weak-scattering limit, in which the potential strength of the slices is very weak and their linear density is very large, so that the resulting mean free paths are fixed, the corresponding statistical properties of the full system depend only on the mean free paths and on no other property of the slice distribution.

We first consider the left end of the sample, \( x = 0 \). The quantity \( \langle \ln(W(0)/N) \rangle \) was calculated numerically by a computer simulation for \( N = 2 \) open channels, from the ballistic to the localized regime, as a function of \( L/\ell \), as can be seen in Fig. 2a. In the ballistic regime, corresponding to \( s \ll 1 \), the result is in excellent agreement with \( \ln(\kbar/k) + (2/3)L/\ell \), as
predicted by Eqs. (3.5) for \( x = 0 \). The result is not \( L \) independent, as it is for \( N = 1 \). In the localized regime, \( s \gg 1 \), the simulation tends asymptotically to \( \approx 0.61 \), while theory gives 0.57, a constant independent of \( s \), just as for \( N = 1 \). The cause for the slight discrepancy is probably due to the approximations involved in the DMPK equation, like the isotropy assumption, whose influence is more apparent in the localized regime; this assumption has been relaxed in improved models, like Refs. [26–28]. Results for the right end of the sample, \( x = L \), \( \langle \ln(W(L)/N) \rangle \), are given in panel b), and for \( \langle \ln(T/N) \rangle \), in panel c). The difference between the results in the two panels, not noticeable in the figure, is due to the two channels not having the same longitudinal momentum \( k_n \): compare \( W(x) \) in Eq. (2.9b) with \( T \) in Eq. (2.10).

Fig. 3 shows, in panel a), the profile \( \langle \ln(W(x)/N) \rangle \) for \( N = 2 \) open channels as a function of \( x \) for various values of \( s \), for the ballistic regime, \( s \ll 1 \), and slightly beyond, \( s \sim 1 \). The theoretical prediction of Eq. (3.5), which takes into account, approximately, that the two longitudinal momenta \( k_n \) are not equal, agrees well with the computer simulations up to \( s \sim 0.3 \). The simulations show oscillations as a function of \( x/\ell \), which are not reproduced by the DMPK treatment. On the other hand, these interference oscillations are described well by an analysis based on Born’s approximation, up to second order [29]. Fig. 3 shows the behavior of the profile \( \langle \ln W(x)/N \rangle \) for \( N = 2 \) channels as a function of \( x \), for various values of \( s \), from the ballistic to the localized regime. In the localized regime, as \( x \) goes from the left to the right edge of the sample, \( x = 0 \) to \( x = L \), the discrepancy between theory and simulations increases. For \( x = 0 \), a slight difference was already noted in Fig. 2a). The discrepancy for \( x = L \) is quite noticeable in Fig. 3b). A useful example to understand this discrepancy is to compare the behavior of \( \langle \ln R \rangle \) with that of \( \langle \ln T \rangle \) as \( s \) increases. Although one always has \( R + T = N \), the quantities \( \langle \ln R \rangle \) and \( \langle \ln T \rangle \) behave very differently. As \( s \) increases, \( \langle \ln R \rangle \) tends to \( \ln N \) for both theory and simulations. On the other hand, \( \langle \ln T \rangle \) is not bounded as \( s \) increases: it tends to \( -\infty \), so the difference between theory and simulations cannot be assessed a priori. In the present case, there seems to be a difference in slope of \( \langle \ln T \rangle \) vs \( s \) between theory and simulations, which becomes larger for larger values of \( s \). In order to be more specific, one would need an improved theory beyond DMPK, such as the theories developed in Refs. [26–28].

