Geometric Crystals on Schubert Varieties

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Abstract

We define geometric crystals and unipotent crystals for arbitrary Kac-Moody groups and describe geometric and unipotent crystal structures on the Schubert varieties.

1 Introduction

The theory of crystal base introduced by Kashiwara succeeds in being applied to many areas in mathematics and mathematical physics to clarify their combinatorial behavior. One of the reasons why it can be well-applied is that it contains not only “real crystals” but also “virtual crystals”, e.g., $B_i$ (see Sect.6), $t_\lambda$ (see [6]), etc, where “virtual crystals” mean certain purely combinatorial objects not having the corresponding $U_q(g)$-modules. They are deeply related to each other by some ‘limit’ operations and then they describe many combinatorial phenomena together. Indeed, some “real crystal” is obtained as subcrystal in infinitely many tensor products of “virtual crystals” (see [6] and the references therein) and the crystal $B_i$ and $\tilde{B}$ in Sect.5 are also obtained as some limit of real crystals. In this sense, the theory of crystals would be wider than usual representation theory of the quantum algebra $U_q(g)$. Roughly, we can say that real crystal bases are obtained by taking the limit $q \to 0$ from some bases of $U_q(g)$-modules, which is called “crystallization”. But, “virtual” ones are not gotten by such crystallizations from $U_q(g)$-modules.

Berenstein and Kazhdan clarify ([1]) that such “virtual crystals” also have some “real” backgrounds as the “tropicalization/ultra-discritization” of “geometric crystals” for semi-simple(reductive) groups.

\[
\begin{align*}
\{ \text{Global Bases of } U_q(g) \text{-modules} \} & \xrightarrow{\text{crystallization}} \{ \text{Geometric Crystals} \} \\
\{ \text{Geometric Crystals} \} & \xrightarrow{\text{ultra-discritization}} \{ \text{Crystals} \} \\
\{ \text{Crystals} \} & \xrightarrow{\text{tropicalization}} \{ \text{Global Bases of } U_q(g) \text{-modules} \}
\end{align*}
\]

Recently, by the ultra-discritization/tropicalization method, the relations between soliton cellular automaton and crystals are revealed (see e.g., [3] [4]). In the meanwhile, it is well-known that flag varieties $G/B$ (reps. $G/P$) plays a significant role in the soliton theory, where $G$ is an affine Kac-Moody group and $B$ (resp. $P$) is its Borel (resp. parabolic) subgroup. We would like to find the connection of affine flag varieties and geometric crystals. For the purpose, we shall extend the theory of geometric/unipotent crystals [1] to Kac-Moody...
setting. And then we shall define geometric/unipotent crystals on finite Schubert cells/varieties and consider some positive structures on them. Finally, we show that some ultra-discritizations of the geometric crystals on Schubert varieties are isomorphic to tensor products of some Kashiwara’s crystals. But in general, it is still unclear to define geometric crystal structures on flag varieties as mentioned in the remark in Sect.4.

The organization of the article is as follows: in Sect.2 we review briefly the theory of Kac-Moody groups, ind-varieties and ind-groups. In Sect.3, we define the notion of unipotent crystals in Kac-Moody setting and their product structures. We also define the notion of geometric crystals and give a recipe for obtaining canonically geometric crystals from unipotent crystals following to [1]. In Sect.4, on finite Schubert cells/varieties we induce the structure of unipotent/geometric crystals. In Sect.5, we recall the notion of positive structure on geometric crystals and define ultra-discritization/tropicalization operations. We also consider certain positive structures of geometric crystals on Schubert cells and show that its ultra-discritization is isomorphic to (Langlands dual of) Kashiwara’s crystal $B_{i_1} \otimes \cdots \otimes B_{i_t}$. In the last section, we apply the result in Sect.5 to give a new proof of braid-type isomorphisms ([12]).

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2 Kac-Moody groups and Ind-varieties

In this section, we review on Kac-Moody groups following to [9,11,14].

2.1 Kac-Moody algebras and Kac-Moody groups

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, where $I$ be a finite index set. Let $(\{\alpha_i\}_{i \in I}, \{\beta_i\}_{i \in I})$ be the associated root data, where $t$ be the vector space over $\mathbb{C}$ with dimension $|I| + \text{corank}(A)$, and $\{\alpha_i\}_{i \in I} \subset t^*$ and $\{\beta_i\}_{i \in I} \subset t$ are linearly independent indexed sets satisfying $\alpha_i(\beta_j) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with $A$ is the Lie algebra over $\mathbb{C}$ generated by $t$, the Chevalley generators $e_i$ and $f_i$ ($i \in I$) with the usual defining relations ([9,11]). There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \rho^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in t^* | \alpha \neq 0, \ \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$ and $\Delta_+ := \Delta \cap Q_+$. An element of $\Delta_+$ is called a positive root.

Define simple reflections $s_i \in \text{Aut}(t)$ ($i \in I$) by $s_i(h) := h - \alpha_i(h)\alpha_i$, which generate the Weyl group $W$. We also define the action of $W$ on $t^*$ by $s_i(\lambda) := \lambda - \alpha(h_i)\alpha_i$. Set $\Delta^{re} := \{w(\alpha_i) | w \in W, i \in I\}$, whose element is called a real root.

Let $\mathfrak{g}^e$ be the derived Lie algebra of $\mathfrak{g}$ and $G^e$ be the free group generated by the free product of the additive groups $\mathfrak{g}_\alpha$ ($\alpha \in \Delta^{re}$), with the canonical inclusion $i_\alpha : \mathfrak{g}_\alpha \hookrightarrow G^e$. For any integrable $\mathfrak{g}^e$-module $(V, \pi)$, a homomorphism $\pi^*_V : G^e \rightarrow \text{Aut}_\mathbb{C}(V)$ is defined by $\pi^*_V(i_\alpha(e)) = \exp(\pi(e))$. Set $N^* := \cap_{V: \text{integrable}} \text{Ker}(\pi^*_V)$ and $G := G^*/N^*$, which is called a Kac-Moody group associated with the Kac-Moody Lie algebra $\mathfrak{g}^e$. Let $\rho : G^e \rightarrow G$ be the canonical homomorphism. For $e \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta^{re}$), define $\exp e := \rho(i_\alpha(e))$.
and $U_{\alpha} := \exp \mathfrak{g}_{\alpha}$, which is an one-parameter subgroup of $G$. The group $G$ is generated by $U_{\alpha}$ ($\alpha \in \Delta^o$). Let $U^\pm$ be the subgroups generated by $U_{\pm \alpha}$ ($\alpha \in \Delta^o = \Delta^+ \cap Q^+$), i.e., $U^\pm := \{U_{\pm \alpha}| \alpha \in \Delta^+\}$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$\phi_i \left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp t e_i, \quad \phi_i \left( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp t f_i (t \in \mathbb{C})$$

Set $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{\text{diag}(t, t^{-1})| t \in \mathbb{C}\})$ and $N_i := N_{G_i}(T_i)$. Let $T$ (resp. $N$) be the subgroup of $G$ generated by $T_i$ (resp. $N_i$), which is called a maximal torus in $G$. We have the isomorphism $\phi : W \rightarrow N/T$ defined by $\phi(s_i) = N_iT/T$. An element $\mathfrak{s}_i := x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \cdots, i_l) \in I^l | w = s_{i_1}s_{i_2}\cdots s_{i_l}\},$$

where $l$ is the length of $w$. We associate to each $w \in W$ its standard representative $\bar{w} \in N_G(T)$ by

$$\bar{w} = \bar{s}_{i_1}\bar{s}_{i_2}\cdots \bar{s}_{i_l},$$

for any $(i_1, i_2, \cdots, i_l) \in R(w)$.

### 2.2 Ind-variety and Ind-group

Let us recall the notion of ind-varieties and ind-groups. (see [8]).

**Definition 2.1.** Let $k$ be an algebraically closed field.

(i) A set $X$ is an ind-variety over $k$ if there exists a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that

(a) $\bigcup_{n \geq 0} X_n = X$.

(b) Each $X_n$ is a finite-dimensional variety over $k$ such that the inclusion $X_n \hookrightarrow X_{n+1}$ is a closed embedding.

The ring of regular functions $k[X]$ is defined by

$$k[X] := \lim_{\longrightarrow} k[X_n].$$

(ii) A Zariski topology on an ind-variety $X$ is defined as follows; a set $U \subset X$ is open if and only if $U \cap X_n$ is open in $X_n$ for any $n \geq 0$.

(iii) Let $X$ and $Y$ be two ind-varieties with filtrations $\{X_n\}$ and $\{Y_n\}$ respectively. A map $f : X \rightarrow Y$ is a morphism if for any $n \geq 0$, there exists $m$ such that $f(X_n) \subset Y_m$ and $f|_{X_n} : X_n \rightarrow Y_m$ is a morphism. A morphism $f : X \rightarrow Y$ is said to be an isomorphism if $f$ is bijective and $f^{-1} : Y \rightarrow X$ is also a morphism.

(iv) Let $X$ and $Y$ be two ind-varieties. A rational morphism $f : X \rightarrow Y$ is an equivalence class of morphisms $f_U : U \rightarrow Y$ where $U$ is an open dense subset of $X$, and two morphisms $f_U : U \rightarrow Y$ and $f_V : V \rightarrow Y$ are equivalent if they coincide on $U \cap V$. 

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Lemma 2.2.  (i) A finite dimensional variety over $k$ holds canonically an ind-variety structure.
(ii) If $X$ and $Y$ are ind-varieties, then $X \times Y$ is canonically an ind-variety by taking the filtration
\[(X \times Y)_n := X_n \times Y_n.\]

Definition 2.3. An ind-variety $H$ is called an ind (algebraic) group if the underlying set $H$ is a group and the maps
\[H \times H \rightarrow H \quad H \rightarrow H \quad (x, y) \mapsto xy \quad x \mapsto x^{-1}\]
are morphisms of ind-varieties.

Proposition 2.4 ([8]).  (i) Let $G$ be a Kac-Moody group and $U^\pm, B^\pm$ be its subgroups as above. Then $G$ is an ind-group and $U^\pm, B^\pm$ are its closed ind-subgroups.
(ii) The multiplication maps
\[T \times U \rightarrow B \quad U^- \times T \rightarrow B^- \quad (t, u) \mapsto tu \quad (v, t) \mapsto vt\]
are isomorphisms of ind-varieties.

3 Geometric Crystals and Unipotent Crystals

In this section, we define geometric crystals and unipotent crystals associated with Kac-Moody groups, which is just a generalization of [1] to a Kac-Moody setting.

