Uncertainty Quantification of Multi-Scale Resilience in Nonlinear Complex Networks using Arbitrary Polynomial Chaos

Mengbang Zou, Luca Zanotti Fragonara, Weisi Guo, Senior Member, IEEE

Abstract—In an increasing connected world, resilience is an important ability for a system to retain its original function when perturbations happen. Even though we understand small-scale resilience well, our understanding of large-scale networked resilience is limited. Recent research in network-level resilience and node-level resilience pattern has advanced our understanding of the relationship between topology and dynamics across network scales. However, the effect of uncertainty in a large-scale networked system is not clear, especially when uncertainties cascade between connected nodes. In order to quantify resilience uncertainty across the network resolutions (macro to micro), we develop an arbitrary polynomial chaos (aPC) expansion method to estimate the resilience subject to parameter uncertainties with arbitrary distributions. For the first time and of particular importance, is our ability to identify the probability of a node in losing its resilience and how the different model parameters contribute to this risk. We test this using a generic networked bistable system and this will aid practitioners to both understand macro-scale behaviour and make micro-scale interventions.

Index Terms—Uncertainty; Resilience; Arbitrary Polynomial Chaos Expansion; Dynamic Complex Network

I. INTRODUCTION

ORGANIZED behaviors in economics, infrastructure, ecology and human society often involve large-scale networked dynamical systems. These networked systems connect together relatively simple local component dynamics to achieve sophisticated multi-scale network wide behaviour [1]. Example include a water distribution network couples local pumps and reservoirs to deliver supply via Navier-Stokes dynamics [2], an electric grid that uses power-flow equations, a fully loaded structure that connects beams and joints via the Ramberg–Osgood equation, a spatially stochastic wireless network that performs traffic load balancing [3], or a fibre optic network that connects optic switches via the Nonlinear Schrodinger’s dynamic.

A. Network Resilience Modeling

A critical part of the organized behavior is the ability for a system to stay resilient - defined as the ability to retain original functionality after a perturbation or failure. A system’s resilience is a key property and plays a crucial role in reducing risks and mitigating damages [4], [5]. Research on resilience of dynamic networks has been widely applied in different fields including blackout in power systems [6] to loss of biodiversity in ecology [7]. Whilst we understand how a few interacting components (small networks) work [5], the loss of resilience in large-scale networked systems (e.g. $10^5$ nodes) is difficult to predict and analyse analytically.

These analytical limitations are rooted in a theoretical gap: most current analytical framework of resilience is designed to treat models with a high degree of homogeneity which enables mean field to be applied [1]. Whilst this has advanced our understanding of the coupling relationship between topology and dynamics, it doesn’t enable heterogeneous prediction of node level dynamics. Node level is important to make critical interventions to specific components whilst preserving our multi-scale understanding of general system behaviour. In order to precisely identify the resilience function at the node-level, a sequential heterogeneous mean field estimation approach is proposed recently [3].

B. Uncertainty in Network Resilience

To simulate the dynamics and estimate resilience of complex networks with dynamical effects, we need to define dynamical models with parameter values. However, in practice, uncertainty on the model form and parameters are inherently present. Uncertainty can originate from latent process variables (process noise), e.g., inherent biological variability between cells which are genetically identical [9] or from a parameter estimation procedure based on noisy measurements (measurement or inference noise). In order to know the effect of arbitrary parameters uncertainty on the network-level resilience, our previous work introduced a polynomial chaos (PC) method [10] to understand macro-scale network wide resilience loss uncertainty. However, we still do not know the effect of parameters uncertainty on node-level resilience, especially as parameters uncertainty may cause different effects on nodes in a network. This can paint a different picture to that of the overall macro-scale network behaviour. That is to say, a macro-scale resilient network may hide non-resilient behaviour at the micro-level, which if not addressed in time can cause long term issues.

1) Uncertainty Modeling Review: In recent years, the modeling and numerical simulation of practical problems with uncertainty have received unprecedented attention, which is
generally called Uncertainty Quantification (UQ). UQ methods mainly includes: Monte Carlo Methods [11], Perturbation Methods [12], Moment Equation Methods [13], Polynomial approximation methods [14].

The Polynomial chaos expansion (PCE) method is a standard method for UQ in singular dynamical systems. The basic idea is to perform polynomial expansion of the exact solution in a random parameter space. This method can potentially solve problems with any type of random parameter inputs. The PCE method can mainly sub-divided into intrusive and non-intrusive approaches for the involved projection integral. The original PCE is based on Hermite polynomials, which are optimal for Gaussian distributed random variables (r.v.). However, uncertainty does not always obey the Gaussian distribution. Whilst a normal score transformation could be used to solve this problem [15], but can lead to slow convergence [16]. To solve this problem, the generalized polynomial chaos (gPC) has been developed [16] [17]. The gPC extends PCE toward a broader range of applications which could be used encompassing the more general Gamma distribution, Beta distribution, and many other flexible distribution functions. This is further advanced to consider stochastic processes represented by r.v. of arbitrary distributions [18].

