Cluster models of $^{6}_{\Lambda\Lambda}$He and $^{9}_{\Lambda}$Be hypernuclei

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Abstract

Configuration space Faddeev calculations are performed for the binding energy of $^{6}_{\Lambda\Lambda}$He and $^{9}_{\Lambda}$Be bound states, here considered as $\alpha\Lambda\Lambda$ and $\alpha\alpha\Lambda$ clusters respectively, in order to study the dependence of the calculated binding energy on the $\alpha\Lambda$ potential input. For $^{6}_{\Lambda\Lambda}$He, using realistic interactions, the uncertainty in extracting the $^{1}\Lambda S_{0}$ interaction strength does not exceed 0.1 MeV, which is a fraction of the order of magnitude derived for other theoretical uncertainties. For $^{9}_{\Lambda}$Be, the dependence of the calculated binding energy on the $\alpha\Lambda$ potential is considerably larger, of order 1 MeV. Our results for $^{9}_{\Lambda}$Be suggest that the odd-state $\alpha\Lambda$ interaction is substantially reduced with respect to the even-state component.

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I. INTRODUCTION

The recent identification of $^6\Lambda\Lambda$He and the measurement of the binding energy of the $\Lambda\Lambda$ pair [1] has provided an experimental benchmark for extracting the strength of the $\Lambda\Lambda$ interaction within perhaps the simplest $\Lambda\Lambda$ hypernuclear system. A measure of the $\Lambda\Lambda$ interaction strength is given by the incremental binding energy:

$$\Delta B_{\Lambda\Lambda}(^{6}\Lambda\Lambda) = B_{\Lambda\Lambda}(^{6}\Lambda\Lambda) - 2B_{\Lambda}(^{5}\Lambda) = B_{\Lambda}(^{6}\Lambda\Lambda) - B_{\Lambda}(^{5}\Lambda) = 1.0\pm0.2^{+0.2}_{-0.1} \text{ MeV},$$

(1)

and is rather small compared with a similar measure of the $\Lambda N$ interaction strength

$$\Delta B_{\Lambda N}(^{5}\Lambda) = B_{\Lambda N}(^{5}\Lambda) - \bar{B}_{\Lambda}(^{4}\Lambda Z) - B_{N}(^{4}\Lambda) = B_{\Lambda}(^{5}\Lambda) - B_{\Lambda}(^{4}\Lambda Z) = 1.73\pm0.04 \text{ MeV},$$

(2)

where $\bar{B}_{\Lambda}$ stands for a proper spin and charge average over the binding energies of the $0^+$ and $1^+$ states in $^{4}\Lambda Z$ and $^{5}\Lambda$He [2]. The corresponding $NN$ interaction strength is considerably larger than either of these values. Several subsequent calculations have discussed and analyzed, within three-body cluster models for the $A = 5, 6$ $\Lambda\Lambda$ hypernuclei, the model dependence of such extraction of the $\Lambda\Lambda$ interaction strength [2–9]. In our latest report of Faddeev calculations for these $\Lambda\Lambda$ hypernuclei [4], the approximation of considering only implicitly the channel coupling $\Lambda\Lambda - \Xi N - \Sigma\Sigma$ was estimated qualitatively, within the semi-realistic NSC97 OBE model, to underestimate the calculated binding energy by about 0.2 MeV. This has been very recently borne out by an explicit calculation [8]. In the present work we study the theoretical uncertainty resulting from the model dependence of the $\alpha\Lambda$ interaction which normally is constrained only by fitting to the experimental $B_{\Lambda}$ value of $^{5}\Lambda$He. We will find that this uncertainty, for realistic potentials, amounts to about 0.1 MeV. In this context, we also study here the sensitivity of the $^{9}\Lambda\Lambda$Be binding energy calculation, within an $\alpha\alpha\Lambda$ model, to the assumed $\alpha\Lambda$ interaction since it is known that $B_{\Lambda}(^{9}\Lambda\Lambda)$ exhibits considerable sensitivity to the unknown $p$-wave component of the $\alpha\Lambda$ interaction [10]. We note several other few-body calculations of $^{9}\Lambda\Lambda$Be [11–15]. Recently a more realistic $\alpha\Lambda$ potential consisting of separate $s$- and $p$-wave components [5] was constructed, inspired by the NSC97 model of the $\Lambda N$ interaction. The resulting $p$-wave component is weaker than the $s$-wave component. In the present work we use this $\alpha\Lambda$ potential for the first time in the $^{9}\Lambda\Lambda$Be binding energy calculation, in order to check the sensitivity to the $p$-wave component.

