Fast Parallel Algorithms for Feature Selection

Sharon Qian
Harvard University
sharonqian@g.harvard.edu

Yaron Singer
Harvard University
yaron@seas.harvard.edu

March 8, 2019

Abstract

In this paper, we analyze a fast parallel algorithm to efficiently select and build a set of $k$ random variables from a large set of $n$ candidate elements. This combinatorial optimization problem can be viewed in the context of feature selection for the prediction of a response variable. Using the adaptive sampling technique, which has recently been shown to exponentially speed up submodular maximization algorithms, we propose a new parallelizable algorithm that dramatically speeds up previous selection algorithms by reducing the number of rounds from $O(k)$ to $O(\log n)$ for objectives that do not conform to the submodularity property. We introduce a new metric to quantify the closeness of the objective function to submodularity and analyze the performance of adaptive sampling under this regime. We also conduct experiments on synthetic and real datasets and show that the empirical performance of adaptive sampling on not-submodular objectives greatly outperforms its theoretical lower bound. Additionally, the empirical running time drastically improved in all experiments without comprising the terminal value, showing the practicality of adaptive sampling.
1 Introduction

In this paper, we study fast parallel algorithms for feature selection in a high-dimensional space. Feature selection is a core problem in modern data analysis and statistics where we are given a set of $n$ variables, but would like to select only $k$ features to comprise a solution set and predict the response variable $y$. This fairly general formulation includes common statistical problems such as regression, classification and maximum likelihood estimation.

A popular heuristic for feature selection is forward stepwise regression, where a variable or element is added to the solution set based on an objective function at each iteration. This approach is known to work well in practice and is comparable with many popular subset selection algorithms such as LASSO [Elenberg et al. (2018)] and has many advantages for model training and development. For example, one such advantage is that the user can specify exactly how many features to select and avoid computing the regularization parameter (in the case of LASSO), which is generally computationally hard [Mairal & Yu (2012)].

In a series of recent works, [Das & Kempe (2011)] and [Elenberg et al. (2018)] extend the canonical result of the greedy algorithm for submodular function maximization and give strong theoretical guarantees for forward regression via relaxed notions of submodularity. Even with a non-submodular objective function, this approach has been shown to work well in practice.

Although it performs well in practice and has strong theoretical guarantees, forward stepwise selection is highly sequential in nature and cannot be parallelized efficiently. Specifically, at each iteration, the algorithm must compute the contribution of each element in the feature set before selecting an element. Thus, the parallel runtime of the forward stepwise algorithm scales linearly with the number of features we want to select. In cases where the computation of the objective function across all elements is expensive or the feature set is large, this can be computationally infeasible.

Adaptive sampling for submodular maximization. In a recent line of work initiated by [Balkanski & Singer (2018a)], adaptive sampling techniques have been used for submodular maximization under varying constraints [Balkanski et al. (2019); Balkanski & Singer (2018b); Chekuri & Quanrud (2015, 2018); Balkanski et al. (2018a); Chen et al. (2018); Ene et al. (2018a); Fahrbach et al. (2019); Balkanski et al. (2018b); Ene & Nguyen (2019); Ene et al. (2018b)]. Intuitively, instead of growing the solution set element-wise, adaptive sampling adds sets to the solution set at each round. This allows the algorithm to be highly parallelizable and decreases the number of rounds until algorithm termination. The main result is that adaptive sampling can give the same approximation guarantee as the greedy algorithm for submodular maximization problems, but exponentially faster.

Non-submodular objectives. Adaptive sampling techniques are tailored to submodular functions and their theoretical guarantees rely heavily on the properties of submodularity to add sets instead of one element at each round. While the feature selection problem has been shown to be weakly submodular [Elenberg et al. (2018)], which is sufficient for the traditional greedy algorithm, it is not a strong enough condition for the theoretical guarantees of adaptive sampling. However, feature selection is close to submodular so perhaps we can still apply this technique.

Are there fast adaptive algorithms for feature selection?

In this paper we give algorithms that are exponentially faster than any known before for the combinatorial optimization problem of feature selection. We achieve our exponential speedups using the recent advancement of the adaptive sampling technique for submodular maximization. The main contribution is a novel characterization of feature selection in terms of a condition that we call differential submodularity that bounds the objective function’s deviation from submodularity. By framing the feature selection problem in this framework, we are able to apply adaptive sampling, which has previously only been used in submodular maximization problems.
Main results. Our main result is presented in the DASH algorithm, which obtains an approximation guarantee arbitrarily close to $1 - 1/e^\alpha$ for maximizing approximate submodular functions under cardinality constraints in $O(\log n)$ adaptive rounds, where $\alpha$ is a measure of how close a function is to being submodular. We are able to attain this guarantee by extending previous results on a general $\ell(\cdot)$ objective function with strong concavity and smoothness properties and showing that the marginal contributions are bound by submodular functions. Furthermore, in our experiments, we show that the empirical performance of adaptive sampling greatly outperforms its theoretical lower bound and is especially computationally efficient compared to traditional methods for large number of elements selected.

Technical overview. Previous adaptive sampling algorithms focus on the maximization of submodular functions. However, these algorithms cannot be applied to problems where the objective function is not submodular. In these cases, the marginal contribution of individual elements is not necessarily subadditive compared to the marginal contribution of the set of elements combined. This lack of submodularity causes difficulty in the adaptive sampling algorithm analysis, where we attempt to add large sets of elements to the solution set by assessing the value of individual elements. In addition, the parameter quantifying closeness to submodularity can propagate throughout the analysis, resulting in a weak approximation guarantee. By bounding the marginal contribution of the objective function by submodular functions, we are able to prevent this propagation and show a stronger approximation guarantee. Utilizing techniques from canonical submodular maximization algorithms in addition to the recent advances of adaptive sampling, we show theoretical guarantees for feature selection problems where not-submodular objectives in exponentially fewer rounds.

Paper organization. We first introduce preliminary definitions in Section 2 followed by introducing our main framework of differential submodularity and its reduction to linear and logistic regression objectives in Section 3. We then introduce an algorithm for selection problems using adaptive sampling in Section 4 and conclude with experiments in Section 5.

2 Preliminaries

For a positive integer $n$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Boldface lower and upper case letters denote vectors and matrices respectively: $\mathbf{a, x, y}$ represent vectors and $A, X, Y$ represent matrices. Unbolded lower and upper case letters present elements and sets respectively: $a, x, y$ represent elements and $A, X, Y$ represent sets. For a matrix $A \in \mathbb{R}^{d \times n}$ and $X \subseteq [n]$, we denote submatrices by column indices by $A_X$. For vectors, we use $x_S$ to denote supports $\text{supp}(x) \subseteq S$. To connect the discrete function $f(S)$ to a continuous function, we let $f(S) = \ell(w(S))$, where $w(S)$ denotes the $w$ that maximizes $\ell(\cdot)$ subject to $\text{supp}(w) \subseteq S$.

Submodularity. A function $f : 2^N \to \mathbb{R}_+$ is submodular if it it exhibits a diminishing returns property, i.e., $f_S(a) \geq f_T(a)$ for all $a \in N \setminus T$ and $S \subseteq N$, where $f_S(a) := f(S \cup a) - f(S)$. A function $f$ is monotone if $f(S) \leq f(T)$ for all $S \subseteq T$. If $f$ is submodular function, it is also subadditive, meaning $f(S \cup T) \leq f(S) + f(T)$ for all $S \subseteq T$. We assume that $f$ is normalized and non-negative, i.e., $0 \leq f(S) \leq 1$ for all $S \subseteq N$.

