Extending the small-ball method

Shahar Mendelson *

September 5, 2017

Abstract

The small-ball method was introduced as a way of obtaining a high probability, isomorphic lower bound on the quadratic empirical process, under weak assumptions on the indexing class. The key assumption was that class members satisfy a uniform small-ball estimate, that is, \( \Pr(|f| \geq \kappa \|f\|_{L_2}) \geq \delta \) for given constants \( \kappa \) and \( \delta \).

Here we extend the small-ball method and obtain a high probability, almost-isometric (rather than isomorphic) lower bound on the quadratic empirical process. The scope of the result is considerably wider than the small-ball method: there is no need for class members to satisfy a uniform small-ball condition, and moreover, motivated by the notion of tournament learning procedures, the result is stable under a ‘majority vote’.

As applications, we study the performance of empirical risk minimization in learning problems involving bounded subsets of \( L_p \) that satisfy a Bernstein condition, and of the tournament procedure in problems involving bounded subsets of \( L_{\infty} \).

1 Introduction

In this article we study a more general version of the following question:

**Question 1.1.** Let \( F \) be a class of functions defined on a probability space \((\Omega, \mu)\), let \( X \) be distributed according to \( \mu \) and consider a sample \( X_1, ..., X_N \), consisting of \( N \) independent copies of \( X \). Find a high probability, lower bound on \( \theta = \theta(r) \), defined by

\[
\theta = \inf_{\{f \in F : \|f\|_{L_2} \geq r\}} \frac{1}{N} \sum_{i=1}^{N} \frac{f^2(X_i)}{\|f\|_{L_2}^2},
\]

for a value of \( r \) that is as small as possible.

The obvious implication of (1.1) is if \( f \in F \) and \( \|f\|_{L_2} \geq r \) then

\[
\frac{1}{N} \sum_{i=1}^{N} f^2(X_i) \geq \theta \|f\|_{L_2}^2,
\]

which is an ‘isomorphic’ lower bound on the quadratic empirical process.

Lower bounds on (1.1) play an important role in applications in probability (e.g., the smallest singular value of a random matrix with iid rows), geometry (for example, estimates

*Department of Mathematics, Technion, I.I.T, and Mathematical Sciences Institute, The Australian National University. Email: shahar@tx.technion.ac.il
on the Gelfand widths of convex bodies \[26, 27, 3, 23\], and statistics. Our main interest in Question 1.1 and the more general version we explore in what follows is its implications in statistical learning theory.

The standard way of estimating (1.1) is by two-sided concentration, that is, by obtaining a high probability upper bound on

\[
\sup_{f \in F : \|f\|_{L_2} \geq r} \left| \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) - 1 \right|
\]

Estimates of this type are called ratio-limit theorems (see [10, 9] and references therein). However, a nontrivial ratio-limit theorem is possible only if class members have well-behaved tails, and even then obtaining the two-sided estimate is rather involved (see, e.g., [23, 22]).

The fact that a high probability, two-sided estimate as in (1.2) is false without assuming that class members have well-behaved tails can be seen by considering what happens for a single function: given a square-integrable function \(f\), the probability that

\[
\frac{1}{N} \sum_{i=1}^{N} f^2(X_i) \leq C \|f\|^2_{L_2}
\]

may be small; it need not be better than the outcome of Chebychev’s inequality. Even if one allows for large values of \(C\) the situation remains the same: for example, it is straightforward to construct a function \(f\) on the unit sphere of \(L_2(\mu)\) such that

\[
Pr \left( \frac{1}{N} \sum_{i=1}^{N} f^2(X_i) \geq N \right) \geq \frac{c_1}{N}.
\]

In contrast, a lower bound of the form

\[
\frac{1}{N} \sum_{i=1}^{N} f^2(X_i) \geq c \|f\|^2_{L_2}
\]

is almost universal and holds with very high probability under minimal assumptions on \(f\).

**Definition 1.2.** The function \(f\) satisfies a small-ball condition with constants \(\kappa > 0\) and \(0 < \delta < 1\) if

\[
Pr(|f| \geq \kappa \|f\|_{L_2}) \geq \delta.
\]

All that a small-ball condition implies is that \(f\) does not assign too much weight to a small neighbourhood of 0; it does not mean that \(f\) has a well behaved tail, and in particular, it does not exclude the possibility that \(f\) does not have any moment beyond the second one. As it happens, a small-ball condition is enough to ensure that the lower bound (1.3) holds with very high probability for a well-chosen constant \(c\). Indeed, a standard binomial estimate shows that with probability at least \(1 - 2 \exp(-c_1 \delta N)\),

\[
|\{i : |f(X_i)| \geq \kappa \|f\|_{L_2}\}| \geq \frac{\delta N}{2};
\]

therefore, on that event,

\[
\frac{1}{N} \sum_{i=1}^{N} f^2(X_i) \geq \frac{\delta}{2} \kappa^2 \|f\|^2_{L_2}.
\]
This overwhelming difference between the upper and lower bounds on $\frac{1}{N} \sum_{i=1}^{N} f^2(X_i)$ motivated the introduction of the small-ball method [21, 18]. The small-ball method led to a lower bound on (1.1), under the assumption that class in question satisfies a small-ball property—that there are constants $\kappa > 0$ and $0 < \delta < 1$ such that for every $f \in F$, $Pr(|f| \geq \kappa \|f\|_{L_2}) \geq \delta$. To formulate this fact, recall that $\text{star}(H, f)$ denotes the star-shaped hull of $H$ with $f$, that is,

$$\text{star}(H, f) = \{ \lambda h + (1 - \lambda) f : h \in H, \ 0 \leq \lambda \leq 1 \}. $$

Also, from here on we denote by $(\epsilon_i)_{i=1}^{N}$ independent, symmetric $\{-1, 1\}$-valued random variables that are also independent of $(X_i)_{i=1}^{N}$; $D$ is the unit ball in $L_2(\mu)$; and $S$ is the corresponding unit sphere.

**Theorem 1.3.** [18] There exist absolute constants $c_1$ and $c_2$ for which the following holds. Let $H \subset L_2(\mu)$ and assume that for every $h \in H$, $Pr(|h| \geq \kappa \|h\|_{L_2}) \geq \delta$. If $r > 0$ satisfies

$$\mathbb{E} \sup_{h \in \text{star}(H, 0) \cap rS} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \epsilon_i h(X_i) \right| \leq c_1 \kappa \delta r \sqrt{N},$$

then with probability at least $1 - 2 \exp(-c_2 \delta N)$

$$\inf_{\{h \in H : \|h\|_{L_2} \geq r\}} \left\{ \left| \left\{ i : |h(X_i)| \geq \frac{\kappa}{2} \|h\|_{L_2} \right\} \right| \right\} \geq \frac{N \delta}{4}.$$ 

In particular, on the same event,

$$\inf_{\{h \in H : \|h\|_{L_2} \geq r\}} \frac{1}{N} \sum_{i=1}^{N} \frac{h^2(X_i)}{\|h\|^2_{L_2}} \geq \frac{\kappa^2 \delta}{16}.$$ 

Theorem 1.3 has many applications, but our focus here is on three directions in which it should be extended:

- Although a small-ball property is a rather minimal condition on a class, there are still natural situations in which it is not satisfied. For example, if the class $H$ is a bounded subset of $L_p$ for some $p > 2$, it need not satisfy a small-ball property. Indeed, even if $p = \infty$ and class members are bounded almost surely by 1, the best possible choice of $\delta$ for a function $h$ may be as bad as $\|h\|_{L_2}$ (e.g. if $h$ is $\{0, 1\}$-valued).

  With that in mind, one would like to find a version of Theorem 1.3 that is strong enough to deal with more general situations than classes that satisfy a small-ball property.

- Results that are based on a small-ball property are of an ‘isomorphic’ nature. The best that one can hope for is that if $h \in H$ and $\|h\|_{L_2} \geq r$, then

$$\frac{1}{N} \sum_{i=1}^{N} h^2(X_i) \geq c(\kappa, \delta) \|h\|^2_{L_2},$$

where $c(\kappa, \delta)$ depends only on the small-ball parameters $\kappa$ and $\delta$. The fact that $c$ is not close to 1 is unfortunate but unavoidable. On the other hand, as we explain in what follows, some applications require an almost isometric lower bound, with $c = 1 - \xi$ for a
small $\xi$. Therefore, the second extension of the small-ball method is to ensure that for a fixed $0 < \xi < 1$ that can be (almost) arbitrarily small, and with high probability,

$$\inf_{(h \in H : \|h\|_{L_2} \geq r)} \frac{1}{N} \sum_{i=1}^{N} \frac{h^2(X_i)}{\|h\|_{L_2}^2} \geq 1 - \xi.$$  

- The final extension is motivated by tournaments \[16, 19\]. Tournaments are statistical procedures that attain the optimal accuracy/confidence tradeoff for (almost) any prediction problem relative to the squared loss. Roughly and inaccurately put, the idea behind a tournament is to split the given sample $(X_i, Y_i)_{i=1}^{N}$ to $n$ coordinate blocks $(I_j)_{j=1}^{n}$, each one of cardinality $m$. For any $f, h$ that belong to the underlying class one compares the $n$ empirical risks

$$\frac{1}{m} \sum_{i \in I_j} (f(X_i) - Y_i)^2 \quad \text{and} \quad \frac{1}{m} \sum_{i \in I_j} (h(X_i) - Y_i)^2,$$

and based on the outcomes nominates the winner in this ‘statistical match’ between $f$ and $h$. The procedure selects a function that wins all of its matches\[1\]. As it happens, at the heart of the analysis of tournament procedures is the following question:

**Question 1.4.** Let $H \subset L_2(\mu)$. Fix an integer $n$ and set $(I_j)_{j=1}^{n}$ to be the decomposition of \{1,...,N\} to $n$ blocks of equal size. Given $0 < \xi < 1$ and $0 < \eta < 1$, find $r > 0$ that is as small as possible such that with high probability, for any $h \in H$ with $\|h\|_{L_2} \geq r$

$$\left\| \left\{ j : \frac{1}{m} \sum_{i \in I_j} h^2(X_i) \geq (1 - \xi)\|h\|_{L_2}^2 \right\} \right\| \geq (1 - \eta)n.$$  

Question 1.4 is significantly harder than Question 1.1 for every function in the class whose $L_2$ norm is not too small one must show an almost isometric lower bound that holds for a large majority of the coordinate blocks $I_j$. Clearly, when $n = 1$ and $\eta = 0.5$ (or any other constant smaller than 1) and $\xi$ is a constant that need be small, Question 1.4 reverts to Question 1.1.

