ISOLATED SINGULARITIES OF HYPERSURFACES

RUSTAM SADYKOV AND STANISLAV TRUNOV

Abstract. Introduced by Seifert and Threlfall, cylindrical neighborhoods is an essential tool in the Lusternik-Schnirelmann theory. We conjecture that every isolated critical point of a smooth function admits a cylindrical ball neighborhood. We show that the conjecture is true for cone-like critical points, Cornea reasonable critical points, and critical points that satisfy the Rothe $H$ hypothesis. In particular, the conjecture holds true at least for those critical points that are not infinitely degenerate.

1. Introduction

Given a smooth function $f: M \to \mathbb{R}$ on a manifold $M$ without boundary, a point $x$ in $M$ is said to be critical if the differential $d_x f$ of $f$ at $x$ is trivial. A point that is not critical is said to be regular. We say that a value $c$ of $f$ is critical if the fiber $f^{-1}(c)$ contains a critical point. Critical points of functions could be extremely complicated. For example, the fiber $f^{-1}(c)$ over a regular value $c$ is a hypersurface of $M$ with no singularities. On the other hand, every closed subset $V \subset M$ is the hypersurface level $f^{-1}(c)$ of an appropriately chosen smooth function $f$ on $M$. In other words, critical points of smooth functions are at least as complicated as closed subsets of a manifold.

Let $x_0$ be a critical point of a smooth function $f$ with critical value $f(x_0) = c$. We say that an isolating neighborhood $U$ of $x_0$ is cylindrical if it consists of trajectories in $f^{-1}[c - \varepsilon, c + \varepsilon]$ of the gradient vector field of $f$. We will always assume that $U$ is a smooth manifold of dimension $\dim M$ with corners such that $U \cap f^{-1}(c - \varepsilon)$ and $U \cap f^{-1}(c + \varepsilon)$ are smooth manifolds with (smooth) boundary. Cylindrical neighborhoods were introduced by Seifert and Threlfall [16] in 1938, and were used in [13, 19, 4] and [3]. The existence of a cylindrical ball neighborhood, i.e., a cylindrical neighborhood homeomorphic to a ball, is an important property as it allows one to apply, for example, the Lusternik-Schnirelman type argument and deduce Lusternik-Schnirelman inequalities [19, 3, 15].
In this paper we study cylindrical ball neighborhoods and give evidence supporting Conjecture 1. We note that in dimensions $\neq 4, 5$ all smooth manifolds homeomorphic to a ball are diffeomorphic to a ball.

**Conjecture 1.** Every isolated critical point of a smooth function admits a cylindrical ball neighborhood.

If there is an open subset $U \subset \mathbb{R}^m$, and a coordinate neighborhood $\tau : U \to M$ of a point $x \in M$ such that $f \circ \tau$ is the restriction of a polynomial, we say that the critical point $x$ of $f$ is algebraic. In this case, the singular hypersurface $V = f^{-1}(c)$ is locally an algebraic set. It is known (e.g. see [10]) that for every algebraic critical point $x_0$ of $f$ with $f(x_0) = c$, there is a closed disc neighborhood $B_x$ of $x_0$ in $M$ such that $\partial B_x$ is transverse to $V$, and the pair $(B_x, V \cap B_x)$ is homeomorphic to the cone over the pair $(\partial B_x, V \cap \partial B_x)$. We say that $(B_x, V \cap B_x)$ is a cone neighborhood pair for $x_0$.

This essential property of algebraic singular points is a starting point for the theory of hypersurface singularities, see [10]. It follows (see [6, 7] and Proposition 17) that a critical point $x_0$ admits a cone neighborhood pair if and only if it is cone-like, i.e., if it admits a cone neighborhood in $f^{-1}(f(x_0))$.

**Theorem 2.** Every cone-like critical point of a smooth function $f : M \to \mathbb{R}$ admits a cylindrical ball neighborhood.

A version of Theorem 2 appears in [6, p. 396] for manifolds $M$ of dimension $m \neq 4$.

In [3], Cornea defined a reasonable critical point as an isolated critical point $x_0$ of $f$ such that $f^{-1}(f(x_0))$ admits a Whitney stratification into the stratum $\{x_0\}$ and its complement. By [3, Lemma 2] every reasonable critical point admits a cone neighborhood pair, and, therefore, it is cone-like.

**Corollary 3.** Every reasonable critical point admits a cylindrical ball neighborhood.

Another large class of critical points was studied by E. Rothe in [13] and [14]. We say that an isolated critical point $x_0$ satisfies the hypothesis $H$ if there is an isolating coordinate neighborhood $U$ of $x_0$ such that for all $x \neq x_0$ in $U \cap f^{-1}(x_0)$ the vectors $x - x_0$ and $\text{grad } f(x)$ are linearly independent.

**Corollary 4.** Every critical point satisfying the hypothesis $H$ admits a cylindrical ball neighborhood.

We will show that the set of jets of non cone-like critical points is of infinite codimension in the space of all jets, see Corollary 20. In other words, non cone-like critical points are extremely rare and infinitely degenerate. We also deduce that the critical point of any infinitely determined map germ is cone-like, see Proposition 23.

Finally, we show that the class of critical points that admit a cylindrical ball neighborhood diffeomorphic to a ball is larger than the class of cone-like critical points.
Theorem 5. For every $m \geq 4$ there is a smooth function $f: \mathbb{R}^m \to \mathbb{R}$ with an isolated critical point $x_0$ such that $x_0$ is not cone-like and such that $x_0$ admits a cylindrical neighborhood diffeomorphic to a ball.

The functions $f$ that appear in Theorem 5 are constructed by Takens in [18]. Theorem 5 implies that critical points of Takens functions are removable, see Corollary 27. This is closely related to the Funar theorem [5, Proposition 2.1] asserting that cone-like critical points of smooth maps $f: M \to N$ of a manifold of dimension $m \leq 2k - 1$ to a manifold of dimension $k \geq 2$ are topologically removable except possibly when $(m, k) \in \{(2, 2), (4, 3), (8, 5), (16, 9)\}$, and under an additional condition that the critical point admits an adapted neighborhood diffeomorphic to a ball when $(m, k) = (5, 3)$.

Theorem 2 and its corollaries are related to the Lusternik-Schnirelmann category [3, 17, 19]. In fact, in [15] we will show that a closed manifold of dimension at least 6 admits a Singhof-Takens filling by $n$ smooth balls if and only if it admits a function with $n$ critical points each of which admits a cylindrical ball neighborhood.

