Vortex-lattice structures in rotating Bose-Fermi superfluid mixtures

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Abstract

The system of Bose-Fermi superfluid mixture offers a playground to explore rich macroscopic quantum phenomena. In a recent experiment of Yao et al. [Phys. Rev. Lett. \textbf{117}, 145301 (2016)], $^{41}\text{K}^{6}\text{Li}$ superfluid mixture is implemented. Coupled quantized vortices are generated via rotating the superfluid mixture, and a few unconventional behaviors on the formations of vortex numbers are observed, which can be traced to boson-fermion interactions. Here we provide a theoretical insight into the unconventional behaviors observed in the experiment. To this end, the orbital-free density functional theory is hired, and its utility is validated by making comparison of the numerical results and a microscopic theory for vortex lattices in strongly interacting Fermi superfluids alone. We also predict interesting phenomena which can be readily explored experimentally, including the novel structures of bosonic-fermionic vortices, and attractive interactions between the bosonic and fermionic vortices.

Keywords: Rotating Bose-Fermi superfluid mixture; Miscible-immiscible transition; Bosonic-fermionic vortices;

1. Introduction

One of the characteristic properties of superfluids is that they respond to rotation by forming quantized vortices \cite{1}. Systems of ultracold quantum gases stand out as they provide an ideal platform to study quantized vortices with precise controls and broad tunability \cite{2, 3}. Since the experimental
realization of Bose-Einstein condensation (BEC) in atomic clouds, triangular vortex lattice being a conventional structure has been created in rotating Bose superfluid [4, 5], and widely investigated theoretically [6, 7, 8].

Like bosons, ultracold fermions also offer insights into macroscopic quantum phenomena. A tunable Feshbach resonance provides a unique opportunity to access pairing and superfluidity in the crossover between a Bardeen-Cooper-Schrieffer (BCS) state of largely overlapping pairs of fermions and a BEC state of fermion dimers [9, 10]. On the cusp of this BCS-BEC crossover, there exists a strongly interacting regime—the so-called unitary limit [11], which becomes the subject of numerous experiments [12, 13, 14, 15, 16, 17]. Convincing proof of superfluidity along the BCS-BEC crossover is obtained through the observation of quantized vortices [18, 19]. In contrast to weakly interacting Bose superfluids well understood in terms of the Gross-Pitaevskii (GP) theory [6, 7, 8], interacting fermions requires a complete microscopic approach which is more complicated and computationally intensive [20, 21].

To account for Pauli exclusion, one should resort to the orbital-based density functional theory (DFT), such as Bogoliubov-de Gennes (BdG) equations [22, 23, 24, 25, 26] or superfluid local density approximation (SLDA) [27, 28, 29, 30, 31, 32]. Both of the two methods require to solve a huge system of non-linear equations in a self-consistent way [22, 23, 31, 27, 24, 25, 26, 28, 29, 30, 32]. To reveal the properties and configurations of vortex lattices under experimental conditions, a spatial coarse graining of the BdG equations was performed [23]. Later on, an asymmetric SLDA was applied to study the formations of vortex lattices in spin-imbalanced unitary Fermi gases harmonically confined in two-dimensional (2D) traps [31].

While the long-sought goal of simultaneous superfluidity in mixtures of $^4$He-$^3$He still remains elusive due to strong interisotope interactions, Bose-Fermi superfluidity has been realized in atomic gas mixtures of $^7$Li-$^6$Li [33, 34], $^{41}$K-$^6$Li [35] and $^{174}$Yb-$^6$Li [36]. In the recent experiment of $^{41}$K-$^6$Li atoms [35], the conclusive evidence of the double superfluidity is first provided by producing coupled vortex lattices. By carrying out a series of vortex-number measurements, the significant effects of boson-fermion interactions on the formation and decay of the coupled vortices are studied. However, there is no theoretical interpretation thus far.

One can envisage that the vortex lattices in Bose-Fermi superfluid mixtures can possess novel properties compared to a single superfluid. Apart from the intercomponent interactions as in two-component BECs, the distinct quantum statistical properties of the two atom species give rise to far
richer vortex-lattice structures. The intercomponent interaction drives the two-component BECs \[37, 38, 39, 40\] going through transition from miscible to immiscible phases \[41\]. For equal masses and equal intracomponent interactions, vortex lattices in the miscible phase have triangular, rectangular, square, and double-core structures \[42, 43\], while the lattices in the immiscible phase are featured by stripes and interwoven vortex sheets \[43, 44, 45, 46\]. Asymmetric systems where the two components have different masses and different intracomponent interactions, support coreless vortex lattices in the miscible phase, and rotating droplets and giant skyrmions in the immiscible phase \[46, 47, 48\]. In addition, unequal masses can result in homogeneous infinite vortex lattices having some notable geometries \[50, 49\]. These studies have been focused on the somewhat less computationally intensive 2D realm.

It is thus natural to question what new features come about \[51, 52, 53, 54\], if we couple a rotating, weakly interacting Bose superfluid and a rotating, strongly interacting Fermi superfluid as in the experiment. Within the lowest Landau level approximation \[52\], the structures of vortex lattices in the BCS-BEC crossover is determined by minimizing the total free energy of Bose-Fermi mixtures. The transition of the vortex-core structure of the fermionic component on the BCS side as a function of repulsive Bose-Fermi interactions is investigated by solving coupled BdG and GP equations self-consistently \[53\].

Our work aims at studying vortex lattices in an experimentally relevant trapped rotating Bose-Fermi mixture \[35\] in the unitary limit, and unravelling the structures of the coupled vortices through a miscible-immiscible transition. To this end, our approach relies on the so-called orbital-free DFT \[55, 56, 57, 58, 59\], which for fermions is not written in terms of single-particle orbitals as the orbital-based DFT aforementioned, but only in terms of a single macroscopic wavefunction \[59\]. The orbital-free DFT provides an attractive computationally practical method for simulating huge systems under realistic conditions. In the framework of the orbital-free DFT, the rotational properties of Bose-Fermi superfluid mixtures confined in a 2D harmonic trap \[51\] and in a tight toroidal trap \[54\] have been recently investigated.

