On Some Generalized Vertex Folkman Numbers

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Abstract
For a graph $G$ and integers $a_i \geq 1$, the expression $G \rightarrow (a_1, \ldots, a_r)^v$ means that for any $r$-coloring of the vertices of $G$ there exists a monochromatic $a_i$-clique in $G$ for some color $i \in \{1, \ldots, r\}$. The vertex Folkman numbers are defined as $F_v(a_1, \ldots, a_r; H) = \min\{|V(G)| : G$ is $H$-free and $G \rightarrow (a_1, \ldots, a_r)^v\}$, where $H$ is a graph. Such vertex Folkman numbers have been extensively studied for $H = K_s$ with $s > \max\{a_i\}_{1 \leq i \leq r}$. If $a_i = a$ for all $i$, then we use notation $F_v(a^r; H) = F_v(a_1, \ldots, a_r; H)$.

Let $J_k$ be the complete graph $K_k$ missing one edge, i.e. $J_k = K_k - e$. In this work we focus on vertex Folkman numbers with $H = J_k$, in particular for $k = 4$ and $a_i \leq 3$. A result by Nešetřil and Rödl from 1976 implies that $F_v(3^r; J_4)$ is well defined for any $r \geq 2$. We present a new and more direct proof of this fact. The simplest but already intriguing case is that of $F_v(3, 3; J_4)$, for which we establish the upper bound of 135 by using the $J_4$-free process. We obtain the exact values and bounds for a few other small cases of $F_v(a_1, \ldots, a_r; J_4)$ when $a_i \leq 3$ for all $1 \leq i \leq r$, including...
$F_v(2, 3; J_4) = 14$, $F_v(2^4; J_4) = 15$, and $22 \leq F_v(2^5; J_4) \leq 25$. Note that $F_v(2^r; J_4)$ is the smallest number of vertices in any $J_4$-free graph with chromatic number $r + 1$. Most of the results were obtained with the help of computations, but some of the upper bound graphs we found are interesting by themselves.

**Keywords**  Folkman number · Vertex coloring

**Mathematics Subject Classification** 05C55

1 Introduction

1.1 Notation and Background

All graphs considered in this paper are finite undirected simple graphs. The order of the largest independent set in graph $G$ will be denoted by $\alpha(G)$, and the chromatic number of $G$ by $\chi(G)$. Let $J_k$ be the complete graph $K_k$ missing one edge, i.e., $J_k = K_k - e$. Note that $J_3$ is the path on 3 vertices, and the diamond graph $J_4$ is formed by two triangles sharing an edge.

For a graph $G$ and integers $a_1, \ldots, a_r$, such that $a_i \geq 1$ for $1 \leq i \leq r$, the expression $G \rightarrow (a_1, \ldots, a_r)^v$ means that for any $r$-coloring of the vertices of $G$ there exists a monochromatic $a_i$-clique in $G$ for some color $i \in \{1, \ldots, r\}$. In this paper, we call this property vertex-arrowing, or simply arrowing. It should be noted that an analogous edge-arrowing property $G \rightarrow (a_1, \ldots, a_r)^e$ is the basis of widely studied Ramsey edge-arrowing problems. In particular, the Ramsey number $R(a_1, \ldots, a_r)$ is defined as the smallest integer $n$ such that $K_n \rightarrow (a_1, \ldots, a_r)^e$. All our results address vertex-arrowing, but in some places we will refer to edge-arrowing for comparison, context, or when used as a tool. The vertex Folkman numbers $F_v$ are defined as

$$F_v(a_1, \ldots, a_r; H) = \min\{|V(G)| : G \text{ is } H\text{-free and } G \rightarrow (a_1, \ldots, a_r)^v\},$$

where $H$ is a graph and $G$ is called $H$-free if it does not contain $H$ as a subgraph (in contrast to a commonly used definition of $H$-free where it is required that $G$ does not contain $H$ as an induced subgraph). If $a_i = a$ for all $i$, then we use a more compact notation, $F_v(a^r; H) = F_v(a_1, \ldots, a_r; H)$. The set of all $H$-free graphs satisfying the arrowing $G \rightarrow (a_1, \ldots, a_r)^v$ will be denoted by $F_v(a_1, \ldots, a_r; H)$ and we will call it the set of Folkman graphs for the corresponding parameters. Further, $F_v(a_1, \ldots, a_r; H; n)$ will denote a subset of the latter when restricted to graphs on $n$ vertices.

Let us set $m = 1 + \sum_{i=1}^r (a_i - 1)$. We will often use the following lemma by Nenov [20] stating a simple necessary condition for vertex arrowing to hold.

**Lemma 1** (Nenov 1980 [20]) If $G \rightarrow (a_1, \ldots, a_r)^v$, then $\chi(G) \geq m$.

Observe that if $a_i = 2$ for all $i$, $1 \leq i \leq r$, then the converse is also true. In the terms of Folkman numbers, this means that $F_v(2^r; H)$ is equal to the smallest number of
vertices in any $H$-free graph $G$ with $\chi(G) = r + 1 = m$. Note also that if $a_i = 1$ for any color $i$, then essentially this color can be disregarded.

The vertex Folkman numbers have been extensively studied when the avoided graph is complete, i.e., when $H = K_s$ (see [7, 9, 16, 19, 21, 22, 33]). They are well defined when $s > \max\{a_i | 1 \leq i \leq r\}$, since it is known that for such $s$ the minimum in the definition ranges over a nonempty set of graphs. The situation is easy for $s \geq m$. Moreover, much is known about vertex Folkman numbers and the corresponding Folkman graphs when $s$ is close to, but less than, $m$. However, even some of the basic questions become difficult for small $s$, such as $s = 3$ or $s = 4$. One of the famous problems which can be stated in these terms is the task of finding the smallest triangle-free graph with given chromatic number $r$, which is equal to $F_v(2^{r-1}; K_3)$. See the following subsection for references and more details about this problem in relation to our current work. A recent Ph.D. thesis by Bikov [1] presents a variety of Folkman problems, focusing on a computational approach together with the known values and bounds for Folkman numbers.

