On the integral representation of \( g \)-expectations with terminal constraints

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Abstract. In this paper, we study the integral representation of \( g \)-expectations with two kinds of terminal constraints, and obtain the corresponding necessary and sufficient conditions.

Keywords: Backward stochastic differential equations, \( g \)-expectations, Conditional \( g \)-expectations.

MSC-classification: 60H10, 60H30

1 Introduction

Pardoux and Peng [15] showed that the following type of nonlinear backward stochastic differential equation (BSDE for short)

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s
\]

has a unique solution \((Y, Z)\) under some conditions on \( g \), where \( \xi \) is called terminal value and \( g \) is called the generator. Based on the solution of BSDEs, Peng [17] introduced the notion of \( g \)-expectations \( \mathbb{E}_g[\cdot] : L^2(\mathcal{F}_T) \to \mathbb{R} \), which is the first kind of dynamically consistent nonlinear expectations. Moreover, Coquet et al. [7] proved that any dynamically consistent nonlinear expectation on \( L^2(\mathcal{F}_T) \) under certain conditions is \( g \)-expectation.

One problem of \( g \)-expectation is to find the condition of \( g \) under which the following integral representation

\[
\mathbb{E}_g[\xi] = \int_{-\infty}^0 (\mathbb{E}_g[I_{\{\xi \geq \alpha \}}] - 1) d\alpha + \int_0^\infty \mathbb{E}_g[I_{\{\xi \geq \alpha \}}] d\alpha
\]

holds. Chen et al. [3] proved that the integral representation holds for each \( \xi \in L^2(\mathcal{F}_T) \) if and only if \( \mathbb{E}_g[\cdot] \) is a classical linear expectation under the assumptions: \( g \) is continuous in \( t \) and \( W \) is 1-dimensional Brownian motion. Without these assumptions on \( g \) and \( W \), Hu [12, 13] showed that the above

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result on integral representation (1) for each \( \xi \in L^2(\mathcal{F}_T) \) still holds. For the integral representation (1) with terminal constraints on \( \xi = \Phi(X_T) \), where \( \Phi \) is a monotonic function and \( X \) is a solution of stochastic differential equation (SDE for short), Chen et al. \cite{5,4} obtained a necessary and sufficient condition under the above assumptions on \( g \) and \( W \), and gave a sufficient condition for multi-dimensional Brownian motion.

In this paper, we want to study the integral representation (1) with the following two kinds of terminal constraints on \( \xi = \Phi(X_T) \): one is for the monotonic \( \Phi \), the other is for the measurable \( \Phi \). Specially, we make further research to the structure of \( Z \) in the BSDE and apply it to obtain the corresponding necessary and sufficient conditions without the above assumptions on \( g \) and \( W \), which is weaker than the sufficient condition in \cite{4} (see Remark 9 in Section 3 for detailed explanation). Furthermore, this method can be extended to solve more general terminal constraints on \( \xi \).

This paper is organized as follows: In Section 2, we recall some basic results of BSDEs and \( g \)-expectations. The main result is stated and proved in Section 3.

2 Preliminaries

Let \((W_t)_{t\geq 0} = (W^1_t, \ldots, W^d_t)_{t\geq 0}\) be a \( d \)-dimensional standard Brownian motion defined on a completed probability space \((\Omega, \mathcal{F}, P)\) and \((\mathcal{F}_t)_{0 \leq t \leq T}\) be the natural filtration generated by this Brownian motion, i.e.,

\[ \mathcal{F}_t := \sigma\{W_s : s \leq t\} \vee \mathcal{N}, \]

where \( \mathcal{N} \) is the set of all \( P \)-null subsets. Fix \( T > 0 \), we denote by \( L^2(\mathcal{F}_t; \mathbb{R}^m) \), \( t \in [0,T] \), the set of all \( \mathbb{R}^m \)-valued square integrable \( \mathcal{F}_t \)-measurable random vectors and \( L^2(0,T; \mathbb{R}^m) \) the space of all progressively measurable, \( \mathbb{R}^m \)-valued processes \((a_t)_{t \in [0,T]}\) with \( E\left[ \int_0^T |a_t|^2 \, dt \right] < \infty \).

We consider the following forward-backward stochastic differential equations:

\[
\begin{cases}
    dX^{t,x}_s = b(s, X^{t,x}_s) \, ds + \sigma(s, X^{t,x}_s) \, dW_s, \quad s \in [t,T], \\
    X^{t,x}_t = x \in \mathbb{R}^n,
\end{cases}
\]

(2)

\[
y^{t,x}_s = \Phi(X^{t,x}_T) + \int_s^T g(r, y^{t,x}_r, z^{t,x}_r) \, dr - \int_s^T z^{t,x}_r \, dW_r.
\]

(3)

In this paper, we use the following assumptions:

(S1) \( b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d} \) are measurable.

