On the $K(\pi, 1)$-property for rings of integers in the mixed case

by Alexander Schmidt

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Abstract

We investigate the Galois group $G_S(p)$ of the maximal $p$-extension unramified outside a finite set $S$ of primes of a number field in the (mixed) case, when there are primes dividing $p$ inside and outside $S$. We show that the cohomology of $G_S(p)$ is ‘often’ isomorphic to the étale cohomology of the scheme $\text{Spec}(\mathcal{O}_K \setminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then. We deduce this from the results in our previous paper [Sch2], which mainly dealt with the tame case.

1 Introduction

Let $Y$ be a connected locally noetherian scheme and let $p$ be a prime number. We denote the étale fundamental group of $Y$ by $\pi_1^{\text{ét}}(Y)$ and its maximal pro-$p$ factor group by $\pi_1(Y)(p)$. The Hochschild-Serre spectral sequence induces natural homomorphisms

$$\phi_i : H^i(\pi_1^{\text{ét}}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{\text{ét}}(Y, \mathbb{Z}/p\mathbb{Z}), \; i \geq 0,$$

and we call $Y$ a ‘$K(\pi, 1)$ for $p$’ if all $\phi_i$ are isomorphisms; see [Sch2] Proposition 2.1 for equivalent conditions. See [Wi2] for a purely Galois cohomological approach to the $K(\pi, 1)$-property. Our main result is the following

**Theorem 1.1.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ does not contain a primitive $p$-th root of unity and that the class number of $k$ is prime to $p$. Then the following holds:

Let $S$ be a finite set of primes of $k$ and let $T$ be a set of primes of $k$ of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $\text{Spec}(\mathcal{O}_k \setminus (S \cup T_1))$ is a $K(\pi, 1)$ for $p$.

**Remarks.** 1. If $S$ contains the set $S_p$ of primes dividing $p$, then Theorem 1.1 holds with $T_1 = \emptyset$ and even without the condition $\zeta_p \not\in k$ and $C(k)(p) = 0$, see [Sch2], Proposition 2.3. In the tame case $S \cap S_p = \emptyset$, the statement of Theorem 1.1 is the main result of [Sch2]. Here we provide the extension to the ‘mixed’ case $\emptyset \subset S \cap S_p \subset S_p$.  

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2. For a given number field \( k \), all but finitely many prime numbers \( p \) satisfy the condition of Theorem 1.1. We conjecture that Theorem 1.1 holds without the restricting assumption on \( p \).

Let \( S \) be a finite set of places of a number field \( k \). Let \( k_S(p) \) be the maximal \( p \)-extension of \( k \) unramified outside \( S \) and put \( G_S(p) = \text{Gal}(k_S(p)|k) \). If \( S_R \) denotes the set of real places of \( k \), then \( G_{S,R}(p) \cong \pi_1(Spec(\mathcal{O}_k)\setminus S)(p) \) (we have \( G_S(p) = G_{S,R}(p) \) if \( p \) is odd or \( k \) is totally imaginary). The following Theorem 1.2 sharpens Theorem 1.1.

**Theorem 1.2.** The set \( T_1 \subset T \) in Theorem 1.1 may be chosen such that

1. \( T_1 \) consists of primes \( p \) of degree 1 with \( N(p) \equiv 1 \mod p \),
2. \( (k_{S,T_1}(p))_p = k_p(p) \) for all primes \( p \in S \cup T_1 \).

Note that Theorem 1.2 provides nontrivial information even in the case \( S \supset S_p \), where assertion (ii) was only known when \( k \) contains a primitive \( p \)-th root of unity (Kuz'min's theorem, see [Kuz] or [NSW], 10.6.4 or [NSW2], 10.8.4, respectively) and for certain CM fields (by a result of Mukhamedov, see [Muk] or [NSW], X §6 exercise or [NSW2], X §8 exercise, respectively).

By Theorem 5.3 below, Theorem 1.2 provides many examples of \( G_S(p) \) being a duality group. If \( \zeta_p \notin k \), this is interesting even in the case that \( S \supset S_p \), where examples of \( G_S(p) \) being a duality group were previously known only for real abelian fields and for certain CM-fields (see [NSW], 10.7.15 and [NSW2], 10.9.15, respectively, and the remark following there).

