Repulsive Casimir forces at quantum criticality

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Abstract – We study the Casimir effect in the vicinity of a quantum-critical point. As a prototypical system we analyze the $d$-dimensional imperfect (mean-field) Bose gas enclosed in a slab of extension $L^{d-1} \times D$ and subject to periodic boundary conditions. The thermodynamic state is adjusted so that $L \gg \lambda \gg D \gg l_{mic}$, where $\lambda \sim T^{-1/2}$ is the thermal de Broglie length, and $l_{mic}$ denotes microscopic length scales. Our exact analysis indicates that the Casimir force in the above-specified regime is generically repulsive and decays either algebraically or exponentially, with a non-universal amplitude.

Motivation. – Casimir-type interactions [1–8] are nowadays recognized in a multitude of systems spanning from biological membranes to cosmology. The QED and condensed-matter contexts are those where the predictions of the theory of Casimir forces found experimental confirmation [9–13]. Herein we specify to the latter context, where the Casimir force is attractive. A well-known exception is the case of two bodies characterized by electric permittivities $\epsilon_1, \epsilon_2$ embedded in a medium of permittivity $\epsilon_m$, such that $\epsilon_1 < \epsilon_m < \epsilon_2$ [14], experimentally verified in ref. [15]. On the theory side, exact results place severe restrictions on the possibility of obtaining Casimir repulsion in QED models [16–18]. Detours around these restrictions invoke out-of-equilibrium systems [19–21]. The situation is more complex in the case of condensed-matter systems. Theoretically one may change the character of the force by varying the boundary condition. Notably, the boundary condition may also be tuned experimentally by engineering the surface properties (see, e.g., [13,22–24]). We note that the (theoretically appealing) periodic boundary conditions applied in the present work were not realized experimentally.

In the present study, the fluctuating medium enclosed in a slab is an imperfect Bose gas at finite, but asymptotically low temperature, in a thermodynamic state corresponding to the vicinity of a bulk quantum-critical point. The microscopic interparticle interactions are repulsive. We show that in a specific limit the effective Casimir force acting between the walls enclosing the Bose gas is repulsive and decays as a power of the separation $D$ even for the periodic boundary conditions which typically yield Casimir attraction. This gives a hint on the possible regime of parameters, where a repulsive Casimir force might potentially be detectable experimentally in a system involving Bose-Einstein condensation, and, hypothetically a wider class of quantum-critical systems.

The essential ingredient of the critical Casimir effect is the interplay between two large length scales: $D$ and the bulk correlation length $\xi$. As long as $\xi \ll D$, the effective interaction between the walls decays exponentially $\omega_s \sim e^{-D/\xi}$ with $\xi$ setting the decay scale. If, however, the system is tuned sufficiently close to a (bulk) critical point, or is in a phase exhibiting soft excitations, one has $\xi \gg D$, and $\omega_s \sim D^{-d+1}$. The crossover between the above two regimes ($D/\xi \ll 1$ and $D/\xi \gg 1$) is governed by a scaling function, showing universal properties.

The situation becomes more complex for $T \to 0$, where, in addition to $L, D$ and $\xi$, the thermal de Broglie length $\lambda = \frac{\hbar}{\sqrt{2\pi m k_B T}}$ becomes macroscopic. We assume here that a phase transition may be tuned by a non-thermal control parameter (such as density, pressure, or chemical conditions...}
composition). Considering the Casimir forces in the low-$T$ limit one identifies three regimes differentiated by the hierarchy of the macroscopic scales $L$, $D$, and $\lambda$. The standard thermal regime is recovered for $L \gg D \gg \lambda$. For the case $\lambda \gg L \gg D$, where one performs the $T \to 0$ limit before sending the system size to infinity, by virtue of the quantum-classical mapping [25], one expects the system properties to be similar to those of the thermal regime, albeit in elevated dimensionality. Finally, there is the possibility of the thermal length being squashed between the scales characterizing the system size, namely
\[
L \gg \lambda \gg D \gg l_{\text{mic}}. \tag{1}
\]
Here $l_{\text{mic}}$ denotes any microscopic length present in the system. To our knowledge, the limit defined by eq. (1) was not addressed so far, and it is not very simple to give a prediction for the asymptotics of the Casimir force relying solely on general arguments. Note that eq. (1) implies that the thermodynamic limit cannot be taken the usual way, keeping the temperature fixed. Instead, while increasing $D$ the temperature has to be reduced so that the condition $\lambda \gg D$ remains fulfilled. On the other hand, from a realistic (and experimental) point of view, the hierarchy of eq. (1) corresponds to a perfectly well-defined regime.

