Distributed Computation of Sparse Cuts

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Abstract—Finding sparse cuts is an important tool in analyzing large-scale distributed networks such as the Internet and Peer-to-Peer networks, as well as large-scale graphs such as the web graph, online social communities, and VLSI circuits. Sparse cuts are useful in graph clustering and partitioning among numerous other applications. In distributed communication networks, they are useful for topology maintenance and for designing better search and routing algorithms.

In this paper, we focus on developing a fast distributed algorithm for computing sparse cuts in networks. Given an undirected \( n \)-node network \( G \) with conductance \( \phi \), the goal is to find a cut set whose conductance is close to \( \phi \). We present a distributed algorithm that finds a cut set with sparsity \( O(\sqrt{n}) \) (\( O \) hides polylog \( n \) factors). Our algorithm works in the CONGEST distributed computing model and outputs a cut of conductance at most \( O(\sqrt{\phi}) \) with high probability, in \( O(\frac{1}{\phi}(\frac{1}{\phi} + n)) \) rounds, where \( b \) is balance of the cut of given conductance. In particular, to find a sparse cut of constant balance, our algorithm takes \( O(\frac{1}{\phi} + n) \) rounds. Our algorithm can also be used to output a local cluster, i.e., a subset of vertices near a given source node, and whose conductance is within a quadratic factor of the best possible cluster around the specified node. Our distributed algorithm can work without knowledge of the optimal \( \phi \) value and hence can be used to find approximate conductance values both globally and with respect to a given source node. We also give a lower bound on the time needed for any distributed algorithm to compute any non-trivial sparse cut — any distributed approximation algorithm (for any non-trivial approximation ratio) for computing sparsest cut will take \( \Omega(\sqrt{n} + D) \) rounds, where \( D \) is the diameter of the graph.

Our algorithm can be used to find sparse cuts (and their conductance values) and to identify well-connected clusters and critical edges in distributed networks. This in turn can be helpful in the design, analysis, and maintenance of topologically-aware networks.

Keywords—Distributed Algorithm; Sparse Cut; Conductance; Random Walks

I. Introduction

Developing distributed algorithms for computing key metrics of a communication network is an important research goal with various applications. Network properties — which depend on the collective behavior of nodes and links — characterize global network performance such as routing, sampling, information dissemination, etc. These in turn depend on topological properties of the network such as high connectivity, low diameter, high conductance, and good spectral properties [14]. The above properties, all of which are critical, need to be measured periodically. Having a highly-connected network is good for fault-tolerance and reliable routing, since a packet can be routed via many disjoint paths. Low diameter ensures that packets can be routed quickly with short delay. Conductance (formally defined in Section I-A2) measures how “well-knit” the network is; it determines how fast a random walk converges to the stationary distribution — known as the mixing time. Conductance is related to the expansion, spectral gap, and mixing time of a graph. High expansion and spectral gap means that the graph has fast mixing time. Such a network supports fast random sampling (which has many applications [13]) and low-congestion routing [14].

Sparse cuts are those cuts that have low conductance and can be used to determine well-connected clusters\(^1\) and thus also identify potential “bottlenecks” in the network. In particular, the edges crossing the cut can be considered as critical edges and they have been used in designing algorithms to improve searching, topology maintenance (i.e., maintaining a well-connected topology), and reducing routing congestion in networks [14].

In this paper, we focus on developing a fast distributed algorithm for computing sparse cuts in networks. Given an undirected \( n \)-node network \( G \) with conductance \( \phi \) (a quantity less than 1), the goal is to find a cut set whose conductance is close to \( \phi \). (We note that computing the minimum conductance cut — the one with conductance \( \phi \) of the network—is NP-hard [26].) We present a fast distributed algorithm that finds a cut set with sparsity \( O(\sqrt{\phi}) \). Our algorithm uses small-sized messages and works in the CONGEST distributed computing model. Our algorithm builds on previous work [23], [31] on classical algorithms for sparse

\(^1\) A cut \( (S, V - S) \) is a partition of the set of nodes \( V \) into \( S \) (assume \( |S| \leq |V|/2 \)) and \( V - S \). A low conductance cut has lot more edges within \( S \) than those going outside \( S \); and hence \( S \) is relatively well (intra)connected.
Our algorithm outputs a cut of conductance at most $O(\sqrt{\phi})$ with high probability, in $O(\frac{1}{\phi} (\frac{1}{\lambda_2} + n))$ rounds, where $b$ is balance of the cut of given conductance (cf. Section I-A2). In particular, to find a cut of constant balance (i.e., $b$ is balance of the cut of given conductance (cf. Section I-A2)), our algorithm can also be used to output a well-connected local cluster (cf. Section I-A2), i.e., a subset $S$ of vertices containing the given source node such that the internal edge connections in $S$ are significantly higher than the outgoing edges from $S$. Our distributed algorithm works without knowledge of the optimal $\phi$ value and hence can be used to find approximate conductance values both globally and locally with respect to a given source node. We also show a lower bound on the time needed for any distributed algorithm to compute any non-trivial sparse cut. In particular, we show that there is a graph in which any distributed approximation algorithm (for any non-trivial approximation ratio, not just quadratic approximation) for computing sparsest cut will take $\tilde{O}(\sqrt{n} + D)$ rounds, where $D$ is the diameter of the graph.