3.1 Geometric Crystals

Let $(a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix and $G$ be the associated Kac-Moody group with the maximal torus $T$. An element in $\text{Hom}(T, \mathbb{C}^\times)$ (resp. $\text{Hom}(\mathbb{C}^\times, T)$) is called a character (resp. co-character) of $T$. We define a simple co-root $\alpha_i^\vee \in \text{Hom}(\mathbb{C}^\times, T)$ $(i \in I)$ by $\alpha_i^\vee(t) := T_i$. We have a pairing $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Definition 3.1.  (i) Let $X$ be an ind-variety over $\mathbb{C}$, $\gamma : X \rightarrow T$ be a rational morphism and a family of rational morphisms $e_i : \mathbb{C}^\times \times X \rightarrow X$ $(i \in I)$;
\[e_i^c : \mathbb{C}^\times \times X \rightarrow X \quad (c, x) \mapsto e_i^c(x).\]
The triplet $\chi = (X, \gamma, \{e_i\}_{i \in I})$ is a geometric pre-crystal if it satisfies $e_i^1(x) = x$ and $\gamma(e_i^c(x)) = \alpha_i^\vee(c)\gamma(x)$.

(3.1)
Definition 3.2. Let \((X, \gamma, \{e_i^X\}_{i \in I})\) and \((Y, \gamma, \{e_i^Y\}_{i \in I})\) be geometric pre-crystals. A rational morphism \(f : X \rightarrow Y\) is called a geometric pre-crystal if it is a morphism of geometric pre-crystals. In particular, if a morphism \(f\) is a birational isomorphism of ind-varieties, it is called an isomorphism of geometric pre-crystals.

Let \(\chi = (X, \gamma, \{e_i^X\}_{i \in I})\) be a geometric pre-crystal. For a word \(i = (i_1, i_2, \ldots, i_l) \in R(w)\) \((w \in W)\), set \(\alpha^{(l)} : = \alpha_{i_l}, \alpha^{(l-1)} : = s_{i_l}(\alpha_{i_{l-1}}), \ldots, \alpha^{(1)} : = s_{i_l} \cdots s_{i_2}(\alpha_{i_1})\). Now for a word \(i = (i_1, i_2, \ldots, i_l) \in R(w)\) we define a rational morphism \(e_i : T \times X \rightarrow X\) by

\[
(t, x) \mapsto e_i(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x).
\]

Definition 3.2. (i) A geometric pre-crystal \(\chi\) is called a geometric crystal if for any \(w \in W\), and any \(i, i' \in R(w)\) we have

\[
e_i = e_{i'}.
\] (3.2)

(ii) Let \((X, \gamma, \{e_i^X\}_{i \in I})\) and \((Y, \gamma, \{e_i^Y\}_{i \in I})\) be geometric crystals. A rational morphism \(f : X \rightarrow Y\) is called a morphism (resp. an isomorphism) of geometric crystals if it is a morphism (resp. an isomorphism) of geometric pre-crystals.

The following lemma is a direct result from [1][Lemma 2.1] and the fact that the Weyl group of any Kac-Moody Lie algebra is a Coxeter group [5][Proposition 3.13].

Lemma 3.3. The relations (3.2) are equivalent to the following relations:

\[
e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} \quad \text{if} \quad \langle \alpha_i^\vee, \alpha_j \rangle = 0,
\]

\[
e_i^{c_1} e_j^{c_1} e_i^{c_2} = e_j^{c_2} e_i^{c_1} e_j^{c_1} \quad \text{if} \quad \langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle = -1,
\]

\[
e_i^{c_1} e_j^{c_1} e_i^{c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1} e_j^{c_2} e_i^{c_1} \quad \text{if} \quad \langle \alpha_i^\vee, \alpha_j \rangle = -2, \quad \langle \alpha_j^\vee, \alpha_i \rangle = -1,
\]

\[
e_i^{c_1} e_j^{c_2} e_i^{c_1} e_j^{c_2} e_i^{c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1} e_j^{c_2} e_i^{c_2} e_j^{c_2} e_i^{c_1} \quad \text{if} \quad \langle \alpha_i^\vee, \alpha_j \rangle = -3, \quad \langle \alpha_j^\vee, \alpha_i \rangle = -1,
\]

Remark. If \(\langle \alpha_i^\vee, \alpha_j \rangle \langle \alpha_j^\vee, \alpha_i \rangle \geq 4\), there is no relation between \(e_i\) and \(e_j\).

3.2 Unipotent Crystals

In the sequel, we denote the unipotent subgroup \(U^+\) by \(U\). We define unipotent crystals (see [1]) associated to Kac-Moody groups.

The definitions below are followed to [1].

Definition 3.4. Let \(X\) be an ind-variety over \(\mathbb{C}\) and \(\alpha : U \times X \rightarrow X\) be a rational \(U\)-action such that \(\alpha\) is defined on \(\{e\} \times X\). Then, the pair \(X = (X, \alpha)\) is called a \(U\)-variety. For \(U\)-varieties \(X = (X, \alpha_X)\) and \(Y = (Y, \alpha_Y)\), a rational morphism \(f : X \rightarrow Y\) is called a \(U\)-morphism if it commutes with the action of \(U\).
Now, we define the $U$-variety structure on $B^- = U^- T$. By Proposition 2.3, $B^-$ is an ind-subgroup of $G$ and then is an ind-variety over $C$. The multiplication map in $G$ induces the open embedding: $B^- \times U \hookrightarrow G$, then this is a birational isomorphism. Let us denote the inverse birational isomorphism by $g$:

$$g : G \rightarrow B^- \times U.$$  

Then we define the rational morphisms $\pi^- : G \rightarrow B^-$ and $\pi : G \rightarrow U$ by $\pi^- := \text{proj}_{B^-} \circ g$ and $\pi := \text{proj}_U \circ g$. Now we define the rational $U$-action $\alpha_{B^-}$ on $B^-$ by

$$\alpha_{B^-} := \pi^- \circ m : U \times B^- \rightarrow B^-,$$

where $m$ is the multiplication map in $G$. Then we obtain $U$-variety $B^- = (B^-, \alpha_{B^-})$.

**Definition 3.5.** (i) Let $X = (X, \alpha)$ be a $U$-variety and $f : X \rightarrow B^-$ be a $U$-morphism. The pair $(X, f)$ is called a unipotent $G$-crystal or, for short, unipotent crystal.

(ii) Let $(X, f_X)$ and $(Y, f_Y)$ be unipotent crystals. A $U$-morphism $g : X \rightarrow Y$ is called a morphism of unipotent crystals if $f_X = f_Y \circ g$. In particular, if $g$ is a birational isomorphism of ind-varieties, it is called an isomorphism of unipotent crystals.

We define a product of unipotent crystals following to [1]. For unipotent crystals $(X, f_X)$, $(Y, f_Y)$, define a morphism $\alpha_{X \times Y} : U \times X \times Y \rightarrow X \times Y$ by

$$\alpha_{X \times Y}(u, x, y) := (\alpha_X(u, x), \alpha_Y(\pi(u \cdot f_X(x)), y)) \quad (3.3)$$

If there is no confusion, we use abbreviated notation $u(x, y)$ for $\alpha_{X \times Y}(u, x, y)$.

**Theorem 3.6 ([1]).** (i) The morphism $\alpha_{X \times Y}$ defined above is a rational $U$-morphism on $X \times Y$.

(ii) Let $m : B^- \times B^- \rightarrow B^-$ be a multiplication morphism and $f = f_{X \times Y} : X \times Y \rightarrow B^-$ be the rational morphism defined by

$$f_{X \times Y} := m \circ (f_X \times f_Y).$$

Then $f_{X \times Y}$ is a $U$-morphism and then, $(X \times Y, f_{X \times Y})$ is a unipotent crystal, which we call a product of unipotent crystals $(X, f_X)$ and $(Y, f_Y)$.

(iii) Product of unipotent crystals is associative.

### 3.3 From unipotent crystals to geometric crystals

For $i \in I$, set $U_{i}^{\pm} := U^{\pm} \cap \bar{s}_i U^{\pm} \bar{s}_i^{-1}$ and $U_{i}^{i} := U^{\pm} \cap \bar{s}_i U^{\pm} \bar{s}_i^{-1}$. Indeed, $U_{i}^{\pm} = U_{\pm \alpha_i}$. Set

$$Y_{\pm \alpha_i} := \langle x_{\pm \alpha_i}(t)U_{\alpha_i}x_{\pm \alpha_i}(-t) | t \in \mathbb{C}, \alpha \in \Delta_{\pm}^{re} \setminus \{ \pm \alpha_i \} \rangle.$$

**Lemma 3.7 ([8], [10]).** For a simple root $\alpha_i$ $(i \in I)$, we have:

(i) $Y_{\pm \alpha_i} = U_{i}^{\pm}$.
(ii) \( U^\pm = U^\pm_i \cdot Y_{\pm \alpha_i} \) (semi-direct product).

(iii) \( \bar{s}_i Y_{\pm \alpha_i} \bar{s}_i^{-1} = Y_{\pm \alpha_i} \).

By this lemma, we have the unique decomposition:
\[
U^- = U^-_i \cdot Y_{\pm \alpha_i} = U^-_{\alpha_i} \cdot U^-_i.
\]

By using this decomposition, we get the canonical projection \( \xi_i : U^- \to U^-_{\alpha_i} \).

Now, we define the function on \( U \) by
\[
\varphi_i := \chi_i \circ f_\mathbf{X},
\]
and a rational morphism \( \gamma_X : X \to T \) by
\[
\gamma_X := \text{proj}_T \circ f_\mathbf{X} : X \to B^- \to T,
\]
where \( \text{proj}_T \) is the canonical projection. Suppose that the function \( \varphi_i \) is not identically zero on \( X \). We define a morphism \( e_i : \mathbb{C}^x \times X \to X \) by
\[
e_i(x) := x \left( \frac{c - 1}{\varphi_i(x)} \right) (x).
\]

**Theorem 3.8** ([1]). For a unipotent \( \mathbb{G} \)-crystal \( (\mathbf{X}, f_\mathbf{X}) \), suppose that the function \( \varphi_i \) is not identically zero for any \( i \in I \). Then the rational morphisms \( \gamma_X : X \to T \) and \( e_i : \mathbb{C}^x \times X \to X \) as above define a geometric \( \mathbb{G} \)-crystal \( (X, \gamma_X, \{e_i\}_{i \in I}) \), which is called the induced geometric \( \mathbb{G} \)-crystals by unipotent \( \mathbb{G} \)-crystal \( (\mathbf{X}, f_\mathbf{X}) \).

