Imperfections and Corrections

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Abstract
After a review of linear imperfections and their causes, we discuss how to model them, the diagnostic equipment needed to monitor them, and the correction algorithms to fix the problem they cause. We first address linear systems—beam lines or linear accelerators. In a later part we cover circular systems, such as storage rings.

1 Introduction
When starting up a newly-built accelerator, we often find the beam not quite where we designed it to be and its beam size is not quite what the computer model had predicted. The reason for these discrepancies are, of course, additional magnetic fields that affect the charged particles, which constitute the beam. The cause of these undesired fields are often misaligned magnets or stray fields from adjacent components. Other reasons are intentionally installed components that were not accounted for during the design phase and are not in the computer model, for example, undulators in synchrotron light sources.

In the first part of these lectures we characterize the imperfections and discuss methods of how to include them in computer models. In the second part we discuss how the imperfections show up in linear accelerators, how to diagnose what’s wrong, and then how to correct them. In the third part we do likewise for circular, or periodic, systems, such as storage rings.

As a prerequisite the reader should be familiar with the material from [1] and [2].

2 Imperfections
We will predominantly deal with linear imperfections; they affect the linear optics of the accelerator. The types of fields that cause these imperfections are schematically shown in Figure 1, where the beam is depicted as a shaded blue circle. The (transverse) field can be constant across the beam, as shown on the left-hand side, which causes all particles to receive the same transverse change of angle—a kick—\(\Delta x'\) or \(\Delta y'\). This type of field resembles that of a dipole corrector. A second type of field can vary linearly across the beam, such that the particles receive a kick that is proportional to their transverse position, as shown on the middle in Figure 1. This type of field resembles that of a quadrupole. A third option is shown on the right-hand side and resembles that of a quadrupole that is rotated by \(45^\circ\)—a skew-quadrupole. Note

![Fig. 1: Types of fields (red) that cause imperfections; either constant across the beam (left), with a gradient (middle) or with a skew-gradient (right).](https://cas.web.cern.ch/previous-schools)
that these fields correspond to the lowest-order terms of a multipole expansion. Apart from the transverse fields, shown in Figure 1, can solenoids, which may be part of high-energy physics detectors or electron coolers, cause longitudinal fields, which we, however, do not discuss further.

2.1 Alignment

Misaligned magnets are the prime sources of imperfections. The problems they cause are mitigated by placing the magnets on alignment tables with attached pods that are aligned to the magnetic centers of the magnets. Surveyors then use triangulation with respect to reference points in the tunnel to correct the positions of the magnets. The achievable tolerances are on the order of 0.2 mm or, with additional effort, somewhat better. Significantly better alignment, down to the resolution of the beam-monitoring system, requires beam-based methods.

Apart from transversely displacing magnets, the magnets can be tilted in the $x$-$s$–plane, where the entrance of a magnet is displaced towards one side and the exit towards the other. And yet another misalignment is caused by a roll angle around the direction of propagation $s$.

In the following section we discuss how to model these imperfections, which is necessary in order to understand them and develop correction methods in later sections.

2.2 Modeling misalignment

Since we cannot place the magnets with infinite precision, we need to be able to simulate their misalignment in computer codes. Let us consider one transverse direction $x$ only. A particle with initial coordinates $\vec{x}_i = (x_i, x'_i)$, passing an element, characterized by transfer matrix $\hat{R}$ that is displaced by $d_x$, is modeled by first displacing the particle by $d_x$, then passing through the element, and finally adding the displacement $-d_x$ to the particle coordinates. This is illustrated in the following sketch and equation.

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = \begin{pmatrix} -d_x \\ 0 \end{pmatrix} + \hat{R} \begin{pmatrix} d_x \\ 0 \end{pmatrix} + \begin{pmatrix} x_i \\ x'_i \end{pmatrix}$$

The algebraic manipulations show that the final coordinates $\vec{x}_f = (x_f, x'_f)$ are given by $\vec{x}_f = \vec{q} + \hat{R} \vec{x}_i$, which equals the un-misaligned propagation $\hat{R} \vec{x}_i$ and an additional term $\vec{q}$, which describes an additional kick. For a thin-lens quadrupole with focal length $f$ it is easy to show that $\vec{q} = (0, -d_x/f)$. We also note that the focusing of a quadrupole is not affected by the misalignment, only the steering is, because $\vec{q}$ does not depend on $\vec{x}_i$ and all particles receive the same kick.

The effect on the beam of a magnet with length $L$ and transfer matrix $\hat{R}$, tilted by $d'_x$ in the $x$–$s$–plane, is described by first adding $(-d_x L/2, d'_x)$ to $\vec{x}_i$ before passing through the magnet and finally adding $(-d_x L/2, -d'_x)$ to the particle coordinates. Performing these step algebraically, which is left as an exercise, shows that the result is again $\vec{x}_f = \vec{q} + \hat{R} \vec{x}_i$, where $\vec{q}$ depends on the transfer matrix $\hat{R}$, the length $L$, and the misalignment angle $d'_x$.

Magnets that are rolled around the $s$–direction are modeled with the help of a coordinate rotation in the $x$ -- $y$–plane, as shown in the following figure and equation.
We denote the matrix in the previous equation by \( R(\phi) \). A rolled magnet can then be described by first rotating the coordinate system with \( R(\phi) \), applying the transfer matrix of the magnet \( M \), and then rotating the coordinate system back with \( R(-\phi) \). Thus, the transfer matrix of a rolled element is given by \( R(-\phi)MR(\phi) \).

2.3 Focusing errors

A further class of imperfections are caused by incorrectly powered quadrupoles. For example, a focusing quadrupole that excited too strongly, will focus the particles to a point closer to the quadrupole. This will cause the beam (or sigma) matrix to differ from its design values. Consequently, the beta functions will be “wrong” and, in a ring, the tunes will differ from their design values. Modeling incorrectly powered quadrupoles is accomplished by simply changing their gradient, usually given as \( k_1 \), in the optics codes.

Undulators and wigglers have a vertical magnetic field \( B_y \) component that varies along \( s \), the direction of propagation. Maxwell’s equations therefore cause the longitudinal component \( B_s \) to vary vertically, because of \( \frac{\partial B_y}{\partial s} = \frac{\partial B_s}{\partial y} \). The horizontally undulating particles therefore cross a non-zero longitudinal field and experience a vertical force, which can be shown to be focusing. This is a weak effect but can in some circumstances affect the orbit and the focusing of the particles, especially when changing the field by adjusting the gap of the undulator.

2.4 Dispersion and Chromaticity

Yet another class of imperfections is caused by the unavoidable spread of relative momenta \( \delta = \Delta p/p \), because the deflection of the particles is proportional to \( B/p \), thus inversely proportional to the momentum \( p \). Therefore, every dipole magnet acts like a spectrometer and separates the particles dependent on their momentum. The position of particles is therefore to first order proportional to their relative momentum deviation \( \delta = \Delta p/p \) and given by \( x = D(s)\Delta p/p \) with the dispersion function \( D(s) \). Note that the dispersion varies along the accelerator and depends on the position \( s \). A finite value of the dispersion increases the beam size. In planar accelerators, this effect only affects the horizontal beam size, but finite alignment tolerances can also cause vertical dispersion to appear.