Our algorithm can be useful in efficiently finding sparse cuts (and their conductance values) and critical edges (the edges crossing sparse cuts) in distributed networks. In particular, the work of [14] shows how critical edges can be used to design algorithms to improve search, reduce congestion in routing, and for keeping the graph well-connected (topology maintenance). Such algorithms can be useful in the design and deployment of reconfigurable networks (whose topology can be changed by rewiring edges) such as peer-to-peer networks and wireless mesh networks. The paper [10] study information spreading where they used a generalized notion of conductance as a key tool. In fact, the conductance helps to identify bottlenecks in the network and thus achieves fast information spreading.

The focus of distributed computation of spectral properties that we are interested here, in particular, conductance and sparse cuts, is relatively new. The work of [13] presented a fast decentralized algorithm for estimating mixing time, conductance, and spectral gap of the network. The work of Kempe and McSherry [19] gives a decentralized algorithm for computing the top eigenvectors of a weighted adjacency matrix that runs in $O(\tau_{mix} \log^2 n)$ round, where $\tau_{mix}$ is the mixing time of the network$^2$.

While the above works give distributed algorithms to estimate the conductance $\phi$, they do not give an efficient distributed algorithm to compute sparse cuts. Sparse cuts have low conductance (i.e., close to $\phi$) and, in particular, the sparest cut is a cut that achieves the network conductance. Since there are exponential number of cuts in the network, it is significantly more challenging to efficiently find the sparsest cut or approximate it in a distributed fashion. Hence computing sparse cuts needs a different approach compared to computing conductances and mixing time as in the works of [13], [19].

Our approach, on a high-level, is based on efficiently implementing the methods of Lovász and Simonovits [23], [31]. This method uses random walks to estimate the probability distribution of such walks terminating at nodes. This probability distribution can then be used to identify sparse cuts. Our algorithm uses standard random walks as a key subroutine (cf. Section II).

A. Model and Definitions

1) Distributed Computing Model

We model the communication network as an undirected, unweighted$^3$, connected $n$-node graph $G = (V, E)$. Every node has limited initial knowledge. Specifically, assume that each node is associated with a distinct identity number (e.g., its IP address). At the beginning of the computation, each node $v$ accepts as input its own identity number and the identity numbers of its neighbors in $G$. The node may also accept some additional inputs as specified by the problem at hand. The nodes are allowed to communicate through the edges of the graph $G$. We assume that the communication occurs in synchronous rounds. We will use only small-sized messages. In particular, in each round, each node $v$ is allowed to send a message of size $O(\log n)$ through each edge $e = (v, u)$ that is adjacent to $v$. The message will arrive to $u$ at the end of the current round. This is a widely used standard model known as the CONGEST model to study distributed algorithms (e.g., see [29], [28]) and captures the bandwidth constraints inherent in real-world computer networks. (We note that if unbounded-size messages were allowed through every edge in each time step, then the problem addressed here can be trivially solved in $O(D)$ time by collecting all the topological information at one node, solving the problem locally, and then broadcasting the results back to all the nodes [29].)

There are several measures of efficiency of distributed algorithms, but we will focus on one of them, specifically, the running time, i.e. the number of rounds of distributed communication. Note that the computation that is performed by the nodes locally is “free”, i.e., it does not affect the number of rounds; however, we will only perform polynomial cost computation locally in any node. We note that in the CONGEST model, it is rather trivial to solve a problem in $O(m)$ rounds, where $m$ is the number of edges in the

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$^2$Estimating mixing time also allows one to estimate conductance $\phi$ (upto a quadratic factor) and spectral gap of the graph. The spectral gap is $1 - \lambda_2$ where $\lambda_2$ is the second eigenvalue of the connected transition matrix. It is known that conductance, mixing time, and spectral gap are related to each other [15]: $\frac{1}{1 - \lambda_2} \leq \tau_{mix} \leq \frac{\log n}{1 - \lambda_2}$ and $\Theta(1 - \lambda_2) \leq \Phi \leq \Theta(\sqrt{1 - \lambda_2})$.

$^3$We restrict our attention to unweighted graphs for the upper bound analysis, however, our algorithm can be extended to weighted graphs as well.
network, since the entire topology (all the edges) can be collected at one node and the problem solved locally. The goal is to design faster algorithms.

2) Definitions

We present notations that we use throughout the paper. Consider a graph $G = (V, E)$ with conductance $\phi$ and let $|V| = n, |E| = m$. Let $p_t(s, t)$ denote the probability that a random walk of length $\ell$ starting from $s$ ends in $t$. In fact, $p_t(s, t)$ is the probability distribution over the nodes after a walk of length $\ell$ starting from $s$. We simply use $p(t)$ instead of $p_t(s, t)$ when source node and length is clear from the text. Let $S$ be a subset of $V$. We denote a partition or cut by $(S, \bar{S})$ or sometimes by $(S, V \setminus S)$ interchangeably throughout the paper. For a probability distribution $p(i)$ on nodes, let $p_p(i) = p(i)/d(i)$. Let $\pi_p$ denote the ordering of nodes in decreasing order of $p_p(i)$.

Definition I.1 (Conductance and Sparsity). The conductance of a cut $(S, \bar{S})$ (also called as sparsity) is $\phi(S) = \frac{|E(S, \bar{S})|}{\min\{|V|, |S|\} |\bar{S}|}$ where $|V|$ is the sum of the degrees of nodes in $S$. The conductance of the graph $G$ is $\phi(G) = \min_{S \subseteq V} \phi(S)$. We denote it by only $\phi$, if it is clear from the text.

Definition I.2 (Balance). The balance of a cut $(S, \bar{S})$ is defined as $\min\{|S|/|V|, |\bar{S}|/|V|\}$ and is denoted by $b$.

