The Cameron-Martin Theorem for (p-)Slepian processes

Wolfgang Bischoff∗ and Andreas Gegg
Faculty of Mathematics and Geography,
Catholic University of Eichstätt-Ingolstadt,

Abstract

We show a Cameron-Martin theorem for Slepian processes

\[ W_t := \frac{1}{\sqrt{p}} (B_t - B_{t-p}), t \in [p, 1], \]

where \( p \geq \frac{1}{2} \) and \( B_t \) is Brownian motion. More

exactly, we determine the class of functions \( F \) for which a density of

\[ F(t) + W_t \]

exists with respect to \( W_t \). Moreover, we prove an explicit for-

mula for this density. p-Slepian processes are closely related to Slepian

processes. p-Slepian processes play a prominent role among others in scan

statistics and in testing for parameter constancy when data are taken from

a moving window.

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derivative

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1 Introduction

A Cameron-Martin theorem for a stochastic process is one of the most useful
tools to solve problems the process is involved in. Let a stochastic process

\[ X_{[a,b]} = (X_t)_{t \in [a,b]} \]

with paths in \( C[a,b] \), the set of real-valued and continuous

functions on \( [a,b] \subseteq \mathbb{R} \), and a deterministic function \( F \in C[a,b] \) be given. Let

\[ P^{X_{[a,b]}} \] and \( P^{(X_t + F(t))_{t \in [a,b]}} \) denote the distribution of \( X_{[a,b]} \) and of \((X_t + F(t))_{t \in [a,b]} \)
on \( C[a,b] \), respectively. Then a Cameron-Martin theorem gives conditions on

\( F \) under which a density \( \frac{dP^{(X_t + F(t))}}{dP^{X_t}} \) exists and, additionally, it gives an explicit
formula for this density.

The first and best known result of this type is by [6]. They proved it for
the standard Brownian motion \( B_{[0,1]} := (B_t)_{t \in [0,1]} \) with continuous paths. This
Cameron-Martin theorem can be used, for example, to calculate optimal tests,
see [4], and to estimate boundary crossing probabilities, see [5]. Therefore, such
a result of Cameron-Martin type is also of great interest for other stochastic
processes. The following theorem, see Lifshits [12, Theorem 5.1], can be used
as basis to get results of Cameron-Martin type for centered Gaussian processes.
To this end, let \( L^2(C[a,b], P) \) be the set of all real-valued and square-integrable

∗corresponding author: Wolfgang Bischoff, Faculty of Mathematics and Geography, D-
85071 Eichstätt, Germany; E-Mail: wolfgang.bischoff@ku-eichstaett.de
functions on $C[a, b]$ with respect to a measure $P$ defined on the Borel-σ-Algebra of $C[a, b]$.

**Theorem 1.1 (An abstract Cameron-Martin theorem)**

Let $X_{[a, b]} := (X_t)_{t \in [a, b]}$ be a centered Gaussian process with paths in $C[a, b]$, let $\mathcal{H} = \mathcal{H}_{X_{[a, b]}} \subseteq C[a, b]$ be the kernel of $P_{X_{[a, b]}}$ and $\| \cdot \|_{\mathcal{H}}$ its inherent norm. Then,

$$P^{(X+h)_{[a, b]}}(\cdot) \text{ is absolutely continuous with respect to } P_{X_{[a, b]}} \Leftrightarrow h \in \mathcal{H}.$$  

If $h \in \mathcal{H}$, then

$$\frac{dP^{(X+h)_{[a, b]}}}{dP_{X_{[a, b]}}}(g) = \exp \left(-\frac{1}{2}\|h\|_{\mathcal{H}}^2 + z(g)\right) \text{ for } P_{X_{[a, b]}}\text{-almost all } g \in C([a, b]),$$

where $z = z_{X_{[a, b]}} \in L^2(C[a, b], P_{X_{[a, b]}})$ is a linear functional fulfilling the equation

$$E\left(X_t \cdot z_{X_{[a, b]}}\right) = h(t), \quad t \in [a, b].$$

Hence, the kernel $\mathcal{H} = \mathcal{H}_{X_{[a, b]}}$ of the stochastic process $X_{[a, b]}$ together with its inherent norm $\| \cdot \|_{\mathcal{H}}$ and the functional $z_{X_{[a, b]}}$ must be determined to obtain an applicable result.