Not only the kick that the particles receive depends on their momentum, also their focusing is affected. This momentum-dependent focusing is called chromaticity and affects beam matrix and beta functions. In rings, also the tunes become momentum-dependent and instead of a single value for the entire beam, chromaticity causes a spread of tune values.

We can measure dispersion and chromaticity by changing the beam energy and observing the ensuing change in the beam position (dispersion: \( D = \Delta x/(\Delta p/p) \)). In rings, we can change the frequency \( f_{RF} \) system, which causes the beam to adjust its energy to remain synchronous with the RF system and in a linear accelerator we can change the amplitude or phase of part of the accelerator. Optionally, we can scale all magnets by the same factor, which is equivalent to changing the beam energy, because all observable effects are proportional to \( B/p \).
2.5 Multipoles and feed-down

Sextupoles and other higher-order multipoles are included in accelerator lattices in order to correct undesirable aberration. This works nicely, if they are aligned properly. It turns out that misaligned multipoles cause additional multipoles to appear. To quantify this effect, we remember that transverse magnetic fields are described by the multipole expansion

\[ B_y + iB_x = B_0 \sum_{m=1}^{\infty} (b_m + ia_m) \left( \frac{x + iy}{R_0} \right)^{m-1}, \]

where \( B_0 \) and \( R_0 \) are reference values and \( b_m \) and \( a_m \) characterize the magnitude of the multipole component. The \( b_m \) describe magnets, which only have a vertical field component \( B_y \) along the \( x \)-axis. They are called upright multipoles, whereas the \( a_m \) describe magnets which are rolled by \( \phi = \pi/m \) and are called skew multipoles.

Assuming that the magnets are short, such that they only affect the angles \( x' \) and \( y' \) of the particles, the kicks can be written as

\[ \Delta x' - i\Delta y' = \frac{(B_y + iB_x)L}{B\rho} = -\sum_{n=0}^{\infty} \frac{k_n L}{n!} (x + iy)^n. \]

Here \( L \) is the length of the magnet. It is easy to show that \( k_n L = L(\partial^n B_y/\partial x^n)_{y=0}/B\rho \) for an upright magnet, where we use \( B\rho = p/e \) to express the momentum \( p \).

For a magnet with a single multipole component the kick from Equation 2 simplifies to \( \Delta x' - i\Delta y' = (k_n L/n!)(x + iy)^n \) and, if the magnet is horizontally displaced by \( d_x \), the kick becomes

\[ \Delta x' - i\Delta y' = -\frac{k_n L}{n!}(x + d_x + iy)^n \]

\[ = -\frac{k_n L}{n!}(x + iy)^n - \frac{k_n L}{n!} \sum_{k=0}^{n-1} \left( \frac{n}{k} \right) d_x^{n-k} (x + iy)^k, \]

where the second equality derives from a binomial expansion of \( (x + d_x + iy)^n \). The first term shows that the displaced multipole still does what it was supposed to do. But additionally all lower-order multipoles \( k = 0, \ldots, n - 1 \) appear. Their magnitude can be read off from Equation 3.

These lower-order multipoles have an intuitive interpretation, which becomes apparent when considering a horizontally displaced sextupole, whose kick is given by

\[ \Delta x' - i\Delta y' = -\frac{k_2 L}{2} \left[ (x + iy)^2 + 2d_x(x + iy) + d_y^2 \right]. \]

The terms in Equation 4 are illustrated in Figure 2, which shows the absolute value of the horizontal kick \( \Delta x' \) from an upright sextupole as a function of the horizontal position \( x \) as the black parabola. The displaced beam is shown as the red Gaussian with the red dot denoting its center. The last term in Equation 4, proportional to \( d_y^2 \), describes a constant kick that affects all particles equally. It is illustrated in Figure 2 by the vertical dot-dashed line under the bunch center. The term in the middle, proportional to \( 2d_x(x + iy) \), describes the slope of the parabola and illustrates that the left-hand part of the bunch experiences a smaller kick than the right-hand part. This is just what quadrupoles do. The first term, proportional to \( (x + iy)^2 \), describes the curvature of the parabola at the position of the red dot turns out to be equal to the one in the center of the parabola.

A vertically misaligned sextupole causes the particles to be kicked by

\[ \Delta x' - i\Delta y' = -\frac{k_2 L}{2} \left[ (x + iy)^2 + 2id_y(x + iy) - d_y^2 \right]. \]
The term, proportional to \((x + iy)^2\), describes the sextupolar kick and the constant term \(-d_y^2\) describes a constant kick, as before. The linear term, proportional to \(d_y(x + iy)\) is now multiplied by an imaginary unit, which therefore describes a skew-quadrupolar field. This, in turn, couples the transverse planes, because, for example, a horizontal beam position \(x\) gives rise to a vertical kick \(\Delta y'\). In synchrotron light sources the vertical offset of the often very strong sextupoles is one of the main causes of vertical dispersion, which spoils the vertical emittance.

So far, we discussed the imperfections of the magnets. In the next section we briefly touch upon the imperfections of the diagnostic equipment that we will use to identify and correct the imperfections.

2.6 Imperfections of diagnostic components

Beam position monitors (BPM) are based on electronically comparing signals from four electrodes exposed to fields that the beam generates. Tolerances in the electronics or slight differences of their mechanical assembly can result in non-zero BPM readings, despite the beam being physically centered in the BPM. Tracking down these BPM offsets is often tricky, unless the BPMs are rigidly mounted next to a quadrupole. Figure 3 illustrates the idea. We slightly perturb the quadrupole with an additional

![Fig. 2: The black parabola denotes the kick from a sextupole that particles in a displaced beam, shown as the red distribution, receive. Locally, a constant offset, shown by the blue dashed vertical line, and a slope, shown by the dot-dashed blue line, are added.](image1)

![Fig. 3: We can determine the distance between the center of the quadrupole and BPM1 by K-modulating the quadrupole gradient and scanning the beam with a local bump across the quadrupole until the signal at the modulation frequency vanishes on BPM2.](image2)
sinusoidal current $\Delta I \cos(\omega_{\text{mod}} t)$ and use a local corrector bump (more on them later) to change the position of the beam in the quadrupole. No signal with the modulating frequency $\omega_{\text{mod}}$ will show up on the second BPM2, once the beam is centered in the quadrupole. In this state, the reading of BPM1 reveals its offset with respect to the center of the quadrupole.

Screens, inserted in the beam’s path and observed by a camera, are used to determine the beam transverse width. Fluorescent screens often have blind spots, because they are burnt out when the beam was unintentionally parked on the screen for extended time. The response of screens—the signal generated per nC—is often non-linear and makes careful calibration necessary if the screens are used for quantitative measurements. Moreover, the magnification of the optical system, consisting of lenses, between the screen and the camera needs to be determined, which is often accomplished by placing fiducial markers with fixed separation on the screens. This allows to relate the pixels from the camera to the mm on the screen. This also helps to calibrate the different scales in the horizontal and vertical direction, if the screen is mounted at an angle.

