Semi-classical edge states for the Robin Laplacian

Bernard Helffer\textsuperscript{1} | Ayman Kachmar\textsuperscript{2}

\textsuperscript{1}Laboratoire de Mathématiques Jean Leray, Nantes Université, Nantes, France
\textsuperscript{2}Department of Mathematics, Lebanese University, Nabatieh, Lebanon

Correspondence
Ayman Kachmar, Department of Mathematics, Lebanese University, Nabatieh, Lebanon.
Email: akachmar@ul.edu.lb

Funding information
Lebanese University

\textbf{Abstract}
Motivated by the study of high-energy Steklov eigenfunctions, we examine the semi-classical Robin Laplacian. In the two-dimensional situation, we determine an effective operator describing the asymptotic distribution of the negative eigenvalues, and we prove that the corresponding eigenfunctions decay away from the boundary, for all dimensions.

\textbf{MSC (2020)}
35P15 (primary), 47F05 (secondary)

\section{INTRODUCTION}

\subsection{Motivation: Generalized Steklov eigenfunctions}

Let us consider an open bounded set $\Omega \subset \mathbb{R}^n$ with a smooth connected boundary $\Gamma$. Let $-\Delta^D$ be the Dirichlet Laplace operator on $\Omega$ with spectrum $\sigma(-\Delta^D)$. We fix a constant $w \in \mathbb{R} \setminus \sigma(-\Delta^D)$. For every function $\psi \in H^{1/2}(\Gamma)$, we assign the unique function $u = u_{w,\psi}$ as follows

$$-\Delta u = w u \text{ on } \Omega \quad \text{and} \quad u = \psi \text{ on } \Gamma.$$ (1.1)

The operator

$$\psi \in H^{1/2}(\Gamma) \mapsto \Lambda(w)\psi := \frac{\partial u_{w,\psi}}{\partial \nu} \in H^{-1/2}(\Gamma)$$ (1.2)

is the Dirichlet-to-Neumann (DN) operator. Here $\nu$ denotes the unit outward normal vector of $\Gamma$. The DN operator is a boundary pseudo-differential operator of order 1. Its spectrum consists
of a non-decreasing sequence of eigenvalues \((\mu_m(w))_{m \geq 1}\) counting multiplicities, known as the (generalized) Steklov eigenvalues\(^†\). More precisely,

\[
\sigma(\Lambda(w)) = \sigma^\delta,
\]

where \(\sigma^\delta\) is the Steklov spectrum defined as the set of real numbers \(\mu\) such that a non-trivial solution \(u\) exists for the following Robin problem

\[
-\Delta u = w u \text{ on } \Omega, \quad u \in H^2(\Omega) \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \mu u \text{ on } \Gamma.
\] (1.3)

The study of the localization of the normalized solutions \(u^\mu\) of (1.3) in the limit \(^‡\) \(\mu \to +\infty\) is connected with the semi-classical Robin Laplacian studied in [16].

Let us formulate the Steklov problem in the framework of [16]. We introduce the semi-classical parameter \(h = \mu^{-2}\) and denote by \(u_h\) a non-trivial solution of (1.3); the eigenfunction \(u_h\) satisfies\(^§\)

\[
\begin{cases}
-\Delta u_h = w u_h & \text{in } \Omega \\
\frac{\partial u_h}{\partial \nu} = h^{-1/2} u & \text{on } \Gamma.
\end{cases}
\] (1.4)

We introduce the self-adjoint operator \(\mathcal{T}_h\) with domain \(D(\mathcal{T}_h)\) as follows

\[
\mathcal{T}_h = -h^2 \Delta, \quad D(\mathcal{T}_h) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = h^{-1/2} u \text{ on } \Gamma\}.
\]

Then (1.4) can be rewritten in the form

\[
\mathcal{T}_h u_h = w_h u_h, \quad u_h \in D(\mathcal{T}_h) \setminus \{0\}, \quad w_h := h^2 w.
\] (1.5)

By [16], in the planar situation \(n = 2\), if \(w_h < 0\) (see below the more precise condition), \(u_h\) decays exponentially as follows.

Given \(M \in (0, 1)\) and \(\alpha \in (0, \sqrt{M})\), there exist \(h_0, C > 0\) such that

\[
\int_{\Omega} (|u_h|^2 + h|\nabla u_h|^2) \exp \left( \frac{2\alpha d(x, \Gamma)}{h^{1/2}} \right) dx \leq C \|u_h\|^2_{L^2(\Omega)},
\] (1.6)

for \(h \in (0, h_0]\) and \(w_h < -Mh\).

Here \(d(\cdot, \Gamma)\) is the normal distance to the boundary

\[
d(x, \Gamma) = \inf\{|x - y| : y \in \Gamma\} \quad (x \in \mathbb{R}^n).
\] (1.7)

\(^†\) The Steklov eigenvalues correspond to the case where \(w = 0\).

\(^‡\) This amounts to the study of the Steklov eigenpair \((u^{\mu_m}, \mu_m)\) as \(m \to +\infty\).

\(^§\) Introducing \(h = \mu^{-1}\) leads to (1.4) with \(h^{-1}\) instead of \(h^{-1/2}\) in the boundary condition, which is the scale on most papers on the Steklov eigenfunctions [10]. However, we choose the scaling in (1.4) to match with the papers on the semi-classical Robin Laplacian [16].
This decay is a consequence of Agmon type estimates. If we note that the ground state energy of the operator $T_h$ satisfies $\lambda_1(T_h) = -h + o(h)$ as $h \to 0_+$, the theorem applies with $\alpha < 1$. This decay result can be easily extended to the $n$-dimensional situation [24] from which we can deduce pointwise estimates on $u_h$ (see Theorem 2.1).

Examining the case of the annulus, $\Omega = \{x \in \mathbb{R}^2 : r_0 < |x| < 1\}$, we observe that the constant $\alpha$ and the distance function $d(x, \Gamma)$ in (1.6) are non-optimal. The example of the annulus suggests the optimal decay rate is achieved with $\alpha \approx 1$ and a distance function $\hat{d}_\Gamma$ that depends on the curvature of the boundary (see [10, section 1.1.3] and [6]).

Returning to the problem in (1.3), we see that a consequence of (1.6) is that the Steklov eigenfunction decays away from the boundary provided the Steklov eigenvalue $\lambda$ satisfies $w \leq -M\lambda^2$ and $\lambda \gg 1$ (i.e., $|w| \geq M\lambda^2 \gg 1$).

Our aim is to relax this strong assumption imposed on $w$. This question is motivated by the paper by Galkowski–Toth [10] (who also refer to Hislop–Lutzer [18] and Polterovich–Sher–Toth [25]) and by the Ph.D. thesis of Gendron [11] discussing for special manifolds with boundary the correspondence between the spectrum of the Steklov and the metric given on the manifold. In the first contribution, it is assumed that $w = 0$, and the above decay is obtained with $\alpha = 1$, but under the condition that the boundary is analytic. Although not written explicitly, the computations by Gendron can also lead to the same result (but for a particular case). This has been developed in the recent work [6].

In the semi-classical framework, we will study the spectral properties of the eigenvalues of the Robin Laplacian $T_h$ below the energy level $h^2\lambda_1^D(\Omega)$, where $\lambda_1^D(\Omega)$ is the ground state energy of the Dirichlet Laplacian. We obtain a boundary effective operator that describes the asymptotic distribution of the eigenvalues in the semi-classical limit (see Theorem 5.1). The corresponding eigenfunctions (which can be viewed as interior Steklov eigenfunctions in the sense of [10, 18]) are expected to be localized near the domain’s boundary (thereby called edge states in the literature). We confirm this property in Theorem 1.1, which is valid for any dimension $n \geq 2$.

Our results are stated for bounded domains in $\mathbb{R}^n$ to avoid technical difficulties in the proofs, but they can be generalized to compact manifolds with boundaries (the boundary must be $C^\infty$ in Theorem 1.1 and analytic in Theorem 1.2).

### 1.2 Decay of eigenfunctions

Using the boundary pseudo-differential calculus (as in [18]), we obtain that all eigenfunctions corresponding to non-positive eigenvalues of the Robin Laplacian $T_h$ decay away from the boundary, uniformly with respect to the non-positive eigenvalues. This extends the result of [18] up to the boundary, and presents a weaker version of the result of [10] but valid for the non-zero modes of $T_h$.

**Theorem 1.1.** Let $\lambda_1^D(\Omega)$ the principal eigenvalue of the Dirichlet Laplacian $-\Delta$ on $\Omega$.

For any $p \in \mathbb{N}$ and $\zeta < \lambda_1^D(\Omega)$, there exist positive constants $C_{p,\zeta}$, $h_{p,\zeta}$ and such that if $(h, u_h, w)$ is a solution of (1.8)

\[
\begin{align*}
-\Delta u_h &= w u_h \quad \text{in } \Omega, \\
\partial u_h &= h^{-1/2} u_h \quad \text{on } \Gamma',
\end{align*}
\]

(1.8)
with $h \in (0, h_{p, \zeta}]$, $w \leq \zeta$, and $\|u_h\|_{L^2(\partial\Omega)} = 1$ then it satisfies

$$|u_h(x)| \leq C_{p, \zeta} \left( \frac{h}{(d(x, \Gamma))^2} \right)^p, \forall x \in \Omega,$$

(1.9)

where $d(x) = d(x, \Gamma)$ is the distance to the boundary introduced in (1.7).

It would be desirable to control the constant $C_{p, \zeta}$ in (1.9) with respect to $p$. In the case of an analytic boundary, where analytic pseudo-differential calculus is handy, we anticipate an estimate of the following form

$$|u_h(x)| \leq C_0 d(x)^{2-n-h^{-\frac{1}{2}}} \exp \left( -\frac{1}{C_1} d(x) h^{-\frac{1}{2}} \right), \forall x \text{ such that } d(x) \leq \frac{1}{C_2},$$

(1.10)

for some constants $C_0, C_1, C_2 > 0$, which could be difficult to determine explicitly. We will discuss this in Subsection 3.5. Note that, for $w = 0$, (3.25) is established with $C_1 = 1 + \eta$ and $\eta$ arbitrarily small in [10] by using analytic micro-local methods. This was improving the non-optimal exponential bound of [25] in the two-dimensional case.

In the case of an analytic boundary, based on the analysis in [10], we are able to improve the decay in Theorem 1.1 for $w \neq 0$.

**Theorem 1.2.** Assume that $\Gamma$, the boundary of $\Omega$, is analytic. For any $\zeta < \lambda_D^1(\Omega)$ and $\eta > 0$, there exist positive constants $\varepsilon, C, h_0$ such that if $(h, u_h, w)$ is a solution of (1.8) with $h \in (0, h_0]$, $w \leq \zeta$, and $\|u_h\|_{L^2(\partial\Omega)} = 1$, then the following estimate holds

$$\forall x \in \Omega, |u_h(x)| \leq C h^{-\frac{n}{2} + \frac{1}{3}} \exp \left( -\frac{(1 - \eta) \inf(d(x, \Gamma), \varepsilon)}{h^{1/2}} \right).$$

(1.11)

The constant $\varepsilon$ in Theorem 1.11 depends on $\Omega$ and cannot be chosen independently of $\eta$ (even in the case $w = 0$ of [10, Theorem 1]).

At the moment, it is unclear if the analytic assumptions are important for the validity of the estimates (and more accurate estimates discussed around (3.25)). Note that in the $C^\infty$ case, the micro-local approach proposed in [10] could at most give an information modulo $\mathcal{O}(h^\infty)$ leading perhaps to (1.9) with $\varepsilon = C h^{\frac{1}{2}} |\log h|$, for some $C > 0$, to compare with (1.9).

