All-optical signatures of Strong-Field QED in the vacuum emission picture

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(Dated: September 20, 2018)

We study all-optical signatures of the effective nonlinear couplings among electromagnetic fields in the quantum vacuum, using the collision of two focused high-intensity laser pulses as an example. The experimental signatures of quantum vacuum nonlinearities are encoded in signal photons, whose kinematic and polarization properties differ from the photons constituting the macroscopic laser fields. We implement an efficient numerical algorithm allowing for the theoretical investigation of such signatures in realistic field configurations accessible in experiment. This algorithm is based on a vacuum emission scheme and can readily be adapted to the collision of more laser beams or further involved field configurations. We solve the case of two colliding pulses in full 3+1 dimensional spacetime, and identify experimental geometries and parameter regimes with improved signal-to-noise ratios.

I. INTRODUCTION

The fluctuations of virtual particles in the quantum vacuum give rise to effective interactions among electromagnetic fields, supplementing Maxwell’s linear theory of vacuum electrodynamics with effective nonlinearities \[\mathcal{O}(\alpha^3)\]; for reviews, see Refs. \[4–13\]. Prominent signatures of quantum vacuum nonlinearities are vacuum magnetic birefringence (VMB) \[14, 15\] and direct light-by-light scattering \[16, 17\].

Being of quantum nature, the latter are typically tiny and rather elusive in experiment. In quantum electrodynamics (QED), they are suppressed parametrically with inverse powers of the electron mass \(m_e\). This mass scale serves as the typical energy to be compared with the scales of the applied fields, and defines the critical field strengths

\[E_{cr} := \frac{\alpha^3 m_e^2}{2} \approx 1.3 \times 10^{16} \text{ V/cm} \quad \text{and} \quad B_{cr} := \frac{\alpha^3}{\epsilon_0} \approx 4 \times 10^{18} \text{T}.\]

In the laboratory, field strengths of this order are only reached in strong Coulomb fields of highly charged ions. Hence, experimental verifications of QED vacuum nonlinearities have so far been limited to high-energy experiments with highly charged ions \[18–22\]. Note, that VMB is potentially also relevant for the optical polarimetry of neutron stars \[23–25\]. Even though QED vacuum nonlinearities in macroscopic electromagnetic fields have not been directly verified yet, laboratory searches of VMB in macroscopic magnetic fields \[26–28\] have already demonstrated the need for high field strengths and, at the same time, a high signal detection sensitivity, see also \[29, 30\]. The demand for strong fields together with the recent advances in the development of high-intensity laser systems has opened up an alternative route to access the extreme-field territory in the laboratory. The overarching key idea is to combine high-intensity lasers with polarization sensitive single photon detection schemes.

State-of-the-art high-intensity lasers reach peak field strengths of the order of \(10^6\)T and \(10^{12}\)V/cm in micron sized focal spots. Laser pulses achieving these field strengths are typically made up of \(\mathcal{O}(10^2)\) photons, constituting a challenging background for the detection of the generically tiny signals of QED vacuum nonlinearities in experiment. In this context, theoretical proposals specifically focused on VMB \[31–33\], photon-photon scattering in the form of laser-pulse collisions \[34, 41\], quantum reflection \[42, 43\], photon merging \[44–47\] and splitting \[48–52\], and optical signatures of QED vacuum nonlinearities based on interference effects \[53–55\].

In this article, we introduce and benchmark an efficient numerical algorithm tailored to the study of all-optical signatures of QED vacuum nonlinearities. Reformulating the signatures in terms of vacuum emission processes \[56\], the effects of quantum vacuum nonlinearities are encoded in signal photons emitted from the strong-field region. As no signal photons are induced in the absence of vacuum nonlinearities, these photons generically constitute a distinct signal. However, in order to allow for their detection in experiment, they have to differ from the photons constituting the high-intensity laser pulses driving the effects, e.g., by their kinematic and polarization properties. Correspondingly, one central objective is to identify scenarios where such effects are most pronounced.

A standard approach of dealing with this challenge is to solve the nonlinear photon wave equation, i.e. a partial differential equation, by suitable numerical techniques. Successful examples can be found, e.g., in \[41\], where the nonlinearities of the field equations have been treated as source terms and Green’s function methods are used for an iterative solution strategy; see also \[57, 58\] for an advanced implementation based on the pseudo-characteristic method of lines. For large-scale simulation purposes, an implicit ODE-based solver has been specifically designed in \[59\], as well as in \[60\] using a finite-difference time-domain solver.

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As demonstrated in the following, the vacuum emission picture advocated in this work is particularly suited for a numerical implementation. In our formalism, the essential numerical ingredients are reduced to one standard and easy-to-use algorithm: fast Fourier transformation. Space- or time-integrated observables may additionally require simple low-dimensional integration techniques. This numerical simplicity parallels the conceptual adaption of the vacuum emission scheme to the physical situation: in this picture, all macroscopically controlled fields such as high-intensity laser pulses are treated as classical fields, whereas the fluctuation-induced signal photons are dealt with on the level of the quantum Fock space.

Our article is organized as follows: In Sec. II we outline the theoretical foundations of our approach. We apply our methods in Sec. III to the collision of two focused, linearly polarized high-intensity laser pulses in vacuum [41]. In Sec. IV we introduce our numerical algorithm in detail. Section V is devoted to the discussion of explicit results. Here, we first benchmark our numerical algorithm with analytical results for the limit of infinite Rayleigh ranges of the two beams, where analytical results are available. Subsequently, we use it to obtain new results: in Sec. VI we study the collision of two petawatt class laser pulses of identical frequency, continuing with fundamental and doubled frequency in Sec. VII. Considering the fundamental frequency laser beam as focused down to the diffraction limit, the latter scenario allows for the study of two limiting cases of specific interest, differing in the focusing of the frequency-doubled beam. In the first case, it is focused to the diffraction limit of the fundamental frequency beam, maximizing the beam overlap in the focus, and in the second case to its own diffraction limit, resulting in a narrower beam waist and thus in a considerably smaller overlap region of the beams but higher intensity in the focus. Finally, we end with conclusions and an outlook in Sec. VIII.

II. THEORETICAL FOUNDATIONS

In Ref. [56], it has been argued that all-optical signatures of quantum vacuum nonlinearities can be efficiently analyzed by reformulating them in terms of vacuum emission processes. This approach has meanwhile been successfully employed to obtain experimentally realistic predictions for the phenomenon of VMB, particularly in the combination of x-ray free electron and high-intensity lasers [34, 36, 61].

The central idea is to consider all applied macroscopic electromagnetic fields as constituting the external background field; cf also Ref. [62]. This implies, that the quantum character of the applied fields is not resolved, and effects like, e.g., QED-induced beam depletion are neglected. We emphasize that this is typically well-justified for scenarios where the strong electromagnetic fields E and B are provided by high-intensity lasers and fulfill E ≪ Ec and B ≪ Bc. Due to the parametric suppression of QED vacuum nonlinearities by powers of the electron mass, the pulses delivered by such lasers can be considered as traversing each other in vacuum essentially unaltered.

At one-loop order, but fully nonperturbative, the vacuum emission picture advocated in this work is particularly suited for a numerical implementation. The double line denotes the dressed fermion propagator accounting for arbitrarily many couplings to the external field A, represented by the wiggly lines ending at crosses.

\[ \gamma_p(k) \]

\[ = \sum \] + \[ \sum \] + \[ \sum \] + \ldots \]

FIG. 1. Diagrammatic representation of the single photon vacuum emission process [29]. The double line denotes the dressed fermion propagator accounting for arbitrarily many couplings to the external field A, represented by the wiggly lines ending at crosses.

Outside the interaction region is then given by

\[ \Gamma_{\text{int}}[\hat{A}(x)] = \int d^4x \frac{\delta \Gamma^{\text{1-loop}}_{\text{HE}}[A]}{\delta A^\mu}_{A=\hat{A}(x)} \ a^\mu(x) , \] (1)

where \( \Gamma^{\text{1-loop}}_{\text{HE}}[A] = -i \ln \det(-i\partial - eA + m_e) \) is the one-loop Heisenberg-Euler action evaluated in the generic external field \( \hat{A} \equiv A(x) \). Our metric convention is \( g_{\mu\nu} = \text{diag}(-1,+1,+1,+1) \), and we use the Heaviside-Lorentz System with \( c = \hbar = 1 \).