For graphs $F$ and $G$, the Ramsey number $R(F, G)$ is the smallest integer $n$ such that if the edges of $K_n$ are 2-colored, say red and blue, then necessarily this coloring includes a copy of red-colored $F$ or a copy of blue-colored $G$. If $F$ and $G$ are complete graphs, then we write $R(s, t)$ instead of $R(K_s, K_t)$. A regularly updated survey *Small Ramsey Numbers* [30] contains the known bounds and values for a variety of Ramsey numbers.

In this work we focus on the vertex Folkman numbers for graphs avoiding $H = J_k$, in particular for $k = 4$ and $2 \leq a_i \leq 3$. Note that this special case of $J_4$-free graphs admits some triangles, but not too many, since in $J_4$-free graphs each edge can belong to at most one triangle. We observe that avoiding $J_4$ falls in-between the two extensively studied classical cases of avoiding $K_3$ and $K_4$. Also for $H = J_4$, we see that $F_v(2^r; J_4)$ is the smallest number of vertices in any $J_4$-free graph with chromatic number $r + 1$. These numbers are clearly well defined since any $K_3$-free graph is also $J_4$-free, and $F_v(2^r; K_3)$ is well defined for every $r \geq 1$. Furthermore, the classical results for multicolor Folkman numbers by Nešetřil and Rödl [25, 26] guarantee the existence of $F_v(3^s; K_s)$ for $s \geq 4$ and of $F_v(3, 3; J_4)$, while little less known results by Nešetřil and Rödl [24] and Rödl and Sauer [31] imply also the existence of $F_v(3^r; J_4)$ with $r > 2$.

1.2 Summary of Contributions

This subsection summarizes our contributions. They consist mainly of new lower and upper bounds for some concrete Folkman numbers, which were obtained with the help of computations (Theorems 2–6), but also new interesting graph constructions. Several special graphs were found during our computations, but we think that some of them (see Figs. 1–4) are interesting just by themselves.

We start with Theorem 1, which is theoretical. This result is not new, as it is implied by more general old results by Nešetřil and Rödl [24–26], and more directly it follows from Theorem 1.2 in the work by Rödl and Sauer [31]. In the next section, we include our proof of Theorem 1 addressing directly the case of avoiding $J_4$. 
Table 1 Known values and bounds for $F_v(2^r; H)$, for $r \leq 5$ and $H \in \{K_3, J_4, K_4\}$

| $r$ | $K_3$ | Ref. | $J_4$ | $K_4$ | Refs. |
|-----|-------|------|-------|-------|-------|
| 2   | 5     | $C_5$| 3     | 3     | $K_3$|
| 3   | 11    | [4]  | 9     | 6     | $W_6$|
| 4   | 22    | [12] | 15    | 11    | [19]  |
| 5   | 32–40 | [10] | 22–25 | 16    | [15, 23] |

The bold entries in the $J_4$ column were obtained in this work. For easy entries we give the upper bound witness graph. The unique witness for $F_v(2^3; K_4) = 6$ is the wheel graph $W_6 = K_1 + C_5 = K_6 - C_5$.

Theorem 1 $F_v(3^r; J_4)$ is well defined for all $r \geq 1$.

The $J_4$-free graphs satisfying the required arrowing property in Theorem 1 quickly become very large as $r$ grows. The simplest, but already intriguing case, is that of $F_v(3; 3; J_4)$, for which we establish the upper bound of 135 in Theorem 6. Note that, by monotonicity, Theorem 1 implies the existence of Folkman numbers of the form $F_v(a_1, \ldots, a_r; J_4)$ with $a_i \leq 3$ for all $1 \leq i \leq r$.

A $J_4$-free graph $G$ is called **maximal $J_4$-free** if the addition of any edge creates a $J_4$ in $G$. A graph $G$ for which $G \rightarrow (a_1, \ldots, a_r)$ is called **minimal** if after the deletion of any edge this arrowing does not hold. If $G$ is maximal and minimal, it is referred to as **bicritical**. Using computational methods, we obtain the exact values and bounds for several small cases, as stated in Theorems 2–6 below.

Theorem 2 $F_v(2^3; J_4) = 9$, and there are exactly 3 graphs in $F_v(2^3; J_4; 9)$, of which 1 is maximal and 1 is minimal.

Theorem 3 $F_v(2^4; J_4) = 15$, and there are exactly 5 graphs in $F_v(2^4; J_4; 15)$, of which 1 is maximal and 2 are minimal.

Theorem 4 $22 \leq F_v(2^5; J_4) \leq 25$.

Theorem 5 $F_v(2, 3; J_4) = 14$ and there are exactly 212 graphs in $F_v(2, 3; J_4; 14)$, of which 24 are maximal, 26 are minimal, and 1 is bicritical.

Theorem 6 $F_v(3, 3; J_4) \leq 135$.

For the context and comparison with the cases involving $K_3$ and $K_4$ instead of $J_4$, we collect the values and bounds from Theorems 2–4 in Table 1. Observe that since $K_3 \subset J_4 \subset K_4$, we must have $F_v(2^r; K_3) \geq F_v(2^r; J_4) \geq F_v(2^r; K_4)$, for each $r$.

The following sections contain our proof of Theorem 1, the description of computations leading to Theorems 2–6, and graph constructions. The closing section states some open problems and it contains a few remarks for parameters beyond those studied in this paper. The witness graphs for Theorems 2–6, as well as the code implementing algorithm $A$ in Sect. 3.1 are available at https://www.cs.rochester.edu/~dnarvaez/folkmanj4/.

We found a few discrepancies between our results and those claimed in the paper [13]. We investigated all such differences, and we arrived to the conclusion that the computations reported in [13] were incomplete. The computational results reported in...
this paper were obtained by independent implementations which agreed on the final
and intermediate claims. The main implementation described throughout this paper
produced all reported results, except those involving SAT-solvers in Sect. 3.4. Two
others implementations were used to corroborate various parts of these results and to
manage the interface in Sect. 3.4.

