RISK-SENSITIVE CONTROL OF REFLECTED DIFFUSION PROCESSES ON ORTHRANT

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Abstract. In this article, we prove the existence of optimal risk-sensitive control with state constraints. We use near monotone assumption on the running cost to prove the existence of optimal risk-sensitive control.

Key words: Risk sensitive control, discounted risk-sensitive control, diffusion in the orthrant,

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1. Introduction and Problem Description

In this paper we study the risk-sensitive control problem when the state dynamics is governed by a controlled reflecting stochastic differential equation in $d$-dimensional orthant. We prove that the risk-sensitive value is an eigenvalue of the nonlinear eigenvalue problem with oblique boundary conditions (see, the equation (3.2) ) which is the so called Hamilton Jacobi Bellman (HJB) equation of the risk-sensitive control problem with state constraints. We also show that any minimizing selector in (3.2) corresponding to the eigen function of the risk-sensitive value is a risk-sensitive optimal control. We use near monotone structural condition on the running cost and a blanket recurrence condition for the state dynamics for proving this result.

The paper is organized as follows. The remaining part of Section 1 contains the detailed description of the problem and some results on controlled reflected stochastic differential equations which are used in subsequent sections. In Section 2, we discuss an auxiliary risk-sensitive control problem with discounted cost structure. We prove the existence of optimal value and control without the structural condition near monotonicity on the running cost. In the final section, we prove our main theorem, i.e. Theorem 3.2. The proof is based on the so-called vanishing discounting method.

Let $U$ be a compact metric space and $D$ denote the positive orthrant of $\mathbb{R}^d$, i.e.,

$$D := \{ x \in \mathbb{R}^d : x_i > 0, \forall i = 1, 2, \cdots, d \}.$$
Let \( \overline{A}, \partial A \) denote the closure and boundary of the set \( A \), for any subset \( A \) of \( \mathbb{R}^d \) respectively.

For the given functions \( b : \overline{D} \times U \rightarrow \mathbb{R}^d \), \( \sigma : \overline{D} \rightarrow \mathbb{R}^{d \times d} \) and \( \gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d \), consider the controlled reflected diffusion in \( \overline{D} \), given by the solution of the reflected stochastic differential equation (in short RSDE)

\[
\begin{align*}
  dX_t &= b(X_t, v_t)dt + \sigma(X_t)dW_t - \gamma(X_t)d\xi_t, \\
  d\xi_t &= I\{X_t \in \partial D\}d\xi_t, \\
  \xi_0 &= 0, \quad X_0 = x \in \overline{D},
\end{align*}
\]

where \( W = (W_1, \cdots, W_d) \) is an \( \mathbb{R}^d \)-valued standard Wiener process, \( v(\cdot) \) is a \( U \)-valued measurable process non-anticipative with respect to \( W(\cdot) \), called an admissible control. In fact the pair \((v(\cdot), W(\cdot))\) defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) satisfying the usual hypothesis is an admissible control if and only if \( v(\cdot) \) is measurable and \( \{\mathcal{F}_t\} \)-adapted, see Remark 2.1, p.31 of [1]. Henceforth, all filtered probability spaces are assumed to satisfy usual hypothesis. The set of all admissible control is denoted by \( \mathcal{A} \).

By a solution to (1.1) we mean a pair of continuous time processes \((X(\cdot), \xi(\cdot))\) satisfying (1.1) such that the process \( X(\cdot) \) is \( \overline{D} \)-valued and \( \xi(\cdot) \) is a non-decreasing process which increases only when \( X(\cdot) \) hits the boundary \( \partial D \).

The above is a special case of the more general definition of solutions of SDEs with reflection, see [8]. In fact we consider the case when the direction of reflection is single valued.

We use the relaxed control framework given as follows. The compact metric space \( U = \mathcal{P}(S) \) for some compact metric space \( S \), where \( \mathcal{P}(S) \) denote the space of probability measures on \( S \) endowed with the Prohorov topology, i.e. the topology induced by weak convergence. The drift coefficient \( b \) takes the form

\[
b(x,v) = \int_S \bar{b}(x,s)v(ds), \quad v \in U, x \in \overline{D}.
\]

For \( l = 1, 2, \cdots \), set

\[
D'_l = D \cap B(0,l), \quad B(0,l) = \{x \in \mathbb{R}^d ||x|| < l\}.
\]

From the proof of Theorem A2 (ii) and the remark in p. 28 of [2] there exists open domains \( D_{lm} \subseteq \mathbb{R}^d \) with \( C^\infty \) boundary such that

- The distance between \( \partial D'_l \) and \( D_{lm} \) satisfies,

\[
d(D_{lm}, \partial D'_l) < \frac{1}{m}, \quad l \geq 1,
\]

- \( D_{lm} \subseteq D_{ln}, \quad n \geq m, \quad l \geq 1.\)

Set

\[
D_m = \bigcup_{l=1}^\infty D_{lm}, \quad m \geq 1.
\]

Then we have

(i) For each \( m \geq 1 \), \( D_m \) is with \( C^\infty \) smooth boundary and \( D_m \downarrow \tilde{D} \).

(ii) For any compact set \( C \subset \tilde{D} \), we have \( C \subset \overline{D_{lm}} \) for \( m \geq 1 \) and \( l \) sufficiently large.
We make the following assumption which is sufficient to ensure the existence of unique solution to the equation (1.1)

**A1**  
(i) The function \( \tilde{b} \) is bounded continuous, Lipschitz continuous in its first argument uniformly with respect to the second argument.  
(ii) The functions \( \sigma_{ij} \in C^2(\bar{D}), i,j = 1, \ldots, d \) and bounded.  
(iii) The function \( a \) is uniformly elliptic with ellipticity constant \( \delta \), i.e.,  
\[
xa(x)x^\perp \geq \delta \|x\|^2, \quad x \in \overline{D},
\]
where \( x^\perp \) denote the transpose of the vector \( x \).

**A2**  
(i) The function \( \gamma = (\gamma_1, \ldots, \gamma_d) \) is such that \( \gamma_i \in C^2_b(\mathbb{R}^d) \), and there exists \( \eta > 0 \) such that  
\[
\gamma(x) \cdot n_m(x) \geq \eta \quad \text{for all} \quad x \in \partial D_m,
\]
here \( n_m(\cdot) \) denote the outward normal to \( \partial D_m \).  
(ii) There exists a symmetric matrix valued map \( M : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) the set of all \( d \times d \) real valued matrices with usual metric, such that \( M = (m_{ij}), m_{ij} \in C^2_b(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d) \) for \( i,j = 1, 2, \ldots, d \) and satisfies the following  
(a) there exists \( \delta_1 \) such that  
\[
x^\perp Mx \geq \delta_1 \|x\|^2, \quad x \in \mathbb{R}^d;
\]
(b) there exists \( C_0 > 0 \) such that  
\[
C_0 \|x - y\|^2 + \sum_{i,j} m_{ij}(x)(x_i - y_i)\gamma_j(x) \geq 0, \quad \text{for all} \quad x \in \partial D, y \in \overline{D};
\]
(c) Let \( z \in \overline{D} \) and if for some \( C_0 > 0 \)  
\[
C_0 \|x - y\|^2 + \sum_{i,j} m_{ij}(x)(x_i - y_i)z_j(x) \geq 0, \quad \text{for all} \quad x \in \partial D, y \in \overline{D};
\]
then \( z = \theta \gamma(x) \) for some \( \theta > 0 \).

The existence of a unique weak solution of (1.1) for an admissible control has been proved in [10],[14] using the following programme. First establish the existence of unique strong solution with zero drift as follows.

- Establish the existence of a solution to (1.1) in the smooth domain \( \overline{D}_m, m \geq 1 \),
- use convergence arguments to obtain a solution of (1.1) in \( \overline{D} \),
- establish pathwise uniqueness, see Lemma 3.3 of [2].

Now with non zero drift, using Girsanov transformation method to establish existence of unique weak solution under admissible controls, see [1], pp-42-44. For a Markov control, one can prove the existence of unique strong solution by adapting the approach by Zovokin and Veretenikov, see [1],
pp.45-46] for the analogous proof for the unconstrained diffusions. See Theorem 3.2 of [2] for details.

The running cost function \( r : \overline{D} \times U \rightarrow [0, \infty) \) is given in the relaxed frame work as

\[
r(x, v) = \int_S \bar{r}(x, s)v(ds), \quad x \in \overline{D}, \ v \in U.
\]

Throughout this paper we assume that the cost function \( \bar{r} \) is continuous in \( (x, s) \) and Lipschitz continuous in the first argument uniformly with respect to the second. We consider two risk-sensitive cost criteria, discounted cost and ergodic cost criteria which is described below.

1.1. **Discounted cost criterion.** Let \( \theta \in (0, \Theta) \) be the risk-aversion parameter. In the \( \alpha \)-discounted cost criterion, controller chooses his control \( v(\cdot) \) from the set of all admissible controls \( \mathcal{A} \) to minimize his \( \alpha \)-discounted risk-sensitive cost given by

\[
J^\alpha_\theta(\theta, x) := \frac{1}{\theta} \ln E^x_v \left[ e^{\theta \int_0^\infty e^{-\alpha t} r(X_t, v_t) dt} \right], \ x \in \overline{D},
\]

where \( \alpha > 0 \) is the discount parameter, \( X(\cdot) \) is the solution of the s.d.e. (1.1) corresponding to \( v(\cdot) \in \mathcal{A} \) and \( E^v_x \) denote the expectation with respect to the law of the process (1.1) corresponding to the admissible control \( v \) with the initial condition \( X_0 = x \). An admissible control \( v^*(\cdot) \in \mathcal{A} \) is called optimal control if

\[
J^\alpha_\theta(\theta, x) \leq J^\alpha_\theta(\theta, x), \ \text{for all} \ v(\cdot) \in \mathcal{A} \ \text{and} \ x \in \overline{D}.
\]

1.2. **Ergodic cost criterion.** In this criterion controller chooses his control \( v(\cdot) \in \mathcal{A} \) so as to minimize his risk-sensitive accumulated cost given by

\[
\rho^v(\theta, x) = \limsup_{T \to \infty} \frac{1}{\theta T} \ln E^x_v \left[ e^{\theta \int_0^T r(X_t, v_t) dt} \right], \ x \in \overline{D}.
\]

The definition of optimal control is analogous. From now onwards, we take \( \Theta = 1 \) without any loss of generality.

