Pion masses under intense magnetic fields within the NJL model

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The behavior of charged and neutral pion masses in the presence of a static uniform magnetic field is studied in the framework of the two-flavor Nambu-Jona-Lasinio (NJL) model. Analytical calculations are carried out employing the Ritus eigenfunction method. Numerical results are obtained for definite model parameters, comparing the predictions of the model with present lattice QCD (LQCD) results.

The study of the behavior of strongly interacting matter under intense external magnetic fields has gained increasing interest in the last few years, especially due to its applications to the analysis of relativistic heavy ion collisions and the description of compact objects like magnetars [1]. In this work we concentrate on the effect of an intense external magnetic field on π meson properties. This issue has been studied in the last years following various theoretical approaches for low-energy QCD, such as NJL-like models, chiral perturbation theory, path integral Hamiltonians and LQCD calculations (see e.g. [2] and refs therein). In the framework of the NJL model, mesons are usually described as quantum fluctuations in the random phase approximation (RPA) [2]. In the presence of a magnetic field, the corresponding calculations require some special care, due to the appearance of Schwinger phases [3] associated with quark propagators. For the neutral pion these phases cancel out, and as a consequence the usual momentum basis can be used to diagonalize the corresponding polarization function [4–7]. On the other hand, for charged pions Schwinger phases do not cancel, leading to a breakdown of translational invariance that prevents to proceed as in the neutral case. In this contribution we present a method based on the Ritus eigenfunction approach [8] to magnetized relativistic systems, which allows us to fully diagonalize the charged pion polarization function. Further details of this work can be found in Ref. [9].

We start by considering the Euclidean Lagrangian density for the NJL two-flavor model in the presence of an electromagnetic field. One has

\[ \mathcal{L} = \bar{\psi} (-i \not{D} + m_0) \psi - G \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \not{F} \psi) \right], \]

where \( \psi = (u \ d)^T \), \( \tau_i \) are the Pauli matrices, and \( m_0 \) is the current quark mass, which is assumed to be equal for \( u \) and \( d \) quarks. The interaction between the fermions and the electromagnetic field \( A_\mu \) is driven by the covariant derivative \( D_\mu = \partial_\mu - i \bar{Q} A_\mu \) where \( \bar{Q} = \text{diag}(g_u, g_d) \), with \( g_u = 2e/3 \) and \( g_d = -e/3 \), \( e \) being the proton electric charge. We consider here an homogeneous stationary magnetic field along the 3 axis in the Landau gauge, \( A_\mu = B x_1 \delta_{\mu 2} \).

To study meson properties it is convenient to introduce scalar and pseudoscalar fields \( \sigma(x) \) and \( \pi(x) \), integrating out the fermion fields. The bosonized Euclidean action is given by [2]

\[ S_{\text{bos}} = - \log \det D + \frac{1}{4G} \int d^4x \left[ \sigma(x) \sigma(x) + \pi(x) \pi(x) \right]. \]

(2)

We proceed by expanding this effective action in powers of the fluctuations \( \delta \sigma(x) \) and \( \delta \pi(x) \) around the corresponding mean field (MF) values. As usual, we assume that the field \( \sigma(x) \) has a nontrivial translational invariant MF value \( \bar{\sigma} \), while the vacuum expectation values of pseudoscalar fields are zero. In this way one has

\[ S_{\text{bos}} = S_{\text{bos}}^{\text{MF}} + S_{\text{bos}}^{\text{quad}} + \ldots \]

(3)

Here, the mean field action per unit volume reads

\[ S_{\text{bos}}^{\text{MF}} = \frac{\bar{\sigma}^2}{4G} - \frac{N_c}{V(4)} \sum_j d^4x d^4x' \tr \ln \left( S_{x-x'}^{\text{MF},f} \right)^{-1}, \]

(4)

where \( \tr \) stands for the trace in Dirac space. The quadratic contribution can be written as

\[ S_{\text{bos}}^{\text{quad}} = \frac{1}{2} \sum_{M=\sigma,\pi} \int d^4x d^4x' \delta M(x)^* G_M(x,x') \delta M(x'), \]