2 The Existence of $F_v(3^r; J_4)$ and $F_e(3, 3; H)$

Two seminal papers by Nešetřil and Rödl [25, 26] lay the foundation for our reason-
ing in this section: the first one from 1976 implies that the edge Folkman numbers
$F_e(3^r; K_4)$ are well defined for all $r \geq 1$, and the second paper from 1981 shows that
$F_v(3, 3; J_4)$ is well defined. These, together with a technique developed by Dudek and
Rödl in 2008 [6], permit us to give a rather elementary proof that $F_v(3^r; J_4)$ is well
defined for all $r \geq 1$.

For graph $G = (V_G, E_G)$, we define the graph $F = DR(G)$ as in the construction
by Dudek-Rödl [6], as follows:

$$DR(G) = F = (E_G, E_F),$$

where the set of vertices of $F$ consists of the edges of $G$, and the edge set $E_F$ contains
the edges $\{ef, fg, eg\}$ for every edge-triangle $efg$ in $E_G$, and no other edges. Note
that each pair of edges from triangle $\{e, f, g\} \subseteq E_G$ spans the same three vertices in
$G$, and thus the same three vertices in the corresponding vertex-triangle in $F$. Such
triangles in $F$ will be called images of triangles from $G$, other triangles in $F$ will be
called spurious. Note that for any edge $ef$ in $E_F$, the edges $e$ and $f$ in $E_G$ must share
one vertex. It is easy to observe (see the proof of Lemma 2(2) below) that two image
triangles may share vertices but no edges.

Example 1 For $G = K_4$, $DR(G)$ has 6 vertices, 12 edges (it is 4-regular), and 8
triangles. These 8 triangles are split into 4 images of triangles from $G$ and 4 spurious
triangles.

Lemma 2 Let $G$ be any $K_4$-free graph, and let $F$ denote the graph $DR(G)$. Then we
have that:

1. $F$ has no spurious triangles,
2. $F$ is $J_4$-free, and
3. $G \rightarrow (3^r)^e$ if and only if $F \rightarrow (3^r)^v$, for every $r \geq 1$.

Proof First, we will show that the graph $F = DR(G)$ has no spurious triangles. For
contradiction, suppose that $efg$ is a spurious triangle in $F$, where $e = \{A, B\}$ and
$f = \{A, C\}$ for some vertices $A, B, C \in V_G$. Since edge $g$ is incident to both $e$ and $f$,
but $efg$ is not an image of a triangle in $G$, we must have $g = \{A, D\}$ for another vertex
$D \in V_G$. This implies that $ABC, ABD, ACD$ are triangles, and thus also $BCD$, in
$G$. Hence $ABCD$ forms a $K_4$ in $G$, contradicting the assumption. This shows part (1).

If $F$ contains $J_4$ with the vertices $\{e, f, g, h\}$ and formed by two triangles $\{efg\}$
and $\{efh\}$, then we claim that at most one of them is an image triangle. In order to
see this, let $e = \{A, B\}$, $f = \{A, C\}$ and note that the unique image triangle in $F$ containing $e$ and $f$ is the one implied by the triangle $ABC$ in $G$. Hence, at least one of $\{efg\}$ and $\{efh\}$ must be spurious, but by (1) this is impossible in $F = DR(G)$ obtained from a $K_4$-free graph $G$. Thus (2) follows.

For (3), consider the natural bijection between all $r$-vertex-colorings of $F$ and $r$-edge-colorings of $G$. This bijection preserves the number of colors used in any edge-triangle in $G$ when mapped to its image triangle in $F$. Hence, because of (1) and (2), we can conclude (3).

Using Lemma 2, we can give a simple proof that for every $r$ there exists a $J_4$-free graph such that in any $r$-coloring of its vertices there must be a monochromatic triangle, or equivalently, $\mathcal{F}_v(3' ; J_4) \neq \emptyset$.

**Proof of Theorem 1.** A general result by Nešetřil and Rödl [25] implies that the sets $\mathcal{F}_e(3' ; K_4)$ are nonempty for all $r \geq 1$. This, together with the claim that $\mathcal{F}_v(3^3 ; J_4) \neq \emptyset$, was also discussed in [34]. For general $r$, consider any graph $G \in \mathcal{F}_e(3' ; K_4)$. Then by Lemma 2(3), we have that $DR(G) \in \mathcal{F}_v(3' ; J_4)$, and thus the numbers of the form $\mathcal{F}_v(3' ; J_4)$ are well defined for all $r \geq 1$.

Note that, by monotonicity, Theorem 1 implies the existence of Folkman numbers of the form $\mathcal{F}_v(a_1, \ldots, a_r ; J_4)$ with $1 \leq a_i \leq 3$ for all $1 \leq i \leq r$. The orders of graphs in $\mathcal{F}_e(3' ; K_4)$ and $\mathcal{F}_v(3' ; J_4)$ can be expected to be quite large, even for small $r$. In the trivial case for $r = 1$ we have $\mathcal{F}_e(3 ; K_4) = \mathcal{F}_v(3 ; J_4) = 3$, but both problems become very difficult already for $r = 2$. For the edge problem, the best known bounds are $21 \leq \mathcal{F}_e(3, 3 ; K_4) \leq 786 [2, 14]$, while for the vertex problem we establish the bound $\mathcal{F}_v(3, 3 ; J_4) \leq 135$ in Theorem 6. We are not aware of any reasonable bounds for $r \geq 3$ in either case.