The method of Agmon estimates, recalled in (1.6), is on one hand advantageous since it does not require the analytic hypothesis of the boundary, but on the other hand its drawback is that it becomes weaker, due to the condition on $\alpha$, as the eigenvalue $w$ approaches 0. However, Theorem 1 of Galkowski–Toth [10] and Theorem 1.1 show that all eigenfunctions decay with a constant exponential profile under the analytic boundary hypothesis. It would then be interesting to extend these estimates to the case of a $C^\infty$- boundary.

Positive indications will be given in the two-dimensional case that we will discuss in the next section. Let us denote by

$$L = \frac{|\Gamma|}{2},$$

(1.12)
where $|\Gamma|$ is the length of the boundary $\Gamma$. Assuming $\Gamma$ is connected, we will encounter quasi-modes with trace on the boundary normalized in $L^2(\Gamma)$ and having the following profile

$$u_h \approx (2L)^{-1/2} \exp \left( -h^{-1/2} d(x, \Gamma) \right) e^{ik\pi s/L}$$

with $k \in \mathbb{Z}$. Such quasi-modes appear also in Polterovich–Sher–Toth’s paper [25] for the eigenvalue $\omega = 0$, where it is proved, in the case of an analytic boundary, that they are close to the actual zero-modes of the operator $T_h$. In the case where $\Gamma$ is not connected [25], we still encounter the foregoing quasi-modes on each connected component of $\Gamma$ and their linear combinations.

### 1.3 Asymptotic distribution of eigenvalues

There is a one-to-one correspondence between the negative eigenvalues of $T_h$ and the Steklov eigenvalues below the energy level $h^{-1/2}$ (see [2, Lemma 1; [5]] in a slightly different context). The correspondence being not explicit, it does not yield a precise description of the eigenvalues of the operator $T_h$, based on the existing eigenvalue asymptotics for the Steklov eigenvalues, but it does allow to deduce the asymptotics for the counting function of the operator $T_h$ from that of the DN operator $\Lambda(0)$. Our result on the Robin eigenvalues (Theorem 1.3) is new and within our approach we can quantify the correspondence between the Robin and Steklov eigenvalues, and also to derive Weyl laws for the operator $T_h$ (and consequently for the DN operator) in Theorem 1.4.

Let us consider the case $n = 2$ for the sake of simplicity and assume that $\Omega$ is simply connected. We denote by $(\lambda_n(T_h))_{n \geq 1}$ the sequence of min–max eigenvalues of the operator $T_h$. We will determine the asymptotic behavior of $\lambda_n(T_h)$ in the regime $h \to 0_+$ thereby describing the distribution of all the negative eigenvalues of $T_h$.

For all $n \geq 2$ and $L$ introduced in (1.12), we introduce the eigenvalues

$$\lambda_n^F(L) = \frac{\pi^2 k^2}{L^2} \quad \text{for } n \in \{2k, 2k+1\} \& k \in \mathbb{N},$$

which correspond to the Fourier modes $e^{\pm i\pi ks/L}$ on $\mathbb{R}/2L\mathbb{Z}$.

**Theorem 1.3.** Let $\lambda_2^N(\Omega)$ denote the second eigenvalue of the Neumann Laplacian $-\Delta$ on $\Omega$ and consider a positive constant $\varepsilon < \lambda_2^N(\Omega)$. Assume furthermore that $\Omega$ is simply connected. Then, there exist positive constants $C$ and $h_0$ such that, for all $h \in (0, h_0)$, the following estimates hold

$$|\lambda_n(T_h) + h - h^2 \lambda_n^F(L)| \leq C h^{3/2} \left( 1 + h^{3/4} \lambda_n^F(L) \right),$$

provided that $\lambda_n(T_h) < \varepsilon h^2$.

The proof of Theorem 1.3 follows by deriving an effective operator approximating the operator $T_h$. The precise statement is given in Theorem 5.1.

The estimates of Theorem 1.3 are interesting when $n \gg h^{-1/4}$, since by [16, Theorem 2.1; 17, Proposition 7.4], $\lambda_n(T_h) \leq -h - h^{3/2} x_{\max} + O(h^{7/4})$ for $n \lesssim h^{-1/4}$. In particular, the negative
eigenvalues of $T_h$ satisfy

$$\lambda_{2k}(T_h) \sim -h + \pi^2 L^{-2} k^2 h^2 \quad \text{and} \quad \lambda_{2k+1}(T_h) - \lambda_{2k}(T_h) = \mathcal{O}(h^{3/2})$$

provided $k \gg h^{-1/4}$. This is consistent with the results in [7, 12, 28] and [25, section 3.1] dealing with the spectrum of the DN operator $\Lambda(0)$, whose principal symbol coincides with $\sqrt{-\Delta_\Gamma}$, the square root of the Laplace-Beltrami operator on $\Gamma$. In fact, the Steklov eigenvalues $(\mu_n)_{n \geq 1}$ of $\Lambda(0)$ satisfy the following asymptotics [28]

$$\mu_{2k+1} = \mu_{2k} + O(k^{-\infty}) = \frac{\pi}{L} k + O(k^{-\infty}) \quad (k \to +\infty). \quad (1.13)$$

So, we get the following correspondence between the negative Robin eigenvalues $\{\lambda_n(T_h) < 0\}$ and the Steklov eigenvalues $\{\mu_n < h^{-1/2}\}$:

$$\mu_n \sim h^{-1/2} \sqrt{h + \lambda_n(T_h)} \quad \text{for} \quad h^{-1/4} \ll n \leq \frac{L}{\pi} h^{-1/2} + \mathcal{O}(h^{1/2}).$$

As a direct consequence of Theorem 1.3, we obtain a Weyl law extending earlier results [17, 19, 20].

**Theorem 1.4.** Assume that $\Omega$ is simply connected. Let $\varepsilon \in [0, \lambda_2^N(\Omega))$. For all $h > 0$ and $\lambda \in \mathbb{R}$, we denote by

$$N(T_h, \lambda) := \text{tr}(1_{(-\infty, \lambda)}(T_h)).$$

Then we have the following asymptotics as $h \to 0$,

$$N(T_h, \varepsilon h^2) = \frac{|\Gamma|}{\pi} h^{-1/2} + \mathcal{O}(h^{-1/4}).$$

Furthermore,

$$N(T_h, \lambda h) = \frac{|\Gamma|}{\pi} \sqrt{1 + \lambda h^{-1/2} + \mathcal{O}(h^{-1/4})}$$

holds for all $\lambda \in (-1, 0)$.

The asymptotics of $N(T_h, \lambda h)$ and $N(T_h, \varepsilon h^2)$ hold uniformly with respect to $\lambda \in (-1, 0)$ and $\varepsilon \in [0, \lambda_2^N(\Omega))$ respectively. Noting that

$$N(T_h, 0) = N(\Lambda(0), h^{-1/2}),$$

we recover the leading order term for the existing results on the DN operator (see [13, eq. (2.1.4)])

$$N(\Lambda(0), h^{-1/2}) = \frac{|\Gamma|}{\pi} h^{-1/2} + \mathcal{O}(1). \quad (1.14)$$

The asymptotics in (1.14) continues to hold for the generalized DN operator $\Lambda(\varepsilon)$ introduced in (1.2) if $\varepsilon < \lambda_2^D(\Omega)$ is fixed (or in a compact interval of $(-\infty, \lambda_2^D(\Omega))$, see [23]). Moreover,
\[ N(\Lambda(\epsilon), h^{-1/2}) = N(\mathcal{T}_h, \epsilon h^2), \] hence we get
\[ N(\mathcal{T}_h, \epsilon h^2) = \frac{|\Gamma|}{\pi} h^{-1/2} + \mathcal{O}(1) \]
which gives a more accurate estimate of the remainder than the one appearing in Theorem 1.4.

**Organization of the paper**

- In Section 2, we show how we can extract pointwise bounds on the eigenfunctions from the Agmon decay estimates.
- In Section 3, we use a pseudo-differential calculus to prove Theorems 1.1 and 1.2.
- In Section 4, we analyze one-dimensional operators that we use later in Section 5 to derive an effective operator for the Robin Laplacian and prove Theorem 5.1.

## 2 POINTWISE BOUNDS VIA AGMON ESTIMATES

Using the elliptic and Agmon estimates, we can derive pointwise bounds on the low-energy eigenfunctions of the semi-classical Robin Laplacian operator \( \mathcal{T}_h \). This was standard in the case of Dirichlet case but because the Robin condition includes the parameter inside the boundary condition, we feel that it is useful to give the details in this new case.

**Theorem 2.1.** Given \( M \in (0, 1) \) and \( \alpha \in (0, \sqrt{M}) \), there exist positive constants \( \epsilon_0, h_0, C > 0 \) such that, if \( h \in (0, h_0) \) and \( u_h \) is a solution of
\[
\begin{align*}
-\Delta u_h &= w u_h \quad \text{in } \Omega \\
\frac{\partial u_h}{\partial \nu} &= h^{-1/2} u_h \quad \text{on } \Gamma
\end{align*}
\]
with \( w < -M h^{-1} \) and \( \|u_h\|_{L^2(\Gamma)} = 1 \), then
\[
|u_h(x)| \leq C h^{-\frac{n}{2} - \frac{1}{2}} \exp \left( -\frac{\alpha \min(d(x, \Gamma), \epsilon_0)}{h^{1/2}} \right) \quad (x \in \Omega). \quad (2.1)
\]

**Proof.** For all \( \epsilon > 0 \), we introduce the tubular neighborhood of the boundary,
\[
\Omega_{\epsilon} := \{ x \in \Omega, d(x, \Gamma) < \epsilon \}. \quad (2.2)
\]
Choose \( \epsilon_0 > 0 \) so that the function \( x \mapsto d(x, \Gamma) \) is smooth on \( \Omega_{2\epsilon_0} \). We extend this function to a smooth function \( \bar{d} \) on \( \Omega \) as follows
\[
\bar{d}(x) = \begin{cases} 
  d(x, \Gamma) & \text{if } x \in \Omega_{\epsilon_0} \\
  2\epsilon_0 & \text{if } x \in \Omega \setminus \Omega_{2\epsilon_0} 
\end{cases}
\]
and \( \epsilon_0 \leq \bar{d}(x) \leq 2\epsilon_0 \) if \( x \in \Omega_{2\epsilon_0} \setminus \Omega_{\epsilon_0} \).
We introduce the function \( v_h(x) = u_h(x) \exp\left(\frac{\alpha(x)}{h^{1/2}}\right) \). We select \( \alpha \in (0, \sqrt{M}) \) and \( h_0 > 0 \) so that, for all \( h \in (0, h_0) \), (1.6) holds, which in turn yields
\[
\|v_h\|_{H^1(\Omega)} + h^{-1/2}\|v_h\|_{L^2(\Omega)} \leq C_1 h^{-1/2}\|u_h\|_{L^2(\Omega)}.
\]

The function \( v_h \) satisfies the non-homogeneous Neumann problem:
\[
\Delta v_h = f_h \text{ in } \Omega \text{ and } \frac{\partial v_h}{\partial \nu} = g_h \text{ on } \Gamma,
\]
where
\[
f_h(x) = \left(\alpha h^{-1/2}\Delta\bar{t} - 2\alpha^2 h^{-1} |\nabla\bar{t}|^2 - w\right)v_h + 2\alpha h^{-1/2}\nabla\bar{t} \cdot \nabla v_h,
\]
and
\[
g_h(x) = (\alpha + 1) h^{-1/2}v_h(x).
\]

By the elliptic estimates for the Neumann non-homogeneous problem, we get
\[
\|v_h\|_{H^2(\Omega)} \leq C_2 \left(\|f_h\|_{L^2(\Omega)} + \|v_h\|_{L^2(\Omega)} + \|g_h\|_{H^{1/2}(\Gamma)}\right) \leq C_2 h^{-1}\|u_h\|_{L^2(\Omega)}.
\]

In the cases \( n = 2, 3 \) and by Sobolev embedding, we deduce an estimate in the Hölder norm. For the case \( n \geq 4 \), we pick the smallest integer \( k_\ast > \frac{n}{2} \) and we iterate the previous estimate so that
\[
\|v_h\|_{H^{k_\ast}(\Omega)} \leq C_\ast \left(\|f_h\|_{H^{k_\ast-2}(\Omega)} + \|v_h\|_{H^{k_\ast-2}(\Omega)} + \|g_h\|_{H^{k_\ast-2+1/2}(\Gamma)}\right) \leq C_\ast h^{-k_\ast/2}\|u_h\|_{L^2(\Omega)}.
\]

We use Sobolev embedding of \( H^{k_\ast}(\Omega) \) in \( C(\overline{\Omega}) \) and that \( k_\ast \leq \frac{n}{2} + 1 \). To finish the proof, we note that due to our normalization of \( u_{h_{1/}} \), the norm of \( u_h \) in \( \Omega \) satisfies \( \|u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{1/4}) \) since
\[
-Mh^{-1}\|u_h\|_{L^2(\Omega)}^2 \
\geq -w\|u_h\|_{L^2(\Omega)}^2 = \|\nabla u_h\|_{L^2(\Omega)}^2 - h^{-1/2}\|u_h\|_{L^2(\Gamma)}^2.
\]

3 \ Boundary Pseudo-Differential Calculus and Decay of Eigenfunctions

3.1 \ Decay in the interior

Here we discuss (and improve) the weaker result of [18] leading to the conjecture proved by [10].