In what follows, we analyze the Casimir forces in the limit of eq. (1), employing a specific microscopic model of interacting bosons, the so-called imperfect Bose gas (IBG), which exhibits a phase transition to a Bose-Einstein condensed phase for $d > 2$ at any $T \geq 0$. The transition can be tuned by varying the chemical potential, which acts as the non-thermal control parameter. The model is susceptible to an exact analytical treatment within the grand-canonical formalism.

Model. – We consider a system of spinless, interacting bosons at a fixed temperature $T$ and the chemical potential $\mu$. The system is enclosed in a slab of volume $V = L^{d-1} \times D$ and is governed by the Hamiltonian
\[
\hat{H} = \sum_{k} \frac{\hbar^2 k^2}{2m} \hat{n}_k + \frac{a}{2V} \hat{N}^2, \tag{2}
\]
where we use the standard notation. The symbol $\hat{N} = \sum_{k} \hat{n}_k$ denotes the total particle number operator. The repulsive interaction term $H_{\text{int}} = \frac{a}{2V} \hat{N}^2$ ($a > 0$) may be recovered from the long-range repulsive part $v(r)$ of a 2-particle interaction potential in the Kac limit $\lim_{r \to 0} \frac{\alpha v(\kappa r)}{r^2} \equiv \alpha_{\kappa} v(\kappa a)$, i.e. for vanishing interaction strength and diverging range. The constant $a > 0$ is closely related to the interaction energy [26]. After imposing periodic boundary conditions, the grand-canonical partition function is cast in the convenient form [26]:
\[
\Xi(T, L, D, \mu) = -ie^{\frac{\beta a}{2V}} V \left( \frac{V}{2\pi a \beta} \right)^{1/2} \int_{\alpha \beta - i \infty}^{a \beta + i \infty} ds e^{-s \phi(s)}, \tag{3}
\]
where
\[
\phi(s) = -\frac{s^2}{2a\beta} + \frac{\mu}{a} + \frac{1}{V} \sum_{n_d} \ln(1 - e^{s - \beta \epsilon_{k_d}}) - \frac{1}{2D^{d-1}} \sum_{n_d} (e^{s - \beta \epsilon_{k_d}} - 1).
\]
Here $k_d = \frac{2\pi n_d}{L}$, $n_d \in \mathbb{Z}$, $\beta \epsilon_{k_d} = \frac{\lambda^2}{2} n_d^2$, $g_0(z) = \sum_{k=1}^{\infty} \frac{1}{z^k}$ are the Bose functions, and the contour parameter $\alpha$ is negative. The representation of the grand-canonical partition function given by eq. (3) is derived by a Hubbard-Stratonovich–type transformation (for details see ref. [26]). The simplicity of eq. (3) stems from the occurrence of the factor $V$ in the exponential in eq. (3), which appears due to the long-ranged type of interactions, assures that the saddle-point approximation becomes exact for $V \to \infty$:
\[
\lim_{V \to \infty} \frac{1}{V} \log \Xi(T, L, D, \mu) = \frac{\beta \mu^2}{2a} - \phi(\bar{s}). \tag{5}
\]
Here $\bar{s}$ is the value of $s$ corresponding to the minimum of $\phi(s)$. It follows that for $L \to \infty$ the problem of evaluating the partition function becomes reduced to solving the stationary-point equation
\[
\phi'(\bar{s}) = 0 \tag{6}
\]
for $\bar{s} \leq 0$. More explicitly eq. (6) reads
\[
\frac{\mu}{a} - \frac{\bar{s}}{a \beta} = \frac{1}{L^{d-1}D} \sum_{k=1}^{\infty} \frac{e^{s - \beta \lambda^2 k^2}}{1 - e^{s - \beta \lambda^2 k^2}} + \frac{1}{V} \sum_{n=1}^{\infty} \frac{e^{ks}}{1 - e^{s}} + \frac{2}{L^{d-1}D} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} e^{-k \pi r^2 / \beta \lambda^2 n^2}.
\]
\[
\tag{7}
\]
Bulk properties of the model defined by eq. (2) were studied rigorously since 1980s [27–30]. The limit $T \to 0$ and the (bulk) quantum-critical behavior were addressed in ref. [31]. For $d > 2$ in the phase diagram spanned by $\mu$ and $T$ there is a line of second-order phase transitions to the phase hosting the Bose-Einstein condensate. The lower critical dimension $d_l = 2$ for the formation of the condensate is not affected by the long-ranged character of the interactions. Nor is the upper critical dimension $d_u = 4$ above which the critical exponents acquire classical (Landau) values. The critical line extends down to $T = 0$, where it ends with a quantum-critical point. The transition at $T > 0$ falls into the universality class of the spherical model [32], which also corresponds to the $N \to \infty$ limit of the $O(N)$-symmetric models. The shape of the critical line $T_c(\mu)$ [31] is in the limit $T \to 0$ governed by the universal shift exponent $\psi = z/(d + z - 2)$ in full agreement with the renormalization-group predictions of effective order-parameter models (see, e.g., [25,33,34]) where the dynamical exponent $z = 2$. We refer to [27–31] for a comprehensive discussion of the bulk properties of
the imperfect Bose gas, including its critical features. The Casimir forces corresponding to the above model were investigated in refs. [26,35] in the thermal regime, where \( D \gg \lambda \). Here we analyze the opposite case defined by the relation (1).