In the following, we use the notation stated in the next definition.

**Definition 1.2**

Let $X_{[a, b]} := (X_t)_{t \in [a, b]}$ be a centered Gaussian process with paths in $C[a, b]$ and let $\mathcal{H} = \mathcal{H}_{X_{[a, b]}} \subseteq C[a, b]$ be the kernel of $P_{X_{[a, b]}}$. Then, we say that a function

$$z = z_{X_{[a, b]}} : \mathcal{H}_{X_{[a, b]}} \times C[a, b] \to \mathbb{R}$$

fulfills condition $A$ for $X_{[a, b]}$, if and only if for all $h \in \mathcal{H}_{X_{[a, b]}}$ the function $z(h, \cdot) \in L^2(C[a, b], P_{X_{[a, b]}})$ is a linear functional fulfilling

$$E\left(X_t \cdot z(h, X_{[a, b]}(t))\right) = h(t), \quad t \in [a, b]. \quad (1)$$

For the Brownian motion, we have the following result by [6]. To this end, let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$ and let $L^2[a, b] := L^2([a, b], \lambda)$ be the square-integrable functions on $[a, b]$ with respect to $\lambda$.

**Theorem 1.3**

The kernel of $B_{[0, 1]}$ is given by

$$\mathcal{H}_{B_{[0, 1]}} := \{ s_f \in C[0, 1] \mid f \in L^2[0, 1] \},$$

where

$$s_f(t) := \int_{[0, t]} f \, d\lambda , \quad t \in [0, 1], \text{ for } \lambda\text{-integrable } f : [0, 1] \to \mathbb{R}.$$  

It is furnished with the norm

$$\|s_f\|_{\mathcal{H}_{B_{[0, 1]}}} := \|f\|_{L^2[0, 1]}, \quad f \in L^2[0, 1].$$
A function \( z := z_{B_{[0,1]}} \) fulfilling condition \( A \) for \( B_{[0,1]} \) is given by the Wiener integral, see, for instance, [14].

\[
z : \mathcal{H}_{B_{[0,1]}} \times C[0,1] \to \mathbb{R}, \quad (s_f, b) \to \int_0^1 f \, db.
\]

Slepian and \( p \)-Slepian processes are defined and their close relations are discussed in the next section. In Sect. 3, results of Cameron-Martin type are established for \( p \)-Slepian processes. Several proofs are postponed to an appendix.

## 2 Slepian and \( p \)-Slepian processes

The Slepian process \( X \) is the centered stationary Gaussian process with covariance function

\[
C_X(s', s' + u) = (1 - u)^+ , \quad 0 \leq s' \leq s' + u,
\]

where \( t^+ = \max(0, t), t \in \mathbb{R} \). This process was introduced and studied in [18] and later in [16,17]. Afterward, it was handled in numerous theoretical and applied probabilistic models; see, e.g., [1-3,9-11,13,15].

Let us consider the Slepian process on time intervals \([1, b]\), where \( 1 < b \) is any fixed constant. Let \( \nabla_p, p > 0 \), be the backward difference operator with lag \( p \), i.e., \( \nabla_p F(t) = F(t) - F(t - p), t \in \mathbb{R}, \) for functions \( F : \mathbb{R} \to \mathbb{R} \). Note that the Slepian process coincides in distribution with

\[
(\nabla_1 B)_{[1,b]} := (\nabla_1 B_t)_{t \in [1,b]} = (B_t - B_{t-1})_{t \in [1,b]},
\]

where \( B_t \) is the standard Brownian motion with continuous paths in \([0, b]\). In Cressie [8, pp. 834], a slightly different expression of Slepian processes appeared in connection with scan statistics. They were considered the processes

\[
\frac{1}{\sqrt{p}}(\nabla_p B)_{[p,1]} := \frac{1}{\sqrt{p}}(\nabla_p B_t)_{t \in [p,1]} = \frac{1}{\sqrt{p}}(B_t - B_{t-p})_{t \in [p,1]}
\]

for any fixed constant \( p \in (0, 1) \). We call this process \( p \)-Slepian process. Another application of \( p \)-Slepian processes is given by [7]. They consider moving sums of recursive residuals which are taken from windows of length \( p \). By letting the number of residuals to \( \infty \), they get the \( p \)-Slepian process \( \frac{1}{\sqrt{p}}(\nabla_p B)_{[p,1]} \).