Definition I.3 (Local Cluster). A local cluster with respect to a given vertex $v$ is a subset $S \subseteq V$ containing $v$ such that the conductance of $(S, \bar{S})$ is within a quadratic factor of the best possible local cluster containing $v$.

B. Problem Statement and Our Results

1) Problem Statement

In this paper, we consider the problem of finding a sparse cut in an undirected graph. Formally, given a graph $G = (V, E)$ with conductance $\phi$, we want to find a cut set whose conductance is close to $\phi$. Our goal is to design a distributed algorithm which finds a cut set with sparsity $O(\sqrt{\phi})$.

2) Our Results

Our main contribution is a distributed algorithm (in the CONGEST model) to find sparse cuts with approximation guarantees. Our algorithm crucially uses random walks.

Theorem I.4. (cf. Section II) Given an $n$-node network $G$ with a cut of balance $b$ and conductance at most $\phi$, there is a distributed algorithm SPARSECUT (cf. Algorithm 2) that outputs a cut of conductance at most $O(\sqrt{\phi})$ with high probability, in $O\left(\frac{1}{\phi} \left(\frac{n}{\sqrt{\phi}} + n\right)\right)$ rounds. In particular, to find a cut of constant balance, the SPARSECUT algorithm takes $O\left(\frac{1}{\phi} + n\right)$ rounds and finds a cut (if it exists) with similar approximation.

Using the above result, we also show:

Theorem I.5. Given an $n$-node network $G$ and source node $s$, there is a distributed algorithm that outputs a local cluster in $O\left(\frac{1}{\phi} + n\right)$ rounds, where $\phi$ is the conductance of the graph.

To prove the running time bound, we derive a technical result on computing conductances of $n$ (different) cuts in linear time (cf. Lemma II.4).

We note that the time bound of $O(\frac{1}{\phi} + n)$ is linear in $n$ (the number of nodes) and $1/\phi$. From the definition of conductance (cf. Definition I.1), it is clear that for every graph, $1/\phi = O(m)$ (where $m$ is the number of edges) and for many graphs it can be much smaller, e.g., for expanders it is $O(1)$. Hence, the running time of our algorithms can be significantly faster than the naive bound of $O(m)$ (cf. Section I-A1), especially in well-connected dense graphs. We next show a lower bound on the time needed for any distributed algorithm to compute a (non-trivial) sparse cut.

Theorem I.6. (cf. Section III) There is a $n$-node graph in which any distributed approximation algorithm for computing sparsest cut (within any non-trivial approximation ratio) will take $\tilde{O}(\sqrt{n} + D)$ rounds, where $D$ is the diameter of the graph.

Since $1/\phi = \Omega(D)$ for any graph, the above lower bound says that in general, one cannot hope to improve on the $1/\phi$ term of our upper bound.

C. Related Work

The problem of finding sparse cuts on graphs has been studied extensively [8], [7], [18], [6], [31], [25], [11]. Sparse cuts form an important tool for analyzing large-scale distributed networks such as the Internet and Peer-to-Peer networks, as well as large-scale graphs such as the web graph, online social communities, click graphs from search engine query logs and VLSI circuits. Sparse cuts are useful in graph clustering and partitioning among numerous other applications [31], [3].

The second eigenvector of the transition matrix is an important quantity to analyze many properties of a graph. A simple way of graph partitioning is by ordering the nodes in increasing order of coordinate values in the eigenvector. This partition can be used to compute sparse cut. This is a well known approach studied in [23], [24], [31], [3], [11]. We use this approach in this paper. The second eigenvector technique has been analyzed in many papers [1], [9], [16].

Lovász and Simonovits [23], [24] first show how random walks can be used to find sparse cuts. Specifically, they show that random walks of length $O(1/\phi)$ can be used to compute a cut with sparsity at most $O(\sqrt{\phi})$ if the sparsest cut has conductance $\phi$. Spielman and Teng [31] mostly follow the work of Lovász and Simonovits, but they implement it more efficiently by sparsifying the graph. They propose a nearly linear time algorithm for finding an approximate sparsest cut with approximate balance. Andersen, Chung, and Lang [3] proposed a local partitioning algorithm using PageRank
vector (instead of second eigenvector) to find cuts near a specified vertex and global cuts. The running time of their algorithm was proportional to the size of small side of the cut. Das Sarma, Gollapudi and Panigrahy [11] present an algorithm for finding sparse cut in graph streams. Their algorithm requires sub-linear space for a certain range of parameters, but provides much a weaker approximation to the sparsest cut compared to [3], [31]. Arora, Rao, and Vazirani [6] provide $O(\sqrt{\log n})$-approximation algorithm using semi-definite programming techniques. Their algorithm gives good approximation ratio, however it is slower than algorithms based on spectral methods and random walks. Kannan, Vempala, and Vetta [17] studied variants of spectral algorithm for clustering or partitioning a graph.

Graph partitioning or rather clustering is an well studied optimization problem. Suppose we are given an undirected graph and a conductance parameter $\phi$. The problem of finding a partition $(S, \overline{S})$ such that $\phi(S) \leq \phi$, or conclude no such partition exits is NP-complete problem (see, [22],[30]). As a result, several approximation algorithms exits in literature. Leighton and Rao presents $O(\log n)$ approximation of the sparsest cut algorithm in [22] where they used linear programming. Later Arora, Rao, and Vazirani [6] improved this to $O(\sqrt{\log n})$ using semi-definite programming techniques. This is the best known approximation of the sparsest cut computation problem. Further, several works obtains algorithm with similar approximation guarantees algorithm but better running time such as [4], [21], [5], [27]. However, unfortunately no work have been found in distributed computing model. Our paper is the first to attempt in distributed setting for sparse cuts computation.

