A strong law of large numbers for simultaneously testing parameters of Lancaster bivariate distributions

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Abstract

We prove a strong law of large numbers for simultaneously testing parameters of a large number of dependent, Lancaster bivariate random variables with infinite supports, and discuss its implications.

Keywords: False discovery proportion, Lancaster bivariate distributions, orthogonal polynomials, strong law of large numbers.

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1 Introduction

Multiple hypothesis testing with false discovery rate (FDR, [2]) control has been widely applied to various scientific endeavors, and it can often be stated as follows. There are \( m \in \mathbb{N} \) test statistics \( \{\zeta_i\}_{i=1}^m \) such that \( \zeta_i \) has parameter \( \mu_i \), and the \( i \)th null hypothesis is \( H_{i0} : \mu_i = \mu_0 \) (versus its alternative hypothesis \( H_{i1} : \mu_i \neq \mu_0 \)) for a fixed, known \( \mu_0 \in \Theta \subseteq \mathbb{R} \), where \( \Theta \) is the parameter space for the \( m \) \( \zeta_i \)'s. Define \( p_i = 1 - F_i(\zeta_i) \) as the one-sided p-value for \( \zeta_i \), where \( F_i \) is the cumulative distribution function (CDF) of \( \zeta_i \) when \( \mu_i = \mu_0 \). Let \( I_{0,m} \) be the set of indices of the true null hypotheses, and denote its cardinality (often being positive) by \( m_0 \). Consider the multiple testing procedure (MTP) with a fixed rejection threshold \( t \in [0,1] \) that rejects \( H_{i0} \) if and only if \( (\text{iff}) \) \( p_i \leq t \). Then the MTP induces \( R_m(t) = \sum_{i=1}^m 1 \{p_i \leq t\} \), the number of rejections, and \( V_m(t) = \sum_{i \in I_{0,m}} 1 \{p_i \leq t\} \), the number of false discoveries, where \( 1A \) is the indicator function of a set \( A \). Further, the false discovery proportion (FDP) and FDR of the MTP are

\[
\text{FDP}_m(t) = \frac{V_m(t)}{R_m(t) \lor 1} \quad \text{and} \quad \text{FDR}_m(t) = \mathbb{E} \left[ \text{FDP}_m(t) \right]
\]

respectively, where the operator \( \lor \) returns the maximum of its two arguments. When \( m \), the number of tests to conduct, is large, we aim to control the FDR of the MTP at a given level \( \theta \in (0,1) \) by choosing an appropriate \( t \) or to estimate the FDP or FDR of the MTP at a given threshold \( t \).

However, the test statistics \( \{\zeta_i\}_{i=1}^m \) are often dependent on each other, and under dependence the behavior of the FDP is usually unstable and can sometimes be unpredictable; see, e.g., [6],

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This can make irreproducible and untrustable the inferential results from the MTP. The very few works of [1], [3], [4] and [5] studied the asymptotic behavior of $R_m(t)$ or $m^{-1}R_m(t)$ under dependence by utilizing conditions on the correlation matrix $R = (\rho_{ij})$ of $\zeta = (\zeta_1, \ldots, \zeta_m)$. However, they all considered the setting where each $\zeta_i$ is a Gaussian random variable. Specifically, when each pair $(\zeta_i, \zeta_j), i \neq j$ is bivariate Gaussian, the authors of [3] proved “a SLLN for $R_m(t)$ and $V_m(t)$”, i.e.,

C1) If

$$m^{-2} \|R\|_1 = O(m^{-\delta}) \text{ for some } \delta > 0,$$

then

$$\lim_{m \to \infty} \left| m^{-1}R_m(t) - \mathbb{E}[m^{-1}R_m(t)] \right| = 0 \text{ almost surely},$$

$$\lim_{m \to \infty} \left| m^{-1}V_m(t) - \mathbb{E}[m^{-1}V_m(t)] \right| = 0 \text{ almost surely}. \quad (3)$$

C2) If $\lim \inf_{m \to \infty} m_0m^{-1} > 0$ and (2) hold, then

$$\lim_{m \to \infty} \left| m_0^{-1}V_m(t) - \mathbb{E}[m_0^{-1}V_m(t)] \right| = 0 \text{ almost surely}. \quad (4)$$

Here “the $l_1$-norm $\|R\|_1$” of $R$ is defined as $\|R\|_1 = \sum_{i,j=1}^{m} |\rho_{ij}|$. We remark that, even though the assertion (4) is not explicitly stated by Theorem 1 of [3], it is written in the proof of this theorem.