Here we answer Question 1.4 without assuming that the class satisfies a small-ball property, thus extending Theorem 1.3 in all the three directions we outlined. The estimate we obtain holds, for example, for bounded subsets of $L_p$; for classes that satisfies an $L_q - L_2$ norm equivalence for some $q > 2$; and when the class satisfies a uniform integrability condition as is \[19\]. We then show how the result may be used in the study of Empirical Risk Minimization (ERM) and briefly sketch its importance in the analysis of the tournaments. We also derive an almost optimal estimate on the smallest singular value of a random matrix with iid rows. Despite a logarithmic looseness, it is the first estimate of its kind that can be obtained using general principles, without taking into account the special structure of the indexing class, which for the problem of the smallest singular value is the unit sphere in $\mathbb{R}^d$.

We end this introduction with some notation. Throughout the article, absolute constants are denoted by $c, c_1, ...$ and $C, C_1, ....$. Their value may change from line to line. $c_p$ or $c(p)$

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1In actual fact, the choice of a winner of a tournament is more involved. The reason is that when the functions are ‘too close’, the outcome of the statistical match between them is unreliable.
means that the constants depend only on the parameter $p$. We write $a \sim b$ when there are absolute constants $c$ and $C$ such that $ca \leq b \leq Ca$, and $a \precsim b$ if only a one-sided inequality holds; $a \sim_p b$ and $a \precsim_p b$ implies that the constants depend only on the parameter $p$.

2 Beyond the small-ball condition

Before we can extend the small-ball method we must first identify a notion that can replace the small-ball condition. To that end, let us examine the way in which a small-ball condition is used in the proof of Theorem 1.3.

Given $X_1, ..., X_m$, a small-ball condition with constants $\kappa$ and $\delta$ implies that with very high probability $(1 - 2\exp(-c\delta m))$, there are at least $\delta m / 2$ coordinates $i$ such that $|h(X_i)| \geq \kappa \|h\|_{L_2}^2$. Thus, not only is

$$\frac{1}{m} \sum_{i=1}^{m} h^2(X_i) \geq (1 - \frac{\xi}{\sqrt{m}}) \|h\|_{L_2}^2,$$

but (2.1) is stable: discarding a small proportion of the coordinates $\{1, ..., m\}$ does not ruin the lower bound.

The notion we use in what follows captures these features: not only is $\frac{1}{m} \sum_{i=1}^{m} h^2(X_i)$ large enough with high probability, it remains large if any subset of $\{1, ..., m\}$ of a certain cardinality is discarded from the sum.

Definition 2.1. A function $h$ satisfies a stable lower bound with parameters $(\xi, \ell, k)$ for a sample of cardinality $m$ if with probability at least $1 - 2\exp(-k)$, for any $J \subset \{1, ..., m\}$, $|J| \leq \ell$ one has

$$\frac{1}{m} \sum_{i \in J^c} h^2(X_i) \geq (1 - \xi) \|h\|_{L_2}^2.$$

In what follows we do not specify the cardinality of the coordinate block in question (it is denoted by $m$ throughout the article); instead we just say that $h$ satisfies a stable lower bound with parameters $(\xi, \ell, k)$.

Stability and geometry

The notion of a stable lower bound has a geometric interpretation. Observe that

$$\frac{1}{m} \sum_{i=1}^{m} h^2(X_i) \geq (1 - \xi) \|h\|_{L_2}^2$$

means that the random vector $v = (h(X_i))_{i=1}^{m}$ is located outside the Euclidean ball $(1 - \xi)^{1/2} \sqrt{m} \|h\|_{L_2} B_2^m$. Also, with constant probability and in expectation, $\|v\|_2 \precsim \sqrt{m} \|h\|_{L_2}$. However, all that information says very little about the coordinate distribution of the vector: the fact that $v$ has a Euclidean norm of order at least $\sqrt{m} \|h\|_{L_2}$ does not rule out the possibility that all of its ‘mass’ is concentrated at a single coordinate. In contrast, a stable lower bound implies that the vector $v$ is well-spread in the sense that its $m - \ell$ smallest coordinates still carry significant mass. As a result, the (conditional) Bernoulli random variable $\sum_{i=1}^{m} \varepsilon_i h(X_i) = \sum_{i=1}^{m} \varepsilon_i v_i$ exhibits a gaussian-like behaviour in the following sense: It is well
known [11] that for every \( p \geq 2 \) and every \( x \in \mathbb{R}^m \),
\[
\left\| \sum_{i=1}^{m} \varepsilon_i x_i \right\|_{L_p} \sim \sum_{i \leq p} |x_i^*| + \sqrt{p} \left( \sum_{i > p} (x_i^*)^2 \right)^{1/2},
\]
where \((x_i^*)^m_{i=1}\) denotes the nonincreasing rearrangement of \((|x_i|)^m_{i=1}\). If all the mass of \( x \) is concentrated at a single coordinate then \( \| \sum_{i=1}^{m} \varepsilon_i x_i \|_{L_p} \sim \| x \|_2 \), whereas for a gaussian like behaviour one would expect to have \( \| \sum_{i=1}^{m} \varepsilon_i x_i \|_{L_p} \sim \sqrt{p} \| x \|_2 \). Thanks to stability it follows that with probability at least \( 1 - 2 \exp(-k) \),
\[
\sum_{i \geq \ell} (v_i^*)^2 \geq (1 - \xi)m \| h \|^2_{L_2},
\]
and on that event,
\[
\left\| \sum_{i=1}^{m} \varepsilon_i v_i \right\|_{L_p} \gtrsim \sqrt{p} \sqrt{m} \| h \|^2_{L_2} \sim \sqrt{p} \| v \|^2_{L_2} \quad \text{for} \quad 2 \leq p \leq \ell. \tag{2.2}
\]

In other words, stability—for any fixed \( \xi \)—implies that with high probability, conditioned on \( X_1, \ldots, X_m \), the random variable \( \sum_{i=1}^{m} \varepsilon_i h(X_i) \) exhibits a gaussian-like behaviour of moments, at least for \( 2 \leq p \leq \ell \).

**Remark 2.2.** Definition 2.1 is very different from concentration. If the ‘smaller coordinates’ \((v_i^*)_{i \geq \ell}\) of a typical realization \( v = (h(X_i))^m_{i=1}\) have ‘enough mass’ then \( h \) satisfies a stable lower bound. However, the ‘larger coordinates’ \((v_i^*)_{i=1}^{\ell}\) can completely destroy any hope of a reasonable upper estimate on \( \frac{1}{m} \sum_{i=1}^{m} h^2(X_i) \), making two-sided concentration impossible.

### 2.1 Examples of a stable lower bound

To put the notion of a stable lower bound in some context, let us show that there are many natural situations in which it holds.

**A bounded function**

Let \( h \) be a function that is bounded almost surely by \( M \). As the next lemma shows, \( h \) satisfies a stable lower bound.

**Lemma 2.3.** There are absolute constants \( c_0 \) and \( c_1 \) for which the following holds. Let \( h \) be a function that is bounded almost surely by \( M \). For any \( 0 < \xi < 1 \), \( h \) satisfies a stable lower bound with parameters \((\xi, \ell, k)\) for
\[
\ell = c_0 m \xi \frac{\| h \|^2_{L_2}}{M^2} \quad \text{and} \quad k = c_1 m \xi^2 \frac{\| h \|^2_{L_2}}{M^2}.
\]

**Proof.** Applying Bernstein’s inequality, it follows that
\[
\Pr \left( \left| \frac{1}{m} \sum_{i=1}^{m} h^2(X_i) - \mathbb{E} h^2 \right| > u \right) \leq 2 \exp \left( -cm \min \left\{ \frac{u^2}{\mathbb{E} h^4}, \frac{u}{\| h \|_{L_2}} \right\} \right).
\]
Note that $\mathbb{E}h^4 \leq M^2 \mathbb{E}h^2$ and $\|h^2\|_{L_\infty} \leq M^2$. Setting $u = (\xi/2)\mathbb{E}h^2$ it is evident that with probability at least $1 - 2\exp(-c_1 m \xi^2 \mathbb{E}h^2 / M^2)$,
\[
\frac{1}{m} \sum_{i=1}^m h^2(x_i) \geq \left(1 - \frac{\xi}{2}\right) \mathbb{E}h^2.
\]

The contribution to the sum of the largest $\ell$ coordinates is at most $\ell M^2 / m$, which is at most $(\xi/2)\mathbb{E}h^2$ provided that $\ell \leq m \xi \mathbb{E}h^2 / 2M^2$, as claimed. \hfill \blacksquare

Lemma 2.3 is not very surprising as empirical means of a bounded function exhibit a two-sided concentration around the true mean, which in return implies a stable lower bound. The situation is somewhat different when leaving the bounded realm and considering functions with empirical means that need not concentrate well. The other examples presented in what follows are of that nature: situations in which a stable lower bound holds but there is no hope for a two-sided concentration.

Tail cutoff

Because our interest lies in obtaining a lower bound, truncating the function is a possible approach. And, there is a natural location in which the function should be truncated:

**Definition 2.4.** For $0 < \xi < 1$, set
\[
M(h, \xi) = \inf \left\{ t : \mathbb{E}h^2 1_{\{|h| > t\}} \leq \frac{\xi}{2}\mathbb{E}h^2 \right\}
\]

In other words, $M(m, \xi)$ is the smallest level at which the $L_2$ norm of the truncated function $w = h 1_{\{|h| \leq t\}}$ still has a significant $L_2$ norm: $\mathbb{E}w^2 \geq (1 - \xi/2)\mathbb{E}h^2$. Applying Lemma 2.3 to the truncated function $h 1_{\{|h| \leq M(\xi)\}}$, one has the following:

**Corollary 2.5.** There are absolute constants $c_0$ and $c_1$ for which the following holds. If $0 < \xi < 1$ and $M = M(h, \xi)$ then $h$ satisfies a stable lower bound with parameters $(\xi, \ell, k)$ for
\[
\ell = c_0 m \xi \frac{\mathbb{E}h^2}{M^2}, \quad \text{and} \quad k = c_1 m \xi^2 \frac{\mathbb{E}h^2}{M^2}.
\]

An important example of a tail cutoff which has been studied in [19] in the context of tournaments is when one is given a class of functions $H$ such that for any $h \in H$, $M(h, \xi) \leq \kappa(\xi)\|h\|_{L_2}$.

**Definition 2.6.** A class $H$ satisfies a uniform integrability condition if for every $0 < \xi < 1$ there is $\kappa(\xi)$ such that for any $h \in H$, $M(h, \xi) \leq \kappa(\xi)\|h\|_{L_2}$.