(S2) There exists a constant \( K_1 \geq 0 \) such that

\[
|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq K_1 |x - x'|, \quad \forall t \leq T, x, x' \in \mathbb{R}^n.
\]

(S3) \( \int_0^T (|b(t,0)|^2 + |\sigma(t,0)|^2) \, dt < \infty \).
(H1) \( g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is measurable.

(H2) There exists a constant \( K_2 \geq 0 \) such that
\[
|g(t, y, z) - g(t, y', z')| \leq K_2(|y - y'| + |z - z'|), \quad \forall t \leq T, y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d.
\]

(H3) \( g(t, y, 0) \equiv 0 \) for each \((t, y) \in [0, T] \times \mathbb{R}\).

(H3') \( \int_0^T |g(t, 0, 0)|^2 dt < \infty \).

(H4) \( \Phi : \mathbb{R}^n \to \mathbb{R} \) is measurable and satisfies \( \Phi(X_t^{t,x}) \in L^2(F_T) \).

Remark 1 Obviously, (H3) implies (H3').

It is well-known that the SDE (2) has a unique solution \((X_t^{t,x})_{s \in [t, T]} \in L^2(t, T; \mathbb{R}^n)\) under the assumptions (S1)-(S3). Under the assumptions (H1), (H2), (H3') and (H4), Pardoux and Peng [15] showed that the BSDE (3) has a unique solution \((y_s^{t,x}, z_s^{t,x})_{s \in [0, T]} \in L^2(0, T; \mathbb{R}^{1+d})\). Moreover, the following result holds.

Theorem 2 (Theorem 7) Suppose (S1)-(S3), (H1), (H2), (H3') and (H4) hold. If \( b, \sigma, g \) and \( \Phi \in C^{1,3} \), then

(i) \( u(t, x) := y_t^{t,x} \in C^{1,2}([0, T] \times \mathbb{R}^n) \) and solves the following PDE:
\[
\begin{align*}
\partial_t u(t, x) + Lu(t, x) + g(t, u(t, x), \sigma^T(t, x)\partial_x u(t, x)) &= 0, \\
u(T, x) &= \Phi(x),
\end{align*}
\]

where
\[
Lu(t, x) = \frac{1}{2} \sum_{i,j=1}^n (\sigma^T)_{ij}(t, x)\partial_{x_i x_j}^2 u(t, x) + \sum_{i=1}^n b_i(t, x)\partial_{x_i} u(t, x).
\]

(ii) \( z_s^{t,x} = \sigma^T(s, X_s^{t,x})\partial_x u(s, X_s^{t,x}), s \in [t, T] \).

Remark 3 For notation simplicity, when \( t = 0 \) and only one \( x \), we write \((X_t, y_t, z_t)_{t \in [0, T]}\) for the solution of SDE (2) and BSDE (3) in the following.

Using the solution of BSDE, Peng [17] proposed the following consistent nonlinear expectations.

Definition 4 Suppose \( g \) satisfies (H1)-(H3). Let \((y_t, z_t)_{t \in [0, T]}\) be the solution of BSDE (3) with terminal value \( \xi \in L^2(F_T) \), i.e.,
\[
y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dW_s.
\]

Define
\[
\mathcal{E}_g[\xi|\mathcal{F}_t] := y_t \quad \text{for each } t \in [0, T].
\]

\( \mathcal{E}_g[\xi|\mathcal{F}_t] \) is called the conditional g-expectation of \( \xi \) with respect to \( \mathcal{F}_t \). In particular, if \( t = 0 \), we write \( \mathcal{E}_g[\xi] \) which is called the g-expectation of \( \xi \).
Remark 5 The assumption \((H3)\) is important in the definition of \(g\)-expectation. In particular, under the assumptions \((H1)-(H3)\), if \(\xi \in L^2(\mathcal{F}_{t_0})\) with \(t_0 < T\), then \(\mathcal{E}_g[\xi | \mathcal{F}_t] = \xi\) for \(t \in [t_0, T]\).

The following standard estimates of BSDEs can be found in [10, 7, 1].

Proposition 6 Suppose \(g_1\) and \(g_2\) satisfy \((H1), (H2)\) and \((H3')\). Let \((y^i_t, z^i_t)_{t \in [0,T]}\) be the solution of BSDE \((3)\) with the generator \(g_i\) and terminal value \(\xi_i \in L^2(\mathcal{F}_T), i = 1, 2\). Then there exists a constant \(C > 0\) depending on \(K_2\) and \(T\) such that

\[
\mathbb{E}[\sup_{0 \leq t \leq T} |y^1_t - y^2_t|^2 + \int_0^T |z^1_t - z^2_t|^2 dt] \leq C \mathbb{E}[|\xi^1 - \xi^2|^2 + \int_0^T |\tilde{g}_t|^2 dt],
\]

where \(\tilde{g}_t = g_1(t, y^1_t, z^1_t) - g_2(t, y^1_t, z^1_t)\).