(5)

where \( r = 0, \pm \) with \( \pi^\pm = (\pi_1 \mp i\pi_2)/\sqrt{2} \), and

\[ G_M(x,x') = \frac{1}{2G} \delta^{(4)}(x-x') - J_M(x,x'), \]

\[ J_{\pi^\pm}(x,x') = N_c \sum_f \tr \left[ S_{x-x'}^{\text{MF},f} \gamma_5 S_{x-x'}^{\text{MF},f} \gamma_5 \right], \]

\[ J_{\pi^\pm}(x,x') = 2N_c \tr \left[ S_{x-x'}^{\text{MF},u} \gamma_5 S_{x-x'}^{\text{MF},d} \gamma_5 \right]. \]

(6)

The expression for \( J_{\pi^\pm} \) is obtained from \( J_{\pi^\pm} \) replacing \( \gamma_5 \) matrices with unit matrices in Dirac space. In these expressions we have introduced the mean field quark propagators \( S_{x-x'}^{\text{MF},f} \). As is well known, their explicit form can be written in different ways [1]. For convenience we take...
here a form given by a product of a phase factor and a translational invariant function, namely

$$S_{x,x'}^{\text{MF,f}} = e^{i\Phi_f(x,x')} \int \frac{d^4p}{(2\pi)^4} e^{i\rho(p-x')} \tilde{S}_p^f,$$  \hspace{1cm} (7)

where $\Phi_f(x,x') = \exp [iq_f(Bx_1 + x_1')(x_2 - x_2')/2]$ is the so-called Schwinger phase. We express now $\tilde{S}_p^f$ in the Schwinger form \[1\]

$$\tilde{S}_p^f = \int_0^\infty d\tau \exp \left[ -\tau \left( M^2 + p_0^2 + p_\perp^2 \tan \frac{\tau B_f}{\tau B_f} \right) \right] \times \left( (M - p_0 \gamma_0^0) (1 + is_f \gamma_1 \gamma_2 \tanh \frac{\tau B_f}{\tau B_f}) - \frac{p_\perp \gamma_3 \tanh \tau B_f}{\cosh^2 \tau B_f} \right),$$  \hspace{1cm} (8)

where we have introduced some definitions. The perpendicular and parallel gamma matrices are collected in vectors $\gamma_\perp = (\gamma_1, \gamma_2)$ and $\gamma_\parallel = (\gamma_3, \gamma_4)$. Similarly, $p_\perp = (p_1, p_2)$ and $p_\parallel = (p_3, p_4)$. The quark effective mass $M$ is given by $M = m_0 + \bar{s}$, while $s_f = \text{sign}(q_f B)$ and $B_f = |q_f| B$. Notice that the integral in Eq. (8) is divergent and has to be properly regularized, as we discuss below.

At the MF level, one arrives to the usual gap equation by replacing in Eq. (4) the above expression for the quark propagator and minimizing with respect to $M$. It can be seen that if we regularize this equation using the Magnetic Field Independent Regularization (MFIR) scheme \[10, 11\] together with a 3D cutoff, the resulting equation for the quark propagator different from the Magnetic Field Independent Regularization (MFIR) scheme has already been carried out in Ref. \[2\]. Moreover, it also matches the result obtained in Ref. \[10\], where the propagator is expressed in terms of a sum over Landau levels.

As for the pion masses, we notice that the analysis of the $\pi^0$ pole mass in the presence of a magnetic field within the MFIR scheme has already been carried out in Refs. \[3, 4\]. However, in those works the authors use a representation of the quark propagator different from the Schwinger one in Eqs. \[7, 8\]. Thus, we find it opportune to verify that both representations lead to the same results for the $\pi^0$ mass. We start by replacing Eq. (4) into the expression for the polarization function $J_{\pi^0}(x,x')$ in Eq. (6). The contributions of the Schwinger phases to each term of the sum correspond to the same quark flavor, hence, they cancel out. As a consequence, the polarization function depends only on the difference $x - x'$ (i.e., it is translational invariant), which leads to the conservation of $\pi^0$ momentum. If we take now the Fourier transform of the $\pi^0$ fields to the momentum basis, the corresponding transform of the polarization function will be diagonal in $q, q'$ momentum space. Thus, the $\pi^0$ contribution to the quadratic action in the momentum basis can be written as

$$S_{\pi^0}^{\text{quad}} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \delta \pi^0(-q) \left[ \frac{1}{2G} - J_{\pi^0}(q_\perp^2, q_\parallel^2) \right] \delta \pi^0(q),$$  \hspace{1cm} (9)

