Commensurability oscillations in the SAW induced acousto-electric effect in a 2DEG.

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We study the acousto-electric (AE) effect generated by surface acoustic waves (SAW) in a high mobility 2D electron gas (2DEG) with isotropic and especially small-angle impurity scattering. In both cases the acousto-electric effect exhibits Weiss oscillations periodic in $B^{-1}$ due to the commensurability of the SAW period with the size of the cyclotron orbit and resonances at the SAW frequency $\omega = k_0\omega_c$, multiple of the cyclotron frequency. We describe how oscillations in the acousto-electric effect are damped in low fields where $\omega_c\tau < 1$ (with the time scale $\tau$, dependent on the type of scattering) and find its non-oscillatory part which remains finite to the lowest fields.

Due to a finite wave number carried by surface acoustical waves (SAWs) their application enables one to access the properties of low-dimensional electron systems that cannot be studied with the standard microwave absorption techniques. The observation of magneto-oscillations in the shift of SAW velocity caused by its interaction with the 2D electrons in the vicinity of filling factor $\nu = 1/2$ and its comparison with Weiss geometrical oscillations for electrons has enabled Willett et al. to establish the existence of a composite fermion Fermi surface. Also, the additional length scale in the system permits transitions otherwise forbidden by Kohn’s theorem, thus making possible detection of cyclotron transitions in a gas of composite quasiparticles.

In this Communication we extend the analysis of the phenomenon of geometrical commensurability onto the acousto-electric (drag) effect which has been studied experimentally in semiconductor structures by Esslinger et al. and Shilton et al. We show that by measuring magneto-oscillations and study the frequency dependence of the DC electric field induced in a 2DEG by a propagating SAW, one can access the same information about the Fermi gas as was previously studied in absorption and SAW propagation experiments. In the present publication we also compare and contrast two types of high-mobility structures: with isotropic and small-angle impurity scattering.

Here we use an approach recently applied to the studies of another DC effect produced by a dynamical acoustic wave field, the SAW-induced magneto-resistance. We investigate the AE effect in the linear order in the SAW power and find the parametric dependences of the steady-state electric field $E_{AE}$ generated by the SAW in the direction of its propagation, $E_{AE} = \hat{q}E_{AE}(qR_c, \omega_c\tau, \omega/\omega_c)$, where $\hat{q}$ is the SAW wave vector, $R_c$ and $\omega_c$ are the electron cyclotron radius and frequency, $\omega$ the SAW frequency, and $\tau$ is the effective electron scattering time crucially dependent on the type of impurity scattering. In the high-field limit, $\omega_c\tau \gg 1$, we describe Weiss oscillations of the AE effect as a function of $qR_c$. In addition, for low-density structures (as well as for heavy-mass carriers, such as composite fermions) we predict resonances in $E_{AE}$ at frequencies $\omega = k_0\omega_c$. For a small magnetic field (although large enough to ensure that $v_F B > E_{AE}$), we show that while commensurability oscillations are damped, there is a finite field-independent contribution to the AE drag.

Our theory consists of the analysis of the Boltzmann equation,

$$\dot{\hat{f}}[\mathbf{p},\mathbf{x},t] = \hat{C}[\mathbf{p},\mathbf{x},t],$$

$$\hat{C} = \partial_t + \omega_c R_c \cos \varphi \partial_\varphi + \omega_c \partial_\varphi + eE\hat{P},$$

$$\hat{P} = v \cos \varphi \partial_\varphi - \frac{\sin \varphi \partial_\varphi}{p},$$

where $\hat{C}$ is the collision integral and the momentum dependent part in $\hat{C}$ and $\hat{P}$ is written in terms of the electron kinetic energy $\epsilon = p^2/2m$ and angle $\varphi$, characterizing the direction of electron propagation with respect to the direction of propagation of the SAW. Here, $v$ is the electron velocity, $\omega_c$ and $R_c$ are the cyclotron frequency and radius respectively, $E = l e^i(\omega t - qx)$ is the longitudinally polarized SAW field, screened by the 2DEG.