In turn, the amplitude for emission of a single signal photon with momentum \( \hat{k} \) from the QED vacuum subject to the external field \( \hat{A} \) is given by [62] (cf. also Fig. 11)

\[ S_{(p)}(\hat{k}) = \langle \gamma_p(\hat{k})|\Gamma_{\text{int}}^{(1)}[\hat{A}(x)]|0\rangle . \] (2)

Here \( |\gamma_p(\hat{k})\rangle \equiv a^\dagger_{k,p} |0\rangle \) denotes the single signal photon state, and \( p \) labels the polarization of the emitted photons. Transition amplitudes to final states with more photons can be constructed along the same lines, but are typically suppressed because of a significantly larger phase space for the signal photons; cf. the photon splitting process in Ref. [47]. The differential number of signal photons with polarization \( p \) to be measured far outside the interaction region is then given by

\[ d^3N_{(p)}(\hat{k}) = \frac{d^3k}{(2\pi)^3} |S_{(p)}(\hat{k})|^2 . \] (3)

Representing the photon field in Lorenz gauge as

\[ a^\mu(x) = \sum_p \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k^0}} \times \left( e_{(p)}^\mu(k) e^{-\text{i}kx} a^\dagger_{k,p} + e_{(p)}^\mu(k) e^{\text{i}kx} a_{k,p} \right) , \] (4)
where \(k^0 = |\vec{k}|, k_x := k^\mu x_\mu\) and the sum is over the two physical (transverse) photon polarizations, Eq. \([2]\) can be expressed as

\[
\mathcal{S}_{(p)}(\vec{k}) = \frac{\epsilon^\mu_{(p)}(k)}{\sqrt{2k^0}} \int d^4x e^{ikx} \frac{\delta\mathcal{L}_{\text{HE}}}{\delta A^\mu} \bigg|_{A = \tilde{A}(x)}. \tag{5}
\]

No closed-form expressions of Eq. \([3]\) for generic background field profiles are available. For the field configurations generated by high-intensity lasers, which vary on length (time) scales much larger than the Compton wavelength (time) of the electron \(\chi_C \approx 3.86 \cdot 10^{-13}\) m \((\chi_C \approx 1.29 \cdot 10^{-21}\) s), analytical insights are nevertheless possible by means of a locally constant field approximation (LCFA).

The LCFA amounts to first obtaining the Heisenberg-Euler action in constant electromagnetic fields, \(F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \text{const.}\), resulting in a closed-form expression \(\Gamma_{\text{HE}}(\bar{F})\). As already determined in the original works \([1, 3]\), \(\Gamma_{\text{HE}}(\bar{F})\) is a function of the two field invariants \(\mathcal{F} := \frac{1}{4} \bar{F}_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\vec{B}^2 - \vec{E}^2)\) and \(\bar{G} = \frac{1}{4} \bar{F}_{\mu\nu} \ast F^{\mu\nu} = -\vec{B} \cdot \vec{E},\) where \(* F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}\). Adopting this result for inhomogeneous fields, yields the LCFA approximation for the action functional,

\[
\Gamma_{\text{HE}}(\bar{F}) = \int d^4x \mathcal{L}_{\text{HE}}(\bar{F})\bigg|_{\bar{F} = \bar{F}(x)} = \int d^4x \mathcal{L}_{\text{HE}}(\bar{F}(x)). \tag{6}
\]

Due to parity invariance of QED, the dependency of the Heisenberg-Euler Lagrangian is actually even in \(\bar{G}\), such that \(\mathcal{L}_{\text{HE}}(\bar{F}) = \mathcal{L}_{\text{HE}}(\bar{F}, \bar{G}^2)\) for constant fields as well as for the LCFA. As has been argued, e.g., in Refs. \([22, 23]\), the deviations of the LCFA result from the corresponding exact expression for \(\Gamma_{\text{HE}}\) are of order \(\mathcal{O}(\alpha^2/m_e^2)\), where \(\alpha\) delimits the moduli of the frequency and momentum components of the considered inhomogeneous field from above.

Within the LCFA, we obtain \([3, 22, 23]\)

\[
\mathcal{S}_{(p)}(\vec{k}) = \frac{\epsilon^\mu_{(p)}(k)}{\sqrt{2k^0}} \int d^4x e^{ikx}
\times \left[ (k \vec{F})_{\mu} \frac{\partial\mathcal{L}_{\text{HE}}}{\partial F^{\mu\nu}} + (k \ast \vec{F})_{\mu} \frac{\partial\mathcal{L}_{\text{HE}}}{\partial \bar{G}} \right], \tag{7}
\]

where \((k \vec{F})_{\mu} := k^\nu \bar{F}_{\nu\mu}(x), (k \ast \vec{F})_{\mu} := k^\nu \ast \bar{F}_{\nu\mu}(x)\) and

\[
\frac{\partial\mathcal{L}_{\text{HE}}}{\partial F^{\mu\nu}} = \frac{\alpha}{2\pi} \int_0^\infty \frac{ds}{s} e^{-\frac{m^2}{s}} \left[ \frac{ab}{a^2 + b^2 \sinh^2(bs)} (a \leftrightarrow ib) + \frac{2}{3} \right], \tag{8}
\]

\[
\frac{\partial\mathcal{L}_{\text{HE}}}{\partial \bar{G}} = \frac{\alpha}{2\pi} \int_0^\infty \frac{ds}{s} e^{-\frac{m^2}{s}} \left[ \frac{1}{s} \frac{1}{a^2 + b^2 \sinh^2(bs)} (a \leftrightarrow ib) \right] \tag{9}
\]

with \(a := \sqrt{\mathcal{F}(x) + \mathcal{G}^2(x)} - \mathcal{F}(x)^{1/2}\) and \(b := \sqrt{\mathcal{F}(x) + \mathcal{G}^2(x) - \mathcal{F}(x)^{1/2}}\).

Using spherical coordinates \(k = \sqrt{k_x^2 + k_y^2 + k_z^2}\) and \(\kappa = (\cos \vartheta \sin \vartheta, \sin \vartheta \sin \vartheta, \cos \vartheta),\) the vectors perpendicular to \(k\) can be parameterized by a single angle \(\beta,\)

\[
\vec{e}_\beta = \begin{pmatrix} \cos \vartheta \cos \beta - \sin \vartheta \sin \beta \\ \sin \vartheta \cos \beta + \cos \vartheta \sin \beta \\ -\sin \beta \end{pmatrix}. \tag{10}
\]

Correspondingly, the transverse polarization modes of photons with wave vector \(\vec{k}\) can be spanned by two orthonormal four-vectors, e.g.,

\[
\epsilon^\mu_{(1)}(\vec{k}) := (0, \hat{\epsilon}_\beta) \quad \text{and} \quad \epsilon^\mu_{(2)}(\vec{k}) := (0, \hat{\epsilon}_{\beta + \pi}), \tag{11}
\]

for a suitable choice of \(\beta\). With these definitions, we obtain

\[
\mathcal{S}_{(1)}(\vec{k}) = \frac{1}{i} \sqrt{\frac{k^0}{2}} \int d^4x e^{ikx} \times \left\{ \left[ \vec{e}_\beta \cdot \vec{E}(x) - \hat{\epsilon}_{\beta + \pi} \cdot \vec{B}(x) \right] \frac{\partial\mathcal{L}_{\text{HE}}}{\partial \bar{F}} + \left[ \vec{e}_\beta \cdot \vec{B}(x) + \hat{\epsilon}_{\beta + \pi} \cdot \vec{E}(x) \right] \frac{\partial\mathcal{L}_{\text{HE}}}{\partial \bar{G}} \right\}, \tag{12}
\]

and \(\mathcal{S}_{(2)}(\vec{k}) = \mathcal{S}_{(1)}(\vec{k})|_{\beta \rightarrow \beta + \pi},\) using \(\hat{\epsilon}_{\beta + \pi} = -\hat{\epsilon}_\beta.\) In the limit of weak electromagnetic fields, \(\epsilon F^{\mu\nu} \ll m_e^2,\) Eq. \([9]\) results in

\[
\left\{ \frac{\partial\mathcal{L}_{\text{HE}}^{\text{1-loop}}}{\partial \mathcal{F}} \right\}_{\beta \rightarrow \beta + \pi} = \frac{\alpha}{4\pi} \left( \frac{e}{m_e^2} \right)^2 \left\{ 4\mathcal{F}(x) \Rightarrow 7\bar{G}(x) \right\} + \mathcal{O}(\left( \frac{e}{m_e^2} \right)^5), \tag{13}
\]

such that Eq. \([12]\) becomes

\[
\mathcal{S}_{(1)}(\vec{k}) = \frac{1}{i} \frac{e}{4\pi^2} \left( \frac{e}{m_e^2} \right)^3 \frac{k^0}{2} \int d^4x e^{ikx} \times \left\{ \left[ 4\vec{e}_\beta \cdot \vec{E}(x) - \hat{\epsilon}_{\beta + \pi} \cdot \vec{B}(x) \right] \mathcal{F}(x) + 7 \left[ \vec{e}_\beta \cdot \vec{B}(x) + \hat{\epsilon}_{\beta + \pi} \cdot \vec{E}(x) \right] \mathcal{G}(x) \right\}, \tag{14}
\]

where we neglected higher-order terms of \(\mathcal{O}(\left( \frac{e}{m_e^2} \right)^5).\) The corresponding Feynman diagram is depicted in Fig. \([2]\). Because of Furry’s theorem, in QED the total number of couplings of fermion loops to electromagnetic fields (i.e., including the signal photon) is always even. For single
signal photon emission, the number of couplings to the external field is odd.

