DEFECT AND HODGE NUMBERS OF HYPERSURFACES

SLAWOMIR RAMS

ABSTRACT. We define defect for hypersurfaces with A-D-E singularities in complex projective normal Cohen-Macaulay fourfolds having some vanishing properties of Bott-type and prove formulae for Hodge numbers of big resolutions of such hypersurfaces. We compute Hodge numbers of Calabi-Yau manifolds obtained as small resolutions of cuspidal triple sextics and double octics with higher $A_j$ singularities.

1. Introduction

The starting point of this considerations is the computation of Hodge numbers of double solids, i.e. double covers $Y_d$ of the three-dimensional projective space $\mathbb{P}_3(\mathbb{C})$ branched along a degree-$d$ surface $B_d$. If the branch divisor $B_d$ is smooth, then $Y_d$ has always the Hodge number $h^{1,1}(Y_d) = 1$. Let us assume that the branch locus $B_d$ has $\mu$ ordinary double points as its only singularities. We call the blow-up of $Y_d$ along $\text{sing}(Y_d)$ the big resolution of $Y_d$ and denote it by $\tilde{Y}_d$. In [6] Clemens showed the following formula for the Hodge number of the big resolution

$$h^{1,1}(\tilde{Y}_d) = 1 + \mu + \delta.$$ 

Here the first summand comes from the pull-back of the hyperplane section $\mathcal{O}_{\mathbb{P}_3}(1)$. The number $\mu$ is also expected as it counts the number of exceptional divisors in $\tilde{Y}_d$. The integer $\delta$, called the defect by Clemens, however, is a very subtle invariant of the threefold $Y_d$. It can be defined as the number of dependent conditions imposed on homogenous forms of degree $(3/2 \cdot d - 4)$ on $\mathbb{P}_3$ by the vanishing in the nodes of $B_d$:

$$\delta := h^0(\mathcal{O}_{\mathbb{P}_3}(3/2 \cdot d - 4) \otimes \mathcal{I}_{\text{sing}(B_d)}) - \left[h^0(\mathcal{O}_{\mathbb{P}_3}(3/2 \cdot d - 4)) - \mu\right].$$

Later, the defect was defined for nodal hypersurfaces in $\mathbb{P}_4$ in [38] (see also Concluding Remarks). Defects of linear systems were also used to compute Betti numbers of singular hypersurfaces in weighted projective spaces (see § 6.4).

Cynk [8] gave another proof of Clemens’ formula and generalized it to ample three-dimensional hypersurfaces $Y$ with ordinary double points in a smooth projective ambient variety $X$ sharing with the projective space some vanishing properties of Bott-type ([9, Thm 1]). He defined $V_Y$ to be the vector space of the global sections of the line bundle $\mathcal{O}_X(2Y) \otimes K_X$ vanishing in the singularities of $Y$. In [9] the defect of $Y$ is given by the formula:

$$\delta_Y := \dim V_Y - \left[h^0(\mathcal{O}_X(2Y) \otimes K_X) - \mu\right].$$

By [9, Thm 1], if we assume $h^2(\Omega^1_X) = h^3(\Omega^1_X(-Y)) = 0$, then the Hodge number of $\tilde{Y}$ is

$$h^{1,1}(\tilde{Y}) = h^{1,1}(X) + \mu + \delta_Y.$$ 

The double solid $Y_d$ can be embedded as an ample hypersurface in the weighted projective space $\mathbb{P} := \mathbb{P}(1,1,1,d/2)$ in such a way that it does not meet the set $\text{sing}(\mathbb{P})$. However, if we resolve...

2000 Mathematics Subject Classification. Primary: 14J30, 14C30; Secondary 14Q10.

Research partially supported by KBN grant no 2P03A 016 25 and the DFG Schwerpunktprogramm "Global methods in complex geometry".
the singularity of \( P \), the proper transform of \( Y_d \) is no longer ample. Thus Clemens’ formula cannot be directly derived from [9, Thm 1].

Here we consider the following more general situation: \( X \) is a projective normal Cohen-Macaulay fourfold and \( Y \subset X \) is a hypersurface such that \( \text{sing}(X) \cap Y = \emptyset \). We assume that \( Y \) has only A-D-E singularities, i.e. for every \( P \in \text{sing}(Y) \) there exists local (analytic) coordinates \( x_1, \ldots, x_4, P \) centered at \( P \) such that the germ of \( Y \) at \( P \) is given by an equation

\[
H(Y) = 0,
\]

where \( n(x_1, x_2, x_3, x_4) \) is the normal form of the equation of a two-dimensional A-D-E singularity (see the table (3.1)). Let \( a_m, d_m, e_m \) stand for the number of the singularities of \( Y \) of the type \( A_m \) (resp. \( D_m, E_m \)).

We define the big resolution \( \tilde{\pi} : \tilde{Y} \to Y \) as the composition \( \tilde{\pi} = \sigma_n \circ \cdots \circ \sigma_1 \), where \( \sigma_j : \tilde{Y}^j \to \tilde{Y}^{j-1} \), for \( j = 1, \ldots, n \), is the blow-up with the center \( \text{sing}(\tilde{Y}^j), \tilde{Y}^0 := Y \), and \( \tilde{Y} = \tilde{Y}^n \) is smooth. The main purpose of this paper is to define the defect of a hypersurface with A-D-E singularities and obtain a formula analogous to [9, Thm 1] for the Hodge numbers of the big resolution \( \tilde{Y} \).

The definition of the integer \( m \mu_Y \) has to be adapted as follows:

\[
\mu_Y := \sum_{m \geq 1} a_m \cdot \lfloor m/2 \rfloor + \sum_{m \geq 2} 2 \cdot d_m \cdot \lfloor m/2 \rfloor + 4 \cdot e_6 + 7 \cdot e_7 + 8 \cdot e_8 .
\]

Let \( \mathcal{Y}_Y \) be the space of global sections \( H \) of the sheaf \( \mathcal{O}_X(2Y + K_X) \) that vanish in all points \( P \in \text{sing}(Y) \) and satisfy the conditions:

- if \( P \) is an \( A_m \) point, \( m \geq 1 \), then \( \frac{\partial H}{\partial x_1}_P = 0 \) for \( j \leq \lfloor m/2 \rfloor - 1 \),
- if \( P \) is a \( D_m \) point, \( m \geq 4 \), then \( \frac{\partial H}{\partial x_2}_P = \frac{\partial H}{\partial x_1}_P = 0 \) for \( j \leq \lfloor m/2 \rfloor - 1 \),
- if \( P \) is an \( E_m \) point, \( m = 6, 7, 8 \), then \( \frac{\partial H}{\partial x_2}_P = \frac{\partial H}{\partial x_1}_P = 0 \) for \( j \leq m - 5 \).

The notion of defect has to be adapted in the following way:

\[
\delta_Y := \dim(\mathcal{Y}_Y) - \lfloor h^0(\mathcal{O}_X(2Y + K_X)) - \mu_Y \rfloor .
\]

Since we work on a singular ambient variety \( X \), we consider the Zariski sheaf of germs of 1-forms \( \Omega_X^1 = j_! \Omega_{\text{reg}X}^1 \), where \( j \) stands for the inclusion \( \text{reg}(X) \to X \), in place of the sheaf of differentials \( \Omega_X^1 \). The Bott-type assumptions read as follows:

- \([A1]\): \( H^i(\mathcal{O}_X(-Y)) = 0 \) for \( i \leq 3 \) and \( H^j(\mathcal{O}_X(-2Y)) = 0 \) for \( j \leq 2 \),
- \([A2]\): \( H^2(\Omega_X^1) = 0 \),
- \([A3]\): \( H^i(\Omega_X^1 \otimes \mathcal{O}_X(-Y)) = 0 \), for \( i = 1, 2, 3 \).

Here we show (Thm 4.1) that

\[
h^{1,1}(\tilde{Y}) = h^1(\Omega_X^1) + \mu_Y + \delta_Y + h^3(\mathcal{O}_X(-2Y)),
\]

and, if \( h^2(\mathcal{O}_X) = 0 \), then

\[
h^{1,2}(\tilde{Y}) = h^0(\mathcal{O}_X(2Y + K_X)) + h^4(\Omega_X^1) - h^0(\mathcal{O}_X(Y + K_X)) - h^3(\Omega_X^1) - h^4(\Omega_X^1 \otimes \mathcal{O}_X(-Y)) - \mu_Y + \delta_Y.
\]

In this way we obtain formulae that can be applied to a large class of ambient spaces and to hypersurfaces with higher singularities. In particular, Thm 4.1 implies [6, Cor. 2.32] and [9, Thm 1].

The assumptions \([A1]\), \([A2]\), \([A3]\) are satisfied when \( Y \) is an ample hypersurface in a complete simplicial toric fourfold (Cor. 4.2). As an application of Thm 4.1 we derive formulae for the Hodge numbers of various covers of \( \mathbb{P}_3 \): double solids branched along surfaces with Du Val singularities,
cyclic n-fold covers branched along nodal hypersurfaces and triple solids whose branch divisor has $A_2$ singularities.

Sect. 6 is devoted to the study of the relation between the Hodge numbers of a Kähler small resolution and the big one. In the last section of the paper we compute the Hodge numbers of the Calabi-Yau manifolds obtained as Kähler small resolutions of the triple solids branched along the sextics studied in [1, 2, 25].

Notations and conventions: All varieties are defined over the base-field $\mathbb{C}$. By a divisor we mean a Weil divisor, and "$\sim$" stands for the linear equivalence. The round-up, resp. the round-down is denoted by $[\cdot]$, resp. $\lfloor \cdot \rfloor$.

2. Technical preliminaries

In this section we assume $X$ to be a projective normal Cohen-Macaulay fourfold and consider a hypersurface $Y \subset X$ with isolated double points as only singularities. Moreover, we require that

$$\text{sing}(X) \cap Y = \emptyset.$$ (2.1)

Here we modify several results from [9] in order to apply them in our set-up. Since we are going to work on a singular (normal) variety $X$, let us recall that for a Weil divisor $D := \sum \Gamma n_{\Gamma} \Gamma$ on $X$ one defines the sheaf $\mathcal{O}_X(D)$ by putting

$$\mathcal{O}_X(D)(U) := \{ f \in \text{Rat}(X) : v_{\Gamma}(f) + n_{\Gamma} \geq 0 \text{ for every } \Gamma \cap U \neq \emptyset \},$$

where $v_{\Gamma}(\cdot)$ is the discrete valuation given by the prime divisor $\Gamma \subset X$. Then, the map $D \to \mathcal{O}_X(D)$ gives one-to-one correspondence between the linear equivalence classes of Weil divisors and isomorphism classes of rank-1 reflexive sheaves on $X$ (see [31, p. 281] for the details).

Moreover, if $D_1$ is Cartier, then one can show that

$$\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2),$$ (2.2)

(this equality does not hold in general - see [31, Remark (5)]).

Let $j$ stand for the inclusion $\text{reg}(X) \to X$. Observe that $X$ is endowed with the dualising sheaf $\omega_X = j_* \omega_{\text{reg}(X)}$, that is reflexive of rank one (see e.g. [31, Prop. 5.75]), and we have $\omega_X = \mathcal{O}_X(K_X)$. For the definition of the canonical divisor of a normal variety see e.g. [24, Def. 0-2.1].

We define

$$V_Y := \text{H}^0(\mathcal{O}_X(2Y + K_X) \otimes \mathcal{I}_{\text{sing}(Y)})$$

to be the vector space of global sections of the sheaf $\mathcal{O}_X(2Y + K_X)$ on $X$ vanishing in all singularities of $Y$. Let $\nu$ be the number of points in $\text{sing}(Y)$. We put

$$r_Y = \text{dim} V_Y - \left[ \text{h}^0(\mathcal{O}_X(2Y + K_X)) - \nu \right].$$ (2.3)

Note that the integer $r_Y$ is non-negative because it is the difference between the actual dimension of the space $V_Y$ and the one expected for a hypersurface with singular points in general position.

Let $\sigma : \check{X} \to X$ be the blow-up of $X$ along $\text{sing}(Y)$ and let $\check{Y}$ (resp. $E$) stand for the strict transform of $Y$ (resp. the exceptional divisor of $\sigma$ in $\check{X}$). Since we blow up points in $\text{reg}(X)$, we have the equalities:

$$\sigma_*(\mathcal{O}_X(\check{Y})) = \mathcal{I}_{\text{sing}(Y)}, \quad R^i \sigma_*(\mathcal{O}_X(\check{Y})) = 0, \quad \text{for } i > 0,$$ (2.4)

$$\sigma_*(\mathcal{O}_X(kE)) = \mathcal{O}_X, \quad R^k \sigma_*(\mathcal{O}_X(kE)) = 0, \quad \text{for } k = 0, 1, 2, 3 \text{ and } i > 0,$$ (2.5)

$$\sigma_*(\mathcal{O}_X(4E)) = \mathcal{O}_X, \quad R^j \sigma_*(\mathcal{O}_X(4E)) = 0, \quad \text{for } j = 1, 2,$$

$$R^3 \sigma_*(\mathcal{O}_X(4E)) = \mathbb{C}^{r_Y},$$ (2.6)
where $\mathbb{C}^\nu$ stands for the skyscraper sheaf with stalk $\mathbb{C}$ over the centers of blow-up.

Indeed, observe that the first equality in (2.4) and the equalities (2.5) for $k = 0$ are basic properties of blow-up. To prove the second assertion of (2.4) apply the direct image $\sigma_*$ to the exact sequence
\[(2.7)\]
\[0 \to \mathcal{O}_X(-E) \to \mathcal{O}_\tilde{X} \to \mathcal{O}_E \to 0.\]

To show (2.5) for $k = 1, 2, 3$ and (2.6) consider the tensor product of (2.7) with the locally free sheaf $\mathcal{O}_X(kE)$ (see [9] the proof of Lemma 1 for details).

One can see that if $\mathcal{F}$ is a coherent sheaf on $\tilde{X}$ and $\mathcal{E}$ is a coherent sheaf on $X$ such that $\mathcal{E}|_{\text{reg}(X)}$ is locally free, then
\[(2.8)\]
\[R^i\sigma_*(\mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \sigma^*\mathcal{E}) = R^i\sigma_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E} \text{ for } i \geq 0.\]

Indeed, the blow-up $\sigma$, when restricted to an appropriate neighbourhood of singular points of $X$ and $\tilde{X}$, is an isomorphism, so the stalks of both sheaves vanish for a point in such a neighbourhood and $i \geq 1$. To complete the proof of (2.8) for $i \geq 1$ apply the projection formula [21] III. Ex. 8.3] on $\text{reg}(X)$. Finally, by the assumption (2.1), one can literally repeat the proof of the projection formula (see e.g. [22] Lemma 2.29) to show that the formula in question holds when $i = 0$.