Weak submodularity. The concept of weak submodularity is a relaxation of submodularity. We first define the submodularity ratio, introduced by [Das & Kempe 2011].

**Definition 1.** The submodularity ratio of a function $f : 2^N \to \mathbb{R}_+$ is defined as, for all $A \subseteq N$

$$\gamma_k = \min_{A \subseteq N, S : |A| \leq k} \sum_{a \in A} \frac{f_S(a)}{f_S(A)}.$$
Clearly, $\gamma_k \geq 1$ for all $k$, if and only if $f$ is submodular. We say functions with submodularity ratios $\gamma = \min_k \gamma_k < 1$ are weakly submodular with parameter $\gamma$.

**Restricted strong concavity and smoothness.** [Elenberg et al. 2018](#) introduces the notion of concavity and smoothness of the objective function and connects it to submodularity-like properties.

**Definition 2.** [Elenberg et al. 2018](#) Let $\Omega$ be a subset of $\mathbb{R}^n \times \mathbb{R}^n$ and $\ell : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. A function $\ell$ is restricted strong concave (RSC) with parameter $m_\Omega$ and restricted smooth (RSM) with parameter $M_\Omega$ if, for all $(\mathbf{y}, \mathbf{x}) \in \Omega$,

$$-\frac{m_\Omega}{2} \| \mathbf{y} - \mathbf{x} \|^2 \geq \ell(\mathbf{y}) - \ell(\mathbf{x}) - \langle \nabla \ell(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq -\frac{M_\Omega}{2} \| \mathbf{y} - \mathbf{x} \|^2.$$

**Feature selection.** In a feature selection problem, we wish to select the best subset of size $k$ from a large feature set that can predict the continuous response variable $y$. More formally, for a response variable $y \in \mathbb{R}^d$, we wish to select $k$ columns from $A \in \mathbb{R}^{d \times n}$, which represents our feature set of size $n$. We can assess the quality of $A_S$, where $S$ is our set of selected features, by a given function $\ell : \mathbb{R}^n \to \mathbb{R}$. We can now define our maximization problem:

$$\max_{S : |S| \leq k} f(S) = \max_{S : |S| \leq k} \ell(w(S)).$$

To regularize for diverse features, we can formulate the problem as follows

$$\max_{S : |S| \leq k} f_{\Delta V}(S) = \max_{S : |S| \leq k} \ell(w(S)) + g(S),$$

where $g : 2^N \to \mathbb{R}_+$ is a “diverse” submodular function for “diversity-promoting” regularization [Das et al. 2012](#).

The feature selection objective has been shown to be weakly submodular and with the traditional greedy algorithm, we get a $1 - 1/e^\gamma$ approximation guarantee, where $\gamma \geq \frac{m_\Omega}{M_\Omega}$ [Elenberg et al. 2018](#).

**Adaptivity.** The adaptivity of algorithms refers to the number of sequential rounds of queries it makes, where each round allows for polynomial-many parallel queries. We are interested in adaptive algorithms that can be easily parallelizable and scalable for large datasets.

**Definition 3.** For a function $f$, an algorithm is $r$-adaptive if every query $f(S)$ given a set $S$ occurs at a round $i \in [r]$ such that $S$ is independent of the values $f(S')$ of all other queries at round $i$.

## 3 Differential Submodularity

In this section, we introduce a measure to quantify the deviation from submodularity and prove our main theorem by connecting our notion of differential submodularity to feature selection in regression and classification problems. Specifically we first define differential submodularity and prove a bound of feature selection objectives which is critical in showing theoretical approximation guarantees.

We will show that, for certain properties of the feature selection loss function $\ell(\cdot)$, its discrete analog $f$ has properties close to submodularity. We first introduce a novel notion of how close a function is to submodularity by the following property on the marginals. This property can be seen as an extension of weak submodularity that bounds the differential behavior of the objective.
**Definition 4.** A function \( f : 2^N \to \mathbb{R}_+ \) is \( \alpha \)-differentially submodular on the marginals for \( \alpha \in (0, 1] \), if there exist two submodular functions \( h^{(1)}, h^{(2)} \) s.t. for any \( S, A \subseteq N \), we have that \( h^{(1)}_S(A) \leq f_S(A) \leq h^{(2)}_S(A) \) and \( h^{(1)}_S(A) \geq \alpha \cdot h^{(2)}_S(A) \).

If \( f \) is a 1-differentially submodular function, then it is submodular and we have the desirable properties of submodularity for provable guarantees. If \( f \) is 0-differentially submodular, then the function strays far from submodularity.

### 3.1 RSC/RSM Implies Differential Submodularity

Before connecting our notion of differential submodularity to RSC/RSM properties, we first define concavity and smoothness parameters on subsets of \( \Omega \). If \( \Omega' \subseteq \Omega \), then \( M_{\Omega'} \leq M_\Omega \) and \( m_{\Omega'} \geq m_\Omega \).

**Definition 5.** We define the domain of \( s \)-sparse vectors as \( \Omega_s = \{ (x, y) : \|x\|_0 \leq s, \|y\|_0 \leq s, \|x - y\|_0 \leq s \} \). If \( s \geq t \), \( M_t \leq M_s \) and \( m_t \geq m_s \).

We now introduce our framework to measure the deviation from submodularity. This differential property is crucial in allowing us to analyze adaptive sampling techniques.

**Theorem 6.** Suppose \( \ell(\cdot) \) is \( m_s \)-strongly concave and \( M_s \)-smooth on \( \Omega_s \) and \( m_t \)-strongly concave and \( M_t \)-smooth on \( \Omega_t \). Then, the objective defined by \( f(S) = \ell(w^{(S)}) \) is a differentially submodular function such that for \( S, A \subseteq N, |A| \leq k, \frac{m_s}{M_t} f_S(A) \leq f_S(A) \leq \frac{M_t}{m_s} f_S(A) \), where \( f_S(A) = \sum_{a \in A} f_S(a) \) and \( t = |S| + k, s = |S| + 1 \).

**Proof.** We first prove the lower bound of the inequality. We define \( x_{(S \cup A)} = \frac{1}{M_t} \nabla \ell(w^{(S)}) A + w^{(S)} \) and use the strong concavity of \( \ell(\cdot) \) to lower bound \( f_S(A) \):

\[
\begin{align*}
f_S(A) & \geq \ell(w^{(S \cup A)}) - \ell(w^{(S)}) \\
& \geq \ell(x_{(S \cup A)}) - \ell(w^{(S)}) \\
& \geq \langle \nabla \ell(w^{(S)}), x_{(S \cup A)} - w^{(S)} \rangle \\
& \quad - \frac{M_t}{2} \|x_{(S \cup A)} - w^{(S)}\|_2^2 \\
& \geq \frac{1}{2M_t} \|\nabla \ell(w^{(S)})\|_2^2
\end{align*}
\]

where the first inequality follows from the optimality of \( \ell(w^{(S \cup A)}) \) for vectors with support \( S \cup A \) and the last inequality is by the definition of \( x_{(S \cup A)} \).

