FOLIATION OF A SPACE ASSOCIATED TO VERTEX OPERATOR ALGEBRA

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Abstract. We construct the foliation of a space associated to correlation functions of vertex operator algebras on considered on Riemann surfaces. We prove that the computation of general genus \( g \) correlation functions determines a foliation on the space associated to these correlation functions a sewn Riemann surface. Certain further applications of the definition are proposed.

1. Introduction

The theory of vertex operator algebras correlation functions considered on Riemann surfaces is a rapidly developing field of studies. Algebraic nature of methods applied in this field helps to understand and compute the structure of vertex operator algebras correlation functions. On the other hand, the geometric side of vertex operator algebra correlation functions is in associating their formal parameters with local coordinates on a manifold. Depending on the geometry, one can obtain various consequences for a vertex operator algebra and its space of correlation functions. One is able to study the geometry of a manifold using the algebraic structure of a vertex operator algebra defined on it, and, in particular, it is important to consider foliations of associated spaces.

In this paper we introduce the formula for an \( n \)-point function for a vertex operator algebra \( V \) on a genus \( g \) Riemann surface \( S^{(g)} \) obtained as a result of sewing of lower genus surfaces \( S^{(g_1)} \) and \( S^{(g_2)} \) of genera \( g_1 \) and \( g_2 \), \( g = g_1 + g_2 \). Using this formulation, we then introduce the construction of a foliation for the space of correlation functions for vertex operator algebra with formal parameters defined on general sewn Riemann surfaces. Computations of a vertex operator algebra correlation functions allow us to define foliation of the space associated to a vertex operator algebra with formal parameter associated to a local coordinate on a genus \( g \) Riemann surface sewn from lower genus Riemann surfaces. Such foliations are important both for studies of the space of correlation function for a vertex operator algebras and possibly for studies of smooth manifolds in the frames of Losik’s approach [9].

Key words and phrases. Vertex operator algebras; Riemann surfaces; correlation functions; foliations.
2. Vertex operator algebras

First, we recall the definitions of the standard formal series

$$\delta \left( \frac{x}{y} \right) = \sum_{n \in \mathbb{Z}} x^n y^{-n},$$  \hfill (2.1)

$$(x + y)^\kappa = \sum_{m \geq 0} \left( \frac{\kappa}{m} \right)x^{\kappa-m}y^m,$$  \hfill (2.2)

for any formal variables $x, y, \kappa$ where $\left( \frac{\kappa}{m} \right) = \frac{\kappa (\kappa-1) \ldots (\kappa-m+1)}{m!}$.

A vertex operator algebra $[2, 4] (V, Y, 1, \omega)$ consists of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ where $\dim V_n < \infty$ for each $n \in \mathbb{Z}$, a linear map $Y : V \to \text{End}(V)[[z, z^{-1}]]$ for a formal parameter $z$ and pair of distinguished vectors: the vacuum $1 \in V_0$ and the conformal vector $\omega \in V_2$. For each $v \in V$, the image under the map $Y$ is the vertex operator

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1},$$

with modes $v(n) \in \text{End}(V)$, where $Y(v, z)1 = v + O(z)$.

The linear operators (modes) $u(n) : V \to V$ satisfy creativity

$$Y(u, z)1 = u + O(z)$$  \hfill (2.3)

and lower truncation

$$u(n)v = 0,$$  \hfill (2.4)

for each $u, v \in V$ and $n \gg 0$. Each vertex operator satisfies the translation property

$$Y(L(-1)u, z) = \partial_z Y(u, z).$$  \hfill (2.5)

Finally, the vertex operators satisfy the Jacobi identity

$$z_0^{-1}\delta \left( \frac{z_1 - z_2}{z_0} \right)Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta \left( \frac{z_2 - z_1}{-z_0} \right)Y(v, z_2)Y(u, z_1)$$

$$= z_0^{-1}\delta \left( \frac{z_1 - z_0}{z_2} \right)Y(Y(u, z_0)v, z_2).$$  \hfill (2.6)

