On Clifford-Algebraic “Holoraumy,”
Dimensional Extension, and SUSY Holography

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ABSTRACT

We analyze the group of maximal automorphisms of the $N$-extended world-line
supersymmetry algebra, and its action on off-shell supermultiplets. This defines
a concept of “holoraumy” that extends the notions of holonomy and curvature
in a novel way and provides information about the geometry of the supermult-
iple field-space. In turn, the “holoraumy” transformations of 0-brane dimension-
ally reduced supermultiplets provide information about Lorentz transforma-
tions in the higher-dimensional spacetime from which the 0-brane supermult-
iplelets are descended. World-line supermultiplets are thus able to holographi-
cally encrypt information about higher dimensional spacetime geometry.

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If the facts are right, then the proofs are a matter of
playing around with the algebra correctly.
—Richard P. Feynman
1 Introduction, Results and Summary

In this paper, we discuss a novel structure contained within off-shell representations of spacetime supersymmetry, which we dub “holoraumy\(^1\).” This structure reveals itself upon following the line of work that began in Refs. [1, 2] and eventually coalesced with the idea of “SUSY holography” [3, 4]—the proposal that supermultiplets retain enough information in the process of dimensional reduction (even to world-line supersymmetric quantum mechanics!) so as to enable a full reconstruction of the original, higher-dimensional spacetime supersymmetry structure.

In general, these holoraumy transformations compose linear homogeneous transformations in the target space with the full Poincaré group of transformations or perhaps even the conformal transformations and central charge action in the domain space. Herein, we focus on the simplest version (see Eqs. (2.4) below), exhibited on unitary, finite-dimensional, off-shell representations of world-line \(N\)-extended supersymmetry. This framework is the common denominator in all physics applications of supersymmetry, and will provide building blocks in all developments of a fully quantum description. Also, in the special case of so-called valise adinkras, the simplest off-shell linear

\(^1\)We ask the forbearance of linguistic purists who will decry the mixing of the Greek word “holos” (complete) with the German word “Raum” (space); the linguistically pure “holochory” (from holos + horos) seems considerably less euphonious, at least to our ears, and much more prone to misunderstanding.
representations of world-line $N$-extended supersymmetry, these holoraumy transformations turn out to be a uniform composition of target-space $\text{SO}(n)$ rotations and domain-space $\tau$-translations. Fixed points of a similar type of composition of domain- and target-space transformations comprise “orientifolds,” which have been studied for well over two decades [5, 6, 7, 8]. It is then a little surprising that this type of transformation, existing in all of field theory, has not been studied more systematically, and seems to have remained nameless.

Being a composition of target- and domain-space transformations, holoraumy differs significantly from holonomy in the field-space (thought of as the total space of the bundle of vector spaces spanned by all the fields and fibered over the domain-space), which is a linear combination of those two types of transformation. Also, the very definition and computation of holoraumy differ significantly from holonomy, as will be shown herein. It is this inherent novelty that motivates our present study.

In the following, we introduce two holoraumy tensors that naturally appear in the context of so-called valise supermultiplets. The definition of these tensors is reminiscent of the familiar definition of curvature and torsion tensors in differential geometry: On a curved general manifold, the commutator of two covariant derivatives produces the Riemann curvature and the torsion tensors. In turn, applying commutators (rather than anti-commutators!) of the covariant super-derivatives on supermultiplets produces specific transformations among the bosonic and fermionic components separately. These transformations generate a group and the objects encoding them we dub the quadratic holoraumy tensors.

The first of the two particular holoraumy tensors we will study herein has implicitly appeared in our previous work [9, 10]. Here we formalize its definition in such a way that it can be easily generalized. This holoraumy tensor seems particularly suited to play a fundamental role in understanding SUSY holography. Just as the Riemann curvature tensor may be used to define holonomy groups for curved manifolds, the holoraumy tensors likewise provide a similar characteristic for supermultiplets, as well as the supersymmetric field theory models built from such supermultiplets. Just as the holonomy group of a real $n$-dimensional Riemannian manifold must be a subgroup of $\text{Spin}(n)$ and that of a complex $n$-dimensional Kähler manifold a subgroup of $\text{U}(n)$, the holoraumy groups acting on both bosonic and fermionic component fields of any supermultiplet must be subgroups of $2\text{Pin}(N) = \text{Aut}(\text{Sp}^{1|N})$, the maximal group of outer automorphisms of the $N$-extended world-line supersymmetry without central extensions (1.1); see below.

Perhaps the most remarkable feature about the holoraumy tensors is that they are inherent characteristics of representations of world-line supersymmetry. Yet, as we show below, they provide ample information about higher-dimensional supermultiplets to which those world-line supermultiplets may extend, as well as obstructions that prevent extension to certain supermultiplets in certain higher-dimensional spacetimes. In particular, the fermionic holoraumy tensor contains information about Fierz identities that hold within the 3+1-dimensions, $\mathcal{N} = 1$ chiral and vector supermultiplets when they are dimensionally reduced to the world-line 0-brane and upon a type of field redefinitions called “node-lowering” [11].

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\[^2\]Pin$(p, q)$ extends Spin$(p, q)$ by elements of negative determinant, and is the double-cover of the orthogonal group $O(p, q)$ wherein spinors are faithful representations; $O(p, q)$ preserves a metric with $p$ negative and $q$ positive eigenvalues. By definition, Pin$(n) := \text{Pin}(0, n)$; Spin$(p, q) \approx \text{Spin}(q, p)$ but Pin$(p, q) \not\approx \text{Pin}(q, p)$.
This paper is organized as follows: the basic definitions and notation is provided in the remainder of this introduction, and Section 2 provides the general framework, definitions and results pertaining to the holoraumy tensors. These ideas are then applied to several 3+1-dimensional $\mathcal{N}=1$ supermultiplets and their world-line dimensional reduction in Section 3. Finally, Section 6 collects our concluding comments, while the technical details are deferred to the Appendices.

**Notation and Definitions:** We focus on world-line $N$-extended supersymmetry without central extensions, for which the covariant super-derivatives and the $\tau$-derivative satisfy the algebra

$$\mathfrak{Sp}^{1|N} : \{ D_I, D_J \} = 2i \delta_{IJ} \partial_\tau, \quad \left[ \partial_\tau, D_I \right] = 0, \quad I, J = 1, \cdots, N. \tag{1.1}$$

Unitary and finite-dimensional representations of this algebra—supermultiplets—are provided by collections of intact superfields, related by first order super-differential relations.

It has been proven recently [17] that every unitary, finite-dimensional and engineerable off-shell supermultiplet of world-line $N$-extended supersymmetry without central charges (1.1):

1. becomes, through an iterative sequence of local component field transformations (so-called “node-raising” [11], i.e., “dressing” [18], and simple linear combinations), a direct sum of minimal “valise” supermultiplets [19], and conversely
2. may be “synthesized” from this direct sum of minimal “valise” supermultiplets, by reversing the linear combination and node-raising procedure.

For this reason, we focus herein on these valise supermultiplets, wherein the component superfields, $(\Phi_i | \Psi_j)$, are related by means of the first order super-differential system of equations:

$$D_I \Phi_i = i (\mathbb{L}_I)_{i}^{\ell} \Psi_j, \quad j = 1, \cdots, d; \tag{1.2a}$$

$$D_I \Psi_j = (\mathbb{R}_I)_{j}^{\ell} \partial_\tau \Phi_i, \quad i = 1, \cdots, d. \tag{1.2b}$$

Within such a valise supermultiplet, the engineering dimensions (physical units) of all bosons $\Phi_i$ are the same, as are those of the fermions $\Psi_j$, and the two are related by $[\Psi_j] = [\Phi_i] + \frac{1}{2}$, with $[\partial_\tau] = 1 = 2[D_I]$. For this to be an off-shell representation of the algebra (1.1), the second order super-differential identities (1.1) must continue to hold on each of the component superfields, $(\Phi_i | \Psi_j)$, without requiring any of these superfields to satisfy any $\tau$-differential equation, which could be derived as an equation of motion from some Lagrangian. In turn, this requires the $\mathbb{L}_I$- and $\mathbb{R}_I$-matrices to close the so-called $\mathcal{GR}(d, N)$ algebra [20]:

$$\left( \mathbb{L}_I \right)_{i}^{\ell} (\mathbb{R}_J)_{j}^{k} + (\mathbb{L}_J)_{j}^{\ell} (\mathbb{R}_I)_{i}^{k} = 2 \delta_{IJ} \delta_{ik}, \tag{1.3a}$$

$$(\mathbb{R}_I)_{j}^{\ell} (\mathbb{L}_J)_{i}^{\hat{k}} + (\mathbb{R}_J)_{j}^{\ell} (\mathbb{L}_I)_{i}^{\hat{k}} = 2 \delta_{IJ} \delta_{\hat{j}\hat{k}}. \tag{1.3b}$$

Following Refs. [21, 22], we note that the $I = J$ cases of (1.3) imply the identity $\mathbb{R}_I = \mathbb{L}_I^{-1}$, and we use this result hereafter.

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3 Throughout, and without loss of generality [12], we use superspace methods and notation [13, 14, 15].

4 By “intact superfields,” we mean un-constrained, un-projected, un-gauged and in no other way restricted Salam-Strathdee superfields [16], but will assume them to be real unless explicitly stated otherwise.

5 A supermultiplet is engineerable if all component fields have a consistent assignment of engineering dimension, i.e., physical units; this is a natural requirement in all physics applications.
Using the well-known relationship between the covariant super-derivatives and the super-charges, \( Q_I = -i(D_I + 2i\delta I, \theta^J\partial_J) \) and the projection to the purely bosonic part of superspace, the system (1.2) produces the supersymmetry transformations within the supermultiplet \( (\Phi_i|\Psi_j) \). Herein, however, we explore (1.2) as a system of super-differential equations, which restrict the geometry of the field-space spanned by the component fields \( \phi_i := \Phi_i \) and \( \psi_j := \Psi_j \).

## 2 Clifford-Algebraic Structure of World-Line Supermultiplets

Writing the super-differential system (1.2) as

\[
(\Phi_i|\Psi_j) : D_I \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{L}_I \\ \mathbb{L}_I^{-1} & 0 \end{bmatrix} \begin{bmatrix} \partial_\tau \Phi \\ i\partial_\tau \Psi \end{bmatrix},
\]

(2.1)
defines:

\[
\Gamma_I := \begin{bmatrix} 0 & \mathbb{L}_I \\ \mathbb{L}_I^{-1} & 0 \end{bmatrix} \quad \text{(1.3)}
\]

and the matrices \( \Gamma_I \) generate the \( \mathfrak{Cl}(0, N) \) Clifford algebra. Refs. [21, 22] also introduce the fermion number operator, \( \Gamma_0 := (-1)^F \), which acts on the supermultiplet \( (\Phi_i|\Psi_j) \) as a diagonal matrix \( \Gamma_0 = \text{diag}(+1, \cdots, +1|−1, \cdots, −1) \). It is easy to show that \( \{\Gamma_0, \Gamma_I\} = 0 \) and \( (\Gamma_0)^2 = \mathbb{1} \) and solely by virtue of (2.2), so that \( \Gamma_0 \) and the \( \Gamma_I \)'s jointly generate the \( \mathfrak{Cl}(0, N+1) \) Clifford algebra.

### 2.1 General Facts about World-Line Holoraumy

**The Enveloping System:** The realization that (1.2) is equivalent to (2.1) where \( \Gamma_I \) generate the \( \mathfrak{Cl}(0, N) \) Clifford algebra (2.2) has a standard but important consequence [23, 24]: The Clifford algebra \( \mathfrak{Cl}(0, N) := \otimes^\ast \text{Span}(\Gamma_I)/(2.2) \) and the exterior algebra \( \wedge^\ast \Gamma \) are isomorphic as vector spaces, and so are both spanned by the matrices familiar to physicists as the Dirac algebra\(^6\):

\[
\mathbb{1}, \quad \Gamma_I, \quad \Gamma_{IJ}, \quad \Gamma_{IJK}, \quad \cdots \quad \Gamma_{I_1I_2\cdots I_N},
\]

(2.3)

which is a basis that is canonically induced from the choice of \( \Gamma_1, \cdots, \Gamma_N \)—using only Eq. (2.2) to reduce un-symmetrized tensor products to the antisymmetric products, \( \Gamma_{I_1}\cdots\Gamma_{I_n} \xrightarrow{(2.2)} \Gamma_{[I_1}\cdots\Gamma_{I_n]} + \text{lower-order terms} \). This prompts us to consider the “enveloping system” of (2.1):

\[
D_{[I}D_{J]} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = - \begin{bmatrix} \mathbb{L}_{[I} \mathbb{L}_{J]}^{-1} & 0 \\ 0 & \mathbb{L}_{[I}^{-1} \mathbb{L}_{J]} \end{bmatrix} \begin{bmatrix} i\partial_\tau \Phi \\ i\partial_\tau \Psi \end{bmatrix},
\]

(2.4a)

\[
D_{[I}D_{J}D_{K]} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = - \begin{bmatrix} 0 & \mathbb{L}_{[I} \mathbb{L}_{J]}^{-1} \mathbb{L}_{K]}^{-1} \\ \mathbb{L}_{[I} \mathbb{L}_{J]}^{-1} \mathbb{L}_{K]} & 0 \end{bmatrix} \begin{bmatrix} i\partial_\tau^2 \Phi \\ -\partial_\tau^2 \Psi \end{bmatrix},
\]

(2.4b)

\[
D_{[I}D_{J}D_{K}D_{L]} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} \mathbb{L}_{[I} \mathbb{L}_{J]}^{-1} \mathbb{L}_{K]}^{-1} \mathbb{L}_{L]}^{-1} & 0 \\ 0 & \mathbb{L}_{[I} \mathbb{L}_{J]}^{-1} \mathbb{L}_{K]}^{-1} \mathbb{L}_{L]} \end{bmatrix} \begin{bmatrix} -\partial_\tau^2 \Phi \\ -\partial_\tau^2 \Psi \end{bmatrix},
\]

(2.4c)

and so on. The antisymmetrized products of the \( \mathbb{L}^{-} \) and \( \mathbb{L}^{-1} = \mathbb{R} \)-matrices appearing in (2.4) have been studied previously in Ref. [4], without exploring the holoraumy that they generate.

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\(^6\)Throughout this paper, square brackets indicate normalized antisymmetrization of the enclosed indices: \( \Gamma_{IJ} := \frac{1}{2!}(\Gamma_I \Gamma_J - \Gamma_J \Gamma_I), \Gamma_{IJK} := \frac{1}{3!}(\Gamma_I \Gamma_J \Gamma_K - \Gamma_J \Gamma_K \Gamma_I + \cdots), \text{etc.} \)
In precise analogy with (2.3), un-symmetrized composition of the super-derivatives is reduced to the antisymmetric products using only the defining relations of the supersymmetry algebra, \( D_{I_1} \cdots D_{I_n} \overset{(1.1)}{\rightarrow} D_{[I_1} \cdots D_{I_n]} + \text{lower-order super-derivatives} \). We emphasize that both sides of the super-differential systems (2.4) have been obtained solely by using the defining equations, (1.1) and (2.2) respectively. Therefore, they apply equally to all valise supermultiplets (1.2) and their geometric content is independent of any concrete choice of the \( L_I \)-matrices. The list of such higher-order super-differential relations is straightforwardly generated by iterating the relations from the original super-differential system (1.2), and is finite: the progression effectively stops with the \( N \)th order super-derivative. Every product of more than \( N \) super-differential operators \( D_{I_n} \) necessarily reduces to one of an order at most equal to \( N \), composed with a suitable power of \( \partial_\tau \). Thereby, every super-differential relation (2.4) of order higher than \( N \) simply produces a (multiple) \( \partial_\tau \)-derivative of a lower-order relation.

Given the superspace relation \( Q_I = -i(D_I + 2i\delta_{IJ}\theta^J\partial_\tau) \), the Taylor expansion of the “finite” supersymmetry transformation operator, \( \exp\{i\epsilon \cdot Q\} = \exp\{\epsilon^I[D_I + 2i\delta_{IJ}\theta^J\partial_\tau]\} \), straightforwardly reproduces the enveloping system (2.4), order-by-order and ignoring the terms explicitly containing \( \theta^I \) as they vanish upon component evaluation. The enveloping system of super-differential relations (2.4) in addition to (2.1) therefore spans the closed orbit of the supersymmetry transformations. Considering (2.1) together with (2.4) is therefore akin to considering the “finite” (super)symmetry transformation instead of just its infinitesimal generator.

Amongst the differential relations (2.4)—and very much shadowing the situation in the structure of Clifford algebras—the even-order super-derivatives, \( D_{[I}D_{J]} \), \( D_{[I}D_{J}D_{K]}D_{L]} \), etc., are of special interest in that their application on the supermultiplet clearly elicits transformations that are uniform compositions of:

1. (possibly higher-order) translations in the domain space (here, time \( \tau \)), and
2. spin/statistics-preserving, linear and homogeneous transformations in the field-space.

This general nature of these transformations makes it possible to regard them as a natural generalization of the concept of holonomy. In standard (bosonic/commutative) differential geometry, covariant derivatives generate parallel translations; their commutator defines the torsion \( T_{\mu\nu\rho} \) and the curvature \( \hat{R}_{\mu\nu} \) tensors:

\[
[\nabla_\mu, \nabla_\nu] = T_{\mu\nu}^\rho \nabla_\rho + \hat{R}_{\mu\nu}, \tag{2.5}
\]

where \( \hat{R}_{\mu\nu} \) is valued in the Lie algebra of transformations of the objects (vectors, tensors,...) upon which it acts. For example, when acting on covariant vectors, \( \hat{R}_{\mu\nu} \) takes on the familiar form of the Riemann tensor, \( \hat{R}_{\mu\nu}(V)_\rho : R_{\mu\nu\rho\sigma}V_\sigma \). Algebraically, the curvature tensor \( \hat{R}_{\mu\nu} \) is valued in the Lie algebra of linear, homogeneous transformations of the tensors upon which it acts.

In Eq. (1.1) and all its applications, the operators \( D_I \) and \( \partial_\tau \) both generate (twisted) translations in superspace, and the \( D_I \)'s anticommute with the supercharges and so are supersymmetry-covariant. Owing to the identity \( D_I D_J = \frac{1}{2}[D_I, D_J] \), these facts justify comparing the relation (2.4a) with the standard differential geometry definition (2.5). However, it is crucial that (2.4a) produces not a linear combination of translations and “rotations” as does (2.5), but a composition of the two. The higher even-order operators in the progression (2.4) all produce similar types of transformations, and we dub this entire class of geometric effects holoraumy; we will specifically refer
to “quadratic holoraumy” in distinction from the quartic and higher order operators as producing “higher order holoraumy.”

The geometric interpretation of the commutator $[\nabla_\mu, \nabla_\nu]$ is the comparison of a concatenation of two infinitesimal parallel translations with the oppositely ordered concatenation of those same parallel translations. When acting on tangent vectors on a manifold, the curvature term determines the linear, homogeneous transformation, i.e., holonomy associated to the quadrilateral loop formed by the alternating concatenations of infinitesimal parallel transports.

The identity $D_I D_J = \frac{1}{2} [D_I, D_J]$ shows that the quadratic super-differential operator (2.4a) similarly compares the concatenation of two infinitesimal parallel translations—except that (2.4a) crucially uses the wrong type of bracket operation: The standard bracket operation comparing two anticommutative translations in superspace is the anticommutator, $\{D_I, D_J\}$—which must evaluate identically to $2i\delta_{IJ}\partial_\tau$ for every off-shell supermultiplet, by its very definition. It is thus the operators constructed using the “wrong-type” brackets,

$$D_I D_J = \frac{1}{2} [D_I, D_J],$$

$$D_I D_J D_K D_L = \frac{1}{4!} \left( \{D_I, D_J\} \{D_K, D_L\} + \{D_I, D_K\} \{D_J, D_L\} + \{D_I, D_L\} \{D_J, D_K\} \right),$$

etc., that elicit the non-trivial information. That is, only jointly do the standard bracket and the “wrong-type” bracket provide the complete information on the (un-symmetrized) concatenation of two (superspace-twisted) translations $D_I$ and $D_J$. Using the standard bracket relation (1.1) implicitly, the holoraumy (2.4) then exhibits this complete information.

The information elicited by the non-linear super-derivative operators (2.4) is expected to strongly depend on the representation—which is precisely the feature that makes it useful. Indeed, this is not at all unexpected: In the familiar case of $\mathfrak{su}(2)$, the defining relation $[\hat{J}_\alpha, \hat{J}_\beta] = \imath \varepsilon_{\alpha\beta\gamma} \hat{J}_\gamma$ holds identically in all representations. On the other hand, the results of the wrong-type bracket relations depend strongly on the representation: The familiar relation $\{\frac{1}{2} \sigma_\alpha, \frac{1}{2} \sigma_\beta\} = \frac{1}{2} \delta_{\alpha\beta} \mathbb{I}$ holds only for the spin-$\frac{1}{2}$ representation $\hat{J}_\alpha = \frac{1}{2} \sigma_\alpha$; for larger spin, where $\hat{J}_\alpha$ are represented by matrices of larger size, the results of the anticommutators $\{\hat{J}_\alpha, \hat{J}_\beta\}$ are no longer even linear combinations of the generators $\hat{J}_\alpha$ and the identity matrix, $\mathbb{I}$.