Assume \(g\) satisfies \((H1)-(H3)\), set

\[V_g(A) := \mathcal{E}_g[I_A]\]

for each \(A \in \mathcal{F}_T\).

It is easy to verify that \(V_g(\cdot)\) is a capacity, i.e., (i) \(V_g(\emptyset) = 0, V_g(\Omega) = 1\); (ii) \(V_g(A) \leq V_g(B)\) for each \(A \subset B\). The corresponding Choquet integral (see [6]) is defined as follows:

\[C_g[\xi] := \int_{-\infty}^0 V_g(\xi \geq t) - 1]dt + \int_0^\infty V_g(\xi \geq t)dt \quad \text{for each} \ \xi \in L^2(\mathcal{F}_T).\]

It is easy to check that \(C_g[I_A] = \mathcal{E}_g[I_A]\) for each \(A \in \mathcal{F}_T\). Moreover, \(|C_g[\xi]| < \infty\) for each \(\xi \in L^2(\mathcal{F}_T)\) (see [11]).

Definition 7 Two random variables \(\xi\) and \(\eta\) are called comonotonic if

\[\{\xi(\omega) - \xi(\omega')\} \{\eta(\omega) - \eta(\omega')\} \geq 0 \quad \text{for each} \ \omega, \omega' \in \Omega.\]

The following properties of Choquet integral can be found in [6] [8] [9].

(1) Monotonicity: If \(\xi \geq \eta\), then \(C_g[\xi] \geq C_g[\eta]\).
(2) Positive homogeneity: If \(\lambda \geq 0\), then \(C_g[\lambda \xi] = \lambda C_g[\xi]\).
(3) Translation invariance: If \(c \in \mathbb{R}\), then \(C_g[\xi + c] = C_g[\xi] + c\).
(4) Comonotonic additivity: If \(\xi\) and \(\eta\) are comonotonic, then \(C_g[\xi + \eta] = C_g[\xi] + C_g[\eta]\).
3 Main result

Suppose \( n = 1 \), we define

\[
\begin{align*}
    \mathcal{H} & := \{ \xi : \exists b, \sigma \text{ satisfying (S1)-(S3) and } x \text{ such that } \xi = X_t^x \}, \\
    \mathcal{H}_1 & := \{ \Phi(\xi) \in L^2(\mathcal{F}_T) : \Phi \text{ is monotonic and } \xi \in \mathcal{H} \}, \\
    \mathcal{H}_2 & := \{ \Phi(\xi) \in L^2(\mathcal{F}_T) : \Phi \text{ is measurable and } \xi \in \mathcal{H} \}.
\end{align*}
\]

The elements in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) can be seen as the contingent claims of European option. Now we give our main theorem.

**Theorem 8** Suppose \( \mathcal{H}_1 \) satisfies (H1)-(H3). Then

(i) \( \mathcal{E}_g(\cdot) = C_g(\cdot) \) on \( \mathcal{H}_1 \) if and only if \( g \) is independent of \( y \), and \( g(t, \lambda z) = \lambda g(t, z) \) for all \( \lambda \geq 0 \);

(ii) \( \mathcal{E}_g(\cdot) = C_g(\cdot) \) on \( \mathcal{H}_2 \) if and only if \( g \) is independent of \( y \) and \( g(t, \lambda z) = \lambda g(t, z) \) for all \( \lambda \in \mathbb{R} \).

**Remark 9** In [4], Chen et al. showed that \( \mathcal{E}_g(\cdot) = C_g(\cdot) \) on \( \mathcal{H}_1 \) under the assumption that \( g \) is positively additive, i.e., \( g(t, z_1+z'_1, \ldots, z_d+z'_d) = g(t, z_1, \ldots, z_d) + g(t, z'_1, \ldots, z'_d) \) for \( z_i, z'_i \geq 0 \), \( i = 1, \ldots, d \). Obviously, this condition on \( g \) is stronger than positive homogeneity. For example, \( g(z) = |z| \) is not positively additive, but is positively homogeneous.

In order to prove this theorem, we need the following lemmas.

**Lemma 10** Suppose \( g \) satisfies (H1)-(H3). Then for each given \( p \in (1, 2) \), there exists a constant \( L > 0 \) depending on \( p \), \( K_2 \) and \( T \) such that for each \( \xi, \eta \in L^2(\mathcal{F}_T) \),

\[
|C_g[\xi] - C_g[\eta]| \leq L(1 + (E[|\xi|^2] + |\eta|^2))^{\frac{1}{2}} (E[|\xi - \eta|^2])^{\frac{1}{2}}.
\]

In particular, for each \( \xi \in L^2(\mathcal{F}_T) \), we have \( C_g([\xi \wedge N] \vee (-N)) \rightarrow C_g[\xi] \) as \( N \rightarrow \infty \).