Using the ($\epsilon, \varphi$) parametrization of momentum space, we expand the distribution function $f(\mathbf{p},\mathbf{x},t)$ into

$$f(\epsilon, \varphi, x, t) = f_T(\epsilon) + \sum_m \sum_{\Omega Q} f_{\Omega Q}^m(\epsilon) e^{-i\Omega t + iQx} e^{im\varphi},$$

where $f_T(\epsilon)$ is the equilibrium Fermi function and $f_{\Omega Q}^m$ characterize the non-equilibrium part caused by the SAW. We determine $g_m = \int_0^\infty df f^m$, so that $g^0$ would characterize the electron density and $g^\pm$ combine into electric current. We separate the collision integral

$$\dot{\hat{C}}[\mathbf{p},\mathbf{x},t] = \hat{C}_{\sigma/\eta}[\mathbf{p},\mathbf{x},t] - \frac{f^0(\epsilon) + (\partial_\epsilon f_T)g^0}{\tau_{in}},$$

into the elastic and inelastic parts. Relaxation of the non-equilibrium part of the distribution function towards an isotropic distribution is described by the term $\hat{C}_{\sigma/\eta}$:

$$\dot{\hat{C}}[\mathbf{p},\mathbf{x},t] = \frac{f^0 - f}{\tau}, \quad \text{and} \quad \hat{C}_{\sigma}[f] = \frac{1}{\tau} \partial_\varphi^2 f.$$
with $\tau_{\text{rel}}^{-1} \ll \tau^{-1}$, is taken into account by the last term in Eq. (4) using the relaxation time approximation.

The rectified (acousto-electric) current can be described using

$$J = j_x - i j_y = e \gamma \int_0^\infty d\epsilon f_{00}(\epsilon),$$  \hspace{1cm} (5)

where $\gamma$ is the 2D density of states, and $f_{00}(\epsilon, \varphi)$ is the steady state homogenous part of the non-equilibrium distribution. Below, we restrict the analysis to effects linear in the SAW power and perform a perturbative analysis. We assume that the force from the SAW field is much less than the Lorentz force, $E_{\omega q} \ll \nu_F B$, whereby electron cyclotron orbits are not destroyed by the SAW and no channelling of electron trajectories occurs. To describe the AE effect we relate the steady state term $f_{00}(\epsilon, \varphi)$ to the SAW field and $f_{\omega q}(\epsilon, \varphi)$ at the SAW frequency by taking the $Q = 0, \Omega = 0$ harmonics of Eq. (1),

$$\partial_\varphi f_{00}(\epsilon, \varphi) = \frac{\partial \epsilon}{\omega_c} f_{00}(\epsilon, \varphi) - \sum_{\pm q} e E_{\omega q} \hat{\rho} f_{\omega q}(\epsilon, \varphi).$$

We then evaluate the complex current, $J_c$,

$$J = -\sum_{\pm q} e^2 E_{\omega q} \omega_c \int_0^\infty d\epsilon \frac{\gamma \omega_c}{1 + i \omega_c \tau} \times$$

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\epsilon \varphi} \left[ \frac{2e}{m} \cos \varphi \partial_\epsilon - \frac{\sin \varphi}{m} \partial_\varphi \right] f_{\omega q}(\epsilon, \varphi).$$

Assuming energy independence of $\tau$ and density of states, $\gamma$, we arrive at

$$J = e^2 \gamma \tau / m \sum_{\pm q} E_{\omega q} g_{00}(\epsilon, \varphi)$$

thus reducing the problem to that of finding the AC density modulation $n_{\omega q} = \gamma \tau g_{00}(\epsilon, \varphi)$ excited by the SAW. Note that $(g_{00}(\epsilon, \varphi))^* = g_{00}^{\omega q - q}$, while summation over the SAW harmonics satisfies $\omega = s \cdot q$.