The work of [14] discusses spectral algorithms for enhancing the topology awareness, e.g., by identifying and assigning weights to critical edges of the network. Critical edges are those that cross sparse cuts. They discuss centralized algorithms with provable performance, and introduce decentralized heuristics with no provable guarantees. These algorithms are based on distributed solutions of convex programs and assign special weights to links crossing or directed towards small cuts by minimizing the second eigenvalue. It is mentioned that obtaining provably efficient decentralized algorithm is an important open problem. Our algorithm is fully decentralized and based on performing random walks, and so more amenable to dynamic and self-organizing networks.

II. A Distributed Algorithm for Sparse Cut

In this section, we present an algorithm to find a cut that approximates the minimum conductance $\phi$. We are given a network, $G = (V, E)$, that has cut of conductance $\phi$ and balance $b$. We design a distributed algorithm running on $G$ to compute a cut set $S$ with conductance $O(\sqrt{\phi})$. At the end of the algorithm, every node in $G$ will know whether it is in $S$ or $\overline{S}$. Further, each node will also know all other nodes in $S$ or $\overline{S}$. Our algorithm works in the standard CONGEST model of distributed computing (cf. Section I-A). Without loss of generality, we will assume that our algorithm knows $\phi$ and $b$. Otherwise, we can do the following. Suppose we want to find a cut with the required sparsity, i.e., $O(\sqrt{\phi})$, without knowing $\phi$, but assume that we know $b$ (the balance of such a cut). Then we can guess the value of $\phi$ starting from a constant (say 1/2, which is essentially the highest possible) and then run our algorithm and check whether the output cut value satisfies the quadratic factor approximation (and the given balance). If yes, we stop; otherwise, we halve our guess and continue. If we don’t know $b$ as well, then our algorithm (with some assumed balance) will still work and will give a cut with similar quadratic approximation to the minimum conductance cut that is minimum among all possible cuts with the assumed balance. Thus, henceforth we will assume that our algorithm knows both $\phi$ and $b$.

The outline of our approach (cf. Theorem II.3) is to try several different cuts obtained by various distributions of random walks. Further these distributions need to be computed from a good source node. A good source node is one from the smaller side of the desired cut. In this approach, instead of computing the exact distribution after the chosen length of walk, it suffices to have an approximate distribution of sufficiently high accuracy. Assuming that a good source is used, one needs to estimate the distribution after doing a random walk of length $\ell$ that is sampled uniformly in the range of $\{1, 2, \ldots, O(1/\phi)\}$. For the sampled length $\ell$, estimate the landing probability $p(i)$ at every node $i$. Assume the estimation is $\tilde{p}(i)$. Then arrange the nodes according to decreasing order of $p_\ell = \tilde{p}(i)/d(i)$. Suppose the order is $\pi_\ell = (1, 2, \ldots, n)$. Then, with constant probability, at least one of the $n$ cuts $(S_j, \overline{S}_j)$ has the given conductance (approximated), where $S_j = \{1, 2, \ldots, j\}$. This algorithm and its proof of correctness was given in Spielman and Teng [31]. To get the required cut with high probability, we run our algorithm for $\Theta(\log n)$ different lengths $\ell$, each chosen independently and uniformly at random in the range of $\{1, 2, \ldots, O(1/\phi)\}$. For a particular $\ell$, there are $n-1$-partitions and so $n-1$ different conductances. The minimum conductance cut among all the $\Theta(n\log n)$ cuts would be the output of our algorithm. Before going to the main algorithm SPARSECut, we first present an algorithm to estimate the probability distribution $\tilde{p}(i)$ of $p(i)$ using random walks.

A. Estimating Random Walk Probability Distribution

We focus on estimating $p_\ell(s, i)$ which is the probability of landing at node $i$ after a random walk of length $\ell$ from a specific source node $s$. As we noted above, we denote it by simply $p(i)$. The basic idea is to perform several random walks of length $\ell$ from $s$ and at the end, each node $i$ computes the fraction of walks that land at node $i$. It is easy to see that the accuracy of estimation is dependent on the number of random walks that are performed from
s. Let us parameterize the number as $K$. We show (cf. Lemma II.2) that we can perform a polynomial in $n$ number of random walks without any congestion in the network. We first present the algorithm EstimateProbability, and then describe the result on accuracy of the estimation (cf. Lemma II.1). The pseudocode of the algorithm EstimateProbability is given below in Algorithm 1.

Algorithm 1 EstimateProbability

Input: Starting node $s$, length $\ell$, and number of walks $K$.
Output: $\tilde{p}(i)$ for each node $i$, which is an estimate of $p(i)$ with explicit bound on additive error.

1: Node $s$ creates $K$ tokens of random walks and performs them simultaneously for $\ell$ steps as follows.
2: for each round from 1 to $\ell$ do
3: A node holding random walk tokens, samples a random neighbor corresponding to each token and subsequently sends the appropriate count to each neighbor. (Note that tokens do not contain any node IDs.)
4: end for
5: Each node $i$ counts the number of tokens that landed on it — let this count be $\eta_i$.
6: Each node estimates the probability $\tilde{p}(i)$ as $\frac{\eta_i}{K}$.

