Ultrametric random field

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Abstract

Gaussian random field on general ultrametric space is introduced as a solution of pseudodifferential stochastic equation. Covariation of the introduced random field is computed with the help of wavelet analysis on ultrametric spaces.

Notion of ultrametric Markovianity, which describes independence of contributions to random field from different ultrametric balls is introduced. We show that the random field under investigation satisfies this property.

Keywords: ultrametric analysis, random fields, pseudodifferential operators

AMS Subject Classification: 60G15, 60G60, 60H15, 60H40, 60J99

1 Introduction

In the present paper we apply analysis of ultrametric pseudodifferential operators and ultrametric wavelets to investigation of Gaussian random fields, which are solutions of stochastic pseudodifferential equations on general ultrametric spaces (actually, these spaces obey some properties which guarantee, for instance, local compactness of these spaces). This construction is a far generalization of the \( p \)-adic Brownian motion of Bikulov and Volovich [1], which is a solution of \( p \)-adic stochastic pseudodifferential equation

\[
D^\alpha \psi(x) = \phi(x)
\]

Here \( D^\alpha \) is the Vladimirov operator of \( p \)-adic fractional derivation [2] and \( \phi \) is the white noise on the field of \( p \)-adic numbers \( Q_p \).

The main result of the present paper is theorem 16 (see Section 3), which schematically can be formulated as follows.

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Theorem  Let $X$ be an ultrametric space (satisfying some natural properties), $\nu$ a measure on $X$, for which measures of all balls are positive, $T$ be positive pseudodifferential operator on this space of the form

$$Tf(x) = \int T(\sup(x, y))(f(x) - f(y))d\nu(y)$$

and let some conditions of convergence be satisfied.

Let $\phi(x)$ be white noise on $X$ (i.e. Gaussian mean zero real valued $\delta$–correlated generalized random field).

Then the solution of the stochastic pseudodifferential equation

$$T\psi(x) = \phi(x)$$

is a Gaussian mean zero random field with covariation

$$\langle \psi(x)\psi(x') \rangle = -\lambda_{\sup(x, x')}^{-2}\nu^{-1}(\sup(x, x')) + \sum_{I > \sup(x, x')} \lambda_{I}^{-2} \left( \nu^{-1}(I - 1, \sup(x, x')) - \nu^{-1}(I) \right)$$

The corresponding Gaussian measure is $\sigma$–additive on the $\sigma$–algebra of weak Borel subsets of the space of distributions $D'(X)$.

We also introduce the definition of ultrametric Markovianity for random fields of ultrametric argument, and show that the described by the theorem above random field is ultrametric Markovian.

In the theorem above ultrametric space $X$ should satisfy some natural properties (which imply, for instance, local compactness). The construction of solutions of stochastic pseudodifferential equations on ultrametric spaces given in this article can be generalized on arbitrary ultrametric spaces. In the general case the space $D(X)$ of test functions is not nuclear and one could not apply the theorem of Minlos–Sazonov. However, even in this case it is possible to show $\sigma$–additivity of the corresponding Gaussian measure on $D'(X)$ by using results of the paper by O.G.Smolyanov and S.V.Fomin [3].

The present paper continues development in the field of ultrametric analysis and $p$–adic mathematical physics, see [2] for other results in this direction. We use analysis of general ultrametric pseudodifferential operators and wavelets, developed in [4]–[6].

For exposition of theory of $p$–adic pseudodifferential operators (PDO) see [2]. It was found [2] that $p$–adic PDO (which can be diagonalized by the Fourier transform) have also bases of locally constant eigenvectors with compact support (i.e. vectors of these bases are functions in the space of test functions $D(Q_p)$). In [7] an example of such a basis was found, which is gives the construction of basis of wavelets on $Q_p$. In [8], [9] generalizations of construction of $p$–adic wavelets [7] were built for locally compact abelian groups. In [10]–[12] $p$–adic wavelet bases were used for diagonalization of $p$–adic PDO, which can not be diagonalized by the Fourier transform. In [13]–[16] theory of wavelets and pseudodifferential operators on general ultrametric spaces (which do not possess any group structure) was constructed. Introduced there ultrametric wavelets are eigenvectors of ultrametric PDO. Therefore,
on general ultrametric space there exists an analysis of PDO and of wavelets, and wavelet transform play the role of the Fourier transform, used in the analysis on abelian groups.

