KÄHLER REDUCTION OF METRICS WITH HOLOMONY G₂

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INTRODUCTION

There are now many explicitly known examples of metrics with holonomy group equal to \( G₂ \), the simplest of which admit an isometry group with orbits of codimension one. A metric with holonomy \( G₂ \) on a smooth 7-manifold \( Y \) is characterized by a 3-form and a 4-form that are interrelated and both closed. If this structure is preserved by a circle group \( S¹ \) acting on \( Y \) (which cannot then be compact), the quotient \( Y/S¹ \) has a natural symplectic structure. A geometrical description of such quotients was carried out by Atiyah and Witten [8] when \( Y \) is one of three original manifolds with a complete metric of holonomy \( G₂ \) described in [14]. The symplectic structure is also defined on the image of the fixed points, which in each of these cases is a Lagrangian submanifold \( L \). The embedding of \( L \) in a neighborhood of \( Y/S¹ \) is believed to approximate the geometry of a special Lagrangian submanifold of a Calabi-Yau manifold, and this is consistent with models of special Lagrangian submanifolds of \( \mathbb{R}^6 \) described for example in [24].

This work motivates a general investigation of the quotient \( N = Y/S¹ \) of a manifold with holonomy \( G₂ \) by an \( S¹ \) action. Initial results in this direction can be found in [15, 16], and in this paper we pursue the theory under the simplifying assumption that the \( S¹ \) action is free. It is an elementary fact that \( N \) has, in addition to its symplectic 2-form, a natural reduction to \( SU(3) \). Whilst this structure cannot be torsion-free (i.e. Calabi-Yau) if \( Y \) is irreducible, there are non-trivial examples in which the associated almost-complex structure is integrable, so that \( N \) is Kähler. This paper is devoted to an investigation of such a situation, which turns out to be surprisingly rich. Our study began with the realization that in this case, a parameter measuring the size of the fibers of \( Y \) generates both a Killing and a Ricci potential for \( N \) (Proposition 1 and Corollary 2), exhibiting a link with the theory of the so-called Hamiltonian 2-forms [4].

We first describe the induced \( SU(3) \) structure on \( N \) in §1, and then pursue consequences of the integrability condition. In particular, we prove that when \( N \) is Kähler, an infinitesimal isometry \( U \) of the \( SU(3) \) structure is inherently defined. The situation is reminiscent of the study of Einstein-Hermitian 4-manifolds in which Killing vector fields appear automatically [6, 17, 27]. We explain in §2 that \( U \) can be used to obtain a Kähler quotient of \( N \), consisting of a 4-manifold equipped with a 1-parameter family of smooth functions and 2-forms, satisfying a coupled second-order evolution equation (Theorem 1). The procedure can be reversed so as to construct metrics with holonomy (generically equal to) \( G₂ \) from a 4-manifold \( M \) with the appropriate structure.

A fundamental construction of a holonomy \( G₂ \) metric starting from a \( \mathbb{T}² \) bundle over a hyperkähler 4-manifold was discovered by Gibbons, Lü, Pope and Stelle [20],
and towards the end of §3 we exhibit our reverse procedure as a generalization of the above. It can be further improved by introducing an anti-self-dual 2-form with the opposite orientation to the Kähler one (Theorem 2). This leads to the construction of new examples of $G_2$ metrics based on the examples of Ricci-flat almost-Kähler metrics $[5, 7, 28]$. We point out that there also exist constructions of holonomy $G_2$ metrics from higher dimensional hyperkähler manifolds (see for example $[3]$).

When $M$ is $T^4$, the basic examples are modelled on nilpotent Lie groups and fall into three types, special cases of which were also mentioned in $[15]$. We explain how to show that these are irreducible and therefore have holonomy equal to $G_2$. In §4, we analyse the holonomy of these metrics by restricting them to hypersurfaces so as to induce an $SU(3)$ structure that evolves according to equations studied by Hitchin $[21]$. Our examples provide perhaps the simplest instance of this phenomenon other than the case of nearly-Kähler manifolds. The different $SU(3)$ structures obtained in this way, an integrable one on a quotient and non-integrable ones on hypersurfaces, are naturally linked via the common 7-manifold.

In the final section, we study a more general class of solutions of our system, formulated in terms of the complex Monge-Ampère equation. This leads to both an abstract existence theorem, some special solutions, and an explicit final example.

Our assumption that the symplectic manifold $Y/S^1$ be Kähler leads to a (local) action on $Y$ not just by $S^1$ but by the torus $T^2$ (Corollary 3), though examples in §4 exhibit $T^2$ actions for which the Kähler assumption fails (see also $[11, 30]$). We hope nonetheless that similar methods will lead to the classification of metrics with holonomy $G_2$ admitting a $T^2$ action.

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1. **The First Reduction**

In this section we study general properties of a 7-dimensional Riemannian manifold $(Y, k)$ endowed with a torsion-free $G_2$ structure and an infinitesimal isometry $V$. The $G_2$ structure on $Y$ is defined by an admissible 3-form $\varphi$, which itself determines the Riemannian metric $k$, an orientation and Hodge operator $\ast$. The torsion-free condition is equivalent to the closure of both $\varphi$ and the 4-form $\ast\varphi$, and amounts to asserting that the holonomy group of $k$ is contained in $G_2$. This theory is elaborated in the standard references $[13, 19, 25, 29]$

We denote by $(N, \check{k})$ the Riemannian quotient of $(Y, k)$, so that $N$ is a 6-dimensional manifold formed from the orbits of the Killing vector field $V$. The considerations are local and hold in a suitable neighborhood of any point of $Y$ where the vector field $V$ is non-zero. The Riemannian metric $\check{k}$ is induced by the
formula

\[ k = \dot{k} + t^{-2} \eta \otimes \eta, \]

where \( t = \|V\|^{-1} = k(V, V)^{-1/2} \) and \( \eta = k(\cdot, t^2 V) \) is the 1-form dual to \( t^2 V \) so that \( \eta(V) = 1 \). In the case when \( Y \) can be realized as a principal \( S^1 \) bundle over \( N \) (the fibers being the closed orbits of \( V \)), \( \eta \) is nothing but a connection 1-form.

Since \( V \) is a Killing vector field, \( dt(V) = 0 \) and \( t \) descends to a function on \( N \).

The 2-form

\[ \sigma = i_V \varphi \]

is horizontal in the sense that \( i_V \sigma = 0 \), and since \( V \) preserves the \( G_2 \) structure we have

\[ d\sigma = d(i_V \varphi) = \mathcal{L}_V \varphi = 0, \]

\[ \mathcal{L}_V \sigma = d(i_V \sigma) = 0. \]

Thus, \( \sigma \) is closed and \( V \)-invariant, and therefore the pullback of a closed 2-form on \( N \), again denoted by \( \sigma \). (We identify functions and forms on \( N \) with their pullbacks to \( Y \) throughout, and \( \mathcal{L} \) and \( i \) stand for the Lie derivative and the interior product.) Since \( \sigma \) is non-degenerate transverse to the fibers of \( Y \), it defines a symplectic form on \( N \).

We now choose to express the forms characterizing the \( G_2 \) structure on \( Y \) as

\[ \varphi = \sigma \land \eta + t \Psi^+ \]

(1)

\[ *\varphi = \Psi^- \land \eta + \frac{1}{2} t^2 \sigma \land \sigma, \]

where

\[ \Psi^- = -i_V (*\varphi). \]

If \( \eta \) were replaced by the unit form \( t^{-1} \eta \) then both \( \sigma \) and \( \Psi^- \) would need to be multiplied by a compensatory factor of \( t \) if the left-hand sides of (1) are to remain the same. This explains why \( \sigma \land \sigma \) appears in (1) with a coefficient of (one half) \( t^2 \), and \( \Psi^+ \) appears with a coefficient of \( t \) so as to have the same norm as \( \Psi^- \). The rescaled Riemannian metric

\[ h = t^{-1} \dot{k} \]

(2)

is compatible in the sense that the skew-symmetric endomorphism \( J \) defined by

\[ h(J \cdot, \cdot) = \sigma(\cdot, \cdot) \]

(3)

is an almost-complex structure on \( N \). The triple \((h, \sigma, J)\) then defines an almost-Kähler structure on \( N \), though the qualification ‘almost’ can be deleted when \( J \) is integrable.