In section 2 we give a detailed proof of Theorem 2. Next, in section 3 we prove Corollary 4 and show that every cone-like critical point admits a cone neighborhood pair. In section 4 we review relevant theorems from singularity theory and deduce that the set of jets of map germs with non cone-like critical points is a subset of a proalgebraic set of infinite codimension in the space of all jets. We also show that critical points of infinitely determined map germs are cone-like. Finally, in section 5 we list properties of Takens critical points, namely: the Takens function germs are flat (Proposition 24), the critical points of Takens functions admit cylindrical neighborhoods (Proposition 25) diffeomorphic to a ball, and the critical points of Takens functions are removable (Corollary 27). The proof of Theorem 5 can also be found in section 5.

We are grateful to Louis Funar for valuable comments, and references.

2. Proof of Theorem 2

Let $f$ be a smooth function on a manifold $M$ without boundary. We choose a Riemannian metric on $M$. It defines a gradient flow $\gamma_t$ on $M$ such that the trajectory $t \mapsto \gamma_t(y)$ of any point $y$ is a curve in $M$ parametrized by an open interval $(t_1, t_2)$ where $-\infty \leq t_1 < t_2 \leq \infty$. Every trajectory $t \mapsto \gamma_t(y)$ is either closed in $M$ or the limit of $\gamma_t(y)$ as $t \to \infty$ or $t \to -\infty$ is a critical point of $f$. For example, for every critical point $x \in M$, its trajectory is a constant curve $t \mapsto x$ parametrized by $t \in (-\infty, \infty)$. The union $D$ of all non-closed trajectories of the gradient flow as well as all critical points of $f$ is a closed subset of $M$. The closed set $D$ is a union of closed subsets $D(x)$ of points $y$ such that for the trajectory $t \mapsto \gamma_t(y)$ the limit of $\gamma_t(y)$ as $t \to \infty$ or $t \to -\infty$ is the critical point $x$. 
For any value \( a \) of the function \( f \), let \( M_a \) denote the level \( f^{-1}(a) \). The intersection of \( D \) with \( M_a \) will be denoted by \( D_a \). Similarly, \( D_a(x) \) denotes the intersection of \( D(x) \) and \( M_a \).

The gradient flow \( \gamma_t \) defines a diffeomorphism

\[
h_{a,b} : M_a \setminus D_a \to M_b \setminus D_b
\]

for every pair \((a, b)\) of real numbers such that \( a < b \) by associating to a point \( x \) a unique point in \( M_b \setminus D_b \) on the trajectory \( t \mapsto \gamma_t(x) \) of \( x \). We will also write \( h_{b,a} \) for the inverse of \( h_{a,b} \).

Suppose now that \( x_0 \) is a cone-like singular point of a smooth function \( f \). Since \( x_0 \) is isolated, we may assume that \( f(x_0) = 0 \), and \( x_0 \) is the only critical point on the singular hypersurface \( M_0 \). Similarly, we may assume that there are regular values \( c \) and \( -c \) of \( f \) such that \( 0 \) is a unique critical value in the interval \( (-c - \epsilon, c + \epsilon) \) for some \( \epsilon > 0 \).

Let \((B_z, M_0 \cap B_z)\) be an isolating cone neighborhood pair for \( x_0 \). We recall that \( \partial B_z \) intersects \( M_0 \) transversally. We will write \( V \) for \( M_0 \cap B_z \). Let \( H \) denote the subset of \( M_{[-c,c]} \) of points \( x \) such that either the gradient curve through \( x \) intersects \( V \) or one of the limits of the gradient curve through \( x \) is \( x_0 \), where \( M_{[-c,c]} \) is the submanifold of \( M \) of points \( x \) such that \( f(x) \in [-c, c] \). We write \( H_a \) for the intersection of \( H \) and \( M_a \). By choosing \( V \) and \( c \) sufficiently small, we may assume that \( H \subset B_z \).

**Lemma 6.** The subsets \( H_c \subset M_c \) and \( H_{-c} \subset M_{-c} \) are smooth submanifolds with (smooth) boundary. The subset \( H \subset M \) is a smooth manifold of codimension 0 with corners \( \partial H_{-c} \sqcup \partial H_c \). In other words, \( H \) is a smooth cylindrical neighborhood of \( x_0 \).

**Proof.** Let \( x \) be a point in \( H \setminus D \). Suppose that \( f(x) \in (-c, c) \). We will omit the cases \( f(x) = c \) and \( f(x) = -c \) as the argument in these cases is similar. Without loss of generality we may assume that \( f(x) \leq 0 \); the case \( f(x) \geq 0 \) is similar.

Let \( \delta \) denote the gradient flow of \( f \). Then for some \( t \geq 0 \), the point \( \delta_t(x) \) is in \( V \). Suppose \( x \in H \) is a point with \( \delta_t(x) \in \partial V \). Then for a small neighborhood \( U \) of \( x \) in \( M \), there is a smooth map \( \varphi : U \to M_0 \) that takes \( y \in U \) to a unique point \( \varphi(y) \in M_0 \) on the trajectory of \( y \). In view of the diffeomorphism \( h_{f(x), 0} \), the map \( \varphi \) is a submersion onto a small neighborhood of \( \delta_t(x) \). Therefore \( x \) admits a half disc neighborhood in \( H \). The case where \( \delta_t(x) \) is in the interior of \( V \setminus \{x_0\} \) is similar.

Suppose now that \( x \in D \cap H \). We still assume that \( f(x) \leq 0 \). We need to show that for every point \( y \) sufficiently close to \( x \), either \( y \in D \) or the trajectory of the gradient flow of \( f \) through \( y \) either intersects \( V \). Assume that \( y \) is not in \( D \), and the trajectory through \( y \) does not intersect \( V \). Then there is a sequence of points \( y_t \) approaching \( x \) such that each trajectory \( t \mapsto \delta_t(y_t) \) passes through a point \( z_t \) in \( M_0 \setminus V \) at some moment \( t_t \). We may assume \( f(y_t) = f(x) \). Since \( M_0 \) is compact, there is an accumulating point \( z \) of the points \( z_t \) in \( M_0 \). In fact, by dropping some
of the points $y_i$ (and $z_i$), we may assume that $z$ is a limit point of $z_i$. We claim that the trajectory of the gradient flow of $f$ through the point $z$ passes through the point $x$, which contradicts the assumption that the trajectory of the gradient flow through $x$ has a limit point at $x_0 \in V$. To prove the claim, let $y$ denote the point on the trajectory $t \mapsto \delta_t(z)$ such that $f(y) = f(x)$. Then the trajectories through the points $z_i$ sufficiently close to $z$ intersect the level set $f^{-1}(f(y))$ at points close to $y$, i.e., the points $y_i$ approach the point $y$. Thus, $y$ coincides with $x$.

