Gap Function and Global Error Bounds for Generalized Mixed Quasi-variational Inequality

C. Wang, Y.L. Zhao and L. Shen

ABSTRACT

In this paper, we introduced a new class of generalized mixed quasi-variational inequality in Hilbert space. We get some gap functions for the generalized mixed quasi-variational inequality in term of regularized gap function and D-gap function. Further, we obtain global error bounds for the solution of the generalized mixed quasi-variational inequality.

KEYWORDS

Strong monotone mapping; Residual vector; Regularized gap function; D-gap function; Error bounds

INTRODUCTION

Variational inequality in mathematical programming, game theory, economics and engineering and other fields has a broad application and research, the variational inequality mainly research the existence of solutions, stability and sensitivity analysis of the solution mapping, etc. The global error bounds of solution for variational inequalities is a precondition of stability and sensitivity analysis for the solution mapping, gap function is an effective method about obtaining error bounds of the solution for variational inequalities. In recent years, many researchers get the error bounds of introduced variational inequalities[1-10]. In 2010, Charitha and Dutta[1] studied regularized gap function and error bounds of vector variational inequality; In 2011, Aussel and Dutta[2] studied multivalued Stampacchia variational inequalities by gap functions; Aussel, Correa and Marechal[3] studied error bounds of solution for quasi-variational inequalities and generalized Nash equilibrium problems in the same year.

C. Wang, Y.L. Zhao, L. Shen, College of Mathematics and Physics, Bohai University, Jinzhou, Liaoning, China.
In 2013, Aussel, Gupta and Mehra[4] studied gap functions and error bounds of inverse quasi-variational inequality problems and Tang and Huang[5] studied gap functions and error bounds of set-valued mixed variational inequalities; In 2015, Khan and Chen[6] obtained global error bound of solution for generalized mixed quasi-variational inequalities by using regularized gap function.

Inspired and motivated by the recent research work above, we introduce a new class of generalized mixed quasi-variational inequality in this paper. With the suitable conditions of strongly mixed monotonicity, inverse monotonicity, nonexpacidity and Lipschitz continuity, we obtain the global error bound of the generalized mixed quasi-variational inequality by using the introduced regularized gap function and D-gap function. The results obtained in this paper generalize some corresponding results in literature [6].

PRELIMINARIES

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ respectively. Let $C(H)$ be a family of nonempty compact subsets of $H$.

$ABC,D,H\rightarrow \mathcal{K}(H)$ be the set-valued mappings and $g:H\rightarrow H$ be a single-valued mapping, $\phi:H\times H\rightarrow \mathbb{R}$ be a continuous bifunction with respect to both arguments, $F:H\times H\rightarrow \mathbb{R}$ be a bifunction satisfying $F(x,x)=0$, for all $x\in H$. For given nonlinear mappings $N,H,M,N:H\times H\rightarrow H$, we consider the following generalized mixed quasi-variational inequality problem, for short, denoted by (GMQVIP), which consists in finding $x\in H,u\in A(x),v\in B(x),w\in C(x),z\in D(x)$ such that

$$F(g(x),g(y))+\langle N(u,v)-M(w,z),g(y)-g(x)\rangle +\phi(g(x),g(y))-\phi(g(x),g(x))\geq 0, \forall y\in H.$$ 

In order to establish resolvent equations for the (GMQVIP), we need the following definitions and results.

Definition 2.1 Let $F:H\times H\rightarrow \mathbb{R}$ and $\phi:H\times H\rightarrow \mathbb{R}$ be two bifunctions. Then

(a) $F$ is monotone, if $F(x,y)+F(y,x)\leq 0, \forall x,y\in H$;
(b) $\phi$ is skew-symmetric, if $\phi(x,x)=\phi(x,y)-\phi(y,x)+\phi(y,y)\geq 0, \forall x,y\in H$;

Definition 2.2 Let $ABC,D,H\rightarrow \mathcal{K}(H)$ be the set-valued mappings, $N,H,M:H\times H\rightarrow H$ be the nonlinear mappings and $g:H\rightarrow H$ be a single-valued mapping, then for all $x,y,z\in H,u\in A(x),v\in B(x),w\in C(x),z\in D(x)$,