We also can use smoothness of \( \ell(\cdot) \) to upper bound the marginal contribution of each element in \( A \) to \( S \), \( f_S(a) \). We define \( x_{(S \cup a)} = \frac{1}{m_s} \nabla \ell(w^{(S)}) a + w^{(S)} \). For \( a \in A \),

\[
\begin{align*}
f_S(a) & \leq \ell(w^{(S \cup a)}) - \ell(w^{(S)}) \\
& \leq \langle \nabla \ell(w^{(S)}), w_{(S \cup a)} - w^{(S)} \rangle \\
& \quad - \frac{m_s}{2} \|w_{(S \cup a)} - w^{(S)}\|_2^2 \\
& \leq \langle \nabla \ell(w^{(S)}), x_{(S \cup a)} - w^{(S)} \rangle \\
& \quad - \frac{m_s}{2} \|x_{(S \cup a)} - w^{(S)}\|_2^2 \\
& \leq \frac{1}{2m_s} \|\nabla \ell(w^{(S)})\|_2^2
\end{align*}
\]

where the last inequality follows from the definition of \( x_{(S \cup a)} \).
Summing across all $a \in A$, we get
\[
\sum_{a \in A} f_S(a) \leq \sum_{a \in A} \frac{1}{2m_s}\|\nabla \ell(w^1(S))_a\|_2^2
= \frac{1}{2m_s}\|\nabla \ell(w^1(S))_A\|_2^2
\] (2)

By combining (1) and (2), we can get the desired lower bound of $f_S(A)$.

To get the upper bound on the marginals, we can use the lower bound of submodularity ratio $\gamma_{S,t}$ of $f$ from [Elenberg et al. 2018], which is no less than $\frac{\gamma_{S,t}}{\gamma_{S,k}}$. Then, by letting $f_S(A) = \sum_{a \in A} f_S(a)$, we can complete the proof and show that the marginals can be bounded.

We can further generalize the previous lemma to all sets $S, A \subseteq N$, by using the general concavity and smoothness parameters $m, M$ associated with $\Omega_U$, where $U \geq t, s$. To obtain bounds that are independent of the size of $S$ and $A$, we can use properties of strong concavity and smoothness to produce weaker bounds. For $k \geq 1, t \geq s$, Since $\Omega_s \subseteq \Omega_t \subseteq \Omega_U$, $M_s \leq M_t \leq M$ and $m_s \geq m_t \geq m$. Thus, we can weaken the bounds from Lemma [6] to get $\frac{\gamma_{S,t}}{\gamma_{S,k}}f_S(A) \leq f_S(A) \leq \frac{\gamma_{S,k}}{\gamma_{S,t}}f_S(A)

3.2 Feature Selection for Regression

We now state the objective used in linear regression for the problem of feature selection.

Linear regression. For a response variable $y \in \mathbb{R}^d$, the objective in selecting the elements to form a solution set is the maximization of the $\ell^2$-utility function that represents the variance reduction of $y$ given the feature set $S$:

$$
\ell_{reg}(y, w^{(S)}) = \|y\|_2^2 - \|y - A_S w\|_2^2
$$

In linear regression, we can bound the marginals by the eigenvalues of the feature covariance matrix. We denote the minimum and maximum eigenvalues of the k-sparse feature covariance matrix by $\lambda_{min}(k)$ and $\lambda_{max}(k)$.

Corollary 7. Suppose that $g : 2^N \rightarrow \mathbb{R}_+$ is a submodular function, then the objectives of linear regression defined by $f(S) = \ell_{reg}(w^{(S)})$ and diverse linear regression defined by $f_{div}(S) = \ell_{reg}(w^{(S)}) + g(S)$ are $\frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}$-differentially submodular functions, where $t = |S| + k, s = |S| + 1$.

Proof. In the case where there is no diversity regularization term, the concavity and smoothness parameters correspond to the sparse eigenvalues of the covariance matrix, i.e., $m_k = \lambda_{min}(k)$ and $M_k = \lambda_{max}(k)$ [Elenberg et al. 2018]. Thus, by Theorem [6] we can also write the bounds for $f_S(A)$ in terms of eigenvalues $\frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}f_S(A) \leq f_S(A) \leq \frac{\lambda_{max}(s)\lambda_{max}(t)}{\lambda_{min}(s)\lambda_{min}(t)}f_S(A)$. With $h_S^{(1)}(A) = \frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}f_S(A)$ and $h_S^{(2)}(A) = \frac{\lambda_{max}(s)\lambda_{max}(t)}{\lambda_{min}(s)\lambda_{min}(t)}f_S(A)$, we get that the objective is a $\frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}$-differentially submodular function.

In the case where there is a diversity regularization term in the objective $f_{div}(S) = \ell_{reg}(w^{(S)}) + g(S)$, we have $\frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}f_S(A) + g(A) \leq (f_{div})_S(A) \leq \frac{\lambda_{max}(s)\lambda_{max}(t)}{\lambda_{min}(s)\lambda_{min}(t)}f_S(A) + g(A)$. With $h_S^{(1)}(A) = \frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}f_S(A) + g(A)$ and $h_S^{(2)}(A) = \frac{\lambda_{max}(s)\lambda_{max}(t)}{\lambda_{min}(s)\lambda_{min}(t)}f_S(A) + g(A)$, we get that $h_S^{(1)}(A)/h_S^{(2)}(A) \geq \frac{\lambda_{min}(s)\lambda_{min}(t)}{\lambda_{max}(s)\lambda_{max}(t)}$ since $g(A) \geq 0$, which concludes the proof.

We note that [Das & Kempe 2011] use a different objective function to measure the goodness of fit $R^2$. In Appendix [6] we directly show an analogous bound on the marginals for the $R^2$ objective function with the eigenvalues of the feature covariance matrix.
3.3 Feature Selection for Classification

In the classification problem, we wish to select the best \( k \) columns from \( A \in \mathbb{R}^{d \times n} \) to predict a categorical variable \( y \in \mathbb{R}^d \). We use the following log-likelihood objective in logistic regression to select features.

**Logistic regression.** For a categorical variable \( y \in \mathbb{R}^d \), the objective in selecting the elements to form a solution set is the maximization of the log-likelihood function for a given \( S \):

\[
\ell_{\text{class}}(y, w^{(S)}) = \sum_{i=1}^{d} y_i (A_S w) - \log(1 + e^{A_S w})
\]

**Corollary 8.** Suppose that \( g : 2^N \to \mathbb{R}_+ \) is a submodular function, then the objectives of logistic regression defined by \( f(S) = \ell_{\text{class}}(w^{(S)}) \) and of diverse logistic regression defined by \( f_{\text{div}}(S) = \ell_{\text{class}}(w^{(S)}) + g(S) \) is an \( \frac{m^2}{M^2} \)-differentially submodular function for some \( m, M > 0 \).

**Proof.** In the case where there is no diversity regularization term, we show that the logistic objective can be \( m \)-RSC and \( M \)-RSM for some \( m, M > 0 \). Then the result follows directly from Theorem 6. In general, log-likelihood functions of generalized linear models (GLMs) are not RSC/RSM. However, Elenberg et al. 2018 showed that log-likelihoods of GLMs are indeed RSC/RSM under mild conditions of the feature matrix. Thus, the logistic regression, a GLM with the logit as the link function, has a log-likelihood objective with differentially submodular properties. The case where there is a diversity regularization term then follows similarly as for Corollary 7.

In the next section, we introduce an adaptive sampling algorithm for differential submodular objective functions and show theoretical guarantees.

4 Adaptive Algorithm

We now present the DASH (DIFFERENTIAL-ADAPTIVE-SHAMPLING) algorithm for adaptively selecting features from large datasets. We show that, for an \( \gamma^2 \)-differential submodular function, the approximation ratio of the algorithm output is arbitrarily close to \( 1 - 1/e^\gamma \), where \( \gamma = m/M \).