**Definition 2.6.** A class $H$ satisfies a uniform integrability condition if for every $0 < \xi < 1$ there is $\kappa(\xi)$ such that for every $h \in H$,
\[
\mathbb{E}h^2 1_{\{|h| \geq \kappa(\xi)\|h\|_{L_2}\}} \leq \frac{\xi}{2}\|h\|^2_{L_2}.
\]

By Corollary 2.5 each $h \in H$ satisfies a stable lower bound with constants
\[
\ell \sim m \frac{\xi}{\kappa^2(\xi)}, \quad \text{and} \quad k \sim m \frac{\xi^2}{\kappa^2(\xi)}.
\]

When one is given more information on the function $h$, that results in an improved estimate on the tail cutoff point $M(h, \xi)$ and also affects the way $\sum_{i=1}^m h^2 1_{\{|h| \leq M\}}(X_i)$ concentrates around its mean. Two such examples are when $\|h\|_{L_q} \leq L$ and when there is norm equivalence between the $L_q$ and $L_2$ norms, i.e., when $\|h\|_{L_q} \leq L\|h\|_{L_2}$. 

7
A function bounded in $L_p$

Let $h \in L_p$ for some $p > 2$. To identify its cutoff point, let $q = p/2$ and set $q'$ to be the conjugate index of $q$. Then

$$
E h^2 1_{(|h| > t)} \leq (E h^2 q)^{1/q} (P r(|h| > t))^{1/q'} \leq (E |h|^p)^{1/q'} \cdot \frac{(E |h|^p)^{1/q}}{t^{p/q}} = \frac{E |h|^p}{t^{p-2}}.
$$

Therefore,

$$
M(h, \xi) \leq \left( \frac{2 ||h||_{L_p}^p}{\xi ||h||_{L_2}^2} \right)^{1/(p-2)} = 2^{1/(p-2)} ||h||_{L_p} \cdot \left( \frac{||h||_{L_p}^2}{\xi ||h||_{L_2}^2} \right)^{1/(p-2)},
$$

and one has

$$
\ell = c_0 m^2 \frac{E h^2}{M^2} \geq c_1 (p) \left( \frac{\xi ||h||_{L_2}^2}{\|h\|_{L_p}^2} \right)^{p/(p-2)}.
$$

To identify $k$, set $Z = h^2 1_{(|h| \leq M)}(X)$, and observe that

$$
E Z^2 \leq \begin{cases} 
  c_2(p) \|h\|_{L_p}^4 \left( \frac{\|h\|_{L_p}^2}{\xi ||h||_{L_2}^2} \right)^{(4-p)/(p-2)} & \text{if } 2 < p \leq 4, \\
  \|h\|_{L_4}^4 & \text{if } p \geq 4.
\end{cases}
$$

Let $Z_1, \ldots, Z_m$ be independent copies of $Z$. Applying Bernstein's inequality it follows that

$$
\left| \frac{1}{m} \sum_{i=1}^m Z_i - E Z \right| > \frac{\xi}{2} ||h||_{L_2}^2
$$

with probability at most

$$
\begin{cases} 
  2 \exp \left( -c_3(p) m \left( \frac{\xi ||h||_{L_2}^2}{\|h\|_{L_p}^2} \right)^{p/(p-2)} \right) & \text{if } 2 < p < 4, \\
  2 \exp \left( -c_4 m \min \left\{ \left( \frac{\xi^2 ||h||_{L_4}^2}{\|h||_{L_2}^4} \right)^{2}, \left( \frac{\xi ||h||_{L_2}^2}{\|h\|_{L_p}^2} \right)^{p/(p-2)} \right\} \right) & \text{if } p \geq 4.
\end{cases}
$$

implying that one may set

$$
k = \begin{cases} 
  c_3(p) m \left( \frac{\xi ||h||_{L_2}^2}{\|h||_{L_p}^2} \right)^{p/(p-2)} & \text{if } 2 < p < 4, \\
  c_4 m \min \left\{ \left( \frac{\xi^2 ||h||_{L_4}^2}{\|h||_{L_2}^4} \right)^{2}, \left( \frac{\xi ||h||_{L_2}^2}{\|h\|_{L_p}^2} \right)^{p/(p-2)} \right\} & \text{if } p \geq 4.
\end{cases}
$$

**Remark 2.7.** Note that if $p > 4$ then $h^4 = h^4 h^{4-\alpha}$ for $\alpha = 2(p-4)/(p-2)$. By Hölder’s inequality for $q = (p-2)/(p-4)$ and $q' = (p-2)/2$, it follows that

$$
Eh^4 \leq (E h^2)^{(p-4)/(p-2)} \cdot (E |h|^p)^{2/(p-2)};
$$

therefore,

$$
\frac{||h||_{L_4}^2}{||h||_{L_p}^2} \leq \left( \frac{||h||_{L_2}^2}{||h||_{L_p}^2} \right)^{p/(p-2)},
$$

and for $p \geq 4$ one may take

$$
k \sim m \xi^2 \left( \frac{||h||_{L_2}^2}{||h||_{L_p}^2} \right)^{p/(p-2)}.
$$
Norm equivalence

Another useful example is when $h$ satisfies an $L_q - L_2$ norm equivalence, i.e., when $\|h\|_{L_q} \leq L \|h\|_{L_2}$ for some constant $L$. It follows that

$$M(h, \xi) \leq \left( \frac{2L^2}{\xi} \right)^{1/(q-2)} \|h\|_{L_q},$$

and one may set

$$\ell = c_1(q) \left( \frac{\xi}{L^2} \right)^{q/(q-2)} \quad \text{and} \quad k = \begin{cases} c_2(q) m \left( \frac{\xi}{L^2} \right)^{q/(q-2)} & \text{if } 2 < q < 4, \\ c_3 m \left( \frac{\xi}{L^2} \right) & \text{if } q \geq 4. \end{cases}$$

3 The main result

With the notion of a stable lower bound set in place we can now formulate our main result. To that end, fix integers $m, n$ such that $N = mn$ and let $(I_j)_{j=1}^n$ be the natural partition of $\{1, \ldots, N\}$ to coordinate blocks of cardinality $m$. Recall that $D$ is the unit ball in $L_2(\mu)$ and $S$ is the corresponding unit sphere. For $H \subset L_2(\mu)$ denote by $M(H, \rho D)$ the cardinality of a maximal $\rho$-separated subset of $H$ with respect to the $L_2(\mu)$ norm.

**Theorem 3.1.** There exist absolute constants $c_0, c_1$ and $c_2$ for which the following holds.

Let $H$ be star-shaped around 0 (i.e., $\text{star}(H, 0) = H$) and for $r > 0$ set $H_r = H \cap rD$. Fix $0 < \eta, \xi < 1$ and let $r > 0$ such that

1. Every $h \in H \cap rS$ satisfies a stable lower bound with parameters $(\xi/2, \ell, k)$, for $k \geq \max\{4, 2 \log(4/\eta)\}$.

2. $\log M(H \cap rS, c_0 \sqrt{\xi} r D) \leq \frac{\eta N}{16} \cdot \frac{k}{m}$.

3. $\mathbb{E} \sup_{u \in (H_r - H_r) \cap c_0 \sqrt{\xi} r D} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i u(X_i) \right| \leq c_1 \eta \xi r \cdot \sqrt{\frac{k}{m}}$.

Then with probability at least

$$1 - 2 \exp \left( -c_2 \eta N \min \left\{ \frac{\ell}{m}, \frac{k}{m} \right\} \right)$$

we have

$$\inf_{\{h \in H : \|h\|_{L_2} \geq r\}} \left\{ j : \frac{1}{m} \sum_{i \in I_j} h^2(X_i) \geq (1 - \xi)\|h\|_{L_2}^2 \right\} \geq (1 - \eta)n.$$

Moreover, the same assertion holds if we replace Conditions (2) and (3) with

4. $\mathbb{E} \sup_{u \in H \cap rD} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i u(X_i) \right| \leq c_3 \eta \xi r \cdot \min \left\{ \sqrt{\frac{\ell}{m}}, \sqrt{\frac{k}{m}} \right\}$, where $c_3$ is an absolute constant.

5. If $h_1, h_2 \in H \cap rS$ and $\|h_1 - h_2\|_{L_2} \geq c_0 \sqrt{\xi} r$ then $h_1 - h_2$ satisfies a stable lower bound with parameters $(1/2, \ell, k)$.

9
Remark 3.2. In what follows we only consider the more difficult case, in which $0 < \xi \leq 1/2$. We omit the proof of Theorem 3.1 when $1 - \xi$ is closer to 0 (e.g., in the situation explored in Theorem 1.3 using the standard small-ball method). The proof when $\xi$ is larger than $1/2$ requires a minimal modification of the argument we do present. Also, it is straightforward to verify that Theorem 1.3 is an outcome of Theorem 3.1 (with slightly different constants) when $H$ is assumed to satisfy a small-ball property.

The sufficient condition described in the ‘moreover’ part of Theorem 3.1 can be far from optimal because Condition (4) is significantly more restrictive than the combination of Conditions (2) and (3), forcing one to consider larger values of $r$. Indeed, standard examples of a stable lower bound indicate that often $k \sim \xi \ell$. Therefore, taking the minimum between $\sqrt{\ell/m}$ and $\sqrt{k/m}$ comes at a cost of $\sim \sqrt{\xi}$. Moreover, the indexing set $H \cap rS$ may be much larger than $(H - H) \cap c\sqrt{r}$. Both factors affect the outcome of Theorem 3.1 when one is looking for a sharp dependence on $\xi$ or when $\xi$ is very small—tending to 0 with $N$ (see the example of the smallest singular value of a random matrix with iid rows, described below). However, when $\xi$ happens to be a fixed constant or when the dependence on $\xi$ is of secondary importance as is the case in most statistical applications, the combination of Condition (4) and Condition (5) is a suitable replacement for Conditions (2) and (3).

3.1 Some examples

To give a flavour of the ways in which Theorem 3.1 may be applied, let us present some of its implications, starting with the question of a lower bound on the smallest singular value of a random matrix with iid rows.

The smallest singular value

Let $X$ be an isotropic random vector in $\mathbb{R}^d$; that is, for every $t \in \mathbb{R}^d$, $E \langle X, t \rangle^2 = \|t\|^2$. Let $\Gamma = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \langle X_i, e_i \rangle$ be the random matrix whose rows are $N$ independent copies of $X$. Assume that the class of linear functionals $\{\langle t, \cdot \rangle : t \in \mathbb{R}^d\}$ satisfies an $L_q$–$L_2$ norm equivalence for some $2 < q \leq 4$, namely, that for every $t \in \mathbb{R}^d$, $\|\langle X, t \rangle\|_{L_q} \leq L \|\langle X, t \rangle\|_{L_2} = L \|t\|_2$.