Wire scanners record the secondary emission electrons that the beam knocks out from a wire scanned across its path. They require carefully calibrating the position of the wire. In a SEM grid the currents from multiple wires are read out simultaneously, which requires multiple well-balanced current amplifiers.

After having discussed the different imperfections, let us turn to linear systems, beam lines and linear accelerators and discuss how these effects disturb the system and how to correct it.

3 Linear accelerators and beam lines

In this section we center our discussion on straight systems. The key quantities for much of the following discussion are transfer matrix elements, especially $R_{12}$. It describes the dependence of the position $x$, which we observe, on the cause of the change, which is an angle $x'$. The first index in $R_{12}$ denotes the quantity we observe, here $x$, which is the first element in the state vector $(x, x', y, y')$. The second index denotes what is causing the change, here it is 2, because $x'$ is at the second place in the state vector. You might want to work out which transfer matrix element describes the change of the vertical position $y$ due to varying the vertical angle $y'$.

3.1 Transfer matrices in linear accelerators

When calculating transfer-matrix elements in a linear accelerator, where beam energy and momentum at the observation point—the first index—and at the “kicking point”—the second index—are different. Under acceleration, the longitudinal momentum $p_s$ increases, while the transverse momentum $p_x$ remains unchanged. This causes the beam angle $x' = p_x/p_s$ to decrease by the relativistic factor $\beta \gamma = p_s/mc$. This effect is called adiabatic damping because it decreases the emittance under acceleration. Moreover, $R_{12}$ scales with $(\beta \gamma)_{\text{kick}}/(\beta \gamma)_{\text{look}}$, which we need to take into account, when considering linear accelerators.

3.2 Dipole errors and steering magnets

Remember from Section 2.2 that misalignments can be described by applying an operator $O = \tilde{q} + R$, consisting of a misalignment vector $\tilde{q}$ and the transfer matrix $R$ of the element. Here we interpret the operation of $O$ on a state vector $\vec{x}$ as first applying the transfer matrix $R$ and the adding $\tilde{q}$ to that vector. Multiple misalignments can therefore represented by sequentially applying operators $O_k$. The particle coordinates at the end of the beam line, the state vector $\vec{x}_n$ is the given by

$$\vec{x}_n = R_n \cdots (\tilde{q}_{k+1} + R_{k+1})(\tilde{q}_k + R_k) \cdots (\tilde{q}_1 + R_1)\vec{x}_0$$

$$= R_n \cdots R_1\vec{x}_0 + \sum_{j=1}^{n-1} (R_n \cdots R_{j+1})\tilde{q}_j.$$  \hspace{1cm} (6)
Inspecting the expression in the second line, we see that the final position \( \vec{x}_n \) is given by propagating the initial state vector \( \vec{x}_0 \) with the product of all transfer matrices from start to end, which equals what a beam line without misalignments would cause. The sum extends over the perturbations \( \vec{q}_j \), weighted with the transfer matrices from the respective perturbation to the end of the beam line, which is illustrated on the left-hand side in Figure 4. We can use this method to find the influence of each misalignment vector \( \vec{q}_j \) on the beam position at the end \( \vec{x}_n \).

We correct these perturbations by introducing dipole-corrector magnets, such as the one shown on the right-hand side in Figure 4. They apply the same kick to all particles. The effect of a steering magnet on the beam is given by

\[
\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \end{pmatrix} + \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix},
\]

which can be cast into the same form as the misalignments, namely \( \vec{x}_1 = \vec{q} + \vec{R}\vec{x}_0 \). We therefore can treat them like any other perturbation.

### 3.3 Bumps and Knobs

Often we need to combine several steering magnets to cause a well-defined change of the beam trajectory, such as the parallel displacement to bring the blue and the red counter-propagating beams into collision, as shown on the left-hand side in Figure 5. A second example is a so-called closed bump with three steering magnets, shown on the right-hand side in Figure 5, where we can adjust the position \( \Delta x_0 \) without perturbing the accelerator after the third corrector magnet, where the trajectory is steered back onto the original one.

We now seek linear combinations of steering-magnet excitations that achieve the desired objective, for example, parallel displacement, which requires to adjust \( \Delta x_0 \) without changing \( \Delta x'_0 \). We note that the first steering magnet changes the position by \( \Delta x_0 = R_{12}^0 \theta_1 \) and the angle by \( \Delta x'_0 = R_{22}^0 \theta_1 \). Here the
Fig. 6: Using four steerers to change position and angle without affecting the trajectory after the last steerer, which called a four-bump.

subscripts denote the respective transfer-matrix element and the first superscript denotes the objective point, here labeled “0.” The second superscript denotes the position of the steerer, here labeled “1.” Combining this and an equivalent equation for the second steerer, we arrive at the following equation

$$
\begin{pmatrix}
\Delta x_0 \\
\Delta x_0'
\end{pmatrix} =
\begin{pmatrix}
R_{01}^{12} & R_{02}^{12} \\
R_{01}^{22} & R_{02}^{22}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix}.
$$

(8)

Since the matrix describes the response of the observables $\Delta x_0$ and $\Delta x_0'$ to a change in steerer—the actuator—it is called the response matrix for this particular problem. Inverting the equation results in

$$
\begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} =
\begin{pmatrix}
R_{01}^{12} & R_{02}^{12} \\
R_{01}^{22} & R_{02}^{22}
\end{pmatrix}^{-1}
\begin{pmatrix}
\Delta x_0 \\
\Delta x_0'
\end{pmatrix}.
$$

(9)

and gives us a way to determine the required steerer excitations $\theta_1$ and $\theta_2$ to cause a particular change in $\Delta x_0$ and $\Delta x_0'$. In particular, changing the trajectory by $\Delta x_0$ without changing $\Delta x_0'$ gives us a linear combination of steerer excitations to fulfill this objective, which is often called multi-knob. In short, the columns of the inverse of the response matrix yield the knobs to change one of the objective parameters.

Let us consider the slightly more advanced example of a four-bump, which is shown in Figure 6. The objective is to independently control the position $\Delta x_0$ and $\Delta x_0'$ at the indicated point without affecting the trajectory after the last of the four steering magnets. To do so, we first determine the response matrix, which is given as follows

$$
\begin{pmatrix}
\Delta x_0 \\
\Delta x_0'
\end{pmatrix} =
\begin{pmatrix}
R_{12}^{01} & R_{12}^{02} \\
R_{22}^{01} & R_{22}^{02}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix}.
$$

(10)

The top left $2 \times 2$ matrix equals that from the previous example. The downstream steerers $\theta_3$ and $\theta_4$ can affect neither position $\Delta x_0$ nor angle $\Delta x_0'$, which accounts for the $2 \times 2$ matrix of zeros in the top right corner. The third row contains the $R_{12}$ matrix elements from each steerer to the final point after the last steerer. Likewise the fourth row contains the $R_{22}$ elements. The vector on the left-hand side contains the desired objectives, namely to adjust $\Delta x_0$ and $\Delta x_0'$ while closing the bump requires both the final position $x_f$ and angle $x'_{f}$ to be zero. The first and second column of the inverse response matrix respectively are the knobs to vary position and angle independently.

