Cosmological Constraints on the Very Low Frequency Gravitational-Wave Background

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The curl modes of cosmic microwave background polarization allow one to indirectly constrain the primordial background of gravitational waves with wavelengths roughly the horizon size or larger with frequencies below $10^{-16}$ Hz. The planned high precision timing observations of a large sample of millisecond pulsars with the Pulsar Timing Array or with the Square Kilometer Array can either detect or constrain the stochastic gravitational wave background at frequencies greater than roughly 0.1 years$^{-1}$. While there are no strong observational constraints on the gravitational wave background across six or more orders of magnitude between $10^{-16}$ Hz and $10^{-10}$ Hz and it is difficult to get a constraint below $10^{-12}$Hz using objects in our Galaxy, we suggest that the anisotropy pattern of time variation of the redshift related to a sample of high redshift objects can be used to constrain the gravitational wave background around $10^{-12}$ Hz. Useful observations for the monitoring of an anisotropy signal in a global redshift change include spectroscopic observations of the Ly-$\alpha$ forest in absorption towards a sample of quasars, redshifted 21 cm line observations either in absorption or emission towards a sample of neutral HI regions before or during reionization, and high frequency (0.1 Hz to 1 Hz) gravitational wave analysis of a sample of neutron star—neutron star binaries detected with gravitational wave instruments such as the Decihertz Interferometer Gravitational Wave Observatory (DECIGO). The low frequency background can also be constrained by arcsecond-scale anisotropy observations of the CMB. For reasonable observations in the future involving extragalactic sources, we find best limits at the level of $\Omega_{GW} < 10^{-7}$ at a frequency around $10^{-12}$ Hz while the eventual ultimate limit one cannot beat is $\Omega_{GW} < 10^{-11}$.

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I. INTRODUCTION

The observation of cosmic microwave background (CMB) anisotropies, especially using a map of the curl modes of polarization, allows one to constrain the primordial background of gravitational waves with wavelengths roughly the horizon size or larger with frequencies below $10^{-16}$ Hz\textsuperscript{1}. Recent temperature anisotropy observations at large angular scales with Wilkinson Microwave Anisotropy Probe (WMAP;\textsuperscript{2}) yields a limit (at the 2$\sigma$ level) of $\Omega_{GW} < 5 \times 10^{-11}$\textsuperscript{3,4} with a maximal sensitivity around $10^{-17}$ Hz. Here, $\Omega_{GW}$ is the fractional density contribution from a background of stochastic gravitational waves with the density $\rho_{GW}$:

$$\Omega_{GW}(f) = \frac{1}{\rho_c} \frac{d\rho_{GW}}{d\log f},$$

when the closure density is $\rho_c = 3H_0^2/8\pi G$. Here, $H_0$ is the Hubble constant. At these super-horizon and horizon-size scales, a stochastic background of gravitational waves is expected from inflationary physics\textsuperscript{3,4}. The direct detection of this signal is now considered to be one of the primary goals of upcoming CMB polarization anisotropy observations both from ground and space with planned missions such as the Inflation Probe of Inflationary Cosmology (EPIC), to be launched around 2012 as part of NASA’s Beyond Einstein Foundation Science program, will be optimized to reach this level.