Just as for \( \sigma \), the 3-form \( \Psi^- \) is closed and basic (meaning \( i_V \Psi^- = 0, \mathcal{L}_V \Psi^- = 0 \)). Thus, it too is the pullback of a real form on \( N \). This has type \((3, 0) + (0, 3)\) with respect to \( J \), so that \( \Psi^- (J \cdot, J \cdot, \cdot) = -\Psi^- (\cdot, \cdot, \cdot). \) The theory of \( G_2 \) structures then implies that

\[ \Psi^+(\cdot, \cdot, \cdot) = \Psi^- (J \cdot, \cdot, \cdot), \]

(4)

and the two real 3-forms combine to define a complex form

\[ \Psi = \Psi^+ + i \Psi^- \]

(5)

of type \((3, 0)\) with respect to \( J \).
Unlike $\Psi^-$, the 3-form $\Psi^+$ is not in general closed; indeed, $d\Psi^+$ can be identified with the Nijenhuis tensor of $J$, the obstruction to the integrability of $J$:

**Lemma 1.** $\Psi^+$ is closed if and only if the almost-complex structure $J$ is integrable.

**Proof.** Since $\Psi^-$ is already closed, the exterior derivative $d\Psi^+$ of (5) is real. But if $J$ is integrable so that $(N,J)$ is a complex manifold, this real 4-form has type $(3,1)$ and therefore vanishes. Conversely, if $d\Psi^+ = d\Psi^- = 0$ then writing (5) locally as a wedge product of $(1,0)$-forms $\alpha^i$ shows that $(d\alpha^i)^{0,2} = 0$ for $i = 1, 2, 3$, and $J$ is integrable. $\square$

The almost-Kähler structure corresponds to a reduction to $U(3)$ at each point of $N$. Specification of the non-zero element (5) in the space $\Lambda^{3,0}$ is precisely what is needed to reduce the structure to $SU(3)$, but one should first rescale given that $\|\Psi^\pm\|_h$ is not in general constant. The 3-forms $\psi^\pm = t^{-1/2}\Psi^\pm$ have norm equal to 2. They are subject to the compatibility equations

(6) \[ \sigma \wedge \psi^\pm = 0, \]

and

(7) \[ \psi^+ \wedge \psi^- = \frac{2}{3} \sigma^3 = 4\text{vol}_h, \]

consistent with (3), and give rise to an $SU(3)$ structure with underlying metric $h$. Notation for the 3-forms is now consistent with that of [15]. Since $\psi^+ + i\psi^- \in \Lambda^{3,0}$, we have

(8) \[ (\gamma + iJ\gamma) \wedge (\psi^+ + i\psi^-) = 0 \]

for any 1-form $\gamma$, where $J$ acts on 1-forms by $J\gamma(\cdot) = -\gamma(J\cdot)$. Whence

(9) \[ \gamma \wedge \psi^+ = J\gamma \wedge \psi^- . \]

This last equation will be useful for calculations.

The real 3-forms $\psi^+, \psi^-$ have a common stabilizer group $SL(3,\mathbb{C})$ at each point of $N$, and either one determines the almost-complex structure $J$ [22]. From this point of view, the further reduction to $SU(3)$ is achieved by a 2-form $\sigma \in \Lambda^{1,1}$ such that $\sigma(\cdot,J\cdot) = h(\cdot,\cdot)$ is positive definite. The $SU(3)$ structure is therefore fully determined by (say) $\psi^+$ and $\sigma$; in the new notation Lemma 1 reads as

**Corollary 1.** The almost-complex structure $J$ is integrable if and only if

\[ \nabla \psi^+ = -\frac{1}{2}(df^c \log t) \otimes \psi^-, \quad \nabla \psi^- = \frac{1}{2}(df^c \log t) \otimes \psi^+, \]

where $\nabla$ denotes the Levi-Civita connection of $h$ and $df^c = Jdf$.

**Proof.** Given the above equations,

\[ d\Psi^+ = d(t^{1/2}\psi^+) = \frac{1}{2}t^{1/2}(d\log t \wedge \psi^+ - df^c \log t \wedge \psi^-), \]

but the right-hand side is zero by (8). Thus, $J$ is integrable by Lemma 1.
Conversely, if $J$ is integrable then $(h, J)$ is Kähler and $\nabla_U \psi^\pm$ is a form of type $(3,0)+(0,3)$ for any vector field $U$. Since $\psi^+$ and $\psi^-$ are mutually orthogonal and of constant norm 2,

$$\nabla \psi^+ = \gamma \otimes \psi^-, \quad \nabla \psi^- = -\gamma \otimes \psi^+$$

for some 1-form $\gamma$. Using (8) again, we obtain

$$d \Psi^+ = d t^{1/2} \psi^+ = \tfrac{1}{2} t^{1/2} (d \log t \wedge \psi^+ + 2 \gamma \wedge \psi^-)$$
$$= \frac{1}{2} t^{1/2} (\gamma \otimes \psi^- + 2J \gamma)$$

The claim follows by Lemma 1 and the injectivity of the linear map $\Lambda^1 \to \Lambda^4$ defined at each point by wedging with $\psi^+$.

**Corollary 2.** If $J$ is integrable, the Ricci form $\kappa$ of the Kähler manifold $(N, h, J)$ is given by

$$\kappa = \frac{1}{2} dd^c \log t.$$  

**Proof.** This is again an immediate consequence of the integrability criterion, which implies that the $(3,0)$-form $\Psi = \Psi^+ + i \Psi^-$ is closed. It follows that $\Psi$ is a holomorphic section of the canonical bundle $\Lambda^{3,0}$ and the Ricci form is given by

$$\kappa = i \partial \bar{\partial} \log (\|\Psi\|^2_h) = \frac{1}{2} dd^c \log (\|\Psi\|^2_h),$$

and the result follows from (7).  

In the case when $J$ is integrable, it is now possible to formulate a complete system of conditions for the $SU(3)$ structure on $N$ to arise from the quotient of a torsion-free $G_2$ structure.

**Proposition 1.** Let $(N, h, J, \sigma)$ be a Kähler manifold of real dimension 6, endowed with a compatible $SU(3)$ structure defined by a $(3,0)$-form $\psi^+ + i \psi^-$. Then this structure is obtained as the quotient of a 7-dimensional manifold $Y$ with a torsion-free $G_2$ structure by a nontrivial infinitesimal isometry $V$ if and only if

(i) $d \psi^+ = -\frac{1}{2} (d^c \log t) \wedge \psi^-$ and $d \psi^- = \frac{1}{2} (d^c \log t) \wedge \psi^+$ for a smooth positive function $t$, and

(ii) the Hamiltonian vector field $U$ on $(N, \sigma)$, corresponding to $-t$, is an infinitesimal isometry of the $SU(3)$ structure.

In this case, the corresponding 7-manifold $Y$ is locally $\mathbb{R} \times N$ and the metric $k$, infinitesimal isometry $V$ and $G_2$ invariant forms $\varphi, \ast \varphi$ are given by

$$k = th + t^{-2} \eta \otimes \eta, \quad V = \frac{\partial}{\partial y}, \quad \varphi = \sigma \wedge \eta + t^{3/2} \psi^+, \quad \ast \varphi = t^{1/2} \psi^- \wedge \eta + \tfrac{1}{2} t^2 \sigma \wedge \sigma,$$

where $y$ is a variable for the $\mathbb{R}$ factor, and $\eta = dy + \eta_N$ is a 1-form on $\mathbb{R} \times N$ for which

$$d \eta_N = -t^{1/2} (\varphi_U \psi^+).$$
Proof. Condition (i) is necessary by Corollary 1, and the equalities (9) reflect the earlier definitions of \( h, \sigma, \psi^\pm \). Moreover, \( \partial/\partial y \) is identified with the Killing field \( V \) (so that \( k(V, V) = t^{-2} \)) and \( \eta \) is the corresponding connection form. Using (i) and (8) in that order, we obtain

\[
0 = d\varphi = \sigma \wedge d\eta + \frac{3}{2} t^{1/2} dt \wedge \psi^+ - \frac{1}{2} t^{1/2} J dt \wedge \psi^-
\]

Consequently, \( (d\eta)^{1,1} = 0 \) and (10) follows from the fact that \( \iota_U \sigma = -dt \).