We compare, in Fig. 4, theoretical results with computer simulations for the regime in which the number of open channels is very large, \( N \gg 1 \), and for \( 0 < s = L/\ell \ll N \). The
FIG. 3: Computer simulations of the profile $\langle \ln W(x)/N \rangle$ for $N = 2$ open channels and some theoretical predictions. a) Numerical results for the ballistic regime and slightly beyond, and comparison with theory, Eq. (3.5), shown as full lines. b) Numerical results for the localized regime. Theory is given only for $x = 0$ and $x = L$, and is indicated by big dots. The origin of the discrepancy is probably the same as discussed in the text in relation with Fig. 2.

Theoretical description is given in Sec. 3.6a. In the left panels, the theoretical results are taken from Eq. (3.9b) for panel a), Eq. (3.13a) for panel c), and $1/(1 + s)$ for panel e). In the panels on the right, the simulations represent the average of a logarithm, while the theoretical results are the logarithm of an average: the latter are taken from Eq. (3.10) for panel b), Eq. (3.13b) for panel d), and $-\ln(1 + s)$ for panel f). That the two calculations give almost the same results is an indication of the self-averaging property for the logarithm of the various quantities considered (see Ref. [17], Supplemental Material).

Fig. 3 shows computer simulations for the profiles $\langle W(x)/N \rangle$ and $\langle (\ln W(x)/N) \rangle$ as a function of $x/\ell$, for the same data as Fig. 4 $N = 50$ and for $s = 8.44$ (the maximum value of $s$ in Fig. 4), so we are approximately in the diffusive regime. The equivalent of panel a) for the 1D case [20] would be an s-shaped curve going from 2 to zero, being antisymmetrical with respect to the intersection of the two thin lines shown in a). The equivalent of panel b) for the 1D case [17] would be a 45-degree straight line starting from zero on the left.
V. SUMMARY AND CONCLUSIONS

In this paper we studied the statistical properties of the electron density inside a q1D multiply-scattering disordered electric conductor which may support more than one propagating mode ($N \geq 1$).

The physical quantity that was mainly considered was the logarithm of the electron density, $\ln W(x)$, since its statistical properties possess a self-averaging behavior, as was explained, for $N = 1$, in the supplemental material to Ref. [17]. The theoretical analysis in the present work was based on the DMPK model [1], and the results were compared with
**Diffusive Regime:** $0 < s < < N=50$, $s=8.44$, $kl=100$, $d/l=10^{-3}$, 20 realizations.

![Graph](image)

**FIG. 5:** For the same data as in Fig. 4 and for $s = 8.44$, which is the maximum value of $s$ in Fig. 4 (so we are approximately in the diffusive regime), computer simulations for the profile of a) $\langle W(x)/N \rangle$ as a function of $x/\ell$; b) $\langle \ln(W(x)/N) \rangle$ as a function of $x/\ell$. Theoretical results of $\langle W(x)/N \rangle$ and $\ln(W(x)/N)$ for $x = 0$ and $x = L$ are indicated as big dots. Points marked as C, D, E, F correspond to the ones marked similarly in Fig. 4.

We first addressed the case in which the system is in the ballistic regime, where we found a good agreement between theory and simulations. However, we should point out that the latter exhibit oscillations as a function of $x/\ell$, which are missed by the DMPK treatment. In contrast, these oscillations are reproduced by an analysis based on Born’s approximation, as studied in Ref. [29].

We then dealt with systems possessing a large number of propagating modes, or open channels, $N \gg 1$, when the parameters $s$ and $L$ fulfill $0 < s \equiv L/\ell \ll N$, this inequality including the ballistic and the diffusive regimes. The theoretical results for this regime are in very good accordance with the computer simulations. We found evidence of a self-averaging property, judging from the good agreement between the expectation value of the logarithm of the electron density and the logarithm of the expectation value, the latter being the quantity that was amenable to a theoretical calculation.