Note that in [1], the cases \( \varphi_i \equiv 0 \) for some \( i \in I \) are treated by considering Levi subgroups of \( \mathbb{G} \). But here we do not treat such things.

The following product structure on geometric crystals are most important results in the sense of comparison with the tensor product theorem in Kashiwara’s crystal theory.

**Proposition 3.9.** For unipotent \( \mathbb{G} \)-crystals \( (\mathbf{X}, f_\mathbf{X}) \) and \( (\mathbf{Y}, f_\mathbf{Y}) \), set the product \( (\mathbf{Z}, f_\mathbf{Z}) := (\mathbf{X}, f_\mathbf{X}) \times (\mathbf{Y}, f_\mathbf{Y}) \), where \( Z = X \times Y \). Let \( (Z, \gamma_Z, \{e_i\}) \) be the induced geometric \( \mathbb{G} \)-crystal from \( (\mathbf{Z}, f_\mathbf{Z}) \). Then we obtain:

(i) \( \gamma_Z = \text{m} \circ (\gamma_X \times \gamma_Y) \).

(ii) For each \( i \in I \), \( (x, y) \in Z \), we obtain
\[
\varphi_i^Z(x, y) = \varphi_i^X(x) + \frac{\varphi_i^Y(y)}{\alpha_i(\gamma_X(x))}.
\]

(iii) For any \( i \in I \), the action \( e_i : \mathbb{C}^x \times Z \to Z \) is given by:
\[
e_i(x, y) = (e_i^1(x), e_i^2(y)),
\]
where
\[
c_1 = \frac{c \alpha_i(\gamma_X(x)) \varphi_i^X(x) + \varphi_i^Y(y)}{\alpha_i(\gamma_X(x)) \varphi_i^X(x) + \varphi_i^Y(y)}, \quad c_2 = \frac{\alpha_i(\gamma_X(x)) \varphi_i^X(x) + \varphi_i^Y(y)}{\alpha_i(\gamma_X(x)) \varphi_i^X(x) + \varphi_i^Y(y)}.
\]
Here note that \( c_1c_2 = c \). The formula \( c_1 \) and \( c_2 \) in [1] seem to be different from ours. Thus, we give the proof of (iii). Others are obtained by the same way as [1].

Proof. By using the result (ii), we have

\[
\varphi_i^Z(x, y) = \varphi_i^X(x) + \frac{\varphi_i^Y(y)}{\alpha_i(\gamma_X(x))}.
\]

Here we set \( A := \frac{c - 1}{\varphi_i^Y(x, y)} \) for \((x, y) \in Z\). Since by [3] we have

\[
e_i^c(x, y) = x_i(A)(x, y) = (x_i(A)(x), \pi(x_i(A) \cdot f_X(x))(y)),
\]

we get \( \frac{c - 1}{\varphi_i^Y(x, y)} = A \), and then we obtain \( c_1 \) in [3].

Let us see \( c_2 \). Writing \( f_X(x) = u \cdot t \ (u \in U^-, t \in T) \), by Lemma 3.1 (3.2) in [1], we get

\[
\pi(x_i(A) \cdot f_X(x)) = x_i((A^{-1} + \chi_i(u)^{-1})^{-1} \alpha_i(t^{-1}))
\]

Since \( \chi_i(u) = \varphi_i(x) \) and \( \alpha_i(t) = \alpha_i(\gamma_X(x)) \), we obtain

\[
\pi(x_i(A) \cdot f_X(x)) = x_i \left( \frac{A}{(1 + A \varphi_i(x))\alpha_i(\gamma_X(x))} \right).
\]

Now, set \( B = \frac{A}{(1 + A \varphi_i(x))\alpha_i(\gamma_X(x))} \). Substituting \( A = \frac{c - 1}{\varphi_i^Y(x, y)} \) and \( \frac{c_2 - 1}{\varphi_i^Y(y)} = B \), we obtain the formula \( c_2 \) in [3]. \( \square \)

4 Crystal structure on Schubert varieties

4.1 Highest weight modules and Schubert varieties

As in Sect.2, let \( G \) be a Kac-Moody group, \( B^\pm = U^\pm T \) (resp. \( U^\pm \)) be the Borel (resp. unipotent) subgroups in \( G \) and \( W \) be the associated Weyl group. Here, we have the following Bruhat decomposition and Birkhoff decomposition;

Proposition 4.1 ([8], [11], [14]). We have

\[
G = \bigcup_{w \in W} B^+ w B^+ = \bigcup_{w \in W} U^+ w B^+ \quad \text{(Bruhat decomposition),} \tag{4.1}
\]

\[
G = \bigcup_{w \in W} B^- w B^+ = \bigcup_{w \in W} U^- w B^+ \quad \text{(Birkhoff decomposition).} \tag{4.2}
\]

Let \( J \subset I \) be a subset of the index set \( I \) and \( W_J := \langle s_i \mid i \in J \rangle \) be the subgroup of \( W \) associated with \( J \). Set \( P_J := B^+ w_J B^+ \) and call it a \( \text{(standard) parabolic subgroup} \) of \( G \) associated with \( J \subset I \). The following lemma is well-known;

Lemma 4.2. Any coset in \( W/W_J \) contains a unique element \( w^* \) of minimal length, and for any \( w' \in W_J \), we have \( l(w^* w') = l(w^*) + l(w') \).

We denote the set of the elements \( w^* \) as in Lemma 1.2 by \( W^J \), which is a set of representatives of \( W/W_J \) in \( W \). There exist the following parabolic Bruhat/Birkhoff decompositions:
Proposition 4.3 ([8],[11],[14]). Let $J$ be a subset of $I$ and, $W_J$ and $W^J$ be as above. Then we have

$$G = \bigcup_{w^* \in W^J} U^+ w^* P_J,$$
$$G = \bigcup_{w^* \in W^J} U^- w^* P_J.$$

4.2 Unipotent crystal structure on Schubert variety

For $\Lambda \in P_+$ ($P_+$ is the set of dominant integral weight), let us denote an integral highest weight simple module with the highest weight $\Lambda$ by $L(\Lambda)$ ([5]) and its projective space by $P(\Lambda) := (L(\Lambda) \setminus \{0\})/\mathbb{C}^\times$. Let $v_{\Lambda} \in P(\Lambda)$ be the point corresponding to the line containing the highest weight vector of $L(\lambda)$ and set

$$X(\Lambda) := G \cdot v_{\Lambda} \subset P(\Lambda).$$

Set $J_{\Lambda} := \{i \in I \mid \langle h_i, \Lambda \rangle = 0\}$. By Proposition 4.3 and the fact that $P_{J_{\Lambda}}$ is the stabilizer of $v_{\Lambda}$, we have the isomorphism between $X(\Lambda)$ and the flag variety $G/P_{J_{\Lambda}}$:

Proposition 4.4 ([11],[14]). There is the following isomorphism and the decomposition;

$$\rho : G/P_{J_{\Lambda}} = \bigcup_{w \in W_{J_{\Lambda}}} U^\pm \tilde{w}P_{J_{\Lambda}}/P_{J_{\Lambda}} \xrightarrow{\sim} X(\Lambda) \quad \mapsto \quad g \cdot v_{\Lambda}$$

Definition 4.5. We denote the image $\rho(U^+ \tilde{w}P_{J_{\Lambda}}/P_{J_{\Lambda}})$ (resp. $\rho(U^- \tilde{w}P_{J_{\Lambda}}/P_{J_{\Lambda}})$) by $X(\Lambda)_w$ (resp. $X(\Lambda)^w$) and call it a finite (resp. co-finite) Schubert cell and its Zariski closure in $P(\Lambda)$ by $\overline{X(\Lambda)}_w$ (resp. $\overline{X(\Lambda)}^w$) and call it a finite (resp. co-finite) Schubert variety.

The names “finite” and “co-finite” come from the fact

$$\dim X(\Lambda)_w = l(w), \quad \text{codim}_{X(\Lambda)} X(\Lambda)^w = l(w),$$

Indeed, $X(\Lambda)_w \cong \mathbb{C}^{l(w)}$. There exist the following closure relations;

$$\overline{X(\Lambda)}_w = \bigcup_{y \leq w, y \in W^{J_{\Lambda}}} X(\Lambda)_y, \quad \overline{X(\Lambda)}^w = \bigcup_{y \geq w, y \in W^{J_{\Lambda}}} X(\Lambda)^y. \quad (4.3)$$

Indeed, by [5] 7.1,7.3],

$$\overline{X(\Lambda)}_w \text{ and } \overline{X(\Lambda)}^w \text{ are ind-varieties.} \quad (4.4)$$

Let us associate a unipotent crystal structure with $X(\Lambda)_w$. Since by the definition of $X(\Lambda)_w$ and Proposition 4.4 we have $X(\Lambda)_w = U^+ \tilde{w} \cdot v_{\Lambda}$, which implies;

Lemma 4.6. Schubert cell $X(\Lambda)_w$ is a $U$-variety.
Next, let us construct a $U$-morphism $X(\Lambda)_w \to B^-$. For that purpose, we consider the following: let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression and set $U_w = U \cap \bar{w}U \bar{w}^{-1}$ and $U^w = U \cap \bar{w}U \bar{w}^{-1}$. Define

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_k),$$

then we have

$$U_w := U_{\beta_1} \cdot U_{\beta_2} \cdots U_{\beta_k}.$$  

This is a closed subgroup of $U$ and we have an isomorphism of ind (algebraic)-varieties (4.9),

$$U_w \cong U_{\beta_1} \times U_{\beta_2} \times \cdots \times U_{\beta_k} \cong \mathbb{C}^k,$$  

by

$$U_w \cdot \bar{w} = U_{\alpha_{i_1}} \bar{s}_{i_1} \cdot U_{\alpha_{i_2}} \bar{s}_{i_2} \cdots U_{\alpha_{i_k}} \bar{s}_{i_k} \mathbb{C}^k,$$  

$$x_{i_1}(a_1) \bar{s}_{i_1} \cdot x_{i_2}(a_2) \bar{s}_{i_2} \cdots x_{i_k}(a_k) \bar{s}_{i_k} \mapsto (a_1, a_2, \cdots, a_k).$$

Lemma 4.7 (14, 2.2). (i) We have a decomposition

$$U = U_w \cdot U^w,$$  

and the decomposition (4.7) is unique in the sense ;if $u_1 v_1 = u_2 v_2$ ($u_1 < U_w$, $v_1 < U^w$), then $u_1 = u_2$ and $v_1 = v_2$.