In addition to \( D \) and \( \lambda \), the model involves the lengthscale
\[
L_{\mu} = (a|\mu|^{-1})^{1/d},
\]
which may be considered large or small compared to \( D \) and \( \lambda \). The microscopic scale is set by
\[
l = \left( \frac{2\pi ma}{\hbar^2} \right)^{1/(d-2)},
\]
which is defined for \( d \neq 2 \). The quantity \( l \) plays the role of the microscopic scale \( l_{mic} \) occurring in eq. (1). Let us observe that both the above length scales involve the interaction coupling \( a \) and, therefore, are not present for the perfect Bose gas. The Casimir forces in the perfect Bose gas [36] were first addressed in [37] (see also [38]).

In the thermal regime the excess surface grand-canonical free energy (per unit area) \( \omega_s(T,D,\mu) \) is extracted by a subtraction of the bulk contribution \( \omega_b(T,\mu) \) from the full grand-canonical free energy \( \Omega(T,L,D,\mu) = -\beta^{-1} \ln \Xi(T,L,D,\mu) \). One obtains
\[
\omega_s(T,D,\mu) = \lim_{L \to \infty} \left[ \frac{\Omega(T,L,D,\mu)}{L^{d-1}} - D\omega_b(T,\mu) \right].
\]
The Casimir force (per unit area) is then evaluated via
\[
F = -\frac{\partial \omega_s(T,D,\mu)}{\partial D}.
\]
In the regime considered in the present paper (eq. (1)) one can take the alternative approach amounting to calculating the derivative \( \partial_D \lim_{L \to \infty} \Omega(T,L,D,\mu)/L^{d-1} \) and neglecting a constant (\( D \)-independent) term identified in the grand-canonical potential with the bulk contribution.