In spherical coordinates, the differential number of signal photons of Eq. (3) can finally be expressed as

\[
d^3N_{(p)}(\hat{k}) = dk d\varphi d\cos \vartheta \frac{1}{(2\pi)^3} |kS_{(p)}(\hat{k})|^2. \tag{15}
\]

Moreover, it is convenient to introduce the total number density of induced signal photons polarized in mode \( p \) and emitted in the direction \((\varphi, \vartheta)\) as follows \(56\),

\[
\rho_{(p)}(\varphi, \vartheta) := \frac{1}{(2\pi)^3} \int_0^\infty dk |kS_{(p)}(\hat{k})|^2. \tag{16}
\]

The total number of signal photons of polarization \( p \) is then obtained as \( N_{(p)} := \int_0^{2\pi} d\varphi \int_0^\pi d\cos \vartheta \rho_{(p)}(\varphi, \vartheta) \). Accordingly, the total number of signal photons of any polarization is given by \( N := \sum_{p=1}^2 N_{(p)} \), and the associated number density by \( \rho := \sum_{p=1}^2 \rho_{(p)} \).

### III. COLLISION OF TWO HIGH-INTENSITY LASER PULSES

In the present work, we consider the collision of two high-intensity laser pulses as a concrete example for our computational scheme. On the one hand, this configuration already features a high degree of complexity due to a substantial set of experimentally tunable laser and geometry parameters. On the other hand, this case is sufficiently simple to allow for analytically or semi-analytically insights which are essential for reliably benchmarking our numerical procedures.

Let us thus assume the background electric and magnetic fields to be generated by the superposition of two linearly polarized laser beams. In leading-order paraxial approximation, each of these laser beams is characterized by a single, globally fixed wave vector and its electric and magnetic fields. We define the normalized wave vectors of the two laser beams \( b \in \{1, 2\} \) as \( \hat{k}^b = (1, \hat{e}_{k_b}) \). The associated electric and magnetic fields are characterized by an overall amplitude profile \( E_b \) and point in \( \hat{e}_{E_b} \) and \( \hat{e}_{B_b} \) directions. These unit vectors are independent of \( x \) for linear polarization. They fulfill \( \hat{e}_{E_b} \cdot \hat{e}_{B_b} = \hat{e}_{k_b} \cdot \hat{e}_{k_b} = 0 \) and \( \hat{e}_{E_b} \times \hat{e}_{B_b} = \hat{e}_{k_b} \).

Hence, in this case Eq. (14) can be expressed as

\[
S_{(1)}(\hat{k}) = \frac{1}{i} \frac{e}{4\pi^2} \frac{m_e^2}{45} \sqrt{\frac{\alpha}{2}} \frac{e}{m_e^2} \int d^4x e^{i\hat{k} \cdot \hat{x}} \mathcal{E}_1^2(x) \mathcal{E}_2(x) \\
\times \left[ 4\left( \hat{e}_\beta \cdot \hat{e}_{E_1} - \hat{e}_\beta \cdot \hat{e}_{B_1} - \hat{e}_\beta \cdot \hat{e}_{E_2} + \hat{e}_\beta \cdot \hat{e}_{B_2} \right) \right] (17)
\]

The generalization of Eq. (17) to background fields generated by more laser beams is straightforward. Without loss of generality we assume the beam axes of the two lasers to be confined to the \( xz \)-plane and parameterize the unit wave and field vectors as \( \hat{e}_{k_b} = \left( \sin \vartheta_b, \cos \vartheta_b \right) \), \( \hat{e}_{E_b} = \left( \cos \vartheta_b \cos \beta_b, \sin \vartheta_b, -\sin \vartheta_b \cos \beta_b \right) \), and \( \hat{e}_{B_b} = \hat{e}_{E_1} \big|_{\beta_b \rightarrow \beta_b + \pi} \), where the choice of \( \beta_b \) fixes the polarization of the beam. Throughout this article, we assume \( \vartheta_1 = 0 \), such that the first laser beam propagates along the positive \( z \) axis. In turn, the angle \( \vartheta_2 \) parameterizes the tilt of the beam axis of the second laser beam with respect to the first. With these definitions, the terms written explicitly in Eq. (17) can be expressed as

\[
S_{(1)}(\hat{k}) = \frac{i}{\sqrt{\alpha}} \frac{1}{(2\pi)^3/2} \left( \frac{e}{m_e^2} \right)^3 \left( (1 - \cos \vartheta_2) \sqrt{k} \right) \\
\times \begin{cases}
I_{21}(k)(1 - \cos \vartheta) f(\beta_1 + \beta_2, \beta + \beta - \varphi) \\
+ I_{12}(k) \left[ (1 - \cos \vartheta \cos \vartheta_2) \cos \varphi - \sin \vartheta \sin \vartheta_2 \right] \times f(\beta_1 + \beta_2, \beta + \beta_2) \\
- \sin \varphi (\cos \vartheta - \cos \vartheta_2) \frac{e}{m_e^2} \mu \sin \nu \mu \sin \nu \right), \tag{19}
\end{cases}
\]

where we have made use of the shorthand notations

\[
f(\mu, \nu) := 4 \cos \mu \cos \nu + 7 \sin \mu \sin \nu, \\
g(\mu, \nu) := 4 \cos \mu \sin \nu - 7 \sin \mu \cos \nu, \tag{20}
\]

and

\[
I_{mn}(k) := \int d^4x e^{i\hat{k} \cdot \hat{x}} \mathcal{E}_1^m(x) \mathcal{E}_2^n(x). \tag{21}
\]

Hence, the only remaining nontrivial task in determining the single photon emission amplitude is to compute the Fourier transforms (21). As it is linear in \( E_1 (\mathcal{E}_2) \), the contribution \( \sim I_{12} \sim I_{21} \) in Eq. (19) can, for instance, be interpreted as signal photons originating from the laser beam characterized by the field profile \( E_1 (\mathcal{E}_2) \), which are scattered into a different kinematic and polarization mode due to interactions with the other laser beam described by \( \mathcal{E}_2 (\mathcal{E}_1) \).
In a next step we specify the amplitude profiles $\mathcal{E}_b$ of the two laser beams, which we assume to be well-described by pulsed Gaussian laser beams of the following amplitude profile (cf., e.g., Refs. 63, 65):

$$\mathcal{E}_b(x) = \mathcal{E}_{0,b} e^{-\frac{(x - x_{b0})^2}{w_{0,b}^2}} w_{0,b} e^{-\frac{x^2}{w_0^2(x)}}$$

$$\times \cos\left(\omega_b (z_b - t_b) + \frac{2 \omega_b}{z_{R,b}} w_{0,b}^2 (z_b) - \arctan\frac{x - x_{b0}}{w_{0,b}} + \varphi_{0,b}\right),$$

(22)

with $z_{b0} := \frac{\tilde{z}_{R,b}}{\lambda_b} \cdot (\tilde{x} - \tilde{x}_{b0})$, $t_b := t - t_{0,b}$ and $r_b := \sqrt{(\tilde{x} - \tilde{x}_{b0})^2 + z_{b0}^2}$. Here, $\mathcal{E}_{0,b}$ is the peak field strength, $\omega_b = \frac{2 \pi}{\lambda_b}$ the photon energy and $\tau_b$ the pulse duration. The beam is focused at $\tilde{x} = \tilde{x}_{b0}$, where the peak field is reached for $t = t_{0,b}$. Its waist size is $w_{0,b}$ and its Rayleigh range is $z_{R,b} = \frac{\pi w_{0,b}^2}{\lambda_b}$. The widening of the beam's transverse extent as a function of $z_b$ is encoded in the function $w_{0,b}(z_b) = w_{0,b} \sqrt{1 + (z_b/z_{R,b})^2}$, arctan$(x - x_{b0})$ is the Gouy phase of the beam and $\varphi_{0,b}$ determines its phase in the focus. The total angular spread $\Theta_b$ and the radial beam divergence $\theta_b$ far from the beam waist are given by $\Theta_b = 2 \theta_b \simeq 2 \frac{w_{0,b}}{z_{R,b}}$.

