Anomalous currents in a driven $XXZ$ chain with boundary twisting at weak coupling or weak driving

Vladislav Popkov$^{1,2}$ and Mario Salerno$^3$

$^1$ Dipartimento di Fisica e Astronomia, Università di Firenze, via G Sansone 1, I-50019 Sesto Fiorentino, Italy
$^2$ Max Planck Institute for Complex Systems, Nöthnitzer Straße 38, D-01187 Dresden, Germany
$^3$ Dipartimento di Fisica ‘ER Caianiello’, Università di Salerno, via ponte don Melillo, I-84084 Fisciano (SA), Italy
E-mail: popkov@fi.infn.it and salerno@sa.infn.it

Received 5 December 2012
Accepted 21 January 2013
Published 27 February 2013

Abstract. The spin $1/2$ $XXZ$ chain driven out of equilibrium by coupling with boundary reservoirs targeting perpendicular spin orientations in the $XY$ plane is investigated. The existence of an anomaly in the nonequilibrium steady state (NESS) at the isotropic point $\Delta = 1$ is demonstrated in both the weak coupling and weak driving limits. The nature of the anomaly is studied analytically by calculating exact NESSs for small system sizes, and investigating steady currents. The spin current at the points $\Delta = \pm 1$ has a singularity which leads to a current discontinuity when either driving or coupling vanishes, and the current of energy develops a twin peak anomaly. The character of the singularity is shown to depend qualitatively on whether the system size is even or odd.

Keywords: exact results, transport processes/heat transfer (theory), stationary states, quantum transport

ArXiv ePrint: 1212.1088
1. Introduction

The coupling of a quantum system with an environment or under a continuous measurement is often modelled via a quantum master equation in the Lindblad form [1, 2]. Unlike a quantum system evolving coherently, the time evolution of which depends on the initial state, a quantum system with pumping tends to a nonequilibrium steady state, independent of the initial conditions. Recently, there has been a great deal of interest in investigating chains of quantum spins driven towards nonequilibrium steady states by applying gradients (pumping) at the edges. An important role in these studies is played by the one-dimensional driven $XXZ$ chain. The equilibrium $XXZ$ model remains a subject of intensive investigation via various methods [4]–[8]. In experiments, many transport characteristics in quasi-one-dimensional spin chain materials can be measured [9]. In the context of dissipative dynamics, recent numerical and analytic studies of a driven $XXZ$ chain show a number of remarkable features like long range order and quantum phase transitions far from equilibrium [10], negative differential conductivity of spin and energy [11, 12], solvability [13]–[15] etc.

Recently, it was shown that at an isotropic point $\Delta = 1$, a weakly driven $XXZ$ spin chain has anomalous spin transport [16]. The current-carrying nonequilibrium steady state was obtained by bringing an $XXZ$ spin chain into contact with reservoirs aligning the boundary spins along the $Z$-axis. The boundary gradient was small, which allows one to think in terms of the linear response of a system to a weak perturbation along the anisotropy axis.

From the standard theory of linear response [17] one expects that the result of linear response will not depend on the form of boundary perturbation as long as it remains sufficiently weak. Respectively, an anomaly seen with one type of perturbation should be visible with another type of perturbation as well. On the other hand, the linear response theory is valid for a system in thermodynamic equilibrium at a certain temperature, while driven chains with dissipative boundary driving are intrinsically nonequilibrium systems.
At ‘zero gradient’ the nonequilibrium steady state (NESS) describes a system without currents, in a maximally mixed quantum state, which corresponds to a Gibbs ensemble with an infinite temperature. To test further the linear response regime in the situation of a nonequilibrium, we investigate a weakly driven XXZ chain, where the weak boundary gradient at the boundaries is applied in the XY direction, transverse to the anisotropy axis [18, 19]. Note that there are two complementary ways of realizing a weak drive in a master equation formalism: (i) set a weak boundary gradient, but finite or strong coupling to boundary reservoirs, (ii) set a weak coupling to reservoirs, but finite or strong boundary gradient.

Interestingly, in both contexts (either weak gradient $\kappa \ll 1$ or weak coupling $\Gamma \ll 1$) we see a distinct anomaly in the NESS. The anomaly becomes a singularity as $\Gamma \rightarrow 0$ or $\kappa \rightarrow 0$ of the magnetization current at the isotropic point $\Delta = 1$. For weak coupling the anomaly sets in for $N \geq 2$ and for weak driving for $N \geq 4$. The presence of an anomaly is seen in practically all observables. In this paper we focus on the singular behaviour of the steady currents of spin $j$ and energy $J^E$, and one-point correlations at the boundaries. The full analytic treatment is reported for spin current. In the limit $\kappa \rightarrow 0$ or $\Gamma \rightarrow 0$ we observe a non-analyticity in the dependence $j(\Delta)$: the spin current is an analytic smooth function of $\Delta$ everywhere except at the isotropic point $\Delta = 1$ where its value is different. The nature of the singularity, in addition, depends on the parity of the system size, namely on whether the system size is an even or an odd number.

Other observables, e.g. current of energy, density profiles, etc, also have a singular behaviour at $\Delta = \pm1$. Thus, we can complement the result of Znidaric [16] with prediction of further anomalies at the isotropic point. In particular, we demonstrate that the energy current also develops an anomaly at the isotropic point, but with a different scaling and of a qualitatively different form.