We finally analyzed the 1D systems in the localized regime. In an equivalent-channel approximation, we obtained a rather general expression for $\langle \ln W(x) \rangle$ for $x = 0$, i.e., near
the incident beam entrance on the left side of the sample, which we could compute explicitly for two open channels, $N = 2$, and compare with computer simulations. We also found a rather general expression for $\langle \ln W(x) \rangle$ for $x = L$, as in this case it is directly connected with the transmittance. We noticed that, in the localized regime, as $x$ goes from the left to the right edge of the sample, $x = 0$ to $x = L$, the discrepancy between theory and simulations increases. The slight discrepancy shown in Fig. 2 a for $x = 0$ is even more noticeable for $x = L$, as exhibited in Fig. 3 b. We conjectured that such a discrepancy is due to the approximations involved in the DMPK equation, like the isotropy assumption, whose influence is more apparent in the localized than in the diffusive regime; such an assumption has been relaxed in a number of models, like those presented in Refs. 26–28.

The multichannel problem is more complicated than the 1D case that was studied in Refs. 17, 20. As a matter of fact, because of technical difficulties we have not been able to compute theoretically, for any number of propagating modes $N \geq 1$ (as was done in Ref. 17 for $N = 1$): i) the profile $\langle \ln W(x) \rangle$ for arbitrary positions $x$ and for arbitrary $s = L/\ell$, except for the particular positions and for the regimes $s$ mentioned above; on the other hand, we reported on computer simulations for arbitrary values of these parameters; ii) the profile of the variance of $\ln W(x)$, again for any $x$ and $s$. These problems remain open for future studies.

As was indicated in the Introduction, a real electrical conduction problem is modeled by placing the system between two reservoirs at different chemical potentials 2, 3. The electrons fed by the reservoirs at all energies, weighted by the respective Fermi function, would then contribute to the electron density inside the system. In contrast, in the present paper we restricted the analysis to the system being fed with electrons of a given energy from one end of the disordered conductor, and the electron density was then evaluated for that energy along the conductor and outside. The more complete calculation will be performed in a future publication.

We conclude by encouraging the development of appropriate methods towards the experimental verification of the present results for electron systems. In order to cover a variety of physical domains, we would also like to suggest the realization of laboratory experiments with electro-magnetic waves, extending the experiments reported in Ref. 17, and with elastic waves, as an extension of the studies performed in Refs. 18, 19.
Appendix A: Technical details

1. Some properties of the transfer matrix

By definition, the transfer matrix $M$ for the full system relates the amplitudes on the RHS, $a^{(2)}, b^{(2)}$, to those on the LHS, $a^{(1)}, b^{(1)}$, as

$$
M \begin{bmatrix} a^{(1)} \\ b^{(1)} \end{bmatrix} = \begin{bmatrix} a^{(2)} \\ b^{(2)} \end{bmatrix};
$$

(A1)

$M$ is a $2N \times 2N$ matrix, with the structure

$$
M = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix},
$$

(A2)

satisfying the properties of time-reversal and flux conservation, as

$$
M^\dagger \Sigma_z M = \Sigma_z,
$$

(A3a)

$$
M^* = \Sigma_x M \Sigma_x,
$$

(A3b)

respectively. Here, $\Sigma_z$ and $\Sigma_x$ are the $2N \times 2N$ generalization of Pauli matrices $\sigma_z$ and $\sigma_x$. The $N$-dimensional blocks of the $M$ matrix are related to the reflection and transmission matrices $r, t$ for left incidence and $r', t'$ for right incidence as

$$
r = -(\alpha^*)^{-1} \beta^*, \quad t' = (\alpha^*)^{-1},
$$

(A4a)

$$
t = (\alpha^*)^{-1}, \quad r' = \beta (\alpha^*)^{-1}.
$$

(A4b)

2. Calculation of the amplitudes $a_n, b_n$ appearing in Eq. (2.4)

The transfer matrices of the two portions of the wire are denoted by

$$
M_i = \begin{bmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{bmatrix}, \quad i = 1, 2.
$$

(A5)

From the definition of the transfer matrix, we have

$$
M_2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} t a^{(1)} \\ 0 \end{bmatrix},
$$

(A6)
so that the amplitudes between the two portions are given by

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} = \begin{bmatrix}
(\alpha_2^t \alpha_1^{(1)}) \\
-(\beta_2^t \alpha_1^{(1)})
\end{bmatrix}, \quad t = \frac{1}{\alpha_1^\dagger} = \frac{1}{\alpha_1^\dagger \alpha_2^\dagger + \beta_1^\dagger \beta_2^\dagger}.
\] (A7)

This result is used in Eq. (2.5).