Our methodology consists of using the Faddeev equations formalism in configuration space, describing $^{6}\Lambda\Lambda$He and $^{9}\Lambda\Lambda$Be in terms of three-body systems made out of $\alpha$ clusters and $\Lambda$ particles. The bound-state Faddeev equations for $^{9}\Lambda\Lambda$Be, viewed as a three-cluster $\alpha\alpha\Lambda$ system, are discussed in Sect. II, in close analogy to the discussion of the Faddeev equations for $^{6}\Lambda\Lambda$He ($\alpha\Lambda\Lambda$) in Ref. [4]. Aspects of the partial-wave decomposition of the configuration-space Faddeev equations are relegated to an Appendix. In Sect. III we specify the $\Lambda\Lambda$ and $\alpha\Lambda$ interaction potentials used in the present work to test the model dependence of the Faddeev calculations for binding energies. The results of our calculations are given in Sect. IV. For $^{6}\Lambda\Lambda$He, using a sufficiently expanded range of partial waves [4], we calculated the $\Lambda\Lambda$ binding energy $B_{\Lambda\Lambda}$ for several $\alpha\Lambda$ potentials. Defining a measure $\delta B_{\Lambda\Lambda}$ of the $\Lambda\Lambda$ ‘pairing’ energy, the variation of $\delta B_{\Lambda\Lambda}$ with the input $\alpha\Lambda$ potential is used to place limits on the model dependence of choosing $V_{\Lambda\Lambda}$ from binding energy calculations. Finally, we
give results for $^9\Lambda$Be (\(\frac{1}{2}\))\(^+\) ground state and for the first excited (\(\frac{3}{2}\), \(\frac{5}{2}\)\(^+\)) doublet of levels, for a variety of $\alpha\Lambda$ interaction potentials. Here the $p$-wave component, particularly, plays a large role, considerably larger than for $^6\Lambda\Lambda\Lambda\Lambda\Lambda$He. The conclusion to be drawn from the present Faddeev calculations of $^9\Lambda$Be is that, unless three-body $\alpha\alpha\Lambda$ repulsive interactions contribute substantially (of order 1 MeV), the $p$-wave component of the $\alpha\Lambda$ interaction must be very weak, close to zero, with respect to the $s$-wave component. The paper is concluded with a summary in Sect. V.

II. BOUND-STATE FADDEEV EQUATIONS FOR $\alpha\alpha\Lambda$

Bound states of three-body systems are calculated by solving the differential Faddeev equations [16], which for a combination of short-range nuclear forces and Coulomb forces have the following form:

\[
(H_0 + V_\gamma^\alpha(|\vec{x}_{\gamma}|) + \sum_{\beta=1}^{3} V_\beta^c(|\vec{x}_{\beta}|) - E)\Psi_{\gamma}(\vec{x}_{\gamma}, \vec{y}_{\gamma}) = -V_\gamma^\alpha(|\vec{x}_{\gamma}|) \sum_{\beta \neq \gamma}^{3} \Psi_{\beta}(\vec{x}_{\beta}, \vec{y}_{\beta}) ,
\]

where $V_\beta^c$ is the Coulomb force between the particles of the pair denoted by channel $\beta$ ($\beta=1,2,3$) and $V_\gamma^\alpha$ is the short-range pair interaction in the channel $\gamma$ ($\gamma=1,2,3$). The kinetic energy operator is given by $H_0 = -\Delta_{\vec{x}_{\gamma}} - \Delta_{\vec{y}_{\gamma}}$ and $E$ is the total energy. The variables $\vec{x}_{\gamma}, \vec{y}_{\gamma}$ are properly normalized relative Jacobi coordinate vectors, as defined in Ref. [4]. The wave function of the three-body system $\Psi$ is given by a sum over the three Faddeev components, $\Psi = \sum_{\gamma=1}^{3} \Psi_{\gamma}$.

When two particles of the three-body system are identical, as the $\alpha$ particles are in $^9\Lambda$Be ($\alpha\alpha\Lambda$), the coupled set of Faddeev equations reduces to two equations:

\[
(H_0 + V_{\alpha\alpha} + V_3^c - E)U_3 = -V_{\alpha\alpha}(U_1 + P_{12}U_1) ,
(H_0 + V_{\alpha\Lambda} + V_1^c - E)U_1 = -V_{\alpha\Lambda}(U_3 + P_{12}U_1) ,
\]

where $P_{12}$ is the permutation operator for the $\alpha$ bosons (particles 1,2), $V_{\alpha\alpha}$ and $V_{\alpha\Lambda}$ are nuclear potentials for the $\alpha\alpha$ and $\alpha\Lambda$ interactions respectively, $U_3$ is the Faddeev component corresponding to the rearrangement channel ($\alpha\alpha$) - $\Lambda$ and $U_1$ corresponds to the rearrangement channel ($\alpha\Lambda$) - $\alpha$, $V_3^c$ is the Coulomb potential for channels $\gamma = 1, 3$. The total wave function is expressed by the components $U_1$ and $U_3$: $\Psi = U_3 + (1 + P_{12})U_1$. The total orbital angular momentum is given by $\ell = \ell_{\alpha\alpha} + \lambda_{(\alpha\alpha) - \Lambda} = \ell_{\alpha\Lambda} + \lambda_{(\alpha\Lambda) - \alpha}$, where $\ell_{\alpha\alpha}$ ($\ell_{\alpha\Lambda}$) is the orbital angular momentum of the $\alpha\alpha$ ($\alpha\Lambda$) pair and $\lambda_{(\alpha\alpha) - \Lambda}$ ($\lambda_{(\alpha\Lambda) - \alpha}$) is the orbital angular momentum of the $\Lambda$ hyperon ($\alpha$ particle) with respect to the center of mass of the $\alpha\alpha$ ($\alpha\Lambda$) pair. Bose symmetry for the $\alpha\alpha$ pair requires that $\ell_{\alpha\alpha}$ assumes non-negative even values, and the positive parity of the $^9\Lambda$Be states considered here requires that $\ell_{\alpha\alpha} + \lambda_{(\alpha\alpha) - \Lambda}$ and $\ell_{\alpha\Lambda} + \lambda_{(\alpha\Lambda) - \alpha}$ are non-negative even numbers. For convenience we have assembled in the Appendix of the present paper details related to the partial-wave decomposition of Eqs.(4).