We now have

\[
0 = -d(d\eta) = d(t^{1/2} \iota_U \psi^+) = d(\iota_U \Psi^+) = L_U \Psi^+ = t^{1/2} L_U \psi^+.
\]

Since \( U \) is Hamiltonian, we also have

\[
L_U \sigma = 0,
\]

i.e. \( U \) is an infinitesimal isometry for the pair \( (\sigma, \psi^+) \), and therefore for the \( SU(3) \) structure (using [22]). Reversing the above arguments, one can check directly that (9) and (10) define a torsion-free \( G_2 \) structure on \( Y = \mathbb{R} \times N \). \( \square \)

Corollary 3. Under the hypothesis of Proposition 1, the horizontal lift of \( U \) to \((Y, k)\) is an infinitesimal isometry of the \( G_2 \) structure \( \varphi \), which commutes with \( V \).

Proof. This is an immediate consequence from Proposition 1. \( \square \)

Remark 1. It follows from Proposition 1 that when \( t \) is constant, \((Y, k)\) is locally the Riemannian product of a Calabi-Yau 6-manifold with \( \mathbb{R} \). In this case the holonomy group of \( h \) lies in \( SU(3) \). In general, the failure of the holonomy to reduce to \( SU(3) \) is measured by a torsion tensor

\[
\tau \in T^*_N \otimes \frac{\mathfrak{so}(6)}{\mathfrak{su}(3)}
\]

determined by \( d\sigma, d\psi^+, d\psi^- \). Whereas \( \tau \) has a total of 42 components, exactly two thirds of these will always vanish on \( N \) and \( \tau \) is determined by the remaining 14 tracefree components of \( d\psi^+ \), or 6 in the Kähler case [15].

2. A second reduction

Proposition 1 is the key ingredient for performing a further quotient via the infinitesimal isometry \( U \). To carry this out, we shall assume from now on that \( t \) is not constant and that \( J \) is integrable. Thus, \( U \) is a non-trivial infinitesimal isometry of the \( SU(3) \) structure on \( N \) determined by the pair \( (h, \psi^+) \).

Denote by \((M, J_1)\) the ‘stable’ or holomorphic quotient of \((N, J)\), defined at least locally as the complex two-dimensional manifold of holomorphic leaves of the foliation generated by \( \Xi = U - iJU \). Here we exploit the fact that \( \Xi \) is a holomorphic vector field on \((N, J)\). Since \( J \) is integrable, \( \Psi = \Psi^+ + i\Psi^- \) is a closed \((3,0)\)-form, and we set

\[
\Omega = \frac{1}{2} \iota_\Xi \Psi.
\]
Since \( r_2 \Omega = 0 \), it follows from (4) that \( \Omega \) is closed and the pullback of a holomorphic symplectic form on \( M \), again denoted by \( \Omega \). The real closed 2-forms

\[
\omega_2 = \mathfrak{Re} \Omega, \quad \omega_3 = \mathfrak{Im} \Omega
\]
on \( M \) (that pull back to \( \iota_U \Psi^+, \iota_U \Psi^- \) respectively on \( N \)) satisfy

(11) \[
\omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3, \quad \omega_2 \wedge \omega_3 = 0.
\]

The complex structure \( J_1 \) can now be determined by the formula

\[
\omega_2(\cdot, \cdot) = \omega_3(J_1 \cdot, \cdot),
\]

and equation (10) in Proposition 1 reads

(12) \[
d\eta = -\omega_2.
\]

Set \( u = \|U\|_h^{-2} \) and let \( \xi \) be the 1-form \( h \)-dual to the vector field \( uU \), so that \( \iota_U \xi = 1 \) and \( \xi = uJdt \). The 3-forms \( \Psi^\pm \) are completely determined by (10) which forces them to be the real and imaginary components of \( (\omega_2 + i\omega_3) \wedge (\xi + iJ\xi) \). Thus

(13) \[
\psi^+ = t^{-1/2}(\omega_2 \wedge \xi + u\omega_3 \wedge dt),
\]

\[
\psi^- = t^{-1/2}(\omega_3 \wedge \xi - u\omega_2 \wedge dt).
\]

They are \( U \)-invariant in accordance with (ii) in Proposition 1.

For any regular value of the momentum map \(-t\), the stable quotient \((M, J_1)\) of \((N, J)\) can be identified with the symplectic quotient \((M, \tilde{\omega}_1(t))\) of \((N, \sigma)\) generated by the vector field \( U \). In this way, we obtain the Kähler quotient \((M, g(t), \tilde{\omega}_1(t), J_1)\).

In this correspondence,

(14) \[
\sigma = \tilde{\omega}_1(t) + dt \wedge \xi,
\]

\[
h = g(t) + u^{-1} \xi \otimes \xi + u dt \otimes dt,
\]

so that the equation (7) reduces to

(15) \[
t\tilde{\omega}_1(t) \wedge \tilde{\omega}_1(t) = \frac{1}{2} u \Omega \wedge \overline{\Omega} = u\omega_2 \wedge \omega_2 = u\omega_3 \wedge \omega_3.
\]

To ease the notation, we shall below omit the explicit dependence of \( \tilde{\omega}_1 = \tilde{\omega}_1(t) \) on \( t \) except on occasions for emphasis.

We now denote by \( P \) the (locally defined) space of orbits of \( U \), so that \( N \) can be thought as an \( \mathbb{R} \) bundle over \( P \) with connection 1-form \( \xi \). Locally, \( N = \mathbb{R} \times P \), and introducing a variable \( x \) for the \( \mathbb{R} \) factor, we may write

\[
\xi = dx + \xi_P, \quad U = \frac{\partial}{\partial x}
\]

for some 1-form \( \xi_P \) on \( P \). The space of orbits of the vector field \( JU \) on \( P \) is the stable quotient of \((N, J)\), whereas the symplectic quotients of \((N, \sigma)\) are identified with the level sets of \( t \) in \( P \). Using a local description \( P = \mathbb{R}^+ \times M \) in which the \( \mathbb{R}^+ \)-factor corresponds to \( t \), we may regard \( u \) as a function on \( M \) for each value of \( t \). In these terms, we have

\[
d\xi = dp \xi_P = \alpha_M \wedge dt + \beta_M,
\]

\[
d_P u = u'dt + d_M u,
\]

\[
d\tilde{\omega}_1 = \tilde{\omega}_1' \wedge dt,
\]
where $\alpha_M = \alpha_M(t), \beta_M = \beta_M(t)$ are 1-parameter families of forms on $M$, $'$ denotes $\partial / \partial t$ and $d, d_P, d_M$ denote exterior derivative on $N, P, M$.

Differentiating the first relation gives

$$d_M \beta_M = 0, \quad \beta'_M = -d_M \alpha_M.$$  

Using the formula (14) for the symplectic form $\sigma$ yields

$$\tilde{\omega}'_1 = \beta_M,$$

and it follows that $\beta_M$ has type $(1, 1)$ relative to $J_1$. Using (13) gives

$$d\psi^+ = -\frac{1}{2}t^{-1/2}dt \wedge \omega_2 \wedge \xi + t^{-1/2}\omega_2 \wedge d\xi + t^{-1/2}d_M u \wedge \omega_3 \wedge dt$$


$$= -\frac{1}{2}t^{-1}Jdt \wedge \psi + t^{-1/2}(\omega_2 \wedge \alpha_M + d_M u \wedge \omega_3) \wedge dt,$$

since $\omega_2 \wedge \beta_M = 0$. In view of Proposition 1(i), $\omega_2 \wedge \alpha_M + d_M u \wedge \omega_3 = 0$, whence

$$\alpha_M = J_1 du = d_M^c u,$$

and everything can be expressed in terms of $\tilde{\omega}_1$ and $u$.