In particular, we get
\[(2.9)\]
\[\sigma_*(\sigma^*\mathcal{O}_X(K_X) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_X(-E))) = \mathcal{O}_X(K_X) \otimes_{\mathcal{O}_X} \mathcal{I}_{\text{sing}(Y)}.\]

Let $D$ be a Weil divisor on $X$. Since the centers of the blow-up $\sigma$ are smooth on $X$, the pull-back $\sigma^*D$ is well-defined. Moreover, if we choose a divisor $K_X$ (resp. $\tilde{K}_X$) in the canonical class of $X$ (resp. $\tilde{X}$), then we have
\[(2.10)\]
\[K_{\tilde{X}} \sim \sigma^*K_X + 3E.\]

In the rest of this section we put
\[\mathcal{M} := \mathcal{O}_X(Y), \quad \mathcal{L} := \mathcal{O}_{\tilde{X}}(\tilde{Y}).\]

**Lemma 2.1.** Let $\tilde{Y}$ be the proper transform of $Y$ under the blow-up with center $\text{sing}(Y)$. If
\[h^i(\mathcal{M}^{-j}) = 0 \text{ for } i \leq 2, j = 1, 2, \text{ and } h^3(\mathcal{M}^{-1}) = 0,
\]
then
a) $h^i(\mathcal{L}^{-j}) = 0$ for $i \leq 2, j = 1, 2$, and $h^3(\mathcal{L}^{-1}) = 0$,
\nb) $h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_X)$, for $i \leq 2$,
\nc) $h^0(\mathcal{O}_{\tilde{Y}} \otimes \mathcal{L}^{-1}) = h^1(\mathcal{O}_{\tilde{Y}} \otimes \mathcal{L}^{-1}) = 0$,
\nd) $h^4(\mathcal{L}^{-2}) = \dim V_Y$,
\ne) $h^3(\mathcal{L}^{-2}) = h^3(\mathcal{M}^{-2}) + r_Y$.

**Proof.** By (2.1) the sheaf $\mathcal{L}$ is locally free, so we can apply the projection formula and (2.5) to the bundle $\mathcal{L}^{-1} = \sigma^*\mathcal{M}^{-1} \otimes \mathcal{O}_X(2E)$. In this way we obtain the equalities
\[(2.11)\]
\[\sigma_*(\mathcal{L}^{-1}) = \mathcal{M}^{-1}, \quad R^i\sigma_*(\mathcal{L}^{-1}) = 0 \text{ for } i > 0.\]

Similarly, (2.6) yields
\[(2.12)\]
\[\sigma_*(\mathcal{L}^{-2}) = \mathcal{M}^{-2}, \quad R^i\sigma_*(\mathcal{L}^{-2}) = 0, \text{ for } i = 1, 2, \text{ and } R^3\sigma_*(\mathcal{L}^{-2}) = \mathbb{C}^\nu.\]

Moreover, by (2.4) and (2.8) we have
\[(2.13)\]
\[\sigma_*(\sigma^*\mathcal{M}^2 \otimes \sigma^*\mathcal{O}_X(K_X) \otimes \mathcal{O}_{\tilde{X}}(-E)) = \mathcal{M}^2 \otimes \mathcal{O}_X(K_X) \otimes \mathcal{I}_{\text{sing}(Y)},\]
\[(2.14)\]
\[R^i\sigma_*(\sigma^*\mathcal{M}^2 \otimes \sigma^*\mathcal{O}_X(K_X) \otimes \mathcal{O}_{\tilde{X}}(-E)) = 0, \text{ for } i \geq 1.\]

The Leray spectral sequence for the map $\sigma$ and the sheaf $\mathcal{L}^{-1}$, combined with (2.11), yields for all $i$
\[(2.15)\]
\[h^i(\mathcal{L}^{-1}) = h^i(\mathcal{M}^{-1}).\]
Let us consider the Leray spectral sequence for \( \sigma \) and \( \mathcal{L}^{-2} \). Since \( E_{2}^{p,q} = H^{p}(X, R^{q}\sigma \mathcal{L}^{-2}) \), we have \( E_{2}^{0,3} = \mathbb{C}^{\nu} \) and, by (2.12), the \( E_{2}^{p,q} \)-terms vanish for \( q > 0 \) and \( q \neq 3 \). Therefore, (see e.g. [28, Ex. 1.D]) we have the equalities

\[
(2.16) \quad h^{i}(\mathcal{L}^{-2}) = h^{i}(\mathcal{M}^{-2}) \text{ for } i \leq 2
\]

and the exact sequence

\[
(2.17) \quad 0 \longrightarrow H^{3}(\mathcal{M}^{-2}) \longrightarrow H^{3}(\mathcal{L}^{-2}) \longrightarrow \mathcal{C}'' \longrightarrow H^{4}(\mathcal{M}^{-2}) \longrightarrow H^{4}(\mathcal{L}^{-2}) \longrightarrow 0.
\]

Finally, we have

\[
(2.18) \quad h^{0}(L^{2} \otimes \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})) = h^{0}(\sigma^{*}\mathcal{M}^{2} \otimes \mathcal{O}_{\tilde{X}}(-4E) \otimes \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})) = h^{0}(\mathcal{M}^{2} \otimes \mathcal{O}_{X}(K_{X}) \otimes \mathcal{O}_{\tilde{X}}(-E)) = h^{0}(\mathcal{M}^{2} \otimes \mathcal{O}_{X}(K_{X}) \otimes \mathcal{I}_{\text{sing}(\mathcal{Y})}),
\]

where the last equality results from the Leray spectral sequence, (2.13) and (2.14).

Now we are in position to prove the claims (a) - (e).

(a) results immediately from (2.15) - (2.16).

(b) Obviously \( h^{i}(\mathcal{O}_{\tilde{X}}) = h^{i}(\mathcal{O}_{X}) \) for all \( i \). For \( i \leq 2 \), the equality \( h^{i}(\mathcal{O}_{\tilde{X}}) = h^{i}(\mathcal{O}_{X}) \) results from (a) and the long cohomology sequence associated to the short exact sequence

\[
0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow 0.
\]

(c) results from (a) and the long cohomology sequence associated to the sequence

\[
0 \longrightarrow \mathcal{L}^{-2} \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{L}^{-1} \longrightarrow 0.
\]

(d) The sheaf \( \mathcal{L}^{-2} \) is locally free and \( \tilde{X} \) is Cohen-Macaulay, so we can use Serre duality ([21, Cor. 7.7]) to prove that

\[
(2.19) \quad h^{4}(\mathcal{L}^{-2}) = h^{0}(L^{2} \otimes \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})) \overset{2.13}{=} \dim V_{\mathcal{Y}}.
\]

e) By the same argument we have

\[
(2.20) \quad h^{0}(\mathcal{M}^{2} \otimes \mathcal{O}_{X}(K_{X})) = h^{4}(\mathcal{M}^{-2}).
\]

From the exact sequence

\[
0 \longrightarrow \mathcal{M}^{-2} \longrightarrow \mathcal{M}^{-1} \longrightarrow \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{M}^{-1} \longrightarrow 0,
\]
we obtain

\[
(2.21) \quad h^{2}(\mathcal{O}_{\mathcal{Y}} \otimes \mathcal{M}^{-1}) = h^{3}(\mathcal{M}^{-2}),
\]

\[
(2.22) \quad h^{3}(\mathcal{O}_{\mathcal{Y}} \otimes \mathcal{M}^{-1}) = h^{4}(\mathcal{M}^{-2}) - h^{4}(\mathcal{M}^{-1}).
\]

Finally, we have the equalities

\[
h^{3}(\mathcal{M}^{-2}) + r_{\mathcal{Y}} \overset{2.19}{=} h^{3}(\mathcal{M}^{-2}) + h^{4}(\mathcal{L}^{-2}) + \nu - h^{0}(\mathcal{M}^{2} \otimes \mathcal{O}_{X}(K_{X})) \overset{2.20}{=} h^{3}(\mathcal{M}^{-2}) + h^{4}(\mathcal{L}^{-2}) + \nu - h^{4}(\mathcal{M}^{-2}) \overset{2.17}{=} h^{3}(\mathcal{L}^{-2}).
\]

\[\square\]

It should be pointed out that if \( X \) has isolated rational singularities and \( \mathcal{M} \) is ample, then the assumptions of Lemma [2.1] are satisfied by [33, VII, Thm. 7.80].

We intend to apply [30, Bott’s Vanishing, p. 130] (see also [15, Thm 2.3.2]) in Sect. 3 so in the remainder of this section we focus our attention on the Zariski sheaf of germs of 1-forms \( \Omega^{1}_{\text{reg}X} \), where \( j : \text{reg}(X) \rightarrow X \) stands for the inclusion. Recall that the sheaf \( \Omega^{1}_{X} \) does not have to be locally free.

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Let $C_l$, where $l = 1, \ldots, \nu$, stand for a component of the exceptional divisor $E$. We have the following exact sequence

\[(2.23) \quad 0 \rightarrow \sigma^*\overline{\Omega}^1_X \rightarrow \overline{\Omega}^1_X \rightarrow \bigoplus_{i=1}^\nu \Omega^1_{C_i} \rightarrow 0,\]

Indeed, away from the singularities of $X$ the sheaves $\overline{\Omega}^1_X$ and $\Omega^1_X$ coincide, whereas around the singular locus $\text{sing}(X)$ the blow-up is an isomorphism.

**Lemma 2.2.** We have $\sigma_*(\sigma^*\overline{\Omega}^1_X) \cong \overline{\Omega}^1_X$ and $R^i\sigma_*(\sigma^*\overline{\Omega}^1_X) = 0$ for $i > 0$.

**Proof.** Apply (2.3) and (2.5) for $k = 0$. \hfill \Box

Now, we are in position to prove

**Lemma 2.3.** a) If $h^2(\overline{\Omega}^1_X) = 0$, then $h^1\overline{\Omega}^1_X = h^1\overline{\Omega}^1_X + \nu$ and $h^i\overline{\Omega}^1_X = h^i\overline{\Omega}^1_X$ for $i \neq 1$.

b) The equality $h^i(\overline{\Omega}^1_X \otimes \mathcal{L}^{-1}) = h^i(\overline{\Omega}^1_X \otimes \mathcal{M}^{-1})$ holds for every $i$.

**Proof.** a) (c.f. [9, Lemma 2]) Recall that all centers of the blow-up $\sigma$ are smooth on $X$, so $C_l \cong \mathbb{P}_3$ for $l = 1, \ldots, \nu$. Apply the direct image $\sigma_*$ to the Euler sequence to see that $R^1\sigma_*\Omega^1_{C_l}$ is the skyscraper sheaf with stalk $\mathbb{C}$ at each center of the blow-up, whereas $R^i\sigma_*\Omega^1_{C_l} = 0$ for $i \neq 1$. Consider the direct image $\sigma_*$ of the exact sequence (2.24). Lemma 2.2 yields

\[(2.24) \quad \sigma_*\overline{\Omega}^1_X = \overline{\Omega}^1_X, \quad R^1\overline{\Omega}^1_X = \mathbb{C}, \quad R^i\overline{\Omega}^1_X = 0 \text{ for } i \geq 2.\]

Now we consider the Leray spectral sequence $E^{p,q}_2 = H^p(X, R^q\overline{\Omega}^1_X) \Rightarrow H^{p+q}(\tilde{X}, \overline{\Omega}^1_{\tilde{X}})$. By (2.24) we have the vanishing

\[E^{2,0}_2 = H^2(\sigma_*\overline{\Omega}^1_X) = H^2(\overline{\Omega}^1_X) = 0\]

The latter implies that the differential $d_2 : E^{0,1}_2 \rightarrow E^{2,0}_2$ is the zero map and we can compute all $E^{p,q}_\infty$-terms.

b) In view of the Leray spectral sequence $E^{p,q}_2 = H^p(X, R^q(\overline{\Omega}^1_X \otimes \mathcal{L}^{-1})) \Rightarrow H^{p+q}(\tilde{X}, \overline{\Omega}^1_{\tilde{X}} \otimes \mathcal{L}^{-1})$, it suffices to show that

\[(2.25) \quad \sigma_*(\overline{\Omega}^1_X \otimes \mathcal{L}^{-1}) = \overline{\Omega}^1_X \otimes \mathcal{M}^{-1} \text{ and } R^i\sigma_*(\overline{\Omega}^1_X \otimes \mathcal{L}^{-1}) = 0 \text{ for } i > 0.\]

Tensoring (2.24) with $\mathcal{L}^{-1}$ we get

\[(2.26) \quad 0 \rightarrow \sigma^*\overline{\Omega}^1_X \otimes \sigma^*\mathcal{M}^{-1} \otimes \mathcal{O}_{\tilde{X}}(2E) \rightarrow \overline{\Omega}^1_{\tilde{X}} \otimes \mathcal{L}^{-1} \rightarrow \bigoplus_{l=1}^\nu \Omega^1_{C_l}(-2) \rightarrow 0.\]

For $i \geq 0$, we have the vanishing $R^i\sigma_*\Omega^1_{C_l}(-2) = 0$, so applying the direct image functor to (2.26) yields the exact sequences

\[(2.27) \quad 0 \rightarrow R^i\sigma_*(\sigma^*\overline{\Omega}^1_X \otimes \sigma^*\mathcal{M}^{-1} \otimes \mathcal{O}_{\tilde{X}}(2E)) \rightarrow R^i\sigma_*(\overline{\Omega}^1_X \otimes \mathcal{L}^{-1}) \rightarrow 0.\]

Now (2.8), (2.5) and Lemma 2.2 imply $\sigma_*(\sigma^*(\overline{\Omega}^1_X) \otimes \mathcal{O}_{\tilde{X}}(2E)) \cong \overline{\Omega}^1_X$. From the projection formula we obtain

\[\sigma_*(\sigma^*(\overline{\Omega}^1_X) \otimes \sigma^*(\mathcal{M}^{-1} \otimes \mathcal{O}_{\tilde{X}}(2E))) \cong \mathcal{M}^{-1} \otimes \sigma_*(\sigma^*(\overline{\Omega}^1_X) \otimes \mathcal{O}_{\tilde{X}}(2E)),\]

which completes the proof of the first claim in (2.25).