**Quadratic Holoraumy Tensors:** The realization that (1.2) is equivalent to (2.1) where $\Gamma_I$ generate the $\mathfrak{Cl}(0, N)$ Clifford algebra (2.2) also has another standard and just as important consequence [23, 24]: Owing solely to the Clifford algebra defining relation (2.2), the quadratic matrices $\Gamma_{IJ} := \frac{1}{2} [\Gamma_I, \Gamma_J]$ canonically generate the Spin$(0, N)$ group:

$$[\Gamma_{IJ}, \Gamma_{KL}] = 2 \delta_{IL} \Gamma_{JK} - 2 \delta_{IK} \Gamma_{JL} + 2 \delta_{JK} \Gamma_{IL} - 2 \delta_{JL} \Gamma_{IK}. \quad (2.7)$$

This gives a special interpretation to the numerical matrices appearing in the quadratic holoraumy relation (2.4a). Let us define:

$$\mathcal{B}_{IJ} := L_{[I} L_{J]}^{-1} \quad \text{and} \quad \mathcal{F}_{IJ} := L_{[I}^{-1} L_{J]},$$

so

$$\Gamma_{IJ} = \begin{bmatrix} \mathcal{B}_{IJ} & 0 \\ 0 & \mathcal{F}_{IJ} \end{bmatrix}, \quad (2.8)$$

that is,

$$D_I D_J \Phi_i = -i (\mathcal{B}_{IJ})_{ik} \partial_k \Phi_i \quad \text{and} \quad D_I D_J \Psi_j = -i (\mathcal{F}_{IJ})_{jk} \partial_k \Psi_j. \quad (2.9)$$
The block-diagonal nature of the canonical result (2.7) and the definitions (2.8) then imply that

\[
\begin{align*}
[\mathcal{B}_{IJ}, \mathcal{B}_{KL}] &= 2\delta_{IL}\mathcal{B}_{JK} - 2\delta_{IK}\mathcal{B}_{JL} + 2\delta_{JK}\mathcal{B}_{IL} - 2\delta_{JL}\mathcal{B}_{IK}, \\
[\mathcal{F}_{IJ}, \mathcal{F}_{KL}] &= 2\delta_{IL}\mathcal{F}_{JK} - 2\delta_{IK}\mathcal{F}_{JL} + 2\delta_{JK}\mathcal{F}_{IL} - 2\delta_{JL}\mathcal{F}_{IK}.
\end{align*}
\]

(2.10a)
(2.10b)

That is, both (quadratic) holoraumy tensors \(\mathcal{B}_{IJ}\) and \(\mathcal{F}_{IJ}\) generate two separate Lie groups that are isomorphic to subgroups of Spin(\(N\)):

\[
\begin{align*}
\mathcal{B}_{IJ} : \quad \Phi_i &\rightarrow (\mathcal{B}_{IJ})_i^k (i\partial_j \Phi_i) = (\mathbb{I}_{IJ})_i^j (\mathbb{L}_{ij}^{-1})_j^k (i\partial_j \Phi_k), \\
\mathcal{F}_{IJ} : \quad \Psi_j &\rightarrow (\mathcal{F}_{IJ})_j^\ell (i\partial_i \Psi_j) = (\mathbb{L}_{ij}^{-1})_j^i (\mathbb{L}_{ij})_i^\ell (i\partial_i \Psi_\ell).
\end{align*}
\]

(2.11a)
(2.11b)

That is, the \(\mathcal{B}_{IJ}\)-tensors generate Spin(\(N\))-transformations upon the bosons \(\Phi_i\), while the \(\mathcal{F}_{IJ}\)-tensors generate the action of this group upon the fermions \(\Psi_j\). In both cases, these field-space transformations are composed with a first order domain-space (here, world-line) translation.

2.2 Quadratic Holoraumy Group and Characteristics

Looking back at (1.1), we see that the extension of Spin(\(N\)) by the \(\mathbb{Z}_2\)-group of sign-changes \(D_I \rightarrow -D_I\) forms the Lie group Pin(\(N\))—the maximal group of outer automorphisms, Aut(\(\mathfrak{sp}^{1|N}\)), of the supersymmetry algebra (1.1). This then provides the geometric significance of the quadratic holoraumy tensors (2.8) as generating the Aut(\(\mathfrak{sp}^{1|N}\))-action on the component fields of the supermultiplet. Indeed, the higher even-order holoraumy tensors, such as in (2.4c) and higher, all appear in the power expansion of the formal exponential group-elements (Eqs. (2.9) imply that \(\mathcal{B}_{IJ}\) and \(\mathcal{F}_{IJ}\) are antihermitian)

\[
\exp\{\frac{1}{2}\Lambda^{[IJ]} \mathcal{B}_{IJ}\} \in \mathcal{H}_\mathfrak{g}, \quad \text{and} \quad \exp\{\frac{1}{2}\Lambda^{[IJ]} \mathcal{F}_{IJ}\} \in \mathcal{H}_\mathfrak{g},
\]

(2.12)

owing to the relations (2.6). Using again only (2.2), powers of \(\Gamma_{IJ}\) can always be expressed as their totally antisymmetrized products plus lower-order products; decomposing into blocks, the same follows for \(\mathcal{B}_{IJ}\) and of \(\mathcal{F}_{IJ}\).

Algebraic Invariants: On the other hand, the index structure in \((\mathcal{B}_{IJ})_i^k\) and \((\mathcal{F}_{IJ})_j^\ell\) reminds of the gauge algebra-valued field strengths in Yang-Mills theory. Indeed, we have shown above that \((\mathcal{B}_{IJ})_i^k\) and \((\mathcal{F}_{IJ})_j^\ell\) take values in the Lie (sub)algebra of Spin(\(N\)), given respectively in the matrix representations acting on the bosons \(\Phi_i\) and the fermions \(\Psi_j\). It is then reasonable to also consider evaluating and comparing various characteristics of these holoraumy tensors, obtained using the methods of Ref. [25]. Here, we list such monomial invariants that are up to quartic in \((\mathcal{B}_{IJ})_i^j\):

\[
\begin{align*}
& (\mathcal{B}_{IJ})_i^j (\mathcal{B}_{IJ})_j^i, \quad (\mathcal{B}_{IJ})_i^j (\mathcal{B}_{JK})_j^k (\mathcal{B}_{KL})_k^i, \\
& (\mathcal{B}_{IJ})_i^j (\mathcal{B}_{JK})_j^k (\mathcal{B}_{KL})_k^\ell (\mathcal{L}_{ij}^{-1})_j^\ell (\mathcal{L}_{ij})_i^k, \\
& (\mathcal{B}_{IJ})_i^j (\mathcal{B}_{JK})_j^k (\mathcal{B}_{KL})_k^\ell (\mathcal{L}_{ij}^{-1})_j^i (\mathcal{L}_{ij})_i^\ell (\mathcal{L}_{ij})_i^k, \\
& (\mathcal{B}_{IJ})_i^j (\mathcal{B}_{IJ})_j^i (\mathcal{B}_{KL})_k^\ell (\mathcal{L}_{ij}^{-1})_j^i (\mathcal{L}_{ij})_i^\ell (\mathcal{L}_{ij})_i^k, \\
& (\mathcal{B}_{IJ})_i^j (\mathcal{B}_{IJ})_j^i (\mathcal{B}_{KL})_k^\ell (\mathcal{L}_{ij}^{-1})_j^i (\mathcal{L}_{ij})_i^\ell (\mathcal{L}_{ij})_i^k.
\end{align*}
\]

(2.13a)
(2.13b)
(2.13c)
(2.13d)
(2.13e)
For \( N = 8 \) even, there also exist “volume” invariants (2.16):

\[
\varepsilon^{I_1I_2I_3I_4I_5I_6I_7I_8} (B_{I_1I_2})_{i_1}^{i_2}(B_{I_3I_4})_{i_3}^{i_4}(B_{I_5I_6})_{i_5}^{i_6}(B_{I_7I_8})_{i_7}^{i_8},
\tag{2.14a}
\]

\[
\varepsilon^{I_1I_2I_3I_4I_5I_6I_7I_8} (B_{I_1I_2})_{i_1}^{i_2}(B_{I_3I_4})_{i_3}^{i_4}(B_{I_5I_6})_{i_5}^{i_6}(B_{I_7I_8})_{i_7}^{i_8},
\tag{2.14b}
\]

\[
\varepsilon^{I_1I_2I_3I_4I_5I_6I_7I_8} (B_{I_1I_2})_{i_1}^{i_2}(B_{I_3I_4})_{i_3}^{i_4}(B_{I_5I_6})_{i_5}^{i_6}(B_{I_7I_8})_{i_7}^{i_8},
\tag{2.14c}
\]

and so on. Besides the analogous invariants constructed from \((F_{IJJ})_{a}^{b}\), there are also \(B-F\) mixed invariants such as

\[
(B_{I})_{i}^{j} (B_{J})_{j}^{i} (F_{K})_{a}^{b} (F_{L})_{b}^{a}, \quad (B_{I})_{j}^{i} (F_{J})_{i}^{b} (B_{K})_{j}^{i} (F_{L})_{b}^{a},
\tag{2.15a}
\]

\[
(B_{I})_{i}^{j} (F_{J})_{j}^{i} (F_{L})_{a}^{b} (B_{K})_{b}^{a}, \quad (B_{I})_{j}^{i} (F_{J})_{i}^{b} (B_{K})_{j}^{i} (F_{L})_{b}^{a},
\tag{2.15b}
\]

as well as

\[
\varepsilon^{I_1I_2I_3I_4I_5I_6I_7I_8} (B_{I_1I_2})_{i_1}^{i_2}(B_{I_3I_4})_{i_3}^{i_4}(F_{I_5I_6})_{i_5}^{i_6}(B_{I_7I_8})_{i_7}^{i_8},
\tag{2.15c}
\]

and so on. Using these, one can construct “polynomial holoraumy invariants,” somewhat akin to the polynomial curvature invariants that provide useful information about the geometry of manifolds.

**Geometric Invariants:** The first of the “volume” invariants (2.14a) in fact has a very suggestive geometric meaning as it may also be calculated as the characteristic fermionic integral,

\[
\int d^N \theta \, \det[B_{IJ} \theta^I \theta^J - \lambda \mathbb{I}] \propto \varepsilon^{I_1 \ldots I_N} \text{Tr}[B_{I_1I_2} \cdots B_{I_N-1I_N}].
\tag{2.16}
\]

In fact, the “characteristic polynomial” superfield \(\det[B_{IJ} \theta^I \theta^J - \lambda \mathbb{I}]\) may also have non-zero fermionic integrals over certain subsets of the \(\theta\)’s, as is the case for chiral representations\(^7\). Clearly, analogous characteristic quantities can just as well be constructed also from \(F_{IJJ}\), from the coefficients of the characteristic polynomials

\[
\det[B_{IJ} \theta^I \theta^J - \lambda \mathbb{I}] \quad \text{and} \quad \det[F_{IJJ} \theta^I \theta^J - \mu \mathbb{I}].
\tag{2.17}
\]

It is curious to see the fermionic coordinates \(\theta^I\) in superspace here play the role usually reserved for the differentials of ordinary (commuting) local coordinates on a manifold in differential geometry.

**Two Remarks:** First, although it is well known that the Riemann tensor in 3+1 dimensional spacetime has twenty independent degrees of freedom, it is not known how to construct a complete basis of twenty algebraically independent polynomial curvature invariants [26]; the situation in higher dimensions is clearly only more complex. We therefore do not expect to be able to reduce the copious list (2.13)–(2.16) by any a priori methods. Rather, these invariants should be evaluated—presumably by computer-aided methods—for as many supermultiplets as possible, so as to determine which of their combinations, if any, provide unambiguous distinction between world-line dimensional reductions of supermultiplets. Second, it is known that polynomial curvature invariants are not sufficient to differentiate between physically distinct spacetimes even in 3+1 dimensions [26], and we do not expect polynomial holoraumy invariants built from monomials.

\(^7\)In this sense, the chiral superspace is akin to a homologically nontrivial sub-superspace inside the full standard superspace.
such as (2.13)–(2.16) to differentiate between the dimensional reductions of all distinct higher-dimensional supermultiplets.

Nevertheless, it would seem prudent to explore this resolving capability of the polynomial holoraumy invariants built from monomials such as (2.13)–(2.16), and we hope to return to this task in a later, computer-aided effort.

**Quadratic Holoraumy Recursions**: The higher holoraumy tensors that occur in (2.4b)–(2.4c) and further may be generated from the quadratic holoraumy tensors and the original \( \mathbb{L} \)-matrices (2.1):

\[
\mathbb{L}_I^{[L]L_J^{[L]L_K]} = \mathcal{B}_{IJJ}^{[L]L_K} = \mathbb{L}_I^{[F]F_K},
\]

\[
\mathbb{L}_I^{-1L_J^{[L]L_K}} = \mathcal{F}_{IJJ}^{-1L_K} = \mathbb{L}_I^{-1L_K},
\]

(2.18a)

In turn, these recursive identities also provide relationships between the bosonic and the fermionic quadratic holoraumy tensors, \( \mathcal{B}_{IJJ} \) and \( \mathcal{F}_{IJJ} \)—represented by the second equations (2.18).

In turn, another set of relationships between the \( \mathcal{B}_{IJJ} \) and \( \mathcal{F}_{IJJ} \) matrices emerges from the very definition of the matrix \( \mathcal{B}_{IJJ} \),

\[
(\mathcal{B}_{IJJ})^k_i := \left(\mathbb{L}_I^{-1L_JL_K}\right)^k_i := \frac{1}{2}(\mathbb{L}_I^{-1L_JL_K})^k_i - \frac{1}{2}(\mathbb{L}_I^{-1L_JL_K})^k_i,
\]

(2.19)

upon multiplying the first term from the right and the second term from the left by \( \mathbb{1} = \mathbb{L}_I^{-1L_J} \) (for each \( \mathbb{L}_I \), \( \mathbb{L}_I^{-1} \) is the matrix-inverse of \( \mathbb{L}_I \), whence there is no summation on \( I \)):

\[
= \frac{1}{2}(\mathbb{L}_I^{-1L_JL_K})^k_i - \frac{1}{2}(\mathbb{L}_I^{-1L_JL_K})^k_i = -(\mathbb{L}_I)^i^j \frac{1}{2}(\mathbb{L}_I^{-1L_JL_K})^k_i\mathbb{L}_I^{-1L_JL_K})^k_i,
\]

from which we have:

\[
(\mathcal{B}_{IJJ})^k_i = -(\mathbb{L}_I^{-1L_JL_K})^k_i.
\]

(2.20)

The analogous computations using \( \mathbb{1} = \mathbb{L}_J^{-1L_J} \) (no summation on \( J \)), as well as starting from \( \mathcal{F}_{IJJ} = \mathbb{L}_I^{-1L_JL_K} \) then produces the \( 4 \times \left(\begin{array}{c} N \\ 2 \end{array}\right) \) matrix relations:

\[
(\mathcal{B}_{IJJ})^k_i = -(\mathbb{L}_I^{-1L_JL_K})^k_i, \quad (\mathcal{B}_{IJJ})^k_i = -(\mathbb{L}_I^{-1L_JL_K})^k_i,
\]

(2.21a)

\[
(\mathcal{F}_{IJJ})^k_i = -(\mathbb{L}_J^{-1L_JL_K})^k_i, \quad (\mathcal{F}_{IJJ})^k_i = -(\mathbb{L}_J^{-1L_JL_K})^k_i,
\]

(2.21b)

for each of the \( \left(\begin{array}{c} N \\ 2 \end{array}\right) \) choices of \( I, J = 1, 2, \cdots, N \), with no summation over either \( I \) or \( J \).

Finally, there exists also a sum-rule obtained by iterative use of the relations (2.2):

\[
\Gamma_K \Gamma_IJ \Gamma_K = \frac{1}{2}(\Gamma_K \Gamma_IJ \Gamma_K - \Gamma_K \Gamma_IJ \Gamma_K) = (N-4)\Gamma_IJ.
\]

(2.22)

Reading off the blocks of these matrices, this implies:

\[
\mathbb{L}_K^{-1L_JL_K} = (N-4)\mathcal{F}_{IJJ} \quad \text{and} \quad \mathbb{L}_K \mathcal{F}_{IJJ} \mathbb{L}_K^{-1} = (N-4)\mathcal{B}_{IJJ},
\]

(2.23)

which provide additional useful relations between \( \mathcal{F}_{IJJ} \) and \( \mathcal{B}_{IJJ} \) for \( N \neq 4 \), and a constraining sum-rule for \( N = 4 \).
2.3 The $N = 4$ Valises

Motivated by the most familiar and most often employed framework of 3+1-dimensional simple $(N = 1)$ supersymmetry, we now focus on the case of its 0-brane dimensional reduction, the world-line $(N = 4)$-supersymmetry. Furthermore, in view of the main theorem of Ref. [17] we focus on the minimal supermultiplets which have 4+4 components. The Dirac algebra\(^8\) (2.3) then reduces to the sixteen matrices

\[ 1, \quad \Gamma_I, \quad \Gamma_{IJ}, \quad \Gamma_{IJK} =: \varepsilon_{IJKLM} \hat{\Gamma}, \quad \Gamma_{IJKL} =: \varepsilon_{IJKLM} \hat{\Gamma}. \tag{2.24} \]

The minimal $N = 4$ supermultiplets all have four bosonic and four fermionic components, so that the list (2.3) involves an $8 \times 8$-dimensional matrix representation of the $\Gamma$-matrices (2.2)—and in the block off-diagonal form (2.1). That is, the list (2.3) is regenerated entirely from the “half-sized,” $4 \times 4$-dimensional, invertible $\mathcal{L}$-matrices defined in (2.1). The rank-2 matrices $\Gamma_{IJ}$ generate the $\text{spin}(4) = \text{spin}(3)_- \oplus \text{spin}(3)_+$ algebra of the connected component of the group of automorphisms, $\text{Aut}(\mathfrak{sp}_{114}) = \text{Pin}(4)$, of the world-line supersymmetry algebra (1.1). Notice that the notation in (2.24)—as well as throughout Section 2—is chosen to explicitly “forget” all (the?) structure inherited from the action of the higher-dimensional Lorentz symmetry; we will return to this below.

**The Reference Algebra:** Explicit calculation shows that the fifteen real, non-identity matrices (2.24) in fact close an irreducible Lie algebra, which then must be $\text{spin}(3, 3) \approx \mathfrak{sl}(4, \mathbb{R})$ [24, 27]. Together with $1$, this provides a complete basis of real, $4 \times 4$, invertible matrices, for which we use the basis from Ref. [28] and which are displayed in Table 3 in Appendix A for convenience.

The list of the fifteen non-identity $8 \times 8$-dimensional matrices (2.24) is therefore constructed from the $4 \times 4$-dimensional matrix generators of $\text{spin}(3, 3)$ and their inverses, and so explicitly provide a real $8$-dimensional representation\(^9\) of $\text{spin}(3, 3)$. These matrices may alternatively be regarded as spanning either of the algebras

\[ \mathfrak{Cl}(1, 3) \approx 1 \oplus \text{spin}(3, 3), \quad \text{spin}(3, 3) \approx \mathfrak{sl}(4, \mathbb{R}) \approx \mathfrak{su}(4). \tag{2.25} \]

We are thus led to inquire how are the holoraumy tensors (2.8)—and the two copies of the $\text{spin}(N = 4) = \text{spin}(3)_- \oplus \text{spin}(3)_+$ Lie algebra (2.10) that they generate—embedded, respectively, in this $\text{spin}(3, 3)$ reference algebra (2.25). That is to say, given two 4+4-component off-shell supermultiplets of $N = 4$-extended world-line supersymmetry, we can construct the list of sixteen $4 \times 4$-dimensional matrices for both, and compare these two lists to discern the transformation required to relate the two supermultiplets.

The “location” of the $\Gamma_{IJ} = (\mathcal{B}_{IJ} \oplus \mathcal{F}_{IJ})$ holoraumy tensors of one supermultiplet as compared to the like tensors of another supermultiplet—within any particular fixed basis (such as Table 3) of $\mathfrak{Cl}(1, 3) \approx 1 \oplus \text{spin}(3, 3)$—then provides valuable relative information about the two supermulti-

---

8 Most importantly, this is not the familiar Dirac algebra used 3+1-dimensional spacetime physics for which we will use lower-case $\gamma$-matrices, but refers to the similar algebra of matrices defined in (2.2).

9 Since $\text{spin}(3, 3) \approx \mathfrak{su}(4)$, this real 8-dimensional representation is identified with the complex 4 of $\mathfrak{su}(4)$. 