Finally, the maps $h_{-c,0}$ and $h_{0,c}$ define diffeomorphisms $\partial H_{-c} \to \partial V$ and $V \to \partial H_c$, and therefore, the subsets $H_c$ and $H_{-c}$ are smooth submanifolds with smooth boundary. \hfill \Box

**Proposition 7.** The manifold $H$ is a cone over $\partial H$ with vertex at $x_0$.

**Proof.** Since $V$ is a cone over its boundary with vertex at $x_0$, it is a union of $x_0$ and $\partial V \times [0,1]$. Let $v'$ denote a vector field along the second component in $V \times [0,1]$. Let $\lambda$ denote a monotonic function on $[0,1]$ that is 0 only at 0 and 1 at 1. Let $\pi$ denote the projection $V \times [0,1] \to [0,1]$. Then $(\lambda \circ \pi)v'$ extends to a vector field $v$ on $V$.

The vector field $v$ restricts to a tangent radial vector field over the manifold $V \setminus \{x_0\}$, it is trivial only at 0, and any point in $V \setminus \{x_0\}$ escapes $V$ along $v$ in finite time.

We will now use the gradient vector field $\text{grad}(f)$ to extend the vector field $v$ over $H$, see Lemma \[\Box\] For any regular point $x \in H$ of $f$, the tangent space $T_xM$ is the direct sum of the horizontal component $T_xM_{f(x)}$ and the vertical component generated by $\text{grad}(f)$. We say that a vector field $u$ over $H \setminus \{x_0\}$ is horizontal if the vector $u(x)$ is in $T_xM_{f(x)}$ for all $x \in H \setminus \{x_0\}$.

**Lemma 8.** The vector field $v$ extends to a continuous vector field over $H$ such that $v|H \setminus \{x_0\}$ is a nowhere zero horizontal vector field.

**Proof.** For any point $x \in H \setminus D(x_0)$ with $f(x) < 0$, there is a unique value $t > 0$ such that $\delta_t(x)$ is a point $y$ in $V$, where $\delta_t$ is the gradient flow of $f$. We define $v(x)$ by

$$v(x) = \pi \circ (d\delta_t)^{-1}(v(y)),$$

where the map $\pi$ projects the tangent space $T_yM$ at $y \in M_{[-c,c]} \setminus x_0$ to the tangent space $T_yM_{f(y)}$ of the level manifold.

We claim that $v(x)$ is non-zero. Since $\delta_t$ takes gradient curves of $f$ to gradient curves of $f$, it takes $\nabla f(y)$ to $\nabla f(x)$, and therefore it takes $v(y)$ to a vector linearly independent from $\nabla f(x)$, i.e., to a vector that is not in the kernel of $\pi$.

Similarly, for $x$ in $H \setminus D(x_0)$ with $f(x) > 0$ we define

$$v(x) = (\pi \circ d\delta_t)(v(y)),$$

where $x = \delta_t(y)$ for some $t > 0$ and $y \in V$. Finally, we set $v(x) = 0$ for all $x \in \overline{D(x_0)}$. Then $v$ is a desired continuous vector field over $H$. \hfill \Box
Let $w$ denote $f \cdot \nabla f$. Then $w(x)$ vanishes precisely over the set $M_0$ as well as over the critical points of $f$.

**Lemma 9.** The vector field $v + w$ is a continuous vector field on $H$ which is nowhere zero on $H \setminus \{x_0\}$.

**Proof.** Since $w$ and $v$ are continuous vector fields, we deduce that $w + v$ is a continuous vector field over $H$. It is easily verified that $w + v$ is trivial only at $x_0$. \hfill $\Box$

**Lemma 10.** The corners of $H$ can be smoothened so that $v + w$ is still defined on the modified manifold $H$ and $v + w$ is nowhere tangent to $\partial H$.

**Proof.** Recall that the vector field $w(x)$ is a positively scaled gradient vector field over $M_{[0,c]}$ and it is a negatively scaled gradient vector field over $M_{[-c,0]}$. For all non-corner points $x$ of $\partial H$ on the level $\pm c$, the tangent space of $\partial H$ is tangent to the level $M_{\pm c}$, the vector $v(x)$ is also tangent to the level $M_{\pm c}$ while $w(x)$ is normal to $\partial H$ directed outward. Therefore the vector field $v + w$ over $\partial H \cap M_{\pm c}$ is outward normal.

For all non-corner points $x$ in $\partial H$ that are not on the levels $\pm c$, the vector $v(x)$ is outward normal, while $w(x)$ is tangent to $\partial H$. Again, the vector field $v + w$ is outward normal over $\partial H \setminus M_{\pm c}$.

Finally, at any corner point $x$, locally $H$ is a quarter subspace of $\mathbb{R}^m$ bounded by the horizontal space $f^{-1}(c_+)$ or $f^{-1}(c_-)$, and a vertical space composed of gradient flow curves. It follows that the corner of $H$ can be smoothened, and the vector field $v + w$ can be modified near the smoothened corner so that it is outward normal everywhere over $\partial H$, see [12]. \hfill $\Box$

To summarize, we smoothened the corners of $H$ so that $H$ is a manifold with smooth boundary $\partial H$. We also constructed a vector field $v + w$ with a unique critical point at $x_0$. The vector field $v + w$ is outward normal everywhere over $\partial H$.

**Lemma 11.** The manifold $H \setminus \{x_0\}$ is diffeomorphic to $\partial H \times [0, \infty)$.

**Proof.** Let $\delta_{t}^{-v-w}$ denote the flow along the vector field $-v - w$. Suppose that the trajectory $t \mapsto \delta_{t}^{-v-w}(z)$ is well-defined for $t \in [0, a]$. Since $-v - w$ is inward normal over $\partial H$, the trajectory of any point $z$ does not escape $H$, and therefore $\delta_{a}^{-v-w}(z)$ is in $H$. Consequently, the curve $t \mapsto \delta_{t}^{-v-w}(z)$ is well-defined for $t \in [0, a + \varepsilon)$ for some $\varepsilon$. Suppose now that $t \mapsto \delta_{t}^{-v-w}(z)$ is well-defined for $t \in [0, a]$ and it is not well-defined for $t \in [0, a]$. Since $-v - w$ is nowhere zero, we deduce that the limit of $\delta_{t}^{-v-w}(z)$ as $t \to a$ is $x_0$.