We first apply the orbital-free DFT to study vortex lattices in strongly interacting Fermi superfluids alone. The obtained vortex numbers and critical frequencies in the BCS-BEC crossover are consistent with those from the BdG equations \[23\]. Subsequently, we carry out extensive numerical simulations on the structures of vortex lattices as functions of repulsive boson-fermion interactions and rotation frequencies. It is found that the effects
of the repulsive interspecies interaction on the vortex lattices of the Bose and Fermi superfluids are very different, which may provide a theoretical insight into the experiment. Various structures of bosonic-fermionic vortices in phase-separated states under various rotation frequencies are further revealed. This study is of particular interest under the experimentally accessible conditions as it can present a comparison between the experiment and the numerical results of the simplified DFT, as well as illustrating unique features of Bose-Fermi superfluid mixture.

The paper is organized as follows. In Sec. 2 we present the theoretical formalism for a rotating, harmonically trapped Bose-Fermi superfluid mixture, and the numerical methods. First in Sec. 3 we study the properties of vortex lattices in strongly interacting Fermi superfluids alone numerically and analytically. Next in Sec. 4 we study the effects of repulsive boson-fermion interactions on the vortex-lattice structures. Finally, we conclude in Sec. 5 with a summary of our results and an outlook to future research.

2. theoretical model and numerical approach

We consider a mixture of a single-component bosonic superfluid and a fermionic superfluid paring between two spin components, which rotates around the $z$ axis with the same rotation frequency $\Omega$ for both superfluids. We consider the experimental situation of a large number of particles in realistic geometries. To find out the equilibrium vortex states in the rotating Bose-Fermi superfluid mixtures, we use a relatively computationally simple model, i.e. the orbital-free DFT \cite{60, 61, 62, 63, 64}. In terms of two complex-valued order parameters $\Psi_b$ for condensed bosons \cite{1} and $\Psi_p$ for condensed fermionic pairs \cite{65}, the energy functional associated with Bose-Fermi mixtures in a rotating frame of reference is written, within the mean-field approximation, as

$$E = \int \mathcal{E}[\Psi_b, \Psi_p] \, dr,$$

where the energy density is

$$\mathcal{E}[\Psi_b, \Psi_p] = \frac{\hbar^2}{2m_b} |\nabla \Psi_b|^2 + V_b(r)|\Psi_b|^2 + \frac{1}{2} g_{bb}|\Psi_b|^4 - \Psi_b^\dagger \Omega L_z \Psi_b$$

$$+ \frac{\hbar^2}{4m_f}|\nabla \Psi_p|^2 + 2V_f(r)|\Psi_p|^2 + \frac{6}{5} \epsilon_f|\Psi_p|^2 \sigma(\eta) - \Psi_p^\dagger \Omega L_z \Psi_p$$

$$+ g_{bf}|\Psi_b|^2|\Psi_p|^2.$$
The following calculations are performed in 3D formalism with \( r = \{x, y, z\} \). Here \( m_b(m_f) \) is the mass of a bosonic (fermionic) atom, and \( L_z = \hbar(y\partial/\partial x - x\partial/\partial y) \) is the \( z \) component of the angular momentum. The disk-shaped trapping potentials acting on bosons and fermions are given by \( V_{b,f}(r) = m_{b,f} [\omega_{b,f}^2(x^2 + y^2) + \omega_{b,f}^2 z^2]/2 \). The total numbers of bosonic and fermionic atoms are determined, by \( N_b = \int d\mathbf{r} \, n_b = \int d\mathbf{r} \, |\Psi_b|^2 \) and \( N_f = \int d\mathbf{r} \, n_f = 2 \int d\mathbf{r} \, |\Psi_f|^2 \), respectively.

Although it does not incorporate fermionic degrees of freedom, the orbital-free approach in Eq. (1) has computational advantage, and very recently has been used to be as a benchmark for experimental observations [12, 13, 16, 17]. Furthermore, by comparing with the fully microscopic theory [29, 66, 30, 67], the orbital-free DFT has been proved to be a good description for static properties and low-frequency linear dynamics, in which pair-breaking effects play a negligible role [66].

The quantities \( g_{bb} = 4\pi \hbar^2 a_{bb}/m_b \) and \( g_{bf} = 4\pi \hbar^2 a_{bf}(m_b + m_f)/(m_b m_f) \) are the bosonic intraspecies and the boson-fermion interspecies interaction constants [66]. In contrast, the strength of the two-spin fermionic interaction in the BCS-BEC crossover is characterized by the equation of state \( \mu(n_f) = \partial[n_f^2 \epsilon_f \sigma(\eta)]/\partial n_f = \epsilon_f [\sigma(\eta) - (\eta/5) \partial \sigma(\eta)/\partial \eta] \), depending on the Fermi energy \( \epsilon_f = (\hbar k_f)^2/(2m_f) \) and the interaction parameter \( \eta = 1/(k_f a_f) \), with the Fermi wave vector \( k_f = (3\pi^2 n_f)^{1/3} \) and scattering length of fermions \( a_f \). \( \sigma(\eta) \) is the fitting function that is a Padé-type parametrization of the experimental data [68] for a two-component Fermi gas at zero temperature in the BCS-BEC crossover. To obtain a further analysis, we treat the equation of state by a polytropic approximation [69, 70]

\[
\mu(n_f) = \mu^0 \left( \frac{n_f}{n_0} \right)^\gamma
\]

\[
\gamma = \gamma(\eta^0) = \left( \frac{n_f}{n_0} \right) \frac{\partial \mu}{\partial n_f} |_{\eta=\eta^0}
\]

\[
= \frac{2}{5} \sigma(\eta^0) - \frac{2\eta^0}{5} \sigma'(\eta^0) + \frac{4\eta^{02}}{5} \sigma''(\eta^0)
\]