**Theorem 3.1** ([Hislop–Lutzer [18]]) \( \frac{1}{2} \). For any \( p \in \mathbb{N} \), any \( K \subset \Omega \) compact, there exists \( C_{K,p} > 0 \) and \( h_{K,p} > 0 \) such that if \( (h, u_h) \) is a solution of (3.1)

\[
\begin{cases}
-\Delta u_h = 0 & \text{in } \Omega \\
\frac{\partial u_h}{\partial \nu} = h^{-1/2}u_h & \text{on } \Gamma,
\end{cases}
\]
with \( h \in (0, h_{K, p}] \) and \( \|u_h\|_{L^2(\partial \Omega)} = 1 \), then it satisfies
\[
|u_h(x)| \leq C_{K, p} h^{p/2}, \forall x \in K.
\] (3.2)

Note that our Theorem 1.1 extends the result of Theorem 3.1 up to the boundary. The idea is to use the properties of the Poisson kernel of the operator \(-\Delta - w\) up to the boundary, while in [18], the properties of the Poisson kernel were used in the interior of the domain.

### 3.2 Proof of Theorem 1.1 for \( w = 0 \)

The proof of [18] \((w = 0)\) is based on the classical Green-Representation Formula for \( u_h \) (see [9, chapter 2, section 2.2.4] for the basic theory)
\[
u_h(x) = \int_{\partial \Omega} u_h(\cdot)P(x, \cdot) d\sigma ,
\] (3.3)
where \( P(x, \cdot) \) is the Poisson kernel defined as follows
\[
P(x, \cdot) = -\partial_y G(x, \cdot),
\] (3.4)
where the distribution \( G(x, y) \in \mathcal{D}'(\Omega \times \Omega) \) is, given \( x \in \Omega \), the solution of the inhomogeneous Dirichlet problem
\[
-\Delta_y G(x, \cdot) = \delta_x , G(x, y) = 0 \text{ for } y \in \partial \Omega.
\] (3.5)
The properties of \( G \) (which is called the Green function) are rather well-known in the case of a smooth boundary (see [18, Theorem 2.3]) but for the proof of the conjecture, we will need a more precise information for the Poisson kernel for \( y \in \partial \Omega \) and \( x \) close to \( \partial \Omega \). This is done, at least for \( w = 0 \) in [8] (see also [22]).

The proof is based on the connection with the DN operator \( \Lambda(w) \). Indeed, \( u_{h/\partial \Omega} \) is an eigenfunction of \( \Lambda(w) \) associated with the eigenvalue \( h^{-\frac{1}{2}} \). We can then write
\[
u_h(x) = h^{\frac{p}{2}} (P \circ \Lambda(w)^p)(u_{h/\partial \Omega}).
\] (3.6)
For \( w = 0 \), (3.6) reads as follows
\[
u_h(x) = h^{\frac{p}{2}} \int_{\partial \Omega} u_h(y) \cdot (\Lambda(0)^p(y, D_y)P(x, y)) d\sigma .
\] (3.7)
For \( x \in K \), it is then easy to get the result obtained in [18], that is, the interior decay estimate of Theorem 3.1. As for the estimate of Theorem 1.1 up to the boundary, we recall the estimate given by Englis in [8].
Theorem 3.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Then the Poisson kernel $P(x, y)$ admits the following decomposition

$$P(x, y) = \frac{c_n d(x)}{|x - y|^n} \left[ F(y, |x - y|, \frac{x - y}{|x - y|}) + H(x, y)|x - y|^n \log |x - y| \right], \quad (3.8)$$

where

- $d \in C^\infty(\bar{\Omega}), d > 0$ on $\Omega$,
- $d(x) = d(x, \partial \Omega)$ for $x$ near $\partial \Omega$,
- $c_n = \frac{\Gamma(n/2)}{\pi^{n/2}}$,
- $F \in C^\infty(\partial \Omega \times \mathbb{R}^+ \times \mathbb{S}^{n-1}), F(y, 0, \omega) = 1$ for $y \in \partial \Omega, \omega \in \mathbb{S}^{n-1},$
- $H \in C^\infty(\bar{\Omega} \times \partial \Omega)$.

This implies in particular the weak version mentioned by Krantz [22] which reads, for $n \geq 2$,

$$|\partial^\alpha_y P(x, y)| \leq C_\alpha d(x)|x - y|^{-n-|\alpha|}, \forall y \in \partial \Omega, x \in \Omega. \quad (3.9)$$

This last estimate directly implies

$$|\partial^\alpha_y P(x, y)| \leq C_\alpha d(x)^{1-n-|\alpha|}, \forall y \in \partial \Omega, x \in \Omega. \quad (3.10)$$

Coming back to (3.7), we can write for $p$ even (if we do not want to use the complete Boutet de Monvel calculus)

$$u_h(x) = h^{\frac{p}{2}} \int_{\partial \Omega} u_h(y) \cdot (\Lambda(0)^p \cdot (-\Delta_y)^{-p/2})((-\Delta_y)^{p/2})P(x, y)d\sigma$$

$$= h^{\frac{p}{2}} \int_{\partial \Omega} ((-\Delta_y)^{-p/2} \cdot \Lambda(0)^p)u_h(y) \cdot ((-\Delta_y)^{p/2})P(x, y)d\sigma. \quad (3.11)$$

We now observe that $(-\Delta_y)^{-p/2} \cdot \Lambda(0)^p$ is a boundary pseudo-differential operator of degree 0 (with constant principal symbol) and using (3.10) we obtain, for any $p \geq 1$,

$$|u_h(x)| \leq C_p h^{\frac{p}{2}} d(x)^{1-n-p}. \quad (3.12)$$

This proves Theorem 1.1 for $w = 0$.

### 3.3 Proof of Theorem 1.1 for $w \in [-\pi^2, \lambda_1^D(\Omega))$

Now assume that $-\pi^2 \leq w < \lambda_1^D(\Omega)$ and $w \neq 0$. The proof is similar to the case $w = 0$ but we should replace the Green function $G$ by $G_w$ and the ND operator $\Lambda(0)$ by $\Lambda(w)$. There is no problem of definition if $w$ is not an eigenvalue of the Dirichlet Laplacian. To avoid to analyze if the proof written for $w = 0$ goes on, we can use a weaker theorem which holds for general potential operators (or Poisson like operators). See [8, p. 18, Theorem 8] with $n \geq 2, d = p$ and observe that $|x - \xi|^{-1} \leq d(x)^{-1}$. The aforementioned result of [8] reads as follows.
Theorem 3.3 (Englis [8]). If $K$ is a potential operator in $K^d(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary (or $\Omega = \mathbb{R}^n_+$), then the Schwartz kernel $k_K$ satisfies, if $d \in \mathbb{Z}, d > 1 - n$

$$k_K(x, y) = |x - y|^{1-n-d} F(y, |x - y|, \frac{x - y}{|x - y|}) + H(x, y) \log |x - y|, \quad (3.13)$$

where $F$ and $H$ have the same property as in the previous theorem.

In our application, we use that the Poisson operator (associated with $(-\Delta - w)$) is a potential operator $P(w)$ if $w$ is not an eigenvalue of the Dirichlet Laplacian. We also use the property that the DN operator $\Lambda(w)$ is a boundary pseudo-differential operator of degree 1 with elliptic principal symbol.

We apply Theorem 3.3 to $K = (P \circ \Lambda(w)^p)$ and use (3.7). Everything depends continuously of $w$ in the interval $I := [-\pi^2, \lambda^D_1(\Omega))$ and the control is uniform in any compact interval in $I$. This is clear for the computation (symbolic calculus) of an approximate Poisson operator $P^\text{app}(w)$ modulo regularizing operators $R^\text{reg}(w)$ and $r^\text{reg}(w)$ without additional assumptions. One gets

$$(-\Delta - w)P^\text{app}(w) = R^\text{reg}(w), \quad \gamma \circ P^\text{app}(w) = I + r^\text{reg}(w).$$

For eliminating the remainder, we use the resolvent and this is there that the assumption that $w$ is not in the spectrum of the Dirichlet Laplacian is used. More precisely, we first eliminate $r(w)$ by using simply an extension operator $\varepsilon$ from $C^\infty(\partial \Omega)$ into $C^\infty(\Omega)$. We note that $\varepsilon \circ r^\text{reg}(w)$ is regularizing.

Then, we compute

$$(-\Delta - w)(P^\text{app}(w) - \varepsilon \circ r^\text{reg}(w)) = R^\text{reg}(w) - (-\Delta - w)\varepsilon \circ r^\text{reg}(w) := R^\text{reg}(w).$$

Finally, we get for the Poisson kernel

$$P(w) = P^\text{app}(w) - \varepsilon \circ r^\text{reg}(w) - (-\Delta - w)^{-1} R^\text{reg}(w).$$

At this stage we get (3.12) from (3.6) in the case where $w \neq 0$ is fixed in the interval $[-\pi^2, \lambda^D_1(\Omega))$, the estimate being uniform in $w$ for any compact subinterval. We have the same result for any compact interval in the resolvent set of the Dirichlet Laplacian in $\Omega$. The choice of $-\pi^2$ is only motivated by the next step.

3.4 | Proof of Theorem 1.1 for $w < -\pi^2$

The problem here is that we lose in the previous approach the control of the uniformity with respect to $w$ in the estimates of the Poisson kernel $P(w)$. Actually, since $h^{-1}w \in \sigma(T_h), w = w(h)$ may approach $-\infty$ in the semi-classical limit, by Theorem 5.1.

Pick the unique integer $k \geq 1$ such that

$$k\pi \leq \sqrt{-w} < (k + 1)\pi$$
and set

\[ a = \frac{k\pi}{\sqrt{-w}}. \]

Then,

\[ w + \frac{k^2\pi^2}{a^2} = 0, \quad k \in \mathbb{N}, \quad \frac{1}{2} \leq a \leq 1. \]

(3.14)

We introduce a weighted Laplace operator \(-\Delta_{\hat{\Omega}, a}\) in the cylinder \(\hat{\Omega} := \Omega \times \mathbb{T}^1\), where \(\mathbb{T}^1 = [0, 1)\) is the one-dimensional torus. That is

\[ \Delta_{\hat{\Omega}, a} = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} + \frac{1}{a^2} \frac{\partial^2}{\partial \theta^2}, \]

(3.15)

where \((x_1, x_2)\) denote the coordinates in \(\Omega\) and \(\theta\) denotes the coordinate in \(\mathbb{T}^1 = [0, 1)\); these coordinates represent a point \(\hat{x} = (x, \theta)\) of \(\hat{\Omega}\). We introduce the following function

\[ v_h(\hat{x}) = e^{ik\pi\theta}u_h(x) (\hat{x} = (x, \theta)), \]

(3.16)

which satisfies

\[ -\Delta_{\hat{\Omega}, a}v_h = 0 \text{ on } \hat{\Omega}, \quad \nu_{\hat{\Gamma}} \cdot \nabla_{\hat{x}} v_h = h^{-1/2}v_h \text{ on } \hat{\Gamma}. \]

(3.17)

Here \(\hat{\Gamma} = (\partial \Omega) \times \mathbb{T}^1\) is the boundary of \(\hat{\Omega}\), and \(\nu_{\hat{\Gamma}}\) its unit outward normal vector; we have \(\nu_{\hat{\Gamma}} = (\nu_1, \nu_2, 0)\) where \(\nu = (\nu^1, \nu^2)\) is the outward unit normal vector of \(\Gamma = \partial \Omega\).