1.3. **Various subclasses of controls.** An admissible control \( v(\cdot) \) is said to be a Markov control if there exists a measurable map \( \bar{v} : [0, \infty) \times \overline{D} \rightarrow U \), where \( X(s, \cdot, \xi) \) denote the solution of (1.1) and \( F^X_t \) and \( F^{X, \xi}_t \) denote respectively the set of all Markov control and stationary Markov control by \( \mathcal{M} \) and \( \mathcal{S} \) respectively. An admissible control \( v(\cdot) \) is said to be a feedback control if it is progressively measurable with respect to \( \{F^X_t\} \), where \( \sigma \left\{ [0, \infty) : \mathcal{D} \right\} \times \{0, \infty) \) denote the solution of (1.1) and \( F^X_t \) denote sigma field generated by \{\( X_s, \xi_s | s \leq t \}, t \geq 0 \). This is equivalent to saying that there exists a progressively measurable map \( \bar{v} : [0, \infty) \times C([0, \infty) : \mathcal{D}) \times C([0, \infty) : \mathcal{D}) \rightarrow U \) such that \( v(t) = \bar{v}(t, X[0, t], \xi[0, t]), t \geq 0 \), where \( X[0, t], \xi[0, t] \) denote respectively \{\( X_s, 0 \leq s \leq t \}, \{\xi_s, 0 \leq s \leq t \}. \) Hence by an abuse of notation, we denote
the set of feedback controls by all progressively measurable maps. The following lemma tells that we can restrict ourselves to feedback controls. Its proof is a straightforward adaptation of Theorem 2.3.4 (a), p.52 of [1].

Lemma 1.1. Let \((v(\cdot), W(\cdot))\) be an admissible control and \((X(\cdot), \xi(\cdot))\) be a solution pair to (1.1) on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\). Then on an augmentation \((\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})\) with a \(\{\tilde{\mathcal{F}}_t\}\)-Wiener process \(\tilde{W}(\cdot)\) and a feedback control \(\tilde{v}(\cdot)\) such that \((\tilde{X}(\cdot), \tilde{\xi}(\cdot))\) solves (1.1) for the pair \((\tilde{v}(\cdot), \tilde{W}(\cdot))\) on \((\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})\).

1.4. Properties of Controlled RSDEs. We prove some results about the controlled RSDE (1.1) which are used in the subsequent sections. To the best of our knowledge these results are not available the controlled RSDEs we are considering.

First result is about the equivalence of weak solution and martingale problem for reflected diffusions. For a feedback control \(v(\cdot)\), we say that the RSDE (1.1) admits a weak solution if there exists a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\), a \(\{\mathcal{F}_t\}\)-Wiener process \(W(\cdot)\) and a pair of \(\{\mathcal{F}_t\}\)-adapted processes \((X(\cdot), \xi(\cdot))\) with a.s. continuous paths such that \(X(\cdot)\) is \(D\)-valued, \(\xi(\cdot)\) is non decreasing and satisfy

\[
\begin{align*}
    dX(t) &= b(X(t), v(t, X[0,t], \xi[0,t])dt + \sigma(X(t))dW(t) - \gamma(X(t))d\xi(t) \\
    d\xi(t) &= I\{X(t) \in \partial D\}d\xi(t), X(0) = x, \xi(0) = 0 \text{ P a.s.}
\end{align*}
\]

Set

\[
(1.4) \quad \mathcal{H} = \{f \in C^2_0(D) | \nabla f \cdot \gamma \geq 0 \text{ on } \partial D\}
\]

and

\[
(1.5) \quad \mathcal{L}f(x, v) = b(x, v) \cdot \nabla f(x) + \frac{1}{2} \text{trace}(a(x)\nabla^2 f(x)), f \in \mathcal{D}(\mathcal{L}),
\]

where the domain \(\mathcal{D}(\mathcal{L})\) of the oblique elliptic operator \(\mathcal{L}\) contains \(C^2_{b,\gamma}(\mathcal{D})\), the set of all bounded twice continuously differentiable functions satisfying \(\nabla f \cdot \gamma \geq 0 \text{ on } \partial D\).

**Constrained controlled martingale problem:** A pair of \(\{\mathcal{F}_t\}\)-adapted processes \((X(\cdot), \xi(\cdot))\) defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) is said solve the constrained controlled martingale problem to the RSDE (1.1) corresponding to the admissible control \(v(\cdot)\) and initial condition \(x \in \overline{D}\) if the following holds.

(i) \(X(\cdot)\) is \(\overline{D}\)-valued and \(\xi(\cdot)\) is non decreasing and \(X(0) = x, \xi(0) = 0\) a.s.

(ii) \(\int_0^t I\{X(s) \in \partial D\}d\xi(s) = \xi(t), P \text{ a.s. for all } t \geq 0,\)
(iii) For all \( f \in \mathcal{H} \),
\[
M_f(t) = f(X(t)) - \int_0^t \mathcal{L}(X(s), v(s))ds + \int_0^t \nabla f \cdot \gamma(X(s))d\xi(s), \quad t \geq 0
\]
is an \( F_t \)-martingale in \((\Omega, F, P)\).

**Theorem 1.1.** For a feedback control \( v(\cdot) \), the pair of processes \((X(\cdot), \xi(\cdot))\) defined on a filtered probability space \((\Omega, F, \{F_t\}, P)\) solves the constrained controlled martingale problem if and only if there exists a filtered probability space \((\tilde{\Omega}, \tilde{F}, \{\tilde{F}_t\}, \tilde{P})\) and a Wiener process \( \tilde{W}(\cdot) \) such that \((\tilde{X}(\cdot), \tilde{\xi}(\cdot))\) which is a weak solution to (1.1) such that \((X(\cdot), \xi(\cdot))\) and \((\tilde{X}(\cdot), \tilde{\xi}(\cdot))\) agree in law.

**Proof.** Suppose \((X(\cdot), \xi(\cdot))\) solves the constrained controlled martingale problem. Hence the law of \( X(\cdot) \) solves the corresponding submartingale problem. Now using Theorem 1 of [12], there exists a filtered probability space \((\tilde{\Omega}, \tilde{F}, \{\tilde{F}_t\}, \tilde{P})\) and \( \{\tilde{F}_t\} \)-adapted processes with continuous paths \((\tilde{X}(\cdot), \tilde{\xi}(\cdot))\) and a Wiener process \( \tilde{W}(\cdot) \) such that \((\tilde{X}(\cdot), \tilde{\xi}(\cdot))\) is a weak solution to (1.1) and law of \( X(\cdot) \) is same as law of \( \tilde{X}(\cdot) \). Now since (1.1) has a unique weak solution, law of \((X(\cdot), \xi(\cdot))\) equals the law of \((\tilde{X}(\cdot), \tilde{\xi}(\cdot))\). Converse follows from Itô’s formula. \(\square\)

**Remark 1.1.** Under suitable \( C^2 \) smoothness assumption on the domain and bounded continuity assumption on direction of reflection \( \gamma \), the equivalence is shown in [18]. The case of domains with piecewise smooth boundaries and with constant direction of reflections is treated in [7].

For an admissible control \( v(\cdot) \), if \((X(\cdot), \xi(\cdot))\) denote a unique weak solution pair to the RSDE (1.1) on \((\Omega, F, \{F_t\}, P)\) and \( \tau \) a \( \{F_t\} \)-stopping time, then \( F_\tau \) is finitely generated and hence using Theorem 1.3.4, p.34 of [18], it follows that regular conditional probability distribution (rcpd) \( P_\omega \) of \( P \) given \( F_\tau \) exists. Now we prove a result analogous to Lemma 2.3.7 of [1].

**Lemma 1.2.** Let \((X(\cdot), \xi(\cdot))\) denote a weak solution pair corresponding to an admissible feedback control \( v(\cdot) \) and defined on \((\Omega, F, \{F_t\}, P)\) and \( \tau \) be a finite \( \{F_t\} \)-stopping time. Then the conditional law \( \mu_\omega \) of the process \( X(\tau+) \) given \( F_\tau \) is a.s. the law of the process \( X_\omega(\cdot) \), where \( X_\omega(\cdot) \) is a unique weak solution to the RSDE (1.1) on a probability space \((\Omega_\omega, F_\omega, \{F_{\omega,t}\}, P_\omega)\) for an admissible control given by \( v_\omega(t) = v(t + \tau(\omega), X[0, \tau(\omega) + t], \xi[0, \tau(\omega) + t]) \), \( t \geq 0 \).

**Proof.** For \( f \in \mathcal{H} \), since
\[
M_f(t) = f(X(t)) - f(X(0)) - \int_0^t \mathcal{L}(X(s), v_s)ds + \int_0^t \nabla f \cdot \gamma(X(s))d\xi(s), \quad t \geq 0,
\]
where \( \mathcal{L} \) is given by (1.5) is an \( F_t \)-martingale on \((\Omega, F, P)\), it follows from Theorem 1.2.10, p.28 of [18] that there exist a \( P \)-null set \( N \) such that for \( \omega \notin N \), \( M_f^{\omega}(t) = M_t - M_{t \wedge \tau(\omega)}, t \geq 0 \) is a Martingale with respect to \( \{F_t\} \).
Lemma implies that any bounded non-negative solution $\phi$ satisfies

$$M^\tau(\omega)(t) = f(X_t) - f(X_{\tau(\omega)}) - \int_{\tau(\omega)}^t \mathcal{L} f(X_s, v_s) ds + \int_{\tau(\omega)}^t \nabla f \cdot \gamma(X_s) d\xi_s, \quad t \geq \tau(\omega)$$

is a Martingale under $P_\omega$, $\omega \notin N$, i.e.,

$$M^\tau(\omega)(t) = f(X_t) - f(X_{\tau(\omega)}) - \int_0^t \mathcal{L} f(X(\tau(\omega) + s, v_s+\tau(\omega)) ds \quad + \int_0^t \nabla f \cdot \gamma(X_{s+\tau(\omega)}) d\xi_{s+\tau(\omega)}, \quad t \geq 0$$

is a Martingale under $P_\omega$, $\omega \notin N$, i.e. $(X_\omega(\cdot), \xi_\omega(\cdot)) := (X(\cdot + \omega), \xi(\cdot) + \tau(\omega) - \xi(\tau(\omega)))$ solves the constrained controlled martingale problem for the admissible control $v_\omega$ and initial distribution $X(\tau(\omega))$. This completes the proof.

Now we give a characterization for recurrence of the RSDE (1.1) corresponding to a stationary Markov control in the following lemma.

**Lemma 1.3.** Let $v(\cdot) \in S$ and $X(\cdot)$ be a solution to the RSDE (1.1) corresponding to $v(\cdot)$ and $B$ be a ball in $D$. Then $X(\cdot)$ is recurrent iff the PDE

$$\mathcal{L} \varphi(x, v(x)) = 0, \quad \varphi \equiv 1 \text{ on } \partial B, \quad \nabla \varphi \cdot \gamma \equiv 0 \text{ on } \partial D.$$  

has a unique non-negative bounded solution in $W^{2,d+1}_{loc}(\overline{B}^c) \cap C(B^c)$.