\[
\sigma(\eta) - \frac{\eta^0}{5} \sigma'(\eta^0)
\]

where \( \gamma \) is an effective polytropic index. The reference particle number density \( n_0 = (2m_f \epsilon_f^0)^{3/2}/(3\pi^2 \hbar^3) \) is taken to be the density of the noninteracting Fermi gas at trap center, with the Fermi energy \( \epsilon_f^0 = (\hbar k_f^0)^2/(2m_f) = \hbar(3N_f \omega_{\perp}^2 \omega_z)^{1/3} \), and the reference chemical potential \( \mu^0 = \epsilon_f^0 [\sigma(\eta^0) - (\eta^0/5) \partial \sigma(\eta^0)/\partial \eta^0] \) with \( \eta^0 = 1/(k_f^0 a_f) \) and \( k_f^0 = (3\pi^2 n_0)^{1/3} \) [70].

Minimizing the energy functional with respect to variations of \( \Psi_b \) and \( \Psi_f \), and introducing the chemical potentials \( \mu_b \) and \( \mu_f \) to fix the particle numbers
\( N_b \) and \( N_f \), yields the following equations

\[
\begin{align*}
\mu_b \Psi_b &= \left[ -\frac{\hbar^2 \nabla^2}{2m_b} + V_b + g_b |\Psi_b|^2 + g_{bf} |\Psi_p|^2 - \Omega L_z \right] \Psi_b \quad (3a) \\
\mu_p \Psi_p &= \left[ -\frac{\hbar^2 \nabla^2}{4m_f} + 2V_f + 2\mu(n_f) + g_{bf} |\Psi_b|^2 - \tilde{\Omega} L_z \right] \Psi_p, \quad (3b)
\end{align*}
\]

with \( \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \). We introduce the energy, time and length scales, given by \( \hbar \omega_{b,\perp}, \omega_{b,\perp}^{-1} \) and \( \ell_{b,\perp} = \sqrt{\hbar/(m_b \omega_{b,\perp})} \), respectively. The order parameters are normalized by the atomic numbers in 3D as \( \Psi_b \rightarrow \sqrt{N_b} \psi_b/\ell_{b,\perp}^{3/2} \) and \( \Psi_p \rightarrow \sqrt{N_f} \psi_p/\ell_{b,\perp}^{3/2} \), and \( \int \text{d}r |\psi_{b,f}|^2 = 1 \). To find the stationary states of Eqs. (3), we use the imaginary time propagation of the time-dependent version of Eqs. (3) after sufficient convergence \( 43, 44 \). The time-dependent version of Eqs.(3) takes the following dimensionless form

\[
\begin{align*}
i \frac{\partial \Psi_b}{\partial t} &= \left[ -\frac{1}{2} \nabla^2 + \tilde{V}_b + u_b |\Psi_b|^2 + u_{fb} |\Psi_p|^2 - \tilde{\Omega} \tilde{L}_z \right] \Psi_b, \quad (4a) \\
i \frac{\partial \Psi_p}{\partial t} &= \left[ -\frac{\alpha}{2} \nabla^2 + \tilde{V}_f + u_f |\Psi_p|^2 + u_{bf} |\Psi_b|^2 - \tilde{\Omega} \tilde{L}_z \right] \Psi_p, \quad (4b)
\end{align*}
\]

where the mass ratio is defined by \( \alpha = m_b/(2m_f) \). Here the rotation frequency is \( \tilde{\Omega} = \Omega/\omega_{b,\perp} \) and \( \tilde{L}_z = L_z/\hbar \), the trapping potentials for bosons and fermions are \( \tilde{V}_b = (x^2 + y^2 + \omega_{b,\perp}^2 z^2/\omega_{b,\perp}^2)/2 \) and \( \tilde{V}_f = [\omega_{f,\perp}^2 (x^2 + y^2) + \omega_{f,z}^2 z^2]/(2\alpha \omega_{b,\perp}^2) \), and the dimensionless parameters for the intra- and interspecies interactions are \( u_{fb} = 4\pi N_b a_b/\ell_{b,\perp} \), \( u_f = 2\mu^3 (N_f/2m_b \ell_{b,\perp}^3)^{\gamma}/(\hbar \omega_{b,\perp}) \), \( u_{fb} = 2\pi N_f m_b a_{bf}/(m_{bf} \ell_{b,\perp}) \), and \( u_{bf} = 4\pi N_b m_b a_{bf}/(m_{bf} \ell_{b,\perp}) \), respectively.

We consider the parameters of the experiment \( 32 \), in which vortex lattices are created in a rotating \( ^{41}\text{K}-^{6}\text{Li} \) superfluid mixture with an imbalanced mass \( \alpha = 3.4 \). The bosons and fermions feel different radial frequencies \( \omega_{b,\perp} = 2\pi \times 20\text{Hz} \) (\( \omega_{f,\perp} = 2\pi \times 40\text{Hz} \)) and axial frequencies \( \omega_{b,z} = 2\pi \times 85\text{Hz} \) (\( \omega_{f,z} = 2\pi \times 237\text{Hz} \)) of the disk-shaped trapping potentials. In order to enhance the contrast of vortices, the particle numbers are chosen \( N_b = 1 \times 10^4 \) and \( N_f = 2 \times 10^5 \), respectively, both to be an order of magnitude smaller than the experiment. The scattering lengths for bosons is \( a_b = 60.5 a_0 \) with \( a_0 \) being the Bohr radius, and the boson-fermion scattering length \( a_{bf} \) varies positive over a large range to realize a miscibility-immiscibility transition.