10
where \( \eta \) world-line supermultiplet equivalences bijective transformations (2.30) then define

\[
\text{As standard in field theory, } \mathcal{X}.
\]

hermitian generators of Spin(3) component (super)field redefinitions \( X \). Clearly, the quadratic holoraumy tensors spinplets. This “location” can be made more precise by means of the diagram of maps:

\[
\text{(reference) } \text{Spin}(3, 3) \left\{ \begin{array}{c}
\supset \text{Spin}(3)_{\text{rot}} \times \text{Spin}(3)_{\text{exR}} \\
\supset \text{Spin}(1, 3)_{\text{Lorentz}} \times U(1)_{R}
\end{array} \right.
\]

where \( \eta_B \) and \( \eta_F \) map the holoraumy groups \( \mathcal{H}_B \) and \( \mathcal{H}_F \), respectively, to \( \text{Spin}(3)_{\text{rot}} \times \text{Spin}(3)_{\text{exR}} \subset \text{Spin}(3, 3) \) and so specify the “location” of \( B_{IJ} \) and \( F_{IJ} \) respectively in the reference algebra \( \text{spin}(3, 3) \). The bottom part of the mapping diagram (2.26) may be visualized by realizing the hermitian generators of \( \text{Spin}(3, 3) \) as \( 6 \times 6 \) antisymmetric real matrices:

\[
\text{spin}(3)_{\text{exR}} = \{0, 3\} = \{\gamma^0, \gamma^{123}, \gamma^{0123}\}
\]

\[
\text{Spin}(2, 0) \rightarrow U(1)_{R} : \{\gamma^5 = i \gamma^{0123}\}
\]

\[
\text{spin}(1, 3)_{\text{Lorentz}} \rightarrow U(1)_{R} : \{\gamma^\mu = 0, 1, 2, 3\}
\]

\[
\text{Spin}(3)_{\text{rot}} = \{0, 3\} = \{\gamma^12, \gamma^23, \gamma^31\}
\]

\[
\text{spin}(2, 0) \rightarrow U(1)_{R} : \{\gamma^5 = i \gamma^{0123}\}
\]

\[
\text{spin}(2, 0) \rightarrow U(1)_{R} : \{\gamma^5 = i \gamma^{0123}\}
\]

The corresponding \( \text{Spin}(3)_{\text{rot}} \) group is both the maximal compact subgroup of the 3+1-dimensional Lorentz group \( \text{Spin}(1, 3) = \text{SL}(2, \mathbb{C}) \) and also one of the two factors in the maximal compact connected subgroup of \( \text{Aut}(\mathfrak{sp}^{1|4}) = \text{Pin}(4) \). The complementary \( \text{Spin}(3)_{\text{exR}} \) factor in turn contains the well-known 3+1-dimensional \( U(1) \) \( R \)-symmetry generated by the \( \gamma^3 \)-matrix. To disambiguate between the several \( \text{spin}(3) \) algebras, we summarize:

\[
\text{spin}(3)_{\text{rot}} \oplus \text{spin}(3)_{\text{exR}} = \text{spin}(0, 3) \oplus \text{spin}(3, 0) \subset \text{spin}(3, 3), \text{ refers to Table } 3.
\]

\[
\text{spin}(3)_{\text{rot}} \oplus \text{spin}(3)_{\text{exR}} = \text{spin}(0, 3) \oplus \text{spin}(3, 0) \subset \text{spin}(3, 3), \text{ refers to Table } 3.
\]

\[
\text{spin}(3)_{\text{rot}} \oplus \text{spin}(3)_{\text{exR}} = \text{spin}(0, 3) \oplus \text{spin}(3, 0) \subset \text{spin}(3, 3), \text{ refers to Table } 3.
\]

World-Line Equivalence: The \( \mathbb{L}_I \)-matrices (2.1) are specified with respect to a chosen basis of component (super)fields, \( \mathcal{X} \) and \( \mathcal{Y} \) respectively:

\[
\mathcal{X} \oplus \mathcal{Y} : \mathbb{L}_I \rightarrow \mathcal{X}_I \mathbb{L}_I \mathcal{Y}^{-1}, \text{ so that } \mathcal{X} \oplus \mathcal{Y} : \mathcal{B}_{IJ} \rightarrow \mathcal{X} \mathcal{B}_{IJ} \mathcal{X}^{-1} \text{ and } \mathcal{F}_{IJ} \rightarrow \mathcal{Y} \mathcal{F}_{IJ} \mathcal{Y}^{-1} \text{ for } I, J = 1, 2, 3, 4.
\]

Clearly, the quadratic holoraumy tensors \( \mathcal{B}_{IJ} \) and \( \mathcal{F}_{IJ} \) transform separately by bosonic and fermionic component (super)field redefinitions \( \mathcal{X} \) and \( \mathcal{Y} \), respectively.

Now, let \( \mathcal{B}_{IJ}^{(a)}, \mathcal{F}_{IJ}^{(a)} \) denote the quadratic holoraumy tensors of the \( A^4 \) supermultiplet. If the two supermultiplets are equivalent, there must exist a component (super)field basis change that relates their \( \mathbb{L}_I \)-matrices:

\[
\mathcal{X} \oplus \mathcal{Y} : \mathbb{L}_I^{(2)} = \mathcal{X}_I \mathbb{L}_I^{(1)} \mathcal{Y}^{-1}, \text{ for each } I = 1, 2, 3, 4 \text{ so that } \mathcal{X} \oplus \mathcal{Y} : \mathcal{B}_{IJ}^{(2)} = \mathcal{X} \mathcal{B}_{IJ}^{(1)} \mathcal{X}^{-1} \text{ and } \mathcal{F}_{IJ}^{(2)} = \mathcal{Y} \mathcal{F}_{IJ}^{(1)} \mathcal{Y}^{-1} \text{ for each } I, J = 1, 2, 3, 4.
\]

As standard in field theory, \( \mathcal{X}, \mathcal{Y} \) and their inverses must be local field redefinitions. Such local bijective transformations (2.30) then define world-line supermultiplet equivalences.
Since the maximal compact subgroup of Spin(3, 3) is Spin(3, 0) × Spin(0, 3) ≈ Spin(4), it is tempting to identify this abstract subgroup with the connected part of the Aut(.Sp^{14}) = Pin(4) group generated by the \( \Gamma_{IJ} \) from among (2.3), which combines in a block-diagonal form the two copies of Spin(4) generated by \( \mathcal{B}_{IJ}, \mathcal{F}_{IJ} \) in (2.10). In turn, it is a standard result in Lie group theory that every regular subgroup \( H \subset G \) of a simple Lie group \( G \) in fact has a continuum of distinct embeddings in \( G \), but that they are all equivalent by \( G \)-conjugation [27, 29, 30]. This makes it tempting to conclude that every two possible holoraumy Spin(4) subgroups (2.7) of Spin(3, 3), generated by (2.25) are isomorphic. The same would then seem to follow also for (2.10), i.e., that the holoraumy groups computed for any two 4+4-component supermultiplets are isomorphic to each other by way of (2.30), and so cannot distinguish between inequivalent supermultiplets.

Explicit calculations in Section 3 however show that this does not hold, and that concrete results for \( \Gamma_{IJ} = \mathcal{B}_{IJ} \oplus \mathcal{F}_{IJ} \) can—and indeed do distinguish between several world-line supermultiplets that were dimensionally reduced from higher-dimensional spacetime and for which we have carried out the computations explicitly. Reverse-engineering this dimensional reduction, we are then able to exhibit some necessary conditions (i.e., obstructions, conversely) for dimensionally extending a given world-line supermultiplet into a desired 3+1-dimensional supermultiplet. Both in concept and in practice then, the present results on holoraumy expand on the 1 \( \rightarrow \) 2-dimensional extension results of Ref. [31] and complements the analysis of Refs. [32, 33].

Results: While we have so far not been able to provide a mathematically rigorous proof of the precise extent to which holoraumy tensors can be used to distinguish off-shell supermultiplets, this preliminary study and the explicit examples in Section 3 do demonstrate its potential. For example, we will show below in detail that the holoraumy—defined and calculated solely from world-line physics—faithfully discerns between the inequivalent dash-chromotopologies [21, 22, 34] and also “twisting” [35, 36], originally defined for supersymmetry in 1+1-dimensional spacetime. In addition, however, it also signals the existence of a complex structure, which is necessary in a chiral supermultiplet but obstructs the extension to a vector supermultiplet, for example. This also clears up a minor conundrum, provided by the fact that the so-called twisted-chiral superfield is manifestly complex, albeit being a dimensional reduction of the 3+1-dimensional vector superfield [35, 36], which is manifestly real. Also, a finer distinctive property of the holoraumy tensors correlates with the dimensional extension to different real 3+1-dimensional supermultiplets, such as the vector vs. the tensor supermultiplet (both in the Wess-Zumino gauge).

The possible reasons that—contrary to the above-cited suspicion—certain of the possible holoraumy subgroups of Spin(4) \( \subset \) Spin(3, 3) are not isomorphic to each other, and so can distinguish between 4+4-component supermultiplets include:

1. The detailed comparison of the holoraumy tensors and where they are within the spin(3, 3) algebra (2.4) crucially depends on the details of the real forms of the respective groups.
2. The maximal compact subgroup of Spin(3, 0) × Spin(0, 3) \( \subset \) Spin(3, 3) is singled out by the signature of the metrics that these groups preserve.
3. The main theorem of Ref. [17] permits restricting to monomial\(^{10} \) \( 4 \times 4 \) \( L_I \)-matrices (2.1); to preserve this, the component field redefinitions (2.30a) must be significantly restricted.

\(^{10}\)A matrix is monomial if it has a single nonzero entry in every row and every column.
Needless to say, the definitive determination (and rigorous proof) of precisely how “resolving” the holoraumy concept is in reconstructing the higher-dimensional spacetime symmetry structures from the world-line dimensional reduction of supermultiplets hinges on a complete understanding of the precise relationship between the maximal group of outer automorphisms, \( \text{Aut}(\mathfrak{sp}^{1|N}) \approx \text{Pin}(0, N) \)— which is what features prominently throughout this work, and the full Poincaré group in the intended higher-dimensional spacetime. This ultimate goal of uncovering the precise workings of an as yet conjectured “supersymmetry holography” is quite beyond our present scope. However, the subsequent skein of sample supermultiplets should provide a good starting point for such a more complete study.

3 Several Examples

We now turn to exhibit a few well known 3+1-dimensional supermultiplets, dimensionally reduced to the world-line in Refs. [28, 37], for which we calculate and analyze the holoraumy tensors (2.8).

3.1 The Chiral Supermultiplet Valise

We begin with the familiar example of the chiral supermultiplet, of which the 0-brane dimensional reduction to the coordinate time world-line and in terms of real component superfields \( (A, B, F, G|\psi_a) \) is given by the (2.1)-like super-differential system [28]

\[
\begin{align*}
D_a A &= \psi_a, & D_a B &= i(\gamma^5)_{ab} \psi_b, \\
D_a F &= (\gamma^0)_{ab} \psi_b, & D_a G &= i(\gamma^5 \gamma^0)_{ab} \psi_b, \\
D_a \psi_b &= i(\gamma^0)_{ab} \partial_\tau A - (\gamma^5 \gamma^0)_{ab} \partial_\tau B - iC_{ab} \partial_\tau F + (\gamma^5)_{ab} \partial_\tau G,
\end{align*}
\]

(3.1)

where the component superfields \( A, B, F, G \) all have the same engineering dimension, \( \frac{1}{2} \) lower than the fermions \( \psi_a \), and are all real, as are their lowest component fields obtained by standard projection [13, 15].

Using the matrices in Tables 3 and 4, the system (3.1a)–(3.1c) may also be tabulated as

\[
\begin{array}{|c|cccc|cccc|}
\hline
\text{vCS} & A & B & F & G & \psi_1 & \psi_2 & \psi_3 & \psi_4 \\
\hline
D_1 & \psi_1 & -\psi_4 & \psi_2 & -\psi_3 & i\partial_\tau A & i\partial_\tau F & -i\partial_\tau G & -i\partial_\tau B \\
D_2 & \psi_2 & \psi_3 & -\psi_1 & -\psi_4 & -i\partial_\tau F & i\partial_\tau A & i\partial_\tau B & -i\partial_\tau G \\
D_3 & \psi_3 & -\psi_2 & -\psi_4 & \psi_1 & i\partial_\tau G & -i\partial_\tau B & i\partial_\tau A & -i\partial_\tau F \\
D_4 & \psi_4 & \psi_1 & \psi_3 & \psi_2 & i\partial_\tau B & i\partial_\tau G & i\partial_\tau F & i\partial_\tau A \\
\hline
\end{array}
\]

(3.1d)

This makes it clear that the system (3.1) is monomial: Appearances in (3.1c) to the contrary, the result of applying each super-derivative to each single component superfield is again a single component superfield or its derivative, not a linear combination of such terms. Such supermultiplets are faithfully depicted by graphs called Adinkras [11]. In particular, the system (3.1) is then depicted as

(3.2)
where the nodes depict the component superfields, are drawn at the height proportional to their engineering dimension and where the edges depict the $D_a$-relations in (3.1). The corresponding monomial $L$-matrices, as defined in (1.2), are:

\[
L_{CS}^1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix},
L_{CS}^2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
L_{CS}^3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0
\end{bmatrix},
L_{CS}^4 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\] (3.3)

With the supermultiplet thus specified both in the tensorial representation (3.1) and in terms of its $L$-matrices (3.3), we can compute straightforwardly the quadratic holoraumy tensors both ways, for a comparative illustration.

**Tensorial Computation:** Iterating (3.1), we obtain\(^{11}\):

\[
D_a D_b \psi_c = \frac{1}{2} [D_a, D_b] \psi_c = -i (F_{ab}^{(CS)})_c^d (\partial_\tau \psi_d),
\] (3.4)

where the quadratic fermionic holoraumy tensor $(F_{ab}^{(CS)})_c^d$ may be written, utilizing a series of Fierz identities, as:

\[
(F_{ab}^{(CS)})_c^d = \frac{1}{2} (\gamma^0 \gamma_{mn})_{ab} (\gamma^{mn})_c^d, \quad m, n, p = 1, 2, 3.
\] (3.5)

The fermionic holoraumy is thus generated by $\gamma^{12}$, $\gamma^{23}$ and $\gamma^{31}$, the generators of the (spatial) rotation subgroup $\text{Spin}(3)_{\text{rot}} = \text{Spin}(0,3) \subset \text{Spin}(1,3)$ of the Lorentz group, and “rotates” the “direction” of $(\partial_\tau \psi)$ relative to the original “direction” of $\psi$. As stated above for the general case, the holoraumy (3.4) indeed composes $\tau$-translations with homogeneous linear transformations in the $\psi$-space—the latter of which in fact really are spatial rotations within the Lorentz group acting on the fermions $\psi$. That is to say, the fermionic quadratic holoraumy tensor $F_{ab}^{(CS)}$ is a $\text{Spin}(3)_{\text{rot}}$-valued 2-form in the 4-component space of fermions.

The bosonic holoraumy tensor is computed in the same way—again utilizing a series of Fierz identities, producing:

\[
B_{ab}^{(CS)} \eta \mapsto \text{spin}(3)_{\text{rot}}, \quad \text{and} \quad F_{ab}^{(CS)} \eta \mapsto \text{spin}(3)_{\text{rot}},
\] (3.7)

with the concrete results (3.5) and (3.6) specifying the details of these maps.

Having exhibited the first concrete example, we are in position to draw the Reader’s attention to an important feature: The computations resulting in (3.5) and (3.6) strongly depend on a particular series of 3+1-dimensional Fierz identities for the $\gamma$-matrices, the structure of which in turn strongly depends on the dimension and signature of the higher-dimensional spacetime for which the used $\gamma$-matrices have been defined.

\(^{11}\)Unlike in the general definition (2.11b), both the fermions and the super-derivatives are here counted by indices of the same type by virtue of the fact that their numbers equal in the chiral supermultiplet.
In the subsequent “matrix computation” below, we will recover the algebraic structure of these results from the purely world-line information (3.2)–(3.3). As mentioned in the comment after Eq. (2.24), this computational framework and notation were chosen to explicitly (try to) “forget” the structure of the action of the higher-dimensional Lorentz symmetry. The very fact that the algebraic structure of the results (3.5) and (3.6) can be recovered from the subsequent “matrix computation” implies that this purely world-line representation of the 0-brane dimensional reduced supermultiplet nevertheless does retain—holographically—the structure of the higher-dimensional Lorentz symmetry action within the supermultiplet from which (3.2)–(3.3) was obtained. In the rest of the article, we will show that the same holography persists throughout all of the examples.

**Matrix Computation:** In turn, with the explicit $L$-matrices (3.3) identified, it is a straightforward matter to compute:

\[
B^{(CS)}_{12} = -B^{(CS)}_{34} = -\gamma^{12}, \quad B^{(CS)}_{23} = -B^{(CS)}_{14} = +\gamma^{13}, \quad B^{(CS)}_{31} = -B^{(CS)}_{24} = -\gamma^{23},
\]

\[
F^{(CS)}_{12} = +F^{(CS)}_{34} = -\gamma^{13}, \quad F^{(CS)}_{23} = +F^{(CS)}_{14} = +\gamma^{23}, \quad F^{(CS)}_{31} = +F^{(CS)}_{24} = -\gamma^{12},
\]

(3.8a)

(3.8b)

where we have identified the results in terms of the reference matrices as given in Table 3.

The first of each of these triple equalities produce the basis-independent results

\[
B^{(CS)}_{ab} = B^{(CS)}_{ab} + \frac{1}{2} \varepsilon_{ab}^{\ cd} B^{(CS)}_{cd} = 0 \quad \text{and} \quad F^{(CS)}_{ab} = F^{(CS)}_{ab} - \frac{1}{2} \varepsilon_{ab}^{\ cd} F^{(CS)}_{cd} = 0,
\]

(3.9)

which are a consequence of the fact that the chiral superfield is annihilated by a pair of complex combinations of the super-derivatives tantamount to the result (3.27), the product of which yields the annihilating quasi-projection operators $[D_a D_b] + \frac{1}{2} \varepsilon_{ab}^{\ cd} D_c D_d]$. A similar reduction holds all minimal supermultiplets and will be referred as the minimality condition.

The results (3.8) further show that not only do both $B^{(CS)}_{ab}$ and $F^{(CS)}_{ab}$ generate only a Spin(3) subset of Spin(4), but have the same $\eta$-images within our reference Spin(3). Eqs. (3.8a)–(3.8b) clearly imply that

\[
\varpi : \eta \{ F^{(CS)}_{12}, F^{(CS)}_{23}, F^{(CS)}_{31} \} = \eta \{ -B^{(CS)}_{23}, -B^{(CS)}_{31}, B^{(CS)}_{12} \} \supset \text{spin}(3)_{\text{rot}},
\]

(3.12)

which is an even (det = +1) signed permutation of the generators, and does not depend on the concrete matrix realizations (3.8). $\eta$ denotes the mapping (2.26) that identifies the elements of $B_{ab}$ and
acting on different spaces, with elements from the reference algebra, \(\text{spin}(3, 3)\) and its maximal compact subalgebra \(\text{spin}(3)_{\text{rot}} \oplus \text{spin}(3)_{\text{exR}}\). Summarizing the composition of the result (3.7) following (3.11),

\[
(\eta_B; \eta_F)^{(CS)} : \begin{cases}
\mathcal{F}_{ab}^{(CS)} \in \text{spin}(3)_- \overset{\mathcal{F}_{ab}}{\rightarrow} \text{spin}(3)_{\text{rot}}, \\
\mathcal{F}_{ab}^{(CS)} \in \text{spin}(3)_+ \overset{\mathcal{F}_{ab}}{\rightarrow} \text{spin}(3)_{\text{rot}},
\end{cases}
\]

where \(\varpi = (12, 23, 31, 23, 31, 12)\) is the even \((\det = +1)\) relative signed permutation (3.12) relating the images of the fermionic and bosonic holonomy within the reference algebra \(\text{spin}(3, 3)\).

### 3.2 The Vector Supermultiplet Valise

The 3+1-dimensional vector supermultiplet, given in the Wess-Zumino gauge and our Majorana (real component) framework and dimensionally reduced to the coordinate time world-line, is specified by the super-differential system [28]:

\[
\begin{align*}
D_a A_m &= (\gamma_m)_a^\lambda \lambda_b, \\
D_a d &= i(\gamma^5 \gamma^0)_a^\lambda \lambda_b = -(\gamma^{123})_a^\lambda \lambda_b, \\
D_a \lambda_b &= -i(\gamma^0 \gamma^n)_a^\lambda_b (\partial_\tau A_n) + (\gamma^n)_a^\lambda_b (\partial_\tau d),
\end{align*}
\]

where the \(\partial_\tau d\) is to be identified with the usual auxiliary field in the vector supermultiplet. In tabular format, using the matrices in Tables 3 and 4, these produce:

| vVS | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_3\) | \(\lambda_4\) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(D_1\) | \(\lambda_2\) | \(-\lambda_4\) | \(\lambda_1\) | \(-\lambda_3\) | \(i\partial_\tau A_3\) | \(i\partial_\tau A_1\) | \(-i\partial_\tau d\) | \(-i\partial_\tau A_2\) |
| \(D_2\) | \(\lambda_1\) | \(\lambda_3\) | \(-\lambda_2\) | \(-\lambda_4\) | \(i\partial_\tau A_1\) | \(-i\partial_\tau A_3\) | \(i\partial_\tau A_2\) | \(-i\partial_\tau d\) |
| \(D_3\) | \(\lambda_4\) | \(\lambda_2\) | \(\lambda_3\) | \(\lambda_1\) | \(i\partial_\tau d\) | \(i\partial_\tau A_2\) | \(i\partial_\tau A_3\) | \(i\partial_\tau A_1\) |
| \(D_4\) | \(\lambda_3\) | \(-\lambda_1\) | \(-\lambda_4\) | \(\lambda_2\) | \(-i\partial_\tau A_2\) | \(i\partial_\tau d\) | \(i\partial_\tau A_1\) | \(-i\partial_\tau A_3\) |

and are depicted as:

![Diagram](image)

and we read off the \(L\)-matrices:

\[
\begin{align*}
L_{VS}^{\lambda_1} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, &
L_{VS}^{\lambda_2} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, &
L_{VS}^{\lambda_3} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, &
L_{VS}^{\lambda_4} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

With the supermultiplet now given both in the tensorial representation (3.14) and in terms of its \(L\)-matrices (3.16), we can compute straightforwardly the quadratic holonomy tensors both ways, for a comparative illustration.
Tensorial Computation: Iterating (3.14), we compute straightforwardly:

$$D_a D_b \lambda_c = \frac{1}{2} [D_a, D_b] \lambda_c = -i (\mathcal{F}_{ab}^{(VS)})_c^d (\partial_r \lambda_d),$$  (3.17)

where the quadratic fermionic holoraumy tensor $(\mathcal{F}_{ab})_c^d$ may be written, utilizing again a series of Fierz identities, as:

$$(\mathcal{F}_{ab}^{(VS)})_c^d = C_{ab} (\gamma^0)_c^d + (\gamma^5)_c^d (\gamma^5 \gamma^0)_c^d + (\gamma^5 \gamma^0)_c^d (\gamma^5)_c^d.$$  (3.18)

As in the case of the chiral super multiplet, the fermionic quadratic holoraumy tensor $\mathcal{F}_{ab}^{(VS)}$ is again a Spin(3)-valued 2-form in the 4-component space of fermions. However, this time the Lie group is generated by $\gamma^0, \gamma^{123}$ and $\gamma^{0123}$, was dubbed the “extended R-symmetry” in Ref. [10], and we denote it Spin(3)$_{\text{exR}}$. The fermionic holoraumy (generated by $\gamma^0, \gamma^{123} = -i \gamma^5 \gamma^0$ and $\gamma^{0123} = -i \gamma^5$), thus “rotates” the “direction” of $(\partial_r \lambda)$ relative to the original “direction” of $\lambda$ by means of a Spin(3)$_{\text{exR}}$-action within the fermionic $\mathbb{R}^4$-like field-space.

