The question of how one can distinguish quark model states from 2-hadron states near an S-wave threshold is discussed, and the usefulness of the running mass is emphasized as the meeting ground for experiment and theory and for defining resonance parameters.

1. Introduction. A current problem of fundamental importance in hadron spectroscopy is: How to distinguish composites formed of two hadrons from normal quark model hadrons? There are now a handful of good experimental candidates [1] which have great difficulties in finding a place within the normal $q\bar{q}$ model. Examples of such states are the $f_1(1420)$, $f_0(1520)$, $f_2(1520)$, $f_0(1710)$, and $\Lambda(1405)$ and a longstanding problem has been the question whether the $f_0(980)$ and the $a_0(980)$ are $q\bar{q}$ or $KK$ states. All these resonances appear near an important S-wave threshold.

Recently Morgan and Pennington [2] and Zou and Bugg [3] have discussed the structure of the $f_0(980)$, noticing that one needs two nearby poles to describe the $f_0(980)$. Morgan and Pennington [2] made the important observation that this two pole structure is what is expected from a normal $q\bar{q}$-meson near an S-wave threshold ($KK$) in contrast to a $KK$ bound state, for which only one pole is expected.

This seems to provide a nice clear-cut method to almost model independently distinguish hadron-hadron bound states from normal quark model states. I shall here show that the question of whether one has one pole or two poles depends on the effective distance to left hand cuts, or on the range of the binding forces. I shall clarify the issues involved, first through some remarks of general nature, and then by emphasizing the usefulness of the concept of the running mass $m(s)$ [4], which also
provides a good way to distinguish between a $q\bar{q}$ and a hadron-hadron state. The shape of this running mass function allows one to in a simple single picture discuss most of the problems involved, and at the same time to see the origin of the one pole - two pole dichotomy.

We shall be concerned with S-wave thresholds, since for higher orbital momenta the centrifugal factor makes the important S-wave square root cusp disappear. Almost all cases of practical interest are, in fact, in S-wave channels. The fact that a $q\bar{q}$ state sometimes requires two poles brings to the fore another old problem of great importance for experiment, which I discuss at the end: How should one parametrize a resonance near threshold? - by the pole positions or by Breit-Wigner (BW) parameters?

2. Poles from $q\bar{q}$ resonances. Let us start the discussion with normal $q\bar{q}$ states. Since these have their origin in the confined sector owing their binding to gluonic exchange, they are CDD poles. A school example of a CDD pole is the $K^0$ pole in $\pi\pi \rightarrow \pi\pi$ due to the weak interaction $K^0 \rightarrow \pi\pi$. Although usually disregarded this makes, in principle, the $\pi\pi$ phase shift jump by 180° at the $K^0$ mass. There is a formal similarity with this $K^0$ pole and a normal $q\bar{q}$ state, which couples by $q\bar{q}$ pair creation to hadron-hadron channels. Both are CDD poles and come from another sector of Hilbert space, than that spanned by the decay channels. Only the magnitude of the coupling to hadrons, is orders of magnitudes stronger for $q\bar{q}$ than for $K^0$. If one could make the quark pair creation very small, like in a zero-width approximation or in a quenched approximation of lattice QCD, the $q\bar{q}$ poles would still remain as spikes in hadron-hadron amplitudes, albeit shifted in mass from their normal positions. In other words, the quark loops (which together with the initial quarks make hadron loops) will generally shift down the ”bare $q\bar{q}$ masses” at the same time as the resonances acquire finite widths. This is quite different from the case of hadron-hadron bound states. There, if the coupling is decreased the state disappears completely out of existence.
Let us imagine that we could tune the bare $q\bar{q}$ mass so that the true resonance position passes the threshold. To be concrete, let us chose the $K\bar{K}$ threshold. The I=1 channel is slightly simpler than I=0, since we expect only one resonance, the $a_0(980)$, whereas for I=0 one also has energy dependent complex mixing \[\text{between the } s\bar{s} \text{ and } (u\bar{u} + d\bar{d})/\sqrt{2} \text{ resonances.} \]