The methods mentioned above need to know the exact knowledge of the involved probability density functions. While, information about distribution is usually limited or incomplete in practical applications. Moreover, the statistical distribution of model parameters can be non trivial, e.g., bounded, skewed, multi-modal, discontinuous, etc. Furthermore, the dependence between several uncertain input parameters might be unknown, compare [19]. Depending on the modeling task and circumstances, statistical information on model parameters may be available either discrete, continuous, or discretized continuous, they could exist analytically in the probability density distribution (PDF) or numerically as a histogram. The key shortcoming of current PCE approaches in this context are two-fold. First, they are heavily restricted in handling most of these conditions, and second they assume that this information is complete and perfect [20].

2) Arbitrary Polynomial Chaos: Arbitrary Polynomial Chaos (aPC) is proposed in [20], [21] to solve this problem. The statistical moments are the only source of information that is propagated in all polynomial expansion-based stochastic approaches. The exact probability density functions do not have to be known and do not even have to exist. For finite-order expansion, only a finite number of moments has to be known. Therefore, considering the fact that uncertainty in large-scaled networked system is not always known or fit existing distributions, aPC is applied in this paper to analysis the effect of uncertainty on node-level resilience.

C. Novelty and Contribution

Even though recent research about resilience of network is prevalent, research in estimating node-level resilience is rarely, and estimating node-level resilience considering uncertainty is lacking. In practical problems, not taking this uncertainty into account possibly leads to deviation when estimating resilience of a system as well as a node. Therefore, considering uncertainty when estimating resilience of each node in network with nonlinear dynamics have great significance.

This paper addresses the lack of uncertainty quantification in the multi-scale resilience of complex networks with nonlinear dynamics. The novelty is to enable parameter uncertainty that follow arbitrary distributions and estimate the resilience of the whole network and each node. To achieve this, we propose a method with multi-dimensional arbitrary polynomial chaos (aPC) to quantify these uncertain factors to reduce the risk of uncertainty when estimating the resilience of each node. We then analyze how parameters and network topology with uncertainty affect the multi-scale resilience of dynamic network, which would give us more insight of large-scale dynamic networks.

II. System Setup

A. Node Level Nonlinear Dynamics and Resilience

The traditional mathematical treatment of resilience used from ecology [22] to engineering [23] approximates the behavior of a complex system with a one-dimensional nonlinear dynamic equation

$$\dot{x} = f(\beta, x)$$  \hspace{1cm} (1)

The functional form of $f(\beta, x)$ represents the system’s dynamics, and the parameter $\beta$ captures the changing control or environment conditions (show in Figure 1). The system is assumed to be in one of the stable fixed points, $x_0$ of equation 1, extract from

$$f(\beta, x_0) = 0$$  \hspace{1cm} (2)

$$\frac{df}{dx} \bigg|_{x=x_0} < 0$$  \hspace{1cm} (3)

where $f$ is smooth and equation (2) provides the system’s steady state and equation (3) guarantees its linear stability. We will assume that this system always has a stable equilibrium $x_d > 0$ that is not close to the origin and the saddle-node bifurcation can happen close to the origin - see Figure 2. The stable equilibrium away from the origin is a desirable state of the system and will it be called healthy. Resilience in this general case is defined by a healthy and an unhealthy equilibrium. The possible stable equilibrium close to the origin is an undesirable state of the system and it will be called unhealthy. If in the system the unhealthy equilibrium is absent, then we say that the system is resilient.

B. Network Level

Real networked systems are usually composed of numerous components linked via a complex set of weighted, often directed, interactions (show in Figure 1 (b)).

$$\dot{x}_i = f(x_i, a_i) + \sum_{j=1}^{n} a_{ji}g(x_i, x_j, b_{ij})$$  \hspace{1cm} (4)

where each connected node $i$‘s behavior is described by a self-dynamic $f(\cdot)$ and a coupling dynamic $g(\cdot)$ with node $j$ via the connectivity matrix $M_{ij}$. $A$ and $B$ both are vectors of
3

Fig. 1. It shows the dynamics of a single node and the coupled dynamics in a complex network. In 1D systems resilience is captured by the resilience function $x(\beta)$, which describes the state(s) of the system as a function of the tunable parameter $\beta$. The system exhibits a single stable fixed point for $\beta > \beta_c$ and two (or more) stable fixed points, a desired state and an undesired state for $\beta < \beta_c$. (b) In a coupled dynamic system, the single parameter $\beta$ is replaced by the complex weighted network $w_i$, whose characteristics depend on both environmental conditions and the specific pairwise interaction strengths. Consequently, the resilience function, now capturing the behaviour of the vector state $x(w_i)$.