Vertex operators satisfy locality, i.e., for all $u, v \in V$ there exists an integer $k \geq 0$ such that

$$(z_1 - z_2)^k [Y(u, z_1), Y(v, z_2)] = 0.$$

The vertex operator of the conformal vector $\omega$ is

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},$$

where the modes $L(n)$ satisfy the Virasoro algebra with central charge $c$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} - \delta_{m,-n}c \text{Id}_V.$$  

We define the homogeneous space of weight $k$ to be

$$V(k) = \{ v \in V | L(0)v = kv \}.$$
and we write \( (v) = k \) for \( v \in V_{(k)} \). Amongst other properties, these axioms imply locality, associativity, commutativity and skew-symmetry:

\[
(z_1 - z_2)^m Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^m Y(v, z_2) Y(u, z_1),
\]
\[
(2.7)
\]

\[
(\omega_n + z_2)^n Y(u, \omega_n + z_2) Y(v, z_2) w = (\omega_n + z_2)^n Y(Y(u, \omega_n) v, z_2) w,
\]
\[
(2.8)
\]

\[
u(k) Y(v, z) - Y(v, z) u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u(j) v, z) z^{k-j},
\]
\[
(2.9)
\]

\[
Y(u, z) v = e^{z L(-1)} Y(v, -z) u,
\]
\[
(2.10)
\]

for \( u, v, w \in V \) and integers \( m, n \gg 0 \).

In \[23\] Zhu introduced a second "square-bracket" vertex operator algebra \( (V, Y[,], 1, \tilde{\omega}) \) associated to a given vertex operator algebra \( (V, Y(\cdot), 1, \omega) \). The new square bracket vertex operators are defined by a change of parameters, namely

\[
Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z V(0), q_z - 1),
\]
\[
(2.11)
\]

with \( q_z = \exp(z) \), while the new conformal vector is \( \tilde{\omega} = \omega - \frac{c}{24} \mathbb{1} \). For \( v \) of \( L(0) \) weight \( wt(v) \in \mathbb{R} \) and \( m \geq 0 \),

\[
v[m] = m! \sum_{i \geq m} c(wt(v), i, m) v(i),
\]
\[
(2.12)
\]

\[
\sum_{m=0}^{i} c(wt(v), i, m) x^m = \binom{wt(v) - 1 + x}{i}.
\]
\[
(2.13)
\]

In particular we note that \( v[0] = \sum_{i \geq 0} \binom{wt(v) - 1}{i} v(i) \).

2.1. The invariant form. The subalgebra \( \{L(-1), L(0), L(1)\} \cong SL(2, \mathbb{C}) \) associated with Möbius transformations on \( z \) naturally acts on a vertex algebra \[4\]. In particular,

\[
\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto w = -\frac{1}{z},
\]
\[
(2.14)
\]

is generated by

\[
T = \exp(L(-1)) \exp(L(1)) \exp(L(-1)),
\]

where

\[
TY(u, z) T^{-1} = Y \left( (\exp(-z)L(1)) (-z)^{-2L(0)} u, -z^{-1} \right).
\]
\[
(2.15)
\]

Following \[13\], we therefore define

\[
Y^\dagger(u, z) = \sum_n u^\dagger(n) z^{-n-1} = TY(u, z) T^{-1}.
\]
\[
(2.16)
\]

For a quasi-primary vector \( u \) (i.e., satisfying the condition \( L(1) u = 0 \)) of weight \( wt(u) \), we have

\[
u^\dagger(n) = (-1)^{n+1} u(2wt(u) - n - 2),
\]
\[
(2.17)
\]

and \( L^\dagger(n) = (-1)^n L(-n) \).
Definition 1. We call a bilinear form \( \langle . , . \rangle \) on \( V \) invariant if for all \( a, b, u \in V \),
\[
\langle Y(u, z)a, b \rangle = \langle a, Y^\dagger(u, z)b \rangle,
\]
i.e.,
\[
\langle u(a)n, b \rangle = \langle a, u^\dagger(n)b \rangle.
\]
Thus it follows that
\[
\langle L(0)a, b \rangle = \langle a, L(0)b \rangle,
\]
so that \( \langle a, b \rangle = 0 \) if \( wt(a) \neq wt(b) \) for homogeneous \( a, b \). One also finds \( \langle a, b \rangle = \langle b, a \rangle \).