As a SLLN is perhaps the strongest characterization of the stability of a sequence random variables, in this work we continue the line of research of [3], and characterize the type of dependence (via the order of $\|R\|_1$) under which (3) and (4) hold when $(\zeta_i, \zeta_j), i \neq j$ follows a Lancaster (but non-Gaussian) bivariate distribution with an infinite support. It turns out that the strategy of [3] applies to the settings here. Specifically, to prove (3) we only need to implement the following two steps: first, obtain a “comparison inequality”, i.e.,

$$|\text{cov}(1\{p_i \leq t\}, 1\{p_j \leq t\})| \leq C|\rho_{ij}| \text{ for all } i \neq j \text{ and a constant } C > 0; \quad (5)$$

second, apply Theorem 1 of [9], under the condition (2), to the indicators $X_i = 1\{p_i \leq t\}$ with $1 \leq i \leq m$ (or $i \in I_{0,m}$) that induces $R_m(t)$ (or $V_m(t)$). Once (3) is proved and $\lim \inf_{m \to \infty} m_0m^{-1} > 0$ holds, (4) follows as an easy corollary.

Our main result is the following:

**Theorem 1** Suppose that each pair $(\zeta_i, \zeta_j), i \neq j$ follows any of the following four Lancaster bivariate distributions with correlation $\rho_{ij}$:

1. A Lancaster bivariate gamma distribution (defined by (12)) with shape parameter $\alpha \in (0, 1]$;
2. A Lancaster bivariate Poisson distribution (defined by (14)) with parameter $\alpha > 0$;
3. A Lancaster negative binomial distribution (defined by (15)) with parameter $(\beta, c)$ such that $\beta > 0$ and $0 < c < 1$;
4. A Lancaster bivariate exponential distribution (defined by (16)) with parameter $\beta > 0$.

Here $I_{0,m} = \{1, 2, \ldots, m\}$ and $m \geq 2$. Theorem 1 is illustrated in [9].
4. A Lancaster bivariate gamma-negative binomial distribution (defined by (16)) with parameter \((\beta, c, \alpha)\) such that \(\beta > 0, 0 < c < 1\) and \(\alpha > 0\).

Then (5) holds. If (2) holds, then (3) holds. If in addition \(\lim_{m \to \infty} m_0m^{-1} > 0\), then (4) holds.

The definitions of the four Lancaster bivariate distributions covered by Theorem 1 can be found in [8] and will be provided in the proof of this theorem. Our findings seem to suggest that the inequality (5) is universal for Lancaster bivariate distributions with infinite supports. On the other hand, the Lancaster distributions considered by Theorem 1 are often associated with the true null hypotheses in a multiple testing scenario. For example, the Lancaster bivariate gamma distribution includes the Lancaster bivariate central chi-square distribution as a special case, and the latter distribution corresponds to the true null hypothesis that its two marginal distributions have a zero centrality parameter and the same degrees of freedom. Further, bivariate Poisson or bivariate negative binomial distributions are widely used to model count data, and the Lancaster bivariate Poisson or negative binomial distribution corresponds to the true null hypothesis that its two marginal distributions have identical parameters.

In view of the above discussion, Theorem 1 has the following implication. Consider the slightly extended multiple testing scenario, where

- There are \(\tilde{m} (\geq m)\) null hypotheses, \(H_{i0} : \mu_i = \mu_0\) with \(1 \leq i \leq \tilde{m}\), to test simultaneously, each of which has an associated test statistic \(\zeta_i\);
- Each \(H_{i0}\) with \(1 \leq i \leq m\) is a true null hypothesis, and the rest \(\tilde{m} - m\) null hypotheses are false;
- The MTP rejects an \(H_{i0}\) iff its associated p-value \(p_i \leq t\) for a fixed rejection threshold \(t \in (0, 1)\).