We show that for $K = \Theta(n^2 \log n/\epsilon^2)$, the algorithm EstimateProbability (cf. Algorithm 1) gives an estimation of $p(i)$ with accuracy $p(i) \pm \epsilon/n$ for each node $i$. In other words, by performing $\Theta(n^4 \log n/\epsilon^2)$ random walks, if $\tilde{p}(i)$ is an estimation for $p(i)$, then $|\tilde{p}(i) - p(i)| \leq \epsilon/n$. This follows directly from the following lemma.

Lemma II.1. If the probability of an event $X$ occurring is $p$, then in $t = 4n^2 \log n/\epsilon^2$ trials, the fraction of times the event $X$ occurs is $p \pm \frac{\epsilon}{n}$ with high probability.

Proof: The proof follows from a Chernoff bound:

$$\Pr \left[ \frac{1}{t} \sum_{i=1}^{t} X_i < (1 - \delta)p \right] < \left( \frac{e^{-\delta}}{(1 - \delta)(1 - \delta)} \right)^{tp} < e^{-tp\delta^2/2}$$

and

$$\Pr \left[ \frac{1}{t} \sum_{i=1}^{t} X_i > (1 + \delta)p \right] < \left( \frac{e^{\delta}}{(1 + \delta)(1 + \delta)} \right)^{tp}.$$

Where $X_1, X_2, \ldots, X_t$ are $t$ independent identically distributed 0 - 1 random variables such that $Pr[X_1 = 1] = p$ and $Pr[X_1 = 0] = (1 - p)$. The right hand side of the upper tail bound further reduces to $2^{-\delta p}$ for $\delta > 2e - 1$ and for $\delta < 2e - 1$, it reduces to $e^{-tp\delta^2/4}$.

Let us choose $t = 4n^2 \log n/\epsilon^2$, and $\delta = \frac{\epsilon}{pn}$. Consider two cases, when $pn \leq \epsilon$ and when $pn > \epsilon$. When $pn \leq \epsilon$, the lower tail bound automatically holds as $pn - \epsilon < 0$. In this case, $\delta > 1$, so we consider the weaker bound of the upper tail bound which is $2^{-\epsilon p}$. We get $2^{-\epsilon p} - 2^{-\delta p/n} = 2^{-4n \log n/\epsilon} = \frac{1}{n^{2n/\epsilon}}$. Now consider the case when $pn > \epsilon$.

Here, $\delta < 1$ is small and hence the lower and upper tail bounds are $e^{-tp\delta^2/2}$ and $e^{-tp\delta^2/4}$. Therefore, between these two, we go with the weaker bound of $e^{-tp\delta^2/4} = e^{-\frac{\epsilon^2}{tp^2}} = e^{-\frac{1}{2} \log n} = 1/n^{\Theta(1)}$. 

Lemma II.2. Algorithm EstimateProbability (cf. Algorithm 1) finishes in $O(\ell)$ rounds, if the number of walks $K$ is at most polynomial in $n$.

Proof: To prove this, we first show that there is no congestion in the network if we perform at most a polynomial number of random walks from $s$. This follows from the algorithm that each node only needs to count the number of random walk tokens that end on it. Therefore nodes do not need to know from which source node or rather from where it receives the random walk tokens. Hence it is not needed to send the ID of the source node with the token. Since we consider CONGEST model, a polynomial in $n$ number of token’s count (i.e., we can send count of up to a polynomial number) can be sent in one message through each edge without any congestion. Therefore, one round is enough to perform one step of random walk for all $K$ walks in parallel, where $K$ is at most polynomial in $n$. This implies that $K$ random walks of length $\ell$ can be performed in $O(\ell)$ rounds. Hence the lemma.

B. Computation of Sparse Cut

With the probability approximation result (cf. Lemma II.1) and results from the algorithm Nibble in [31], a key technical result follows (stated below). The result guarantees that one of the cuts formed by $n$-prefixes in a specific sorted order of the probability distribution $\tilde{p}(i)$ has sparsity $O(\sqrt{\phi})$ [24], [31].

Theorem II.3. Let $(U, \bar{U})$ be a cut of conductance at most $\phi$ such that $|U| \leq |V|/2$. Let $\tilde{p}(i)$ be an estimate for the probability $p(i)$ of a random walk of length $\ell$ from a source node $s$ from $U$. Assume that $|\tilde{p}(i) - p(i)| \leq \epsilon(\sqrt{p(i)}/n + 1/n)$, where $\epsilon \leq o(\phi)$. Consider the $n - 1$ candidate cuts obtained by ordering the vertices in decreasing order of $p(i)$; each candidate cut $(S_j, \bar{S}_j)$ is obtained by setting $S_j$ equal to the set $(1, 2, \ldots, j)$, if the source node is randomly chosen from $U$ and the length is chosen randomly in the range $\{1, 2, \ldots, O(1/\phi)\}$, then with constant probability, one of these $n - 1$ candidate cuts has conductance at most $O(\sqrt{\phi})$, i.e. $\phi(S_j) \leq O(\sqrt{\phi})$.

Proof: The proof is shown in [11] and is implicit in [31] and uses a random walk mixing result from [24].