Fig. 2. In (a) we can see a system before the saddle-node bifurcation, where both the unhealthy and the healthy equilibria are present. In (b), we see a system after the saddle-node bifurcation, where the unhealthy equilibrium has been annihilated.

parameters. $A = \{a_1, ..., a_i\}$, $B = \{b_{11}, ..., b_{ij}\}$. We rewrite equation (4) in the compact form:

$$\dot{X} = F(X, A, B),$$

(5)

where $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by equation (4). Let $w_i$ be the weighted in-degree of vertex $v_i$, i.e.

$$w_i = \sum_{j=1}^{N} a_{ji},$$

(6)

and we denote by $w_{av}$ the average of all weighted in-degrees. We denote by $w_{out}$ the weighted out-degree of vertex $v_j$, i.e.

$$w_{out} = \sum_{i=1}^{N} a_{ij}.$$  

(7)

Similarly, let $d_i$ be the in-degree of $v_i$ and $d_{out}$ be its out-degree. In general, we do not know very well how functional resilience maps to the topological resilience (e.g. properties of $M_{ij}$) in connected ecosystems. Indeed, recent research has begun to address this by mapping the overall effective dynamics of a networked system to its topological structure and individual dynamics $\dot{x}_{\text{eff}}(\beta_{\text{eff}}, x_{\text{eff}})$, where $x_{\text{eff}}$ yields the effective mean network dynamics and $\beta_{\text{eff}}$ captures effective aspects of the network topology. Many systems can exhibit a common network-level effective dynamics, but have different node-level dynamics (shown in Figure 3). In order to understand the resilience and dynamics of individual nodes, a sequential estimation approach is proposed to solve this problem. However, we still do not know the affect of uncertainty parameters on the resilience at node-level.
III. APPROACH AND METHODOLOGY

To answer this question, an arbitrary polynomial chaos expansion method is proposed to estimate the resilience at node-level with uncertainty. We do so by defining arbitrary uncertainty distributions on the network dynamic parameters.

A. Dynamic network with uncertainty

Uncertainty in a dynamic network may exit in self-dynamics of each component in \( f(x, A) \) and each component in coupling term \( g(x, y, B) \) as well as the network topology. We assume that each parameter can be represented by a random variable that gets a different realization on each node and moreover the value of any parameters has to be within a range of its true value. So we have \( a_i = a_i(1 + e_1 u_i), b_{ij} = b_{ij}(1 + e_2 v_{ij}) \), \( M = M(1 + e_3 r) \), where \( u_i, v_{ij}, r \) are r.v. uniform in \([a, b]\) and \( e_1, e_2, e_3 \) are constants. \( U = \{u_1, ..., u_i\}, V = \{v_{11}, ..., v_{ij}\} \). The mathematics model of a dynamic network with uncertainty is showed as:

\[
\dot{x}_i = f(x_i, a_i(1 + e_1 u_i)) + \sum_{j} a_{ji}(1 + e_3 r)g(x_i, x_j, b_{ij}(1 + e_2 v_{ij}))
\]  

B. Sequential Heterogeneous Mean Field Estimation

The proposed framework utilizes an initial homogeneous mean field approximation of the system, By using either a homogeneous average degree \( w_{av} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \) or a weighted average degree \( w_{av} = \frac{< w^{out} >}{< w^{out} >} \) where \( w^{out} = (w_{1}^{out}, w_{2}^{out}, ..., w_{N}^{out}) \) is the vector of weighted out-degrees and \( w^{in} = (w_{1}^{in}, w_{2}^{in}, ..., w_{N}^{in}) \) is the vector of weighted in-degrees, \( < w^{out} > = \frac{1}{N} \sum_{i=1}^{N} w_{i}^{out} \) is the average weighted out-degree. We can calculate the equilibrium \( e(\theta) \) of the dynamical system. In order to find the mean field approximation of the equilibrium of the system, we define \( X := Mean[F(x, A, B)] = \frac{1}{N} \sum_{i=1}^{N} (f(x, a_i)) + \frac{1}{N} \sum_{i=1}^{N} w_{av} g(x, x, b_{ij}) \) 

Note that \( X(x) \) depends on \( A \) and \( B \). Since \( A \) and \( B \) are vectors of r.v., for any \( x, X(x) \) is a function depending on the random variable \( x \). Then we search for \( x \) such that \( X(x) = 0 \).

Because the parameters \( a_i \) are assumed to be iid r.v., for fixed \( x, f(x, a_i) \) are also iid r.v. We define

\[
\mu_f(x) := E[f(x, a_i)]
\]
\[
\delta_f(x) := \sqrt{Var[f(x, a_i)]}
\]

This means that by Central Limit Theorem (CLT), for big enough \( n, \frac{1}{n} \sum_{i=1}^{n} f(x, a_i) \) can be approximated by a normally distributed random variable with mean \( \mu_f(x) \) and standard deviation \( \frac{1}{\sqrt{n}} \delta_f(x) \), i.e

\[
\frac{1}{n} \sum_{i=1}^{n} f(x, A_i) \sim N(\mu_f(x), \frac{1}{n} \delta_f(x)^2)
\]

Similarly, the function \( g(x, x, b_{ij}) \) depending on random variable \( x \) is i.i.d, we define

\[
\mu_g(x) := E[g(x, x, b_{ij})]
\]
\[
\delta_g(x) := \sqrt{Var[g(x, x, b_{ij})]}
\]