When considering a subhorizon-scale wavelengths, the planned high precision timing observations of a large sample of millisecond pulsars with the Pulsar Timing Array or with the Square Kilometer Array can either detect or constrain the stochastic gravitational wave background at frequencies greater than roughly $1/100$ years where $T_{\text{obs}}$ is the total observational duration of the pulsar sample\textsuperscript{5}. While the waves with wavelengths lower than $1/T_{\text{obs}}$ linearly change the observed interval of the pulse, the lower frequency limit of $1/T_{\text{obs}}$ for the pulsar timing method comes from the fact that one cannot distinguish their effect from the long-term intrinsic spin evolution of pulsars. Since observations are likely to be restricted to less than 100 years, the pulsar timing arrays cannot constrain the gravitational wave background below $10^{-10}$ Hz. Observations of two millisecond pulsars over 17 years limit the background to be $\Omega_{GW}h^2 < 2 \times 10^{-9}$ at the 95% confidence level\textsuperscript{6,11}, while with the Pulsar Timing Array, using a larger sample of pulsars, the limit is expected to reach the level of $2 \times 10^{-13}$\textsuperscript{8}. At these frequencies, the stochastic background of gravitational waves is expected to be primarily dominated by massive black hole binaries\textsuperscript{12} and either a detection or a tight constrain on the background is useful to understand the extent to which massive black holes merge at redshifts around unity.
Below frequencies probed by the millisecond pulsar timing method, a background of stochastic waves is expected from models related to global cosmic strings, phase transitions in the early universe, and cosmic turbulence. Note also that the primordial background below \(10^{-10}\) Hz cannot be constrained by the abundance of the light elements predicted by big bang nucleosynthesis. At present, there are no strong observational constraints on the gravitational wave background across six or more orders of magnitude between \(10^{-16}\) Hz and \(10^{-10}\) Hz. The binary pulsars give a limit \(\Omega_{GW} h^2 < 0.04\) for \(10^{-12}\) Hz < \(f < 4.4 \times 10^{-9}\) Hz, and \(\Omega_{GW} h^2 < 0.5\) for \(10^{-12}\) Hz < \(f < 10^{-11}\) Hz from the time variation of their orbital period compared with prediction of general relativity. In future they might give a limit at the \(\Omega_{GW} h^2 \sim 10^{-4}\) level that is determined by uncertainties in the local acceleration of these binaries. Using Galactic objects, it is difficult to get information on the background below \(f < 10^{-12}\) Hz as this corresponds roughly to the typical distance to galactic objects. The longer waves act on the target object and the observer in the same manner, and the measurement sensitivity for these low frequency waves is strongly suppressed due to a cancellation effect. Therefore, it is essential that we use extra-Galactic objects to set a limit for the low frequency background below \(10^{-12}\) Hz. The current limit comes from observations of quasar proper motions across the sky (transverse motions). For data collected over a 10 year span, the resulting constraint on the stochastic background at a frequency below \(2 \times 10^{-9}\) Hz is \(\Omega_{GW} h^2 < 0.11\) (see also [18] for prospects with the Square Kilometer Array).

In this paper, to probe the background at frequencies below \(10^{-12}\) Hz, we propose measuring the anisotropy pattern of time variation of the redshift related to a sample of high redshift objects. Previously, observations of the time variation of the redshift have been proposed to study global cosmological parameters [19]. While cosmology can induce either an acceleration or a deceleration to the redshift of an object, the resulting change can be extracted through the monopole change from a sample of objects spread over the sky since all sources are equally affected by cosmology. The resulting change due to a gravitational wave background, at the lowest order in spherical moments, is present in the quadrupole moment of the time evolving redshift [20]. The use of the anisotropy pattern also allows one to separate secondary effects that can cause a time-dependent redshift, such as the peculiar acceleration of the observer which is present via a dipole at the lowest order. We focus on the quadrupole measurement, but the measurement of additional multiple moments of the anisotropy can be used to increase the confidence level of the derived limit. As we discussed, the proposed method with using the time variation of redshifts would be performed as a byproduct of the dark energy study, in contrast to the method based on the transverse motions that is basically caused by perturbative effects. Therefore we will pay special attention to relate the sensitivities of the measurement for the dark energy and the gravitational wave background. As we are studying cosmology, we can expect, in order of magnitude sense, that there is a simple relation between their sensitivities that are properly normalized by cosmological parameters such as \(H_0\). It is, however, important to calculate the relevant coefficients in discussing future prospects. In fact, we find that these coefficients are largely different from unity.

Here, extending the discussion in Ref. [19], we consider a measurement of the quadrupole related to the anisotropy of the time derivative of Ly-\(\alpha\) forest redshifts seen in absorption towards a sample of quasars. Additional probes for the same purpose include low-frequency redshifted-21 cm observations either in absorption or emission towards a sample of neutral HI regions before or during reionization [21]. Another possibility is the high frequency (0.1 Hz to 1 Hz) gravitational wave analysis of a sample of neutron star-neutron star binaries detected with gravitational wave instruments such as the Decihertz Interferometer Gravitational Wave Observatory (DECIGO) [22]. These observations have been suggested for measurement of cosmological parameters based on the phase variation induced during the propagation of gravitational waves in a cosmologically expanding background [22, 23]. While cosmologically induces a global phase shift, a low frequency gravitational wave background induces an anisotropy to this phase shift and its presence can be extracted from a sample of binaries spread over the sky with well-modeled wave forms. For reasonable future observations along these lines, we find best constraints at the level of \(\Omega_{GW} < 10^{-5}\) at a frequency around \(10^{-12}\) Hz, though, the eventual limit is probably at \(\Omega_{GW} < 10^{-11}\). Reaching this level is challenging as it requires high resolution observations of 21 cm background to resolve all halos spatially with follow up high resolution spectral measurements over a 10 year span over a bandwidth less than a kHz. In figure 1, we plot sensitivities of various current and future constraints for the low frequency gravitational wave background. From this figure, the importance of the extra-Galactic probes to study waves below \(10^{-12}\) Hz, but above the regime studied by CMB experiments, is clear.