In summary,

**Theorem 1.** Let $(Y, \varphi)$ be a 7-manifold with a torsion-free $G_2$ structure, admitting an infinitesimal isometry $V$. Suppose that the norm of $V$ is not constant, and let $y \in Y$ be a point where $V$ does not vanish. Let $V$ be a neighborhood of $y$, such that the space of orbits of $V$ in $V$ is a manifold $N$ and suppose that the almost-Kähler structure on $N$ is in fact Kähler. Then, there exists a 4-dimensional manifold $M$ endowed with a complex structure $J_1$, a complex symplectic form $\Omega = \omega_2 + i\omega_3$, and 1-parameter families of Kähler 2-forms $\tilde{\omega}_1 = \tilde{\omega}_1(t)$ and positive functions $u = u(t)$ on $M$, satisfying the relations

$$\tilde{\omega}_1'' = -d_M d_M^c u$$

$$t\tilde{\omega}_1 \wedge \tilde{\omega}_1 = \frac{1}{2}u \Omega \wedge \overline{\Omega}.$$

On a sufficiently small neighborhood of $y$, $(\varphi, V)$ is equivariantly isometric to the torsion-free $G_2$ structure

$$\varphi = \tilde{\omega}_1 \wedge (dy + \eta_N) + dt \wedge (dx + \xi_P) \wedge (dy + \eta_N)$$

$$+ t(\omega_2 \wedge (dx + \xi_P) + u\omega_3 \wedge dt),$$

on $\mathbb{R}_t^+ \times \mathbb{R}_{x,y}^2 \times M$ endowed with the infinitesimal isometry $V = \partial / \partial y$, where

$d_M$ denotes the differential on $M$, and $d_M^c = J_1 \circ d_M$;

$t > 0$ is the variable on the $\mathbb{R}_t^+$-factor;

$(x, y)$ are standard coordinates on $\mathbb{R}_{x,y}^2 = \mathbb{R}_x \times \mathbb{R}_y$;

$\eta_N$ is a 1-form on $N = \mathbb{R}_t^+ \times \mathbb{R}_x \times M$ with $d\eta_N = -\omega_2$;

$\xi_P$ is a 1-form on $P = \mathbb{R}_t^+ \times M$ with $d\xi_P = (d_M^c u) \wedge dt + \tilde{\omega}_1'$.

**Remark 2.** By redefining the local coordinate $x$, one can assume (without loss) that $\eta_N$ is in fact a 1-form on $M$. In this case the $G_2$ structure $\varphi$ is invariant under $\frac{\partial}{\partial x}$ as well, and $\frac{\partial}{\partial x}$ is identified with the Killing vector field defined in Corollary 3.

It is not difficult to see that for generic data on $M$, the holonomy group of the $G_2$ structure (18) is equal to $G_2$. Indeed, the general theory of holonomy groups
(see e.g. [29]) implies that if the holomomy group of \((Y, k)\) were strictly less than \(G_2\), then there would exist a non-trivial parallel vector field \(X\) on \((Y, k)\) commuting with \(V = \frac{\partial}{\partial y}\); it would therefore come from an infinitesimal isometry of the \(SU(3)\) structure on \(N\) (still denoted by \(X\)), which preserves the level sets of \(t\) and commutes with \(U = \frac{\partial}{\partial x}\). The equations for \(d\eta_N\) and \(d\xi_P\) imply that there is no parallel vector field in span\(\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}\) unless \(d_M u = 0\) and \(\tilde{\omega}' = 0\) (a situation which we shall exclude below), thus showing that \(X\) is in general different from \(U\). Therefore, \(X\) must come from a (real) holomorphic vector field on \((M, J)\) which preserves the Kähler metrics \(\omega(t)\) for each \(t\). In particular, if we assume that \((M, J_1, \Omega, \tilde{\omega}_1(t), u(t))\) does not admit any infinitesimal isometry and either \(d_M u \neq 0\) or \(\tilde{\omega}' \neq 0\), then we know that the holonomy group of \((Y, \varphi)\) must equal \(G_2\).

It is important to note that the triple \((\tilde{\omega}_1(t), \omega_2, \omega_3)\) appearing in Theorem 1 does not in general constitute a hyperkähler structure. Indeed, by (17), the holomorphic section \(\Omega\) of the canonical bundle \(\Lambda^{2,0} M\) satisfies

\[
\|\Omega\|^2_g = tu^{-1},
\]

and the Ricci form \(\kappa\) of the Kähler metric \(\tilde{\omega}_1(t)\) is given by

\[
\kappa = \frac{1}{2} d_M d_M^* (\log t - \log u) = -\frac{1}{2} d_M d_M^* \log u.
\]

In the next section, we shall however explain how the simplest case does correspond to a hyperkähler situation.

### 3. Constant solutions

A careful inspection of the proof of Theorem 1 shows that the process can be inverted so as to construct a torsion-free \(G_2\) structure from a 4-manifold \(M\) with a complex symplectic structure \((J_1, \Omega)\) together with a 1-parameter family \((\tilde{\omega}_1(t), u(t))\) of Kähler forms and smooth functions satisfying (16) and (17). In this section, we shall carry out this inverse construction explicitly in the case in which \(u\) really is just a function of \(t\), so that \(d_M u = 0\).

The above assumption reduces (16) to

\[
\tilde{\omega}_1 = \omega + t\omega'
\]

for some closed \((1,1)\)-forms \(\omega, \omega'\) on \(M\). Consider the real symmetric bilinear form \(B\) defined by

\[
\alpha \wedge \beta = \frac{1}{4} B(\alpha, \beta) \Omega \wedge \overline{\Omega}
\]

on the space of 2-forms. Restricting \(B\) to the subspace \(\langle \omega, \omega' \rangle\) and diagonalizing, we may write

\[
\tilde{\omega}_1 = (p + qt)\omega_0 + (r + st)\omega_1,
\]

where

\[
\omega_0 \wedge \omega_0 = -\frac{1}{2} \varepsilon \Omega \wedge \overline{\Omega}, \quad \omega_1 \wedge \omega_1 = \frac{1}{2} \Omega \wedge \overline{\Omega}, \quad \omega_0 \wedge \omega_1 = 0,
\]

\(\varepsilon\) is 0 or 1, and \(p, q, r, s\) are constants satisfying

\[
r + st > |p + qt|,
\]
to ensure the overall positivity of \( \tilde{\omega}_1 \). From (21) and (17) we have

\[
(24) \quad u = t \left( (r + st)^2 - (p + qt)^2 \right).
\]

Note that \( \varepsilon = 0 \) in (22) if and only if \( \omega_0 = 0 \); to avoid redundancy we declare that \( p = q = 0 \) in this case.

The real symplectic forms \( \omega_1, \omega_2, \omega_3 \) satisfy the usual compatibility relations

\[
\begin{align*}
\omega_i \wedge \omega_i &= \omega_j \wedge \omega_j, \\
\omega_i \wedge \omega_j &= 0, \quad i \neq j,
\end{align*}
\]

extending (11). It is well known ([23, 29]) that they then determine a hyperkähler structure, consisting of

(i) complex structures \( J_1, J_2, J_3 \) satisfying \( J_i \circ J_j + J_j \circ J_i = -2\delta_{ij} \text{Id} \);
(ii) a Riemannian metric \( g_0 \) and associated Levi-Civita connection relative to which the \( J_i \) are all orthogonal and parallel.

When \( p^2 + q^2 > 0 \), in addition to the hyperkähler structure, the \((1,1)\) form \( \omega_0 \) defines an almost-complex structure \( I \) on \( M \), such that \( \omega_0(\cdot, \cdot) = g_0(I \cdot, \cdot) \) as in (3). It follows that \((g_0, I, \omega_0)\) is a Ricci-flat almost-Kähler metric on \( M \), compatible with the opposite orientation to the one induced on \( M \) by the hyperkähler structure \((\omega_1, \omega_2, \omega_3)\). The integrability of the almost-complex structure \( I \) is equivalent to the flatness of the metric \( g_0 \), and this is the only possibility when \( M \) is compact, see [31]. However, completely explicit local examples of 4-dimensional hyperkähler manifolds admitting a non-integrable almost-Kähler structure \( I \) are now known [5, 7, 28].

**Theorem 2.** Let \((M, g_0, \omega_1, \omega_2, \omega_3)\) be a hyperkähler 4-manifold. Let \( r, s \) be real constants with \( r + st > 0 \) for any \( t \in (a, b) \) with \( a > 0 \), and set

\[
\tilde{\omega}_1 = (r + st)\omega_1, \quad u = t(r + st)^2.
\]

Then the 3-form (18) defines a torsion-free \( G_2 \) structure on \((a, b) \times \mathbb{R}^2_{x,y} \times M'\) where \( M' \) is a suitable open subset of \( M \).