For other results in ultrametric an $p$–adic mathematical physics see [13]–[21]. For instance, in cite [13], [14] $p$–adic strings were investigated. In [15], [16] $p$–adic analysis was applied to investigation of spin glasses in the replica approach.

The exposition of the present paper is as follows.

In Section 2 we put the exposition of results in ultrametric PDO, wavelets, and distribution theory.

In Section 3, using the analysis of ultrametric wavelets, we construct Gaussian random field, which is the solution of pseudodifferential stochastic equation on general ultrametric space. We introduce the definition of ultrametric Markovianity and show that the obtained random field is ultrametric Markovian.

In Section 4 (the Appendix) we discuss the Minlos–Sazonov theorem about $\sigma$–additivity of measures on topological vector spaces.

## 2 Ultrametric analysis

In this Section we put the results on ultrametric analysis, which mainly may be found in [4], [5], [6].

We introduce ultrametric wavelet analysis, analysis of ultrametric pseudodifferential operators (PDO), and ultrametric distribution theory.

Rich theory of ultrametric wavelets and pseudodifferential operators, and especially the fact that ultrametric PDO can be diagonalized by ultrametric wavelet transform shows that analysis of ultrametric wavelets plays the role of the Fourier analysis for general ultrametric spaces. Since the existence of the Fourier transform is related to structure of abelian group, this shows that that the ultrametricity property actually is the analogue of the group property.

### 2.1 Ultrametric PDO

**Definition 1** An ultrametric space is a metric space with the ultrametric $|xy|$ (where $|xy|$ is called the distance between $x$ and $y$), i.e. the function of two variables, satisfying the properties of positivity and non degeneracy

$$|xy| \geq 0, \quad |xy| = 0 \implies x = y;$$

symmetricity

$$|xy| = |yx|;$$

and the strong triangle inequality

$$|xy| \leq \max(|xz|, |yz|), \quad \forall z.$$

We say that ultrametric space $X$ is regular, if this space satisfies the following properties:
1) The set of all the balls of nonzero diameter in $X$ is no more than countable;

2) For any decreasing sequence of balls $\{D^{(k)}\}$, $D^{(k)} \supset D^{(k+1)}$, diameters of the balls tend to zero;

3) Any ball is a finite union of maximal subballs.

For ultrametric space $X$ consider the set $T(X)$, which contains all the balls in $X$ of nonzero diameter, and the balls of zero diameter which are maximal subballs in balls of nonzero diameter. Remind that non–maximal vertex $I$ of a directed tree has the branching index $p_I$, if this vertex is connected to $p_I + 1$ other vertices by links (and to $p_I$ other vertices in the case of the maximal vertex). The following theorem can be found in [6] (the analogous result was obtained in [22], see also [23] where it was presented).

**Theorem 2** The set $T(X)$ which corresponds to the regular ultrametric space $X$ with the partial order, defined by inclusion of balls, is a directed tree where all neighbor vertices are comparable.

Branching index for vertices of this tree may take finite integer non–negative values not equal to one, and the maximal vertex (if exists) has the branching index $\geq 2$. Balls of nonzero diameter in $X$ correspond to vertices of branching index $\geq 2$ in $T(X)$, and the balls of zero diameter which are maximal subballs in balls of nonzero diameter correspond to vertices of branching index 0 in $T(X)$.

Remind that a directed set is a partially ordered set, where for any pair of elements there exists the unique supremum with respect to the partial order. The ball $I$ the ball in $X$ is a vertex in $T(X)$.