The form \( \langle . , . \rangle \) is unique up to normalization if \( L(1)V_1 = V_0 \). We choose the normalization \( \langle 1_V, 1_V \rangle = 1 \). It is non-degenerate if and only if \( V \) is simple \([8]\). Given any \( V \) basis \( \{u^\alpha\} \) we define the dual basis \( \{\overline{u}^\beta\}_\lambda = \delta^\alpha_\beta \).

3. Construction of an \( n \)-point function at a genus \( g \) Riemann surface

Recall that a conformal field theory defined on a Riemann surface \( S^{(g)} \) of genus \( g \) \([3, 10, 16, 17, 18, 20]\) is determined by the set \( \{Z^{(g)}_V(v_0, z_1; \ldots; v_n, z_n; \Omega^{(g)})\} \) of all correlation functions for all \( n \), and all choices of points \( z_i \in S^{(g)} \), and all choices of elements \( v_i \) of corresponding vertex operator algebra \( V \).

In this section, extending the genus two results of \([16, 19, 20]\), we introduce the definition of an \( n \)-point correlation function for a vertex operator algebra \( V \) on a genus \( g \) Riemann surface \( S^{(g)} \) obtained as a result of sewing of lower genus surfaces \( S^{(g_i)} \) and \( S^{(g_2)} \) of genera \( g_1 \) and \( g_2 \), \( g = g_1 + g_2 \). The genus \( g \) \( n \)-point correlation function for \( a_1, \ldots, a_L \in V \) and \( b_1, \ldots, b_R \in V \), \( L + R = n \) inserted at \( x_1, \ldots, x_L \in S^{(g_1)} \) and \( y_1, \ldots, y_R \in S^{(g_2)} \), respectively, can be defined by
\[
Z^{(g)}_V(a_1, x_1; \ldots; a_L, x_L|b_1, y_1; \ldots; b_R, y_R; \Omega_1^{(g_1)}, \Omega_2^{(g_2)}; \epsilon, z_1, z_2) \\
= \sum_{n \geq 0} \sum_{u \in V} \epsilon^n Z^{(g_1)}_V(Y(a_1, x_1) \ldots Y(a_L, x_L)u; \Omega_1^{(g_1)}, z_1) \\
\cdot Z^{(g_2)}_V(Y(b_R, y_R) \ldots Y(b_1, y_1); \Omega_2^{(g_2)}, z_2).
\]
Note that this construction of the correlation functions depends on the choice of insertion points \( z_1 \in S^{(g_1)} \) and \( z_2 \in S^{(g_2)} \) of the \( \epsilon \)-sewing construction \([22]\). Here \( \epsilon \) is the sewing complex parameter \([22]\) and \( \Omega_i^{(g_i)}, i = 1, 2 \) are period matrices for Riemann surfaces \( S^{(g_i)} \). To avoid misunderstanding, we say here, that the lower genus correlation functions \( Z^{(g_i)}_V(Y(a_1, x_1) \ldots Y(a_L, x_L)u; \Omega_1^{(g_1)}, z_1) \) and \( Z^{(g_2)}_V(Y(b_R, y_R) \ldots Y(b_1, y_1); \Omega_2^{(g_2)}, z_2) \) are supposed to be known for given fixed \( g_1, g_2 \), or obtained via \([5, 1]\) recursively, from known lower genus correlation functions (e.g., starting from the partition functions, or torus correlation functions). The explicit dependence on \( \Omega^{(g_i)}, i = 1, 2 \) is assumed. Not all vertex operator algebras admit the notion of dual states. Thus we assume that there exists non-degenerate bilinear form on \( V \) \([4]\). On the right hand side the form is present in the definition of the dual state \( \overline{\pi} \) with respect to such form.
There exists an algebraic procedure [23] relating \( n \)-point correlation functions to a sum of \((n - 1)\)-point functions for vertex operator algebras on the torus. In [23] we find that the genus one \( n \)-point correlation functions obey