It is well known that an $L_q$–$L_2$ norm equivalence is not a strong enough condition for a sharp estimate on the largest singular value of a typical $\Gamma$. It certainly does not suffice for obtaining the optimal Bai-Yin type estimate of

$$1 - c(L,q) \sqrt{\frac{d}{N}} \leq \lambda_{\min}(\Gamma) \leq \lambda_{\max}(\Gamma) \leq 1 + C(L,q) \sqrt{\frac{d}{N}}$$

(see [4] and [1, 24, 30] and references therein for more details on the Bai-Yin Theorem and its non-asymptotic versions). However, it turns out that the smallest singular value can be controlled, and in an optimal way. Indeed, following the progress in [28, 25, 13], Yaskov showed in [35] that if $\sup_{t \in S^d-1} \|\langle t, X \rangle\|_{L_q} \leq L$, then with probability at least $1 - 2 \exp(-cd)$,

$$\lambda_{\min}(\Gamma) \geq \begin{cases} 1 - c(L) \left(\frac{d}{N}\right)^{1/2} & \text{if } q = 4 \\ 1 - c(L,q) \left(\frac{d}{N}\right)^{1 - 2/q} & \text{if } 2 < q < 4 \end{cases}$$

(3.1)

All the arguments previously used to obtain a lower bound on the smallest singular value were based on the specific nature of the problem in a very strong way: the fact that one was
looking for a lower bound on $\inf_{t \in S^{d-1}} \| \Gamma t \|_2$. In fact, almost all the methods of proof used to obtain non-asymptotic Bai-Yin type estimates were limited to bounds on

$$\inf_{t \in S^{d-1}} \frac{1}{N} \sum_{i=1}^{N} \langle X_i, t \rangle^2$$

and could not be extended to indexing sets other than the Euclidean unit ball.

Let us show that Theorem 3.1, which is completely general and does not rely on a special structure of the indexing set, leads to an almost optimal estimate on $\lambda_{\min}(\Gamma)$ with just a logarithmic looseness.

**Theorem 3.3.** Let $X$, $q$ and $L$ be as in (3.1) and let $N \geq c_0 d$. Then with probability at least

$$1 - 2 \exp \left( -c_1(q,L)d \log \left( \frac{eN}{d} \right) \right),$$

we have

$$\lambda_{\min}(\Gamma) \geq 1 - c_2(L,q) \left( \frac{d}{N} \log \left( \frac{eN}{d} \right) \right)^{1-2/q}.$$

**Proof.** Clearly, a lower bound on $\lambda_{\min}(\Gamma) = \inf_{t \in S^{d-1}} \frac{1}{N} \sum_{i=1}^{N} \langle X_i, t \rangle^2$ falls within the scope of Theorem 3.1 by setting $H = \{ \langle t, \cdot \rangle : \| t \|_2 \leq 1 \}$ and with the choices of $m = N$ (a single coordinate block), and $\eta = 0.5$. Let

$$\xi = \left( \alpha \frac{d}{N} \cdot \log \left( \frac{eN}{d} \right) \right)^{1-2/q}$$

for $\alpha \geq 1$ to be named later.

Fix $t \in \mathbb{R}^d$ and recall that by the $L_q - L_2$ norm equivalence, the function $\langle t, X \rangle$ satisfies a stable lower bound for parameters $(\xi/2, \ell, k)$, where one may set

$$\ell, k \sim_{L,q} \xi^{q/(q-2)} N = c(q,L) \alpha d \log \left( \frac{N}{d} \right).$$

Note that $k$ satisfies Condition (1) if $\alpha$ is larger than a suitable absolute constant. To verify Condition (2) for $r = 1$, recall that $X$ is isotropic, and thus the $L_2(\mu)$ distance coincides with the Euclidean one. Therefore, by a standard volumetric estimate,

$$\mathcal{M}(H \cap S, c\xi D) \leq \left( \frac{c_1}{\xi} \right)^d \leq \exp \left( c_2(q) d \log \left( \frac{N}{\alpha d} \right) \right),$$

and, since $m = N$ and $\eta = 0.5$,

$$\exp \left( \frac{\eta N}{16} \cdot \frac{k}{m} \right) = \exp \left( c_3(q,L) \alpha d \log \left( \frac{N}{d} \right) \right).$$

Clearly, $c_2(q) d \log \left( \frac{N}{\alpha d} \right) \leq c_3(q,L) \alpha d \log \left( \frac{N}{d} \right)$ for $\alpha = \alpha(q,L)$ large enough.

Turning to Condition (3), observe that for $r = 1$ and $\rho > 0$

$$(H_r - H_r) \cap \rho D \subset \{ \langle t, \cdot \rangle : \| t \|_2 \leq \rho \}$$
and that $E\|X\|^2 = d$. Hence,

$$E \sup_{u \in (H_r - H_r) \cap \rho D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq \frac{\rho}{N} E\| \sum_{i=1}^{N} \varepsilon_i X_i \|_2 \leq \rho \sqrt{\frac{d}{N}},$$

and to confirm Condition (3) one has to show that for $\rho = c \xi, c \xi \sqrt{\frac{d}{N}} \leq c_4 \xi \frac{\sqrt{d}}{N}$. By (3.3) it suffices that

$$\xi \geq c_5(q, L) \left( \frac{d}{N} \right)^{1-2/q}.$$

Hence, if $\alpha = \alpha(q, L)$ is a well-chosen constant then the conditions of Theorem 3.1 hold, and therefore, with probability at least

$$1 - 2 \exp \left( -c_6(q, L) d \log \left( \frac{N}{d} \right) \right),$$

$$\inf_{t \in S} \frac{1}{N} \sum_{i=1}^{N} \langle X_i, t \rangle^2 \geq 1 - c_7(q, L) \left( \frac{d}{N} \log \left( \frac{eN}{d} \right) \right)^{1-2/q}.$$

Remark 3.4. Note that invoking the ‘moreover’ part of Theorem 3.1 would result in a weaker estimate, since

$$(H_r - H_r) \cap c \xi D = \{ \langle t, \cdot \rangle : \|t\|_2 \leq c \xi \},$$

while

$$H \cap r D = \{ \langle t, \cdot \rangle : \|t\|_2 \leq 1 \}$$

is a much larger set.

It is straightforward to apply Theorem 3.1 to any class of functions whose members satisfy a stable lower bound. We chose to focus on two cases: classes that satisfy a uniform integrability condition following the path of [19], and bounded classes in $L_p$, most notably, classes consisting of uniformly bounded functions.

Uniform integrability

Corollary 3.5. There are absolute constants $c_0, ..., c_4$ for which the following holds. Assume that for $0 < \xi < 1$ there is a constant $\kappa(\xi)$ such that for every $h \in H$, $M(h, \xi) \leq \kappa(\xi) \|h\|_{L_2}$. Let $r > 0$ satisfy that

$$\log \mathcal{M}(H \cap r S, c_0 \xi r D) \leq c_1 N \frac{\xi^2}{\kappa^2(\xi)}$$

and

$$E \sup_{u \in (H_r - H_r) \cap c \xi r D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_3 \frac{\xi^{3/2}}{\kappa(\xi)}.$$

Then with probability at least

$$1 - 2 \exp \left( -c_4 N \frac{\xi^2}{\kappa^2(\xi)} \right),$$

$$\inf_{\{h \in H : \|h\|_{L_2} \geq r\}} \left\{ j : \frac{1}{m} \sum_{i \in I_j} h_i^2(X_i) \geq (1 - \xi) \|h\|_{L_2}^2 \right\} \geq 0.99 n.$$

(3.4)

A version of Corollary 3.5 was used in [19] to analyze the performance of a tournament procedure. However, though the proof presented in [19] is weaker than the proof of Theorem 3.1 as it does not extend to other situations in which a stable lower bound holds.
Bounded subsets of $L_p$

The second corollary is based on the ‘moreover’ part of Theorem 3.1:

**Corollary 3.6.** Let $p > 2$ and assume that $H$ is a bounded class in $L_p$, by $M_p$. Set $r > 0$ such that

$$
\mathbb{E} \sup_{h \in H \cap c_0 r D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_1 r \frac{r^{p/(p-2)}}{M_p}
$$

Then with probability at least

$$
1 - 2 \exp \left( -c_2 N \left( \frac{r^2}{M_p^2} \right)^{p/(p-2)} \right),
$$

(3.4) holds; here $c_0$ is a constant that depends only on $\xi$ and $c_1$ and $c_2$ depend only on $\xi$ and $p$.

Moreover, for $p = \infty$, i.e., if every $h \in H$ satisfies that $\|h\|_{L_{\infty}} \leq M$ and if

$$
\mathbb{E} \sup_{h \in H \cap c_0 r D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_1 \frac{r^2}{M},
$$

then with probability at least $1 - 2 \exp(-c_2 N \left( \frac{r^2}{M^2} \right))$, (3.4) holds; here $c_0, c_1$ and $c_2$ depend only on $\xi$.

In what follows we use Corollary 3.6 to study the performance of statistical procedures—mainly of Empirical Risk Minimization (ERM) but also of the tournament procedure from [19].

### 4 Proof of Theorem 3.1

Recall that the proof of Theorem 1.3 is based on three components. Firstly, an individual estimate that holds with high probability—specifically, that with probability at least $1 - 2 \exp(-c_0 m \delta)$,

$$
\frac{1}{m} \sum_{i=1}^{m} h^2(X_i) \geq c_1(\kappa, \delta) \|h\|_{L_2}^2;
$$

secondly, that this estimate is stable, in the sense that discarding a reasonable number of coordinates does not significantly affect the sum; and finally, a second type of stability: if $f, h$ are close then the vector $((f - h)(X_i))_{i=1}^{N}$ does not have many large coordinates. Once these properties are established, the high probability individual estimate leads to uniform control over a net, and the two notions of stability allow one to pass from the net to the entire class.

The same ideas are used in the proof of Theorem 3.1. Because the claim is homogeneous and $H$ is star-shaped around 0, it suffices to prove Theorem 3.1 only for $H \cap r S$. And to deal with $H \cap r S$, one proceeds with the following steps:

1. For the given choice of $0 < \xi < 1$, each individual function $h$ satisfies a stable lower bound with parameters $(\xi/2, \ell, k)$. 

14
(2) Given the \( n \) coordinate blocks \( (I_j)_{j=1}^n \) of cardinality \( m \), with probability at least \( 1 - 2 \exp(-c_0 \eta nk) \), the stable lower bound in (1) holds for at least \( (1 - \eta/2)n \) coordinate blocks.

(3) The high probability estimate in (2) combined with the union bound allows one to obtain (2) for a net in \( H \cap rS \), as long as its cardinality is at most \( \exp(c_1 \eta nk) \) for \( c_1 = c_0/2 \).