Suppose, furthermore, that \((M, g_0)\) admits an almost-Kähler structure \((\omega_0, I)\) compatible with the opposite orientation to the one induced by \((\omega_1, \omega_2, \omega_3)\). Let \( p, q, r, s \) be real constants satisfying (23) for \( t \in (a, b) \), \( a > 0 \), and set

\[
\tilde{\omega}_1 = (p + qt)\omega_0 + (r + st)\omega_1, \quad u = t \left( (r + st)^2 - (p + qt)^2 \right).
\]

Then (18) again defines a torsion-free \( G_2 \) structure on a manifold of the form \((a, b) \times \mathbb{R}^2_{x,y} \times M'\).

**Remark 3.** It suffices to take \( M' \) to be any contractible open subset of \( M \), in which case the 1-forms \( \xi_P \) and \( \eta_N \) of Theorem 1 can always be defined on \( P = \mathbb{R}^+ \times M' \) and \( N = \mathbb{R}^+ \times \mathbb{R}^4 \times M' \) respectively. However, as we shall see below, we can alternatively keep the 4-manifold \( M \) fixed and think of the 1-forms \((\xi, \eta)\) in Theorem 1 as connection forms of a principal \( T^2 \) bundle \( W \) over \( M \). Since

\[
(26) \quad (d\xi, d\eta) = (q\omega_0 + s\omega_1, -\omega_2)
\]
is the curvature of the principal connection, we obtain integrality constraints for
the cohomological classes $[\frac{1}{2\pi}(q\omega_0 + s\omega_1)]$ and $[\frac{1}{2\pi}\omega_2]$ of $M$, in the sense that they
must be contained in the image of the universal morphism $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$.

Note also that in general only two of the four real parameters $(p,q,r,s)$ are
effective in the sense that one can fix two of them, by rescaling $\varphi$ and $V$ on $Y$.
Moreover, if $q = s = 0$, (18) corresponds to the product of a Calabi-Yau 6-manifold
with $\mathbb{R}_x$, while generically the corresponding metric has holonomy equal to $G_2$ in
accordance with Remark 2.

By way of proving Theorem 2, we shall describe the reverse construction in the
case $(p,q,r,s) = (0,0,0,1)$ for simplicity, so that $u = t^3$. This is also the case in
which the family $g(t)$ consists of homothetic hyperkähler metrics on $M$. The
resulting metrics with holonomy $G_2$ were first described in [20] in the context of $T^2$
bundles. To duplicate this situation, we assume that the 2-forms $\frac{1}{2\pi}\omega_1$ and $\frac{1}{2\pi}\omega_2$
are integral. This is always true locally, though if $M$ is compact (i.e. a K3 surface
or a torus) the assumptions imply that $(M,J_3)$ is exceptional – its Picard number
is maximal, see e.g. [9]. On any exceptional K3 surface, Yau’s theorem implies the
existence of hyperkähler metrics satisfying these integrality assumptions.

Let $P$ be the total space of the principal $S^1$ bundle over the hyperkähler 4-
manifold $M$ classified by $[\frac{1}{2\pi}\omega_1]$, and $\xi$ a connection 1-form on $P$ such that $d\xi = \omega_1$
in accordance with (26). Attached to $P$ is also a principal $\mathbb{C}^*$ bundle over $M$, whose
total space $N$ is the real 6-manifold manifold $N \cong \mathbb{R}^+_t \times P$. Pulling back forms to
this new total space, we define on $N$ an almost-Hermitian structure $(h,\sigma,J)$ by

$$
\sigma = tw_1 + dt \wedge \xi
$$
$$
h = tg_0 + t^{-3}\xi \otimes \xi + t^3 dt \otimes dt
$$

and

$$
J\alpha = J_1\alpha, \quad \alpha \in \Lambda^1 M, \quad J dt = t^{-3}\xi.
$$

The 2-form $\sigma$ is closed, satisfies (3), and it is easy to check that $J$ is integrable.
Thus, $(h,\sigma,J)$ defines a Kähler structure on $N$.

Now let $U$ be vector field which is $h$-dual to $t^{-3}\xi$ so that $U$ is tangent to the fibers
of $P$ and $\xi(U) = 1$; it follows that $U$ is the generator of the natural $S^1$ action on
$N$, acting as a rotation on each fibre of $P$. Moreover, $U$ is a Hamiltonian isometry
of the Kähler structure (27) and the corresponding momentum map is $-t$.

The integrality assumption for $\omega_2$ implies that there exists a principal $S^1$ bundle
over $N$, classified by $[-\frac{1}{2\pi}\omega_2] \in H^2(N,\mathbb{Z})$. We denote by $Y$ the corresponding
7-dimensional total space and take $\eta$ to be a connection 1-form satisfying (12).

**Corollary 4.** With the above assumptions, the 3-form

$$
\varphi = tw_1 \wedge \eta + dt \wedge \xi \wedge \eta + t^4\omega_3 \wedge dt + tw_2 \wedge \xi
$$

defines a torsion-free $G_2$ structure on $Y$. The corresponding 4-form is

$$
*\varphi = \omega_3 \wedge \xi \wedge \eta - t^3\omega_2 \wedge dt \wedge \eta + t^3\omega_1 \wedge dt \wedge \xi + \frac{1}{2}t^4\omega_1 \wedge \omega_1.
$$
Remark 4. In the above situation, the Kähler structure \((h, \sigma, J)\) on \(N\) belongs to the classes of metrics recently studied in \([4]\) and \([18]\). In the terminology of the former, \((h, J)\) arises from a Hamiltonian 2-form of order 1 with one non-constant eigenvalue (equal to \(t\)) and one zero constant eigenvalue of multiplicity 2; the structure function \(F(t)\) of \(h\) is \(t^{-1}\).

Moreover,
\[
\sigma = t \omega_1 + dt \wedge \xi = d(t \xi) = d(t^4 J dt) = dd^e(t^5),
\]

using the definition of \(\xi\) after (12), and (26), (27). Not only then is \(f(t) = \frac{1}{2} \log t\) a Ricci potential for \(h\) (see Corollary 2), but \(\frac{24}{5} t^5 = 2 e^{5f}\) acts as a Kähler potential for the same metric.

We now return to the general case of Theorem 2, when \(p\) and \(q\) are not both zero. Now the hyperkähler metrics \(g(t)\) are not homothetic, though for the examples in \([5, 7, 28]\) the family \(g(t)\) is an isotopy (i.e. \(g(t)\) are all isometric to an initial hyperkähler metric, under the flow of a vector field on \(M\)).

If the cohomology classes of \(\frac{1}{2\pi}(q \omega_0 + s \omega_1)\) and \(\frac{1}{2\pi}\omega_2\) are integral, the \(G_2\) structure is defined on the product of an interval and the principal \(T^2\) bundle \(W\) over \(M\), classified by \([\frac{1}{2\pi}(q \omega_0 + s \omega_1)]\) and \([-\frac{1}{2\pi}\omega_2]\). The action of the 2-torus \(T^2\) is generated by the commuting vector fields \(U = k(\cdot, t^{-2} \xi)\) and \(V = k(\cdot, t^{-2} \eta)\), which preserve the \(G_2\) structure.

Take \(M = T^4\) to be the 4-torus with a flat hyperkähler metric. There are three cases according to the signature of the bilinear form \(B\) of (20) restricted to the 2-dimensional subspace generated by (26):

(i) \(s > q\), so \(B\) is positive definite and there is a flat metric \(g_0\) on \(T^4\) with \(d\xi, d\eta \in \Lambda^2_T\). Thus, there exists a basis \((e^i)\) of 1-forms on \(T^4\) such that \(d\xi = \omega_1\) and \(d\eta = -\omega_2\) with

\[
\omega_1 = e^{14} + e^{23}, \quad \omega_2 = e^{13} + e^{42}, \quad \omega_3 = -(e^{12} + e^{34}),
\]

reflecting the structure of the real Lie algebra \(\mathfrak{h}\) associated to the complex Heisenberg group \(H\). Then \(W = \Gamma\backslash H\) is the Iwasawa manifold \([1, 2]\).