Since the structure of $J$ repeats that of the Drude conductivity tensor, it is natural to work with the DC acousto-electric field generated by the SAW in the direction of its propagation,

$$E_{AE} = \frac{\partial}{\epsilon_F} \{ E_{\omega q} g_{00}(\epsilon, \varphi) \}. \hspace{1cm} (6)$$

As we show in the following calculation, the quantity of interest, $E_{AE}$, depends on the relaxation rate, $\tau_{\text{rel}}^{-1}$, determined by the type of electron scattering which is quite different for small-angle ($\sigma$) scattering from the momentum relaxation rate measured in conductivity.

The dynamical perturbation $g_{\omega q}(\varphi)$ can be found by taking Fourier harmonics of Eq. (1) at the frequency and wave number of the SAW,

$$\partial_\varphi - \frac{\partial f_{\omega q}}{\omega_c} - i \frac{\omega}{\omega_c} + iq R_e \cos \varphi \right] f_{\omega q}(\epsilon, \varphi) =$$

$$= \frac{f_{\omega q}(\epsilon, \varphi) f_{00}(\epsilon) + f_{\omega q}(\epsilon)}{\omega_c r_m} - \frac{e E_{\omega q}}{\omega_c} \hat{\rho} f_{00}(\epsilon, \varphi) + f_{\omega q}(\epsilon).$$

Here we neglect the term $f_{00} \propto |E_{\omega q}|^2$ since any resulting corrections in $f_{\omega q}(\epsilon, \varphi)$ would be non-linear in the SAW power. Assuming a low temperature regime, $kT \ll \omega q / q R_e$, we integrate the above over energy approximating all energy dependent parameters (such as $R_e$) by their respective values at the Fermi level, $\partial f_T \approx -\delta (\epsilon - \epsilon_F)$, and arrive at

$$\left[ \partial_\varphi - \frac{\partial f_{\omega q}}{\omega_c} - i \frac{\omega}{\omega_c} + iq R_e \cos \varphi \right] g_{\omega q}(\varphi) = \frac{e F_{\omega q}}{\omega_c} \cos \varphi. \hspace{1cm} (8)$$

In the limit of $q R_e \gg 1$, the solution to Eq. (8) displays a fast-oscillating angular dependence, $e^{-iq R_e \sin \varphi}$, caused by the last term in brackets on the LHS of Eq. (8). To take those fast oscillations into account we use the method proposed by Rudin, Aleiner and Glazman and write $g_{\omega q}(\varphi) = h_{\omega q}(\varphi) e^{-iq R_e \sin \varphi}$. Using angular Fourier harmonics of $h_{\omega q}(\varphi)$, this reads

$$g_{\omega q} = \sum_{k=\omega q}^{\infty} h_{\omega q} J_{k-n} (q R_e). \hspace{1cm} (9)$$

Having multiplied Eq. (8) by $e^{i q R_e \sin \varphi - i k \varphi}$ and integrated over angle $\varphi$, we arrive at the system of coupled equations for Fourier coefficients $h_{\omega q}$:

$$\left( i k - \frac{\omega}{\omega_c} \right) h_{\omega q} - \frac{1}{\omega_c} \left\langle e^{i q R e \sin \varphi - i k \varphi} \hat{\sigma}_{\omega q} \right\rangle = \frac{e F_{\omega q}}{q} J_k (q R_e), \hspace{1cm} (10)$$

where $\langle ... \rangle = \int_0^{2\pi} d\varphi / 2\pi$ stands for averaging over the angle $\varphi$.