Near the threshold the inverse propagator has the general form:

\[
P^{-1}(s) = m_0^2 - s + \Pi(s) = m^2(s) - s - i\gamma' \sqrt{s - 4m_K^2} G(s) \Theta(s - 4m_K^2) - im\Gamma_{\eta\pi}, \tag{1}
\]

where $m_0^2 - s$ comes from the bare $q\bar{q}$ propagator and $\Pi(s)$ is the "correction" term due to $K\bar{K}$ and other hadronic loops, whose imaginary part is given in the second expression. The constant $\gamma'$ measures the strength of the coupling to $K\bar{K}$ and $G(s)$ is a form factor which includes left hand cuts, and if one wishes, a factor $2m_K/\sqrt{s}$ for relativistic phase space. The simplest parametrization of $G(s)$ is by a pole such that

\[
G(s) = [1 + (s - 4m_K^2)/\mu^2]^{-1}. \tag{2}
\]

The constant $\mu$ gives a cutoff, and is in order of magnitude given by the energy of the t-channel exchanges. Normally for a $q\bar{q}$ state this cutoff should be large. The squared running mass $m^2(s)$ in eq. (2) is $m_0^2$ plus the mass shift function $\text{Re}\Pi(s)$, which is given by $\text{Im}\Pi(s)$ through a dispersion relation

\[
m^2(s) = m_0^2 + \frac{1}{\pi} \mathcal{P} \int \frac{\text{Im}\Pi(s')}{s' - s} ds' = m_0^2 + \gamma'[-\mu + (s - 4m_K^2)/\mu + \sqrt{4m_K^2 - s} \Theta(4m_K^2 - s)] + \mathcal{O}(s - 4m_K^2)^{3/2}. \tag{3}
\]

The negative slope of this running mass, $\alpha = -dm^2/ds$, evaluated at the bound state pole is proportional to the probability ($Z$) to find the state as $K\bar{K}$, when the $q\bar{q}$

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\[\text{If the factor } 2m_K/\sqrt{s} \text{ from relativistic phase space is included in eq. (2) the slope of linear term in (3) is increased by } \gamma'/(m_K\pi). \text{ As in the discussion following eq. (3) this linear term can be absorbed by renormalizing the coupling } \gamma'.\]

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probability is normalized to 1, i.e., \( Z = \alpha/(1+\alpha) \). Thus for a \( q\bar{q} \) state, shifted in mass a little below an S-wave threshold, \( Z \) can be close to unity, since the slope diverges at the threshold. Thus most of the time such a "unitarized remnant of a \( q\bar{q} \) state" is found to be in a \( K\bar{K} \) state, while the \( q\bar{q} \) probability, \( 1 - Z = 1/(1+\alpha) \), is small although the state owes its existence to the \( q\bar{q} \) sector. This factor, \( 1/(1+\alpha) \), also appears in the \( K\bar{K} \) coupling constant defined by the pole residue, \( g^2_{K\bar{K}}/4\pi = \gamma/(1+\alpha) \), making the \( g_{K\bar{K}} \) very sensitive to the exact pole position.

The function \( \text{Re}\Pi(s) \) shifts the mass down by \( \approx \gamma'\mu \) and includes a linear term, \( s\gamma'/\mu \). This constant and linear term can, however, be absorbed by the corresponding terms of the inverse propagator eq. (1) (at least for small \( \gamma'/\mu < 1 \), see discussion below), by the redefinitions: \( m^2_{BW} = (m_0^2 - \gamma'/\mu)/(1 - \gamma'/\mu) \) and \( \gamma = \gamma/(1 - \gamma'/\mu) \), since the \( T \) matrix element, \( \gamma'G(s)P(s) \), actually depends on only two independent parameters, not on all three of eq. (3). Thus all the essential features of an S-wave \( q\bar{q} \) propagator near the threshold is described by the form Flatté used long time ago [3]:

\[
m^2(s) = m^2_{BW} + \gamma\sqrt{4m^2_K - s} + \Theta(4m^2_K - s) \tag{4}
\]