Within this framework, we consider the problem by solving two coupled nonlinear Schrödinger equations \( 1 \), with the polytropic equation of state
Eq. (2a) treating exactly several important regimes of interacting Fermi superfluids. Therefore, one can study the properties of a Bose-Fermi mixture through the BCS-BEC crossover in a unified way, i.e. from a mixture of a weakly interacting Bose superfluid and a strongly interacting Fermi superfluid to a weakly interacting two-component BECs. For instance, at the unitary point ($\eta = 0$), the equation of state is characterized by the parameters $\mu_0 = 0.412 e_f^0$ and $\gamma = 2/3$, which takes a universal density dependence $\mu^0/n_0^\gamma = 0.412\hbar^2(3\pi^2)^{2/3}/(2m_f)$ [11]. In the deep BEC regime ($\eta^0 = 6, a_f = 1168 a_0$) characterized by $\mu_0 = 0.01 e_f^0$ and $\gamma = 1.01$, the equation of state in terms of the above parameters takes $2\mu^0/n_0^\gamma = 0.66 g_M$, slightly different from the mean-filed interaction $g_M = 4\pi\hbar^2 a_M/(2m_f)$ of a BEC with the molecule-molecule scattering length $a_M = 0.6 a_f$. It is because that the used equation of state includes the beyond-mean-field correction [68]. In the BEC limit ($\eta^0 = 16(70), a_M = 263(61) a_0$), the equation of state characterized by $\mu_0 = 0.004(0.0009) e_f^0$ and $\gamma = 1.003(1.0003)$ can reproduce the expected mean-field interaction $2\mu^0/n_0^\gamma = 0.9(1.0)g_M$ well.

For initial conditions, we consider the ones with a single vortex aligned with $z$ axis at the center, modulated by a random phase at different space points [71]

$$\Psi_{b,0} = \left(\frac{\omega_b}{\pi \omega_{b\perp}}\right)^{1/4} x + iy \exp\left[-\frac{x^2 + y^2}{2\omega_{b\perp}} - \frac{\omega_b z^2}{2\omega_{b\perp}} + 2\pi i \Re(x, y)\right]$$  \hspace{1cm} (5a)

$$\Psi_{p,0} = \left(\frac{\omega_f}{\alpha \pi \omega_{b\perp}}\right)^{1/4} x + iy \exp\left[-\frac{\omega_f (x^2 + y^2)}{2\alpha \omega_{b\perp}} - \frac{\omega_f z^2}{2\alpha \omega_{b\perp}} + 2\pi i \Re(x, y)\right].$$  \hspace{1cm} (5b)

$\Re(x, y)$ is a randomly generated number distributed uniformly between 0 and 1. The included random phase term breaks the underlying symmetries and prevents the simulation from getting stuck in any metastable states. A combination of angular harmonics with the randomly generated numbers has been successfully used to generate vortex lattices in dipolar two-component BECs [72]. The numerical method is based on the split-step Crank-Nicolson scheme [73]. Vortices arrange themselves inside the trap, and the system is closer to the equilibrium configuration for long times. After each time step of computations, the wavefunctions for both superfluids are renormalized to one. Imaginary-time propagation is conducted until the desired precision is reached for the energy or chemical potential [71]. The numerical simulations are conducted in a grid with a maximum of $600 \times 600 \times 48$ points along the $x$, $y$ and $z$ directions respectively, with a spatial step of 0.1 in both $x$ and $y$.
directions, and 0.5 in the z direction, and a time step of 0.001. To speed up the calculation, the programs are parallelized using Open Multi-processing (OpenMP) interface \[74\] run on supercomputing system.

3. vortex lattices in strongly interacting Fermi superfluids

Vortex lattices in a strongly interacting Fermi superfluid have been studied extensively by using the orbital-based DFT \[22, 23, 31\]. In particular, by solving non-uniform BdG equations in a local phase density approximation, vortex lattices in the BCS-BEC crossover in a real 3D trap have been investigated \[23\], addressing a comparison with experimental data. In this section, we first study the formation of vortex lattices in Fermi superfluids alone in the BCS-BEC crossover. Because of a similar configuration, it is of interest and necessary to compare our results with the full microscopic theory \[23\]. In Fig. 1 we present our results for the vortex lattices in the unitary Fermi superfluid (\(\eta^0 = 0\)) as a function of the rotation frequency \(\Omega\). It should be noticed that the numerical simulations are actually carried out in enough larger cuboid computational domains to avoid the boundary effects, and the extra boundary areas are cut for a better presentation. One can find the generated vortices emerging in the cross-sectional density profiles \(n_f(x, y, z = 0)\), verified them in corresponding cross-sectional phases, i.e. \(\arctan[\text{Im}\Psi_p(x, y, z = 0)/\text{Re}\Psi_p(x, y, z = 0)]\), by the varying value from 0 to \(2\pi\).

In regard to the properties of vortex lattices in a rotating strongly interacting Fermi superfluid, one is the dependence of the vortex number \(N_v\) on the rotation frequency \(\Omega\). Fig. 2 shows \(N_v\) versus \(\Omega\) in the different interaction regimes, and the case of the weakly interacting Bose superfluid is also plotted in the lower inset of Fig. 2 for comparison. It is seen that the number of vortices follows a linear dependence on \(\Omega\) at the low rotation frequency, and the rate of increase is larger and larger as \(\Omega\) increases. At the large rotation frequency, when entering into the strongly interacting regimes (i.e. \(\eta^0 = 0\) and 1), however, the increase of the vortex number is suppressed displaying a nonlinear dependence. At the very high frequencies, a rotating superfluid mimics rigid body rotation with the average curl of the velocity field \(\nabla \times \vec{v} = 2\vec{\Omega}\). The areal density of vortices \(n_v\) obeys the Feynman’s relation \[75\] applied to a uniform vortex distribution

\[n_v = \frac{2m_f \Omega}{\pi \hbar},\]  \(6\)
Figure 1: The formations of vortex lattices in a strongly interacting Fermi superfluid as the rotation frequency from (a) $\Omega = 0.05 \omega_{f\perp}$, to (b) $0.15 \omega_{f\perp}$, to (c) $0.7 \omega_{f\perp}$. Shown in panels (a1)-(c1) are the corresponding cross-sectional densities (in units of $10^{-4}$) at $z = 0$ plane, and (a2)-(c2) are the corresponding cross-sectional phases.