Incidently, one can also, in principle, use the monopole to constrain the energy density of gravitational background. The resulting correction comes from the fact that a subhorizon-scale gravitational wave background acts as an additional radiation field in the universe and the expansion rate of the universe is modified to become \(H^2(z) = H_0^2\left[\Omega_m(1+z)^3 + (\Omega_R + \Omega_{GW})(1+z)^4 + \Omega_{\Lambda}\right]\), for a spatially-flat universe with cosmological constant \(\Lambda\), the matter density parameter \(\Omega_M\) and the radiation content \(\Omega_R\), all relative to the critical density. Since matter and dark energy densities dominate the expansion rate over the redshift ranges one can probe with current observational techniques, we do not pursue a limit based on this argument further. An interesting constraint, however, could potentially be derived if one were to study...
and can derive an improved constraint on the subhorizon gravitational background that existed between an age for the Universe of a minute to a few hundred thousand years. We will return to this topic in an upcoming paper.

Just as horizon-size waves produce CMB anisotropies, a gravitational wave background at frequencies between $10^{-14}$ to $10^{-12}$ Hz can produce fluctuations in the CMB temperature at arcsecond angular scales. Currently most CMB anisotropy observations probe down to arcminute scales from acoustic peak structure at the degree scale, and prospects for even further small-scale CMB anisotropies observations are limited. These observations are also likely to be challenging given additional secondary effects such as the Sunyaev-Zel’dovich effect \[24\]. While most secondaries can be separated, through properties such as the frequency spectrum relative to CMB \[25\], the foreground contamination from unresolved point-sources is expected to be the biggest confusion. We make an estimate on the expected constraint by extrapolating the foreground contamination seen in current anisotropy data at arcminute scales.

The discussion is organized as follows: In the next Section, we outline the calculation related to an anisotropy pattern in the time evolving redshift of a sample of sources in the presence of a gravitational wave background. In § III, we discuss various cosmological probes related to this effect and estimate resulting limits that one can potentially obtain in the future related to the presence of a low frequency, but sub-horizon, background of gravitational waves. We conclude with a summary in § IV.

II. FORMULATION

We first study how a background of stochastic gravitational waves affect the redshift of a photon, or a massless particle, propagating to an observer today from a source at a high redshift. We consider a spatially-flat Universe and write the metric that is perturbed by gravitational waves as

$$ds^2 = a(\eta)^2 \left\{ -d\eta^2 + [\delta_{ij} + h_{ij}(\eta, x)]dx^idx^j \right\},$$

where $\eta$ is the conformal time and the scale factor $a(\eta)$ is normalized such that $a(\eta_0) = 1$ at the present epoch with $\eta_0$. We write the conformal time interval and the normal time interval today as $\Delta \eta$ and $\Delta T$, respectively. Comparing to a background universe with $h_{ij} = 0$, the propagation time of a photon changes by $\delta \eta$ due to the presence of gravitational waves. We can formally write down $\delta \eta$ as a line integral along the photon path:

$$\delta \eta = \frac{\eta_1 - \eta_0}{2} \int_{\eta_0-L}^{\eta_1} h_{ij}[\eta, (\eta - \eta_0)n]d\eta,$$

where $n$ is the unit directional vector to the high redshift source and $L$ is its comoving distance from the observer. This shift $\delta \eta$ will result in an apparent time variation,
\[\Delta z, \text{ to the redshift of the source, } z, \text{ in a time interval } \Delta T:\]

\[
\frac{\Delta z}{\Delta T} = \frac{n_in_j}{2} \int_{\eta_0-L}^{\eta_0} \frac{\partial h_{ij}(\eta, (\eta - \eta_0)n)}{\partial \eta} d\eta. \tag{4}
\]

Since we are interested in wave lengths (\(\lesssim 10\)Mpc) smaller than the horizon scale, we can safely express the gravitational waves for our study as

\[h_{ij}(f, p, \eta, x) = \exp[i2\pi f(\eta - p \cdot x)]D(\eta, f)b_{ij}(fp), \tag{5}\]

where \(p\) is the unit vector for the propagation direction of the wave, \(D(\eta, f)\) represents the cosmological evolution of the wave amplitude and \(b_{ij}(fp)\) is the random number for each wave characterized by the vector \(fp\).

