On quadratic variation in the Skorokhod space

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Abstract

We perceive an elegant connection between the notion of pathwise quadratic variation defined in Föllmer’s (1981) classic piece [1] and the Skorokhod topology. As a result, we are able to obtain an equivalent definition of this notion in its canonical form. The result is extended coherently to any càdlàg path on \([0, \infty)\) taking values in \(\mathbb{R}^n\).

1. Introduction

Quadratic variation is a fundamental object of study in stochastic analysis, if only because it sits in the celebrated Itô’s formula. It was later discovered by Hans Föllmer during 1979-80 that Itô’s calculus can be applied pathwise to any \(C^2\) function driven by càdlàg paths without using any probabilistic machinery.

To do so, it is required that the càdlàg path be of finite quadratic variation, a fairly general analytic notion which was nonetheless, complicated by the defining properties what a "good" quadratic variation ought to possess. For example, the quadratic variation must admit a specific form of Lebesgue decomposition (LD), an essential property that was however, imposed on, rather than obtained from, the definition. Moreover, the extension of this notion to the multidimensional setting was subtle, i.e. the definition [1, Rem.(1)] reduces to a tautological statement in the one dimensional case.

The aim of this article is to shed light on what this notion really is, in its minimal form! We established an equivalent definition such that the two defining properties adopted in [1] become consequences rather than the definition. In particular, we prove that quadratic variation is merely the natural limit of the quadratic sums in the Skorokhod space and vice versa without imposing any form of (LD) on the limit. This results in a succinct and coherent definition of quadratic variation for any càdlàg path on \([0, \infty)\) taking values in \(\mathbb{R}^n\).

The layout of this article is as follows: In section 2, we show how quadratic variation was defined in the semimartingale context and where (LD) comes into
play. In section 3, we introduce Föllmer’s notion of pathwise quadratic variation and explain why a specific form of (LD) was imposed to the original definition. In section 4, we show how quadratic variation is naturally connected to the Skorokhod topology (Theorems A&B) and obtain an equivalent definition (LD free). In section 5, we extend our results to the multidimensional setting.

2. Quadratic variation of a semimartingale

Throughout this article, \( \pi := \left( \pi_n \right)_{n \geq 1} \) is a fixed sequence of partitions \( \pi_n = (t_0, \ldots, t_{n+1}) \) of \([0, \infty)\) into intervals \( 0 \leq t_0 < \ldots < t_{n+1} < \infty; \ t_{n+1} \uparrow \infty \) with vanishing mesh \( |\pi_n| \downarrow 0 \) on compacts.

Before we proceed to the purely deterministic pathwise setting, let us recap a few observations in the semimartingale framework, from which the term "Quadratic variation" was originated.

Let \( X \) be a real-valued semimartingale, the Itô’s formula gives the quadratic variation \( [X] \) of \( X \):

\[
[X]_t = X_t^2 - X_0^2 - 2 \int_0^t X_s dX_s
\]

and by the properties of the Itô’s integral,

\[
S_n(t) := \sum_{t_i \in \pi_n} \left( X_{t_{i+1}} \wedge t - X_{t_i} \wedge t \right)^2
\]

converge uniformly to \( [X] \) in probability on compacts (ucp) [2, Prop.I.4.44]. Thus, there exists a sub-sequence \( (n_k) \) such that (1) converges uniformly to \( [X] \) on compacts almost surely (a.s.). As a direct consequence of uniform convergence, we observe that

\[
\Delta[X]_t = \Delta \left( \lim_{k \to \infty} S_{n_k} \right)(t) = \lim_{k \to \infty} \Delta S_{n_k}(t) = (\Delta X_t)^2.
\]

It follows that a.s. sample paths of \( [X] \) admit a specific form of Lebesgue decomposition, which relates the jumps of \( [X] \) to those of \( X \):

\[
[X]_t = [X]_t^c + \sum_{s \leq t} (\Delta X_s)^2.
\]