Without loss of generality, in the remainder of this article we will assume $x_{b0}^2 = (0, 0)$, such that the temporal and spatial offsets of the two beams are fully controlled by $x_{b1,2}^2 = (t_0, \tilde{x}_0)$. 

With regard to the Fourier integrals (22), it is particularly helpful to note that the $m$-th power of the field profile (22) can be expressed as

$$\mathcal{E}_b^m(x) = \left(\mathcal{E}_{0,b}\right)^m \sum_{l=0}^{m} \binom{m}{l} c_{b;lm}(z_{b}, r_{b})$$

$$\times e^{i(m-2l)[\omega(z_{b} - t_{b}) + \varphi_{0,b}]} e^{-4m(z_{b} - t_{b})^2/\tau_b^2},$$

(23)

where

$$c_{b;lm}(z_{b}, r_{b}) = \frac{-\left(i/\omega_{0,b}\right)^2 \frac{m!}{m - l!} \frac{l!}{\tau_{R,b}^2}}{(1 + i \frac{x - x_{b0}}{w_{0,b}})^{m-l} (1 - i \frac{x - x_{b0}}{w_{0,b}})^{l}}.$$  

(24)

which can be derived straightforwardly from Eq. (22) of Ref. 63 by employing the binomial theorem. Note that the entire dependence of Eq. (23) on the Rayleigh range $z_{R,b}$ and the transverse structure of the laser fields is encoded in the function $c_{b;lm}(z_{b}, r_{b})$.

The integration over time in Eq. (21) can be easily performed analytically for generic values of $z_{R,b}$, resulting in

$$I_{mn}(k) = m \left(\frac{\mathcal{E}_{0,1}}{2}\right)^m \left(\frac{\mathcal{E}_{0,2}}{2}\right)^n \frac{\sqrt{\pi} \tau_{1} \tau_{2}}{2 \sqrt{m \tau_b^2 + n \tau_b^2}}$$

$$\times \sum_{l=0}^{m} \sum_{j=0}^{n} \binom{m}{l} \binom{n}{j} \int d^3 \vec{x} e^{i \vec{k} \cdot \vec{x}}$$

$$\times c_{1;lm}(z_1, r_1) c_{2;jm}(z_2, r_2) e^{-4m(z_{b1} + z_{b2})^2/\tau_b^2}$$

$$\times e^{-\frac{1}{8m(\tau_b^2 + n \tau_b^2)}}$$

$$\times e^{i\{m (m-2l)(\omega_1 z_{b1} + \varphi_{0,1}) + n (n-2j)(\omega_2 z_{b2} + \varphi_{0,2})\}}.$$  

(25)

Let us now briefly focus on the limit of infinitely long pulse durations, $\{\tau_1, \tau_2\} \to \infty$. To this end, we first set $\tau_2 = \tau_1$ and subsequently send $\tau_1 \to \infty$. This results in the following expression,

$$\lim_{\tau_1, \tau_2 \to \infty} I_{mn}(k) = \delta(k + m - 2l) \omega_1 + (n - 2j) \omega_2$$

$$\times \frac{2\pi}{\mathcal{E}_{0,1}} m \left(\frac{\mathcal{E}_{0,2}}{2}\right)^n \sum_{l=0}^{m} \sum_{j=0}^{n} \binom{m}{l} \binom{n}{j}$$

$$\times \int d^3 \vec{x} e^{i \vec{k} \cdot \vec{x}} c_{1;lm}(z_1, r_1) c_{2;jm}(z_2, r_2)$$

$$\times e^{i\{m (m-2l)(\omega_1 z_{b1} + \varphi_{0,1}) + n (n-2j)(\omega_2 z_{b2} + \varphi_{0,2})\}}$$

$$\times e^{-\frac{1}{8m(\tau_b^2 + n \tau_b^2)}}$$

$$\times e^{i\{m (m-2l)(\omega_1 z_{b1} + \varphi_{0,1}) + n (n-2j)(\omega_2 z_{b2} + \varphi_{0,2})\}}.$$  

(26)

where we have employed the identity $\lim_{\tau \to \infty} \tau e^{-\frac{\tau^2}{4\pi}} \chi(\tau) = \sqrt{2\pi} \delta(\chi)$. The argument of the Dirac delta function in Eq. (26) reflects the various possibilities of energy transfer from the laser beams to the signal photons. Due to the strictly harmonic time dependences of the beams in the limit $\{\tau_1, \tau_2\} \to \infty$, implying sharp laser photon energies $\{\omega_1, \omega_2\}$, only signal photons with sharp energies $k \geq 0$. Hence, particularly for $\{\tau_1, \tau_2\} \to \infty$, the $I_{mn}(k)$ in Eq. (19) generically give rise to signal photons of energy

$$k = \begin{pmatrix} \omega_1 \\ \omega_1 + 2\omega_2 \\ |\omega_1 - 2\omega_2| \\ \omega_2 \\ \omega_2 + 2\omega_1 \\ |\omega_2 - 2\omega_1| \end{pmatrix}.$$  

(27)

For finite pulse durations the time dependences of the beams are no longer purely harmonic, and correspondingly the signal frequencies in general no longer sharp and discrete, but rather smeared and continuous. However, for pulse durations fulfilling $\{\omega_1 \tau_1, \omega_2 \tau_2\} \gg 1$, the signal frequencies should still be strongly peaked around the values listed in Eq. (27).

In the limit of infinite Rayleigh ranges $\{z_{R,1}, z_{R,2}\} \to \infty$, also the spatial Fourier integral in Eq. (25) can be performed analytically; cf. also Ref. 30. For this, we
for more details. The given above. In the following, we use the limit \((\text{FFT})\). For the laser pulses under consideration, we

arbitrary spacetime-dependent fields are straightforward

on the basis of a 4-dimensional fast Fourier transform

signal. For an illustration of the beam profiles used, see

information about the asymptotic signal photon,

region of the focused high-intensity laser pulses where

approximation can still be justified by the following

observation: The emission of signal photons from the

QED vacuum becomes substantial only in the overlap

beam diameter increases by a factor of \(\sqrt{2}\).
The radial momentum is already fixed by the constraint \( k \), rather than in Cartesian coordinates (but also a mapping to a polar and azimuthal angle photon in spherical momentum coordinates (\( k \) scattering center, it is useful to characterize the signal requires Nevertheless, the advantage is that the spatial grid determination of the directional emission characteristics of the signal photons. This specific design allows for building highly flexible code enabling, e.g., efficient parallelization. For the sake of convenience, we have summarized the scheme in Proc. I.

As the present collision set-up has a well-defined scattering center, it is useful to characterize the signal photon in spherical momentum coordinates \((k, \varphi, \vartheta)\) rather than in Cartesian coordinates \((k_x, k_y, k_z)\). Hence, step (i) does not only involve the FFT to \(k_{x,y,z}\) space, but also a mapping to a polar and azimuthal angle grid discretized into \(N_\varphi\) and \(N_\vartheta\) intervals, respectively. The radial momentum is already fixed by the constraint \( k = \sqrt{k_x^2 + k_y^2 + k_z^2}. \) This mapping is sketched in Fig. 4.

**Code:**

```plaintext
Initialization
for all \(k_u\) do
  for all \(I_m\) do
    Fourier transform from \((x, y, z)\) to \((k_x, k_y, k_z)\)
    Map from \((k_u, k_y, k_z)\) to \((\varphi, \vartheta)\)
  end for
end for
for all \(\varphi_u, \vartheta_w\) do
  Specify the polarization \(\beta\) of the signal photons
  Calculate emission rates \(S_\beta, \rho_\beta\)
end for
Post processing
```

**Notation:**

- \(x, y, z, k_x, k_y, k_z, \varphi, \vartheta\) discrete variables
- \(k_u, \varphi_u, \vartheta_w\) index denotes the loop variable
- \((\ldots)\) denotes a domain

**Procedure 1:** Pseudocode showing the general evaluation routine. The blocks are called consecutively, taking as input arguments only the results from the previous task.

Upon combination with the functions encoding the collision geometry and the polarization properties of the driving laser fields in Eq. (29), it is straightforward to obtain the discretized version of the differential number of signal photons with energy \(k_u\), emitted in the direction \((\varphi_u, \vartheta_w)\) from Eq. (29), where \(u = 0, \ldots, N_{\Delta(k)}\), \(v = 1, \ldots, N_\varphi, w = 1, \ldots, N_\vartheta\). Throughout this article we use \(N_{\Delta(k)} = 31\), \(N_\varphi = 257\) and \(N_\vartheta = 513\). Note, that at this point the polarization properties of the signal photons have to be specified.