The plan of this paper is the following. We introduce the model in section 2. The cases of weak coupling and weak driving are treated separately in sections 3 and 4. In the conclusion we summarize our findings.

2. The model

We study an open chain of $N$ quantum spins in contact with boundary reservoirs. The time evolution of the reduced density matrix $\rho$ is described by a quantum master equation in the Lindblad form [1, 2] (here and below we set $\hbar = 1$)

$$\frac{\partial \rho}{\partial t} = -i[H,\rho] + \Gamma_L \mathcal{L}_L[\rho] + \Gamma_R \mathcal{L}_R[\rho],$$  \hspace{1cm} (1)

where $H$ is the Hamiltonian of an open XXZ spin chain with an anisotropy $\Delta$,

$$H = \sum_{k=1}^{N-1} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z),$$  \hspace{1cm} (2)

while $\mathcal{L}_L$ and $\mathcal{L}_R$ are Lindblad dissipators favouring a relaxation of boundary spins $k = 1$ and $N$ towards states described by one-site density matrices $\rho_L$ and $\rho_R$, i.e. $\mathcal{L}_L[\rho_L] = 0$ and $\mathcal{L}_R[\rho_R] = 0$. 

\[\text{doi:10.1088/1742-5468/2013/02/P02040}\]
and $L_R[\rho_R] = 0$. The Lindblad actions $L_L$ and $L_R$ are chosen in the form

$$L_L[\rho] = -\frac{1}{2} \sum_{m=1}^{2} \{ \rho, W_m^\dagger W_m \} + \sum_{m=1}^{2} W_m \rho W_m^\dagger,$$

$$L_R[\rho] = -\frac{1}{2} \sum_{m=1}^{2} \{ \rho, V_m^\dagger V_m \} + \sum_{m=1}^{2} V_m \rho V_m^\dagger,$$

where $W_m$ and $V_m$ are polarization targeting Lindblad operators, which act on the first and the last spin respectively, $W_1 = \sqrt{(1 - \kappa)/2}(\sigma_1^z + i\sigma_1^x)$, $W_2 = \sqrt{(1 + \kappa)/2}(\sigma_1^z - i\sigma_1^x)$, $V_1 = \sqrt{(1 + \kappa)/2}(\sigma_N^y + i\sigma_N^z)$, $V_2 = \sqrt{(1 - \kappa)/2}(\sigma_N^y - i\sigma_N^z)$. In the absence of the unitary term in (1) the boundary spins relax with a characteristic time $\Gamma^{-1}$, $\Gamma_R^{-1}$ to 4 specific states described via the one-site density matrices $\rho_L$ and $\rho_R$, satisfying $L_L[\rho_L] = 0$ and $L_R[\rho_R] = 0$, where

$$\rho_L = \frac{1}{2} (I - \kappa \sigma_N^y),$$

$$\rho_R = \frac{1}{2} (I + \kappa \sigma_N^y).$$

From the definition of a one-site density matrix $\rho_{\text{one-site}} = \frac{1}{2} (I + \sum (\sigma^\alpha) \sigma^\alpha)$, we see that the Lindblad superoperators $L_L$ and $L_R$ indeed try to impose a twisting angle of $\pi/2$ in the $XY$ plane between the first and the last spins. The twisting gradient drives the system in a steady state with currents. In the following we restrict to a symmetric choice

$$\Gamma_L = \Gamma_R = \Gamma$$

for the spin current while $\Gamma_L \neq \Gamma_R$ is chosen for studying the energy current. The parameter $\kappa$ determines the amplitude of the gradient between the left and right boundaries, and therefore plays a fundamentally different role from the coupling strength $\Gamma$. The limits $\kappa = 1$ and $\kappa \ll 1$ will be referred to as the strong driving and weak driving cases respectively. The two limits describe different physical situations as exemplified in other NESS studies. For the symmetric choice (6) the steady state has a global symmetry [19, 22]

$$\rho(N, \Delta) = \Sigma_x U_{rot} R \rho(N, \Delta) R U_{rot}^\dagger \Sigma_x,$$

where $R(A \otimes B \otimes \cdots \otimes C) = (C \otimes \cdots \otimes B \otimes A) R$ is a left–right reflection, and the diagonal matrix $U_{rot} = \text{diag}(1, i)^{\otimes N}$ is a rotation in the $XY$ plane, $U_{rot} \sigma_n^x U_{rot}^\dagger = \sigma_n^y$, $U_{rot} \sigma_n^y U_{rot}^\dagger = -\sigma_n^x$ and $\Sigma_x = (\sigma^z)^{\otimes N}$. In addition, there are further transformations, see [19] for details, relating the NESSs for positive and negative $\Delta$,

$$\rho(2N, -\Delta) = U \rho^*(2N, \Delta) U,$$

$$\rho(2N + 1, -\Delta) = \Sigma_y U \rho^*(2N + 1, \Delta) U \Sigma_y,$$

where $\Sigma_y = (\sigma^y)^{\otimes N}$ and $U = \prod_{n \text{ odd}} \otimes \sigma_n^z$ and the conjugate is taken in the basis where $\sigma^z$ is diagonal. In view of the above symmetries, below we restrict to the case of positive $\Delta \geq 0$.

4 The relaxation times depend on the spin components. At the left boundary, the $x$- and $z$-spin components of $\rho$ relax as $e^{-4t\kappa}$ while the $z$-spin component relaxes as $e^{-2t\kappa}$, see [18].