which has a factor 2 for a Fermi superfluid compared with that of a Bose superfluid with the same value of atomic mass. In the upper inset of Fig. 2 for $\Omega/\omega_{f\perp} = 0.95$, we shows the ratio between the number of vortices $N_v(r)$ obtained numerically within a circle with the radius $r = \sqrt{x^2 + y^2}$, and the corresponding results $N_f(r) = n_v \pi r^2$ expected from Feynman’s theorem. One can find that they are in good agreement near the trap center, but the density inhomogeneity from the numerical results reduces the vortex density away from the center [23]. Up to the boundary of the superfluid ($r/R_{f\perp} = 1$), the ratio $\frac{N_v}{N_f} = \frac{270}{502} = 0.54$ in the unitary is smaller than $\frac{N_v}{N_f} = \frac{100}{125} = 0.8$ in the BEC regime. It suggests that the inhomogeneity correction in the strongly interaction regime is larger, and explains the increase suppression and nonlinear trend of the vortex numbers at the large rotation frequency.

In Ref. [23] based on the BdG calculations, the vortex number is obtained $N_v = \{85, 112, 137\}$ for $\Omega = \{0.4, 0.6, 0.8\} \omega_{f\perp}$ in the unitary limit, which is smaller than our results $N_v = \{108, 175, 236\}$ as shown in Fig. 2. In addition, the maximum of $N_v$ for a fixed $\Omega$ is at the BEC side of the crossover, shifting
The number of vortices \( N_v \) as a function of the rotation frequency \( \Omega \) for the different interaction regimes: the unitary limit \( (\eta^0 = 0) \), the BEC side \( (\eta^0 = 1) \), and the BEC regime \( (\eta^0 = 6) \). Lower inset: the behavior of the vortex number in the weakly interacting Bose superfluid. Upper inset: the ratio between the number of vortices \( N_v \) obtained numerically and the prediction \( N_f \) from Feynman’s theorem as a function of the radius \( r = \sqrt{x^2 + y^2} \) for \( \Omega/\omega_{f\perp} = 0.95 \).

Towards the BEC side as \( \Omega \) increases. This is in contrast to our results showing a monotonic increase of \( N_v \) from the BEC regime to the unitary limit. Such discrepancies are attributed to that the microscopic theory can account for the filling of fermionic vortex cores with a normal component \[31, 27, 78\], and the density depletion at the vortex core is not completed around the unitary limit even at zero-temperature. Our formulation only in terms of the order parameter misses the normal state. As a result, a vanishing order parameter yields a vanishing density \[67, 76\], which magnifies the contrast of the vortex core and the vortex number. In the BEC regime \( (\eta^0 = 1) \) where the density depletion of the vortex core is also completed from the microscopic theory, therefore our results \( N_v = \{53, 92\} \) for \( \Omega = \{0.4, 0.6\} \omega_{f\perp} \) are in better agreement with the microscopic theory \( N_v = \{60, 100\} \[23\].

The other property is the critical frequency \( \Omega_c \) of the vortex nucleation in the strongly interacting Fermi superfluids \[77, 78\]. The thermodynamic critical frequency can be calculated as \( \Omega_c = (E_1 - E_0)/\langle L_z \rangle \) analytically, where \( E_1 \) and \( E_0 \) are the energy of the single-vortex state and the vortex-free energy, respectively, and \( \langle L_z \rangle \) is the mean angular momentum of a vortex state. We have previously obtained the extra energy for per unit length of a uniform Fermi system \[76\], with a single quantum of circulation lying long
the axis of a cylinder of radius $R_{f\perp}$. Dividing the extra energy by the angular momentum yields the characteristic frequency of the first vortex nucleation

$$\Omega_c = \frac{\hbar}{2m_f R_{f\perp}^2} \ln(1.464 \gamma^\frac{5}{2} \frac{R_{f\perp}}{\xi_f}), \quad (7)$$

with the Thomas-Fermi (TF) radius $R_{f\perp} = \sqrt{\frac{2\mu_f}{(2m_f\omega_{f\perp})^3} u_f \Gamma(\frac{1}{\gamma} + \frac{5}{2})/(\omega_{f\perp}\Gamma(\frac{1}{\gamma} + 1)\pi^\frac{3}{2})}$ of the Fermi superfluid. The coherence length $\xi_f = \hbar/\sqrt{4m_f \mu_f}$ evaluated by the central density. They are both determined by the chemical potential $\mu_f = \hbar \omega_{b\perp} [\omega_{f\perp}^2/(2\alpha \omega_{f\perp})]^{\frac{3}{2}} u_f \Gamma(\frac{1}{\gamma})$, $\xi_b = \hbar/\sqrt{2m_b \mu_b}$, and $\mu_b = \hbar \omega_{b\perp} [15u_{bf}\omega_{b\perp}/(16\sqrt{2} \pi \omega_{b\perp})]^{\frac{5}{2}}$.

For our chosen parameters, the critical frequencies from Eq. (7) for a uniform rotating system are given by $\Omega_c = \{0.016, 0.03, 0.06\} \omega_{f\perp}$ for the unitary limit ($\eta^0 = 0$), the BEC side ($\eta^0 = 1$) and the BEC regime ($\eta^0 = 6$), respectively, which are smaller than the numerical results $\Omega_c = \{0.0325, 0.05, 0.1\} \omega_{f\perp}$ (see Fig. 2). This is because that the nonuniform density in the axisymmetric trap reduces the total angular momentum relative to that of a uniform system. In the case of BEC, by taking into account inhomogeneity [1] the critical frequency is $\Omega_c = 5\hbar/(2m_b R_{b\perp}^2) \ln(0.671 R_{b\perp}/\xi_b) = 0.42 \omega_{b\perp}$, larger than $\Omega_c = 0.2 \omega_{b\perp}$ from Eq. (7), and in good agreement with the numerical result $\Omega_c = 0.5 \omega_{b\perp}$ (see the lower inset of Fig. 2). The monotonic increase of the critical frequency from the unitary limit to the BEC side of the crossover is also found from the BdG calculations. The critical frequency $\Omega_c = 0.069 \omega_{f\perp}$ in the unitary limit is reported [23], which is a little larger than ours traced to the same reason as before.