We see that (2) is a fundamental property of quadratic variation \( [X] \), as it infers regularity i.e. \( [X] \) is continuous \( \iff \) \( X \) is, that of which the validity of the Itô’s formula relies on. Nonetheless, (2) is not a defining property of \( [X] \), but rather, a consequence of the definition of \( [X] \) i.e. the ucp limit of the quadratic sums in (1).
3. Prequel

Denote $\mathcal{D} := \mathcal{D}([0, \infty), \mathbb{R})$ to be the Skorokhod space and $\mathcal{C} := \mathcal{C}([0, \infty), \mathbb{R})$ the subspace of real-valued continuous functions. In this section, $\mathcal{D}$ shall be equipped with the (local) uniform topology \cite[VI.1.2]{2}.

Put $\tilde{\pi}_k := \pi_{n_k}$ and denote $Q_{\infty}^{\tilde{\pi}} \subset \mathcal{D}$ to be the subspace of those $x \in \mathcal{D}$ such that

$$s_k(t) := \sum_{t_i \in \tilde{\pi}_k} (x_{t_i+1} \wedge t - x_{t_i} \wedge t)^2$$

converges in the (local) uniform topology to an $[x] \in \mathcal{D}$. From the previous section, we see that a.s. paths of the semimartingale $X$ live in $Q_{\infty}^{\tilde{\pi}}$.

Since the Lebesgue decomposition (2), a sufficient condition to apply Itô's formula pathwise \cite{1}, holds on $Q_{\infty}^{\tilde{\pi}}$ per (local) uniform convergence of (3), i.e.

Proposition 1 \quad $\Delta[x]_t = (\Delta x_t)^2$.

Proof. $t_i^{(k)} := \max \{ \tilde{\pi}_k \cap [0, t] \}$ then

$$\Delta[x]_t = \Delta(\lim_{k \to \infty} s_k)(t) = \lim_{k \to \infty} \Delta s_k(t) = (\Delta x_t)^2 + 2\Delta x_t \lim_{k \to \infty} (x_{t-} - x_{t_i^{(k)}})$$

we may as well be content with a pathwise theory of Itô's calculus in $Q_{\infty}^{\tilde{\pi}}$. Indeed, as we shall reveal in the sequel (4.Cor.(iii)), it makes little sense to move beyond $Q_{\infty}^{\tilde{\pi}}$ when the underlying space is $\mathcal{C}$! Observe that, in this pathwise setting, $Q_{\infty}^{\tilde{\pi}}$ is also exclusively defined by the existence of (local) uniform limit of the quadratic sums in (3).

When $X$ is a general process whose paths live a.s. in $\mathcal{D}$ as opposed to $\mathcal{C}$, it becomes necessary to extend $Q_{\infty}^{\tilde{\pi}}$ further, for example, by weakening the convergence topology in the definition of $[x]$. Although one obtains, potentially, a larger domain of application than $Q_{\infty}^{\tilde{\pi}}$, the pathwise Itô's formula \cite{1}

$$f(x_t) = f(x_0) + \int_0^t f'(x_{s-}) dx_s + \frac{1}{2} \int_0^t f''(x_{s-}) d[x]_s$$

$$+ \sum_{s \leq t} \left[ f(x_s) - f(x_{s-}) - f'(x_{s-}) \Delta x_s - \frac{1}{2} f''(x_{s-})(\Delta x_s)^2 \right]$$

breaks down whenever (2) does i.e. $\Delta[x]_s \neq (\Delta x_s)^2$. Indeed, there exists continuous $x$ such that (3) converges pointwise to a discontinuous $[x] \in \mathcal{D}$ \cite[Appendix 5.2]{3}. It is then clear as to why property (2) was imposed in the the following
Definition 2 (Föllmer 1981) We say that \( x \in \mathcal{D} \) has finite quadratic variation \([x]\) along \( \pi \) if the discrete measures
\[
\mu_n := \sum_{t_i \in \pi_n} (x_{t_{i+1}} - x_{t_i})^2 \delta_{t_i}
\]
converges vaguely to a Radon measure \( \mu \) on \([0, \infty)\) and such that the distribution \([x]\) of \( \mu \) admits the Lebesgue decomposition (2), i.e.
\[
[x]_t = [x]_c^t + \sum_{s \leq t} (\Delta x_s)^2.
\]
Denote \( Q_0^\pi \) to be the subset of those \( x \in \mathcal{D} \) with finite quadratic variation.