We wish to point to a study of the existence of edge Folkman numbers for some small parameters [34]. While a simple argument easily shows that $\mathcal{F}_e(3, 3 ; J_4)$ does not exist, for other cases with $|V(H)| \leq 5$ one can prove or disprove the existence of $\mathcal{F}_e(3, 3 ; H)$ with some work, leaving only two open cases. Namely, the following is known: the sets $\mathcal{F}_e(3, 3 ; H)$ are nonempty for all connected graphs $H$ containing $K_4$, and for some graphs not containing $K_4$. If $H$ is any connected $K_4$-free graph on 5-vertices containing $K_3$, then $\mathcal{F}_e(3, 3 ; H) = \emptyset$ except for two possible cases: the wheel graph $W_5 = K_1 + C_4$ and its subgraph $P_2 \cup P_3 \subset W_5$. The latter two cases remain open.

### 3 Computational Proofs

#### 3.1 Overview and Algorithms

In this section we describe the computations which were performed to obtain the proofs of Theorems 2–6 stated in Sect. 1.2. First, we give an overview of the algorithms that were used or developed for this work, including some details of their implementation. In the following subsections we summarize the results of our computations. We present graphs establishing the upper bounds in Theorems 2–5, and give counts for several intermediate graph families which were obtained. All graphs involved in the
computations were $J_4$-free. The target sets of graphs had additional constraints consisting of the number of vertices, independence number, chromatic number, and the desired parameters of arrowing, $\{a_1, \ldots, a_r\}$, where $2 \leq a_i \leq 3$ for $1 \leq i \leq r$.

The basis of our software framework consisted of the package nauty developed by McKay [17], which includes a powerful graph generator geng, tools to remove graph isomorphs, and several other utilities for graph manipulation. In the following, we will list some of the graphs in their g6-format, a compact string representation of graphs in nauty. These graphs are also available at https://www.cs.rochester.edu/~dnarvaez/folkmanj4/.

The template of our main Extension Algorithm A is presented and commented on below. We also implemented filters for extracting graphs with specified chromatic number, graph which are maximal $J_4$-free, those which arrow $(2, 3)^v$ and $(3, 3)^v$, and other utilities. Observe that by Lemma 1 the test for arrowing $(2^r)^v$ is the same as for chromatic number.

The graph families pointed to in Table 2 were obtained by using geng with filters for $J_4$-free graphs and for graphs with given chromatic number. For graph families on 13 or more vertices, we used mainly algorithm A together with other utilities, as described in the notes to Table 3.

Our custom filter for graphs with specified chromatic number range was tuned to process large number of graphs with small $\chi(G)$. For a given graph $G$, first we find all maximal independent sets, and then determine $\chi(G)$ as the minimum number of these independent sets which cover $V_G$. The custom filter for maximal $J_4$-free graphs has two modes: a full test detecting graphs for which addition of any edge forms a $J_4$, and a partial test for graphs being constructed within algorithm A which cannot be maximal $J_4$-free after A terminates. The latter permitted to significantly prune the output of A, which was then filtered through the full test.

Testing whether $G \rightarrow (2, 3)^v$ was applied only to graphs with $\chi(G) \geq 4$, since by Lemma 1 this arrowing does not hold if $\chi(G) \leq 3$. This test was done by checking that for every maximal independent set $I \subseteq V_G$ the set of vertices $(V_G \setminus I)$ does not induce any triangle. The test for $G \rightarrow (2, 2, 3)^v$ was accomplished similarly by checking that the set of vertices $(V_G \setminus (I_1 \cup I_2))$ does not induce any triangle, for every pair of maximal independent sets $I_1$ and $I_2$. The test for $G \rightarrow (3, 3)^v$ was accomplished with a totally distinct approach involving SAT-solvers as described in Sect. 3.4. This was necessary since for arrowing $(3, 3)^v$ we were processing a large number of graphs on 100 to 200 vertices, for which an effective handling of their chromatic number and enumeration of maximal triangle-free sets would be computationally very expensive.

Next, we give a high-level pseudocode of the Extension Algorithm A followed by comments explaining some of its components in more detail.
Extension Algorithm A

Input: \( G \) - a set of \( J_4 \)-free graphs, each on \( n \)-vertices, \( q \) - extension degree, \( \chi \) - target chromatic number, \( \delta \) - minimum cone size.

Output: \( \mathcal{H} \) - a set graphs which are extensions of graphs from \( G \), \( H \in \mathcal{H} \) if and only if \( H \) is a \( q \)-vertex extension of any graph \( G \in \mathcal{G} \), \( q \) new vertices in \( H \) form an independent set, \( |V_H| = n + q \), new vertices have degree \( \geq \delta \), and such that \( H \) is maximal \( J_4 \)-free and \( \chi(H) \geq \chi \).

\[ \mathcal{H} = \emptyset; \]

for every graph \( G \in \mathcal{G} \) do

(for definitions of feasible cones and \( \tau \) see Comments part in the main text below)

Compute and store \( C_G \), the set of feasible cones in \( G \) of size at least \( \delta \);

Compute and store the values of \( \tau(C, D) \), for all \( C, D \in C_G \);

If \( q = 1 \) then form \( \mathcal{H} \) from \( G \) and \( C_G \);

If \( q = 2 \) then form \( \mathcal{H} \) from \( G \) and pairs of cones in \( C_G \) using \( \tau \);

If \( q \geq 3 \) then

for \( k = 3 \) to \( q \) do

Using known \((k - 1)\)-tuples, make all \( k \)-tuples (\( k \)-multisets) of feasible cones \( \{C_1, \ldots, C_k\} \) such that \( \tau(C_i, C_j) \) is true for all \( 1 \leq i, j \leq k \).

If \( k = q \), then for each such \( q \)-tuple make graph \( H \) from \( G \) and \( \{C_1, \ldots, C_q\} \).

If \( H \) is maximal \( J_4 \)-free, then add \( H \) to \( \mathcal{H} \).

end for

end for

If \( q < 3 \) then remove from \( \mathcal{H} \) graphs \( H \) which are not maximal \( J_4 \)-free;

Remove isomorphs from \( \mathcal{H} \);

Remove from \( \mathcal{H} \) graphs \( H \) with \( \chi(H) < \chi \).