**Proof.** Note that $\varphi \equiv 1$ is always a positive bounded solution of (1.6) in $W^{2,d+1}_{loc}(\overline{B}^c) \cap C(B^c)$. Also an application of Itô-Dynkin formula and Fatou’s lemma implies that any bounded non-negative solution $\varphi \in W^{2,d+1}_{loc}(\overline{B}^c) \cap C(B^c)$ satisfies

$$\varphi(x) \geq P_x(\tau(\overline{B}^c) < \infty), \quad x \in \overline{D}.$$  

Hence the result follows, since non degeneracy of the RSDE implies that $X(\cdot)$ recurrent iff it is $B$-recurrent for some ball $B$ in $D$.

**1.5. Notations.** In this subsection, we introduce various frequently used notations in this paper. We denote $\sup_{v,x} |r(x,v)|$ by $\|r\|_\infty$. For $\varphi \in C_b(\overline{D})$, the space of all real-valued bounded continuous functions, we denote for each $B$, a Borel subset of $\overline{D}$,

$$\|\varphi\|_{\infty,B} = \sup_{x \in B} |\varphi(x)|, \quad \|\varphi\|_\infty = \sup_{x \in D} |\varphi(x)|.$$  

For a Banach space $\mathcal{X}$ with norm $\| \cdot \|_\mathcal{X}$, $1 \leq p < \infty$, define for $\kappa \geq 0$

$$L^p(\kappa, T; \mathcal{X}) = \{ \varphi : (\kappa, T) \to \mathcal{X} | \varphi \text{ is Borel measurable and } \int_0^T \|\varphi(t)\|_\mathcal{X}^p dt < \infty \}$$
are compactly supported. The spaces\( C^\infty_c((\kappa, 1) \times D)\) denote the space of all functions in \( C^\infty((\kappa, 1) \times D)\) which are compactly supported. The spaces \( C^\infty_c((\kappa, 1) \times \overline{D}),\ C^\infty_c((\kappa, 1) \times \overline{D})\) are similarly defined.

For \( \kappa < T < \infty \) and an open bounded set \( B \) in \( \mathbb{R}^d\), \( H^{\beta/2, \beta}([\kappa, T] \times \overline{B}), \beta \geq 0\), denotes the set of all continuous functions \( \varphi(t, x) \) in \([\kappa, T] \times \overline{B}\) together with all the derivatives of the from \( D^r_tD^s_x\varphi(t, x) \) for \( 2r + s < \beta \), have a finite norm

\[
\|\varphi\|_{H^{(\beta)}([\kappa, T] \times \overline{B})} = \|\varphi\|_{C^\infty([\kappa, T] \times \overline{B})} + \sum_{j=1}^{[\beta]} H^j_{[\kappa, T] \times \overline{B}}(\varphi),
\]

where

\[
H^j_{[\kappa, T] \times \overline{B}}(\varphi) = \sum_{2r+s=j} \|D^r_tD^s_x\varphi\|_{C^\infty([\kappa, T] \times \overline{B})},
\]

\[
H^\beta_{[\kappa, T] \times \overline{B}}(\varphi) = H^\beta_{x, [\kappa, T] \times \overline{B}}(\varphi) + H^{\beta/2}_{t, [\kappa, T] \times \overline{B}}(\varphi),
\]

\[
H^\beta_{x, [\kappa, T] \times \overline{B}}(\varphi) = \sum_{2r+s=[\beta]} H^s_{x, [\kappa, T] \times \overline{B}}(D^r_tD^s_x\varphi),
\]

\[
H^{\beta/2}_{t, [\kappa, T] \times \overline{B}}(\varphi) = \sum_{0<\beta-2r-s<2} H^{(\beta-2r-s)}_{t, [\kappa, T] \times \overline{B}}(D^r_tD^s_x\varphi),
\]

\[
H^{(\alpha)}_{x, [\kappa, T] \times \overline{B}}(\varphi) = \sup_{(t,x), (t,x) \in [\kappa, T] \times \overline{B}} \frac{|\varphi(t, x) - \varphi(t, \bar{x})|}{\|x - \bar{x}\|^\alpha}, \quad 0 < \alpha < 1,
\]

\[
H^{(\alpha)}_{t, [\kappa, T] \times \overline{B}}(\varphi) = \sup_{(t,x), (t,x) \in [\kappa, T] \times \overline{B}} \frac{|\varphi(t, x) - \varphi(t, \bar{x})|}{|t - \bar{t}|^\alpha}, \quad 0 < \alpha < 1.
\]

We denote

\[
C^{\beta, \beta}_{([\kappa, T] \times \overline{B})} = \{ \varphi \in C([\kappa, T] \times \overline{B})|\varphi \in C^{\beta/2, \beta}([\kappa, T] \times K)\}, \text{ for some compact subset of } \overline{B}.
\]

The space \( \mathcal{W}^{1,2,p}([\kappa, T] \times \overline{D})\), \( \kappa \geq 0\), denotes the set of all \( \varphi \in L^p([\kappa, T]; W^{2,p}(\overline{D}))\) such that \( \frac{\partial \varphi}{\partial t} \in L^p([\kappa, T]; L^p(\overline{D}))\) with the norm given by

\[
\|\varphi\|_{1,2,p; W^{2,p}(\overline{D})} = \|\varphi\|_{p; W^{2,p}(\overline{D})} + \|\frac{\partial \varphi}{\partial t}\|_{p; L^p(\overline{D})}, \quad 1 \leq p < \infty.
\]

Also the local Sobolev spaces \( \mathcal{W}^{1,2,p}_{\text{loc}}([\kappa, T] \times \overline{D})\) are defined by

\[
\mathcal{W}^{1,2,p}_{\text{loc}}([\kappa, T] \times \overline{D}) = \left\{ \varphi : ([\kappa, T] \times \overline{D} \to \mathbb{R} | \varphi \text{ is measurable and } \varphi \in W^{1,2,p}([\kappa, T] \times K), \text{ for some } K \text{ is a compact subset of } \overline{D} \right\}.
\]
To any domain $B$ in $D$, define

$$W^{1,2,p}((\kappa, T) \times B) = \left\{ \varphi : (\kappa, T) \times B \to \mathbb{R} \mid \|\varphi\|_{1,2,p;((\kappa, T) \times B)} < \infty \right\},$$

where the norm $\|\cdot\|_{1,2,p;((\kappa, T) \times B)}$ is defined as

$$\|\varphi\|_{1,2,p;((\kappa, T) \times B)} = \int_{\kappa}^{T} \int_{B} |\varphi(t, x)|^p dx dt + \sum_i \int_{\kappa}^{T} \int_{B} \left| \frac{\partial \varphi(t, x)}{\partial x_i} \right|^p dx dt + \sum_{ij} \int_{\kappa}^{T} \int_{B} \left| \frac{\partial^2 \varphi(t, x)}{\partial x_i \partial x_j} \right|^p dx dt.$$

### 2. Analysis of the Discounted Cost criterion

In this section, we study the discounted risk-sensitive control problem with the state dynamics (1.1) and cost criterion

$$J^v_{\alpha}(\theta, x) = \frac{1}{\theta} \ln E^v_x \left[ e^{\theta \int_0^\infty e^{-\alpha t} r(X_t, v_t) dt} \right].$$

The $\alpha$-discounted risk-sensitive control problem is to minimize (1.2) over all admissible controls. We define the so-called ‘value function’ for the cost (1.2) as

$$\phi_{\alpha}(\theta, x) = \inf_{v \in A} J^v_{\alpha}(\theta, x).$$

Set

$$\bar{J}^v_{\alpha}(\theta, x) = E^v_x \left[ e^{\theta \int_0^\infty e^{-\alpha t} r(X_t, v_t) dt} \right].$$

Since logarithm is an increasing function for fixed $\theta > 0$, a minimizer of $\bar{J}^v_{\alpha}(\theta, x)$ if it exists will be a minimizer of $J^v_{\alpha}(\theta, x)$). Corresponding to the cost (2.2), the value function is defined as

$$u_{\alpha}(\theta, x) = \inf_{v \in A} \bar{J}^v_{\alpha}(\theta, x).$$

Note that

$$\phi_{\alpha}(\theta, x) = \frac{1}{\theta} \ln u_{\alpha}(\theta, x).$$

Since we are dealing with exponential cost we need multiplicative version of DPP in place of additive DPP, see [6], pp. 53-59. We mimic the arguments as in [15] to prove DPP for the value function $u_{\alpha}(\theta, x)$.

**Theorem 2.1** (DPP). Let $\tau$ be any bounded stopping time with respect to the natural filtration of process $X(\cdot)$, i.e., $\{F_t^X\}$. Then

$$u_{\alpha}(\theta, x) = \inf_{v(\cdot)} E^v_x \left[ e^{\theta \int_0^{\tau} e^{-\alpha t} r(X_t, v_t) dt} u_{\alpha}(\theta e^{-\alpha \tau}, X(\tau)) \right].$$

where infimum is taken over all feedback controls.
Thus

Conversely, let \( \tau, r \) and \( v \) be given as above, and let \( \tau \) be defined as

\[
(2.6) \quad v(t) = v_1(t)I_{t < \tau} + v_2(t - \tau)I_{t \geq \tau}, \quad t \geq 0,
\]

is also a feedback control. Indeed, we are given pairs of processes \( (X_1(\cdot), \xi_1(\cdot), v_1(\cdot)) \) and \( (X_2(\cdot), \xi_2(\cdot), v_2(\cdot)) \) satisfying (1.1) on some, possibly distinct, probability spaces \( (\Omega_1, \mathcal{F}_1, P_1), (\Omega_2, \mathcal{F}_2, P_2) \) respectively, with \( v_1(\cdot), v_2(\cdot) \) in feedback from. Also, \( X_1(0) = x \) and the law of \( X_2(0) = \text{law of } X(\tau) \), where \( \tau \) is a prescribed stopping time with respect to the natural filtration of process \( X(\cdot) \). By augmenting \( (\Omega, \mathcal{F}, P) \) suitably, one can construct a processes \( (X(\cdot), \xi(\cdot)) \) and \( v(\cdot) \) satisfying (1.1) such that they coincide with \( (X_1(\cdot), \xi_1(\cdot)) \) and \( v_1(\cdot) \) on \( [0, \tau] \), and \( (X(\tau + \cdot), \xi(\tau + \cdot)) \) and \( v(\tau + \cdot) \) agree in law with \( (X_2(\cdot), \xi_2(\cdot)) \) and \( v_2(\cdot) \). Also the conditional law of \( X(\tau + \cdot) \) of given \( \mathcal{F}_\tau \) is the same as its conditional law given \( X(\tau) \) and agrees with the conditional law of \( X(\tau + \cdot) \) given \( X_2(0) \) a.s. with respect to the common law of \( X_2(0), X(\tau) \). The above construction uses Lemma 1.2.

