KNUTSON IDEALS OF GENERIC MATRICES

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Abstract. In this paper we show that determinantal ideals of generic matrices are Knutson ideals. This fact leads to a useful result about Gröbner bases of certain sums of determinantal ideals. More specifically, given $I = I_1 + \ldots + I_k$ a sum of ideals of minors on adjacent columns or rows, we prove that the union of the Gröbner bases of the $I_j$’s is a Gröbner basis of $I$.

1. Introduction

Let $\mathbb{K}$ be a field of any characteristic. Fix $f \in S = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial such that its leading term in $\prec(f)$ is a squarefree monomial for some term order $\prec$. We can define many more ideals starting from the principal ideal $(f)$ and taking associated primes, intersections and sums. Thereby, if $\mathbb{K}$ has characteristic $p$, we obtain a family of ideals which are compatibly split with respect to $\text{Tr}(f^{p-1} \bullet)$ (see [Kn] for more details).

Geometrically this means that we start from the hypersurface defined by $f$ and we construct a family of new subvarieties $\{Y_i\}_i$ by taking irreducible components, intersections and unions.

Definition 1 (Knutson ideals). Let $f \in S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial such that its leading term in $\prec(f)$ is a squarefree monomial for some term order $\prec$. Define $C_f$ to be the smallest set of ideals satisfying the following conditions:

1. $(f) \in C_f$;
2. If $I \in C_f$ then $I : J \in C_f$ for every ideal $J \subseteq S$;
3. If $I$ and $J$ are in $C_f$ then also $I + J$ and $I \cap J$ must be in $C_f$.

This class of ideals has some interesting properties which were first proved by Knutson in the case $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$ and then generalized to fields of any characteristic in [Se]:

i) Every $I \in C_f$ has a squarefree initial ideal, so every Knutson ideal is radical.
ii) If two Knutson ideals are different their initial ideals are different. So $C_f$ is finite.
iii) The union of the Gröbner bases of Knutson ideals associated to $f$ is a Gröbner basis of their sum.

Remark 1. Actually, assuming that every ideal of $C_f$ is radical, the second condition in Definition 1 can be replaced by the following:

2’. If $I \in C_f$ then $\mathcal{P} \in C_f$ for every $\mathcal{P} \in \text{Min}(I)$.

In this paper we will continue the study undertaken in [Se] about Knutson ideals.

So far, it has been proved that determinantal ideals of Hankel matrices are Knutson ideals for a suitable choice of $f$ ([Se, Theorems 3.1,3.2]). As a consequence of these results, one can derive an alternative proof (see [Se, Corollary 3.3]) of the $F$-purity of...
Hankel determinantal rings, a result recently proved in [CMSV].

In this paper we are going to show that also determinantal ideals of generic matrices are Knutson ideals (see Theorem 2.1). In particular, they define $F$-split rings. This was already known since the 1990’s from a result by Hochster and Huneke ([HH]).

As a corollary we obtain an interesting result about Gröbner bases of certain sums of determinantal ideals. More specifically, given $I = I_1 + \ldots + I_k$ a sum of ideals of minors on adjacent columns or rows, we will prove that the union of the Gröbner bases of the $I_j$’s is a Gröbner basis of $I$ (see Corollary 2.4).

**Example 1.** Let $X = (x_{ij})$ be the generic square matrix of size 6 and consider the ideal $J = I_3(X_{[1,3]}) + I_3(X_{[1,3]})$ in the polynomial ring $S = \mathbb{K}[X]$. Then $J$ is the ideal generated by the 3-minors of the following highlighted ladder

$$X = \begin{bmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
  x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
  x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
  x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{bmatrix}.$$

From Corollary 2.4, we get that set of 3-minors that generate $J$ is a Gröbner basis of $J$ with respect to any diagonal term order. Actually, this result was already known for ladder determinantal ideals (see [Na, Corollary 3.4]). Nonetheless, Corollary 2.4 can be applied to more general sums of ideals. Consider for instance the ideal $J = I_2(X_{[1,2]}) + I_2(X_{[1,2]}) + I_2(X_{[5,6]}) + I_2(X_{[5,6]})$

that is the ideal generated by the 2-minors inside the below coloured region of $X$

$$X = \begin{bmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
  x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
  x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
  x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{bmatrix}.$$

In this case, $J$ is not a ladder determinantal ideal but we can use Corollary 2.4 to prove that the 2-minors that generate $J$ form a Gröbner basis for the ideal $J$ with respect to any diagonal term order. In fact, $J$ is a sum of ideals of the form $I_t(X_{[a,b]})$ or $I_t(X_{[c,d]})$ which are Knutson ideals from Theorem 2.1. Then a Gröbner basis for $J$ is given by the union of their Gröbner bases.