Similarly computed, the bosonic holoraumy tensor is:

$$(\mathcal{B}_{ab}^{(VS)})_i^j = -i (\gamma^5 \gamma^1)_{ab} (\gamma^3)_{i}^j + i (\gamma^5 \gamma^2)_{ab} (\gamma^2)_{i}^j + i (\gamma^5 \gamma^3)_{ab} (\gamma^1)_{i}^j.$$  (3.19)

To summarize, the mapping (2.26) locates:

$$\mathcal{B}_{ab}^{(VS)} \xrightarrow{\eta} \text{spin}(3)_{\text{rot}}, \quad \text{and} \quad \mathcal{F}_{ab}^{(VS)} \xrightarrow{\eta} \text{spin}(3)_{\text{exR}},$$  (3.20)

with the concrete results (3.18) and (3.19) again specifying the details of these maps. The algebraic structure of these results is shown below to be encoded just as well in the purely world-line description of the 0-brain dimensional reduction (3.15)–(3.16) of this supermultiplet.

Matrix Computation: Given the explicit $\mathbb{L}$-matrices (3.16), straightforward matrix algebra produces the quadratic holonomy matrices:

$$\mathcal{B}_{12}^{(VS)} = + \mathcal{B}_{34}^{(VS)} = + \gamma^{12}, \quad \mathcal{B}_{23}^{(VS)} = + \mathcal{B}_{14}^{(VS)} = + \gamma^{23}, \quad \mathcal{B}_{31}^{(VS)} = + \mathcal{B}_{24}^{(VS)} = + \gamma^{13},$$

$$\mathcal{F}_{12}^{(VS)} = - \mathcal{F}_{34}^{(VS)} = - \gamma^0, \quad \mathcal{F}_{23}^{(VS)} = - \mathcal{F}_{14}^{(VS)} = + \gamma^{0123}, \quad \mathcal{F}_{31}^{(VS)} = - \mathcal{F}_{24}^{(VS)} = - \gamma^{123},$$  (3.21a)

where we have again identified the results in terms of the reference matrices as given in Table 3.

The first of these equalities produce the basis-independent results

$$\mathcal{B}_{ab}^{(VS)} := \mathcal{B}_{ab}^{(VS)} - \frac{1}{2} \varepsilon_{abcd} \mathcal{B}_{cd}^{(VS)} = 0 \quad \text{and} \quad \mathcal{F}_{ab}^{(VS)} := \mathcal{F}_{ab}^{(VS)} + \frac{1}{2} \varepsilon_{abcd} \mathcal{F}_{cd}^{(VS)} = 0,$$  (3.22)

which are a consequence of the fact that the vector superfield in the Wess-Zumino gauge is annihilated by quasi-projection operators [34, 38]

$$[D_a D_b] - \frac{1}{2} \varepsilon_{abcd} D_{[c} D_{d]},$$  (3.23)

which has the relative sign opposite to the chiral supermultiplet result (3.10). Equivalently, the Adinkra (3.31) exhibits closed (red-green-blue-orange) 4-color cycles with $CP(3.15) = -1 = \chi_o$ [38, 28]. That (3.23) has the relative sign opposite to that one in (3.10) follows from the following facts: (1) The vector superfield in the Wess-Zumino gauge is the complement within a real intact
superfield \( U \) of the gauged-away formal imaginary part of a chiral superfield. (2) Cycle parity is additive: \( CP(U) = 0, CP(A) = +1 = CP(A^\dagger) \), and \( CP(U/\Im(A)) = 0 - 1 = -1 \).

The general result (2.10) of course again holds, so that \( \mathcal{R}^{(VS)}_{ab} \) and \( \mathcal{F}^{(VS)}_{ab} \) generate rotations in the \( \mathbb{R}^4 \)-like sectors of the field-space, respectively \( (A_1, A_2, A_3, d) \) and \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \). The minimality relations (3.22)–(3.23) reduce these rotations

\[
\begin{align*}
\text{spin}(4)_B & \xrightarrow{(3.22)} \text{spin}(3)^+ \\
\text{spin}(4)_F & \xrightarrow{(3.22)} \text{spin}(3)_- 
\end{align*}
\]

\( \subset \text{spin}(3)^- \oplus \text{spin}(3)^+ = \text{spin}(4) \) (3.24)

to complementary parts of the algebra of \( \text{Pin}(4) = \text{Aut}(\mathfrak{Sp}^{14}) \), but in a way opposite of that in the chiral supermultiplet; see (3.11).

Also unlike the case of the chiral supermultiplet, the relations (3.21) do not reduce the direct sum of these rotations from a full \( \text{spin}(4) \subset \text{spin}(3,3) \): the \( \mathcal{R}^{(VS)}_{ab} \) and \( \mathcal{F}^{(VS)}_{ab} \) tensors are each valued in a separate and mutually commuting \( \text{Spin}(3) \) subgroup of \( \text{Spin}(4) \). In particular, the mapping (2.26) locates:

\[
\begin{align*}
\mathcal{R}^{(VS)}_{ab+} & \eta \rightarrow \text{spin}(3)_{\text{rot}}, \quad \text{while} \quad \mathcal{F}^{(VS)}_{ab-} \eta \rightarrow \text{spin}(3)_{\text{exR}},
\end{align*}
\]

where \( \text{spin}(3)_{\text{rot}} \) is the rotational subgroup of the Lorentz group \( \text{Spin}(1,3) \), and \( \text{spin}(3)_{\text{exR}} \) is the “extended \( R \)-symmetry” algebra generated by \( \gamma^0, \gamma^{123} \) and \( \gamma^{0123} \). Referring again to the diagram (2.26), we summarize the results (3.24) and (3.25) as

\[
(\eta_B; \eta_F)^{\text{(VS)}} : \begin{cases} 
\mathcal{R}^{(VS)}_{ab} \in \text{spin}(3)^+ & \rightarrow \text{spin}(3)_{\text{rot}}, \\
\mathcal{F}^{(VS)}_{ab} \in \text{spin}(3)^- & \rightarrow \text{spin}(3)_{\text{exR}}.
\end{cases}
\]

Since \( \mathcal{R}_{ab+} \) and \( \mathcal{F}_{ab-} \) map to mutually commuting factors of the reference algebra \( \text{spin}(3,3) \), there is no relative permutation—unlike the case of the chiral supermultiplet (3.13).

### 3.3 Complex Structure vs. Absence Thereof

This comparison between the chiral and the vector supermultiplet is best highlighted by recalling that the chiral supermultiplet is complex, while the vector supermultiplet is inherently real.

To this end, note that the components (3.1) are also found in the superspace expansion of the (complex) chiral superfield \( \Phi \), which is defined to satisfy the super-differential constraints \( \overline{D}_a \Phi = 0 \). Since the Weyl spinor super-derivatives \( \overline{D}_a \) may be identified with the Majorana super-derivative expressions \( \frac{1}{2}(\mathbb{1} + \gamma^5)D_a \) \[13, 14, 36, 15\], the lowest components of the condition \( \overline{D}_a \Phi = 0 \) translate into

\[
\left( \frac{1}{2}(\mathbb{1} + \gamma^5)D_a \right) (A + iB) = \frac{1}{2}\left[ \delta_{a}^{\,b} + i(\gamma^{0123})_{a}^{\,b} \right] D_b (A + iB) = 0.
\]

Applying the conjugate super-derivatives \( \frac{1}{2}(\mathbb{1} - \gamma^5)D \) repeatedly on \( (A+iB) \) then generates the supersymmetric completion of this complex pairing throughout the rest of the supermultiplet:

\[
\left\{ (A+iB, F-iG| \psi_1-i\psi_4, \psi_2+i\psi_3) ; [D_1 - iD_4], [D_2 + iD_4] \right\},
\]

which is therefore a complex supermultiplet with respect to the complex supersymmetry action generated by \( \frac{1}{2}(\mathbb{1} - \gamma^5)D_a \simeq D_\alpha \) and \( [1 - \gamma^5]\psi \simeq \psi_\alpha \). Since \( \gamma^5 := i\gamma^{0123} \) is purely imaginary, it
defines the supersymmetric complex structure on all fermions, as indicated in (3.28). Also, the combination $\partial_\tau(F - iG)$ may be identified with the usually auxiliary component of $\Phi$. Thus, the complex structure of the whole supermultiplet is completely determined by the initial choices $A+iB$ and $D_\alpha = \frac{1}{2}([1-\gamma^5]D)_\alpha$: the super-differential system (3.1) then determines the rest of (3.28). In general, we define:

**Definition 3.1** A supermultiplet has a **supersymmetric complex structure** if the action of a fixed complex pairing of all super-derivatives (and supercharges) consistently combines all component (super)fields into complex pairs throughout the supermultiplet.

In any $\sigma$-model, the dynamical bosons provide local coordinates in the target manifold, while the dynamical fermions span the local tangent space. The simultaneous and coinciding (aligned) reduction $\text{spin}(4)_B \oplus \text{spin}(4)_F \rightarrow (\text{spin}(3) \approx \text{su}(2)) \subset \text{spin}(3,3)$ of these holoraumic rotations in both the fermionic (tangential) and the bosonic (coordinate) target-space of the chiral supermultiplet is consistent with the chiral supermultiplet admitting a supersymmetric complex structure (3.28).

This mirrors the maximal holonomy group of a (real $2n$-dimensional) complex Kähler $n$-fold being $\text{SU}(n) \subset \text{SO}(2n)$, rather than the full $\text{SO}(2n)$ of an orientable real $2n$-dimensional manifold: the existence of a complex structure reduces the holonomy group. Much the same, the existence of the supersymmetric complex structure (3.28) reduces the holoraumy group (2.12) from $\text{Spin}(4)$ to the $\text{Spin}(3)_\text{rot} \approx \text{SU}(2)_\text{rot}$ subgroup of field-space “rotations,” composed with $\tau$-translations.

The careful Reader will have noticed that it is the existence of a Kähler metric on a complex $n$-dimensional manifold that reduces the holonomy $\text{Spin}(2n) \rightarrow \text{U}(n)$, and the Ricci-flatness of such a Kähler metric that further reduces the holonomy $\text{U}(n) \rightarrow \text{SU}(n)$. The type and characteristics of the target-space metric is ultimately determined by the action of the $\sigma$-model built from this supermultiplet. The above considerations refer only to the supermultiplet itself, and to the local target space patch spanned by its components and equipped with the canonical flat supersymmetric metric; see appendix C for a proof and construction. As the flat metric is both Kähler and Ricci-flat, the holoraumy reduction

$$\left( \text{Spin}(2n)_B \times \text{Spin}_F(2n) \right) \circ \mathbb{R}^1_{\partial_\tau} \rightarrow \text{SU}(n) \circ \mathbb{R}^1_{\partial_\tau}$$

is indeed a consequence of only the complex structure. Generalizing this, we have:

**Conclusion 3.1** In a valise supermultiplet with $2n$ real bosons and $2n$ real fermions, an isomorphism of the bosonic and the fermionic holoraumy groups to the same $\text{SU}(n)$ subgroup within any reference framework implies the existence of a supersymmetric complex structure in the entire supermultiplet.

Compare now the corresponding characteristics of the chiral and vector supermultiplet:

**Chiral supermultiplet:** The valise version (3.1) of this supermultiplet has a $(\mathbb{C}^2_B|\mathbb{C}^2_F)$-like field-space, spanned by the complex components indicated in (3.28).
1. By virtue of the supermultiplet minimality relations (3.9), $\mathcal{B}_{ab}(CS)$- and $\mathcal{F}_{ab}(CS)$-rotations both span SU(2) \approx Spin(3) subgroups of the maximal field-space rotations.

2. The images of these two a priori separate Spin(3)-groups in Spin(3, 3) coincide, and span the same SU(2)$_{rot}$ \approx Spin(3)$_{rot}$, which is the common subgroup of the Lorentz group Spin(1, 3) and the maximal connected component Spin(4) \subset Aut(\mathfrak{Sp}^{1|4})

3. This coincidence implies that the complex structures on $\mathbb{C}^2_B$ and $\mathbb{C}^2_F$ are related by supersymmetry, and provide a supersymmetric complex structure. Indeed, the two complex structures are generated by supersymmetry from $(A, B) \rightarrow (A+iB)$.

**Vector supermultiplet** (in the Wess-Zumino gauge): The valise version (3.14) of this supermultiplet has an $(\mathbb{R}^4|\mathbb{R}^4)$-like field-space, spanned by $(A_1, A_2, A_3, d|\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

1. By virtue of the supermultiplet minimality relations (3.22), $\mathcal{B}_{ab}(VS)$- and $\mathcal{F}_{ab}(VS)$-rotations both span SU(2) \approx Spin(3) subgroups of the maximal field-space rotations.

2. The images of these two a priori separate Spin(3)-groups in Spin(3, 3) mutually commute, and jointly span its maximal compact subgroup Spin(3)$_{rot}$ \times Spin(3)$_{exR}$, isomorphic to Spin(4), the maximal connected component of Aut(\mathfrak{Sp}^{1|4}), the outer group of automorphisms of the supersymmetry algebra (1.1).

3. The perfect mis-alignment and mutual commutation of $\eta(\mathcal{B}_{ab}(VS))$- and $\eta(\mathcal{F}_{ab}(VS))$-transformations within the reference Spin(3, 3) correlates with the absence of a supersymmetric complex structure in the vector supermultiplet.

That is, although the 4-plets $(A_1, A_2, A_3, d)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(D_1, D_2, D_3, D_4)$ separately can be combined into complex pairs, no such choice gives the field-space the $(\mathbb{C}^2_B|\mathbb{C}^2_F)$ structure compatible with the supersymmetry (3.14); see also Eqs. (B.1)-(3.49) below.

### 3.4 The Tensor Supermultiplet

The 3+1-dimensional tensor supermultiplet, again in the Wess-Zumino gauge, our Majorana (real component) framework and dimensionally reduced to the coordinate time world-line, is specified by the super-differential system [28]:

\[
\begin{align*}
D_a \varphi &= \chi_a, \\
D_a B_{mn} &= -\frac{1}{2} (\gamma_{mn})^a_b \chi_b, & m, n = 1, 2, 3, \\
D_a \chi_b &= i (\gamma^0)_{ab} \partial_\tau \varphi - (\gamma^5 \gamma^m)_{ab} \varepsilon_m \partial_\tau B_{rs}.
\end{align*}
\]  

$(3.30a)$ $(3.30b)$

Using the matrices in Tables 3 and 4, we tabulate these results:

| TS | $\varphi$ | $2B_{12}$ | $2B_{23}$ | $2B_{31}$ | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ |
|----|----------|----------|----------|----------|----------|----------|----------|----------|
| $D_1$ | $\chi_1$ | $-\chi_3$ | $-\chi_4$ | $-\chi_2$ | $i \partial_\tau \varphi$ | $-2i \partial_\tau B_{31}$ | $-2i \partial_\tau B_{12}$ | $-2i \partial_\tau B_{23}$ |
| $D_2$ | $\chi_2$ | $\chi_4$ | $-\chi_3$ | $\chi_1$ | $2i \partial_\tau B_{31}$ | $i \partial_\tau \varphi$ | $-2i \partial_\tau B_{23}$ | $2i \partial_\tau B_{12}$ |
| $D_3$ | $\chi_3$ | $\chi_1$ | $\chi_2$ | $-\chi_4$ | $2i \partial_\tau B_{12}$ | $2i \partial_\tau B_{23}$ | $i \partial_\tau \varphi$ | $-2i \partial_\tau B_{31}$ |
| $D_4$ | $\chi_4$ | $-\chi_2$ | $\chi_1$ | $\chi_3$ | $2i \partial_\tau B_{23}$ | $-2i \partial_\tau B_{12}$ | $2i \partial_\tau B_{31}$ | $i \partial_\tau \varphi$ |

$(3.30c)$
and verify that the transformations are again monomial, whereby the supermultiplet can be depicted by the Adinkra:

![Adinkra Diagram](image)

(3.31)

and we read off the $L$-matrices:

$$L^{TS}_{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L^{TS}_{2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad L^{TS}_{3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad L^{TS}_{4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3.32)$$

With the supermultiplet now given both in the tensorial representation (3.30) and in terms of its $L$-matrices (3.32), we can compute straightforwardly the quadratic holoraumy tensors both ways, for a comparative illustration.

**Tensorial Computation:** Iterating (3.14), we compute straightforwardly:

$$D_{[a}D_{b]} \lambda_c = \frac{1}{2} [D_a, D_b] \lambda_c = -i (F_{TS})^{(ab)}_{cd} ( \partial_c \lambda_d ), \quad (3.33)$$

where the quadratic fermionic holoraumy tensor $(F_{ab})_{cd}$ may be written, utilizing again a series of Fierz identities, as:

$$(F_{ab})^{(TS)}_{cd} = -C_{ab} (\gamma^0)_{cd} + (\gamma^5)_{ab} (\gamma^5 \gamma^0)_{cd} - (\gamma^5 \gamma^0)_{ab} (\gamma^5)_{cd}. \quad (3.34)$$

As in the case of the chiral and the vector supermultiplet, the fermionic quadratic holoraumy tensor $F_{ab}^{(TS)}$ is again a Spin(3)-valued 2-form in the 4-component space of fermions. Just as for the vector supermultiplet, $\{F_{ab}^{(TS)}\} \in \text{spin}(3)_{\text{ext}}$. Similarly computed, the bosonic holoraumy tensor is now:

$$(B_{ab}^{(TS)})_{ij} = -i (\gamma^5 \gamma^1)_{ab} (\gamma^{12})_{ij} - i (\gamma^5 \gamma^2)_{ab} (\gamma^{23})_{ij} + i (\gamma^5 \gamma^3)_{ab} (\gamma^{13})_{ij}. \quad (3.35)$$

To summarize,

$$B_{ab}^{(TS)} \eta \rightarrow \text{spin}(3)_{\text{rot}}, \quad \text{and} \quad F_{ab}^{(TS)} \eta \rightarrow \text{spin}(3)_{\text{ext}}, \quad (3.36)$$

with the concrete results (3.34) and (3.35) again specifying the details of these maps. The algebraic structure of these results is shown below to again be encoded just as well in the purely world-line description of the 0-brane dimensional reduction (3.31)–(3.32) of this supermultiplet.

**Matrix Computation:** Given the explicit $L$-matrices (3.32), straightforward matrix algebra produces the quadratic holonomy matrices:

$$B_{12}^{(TS)} = +B_{34}^{(TS)} = +\gamma^{23}, \quad B_{23}^{(TS)} = +B_{14}^{(TS)} = +\gamma^{12}, \quad B_{31}^{(TS)} = +B_{24}^{(TS)} = +\gamma^{13}, \quad (3.37a)$$

$$F_{12}^{(TS)} = -F_{34}^{(TS)} = +\gamma^{0}, \quad F_{23}^{(TS)} = -F_{14}^{(TS)} = -\gamma^{0123}, \quad F_{31}^{(TS)} = -F_{24}^{(TS)} = -\gamma^{123}, \quad (3.37b)$$

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where we have again identified the results in terms of the reference matrices as given in Table 3.

The first of these equalities produce the basis-independent results
\[ B_{ab-}^{(TS)} := B_{ab}^{(TS)} - \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} B_{cd}^{(TS)} = 0 \quad \text{and} \quad B_{ab+}^{(TS)} := B_{ab}^{(TS)} + \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} B_{cd}^{(TS)} = 0, \]  
(3.38)
which are a consequence of the fact that the tensor superfield in the Wess-Zumino gauge is annihilated by quasi-projection operators \[ [D_{[a} D_{b]} - \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} D_{[c} D_{d]}], \]  
(3.39)
where the relative sign is opposite from the one in (3.10) and same as the one in (3.23). Equivalently, the Adinkra (3.31) exhibits closed (red-green-blue-orange) 4-color cycles with \( CP(3.31) = -1 = \chi_0 \) [38, 28].

Again, \( B_{ab}^{(TS)} \) and \( F_{ab}^{(TS)} \) generate rotations in the \( \mathbb{R}^4 \)-like sectors of the field-space, respectively (\( \varphi, 2B_{12}, 2B_{23}, 2B_{31} \)) and (\( \chi_1, \chi_2, \chi_3, \chi_4 \)). The minimality relations (3.38) again reduce these rotations
\[ \begin{array}{c}
\text{spin}(4)_B \xrightarrow{(3.38)} \text{spin}(3)_+ \\
\text{spin}(4)_F \xrightarrow{(3.38)} \text{spin}(3)_-
\end{array} \]  
\( \subset \text{spin}(3)_- \oplus \text{spin}(3)_+ = \text{spin}(4) \)  
(3.40)
Unlike the case of the chiral supermultiplet and just as in the vector supermultiplet, the relations (3.38) do not reduce these rotations from \( \text{Spin}(4) \): the \( B_{ab}^{(TS)} \) and \( F_{ab}^{(TS)} \) tensors are each valued in a separate and mutually commuting \( \text{Spin}(3) \) subgroup of \( \text{Spin}(4) \). In particular,
\[ B_{ab+}^{(TS)} \xrightarrow{\eta} \text{spin}(3)_\text{rot}, \quad \text{while} \quad F_{ab-}^{(TS)} \xrightarrow{\eta} \text{spin}(3)_\text{exR}, \]  
(3.41)
and referring again to the diagram (2.26), we summarize the results (3.40) and (3.41) as
\[ \{ \eta_B, \eta_F \}^{(TS)} : \quad \begin{array}{c}
B_{ab}^{(TS)} \in \text{spin}(3)_+ \xrightarrow{1} \text{spin}(3)_\text{rot}, \\
F_{ab}^{(TS)} \in \text{spin}(3)_- \xrightarrow{1} \text{spin}(3)_\text{exR},
\end{array} \]  
(3.42)
precisely as is the case (3.26) with the vector supermultiplet.