Comments on the Extension Algorithm A. The inputs are: a family of graphs \( \mathcal{G} \) consisting of \( n \)-vertex \( J_4 \)-free graphs, an integer \( q \), which is the extension degree, the target chromatic number \( \chi \), and the minimum degree \( \delta \) of new vertices. For each \( G \in \mathcal{G} \), algorithm A outputs all maximal \( J_4 \)-free graphs \( H \) with \( \chi(H) \geq \chi \) such that they can be obtained from \( G \) by adding an independent set \( I = \{v_1, \ldots, v_q\} \) and some edges between \( I \) and \( V_G \). New vertices have degree at least \( \delta \). These output graphs \( H \) will be called \( q \)-vertex extensions of the input graph \( G \).

The new edges of \( H \) are defined by \( q \) cones \( \{C_1, \ldots, C_q\} \), \( C_i \subseteq V_G \), where the set of edges connecting \( v_i \) to \( V_G \) is \( \{v_i, u \mid u \in C_i\} \). First, we precompute the set of all feasible cones \( C_G \) such that for each \( C \in C_G \) the 1-vertex extension of \( G \) using \( C \) is \( J_4 \)-free and \( |C| \geq \delta \). We also precompute a binary predicate \( \tau(C, D) \) on pairs of feasible cones which is false if \( C \cap D = \emptyset \), \( C \cap D = \{x\} \) and there is no vertex \( y \in (C\setminus D) \cup (D\setminus C) \) connected to \( x \), or \( C \cap D \) induces an edge in \( G \). Otherwise, \( \tau(C, D) \) is set to true. One can easily see that if \( \tau(C, D) \) is false and both cones \( C \) and \( D \) are used in the extension, then \( H \) is not maximal \( J_4 \)-free. This test significantly prunes the search space. Next, we assemble \( q \)-tuples of feasible cones such that each pair of cones used passes the \( \tau \)-test. Each such \( q \)-tuple defines one graph \( H \). Finally, the isomorphic copies of graphs are removed, and the remaining graphs \( H \) are tested for \( \chi(H) \).

The cases for small \( q \leq 2 \) do not require the inner for-loop, since they essentially reduce to the computation of feasible cones \( (q = 1) \) and \( \tau (q = 2) \), respectively. \( \square \)

Note that larger values of input \( \delta \) in A produce fewer cones and thus allow for faster computation. Maximal \( J_4 \)-free graphs \( H \) with \( \delta = 1 \) are easy to characterize theoretically. Such graphs on \( n + 1 \) vertices exist only for even \( n = 2m \), and for all of them the new vertex cone induces the fan graph \( F_m = K_1 + mK_2 \) with \( \chi(F_m) = 3 \).
Table 2  The number of nonisomorphic graphs $G$ by their type and the number of vertices $n$, $6 \leq n \leq 12$

| $n$ | All graphs | $J_4$-free | $\chi = 2$ | $J_4$-free, $\chi = 3$ | $J_4$-free, $\chi = 4$ |
|-----|------------|------------|-------------|----------------------|----------------------|
| 6   | 156        | 69         | 34          | 34                   | 0                    |
| 7   | 1044       | 255        | 87          | 167                  | 0                    |
| 8   | 12,346     | 1301       | 302         | 998                  | 0                    |
| 9   | 274,668    | 9297       | 1118        | 8175                 | 3                    |
| 10  | 12,005,168 | 97,919     | 5478        | 92,379               | 61                   |
| 11  | 1,018,997,864 | 1,519,456 | 32,302     | 1,484,866           | 2287                 |
| 12  | 165,091,172,592 | 34,270,158 | 251,134   | 33,888,537          | 130,486              |

The corresponding sets of graphs were obtained by using graph generator `geng` of `nauty` with tests for $J_4$-free graphs and chromatic number $\chi(G)$.

We confirmed this claim by running A. In all further computations we set $\delta \geq 2$. For most of our computations we use $\delta = 2$, but we also observe that when constructing graph families which are known to be $\chi$-vertex-critical, it is sufficient to set $\delta = \chi - 1$.

The values of the Ramsey numbers of the form $R(J_4, K_q)$ are known for all $q \leq 6$ (cf. [30]). In particular, the values of importance to our computations are $R(J_4, K_4) = 11$, $R(J_4, K_5) = 16$ and $R(J_4, K_6) = 21$. We use these values to determine the value of parameter $q$ for algorithm A by applying an observation that any $J_4$-free graph $H$ must have $\alpha(H) \geq q$, provided $|V_H| = |V_G| + q \geq R(J_4, K_q)$.

Sections 3.2 and 3.3 list several sets of parameters for A which were used: $G$ is taken from the cases reported in Table 2, or equal to $F_v(2, 3; J_4; 14)$ or $F_v(2^4; J_4; 15)$, and the ranges of other parameters are $1 \leq q \leq 7, 4 \leq \chi \leq 6, \text{ and } 2 \leq \delta \leq \chi - 1$.

### 3.2 Enumerations for Small Cases

The results of our computations for small cases are summarized in Tables 2 and 3. The special graphs, which are witnesses for the exact values in Theorems 2, 3 and 5, are presented in Figs. 1, 2, 3 and 4.

#### Notes to Table 2

- In each row $n$, the sum of entries for $\chi = 2, 3, 4$ is one less than the count of $J_4$-free graphs (because the only missed graph has $\chi = 1$). Note that $\chi(G) = 2$ implies that $G$ is $J_4$-free.
- The entries 0 and 3 in rows 8 and 9, respectively, of the last column show that $F_v(2, 2, 2; J_4) = 9$ and $|F_v(2, 2, 2; J_4; 9)| = 3$; the three witnesses have 16, 17 and 18 edges, respectively, and they are presented in Fig. 1. This part proves Theorem 2.
- Obtaining the next row of Table 2 (for $n = 13$) by the same approach is doable but only at an extraordinary computational cost. Thus, for $n \geq 13$ we first targeted only maximal $J_4$-free graphs. The results are reported in Table 3.
Table 3  Counts of nonisomorphic maximal $J_4$-free graphs $G$ by their chromatic number $\chi = \chi(G)$ and number of vertices $n$, for $13 \leq n \leq 15$

| Type of graphs | $n = 13$ | $n = 14$ | $n = 15$ |
|---------------|---------|---------|---------|
| Maximal $J_4$-free, $\chi = 2$ | 5      | 6       | 6       |
| Maximal $J_4$-free, $\chi = 3$ | 15684  |         |         |
| Maximal $J_4$-free, $\chi = 4$ | 4750   | 74738   |         |
| Maximal $J_4$-free, $\chi = 5$ | 0      | 0       | 1       |

The results for $n \geq 14$ and $\chi \geq 4$ required significant computational resources.