We can introduce the DN operator of \(\hat{\Omega}\), \(\Lambda_{\hat{\Omega}, a}(0)\), defined on \(H^{1/2}(\hat{\Gamma})\) as in (1.2) (with \(\hat{\Omega}, \hat{\Gamma}\) replacing \(\Omega, \Gamma\) and \(-\Delta_{\hat{\Omega}, a}\) replacing \(-\Delta\)). Consequently, the function \(v_h\) in (3.17) is an eigenfunction of \(\Lambda_{\hat{\Omega}, a}(0)\) with eigenvalue \(h^{-1/2}\). We will use the Poisson kernel \(P_{\hat{\Omega}, a}\) corresponding to \(-\Delta_{\hat{\Omega}, a}\). Using Theorem 3.3 for the domain \(\hat{\Omega}\) and the operator \(-\Delta_{\hat{\Omega}, a}\), we get the following Poisson kernel estimates (as in (3.9) and (3.10))

\[ |\partial_{\hat{x}}^\alpha P_{\hat{\Omega}, a}(\hat{x}, \hat{y})| \leq C_\alpha d(\hat{x}, \hat{\Gamma})^{-n-|\alpha|} \quad (\hat{x} \in \hat{\Omega}, \hat{y} \in \hat{\Gamma}), \]

(3.18)

where the constant \(C_\alpha\) can be chosen independently of \(a \in [\frac{1}{2}, 1]\).

With (3.17) in hand, we write, for \(\hat{x} = (x, \theta) \in \hat{\Omega},\)

\[ v_h(\hat{x}) = \int_{\hat{\Gamma}} v_h(\hat{y})P_{\hat{\Omega}, a}(\hat{x}, \hat{y})d\sigma_{\hat{\Gamma}}(\hat{y}) = h^{p/2} \int_{\hat{\Gamma}} \Lambda_{\hat{\Omega}, a}(0)^p v_h(\hat{y})P_{\hat{\Omega}, a}(\hat{x}, \hat{y})d\sigma_{\hat{\Gamma}}(\hat{y}). \]

Using the Poisson kernel estimate in (3.18) and the pseudo-differential calculus as in (3.11), we get, for any positive even integer \(p\), any \(a \in [\frac{1}{2}, 1]\), the existence of \(C_p\) and \(h_p > 0\) such that, for
\( h \in (0, h_p]. \)

\[
|v_h(\hat{x})| \leq C_p \ h^{b/2} d(\hat{x}, \hat{\Gamma})^{-n-p}.
\]

Since \(|v_h(\hat{x})| = |u_h(x)|\) by (3.16) and \(d(\hat{x}, \hat{\Gamma}) = d(x, \partial \Omega) = d(x)\) for \(\hat{x} = (x, \theta) \in \hat{\Omega}\), this implies

\[
|u_h(x)| \leq C_p \ h^{b/2} d(x)^{-n-p}, \forall x \in \Omega,
\]  

(3.19)

as stated in Theorem 1.1 for \(w \leq -\pi^2\).

### 3.5 Analytic case

We now consider the case when \(\partial \Omega\) is analytic and handle the case where \(w < \lambda_1^D(\Omega)\).

#### 3.5.1 Using analytic pseudo-differential calculus

At a heuristic level, one could hope from the Boutet de Monvel analytic pseudo-differential calculus [3] that we will get an estimate in the form

\[
|u_h(x)| \leq C_p+1 \ p! \ h^p \ d(x)^{1-n-p}.
\]  

(3.20)

A first step could be the following (to our knowledge unproved) result: If \(A\) is an analytic pseudo-differential operator on \(\partial \Omega\) (or more generally a compact analytic manifold) of degree 1 and \(u\) is an analytic function on \(\partial \Omega\), then \(A^p \ u\) satisfies

\[
|(A^p u)(y)| \leq C_{p+1} \ p!.
\]

This kind of estimate (with additional control with respect to the distance of \(x\) to \(\partial \Omega\)) should be applied to the distribution kernel of the Poisson operator of \(- \Delta - w\).

Assuming that the estimate (3.20) is true we can try to optimize over \(p\). Using Stirling Formula, we get the simpler

\[
|u_h(x)| \leq C_{p+1} \ p^{p+1} \ h^p \ d(x)^{1-n-p}.
\]  

(3.21)

Optimizing over \(p\) will give an estimate of the form (1.10).

It seems difficult by this approach to have the optimal result of Galkowski–Toth [10], that is, to have a control of the constant \(C_1\) appearing in (1.10).

We also refer the reader to an interesting discussion at the end of [8, subsection 7.4] and to [25].

#### 3.5.2 Using Galkowski–Toth

In this section, we prove Theorem 1.2. To keep tracking the uniformity with respect to \(w\) of the estimates, we introduce a fixed positive constant \(0 < \zeta < \lambda_1^D(\Omega)\) and assume that \(w\) varies as follows, \(-\infty < w \leq \zeta\).
We recall Theorem 1 of Galkowski–Toth [10].

**Theorem 3.4** [10]. For all $\delta > 0$ and $\alpha \in \mathbb{N}^n$, there exist $\varepsilon > 0$, $C$ and $h_0$ such that, for $h \in (0, h_0]$, any solution $u_h$ of

\[
\begin{cases}
-\Delta u_h = 0 & \text{in } \Omega \\
\frac{\partial u_h}{\partial \nu} = h^{-1/2} u_h & \text{on } \Gamma \\
\|u_h\|_{L^2(\partial \Omega)} = 1
\end{cases}
\]  

(3.22)

satisfies the following estimate in $\{d(x, \Gamma) < \varepsilon\}$,

\[
|\partial^{\alpha}_x u_h(x)| \leq C h^{-\frac{n}{2} + \frac{1}{8} - \frac{|\alpha|}{2}} \exp \left( - \frac{d(x, \Gamma)(1 + (C_\Omega - \delta)d(x, \Gamma))}{h^{1/2}} \right).
\]  

(3.23)

Here $C_\Omega = -\frac{3}{2} + \inf_{(x', \xi') \in S^* \Gamma} Q(x', \xi')$, $Q$ is the symbol of the second fundamental form of the boundary $\Gamma$.

It results from Theorem 3.4 the following weaker estimate. There exist constants $\varepsilon, C, \hat{C}$ such that, for $d(x, \Gamma) < \varepsilon$, we have

\[
|\partial^{\alpha}_x u_h(x)| \leq C h^{-\frac{n}{2} + \frac{1}{8} - \frac{|\alpha|}{2}} \exp \left( - \frac{d(x, \Gamma)(1 - \hat{C}d(x, \Gamma))}{h^{1/2}} \right).
\]  

(3.24)

Looking at the proof, Theorem 3.4 can be generalized in two different ways.

• When replacing $-\Delta$ by $-\Delta - w$, the constants in the estimates can be controlled uniformly with respect to $w$ in any compact interval of $(-\infty, \lambda^1_D(\Omega))$.

• When replacing $-\Delta$ by $\text{div}(c \nabla)$ with $c \in \mathbb{R}^n$ a constant vector with positive components, the constants in the estimates can also be controlled uniformly with respect to $|c|$ when it varies in a compact interval in $(0, +\infty)$.

In the two aforementioned situations, (3.23) continues to hold, which also yields that, for all $\eta > 0$, there exist positive constants $\varepsilon, C, h_0$ such that, for $h \in (0, h_0]$, any solution $u_h$ of (3.22) satisfies the following estimate in $\{d(x, \Gamma) < \varepsilon\}$,

\[
|u_h(x)| \leq C h^{-\frac{n}{2} + \frac{1}{8}} \exp \left( - (1 - \eta) \frac{d(x, \Gamma)}{h^{1/2}} \right).
\]  

(3.25)

Note that we just keep (3.24) which is the weaker version of (3.23) for simplification. In the procedure of addition of one variable described below, we cannot keep the additional information related to the curvature of $\Gamma$, but we can always write the following estimate (which also leads to (3.25)):

There exist positive constants $C, \hat{C}, h_0$ such that, for all $h \in (0, h_0]$,

\[
|u_h(x)| \leq C h^{-\frac{n}{2} + \frac{1}{8}} \exp \left( - \frac{d_c(x)}{h^{1/2}} \right),
\]  

(3.26)
with

\[ d_C(x) = (d(x, \Gamma) - \hat{C} d(x, \Gamma)^2) 1_{y, d(y, \Gamma) < \frac{1}{2\alpha}}(x) + \frac{1}{2\hat{C}} \left( 1 - 1_{y, d(y, \Gamma) < \frac{1}{2\alpha}}(x) \right). \]

We proceed with the proof of Theorem 1.2. We start with the case \( w < -\pi^2 \) and apply the Galkowski–Toth estimate (3.25) for the solution \( v_h \) of (3.17). We get

\[ |v_h(\hat{x})| \leq C \alpha^{-1/2} \exp \left( \frac{-1}{h^{1/2}} \right), \]

in a tubular neighborhood \( \hat{\Omega}_\varepsilon = \{ x \in \hat{\Omega} : \text{dist}(\hat{x}, \hat{\Gamma}) < \varepsilon \} \). Note that the second fundamental form of \( \hat{\Omega} \) vanishes so the estimate does not display the effect of the curvature of \( \Omega \) as in (3.22).

Remarking that \( |v_h(\hat{x})| = |u_h(x)| \) and \( d(\hat{x}, \hat{\Gamma}) = d(x, \Gamma) = d(x) \) for \( \hat{x} = (x, \theta) \in \hat{\Omega} \), we get

\[ |u_h(x)| \leq C h^{-\frac{1}{2}} \exp \left( \frac{-1}{h^{1/2}} \right), \]

in \( \Omega_\varepsilon \). To get the interior estimate

\[ |\hat{v}_h(\hat{x})| \leq \hat{C} \exp(-\hat{c} h^{-1/2}) \text{ in } \Omega \setminus \hat{\Omega}_\varepsilon, \]

we use the maximum principle, for the operator \(-\Delta_{\hat{\Omega}, a}\) and the solution \( \hat{v}_h \), in \( \Omega \setminus \hat{\Omega}_\varepsilon \) (see [25, Lemma 3.2.9] for the details of the argument). This finishes the proof of (3.25) for \( w < -\pi^2 \).

We move now to the case where \(-\pi^2 \leq w \leq \zeta\). We use the estimate (3.25) for the solution of \(-\Delta u_h = w u_h\) and get

\[ |u_h(x)| \leq C h^{\frac{n+1}{4}} \exp \left( \frac{-1}{h^{1/2}} \right) \text{ in } \Omega_\varepsilon. \]

If moreover \( w \leq 0 \), we use the maximum principle, as in [10, 25] to get the interior estimates. Note that we use the maximum principle for the operator \(-\Delta - w\) with \( w \leq 0 \) so that the arguments of [10, 25] hold.