Motivated by this holoraumy isomorphism, \( \{ \eta_B, \eta_F \}^{(TS)} \simeq \{ \eta_B, \eta_F \}^{(VS)} \), we find the component field identification
\[ \{ \varphi, 2B_{12}, 2B_{23}, 2B_{31} | \chi_1, \chi_2, \chi_3, \chi_4 \} \cong (A_1, A_2, A_3, d, -A_3 | \lambda_2, \lambda_1, \lambda_4, \lambda_3), \]  
(3.43a)
which may be written, in terms of (2.30), as
\[ \begin{bmatrix}
\varphi \\
2B_{12} \\
2B_{23} \\
2B_{31}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} \begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
d
\end{bmatrix}, \quad \begin{bmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\chi_4
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{bmatrix}, \quad \begin{bmatrix}
\det[X^{(TS)}_{(VS)}] = +1, \\
\det[Y^{(TS)}_{(VS)}] = +1
\end{bmatrix}, \]  
(3.43b)
Since \( d = \int d\tau \, \mathcal{D} \) is a non-local redefinition of the auxiliary component \( \mathcal{D} \) in the vector supermultiplet, the component field identification (3.43) is a non-local relationship between the world-line dimensional reductions of the tensor and the vector supermultiplets, (3.30) and (4.8), respectively.
3.5 The Twisted-Chiral Supermultiplet Valise

The twisted-chiral supermultiplet was first constructed in Ref. [35], by dimensionally reducing the vector supermultiplet to 1+1-dimensional spacetime. For completeness, this derivation is re-traced in Appendix B, dimensionally reducing however straight to 1-dimensional the world-line. Here, we cite the end-result:

\[
\begin{array}{c|cccc|cccc}
\text{vtCS} & \tilde{A} & \tilde{B} & \tilde{F} & \tilde{G} & \tilde{\psi}_1 & \tilde{\psi}_2 & \tilde{\psi}_3 & \tilde{\psi}_4 \\
D_1 & \tilde{\psi}_1 & -\tilde{\psi}_1 & \tilde{\psi}_2 & \tilde{\psi}_3 & i\partial_\tau \tilde{A} & i\partial_\tau \tilde{F} & i\partial_\tau \tilde{G} & -i\partial_\tau \tilde{B} \\
D_2 & \tilde{\psi}_2 & -\tilde{\psi}_3 & -\tilde{\psi}_1 & -\tilde{\psi}_4 & -i\partial_\tau \tilde{F} & i\partial_\tau \tilde{A} & -i\partial_\tau \tilde{B} & -i\partial_\tau \tilde{G} \\
D_3 & \tilde{\psi}_3 & \tilde{\psi}_2 & -\tilde{\psi}_4 & -\tilde{\psi}_1 & -i\partial_\tau \tilde{G} & i\partial_\tau \tilde{B} & i\partial_\tau \tilde{A} & i\partial_\tau \tilde{F} \\
D_4 & \tilde{\psi}_4 & \tilde{\psi}_1 & -\tilde{\psi}_3 & \tilde{\psi}_2 & i\partial_\tau \tilde{B} & i\partial_\tau \tilde{G} & -i\partial_\tau \tilde{F} & i\partial_\tau \tilde{A} \\
\end{array}
\]

and in tensorial notation:

\[
\begin{align*}
D_a \tilde{A} &= \tilde{\psi}_a, & D_a \tilde{B} &= - (\gamma_{23})_a^b \tilde{\psi}_b, & D_a \tilde{F} &= - (\gamma_{13})_a^b \tilde{\psi}_b, & D_a \tilde{G} &= (\gamma_{12})_a^b \tilde{\psi}_b, \\
D_a \tilde{\psi}_b &= i (\gamma^0)_{ab} (\partial_\tau \tilde{A}) - i (\gamma^{023})_{ab} (\partial_\tau \tilde{B}) - i (\gamma^{013})_{ab} (\partial_\tau \tilde{F}) + i (\gamma^{012})_{ab} (\partial_\tau \tilde{G}).
\end{align*}
\]

The transformation rules (3.44) specify the world-line valise twisted-chiral supermultiplet that differs from the original definition [35] only through the component (super)field redefinitions (B.1) and (B.3) detailed in the Appendix B, both of which have det = +1.

For a comparison with (3.1d) however, we perform one additional redefinition:

\[
\tilde{\psi}_3 \rightarrow \tilde{\psi}_3 := -\tilde{\psi}_3,
\]

and obtain:

\[
\begin{array}{c|cccc|cccc}
\text{vtCS} & \tilde{A} & \tilde{B} & \tilde{F} & \tilde{G} & \tilde{\psi}_1 & \tilde{\psi}_2 & \tilde{\psi}_3 & \tilde{\psi}_4 \\
D_1 & \tilde{\psi}_1 & -\tilde{\psi}_1 & \tilde{\psi}_2 & \tilde{\psi}_3 & i\partial_\tau \tilde{A} & i\partial_\tau \tilde{F} & -i\partial_\tau \tilde{G} & -i\partial_\tau \tilde{B} \\
D_2 & \tilde{\psi}_2 & -\tilde{\psi}_3 & -\tilde{\psi}_1 & -\tilde{\psi}_4 & -i\partial_\tau \tilde{F} & i\partial_\tau \tilde{A} & i\partial_\tau \tilde{B} & -i\partial_\tau \tilde{G} \\
D_3 & -\tilde{\psi}_3 & \tilde{\psi}_2 & -\tilde{\psi}_4 & -\tilde{\psi}_1 & -i\partial_\tau \tilde{G} & i\partial_\tau \tilde{B} & -i\partial_\tau \tilde{A} & i\partial_\tau \tilde{F} \\
D_4 & \tilde{\psi}_4 & \tilde{\psi}_1 & -\tilde{\psi}_3 & \tilde{\psi}_2 & i\partial_\tau \tilde{B} & i\partial_\tau \tilde{G} & i\partial_\tau \tilde{F} & i\partial_\tau \tilde{A} \\
\end{array}
\]

which now differs from the valise version of the world-line dimensionally reduced chiral supermultiplet supersymmetry transformation pattern in (3.1d) in—and only in—the signs of each resulting term in the D_3-row, and so is the world-line twisted-chiral supermultiplet as defined in Ref. [21, 22, 32].

Quite clearly, the “one additional redefinition” of the fermions (3.45) has a negative determinant, and so does not belong to the connected component Spin(4) ⊂ Aut(osp^{1|4}), but its negative-determinant complement within the full Aut(osp^{1|4}) = Pin(4). It should be noted that the classification of world-line supermultiplets in Refs. [39, 40, 41, 42, 18, 11, 43, 21, 19, 34, 44, 22, 17] generally proceeds up to general component field redefinitions—which then include negative-determinant component field transformations such as (3.45). While perfectly reasonable within the framework of purely world-line physics, we will show that it behooves us to trace these characteristics when inquiring whether a given world-line supermultiplet is the dimensional reduction of a supermultiplet from higher-dimensional spacetime.
The tabular representations of the super-differential relations (3.44) and (3.46) make it clear that both versions of this system are also monomial, and may be depicted, respectively, by the Adinkras:

the only difference between them being dashedness of the edges adjacent to $\tilde{\psi}_3$, i.e., $\tilde{\psi}_3$, as implied by the “one additional redefinition” (3.45). The corresponding monomial $\mathbb{L}$-matrices, as defined in (1.2), are:

$$
\mathbb{L}^\text{CS}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbb{L}^\text{CS}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbb{L}^\text{CS}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{L}^\text{CS}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
$$

for (3.44), whereas (3.46) produces:

$$
\mathbb{L}^\text{CS}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbb{L}^\text{CS}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbb{L}^\text{CS}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{L}^\text{CS}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
$$

**Complex Structures:** Given the almost perfect identity between the super-differential system (4.3) and (3.46), where $D_3^{(\text{CS})} = -D_3^{(\text{cs})}$ being the only difference, we readily conclude that

$$
\text{vtCS}: \quad (\tilde{A}+i\tilde{B}), (\tilde{F}-i\tilde{G}) | (\tilde{\psi}_1-i\tilde{\psi}_4) \ (\tilde{\psi}_2-i\tilde{\psi}_3) \ \text{and} \ \mathbb{D}_1 - i\mathbb{D}_4, \mathbb{D}_2 - i\mathbb{D}_3), \quad (3.49a)
$$

$$
\text{vtCS}: \quad (\tilde{A}+i\tilde{B}), (\tilde{F}-i\tilde{G}) | (\tilde{\psi}_1-i\tilde{\psi}_4) \ (\tilde{\psi}_2+i\tilde{\psi}_3) \ \text{and} \ \mathbb{D}_1 - i\mathbb{D}_4, \mathbb{D}_2 - i\mathbb{D}_3), \quad (3.49b)
$$

form two versions of the complex valise world-line twisted-chiral supermultiplets (3.44) and (3.46).

The geometric meaning of this is as follows:

1. As in the complex chiral supermultiplet (3.28), so also in both versions of the twisted-chiral supermultiplet (3.49a) and (3.49b), the four bosons span a $\mathbb{C}^2$-like target-space in like complex combinations: $(A+iB), (F-iG)$ and $(\tilde{A}+i\tilde{B}), (\tilde{F}-i\tilde{G})$.

2. Both in the complex chiral supermultiplet (3.28), and in the “redefined” [21, 22, 32] twisted-chiral supermultiplet (3.49b), the four fermions jointly span a $\mathbb{C}^2$-like tangent space, all four in like complex combinations: $(\psi_1-i\psi_4) \sim (\psi_1-i\tilde{\psi}_4)$, as well as $(\psi_2+i\psi_3) \sim (\psi_2+i\tilde{\psi}_3)$, respectively.

3. In the “original” [35] twisted-chiral supermultiplet (3.49a) however, the four fermions span a $\mathbb{C} \oplus \mathbb{C}^*$-like (holomorphic+antiholomorphic) tangent space in the opposite complex combinations: $(\psi_1-i\psi_4) \sim (\psi_1-i\tilde{\psi}_4)$, but $(\psi_2+i\psi_3) \sim (\psi_2-i\tilde{\psi}_3)$, respectively.
Indeed, the “one additional redefinition” \((3.45)\) serves to map the fermionic space \(\mathbb{C} \oplus \mathbb{C}^* \to \mathbb{C}^2\), and so “attune” the component field complex structures in the manner of the complex chiral supermultiplet.

Having obtained both the tensorial and the \(L\)-matrix representation \((3.44)\)--\((3.48)\), we proceed computing the quadratic holoraumy tensors.

**Tensorial Computation:** Iterating \((3.44b)\)--\((3.44c)\), we compute

\[
D[a]D[b] \tilde{\psi}_c = \frac{1}{2}[D_a, D_b] \tilde{\psi}_c = -i(\mathcal{F}^{(\text{tCS})})_c^d (\partial_d \tilde{\psi}_a),
\]

where

\[
(\mathcal{F}^{(\text{tCS})})_c^d = -C_{ab}(\gamma^0)_c^d + (\gamma^5)_{ab}(\gamma^5 \gamma^0)_c^d - (\gamma^5 \gamma^0)_{ab}(\gamma^5)_c^d.
\]

Similarly computed, the bosonic holoraumy tensor is:

\[
(\mathcal{B}^{(\text{tCS})})_i^j = +i(\gamma^5 \gamma^1)_{ab}(\gamma^3)_i^j + i(\gamma^5 \gamma^2)_{ab}(\gamma^2)_i^j + i(\gamma^5 \gamma^3)_{ab}(\gamma^3)_i^j.
\]

To summarize,

\[
\mathcal{F}^{(\text{tCS})} \xrightarrow{\eta} \text{spin}(3)_{\text{rot}}, \quad \text{and} \quad \mathcal{F}^{(\text{tCS})} \xrightarrow{\eta} \text{spin}(3)_{\text{exR}},
\]

with the concrete results \((3.51)\) and \((3.52)\) again specifying the details of these maps. The algebraic structure of these results is shown below to again be encoded just as well in the purely world-line description of the 0-brain dimensional reduction \((3.47)\)--\((3.48a)\) of this supermultiplet.

**Matrix Computation:** Given the \(L\)-matrices \((3.48a)\), straightforward matrix algebra produces:

\[
\begin{align*}
\mathcal{B}_{12}^{(\text{tCS})} &= +\mathcal{B}_{34}^{(\text{tCS})} = -\gamma^{12}, & \mathcal{B}_{23}^{(\text{tCS})} &= +\mathcal{B}_{14}^{(\text{tCS})} = -\gamma^{13}, & \mathcal{B}_{31}^{(\text{tCS})} &= +\mathcal{B}_{24}^{(\text{tCS})} = +\gamma^{23}, \\
\mathcal{F}_{12}^{(\text{tCS})} &= -\mathcal{F}_{34}^{(\text{tCS})} = +\gamma^0, & \mathcal{F}_{23}^{(\text{tCS})} &= -\mathcal{F}_{14}^{(\text{tCS})} = +\gamma^{0123}, & \mathcal{F}_{31}^{(\text{tCS})} &= -\mathcal{F}_{24}^{(\text{tCS})} = -\gamma^{123},
\end{align*}
\]

where we have again identified the results in terms of the reference matrices as given in Table 3.

The first of these equalities produce the basis-independent results

\[
\mathcal{B}_{ab}^{(\text{tCS})} := \mathcal{B}_{ab}^{(\text{tCS})} - \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} \mathcal{B}_{cd}^{(\text{tCS})} = 0 \quad \text{and} \quad \mathcal{F}_{ab}^{(\text{tCS})} := \mathcal{F}_{ab}^{(\text{tCS})} + \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} \mathcal{F}_{cd}^{(\text{tCS})} = 0,
\]

which are a consequence of the fact that the twisted-chiral supermultiplet is annihilated by quasi-projection operators \([34, 38]\)

\[
[D[a]D[b] - \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} D[c]D[d]],
\]

where the relative sign is opposite from the one in \((3.10)\) and same as the one in \((3.39)\), as well as the one in \((3.23)\) from which \((3.44a)\) was derived. Equivalently, the Adinkras \((3.47)\) exhibit closed \((\text{red-green-blue-orange}) 4\)-color cycles with \(CP(3.47) = -1 = \chi_o [38, 28]\).

Again, \(\mathcal{B}^{(\text{tCS})}\) and \(\mathcal{F}^{(\text{tCS})}\) generate rotations in the \(\mathbb{R}^4\)-like sectors of the field-space, respectively \((\tilde{A}, \tilde{B}, \tilde{F}, \tilde{B})\) and \((\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4)\). The minimality relations \((3.55)\) again reduce these rotations

\[
\begin{align*}
\text{spin}(4)_B & \xrightarrow{(3.55)} \text{spin}(3)_+ \quad \text{and} \quad \text{spin}(4)_F \xrightarrow{(3.55)} \text{spin}(3)_-
\end{align*}
\]

\[
\supset \text{spin}(3)_- \oplus \text{spin}(3)_+ = \text{spin}(4)
\]

\(25\)
Unlike the case of the chiral supermultiplet and just as in the vector supermultiplet, the relations (3.55) do not reduce the direct sum of these rotations from Spin(4): the $\mathcal{B}_{ab}^{(\text{tCS})}$ and $\mathcal{F}_{ab}^{(\text{tCS})}$ tensors are each valued in a separate and mutually commuting Spin(3) subgroup of Spin(4). In particular,

$$\mathcal{B}_{ab+}^{(\text{tCS})} \xrightarrow{\eta} \mathfrak{spin}(3)_{\text{rot}}, \quad \text{while} \quad \mathcal{F}_{ab-}^{(\text{tCS})} \xrightarrow{\eta} \mathfrak{spin}(3)_{\text{exR}}, \quad (3.58)$$

and referring again to the diagram (2.26), we summarize the results (3.57) and (3.58) as

$$\left(\eta_B; \eta_F\right)^{(\text{tCS})} : \begin{cases} \mathcal{B}_{ab}^{(\text{tCS})} & \in \mathfrak{spin}(3)_+ \xrightarrow{1} \mathfrak{spin}(3)_{\text{rot}}, \\ \mathcal{F}_{ab}^{(\text{tCS})} & \in \mathfrak{spin}(3)_- \xrightarrow{\alpha} \mathfrak{spin}(3)_{\text{exR}}, \end{cases} \quad (3.59)$$

precisely as is the case (3.26) with the vector supermultiplet and (3.42) with the tensor supermultiplet.

However, using the $\mathbb{L}$-matrices (3.48b) produces instead:

$$\mathcal{B}_{12}^{(\text{tCS})} = +\mathcal{B}_{34}^{(\text{tCS})} = -\gamma_{12}^2, \quad \mathcal{B}_{23}^{(\text{tCS})} = +\mathcal{B}_{14}^{(\text{tCS})} = -\gamma_{13}^3, \quad \mathcal{B}_{31}^{(\text{tCS})} = +\mathcal{B}_{24}^{(\text{tCS})} = +\gamma_{23}^3,$$

$$\mathcal{F}_{12}^{(\text{tCS})} = -\mathcal{F}_{34}^{(\text{tCS})} = -\gamma_{13}^3, \quad \mathcal{F}_{23}^{(\text{tCS})} = -\mathcal{F}_{14}^{(\text{tCS})} = -\gamma_{23}^3, \quad \mathcal{F}_{31}^{(\text{tCS})} = -\mathcal{F}_{24}^{(\text{tCS})} = +\gamma_{12}^3. \quad (3.60a)$$

The first of these equalities produce the basis-independent results

$$\mathcal{B}_{ab-}^{(\text{tCS})} := \mathcal{B}_{ab}^{(\text{tCS})} - \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} \mathcal{D}_{cd}^{(\text{tCS})} = 0 \quad \text{and} \quad \mathcal{F}_{ab+}^{(\text{tCS})} := \mathcal{F}_{ab}^{(\text{tCS})} + \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} \mathcal{F}_{cd}^{(\text{tCS})} = 0, \quad (3.61)$$

just like (3.61), since (3.44a) and (3.46) differ only by a component field redefinition (3.45), and so are both annihilated by the same quasi-projection operators [34, 38]

$$\left[ D_{[a} D_{b]} - \frac{1}{2} \varepsilon_{ab}^{\phantom{ab}cd} D_{[c} D_{d]} \right]. \quad (3.62)$$

The relative sign is opposite from the one in (3.10), and same as the one in (3.39), (3.23) and (3.44a), thus implying $CP(3.47) = -1 = \chi_o [38, 28]$.

Again, $\mathcal{B}_{ab}^{(\text{tCS})}$ and $\mathcal{F}_{ab}^{(\text{tCS})}$ generate rotations in the $\mathbb{R}^4$-like sectors of the field-space, respectively ($\tilde{A}, \tilde{B}, \tilde{F}, \tilde{\bar{B}}$) and ($\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4$). The relations (3.61) reduce these rotations

$$\begin{align*}
\mathfrak{spin}(4)_B & \xrightarrow{(3.61)} \mathfrak{spin}(3)_+ \\
\mathfrak{spin}(4)_F & \xrightarrow{(3.61)} \mathfrak{spin}(3)_- \\
\end{align*} \subset \mathfrak{spin}(3)_- \oplus \mathfrak{spin}(3)_+ = \mathfrak{spin}(4). \quad (3.63)$$

Unlike the case of the vector, tensor and “original” twisted-chiral supermultiplet and just as in the case of the chiral supermultiplet, the relations (3.60) do reduce these rotations from Spin(4): the $\mathcal{B}_{ab}^{(\text{tCS})}$ and $\mathcal{F}_{ab}^{(\text{tCS})}$ tensors are both valued in the same Spin(3) subgroup of Spin(4). In particular,

$$\mathcal{B}_{ab+}^{(\text{tCS})} \xrightarrow{\eta} \mathfrak{spin}(3)_{\text{rot}}, \quad \text{while} \quad \mathcal{F}_{ab-}^{(\text{tCS})} \xrightarrow{\eta} \mathfrak{spin}(3)_{\text{rot}}, \quad (3.64)$$

and referring again to the diagram (2.26), we summarize the results (3.63) and (3.64) as

$$\left(\eta_B; \eta_F\right)^{(\text{tCS})} : \begin{cases} \mathcal{B}_{ab}^{(\text{tCS})} & \in \mathfrak{spin}(3)_+ \xrightarrow{1} \mathfrak{spin}(3)_{\text{rot}}, \\ \mathcal{F}_{ab}^{(\text{tCS})} & \in \mathfrak{spin}(3)_- \xrightarrow{\alpha} \mathfrak{spin}(3)_{\text{rot}}, \end{cases} \quad (3.65)$$
just as in the case (3.13) of the chiral supermultiplet, but with the roles of spin(3)_− and spin(3)_+ swapped, and where \( \varpi' = \left( \frac{12, 23, 31}{23, 31, 12} \right) \) is the even (det = +1) relative signed permutation

\[
\varpi' : \eta\{ \mathcal{F}_{12}^{(tCS)}, \mathcal{F}_{23}^{(tCS)}, \mathcal{F}_{31}^{(tCS)} \} = \eta\{ \mathcal{D}_{23}^{(tCS)}, -\mathcal{D}_{31}^{(tCS)}, -\mathcal{D}_{12}^{(tCS)} \} \supset \text{spin}(3)_{\text{rot}}, \tag{3.66}
\]
relating the images of the fermionic and bosonic holonomy in the reference algebra \( \text{spin}(3, 3) \). This permutation is the same as (3.12), up to an even number of sign-changes in the generators, which is an outer automorphism of all \( \text{spin}(3) \) algebras.