We decompose eq. \([6]\) as

\[h_{ij}(f, p, \eta, x) = \exp[i2\pi f(\eta - p \cdot x)]\{D(\eta, f) - D(\eta_0, f)\} + \exp[i2\pi f(\eta - p \cdot x)]D(\eta_0, f)b_{ij}(fp), \tag{6}\]

and substitute this in Eq. \([7]\) \([8]\). Our basic aim is to extract effects that are common to many sources. Here, the contribution of the first term in Eq. \([8]\) is negligible since (i) we are considering a wave with a wavelength shorter than the horizon scale, (ii) the common coherent effect is made by the perturbation close to us, or the observer, and (iii) the evolution time scale of the amplitude \((D/D)^{-1}\) is order of the age of the Universe. As is well known \([26]\), the second term in Eq. \([6]\) can be expressed as a sum of two quantities, the information of the gravitational wave at the source and at the observer. The former can be dropped for the same reason as in the previous case. In summary, we can apply the standard argument for the low frequency gravitational background by using objects within our galaxy \((e.g. \text{ timing residual for a galactic millisecond pulsar})\). Random variations in \(\Delta z/\Delta T\) for each binary due to the gravitational wave background can be regarded as a noise for measurement of the common signal. For these binaries, however, the expected magnitude of variation is smaller than other sources of noise \((e.g. \text{ local acceleration, detector noise})\), and hereafter we neglect this noise component.

Following the calculation of Burke \([20]\), we obtain the angular pattern of the common redshift variation \(\Delta z\) due to the background in the time interval \(\Delta T\) as

\[
\dot{z}_{GW}(\theta, \phi, \Delta z) = \frac{1}{H_0} \left(\frac{\Delta z}{\Delta T}\right)_{GW} = \sum_{l \geq 2} \dot{z}_{lmGW}Y_{lm}(\theta, \phi), \tag{7}\]

where \(\dot{z}_{GW}(\theta, \phi)\), given through Eq. \([7]\) has the angular dependence of \((1 - \cos \theta) \cos 2\phi\). This angular dependence can be written as a combination of \(Y_{l2}\) and \(Y_{l-2}\), when \(l \geq 2\) such that

\[
(1 - \cos \theta) \cos 2\phi = \sum_{l=2}^{\infty} (-1)^l \sqrt{(l - 2)! (l + 2)!} [Y_{l2} + Y_{l-2}]. \tag{8}\]

The angular power spectrum of coefficient \(\dot{z}_{lmGW}\) is

\[
\langle \dot{z}_{lmGW} \dot{z}_{l'm'm'}GW \rangle = \frac{6\Omega_{GW}}{\pi(l + 2)(l + 1)(l - 1)} \delta_{ll'}\delta_{mm'}, \tag{9}\]

and the parameter \(\Omega_{GW}\) here is defined by the normalized energy density \(\Omega_{GW}(f)\) per unit log frequency interval as

\[
\Omega_{GW} = \int_{f_{cut}}^{(\Delta f)^{-1}} \Omega_{GW}(f) \frac{df}{f}. \tag{10}\]

We have explicitly included a lower frequency cut-off since, below a certain frequency, energy density of the gravitational wave background will be severely constrained by CMB polarization observations.

We also define the angular power spectrum \(C_l\) that is useful for a statistical study as

\[
C_l = \frac{1}{2l + 1} \sum_m |\dot{z}_{lm}|^2 = \frac{6\Omega_{GW}}{\pi(l + 2)(l + 1)(l - 1)} \tag{11}\]

In eq. \([7]\) it is important to note that the effect of the gravitational wave background starts form quadrupole \((l = 2)\) mode and independent on the source redshift \(z\), while the redshift change \(\Delta z\) due to the cosmic acceleration/deceleration is a monopole \((l = 0)\) effect and depends on the redshift \(z\) \([19]\) that corresponds to the coefficient \(\dot{z}_{00}\) as

\[
\dot{z}_{00,acc} = - \sqrt{4\pi} \left\{ \left[ \Omega_M(1 + z) + \Omega_R(1 + z)^2 + \Omega_\Lambda(1 + z)^{-2} \right]^{1/2} - 1 \right\}. \tag{12}\]

We define a parameter \(X\) that characterizes the second order correction to the relation between time intervals at the observer, \(\Delta t\), and the source, \(\Delta t_z\) due to effective acceleration as

\[
(1 + z)\Delta t_z = \Delta t - X\Delta t^2 + O(\Delta t^3). \tag{12}\]