The analysis for this guarantee uses similar techniques as the adaptive sampling algorithm for submodular function maximization introduced by Balkanski & Singer 2018a. Since their paper, variations of the algorithm for multilinear extensions, non-monotonicity, matroid constraints and packing constraints have been introduced using similar techniques Balkanski et al. (2018); Balkanski & Singer (2018); Chekuri & Quanrud (2014, 2018); Balkanski et al. (2018); Chen et al. (2018); Ene et al. (2018a); Fahrbach et al. (2019); Balkanski et al. (2018); Ene et al. (2018b). All of these adaptations heavily rely on properties of submodular functions to show theoretical guarantees.

While the weakly submodular property of feature selection is sufficient for the standard greedy algorithm, it is insufficient for adaptive sampling methods and the original algorithm cannot be applied to non-submodular functions. Now, with our adjustment to the algorithm and framing the problem in our first-order submodularity framework, we can extend the popular adaptive sampling technique to maximize non-submodular functions and significantly speedup the problem of feature selection.

**Algorithm overview.** At each round, the DASH algorithm selects good elements determined by their individual marginal contributions and attempts to add a set of \( k/r \) elements to the solution set \( S \). The decision to label elements as “good” or “bad” depends on the threshold \( t \) which quantifies the distance between the elements that have been selected and \( \text{OPT} \). This elimination step takes
Theorem 9. Let $f$ be a monotone, $\alpha$-first-order submodular function where $\alpha \leq 1$, then, for any $\epsilon > 0$, DASH is a $\log_{1+\epsilon/2}(n)$ adaptive algorithm that obtains the following approximation for the set $S$ that is returned by the algorithm

$$f(S) \geq (1-1/e^{\alpha^2} - \epsilon) f(O).$$

Proof Sketch. We first show that the marginal contribution of the set added in iteration $\rho$ can be lower bounded. At iteration $\rho$, we consider a subset of the optimal solution set $T \subseteq O$. By using properties of first-order submodularity and by bounding the objective function $f$ with its submodular analog $\hat{f}$, we can show that the individual elements of $T$ survive $\rho$ iterations of the algorithm, by showing, for $o \in T$, $\mathbb{E}_{R_i \sim \mathcal{U}(X)}[f_{S \cup (\cup_{i=1}^{\rho} R_i \setminus \{a\})}(o)] \geq \alpha(1 + \frac{\epsilon}{\alpha})t/k$.

Since the elements of $T$ survive the elimination process, $T \subseteq X_{\rho}$. To complete the proof of the lemma, we can use monotonicity and lower bound $f_S(T)$ by $\frac{\epsilon^2}{2}(1-\epsilon)(f(O) - f(S))$, to show the termination of the while loop (Lemma 12 in Appendix B).

We can show that the algorithm terminates in $\log_{1+\epsilon/2}(n)$ rounds (Lemma 14 in Appendix B). Then, using the lower bound of the marginal contribution of a set at each round $f_S(X_{\rho})$ in conjunction with an inductive proof, we get the desired result. \hfill $\square$

| Algorithm 1 DASH ($N, S, r, \alpha$) |
|------------------------------------|
| 1: **Input** Remaining elements of $X$, current solution $S$, bound on number of outer-iterations $r$, first-order submodularity parameter $\alpha$ |
| 2: $X \leftarrow N$, $S \leftarrow \emptyset$ |
| 3: **for** $r$ iterations **do** |
| 4: $t := (1 - \epsilon)(f(O) - f(S))$ |
| 5: **while** $\mathbb{E}_{R \sim \mathcal{U}(X)}[f_S(R)] < \alpha^2 \frac{2}{t}$ **do** |
| 6: $X \leftarrow X \setminus \{a: \mathbb{E}_{R \sim \mathcal{U}(X)}[f_{S \cup (R \setminus \{a\})}](a) < \alpha(1 + \frac{\epsilon}{\alpha})t/k\}$ |
| 7: **end while** |
| 8: $S \leftarrow S \cup R$ where $R \sim \mathcal{U}(X)$ |
| 9: **end for** |
| 10: **return** $S$ |

place in the **while** loop and effectively filters out elements with low marginal contributions. The algorithm terminates when sufficient number of elements have been added to the set $S$ or when the value of $f(S)$ is sufficiently close to $OPT$.

The adaptive sampling algorithm for submodular functions, where $\alpha = 1$, is not guaranteed to terminate for non-submodular functions. Thus, the key adaptation for $\alpha$-differential submodular functions appears in the thresholds of the algorithm, one to filter out elements and another to lower bound the marginal contribution of the set added in each round. The additional $\alpha$ factor in the **while** condition compared to the single element marginal contribution threshold is a result of first-order submodularity properties, and guarantees termination.

The algorithm presented is an idealized version because we cannot exactly calculate expectations and the optimal solution and first-order submodularity parameter $\alpha$ is unknown. We can estimate the expectations by sampling at the cost of increased calls to the oracle and we can guess $OPT$ and $\alpha$ through parallelizing multiple guesses.

Algorithm analysis. We present proof sketches for theorems and all omitted proof details can be found in the Appendix. In our analysis, we denote the optimal solution as $OPT = f(O)$ where $O = \arg\max_{|S|\leq k} f(S)$ and $k$ is a cardinality constraint chosen by the user.

We now outline the proof sketch of the approximation guarantee of $f(S)$ using DASH. The details of the proof can be found in Appendix B.

Theorem 9. Let $f$ be a monotone, $\alpha$-first-order submodular function where $\alpha \leq 1$, then, for any $\epsilon > 0$, DASH is a $\log_{1+\epsilon/2}(n)$ adaptive algorithm that obtains the following approximation for the set $S$ that is returned by the algorithm

$$f(S) \geq (1-1/e^{\alpha^2} - \epsilon) f(O).$$

Proof Sketch. We first show that the marginal contribution of the set added in iteration $\rho$ can be lower bounded. At iteration $\rho$, we consider a subset of the optimal solution set $T \subseteq O$. By using properties of first-order submodularity and by bounding the objective function $f$ with its submodular analog $\hat{f}$, we can show that the individual elements of $T$ survive $\rho$ iterations of the algorithm, by showing, for $o \in T$, $\mathbb{E}_{R_i \sim \mathcal{U}(X)}[f_{S \cup (\cup_{i=1}^{\rho} R_i \setminus \{a\})}(o)] \geq \alpha(1 + \frac{\epsilon}{\alpha})t/k$.

Since the elements of $T$ survive the elimination process, $T \subseteq X_{\rho}$. To complete the proof of the lemma, we can use monotonicity and lower bound $f_S(T)$ by $\frac{\epsilon^2}{2}(1-\epsilon)(f(O) - f(S))$, to show the termination of the **while** loop (Lemma 12 in Appendix B).

We can show that the algorithm terminates in $\log_{1+\epsilon/2}(n)$ rounds (Lemma 14 in Appendix B). Then, using the lower bound of the marginal contribution of a set at each round $f_S(X_{\rho})$ in conjunction with an inductive proof, we get the desired result. \hfill $\square$
We have seen in Corollary 7 and Corollary 8 that the linear regression and logistic regression feature selection problems are differentially submodular. Thus, we can apply DASH to these problems to obtain the \( f(S) \geq (1 - \frac{1}{e^\alpha - 1} - \epsilon) f(O) \) guarantee from Theorem 9, where \( \alpha = \frac{ms \cdot Ms \cdot Mt}{Mr} \).

We note that for the special case where there is no diversity term, we can get a better worst case approximation guarantee of \( \gamma^2 \) for feature selection in a single round by selecting the top \( k \) elements with the highest objective value (Appendix C). In our experiments on feature selection, we compare the performance of both DASH and Top-k and in most settings, DASH performs significantly better.