Let \( \epsilon > 0 \). Let \( X(\cdot) \) be a process (1.1) controlled by \( v(\cdot) \) as above with \( v_1(\cdot) \) an arbitrary feedback control and \( v_2(\cdot) \) an \( \epsilon \)-optimal feedback control for initial data \( X(0) = x \). By (2.3) we have

\[
u_\alpha(\theta, x) \leq E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt + \theta \int_\tau^\infty e^{-\alpha t} r(X_t, v_t)dt} \right] \]

\[
eq E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} \cdot e^{\theta \int_\tau^\infty e^{-\alpha t} r(X_t, v_t)dt} \right] \]

\[
\leq E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} \cdot u_{\alpha}(\theta e^{-\alpha t}, X(\tau)) + \epsilon \right] \]

\[
= E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} u_{\alpha}(\theta e^{-\alpha t}, X(\tau)) \right] + \epsilon E_x^v \left[ e^{\theta \int_\tau^\infty e^{-\alpha t} r(X_t, v_t)dt} \right].
\]

Since \( \tau, r \) are bounded and \( \epsilon \) is arbitrary we get

\[
u_{\alpha}(\theta, x) \leq \inf_{v(\cdot)} E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} u_{\alpha}(\theta e^{-\alpha t}, X(\tau)) \right].
\]

Conversely, let \( \epsilon > 0 \) and \( v(\cdot) \) is an \( \epsilon \)-optimal feedback control for initial data \( X(0) = x \). Then

\[
u_{\alpha}(\theta, x) + \epsilon \geq E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt + \theta \int_\tau^\infty e^{-\alpha t} r(X_t, v_t)dt} \right] \]

\[
= E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} E \left[ e^{\theta e^{-\alpha t} \int_0^\tau e^{-\alpha t} r(X_t, v_t)dt} \right] X(\tau) \right] \]

\[
\geq E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} \inf_{v(\cdot)} E \left[ e^{\theta e^{-\alpha t} \int_0^\tau e^{-\alpha t} r(X_t, v_t)dt} \right] X(\tau) \right] \]

\[
= E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} u_{\alpha}(\theta e^{-\alpha t}, X(\tau)) \right].
\]

Thus

\[
u_{\alpha}(\theta, x) + \epsilon \geq \inf_{v(\cdot)} E_x^v \left[ e^{\theta \int_0^\tau r(X_t, v_t)dt} u_{\alpha}(\theta e^{-\alpha t}, X(\tau)) \right].\]
Using dynamic programming heuristics, the HJB equations for discounted cost criterion is given by

\[
\frac{\partial u}{\partial \theta} = \inf_{v \in \mathcal{U}} \left[ b(x, v) \cdot \nabla u + \theta r(x, v) u \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 u) \\
u(0, x) = 1 \text{ on } \partial D, \quad \nabla u \cdot \gamma(x) = 0 \text{ on } (0, 1) \times \partial D.
\]

First we show that (2.7) has unique a solution. There are two technical difficulties in solving the p.d.e. (2.7). First is the singularity in \( \theta \) at 0 and the second is the unbounded non smooth nature of the orthant. We circumvent these difficulties by suitable approximation arguments as follows. For each \( m, l \geq 1 \) and \( 0 < \kappa < 1 \), consider the p.d.e.

\[
\frac{\partial u_{\kappa, lm}}{\partial \theta} = \inf_{v \in \mathcal{U}} \left[ b(x, v) \cdot \nabla u_{\kappa, lm} + \theta r(x, v) u_{\kappa, lm} \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 u_{\kappa, lm}) \\
u_{\kappa, lm}(\kappa, x) = e^{\kappa \|r\|_\infty} \text{ on } D, \quad \nabla u_{\kappa, lm} \cdot \gamma = 0 \text{ on } (\kappa, 1) \times \partial D_{lm}.
\]

**Lemma 2.1.** Assume (A1) and (A2). Then the p.d.e. (2.8) has a unique solution \( u_{\kappa, lm} \in H^{3,2}(\kappa, 1) \times D_{lm} \), and

\[
\|u_{\kappa, lm}\|_{\infty; [\kappa, 1] \times D_{lm}} \leq e^{\frac{\theta \|r\|_\infty}{\alpha}}, \quad \text{for all } \kappa > 0, \; m, l \geq 1,
\]

\[
\left\| \frac{\partial u_{\kappa, lm}}{\partial \theta} \right\|_{\infty; [\kappa, 1] \times D_{lm}} \leq 3 e^{\frac{(\theta + 3) \|r\|_\infty}{\alpha}}, \quad \text{for all } \kappa > 0, \; m, l \geq 1.
\]

**Proof.** For the existence and uniqueness result we use Theorem 7.4 from [13, p. 491]. Set \( \theta = 1 - t \), \( u_{\kappa, lm}(\theta, x) = u(t, x) \).

Then equation (2.8) reduce to

\[
-(1 - t) \frac{\partial u}{\partial t} = \inf_{v \in \mathcal{U}} \left[ b(x, v) \cdot \nabla u + (1 - t) r(x, v) u \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 u) \\
u(1 - \kappa, x) = e^{\kappa \|r\|_\infty} \text{ for } x \in D_m, \quad \nabla u(t, x) \cdot \gamma = 0 \text{ on } (1 - \kappa) \times \partial D_m.
\]

Rewrite the above equation as

\[
0 = \frac{\partial u}{\partial t} + \inf_{v \in \mathcal{U}} \left[ \frac{b(x, v) \cdot \nabla u}{\alpha(1 - t)} + \frac{1}{\alpha} r(x, v) u \right] + \frac{1}{2} \text{trace} \left( \frac{a(x)}{\alpha(1 - t)} \nabla^2 u \right) \\
u(1 - \kappa, x) = e^{\kappa \|r\|_\infty} \text{ for } x \in D_{lm}, \quad \nabla u(t, x) \cdot \gamma = 0 \text{ on } (1 - \kappa) \times \partial D_{lm}.
\]

Set

\[
b(t, x, u, p) = \inf_{v \in \mathcal{U}} \left[ \frac{b(x, v) \cdot p}{\alpha(1 - t)} + \frac{1}{\alpha} r(x, v) u \right]
\]
Note that $b(t, x, u, p)$ and $a_{ij}(x, t)$ are Lipschitz continuous in $x$, since $b(x, v)$, $r(x, v), a_{ij}(x)$ are Lipschitz continuous in the first argument uniformly with respect to the second.

Therefore from [13, Theorem 7.4, p. 491] it follows that (2.11) has a unique solution in $H^{2,1}([\kappa, 1] \times \overline{D}_{lm})$.

Let $v(\cdot)$ be an admissible control and $X(\cdot)$ be the process given by

$$dX_t = (X_t, v_t)dt + \sigma(X_t)dW_t - \gamma(X_t)d\xi_t$$

Applying Itô’s formula to $e^{\int_0^t \theta r(x, v_s)ds} u_{\alpha,lm}^\kappa(\theta_t, X_t), \theta_t = e^{-\alpha t}$, we get

$$d \left( e^{\int_0^t \theta r(x, v_s)ds} u_{\alpha,lm}^\kappa(\theta_t, X_t) \right) = e^{\int_0^t \theta r(x, v_s)ds} d u_{\alpha,lm}^\kappa(\theta_t, X_t)$$

$$+ \theta_t e^{\int_0^t \theta r(x, v_s)ds} t(X_t)e^{\int_0^t \theta r(x, v_s)ds} r(X_t, v_t) dt,$$

where

$$d u_{\alpha,lm}^\kappa(\theta_t, X_t) = (\nabla u_{\alpha,lm}^\kappa(\theta_t, X_t)) \sigma(X_t) dW_t - \left[ \gamma(X_t) \cdot \nabla u_{\alpha,lm}^\kappa(\theta_t, X_t) \right] I_{\{X_t \in \partial D_{lm}\}} d\xi_t$$

$$+ \left[ \alpha u_{\alpha,lm}^\kappa(\theta_t, X_t, v_t) - \alpha \theta_t \frac{\partial}{\partial \theta} u_{\alpha,lm}^\kappa(\theta_t, X_t) \right] dt.$$ 

Using the fact that $u_{\alpha,lm}^\kappa$ satisfy the equation (2.8), we have

$$u_{\alpha,lm}^\kappa(\theta, x) \leq E_x^\nu \left[ e^{\frac{\|r\|_\infty}{\alpha} \int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s, v_s) ds} \right],$$

where $T_\kappa = \frac{\ln(\frac{\theta}{\epsilon})}{\alpha}$. Repeating the above argument with a minimizing selector in (2.8), we get

$$u_{\alpha,lm}^\kappa(\theta, x) = \inf_{v(\cdot)} E_x^\nu \left[ e^{\frac{\|r\|_\infty}{\alpha} \int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s, v_s) ds} \right].$$

From (2.13), we have

$$|u_{\alpha,lm}^\kappa| \leq E_x^\nu \left[ e^{\frac{\|r\|_\infty}{\alpha} \int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s, v_s) ds} \right] \leq e^{\frac{\|r\|_\infty}{\alpha} e^{\int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s, v_s) ds}} \leq e^{\frac{\|r\|_\infty}{\alpha} e^{\|\theta\|_{\kappa} \frac{(\theta - \kappa)}{\alpha}}},$$

which proves the estimate (2.9).