5 If $\kappa < 1$, the situation is more subtle since then the target states $\rho_L$ and $\rho_R$ are mixed.

6 For a symmetric choice of the couplings (6) the energy current disappears as a consequence of the symmetry (7), see [19].

doi:10.1088/1742-5468/2013/02/P02040
We shall consider two particular limits:

(A) weak coupling, \( \Gamma \ll 1 \); strong driving (large gradient), \( \kappa = 1 \);

(B) weak gradient, \( \kappa \ll 1 \); strong coupling, \( \Gamma^{-1} \ll 1 \).

Both limits show qualitatively similar singular behaviour as \( \kappa \to 0 \) or \( \Gamma \to 0 \) at the isotropic point \( \Delta = 1 \), as demonstrated below.

### 3. The case of strong driving, \( \kappa = 1 \), and weak coupling, \( \Gamma \ll 1 \)

To investigate the system in the case of large boundary gradients, \( \kappa = 1 \), and weak coupling, \( \Gamma \ll 1 \), we solve the linear system of equations that determines the full steady state, analytically, making use of the global symmetry (7) [22], which decreases the number of unknown variables by roughly a factor of 2. For \( N = 2, 3, 4 \) we have indeed 9, 31 and 135 real unknowns, respectively, instead of \( 2^N - 1 \) real unknowns for the full Hilbert space. For \( N = 2 \) the current \( j(\Gamma, \Delta) \) is readily obtained as

\[
j(\Gamma, \Delta) = \frac{8\Gamma^4(3 + 6\Gamma^2 + 2\Delta^4)}{24\Gamma^6 + 3(\Delta^2 - 1)^2 + 2\Gamma^4(7\Delta^4 + 30) + \Gamma^2(2\Delta^4 + 13\Delta^2 + 30)},
\]

from which we see that at the points \( \Delta = \pm 1 \) and in the small coupling limit, it scales quadratically with \( \Gamma \),

\[
\lim_{\Gamma \to 0} \lim_{\Delta \to 1} \frac{j(\Gamma)}{\Gamma^2} \bigg|_{N=2} = \frac{8}{9}.
\]

Note, however, that for \( \Delta \neq \pm 1 \) the current scales as \( \Gamma^4 \), i.e.

\[
\lim_{\Gamma \to 0} \frac{j(\Gamma)}{\Gamma^4} \bigg|_{N=2} = \frac{8(2\Delta^2 + 3)}{3(\Delta^2 - 1)^2}.
\]

For \( N = 3 \) the current \( j(\Gamma, \Delta) \) at \( \Delta = 1 \) still scales as \( \Gamma^2 \),

\[
\lim_{\Gamma \to 0} \lim_{\Delta \to 1} \frac{j(\Gamma)}{\Gamma^2} \bigg|_{N=3} = \frac{16}{9},
\]

but a different scaling behaviour, with respect to the \( N = 2 \) case, is obtained for other \( \Delta \) points, where the \( j \) is found to also scale quadratically,

\[
\lim_{\Gamma \to 0} \frac{j(\Gamma)}{\Gamma^2} \bigg|_{N=3} = \frac{6\Delta (9\Delta^2 + 59)}{9\Delta^4 + 155\Delta^2 + 416}.
\]

Taking the limit \( \Delta \to 1 \) in the above, we obtain

\[
\lim_{\Delta \to 1} \lim_{\Gamma \to 0} \frac{j(\Gamma)}{\Gamma^2} \bigg|_{N=3} = \frac{102}{145},
\]

different from (13). Consequently, the limits \( \Delta \to 1 \) and \( \Gamma \to 0 \) do not commute. In more detail, we have

\[
\lim_{\Delta \to 1} \frac{j(\Gamma)}{\Gamma^2} \bigg|_{N=3} = \frac{16(3 + \Gamma^2)}{27 + 52\Gamma^2 + 12\Gamma^4},
\]
from which equation (13) is straightforwardly obtained. Alternatively, one can see the presence of a singularity at all even orders of expansion of $j(\Gamma)$. Starting from the order 4 of the expansion, all even terms \( \left( \frac{\partial^m j_0(\Gamma, \Delta)}{\partial \Gamma^m} \right)_{\Gamma=0} \) contain poles at $\Delta = \pm 1$ of the type $1/(\Delta^2 - 1)^{m-2}$, e.g.

\[
\left. \frac{\partial^m j_0(\Gamma, \Delta)}{\partial \Gamma^m} \right|_{\Gamma=0} = \frac{(-1)^{m/2+1}}{(\Delta^2 - 1)^{m/2-2}} Q(\Delta),
\]

(17)

with $Q(\Delta)$ a rational function which is regular at points $\Delta = \pm 1$.