(a) $N$ is said to be $\alpha$-strongly mixed $\beta$-monotone, there exists a constant $\alpha>0$ such that $\langle N(u,v)-N(u_o,v_o),g(x)-g(x_o)\rangle \geq \alpha\|g(x)-g(x_o)\|^2$.
(b) $N$ is said to be $\beta,\delta$-mixed Lipschitz continuous, if there exist constants $\beta,\delta>0$ such that $\|N(u,v)-N(u_o,v_o)\| \leq \beta\|u-u_o\|+\delta\|v-v_o\|$.
(c) $M$ is said to be $\rho$-inverse strongly monotone, if there exists a constant $\rho>0$ such that $\langle M(w,z)-M(w_o,z_o),g(x)-g(x_o)\rangle \geq \rho\|M(w,z)-M(w_o,z_o)\|^2$.
(d) $M$ is said to be $\lambda$-strongly nonexpanding, if there exists a constant $\lambda>0$ such that $\|M(w,z)-M(w_o,z_o)\| \leq \lambda\|x-x_o\|$.
(e) $g$ is said to be $L$-Lipschitz continuous, if there exists a constant $L>0$ such that $\|g(x)-g(x_o)\| \leq L\|x-x_o\|$.

205
(f) $g$ is said to be $\tau-$strongly nonexpanding, if there exists a constant $\tau>0$ such that $\|g(x)-g(x_0)\|\geq\|x-x_0\|$.

g) $A$ is said to be $\eta-$Lipschitz continuous, if there exists a constant $\eta>0$, such that $T(A(x), A(x_0)) \leq \eta\|x-x_0\|$

where $T(\cdot, \cdot)$ is the Hausdorff metric on $C(H)$.

**Definition 2.3** Resolvent operator $J_\theta^F : H \to H$ is nonexpansive, such that $\|J_\theta^F(x)-J_\theta^F(y)\| \leq \|x-y\|, \forall x, y \in H$.

**Lemma 2.1**[11,12] Let $X$ be a closed convex subset of a Hausdorff topological vector space $E$ and $G : X \times X \to R$ be a bifunction. Assume that the following conditions hold:

(i) $G(x, x) \geq 0, \forall x \in X$;

(ii) $G$ is monotone;

(iii) For each fixed $y \in X$, the function $x \mapsto G(x, y)$ is upper-semicontinuous, that is,

$$\limsup_{t \to 0} G(tw+(1-t)x, y) \leq G(x, y), \forall x, y, w \in X, t \in [0,1]$$

(iv) For each fixed $x \in X$, the function $y \mapsto G(x, y)$ is convex and lower semicontinuous;

(v) There exists a compact subset $K$ of $E$ and there exists $y_0 \in K \cap X$ such that $G(x, y_0) < 0$ for each $x \in X \setminus K$.

Then the set $\{x \in X : G(x, y) \geq 0, \forall y \in X\}$ is nonempty and compact.

**Lemma 2.2**[6] Let $H$ be a real Hilbert space, $F : H \times H \to R$ and $\varphi : H \times H \to R$ be nonlinear bifunctions and $\theta > 0$. Suppose that the following conditions are satisfied:

(i) $F$ satisfies condition (i)-(iv) in Lemma 2.1;

(ii) $\varphi$ is skew-symmetric, convex in second argument and continuous;

(iii) For each fixed $z \in H$, there exists a compact subset $K$ of $E$ and $y_0 \in K \cap H$ such that $\varphi(F(x, y_0) + \langle x-z, y_0-x \rangle + \theta \varphi(x, y_0) - \theta \varphi(x, x) < 0, \forall x \in H \setminus K$.

Then for each fixed $z \in H$, find $x \in H$ such that

$$\varphi(F(x, y) + \langle x-z, y-x \rangle + \theta \varphi(x, y) - \theta \varphi(x, x) \geq 0, \forall y \in H,$$

has a unique solution if and only if $x = J_{\theta \varphi(x)}^F[z]$, that is,

$$\varphi(J_{\theta \varphi(x)}^F[z], y) + \langle J_{\theta \varphi(x)}^F[z] - z, y - J_{\theta \varphi(x)}^F[z] \rangle + \theta \varphi(J_{\theta \varphi(x)}^F[z], y) - \theta \varphi(J_{\theta \varphi(x)}^F[z], J_{\theta \varphi(x)}^F[z]) \geq 0, \forall y \in H.$$  \hspace{1cm} (2.1)

**Lemma 2.3** Any $x \in H$ is a solution of the (GMQVIP), if and only if, $x \in H$ satisfies the relation:

$$g(x) = J_{\theta \varphi(x)}^F[g(x) - \theta N(u, v) + \theta M(w, z)],$$

where $\theta > 0$ is a constant and $J_{\theta \varphi(x)}^F$ is resolvent operator.