We mimic the arguments of [14, Theorem 3.1], to prove the estimate (2.10). For $\epsilon$ with $|\epsilon|$ sufficiently small, set

$$T_\kappa^\epsilon = \frac{1}{\alpha} \log \left( \frac{\theta + \epsilon}{\kappa} \right).$$
Now consider for each \(v(\cdot)\) admissible
\[
\left| E^v_x \left[ e^{(\theta + \varepsilon) \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] - E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] \right|
\leq \left| E^v_x \left[ e^{(\theta + \varepsilon) \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] - E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] \right|
+ \left| E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] - E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] \right|
\leq \frac{e^{\theta T^x}}{\alpha} \epsilon \frac{\kappa e}{\theta + \varepsilon} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \times E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} - 1 \right]
\leq e^{\theta T^x} \epsilon \frac{\kappa e}{\theta + \varepsilon} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right)
\] (2.14)

Now
\[
\left| E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] - E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] \right|
\leq \frac{e^{\theta T^x}}{\alpha} \epsilon \frac{\kappa e}{\theta + \varepsilon} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \times e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} - 1
\leq e^{\theta T^x} \epsilon \frac{\kappa e}{\theta + \varepsilon} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right)
\] (2.15)

and
\[
\left| E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] - E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] \right|
\leq \frac{e^{\theta T^x}}{\alpha} \epsilon \frac{\kappa e}{\theta + \varepsilon} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \times e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} - 1
\leq e^{\theta T^x} \epsilon \frac{\kappa e}{\theta + \varepsilon} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right)
\] (2.16)

Note that for each \(\theta > 0\), when \(\varepsilon\) is positive, then \(e^{\theta e^{\frac{\kappa e}{\theta + \varepsilon}}} \leq 1\) and for \(\varepsilon < 0\) we can choose a \(0 < \varepsilon_\theta < 1\) such that \(e^{\frac{\kappa e}{\theta + \varepsilon}} \leq 2\) whenever \(|\varepsilon| \leq \varepsilon_\theta\). Hence we have
\[
\left| E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] - E^v_x \left[ e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right] \right|
\leq e^{\frac{\theta T^x}{\alpha}} \epsilon \frac{2 |r_\infty|}{\alpha} \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right)
\] (2.16)

From (2.13), (2.14), (2.15) and (2.16) we have
\[
|u^\kappa_{\alpha, l_m}(\theta + \varepsilon, x) - u^\kappa_{\alpha, l_m}(\theta, x)| \leq e^{\frac{\theta T^x}{\alpha}} \epsilon \left| e^{\theta \int_0^{T^x} e^{-\alpha t r(X_t, v_t)} dt} \right|
\leq 3 e^{\frac{\theta T^x}{\alpha}} \epsilon |r_\infty| \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right) \left( 1 - \frac{\kappa e}{\theta + \varepsilon} \right)
\]
This completes the proof of the lemma.

Theorem 2.2. Assume (A1) and (A2). Then equation (2.7) has a solution
\( u_\alpha \in W^{1,2,p}_{\text{loc}}((0, 1) \times D) \), \( p \geq 2 \).

Proof. Let \( C \) be an open bounded set with \( C^\infty \) boundary such that \( \overline{C} \subseteq \overline{D} \). Let \( N \) be a positive integer such that
\[ C \subseteq \overline{D}_m, \text{ for all } m \geq 1, \ l \geq N. \]
From Lemma 2.1, p.d.e. (2.8) has a unique solution \( u_{\alpha,m}^\kappa \in H^{\frac{3}{2},3}_\alpha ([\kappa, 1] \times D_m) \) and
\[ \| u_{\alpha,m}^\kappa \|_{\infty; (\kappa, 1) \times D_m} \leq e^{\frac{\| r \|_\infty}{\alpha}}, \ \forall \ \kappa > 0 & m, l \geq 1. \]
Thus from Theorem 9.11, p.235 of [11], we get
\[ (2.17) \quad \| u_{\alpha,m}^\kappa \|_{1,2,p;((\kappa, 1) \times C)} < K, \text{ for all } m \geq 1, l \geq N, p \geq 2, \]
where \( K \) does not depend on \( l \) and \( m \). Now choose a sequence of open domains \( \{C_n\} \) from \( \overline{D} \) such that \( \bigcup_n C_n = \overline{D}. \) Now by a standard diagonalization procedure there exists \( u_{\alpha}^\kappa \in W^{1,2,p}_{\text{loc}}((\kappa, 1) \times \overline{D}) \) such that along a subsequence in \( l \to \infty, \)
\[ (2.18) \quad u_{\alpha,m}^\kappa \to u_{\alpha}^\kappa \text{ weakly in } W^{1,2,p}((\kappa, 1) \times C). \]
Now from (2.17), we have
\[ (2.19) \quad \| u_{\alpha,m}^\kappa \|_{1,2,p;((\kappa, 1) \times \overline{C})} < K, \text{ for all } m \geq 1. \]
Now by repeating the diagonalization argument there exists \( u_{\alpha}^\kappa \in W^{1,2,p}_{\text{loc}}((\kappa, 1) \times \overline{D}) \) such that along a subsequence in \( m \to \infty \)
\[ (2.20) \quad u_{\alpha,m}^\kappa \to u_{\alpha}^\kappa \text{ weakly in } W^{1,2,p}((\kappa, 1) \times \overline{C}). \]
Using parabolic version of the Morrey’s lemma, see [19], pp.26-27, \( W^{1,2,p}((\kappa, 1) \times C) \) is compactly embedded in \( H^{\frac{3}{2},\hat{\alpha}}((\kappa, 1] \times \overline{C}), 0 < \hat{\alpha} < 2 - \frac{d-2}{p}. \) Hence along a subsequence of \( l \to \infty, m \to \infty \), we get
\[ (2.21) \quad \lim_{m \to \infty} \lim_{l \to \infty} u_{\alpha,m}^\kappa = u_{\alpha}^\kappa \text{ where the convergence is in } H^{\frac{3}{2},\hat{\alpha}}((\kappa, 1] \times \overline{C}). \]
Now (2.21) implies (along a subsequence in \( l, m \to \infty \))
\[ (2.22) \quad \lim_{m \to \infty} \lim_{l \to \infty} \inf_{\nu} \left[ b(x,v) \cdot \nabla u_{\alpha,m}^\kappa + \theta r(x,v) u_{\alpha,m}^\kappa \right] = \inf_{\nu} \left[ b(x,v) \cdot \nabla u_{\alpha}^\kappa + \theta r(x,v) u_{\alpha}^\kappa \right] \text{ in } (\kappa, 1] \times \overline{C}. \]
By letting (along a subsequence) \( l \to \infty \) and then \( m \to \infty \) in (2.8), with the help of (2.18) and (2.22), we get
\[ (2.23) \quad \alpha \theta \frac{\partial u_{\alpha}^\kappa}{\partial \theta} = \inf_{\nu} \left[ b(x,v) \cdot \nabla u_{\alpha}^\kappa + \theta r(x,v) u_{\alpha}^\kappa \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 u_{\alpha}^\kappa) \text{ in } (\kappa, 1) \times D \]
in the sense of distribution and \( u_{\alpha}^\kappa \in W^{1,2,p}((\kappa, 1] \times C) \) for any compact subset \( \overline{C} \) of \( \overline{D} \) with \( C^\infty \) smooth boundary. Also from (2.21) it follows that
\( u_\alpha^\kappa(\kappa, x) = e^{\frac{\alpha \|r\|_\infty}{\kappa}}. \)

Since
\[ \nabla u_{\alpha,m} \cdot \gamma \equiv 0 \text{ on } \partial C \]
it follows that
\[ \nabla u_\alpha^\kappa \cdot \gamma \equiv 0 \text{ on } \partial C \cap \partial D. \]
Since \( C \) is arbitrarily chosen, it follows that
\[ \nabla u_\alpha^\kappa \cdot \gamma \equiv 0 \text{ on } \partial D \text{ a.e.} \]
This proves that \(2.7\) has a solution \( u_\alpha^\kappa \in W^{1,2,p}_{\text{loc}}((\kappa, 1) \times \overline{D}) \cap C^{\alpha/2,\delta}((0, 1) \times \overline{D}), \ p \geq 2. \)

Following the arguments in \([15], \text{Proposition 3.2}\), extend the function \( u_\alpha^\kappa \) to whole of \((0, 1)\) as follows:
\[ \tilde{u}_\alpha^\kappa(\theta, x) = \begin{cases} u_\alpha^\kappa(\theta, x) & \text{if } \theta > \kappa \\ e^{\frac{\alpha \|r\|_\infty}{\kappa}} & \text{if } 0 \leq \theta \leq \kappa. \end{cases} \]

Then it follows that, \( \tilde{u}_\alpha^\kappa \) is nonnegative, bounded, continuous,
\[ \sup_{0 < \kappa < 1} \left\| \frac{\partial \tilde{u}_\alpha^\kappa}{\partial \theta} \right\|_{\infty; (0,1) \times \overline{D}} < \infty, \]
and for each compact \( \overline{C} \subset \overline{D}, \)
\[ \sup_{0 < \kappa < 1} \| \tilde{u}_\alpha^\kappa \|_{2,p,\overline{C}} < \infty, \]
for each \( 0 < \theta < 1 \). The function \( \tilde{u}_\alpha^\kappa \) is a solution in almost everywhere sense to the following p.d.e
\[ \alpha \frac{\partial \tilde{u}_\alpha^\kappa}{\partial \theta} = \inf_{v} \left\{ b(x, v) \cdot \nabla \tilde{u}_\alpha^\kappa + \theta r(x, v) \tilde{u}_\alpha^\kappa + \frac{1}{2} \text{trace}(a(x) \nabla^2 \tilde{u}_\alpha^\kappa) \right\} \]
\begin{equation}
\tilde{u}_\alpha^\kappa(0, x) = 1, \quad \nabla \tilde{u}_\alpha^\kappa \cdot \gamma = 0 \text{ on } \partial D.
\end{equation}
Hence \( \tilde{u}_\alpha^\kappa \in W^{1,2,p}_{\text{loc}}((0, 1) \times \overline{D}) \) is a weak solution to \(2.24\). So multiply equation \(2.24\) with a test function \( \hat{\phi} \in C^\infty_c((0, 1) \times \overline{D}) \) and integrate over \((0, 1) \times \overline{D}\) we get
\[ -\alpha \int_{0}^{1} \frac{1}{\theta} \left\langle \frac{\partial \tilde{u}_\alpha^\kappa}{\partial \theta}, \hat{\phi} \right\rangle d\theta + \int_{0}^{1} \left\langle \inf_{v \in U} \left\{ b(x, v) \cdot \nabla \tilde{u}_\alpha^\kappa + \theta r(x, v) \tilde{u}_\alpha^\kappa \right\}, \hat{\phi} \right\rangle d\theta \]
\[ + \frac{1}{2} \int_{0}^{1} \left\langle \text{trace}(a(x) \nabla^2 \tilde{u}_\alpha^\kappa), \hat{\phi} \right\rangle d\theta = \int_{0}^{1} \left\langle \inf_{v \in U} \left\{ \theta r(x, v) e^{\frac{\alpha \|r\|_\infty}{\kappa}} \right\}, \hat{\phi} \right\rangle d\theta, \]
\begin{equation}
(2.25)
\end{equation}
where \( \langle \cdot, \cdot \rangle \) is inner product on \( L^2(\overline{D}) \). By letting \( \kappa \to 0 \) in above, we obtain
\[ -\alpha \int_{0}^{1} \frac{1}{\theta} \left\langle \frac{\partial u_\alpha}{\partial \theta}, \hat{\phi} \right\rangle d\theta + \int_{0}^{1} \left\langle \inf_{v \in U} \left\{ b(x, v) \cdot \nabla u_\alpha + \theta r(x, v) u_\alpha \right\}, \hat{\phi} \right\rangle d\theta \]