The lowest states of $^9\Lambda$Be are expected to follow the structure of the corresponding low-lying states of $^8$Be. Thus, the (\(\frac{1}{2}\))\(^+\) ground state of $^9\Lambda$Be, considered as an $\alpha\alpha\Lambda$ system, is assumed to have $L = 0$ total orbital angular momentum. The allowed combinations of the relative angular momenta ($\ell_{\alpha\alpha}, \lambda_{(\alpha\alpha) - \Lambda}$) are (0,0), (2,2), (4,4), \ldots; and for ($\ell_{\alpha\Lambda}, \lambda_{(\alpha\Lambda) - \alpha}$) are
The quantum numbers $\ell_{\alpha\alpha}$ and $\ell_{\alpha\Lambda}$ specify completely the orbital angular momentum states of the corresponding subsystems.

The first excited \((3^{+}, 5^{+})\) doublet of \(^{9}_{\Lambda}\)Be is assumed to have \(L = 2\) total orbital angular momentum. The allowed combinations of the relative angular momenta \((\ell_{\alpha\alpha}, \lambda_{(\alpha\alpha)-\Lambda})\) are \((0,2), (2,0), (2,2), (4,4), \ldots\); and for \((\ell_{\alpha\Lambda}, \lambda_{(\alpha\Lambda)-\alpha})\) are \((0,2), (1,1), (1,3), (2,0), (2,2), (2,4), \ldots\).

For the $\Lambda\alpha\alpha$ system the Faddeev equations (4) have a similar form (but with a minus sign in front of $P_{12}$ for the $\Lambda$ fermions, and without the Coulomb potential) which has been given in Refs. [2,4] for the $^{6}_{\Lambda}\Lambda\Lambda$He hypernucleus.

### III. POTENTIALS

To describe interactions in the $\alpha\Lambda\Lambda$ and $\alpha\alpha\Lambda$ systems, local pairwise potentials are used. The input $\alpha\alpha$ nuclear interaction is given by version $a$ of the phenomenological Ali-Bodmer (AB) potential [17] (as modified in Ref. [18]) or by the Chien-Brown (CB) potential [19] which include $s$, $d$ and $g$-wave components, fitted to the low-energy phase shifts in the $\alpha\alpha$ system. The resulting $\alpha\alpha$ potential has the following form:

$$V_{\alpha\alpha}(r) = \sum_{l=0,2,4} V_{\alpha\alpha}^{l}(r) P_{l},$$

where $P_{l}$ is a projector onto the state of the $\alpha\alpha$ pair with orbital angular momentum $l$. The $s$-wave component $V_{\alpha\alpha}^{0}(r)$ has a strong repulsive core which simulates a Pauli blocking effect for the $\alpha\alpha$ pair at short distances. The $l$ dependence of the CB potential is demonstrated in Fig. 1, where we added for comparison an $l$-independent potential (WTB) due to Ref. [20].

This latter potential is purely attractive, yet at short distances the attraction is moderated by the Coulomb repulsion between the two $\alpha$s which is always included in the present calculation.

For the $\alpha\Lambda$ interaction, several potentials used in previous calculations [11–14] are listed in Table I. These potentials have different shapes, while reproducing closely the experimental value of the binding energy for the $^{5}_{\Lambda}\Lambda$He hypernucleus [21] which is considered to be an $s$-wave bound state of the $\alpha\Lambda$ system. The Tang-Herndon (TH) potential [22] used in Ref. [13] and the Gibson I (Gibson) potential [23] used in Ref. [24] are purely attractive Gaussian $s$-wave potentials. The Maeda-Schmid $s$-wave potential (MS) [25] is a sum of two Woods-Saxon functions, where the inner function is repulsive, moderating at short distances the attraction due to the outer function:

$$V_{\alpha\Lambda}(r) = \frac{V_{\text{rep}}}{1 + \exp((r - R_{\text{rep}})/a_{\text{rep}})} - \frac{V_{\text{att}}}{1 + \exp((r - R_{\text{att}})/a_{\text{att}})}.$$  

The Isle $s$-wave potential [26] is a sum of two Gaussians where the inner Gaussian is repulsive, outweighing at short distances the attractive outer Gaussian. In addition to these $s$-wave $\alpha\Lambda$ potentials, we used for the first time ever in $^{9}_{\Lambda}\Lambda$Be binding energy calculations a potential (MSA) proposed in Ref. [5] that includes also a $p$-wave component inspired by the NSC97 model of the $\Lambda N$ interaction. Generally, the contribution of the various partial waves to the $\alpha\Lambda$ potential is given by:
\[ V_{\alpha\Lambda}(r) = \sum_{l=0,1,...} V^{l}_{\alpha\Lambda}(r) P_l, \quad (7) \]

where \( P_l \) is a projector onto the state of the \( \alpha\Lambda \) system with orbital angular momentum \( l \), and
\[ V^{l}_{\alpha\Lambda}(r) = V^{l}_{\text{rep}} \exp\left(-\left(\frac{r}{\beta^{l}_{\text{rep}}}\right)^2\right) - V^{l}_{\text{att}} \exp\left(-\left(\frac{r}{\beta^{l}_{\text{att}}}\right)^2\right), \quad (8) \]
are the partial-wave components of the potential. The calculated \( s \)-wave scattering lengths and effective ranges, and the \( \Lambda^6 \text{He} \) binding energy for all of these potentials, are listed in Table II.