(ii) For any $w \in W^{J_\Lambda} (\Lambda \in P_+)$, there exists an isomorphism of ind (algebraic)-varieties

$$\delta : U_w \sim \to X(\Lambda)_w \quad \varepsilon_u \mapsto u \cdot v_{\Lambda}$$

The following lemma is the first step for our purpose.

Lemma 4.8. For any $u \in U$ and $w \in W$, there exist unique $u' \in U_w \cdot \bar{w}$ and $v \in U$ such that $uw = u'v$.

Proof. By Lemma 4.7(i), there are unique $u'' \in U_w$ and $v'' \in U^w$ such that $u = u'' v''$. By the definition $U_w = U \cap \bar{w}U \bar{w}^{-1}$, we have $\bar{w}^{-1} v'' \bar{w} \in U$. Thus, setting $u' = u'' \bar{w}$ and $v = \bar{w}^{-1} v'' \bar{w}$, we get the desired result.

By using this decomposition, we define the following rational morphisms:

$$p_w : U \cdot \bar{w} \longrightarrow U_w \cdot \bar{w}$$

$$\bar{u} \bar{w} \mapsto u'$$

$$p^w : U \cdot \bar{w} \longrightarrow U$$

$$\bar{u} \bar{w} \mapsto v$$

Define a rational $U$-action on $U_w \cdot \bar{w}$ by

$$U \times U_w \cdot \bar{w} \longrightarrow U_w \cdot \bar{w}$$

$$(x, \bar{u} \bar{w}) \mapsto x(\bar{u} \bar{w}) := p_w(x \bar{u} \bar{w}) = xu \bar{w} \cdot p^w(xu \bar{w})^{-1}$$

Next, we show the following lemma:

Lemma 4.9. Let $\pi^- : G \to B^-$ be as above. For $x \in U$ and $u \bar{w} \in U_w \bar{w}$, we have

$$\alpha^- B^- (x, \pi^-(u \bar{w})) = \pi^- (x(u \bar{w})).$$
Proof. We have
\[
\pi^-(x(u\bar{w})) = x(u\bar{w}) \cdot \pi(x(u\bar{w}))^{-1} = xu\bar{w} \cdot p^w(xu\bar{w})^{-1} \pi(xu\bar{w} \cdot p^w(xu\bar{w})^{-1})^{-1} = xu\bar{w} \cdot p^w(xu\bar{w})^{-1} p^w(xu\bar{w}) \pi(xu\bar{w})^{-1} \quad \text{(since } p^w(xu\bar{w}) \in U)\]
\[
= xu\bar{w} \cdot \pi(xu\bar{w})^{-1} = \pi^-(xu\bar{w})
\]
On the other hand,
\[
\alpha_{B^-}(x, \pi^-(u\bar{w})) = \pi^-(x\pi^-(u\bar{w})) = x\pi^-(u\bar{w}) \cdot \pi(x\pi^-(u\bar{w}))^{-1} = xu\bar{w} \cdot \pi(u\bar{w})^{-1} \cdot \pi(xu\bar{w} \cdot \pi(u\bar{w})^{-1})^{-1} = xu\bar{w} \cdot \pi(u\bar{w})^{-1} \cdot \pi(xu\bar{w}) \cdot \pi(xu\bar{w})^{-1} \quad \text{(since } \pi(u\bar{w}) \in U)\]
\[
= xu\bar{w} \cdot \pi(xu\bar{w})^{-1} = \pi^-(xu\bar{w}),
\]
which completes the proof. \[\square\]

Define an isomorphism of ind (algebraic)-varieties
\[
\zeta : X(\Lambda)_w \xrightarrow{\sim} \text{ U}_w\bar{w}
\]
\[
v \mapsto \zeta(v) := \delta^{-1}(v)\bar{w},
\]
where \( w \in W^J\Lambda \) and \( \Lambda \in P_+ \). Since \( X(\Lambda)_w \) is \( U \)-orbit of \( \rho(\bar{w} \cdot P_{Jw}/P_{Jw}) \), \( U \) acts rationally on \( X(\Lambda)_w \). We denote the action of \( x \in U \) on \( v \in X(\Lambda)_w \) by \( x(v) \).

Lemma 4.10. The isomorphism \( \zeta : X(\Lambda)_w \to \text{ U}_w\bar{w} \) is a \( U \)-morphism.

Proof. It is sufficient to show that \( \zeta(x(v)) = x(\zeta(v)) \) for \( x \in U \) and \( v \in X(\Lambda)_w \). Set \( u = \delta^{-1}(v) \) and then we have \( v = u\bar{w}v_{\Lambda} \). Since \( v_{\Lambda} \) is stable by the action of \( U \), i.e., \( U \cdot v_{\Lambda} = v_{\Lambda} \), we get
\[
x(v) = p_w(xuu\bar{w})(v_{\Lambda}).
\]
Since \( p_w(xuu\bar{w}) \in \text{ U}_w\bar{w} \), we get
\[
\zeta(x(v)) = p_w(xu\bar{w}).
\]
We also have \( x(\zeta(v)) = x(u\bar{w}) = p_w(xuu\bar{w}) \) and then \( \zeta(x(v)) = x(\zeta(v)) \). \[\square\]

Define a rational morphism \( f_w : X(\Lambda)_w \to B^- \) by \( f_w = \pi^- \circ \zeta \). The following is one of the main results of this article.

Theorem 4.11. For \( \Lambda \in P_+ \) and \( w \in W^J\Lambda \), let \( X(\Lambda)_w \) be a finite Schubert cell and \( f_w : X(\Lambda)_w \to B^- \) be as defined above. Then the pair \( (X(\Lambda)_w, f_w) \) is a unipotent \( G \)-crystal.

Proof. We see that \( X(\Lambda)_w \) is a \( U \)-variety in Lemma 4.10. So, we may show that \( f_w \) is a \( U \)-morphism. For \( x \in U \) and \( v \in X(\Lambda)_w \), we get
\[
f_w(x(v)) = \pi^-(\zeta(x(v))) = \pi^-(x(\zeta(v))) = \pi^-(x(u\bar{w})),
\]
where \( u = \delta^{-1}(v) \). On the other hand,
\[
x(f_w(v)) = x(\pi^-(\zeta(v))) = x(\pi^-(u\bar{w})) = \alpha_{B^-}(x, \pi^-(u\bar{w})).
\]
By Lemma 4.10 we obtain \( f_w(x(v)) = x(f_w(v)) \), which implies that \( f_w \) is a \( U \)-morphism. \[\square\]
In the sense of Definition 3.5 ii), $\zeta$ is an isomorphism of unipotent crystals on $X(\Lambda)_w$ and $U_w$. Since $X(\Lambda)_w \hookrightarrow \overline{X}(\Lambda)_w$ is an open embedding, they are birationally equivalent. Let $\omega: \overline{X}(\Lambda)_w \to X(\Lambda)_w$ be the inverse birational isomorphism. Thus, $f_w := f_w \circ \omega: \overline{X}(\Lambda)_w \to B^-$ is a $U$-morphism. Then we have

**Corollary 4.12.** Let $\overline{X}(\Lambda)_w$ be a finite Schubert variety and $f_w$ be defined as above. Then the pair $(\overline{X}(\Lambda)_w, f_w)$ is a unipotent $G$-crystal.

**Remark.** Note that for all $w \leq w'$, we have the closed embedding $\overline{X}(\Lambda)_w \hookrightarrow \overline{X}(\Lambda)_{w'}$ (14), and then isomorphism

$$X(\Lambda) \sim \lim_{w \in W/\Lambda} \overline{X}(\Lambda)_w.$$  

Nevertheless, in general, we do not obtain a unipotent crystal structure on $X(\Lambda)$ by using this direct limit since for $y < w$, the rational morphism $f_w: \overline{X}(\Lambda)_w \to B^-$ is not defined on $\overline{X}(\Lambda)_y$.

### 4.3 Geometric Crystal structure on $X(\Lambda)_w$

As we have seen in 3.3, we can associate geometric crystal structure with the finite Schubert cell (resp. variety) $X(\Lambda)_w$ (resp. $\overline{X}(\Lambda)_w$) since we have shown that they are unipotent $G$-crystals.

Now, let us verify the condition by which the function $\varphi_i: X(\Lambda)_w \to \mathbb{C}$ is not identically zero.

We recall the formula:

$$x_i(a)y_j(b) = \begin{cases} y_i\left(\frac{b}{1+ab}\right)\alpha_j^\vee(1+ab)x_i\left(\frac{a}{1+ab}\right) & \text{if } i = j \\ y_j(b)x_i(a) & \text{if } i \neq j \end{cases} (4.8)$$

Hence, we have

$$x_i(c)\bar{s}_i = y_i\left(\frac{1}{c}\right)\alpha_i^\vee(c)x_i\left(-\frac{1}{c}\right),$$  

where $\bar{s}_i = x_i(-1)y_i(1)x_i(-1)$. We also have

$$\alpha_i^\vee(a)x_j(b) = x_j(a^{-a_i}b)\alpha_i^\vee(a), \quad \alpha_i^\vee(a)y_j(b) = y_j(a^{-a_i}b)\alpha_i^\vee(a) (4.10)$$

By the formula (4.8), (4.9) and (4.10), we obtain

$$x_i(a) \cdot \left( y_j\left(\frac{1}{c}\right)\alpha_j^\vee(c) \right) = \left( y_j\left(\frac{1}{c}\right)\alpha_j^\vee(c) \right) \cdot x_i\left(c^{-a_i}a\right), \quad (i \neq j) (4.11)$$

$$x_i(a) \cdot \left( y_i\left(\frac{1}{a+c}\right)\alpha_i^\vee(c) \right) = \left( y_i\left(\frac{1}{a+c}\right)\alpha_i^\vee(c) \right) \cdot x_i\left(\frac{a}{ac+c^2}\right). (4.12)$$

Due to these formula, we get the following lemma;

**Lemma 4.13.** For $w = s_{i_1}s_{i_2} \cdots s_{i_k} \in W$ (reduced expression) and $c_1, c_2, \ldots, c_k \in \mathbb{C}^\times$, there exist $c'_1, c'_2, \ldots, c'_k$ such that

$$\pi^-(x_{i_1}(c_1)\bar{s}_{i_1} \cdot x_{i_2}(c_2)\bar{s}_{i_2} \cdots x_{i_k}(c_k)\bar{s}_{i_k}) = y_{i_1}\left(\frac{1}{c'_1}\right)\alpha_{i_1}(c'_1) \cdot y_{i_2}\left(\frac{1}{c'_2}\right)\alpha_{i_2}(c'_2) \cdots y_{i_k}\left(\frac{1}{c'_k}\right)\alpha_{i_k}(c'_k) (4.13)$$
For $w \in W$, let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ be one reduced expression and set

$$I(w) := \{i_1, i_2, \cdots, i_k\}.$$  

Indeed, this does not depend on the choice of reduced expressions since $W$ is a Coxeter group. By Lemma 4.13 we get

**Lemma 4.14.** For $w \in W$ and $i \in I$, if $i \in I(w)$, then the function $\varphi_i : X(\Lambda)_w \to \mathbb{C}$ is not identically zero.