3.4 Orbit correction

If the trajectory differs from some previously determined “golden orbit” we measure the differences of the position recorded by the BPM and adjust steerers to zero this difference. This process is called orbit
correction. The simplest version is illustrated in Figure 7, where the first quadrupole is misaligned and gives the particles a transverse kick such that the position $x_1$ on the first BPM will be back to zero. Next we use the second steerer to correct the trajectory on the second BPM and the third steerer to correct the third BPM. In this way we correct one BPM at a time. This method is commonly called one-to-one steering.

We can formalize the trajectory correction by introducing the response matrix between all BPM and steerers. Let us consider the following setup

$$
\begin{pmatrix}
-x_1 \\
-x_2 \\
-x_3
\end{pmatrix}
= \begin{pmatrix}
R_{11} & 0 & 0 \\
R_{21} & R_{22} & 0 \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix}
$$

where we show the beam line with correctors and BPM on the left-hand side and the corresponding equation on the right-hand side. Note that the $3 \times 3$ response matrix has zeros in the top right corner, because the downstream steerers cannot affect the upstream BPMs. Otherwise the $R_{12}$ elements of the transfer-matrices between the respective corrector and BPM appear. The vector on the left-hand side contains the BPM readings, but with a negative sign, because we want to find corrector values that undo the BPM readings and make them zero. Solving this equation involves inverting the matrix, which is possible unless the response matrix is degenerate, and gives us the steering magnet excitations $\theta_k$ to achieve this.

We can calculate the response matrix with beam optics codes such as MADX [4] but then the matrix is based on the model of the accelerator and may be somewhat idealized, Moreover, neither BPM scale errors nor corrector scale errors, for example, due to badly calibrated power supplies, are included. A second option is therefore to determine the response matrix experimentally by first recording a reference trajectory and observing changes of the BPM readings while changing one steering magnet at a time.

Since inverting response matrices is a very frequent task, we will look at a number of different cases in the next section.

### 3.5 Digression on linear algebra

In general the systems we need to invert can be written as $-\vec{x} = A\vec{\theta}$ with the $n \times m$ response matrix $A$ for $n$ BPMs and $m$ steerers. In the previous example we had $n = m = 3$ and could simply invert the response matrix, provided it is non-degenerate.

If we have an accelerator with more BPMs than steerers, such that $n > m$ the system of equations, as defined by the response matrix, is over-determined and we do not have enough steerers to correct
the trajectory on all BPMs. We can, however, do our best to minimize the rms trajectory, given by
\[ \chi^2 = | - \vec{x} - A \vec{\theta} |^2, \]
which results in the well-known pseudo-inverse
\[ \vec{\theta} = - (A^t A)^{-1} A^t \vec{x}. \]  
(11)

In very big accelerators with many BPMs and correctors, the inversion of large matrices is numerically very sensitive, which makes using the MICADO [5] algorithm attractive. It is based on finding the corrector that minimizes the rms orbit by the largest amount and then implement that corrector change. In the next step the second-most effective corrector is found and its correction applied. This process is repeated until the trajectory is below a predetermined threshold. An added bonus is that efficient numerical methods are used to minimize the number of computations.

Finally, if the accelerator contains more steerers than BPM, the response matrix is under-determined and cannot be inverted. In such cases singular-value decomposition (SVD) is used. It decomposes \( A = O \Lambda U^t \) into a diagonal matrix \( \Lambda \) and two orthogonal matrices \( O \) and \( U \). SVD has a very intuitive interpretation, because the orthogonal matrices are generalized rotations and the entries on the diagonal of \( \Lambda \) are stretching factors along the rotated axes. The action of \( A \) on a vector \( \vec{\theta} \) can thus be described by first rotating \( \vec{\theta} \) with \( U^t \) into a coordinate system, where the axes are stretched with the factors on the diagonal of \( \Lambda \). Finally the result is rotated by \( O \) into a coordinate system which may be different from the one, where \( \vec{\theta} \) is defined. But this is no surprise, because the \( A \vec{\theta} \) maps \( \vec{\theta} \) onto a space where the BPM positions \( \vec{x} \) “live.”

The decomposition of \( A \) now allows us to analyze where the inversion of \( A \) fails, which is the case when one or several of the stretching factors on the diagonal of \( \Lambda \) are zero. These subspaces are thus projected out and cannot be recovered. But we can still invert the matrix on the subspaces, where the diagonals are non-zero. This entails to also project out the degenerate subspace when calculating the inverse, which we can do by writing \( "A^{-1}" = U "\Lambda^{-1}\" O^t \), where the quotes indicate that the inverse is an inverse with a twist. And twist is to invert the diagonal matrix \( \Lambda \) only where we can, namely by inverting the entries on the diagonal where they are non-zero and project out where they are zero. This procedure implies that wherever there is a zero on the diagonal, we invert it by replacing 1/0 by 0. Finally we multiply the three inverted matrices \( U "\Lambda^{-1}\" O^t \) and obtain \( "A^{-1}\" \), the inverse with a twist. See the chapter on SVD in [6] for a more elaborate discussion.

We emphasize the usefulness of the different methods to invert matrices, because it appears in many contexts where we can calculate the effect of control variables on observables, such that we can calculate the response matrix \( C^{ij} = \partial \text{Observable}_i / \partial \text{Controller}_j \). But then we need to figure out how to set the control variables to minimize or to change the observables by a specific amount. And that involves inverting the response matrix \( C^{ij} \).

### 3.6 Gradient errors and filamentation

Often magnetic lattices are designed to produce regular and repetitive beta functions such as the one shown in the upper plot in Figure 8, which shows \( \beta_x \) (solid) and \( \beta_y \) (dashed) for eight 90°–FODO cells. Incorrectly powered quadrupoles or other sources of magnetic gradients, feed down is an example, causes the beating of the beta functions, shown on the lower plot in Figure 8 where the first quadrupole has a gradient 10 % too low. Note the beating pattern of the red dots that indicate the maxima of \( \beta_x \). Blue dots mark the maxima of \( \beta_y \). It can be shown [3] that the beam size \( \bar{\sigma}_x \) at a location downstream of the error can be described by
\[ \bar{\sigma}^2_x = \varepsilon \beta \left[ B_{mag} + \sqrt{B_{mag}^2 - 1 \cos(2\mu - \varphi)} \right], \]
(12)
Fig. 8: Top: the horizontal (solid) and vertical (dashed) beta functions for a beam line of eight 90° FODO cells. Bottom: changing the first quadrupole by 10% causes the beta functions to oscillate in a regular fashion, which is called beta beating.

where $\varepsilon$ is the emittance, $\bar{\beta}$ is the beta function at the observation point, $\mu$ is the betatron phase advance, and $\phi$ is the starting phase. $B_{mag}$ is called the mismatch parameter [7] and is given by

$$B_{mag} = \frac{1}{2} \left[ \left( \frac{\hat{\beta}}{\beta} + \frac{\beta}{\hat{\beta}} \right) + \beta \hat{\beta} \left( \frac{\alpha}{\beta} - \frac{\hat{\alpha}}{\hat{\beta}} \right) \right]^{2}$$

where $\beta$ and $\alpha$ are the unperturbed Twiss parameters and and $\hat{\beta}$ and $\hat{\alpha}$ the corresponding values with perturbation. Since $B_{mag}$ is always larger than unity, Equation 12 implies that the average beam size is increased by $B_{mag}$ and beats with amplitude $(B_{mag}^{2} - 1)^{1/2}$ at twice the betatron phase advance $\mu$.