4 Non-Valise Supermultiplets

The straightforward dimensional reduction of higher-dimensional supermultiplets to the coordinate time world-line in fact contains more information than can be discerned from the valise supermultiplets discussed in Section 3. Those particular example valises have been obtained upon the following non-local field redefinitions:

\[
\text{chiral (3.1)} : \ (A, B|\psi_a|\mathcal{F}, \mathcal{G}) \rightarrow (A, B, F, G|\psi_a), \quad (F, G) := (\int d\tau \mathcal{F}, \int d\tau \mathcal{G}); \tag{4.1a}
\]

\[
\text{vector (3.14)} : \ (A_1, A_2, A_3|\lambda_a|\mathcal{D}) \rightarrow (A_1, A_2, A_3, d|\psi_a), \quad d := \int d\tau \mathcal{D}; \tag{4.1b}
\]

\[
\text{twisted-chiral} (3.44) \& (B.3) : \ (A_1, A_2, A_3|\lambda|\mathcal{D}) \rightarrow (\tilde{A}, \tilde{B}, \tilde{F}, \tilde{G}|\tilde{\psi}_a), \quad \tilde{F} := -\int d\tau \mathcal{D}. \tag{4.1c}
\]

In the supermultiplets displayed in the left-hand side of the field redefinition maps (4.1), the action of \( D_{[I}D_{J]} \) still produces a spin-preserving transformation on the fields, but does not always reduce to a uniform factorization of a field-space rotation and domain-space (time) translation. To indicate this distinction, we will label the resulting operators:

\[
D_{[I}D_{J]}(\text{bosons}) =: -i\tilde{\mathcal{D}}_{IJ}(\text{bosons}) \quad \text{and} \quad D_{[I}D_{J]}(\text{fermions}) =: -i\tilde{\mathcal{F}}_{IJ}(\text{fermions}), \tag{4.2}
\]
and examine these results in turn.

4.1 Chiral supermultiplet

Restoring the original component superfields \( \mathcal{F} = \partial_r F \) and \( \mathcal{G} = \partial_r G \) and drawing their nodes above the fermionic ones to indicate their relative engineering dimensions, \( [\mathcal{F}] = [\mathcal{G}] = [\psi_a] + \frac{1}{2} = [A] + 1 = [B] + 1 \), the system (3.1d) becomes:

| CS | \( A \) | \( B \) | \( \mathcal{F} \) | \( \mathcal{G} \) | \( \psi_1 \) | \( \psi_2 \) | \( \psi_3 \) | \( \psi_4 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( D_1 \) | \( \psi_1 \) | \( -\psi_4 \) | \( \partial_r \psi_2 \) | \( -\partial_r \psi_3 \) | \( i\partial_r A \) | \( i\mathcal{F} \) | \( -i\mathcal{G} \) | \( -i\partial_r B \) |
| \( D_2 \) | \( \psi_2 \) | \( \psi_3 \) | \( -\partial_r \psi_1 \) | \( -\partial_r \psi_4 \) | \( -i\mathcal{F} \) | \( i\partial_r A \) | \( i\partial_r B \) | \( -i\mathcal{G} \) |
| \( D_3 \) | \( \psi_3 \) | \( -\psi_2 \) | \( \partial_r \psi_4 \) | \( \partial_r \psi_1 \) | \( i\mathcal{G} \) | \( -i\partial_r B \) | \( i\partial_r A \) | \( -i\mathcal{F} \) |
| \( D_4 \) | \( \psi_4 \) | \( \psi_1 \) | \( \partial_r \psi_3 \) | \( \partial_r \psi_2 \) | \( i\partial_r B \) | \( i\mathcal{G} \) | \( i\mathcal{F} \) | \( i\partial_r A \) |

showing the complex combinations

\[
\left( (A+iB) | (\psi_1-i\psi_4), (\psi_2+i\psi_3) | (\mathcal{F}-i\mathcal{G}) : [D_1-iD_4, [D_2+iD_3]] \right) \tag{4.4}
\]
of our real components that are in precise correspondence with usual the complex and Weyl components \[13, 14, 15\] given the like complex combinations of the super-derivatives; see Eqs. (3.28), as induced by (3.27).

Two of the bosonic holoraumy generators \(\hat{\mathcal{B}}_{IJ}\) now become nontrivial compositions with world-line translations. In particular, we obtain for (4.3):

\[
\hat{\mathcal{B}}^{(\text{CS})}_{12} = -\hat{\mathcal{B}}^{(\text{CS})}_{34} = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\partial^2 & 0 & 0 & 0 \\
0 & -\partial^2 & 0 & 0
\end{bmatrix}, \quad \hat{\mathcal{B}}^{(\text{CS})}_{31} = -\hat{\mathcal{B}}^{(\text{CS})}_{24} = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & \partial^2 & 0 & 0 \\
0 & 0 & \partial^2 & 0
\end{bmatrix},
\]

(4.5a)

while

\[
\hat{\mathcal{B}}^{(\text{CS})}_{23} = -\hat{\mathcal{B}}^{(\text{CS})}_{14} = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix} \circ \partial \tau
\]

(4.5b)

remains a uniform composition of field-space (simultaneous, double) rotations and domain-space translations:

\[
\hat{\mathcal{B}}^{(\text{CS})}_{23}(A, B|F, G) = \partial \tau(-B, A| - G, F).
\]

(4.6)

In turn, all the generators \(\hat{\mathcal{F}}^{(\text{CS})}_{IJ} = \hat{\mathcal{F}}^{(\text{CS})}_{IJ} \circ \partial \tau\) remain a uniform composition. This “algebraic subset” of the holoraumy transformations then is generated by \(\hat{\mathcal{B}}^{(\text{CS})}_{23}\) and \(\hat{\mathcal{F}}^{(\text{CS})}_{IJ}\), and generates the group \(U(1)_B \times SU(2)_F\). This type of effective (dynamical) symmetry reduction, shown in the central column of the summary:

| Chiral Subset of Holoraumy | Table 3 |
|---------------------------|---------|
| original (4.3) | \(SO(4)_B \times SO(4)_F \rightarrow U(1)_B \times SU(2)_F \overset{\eta}{\rightarrow} SU(2)_{\text{rot}}\) |
| valise (3.1) | \(SO(4)_B \times SO(4)_F \rightarrow SU(2)_B \times SU(2)_F \overset{\eta}{\rightarrow} SU(2)_{\text{rot}}\) |

has been systematically explored for the \(N=8\) ultra-multiplet in Ref. [45], so that our present results extend that work both conceptually in Section 2, and also to \(N=4\) supermultiplets.

The generators (4.5a) can no longer form a matrix group in the usual sense with (4.5b), but they do form an \(\mathbb{R}[\partial \tau]\)-module. While this general discussion of holoraumy is well beyond our present scope, we note in passing that when acting on functions expanded in plane-waves, such operators become matrix functions of plane-wave frequencies, and so provide an algebraic (Fourier-dual) approach to these differential operators as bundles over the energy-momentum space.

### 4.2 Vector Supermultiplet

The coordinate time world-line dimensional reduction of the standard vector supermultiplet in the Wess-Zumino gauge is given by the super-differential system [28]:

\[
D_a A_m = (\gamma_m)^a_b \lambda_b, \quad m = 1, 2, 3, \quad \gamma_m = \eta_{mn} \gamma^n = \gamma^m,
\]

(4.8a)

\[
D_a \lambda_b = -i(\gamma^0 \gamma^a)_{ab} (\partial \tau A_m) + (\gamma^5)_{ab} D,
\]

(4.8b)

\[
D_a D = i(\gamma^5 \gamma^0)^a_b (\partial \tau \lambda_b).
\]

(4.8c)
Using the matrices in Tables 3 and 4, we tabulate these results:

| VS  | $A_1$ | $A_2$ | $A_3$ | $D$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|-----|-------|-------|-------|-----|-------------|-------------|-------------|-------------|
| $D_1$ | $\lambda_2$ | $-\lambda_4$ | $\lambda_1$ | $-\partial_r \lambda_3$ | $i\partial_r A_3$ | $i\partial_r A_1$ | $-iD$ | $-i\partial_r A_2$ |
| $D_2$ | $\lambda_1$ | $\lambda_3$ | $-\lambda_2$ | $-\partial_r \lambda_4$ | $i\partial_r A_1$ | $-i\partial_r A_3$ | $i\partial_r A_2$ | $-iD$ |
| $D_3$ | $\lambda_4$ | $\lambda_2$ | $\lambda_3$ | $\partial_r A_1$ | $iD$ | $i\partial_r A_2$ | $i\partial_r A_3$ | $i\partial_r A_1$ |
| $D_4$ | $\lambda_3$ | $-\lambda_1$ | $\lambda_4$ | $\partial_r \lambda_2$ | $-i\partial_r A_2$ | $iD$ | $i\partial_r A_1$ | $-i\partial_r A_3$ |

The holoraumy generators for the vector supermultiplet (in the Wess-Zumino) gauge (3.21a) then become:

$$
\mathcal{B}_{12}^{(VS)} + \mathcal{B}_{34}^{(VS)} = \begin{bmatrix}
0 & 0 & \partial_r & 0 \\
0 & 0 & 0 & -1 \\
-\partial_r & 0 & 0 & 0 \\
0 & \partial_r & 0 & 0
\end{bmatrix}, \quad \mathcal{B}_{31}^{(VS)} + \mathcal{B}_{24}^{(VS)} = \begin{bmatrix}
0 & -\partial_r & 0 & 0 \\
\partial_r & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & \partial_r & 0
\end{bmatrix}, \quad (4.9a)
$$

$$
\mathcal{B}_{23}^{(VS)} + \mathcal{B}_{14}^{(VS)} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & \partial_r & 0 \\
0 & -\partial_r & 0 & 0 \\
-\partial_r & 0 & 0 & 0
\end{bmatrix}, \quad (4.9b)
$$

none of which form a uniform composition of an algebraic matrix and $\partial_r$. In contrast, $\mathcal{F}_{IJ}^{(VS)}$ of course remain algebraic. This is then summarized as:

| vector | Algebraic Subset of Holoraumy | Table 3 |
|--------|-----------------------------|---------|
| original (4.8) | $SO(4)_B \times SO(4)_F \rightarrow 1_B \times SU(2)_F \xrightarrow{\eta} SU(2)_{exR}$ | (4.10) |
| valise (3.14) | $SO(4)_B \times SO(4)_F \rightarrow SU(2)_B \times SU(2)_F \xrightarrow{\eta} SU(2)_{rot} \times SU(2)_{exR}$ | |

4.3 Twisted-Chiral Supermultiplet

Restoring the original component superfields $\tilde{\Phi} = \partial_r \tilde{\Phi} = -D$ and $\tilde{\Theta} = \partial_r \tilde{\Theta} = \partial_r A_1$, the system (3.44a) becomes:

| tCS | $\tilde{A}$ | $\tilde{B}$ | $\tilde{\Phi}$ | $\tilde{\Theta}$ | $\tilde{\psi}_1$ | $\tilde{\psi}_2$ | $\tilde{\psi}_3$ | $\tilde{\psi}_4$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $D_1$ | $\tilde{\psi}_1$ | $-\tilde{\psi}_4$ | $\partial_r \tilde{\psi}_2$ | $\partial_r \tilde{\psi}_3$ | $i\partial_r \tilde{A}$ | $i\tilde{\Phi}$ | $i\tilde{\Theta}$ | $-i\partial_r \tilde{B}$ |
| $D_2$ | $\tilde{\psi}_2$ | $-\tilde{\psi}_3$ | $-\partial_r \tilde{\psi}_1$ | $-\partial_r \tilde{\psi}_4$ | $-i\tilde{\Phi}$ | $i\partial_r \tilde{A}$ | $-i\partial_r \tilde{B}$ | $-i\tilde{\Theta}$ |
| $D_3$ | $\tilde{\psi}_3$ | $\tilde{\psi}_2$ | $\partial_r \tilde{\psi}_4$ | $-\partial_r \tilde{\psi}_1$ | $-i\tilde{\Theta}$ | $i\partial_r \tilde{B}$ | $i\partial_r \tilde{A}$ | $i\tilde{\Phi}$ |
| $D_4$ | $\tilde{\psi}_4$ | $\tilde{\psi}_1$ | $-\partial_r \tilde{\psi}_3$ | $\partial_r \tilde{\psi}_2$ | $i\partial_r \tilde{B}$ | $i\tilde{\Theta}$ | $-i\tilde{\Phi}$ | $i\partial_r \tilde{A}$ |

whereas the (3.45)-“redefined” twisted-chiral supermultiplet is:

| tCS | $\hat{A}$ | $\hat{B}$ | $\hat{\Phi}$ | $\hat{\Theta}$ | $\hat{\psi}_1$ | $\hat{\psi}_2$ | $\hat{\psi}_3$ | $\hat{\psi}_4$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $D_1$ | $\hat{\psi}_1$ | $-\hat{\psi}_4$ | $\partial_r \hat{\psi}_2$ | $-\partial_r \hat{\psi}_3$ | $i\partial_r \hat{A}$ | $i\hat{\Phi}$ | $-i\hat{\Theta}$ | $-i\partial_r \hat{B}$ |
| $D_2$ | $\hat{\psi}_2$ | $\hat{\psi}_3$ | $-\partial_r \hat{\psi}_1$ | $-\partial_r \hat{\psi}_4$ | $-i\hat{\Phi}$ | $i\partial_r \hat{A}$ | $i\partial_r \hat{B}$ | $-i\hat{\Theta}$ |
| $D_3$ | $-\hat{\psi}_3$ | $\hat{\psi}_2$ | $\partial_r \hat{\psi}_4$ | $-\partial_r \hat{\psi}_1$ | $-i\hat{\Theta}$ | $i\partial_r \hat{B}$ | $-i\partial_r \hat{A}$ | $i\hat{\Phi}$ |
| $D_4$ | $\hat{\psi}_4$ | $\hat{\psi}_1$ | $\partial_r \hat{\psi}_3$ | $\partial_r \hat{\psi}_2$ | $i\partial_r \hat{B}$ | $i\hat{\Theta}$ | $i\hat{\Phi}$ | $i\partial_r \hat{A}$ |

(4.11)
Perhaps more so that comparing the two side-by-side Adinkras in (3.47), a comparison of (4.12)
with (4.11) and (4.3) The bosonic holoraumy generators for both versions of the twisted-chiral supermultiplet, (4.11)
and (4.12), are no longer all uniform compositions with $\partial_\tau$:

$$
\hat{B}^{(tCS)}_{12} = + \hat{B}^{(tCS)}_{34} = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\partial_\tau^2 & 0 & 0 & 0 \\
0 & -\partial_\tau^2 & 0 & 0
\end{bmatrix} , \quad \hat{B}^{(tCS)}_{34} = + \hat{B}^{(tCS)}_{24} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -\partial_\tau^2 & 0 & 0 \\
-\partial_\tau^2 & 0 & 0 & 0
\end{bmatrix} ,
$$

(4.13a)

while

$$
\hat{B}^{(tCS)}_{23} = + \hat{B}^{(tCS)}_{14} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} \circ \partial_\tau ,
$$

(4.13b)

remains a uniform composition with $\partial_\tau$. In turn, the fermionic holoraumy generators remain uniform
compositions, and $\hat{F}^{(tCS)}_{IJ}$ equal (3.54b) while $\hat{F}^{(GS)}_{IJ}$ equal (3.60b). To summarize:

| Twisted-Chiral Algebraic Subset of Holoraumy | cf. Table 3 |
|--------------------------------------------|-------------|
| original (4.11) SO(4)$\times$SO(4)$_{F}$ $\rightarrow$ U(1)$_{B}$ $\times$ SU(2)$_{F}$ $\xrightarrow{\eta}$ U(1)$_{rot} \times$ SU(2)$_{exR}$ |
| valise (3.44) SO(4)$_{B}$ $\times$ SO(4)$_{F}$ $\rightarrow$ SU(2)$_{B}$ $\times$ SU(2)$_{F}$ $\xrightarrow{\eta}$ SU(2)$_{rot} \times$ SU(2)$_{exR}$ |
| original (4.12) SO(4)$_{B}$ $\times$ SO(4)$_{F}$ $\rightarrow$ U(1)$_{B}$ $\times$ SU(2)$_{F}$ $\xrightarrow{\eta}$ SU(2)$_{rot}$ |
| valise (3.46) SO(4)$_{B}$ $\times$ SO(4)$_{F}$ $\rightarrow$ SU(2)$_{B}$ $\times$ SU(2)$_{F}$ $\xrightarrow{\eta}$ SU(2)$_{rot}$ |

This clearly distinguishes the “original” [35] twisted-chiral supermultiplet (3.49a) and its “redefined” [21, 22, 32] twisted-chiral supermultiplet (3.49b). A comparison with (4.7) shows that the chiral supermultiplet (4.3) and the “redefined” twisted-chiral supermultiplet (4.12) have isomorphic holoraumy groups, but of course are differentiated by cycle parity, i.e., $\chi_{C}$; compare the discussions after Eqs. (3.10) and (3.56).

These two supermultiplets are transformed into each other by means of a fermion redefinition such as (3.45) that has a negative determinant. Such a redefinition of fermions is compatible with Lorentz transformations only on the world-line and on the world-sheet, where the Lorentz groups are abelian, $\mathbb{Z}_2$ and $\text{Spin}(1, 1) \approx \mathbb{R}^*$, respectively [24]. It follows that this feature may be used to distinguish world-sheet reductions of supermultiplets from $n+1$-dimensional spacetime with $n \geq 2$: Starting with 2+1-dimensional spacetime, minimal spinors have an even number of real components so that Lorentz-compatible sign change transformations cannot have negative determinants.

5 Higher-Dimensional Holoraumy

Superspace methods [13, 15] are applicable without loss of generality [12] in all discussions of
supersymmetry and in spacetimes of all dimensions, albeit the formalism and especially the off-shell representations are completely understood only for a limited number of supersymmetries, $N \leq 8$ [3]. It is nevertheless possible to extend the definitions of holoraumy from Section 2 to higher spacetime dimensions, as follows:
1. Start with the simple fact that every supermultiplet consists of bosonic and fermionic components, and represent each of them by a Salam-Strathdee intact superfield the lowest component of which equals the particular component field.

2. The supermultiplet is then represented by a first order super-differential coupled system

\[ D_\alpha (\text{bosons}) = (\text{fermions} + \text{derivatives thereof}), \]
\[ D_\alpha (\text{fermions}) = (\text{bosons} + \text{derivatives thereof}) \]

where the expressions on the right-hand side of course contain also ordinary spacetime derivatives of the indicated components, and the \( D_\alpha \) symbolically denote the super-differential operators closing the desired supersymmetry algebra of the chosen type (Poincaré, superconformal, with central charges, etc.), and in the spacetime of chosen dimensions and signature.

3. Construct the enveloping super-differential system as in (2.4), and define:

\[ \tilde{B}_{\alpha\beta} (\text{bosons}) := [D_\alpha, D_\beta](\text{bosons}), \]
\[ \tilde{F}_{\alpha\beta} (\text{fermions}) := [D_\alpha, D_\beta](\text{fermions}), \]

and so on, obtaining thus the complete action of \( \wedge^* \text{Span}(D_\alpha) \) on the chosen supermultiplet.

This specifies the action of the complete hierarchy of holoraumy operators on the considered supermultiplet. In full generality, these operators do not form a group, but an \( \mathbb{R}[\nabla] \) (or \( \mathbb{C}[\nabla] \)) module, where \( \nabla \) stands for the list of (bosonic) spacetime derivatives, including all types of gauge-covariant derivatives, as needed.

It is straightforward that the action of so-defined quadratic holoraumy operators \( \tilde{B}_{\alpha\beta} \) and \( \tilde{F}_{\alpha\beta} \) (as well as their higher-order analogues) generally take the form of non-uniform compositions of field-space homogeneous linear transformations and domain-space transformations, as is the case in (4.5), (4.9) and (4.13). The domain-space transformations may well include the inhomogeneous transformations of spacetime translations, but possibly also the other bosonic generators from the considered supersymmetry algebra.

**Holoraumy Invariants:** Since the holoraumy operators \( \tilde{B}_{\alpha\beta} \) and \( \tilde{F}_{\alpha\beta} \) are generally non-uniform compositions of field-space and domain-space transformations, it is not possible to factorize them into a uniform composition of tensors such as \( B_{IJ} \) and \( F_{IJ} \) in (2.8)–(2.11) with a purely field-space action on one hand, and domain-space differential operators on the other. They naturally generate diverse elements in the \( \mathbb{R}[\nabla] \) (or \( \mathbb{C}[\nabla] \)) modules mentioned above.

However—on a case-by-case basis and in a representation-dependent way:

1. Each supermultiplet will have a characteristic restriction of the holoraumy operators to appropriate subsets of components (e.g., those of the same engineering dimension), which factorizes each so-restricted subset of holoraumy operators uniformly, as is the case within (4.5b) within (4.5), (4.9b) within (4.9) and (4.13b) within (4.13).