5 Experiments

In order to empirically evaluate the performance of DASH, we conducted several experiments on both linear and logistic regression on several feature selection applications. While the \( 1 - 1/e^\gamma^4 \) approximation guarantee of DASH is weaker than the \( 1 - 1/e^\gamma \) of the greedy algorithm, we observe that DASH performs comparably, and in some cases outperforms greedy. Most importantly, in all experiments DASH achieves greater than two-fold speedup of parallelized greedy implementations, even for moderate values of \( k \).

Machines. All algorithms were implemented in Python 3.6. Experiments on third-party datasets were conducted on AWS EC2 C4 with 2.9 GHz Intel Xeon E5-2666 v3 Processors on 16 or 36 cores. Experiments on synthetic datasets ran on 3.1 GHz Intel Core i7 processors on 8 cores.

Datasets. We conducted experiments for both linear regression and logistic regression using the \( \ell_{\text{reg}} \) and \( \ell_{\text{class}} \) objective functions. For both settings, we use one synthetic dataset and one clinical dataset, which are described below.

- **D1: Synthetic Dataset for Regression.** We generated 500 features by sampling from a multivariate normal distribution. Each feature is normalized to have mean 0 and variance 1. Furthermore, features have a covariance of 0.4 to guarantee differential submodularity. To generate our response variable \( y \), we sample the coefficient \( \beta \sim U(-2, 2) \) for a subset of size 100 from the feature set and compute \( y \) after adding a small noise term to the coefficients. Our goal is to select features that have coefficients of large magnitude and accurately predict the response variable \( y \);

- **D2: Clinical Dataset for Regression.** We used a publicly available dataset with 53,500 samples from 74 patients with 385 features and want to select a smaller set of features that can accurately predict the location on the axial axis from an image of the brain slice;

- **D3: Synthetic Dataset for Classification.** We generated a synthetic dataset for logistic regression using a similar methodology as the synthetic regression dataset. We select a set of 50 true support features from a set of 200 and generate the coefficients using \( U(-2, 2) \). However, instead of a numerical response variable, we create a two-class classification problem by transforming the continuous \( y \) into probabilities and assigning the class label using a threshold of 0.5. The goal is to select features to perform binary classification on the synthetic dataset by using the log likelihood objective;

- **D4: Biological Dataset for Classification.** We used clinical data that contains the presence or absence of 2,500 genes in 10,633 samples from various patients. In this 5-class multi-classification problem, we want to select a small set of genes that can accurately predict the site of cancer metastasis (spleen, colon, parietal peritonium, mesenteric lymph node and intestine).
Figure 1: Linear regression feature selection results comparing DASH (blue) to baselines on synthetic (top row) and clinical datasets (bottom row). Figure 1a and 1d show the progression of value of the current solution at each round of the algorithms. Figure 1b and 1e show the $R^2$ of the final solution set of the user specified $k$ and Figure 1c and 1f show running time for algorithm completion for each $k$. Dashed line represents approximation for LASSO extrapolated across varying $\lambda$.

**Benchmarks.** We compared DASH to these algorithms:

- **Random.** In one round, this algorithm randomly selects $k$ features to create the solution set;

- **Top-$k$.** In one round, this algorithm selects the $k$ features whose individual objective value is largest;

- **SDS\textsubscript{MA}.** This uses the traditional greedy algorithm to select elements with the largest marginal contribution at each round [Krause & Cevher (2010)]. In each round, the algorithm adds one element to the solution set;

- **Parallel SDS\textsubscript{MA}.** To compare parallel runtime between DASH and greedy, we also implemented a parallelized version of the SDS\textsubscript{MA} algorithm, where queries to the oracle are parallelized across multiple cores;

- **Lasso.** This popular algorithm fits either a linear or logistic regression with an $\ell_1$ regularization term $\gamma$. It is known that for any given instance that is $k$-sparse there exists a regularizer $\gamma_k$ that can recover the $k$ sparse features. Using Lasso to find a fixed set of features is computationally intensive since in general, finding the regularizer is computationally intractable [Mairal & Yu (2012)] and even under smoothed analysis its complexity is at least linear in the dimension of the problem [Li & Singer (2018)]. We therefore used sets of values returned by Lasso for varying choices of regularizers and use these values to benchmark the objective values returned by DASH and the other benchmarks.
5.1 Experimental Setup

In our experiments, we run DASH and the baselines for different values of $k$. We performed two sets of experiments:

- **Accuracy vs. rounds.** In this set of experiments, for each data set we fixed a value of $k$ ($k = 100$ for D1, D2, D3 and $k = 200$ for D4) and ran DASH, RANDOM, Top-$k$, SDS_MA, Parallel SDS_MA to compare accuracy ($R^2$ in the case of linear regression and classification rate in the case of logistic regression) of the solution as a function of the number of parallel rounds. The results are plotted in Figures 1a, 1d and Figures 2a, 2d.

- **Accuracy and time vs. features.** In these experiments, we ran the same benchmarks for varying values of $k$ (in D1, D2, D3 the maximum is $k = 100$ and in D4 the maximum is $k = 200$) and measure both accuracy (Figures 1b, 1e, 2b, 2e) and time (Figures 1c, 1f, 2c, 2f). When measuring accuracy, we also ran LASSO by manually varying the regularization parameter $\lambda$ to select approximately $k$ features.

Throughout our experiments we observed that in practice, the sample complexity needed to effectively run DASH can be much lower than the sample complexity needed for the approximation guarantee, making each round of DASH computationally efficient. In our experiments we implemented DASH by using only 5 samples at every round.

5.2 Experimental Results

**General performance.** We first analyze the performance of DASH compared to several baselines. In both linear and logistic regression feature selection, we can see in Figures 1a, 1d and Figures 2a, 2d...
that the final accuracy of Dash ($R^2$ in the case of linear regression and classification rate in the case of logistic regression) is comparable to SDS$_{MA}$, outperforms and Random, and is able to achieve the solution much fewer rounds. In Figures 1b, 1c and Figures 2b, 2c, we show Dash can be very practical in finding a comparable solution set to SDS$_{MA}$ especially for larger values of $k$. In the synthetic linear regression experiment, Dash significantly outperforms Lasso and is comparable to Lasso performance in other experiments. While Dash outperforms the simple baseline of Random, we note that the performance of Random varies widely depending on properties of the feature set. In cases where a small number of features can give high accuracy and increasing $k$ improves the accuracy marginally, Random can perform well by randomly selecting well-performing features when $k$ is large (Figure 1c). However, in more interesting cases, where the accuracy does not immediately saturate, both Dash and SDS$_{MA}$ significantly outperform Random (Figure 1b).

We can also see in Figures 1d, 1f and Figures 2d, 2f that Dash is computationally efficient compared to the other baselines for large values of $k$. In some cases, for smaller values of $k$, SDS$_{MA}$ is faster (Figure 2c). This is mainly due to the sampling done by Dash to estimate the marginals, which can be more computationally intensive than adding a few elements to the solution set. However, in most experiments, Dash terminates more quickly than the baselines for even small values of $k$. For larger values, Dash completes the solution set in under half the time of the fastest baseline.

**Effect of oracle queries.** Across our experiments, the running times for oracle queries vary largely. In the case where the calculation of the marginal contribution of an element is computationally cheap, the parallelization of SDS$_{MA}$ has a much longer running time than its sequential analog. This is due to the cost of merging the parallelized results outweighing the time saved in parallelizing queries (Figures 1c, 2c). However, in the logistic regression gene selection experiment, calculating the marginal contribution of an element to the solution set can span more than 1 minute. This is due to both the large dataset size and the computational intensity of the gradient descent operation performed to optimize the logistic regression solution. In this experiment, we can see that the sequential version of SDS$_{MA}$ is computationally difficult and even selecting 100 elements would take several days for the algorithm to terminate (Figure 2f). Parallelization of SDS$_{MA}$ drastically improves the algorithm running time, but Dash is still much faster and is able to find a comparable solution set in under half the time of the parallelized SDS$_{MA}$.