For $N = 4$ one can see that the current at $\Delta = \pm 1$ scales with $\Gamma$ as

\[
\lim_{\Delta \to \pm 1} \frac{j(\Gamma)}{\Gamma^2}\bigg|_{N=4} = \frac{8(27 + 46\Gamma^2 + 10\Gamma^4)}{3(27 + 135\Gamma^2 + 144\Gamma^4 + 28\Gamma^6)},
\]

(18)

from which it follows that in the small coupling limit the current reduces to

\[
\lim_{\Gamma \to 0} \lim_{\Delta \to \pm 1} \frac{j(\Gamma)}{\Gamma^2}\bigg|_{N=4} = \frac{8}{3}.
\]

At all other points $\Delta \neq \pm 1$, however, the current scales as for the $N = 2$ case, e.g.

\[
\lim_{\Gamma \to 0} \frac{j(\Gamma)}{\Gamma^4}\bigg|_{N=4} = O\left(\frac{1}{(\Delta^2 - 1)^2}\right).
\]

(20)

Although analytical expressions of the current are very difficult to obtain for larger values of $N$ (e.g. $N > 4$), we can infer from numerical calculations that similar behaviour exists also in these more complicated cases. Indeed, from the exact expressions reported above and from direct numerical calculations it is possible to extrapolate the values of the current for arbitrary $N$ at the points $\Delta \pm 1$ in the small coupling limit as

\[
\lim_{\Gamma \to 0} \lim_{\Delta \to \pm 1} \frac{j(\Gamma)}{\Gamma^2}\bigg|_N = \frac{8}{9}(N-1),
\]

(21)

hinting at $\Gamma = 0$ becoming a singular point in the thermodynamic limit. One can readily see that our conjecture (21) coincides with the exact values reported in equations (11), (13) and (19) for the cases $N = 2, 3$ and 4, respectively. Moreover, numerical results provide us with a high confidence about the validity of equation (21) for arbitrary $N$. More precisely, we find from direct numerical solutions of the LME that the peaks of the current for $N = 7, 8$ and $9$ at $\Gamma = 10^{-5}$ are $5.3333, 6.2222$ and $7.111$, respectively, in perfect agreement with the prediction of equation (21). Also, note from figure 1 that the numerical values of the current peaks depicted in panels (c) and (d) for the cases $N = 54$ and $6$ (3.5556 and 4.4444, respectively) are in excellent agreement with equation (21). Moreover, using the matrix product ansatz [15] we were able to check the conjecture (21) analytically up to sizes $N = 100$ (details to be published elsewhere). From the physical point of view, equation (21) is quite interesting because it implies that in the weak coupling limit the addition of an extra spin to a finite chain contributes to the total current by a ‘quantum’ of $8\Gamma^2/9$. Since the spin current is bounded, equation (21) hints at a singularity in the $\Gamma = 0$ point in the thermodynamic limit $N \to \infty$. Further details and an analytical proof of (21) will be presented elsewhere.
Figure 1. The typical dependence of the renormalized current \( j/\Gamma^2 \) on the anisotropy for small couplings \( \Gamma \) and odd system sizes (panels (a) and (c)) and even system sizes (panels (b) and (d)). The dashed, thin and thick lines correspond to \( \Gamma = 10^{-1}, 10^{-2} \) and \( 10^{-3} \) respectively. The system sizes are \( N = 3, 4, 5 \) and 6 for panels (a), (b), (c) and (d), respectively. The curves are obtained by a numerical solution of the LME.

One would naturally expect that the noticed singular behaviour of the driven XXZ model at the isotropic point would not be restricted to the magnetization current but could be seen for other observables as well. Here, we show that the energy current is also anomalous at the isotropic point. Our purpose here, in addition, is to demonstrate that the global symmetry (7) is not crucial for singularity, because the energy current only flows if the symmetry (7) is broken. Indeed, the energy current operator \( \hat{J}^E_n(\Delta) \), defined through the local Hamiltonian term \( h_{n,n+1} \) as \( \frac{dh_{n,n+1}}{dt} = \hat{J}^E_n(\Delta) - \hat{J}^E_n(\Delta) \),

\[
\hat{J}^E_n(\Delta) = -\sigma^x_n \hat{j}_{n-1,n+1} + \Delta (\hat{j}_{n-1,n} \sigma^z_{n+1} + \sigma^z_{n-1} \hat{j}_{n,n+1}),
\]

where \( \hat{j}_{n,m} = 2(\sigma^+_n \sigma^+_m - \sigma^-_n \sigma^-_m) \), changes sign under the action of (7), \( (RU_{\text{rot}}^+ \Sigma_x) \hat{J}^E_n(\Delta) = -\hat{J}^E_n \), and therefore in the stationary state it is strictly zero, \( \langle \hat{J}^E_n \rangle = J^E = 0 \) (as a local integral of motion, the energy current does not depend on the choice of the bond \( n \)). Lifting up the symmetric choice for the couplings to the left and right reservoirs in (1), i.e., considering the Lindblad equation (1) with \( \Gamma_L \neq \Gamma_R \), we break the symmetry (7) and the energy current becomes permissible, see also [20]. The current of energy for small couplings \( \Gamma_L, \Gamma_R \ll 1 \), with a fixed ratio \( \Gamma_L/\Gamma_R = \text{const} \), shows, again, an approach to a singularity in the isotropic point, but with a different scaling, \( J^E(\Delta = 1) = O(\Gamma_L) \), and
Anomalous currents in a driven XXZ chain with boundary twisting at weak coupling or weak driving

Figure 2. The dependence of the renormalized energy current $j/\Gamma_L$ on the anisotropy for small couplings $\Gamma_L, \Gamma_R$ and system sizes $N = 3$ (a) and $N = 4$ (b). The dashed, thin and thick lines correspond to $\Gamma_L = 0.1, 0.05$ and 0.02 respectively, while $\Gamma_R = \Gamma_L/2$. The curves are obtained by numerical solution of the LME. The peaks for $N = 4$ are at the points $\Delta = \pm 0.5, \pm 1$. (c) A close-up view of (a), demonstrating the double peak structure close to $\Delta = 1$.