The following analysis of Eq. (10) depends on the form of the collision integral. For isotropic scattering,

$$\left\langle e^{i q R e \sin \varphi - i k \varphi} \hat{\sigma}_{\omega q} \right\rangle = \frac{J_k (q R_e)}{\tau} - h_{\omega q} \frac{h_{\omega q}}{\tau}, \hspace{1cm} (11)$$

and the elements $h_{\omega q}$ in Equation Eq. (10) decouple. One therefore finds the $m$-th angular harmonic of $h_{\omega q}(\varphi)$ as

$$g_{\omega q} = \sum_{k=\omega q}^{\infty} J_k (q R_e) J_{k-m} (q R_e) \left\{ \frac{h_{\omega q}^{k}}{\omega_c} + \frac{e F_{\omega q}}{q} \right\}. \hspace{1cm} (12)$$

Setting $m = 0$, we find

$$g_{\omega q} = \frac{e F_{\omega q}}{q} \sum_{k=\omega q}^{\infty} \frac{k J_k^2 (q R_e)}{ik - i \frac{\omega_c}{\omega_c} + \frac{1}{\omega_c \tau}} \frac{1}{1 - K} \hspace{1cm} (13)$$

In the limit of $q R_e \gg 1$, then $K \ll 1$ since $J_k^2 (q R_e) \gg 1$ and $q R_e \rightarrow 0$. Additionally, the linear $k$ dependence in Eq. (12) may be manipulated to read $k = -i (ik - i \omega / \omega_c + 1 / \omega_c \tau) + i (-i \omega / \omega_c + 1 / \omega_c \tau) - \ldots$
first term exactly cancels the resonance denominator, and application of the identity \( \sum_k J_k^2(x) = 1 \) yields an approximate form of \( g_{0q}^0 \):

\[
g_{0q}^0 \approx \frac{eE_{q0}}{q} \left\{ \frac{1}{1} + \frac{\omega}{\omega_c} \sum_{k=-\infty}^{\infty} \frac{J_k^2(qR_c)}{ik - i\frac{\omega}{\omega_c} + \frac{\omega}{\omega_c}} \right\}. \tag{14}
\]

For small-angle scattering the angle-average in Eq. (10) takes the form

\[
\left( e^{iqR_c \sin \varphi - ik_q \varphi} \right)_{\omega, \varphi} = -\left( \frac{qR_c}{2\pi} \right)^2 h_k - \frac{G_k}{\tau}. \tag{15}
\]

\[
G_k,n = \left( \frac{qR_c}{2} \right)^2 \left[ h_{k+2} - \frac{k}{\omega} + 2h_{k+1} - \frac{k}{\omega} + 2h_{k+1} - \frac{k}{\omega} \right].
\]

Coupling between different elements \( h_k \) in a combination of Eq. (10) and (13) occurs with multipliers of \( (qR_c)^2 / \omega < \omega_c \) and \( kqR_c / \omega < \omega_c \), and is now much weaker than the coupling one would obtain from a direct Fourier transform of Eq. (8). This enables us to solve Eq. (10) perturbatively in \( \hat{G}_k \). Note that we also attribute the term \( k^2h_k \) to the perturbative correction \( \hat{G}_k \), since \( k < qR_c \), and inclusion of this term in the leading approximation would exceed the chosen accuracy. Thus, we write

\[
g_{0q}^0 = \frac{eE_{q0}}{q} \left\{ \frac{1}{1} + \frac{\omega}{\omega_c} \sum_{k=-\infty}^{\infty} \frac{J_k^2(qR_c)}{ik - i\frac{\omega}{\omega_c} + \frac{\omega}{\omega_c}} \right\}, \tag{16}
\]

Setting \( m = 0 \), and solving up to second order Eqs. (13) and in \( \hat{G}_k \) we find that in the leading order of the parameters \( \omega/qR_c \omega_c = s/v_F \ll 1 \), \( k/qR_c \ll 1 \) and \( qR_c \ll \omega_c \), the main contribution to the zeroth harmonic \( g_{0q}^0 \) is given by

\[
g_{0q}^0 \approx \frac{eE_{q0}}{q} \left\{ 1 + \frac{\omega}{\omega_c} \sum_{k=-\infty}^{\infty} \frac{J_k^2(qR_c)}{ik - i\frac{\omega}{\omega_c} + \frac{\omega}{\omega_c}} \right\}. \tag{17}
\]