Then we have

\[
\frac{1}{n} \sum_{i,j=1}^{n} M_{ji} g(x, x, b_{ij}) \sim N\left(\frac{m}{n} \mu_g(x), \frac{m}{n^2} \delta_g(x)^2\right)
\]
Since $\Xi(x)$ is the sum of 2 normally distributed r.v., when we combine the above we get

$$\Xi(x) \sim N(\mu_f(x) + \frac{m}{n} \mu_g(x), \frac{1}{n} \delta^2 f(x) + \frac{m}{n^2} \delta^2 g(x))$$ (17)

We can get a realisation of $\Xi_\alpha(x)$ by drawing $\zeta_\alpha$ from $N(0,1)$ and setting

$$\Xi_\alpha(x) = \mu_f(x) + \frac{m}{n} \mu_g(x) + \sqrt{\frac{1}{n} \delta^2 f(x) + \frac{m}{n^2} \delta^2 g(x)} \zeta_\alpha$$ (18)

We assume that every realisation of $\Xi(x)$ has the shape described in Figure 1. We can calculate the equilibrium $e^{(0)}$ from equation 10. Since $\zeta_\alpha$ is a random variable which is normally distributed, we can use a polynomial chaos expansion (PCE) truncate to degree $n$ to approximate the equilibrium $e^{(0)}$. We find the smallest positive root $\rho^{(0)}$ of $\Xi'(x)$. Finally we set $\tau^{(0)} = \Xi(\rho^{(0)})$.

Since $\Xi(x)$ is a random variable, both $\rho^{(0)}$ and $\tau^{(0)}$ are r.v.. Moreover, $\tau^{(0)}$ is an indicator for the saddle-node bifurcation. For a given realization of $\zeta_\alpha$, if $\tau_\alpha > 0$, then there is only one equilibrium and the dynamics is resilient and if $\tau_\alpha < 0$, then there are three equilibria and the dynamics is non-resilient. Thus the probability of the system being resilient is $P(\tau > 0)$.

We can use PCE truncated to degree $n$ to approximate $\tau(\zeta)$, we will denote this PCE by $\tilde{\tau}_n(\zeta)$. We define the function

$$pos(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$ (19)

Then, the probability that the system is resilient is given by
distribution like Gaussian distribution, Binomial distribution to approximate etc. We need to use the arbitrary polynomial chaos (aPC) \[20\] to approximate the effect that the graph has on a single vertex. Given \(g\) we have to notice that the probability of a vertex \(j\) on the other side of the in-edge is proportional to its out-degree. With this in mind we can average over all possibilities and we find the equilibrium of the system can be approximated in a system with uncertain parameters, network and node-level dynamics are uncertain. In (b) \(e\) represents the equilibrium of a node, \(U_1, U_2\) are uncertain parameters.

Then our first order approximation is

\[
\dot{e} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \text{pos}(\tau_n(\zeta)) \, d\zeta \tag{20}
\]

**Step 1:** We use the mean field approximation as an initial guess to bootstrap our approximations. We approximate the dynamics on each node by the dynamical system:

\[
\dot{x}_i = f(x_i, a_i) + w_i g(x_i, e^{(0)}, b_{ij}) = 0 \tag{21}
\]

The solution of this equation is a function of \(w_i\), i.e. \(\chi^{(1)}(w_i)\). Then our first order approximation is \(e^{(1)} = \chi^{(1)}(w_i)\).

Since parameters \(a_i, b_{ij}\) do not always belong to common distribution like Gaussian distribution, Binomial distribution etc. We need to use the arbitrary polynomial chaos (aPC) \[20\] to approximate \(e^{(1)}\) and its distribution.

**Step 2:** We can use the previous approximation to approximate the effect that the graph has on a single vertex. Given a vertex \(i\) an effect an in-edge will have on the dynamics is \(g(x_i, x_j)\). In order to find the average effect of an in-edge, we have to notice that the probability of a vertex \(j\) is on the other side of the in-edge is proportional to its out-degree. With this in mind we can average over all possibilities and we find that the average effect is \(\sum_{j=1}^N d_{ij}^{\text{out}} g(x_i, x_j, b_{ij}) / \sum_{j=1}^N d_{ij}^{\text{out}}\). In order to find their mean effect of the neighbours, each component of the coupling vector \(g(\cdot)\) is weighted by \(d_{ij}^{\text{out}}\). This means that we can use the previous step’s approximation and we find that the equilibrium of the system can be approximated by the equilibrium of

\[
\dot{x}_i = f(x_i) + w_i \sum_{j=1}^N d_{ij}^{\text{out}} g(x_i, e^{(1)}_{ij}) = 0 \tag{22}
\]

The solution of this equation depends on \(w_i\). Then our second order approximation is \(e^{(2)} = \chi^{(2)}(w_i)\). Also, \(e^{(2)}\) and its distribution could be approximated by aPC.