The graph families for $\chi \geq 3$ summarized in Table 3 were constructed using algorithm A. The following lemma was used to determine the initial family of graphs $G$. More details of how each entry with $\chi \geq 3$ was computed are listed in the notes to Table 3.

**Lemma 3**  Let $G$ be any graph with $\chi(G) \geq k$, and let $I \subseteq V(G)$ be any independent set in $G$. Then for $G' = G[V(G) \setminus I]$ we have $\chi(G') \geq k - 1$.

**Proof**  Assume there exists an $I \subseteq V(G)$ such that the graph $G'$ induced in $G$ on $V(G) \setminus I$ has $\chi(G') \leq k - 2$. If $V(G')$ is colored with $k - 2$ colors, then all vertices in $I$ can be colored with the same $(k - 1)$-st color, not used in the coloring of $V(G')$. This implies that $\chi(G) \leq k - 1$, which is a contradiction. $\square$

### 3.2.1 Notes to Table 3

- The row for $\chi = 2$ shows the number of complete bipartite graphs $K_{s,t}$ with $s + t = n$ and $s, t \geq 2$.
- **Graphs with $n = 13$, $\chi \geq 3$.** Using the fact that $R(J_4, K_4) = 11$, we can see that any $J_4$-free graph $G$ of order at least 11 must have $\alpha(G) \geq 4$. The graphs with $n = 13$ were obtained by A in three different ways: via 3-, 2- and 1-vertex extensions of graphs with 10, 11 and 12 vertices, respectively. When $3 \leq \chi \leq 4$, we use $\delta = 2$. When $\chi = 5$, we set $\delta = \chi - 1 = 4$, since the target graphs are known to be $\chi$-vertex-critical. This is because Lemma 3, $R(J_4, K_4) = 11$ and Table 2 imply that there is no $J_4$-free graph with $n = 12$ and $\chi = 5$.
- **Graphs with $n = 14$, $\chi \geq 4$.** These graphs were obtained by computing 4-vertex extensions of the graphs with $n = 10$ and $\chi \geq 3$ using $\delta = 2$. We set $\delta = 4$ when generating graphs with $\chi = 5$ since no graphs with $\chi = 5$ were found on 13 vertices.
- **Graphs with $n = 15$, $\chi \geq 5$.** The unique maximal graph with $n = 15$ and $\chi = 5$ was obtained by performing 3-vertex extensions of all graphs with $n = 12$ and $\chi \geq 4$, and independently by 4-vertex extensions of all graphs with $n = 11$ and $\chi \geq 4$. Since no graphs with $\chi = 5$ were found on 14 vertices, we set $\delta = 4$.
- Empty entries correspond to graphs whose full enumeration was not attempted. These would be difficult to obtain, and they are not relevant for this work. Still, many such graphs were obtained as side result of other computations.
- The entry requiring most CPU time (about one CPU-week if run on a single processor) was that for $\chi = 4, n = 14$. It was obtained by applying algorithm A.
The 18-edge graph, which is formed by all depicted edges, the other two graphs with 17 and 16 edges are obtained by deleting the edges marked in red, in any order.

Fig. 1 $|F_v(2, 2; 2; J_4; 9)| = 3$.

Fig. 2 Unique bicritical graph $G_{14} \in F_v(2, 3; J_4; 14)$. Edges marked in red do not belong to any triangle in $G_{14}$. The graph $G_{14}$, which is H\QVQCG\QQXPQ\QXB\PQ\QW in g6-format, has 33 edges, 9 triangles and just one non-trivial symmetry (left-right swap of the figure) to make $J_4$-maximal 4-vertex extensions of the 97918 graphs with $n = 10$ and $\chi \geq 2$ (though using $\chi \geq 3$ would suffice). Among the resulting 74738 graphs $G$, there are 24 of them for which $G \rightarrow (2, 3)^v$. No graph reported in column 13 satisfies this arrowing (though, by Lemma 1, it would suffice to test only 4750 graphs with $\chi = 4$), and thus $F_v(2, 3; J_4) = 14$.

– The complete set $F_v(2, 3; J_4; 14)$ was obtained from the above 24 maximal $J_4$-free graphs by repeatedly deleting the edges until they did not satisfy the arrowing. This set consists of 212 nonisomorphic graphs, with the number of edges ranging from 31 to 39, the number of triangles from 8 to 10, and the orders of their automorphism groups ranging from 1 to 8. 26 of these graphs are minimal. There exists a unique (up to isomorphisms) bicritical graph, namely the graph $G_{14}$ shown in Fig. 2, for which addition of any edge creates a $J_4$, and deletion of any edge $e$ yields $G_{14} - e \not\rightarrow (2, 3)^v$. This part proves Theorem 5.
Fig. 3 A view of graphs in $\mathcal{F}_v(2^4; J_4; 15)$. The maximal graph on 45 edges, which is N(eC1hJSaWEfIuKqDeG in g6-gormat, is formed by all depicted edges. It is 6-regular. Three vertices of the outer triangle form one orbit, 12 other vertices form the second orbit (the center of the figure is not a vertex). Three edges marked in red can be removed, in any order, to give its subgraphs in $\mathcal{F}_v(2^4; J_4; 15)$. The 5-th graph is a subgraph of one with 44 edges. The minimal graph on 42 edges (formed by the black edges) has 72 automorphisms, more than the other four graphs.