Choosing the frame in which the $\pi^0$ meson is at rest, its mass can be obtained by solving the equation

$$\frac{1}{2G} - J_{\pi^0}^{(\text{reg})}(0, -m_{\pi^0}^2) = 0,$$  \hspace{1cm} (10)

where $J_{\pi^0}^{(\text{reg})}(0, -m_{\pi^0}^2)$ is obtained from $J_{\pi^0}(0, -m_{\pi^0}^2)$ after some regularization procedure. Using the MFIR scheme, it can be shown that —as in the case of the gap equation— our result for $J_{\pi^0}^{(\text{reg})}(0, -m_{\pi^0}^2)$ agrees with the corresponding expression obtained in Ref. \[5\], where the calculation has been done using an expansion in Landau levels for the quark propagators instead of considering the Schwinger form in Eq. \[6\].

Let us focus on the study of charged pion masses. We will consider the $\pi^\pm$ meson, although a similar analysis can be carried out for the $\pi^0$, leading to the same expression for the $B$-dependent mass. Once again, we replace Eq. (4) into the expression for the polarization function $J_{\pi^\pm}(x,x')$ in Eq. (6). Now, in contrast to the $\pi^0$ case, it is seen that the Schwinger phases do not cancel, due to their different quark flavors. Therefore, the $\pi^\pm$ polarization function is not translational invariant, and consequently it will not become diagonal when transformed to the momentum basis. In this situation we find it convenient to follow the Ritus eigenfunction method \[5\]. Namely, we expand the charged pion field as

$$\pi^\pm(x) = \frac{1}{2}\sum_{k=0}^\infty \left[ \prod_{l=2}^4 \int_{2\pi} dq_l \right] F^\pm_{\bar{q}}(x) \pi^\pm_{\bar{q}},$$  \hspace{1cm} (11)

where $\bar{q} = (k, q_2, q_3, q_4)$ and

$$F^\pm_{\bar{q}}(x) = N_k e^{is_1(q_2x_2 + q_3x_3 + q_4x_4)} D_k(\rho_+).$$  \hspace{1cm} (12)

Here $D_k(x)$ are the cylindrical parabolic functions, and we have used the definitions $N_k = (4\pi B_\pi^+)^{1/4}/\sqrt{k!}$ and $\rho_+ = \sqrt{2B_\pi^+ x_1 - s_+ \sqrt{2}/B_\pi^+} q_2$, where $B_\pi^+ = |q_\pi^+ B|$ and $s_+ = \text{sign}(q_\pi^+ B)$, with $q_\pi^+ = q_\pi - q_\delta = e$. In this basis the charged pion polarization function becomes diagonal. The corresponding contribution to the quadratic action in Eq. (5) is given by

$$S_{\pi^\pm}^{\text{quad}} = \frac{1}{4\pi} \sum_{k=0}^\infty \left[ \prod_{l=2}^4 \int dq_l \right] \left( \delta \pi^\pm_{\bar{q}} \right)^2 \frac{1}{2G} - J_{\pi^\pm}(k, \Pi^2) \left( \delta \pi^\pm_{\bar{q}} \right)^2,$$  \hspace{1cm} (13)

where $\Pi^2 = (2k + 1) B_\pi^+ + q_\parallel^2$ and

\[14\]

$$J_{\pi^\pm}(k, \Pi^2) = \frac{N_k}{2\pi^2} \int_0^\infty dz \int_0^1 d\bar{q} \frac{e^{-z\bar{q}(1-y)(\Pi^2-(2k+1)B_\pi^+)}}{\alpha_+} \left( 1 - \frac{t_\parallel^2}{\alpha_+} \right) \left( \frac{1 - (1-t_\parallel^2)}{\alpha_+ + (\alpha_- - \alpha_+) k} \right) \left( M^2 + \frac{1}{z} - y(1-y) (\Pi^2 - (2k+1)B_\pi^+) \right) \left( 1 - t_\parallel t_\parallel \right).$$

\hspace{1cm}
Here we have defined \( t_u = \tanh(B_u y z), t_d = \tanh[B_d(1 - y)z] \) and \( \alpha_{\pm} = (B_d t_u + B_u t_d \pm B_x t_u t_d)/(B_u B_d) \).