We note that, in the present grand-canonical setup, the subtracted bulk term gives rise to a constant (\( D \)-independent) force acting between the boundary walls. For large \( D \) this force certainly dominates over the \( D \)-dependent contribution discussed below. It should not be understood as a fluctuation-mediated iteration (which vanishes for \( D \to \infty \)). The constant contribution is not specific to the model (nor to the scales’ hierarchy of eq. (1)) and generically arises when one computes interactions between system boundaries. In other contexts (such as QED or colloids immersed in a critical fluid) such constant force is absent since the fluctuating medium (or vacuum) exerts pressure on both sides of the immersed object.

The character of the solution to eq. (6) in the asymptotic regime specified by (1) crucially depends on the dimensionality \( d \). Here we primarily focus on the interval \( d \in [2,3] \), where the term \( \frac{\lambda^2}{D^2} \ln \frac{L_{\mu}}{a} \) in eq. (7) is singular at \( \bar{s} \to 0^- \). This guarantees that when \( L \to \infty \) (keeping all the other length scales fixed), the solution \( \bar{s}(T,D,\mu) \) remains separated from zero. As a consequence, the terms \( \sim 1/V \) in eq. (7) can be dropped. The last contribution to eq. (7) is bounded from above by \( \sim \frac{1}{D^2} e^{-\pi\lambda D} \) and is also negligible as compared to the other terms. The solution of eq. (7) then boils down to analyzing the equation
\[
\frac{\mu}{a} - \frac{\bar{s}}{a\beta} = \frac{1}{\lambda^d-1} \frac{\lambda^{d-1}}{D^2} (e^{x})
\]
in the asymptotic regimes, where the correlation length \( \xi \sim |\bar{s}|^{-1/2} \) (see below) is asymptotically large.

We note that the situation is quite different for \( d > 3 \), where the vanishing of \( \bar{s} \) is controlled by \( L^{-1} \). We shall discuss this case separately.

We also observe that the small parameter \( D/\lambda \) naturally arises in eq. (7). This makes the analysis somewhat simpler than in the thermal regime \( D \gg \lambda \), which required an application of the Jacobi identity [37] which swaps the roles of \( D \) and \( \lambda \) in the last sum on the right-hand side of eq. (7). Technically it is precisely this aspect that leads to different signs of the Casimir force in these two regimes.

**Results for \( d = 3 \).** – For the case of \( d = 3 \) the term \( \frac{\lambda^2}{D^2} \ln \frac{L_{\mu}}{a} \) in eq. (12) displays a logarithmic singularity at \( \bar{s} \to 0^- \);
\[
g_1(\bar{s}) \approx -\ln |\bar{s}| \quad \text{for} \quad |\bar{s}| \ll 1.
\]
Equation (12) finds an asymptotic solution in the following form:
\[
\bar{s} \approx \left\{ \begin{array}{ll}
\frac{l}{D} \ln \frac{l}{D}, & \text{in Regime I,} \\
\frac{\lambda^2}{L_{\mu}^2} - \frac{l}{D} g_1(e^{\lambda^2 l/L_{\mu}^2}), & \text{in Regime III.}
\end{array} \right.
\]
Regime I corresponds to the condition \( L_{\mu} \gg (D\lambda^2)^{1/3} \);
Regime II to \( L_{\mu} \ll (D\lambda^2)^{1/3}, \mu > 0 \); and, finally, Regime III is defined by \( L_{\mu} \ll (D\lambda^2)^{1/3}, \mu < 0 \).

The crucial observation is that the quantity \( |\bar{s}| \) is related to the correlation length \( \xi \) by the formula [26,39]
\[
|\bar{s}| = \kappa \lambda |s|^{-1/2},
\]
where \( \kappa \) is a numerical constant.