Let $i > 0$. To prove the second claim of (2.25) apply (2.8) and (2.5) for $k = 2$ to show that

\[R^i\sigma_*(\sigma^*\overline{\Omega}^1_X \otimes \sigma^*\mathcal{M}^{-1} \otimes \mathcal{O}_{\tilde{X}}(2E)) = 0,\]

and use (2.27). \hfill \Box

\[\hfill 6\]
Lemma 2.4. If $h^2(\Omega_X) = 0$ and $h^i(\Omega_X \otimes M^{-1}) = 0$ for $i = 1, 2, 3$, then
\[
\begin{align*}
&h^0(\Omega_X \otimes O_Y) = h^0(\Omega_X) - h^0(\Omega_X \otimes M^{-1}), \\
&h^1(\Omega_X \otimes O_Y) = h^1(\Omega_X) + \nu, \\
&h^2(\Omega_X \otimes O_Y) = 0, \\
&h^3(\Omega_X \otimes O_Y) = h^3(\Omega_X) + h^3(\Omega_X \otimes M^{-1}) - h^3(\Omega_X).
\end{align*}
\]

Proof. Observe that the following short sequence
\[
0 \rightarrow \Omega_X^1 \otimes L^{-1} \rightarrow \Omega_X^1 \rightarrow \Omega_X^0 \otimes O_Y \rightarrow 0
\]
is exact. Apply Lemma 2.2 to the associated long cohomology sequence. \qed

3. Defect for A-D-E Singularities

In this section we define the defect for threefold hypersurfaces with A-D-E singularities and give a formula for this integer. We work with semiquasihomogenous equations of A-D-E germs, instead of their normal forms, because the analytic coordinates in which a germ is given by the former are easier to find, while the formula for defect remains the same.

Let $X$ be a projective normal Cohen-Macaulay fourfold and let $Y \subset X$ be a hypersurface with A-D-E singularities such that $\text{sing}(X) \cap Y = \emptyset$. In particular the singularities of $Y$ are absolutely isolated, i.e. $\text{sing}(Y)$ can be resolved by blowing up (closed) points ([16, p.137]).

We define the big resolution $\tilde{\pi} : \tilde{Y} \rightarrow Y$ as the composition:
\[
(3.1) \quad \tilde{\pi} = \sigma_n \circ \ldots \circ \sigma_1 : \tilde{Y} \rightarrow Y =: \tilde{Y}^0,
\]
where $\sigma_j : \tilde{Y}^j \rightarrow \tilde{Y}^{j-1}$, for $j = 1, \ldots, n$, is the blow-up with the center $\text{sing}(\tilde{Y}^{j-1}) \neq \emptyset$ and $\tilde{Y} = \tilde{Y}^n$ is smooth. By abuse of notation we use the same symbol to denote the composition of the blow-ups of $X$ with the same centers $\tilde{\pi} : \tilde{X} \rightarrow X$.

We put $\tilde{Y} = Y$, $n = 0$, when $Y$ is smooth. The number of singular and infinitely near singular points of $Y$ is denoted by
\[
\mu_Y := \sum_{j=0}^{n-1} \# \text{sing}(\tilde{Y}^j).
\]
Since the singularities of $\tilde{Y}^j$ are isolated for every $j$, we can formulate the following definition

Definition 3.1. We define the defect of $Y$ as the non-negative number
\[
\delta_Y = r_Y + \ldots + r_{Y_{n-1}}.
\]
Observe that if $Y$ has only ordinary double points and $X$ is smooth, then this definition coincides with [9, Def. 1].

Let $j = 1, \ldots, n$ and let $E_j$ stand for the exceptional divisor of the blow-up $\sigma_j : \tilde{X}^j \rightarrow \tilde{X}^{j-1}$. For a divisor $H$ on $\tilde{X}^{j-1}$, we define the (Weil) divisor
\[
\mathfrak{s}_j(H) := \sigma_j^*H - E_j.
\]
To simplify our notation, in the following lemma we use the same letter to denote a section of $O_X(2Y + K_X)$ and the divisor it defines.

Lemma 3.2. Let $\text{sing}(Y)$ consist of A-D-E singularities and let $\tilde{\pi} = \sigma_n \circ \ldots \circ \sigma_1$ be the big resolution of $Y$. Then
\[
\begin{align*}
\delta_Y &= h^0(O_{\tilde{X}}(2\tilde{Y} + K_{\tilde{X}})) - h^0(O_X(2Y + K_X)) + \mu_Y, \\
h^0(O_{\tilde{X}}(2\tilde{Y} + K_{\tilde{X}})) &= \dim \{H \in V_Y : \text{sing}(\tilde{Y}^j) \subset \text{supp}((\mathfrak{s}_j \circ \ldots \circ \mathfrak{s}_1)(H)) \text{ for } j \leq n-1\}.
\end{align*}
\]
Proof. For \( j = 1, \ldots, n \) the singularities of \( \hat{Y}^{j-1} \) are double points. In this case the equality (2.18) (see also (2.2)) reads
\[
(3.2) \quad h^0(\mathcal{O}_X(2\hat{Y}^j + K_X)) = h^0(\mathcal{O}_{\hat{X}^{j-1}}(2\hat{Y}^{j-1} + K_{\hat{X}^{j-1}}) \otimes \mathcal{I}_{\text{sing}(\hat{Y}^{j-1})}).
\]

Now the formula for \( \delta_Y \) follows directly from Def. 3.1.

The second equality is obvious for \( n = 1 \), so we can assume \( n \geq 2 \). Fix \( j = 1, \ldots, n \) and observe that the natural linear map
\[
V_{\hat{Y}^{j-1}} \longrightarrow H^0(\mathcal{O}_{\hat{X}^j}(\sigma_j^*(2\hat{Y}^{j-1} + K_{\hat{X}^{j-1}}) - E_j)) = H^0(\mathcal{O}_{\hat{X}^j}(2\hat{Y}^j + K_{\hat{X}^j}))
\]
that maps a section which defines a divisor \( D \in |2\hat{Y}^{j-1} + K_{\hat{X}^{j-1}}| \) satisfying the condition
\[
\text{sing}(\hat{Y}^{j-1}) \subset \text{supp}(D)
\]
to its lift, which is a section that gives the divisor \( s_j(D) \), is an isomorphism. We obtain the equalities
\[
(3.3) \quad \text{dim} V_{\hat{Y}^{n-1}} = \text{dim} \{ H \in V_{\hat{Y}^{n-2}} : \text{sing}(\hat{Y}^{n-1}) \subset \text{supp}(s_{n-1}(H)) \} = \ldots = \text{dim} \{ H \in V_Y : \text{sing}(\hat{Y}^j) \subset \text{supp}(s_j \circ \ldots \circ s_1)(H) \} \text{ for } j \leq n - 1
\]

Remark: In the proof of Lemma 3.2 we used the apparently weaker assumption that all singularities and infinitely near singularities of \( Y \) are isolated double points. By [34, Thm 1] if \( P \in Y \) is an absolutely isolated double point, then it is an A-D-E singularity.

For an A-D-E point \( P \in \text{sing}(Y) \) we choose (analytic) coordinates \( x_1, p, \ldots, x_4, p \) centered at \( P \) such that the germ of \( Y \) at \( P \) is given by the equation
\[
(3.3) \quad n(x_1, p, x_2, p, x_3, p) + x_4^2 + F(x_1, p, x_2, p, x_3, p, x_4, p) = 0,
\]
where \( n(x_1, p, x_2, p, x_3, p) \) is the normal form of the equation of a two-dimensional A-D-E singularity and \( F(x_1, p, x_2, p, x_3, p, x_4, p) \) is a polynomial of order strictly greater than 1 with respect to the weights \( w_n(x_1, p), w_n(x_2, p), w_n(x_3, p), w_n(x_4, p) \) given in the table below:

| \( m, m \geq 1 \) | \( (x_1, x_2, x_3) \) | \( (w_n(x_1), \ldots, w_n(x_4)) \) |
|-----------------|----------------|----------------|
| \( A_m \) | \( x_1^{m+1} + x_2^2 + x_3^2 \) | \( (\frac{1}{m+1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) |
| \( D_m \), \( m \geq 4 \) | \( x_1 \cdot (x_2^2 + x_1^{m-2}) + x_3^2 \) | \( (\frac{1}{m-1}, \frac{m-2}{2m-1}, \frac{1}{2}, \frac{1}{2}) \) |
| \( E_6 \) | \( x_1^4 + x_2^3 + x_3^2 \) | \( (\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}) \) |
| \( E_7 \) | \( x_1^3 \cdot x_2 + x_2^3 + x_3^2 \) | \( (\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}) \) |
| \( E_8 \) | \( x_1^5 + x_2^3 + x_3^2 \) | \( (\frac{1}{7}, \frac{1}{5}, \frac{1}{2}, \frac{1}{2}) \) |

Recall that, according to [16, Char. C 9], every germ given by an equation of the type (3.3) is A-D-E, so we can assume \( F = 0 \) to check that the number of singular points (different from \( P \)) which are infinitely near \( P \) is as follows
\[
(3.5) \quad \begin{array}{c|c|c|c|c}
A_m, \( m \geq 1 \) & D_m, \( m \geq 4 \) & E_6 & E_7 & E_8 \\
\lceil m/2 \rceil - 1 & 2 \cdot \lceil m/2 \rceil - 1 & 3 & 6 & 7
\end{array}
\]

Now, we are in position to state a lemma that, combined with Lemma 3.2, gives a method of computing \( \delta_Y \) without studying the configuration of infinitely near singular points.
Lemma 3.3. Let $\text{sing}(Y)$ consist of $A$-$D$-$E$ points and let $H \in H^0(O_X(2Y + K_X) \otimes I_{\text{sing}(Y)})$. Then, the following conditions are equivalent:

[I] The inclusion $\text{sing}(\tilde{Y}^j) \subset \text{supp}((s_j \circ \ldots \circ s_1)(H))$ holds for $j \geq 1$.

[II] Every $P \in \text{sing}(Y)$ satisfies one of the following:

(3.6) $P$ is an $A_m$ point, with $m \geq 1$ and $\frac{\partial H}{\partial x_{1,p}^j}(P) = 0$ for $j \leq \lfloor m/2 \rfloor - 1$,

(3.7) $P$ is a $D_m$ singularity, where $m \geq 4$ and $\frac{\partial H}{\partial x_{2,p}^j}(P) = \frac{\partial H}{\partial x_{1,p}^j}(P) = 0$ for $j \leq \lfloor m/2 \rfloor - 1$,

(3.8) $P$ is an $E_m$ point, where $m = 6, 7, 8$ and $\frac{\partial H}{\partial x_{2,p}^j}(P) = \frac{\partial H}{\partial x_{1,p}^j}(P) = 0$ for $j \leq m - 5$,

where $x_{1,p}, \ldots, x_{4,p}$ are analytic local coordinates centered at the point $P$ such that the hypersurface $Y$ is given near $P$ by the semiquasihomogenous equation (3.3).

Proof. We choose a point $P$ and fix the coordinates $x_{1,p}, \ldots, x_{4,p}$. Let $\tilde{\pi} = \sigma_n \circ \ldots \circ \sigma_1$ be the big resolution of $Y$. We claim that for a point $P$ of the type $A_m$ (resp. $D_m$, resp. $E_m$), the condition (3.6) (resp. (3.7), resp. (3.8)) is fulfilled iff for all $j$ such that $\text{sing}(\tilde{Y}^j) \neq \emptyset$ we have the inclusion

(3.9) $\text{sing}(\tilde{Y}^j) \cap (\sigma_j \circ \ldots \circ \sigma_1)^{-1}(P) \subset \text{supp}((s_j \circ \ldots \circ s_1)(H))$.

Suppose that $P$ is a $D_m$ singularity and put $x_j := x_{j,p}$. To simplify our notation we denote the local equation of the divisor $H$ near the point $P$ by $H$. Let

$$F(x_1, \ldots, x_4) = \sum f_{j_1,j_2,j_3,j_4} \cdot x_1^{j_1} \ldots x_4^{j_4}$$ and $H(x_1, \ldots, x_4) = \sum h_{j_1,j_2,j_3,j_4} \cdot x_1^{j_1} \ldots x_4^{j_4}$.

We consider three cases:

**D$_4$:** In this case, $\tilde{Y}^1 \subset X \times \mathbb{P}_3$ has three $A_1$ points as its only singularities. Let $(y_1 : y_2 : y_3 : y_4)$ stand for homogenous coordinates on $\mathbb{P}_3$. Then in the affine set $y_2 = 1$ the blow-up $\tilde{Y}^1$ is given by the equation

$$x_2 \cdot y_1 \cdot (1 + y_1)^2 + y_3^2 + y_4^2 + \sum f_{j_1,j_2,j_3,j_4} \cdot y_1^{j_1} \cdot x_2^{j_2} \cdot y_3^{j_3} \cdot y_4^{j_4}.$$ Observe that if

$$1/3 \cdot j_1 + 1/3 \cdot j_2 + 1/2 \cdot j_3 + 1/2 \cdot j_4 > 1,$$

then the monomial $y_1^{j_1} \cdot x_2^{j_2} \cdot y_3^{j_3} \cdot y_4^{j_4}$ is singular along the set $x_2 = y_3 = y_4 = 0$. Consequently, for every choice of $F(x_1, \ldots, x_4)$ in the semi-quotiashomogenous equation (3.3), the variety $\tilde{Y}^1$ is singular in the points

$$P_{1,1} := (P, (0 : 1 : 0 : 0)), P_{1,2} := (P, (i : 1 : 0 : 0)), P_{1,3} := (P, (-i : 1 : 0 : 0)) \in X \times \mathbb{P}_3.$$

For $y_2 = 1$, the divisor $s_1(H)$ is given by

$$\sum h_{j_1,j_2,j_3,j_4} \cdot y_1^{j_1} \cdot x_2^{j_2} \cdot y_3^{j_3} \cdot y_4^{j_4} = 0.$$ Thus one can easily see that

$$P_{1,1}, \ldots, P_{1,3} \subset \text{supp}(s_1(H)) \text{ iff } h_{1,0,0,0} = h_{0,1,0,0} = 0.$$

**D$_5$:** Now $\tilde{Y}^1 \subset X \times \mathbb{P}_3$ has an $A_1$ point and an $A_3$ point as its only singularities. One can check that

$$P_{1,1} := (P, (1 : 0 : 0 : 0)) \in \text{sing}(\tilde{Y}^1)$$

for every choice of the polynomial $F$ in (3.3). Since the germ of $\tilde{Y}^1$ in $P_{1,1}$ is given by a semi-quotiashomogenous equation with regard to the weights $w(x_1) = w(y_3) = w(y_4) = \frac{1}{2}$ and $w(y_2) = \frac{1}{4}$,
it is an $A_3$ point (see [16, Char. C9]). The other singularity of the variety $\tilde{Y}^1$ is the $A_1$ point $P_{1,2} := (P, (-f_{0,3,0,0} : 1 : 0 : 0))$.

Finally, we blow-up the singularity $P_{1,1}$. Then, for every polynomial $F$ of order $> 1$, the set $\text{sing}(\tilde{Y}^2)$ consists of the unique $A_1$ point $P_{2,1} := (P_{1,1}, (0 : 1 : 0 : 0))$.

By direct computation we obtain:

\begin{align}
(3.10) & \quad P_{1,1} \in \text{supp}(s_1(H)) \iff h_{1,0,0,0} = 0, \\
(3.11) & \quad P_{1,2} \in \text{supp}(s_1(H)) \iff h_{0,1,0,0} - f_{0,3,0,0} \cdot h_{1,0,0,0} = 0, \\
& \quad P_{2,1} \in \text{supp}(s_2(H)) \iff h_{0,1,0,0} = 0.
\end{align}

$D_m$, $m \geq 6$: In this case, $\text{sing}(\tilde{Y}^1)$ consists of the points $P_{1,1}, P_{1,2}$ as for $m = 5$. The point $P_{1,1}$ is a $D_{m-2}$ singularity by [16, Char. C9], so $P_{1,2}$ in an $A_1$ point. We arrive at the conditions (3.10), (3.11).