If we inject the beam at the end of the above transfer line into a ring, which also constitutes a repetitive beam line, its beam size after $n$ turns is given by

$$\sigma_{n}^{2} = \varepsilon \bar{\beta} \left[ B_{mag} + \sqrt{B_{mag}^{2} - 1} \cos(4\pi n(Q + Q' \delta) - \phi) \right]$$

where $Q$ is the tune of the ring and $Q'$ its chromaticity. Since the beam particles have a distribution of relative momenta $\delta$ with width $\sigma_{\delta}$ they all have slightly different tunes. Therefore the oscillations are no longer synchronized and de-cohere. This mechanism is called filamentation. We calculate the beam size $\sigma_{n}$ after $n$ turns by averaging over the momentum distribution $\psi(\delta) = e^{-\delta^{2}/2\sigma_{\delta}^{2}}/\sqrt{2\pi\sigma_{\delta}}$, which gives us

$$\sigma_{n}^{2} = \varepsilon \bar{\beta} \left[ B_{mag} + e^{-2(2\pi Q' \sigma_{\delta})^{2} n^{2}} \sqrt{B_{mag}^{2} - 1} \cos(4\pi nQ - \phi) \right].$$
We find that the beam size shows decaying oscillations towards a value that is given by $B_{mag}$ times the unperturbed value. This can be interpreted as an increase of the emittance by $B_{mag}$ and since $B_{mag}$ is always larger than unity this is a very undesirable effect, especially in hadron rings without a natural damping mechanism. Note that the decay is of type $e^{-n^2}$, which is characteristic for de-coherence.

3.7 Measuring the beam matrix

Since small gradient errors are undesirable, yet unavoidable, we need way to determine the beta functions experimentally before correcting them with additional quadrupoles. A common method to measure the beam matrix and with it the Twiss parameters and the emittance, is a quadrupole scan. It is based on changing the quadrupole excitation and observing the changing beam size on a screen or with a wire scanner. The setup is schematically shown on the left-hand side in Figure 9. The transfer matrix between the quadrupole and the screen is given by

$$ R = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 - L/f & 1 \\ -1/f & 1 \end{pmatrix}, $$

(16)

where $f$ is the focal length of the quadrupole and $L$ is the distance between quadrupole and screen. If we knew the beam matrix $\sigma$ with elements $\sigma_{11}, \sigma_{12},$ and $\sigma_{22}$ we can predict the beam size on the screen $\bar{\sigma}$ to be

$$ \bar{\sigma}_x^2 = R_{x1}^2 \sigma_{11} + 2R_{x1}R_{x2} \sigma_{12} + R_{x2}^2 \sigma_{22} 
\quad = (1 - l/f)^2 \sigma_{11} + 2l(1 - l/f) \sigma_{12} + l^2 \sigma_{22}, $$

(17)

which has a quadratic dependence of $\bar{\sigma}^2$ on $L/f$, which is also visible on the right-hand side in Figure 9. In order to determine the $\sigma_{ij}$ from a number of measurements, we assemble multiple—here five—measurements in a matrix

$$ \begin{pmatrix} \bar{\sigma}_{x,1}^2 \\ \bar{\sigma}_{x,2}^2 \\ \bar{\sigma}_{x,3}^2 \\ \bar{\sigma}_{x,4}^2 \\ \bar{\sigma}_{x,5}^2 \end{pmatrix} = \begin{pmatrix} (1 - L/f_1)^2 & 2L(1 - L/f_1) & L^2 \\ (1 - L/f_2)^2 & 2L(1 - L/f_2) & L^2 \\ (1 - L/f_3)^2 & 2L(1 - L/f_3) & L^2 \\ (1 - L/f_4)^2 & 2L(1 - L/f_4) & L^2 \\ (1 - L/f_5)^2 & 2L(1 - L/f_5) & L^2 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix}, $$

(18)

Finding $\sigma_{11}, \sigma_{12},$ and $\sigma_{22}$ is now only a matter of solving this over-determined system using the pseudo-inverse from Equation 11, albeit without the minus sign. The Twiss parameters and the beta functions can be derived from the beam matrix elements with

$$ \varepsilon = \sqrt{\text{det} \sigma} = \sqrt{\sigma_{11} \sigma_{22} - \sigma_{12}^2}, \quad \beta = \frac{\sigma_{11}}{\varepsilon}, \quad \text{and} \quad \alpha = -\frac{\sigma_{12}}{\varepsilon}, $$

(19)

which follows from the definition of the beam matrix in terms of emittances and Twiss parameters.

Instead of using a quadrupole and a screen, we can also use several, at least three, wire scanners in a beam line and deduce the incoming beam matrix from size measurement on the scanners as follows
If we know all transfer matrices, here from the reference point to the respective wires, we can predict what we would measure, if we knew the incoming beam matrix elements $\sigma_{11}, \sigma_{12},$ and $\sigma_{22}$. This permits us to set up the equations shown on the right-hand side and transform them to a matrix-equation, which we can invert with one of the methods from Section 3.5.

### 3.8 Correction and beta matching

Using the measured beam matrix, we can use it to correct the Twiss parameters $\beta_x, \alpha_x, \beta_y,$ and $\alpha_y$ at a control location, for example, the injection point into a ring, in order to prevent emittance growth due to filamentation. Figure 10 illustrates the setup with a sigma measuring section, shown in red, and four independently powered quadrupoles, shown in blue that can independently adjust the Twiss parameters at the injection, or control, point. We point out that $\beta_x, \alpha_x, \beta_y,$ and $\alpha_y$ at the control location have a non-linear dependence on the quadrupole excitation. Finding these excitations, based on the knowledge of the incoming beam matrix at the reference location, to set the Twiss parameters to their design values involves non-linear optimization, commonly called matching. Beam optics codes, such as MADX [4], provide functions to specify Twiss parameters at the start and end of a section and then suitably adjust the excitations of the quadrupoles to match the specified boundary conditions.

If the discrepancy of the actually measured Twiss parameters at the reference position is not too far
from their design values, we can calculate a linearized response matrix of the dependence of $\beta_x, \alpha_x, \beta_y,$ and $\alpha_y$ on the four quadrupole excitations and calculate knobs to independently adjust one of the four Twiss parameters without affecting the others. These knobs are thus suitable linear combinations of quadrupole excitation patterns to correct one parameter at a time.

An even simpler example is a so-called waist knob that uses two quadrupoles near an interaction point. It uses two quadrupoles to independently control $\alpha_x$ and $\alpha_y$, or equivalently the longitudinal position of the focal point—the waist. As in the previous paragraph, the knob is constructed from the response matrix that relates $\alpha_x$ and $\alpha_y$ to small changes of the two quadrupole excitations. In this case, the incoming beam matrix is assumed to have design values.