Fig. 4 Set view of $\mathcal{F}_v(2^4; J_4; 15)$. The 9-vertex $3 \times 3$ grid on the right has 6 triangles and 6 independent sets of 3 vertices. It is a self-complementary graph. The vertices $\{A,B,C\}$ and $\{1,2,3\}$ on the left are connected to the grid by 18 edges as indicated by the labels. Equivalently, 9 vertices of the grid on the right connect to pairs of vertices (one from each of two triangles) on the left. The red edges can be dropped (in any order) yielding graphs with 44, 43 and 42 edges, respectively. This figure describes 4 graphs isomorphic to those in Fig. 3, but presenting them in a very different way. The vertices of the middle row of the grid form the triangle corresponding to the outer triangle in Fig. 3.
The second most-expensive-to-obtain entry was that for \( n = 15, \chi = 5 \). The entries in the last row of Table 3 prove that \( F_v(2^4; J_4) = 15 \). The unique maximal graph in \( F_v(2^4; J_4; 15) \) was obtained in two ways: as a 4-extension and 3-extension of all graphs with \( \chi = 4 \) on 11 and 12 vertices, respectively.

The complete set \( F_v(2^4; J_4; 15) \) consists of 5 graphs with 45, 44, 43, 43 and 42 edges, respectively. Four of these graphs (except one on 43 edges) form a chain presented in Fig. 3. The graph on 45 edges is formed by all edges, three red edges can be removed (in any order) to give its subgraphs which are also in \( F_v(2^4; J_4; 15) \). The fifth graph is a subgraph of one on 44 edges. This part proves Theorem 3.

Another view of the graphs in \( F_v(2^4; J_4; 15) \) is presented in Fig. 4. The 9-vertex grid on the right has 6 independent sets of 3 vertices. The vertices of triangles ABC and 123, on the left, are connected to the grid as indicated by the labels. The red edges can be dropped (in any order), yielding graphs with 44, 43, 42 edges, also in \( F_v(2^4; J_4; 15) \). In this view we can easily see all 72 symmetries of the minimal graph.

### 3.3 Bounds for \( F_v(2^5; J_4) \) and \( F_v(2, 2, 3; J_4) \)

Easy bounds on the order of the smallest 6-chromatic \( J_4 \)-free graph, \( F_v(2^5; J_4) \), are implied in prior work by others on avoiding \( K_3 \) and \( K_4 \) (see Table 1 in Sect. 1.2), namely:

\[
16 = F_v(2^5; K_4) \leq F_v(2^5; J_4) \leq F_v(2^5; K_3) \leq 40.
\]

We obtain much better bounds stated in Theorem 4:

\[
22 \leq F_v(2^5; J_4) \leq 25
\]

These bounds were computed as follows. Take \( S \) to be the Schläfli graph on 27 vertices [32]: \( S \) is a strongly regular graph of degree 16. Its complement is \( J_4 \)-free and it has \( \chi(S) = 6 \). Removing from \( \overline{S} \) any two adjacent vertices with all incident edges yields a 25-vertex witness to \( F_v(2^5; J_4) \leq 25 \) (the g6-format of this graph is XIPA@CQA_KEBIIHKBHiCBXB_w) auURYbDu_maULkdQTseOfpp?). Removing any further vertices reduces the chromatic number below 6. It is interesting to note that this witness graph is the unique 6-vertex-critical (\( P_6, J_4, K_4 \))-induced-free graph on 25 vertices [11].

For the lower bound, since \( R(J_4, K_6) = 21 \), any 21-vertex \( J_4 \)-free graph must have an independent set of 6 vertices. Thus, any graph \( G \in F_v(2^5; J_4; 21) \) can be obtained by adding a 6-independent set (with 6 cones) to one of the 5 graphs in \( F_v(2^4; J_4; 15) \). This was verified with algorithm A and no suitable graph was found with \( \chi \geq 6 \). Thus \( F_v(2^5; J_4) \geq 22 \). This part proves Theorem 4.

---

1 There is no induced subgraph isomorphic to a \( J_4, K_4 \), or \( P_6 \).
The bounds we have for $F_v(2, 2, 3; J_4)$ are rather weak, namely

$$F_v(2, 3; J_4) + 6 = 20 \leq F_v(2, 2, 3; J_4) \leq F_v(3, 3; J_4) \leq 135.$$ 

The upper bound follows from Theorem 6 and by an easy observation that for any graph $G$, if $G \to (3, 3)^v$, then $G \to (2, 2, 3)^v$. For the lower bound, suppose that $G \in F_v(2, 2, 3; J_4; k)$. Note that by Lemma 1, we must have $\chi(G) \geq 5$. For $k = 19$, since $R(J_4, K_5) = 16$, we have $\alpha(G) \geq 5$, and thus $G$ is a 5-vertex extension of at least one of the 212 graphs in $F_v(2, 3; J_4; 14)$. Using again algorithm A we have found no suitable graph $G$, and hence $k > 19$. We attempted to use A and other ad-hoc methods to construct a witness $G$ for $k \geq 20$, but all such searches failed. The complement of the Schläfli graph $\overline{S}$ does not arrow $(2, 2, 3)^v$. We also tested 8933 $J_4$-free graphs $G$ on 20 vertices with $\chi(G) = \alpha(G) = 5$ [29] and found that none of them arrows $(2, 2, 3)^v$. For comparison with the cases for $K_4$-free graphs, we note that it is not hard to check that $F_v(2, 3; K_4) = 7$ with the unique witness graph $K_7 - C_7$ (cf. Theorem 3 in [16]), and it is known that $F_v(2, 2, 3; K_4) = 14$ [5].

Finding any non-obvious bounds for $F_v(2, 3, 3; J_4)$ or $F_v(3, 3, 3; J_4)$ is an interesting challenge which we pose as a problem to work on.