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where \( u_\alpha \in W^{1,2,p}_{\text{loc}}((0,1) \times D), p \geq 2 \). Therefore we have

\[
\alpha \partial u_\alpha / \partial \theta = \inf_{v \in U} \left[ b(x,v) \cdot \nabla u_\alpha + \theta r(x,v)u_\alpha \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 u_\alpha)
\]

Let \( \theta \in (0,1) \) and \( B \) be an open and bounded subset of \( D \) with Lipschitz boundary such that its closure in \( \overline{D} \) contains the part of the boundary of \( D \). Clearly \( \bar{u}_\alpha^\kappa(\theta,\cdot) \) and \( u_\alpha(\theta,\cdot) \in W^{2,p}(B) \). By Morrey Lemma, see [16], pp. 335-339], we get \( W^{2,p}(B) \) is compactly contained in \( C^{1,\alpha}(\overline{B}) \). Hence for each fixed \( \theta > 0 \), we have

\[
\bar{u}_\alpha^\kappa(\theta,\cdot) \to u_\alpha(\theta,\cdot) \quad \text{in} \quad C^{1,\alpha}(\overline{B}).
\]

Which implies that \( \nabla u_\alpha \cdot \gamma = 0 \) since \( \nabla \bar{u}_\alpha^\kappa \cdot \gamma = 0 \) on \( \partial D \).

Hence we have the existence of a weak solution \( u_\alpha \in W^{1,2,p}_{\text{loc}}((0,1) \times \overline{D}), p \geq 2 \) for the equation (2.7). This completes the proof.

Now we prove the existence of optimal control for the discounted risk-sensitive control problem. From [3], existence of a measurable minimizing subsequence, using dominated convergence theorem, we get

\[
\inf_{v(\cdot) \in A} E_x^v \left[ e^{\int_0^T e^{-\alpha s_r(X_s,v_s)}ds} \right] = 1 \quad \text{on} \quad \overline{D}.
\]

Moreover if \( v_\alpha(\cdot) \) is a minimizing selector in (2.7), then \( v_\alpha(\cdot) \) is optimal for the \( \alpha \)-discounted risk-sensitive control problem.

**Theorem 2.3.** Assume (A1) and (A2). Then equation (2.7) has a unique solution \( u_\alpha \in W^{1,2,p}_{\text{loc}}((0,1) \times \overline{D}), p \geq 2 \), given by

\[
u_\alpha(\theta,x) = \inf_{v(\cdot) \in A} E_x^v \left[ e^{\int_0^T e^{-\alpha s_r(X_s,v_s)}ds} \right].
\]

Moreover if \( v_\alpha(\cdot) \) is a minimizing selector in (2.7), then \( v_\alpha(\cdot) \) is optimal for the \( \alpha \)-discounted risk-sensitive control problem.

**Proof.** From the proof of Theorem 2.2 it is clear that for fixed \( \theta > 0 \), \( \bar{u}_\alpha^\kappa(\theta,x) = u_\alpha^\kappa(\theta,x) \) for sufficiently small \( \kappa \). Mimicking the arguments used to prove (2.10), we have the following stochastic representation

\[
u_\alpha^\kappa(\theta,x) = \inf_{v(\cdot)} E_x^v \left[ e^{\int_0^T e^{-\alpha s_r(X_s,v_s)}ds} \right],
\]

where \( X(\cdot) \) is the process (11) corresponding to an admissible control \( v(\cdot) \).

Since \( u_\alpha^\kappa(\theta,x) \to u_\alpha(\theta,x) \) pointwise and \( T_\kappa \to \infty \) as \( \kappa \to 0 \) along a subsequence, using dominated convergence theorem, we get

\[
u_\alpha(\theta,x) \leq E_x^v \left[ e^{\int_0^T e^{-\alpha s_r(X_s,v_s)}ds} \right].
\]

Since \( v(\cdot) \) is an arbitrary admissible control, we have

\[
u_\alpha(\theta,x) \leq \inf_{v(\cdot)} E_x^v \left[ e^{\int_0^T e^{-\alpha s_r(X_s,v_s)}ds} \right].
\]

In particular we get

\[
u_\alpha(\theta,x) \leq E_x^v \left[ e^{\int_0^T e^{-\alpha s_r(X_s,v_\alpha(X_s))ds} \right],
\]

where \( u_\alpha \in W^{1,2,p}_{\text{loc}}((0,1) \times D), p \geq 2 \). Therefore we have
where \( v_\alpha(\cdot) \) is a minimizing selector in (2.7). To prove other way inequality we argue as follows. The non-negativity of the function \( r \) implies \( u_\alpha^*(\theta, x) \geq 1 \) and hence \( u_\alpha(\theta, x) \geq 1 \). Consider the following s.d.e.

\[
\begin{align*}
\frac{d}{dt} X(t) &= b(X, t, u_\alpha(X(t)))dt + \sigma(X(t))dW(t) - \gamma(X(t))d\xi(t) \\
\frac{d}{dt} \xi(t) &= I_{\{X(t) \in \partial \Omega\}}d\xi(t) \\
\xi(0) &= 0, \quad X(0) = x \in \overline{\Omega}.
\end{align*}
\]

Define a sequence of stopping times as follows:

\[
\tau_k = \begin{cases} 
0 & |x| \geq k, \\
\inf\{t \geq 0 : |X(t)| \geq k\} & |x| < k,
\end{cases}
\]

where \( X(\cdot) \) is the process given by (2.26) and we use the convention that infimum of an empty set is \(+\infty\). The resulting sequence is nondecreasing with \( \lim_{k \to \infty} \tau_k = \infty, \text{ a.s.} \) Apply Ito-Dynkin formula to \( e^{\int_0^{\tau_k} \theta r(X, u_\alpha(X))ds} u_\alpha(\theta_t, X_t) \), we get

\[
e^{\int_0^{\tau_k} \theta r(X, u_\alpha(X))ds} u_\alpha(\theta_{T^{\wedge} \tau_k}, X_{T^{\wedge} \tau_k}) = u_\alpha(\theta, x) + \int_0^{\tau_k} e^{\int_0^t \theta r(X, u_\alpha(X))ds} \partial_\theta u_\alpha(\theta, X_t) dt
\]

where

\[
du_\alpha(\theta_t, X_t) = (\nabla u_\alpha(\theta_t, X_t))^{\perp} \sigma(X_t) I_{\{X_t \in \partial \Omega\}}dW(t) - \alpha \theta_t \nabla u_\alpha(\theta_t, X_t) dt
\]

\[
+ \left[ \nabla u_\alpha(\theta_t, X_t) \cdot b(X_t, u_\alpha(X_t)) + \frac{1}{2} \text{trace}(a(X_t) \nabla^2 u_\alpha(\theta_t, X_t)) \right] dt
\]

\[- \gamma(X_t) \cdot \nabla u_\alpha(\theta_t, X_t) I_{\{X_t \in \partial \Omega\}}d\xi(t).
\]

Using the fact that \( u_\alpha \) satisfy the equation (2.7), we get

\[
e^{\int_0^{\tau_k} \theta r(X, u_\alpha(X))ds} u_\alpha(\theta_{T^{\wedge} \tau_k}, X_{T^{\wedge} \tau_k}) = u_\alpha(\theta, x) + \int_0^{\tau_k} e^{\int_0^t \theta r(X, u_\alpha(X))ds} (\nabla u_\alpha(\theta_t, X_t))^{\perp} \sigma(X_t)dW(t).
\]

Since \( \nabla u_\alpha \) is continuous on \( \overline{B_k \cap \Omega} \) by the Sobolev embedding Theorem, therefore \( \nabla u_\alpha \) is bounded on \( \overline{B_k \cap \Omega} \), which implies that the stochastic integral

\[
\int_0^{\tau_k} e^{\int_0^t \theta r(X, u_\alpha(X))ds} (\nabla u_\alpha(\theta_t, X_t))^{\perp} \sigma(X_t)dW(t)
\]

is a zero mean martingale for each \( k \). Hence we get

\[
u_\alpha(\theta, x) = E_x \left[ e^{\int_0^{T^{\wedge} \tau_k} \theta r(X, u_\alpha(X))ds} u_\alpha(\theta_{T^{\wedge} \tau_k}, X_{T^{\wedge} \tau_k}) \right].
\]
Letting $k \to \infty$, we get
\[ u_\alpha(\theta, x) = E_x^v \left[ e^{\int_0^T \theta r(X_s, v_s(X_s))ds} u_\alpha(\theta_T, X_T) \right] \geq E_x^v \left[ e^{\int_0^T \theta r(X_s, v_s(X_s))ds} \right]. \]

Now taking $T \to \infty$, we obtain
\[ u_\alpha(\theta, x) \geq E_x^v \left[ e^{\int_0^\infty \theta r(X_s, v_s(X_s))ds} \right]. \]
Thus,
\[ u_\alpha(\theta, x) = \inf_{v(\cdot) \in \mathcal{A}} E_x^v \left[ e^{\int_0^\infty \theta e^{-\alpha s} r(X_s, v_s(X_s))ds} \right] = E_x^v \left[ e^{\int_0^\infty \theta e^{-\alpha s} r(X_s, v_s(X_s))ds} \right]. \]

This proves $v_\alpha(\cdot)$ is optimal and $u_\alpha$ is the unique solution to the equation (2.7), which completes the proof. □

3. Risk-sensitive Control with Near Monotone Cost

In this section we prove existence of optimal control for the risk-sensitive control problem described in Section 1, under a condition on the cost function $r(\cdot, \cdot)$, called “near monotonicity”. We also use an additional assumption that the process given by (1.1) is recurrent for each admissible control. If $C \subset \mathcal{D}$, we denote by $\tau(C)$ the first exit time of the process $X(\cdot)$ from $C$,
\[ \tau(C) = \inf \{ t > 0 : X(t) \notin C \}. \]

**Definition 3.1.** Let $X(\cdot)$ be the process given by (1.1) corresponding to an admissible control $v(\cdot)$ with initial condition $x$. We say controlled process $X(\cdot)$ is recurrent, if for any open connected set $O \subset \mathcal{D}$ the first hitting time of the set $O$, i.e., $\tau(O^c)$, satisfies $P(\tau(O^c) < \infty) = 1$, for all $x \in \mathcal{D}$. If $E[\tau(O^c)] < \infty$ for all $x \in \mathcal{D}$, then $X$ is said to be positive recurrent. Correspondingly, the control $v(\cdot)$ is called a stable control. We denote the set of stable, stationary Markov controls by $\mathcal{M}_s$.