Before we proceed, let us draw a link between vague and weak convergence of Radon measures on \([0, \infty)\), the link of which, is well known in the special case where the measures are sub-probability measures:

**Lemma 3** Let \( v_n \) and \( v \) be non-negative Radon measures on \([0, \infty)\) and \( J \subset [0, \infty) \) be the set of atoms of \( v \), the followings are equivalent:

(i) \( v_n \to v \) vaguely on \([0, \infty)\).

(ii) \( v_n \to v \) weakly on \([0, T] \) for every \( T \notin J \).

**Proof.** Let \( f \in C_K([0, \infty)) \) be a compactly supported continuous function. Since \( J \) is countable, \( \exists T \notin J; \text{supp}(f) \subset [0, T] \). Now (ii) \( \Rightarrow \) \( \int_T^\infty f \, dv_n \to \int_0^T f \, dv \Rightarrow (i) \). Suppose (i) holds, let \( T \notin J \) and \( f \in C([0, T], \| \cdot \|_\infty) \).

Since \( f = (f)^+ - (f)^- \), we may take \( f \geq 0 \) and define the following extensions:
\[
\overline{f}(t) := f(t) \mathbb{I}_{[0,T]}(t) + f(T) \left( 1 + \frac{T-t}{\epsilon} \right) \mathbb{I}_{(T,T+\epsilon]}(t),
\]
\[
\underline{f}(t) := f(t) \mathbb{I}_{[0,T-\epsilon)}(t) + f(T) \left( \frac{T-t}{\epsilon} \right) \mathbb{I}_{[T-\epsilon,T]}(t),
\]
then \( \overline{f}, \underline{f} \in C_K([0, \infty)) \), \( 0 \leq \underline{f} \leq f \|_{[0,T]} \leq \overline{f} \leq \| f \|_\infty \) and we have
\[
\int_0^\infty \underline{f}^' \, dv_n \leq \int_0^T f \, dv_n \leq \int_0^\infty \overline{f}^' \, dv_n.
\]
Since \( v_n \to v \) vaguely and \( T \notin J \), thus, as \( n \to \infty \) we obtain
\[
0 \leq \limsup_n \int_0^T f \, dv_n - \liminf_n \int_0^T f \, dv_n \leq \int_T^\infty f - f^' \, dv \leq \| f \|_\infty \mathbb{I}_{(T-\epsilon,T+\epsilon]}(\epsilon) \to 0,
\]
hence by monotone convergence
\[
\lim_n \int_0^T f \, dv_n = \lim_n \int_0^\infty f^' \, dv = \int_0^T f \, dv
\]
and (ii) follows. \( \square \)
4. Quadratic variation in the Skorokhod space $\mathcal{D}([0,\infty), \mathbb{R})$

From this section onwards, $\mathcal{D}$ shall be equipped with a metric $d$ which induces the Skorokhod (a.k.a $J_1$) topology [2, VI]. Denote $\mathcal{D}_0^+ \subset \mathcal{D}$ to be the subset of non-negative increasing càd functions null at 0.

**Proposition 4** If $x \in Q^0_0$, then the pointwise limit $s$ of

$$s_n(t) := \sum_{t_i \in \pi_n} (x_{t_{i+1}} \land t - x_{t_i \land t})^2$$

exists and $s$ admits the Lebesgue decomposition (2):

$$s(t) = s^c(t) + \sum_{s \leq t} (\Delta x_s)^2.$$  \hspace{1cm} (7)

In this case $s = [x]$.