**Definition 2.4** Let $K$ be the domain of the (GMQVIP). A function $P : K \to R$ said to be a gap function for the (GMQVIP), if it satisfies the following properties:

(i) $P(x) \geq 0, \forall x \in K$;

(ii) $P(x) = 0$ if and only if $x$ is a solution of the (GMQVIP).

We now define the residual vector $R(x, \theta)$ by relation

$$R(x, \theta) = g(x) - J_{\theta \varphi(x)}^F[g(x) - \theta N(u, v) + \theta M(w, z)].$$ \hspace{1cm} (2.2)

From Lemma 2.3 that $x \in H$ is a solution of the (GMQVIP), if and only if $x \in H$ is a root of the equation

$$R(x, \theta) = 0.$$ \hspace{1cm} (2.3)

The residual vector $R(x, \theta) = 0$ is a gap function for (GMQVIP).
Now by using residual vector $R(x, \theta)$, we obtain the global error bounds for the solution of (GMQVIP).

**Theorem 2.1** Assume that all conditions of Lemma 2.2 hold. Let $x_0 \in H$ be a solution of the (GMQVIP), $N$ be $\alpha$ - strongly mixed $g$ - monotone and $\beta, \delta_1$, mixed Lipschitz continuous, $M$ be $\rho$ - inverse strongly monotone, $\lambda$ - strongly nonexpanding and $\beta, \delta_2$, mixed Lipschitz continuous. Let $g: H \to H$ be $L$ - Lipschitz continuous and $\tau$ - strongly nonexpanding, $A, B, C, D: H \to \mathcal{G} (H) = \eta_1, \eta_2, \eta_3, \eta_4 - T$ - Lipschitz continuous, respectively. If for any $\mu > 0$, \[ \| J_{\theta_{opt}}^F (w) - J_{\theta_{opt}}^F (x) \| \leq \mu \| y - x \|, \forall x, y, w \in H. \] (2.4)

Then \[ \frac{1}{c_i} \| R(x, \theta) \| \leq \| x - x_0 \| \leq c_i \| R(x, \theta) \|, \forall x \in H, \]

where $R(x, \theta)$ is residual vector defined by (2.2) and $c_1, c_2$ are generic constants.

**Proof.** Let $x_0 \in H, y_0 \in \mathcal{A}(x_0), v_0 \in \mathcal{B}(x_0), v_0 \in \mathcal{C}(x_0), z_0 \in \mathcal{D}(x_0)$ be a solution of the (GMQVIP), then \[ F(g(x_0), g(y)) + \langle N(u, v) - M(w, z), g(y) - g(x_0) \rangle + \phi(g(x_0), g(y)) - \phi(g(x_0), g(x_0)) \geq 0, \forall y \in H. \]

Substituting $g(y) = J_{\theta_{opt}}^F [g(x) - \partial N(u, v) + \partial M(w, z)]$ in above inequality, we get \[ F(g(x_0), J_{\theta_{opt}}^F [g(x) - \partial N(u, v) + \partial M(w, z)]) \] (2.5)

Adding (2.5) and (2.6), since $F$ is monotone and $\varnothing$ is skew-symmetric, we get \[ \| N(u, v) - M(w, z) \| \leq \frac{1}{\theta} \langle J_{\theta_{opt}}^F [g(x) - \partial N(u, v) + \partial M(w, z)], g(x_0) \rangle \]

Adding (2.5) and (2.6), since $F$ is monotone and $\varnothing$ is skew-symmetric, we get