For this simple function of a constant plus a square root cusp the pole traces a path in the complex plane of the kaon c.m. momentum \( k_K \) shown in Fig. 1a by the dotted lines. The filled black squares are special examples. The straight lines are obtained assuming \( \Gamma_{\eta\pi} = 0 \), while one gets the hyperbola for a small but finite \( \Gamma_{\eta\pi} \). The \( K\bar{K} \) coupling \( \gamma \) is given in the figures a rather realistic value of 0.4 GeV, which gives a typical strong BW width of about 200 MeV, if the resonance would be 200 MeV above threshold. For \( \Gamma_{\eta\pi} = 0 \), bound states lie on the positive \( \text{Im}k_K \) axis, and resonance poles lie slightly below the positive real axis (\( \text{Im}k_K = -\gamma/4 \)). The shadow pole lies symmetrically reflected with respect to the point \((0, -\gamma/4)\) and at the unfilled squares for the examples. Normally these are far away from the physical region. (imaginary axis for bound states). But, when the \( q\bar{q} \) state is near the threshold (and in particular if it is a virtual bound state) then the shadow pole creeps up very close to the physical region, and influences the resonance shape, just as the true pole does.
If one allows a finite $\Gamma_{\pi\eta}$ (which is slowly varying in the narrow energy region of interest near the $K\overline{K}$ threshold) then the bound state pole is shifted to the left branch of the hyperbola in Fig. 1a, while the shadow pole is shifted to the right branch. With this open $\pi\eta$ channel there are 4 sheets of the energy plane; two for each threshold. Conventionally [7] these are numbered such that in the $k_K$ plane (which is cut by the $\pi\eta$ threshold along the imaginary axis) each quadrant correspond to sheets with Roman numbers shown in Fig. 1a.

The same poles discussed in Fig. 1a trace in the $s$-plane, Fig. 1b, much more complicated trajectories. With increasing bare $q\bar{q}$ mass the physical $q\bar{q}$ pole position (the solid line for $\Gamma_{\pi\eta} = 0$) increases until one reaches the threshold. Then, it creeps through the cut and becomes a virtual bound state, for which the pole mass actually decreases, although the bare mass increases! For such a virtual state $\text{Im}(\text{pole}) = 0$ and $\text{Re}(\text{pole}) < 4m_K^2$, but it is still seen also as a BW resonance with a finite $K\overline{K}$ width and phase shift passing through $90^\circ$ a little above threshold. Eventually, as the bare mass increases the pole acquires an imaginary part and $\text{Re}(\text{pole})$ increases again, while the resonance pole is along the lower branch of the parabola in Fig. 1b.

The shadow poles are also shown in Fig. 1b, by the dashed lines. These are close to the physical region when the regular pole is near threshold. The BW parameters are also shown (when $\Gamma_{\pi\eta} = 0$) by the filled circles and by the dotted half-parabola. As can be seen the pole parameters [the zeroes of eq. (1)] and the BW parameters [defined at the zeroes of the real part of eq. (1)] can differ considerably. With the value of $\gamma = 0.4$ GeV, as chosen in the figures, the BW mass is 50-100 MeV larger than the pole mass, and in particular a virtual bound state whose $\text{Im}(\text{pole}) = 0$, can have a BW width of 100 MeV!

The remaining dotted curves in Fig. 1b shows how the pole in the $s$-plane is shifted when one adds a small width, $\Gamma_{\pi\eta}$, as was done in Fig. 1a. It is interesting to note that it is the shadow of the bound state, and not the bound state itself, which is continuously connected to the resonance pole when one increases the mass of the state.
though the threshold region. Similarly it is the bound state (or the virtual bound state) pole which turns into the shadow pole of the resonance. Thus, although below and far above the threshold it is obvious which pole should be chosen as the physical pole, a little above the threshold both poles are "equally physical", and it is not clear which of the poles is the shadow pole! Near the threshold it is thus mandatory to give both poles in order to have a complete description of the pole structure.