\[
Z_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau) = \text{Tr}_V \left( \left( (v_1) \omega(t_{v_1}) - 1 \right) Y(q_{z_2}L(0)v_2, q_{z_2}) \ldots Y(q_{z_n}L(0)v_n, q_{z_n}) q^{L(0) - c/24} \right)
+ \sum_{k=2}^{n} \sum_{j \geq 0} P_{1+j}(z_1 - z_k, \tau) Z_V^{(1)}(v_2, z_2; \ldots; v_1[j]v_k, z_k; \ldots; v_n, z_n; \tau),
\]

where \( P_{1}(z, \tau) \) are Weierstrass functions [8], and square bracket on the right hand side denotes the deformed mode for vertex operator algebra on the torus [23].

A generalization [17] of the recursion procedure of [23] allows us to reduce a genus \( g = g_1 + g_2 \) \( n \)-point correlation function to the genus \( g \) zero-point correlation function (the partition function) on any higher genus Riemann surface formed from two lower genus \( g_1, g_2 \) Riemann surfaces in the\( \epsilon \) sewing procedure [22]. Using explicit results of [17][18] for the Heisenberg vertex operator algebra in the Schottky formation of a genus \( g \) Riemann surface, we conjecture the following general form of the reduction formula for a general vertex operator algebra considered on a genus \( g \) Riemann surface obtained as a results of sewing of two lower genera \( g_1 \) and \( g_2 \) Riemann surfaces:

\[
Z_V^{(g)}(v_1, z_1; \ldots; v_n, z_n; \mathcal{T}^{(g)}) = \sum_{k \geq 0} \mathbb{P}_{k,n}^{(g)}(A_1^{(g_1)}, A_2^{(g_2)} \cdot f_{V,k,n}(A) \cdot \mathcal{O}_{k,n}^{(g)}),
\]

where \( \mathcal{T}^{(g)} = (\epsilon_1, \epsilon_2, z_1, z_2, \Omega^{(g_1)}, \Omega^{(g_2)}) \), are parameters of the genus \( g \) \( n \)-point correlation function a sewn genus \( g \) Riemann surface, \( \mathbb{P}_{k,n}^{(g)}(A_1^{(g_1)}, A_2^{(g_2)}) \) is a polynomial of special infinite matrices \( A_1^{(g_1)}, A_2^{(g_2)} \), [10][17] associated to the period matrices for Riemann surfaces \( S^{(g_1)} \) and \( S^{(g_2)} \), \( \mathbb{P}_{k,n}^{(g)} \), \( \mathcal{O}_{k,n}^{(g)} \) are certain generalizations of classical elliptic functions on higher genus Riemann surfaces depending on arguments \((v_1, z_1; \ldots; v_n, z_n)\), and \( f_{V,k,n} \) is a function of the matrix \( A = (I - A_1^{(g_1)} A_2^{(g_2)}) \).

Note that, using results of [15][18] for the multiple-sewn sphere case, and as it is shown in [20], the genus one case trace formulas can be obtain from the higher genus formulas (in particular, from genus two) in the complex parameter \( \rho \)-sewing procedure [22].