Qualitatively different behaviour, see figure 2. Indeed, the anomaly of the energy current at the isotropic point has the shape of a twin peak, which, as $\Gamma_L$ decreases, becomes more and more narrow, see figure 2(c), and for $\Gamma_L \to 0$ develops a twin peak singularity. For $N = 4$ we see a twin peak singularity at $\Delta = 1$ and a single peak singularity at $\Delta = 0.5$, as illustrated in figure 2(b). Unlike the spin current, which is an odd (even) function of $\Delta$ for an odd (even) number of sites, the energy current is always an odd function of $\Delta$. These parity features are direct consequences of the mappings (8) and (9), see also [19]. Further details will be discussed elsewhere. Here, we just mention some other observables which have singularities at the isotropic point, also in the symmetric case $\Gamma_L = \Gamma_R = \Gamma$. These are the actual values of the magnetizations at the boundaries, $\langle \sigma_1^x \rangle, \langle \sigma_N^x \rangle, \langle \sigma_1^y \rangle, \langle \sigma_N^y \rangle$, which, as $\Gamma \to 0$, remain finite at $\Delta = 1$, and attain different values at the points $\Delta \pm 1$, data not shown. Interestingly, these singularities cancel in the differences $\Delta x = \langle \sigma_1^x - \sigma_N^x \rangle$ and $\Delta y = \langle \sigma_1^y - \sigma_N^y \rangle$: the actual magnetization differences $\Delta x, \Delta y$ are regular at $\Delta = 1$.

At this point one might suspect that the singularities in the currents of spin and energy are just artefacts (or direct consequences) of the singular behaviour of the boundary gradients. To eliminate the influence of the irregular behaviour of the boundary gradients, in section 4 we consider another limit, where we control not only the boundary gradients $\Delta x, \Delta y$ but also the individual magnetizations at the boundaries, $\langle \sigma_1^x \rangle, \langle \sigma_1^y \rangle, \langle \sigma_N^x \rangle, \langle \sigma_N^y \rangle$ and $\langle \sigma_N^z \rangle$, which become exactly equal to $0, -\kappa, 0, \kappa, 0$ and 0, respectively.

doi:10.1088/1742-5468/2013/02/P02040
4. The case of weak boundary gradients and strong coupling

In this section we investigate the weak gradient and strong coupling case \((\kappa \ll 1, \Gamma^{-1} \ll 1)\) by means of the perturbative approach for the LME in powers of \(\Gamma^{-1}\), developed in [19]. Note that while in the standard derivation of the Lindblad master equation one usually assumes the coupling to the reservoirs to be weak [1, 3], in the ancilla master equation construction [21] this restriction is lifted and the effective coupling can become arbitrarily strong. We also remark that the uniqueness of the nonequilibrium steady state for any coupling \(\Gamma\) is guaranteed by the completeness of the algebra, generated by the set of operators \(\{H, V_m, W_m, V_m^+, W_m^+\}\) under multiplication and addition [23], and is verified straightforwardly along the lines of [24]. We search for a stationary solution of the Lindblad equation in the form of a perturbative expansion in \(\Gamma^{-1}\),

\[
\rho(\Delta, \Gamma) = \sum_{k=0}^{\infty} \left(\frac{1}{2\Gamma}\right)^k \rho_k(\Delta),
\]

(22)

where \(\rho_0(\Delta) = \lim_{\Gamma \to \infty} \rho(\Delta, \Gamma)\) satisfies \(\mathcal{L}_{LR}[\rho_0(\Delta)] = 0\). Here and below we denote by \(\mathcal{L}_{LR}[\cdot]\) the sum of Lindblad boundary actions \(\mathcal{L}_{LR}[\cdot] = \mathcal{L}_L[\cdot] + \mathcal{L}_R[\cdot]\). This enforces a factorized form

\[
\rho_0(\Delta, \kappa) = \rho_L \otimes \left(\left(\frac{I}{2}\right)^{\otimes N-2} + M_0(\Delta, \kappa)\right) \otimes \rho_R,
\]

(23)

where \(\rho_L\) and \(\rho_R\) are density matrices acting on a single site given by (4) and (5) which satisfy \(\mathcal{L}_L[\rho_L] = 0, \mathcal{L}_R[\rho_R] = 0,\) and \(M_0(\Delta, \kappa)\) is a matrix to be determined self-consistently later. Below we shall drop the \(\Delta\)- and \(N\)-dependence in \(\rho_k\) and \(M_k\) for brevity of notation. Substituting (22) into (1), and comparing the orders of \(\Gamma^{-k}\), we obtain the recurrence relations

\[
i[H, \rho_k] = \frac{1}{2}\mathcal{L}_{LR}\rho_{k+1}, \quad k = 0, 1, 2, \ldots
\]

(24)

A formal solution of the above is \(\rho_{k+1} = -2\mathcal{L}_{LR}^{-1}(Q_{k+1})\), where \(Q_{k+1} = -i[H, \rho_k]\). Note, however, that the operator \(\mathcal{L}_{LR}\) has a nonempty kernel subspace, and is not invertible on the elements from it. The kernel subspace \(\ker(\mathcal{L}_{LR})\) consists of all matrices of type \(\rho_L \otimes A \otimes \rho_R\), where \(A\) is an arbitrary \(2^{N-2} \times 2^{N-2}\) matrix. Therefore, \(\rho_{k+1}\) exists only if \([H, \rho_k] \cap \ker(\mathcal{L}_{LR}) = \emptyset\), which in our case reduces to a requirement of a null partial trace, see [19].