Now, by Theorem 3.3 we have

**Theorem 4.15.** For $w \in W$, suppose that $I = I(w)$. We can associate the geometric $G$-crystal structure with the finite Schubert cell $X(\Lambda)_w$ (resp. variety $\overline{X}(\Lambda)_w$) by setting (see (3.4) and (3.5))

$$\gamma_w := \text{proj}_T \circ f_w \quad \text{(resp. } \nu_w := \text{proj}_T \circ \tilde{f}_w \text{)}, \quad e_i^c(x) = x_i \left(\frac{c-1}{\varphi_i(x)}\right)(x),$$

where $\text{proj}_T : B^- = U^- T \to T$.

We denote this induced geometric crystal by $(X(\Lambda)_w, \gamma_w, \{e_i\}_{i \in I})$ (resp. $(\overline{X}(\Lambda)_w, \nu_w, \{e_i\}_{i \in I})$). This geometric/unipotent crystal $(X(\Lambda)_w, \gamma_w, \{e_i\}_{i \in I})$ is realized in $B^-$ in the following sense.

**Proposition 4.16.** For $w = s_{i_1}\cdots s_{i_k}$, define

$$B^w := \{Y_w(c_1, \cdots, c_k) := y_{i_1}(\frac{1}{c_1})\alpha^\vee_{i_1}(c_{i_1})\cdots y_{i_k}(\frac{1}{c_k})\alpha^\vee_{i_k}(c_{i_k}) \in B^- | c_i \in \mathbb{C}^\times\}.$$  

and $U$-actions on $B^w_w$ by

$$u(Y_w(c_1, \cdots, c_k)) := \pi^-(u \cdot Y_w(c_1, \cdots, c_k)) \quad (u \in U).$$

Then $X(\Lambda)_w$ and $B^w_w$ are birationally equivalent via $f_w$ and isomorphic as unipotent crystals. Moreover, they are isomorphic as induced geometric crystals.

**Proof.** By Lemma 4.13 it is sufficient to show that they are birationally equivalent to each other and then we may show that $U_w \cdot w$ and $B^w_w$ are birationally equivalent via $\pi^-$. For that purpose, since we have the isomorphism $\text{pr}_1$ and the birational isomorphism $B^w_w \cong (\mathbb{C}^\times)^k$, it suffices to show that the correspondence $(c_1, \cdots, c_k) \longleftrightarrow (c'_1, \cdots, c'_k)$ in (4.14) is birational. In (4.14), each $c'_i$ is a rational function in $c_1, c_2, \cdots, c_i$ obtained by composing the birational morphisms defined by (4.11) and (4.12) (in particular, $c_i = c'_i$), which implies that $U_w \cdot w$ and $B^w_w$ are birationally equivalent.

**Example 4.17.** We consider the case $G = SL_{n+1}(\mathbb{C})$, i.e., the Cartan matrix $A = (a_{ij})_{i,j \in I}$ is given by: $a_{ii} = 2$, $a_{ii+1} = -1$ and $a_{ij} = 0$ otherwise. Here $I = \{1, 2, \cdots, n\}$. Take $\bar{w} = s_1s_2\cdots s_n \in W$. In this case, we can easily find that $I = I(\bar{w})$ and

$$\pi^-(x_1(c_1)s_1x_2(c_2)s_2\cdots x_n(c_n)s_n) = y_1(\frac{1}{c_1})\alpha^\vee_1(c_1)y_2(\frac{1}{c_2})\alpha^\vee_2(c_2)\cdots y_n(\frac{1}{c_n})\alpha^\vee_n(c_n).$$
Here changing the coordinate by \( c_i = a_1 a_2 \cdots a_i \) and identifying \( y_i (a) = I_n + a E_{i+1} \), we obtain

\[
\begin{aligned}
f_{\tilde{w}}(X(\Lambda)_{\tilde{w}}) &= \left\{ u(a) : \begin{pmatrix} a_1 \\ 1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \\ 1 \end{pmatrix} ; a_i \in \mathbb{C}^* \right\}
\end{aligned}
\]

where \( a = (a_1, \ldots, a_{n+1}) \) and \( a_1 a_2 \cdots a_{n+1} = 1 \). By using this explicit presentation, we describe the geometric crystal structure of \( f_{\tilde{w}}(X(\Lambda)_{\tilde{w}}) \). Since \( \varphi_i (u(a)) = \frac{1}{a_i} \), we have

\[
e_i^\vee (u(a)) = x_i (a_i (c-1)) \cdot u(a) \cdot x_i (a_{i+1} (c^{-1} - 1)) = u(a_1, \ldots, a_i, c^{-1} a_{i+1}, \ldots, a_{n+1}).
\]

Further, we have \( \gamma_{\tilde{w}} \) by

\[
\gamma_{\tilde{w}} (x_1 (c_1) s_1 x_2 (c_2) s_2 \cdots x_n (c_n) s_n) = \alpha_1^\vee (c_1) \alpha_2^\vee (c_2) \cdots \alpha_n^\vee (c_n).
\]

## 5 Tropicalization of Crystals and Schubert Varieties

We use the same notations as in the previous sections otherwise stated. We introduce a positive structure on geometric crystals and their ultra-discritizations following to [14 2.5].

Let \( T \) be an algebraic torus over \( \mathbb{C} \) and \( X^* (T) \) (resp. \( X_+ (T) \)) be the lattice of characters (resp. co-characters) of \( T \). Let \( R := \mathbb{C}[|c|][c^{-1}] \) and set \( L(T) := \{ \phi \in \text{Hom}(\mathcal{O}_T, R) \} \) (\( \mathcal{O}_T \) is the ring of regular functions on \( T \)), which is called a set of formal loops on \( T \). Here we specify the discrete valuation

\[
v : R \setminus \{0\} \to \mathbb{Z} \quad \sum_{n > -\infty} a_n c^n \mapsto -\min \{ n \in \mathbb{Z} \} \{ a_n \neq 0 \}.
\]

For any \( \phi \in L(T) \), set \( \text{deg}_T (\phi) := v \circ \phi |_{X_+ (T)} \). Since for \( f_1, f_2 \in R \setminus \{0\} \)

\[
v(f_1 f_2) = v(f_1) + v(f_2),
\]

\( \text{deg}_T (\phi) \) can be considered as an element in \( X_+ (T) = \text{Hom}(X^* (T), \mathbb{Z}) \). Hence, \( \text{deg}_T \) can be seen as a map \( \text{deg}_T : L(T) \to X_+ (T) \). For any \( \lambda^\vee \in X_+ (T) \), define \( L_{\lambda^\vee} (T) := \text{deg}_T^{-1} (\lambda^\vee) \subset L(T) \). Since \( \text{deg}_T^{-1} (\lambda^\vee) \) has an irreducible \( \mathbb{C} \) variety structure and \( L(T) = \bigsqcup_{\lambda^\vee \in X_+ (T)} L_{\lambda^\vee} (T) \), the set of irreducible components \( \pi_0 (L(T)) = \{ L_{\lambda^\vee} (T) | \lambda^\vee \in X_+ (T) \} \) can be identified with \( X_+ (T) \), i.e., \( \text{deg}_T \) induces the bijection \( \text{deg}_T : \pi_0 (L(T)) \overset{1:1}{\to} X_+ (T) \).

More explicitly, set \( T = (\mathbb{C}^*)^l \) and identify \( L(T) \) with \( (R^*)^l \). For \( \lambda^\vee (c) = (c^{m_1}, c^{m_2}, \ldots, c^{m_l}) \) (\( m_j \in \mathbb{Z} \)), we have

\[
L_{\lambda^\vee} (T) = \left\{ \left( b_1 c^{-m_1} + \sum_{n > -m_1} a_n c^n, \ldots, b_l c^{-m_l} + \sum_{n > -m_l} a_n c^n \right) ; b_1, \ldots, b_l \neq 0 \right\}.
\]
Let \( f : T \to T' \) be a rational morphism between two algebraic tori \( T \) and \( T' \). The morphism \( f \) induces the rational morphism \( \tilde{f} : L(T) \to L(T') \) and then the map \( \pi_0(f) : \pi_0(L(T)) \to \pi_0(L(T')) \), which defines the map \( \deg(f) : X_*(T) \to X_*(T') \).

\[
\begin{array}{ccc}
\pi_0(L(T)) & \xrightarrow{\pi_0(\tilde{f})} & \pi_0(L(T')) \\
\downarrow{\tilde{de}_T} & & \downarrow{\tilde{de}_{T'}} \\
X_*(T) & \xrightarrow{\deg(f)} & X_*(T')
\end{array}
\]

A rational function \( f(c) \in \mathbb{C}(c) \) \((f \neq 0)\) is positive if \( f \) can be expressed as a ratio of polynomials with positive coefficients.

Remark. A rational function \( f(c) \in \mathbb{C}(c) \) is positive if and only if \( f(a) > 0 \) for any \( a \) > 0 \((\text{pointed out by M.Kashiwara})\).

If \( f_1, f_2 \in \mathbb{C}(c)(\subset R) \) are positive, then we have

\[
\begin{align*}
\nu(f_1 f_2) &= \nu(f_1) + \nu(f_2), \\
\nu\left(\frac{f_1}{f_2}\right) &= \nu(f_1) - \nu(f_2), \\
\nu(f_1 + f_2) &= \max(\nu(f_1), \nu(f_2)).
\end{align*}
\]

**Definition 5.1 (II).** A rational morphism \( f : T \to T' \) between two algebraic tori \( T, T' \) is called positive, if the following two conditions are satisfied:

(i) For any co-character \( \lambda^\vee : \mathbb{C}^\times \to T \), the image of \( \lambda^\vee \) is contained in \( \text{dom}(f) \).

(ii) For any co-character \( \lambda^\vee : \mathbb{C}^\times \to T \) and any character \( \mu : T' \to \mathbb{C}^\times \), the composition \( \mu \circ f \circ \lambda^\vee \) is a positive rational function.