2. The reduction of the holoraumy operators to the (on-shell) physical degrees of freedom within every supermultiplet will factorize uniformly, for dimensional reasons.

3. The so-restricted (so-reduced) holoraumy operators will span an algebraic structure that is constrained by the process of restriction (reduction), and form substructures of the \( \mathbb{R}[\nabla] \) (or \( \mathbb{C}[\nabla] \)) modules mentioned above.
The representation-dependent structure obtained in this way then provides the foundation for constructing representation-dependent invariants generalizing (2.13)–(2.16). Within the restricted (reduced) subset of holoraumy invariants that do factorize, these invariants are still numerical. When computed with the full, un-restricted (un-reduced) holoraumy operators, these invariants are specific spacetime operators, characteristic of the particular representation for which they were calculated.

It is this representation-dependent and highly hierarchical holoraumy structure and its computable invariants that we hope will facilitate in classifying and identifying supermultiplets.

**SuSy-Holography:** This brings us to an important topic, having to do with specifying which component (super)field redefinitions are regarded as equivalence relations when assessing whether or not two supermultiplets are to be regarded as inequivalent or equivalent. In turn, this is related to the question whether or not a particular supermultiplet from a spacetime of one dimension is the dimensional reduction of a supermultiplet from a spacetime of a larger dimension.

For example, the particular field redefinition (3.45) may certainly be regarded as an equivalence relation when discussing world-line supermultiplets within the context of world-line models all by themselves. This is indeed the framework of Refs. [39, 40, 41, 42, 18, 11, 43, 21, 19, 34, 44, 22, 17], wherein real, finite-dimensional, unitary and off-shell supermultiplets are classified up to all component field redefinitions. This includes the negative-determinant ones that include (3.45), which are—within this purely world-line framework—also to be regarded as equivalence relations. However, the determination whether or not two supermultiplets are to be regarded as (in)equivalent changes considerably once additional structure is included, and is showcased already within the simple examples of Section 3 as summarized in Table 1.

![Table 1](image)

**Table 1:** Some of the key computational results from Section 3

For one, the middle column in Table 1 clearly indicates the difference in the supersymmetric complex structures resulting from the field redefinition (3.45) transforming the twisted-chiral supermultiplet, “vtCS” (3.44a) into its redefined form, “v̂tCS” (3.46); see also Sections 3.3, 3.5 and 4.3. Thus, “vtCS” (3.44a) and “v̂tCS” (3.46) are inequivalent complex supermultiplets.

This distinction of course remains when the component fields of these valise supermultiplets are redefined so as to avoid the obstruction to dimensional extension to 1+1-dimensional worldsheet [31], as was done in (4.11) and (4.12), respectively. The holoraumy structure of the resulting
supermultiplets continues to track this distinction in the supersymmetric complex structures of the two supermultiplets; see (4.14) and Table 1. Thus, “tCS” (4.11) and “tCS” (4.12) are inequivalent complex supermultiplets, both on the world-line and on the world-sheet.

In addition, in spacetimes where the total number of dimensions is at least three, all fermionic representations of the Lorentz group have an even number of components. Therefore, Lorentz-compatible fermionic field redefinitions cannot have negative determinants, and fermionic component field transformations such as (3.45) cannot possibly be regarded as equivalence relations in such higher-dimensional spacetimes.

Denote by \( \varrho_0 \) the coordinate world-line dimensional reduction, and let \( M_i \) be the \( i \)th supermultiplet in any fixed spacetime \( X \) of at least three dimensions and any (fixed) signature. Then, if \( \varrho_0(S_1) \) and \( \varrho_0(M_2) \) differ by a negative-determinant fermionic field redefinition, \( M_1 \) and \( M_2 \) are inequivalent in the higher-dimensional spacetime. In fact, if \( \nu_x \) is a negative-determinant fermionic field redefinition, the world-line supermultiplet \( \nu_x[\varrho_0(M_1)] \) need not be the dimensional reduction of any supermultiplet in \( X \).

We finally turn to an even subtler distinction that is detected by holoraumy: The vector supermultiplet (3.14c), the tensor supermultiplet (3.30c) and the “original” twisted-chiral supermultiplet (3.44a) have isomorphic holoraumy algebras. The precise nature of the isomorphisms between these three pairs of holoraumy groups and algebras may be read off from the bottom three rows of the left-hand half of Table 1. The distinctions are exhibited in Table 2, which specifies the isomorphisms in matrix form, with respect to the reference basis in Table 3. All of these isomorphisms involve odd (negative-determinant) signed permutations, although they are achieved by means of positive-determinant component field redefinitions; see (3.43) and Appendix B.

| \( \mathcal{B} \) | \( \mathcal{F} \) | \( (\text{det}(b), \text{det}(f)) \) |
|----------------|----------------|----------------|
| \( vVS \to vTS \): | \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} | \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | \( (-1, +1) \) |
| \( vVS \to vtCS \): | \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} | \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | \( (-1, -1) \) |
| \( vTS \to vtCS \): | \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} | \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | \( (+1, -1) \) |

Table 2: The isomorphism transformations \( b \in \text{Aut}(\mathcal{H}_B) \) and \( f \in \text{Aut}(\mathcal{H}_F) \)

6 Conclusions

To summarize, we have defined, exhibited and discussed a novel super-differential algebraic structure inherent in all supermultiplets, which we dub the holoraumy. In full generality, it is a nontrivial composition of:

- homogeneous, linear transformations in the field-space
- inhomogeneous transformations in the domain space

as realized on the component superfields spanning the specified representation of supersymmetry. This structure is revealed and is computable within the enveloping system of the system of super-differential equations that specify the supersymmetry transformations within the supermultiplet.

As a proof of concept, we have explored this structure within the simple framework of world-line $N$-extended supersymmetry without central charges, and have studied it in the illustrative cases of several well-known 3+1-dimensional supermultiplets, dimensionally reduced to the coordinate time world-line. The results may be gleaned from Tables 1 and 2.

The holoraumy tensors, their algebras and the groups they generate detect the following properties of supermultiplets:

1. The “(un)twistedness” of minimal supermultiplets (agreeing with the characteristic $\chi_o$ [28] and the cycle parity [38]), via the “minimality relations,” such as (3.9), (3.22) and (3.38).
2. The supersymmetric complex structure, by means of the even (positive-determinant) signed permutation isomorphism $\eta(\mathcal{B}_{IJ}) \approx \eta(\mathcal{F}_{IJ})$—and lack thereof; see Table 1.
3. The subtler distinction between the valise versions of the world-line dimensional reductions of all three of the $(a)$ vector and $(b)$ tensor supermultiplets in Wess-Zumino gauge and $(c)$ the “original” twisted-chiral supermultiplet of Ref. [35].

Although the number of examples we have examined is relatively small, we find it telling that the structure of even just the quadratic holoraumy tensors is in fact sufficient to distinguish every one of them. Having introduced this new concept and having described its features in considerable computational detail in the case of a handful of well-known supermultiplets, we conclude that holoraumy ought to be further explored, and from both aspects:

1. both computationally, by dimensionally reducing all other known supermultiplets to the world-line from the various spacetimes wherein they are defined and assessing the utility of holoraumy in telling inequivalent representations apart,
2. as well as by more general means of rigorous mathematics.

We are looking forward to both of these future efforts.

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A Matrix Conventions

Throughout, we have used the $\gamma$-matrices as defined in Ref. [28], but have clearly indicated the results that are independent of any such choice; these basis-independent results were also confirmed by repeating the calculations using a handful of different $\gamma$-matrix bases. In particular, the four $\gamma$-matrices associated with the 3+1-dimensional spacetime are chosen as

\[
\gamma^0_a \;=\; i[\sigma^3 \otimes \sigma^2]_a^b, \quad \gamma^1_a \;=\; [\mathbb{1} \otimes \sigma^1]_a^b, \quad \gamma^2_a \;=\; [\sigma^2 \otimes \sigma^2]_a^b, \quad \gamma^3_a \;=\; [\mathbb{1} \otimes \sigma^3]_a^b, \quad \text{(A.1)}
\]

which clearly implies that all elements of the Dirac algebra are real, i.e., we work in a Majorana representation. The fact that $(\gamma^0)^2 = -\mathbb{1}$ while $(\gamma^m)^2 = \mathbb{1}$ for $m = 1, 2, 3$ implies that we use the $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ metric. The $\gamma^5$-matrix

\[
(\gamma^5)_a^b := i\gamma^{0123} := i\gamma^0\gamma^1\gamma^2\gamma^3 = -(\sigma^1 \otimes \sigma^2)_a^b
\]

is then purely imaginary and antisymmetric. As usual, we employ the fact that the vector space of the Clifford-Dirac algebra is isomorphic to the exterior algebra

\[
\mathfrak{Cl}(1, 3) = (\otimes^* \text{Span} (\gamma^\mu)) / \{ (\gamma^\mu, \gamma^\nu) = 2\eta^{\mu\nu} \mathbb{1} \} \approx \wedge^* \text{Span} (\gamma^\nu), \quad \text{(A.3)}
\]

and use the weighted antisymmetrized products

\[
\gamma^{\mu\nu} := \frac{1}{2}[\gamma^\mu, \gamma^\nu], \quad \gamma^{\mu\nu\rho} := \frac{1}{3}(\gamma^{\mu\nu}\gamma^\rho + \gamma^{\nu\rho}\gamma^\mu + \gamma^{\rho\mu}\gamma^\nu), \quad \text{etc.} \quad \text{(A.4)}
\]

as the induced basis elements. For convenience, these matrices are tabulated in Table 3.

**Table 3:** A basis of sixteen real invertible matrices [28], spanning $\wedge^* \text{Span}(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, shown both in terms of the outer product of the usual 2×2 identity and Pauli matrices, and also explicitly

| $(\mathbb{1})_a^b$ | $(\gamma^0)_a^b$ | $(\gamma^1)_a^b$ | $(\gamma^2)_a^b$ | $(\gamma^3)_a^b$ | $(\gamma^{23})_a^b$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $[\mathbb{1} \otimes \mathbb{1}]$ | $[\mathbb{1} \otimes \mathbb{1}]$ | $[\mathbb{1} \otimes \mathbb{1}]$ | $[\mathbb{1} \otimes \mathbb{1}]$ | $[\mathbb{1} \otimes \mathbb{1}]$ | $[\mathbb{1} \otimes \mathbb{1}]$ |
| $[\sigma^3 \otimes \sigma^2]$ | $[\sigma^3 \otimes \sigma^2]$ | $[\sigma^3 \otimes \sigma^2]$ | $[\sigma^3 \otimes \sigma^2]$ | $[\sigma^3 \otimes \sigma^2]$ | $[\sigma^3 \otimes \sigma^2]$ |
| $i[\sigma^2 \otimes \sigma^2]$ | $i[\sigma^2 \otimes \sigma^2]$ | $i[\sigma^2 \otimes \sigma^2]$ | $i[\sigma^2 \otimes \sigma^2]$ | $i[\sigma^2 \otimes \sigma^2]$ | $i[\sigma^2 \otimes \sigma^2]$ |

On the space of Majorana, real 4-component spinors, we choose the metric

\[
C_{ab} := -i[\sigma^3 \otimes \sigma^2]_{ab}, \quad C_{ab} = -C_{ba}, \quad \text{(A.5)}
\]

and which numerically equals $-(\gamma^0) = (\gamma^0)^{-1}$. Therefore, $(\gamma^0)_{ab} = (\gamma^0)_a^c C_{cb} = \delta_{ab}$. In turn, the inverse spinorial metric is defined by the condition $C_{ac} C^{ca} = \delta_c^b$. Upon lowering the second index

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Also, we have that:

\[
\psi = \sum_{\lambda} \lambda \otimes \sigma^\lambda, \quad \psi^\dagger = \sum_{\lambda} \bar{\lambda} \otimes \sigma^\lambda.
\]

so that, in particular:

\[
\psi^\dagger = \sum_{\lambda} \bar{\lambda} \otimes \sigma^\lambda, \quad \psi = \sum_{\lambda} \lambda \otimes \sigma^\lambda.
\]

Also, we have that:

\[
\gamma_5 = \sum_{\lambda} \bar{\lambda} \otimes \sigma^\lambda, \quad \gamma^\dagger_5 = \sum_{\lambda} \bar{\lambda} \otimes \sigma^\lambda.
\]

\[
\gamma_5 = \sum_{\lambda} \bar{\lambda} \otimes \sigma^\lambda, \quad \gamma^\dagger_5 = \sum_{\lambda} \bar{\lambda} \otimes \sigma^\lambda.
\]

B Twisted-Chiral Supermultiplet

This is an abridged derivation of the world-line dimensional reduction of the “original” twisted-chiral supermultiplet [35], since we start from the world-line dimensional reduction of the vector supermultiplet in the Wess-Zumino gauge (4.8). For reasons that will soon become clear, we define \( \tilde{\psi}_a := (\gamma^a)_{\tilde{a}} \), so that:

\[
(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4) := (-\lambda_4, \lambda_3, \lambda_2, -\lambda_1),
\]

\[
\det [(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (-\lambda_4, \lambda_3, \lambda_2, -\lambda_1)] = +1.
\]

which turns (4.8d) into:

\[
| A_1 \rangle \langle A_2 | A_3 \rangle \langle \mathcal{D} | \tilde{\psi}_1 \rangle \langle \tilde{\psi}_2 | \tilde{\psi}_3 \rangle \langle \tilde{\psi}_4 | \mathcal{D} | -i \partial_\tau A_2 - i \partial_\tau A_1 - i \partial_\tau A_3 - i \partial_\tau A_3 - i \partial_\tau A_1 - i \partial_\tau A_2 - i \partial_\tau A_2 - i \partial_\tau A_1
\]

\[
| D_1 | \tilde{\psi}_2 \rangle \langle \tilde{\psi}_1 | - \tilde{\psi}_2 \rangle \langle - \tilde{\psi}_1 | - \partial_\tau \tilde{\psi}_2 \rangle \langle - \partial_\tau \tilde{\psi}_1 | i \partial_\tau A_2 - i \partial_\tau A_1 - i \partial_\tau A_3 - i \partial_\tau A_3 - i \partial_\tau A_1 - i \partial_\tau A_2 - i \partial_\tau A_2 - i \partial_\tau A_1
\]

\[
| D_2 | - \tilde{\psi}_2 \rangle \langle \tilde{\psi}_3 | \tilde{\psi}_2 \rangle \langle \tilde{\psi}_3 | - \partial_\tau \tilde{\psi}_2 \rangle \langle - \partial_\tau \tilde{\psi}_3 | i \partial_\tau A_2 - i \partial_\tau A_1 - i \partial_\tau A_3 - i \partial_\tau A_3 - i \partial_\tau A_1 - i \partial_\tau A_2 - i \partial_\tau A_2 - i \partial_\tau A_1
\]

\[
| D_3 | - \tilde{\psi}_1 \rangle \langle \tilde{\psi}_3 | \tilde{\psi}_2 \rangle \langle \tilde{\psi}_4 | - \partial_\tau \tilde{\psi}_1 \rangle \langle - \partial_\tau \tilde{\psi}_4 | i \partial_\tau A_1 - i \partial_\tau A_3 - i \partial_\tau A_3 - i \partial_\tau A_3 - i \partial_\tau A_1 - i \partial_\tau A_2 - i \partial_\tau A_2 - i \partial_\tau A_1
\]

\[
| D_4 | \tilde{\psi}_2 \rangle \langle \tilde{\psi}_4 | \tilde{\psi}_1 \rangle \langle \tilde{\psi}_3 | i \partial_\tau A_3 - i \partial_\tau A_1 - i \partial_\tau A_3 - i \partial_\tau A_3 - i \partial_\tau A_1 - i \partial_\tau A_2 - i \partial_\tau A_2 - i \partial_\tau A_1
\]
or, in tensorial notation:

\[
D_a A_1 = (\gamma^{12})_a^b \tilde{\psi}_b, \quad D_a A_2 = \tilde{\psi}_a, \quad D_a A_3 = -(\gamma^{23})_a^b \tilde{\psi}_b, \quad D_a \mathcal{D} = (\gamma^{13})_a^b \partial_\tau \tilde{\psi}_b, \quad (B.2b)
\]

\[
D_a \tilde{\psi}_b = i(\gamma^{012})_{ab} \partial_\tau A_1 + i(\gamma^0)_{ab} \partial_\tau A_2 - i(\gamma^{023})_{ab} \partial_\tau A_3 + i(\gamma^{013})_{ab} \mathcal{D}. \quad (B.2c)
\]

Aiming to match the unsigned pattern of resulting fields in the transformation of the (now fixed) fermions in (B.2a) to that found in (3.1d), we define:

\[
(\tilde{A}, \tilde{B}, \tilde{F}, \tilde{G}) := (A_2, A_3, -(f d\tau \mathcal{D}), A_1), \quad \det [(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (\tilde{\lambda}_4, \lambda_3, \lambda_2, -\lambda_1)] = +1. \quad (B.3)
\]

This turns (B.2a) into the results (3.44) used above.

Analogously to (4.3), we revert to the original component superfield \( \tilde{\mathcal{F}} = \partial_\tau \tilde{F} = -\mathcal{D}, \) define \( \tilde{\mathcal{G}} = \partial_\tau \tilde{G} = \partial_\tau A_1, \) and draw their nodes above the fermionic ones to indicate their relative engineering dimensions, \( [\tilde{\mathcal{F}}] = [\tilde{\mathcal{G}}] = [\tilde{\psi}_a] + \frac{1}{2} = [\tilde{A}] + 1 = [\tilde{B}] + 1, \) the system (3.46) becomes:

\[
| tCS | \tilde{A} \quad \tilde{B} \quad \tilde{\mathcal{F}} \quad \tilde{\mathcal{G}} | \tilde{\psi}_1 \quad \tilde{\psi}_2 \quad \tilde{\psi}_3 \quad \tilde{\psi}_4 |
\]

\[
D_1 \quad \tilde{\psi}_1 - \tilde{\psi}_4 \quad \partial_\tau \tilde{\psi}_2 - \partial_\tau \tilde{\psi}_3 \quad i\partial_\tau \tilde{A} \quad i\tilde{\mathcal{F}} \quad -i\tilde{\mathcal{G}} \quad -i\partial_\tau \tilde{B}
\]

\[
D_2 \quad \tilde{\psi}_2 \quad \tilde{\psi}_3 - \partial_\tau \tilde{\psi}_1 \quad -\partial_\tau \tilde{\psi}_4 \quad -i\tilde{\mathcal{F}} \quad i\partial_\tau \tilde{A} \quad i\tilde{\mathcal{G}} \quad -i\tilde{\mathcal{G}}
\]

\[
D_3 \quad -\tilde{\psi}_3 \quad \tilde{\psi}_2 \quad \partial_\tau \tilde{\psi}_4 \quad -\partial_\tau \tilde{\psi}_1 \quad -i\tilde{\mathcal{G}} \quad i\partial_\tau \tilde{B} \quad -i\partial_\tau \tilde{A} \quad i\tilde{\mathcal{F}}
\]

\[
D_4 \quad \tilde{\psi}_4 \quad \tilde{\psi}_1 \quad \partial_\tau \tilde{\psi}_3 \quad \partial_\tau \tilde{\psi}_2 \quad i\partial_\tau \tilde{B} \quad i\tilde{\mathcal{G}} \quad i\tilde{\mathcal{F}} \quad i\partial_\tau \tilde{A}
\]

In our Majorana basis, the twisted-complex chiral supermultiplet is

\[
(\tilde{A} + i\tilde{B}) \mid (\tilde{\psi}_1 - i\tilde{\psi}_4), (\tilde{\psi}_2 + i\tilde{\psi}_3) \mid (\tilde{\mathcal{F}} - i\tilde{\mathcal{G}}) : [D_1 - iD_4], [D_2 - iD_3], \quad (B.4)
\]

which indeed differs from (4.4) only in the sign of \( D_3. \) Besides clustering the edges depicting the actions of the complex super-derivatives \([D_1 - iD_4]\) and \([D_2 - iD_3],\) the right-hand side Adinkra (??) is also drawn in a way that immediately permits dimensionally extending this world-line supermultiplet to the world-sheet, by defining:

\[
D_- := [D_1 - iD_4], \quad \tilde{D}_+ := [D_2 - iD_3], \quad \tilde{\psi}_- := (\tilde{\psi}_1 - i\tilde{\psi}_4), \quad \tilde{\psi}_+ := (\tilde{\psi}_2 + i\tilde{\psi}_3), \quad (B.6)
\]

where the \( \pm \) subscripts indicate spin, in units of \( \frac{1}{2}h; \) \((\tilde{A} + i\tilde{B})\) and \((\tilde{\mathcal{F}} - i\tilde{\mathcal{G}})\) have spin 0. Note that the definitions (B.6) are in perfect agreement with (4.4) and so provide for a direct comparison between (??)–(B.5) and (4.3)–(4.4). In particular, the annihilation condition specifying the twisted chiral supermultiplet, the analogue of (3.27), becomes the well-known defining equation [36]:

\[
D_- (\tilde{A} + i\tilde{B}) = 0 = D_+ (\tilde{A} + i\tilde{B}). \quad (B.7)
\]

In fact, the same holds also for (3.44) and (3.49b), the Adinkra of which differs from (??) only in the change \( \tilde{\psi}_3 \to \tilde{\psi}_3 = -\tilde{\psi}_3, \) so that the edges adjacent to the (now) \( \tilde{\psi}_3\)-node have flipped dashedness. This then ruins the mirror symmetry evident in (??) and so obscures the presence of a supersymmetric complex structure [46]. Indeed, the fermionic identifications (B.6) would become

\[
D_- := [D_1 - iD_4], \quad D_+ := [D_2 - iD_3], \quad \tilde{\psi}_- := (\tilde{\psi}_1 - i\tilde{\psi}_4), \quad \tilde{\psi}_+ := (\tilde{\psi}_2 - i\tilde{\psi}_3), \quad (B.8)
\]

where the conjugation in \( \tilde{\psi}_+^* \) is necessary in direct comparison with (4.4).
C The Flat Metric of Linear Supersymmetry Representations

Upon dimensional reduction to the world-line, all supersymmetric models result in models of supersymmetric quantum mechanics. After a judicious field redefinition and renaming, the free-field kinetic term for standard off-shell propagating (physical) real bosons $\phi_a(\tau)$ and real fermions $\psi_\alpha(\tau)$ is of course:

$$KE[(\phi|\psi)] = \frac{\kappa}{2} \int d\tau \left[ \delta^{ab} \dot{\phi}_a \dot{\phi}_b + \frac{i}{2} \delta^{\alpha\beta} (\dot{\psi}_\alpha \dot{\psi}_\beta - \dot{\psi}_\alpha \psi_\beta) \right]$$  \hspace{1cm} (C.1)

with a suitable parameter $\kappa$. It is of course supersymmetric with respect to the world-line dimensional reduction of the original, higher-dimensional supersymmetry. Conversely, we have:

**Theorem C.1** The free-field action (C.1) is supersymmetric with respect to the maximally $N$-extended world-line supersymmetry transformations

$$Q_I \phi_a = (\mathbb{L}_I)_a^\alpha \psi_\alpha, \quad Q_I \psi_\alpha = i(\mathbb{L}_I^{-1})_a^\alpha \phi_a,$$  \hspace{1cm} (C.2)

without central charges, where

$$\left\{ Q_I, Q_J \right\} = 2i\delta_{IJ} \partial_\tau, \quad \left[ \partial_\tau, Q_I \right] = 0, \quad I, J = 1, \cdots, N,$$  \hspace{1cm} (C.3a)

$$(Q_I)^\dagger = Q_I, \quad H^\dagger = i\partial_\tau,$$  \hspace{1cm} (C.3b)

provided there are as many bosons as fermions, $d_B = d_F = d$, and they can be partitioned into collections, each satisfying (C.18), given below. Moreover, the field-space flat metric $\delta^{ab} \oplus \delta^{\alpha\beta}$ is induced canonically from the $\delta_{IJ}$ specified by the algebra (C.3).