Next, we claim that for every point $x \in H \setminus \{x_0\}$, its flow curve along $-v - w$ approaches $x_0$. Indeed, along the vector field $-w$ every point of $H$ flows towards $V$, while along the vector field $-v$ the points of $H$ stay on the same level of $f$. Consequently, any point in $H \setminus \{x_0\}$ appears arbitrarily close to $V$ after flowing along
For sufficiently long time. On the other hand, by the definition of \( v \) any point on \( V \) flows towards \( x_0 \) along \(-v\).

Let \( \lambda > 0 \) denote a function on \( H \setminus \{x_0\} \) such that \( \lambda(x) \) and all its derivatives tend to 0 as \( x \to x_0 \). We may choose the function \( \lambda \) so that \( \lambda \equiv 1 \) near \( \partial H \), and every trajectory \( \delta_t^{-\lambda(v+w)}(z) \) of \(-\lambda(v + w)\) starting from any point \( z \in \partial V \) is well-defined for \( t \in [0, \infty) \). Then the vector field \(-\lambda(v + w)\) defines a diffeomorphism \( \varphi: H \setminus \{x_0\} \to \partial H \times [0, \infty) \) by \( \varphi(x) = (y, t) \) for a unique point \( y \in \partial H \) and a unique value \( t \geq 0 \) such that \( \delta_t^{-\lambda(v+w)}(y) = x \). \( \square \)

Thus, for every point \( x \in H \setminus \{x_0\} \), the flow curve through \( x \) passes through a unique point of \( \partial H \), and has \( x_0 \) as its limit point. This implies that \( H \) is a cone over \( \partial H \). \( \square \)

**Corollary 12.** The manifold \( H \) is a smooth manifold homeomorphic to a ball.

**Proof.** Since \( H \) is a cone over a manifold, and \( H \) itself is a manifold, it follows that \( H \) is a disc, see [15, Lemma 12]. \( \square \)

Since the vector field \(-v - w\) is tangent to \( V \setminus \{x_0\} \) over \( V \setminus \{x_0\} \), it follows that \((H, V)\) is a cone neighborhood pair for the critical points \( x_0 \). This completes the proof of Theorem 2.

**Remark 13.** In the theory of maps with isolated cone-like critical points to manifolds of arbitrary dimension adapted neighborhoods [5] play an important role. In general the definition involves a list of assumptions, but in the case of a map to \( \mathbb{R} \) the definition is equivalent to the following one. An adapted neighborhood is a closed isolated connected neighborhood \( U \) of a critical point of a continuous function such that the level set \( f^{-1}(a) \) is transverse to \( \partial U \) for each \( a \) in the interior of \( f(U) \). Clearly, every cylindrical neighborhood is an adapted neighborhood.

**3. Proof of Corollary 4**

Let \( x_0 \) be an isolated critical point of a function \( f: M \to \mathbb{R} \) on a smooth manifold of dimension \( n \). By restricting the function \( f \) to a coordinate neighborhood about \( x_0 \), we may assume that \( M \) is \( \mathbb{R}^n \). In [13] E. Rothe studied the class of critical points satisfying the hypothesis \( \mathcal{H} \).

**Hypothesis \( \mathcal{H} \).** There exists an isolating neighborhood \( U \) of \( x_0 \) such that for all \( x \neq 0 \) in \( U \cap f^{-1}(x_0) \) the vectors \( x - x_0 \) and \( \nabla f(x) \) are linearly independent.

The hypothesis \( \mathcal{H} \) is closely related to the hypothesis \( \mathcal{H}' \).
Hypothesis $\mathcal{H}$. There exists an isolating neighborhood $U$ of $x_0$ such that over the complement to $x_0$ in $V = U \cap f^{-1}(x_0)$ there is a smooth vector field $v$ which is nowhere zero and which is outward normal over $\partial V$.

**Lemma 14.** The hypothesis $\mathcal{H}$ implies the hypothesis $\mathcal{H}'$.

**Proof.** Let $f$ be a smooth function with an isolated critical point $x_0$ that satisfies the hypothesis $\mathcal{H}$, i.e., there exists an isolating neighborhood $U$ of $x_0$ with a certain property. Let $r: U \to \mathbb{R}$ denote the function that associates with $x$ the distance between $x$ and $x_0$ with respect to the Euclidean metric on $\mathbb{R}^n$. We may choose a ball neighborhood $U_\varepsilon$ of $x_0$ of radius $\varepsilon$ so that $U_\varepsilon \subset U$ and $\varepsilon$ is a regular value of $r$. Put $\tilde{V} = U_\varepsilon \cap f^{-1}(x_0)$.

Since the vectors $x - x_0$ and $\nabla f(x)$ are linearly independent, the projection $v(x)$ of the vector $x - x_0$ to $T_x V$ is non-zero for all $x \in \tilde{V} \setminus \{x_0\}$. Then the vector field $v$ satisfies the hypothesis $\mathcal{H}'$. □

The proof of the following Lemma 15 is similar to one of Lemma 11.

**Lemma 15.** The manifold $\tilde{V} \setminus \{x_0\}$ is diffeomorphic to $\partial \tilde{V} \times [0, \infty)$.

We also note that the limit of $\delta t^{-\lambda u}(z)$ as $t \to \infty$ is $x_0$ for all $z \in \partial \tilde{V}$. Therefore we deduce Corollary 16.

**Corollary 16.** The set $V$ is homeomorphic to the cone over its boundary.

Then the proof of Proposition 7 shows that a critical point satisfying the hypothesis $H$ admits a cylindrical ball neighborhood.

The above argument proves a characterization of a cone neighborhood pair for an isolated critical point.

**Proposition 17.** Let $B_\varepsilon$ be an $\varepsilon$-disc neighborhood of $x_0$ such that $\partial B_\varepsilon$ is transverse to $M_0$. Suppose that $V = B_\varepsilon \cap M_0$ is a cone over its boundary with vertex at $x_0$. Then there exists a cone neighborhood pair for $x_0$.

**Proof.** Since $V$ is a cone over its boundary with vertex at $x_0$, the complement $V \setminus \{x_0\}$ is diffeomorphic to $V \times (-\infty, 0]$. Consequently, there is a nowhere zero vector field $v$ on $V \setminus \{x_0\}$ that is outward normal over $\partial V$. By an argument as in the proof of Proposition 7, we conclude that the critical point $x_0$ admits a cylindrical ball neighborhood $H$. In fact, the construction in the proof of Proposition 7 also produces a nowhere zero vector field $w + v$ over $H \setminus \{0\}$ that restricts to $v$ over $V$. Consequently, the pair $(H, H \cap V)$ is a cone neighborhood pair for $x_0$. □

4. **Algebraic singular points**

In this section we make precise the statement that the set of Taylor series of map germs with non cone-like critical points is of infinite codimension in the space
of all Taylor series. In subsection 4.1 we recall the Tourgeron theorem on finitely determined map germs and deduce that map germs with non cone-like critical points are extremely rare, see Corollary 20. In subsection 4.2 we will discuss infinitely determined map germs, which is a larger class than the class of finitely determined map germs, and show that if a singular map germ is infinitely determined, then its critical point is cone-like, see Proposition 23.