Therefore, it follows from the above Theorem II.3 that if we can estimate the probability $p(i)$ in such a way that it satisfies all the conditions as stated, then we can find a cut with sparsity $O(\sqrt{\phi})$. We see that the algorithm
The ordering \( \pi \) of vertices, in linear time. In particular, in this paper we use the ordering of the vertices in decreasing order of degree. Therefore, node 1 computes its neighbor’s position (i.e., whether its neighbor is in the left side or in the right side of the cut) in constant number of rounds (cf. proof of the Lemma II.5). Then, we arrange the set of vertices in decreasing order of degree. Therefore, node 1 computes its neighbor’s position (i.e., whether its neighbor is in the left side or in the right side of the cut) in constant number of rounds (cf. Figure 1). Every node can do this computation in parallel. It will take constant number of rounds for every node to compute \( L_j^\pi \) and \( R_j^\pi \). Then each node \( j \) sends the information which contains its ID, \( L_j^\pi \) and \( R_j^\pi \) to node 1 by upcast [29]. This will take at most \( O(n + D) \) rounds. Then node 1 can compute conductances locally as discussed above. Therefore, total time taken is \( O(n + D) \) rounds to compute all \( n \) conductances. This is actually \( O(n) \) rounds since \( D = O(n) \).

The algorithm and description of each step is given below. Complete pseudocode is given in Algorithm 2 which computes an approximate cut. At the end of our algorithm, each node knows the cut set which has sparsity \( O(\sqrt{\phi}) \).

C. Description and Analysis of the Algorithm

We describe the algorithm \textsc{SparseCut} in detail here. First we want to compute the probability distribution of random walks starting from a source node. It is shown in [31] (cf. Theorem II.3) that the source node should be from the smaller side of the cut of given conductance \( \phi \). Since we do not know about the cut set, we cannot choose such a source node. However, if the balance of the cut is \( b \), if we choose \( \log n/b \) source nodes uniformly at random from \( V \), then with high probability at least one node should be from the smaller side of the cut.

For each source node, we compute landing probability distribution of random walks of length \( \ell \). The length could be at most \( O(1/\phi) \) (cf. Theorem II.3) in the range of \( \{1, 2, \ldots, O(1/\phi)\} \). As mentioned earlier, we run our algorithm for \( \log n \) different lengths \( \ell \), chosen uniformly at random in this range. For simplicity, we break the remaining portion of the algorithm into two parts: Phase 1 and Phase 2. We run these two phases for each length \( \ell \) and for every (chosen) source node. In Phase 1, we partition the vertex set \( V \) according to the prefixes of decreasing order of the ratio of node probability to its degree. First, source node calls the algorithm \textsc{EstimateProbability} with input \( \ell \) and \( K \) to estimate landing probability over nodes. After computing approximate probability distribution \( \tilde{p}(i) \), each node sends the value \( p(i) = \tilde{p}(i)/d(i) \) to all other nodes in the network (cf. proof of the Lemma II.5). Then, we arrange the set of vertices in decreasing order of \( p(i) \), say, the ordered set is \( \pi_p = \{1, 2, \ldots, n\} \). At the end of this phase, each node knows all the partitions \( (S_j, \bar{S}_j) \), for all \( j = 1, 2, \ldots, n - 1 \). In phase 2, we compute the conductances of these \( n - 1 \) partitions. We describe in Lemma II.4 on how to compute conductances of all these cuts in linear time.

There are \( n - 1 \) partitions (cut sets) corresponding to each
Algorithm 2 \textsc{SparseCut}

\textbf{Input:} Graph $G = (V, E)$, a conductance $\phi$ of a cut and balance $b$ of the cut (as mentioned in the beginning of Section II, we assume knowledge of $\phi$ and $b$, without loss of generality).

\textbf{Output:} A sparse cut $C = (S, \overline{S})$ with conductance at most $O(\sqrt{b})$.

1: \textbf{for} $k = 1$ to $\log n/b$ \textbf{do}
2: \quad Choose a source node $s_k$ uniformly at random from $V$.
3: \textbf{for} $h = 1$ to $\log n$ \textbf{do}
4: \quad Choose a length $\ell$ uniformly at random in the range of $\{1, 2, \ldots, O(1/\phi)\}$.
5: \{\textbf{Phase 1: Finding partition of nodes using probability distribution by random walks.}\}
6: \quad Source node $s_k$ calls algorithm \textsc{EstimateProbability} with input $\ell$ and $K = \Theta(n^\ell \log n)$ to compute $p(i)$ for all nodes $i$.
7: \quad Each node sends the value $\rho(i) = \tilde{p}(i)/d(i)$ to all other nodes in the network.
8: \quad Let (without loss of generality) $\pi_p = \{1, 2, \ldots, n\}$ be the ordering of nodes in decreasing order of the set $\{\rho(i) : i \in V\}$. Each node knows $\pi_p$.
9: \{\textbf{Phase 2: Finding conductance of the cuts $(S_j, \overline{S}_j)$ where $S_j = \{1, 2, \ldots, j\}$ for $j = 1, 2, \ldots, n - 1$ in $\pi_p$.}\}
10: \quad Consider node $1$ as master node which collects information of all $n - 1$ cuts $(S_j, \overline{S}_j)$ one by one and computes conductances locally as follows.
11: \quad Each node $j$ sends the information which contains their ID, $L_j^\ell$ and $R_j^\ell$, to node $1$.
12: \quad Node $1$ computes the conductance of all $n - 1$ cuts locally using this information.
13: \textbf{end for}
14: \quad Node $1$ chooses the cut $C_{s_k}$ of minimum conductance among all $C_{\ell}$ i.e., $\phi(C_{s_k}) = \min_{j=1,2,\ldots,n-1} \phi(S_j)$.
15: \textbf{end for}
16: Node $1$ broadcasts the cut which has minimum conductance among all $C_{s_k}$, to all the nodes in the network.