**Proof.** If $x \in Q^0_0$, define

$$q_n(t) := \sum_{\pi_n \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2,$$

the distribution function of $\mu_n$ in (4). Since $\mu_n \to \mu$ vaguely, we have $q_n \to [x]$ pointwise at all continuity points of $[x]$ (Lemma 3 & [4, X.11]). Let $I$ be the set of continuity points of $[x]$. Observe $(q_n)$ is monotonic in $[0, \infty)$ and $I$ is dense in $[0, \infty)$, if $t \not\in I$, it follows [4, X.8] that

$$[x]_{t-} \leq \liminf_{n} q_n(t) \leq \limsup_{n} q_n(t) \leq [x]_{t+} = [x]_t.$$

Thus, we may take any sub-sequence $(n_k)$ such that $\lim_k q_{n_k}(t) =: q(t)$. Since $x \in Q^0_0$ and the Lebesgue decomposition (5) holds on $[x]$, we have

$$([x]_{t+} - q(t)) + (q(t) - [x]_{t-}) = [x]_{t+} - [x]_{t-} \to (\Delta x_t)^2.$$ \hspace{1cm} (8)

If $t \pm \epsilon \in I$, $\tilde{\pi}_k := \pi_{n_k}$ and $t^{(k)}_j := \max\{\tilde{\pi}_k \cap [0, t]\}$, the second sum in (8) is

$$\lim_k \sum_{t_i \in \tilde{\pi}_k, t - \epsilon < t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2 = \lim_k \sum_{t_i \in \tilde{\pi}_k, t - \epsilon < t_i < t^{(k)}_j} (x_{t_{i+1}} - x_{t_i})^2 + (\Delta x_t)^2 \geq (\Delta x_t)^2$$

by the fact that $x$ is càdlàg and that $t \not\in I$.

We see immediately from (8) that $q(t) = [x]_t$ as $\epsilon \to 0$. Since the choice of the sub-sequential limit $q(t)$ is arbitrary, we conclude that $q_n \to [x]$ pointwise on $[0, \infty)$. Observe that the pointwise limits of $(s_n)$ and $(q_n)$ coincide (11) by the right-continuity of $x$ and that $x \in Q^0_0$, (Prop. 4) follows.

\hfill $\blacksquare$
Denote $Q^x_1$ to be the subset of those $x \in D$ such that the pointwise limit $s$ of the quadratic sums $(s_n)$ in (6) exists and the Lebesgue decomposition in (7) holds on $s$. Then $Q^x_0 \subset Q^x_1$ and we have:

**Proposition 5** If $x \in Q^x_1$, then the pointwise limit $q$ of

$$q_n(t) := \sum_{\pi_n \ni t \leq t} (x_{t+1} - x_t)^2$$

exists and $q$ admits the Lebesgue decomposition (2):

$$q(t) = q^c(t) + \sum_{s \leq t} (\Delta x_s)^2. \quad (10)$$

In this case $q = s$.

**Proof.** Define $t_i^{(n)} := \max \{\pi_n \cap [0,t]\}$. Since the pointwise limits of $(s_n)$ and $(q_n)$ coincide i.e.

$$|s_n(t) - q_n(t)| = (x_{t_i^{(n)} - x_t})^2 + 2(x_{t_i^{(n)} - x_t})(x_t - x_{t_i^{(n)})} \quad (11)$$

converges to 0 by the right-continuity of $x$. (Prop. 5) now follows from $x \in Q^x_1$. 

Denote $Q^x_2$ to be the subset of those $x \in D$ such that the pointwise limit $q$ of the quadratic sums $(q_n)$ in (9) exists and the Lebesgue decomposition in (10) holds on $q$. Then $Q^x_0 \subset Q^x_1 \subset Q^x_2$ and we have:

**Theorem A** If $x \in Q^x_2$, then $q_n \rightarrow q$ in the Skorokhod topology.

**Proof.** Since $x \in Q^x_2$, we have $q_n \rightarrow q$ pointwise on $[0,\infty)$ and that $(q_n)$, $q$ are elements in $D^0_\infty$. By [2, Thm.VI.2.15], it remains to show that

$$\sum_{s \leq t} (\Delta q_n(s))^2 \rightarrow \sum_{s \leq t} (\Delta q(s))^2$$

on a dense subset of $[0,\infty)$. Let $t > 0$, define $J^t := \{s \geq 0, (\Delta X_s)^2 \geq \frac{\varepsilon}{t}\}$, $J_n^t := \{\pi_n \ni \exists s \in (t_i, t_i+1); (\Delta X_s)^2 \geq \frac{\varepsilon}{t}\} \subset \pi_n$ and observe that

$$\sum_{s \leq t} (\Delta q_n(s))^2 = \sum_{\pi_n \ni t \leq t} (x_{t+1} - x_t)^4 = \sum_{J_n^t \ni t \leq t} (x_{t+1} - x_t)^4 + \sum_{(J_n^t)^c \ni t \leq t} (x_{t+1} - x_t)^4. \quad (12)$$

Since $x$ is càdlàg and that $|\pi_n| \downarrow 0$ on compacts, the first sum in (12) converges to $\sum_{J^t \ni t \leq t} (\Delta x_s)^4$ and the second sum in (12)

$$\sum_{(J_n^t)^c \ni t \leq t} (x_{t+1} - x_t)^4 \leq \sup_{(J_n^t)^c \ni t \leq t} (x_{t+1} - x_t)^2 \sum_{(J_n^t)^c \ni t \leq t} (x_{t+1} - x_t)^2 \leq \varepsilon q(t).$$
for sufficiently large \( n \) [5, Appendix A.8] hence

\[
\lim_n \sum_{s \leq t} (\Delta q_n(s))^2 = \sum_{J_n \ni s \leq t} (\Delta x_s)^4 + \limsup_n \sum_{(J_n') \ni t_{i+1} \leq t} (x_{t_{i+1}} - x_{t_i})^4.
\]

By the Lebesgue decomposition (10), we observe \( \sum_{J_n \ni t \leq t} (\Delta x_s)^4 \leq q(t)^2 \) and that

\[
\lim_n \sum_{s \leq t} (\Delta q_n(s))^2 = \sum_{s \leq t} (\Delta x_s)^4 = \sum_{s \leq t} (\Delta q(s))^2
\]

as \( \epsilon \to 0 \).

Denote \( Q^\pi \) to be the subset of those \( x \in D \) such that the limit \( \bar{q} \) of \( (q_n) \) exists in \( (D, d) \). Then \( Q^\pi_0 \subset Q^\pi_1 \subset Q^\pi_2 \subset Q^\pi \) and we have:

**Theorem B** \( Q^\pi \subset Q^\pi_0 \) and \( \bar{q} = [x] = s = q \).

**Proof.** Let \( x \in Q^\pi \) and \( I \) be the set of continuity points of \( \bar{q} \). [2, VI.2.1(b.5)] implies that \( q_n \to \bar{q} \) pointwise on \( I \). Since \( q_n \in D^+_0 \) and \( I \) is dense on \([0, \infty)\), it follows \( \bar{q} \in D^+_0 \). Denote \( \mu \) to be the Radon measure of \( \bar{q} \) on \([0, \infty)\), observe the set of atoms of \( \mu \) is \( J := [0, \infty) \setminus I \) and that \( (q_n) \) are the distribution functions of the discrete measures \( (\mu_n) \) in (4). Thus, by (Lemma 3 & [4, X.11]), we see that \( \mu_n \to \mu \) vaguely on \([0, \infty)\).

If \( t > 0 \), put \( t^{(n)}_i := \max\{\pi_n \cap [0, t]\} \). Since \( |\pi_n| \downarrow 0 \) on compacts, we have \( t^{(n)}_i < t \), \( t^{(n)}_i \uparrow t \) and \( t^{(n)}_{i+1} \downarrow t \). Observe that

\[
\Delta q_n(t) = \begin{cases} 
(x_{t_{i+1}} - x_{t_i})^2, & \text{if } t = t_i \in \pi_n, \\
0, & \text{otherwise}.
\end{cases}
\]

(13)