By a suitable scaling in time the Slepian process \( (\nabla_1 B)_{[1,b]}, b \in (1, \infty) \), can be transferred to the \( p \)-Slepian process \( \frac{1}{\sqrt{p}}(\nabla_p B)_{[p,1]}, p \in (0, 1) \), by putting \( p = \frac{1}{b} \). More exactly it holds \( (\nabla_1 B_u)_{u \in [1,b]} = \sqrt{b}(\nabla_{\frac{1}{b}} B_{\frac{1}{b}u})_{u \in [1,b]} \) in distribution. The covariance function of a \( p \)-Slepian process is given by

\[
C_{\frac{1}{\sqrt{p}}(\nabla_p B)_{[p,1]}}(s, s + t) = \left(1 - \frac{t}{p}\right)^+, \quad p \leq s \leq s + t \leq 1.
\]
3 Cameron-Martin Theorem for p-Slepian processes

The function $\frac{1}{\sqrt{p}} \nabla_p$, $0 < p \leq 1$, is linear. Hence, by [12] proposition 4.1, the kernel $\mathcal{H}_{p\text{-SI}}$ of the p-Slepian process $\frac{1}{\sqrt{p}}(\nabla_p B)_{[p,1]}$ is given by $\mathcal{H}_{p\text{-SI}} = \{ \frac{1}{\sqrt{p}} \nabla_p h | h \in \mathcal{H}_{B[0,1]} \}$. By some calculations we get

$$\mathcal{H}_{p\text{-SI}} = \left\{ \frac{1}{\sqrt{p}} \nabla_p s_f : [p,1] \to \mathbb{R} | f \in L^2[0,1] \right\} = \{ c + s_g : [p,1] \to \mathbb{R} | c \in \mathbb{R}, g \in L^2[p,1] \}.$$

More exactly, we have for $f \in L^2[0,1]$

$$\frac{1}{\sqrt{p}} \nabla_p s_f(t) = \frac{1}{\sqrt{p}} s_f(p) + s_g \nabla_p f(t), t \in [p,1]. \quad (3)$$

It is much more complicated to determine the inherent norm of $\mathcal{H}_{p\text{-SI}}$ since $\nabla_p$ is not injective. For the following, the information of $\nabla_p s_f$ is important. For $f, g \in L^2[0,1]$, it holds true

$$\forall t \in [p,1]: \nabla_p s_f(t) = s_f(p) + \int_p^t f(s) - f(s - p) ds = \nabla_p s_g(t) \iff f(t) - f(t - p) = g(t) - g(t - p) \text{ a.s. for all } t \in [p,1], s_f(p) = s_g(p).$$

We prove the following result in the appendix.

**Lemma 3.1** Let $1/2 \leq p \leq 1$. Then the kernel $\mathcal{H}_{p\text{-SI}}$ is furnished with its inherent norm

$$\left\| \frac{1}{\sqrt{p}} \nabla_p s_f \right\|_{p\text{-SI}}^2 = \frac{1}{p} \inf_{g \in L^2[0,1]} \left\| g \right\|_{L^2[0,1]}^2 = \frac{1}{2p(3p - 1)} (2s_f(p)^2 + \delta(1 - p)) + \frac{1}{2p} \left\| (f(t) - f(t - p))_{t \in [p,1]} \right\|_{L^2[0,1]}^2,$$

where $\delta = \frac{1}{1 - p} (s_f(1) - s_f(p) - s_f(1 - p))$. This minimum is attained at the function

$$f^*(t) = \left( \frac{1}{3p - 1}(s_f(p) + \frac{1 - p}{2} \delta + \frac{1}{2} (-f(t + p) + f(t)) \right) 1_{[0,1-p]}(t) + \frac{1}{2p - 1} (s_f(p)) - \frac{1 - p}{3p - 1} (s_f(p) - (2p - 1) \delta)) 1_{(1-p,1]}(t) \right.$$

