SHARP SOBOLEV INEQUALITIES FOR VECTOR VALUED MAPS

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Let \((M, g)\) be a smooth compact Riemannian \(n\)-manifold, \(n \geq 3\). Let also \(p \geq 1\) be an integer, and \(M^p_s(\mathbb{R})\) be the vector space of symmetrical \(p \times p\) real matrix. Namely the vector space of \(p \times p\) real matrix \(S = (S_{ij})\) which are such that \(S_{ij} = S_{ji}\) for all \(i, j = 1, \ldots, p\). We regard the elements \(S = (S_{ij})\) in \(M^p_s(\mathbb{R})\) as bilinear forms on \(\mathbb{R}^p\) by letting \(S(X, Y) = \sum_{i,j=1}^p S_{ij} X_i Y_j\), where \(X = (X^1, \ldots, X^p)\) and \(Y = (Y^1, \ldots, Y^p)\). We let \(H^2_p(M)\) be the Sobolev space consisting of \(p\)-maps \(U : M \to \mathbb{R}^p, U = (u_1, \ldots, u_p)\), which are such that the \(u_i\)'s are all in the standard Sobolev space \(H^2_1(M)\) consisting of functions in \(L^2(M)\) with one derivative in \(L^2\).

We also let \(A : M \to M^p_s(\mathbb{R})\) smooth, \(A = (A_{ij})\), be such that \(A(x)\) is positive for all \(x \in M\) as a bilinear form. Then we consider Sobolev inequalities like

\[
\left( \int_M |U|^2^\ast \, dv_g \right)^{2/2^\ast} \leq K \int_M |\nabla U|^2 \, dv_g + \Lambda \int_M A(U, U) \, dv_g ,
\]

(0.1)

where the inequality is required to hold for all \(U \in H^2_p(M)\), \(K, \Lambda > 0\) are positive constants, \(A\) is regarded as a bilinear form, \(dv_g\) is the Riemannian volume element of \(g\), and the exponent \(2^\ast = 2n/(n - 2)\) is critical from the Sobolev viewpoint. Here, in (0.1), \(|U|^{2^\ast} = \sum_{i=1}^p |u_i|^{2^\ast}\), \(|\nabla U|^2 = \sum_{i=1}^p |\nabla u_i|^2\), and \(A(U, U) = \sum_{i,j=1}^p A_{ij} u_i u_j\) when we write that \(U = (u_1, \ldots, u_p)\). The questions we ask in this paper are: what is the value \(K_n\) of the sharp \(K\) in (0.1), does the corresponding sharp inequality hold true, and if yes, does its saturated version (where \(\Lambda\) is lowered to its minimum value under the constraint \(K = K_n\)) possess extremal functions. When \(p = 1\), we are back to the classical setting of the Sobolev inequality for functions. Inequality (0.1) when \(p \geq 2\), and the above sequence of questions, are the natural extensions to vector valued maps of the \(AB\)-program which was developed in the case of functions. Possible references in book form for the problem in the case of functions and the \(AB\)-program are Druet and Hebey [10], and Hebey [13]. In what follows we let \(K_n\) be the sharp constant \(K\) in the Euclidean Sobolev inequality \(\|u\|_{2^\ast} \leq K_n \|\nabla u\|_2\).

Then

\[
K_n = \sqrt[2^\ast]{\frac{4}{n(n - 2)\omega_n^{2/n}}} ,
\]

(0.2)

where \(\omega_n\) is the volume of the unit \(n\)-dimensional sphere. In the sequel, we say that a matrix \(A = (A_{ij})\) is cooperative if \(A_{ij} \geq 0\) for all \(i \neq j\). When \(A : M \to M^p_s(\mathbb{R})\) is a map, \(A\) is said to be cooperative in \(M\) if \(A_{ij}(x) \geq 0\) for all \(i \neq j\), and all \(x \in M\).

We also say that \(A\) is globally irreducible if the index set \{1, \ldots, \(p\)\} does not split in two disjoint subsets \(\{i_1, \ldots, i_k\}\) and \(\{j_1, \ldots, j_{k'}\}\), \(k + k' = p\), such that \(A_{i\alpha j\beta} \equiv 0\) for all \(\alpha = 1, \ldots, k\) and \(\beta = 1, \ldots, k'\). A \(p\)-map \(U = (u_1, \ldots, u_p)\) is said to be nonnegative if the \(u_i\)'s are all nonnegative functions (i.e. \(u_i \geq 0\) for all \(i\)), weakly...
positive if the \(u_i\)'s are all positive functions unless they are identically zero (i.e., for any \(i\), either \(u_i > 0\) or \(u_i = 0\)), and strongly positive if the \(u_i\)'s are all positive functions (i.e \(u_i > 0\) for all \(i\)). The map \(U\) is said to be of undeterminate sign if neither \(U\) nor \(-U\) is nonnegative. For \(0 < \theta < 1\), we let \(C_{p,\theta}^2(M)\) be the space of \(p\)-maps with components in \(C^{2,\theta}(M)\). If \(\mathcal{H}\) is a subset of \(C_{p,\theta}^2(M)\), invariant under the scaling \(U \rightarrow \lambda U\) for \(\lambda\) real, we refer to the \(L^2\)-normalized subset of \(\mathcal{H}\) as the subset of \(\mathcal{H}\) consisting of the \(U\) in \(\mathcal{H}\) such that \(\int_M |U|^2 \, dv_g = 1\). At last, we let \(S_g\) be the scalar curvature of \(g\). Our result states as follows.

**Theorem 0.1.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\), \(p \geq 1\) an integer, and \(A : M \rightarrow M^*_p(\mathbb{R})\) smooth and such that \(A(x)\) is positive as a bilinear form for all \(x \in M\). The value \(K_s\) of the sharp constant \(K\) in (0.1) is \(K_s = K^2_n\), where \(K_n\) is given by (0.2), and there exists \(\Lambda > 0\) such that the sharp inequality

\[
\left( \int_M |U|^2 \, dv_g \right)^{2/2^*} \leq K_n^2 \int_M |\nabla U|^2 \, dv_g + \Lambda \int_M A(U, U) \, dv_g
\]

(0.3)

holds for all \(U \in H^2_{1,p}(M)\), where \(A\) is regarded as a bilinear form. Moreover, if \(\Lambda_0(g)\) stands for the lowest \(\Lambda\) in (0.2), then \(\Lambda_0(g) > 0\) and, when \(n \geq 4\),

\[
A_{ii}(x) \geq \frac{(n - 2)K_n^2}{4(n - 1)\Lambda_0(g)} S_g(x)
\]

(0.4)

for all \(i = 1, \ldots, p\) and all \(x \in M\), where the \(A_{ij}\)'s are the components of \(A\). At last, if the inequality in (0.4) is strict for all \(i\) and all \(x\), then the sharp and saturated inequality

\[
\left( \int_M |U|^2 \, dv_g \right)^{2/2^*} \leq K_n^2 \int_M |\nabla U|^2 \, dv_g + \Lambda_0(g) \int_M A(U, U) \, dv_g
\]

(0.5)

possesses extremal maps, namely nontrivial \(p\)-maps in \(C_{p,\theta}^2(M)\) which realize the equality in (0.4), and the \(L^2\)-normalized set of such extremal maps is precompact in the \(C_{p,\theta}^2\)-topology, where \(0 < \theta < 1\). When no specific assumption is made on \(A\), extremal maps might be of undeterminate sign. They can be chosen weakly positive if \(A\) is cooperative, and strongly positive if \(A\) is also globally irreducible.

When \(p = 1\), as already mentioned, we are back to the classical Sobolev inequality for functions. The validity of the classical sharp inequality for functions on arbitrary manifolds was proved in Hebey and Vaugon [14, 15]. The existence of extremal functions (and the above result when \(p = 1\)) was studied in Djadli and Druet [8], and Hebey and Vaugon [16]. Possible references in book form on the sharp classical Sobolev inequality are Druet and Hebey [10], and Hebey [12]. We refer also to Collion, Hebey and Vaugon [4], Ghoussoub and Robert [11], Humbert and Vaugon [13], and Robert [21]. An easy corollary to Theorem 0.1 is that sharp and saturated inequalities like (0.5) always possess extremal maps when \((M, g)\) has nonpositive scalar curvature and \(n \geq 4\). Another possible corollary, which follows from the theorem and the resolution of the Yamabe problem by Aubin [2] and Schoen [22], see Section 6, is that if \(n \geq 4\), \(A\) does not depend on \(x\), and \((M, g)\) has constant scalar curvature, then (0.5) possesses extremal maps.

We prove Theorem 0.1 in Sections 1, 2, and 4 below. That \(K_s = K^2_n\), and that (0.3) and (0.4) are true, easily follows from what has been done in the scalar
case of Sobolev type inequalities. This is discussed in Section 1. The difficult part is to prove existence and compactness of extremal maps. This is discussed in Sections 2 and 3. Section 4 is devoted to the proof of the above mentioned corollaries to Theorem 0.1. When \( p \geq 2 \), contrary to the scalar case where the maximum principle for functions can be applied, there are no maximum principle for the equations associated to inequalities like (0.3). We have to deal with maps of undeterminate sign, and not only with positive, or even nonnegative maps. The case of maps is more involved than the case of functions.

1. Proof of the first part of Theorem 0.1

We let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 3 \), \( p \geq 1 \) an integer, and \( A : M \to M_1^p(\mathbb{R}) \), \( A = (A_{ij}) \), be smooth and such that \( A(x) \) is positive as a bilinear form for all \( x \in M \). We know from Hebey and Vaugon [14, 15] that there exists \( \delta > 0 \) such that for any \( U \in \mathcal{B}_H \) where \( u \) such that the components \( u_{ij} \) of bilinear forms. Letting \( \Lambda = \Lambda_0 \), then, by the developments in Aubin [2], we can take \( \Lambda = \Lambda_0 \), and it follows from (1.1) that for any \( U \in H^2_1(M) \),

\[
\left( \int_M |U|^2 \, dg \right)^{2/2^*} \leq K^2_n \int_M |\nabla U|^2 \, dg + B \int_M |U|^2 \, dg ,
\]

where \( H^2_1(M) \) is the Sobolev space of functions in \( L^2(M) \) with one derivative in \( L^2 \). Since \( 2/2^* \leq 1 \), \((a+b)/2^* \leq a^{2/2^*} + b^{2/2^*} \) for \( a, b \geq 0 \), and it follows from (1.1) that for any \( U \in H^2_1(M) \),

\[
\left( \int_M |U|^2 \, dg \right)^{2/2^*} \leq K^2_n \int_M |\nabla U|^2 \, dg + B \int_M |U|^2 \, dg ,
\]

where \( H^2_1(M) \) is the space of \( p \)-maps \( U : M \to \mathbb{R}^p \), \( U = (u_1, \ldots, u_p) \), which are such that the components \( u_i \) are all in \( H^2_1(M) \). Since we assumed that \( A(x) \) is positive for all \( x \) as a bilinear form, there exists \( t > 0 \) such that \( \delta_{ij} \leq t A_{ij}(x) \) for all \( x \), in the sense of bilinear forms. Letting \( \Lambda = Bt \), we get that for any \( U \in H^2_1(M) \),

\[
\left( \int_M |U|^2 \, dg \right)^{2/2^*} \leq K^2_n \int_M |\nabla U|^2 \, dg + \int_M A_\Lambda(U, U) \, dg ,
\]

where \( A_\Lambda = \Lambda A \), and \( A \) is regarded as a bilinear form. Conversely, taking \( U \) in (0.1) such that the components \( u_j \) of \( U \) are all zero if \( j \neq i \), and \( u_i = u \) is arbitrary, it clearly follows from an inequality like (0.1) that for any \( i \), and any \( u \in H^2_1(M) \),

\[
\left( \int_M |u|^2 \, dg \right)^{2/2^*} \leq K \int_M |\nabla u|^2 \, dg + \Lambda \int_M A_{ii} u^2 \, dg .
\]

In particular, see for instance Hebey [13], we get from (1.4) that we necessarily have that \( K \geq K^2_n \). This inequality, together with (1.3), gives that \( K_n \geq K^2_n \). By (1.3) we also have that (0.3) is true, and by the definition of \( \Lambda_0(g) \), we can take \( \Lambda = \Lambda_0(g) \) in (0.3). In particular, (0.5) is true. Like when passing from (0.1) to (1.4), it follows from the sharp and saturated (0.5) that for any \( i \), and any \( u \in H^2_1(M) \),

\[
\left( \int_M |u|^2 \, dg \right)^{2/2^*} \leq K^2_n \int_M |\nabla u|^2 \, dg + \Lambda_0(g) \int_M A_{ii} u^2 \, dg .
\]

Taking \( u = 1 \) in (1.5), we get that \( \Lambda_0(g) \int_M A_{ii} \, dv_g \geq V^{2/2^*}_g \) for all \( i \), where \( V_g \) is the volume of \( M \) with respect to \( g \). Since \( A(x) \) is positive for all \( x \), the \( A_{ii} \)’s are positive functions, and \( \Lambda_0(g) \) has to be positive. By the developments in Aubin [2], we also get with (1.5) that when \( n \geq 4 \), (0.5) has to be true for all \( x \in M \), and all
i. More precisely, given \( \delta > 0 \) small, \( \varepsilon > 0 \) small, and \( x_0 \in M \), we let \( u_{x_0}^\varepsilon \) be the function defined by
\[
u_{x_0}^\varepsilon = (\varepsilon + r^2)^{-1-n/2} - (\varepsilon + \delta^2)^{-1-n/2}
\]
if \( r \leq \delta \), and \( u_{x_0}^\varepsilon = 0 \) if not, where \( r = d_g(x_0, \cdot) \). Then, see Aubin [2],
\[
\frac{\int_M |\nabla u_{x_0}^\varepsilon|^2 dv_g + \frac{\Delta_0(g)}{K_n} \int_M A_{ij}(u_{x_0}^\varepsilon)^2 dv_g}{(\int_M |u_{x_0}^\varepsilon|^2^* dv_g)^{2/2^*}} < \frac{1}{K_n^2}
\]
if \( n \geq 4 \), \( \varepsilon > 0 \) is sufficiently small, and (0.3) is not satisfied at \( x_0 \). This proves (0.4).