One can check that, for $j \geq 2$, the configuration of singularities of $\tilde{Y}^j$ is independent of the polynomial $F$. Proceeding by induction one shows that (3.9) holds.

The proof of (3.9) for $A_m$ and $E_m$ points is analogous, so we leave it to the reader. \qed

Remark: If $\text{sing}(Y)$ consists of $D_4$ points, then the tangent cone $C_P Y$ of $Y$ in $P \in \text{sing}(Y)$ consists of two 3-planes that meet along a 2-plane $\Pi_P$ in this case, Lemma 3.3 reads

\begin{align}
(3.12) & \quad \delta_Y = \dim \{H \in V_Y : \Pi_P \subset T_PH \text{ for every } P \in \text{sing}(Y)\} - h^0(O_X(2Y + K_X)) + 4 \cdot \nu,
\end{align}

where $T_PH$ stands for the Zariski tangent space and $\# \text{sing}(Y) = \nu$.

4. HODGE NUMBERS OF BIG RESOLUTIONS

Let $Y \subset X$ be a three-dimensional hypersurface such that all its singularities are A-D-E and $\text{sing}(X) \cap Y = \emptyset$. Let $a_m$ (resp. $d_m$, resp. $e_m$) stand for the number of singularities of $Y$ of the type $A_m$ (resp. $D_m$, resp. $E_m$). In this case (see (3.5)), the number of singular and infinitely near singular points of the variety $Y$ is the sum

\begin{align}
(4.1) & \quad \sum_{m \geq 1} a_m \cdot [m/2] + \sum_{m \geq 4} 2 \cdot d_m \cdot [m/2] + 4 \cdot e_6 + 7 \cdot e_7 + 8 \cdot e_8.
\end{align}

For every $P \in \text{sing}(Y)$ we choose local (analytic) coordinates $x_{1,P}, x_{2,P}, x_{3,P}, x_{4,P}$ such that the equation of the germ of $Y$ at $P$ in those coordinates is of the form (3.3) and put (see Lemma 3.3)

\begin{align}
\mathfrak{Q}_Y := \{H \in H^0(O_X(2Y + K_X)) \otimes J_{\text{sing}(Y)}\} \text{ such that the condition [II] is fulfilled}\).
\end{align}

Then, by Lemma 3.3, the defect of $Y$ can be expressed as

\begin{align}
(4.2) & \quad \delta_Y := \dim(\mathfrak{Q}_Y) - (h^0(O_X(2Y + K_X)) - \mu_Y).
\end{align}

We have the following generalization of [9 Thm 1] for hypersurfaces with A-D-E singularities.

**Theorem 4.1.** Let $X$ be a projective normal Cohen-Macaulay fourfold and let $Y \subset X$ be a hypersurface with A-D-E singularities such that $\text{sing}(X) \cap Y = \emptyset$. Let $\tilde{\pi} : \tilde{Y} \rightarrow Y$ be the big resolution of $Y$. Assume that the following conditions are satisfied:

- **[A1]**: $H^i(O_X(-Y)) = 0$ for $i \leq 3$ and $H^j(O_X(-2Y)) = 0$ for $j \leq 2$,
- **[A2]**: $H^2(O_X(-Y)) = 0$,
- **[A3]**: $H^i(O_X \otimes O_Y(-Y)) = 0$, for $i = 1, 2, 3$,

then

\begin{align}
(4.3) & \quad h^{1,1}(\tilde{Y}) = h^1(O_X^\perp) + \mu_Y + \delta_Y + h^3(O_X(-2Y)).
\end{align}
Moreover, if $h^2(O_X) = 0$, then
\begin{equation}
(4.4) \quad h^{1,2}(\tilde{Y}) = h^0(O_X(2Y + K_X)) + h^4(\Omega^1_X) - h^0(O_X(Y + K_X)) - \mu_Y + \delta_Y,
\end{equation}
where $\mu_Y$ is the number of singularities and infinitely near singularities of $Y$ and $\delta_Y$ is the defect of $Y$.

**Proof.** We maintain the notation of Sect. 3. In particular, we consider the blow-ups $\sigma_j : \tilde{X}^j \rightarrow \tilde{X}^{j-1}$, where $j = 1, \ldots, n$, and the big resolution $\tilde{\sigma} = \sigma_n \circ \cdots \circ \sigma_1$. We put $\mathcal{M} = O_X(Y)$ and proceed by induction on $n$.

$n=0$: Here $\tilde{Y} = Y, \tilde{X} = X$. By (2.4), we have the isomorphism $N^/_\gamma_X \cong O_Y \otimes \mathcal{M}^{-1}$ and the conormal exact sequence
\begin{equation}
(4.5) \quad 0 \rightarrow O_Y \otimes \mathcal{M}^{-1} \rightarrow \Omega^1_X \otimes O_Y \rightarrow \Omega^1_Y \rightarrow 0,
\end{equation}
so we can almost verbatim follow the proof of [9, Thm 1]:

Lemma 2.4 and the assumptions [A2], [A3] imply
\begin{equation}
(4.6) \quad h^2(\Omega^1_X \otimes O_Y) = 0.
\end{equation}
By Lemma 2.1c and (4.6) the cohomology sequence associated to the conormal sequence (4.5) splits and we obtain the short exact sequence
\begin{equation}
0 \rightarrow H^1(\Omega^1_X \otimes O_Y) \rightarrow H^1(\Omega^1_Y) \rightarrow H^2(O_Y \otimes \mathcal{M}^{-1}) \rightarrow 0.
\end{equation}

Use Lemma 2.4 to compute $h^1(\Omega^1_X \otimes O_Y)$. The first formula results immediately from (2.21).

Assume $h^2(O_X) = 0$. Lemma 2.1b implies $h^3(\Omega^1_Y) = h^2(O_Y) = 0$. Thus, from (4.5) and (4.6), we have the exact sequence
\begin{equation}
0 \rightarrow H^2(\Omega^1_Y) \rightarrow H^3(O_Y \otimes \mathcal{M}^{-1}) \rightarrow H^3(\Omega^1_X \otimes O_Y) \rightarrow 0.
\end{equation}

Lemma 2.4 and (2.22) give the second formula for $n = 0$.

$n-1 \sim n$: By Lemma 2.1a (resp. Lemma 2.1b) the pair $(\tilde{X}^1, \tilde{Y}^1)$ satisfies [A1] (resp. [A3]). Lemma 2.3a yields that $\tilde{X}^1$ fulfills [A2].

We put $\mathcal{L} := O_{\tilde{X}^1}(\tilde{Y}^1)$ and $\nu := \# \text{sing}(Y)$. Since $\mu_Y = \mu_{\tilde{Y}^1} + \nu$, Lemma 2.3a implies the equality
\begin{equation}
(4.7) \quad h^1(\Omega^1_{\tilde{X}^1}) + \mu_{\tilde{Y}^1} = h^1(\Omega^1_X) + \mu_Y.
\end{equation}

From the inductive hypothesis we obtain
\begin{equation}
\begin{align*}
h^{1,1}(\tilde{Y}) &= h^1(\Omega^1_{\tilde{X}^1}) + \mu_{\tilde{Y}^1} + \delta_{\tilde{Y}^1} + h^3(\mathcal{L}^{-2}) \quad \text{(4.7)} \\
&= h^1(\Omega^1_X) + \mu_Y + \delta_{\tilde{Y}^1} + h^3(\mathcal{L}^{-2}) \quad \text{Lemma 2.1c} \\
&\geq h^1(\Omega^1_X) + \mu_Y + \delta_Y + h^3(\mathcal{M}^{-2}),
\end{align*}
\end{equation}
which completes the proof of the formula for $h^{1,1}(\tilde{Y})$.

If $h^2(O_X) = 0$, then we have $h^2(O_{\tilde{X}^1}) = 0$. Serre duality and (2.15) yield
\begin{equation}
(4.8) \quad h^0(\mathcal{L} \otimes O_{\tilde{X}^1}(K_{\tilde{X}^1})) = h^0(\mathcal{M} \otimes O_X(K_X)).
\end{equation}
Moreover, the following equality holds
\begin{equation}
(4.9) \quad h^0(\mathcal{L}^2 \otimes O_{\tilde{X}^1}(K_{\tilde{X}^1})) = \dim V_Y \geq h^0(\mathcal{M}^2 \otimes O_X(K_X)) - \nu + \delta_Y.
\end{equation}
By the inductive hypothesis we have

\[ h^{1,2}(\tilde{Y}) = h^0(\mathcal{L}^2 \otimes \mathcal{O}(K_{\tilde{X}1})) + h^4(\tilde{\Omega}_{\tilde{X}1}) - h^0(\mathcal{L} \otimes \mathcal{O}(K_{\tilde{X}1})) - h^3(\tilde{\Omega}_{\tilde{X}1} - h^4(\tilde{\Omega}_{\tilde{X}1} \otimes \mathcal{L}^{-1}) - \mu_{\tilde{Y}1} + \delta_{\tilde{Y}1} \]

\[ \text{Lemma 2.3} \]

\[ h^0(\mathcal{L}^2 \otimes \mathcal{O}(K_{\tilde{X}1})) + h^4(\tilde{\Omega}_{\tilde{X}1}) - h^0(\mathcal{L} \otimes \mathcal{O}(K_{\tilde{X}1})) - h^3(\tilde{\Omega}_{\tilde{X}1} - h^4(\tilde{\Omega}_{\tilde{X}1} \otimes \mathcal{L}^{-1}) - \mu_{\tilde{Y}1} + \delta_{\tilde{Y}1} \]

Use (4.8) and (4.9) to complete the proof. \qed

Observe that, as in the nodal case (see [9]), using Lemma 2.1 one can easily compute the other Hodge numbers of \( \tilde{Y} \):

\[(4.10) \quad h^1(\mathcal{O}_{\tilde{Y}}) = h^1(\mathcal{O}_X), \quad h^2(\mathcal{O}_{\tilde{Y}}) = h^2(\mathcal{O}_X) \]

\[(4.11) \quad h^3(\mathcal{O}_{\tilde{Y}}) = h^3(\mathcal{O}_X) + h^4(\mathcal{O}_X(-Y)) - h^4(\mathcal{O}_X) . \]

Moreover, for a smooth \( X \) and an ample \( Y \) with A-D-E singularities, [17, Cor. 6.4] reduces the assumptions [A1], [A2], [A3] to the vanishing \( h^2(\Omega^{-1}_X) = h^4(\Omega^{-1}_X(-Y)) = 0 \).

The conditions [A1], [A2], [A3] are satisfied for a simplicial, toric variety \( X \) and an ample hypersurface \( Y \) (for an exposition of the theory of toric varieties see [30]).

**Corollary 4.2.** Let \( X \) be a complete simplicial toric fourfold and let \( Y \subset X \) be a hypersurface with A-D-E singularities such that \( \text{sing}(X) \cap Y = \emptyset \). If \( \mathcal{O}_X(Y) \) is ample, then

\[ h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_X), \quad h^2(\mathcal{O}_Y) = h^2(\mathcal{O}_X) \]

\[ h^3(\mathcal{O}_Y) = h^3(\mathcal{O}_X) + h^4(\mathcal{O}_X(-Y)) - h^4(\mathcal{O}_X) . \]

where \( \mu_Y \) (resp. \( \delta_Y \)) is given by the formula (4.1) (resp. (4.2)).

**Proof.** According to [30, Cor. 3.9], the variety \( X \) is Cohen-Macaulay. From [30, Bott’s Vanishing, p. 130] (see also [3 §. 3]) and Serre duality, we obtain

\[ h^i(\mathcal{O}_X(-Y)) = h^i(\mathcal{O}_X(-2Y)) = 0 \text{ for } i \leq 3. \]

By the same argument \( h^i(\Omega^{-1}_X(-Y)) = 0 \), for \( i = 1, 2, 3 \). Now [30, Cor. 2.8] implies the equality \( h^1(\mathcal{O}_X) = 0 \) for \( i \geq 1 \). Finally, [30, Thm. 3.11] yields the vanishing \( h^i(\Omega^{-1}_X) = 0 \) for \( i \neq 1 \). \qed

**Remark:** Let \( (N, \Delta) \) be a fan and let \( \#\Delta(j) \) stand for the number of \( j \)-dimensional cones in \( \Delta \). For \( X = T_N \text{emb}(\Delta) \), by [30, Thm. 3.11], we have the equality

\[ h^3(\Omega^{-1}_X) = \#\Delta(1) - 4 . \]

For a discussion how to compute the value of \( h^4(\Omega^{-1}_X(-Y)) \) see [27, Thm. 2.14].

**Example 4.1.** Let \( Y \subset \mathbb{P}_4 \) be a degree-\( d \) hypersurface with A-D-E singularities, where \( d \geq 3 \). Then \( \mathfrak{D}_Y \) is the space of degree-\((2d - 5)\) polynomials that vanish along \( \text{sing}(Y) \) and satisfy the condition (3.6) (resp. (3.7), (3.8)) in every \( A_m \) (resp. \( D_m, E_m \)) point of \( Y \). Thm 4.1 implies the equalities

\[ h^{1,1}(\tilde{Y}) = 1 + 2 \cdot \mu_Y + \dim(\mathfrak{D}_Y) - \left( \frac{2d - 1}{4} \right), \quad h^{1,2}(\tilde{Y}) = \dim(\mathfrak{D}_Y) - 5 \cdot \left( \frac{d}{4} \right) . \]

In particular, let \( S_n(y_0, \ldots, y_3) = 0 \) be a quintic in \( \mathbb{P}_3 \) with ordinary double points \( P_1, \ldots, P_\nu \) as its only singularities and let \( L(y_0, \ldots, y_3) = 0 \) be a plane that meets \( S_5 \) transversally. One can check that the only singularities of the threefold quintic

\[ Y_5 : S_5 + y_4 \cdot L = 0 . \]
are the A₃ points Pⱼ, j = 1, . . ., ν. Observe that, for a given equation of a space quintic and a plane, the latter condition can be checked with help of Gröbner bases (see Remark 7.3).

Let us fix a basis g₁, . . ., g₁₂₆ of H⁰(𝓞₋₄(5)) and define the matrix
\[ M₅ := \begin{bmatrix} g₁(P₁) & \cdots & g₁(P₁) & (\partial g₁/\partial y₁)(P₁) & \cdots & (\partial g₁/\partial y₄)(P₁) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g₁₂₆(P₁) & \cdots & g₁₂₆(P₁) & (\partial g₁/\partial y₁)(P₁) & \cdots & (\partial g₁/\partial y₄)(P₁) \end{bmatrix}. \]

By Cor. 4.2 the Hodge numbers of the big resolution of Y₅ are
\[ h^{1,1}(Y₅) = 1 + 4 \cdot ν - \text{rank}(M₅) \quad \text{and} \quad h^{1,2}(Y₅) = 101 - \text{rank}(M₅). \]

5. N-FOLD SOLIDS

In this section we prove various generalizations of the formula [6 Cor. 2.32] for Hodge numbers of big resolutions of double covers of P₃ branched along nodal hypersurfaces.