The discretized version of the directional emission rate is obtained by summing over all \(k_u\) and is given by

\[
\rho_\beta(\varphi, \vartheta) \approx \rho_\beta(\varphi_u, \vartheta_w) = \frac{1}{(2\pi)^3} \sum_{u=0}^{N_{\Delta(k)}} W_{k_u} |k_u S_\beta(k_u, \varphi_u, \vartheta_w)|^2, \quad (29)
\]

where \(W_{k_u}\) denotes a weight function that is specified by the integration algorithm. Already simple integration routines give a good rate of convergence. For maximum simplicity, we hence apply the trapezoidal rule, resulting...
\[ \rho(p)(\varphi_v, \vartheta_w) = \frac{1}{(2\pi)^2} \frac{k_{N_{\Delta(k)}} - k_0}{2N_{\Delta(k)}} \left[ |k_0 S(p)(k_0)|^2 + 2 \sum_{u=1}^{N_{\Delta(k)}} |k_u S(p)(k_u)|^2 + |k_{N_{\Delta(k)}} S(p)(k_{N_{\Delta(k)}})|^2 \right]. \] (30)

The total number of signal photons polarized in mode \( p \) is then approximately given by

\[ N(p) \approx \sum_{v=0}^{N_v} \sum_{w=0}^{N_w} W_{\varphi_v} W_{\vartheta_w} \sin(\vartheta_w) \rho(p)(\varphi_v, \vartheta_w), \] (31)

with weights \( W_{\varphi_v} \) and \( W_{\vartheta_w} \). Similarly to Eq. (29), even simple routines provide a good rate of convergence. Hence, the trapezoidal rule is used again as the simplest method.

V. RESULTS

In the following, we provide explicit results for the prospective numbers of signal photons attainable in the collision of two high-intensity laser pulses characterized by the field profiles introduced in Sec. III. More specifically, we consider two identical lasers of the one petawatt (PW) class, delivering pulses of duration \( \tau = 25 \mathrm{fs} \) and energy \( W = 25 \mathrm{J} \) at a wavelength of \( \lambda = 800 \mathrm{nm} \) (photon energy \( \omega = \frac{2\pi}{\lambda} \approx 1.55 \mathrm{eV} \)). The peak intensity of a given laser pulse in the focus is then given by [66]

\[ I_{0,b} = \mathcal{E}_{0,b}^2 \approx 8 \sqrt{\frac{2}{\pi}} \frac{W}{\pi \omega_{0,b}^2}. \] (32)

As the effects of QED vacuum nonlinearities become more pronounced for higher field strengths, we aim at minimizing the beam waists \( \omega_{0,b} \) of the driving laser beams to maximize their peak field strengths. The minimum value of the beam waist \( \omega_{0,b} \) is obtained when focusing the Gaussian beam down to the diffraction limit. The actual limit is given by \( \omega_{0,b} = \lambda_b f^\#, \) where \( f^\# \) is the so-called \( f\)-number, defined as the ratio of the focal length and the diameter of the focusing aperture [65]; \( f\)-numbers as low as \( f^\# = 1 \) can be realized experimentally. Being particularly interested in the maximum number of signal photons, we mainly consider the case of an optimal overlap of the colliding laser pulses and set the offset parameters \( x_{0,1,2} = (t_0, x_0) \) to zero. Furthermore, in the remainder of this article we assume the two lasers to be polarized perpendicularly to the collision plane, corresponding to the choice of \( \beta_1 = \beta_2 = \frac{\pi}{2} \), and to deliver pulses of the same pulse duration, \( \tau_1 = \tau_2 = \tau \).

A. Collision of laser pulses of identical frequency

In a first step, we adopt the choice of \( \omega_1 = \omega_2 = \frac{2\pi}{\lambda} \) and assume that both lasers are focused down to the diffraction limit with \( f^\# = 1 \). Correspondingly, we have \( \omega_{0,1} = \omega_{0,2} = \lambda \). For a sketch of the considered collision geometry, see Fig. 5. Note that the specific scenario considered here is reminiscent of the one studied in Ref. [56]. However, here we go substantially beyond this initial study, which only focused on exactly counter propagating beams and resorted to various additional simplifications, grasping only the most elementary features of Gaussian laser beams.

Figure 5 shows the total number of signal photons \( N \) as a function of the collision angle \( \vartheta_2 \). Here, we depict the results for pulsed Gaussian beams with Rayleigh ranges \( z_{R,b} \) given self-consistently by \( z_{R,b} = (\pi \omega_{0,b}^2)/\lambda = \pi \lambda \) (dashed line). We also compare it to the toy-model benchmark scenario, where \( z_{R,b} \) is treated as an independent parameter, which is formally sent to infinity; cf. Sec. III above. This figure also demonstrates that the results obtained with our numerical algorithm (solid line) for the toy-model scenario with \( z_{R,b} \rightarrow \infty \) are in satisfactory agreement with benchmark data points (cross symbols). The latter are obtained by performing the Fourier transform from position to momentum space analytically, and the integration over the signal photon momenta numerically using MapleTM. We infer that the maximum number of signal photons is obtained for a head-on collision of the two high-intensity laser pulses, while no signal photons are induced for co-propagating beams. This fact is well-known from the study of probe photon propagation in constant crossed and plane wave fields; cf., e.g., Ref. [5]. Even though for collision angles in the range of 120°...180° signal photon numbers of \( N \approx 100 \) per shot are attainable,

![FIG. 5. Sketch of the collision geometry considered in Sec. V A. Two Gaussian laser pulses collide under an angle \( \vartheta_2 \) with respect to their beam axes; the offset between the beam foci is \( x_0 = 0 \). Note that an angle of \( \vartheta_2 = 0°(180°) \) corresponds to co(counter)-propagating laser beams.](image-url)
we exemplarily depict the directional emission of signal photons. For comparison, we have depicted the forward cones of the colliding Gaussian laser beams focused down to a beam divergence and delimited by the beams’ divergence of \(\theta_0 = \frac{\pi}{2}\).

In order to separate a signal – which is detectable at least in principle – from background, we turn to a different observable, namely the fraction of signal photons polarized perpendicularly to the high-intensity laser beams. Due to their distinct polarization, these photons constitute a viable signal that could be extracted with high-purity polarimetry. Recall, that both high-intensity laser beams are polarized perpendicularly to the collision plane (\(\beta_1 = \beta_2 = \frac{\pi}{2}\)).

In Fig. 6 we plot the number of signal photons polarized perpendicularly to the high-intensity laser beams \(N_\perp\) as a function of \(\vartheta_2\). For the particular collision scenario considered here, this number follows from the integration of

\[
\rho_\perp(\varphi, \vartheta) := \frac{1}{(2\pi)^3} \int_0^\infty dk S(k_1) \left| \beta = -\text{arctan}(\cos \vartheta \tan \varphi) \right|
\]

over the spherical angles, i.e.,

\[
N_\perp := \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos \vartheta \rho_\perp(\varphi, \vartheta).
\]

Note that the polarization-angle parameter \(\beta\) has to be adjusted as a function of the emission direction \(\vec{k}\) parameterized by \(\{\varphi, \vartheta\}\). In order to project on the perpendicular polarization, \(\hat{e}_{E_1} \cdot \hat{e}_\beta = \hat{e}_{E_2} \cdot \hat{e}_\beta = 0\) for all \(\vec{k}\). As in Fig. 4 we present results for the collision of pulsed Gaussian laser beams, as well as for the toy model scenario with \(z_{R,b} \to \infty\). Again the latter scenario is used to benchmark the performance of our numerical algorithm by comparing data points obtained for both strategies.