4.1. Finitely determined map germs. Let \( \mathcal{E}(n, p) \) denote the vector space of map germs at 0 of smooth functions \( \mathbb{R}^n \to \mathbb{R}^p \). The vector space \( \mathcal{E}(n) = \mathcal{E}(n, 1) \) of function germs is a ring with multiplication and addition induced by multiplication and addition of functions. Let \( \mathcal{B}(n) \) denote the subset of invertible map germs \( f \) in \( \mathcal{E}(n, n) \) with an additional condition \( f(0) = 0 \). The set \( \mathcal{B}(n) \) is a group with operation given by the composition. We say that map germs \( f_0, f_1 \in \mathcal{E}(n, p) \) are right equivalent if there is a map germ \( h \in \mathcal{B}(n) \) such that \( f_0 \circ h = f_1 \). For a non-negative integer \( k \), the \( k \)-jet \( \hat{f} \) of a map germ \( f \) at 0 is the Taylor polynomial of \( f \) at 0 of order \( k \). Similarly, for \( k = \infty \), the \( k \)-jet of \( f \) is the Taylor series of \( f \) at 0. The vector space of all \( k \)-jets at 0 is denoted by \( \hat{\mathcal{E}}_k(n, p) \). The vector space of Taylor series of map germs \( \mathbb{R}^n \to \mathbb{R}^p \) at 0 is called the (infinite) jet space. It is denoted by \( \hat{\mathcal{E}}(n, p) = \mathcal{E}_\infty(n, p) \). There is a sequence of projections of Euclidean spaces

\[
\hat{\mathcal{E}}(n, p) \to \cdots \to \hat{\mathcal{E}}_{k+1}(n, p) \to \hat{\mathcal{E}}_k(n, p) \to \cdots
\]

that truncate the series/polynomials. The truncation \( \hat{\mathcal{E}}(n, p) \to \hat{\mathcal{E}}_k(n, p) \) is denoted by \( \pi_k \). We say that a subset \( X \) in \( \hat{\mathcal{E}}(n, p) \) is proalgebraic if it is of the form \( \cap \pi_k^{-1} X_k \) for some algebraic sets \( X_k \subset \hat{\mathcal{E}}_k(n, p) \). By definition, the codimension of \( X \) is the upper limit of codimensions of the algebraic subsets \( X_k \).

A map germ \( f \) is said to be \( k \)-definite if every map germ \( g \) with the same \( k \)-jet \( \hat{g} = \hat{f} \) is right equivalent to \( f \). For example, a function germ \( f : \mathbb{R}^n \to \mathbb{R} \) is 1-determined at 0 if and only if it is non-singular at 0, i.e., \( df(0) \neq 0 \). A function germ \( f : \mathbb{R}^n \to \mathbb{R} \) is 2-determined at a critical point 0 if and only if 0 is a Morse critical point. We say that a map germ \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) is finite if \( \mathcal{E}(n)/\langle f_1, ..., f_p \rangle \) is finite dimensional, where \( \langle f_1, ..., f_p \rangle \) is the ideal generated by the components \( f_1, ..., f_p \) of \( f \). We note that the condition \( \mathcal{E}(n)/\langle f_1, ..., f_p \rangle = k \) implies

\[
\dim \mathcal{E}(n)/\langle f_1, ..., f_p \rangle + m(n)k^k = \dim \hat{\mathcal{E}}_k(n)/\langle \hat{f}_1, ..., \hat{f}_p \rangle \leq k,
\]

where \( m(n) \) is the maximal ideal in \( \mathcal{E}(n) \). The converse is also true, i.e., if we have \( \dim \hat{\mathcal{E}}_k(n)/\langle \hat{f}_1, ..., \hat{f}_p \rangle \leq k \) for some \( k \), then \( f \) is finite. It is known that if \( df : \mathbb{R}^n \to \mathbb{R} \) is a finite map germ at 0, then the map germ \( f : \mathbb{R}^n \to \mathbb{R} \) at 0 is finitely determined, e.g., see [7, §11.10].
On the other hand, let $Y$ denote the set of Taylor series of non-finite map germs. Then $Y = \cap_k Y_k$ for the sets
\[ Y_k = \{ \hat{g} \in \hat{E}_k(n, p) \mid \dim \hat{E}_k(n)/\langle \hat{g}_1, \ldots, \hat{g}_p \rangle > k \}, \]
where $\hat{g}_1, \ldots, \hat{g}_p$ are components of $\hat{g}$.

**Theorem 18 (Tourgeron Theorem).** The sets $Y_k$ are algebraic. In fact, if $n \leq p$, then $Y$ is a proalgebraic set of infinite codimension.

**Corollary 19.** The set of jets of map germs $f \in \mathcal{E}(n)$ for which $df$ is not finite is a subset of infinite codimension. Thus, the set of jets of non-finitely determined map germs is a subset of a proalgebraic set of infinite codimension.

For proofs of the Tourgeron Theorem and its corollary we refer to [1, Theorem 13.4 and Remark 13.6] respectively.

We note that a $k$-determined map germ $f$ is right equivalent to a polynomial $g$ of degree $k$. Since the critical point of a polynomial $g$ is algebraic, we deduce that the critical point of any finitely determined map germ $f$ is cone-like.

**Corollary 20.** The set of jets of map germs with non cone-like critical points is a subset of a proalgebraic set of infinite codimension.

4.2. Infinitely determined map germs. Next, we turn to infinitely determined map germs $f : \mathbb{R}^n \to \mathbb{R}^p$ at 0. We say that a map germ $f$ at 0 is infinitely determined if any map germ $g$ at 0 that has the same Taylor series as $f$ is right equivalent to $f$.

Let $J_f$ denote the ideal in $\mathcal{E}(n)$ generated by $p \times p$ minors of the differential $d_0 f$ of $f$ at 0, where $f$ is a map germ at 0 of a function $\mathbb{R}^n \to \mathbb{R}^p$. We say that a finitely generated ideal $I$ in $\mathcal{E}(n)$ is elliptic if it contains $m_n$, where $m_n$ is the maximal ideal in $\mathcal{E}(n)$. Part of Theorem 21 was announced in [8] and for the proof, we refer to [2], [11], and [21], see also [20, Theorem 6.1 and Lemma 6.2].