**Proof.** For each given \( p \in (1, 2) \), by Proposition 3.2 in Briand et al. [2], there exists a constant \( L_1 > 0 \) depending on \( p \), \( K_2 \) and \( T \) such that for each \( \xi, \eta \in L^2(\mathcal{F}_T) \),

\[
|\mathcal{E}_g[\xi] - \mathcal{E}_g[\eta]| \leq L_1 (E[|\xi - \eta|^p])^{\frac{1}{p}}.
\]

Set \( \bar{g}(t, y, z) = -g(t, 1 - y, -z) \), it is easy to check that \( 1 - V_{\bar{g}}(A) = V_g(A^c) \). Thus \( C_g[\xi] = C_g[\xi^+] - C_g[\xi^-] \). From this we only need to prove the result for \( \xi \geq 0 \) and \( \eta \geq 0 \). We have

\[
|C_g[\xi] - C_g[\eta]| \leq \int_0^\infty |\mathcal{E}_g[I_{\{t \geq t_1\}}] - \mathcal{E}_g[I_{\{t \geq t_2\}}]| \, dt
\]

\[
\leq L_1 \int_0^\infty (E[|I_{\{t \geq t_1\}}|] - E[I_{\{t \geq t_2\}}])^\frac{1}{p} \, dt
\]

\[
= L_1 \int_0^\infty (E[|I_{\{t \geq t_1\}}| - I_{\{t \geq t_2\}}])^\frac{1}{p} \, dt,
\]

5
Proof.

\[
\int_0^1 (E[|I_{\{\xi \geq t\}} - I_{\{\eta \geq t\}}|])^{\frac{1}{p}} dt \leq (E[\int_0^1 |I_{\{\xi \geq t\}} - I_{\{\eta \geq t\}}| dt])^{\frac{1}{p}} = (E[\int_0^1 I_{\{\xi \wedge \eta \leq t \leq \xi \vee \eta\}} dt])^{\frac{1}{p}} \leq (E[|\xi - \eta|])^{\frac{1}{p}} \leq (E[|\xi - \eta|^2])^{\frac{1}{2p}},
\]

\[
\int_1^\infty (E[|I_{\{\xi \geq t\}} - I_{\{\eta \geq t\}}|])^{\frac{1}{p}} dt \leq \left( \int_1^\infty t^{\frac{1}{p}} dt \right)^{\frac{1}{p}} (E[\int_1^\infty t |I_{\{\xi \geq t\}} - I_{\{\eta \geq t\}}| dt])^{\frac{1}{p}} = \frac{p-1}{2-p} \left( \int_1^\infty t^{\frac{1}{p}} dt \right)^{\frac{1}{p}} (E[|\xi - \eta|^2])^{\frac{1}{p}} \leq \frac{p-1}{2-p} \left( \frac{1}{2} \right)^{\frac{1}{p}} (E[|\xi|^2 + |\eta|^2])^{\frac{1}{p}} (E[|\xi - \eta|^2])^{\frac{1}{2p}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \). Thus we obtain the result. \( \square \)

**Lemma 11** Let \( b, \sigma \) satisfy (S1)-(S3), \( g \) satisfy (H1)-(H3) and \( \Phi \in C^3_0 \). Then there exist \( b_k, \sigma_k, g_k \in C^{1,3}_b, k \geq 1 \), such that

\[
E[ \sup_{t \in [0, T]} |X^k_t - X_t|^2 + \int_0^T (|\sigma_k(t, X^k_t) - \sigma(t, X_t)|^2 + |z^k_t - z_t|^2) dt] \to 0,
\]

where \((X_t, y_t, z_t)_{t \in [0, T]}\) is the solution corresponding to \( b, \sigma, g \) and \((X^k_t, y^k_t, z^k_t)_{t \in [0, T]}\) is the solution corresponding to \( b_k, \sigma_k, g_k \).

**Proof.** By the standard estimates of SDEs and Proposition [4] we only need to prove the result for bounded \( b, \sigma \) and \( g \). For any function \( h(u), u \in \mathbb{R}^m \), we will denote, for each \( \varepsilon > 0 \),

\[
h_{\varepsilon}(u) = \int_{\mathbb{R}^m} h(u-v) \varepsilon^{-m} \varphi(\frac{v}{\varepsilon}) dv,
\]