### 3.9 Skew-gradient errors

In accelerators with very flat beams, having $\varepsilon_y \ll \varepsilon_x$, skew quadrupoles couple the large amplitude horizontal oscillations into the vertical plane and spoil the small vertical emittance. In order to quantify this effect we consider the effect of an additional thin skew-quadrupole with transfer matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/f & 0 \\ 0 & 0 & 1 & 0 \\ 1/f & 0 & 0 & 1 \end{pmatrix}$$

(20)

on the vertical emittance of an initially uncouple beam matrix. After the skew quadrupole, the vertical lower-right $2 \times 2$ part of the beam matrix is

$$\left( \begin{array}{cc} \hat{\sigma}_{33} & \hat{\sigma}_{34} \\ \hat{\sigma}_{43} & \hat{\sigma}_{44} \end{array} \right) = \left( \begin{array}{cc} \sigma_{33} & \sigma_{34} \\ \sigma_{43} & \sigma_{44} + \sigma_{11}/f^2 \end{array} \right)$$

(21)

and its (projected) emittance $\hat{\varepsilon}_y$ is given by the determinant

$$\hat{\varepsilon}_y^2 = \varepsilon_y^2 + \frac{\sigma_{11}\sigma_{33}}{f^2} = \varepsilon_y^2 \left( 1 + \frac{\varepsilon_x \beta_x \beta_y}{\varepsilon_y f^2} \right).$$

(22)

We observe that the vertical emittance $\hat{\varepsilon}_y$ increases with $\varepsilon_x / \varepsilon_y \gg 1$ and with $\beta_x \beta_y / f^2$, such that a large emittance ratio is particularly detrimental; as are large beta functions $\beta_x$ and $\beta_y$ at the location of the skew quadrupole.

### 4 Circular accelerators

In rings the beam “bites its tail;” it has to satisfy periodic boundary conditions. This poses additional constraints on the motion. We first address the consequence of a dipole error on the closed orbit in a ring.

#### 4.1 Dipole errors

We consider a dipole error, represented as the small wedge in Figure 11. It causes a perturbation of the closed orbit, which is shown as the red line oscillating around the unperturbed orbit. We characterize the perturbing kick by the vector $\vec{q} = (0, \theta, 0, 0)$, here for a horizontal kick $\theta$, and the one-turn transfer matrix that starts at the location of the perturbation by the $4 \times 4$–matrix $R_{ij}^{ij}$. The perturbed closed orbit $\vec{x}_j$, immediately after the perturbation, is given by $\vec{x}_j = R_{ij}^{ij} \vec{x}_j + \vec{q}_j$, which requires $\vec{x}_j$ to reproduce after one turn. Solving for $\vec{x}_j$ yields $\vec{x}_j = (1 - R_{ij}^{ij})^{-1} \vec{q}_j$, which requires $\vec{x}_j$ to the location of a BPM, with the $4 \times 4$ transfer matrix $R_{ij}^{ij}$, results in the response of the BPM, to the kick at location $j$

$$\vec{x}_i = R_{ij}^{ij} \vec{x}_j = R_{ij}^{ij} (1 - R_{ij}^{ij})^{-1} \vec{q}_j = C^{ij} \vec{q}_j .$$

(23)
Fig. 11: Closed orbit (red), perturbed by a small dipole.

Here $C^{ij} = R^{ij} (1 - R^{jj})^{-1}$ is a $4 \times 4$-matrix that describes the response of the closed orbit at BPM$_j$ to a perturbation at a location labeled $j$. In this sense it takes the role that the transfer matrix has in a beam line, but has the closed orbit constraint built in through the factor $(1 - R^{jj})^{-1}$.

It can be shown that in uncoupled rings $C^{ij}_{12}$ can be written with the help of the Twiss parameters and the phase advance between the two locations as

$$C^{ij}_{12} = \sqrt{\beta_i \beta_j} \cos(\mu_{ij} - \pi Q),$$

(24)

where $Q$ is the tune of the ring. Note that the expression diverges at integer values of the tune, because the sine in the denominator becomes zero.

A horizontal kick will cause the closed orbit to become slightly longer; it increases the circumference $C$ by $\Delta C = D_j \theta$ compared to the unperturbed orbit. Here $D_j$ is the dispersion at the location of the perturbation. In the presence of a radio-frequency (RF) system, the beam therefore has to adjust its relative momentum by $\delta = -D_j \theta / \eta C$ to remain synchronous with the RF. This small change of momentum will show up on BPM$_i$ as an additional displacement of the orbit by $D_i \delta$, where $D_i$ is the dispersion at BPM$_i$, such that the response coefficient $C^{ij}_{12}$ that includes this effect is given by

$$C^{ij}_{12} = \left[ \sqrt{\beta_i \beta_j} \cos(\mu_{ij} - \pi Q) - \frac{D_i D_j}{\eta C} \right],$$

(25)

where $\eta = \alpha - 1/\gamma^2$ is the phase-slip factor and $\alpha$ is the momentum compaction factor. This additional factor in $C^{ij}_{12}$ is often neglected, but plays a role in small rings that ramp their energy, thus changing the relativistic factor $\gamma$ to become close or even equal to $1/\sqrt{\alpha}$, a condition called transition.

### 4.2 Quadrupole alignment tolerances

The alignment tolerances for the quadrupoles can be specified by calculating the rms orbit displacement caused by quadrupoles, transversely displaced by $d_j$, which kick the beam by $\theta_j = d_j / f$, where $f$ is the focal length of the quadrupole. If we assume that the displacements $d_j$ are independent, have zero mean and rms value $\sigma_d$, we can specify their statistics by $\langle d_j \rangle = 0$ and $\langle d_j d_k \rangle = \sigma_d^2 \delta_{jk}$, where $\delta_{jk}$ is unity for $j = k$ and zero otherwise. Using the response coefficients from Equation 24 we find the rms orbit displacement from summing over all quadrupoles and averaging over the random distribution of
displacements with the following result
\[
(x_i^2) = \left[ \sum_j \frac{\beta_j \beta_i}{2 \sin \pi Q} \cos(\mu_{ij} - \pi Q) \frac{d_j}{f_j} \right] \left[ \sum_k \frac{\beta_k \beta_i}{2 \sin \pi Q} \cos(\mu_{ik} - \pi Q) \frac{d_k}{f_k} \right]
\]
\[
= \sum_j \frac{\beta_j \beta_i}{(2 \sin \pi Q)^2} \cos^2(\mu_{ij} - \pi Q) \frac{d_j}{f_j}^2,
\]
where we assumed that the phases are evenly distributed, such that we can use \(\langle \cos^2 \rangle \rightarrow 1/2\). Furthermore, introducing the average beta function \(\bar{\beta}\) at the \(N_q\) quadrupole locations and the average focal length \(\bar{f}\), we finally arrive at
\[
\sqrt{(x_i^2)} \approx \sqrt{\frac{N_q \bar{\beta} \bar{f}}{2 \sqrt{2} \sin \pi Q} \sigma_d},
\]
where we see that very large rings with a large number of quadrupoles \(N_q\) will cause large rms orbit deviations, unless very tight alignment tolerances \(\sigma_d\) are enforced.