Now, in order to end the proof of Theorem 0.1, it remains to prove the assertions in the theorem concerning extremal maps. This is the subject of Sections 2 and 4.

2. PROOF OF THE SECOND PART OF THEOREM 0.1

As in Section 1 we let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 3 \), \( p \geq 1 \) an integer, and \( A : M \to M_p^s(\mathbb{R}) \), \( A = (A_{ij}) \), be smooth and such that \( A(x) \) is positive as a bilinear form for all \( x \in M \). We know from Section 1 that \( K_s = K_n^s \), and that (0.3) and (0.4) are true. It remains to prove that if the inequality in (0.4) is strict for all \( i \) and all \( x \), then the sharp and saturated inequality (0.5) possesses extremal maps, and the \( L^{2^*} \)-normalized set of such extremal maps is precompact in the \( C_{p,\theta}^{2,\theta} \)-topology, where \( 0 < \theta < 1 \). It remains also to prove that extremal maps, when they exist, are in general of undeterminate sign, but that they can be chosen weakly positive if \(-A\) is cooperative, and strongly positive if \( A \) is also globally irreducible. We claim that the existence of extremal maps when the inequality in (0.4) is strict, and compactness of extremal maps, follow from Lemma 2.1 below. In the sequel, \( \Delta_g = -\text{div}_g \nabla \) is the Laplace-Beltrami operator with respect to \( g \).

**Lemma 2.1.** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( n \geq 4 \), \( p \geq 1 \) an integer, and \( A_0 : M \to M_p^s(\mathbb{R}) \), \( A_0 = (A_{ij}^0) \), be smooth, such that \( A_0(x) \) is positive as a bilinear form for all \( x \in M \), and such that for any \( i \) and any \( x \), \( A_{ij}^0(x) > \frac{2^*}{4(n-1)} S_g(x) \), where \( S_g \) is the scalar curvature of \( g \). Let \( (\Lambda_\alpha)_{\alpha \in \mathbb{N}} \), \( \alpha \in \mathbb{N} \), be a sequence of smooth maps \( A(\alpha) : M \to M_p^s(\mathbb{R}) \) such that \( A_{ij}^0 \to A_{ij}^0 \) in \( C^{0,\theta}(M) \) as \( \alpha \to +\infty \), for all \( i, j \), where the \( A_{ij}^0 \)'s are the components of \( A(\alpha) \), and \( 0 < \theta < 1 \). Let also \((U_\alpha)_{\alpha \in \mathbb{N}}\) be a sequence of \( C^{2,\theta}_p \)-solutions of the \( p \)-systems
\[
\Delta_g u_{\alpha}^i + \sum_{j=1}^p A_{ij}^\alpha(x) u_{\alpha}^j = \lambda_{\alpha} |u_{\alpha}^i|^{2^*-2} u_{\alpha}^i \tag{2.1}
\]
for all \( i \) and all \( \alpha \), such that \( \int_M |U_\alpha|^2 dv_g = 1 \) and \( 0 < \lambda_{\alpha} \leq K_n^{-2} \) for all \( \alpha \), where the \( u_{\alpha}^i \)'s are the components of \( U_\alpha \). Then, up to a subsequence, \( U_\alpha \to U^0 \) in \( C^{2,\theta}_p(M) \) as \( \alpha \to +\infty \), where \( U^0 \) is a nontrivial \( p \)-map in \( C^{2,\theta}_p(M) \).

The proof of Lemma 2.1 is postponed to Section 4. We prove here, in Section 2, that when the inequality in (0.4) is strict for all \( i \), the existence of extremal maps, and compactness of extremal maps, follow from the lemma. The \( A(\alpha) \)'s in our context are either like \( A(\alpha) = \Lambda_\alpha K_n^{-2} A \), where the \( \Lambda_\alpha \)'s are real numbers converging to \( \Lambda_0(g) \), or like \( A(\alpha) = \Lambda_\alpha(g) K_n^{-2} A \) for all \( \alpha \), where \( \Lambda_0(g) \) and \( A \) are as in Theorem 0.1. Extensions of Lemma 2.1 to higher energies, in the case of conformally flat manifolds, are in Hebey [12]. The manifold in Lemma 2.1 needs
the energies \( E \) converge in such that \( \Phi \) in (0.4) is strict for all \( i \) \( A \) \( \alpha \) when we let \( \Lambda = \Lambda \alpha \) all \( < \) when \( \Lambda \) not to be conformally flat. Possible references on elliptic systems, not necessarily like [24], are Amster, De Nápoli, and Mariani [11], De Figueiredo and Ding [10], De Figueiredo and Felmer [7], Hulshof, Mitidieri and Vandervorst [17], Mitidieri and Sweers [19], and Sweers [23].

We assume that Lemma 2.1 is true. Given \( \Lambda > 0 \), and \( \mathcal{U} \in H^2_{1,p}(M) \), we define the energies \( E^\Lambda_g(\mathcal{U}) \) and \( \Phi_g(\mathcal{U}) \) by

\[
E^\Lambda_g(\mathcal{U}) = \int_M |\nabla \mathcal{U}|^2 dv_g + \frac{\Lambda}{K_n^2} \int_M A(\mathcal{U}, \mathcal{U}) dv_g \tag{2.2}
\]

and \( \Phi_g(\mathcal{U}) = \int_M |\mathcal{U}|^2 dv_g \). By definition of \( \Lambda_0(g) \),

\[
\inf_{\mathcal{U} \in \mathcal{H}} E^\Lambda_g(\mathcal{U}) < \frac{1}{K_n^2} \tag{2.3}
\]

when \( \Lambda < \Lambda_0(g) \), where \( \mathcal{H} \) is the set consisting of the \( \mathcal{U} \in H^2_{1,p}(M) \) which are such that \( \Phi_g(\mathcal{U}) = 1 \). Let \( (\Lambda_\alpha)_\alpha \) be a sequence of positive real numbers such that \( \Lambda_\alpha < \Lambda_0(g) \) for all \( \alpha \), and \( \Lambda_\alpha \to \Lambda_0(g) \) as \( \alpha \to +\infty \). Let also \( \lambda_\alpha \) be the infimum in (2.3) when we let \( \Lambda = \Lambda_\alpha \). Since \( \Lambda > 0 \) as a bilinear form, \( \lambda_\alpha \) is positive for all \( \alpha \). By the strict inequality in (2.3), see Hebey [12], for any \( \alpha \), there exists \( \mathcal{U}_\alpha = (u_\alpha^1, \ldots, u_\alpha^n) \) a minimizer for \( \lambda_\alpha \). In particular, the \( \mathcal{U}_\alpha \)'s are solutions of the \( p \)-systems

\[
\Delta_g u_\alpha^i + \frac{\Lambda_\alpha}{K_n^2} \sum_{j=1}^p A_{ij}(x)u_\alpha^j = \lambda_\alpha |u_\alpha^i|^{2^*-2}u_\alpha^i \tag{2.4}
\]

for all \( i \), and such that \( \Phi_g(\mathcal{U}_\alpha) = 1 \) and \( \mathcal{U}_\alpha \in C^2_{p,\theta}(M) \) for all \( \alpha \), where \( 0 < \theta < 1 \).

Up to a subsequence, we may assume that \( \lambda_\alpha \to \lambda_0 \) as \( \alpha \to +\infty \). If the inequality in (2.3) is strict for all \( i \), we can apply Lemma 2.1 with \( A(\alpha) = \Lambda_\alpha K_n^{-2}A \), and \( A_0 = \Lambda_0(g)K_n^{-2}A \). By Lemma 2.1, we then get that, up to a subsequence, the \( \mathcal{U}_\alpha \)'s converge in \( C^2_{p,\theta}(M) \) to some \( \mathcal{U}^0 \). Then \( \Phi_g(\mathcal{U}^0) = 1 \), and, by (2.4),

\[
\Delta_g u_0^i + \frac{K_n^2}{\lambda_0} \sum_{j=1}^p A_{ij}^0(x)u_0^j = \lambda_0 |u_0^i|^{2^*-2}u_0^i \tag{2.5}
\]

for all \( i \), where the \( A_{ij}^0 \)'s are the components of the matrix \( A_0(g) = \Lambda_0(g)A \). Since we have that \( \lambda_\alpha < K_n^{-2} \) for all \( \alpha \), we can write that \( \lambda_0 \leq K_n^{-2} \). On the other hand, multiplying (2.5) by \( u_0^i \), integrating over \( M \), and summing over \( i \), we get that

\[
E^{\Lambda_0(g)}(\mathcal{U}^0) = \lambda_0 ,
\]

where \( E^{\lambda}_g \) is given by (2.2). By the definition of \( \Lambda_0(g) \), it follows that \( \lambda_0 \geq K_n^{-2} \).

In particular, \( \lambda_0 = K_n^{-2} \), and \( \mathcal{U}^0 \) is a nontrivial extremal map for (0.5). This proves the above claim that if the inequality in (0.4) is strict for all \( i \), then the existence of extremal maps follows from Lemma 2.1.

Concerning compactness, let \( \mathcal{H}_0 \) be the \( L^2 \)-normalized set of extremal maps for (0.5). Then \( \mathcal{H}_0 \) consists of the \( \mathcal{U}^0 \in H^2_{1,p}(M) \) such that \( \Phi_g(\mathcal{U}^0) = 1 \), and

\[
E^{\Lambda_0(g)}(\mathcal{U}^0) = \inf_{\{\Phi_g(\mathcal{U})=1\}} E^{\Lambda_0(g)}(\mathcal{U}) = \frac{1}{K_n^2} ,
\]
where \( E_i^0 \) is given by (2.2). In particular, the extremal maps \( U^0 \) in \( H_0 \) are solutions of the \( p \)-system
\[
\Delta_g u_i^0 + \frac{1}{K_n^2} \sum_{j=1}^{p} A_{ij}(x)u_j^0 = K_{n}^{-2}|u_i^0|^2 - 2u_i^0
\]  
(2.6)
for all \( i \), and such that \( \Phi_0(\partial U^0) = 1 \), where the \( A_{ij}^0 \)'s are the components of the matrix \( A_0(g) = \Lambda_0(g)A \), and the \( u_j^0 \)'s are the components of \( U \). Such \( U^0 \)'s, see Hebey [12] are in \( C^{2,\theta}_p(M) \), where \( 0 < \theta < 1 \). If the inequality in (0.4) is strict for all \( i \), we can apply Lemma 2.1 with \( A(\alpha) = A_0 = \Lambda_0(g)K_{n}^{-2}A \). We get that any sequence in \( H_0 \) possesses a converging subsequence in \( C^{2,\theta}_p(M) \). In particular, \( H_0 \) is precompact in the \( C^{2,\theta}_p \)-topology. This proves the above claim that if the inequality in (0.4) is strict for all \( i \), then the compactness of the set of extremal maps in Theorem 0.1 follows from Lemma 2.1.

Now, in order to end this section, we discuss the assertions in Theorem 0.1 concerning the sign of extremal maps. Extremal maps for (0.5) are solutions of systems like (2.7), \( p \geq 2 \), and such that the system possesses solutions with the property that the factors of the solutions are nonnegative, nonzero, but with zeros in \( M \). Such a phenomenon does not occur when \( p = 1 \) since, when \( p = 1 \), the maximum principle can be applied and nonnegative solutions are either identically zero or everywhere positive. On the other hand, we recover the maximum principle for (2.7) if we assume that \( -A_0 \) is cooperative. Indeed, when \( -A_0 \) is cooperative, nonnegative solutions of (2.7) are such that
\[
\Delta_g u_i + A_{ij}^0 u_j \geq \lambda u_i^2 - 1
\]
for all \( i \), and the classical maximum principle for functions can be applied so that either \( u_i > 0 \) everywhere in \( M \), or \( u_i \equiv 0 \). In particular, in this case, nonnegative solutions of (2.7) are weakly positive. Still when \( -A_0 \) is cooperative, if \( U \) is a weakly positive solution of the system, with zero factors, then \( A_0 \) can be factorized in blocs with respect to the zero and nonzero components of \( U \). More precisely, if we write \( U = (u_1, \ldots, u_k, 0, \ldots, 0) \) with \( k < p \), and \( u_i > 0 \) for all \( i \), then
\[
A_0 = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix},
\]  
(2.8)
where \( S : M \to M^*_k(\mathbb{R}) \), \( T : M \to M^*_{p-k}(\mathbb{R}) \), and the \( 0 \)'s are null matrix of respective order \( k \times (p-k) \) and \( (p-k) \times k \). This easily follows from the equations \( \sum_{j=1}^{k} A_{ij}^0 u_j = 0 \) for all \( i \geq k + 1 \), so that we necessarily have that \( A_{ij}^0 = 0 \) for all \( i \geq k + 1 \) and \( j \leq k \). In this case, the \( p \)-system (2.7) splits into two independent systems – a \( k \)-system where \( A_0 \) is replaced by \( S \), and a \((p-k)\)-system where \( A_0 \) is
replaced by $T$. In particular, if $-A_0$ is cooperative and $A_0$ is globally irreducible, so that (2.8) cannot be true, then any weakly positive solution of the system is also strongly positive.