(4) If we denote by \( \pi h \) the nearest element to \( h \) in the net, the stability implies that for at least \( (1 - \eta/2)n \) of the coordinate blocks, one may discard the set \( J_j(h) \) consisting of the \( \ell \) largest values of the oscillation term \(|h - \pi h|(X_i)_{i \in I_j}\) and still have

\[
\frac{1}{m} \sum_{i \in I_j \setminus J_j(h)} (\pi h)^2(X_i) \geq \left(1 - \frac{\xi}{2}\right) \|\pi h\|_{L^2}^2 = \left(1 - \frac{\xi}{2}\right) r^2.
\]

Hence, for every \( h \in H \cap rS \) there are at least \( (1 - \eta/2)n \) coordinate blocks such that

\[
\left(\frac{1}{m} \sum_{i=1}^m h^2(X_i)\right)^{1/2} \geq \left(\frac{1}{m} \sum_{i \in I_j \setminus J_j(h)} (\pi h)^2(X_i)\right)^{1/2} - \left(\frac{1}{m} \sum_{i \in I_j \setminus J_j(h)} (h - \pi h)^2(X_i)\right)^{1/2}
\]

\[
\geq (1 - \xi/2)^{1/2} r - \left(\frac{1}{m} \sum_{i \in I_j \setminus J_j(h)} (h - \pi h)^2(X_i)\right)^{1/2} \geq (1 - \xi)^{1/2} r \quad (4.1)
\]

if there is sufficient control over the last term.

Out of this list, (1) is just the stable lower bound; (2) is an immediate outcome of Bennett’s inequality; and (3) is the reason for the entropy condition in Theorem 3.1. This leaves us with the crucial point in the proof of Theorem 3.1 which is controlling (4).

To that end, let \( \theta_1 \) be a constant that is specified in what follows, and let \( H' \) be a maximal \( \theta_1 \xi r \)-separated subset of \( H \cap rS \). Given a sample \( (X_i)_{i=1}^N \), let

\[
V = \left\{ v = ((h - \pi h)(X_i))_{i=1}^N : h \in H \cap rS \right\}
\]

and put \( P_{I_j}v = (v_i)_{i \in I_j} \). The aim is to ensure that for every \( v \in V \) there are at least \( (1 - \eta/2)n \) coordinate blocks \( I_j \) such that

\[
\frac{1}{m} \sum_{i > \ell} \left((P_{I_j}v)^*\right)^2 \leq \frac{\xi^2}{4} r^2,
\]

implying that for every \( h \in H \cap rS \), (4.1) holds for \( (1 - \eta)n \) coordinate blocks.

In other words, if for \( h \in H \cap rS \) and \( v = ((h - \pi h)(X_i))_{i=1}^N \) we set

\[
\mathcal{P}_h = \left\{ j : \frac{1}{m} \sum_{i > \ell} \left((P_{I_j}v)^*\right)^2 > \frac{\xi^2}{4} r^2 \right\}
\]

then the main component of the proof of Theorem 3.1 is to show that with high probability,

\[
\sup_{h \in H \cap rS} \mathcal{P}_h \leq \frac{nm}{2}.
\]
Lemma 4.1. There exist absolute constants \(c_1, c_2\) and \(c_3\) for which the following holds. Let \(H\) be star-shaped around 0, set \(\theta^2 \leq c_1\eta\) and let \(H'\) to be a maximal \(\theta_1\xi r\)-separated subset of \(H \cap rS\) with respect to the \(L_2(\mu)\) norm. If

\[
\mathbb{E} \sup_{h \in H \cap rS} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i(h - \pi h)(X_i) \leq c_2\eta \sqrt{\frac{\ell}{m}} r,
\]

then

\[
Pr \left( \sup_{h \in H \cap rS} \|h\| \geq \frac{\eta m}{2} \right) \leq 2 \exp \left( -c_3 N \frac{\ell}{m} \min \left\{ \eta, \frac{\eta^2}{\theta^2} \right\} \right).
\]

The proof of Lemma 4.1 is based on Talagrand’s concentration inequality for empirical processes indexed by classes of uniformly bounded functions \([29]\), see also \([7]\).

Theorem 4.2. There exists an absolute constant \(C_0\) for which the following holds. Let \(F\) be a class of functions and set \(\sigma_F^2 = \sup_{f \in F} \mathbb{E} f^2\) and \(b_F = \sup_{f \in F} \|f\|_{L_\infty}\). Then, for any \(x > 0\), with probability at least \(1 - 2 \exp(-x)\),

\[
\sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \mathbb{E} f \right| \leq C_0 \left( \mathbb{E} \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i f(X_i) \right| + \sigma_F \sqrt{\frac{x}{N}} + b_F \frac{x}{N} \right),
\]

(4.2)

Proof of Lemma 4.1. Let \(A = (m/\ell)^{1/2} \xi r\) and set

\[
\phi_A(t) = \begin{cases} A \cdot \text{sgn}(t) & \text{if } |t| > A, \\ t & \text{if } |t| \leq A. \end{cases}
\]

Given \(h \in H \cap rS\), \(\pi h \in H'\) and \(v_i = (h - \pi h)(X_i)\) as above, let

\[
u_i = \phi_A(|v_i|) \quad \text{and} \quad w_i = |v_i| 1_{\{|v_i| > A\}}.
\]

Note that for every coordinate block \(I_j\),

\[
\min_{J_j \subset I_j, |J_j| = \ell} \frac{1}{m} \sum_{i \in I_j \setminus J_j} (P_{I_j} v_i)^2 \leq \min_{J_j \subset I_j, |J_j| = \ell} \frac{2}{m} \sum_{i \in I_j \setminus J_j} u_i^2 + w_i^2,
\]

and if

\[
\frac{1}{m} \sum_{i \geq \ell} (P_{I_j} v_i^*)^2 = \min_{J_j \subset I_j, |J_j| = \ell} \frac{1}{m} \sum_{i \in I_j \setminus J_j} (P_{I_j} v_i)^2 \geq \xi^2 r^2 / 4
\]

then either \(\frac{1}{m} \sum_{i \in I_j} u_i^2 \geq \xi^2 r^2 / 16\), or, there are at least \(\ell\) coordinates in \(I_j\) such that \(w_i^2 \geq \xi^2 r^2 / 16\). Therefore, if we set

\[
\sharp_h^1 = \left\{ j : \frac{1}{m} \sum_{i \in I_j} \phi_A^2(|h - \pi h|(X_i)) \geq \frac{\xi^2 r^2}{16} \right\} \quad \text{and} \quad \sharp_h^2 = \left\{ j : (|h - \pi h|(X_i))^* \geq \frac{\xi r}{4} \right\},
\]

then

\[
\sharp_h \leq \sharp_h^1 + \sharp_h^2.
\]
Observe that if \( \sup_{h \in H \cap rS} \frac{1}{h} \geq \eta m/4 \) then
\[
\sup_{h \in H \cap rS} \sum_{j=1}^{n} \frac{1}{m} \sum_{i \in I_j} \phi^2_A(\|h - \pi h\|(X_i)) \geq \frac{\eta m}{4} \cdot \xi^2 r^2 / 16;
\]
that is,
\[
(*)_1 \equiv \sup_{h \in H \cap rS} \frac{1}{N} \sum_{i=1}^{N} \phi^2_A(\|h - \pi h\|(X_i)) \geq \frac{1}{64} \eta^2 \xi^2 r^2. \tag{4.3}
\]

Invoking Theorem 4.2 let us show that with high probability, \((*)_1 < \frac{1}{16} \eta^2 \xi^2 r^2\), and therefore, on that event, \( \sup_{h \in H \cap rS} \frac{1}{h} < \eta m/4 \).

Clearly, \( \phi_A(t) \leq |t| \), and for \( \theta^2 \leq c_1 \eta \) we have that
\[
\mathbb{E} \phi^2_A(\|h - \pi h\|(X_i)) \leq \mathbb{E} |h - \pi h|^2 \leq \theta^2 \eta^2 r^2 \leq \frac{1}{256 C_0} \eta^2 \xi^2 r^2,
\]
where \( C_0 \) is the constant from (4.2).

Also, \( \phi^2_A \) is a Lipschitz function with a constant \( 2A \) and satisfies \( \phi^2_A(0) = 0 \). Thus, by the contraction inequality for Bernoulli processes [15],
\[
\mathbb{E} \sup_{h \in H \cap rS} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \phi^2_A(\|h - \pi h\|(X_i)) \right| \leq 2A \mathbb{E} \sup_{h \in H \cap rS} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (h - \pi h)(X_i) \right| \leq \frac{1}{256 C_0} \eta^2 \xi^2 r^2
\]
provided that
\[
\mathbb{E} \sup_{h \in H \cap rS} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i |h - \pi h|(X_i) \right| \lesssim \eta \xi^2 r^2 / A \sim \left( \frac{\xi}{m^2} \right)^{1/2} \eta \xi r \tag{4.4}
\]
by our choice of \( A \).

Next, note that for \( F = \{ \phi^2_A(\|h - \pi h\|(X)) : h \in H \cap rS \} \),
\[
\sigma^2_F \leq \sup_{h \in H \cap rS} A^2 \|h - \pi h\|_{L^2}^2 \leq A^2 (\theta_1 \xi r)^2 \quad \text{and} \quad b_F \leq A^2.
\]
By Theorem 4.2 \((*)_1 \leq \frac{1}{16} \eta^2 \xi^2 r^2\) with probability at least \( 1 - 2 \exp \left( -c_3 N \xi^2 r^2 \min \left\{ \eta, \theta^2 \right\} \right) \), implying that
\[
Pr \left( \sup_{h \in H \cap rS} \frac{1}{m} \leq \frac{\eta m}{4} \right) \geq 1 - 2 \exp \left( -c_3 N \frac{\ell}{m} \min \left\{ \eta, \theta^2 \right\} \right). \tag{4.5}
\]

Next, note that if \( \frac{1}{m} \geq \frac{\eta m}{4} \), then at least \( \ell \eta m/4 \) of the values \( (|h - \pi h|(X_i))_{i=1}^{N} \) are larger than \( A \). To conclude the proof, let us show that with high probability,
\[
\sup_{h \in H \cap rS} |\{ i : |h - \pi h|(X_i) \geq A \}| \leq \frac{1}{8} \ell \eta m.
\]

Define \( \Psi_A : \mathbb{R}_+ \to \mathbb{R}_+ \) by
\[
\Psi_A(t) = \begin{cases} 
1 & \text{if } t \geq A, \\
\frac{2}{A} (t - \frac{A}{2}) & \text{if } t \in [A/2, A), \\
0 & \text{if } t \in [0, A/2].
\end{cases}
\]
It is evident that
\[ \sum_{i=1}^{N} 1\{ |h - \pi h| > A \}(X_i) \leq \sum_{i=1}^{N} \Psi_A(|h - \pi h|(X_i)), \]
and therefore, it suffices to show that
\[ \sup_{h \in H \cap rS} \frac{1}{N} \sum_{i=1}^{N} \Psi_A(|h - \pi h|(X_i)) \leq \frac{1}{8} \eta \frac{\ell}{m}. \]
Again, we invoke Theorem 4.2. Observe that
\[ \mathbb{E} \Psi_A(|h - \pi h|(X_i)) \leq \Pr(|h - \pi h|(X) \geq A/2) \leq \frac{4 \| h - \pi h \|_{L_2}^2}{A^2} \leq \frac{4 \theta_1^2 \xi r^2}{A^2}. \]
Therefore, \( \mathbb{E} \Psi_A(|h - \pi h|(X_i)) \leq (\eta/24) \cdot (\ell/m) \) provided that
\[ \frac{\theta_1^2 \xi r^2}{A^2} \leq \eta \frac{\ell}{m}, \tag{4.6} \]
which, by our choice of \( A \), holds if \( \theta_1^2 \leq \eta. \)