Then node $1$ chooses the minimum of the $\Theta(n \log n)$ (among all $\ell$) minimum conductance cuts. Say, the cut is $C_{s_k}$, where $\phi(C_{s_k}) = \min_k \{ \min_j \phi(S_j) \}$. Then node $1$ chooses the minimum conductance cut among all source nodes $s_k$ (there are total $\log n/b$) and broadcasts it to all the nodes in the network. Let the minimum cut be $C = (C_t, C_{\overline{t}})$, then it is enough for node $1$ to broadcast the node $t$ only as all nodes know the ordered set $\pi_{\overline{t}}$.

D. Time Complexity Analysis

We now analyze the running time of the algorithm \textsc{SparseCut}. The following lemmas are required to prove the time complexity of \textsc{SparseCut}.

Lemma II.5. Phase 1 of \textsc{SparseCut} (cf. Algorithm 2) takes $O(1/\phi + n)$ rounds.

Proof: In phase 1, we estimate probability distribution $\rho(i)$ using the \textsc{EstimateProbability} Algorithm. The running time of \textsc{EstimateProbability}, following Lemma II.2, is $O(\ell)$ rounds. After estimating the landing probability, each vertex sends the quantity $\rho(i) = \tilde{p}(i)/d(i)$ to all vertices in the network. A simple way of sending these $n$ value to $n$ nodes can be done by constructing a BFS tree (e.g., by first electing a leader). We first construct a BFS tree using the value $\rho(t)$ of each node as its rank. Then the node of highest $\rho$ value would be the root of the tree. Each node upcasts its $\rho$ value to the root node through tree edges. Then the root node floods all $\rho(t)$ to reach all the nodes through the tree edges. It is shown in [29] that the upcast and then flooding $n$ values through tree edges can all be done in $O(n + D)$ rounds, where $D$ is the diameter of the graph. Also constructing BFS can be done in $O(D)$ rounds (e.g., [20]).

All other computations are done locally. Therefore, the total time required for Phase 1 is $O(\ell + n + D)$ rounds. However, the algorithm \textsc{EstimateProbability} is called for $\Theta(\log n)$ different random walk lengths, where each length value is at most $O(1/\phi)$. Also the diameter $D$ is at most $O(n)$ for any graph. Therefore, phase 1 finishes in $O(1/\phi + n)$ rounds.

Lemma II.6. Phase 2 of \textsc{SparseCut} (cf. Algorithm 2) takes $O(n)$ rounds.

Proof: Phase 2 is for computing conductance of $n - 1$ cuts $(S_j, \overline{S}_j), j = 1, 2, \ldots, n - 1$ where $S_j = \{1, 2, \ldots, j\}$ according to the ordering in $\pi_{\overline{t}}$. Therefore, it follows from the proof of Lemma II.4 that node $1$ can compute these $(n - 1)$ conductances in $O(n)$ rounds.

Theorem II.7. The running time of \textsc{SparseCut} (cf. Algorithm 2) is $O(1/\phi + n) \log^2 n)$ rounds where $\phi$ is the conductance of the graph and $b$ is balance of the cut.

Proof: The algorithm \textsc{SparseCut} essentially runs in two phases inside first two for loops, one is for choosing source nodes and other is for choosing length of random walks. Then at the end, node $1$ performs some local computation to choose the minimum conductance cut and sends it to all other nodes. Sending this to all the nodes in the network can be done in $O(D)$ rounds, which follows from the above discussion of the algorithm. Now, for phase 1 and
phase 2, we already have calculated the running time (cf. Lemma II.5 and Lemma II.6). Therefore, adding all these time together, we get \( O\left(\frac{\log n}{\phi} (1/\phi + n) \log n + D\right) \) rounds, where the factor \( \frac{\log n}{\phi} \) is for the first for loop, the factor \( \log n \) for the second for loop and last \( D \) is for sending the cut information to all nodes. All other computations are dominated by this bound. Since \( D \) is dominated by \( n \), therefore the running time of the algorithm \textsc{SparseCut} reduces to \( O\left(\frac{1}{\phi} (1/\phi + n) \log^2 n\right) \) rounds.

Combining the above running time lemmas, we can prove the main result of this section — Theorem I.4 (cf. Section I-B).

**Proof:** (of Theorem I.4) The approximation guarantee of algorithm \textsc{SparseCut}, i.e., it computes a cut with sparsity \( O(\sqrt{\phi}) \) follows from Theorem II.3. We choose \( \epsilon = O(\phi^2) \). Moreover, we are performing random walks up to length \( O(1/\phi) \). Therefore, it follows from Theorem II.3 that our algorithm computes a cut with conductance \( O(\sqrt{\phi}) \). The running time of the algorithm follows from the above Theorem II.7 which is \( O\left(\frac{1}{\phi} (\frac{1}{\phi} + n)\right) \) rounds.

In the \textsc{SparseCut} algorithm, we are required to compute probability distributions by performing random walks from a good source node to satisfy the condition of Theorem II.3. A source node is good if it is from the smaller side of a desired cut (as shown in [31]). If we are interested in finding a cut of constant balance, then \( \bar{b} \) is constant. Therefore, as an immediate corollary, computing a sparse cut of constant balance takes \( O(\frac{1}{\phi} + n) \) rounds.

The analysis of our algorithm is tight. Consider the barbell graph \( B_n \), which is a graph consisting of two cliques of size \( (n - 1)/2 \) connected by a path of length 2 (see, figure 2 in [2]). Consider a source node \( s \) in one clique. Then to compute the smallest conductance cut (one set of which would be the clique containing \( s \)), the random walk starting from \( s \) should reach the second clique. This will take at least \( \Theta(n^2) \) rounds, which is bounded by \( \Omega(1/\phi) \). Then to collect all the information as in Lemma II.4 at the node \( s \) will take \( \Omega(n) \) rounds. Hence, total time required is \( \Omega(1/\phi + n) \).