It follows that the singularity of \( \xi \) occurring at the quantum-critical point (in the limit \( T,\mu,D^{-1} \to 0 \)) is effectively cut off by the system width \( D \) in Regime I, by the thermodynamic fields \( T,\mu \) in Regime III, and by a combination of the thermodynamic and geometric parameters in Regime II. This gives rise to the rich behavior predicted for the Casimir force (see below).

We now use eq. (5) to compute the grand-canonical free energy \( \Omega(T,L,D,\mu) \) and take the derivative \( \partial_D \lim_{L \to \infty} \Omega(T,L,D,\mu) \). Neglecting a constant, which is attributed to the bulk term in the free energy, we obtain
the microscopic length $l$ of the system, and the Casimir forces in particular, when continuously reducing the dimensionality parameter $d$ from 3 towards the other physical value $d = 2$. The case $d = 3$ is special because the function $g_{\text{eq}}(e^\mu)$ exhibits a logarithmic singularity at $s \to 0^-$.

The asymptotic form of the saddle-point equation (12) admits the following solutions:

\[
\bar{s} \approx \begin{cases}
- C_1 \left( \frac{d-2}{d} \right)^{\frac{d-2}{2}} \lambda^{-2(d-3)/d} D^{-\frac{d}{d-2}}, & \text{in Regime I,} \\
- C_2 \left( \frac{d-2}{d} \right) \lambda^{-2(d-3)/d} D^{-\frac{d}{d-2}}, & \text{in Regime II,} \\
- l^{d-2} \left( \lambda^2 L_\mu^{-d} + \lambda^{3-d} D^{-1} g \right), & \text{in Regime III,}
\end{cases}
\]

where we introduced $C_1 = (2\pi)^{-2/(5-d)} \left[ \Gamma \left( \frac{3-d}{d} \right) \right]^{\frac{d}{d-2}}$, $C_2 = \left[ \Gamma \left( \frac{3-d}{d} \right) \right]^{\frac{d}{d-2}}$, and $g = g_{\text{eq}}(e^{-\lambda^2 l^2/2 L_\mu^2})$.

The three emergent asymptotic regimes are defined by the condition

\[
L_\mu \gg (\lambda^{d-1} D)^{1/d}, \quad \text{Regime I,}
\]
\[
L_\mu \ll (\lambda^{d-1} D)^{1/d}, \quad \mu > 0, \quad \text{Regime II},
\]
\[
L_\mu \ll (\lambda^{d-1} D)^{1/d}, \quad \mu < 0, \quad \text{Regime III.}
\]

For $d = 3$ this reduces to the regimes studied in the previous section.

From eq. (5) we evaluate the free energy and extract the Casimir force by taking the $D$-derivative. The result reads

\[
\beta F(T, D, \mu) \approx \begin{cases}
\frac{C_1^2}{21^2} \left( \frac{D}{\lambda} \right)^{2(\frac{d}{d-1})} \left( \frac{l}{D} \right)^{2(d-1)} D^{\frac{d}{d-1}}, & \text{(I)} \\
C_2 \left( \frac{L_\mu}{\lambda^{d-1} D} \right)^{\frac{d}{d-1}} \left( \frac{D}{\lambda} \right)^{d-1} \left( \frac{l}{D} \right)^{2(d-1)} D^{\frac{d}{d-1}}, & \text{(II)} \\
\frac{1}{21^2} \left( \frac{D}{\lambda} \right)^{2(d-2)} \left( \frac{l}{D} \right)^{2(d-1)} g_{\text{eq}}^2(\bar{s}^{\beta \mu}), & \text{(III)}
\end{cases}
\]