The function \( \Psi_A(t) \) is Lipschitz with constant \( 2/A \) and \( \Psi_A(0) = 0. \) By the contraction inequality for Bernoulli processes,
\[ \mathbb{E} \sup_{h \in H \cap rS} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \Psi_A(|h - \pi h|(X_i)) \leq \frac{2}{A} \mathbb{E} \sup_{h \in H \cap rS} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i (h - \pi h)(X_i) \leq \frac{\eta}{24C_0} \frac{\ell}{m}, \]
if
\[ \mathbb{E} \sup_{h \in H \cap rS} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i (h - \pi h)(X_i) \right| \leq \eta \frac{\ell}{m} \cdot A \sim \eta \sqrt{\frac{\ell}{m}} \xi r, \tag{4.7} \]
and (4.7) follows for our choice of \( r. \)

Moreover, for every \( h \in H \cap rS \) we have
\[ \mathbb{E} \Psi_A^2(|h - \pi h|(X)) \leq \Pr(|h - \pi h|(X) \geq A/2) \leq \frac{4 \theta_1^2 \xi r^2}{A^2} \] and \( \| \psi_A(|h - \pi h|) \|_{L_\infty} \leq 1, \)
and by Theorem 4.2,
\[ \Pr \left( \sup_{h \in H \cap rS} \frac{1}{N} \sum_{i=1}^{N} \Psi_A(|h - \pi h|(X_i)) \leq \frac{1}{8} \eta \frac{\ell}{m} \right) \geq 1 - 2 \exp \left( -c_0 N \frac{\ell}{m} \min \left\{ \eta, \frac{\eta^2}{\theta_1^2} \right\} \right), \]
as claimed.

Thanks to Lemma 4.3, the proof of the first part of Theorem 3.1 follows by showing that there is a net \( H' \) of \( H \cap rS \) whose mesh is \( \theta_1 \xi r, \) and with high probability, each \( h' \in H' \) satisfies an appropriate stable lower bound on at least \( (1 - \eta/2)n \) of the coordinate blocks.

**Lemma 4.3.** Let \( k \geq \max\{4, 2 \log(4/\eta)\} \) and let \( h \) satisfy a stable lower bound with parameters \((\xi/2, \ell, k). \) Then with probability at least \( 1 - 2 \exp(-\eta nk/8) \) there are at least \( (1 - \eta/2)n \) coordinate blocks \( I_j \) such that for any \( J_j \subset I_j \) of cardinality \( \ell, \)
\[ \frac{1}{m} \sum_{i \in I_j \setminus J_j} h^2(X_i) \geq \eta \frac{\ell}{2} \text{E} h^2. \tag{4.8} \]
Moreover, if \( H' \) is a class of functions that satisfy such a stable lower bound and \( \log |H'| \leq \eta nk/16, \) then with probability at least \( 1 - 2 \exp(-\eta nk/16), \tag{4.8} \) holds for every \( h' \in H'.\)
Proof. Let \((\delta_j)_{j=1}^n\) be independent selectors that take the value 0 on the ‘good event’ we are interested in; that is, each \(\delta_j\) is a \(\{0, 1\}\)-valued random variable, defined by \(\delta_j = 0\) if for every \(J_j \subset I_j\) of cardinality at most \(\ell\) one has

\[
\frac{1}{m} \sum_{i \in I_j \setminus J_j} h^2(X_i) \geq (1 - \xi) \mathbb{E}h^2.
\]

Therefore, \(\mathbb{E}\delta_j = \delta \leq 2 \exp(-k)\). If \(\eta \geq 4 \exp(-k/2)\) and \(k \geq 4\) then by Bennett’s inequality,

\[
\Pr \left( \sum_{j=1}^n \delta_i \geq \frac{\eta}{2} n \right) \leq \exp \left( -\frac{\eta}{2} n (\log(1 + \eta/2\delta) - 1) \right) \leq \exp(-\eta nk/8),
\]

as required.

The second part of the claim is evident from the union bound.

Proof of Theorem 3.1, part I. As noted previously, the claim is positive homogeneous, and since \(H\) is star-shaped around 0, it suffices to prove it for \(H \cap rS\). For that class, the combination of Lemma 4.3 and Lemma 4.1 leads to the wanted conclusion. Indeed, setting \(\theta_1 \sim \sqrt{\eta}\), by Conditions (1) and (2) there is \(H' \subset H \cap rS\) that is \(\theta_1\xi r\)-maximal separated and \(\log |H'| \leq \eta nk/16\). Hence, by Lemma 4.3 with probability at least

\[
1 - 2 \exp(-\eta nk/4) = 1 - 2 \exp \left( -c_1 \eta N \frac{k}{m} \right),
\]

for every \(u \in H'\) there are at least \((1 - \eta/2)\) coordinate blocks \(I_j\) such that for every \(J_j \subset I_j\) of cardinality at most \(\ell\),

\[
\frac{1}{m} \sum_{i \in I_j \setminus J_j} u^2(X_i) \geq (1 - \xi/2)\|u\|^2_{L^2} = (1 - \xi/2)r^2.
\]

(4.9)

Recall that \(\pi h\) is the nearest point to \(h\) in \(H'\) relative to the \(L^2(\mu)\) distance and set \(v = ((h - \pi h)(X_i))_{i=1}^N\). By Lemma 4.1 with probability at least

\[
1 - 2 \exp \left( -c_2 \eta N \frac{\ell}{m} \right),
\]

for every \(h \in H \cap rS\), there are at most \(\eta m/2\) coordinate blocks \(I_j\) such that

\[
\frac{1}{m} \sum_{i > \ell} \left( (P_{I_j}v)^* \right)^2 > \frac{\xi^2 r^2}{2},
\]

(4.10)

and by (4.1), if (4.9) and (4.10) hold then for every \(h \in H \cap rS\) there are at least \((1 - \eta)n\) coordinate blocks \(I_j\) such that

\[
\frac{1}{m} \sum_{i \in I_j} h^2(X_i) \geq (1 - \xi)r^2 = (1 - \xi)\|h\|^2_{L^2}.
\]
Let us turn to the proof of the second part of Theorem 3.1 showing that Conditions (2) and (3) can be replaced by Conditions (4) and (5).

Clearly, if \( r \) satisfies Condition (4) for the right choice of constant then it satisfies Condition (3) as well. This is evident because \( 0 \in H \) and therefore the indexing set in Condition (4) is larger than in Condition (3); moreover, the RHS of Condition (4) is smaller than the RHS of Condition (3). Therefore, all that is left is to show that Conditions (4) and (5) also imply Condition (2); in particular, that if

\[
\mathbb{E} \sup_{u \in H \cap rD} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c \eta \sqrt{\xi r} \min \left\{ \sqrt{\frac{\ell}{m}}, \sqrt{\frac{k}{m}} \right\},
\]

for the right choice of \( c \) then

\[
\log \mathcal{M}(H \cap rS, c_0 \sqrt{\eta \xi r} D) \leq \frac{\eta nk}{16}.
\]

**Theorem 4.4.** There exist absolute constants \( c_0, c_1 \) and \( c_2 \) for which the following holds. Let \( \rho > 0 \) and \( 0 < \eta < 1 \). Assume that for any \( h_1, h_2 \in H \cap rS \) that are \( \rho \)-separated, \( h_1 - h_2 \) satisfies a stable lower bound with constants \((1/2, \ell, k)\) for \( k \geq c_0 \). Assume further that

\[
\log \mathcal{M}(H \cap rS, \rho D) \geq \frac{\eta nk}{16}.
\]

Then with probability at least \( 1 - 2 \exp(-c_1 \eta nk) \),

\[
\mathbb{E}_\varepsilon \sup_{h \in H \cap rS} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i h(X_i) \right| \geq c_2 \rho \min \left\{ \sqrt{\frac{\ell}{m}}, \sqrt{\frac{k}{m}} \right\}.
\]

**Remark 4.5.** The constant \( 1/2 \) in the stable lower bound may be replaced by any number in \((0,1)\), and that only affects the value of \( c_2 \) in Theorem 4.4.

Applying Theorem 4.4 for the choice of \( \rho = c_0 \sqrt{\eta \xi r} \) shows that Conditions (4) and (5) imply Condition (2). With that, the second part of the theorem follows from the first one.

The proof of Theorem 4.4 is based on Sudakov’s inequality for Bernoulli processes [15] in its scale-sensitive formulation (see, e.g., [14]):

**Theorem 4.6.** There exists an absolute constant \( c \) for which the following holds. Let \( V \subset \mathbb{R}^N \) and for every \( v \in V \) set \( Z_v = \sum_{i=1}^{N} \varepsilon_i v_i \). If \( |V| \geq \exp(p) \) and \( \{Z_v : v \in V\} \) is \( \varepsilon \)-separated in \( L_p \) then

\[
\mathbb{E} \sup_{v \in V} \sum_{i=1}^{N} \varepsilon_i v_i \geq c \varepsilon.
\]

**Proof of Theorem 4.4.** Let \( h_1, h_2 \in H \cap rS \) such that \( \|h_1 - h_2\|_{L_2} \geq \rho \), implying that \( h_1 - h_2 \) satisfies a stable lower bound with parameters \((1/2, \ell, k)\). Hence, by Lemma 4.3 with probability at least \( 1 - 2 \exp(-c_1 \eta nk) \), there are at least \( n/2 \) coordinate blocks \( I_j \) such that for any \( J_j \subset I_j \) of cardinality \( \ell \),

\[
\frac{1}{m} \sum_{i \in I_j \setminus J_j} (h_1 - h_2)^2(X_i) \geq \frac{1}{2} \mathbb{E}(h_1 - h_2)^2. \tag{4.11}
\]
Without loss of generality assume that the first $n/2$ coordinate blocks are among the ‘good blocks’, and in particular their union contains the coordinates $\{1, \ldots, N/2\}$. Set $v = (h_1(X_i))_{i=1}^N$ and $u = (h_2(X_i))_{i=1}^N$ and consider the random variable

$$Z_v - Z_u = \sum_{i=1}^N \varepsilon_i (v_i - u_i).$$

By a standard contraction inequality [15] and the characterization of the $L_p$ norm of the random variable $Z_a = \sum_{i=1}^N \varepsilon_i a_i$ from [14], it follows that

$$\|Z_v - Z_u\|_{L_p} \geq \left\| \sum_{i=1}^{N/2} \varepsilon_i (u_i - v_i) \right\|_{L_p} \sim \max_{I'} \left( \sum_{i \in I'} |u_i - v_i| + \sqrt{p} \left( \sum_{i \in \{1, \ldots, N/2\} \setminus I'} (v_i - u_i)^2 \right)^{1/2} \right),$$

where the maximum is taken over all the sets $I' \subset \{1, \ldots, N/2\}$ of cardinality at most $p$. Let us obtain a lower bound by selecting $I'$: we take any set $J_j \subset I_j$ of cardinality $\ell$, and $I'$ is the union of these sets. Hence, $|I'| = \ell n/2$ and by (4.11),

$$\sum_{i \in \{1, \ldots, N/2\} \setminus I'} (v_i - u_i)^2 \geq \frac{n}{2} \cdot \frac{m}{2} \|h_1 - h_2\|_{L_2}^2 \gtrsim N\|h_1 - h_2\|_{L_2}^2,$$