Note that the above arrangement of the indices for the true and false null hypotheses is unrestrictive. In this setting, the number of false rejections of the MTP is \(V_{\tilde{m}} (t) = \sum_{i=1}^{\tilde{m}} 1 \{p_i \leq t\}\), and the FDP of the MTP is

\[
\text{FDP}_{\tilde{m}} (t) = \frac{V_{\tilde{m}} (t)}{R_{\tilde{m}} (t) \vee 1} \quad \text{with} \quad R_{\tilde{m}} (t) = \sum_{i=1}^{\tilde{m}} 1 \{p_i \leq t\}.
\]

Let \(S\) be the correlation matrix of \(\{\zeta_i\}_{i=1}^{\tilde{m}}\) and \(\pi_{0, \tilde{m}} = m^{-1} \tilde{m}\). When \(\lim_{m \to \infty} \pi_{0, \tilde{m}} > 0\),

\[
\tilde{m}^{-2} \|S\|_1 = O(\tilde{m}^{-\delta}) \quad \text{for some} \ \delta > 0
\]

and the \(p_i\)'s associated with \(H_{i0}\) for \(1 \leq i \leq m\) are identically distributed as \(p_0\), Theorem 1 implies

\[
\lim_{m \to \infty} \left| m^{-1} V_{\tilde{m}} (t) - \mathbb{P} (\{p_0 \leq t\}) \right| = 0 \text{ almost surely.} \tag{7}
\]
Let \( \hat{\pi}_{0,\hat{m}} \) be an estimator of \( \pi_{0,\hat{m}} \) and set
\[
\vartheta_{\hat{m}}(t) = \frac{\hat{\pi}_{0,\hat{m}} \mathbb{P} \left( \{ p_0 \leq t \} \right)}{m^{-1}R_{\hat{m}}(t)}.
\] (8)

It is easy to verify that, if \( \hat{\pi}_{0,\hat{m}} \mathbb{P} \left( \{ p_0 \leq t \} \right) \sim 1 \) as \( \hat{m} \to \infty \), \( \liminf_{\hat{m} \to \infty} \hat{m}^{-1}R_{\hat{m}}(t) > 0 \) almost surely and (6) holds, then \( |\vartheta_{\hat{m}}(t) - \text{FDP}_{\hat{m}}(t)| \sim 0 \) as \( \hat{m} \to \infty \), where \( \sim \) denotes “convergence in probability”. Namely, \( \vartheta_{\hat{m}}(t) \) consistently estimates \( \text{FDP}_{\hat{m}}(t) \) for each fixed \( t \in (0,1) \). Note that \( \vartheta_{\hat{m}}(t) \) in (8) can be regarded as a slight extension of the FDR estimator proposed by [12].

A second implication of Theorem 1 is as follows. The “weak dependence” assumption, proposed in [12] and widely used in the multiple testing literature, requires that there exist two continuous functions \( G_0 \) and \( G_1 \) such that for each \( t \in (0,1) \),
\[
\lim_{m \to \infty} m_0^{-1}V_m(t) = G_0(t) \quad \text{and} \quad \lim_{m \to \infty} (m - m_0)^{-1}[R_m(t) - V_m(t)] = G_1(t)
\] (9)
almost surely. However, to check whether (9) holds is often very hard (even after the continuity requirement on \( G_0 \) and \( G_1 \) is removed). Theorem 1 here and Theorem 1 in [3] together provide a way to check whether this assumption holds in the scenario of simultaneously testing the parameters of a larger number of dependent random variables, each pair of which follows any of the five Lancaster bivariate distributions that are studied in [8]. Specifically, a check on the order of the \( l_1 \)-norm of the correlation matrix of these random variables suffices for this purpose. We will report in another article on how to consistently estimate \( m^{-2} \| \mathbf{R} \|_1 \) or efficiently test the order of \( \| \mathbf{R} \|_1 \).

The rest of the article is devoted to the proof of Theorem 1.

## 2 Proof of Theorem 1

In the proof, \( \mathbb{V}[ \cdot ] \) and \( \text{cov}[\cdot,\cdot] \) are the variance and covariance operators, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and \( C \) denotes a positive constant that can assume different (and appropriate) values at different occurrences. We need Theorem 1 of [9] in the proof, which reads “Let \( \{\chi_n\}_{n=1}^{\infty} \) be a sequence of complex-valued random variables such that \( \mathbb{E}[|\chi_n|^2] \leq 1 \). Set \( Q_N = N^{-1}\sum_{n=1}^{N}\chi_n \). If \( |\chi_n| \leq 1 \) a.s. and
\[
\sum_{N=1}^{\infty} N^{-1}\mathbb{E}[|Q_N|^2] < \infty,
\] (10)
then \( \lim_{N \to \infty} Q_N = 0 \) a.s.” A sufficient condition for the SLLN to hold for \( \{\chi_n\}_{n=1}^{\infty} \) is that \( \mathbb{E}[|Q_m|^2] = O(m^{-\delta}) \) for some \( \delta > 0 \), which implies (10).