If \( \Delta \bar{q}(t) = 0 \), [2, VI.2.1(b.5)] implies that \( \Delta q_n(t^{(n)}_i) \to \Delta \bar{q}(t) \). Hence, by the fact that \( x \) is càdlàg, \( (\Delta x_i)^2 = \lim_n \Delta q_n(t^{(n)}_i) = \Delta \bar{q}(t) \). If \( \Delta \bar{q}(t) > 0 \), there exists [2, VI.2.1(a)] a sequence \( t'_n \to t \) such that \( \Delta q_n(t'_n) \to \Delta \bar{q}(t) > 0 \). Using the fact that \( x \) is càdlàg, \( t'_n \to t \) and (13), we deduce that \( (t'_n) \) must coincide with \( (t^{(n)}_i) \) for all \( n \) sufficiently large, else we will contradict \( \Delta \bar{q}(t) > 0 \). Thus, \( (\Delta x_i)^2 = \lim_n \Delta q_n(t^{(n)}_i) = \lim_n \Delta q_n(t'_n) = \Delta \bar{q}(t) \) and the Lebesgue decomposition (2) holds on \( \bar{q} \).

By (Def. 2), we have \( \bar{q} = [x] \) hence \( Q^\pi \subset Q^\pi_0 \). \( [x] = s = q \) now follows from \( Q^\pi_0 \subset Q^\pi_1 \subset Q^\pi_2 \) and (Prop. 4 & 5).
Backed by Theorems A and B, we arrive at the following equivalent

**Definition 6** We say that $x \in D$ has finite quadratic variation $[x]$ along $\pi$ if the following quadratic sums:

$$q_n(t) := \sum_{\pi_n \ni t_i \leq t} (x_{t_{i+1}} - x_{t_i})^2$$

converges to $[x]$ in $(D, d)$.

We immediately see that the two defining properties of $[x]$ in (Def. 2) are consequences per Theorem B. The following corollary treats the special case when the underlying space is $C$. Observe that if $(n_k)$ is any sub-sequence, $\tilde{\pi}_k := \pi_{n_k}$ then obviously $Q^\tilde{\pi} \subset Q^\pi$. Thus, we prove instead the following starker

**Corollary 7** Let $x \in Q^\pi$, then

(i) $q_k \to [x]$ (local) uniformly on $[0, \infty)$ if and only if $x \in C$.
(ii) If $q_k \to [x]$ (local) uniformly on $[0, \infty)$, so would $(s_k)$ in (3).
(iii) $Q^\pi_\infty \subset Q^\pi$, $C \cap Q^\pi_\infty = C \cap Q^\pi$.

**Proof.** (i): It is an immediate consequence of (Thm. B), (5) and [2, VI.1.17(b)].
(ii): Let $T > 0$, $\| \cdot \|_T$ the supremum norm on $D([0, T])$ and observe that

$$\|s_k - [x]\|_T \leq \|q_k - [x]\|_T + \|s_k - q_k\|_T.$$

Since (i) implies $x \in C$, (ii) now follows from uniform continuity of $x$ and (11).
(iii): By (Prop. 1) and that (local) uniform convergence implies pointwise convergence, we have $Q^\pi_\infty \subset Q^\pi_1 \subset Q^\pi$. (iii) now follows from (i) & (ii).

**Remark 8**

1.) The reverse of (ii) is in general not true. To see this, take any semimartingale $X$ whose paths live a.s. in $D \setminus C$. Then a.s. paths of $X$ actually live in $Q^\pi_\infty \setminus C \neq \emptyset$ (Section 2 & 3). If $x \in Q^\pi_\infty \setminus C$ then $s_k \to [x]$ (local) uniformly per definition of $Q^\pi_\infty$ (Section 3) while $q_k \not\to [x]$ (local) uniformly according to (i).