The \( \Lambda\Lambda \) interaction potentials in the \( ^1S_0 \) channel which are used as input to the Faddeev equations are of a three-range Gaussian form
\[ V_{\Lambda\Lambda} = \sum_{i} v^{(i)} \exp(-r^2/\beta_{i}^2), \quad (9) \]
following the work of Hiyama et al. [10] where a phase-equivalent \( \Lambda\Lambda \) potential of this soft-core form was fitted to the Nijmegen model D (ND) hard-core interaction [27] assuming the same hard core for the \( NN \) and \( \Lambda\Lambda \) potentials in the \( ^1S_0 \) channel. For other interactions, notably the Nijmegen soft-core NSC97 model \( \Lambda\Lambda \) potentials [28], we have renormalized the strength of the medium-range attractive component \( (i = 2) \) of this potential fitting as closely as possible the scattering length and the effective range. The appropriate range and strength parameters are listed in Tables 1 and 2 of Ref. [2]. For the NSC97e interaction, Myint et al. [5] have used a different parameterization which is listed in Table 3 of their paper. These soft-core-repulsion \( \Lambda\Lambda \) potentials, as well as a purely attractive Gaussian potential
\[ V^{1}_{\alpha\Lambda} = \frac{1}{4}(-52.25 \exp(-r^2/1.034^2)), \]
are shown in Fig. 2. This latter potential (marked Ikeda) was obtained by taking 1/4 of the potential used by Ikeda et al. [29].

IV. RESULTS

The Faddeev coupled equations (4) for the \( \alpha\Lambda\Lambda \) and \( \alpha\alpha\Lambda \) systems were solved numerically in configuration space, using the method given in Ref. [30] and applied by us in Ref. [4]. The partial-wave decomposition of these equations is relegated to the Appendix. We applied different values for the cutoff radius used for the \( \alpha\Lambda\Lambda \) and \( \alpha\alpha\Lambda \) systems. For the \( \alpha\Lambda\Lambda \) system a cutoff radius value \( \rho_{\text{cutoff}} = 25 \) fm was used, whereas for the \( \alpha\alpha\Lambda \) system a value \( \rho_{\text{cutoff}} = 40 \) fm was used (considering the long-range Coulomb force).

A. \( \Lambda^6 \text{He} \)

The \( \Lambda\Lambda \) binding energies \( E_B \) (\( E_B = -B_{\Lambda\Lambda} \)) calculated for the \( 0^+ \) ground state of \( \Lambda^6 \text{He} \), viewed as a three-body system \( \alpha\Lambda\Lambda \) with \( L^\pi = 0^+, S = 0 \) quantum numbers, are listed in Table III for several combinations of \( \Lambda\Lambda \) and \( \alpha\Lambda \) interaction potentials discussed in Sect. III. The orbital angular momenta included are \( l_{\alpha\Lambda} = 0,1,2,3,4,5,6, l_{\Lambda\Lambda} = 0,2,4,6, \) ensuring convergence within few keV [4]. Here an ‘s-wave’ model is used for the \( \alpha\Lambda \) interaction, meaning that \( V^{l}_{\alpha\Lambda} \) is independent of \( l \) and equals \( V^{l=0}_{\alpha\Lambda} \). The high partial-wave contributions of the \( \alpha\Lambda \)
potential for this system are small and in total do not exceed about 0.2 MeV [4]. Note that for \( V_{\Lambda\Lambda} = 0 \), the calculated binding energy is larger (in absolute values) than twice the \( ^5\Lambda\Lambda\) binding energy for all of the \( \alpha\Lambda \) potentials listed in Table II. Hence \( \Delta B_{\Lambda\Lambda}(V_{\Lambda\Lambda} = 0) > 0 \). This nonzero value is due to the mass-polarization term contained in the kinetic energy operator [3,31]. In order to eliminate the contribution from this term, we have listed in Table III, in parentheses, the \( \Lambda\Lambda \) ‘pairing’ energies corresponding to a given \( V_{\Lambda\Lambda} \), using the definition

\[
\delta B_{\Lambda\Lambda} = E_B(V_{\Lambda\Lambda} = 0) - E_B(V_{\Lambda\Lambda}).
\] (10)

The calculated \( \delta B_{\Lambda\Lambda} \) values are also plotted in Fig. 3. For a given \( \Lambda\Lambda \) interaction, a clear dependence of \( \delta B_{\Lambda\Lambda} \) on the \( \alpha\Lambda \) potential is observed, and it is related to the hard core of the \( \Lambda\Lambda \) interaction which acts more effectively for purely attractive \( \alpha\Lambda \) potentials (TH, Gibson) that ‘compress’ the three-body system, leading to a smaller value of \( \delta B_{\Lambda\Lambda} \) than for \( \alpha\Lambda \) repulsive-core potentials (MS, MSA, Isle). This dependence on the \( \alpha\Lambda \) potential is reversed, as shown by the added dashed-line histogram in Fig. 3, for the purely attractive (Ikeda) \( \Lambda\Lambda \) interaction potential. Anticipating some kind of repulsive cores in the \( \Lambda\Lambda \) and \( \alpha\Lambda \) interaction potentials, one may conclude that the uncertainty in \( \delta B_{\Lambda\Lambda} \) due to the type of the \( \alpha\Lambda \) potential is less than 0.1 MeV. We note that the uncertainty in \( \Delta B_{\Lambda\Lambda} \) is larger, amounting for the Myint (e) potential in Table III to 0.4 MeV which is close to the uncertainty found by Carr et al. [32] in their \( \alpha\Lambda\Lambda \) study of \( ^6\Lambda\Lambda\)He (surveying a range of considerably stronger \( \Lambda\Lambda \) interactions prior to the NAGARA event [1] report).