**Step 3 to n:** We repeat the above, using each time the approximation we calculate in the previous step.

### C. Arbitrary Polynomial Chaos Expansion

1) One-Dimensional aPC: Let \(\Xi\) be random variable with PDF \(w\). Moreover let \(X = \phi(\Xi)\), with \(\phi\) a function that is square integrable on \(\mathbb{R}\) with \(w\) as weight function, let us call this space \(L_w^2\). Our goal is to approximate \(X\) by a polynomial series of \(\Xi\). For a stochastic analysis of \(X\), the model \(\phi(\Xi)\) may be expanded as follows:

\[
\phi(\Xi) = \sum_{n=0}^{\infty} c_n P_n^{(0)}(\Xi) \tag{23}
\]

where \(c_n\) are the expansion coefficients that are determined by Galerkin projection, numerical integration or collection, and \(P_n^{(0)}(\Xi)\) are the polynomials forming the basis \(\{P_0^{(0)}, P_1^{(1)}, P_2^{(2)}, ..., P_n^{(n)}\}\) that is orthogonal with respect to \(w\). The only difference between aPC and previous PCE methods is that the measure \(w\) can have an arbitrary form, and thus the basis \(\{P_0^{(0)}, P_1^{(1)}, P_2^{(2)}, ..., P_n^{(n)}\}\) has to be found specifically for the probability measure \(w\) appearing in the respective application.

2) Multi-Dimensional aPC: Most realistic applications feature multi-dimensional model input \(\Xi\), i.e. \(\Xi = \{\Xi_1, \Xi_2, ..., \Xi_N\}\). Here, the total number of input parameters is equal to \(N\). The number \(M\) of in equation \[24\] depends on parameter \(N\) and the order \(d\) of the expansion, according to the formula \(M = (N + d)!/(N!d!)\). The model output \(X\) can be represented by a multivariate polynomial expansion as follows:

\[
\phi(\Xi_1, \Xi_2, ..., \Xi_N) = \sum_{i=1}^{M} c_i \Phi_i(\Xi_1, \Xi_2, ..., \Xi_N). \tag{24}
\]

The function \(\Phi_i\) is a simplified notation of the multi-variate orthogonal polynomial basis for \(\Xi_1, \Xi_2, ..., \Xi_N\). Assuming that the input parameters are independent, the multi-dimensional
basis can be constructed as a simple product of the corresponding univariate polynomials

\[ \Phi_j(\Xi_1, \Xi_2, ..., \Xi_N) = \prod_{j=1}^{N} p_j^{(\alpha_j)}(\Xi_1, \Xi_2, ..., \Xi_N), \]

\[ \sum_{j=1}^{N} \alpha_j^i \leq M, \quad i = 1, ..., N, \quad (25) \]

where \( \alpha_j^i \) is a multivariate index that contains the combinatorial information how to enumerate all possible products of individual univariate basis functions. In other words, the index \( \alpha \) can be seen as an \( M \times N \) matrix, which contains the corresponding degree for parameter number \( j \) in expansion term \( k \).

Let us define the polynomial \( P^{(k)}(\Xi) \) of degree \( k \) in the random variable \( \Xi \):

\[ P^{(k)}(\Xi) = \sum_{i=0}^{k} P^{(k)}_i \Xi^i, \quad k \in [0, d] \quad (26) \]

where \( P^{(k)}_i \) are coefficients in \( P^{(k)}(\Xi) \).

Our goal is to construct the polynomials in equation (26) to form an orthonormal basis for arbitrary distributions. The arbitrary distribution could be discrete, continuous, raw data sets or by their moments. Orthonormality for polynomials \( P^{(k)} \) of degree \( k \) and \( P^{(l)} \) of degree \( l \) is defined as

\[ \int P^{(k)}(\Xi)P^{(l)}(\Xi)dw(\Xi) = \begin{cases} 0 & \forall k \neq l \\ 1 & \text{else} \end{cases} \quad (27) \]

Here we need to introduce an intermediate auxiliary condition by demanding that the leading coefficients of all polynomials be equal to 1: \( P^{(k)}_i = 1 \) \( \forall k \). The \( k \)th raw (crude) moment of the random variable \( \Xi \) is defined as

\[ \mu_k = \int \Xi^k dw(\Xi) \quad (28) \]

The relationship between raw moments of \( \Xi \) and their coefficients can be written in matrix form (the detail process could be seen in [20]):

\[ \begin{bmatrix} \mu_0 & \mu_1 & \ldots & \mu_k \\ \mu_1 & \mu_2 & \ldots & \mu_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k-1} & \mu_k & \ldots & \mu_{2k-1} \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} P^{(k)}_{0,k} \\ P^{(k)}_{1,k} \\ \vdots \\ P^{(k)}_{k-1,k} \\ P^{(k)}_{k,k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (29) \]

For multi-dimensional r.v., the polynomial \( P^{(k)}_{ij}(\Xi_j) \) is defined as:

\[ P^{(k)}_{ij}(\Xi_j) = \sum_{i=0}^{k} p^{(k)}_{ij} \Xi_j^i \quad (30) \]

and the unknown polynomial coefficients \( P^{(k)}_{ij} \) can be defined from the following matrix equation [24]:

\[ \begin{bmatrix} \mu_{0,j} & \mu_{1,j} & \ldots & \mu_{k,j} \\ \mu_{1,j} & \mu_{2,j} & \ldots & \mu_{k+1,j} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k-1,j} & \mu_{k,j} & \ldots & \mu_{2k-1,j} \\ 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} P^{(k)}_{0,j} \\ P^{(k)}_{1,j} \\ \vdots \\ P^{(k)}_{k-1,j} \\ P^{(k)}_{k,j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (31) \]

We now show the results of a real system case study to illustrate how the aPC framework can be used.