4. The impact of boson-fermion interaction

In this section, we next investigate the effects of repulsive boson-fermion interactions on the rotating Bose-Fermi superfluid mixtures, focusing on the unitary limit ($\eta^0 = 0$) in the experimentally accessible parameters. To evaluate the strength of the repulsive boson-fermion interaction, in Fig. 3 we present the density distributions of nonrotating Bose and Fermi superfluids ($\Omega = 0$) for various boson-fermion interactions. As a reference, we introduce a dimensionless scattering length $\tilde{a}_{bf} = a_{bf}/(60.9a_0)$ scaled by the case...
studied in the experiment [35]. A relevant characteristic of coupled mixtures is the miscibility of the components [79]. In the miscible phase, the densities of two components overlap with each other; whereas, they get spatially separated in immiscible phase. For a weak repulsive interspecies interaction ($\tilde{a}_{bf} = 0.05$) in Fig. 3(a), it is shown that the density distributions of the Bose and Fermi superfluids are almost identical to the uncoupled densities ($\tilde{a}_{bf} = 0$) by the dashed lines. Increasing the repulsive interaction ($\tilde{a}_{bf} = 1$) in Fig. 3(b) which is for the experimental case, the density of the Fermi superfluid reduces at the center pronouncedly, but the mixture is still overlapping in the miscible phase. Further increasing the interaction ($\tilde{a}_{bf} = 6$) in Fig. 3(e) to induce immiscibility transition, there is a shell structured geometry, in which the bosonic atoms occupy the small central region as the core-part, and the density of the fermions is zero at the center forming the large shell-part.

A parameter to measure the spatial overlapping between densities of the
components can be given by
\[ \Lambda = \int d\mathbf{r} \sqrt{|\Psi_b|^2 |\Psi_p|^2}, \] (8)
where the order parameters \( \Psi_b \) and \( \Psi_p \) are both normalized to one. Even without interspecies interaction, the overlap parameter in our case is merely up to \( \Lambda_0 = 0.135 \). In the presence of the interaction \( \tilde{a}_{bf} = 1 \), we then obtain a smaller value of \( \Lambda = 0.129 \) indicating partial overlapping for the ratio \( \tilde{\Lambda} = \Lambda / \Lambda_0 = 0.95 \). Fig. 4 shows the scaled overlap parameter \( \tilde{\Lambda} \) as a function of \( \tilde{a}_{bf} \). It is clearly that the system undergoes a miscible-immiscible transition in the parameter range of \( \tilde{a}_{bf} = 3 \sim 4 \) corresponding to \( \Lambda = 0.1 \sim 0.07 \).

The energy density of a homogenously mixed phase of Bose-Fermi superfluid mixture can be obtained from Eq. (1) by neglecting the kinetic energy terms, that is \( \mathcal{E}_r = 3\hbar^2 (3\pi^2)^{\frac{2}{3}} n_f^{\frac{2}{3}} \sigma(\eta)/(10m_f) + g_{bb} n_b^2/2 + g_{bf} n_b n_f/2 \). A condition for miscibility is that the Hessian matrix of \( \mathcal{E}_r \) is positive semidefinite \([80]\), i.e. \( (\partial^2 \mathcal{E}_r / \partial n_b^2)(\partial^2 \mathcal{E}_r / \partial n_f^2) - (\partial^2 \mathcal{E}_r / \partial n_b \partial n_f)^2 > 0 \). The solution of this inequality gives the parameter regime of boson-fermion scattering length
\[ \tilde{a}_{bf}^2 < \frac{3m_{bf}^2 a_{bb} (3\pi^2)^{\frac{2}{3}}}{10\pi m_b m_f} \frac{\partial^2}{\partial n_f^2} [n_f^{\frac{2}{3}} \sigma(\eta)], \] (9)
where the homogeneous mixed phase is energetically stable. In the unitary limit $\sigma(\eta)$ is taken as the universal factor $\xi = 0.41$ \cite{68}, and $n_f$ is approximated by the Fermi density at the trap center without the interspecies interaction. The critical value $\tilde{a}_{bf} = 3.4$ for miscibility of the Bose and Fermi superfluids is predicted analytically, which is denoted by the vertical dashed line. Fig. 4 also compares the behavior of the miscibility as a function of the boson-fermion interaction under different rotations. One can find that the miscibility of the mixtures enhances due to the centrifugal force and increases as the rotation frequency. In addition, as the interspecies interaction increases, the discrepancies of the miscibility for different rotation frequencies in the phase-separated regime are narrowed.

Here we start from a very small value of the rotation frequency, and examine the effect of boson-fermion interactions on the vortex nucleation. We find that the boson-fermion interaction decreases the critical frequency of the Fermi superfluid from $\Omega_c = 0.0325\omega_{f\perp}$ to $0.031\omega_{f\perp}$ slightly. In the presence of a very weak boson-fermion interaction $\tilde{a}_{bf} = 0.05$, the first vortex appears at $\Omega = 0.031\omega_{f\perp}$.