We assume that for some admissible control $v(\cdot) \in \mathcal{A}$ and initial condition $x \in \mathcal{D}$,
\[ \limsup_{T \to \infty} \frac{1}{T} \theta \ln E_x^v \left[ e^{\int_0^T r(X_t, v_t)dt} \right] < \infty, \]
where $X(\cdot)$ is the process (1.1) corresponding to $v(\cdot)$. Define the optimal risk-sensitive values as follows
\[ \beta = \inf_{v(\cdot) \in \mathcal{A}} \limsup_{T \to \infty} \frac{1}{T} \theta \ln E_x^v \left[ e^{\int_0^T r(X_t, v_t)dt} \right]. \]

Now we state the near-monotonicity assumption.

**A3** The cost function $r$ satisfy the following
\[ \liminf_{|x| \to \infty} \inf_{v \in \mathcal{V}} r(x, v) > \beta, \quad x \in \mathcal{D}, \]
i.e., $r$ is near monotone with respect to $\beta$.

Also we use the following recurrent condition.

**A4** For each stationary Markov control $v(\cdot)$, the corresponding the process $X(\cdot)$ given by (1.1) is recurrent.
See Lemma 1.3 for a characterization of (A4).

**Remark 3.1.** (i) Note that if \( r \) is bounded then \( \beta \leq \| r \|_\infty \).
(ii) It may seem at first that (3.1) cannot be verified unless \( \beta \) is known. However there are two important cases where (3.1) always holds. The first is the case where \( \inf_{v \in A} r(x, v) \) grows asymptotically unbounded in \( x \), and \( \beta < \infty \). The second covers problems in which \( r(x, v) = r(x) \) does not depend on \( v \) and \( r(x) < \lim_{|y| \to \infty} r(y) \) for all \( x \in \overline{D} \).

We adapt the vanishing discount approach to prove the existence of optimal risk-sensitive ergodic control under the near-monotonicity assumption. To prove existence of solution for risk-sensitive ergodic HJB, we study the limiting behaviour of the equation (2.7) as \( \alpha \to 0 \).

**Theorem 3.1.** Assume (A1) and (A2). Then there exist a solution \( (\rho, \hat{u}) \in \mathbb{R} \times W_{loc}^{2,p}(\overline{D}) \) to the equation

\[
\begin{align*}
\theta \rho \hat{u} &= \inf_{v \in U} \left[ b(x, v) \cdot \nabla \hat{u} + \theta r(x, v) \hat{u} \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 \hat{u}) \\
\nabla \hat{u} \cdot \gamma &= 0 \text{ on } \partial D, \quad \hat{u}(x_0) = 1.
\end{align*}
\]

Moreover \( \rho \leq \beta \).

**Proof.** Let \( \chi_k \) denote a nonnegative smooth function such that \( \chi_k \equiv 1 \) in \( B_k \), \( \chi_k \equiv 0 \) in \( B_k^\circ \) and \( 0 \leq \chi_k \leq 1 \). Let \( r_k = r \chi_k \). Then

\[
\| r_k \|_\infty \leq \| r \|_\infty, B_{k+1}.
\]

Define for \( \alpha > 0 \)

\[
(3.3) \quad u^k_\alpha(\theta, x) := \inf_{v(\cdot) \in M_{sr}} \mathbb{E}_x^v \left[ e^{\theta \int_0^\infty e^{-\alpha t} r_k(X_t, v_t)dt} \left| X_0 = x \right. \right].
\]

Consider the p.d.e.

\[
\begin{align*}
\alpha \theta \frac{\partial u^k_\alpha}{\partial \theta} &= \inf_{v \in U} \left[ b(x, v) \cdot \nabla u^k_\alpha + \theta r_k(x, v) u^k_\alpha \right] + \frac{1}{2} \text{trace}(a(x) \nabla^2 u^k_\alpha) \\
\nabla u^k_\alpha \cdot \gamma(x) &= 0 \text{ on } \partial D, \quad u^k_\alpha(0, x) = 1.
\end{align*}
\]

Mimicking the arguments as in Theorem 2.2 and 2.3 one can see that p.d.e. (3.4) has a solution \( u^k_\alpha \) in \( W_{loc}^{1,2,p}((0, 1) \times D) \) with \( p \geq 2 \), and \( u^k_\alpha \) has the representation (3.3).

Set

\[
(3.5) \quad \phi^k_\alpha(\theta, x) = \frac{1}{\theta} \ln u^k_\alpha(\theta, x), \quad \phi^k_\alpha(\theta, x) = \alpha \phi^k_\alpha + \alpha \theta \frac{\partial \phi^k_\alpha}{\partial \theta}
\]

Mimicking the arguments as in [4], Lemma 2.1 we have

\[
(3.6) \quad \| \alpha \phi^k_\alpha \|_\infty + \left\| \alpha \theta \frac{\partial \phi^k_\alpha}{\partial \theta} \right\|_\infty \leq 3 \| r_k \|_\infty, \forall 0 < \alpha < 1, \ 0 < \theta \leq 1.
\]
Let \( \tau \) denote the entrance time of the process (1.1) to the set \( B_{k+1} \) under the admissible control \( v(\cdot) \in \mathcal{A} \). Let \( x \in B_{k+1}^c \). Then dynamic programming principle (2.5) gives

\[
\begin{align*}
    u_k^\alpha(\theta, x) &= \inf_{v(\cdot)} E_x^v \left[ e^{\theta \int_0^{\tau(v)} e^{-\alpha t} r_k(X_t, v_t) dt} \right] \\
    &= \inf_{v(\cdot)} E_x^v \left[ u_k^\alpha(\theta e^{-\alpha \tau}, X_{\tau}) \right] \quad (\because r_k \equiv 0 \text{ on } B_{k+1}^c) \\
    &\leq \inf_{v(\cdot)} E_x^v \left[ u_k^\alpha(\theta, X_{\tau}) \right] \quad (\because e^{-\alpha \tau} < 1 \text{ a.s. and } u_k^\alpha \text{ is increasing in } \theta) \\
    &\leq \sup_{y \in \partial B_{k+1}} u_k^\alpha(\theta, y)
\end{align*}
\]

Using Harnack’s inequality, from [11], Theorem 8.20, pp. 199 we have

\[
(3.7) \quad \sup_{x \in B_{k+1}} u_k^\alpha(\theta, x) \leq K_{4,1}(k),
\]

where \( K_{4,1}(k) \) is independent of \( \alpha \).

Set

\[
\tilde{u}_k^\alpha(\theta, x) := \frac{u_k^\alpha(\theta, x)}{u_k^\alpha(\theta, x_0)} \quad \text{for some } x_0 \in D.
\]

Then \( \tilde{u}_k^\alpha \) is solution to

\[
(3.8) \quad 0 = \inf_{v \in U} \left[ \mathcal{L} \tilde{u}_k^\alpha(\theta, x, v) + \theta (r_k(x, v) - g_k^\alpha) \tilde{u}_k^\alpha \right] + \nabla \tilde{u}_k^\alpha \cdot \gamma(x) = 0 \text{ on } \partial D, \quad \tilde{u}_k^\alpha(\theta, x_0) = 1.
\]

From (3.7) it follows that

\[
\sup_{x \in B_{k+1}} \tilde{u}_k^\alpha(\theta, x) \leq K_{4,1}(k).
\]

But the foregoing arguments show that for \( x \in B_{k+1}^c \),

\[
\tilde{u}_k^\alpha(\theta, x) \leq \sup_{y \in \partial B_{k+1}} \left[ \frac{u_k^\alpha(\theta, y)}{u_k^\alpha(\theta, x_0)} \right] \leq K_{4,1}(k),
\]

where \( K_{4,1}(k) \) can be chosen independent of \( x, \alpha \). Now using [11], Theorem 9.11, pp. 235] we have for each \( R < k + 1 \)

\[
(3.9) \quad \|\tilde{u}_k^\alpha(\theta, \cdot)\|_{W^{2,p}(B_R)} \leq K_{4,2},
\]

where \( K_{4,2} > 0 \) is independent of \( \alpha > 0 \). Now using compact and continuous Sobolev embedding theorem, for each fixed \( \theta > 0 \), without loss of generality \( \theta = 1 \), there exists \( \bar{u}^k \in W^{2,p}_{\text{loc}}(D) \) such that

\[
\tilde{u}_k^\alpha(1, \cdot) \rightarrow \bar{u}^k \text{ strongly in } W^{1,p}_{\text{loc}}(D),
\]

\[
\tilde{u}_k^\alpha(1, \cdot) \rightharpoonup \bar{u}^k \text{ weakly in } W^{2,p}_{\text{loc}}(D),
\]

Using compactness argument, it follows that

\[
\bar{u}^k(1, \cdot) \rightharpoonup \bar{u} \text{ weakly in } W^{2,p}_{\text{loc}}(D),
\]

\[
\bar{u}^k(1, \cdot) \rightarrow \bar{u} \text{ strongly in } L^p(\Omega) \text{ and } W^{1,p}_{\text{loc}}(D).
\]
along a subsequence as $\alpha \downarrow 0$. By Sobolev embedding theorem, the convergence is uniform on compact subsets of $\overline{D}$, hence we have $\hat{u}^k$ is bounded above by $K_{4.1}(k)$. Now we show that

$$g^k_\alpha(1, x) \longrightarrow \rho_k \in \mathbb{R}. $$

From (3.6), along a further subsequence,

$$\alpha \phi^k_\alpha(\theta, x) \longrightarrow \rho^k_1(\theta, x), \text{ in weak* topology of } L^\infty((0, 1) \times D).$$

We show that $\rho^k_1$ is a function of $\theta$ alone. From (3.9) and

$$\frac{1}{\theta} \nabla \ln \bar{u}^k_\alpha(\theta, x) = \nabla \phi^k_\alpha(\theta, x),$$

we have for any $R > 0$

$$\|\nabla \phi^k_\alpha(\theta, \cdot)\|_{W^{1,p}(B_R)} \leq K_{4.3},$$

where $K_{4.3} > 0$ is independent of $\alpha > 0$. By (3.11),

$$\lim_{\alpha \downarrow 0} \int_D \alpha \nabla \phi^k_\alpha(\theta, x) f(x) = 0,$$

for each $f \in C_c^\infty(D)$. Thus the distributional derivative of $\rho^k_1$ in $x$ is identically zero, proving the claim. Also by (3.11), for each fixed $\theta = \theta_0 > 0$,

$$\left\{ \alpha \frac{\partial \phi^k_\alpha}{\partial \theta} \mid \alpha > 0 \right\}$$

is bounded in $L^\infty([\theta_0, 1] \times D)$. Hence along a further subsequence

$$\alpha \frac{\partial \phi^k_\alpha}{\partial \theta} \longrightarrow \rho^k_2(\theta, x), \text{ weakly in } L^2_{loc}([\theta_0, 1] \times D).$$