### 4.3 Orbit correction

Correcting the orbit in a ring is based on calculating—or measuring—the response matrix of how the BPM positions change as a consequence of changing steering magnets. Since the steerers change the closed orbit, the readings of all BPM will be affected as described by the response coefficients in \(C_{ij} = R_{ij}(1 - R_{jj})^{-1}\) between BPM \(i\) and corrector \(j\). For the horizontal plane, we use the 12-elements, such that the full response matrix is
\[
\begin{pmatrix}
-x_1 \\
-x_2 \\
\vdots \\
-x_n
\end{pmatrix} =
\begin{pmatrix}
C_{11}^{12} & C_{12}^{12} & \cdots & C_{im}^{12} \\
C_{21}^{12} & C_{12}^{22} & \cdots & C_{2m}^{12} \\
\vdots & \vdots & \ddots & \vdots \\
C_{n1}^{12} & C_{12}^{n2} & \cdots & C_{nm}^{12}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_m
\end{pmatrix},
\]
which is analogous to the response matrix discussed in Section 3.4. Keep in mind that the superscripts label the respective BPM and steerer, whereas the subscripts label the matrix element of the matrix \(C_{ij}\).

Finding the steerer excitations \(\theta_j\) that zero the orbit \(x_i\) thus involves inverting the matrix using one of the methods discussed in Section 3.5.

Physically, the steering magnets used for orbit correction are dipole magnets and they also generate dispersion. Since we normally do not want to generate additional dispersion, we include the effect in the response matrix through the dispersion-response coefficient \(S_{ij} = \partial D_i/\partial \theta_j\), where \(D_i\) is the dispersion at BPM \(i\) that can be either calculated from the model or measured by changing the RF frequency, which causes beam momentum to change, as already discussed near the end of Section 4.1. In order to correct the orbit, while minimizing the generated dispersion, which is called dispersion-free steering, we use the following augmented response matrix
\[
\begin{pmatrix}
\vdots \\
-x_i \\
\vdots \\
-D_i
\end{pmatrix} =
\begin{pmatrix}
\vdots \\
C_{11}^{i1} & C_{12}^{i2} & \cdots & C_{im}^{i2} \\
C_{21}^{i2} & C_{12}^{i2} & \cdots & C_{2m}^{i2} \\
\vdots & \vdots & \ddots & \vdots \\
S_{11}^{i2} & S_{12}^{i2} & \cdots & S_{im}^{i2}
\end{pmatrix}
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_m
\end{pmatrix},
\]
in which the dispersion-response coefficients \(S_{ij}\) are added below the matrix from Equation 28.
4.4 Gradient errors

A gradient error in a ring with tune $Q = \mu/2\pi$ can be modeled by adding a thin-lens quadrupole at the position in the ring where the error is located, here assumed to have Twiss parameters $\alpha$ and $\beta$. The perturbed full-turn matrix thus can be calculated by evaluating

$$
R_Q R = \begin{pmatrix}
1 & 0 \\
-1/f & 1
\end{pmatrix}
\begin{pmatrix}
\cos \mu + \alpha \sin \mu & \beta \sin \mu \\
-\frac{1+\alpha^2}{2} \sin \mu & \cos \mu - \alpha \sin \mu
\end{pmatrix}
$$

(30)

$$
= \begin{pmatrix}
\cos \mu + \alpha \sin \mu & \beta \sin \mu \\
-(\cos \mu + \alpha \sin \mu)/f + \gamma \sin \mu & \cos \mu - \alpha \sin \mu - (\beta/f) \sin \mu
\end{pmatrix}
$$

The perturbed tune $Q + \Delta Q$ is determined by the sum of the diagonal elements

$$
2 \cos(2\pi(Q + \Delta Q)) = 2 \cos(2\pi Q) - \frac{\beta}{f} \sin(2\pi Q)
$$

(31)

Assuming that $\Delta Q$ is small, expanding the left-hand side to first order gives us an approximate equation for the tune-shift $\Delta Q$, given by $\Delta Q \approx \beta/4\pi f$, an equation that is of great practical use as we shall see.

Not only the tune, but also the beta functions change as a consequence of a gradient error. From the $12$–element of the transfer matrix in Equation 30 we see that the perturbed beta function $\bar{\beta}$ is given by

$$
\bar{\beta} = \frac{\beta \sin(2\pi Q + \Delta Q)}{\sin(2\pi(Q + \Delta Q))} \approx \beta \left[1 + 2\pi \Delta Q \cot(2\pi Q)\right]
$$

(32)

where we see that $\bar{\beta}$ diverges at half-integer values of the tune $Q$.

Actually, a region around the half-integer tune values does not permit stable oscillations, because Equation 31 requires to calculate an inverse cosine of a quantity that has magnitude larger than unity. The range of tune values $Q$ for which

$$
|\cos(2\pi Q) - 2\pi \Delta Q \sin(2\pi Q)| > 1
$$

(33)

exceeds unity, defines the half-integer stop bands, which depend on the magnitude of the gradient perturbation, as quantified by $\Delta Q = \beta/4\pi f$.

4.5 Measuring and correcting the tune and beta functions

The simplest way to measure the tune is to give the beam a small kick and observe the position signal from a BPM on a spectrum analyzer, which reveals the betatron sidebands of the revolution harmonics. Since many modern BPM provide turn-by-turn position information, Fourier-transforming this data yields the tunes directly. Figure 12 illustrates the process. The code on the left-hand side defines the tune $Q_x$, and then tracks the particle for 1024 turns, before plotting the absolute value of the FFT. Note that the initially chosen tune is above $1/2$ such that both the original tune and its alias $1 - Q_x$ appear. This ambiguity can be resolved by slightly increasing the excitation of a horizontally focusing quadrupole and observing the line below the half-integer. If it moves to the right, the tune is below that half-integer, if it moves to a lower value, the “real” tune is above the half-integer.

Once we can measure the tunes, we can also correct it with two suitably chosen quadrupoles. Since the quadrupoles affect both planes, the horizontal and vertical tunes will change with the focal length $f_1$ of the first quadrupole according to $\Delta Q_x = \beta_{1x}/4\pi f_1$ and $\Delta Q_y = -\beta_{1y}/4\pi f_1$. Using a second quadrupole with focal length $f_2$ their combined effect on the tunes is given by

$$
\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} + \frac{\beta_{2x}}{4\pi f_2} \quad \text{and} \quad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1} - \frac{\beta_{2y}}{4\pi f_2}
$$

(34)
Assembling these equations into a matrix-valued equation

\[
\begin{pmatrix}
\Delta Q_x \\
\Delta Q_y
\end{pmatrix} =\frac{1}{4\pi} \begin{pmatrix}
\beta_{1x} & \beta_{2x} \\
-\beta_{1y} & -\beta_{2y}
\end{pmatrix} \begin{pmatrix}
1/f_1 \\
1/f_2
\end{pmatrix}
\]

(35)

makes it obvious that the inverse of the matrix gives the excitations of the quadrupoles that will change the tunes by \(\Delta Q_x\) and \(\Delta Q_y\), respectively.