Coming back to minimizers, and to Theorem 0.1 the first assertion concerning the sign of extremal maps in Theorem 0.1 is that extremal maps might be of undeterminate sign when no specific assumption is made on $A$. Of course this has to be understood when $p \geq 2$ since, when $p = 1$, the maximum principle for functions can be applied. When $p = 1$, extremal functions are either positive or negative. We assume in what follows that $\beta \geq 0$. For $\beta \geq 0$, $\beta \neq 0$, be such that it is nontrivial and nonnegative. Noting that $A(U, U) = A'(U', U')$, we easily get that if $U_0 = (u_0, v_0)$ is an extremal map for the sharp and saturated inequality (2.3), then $U'_0$ is an extremal map for the modified problem we get by replacing $A$ by $A'$, where $A, A'$ are as in (2.4). Since $U_0$ is an extremal map for (2.3), it is also a minimizer for $F = E/\Phi_g^{2/\alpha}$, where $\Phi_g(U) = \int_M |U|^2 dv_g$, $E(U) = E_g^0(U)$ as in (2.2) and $\Lambda = \Lambda_0(g)$. In particular, $F(U_0) \leq F(U'_0)$, and it follows that

$$\int_M \beta u_0 v_0 dv_g \leq 0.$$  \quad (2.10)

Since $\beta \geq 0$, $-A'$ is cooperative, and we can also write that $A'(U_0, U_0) \leq A'(U'_0, U'_0)$, where $U_0$ is given by $U_0 = (|u_0|, |v_0|)$. In particular, $U_0$ is also an extremal map for the modified problem we get by replacing $A$ by $A'$. Since $\beta \neq 0$, $A'$ is globally irreducible, and it follows from the above discussion that $|u_0|$ and $|v_0|$ are positive functions. Then, by (2.4), $U_0$ is like $U_0 = (u_0, -v_0)$ or $U_0 = (-u_0, v_0)$ where $u_0$ and $v_0$ are positive functions. In particular, neither $U_0$ nor $-U_0$ are nonnegative. Clearly, this type of discussion extends to integers $p \geq 2$. For instance, when $p = 3$, choosing $A$ such that $A_{12}, A_{23} \geq 0$ and $A_{13} \leq 0$, we easily construct minimizers like $U_0 = (u_0, -v_0, w_0)$ or $U_0 = (-u_0, v_0, -w_0)$, where $u_0, v_0, w_0$ are positive functions. This proves the above claim that, when no specific assumption is made on $A$, extremal maps for (2.3) might be of undeterminate sign. On the contrary, if we assume that $-A$ is cooperative, then $A(U, U) \leq A(U', U')$ for all $U = (u_1, \ldots, u_p)$, where $U = (|u_1|, \ldots, |u_p|)$. In particular, if $U_0$ is an extremal map for (2.3), then $U_0$ is also an extremal map for (2.5). By the above discussion for systems like (2.7), $U_0$ has to be weakly positive since $-A$ is cooperative. It is even strongly positive if $A$ is also globally irreducible. In particular, extremal maps for (2.3) can be chosen weakly positive when $-A$ is cooperative, and even strongly positive $A$ is also globally irreducible. This proves the assertions in Theorem 0.1 concerning the sign of extremal maps. Up to Lemma 2.1 Theorem 0.1 is proved.

When $-A$ is cooperative, and $A$ is globally irreducible, we can prove the stronger result that any extremal map $U$ for (2.5) has to be such that either $U$ or $-U$ is strongly positive. In order to see this we first note that, according to the above proof, when $-A$ is cooperative, and $A$ is globally irreducible, the components of an extremal map for (2.5) are either positive or negative functions. By contradiction,
up to permuting the indices, we write that $U = (u_1, \ldots, u_k, -u_{k+1}, \ldots, -u_p)$ is an extremal map for (1.3), where the $u_i$'s are positive functions. We let $U'$ be given by $U' = (u_1, \ldots, u_p)$. Writing that $E^\Lambda_g(U) \leq E^\Lambda_g(U')$, where $E^\Lambda_g$ is as in (2.2) and $\Lambda = \Lambda_0(g)$, we get that

$$
\sum_{i \in \mathcal{H}_k, j \in \mathcal{H}_{k+1}} A_{ij} u_i u_j \geq 0,
$$

where $\mathcal{H}_k = \{1, \ldots, k\}$, and $\mathcal{H}_{k+1} = \{k + 1, \ldots, p\}$. The contradiction follows since $-A$ is cooperative, $A$ is globally irreducible, and the $u_i$'s are positive functions. This proves that when $-A$ is cooperative, and $A$ is globally irreducible, extremal maps $U$ for (0.5) are such that either $U$ or $-U$ is strongly positive.

3. Applications of Theorem 0.1

We discuss the two corollaries, or applications, of Theorem 0.1 we briefly mentioned in the introduction. The first application, stating that the sharp and saturated scalar Sobolev inequality on the unit sphere (7.1) possesses extremal maps when $(M, g)$ has nonpositive scalar curvature and $n \geq 4$, is easy to get. Indeed, since $A$ in Theorem 0.1 is such that $A(x)$ is positive in the sense of bilinear forms for all $x$, we clearly have that $A_{ij}(x) > 0$ for all $x$ and all $i$. In particular, (0.4) is always true when $(M, g)$ has nonpositive scalar curvature.

A less obvious result is the second application stating that if $n \geq 4$, $A$ does not depend on $x$, and $(M, g)$ has constant scalar curvature, then (0.5) possesses extremal maps. When $(M, g)$ is not conformally diffeomorphic to the unit sphere, the result easily follows from the developments in Aubin [2] and Schoen [22]. The energy estimates in Aubin [2] and Schoen [22] give that, in this case, when $(M, g)$ is not conformally diffeomorphic to the unit sphere, the inequality in (0.4) has to be strict. Then we can apply Theorem 0.1. When $(M, g)$ is the unit sphere, or conformally diffeomorphic to the unit sphere, the only problem is when equality holds in (0.4) for one, or at least one $i$. For such an $i$, we claim that we necessarily have that $A_{ij} = 0$ for all $j \neq i$. Assuming for the moment that the claim is true, we easily get with such a claim that there exist extremal maps for (0.5). The sharp and saturated scalar Sobolev inequality on the unit sphere $(S^n, g_0)$ reads as

$$
\left( \int_{S^n} |u|^2 \, dv_{g_0} \right)^{2/2'} \leq K_n^2 \int_{S^n} |\nabla u|^2 \, dv_{g_0} + \omega_n^{-2/n} \int_{S^n} u^2 \, dv_{g_0},
$$

where $\omega_n$ is the volume of the unit sphere. In particular, see for instance Hebey [13], for a reference in book form, there is a whole family of extremal functions for the inequality, including constant functions. Let $u_0$ be one of these functions. We choose $u_0$ such that $u_0$ is positive and $\|u_0\|_{2'} = 1$. When $(M, g)$ is the unit sphere, equality holds in (1.4) for one $i$, and $A_{ij} = 0$ for all $j \neq i$, the $p$-map $U = (u_1, \ldots, u_p)$, where $u_i = u_0$ and $u_j = 0$ for $j \neq i$, is clearly an extremal map for (1.5). In particular, (0.5) possesses an extremal map. It remains to prove the above claim that when $(M, g)$ is the unit sphere, and equality holds in (1.4) for one $i$, we necessarily have that $A_{ij} = 0$ for all $j \neq i$. In order to prove this, we proceed by contradiction. We assume that $(M, g)$ is the unit sphere, that equality holds in (1.4) for one $i$, and that there exists $j \neq i$ such that $A_{ij} \neq 0$. We let $U_\varepsilon = (u_1^\varepsilon, \ldots, u_p^\varepsilon)$ be given by $u_i^\varepsilon = u_0$, where $u_0$ is as above, $u_j^\varepsilon = -\varepsilon A_{ij}$, and...
$u^k = 0$ if $k \neq i, j$, where $\varepsilon > 0$ is small. Then, with the notations in Theorem \ref{thm:main1}

\begin{align*}
K_n^2 \int_{S^n} |\nabla U_\varepsilon|^2 dv_{g_0} + \Lambda_0(g) \int_{S^n} A(U_\varepsilon, U_\varepsilon) dv_{g_0} \\
= K_n^2 \int_{S^n} |\nabla u_0|^2 dv_{g_0} + \omega_n^{-2/n} \int_{S^n} u_0^2 dv_{g_0} \\
- 2\Lambda_0(g_0) A^2_{ij} \varepsilon \int_{S^n} u_0 dv_{g_0} + O(\varepsilon^2) \\
= 1 - 2\Lambda_0(g_0) A^2_{ij} \varepsilon \int_{S^n} u_0 dv_{g_0} + O(\varepsilon^2)
\end{align*}

and since \( \int_{S^n} |U_\varepsilon|^2 dv_{g_0} \geq 1 \), we get a contradiction with \[ \ref{eq:main3} \] by choosing $\varepsilon > 0$ sufficiently small. This proves the above claim that when $(M, g)$ is the unit sphere, and equality holds in \[ \ref{eq:main3} \] for one $i$, we cannot have that there exists $j \neq i$ such that $A_{ij} \neq 0$. This also ends the proof of the second application of Theorem \ref{thm:main1} stating that if $n \geq 4$, $A$ does not depend on $x$, and $(M, g)$ has constant scalar curvature, then $(M, g)$ possesses extremal maps.

4. Proof of Lemma \ref{thm:main2}

We prove Lemma \ref{thm:main2} in this Section. We let $(M, g)$ be a smooth compact Riemannian manifold of dimension $n \geq 3$, $p \geq 1$ an integer, and $A_0 : M \rightarrow M_p^*(\mathbb{R})$, $A_0 = (A^0_{ij})$, be smooth and such that $A_0(x)$ is positive as a bilinear form for all $x \in M$. We let $(A(\alpha))_\alpha$, $\alpha \in \mathbb{N}$, be a sequence of smooth maps $A(\alpha) : M \rightarrow M_p^*(\mathbb{R})$ such that $A^0_{ij} \rightarrow A^0_{ij}$ in $C^{0,\theta}(M)$ as $\alpha \rightarrow +\infty$, for all $i, j$, where the $A^0_{ij}$ is the components of $A(\alpha)$, and $0 < \theta < 1$. We let also $(U_\alpha)_\alpha$ be a sequence of $C^{2,\theta}$-solutions of the $p$-systems

\begin{equation}
\Delta_g u^i_\alpha + \sum_{j=1}^p A^\alpha_{ij}(x) u^j_\alpha = \lambda_\alpha |u^i_\alpha|^{2^*-2} u^i_\alpha \tag{4.1}
\end{equation}

for all $i$ and all $\alpha$, such that $\int_M |U_\alpha|^2 dv_g = 1$ and $\lambda_\alpha \leq K_n^{-2}$ for all $\alpha$, where the $u^i_\alpha$ is the components of $U_\alpha$. Since $A_0(x)$ is positive as a bilinear form for all $x$, and since $A^\alpha_{ij} \rightarrow A^0_{ij}$ in $C^{0,\theta}(M)$ as $\alpha \rightarrow +\infty$ for all $i, j$, there exists $K > 0$ such that $A^\alpha_{ij}(x) \geq K \delta_{ij}$ in the sense of bilinear forms, for all $x$ and all $\alpha$ sufficiently large. Multiplying \[ \ref{eq:main1} \] by $u^i_\alpha$, integrating over $M$, and summing over $i$, we then get with the Sobolev inequality that there exists $\lambda > 0$ such that $\lambda_\alpha \geq \lambda$ for all $\alpha$ sufficiently large. Now we define $\tilde{U}_\alpha$ by $\tilde{U}_\alpha = (\tilde{u}^1_\alpha, \ldots, \tilde{u}^p_\alpha)$, where

\begin{equation}
\tilde{u}^i_\alpha = \lambda_\alpha u^i_\alpha \tag{4.2}
\end{equation}

for all $\alpha$ and all $i$. Then, for any $\alpha$, $\tilde{U}_\alpha$ is a solution of the $p$-system

\begin{equation}
\Delta_g \tilde{u}^i_\alpha + \sum_{j=1}^p A^\alpha_{ij}(x) \tilde{u}^j_\alpha = |\tilde{u}^i_\alpha|^{2^*-2} \tilde{u}^i_\alpha \tag{4.3}
\end{equation}