The parameter \(X\) can have a finite value due to the cosmological acceleration, gravitational wave background or the local acceleration. For each object we fit the parameter \(X\) to statistically extract various information. For the cosmological acceleration we have

\[
X(z) = \frac{1}{2} \left[ H_0 - \frac{H(z)}{1 + z} \right] = \frac{H_0\dot{z}_{00,acc}}{4\sqrt{\pi}} \tag{13}\]

when the Hubble parameter at redshift \(z\) is \(H(z)\). The parameter \(X(z)\) can also be written as \(X(z) = 0.5[\dot{a}(0) - \dot{a}(z)]\), where the overdot represents a derivative with respect to the proper time. At small redshifts, when \(z << 1\), \(X(z) = -0.5\dot{a}(0)z\), where \(\dot{a}(0)\) is the global deceleration parameter, \(\Omega_m - \Omega_\Lambda/2\). In addition to time variation in the redshift, for gravitational waves emitted by the source, the resulting effect related to global dynamics is an additional change to the phase of the Fourier
transformed gravitational waveform of a chirping binary by an amount \( \propto f^{-13/3}X \) whose frequency dependence is largely different from the intrinsic binary evolution predicted by the post Newtonian expansion. This phase correction has been used as a probe of cosmological parameters in Ref. [22–23]. The anisotropy of the phase correction, or anisotropy measurement of \( X \) from a large sample of binaries spread over the sky, can be used as a probe of large wavelength gravitational waves.

We assume that the number of the probes (e.g., neutron star binaries, QSO line-of-sights) is sufficient and the effects caused by a finite number of such sources, such as Poisson fluctuations, is not important for measurement of \( z_{lm} \) at low \( l \). We also introduce the ratio \( R \equiv X(z)/H_0 \) to relate the resolution of the acceleration measurement, from which \( X(z) \) is derived, and the resulting constraint on the low-frequency gravitational wave background. In the case of the measurement related to the cosmic acceleration, in terms of the monopole variation associated with \( X(z) \), the redshift dependence of \( R \) is important and we analyze probes binned over some finite width in redshift. From the same data, but using higher order anisotropies, we get the constraint for \( \Omega_{GW} \). With a statistical relation between the measurement error of the spherical harmonic coefficients, \( z_{lm} \), in terms of an overall error in the power spectrum, we have

\[
\Delta C_l = \sqrt{\frac{2}{2l+1}} [C_l + \sigma^2 z_{00}],
\]

(14)

where \( \sigma^2 z_{00} \) is the variance related to the monopole measurement. The minimum energy density of the gravitational wave background measurable by any of the methods discussed below can be obtained with

\[
\sigma_{GW}^{-2} = \sum_l \frac{1}{(\Delta C_l)^2} \left( \frac{\partial C_l}{\partial \Omega_{GW}} \right)^2.
\]

(15)

To obtain the limit, under the null hypothesis of no gravitational wave background, we set \( C_l = 0 \) in Eq. [14]. Since the measurement of the whole power spectrum related to redshift derivatives, or \( X(z) \), may not be achievable in the near future, we only concentrate on the variance related to the quadrupole, \( C_2 \). Note that the uncertainty in this measurement is now determined by the error to which the monopole can be established, with \( \sigma^2 z_{00} \), given by the error in \( R \equiv X(z)/H_0 \). In figure 2, we plot the angular spectrum \( C_l \) for \( \Omega_{GW} \). We also put the measurement error for each \( C_l \), if the sensitivity for the monopole mode \( R \) is \( 10^{-3} \) for a given redshift shell.

Making use of the quadrupole anisotropy only, we obtain the 1-\( \sigma \) constraint for the background as

\[
\Omega_{GW} < 4.0 \times 10^2 (\Delta R)^2_{\text{shell}}
\]

(16)

from probes at each redshift shell, where \( \Delta R \) is the error to which \( R(z) \) can be established for such a shell. When we use the information of anisotropies up to \( l = 5 \), the coefficient \( 4.0 \times 10^2 \) is reduced to \( 3.9 \times 10^2 \). As we commented earlier, the signal due to the gravitational wave background does not depend on the redshift. This means that the actual constraint is better than the result using a single shell as one can combine shells at different redshifts, independent of the exact redshift. For cosmological model with \( h = 0.7, \Omega_m = 0.3, \) and \( \Omega_{\Lambda} = 0.7 \) the magnitude of \( R \) is \( \sim 0.05 \) around \( z = 1 \). Therefore, if we want to study the evolution of the cosmic acceleration in a precious manner, we need to have a resolution with an accuracy level \( \Delta R \sim 0.005 \), assuming a signal-to-noise ratio of 10, around a redshift interval \( dz \sim 0.1 \). With this sensitivity for the redshift change, \( \Delta z \), we can set the upper limit for \( \Omega_{GW} \) as

\[
\Omega_{GW} \sim 4.0 \times 10^2 \times 10^{-1} \times (5 \times 10^{-3})^2 \sim 10^{-3},
\]