Let Bᵰ ⊂ P₃(C) be a degree-d surface given by the equation bᵰ = 0, where d > n is divisible by n, and let Yᵰ be the n-fold cyclic cover of P₃ branched along Bᵰ. Then, Yᵰ is a degree-d hypersurface in the weighted projective space \( \mathbb{P} := \mathbb{P}(1, 1, 1, \frac{d}{n}) \). It is defined by
\[ (5.1) \quad y₅^n - bᵰ(y₀, \ldots, y₃) = 0, \]
where \( \mathbb{P} \) is embedded in \( \mathbb{P} \) by the map \( \mathbb{P} \ni (y₀ : \ldots : y₃) \mapsto (y₀ : \ldots : y₃ : 0) \in \mathbb{P}(1, 1, 1, \frac{d}{n}). \)

Recall that \( \mathbb{P} \) has the unique singularity (0 : 0 : 0 : 1) and \( K_{\mathbb{P}} = \mathbb{O}(-4 - \frac{d}{n}). \) Obviously Yᵰ does not contain the singular point of \( \mathbb{P} \). One can easily see that \( \mathbb{P} \) is a simplicial toric variety (see [19, p. 35]) and the hypersurface Yᵰ is ample. Thus the pair (\( \mathbb{P}, Yᵰ \)) satisfies the assumptions [A₁], [A₂], [A₃] (see also [15 Thm 1.4.1] and [15 Thm 2.3.2]). Therefore, if the hypersurface Yᵰ has A-D-E singularities, then we can apply Cor. 4.2.

By [15 Thm 2.3.2] and [15 Cor 2.3.5] we have \( h^{1,1}(\mathbb{O}_{\mathbb{P}}) = 1 \) and the equality
\[ h^{4}(\Omega^{1}_{\mathbb{P}}(-d)) = 4 \cdot h^{0}(\mathbb{O}_{\mathbb{P}}((n - 1) \cdot \frac{d}{n} - 3)) + h^{0}(\mathbb{O}_{\mathbb{P}}(d - 4)) - h^{0}(\mathbb{O}_{\mathbb{P}}((n - 1) \cdot \frac{d}{n} - 4)). \]

Thus the second formula of Cor. 4.2 reads
\[ (5.2) \quad h^{1,2}(Yᵰ) = \sum_{j=1}^{n-1} \left( d + j \cdot \frac{d}{3} - 1 \right) - 4 \sum_{j=1}^{n-1} \left( j \cdot \frac{d}{3} \right) - \mu Yᵰ + \delta Yᵰ, \]
whereas (4.10) yields
\[ (5.3) \quad h^{1}(\mathbb{O}_{Yᵰ}) = h^{2}(\mathbb{O}_{Yᵰ}) = 0 \quad \text{and} \quad h^{3}(\mathbb{O}_{Yᵰ}) = h^{0}(\mathbb{O}_{\mathbb{P}}((n - 1) \cdot \frac{d}{n} - 4)). \]

Let Bᵰ be a surface with Du Val singularities and let aₘ, m ≥ 1 (resp. dₘ, m ≥ 4, resp. eₘ, m = 6, 7, 8) stand for the number of singularities of Bᵰ of the type Aₘ (resp. Dₘ, resp. Eₘ). We define the integer \( \mu Bᵰ \) as the sum (4.11).

For every point \( P \in \text{sing}(Bᵰ) \), we choose such local coordinates \( x₁P, x₂P, x₃P \) on \( \mathbb{P}_3 \) that the germ of \( Bᵰ \) at \( P \) is locally given by the equation
\[ (5.4) \quad n(x₁P, x₂P, x₃P) + F(x₁P, x₂P, x₃P) = 0, \]
where \( n(x₁P, x₂P, x₃P) \) is the normal form from the table (3.3) and \( F(x₁P, x₂P, x₃P) \) is a polynomial of order > 1 with respect to the corresponding weights.

Let \( d ≥ 4 \) be an even integer. We define (see Lemma 3.3)
\[ \mathfrak{M}_{Bᵰ,2} := \{ H ∈ H^{0}(\mathbb{O}_{\mathbb{P}}(3d/2 - 4) ⊗ I_{\text{sing}}(Bᵰ)) \} \]
\[ \delta_{Bᵰ,2} := \dim(\mathfrak{M}_{Bᵰ,2}) - \left( \frac{3d/2 - 1}{3} \right) + \mu Bᵰ. \]
With this notation we have:

**Corollary 5.1.** If $Y_d$ is a double solid (i.e. $n = 2$) branched along a surface $B_d$ with Du Val singularities, then

$$h^{1,1}(\tilde{Y}_d) = 1 + \mu_{B_d} + \delta_{B_d,2}, \quad \text{and} \quad h^{1,2}(\tilde{Y}_d) = \left(\frac{3d/2 - 1}{3}\right) - 4 \left(\frac{d/2}{3}\right) - \mu_{B_d} + \delta_{B_d,2}. $$

**Proof.** If we put $x_{4,P} := y_4$ for $P \in \text{sing}(B_d)$, then in the local coordinates $x_1, x_2, x_3, x_{4,P}$ the hypersurface $Y_d$ is given by the equation (3.2).

Observe that every $H \in \mathbb{H}^0(\mathcal{O}_\mathbb{P}(2Y_d + K_\mathbb{P}))$ that is divisible by $y_4$ belongs to $\mathfrak{Y}_{Y_d}$. The latter implies the equality

$$h^0(\mathcal{O}_\mathbb{P}(2Y_d + K_\mathbb{P})) - \dim(\mathfrak{Y}_{Y_d}) = h^0(\mathcal{O}_\mathbb{P}(3 \cdot d/2 - 4)) - \dim(\mathfrak{Y}_{B_d,2}),$$

so we get $\delta_{Y_d} = \delta_{B_d,2}$ (see Def. 3.1 and (2.3)). Now the corollary results from (5.2).

**Suppose** that $Y_d$ is the cyclic $n$-fold cover of $\mathbb{P}_3$ branched along a nodal hypersurface $B_d$ (i.e. all singularities of $B_d$ are $A_1$ points), where $d > n$. Since each singularity of $B_d$ endows $Y_d$ with an $A_{n-1}$ point, we have $\mu_{Y_d} = [(n - 1)/2] \cdot a_1$. We define

$$\delta_{B_d,n} := \sum_{j=n/2}^{n-1} (h^0(\mathcal{O}_{\mathbb{P}_3}(d + j \cdot \frac{d}{n} - 4) \otimes \mathcal{I}_{\text{sing}(B_d)}) - (d + j \cdot \frac{d}{n} - 1)) + [(n - 1)/2] \cdot a_1.$$ 

In this case, Thm 4.1 yields

**Corollary 5.2.** If $Y_d$ is the cyclic $n$-fold cover of $\mathbb{P}_3$ branched along a nodal hypersurface $B_d$ with $a_1$ singularities, then

$$h^{1,1}(\tilde{Y}_d) = 1 + \frac{(n - 1)}{2} \cdot a_1 + \delta_{B_d,n},$$

$$h^{1,2}(\tilde{Y}_d) = \sum_{j=1}^{n-1} (d + j \cdot \frac{d}{n} - 3) - 4 \sum_{j=1}^{n-1} (\frac{j^n}{3}) - [(n - 1)/2] \cdot a_1 + \delta_{B_d,n}.$$ 

**Proof.** Put $x_{1,P} := y_4$. Observe that for $j \geq [(n - 1)/2]$ the inclusion

$$y_4^j \cdot H^0(\mathcal{O}_{\mathbb{P}_3}((2n - 1 - j) \cdot \frac{d}{n} - 4)) \subset \mathfrak{Y}_{Y_d}$$

holds, whereas for $j \leq [(n - 1)/2] - 1$ we have

$$y_4^j \cdot H^0(\mathcal{O}_{\mathbb{P}_3}((2n - 1 - j) \cdot \frac{d}{n} - 4) \cap \mathfrak{Y}_{Y_d} = y_4^j \cdot H^0(\mathcal{O}_{\mathbb{P}_3}((2n - 1 - j) \cdot \frac{d}{n} - 4) \otimes \mathcal{I}_{\text{sing}(B_d)}) .$$

This implies the equality $\delta_{Y_d} = \delta_{B_d,n}$. Now (5.2) completes the proof.

Finally, assume that all singularities of $B_d$ are ordinary cusps (i.e. $A_2$ points). Recall that for $P \in \text{sing}(B_d)$, the tangent cone $C_P B_d$ consists of two planes meeting along a line. We denote this line by $L_P$.

Let $Y_d$ be the triple cyclic cover of $\mathbb{P}_3$ branched along the hypersurface $B_d$. Every singular point of the surface $B_d$ endows the threefold $Y_d$ with a singularity of the type $D_4$ (see [16, Char. C. 9]). By (3.5) we have $\mu_{Y_d} = 4 \cdot a_2$. We define

$$\mathfrak{Y}_{B_d,3} := \{ H \in \mathbb{H}^0(\mathcal{O}_{\mathbb{P}_3}(5d/3 - 4) \otimes \mathcal{I}_{\text{sing}(B_d)}): L_P \subset T_P H \text{ for every } P \in \text{sing}(B_d) \},$$

where $T_P H$ stands for the Zariski tangent space, and

$$\delta_{B_d,3} := \dim(\mathfrak{Y}_{B_d,3}) - (\frac{5d/3 - 1}{3}) + h^0(\mathcal{O}_{\mathbb{P}_3}(4d/3 - 4) \otimes \mathcal{I}_{\text{sing}(B_d)}) - (\frac{4d/3 - 1}{3}) + 4a_2.$$

In this case, Thm 4.1 implies
Corollary 5.3. Let $Y_d$ be the triple cover of $\mathbb{P}_3$ branched along a hypersurface $B_d$, where $d \geq 6$, with $a_2$ ordinary cusps as its only singularities. Then

\begin{align*}
 h^{1,1}(\tilde{Y}_d) &= 1 + 4 \cdot a_2 + \delta_{B_d,3}, \\
 h^{1,2}(\tilde{Y}_d) &= \left(\frac{4d+3}{3} - 1\right) \cdot \left(\frac{5d+3}{3} - 1\right) - 4 \cdot \left(\frac{2d+3}{3}\right) - 4a_2 + \delta_{B_d,3}.
\end{align*}

Proof. We define $x_{4,P} := y_4$ for every $P \in \text{sing}(Y_d)$. We are to compute the defect $\delta_{Y_d}$.

By (5.12) the condition (5.1) reads

\begin{equation}
 H(P) = \frac{\partial H}{\partial x_{4,P}}(P) = \frac{\partial H}{\partial x_{4,P}}(P) = 0.
\end{equation}

Therefore, we have the equality

\begin{equation}
 H^0(\mathcal{O}_{\mathbb{P}_3}(5d/3 - 4)) \cap \mathfrak{M}_{Y_d} = \mathfrak{M}_{B_d,3}.
\end{equation}

Observe that a hypersurface $H \in y_4 \cdot H^0(\mathcal{O}_{\mathbb{P}_3}(4d/3 - 4))$ satisfies (5.3) for every $P \in \text{sing}(Y_d)$ iff we have $H \in y_4 \cdot H^0(\mathcal{O}_{\mathbb{P}_3}(4d/3 - 4)) \otimes \mathcal{J}_{\text{sing}(B_d)}$.

Finally, one can easily see that every $H \in H^0(\mathcal{O}_{\mathbb{P}_3}(5d/3 - 4))$ that is divisible by $y_4^2$ fulfills the condition (5.5). We obtain the equality $\delta_{Y_d} = \delta_{B_d,3}$ and the proof is complete. \hfill \square

We end this section with an example which shows that working on a singular ambient variety $X$ is more efficient than dealing with a desingularization of $X$.

Example 5.1. Let $Y_6 \subset \mathbb{P}_3 := \mathbb{P}(1,1,1,1,2)$ be the triple cover of $\mathbb{P}_3$ branched along a sextic surface and let $\overline{\mathbb{P}}$ be the blow-up of $\mathbb{P}(1,1,1,1,2)$ in the singular point $(0 : \ldots : 1)$. Since $Y_6$ does not pass through the center of the blow-up, its proper transform is no longer ample. By abuse of notation we use $Y_6$ to denote the proper transform in question. We claim that the pair $(\overline{\mathbb{P}},Y_6)$ does not satisfy the assumption [A3] of Thm. [4.1] i.e.

\begin{equation}
 h^1(\Omega^1_{\overline{\mathbb{P}}}(Y_6)) = 1.
\end{equation}

One can show that $\overline{\mathbb{P}} = \mathbb{P}(\mathcal{E})$ with $\mathcal{E} := \mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(2)$ (we maintain the notation of [21] Ex. III.8.4]).

Then, we have $K_{\overline{\mathbb{P}}} = (\pi^*\mathcal{O}_{\mathbb{P}_3}(-2))(-2)$, where $\pi : \overline{\mathbb{P}} \to \mathbb{P}_3$ stands for the bundle projection and $\text{Pic}(\overline{\mathbb{P}}) = \mathbb{Z}$. From [21] Def., p. 429], we get $\mathcal{O}_{\overline{\mathbb{P}}}(1)^2 = 2\pi^*\mathcal{O}_{\mathbb{P}_3}(1) \cdot \mathcal{O}_{\overline{\mathbb{P}}}(1)$, which yields

\begin{equation}
 \pi^*\mathcal{O}_{\mathbb{P}_3}(1)^4 = 0 \quad \text{and} \quad \mathcal{O}_{\overline{\mathbb{P}}}(1)^k \cdot \pi^*\mathcal{O}_{\mathbb{P}_3}(1)^{4-k} = 2^{k-1} \quad \text{for } k = 1, \ldots, 4.
\end{equation}

By studying the canonical quotient map $\mathbb{P}_4 \to \mathbb{P}$ [see [14] App. B]) we obtain the equality

\begin{equation}
 Y_6^k(\pi^*\mathcal{O}_{\mathbb{P}_3}(1)^{4-k}) = 3^k \cdot 2^{k-1} \quad \text{for } k = 1, \ldots, 4.
\end{equation}

We claim that $Y_6 = 3\mathcal{O}_{\overline{\mathbb{P}}}(1)$. Indeed, let $Y_6 = a \pi^*\mathcal{O}_{\mathbb{P}_3}(1) + b\mathcal{O}_{\overline{\mathbb{P}}}(1)$. Since $Y_6 \cdot \pi^*\mathcal{O}_{\mathbb{P}_3}(3)^3 = 3$, the equality (5.7) with $k = 1$ implies $b = 3$. Now, (5.7) with $k = 4$ and (5.8) give $a = 0$.