for all $i$. Moreover, since $\lambda_\alpha \leq K_n^{-2}$, we also have that

\begin{equation}
\int_M |\tilde{U}_\alpha|^2 dv_g \leq K_n^{-n} \tag{4.4}
\end{equation}

for all $\alpha$. Lemma \ref{thm:main2} states that, up to a subsequence, $U_\alpha \rightarrow U^0$ in $C^{2,\theta}_p(M)$ as $\alpha \rightarrow +\infty$, where $U^0$ is a nontrivial $p$-map in $C^{2,\theta}_p(M)$. By standard elliptic theory,
and (4.1), it suffices to prove that, up to a subsequence, the \( U_\alpha \)'s are uniformly bounded in \( L^\infty(\mathcal{M}) \). Since \( \lambda_\alpha \in [\lambda, K_\alpha^{-2}] \) for \( \alpha \) large, the \( U_\alpha \)'s are uniformly bounded in \( L^\infty(\mathcal{M}) \) if and only if the \( \tilde{U}_\alpha \)'s are uniformly bounded in \( L^\infty(\mathcal{M}) \). In particular, Lemma 2.1 reduces to proving that, up to a subsequence, there exists \( C > 0 \) such that

\[
|\tilde{U}_\alpha| \leq C \tag{4.5}
\]

in \( \mathcal{M} \), for all \( \alpha \), where \( |\tilde{U}_\alpha| = \sum_i |\tilde{u}_i^\alpha| \). We prove (4.5) in what follows. Up to a subsequence, by compactness of the embedding \( H^1_\lambda \subset L^2 \), we may assume that \( \tilde{U}_\alpha \rightarrow \tilde{U}^0 \) in \( L^2 \) for some \( \tilde{U}^0 \in H^1_{\lambda, \rho}(\mathcal{M}) \). In other words, we can assume that there are functions \( \tilde{u}_i^0 \) in \( H^1_\lambda(\mathcal{M}) \) such that

\[
\tilde{u}_i^\alpha \rightarrow \tilde{u}_i^0 \quad \text{in} \quad L^2(\mathcal{M}) \tag{4.6}
\]

for all \( i \), as \( \alpha \rightarrow +\infty \). We may also assume that \( \tilde{u}_i^\alpha \rightharpoonup \tilde{u}_i^0 \) weakly in \( H^1_\lambda(\mathcal{M}) \), that \( \tilde{u}_i^\alpha \rightharpoonup \tilde{u}_i^0 \) a.e in \( \mathcal{M} \), and that \( |\tilde{u}_i^\alpha|^{2^*-2}\tilde{u}_i^\alpha \rightharpoonup |\tilde{u}_i^0|^{2^*-2}\tilde{u}_i^0 \) weakly in \( L^{2^*/(2^*-1)}(\mathcal{M}) \) for all \( i \). In particular, \( \tilde{U}^0 \) is a solution of the limit equation

\[
\Delta \tilde{u}_i^0 + \sum_{j=1}^p A_{ij}(x)\tilde{u}_j^0 = |\tilde{u}_i^0|^{2^*-2}\tilde{u}_i^0 \tag{4.7}
\]

for all \( i \). Then, see, for instance, Hebey [12], we can prove that \( \tilde{U}^0 \) is in \( C^{2,\theta}_p(\mathcal{M}) \). For \( (x_\alpha)_\alpha \) a converging sequence of points in \( \mathcal{M} \), and \( (\mu_\alpha)_\alpha \) a sequence of positive real numbers converging to zero, we define a 1-bubble as a sequence \( (B_\alpha)_\alpha \) of functions in \( \mathcal{M} \) given by

\[
B_\alpha(x) = \left( \frac{\mu_\alpha}{\mu_\alpha^2 + d(x_\alpha, x)^2 / n(n-2)} \right)^{\frac{n-2}{4}}. \tag{4.8}
\]

The \( x_\alpha \)'s are referred to as the centers and the \( \mu_\alpha \)'s as the weights of the 1-bubble \( (B_\alpha)_\alpha \). We define a \( p \)-bubble as a sequence \( (B_\alpha)_\alpha \) of \( p \)-maps such that, if we write that \( B_\alpha = (B_{\alpha, 1}^p, \ldots, B_{\alpha, p}^p) \), then \( (B_{\alpha, i}^p)_\alpha \) is a 1-bubble for exactly one \( i \), and for \( j \neq i \), \( (B_{\alpha, j}^p)_\alpha \) is the trivial zero sequence. In other words, a \( p \)-bubble is a sequence of \( p \)-maps such that one of the components of the sequence is a 1-bubble, and the other components are trivial zero sequences. One remark with respect to the definition (4.8) is that if \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) is given by

\[
u(x) = \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-\frac{n-2}{2}}, \tag{4.9}
\]

then \( u \) is a positive solution of the critical Euclidean equation \( \Delta u = u^{2^*-1} \), where \( \Delta = -\sum \partial^2 / \partial x_i^2 \). More precisely, \( u \) is the only positive solution of the equation in \( \mathbb{R}^n \) which is such that \( u(0) = 1 \) and \( u \) is maximum at 0. All the other positive solutions of the equation \( \Delta u = u^{2^*-1} \) in \( \mathbb{R}^n \), see Caffarelli, Gidas and Spruck [8] and Obata [20], are then given by

\[
u(x) = (n-2)/2 u (\lambda(x - a)), \tag{4.9a}
\]

where \( \lambda > 0 \), and \( a \in \mathbb{R}^n \). We prove (4.9a), and thus Lemma 2.1, in several steps. The first step in the proof is as follows.
Step 4.1. Let $\tilde{U}_\alpha$ and $\tilde{U}^0$ be given by (4.2) and (4.6). If $\tilde{U}^0 \not\equiv 0$, then (4.9) is true. If, on the contrary, $\tilde{U}^0 \equiv 0$, then there exists a $p$-bubble $(B_\alpha)_\alpha$ such that, up to a subsequence,

$$\tilde{U}_\alpha = B_\alpha + R_\alpha$$

(4.10)

for all $\alpha$, where $R_\alpha \to 0$ strongly in $H^2_{1,p}(M)$ as $\alpha \to +\infty$. There also exists $C > 0$ such that

$$d_g(x_\alpha, x) \geq \frac{2}{\alpha} \sum_{i=1}^p |\tilde{u}_i^\alpha(x)| \leq C$$

(4.11)

for all $\alpha$ and all $x \in M$, where the $x_\alpha$‘s are the centers of the 1-bubble from which the $p$-bubble $(B_\alpha)_\alpha$ is defined. In particular, the $|\tilde{U}_\alpha|$‘s are uniformly bounded in any compact subset of $M \setminus \{x_0\}$, and $\tilde{u}_i^\alpha \to 0$ in $C^0_{\text{loc}}(M \setminus \{x_0\})$ for all $i$ as $\alpha \to +\infty$, where $x_0$ is the limit of the $x_\alpha$‘s.

Proof of Step 4.1. By the $H^2_1$-theory for blow-up, see Hebey [12], there are generalized $p$-bubbles $(\tilde{B}_{j,\alpha})$, $j = 1, \ldots, k$, such that, up to a subsequence,

$$\tilde{U}_\alpha = \tilde{U}^0 + \sum_{j=1}^k \tilde{B}_{j,\alpha} + R_\alpha$$

(4.12)

and such that

$$\frac{1}{n} \int_M |\tilde{U}_\alpha|^2\ dv_g = \frac{1}{n} \int_M |\tilde{U}^0|^2\ dv_g + \sum_{j=1}^k E_f(\tilde{B}_{j,\alpha}) + o(1)$$

(4.13)

for all $\alpha$, where $R_\alpha \to 0$ strongly in $H^2_{1,p}(M)$ as $\alpha \to +\infty$, $E_f(\tilde{B}_{j,\alpha})$ is the energy of the generalized $p$-bubble $(\tilde{B}_{j,\alpha})$, and $o(1) \to 0$ as $\alpha \to +\infty$. Generalized $p$-bubbles are rescaling of solutions of the critical equation $\Delta u = |u|^{2^*-2}u$, and the energy of the generalized $p$-bubble is the energy of $u$. In particular, the energy $E_f(\tilde{B}_{j,\alpha})$ does not depend on $\alpha$. It is always greater than or equal to $K_\alpha^{-n}/n$, and if equality holds, then, up to lower order terms, the generalized $p$-bubble has to be a $p$-bubble. Namely, we always have that $E_f(\tilde{B}_{j,\alpha}) \geq K_\alpha^{-n}/n$, and if equality holds, then

$$\tilde{B}_{j,\alpha} = B_{j,\alpha} + R_\alpha,$$

where $(B_{j,\alpha})$ is a $p$-bubble, as defined above, and $R_\alpha \to 0$ strongly in $H^2_{1,p}(M)$ as $\alpha \to +\infty$. By (4.1), it follows from (4.13) that if $\tilde{U}^0 \not\equiv 0$, then $k = 0$, and that if $\tilde{U}^0 \equiv 0$, then $k = 0$ or $k = 1$. When $\tilde{U}^0 \equiv 0$, and $k = 0$, we get with (4.12) that $\tilde{U}_\alpha \to 0$ strongly in $H^2_{1,p}(M)$ as $\alpha \to +\infty$. This is impossible since, by construction of the $\tilde{U}_\alpha$‘s, we also have that there is a uniform positive lower bound for the left hand side in (4.13). In particular, $k = 1$ when $\tilde{U}^0 \equiv 0$. When $\tilde{U}^0 \equiv 0$, and $k = 1$, we also get from (4.1), (4.13), and the above discussion, that the generalized $p$-bubble in (4.12) has to be a $p$-bubble, and that $\lambda_\alpha \to K_\alpha^{-2}$ as $\alpha \to +\infty$. Summarizing, we get with the $H^2_1$-theory for blow-up that if $\tilde{U}^0 \not\equiv 0$, then, up to a subsequence,

$$\tilde{U}_\alpha = \tilde{U}^0 + R_\alpha$$

(4.14)

for all $\alpha$, and that if $\tilde{U}^0 \equiv 0$, then there exists a $p$-bubble $(B_\alpha)_\alpha$ such that, up to a subsequence,

$$\tilde{U}_\alpha = B_\alpha + R_\alpha$$

(4.15)
for all $\alpha$, where, in (4.14) and (4.15), $R_\alpha \to 0$ strongly in $H^2_{\tilde{p}}(M)$ as $\alpha \to +\infty$. We let the $x_\alpha$’s and $\mu_\alpha$’s be the centers and weights of the 1-bubble from which the $p$-bubble $(B_\alpha)_\alpha$ in (4.15) is defined. We claim that

$$\text{(4.15)} \quad \text{is true if } \tilde{U}^0 \neq 0, \quad \text{while } \text{(4.11)} \quad \text{is true if } \tilde{U}^0 \equiv 0. \quad (4.16)$$

In order to prove (4.16), we let $\Phi_\alpha$ be the function given by $\Phi_\alpha(x) = 1$ if $\tilde{U}^0 \neq 0$, and $\Phi_\alpha(x) = d_g(x_\alpha, x)$ if $\tilde{U}^0 \equiv 0$. We let also $\Psi_\alpha$ be the function given by

$$\Psi_\alpha(x) = \Phi_\alpha(x) \frac{p}{2} \sum_{i=1}^{\mu_0} |\tilde{u}_\alpha^i(x)|. \quad (4.17)$$

Then (4.16) is equivalent to the statement that the $\Psi_\alpha$’s are uniformly bounded in $L^\infty(M)$. Now we proceed by contradiction. We let the $y_\alpha$’s be points in $M$ such that the $\Psi_\alpha$’s are maximum at $y_\alpha$, and $\Psi_\alpha(y_\alpha) \to +\infty$ as $\alpha \to +\infty$. Up to a subsequence, we may assume that $|\tilde{u}_\alpha^{i_0}(y_\alpha)| \geq |\tilde{u}_\alpha^{i_0}(y_\alpha)|$ for some $i_0 = 1, \ldots, p$, and all $i$. We set $\mu_\alpha = |\tilde{u}_\alpha^{i_0}(y_\alpha)|^{-2/(n-2)}$. Then $\mu_\alpha \to 0$ as $\alpha \to +\infty$, and by (4.17) we also have that

$$d_g(x_\alpha, y_\alpha) \to +\infty \quad \text{if } \tilde{U}^0 \equiv 0, \quad \text{as } \alpha \to +\infty. \quad (4.18)$$