It follows from (3.10) and (3.12) that $\rho^k_2(\cdot, \cdot) = (\rho^k_1)'$ in the sense of distribution, where $(\rho^k_1)'$ is the distributional derivative (in $\theta$) of $\rho^k_1$. Hence $\rho^k_2(\cdot, \cdot)$ is also a function of $\theta$ alone. Thus we have: for each $\theta > 0$ there exists a constant $\rho_k$ such that along a subsequence

$$\alpha \phi^k_\alpha + \alpha \theta \frac{\partial \phi^k_\alpha}{\partial \theta} \longrightarrow \rho_k.$$

Now letting $\alpha \longrightarrow 0$ in (3.8) along the subsequence, we have $(\hat{u}^k, \rho_k) \in W^{2,p}_{loc}(D) \times \mathbb{R}$ satisfying the following equation

$$\rho_k \hat{u}_k = \inf_{v \in U} \left[ b(x, v) \cdot \nabla \hat{u}_k + r_k(x, v) \hat{u}_k + \frac{1}{2} \text{trace}(a(x) \nabla^2 \hat{u}_k) \right]$$

$$\nabla \hat{u}_k \cdot \gamma(x) = 0 \text{ on } \partial D, \quad \hat{u}_k(x_0) = 1.$$

Applying Itô’s formula to the process (1.1) corresponding to $v(\cdot) \in A,$

$$d \left( e^{\int_0^t (r_k(X_s, v_s) - \rho_k) ds} \hat{u}^k(X_t) \right) = e^{\int_0^t (r_k(X_s, v_s) - \rho_k) ds} d\left( \hat{u}^k(X_t) \right)$$

$$+ (r_k(X_t, v_t) - \rho_k) e^{\int_0^t (r_k(X_s, v_s) - \rho_k) ds} \hat{u}^k(X_t) dt,$$
where
\[
d(a^k(X_t)) = \left[ b(X_t, v_t) \cdot \nabla u^k(X_t) + \frac{1}{2} \text{trace}(a(X_t) \nabla^2 u^k(X_t)) \right] I\{X_t \in \partial D\} dt
\]

Hence it follows that
\[
e^{\int_0^{T \wedge \tau_R} (r_k(X_s, v_s) - \rho_k) ds} \hat{u}^k (X_{T \wedge \tau_R}) - \hat{u}^k (x)
\]  \tag{3.14}
\[
\geq \int_0^{T \wedge \tau_R} e^{\int_0^t (r_k(X_s, v_s) - \rho_k) ds} \nabla \hat{u}^k (X_t) \sigma(X_t) dW_t.
\]

Since we have \( \hat{u}^k \in W^{2,p}_{\text{loc}}(D), p \geq d \), which implies \( \nabla \hat{u}^k \) is bounded on each compact subset \( D_R \), hence
\[
\int_0^{T \wedge \tau_R} e^{\int_0^t (r_k(X_s, v_s) - \rho_k) ds} (\nabla \hat{u}^k (X_t)) \sigma(X_t) dW_t,
\]
is a zero mean martingale. Taking expectation in \( 3.14 \) we obtain
\[
E^u \left[ e^{\int_0^{T \wedge \tau_R} (r_k(X_s, v_s) - \rho_k) ds} \hat{u}^k (X_{T \wedge \tau_R}) \right] - \hat{u}^k (x) \geq 0.
\]

Since \( \hat{u}^k \) is bounded above, we have
\[
\hat{u}^k (x) \leq K_{4\cdot1}(k) E^u \left[ e^{\int_0^{T \wedge \tau_R} (r_k(X_s, v_s) - \rho_k) ds} \right] \leq K_{4\cdot1}(k) E^u \left[ e^{\int_0^T (r_k(X_s, v_s) - \rho_k) ds} \right].
\]

Taking ln and divide by \( T \) we get
\[
\frac{1}{T} \ln \hat{u}^k (x) \leq \frac{1}{T} \ln K(k) + \frac{1}{T} \ln E^u \left[ e^{\int_0^T (r_k(X_s, v_s) - \rho_k) ds} \right].
\]

Since \( u^k \geq 1 \) by definition \( \hat{u}^k \) is bounded below, hence uniform convergence on compact sets gives that \( \hat{u}^k \) is bounded below say by \( K_{4\cdot3}(k) > 0 \). Hence
\[
\frac{1}{T} \ln K_{4\cdot3}(k) \leq \frac{1}{T} \ln K(k) + \frac{1}{T} \ln E^u \left[ e^{\int_0^T r_k(X_s, v_s) ds} \right] - \rho_k.
\]

Now taking \( T \to \infty \) we get
\[
\rho_k \leq \limsup_{T \to \infty} \frac{1}{T} \ln E^u \left[ e^{\int_0^T r_k(X_s, v_s) ds} \right].
\]

Since \( |r_k| \leq |r| \),
\[
\rho_k \leq \limsup_{T \to \infty} \frac{1}{T} \log E^u \left[ e^{\int_0^T r(X_s, v_s) ds} \right].
\]

Taking infimum over all stable stationary Markov controls in the right hand side of above, we get
\[
\rho_k \leq \beta, \ \forall \ k. \tag{3.15}
\]

Since the coefficients of \( 3.13 \) are bounded, we have \( |\hat{u}^k| \) is bounded uniformly in \( k \) on compact sets by Harnack’s inequality. Thus we have \( \hat{u}^k \to \hat{u} \) in \( W^{1,p}_{\text{loc}}(D) \) and \( \rho_k \to \rho \) along a subsequence. Furthermore, it follows from
Harnack’s inequality that \( \hat{u} > 0 \) on compacts, in fact one has uniform positive lower bounds for \( \hat{u}^k \) on compacts. Letting \( k \to \infty \) in (3.13), \( (\rho, \hat{u}) \) satisfy

\[
\begin{align*}
\rho \hat{u} &= \inf_{v \in U} \left\{ b(x, v) \cdot \nabla \hat{u} + r(x, v) \hat{u} + \frac{1}{2} \text{trace} (a(x) \nabla^2 \hat{u}) \right\} \\
\nabla \hat{u} \cdot \gamma(x) &= 0 \text{ on } \partial D, \hat{u}(x_0) = 1,
\end{align*}
\]

where for the boundary condition it is same argument as in Theorem 2.2. In view of (3.15) it follows that \( \rho \leq \beta \), which completes the proof. \( \square \)

**Theorem 3.2.** Assume (A1)-(A4). Then ergodic risk-sensitive HJB equation (3.2) has a solution \( (\rho, \hat{\phi}) \) such that \( \rho \) is unique and is characterized by \( \rho = \beta \). Also, minimizing selector in (3.2) is an optimal control.

**Proof.** In view of Theorem 3.1 it remains to show \( \beta \leq \rho \). By assumption (3.1) we have

\[
\inf_{v} r(\cdot, v) \geq \beta > \rho \quad \text{outside } O \subset \overline{D}.
\]

We know that for some \( \nu > 0, \hat{u} \geq \nu > 0 \) in \( O \). Let \( x \in O \cap \overline{D} \). Let \( R > 0 \) be large enough so that \( B_R \) contains the set \( O \) and initial point \( x \). Set \( T_R = R \land \tau(B_R) \). Let \( v^*(\cdot) \) be minimizing selector in (3.2), applying Dynkin’s formula

\[
e^{\int_{0}^{\tau(O^c) \wedge T_R} (r(x, v^*_x) - \rho)ds} \hat{u}(X_{\tau(\Omega^c) \wedge T_R}) - \hat{u}(x) = 0
\]

Since \( \hat{u} \in W^{2,1}_{\text{loc}}(\overline{D}) \), it follows that \( \nabla \hat{u} \) is locally bounded and using the boundedness of \( r, \sigma \),

\[
\int_{0}^{\tau(O^c) \wedge T_R} e^{\int_{0}^{t} (r(x, v^*_x) - \rho)ds} (\nabla \hat{u}(X_t)) \sigma(X_t) dW_t,
\]

is zero mean martingale. Then we have

\[
E_x^v \left[ e^{\int_{0}^{\tau(O^c) \wedge T_R} (r(x, v^*_x) - \rho)ds} \hat{u}(X_{\tau(O^c) \wedge T_R}) \right] - \hat{u}(x) = 0
\]

Using the Fatou’s lemma, letting \( R \to \infty \) we get

\[
\hat{u}(x) \geq E_x^v \left[ e^{\int_{0}^{\tau(O^c)} (r(x, v^*_t) - \rho)dt} \hat{u}(X_{\tau(O^c)}) \right].
\]

Using (A4), it follows that \( \tau(O^c) < \infty \) a.s. Hence

\[
E_x^v \left[ e^{\int_{0}^{\tau(O^c)} (r(x, v^*_t) - \rho)dt} \hat{u}(X_{\tau(O^c)}) \right] \geq \nu.
\]

This proves that \( \hat{u} \) is bounded below by \( \nu \). Repeating the previous argument, we also have for any \( T > 0 \),

\[
\hat{u}(x) \geq E_x^v \left[ e^{\int_{0}^{T} (r(x, v^*_t) - \rho)dt} \hat{u}(X_T) \right] \geq \nu E_x^v \left[ e^{\int_{0}^{T} (r(x, v^*_t) - \rho)dt} \right].
\]
Taking logarithm and dividing by $T$

$$\frac{1}{T} \ln E^v_x \left[ e^{\int_0^T (r(X_t,v^*_t)-\rho)dt} \right] + \frac{1}{T} \ln \nu \leq \frac{1}{T} \ln \hat{u}(x)$$

Letting $T \to \infty$ on both sides, we have

$$\limsup_{T \to \infty} \frac{1}{T} \ln E^v_x \left[ e^{\int_0^T (r(X_t,v^*_t)-\rho)dt} \right] \leq 0$$

i.e.,

$$\limsup_{T \to \infty} \frac{1}{T} \ln E^v_x \left[ e^{\int_0^T r(X_t,v^*_t)dt} \right] \leq \rho.$$ 

Thus $\beta \leq \rho$. This completes the proof of the theorem. $\square$

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