Using the similarity of Eqs. (14) and (17), we express the SAW induced electric field, \( E_{\text{AE}} = q\hat{E}_{\text{AE}} \), for both isotropic and small-angle scattering cases in terms of the effective scattering rate \( \tau_*^{-1} \):

\[
E_{\text{AE}} = \frac{2\epsilon s |E_{q0}|^2}{\epsilon_F} \sum_{k=-\infty}^{\infty} \frac{\tau_* J_k^2(qR_c)}{\tau_*^2 (\omega - k\omega_c)^2 + 1}, \tag{18}
\]

\[
\tau_*^{-1} = \begin{cases} \tau_0^{-1}, & \text{for isotropic scattering,} \\ \frac{(qR_c)^2}{2\tau_0^2}, & \text{for small-angle scattering}. \end{cases} \tag{19}
\]

Geometrical commensurability manifests itself in Eq. (13) through the appearance of the Bessel function \( J_k(qR_c) \). Additionally, the presence of a finite wave vector lifts selection rules for electron transitions, allowing transitions otherwise forbidden by Kohn’s theorem, such as resonances at multiples of the cyclotron frequency.

The dynamical redistribution of electrons leading to screening of the external SAW field \( E_{\text{SAW}} \) by the 2DEG. We relate \( E_{q0} \) to the unscreened field by inclusion of the dielectric function, \( E_{q0} = E_{\text{SAW}}/\kappa(\omega, q) \). In the Thomas-Fermi approximation, and in the limit of \( qR_c > 1 \) we find that \( \kappa^{-1}(\omega, q) = \omega_{\text{sc}}qR_c \), where \( \omega_{\text{sc}} = \chi/2\pi e^2 / \gamma \) is the donor-related Bohr radius (\( \chi \) is the background dielectric constant) and introduce the dimensionless parameter which is a measure of the amplitude of the screened SAW field normalized by the Fermi energy,

\[
\mathcal{E} = \left( e\epsilon_F E_{\text{SAW}}/\epsilon_F \right)^2. \tag{20}
\]

In the limit of \( \omega_c \tau_* \ll 1 \), we discuss two extreme cases, \( v_F \gg s \) and \( v_F \ll s \), where \( s \) is the SAW speed in GaAs, \( s = 2.8 \times 10^8 \text{ cm s}^{-1} \). At electron densities \( n_e \sim 10^{10} / \text{ cm}^{-2} \), \( v_F \gg s \) and the relevant frequency regime in structures with realistic mobility appears to be \( \omega_c \sim (s/v_F)qR_c \ll 1 \). In this situation, the largest contribution to the DC field then comes from the term in Eq. (13) with \( k = 0 \), and

\[
E_{\text{AE}} \approx \frac{2\epsilon s |E_{q0}|^2}{\epsilon_F} \frac{J_0^2(qR_c)}{1 + (\omega_\tau^2)^{-1}}. \tag{21}
\]

It is interesting to note that the magnetic field dependence of the amplitude of geometrical oscillations described by Eqs. (21) strongly differs for the two limiting types of scattering considered above. In the case of isotropic (short-range) scatters the oscillation amplitude decreases as \( N^{-1} \) with the oscillation number \( N \sim qR_c / \pi \). In contrast, for low-angle scattering it is non-monotonic. It increases linearly in \( N \) up to \( N^* \sim (2\omega_\tau / \gamma) \) where the oscillations amplitude has a maximum followed by a gradual \( N^{-3} \) decrease.

The result in Eq. (13) also shows that in a low-density 2DEG or in a gas of “heavy” composite fermions, such that \( v_F \ll s \), resonances in the AE effect at \( \omega = k\omega_c \) become possible. Due to the oscillatory behaviour of Bessel functions \( J_k(qR_c) \) at \( qR_c \gg 1 \) and since \( \omega_\tau = s/v_F qR_c \), resonances would appear in the experiment as a sequence of Lorentzians of apparently random height.