B. \( ^9\Lambda\Lambda\)Be

The binding energies \( E_B \) calculated for the \((\frac{1}{2})^+ \) ground state of \( ^9\Lambda\Lambda\)Be, viewed as a three-body system \( \alpha\alpha\Lambda \), are listed in Table IV for several combinations of \( \alpha\alpha \) and \( \alpha\Lambda \) interaction potentials discussed in Sect. III. We have studied two different potential models. In the first one, the ‘s-wave’ model as described above for \( ^5\Lambda\Lambda\)He, the s-wave \( \alpha\Lambda \) potential acts in all partial waves of the \( \alpha\Lambda \) subsystem: \( V_{\alpha\Lambda}^l(r) = V_{\alpha\Lambda}^0(r) \), with \( l=1,2,3,4,5 \). In the second model, the ‘s and p-wave’ model, we retained only the \( l = 0,1 \) partial-wave components (different from each other) of the MSA \( \alpha\Lambda \) potential (7). For the first model, the agreement between our results (FGS) and those of Cravo et al. [14] which were derived solving integral equations is sufficiently good. Nevertheless it should be noted that in the latter approach the nuclear potentials are approximated by separable potentials which might explain some small differences between our results and those of Ref. [14].

All partial waves up to \( l = 5 \) were taken into account in our calculations, with the partial waves \( l_{\alpha\Lambda} \leq 2 \), \( l_{\alpha\alpha} \leq 2 \) dominating the total contribution. The Coulomb repulsion between the \( \alpha \) clusters is included. The results are insensitive to which realistic \( \alpha\alpha \) interaction (AB or CB) is used. Comparing the first and the second models with each other one observes that in the first model, the ‘s-wave’ model, the p-wave contribution to the \( \alpha\alpha\Lambda \) binding energy is substantial, exceeding 1 MeV for the repulsive-core MSA and Isle potentials. Using the more realistic MSA \( \alpha\Lambda \) potential [5] in the second model, the ‘s and p-wave’ model, with a weaker p-wave component, the calculated binding energy of \( ^9\Lambda\Lambda\)Be is reduced significantly by about 0.75 MeV. The fairly wide spectrum of calculated binding energies for the \( \alpha\alpha\Lambda \) system in the
's-wave' model, from 6 MeV to 8 MeV, may be explained in a similar way to the explanation of the considerably weaker dependence on the $\alpha\Lambda$ potential of the calculated binding energies for the $\alpha\Lambda\Lambda$ system in Table III. The purely attractive TH and Gibson potentials ‘compress’ the system, making the strongly repulsive core of the $s$-wave $\alpha\alpha$ interaction quite effective. Therefore, the calculated binding energy for these potentials is smaller (in absolute value) than for the Isle and MSA potentials. On the other hand, using a purely attractive unrealistic $\alpha\alpha$ potential (WTB, first row of Table IV), the dependence on the $\alpha\Lambda$ potential is negligible, not exceeding 0.1 MeV, except for MSA which deviates by 0.3 MeV.

The effect of the $\alpha\alpha$ interaction may be demonstrated by switching it off, while keeping on the Coulomb repulsion in these $\alpha\alpha$ binding energy calculations. The corresponding binding energies given in the last row of Table IV vary by less than 1 MeV of each other, but in reverse order to that when $V_{\alpha\alpha}$ is on. Consequently, if one defines in analogy to Eq. (10) the $\alpha\alpha$ ‘pairing’ energy corresponding to a given $V_{\alpha\alpha}$,

$$\delta B_{\alpha\alpha} = E_B(V_{\alpha\alpha} = 0) - E_B(V_{\alpha\alpha}), \quad (11)$$

it displays quite a strong dependence on the $\alpha\Lambda$ interaction used in the calculation, extending over a range of 3 MeV. This is shown in Fig.4. The attractive nature of the $\alpha\alpha$ interaction comes out clearly for $\alpha\Lambda$ interaction potentials that contain inner repulsion. The magnitude of $\delta B_{\alpha\alpha}$ is modest with respect to what it would have been, had the Pauli principle been implicitly ignored in the construction of the $\alpha\alpha$ potentials. There is hardly a difference between the $\alpha\alpha$ potentials AB and CB, whereas the variation of $\delta B_{\alpha\alpha}$ for the purely attractive WTB $\alpha\alpha$ potential is more moderate. In summary, $^9\Lambda$Be, with a relatively light $\Lambda$ ‘core’ particle, does not provide a useful hadronic medium in which to determine the $\alpha\alpha$ pairing energy.