Furthermore, we can also consider sums of ideals of minors of different sizes, such as $J = I_2(X_{[2,4]}) + I_3(X_{[2,5]})$. 
In this case, $J$ is generated by the 2-minors of the blue rectangular submatrix and the 3-minors of the red rectangular submatrix illustrated below

$$X = \begin{bmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
  x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
  x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
  x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} 
\end{bmatrix}.$$  

Again from Corollary 2.4, being $I_2(X_{[2,4]})$ and $I_3(X^{[2,5]})$ Knutson ideals, the union of their Gröbner bases is a Gröbner basis for $J$. So, a Gröbner basis of $J$ is given by the 2-minors of $X_{[2,4]}$ and the 3-minors of $X^{[2,5]}$.

Unlike in the case of Hankel matrices, a characterization of all the ideals belonging to the family $C_f$ has not been found yet. A first step towards this result would be to understand the primary decomposition of certain sums belonging to the family. Some known results (see [HS], [MR]) suggest us what these primary decompositions might be and computer experiments seem to confirm this guess. Finding this characterization could lead to interesting properties on the Gröbner bases of determinantal-like ideals and it would also answer to the question asked by F. Mohammadi and J. Rhau in [MR].

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2. Knutson ideals and determinantal ideals of generic matrices

Let $m, n$ be two positive integers with $m < n$, we will denote by $X_{mn}$ the generic matrix of size $m \times n$ with entries $x_{ij}$, that is

$$X_{mn} = \begin{bmatrix}
  x_{11} & x_{12} & x_{13} & \ldots & x_{1n} \\
  x_{21} & x_{22} & x_{23} & \ldots & x_{2n} \\
  x_{31} & x_{32} & x_{33} & \ldots & x_{3n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{m1} & x_{m2} & x_{m3} & \ldots & x_{mn} 
\end{bmatrix}.$$  

Moreover, for any $1 \leq i < j \leq n$ and $1 \leq k < l \leq m$, we denote by $X^{[k,l]}_{[i,j]}$ the submatrix of $X_{mn}$ with column indices $i, i + 1, \ldots, j$ and row indices $k, k + 1, \ldots, l$. In the case $[k,l] = [1,m]$, we omit the superscript and we simply write $X_{[i,j]}$.

Given a generic matrix $X_{mn}$ and an integer $t \leq \min(m, n)$, we will denote by $I_t(X)$ the determinantal ideal in $S = \mathbb{K}[X] = \mathbb{K}[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ generated by all the $t$-minors of $X$.

We are going to prove that determinantal ideals of a generic matrix are Knutson ideals for a suitable choice of the polynomial $f$.  

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Theorem 2.1. Let $X = X_{mn}$ be the generic matrix of size $m \times n$ with entries $x_{ij}$ and $m < n$. Consider the polynomial

$$f = \prod_{k=0}^{m-2} \left( \det X_{[m-k,m]}^{i,k+1} \right) \cdot \det X_{[n-k,n]}^{i,k+1} \cdot \prod_{k=1}^{n-m+1} \left( \det X_{[k,m+k-1]}^{i} \right)$$

in $S = \mathbb{K}[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$. Then $I_t(X) \in C_f$ for $t = 1, \ldots, m$.

Furthermore, $I_t(X_{[a,b]})$ (respectively, $I_t(X_{[a,b]})$) are Knutson ideals associated to $f$ for $t = 1, \ldots, m$ and $0 \leq a < b \leq n$ (respectively, $0 \leq a < b \leq m$) with $b - a + 1 \geq t$.

A first step towards the proof of Theorem 2.1 is showing that all the ideals generated by the $t$-minors on $t$ adjacent columns are in $C_f$. This fact is formally stated in the lemma below.

Lemma 2.2. Let $X = X_{m \times n}$ be the generic square matrix of size $m \times n$ with entries $x_{ij}$ and let $f$ to be as in Theorem 2.1. If we fix $t \leq m$, then:

$$I_t(X_{[t,i,i+1]}) \in C_f \quad \forall i = 1, \ldots, n - t + 1.$$

Proof. It is known (e.g. see [CG] and [BC]) that every determinantal ideal of a generic matrix $X = X_{m \times n}$ is prime and its height is given by the following formula:

$$\text{ht}(I_t(X)) = (n - t + 1)(m - t + 1).$$

Therefore

$$\text{ht}(I_t(X_{[t,i,i+1]})) = (t + i - 1 - i + 1 - t + 1)(m - t + 1) = m - t + 1.$$

We have three possibilities for $i$.