4. CONSTRUCTION OF A FOLIATION OF THE SPACE OF CORRELATION FUNCTIONS OVER RIEMANN SURFACES FOR A VERTEX OPERATOR ALGEBRA

The construction of a vertex operator algebra \( n \)-point function of arbitrary genus gives us a hint how to define a special-type foliation of a space related to a vertex operator with the formal parameter associated to coordinates on a Riemann surface (formed from two lower genus surfaces) of genus \( g \) by means of vertex operator algebra correlation functions.
A \( p \)-dimensional foliation \( \mathcal{F} \) of an \( n \)-dimensional manifold \( \mathcal{M} \) is a covering by a system of domains \( \{ U_i \} \) of \( \mathcal{M} \) together with maps 
\[
\phi_i : U_i \to \mathbb{C}^n,
\]
such that for overlapping pairs \( U_i, U_j \) the transition functions 
\[
\varphi_{ij} : \mathbb{C}^n \to \mathbb{C}^n,
\]
defined by 
\[
\varphi_{ij} = \phi_j \phi_i^{-1},
\]
take the form 
\[
\varphi_{ij}(x, y) = (\varphi_{ij}^1(x), \varphi_{ij}^2(x, y)),
\]
where \( x \) denotes the first \( n - p \) coordinates, and \( y \) denotes the last \( p \) co-ordinates. That is, 
\[
\varphi_{ij}^1 : \mathbb{C}^{n-p} \to \mathbb{C}^{n-p},
\]
\[
\varphi_{ij}^2 : \mathbb{C}^p \to \mathbb{C}^p.
\]

In this paper, we would like to foliate the space 
\[
\mathcal{M}_n = V^\otimes n \times \mathcal{S}^{(g)},
\]
where \( V \) is a vertex operator algebra, and \( \mathcal{S}^{(g)} \) is a Riemann surface of genus \( g \).

**Proposition 1.** The correlation functions \([33]\) determine a foliation on the space 
\[
\mathcal{M} = \bigoplus_{n \geq 0} \mathcal{M}_n.
\]

**Proof.** We consider a system of charts \( \{ U_m \} \), \( m \in \mathbb{N} \), covering the Riemann surface part of the space \( \mathcal{M}_n \) together with a filtration of the space \( V^\otimes n \) with respect to \( z_i, i = 1, \ldots, n \) belonging to \( \{ U_m \} \). This defined an infinite-dimensional analog of charts for \( \mathcal{M} \). Suppose certain points \( x_i, 1 \leq i \leq p \leq n \) are situated on \( \mathcal{S}^{(g_1)} \)-part and other \( x_i, p + 1 \leq i \leq n \) are on \( \mathcal{S}^{(g_2)} \)-part of the Riemann surface \( \mathcal{S}^{(g)} \). On the Riemann surface \( \mathcal{S}^{(g_1)} \) of genus \( g_1 \), let us choose a particular point \( x_i, 1 \leq i \leq p \leq n \) with associated local coordinate \( z_i \) around \( x_i \) which belongs to the domain \( U_i \) of the system \( \{ U_m \} \). For another point \( x_j, 1 \leq j \neq i \leq p \) on \( \mathcal{S}^{(g_1)} \)-part we associate a domain \( U_j \) intersecting with \( U_i \). It can be always done since we can move points \( x_i, 1 \leq i \leq p \) around \( \mathcal{S}^{(g_1)} \)-part of the Riemann surface \( \mathcal{S}^{(g)} \).

Now let us compute a \( p \leq n \)-point \( V \)-correlation functions \( Z^{(g_1)}(v_1, z_1; \ldots; v_p, z_p) \) by means of the recursion procedure described above reducing \( p \)-point correlation functions to a one-point correlation function \( Z^{(g_1)}(v_i, z_i) \) associated to our chosen point \( x_i \) in the domain \( U_i \). Due to properties of vertex operators, the vertex algebra elements \( (v_1, \ldots, v_p) \) can be chosen so that the expansion of the correlation function \( Z^{(g_1)}(v_1, z_1; \ldots; v_p, z_p) \) has a dimension \( p \) polynomial nature. \([33]\). The procedure of computation of \( Z^{(g_1)}(v_1, z_1; \ldots; v_p, z_p) \) by the reduction to one-point correlation function \( Z^{(g_1)}(v_i, z_i) \) defines a map 
\[
\phi_i : V^\otimes n \times \mathcal{S}^{(g)} \to \mathbb{C}^p.
\]
The result of computation of $Z^{(g_1)}(v_1, z_1; \ldots; v_p, z_p)$ can be rewritten to reduce to another one-point function $Z^{(g_1)}(v_j, z_j)$ associated to another coordinate $z_j$ around a point $x_j$ in the domain $U_j$. Thus, we can define the inverse map