where \( \varphi \) is the mollifier in \( \mathbb{R}^m \) defined by \( \varphi(u) = \exp(-\frac{1}{1-|u|^2})I_{\{|u|<1\}} \). By this definition, it is easy to check that \( b_\varepsilon, \sigma_\varepsilon \) and \( g_\varepsilon \) satisfy (S2) and (H2) with the same Lipschitz constant. Also, we have \( b_\varepsilon, \sigma_\varepsilon, g_\varepsilon \in C^{1,3}_b \) and \((b_\varepsilon, \sigma_\varepsilon, g_\varepsilon) \to (b, \sigma, g) \) a.e. in \( t \) for each fixed \((x, y, z) \in \mathbb{R}^{2+d}\). Thus by the diagonal method, we can choose a sequence \( b_k, \sigma_k, g_k \in C^{1,3}_b \) such that \((b_k, \sigma_k, g_k) \to (b, \sigma, g) \) for every \((x, y, z) \in \mathbb{Q}^{2+d}\) a.e. in \( t \). By the Lipschitz condition, we get \((b_k, \sigma_k, g_k) \to (b, \sigma, g) \) for every \((x, y, z) \in \mathbb{R}^{2+d}\) a.e. in \( t \). By the estimates of SDEs, we obtain

\[
E[ \sup_{t \in [0, T]} |X^k_t - X_t|^2] \leq L_2 E[\int_0^T (|b_k(t, X_t) - b(t, X_t)|^2 + |\sigma_k(t, X_t) - \sigma(t, X_t)|^2) dt],
\]
where the constant \( L_2 \) depending on \( K_1 \) and \( T \). By the bounded dominated convergence theorem, we can get \( E[\sup_{t \in [0, T]} |X_t^k - X_t^i|^2] \to 0 \). From this, it is easy to deduce that \( E[\int_0^T |\sigma_k(t, X_t^k) - \sigma(t, X_t)|^2 dt] \to 0 \). By Proposition 8 we can easily obtain \( E[\int_0^T |z_t^i - z_t|^2 dt] \to 0 \).

We now prove the main theorem.

**Proof of Theorem 8.** We first prove that the condition on \( g \) is necessary, and then it is sufficient.

(i) Necessity. We first prove the result for the case \( d = 1 \). For this we choose \( b(s, x) = 0, \sigma(s, x) = zI_{[t, t+\varepsilon]}(s) \) and \( \Phi(x) = x \), where \( z \in \mathbb{R}, t < T \) and \( \varepsilon > 0 \) are given. Then

\[
\mathcal{H}_1 \ni \{ y + z(W_{t+\varepsilon} - W_t) : \forall y, z \in \mathbb{R}, t < T, \varepsilon > 0 \}.
\]

Since \( \mathcal{E}_g[\cdot] = \mathcal{C}_g[\cdot] \) on \( \mathcal{H}_1 \) and \( g \) is deterministic, by the properties of \( \mathcal{C}_g[\cdot] \) we can get

\[
\mathcal{E}_g[y + z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] = \mathcal{E}_g[y + z(W_{t+\varepsilon} - W_t)] = \mathcal{E}_g[z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] + y,
\]

\[
\mathcal{E}_g[\lambda z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] = \lambda \mathcal{E}_g[z(W_{t+\varepsilon} - W_t)|\mathcal{F}_t] \text{ for } \lambda \geq 0.
\]

By Lemma 2.1 in Jiang [14], we can obtain that \( g \) is independent of \( y \) and \( g(t, \lambda z) = \lambda g(t, z) \) for all \( \lambda \geq 0 \). For the case \( d > 1 \). For each given \( a \in \mathbb{R}^d \) with \( |a| = 1 \), we define \( W^a \) by \( W^a_t = a \cdot W_t \) and \( g^a : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by \( g^a(t, y, z) = g(t, y, az) \). It is easy to check that \( \mathcal{E}_g[\xi] = \mathcal{E}_{g^a}[\xi] \) and \( \mathcal{C}_g[\xi] = \mathcal{C}_{g^a}[\xi] \) for \( \xi \in L^2(\mathcal{F}^T_t) \), where \( \mathcal{F}^a_t := \sigma\{W^a_t : t \leq T\} \vee \mathcal{N} \). Thus by applying the method of \( d = 1 \), we can obtain \( g^a \) is independent of \( y \) and is positively homogeneous in \( z \) for each given \( a \in \mathbb{R}^d \) with \( |a| = 1 \), which implies the necessary condition on \( g \).

Sufficiency. By Proposition 6 and Lemma 10, we only need to prove the result for bounded and monotonic \( \Phi \). The proof is divided into two steps.