2.) We note that a third form (instead of $s_n$ and $q_n$) of quadratic sums

$$\tilde{q}_n(x, t) := \sum_{\pi_n \ni t_{i+1} \leq t} (x_{t_{i+1}} - x_{t_i})^2$$

has sometimes been "mistakenly" adopted in the literature in the definition of $[x]$ (i.e. by the pointwise limit of $\tilde{q}_n$ on $[0, \infty)$ that admits LD). To see the problem, take any $t_0 \notin \pi$, put $x_t := \mathbb{I}_{[t_0, \infty)}(t)$ then obviously $[x](t_0) = \lim s_n(t_0) = \lim q_n(t_0) = 1$ but $\lim \tilde{q}_n(t_0) = 0$. The same conclusion can be drawn in the
probabilistic setting e.g. if one puts the entire probability mass on one path \(\mathbb{I}_{[t_0, \infty)}\) in \(\mathcal{D}\), the resulting canonical process \(X\) on \(\mathcal{D}\), strictly speaking, is a semimartingale whose \([X](t_0) = 1\) a.s. but now \(\lim \bar{q}_n(X, t_0) = 0\) a.s.

5. Quadratic variation in the Skorokhod space \(\mathcal{D}([0, \infty), \mathbb{R}^n)\)

Denote \(\mathcal{D}^n := \mathcal{D}([0, \infty), \mathbb{R}^n)\) and \(\mathcal{D}^{n\times n} := \mathcal{D}([0, \infty), \mathbb{R}^{n\times n})\) to be the Skorokhod spaces, each of which equipped with a metric \(d\) which induces the corresponding Skorokhod (a.k.a \(J_1\)) topology [2, VI]. \(\mathcal{C}^n := \mathcal{C}([0, \infty), \mathbb{R}^n)\) the subspace of continuous functions in \(\mathcal{D}^n\). We recall (4. Thm. A & B) from the one dimensional case \(n = 1\) that (Def. 2) and (Def. 6) are equivalent.

As is well known, if \(x, y \in \mathcal{Q}^n\), it does not, in general, imply \(x + y \in \mathcal{Q}^n\). An example can be found in [6]. Therefore, the notion of quadratic variation in the multidimensional setting was originally defined in [1, Rem.(1)] as follows:

**Definition 9 (Föllmer 1981)** We say that \(x := (x^1, \ldots, x^n)^T \in \mathcal{D}^n\) has finite quadratic variation along \(\pi\) if all \(x^i, x^i + x^j\) \((1 \leq i, j \leq n)\) have finite quadratic variation.

In this case, note that the quadratic covariation \([x^i, x^j]\) of \(x^i\) and \(x^j\) may be defined by
\[
[x^i, x^j]_t := \frac{1}{2} \left( [x^i + x^j]_t - [x^i]_t - [x^j]_t \right),
\]
which admits the following specific form of Lebesgue decomposition:
\[
[x^i, x^j]_t = [x^i, x^j]_t^c + \sum_{s \leq t} \Delta x^i_s \Delta x^j_s.
\]

We shall call \([x] := ([x^i, x^j])_{1 \leq i \leq j \leq n}\) the quadratic variation of \(x\).

Be aware that in the special case \(n = 1\), the above definition becomes a tautological statement i.e. it is not possible to work with the above definition alone without the company of a one dimensional definition, for example (Def. 2) or equivalently (Def. 6). The following

**Definition 10** We say that \(x \in \mathcal{D}^n\) has finite quadratic variation \([x]\) along \(\pi\) if the following quadratic sums:
\[
q_n(t) := \sum_{\pi_n \ni t \leq s} (x_{t+1} - x_t)(x_{t+1} - x_t)^T
\]
converges to \([x]\) in \((\mathcal{D}^{n\times n}, d)\).
is the natural extension of (Def. 6) to the multidimensional setting and we now prove its equivalence to that of Föllmer’s.

For \( u, v, w \in D \), let us write
\[
q_n(u,v) := \sum_{\pi_n \ni t_i \leq t} (u_{t_i+1} - u_{t_i})(v_{t_i+1} - v_{t_i})
\]
and \( q_n(w) := q_n(w,w) \). Note that the Skorokhod topology on \((D^n, d)\) is strictly finer than the product topology on \((D, d)^n\) [2, VI.1.21] and that \((D, d)\) is not a topological vector space [2, VI.1.22], hence the following would be an essential lemma.