IV. RESULTS FOR BI-STABLE SYSTEMS

Bi-stable dynamical systems are common across social (e.g. population logistic model [25]), ecological (e.g. soil health [26]), climate (e.g. ocean circulation [27]), and human conflict systems [28]. There exists a stable undesirable state (e.g. population collapse or conflict) and a stable desirable state (e.g. healthy population with collaboration [29]), with an unstable transition brink in between, and this is ideal for demonstrating the concept of resilience and uncertainty. Networks that connect such systems represent a wider interacting ecosystem and often a mutualistic coupling represents positive reinforcing interactions. Interaction examples include gravity, radiation, or Boltzmann Lotka Volterra (BLV) models [30] frequently use a \( x_i \times x_j \) mutualistic attractor component.

A. Case Study: Ecological Network

We use a well studied case of pollinator networks [31]. The abundance of species \( i, x_i \) is given by:

\[ \frac{dx_i}{dt} = B_i + x_i(1 - \frac{x_i}{K}) (\frac{x_i}{C} - 1) + \sum_{j=1}^{N} a_{ij} \frac{x_i x_j}{D_i E_i x_i + H_j x_j} \quad (32) \]

The first term on the right hand side of equation (32) accounts for the incoming migration of \( i \) at a rate \( B_i \) from neighboring ecosystems. The second term describes logistic growth with the system carrying capacity \( K \), and the Allee effect, according to which for low abundance \( \langle x_i < C_i \rangle \) the system features negative growth [32]. The third term describes mutualistic interactions, captured by a response function that saturates for large \( x_i \) or \( x_j \), indicating that\( j \)’s positive contribution to \( x_i \) is bounded.

For simplicity, we use homogeneous parameters: \( B = 0.1, C = 1, K = 5, D = 5, E = 0.9, H = 0.1 \). We moreover assume that the value of some parameter has to be within 10% of its mean, so we have \( C = E[C][1 + 0.1U_1], E = E[E][1 + 0.1U_2] \), where \( U_1 \) is a random variable uniform in \([-1, 1]\) (\( U_1, U_2 \) could be r.v. that follow arbitrary distributions.). In this case study, system resilience could be defined by the ability of the system to recover all the populations after extinction [8]. In order for this to happen, the system should be in the regime where there is only one equilibrium. This because if there are two stable equilibria, the system will be trapped in the one with low population density, in which the system can not recover to the original status.
In Figure 4, we show what happened when a network becomes less connected by removing edges. In this case, parameters are certain and the figure explicitly shows the bounds of equilibrium under different perturbation and the regime where loss of resilience happens. Critical function defines resilience regimes mapping network properties (average weighted degree $w_{av}$ to local properties (critical resilience value $w_{crit}$)). For each $w_{av}$, corresponding $w_{crit}$ could be calculated from equation (33). The critical weight, $w_{crit}$, is a function of $w_{av}$ since it is a function of $e^{(0)}$ and $e^{(0)}$ is a function of $w_{av}$. In Figure 4 (b), we see the graph of $w_{crit}$ versus $w_{av}$. Since $e^{(0)}$ is discontinuous, $w_{crit}$ is also discontinuous.

$$\dot{x}_i = f(x_i) + w_i g(x_i, e^{(0)}(w_{av}))$$ (33)

In this case, a critical average weight $w_*$ is about 7 where bifurcation will happen. When average weight is greater than 7, the system is resilient and almost every node in this system is resilient. The critical weight can reveal some basic properties for the dynamics on a nodal level. For example we see in Figure 4 (b) that when when $w_{av} > w_*$, $w_{crit}$ is almost 0. This implies that if the system on average is in the resilient region, a vertex will also be in the resilient region even if it is very weakly connected to the rest of the network. However, in the case with uncertain parameters, even if the average weight is greater than 7, the system is possibly not resilient. We use aPC to analysis what will happen in the regime where loss of resilience may exist. In Figure 5, it shows dynamics of the system with uncertain parameters when average weight is 7. We use aPC to approximate the minimum value of the function and whether the system is resilient.

**B. Analysis on the Effect of Uncertainty**

Firstly, we use the method described in [Step 0](#) to analyse the probability of system to be resilient and approximate the equilibrium. We truncate the series to arbitrary orders $N$ from 2 to 5 shown in Figure 6. Increasing the order $N$ of the polynomial improves the convergence of the function. However, increasing the order of the polynomial means that a substantially higher number of simulations is required. Therefore, a compromise between accuracy and required computational time is necessary.