(17)

where the factor \( 10^{-1} \) represents an average estimate on the improvement due to the increase in the effective number of the shells. If shells between redshift of 1 and 3 can be divided at intervals of 0.1, this increase is a factor of a few larger and one further improves the limit on \( \Omega_{GW} \).

\[\text{FIG. 2: The angular power spectrum } C_l \text{ of a redshift shell due to the low frequency gravitational wave background with } \Omega_{GW} = 10^{-4}. \text{ For the error bars we assume that we can measure the monopole mode } R \text{ with sensitivity } 10^{-3} \text{ for the shell.}\]

III. COSMOLOGICAL PROBES

A. Redshift variations in the line emission

The use of time variation associated with the redshift of distant sources to measure global cosmological param-
etters is described in Ref. 19. The resulting change is an overall common shift to the redshift distribution and the resulting change can be obtained by data separated in time. For this purpose, the most useful observations are the ones that involve narrow lines either in emission or absorption since the precise redshift can easily be ascertained from spectroscopic observations. In Ref. 19, the use of the Lyman-α forest in absorption towards luminous quasars has been discussed. The expected mean shift in the Lyman-α forest, at a redshift \( \sim 3 \), is roughly \( 2 \, \text{m s}^{-1} \) over a time period of 100 years, and for presently favorable cosmological parameters. While this is a small number, spectroscopic variations at the level of \( \sim 1 \, \text{m s}^{-1} \) are now routinely measured with 10 meter class telescopes towards bright stars in extra-Solar planetary searches using radial velocity measurements 27. The advent of 30 meter class telescopes over the next decade or more and the resulting increase in spectral resolution will improve searches for small velocity shifts in high-redshift quasar spectra.

While the precise measurement of the overall cosmological shift in a single quasar will require observations that span many tens of years, a detection of the global change can be obtained through a statistical study of a large number of quasars over a short time interval. While this will allow a measurement of the cosmological parameters 19, the effect due to a gravitational wave background is high order and requires a study related to the quadrupolar pattern of redshift shift. While the use of a quadrupolar pattern allows one to separate the effect related to cosmological acceleration, which is present in the monopole, and that due to local motion, present in the dipole, with that expected from a background of gravitational waves, the measurement, especially for an expected low energy density for the gravitational wave background is harder. For \( \Omega_{GW} \) at the level of \( 10^{-10} \), the anisotropy is at the level of \( 10^{-5} \) to \( 10^{-6} \), so the direct constrain one can put on \( \Omega_{GW} \) depends how well the monopole can be established.

Using estimates in Ref. 19 for observations of 1000 quasar sight lines with improved spectroscopic data with a pixel scale of 0.5 km sec\(^{-1}\) over a 10 year span, and assuming no systematic or additional statistical errors such as in the calibration of the spectrum, we find that one can constrain the monopole to an accuracy of \( \sim 1\% \). The resulting constrain on the low frequency, with wavelengths out to \( z \sim 3 \), background of gravitational waves is at the level \( \Omega_{GW} < 10^{-3} \). While the constrain is not significant, it is still useful to obtain this directly from the data as this will be one of the few methods to constrain the energy density of gravitational waves around \( 10^{-12} \) Hz.

In addition to the Lyman-α forest, one can constrain the energy density of gravitational waves using the redshift distribution of 21 cm emitter. Here, the line emission is at low radio wavelengths and statistics are expected to significantly improve given the expected narrow width of these lines, especially before the era of reionization, and the narrow band observations one can, in principle, achieve at low radio frequencies. Expected statistics of HI emitters before reionization is not well known and the complication here will be more related to foreground contamination 28 more than the number of objects one can use to make these measurements. Improvements in foreground removal techniques suggest that the contamination can be mostly removed 29.