Once again, we carry out a regularization within the MFIR scheme, using a 3D cutoff. We obtain

\[
J_{\pi+}^{(reg)}(k, \Pi^2) = J_{\pi+, \Pi^2=0}^{(reg)}(\Pi^2) + J_{\pi+}^{(mag)}(k, \Pi^2),
\]

where \( J_{\pi+}^{(mag)}(k, \Pi^2) \) is finite and \( J_{\pi+, \Pi^2=0}^{(mag)}(\Pi^2) \) corresponds to the usual pion polarization function in the absence of magnetic field evaluated at \( q^2 = \Pi^2 \). It can be easily seen that the same polarization function is obtained for the case of the \( \pi^- \) meson.

For a point-like pion in Euclidean space, the two-point function will vanish when \( \Pi^2 = -m_{\pi+}^2 \) or, equivalently, \( q_i^2 = -[m_{\pi+}^2 + (2k + 1)eB] \), for a given value of \( k \). Therefore, in our framework the charged pion pole mass can be obtained for each Landau level \( k \) by solving the equation

\[
1 - J_{\pi+}^{(reg)}(k, -m_{\pi+}^2) = 0.
\]

Of course, while for a point-like pion \( m_{\pi+} \) is a \( B \)-independent quantity (the \( \pi^+ \) mass in vacuum), in the present model—which takes into account the internal quark structure of the pion—it depends on the magnetic field. Instead of dealing with this quantity, it has become customary in the literature to define the \( \pi^+ \) “magnetic field-dependent mass” (MFDM) as the lowest quantum-mechanically allowed energy of the \( \pi^+ \) meson, namely

\[
E_{\pi+}(eB) = \sqrt{m_{\pi+}^2 + \Pi^2 - q_3^2} \bigg|_{q_3 = 0} = \sqrt{m_{\pi+}^2 + eB},
\]

(see e.g. Ref. [12]). Notice that this “mass” is magnetic field-dependent even for a point-like particle. In fact, owing to zero-point motion in the 1-2 plane, even for \( k = 0 \) the charged pion cannot be at rest in the presence of the magnetic field.

To get numerical predictions we consider some model parameterizations that reproduce not only low-energy phenomenological vacuum properties but also LQCD results for the behavior of quark-antiquark condensates under an external magnetic field. Let us consider the parameter set \( m_0 = 5.66 \text{ MeV}, \Lambda = 613.4 \text{ MeV} \) and \( GA^2 = 2.250 \), which (for vanishing external field) corresponds to an effective mass \( M = 350 \text{ MeV} \) and a quark-antiquark condensate \( \langle \bar{q}q \rangle (B = 0) = (-243.3 \text{ MeV})^3 \). We denote this parameterization as Set I. To test the sensitivity of our results with respect to the model parameters we will consider two alternative parameterizations, denoted as Set II and Set III, which correspond to \( M = 320 \) and 380 MeV, respectively. All these parameter sets properly reproduce the empirical values of the pion mass and decay constant in vacuum, \( m_\pi = 138 \text{ MeV} \) and \( f_\pi = 92.4 \text{ MeV} \). As discussed in Ref. [8], they also provide a very good agreement with the lattice results quoted in Ref. [13] for the quark condensates under an external magnetic field. In fact, it is seen that the predictions are not significantly affected by the parameter choice.