(ii) \(s = q\) so that \(d\xi\) is a simple 2-form. In this case, we may assume that \(d\xi = e^{24}\) and either \(d\eta = e^{14} + e^{23}\) or \(d\eta = e^{14}\). Then \(W\) is a \(T^2\) bundle over \(T^4\) corresponding to one of two other nilmanifolds. The simplest holonomy \(G_2\) metric in the first case is obtained by setting \((p, q, r, s) = (-1, 1, 1, 1)\) so that \(u = 4t^2\). The resulting metric coincides with that described in Example 2 of \([15, \S 4]\) (except that \(t^2\) there has now become \(t > 0\)).

(iii) \(s < q\) and there is a basis with \(d\xi = e^{14} - e^{23}\) and \(d\eta = e^{14} + e^{23}\), so \(\langle d\xi, d\eta\rangle = \langle d\tilde{\xi}, d\tilde{\eta}\rangle\) with \(d\tilde{\xi} = e^{14}\) and \(d\tilde{\eta} = e^{23}\). In this case, we may take \(W\) to be a discrete quotient of \(H_3 \times H_3\), where \(H_3\) is the real Heisenberg group.

Example 1. To make (i) more explicit, we may take coordinates \(\lambda, \mu, \ell, m\) on \(T^4\) and fibre coordinates \(x, y\) on \(W\) so that

\[
e^5 = -\eta = dx - \lambda d\ell + \mu dm, \quad e^6 = \xi = dy - \mu d\ell - \lambda dm
\]
are the corresponding connection 1-forms. The resulting $G_2$ metric
\[ k = t^2(d\lambda^2 + d\mu^2 + dt^2 + dm^2) + t^{-2}(dx - \lambda dl + \mu dm)^2 + t^{-2}(dy - \mu dl - \lambda dm)^2 + t^4 dt^2 \]
is defined on $(0, \infty) \times W$. It is shown in [20] to arise from an $SO(5)$ invariant $G_2$ metric on the total space of $\Lambda^2_+ \to S^4$ by a contraction of the isometry group.

The explicit form above makes it easy to compute the Riemann tensor $R_{ijkl}$ of $k$. The package GRTensor (available from http://grtensor.phy.queensu.ca) was in fact used to verify that (a) the Ricci tensor $R_{ijil}$ is zero, (b) the matrix $R_{ijkl}$ with rows labelled by $i$ has rank 7 everywhere, and (c) the matrix $R_{ijkl}$ with rows labelled by $(i, j)$ has rank 14 everywhere. Point (b) confirms that $k$ is irreducible and therefore has holonomy equal to $G_2$, which is consistent with (c). The same technique can be used to analyse metrics arising from (ii) and (iii).

4. Hypersurface structures

We shall now investigate the metrics with holonomy $G_2$ constructed in Theorem 2 by restricting them to hypersurfaces on which $t = \|\eta\|_k$ is constant before taking an $S^1$ quotient. These hypersurfaces correspond to the total spaces $W$ of the $T^2$ bundles of Example 1.

We can write the 3-form (18) as
\[ \varphi = (\tilde{\omega}_1 + dt \wedge \xi) \wedge \eta + t(\omega_3 \wedge dt + \omega_2 \wedge \xi) \]
where $\xi, \eta$ are the corresponding connection 1-forms on $N, Y$ respectively. Since $\|dt\|^2_k = u^{-1}$, it follows from (2) that $\|dt\|^2_k = z^{-1}$ where $z = ut$, so
\[ z = t^2 \left( (r + st)^2 - (p + qt)^2 \right) \]
The 1-form $z^{1/2} dt$ therefore has unit norm relative to the $G_2$ metric $k$, and we may write
\[ z^{1/2} dt = d\tau \]
where
\[ \tau = \int t \left( (r + st)^2 - (p + qt)^2 \right)^{1/2} dt. \]
The important point here is that $u$ is constant as a function on $M$, and so $z$ is really just a function of $t$.

As a consequence,
\[ \varphi = \rho \wedge d\tau + \phi^+ \]
\[ *\varphi = \phi^- \wedge d\tau + \frac{1}{2} \rho \wedge \rho, \]
where
\[ \rho = z^{1/2} \omega_3 + z^{-1/2} \xi \wedge \eta, \]
\[ \phi^+ = \tilde{\omega}_1 \wedge \eta + t \omega_2 \wedge \xi, \]
\[ \phi^- = t^{-1} z^{1/2} \omega_2 \wedge \eta - t^2 z^{-1/2} \tilde{\omega}_1 \wedge \xi. \]
Restricting to an interval where the function $t \mapsto \tau$ is bijective, let $Y_\tau$ denote the hypersurface of $Y$ for which $\tau$ has the constant value ‘$\tau$’ (excusing the abuse of
notation). Whereas $\sigma, \psi^+, \psi^-$ characterize the $SU(3)$ structure on $N$, we are using $\rho, \phi^+, \phi^-$ (a lexicographic shift) for the corresponding objects on $Y_\tau$.

The closure of the forms (30) is equivalent to asserting that

$$d\phi^+ = 0, \quad d(\rho^2) = 0,$$

($\rho^2$ denotes $\rho \wedge \rho$) for every fixed value of $\tau$, and that the $SU(3)$ structures on $Y_\tau$ satisfy the equations

$$\frac{\partial \phi^+}{\partial \tau} = dp, \quad \frac{\partial}{\partial \tau}(\frac{1}{2}\rho^2) = -d\phi^-.$$

An $SU(3)$ structure satisfying (32) is called half-integrable or half-flat in the formalism of [15].

To verify (33) directly in our situation, first observe that

$$d(\xi \wedge \eta) = (q\omega_0 + s\omega_1) \wedge \eta + \xi \wedge \omega_2$$

(recall (26)), and

$$\phi^+ = (p\omega_0 + r\omega_1) \wedge \eta + t d(\xi \wedge \eta)$$

$$\frac{1}{2}\rho^2 = \xi \wedge \eta \wedge \omega_3 + \frac{1}{2}z\omega_3 \wedge \omega_3.$$

These equations explain the significance of the coordinates $(t, z)$ – the above forms are linear in $t$ and $z$ respectively. The reader may now check that (33) implies both (28) and (29); a constant of integration may be absorbed into the term $r^2 - p^2$.

Hitchin discovered that (33) leads to a Hamiltonian system in the symplectic vector space $V \times V^*$ where $V$ is the space of exact 3-forms on the compact manifold $Y_\tau$, whose dual $V^*$ can be identified with the space of exact 4-forms. The Hamiltonian function $H$ is derived from integrating volume forms determined algebraically by $\phi^+$ and $\rho^2$, and this also enables $\phi^-$ to be determined from $\phi^+$ in (31). Elements of $V$ represent deformations of $\phi^+$ in a fixed cohomology class, and those of $V^*$ deformations of $\rho^2$. Given a solution of (32) for $\tau = a$, a solution of (33) can then be found on some interval $(a, b)$ [21]. This approach also underlies some of the newly constructed metrics (such as [12]) with reduced holonomy.

The function $H$ is already implicit in our calculations above, which may be summarized in the following way:

**Proposition 2.** The solution (31) can be expressed in the form $H = 0$ where the function $H = 2t \left( (r + st)^2 - (p + qt)^2 \right)^{1/2} - 2z^{1/2}$ satisfies

$$\frac{dt}{d\tau} = -\frac{\partial H}{\partial z}, \quad \frac{dz}{d\tau} = \frac{\partial H}{\partial t}.$$

It is an important consequence of the Kähler assumption that, for each choice $(p, q, r, s) \in \mathbb{R}^4$, there is only one valid solution curve in the $(t, z)$ plane.

**Example 2.** In the light of Proposition 2, a variant of (31) and Example 1 is provided by setting

$$\rho = \pm z^{1/2}(e^{12} + e^{34}) + z^{-1/2}e^{56},$$

$$\phi^+ = \phi_0^+ + t d(e^{56}),$$

$$\phi^- = \phi_0^- \pm t(e^5 \wedge de^5 + e^6 \wedge de^6),$$
with

\[ z = (t^2 - \frac{1}{2} H)^2. \]

We assume that \( z^{1/2} > 0 \), so that the orientation \( \rho^3 \) remains fixed. The notation is consistent with that of [1], with \( de^5 = e^{13} + e^{42} \) and \( de^6 = e^{14} + e^{23} \). Any value of the constant \( H \) gives a valid solution and so a holonomy \( G_2 \) metric on the product \((a, b) \times W\) of some interval with the Iwasawa manifold \( W \), though \( \phi_0^\pm \) and the signs must be chosen to ensure compatibility (essentially (7)) and positive definiteness of the resulting metric.