**Theorem 21.** The following conditions are equivalent:

- A map germ $f$ is infinitely determined.
- $J_f$ is elliptic.
- $||d_x f||^2 \geq C||x||^\alpha$ holds on some neighborhood of 0 for some constants $C$ and $\alpha > 0$, where $||d_x f||^2$ is the sum of squares of $p \times p$ minors of $d_x f$.

By Theorem 21 if $f : \mathbb{R}^n \to \mathbb{R}$ is not an infinitely determined function germ at 0, then there is a sequence of points $z_1, z_2, \ldots$ converging to 0 such that
\[ \left( \frac{\partial f}{\partial x_1}(z_i) \right)^2 + \cdots + \left( \frac{\partial f}{\partial x_n}(z_i) \right)^2 = o||z_i||^N \]
for all $N$. 
Example 22. The map germ \( f(x, y) = (x^2 + y^2)^2 \) is not finitely determined \cite[p. 237]{22}, but it is infinitely determined. The map germ \( g(x, y) = (x, y^4 + xy^2) \) is not infinitely determined.

Proposition 23. If a singular map germ is infinitely determined, then its critical point is cone-like.

If \( E \) is an equivalence relation on map germs, then we say that a map germ \( f \) is \( k \)-\( E \)-determined if every map germ \( g \) with the same \( k \)-jet \( \tilde{g} \) as the \( k \)-jet \( \tilde{f} \) of \( f \) is \( E \)-equivalent to \( f \). An important example of an \( E \)-equivalence is a \( K \)-equivalence. If \( f \) and \( g \) are \( K \)-equivalent, then there is a diffeomorphism of source spaces of \( f \) and \( g \) that takes \( f^{-1}(0) \) to \( g^{-1}(0) \).

Proof. Suppose that \( f \) is infinitely determined. Then it is infinitely \( K \)-determined. Then, by the Brodersen theorem \cite{2}, it is finitely \( K \)-determined. In particular, the map germ \( f \) is \( K \)-equivalent to a polynomial. Thus, there is a diffeomorphism of the source spaces of \( f \) and \( g \) that takes \( f^{-1}(0) \) to an algebraic set \( g^{-1}(0) \).

5. The Takens functions

In \cite{19}, Takens constructed functions on manifolds of dimension \( \geq 3 \) with non-cone-like critical points. In this section we study Takens functions, and show that the critical points of these functions still admit cylindrical neighborhoods diffeomorphic to a ball.

Given a smooth manifold \( X \) with corners and a smooth manifold \( Y \) with smooth boundary, a map \( f : X \to Y \) as well as its inverse is said to be a special almost diffeomorphism if it is continuous and restricts to a diffeomorphism \( X \setminus \Sigma \to Y \setminus f(\Sigma) \) where \( \Sigma \) is the set of points of the boundary of \( X \) at which \( \partial X \) is not smooth. An almost diffeomorphism of manifolds with corners is a composition of special almost diffeomorphisms.

The Takens construction is based on the existence for \( n \geq 4 \) of a set \( Q \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n \) such that

- there exists a continuous map \( \tau : \mathbb{R}^n \to \mathbb{R}^n \) with \( \tau(Q) = 0 \) such that the restriction \( \tau|_{\mathbb{R}^n \setminus Q} \) is a diffeomorphism onto \( \mathbb{R}^n \setminus \{0\} \), and
- there is no closed neighborhood \( U \) of \( Q \) in \( \mathbb{R}^{n-1} \) such that \( U \setminus Q \) is homeomorphic with \( \partial U \times (0, 1] \).

When \( n = 4 \), we may choose \( Q \subset \mathbb{R}^3 \subset \mathbb{R}^4 \) to be the Fox-Artin arc, which is an embedded segment in \( \mathbb{R}^3 \) with non-trivial fundamental group \( \pi_1(\mathbb{R}^3 \setminus Q) \). When \( n \geq 5 \), the space \( Q \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n \) can be constructed by means of the Newman-Mazur compact contractible manifold \( M \) of dimension \( n - 1 \) with non-trivial \( \pi_1(\partial M) \). The Newman-Mazur compact manifold \( M \) has the properties that

- \( M \times [0, 1] \) is almost diffeomorphic to a round ball,
the double of \( M \) is diffeomorphic to \( S^n \), and

- the disc \( D^{n-1} \) can be decomposed into a union \( A_1 \cup A_2 \) of two manifolds with corners each of which is almost diffeomorphic to the Newman-Mazur manifold \( M \).

Namely, starting with a disc \( B_0 \) of dimension \( n - 1 \), Takens defines a sequence of compact \((n - 1)\)-manifolds \( B_i \), where \( B_{i+1} \) is obtained from \( B_i \) by removing from \( B_i \) a collar neighborhood of \( B_i \) as well as a subset diffeomorphic to \( A_1 \) so that \( B_{i+1} \) is diffeomorphic to the boundary connected sum \( B_i \# b A_2 \), which is almost diffeomorphic to \( B_i \# b M \). Then \( Q = \cap B_i \). It follows that in all these cases \( Q \) is a compact subset of \( \mathbb{R}^{n-1} \).

To construct a smooth function \( f : \mathbb{R}^n \to \mathbb{R} \) with an isolated non cone-like critical point 0, we first construct a smooth function \( F : \mathbb{R}^n \to \mathbb{R} \) with \( F^{-1}(0) = \mathbb{R}^{n-1} \) and \( \partial F / \partial x_n \equiv 0 \) precisely over \( Q \). Then \( f \) is defined by \( F = f \circ \tau \).