In Fig. 2a $m^2(s) - m_{BW}^2$ of eq. (4) is shown with the same parameters as in Figs. 1a-b, when $\Gamma_{\eta\pi} = 0$. The crossing points with the line $s - m_{BW}^2$ gives the pole mass of the state if it is below threshold, while above threshold it simply gives the BW mass of a resonance. If the line crosses the shadow branch of the running mass (dashed in Fig. 2a) one finds the shadow pole, and if there are two crossings also the virtual bound state pole. Thus one can find from the same plot the positions of both pole and shadow pole, and for a virtual bound state one can furthermore also find the BW mass. For a resonance well above threshold the graph gives of course only the BW mass. One sees easily why two poles appear near threshold, and why the shadow pole comes equally close to the physical region as the true pole.

3. Deuteronlike states. In the discussion above I have up til now assumed that the linear term in (3) is small enough ($\gamma'/\mu < 1$), so that it can be removed by the redefinitions of $m_0$ and $\gamma'$. For a sufficiently steep form factor this condition is violated, and the nature of the solution changes dramatically. The shorter the distance ($\mu$) to the left hand singularities is, the further moves the shadow pole from the true pole until for $\gamma'/\mu = 1$ it is at $-\infty$.

This situation resembles, in fact, that of the deuteron (Fig. 2a), or in general that of any 2-hadron bound state. There, the bare $q\bar{q}$ term of eq. (1), $m_0^2 - s$, is of course absent and the inverse propagator is given by an analytic function $D(s)$, which like $\Pi(s)$ above, is assumed to approach a constant at $s \rightarrow \pm \infty$. When this function develops a zero, $D(m_d^2) = 0$, below threshold one has a bound state at $s = m_d^2$. The
running mass can be written:

\[ m_d^2(s) = s - D(s)/D'(m_d^2) = m_d^2 - \frac{(s - m_d^2)^2}{\pi D'(m_d^2)} P \int \frac{\text{Im} D(s') ds'}{(s' - m_d^2)^2(s' - s)}. \]  

(5)

In Fig. 2a the parameters for the deuteron \( m_d^2(s) \) are obtained from the scattering length \( a = 3.82 m_\pi^{-1} \) and effective range parameter \( r = 1.2 m_\pi^{-1} \) of the proton-neutron \( ^3S_1 \) wave phase shifts above threshold. With a left hand cut parametrized as \( G(s) \) of eq. (2) the parameter \( \mu = 2.68 m_\pi \) and the deuteron pole (more precisely, \( (4m_N^2 - m_d^2)^2 \)) are then fixed by \( a \) and \( r \) through \( 2[1 \pm (1 - 2r/a)^\frac{1}{2}] / r \). Up to terms of \( \mathcal{O}(s - 4m_N^2) \)

one has:

\[ m_d^2(s) = s + \frac{g_{dNN}^2}{4\pi} \left[ -\frac{2}{a} + (s - 4m_N^2) \left( \frac{r}{4} + \frac{1}{a\mu^2} \right) + \sqrt{4m_N^2 - s} \Theta(4m_N^2 - s) \right], \]  

(6)

where \( m_N \) is the nucleon mass, and \( g_{dNN}^2/(4\pi) \) is also fixed by \( a \) and \( r \) through the condition \( dm^2(s)/ds = 0 \) at the deuteron pole.

Notice that for a deuteronlike state the slope of the running mass above threshold must be larger than unity, as can be seen from eq. (5). This gives the condition, \( r + 4/(a\mu^2) > 0 \), or in practice that the effective range parameter \( r \) is positive \( \lbrack 3 \rbrack \), in contrast to the case of a \( q\bar{q} \) state where \( r = -8/\gamma \) is negative. This also implies that there is no crossing with \( s \) and the shadow branch, i.e. there is no shadow pole connected with a deuteronlike state, in agreement with the result of Ref. \( \lbrack 2 \rbrack \). Instead, above threshold there is a second crossing of \( s \) with the linear part of the running mass (Fig. 2b). This crossing is from below, which means that the phase shift decreases slowly through 90°. More generally, the phase shift obtained from the running mass \( m_d^2(s) \) is in accord with Levinson’s theorem (which holds for single channel potential scattering), and which requires the phase shift to drop by 180° from threshold to \( s = \infty \) to compensate for the existence of the deuteron pole.