$$\phi_j^{-1} : \mathbb{C}^p \to V^{\otimes p} \times S^{(g_1)}.$$  

For any set $x$ of pairs of non-coinciding points among $x_1, \ldots, x_p$ on $S^{(g_1)}$ we then define the function for the intersecting domains $U_i$ and $U_j$ on the Riemann surface $S^{(g_1)}$

$$\varphi_{ij}^{(g_1)}(x) = \phi_i \phi_j^{-1}(x).$$  

Exactly similar procedure is then performed for any set of $y$ non-coinciding points among $x_{p+1}, \ldots, x_n$ on the Riemann surface $S^{(g_2)}$ of genus $g_2$ to define $\varphi_{ij}^{(g_2)}(y)$. Then the transition function is given by the map

$$\varphi_{ij} : \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi_{ij}(x, y) = \left(\varphi_{ij}^{(g_1)}(x), \varphi_{ij}^{(g_2)}(y)\right).$$

As a result, we have defined a foliation of the space $\mathcal{M}_n$ by means of correlation functions for corresponding vertex operator algebras with the formal parameter associated to a coordinate on a Riemann surface. The total space $\mathcal{M} = \bigoplus_{n \geq 0} \mathcal{M}_n$ is foliated similar to $\mathcal{M}_n$.

The general situation is more complicated. Namely, we have to vary $n, p$, the set of vertex operator elements $\{v_i\}, i = 1, \ldots, n$, and $g_1, g_2$ with $g = g_1 + g_2$ (note also, that another parameter can be provided by the type of grading of the vertex operator algebra).

5. Further Directions

The recursion procedure [23] plays a fundamental role in the theory of correlation functions for vertex operator algebras. It provides relations between $n$- and $n-1$-point correlation functions. In our foliation picture, the recursion procedure brings about relations among leaves of foliation. We work in the approach of formulation and computation of cohomologies of vertex operator algebras. Taking into account the above definitions and construction, we would like to develop a theory of characteristic classes for vertex operator algebras, and, in particular, for the space $\mathcal{M} = \bigoplus_{n = 0}^\infty V^{\otimes n} \times S^{(g)}$. This can have a relation with Losik’s work [9] proposing a new framework for singular spaces and new kind of characteristic classes. Losik defines a smooth structure on the leaf space $\mathcal{M}/F$ of a foliation $\mathcal{F}$ of codimension $n$ on a smooth manifold $M$ that allows to apply to $\mathcal{M}/F$ the same techniques as to smooth manifolds. Losik defined the characteristic classes for a foliation as elements of the cohomology of certain bundles over the leaf space $\mathcal{M}/F$. Similar to Losik’s theory, we use certain bundles (of correlation functions) over a foliated space. Relations to [1] on Reeb foliations modified Godbillon-Vey class and can be also considered.

On the other hand, we would like to apply methodology of vertex algebras in order to complete Losik’s theory of characteristic classes [9]. One can mention a possibility to derive differential equations for correlation functions on separate leaves
of foliation. Such equations are derived for various genera and can be used in frames of Vinogradov theory [21]. The structure of foliation (in our sense) can be also studied from the automorphic function theory point of view. Since on separate leaves one proves automorphic properties of correlation functions, one can think about "global" automorphic properties for the whole foliation.

Since we consider multipoint correlation function for vertex algebras on Riemann surfaces of arbitrary genus, there exists also a connection to Krichever-Novikov type algebras [5, 6, 11–13]. These are generalizations of the Witt, Virasoro, affine Lie algebras. In particular, one is able to use the structure of Krichever-Novikov type algebras (as higher-genus generalizations of algebras related to vertex algebras) to study foliated spaces of correlation functions introduced in this paper. The construction of $n$-point functions on genus $g$ sewn Riemann surfaces can be used for introduction and computation of cohomology of Krichever-Novikov type algebras in appropriate setup. We plan to fully shed light on this subject in a future paper.

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