Fig. 5 illustrates the case of slowly rotating $\Omega = 0.05\omega_{f\perp}$. Fig. 5(a) corresponds to a very weak boson-fermion strength $\tilde{a}_{bf} = 0.05$. By comparing with Fig. 1(a) only two vortices in the single Fermi superfluid, one can see that one extra vortex emerges instantaneously in Fig. 5(a3) even in the presence of the weak interaction. With further increasing $\tilde{a}_{bf} = 1$ in Fig. 5(b) and $\tilde{a}_{bf} = 3$ in Fig. 5(c) that are both in the miscible regime, the vortex number keeps invariant but the vortex configurations evolves. As the central density becomes more depressed, the three vortices are attracted spirally to the center areas, which can be seen more clearly from the corresponding phases in Fig. 5(a4)-(c4). In this experimental relevant system, the trap frequencies for bosons and fermions are not equal, i.e. $\omega_{f\perp}/\omega_{b\perp} = 2$. The rotation frequency is given by $\Omega = 0.05\omega_{f\perp} = 0.1\omega_{b\perp}$, which is much smaller than the critical frequency $0.5\omega_{b\perp}$ of the Bose superfluid. The Bose superfluid is thus without vortex nucleation, and the density profiles shown in Fig. 5(a1)-(c1) are like nonrotation and the corresponding phases shown in Fig. 5(a2)-(c2). The numerical results are consistent with the experimental observation that the boson-fermion interaction leads to unexpected vortex formation compared to a single Fermi superfluid and increases the vortex number.

Instead of the cross-sectional densities shown in Fig. 5(a)-(c), Fig. 5(d) and 5(e) present the integrated densities of the cases in the immiscible regime.
Figure 5: Vortex-lattice structures in the slowly rotating Bose-Fermi superfluid mixtures ($\Omega = 0.05\omega_f\perp$), for (a) $\tilde{a}_{bf} = 0.05$, (b) 1, and (c) 3 in the miscible regime. Panels (a1)-(c1) show the cross-sectional densities (in units of $10^{-2}$) of the Bose superfluid at $z = 0$ plane, and (a3)-(c3) show the cross-sectional densities (in units of $10^{-4}$) of the Fermi superfluid. In contrast, (d) $\tilde{a}_{bf} = 4$ and (e) 6 correspond to the immiscible phases. (d1) and (e1) show the integrated densities $\int n_b(x, y, z)\,dz$ (in units of $10^{-2}$) of the Bose superfluid, (d3) and (e3) are the integrated densities $\int n_f(x, y, z)\,dz$ (in units of $10^{-4}$) of the Fermi superfluid. Panels (a2)-(e2) and (a4)-(e4) correspond to the cross-sectional phases of the Bose and Fermi superfluids at $z = 0$ plane, respectively.

It is because that in the immiscible state the cross-sectional density of the Fermi superfluid at the center is zero and the vortex is invisible. For $\tilde{a}_{bf} = 4$, we observe the formation of the coreless vortex in Fig. 5(d). One of vortices in the Fermi superfluid enters into the overlapping center area (Fig. 5(d3)), which creates a density peak in the Bose superfluid seen as a dark spot in Fig. 5(d1) (denoted by the arrow). The coreless vortex was first experimentally created in two-component BECs [37]. In terms of a pseudospin representation [40], an axisymmetric vortex is interpreted as skyrmions, in which the vortex core of one component is filled with the other nonrotating component, while a nonaxisymmetric one is regarded as meron pairs, in which each component has one off-centered vortex [81]. More recently, the
static and dynamical properties of massive vortices (coreless vortex) have been studied by means of a massive point-vortex model \[82, 83\]. For a larger value $\tilde{a} = 6$ in Fig. 5(e) where the Bose and Fermi superfluids are well separated, however, the coreless vortex disappears since the vortex is repelled from the overlapping center area of the Fermi superfluid (Fig. 5(c3)). Our results indicate that a coreless vortex state with distinct quantum statistics can be observed in the slowly rotating Bose-Fermi superfluid mixtures around the miscible-immiscible transition.

Increasing the rotation frequency above the threshold for the appearance of the first vortex in the Bose superfluid, we can study the interplay between the vortex lattices emerging in two different superfluids. We find that for a very weak interspecies interaction ($\tilde{a}_{bf} = 0.05$) the critical frequency of the Bose superfluid decreases $\Omega_c = 0.45\omega_{b\perp}$ compared with $0.5\omega_{b\perp}$ for a single one.

In Fig. 6, we present the structural variations of vortex lattices in a moderately rotating Bose-Fermi superfluid mixture ($\Omega = 0.35\omega_{f\perp}$) through the miscible-immiscible transition. In the absence of the interaction $\tilde{a}_{bf} = 0$ in Fig. 6(a), the Bose and Fermi superfluids behave independently and the vortex lattices in two superfluids are uncoupled. With the onset of the very weak interaction $\tilde{a}_{bf} = 0.05$ in Fig. 6(b), one extra vortex is found to appear immediately in the Bose superfluid, and the vortex number increases from 3 to 4 distinctly in Fig. 6(b1). The interspecies interaction also affects the arrangement of the vortices in the Fermi superfluid, which distribute spirally from the center more regularly (Fig. 6(b3)). Continually turning $\tilde{a}_{bf} = 2$ up in Fig. 6(c1) still in the miscible regime, we observe the annihilation of the vortex lattice with one of the vortices disappearing, and the vortex number is the same as that in Fig. 6(a1) for no interaction.

In the experiment a few unconventional behaviors are observed \[35\]. The number and lifetime of vortices in the Fermi superfluid are greater than that in a single superfluid. In contrast, the effects of the boson-fermion interaction are less pronounced on the Bose superfluid. However, our numerical results in Fig. 6(a)-(c) indicate a different behavior of the vortex number of the Bose superfluid. By comparing with Fig. 4 of Ref. \[35\], more vortices in the Bose superfluid are observed, and the effects are measured at $\tilde{a}_{bf} = 1$ smaller than 2 where we find the decrease of the vortex number. The particle number of the bosons in the experiment is an order of magnitude larger than ours. Therefore, more particles result in more vortices and enhance the miscible-immiscible transition that is $\tilde{a}_{bf}^* = 2.8$ from Eq. (9).
Figure 6: Vortex-lattice structures in the moderately rotating Bose-Fermi superfluid mixtures ($\Omega = 0.35\omega_{f\perp}$), for (a) $\tilde{a}_{bf} = 0$, (b) 0.05, and (c) 2 in the miscible regime, and (d) $\tilde{a}_{bf} = 4$ and (e) 6 in the immiscible regime. As the same as in Fig. 4, panels (a1)-(c1) and (d1)-(e1) show the cross-sectional and integrated densities of the Bose superfluid, respectively, and correspondingly (a3)-(c3) and (d3)-(e3) present the Fermi superfluid. Panels (a2)-(e2) and (a4)-(e4) are the corresponding cross-sectional phases.