Consider the set $X \cup T(X)$, where we identify the balls of zero diameter from $T(X)$ with the corresponding points in $X$. We call $T(X)$ the tree of balls in $X$, and $X \cup T(X)$ the extended tree of balls. One can say that $X \cup T(X)$ is the set of all the balls in $X$, of nonzero and zero diameter.

Introduce the structure of a directed set on $X \cup T(X)$. At the tree $T(X)$ this structure is the following: $I < J$ if for the corresponding balls $I \subset J$.

The supremum

$$\sup(x, y) = I$$

of points $x, y \in X$ is the minimal ball $I$, containing the both points.

Analogously, for $J \in T(X)$ and $x \in X$ the supremum

$$\sup(x, J) = I$$

corresponds to the minimal ball $I$, which contains the ball $J$ and the point $x$.

Conversely, starting from a directed tree one can reproduce the corresponding ultrametric space [4, 6, 22, 24], which will be the absolute of the directed tree.

Consider a $\sigma$–additive Borel measure $\nu$ with countable or finite basis on regular ultrametric space $X$, such that for arbitrary ball $D$ its measure $\nu(D)$ is positive number (i.e. is not equal to zero).
We study the ultrametric pseudodifferential operator (or the PDO) of the form considered in [5], [6]
\[ Tf(x) = \int T(\sup(x, y))(f(x) - f(y))d\nu(y) \]

Here \( T(I) \) is some nonnegative function on the tree \( \mathcal{T}(X) \). Thus the structure of this operator is determined by the direction on \( X \cup \mathcal{T}(X) \). This kind of ultrametric PDO we call the sup-operator.

### 2.2 Ultrametric wavelets

Build a basis in the space \( L^2(X, \nu) \) of quadratically integrable with respect to the measure \( \nu \) functions, which we will call the basis of ultrametric wavelets.

Denote \( V_I \) the space of functions on the absolute, generated by characteristic functions of the maximal subballs in the ball \( I \) of nonzero radius. Correspondingly, \( V_I^0 \) is the subspace of codimension 1 in \( V_I \) of functions with zero mean with respect to the measure \( \nu \). Spaces \( V_I^0 \) for the different \( I \) are orthogonal. Dimension of the space \( V_I^0 \) is equal to \( p_I - 1 \).

We introduce in the space \( V_I^0 \) some orthonormal basis \( \{\psi_{Ij}\}, j = 1, \ldots, p_I - 1 \). The next theorem shows how to construct the orthonormal basis in \( L^2(X, \nu) \), taking the union of bases \( \{\psi_{Ij}\} \) in spaces \( V_I^0 \) over all non minimal \( I \).

**Theorem 3**

1) Let the ultrametric space \( X \) contains an increasing sequence of embedded balls with infinitely increasing measure. Then the set of functions \( \{\psi_{Ij}\} \), where \( I \) runs over all non minimal vertices of the tree \( \mathcal{T} \), \( j = 1, \ldots, p_I - 1 \) is an orthonormal basis in \( L^2(X, \nu) \).

2) Let for the ultrametric space \( X \) there exists the supremum of measures of the balls, which is equal to \( A \). Then the set of functions \( \{\psi_{Ij}, A^{-\frac{1}{2}}\} \), where \( I \) runs over all non minimal vertices of the tree \( \mathcal{T} \), \( j = 1, \ldots, p_I - 1 \) is an orthonormal basis in \( L^2(X, \nu) \).

The introduced in the present theorem basis we call the basis of ultrametric wavelets.

The next theorem shows that the basis of ultrametric wavelets is the basis of eigenvectors for ultrametric PDO.

**Theorem 4**

Let the following series converge:
\[ \sum_{J > R} T(J)(\nu(J) - \nu(J - 1, R)) < \infty \]  \( \text{(1)} \)

Then the operator
\[ Tf(x) = \int T(\sup(x, y))(f(x) - f(y))d\nu(y) \]
is self-adjoint, has the dense domain in $L^2(X, \nu)$, and is diagonal in the basis of ultrametric wavelets from the theorem 3:

$$T\psi_{Ij}(x) = \lambda_I \psi_{Ij}(x) \quad (2)$$

with the eigenvalues:

$$\lambda_I = T(I)\nu(I) + \sum_{J > I} T(J)(\nu(J) - \nu(J - 1, I)) \quad (3)$$

Here $(J - 1, I)$ is the maximal vertex which is less than $J$ and larger than $I$.