If \( 0 < w \leq \zeta < \lambda_1^D(\Omega) \), we apply the maximum principle to the function \( f_h \) defined by \( u_h = f_h u^D \), where \( u^D \) is the normalized positive ground state of the Dirichlet Laplacian on \( \Omega \). The function \( f_h \) satisfies

\[ -\frac{1}{(u^D)^2} \text{div}((u^D)^2 \nabla f_h) + cf_h = 0 \text{ with } c := \lambda_1^D(\Omega) - w > 0. \]

4 ONE-DIMENSIONAL ROBIN LAPLACIANS

We revisit one-dimensional model operators appearing in [16].

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\[ \text{See, for example, Stampacchia [29, Theorem 3.8] for the maximum principle for } -\Delta - w \text{ when } w \leq 0 \]
4.1 On the half line

We start with the self-adjoint operator in $L^2(\mathbb{R}_+)$ defined by

$$\mathcal{H}_0 = -\partial_x^2$$

(4.1)

on the domain

$$\text{Dom}(\mathcal{H}_0) = \{ u \in H^2(\mathbb{R}_+) : u'(0) = -u(0) \}. \quad (4.2)$$

The quadratic form associated with this operator is

$$V_0 \ni u \mapsto \int_0^{+\infty} |u'(\tau)|^2 \, d\tau - |u(0)|^2,$$

with form domain $V_0 = H^1(0, +\infty)$. The spectrum of this operator is (see [16])

$$\sigma(\mathcal{H}_0) = \{-1\} \cup [0, +\infty)$$

and the eigenvalue $-1$ has multiplicity one with the corresponding $L^2$-normalized positive eigenfunction

$$u_1(\tau) = \sqrt{2} \exp(-\tau). \quad (4.3)$$

4.2 On an interval

Let us consider $T \geq 1$ and the self-adjoint operator acting on $L^2(0, T)$ and defined by

$$\mathcal{H}^{T,D}_0 = -\partial_x^2,$$

(4.4)

with domain,

$$\text{Dom}(\mathcal{H}^{T,D}_0) = \{ u \in H^2(0, T) : u'(0) = -u(0) \text{ and } u(T) = 0 \}. \quad (4.5)$$

This operator is associated with the quadratic form

$$V^{T,D}_0 \ni u \mapsto \int_0^T |u'(\tau)|^2 \, d\tau - |u(0)|^2,$$

with $V^{T,D}_0 = \{ v \in H^1(0, T) : v(T) = 0 \}$. The spectrum of the operator $\mathcal{H}^{T,D}_0$ is purely discrete. We denote by $(\lambda^{D}_n(T))_{n\geq 1}$ the sequence of min–max eigenvalues and by $(u^{T,D}_n)_{n\geq 1}$ some associated $L^2(0, T)$ Hilbert basis of eigenfunctions. We can localize the spectrum in the large $T$ limit [16, Lemma 4.1 and Remark 4.3; 19, Lemma A.2].
Lemma 4.1. As $T \to +\infty$, it holds

\[ \lambda_{1}^{T,D}(T) = -1 + 4(1 + o(1)) \exp(-2T), \]  

(4.6)

and the eigenfunction $u_{1}^{T,D}$ satisfies

\[ \|e^{\tau}(u_{1}^{T,D} - u_{1})\|_{W^{1,\infty}(0,T)} = \mathcal{O}(T), \]  

(4.7)

where $u_{1}$ is the eigenfunction in (4.3).

Furthermore, for all $T > 1$ and $n \geq 2$, we have

\[ \left( \frac{(2n-3)\pi}{2T} \right)^{2} < \lambda_{n}^{D}(T) < \left( \frac{(n-1)\pi}{T} \right)^{2}. \]

Also we consider the Neumann realization at the boundary $t = T$,

\[ H_{0}^{T,N} = -\partial_{t}^{2}, \]  

(4.8)

with domain,

\[ \text{Dom}(H_{0}^{T,N}) = \{ u \in H^{2}(0,T) : u'(0) = -u(0) \text{ and } u'(T) = 0 \}. \]  

(4.9)

The spectrum of the operator $H_{0}^{T}$ is purely discrete, consisting of the sequence of min–max eigenvalues $(\lambda_{n}(H_{0}^{T,N}))_{n \geq 1}$. We denote by $(u_{n}^{T,N})_{n \geq 1}$ the corresponding Hilbert basis of eigenfunctions. We can localize the spectrum in the large $T$ limit.

Lemma 4.2. As $T \to +\infty$, it holds

\[ \lambda_{1}^{N}(T) = -1 + 4(1 + o(1)) \exp(-2T) \]  

(4.10)

and

\[ \|e^{\tau}(u_{1}^{T,N} - u_{1})\|_{W^{1,\infty}(0,T)} = \mathcal{O}(T), \]  

(4.11)

where $u_{1}$ is the eigenfunction in (4.3).

Furthermore, for all $T > 1$ and $n \geq 2$, we have

\[ \left( \frac{(2n-3)\pi}{2T} \right)^{2} < \lambda_{n}^{N}(T) < \left( \frac{(n-1)\pi}{T} \right)^{2}. \]

Proof. The proof is similar to that of Lemma 4.1 but we give the main points for the convenience of the reader. Let $\lambda \leq 0$ be a non-positive eigenvalue of the operator $H_{0}^{T,N}$ with an eigenfunction $f$. Solving the ordinary differential equation (ODE) $f'' = \lambda f$ with the boundary conditions $f'(0) = -f(0)$ and $f'(T) = 0$ yields that the only possible value of $\lambda$ is

\[ \lambda = -1 + 4(1 + o(1)) \exp(-2T), \]
which corresponds to the first eigenvalue (see [16]). The corresponding normalized eigenfunction is

\[ u_{1}^{T, N}(\tau) = A_{T} (e^{-\tau} + e^{-2\alpha T} e^{\tau}) \]

with \( A_{T} = \sqrt{2} + O(T e^{-T}) \) and \( \alpha = 1 - 2(1 + o(1))e^{-2T} \), so that

\[ e^{-2\alpha T} e^{\tau} = O(e^{\tau-2T}) = o(e^{-\tau}). \]

We then have the following uniform estimate,

\[ |u_{1}^{T, N}(\tau) - A_{T} e^{-\tau}| = O(e^{-2T}) \]

which also yields

\[ |e^{\tau}(u_{1}^{T, N}(\tau) - \sqrt{2} e^{-\tau})| = O(T). \]

Although not needed here, note that we have the much more accurate approximation

\[ |A_{T}^{-1} u_{1}^{T, N}(\tau) - (e^{-\tau} + e^{\tau-2T})| = O(T e^{-3T}). \]

Now we study the positive eigenvalues. Let \( \ell > 0 \) and \( \lambda = \ell^{2} \) be a non-negative eigenvalue of the operator \( H_{0}^{T, N} \) with an eigenfunction \( u \), which will have the form

\[ u(\tau) = A \cos(\ell \tau) + B \sin(\ell \tau), \]

for some constants \( A, B \in \mathbb{R} \) that depend on \( T \), with \( A = -B \ell \), \( \sin(\ell T) \neq 0 \), and \( \cot(\ell T) = -\ell \), to respect the boundary conditions. The positive fixed points of the \( \pi/T \)-periodic function \( x \mapsto -\cot(xT) \) must belong to the intervals \( I_{k} := (\frac{\pi}{2T}, \frac{\pi}{T}) + k\pi T \), \( k = 0, 1, \ldots \). In each interval \( I_{k} \), there exists a unique fixed point \( \ell_{k} \) because the function \( g(x) = \cot(xT) + x \) satisfies \( g'(x) = -T(1 + \cot^{2}(xT)) + 1 < 0 \) for \( T > 1 \). For each \( k = 0, 1, 2, \ldots \), the fixed point \( \ell_{k} \in I_{k} \) is equal to

\[ \sqrt{\frac{\lambda_{k}^{N}}{k+2}(T)}. \]

\[ \square \]

4.3 On a weighted space

Now we consider operators with weight terms, which can be viewed as perturbations of the operators studied previously on the interval \((0, T)\) with Dirichlet or Neumann realizations at the endpoint \( t = T \).

In the sequel, \( \rho \in (\frac{1}{3}, \frac{1}{2}) \) and \( M > 0 \) are fixed constants, and

\[ T = T_{h} := h^{\rho - \frac{1}{2}}. \]

We pick \( h_{0} = h_{0}(\rho, M) > 0 \) such that, for all \( h \in (0, h_{0}] \) and \( \beta \in [-M, M] \), we have \( \frac{1}{2} < 1 - h^{1/2} \beta \tau < 1 \) for all \( \tau \in (0, T) \).
Consider the differential operator
\[ \mathcal{H}_{h,\beta} = -(1 - h^{1/2}\beta \tau)^{-1} \frac{d}{d\tau} \left( (1 - h^{1/2}\beta \tau) \frac{d}{d\tau} \right) = -\frac{d^2}{d\tau^2} + \beta h^{1/2}(1 - h^{1/2}\beta \tau)^{-1} \frac{d}{d\tau}. \]

We work in the Hilbert space
\[ X_{h,\beta} = L^2 \left( (0, T_h); (1 - h^{1/2}\beta \tau) d\tau \right), \]
with inner product and norm defined by
\[ \langle u, v \rangle_{h,\beta} = \int_0^T u(\tau)v(\tau)(1 - h^{1/2}\beta \tau)d\tau, \quad \|u\|_{h,\beta} = \langle u, u \rangle_{h,\beta}^{1/2}. \]

Consider the two self-adjoint realizations of \( \mathcal{H}_{h,\beta} \) in \( X_{h,\beta} \), \( \mathcal{H}_N^{h,\beta} \) and \( \mathcal{H}_D^{h,\beta} \), with domains
\[ \mathcal{D}_h^N = \{ u \in H^2(0, T), u'(0) = -u(0) \text{ and } u'(T) = 0 \} \]
and
\[ \mathcal{D}_h^D = \{ u \in H^2(0, T), u'(0) = -u(0) \text{ and } u(T) = 0 \}. \]

We denote the sequences of min–max eigenvalues by \( (\lambda_{n,h}^N(\beta))_{n \geq 1} \) and \( (\lambda_{n,h}^D(\beta))_{n \geq 1} \) respectively. By the min–max principle, we can localize the foregoing eigenvalues as follows
\[ |\lambda_{n,h}^\#(\beta) - \lambda_{n,h}^{T,\#}| \leq \left( 1 + \lambda_{n,h}^{T,\#} \right) h^\rho, \]
uniformly with respect to \( \beta \in [-M, M] \) and \( h \in (0, h_0] \). Here \( \# \in \{N, D\} \) and \( \lambda_{n,h}^{T,\#} \) are the eigenvalues of the operators introduced in (4.4) and (4.8) with \( T = h^{\rho - \frac{1}{2}} \gg 1 \). We deduce then that there exists \( h_0 > 0 \) such that for \( h \in (0, h_0] \)
\[ \lambda_{2,h}^\#(\beta) \geq \frac{\pi^2}{4} h^{1-2\rho} - O(h^\rho) \geq \frac{\pi^2}{8} h^{1-2\rho} > 0, \]

since \( \rho \in \left( \frac{1}{3}, \frac{1}{2} \right) \). The first eigenvalue \( \lambda_{1,h}^\# \) was analyzed in [16, Proposition 4.5; 21, Lemma 2.5] for the Dirichlet case \( (\# = D) \). The same analysis applies for the Neumann case \( (\# = N) \). We have
\[ |\lambda_{1,h}^\#(\beta) - \left(-1 - \beta h^{1/2} - \frac{\beta^2}{2} h \right)| \leq C(|\beta|^3 + 1) h^{3/2}, \]
uniformly with respect to \( \beta \in [-M, M] \) and \( h \in (0, h_0] \).