Let $\delta > 0$ be less than the injectivity radius of $(M, g)$. For $i = 1, \ldots, p$, we define the function $\tilde{v}_\alpha^i$ in $B_0(\delta \mu_\alpha^{-1})$ by

$$\tilde{v}_\alpha^i(x) = \mu_\alpha^{\frac{n-2}{2}} \tilde{u}_\alpha^i \left( \exp_{y_\alpha}(\mu_\alpha x) \right) \quad (4.19)$$

where $B_0(\delta \mu_\alpha^{-1})$ is the Euclidean ball of radius $\delta \mu_\alpha^{-1}$ centered at 0, and $\exp_{y_\alpha}$ is the exponential map at $y_\alpha$. Given $R > 0$ and $x \in B_0(R)$, the Euclidean ball of radius $R$ centered at 0, we can write with (4.17) and (4.18) that

$$|\tilde{v}_\alpha^i|(x) \leq \frac{\tilde{\mu}_\alpha^{\frac{n-2}{2}} \Psi_\alpha(\exp_{y_\alpha} (\mu_\alpha x))}{\Phi_\alpha(\exp_{y_\alpha} (\mu_\alpha x))^{\frac{n-2}{2}}} \quad (4.20)$$

for all $i$, when $\alpha$ is sufficiently large. For any $x \in B_0(R)$, when $\tilde{U}^0 \equiv 0$,

$$d_g \left( x_\alpha, \exp_{y_\alpha}(\mu_\alpha x) \right) \geq d_g \left( x_\alpha, y_\alpha \right) - R \tilde{\mu}_\alpha \geq \left( 1 - \frac{R \tilde{\mu}_\alpha}{\Phi_\alpha(y_\alpha)} \right) \Phi_\alpha(y_\alpha) \quad \text{when } \alpha \text{ is sufficiently large so that, by (4.18), the right hand side of the last equation is positive.}$$

Coming back to (4.20), thanks to the definition of the $y_\alpha$’s, we then get that for any $i$, and any $x \in B_0(R)$,

$$|\tilde{v}_\alpha^i(x)| \leq \frac{\tilde{\mu}_\alpha^{\frac{n-2}{2}} \Psi_\alpha(y_\alpha)}{\Phi_\alpha(\exp_{y_\alpha} (\mu_\alpha x))^{\frac{n-2}{2}}} \leq \left( 1 - \frac{R \tilde{\mu}_\alpha}{\Phi_\alpha(y_\alpha)} \right)^{-\frac{n-2}{2}} \quad (4.21)$$

when $\alpha$ is sufficiently large. In particular, by (4.18) and (4.21), up to passing to a subsequence, the $\tilde{v}_\alpha^i$’s are uniformly bounded in any compact subset of $\mathbb{R}^n$ for all
Let \( \tilde{V}_\alpha = (\tilde{v}_1^\alpha, \ldots, \tilde{v}_p^\alpha) \). The \( \tilde{V}_\alpha \)'s are solutions of the system

\[
\Delta g_\alpha \tilde{v}_i^\alpha + \sum_{j=1}^p \tilde{A}_{ij}^\alpha \tilde{v}_j^\alpha = |\tilde{v}_i^\alpha|^{2^*-2} \tilde{v}_i^\alpha ,
\]

for all \( i \), where

\[
\tilde{A}_{ij}^\alpha(x) = A_{ij}^\alpha \left( \exp_{g_\alpha}(\tilde{\mu}_\alpha x) \right) ,
\]

and

\[
g_\alpha(x) = (\exp_{g_\alpha} g)(\tilde{\mu}_\alpha x) .
\]

Let \( \xi \) be the Euclidean metric. Clearly, for any compact subset \( K \) of \( \mathbb{R}^n \), \( g_\alpha \to \xi \) in \( C^2(K) \) as \( \alpha \to +\infty \). Then, by standard elliptic theory, and (4.22), we get that the \( \tilde{v}_i^\alpha \)'s are uniformly bounded in \( C^{2,\theta}_{\text{loc}}(\mathbb{R}^n) \) for all \( \theta < 1 \). In particular, up to a subsequence, we can assume that \( \tilde{v}_i^\alpha \to \tilde{v}_i \) in \( C^2_{\text{loc}}(\mathbb{R}^n) \) as \( \alpha \to +\infty \) for all \( i \), where the \( \tilde{v}_i \)'s are functions in \( C^2(\mathbb{R}^n) \). The \( \tilde{v}_i \)'s are bounded in \( \mathbb{R}^n \) by (4.24), and such that \( |\tilde{v}_i(0)| = 1 \) by construction. Without loss of generality, we may also assume that the \( \tilde{v}_i \)'s are in \( D_1^2(\mathbb{R}^n) \) and in \( L^{2^*}(\mathbb{R}^n) \) for all \( i \), where \( D_1^2(\mathbb{R}^n) \) is the Beppo-Levi space defined as the completion of \( C_0^\infty(\mathbb{R}^n) \), the space of smooth functions with compact support in \( \mathbb{R}^n \), with respect to the norm \( \|u\| = \|\nabla u\|_2 \).

We let \( \tilde{V} = (\tilde{v}_1, \ldots, \tilde{v}_p) \). According to the above, \( \tilde{V} \neq 0 \). By construction, for any \( R > 0 \),

\[
\int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{U}_\alpha|^2 dv_{\tilde{g}_\alpha} = \int_{B_{\alpha}(R)} |\tilde{V}_\alpha|^2 dv_{\tilde{g}_\alpha} .
\]

It follows that for any \( R > 0 \),

\[
\int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{U}_\alpha|^2 dv_{\tilde{g}_\alpha} = \int_{\mathbb{R}^n} |\tilde{V}|^2 dx + \varepsilon_R(\alpha) ,
\]

where \( \varepsilon_R(\alpha) \) is such that \( \lim_{\alpha \to +\infty} \varepsilon_R(\alpha) = 0 \), and the limits are as \( \alpha \to +\infty \) and \( R \to +\infty \). When \( \tilde{U}_0 \equiv 0 \), see for instance Heby [12], we also get with (4.17) that

\[
\lim_{\alpha \to +\infty} \int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{B}_\alpha|^2 dv_{\tilde{g}_\alpha} = 0
\]

for all \( R > 0 \), where \( \tilde{B}_\alpha \) is the \( p \)-bubble in (4.15). By (4.14) and (4.15),

\[
\int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{U}_\alpha|^2 dv_{\tilde{g}_\alpha} \leq C \int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{U}_0|^2 dv_{\tilde{g}_\alpha} + o(1) = o(1)
\]

for all \( \alpha \) and \( R > 0 \) if \( \tilde{U}_0 \neq 0 \), while

\[
\int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{U}_\alpha|^2 dv_{\tilde{g}_\alpha} \leq C \int_{B_{R\alpha}(R\tilde{\mu}_\alpha)} |\tilde{B}_\alpha|^2 dv_{\tilde{g}_\alpha} + o(1)
\]

for all \( \alpha \) and \( R > 0 \) if \( \tilde{U}_0 \equiv 0 \), where \( C > 0 \) is independent of \( \alpha \) and \( R \), and \( o(1) \to 0 \) as \( \alpha \to +\infty \). Combining (4.23)–(4.26), letting \( \alpha \to +\infty \), and then \( R \to +\infty \), we get that

\[
\int_{\mathbb{R}^n} |\tilde{V}|^{2^*} dx = 0 ,
\]

and this is in contradiction with the equation \( \tilde{V} \neq 0 \). In particular, the \( \Psi_\alpha \)'s are uniformly bounded in \( L^\infty(M) \). This proves (4.16). When \( \tilde{U}_0 \equiv 0 \), (4.16) gives that (4.11) is true, and if \( x_0 \) is the limit of the \( x_\alpha \)'s, (4.11) gives that the \( |\tilde{U}_\alpha|^2 \)'s are uniformly bounded in any compact subset of \( M \setminus \{x_0\} \). By standard elliptic theory,
According to Step 4.1 in order to prove (4.3), it suffices to prove that the $p$-map $\hat{U}$ given by (4.6) is not identically zero. We proceed here by contradiction and assume that $\hat{U} \equiv 0$. The next step in the proof of (4.3) consists in proving that the $\hat{U}_\alpha$'s satisfy perturbed De Giorgi-Nash-Moser type estimates. Step 4.2 in the proof of (4.3) is as follows.

**Step 4.2.** Let $\hat{U}_\alpha$ and $\hat{U}^0$ be given by (4.2) and (4.6). Assume $\hat{U}^0 \equiv 0$. For any $\delta > 0$, there exists $\alpha > 0$ such that, up to a subsequence,

$$ \max_{M \setminus B_j} |\hat{U}_\alpha| \leq C \int_M \left(1 + |\hat{U}_\alpha|^{2^*-2}\right) |\hat{U}_\alpha| dv_g $$

(4.27)

for all $\alpha$, where $B_\delta = B_{x_0}(\delta)$ is the ball centered at $x_0$ of radius $\delta$, $|\hat{U}_\alpha| = \sum_{i=1}^p |\tilde{u}_\alpha^i|$, $|\tilde{u}_\alpha^i|^{2^*-2} = \sum_{i=1}^p |\tilde{u}_\alpha^i|^{2^*-2}$, and $x_0$ is the limit of the centers of the $1$-bubble from which the $p$-bubble $(B_\alpha)_{\alpha}$ in (4.10) is defined.

**Proof of Step 4.2.** Let $B = B_{x}(r)$ be such that $B_{x}(2r) \subset M \setminus \{x_0\}$. By (4.3), and Step 4.1 $|\Delta_\gamma \tilde{u}_\alpha^i| \leq C|\hat{U}_\alpha|$ in $B$, for all $i$ and all $\alpha$, where $C > 0$ is independent of $\alpha$ and $i$. Then we also have that

$$ |\Delta_\gamma \hat{u}_\alpha^i + \tilde{u}_\alpha^i| \leq C|\hat{U}_\alpha| $$

(4.28)

in $B$, for all $i$ and all $\alpha$, where $C > 0$ is independent of $\alpha$ and $i$. We define the $\hat{u}_\alpha^i$'s by

$$ \Delta_\gamma \hat{u}_\alpha^i + \tilde{u}_\alpha^i = |\Delta_\gamma \tilde{u}_\alpha^i + \tilde{u}_\alpha^i| $$

(4.29)

in $M$, for all $\alpha$ and all $i$. Since

$$ \Delta_\gamma \left( \hat{u}_\alpha^i + \tilde{u}_\alpha^i \right) + \left( \hat{u}_\alpha^i + \tilde{u}_\alpha^i \right) \geq 0, $$

we can write that $\hat{u}_\alpha^i \geq |\tilde{u}_\alpha^i|$ in $M$, for all $\alpha$ and all $i$. In particular, the $\hat{u}_\alpha^i$'s are nonnegative. By (4.28) and (4.29) we also have that

$$ |\Delta_\gamma \hat{U}_\alpha| \leq C|\hat{U}_\alpha| $$

(4.30)

in $B$, for all $\alpha$, where $\hat{U}_\alpha$ is the $p$-map of components the $\hat{u}_\alpha^i$'s, $|\hat{U}_\alpha| = \sum_{i=1}^p |\hat{u}_\alpha^i|$. Since the $\hat{u}_\alpha^i$'s are nonnegative, and $C > 0$ is independent of $\alpha$. It easily follows from (4.28) and (4.29), and Step 4.1 that the $|\hat{U}_\alpha|'$'s are uniformly bounded in $L^\infty(B)$. Then, by (4.30), we can apply the De Giorgi-Nash-Moser iterative scheme for functions to the $|\hat{U}_\alpha|$'s. In particular, we can write that

$$ \max_{B_{x}(r/4)} |\hat{U}_\alpha| \leq C \int_{B_{x}(r/2)} |\hat{U}_\alpha| dv_g, $$

(4.31)

where $C > 0$ is independent of $\alpha$. With the notations in the statement of Step 4.2, since $B$ is basically any ball in $M \setminus \{x_0\}$, it easily follows form (4.30) that for any $\delta > 0$, \begin{align*}
\max_{M \setminus B_\delta} |\hat{U}_\alpha| & \leq C \int_M |\hat{U}_\alpha| dv_g, \\
\max_{M \setminus B_\delta} |\hat{U}_\alpha| & \leq C \int_M |\hat{U}_\alpha| dv_g,
\end{align*}

(4.32)

where $C > 0$ is independent of $\alpha$. By (4.33),

$$ |\Delta_\gamma \hat{u}_\alpha^i + \tilde{u}_\alpha^i| \leq C \left(1 + |\hat{U}_\alpha|^{2^*-2}\right) |\hat{U}_\alpha| $$

(4.33)
in $M$, for all $i$ and $\alpha$, where $C > 0$ is independent of $\alpha$ and $i$. Integrating \[ \int_M (\Delta g \hat{u}_\alpha^i) dv_g = 0 \] for all $i$ and all $\alpha$, we get with \[ (4.32) \] that
\[
\int_M |\hat{u}_\alpha| dv_g \leq C \int_M \left(1 + |\hat{U}_\alpha|^2 + 2 - 2\right) |\hat{U}_\alpha| dv_g
\]
(4.34)

for all $\alpha$, where $C > 0$ is independent of $\alpha$. As already mentioned, $|\hat{U}_\alpha| \leq |\hat{U}_\alpha|$ in $M$. In particular, we get with \[ (4.32) \] and \[ (4.34) \] that \[ (4.27) \] is true. Step \[ 4.2 \] is proved.

Step \[ 4.3 \] in the proof of \[ 4.5 \] is concerned with the $L^1/L^{2^*-1}$-controlled balance property of the $\hat{U}_\alpha$’s. Step \[ 4.3 \] is as follows.

**Step 4.3.** Let $\hat{U}_\alpha$ be given by \[ 4.2 \]. There exists $C > 0$ such that, up to a subsequence,
\[
\int_M |\hat{u}_\alpha| dv_g \leq C \int_M |\hat{U}_\alpha|^{2^*-1} dv_g
\]
(4.35)

for all $\alpha$, where $|\hat{u}_\alpha| = \sum_{i=1}^p |\hat{u}_\alpha^i|$, and $|\hat{U}_\alpha|^{2^*-1} = \sum_{i=1}^p |\hat{u}_\alpha^i|^{2^*-1}$.