Also the operator $T$ kills constants.

### 2.3 Distributions

In present Section we introduce the spaces of (complex valued) test and generalized functions (or distributions) on ultrametric space $X$. This construction is an analogue of the construction of the Bruhat–Schwartz space in $p$–adic case, which can be found in [2].

**Definition 5** Function $f$ on (ultrametric) space $X$ is called locally constant, if for arbitrary point $x \in X$ there exists a positive number $r$ (depending on $x$), such that the function $f$ is constant on the ball with center in $x$ and radius $r$:

$$f(x) = f(y), \quad \forall y : |xy| \leq r$$

The next definition is an analogue of definition of the Bruhat–Schwartz space in $p$–adic case.

**Definition 6** The space of test functions $D(X)$ on ultrametric space $X$ is defined as the space of locally constant functions with compact support.

Remind that a set (in particular, linear space) is filtrated by a directed family of subsets (in particular, linear subspaces), if:

1) To any relation of order in this family corresponds inclusion of subsets. This means that if $A < B$, then exists the embedding of $A$ into $B$;

2) Any element of the set lies in some set of the family.

In the case of filtrated linear spaces all embeddings are linear.

Consider the tree $T = T(X)$ of balls in ultrametric space $X$. Introduce the filtration of the space $D(X)$ of test functions by finite dimensional linear subspaces $D(S)$, where $S \subset T$ is a finite subtree of the following form.

**Definition 7** The subset $S$ in a directed tree $T$ is called of the regular type, iff:

1) $S$ is finite;

2) $S$ is a directed subtree in $T$ (where the direction in $S$ is the restriction of the direction in $T$ onto $S$);

3) The directed subtree $S$ obey the following property: if $S$ contains a vertex $I$ and a vertex $J$: $J < I$, $|IJ| = 1$, then the subtree $S$ contains all the vertices $L$ in $T$: $L < I$, $|IL| = 1$. 

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Remind that the distance between vertices of a tree is the number of edges of the path connecting these vertices. The maximal vertex in \( \mathcal{S} \) we will denote \( K \).

We denote by \( J \) the ball in \( X \), corresponding to vertex \( J \) of \( \mathcal{T}(X) \), and by \( \chi_J \) we denote the characteristic function of this ball. In this language we consider a finite set \( \mathcal{S} \) of balls in ultrametric space \( X \), characterized by the properties:

1) If \( \mathcal{S} \) contains balls \( I \) and \( J \), then it contains \( \text{sup}(I, J) \);
2) If \( \mathcal{S} \) contains balls \( I \) and \( J \): \( I \subset J \), then \( \mathcal{S} \) contains all the balls \( L: I \subset L \subset J \);
3) If \( \mathcal{S} \) contains balls \( I \) and \( J \), where \( J \) is a maximal subball in \( I \), then it contains all the maximal subballs in \( I \).

**Definition 8** For the finite subtree \( \mathcal{S} \subset \mathcal{T} \) of the regular type consider the space \( D(\mathcal{S}) \), which is the linear span of characteristic functions \( \chi_J \) with \( J \in \mathcal{S} \).

We consider this space as the subspace in the space \( L^2(X, \nu) \) of quadratically integrable with respect to the measure \( \nu \) functions on \( X \).

Dimension of \( D(\mathcal{S}) \) is equal to the number of minimal elements in \( \mathcal{S} \).