For the convenience of the reader, we present the outline of the proof of (4.17). The idea is to look for a formal eigenpair of the form
\[ u_{h,\beta}^{\text{app}} = v_0 + h^{1/2}v_1 + hv_2 \text{ and } \mu_{h,\beta}^{\text{app}} = \mu_0 + \mu_1 h^{1/2} + \mu_2 h. \]
We expand \((H_{h,\beta} - \mu_{h,\beta}^{\text{app}})u_{h,\beta}^{\text{app}}(\tau)\) as \(L_0 + h^{1/2}L_1 + hL_2 + h^{3/2}r_\beta(\tau)\) with

\[
L_0 = \left(-\frac{d^2}{d\tau^2} - \mu_0\right)v_0, \quad L_1 = \left(-\frac{d^2}{d\tau^2} - \mu_0\right)v_1 + \left(\beta \frac{d}{d\tau} - \mu_1\right)v_0,
\]
\[
L_2 = \left(-\frac{d^2}{d\tau^2} - \mu_0\right)v_2 + \left(\beta \frac{d}{d\tau} - \mu_1\right)v_1 + \left(\beta^2 \frac{d}{d\tau} - \mu_2\right)v_0
\]

\[|r_\beta(\tau)| \leq C(|\beta|^3 + 1)(\tau^2 + 1)\sum_{i=1}^2 |v_i(\tau)|.\]

We choose the pairs \((v_i, \mu_i)\) so that the coefficients \(L_0, L_1, L_2\) vanish [21, Lemma 2.5]. Eventually we get the approximate eigenfunction

\[\mu_{h,\beta}^{\text{app}} := -1 - \beta h^{1/2} - \frac{\beta^2}{2} h,\]

and the following quasi-mode

\[u_{h,\beta}^{\text{app}}(\tau) := \left(1 + \beta^2 h \left(\frac{\tau^2}{4} - \frac{1}{8}\right)\right)u_1(\tau),\] (4.18)

where \(u_1\) the eigenfunction in (4.3). The following estimate holds, for all \(\tau \in (0, T)\),

\[\left|\left(H_{h,\beta} - \mu_{h,\beta}^{\text{app}}\right)u_{h,\beta}^{\text{app}}(\tau)\right| \leq Ch^\frac{3}{2}(|\beta|^5 + 1)(\tau^2 + 1)^2|u_1(\tau)|\] (4.19)

uniformly with respect to \(\beta \in [-M, M]\) and \(\tau \in (0, T)\).

We introduce the following quasi-mode (it belongs to \(\mathcal{D}(H_{h,\beta}^{\#})\))

\[v_{h}(t) = c_h \chi(T^{-1}\tau)u_{h,\beta}^{\text{app}}(\tau),\] (4.20)

where \(\chi \in C^\infty_c(\mathbb{R})\) satisfies \(0 \leq \chi \leq 1\), supp \(\chi \subset (-1, 1)\), \(\chi/\chi_{[-\frac{1}{2}, \frac{1}{2}]} = 1\), and where \(c_h\) is selected so that \(\|v_{h}\|_{h,\beta} = 1\). By the exponential decay of \(u_1\) (see (4.3)), the constant \(c_h\) and the quasi-mode \(v_{h}\) satisfy

\[c_h = 1 + \mathcal{O}(h^{1/2})\]

and

\[\left|\left(H_{h,\beta} - \mu_{h,\beta}^{\text{app}}\right)v_{h}(\tau)\right| \leq \tilde{C}h^\frac{3}{2}(|\beta|^5 + 1)(\tau^2 + 1)^2|u_1(\tau)|.\] (4.21)

The spectral theorem and (4.16) yield the estimate in (4.17).

We will need the following lemma on the “energy” of functions orthogonal to the quasi-mode \(v_{h}\) in the space \(X_{h,\beta}\) introduced in (4.12).
Lemma 4.3. Then there exist positive constants $m, h_0$ such that, if $h \in (0, h_0]$ and $g_h \in H^1(0, T)$ is orthogonal to $v_h$ in $X_h, \beta$, then

$$\|g_h'\|_{h, \beta}^2 - |g_h(0)|^2 \geq m h^{1-2\rho} \|g_h\|_{h, \beta}^2.$$ 

Proof. Let $u_{h, \beta}^{gs} \in \mathfrak{D}_h^N$ be the normalized (in $X_h, \beta$) ground state of the operator $\mathcal{H}_{h, \beta}^N$:

$$\mathcal{H}_{h, \beta}^N u_{h, \beta}^{gs} = \lambda_{1, h}^N(\beta) u_{h, \beta}^{gs}.$$ (4.22)

By the min–max principle, (4.15) and Lemma 4.2, if $f$ belongs to the form domain of $\mathcal{H}_{h, \beta}$ and satisfies $\langle f, u_{h, \beta}^{gs} \rangle_{h, \beta} = 0$, then

$$\langle \mathcal{H}_{h, \beta} f, f \rangle_{h, \beta} \geq \lambda_{2, h}^N(\beta) \|f\|_{h, \beta}^2 \geq (\lambda_{1, h}^N(\beta) + c h^{1-2\rho}) \|f\|^2,$$ (4.23)

where $c$ is a positive constant.

Now consider a function $g_h \in H^1(0, T)$ such that $\langle g_h, v_h \rangle_{h, \beta} = 0$. We decompose $v_h$ and $g_h$ as follows

$$v_h = \alpha_h u_{h, \beta}^{gs} + f_h \text{ and } g_h = \gamma_h u_{h, \beta}^{gs} + e_h,$$ (4.24)

with

$$\alpha_h = \langle v_h, u_{h, \beta}^{gs} \rangle_{h, \beta}, \quad \gamma_h = \langle g_h, u_{h, \beta}^{gs} \rangle_{h, \beta} \quad \text{and} \quad \langle f_h, u_{h, \beta}^{gs} \rangle_{h, \beta} = \langle e_h, u_{h, \beta}^{gs} \rangle_{h, \beta} = 0.$$ (4.25)

We infer from (4.17), (4.21), and (4.22) that

$$\|(\mathcal{H}_{h, \beta} - \lambda_{1, h}^N(\beta)) f_h\|_{h, \beta} = O\left(h^{\frac{3}{2}}\right).$$

Consequently,

$$q_{h, \beta}(f_h) := \langle (\mathcal{H}_{h, \beta} - \lambda_{1, h}^N(\beta)) f_h, f_h \rangle_{h, \beta} = O\left(h^{\frac{3}{2}}\right) \|f_h\|_{h, \beta}^2,$$

and by (4.23),

$$q_{h, \beta}(f_h) \geq c h^{1-2\rho} \|f_h\|^2.$$

Eventually we get that

$$\left(1 - |\alpha_h|^2\right)^{1/2} = \|f_h\|_{h, \beta} = O\left(h^{\frac{1}{2} + 2\rho}\right),$$

where $\alpha_h$ is introduced in (4.24).

We return to the function $g_h$ in (4.24). Since $e_h \perp u_{h, \beta}^{gs}$, we get by (4.23),

$$\|g_h'\|_{h, \beta}^2 - |g_h(0)|^2 = q_{h, \beta}(e_h) \geq c h^{1-2\rho} \|e_h\|^2.$$ (4.26)
Since \( \langle g_h, v_h \rangle_{h,\beta} = 0 \), we get from (4.24),

\[
\gamma_h \overline{\alpha_h} + \langle e_h, f_h \rangle_{h,\beta} = 0
\]

which yields that

\[
|\gamma_h| \leq \frac{1}{|\alpha_h|} \| e_h \| \| f_h \| = O(h^{\frac{1}{2}+2\rho}) \| e_h \|_{h,\beta}
\]

and consequently

\[
\| g_h \|_{h,\beta}^2 = |\gamma_h|^2 + \| e_h \|_{h,\beta}^2 = (1 + O(h^{1+4\rho})) \| e_h \|_{h,\beta}^2.
\]

Inserting this into (4.26), we finish the proof of Lemma 4.3.

\[\square\]

5 | THE EFFECTIVE OPERATOR

5.1 | The operator near the boundary

Assume that \( \Omega \) is simply connected, hence \( \Gamma \) consists of a single connected component. In the case of a multiply connected domain, with \( \Gamma \) having a finite number of connected components, we can do the constructions below in each connected component of \( \Gamma \).

We introduce the coordinates \((s, t)\) valid in a tubular neighborhood of the boundary, \(\Omega_\varepsilon := \{ x \in \Omega, \text{dist}(x, \partial \Omega) < \varepsilon \}\), and defined as follows: \(t(x) = \text{dist}(x, \Gamma)\) measures the transversal distance to \(\Gamma - \partial \Omega\), and \(s(x) \in [-L, L)\) measures the (arc-length) tangential distance along \(\Gamma\), with \(2L = |\Gamma|\) is the length of the boundary. More precisely, we denote by \([−L, L[\ni s ↦ M(s)\) the arc-length parameterization of \(\Gamma\) oriented counter-clockwise and consider the transformation

\[
\Phi : (s, t) \mapsto M(s) - t \nu(s)
\]

where \(\nu(s)\) is the unit outward normal of \(\partial \Omega\).

The \(L^2\)-norm of \(u\) in \(\Omega_\varepsilon\) is

\[
\|u\|_{L^2(\Omega_\varepsilon)}^2 = \int_{-L}^{L} \int_0^\varepsilon |u(s, t)|^2 a(s, t) \, dt \, ds
\]

and the operator \(T_h\) is expressed as follows

\[
T_h = -a^{-1} \partial_s (a \partial_s) + a^{-1} \partial_t (a^{-1} \partial_t)
\]

where

\[
a(s, t) = 1 - t \kappa(s)
\]

and \(\kappa(s)\) is the curvature of \(\Gamma\) at the point \(M(s)\).

For every \(c \in \mathbb{R}\), let \(L^c_h\) denote the operator (on \(\mathbb{R}/2L\mathbb{Z}\))

\[
L^c_h = -(h^{1/2} + ch^{3/4}) \frac{d^2}{ds^2} - \kappa(s) - \frac{1}{2} h^{1/2} (\kappa(s))^2 + ch^{7/8},
\]
with domain
\[ \mathcal{D} = \{ u \in H^2([-L, L]): u(-L) = u(L) \text{ and } u'(-L) = u(L) \}. \]

For a self-adjoint semi-bounded operator \( P \), we denote by \( (\lambda_n(P))_{n \geq 1} \) the sequence of min–max eigenvalues. For all \( h > 0 \) and \( \epsilon \in \mathbb{R} \), we introduce the following subset of \( \mathbb{N} \)
\[ I^\epsilon_h = \{ k \geq 1 : \lambda_k(T_h) < \epsilon h \}. \] (5.1)

**Theorem 5.1.** Given \( 0 \leq \epsilon < \lambda^N_2(\Omega) \), there exist positive constants \( \epsilon, h_0 \), such that, for all \( h \in (0, h_0] \) and \( n \in I^\epsilon_h \),
\[ h^{3/2} \min \left( \frac{\lambda_n(L^{-\epsilon}_h)}{h}, h^{-1/2} \right) \leq \lambda_n(T_h) + h \leq h^{3/2} \min \left( \frac{\lambda_n(L^\epsilon_h)}{h}, \epsilon h^{1/2} \right). \] (5.2)
In particular, for \( \lambda_n(T_h) < 0 \), we have,
\[ h^{3/2} \lambda_n(L^{-\epsilon}_h) \leq \lambda_n(T_h) + h \leq h^{3/2} \min \left( \lambda_n(L^\epsilon_h), 0 \right). \] (5.3)

**Remarks 5.2.**

(1) Note that for \( \epsilon < 0 \) a stronger result is proven in [17].

(2) Let \( \lambda^D_1(\Omega) \) be the first eigenvalue of the Dirichlet Laplacian on \( \Omega \). It follows from [26, 30] that \( \lambda^N_2(\Omega) < \lambda^D_1(\Omega) \) (see also [1, eq. (2.2)]). The upper bound in (5.2) actually holds for \( \epsilon < \lambda^D_1(\Omega) \).