1st case: $m - t + 1 \leq i \leq n - m + 1$. Then

$$I_t(X_{[t,i,i+1]}) \supseteq \left( \det X_{[i,i+m-1]}, \det X_{[i-1,i+m-2]}, \ldots, \det X_{[i-m+t,i+t-1]} \right).$$

2nd case: $i \leq m - t$

$$I_t(X_{[t,i,i+1]}) \supseteq \left( \det X_{[1,m]}, \det X_{[2,m+1]}, \ldots, \det X_{[i,i+m-1]}, \det X_{[m-t-i+2,m]}, \det X_{[m-t-i+1,m]}, \ldots, \det X_{[1,m-i]} \right).$$

3rd case: $i \geq n - m + 2$

$$I_t(X_{[t,i,i+1]}) \supseteq \left( \det X_{[n-m+1,n]}, \det X_{[n-m,n-1]}, \ldots, \det X_{[t+i-m,t+i-1]}, \det X_{[1,n-i+1]}, \det X_{[1,n-i+2]}, \ldots, \det X_{[n-m+1,n-m+2,n]} \right).$$

Define $H$ to be the right hand side ideal for each of the previous cases. Note that the initial ideal of $H$ is given by some of the diagonals of the matrix $X$. Since these monomials are coprime, this ideal is a complete intersection and

$$\text{ht}(H) = m - t + 1$$

in each of the above mentioned cases. So $I_t(X_{[t,i,i+1]})$ is minimal over $H$.

By Definition [f] : $J \in C_f$ for every ideal $J \subseteq S$. Taking $J$ to be the principal ideal generated by the product of some of the factors of $f$, we have that all the principal ideals
generated by one of the factors of \( f \) are Knutson ideal associated to \( f \). Being \( H \) a sum of these ideals, \( H \in C_f \).
In conclusion, we get that \( I_t(X_{[t,t+1]}) \) is a minimal prime over an ideal of \( C_f \). So it is in \( C_f \).

Using Lemma 2.2, we can then prove Theorem 2.1

\( \textbf{Proof.} \) Fix \( t \in \{1, \ldots , m \} \). We want to prove that \( I_t(X) \in C_f \). By lemma 2.2, we know that \( I_t(X_{[1,t]}), I_t(X_{[2,t+1]}) \in C_f \) and so their sum.

We claim that that the minimal prime decomposition of the sum is given by

\[ I_t(X_{[1,t]}) + I_t(X_{[2,t+1]}) = I_t(X_{[1,t+1]}) \cap I_{t-1}(X_{[2,t]}). \]

To simplify the notation, we set \( I_1 := I_t(X_{[1,t]}), I_2 := I_t(X_{[2,t+1]}), P_1 = I_t(X_{[1,t+1]}) \) and \( P_2 = I_{t-1}(X_{[2,t]}) \). We want to prove that the minimal prime decomposition is given by:

\[ I_1 + I_2 = P_1 \cap P_2. \]

We already know that \( I_1 + I_2 \subseteq P_1 \cap P_2 \). Passing to the correspondent algebraic varieties, we get the reverse inclusion

\[ \mathcal{V}(I_1 + I_2) \supseteq \mathcal{V}(P_1 \cap P_2) \]

If we prove that \( \mathcal{V}(I_1 + I_2) \subseteq \mathcal{V}(P_1 \cap P_2) \), then

\[ \mathcal{V}(I_1 + I_2) = \mathcal{V}(P_1 \cap P_2) \]

and this is equivalent to say that \( \sqrt{I_1 + I_2} = \sqrt{P_1 \cap P_2} \). Since \( I_1 + I_2 \in C_f \), it is radical and \( P_1 \cap P_2 \) is radical because \( P_1 \) and \( P_2 \) are both radical ideals, then

\[ I_1 + I_2 = P_1 \cap P_2 \]

and we are done.

For this aim, let \( X \in \mathcal{V}(I_1 + I_2) = \mathcal{V}(I_1) \cap \mathcal{V}(I_2) \). This means that \( X_{[1,t]} \) and \( X_{[2,t+1]} \) have rank less or equal than \( t-1 \). Now we consider two cases:

\textbf{Case 1.} Suppose that \( X_{[2,t]} \) has rank less or equal than \( t-1 \). This implies that all the \( (t-1) \times (t-1) \)-minors corresponding to this interval vanish on \( X \). So \( X \in \mathcal{V}(P_2) \).