The space \( D(\mathcal{S}) \) has the natural topology (which can be described as the topology of pointwise convergence), since \( D(\mathcal{S}) \) is finite dimensional. The space \( D(X) \) of test functions on \( X \) is the inductive limit of spaces \( D(\mathcal{S}) \):

\[
D(X) = \lim \text{ind}_{\mathcal{S} \to \mathcal{T}} D(\mathcal{S}) \tag{4}
\]

Since all spaces \( D(\mathcal{S}) \) are finite dimensional, the restriction of the topology of \( D(\mathcal{S}) \) on any subspace \( D(\mathcal{S}_0), \mathcal{S} \supset \mathcal{S}_0 \), coincides with the original topology of \( D(\mathcal{S}_0) \). Thus the inductive limit \( D(X) \) is, in fact, the strict inductive limit \cite{25}, p.57. By proposition 6.5, \cite{25}, p.59, a set \( B \) in a strict inductive limit of a countable family of locally convex spaces \( \{E_n\} \) is bounded iff there exists \( n \) such that \( B \subset E_n \) and bounded in it. This implies the proposition below.

**Proposition 9** The sequence \( \{f_n\} \in D(X) \) converge, if this sequence lies in some subspace \( D(\mathcal{S}) \) of the filtration and converges (and convergence in finite dimensional space \( D(\mathcal{S}) \) is defined uniquely).

We also pay attention that by corollary of theorem 7.4, \cite{25}, p.103, we have the following proposition.

**Proposition 10** The space \( D(X) \) is nuclear.

**Definition 11** Distribution (or generalized function) on \( X \) is a linear functional on the space \( D(X) \) of test functions.

It is easy to see that this functional automatically will be continuous (since convergence in the space of test functions is defined through the convergence in finite dimensional subspaces). The linear space of generalized functions we denote \( D'(X) \). The convergence in \( D'(X) \) is defined as a weak convergence of functionals. Thus \( D'(X) \) is conjugated to \( D(X) \) with the weak topology.

We remind that

\[
D'(X) = \bigcap D'(\mathcal{S})
\]

since \( D(X) \) is the strict inductive limit.

\(^1\)Such subspaces are important in relation to replica method.
3 Construction of the random field

Let \( X \) be ultrametric space, satisfying the properties 1–3 of the previous Section (and therefore we have the analysis of pseudodifferential operators and distribution theory on this space).

We define the white noise \( \phi(x) \) as the generalized Gaussian real valued random field on \( X \) with mean zero and the \( \delta \)–like covariation
\[
\langle \phi(x)\phi(x') \rangle = \delta(x - x')
\]
where \( \delta \)–function is understood in the sense of distributions in ultrametric space \( X \).

Then the Minlos–Sazonov theorem, see the Appendix, implies the following proposition.

**Proposition 12** The Gaussian measure, corresponding to the white noise \( \phi(x) \), is \( \sigma \)–additive on the \( \sigma \)–algebra of weak Borel subsets of the space of distributions \( D'(X) \).

Assume we have an ultrametric space \( X \), satisfying the necessary properties, and \( T \) is an ultrametric PDO on this space:
\[
Tf(x) = \int T(\sup(x, y))(f(x) - f(y))d\nu(y)
\]
Consider the stochastic equation
\[
T\psi(x) = \phi(x)
\]
where \( \phi(x) \) is the white noise.

Remind that the operator \( T \) is diagonal in the basis of ultrametric wavelets \( \Psi_{Ij} \):
\[
T\Psi_{Ij} = \lambda_I\Psi_{Ij}
\]
Consider the expansion of the white noise over the wavelets
\[
\phi = \sum_{Ij} d_{Ij}\Psi_{Ij}, \quad d_{Ij} = \int \phi(x)\Psi_{Ij}(x)d\nu(x)
\]
The coefficients \( d_{Ij} \) are mean zero Gaussian random variables, with the quadratic correlation
\[
\langle d_{Ij}^*d_{I'j'} \rangle = \int \int \langle \phi(x)\phi(x') \rangle\overline{\Psi_{Ij}(x)}\Psi_{I'j'}(x')d\nu(x)d\nu(x') = \delta_{II'}\delta_{jj'}
\]
since the wavelets are orthonormal, i.e. \( d_{Ij} \) are independent \( \delta \)–correlated random variables.