(3) A comparison similar to the one in Theorem 5.1 has been proved in [17] when \( n \in I^{-\epsilon}_h := \{ k \geq 1 : \lambda_k(T_h) < -\epsilon h \} \) with \( 0 < \epsilon < 1 \) a fixed constant. More precisely, the effective operator in [17] is of the form
\[ -\left( h^{1/2} + \epsilon b(s) \right) \frac{d^2}{ds^2} - \kappa(s), \]
with \( b(s) = \mathcal{O}(1) \) uniformly with respect to \( s \). Our result extends that in [17] all the way up to \( \epsilon = 0 \), but with a worse remainder term for the coefficient of \( \frac{d^2}{ds^2} \), in order to consider all the non-positive eigenvalues.

(4) Note that for the realization of \(-\partial^2_s\) on \( \mathbb{R}/2L\mathbb{Z} \), the spectrum is
\[ \{ \pi^2 L^{-2}(n - 1)^2, \ n \geq 1 \} \]
with the first eigenvalue being simple and the others being of multiplicity 2, hence
\[ \lambda_1(-\partial^2_s) = 0 \]
and
\[ \lambda_{2k}(-\partial^2_s) = \lambda_{2k+1}(-\partial^2_s) = \pi^2 L^{-2} k^2, \ k = 1, 2, \ldots. \]
Theorem 5.1 then yields the existence of \( c > 0 \) and \( h_0 > 0 \) such that for \( h \in (0, h_0] \) and \( n \in \{2k, 2k + 1\} \) with \( \lambda_n(T_h) < \epsilon h^2 \), we have

\[
h^{-3/2} \lambda_n(T_h) \leq -h^{-1/2} + \frac{\pi^2 k^2}{L^2} \left( 1 + ch^{1/4} \right) h^{1/2} - \kappa_{\min} + M_+ h^{1/2} + ch^{7/8},
\]

and

\[
h^{-3/2} \lambda_n(T_h) \geq -h^{-1/2} + \frac{\pi^2 k^2}{L^2} \left( 1 - ch^{1/4} \right) h^{1/2} - \kappa_{\max} + M_- h^{1/2} - ch^{7/8},
\]

where

\[
\kappa_{\min} = \min_{s \in [-L, L)} \kappa(s), \quad \kappa_{\max} = \max_{s \in [-L, L)} \kappa(s),
\]

\[
M_- = -\max_{s \in [-L, L)} |\kappa(s)|^2, \quad M_+ = -\min_{s \in [-L, L)} |\kappa(s)|^2.
\]

These estimates yield Theorem 1.3.

(5) For a positive integer \( k = k(h) \gg h^{-1/4} \) satisfying

\[
k^2 \leq (1 + ch^{1/4})^{-1} \left( 1 + h^{1/2} \kappa_{\max} + \frac{1}{2} h \kappa_{\min}^2 - ch^{7/8} \right) h^{-1},
\]

we get

\[
\lambda_{2k}(T_h) \sim -h + \pi^2 L^2 (2k - 1)h^2 \quad \text{and} \quad \lambda_{2k+1}(T_h) - \lambda_{2k}(T_h) = \mathcal{O}(h^{3/2}).
\]

5.2 Decomposition of \( L^2(\Omega) \)

Let \( \rho \in \left( \frac{1}{3}, \frac{1}{2} \right) \) and consider the domain \( \Omega_h^\rho \) defined by (2.2).

We decompose the Hilbert space \( L^2(\Omega) \) as \( L^2(\Omega_h^\rho) \oplus L^2(\Omega \setminus \overline{\Omega_h^\rho}) \). We will decompose further the space \( L^2(\Omega_h^\rho) \) by considering the orthogonal projection on the function

\[
u_h^{\text{tran}}(s, t) = c_h h^{-1/4} \chi(h^{-\rho} t) u_{h,\kappa(s)}^{\text{app}}(h^{-1/2} t),
\]

where \( u_{h,\kappa(s)}^{\text{app}} \) is the function defined by (4.18), and \( \chi \in C^\infty_c(\mathbb{R}) \) satisfies \( 0 \leq \chi \leq 1 \), \( \text{supp} \chi \subset (-1, 1) \) and \( \chi/\left[ -\frac{1}{2}, \frac{1}{2} \right] = 1 \). The coefficient \( c_h \) is determined by \( \|u_h^{\text{tran}}\|_{L^2(\Omega)} = 1 \) and satisfies \( c_h = 1 + \mathcal{O}(h^\infty) \).

Note that \( u_h^{\text{tran}} \in D(T_h) \) and by (4.19)

\[(-a^{-1} \partial_i (a \partial_i) - \lambda_h(s)) u_h^{\text{tran}}(s, t) = \mathcal{O}(h^{3/2 + 2\rho}),\]

with

\[
\lambda_h(s) = -h - h^{3/2} \kappa(s) - \frac{1}{2} h^2 \kappa(s)^2.
\]

We introduce the projections in the space \( L^2(\Omega_h^\rho) \),

\[
\Pi_s \psi = \langle \psi, u_h^{\text{tran}} \rangle u_h^{\text{tran}} \quad \text{and} \quad \Pi_s^\perp \psi = \psi_{/\Omega_h^\rho} - \Pi_s \psi,
\]
and the isometry

$$\psi \ni L^2(\Omega_{h^\varepsilon}) \mapsto (\Pi_s \psi, \Pi^\perp_s \psi) \in V_h \oplus W_h,$$

where

$$V_h = L^2([-L, L]) \otimes \{\text{span}(u_{h}^{\text{tran}})\}
= \{v \in L^2(\Omega_{h^\varepsilon}), \exists k \in L^2([-L, L]), v(s, t) = k(s)u_{h}^{\text{tran}}(s, t)\}$$

and

$$W_h = \{v \in L^2(\Omega_{h^\varepsilon}), \int_0^{h^\varepsilon} v(s, t)u_{h}^{\text{tran}}(s, t)(1 - t\varkappa(s))dt = 0\}.$$  

Using (5.6) and the decomposition of $L^2(\Omega)$ as $L^2(\Omega_{h^\varepsilon}) \oplus L^2(\Omega \setminus \Omega_{h^\varepsilon})$, we construct the following isometry

$$\psi \ni L^2(\Omega) \mapsto \chi_\psi := (\Pi_s \psi, \Pi^\perp_s \psi, \psi/\Omega \setminus \Omega_{h^\varepsilon}) \in V_h \oplus W_h \oplus L^2(\Omega \setminus \Omega_{h^\varepsilon}).$$

Note that

$$\|\psi\|_{L^2(\Omega)}^2 = \|\chi_\psi\|^2 = \int_{-L}^{L} |k_\psi(s)|^2 ds + \int_{\Omega_{h^\varepsilon}} |\Pi^\perp_s \psi|^2 dx + \int_{\Omega \setminus \Omega_{h^\varepsilon}} |\psi|^2 dx,$$

where

$$k_\psi(s) := \langle \psi, u_{h}^{\text{tran}} \rangle = \int_0^{h^\varepsilon} \psi(s, t)u_{h}^{\text{tran}}(s, t)(1 - t\varkappa(s))dt.$$  

5.3 | Decomposition of the quadratic form

We examine the quadratic form

$$q_h^\Omega(\psi) := h^2 \int_{\Omega} |\nabla \psi|^2 dx - h^{3/2} \int_{\partial \Omega} |\psi|^2 ds(x)$$

$$= q_h^{\Omega_{h^\varepsilon}}(\psi) + q_{h, \varepsilon}^{\text{int}}(\psi),$$

where

$$q_{h, \varepsilon}^{\text{int}}(\psi) = \int_{\Omega \setminus \Omega_{h^\varepsilon}} |\nabla \psi|^2 dx.$$  

Working in the $(s, t)$ coordinates, we express the quadratic form $q_h^{\Omega_{h^\varepsilon}}(\psi)$ as follows

$$q_h^{\Omega_{h^\varepsilon}}(\psi) = h^2 \int_{-L}^{L} \left( \int_0^{h^\varepsilon} \left( |\partial_t \psi|^2 + a^{-2} |\partial_s \psi|^2 \right) a dt - h^{-1/2}|\psi(s, t = 0)|^2 \right) ds.$$
Freezing the $s$-variable, the $\Pi_s$ is an orthogonal projection in the weighted Hilbert space $L^2((0, h^\rho); a(s, t) dt)$; consequently,

\[
q_{h}^{\text{tran}}(\psi) := h^2 \int_0^{h^\rho} |\partial_t \psi|^2 (1 - t \kappa(s)) dt - h^{3/2} |\psi(s, t = 0)|^2
\]

\[
= q_{h}^{\text{tran}}(\Pi_s \psi) + q_{h}^{\text{tran}}(\Pi_s^\bot \psi)
\]

and

\[
\int_0^{h^\rho} |\partial_s \psi|^2 (1 - t \kappa(s)) dt = \int_0^{h^\rho} (|\Pi_s \partial_s \psi|^2 + |\Pi_s^\bot \partial_s \psi|^2) (1 - t \kappa(s)) dt
\]

We have (see (5.4))

\[
q_{h}^{\text{tran}}(\Pi_s \psi) = \left( \lambda_{h}(s) + \mathcal{O}(h^{\frac{3}{2} + 2\rho}) \right) \int_0^{h^\rho} |\Pi_s \psi|^2 (1 - t \kappa(s)) dt
\]

\[
= \left( \lambda_{h}(s) + \mathcal{O}(h^{\frac{3}{2} + 2\rho}) \right) |k_{\psi}(s)|^2,
\]

\[
q_{h}^{\text{tran}}(\Pi_s^\bot \psi) \gtrsim h^{2-2\rho},
\]

and, setting $\Re = 8||\kappa||_{\infty}$,

\[
1 - t \Re \leq a^{-2} \leq 1 + t \Re.
\]

Therefore, we end up with the following upper bound of the quadratic form

\[
q_{h}^{\Omega_{h^\rho}}(\psi) \leq h^2 \int_{-L}^{L} \left( \lambda_{h}(s) + \mathcal{O}(h^{\frac{3}{2} + 2\rho}) \right) |k_{\psi}(s)|^2 + \left( 1 + \Re t \right) |\Pi_s \partial_s \psi|^2 dt \right) ds
\]

\[
+ h^2 \int_{-L}^{L} \int_0^{h^\rho} (|\partial_t \Pi_s^\bot \psi|^2 + (1 + \Re t) |\Pi_s^\bot \partial_s \psi|^2) a dt ds
\]

\[
- h^{3/2} \int_{-L}^{L} |\Pi_s^\bot \psi(s, t = 0)|^2 ds.
\]

(5.11)

The same argument yields the following lower bound

\[
q_{h}^{\Omega_{h^\rho}}(\psi) \geq h^2 \int_{-L}^{L} \left( \lambda_{h}(s) + \mathcal{O}(h^{\frac{3}{2} + 2\rho}) \right) |k_{\psi}(s)|^2 + \left( 1 - \Re t \right) |\Pi_s \partial_s \psi|^2 dt \right) ds
\]

\[
+ h^2 \int_{-L}^{L} \int_0^{h^\rho} (|\partial_t \Pi_s^\bot \psi|^2 + (1 - \Re t) |\Pi_s^\bot \partial_s \psi|^2) a dt ds
\]

\[
- h^{3/2} \int_{-L}^{L} |\Pi_s^\bot \psi(s, t = 0)|^2 ds.
\]

(5.12)
Let us now handle the term $|\Pi_s \partial_s \psi|$. Let us introduce $u = \partial_s \psi$. It is easy to check the following identities,

$$
\partial_s \Pi_s \psi = \partial_s \left( k\psi(s)u^\text{ran}_h \right)
= k'_\psi(s)u^\text{ran}_h + k\psi(s)\partial_s u^\text{ran}_h
= (k_u - \chi'(s)k_{t\psi} + \langle \psi, \partial_s u^\text{ran}_h \rangle)u^\text{ran}_h + k\psi(s)\partial_s u^\text{ran}_h.
$$

Therefore,

$$
\Pi_s \partial_s \psi = k'_\psi(s)u^\text{ran}_h + k\psi(s)\partial_s u^\text{ran}_h + (\chi'(s)k_{t\psi}(s) - \langle \psi, \partial_s u^\text{ran}_h \rangle)u^\text{ran}_h := w_\psi(s,t).
$$