Denote by \( \text{Mor}^+(T, T') \) the set of positive rational morphisms from \( T \) to \( T' \).

**Lemma 5.2 (II).** For any positive rational morphisms \( f \in \text{Mor}^+(T_1, T_2) \) and \( g \in \text{Mor}^+(T_2, T_3) \), the composition \( g \circ f \) is in \( \text{Mor}^+(T_1, T_3) \).

By this lemma, we can define a category \( \mathcal{T}_+ \) whose objects are algebraic tori over \( \mathbb{C} \) and arrows are positive rational morphisms.

**Lemma 5.3 (II).** For any algebraic tori \( T_1, T_2, T_3 \), and positive rational morphisms \( f \in \text{Mor}^+(T_1, T_2) \), \( g \in \text{Mor}^+(T_2, T_3) \), we have

\[
\deg(g \circ f) = \deg(g) \circ \deg(f).
\]

By this lemma, we obtain a functor

\[
UD : \mathcal{T}_+ \to \text{Set}
\]

\[
(f : T \to T') \mapsto (\deg(f) : X_*(T) \to X_*(T'))
\]

**Definition 5.4 (II).** (i) Let \( \chi = (X, \gamma, \{e_i\}_{i \in I}) \) be a geometric pre-crystal, \( T \) be an algebraic torus and \( \theta : T \to X \) be a birational isomorphism. The isomorphism \( \theta \) is called positive structure on \( \chi \) if it satisfies

(a) the rational morphism \( \gamma \circ \theta : T' \to T \) is positive.
(b) For any $i \in I$, the rational morphism $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ given by
\[
e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^e \circ \theta(t)
\]
is positive.

(ii) Two positive structures $\theta_1, \theta_2$ on a geometric pre-crystal are equivalent if the rational morphisms $\theta_1^{-1} \circ \theta_2$ and $\theta_2^{-1} \circ \theta_1$ are positive.

Applying the functor $\mathcal{U}D$ to positive rational morphisms $e_{\iota,\theta} : \mathbb{C}^\times \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain
\[
\tilde{e}_i := \mathcal{U}D(e_{\iota,\theta}) : \mathbb{Z} \times X_+(T) \to X_+(T)
\]
\[
\tilde{\gamma} := \mathcal{U}D(\gamma \circ \theta) : X_+(T') \to X_+(T).
\]

Now, for given positive structure $\theta : T' \to X$ on a geometric pre-crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$, we associate the triplet $(X_+(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i \in I})$ with a pre-crystal structure (see [1, 2.2]) and denote it by $\mathcal{U}D_{\theta, T'}(\chi)$. By Lemma B3 we have the following theorem:

**Theorem 5.5.** For any geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $\mathcal{U}D_{\theta, T'}(\chi) = (X_+(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i \in I})$ is a free $\mathcal{W}$-crystal (see [1, 2.2]).

We call the functor $\mathcal{U}D$ “ultra-discritization” instead of “tropicalization” unlike in [1]. The term “tropicalization” here means the inverse tropicalization in [1]. More precisely, for an object $B$ in $\mathcal{G}et$, if there exists an object $T$ in $\mathcal{T}_+$ such that $\mathcal{U}D(T) \cong B$ as crystals, we call $T$ a tropicalization of $B$.

Now, we define certain positive structure on geometric crystal $B^-_w$ ($I = I(w)$, and $w \in W^{J(x)}$) and see that it turns out to be a tropicalization of (Langlands dual of) some Kashiwara’s crystal.

For $i \in I$, let $B_i$ be the crystal defined by (see e.g. [3])
\[
B_i := \{(x)_i | x \in \mathbb{Z}\},
\]
\[
\tilde{e}_i(x)_i = (x + 1)_i, \quad \tilde{f}_i(x)_i = (x - 1)_i, \quad \tilde{e}_i(x)_i = \tilde{f}_j(x)_i = 0 \quad (i \neq j)
\]
\[
wt(x)_i = x\alpha_i, \quad \varphi_i(x)_i = -\infty, \quad \varphi_j(x)_i = -\infty \quad (i \neq j).
\]

For $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in W$ and $i = (i_1, i_2, \cdots, i_k) \in R(w)$, we define the morphism $\theta_i : (\mathbb{C}^\times)^k \to B^-_w$ by
\[
\theta_i(c_1, c_2, \cdots, c_k) := y_{i_1}(\frac{1}{c_1})\alpha_{i_1}^\vee(c_1) \cdots y_{i_k}(\frac{1}{c_k})\alpha_{i_k}^\vee(c_k)
\]
(5.5)

Similar statements to the following proposition are given in [4] Theorem 2.11] for reductive cases. Here we show it for arbitrary Kac-Moody cases by direct methods.

**Proposition 5.6.** (i) For any $i \in R(w)$ ($w \in W$), the morphism $\theta_i$ defined in (5.5) is a positive structure on the geometric crystal $B^-_w$.

(ii) Geometric crystal $B^-_w$ is a tropicalization of the Langlands dual of the crystal $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ with respect to the positive structure $\theta_i(c_1, c_2, \cdots, c_k)$, or equivalently $\mathcal{U}D(B^-_w) \cong$ Langlands dual$(B_{i_1} \otimes \cdots \otimes B_{i_k})$ as crystals.
Proof. It is clear that $\theta_i$ is a birational isomorphism. Since the rational morphism $\gamma : B_w \to T$ is given by

$$\gamma \left( y_i \left( \frac{1}{c_1} \alpha_{i_1}^\vee (c_1) \cdots y_k \left( \frac{1}{c_k} \alpha_{i_k}^\vee (c_k) \right) \right) = \alpha_{i_1}^\vee (c_1) \cdots \alpha_{i_k}^\vee (c_k),$$

we have that $\gamma \circ \theta_i$ is positive. In order to show that $e_i \theta_1 : C^\times \times T' \to T'$ is positive, we see the explicit action of $e_i^\vee$ on $Y_w(c_1, \cdots, c_k)$. First let us evaluate $\varphi_i(Y_w(c_1, \cdots, c_k))$.

**Lemma 5.7.** For $Y := y_i(a_1) \cdots y_k(a_k) \in U^-$, we have

$$\varphi_i(Y) = \sum_{j_i = i} a_{ij_i}.$$  \hfill (5.6)

**Proof.** Let $\{j_1, j_2, \cdots, j_r\}$ be the set of indecies such that $i_{jm} = i$. Then we can write

$$Y = A_0 \cdot y_i(a_{i_1}) \cdot A_1 \cdot y_i(a_{i_2}) \cdot A_2 \cdot y_i(a_{i_3}) \cdots A_{r-1} \cdot y_i(a_{i_r}) \cdot A_r,$$

where $A_s := \prod_{j_s < p < j_{s+1}} y_{i_p}(a_{i_p})$ ($j_0 = 0$, $j_{r+1} = k + 1$). Here we set

$$B_m := y_i(- \sum_{m < s \leq r} a_{ij_s}) \cdot A_m \cdot y_i(\sum_{m < s \leq r} a_{ij_s}),$$

Then we have

$$Y = y_i(\sum_{0 < s \leq r} a_{ij_s}) \cdot (B_0 \cdot B_1 \cdots B_r).$$ \hfill (5.7)

Since $B_0 \cdot B_1 \cdots B_r$ is in $Y_{-\alpha_i}$ and the decomposition (5.7) is unique by Lemma 3.7, we have

$$\varphi_i(Y) = \sum_{0 < s \leq r} a_{ij_s} = \sum_{i_j = i} a_{ij},$$

which is the desired result. \quad \Box

Set

$$C_j^* := (c_1^{a_{i_1,i}}, c_2^{a_{i_2,i}}, \cdots, c_{j-1}^{a_{i_{j-1},i}}, c_j^{a_{i_j,i}})^{-1}$$

where $a_{i,j}$ is an $(i,j)$-entry of the generalized Cartan matrix $A$. By (4.10), we have

$$Y_w(c_1, \cdots, c_k) = y_i(c_1^* \cdots y_k(c_k^*) \alpha_i^\vee (c_1) \cdots \alpha_k^\vee (c_k)).$$

Then by Lemma 5.7 we obtain

$$\varphi_i(Y_w(c_1, \cdots, c_k)) = \sum_{i_j = i} C_{ij}^* = \sum_{j=1}^k \frac{\delta_{i,j}}{c_1^{a_{i_1,i}} \cdots c_{j-1}^{a_{i_{j-1},i}} c_j},$$ \hfill (5.8)

For $c \in \mathbb{C}$ and $i \in I$, define $\{\tilde{C}_j\}_{1 \leq j \leq k}$ and $\{C_j\}_{0 \leq j \leq k}$ recursively by

$$\tilde{C}_0 = c, \quad \tilde{C}_j = c_j + \delta_{i,j,i} \tilde{C}_{j-1}, \quad C_j = \frac{C_{j-1} - a_{i_j,i}}{C_j}.$$
Then, by using (4.11) and (4.12) repeatedly, we obtain
\[ x_i(c)(Y_w(c_1, \ldots, c_k)) = Y_w(C_1, \ldots, \tilde{C}_k). \]  
\[ (5.9) \]

It is easy to get the explicit form of \( \overline{C}_j \):
\[ \overline{C}_j = \frac{c \prod_{m=1}^{j} c_m^{1-a_{im,i}}}{\sum_{1 \leq m \leq j, i_m = i} c \cdot D_m + \prod_{m=1}^{j} c_m}, \]
where
\[ D_m := c_1^{1-a_{1,i}} \cdots c_{m-1}^{1-a_{m-1,1}} \cdots c_{m+1} \cdots c_{j-1}c_j. \]
Now, in (5.9) replacing \( c \) with \( (c - 1)/\varphi_i(Y_w(c_1, \ldots, c_k)) \) and using (5.8), we obtain
\[ e^c_i(Y_w(c_1, \ldots, c_k)) = x_i \left( \frac{c - 1}{\varphi_i(Y_w(c_1, \ldots, c_k))} \right)(Y_w(c_1, \ldots, c_k)) =: Y_w(C_1, \ldots, C_k), \]
where
\[ C_j := c_j \cdot \frac{\sum_{1 \leq m \leq j, i_m = i} c_1^{a_{1,i}} \cdots c_{m-1}^{a_{m-1,1}} c_m + \sum_{j \leq m \leq k, i_m = i} c_1^{a_{1,i}} \cdots c_{m-1}^{a_{m-1,1}} c_m + \sum_{1 \leq m < j, i_m = i} c_1^{a_{1,i}} \cdots c_{m-1}^{a_{m-1,1}} c_m}{\sum_{1 \leq m \leq j, i_m = i} c_1^{a_{1,i}} \cdots c_{m-1}^{a_{m-1,1}} c_m + \sum_{j \leq m \leq k, i_m = i} c_1^{a_{1,i}} \cdots c_{m-1}^{a_{m-1,1}} c_m + \sum_{1 \leq m < j, i_m = i} c_1^{a_{1,i}} \cdots c_{m-1}^{a_{m-1,1}} c_m}. \]  
\[ (5.10) \]

By this formula, it is clear that \( e_i(c_\theta) \) is positive. We have shown (i).