The Figure plots the quartic curves (35) in the \((t, z)\) plane for various values of \( H \); two pass through each point because of the sign ambiguity implicit in the definition of \( H \).

(i) \( H = 2 \) gives \( z = (1 - t^2)^2 \), including the bell-shaped segment. To satisfy (33) for \(|t| < 1\) we need to choose the plus sign in \( \phi^- \). We may then define the 3-forms \( \phi_0^\pm \) by setting

\[ \phi^+ + i\phi^- = i((e^5 + ie^6) + t(e^5 - ie^6)) \wedge d(e^5 + ie^6). \]

It follows that \( \phi_0^+ + i\phi_0^- \) is a form of type \((3, 0)\) relative to the standard complex structure on \( W \), and taking \( z = 1 \) and the plus sign in \( \rho \) determines a compatible Hermitian metric via (3). As one flows away from the point \((0, 1)\), the almost-complex structure induced on the hypersurface becomes non-integrable; \( \rho, \phi^\pm \) degenerate simultaneously when one reaches \( t = \pm 1 \). Furthermore, we may take \( \tau = \frac{1}{3} t^3 + t \), a cubic equation with solution

\[ t = \alpha^{1/3} - \alpha^{-1/3}, \quad 2\alpha = 3\tau + (9\tau^2 - 4)^{1/2}, \]

in contrast with the Kähler scheme in which \( \tau = c + \frac{1}{2} rt^2 + \frac{1}{3} st^3 \) in the case \( p = q = 0 \).
(ii) $H = -2$ gives $z = (t^2 + 1)^2$ (the curve above and touching the bell). This requires the minus sign in $\phi^-$ and provides a different deformation of the standard Hermitian structure on $W$. Indeed, $\phi^+_0 + i\phi^-_0$ is modified by the addition of a form of type $(1,2)$ rather than $(2,1)$, and the resulting almost-complex structure is undefined when $|t|$ reaches 1. However, $\rho$ remains non-degenerate for all $t$, and this corresponds to a different singular behaviour of the $G_2$ metric.

(iii) $H = 0$ and $z = t^4$ (with the flattened base) requires $\phi^+_0 = 0$ and the minus signs in $\phi^-, \rho$, reproducing exactly the solution of Example 1. The almost-complex structure induced on $W$ by $\phi^+$ is constant and was first singled out for study in [2], where it is called $J_3$. The 2-form $\rho$ degenerates only for $t = 0$, and the resulting metric has the advantage of being ‘half-complete’.

We conclude this section by providing an explanation of why the system (33) often reduces to one variable. Let $\phi^+$ be an invariant closed 3-form with stabilizer $SL(3,\mathbb{C})$ on either of the four nilmanifolds $\Gamma \backslash H$ described just before Example 1. Let $\phi^-$ be the unique 3-form for which $\phi^+ + i\phi^-$ has type $(3,0)$ relative to the almost-complex structure $J$ determined by $\phi^+$. Let $\ker d$ denote the space of invariant closed 1-forms (equivalently, the kernel of the natural mapping $d : h^* \to \Lambda^2 h^*$).

**Lemma 2.** In the above situation, $d\psi_+ = 0$ implies that $J(\ker d) = \ker d$. 
This can be proved by generalizing an argument in the proof of [26, Theorem 1.1],
which draws the same conclusion if \( J \) is integrable. In view of the structure equa-
tions for the Lie algebra \( \mathfrak{h} \), the condition that \( J \) leave \( \ker d \) invariant is equivalent
to asserting that \( d\phi^- \) belong to the 1-dimensional space
\[
\langle e^{1234\lambda} \rangle = \bigwedge^4(\ker d),
\]
but it is this fact that simplifies the equations.

5. Solutions varying on \( M \)

As another ramification of the evolution equations for \( (\tilde{\omega}_1 = \tilde{\omega}_1(t), u = u(t)) \), we
consider the case when \((M, g_0, \omega_1, \omega_2, \omega_3)\) is a hyperkähler manifold, and
\[
\tilde{\omega}_1 = \omega_1 - \frac{1}{2}d_M d_M^* G,
\]
\[
2u = G''
\]
for a smooth function \( G \) on \((a, b) \times M\), where \( ' \) continues to denote \( \partial/\partial t \). We
assume here that \( \tilde{\omega}_1 \) is a positive definite (1,1)-form on \((M, J_1)\) for each fixed \( t \) in
the interval \((a, b)\). Whilst (16) is automatically satisfied, (17) becomes
\[
2t \mathcal{M}(G) = G''
\]
where \( \mathcal{M} \) denotes the complex Monge-Ampère operator on \((M, g_0, J_1)\), defined by
\[
\left( \omega_1 - \frac{1}{2}d_M d_M^* f \right)^{\wedge 2} = \mathcal{M}(f) \omega_1 \wedge \omega_1,
\]
for all \( f \in C^\infty(M) \). Note that the 1-form \( \xi_P \) of Theorem 1 is automatically defined
on \( P = \mathbb{R}_t^+ \times M \) by
\[
\xi_P = -\frac{1}{2}d_M^*(G').
\]

Since the hyperkähler metric \( g_0 \) is Ricci-flat and the \( \omega_i \)'s are parallel 2-forms,
there exists a real-analytic structure on \( M \), compatible with \((g_0, J_1, \omega_1, \omega_2, \omega_3)\).
Thus, applying the Cauchy-Kowalewski theorem, one obtains a general existence
result.

**Corollary 5.** Let \((M, g_0, \omega_1, \omega_2, \omega_3)\) be a hyperkähler, real-analytic 4-manifold and
\( J_1 \) be the Kähler structure compatible with \( \omega_1 \). Suppose that \( G^0, G^1 \) are real-analytic
functions on \( M \) with \( \omega_1 - \frac{1}{2}d_M d_M^* G^0 \) positive definite with respect to \( J_1 \). Then there
exist a real number \( a > 0 \) and a real-analytic solution \( G(t, \cdot) \) of (36) defined on
\((0, a) \times M \) with \( G(0) = G^0, \ G'(0) = G^1 \), and such that \( \omega_1 - \frac{1}{2}d_M d_M^* G \) is positive
definite for any \( t \in (0, a) \). Thus, \( \tilde{\omega}_1 = \omega_1 - \frac{1}{2}d_M d_M^* G \) and \( u = \frac{1}{2}G'' \) define, via
Theorem 1, a torsion-free \( G_2 \) structure on a manifold of the form \((0, a) \times \mathbb{R}_{x,y}^2 \times M' \) where \( M' \) is a suitable open subset of \( M \).

In the above construction \( M' \) should be taken so as to solve \( d\eta_N = -\omega_2 \) for a
1-form \( \eta_N \) on \( N = \mathbb{R}_t^+ \times \mathbb{R}_x \times M' \) (see Theorem 1 and (37)). Alternatively,
we may assume that the cohomology class \([\frac{1}{2\pi} \omega_2]\) of \( M \) is integral and \( \eta \) is a principal
connection of the principal \( S^1 \) bundle \( Q \) over \( M \), classified by \([-\frac{1}{2\pi} \omega_2]\). In this
case Corollary 5 produces examples of torsion-free \( G_2 \) structures with a \( \mathbb{R} \times S^1 \) symmetry on \( Y = (0, a) \times \mathbb{R}_x \times Q \). One has no control over the real number \( a \).
It is tempting to spot some special solutions to (36), by reducing the problem to a linear (elliptic) equation. This can be done by assuming that for each \( t > 0 \) the function \( G \) generates a complex Monge-Ampère foliation on \((M,J_1)\) (see e.g. [10]), meaning that

\[
d_M d_M^c G \wedge d_M d_M^c G = 0
\]

and \( d_M d_M^c G \) has constant rank. (The integral curves of \( d_M d_M^c G \) then foliate \( M \) by complex submanifolds.) The point is that in this case we have

\[
\mathcal{M}(G) = 1 + \frac{1}{2} \Delta G,
\]

where \( \Delta \) is the Riemannian Laplacian of the hyperkähler metric \( g_0 \).