It follows that for $p = \ell n/2$ and every sample $(X_i)_{i=1}^N$ in an event with probability at least $1 - 2 \exp(-c_1 kn)$,

$$\|Z_v - Z_u\|_{L_p} \gtrsim \sqrt{p} \cdot \sqrt{N\|h_1 - h_2\|_{L_2}}.$$

The same argument holds for any $2 \leq p \leq \ell n/2$, as one may choose $\tilde{I} \subset I'$ and

$$\sum_{i \in \{1, \ldots, N/2\} \setminus \tilde{I}} (v_i - u_i)^2 \geq \sum_{i \in \{1, \ldots, N/2\} \setminus I'} (v_i - u_i)^2.$$

Now, let $H'$ be a maximal $\rho$-separated subset of $H \cap rS$ and recall that $\log |H'| \gtrsim \eta nk$. By the union bound, with probability at least $1 - 2 \exp(-c_2 \eta nk)$ the set $V = \{(h(X_i))_{i=1}^N : h \in H'\}$ contains at least $\exp(c_3 \eta nk)$ vectors $v$, for which the random variables $\sum_{i=1}^N \varepsilon_i v_i$ are $\sim \sqrt{p} \cdot \sqrt{N\rho}$ separated in $L_p$ for any $2 \leq p \leq \ell n/2$. Set

$$p \sim n \min\{k, \ell\} = N \min\left\{ \frac{k}{m}, \frac{\ell}{m} \right\},$$

and by Theorem [4.6] with probability at least $1 - 2 \exp(-c_4 \eta nk)$ relative to $X_1, \ldots, X_N$,

$$\mathbb{E}_v \sup_{v \in V} Z_v \geq c_5 N \rho \min\left\{ \sqrt{\frac{\ell}{m}}, \sqrt{\frac{k}{m}} \right\},$$

i.e.,

$$\mathbb{E}_h \sup_{v \in V} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i h(X_i) \right| \geq c_5 \rho \min\left\{ \sqrt{\frac{\ell}{m}}, \sqrt{\frac{k}{m}} \right\}.$$
5 Applications in Statistical Learning Theory

Finally, we would like to apply Theorem 3.1 to problems in statistical learning theory—specifically in the context of empirical risk minimization (ERM) and of the recently introduced tournament procedure [16, 19].

Unfortunately, because the analysis of tournament procedures is rather involved, it would be impossible to present an accurate description of the role Theorem 3.1 plays in the study of tournaments. Therefore, we only formulate one tournament-related outcome of Theorem 3.1—regarding a tournament procedure performed in a class of uniformly bounded functions, and without presenting the proof (see Appendix A). For more details on tournament procedures we refer the reader to [16, 19], and especially to Section 5 in [19] which explains the need for a result like Theorem 3.1 in the study of tournaments.

In what follows we focus on applications of Theorem 3.1 to ERM relative to the squared loss—a procedure that has been of central importance in statistical learning theory from the very first days of the area (see the books [8, 33, 34, 2, 32, 6, 17, 12, 31] for some information on the history of ERM).

Definition 5.1. For a class of functions $F$ and an iid sample $(X_i,Y_i)_{i=1}^N$ selected according to the joint distribution of $X$ and $Y$, ERM selects a function $\hat{f} \in F$ that minimizes the empirical squared risk functional

$$f \rightarrow \frac{1}{N} \sum_{i=1}^N (f(X_i) - Y_i)^2.$$

It turns out that the study of ERM requires an almost isometric lower bound for a fixed value of $\xi$, and on a single block ($m = N$). To explain why that is the case, one must first explore the so-called Bernstein condition and its connection with the squared loss.

The success of any learning procedure is measured according to its ability to produce a function that has almost the same predictive capabilities as $f^* = \arg\min_{f \in F} \mathbb{E}(f(X) - Y)^2$. In other words, one would like to show that with probability at least $1 - \delta$,

$$\mathbb{E}\left(\left(\hat{f}(X) - Y\right)^2 \mid (X_i,Y_i)_{i=1}^N\right) \leq \mathbb{E}(f^*(X) - Y)^2 + r^2,$$

and the tradeoff between the accuracy parameter $r$ and the confidence with which a procedure attains the accuracy $r^2$ quantifies the procedure’s performance.

One of the main ingredients needed in the analysis of the accuracy/confidence tradeoff exhibited by ERM is a natural convexity type condition which we now describe.

Definition 5.2. The triplet $(F, X, Y)$ satisfies a Bernstein condition with constant $B$ if there is some $f^* \in F$ such that for any $f \in F$,

$$\|f - f^*\|_{L^2}^2 \leq B \left(\mathbb{E}(f(X) - Y)^2 - \mathbb{E}(f^*(X) - Y)^2\right).$$

(5.1)

Since (5.1) remains true if $B$ increases, one may assume without loss of generality that $B \geq 2$.

Note that if $(F, X, Y)$ satisfies a Bernstein condition then the risk functional $f \rightarrow \mathbb{E}(f(X) - Y)^2$ has a unique minimizer in $F$, and that minimizer is $f^*$. Setting

$$L_f(X, Y) = (f(X) - Y)^2 - (f^*(X) - Y)^2$$

21
to be the excess loss associated with \( f \in F \), \(^{[5.1]}\) becomes
\[
\| f - f^* \|_{L^2}^2 \leq B \mathbb{E} L_f
\]
and thus, for every \( f \in F \),
\[
\mathbb{E}( f - f^* ) ( X ) \cdot ( f^*(X) - Y ) \geq -\frac{1}{2} \left( 1 - \frac{1}{B} \right) \| f^* - f \|_{L^2}^2.
\]
(5.3)

If \( \mathcal{B} \) denotes the \( L^2 \) ball centred at \( Y \) and of radius \( \| f^*(X) - Y \|_{L^2} \), then the linear functional
\[
h \rightarrow \mathbb{E}_h(X)(f^*(X) - Y)
\]
supports \( \mathcal{B} \) at \( f^*(X) - Y \). Since the LHS of \(^{[5.3]}\) is the action of that functional on \( ( f - f^*) ( X ) \), it follows that if \( F \) is convex and closed in \( L^2(\mu) \) then the triplet \( ( F, X, Y ) \) satisfies \(^{[5.2]}\) with constant 1. Indeed, by the characterization of the nearest point map onto a closed convex set in a Hilbert space, \( h \rightarrow \mathbb{E}_h(X)(f^*(X) - Y) \) separates \( \mathcal{B} \) and \( F \).

Another example in which a Bernstein condition is satisfied even when \( F \) need not be convex is when \( Y = f_0(X) + W \) for some \( f_0 \in F \) and \( W \) that is independent of \( X \)—the so-called independent additive noise model. In that case \( f^* = f_0 \) and again \(^{[5.2]}\) holds with constant 1.

In general, the Bernstein condition is related to the distance between the target \( Y \) and the set of targets that have multiple minimizers in \( F \) \(^{[20]}\).

The main fact about the performance of ERM for triplets \( ( F, X, Y ) \) that satisfy a Bernstein condition is the following:

**Theorem 5.3.** Let \( ( F, X, Y ) \) be a triplet that satisfies a Bernstein condition with constant \( B \) and without loss of generality assume that \( B \geq 2 \). Set \( \xi(X, Y) = f^*(X) - Y \) and for \( 1 \leq i \leq N \), put \( \xi_i = \xi(X_i, Y_i) \). Let \( \rho > 0 \) and \( r > 0 \) and set \( \mathcal{A} \) to be an event for which the following holds: for every \( f \in F \) and every \( ( X_i, Y_i )_{i=1}^N \),
\[
\begin{align*}
(1) \text{ if } \| f - f^* \|_{L^2} \geq r \text{ then } & \quad \frac{1}{N} \sum_{i=1}^N (f - f^*)^2(X_i) \geq (1 - \gamma) \| f - f^* \|_{L^2}^2 \text{ for } \gamma = \rho + 1/B \text{ such that } 0 < \gamma < 1, \\
(2) \text{ if } \| f - f^* \|_{L^2} \geq r \text{ then } & \quad \frac{1}{N} \sum_{i=1}^N \xi_i (f - f^*)(X_i) - \mathbb{E} \xi(f - f^*)(X) \leq \frac{\rho}{2} \| f - f^* \|_{L^2}^2, \\
(3) \text{ if } \| f - f^* \|_{L^2} \leq r \text{ then } & \quad \frac{1}{N} \sum_{i=1}^N \xi_i (f - f^*)(X_i) - \mathbb{E} \xi(f - f^*)(X) \leq r^2.
\end{align*}
\]
Let \( \hat{f} \) denote the outcome of ERM. Then on the event \( \mathcal{A} \) one has
\[
\| \hat{f} - f^* \|_{L^2} \leq r^2 \quad \text{and} \quad \mathbb{E} \left( \hat{f} - Y \vert (X_i, Y_i)_{i=1}^N \right) \leq 3r^2.
\]

The proof of Theorem \(^{5.3}\) is standard, and is presented in an Appendix \(^{[3]}\) for the sake of completeness.

Condition (1) in the definition of the event \( \mathcal{A} \) follows from Theorem \(^{3.1}\) for a single block and with a constant \( \xi = \rho + 1/B \). Establishing Conditions (2) and (3) is in some sense trivial, as the two follow from estimates on a multiplier process rather than on a quadratic one like Condition (1). Indeed, set
\[
\phi(r) = \sup_{u \in \star ( F - f^*, 0 ) \cap D} \left| \frac{1}{N} \sum_{i=1}^N \varepsilon_i \xi_i u(X_i) \right|
\]

22
and let
\[ r_1(\delta) = \inf \left\{ r > 0 : \Pr \left( \phi(r) \geq \frac{\rho}{2} r'^2 \right) \leq \delta \right\} \] (5.4)
where \( \rho < 1 \) as above. It follows that for \( r > r_1(\delta) \), with probability at least \( 1 - \delta \) both Condition (2) and (3) hold.

Let us use Theorem 5.3 together with Theorem 5.4 to analyze the performance of ERM when \( F \) is a bounded class in \( L_p \) for some \( p > 2 \) and \( (F, X, Y) \) satisfies a Bernstein condition with constant \( B \geq 2 \).