Step 1. Let \((X_t)_{t \in [0, T]} \) be the solution of SDE (2) corresponding to \( b \) and \( \sigma \) satisfying (S1)-(S3) and let \( \phi_i \in C^1_b(\mathbb{R}), i = 1, \ldots, N, \) be non decreasing functions. We assert that

\[
\mathcal{E}_g[\sum_{i=1}^N \phi_i(X_T)] = \sum_{i=1}^N \mathcal{E}_g[\phi_i(X_T)].
\]

(4)

Let \((y_t^i, z_t^i)_{t \in [0, T]}, i = 1, \ldots, N, \) be the solution of the following BSDEs:

\[
y_t^i = \phi_i(X_T) + \int_t^T g(s, z_s^i) ds - \int_t^T z_s^i dW_s.
\]

(5)

By Lemma 11 we can choose \( b_k, \sigma_k, g_k \in C^{1,3}_b, k \geq 1 \), such that

\[
E[\int_0^T (|\sigma_k(t, X_t^k) - \sigma(t, X_t)|^2 + |z_t^{i,k} - z_t^i|^2 | dt] \to 0, i = 1, \ldots, N,
\]


\[
\mathcal{E}_g[\sum_{i=1}^N \phi_i(X_T)] = \sum_{i=1}^N \mathcal{E}_g[\phi_i(X_T)].
\]
where \((X^k_t, y^i,t, z^i,t)_{t \in [0,T]}\) is the solution corresponding \(b_k, \sigma_k, g_k\) and terminal value \(\phi_t(X^k_T)\). From this we can get
\[
z^i_t \rightarrow z^i_t, \quad \sigma_k(t, X^k_t) \rightarrow \sigma(t, X_t) \quad dP \times dt - \text{a.s.} \tag{6}
\]
On the other hand, it follows from Theorem 2 that
\[
z^i_t = \sigma^T_k(t, X^k_t) \partial_x u^i,k(t, X^k_t), \tag{7}
\]
where \(u^i,k(t, x) := y^i,t, \quad t, x \in [0,T].\) By comparison theorem of SDE and BSDE, it is easy to verify that \(u^i,k(t, x)\) is non decreasing in \(x\), which implies \(\partial_x u^i,k(t, X^k_t) \geq 0.\)

Thus by combining equation (6) and (7), we obtain that there exist progressive processes
\[
D^i_t \geq 0, \quad i = 1, \ldots, N, \quad i \leq \sum_{i=1}^N D^i_t.
\tag{8}
\]

Set
\[
Y_t = \sum_{i=1}^N y^i_t, \quad Z_t = \sum_{i=1}^N z^i_t,
\]
then by combining equation (5) and (8), we can get
\[
Y_t = \sum_{i=1}^N \phi_i(X_T) + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s.
\]

By the definition of \(g\)-expectation, we obtain equation (4).

Step 2. Let \((X_t)_{t \in [0,T]}\) be as in Step 1 and let \(\Phi\) be a bounded and monotonic function. Note that for each \(\xi \in L^2(F_T)\) and \(c \in \mathbb{R},\)
\[
\mathcal{E}_g[\xi + c] = \mathcal{E}_g[\xi] + c, \quad C_g[\xi + c] = C_g[\xi] + c,
\]
then we only need to prove the result for \(\Phi \geq 0.\) Since the analysis of non increasing \(\Phi\) is the same as in non decreasing \(\Phi,\) we only prove the case for non decreasing \(\Phi\) with \(0 \leq \Phi < M,\) where \(M > 0\) is a constant. For each given \(N > 0,\) we set
\[
\Phi_N(x) = \sum_{i=1}^N \frac{(i-1)M}{N} \mathbb{I}_{\left(\frac{(i-1)M}{N} \leq \Phi < \frac{iM}{N}\right)} = \sum_{i=1}^N \frac{M}{N} \mathbb{I}_{\left(\Phi \geq \frac{iM}{N}\right)}.
\]
It is easy to check that \( E[|\Phi_N(X_T) - \Phi(X_T)|^2] \leq \left( \frac{M}{N} \right)^2 \to 0 \) as \( N \to \infty \). Thus by Proposition 6 and Lemma 10 we get
\[
E_g[\Phi_N(X_T)] \to E_g[\Phi(X_T)], \quad C_g[\Phi_N(X_T)] \to C_g[\Phi(X_T)] \quad \text{as} \quad N \to \infty. \quad (9)
\]
For each fixed \( N > 0 \), noting that \( \Phi \) is non decreasing, then \( \{ \Phi \geq \frac{iM}{N} \} \) is \([a_i, \infty)\) or \((a_i, \infty)\), where \( a_i \) is a constant. For each \( \varepsilon > 0 \), we define
\[
\psi_{i,\varepsilon}^0(x) = \int_{[a_i-\varepsilon, \infty)} (x-v) \frac{1}{\varepsilon} \varphi(v) dv, \quad \psi_{i,\varepsilon}^2(x) = \int_{[a_i+\varepsilon, \infty)} (x-v) \frac{1}{\varepsilon} \varphi(v) dv,
\]
where \( \varphi(v) = \exp(-\frac{1}{1-|v|^2})I_{|v|<1} \). It is easy to check that \( \psi_{i,\varepsilon}^0, \psi_{i,\varepsilon}^2 \in C_b^0(\mathbb{R}) \) are non decreasing and satisfy \( \psi_{i,\varepsilon}^0 \downarrow I_{[a_i, \infty)}, \psi_{i,\varepsilon}^2 \uparrow I_{(a_i, \infty)} \) as \( \varepsilon \downarrow 0 \). Thus we can choose non decreasing \( \phi_i^k \in C_b^0(\mathbb{R}), \ k \geq 1, \) such that \( E[|\phi_i^k(X_T) - I_{\{\Phi \geq \frac{iM}{N}\}}(X_T)|^2] \to 0 \) as \( k \to \infty \), which implies
\[
E[|\Phi_N(X_T) - \frac{M}{N} \sum_{i=1}^N \phi_i^k(X_T)|^2] \to 0 \quad \text{as} \quad k \to \infty.
\]
By Step 1, Proposition 6 and properties of Choquet integral, we can obtain
\[
E_g[\Phi_N(X_T)] = \lim_{k \to \infty} E_g[\frac{M}{N} \sum_{i=1}^N \phi_i^k(X_T)] = \lim_{k \to \infty} \frac{M}{N} E_g[\sum_{i=1}^N \phi_i^k(X_T)]
\]
\[
= \frac{M}{N} \sum_{i=1}^N \lim_{k \to \infty} E_g[\phi_i^k(X_T)] = \frac{M}{N} \sum_{i=1}^N E_g[I_{\{\Phi \geq \frac{iM}{N}\}}(X_T)]
\]
\[
= \frac{M}{N} \sum_{i=1}^N C_g[I_{\{\Phi \geq \frac{iM}{N}\}}(X_T)] = C_g[\Phi_N(X_T)].
\]
Thus by (9), we get \( E_g[\Phi(X_T)] = C_g[\Phi(X_T)] \). The proof of (i) is complete.