**Proof of Step 4.3.** Let $f^i_\alpha = \text{sign} (\hat{u}_\alpha^i)$ be the function given by
\[
f^i_\alpha = \chi (\hat{u}_\alpha > 0) - \chi (\hat{u}_\alpha < 0),
\]
(4.36)

where $\chi$ is the characteristic function of $A$. Then $f^i_\alpha \hat{u}_\alpha^i = |\hat{u}_\alpha^i|$ for all $\alpha$ and all $i$. We also have that $|f^i_\alpha| \leq 1$ for all $\alpha$ and all $i$. As already mentioned, up to passing to a subsequence, we can assume that there exists $K > 0$ such that $A^{ij}_\alpha (x) \geq K \delta^{ij}$ in the sense of bilinear forms for all $x$. In particular, if we let $\Delta^p_g$ be the Laplacian acting on $p$-maps, the operators $\Delta^p_g + A(\alpha)$ are (uniformly) coercive in the sense that there exists $C > 0$ such that for any $U \in H^2_{1,p}(M)$, and any $\alpha$,
\[
I_{A(\alpha)} (U) \geq C ||U||^2_{H^2_{1,p}},
\]
(4.37)

where
\[
I_{A(\alpha)} (U) = \int_M |\nabla U|^2 dv_g + \int_M A(\alpha)(U, U) dv_g.
\]
(4.38)

By \[ 4.37 \], and standard minimization technics, there is a solution $U'_\alpha$ to the minimization problem consisting of finding a minimizer for $I_{A(\alpha)} (U)$ under the constraint $\int_M (f_\alpha, U) dv_g = 1$, where $I_{A(\alpha)} (U)$ is as in \[ 4.32 \], $(f_\alpha, U) = \sum_{i=1}^p f^i_\alpha u^i_\alpha$, and the $u^i_\alpha$’s are the components of $U$. If $\lambda_\alpha$ is the minimum of the $I_{A(\alpha)} (U)$’s, where $U \in H^2_{1,p}(M)$ satisfies the constraint $\int_M (f_\alpha, U) dv_g = 1$, it easily follows from \[ 4.37 \] that $\lambda_\alpha > 0$. We let $\hat{U}_\alpha = \lambda_\alpha^{-1} U'_\alpha$. Then $\hat{U}_\alpha$ is a solution of the system
\[
\Delta g \hat{u}^i_\alpha + \sum_{j=1}^p A^{ij}_\alpha \hat{u}^j_\alpha = f^i_\alpha
\]
(4.39)

for all $i$ and all $\alpha$, where the $\hat{u}^i_\alpha$’s are the components of $\hat{U}_\alpha$, and the $f^i_\alpha$’s are as in \[ 4.36 \]. Multiplying \[ 4.39 \] by $\hat{u}^i_\alpha$, integrating over $M$, and summing over $i$, we get with \[ 4.37 \] that the square of the $H^2_{1,p}$-norm of the $\hat{U}_\alpha$’s is uniformly controlled by the $L^1$-norm of the $|\hat{u}_\alpha|$’s. In particular, the $\hat{u}^i_\alpha$’s are uniformly bounded in $L^2$. By standard elliptic theory, the $\hat{u}^i_\alpha$’s are in the Sobolev spaces $H^2_q$ for all $q$. As an easy consequence, the $\hat{u}^i_\alpha$’s are continuous. By the above discussion, and standard
elliptic theory, we then get that there exists $C > 0$ such that $|\tilde{u}_\alpha^i| \leq C$ in $M$, for all $\alpha$ and all $i$. By (4.30) and (4.31) we can now write that

$$\sum_{i=1}^{p} \int_M |\tilde{u}_\alpha^i| dv_g = \sum_{i=1}^{p} \int_M \tilde{u}_\alpha^i f_\alpha^i dv_g$$

$$= \sum_{i=1}^{p} \int_M \left( \Delta_g \tilde{u}_\alpha^i + \sum_{j=1}^{p} A_{ij}^0 \tilde{u}_\alpha^i \right) \tilde{u}_\alpha^i dv_g$$

$$= \sum_{i=1}^{p} \int_M \left( \Delta_g \tilde{u}_\alpha^i + \sum_{j=1}^{p} A_{ij}^0 \tilde{u}_\alpha^i \right) \tilde{u}_\alpha^i dv_g$$

$$= \sum_{i=1}^{p} \int_M |\tilde{u}_\alpha^i|^{2^* - 2} \tilde{u}_\alpha^i \bar{u}_\alpha^i dv_g$$

for all $\alpha$. Since there exists $C > 0$ such that $|\tilde{u}_\alpha^i| \leq C$ in $M$ for all $\alpha$ and all $i$, it follows from (4.30) that

$$\int_M |\tilde{U}_\alpha| dv_g \leq C \int_M |\tilde{U}_\alpha|^{2^* - 1} dv_g$$

for all $\alpha$, where $C > 0$ does not depend on $\alpha$. This proves Step 4.3. $\square$

Step 4.4 in the proof of (4.3) is concerned with $L^2$-concentration. We assume here that $n \geq 4$. When $n = 3$, bubbles do not concentrate in the $L^2$-norm, and $L^2$-concentration turns out to be false in this dimension. Dimension 4 is the smallest dimension for this notion of $L^2$-concentration. Step 4.4 is as follows.

**Step 4.4.** Let $\tilde{U}_\alpha$ and $\tilde{U}^0$ be given by (4.2) and (4.0). Assume $\tilde{U}^0 \equiv 0$ and $n \geq 4$. Up to a subsequence,

$$\lim_{\alpha \to +\infty} \int_{B_\delta} |\tilde{U}_\alpha|^2 dv_g = 1$$

for all $\delta > 0$, where $B_\delta = B_{x_0}(\delta)$ is the ball centered at $x_0$ of radius $\delta$, $x_0$ is the limit of the centers of the 1-bubble from which the $p$-bubble $(B_\alpha)_\alpha$ in (4.10) is defined, and $|\tilde{U}_\alpha|^2 = \sum_{i=1}^{p} |\tilde{u}_\alpha^i|^2$.

**Proof of Step 4.4.** Clearly, Step 4.4 is equivalent to proving that for any $\delta > 0$, $R_\delta(\alpha) \to 0$ as $\alpha \to +\infty$, where $R_\delta(\alpha)$ is the ratio given by

$$R_\delta(\alpha) = \frac{\int_{M \setminus B_\delta} |\tilde{U}_\alpha|^2 dv_g}{\int_M |\tilde{U}_\alpha|^2 dv_g}.$$  \hspace{1cm} (4.42)

We fix $\delta > 0$. By Steps 1.2 and 1.3 we can write that for any $\alpha$,

$$\int_{M \setminus B_\delta} |\tilde{U}_\alpha|^2 dv_g \leq \left( \max_{B_\delta} |\tilde{U}_\alpha| \right) \int_{M \setminus B_\delta} |\tilde{U}_\alpha| dv_g$$

$$\leq C \int_M |\tilde{U}_\alpha|^2 dv_g \int_M |\tilde{U}_\alpha|^{2^* - 1} dv_g,$$

where $C > 0$ is independent of $\alpha$. In particular,

$$R_\delta(\alpha) \leq C \frac{\int_M |\tilde{U}_\alpha|^{2^* - 1} dv_g}{\sqrt{\int_M |\tilde{U}_\alpha|^2 dv_g}}$$  \hspace{1cm} (4.43)
for all \( \alpha \), where \( C > 0 \) is independent of \( \alpha \), and \( R_\delta(\alpha) \) is given by (4.12). If we assume now that \( n \geq 6 \), then \( 2^* - 1 \leq 2 \), and we can write with Hölder’s inequality that

\[
\int_M |\tilde{u}_\alpha^i|^{2^*-1} dv_g \leq V_g^{\frac{2}{2^*}} \left( \int_M |\tilde{u}_\alpha^i|^2 dv_g \right)^{\frac{2^*-1}{2}}
\]

for all \( i \), where \( V_g \) is the volume of \( M \) with respect to \( g \). In particular, there exists \( C > 0 \) such that

\[
\int_M |\tilde{U}_\alpha|^{2^*-1} dv_g \leq C \left( \int_M |\tilde{U}_\alpha|^2 dv_g \right)^{\frac{2^*-1}{2}}, \tag{4.44}
\]

for all \( \alpha \). Since \( \tilde{U}^0 \equiv 0 \), it follows from (4.40) that \( \tilde{U}_\alpha \to 0 \) in \( L^2 \) as \( \alpha \to +\infty \). Since \( 2^* > 2 \), we then get with (4.43) and (4.44) that \( R_\delta(\alpha) \to 0 \) as \( \alpha \to +\infty \). This proves (4.41) when \( n \geq 6 \). If we assume now that \( n = 5 \), then \( 2 \leq 2^* - 1 \leq 2^* \), and we can write with Hölder’s inequality that

\[
\left( \int_M |\tilde{u}_\alpha^i|^{2^*-1} dv_g \right)^{\frac{1}{\theta}} \leq \left( \int_M |\tilde{u}_\alpha^i|^2 dv_g \right)^{\frac{1}{2}} \left( \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \right)^{\frac{1}{2^*-1}},
\]

where \( \theta = \frac{3}{2(2^*-1)} \). By (4.41) we then get that

\[
\int_M |\tilde{U}_\alpha|^{2^*-1} dv_g \leq C \left( \int_M |\tilde{U}_\alpha|^2 dv_g \right)^{\frac{2}{3}},
\]

where \( C > 0 \) does not depend on \( \alpha \). Since \( \frac{3}{2} > \frac{1}{2} \) and \( \tilde{U}_\alpha \to 0 \) in \( L^2 \), we get with (4.40) that \( R_\delta(\alpha) \to 0 \) as \( \alpha \to +\infty \). This proves (4.41) when \( n = 5 \). Now it remains to prove (4.41) when \( n = 4 \). The argument when \( n = 4 \) is slightly more delicate. We start writing that

\[
\frac{\int_M |\tilde{U}_\alpha|^{2^*-1} dv_g}{\sqrt{\int_M |\tilde{U}_\alpha|^2 dv_g}} = \sum_{i=1}^p \frac{\int_M |\tilde{u}_\alpha^i|^{2^*-1} dv_g}{\sqrt{\int_M |\tilde{u}_\alpha^i|^2 dv_g}} \leq \sum_{i=1}^p \frac{\int_M |\tilde{u}_\alpha^i|^{2^*-1} dv_g}{\sqrt{\int_M |\tilde{u}_\alpha^i|^2 dv_g}}, \tag{4.45}
\]

We let the \( x_\alpha \)'s and \( \mu_\alpha \)'s be the centers and weights of the 1-bubble \((B_\alpha)_\alpha\) from which the \( p \)-bubble \((B_\alpha)_\alpha\) in (4.10) is defined. We let \( i_0 = 1, \ldots, p \), be such that \( B_{\alpha}^{i_0} = B_\alpha \) for all \( \alpha \). For \( R > 0 \), we let also \( \Omega_{i_0, \alpha}(R) \) be given by

\[
\Omega_{i_0, \alpha}(R) = B_{x_\alpha}(R:\mu_\alpha). \tag{4.46}
\]

Since \( n = 4 \), we have that \( 2^* = 4 \). If \( i \neq i_0 \), we can write, thanks to Hölder’s inequalities, that

\[
\int_M |\tilde{u}_\alpha^i|^{2^*-1} dv_g \leq \sqrt{\int_M |\tilde{u}_\alpha^i|^2 dv_g} \sqrt{\int_M |\tilde{u}_\alpha^i|^2 dv_g}. \tag{4.47}
\]

By (4.10), when \( i \neq i_0 \), \( \tilde{u}_\alpha^i \to 0 \) in \( H^2_\alpha(M) \) as \( \alpha \to +\infty \). It follows that for any \( i \neq i_0 \),

\[
\frac{\int_M |\tilde{u}_\alpha^i|^{2^*-1} dv_g}{\sqrt{\int_M |\tilde{u}_\alpha^i|^2 dv_g}} = o(1),
\]
where \(o(1) \to 0\) as \(\alpha \to +\infty\). On the other hand, when \(i = i_0\), we get with Hölder’s inequalities that

\[
\int_M |\tilde{u}_\alpha|^{2^*-1} dv_g \leq \int_{\Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} dv_g + \sqrt{\int_{M \setminus \Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} dv_g} \sqrt{\int_M |\tilde{u}_\alpha|^{2} dv_g},
\]

and we can write that

\[
\frac{\int_M |\tilde{u}_\alpha|^{2^*-1} dv_g}{\sqrt{\int_M |\tilde{u}_\alpha|^{2} dv_g}} \leq \sqrt{\int_{M \setminus \Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} dv_g + \frac{\int_{\Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} dv_g}{\sqrt{\int_M |\tilde{u}_\alpha|^{2} dv_g}}},
\]

where \(\Omega_{i_0,\alpha}(R)\) is as in (4.40). For \(\varphi \in C^\infty_0(\mathbb{R}^n)\), where \(C^\infty_0(\mathbb{R}^n)\) is the set of smooth functions with compact support in \(\mathbb{R}^n\), we let \(\varphi_i^0\) be the function defined by the equation

\[
\varphi_i^0(x) = (\mu_\alpha)^{-\frac{n-2}{2}} \varphi((\mu_\alpha)^{-1} \exp_{x_\alpha}(x)).
\]