**Corollary 5.4.** For \( B \geq 2 \) and \( p > 2 \) there are constants \( c_0, c_1 \) and \( c_2 \) that depend only on \( B \) and \( p \) for which the following holds. Let \( F \) be a class of functions whose diameter in \( L_p(\mu) \) is at most \( M_p \) and assume that the triplet \( (F, X, Y) \) satisfies a Bernstein condition with constant \( B \). Fix \( 0 < \delta < 1 \), set \( r_1(\delta) \) as above and put
\[ r_2 = \inf \left\{ r > 0 : \mathbb{E} \sup_{u \in \text{star}(F - f^*, 0) \cap \mathcal{R}D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_1 r \left( \frac{r}{M_p} \right)^{p/(p-2)} \right\}. \]
Then with probability at least
\[ 1 - \delta - 2 \exp(-c_2 N(r^2/M_p^2)^{p/(p-2)}) \]
\[ \mathbb{E} \left( L_f \mid (X_i, Y_i)_{i=1}^{N} \right) \leq 3 \max\{r_1^2(\delta), r_2^2\}. \] (5.5)

The proof is an immediate outcome of Theorem 3.1 for the class \( \text{star}(F - f^*, 0) \) combined with Theorem 5.3 and the definition of \( r_1(\delta) \).

In the case \( p = \infty \), Corollary 5.4 implies that one may take
\[ r_2 = \inf \left\{ r > 0 : \mathbb{E} \sup_{u \in \text{star}(F - f^*, 0) \cap \mathcal{R}D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_1 r^2 \right\} \]
for constants \( c_0 \) and \( c_1 \) that depend only on \( B \). Then, (5.3) holds with probability at least
\[ 1 - \delta - 2 \exp(-c_2 Nr^2/M^2) \]
for a constant \( c_2 \) that depends only on \( B \).

As it happens, the case \( p = \infty \) is special because there is a natural way of controlling \( r_1(\delta) \) for \( \delta \) that is very small. Indeed, assume that the target \( Y \) is also bounded by \( M \) and observe that by Talagrand’s concentration inequality for bounded empirical processes, with probability at least \( 1 - \exp(-x) \)
\[ \phi(r) \leq c_0 \left( \mathbb{E} \sup_{u \in \text{star}(F - f^*, 0) \cap \mathcal{R}D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| + \sigma \sqrt{\frac{x}{N} + \frac{bx}{N^2}} \right), \]
where
\[ \sigma^2 = \sup_{u \in \text{star}(F - f^*, 0) \cap \mathcal{R}D} \mathbb{E} \xi^2 u^2 \leq \| \xi \|_{L_\infty}^2 r^2 \leq 4M^2 r^2, \]
and
\[ b = \sup_{u \in \text{star}(F - f^*, 0) \cap \mathcal{R}D} \| \xi u \|_{L_\infty} \leq 4M^2. \]
Set \( x = c_1 \lambda^2 N r^2 / M^2 \) and \( \rho = 1/B \). By the contraction inequality for Bernoulli processes,

\[
E \sup_{u \in \text{star}(F-f^*,0) \cap rD} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i \xi_i u(X_i) \right| \leq 2M E \sup_{u \in \text{star}(F-f^*,0) \cap rD} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| .
\]

Hence, if

\[
E \sup_{u \in \text{star}(F-f^*,0) \cap rD} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_2 \lambda^2 \frac{r^2}{M}
\]

then with probability at least \( 1 - 2 \exp(-c_1 \lambda^2 N r^2 / M^2) \), \( r_1(\delta) \leq \lambda r^2 \leq (\rho/2)r^2 \) provided that \( \lambda \) is a well-chosen constant that depends only on \( B \). Combining this observation with Corollary 5.4 for \( p = \infty \) one recovers the optimal estimate on the performance of ERM in the bounded framework from \( [5] \) (see also \( [12] \)).

**Corollary 5.5.** For \( B \geq 2 \) there exist constants \( c_0, c_1, c_2 \) that depend only on \( B \) for which the following holds. Let \( F \) be a class of functions bounded almost surely by \( M \), assume that the target \( Y \) is also bounded almost surely by \( M \) and that \( (F,X,Y) \) satisfies a Bernstein condition with constant \( B \). If we set

\[
r = \inf \left\{ E \sup_{u \in \text{star}(F-f^*,0) \cap cD} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_1 \frac{r^2}{M} \right\}
\]

then with probability at least \( 1 - 2 \exp(-c_2 N r^2 / M^2) \)

\[
E \left( \mathcal{L}_f|X_i,Y_i|_{i=1}^{N} \right) \leq 3r^2.
\]

**Remark 5.6.** Corollary 5.4 does not follow from the standard concentration/contraction type argument that may be used when \( p = \infty \). Unless \( p = \infty \), class members need not be bounded; the squared loss does not satisfy a Lipschitz condition on the range on \( F \); and the contraction argument for Bernoulli processes cannot be used. Moreover, Talagrand’s concentration inequality is true only for classes of uniformly bounded functions, which is not the case when \( p < \infty \).

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A Tournaments involving bounded classes

It is well known that ERM is far from being an optimal procedure unless the triplet $(F, X, Y)$ satisfies a Bernstein condition. Moreover, even when a Bernstein condition is true, ERM performs poorly in heavy-tailed situations. This is indicated in the argument presented in the previous section by the fact that for ERM to perform with accuracy $\sim r^2$ and confidence $1 - \delta$, one must ensure that $\phi(r) \lesssim r^2$ with probability $1 - \delta$. However, when class members or the ‘noise’ $\xi = Y - f^*(X)$ happen to be heavy-tailed, the tradeoff between $r$ and $\delta$ is not satisfactory, and considerably weaker than the optimal tradeoff one expects (see the discussion in [16] for more details). The suboptimal behavior of ERM was the reason for the introduction of the tournament procedure in [16] for convex classes and in [19] for general prediction problems even when the underlying class need not be convex.

Due to the technical nature of the definition of a tournament procedure we will not describe it here. As it happens, a key component in the analysis of tournaments is to identify $r > 0$ and a high probability event such that for any $f \in F$ with $\|f - f^*\|_{L^2} \geq r$

$$\frac{1}{m} \sum_{i \in I_j} (f - f^*)^2(X_i) \geq (1 - \varepsilon)\|f - f^*\|^2_{L^2}$$

for at least $0.99n$ of the coordinate blocks $I_j$; here $n \sim N \min\{r^2/\|\xi\|_{L^2}^2, 1\}$ and $\varepsilon < 1$ is some fixed constant.

The following can be obtained by combining Theorem 3.1 with the methods developed in [19]:

**Corollary A.1.** There exist absolute constants $c_0, c_1, c_2$ and $c_3$ for which the following holds. Let $F$ be a class consisting of functions bounded almost surely by $M$ and assume that $Y$ is also bounded almost surely by $M$. Let $H = \text{star}(F - F, 0)$, set

$$r_0 = \inf \left\{ r : \mathbb{E} \sup_{f \in H \cap \text{core} D} \left| \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i u(X_i) \right| \leq c_1 \frac{r^2}{M} \right\}$$

and put $r > r_0$. There is a tournament procedure $\tilde{f}$ such that with probability at least

$$1 - 2 \exp \left( -c_2 N \min \left\{ \frac{r^2}{M^2}, 1 \right\} \right),$$

satisfies

$$\mathbb{E} \left( (\tilde{f}(X) - Y)^2 \right) \leq \inf_{f \in F} \mathbb{E} (f(X) - Y)^2 + c_3 r^2.$$
B Proof of Theorem 5.3

Let \((F,X,Y)\) satisfy a Bernstein condition with constant \(B\) and without loss of generality assume that \(B \geq 2\). Setting \(\xi(X,Y) = f^*(X) - Y\) it follows that

\[
\mathcal{L}_f = (f(X) - Y)^2 - (f^*(X) - Y)^2 = (f - f^*)^2(X) + 2(f(X) - f^*(X)) \cdot (f^*(X) - Y) \\
= (f - f^*)^2(X) + 2(\xi(f(X) - f^*(X)) - \mathbb{E}\xi(f(X) - f^*(X))) + 2\mathbb{E}(\xi(f(X) - f^*(X)) \\
\geq (f - f^*)^2(X) - \left(1 - \frac{1}{B}\right) \|f^* - f\|_{L^2}^2 + 2(\xi(f(X) - f^*(X)) - \mathbb{E}\xi(f(X) - f^*(X))).
\]

Observe that if \(\hat{f}\) minimizes the empirical risk then it also minimizes the empirical excess risk; therefore, \(\sum_{i=1}^N \mathcal{L}_f(X_i, Y_i) \leq 0\), simply because \(\mathcal{L}_{f^*} \equiv 0\) is a potential minimizer.

Let us begin by showing that on the event \(\mathcal{A}\), for any \(f \in F\) for which \(\|f - f^*\|_{L^2} \geq r\), one also has \(\sum_{i=1}^N \mathcal{L}_f(X_i, Y_i) > 0\)—thus excluding functions in the set \(\{f \in F : \|f - f^*\|_{L^2} \geq r\}\) from being potential empirical minimizers. Indeed, by Condition (1) and (2), on the event \(\mathcal{A}\),

\[
\frac{1}{N} \sum_{i=1}^N \mathcal{L}_f(X_i, Y_i) \geq \frac{\rho}{2} \|f - f^*\|_{L^2}^2
\]

if \(\|f - f^*\|_{L^2} \geq r\). Thus, \(\hat{f} \notin \{f \in F : \|f - f^*\|_{L^2} \geq r\}\) for a sample in \(\mathcal{A}\), proving the first part of the claim.

Next, observe that if \(f \in F\) satisfies that \(\mathbb{E}\mathcal{L}_f \geq 3r^2\) and \(\|f - f^*\|_{L^2} \leq r\), then

\[
3r^2 \leq \mathbb{E}\mathcal{L}_f = \|f - f^*\|_{L^2}^2 + 2\mathbb{E}\xi(f - f^*)(X) \leq r^2 + 2\mathbb{E}\xi(f - f^*)(X),
\]

implying that \(2\mathbb{E}\xi(f - f^*)(X) \geq 2r^2\). Hence, by Condition (3),

\[
\frac{1}{N} \sum_{i=1}^N \mathcal{L}_f(X_i, Y_i) \geq \frac{2}{N} \sum_{i=1}^N (\xi_i(f - f^*)(X_i) - \mathbb{E}\xi(f - f^*)(X_i)) + 2\mathbb{E}(\xi(f(X) - f^*(X)) \\
\geq -r^2 + 2\mathbb{E}\xi(f - f^*)(X) \geq r^2,
\]

and \(\hat{f} \in \{f \in F : \mathbb{E}\mathcal{L}_f \leq 3r^2\}\).