Note that changing the excitation of a quadrupole by a small amount, characterized by a small additional thin-lens quadrupole with focal length \(f\) will cause a tune shift by \(\Delta Q = \beta/4\pi f\), which is proportional to the beta function at the location of the quadrupole, thus providing a measurement of the beta function. Often this is, however, difficult to implement, because multiple quadrupoles are powered in series by the same power supply.

4.6 Model calibration, LOCO

An elaborate method to determine the differences of the accelerator in the tunnel to the computer model is based on comparing the response coefficients \(\hat{C}^{ij}\) obtained from measuring orbit changes as a consequence of changing the excitation of steerers one at a time to the response coefficients \(C^{ij}\) from the model. We express the measured coefficients as the first-order Taylor expansion of the model coefficients in the gradients \(g_k\) of the quadrupoles

\[
\hat{C}^{ij} = C^{ij} + \sum_k \frac{\partial C^{ij}}{\partial g_k} \Delta g_k,
\]

(36)

where the derivatives \(\partial C^{ij}/\partial g_k\) are calculated from the model. Note that there are \(2N_{bpm}N_{cor}\) response coefficients in the two planes, which is normally a very large number to determine the \(N_{quad}\) gradients, which is a much smaller number. The fit is therefore vastly over-determined.

It is straightforward to include additional parameters, such as the BPM scale errors \(\Delta x^i\) and the corrector scale errors \(\Delta y^j\) which turn the equation into

\[
\hat{C}^{ij} = C^{ij} + \sum_k \frac{\partial C^{ij}}{\partial g_k} \Delta g_k + C^{ij} \Delta x^i - C^{ij} \Delta y^j.
\]

(37)

This allows to reduce many systematic errors from the measurement system and even adding further parameters is possible. These methods were first used in SPEAR [8] and later refined at NSLS [9] with remarkable success. Today, most synchrotron light sources use response-matrix based method to debug their accelerator optics.
4.7 Coupling and its correction

Quadrupoles that are accidentally mounted with a roll angle or fields from solenoids can couple the beta-tron oscillations in the transverse planes, which has an influence on the tunes $Q_x$ and $Q_y$. Qualitatively this behavior is easily understood by considering a mechanical equivalent system of two mass points connected by springs, as shown on the top left in Figure 13. The deviations $x$ and $y$ from their equilibrium correspond to the betatron oscillations amplitudes and the unperturbed tunes correspond to the eigenfrequencies $Q^{2}_x = k_x/m$ and $Q^{2}_y = k_y/m$, while the coupling between these oscillations originates from the weak coupling constant $c$, the spring constant that connects the two mass points. It is straightforward to obtain the equations of motion, shown at the bottom left of Figure 13. With standard methods to solve coupled linear differential equations, we find the eigenfrequencies $\omega_{\pm}$—corresponding to the two eigentunes of the coupled system—to be

$$\omega_{\pm}^2 = \frac{k_x + k_y + 2c}{2m} \pm \sqrt{\left(\frac{k_x - k_y}{2m}\right)^2 + \frac{c^2}{m^2}}.$$ \hspace{1cm} (38)

We see that the root can never vanish, unless the coupling $c$ is zero. The plot on the right-hand side in Figure 13 shows the eigentunes $\omega_{\pm}$ from Equation 38 plotted as a function of the difference between $Q_x \sim k_x$ and $Q_y \sim k_y$ for $c = 0.05$ and $0.01$. The larger value of $c$ causes the eigentunes to “repel” each other more. This observation is exploited operationally by adjusting upright quadrupoles to make the tunes $Q_x \sim k_x$ and $Q_y \sim k_y$ as close as possible and then adjusting one or more additional skew quadrupoles to minimize the tune separation and thereby the coupling $c$. This procedure is commonly referred to as correction of the closest tune.

4.8 Chromaticity measurement and correction

In order to measure the chromaticity $Q'$ of a ring we have to change the relative momentum $\delta = \Delta p/p$ of the beam and observe the corresponding change of the tune $Q = Q_0 + Q' \delta$. As mentioned towards the end of Section 4.1 can we change the momentum by changing the RF frequency $f_{RF}$ by $\Delta f_{RF}$ as given by

$$-\frac{\Delta f_{rf}}{f_{rf}} = \frac{\Delta T}{T} = \eta \delta = \left(\alpha - \frac{1}{\gamma^2}\right) \delta \quad \text{such that} \quad \delta = -\frac{1}{\eta} \frac{\Delta f_{rf}}{f_{rf}}.$$ \hspace{1cm} (39)

For a number of different relative momenta $\delta$ we then measure the tune, for example, by exciting a betatron oscillation and Fourier-transforming a position signal from a BPM. From a plot of the measured tune versus $\delta$ we can derive the chromaticity from a straight-line fit.
The chromaticity is a consequence of the momentum-dependence of the focusing of quadrupoles and, in order to correct it, we use sextupoles, placed at a location with non-zero horizontal dispersion $D_x$. The dispersion causes the particles with relative momentum $\delta$ to have an additional transverse offset $D_x \delta$, which is equivalent to transversely displacing the sextupole by $d_x = D_x \delta$. Inserting in Equation 4 we read off that the sextupole with integrated strength $k_2 L$ produces the field of a momentum dependent quadrupole with focal length $f_\delta$ given by

$$\frac{1}{f_\delta} = k_2 L D_x \delta,$$

which causes momentum-dependent tune shifts $\beta/4\pi f_\delta$ in the respective planes

$$\Delta Q_x = \frac{k_2 L D_x \beta_x}{4\pi} \delta \quad \text{and} \quad \Delta Q_y = -\frac{k_2 L D_x \beta_y}{4\pi} \delta.$$

(40)

Using two sextupoles at a location with different dispersion and beta functions, we can create a system that allows us to independently control the chromaticities $Q'_{x,y} = \Delta Q_{x,y}/\delta$ independently

$$\begin{pmatrix} \Delta Q'_x \\ \Delta Q'_y \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} D_{1x} \beta_{1x} & D_{2x} \beta_{2x} \\ -D_{1x} \beta_{1y} & -D_{2x} \beta_{2y} \end{pmatrix} \begin{pmatrix} (k_2 L)_1 \\ (k_2 L)_2 \end{pmatrix}.$$

(41)

Finding the sextupoles excitations $(k_2 L)_1$ and $(k_2 L)_2$ to change the two chromaticities by $\Delta Q'_x$ and $\Delta Q'_y$ is now a matter of inverting the matrix in Equation 41.

5 Further reading

Hopefully, reading these pages of introductory material whets your appetite for more, such as Zimmermann and Minty’s book [10] or the chapter on operational considerations in the Accelerator Physics Handbook [11]. Moreover, several textbooks cover corrections, see for example chapter 6 in [12], chapter 3 in [13], chapter 7 in [14], and chapter 8 in [3]. In previous CERN accelerator schools the same topic was covered, see for example [15] in the proceedings of the 2009 Diagnostic school, which also contains contributions on related topics. In general, it is worth to go poaching in the CAS archives [16] and hunt down the slides of colleagues who covered similar topics.

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