For the case of $\tilde{a}_{bf} = 4$ in Fig. 6(d), the Bose-Fermi superfluid mixture in the immiscible state and no vortices are left to be visible in the Bose superfluid (see Fig. 6(d1) and the corresponding phase Fig. 6(d2)). Similar to Fig. 5(d), the vortex-lattice structure is also featured by a coreless vortex. But differently from its disappearance in Fig. 5(e), the coreless vortex can still exist in Fig. 6(e) for $\tilde{a}_{bf} = 6$. The reason is that as the rotation frequency increases, more and more vortices entering in Fig. 6(e3) prevent the vortex leaving from the overlapping area.

In Fig. 7, we finally study the coupling between the two superfluids containing more vortices for a fast rotation $\Omega = 0.45\omega_{f\perp}$. A very weak boson-fermion interaction ($\tilde{a}_{bf} = 0.05$) in Fig. 7(b) can also lead to one extra vortex emerging in the Bose superfluid, compared with the single one in Fig. 7(a). As the boson-fermion interaction increases in Fig. 7(c) and 7(d), similar to
Figure 7: The same case as Fig. 6 but for a larger rotation $\Omega = 0.45 \omega_f \perp$, and (a) $\tilde{a}_{bf} = 0$, (b) 0.05, (c) 3.0, (d) 4.0, and (e) 6.0.

Fig. 6(c) and 6(d), the Bose superfluids are shrinking with a decrease of the vortex number, reducing the space of the overlap with the Fermi superfluids and achieving the lower energy. Such decrease of the vortex number, but a slower one, can be also observed for a mixture with the number of fermionic pairs decreased to be the same as the bosons (which are not shown here).

In the phase-separated state ($\tilde{a}_{bf} = 6$), instead of the coreless vortex in Fig. 6(e), the vortex lattice is featured by a new structure as shown in Fig. 7(e). It is shown in Fig. 7(e1) that one of the vortices remains in the Bose superfluid, which can be verified by the corresponding phase in Fig. 7(e2). This vortex is surrounded by three nearest-neighbor vortices in the Fermi superfluid (Fig. 7(e3)), which locate at three vertices of the resultant triangular boundary of the Bose superfluid.

Furthermore, the 3D visualization of the new structure is illustrated in Fig. 8(b). The vortex lines in the Fermi superfluid pass through the Bose superfluid, bending towards to the vortex of the Bose superfluid featured by a hole at the center. It indicates an attractive interaction between the
bosonic and fermionic vortices. The bending of the straight vortex lines in turn result in the triangular shape of the Bose superfluid. For comparison, we also plot the 3D visualization of the coreless vortex in Fig. 8(a), which is characterized by a straight vortex line of the Fermi superfluid across the vortex-free Bose superfluid without any deformation.

5. Conclusions

We have revealed the equilibrium states of the rotating oblate Bose-Fermi superfluid mixtures in the unitary limit by varying the rotation frequency and the repulsive boson-fermion interaction. In contrast to the well-known rotating two-component BECs, the $^{41}$K-$^6$Li mixture realized experimentally is a highly asymmetric system. The ratio $0.07/26 = 0.0027$ between the healing length and radius of the unitary Fermi superfluid is much smaller than $0.28/3.9 = 0.08$ of the weakly interacting Bose superfluid. The critical frequency of the Fermi superfluid $\Omega_c = 0.0325\omega_{f\perp}$ is also much smaller than $0.25\omega_{f\perp}$ of the Bose superfluid. Therefore, the interplay between the bosonic and fermionic vortices with the significant differences may lead to unique behaviors in quantum instabilities.

In the very slowly rotating regimes below the critical frequency of the Bose superfluid, it is clearly that one more vortex can emerge in the Fermi superfluid immediately, when a very small repulsive boson-fermion interaction is
turned on. The vortex number keeps invariant as the repulsive boson-fermion interaction increases, but the vortex configuration is affected that the vortices move spirally to the overlapping center area. Nearby the miscible-immiscible transition one axisymmetric coreless vortex forms, featuring by the vortex core of the Fermi superfluid filling with the nonrotating Bose superfluid. In the phase-separated state, the coreless vortex disappears. When entering into the moderate rotating regime, for a very small repulsive interaction one extra vortex also emerges in the Bose superfluid as the Fermi superfluid. As the repulsive boson-fermion interaction further increases, however, differently from the Fermi superfluids, the Bose superfluids are shrinking with a decrease of the vortex number to achieve the lower energy. In the phase-separated state the vortex-lattice structure is finally featured by the coreless vortex. For the larger rotation frequency, the vortex lattice in the phase-separated state is instead characterized by a new pattern, which is a single vortex remaining in the Bose superfluid surrounded by three vortices in the Fermi superfluid. An attractive interaction between the bosonic and fermionic vortices is illustrated through the 3D visualization. This study not only sheds light on unique phenomena in rotating Bose-Fermi superfluid mixtures, but also provides a theoretical insight into the unconventional behaviors of the vortex numbers arising from the interplay between Bose and Fermi superfluidity observed experimentally.

It should be noted that vortex lattices in strongly interacting Fermi superfluids cannot be resolved in situ, but detected after ramping the magnetic field to the BEC regime and performing time-flight imaging \cite{19,35}. The target value of the magnetic field, rate of the ramping, as well as expansion time after the release may profoundly impact the configurations of vortex lattices in the final state. In a future work it is deserved to study the effects on vortex-lattice expansions by means of orbital-free DFT including dissipation \cite{30,84,85}.

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