Since $N$ is $\alpha$ - strongly mixed $g$ - monotone, $M$ is $\rho$ - inverse strongly monotone and $\lambda$ - strongly nonexpanding, $g$ is $\tau$ - strongly nonexpanding, and by using the mixed Lipschitz continuity of $N, M, g$ and $T$ - Lipschitz continuity of $A, B, C, D$, respectively, we obtain
\[(\alpha s + \beta \tilde{\alpha})\|x_0 - x\|^2 = \alpha s^2 \|x_0 - x\|^2 + \beta \tilde{\alpha} \|x_0 - x\|^2 \leq \|g(x_0) - g(x)\|^2 + \|M(w_0) - M(w_0, z_0)\|^2\]
\[
\leq \frac{1}{\theta^2} \|g(x) - J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)]\|
\]
\[
J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)] - g(x)\]
\[
\leq \frac{1}{\theta^2} \|g(x) - J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)]\|
\]
\[
+ \|M(w_0) - M(w_0, z_0)\| \|J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)] - g(x)\|
\]
\[
\leq \frac{1}{\theta^2} \|R(x, \theta)\|^2 + \frac{1}{\theta^2} \|R(x, \theta)\| \|g(x_0) - g(x_0)\| + \|M(u_0, v_0) - M(u_0, v_0)\| \|R(x, \theta)\|
\]
\[
+ \|M(u_0, v_0) - M(u_0, v_0)\| \|R(x, \theta)\| \|g(x) - g(x)\| + \|M(u_0, v_0) - M(u_0, v_0)\| \|R(x, \theta)\| \|g(x) - g(x)\|
\]
\[
\leq \frac{L}{\theta^2} + \|M(u_0, v_0) - M(u_0, v_0)\| \|R(x, \theta)\| \|x - x_0\|
\]
\[
which implies that
\[
\|x - x_0\| \leq \frac{1}{\alpha s^2 + \beta \tilde{\alpha}^2} \left( \frac{L}{\theta^2} + \beta \tilde{\eta}_1 + \delta \tilde{\eta}_2 + \beta \tilde{\eta}_3 + \delta \tilde{\eta}_4 \right) \|R(x, \theta)\|
\]
\[
from which we have
\[
\|x_0 - x\| \leq c_2 \|R(x, \theta)\|
\]
\[
where c_2 = \frac{1}{\alpha s^2 + \beta \tilde{\alpha}^2} \left( \frac{L}{\theta^2} + \beta \tilde{\eta}_1 + \delta \tilde{\eta}_2 + \beta \tilde{\eta}_3 + \delta \tilde{\eta}_4 \right).
\]

By the definition of residual vector (2.2), we have
\[
\|R(x, \theta)\| = \|g(x) - J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)]\|
\]
\[
\leq \|g(x) - g(x)\| + \|J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)]\|
\]
\[
- \|J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)]\|
\]
\[
+ \|J_{\partial u,v}^F[g(x) - \partial N(u,v) + \partial M(w,z)]\|
\]
\[
\leq \|R(x, \theta)\| \|x - x_0\|
\]
\[
From the Lipschitz continuity of \( g \), nonexpansion of \( J_{\partial u,v}^F \), mixed Lipschitz continuity of \( N, M \) and \( T \) — Lipschitz continuity of \( A, B, C, D \), we have
\[
\|R(x, \theta)\| \leq \|x - x_0\| + \|g(x) - \partial N(u,v) + \partial M(w,z) - g(x)\|
\]
\[
+ \|\partial N(u,v) - \partial M(w,z)\| + \|x - x_0\|
\]
\[
\leq \|x - x_0\| + \|g(x) - \partial N(u,v) + \partial M(w,z) - g(x)\|
\]
\[
+ \|\partial N(u,v) - \partial M(w,z)\| + \|x - x_0\|
\]
\[
\leq \|x - x_0\| + \|g(x) - \partial N(u,v) + \partial M(w,z) - g(x)\|
\]
\[
+ \|\partial N(u,v) - \partial M(w,z)\| + \|x - x_0\|
\]
\[
\leq \|x - x_0\| + \|g(x) - \partial N(u,v) + \partial M(w,z) - g(x)\|
\]
\[
+ \|\partial N(u,v) - \partial M(w,z)\| + \|x - x_0\|
\]
\[
\leq \|x - x_0\| + \|g(x) - \partial N(u,v) + \partial M(w,z) - g(x)\|
\]
\[
+ \|\partial N(u,v) - \partial M(w,z)\| + \|x - x_0\|
\]
\[
\|x - x_0\| \geq \frac{1}{c_1} \|R(x, \theta)\|
\]
\[
where c_1 = 2L + \|\beta \tilde{\eta}_1 + \delta \tilde{\eta}_2 + \beta \tilde{\eta}_3 + \delta \tilde{\eta}_4 \| + \mu \|x - x_0\|
\]
This completes the proof.