\[
\text{Tr}_{1, N}([H, \rho_k]) = 0, \quad k = 0, 1, 2, \ldots
\]

(25)

which we call secular conditions. Finally, \(\rho_{k+1}\) is defined up to an arbitrary element from \(\ker(\mathcal{L}_{LR})\), so we have

\[
\rho_{k+1} = 2\mathcal{L}_{LR}^{-1}(i[H, \rho_k]) + \rho_L \otimes M_{k+1} \otimes \rho_R, \quad k = 0, 1, 2, \ldots
\]

(26)

Equations (23), (25) and (26) define a perturbation theory for the Lindblad equation (1) for strong couplings. At each order of the perturbation theory the secular conditions (25) must be satisfied, which imposes constraints on \(M_k\). Our aim here is to determine the exact NESS in the limit \(\Gamma \to \infty\) for which considering the two first orders \(k = 0, 1\) has proved to be sufficient. The case \(N = 2\) is trivial since \(\rho_k = \rho_L \otimes \rho_R\). For \(N = 3\), the most general form of the matrices \(M_0(\Delta), M_1(\Delta)\), by virtue of the symmetry (7), is \(M_0 = b_0(\sigma^x - \sigma^y)\),

\[
doi:10.1088/1742-5468/2013/02/P02040
\]

J. Stat. Mech. (2013) P02040

9
$M_1 = b_1(\sigma^x - \sigma^y)$, where $b_0, b_1$ are unknown constants. The secular conditions (25) for $k = 0$ do not give any constraints on $b_0$, while those for $k = 1$ give two nontrivial relations $b_1 = 0$ and $-4\Delta\kappa^3 + (3 + 4\Delta^2 - \kappa^2)b_0 = 0$ from which both $M_0$ and $M_1$ are determined. For $N = 4$, each of the matrices $M_0(\Delta), M_1(\Delta)$ compatible with the symmetry (7) contains nine unknowns, all fixed by the secular conditions (25) for $k = 0, 1$, and so on. For $N = 3$, we obtain

$$M_0(\Delta)|_{N=3} = \frac{4\Delta\kappa^2}{3 + 4\Delta^2 - \kappa^2}(\sigma^x - \sigma^y).$$

(27)

The respective stationary current $j_0(\kappa) = \lim_{\Gamma \to \infty} j(\kappa, \Gamma)$,

$$j_0(\kappa)|_{N=3} = \frac{8\Delta\kappa^2}{3 + 4\Delta^2 - \kappa^2},$$

(28)

does not have any non-analyticity as $\kappa \to 0$, unlike in the case of weak coupling. However, for $N \geq 3$ the singularity sets in. For $N = 4$, the exact expression for the current is

$$j_0(\kappa)|_{N=4} = \frac{8\Delta^2\kappa^4 (16\Delta^4 + 2\Delta^2 (5\kappa^4 - 11\kappa^2 + 24) - 8\kappa^6 + 25\kappa^4 - 36\kappa^2 + 27)}{D(\Delta, \kappa)},$$

(29)

where

$$D(\Delta, \kappa) = 16\Delta^8 (\kappa^2 + 3) + 4\Delta^6 (\kappa^6 + \kappa^4 + 48\kappa^2 - 6) + \Delta^4 (-6\kappa^8 + 21\kappa^6 - 133\kappa^4 + 339\kappa^2 - 69) \Delta^2 (9\kappa^8 - 25\kappa^6 - 7\kappa^4 + 117\kappa^2 + 18) - 2\kappa^8 - 6\kappa^6 + 3\kappa^4 - 27\kappa^2 + 27,$$

(30)

and it does contain a singularity at $\Delta = 1$. Indeed, for $N = 4$ we observe a non-commutativity of the limits $\Delta \to 1$ and $\kappa \to 0$, signalling the presence of a singularity,

$$\lim_{\Delta \to 1} \lim_{\kappa \to 0} \frac{j_0(\kappa)}{\kappa^2} |_{N=4} = 0,$$

(31)

$$\lim_{\kappa \to 0} \lim_{\Delta \to 1} \frac{j_0(\kappa)}{\kappa^2} |_{N=4} = \frac{8}{7},$$

(32)

easily verifiable from (29). Indeed, taking the $\lim_{\kappa \to 0}$ first, we expand $j(\kappa)$ in orders of $\kappa$ and find the first nonzero contribution at the fourth order,

$$\lim_{\kappa \to 0} \frac{j_0(\kappa)}{\kappa^4} |_{N=4} = \frac{8\Delta^2 (4\Delta^2 + 9)}{3 (\Delta^2 - 1)^2 (4\Delta^2 + 3)},$$

(33)

which is singular at $\Delta = 1$. As a consequence, the current at the point $\Delta = 1$ (and, in virtue of the symmetry (8), also $\Delta = -1$) has a different scaling $(j_0(\kappa, \Delta = \pm 1) \sim \kappa^2)$ from all other points $\Delta \neq \pm 1$, where the current scales as $\kappa^4$, see figure 3. Note that the singularity type for weak driving $\kappa$ is exactly the same as the one described in section 3 for an even number of sites and weak coupling $\Gamma$.