Finally, in Table V we show the calculated excitation energies of the excited $(\frac{3}{2}^+, \frac{5}{2}^+)$ spin-flip doublet of $^9\Lambda$Be levels based on the first excited $2^+ \ ^8\Lambda$Be level. The agreement between our results and those by Cravo et al. [14], and also with respect to the observed levels which are split by less than 50 keV [33], is particularly good for the choice of CB $\alpha\alpha$ interactions. In contrast, the non-Faddeev calculations by Portilho and Coon [12] overestimate substantially the Faddeev results and should be looked at with a grain of salt. In our calculation, $l_{\alpha\Lambda}$ = 0, 1, 2 partial waves are included, except for the ‘$s$ and $p$-wave’ results which used $l_{\alpha\Lambda} = 0, 1$, together with $l_{\alpha\alpha} = 0, 2, 4$ everywhere.

V. SUMMARY

The main question addressed in this work was the model dependence of $\delta B_{\alpha\alpha}$, extracted from the calculated $^6\Lambda$He binding energy, due to the $\alpha\Lambda$ potential used in the configuration-space Faddeev $\alpha\alpha\Lambda$ calculation. We have found that this model dependence does not exceed 0.1 MeV, provided realistic $\alpha\Lambda$ potentials that consist of both short-range repulsion and longer-range attraction are used. The $\alpha\Lambda$ potential manifests itself in the Faddeev calculation mostly through its $s$-wave component which is regulated by fitting to $^5\Lambda$He binding energy, the higher partial waves adding up to 0.2 MeV [4]. The dependence on the particular structure of the $p$-wave component is then less than 0.1 MeV. In contrast, the calculated $^9\Lambda$Be binding energy is very sensitive to the $p$-wave contents of the $\alpha\Lambda$ potential. In agreement with
Hyama et al. [10] we find effects of up to 1 MeV due to the p-wave component of the αΛ interaction. We have found that the p-wave component must be weakened considerably with respect to what an l-independent αΛ potential constrained only by the $^{5}_\Lambda$He binding energy yields. Using the MSA l-dependent potential [5], the calculated binding energy of $^{9}_\Lambda$Be is still overestimated by 0.6-0.7 MeV, an amount which could possibly arise from three-body forces [34] unaccounted for in these Faddeev calculations.

VI. APPENDIX

The partial-wave decomposition of Eq.(3) for the ααΛ system leads to the following set of partial differential equations for the Faddeev components $\Psi_\alpha = \rho^{-1/2}U_\alpha$, in polar coordinates $\rho^2 = x_\alpha^2 + y_\alpha^2$, $\tan \theta = |y_\alpha|/|x_\alpha|$:

\[
\begin{align*}
\{ - \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + V_\alpha^l(\rho, \theta) + \frac{l(l+1)}{\rho^2 \cos^2 \theta} + \frac{\lambda(\lambda+1)}{\rho^2 \sin^2 \theta} - \frac{1}{4\rho^2} - E \} U_1^{l\lambda}(\rho, \theta) \\
+ \frac{q}{\rho} (-1)^L \sum_{l', \lambda'} \Pi_{l\lambda \lambda'} U_2^{l', \lambda'}(\rho, \theta) \sum_{\tau = \min(|l-l'|, |\lambda-\lambda'|)} C_{l00}^{l00} C_{\lambda00}^{\tau0} \{ \lambda \lambda' \tau \} \frac{b_\tau}{2\tau + 1} \frac{V_0^{l0}}{V_0^{l0}} \{ \lambda \lambda' \tau \} \}
\end{align*}
\]

\[
\begin{align*}
= -\frac{1}{2} V_3^{l}(\rho, \theta) \sum_{l', \lambda'} \{ (-1)^{l'} (h_{l \lambda \lambda'}^{2l})(\rho, \theta) + (-1)^{l+l'} (h_{l \lambda \lambda'}^{13})(\rho, \theta) \},
\end{align*}
\]

where $\Pi_{l\lambda \lambda'} = \sqrt{(2l+1)(2\lambda+1) \cdots}$, $C_{l00}^{l00}$ are Clebsch-Gordan coefficients and $\{ \cdots \}$ is a $6j$ symbol. Here $b_\tau$ stand for

\[
b_\tau = \begin{cases} 
\frac{1}{|C_{31}| \cos \theta \left( \frac{C_{31} \sin \theta}{C_{31} \cos \theta} \right) \tau}, & \tan \theta < \frac{C_{31}}{\bar{S}_{31}}, \\
\frac{1}{|S_{31}| \sin \theta \left( \frac{C_{31} \sin \theta}{S_{31} \cos \theta} \right) \tau}, & \tan \theta \geq \frac{C_{31}}{\bar{S}_{31}},
\end{cases}
\]

and $q$ denotes the mass-scaled charge

\[
q = 4e^2 \hbar \sqrt{m_\alpha m_\Lambda}.
\]

We used the scaled Jacobi vectors $\vec{x}_\alpha, \vec{y}_\alpha$, related to the standard particle vector coordinates by

\[
\begin{align*}
\vec{x}_\alpha &= \left( \frac{2m_\beta m_\gamma}{m_\beta + m_\gamma} \right)^{1/2} \left( \vec{r}_\beta - \vec{r}_\gamma \right), \\
\vec{y}_\alpha &= \left( \frac{2m_\alpha (m_\beta + m_\gamma)}{M} \right)^{1/2} \left( \frac{m_\beta \vec{r}_\beta + m_\gamma \vec{r}_\gamma}{m_\beta + m_\gamma} - \vec{r}_\alpha \right),
\end{align*}
\]