By [21] Ex. III.8.4.c] we have

\begin{equation}
 \pi_* (\pi^*\mathcal{O}_{\mathbb{P}_3}(-2)(1)) = \mathcal{O}_{\mathbb{P}_3}(-2) \oplus \mathcal{O}_{\mathbb{P}_3} \quad \text{and} \quad R^1 \pi_* (\mathcal{O}_{\mathbb{P}_3}(-2)(1)) = 0,
\end{equation}

so the Leray spectral sequence implies $h^i(\overline{\mathbb{P}}, \pi^*\mathcal{O}_{\mathbb{P}_3}(-2)(1)) = h^i(\mathbb{P}_3, \mathcal{O}_{\mathbb{P}_3}(-2) \oplus \mathcal{O}_{\mathbb{P}_3})$, where $i \geq 0$.

Serre duality yields the equalities

\begin{equation}
 h^4(\mathcal{O}_{\overline{\mathbb{P}}}(-3)) = 1 \quad \text{and} \quad h^j(\mathcal{O}_{\overline{\mathbb{P}}}(-3)) = 0 \quad \text{for } j \leq 3.
\end{equation}

In similar way we show that

\begin{equation}
 h^1(\pi^*\mathcal{E})(-4) = 1, \quad h^4(\pi^*\mathcal{E})(-4) = 12, \quad h^j(\pi^*\mathcal{E})(-4) = 0 \quad \text{for } j = 0, 2, 3.
\end{equation}

Thus the exact sequence [21] Ex. III.8.4.b] tensored with $\mathcal{O}_{\overline{\mathbb{P}}}(-3)$:

\begin{equation}
 0 \to \Omega^3_{\overline{\mathbb{P}}/\mathbb{P}_3}(-3) \to (\pi^*\mathcal{E})(-4) \to \mathcal{O}_{\overline{\mathbb{P}}}(-3) \to 0
\end{equation}
the threefold \( Y \) is smooth, \( \hat{\pi} \) is small resolution is Kähler, then it is projective.

\[ \mathcal{O}_\hat{Y}(-3) \]

In order to compute \( h^j(\pi^*\mathcal{O}_\hat{Y}(-3)) \), we consider the pull-back of the Euler sequence under the map \( \pi \) and tensor it with \( \mathcal{O}_\hat{Y}(-3) \):

\[ 0 \longrightarrow (\pi^*\mathcal{O}_\hat{Y})(-3) \longrightarrow \oplus_1^4 (\pi^*\mathcal{O}_\hat{Y}(-1))(-3) \longrightarrow \mathcal{O}_\hat{Y}(-3) \longrightarrow 0. \]

We use Serre duality, \[21\] Ex. III.8.4.a and the Leray spectral sequence to show that

\[ h^j(\pi^*\mathcal{O}_\hat{Y}(-1))(-3) = 0 \] for \( j \leq 3 \), and \( h^4(\pi^*\mathcal{O}_\hat{Y}(-1))(-3) = 4 \).

The latter, combined with \( 5.9 \) and \( 5.12 \), yields

\[ h^j(\pi^*\mathcal{O}_\hat{Y}(-3)) = 15 \text{ and } h^j((\pi^*\mathcal{O}_\hat{Y})(-3)) = 0 \] for \( j \leq 3 \).

Finally, we tensor the exact sequence

\[ 0 \longrightarrow \pi^*\mathcal{O}_\hat{Y} \longrightarrow \mathcal{O}_\hat{Y} \longrightarrow \mathcal{O}_\hat{Y}/\mathcal{O}_\hat{Y} \longrightarrow 0 \]

with \( \mathcal{O}_\hat{Y}(-3) \), and apply \( 5.11 \), \( 5.13 \) to see that \( 5.6 \) holds.

6. SMALL RESOLUTIONS VERSUS BIG RESOLUTIONS

Let us assume that \( \text{sing}(Y) = \{P_1, \ldots, P_{\nu}\} \) consists of Gorenstein singularities. Suppose that the threefold \( Y \) has a small resolution \( \hat{\pi} : \hat{Y} \rightarrow Y \), i.e. \( \hat{\pi} \) is a proper holomorphic map such that \( \hat{Y} \) is smooth, \( \hat{\pi}|_{\hat{Y} \setminus \hat{\pi}^{-1}(\text{sing}(Y))} \) is an isomorphism onto the image and the exceptional set

\[ \hat{E} := \hat{\pi}^{-1}(\text{sing}(Y)) \]

is a curve. By \[18\] Thm 1.3 and \[18\] Thm 1.5 (see also \[32\]) the exceptional set \( \hat{\pi}^{-1}(P_l) \), where \( l = 1, \ldots, \nu \), consists of smooth rational curves meeting transversally in seven possible configurations. In particular, the fibers of \( \hat{\pi} \) are connected.

It should be pointed out that a small resolution does not have to be Kähler (\[38\]). However, since the algebraic dimension of \( \hat{Y} \) is maximal, i.e. it equals three, by \[21\] Appendix B, Thm 4.2] if a small resolution is Kähler, then it is projective.

Let \( \hat{Y} \) be a smooth projective variety and let \( \hat{\pi} : \hat{Y} \rightarrow Y \) be a projective morphism such that \( \hat{\pi}|_{\hat{Y} \setminus \hat{\pi}^{-1}(\text{sing}(Y))} \) is an isomorphism onto the image. Moreover, we assume that the exceptional set \( \hat{E} := \text{Ex}(\hat{\pi}) \) is a divisor in \( \hat{Y} \). Observe that the big resolution of a hypersurface with A-D-E singularities (see Sect. 3) satisfies the above conditions.

In this section we compare Hodge numbers of the manifolds \( \hat{Y}, \hat{Y} \) in the case when the former is Kähler and certain cohomology groups vanish.

By \[18\] Thm 1.5 and Mayer-Vietoris we have

\[ H^j(\hat{E}, \mathbb{C}) = 0 \text{ for } j \neq 0, 2. \]

We consider the Leray spectral sequence (\[20\] § 4.17 or \[28\] Thm. 12.13) for the constant sheaf \( \mathbb{C}_{\hat{Y}} \) and the map \( \hat{\pi} \):

\[ E_2^{p,q} = H^p(\hat{Y}, R^q\hat{\pi}_*\mathbb{C}_{\hat{Y}}) \Rightarrow H^{p+q}(Y, \mathbb{C}_{\hat{Y}}). \]

The fibers of \( \hat{\pi} \) are connected, so \( \hat{\pi}_*\mathbb{C}_{\hat{Y}} = \mathbb{C}_Y \) and we get \( E_2^{0,0} \cong H^1(Y, \mathbb{C}). \)

Since \( \hat{Y} \) and \( Y \) are locally compact and \( \hat{\pi} \) is proper, \[20\] Remarque 4.17.1 (see also \[20\] Thm 4.11.1) yields that, for \( j > 0 \), the sheaf \( R^j\hat{\pi}_*\mathbb{C}_{\hat{Y}} \) is a sky-scraper sheaf concentrated in the singularities of \( Y \). Furthermore, we have

\[ (R^j\hat{\pi}_*\mathbb{C}_{\hat{Y}})_{P_l} \cong H^j(\hat{\pi}^{-1}(P_l), \mathbb{C}) \]
for \( l = 1, \ldots, \nu \). Thus (6.1) means that the only non-zero \( E^{p,q}_2 \)-term for \( q > 0 \) is
\[
E^{0,2}_2 \cong H^2(\hat{E}, \mathbb{C}) ,
\]
and we have the long exact sequence (see e.g. [28, Ex. 1.D])
\[
\ldots \rightarrow H^{j+2}(\hat{Y}, \mathbb{C}) \rightarrow E^{j,2}_2 \rightarrow E^{j+3,0}_2 \rightarrow H^{j+1+2}(\hat{Y}, \mathbb{C}) \rightarrow E^{j+1,2}_2 \rightarrow E^{j+1+3,0}_2 \rightarrow \ldots .
\]
The latter yields the equalities
(6.3)
\[
h^j(Y, \mathbb{C}) = h^j(\hat{Y}, \mathbb{C}) \text{ for } j = 0, 1, 4, 5, 6 .
\]

After those preparations we can prove

**Proposition 6.1.** If \( h^1(O_{\hat{Y}}) = 0, h^3(\hat{E}, \mathbb{C}) = 0 \) and \( \hat{Y} \) is Kähler, then
\[
h^{2,2}(\hat{Y}) = h^{2,2}(\hat{Y}) + h^4(\hat{E}, \mathbb{C}) .
\]

**Proof.** The varieties \( \hat{Y} \) and \( \hat{Y} \) are smooth and birationally equivalent (recall that \( \hat{Y} \) is projective), so the equalities \( h^{3,1}(\hat{Y}) = h^2(\hat{Y}) = h^2(\hat{Y}) = h^3(\hat{Y}) \) hold. Therefore, it suffices to show that
(6.4)
\[
h^4(\hat{Y}, \mathbb{C}) = h^4(\hat{Y}, \mathbb{C}) + h^4(\hat{E}, \mathbb{C}) .
\]

We consider the Leray spectral sequence \( E^{p,q}_2 = H^p(Y, R^q\pi_*\mathcal{O}_{\hat{Y}}) \Rightarrow H^{p+q}(\hat{Y}, \mathcal{O}_{\hat{Y}}) \). As in the proof of [6.2] we show that, for \( j > 0 \), the sheaf \( R^j\pi_*\mathcal{O}_{\hat{Y}} \) is a sky-scraper sheaf concentrated in the singularities of \( Y \) and for all \( P_i \in \text{sing}(Y) \) we have
(6.5)
\[
(R^j\pi_*\mathcal{O}_{\hat{Y}})_{P_i} \cong H^j(\pi^{-1}(P_i)), \mathcal{O}_{\mathbb{C}} .
\]

Since the fibers of \( \pi \) are connected, we have \( \pi_*\mathcal{O}_{\hat{Y}} = \mathcal{O}_{\hat{Y}} \) and the isomorphisms \( E^{0,0}_2 \cong H^j(Y, \mathbb{C}) \).

We want to show that
(6.6)
\[
E^{0,4}_\infty \cong E^{0,4}_2 \cong H^4(\hat{E}, \mathbb{C}) .
\]
At first we check that \( E^{r,5-r}_2 \)-terms vanish for \( r \in \mathbb{Z} \). The latter is obvious for \( r \neq 5 \) because all \( E^{p,q}_2 \)-terms vanish for \( p, q \neq 0 \), and \( E^{0,5}_2 = H^5(\hat{E}, \mathbb{C}) = 0 \) (recall that \( \dim(\hat{E}) = 2 \)). It remains to prove that \( E^{5,0}_2 = H^5(\hat{Y}, \mathbb{C}) = 0 \). In view of (6.3), it suffices to show that \( H^5(\hat{Y}, \mathbb{C}) = 0 \). But we have the vanishing \( h^0(\hat{Y}) = h^1(\hat{Y}) = h^2(\hat{Y}) = 0 \). Since \( \hat{Y} \) is Kähler, we obtain the equality \( h^5(\hat{Y}, \mathbb{C}) = 2 \cdot h^3(\hat{Y}) = 0 \).

In particular, we have \( E^{r,5-r}_1 = 0 \) for \( r \geq 2, r \in \mathbb{Z} \), so the differential \( d_r : E^{0,4}_r \rightarrow E^{r,(5-r)}_r \) is the zero map for \( r \geq 2 \). The latter yields (6.6) because the \( E^{p,q}_2 \)-terms vanish for \( p < 0 \).

Now, we claim that
(6.7)
\[
E^{4,0}_\infty \cong E^{4,0}_2 \cong H^4(Y, \mathbb{C}) .
\]
Indeed, all differentials \( d_r : E^{4,0}_r \rightarrow E^{4+r-r,0+1}_r \), where \( r \geq 2, \) are trivial because there are no non-zero \( E^{p,q}_2 \)-terms for \( q < 0 \). In order to control the maps \( d_r : E^{4-r,r-1}_r \rightarrow E^{4}_r \), one has to observe that, for \( r \geq 2 \), we have \( E^{0,3}_r \cong E^{0,3}_2 \cong H^3(\hat{E}, \mathbb{C}) = 0 \). Thus if \( r \geq 2 \), then the \( E^{4-r,r-1}_r \)-term vanishes, and we obtain (6.7).

To complete the proof of (6.4) observe that \( E^{4-r}_\infty = 0 \) for \( r \neq 0, 4 \), and apply (6.3). \( \square \)

In Examples 6.1, 6.2 we assume that \( \hat{Y} \) is Kähler and \( h^1(O_{\hat{Y}}) = 0 \).