(ii) Necessity. For the case \( d = 1 \), since \( \mathcal{H}_2 \supset \mathcal{H}_1 \), we can get that \( g \) is independent of \( y \) and is positively homogeneous in \( z \) by (i). On the other hand,
\[
\{l_1I_{(W_T - W_i \geq a)} + l_2I_{(b \geq W_T - W_i \geq a)} : t < T, a < b, a, b, l_1, l_2 \in \mathbb{R} \} \subset \mathcal{H}_2,
\]
by the proof of Lemma 9 in [12], we can obtain \( g(t, z) = g(t, 1)z \). For the case \( d > 1 \), the proof is the same as (i).

Sufficiency. By the similar analysis as in (i), for each \( \phi_i \in C_b^0(\mathbb{R}), \ i = 1, \ldots, N \), we can get
\[
E_g[\sum_{i=1}^N \phi_i(X_T)] = \sum_{i=1}^N E_g[\phi_i(X_T)].
\]
The same analysis as in (i), we only need to prove the result for
\[
\Phi(x) = \sum_{i=1}^N b_i I_{A_i}(x),
\]
where $b_i \geq 0$, $A_i \in \mathcal{B}(\mathbb{R})$ and $A_i \supset A_{i+1}$. Set
\[ P_{X_T}(A) := P(X_T^{-1}(A)) \quad \text{for } A \in \mathcal{B}(\mathbb{R}), \]
then by Lusin’s theorem, we can choose $\phi^k_i \in C^2_b(\mathbb{R})$, $k \geq 1$, such that
\[ E[|\phi^k_i(X_T) - I_{A_i}(X_T)|^2] = E_{P_{X_T}}[|\phi^k_i(x) - I_{A_i}(x)|^2] \to 0 \text{ as } k \to \infty. \]
Thus we obtain $\mathcal{E}_g[\Phi(X_T)] = C_g[\Phi(X_T)]$ as in (i). The proof is complete. \(\square\)

In the following, we consider the case $n > 1$. We give the following assumptions on $\sigma$ in SDE (2).

(S4) There exists a $k \leq d$ such that $\sigma_i(t, x) = (\tilde{\sigma}(t, x), 0, \ldots, 0)$ for $i = 1, \ldots, n$, where $\sigma_i$ is the $i$-th row of $\sigma$ and $\tilde{\sigma} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{1 \times k}$.

(S5) There exists a $k \leq d$ such that $\sigma_i(t, x) = (\tilde{\sigma}(t, x), \tilde{\sigma}_i(t, x))$ for $i = 1, \ldots, n$, where $\sigma_i$ is the $i$-th row of $\sigma$, $\tilde{\sigma} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{1 \times k}$ and $\tilde{\sigma}_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{1 \times (d-k)}$.