REGULARIZED GAP FUNCTIONS AND ERROR BOUNDS FOR THE (GMQVIP)

In this section, we want to overcome the nondifferentiability of normal residual vector $R(x, \theta)$. By using an approach due to literature [9], we establish another gap function $G_\theta(x)$ associated with the (GMQVIP), which can be viewed as a regularized gap function.

The function $G_\theta$ defined by

$$G_\theta(x) = \max_{y \in \Omega} \left\{ -F(g(x), g(y)) + \left( N(u,v) - M(w,z), g(x) - g(y) \right) \right\}$$

which is finite valued everywhere and is differentiable whenever all operators involved in $G_\theta(x)$ are differentiable.

The function $G_\theta(x)$ can be written as

$$G_\theta(x) = -F(g(x), J_{\text{subobj}}^\varphi [g(x) - \partial N(u,v) + \partial M(w,z)])$$

$$-\varphi(g(x), g(y)) + \varphi(g(x), g(y)) - \frac{1}{2\theta} \left\| g(x) - g(y) \right\|^2$$

Theorem 3.1 Suppose that all conditions of Lemma 2.2 hold and $R(x, \theta)$ is residual vector defined by (2.2), then the function $G_\theta(x)$ for $\theta > 0$ defined by (3.1), is a gap function for (GMQVIP).

Proof. Taking $z = g(x) - \partial N(u,v) + \partial M(w,z)$ and $y = g(x)$, we have

$$F(J_{\text{subobj}}^\varphi [g(x) - \partial N(u,v) + \partial M(w,z)])$$

$$g(x) - J_{\text{subobj}}^\varphi [g(x) - \partial N(u,v) + \partial M(w,z)]$$

which follows from the monotonicity of $F$ and skew-symmetry of $\varphi$ that

$$G_\theta(x) \geq \frac{1}{\theta} \left\| g(x) - J_{\text{subobj}}^\varphi [g(x) - \partial N(u,v) + \partial M(w,z)] \right\|^2$$

Clearly, we have $G_\theta(x) \geq 0$, for all $x \in H$.

From the above conclusion, if $G_\theta(x) = 0$, then $\frac{1}{\theta} \left\| R(x, \theta) \right\| = 0$, obviously, $\left\| R(x, \theta) \right\| = 0$. So by Lemma 2.2, $x \in H$ is a solution of the (GMQVIP). Conversely, if $x \in H$ is a solution of the (GMQVIP), then $g(x) = J_{\text{subobj}}^\varphi [g(x) - \partial N(u,v) + \partial M(w,z)]$ thus, from (3.2) and condition $F(x, \theta) = 0$ for all $x \in H$, we obtain that $G_\theta(x) = 0$. This completes the proof.
Next, we derive the error bounds without using the Lipschitz continuity of the $N$ and $M$.

**Theorem 3.2** Let $x_0$ be a solution of the (GMQVIP). Suppose that $N$ is $\alpha-$strongly mixed $G-$Monotone, $M$ is $\rho-$inverse strongly monotone and $\lambda-$strongly nonexpanding, $F$ is monotone, $\Phi$ is skew-symmetric and $g$ is $\tau-$strongly nonexpanding, then

$$|x - x_0| < \frac{1}{\sqrt{(\alpha - \frac{1}{2}\rho) + \rho \lambda^2}} \sqrt{G_\phi(x)}, \forall x \in H, \theta > \frac{\tau^2}{2(\rho \lambda^2 + \alpha \theta)}.$$ 

**Proof.** From (3.1), it can be written as

$$G_\phi(x) \geq -F(g(x), g(x_0)) + \langle N(u, v) - M(w, z), g(x) - g(x_0) \rangle$$

$$- \varphi(g(x), g(x_0)) + \varphi(g(x), g(x)) - \frac{1}{2\theta} \|g(x) - g(x_0)\|^2.$$ 