Then by (5) and since the wavelets are basis, we get the solution
\[
\psi(x) = T^{-1}\phi(x) = \sum_{Ij} \lambda_I^{-1}d_{Ij}\Psi_{Ij}(x)
\]
This is a Gaussian mean zero generalized random field (at least formally).

Compute the quadratic correlation of (5)
\[
\langle \psi(x)\psi(x') \rangle = \sum_{Ij} \lambda_I^{-2}\overline{\Psi_{Ij}(x)}\Psi_{Ij}(x')
\]
Lemma 13 Correlation function (7) has the expression

\[ \langle \psi(x)\psi(x') \rangle = -\lambda_{\text{sup}(x,x')}^{-2} \nu^{-1}(\text{sup}(x,x')) + \sum_{I > \text{sup}(x,x')} \lambda_I^{-2} \left( \nu^{-1}(I - 1, \text{sup}(x,x')) - \nu^{-1}(I) \right) \]

in the form of the series over the increasing path in the tree \( T(X) \), which begins in \( \text{sup}(x,x') \). This expression is correct if and only if both the series

\[ \sum_{J > R} T(J)(\nu(J) - \nu(J - 1, R)) \]

and the series

\[ \sum_{I > R} \lambda_I^{-2} \left( \nu^{-1}(I - 1, R) - \nu^{-1}(I) \right) \]

where

\[ \lambda_I = T(I)\nu(I) + \sum_{J > I} T(J)(\nu(J) - \nu(J - 1, I)) \]

converge for fixed vertex \( R \).

Proof Taking into account supports of the wavelets, series (7) over the tree \( T \) (indexed by \( I \)) reduces to the series over the increasing path in the tree, starting in \( \text{sup}(x,x') \):

\[ \sum_{I,j: I \geq \text{sup}(x,x')} \lambda_I^{-2} \Psi_{Ij}(x)\Psi_{Ij}(x') \]

Take the sum over \( j \). For fixed \( I \) let us note that

\[ \sum_{j=0}^{p_I-1} \Psi_{Ij}\Psi_{Ij} = \sum_{j=0}^{p_I-1} P_{\chi_{Ij}} \]

where \( P_{\chi_{Ij}} \) is the orthogonal projector in \( L^2(X,\nu) \) onto \( \chi_{Ij} \).

Here \( \Psi_{I0} \) is proportional to \( \chi_I \) and \( \Psi_{Ij}, \ j = 0, \ldots, p - 1 \) is the orthonormal basis of wavelets in the space generated by \( \chi_{Ij}, \ j = 0, \ldots, p - 1 \) of indicators of the maximal subbals in \( I \).

We have for \( I \geq J \)

\[ P_{\chi_I}\chi_J = \frac{\nu(J)}{\nu(I)} \chi_I \]

or equivalently

\[ P_{\chi_I} = \nu^{-1}(I)\chi_I \]

Since

\[ \Psi_{I0} = \nu^{-\frac{1}{2}}(I)\chi_I \]

then

\[ \Psi_{I0}\Psi_{I0} = P_{\chi_I} \]

This implies

\[ \sum_{j=1}^{p_I-1} \Psi_{Ij}\Psi_{Ij} = \sum_{j=0}^{p_I-1} P_{\chi_{Ij}} - P_{\chi_I} \]
Then, using (11) for sufficiently small balls $A$ and $B$, containing $x$ and $x'$ correspondingly, we get

$$\sum_{j=1}^{p-1} \Psi_{Ij}(x)\Psi_{Ij}(x') = \nu^{-1}(A)\nu^{-1}(B)\langle \chi_A, \left[ \sum_{j=0}^{p-1} P_{\chi_{Ij}} - P_{\chi_I} \right] \chi_B \rangle =$$

$$= \nu^{-1}(I-1, \text{sup}(x, x')) \bigg|_{I>\text{sup}(x, x')} - \nu^{-1}(I) \bigg|_{I\geq\text{sup}(x, x')}$$

This implies the expression (8) for the correlation function. Investigate, when this expression will be correct (i.e. when the series will converge).