Now we present the arguments. Recall \( X_i = 1 \{p_i \leq t\} \), for which \( R_m(t) = \sum_{i=1}^{m}X_i \) and \( V_m(t) = \sum_{i \in I_{m,\delta}}X_i \). We aim to show that \( \mathbb{V}[m^{-1}R_m(t)] \) satisfies \( O(m^{-\delta'}) \) with \( \delta' = \min\{\delta,1\} \).

Define two sets
\[
E_{1,m} = \{(i,j) : 1 \leq i,j \leq m, i \neq j, |\rho_{ij}| = 1\}
\]
and

\[ E_{2,m} = \{(i, j) : 1 \leq i, j \leq m, i \neq j, |\rho_{ij}| < 1\}. \]

Namely, \( E_{2,m} \) records pairs \((\zeta_i, \zeta_j)\) with \( i \neq j \) such that \( \zeta_i \) and \( \zeta_j \) are linearly dependent almost surely. Obviously, \( |\text{cov}(X_i, X_j)| \leq C = C |\rho_{ij}| \) for \( (i, j) \in E_{2,m} \). Further,

\[
\mathbb{V}[m^{-1}R_m(t)] \leq O\left(m^{-1}\right) + m^{-2} \sum_{(i,j) \in E_{2,m}} |\text{cov}(X_i, X_j)| + m^{-2} \sum_{(i,j) \in E_{1,m}} |\text{cov}(X_i, X_j)|
\]

\[
\leq O\left(m^{-\min\{\delta, 1\}}\right) + m^{-2} \sum_{(i,j) \in E_{1,m}} |\text{cov}(X_i, X_j)|
\]

(11)

since

\[
m^{-2} \sum_{(i,j) \in E_{2,m}} |\text{cov}(X_i, X_j)| = O\left(m^{-2} \|R\|_1\right) = O\left(m^{-\delta}\right).
\]

So, we only need to upper bound \( B_{1,m} = m^{-2} \sum_{(i,j) \in E_{1,m}} |\text{cov}(X_i, X_j)| \) on the right-hand side of (11).

On the other hand,

\[
\mathbb{V}[m^{-1}V_m(t)] \leq O\left(m^{-1}\right) + m^{-2} \sum_{(i,j) \in \tilde{E}_{2,m}} |\text{cov}(X_i, X_j)| + m^{-2} \sum_{(i,j) \in \tilde{E}_{1,m}} |\text{cov}(X_i, X_j)|
\]

\[
\leq Cm^{-1} + Cm^{-\delta} + CB_{1,m},
\]

where \( \tilde{E}_{k,m} = E_{k,m} \cap (I_{0,m} \times I_{0,m}) \) for \( k \in \{1, 2\} \). So, an upper bound on \( B_{1,m} \) will induce the same upper bound for \( \mathbb{V}[m^{-1}R_m(t)] \) and \( \mathbb{V}[m^{-1}V_m(t)] \).

We will split the rest of the proof into four cases in terms of upper bounding \( B_{1,m} \), each corresponding to a Lancaster bivariate distribution in the statement of Theorem 1 and each occupying a subsection.

2.1 The Lancaster bivariate gamma distribution

The Lancaster bivariate gamma distribution was derived by [7] and [8]. Specifically, if \((X, Y)\) follows this distribution with shape parameter \( \alpha > 0 \) and correlation \( \rho \in [0, 1) \), then its density is

\[
h(x, y; \alpha, \rho) = f(x; \alpha) f(y; \alpha) \sum_{n=0}^{\infty} \frac{\rho^n n!}{\Gamma(\alpha + n) \Gamma(\alpha)} L_{n}^{(\alpha-1)}(x) L_{n}^{(\alpha-1)}(y),
\]

(12)

where

\[
f(x; \alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad \text{for} \quad x > 0
\]

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is the gamma density with shape parameter $\alpha > 0$, and

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \frac{(-x)^k}{k!} \text{ for } n \in \mathbb{N}_0$$

is the $n$th Laguerre polynomial of order $\alpha > 0$.