For an odd-sized system $N = 5$ the exact expression for the current is very complicated and some limiting cases are reported in the appendix. Analysing the analytic expression in the limit $\kappa \ll 1$ we find scaling of the type $j_{N=5}^{(0)}(\kappa) = \alpha(\Delta)\kappa^2$, where the prefactor $\alpha(\Delta)$ is singular at $\Delta = 1$, namely $\alpha(1) \sim 8/7$ and $\alpha(1 \pm 0) \to 64/181$ as $\kappa \to 0$, see also figure 3. This type of singularity is exactly the same as the one seen in section 3 for vanishing $\Gamma$ and odd system sizes.

doi:10.1088/1742-5468/2013/02/P02040
Anomalous currents in a driven XXZ chain with boundary twisting at weak coupling or weak driving

Figure 3. The typical dependence of the renormalized current $j/\kappa^2$ on the anisotropy for small driving $\kappa$ and even system sizes (a) and odd system sizes (b). The dashed, thin and thick lines correspond to $\kappa = 0.2$, $0.1$ and $10^{-2}$ respectively. The system size is $N = 4$ for (a) and $N = 5$ for (b). The curves are given by the exact analytic expressions (29) for $N = 4$ and an analogous analytic expression (not reported) for $N = 5$.

As the system size increases, the order of the polynomials grows with the system size and becomes complicated. However, at the qualitative level we see the same behaviour as discussed above for $N = 4$ ($N = 5$) for systems of even (odd) sizes.

The qualitative similarity in the magnetization current behaviour in the cases of (i) weak coupling and large driving and (ii) weak driving and large coupling is not a trivial one. In case (ii), the amplitude of the effective $x$- and $y$- boundary gradients is exactly equal to $\kappa$, independently of the anisotropy $\Delta$, so in the limit $\kappa \ll 1$ it becomes infinitesimally small. Also, the individual boundary magnetizations $\langle \sigma_1^\alpha \rangle$, $\langle \sigma_N^\alpha \rangle$, $\alpha = x, y, z$ are bounded by $\kappa$.

In the case of weak coupling, $\Gamma \ll 1$, the amplitude of the boundary gradient at the isotropic point also vanishes in the limit $\Gamma \rightarrow 0$, but the individual boundary magnetizations $\langle \sigma_1^\alpha \rangle$, $\langle \sigma_N^\alpha \rangle$ for $\alpha = x, y$ remain finite and are discontinuous at $\Delta = 1$.

The presence of the singularity and non-commutativity of the limits $\lim_{\Delta \rightarrow 1}$ and $\lim_{\Gamma \rightarrow 0}$ (or $\lim_{\kappa \rightarrow 0}$) must be related to the existence of an additional symmetry which the XXZ model acquires at the isotropic point $\Delta = 1$. This symmetry, however, is not the full rotational symmetry of a unitary quantum XXX Hamiltonian, since the dissipative Lindblad terms in the LME do not have the full rotational symmetry. On the other hand, the qualitative differences between the singularities for odd and even system sizes $N$, seen for the spin current, energy current and other observables, are a consequence of the intrinsic properties of the XY-twisted model, which are manifested by (8) and (9).

5. Conclusions

We have investigated an XXZ spin chain coupled at the ends to dissipative boundary reservoirs, which impose a twisting angle between the first and last spins in the XY plane, described by a Lindblad master equation. We pointed out the non-analytic character of the nonequilibrium steady state in the two limits: vanishing coupling $\Gamma$ and vanishing driving
Anomalous currents in a driven XXZ chain with boundary twisting at weak coupling or weak driving

κ. Unlike in the approaches proposed before, where a small but fixed boundary driving (of the order of $10^{-2}$) is typically used, here we access arbitrary small coupling or driving analytically. Our approach allows us to establish the presence of a singularity in the NESS for $\Gamma \to 0$ and $\kappa \to 0$, which is expected to be present in the system for all system sizes. The singularity is evidenced in examples of several observables: the magnetization current, the energy current and the boundary magnetizations. For the magnetization current, the analytic treatment is presented.

The character of the singularity qualitatively depends on the parity of the system size $N$. For odd $N$, we find that the spin current scales for small $\Gamma$ or small $\kappa$ as $j = f(\Delta)\Gamma^2, j = g(\Delta)\kappa^2$, where for vanishing $\Gamma$ or $\kappa$ the functions $f$ and $g$ have different finite values at $\Delta = 1$ and $1 \pm 0$. For even sizes $N$, the spin current still scales quadratically with $\Gamma$ or $\kappa$ at the point $\Delta = 1$, while it scales as $j = f_1(\Delta)\Gamma^4, j = g_1(\Delta)\kappa^4$ at all other points. The energy current $J^E$ scales in the isotropic point linearly with the couplings $\Gamma_L, \Gamma_R$ (which must be different, $\Gamma_L \neq \Gamma_R$), and develops a twin peak singularity. Here, we are not in a position to discuss the conductivity of the spin or energy due to the small system sizes; however, DMRG studies [25] and recently proposed exact approaches might give access to large system sizes and even to the thermodynamic limit.

Acknowledgments

VP acknowledges the Dipartimento di Fisica e Astronomia, Università di Firenze, for support through an FIRB initiative. MS acknowledges support from the Ministero dell’Istruzione, dell’Università e della Ricerca (MIUR) through a Programma di Ricerca Scientifica di Rilevante Interesse Nazionale (PRIN)-2010 initiative.