The function \( F \) in [18] is of the form

\[
F(x_1, \ldots, x_n) = \lambda(x_1, \ldots, x_{n-1}) \cdot x_n + h(x_n),
\]

where \( \lambda(x_1, \ldots, x_{n-1}) \) and \( h(x_n) \) are certain smooth functions such that \( \lambda > 0 \) over the complement to \( Q \), \( h' \geq 0 \), and \( h(0) = 0 \). Namely, let \( \{Q_i\} \) be a basis of compact neighborhoods of \( Q \) in \( \mathbb{R}^{n-1} \) such that \( Q_i \) is in the interior of \( Q_{i-1} \). Let \( \{\lambda_i\} \) denote the partition of unity of \( \mathbb{R}^{n-1} \setminus Q \) with support of \( \lambda_i \) in the complement of \( Q_2 \), and support of all other functions \( \lambda_i \) in \( Q_{i-1} \setminus Q_{i-1} \). Then \( \lambda = \sum a_i \lambda_i \), where the strictly positive constants \( a_i \) are chosen so that \( a_i \) is less than the minimum of 1 and

\[
\frac{||x||^2}{2^{i}||((\lambda_i \cdot x_n) \tau^{-1})||_{C^i}},
\]

where \( ||x||^2 = x_1^2 + \cdots x_n^2 \), and \( ||g||_{C^i} \) is the maximum of values of derivatives of \( g \) of orders at most \( i \) over the compact support of the smooth function \( g \) restricted to the disc \( ||x|| \leq 1 \). Therefore, the series \( \sum (\lambda_i \cdot x_n) \circ \tau^{-1} \) converges to a smooth function \( (\lambda \cdot x_n) \circ \tau^{-1} \) whose derivative of any order is obtained by term by term differentiating the series.

Let \( \{h_j\} \) denote a sequence of functions on \( \mathbb{R}^1 \) parametrized by \( j = 1, 2, \ldots \) such that \( h_j(x) = x \) if \( |x| \geq 1/j \), \( h_i(x) = 0 \) if \( |x| < 1/(i + 1) \) and \( h_j'(x) \geq 0 \). As above, we define a smooth function \( h(x_n) = \sum b_j h_j \circ x_n \), where the strictly positive constants \( b_j \) are chosen so that \( b_j \) is less than the minimum of 1 and

\[
\frac{||x||^2}{2^{i}||h_j \circ x_n \circ \tau^{-1}||_{C^i}}.
\]

We immediately deduce the following proposition.

**Proposition 24.** The Takens function germs \( f \) are flat, i.e., derivatives of all orders of \( f \) at 0 are trivial.
Proposition 25. The critical points of Takens functions admit cylindrical neighborhoods diffeomorphic to a ball.

Proof. To begin with let us show that the critical points of Takens functions admit cylindrical ball neighborhoods.

Since \( F|Q \equiv 0 \), and \( \partial F/\partial x_n > 0 \) over the complement to \( Q \), the implicit function theorem implies that for each \( \varepsilon \neq 0 \) there is a function \( g_\varepsilon(x_1, \ldots, x_{n-1}) \) such that \( F(x_1, \ldots, x_n, g_\varepsilon(x_1, \ldots, x_{n-1})) = \varepsilon \). In other words, the level set \( F(x_1, \ldots, x_n) = \varepsilon \) is the graph of the function \( x_n = g_\varepsilon(x_1, \ldots, x_{n-1}) \). Furthermore, let \( B_r \) denote a closed ball in \( \mathbb{R}^{n-1} \) centered at 0 of radius \( r \) such that \( B_r \) contains the compact set \( Q \) in its interior. Then the absolute value of the restriction \( g_\varepsilon|B_r \) is bounded by a constant \( c_\varepsilon \) such that \( c_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

For any subset \( X \) of \( \mathbb{R}^{n-1} \), let \( C_\varepsilon(X) \) denote the compact rectangular cylindrical subset of \( \mathbb{R}^n \) over \( X \) bounded by the graphs of the functions \( g_{-\varepsilon}|X \) and \( g_\varepsilon|X \). In other words, the set \( C_\varepsilon(X) \) consists of points \( (x_1, \ldots, x_n) \) such that \( (x_1, \ldots, x_{n-1}) \) is a point in \( X \) and

\[
g_{-\varepsilon}(x_1, \ldots, x_{n-1}) \leq x_n \leq g_\varepsilon(x_1, \ldots, x_{n-1}).
\]

Then \( C_\varepsilon(X) \) is diffeomorphic to the cylinder \( X \times [-\varepsilon, \varepsilon] \).

Let \( H(r, \varepsilon) \) denote a neighborhood of \( Q \) that consists of all trajectories of the gradient vector field of \( F \) in \( F^{-1}[-\varepsilon, \varepsilon] \) that either contain a point in \( B_r \) or has a limit point in \( Q \).

Lemma 26. There is a number \( r_3 > 0 \) such that the cylindrical neighborhood \( H(r_3, \varepsilon) \) is diffeomorphic to the rectangular cylinder \( C_\varepsilon(B_{r_3}) \).

Proof. Let \( r_0 < r_1 < r_2 < r_3 < r_4 \) be numbers such that \( B_{r_0} \) contains \( Q \) in its interior, and \( H(r_i, \varepsilon) \subset C_\varepsilon(B_{r_{i+1}}) \) for \( i = 0, \ldots, 3 \). There is a smooth homotopy \( v_t \) parametrized by \( t \in [0, 1] \) of the gradient vector field \( v_0 \) of \( F \) to a vector field \( v_1 \) such that \( v_t \) is constant over \( H(r_0, \varepsilon) \), \( v_t \cdot \partial / \partial x_n > 0 \) over \( C_\varepsilon(B_{r_3}) \) for all \( t \), and \( v_1 \) is vertical up over \( C_\varepsilon(B_{r_4} \setminus B_{r_2}) \).

Given a point \( x \) in \( \mathbb{R}^{n-1} \setminus Q \), there is a unique trajectory \( \gamma_x \) of the vector filed \( v_t \) passing through \( x \). Let \( y \in \mathbb{R}^n \) denote the point in \( \mathbb{R}^n \) on the trajectory \( \gamma_x \) such that \( F(y) = s \). Then \( j_{t,s}: \mathbb{R}^{n-1} \setminus Q \to \mathbb{R}^n \) defined by \( x \mapsto y \) is a smooth map that takes \( B_{r_3} \setminus Q \) to the upper horizontal part \( \{ F \equiv s \} \cap H(r_3, s) \) of the boundary of \( H(r_3, s) \).

Then there is a smooth isotopy \( \varphi_t \) of a neighborhood \( H(r_3, \varepsilon) \) of \( Q \) to \( C_\varepsilon(B_{r_3}) \) defined by

\[
\varphi_t(x_1, \ldots, x_n) = j_{t,F(x_n)}(j_{0,F(x_n)}^{-1}(x_1, \ldots, x_n)).
\]

We claim that \( \varphi_1 \) is a diffeomorphism onto \( C_\varepsilon(B_{r_3}) \). Indeed, let \( (x_1, \ldots, x_n) \) be a point in \( C_\varepsilon(B_{r_3}) \) that is not on the trajectories of \( v_1 \) with limit points on \( Q \). Then the point \( (x_1, \ldots, x_n) \) is the image of the point

\[
j_{0,F(x_n)} \circ j_{1,F(x_n)}^{-1}(x_1, \ldots, x_n).
\]
On the other hand, if \((x_1, ..., x_n)\) is on a trajectory of \(v\) with limit point on \(Q\), then it is in \(H(r_0, \varepsilon)\) where the isotopy \(v_t\) is constant. Therefore, the point \((x_1, ..., x_n)\) in this case is the image of itself. □

Since \(H = H(r_3, \varepsilon)\) is diffeomorphic to \(C_\varepsilon(B_{r_3})\), we conclude that \(H\) is a ball with corners \(B_{r_3} \times [-\varepsilon, \varepsilon]\). Consequently, \(H/Q\) is a cylindrical ball neighborhood of the critical point of \(f\).