**Example 6.1.** Let \( \text{sing}(Y) = \{ P_1, \ldots, P_\nu \} \) consist of \( D_4 \) points. Assume that the variety \( Y \) is given in a neighbourhood \( U_j \) of the point \( P_j \) in the local coordinates \( x_{1,j}, \ldots, x_{4,j} \) by the equation
(6.8)
\[
x_{1,j} \cdot x_{2,j} + x_{3,j}^3 + x_{4,j}^3 = 0 .
\]
Let $\varepsilon$ be a primitive root of unity of order three. We put $f_{j,k} := x_{3,j} + \varepsilon^k \cdot x_{4,j}$. For every $P_j \in \text{sing}(Y)$ and $k = 0, 1, 2$, we define the local Weil divisors on $Y$

$$D_{j,k} : x_{1,j} = 0, \quad f_{j,k} = 0.$$ 

According to [5, §2.7] (see also [38, p. 101]), a small resolution of $U_j$ can be obtained as the projection from the closure of the graph of the meromorphic map

$$U_j \ni (x_{1,j}, \ldots, x_{4,j}) \rightarrow ((x_{1,j} : f_{j,0}), (x_{1,j} : f_{j,0} \cdot f_{j,1})) \in \mathbb{P}_1 \times \mathbb{P}_1.$$ 

One can easily see that $\hat{\pi}^{-1}(P_j)$ consists of the rational curves $(0 : 1) \times \mathbb{P}_1, \mathbb{P}_1 \times (1 : 0)$. Let $\sigma_1 : \hat{Y}^1 \rightarrow Y$ be the blow-up of $Y$ in $\text{sing}(Y)$. By direct computation $\text{sing}($\hat{Y}^1$)$ consists of three ordinary double points on the rational curve where the components of the exceptional divisor $(\sigma_1)^{-1}(P_j)$ meet. After blowing up the nodes, we get the big resolution. Thus the exceptional divisor $E_{P_j} := \hat{\pi}^{-1}(P_j)$ consists of three quadrics and two copies of $\mathbb{P}_2$ blown-up in three points. Therefore, by Mayer-Vietoris, we have

$$h^3(E_{P_j}, \mathbb{C}) = 0, \quad h^4(E_{P_j}, \mathbb{C}) = 5 \quad \text{and} \quad h^{1,1}(\hat{Y}) = h^{1,1}(\hat{Y}) + 5 \cdot \nu.$$ 

**Example 6.2.** Suppose that $\text{sing}(Y)$ consists of $A_m$ points such that all $m$'s are odd. Let $P_j$, where $j = 1, \ldots, \nu$, be locally given by the equation:

$$x_{1,j}^{2(k_j + 1)} + x_{2,j}^2 + x_{3,j}^2 + x_{4,j}^2.$$ 

If we blow up the germ of the surface

$$x_{1,j}^{k_j + 1} - i \cdot x_{2,j} = x_{3,j} - i \cdot x_{4,j} = 0,$$ 

then the proper transform is smooth and the singular point $P_j$ is replaced with a copy of $\mathbb{P}_1$ (see [26, Ex. 2.2], [32] for more details). After performing such blow-ups in every singularity, we obtain a small resolution $\hat{Y}$. Recall that the exceptional divisor $E_{P_j}$ of the big resolution of an $A_{2k_j + 1}$ point consists of $(k_j + 1)$ smooth rational surfaces $\hat{E}_{1,j}, \ldots, \hat{E}_{k_j + 1}$, where $\hat{E}_l, \hat{E}_{l+1}$ meet along a smooth rational curve and $\hat{E}_1 \cap \hat{E}_{k_j + 1} = 0$ for $|l_1 - l_2| > 1$. For $l \leq k_j$ the component $\hat{E}_l$ is a Hirzebruch surface $\mathbb{F}_2$, whereas $\hat{E}_{k_j + 1}$ is a smooth quadric. We have

$$h^3(E_{P_j}, \mathbb{C}) = 0, \quad h^4(E_{P_j}, \mathbb{C}) = k_j + 1 \quad \text{and} \quad h^{1,1}(\hat{Y}) = h^{1,1}(\hat{Y}) + \sum_{j=1}^\nu (k_j + 1).$$

**Remark 6.2.** a) By [38, Satz, p. 103] the variety $Y$ in $\text{Ex 6.1}$ has a small resolution $\hat{Y}$ that is Kähler if and only if for every $P_j$ the local Weil divisors $D_{j,k}$ (see (6.9)), where $k = 0, 1, 2$, can be prolonged to (global) Weil divisors on $Y$ that are smooth at $P_j$. In particular, in this case the global divisors that prolong $D_{j,k}$, for a fixed $j$ and various $k$, have no common component through the point $P_j$.

b) For even $m$, the threefold singularities $A_m$ have no small resolutions by [32, Cor. 1.16].

7. **Triple sextics and double octics**

Here we compute the Hodge numbers of Kähler small resolutions $\hat{Y}$ of triple (resp. double) covers of $\mathbb{P}_3$ branched along various sextics (resp. octics) with $A_2$ singularities (resp. $A_j$, where $j \geq 3$ is odd). We use the same symbol to denote a hypersurface and its defining polynomial.

Consider the manifold $Y_6$ obtained as a small resolution of a triple cover $Y_6$ of $\mathbb{P}_3$ branched along a sextic $B_6$ with $A_2$ singularities. Obviously the canonical class $K_{Y_6}$ is trivial (see [24, Prop. 5.73]).
Since $\hat{Y}_6$ is a crepant resolution, we have $K_{\hat{Y}_6} = 0$. Assume that $\hat{Y}_6$ is Kahler. By \cite{5.3} it carries neither global 1-forms nor global 2-forms. Consequently, $\hat{Y}_6$ is a Calabi-Yau manifold.

Let $\text{sing}(B_6) = \{P_1, \ldots, P_\nu\}$. In order to compute the Hodge numbers of $\hat{Y}_6$, we fix a basis $h_1, \ldots, h_35$ of $H^0(O_{P_3}(4))$ and a basis $f_1, \ldots, f_{84}$ of $H^0(O_{P_3}(6))$. We define the matrix

$$M_4 := [h_i(P_j)]_{i=1, \ldots, 35 \atop j=1, \ldots, \nu}.$$  

For every $P_j \in \text{sing}(B_6)$, the Hessian $H_{B_6}(P_j)$ vanishes (in $\mathbb{C}^4$) along two 3-planes $\Pi_{1,j}, \Pi_{2,j}$. Their common part consists of a 2-plane. For $j = 1, \ldots, \nu$ we choose a vector $v_j \in \mathbb{C}^4$, such that $\text{span}(v_j, (P_{j,1}, \ldots, P_{j,4})) = \Pi_{1,j} \cap \Pi_{2,j}$, and define the matrix

$$M_6 := \begin{bmatrix} f_1(P_1) & \cdots & f_1(P_\nu) & f'_1(P_1)v_1 & \cdots & f'_1(P_\nu)v_\nu \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{84}(P_1) & \cdots & f_{84}(P_\nu) & f'_{84}(P_1)v_1 & \cdots & f'_{84}(P_\nu)v_\nu \end{bmatrix}.$$  

Observe that, by \cite{5.3} and Ex.\cite{6.1} the variety $\hat{Y}_6$ fulfills the assumptions of Prop.\cite{6.1}. From the latter and Cor.\cite{5.3} we obtain

**Corollary 7.1.** The Hodge numbers of the Calabi-Yau manifold $\hat{Y}_6$ are given by the formulae:

$$h^{1,1}(\hat{Y}_6) = 1 + 3 \cdot \nu - \text{rank}(M_4) - \text{rank}(M_6),$$

$$h^{1,2}(\hat{Y}_6) = 103 - \text{rank}(M_4) - \text{rank}(M_6).$$  

**Proof.** We maintain the notation of the previous section. For $F \in H^0(O_{P_3}(6))$, the condition \cite{5.5} means that the point $P$ belongs to $F$ and the (Zariski) tangent space of the cone over $F$ at the point $P_j \in \mathbb{C}^4$ contains the 2-plane $\Pi_{1,j} \cap \Pi_{2,j}$. The latter amounts to the equalities $F(P_j) = F(P_j)v_j = 0$, so $\dim(\mathfrak{M}_{B_6}) = 84 - \text{rank}(M_6)$ and we are in position to compute the defect

$$\delta_{B_6} = 4 \cdot \nu - \text{rank}(M_4) - \text{rank}(M_6).$$  

Thus Cor.\cite{5.3} combined with Prop.\cite{6.1} and \cite{6.10} yields the formula for $h^{1,1}(\hat{Y}_6)$.

Finally, we claim that

$$h^{1,2}(\hat{Y}_6) = h^{1,2}(\hat{Y}_6).$$  

Indeed, since $e(\overline{\pi}^{-1}(P_j)) = 13$ (see Ex.\cite{6.1}), we have the equality $e(\hat{Y}_6) = e(\hat{Y}_6) + 10 \cdot \nu$. From \cite{5.3} we obtain $e(\hat{Y}_6) = 2(h^{1,1}(\hat{Y}_6) - h^{1,2}(\hat{Y}_6))$. Therefore, the formula for $h^{1,1}(\hat{Y}_6)$ implies \eqref{7.1}. \hfill \square

Remark: As an immediate consequence of Cor.\cite{7.1} we obtain the equality $e(\hat{Y}_6) = 6 \cdot \nu - 204$. The latter results also from \cite{5} Lemma 3. Indeed, by Noether’s formula, we have $e(B_6) = 108 - 2 \cdot \nu$. Thus we obtain $e(\mathbb{P}_3 \setminus B_6) = 2 \cdot \nu - 104$, and the equality in question follows.

Now we are in position to compute the Hodge numbers of Kahler small resolutions of triple sextics branched along the surfaces discussed in \cite{1}, \cite{2} and \cite{25}.

**Direct construction of \cite{2}:** We choose surfaces $S_1, \ldots, S_k$ of degrees

$$\text{deg}(S_1) = d_1, \ldots, \text{deg}(S_k) = d_k, \quad d_1 + \ldots + d_k = 6,$$

a quadric $S$ and consider the sextic $B_6 \subset \mathbb{P}_3$ given by the equation

$$S_1 \cdot \ldots \cdot S_k - S^2 = 0.$$  

We require that

\begin{enumerate}
\item[(\text{a}_1)] any three surfaces $S_i, S_j, S$ meet transversally,
\item[(\text{a}_2)] no four surfaces $S_i, S_j, S_m, S$ meet,
\item[(\text{a}_3)] the surface $B_6$ is smooth away from the cusps $P_\nu$ at the intersections $S_i \cap S_j \cap S$. \end{enumerate}
The resulting sextic \((7.2)\) has \(2 \cdot \sum_{i\neq j} d_i \cdot d_j\) cusps and no other singularities.

**Lemma 7.2.** If \(Y_6\) is the triple cover of \(\mathbb{P}^3\) branched along the sextic \(B_6\) obtained by the direct construction \((7.2)\), then there exists a small resolution \(\hat{Y}_6\) of \(Y_6\) that is Kähler.

**Proof.** Let \(Y_6 \subset \mathbb{P}(1,1,1,1,2)\) be given by the equation \((5.1)\) and let \(P_\nu\) be a cusp in \(S_i \cap S_j \cap S\). We consider the (global) Weil divisors
\[
(7.3) \quad W_\nu : S_i = 0, \quad \epsilon^l \cdot S + y_4 = 0,
\]
where \(l = 0, 1, 2\) and \(\epsilon\) is a primitive root of unity of order three. Obviously, \(W_\nu\) prolongs the local divisor \(D_{\nu,l}\) (see (6.3)) and the germ of \(W_\nu\) in the point \(P_\nu\) is smooth. Thus the assumptions of \([38, \text{ Satz, p. 103}]\) (see also Remark 6.2.a) are satisfied.

To check that the conditions \((\mathfrak{o}_1) - (\mathfrak{o}_3)\) are satisfied by given surfaces \(S_1, \ldots, S_k, S\) we will use the following

**Remark 7.3.** In order to show that a polynomial \(g \in K[y_0, y_1, y_2, y_3]\) belongs to an ideal \(I\), one applies the notion of the remainder on division of a polynomial \(g\) by a Gröbner basis \(B\) of the ideal \(I \subset K[y_0, y_1, y_2, y_3]\) (see \([37, \text{ II.3.6}]\)). It is well-known that if the remainder vanishes, then \(g\) is an element of \(I\). The former can be checked e.g. with the Maple command: \text{normalf}(g, B, \text{tdeg}(y_0, y_1, y_2, y_3)).\ If the output is zero, then \(g \in I\).

**Example 7.1.** We consider the quadric
\[
S : y_0 \cdot y_1 - y_2 \cdot y_3 = 0,
\]
the planes
\[
F_1 : y_0 = 0, \quad F_2 : y_1 = 0, \\
F_3 : 4 \cdot y_0 - 2 \cdot y_2 - 2 \cdot y_3 + y_1 = 0, \quad F_4 : y_0 - 2 \cdot y_2 - 2 \cdot y_3 + 4 \cdot y_1 = 0, \\
F_5 : y_0 + y_2 + y_3 + y_1 = 0, \quad F_6 : y_0 - y_2 - y_3 + y_1 = 0,
\]
and define the sextic
\[
B_6 : F_1 \cdot \ldots \cdot F_6 - S^3 = 0.
\]
The lines \(F_i = F_j = 0\), where \(i \neq j \leq 6\), meet the quadric \(S\) in two points, so the condition \((\mathfrak{o}_1)\) is satisfied.

Since a Gröbner basis computation with Maple (see Remark 7.3) shows that the polynomials \(y_0^1, y_1^1, y_2^1, y_3^1\) belong to the ideal generated by \(F_i, F_j, F_k, S\) with \(i \neq j \neq k\), the condition \((\mathfrak{o}_2)\) is fulfilled.

Finally, a similar Gröbner basis argument shows the polynomial \(S^{10}\) belongs to the jacobian ideal of \(B_6\). Therefore, if \(\text{mult}_P(B_6) \geq 2\), then the inequality \(\text{mult}_P(F_1 \cdot \ldots \cdot F_6) \geq 2\) holds. The latter shows that the condition \((\mathfrak{o}_3)\) is satisfied.

Let \(\hat{Y}_6\) be a Kähler small resolution of the triple sextic branched along \(B_6\). A Maple computation yields \(\text{rank}(M_4) = 25\) and \(\text{rank}(M_6) = 55\). The sextic \(B_6\) has 30 singularities, so we obtain the equalities
\[
h^{1,1}(\hat{Y}_6) = 11 \quad \text{and} \quad h^{1,2}(\hat{Y}_6) = 23.
\]

**Example 7.2.** We maintain the notation of Example 7.1. Consider the quadric
\[
R : y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0
\]
and the planes
\[
R_1 : y_0 + 2 \cdot y_2 + 3 \cdot y_3 + 4 \cdot y_1 = 0, \quad R_2 : 4 \cdot y_0 + 3 \cdot y_2 + 2 \cdot y_3 + y_1 = 0.
\]
The Gröbner basis computation shows that $y_0^{10}, \ldots, y_3^{10}$ belong to the jacobian ideals of the surfaces

\[
\begin{align*}
S_{1,2} : & \ F_1 \cdot F_2 + 2 \cdot S = 0, \\
S_{3,4} : & \ F_3 \cdot F_4 + 2 \cdot S = 0, \\
S_{1,2,3} : & \ F_1 \cdot F_2 \cdot F_3 + S \cdot R_1 = 0, \\
S_{3,4,5,6} : & \ F_3 \cdot F_4 \cdot F_5 \cdot F_6 + S \cdot R = 0, \\
S_{2,3,4,5,6} : & \ F_2 \cdot F_3 \cdot F_4 \cdot F_5 \cdot F_6 + R_1 \cdot S \cdot R = 0,
\end{align*}
\]

(7.4)

so the latter are smooth. We obtain the following table:

| Equation of $B_6$ | $\sharp \text{(sing}(B_6))$ | $\text{rk}(M_4)$ | $\text{rk}(M_6)$ | $h^{1,1}(Y_6)$ | $h^{1,2}(Y_6)$ |
|-------------------|----------------|----------------|----------------|----------------|----------------|
| $F_1 \cdot S_{2,3,4,5,6} - S^3$ | 10 | 9 | 19 | 3 | 75 |
| $S_{1,2} \cdot S_{3,4,5,6} - S^3$ | 16 | 15 | 31 | 3 | 57 |
| $S_{1,2,3} \cdot S_{4,5,6} - S^3$ | 18 | 17 | 35 | 3 | 51 |
| $F_1 \cdot F_2 \cdot S_{3,4,5,6} - S^3$ | 18 | 16 | 34 | 5 | 53 |
| $F_1 \cdot S_{2,3} \cdot S_{4,5,6} - S^3$ | 22 | 20 | 42 | 5 | 41 |
| $S_{1,2} \cdot S_{3,4} \cdot S_{5,6} - S^3$ | 24 | 22 | 46 | 5 | 35 |
| $F_1 \cdot F_2 \cdot F_3 \cdot S_{4,5,6} - S^3$ | 24 | 21 | 45 | 7 | 37 |
| $F_1 \cdot F_2 \cdot S_{3,4} \cdot S_{5,6} - S^3$ | 26 | 23 | 49 | 7 | 31 |
| $F_1 \cdot \ldots \cdot F_4 \cdot S_{5,6} - S^3$ | 28 | 24 | 52 | 9 | 27 |

For the sextics in the first column of the table, we check that the conditions $(\varphi_1) - (\varphi_3)$ are fulfilled in the way shown in the previous example. In particular, all singularities of each sextic lie on the quadric $S$ and the surfaces (7.4) are smooth, so $(\varphi_3)$ holds. To find the singular points of the surfaces from the table, we use the fact that they are singularities of the sextic $B_6$ from Example 7.1.