Appendix. Current for $N = 5$ (weak driving limit)

For $N = 5$ and in the limit $\Gamma \to \infty$, the exact expression for the current is given by a complicated expression. We report different limits below. Taking the limit $\lim_{\kappa \to 0}$ first, we find

$$\lim_{\kappa \to 0} \frac{j_0(\kappa, \Delta)}{\kappa^2} = \frac{16\Delta (\Delta^2 + 3)}{28\Delta^4 + 81\Delta^2 + 72},$$

and consequently $\lim_{\Delta \to 1} \lim_{\kappa \to 0} j_0(\kappa, \Delta)/\kappa^2 = 64/181$. On the other hand, taking the limit $\lim_{\Delta \to 1}$ first, we find $\lim_{\Delta \to 1} j_0(\kappa, \Delta)/\kappa^2 = -F/G$, where

$$F = 8(128\kappa^{22} - 2128\kappa^{20} + 5792\kappa^{18} + 44096\kappa^{16} - 245827\kappa^{14} + 182930\kappa^{12} + 980805\kappa^{10} - 1523750\kappa^8 - 1515697\kappa^6 + 3477286\kappa^4 - 1171473\kappa^2 + 2235870),$$

$$G = 576\kappa^{22} + 856\kappa^{20} - 34736\kappa^{18} - 102787\kappa^{16} + 1669223\kappa^{14} - 4033065\kappa^{12} - 2004745\kappa^{10} + 13257113\kappa^8 + 3528813\kappa^6 - 16075891\kappa^4 - 7450779\kappa^2 - 15651090$$

and consequently $\lim_{\kappa \to 0} \lim_{\Delta \to 1} j_0(\kappa, \Delta)/\kappa^2 = 8/7$. Another way to see the discontinuity is to estimate the derivatives $(\partial^m j_0(\kappa, \Delta)/\partial \kappa^m)|_{\kappa=0}$, all even orders of which, starting
Anomalous currents in a driven $XXZ$ chain with boundary twisting at weak coupling or weak driving from $m = 4$, have singularities at $\Delta = 1$ as
\[
\frac{\partial^2 n}{\partial \kappa^2} j_0(\kappa, \Delta) \bigg|_{\kappa=0} = O\left(\frac{1}{(\Delta-1)^{2n-2}}\right).
\]
For odd $m = 2n + 1$, we find $\left(\frac{\partial^2 n+1}{\partial \kappa^2} j_0(\kappa, \Delta)/\partial \kappa^{n+1}\right)|_{\kappa=0} = 0$.

References

[1] Breuer H-P and Petruccione F, 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press)

[2] Plenio M B and Knight P L, 1998 Rev. Mod. Phys. 70 101

[3] Wichterich H, Henrich M J, Breuer H P, Gemmer J and Michel M, 2007 Phys. Rev. E 76 031115

[4] Heidrich-Meisner F, Honecker A and Brenig W, 2007 Eur. Phys. J. Spec. Top. 151 135 and references therein

[5] Zotos X, 2005 J. Phys. Soc. Japan. Supp. 74 173 and references therein

[6] Klumper A, 2004 Lect. Not. Phys. 645 349

[7] Haldane F M D, 1981 J. Phys. C: Solid State Phys. 14 2585

[8] Gogolin A O and Prokof’ev N V, 1994 Phys. Rev. B 50 4921

[9] Schulz H, 1996 Correlated Fermions and Transport in Mesoscopic Systems (Les Arcs, Savoie) ed T Martin, G Montambaux and J Tran Thanh Van (Gif-sur-Yvette: Editions Frontières)

[10] Schollwöck U, 2005 Rev. Mod. Phys. 77 259

[11] Hušek N, Ribeiro P, Saint-Martin R, Revcolevschi A, Roth G, Behr G, Büchner B and Hess C, 2010 Phys. Rev. B 81 020405

[12] Benenti G, Casati G, Prosen T, Rossini D and Žnidarič M, 2009 Phys. Rev. B 80 035110

[13] Žunkovič B and Prosen T, 2010 J. Stat. Mech. P08016

[14] Prosen T, 2008 New. J. Phys. 10 043026

[15] Prosen T, 2011 Phys. Rev. Lett. 106 047201

[16] Žnidarič M, 2011 Phys. Rev. Lett. 106 220601

[17] Kubo R, Toda M and Hashitsume N, Statistical Physics II: Nonequilibrium Statistical Mechanics (Springer Series in Solid-State Sciences) (Berlin: Springer)

[18] Popkov V, Salerno M and Schütz G M, 2012 Phys. Rev. E 85 031137

[19] Prosen T, 2012 J. Stat. Mech. P12015

[20] Popkov V and Livi R, 2013 New J. Phys. 15 023030

[21] Clark S R, Prior J, Hartmann M J, Jaksch D and Plenio M B, 2010 New J. Phys. 12 025005

[22] Popkov V and Salerno M, 2013 Phys. Rev. E 87 022108

[23] Evans D E, 1977 Commun. Math. Phys. 54 293

[24] Prosen T, 2012 Phys. Scr. 86 058511

[25] Prosen T and Žnidarič M, 2009 J. Stat. Mech. P02035

Žnidarič M, 2011 J. Stat. Mech. P12008