For a more quantitative comparison, we exemplarily list explicit values for the total numbers of attainable signal photons \(N\) and \(N_\perp\) for several collision angles \(\vartheta_2\) for the benchmark toy-model scenario in Table I. We find a relative difference typically on the order of \(O(0.01\%)\) and maximally of \(\sim 0.2\%\) between the semi-analytical approach and our numerical algorithm. While the semi-analytical approach that involves numerical integrations with MapleTM, we expect these algorithms to have a higher accuracy, also because the integrations are performed over the full (infinite) spacetime volume. The remaining difference hence serves as an error estimate for the numerical algorithm that works with absolute coordinate and momentum space cutoffs due to the nature of the fast Fourier transformation. Concretely, the fast Fourier algorithm treats the integration kernels as if they were periodic functions. We compensate for this by a careful adaptation of the domain of periodicity, such that all relevant information is preserved and no

### TABLE I. Benchmark calculations for the total numbers of signal photons attainable in the toy model scenario with \(w_{0,1} = w_{0,2} = \lambda\) finite, but \(z_{R,b} \to \infty\); see also Figs. 3 and 4. The good agreement of the results confirms the excellent performance of our numerical code. We only state the mean relative error for the total numbers of signal photons MRE\(_N\), as these numbers generally show the largest deviation.

| \(\vartheta_2[^\circ]\) | \(N\) | \(N_\perp\) | \(N\) | \(N_\perp\) | MRE\(_N\)[\%] |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 90              | 5.03            | 0.33            | 5.04            | 0.33            | 0.2             |
| 135             | 69.40           | 0.59            | 69.43           | 0.60            | 0.04            |
| 180             | 330.19          | 0.15            | 330.24          | 0.15            | 0.02            |

![FIG. 6. Total number of signal photons \(N\) attainable per shot in the collision of two identical high-intensity laser pulses \((w_{0,1} = w_{0,2} = \lambda = \text{800nm}, W = \text{25J}, \tau = \text{25fs})\) plotted as a function of the collision angle \(\vartheta_2\). The dashed line shows the results for the advanced description of the colliding laser fields in terms of pulsed Gaussian beams, evaluated numerically with our algorithm. In addition, we present results for the toy-model benchmark scenario of keeping \(w_{0,1} = w_{0,2} = \lambda\) finite but formally sending \(z_{R,b} \to \infty\). The latter scenario is analyzed in two different ways: By means of a fully numerical calculation with our algorithm (solid line), and by performing the Fourier transform from position to momentum space analytically, and numerically integrating over the outgoing signal photon momenta with MapleTM (cross symbols).]
artificial frequencies are introduced. Additionally, the transformation to spherical coordinates as well as the integrations over momentum space in our algorithm come with their discretization errors. A convergence test is illustrated in App. A. In summary, we consider a systematic error of our algorithm below the 1% level and thus possibly below two-loop corrections [67] as rather satisfactory.

Coming back to the physics results, Fig. 8 clearly demonstrates that the maximum for perpendicularly polarized signal photons $N_\perp$ is shifted to a collision angle of $\vartheta_2 \approx 120^\circ$. Moreover, the perpendicularly polarized signal is significantly smaller than the total one; the maximum number is $N_\perp \approx 0.6$. Analogously to Fig. 7, we also provide the directional emission characteristics of the perpendicularly polarized signal for a collision angle of $\vartheta_2 = 135^\circ$ in Fig. 9.

In addition, we display the analogous emission characteristics for a collision angle of $\vartheta_2 = 175.8^\circ$ in Fig. 10. Here, the formation of additional pronounced emission peaks opposite to the propagation directions of the high-intensity laser pulses for collision angles $\vartheta_2 \to 180^\circ$ is clearly visible. For a counter-propagation geometry reflection symmetry with respect to the xy-plane is restored [63].

Finally, we study the consequences of a spatial displacement $\vec{x}_0$ of the laser foci. Because of jitter, such a displacement is generically expected to occur in experiments in a random fashion. For simplicity, we specialize to the head-on collision of two identical high-intensity laser pulses with exactly coinciding beam axes, i.e., $\vartheta_2 = 180^\circ$, and consider the cases $\vec{x}_0 = (x_0, 0, 0)$ and $\vec{x}_0 = (0, 0, z_0)$ focused to $w_{0,1} = w_{0,2} = \lambda$. We demonstrate in Fig. 11 how the integrated numbers of signal photons $N$ and $N_\perp$ decrease as a function of the relative displacements $x_0$ and $z_0$ between the laser foci transverse to or along the common beam axis. For the present case, we observe that the signal photon number $N$ drops by a factor of 2 for $x_0 \approx 0.76\lambda$ and $z_0 \approx 1.5\pi\lambda$. 

FIG. 7. Directional emission characteristics of signal photons for two identical laser pulses colliding under an angle of $\vartheta_2 = 135^\circ$. Top: Three-dimensional plot of the total number density $\rho(\varphi, \vartheta)$. For illustration, we also include a projection of the emission characteristics onto the xz-plane (gray). Bottom: Projection of the directional emission characteristics (top) onto the collision plane of the laser pulses (xz-plane). For comparison, the forward cones of the colliding Gaussian laser beams with $f^\# = 1$ and delimited by the beams’ divergences $\theta_b = \frac{1}{f}$ representing the background are highlighted in gray.

FIG. 8. Total number of signal photons polarized perpendicularly to the high-intensity laser beams $N_\perp$ plotted as a function of the collision angle $\vartheta_2$. Both laser pulses ($w_{0,1} = w_{0,2} = \lambda = 800\text{nm}$, $W = 25\text{J}$, $\tau = 25\text{fs}$) are polarized perpendicularly to the collision plane. The dashed (solid) curve shows the result obtained from a numerical calculation for pulsed Gaussian beams (the benchmark scenario with $w_{0,b} = \lambda$ finite, but $z_{R,b} \to \infty$). The cross symbols display data for the benchmark scenario obtained by performing the Fourier transform analytically and evaluating the momentum integral numerically with Maple™.
FIG. 9. Directional emission characteristics of perpendicularly polarized signal photons for two identical laser pulses colliding under an angle of $\vartheta_2 = 135^\circ$. Top: Three-dimensional plot of the number density $\rho_\perp(\varphi, \vartheta)$. Bottom: Projection of the directional emission characteristics (top) onto the collision plane of the laser pulses (xz-plane). For comparison, the forward cones of the colliding Gaussian laser beams with $f^\# = 1$ and delimited by the beams’ divergences $\theta_b = \frac{1}{\pi}$ are highlighted in gray.

B. Collision of laser pulses of fundamental and doubled frequency

Here we go beyond the scenario considered in the previous section, subsequently referred to as scenario (o). Differently to Sec. V A, one of the two high-intensity lasers is now assumed to be frequency doubled, such that $\omega_2 = 2\omega_1 = 2\frac{2\pi}{\lambda}$. The energy loss for a frequency-doubling process conserving the pulse duration is estimated conservatively as 50%. Correspondingly, we have $\tau_1 = \tau_2 = \tau$, $W_1 = W$ and $W_2 = W/2$. Keeping the focusing of the fundamental-frequency laser pulse as in the previous section, i.e., $w_{0,1} = \lambda$, we now consider two different scenarios: (i) In order to ensure a maximal spatial overlap of the two laser pulses in their foci, the frequency-doubled laser pulse is focused down to the waist size of the fundamental-frequency laser pulse, i.e., $w_{0,2} = w_{0,1} = \lambda$. This scenario is illustrated in Fig. 12 (ii) For maximizing the peak field strength in the focus, the frequency-doubled pulse is focused down to its...
we have argued that the signal photons

with

perpendicularly to the collision plane. Top: Transverse shift of the waist of \( x_0 = (x_0, 0, 0) \) in units of the waist size \( w_{0,2} = \lambda/2 \). Bottom: Longitudinal shift along the common beam axis with \( z_0 = (0, 0, z_0) \) in units of the Rayleigh range \( z_{R,1} = z_{R,2} = \pi \lambda \).

As detailed in Sec. II for Gaussian beams the Rayleigh range and far-field beam divergence, are intimately related to the wavelength and the waist size. Hence, in case (i) we have \( z_{R,2} = 2z_{R,1} \), \( \theta_2 = \theta_1/2 \), while in case (ii) \( z_{R,2} = z_{R,1}/2 \), \( \theta_2 = \theta_1 \); cf. also Figs. \ref{fig:fig11} and \ref{fig:fig13}. All the results presented in this section are obtained with our algorithm introduced in Sec. IV. In Fig. \ref{fig:fig11} we show the total number of signal photons \( N \) as a function of the collision angle \( \vartheta_2 \) for the cases (o)-(ii).

In Sec. III we have argued that the signal photons should predominantly be emitted at several pronounced frequencies if the criterion \( \{\omega_1 \tau, \omega_2 \tau\} \gg 1 \) holds; cf. Eq. \ref{eq:27}. For the collision of (o) two fundamental frequency beams we have \( \omega_1 \tau = \omega_2 \tau \approx 58.9 \), while for the cases (i) and (ii), both involving a frequency doubled beam, we have \( \{\omega_1 \tau, \omega_2 \tau\} \approx 58.9, 117.7 \).