By using strongly mixed $G-$monotonicity of $N$, inverse strongly monotonicity and strongly nonexpandicity of $M$, we get

$$G_\phi(x) \geq -F(g(x), g(x_0)) + \langle N(u, v), g(x) - g(x_0) \rangle$$

$$+ \langle N(u, v) - N(u, v), g(x) - g(x_0) \rangle + \langle M(w, z), g(x) - g(x_0) \rangle$$

$$- \langle M(w, z), g(x) - g(x_0) \rangle - \varphi(g(x), g(x_0)) + \varphi(g(x), g(x)) - \frac{1}{2\theta} \|g(x) - g(x_0)\|^2$$

$$\geq -F(g(x), g(x_0)) + \langle N(u_0, v_0) - M(w_0, z_0), g(x) - g(x_0) \rangle + \|g(x) - g(x_0)\|^2 + \rho \lambda^2 |x - x_0|^2$$

$$- \varphi(g(x), g(x_0)) + \varphi(g(x), g(x)) - \frac{1}{2\theta} \|g(x) - g(x_0)\|^2.$$ 

Since $x_0 \in H_u \in A(x_0) \in B(x_0) \in C(x_0) \in D(x_0)$ be a solution of the (GMVIP), then

$$F(g(x_0), g(y)) + \langle N(u_0, v_0) - M(w_0, z_0), g(y) - g(x_0) \rangle$$

$$+ \varphi(g(x_0), g(y)) - \varphi(g(x_0), g(x_0)) \geq 0.$$ 

Taking $y = x$ in above inequality

$$F(g(x_0), g(x)) + \langle N(u_0, v_0) - M(w_0, z_0), g(x) - g(x_0) \rangle$$

$$+ \varphi(g(x_0), g(x)) - \varphi(g(x_0), g(x_0)) \geq 0.$$ 

Combining (3.4) and (3.5), by using monotonicity of $F$ and skew-symmetry of $\Phi$, respectively, we have

$$G_\phi(x) \geq \alpha \tau^2 \|x - x_0\|^2 + \rho \lambda^2 \|x - x_0\|^2 - \frac{1}{2\theta} \|g(x) - g(x_0)\|^2.$$ 

Further, using the strongly nonexpandicity of $\phi$, we get

$$G_\phi(x) \geq \alpha \tau^2 \|x - x_0\|^2 + \rho \lambda^2 \|x - x_0\|^2 - \frac{1}{2\theta} \|x - x_0\|^2$$

$$= [\alpha - \frac{1}{2\theta})\tau^2 + \rho \lambda^2] \|x - x_0\|^2,$$

which implies that

$$\|x - x_0\| \leq \frac{1}{\sqrt{\alpha - \frac{1}{2\theta})\tau^2 + \rho \lambda^2}} \sqrt{G_\phi(x)}.$$ 

This completes the proof.

**GLOBAL ERROR BOUNDS FOR THE (GMQVIP)**

In this section, we consider another gap function associated with (GMQVIP), which can be viewed as a difference of two regularized gap functions with distinct parameters, known as $D-$ gap function. The $D-$gap function with $\theta\psi\phi\theta$ is defined as

$${\text{GLOBAL ERROR BOUNDS FOR THE (GMQVIP)}}$$

In this section, we consider another gap function associated with (GMQVIP), which can be viewed as a difference of two regularized gap functions with distinct parameters, known as $D-$ gap function. The $D-$gap function with $\theta\psi\phi\theta$ is defined as
which was introduced and studied by [10,13,14] for solving variational inequalities and complementarity problems.

The $D$-gap function associated with the (GMQVIP) is given by

\[
D_{\psi}(x) \overset{(4.1)}{=} \max_{y \in C(y) \cap \psi D} \left\{ -F(g(x), g(y)) + \{N(u, v) - M(w, z), g(x) - g(y)\} - \varphi(g(x), g(y)) + \varphi(g(x), g(y)) + \frac{1}{2\psi} \|g(x) - g(y)\|^2 - \frac{1}{2\theta} \|g(x) - g(y)\| \right\}, \quad x \in H, \theta > \psi > 0.
\]

(4.1) can be written as

\[
\begin{align*}
D_{\psi}(x) & \overset{(4.2)}{=} -F(J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)), J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))) \\
& \quad + \{N(u, v) - M(w, z), R(x, \psi) - R(x, \theta)\} \\
& \quad - \varphi(J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)), J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))) \\
& \quad + \varphi(J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)), J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))) \\
& \quad + \frac{1}{2\psi} \|R(x, \psi)\|^2 - \frac{1}{2\theta} \|R(x, \theta)\|^2.
\end{align*}
\]

Next, we obtain global error bounds for the (GMQVIP).