**E. Finding Local Cluster Set**

We describe an approach to compute a local cluster, i.e., a subset of vertices containing a given source node \( v \) such that the internal edge connections are significantly higher than the outgoing edges from it.

Suppose a source node \( s \in V \) is given. First, guess a conductance \( \phi \) starting from a constant (say 1/2, which is essentially the best possible) and then run the above \textsc{SparseCut} algorithm for the particular node \( s \), i.e., run the algorithm from Step 3 for source node \( s \). Then check whether the smallest conductance satisfies the quadratic factor approximation. If yes, we stop; otherwise, we halve the (guessed) conductance and continue. Since the minimum conductance value is \( O(1/m) \), we need to do at most \( O(\log n) \) guesses, as \( m = O(n^2) \). The running time bound of the algorithm for computing a local cluster is stated in Theorem I.5 (cf. Section I-B) and the proof is given below.

**Proof:** (of Theorem I.5) We run the \textsc{SparseCut} algorithm only for one specified source node. The running time of \textsc{SparseCut} algorithm for a single source node is \( O(1/\phi + n) \) rounds with high probability. Checking whether the smallest conductance satisfies the quadratic factor approximation can be done locally at the source node \( s \). Then we may have to run the algorithm at most \( O(\log n) \) times for guessing the (best possible) conductance. Therefore, the running time of the algorithm is \( \tilde{O}(1/\phi + n) \) rounds with high probability.

**III. Lower Bound**

We derive a general lower bound for the distributed sparse cut problem. In particular, we show that there is graph in which any approximation algorithm for computing sparsest cut will take \( \Omega(\sqrt{n} + D) \) rounds, where \( D \) is the diameter of the graph. We use the technique of [12] which shows almost tight lower bounds for many distributed verification and optimization problems. Their lower bound proofs rely on a bridge between communication complexity and distributed computing.

We show a reduction from the spanning connected subgraph verification problem to the sparsest cut (optimization) problem. In the spanning connected subgraph verification problem, given a graph \( G = (V,E) \) and a subgraph \( H = (V,E') \) with \( E' \subseteq E \), it is required to check whether the subgraph \( H \) is a spanning connected subgraph of \( G \) via a distributed algorithm. We convert the spanning connected subgraph verification problem to the sparsest cut problem (with edge weights). In particular, we show that an \( \alpha \)-approximation \( \epsilon \)-error algorithm\(^4\) \( \mathcal{A} \) for sparsest cut problem, can be used to solve the spanning connected subgraph verification problem using the same running time. Hence the lower bound proved in [12] (cf. Theorem 5.1) for the spanning connected subgraph verification problem (which is \( \Omega(\sqrt{n} + D) \)), also applies to the sparsest cut computation problem. We use the graph \( G(\Gamma,d,p) \) (this graph with parameters \( \Gamma,d, \) and \( p \) is defined in [12]) to show the lower bounds. We consider the same parameterized graph \( G = G(\Gamma,d,p) \), which is connected by our assumption. The reduction from the spanning connected subgraph verification problem is direct: In \( G \) we assign a weight of 1 to all edges in the subgraph \( H \) and weight 0 to all other edges. Now, observe that if \( H \) is not connected then the conductance of sparsest cut is 0, since we can then partition the whole graph into two components and all the edges crossing the two components has weight 0. On the other hand, if \( H \) is connected then every cut set contains at least one edge from \( H \), which implies that the conductance of the sparsest cut

\(^4\) A randomized algorithm \( \mathcal{A} \) is \( \alpha \)-approximation \( \epsilon \)-error if for any input , the algorithm \( \mathcal{A} \) outputs a solution that is at most \( \alpha \) times the optimal solution of the input with probability at least \( 1 - \epsilon \).
cut would be non-zero. Thus, any algorithm with non-trivial approximation ratio will be able to distinguish the two cases.

Therefore, it follows that the sparsest cut computation problem has a lower bound $\Omega(\sqrt{n} \cdot D)$. 

**IV. Conclusion**

We presented distributed approximation algorithm for computing sparse cuts, with provable guarantees on the conductance. For future work, one can try to improve the running time bound $\tilde{O}(\frac{1}{\phi} + n)$ rounds. There is previous work on performing an $\ell$ length random walk in time $\tilde{O}(\sqrt{\ell D})$ rounds [13]. This can be used to potentially speed up random walks and hence reduce the $\frac{1}{n}$ part of the time bound, since walks of that much length has to be performed. (As mentioned earlier, since $1/\phi = \Omega(D)$, this cannot be improved beyond $\Omega(D)$ because of our lower bound of $\Omega(D + \sqrt{n})$.) However, the technique in [13] may not be applicable directly here because of congestion; we need to perform many random walks to compute the landing probability distribution with high enough accuracy. One might also try to improve the “n” part of the time bound and see if we can match the $\Omega(\sqrt{n})$ lower bound. Improving this seems to depend on computing the conductance of $n-1$ different cuts in time that is sub-linear in $n$, which seems harder; alternatively it may be possible to try significantly fewer than $n$ cuts in each of our distributional orders and still guarantee an approximation bound.

Our sparse cuts computation can be used to identify the crossing edges, which have been used in prior work ([14]) to heuristically improve network search, routing, and connectivity. It will be useful to rigorously show such results with provable guarantees.

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