Hence, as the criterion \( \{\omega_1 \tau, \omega_2 \tau\} \gg 1 \) is obviously fulfilled here, we expect the signal photons to feature primarily frequencies with (i) \( k \approx \{\omega_1, 3\omega_1\} \) and (ii) \( k \approx \{\omega_1, 2\omega_1, 3\omega_1, 4\omega_1, 5\omega_1\} \), respectively. However, inelastic signal photon emission processes are generically suppressed in comparison to the elastic ones. For instance, in Ref. \ref{56} it was already demonstrated for a simplified model of the head-on collision of fundamental frequency laser pulses that the \( 3\omega_1 \) signal is completely negligible in comparison to the \( \omega_1 \) signal. This fully agrees with the results obtained here: In scenario (o) essentially all signal photons are emitted in an energy range \( \Delta(\omega_1) \); here and in the following \( \Delta(\omega) \) denotes an interval of photon energies centered around a frequency \( \omega \) with an energy width being inversely proportional to the temporal pulse duration. For the scenarios (i) and (ii) we encounter sizable numbers of signal photons in the energy segments \( \Delta(\omega_1) \) and \( \Delta(2\omega_1) \).

In Fig. \ref{fig:fig13} we show the partitioning of the emitted signal photons into the dominant frequency channels \( k \approx \{\omega_1, 2\omega_1\} \). We present results for the total number of attainable signal photons \( N \) for all the scenarios (o)-(ii) introduced above. In addition, we provide the number of signal photons \( N_{>\theta} \) emitted outside the forward cones (delimited by the beam divergences \( \theta_0 \)) of the high-intensity lasers.

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FIG. 11. Impact of a relative shift between the laser foci on the integrated numbers of signal photons \( N \) (blue solid line, left scale) and \( N_\perp \) (orange dashed line, right scale) for two identical laser pulses colliding in a counter-propagation geometry, i.e., \( \vartheta_2 = 180^\circ \). Both laser pulses are polarized perpendicularly to the collision plane. Top: Transverse shift with \( x_0 = (x_0, 0, 0) \) in units of the waist size \( w_{0,2} = \lambda/2 \). Bottom: Longitudinal shift along the common beam axis with \( z_0 = (0, 0, z_0) \) in units of the Rayleigh range \( z_{R,1} = z_{R,2} = \pi \lambda \).

FIG. 12. Scenario (i): The two Gaussian beams of fundamental and doubled frequency are focused to a beam waist of \( w_{0,1} = w_{0,2} = \lambda \). In this scenario, the beam divergences fulfill \( \theta_2 = \theta_1/2 \). We depict the case of zero offset, \( \bar{x}_0 = 0 \), and \( \vartheta_2 \in \{0^\circ, 180^\circ\} \).

FIG. 13. Scenario (ii): The Gaussian beam of fundamental (doubled) frequency is focused to a waist size of \( w_{0,1} = \lambda \) \( w_{0,2} = \lambda/2 \). In this scenario, the beam divergences fulfill \( \theta_1 = \theta_2 \). We depict the case of zero offset, \( \bar{x}_0 = 0 \), and \( \vartheta_2 \in \{0^\circ, 180^\circ\} \).
TABLE II. Prospective numbers of signal photons with energies in the segments \( \Delta(\omega_1) \) and \( \Delta(2\omega_1) \) for the example of a collision angle of \( \vartheta_2 = 135^\circ \). Both high-intensity laser pulses are polarized perpendicularly to the collision plane. Apart from (o) the collision of two identical beams of frequency \( \omega_1 \) focused to \( w_{0,1} = w_{0,2} = \lambda \), we consider collisions of fundamental-frequency \( \omega \) and frequency-doubled \( \omega_2 = 2\omega_1 \) beams focused to (i) \( w_{0,1} = w_{0,2} = \lambda \), and (ii) \( w_{0,1} = 2w_{0,2} = \lambda \). We provide values for the total (perpendicularly polarized) number of signal photons \( N(N_\perp) \). Besides, \( n_{>\theta}(n_{>\theta\perp}) \) denotes the fraction of \( N(N_\perp) \) emitted outside the forward divergence of the Gaussian high-intensity lasers.

| scenario | \( N \) | \( n_{>\theta} \) | \( N_\perp \) | \( n_{>\theta\perp} \) |
|----------|--------|-----------|-----------|-----------|
| (o)      | 70.53  | 42%       | 0.66      | 74%       |
| (i)      | 9.20   | 44%       | 0.08      | 75%       |
| (ii)     | 24.02  | 66%       | 0.35      | 90%       |

| scenario | \( N \) | \( n_{>\theta} \) | \( N_\perp \) | \( n_{>\theta\perp} \) |
|----------|--------|-----------|-----------|-----------|
| (o)      | -      | -         | -         | -         |
| (i)      | 34.24  | 40%       | 0.10      | 75%       |
| (ii)     | 53.67  | 24%       | 0.29      | 54%       |

FIG. 14. Integrated numbers of signal photons attainable in the various scenarios (o)-(ii) plotted as a function of the collision angle \( \vartheta_2 \). We depict results for (o) the collision of two fundamental frequency laser pulses focused to \( w_{0,1} = w_{0,2} = \lambda \), and the collision of fundamental and doubled frequency laser pulses focused to waist sizes (i) \( w_{0,2} = \lambda \) and (ii) \( w_{0,2} = w_{0,1}/2 = \lambda/2 \). Top: Total number of signal photons \( N \). Bottom: Number of signal photons emitted outside the forward cones of the colliding Gaussian laser beams \( N_{>\theta} \), delimited by the beams’ radial divergences \( \theta_b \).

FIG. 15. Partitioning of the attainable numbers of signal photons into the energy regimes \( \Delta(\omega_1) \) and \( \Delta(2\omega_1) \) for the various scenarios (o)-(ii). Both high-intensity laser pulses are polarized perpendicularly to the collision plane. The segment with center frequency \( k = \omega_1 (2\omega_1) \) is depicted by \( \bullet (\) symbols. Naturally, there is no \( k \approx 2\omega_1 \) signal for the collision of two fundamental frequency beams. Top: Total number of signal photons \( N \). Bottom: Integrated number of signal photons emitted outside the forward cones of the colliding Gaussian laser beams \( N_{>\theta} \).
and shows results for the number of fundamental-frequency beams. As one can see in Fig. 16, signal photons is to be expected for the collision of two expectations as the maximum number of frequency-beams. The segment with center frequency k = ω₁ (2ω₁) is depicted by • (+) symbols. Naturally, no k ≈ 2ω₁ signal is induced in the collision of two fundamental frequency beams. Top: Total number of perpendicularly polarized signal photons N⊥. Bottom: Integrated number of perpendicularly polarized signal photons emitted outside the forward cones of the colliding Gaussian laser beams N⊥,>θ.

Analogously, Fig. 16 shows results for the number of signal photons polarized perpendicularly to the high-intensity laser beams N⊥ and N⊥,>θ. Besides, in Tab. 11 we exemplarily stick to a collision angle of θ₂ = 135° and provide explicit numerical values for the numbers of signal photons with energies in the ranges Δ(ω₁) and Δ(2ω₁). For a given energy regime Δ, the values for N and N⊥ and analogously nₘ,>θ = N₂,>θ/N₁ and n⊥,>θ = N⊥,>θ/N₁ exhibit similar trends.

Let us first detail on the behavior of N and N⊥. In the energy regime Δ(ω), the largest numbers for N and N⊥ are obtained for scenario (o), followed by (ii) and finally (i). This is completely consistent with our expectations as the maximum number of frequency-ω₁ signal photons is to be expected for the collision of two fundamental-frequency beams. As one can see in Fig. 7, these essentially elastically scattered signal photons are predominantly emitted in the forward directions of the high-intensity laser beams. The finding that the attainable signal photon numbers in scenario (ii) are larger than for scenario (i) hints at the fact that the peak field strength is most decisive for the effect. Recall, that for (ii) the frequency-doubled laser beam is focused down to the diffraction limit with fθ = 1, guaranteeing a maximum peak field, while in (i) it is only focused with fθ = 2; cf. Figs. 12 and 13. In the energy regime Δ(2ω₁), we find similar trends for the behavior of N and N⊥. Generically, no frequency-2ω signal is generated in the collision of (o) two fundamental-frequency laser beams; see Eq. 27.

Secondly, we comment on the trends observed for the relative fractions of signal photons nₘ,>θ and n⊥,>θ scattered outside the beam divergences in forward direction. Again we first discuss the results obtained for the energy regime Δ(ω₁). This signal is mainly induced in the propagation direction of the high-intensity laser with fundamental frequency, which implies that effectively only the divergence of the fundamental-frequency beam matters. While the values of nₘ,>θ and n⊥,>θ are similar for the cases (o) and (i), the result for case (ii) is significantly different. For the cases (o) and (i), the fundamental frequency beam collides with a beam of similar transverse focus profile of width w₀,₁ = w₀,₂ ≈ λ. As the signal photons are predominantly induced in the focus, the similar values obtained for nₘ,>θ and n⊥,>θ are not surprising.