Eigenvalues of the ultrametric PDO in the basis of wavelets are

$$\lambda_I = T(I)\nu(I) + \sum_{J>I} T(J)(\nu(J) - \nu(J-1, I))$$

Since $\lambda_I$ decreases with the increasing of $I$ (remind that we consider the case when all $T(J) \geq 0$), correlation function (8) can be divergent (infrared divergence). The term $\lambda_I^{-2}$ increases with $I$, and $\nu^{-1}(I)$ decreases.

For convergence of the series (8) for all $x, x'$ it is sufficient to have convergence of the series

$$\sum_{J>R} T(J)(\nu(J) - \nu(J-1, R))$$

for fixed $R$ (to have existence of all $\lambda_I$), and to have convergence of the series

$$\sum_{I>R} \lambda_I^{-2} \left(\nu^{-1}(I-1, R) - \nu^{-1}(I)\right)$$

for fixed vertex $R$. Moreover, it is easy to see that these conditions will also be necessary. This finishes the proof of the lemma.

**Lemma 14** Let conditions (9) and (10) be satisfied. Then correlation function (8) of (6) is a positive continuous bilinear form on $D(X)$.

**Proof** Since (7) is a sum of orthogonal projections, multiplied by positive numbers, correlation function $\langle \psi(x)\psi(x') \rangle$ generates positively definite quadratic form in the space $D(X)$ of test functions in the case, when the series in (8) converge for all $x, x'$, i.e. when conditions (9) and (10) are satisfied. Moreover, this quadratic form will be continuous with respect to the topology in $D(X)$ (since convergence in $D(X)$ reduces to convergence in finite dimensional subspaces, generated by finite sets of characteristic functions of balls, and correlation function (8) is locally constant). This finishes the proof of the lemma.

Formula (8) gives the analytic expression for the correlation function of the solution of the stochastic equation (5). Note that the correlation function (8) depends only on $\text{sup}(x, x')$, i.e. on the minimal ball in $X$ containing both $x$ and $x'$. This property of independence of correlations on the details of the process, related to balls which do not intersect the ball $\text{sup}(x, x')$, may be discussed as the ultrametric analogue of the Markov property, see also paper [1] for discussion in the $p$-adic case.

The next definition describes how one can introduce Markovianity for random fields on ultrametric spaces.
Definition 15 Random field $\psi(x)$ on ultrametric space $X$ with values in $D'(X)$ is called ultrametric Markovian, if any random variables $\Psi_I(f)$, $\Psi_J(g)$, where $f, g \in D(X)$ are supported in the balls $I$ and $J$ correspondingly,

$$\Psi_I(f) = \int_I \psi(x)f(x)dv(x), \quad \Psi_J(g) = \int_J \psi(x)g(x)dv(x)$$

the balls $I$ and $J$ have empty intersection, and at least one of the functions $f$ and $g$ has zero mean, are independent.

Remark This is a far generalization of the Markovianity property for $p$–adic Brownian motion, considered in [1]. In [1] the Markovianity for the random field $\psi(x)$ of $p$–adic argument was discussed as Markov property for the sequence $\psi_\gamma$, enumerated by integer $\gamma$, where

$$\psi_\gamma = \int_{|x|_p = p^\gamma} \psi(x)dx$$

Of course, definition 15 can also be considered when the space $X$ is not necessarily ultrametric, but in this general case we will not have natural examples of such Markovian random fields.

Let us formulate the main result of the present paper.

Theorem 16 Let $X$ be an ultrametric space (satisfying the properties 1–3), $\nu$ a measure on $X$, for which measures of all balls are positive, $T$ be positive pseudodifferential operator on this space of the form

$$Tf(x) = \int T(\sup(x,y))(f(x) - f(y))d\nu(y)$$

and let conditions (9) and (10) be satisfied.