Finally, we need the function $z := z \frac{1}{\sqrt{p}}(\nabla_p B)_{[p,1]}$ fulfilling condition A for the p-Slepian process to be in the position to state a result of Cameron-Martin type.
Lemma 3.2 Let $1/2 \leq p \leq 1$. The function $z := z_{\sqrt{p}}(\nabla_p \mathcal{B})_{[0,1]}$ defined by

$$z : \mathcal{H}_{\sqrt{p}}(\nabla_p \mathcal{B})_{[0,1]} \times C[p,1] \to \mathbb{R}, \quad \left( \frac{1}{\sqrt{p}} s_f(p) + s_{\sqrt{p}} \nabla_p f, \frac{1}{\sqrt{p}} \nabla_p b \right) \mapsto$$

$$= \frac{3s_f(p) + s_{\sqrt{p}} f(1)}{2(3p-1)} (\nabla_p b(p) + \nabla_p b(1)) + \frac{1}{2p} \int \nabla_p f(s) d(\nabla_p b(s))$$

fulfills condition A for $\frac{1}{\sqrt{p}}(\nabla_p \mathcal{B})_{[p,1]}$. Note that the integral

$$\int \nabla_p f(s) d(\nabla_p b(s)) = \int \nabla_p f(s) dB(s) - \int \nabla_p f(s) dB(s-p)$$

has to be understood as Wiener Integral.

The proof is given in the appendix. By the above lemmas, we obtain the following result.

Theorem 3.3 (Cameron-Martin theorem for p-Slepian processes)

Let $1/2 \leq p \leq 1$. It holds true

$$\mathbb{P}(\frac{1}{\sqrt{p}} \nabla_p \mathcal{B} + h)_{[p,1]} \text{ is absolutely continuous with respect to } \mathbb{P}(\frac{1}{\sqrt{p}} \nabla_p \mathcal{B})_{[p,1]}$$

if and only if $h \in \mathcal{H}_{p-St}$. For $h = \frac{1}{\sqrt{p}} s_f(p) + s_{\sqrt{p}} f(\cdot) \in \mathcal{H}_{p-St}$ it holds true for $\mathbb{P}(\frac{1}{\sqrt{p}} \nabla_p \mathcal{B})_{[p,1]}$-almost all $\frac{1}{\sqrt{p}} \nabla_p b \in C[p,1]:$

$$\frac{d\mathbb{P}(\frac{1}{\sqrt{p}} \nabla_p \mathcal{B} + h)_{[p,1]}}{d\mathbb{P}(\frac{1}{\sqrt{p}} \nabla_p \mathcal{B})_{[p,1]}} \left( \frac{1}{\sqrt{p}} \nabla_p b \right) = \exp \left( \frac{1}{4p} \left( \frac{(2s_f(p) + \delta(1-p))^2}{3p-1} + \| (f(\cdot) - f(\cdot - p)) \|^2_{L^2[0,1]} \right) + \frac{3s_f(p) + s_{\sqrt{p}} f(1)}{2(3p-1)} (\nabla_p b(p) + \nabla_p b(1)) + \frac{1}{2p} \int \nabla_p f(s) d(\nabla_p b(s)) \right)$$

where $\delta = \frac{1}{1-p}(s_f(1) - s_f(p) - s_f(1-p))$.