(14)
In Eqs. (12,13) the Coulomb repulsion between the \( \alpha \) l summation over account of any number of partial waves. In actual calculations, the number of terms in the expression above may be carried out to obtain a simpler expression of the form

\[
\cos^2 \theta'(u, \theta) = C_{\beta\alpha}^2 \cos^2 \theta + 2C_{\beta\alpha}S_{\beta\alpha} \cos \theta \cdot u + S_{\beta\alpha}^2 \sin^2 \theta.
\]

More explicitly:

\[
\begin{align*}
&h_{\ell \lambda \lambda', \ell'}^{L_{\beta\alpha}}(\theta, \theta') = (-)^{L+I'+L} \Pi_\lambda \Pi_{\ell'} \sqrt{(2l')!(2\lambda')!} \times \sum_{\lambda_1+\lambda_2=\lambda', 1+1'=1'} \left( \frac{\sin \lambda_1+1 \theta \cos \lambda_2+2 \theta}{\sin \lambda' \theta'} \right)^{\lambda_1+1 \theta} \left( \frac{(-)^{\lambda}C_{\beta\alpha}^{\lambda_1+2, \lambda_2+1}}{(2\lambda_1)!(2\lambda_2)!} \right)^{1/2} \times \sum_{\lambda''} \Pi_{\ell''} \left( \begin{array}{ccc} \lambda_1 & 1 & l_1 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \lambda_2 & l_2 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \lambda & \lambda' & \lambda'' \\ l_1 & l_2 & l' \end{array} \right) \left( \begin{array}{ccc} 0 & \lambda & L \\ 0 & \lambda' & k \end{array} \right) P_k(u),
\end{align*}
\]

in terms of 3j, 6j, 9j symbols and Legendre polynomials \( P_k(u) \). The index \( k \) runs in Eq. (16) from zero to \((\lambda + l' + \lambda + l)/2\).

For zero total orbital angular momentum \( L = 0 \) \((\lambda = l, \lambda' = l')\), all the summations in the expression above may be carried out to obtain a simpler expression of the form

\[
h_{\ell \lambda \lambda', \ell'}^{\alpha \beta}(\theta, \theta') = (-1)^{l+l'} \sqrt{(2l+1)(2\ell' + 1)} P_l(u) P_{\ell'}(u'),
\]

where

\[
u = \frac{-\cos(2\theta') + (C_{\beta\alpha}^2 - S_{\beta\alpha}^2) \cos(2\theta')}{2C_{\beta\alpha}S_{\beta\alpha} \sin(2\theta')},
\]

To solve the eigenvalue problem in the region \( \rho \in [0, \infty], \theta \in [0, \pi/2] \), Eqs.(12,13) must be supplemented by the boundary conditions

\[
U_{\gamma}^l(0, \theta) = U_{\gamma}^l(\infty, \theta) = U_{\gamma}^l(\rho, 0) = U_{\gamma}^l(\rho, \pi/2) = 0.
\]

In Eqs.(12,13) the Coulomb repulsion between the \( \alpha \)'s is included rigorously, allowing full account of any number of partial waves. In actual calculations, the number of terms in the summation over \( l' \) is truncated once convergence has been achieved.
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TABLES

TABLE I. Parameters of $\alpha\Lambda$ potentials used in the present work. The notations follow Eq.(8), except for MS where $\beta = (R, a)$ in terms of Eq.(6).

| Potential | $l$ | $V^l_{\text{rep}}$ (MeV) | $\beta^l_{\text{rep}}$ (fm) | $V^l_{\text{att}}$ (MeV) | $\beta^l_{\text{att}}$ (fm) |
|-----------|----|--------------------------|---------------------------|-----------------------|--------------------------|
| TH [13]   | 0  | –                        | –                         | 60.17                 | 1.2729                   |
| Gibson [24]| 0  | –                        | –                         | 43.48                 | 1.5764                   |
| MS [25]   | 0  | 18.09                    | (0.88, 0.2353)            | 35.98 (1.72, 0.3541)  |                           |
| Isle(DA)  [26]| 0  | 450.4                    | 1.25                      | 404.9                 | 1.41                     |
| MSA [5]   | 0  | 91.0                     | 1.3                       | 95.0                  | 1.7                      |
|           | 1  | 33.4                     | 1.3                       | 39.4                  | 1.7                      |

TABLE II. Scattering length $a$, effective range $r_0$ and binding energy $E_B$ of the $^5\Lambda\text{He}$ hypernucleus for various $\alpha\Lambda$ potentials.

| Potential | $a$ (fm) | $r_0$ (fm) | $E_B$ (MeV) |
|-----------|----------|------------|-------------|
| Isle(DA)  | 4.24     | 2.05       | -3.10       |
| MSA       | 4.18     | 1.97       | -3.12       |
| MS        | 4.00     | 1.67       | -2.84       |
| Gibson    | 3.80     | 1.53       | -3.08       |
| TH        | 3.63     | 1.32       | -3.03       |
| exp. [21] | –        | –          | -3.12±0.02  |

TABLE III. $E_B(^4\Lambda\Lambda\text{He})$ (and in parentheses $\delta B_{\Lambda\Lambda}$, Eq. (10)) in MeV, calculated for various $\Lambda\Lambda$ and $\alpha\Lambda$ potentials. Energy is measured with respect to the $\alpha + \Lambda + \Lambda$ threshold. The ‘s-wave’ model is used for the $\alpha\Lambda$ interaction. $E_B^{\text{exp}}(0^+) = -7.25\pm0.19_{-0.11}^{+0.18}$ MeV [1].