Next, in order to show (ii), we see the action of \( \tilde{e}_i \) on \( B_{i_1} \otimes \cdots \otimes B_{i_k} \). Take \( b_i = (b_{i_1})_{i_{1}} \otimes \cdots \otimes (b_{i_k})_{i_k} \) and \( (i_1, \ldots, i_k), b_j \in \mathbb{Z} \). Since the action of \( \tilde{e}_i \) on tensor products is described explicitly in (3), we obtain
\[ \tilde{e}_i^c(b_i) = (\beta_{i_1})_{i_{1}} \otimes \cdots \otimes (\beta_{i_k})_{i_k}, \]
where
\[ \beta_j = b_j + \max \left( \max_{1 \leq m \leq j, \ i_m = i} (c - b_m - \sum_{l < m} b_{l_a_{i,l_i}}), \ \max_{j \leq m \leq k, \ i_m = i} (-b_m - \sum_{l < m} b_{l_a_{i,l_i}}) \right) \]
\[ - \max \left( \max_{1 \leq m < j, \ i_m = i} (c - b_m - \sum_{l < m} b_{l_a_{i,l_i}}), \ \max_{j \leq m \leq k, \ i_m = i} (-b_m - \sum_{l < m} b_{l_a_{i,l_i}}) \right). \]  
\[ (5.11) \]

Now, we know that (5.10) and (5.11) are related to each other by the tropicalization/ultra-
We have completed the proof of (ii).

The formula similar to (5.10), (5.11) are given in [2, 5.2] for the longest element \( s_0 \) (in reductive cases).

The following formulae are an immediate consequence of Proposition 5.4.1 and Lemma 5.6 which are given implicitly in [1] and shown by direct method in [12].

**Corollary 5.8.** On the crystal \( B_{i_1} \otimes \cdots \otimes B_{i_k} \), we have for any \( c_1, c_2 \in \mathbb{Z}_{\geq 0} \)

\[
\begin{align*}
\tilde{e}_i^{c_1} \tilde{e}_j^{c_2} &= \tilde{e}_j^{c_2} \tilde{e}_i^{c_1} & \text{if } & (\alpha_i^\vee, \alpha_j^\vee) = 0, \\
\tilde{e}_i^{c_1} \tilde{e}_j^{c_1+c_2} &= \tilde{e}_j^{c_1} \tilde{e}_i^{c_1+c_2} + c_2 \tilde{e}_j^{c_1} & \text{if } & (\alpha_i^\vee, \alpha_j^\vee) = (\alpha_i^\vee, \alpha_i) = -1, \\
\tilde{e}_i^{c_1} \tilde{e}_j^{2c_1+c_2} &= \tilde{e}_j^{c_2} \tilde{e}_i^{c_1} + c_2 \tilde{e}_j^{c_1} & \text{if } & (\alpha_i^\vee, \alpha_j^\vee) = -1, (\alpha_i^\vee, \alpha_i) = -2, \\
\tilde{e}_i^{c_1} \tilde{e}_j^{c_1+3c_2+c_2} & \text{if } (\alpha_i^\vee, \alpha_j^\vee) = -1, (\alpha_i^\vee, \alpha_i) = -3.
\end{align*}
\]

**Remark.** What we considered in Example 4.17 is a different kind of positive structure on \( B_{w}^- \) where \( w = s_1 s_2 \cdots s_n \). More precisely, we define a rational morphism:

\[
\tilde{\theta} : (\mathbb{C}^\times)^n \longrightarrow B_{w}^- \quad (a_1, \cdots, a_n) \mapsto y_1(\frac{1}{a_1}) \alpha_1^\vee(c_1) \cdots y_n(\frac{1}{a_n}) \alpha_n^\vee(c_n),
\]

where \( c_i = a_1 a_2 \cdots a_i \). Then it is easy to see that \( \tilde{\theta} \) gives a positive structure on \( B_{w}^- \). Indeed, since we have

\[
e_i^c(Y_w(c_1, \cdots, c_n)) = Y_w(c_1, \cdots, c_1, c c_1, c_1+1, \cdots, c_n),
\]

we obtain

\[
e_i^c \tilde{\theta}(c, a_1, \cdots, a_n, a_{n+1}) = (a_1, \cdots, a_i c^{-1} a_{i+1}, \cdots, a_n, a_{n+1}),
\]

where \( a_1 \cdots a_{n+1} = 1 \). The ultra-discritization of the geometric crystal on \( B_{w}^- \) with respect to \( \tilde{\theta} \) is as follows; Set \( \tilde{B} := \{(x_1, \cdots, x_{n+1}) \in \mathbb{Z}^{n+1} | x_1 + \cdots + x_{n+1} = 0 \} \) and for \( x := (x_1, \cdots, x_{n+1}) \in \tilde{B} \), set

\[
\tilde{e}_i^c(x) = (x_1, \cdots, x_i + c, x_{i+1} - c, \cdots, x_{n+1}) \quad (c \in \mathbb{Z}_{\geq 0}),
\]

and \( \tilde{f}_i^c = \tilde{e}_i^{-c} \). Then \( \text{UD}_{\tilde{\theta}, \tilde{e}^c} (B_{w}^-, \gamma, \{e_i\}) \) is the Langlands dual of the crystal \( \tilde{B} \). The crystal \( \tilde{B} \) holds the similar structure to some limit of “crystal base for the symmetric tensor module”.

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6 Tropical Braid-type isomorphisms

As an application of the tropicalization/ultra-discretization given in the previous section, we shall give a new proof of the braid-type isomorphisms of crystals [13]. In order to do it, let us give the "tropical braid-type isomorphism" (similar formula is given in [2]):

**Proposition 6.1.** We have the following identities:

(i) **Type A₂**:

\[ y_i \left( \frac{1}{c_1} \right) \alpha_i^\gamma (c_1) y_j \left( \frac{1}{c_2} \right) \alpha_j^\gamma (c_2) y_i \left( \frac{1}{c_3} \right) \alpha_i^\gamma (c_3) \]

\[ = y_j \left( \frac{c_1}{c_1 c_3 + c_2} \right) \alpha_j^\gamma \left( \frac{c_1 c_3 + c_2}{c_1} \right) y_i \left( \frac{1}{c_1 c_3} \right) \alpha_i^\gamma \left( \frac{c_1 c_3 + c_2}{c_1 c_3 + c_2} \right) \]

(ii) **Type B₂** \( (\langle \alpha_i^\gamma, \alpha_j \rangle = -2, \langle \alpha_j^\gamma, \alpha_i \rangle = -1) \):

\[ y_i \left( \frac{1}{c_1} \right) \alpha_i^\gamma (c_1) y_j \left( \frac{1}{c_2} \right) \alpha_j^\gamma (c_2) y_i \left( \frac{1}{c_3} \right) \alpha_i^\gamma (c_3) y_j \left( \frac{1}{c_4} \right) \alpha_j^\gamma (c_4) \]

\[ = y_j \left( \frac{1}{d_1} \right) \alpha_j^\gamma (d_1) y_i \left( \frac{1}{d_2} \right) \alpha_i^\gamma (d_2) y_j \left( \frac{1}{d_3} \right) \alpha_j^\gamma (d_3) y_i \left( \frac{1}{d_4} \right) \alpha_i^\gamma (d_4), \]

where \( d_1 = c_4 + \frac{1}{c_2} \left( c_3 + \frac{c_3}{c_1} \right)^2 \), \( d_2 = c_1 c_4 + c_3 + \frac{c_1 c_3^2}{c_2} \), \( d_3 = \frac{1}{c_2 + c_3 + \frac{c_1 c_3}{c_1}} \), \( d_4 = \frac{1}{c_1 + c_3 + \frac{c_1 c_3}{c_1}} \).

(iii) **Type G₂** \( (\langle \alpha_i^\gamma, \alpha_j \rangle = -3, \langle \alpha_j^\gamma, \alpha_i \rangle = -1) \):

\[ y_i \left( \frac{1}{c_1} \right) \alpha_i^\gamma (c_1) y_j \left( \frac{1}{c_2} \right) \alpha_j^\gamma (c_2) y_i \left( \frac{1}{c_3} \right) \alpha_i^\gamma (c_3) y_j \left( \frac{1}{c_4} \right) \alpha_j^\gamma (c_4) y_i \left( \frac{1}{c_5} \right) \alpha_i^\gamma (c_5) y_j \left( \frac{1}{c_6} \right) \alpha_j^\gamma (c_6) \]

\[ = y_j \left( \frac{1}{d_1} \right) \alpha_j^\gamma (d_1) y_i \left( \frac{1}{d_2} \right) \alpha_i^\gamma (d_2) y_j \left( \frac{1}{d_3} \right) \alpha_j^\gamma (d_3) y_i \left( \frac{1}{d_4} \right) \alpha_i^\gamma (d_4) y_j \left( \frac{1}{d_5} \right) \alpha_j^\gamma (d_5) y_i \left( \frac{1}{d_6} \right) \alpha_i^\gamma (d_6), \]

where

\[ d_1 = \frac{1}{c_2} \left( c_3 + \frac{c_2}{c_1} \right)^3 + \frac{1}{c_4} \left( c_5 + \frac{c_4}{c_3} \right)^3 + \frac{2 c_4}{c_2} + \frac{3 c_4}{c_1 c_3} + \frac{3 c_5}{c_1} + \frac{3 c_3 c_5}{c_2} + c_6, \]

\[ d_2 = \frac{c_1}{c_4} \left( c_5 + \frac{c_4}{c_3} \right)^3 + \frac{c_1 c_3}{c_2} \left( c_3 + \frac{c_2}{c_1} \right)^3 + \frac{3 c_1 c_3 c_5}{c_2} + \frac{2 c_1 c_4}{c_2} + \frac{2 c_4}{c_3} + c_1 c_6 + 2 c_5, \]

\[ d_5 = \frac{1}{c_2} \left( \frac{1}{c_4} \left( c_5 + \frac{c_4}{c_3} \right)^2 + \frac{c_4}{c_2} + \frac{1}{c_4} \right)^3 + \frac{3 c_3 c_4}{c_4} + \frac{3 c_2 c_4}{c_2} + \frac{3 c_4}{c_3} + \frac{3}{c_2} + \frac{2}{c_2}, \]

\[ d_6 = \frac{1}{c_1 + \frac{c_3}{c_2} + \frac{c_4}{c_1} \left( c_5 + \frac{c_4}{c_3} \right)^2 + \frac{c_4}{c_5}} \text{ and } \frac{c_2 c_4 c_6}{d_1 d_5}, \frac{c_1 c_3 c_5}{d_2 d_6}. \]
Proof. To show the above proposition, we need the following well-known facts.