**Residual construction of [2]:** To construct another example we apply the residual construction of [2]. We choose

- a residual cubic $R$,
- auxiliary planes $R_1, \ldots, R_3$ such that the curves $R_i \cap R$ are smooth and intersect transversally,
- cubic surfaces $S_i : R_i^3 + \lambda_i \cdot R = 0$,
- a cubic $S : R_1 \cdot \ldots \cdot R_3 + \lambda \cdot R = 0$,

where $\lambda, \lambda_i \in \mathbb{C}$. Then, the polynomial $S_1 \cdot S_2 \cdot S_3 - S^3$ always vanishes along the residual cubic $R$ and we can consider the following sextic

\[(7.5)\]

\[B_6 : (S_1 \cdot S_2 \cdot S_3 - S^3)/R = 0.\]

By [2, Sect. 1.2], if we choose the constants $\lambda, \lambda_i$ general enough, then

1. $B_6$ has no singularities along the residual sextic $R$,
2. the cubics $S_i, S_j, S$ intersect transversally outside of $R$,
3. the sextic $B_6$ is smooth away from the points in $S_i \cap S_j \cap S$.

The conditions (1) – (3) imply that the sextic $B_6$ has 27 $A_2$ singularities.

Let $Y_6 \subset \mathbb{P}(1,1,1,1,2)$ be the triple cover of $\mathbb{P}_3$ branched along the sextic (7.5) and let $P$ be a cusp in $S_i \cap S_j \cap S$. As in the proof of Lemma 7.2, we show that the divisors (7.3) are smooth in $P$, so the triple sextic $Y_6$ has a Kähler small resolution $\tilde{Y}_6$. 

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Example 7.3. (c.f. [2 Sect. 2.3]) We take as the residual cubic the Fermat cubic
\[ R : y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0 \]
and put
\[ S_i : y_i^3 + R = 0, \quad i = 1, 2, 3, \quad S : y_1 \cdot y_2 \cdot y_3 = 0. \]
Using Gröbner bases (see Remark 7.3) we show that the above defined hypersurfaces and the sextic \( B_6 \) given by the equation (7.5) satisfy the conditions (τ1) – (τ3). In particular, the surface \( B_6 \) is smooth away from the points \((1 : 0 : ε^{i_1} : ε^{i_2}), \quad (1 : ε^{i_1} : 0 : ε^{i_2}), \quad (1 : ε^{i_1} : ε^{i_2} : 0), \) where \( 0 ≤ i_1, i_2 ≤ 2 \) and \( ε \) is a primitive root of \((-1/3)\) of order three. We obtain rank(M_4) = 24 and rank(M_6) = 51, so
\[ h^{1,1}(\hat{Y}_6) = 7 \text{ and } h^{1,2}(\hat{Y}_6) = 28. \]

Sextic with 36 cusps (see [23]): Recall that, by [37], [29], the number of \( A_2 \) singularities on a sextic in \( \mathbb{P}_3 \) does not exceed 37 and it is not known whether this bound is sharp. As the last example of a triple sextic, we consider the cover branched along the sextic with 36 ordinary cusps that was constructed in [25 App. A].

Example 7.4. Consider the quadric \( S : y_0 \cdot y_1 - y_2 \cdot y_3 = 0 \) and the linear forms
\[ z_0 := y_0, \quad z_2 := y_0 + y_1 - y_2 - y_3, \quad z_1 := y_1, \quad z_3 := 8y_0 + 8y_1 - 64y_2 - y_3, \]
that define the planes tangent to \( S \) in the points
\[ (0 : 1 : 0 : 0), (1 : 0 : 0 : 0), (1 : 1 : 1 : 1), (8 : 8 : 1 : 64). \]
Let \( B_6 \) be the pull-back of \( S \) under the map \( \Omega_3^3 : (z_0 : z_1 : z_2 : z_3) \rightarrow (z_0^3 : z_1^3 : z_2^3 : z_3^3) \). Then, \( B_6 \) is the sextic given by the polynomial
\[(7.6) \quad (z_0 \cdot z_1)^3 - (8/9z_0^3 + 8z_1^2 - 64/63z_2^3 + 1/63z_3^3) \cdot (1/9z_0^3 + 1/9z_1^3 + 1/63z_2^3 - 1/63z_3^3), \]
with 36 \( A_2 \)-points
\[(\Omega_3^3)^{-1}((0 : 1 : 1 : 8), (1 : 0 : 1 : 8), (1 : 1 : 0 : -(49)), (8 : 8 : (-49) : 0)) \]
and no other singularities. In order to show that the triple cover \( Y_6 \subset \mathbb{P}(1, 1, 1, 1, 2) \) branched along \( B_6 \) has a small resolution \( \hat{Y}_6 \) that is Kähler, we use [38 Satz, p. 103] (see also Remark 6.2.a):

Obviously \( B_6 \) is given by the equation of type (7.1) with the quadric \( z_0 \cdot z_1 \). Let \( W_l \) be the divisors defined by (7.3), where \( l = 0, 1, 2 \). As in the proof of Lemma 7.2 we show that \( W_l \) are smooth in the cusps \((\Omega_3^3)^{-1}((0 : 1 : 1 : 8), (1 : 0 : 1 : 8)).\)

If we put
\[ S' = -3^{4/3} \cdot z_2 \cdot z_3, \quad S'_1 = 7 \cdot z_0^3 - 56 \cdot z_1^3 - 8 \cdot z_2^3 - z_3^3, \quad S'_2 = 56 \cdot z_0^3 - 7 \cdot z_1^3 + 8 \cdot z_2^3 + z_3^3, \]
then, by direct computation, the equation (7.6) can be written as
\[-(1/3969) \cdot (S'_1 \cdot S'_2 - (S')^3). \]

One can easily see that the divisors
\[ W'_l : S'_1 = 0, \quad ε^l \cdot S' + y_4 = 0, \quad l = 0, 1, 2, \]
are smooth in the cusps \((\Omega_3^3)^{-1}((1 : 1 : 0 : (-49)), (8 : 8 : (-49) : 0))\), so there exists a Kähler small resolution.

In this case, the Maple computation yields rank(M_4) = 30 and rank(M_6) = 66. Therefore, we have
\[ h^{1,1}(\hat{Y}_6) = 13 \text{ and } h^{1,2}(\hat{Y}_6) = 7. \]
**Remark 7.4.** a) For a Calabi-Yau manifold $\hat{Y}$ both $h^{1,1}(\hat{Y})$ and $h^{1,2}(\hat{Y})$ have a geometric interpretation: the latter is the dimension of the space of infinitesimal deformations of the manifold in question, whereas the former equals the rank of $\text{Pic}(\hat{Y})$.

Observe that the group $\text{Pic}(\hat{Y}_6)$ is free. Indeed, if we repeat the proof of (6.3) for the constant sheaf $\mathbb{Z}\hat{Y}_6$, then we obtain the exact sequence

$$0 \rightarrow H^2(Y_6, \mathbb{Z}) \rightarrow H^2(\hat{Y}_6, \mathbb{Z}) \rightarrow H^2(\hat{Y}_6, \mathbb{Z}) \rightarrow H^3(\hat{Y}_6, \mathbb{Z}) \rightarrow 0.$$ 

Since $H^2(\hat{Y}_6, \mathbb{Z})$ is torsion-free, $\text{Torsion}(H^2(\hat{Y}_6, \mathbb{Z}))$ comes from the group $H^2(Y_6, \mathbb{Z})$. But we have $\text{Torsion}(H^2(\hat{Y}_6, \mathbb{Z})) \cong \text{Torsion}(H^1(\hat{Y}_6, \mathbb{Z}))$. Therefore, if $Y_6$ is simply-connected, then $H^2(\hat{Y}_6, \mathbb{Z})$ is torsion-free. In particular, for a weighted projective hypersurface $Y_6$, by [14 Cor. B.21], we obtain

$$\text{Torsion}(H^2(\hat{Y}_6, \mathbb{Z})) = 0.$$ 

b) Recall that the extended code $\mathcal{E}_{B_6}$ of a sextic $B_6$ with cusps $P_1, \ldots, P_k$ is defined as the kernel of the $\mathbb{F}_3$-linear morphism

$$\mathbb{F}_3^{k+1} \rightarrow H^2(\tilde{B}_6, \mathbb{F}_3), \quad (t_0, t_1, \ldots, t_k) \mapsto \mathcal{O}_{B_6}(t_0) + \sum_{j=1}^k t_j[C'_j - C''_j],$$

where $\tilde{B}_6$ is the minimal resolution of the surface $B_6$ and $C'_j, C''_j$ are the exceptional $(-2)$-curves over the cusp $P_j \in B_6$ (see [2 Sect. 1.3] for more details). By [2 Prop. 2.1] and [2 Thm 2.9] the extended code $\mathcal{E}_{B_6}$ of a sextic $B_6$ given by (7.2) depends only on the partition $d_1, \ldots, d_k$. Furthermore, we have $\dim(\mathcal{E}_{B_6}) = k - 1$. Thus for the sextics in Ex. 7.1, 7.2 the following equality holds

$$h^{1,1}(\hat{Y}_6) = 2 \cdot \dim(\mathcal{E}_{B_6}) + 1.$$ 

The dimension of $\mathcal{E}_{B_6}$ is unknown for the surfaces from Ex. 7.3, 7.4.

Consider a small resolution of the double octic i.e. the manifold $\hat{Y}_8$ obtained as a small resolution of the double cover $Y_8$ of $\mathbb{P}_3$ branched along an octic $B_8$ with $A_m$ singularities such that $m$’s are odd. Assume that $\hat{Y}_8$ is Kähler. As in the case of triple sextics $\hat{Y}_6$, one can show that the projective manifold $\hat{Y}_8$ is a Calabi-Yau manifold: the canonical class $K_{\hat{Y}_8}$ is trivial, and, by (5.3), $\hat{Y}_8$ carries neither global 1-forms nor global 2-forms.

In order to give an explicit formula for Hodge numbers of $\hat{Y}_8$, we choose a basis $g_1, \ldots, g_{165}$ of $H^0(\mathcal{O}_{\mathbb{P}_3}(8))$. For an $A_{2k+1}$ point $P_j \in \text{sing}(B_8)$ we put

$$M_{8,j} := \begin{bmatrix} g_1(P_j) & \cdots & (g_1|_{L_j})^{(k)}(P_j) \\ \vdots & \ddots & \vdots \\ g_{165}(P_j) & \cdots & (g_{165}|_{L_j})^{(k)}(P_j) \end{bmatrix},$$

where the line $L_j$ is the singular locus of the set of zeroes of the Hessian $H_{B_8}(P_j)$ and $(\cdot)^{(k)}$ denotes the $k$-th derivative. Let $a_{2k+1}$ stand for the number of $A_{2k+1}$ points of the octic $B_8$. We define the $(165 \times (\sum a_{2k+1} \cdot (k + 1)))$-matrix

$$M_8 := [M_{8,1}, \ldots, M_{8,a_{2k+1}}].$$

Then, Cor. 5.1 implies

**Corollary 7.5.** *The Hodge numbers of the Calabi-Yau manifold $\hat{Y}_8$ are given by the formulae:*

$$h^{1,1}(\hat{Y}_8) = 1 + \sum a_{2k+1} \cdot (k + 1) - \text{rank}(M_8),$$

$$h^{1,2}(\hat{Y}_8) = 149 - \text{rank}(M_8).$$
Proof. We maintain the notation of Ex 6.2. Observe that Cor. 5.1 combined with Prop. 6.1 and (6.12) gives the first formula. By direct computation (see Ex. 6.2) we have \( e(\tilde{\pi}^{-1}(P_j)) = 4 + 2k_j \) which yields the equality

\[
e(\tilde{Y}_8) = e(\tilde{Y}_8) - \sum a_{2k+1} \cdot (k + 1).
\]

From (5.3) we obtain \( h^{1,2}(\tilde{Y}_8) = h^{1,2}(\tilde{Y}_8) \) (see the proof of Cor 7.1). □

Observe that in this case, [5, Lemma 3] implies the equality \( e(\hat{Y}_8) = 2 \cdot \sum a_{2k+1} \cdot (k + 1) - 296 \).

By [29] there are no octics in \( \mathbb{P}_3 \) with more than 69 \( A_3 \) points. The best known example is an octic \( B_8 \) with 64 such singularities ([25, App. A]). Now we apply Cor. 7.5 to compute Hodge numbers of a Kähler small resolution \( \hat{Y}_8 \) of the double solid branched along the surface \( B_8 \).

Example 7.5. We maintain the notation of Ex. 7.4 and consider the pull-back of the quadric \( S \) under the map

\[
\Omega^2_4 : (z_0 : z_1 : z_2 : z_3) \longrightarrow (z_0^4 : z_1^4 : z_2^4 : z_3^4).
\]

The resulting surface is smooth away from the 64 \( A_3 \) points

\[
(\Omega^2_4)^{-1}(\{(0 : 1 : 1 : 8), (1 : 0 : 1 : 8), (1 : 1 : 0 : (-49)), (8 : 8 : (-49) : 0)\}).
\]

The Maple computation yields \( \text{rank}(M_8) = 122 \). Therefore, if \( Y \) has a small resolution that is Kähler, then the Hodge numbers are

\[
h^{1,1}(\tilde{Y}_8) = 7 \text{ and } h^{1,2}(\tilde{Y}_8) = 27.
\]

Acknowledgement: The author would like to thank Prof. W. P. Barth for numerous fruitful discussions. We thank Prof. D. van Straten for [36] and Prof. S. Cynk for inspiring discussions on the papers [8], [9], [10], [11].

The paper contains results from the author’s Habilitationsschrift. The author would like to thank the Institute of Mathematics of Erlangen-Nürnberg University for creating perfect research conditions.

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Slawomir Rams
Mathematisches Institut, Universität Erlangen-Nürnberg, Bismarckstraße 1 1/2, D-91054 Erlangen, Germany
and
Institute of Mathematics, Jagiellonian University, ul. Reymonta 4, 30-059 Krakow, Poland
E-mail address: rams@mi.uni-erlangen.de, Slawomir.Rams@im.uj.edu.pl