In both cases where oracle queries are cheap and where queries are computationally intensive, Dash terminates much more quickly than the sequential and parallelized version of SDS$_{MA}$ for larger values of $k$. This can be seen in Figures 1c and 2c, where calculation of marginal contribution on synthetic data is fast and in Figures 1f and 2f, where oracle queries on larger datasets are much slower. This shows the incredible potential of using Dash across a wide array of different applications to drastically cut down on computation time in selecting a large number elements across different objective functions. Given access to more processors, we expect even a larger increase in speed up between Dash and other baselines.
A Helpful Lemma

The following lemma will be helpful in our algorithm analysis.

**Lemma 10.** Suppose that for $i \leq n$, $\alpha, \beta \in (0, 1)$, we have the following the condition

$$f(S_i) - f(S_{i-1}) \geq \frac{\alpha}{n} (1 - \epsilon)(f(O) - f(S)) - \frac{\beta}{n}$$

then, for $i = n$, we have

$$f(S) \geq (1 - 1/e^\alpha - \alpha \epsilon)(f(O) - \frac{\beta}{\alpha(1 - \epsilon)})$$

**Proof.** By subtracting $f(O)$ from the given condition, we get

$$f(S) - f(O) \geq (1 - \alpha \frac{1 - \epsilon}{n}) (f(S_{i-1} - f(O)) - \frac{\beta}{n}$$

By induction and rearranging, we have

$$f(S_{i}) - f(O) \geq (1 - \alpha \frac{1 - \epsilon}{n}) f(O) - \frac{1 - (1 - \alpha \frac{1 - \epsilon}{n})^i}{1 - (1 - \alpha \frac{1 - \epsilon}{n}^i) n} \beta$$

By setting $i = n$ and rearranging, we have

$$f(S) \geq (1 - (1 - \alpha \frac{1 - \epsilon}{n})^{i})(f(O) - \frac{\beta}{\alpha(1 - \epsilon)})$$

By setting $i = n$ and rearranging, we have

$$f(S) \geq (1 - e^{-\alpha(1 - \epsilon)})(f(O) - \frac{\beta}{\alpha(1 - \epsilon)})$$

$$f(S) \geq (1 - 1/e^\alpha - \alpha \epsilon)(f(O) - \frac{\beta}{\alpha(1 - \epsilon)})$$

□

B Proof of Theorem 9 for DASH

We first proof several lemmas before proving the theorem.

B.1 Proofs of Lemmas Leading to Theorem 9

We first begin by proving the following lemma to bound the marginal contribution of the optimal set to the solution set.

**Lemma 11.** Let $R_i \sim \mathcal{U}(X)$ be the random set at iteration $i$ of DASH($N, S, r, \delta$). For all $S \subseteq N$ and $r, \rho > 0$, if the algorithm has not terminated after $\rho$ iterations, then

$$\mathbb{E}_{R_i}[f_{S \cup (\bigcup_{i=1}^{\rho} R_i)}(O)] \geq (1 - \frac{\rho}{r})(f(O) - f(S))$$

(3)

Using Lemma 11, we can complete the proof for Lemma 12.
Proof.

\[ \mathbb{E}_{R_t}[f_{S \cup (\cup_{i=1}^r R_i)}(O)] = \mathbb{E}_{R_t}[f_S(O \cup (\cup_{i=1}^r R_i))] - \mathbb{E}_{R_t}[f_{S}(\cup_{i=1}^r R_i)] \]

\[ \geq f(O) - f(S) - \frac{1}{\alpha} \sum_{i=1}^r \mathbb{E}_{R_t}[f_{S}(R_i)] \]

\[ \geq f(O) - f(S) - \frac{1}{\alpha} \rho (1 - \epsilon)(f(O) - f(S)) \]

\[ \geq (1 - \frac{\rho}{r})(f(O) - f(S)) \]

where the first inequality follows from monotonicity and approximate submodularity and the second inequality follows from the while loop in DASH. \(\square\)

Lemma 12. For each iteration of DASH and for all \(S \subseteq N\) and \(\epsilon > 0\), if \(r \geq 20\rho\epsilon^{-1}\) then the marginal contribution of the elements of \(X_\rho\) that survive \(\rho\) iterations satisfy

\[ f_S(X_\rho) \geq \frac{\alpha^2}{r}(1 - \epsilon)(f(O) - f(S)) \]

We now can prove Lemma 12. To prevent the propagation of the \(\alpha\) factor, we bound \(f\) by its submodular analog \(F\) for our analysis, excluding the queries made by the algorithm.

Proof. We want to show a bound on the marginal contribution of the elements that survive \(\rho\) iterations of the algorithm. Let \(O = \{o_1, \ldots, o_k\}\) be the optimal solutions of \(f\) and \(O_t = \{o_1, \ldots, o_t\}\) be a subset of the optimal elements in some arbitrary order. However, we define the thresholds in terms of submodular function \(h^{(2)}\). Then we define

\[ \Delta_t := \mathbb{E}_{R_t}[h^{(2)}_{S \cup O_t \cup (\cup_{i=1}^r R_i \setminus \{o_t\})}(o_t)] \]

\[ \Delta := \frac{1}{k}\mathbb{E}_{R_t}[h^{(2)}_{S \cup (\cup_{i=1}^r R_i)}(O)] \]

Let \(r \geq \frac{20\rho}{\epsilon}\). Let \(T\) be the set of elements surviving \(\rho\) iterations in \(O\), \(T \subseteq X_\rho\), \(T \subseteq O\), where

\[ T = \{o_t | \Delta_t \geq (1 - \frac{\epsilon}{4})\Delta\} \]

For \(o_t \in T\) and using submodularity of \(F\),

\[ \mathbb{E}_{R_t}[f_{S \cup (\cup_{i=1}^r R_i \setminus \{o_t\})}(o_t)] \geq \mathbb{E}_{R_t}[h^{(1)}_{S \cup (\cup_{i=1}^r R_i \setminus \{o_t\})}(o_t)] \]

\[ \geq \mathbb{E}_{R_t}[h^{(2)}_{S \cup (\cup_{i=1}^r R_i \setminus \{o_t\})}(o_t)] \]

\[ \geq \alpha \mathbb{E}_{R_t}[h^{(2)}_{S \cup O_t \cup (\cup_{i=1}^r R_i \setminus \{o_t\})}(o_t)] \]

\[ \geq \alpha (1 - \frac{\epsilon}{4}) \Delta \quad \text{Definition} \quad \text{10} \]

\[ \geq \frac{\alpha}{k}(1 - \frac{\epsilon}{4})\mathbb{E}_{R_t}[f_{S \cup (\cup_{i=1}^r R_i)}(O)] \]

\[ \geq \frac{\alpha}{k}(1 - \frac{\epsilon}{4})(1 - \frac{\rho}{4})(f(O) - f(S)) \quad \text{Lemma} \quad \text{11} \]

\[ \geq \frac{\alpha}{k}(1 + \frac{\epsilon}{2})(1 - \epsilon)(f(O) - f(S)) \quad r \geq \frac{20\rho}{\epsilon} \]

which shows that elements in \(T\) survive the elimination process (as they are not filtered out from set \(X\) in the algorithm definition).
Now we complete the proof by showing $f_S(T)$ is bounded by $\frac{\alpha^2 r}{2}(1 - \epsilon)(f(O) - f(S))$ which effectively terminates the algorithm.