### 3.4 The $J_4$-free process and $F_v(3, 3; J_4)$

The triangle-free process begins with an empty graph of order $n$, and iteratively adds edges chosen uniformly at random, subject to the constraint that no triangle is formed. The triangle-free process has been used to prove that

$$R(3, t) \geq \left(\frac{1}{4} + o(1)\right) \frac{t^2}{\log t},$$

which currently is the best known lower bound for $R(3, t)$ obtained by Bohman and Keevash in 2013/2019 [3] and independently by Fiz Pontiveros, Griffiths and Morris in 2013/2020 [8].

Similarly to the triangle-free process, the $J_4$-free process begins with an empty graph of order $n$, and iteratively adds edges chosen uniformly at random, subject to the constraint that no $J_4$ is formed. The asymptotic properties of this process were analyzed in [27]. We implemented the $J_4$-free process in C++ and generated several graphs for which we then checked the arrowing property. The check was done by turning the arrowing property into a Boolean formula and then using Boolean satisfiability (SAT) solvers on the resulting formula. The formula is computed as follows: for every triple of vertices $(v_1, v_2, v_3)$, if they form a triangle, we output the disjunctions

$$(\overline{v_1} \lor \overline{v_2} \lor \overline{v_3}) \land (v_1 \lor v_2 \lor v_3)$$

A satisfying assignment for this subformula will assign at least one of the vertices in $\{v_1, v_2, v_3\}$ to the value FALSE and at least one of them to the value TRUE. Taking
FALSE and TRUE to be colors, it is clear that for a graph $G$ the formula

$$\bigwedge_{(v_1, v_2, v_3) \in V_G} (\overline{v_1} \lor \overline{v_2} \lor \overline{v_3}) \land (\overline{v_1} \lor v_2 \lor v_3)$$

s.t. $G([v_1, v_2, v_3]) \sim K_3$ is satisfiable if and only if there is a way to assign colors to the vertices of $G$ that avoids monochromatic triangles. We are thus searching for $J_4$-free graphs $G$ that yield unsatisfiable instances, as these witness the bound $F_v(3, 3; J_4) \leq |V_G|$. The smallest such graph we were able to find has 135 vertices, thus establishing that

$$F_v(3, 3; J_4) \leq 135.$$  

This part proves Theorem 6.

It is easy to see that $G \rightarrow (3, 3)^v$ implies $K_1 + G \rightarrow (3, 3)^e$, where the graph $K_1 + G$ is obtained from $G$ by adding one new vertex connected to all of $V_G$. By applying this implication we can also see that $F_e(3, 3; K_1 + H) \leq F_v(3, 3; H) + 1$. The latter inequality is tight for $H = K_4$, as it was shown that $F_e(3, 3; K_5) = 15$ and $F_v(3, 3; K_4) = 14$ (upper bounds in [18], lower bounds in [28]). Now, using similar steps for $H = J_4$, by Theorem 6 we obtain $F_e(3, 3; J_5) \leq 136$. We observe that by the monotonicity with respect to the avoided graph $H$ we have $F_e(3, 3; K_5) \leq F_e(3, 3; J_5) \leq F_e(3, 3; K_4)$.

The software development for this project, pilot runs and experiments spanned several months. Now, the final run of everything is doable in about a day in a standard lab with dozen of machines, each with 16 cores.

### 4 Remarks and Some Open Problems

Obtaining good bounds for concrete vertex Folkman numbers is often difficult. The results presented in this paper are special for the parameters involving $J_4$ but we hope that they may extend to other or more general vertex cases. The techniques used here might be not easily transferable to obtain new bounds on edge Folkman numbers, not the least because just testing edge-arrowing is typically much more difficult than testing vertex-arrowing.

We close this paper by posing some related open problems.

**Problem 1** Give a general lower bound for $F_v(3'; J_4)$, or any nonobvious lower bound for $F_v(3, 3; J_4)$, which are not easily implied by known bounds for other more studied parameters.

Similarly, we do not know much about the cases like $F_v(2'; K_4)$, $F_v(3'; K_4)$, or $F_v(4'; J_5)$: no general methods are known to obtain good lower or upper bounds.

**Problem 2** Does there exist a $J_4$-free graph $G$ such that every set of $|V_G|/2$ vertices induces a triangle?
If not, this could help in the analysis of $F_v(3, 3; J_4)$. We may consider this problem in a more general case. By using density arguments, it is known how to obtain upper bounds on $F_v(3'; K_4)$, but not on $F_v(3, 3; J_4)$ or $F_v(3'; J_4)$.

Except for $F_e(3, 4; K_5) \geq 22$ [35], we do not know of any other nonobvious bounds for: (a) $F_e(3, 4; J_5)$, (b) $F_v(4, 4; J_5)$, (c) $F_e(3, 4; K_5)$, or (d) $F_e(4, 4; J_5)$. Case (a) might be solvable with computational methods. We also think that case (b), while far from easy, is easier than (c) and much easier than (d). For a similar case of $F_v(4, 4; K_5)$, the best known bounds were obtained in [1, 33]:

$$19 \leq F_e(2, 2, 2, 4; K_5) \leq F_v(4, 4; K_5) \leq 23.$$ 

Finally, we state a related existence problem, which was already raised in an earlier work [34], and which is also described at the end of Sect. 2.

**Problem 3** (a) $F_e(3, 3; K_1 + C_4) = \emptyset$? (b) $F_e(3, 3; P_2 \cup P_3) = \emptyset$?

We note that the YES answer to part (a) implies YES answer to part (b), which eliminates one YES/NO combination of answers (out of four possible ones). This problem can be rephrased in some interesting ways. For example, part (a) is equivalent to the following question: *Does there exist any $W_5$-free graph which is not a union of two triangle-free graphs?*

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**Data Availability** The witness graphs for Theorems 2–6, as well as the code implementing algorithm A in Section 3.1 are available at https://www.cs.rochester.edu/folkmanj4/.

**Declarations**

**Conflict of interest** No conflicts.

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