| Potential | TH   | Gibson | MS    | MSA   | Isle |
|-----------|------|--------|-------|-------|------|
| Myint(e)  | -6.853 | -7.084 | -6.853 | -7.107 | -6.992 |
|           | (0.518) | (0.701) | (0.735) | (0.698) | (0.651) |
| NSC97e    | -6.593 | -6.877 | -6.476 | -6.998 | -6.903 |
|           | (0.258) | (0.494) | (0.586) | (0.589) | (0.562) |
| NSC97b    | -6.200 | -6.541 | -6.182 | -6.773 | -6.698 |
|           | (-0.139) | (0.158) | (0.292) | (0.364) | (0.357) |
| $V_{\Lambda\Lambda}=0$ | -6.335 | -6.383 | -5.890 | -6.409 | -6.341 |

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TABLE IV. $E_B^{(0}_\Lambda$Be g.s.) in MeV, calculated for various $\alpha\alpha$ and $\alpha\Lambda$ potentials. Energy is measured with respect to the $\alpha + \alpha + \Lambda$ threshold. $E_B^{\text{exp}}(1^+)$ = $-6.62 \pm 0.04$ MeV [21].

| Ref. | $\alpha\alpha$ | $\alpha\Lambda$ | $l_{\alpha\Lambda}$ | $l_{\alpha\alpha}$ | TH | Gibson | MS | MSA | Isle(DA) |
|------|----------------|----------------|---------------------|-------------------|----|--------|----|-----|-------|
| FGS  | WTB           | ‘s-wave’       | 0,1,2              | 0,2,4             | -6.623 | -6.725 | -6.718 | -6.932 | -6.726 |
| FGS  | AB            | ‘s-wave’       | 0                  | 0,2,4             | -5.043 | -5.598 | -5.530 | -6.542 | -6.581 |
|      |                |                | 0,1                | 0,2,4             | -5.924 | -6.066 | -6.571 | -7.851 | -8.030 |
|      |                |                | 0,1,2              | 0,2,4             | -5.991 | -6.709 | -6.664 | -7.947 | -8.119 |
|      |                |                | 0,1,2,3,4,5        | 0,2,4             | -6.006 | -6.726 | -6.674 | -7.953 | -8.142 |
|      | [14]           |                | 0,1,2              | 0,2,4             | -5.98  | -6.73  | -8.27  |       |       |
| FGS  | ‘s and p-wave’ [5] |          | 0,1,2              | 0,2,4             | -7.116 |       |       |       |       |
| FGS  | AB            | ‘s-wave’       | 0,1,2              | 0,2,4             | -6.033 | -6.785 | -6.760 | -8.079 | -8.266 |
|      |                |                |                    |                   | -6.02  | -6.75  | -8.19  |       |       |
| FGS  | CB            | ‘s-wave’       | 0,1,2              | 0,2,4             | -6.456 | -6.179 | -5.532 | -5.884 | -5.785 |
|      | [14]           |                |                    |                   | 2.515  | 2.593  | 2.658  | 2.847  | 2.901  |
|      |                |                |                    |                   | 2.73   | 2.76   | 2.92   |       |       |
| FGS  | CB            | ‘s and p-wave’ [5] |          | 0,1,2              | 2.804  | 2.919  | 2.956  | 3.095  | 3.144  |
|      | [14]           |                |                    |                   | 2.85   | 2.95   | 3.14   |       |       |
|      | [12]           |                |                    |                   | 4.08   | 3.61   | 3.66   |       |       |

TABLE V. Excitation energy in MeV of the $^9_\Lambda$Be $L = 2$ excited ($\frac{3}{2}^+, \frac{5}{2}^+$) spin-flip doublet, calculated for various $\alpha\alpha$ and $\alpha\Lambda$ potentials. $E_B^{\text{exp}}(2^+) = 3.04 \pm 0.03$ MeV [33].

| Ref. | $\alpha\alpha$ | $\alpha\Lambda$ | TH | Gibson | MS   | MSA  | Isle(DA) |
|------|----------------|----------------|----|--------|------|------|---------|
| FGS  | AB            | ‘s-wave’       | 2.515 | 2.593 | 2.658 | 2.847 | 2.901 |
|      | [14]           |                | 2.73   | 2.76   | 2.92  |       |       |
| FGS  | CB            | ‘s and p-wave’| 2.804 | 2.919 | 2.956 | 3.095 | 3.144 |
|      | [14]           |                | 2.85   | 2.95   | 3.14  |       |       |
|      | [12]           |                | 4.08   | 3.61   | 3.66  |       |       |
| FGS  | ‘s and p-wave’|               |       |        |      |      | 3.366  |
FIG. 1. $\alpha\alpha$ potentials.

FIG. 2. $\Lambda\Lambda$ potentials.
FIG. 3. The ΛΛ pairing energy $\delta B_{\Lambda\Lambda}$, Eq. (10), for several ΛΛ and $\alpha\Lambda$ potentials.

FIG. 4. The $\alpha\alpha$ pairing energy $\delta B_{\alpha\alpha}$, Eq. (11), for the AB and CB $\alpha\alpha$ potentials, for several $\alpha\Lambda$ potentials. The orbital angular momenta included in these calculations are $l_{\alpha\Lambda}=0,1,2,$ and $l_{\alpha\alpha}=0,2,4.$