Straightforward computations give that for any \(R > 0\),

(i) \(\int_{M \setminus \Omega_{i_0,\alpha}(R)} (B_\alpha)^{2^*} dv_g = \varepsilon_R(\alpha),\)

(ii) \(\int_{\Omega_{i_0,\alpha}(R)} (B_\alpha)^{2^*-1} \varphi_i^0 dv_g = \int_{B_0(R)} u^{2^*-1} \varphi dx + o(1),\)

(iii) \(\int_{\Omega_{i_0,\alpha}(R)} (B_\alpha)^2 (\varphi_i^0)^{2^*-2} dv_g = \int_{B_0(R)} u^2 \varphi^{2^*-2} dx + o(1)\)

where \(i_0\) is such that \(\mathcal{B}_i^0 = B_\alpha\) for all \(\alpha\), \((B_\alpha)_\alpha\) is the \(p\)-bubble in (4.10), \(u\) is given by (4.3), \(\Omega_{i_0,\alpha}(R)\) is given by (4.40), \(o(1) \to 0\) as \(\alpha \to +\infty\), and the \(\varepsilon_R(\alpha)\)'s are such that

\[
\lim_{R \to +\infty} \limsup_{\alpha \to +\infty} \varepsilon_R(\alpha) = 0.
\]

By (i) and (4.10) we can write that

\[
\int_{M \setminus \Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} dv_g = \varepsilon_R(\alpha),
\]

where \(\Omega_{i_0,\alpha}(R)\) is as in (4.40), and the \(\varepsilon_R(\alpha)\)'s are such that (4.50) holds. From now on, we let \(\varphi\) in (4.49) be such that \(\varphi = 1\) in \(B_0(R), R > 0\). Then,

\[
\int_{\Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} dv_g = \mu_\alpha^{-\frac{n-2}{2}} \int_{\Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} \varphi_i^0 dv_g
\]

and, by (4.10) and (ii), we can write that

\[
\int_{\Omega_{i_0,\alpha}(R)} |\tilde{u}_\alpha|^{2^*-1} \varphi_i^0 dv_g \
\leq C \int_{\Omega_{i_0,\alpha}(R)} B_\alpha^{2^*-1} \varphi_i^0 dv_g + o(1)
\]

\[
\leq C \int_{B_0(R)} u^{2^*-1} dx + o(1),
\]
where $o(1) \to 0$ as $\alpha \to +\infty$, and $C > 0$ does not depend on $\alpha$ and $R$. In particular, we have that
\[
\int_{\Omega_{\alpha,n}(R)} |\tilde{u}_{\alpha}^i|^2 v_g \leq \left( C \int_{B_0(R)} u^{2^*-1} \, dx + o(1) \right)^{\frac{2^* - 2}{2^*}} ,
\] where $o(1) \to 0$ as $\alpha \to +\infty$, $u$ is as in (4.10), and $C > 0$ does not depend on $\alpha$ and $R$. Independently, we also have that
\[
\int_M |\tilde{u}_{\alpha}^i|^2 \, dv_g \geq \int_{\Omega_{\alpha,n}(R)} |\tilde{u}_{\alpha}^i|^2 \, dv_g \geq \mu_{\alpha}^{n-2} \int_{\Omega_{\alpha,n}(R)} |\tilde{u}_{\alpha}^i|^2 (\varphi_{\alpha}^i)^{2^*-2} \, dv_g .
\]
Here, $2^* - 2 = 2$. As is easily checked, we can write with (4.10) that
\[
\int_{\Omega_{\alpha,n}(R)} |\tilde{u}_{\alpha}^i|^2 (\varphi_{\alpha}^i)^{2^*-2} \, dv_g = \int_{\Omega_{\alpha,n}(R)} B_{\alpha}^2 (\varphi_{\alpha}^i)^{2^*-2} \, dv_g + o(1)
\]
and thanks to (iii) we get that
\[
\int_{\Omega_{\alpha,n}(R)} |\tilde{u}_{\alpha}^i|^2 (\varphi_{\alpha}^i)^{2^*-2} \, dv_g \geq \int_{B_0(R)} u^2 \, dx + o(1) .
\]
In particular,
\[
\int_M |\tilde{u}_{\alpha}^i|^2 \, dv_g \geq \mu_{\alpha}^{n-2} \left( \int_{B_0(R)} u^2 \, dx + o(1) \right) ,
\] where $o(1) \to 0$ as $\alpha \to +\infty$, and $u$ is as in (4.10). By (4.42), (4.45), and (4.47), we can write that
\[
R_\delta(\alpha) \leq C \int_M |\tilde{u}_{\alpha}^i|^2 v_g \frac{1}{\sqrt{\int_M |\tilde{u}_{\alpha}^i|^2 \, dv_g}} + o(1)
\]
for all $\alpha$, where $R_\delta(\alpha)$ is given by (4.11), and $C > 0$ is independent of $\alpha$. Then, by (4.48), and (4.53), (4.54), we get that for any $R > 0$,
\[
\limsup_{\alpha \to +\infty} R_\delta(\alpha) \leq \varepsilon_R + C \frac{\int_{B_0(R)} u^{2^*-1} \, dx}{\sqrt{\int_{B_0(R)} u^2 \, dx}} ,
\]
where $\varepsilon_R \to 0$ as $R \to +\infty$, and $C > 0$ does not depend on $R$. It is easily seen that
\[
\lim_{R \to +\infty} \int_{B_0(R)} u^{2^*-1} \, dx = \int_{\mathbb{R}^n} u^{2^*-1} \, dx < +\infty
\]
On the other hand, when $n = 4$,
\[
\lim_{R \to +\infty} \int_{B_0(R)} u^2 \, dx = +\infty .
\]
Coming back to (4.54), it follows that for any $\delta > 0$, $R_\delta(\alpha) \to 0$ as $\alpha \to +\infty$. In particular, (4.11) is true when $n = 4$. This ends the proof of Step 4.4. \( \square \)

Step 4.5 in the proof of (4.11) is concerned with proving that the off diagonal terms $\int_M \tilde{u}_{\alpha,i} \tilde{u}_{\alpha,j} \, dv_g$, $i \neq j$, are small when compared to the diagonal terms $\int_M (\tilde{u}_{\alpha})^2 \, dv_g$. Step 4.6 is as follows.
Step 4.5. Let $\tilde{U}_\alpha$ and $\tilde{U}_\alpha^0$ be given by (4.2) and (4.10). Assume $\tilde{U}_\alpha^0 \equiv 0$. Up to a subsequence, for any $i, j = 1, \ldots, p$ such that $i \neq j$,

$$\int_{B_{x_0}(\delta)} |\tilde{u}_\alpha^i \tilde{u}_\alpha^j| dv_g \leq \varepsilon_\delta \int_M |\tilde{U}_\alpha|^2 dv_g$$

(4.55)

for all $\delta > 0$ and all $\alpha$, where $x_0$ is the limit of the centers of the 1-bubble from which the $p$-bubble $(B_\alpha)_\alpha$ in (4.10) is defined, $|\tilde{U}_\alpha|^2 = \sum_{i=1}^p |\tilde{u}_\alpha^i|^2$, and $\varepsilon_\delta > 0$ is independent of $\alpha$ and such that $\varepsilon_\delta \to 0$ as $\delta \to 0$.

Proof of Step 4.5. As in the proof of Step 4.4, we let $i_0 = 1, \ldots, p$, be such that $B_{i_0}^\alpha = B_{i_0}$ for all $\alpha$, where $(B_\alpha)_\alpha$ is the 1-bubble from which the $p$-bubble $(B_\alpha)_\alpha$ in (4.10) is defined. Then, by (4.10),

$$\int_M |\tilde{u}_\alpha^i|^2 dv_g = o(1)$$

(4.56)

for all $\alpha$ and all $i \neq i_0$, where $o(1) \to 0$ as $\alpha \to +\infty$. Let $i \neq i_0$. We multiply the $i$th equation in (4.3) by $\tilde{u}_\alpha^i$, and integrate over $M$. Then we can write that

$$\int_M (|\nabla \tilde{u}_\alpha^i|^2 + A_\alpha^2 (\tilde{u}_\alpha^i)^2) dv_g \leq \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g + C \sum_{j \neq i} \int_M |\tilde{u}_\alpha^i||\tilde{u}_\alpha^j| dv_g$$

(4.57)

for all $\alpha$, where $C > 0$ is independent of $\alpha$ and $i$. As already mentioned in the introduction of this section, up to passing to a subsequence, we can assume that there exists $K > 0$ such that $A_{ij}^\alpha \geq K \delta_{ij}$ in the sense of bilinear forms, for all $\alpha$. Then $A_{ij}^\alpha \geq K$ in $M$, for all $\alpha$ and all $i$, and by the Sobolev embedding theorem, we get that there exists $C > 0$ such that

$$\int_M (|\nabla \tilde{u}_\alpha^i|^2 + A_\alpha^2 (\tilde{u}_\alpha^i)^2) dv_g \geq C \left( \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \right)^{2/2^*}$$

(4.58)

for all $\alpha$. Combining (4.57) and (4.58), we get that there exist $C, C' > 0$ such that

$$C \left( \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \right)^{2/2^*} \leq \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g + C' \sum_{j \neq i} \int_M |\tilde{u}_\alpha^i||\tilde{u}_\alpha^j| dv_g$$

(4.59)

for all $\alpha$, and all $i \neq i_0$. By Hölder’s inequality,

$$\int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \leq \sqrt{\int_M |\tilde{u}_\alpha^i|^{2^*} dv_g} \sqrt{\int_M |\tilde{U}_\alpha|^2 dv_g}$$

$$\leq C \left( \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \right)^{1/2^*} \sqrt{\int_M |\tilde{U}_\alpha|^2 dv_g}$$

(4.60)

for all $\alpha$, where $C > 0$ is independent of $\alpha$. Combining (4.59) and (4.60), it follows that

$$C \left( \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \right)^{2/2^*} \leq \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g$$

$$+ C' \left( \int_M |\tilde{u}_\alpha^i|^{2^*} dv_g \right)^{1/2^*} \sqrt{\int_M |\tilde{U}_\alpha|^2 dv_g}$$

(4.61)
for all $\alpha$, and all $i \neq i_0$, where $C, C' > 0$ are independent of $\alpha$ and $i$. By (4.56) we then get that there exists $C > 0$ such that

$$
\left( \int_M |\tilde{u}_i^j|^2 dv_g \right)^{1/2^*} \leq C \sqrt{\int_M |\tilde{U}_\alpha|^2 dv_g}
$$

(4.62)

for all $\alpha$, and all $i \neq i_0$. Now, given $\delta > 0$, and $i \neq j$ arbitrary, we write that

$$
\int_{B_{x_0}(\delta)} |\tilde{u}_i^j|^2 dv_g \leq \sqrt{\int_{B_{x_0}(\delta)} (\tilde{u}_i^j)^2 dv_g} \sqrt{\int_{B_{x_0}(\delta)} (\tilde{u}_j^i)^2 dv_g}
$$

(4.63)

for all $\alpha$, where $x_0$ is the limit of the centers of the 1-bubble from which the $p$-bubble $(B_\alpha)_\alpha$ in (4.10) is defined. Since $i \neq j$, either $i \neq i_0$ or $j \neq i_0$. Suppose $j \neq i_0$. On the one hand we can write that

$$
\int_{B_{x_0}(\delta)} (\tilde{u}_j^i)^2 dv_g \leq \int_M |\tilde{U}_\alpha|^2 dv_g
$$

(4.64)

for all $\alpha$ and $\delta > 0$. On the other hand, by Hölder’s inequality, we can write that

$$
\int_{B_{x_0}(\delta)} (\tilde{u}_j^i)^2 dv_g \leq |B_{x_0}(\delta)|^{2-2^*} \left( \int_{B_{x_0}(\delta)} |\tilde{u}_j^i|^2 dv_g \right)^{2/2^*} \leq |B_{x_0}(\delta)|^{2-2^*} \left( \int_M |\tilde{U}_\alpha|^2 dv_g \right)^{2/2^*}
$$

(4.65)

for all $\alpha$, where $|B_{x_0}(\delta)|$ is the volume of $B_{x_0}(\delta)$ with respect to $g$. By (4.62), since $j \neq i_0$, we then get that

$$
\int_{B_{x_0}(\delta)} (\tilde{u}_j^i)^2 dv_g \leq C |B_{x_0}(\delta)|^{2-2^*} \int_M |\tilde{U}_\alpha|^2 dv_g
$$

(4.66)

for all $\alpha$ and $\delta > 0$, where $C > 0$ is independent of $\alpha$ and $\delta$. Plugging (4.64) and (4.65) into (4.65), since $|B_{x_0}(\delta)| \to 0$ as $\delta \to 0$, we get that (4.66) is true. This ends the proof of Step 4.5.