**Comments:** All curvature tensors of the free-field metric of course vanish. An even number of boson and fermion fields can always be arranged into complex pairs, and a complex-paired set of supersymmetries (C.2) that leave the action (C.1) invariant can also always be found. However, this supersymmetry need not be the result of the dimensional reduction that produced (C.1), and the dimensional reduction of that higher-dimensional supersymmetry may not be compatible with any complex pairings of the component fields; see the vector and tensor supermultiplets described in sections 3.2 and 3.4, respectively.

The same canonically induced field-space metric, $\delta^{ab} \oplus \delta^{\alpha\beta}$, on the component field space $(\phi|\psi)$ also occurs in the supersymmetric super-Zeeman term [47]:

$$SZ[(\phi|\psi), (\varphi|\chi)] = \omega \int d\tau \left[ \frac{1}{2} \delta^{ab} (\dot{\phi}_a \dot{\varphi}_b - \dot{\phi}_a \varphi_b) - i \delta^{\alpha\beta} \psi_\alpha \chi_\beta \right]$$  \hspace{1cm} (C.4)

with a suitable Larmor-like frequency parameter $\omega$, as well as various modifications of (C.1) and (C.4) as discussed in Ref. [47] and below. The combination (C.1)+(C.4), with various additionally imposed boundary conditions that also restrict the supersymmetry action (C.3), is the core world-line framework for all higher-dimensional models, with all additional terms added to the action representing deformations of this core.

C.1 Proof

We now derive the $\delta^{ab} \oplus \delta^{\alpha\beta}$ metric from the metric $\delta_{IJ}$ that occurs in the supersymmetry defining relations (C.3). The proof consists of three stages, relating to: (1) intact supermultiplets, (2) projected supermultiplets, (3) all engineerable supermultiplets, and we proceed in turn.
**Intact supermultiplets:** We begin with defining the *intact* representation of supersymmetry (C.3), by starting with, say, a real boson $\phi_0$, and defining

\[
\begin{align*}
\phi_0, & \quad \psi_I := Q_I \phi_0, \\
F_{[IJ]} & := Q_I Q_J \phi_0, \\
\Psi_{[IJK]} & := Q_I Q_J Q_K \phi_0,
\end{align*}
\]

which clearly terminates after defining $2^{N-1}$ bosonic and $2^{N-1}$ fermionic component fields. Owing to the relations (C.3), the result of an infinitesimal supersymmetry transformation, $\delta_Q(\epsilon) := \epsilon^I Q_I$, acting on any of these fields is a linear combination of these fields and their $\partial_\tau$-derivatives. Therefore, the collection of fields

\[
\mathcal{M}_\diamond := \{ \phi_0 | \psi_I | F_{[IJ]} | \Psi_{[IJK]} | \cdots \}
\]

represents a supermultiplet; it is not hard to see that these component fields are in 1–1 correspondence with the component fields obtained from a Salam-Strathdee superfield [48]; see also [13, 49, 14, 15]. From the definitions (C.5) and the algebra (C.3) alone, it follows that:

\[
\begin{align*}
Q_I \phi_0 & = \psi_I, & Q_I \phi_J & = F_{[IJ]} + i \delta_I J \phi_0, \\
Q_I F_{[JK]} & = \Psi_{[IJK]} + i \delta_{IJ} \psi_K, & Q_I \Psi_{[JKL]} & = \mathcal{F}_{[IJKL]} + i \delta_{IJ} \bar{F}_{KL},
\end{align*}
\]

and so on. A closer examination shows that this multiplet is *adinkraic* [11], i.e., *monomial*: the action of each one supercharge $Q_I$ upon each one of the component fields produces a single other of the component fields (C.6) or a $\partial_\tau$-derivative thereof.

From (C.6), we easily construct the *valise* supermultiplet:\footnote{This has been variously also called an “isoscalar” and a “base” supermultiplet [4].}

\[
\begin{align*}
\mathcal{M}_\ddagger := & \{ \phi_0, \phi_{[IJ]}, \cdots | \psi_I, \psi_{[IJK]}, \cdots \}, \\
\phi_{[IJ]} & := -i \partial_\tau^{-1} F_{[IJ]}, & \psi_{[IJK]} & := -i \partial_\tau^{-1} \Psi_{[IJK]},
\end{align*}
\]

where now all the bosons, $\phi_0, \phi_{[IJ]}, \cdots$, have the same engineering dimension, as do all the fermions, $\psi_I, \psi_{[IJK]}, \cdots$. By construction, within this supermultiplet, the supersymmetry acts straightforwardly:

\[
\epsilon^I Q_I : \left\{ \begin{array}{c}
(\sum_k \alpha_{[I_1 \cdots I_{2k}]} \phi_{[I_1 \cdots I_{2k}]} ) \rightarrow (\sum_k \beta_{[I_1 \cdots I_{2k+1}]} \psi_{[I_1 \cdots I_{2k+1}]}), \\
(\sum_k \beta_{[I_1 \cdots I_{2k+1}]} \psi_{[I_1 \cdots I_{2k+1}]} ) \rightarrow (\sum_k \gamma_{[I_1 \cdots I_{2k}]} \dot{\phi}_{[I_1 \cdots I_{2k}]}),
\end{array} \right.
\]

where the $\alpha, \beta, \gamma$’s are general real coefficients, such that (C.3) is satisfied; indeed, the relations (C.7) need only be corrected to accommodate the redefinitions (C.7b). In particular, $\mathcal{M}_\ddagger$ is also adinkraic, i.e., monomial.

Finally, we note that the metric $\delta_{IJ}$, which is canonical as it is given by the very definition of the supersymmetry algebra (C.3), induces a metric on the component field space:

\[
(f_{[I_1 \cdots I_p]}, f'_{[I_1 \cdots I_p]}) := f_{[I_1 \cdots I_p]} \delta^{I_1 J_1} \cdots \delta^{I_p J_p} f'_{[I_1 \cdots I_p]}, \quad \text{for } p = 0, \cdots, N,
\]
where $f_{[I_1 \cdots I_p]}$ is a bosonic component field for even $p$, and a fermionic one for odd $p$. Since

$$
\sum_{k \text{ even}} \binom{N}{k} = 2^{N-1} = \sum_{k \text{ odd}} \binom{N}{k}, \tag{C.11}
$$

we may order the bosonic (and separately fermionic) component fields lexicographically, and count them using $a, b = 1, \cdots, 2^{N-1}$ for bosons (and $\alpha, \beta = 1, \cdots, 2^{N-1}$ for fermions). It is straightforward that the diagonal bilinear form (C.10), with this counting, may be chosen to simply give

$$
\phi_a \delta_{ab} \phi_b \oplus \psi_\alpha \delta^{ab} \psi_b \oplus \chi_\alpha, \tag{C.12}
$$
as needed in Eqs. (C.1) and (C.4), respectively. The bosonic bilinear and the fermionic bilinear of course have different engineering dimensions, to accommodate the differing number of derivatives in either of the two supersymmetric Lagrangians (C.1) and (C.4).

**Projected supermultiplets:** As originally proven in Refs. [21, 19] (see also Ref. [22]), all valise supermultiplets may be obtained from $\mathcal{M}_+$ by means of projections. To this end, define the operator

$$
\Pi_b := \frac{1}{2} \left[ \partial_{|b|/2} \pm Q_b \right], \quad Q_b := Q_b^1 \cdots Q_b^N, \tag{C.13}
$$

where $b$ is a length-$N$ doubly even binary number, i.e., the sum of its digits, $|b| := \sum_{I=1}^N b_I$, is divisible by 4. Such operators satisfy

$$
\Pi_b \circ \Pi_b = \partial_{|b|/2} \Pi_b, \quad \Pi_b \circ \Pi_b = 0, \quad \Pi_b + \Pi_b = \partial_{|b|/2}. \tag{C.14}
$$

They act as quasi-projection operators, in that

$$
\mathcal{M}_+ = \mathcal{M}_+^b + \mathcal{M}_-^b \simeq \left( \Pi_+^b \mathcal{M}_+ \right) + \left( \Pi_-^b \mathcal{M}_- \right) \tag{C.15}
$$
is a direct decomposition of the supermultiplet $\mathcal{M}_+$ into two half-sized supermultiplets, and the relation “$\simeq$” here denotes that individual component fields on one and the other side may be equated up to a few initial terms in a Taylor expansion in $\tau$. (This reflects the non-locality of the inverses of the field redefinitions (C.7b) and the ensuing introduction of “integration constants”.)

To successively apply two such projections, $\Pi$ and $\Pi'$, the commutator $[\Pi, \Pi']$ must vanish when acting on a supermultiplet annihilated by both $\Pi$ and $\Pi'$. Correspondingly, the bit-wise sum of the corresponding binary exponents must also be a doubly even binary number. Collections of such binary numbers, complete with respect to bit-wise addition, form doubly-even error-detecting and error-correcting binary linear block codes, $\mathcal{C}$.

It follows from the classification of these [21, 19] (see also Ref. [22]) that the number of such simultaneous projections is

$$
k \leq \kappa(N) := \begin{cases} 
0 & \text{for } N < 4; \\
\lfloor \frac{(N-4)^2}{4} \rfloor & \text{for } N = 4, 5, 6, 7; \\
\kappa(N-8) + 4 & \text{for } N > 7, \text{ recursively.} 
\end{cases} \tag{C.16}
$$

Let $\Pi_b^\mathcal{C}_k$ denote the collection of all mutually commuting projections $\Pi_b$ where $b \in \mathcal{C}_k$, where $\mathcal{C}_k$ is the code formed as a collection of all bit-wise sums of $k$ linearly independent generators of the code,
and \( \pi \) is a choice of signs for each individual projector. Since each projection halves the number of component fields, the total number of bosonic+fermionic component fields in a \( k \)-fold projected (halved) supermultiplet

\[
\Pi_{\pi}^{\mathcal{C}_k} (\mathcal{M}_\pm) = \left( \Pi_{\pi}^{\mathcal{C}_k} \{ \phi_0, \cdots, \phi_{[I_1 \cdots I_p]} \} \right) \Pi_{\pi}^{\mathcal{C}_k} \{ \phi_I, \cdots, \phi_{[I_1 \cdots I_q]} \})
\]

\[0 \leq p, q \leq N, \quad p \equiv 0 \text{ mod } 2, \quad q \equiv 1 \text{ mod } 2,
\]

is

\[
(d_B = 2^{N-k-1}) + (d_F = 2^{N-k-1}),
\]

and each such \( \mathcal{C}_k \)-projected distinct valise supermultiplet defines a corresponding chromotopology; the maximally projected \( (\Pi_{\pi}^{\mathcal{C}_\kappa(N)} \mathcal{M}_\pm) \) are the minimal valise supermultiplets, and are classified by the maximal codes \( \mathcal{C}_\kappa(N) \) [21, 19, 22].

Since each projection defines mutually orthogonal linear combination component fields, the metric \( \delta^{ab} \oplus \delta^{\alpha \beta} \) in \( \mathcal{M}_\pm \) reduces to a half-size, still diagonal and still positive-definite metric on the halved component field space; the diagonal values can then always be adjusted to unity by properly normalizing the linear combination component fields such as \( \phi_0^\pm \). For example, for \( N = 4 \), there is a single binary doubly-even linear block code, defining the operators \( (C.13) \)

\[
\Pi^{(1111)} := \frac{1}{2} \left[ \partial_{\tau}^2 \pm Q_1 Q_2 Q_3 Q_4 \right].
\]

Note that, modulo \( \theta \)-dependent terms and up to an overall numerical constant, the operators \( (3.10), (3.23), (3.39), (3.56) \) and \( (3.23) \) all square to \( (C.19) \). Projecting the \( 8 + 8 \)-component valise supermultiplet \( (C.8) \) then produces component fields such as \( \phi_0^\pm := \frac{i}{2} (\phi_0 \pm \phi_{[1234]}) \), for each of which it is straightforward to prove that

\[
\dot{\phi}_0^2 + \dot{\phi}_{[1234]}^2 = 2(\dot{\phi}_0^-)^2 + 2(\dot{\phi}_0^+)^2
\]

Therefore, each of worldline supermultiplets, with any of the \( 10^{12} \) chromotopologies of Refs. [21, 19, 22], has a “free” kinetic action term of the form \( (C.1) \). In addition, for any pair of such supermultiplets there exists a super-Zeeman action term such as \( (C.4) \).

**Engineerable Supermultiplets:** All physical fields have a definite engineering dimension, stemming from their physical units expressed in the natural system where the units \( h, c \) are not written explicitly. Supermultiplets wherein every component field has a definite engineering dimension were called engineerable [11]. It then follows that the world-line reduction of every physically relevant supermultiplet in any higher-dimensional theory must be engineerable.

In turn, Ref. [17] proved that every engineerable, finite-dimensional, off-shell, unitary representation of \( N \)-extended world-line supersymmetry without central charges, upon component field raising\(^{13} \) of a judicious selection of fields

\[
\phi_a \rightarrow F_a := (\partial_{\tau} \phi_a), \quad \text{for some } a \in \{1, \cdots, d_B\},
\]

\(^{13}\)Refs. [2, 39, 4] introduced and used extensively the operation, dubbed variously automorphic duality [2], dressing transformation [18], and node-raising/lowering [11]; we use the latter terms for their precision.
and judicious real linear combination of so-obtained fields, decomposes into a real linear combination of minimal valises for the specified $N$. We have proven above that all valises (including the minimal ones) do have a \textit{canonical} positive quadratic form (metric) (C.1), induced from the quadratic form defined by the supersymmetry algebra itself (C.3).

As real linear combinations and raising/lowering transformations (C.21) cannot change the positive-definiteness of a quadratic form (metric) on the fields (and their derivatives) as used in (C.1), it follows that all engineerable, finite-dimensional, off-shell, unitary representation of $N$-extended world-line supersymmetry without central charges also have a quadratic form (metric) on the fields (and their derivatives) as used in (C.1).

Therefore, the world-line reduction of every physically relevant supermultiplet in any higher-dimensional theory has a supersymmetric kinetic Lagrangian of the form (C.1), and the positive-definite quadratic form (metric) used in (C.1) is \textit{canonical}, in that it is induced from the metric introduced in the supersymmetry algebra (C.3) itself.\hfill\checkmark

### C.2 Completeness and Symmetry

As rigorously proven in Ref. [11], all supermultiplets with the same chromotopology may be obtained one from another through the component field redefinitions that generalize (C.7b) by combinatorially varying the choices of which component fields are transformed by node-raising/lowering. Having obtained the kinetic (C.1) and super-Zeeman (C.4) action terms, the supermultiplets may be adapted to each of the node-raised/lowered variant of the supermultiplets involved. For example, component field raising (C.21) a selection of fields produces

$$\text{KE}[(\phi|\psi|F)] = \frac{\kappa}{2} \int d\tau \left[ \sum_{a \neq a'} \delta^{ab} \dot{\phi}_a \dot{\phi}_b + \delta^{a'b} F_a F_b + i \delta^{\alpha\beta} (\psi_\alpha \dot{\psi}_\beta - \dot{\psi}_\alpha \psi_\beta) \right]$$

(C.22)

as the resulting kinetic action term. This procedure has been systematically explored for the so-called ultramultiplet in Ref. [45].

This brings us finally to an alternate consideration: as discussed above, the supersymmetry algebra itself (C.3) is invariant with respect to the $O(N)$-transformations $Q_I \rightarrow Q'_I := (O)_{I'}Q_J$ and $\delta_{IJ}$ in (C.3a) defines the $O(N)$-invariant metric. It is not difficult to show that $\{\phi_0, \phi_{[IJ]}, \cdots\}$ and $\{\psi_I, \psi_{[IJK]}, \cdots\}$ span two spinor representations of this group, which is the reason for writing $\text{Aut}(\mathfrak{sp}^{1|N}) = \text{Pin}(N)$ rather than $\text{Aut}(\mathfrak{sp}^{1|N}) = O(N)$; see also footnote 2 on p. 2.

In turn, given a collection of $m = 2^{N-k-1}$ real bosonic and real fermionic component fields (C.8), the maximal group of transformations of these component fields is $O(2^{N-k-1})_B \times O(2^{N-k-1})_F$. Of course, not all such transformations will preserve the action of supersymmetry within the supermultiplet. The collection of component field transformations modulo the ones induced by $\text{Aut}(\mathfrak{sp}^{1|N}) = \text{Pin}(N)$ then form the product of cosets

$$\left[ O(2^{N-k-1})_B / \text{Pin}(N) \right] \times \left[ O(2^{N-k-1})_F / \text{Pin}(N) \right].$$

(C.23)

We note that these cosets are discrete only for $N = 1, 2, 4, 8$ and are continuous for all other $N$. As a coarse estimate, this indicates that the level of difficulty in classifying off-shell supermultiplets radically changes outside the $N = 1, 2, 4, 8$ cases. Not coincidentally, the minimal supermultiplets in
those cases have $1+1$, $2+2$, $4+4$ and $8+8$ component fields an exhibit real, complex, quaternionic and octonionic structures, respectively.

Accordingly,

1. by choosing a subset of bosons to redefine as in (C.21), the $\text{Pin}(N)$ symmetry is broken to a subgroup, and different choices of subsets of bosons to redefine correspond to the distinct subgroups of $\text{Pin}(N)$;

2. the projections obtained using the quasi-projection operators such as (C.13) are spinor-halving projections generalizing the familiar $\text{Dirac} \rightarrow \text{Weyl}$ or the $\text{Dirac} \rightarrow \text{Majorana}$ projection of the so-named spinors in 4d physics\footnote{For such projections of Dirac spinors in spacetime physics, one additionally requires Lorentz-covariance, of which there is no analogue in the present context. This then is what ultimately permits the combinatorially vast plethora of projection possibilities [21, 19].}; see also Ref. [22].

The metric $\delta^{ab} \oplus \delta^{a\beta}$ on the so (possibly iteratively) halved bosonic+fermionic component field space is then the standard, maximally symmetric metric. Indeed, the action terms (C.1) and (C.4) exhibit the much larger $\text{O}(2^{N-k-1})$ (dynamical/effective) symmetry rather than just $\text{Pin}(N)$ or a subgroup thereof.

For example, the unprojected valise supermultiplet of $N = 4$ supersymmetry has $8 + 8$ bosonic+fermionic component fields, whereupon the kinetic action term (C.1) for each such supermultiplet, and also the super-Zeeman action term (C.4) for each pair of such supermultiplets exhibits an $\text{O}(8)$ (dynamical/effective) symmetry, enlarging the $\text{Pin}(4)$ group of automorphisms of the supersymmetry algebra $\mathfrak{Sp}^{1|4}_{\text{14}}$. Note that $\text{Pin}(4) \subset \text{O}(8)$ is a special (irregular) subgroup [50], wherein the real vector of $\text{O}(8)$ transforms as the ($\mathbb{C}^4 \simeq \mathbb{R}^8$-like) spinor of $\text{Pin}(4) \subset \text{O}(8)$. This is also the defining property in the general case, $\text{Pin}(N) \subset \text{O}(2^{N-1})$, and why we systematically refer to $\text{Pin}(N) = \text{Aut}(\mathfrak{Sp}^{1|N})$, but to $\text{O}(2^{N-1})$ or $\text{SO}(2^{N-1})$ as the (C.1) and (C.4)-preserving group of linear, homogeneous transformations of the bosons and separately the fermions.

Within the free-field limit of any model, the bilinear action terms (C.1) and (C.4) provide the “supersymmetry-canonical” kinetic and Zeeman terms; any other action term then may be considered as a deformation thereof.

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