Similar to the result in Lemma 2 of Balkanski et al. 2019, from properties of submodularity of $F$, we have

$$\sum_{o_t \in T} \Delta_t \geq k \frac{\epsilon}{4}\Delta$$

(8)

By submodularity,

$$f_S(T) \geq h_S^{(1)}(T)$$
$$\geq \alpha h_S^{(2)}(T)$$
$$\geq \alpha \sum_{o_t \in T} h_{S \cup O o_{i-1}}^{(2)}(o_t)$$
$$\geq \alpha \sum_{o_t \in T} \mathbb{E}[h_{S \cup O o_{i-1} \cup \bigcup_{i=1}^\rho R_i \setminus \{o_t\}}^{(2)}(o_t)]$$
$$= \alpha \sum_{o_t \in T} \Delta_t$$

(1)

$$\geq (1 - \delta)k \frac{\epsilon}{4}\Delta$$

from (8)

$$\geq \alpha \frac{\epsilon}{4}\mathbb{E}_{R_i}[f_{S \cup \bigcup_{i=1}^\rho R_i}(O)]$$

where the second and third inequalities follow from properties of submodularity. Finally,

$$f_S(X_i) \geq f_S(T)$$

monotonicity

$$= \alpha \frac{\epsilon}{4}(1 - \frac{\rho}{r})(f(O) - f(S))$$

Definition (5)

$$\geq \frac{\alpha^2 r}{2}(1 - \epsilon)(f(O) - f(S))$$

$$r \geq \frac{20\rho}{\epsilon}$$

We now present a lemma for the termination of the algorithm in $O(\log n)$ rounds.

**Lemma 13.** Let $X_i$ and $X_{i+1}$ be the sets of surviving elements at the start and end of iteration $i$ of the while loop of DASH. For all $S \subseteq N$ and $r, i, \epsilon > 0$, if the algorithm does not terminate at iteration $i$, then

$$|X_{i+1}| < \frac{|X_i|}{1 + \epsilon/2}$$

Proof. We consider $R_i \cap X_{i+1}$ to bound the number of surviving elements in $X_{i+1}$. To prevent the
propagation of the $\alpha$ factor, we can bound the function $f$ by its submodular analog $F$. 

\[
\mathbb{E}[f_S(R_i \cap X_{i+1})] \geq \mathbb{E}[h_S^{(1)}(R_i \cap X_{i+1})] \\
\geq \alpha \mathbb{E}\left[ \sum_{a \in R_i \cap X_{i+1}} h_S^{(2)}_{R_i \cap X_{i+1}}(a) \right] \\
\geq \alpha \mathbb{E}\left[ \sum_{a \in X_{i+1}} \mathbb{1}_{a \in R_i} \cdot h_S^{(2)}_{R_i \cap X_{i+1}}(a) \right] \\
= \alpha \sum_{a \in X_{i+1}} \mathbb{E}[\mathbb{1}_{a \in R_i} \cdot h_S^{(2)}_{R_i \cap X_{i+1}}(a)] \\
= \alpha \sum_{a \in X_{i+1}} \mathbb{P}[a \in R_i] \cdot \mathbb{E}[h_S^{(2)}_{R_i \cap X_{i+1}}(a) \mid a \in R_i] \\
\geq \alpha \sum_{a \in X_{i+1}} \mathbb{P}[a \in R_i] \cdot \mathbb{E}[f_{S \cup (R_i \setminus a)}(a)] \\
\geq \alpha \sum_{a \in X_{i+1}} \mathbb{P}[a \in R_i] \cdot \frac{\alpha}{k} (1 + \epsilon/2)(1 - \epsilon)(f(O) - f(S)) \\
= \alpha^2 \frac{|X_{i+1}|}{|X_i|} \cdot \frac{1}{k} (1 + \epsilon/2)(1 - \epsilon)(f(O) - f(S)) \\
= \alpha^2 \frac{|X_{i+1}|}{r|X_i|} (1 + \epsilon/2)(1 - \epsilon)(f(O) - f(S)) 
\] 

(9)

where the first and fourth inequalities are due to approximate submodularity.

Since the elements are discarded from the while loop of the algorithm, we can bound $\mathbb{E}[f_S(R_i \cap X_{i+1})]$ using monotonicity so that

\[
\mathbb{E}[f_S(R_i \cap X_{i+1})] \leq \mathbb{E}[f_S(X_i)] < \alpha^2(1 - \epsilon)(f(O) - f(S))/r. 
\] 

(10)

Combining (9) and (10) yields

\[
(1 - \epsilon)(f(O) - f(S))/r > \frac{1}{\alpha^2} \mathbb{E}[f_S(R_i)] \geq \frac{|X_{i+1}|}{r|X_i|} (1 + \epsilon/2)(1 - \epsilon)(f(O) - f(S))
\]

We can conclude that $|X_{i+1}| < |X_i|/(1 + \epsilon/2)$ by simplifying that above inequality. \hfill \Box

Lemma 14. For all $S \subseteq N$, if $r \geq 20\epsilon^{-1}\log(1 + \epsilon/2)(n)$ then DASH($N, S, r, \delta$) terminates after at most $O(\log n)$ rounds.

Proof. If the algorithm has not terminated after $\log_{1+\epsilon/2}(n)$ rounds, then, by Lemma 14 at most $k/r$ elements survived $\rho = \log_{1+\epsilon/2}(n)$ iterations. By Lemma 12, the set of surviving elements satisfies $f_S(X_{\rho}) \geq \frac{\alpha^2}{r}(1 - \epsilon)(f(O) - f(S))$. Since there are only $k/r$ surviving elements, $R = X_{\rho}$ and $f_S(R) = f_S(X_{\rho}) \geq \frac{\alpha^2}{r}(1 - \epsilon)(f(O) - f(S))$ \hfill \Box
C Observation on Worst Case Bound

In the special case of feature selection where there is no diversity term, we can get an improved approximation guarantee of $\gamma^2$, where $\gamma = m/M$.

We can bound the objective function $f(S)$ by the modular function $\sum_{a \in S} f(a)$ so that $f(S) \leq \frac{m^2}{M^2} \sum_{a \in S} f(a)$. Then, for the Top-k algorithm, where we select the best $k$ elements by their value $f(a)$, we get the following approximation guarantee.

\[
f(S) \geq \frac{m}{M} \sum_{a \in S} f(a) \geq \frac{m}{M} \sum_{o \in O} f(o) \geq (\frac{m}{M})^2 f(O) = \gamma^2 f(O)
\]

where the first and last inequalities come from differential submodularity properties and the second inequality follows from selecting the best $k$ elements.

D Extension to $R^2$ Objective

In this section, we introduce a goodness of fit objective function and show that it is approximately submodular on the marginals.

D.1 Notation

For the predictor matrix $X \in \mathbb{R}^{d \times n}$ and $A \subseteq [n]$, we denote submatrices by column indices by $X_A$. $\text{Var}(x_i)$ and $\text{Cov}(x_i, x_j)$ denotes the variance and covariance of random variables and $b$ denotes the covariances between the response variable $y$ and $X$, where $b_i = \text{Cov}(y, x_i)$. For the covariance matrix $C$, we can denote the covariance submatrix of row and column set $S$ as $C_S$ and $b_S$ as the vector with only entries $b_i$ for $i \in S$. We use $\lambda_{\min}(C)$ and $\lambda_{\max}(C)$ to denote the smallest and largest eigenvalue of the covariance matrix and we denote the residuals as $\text{Res}(y, X_S) = y - \sum_{i \in S} \beta_i x_i$.