with $C_1 = \left[ \Gamma \left( \frac{3-d}{d} \right) \right]^{\frac{d}{d-2}}$. The Casimir force is repulsive in all the three asymptotic regimes and decays with a power of $D$. Note a difference as compared to $d = 3$, where the logarithms and exponents appeared as a consequence of the form of the asymptotic behavior of the Bose function $g_1$ at $s \to 0^-$. As $d \to 3^-$, the exponent $(d-1)/d$ present in Regime II (eq. (19)) diverges, which gives rise to the exponential behavior in $d = 3$. The result is translated back to the thermodynamic variables $\mu$ and $T$ and depicted in fig. 1.
Note on the case \(d = 2\). – We now comment on the Casimir force in \(d = 2\). This is a special case since the microscopic length \(l_{\text{mic}} = 1\) defined in eq. (9) does not exist. It makes no physical sense (nor mathematical) to consider the limit of the scales \(D\) and \(\lambda\) becoming macroscopic without specifying the microscopic length. The only choice possible in \(d = 2\) is to take \(l_{\text{mic}} = L_\mu\) as given by eq. (8). The absence of the quantity \(l\) in \(d = 2\) manifests itself in the non-existence of a solution to the saddle-point equation (6) in the parameter range corresponding to Regime I, where \(L_\mu\) is large. Apart from this, the results of the previous section apply to \(d = 2\).

Results for \(d > 3\). – It is also interesting to examine the case \(d > 3\), where the system may host a thermodynamically stable Bose-Einstein condensate for \(L \to \infty\), but finite \(D\). For \(d > 3\) the function \(g_{\lambda^{-1}}(e^s)\) is finite for \(s \to 0^+\). As a consequence, eq. (12) has no solution at sufficiently low \(T\). This is because upon passing to the limit \(L \to \infty\) in eq. (7) \(\hat{s}\) vanishes, and the term \(1/V\) gives a finite contribution. It must therefore be included in the analysis by replacing eq. (12) with

\[
\frac{\mu}{a} \frac{\hat{s}}{a\beta} = \frac{1}{\lambda^{d-3}D} g_{\lambda^{-1}}(e^s) + \frac{1}{V} \frac{e^s}{1-e^s}. \tag{20}
\]

The above equation is equivalent to the one arising in the bulk case for \(D \gg \lambda\) (see ref. [35]) upon making the substitutions \(\lambda^d \to \lambda^{d-1}D\) and \(g_{\lambda}(e^s) \to g_{\lambda^{-1}}(e^s)\). For \(V \to \infty\) we find a finite, unique solution to eq. (20) provided \(\lambda^{d-1}D\mu < a\zeta(\frac{d-1}{2})\). In the opposite case, the last term in eq. (20) gives a finite contribution equal to the condensate density [39]. This leads to the following result for the critical value of the chemical potential:

\[
\mu_c(T, D) = a\zeta\left(\frac{d-1}{2}\right) \frac{1}{\lambda^{d-1}D}, \tag{21}
\]

above which the Bose-Einstein condensate is present in the system. The transition between the phases may be induced by varying any of the parameters \(\{\mu, T, D\}\) so that the geometric quantity \(D\) may (for the presently relevant regime \(D \ll \lambda\)) serve as the tuning parameter on equal footing with the thermodynamic ones. One may also observe, that \(\mu_c(T, D)\) may be related to the standard, thermodynamic critical value \(\mu_c^{(d)}(T)\) of the chemical potential [31] via

\[
\mu_c(T, D) = \mu_c^{(d)}(T) \frac{\lambda}{D} \zeta\left(\frac{d-1}{2}\right) \zeta\left(\frac{d}{2}\right). \tag{22}
\]

Since \(\lambda/D \gg 1\), it follows that \(\mu_c(T, D) > \mu_c^{(d)}(T)\). At \(d \to 3^+\) we have \(\zeta\left(\frac{d-1}{2}\right) \to \infty\) in eq. (22), so that \(\mu_c(T, D)\) diverges and the condensate is ultimately suppressed in the regime \(\lambda \gg D\).