Note that if we perform the change of variable, $t = h^{1/2}\tau$, we can write

$$
\int_0^{h^\rho} t|u^\text{ran}_h(s,t)|^2 dt = O(h^{1/2})
$$

uniformly with respect to $s$. In a similar manner, we can check that

$$
\int_0^{h^\rho} |\partial_s u^\text{ran}_h|^2 dt = O(h).
$$

Consequently, if we introduce the norms

$$
N^\pm(f) = \left( \int_{-L}^L \int_0^{h^\rho} (1 \pm t\Re\chi)|f|^2 a dt ds \right)^{1/2},
$$

we get that

$$
N^\pm(k\psi u^\text{ran}_h)^2 = \left(1 + O(h^{1/2})\right) \int_{-L}^L |k'_\psi(s)|^2 ds
$$

and

$$
N^\pm(w_\psi)^2 = O(h^{1/2}) \int_{-L}^L \int_0^{h^\rho} |\psi|^2 ds dt = O\left(h^{1/2}\|\psi\|^2_{L^2(\Omega_h)}\right).
$$

Armed with the foregoing estimates, and Cauchy’s inequality, we write, for all $\eta \in (0, 1)$,

$$
N^\pm(\Pi_s \partial_s \psi) \geq (1 - \eta)N^\pm(k\psi u^\text{ran}_h)^2 - \eta^{-1}N^\pm(w_\psi)^2
$$

and

$$
N^\pm(\Pi_s \partial_s \psi) \leq (1 + \eta)N^\pm(k\psi u^\text{ran}_h)^2 + (1 + \eta^{-1})N^\pm(w_\psi)^2.
$$
Choosing \( \eta = h^{1/4} \), we eventually get estimates for the energy of \( \Pi_s \partial_s \psi \) as follows

\[
\left| \int_{-L}^{L} \int_{0}^{h^\rho} \int_{0}^{h^\rho} (1 \pm \Re t)|\Pi_s \partial_s \psi|^2 \, adt \, ds - \int_{-L}^{L} |k'_\psi(s)|^2 \, ds \right| \\
\leq M h^{1/4} \left( \int_{-L}^{L} |k'_\psi(s)|^2 \, ds + \| \psi \|^2_{L^2(\Omega_{h,\rho})} \right)
\]

\[
= M h^{1/4} \left( \int_{-L}^{L} |k'_\psi(s)|^2 \, ds + \frac{\| \Pi_s \psi \|^2_{L^2(\Omega_{h,\rho})} + \| \Pi^\perp_s \psi \|^2_{L^2(\Omega_{h,\rho})}}{= \int_{[-L,L)} |k'_\psi(s)|^2 \, ds} \right) \quad \text{(5.13)}
\]

for \( h \in (0, h_0] \), where \( M, h_0 \) are positive constants.

### 5.4 Comparison of eigenvalues

#### 5.4.1 Upper bounds

Consider the self-adjoint operator \( T_h^{K_h} \) in \( L^2(\Omega) \), defined by the quadratic form

\[
K_h \ni u \mapsto h^2 \int_{\Omega} |\nabla \psi|^2 \, dx - h^{3/2} \int_{\partial \Omega} |u|^2 \, ds(x), \quad \text{(5.14)}
\]

where the form domain \( K_h \) consists of functions with zero trace on the boundary of \( \Omega \setminus \overline{\Omega}_{h,\rho} \), that is,

\[
K_h = \{ v \in H^1(\Omega) : v = 0 \text{ for } \text{dist}(x, \Gamma) = h^\rho \}.
\]

By the min–max principle and comparison of the form domains, for all \( n \geq 1 \),

\[
\lambda_n(T_h) \leq \lambda_n(T_h^{K_h}). \quad \text{(5.15)}
\]

For all \( \psi \in K_h \), we investigate the quadratic form

\[
q_h^\Omega(\psi) = h^2 \int_{\Omega} |\nabla \psi|^2 \, dx - h^{3/2} \int_{\partial \Omega} |u|^2 \, ds(x) \leq q_h(\chi_\psi),
\]

where

\[
q_h(\chi_\psi) := q_{h,1}(k_\psi) + q_{h,2}(f_\psi) + q_{h,3}(u_\psi)
\]

and

\[
\chi_\psi = (k_\psi u_h^{\text{tran}}, f_\psi, u_\psi).
\]
The quadratic forms \( q_{h,i} \) are defined as follows

\[
q_{h,1}(k_{\psi}) = \int_{-L}^{L} \left( \left( \lambda_h(s) + C_1 h^{3/2 + 2\rho} \right) |k_{\psi}(s)|^2 + \left( h^2 + C_1 h^2 \right) |k'_{\psi}(s)|^2 \right) ds,
\]

\[
q_{h,2}(f_{\psi}) = h^2 \int_{-L}^{L} \left( \int_0^{h\rho} \left( |\partial_t f_{\psi}|^2 + C_2 h^{1/2} |f_{\psi}|^2 \right) dt - h^{-1/2} |f_{\psi}(s,0)|^2 \right) ds,
\]

\[
q_{h,3}(u_{\psi}) = h^2 \int_{\Omega_{h\rho}} |\nabla u_{\psi}|^2 dx \geq h^2 \lambda_1^D(\Omega) \|u_{\psi}\|_{L^2(\Omega_{h\rho})}^2.
\]

The functions \( C_1, C_2 \) are positive constants. For all \( i \in \{1, 2, 3\} \), let \( \mathcal{Q}_{h,i} \) be the operator defined by the quadratic form \( q_{h,i} \). By the min–max principle,

\[ \lambda_n(\mathcal{T}_{h}^{K_h}) \leq \lambda_n(\mathfrak{Q}_{h,i}). \]

We insert this into (5.15) and choose \( \rho = \frac{7}{16} \in \left( \frac{1}{3}, \frac{1}{2} \right) \). Note that \( \lambda_n(\mathcal{Q}_{h,3}) > 0 \) for all \( n \geq 1 \).

Since \( f_{\psi} \perp u_{\text{tran}} \) in \( L^2((0,h^\rho);(1 - t\kappa(s))dt) \), we get by Lemma 4.3 and our choice of \( \rho = \frac{7}{16} \) that \( \lambda_n(\mathcal{Q}_{h,2}) \geq h^{2-2\rho} = h^{9/8} > 0 \) for all \( n \geq 1 \).

Thus, we end up with

\[ \forall n \in I_h, \lambda_n(\mathcal{T}_h) \leq -h + h^{3/2} \min \left( \lambda_n(\mathcal{L}_h^+), 0 \right), \]  

(5.16)

where, for some constant \( c_+ > 0 \), \( \mathcal{L}_h^+ \) is the operator acting on \( L^2([-L,L]) \) as follows,

\[ \mathcal{L}_h^+ = -h^{1/2}(1 + c_+ h^{1/4}) \frac{d^2}{ds^2} - \kappa(s) - \frac{1}{2} h^{1/2} \kappa(s)^2 + c_+ h^{7/8}. \]  

(5.17)

If we consider the eigenvalues of \( \mathcal{T}_h \) below the energy level \( \epsilon h^2 \), with \( \epsilon < \lambda_1^D(\Omega) \), we still get

\[ \lambda_n(\mathcal{T}_h) \leq -h + h^{3/2} \min \left( \lambda_n(\mathcal{L}_h^+), \epsilon h^2 \right) \]

5.4.2  |  Lower bounds

For all \( \psi \in H^1(\Omega) \), we write the lower bound

\[ q_{h}^D(\psi) = h^2 \int_{\Omega} |\nabla \psi|^2 dx - h^{3/2} \int_{\partial\Omega} |u|^2 ds(x) \geq p_h(\chi_{\psi}), \]

where

\[ p_h(\chi_{\psi}) := p_{h,1}(k_{\psi}) + p_{h,2}(f_{\psi}) + p_{h,3}(u_{\psi}) \]

and

\[ \chi_{\psi} (k_{\psi} u_{\text{tran}}^h, f_{\psi}, u_{\psi}). \]
The quadratic forms $p_{h,i}$ are defined as follows

\[
p_{h,1}(k\psi) = \int_{-L}^{L} \left( (\lambda_h(s) - C_1 h^{3/2}) |k\psi(s)|^2 + (h^2 - C_1 h^2) |k_\psi'(s)|^2 \right) ds,
\]

\[
p_{h,2}(f\psi) = h^2 \int_{-L}^{L} \left( \int_0^{h^2} \left( |\partial_t f\psi|^2 - C_2 h^2 |f\psi|^2 \right) adt - h^{-3/2} |f\psi(s,0)|^2 \right) ds
\]

\[
p_{h,3}(u\psi) = h^2 \int_{\Omega_h} |\nabla u\psi|^2 dx \geq h^2 \lambda_1^N(\Omega_h) \|u\psi\|_{L^2(\Omega_h)}^2 = 0,
\]

where $C_1, C_2$ are positive constants. For all $i \in \{1, 2, 3\}$, let $\mathcal{L}_h$ be the operator defined by the quadratic form $p_{h,i}$. By the min–max principle,

\[
\lambda_n(\mathcal{T}_h) \geq \lambda_n(\mathcal{L}_h) \mathcal{T}_h = \mathcal{L}_h
\]

with $\lambda_1(I_{h,3}) \geq 0$ and $\lambda_n(I_{h,3}) > 0$ for all $n \geq 2$. We choose $\rho = \frac{7}{16}$ and observe that, by Lemma 4.3, $\lambda_n(I_{h,2}) \geq h^{9/8} > 0$ for all $n \geq 1$. Therefore, for some constant $\epsilon_+ > 0$ and $\mathcal{L}_h^-$ the operator (acting on $L^2([-L, L])$)

\[
\mathcal{L}_h^- = -h^{1/2}(1 - \epsilon_- h^{1/4}) \frac{d^2}{ds^2} - \kappa(s) - \frac{1}{2} h^{1/2} \kappa(s)^2 - \epsilon_- h^{7/8},
\]

we get

\[
\forall n \in I_{h,3} \setminus [2 + N_\epsilon^N, +\infty), \lambda_n(\mathcal{T}_h) \geq -h + h^{3/2} \lambda_n(\mathcal{L}_h^-).
\]

When dealing with the eigenvalues of $\mathcal{T}_h$ below $\epsilon h^2$, with $\epsilon < \lambda_2^N(\Omega)$, we still get

\[
\lambda_n(\mathcal{T}_h) \geq -h + h^{3/2} \min \left( \lambda_n(\mathcal{L}_h^-), h^{-1/2} \right),
\]

because

\[
\min \left( -h + h^{3/2} \lambda_n(\mathcal{L}_h^-), 0 \right) = -h + \min \left( h^{3/2} \lambda_n(\mathcal{L}_h^-), h \right).
\]

Remark 5.3. Consider $\epsilon \in (0, \lambda_2^N(\Omega))$. Since $\lambda_1(I_{h,3}) = 0$ is a simple eigenvalue, the min–max principle allows us to extend (5.19) as follows. Set $N_\epsilon(h) = \max\{n \geq 1, -h + h^{3/2} \lambda_n(\mathcal{L}_h^-) < 0\}$. Then, for $h$ small enough, we have

\[
\forall n \in I_{h,3} \cap [2 + N_\epsilon(h), +\infty), \lambda_n(\mathcal{T}_h) \geq -h + h^{3/2} \lambda_n(\mathcal{L}_h^-).
\]

Acknowledgements
The authors would like to thank Gerd Grubb, Thierry Daudé, François Nicoleau, and I. Polterovich for helpful discussions. The first author was inspired by the very interesting talks proposed at the seminar “Spectral geometry in the clouds” organized by A. Girouard and J. Lagacé and initially
due to this terrible COVID period. The second author is supported by the Lebanese University within the project “Analytical and numerical aspects of the Ginzburg Landau model.”

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