We follow estimates in Ref. 30 which assume a model for the cosmological distribution of neutral gas before reionization based on the halo approach 11 and using so-called “mini halos” 32 that are beginning to establish with adiabatically cooled gas with a temperature below that of the CMB thermal radiation at those redshifts. We estimate the line width of each signal to be around 3 km sec\(^{-1}\) and to be dominated by thermal motions within each halo. This compares to line widths of order 20 km sec\(^{-1}\) in the Lyman-α spectrum. Since the characteristic mass scale of each of these mini halos is around \( 10^{6} \) M\(_{sun} \) to \( 10^{6} \) M\(_{sun} \), one expects a total of, in principle, \( 10^{18} \) halos across the whole sky with a size of 1 kpc, or a projected size of 30 milliarcseconds. While the whole set of mini halos are not needed, using a sample of 10\(^{6} \) brightest 21 cm lines over the whole sky, and assuming observations with an observational bandwidth of 0.2 kHz, which corresponds to a velocity of \( \sim 0.4 \) km sec\(^{-1}\), we find that one can, in principle, constrain the background to be below \( \Omega_{GW} < 10^{-5} \). The ultimate limit related to 21 cm anisotropies, using the whole sample of \( 10^{18} \) halos and assuming sufficient resolution to resolve them in redshift space and ignoring source confusion, is \( \Omega_{GW} < 10^{-11} \) with 10yr observation. This is comparable to the current limit from CMB anisotropies, but at frequencies between \( 10^{-18} \) Hz and \( 10^{-16} \) Hz.

### B. Gravitational waves from neutron star binaries around 1Hz

Here we discuss the prospect of the measurement of the background \( \Omega_{GW} \) at the very low frequency regime \(( < 10^{-12} \) Hz\) with proposed space gravitational wave missions, such as, the Big Bang Observer (BBO) or DECIGO, whose optimum sensitivities are between LISA and ground based detectors and around 0.1 Hz to 1 Hz. Note that this frequency range of 0.1 Hz to 1 Hz is largely different from the band at \( f \lesssim 10^{-12} \) Hz.

The most important aim of these detectors is the direct detection of the gravitational wave background from early universe with \( \Omega_{GW} \lesssim 10^{-15} \) level at 0.1 to 1 Hz. For this measurement, the foreground gravitational waves from various astrophysical sources must be subtracted from the data streams down to some appropriate level. The cosmological neutron star binaries are expected to produce a strong foreground, and we need an extensive fitting analysis with accurate templates to remove these sources from the data. In this fitting procedure we can measure not only the intrinsic binary parameters, but
also the total extrinsic effect made by the cosmic or the local acceleration through the phase variation related to $X$. In the presence of a low frequency gravitational wave background, extracted $X$ values for a sample of binaries spread over the sky, over a certain bin width in redshift, are supposed to vary with an anisotropic pattern. As discussed, the quadrupolar pattern can in return be used as a probe of the low frequency gravitational wave background. Therefore cosmological neutron star binaries can be regarded as probes of the dark energy, through the monopole variation related to $X(z)$ [22, 23], or the low frequency gravitational wave background through the anisotropy pattern of $X$. While the discussion is related to neutron star binaries detectable in the 0.1 Hz to 1 Hz band, we can also make a similar argument for stellar mass black hole binaries in the same band, though their intrinsic waveforms would be generally more complicated and the extraction of $X(z)$, through phase changes, would be a complicated procedure. The local acceleration might be generally larger than the neutron star binaries.

At present, it is not clear how well we can actually resolve and subtract the gravitational wave contribution of each merging neutron star binary in BBO or DECIGO data streams. For example, the confusion effect is more important for a binary with a longer time before coalescence, as its gravitational waves are in a lower frequency regime where the number density of binaries per unit frequency increases. In addition to binaries, the background made by other sources such as supernovae could be a fundamental noise [23].

Now we study how well we might set the constraint for $\Omega_{GW}$ at $f \lesssim 10^{-12}$Hz based on the recent analysis for the measurement of the cosmic acceleration $R \equiv X(z)/H_0$ [22]. Following their paper, we use the coalescence rate of neutron star binaries at $10^{-9}$/yr/Mpc assuming it is constant in time. The total number of binaries in a redshift shell $z \sim z + dz$ is $\sim 10^4$/yr for a width $dz \sim 0.1$ at $z \sim 1$. As we mentioned, in addition to the measurement error $\Delta X_{\text{det}}$ caused by detector noises, the random local acceleration becomes an effective noise $\Delta X_{\text{local}}$ for estimating the cosmic acceleration signal or the gravitational wave background. In the case of a neutron star binary, its magnitude would correspond to $\Delta X_{\text{local}}/H_0 \sim 10^{-1}$ in terms of the noise $\Delta R$ for each binary [24]. Therefore, for a shell with a width $dz \sim 0.1$, the estimation of the common signal is limited by the fluctuations due to the local acceleration around $(\Delta R)_{\text{shell}} \sim 10^{-1} \times \sqrt{N_{\text{shell}}} \sim 10^{-3.5}$. With an effective number for shells of $N_{\text{shell}} = 10$, we can set a upper limit of $\Omega_{GW} \sim 5 \times 10^{-5}$ using the relation (14).