Previous results in the mixed case had been achieved by K. Wingberg [Wing], Ch. Maire [Mai] and D. Vogel [Vog]. Though not explicitly visible in this paper, the present progress in the subject was only possible due to the results on mild pro-\( p \) groups obtained by J. Labute in [Lab].

I would like to thank K. Wingberg for pointing out that the proof of Proposition 8.1 in my paper [Sch2] did not use the assumption that the sets \( S \) and \( S' \) are disjoint from \( S_p \). This was the key observation for the present paper. The main part of this text was written while I was a guest at the Department of Mathematical Sciences of Tokyo University and of the Research Institute for Mathematical Sciences in Kyoto. I want to thank these institutions for their kind hospitality.

## 2 Proof of Theorems 1.1 and 1.2

We start with the observation that the proofs of Proposition 8.2 and Corollary 8.3 in [Sch2] did not use the assumption that the sets \( S \) and \( S' \) are disjoint from \( S_p \). Therefore, with the same proof (which we repeat for the convenience of the reader) as in loc. cit., we obtain
Proposition 2.1. Let \( k \) be a number field and let \( p \) be a prime number. Assume \( k \) to be totally imaginary if \( p = 2 \). Put \( X = \text{Spec}(O_k) \) and let \( S \subset S' \) be finite sets of primes of \( k \). Assume that \( X \smallsetminus S \) is a \( K(\pi,1) \) for \( p \) and that \( G_S(p) \neq 1 \). Furthermore, the arithmetic form of Riemann’s existence theorem holds, i.e., setting \( K = k_S(p) \), the natural homomorphism

\[
\otimes_{p \in S' \smallsetminus S(K)} T(K_p(p)|K_p) \longrightarrow \text{Gal}(k_S(p)|K)
\]

is an isomorphism. Here \( T(K_p(p)|K_p) \) is the inertia group and \( \otimes \) denotes the free \( p \)-product of a bundle of \( p \)-groups, cf. [NSW], Ch. IV, §3. In particular, \( \text{Gal}(k_S(p)|k_S(p)) \) is a free \( p \)-group.

Proof. The \( K(\pi,1) \)-property implies

\[
H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H^i_{et}(X \smallsetminus S, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for} \quad i \geq 4,
\]

hence \( cd \ G_S(p) \leq 3 \). Let \( p \in S' \smallsetminus S \). Since \( p \) does not split completely in \( k_S(p) \) and since \( cd G_S(p) < \infty \), the decomposition group of \( p \) in \( k_S(p)|k \) is a non-trivial and torsion-free quotient of \( \mathbb{Z}_p \cong \text{Gal}(k_S^p(p)|k) \). Therefore \( k_S(p)_p \) is the maximal unramified \( p \)-extension of \( k_p \). We denote the normalization of an integral normal scheme \( Y \) in an algebraic extension \( L \) of its function field by \( Y_L \). Then \( (X \smallsetminus S)_{k_S(p)} \) is the universal \( p \)-covering of \( X \smallsetminus S \). We consider the étale excision sequence for the pair \( ((X \smallsetminus S)_{k_S(p)}, (X \smallsetminus S')_{k_S(p)}) \). By assumption, \( X \smallsetminus S \) is a \( K(\pi,1) \) for \( p \), hence \( H^i_{et}((X \smallsetminus S)_{k_S}, \mathbb{Z}/p\mathbb{Z}) = 0 \) for \( i \geq 1 \) by [Sch2], Proposition 2.1. Omitting the coefficients \( \mathbb{Z}/p\mathbb{Z} \) from the notation, this implies isomorphisms

\[
H^i_{et}((X \smallsetminus S')_{k_S(p)}) \cong \bigoplus_{p \in S' \smallsetminus S(k_S(p))} H^{i+1}_p(((X \smallsetminus S)_{k_S})_p)
\]

for \( i \geq 1 \). Here (and in variants also below) we use the notational convention

\[
\bigoplus_{p \in S' \smallsetminus S(k_S(p))} H^{i+1}_p(((X \smallsetminus S)_{k_S})_p) := \lim_{K \subseteq k_S(p)} \bigoplus_{p \in S' \smallsetminus S(K)} H^{i+1}_p(((X \smallsetminus S)_K)_p),
\]