By Steps 4.1 to 4.5, we are now in position to prove (4.5), and hence to prove Lemma 2.1. We use in the process that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, such that for any smooth function $u$ with compact support in $B_{x_0}(\delta_\varepsilon)$,

$$
\left( \int_M |u|^2 dv_g \right)^{2/2^*} \leq K^2_n \int_M |\nabla u|^2 dv_g + B_\varepsilon \int_M u^2 dv_g
$$

(4.66)

where $B_\varepsilon = \frac{n-2}{4(n-1)} K^2_n (S_g(x_0) + \varepsilon)$, $K_n$ is given by (1.2), and $S_g$ is the scalar curvature of $g$. Inequality (4.66) is a straightforward consequence of the local isoperimetric inequality proved in Druet [3]. Step 4.6 is as follows.

**Step 4.6.** Let $\tilde{U}_\alpha$ and $\tilde{U}_0$ be given by (4.2) and (4.6). Assume that for any $i$ and any $x \in M$,

$$
A^0_{i0}(x) > \frac{n-2}{4(n-1)} S_g(x),
$$

(4.67)

and that $n \geq 4$. Then $\tilde{U}_0 \neq 0$. In particular, (4.5) and Lemma 2.1 are true.
Proof of Step 4.4. We proceed by contradiction and assume that \( \tilde{U}_0 \equiv 0 \). We let \( x_0 \) be the limit of the centers of the 1-bubble from which the \( p \)-bubble \((B_\alpha)\) in (4.10) is defined. We fix \( \varepsilon > 0 \), and let \( \eta \) be a smooth cutoff function such that \( \eta = 1 \) in \( B_{x_0}(\delta \varepsilon/4) \), \( \eta = 0 \) in \( M \setminus B_{x_0}(\delta \varepsilon/2) \), and \( 0 \leq \eta \leq 1 \). We plugg the \( \eta \tilde{u}_\alpha \)'s into (4.66), \( i = 1, \ldots, p \), and then sum over \( i \). Noting that

\[
\int_M |\nabla (\eta \tilde{u}_\alpha)|^2 dv_g = \int_M \eta^2 \tilde{u}_\alpha^2 (\Delta_g \tilde{u}_\alpha) dv_g + \int_M |\nabla \eta|^2 (\tilde{u}_\alpha)^2 dv_g ,
\]

and that \( |\nabla \eta| = 0 \) around \( x_0 \), we get with (4.3) and \( L^2 \)-concentration in Step 4.3 that

\[
\sum_{i=1}^p \left( \left( \int_M |\eta \tilde{u}_\alpha|^2 dv_g \right)^{2/2^*} - K_n^2 \int_M \eta^2 |\tilde{u}_\alpha|^2 dv_g \right) \leq -K_n^2 \sum_{i,j=1}^p \int_M \eta^2 A_{ij}^0 \tilde{u}_\alpha^i \tilde{u}_\alpha^j dv_g + (B_\varepsilon + o(1)) \int_M |\tilde{U}_\alpha|^2 dv_g
\]

(4.68)

for all \( \alpha \), where \( o(1) \to 0 \) as \( \alpha \to +\infty \), and \( B_\varepsilon \) is as in (4.69). By Hölder’s inequality, and (4.4),

\[
\int_M \eta^2 |\tilde{u}_\alpha|^2 dv_g \leq \left( \int_M \eta^2 \tilde{u}_\alpha^2 dv_g \right)^{2/2^*} \left( \int_M |\tilde{u}_\alpha|^2 dv_g \right)^{(2^*-2)/2^*} \leq K_n^{-2} \left( \int_M |\eta \tilde{u}_\alpha|^2 dv_g \right)^{2/2^*}
\]

(4.69)

for all \( \alpha \) and \( i \). By (4.69), the left hand side in (4.68) is nonnegative. Since we also have that \( A_{ij}^0 \to A_{ij}^0 \) in \( C^{0,\theta}(M) \), we can write with (4.68) that

\[
K_n^2 \sum_{i,j=1}^p \int_M \eta^2 A_{ij}^0 \tilde{u}_\alpha^i \tilde{u}_\alpha^j dv_g \leq (B_\varepsilon + o(1)) \int_M |\tilde{U}_\alpha|^2 dv_g
\]

(4.70)

for all \( \alpha \), where \( o(1) \to 0 \) as \( \alpha \to +\infty \), and \( B_\varepsilon \) is as in (4.69). By \( L^2 \)-concentration in Step 4.3 and the control of the off diagonal terms in Step 4.3, it follows from (4.70) that for any \( \varepsilon > 0 \), and any \( \delta > 0 \),

\[
\sum_{i=1}^p \int_M \left( A_{ii}^0(x_0) - \frac{n-2}{4(n-1)} S_g(x_0) \right) (\tilde{u}_\alpha^i)^2 dv_g \leq C \sum_{i,j=1}^p \int_{B_{x_0}(\delta)} |\tilde{u}_\alpha^i| |\tilde{u}_\alpha^j| dv_g + C (\varepsilon + o(1)) \int_M |\tilde{U}_\alpha|^2 dv_g
\]

\[
+ \sum_{i=1}^p \left( \sup_{x \in B_{x_0}(\delta)} |A_{ii}^0(x) - A_{ii}^0(x_0)| \right) \int_M |\tilde{U}_\alpha|^2 dv_g \leq C (\varepsilon + \varepsilon_\delta + o(1)) \int_M |\tilde{U}_\alpha|^2 dv_g
\]

(4.71)

for all \( \alpha \), where \( o(1) \to 0 \) as \( \alpha \to +\infty \), \( \varepsilon_\delta \to 0 \) as \( \delta \to 0 \), and \( C > 0 \) does not depend on \( \alpha \), \( \varepsilon \), and \( \delta \). By (4.67) there exists \( \varepsilon_0 > 0 \) such that

\[
A_{ii}^0(x_0) \geq \frac{n-2}{4(n-1)} S_g(x_0) + \varepsilon_0
\]

(4.72)
for all $i$. Then the contradiction easily follows from (4.11) by choosing $\varepsilon > 0$ and $\delta > 0$ sufficiently small such that $C(\varepsilon + \varepsilon_3) \leq \varepsilon_0/2$, where $C > 0$ is the constant in (4.11), and $\varepsilon_0$ is as in (4.12). This proves that for $\tilde{U}_\alpha$ and $\tilde{U}_0$ as in (4.2) and (4.6), we necessarily have that $\tilde{U}_0 \not\equiv 0$ when we assume that $n \geq 4$ and that (4.67) holds. Then, by Step 4.1 we get that (4.5) is also true. By standard elliptic theory, as already mentioned, this implies in turn that Lemma 2.1 is true. \hfill \Box

A possible extension of Lemma 2.1 is to replace the condition in the Lemma that $A_{ij}^0(x) > \frac{n-2}{4(n-1)} S_g(x)$ for all $i$ and all $x$, by the condition that for any $i$, either $A_{ij}^0(x) > \frac{n-2}{4(n-1)} S_g(x)$ for all $x$, or $A_{ii}^0(x) < \frac{n-2}{4(n-1)} S_g(x)$ for all $x$, and hence that for any $i$, and any $x$,

$$A_{ii}^0(x) \neq \frac{n-2}{4(n-1)} S_g(x).$$

If we assume that the convergence of the $A_{ij}^0$'s to the $A_{ij}^0$'s is in $C^1(M)$, and that the manifold is conformally flat, we can prove, with the estimates we obtained in Steps 4.1 to 4.5, this claim that Lemma 2.1 remains true if we only assume (4.73). The proof, based on the Pohozaev identity instead of (4.66), is as follows. We let $\mathcal{U}_\alpha$ and $\tilde{U}$ be given by (4.22) and (4.6). We assume by contradiction that $\tilde{U}_0 \equiv 0$, and let $x_0$ be the limit of the centers of the 1-bubble from which the $p$-bubble $(B_o)_\alpha$ in (4.10) is defined. Since $g$ is conformally flat, there exist $\delta_0 > 0$ and a conformal metric $\hat{g}$ to $g$ such that $\hat{g}$ is flat in $B_{\delta_0}(4\delta_0)$. Let $\hat{g} = \varphi^{(n-2)}g$, where $\varphi$ is smooth and positive, and $\hat{\alpha}_\alpha = \hat{u}_\alpha\varphi^{-1}$ for all $\alpha$ and $i$. By conformal invariance of the conformal Laplacian, and by (4.3),

$$\Delta \hat{u}_\alpha + \sum_{j=1}^p A_{ij}^0 \hat{u}_\alpha = (\hat{u}_\alpha)^{2^*-1}$$

in $B_{\delta_0}(4\delta_0)$ for all $i$ and all $\alpha$, where $\Delta = \Delta_{\hat{g}}$ is the Euclidean Laplacian, and $\varphi^{2^*-2}A_{ij}^0 = A_{ij}^n - \frac{n-2}{4(n-1)} S_g \delta_{ij}$. The Pohozaev identity in the Euclidean space reads as

$$\int_{\Omega} (x^k \partial_k u) \Delta u dx + \frac{n-2}{n-1} \int_{\Omega} u(\Delta u) dx = - \int_{\partial\Omega} (x^k \partial_k u) \partial_\nu u d\sigma + \frac{1}{2} \int_{\partial\Omega} (x, \nu) |\nabla u|^2 d\sigma$$

$$- \frac{n-2}{2} \int_{\partial\Omega} u \partial_\nu u d\sigma,$$

where $\nu$ is the outward unit normal to $\partial\Omega$, $d\sigma$ is the Euclidean volume element on $\partial\Omega$, and there is a sum over $k$ from 1 to $n$. For $\delta > 0$ small, we let $\eta$ be a smooth cutoff function such that $\eta = 1$ in $B_{\varepsilon_0}(\delta)$, $\eta = 0$ in $M \setminus B_{\varepsilon_0}(2\delta)$, and $0 \leq \eta \leq 1$. We plug the $\eta\hat{u}_\alpha$'s into the Pohozaev identity (1.75) and sum over $i$. In the process, we regard the $\hat{u}_\alpha$'s, $\varphi$, $\eta$, and the $A_{ij}^0$'s as defined in the Euclidean space. Thanks to (4.40), to the $C^1$-convergence of the $A_{ij}^0$'s to the $A_{ij}^0$'s, and to $L^2$-concentration, coming back to the manifold, we get after lengthy (but simple) computations that

$$\sum_{i,j=1}^p \int_{B_{\varepsilon_0}(\delta)} \left( A_{ij}^0(x_0) - \frac{n-2}{4(n-1)} S_g(x_0) \delta_{ij} \right) \hat{u}_\alpha \hat{u}_\alpha \partial_\nu g$$

$$= \varepsilon_{\delta} O \left( \int_M |\hat{u}_\alpha|^2 d\nu_g \right) + o \left( \int_M |\hat{u}_\alpha|^2 d\nu_g \right).$$
for all $\alpha$, where $\varepsilon_\delta > 0$ is independent of $\alpha$ and such that $\varepsilon_\delta \to 0$ as $\delta \to 0$, and where the first term in the right hand side of (4.70) depends on $\delta$ only by $\varepsilon_\delta$. Let $i_0 = 1, \ldots, p$, be such that $B_{i_0} = B_{i_0}(\varepsilon_\delta) \subset B_{i_0}$ for all $\alpha$, where $(B_{i_0})_\alpha$ is the $1$-bubble from which the $p$-bubble $(B_{i_0})_\alpha$ in (4.10) is defined. The argument we developed in the proof of Step 14.3 gives that for $i \neq i_0$,

$$
\int_{B_{i_0}(\delta)} (\tilde{u}_\alpha^i)^2 dv_\gamma \leq \varepsilon_\delta \int_M |\tilde{u}_\alpha|^2 dv_\gamma \quad (4.77)
$$

for all $\alpha$, where $\varepsilon_\delta > 0$ is independent of $\alpha$ and such that $\varepsilon_\delta \to 0$ as $\delta \to 0$. Combining the off diagonal estimates (4.55) of Step 14.5, $L^2$-concentration, (4.76), and (4.77), it follows that

$$
\left( A_{i_0i_0}^0(x_0) - \frac{n-2}{4(n-1)} S_\gamma(x_0) \right) \int_M |\tilde{u}_\alpha|^2 dv_\gamma = \varepsilon_\delta O \left( \int_M |\tilde{u}_\alpha|^2 dv_\gamma \right) + o \left( \int_M |\tilde{u}_\alpha|^2 dv_\gamma \right) \quad (4.78)
$$

for all $\alpha$, where $\varepsilon_\delta > 0$ is independent of $\alpha$ and such that $\varepsilon_\delta \to 0$ as $\delta \to 0$, and where the first term in the right hand side of (4.78) depends on $\delta$ only by $\varepsilon_\delta$. Choosing $\delta > 0$ sufficiently small, and letting $\alpha \to +\infty$, we get a contradiction by combining (4.73) with $i = i_0$ and (4.78). This proves that if we assume that the convergence of the $A_{i_0i_0}^0$’s to the $A_{i_0i_0}^0$’s is in $C^1(M)$, and that the manifold is conformally flat, then Lemma 2.1 remains true if we replace the condition $A_{i_0i_0}^0(x) > \frac{n-2}{4(n-1)} S_\gamma(x)$ for all $i$ and all $x$, by the condition that $A_{i_0i_0}^0(x) \neq \frac{n-2}{4(n-1)} S_\gamma(x)$ for all $i$ and all $x$.

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