Referring to the graph in Figure 6, we can easily find the difference among different orders especially $N = 2$. In order to estimate the probability of resilience, we obtain a graph Cumulative Distribution Function (CDF) with different truncation in Figure 6 (b). It can be seen that the results for
(a) Approximate equilibrium of a node by aPC when we truncate the series to arbitrary orders $N$ from 2 to 5

(b) The CDF of equilibrium when we truncate the series to arbitrary orders $N$ from 2 to 5

Fig. 7. Approximate equilibrium of a node in the networked system by aPC. In (a), the four color surfaces, blue, green, red, yellow surface, present different truncation from 2 to 5. $e$ represents the equilibrium of a node, $U_1, U_2$ are uncertain parameters. It can be seen that these surfaces almost overlap which means that their accuracy are similar. In (b), it shows the CFD when we truncate the series from 2 to 5.

(a) Probability of resilience when average weight of network is different

(b) Critical weight of node with different average weight of network

Fig. 8. It shows the effect of uncertainty parameters on resilience of network and each node. From (b), we could know the probability of resilience of each node according to the relationship between average weight of network and node’s weight.

$N = 3, 4, 5$ almost overlap while there is significant difference for $N = 2$.

Therefore, $N = 3$ can be considered as the appropriate choice for the polynomial order since choosing higher order polynomials substantially increase the required simulation time with only minor effects on improving the accuracy of the results. So, we can see the effect of uncertain parameters on system resilience as well as node-level resilience. When parameters are certain and average weight is 7, the system is resilient and nodes are resilient. However, when parameters are uncertain in this case, the probability of resilience of the system is about 0.798. So according to analysis above, some nodes will also possibly lose resilience.

Second, we use the method in Step 1 and Step 2 to estimate the equilibrium of each node. The method aPC mentioned above is used to estimate a node’s equilibrium and we truncate the series to arbitrary orders $N$ from 2 to 5 shown in Figure 7. In Figure 7 (a), it shows that the node has different equilibrium when parameters $U_1, U_2$ have different values and the results for $N = 2, 3, 4, 5$ almost overlap. Meanwhile, it can be seen that the results of CDF also almost overlap. Therefore, $N = 2$ can be considered as the appropriate choice for the polynomial order since choosing higher order polynomials increase required computation with little improvement of accuracy.

In Figure 8, we show the effect of uncertain parameters on the resilience of whole network and each node. Comparing Figure 4 and Figure 8, it is clear that when parameters are certain, network could maintain its resilience when average weight is greater than 7. However, with the effect of uncertain parameters, network could lose resilience even though its average weight is greater than 7. With the growth of average weight, the network has more chance to be resilient. When the average weight is greater than a critical value, the network is absolute resilient. Similarly, Figure 4 (b) shows that when node’s weight is greater than a critical value under certain average
weight, the node could maintain its resilience. While, with the effect of uncertainty shown in Figure (5) (b), a node may lose resilience even though its weight is greater than the critical value in Figure (4) (b). Therefore, the method mentioned above could help us understand the effect of uncertainty on network-level and node-level resilience. Also, it help us to predict whether a node is resilient and the probability of a node to lose resilience.

V. CONCLUSIONS

At present, the research of how to estimate node-level resilience of dynamic networked system is still limited. Node level is important to make critical interventions to specific components whilst preserving our multi-scale understanding of general system behaviour. In this paper, an arbitrary polynomial chaos expansion (aPC) method is used to quantify the uncertainty of arbitrary uncertain distributions. This approach can effectively estimate node-level resilience and analyse the effect of uncertainty on each node. This could help us better make a prediction of the probability that a node loses its resilience and reduce the risk of uncertainty. In the future, we would like to survey how the community structure of network affects network-level and node-level resilience, for example, whether there exists a relationship between modularity of community in network and resilience.

REFERENCES

[1] J. Gao, B. Barzel, and A.-L. Barabási, “Universal resilience patterns in complex networks,” Nature, vol. 530, no. 7590, pp. 307–312, 2016.
[2] Z. Wei, A. Pagani, G. Fu, I. Gruymer, W. Chen, J. McCann, and W. Guo, “Optimal sampling of water distribution network dynamics using graph fourier transform,” IEEE Transactions on Network Science and Engineering, vol. 7, no. 3, pp. 1570–1582, 2020.
[3] G. Moutsinas and W. Guo, “Probabilistic stability of traffic load balancing on wireless complex networks,” IEEE Systems Journal, 2019.
[4] R. Cohen, K. Erez, D. Ben-Avraham, and S. Havlin, “Resilience of the internet to random breakdowns,” Physical Review letters, vol. 85, no. 21, p. 4626, 2000.
[5] R. V. Sole and M. Montoya, “Complexity and fragility in ecological networks,” Proceedings of the Royal Society of London. Series B: Biological Sciences, vol. 268, no. 1480, pp. 2039–2045, 2001.
[6] R. Arghandeh, A. Von Meier, L. Mehrmanesh, and L. Mili, “On the definition of cyber-physical resilience in power systems,” Renewable and Sustainable Energy Reviews, vol. 58, pp. 1060–1069, 2016.