Let $\phi(x)$ be white noise on $X$.

Then the solution of the stochastic pseudodifferential equation

$$T\psi(x) = \phi(x)$$

is a Gaussian mean zero random field with covariation

$$\langle \psi(x)\psi(x') \rangle = -\lambda_{\sup(x,x')}^{-2}(\sup(x,x')) + \sum_{I > \sup(x,x')} \lambda_I^{-2} \left( \nu^{-1}(I - 1, \sup(x,x')) - \nu^{-1}(I) \right)$$

The corresponding Gaussian measure is $\sigma$–additive on the $\sigma$–algebra of weak Borel subsets of the space of distributions $D'(X)$.

Proof By lemma 14 we can apply the Minlos–Sazonov theorem, which provides $\sigma$–additivity of the Gaussian measure defined by (7) which determines the random field (6) taking values in $D'(X)$. Lemma 13 gives the expression for the covariation.

This finishes the proof of the theorem.
**Theorem 17** Random field $\psi(x)$, defined in theorem 16 is ultrametric Markovian in the sense of definition 15.

**Proof** To prove ultrametric Markovianity we use Gaussianity of $\psi$ and the formula (8). By Gaussianity it sufficient to prove Markovianity for the quadratic correlation function (8).

By (8) the correlation function $\langle \psi(x)\psi(x') \rangle$ is locally constant, and conditions of definition 15 can be proved directly. Indeed, let us compute

$$\int_I \int_J f(x)g(x')\langle \psi(x)\psi(x') \rangle d\nu(x)d\nu(x')$$

where $f$ is supported at the ball $I$, $g$ is supported at the ball $J$, balls $I$ and $J$ do not intersect and at least one of the functions $f$ and $g$ has zero mean. Since $\langle \psi(x)\psi(x') \rangle$ is constant for $x \in I$, $x' \in J$, the above correlation function reduces to

$$\langle \psi(x)\psi(x') \rangle \int_I f(x) d\nu(x) \int_J g(x') d\nu(x') = 0$$

This finishes the proof of the theorem.

**Remark** Markovianity property for stochastic processes of real argument, i.e. independence of the future on the past if the present value of the stochastic process is fixed, is related to the fact that real numbers are ordered. On $p$–adic numbers, and, moreover, general ultrametric spaces there is no natural order. The ultrametric Markovianity in the sense of definition 15 is related to order in the directed tree of balls $T(X)$.

### 4 Appendix: the Minlos–Sazonov theorem

The following material is taken from [26].

**Definition 18** Let $X$, $Y$ be conjugated linear spaces with the pairing $\langle \cdot, \cdot \rangle$ and on $X$ there is a quasimeasure $\mu$. Then the integral

$$\chi_\mu(y) = \int_X e^{i\langle x,y \rangle} \mu(dx)$$

is called the characteristic functional of the quasimeasure $\mu$, and the map $\mu \mapsto \chi_\mu$ is called the Fourier transform of the quasimeasure.

The following is the Minlos–Sazonov theorem.

**Theorem 19** Let $X$ be nuclear locally convex space. For the function $\chi(x)$ to be a characteristic functional of some nonnegative Radon measure on the space $X'_\sigma$ it is sufficient, and if $X$ is barrelled (tonnele), also is necessary, that the function $\chi(x)$ is positively defined and continuous in zero in the topology of the space $X$. 
**Definition 20** The correlation form of nonnegative cylindric measure $\mu$ with finite second moments

$$\int_{X'} |\langle y, x \rangle|^2 \mu(dy) < \infty, \quad x \in X$$

is the bilinear functional on $X$, defined by the expression

$$B_\mu(x_1, x_2) = \int_{X'} \langle x_1, y \rangle \langle x_2, y \rangle \mu(dy)$$

**Theorem 21** If the correlation form $B_\mu(x, y)$ of the cylindric measure $\mu$ in the space $X'$ is continuously defined in the nuclear topology $\tau_S(X)$, then $\mu$ is $\sigma$–additive in $X'$.

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