Conversely, the smaller beam waist of the frequency-doubled beam in (ii) naturally gives rise to a larger fraction of photons scattered out of the divergence of the fundamental frequency beam as compared to (o) and (i).

\footnote{Note, that this argument is not invalidated by the fact that in (o) we consider two frequency-ω₁ beams, while there is only a single frequency-ω₁ beam in (i). The reason for this is the fact that the ratios n are insensitive to the absolute numbers.}
For (o) and (ii) both high-intensity laser beams exhibit the same divergence $\theta_1 = \theta_2$. Conversely, for (i) the divergence of the frequency-doubled beam is $\theta_2 = \theta_1/2$, which explains why for (i) also the signal photons are scattered into a narrower far-field angle.

To allow for a comparison of the angular spread of the photons constituting the high-intensity laser beam and the signal photons, we plot the corresponding differential photon numbers in the far-field as a function of the polar angle $\vartheta$ in Fig. 18. The photon distributions of the high-intensity laser beams in the far-field scale as $d^2 N / d\varphi d\cos \vartheta \sim e^{-2\vartheta^2/\theta^2}$ for the beam propagating along the $z$ axis and as $d^2 N / d\varphi d\cos \vartheta \sim e^{-2(\vartheta - \vartheta_2)^2/(\vartheta_1/\cos \vartheta)^2}$ for the other beam; where $f^\# = 1$ for both (o) and (ii), and $f^\# = 2$ for (i). Obviously, the signal photons are scattered asymmetrically. The different decay of the signal photons and the photons constituting the high-intensity laser fields leads to an improved signal to background ratio.

**VI. CONCLUSIONS AND OUTLOOK**

In this article, we have provided further evidence that all-optical signatures of quantum vacuum nonlinearity can be analyzed efficiently in terms of vacuum emission processes. The essence of this concept is that all macroscopically sourced fields are treated as classical, whereas the fields induced by quantum nonlinearity receive a quantum description in terms of signal photons. This concept matches ideally with the physical situation and thus provides direct access to physical observables.

In the present example of colliding laser pulses, this approach facilitates to directly determine the directional emission characteristics and polarization properties of the signal photons encoding the signature of quantum vacuum nonlinearities. Our main goal was to demonstrate that, assisted by a dedicated numerical algorithm, the vacuum emission approach is particularly suited to tackle signatures of strong-field QED in experimentally realistic electromagnetic field configurations generated by state-of-the-art high-intensity laser systems. To this end, we focused on a comparatively straightforward scenario, based upon the collision of two optical high-intensity laser pulses, which we model as pulsed Gaussian beams. Resorting to a locally constant field approximation of the Heisenberg-Euler effective action, our numerical algorithm allows for a numerically efficient and reliable study of the attainable numbers of signal photons for arbitrary collision angles and polarization alignments. Our formalism can be readily extended to the collision of more laser beams, such as the study of photon-merging [47], or equivalently four-wave mixing processes [39,40] induced by QED vacuum nonlinearities in the collision of three focused high-intensity laser beams.
ACKNOWLEDGMENTS

We are grateful to Nico Seegert for many helpful discussions and support during the development phase of the numerical algorithm. The work of C.K. is funded by the Helmholtz Association through the Helmholtz Postdoc Programme (PD-316). We acknowledge support by the BMBF under grant No. 05P15SJFAA (FAIR-APPA-SPARC). Computations were performed on the “Supermicro Server 1028TR-TF” in Jena, which was funded by the Helmholtz Postdoc Programme (PD-316).

Appendix A: Convergence tests

As discussed in the main text, semi-analytical and numerical results fit almost perfectly for a suitable choice of numerical discretization parameters. In the following, we detail this choice of numerical parameters by studying the convergence of the numerical algorithm in comparison to the semi-analytical results for the toy-model benchmark test. Such an analysis is useful, because it (i) helps to improve the stability of the numerical results and (ii) yields systematic checks enabling to run simulations in regions of the parameter space, where no analytical reference values are available. Eventually, it also helps to minimize the program’s runtime as well as its memory requirements.

In this work, we have in total 10 independent parameters controlling the numerical calculation. These are \( N_x, N_y, N_z \) specifying the lattice in the Cartesian grid for spatial/momentum coordinates, \( N_{\varphi}, N_\theta, N_\chi \) yielding the number of grid points in spherical momentum coordinates and \( L_x, L_y, L_z, L_k \) defining the physical interval of length \( 2L_{x,y,z,k} \) (sampling regions) of the corresponding variables centered around the region of interest. For illustration, we focus here on lower dimensional subsets. Similar convergence checks can be performed for each of these parameters.

In the following, we discuss the numerical convergence of our calculations in the context of two parameters, the radial momentum of the signal photons \( k \) and the longitudinal resolution of the pump fields along the \( z \) axis. For this, we first plot the total number of signal photons \( N \) as a function of the number of grid points \( N_{\Delta(k)} \) for various choices of the momentum grid length \( L_k \) in Fig. 19. By comparison with the semi-analytical results we observe, that accuracy of the result increases with the momentum-space resolution as expected. It is also remarkable, that a few grid points in the total momentum \( k, \mathcal{O}(10) \), are sufficient in order to approximate the analytical solution reasonably well. The crucial ingredient is, of course, an appropriate choice for the resolved momentum interval: while the center of the \( L_k \) region can be adapted to the requirements imposed by energy conservation, cf. Eq. (27), being \( k \approx \omega \) in the present example, the size of \( L_k \) has to cover the bandwidth of the outgoing pulse. In the present case, a region with \( L_k \geq 0.3 \text{eV} \) is required, corresponding to \( \geq 20\% \) of the central pulse energy. For instance, a region limited to \( L_k = 0.1 \text{eV} \) is not sufficient to provide a precise estimate of the signal photon number, see Fig. 19.

Secondly, we investigate the spatial resolution needed in order to satisfactorily resolve the applied laser pulses. In this case, the parameters \( L_x \) and \( N_z \) have to meet two different requirements: on the one-hand side, \( L_x \) has to be chosen large enough to cover the region of interest given by the focal and collision region of the two pulses, while \( N_z \) has to be sufficiently large to precisely sample the details of the pulse shape: On the other hand, the nature of the Fourier transform implies that \( \pi/2L_x \) defines an infrared cutoff and \( \pi N_z/(2L_x) \) an ultraviolet cutoff for the \( z \) component of the momentum of the outgoing signal photon. Hence, both have to be chosen sufficiently large also to resolve the sampling region \( 2L_k \) centered around the peak momentum \( k \) of the signal photon appropriately. As a rule of thumb, an increase of the sampling region should go along with an increase of the number of grid points in order to keep the momentum space ultraviolet resolution (at least) constant.

In the present case, the procedure for choosing the discretization parameters is the following: The parameter \( L_x \) should be chosen large enough in order to resolve the focal region of the pump fields, i.e. at least one oscillation of the pump fields in the present case. Signal energy conservation suggests the signal photons to be located at around \( k \approx \omega \), the values for \( L_x \) and \( N_z \) should take on values such that the momentum region around \( \omega \) is with sufficient resolution within the infrared and ultraviolet cutoffs induced by the Fourier transformation. For definiteness, we have fixed the...
longitudinal sampling region to \( z \in 2^q[-0.95\lambda, 0.95\lambda] \) and study the convergence of the result for increasing \( q \) and \( N_z \).

The results for the signal photon number as a function of the size of the sampling region for two different grid resolutions are shown in Fig. 20 and Tab. III. As expected, the spatial sampling region has to be large enough to cover the focal region of the size of a wavelength \( \lambda \) in order to approach the correct result. We observe that even a rather small number of 32 grid points can give an acceptable result with an error on the percent level, if the size of the sampling region is chosen appropriately as to cover the relevant momentum region of the signal photon upon Fourier transformation. For a reliable result with an error well below 1\%, larger numbers of grid points and a sufficiently large sampling region are required – of course, at the expense of computing time.

TABLE III. Benchmark calculations for the total number of signal photons attainable in the toy-model scenario. Two identical laser pulses (\( \lambda = 800\text{nm}, W = 25J, \tau = 25fs \)) are focused to \( w_{0,1} = w_{0,2} = \lambda \) and collide in a counter-propagation geometry. Both pulses are polarized perpendicularly to the collision plane. The benchmark toy model is used here to allow for a comparison with the semi-analytical result (black line).

| \( N_z \) | Runtime [s] | \( 2^{-1} \) | \( 2^0 \) | \( 2^1 \) |
|----------|-------------|--------|--------|--------|
| 32       | 1120        | 15.25  | 2.17   | -      |
| 128      | 3105        | 14.87  | 2.20   | 0.22   |
| 512      | 9400        | 14.86  | 2.20   | 0.22   |

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