where \( K \) runs through the finite extensions of \( k \) inside \( k_S(p) \). As \( k_S(p) \) realizes the maximal unramified \( p \)-extension of \( k_p \) for all \( p \in S' \smallsetminus S \), the schemes \( ((X \smallsetminus S)_{k_S(p)})_p, p \in S' \smallsetminus S(k_S(p)) \), have trivial cohomology with values in \( \mathbb{Z}/p\mathbb{Z} \) and we obtain isomorphisms

\[
H^i((k_S(p))_p) \cong H^{i+1}_p(((X \smallsetminus S)_{k_S(p)})_p)
\]

for \( i \geq 1 \). These groups vanish for \( i \geq 2 \). This implies

\[
H^i_{et}((X \smallsetminus S')_{k_S(p)}) = 0
\]

for \( i \geq 1 \).
for $i \geq 2$. Since the scheme $(X \setminus S')_{k_S}(p)$ is the universal pro-$p$ covering of $(X \setminus S')_{k_S}(p)$, the Hochschild-Serre spectral sequence yields an inclusion

$$H^2(Gal(k_S(p)|k_S(p))) \hookrightarrow H^2_{et}((X \setminus S')_{k_S}(p)) = 0.$$ 

Hence $Gal(k_{S'}(p)|k_S(p))$ is a free pro-$p$-group and

$$H^1(Gal(k_S(p)|k_S(p))) \to H^1_{et}((X \setminus S')_{k_S}(p)) \cong \bigoplus_{p \in S \setminus S(k_S(p))} H^1(k_S(p)_p).$$

We set $K = k_S(p)$ and consider the natural homomorphism

$$\phi : \bigotimes_{p \in S \setminus S(K)} T(K_p(p)|K_p) \to Gal(k_{S'}(p)|K).$$

By the calculation of the cohomology of a free product ([NSW, 4.3.10 and 4.1.4]), $\phi$ is a homomorphism between free pro-$p$-groups which induces an isomorphism on mod $p$ cohomology. Therefore $\phi$ is an isomorphism. In particular, $k_{S'}(p)_p = k_S(p)$ for all $p \in S \setminus S$. Using that $Gal(k_S(p)|k_S(p))$ is free, the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(Gal(k_{S'}(p)|k_S(p)), H^j_{et}((X \setminus S')_{k_S(p)})) \Rightarrow H^{i+j}_{et}((X \setminus S')_{k_S(p)}$$

induces an isomorphism

$$0 = H^2_{et}((X \setminus S')_{k_S(p)}) \cong H^2_{et}((X \setminus S')_{k_{S'}(p)})_{Gal(k_{S'}(p)|k_S)}.$$

Hence $H^2_{et}((X \setminus S')_{k_S(p)}) = 0$, since $Gal(k_{S'}(p)|k_S(p))$ is a pro-$p$-group. Now [Sch2], Proposition 2.1 implies that $X \setminus S'$ is a $K(\pi, 1)$ for $p$. 

In order to prove Theorem 1.1 we first provide the following lemma. For an extension field $K/k$ and a set of primes $T$ of $k$, we write $T(K)$ for the set of prolongations of primes in $T$ to $K$ and $\delta_K(T)$ for the Dirichlet density of the set of primes $T(K)$ of $K$.

**Lemma 2.2.** Let $k$ be a number field, $p$ a prime number and $S$ a finite set of nonarchimedean primes of $k$. Let $T$ be a set of primes of $k$ with $\delta_k(T) = 1$. Then there exists a finite subset $T_0 \subset T$ such that all primes $p \in S$ do not split completely in the extension $k_{T_0}(p)|k$.

**Proof.** By [NSW], 9.2.2(ii) or [NSW2], 9.2.3(ii), respectively, the restriction map

$$H^1(G_{T \cup S \cup S_p \cup S_{\mu}}(p), \mathbb{Z}/p\mathbb{Z}) \to \prod_{p \in S \cup S_p \cup S_{\mu}} H^1(k_p, \mathbb{Z}/p\mathbb{Z})$$

is surjective. A class in $\alpha \in H^1(G_{T \cup S \cup S_p \cup S_{\mu}}(p), \mathbb{Z}/p\mathbb{Z})$ which restricts to an unramified class $\alpha_p \in H^1_{nr}(k_p, \mathbb{Z}/p\mathbb{Z})$ for all $p \in S \cup S_p \cup S_{\mu}$ is contained in $H^1(G_T(p), \mathbb{Z}/p\mathbb{Z})$. Therefore the image of the composite map