To study the damping of geometrical oscillations at low magnetic fields \( \omega_\tau \tau_* < 1 \), we use the method of residues, transforming the summation in Eq. (13) into the integral

\[
E_{\text{AE}} \approx \frac{\omega_F^2/2i}{\epsilon_0 \omega_c \tau_*} \int_C \frac{1 + \sin(2qR_c - \pi z)}{\gamma(\omega_c z) \cot(\pi z)dz} dz
\]

where for \( qR_c \gg 1 \), \( J_0(qR_c) \approx \sqrt{2/\pi qR_c \cos(qR_c - k\pi/2 - \pi/4)} \), and the contour \( C \) consists of two parts: in the upper half-plane, \( C_+ = x + i0 \) and in the lower half-plane, \( C_- = x - i0 \). Each contour, \( C_\pm \) is then moved away from the real axis, \( C_\pm \to C_\pm = x \pm i|y| \).
such that $e^{-2|y|} \ll 1$, when each contour picks up exactly one residue from the poles at $z_\pm = \omega/\omega_c \pm i/\omega_c \tau^*$ (here and below the subscript $\pm$ is determined by the subscript of the contour). The numerator of the integrands in the shifted line integrals $\int_{C^{\pm}_{L}} dz$ are approximated using $e^{-2|y|} \ll 1$, thus yielding $\cot(\pi z) \approx \mp i$ and $\sin(qR_c - z) \approx \pm e^{\pm 2\pi i (qR_c - z)/2i}$. After this, each contour is then moved, $C^{\pm}_{L} \rightarrow C^{\pm}_{F}$, such that $3C^{\pm}_{L} \rightarrow \mp \infty$, passing the real axis as they approach the opposite extremes of the complex plane. Thus, we arrive at

$$E_{AE} = \frac{E_\omega p_F}{e} \left\{ 1 + \sin \left( 2qR_c - \frac{\pi \omega}{\omega_c} \right) e^{\mp \pi / \omega_c \tau^*} \right\}. $$

The latter equation is typical for damped geometrical oscillations. It shows how commensurability oscillations die away when $\pi/\omega_c \tau^* \geq 1$ and that the onset of oscillations occurs at the magnetic field value such that

$$R_c < R^*_c \sim \left\{ \frac{l/\pi}{1 + \sqrt{2/\pi (l \eta)^2}} \right\} \text{ for isotropic scattering,}$$

and

$$R_c < R^*_c \sim \left\{ \frac{l/\pi}{1 + \sqrt{2/\pi (l \eta)^2}} \right\} \text{ for small-angle scattering.}$$

Together with the result in Eq. (21), the latter offset condition shows that the number of $B^{-1}$ oscillations, $N \sim qR^*_c/\pi$ detectable in a sample with the mean free path $l = v_F \tau > 2\pi/q$ is larger when its mobility is limited by short-range scatterers ($N < N^\omega$) than when scattering is due to smooth disorder ($N < N^\sigma$), where

$$N^\omega \sim \frac{ql}{\pi}, \quad \text{vs} \quad N^\sigma \sim \min \left\{ (ql)^{3/2}, \sqrt{\frac{slq}{\pi v_F}} \right\}. \quad (22)$$

In the regime of damped oscillations a finite and apparently field independent AE effect persists up to the field $v_FB > E_{\omega q}$ (when channelling takes it toll),

$$E_{AE} \approx \omega p_F \left( n_{sc} E_{\omega q}^{SAW} / \epsilon_F \right)^2. \quad (23)$$

The above presented analysis explains why the observed magnetic field dependence of the AE effect by Shilton et al.

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18 The average is calculated by turning the operator $\hat{C}_\omega \propto \partial_{\varphi} \propto \varphi$ onto the term $e^{iqR_c \sin \varphi - ik \varphi}$ which produces a constant and sinusoidal coefficients of single and double frequency. Turning these coefficients into complex exponentials and using the relation $J_{k-n}(qR_c) = \int_{0}^{2\pi} d\varphi e^{iqR_c \sin \varphi - i(k-n)\varphi}/2\pi$ gives the matrix elements $h^{k}_{\omega q}, h^{k \pm 1}_{\omega q}$ and $h^{k \pm 2}_{\omega q}$ respectively.