Now let us show that the cylindrical neighborhood \(H/Q\) is not only homeomorphic to a ball, but it is almost diffeomorphic to a ball. Indeed, if \(n \neq 4, 5\), then every smooth manifold homeomorphic to a ball is diffeomorphic to ball. Suppose that \(n = 5\). Since the boundary of \(H/Q\) is diffeomorphic to the boundary of \(H\), which is almost diffeomorphic to a sphere, we conclude that \(H/Q\) is almost diffeomorphic to a ball, see [13, Proposition C]. Suppose now that \(n = 4\). Then there is an ambient isotopy of \(H/Q\) that is trivial near the boundary, and that takes the wild arc \(Q\) into the unknotted standard segment \(I\). Consequently, the manifold \(H/Q\) is diffeomorphic to the manifold \(H/I\), which is almost diffeomorphic to a smooth 4-ball. □

Proof of Theorem 5. It is known that the critical point of a Takens function is not cone-like. On the other hand, by Proposition 25, such a critical point admits a cylindrical neighborhood diffeomorphic to a ball.

Corollary 27. The critical points of Takens functions are removable, i.e., for any neighborhood \(U\) of a critical point \(x_0\) of a Takens function \(f\), there is a smooth function \(g\) such that \(g = f\) outside of \(U\) and \(g\) has no critical points in \(U\).

Proof. In the proof of Proposition 25 we may choose \(\varepsilon, r_0, r_1, r_2, r_3,\) and \(r_4\) so small that \(H/Q \subset U\). We note that the boundary of the ball \(H/Q\) is diffeomorphic to the boundary of the ball \(H\), and the function \(f\) near \(\partial(H/Q)\) can be identified with the function \(F\) near \(\partial H\). On the other hand, the boundary \(\partial H\) consists of the upper and lower horizontal parts \(\partial_b H = \partial H \cap \{F = \pm \varepsilon\}\) and the vertical part \(\partial_v H = \partial H \setminus \partial_b H\) such that \(F\) is constant over each component of \(\partial_b H\) and increasing over \(\partial_v H\). Therefore, there is a function \(f'\) on \(H/Q\) such that \(f'\) agrees with \(f\) near \(\partial(H/Q) = \partial H\) and \(f'\) has no critical points. Finally, we define \(g\) to be a function on \(\mathbb{R}^n\) such that \(g = f'\) over \(H/Q\) and \(g = f\) otherwise. □

References

[1] Th. Bröcker, Differentiable germs and catastrophes. Translated from the German, last chapter and bibliography by L. Lander. London Mathematical Society Lecture Note Series, No. 17. Cambridge University Press, Cambridge-New York-Melbourne, 1975. pp. 179.

[2] H. Brodersen, A note on infinite determinacy of smooth map germs. Bull. London Math. Soc. 13 (1981), no. 5, 397-402.

[3] O. Cornea, Cone-decompositions and degenerate critical points, Proc. London Math. Soc. 77 (1998), 437-461.
[4] E. N. Dancer, Degenerate critical points, homotopy indices and Morse inequalities, J. Reine Angew. Math. 350 (1984), 1-22.
[5] L. Funar, Maps with finitely many critical points into high dimensional manifolds. Rev. Mat. Complut. 34 (2021), no. 2, 585-595.
[6] H. C. King, Topological type of isolated singularities. Ann. Math. 107 (1978), 385-397.
[7] H. C. King, Topology of isolated critical points of functions on singular spaces, Stratifications, singularities and differential equations, II (Marseille, 1990; Honolulu, HI, 1990), 63-72, Trotman, D., Wilson, L.C. eds., Travaux en Cours, 55, Hermann, Paris, (1997).
[8] W. Kucharz, Jets suffisants et fonctions de détermination finie, C. R. Acad. Sci. Paris Sér. A-B 284 (1977).
[9] J. Milnor, Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow Princeton University Press, Princeton, N.J. 1965 v+116 pp.
[10] J. Milnor, Singular points of complex hypersurfaces, Princeton University Press and the Tokyo University Press, 1968.
[11] N. Tù' Cuò'ng et al, Sur les germes de fonctions infiniment déterminés, C. R. Acad. Sci. Paris Sér. A-B 285, 1977, A1045–A1048.
[12] Ch. Pugh, Smoothing a topological manifold. Topology Appl. 124 (2002), no. 3, 487-503.
[13] E. H. Rothe, A relation between the type numbers of a critical point and the index of the corresponding field of gradient vectors. Erhard Schmidt zum 75. Geburtstag gewidmet. Mathematische Nachrichten, 4 (1950), 12-27.
[14] E. H. Rothe, A remark on isolated critical points. Amer. J. Math. 74 (1952), 253-263.
[15] R. Sadykov, S. Trunov, The minimal number of critical points of a smooth function on a closed manifold and the ball category, [arXiv:2208.09939].
[16] H. Seifert, W. Threlfall, Variationsrechnung im Großen. Hamburger math. Einzelschr. Heft 24. Leipzig-Berlin, 1938.
[17] W. Singhof, Minimal coverings of manifolds with balls, Manuscripta Math., 29 (1979), 385-415.
[18] F. Takens, Isolated critical points of $C^\infty$ and $C^\omega$ functions. Nederl. Akad. Wetensch. Proc. Ser. A 70=Indag. Math. 29 (1967), 238-243.
[19] F. Takens, The minimal number of critical points of a function on a compact manifold and the Lusternik-Schnirelmann category, Inventiones Math., 6 (1968), 197-244.
[20] C. T. C. Wall, Finite determinacy of smooth map-germs. Bull. London Math. Soc. 13 (1981), no. 6, 481-539.
[21] L. C. Wilson, Infinitely determined map germs. Canadian J. Math. 33 (1981), no. 3, 671-684.
[22] L. C. Wilson, Mapgerms infinitely determined with respect to right-left equivalence. Pacific J. Math. 102 (1982), no. 1, 235-245.