Let $\tau = F^{-1}_i(1 - t)$. If $(\zeta_i, \zeta_j)$ with $(i, j) \in E_{2,m}$ follows a Lancaster bivariate gamma distribution with shape parameter $\alpha > 0$ and correlation $\rho_{ij} \in [0, 1)$, then

$$\kappa_{ij} = \text{cov}(1\{p_i \leq t\}, 1\{p_j \leq t\}) = \sum_{n=1}^{\infty} \frac{\rho_{ij}^n n!}{\Gamma(\alpha + n) \Gamma(\alpha)} q_n^{(\alpha)}(\tau; \alpha),$$

where

$$q_n(\tau; \alpha) = \int_{-\infty}^{\tau} f(x; \alpha) L_n^{(\alpha-1)}(x) dx = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\tau} x^{\alpha-1} e^{-x} L_n^{(\alpha-1)}(x) dx.$$  

From the Rodrigue’s formula (e.g., on page 101 of [13]), i.e.,

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{x} \frac{d^n}{dx^n} \left( x^{n+\alpha} e^{-x} \right) \text{ for } n \in \mathbb{N}_0 \text{ and } \alpha > -1,$$

we obtain, for $y > 0$ and $n \geq 1$,

$$\int_{-\infty}^{y} x^\alpha e^{-x} L_n^{(\alpha)}(x) dx = \frac{1}{n!} \int_{-\infty}^{y} \left[ \frac{d^n}{dx^n} \left( x^{n+\alpha} e^{-x} \right) \right] dx$$

$$= \frac{y^{\alpha+1} e^{-y} y^{-(\alpha+1)} e^{y}}{n (n-1)!} \left[ \frac{d^{n-1}}{dx^{n-1}} \left( x^{n+\alpha} e^{-x} \right) \right]_{x=y}$$

$$= \frac{y^{\alpha+1} e^{-y}}{n} L_n^{(\alpha+1)}(y).$$

Therefore,

$$\kappa_{ij} = \sum_{n=1}^{\infty} \frac{\rho_{ij}^n n!}{n^2 \Gamma(\alpha + n) \Gamma(\alpha) ^2} \left[ \tau^\alpha e^{-\tau} L_n^{(\alpha)}(\tau) \right]^2.$$

By Watson’s bound on page 21 of [15], i.e.,

$$|L_n^{(\alpha)}(x)| \leq \frac{\Gamma(\alpha + 1 + n)}{\Gamma(\alpha + 1) n!} e^{x^2/2} \text{ for } x \geq 0, \alpha \geq 0 \text{ and } n \in \mathbb{N}_0,$$

we obtain

$$|\kappa_{ij}| \leq C \sum_{n=1}^{\infty} \frac{\rho_{ij}^n \Gamma(\alpha + n)}{n!} \tau^{2\alpha} e^{-\tau}.$$

However, the identity (1) in [14] states that, for distinct real constants $\alpha$ and $\gamma$,

$$\frac{\Gamma(z + \alpha)}{\Gamma(z + \gamma)} = z^{\alpha - \gamma} \left[ 1 + \frac{\alpha - \gamma}{2z} (\alpha + \gamma - 1) + O \left( \frac{1}{z^2} \right) \right] \text{ as } |z| \to \infty.$$

(13)
So, when \( \alpha \leq 1 \),

\[
|\kappa_{ij}| \leq C \sum_{n=1}^{\infty} \frac{\rho_{ij}^n}{n^{1-\alpha}} = C \rho_{ij} \sum_{n=1}^{\infty} \frac{\rho_{ij}^{n-1}}{n^{1-\alpha}} \leq C \rho_{ij},
\]

and (3) holds.

**Remark 1** If \( \zeta_i \) is the central chi-square random variable with \( v \) degrees of freedom and density

\[
f(x; v/2) = \frac{1}{\Gamma(v/2) 2^{v/2}} x^{v/2-1} e^{-x/2} \text{ for } x > 0,
\]

then Theorem 1 is valid when \( v = 1 \) or \( 2 \).