4. How to parametrize a resonance? As we have seen, for S-wave resonances near the threshold resonance parameters depend on their definition. In particular, the mass is quite different when defined as the 90° BW mass or as the pole position.
Furthermore, one may need two poles to describe the same resonance! This, no doubt, is likely to cause headaches for anyone involved with compilations such as Ref. [1].

The running mass $m^2(s)$ contains all the essential information needed for the analytic continuation to the poles. Together with the rather trivial imaginary parts of eq. (3) it defines the propagator. This then determines any of the other parameters one may wish to know: the positions (and residues) of the poles (one or two), the coupling constant, the BW mass, the BW width, the scattering length and the effective range parameter.

Near the threshold $m^2(s)$ depends on two parameters only, as the examples in eqs. (4, 6) demonstrate. Eq. (3) could be replaced by a form like (4) through a reparametrization of the T matrix element, but then $m_{BW}^2 - 4m_K^2$ and $\gamma$ would have the opposite sign and the physical interpretation would be lost. Therefore, for hadron-hadron states one should use a form like (3) (Fig. 2b) with its linear term having a slope $> 1$.

Which two parameters should be chosen to fix $m^2(s)$ is a question of taste, provided they are not linearly dependent\(^4\), but a natural choice would be: (i) For $q\bar{q}$ state above threshold, the conventional Breit Wigner $90^\circ$ mass and the (partial) width; (ii) For $q\bar{q}$ state below threshold, the (second sheet) pole position and partial width coupling parameter $\gamma$; (iii) For a hadron-hadron state (below threshold), the pole mass and the effective range $r$ parameter (or the slope of $m^2(s)$ above threshold). But, may I suggest that anyone making a fit to a resonance near an S-wave threshold should also compute the other relevant parameters mentioned above.

In conclusion, for an S-wave resonance I find the running mass to be the natural quantity to be determined by experiment, which should be the best meeting ground for experiment, phenomenology and theory.

\(^4\)One pole and its residue is not sufficient to fix the two parameters of the running mass, because near the threshold both of these depend to lowest order only on the quantity $(m_{BW}^2 - m_K^2)/\gamma$. Thus to fix the running mass one must also give the second (shadow) pole or another parameter.
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Figure captions.

Fig. 1
(a) The trajectories of the poles in the complex plane of the c.m. meson momentum $k_K$, when the bare meson mass is varied. The dotted straight lines show the position of the bound state pole, or the third sheet resonance pole and their shadow poles when $\Gamma_{\pi\eta} = 0$. A filled square is an example of the physical pole position, while an unfilled square shows the shadow pole. The points on the hyperbola show how the poles are shifted when one adds a small finite $\Gamma_{\pi\eta}$ (15 MeV). See text for details.
(b) The same poles as in (a) but in the $s$-plane. The full drawn curve shows the physical pole positions and the dashed curve the shadow poles when $\Gamma_{\pi\eta} = 0$. The dotted curve show how the poles are shifted when one adds a small $\Gamma_{\pi\eta}$. Note that it is the shadow pole which turns into the ”physical” third sheet resonance pole. The dotted lower half parabola show the BW parameters $(m_{BW}, -m\Gamma)$. See text for details.

Fig. 2.
(a) The running mass $m^2(s)$ of eq. (I) from which one can read off the pole and shadow pole positions for bound states and virtual bound states of Fig. 2b, and in addition the BW mass above threshold. See text.
(b) The running mass for the deuteron fixed by the experimental values for the scattering length $a = 3.82m_\pi^{-1}$ and effective range $r = 1.2m_\pi^{-1}$ parameters. The crossing point below threshold gives the deuteron pole. Note that the slope of the running mass above threshold is $> 1$ which is the signature for a hadron-hadron bound state. See text.
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