For the measurement error $\Delta X_{\text{det}}$ determined by the detector noises, we can reach the level $(\Delta R)_{\text{shell}} \sim 3 \times 10^{-3}$ (corresponding to $\Delta X_{\text{det}}/H_0 \sim 3 \times 10^{-2}$ for each binary) with a 3 year integration by using the ultimate DECIGO (see figures 3 and 4 in Ref. [23]). But with a one year integration, the measurement error becomes $(\Delta R)_{\text{shell}} \sim 3 \times 10^{-3}$ and dominates the fluctuation by the local acceleration. In this estimation we did not include astrophysical confusion noises.

C. Small Angular Scale CMB Anisotropies

If a gravitational wave background exist at frequencies, say, between $10^{-14}$ Hz and $10^{-12}$ Hz, the time evolution of the background would produce a signature in CMB anisotropies through metric perturbations [31]. This is similar to temperature fluctuations produced at large angular scales with horizon-scale waves. In terms of the angular power spectrum of CMB anisotropies, the signature would be at multipole moments between $\sim 6 \times 10^4$ and $6 \times 10^6$. Such small-scale anisotropies, at arcsecond angular scales, are not probed by any of the present observations. In fact, no dedicated plans exist for the CMB anisotropy observations at such small scales though certain radio interferometers planned for purposes other than CMB, such as the Atacama Large Millimeter Telescope (ALMA), can be used to make the required measurements.

The fluctuations at these scales are dominated by unresolved radio point sources, which produce a shot-noise spectrum. For comparison, assuming a radio point-source removal similar of the Cosmic Background Imager (CBI) [28] experiment, where residual fluctuations were at the level of 150 $\mu$K in the smallest angular scale bin in multipole moments between 2000 and 3500, we find that the resulting limit to CMB temperature fluctuations in a bin with multipole moments between $\sim 6 \times 10^4$ and $6 \times 10^6$ is $\Delta T < 3000\mu$K. This results in a limit for the gravitational wave background between $10^{-14}$ Hz and $10^{-12}$ Hz of $\Omega_{GW} < 10^{-9}$. To reach the ultimate level implied by 21 cm observations require a factor of 100 improvement in foreground removal. Since foregrounds are already restricted to a few percent level, this is a challenging task.

IV. SUMMARY

The curl modes of cosmic microwave background polarization allow one to indirectly constrain the primordial background of gravitational waves with wavelengths roughly the horizon size or larger with frequencies below $10^{-16}$ Hz. The planned high precision timing observations of a large sample of millisecond pulsars with the Pulsar Timing Array or with the Square Kilometer Array can either detect or constrain the stochastic gravitational wave background at frequencies greater than roughly 0.1 years$^{-1}$. While there are no strong observational constraints on the gravitational wave background across six or more orders of magnitude between $10^{-16}$ Hz and $10^{-10}$ Hz, we suggest that by monitoring the anisotropy pattern of time variation of the redshift related to a sample
of high redshift objects one can constrain the gravitational wave background below $10^{-12}$ Hz where methods using Galactic objects do not work well. Direct measurement of the time variation of redshift is one of the potential method to get information of the dark energy. Analyses in this paper would be helpful to discuss the prospects of the low frequency gravitational wave background measurement in a given observational condition of the dark energy study. Useful observations for the monitoring of an anisotropy signal in a global redshift change include spectroscopic observations of the Ly-$\alpha$ forest in absorption towards a sample of quasars, redshifted 21 cm line observations, either in absorption or emission, towards a sample of neutral HI regions before or during reionization, and high frequency (0.1 Hz to 1 Hz) gravitational wave analysis of a sample of neutron star–neutron star binaries detected with gravitational wave instruments such as the Decihertz Interferometer Gravitational Wave Observatory (DECIGO). For reasonable observations in the future, we find best limits at the level of $\Omega_{GW} < 10^{-5}$ at a frequency around $10^{-12}$ Hz.

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