C. N. Kaiser-Bunbury, J. Mougal, A. E. Whittington, T. Valentin, R. Gabriel, J. M. Olesen, and N. Blüthgen, “Ecosystem restoration strengthens pollination network resilience and function,” Nature, vol. 542, no. 7640, pp. 223–227, 2017.
[8] G. Moutsinas and W. Guo, “Node-level resilience loss in dynamic complex networks,” Nature Scientific Reports, 2020.
[9] M. Kaern, T. C. Elston, W. J. Blake, and J. J. Collins, “Stochasticity in gene expression: from theories to phenotypes,” Nature Reviews Genetics, vol. 6, no. 6, pp. 451–464, 2005.
[10] G. Moutsinas, M. Zou, and W. Guo, “Uncertainty of resilience in complex networks with nonlinear dynamics,” arXiv preprint arXiv:2004.13198, 2020.
[11] G. Fishman, Monte Carlo: concepts, algorithms, and applications. Springer Science & Business Media, 2013.
[12] C. Zhao, S. Xie, X. Chen, M. P. Jensen, and M. Dunn, “Quantifying uncertainties of cloud microphysical property retrievals with a perturbation method,” Journal of Geophysical Research: Atmospheres, vol. 119, no. 9, pp. 5375–5385, 2014.
[13] D. Zhang, Stochastic methods for flow in porous media: coping with uncertainties. Elsevier, 2001.
[14] N. Wiener, “The homogeneous chaos,” American Journal of Mathematics, vol. 60, no. 4, pp. 897–936, 1938.
[15] H. Wackernagel, Multivariate geostatistics: an introduction with applications. Springer Science & Business Media, 2013.
[16] D. Xiu and G. E. Karniadakis, “The wiener-askey polynomial chaos for stochastic differential equations,” SIAM journal on scientific computing, vol. 24, no. 2, pp. 619–644, 2002.
[17] D. Xiu and Karniadakis, “Modeling uncertainty in flow simulations via generalized polynomial chaos,” Journal of computational physics, vol. 187, no. 1, pp. 137–167, 2003.
[18] X. Wan and G. E. Karniadakis, “Multi-element generalized polynomial chaos for arbitrary probability measures,” SIAM Journal on Scientific Computing, vol. 28, no. 3, pp. 901–928, 2006.
[19] A. Der Kiureghian and P.-L. Liu, “Structural reliability under incomplete probability information,” Journal of Engineering Mechanics, vol. 112, no. 1, pp. 85–104, 1986.
[20] S. Oladyshkin and W. Nowak, “Data-driven uncertainty quantification using the arbitrary polynomial chaos expansion,” Reliability Engineering & System Safety, vol. 106, pp. 179–190, 2012.
[21] J. A. Paulson, E. A. Buehler, and A. Mesbah, “Arbitrary polynomial chaos for uncertainty propagation of correlated random variables in dynamic systems,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 3548–3553, 2017.
[22] R. M. May, “Thresholds and breakpoints in ecosystems with a multiplicity of stable states,” Nature, vol. 269, no. 5626, pp. 471–477, 1977.
[23] A. M. Lyapunov, “The general problem of the stability of motion,” International journal of control, vol. 55, no. 3, pp. 531–534, 1992.
[24] S. Oladyshkin, H. Class, R. Helmig, and W. Nowak, “A concept for data-driven uncertainty quantification and its application to carbon dioxide storage in geological formations,” Advances in Water Resources, vol. 34, no. 11, pp. 1508–1518, 2011.
[25] M. Lundqvist, A. Compte, and A. Lansner, “Bistable, irregular firing and population oscillations in a modular attractor memory network,” PloS Comput Biol, vol. 6, no. 6, 2010.
[26] L. Todman, F. Fraser, R. Corstanje, and et al., “Evidence for functional state transitions in intensively-managed soil ecosystems,” Sci Rep, vol. 8, 2018.
[27] R. Marsh, A. Yool, T. M. Lenton, M. Y. Gulamali, N. R. Edwards, J. G. Shepherd, M. Krzmaric, S. Newhouse, and S. J. Cox, “Bistability of the thermohaline circulation identified through comprehensive 2-parameter sweeps of an efficient climate model,” Climate Dynamics, vol. 23, 2004.
[28] G. Aquino, W. Guo, and A. Wilson, “Nonlinear dynamic models of conflict via multiplexed interaction networks,” Preprint arXiv 1909.12457, 2019.
[29] J. Ron, I. Pinkovezky, E. Fonio, O. Feinerman, and N. Gov, “Bi-stability in cooperative transport by ants in the presence of obstacles,” PLoS Biology, 2018.
[30] A. Wilson, “Boltzmann, lotka and volterra and spatial structural evolution: an integrated methodology for some dynamical systems,” J. R. Soc. Interface, 2008.
[31] J. N. Holland, D. L. DeAngelis, and J. L. Bronstein, “Population dynamics and mutualism: functional responses of benefits and costs,” The American Naturalist, vol. 159, no. 3, pp. 231–244, 2002.
[32] W. C. Allee, O. Park, A. E. Emerson, T. Park, K. P. Schmidt et al., “Principles of animal ecology,” Saunders Company Philadelphia, Pennsylvina, USA, Tech. Rep., 1949.