D.2 Goodness of Fit

We introduce the formal definition of the $R^2$ objective function, which is widely used to measure goodness of fit in statistical applications.

**Definition 15.** Johnson & Wichern (2004) Let $S \subseteq N$ be a set of variables $X_S$ and a linear predictor $\hat{y} = \sum_{i \in S} \beta_i X_i$ of $y$, the squared multiple correlation is defined as

\[
R^2(S) = \frac{\text{Var}(y) - \mathbb{E}[(y - \hat{y})^2]}{\text{Var}(y)}
\]

where $\beta_i = (C_S)^{-1} b_S$ for $i \in S$.

We assume that the predictor random variables are normalized to have mean 0 and variance 1, so we can simplify the definition above to $R^2(S) = 1 - \mathbb{E}[(y - \hat{y})^2]$. Thus, we can rephrase the definition as $R^2(S) = b_S^T (C_S)^{-1} b_S$. Johnson & Wichern (2004).

D.3 Feature Selection

Objective. For a response variable $y \in \mathbb{R}^d$, the objective is the maximization of the $R^2$ goodness of fit for $y$ given the feature set $S$:

\[
f(S) = R^2(S) = b_S^T (C_S)^{-1} b_S
\]

where $b$ corresponds to the covariance between $y$ and the predictors.
To define the marginal contribution of a set $A$ to the set $S$ of the $R^2$ objective function, we can write $R^2_S(A) = (b^S_A)^T(C^S_A)^{-1}b^S_A$, where $b^S$ is the covariance vector corresponding to the residuals of $i \in A$ to $S$, i.e. $\{\text{Res}(x_1, X_S), \text{Res}(x_2, X_S), \ldots, \text{Res}(x_n, X_S)\}$ and $C^S_A$ is the covariance matrix corresponding to the residuals. The marginal contribution of an element is $R^2_S(a) = (b^S_A)^Tb^S_A$.

**Lemma 16.** The feature selection objective defined by $f(S) = R^2(S)$ is an approximately submodular function such that for all $S, A \subseteq N$,

$$\frac{1}{\lambda_{\max}(C^S_A)} \hat{f}_S(A) \leq f_S(A) \leq \frac{1}{\lambda_{\min}(C^S_A)} \hat{f}_S(A),$$

where $\hat{f}_S(A) = \sum_{a \in A} f_S(a)$.

**Proof.** The marginal contribution of set $A$ to set $S$ of the feature selection objective function is defined as $R^2_S(A) = (b^S_A)^T(C^S_A)^{-1}b^S_A$. Because we know that $(C^S_A)^{-1}$ is a symmetric matrix, we can upper and lower bound the marginals using the eigenvalues of $(C^S_A)^{-1}$.

$$\frac{1}{\lambda_{\max}(C^S_A)} \sum_{a \in A} f_S(a) = \frac{1}{\lambda_{\max}(C^S_A)} (b^S_A)^Tb^S_A$$

$$= \lambda_{\min}((C^S_A)^{-1})(b^S_A)^Tb^S_A$$

$$\leq (b^S_A)^T(C^S_A)^{-1}b^S_A$$

$$= f_S(A)$$

$$\leq \lambda_{\max}((C^S_A)^{-1})(b^S_A)^Tb^S_A$$

$$\leq \frac{1}{\lambda_{\min}(C^S_A)} (b^S_A)^Tb^S_A$$

$$= \frac{1}{\lambda_{\min}(C^S_A)} \sum_{a \in A} f_S(a)$$

By letting $\hat{f}_S(A) = \sum_{a \in A} f_S(a)$, we complete the proof and show that the marginals can be bounded by modular functions.

**Remark 17.** This is a more general form of Lemma 3.3 from Das & Kempe 2011. Our result is on the marginals of $f$ and reduces to their result for $S = \emptyset$.

**Remark 18.** If $\lambda_{\min} = \lambda_{\max}$, the matrix has one eigenvalue of multiplicity greater than 1 and the covariance matrix is a multiple of the identity matrix. This implies the set of predictors is uncorrelated and that the objective function for feature selection is submodular. Otherwise, we have $\alpha = \frac{\lambda_{\min}}{\lambda_{\max}} < 1$. 
References

Balkanski, E. and Singer, Y. The adaptive complexity of maximizing a submodular function. *STOC*, 2018a.

Balkanski, E. and Singer, Y. Approximation guarantees for adaptive sampling. In Dy, J. and Krause, A. (eds.), *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pp. 384–393, Stockholmsmassan, Stockholm Sweden, 10–15 Jul 2018b. PMLR.

Balkanski, E., Breuer, A., and Singer, Y. Non-monotone submodular maximization in exponentially fewer iterations. In *Advances in Neural Information Processing Systems*, pp. 2359–2370, 2018a.

Balkanski, E., Rubinstein, A., and Singer, Y. An optimal approximation for submodular maximization under a matroid constraint in the adaptive complexity model. arXiv, 2018b.

Balkanski, E., Rubinstein, A., and Singer, Y. An exponential speedup in parallel running time for submodular maximization without loss in approximation. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA) 2019*, 2019.

Chekuri, C. and Quanrud, K. Parallelizing greedy for submodular set function maximization in matroids and beyond. arXiv preprint arXiv:1811.12568, 2018.

Chekuri, C. and Quanrud, K. Submodular function maximization in parallel via the multilinear relaxation. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pp. 303–322. SIAM, 2019.

Chen, L., Feldman, M., and Karbasi, A. Unconstrained submodular maximization with constant adaptive complexity. arXiv preprint arXiv:1811.06603, 2018.

Das, A. and Kempe, D. Submodular meets spectral: greedy algorithms for subset selection, sparse approximation and dictionary selection. In *Proceedings of the 28th International Conference on International Conference on Machine Learning*, pp. 1057–1064. Omnipress, 2011.

Das, A., Dasgupta, A., and Kumar, R. Selecting diverse features via spectral regularization. In *Advances in neural information processing systems*, pp. 1583–1591, 2012.

Elenberg, E. R., Khanna, R., Dimakis, A. G., Negahban, S., et al. Restricted strong convexity implies weak submodularity. *The Annals of Statistics*, 46(6B):3539–3568, 2018.

Ene, A. and Nguyen, H. L. Submodular maximization with nearly-optimal approximation and adaptivity in nearly-linear time. *SODA*, 2019.

Ene, A., Nguyen, H. L., and Vladu, A. Submodular maximization with packing constraints in parallel. arXiv preprint arXiv:1808.09987, 2018a.

Ene, A., Nguyen, H. L., and Vladu, A. A parallel double greedy algorithm for submodular maximization. arXiv preprint arXiv:1812.01594, 2018b.

Fahrbach, M., Mirrokni, V., and Zadimoghaddam, M. Submodular maximization with optimal approximation, adaptivity and query complexity. *SODA*, 2019.

Johnson, R. A. and Wichern, D. W. Multivariate analysis. *Encyclopedia of Statistical Sciences*, 8, 2004.

Krause, A. and Cevher, V. Submodular dictionary selection for sparse representation. In *Proceedings of the 27th International Conference on Machine Learning*, pp. 567–574, 2010.
Li, Y. and Singer, Y. The well-tempered lasso. In *Proceedings of the 35th International Conference on Machine Learning*, 2018.

Mairal, J. and Yu, B. Complexity analysis of the lasso regularization path. In *Proceedings of the 29th International Conference on Machine Learning*, 2012.