### 2.2 The Lancaster bivariate Poisson distribution

For \( a > 0 \) and \( x, n \in \mathbb{N}_0 \), let

\[
C_n(x; a) = \sqrt{\frac{a^n}{n!}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^k}{k!} x^k
\]

denote the Charlier polynomial of degree \( n \), where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) if \( n \geq k \) and \( \binom{n}{k} = 0 \) if \( n < k \).

The Lancaster bivariate Poisson distribution was derived by [8]. Specifically, if \((X, Y)\) follows such a distribution with parameter \( a > 0 \) and correlation \( \lambda \in [0, 1] \), then it has density

\[
h(x, y; a, \rho) = f(x; a) f(y; a) \sum_{n=0}^{\infty} \rho^n C_n(x; a) C_n(y; a) \text{ for } x, y \in \mathbb{N}_0,
\]

where

\[
f(x; a) = \frac{a^x e^{-a}}{x!} \text{ for } x \in \mathbb{N}_0
\]

is the probability mass function (PMF) for a Poisson random variable with mean \( a \).

Set \( \tau = F_i^{-1}(1 - t) \), and let \( x_0 \) be the integer part of \( \tau \). If \((\zeta_i, \zeta_j)\) with \((i, j) \in E_{2,m}\) follows a Lancaster bivariate Poisson distribution with correlation \( \rho_{ij} \in [0, 1] \), then

\[
\kappa_{ij} = \text{cov} \{1 \{p_i \leq t\}, 1 \{p_j \leq t\} \} = \sum_{n=1}^{\infty} \rho_{ij}^n q_n^2 (x_0; a),
\]

where

\[
q_n (x_0; a) = \sum_{x=0}^{x_0} f(x; a) C_n(x; a) = \sqrt{\frac{a^n}{n!}} \sum_{x=0}^{x_0} \frac{a^x e^{-a}}{x!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a^k}{k!} x^k
\]

It suffices to bound \( q_n (x_0; a) \). Specifically,

\[
|q_n (x_0; a)| \leq \sqrt{\frac{a^n}{n!}} \sum_{x=0}^{x_0} \frac{a^x e^{-a}}{x!} \sum_{k=0}^{n} \frac{a^{-k}}{(x-k)!} \binom{n}{k} \leq C \sqrt{\frac{a^n}{n!} (1 + a^{-1})^n}.
\]
So,

$$|\kappa_{ij}| \leq C \sum_{n=1}^{\infty} \frac{a^n \rho_{ij}^n}{n!} (1 + a^{-1})^{2n} \leq C \rho_{ij}$$

and (3) holds.

### 2.3 The Lancaster bivariate negative binomial distribution

Let $\beta > 0$ and $0 < c < 1$, and $M_{n}^{\beta,c}(x)$ denote the $n$th (normalized) Meixner polynomial, i.e.,

$$M_{n}^{\beta,c}(x) = \sqrt{c_n(\beta)} \frac{n!}{\prod_{k=0}^{n} (\beta)_k} \sum_{k=0}^{n} \frac{(-n)_k (-x)_k}{(\beta)_k k!} (1 - c^{-1})^k \text{ for } x \in \mathbb{N}_0.$$  

Here $(a)_n = \prod_{k=0}^{n-1} (a + k)$ for $a \in \mathbb{R}$ and $n \in \mathbb{N}$, and $(-x)_k = 0$ is set when $x < k$. The Lancaster bivariate negative binomial distribution with parameter $(\beta, c)$ and correlation $\rho \in [0, 1)$ was derived by [8]. Specifically, if $(X, Y)$ follows such a distribution, then it has density

$$h(x, y; \beta, c) = f(x; \beta, c) f(y; \beta, c) \sum_{n=0}^{\infty} \rho^n M_{n}^{\beta,c}(x) M_{n}^{\beta,c}(y) \text{ for } x, y \in \mathbb{N}_0 \text{ and } 0 \leq \rho < 1, \quad (15)$$

where

$$f(x; \beta, c) = (1 - c)^\beta \frac{c^x (\beta)_x}{x!} \text{ for } x \in \mathbb{N}_0$$

is the PMF for a negative binomial random variable.