**Theorem 4.1** Assume that all conditions of Lemma 2.1 hold and $R(x, \theta)$ is residual vector defined by (2.5), then for $x \in H, \theta > \psi > 0$, we have

\[
\frac{1}{2\psi} \left( \frac{1}{\theta} - \frac{1}{\psi} \right) \|R(x, \psi)\|^2 \leq \|D_{\psi}(x)\| \leq \frac{1}{2 \psi} \left( \frac{1}{\theta} - \frac{1}{\psi} \right) \|R(x, \theta)\|^2.
\]

In particular $D_{\psi}(x) = 0$, if and only if $x \in H$ solves the (GMQVIP).

**Proof.** Taking $z = g(x) - \partial \varphi(u, v) + \partial \psi M(w, z)$ and

\[
y = J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)) \text{ in (2.1), we obtain}
\]

\[
F(J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)), J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))) \\
+ \{N(u, v) - M(w, z), R(x, \psi) - R(x, \theta)\} \\
\geq \frac{1}{\theta} \|R(x, \theta) - R(x, \psi)\|^2 \overset{(4.3)}{=} \frac{1}{\theta} \|R(x, \theta) - R(x, \psi)\|^2 \\
- \varphi(J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)), J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))) \\
+ \varphi(J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)), J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z)))
\]

Combining (4.2) and (4.3), by using monotonicity of $F(\cdot, \cdot)$ and skew-symmetry of $\varphi(\cdot, \cdot)$, we have

\[
D_{\psi}(x) \geq \frac{1}{\theta} \|R(x, \theta) - R(x, \psi)\|^2 + \frac{1}{2\psi} \|R(x, \psi)\|^2 - \frac{1}{2\theta} \|R(x, \theta)\|^2 \\
\geq \frac{1}{2} \left( \frac{1}{\psi} - \frac{1}{\theta} \right) \|R(x, \psi)\|^2. \quad (4.4)
\]

Similarly, by taking $x = J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))$, $z = g(x) - yN(u, v) + yM(w, z)$ and $y = J_{\psi}^{\varphi}(g(x) - yN(u, v) + yM(w, z))$, we get
Combining (4.2) and (4.5), we have

\[
D_{\psi}(x) \geq -\frac{1}{\psi} \langle R(x, \psi), R(x, \theta) - R(x, \psi) \rangle - \frac{1}{2\psi} \|R(x, \psi)\|^2 - \frac{1}{2\theta} \|R(x, \theta)\|^2
\]

Combining (4.4) and (4.6), we get the required result.

Finally, we get a global error bound for the (GMQVIP).

**Theorem 4.2**

Let \( x_0 \) be a solution of the (GMQVIP). Suppose that \( N \) is \( \alpha \)-strongly mixed \( \mathcal{G} \)-monotone, \( M \) is \( \rho \)-inverse strongly monotone and \( \lambda \)-strongly nonexpansive, \( \mathcal{G} \) is \( \tau \)-strongly nonexpansive, \( F(\cdot) \) is monotone and \( \varphi(\cdot) \) is skew-symmetric, then

\[
\|x - x_0\| \leq \frac{1}{\sqrt{\alpha + \frac{1}{2\psi} + \frac{1}{2\theta}}} \sqrt{D_{\psi}(x), \forall x \in H, \alpha > 0, \frac{1}{2\psi} + \frac{1}{2\theta} - \frac{\rho \lambda^2}{\tau^2}}.
\]

**Proof.** From (4.1), it can be written as

\[
D_{\psi}(x) \geq -F(g(x), g(x)) + \{N(u, v) - M(w, z), g(x) - g(x)\} - \varphi(g(x), g(x))
\]