4 Appendix

Proof of Lemma Let $\frac{1}{\sqrt{p}} \nabla_p s_f \in \mathcal{H}_{p-St}, f \in L^2[0,1]$, be arbitrary and let $1/2 \leq p \leq 1$. There are several places where the case $p = 1/2$ must be considered separately. We do not do this since the specifications for $p = 1/2$ are simpler than the following considerations for $1/2 < p \leq 1$. By proposition 4.1 of [12], we get

$$\| \frac{1}{\sqrt{p}} \nabla_p s_f \|_{p-St} = \frac{1}{\sqrt{p}} \inf_{g \in L^2[0,1]} \| \nabla_p s_f - \nabla_p g \|_{\mathcal{H}_{p-St}}$$

$$= \frac{1}{\sqrt{p}} \inf_{g \in L^2[0,1]} \| s_g \|_{\mathcal{H}_{p-St}}$$

$$\| g \|_{L^2[0,1]} = \frac{1}{\sqrt{p}} \inf_{g \in L^2[0,1]} \| \nabla_p s_f - \nabla_p g \|_{\mathcal{H}_{p-St}}$$

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where the second equality follows by Theorem 1.3. Each function $f \in L^2[0,1]$ can be written $\lambda - a.s.$ in the form

$$f(t) = \alpha 1_{[0,1-p]}(t) + a(t) + \beta 1_{(1-p,p)}(t) + b(t) + \gamma 1_{[p,1]}(t) + c(t), t \in [0,1],$$

where

$$\alpha = \frac{1}{1-p} s_f(1-p), a(t) = 1_{[0,1-p]}(t)(f(t) - \alpha),$$

$$\beta = \frac{1}{2p-1} (s_f(p) - s_f(1-p)), b(t) = 1_{(1-p,p)}(t)(f(t) - \beta),$$

$$\gamma = \frac{1}{1-p} (s_f(1) - s_f(p)), c(t) = 1_{[p,1]}(t)(f(t) - \gamma).$$

Note that the six summands in (7) build orthogonal functions in $L^2[0,1]$. In the following, we use the consequence of this representation at several places without citing it explicitly. Furthermore, we have

$$s_f(p) = \alpha(1-p) + \beta(2p-1), f(t) - f(t-p) = \gamma - \alpha + c(t) - a(t), t \in [p,1].$$

Next, we consider $g \in L^2[0,1]$ of the specific form

$$g(t) = \alpha 1_{[0,1-p]}(t) + \beta 1_{(1-p,p)}(t) + (\alpha + \delta) 1_{[p,1]}(t), t \in [0,1], \alpha, \beta, \delta \in \mathbb{R}.$$ 

Hence,

$$s_g(p) = \alpha(1-p) + \beta(2p-1) \iff \beta = \frac{1}{2p-1} (s_g(p) - \alpha(1-p))$$

and

$$g(t) - g(t-p) = \delta, t \in [p,1].$$

Thus, $\nabla_p(g)(t) = s_g(p) + \delta t, t \in [p,1]$. Let $g_0(t) = \alpha_0 1_{[0,1-p]}(t) + \beta_0 1_{(1-p,p)}(t) + (\alpha_0 + \delta_0) 1_{[p,1]}(t) \in L^2[0,1]$ be fixed and let $g \in L^2[0,1]$ with $\nabla_p(g)(t) = \nabla_p(g_0)(t) = s_{g_0}(p) + \delta_0 t, t \in [p,1]$. Then,

$$g(t) = \alpha 1_{[0,1-p]}(t) + \frac{1}{2p-1} (s_{g_0}(p) - \alpha(1-p)) 1_{(1-p,p)}(t) + (\alpha + \delta_0) 1_{[p,1]}(t)$$

and the square of its norm is given by

$$\|g\|^2 = \frac{1-p}{2p-1} (\alpha^2(3p-1) - 2\alpha(s_{g_0}(p) - (2p-1)\delta_0)) + \frac{s_{g_0}(p)^2}{2p-1} + \delta_0^2(1-p).$$

This norm is minimal if and only if

$$\alpha = \frac{1}{3p-1} (s_{g_0}(p) - (2p-1)\delta_0).$$

Hence, we obtain after some calculation

$$\|\nabla_p s_{g_0}\|_{p-st}^2 = \frac{1}{3p-1} (2s_{g_0}(p)^2 + 2(1-p)s_{g_0}(p)\delta_0 + \delta_0^2(1-p)).$$