$$H^1(G_T(p), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow H^1(G_{T \cup S \cup S_p \cup S_{\mu}}(p), \mathbb{Z}/p\mathbb{Z}) \to \prod_{p \in S} H^1(k_p, \mathbb{Z}/p\mathbb{Z})$$

is surjective.
contains the subgroup $\prod_{p \in S} H^1_{nr}(k_p, \mathbb{Z}/p\mathbb{Z})$. As this group is finite, it is already contained in the image of $H^1(G_{T_0}(p), \mathbb{Z}/p\mathbb{Z})$ for some finite subset $T_0 \subset T$. We conclude that no prime in $S$ splits completely in the maximal elementary abelian $p$-extension of $k$ unramified outside $T_0$.

**Proof of Theorems 1.1 and 1.2.** As $p \neq 2$, we may ignore archimedean primes. Furthermore, we may remove the primes in $S \cup S_p$ and all primes of degree greater than 1 from $T$. In addition, we remove all primes $p$ with $N(p) \neq 1 \mod p$ from $T$. After these changes, we still have $\delta_{k(\mu_p)}(T) = 1$.

By Lemma 2.2, we find a finite subset $T_0 \subset T$ such that no prime in $S$ splits completely in $k_{T_0}(p)/k$. Put $X = \text{Spec}(\mathcal{O}_k)$. By [Sch2], Theorem 6.2, applied to $T_0$ and $T \setminus T_0$, we find a finite subset $T_2 \subset T \setminus T_0$ such that $X \setminus (T_0 \cup T_2)$ is a $K(\pi, 1)$ for $p$. Then Proposition 2.1 applied to $T_0 \cup T_2 \subset S \cup T_0 \cup T_2$, shows that also $X \setminus (S \cup T_0 \cup T_2)$ is a $K(\pi, 1)$ for $p$. Now put $T_1 = T_0 \cup T_2 \subset T$.

It remains to show Theorem 1.2. Assertion (i) holds by construction of $T_1$. By [Sch2], Lemma 4.1, also $X \setminus (S \cup T_1)$ is a $K(\pi, 1)$ for $p$. By [Sch2], Theorem 3, the field $k_{T_1}(p)$ realizes $k_p(p)$ for $p \in T_1$, showing (ii) for these primes. Finally, assertion (ii) for $p \in S$ follows from Proposition 2.1.

### 3 Duality

We start by investigating the relation between the $K(\pi, 1)$-property and the universal norms of global units.

Let us first remove redundant primes from $S$: If $p \nmid p$ is a prime with $\zeta_p \notin k_p$, then every $p$-extension of the local field $k_p$ is unramified (see [NSW], 7.5.1 or [NSW$^2$], 7.5.9, respectively). Therefore primes $p \notin S_p$ with $N(p) \neq 1 \mod p$ cannot ramify in a $p$-extension. Removing all these redundant primes from $S$, we obtain a subset $S_{\text{min}} \subset S$, which has the property that $G_S(p) = G_{S_{\text{min}}(p)}$. Furthermore, by [Sch2], Lemma 4.1, $X \setminus S$ is a $K(\pi, 1)$ for $p$ if and only if $X \setminus S_{\text{min}}$ is a $K(\pi, 1)$ for $p$.

**Theorem 3.1.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite set of nonarchimedean primes of $k$. Then any two of the following conditions (a) – (c) imply the third.

(a) $\text{Spec}(\mathcal{O}_k) \setminus S$ is a $K(\pi, 1)$ for $p$.

(b) $\lim_{K \subset k_S(p)} \mathcal{O}_K^\times \otimes \mathbb{Z}_p = 0$.

(c) $(k_S(p))_p = k_p(p)$ for all primes $p \in S_{\text{min}}$.

The limit in (b) runs through all finite extensions $K$ of $k$ inside $k_S(p)$. If (a)–(c) hold, then also

$$\lim_{K \subset k_S(p)} \mathcal{O}_{K,S_{\text{min}}}^\times \otimes \mathbb{Z}_p = 0.$$ 

**Remarks:** 1. Assume that $\zeta_p \in k$ and $S \supset S_p$. Then (a) holds and condition (b) holds for $p > 2$ if $\#S > r_2 + 2$ (see [NSW$^2$], Remark 2 after 10.9.3). In the
case \( k = \mathbb{Q}(\zeta_p) \), \( S = S_p \), condition (b) holds if and only if \( p \) is an irregular prime number.