Set $\tau = F_i^{-1}(1 - t)$, and let $x_0$ be the integer part of $\tau$. If $(\zeta_i, \zeta_j)$ with $(i, j) \in E_{2,m}$ follows a Lancaster bivariate negative binomial distribution with parameter $(\beta, c)$ and correlation $\rho_{ij} \in [0, 1)$, then

$$\kappa_{ij} = \text{cov} \left\{ \{ p_i \leq t \}, \{ p_j \leq t \} \right\} = \sum_{n=1}^{\infty} \rho_{ij}^n q_n^2 (x_0; \beta, c),$$

where

$$q_n (x_0; \beta, c) = \sum_{x=0}^{x_0} f(x; \beta, c) M_{n}^{\beta,c}(x)$$

$$= \sqrt{\frac{c^n(\beta)_n}{n!}} \sum_{x=0}^{x_0} (1 - c)^\beta \frac{c^x (\beta)_x}{x!} \sum_{k=0}^{n} \frac{(-n)_k (-x)_k}{(\beta)_k k!} (1 - c^{-1})^k.$$  

It suffices to bound $q_n (x_0; \beta, c)$. Specifically,

$$|q_n (x_0; \beta, c)| \leq C \sqrt{\frac{c^n(\beta)_n}{n!}} \sum_{k=0}^{x_0} \frac{n (n-1) \cdots (n-k+1)}{(\beta)_k k!} |1 - c^{-1}|^k$$

$$\leq C \sqrt{\frac{c^n(\beta)_n}{n!}} x_0 n^{x_0} \leq C e^{n/2} n^{(\beta-1+2x_0)/2},$$

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where we have applied the identity (13) to obtain the last inequality. Since $0 < c < 1$, we have

$$|\kappa_{ij}| \leq C \sum_{n=1}^{\infty} \rho_{ij}^n c^n n^{\beta-1+2x_0} \leq C \rho_{ij}.$$ 

So, (3) holds.

### 2.4 The Lancaster bivariate gamma-negative binomial distribution

Let $\sigma > 0$, $\beta > 0$ and $0 < c < 1$ be three constants. The Lancaster bivariate gamma-negative binomial distribution was derived by [8]. Specifically, if $(X, Y)$ follows such a distribution with parameter $(\alpha, \beta, c)$ and correlation $\rho \in [0, \sqrt{c}]$, then it has density

$$h(x, y; \alpha, \beta, c) = f(x; \beta, c) g(y; \alpha) \sum_{n=0}^{\infty} \rho^n \sqrt{\frac{n!}{(\alpha)_n}} L_n^{(\alpha-1)}(x) M_n^{\beta,c}(y) \quad (16)$$

for $x \in \mathbb{N}_0$ and $y > 0$, for which $X$ is a negative binomial random variable with PMF

$$f(x; \beta, c) = (1 - c)^\beta \frac{c^x (\beta)^x}{x!} \quad \text{for } x \in \mathbb{N}_0,$$

and $Y$ is a gamma random variable with density

$$g(y; \alpha) = \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} \quad \text{for } y > 0.$$

If $(\zeta_i, \zeta_j)$ with $(i, j) \in E_{2,m}$ follows a Lancaster bivariate gamma-negative binomial distribution with parameter $(\alpha, \beta, c)$ and correlation $\rho_{ij} \in [0, 1)$, then

$$\kappa_{ij} = \text{cov}(1\{p_i \leq t\}, 1\{p_j \leq t\}) = \sum_{n=1}^{\infty} \rho_{ij}^n q_n(x_0; \beta, c) r_n(\tau, \alpha),$$

where $x_0$ is the integer part of $F_i^{-1}(1 - t)$, $\tau = F_j^{-1}(1 - t)$,

$$q_n(x_0; \beta, c) = \sum_{x=0}^{x_0} f(x; \beta, c) M_n^{\beta,c}(x)$$

and

$$r_n(\tau, \alpha) = \sqrt{\frac{n!}{(\alpha)_n}} \int_{-\infty}^{\tau} g(y; \alpha) L_n^{(\alpha-1)}(x) \, dx.$$
Using the bounds derived in Section 2.1 and Section 2.3, we obtain
\[ |\kappa_{ij}| \leq \sum_{n=1}^{\infty} \rho_{ij}^n c^n/2n(\beta-1+2x_0)/2n(1-\alpha)/2 \leq C\rho_{ij}. \]

So, (3) holds.

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