3. The polar representation

From Ref. [1], a transfer matrix \( M \) can be expressed as

\[
M = \begin{bmatrix}
u & 0 \\
0 & u^*
\end{bmatrix} \begin{bmatrix}
\sqrt{1 + \lambda} & \sqrt{\lambda} \\
\sqrt{\lambda} & \sqrt{1 + \lambda}
\end{bmatrix} \begin{bmatrix}
v & 0 \\
0 & v^*
\end{bmatrix}.
\] (A8)

Here, \( u \) and \( v \) are \( N \times N \) arbitrary unitary matrices; \( \lambda \) is an \( N \times N \) diagonal matrix with elements \( \lambda_n \geq 0, n = 1, \cdots, N \).

The reflection matrix \( r \) can be expressed as

\[
r = -v^T \sqrt{\frac{\lambda}{1 + \lambda}} v
\] (A9)

These results are used in Sec. III.

Appendix B: Exact result for \( x = 0 \)

For \( x = 0 \), using Eq. (A7) (see also Ref. [1] for the relation between the transfer matrix and the \( S \) matrix) we have

\[
\alpha_2 = \alpha = \frac{1}{t^\dagger}, \quad \alpha_2^t \alpha_1 = I_N, \quad a = a^{(1)},
\] (B1a)

\[
\beta_2 = \beta = -\frac{1}{t^\dagger} r^*, \quad -\beta_2^t \alpha_1 = r^T = r, \quad b = r a^{(1)}.
\] (B1b)

Eq. (2.9a) then gives

\[
\frac{\mathcal{W}(0)}{N} = \frac{k}{k} + \frac{1}{N} \sum_{n=1}^{N} \frac{k}{k_n} (r_{nn} + r_{nn}^*) + \frac{1}{N} \sum_{n,m=1}^{N} \frac{k}{k_n} |r_{nm}|^2,
\] (B2)

where we have defined the “harmonic” channel average of the longitudinal momenta

\[
\frac{1}{k} \equiv \left( \frac{1}{k} \right) = \frac{1}{N} \sum_n \frac{1}{k_n}.
\] (B3)
Using the polar representation parameters introduced in App. A3, we can write the reflection matrix $r$, its matrix elements, and the reflection coefficients as

$$r = -v^T \sqrt{\frac{\lambda}{1 + \lambda}} v,$$

(B4a)

$$r_{nm} = -\sum_{a=1}^{N} v_{an} v_{am} \sqrt{\frac{\lambda_a}{1 + \lambda_a}},$$

(B4b)

$$R_{nm} \equiv |r_{nm}|^2 = \sum_{a,b=1}^{N} \sqrt{\frac{\lambda_a}{1 + \lambda_a}} \sqrt{\frac{\lambda_b}{1 + \lambda_b}} v_{an} v_{am}^* v_{bn} v_{bm}^*.$$  

(B4c)

Substituting in Eq. (B2), $W(0)/N$ becomes

$$W(0) = \frac{\frac{1}{N} \sum_{n=1}^{N} \sum_{a=1}^{N} k_{n} \lambda_{a} (v_{an} v_{an}^* + v_{an}^* v_{an})}{N} + \frac{1}{N} \sum_{n,m=1}^{N} k_{n} \sum_{a,b=1}^{N} \lambda_{a} \lambda_{b} (v_{an} v_{am}^* v_{bn} v_{bm}^*)}.$$  

(B5)

This result is valid for any regime $L/\ell$. It is used in Secs. III B and III C.

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