2. Assume that \( S \cap S_p = \emptyset \) and \( S_{\min} \neq \emptyset \). If condition (a) holds, then either \( G_S(p) = 1 \) (which only happens in very special situations, see \([\text{Sch2}], \text{Proposition 7.4}\)) or (b) holds by \([\text{Sch2}], \text{Theorem 3}\) (or by Proposition 3.2 below).

**Proof of Theorem 3.1.** We may assume \( S = S_{\min} \) in the proof. Let \( K \) run through the finite extensions of \( k \) in \( k_S(p) \) and put \( X_K = \text{Spec}(O_K) \). Applying the topological Nakayama-Lemma \((\text{NSW}, 5.2.18)\) to the compact \( \mathbb{Z}_p \)-module \( \varprojlim O^\times_K \otimes \mathbb{Z}_p \), we see that condition (b) is equivalent to

\[(b) \quad \varprojlim_{K \subset k_S(p)} O^\times_K / p = 0.\]

Furthermore, by \([\text{Sch2}], \text{Proposition 2.1}\), condition (a) is equivalent to

\[(a) \quad \varprojlim_{K \subset k_S(p)} H^i_{et}(X_K, V) = 0 \text{ for } i \geq 1.\]

Condition (a)' always holds for \( i = 1, i \geq 4 \), and it holds for \( i = 3 \) provided that \( G_S(p) \) is infinite or \( S \) is nonempty or \( \zeta_p \notin k \) (see \([\text{Sch2}], \text{Lemma 3.7}\) ). The flat Kummer sequence \( 0 \rightarrow \mu_p \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0 \) induces exact sequences

\[0 \rightarrow O^\times_K / p \rightarrow H^1_{et}(X_K, \mu_p) \rightarrow \varphi_{p}(X) \rightarrow 0\]

for all \( K \). As the field \( k_S(p) \) does not have nontrivial unramified \( p \)-extensions, class field theory implies

\[\varprojlim_{K \subset k_S(p)} \varphi_{p}(X_K) \subset \varprojlim_{K \subset k_S(p)} \varphi_{p}(X_K) \otimes \mathbb{Z}_p = 0.\]

As we assumed \( k \) to be totally imaginary if \( p = 2 \), the flat duality theorem of Artin-Mazur \((\text{MI}, \text{III Corollary 3.2})\) induces natural isomorphisms

\[H^2_{et}(X_K, \mathbb{Z}/p\mathbb{Z}) = H^2_{fl}(X_K, \mathbb{Z}/p\mathbb{Z}) \cong H^1_{et}(X_K, \mu_p)^\vee.\]

We conclude that

\[\varprojlim_{K \subset k_S(p)} H^2_{et}(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \left( \varprojlim_{K \subset k_S(p)} O^\times_K / p \right)^\vee.\]

We first show the equivalence of (a) and (b) in the case \( S = \emptyset \). If (a)' holds, then (a) shows (b)'. If (b) holds, then \( \zeta_p \notin k \) or \( G_S(p) \) is infinite. Hence we obtain (a)' for \( i = 3 \). Furthermore, (b)' implies (a)' for \( i = 2 \) by (a)'. This finishes the proof of the case \( S = \emptyset \).

Now we assume that \( S \neq \emptyset \). For \( p \in S(K) \), a standard calculation of local cohomology shows that

\[H^i_{fl}(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} H^1(K_p, \mathbb{Z}/p\mathbb{Z}) / H^1_{nr}(K_p, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 1, \\ H^2(K_p, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 3, \\ 0 & \text{for } i \geq 4. \end{cases}\]
For \( p \in S = S_{\text{min}} \), every proper Galois subextension of \( k_p(p)|k_p \) admits ramified \( p \)-extensions. Hence condition (c) is equivalent to

\[
(c)\' \quad \lim_{K \subset k_S(p)} \bigoplus_{p \in S(k)} H^i_p(X_K, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for all } i,
\]

and to

\[
(c)'' \quad \lim_{K \subset k_S(p)} \bigoplus_{p \in S(k)} H^2_p(X_K, \mathbb{Z}/p\mathbb{Z}) = 0.
\]

Consider the direct limit over all \( K \) of the excision sequences

\[
\cdots \to \bigoplus_{p \in S(K)} H^i_p(X_K, \mathbb{Z}/p\mathbb{Z}) \to H^i_{\text{et}}(X_K, \mathbb{Z}/p\mathbb{Z}) \to H^i_{\text{et}}((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) \to \cdots.
\]

Assume that (a)' holds, i.e. the right hand terms vanish in the limit for \( i \geq 1 \). Then (a)' shows that (b)' is equivalent to (c)''.