Next, we consider a general function

$$f = \alpha 1_{[0,1-p]}(t) + a(t) + \beta 1_{(1-p,p)}(t) + b(t) + (\alpha + \delta) 1_{[p,1]}(t) + c(t) \in L^2[0,1].$$
By the above considerations we obtain

\[
\| \nabla_p s_f \|^2_{p-\text{SI}} = \frac{1}{3p-1} (2s_f(p)^2 + 2(1-p)s_f(p)\delta + \delta^2(1-p)p) \\
+ \min \|a(t)\|^2_{L^2[0,1]} + \|c(t)\|^2_{L^2[0,1]}
\]

where the minimum is taken over all \(a : [0,1-p] \rightarrow \mathbb{R}, b : (1-p,p) \rightarrow \mathbb{R}, c : [p,1] \rightarrow \mathbb{R}\) with \(s_a(1-p) = s_a(2p-1) = s_c(1) = 0\) and \(c(t) - a(t-p) = f(t) - f(t-p) - \delta\). It holds true for \(t \in [p,1]\) fixed

\[
\min |a(t-p)|^2 + |c(t)|^2 = \frac{1}{2}(-c(t) + a(t-p))^2 + \frac{1}{2}(c(t) - a(t-p))^2 \\
= \frac{1}{2}|(c(t) - a(t-p))|^2 = \frac{1}{2}|f(t) - f(t-p) - \delta|^2,
\]

where the minimum is taken over all \((a(t-p), c(t)) \in \mathbb{R}\) with \((c(t) - a(t-p) = f(t) - f(t-p) - \delta\). Hence,

\[
\| \nabla_p s_f \|^2_{p-\text{SI}} = \frac{1}{3p-1} (2s_f(p)^2 + 2(1-p)s_f(p)\delta + \delta^2(1-p)p) \\
+ \frac{1}{2} \|f(t) - f(t-p) - \delta\|^2_{L^2[0,1]}.
\]

This minimum is obtained at the function

\[
f^*(t) = \left( \frac{1}{3p-1} (s_f(p) + \frac{1-p}{2}\delta) + \frac{1}{2}(-f(t+p) + f(t)) \right) 1_{[0,1-p]}(t) \\
+ \frac{1}{2p-1} (s_f(p) - \frac{1-p}{2}\delta) 1_{[1-p,p]}(t) \\
+ \frac{1}{3p-1} (s_f(p) + \frac{1-p}{2}\delta) + \frac{1}{2} (f(t+p) - f(t)) \right) 1_{[p,1]}(t), t \in [0,1] .
\]

**Proof of Lemma 3.2** We have to prove Eq. (?1) for the p-Slepian-process. To this end, let \(p \leq t \leq 1, f \in L^2[0,1]\) and \((\nabla_p s_f)(\cdot) = s_f(p) + s \nabla_p f(\cdot) \in \mathcal{H}_{p-\text{SI}}\). It
holds true for $t \in [p, 1]$:

$$
E \left( \frac{1}{\sqrt{p}} (\nabla_p B)_{[p, 1]}(t) \cdot \frac{3sf(p) + s\nabla_s f(1)}{2(3p - 1)} ((\nabla_p B)_{[p, 1]}(p) + (\nabla_p B)_{[p, 1]}(1)) \right) \\
+ E \left( \frac{1}{\sqrt{p}} (\nabla_p B)_{[p, 1]}(t) \cdot \frac{1}{2} \int_p^1 \nabla_p f(s) \, d(\nabla_p B)_{[p, 1]}(s) \right) \\
= \frac{3sf(p) + s\nabla_s f(1)}{2\sqrt{p}(3p - 1)} E \left( (B_{[0,1]}(t) - B_{[0,1]}(t - p))(B_{[0,1]}(p) + B_{[0,1]}(1) - B_{[0,1]}(1 - p)) \right) \\
+ \frac{1}{2\sqrt{p}} E \left( (B_{[0,1]}(t) - B_{[0,1]}(t - p)) \int_p^1 \nabla_p f(s) \, d(B_{[0,1]}(s) - B_{[0,1]}(s - p)) \right) \\
= \frac{3sf(p) + s\nabla_s f(1)}{2\sqrt{p}} - \frac{1}{2\sqrt{p}} (s\nabla_p f(1) - s\nabla_p f(p)) = \frac{1}{\sqrt{p}} sf(p) + s\nabla_p f(t).
$$

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