Set
\[ \mathcal{H}_3 := \{ \xi : \exists b, \sigma \text{ satisfying (S1)-(S3), (S4) and } x \in \mathbb{R}^n \text{ such that } \xi = X_T^{0,x} \}. \]
\[ \mathcal{H}_4 := \{ \xi : \exists b, \sigma \text{ satisfying (S1)-(S3), (S5) and } x \in \mathbb{R}^n \text{ such that } \xi = X_T^{0,x} \}. \]
\[ \mathcal{H}_5 := \{ \Phi(\xi) \in L^2(F_T) : \Phi \text{ is measurable on } \mathbb{R}^n \text{ and } \xi \in \mathcal{H}_3 \}. \]
\[ \mathcal{H}_6 := \{ \Phi(\xi) \in L^2(F_T) : \Phi \text{ is measurable on } \mathbb{R}^n \text{ and } \xi \in \mathcal{H}_4 \}. \]

By the same analysis as in the proof of Theorem 8 and the method in the proof of main result in [12, 13], we can obtain the following corollary.

**Corollary 12** Suppose $g$ satisfies (H1)-(H3). Then

(i) $\mathcal{E}_g[\xi] = C_g[\xi]$ on $\mathcal{H}_5$ if and only if $\tilde{g}$ is independent of $y$ and is homogeneous in $\tilde{z}$, where $\tilde{g}(t, y, \tilde{z}) := g(t, y, (\tilde{z}, 0, \ldots, 0))$ for $(t, y, \tilde{z}) \in [0, T] \times \mathbb{R}^{1+k}$;

(ii) $\mathcal{E}_g[\xi] = C_g[\xi]$ on $\mathcal{H}_6$ if and only if $g$ is independent of $y$, $g(t, (\tilde{z}, \tilde{z}')) = g_1(t, \tilde{z}) + g_2(t, \tilde{z}')$ for $\tilde{z} \in \mathbb{R}^k$, $\tilde{z}' \in \mathbb{R}^{d-k}$, $g_1$ is homogeneous in $\tilde{z}$ and $g_2$ is linear in $\tilde{z}'$.

**References**

[1] Briand, P., Coquet, F., Hu, Y., Mémin, J., Peng, S., 2000. A converse comparison theorem for BSDEs and related properties of g-expectation. Electron. Comm. Probab. 5, 101-117.

[2] Briand, P., Delyon, B., Hu, Y., Pardoux, E., Stoica, L., 2003. $L^p$-solutions of backward stochastic differential equations. Stochastic Processes and their Applications 108, 109-129.
[3] Chen, Z.J., Chen, T., Davison, M., 2005. Choquet expectation and Peng’s g—expectation. The Annals of Probability 33(3), 1179-1199.

[4] Chen, Z.J., Kulperger, R., Wei, G., 2005. A comonotonic theorem for BSDEs. Stochastic Processes and their Applications 115, 41-54.

[5] Chen, Z.J., Sulem, A., 2001. An integral representation theorem of g-expectations. Research Report INRIA, No.4284.

[6] Choquet, G., 1953. Theory of capacities. Ann. Inst. Fourier (Grenoble) 5, 131-195.

[7] Coquet, F., Hu, Y., Mémin, J., Peng, S., 2002. Filtration consistent nonlinear expectations and related g-expectations. Probab. Theory and Related Fields 123, 1-27.

[8] Dellacherie, C., 1991. Quelques commentaires sur les prolongements de capacités. In: Strasbourg, V.(Ed.). Seminaire de probabilites. Springer, Berlin, 77-81.

[9] Denneberg, D., 1994. Non-additive Measure and Integral. Kluwer Academic Publishers, Boston.

[10] El Karoui, N., Peng, S., Quenez, M.C., 1997. Backward stochastic differential equations in finance. Math. Finance 7, 1-71.

[11] He, K., Hu, M., Chen, Z.J., 2009. The relationship between risk measures and Choquet expectations in the framework of g-expectations. Statistics and Probability Letters 79, 508-512.

[12] Hu, M., 2009. Choquet expectations and g-expectations with multi-dimensional Brownian motion. arXiv:0910.2519v1.

[13] Hu, M., 2010. On the integral representation of g-expectations. C. R. Acad. Sci. Paris, Ser. I 348, 571-574.

[14] Jiang, L., 2008. Convexity, translation invariance and subadditivity for g-expectations and related risk measures. Annals of Applied Probability 18(1), 245-258.

[15] Pardoux, E., Peng, S., 1990. Adapted solution of a backward stochastic differential equation. Systems and Control Letters 14, 55-61.

[16] Pardoux, E., Peng, S., 1992. Backward stochastic differential equations and quasilinear parabolic partial differential equations. Lecture Notes in CIS, vol. 176, Springer-Verlag, 200-217.

[17] Peng, S., 1997. Backward SDE and related g-expectations. Backward stochastic differential equations, in El N. Karoui and L. Mazliak, eds. Pitman Res. Notes Math. Ser. Longman Harlow, vol. 364, 141-159.