\textbf{Case 2.} Suppose that \( X_{[2,t]} \) has full rank, namely \( t-1 \). Then it generates a vector space \( V \) of dimension \( t-1 \). But by assumption, \( X_{[1,t]} \) and \( X_{[2,t+1]} \) have rank less or equal than \( t-1 \), so they also generate the vector space \( V \). Consequently, \( X_{[1,t+1]} \) generates the vector space \( V \) and this means that all the \( t \times t \)-minors of our matrix \( X \) vanish on \( X \). Therefore we have proved that \( X \in \mathcal{V}(P_1) \).

This proves the claim and shows that \( I_t(X_{[1,t+1]}) \in C_f \), being a minimal prime over a Knutson ideal.

In the same way, simply shifting the submatrices, we get that \( I_t(X_{[k,t+k]}) \in C_f \) for every \( k = 1, \ldots , n-t \).

In particular \( I_t(X_{[2,t+2]}) \in C_f \); therefore the sum \( I_t(X_{[1,t+1]}) + I_t(X_{[2,t+2]}) \) belongs to \( C_f \).

Using a similar argument to that used to prove the claim, it can be shown that the primary decomposition of the latter sum is given by

\[ I_t(X_{[1,t+1]}) + I_t(X_{[2,t+2]}) = I_t(X_{[1,t+2]}) \cap I_{t-1}(X_{[2,t+1]}). \]

Therefore \( I_t(X_{[1,t+2]}) \) is a Knutson ideal associated to \( f \).
Again, shifting the submatrices, the same argument shows that $I_t(X_{k,t+k+1}) \in \mathcal{C}_f$ for every $k = 1, \ldots, n - t - 1$.

Iterating this procedure we get that $I_t(X_{a,b}) \in \mathcal{C}_f$ for every $1 \leq a < b \leq n$ such that $b - a + 1 \geq t$. In particular, $I_t(X_{[1,n-1]}), I_t(X_{[2,n]}) \in \mathcal{C}_f$. Hence their sum belongs to $\mathcal{C}_f$.

Again, one can show that the primary decomposition of the sum is given by

$$I_t(X_{[1,n-1]}) + I_t(X_{[2,n]}) = I_t(X_{[1,n]}) \cap I_{t-1}(X_{[2,n-1]}).$$

This shows that $I_t(X_{[1,n]}) \in \mathcal{C}_f$ and we are done.

Notice that an identical proof shows that $I_t(X_{[a,b]}) \in \mathcal{C}_f$ for every $0 \leq a < b \leq m$ with $b - a + 1 \geq t$.

As an immediate consequence of the previous theorem, we get an alternative proof of $F$-purity of determinantal ideals of generic matrices.

**Corollary 2.3.** Assume that $\mathbb{K}$ is a field of characteristic $p$ and let $X$ be a generic matrix of size $m \times n$. Then $S/I_t(X)$ is $F$-pure.

**Proof.** We may assume that $\mathbb{K}$ is a perfect field of positive characteristic. In fact, we can always reduce to this case by tensoring with the algebraic closure of $\mathbb{K}$ and the $F$-purity property descends to the non-perfect case. Using Lemma 4 in [Kn], we know that the ideal $(f)$ is compatibly split with respect to the Frobenius splitting defined by $\text{Tr}(f^{p-1} \bullet)$ (where $f$ is taken to be as in the previous theorems). Thus all the ideals belonging to $\mathcal{C}_f$ are compatibly split with respect to the same splitting, in particular $I_t(X)$. This implies that such Frobenius splitting of $S$ provides a Frobenius splitting of $S/I_t(X)$. Being $S/I_t(X)$ $F$-split, it must be also $F$-pure.

Furthermore, we obtain an interesting and useful result about Gröbner bases of certain sums of determinantal ideals.

**Corollary 2.4.** Let $X$ be a generic matrix of size $m \times n$ and let $I$ be a sum of ideals, say $I = I_1 + I_2 + \ldots + I_k$, where each $I_i$ is of the form either $I_t(X_{[a_i,b_i]})$ or $I_t(X^{[a_i,b_i]})$. Then

$$\mathcal{G}_I = \mathcal{G}_{I_1} \cup \mathcal{G}_{I_2} \cup \ldots \cup \mathcal{G}_{I_k}$$

where $\mathcal{G}_I$ denotes a Gröbner basis of the ideal $I$.

Furthermore, if $\mathbb{K}$ has positive characteristic, $I$ is also $F$-pure.

**Proof.** By Theorem 2.1 we know that $I_t(X_{[a_i,b_i]})$ and $I_t(X^{[a_i,b_i]})$ are Knutson ideals. From property (iii) of Knutson ideals, we get the thesis.

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