Lemma 6.2. We have the following identities:

(i) Type $A_2$:
\[ y_i(a)y_j(b) = y_{\alpha_i + \alpha_j}(ab)y_j(b)y_i(a). \]  

(ii) Type $B_2$ ($\langle \alpha_i^\vee, \alpha_j \rangle = -2, \langle \alpha_j^\vee, \alpha_i \rangle = -1$):
\[ y_i(a)y_j(b) = y_{2\alpha_i + \alpha_j}(a^2b)y_{\alpha_i + \alpha_j}(ab)y_j(b)y_i(a), \]
\[ y_i(a)y_{\alpha_i + \alpha_j}(b) = y_{2\alpha_i + \alpha_j}(2ab)y_{\alpha_i + \alpha_j}(b)y_i(a). \]

(iii) Type $G_2$ ($\langle \alpha_i^\vee, \alpha_j \rangle = -3, \langle \alpha_j^\vee, \alpha_i \rangle = -1$):
\[ y_i(a)y_j(b) = y_{3\alpha_i + 2\alpha_j}(a^3b^2)y_{3\alpha_i + \alpha_j}(a^3b)y_{2\alpha_i + \alpha_j}(ab)y_j(b)y_i(a), \]
\[ y_{\alpha_i + \alpha_j}(a)y_{2\alpha_i + \alpha_j}(b) = y_{3\alpha_i + 2\alpha_j}(3ab)y_{2\alpha_i + \alpha_j}(b)y_{\alpha_i + \alpha_j}(a), \]
\[ y_j(a)y_{\alpha_i + \alpha_j}(b) = y_{3\alpha_i + 2\alpha_j}(-ab)y_{3\alpha_i + \alpha_j}(b)y_j(a). \]

By using these relations, immediately we obtain the $A_2$ and $B_2$ cases. The $G_2$ case is quite complicated to obtain the explicit form of $d_j$’s. Using (6.10), (6.11), (6.13) and (6.14), we can write the both sides of (6.4) in the form:
\[ y_{3\alpha_i + 2\alpha_j}(A)y_{3\alpha_i + \alpha_j}(B)y_{2\alpha_i + \alpha_j}(C)y_{\alpha_i + \alpha_j}(D)y_j(E)y_i(F)\alpha_i^\vee(G)\alpha_j^\vee(H). \]

Then comparing the both sides, we get (6.5), (6.6), (6.7) and (6.8).

By Proposition 6.1, we easily see that each $d_j$ is a positive rational function in $c_j$’s. Thus, the map
\[ (c_1, c_2, \ldots) \mapsto y_j\left(\frac{1}{d_1}\right)\alpha_i^\vee(d_1)y_i\left(\frac{1}{d_2}\right)\alpha_j^\vee(d_2) \ldots \]
gives rise to positive structures on $B_w^-$, where $w_0$ is the longest element of the Weyl group of type $A_2$, $B_2$ or $G_2$. Then, if we consider the ultra-discritization of this positive structures, we obtain the so-called “braid-type isomorphisms” between the tensor products of the crystal $B_i$’s (13).

Proposition 6.3 ([13]). (i) If $\langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle = 0$,
\[ \phi_j^{(0)}: B_i \otimes B_i \xrightarrow{\sim} B_j \otimes B_i, \]
\[ (x)_i \otimes (y)_j \mapsto (y)_j \otimes (x)_i. \]

(ii) If $\langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle = -1$,
\[ \phi_j^{(1)}: B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_i, \]
\[ (z_1)_i \otimes (z_2)_j \otimes (z_3)_i \mapsto (\max(z_3, z_2 - z_1))_j \otimes (z_1 + z_3)_i \otimes (-\max(-z_1, z_3 - z_2))_j, \]

(6.15)
(iii) If \( \langle \alpha^\vee_i, \alpha_j \rangle = -1, \langle \alpha^\vee_j, \alpha_i \rangle = -2, \) 

\[
\phi^{(2)}_{ij} : B_i \otimes B_j \otimes B_i \otimes B_j \overset{\sim}{\to} B_j \otimes B_i \otimes B_j \otimes B_i,
\]

\[
(z_1)_i \otimes (z_2)_j \otimes (z_3)_i \otimes (z_4)_j \mapsto (Z_1)_j \otimes (Z_2)_i \otimes (Z_3)_j \otimes (Z_4)_i;
\]

\[
\begin{align*}
Z_1 &= \max(z_4, z_2 - 2z_1, 2z_3 - z_2) \\
Z_2 &= \max(z_1 + z_4, z_3, z_1 - z_2 + 2z_3) \\
Z_3 &= -\max(-z_2, -z_4 - 2z_1, -2z_2 + 2z_3 - z_4) \\
Z_4 &= -\max(-z_3 + z_4, -z_1, z_3 - z_2)
\end{align*}
\]

(6.16)

(iv) If \( \langle \alpha^\vee_i, \alpha_j \rangle = -1, \langle \alpha^\vee_j, \alpha_i \rangle = -3, \) 

\[
\phi^{(3)}_{ij} : B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \overset{\sim}{\to} B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i
\]

\[
(z_1)_i \otimes (z_2)_j \otimes (z_3)_i \otimes (z_4)_j \otimes (z_5)_i \otimes (z_6)_j
\]

\[
\mapsto (Z_1)_j \otimes (Z_2)_i \otimes (Z_3)_j \otimes (Z_4)_i \otimes (Z_5)_j \otimes (Z_6)_i;
\]

\[
\begin{align*}
Z_1 &= \max(z_6, 3z_5 - z_4, -3z_3 + 2z_4, -2z_2 + 3z_3, -3z_1 + z_2) \\
Z_2 &= \max(z_1 + z_6, z_1 - z_4 + 3z_5, z_1 - 3z_3 + 2z_4, z_1 - 2z_2 + 3z_3, -z_1 + z_3) \\
Z_3 &= z_2 + z_4 + z_6 - Z_1 - Z_5, \\
Z_4 &= z_1 + z_3 + z_5 - Z_2 - Z_6, \\
Z_5 &= -\max(-z_4 + z_6, -3z_4 + 6z_5 - z_6, -6z_3 + 3z_4 - z_6, \\
&\quad -3z_2 + 3z_3 - z_6, -3z_1 - z_6) \\
Z_6 &= -\max(-z_1, -z_2 + z_3, -z_4 + 2z_5, -2z_3 + z_4, -z_5 + z_6)
\end{align*}
\]

(6.17)

We call \( \phi^{(k)}_{ij} \) \((k = 0, 1, 2, 3)\) a braided-type isomorphism.

**Proof.** The formula in (6.16), (6.16) and (6.17) are obtained by rewriting the ones in [13] Proposition 4.1 by using:

\[
a_1 + (a_2 + (a_3 + (\cdots + (a_k)_+ \cdots )_+)_+) = \max(a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots, a_1 + \cdots + a_k).
\]

In (6.1), the ultra-discritizations of \( \frac{c_1 + \cdots + c_k}{c_1}, c_1 c_3 \) and \( \frac{c_1 + c_2}{c_1 c_3 + c_2} \) are

\[
v(\frac{c_1 + \cdots + c_k}{c_1}) = \max(v(c_1) + v(c_3), v(c_2) - v(c_1)) = \max(v(c_3), v(c_2) - v(c_1)),
\]

\[
v(c_1 c_3) = v(c_1) + v(c_3),
\]

\[
v(\frac{c_1 + c_2}{c_1 c_3 + c_2}) = v(c_1) + v(c_2) - \max(v(c_1) + v(c_3), v(c_2)) = -\max(v(c_3) - v(c_2), -v(c_1))
\]

Thus, replacing \( v(c_1) \) with \( z_i \), we obtain (6.15).

Similarly, considering the ultra-discritizations of \( d_i \)'s in (6.2) and (6.3), we get (6.16). Here note that in Proposition 6.1 (iii), we suppose \( \langle \alpha^\vee_i, \alpha_j \rangle = -1, \langle \alpha^\vee_j, \alpha_i \rangle = -2 \), which is the Langlands dual of the condition in Proposition 6.1 (ii).
In order to get the formula (6.17), we consider, e.g., $v(d_1)$:

\[
v(d_1) = \max(-2z_2 + 3z_3, -3z_1 + z_2, 3z_5 - z_4, -3z_3 + 2z_4, z_6, \\
z_4 - z_2, z_4 - z_1 - z_3, z_5 - z_1, z_3 + z_5 - z_2) \quad (v(c_j) = z_j),
\]

which seems to be different from $Z_1$ in (6.17). But, it is easy to see that both are same by the following simple formula:

For $m_1, \cdots, m_k \in \mathbb{R}$ and $t_1, \cdots, t_k \in \mathbb{R}_{\geq 0}$ satisfying $t_1 + \cdots + t_k = 1$, we have

\[
\max \left( m_1, \cdots, m_k, \sum_{j=1}^{k} t_j m_j \right) = \max(m_1, \cdots, m_k)
\]

Indeed, in (6.18) we have

\[
\begin{align*}
z_4 - z_2 &= \frac{1}{6}A_1 + \frac{1}{2}A_4, \\
z_4 - z_1 - z_3 &= \frac{1}{6}A_1 + \frac{1}{3}A_2 + \frac{1}{2}A_4, \\
z_5 - z_1 &= \frac{1}{6}A_1 + \frac{1}{2}A_2 + \frac{1}{3}A_3 + \frac{1}{2}A_4, \\
z_3 + z_5 - z_2 &= \frac{1}{6}A_1 + \frac{1}{3}A_4 + \frac{1}{2}A_4,
\end{align*}
\]

where $A_1 := -2z_2 + 3z_3$, $A_2 := -3z_1 + z_2$, $A_3 := 3z_5 - z_4$ and $A_4 := -3z_3 + 2z_4$.

Hence we have $Z_1 = v(d_1)$. Others are obtained similarly. Thus, considering the Langlands dual, we get the desired result.

\[\square\]

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