Now assume that (b) and (c) hold. As above, (b) implies the vanishing of the middle term for \( i = 2, 3 \) in the limit. Condition (c)' then shows (a)'.

We have proven that any two of the conditions (a)–(c) imply the third.

Finally, assume that (a)–(c) hold. Tensoring the exact sequences (cf. [NSW], 10.3.11 or [NSW-2], 10.3.12, respectively)

\[
0 \to \mathcal{O}^X_K \to \mathcal{O}^X_{K,S} \to \bigoplus_{p \in S(K)} (K^P_{\mathbb{Z}}/U_p) \to \text{Pic}(X_K) \to \text{Pic}((X \setminus S)_K) \to 0
\]

by (the flat \( \mathbb{Z} \)-algebra) \( \mathbb{Z}_p \), we obtain exact sequences of finitely generated, hence compact, \( \mathbb{Z}_p \)-modules. Passing to the projective limit over the finite extensions \( K \) of \( k \) inside \( k_S(p) \) and using \( \lim_{\leftarrow} \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0 \), we obtain the exact sequence

\[
0 \to \lim_{K \subset k_S(p)} \mathcal{O}^X_K \otimes \mathbb{Z}_p \to \lim_{K \subset k_S(p)} \mathcal{O}^X_{K,S} \otimes \mathbb{Z}_p \to \lim_{K \subset k_S(p)} \bigoplus_{p \in S(K)} (K^P_{\mathbb{Z}}/U_p) \otimes \mathbb{Z}_p \to 0.
\]

Condition (c) and local class field theory imply the vanishing of the right hand limit. Therefore (b) implies the vanishing of the projective limit in the middle.

If \( G_S(p) \neq 1 \) and condition (a) of Theorem 1.1 holds, then the failure in condition (c) can only come from primes dividing \( p \). This follows from the next

**Proposition 3.2.** Let \( k \) be a number field and let \( p \) be a prime number. Assume that \( k \) is totally imaginary if \( p = 2 \). Let \( S \) be a finite set of nonarchimedean primes of \( k \). If \( \text{Spec}(\mathcal{O}_k) \setminus S \) is a \( K(\pi, 1) \) for \( p \) and \( G_S(p) \neq 1 \), then every prime \( \mathfrak{p} \in S \) with \( \zeta_\mathfrak{p} \in k_{\mathfrak{p}} \) has an infinite inertia group in \( G_S(p) \). Moreover, we have

\[
k_S(p)_{\mathfrak{p}} = k_p(p)
\]

for all \( \mathfrak{p} \in S_{\text{min}} \setminus S_p \).
Proof. We may assume $S = S_{\min}$. Suppose $p \in S$ with $\zeta_p \in k_p$ does not ramify in $k_S(p)/k$. Setting $S' = S \setminus \{p\}$, we have $k_{S'}(p) = k_S(p)$, in particular,

$$H^2_{\et}(X \setminus S', \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^2_{\et}(X \setminus S, \mathbb{Z}/p\mathbb{Z}).$$

In the following, we omit the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation. Using the vanishing of $H^2_{\et}(X \setminus S)$, the étale excision sequence yields a commutative exact diagram

$$
\begin{array}{ccc}
H^2(G_S(p)) & \xrightarrow{\sim} & H^2(G_S(p)) \\
\downarrow & & \downarrow 1 \\
H^2_p(X) & \xrightarrow{\alpha} & H^2_{\et}(X \setminus S) \xrightarrow{\sim} H^3_{\et}(X) \xrightarrow{\sim} H^3_{\et}(X \setminus S').
\end{array}
$$

Hence $\alpha$ is split-surjective and $\mathbb{Z}/p\mathbb{Z} \cong H^3_p(X) \xrightarrow{\sim} H^3_{\et}(X \setminus S')$. This implies $S' = \emptyset$, hence $S = \{p\}$, and $\zeta_p \in k$. The same applies to every finite extension of $k$ in $k_S(p)$, hence $p$ is inert in $k_S(p) = k_\emptyset(p)$. This implies that the natural homomorphism

$$\text{Gal}(k_p^\text{nr}(p)/k_p) \longrightarrow G_\emptyset(k)(p)$$

is surjective. Therefore $G_S(p) = G_\emptyset(p)$ is abelian, hence finite by class field theory. Since this group has finite cohomological dimension by the $K(\pi,1)$-property, it is trivial, in contradiction to our assumptions.