The above situation appears in particular when \((M,J_1)\) admits a holomorphic \( \mathbb{C} \) action and we look for equivariant solutions of (36), or when \((M,J_1)\) admits a holomorphic fibration \( p : M \to \mathbb{C} \) over a complex curve \( \mathbb{C} \) and we look for solutions of the form \( G \circ p \) where for each \( t \), \( G \) is a function on \( \mathbb{C} \).

Consider finally the flat hyperkähler metric \( g_0 \) on \((M,J_1) = \mathbb{C}^2 \cong \mathbb{R}^4\), determined by the 2-forms

\[
\omega_1 = \frac{i}{2} (dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \quad \omega_2 = \Re (dz_1 \wedge d\overline{z}_2), \quad \omega_3 = \Im (dz_1 \wedge d\overline{z}_2),
\]

where \( z_1 = \lambda + i\mu \) and \( z_2 = \ell + im \) are the canonical coordinates of \( \mathbb{C}^2 \). Letting \( G(t,z_1) \) be a function on \( \mathbb{C}^2 \) which does not depend on \( z_2 \), the equation (36) reduces to

\[
G'' + t \left( \frac{\partial^2 G}{\partial \lambda^2} + \frac{\partial^2 G}{\partial \mu^2} \right) = 2t.
\]

Separable solutions are given by \( G(t,\lambda,\mu) = \frac{1}{3} t^3 + H(\lambda,\mu)K(t) \), where

\[
\frac{\partial^2 H}{\partial \lambda^2} + \frac{\partial^2 H}{\partial \mu^2} + cH = 0,
\]

\( c \) is a constant and \( K \) a solution of the Airy equation \( K'' = ctK \) (equivalently, \( L = K^{-1}K' \) satisfies the Riccati equation \( L' + L^2 = ct \)).

**Example 3.** Taking \( H \) to be periodic in \( \lambda,\mu \) (which requires \( c > 0 \)) yields solutions of (36) defined on \((0,a) \times \mathbb{T}^4\). One such example is obtained by taking \( H = \sin \lambda \) (so \( c = 1 \)), and

\[
K = \text{Ai}(t) = \frac{1}{3} t^{1/2} (J_{1/3}(\zeta) + J_{-1/3}(\zeta)), \quad \zeta = \frac{2}{3} i t^{3/2}.
\]

Setting \( f = 1 + \text{Ai}(t) \sin \lambda \), the resulting \( G_2 \) metric

\[
t(f d\lambda^2 + f d\mu^2 + d\ell^2 + dm^2) + f^{-1} (dx - \text{Ai}'(t) \cos \lambda d\mu)^2 + t^{-2} (dy - \lambda d\ell + \mu dm)^2 + t^2 f dt^2
\]

is Ricci-flat and irreducible. Since \( \text{Ai}(t) \to 0 \) as \( t \to \pm \infty \), the above metric is asymptotic to a constant solution with \( u = t \) and holonomy equal to \( SU(3) \) (see Remark 3). However, the above construction can be easily modified to provide explicit deformations of the non-trivial metrics constructed in §3.
References

[1] E. Abbena, S. Garbiero and S. Salamon, Almost Hermitian geometry of 6-dimensional nilmanifolds, Ann. Scuola Norm. Sup. Pisa (4) 30 (2001), 147–170.
[2] E. Abbena, S. Garbiero and S. Salamon, Hermitian geometry on the Iwasawa manifold, Boll. Un. Mat. Ital. 11-B (1997), 231–249.
[3] B. Acharya and E. Witten, Chiral fermions from manifolds of $G_2$ holonomy, available at arXiv:hep-th/0109152.
[4] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, Hamiltonian 2-forms in Kähler geometry, I, available at arXiv:math.DG/0202280.
[5] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, The geometry of weakly self-dual Kähler surfaces, Compositio Math. 135 (2003), 279–322.
[6] V. Apostolov and P. Gauduchon, The Riemannian Goldberg-Sachs Theorem, Internat. J. Math. 8 (1997), 421–439.
[7] J. Armstrong, An ansatz for Almost-Kähler, Einstein 4-manifolds, J. reine angew. Math. 542 (2002), 53–84.
[8] M. Atiyah and E. Witten, M-theory dynamics on a manifold of $G_2$ holonomy, Adv. Theor. Math. Phys. 6 (2003), 1–106.
[9] W. Barth, C. Peters and A. Van de Ven, Compact complex surfaces, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.
[10] E. Bedford and M. Kalka, Foliations and the complex Monge-Ampère equation, Commun. Pure Appl. Math. 30 (1977), 543–571.
[11] K. Behrndt, G. Dall’Agata, D. Lüst and S. Mahapatra, Intersecting 6-branes from new 7-manifolds with $G_2$-holonomy, J. High Energy Phys. 8 (2002), no. 27, 24 pp.
[12] A. Brandhuber, J. Gomis, S. S. Gubser and S. Gukov, Gauge theory at large $N$ and new $G_2$ holonomy metrics, Nuclear Phys. B 611 (2001), 179–204.
[13] R. L. Bryant, Metrics with exceptional holonomy, Annals of Math. 126 (1987), 525–576.
[14] R. L. Bryant and S. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989), 829–850.
[15] S. Chiossi and S. Salamon, The intrinsic torsion of SU(3) and $G_2$ structures, in ‘Differential Geometry, Valencia 2001’, pp. 115–133, World Sci. Publishing, River Edge, NJ, 2002.
[16] M. Cvetic, G. W. Gibbons, H. Lü and C. N. Pope, Almost special holonomy in type IIA and M-theory, Nuclear Phys. B 638 (2002), 186–206.
[17] A. Derdziński, Self-dual Kähler manifolds and Einstein manifolds of dimension four, Compositio Math. 49 (1983), 405–433.
[18] A. Derdziński and G. Maschler, Local classification of conformally-Einstein Kähler metrics in higher dimensions, Proc. London Math. Soc. (3) 87 (2003), 779–819.
[19] M. Fernández and A. Gray, Riemannian manifolds with structure group $G_2$, Ann. Math. Pura Appl. 132 (1982), 19–45.
[20] G. W. Gibbons, H. Lü, C. N. Pope and K. S. Stelle, Supersymmetric domain walls from metrics of special holonomy, Nuclear Phys. B 623 (2002), 3–46.
[21] N. J. Hitchin, Stable forms and special metrics, in ‘Global Differential Geometry: The Mathematical Legacy of Alfred Gray’, Contemp. Math. Vol. 288, pp. 70–89, Amer. Math. Soc., 2001.
[22] N. J. Hitchin, The geometry of three-forms in six dimensions, J. Differ. Geom. 55 (2000), 547–576.
[23] N. J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. 55 (1989), 59–126.
[24] D. D. Joyce, $U(1)$-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ and special Lagrangian fibrations, Turkish J. Math. 27 (2003), 99–114.
[25] D. D. Joyce, Compact manifolds with special holonomy, Oxford University Press, 2000.
[26] G. Ketsetzis and S. Salamon, Complex structures on the Iwasawa manifold, Advances in Geometry 4 (2004), 165–179.
[27] C. LeBrun, Einstein metrics on complex surfaces, in ‘Geometry and Physics’ (Aarhus, 1995), Lect. Notes Pure Appl. Math. Vol. 184, pp. 167–176, Dekker, New York, 1997.
[28] P. Nurowski and M. Przanowski, *A four-dimensional example of Ricci flat metric admitting almost Kähler non-Kähler structure*, Classical Quant. Grav. 16 (1999), L9–L13.

[29] S. Salamon, Riemannian geometry and holonomy groups, *Pitman Research Notes in Mathematics* Vol. 201, Longman, 1989.

[30] O. P. Santillan, *A construction of G2 holonomy spaces with torus symmetry*, Nuclear Phys. B 660 (2003), 169–193.

[31] K. Sekigawa, *On some compact Einstein almost-Kähler manifolds*, J. Math. Soc. Japan 36 (1987), 677–684.

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