**Lemma 11** Let \( t > 0 \), there exists a sequence \( t_n \to t \) such that
\[
\lim_n \left( \Delta q_n(u,v)(t_n) \right) = \Delta \left( \lim_n q_n(u,v) \right)(t),
\]
\( \forall u, v \in D \); \( (q_n(u,v)) \) converges in \((D, d)\).

**Proof.** Define \( t_i^{(n)} := \max\{\pi_n \cap [0, t]\} \). Since \( |\pi_n| \downarrow 0 \) on compacts, we have \( t_i^{(n)} < t, t_i^{(n)} \uparrow t \) and \( t_{i+1}^{(n)} \downarrow t \). Observe that
\[
\Delta q_n(u,v)(t) = \begin{cases} (u_{t_{i+1}} - u_{t_i})(v_{t_{i+1}} - v_{t_i}), & \text{if } t = t_i \in \pi_n, \\ 0, & \text{otherwise.} \end{cases}
\]
(16)

Put \( \bar{q} := \lim_n q_n(u,v) \). If \( \Delta \bar{q}(t) = 0 \), [2, VI.2.1(b.5)] implies that \( \Delta q_n(u,v)(t_i^{(n)}) \to \Delta \bar{q}(t) \). If \( \Delta \bar{q}(t) > 0 \), there exists [2, VI.2.1(a)] a sequence \( t_n^{(n)} \to t \) such that \( \Delta q_n(u,v)(t_n^{(n)}) \to \Delta \bar{q}(t) > 0 \). Using the fact that \( u, v \) are càdlàg, \( t_n^{(n)} \to t \) and (16), we deduce that \( t_{i+1}^{(n)} \) must coincide with \( t_i^{(n)} \) for all \( n \) sufficiently large, else we will contradict \( \Delta \bar{q}(t) > 0 \). Put \( t_n := t_i^{(n)} \).

**Proposition 12** Let \( x, y \in \mathcal{Q}^\pi \), then \( (q_n^{(x+y)}) \) converges in \((D, d)\) if and only if \( (q_n^{(x,y)}) \) does. In this case, \( x+y \in \mathcal{Q}^\pi \) and \( \lim_n q_n^{(x,y)} = \frac{1}{2} ([x+y] - [x] - [y]) \).

**Proof.** Since
\[
q_n^{(x+y)} = q_n^{(x)} + q_n^{(y)} + 2q_n^{(x,y)}
\]
and that \( x, y \in \mathcal{Q}^\pi \), we obtain (Prop. 12) immediately from (Lemma 11) and [2, VI.2.2(a)].

**Proposition 13** \( (q_n) \) converges in \((\mathcal{D}^{\times n}, d)\) if and only if it converges in \((\mathcal{D}, d)^{n \times n}\).
Proof. Since the Skorokhod topology on \((D_n \times D_n, d)\) is strictly finer than the product topology on \((D, d)^{n \times n}\) [2, VI.1.21], we have \((D, d)^{n \times n}\) convergence implies \((D, d)^{n \times n}\) convergence. The other direction follows immediately from the observation that

\[ q_n = \left( q_n^{(i \to j)} \right)_{1 \leq i \leq j \leq n} \quad (17) \]

(Lemma 11) and [2. VI.2.2(b)].

Theorem (Def. 9) and (Def. 10) are equivalent.

Proof. It is an immediate consequence of (14), (17), (Prop. 12 & 13) and (4. Thm. A & B).

Corollary 14 If \(x \in D^n\) has finite quadratic variation, then

(i) \(q_n \to [x]\) (local) uniformly on \([0, \infty)\) if and only if \(x \in \mathcal{C}^n\).

(ii) \(F(q_n) \to F([x])\) for all functionals \(F\) continuous at \([x]\).

Proof. It is an immediate consequence of (Thm.), (15) and [2, VI.1.17(b)].

Remark 15 A handful of functionals \(F\), continuous at points, are well catalogued in [2, VI.2].

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