This shows that all $p \in S$ with $\zeta_p \in k_p$ ramify in $k_S(p)$. As this applies to every finite extension of $k$ inside $k_S(p)$, the inertia groups must be infinite. For $p \in S_{\min} \setminus S_p$ this implies $k_S(p)_p = k_p(p)$.

\begin{theorem}
Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite nonempty set of nonarchimedean primes of $k$. Assume that conditions (a)–(c) of Theorem 3.1 hold and that $\zeta_p \in k_p$ for all $p \in S$. Then $G_S(p)$ is a pro-$p$ duality group of dimension 2.
\end{theorem}

Proof. Condition (a) implies $H^3(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^3_{\et}(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0$. Hence $cd G_S(p) \leq 2$. On the other hand, by (c), the group $G_S(p)$ contains $\text{Gal}(k_p^\text{nr}(p)/k_p)$ as a subgroup for all $p \in S$. As $\zeta_p \in k_p$ for $p \in S$, these local groups have cohomological dimension 2, hence so does $G_S(p)$.

In order to show that $G_S(p)$ is a duality group, we have to show that

$$D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) := \lim_{U \subset G_S(p)} H^1(U, \mathbb{Z}/p\mathbb{Z})^\vee$$

vanish for $i = 0, 1$, where $U$ runs through the open subgroups of $G_S(p)$ and the transition maps are the duals of the corestriction homomorphisms; see [NSW], 3.4.6. The vanishing of $D_0$ is obvious, as $G_S(p)$ is infinite. Using (a), we therefore have to show that

$$\lim_{K \subset k_S(p)} H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee = 0.$$
We put $X = \text{Spec}(\mathcal{O}_K)$ and denote the embedding by $j : (X \setminus S)_K \to X_K$. By the flat duality theorem of Artin-Mazur, we have natural isomorphisms

$$H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee \cong H^2_{\text{fl}}((X \setminus S)_K, \mu_p) = H^2_{\text{fl}}(X_K, j_!\mu_p).$$

The excision sequence together with a straightforward calculation of local cohomology groups shows an exact sequence

$$(*) \quad \bigoplus_{p \in S(K)} K^*_p / K^*_p \to H^2_{\text{fl}}(X_K, j_!\mu_p) \to H^2_{\text{fl}}((X \setminus S)_K, \mu_p).$$

As $\zeta_p \in k_p$ and $k_S(p)_p = k_p(p)$ for $p \in S$ by assumption, the left hand term of $(*)$ vanishes when passing to the limit over all $K$. We use the Kummer sequence to obtain an exact sequence

$$(**) \quad \text{Pic}((X \setminus S)_K)/p \to H^2_{\text{fl}}((X \setminus S)_K, \mu_p) \to p\text{Br}((X \setminus S)_K).$$

The left hand term of $(**)$ vanishes in the limit by the principal ideal theorem. The Hasse principle for the Brauer group induces an injection $p\text{Br}((X \setminus S)_K) \hookrightarrow \bigoplus_{p \in S(K)} p\text{Br}(K_p)$.

As $k_S(p)$ realizes the maximal unramified $p$-extension of $k_p$ for $p \in S$, the limit of the middle term in $(**)$, and hence also the limit of the middle term in $(*)$ vanishes. This shows that $G_S(p)$ is a duality group of dimension 2.

**Remark:** The dualizing module can be calculated to

$$D \cong \text{tor}_p(C_S(k_S(p))),$$

i.e. $D$ is isomorphic to the $p$-torsion subgroup in the $S$-idèle class group of $k_S(p)$. The proof is the same as in ([Sch1], Proof of Thm. 5.2), where we dealt with the tame case.

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Alexander Schmidt, NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Deutschland. email: alexander.schmidt@mathematik.uni-regensburg.de