By using strongly mixed \( \mathcal{G} \)-monotonicity of \( N(\cdot) \),

**inverse strongly monotonicity and strongly nonexpansivity of \( M(\cdot) \),** we have

\[
D_{\psi}(x) \geq -F(g(x), g(x)) + \{N(u_0, v_0) - M(w_0, z_0), g(x) - g(x)\} - \varphi(g(x), g(x))
\]

Taking \( y = x \) in above inequality

\[
F(g(x), g(x)) + \{N(u_0, v_0) - M(w_0, z_0), g(x) - g(x)\} + \varphi(g(x), g(x)) \geq 0.
\]

Combining (4.7) and (4.8), then using monotonicity of \( F \) and skew-symmetry of \( \varphi \), respectively, we get

\[
D_{\psi}(x) \geq \|g(x) - g(x_0)\|^2 + \rho \lambda^2 \|x - x_0\|^2 - \frac{1}{2\psi} \|g(x) - g(x_0)\|^2 - \frac{1}{2\theta} \|g(x) - g(x_0)\|^2,
\]

further, using the strongly nonexpansion of \( \mathcal{G} \), we have

\[
D_{\psi}(x) \geq [(\alpha + \frac{1}{2\psi} - \frac{1}{2\theta}) \tau^2 + \rho \lambda^2] \|x - x_0\|^2,
\]

which implies that
\[ \|x - x_0\| \leq \sqrt{\frac{1}{2\eta} \left( x^2 + \rho \lambda^2 \right)} D_{\theta, \tau}(x). \]

Which completes the proof.

**ACKNOWLEDGEMENTS**

This work is supported by the National Natural Science Foundation of China (11371070).

**REFERENCES**

1. C. Charitha & J. Duuta. 2010. Regularized gap functions and error bounds for vector variational inequality. *Pac. J. Optim.* 6: 497-510.
2. D. Aussel & J. Dutta. 2011. On gap functions for multivalued Stampacchia variational inequalities. *J. Optim. Theory Appl.* 149: 513-527.
3. [3] D. Aussel & R. Correa & M. Marechal. 2011. Gap functions for quasi variational inequalities and generalized Nash equilibrium problems. *J. Optim. Theory Appl.* 151: 474-488.
4. [4] D. Aussel & R. Gupta & A. Mehra. 2013. Gap functions and error bounds for inverse quasi-variational inequality problems. *J. Optim. Theory Appl.* 407: 270-280.
5. [5] G.J. Tang & N.J. Huang. 2013. Gap functions and global error bounds for set-valued mixed variational inequalities. *Taiwanese J. Math.* 17: 1267-1286.
6. [6] Suhel Ahmad Khan & Jia-wei Chen. 2015. Gap function and global error bounds for generalized mixed quasi variational inequalities. *Applied Mathematics and Computation* 260: 71-81.
7. [7] R. Gupta & A. Mehra. 2012. Gap functions and error bounds for quasi-variational inequalities. *J. Global Optim.* 53: 737-748.
8. [8] M.A. Noor. 2006. Merit functions for general variational inequalities. *J. Math. Anal. Appl.* 316: 736-752.
9. [9] J. Li & G. Mastroeni. 2010. Vector variational inequalities involving set-valued mappings via scalarization with applications to error bounds for gap functions. *J. Optim. Theory Appl.* 145: 355-372.
10. [10] M.V. Solodov & P. Tseng. 2000. Some methods based on the D-gap function for solving monotone variational inequalities. *Comput. Optim. Appl.* 17: 255-277.
11. [11] S.S. Chang. 1991. *Variational Inequalities and Complementarity Problems.* Theory and Applications, *Shanghai Scientific and Technical Press*, Shanghai.
12. [12] S.S. Chang & S.W. Xiang. 1996. On the existence of solutions for a class of quasi-bilinear variational inequalities. *J. System Sci. Math. Sci.* 16:136-160.
13. [13] M.V. Solodov. 2003. Merit functions and error bounds for generalized variational inequalities. *J. Math. Anal. Appl.* 287: 405-414.
14. [14] N. Yamashita & M. Fukushima. 1997. Equivalent unconstraint minimization and global error bounds for variational inequality problems. *SIAM J. Control Optim.* 35: 273-284.