In Fig. 1 we show our numerical results for the behavior of pion masses, which are plotted as functions of \( eB \). In the case of the \( \pi^+ \), the curves correspond to the MFDM defined by Eq. (17). As can be seen, the model predicts an increasing enhancement of \( E_{\pi+} \) with the magnetic field. For comparison, we also show the behavior of \( E_{\pi+} \) for the case of a point-like meson and the LQCD results quoted in Ref. [14]. These LQCD calculations consider realistic pion masses and values of \( eB \) up to \( \sim 0.4 \text{ GeV}^2 \), while they seem to deviate from them for larger values of the magnetic field. Concerning the \( \pi^0 \) mass, it is seen that it shows a slight decrease with \( eB \), as previously found e.g. in Refs. [8, 9]. Once again the results are in general rather independent of the model parametrization.

![Figure 1. (Color online) \( \pi^0 \) mass and \( \pi^+ \) MFDM as functions of \( eB \) for different model parameter sets. The MDFM of a point-like \( \pi^+ \) pion (dotted line) as well as results from LQCD calculations in Ref. [14] (squares) are included for comparison.](image_url)

Besides the mentioned LQCD calculation in Ref. [14], more recent lattice simulations using Wilson fermions [12, 15] have been carried out, providing results for \( \pi^+ \) and \( \pi^0 \) masses for larger values of \( eB \). In these simulations, however, a heavy pion with \( m_{\pi^+}(0) = 415 \text{ MeV} \) in vacuum has been considered. In order to compare these results with our predictions we follow the procedure carried out in Ref. [9], viz. we consider a new parameter Set Ib in which \( G \) and \( \Lambda \) are the same as in Set I, while \( m_0 \) is increased so as to obtain \( m_{\pi^+}(0) = 415 \text{ MeV} \). In Ref. [9] the authors also consider a magnetic field dependent coupling of the form \( G(eB) = \alpha + \beta \exp[-\gamma(eB)^2] \) in order to reproduce LQCD results for both the behavior of quark condensates and the \( \pi^0 \) mass.

The curves for the normalized charged pion MFDM, \( E_{\pi+}/m_\pi(0) \), and the neutral pion mass, \( m_{\pi^0}/m_\pi(0) \), for
Set Ib are shown in Fig. 2 together with LQCD results obtained for these quantities after an extrapolation of lattice spacing to the continuum. Results corresponding to the parameter Set IV of Ref. [6], with the $B$-dependent coupling $G(eB)$, are also included. It is seen that for the $\pi^+$ meson the results from Set Ib are consistent with lattice data, although the errors in the latter are considerably large to be conclusive (in fact, results obtained for these quantities after an extrapolation of lattice spacing to the continuum). Results corresponding to the parameter Set IV of Ref. [6], with the $B$-dependent coupling $G(eB)$, are also included. It is seen that for the $\pi^+$ meson the results from Set Ib are consistent with lattice data, although the errors in the latter are considerably large to be conclusive (in fact, results obtained for these quantities after an extrapolation of lattice spacing to the continuum).

In our numerical calculations we have used different model parameterizations that satisfactorily describe not only meson properties in the absence of the magnetic field but also the behavior of quark condensates as functions of $B$ obtained in LQCD calculations. We have found that when the magnetic field is enhanced, the $\pi^0$ mass shows a slight decrease, while the MFDM of the charged pion steadily increases, remaining always larger than that of a point-like pion. These results are in agreement with LQCD calculations with realistic pion masses for low values of $eB$ (say $eB \lesssim 0.15 \text{ GeV}^2$), although there seems to be some discrepancy as the magnetic field is increased. For larger values of $eB$, some recent LQCD simulations for $m_{\pi^0}$ and $E_{\pi^+}$ have been carried out considering unphysically large quark masses. In the case of $E_{\pi^+}$ the results are consistent with our calculations (with adequately rescaled parameters), while there is a significant discrepancy in the case of the $\pi^0$ mass. The agreement for $m_{\pi^0}$ gets improved if, as done in Ref. [6], a magnetic field-dependent coupling $G(eB)$ is introduced. In this sense, it is worth noticing that nonlocal NJL-like models, which naturally predict a magnetic field dependence of the quark current-current interaction, are also able to reproduce adequately the $\pi^0$ mass behavior.

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