For the determination of the Casimir force, we focus on the range of parameters, where the condensate is present \((\mu > \mu_c)\), making the setup clearly distinct from that discussed for \(d \leq 3\). The calculation leading to an expression for \(\hat{s}\) is now analogous to the one performed in ref. [35] (where one makes the replacements \(\lambda^d \to \lambda^{d-1}D, g_{\lambda^d}(e^s) \to g_{\lambda^{-1}}(e^s)\) specified above). We obtain

\[
|\hat{s}(T, \mu, D)| \approx \frac{\lambda^{d-1}D}{V} \frac{1}{\zeta\left(\frac{d-1}{2}\right)} \frac{\mu_c}{\mu - \mu_c}. \tag{23}
\]

This leads to the following expression for the Casimir force:

\[
\beta F(T, \mu, D) = 4\pi \left(\frac{D}{\lambda}\right)^{d-3} \frac{1}{\lambda^{d-1}D} \frac{1}{2^{d-1}} \left(e^{-\pi\lambda^2/D^2}\right), \tag{24}
\]

which is again repulsive.

Summary. – We have performed an exact study of Casimir forces occurring in the \(d\)-dimensional imperfect Bose gas (interacting bosons in the Kac limit) in the regime, where the de Broglie length is squashed between the lengthscales \(D\) and \(\lambda\) characterizing the system geometric size (i.e. for \(D \gg \lambda \ll L\)). We scanned the dependence of our results on the system dimensionality \(d\). The obtained behavior of the Casimir force differs substantially from that established before in the thermal regime (i.e. for \(\lambda \ll D \ll L\)) and the one expected in the quantum regime \(D \ll L \ll \lambda\) by virtue of the quantum-classical mapping. The computed Casimir force turns out to be repulsive and decays as a power of the distance \(D\) in most of the cases (with log-corrections in \(d = 3\)). The peculiarity of our results may be traced back to the occurrence of an extra lengthscale \((\lambda)\) which is considered as macroscopic, and which is absent in the standard condensed-matter setup. We emphasise that the present model perfectly fits into the established classification once we restrict to the thermal regime (i.e. treat \(\lambda\) as a microscopic scale). In this case it falls into the universality class of the \(d\)-dimensional spherical model. The interplay between \(\lambda, D,\) and the scale \(L_\mu\) controlling the distance of the system from the (bulk) quantum-critical point leads to the emergence of three regimes showing different asymptotic behavior of the Casimir force. An additional feature appears for \(d > 3\), where the system admits a \((d-1)\)-dimensional ("surface") condensate as a stable thermodynamic phase for any finite \(D\). The transition to this phase may be tuned by varying \(\mu, T\) as well as \(D\). It is interesting to speculate about the generality of our results and their dependence on the details of the microscopic model. Clearly, our results (for \(D > D\)) do depend on the microscopic parameters. These, however, may be expressed via quantities of the dimensionality of length, which find natural analogues in other condensed-matter systems (in particular close to quantum criticality). We believe that (when expressed via these length parameters) our finding should apply at least to other systems belonging to the universality class of the spherical model [40–44]. Another important question
concerns the sensitivity of our results on the boundary conditions. Such a dependence is well known to occur in the thermal regime. We have checked that for Neumann boundary conditions the essential features of our results are unchanged. The analysis of other cases, such as Dirichlet or free-boundary conditions is however beyond the scope of the present paper.

We close the paper by emphasising some differences between the QED and condensed-matter contexts. In the former case the occurrence of repulsive Casimir forces between reflection-related bodies is strictly forbidden, also in the regime \( D < \lambda \) (see, e.g., [45]). For the present setup, where the finite-size effect yields an effective Casimir-type interaction between the system “boundaries” (after subtracting the constant term), we are not aware of any \textit{a priori} reason that fixes the sign of the force. For example, in the case of the perfect Fermi gas one finds a Casimir-like force that oscillates around zero as a function of the distance \( D \) and becomes non-differentiable in the limit \( T \to 0 \) for periodic, Dirichlet and Neumann boundary conditions [46].

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