The quiver approach to the BPS spectrum of a 4d $\mathcal{N} = 2$ gauge theory

Sergio Cecotti

Dedicated to the Memory of Professor Friedrich Hirzebruch

Abstract. We present a survey of the computation of the BPS spectrum of a general four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in terms of the Representation Theory of quivers with superpotential. We focus on SYM with a general gauge group $G$ coupled to standard matter in arbitrary representations of $G$ (consistent with a non-positive beta–function). The situation is particularly tricky and interesting when the matter consists of an odd number of half–hypermultiplets: we describe in detail $SU(6)$ SYM coupled to a $\frac{1}{2} 20$, $SO(12)$ SYM coupled to a $\frac{1}{2} 32$, and $E_7$ SYM coupled to a $\frac{1}{2} 56$.

1. Introduction

In the last few years many new powerful methods were introduced to compute the exact BPS spectrum of a four–dimensional $\mathcal{N} = 2$ supersymmetric QFT. We may divide the methods in two broad classes: i) geometric methods [1–5] and ii) algebraic methods [6–14]. The geometric methods give a deep understanding of the non–perturbative physics, while the algebraic ones are quite convenient for actual computations. In the algebraic approach the problem of computing the BPS spectrum is mapped to a canonical problem in the Representation Theory (RT) of (basic) associative algebras. A lot of classical results in RT have a direct physical interpretation and may be used to make the BPS spectral problem ‘easy’ for interesting classes of $\mathcal{N} = 2$ theories. Besides, by comparing RT and physics a lot of interesting structures emerge which shed light on both subjects.

1.1. From $\mathcal{N} = 2$ QFT to quiver representations. To fix the notation, we recall how the BPS states are related to quiver representations, referring to [10] for more details. The conserved charges of the theory (electric, magnetic, and flavor) are integrally quantized, and hence take value in a lattice $\Gamma = \oplus_v \mathbb{Z} e_v$. On $\Gamma$ we have a skew–symmetric integral pairing, $\langle \gamma, \gamma' \rangle_{\text{Dirac}} \in \mathbb{Z}$, given by the Dirac electromagnetic pairing; the flavor charges then correspond to the zero–eigenvectors of the matrix $B_{uv} = \langle e_u, e_v \rangle_{\text{Dirac}} \in \mathbb{Z}$.

2010 Mathematics Subject Classification. 81T60.

©0000 (copyright holder)
Following \textsuperscript{[7]} we say that our $\mathcal{N} = 2$ model has the \textit{quiver property} if we may find a set of generators $\{e_\gamma\}$ of $\Gamma$ such that the charge vectors $\gamma \in \Gamma$ of all the BPS particles satisfy

\begin{equation}
\gamma \in \Gamma_+ \quad \text{or} \quad -\gamma \in \Gamma_+,
\end{equation}

where $\Gamma_+ \equiv \oplus_\gamma \mathbb{Z}_+ e_\gamma$ is the \textit{positive cone} in $\Gamma$. Given a $\mathcal{N} = 2$ theory with the quiver property, we associate a 2–acyclic quiver $Q$ to the data $(\Gamma_+, \langle \cdot, \cdot \rangle_{\text{Dirac}})$: to each positive generator $e_\nu$ of $\Gamma_+$ we associate a node $\nu$ of $Q$ and we connect the nodes $u \to v$ with $B_{uv}$ arrows $u \to v$ (a negative number meaning arrows in the opposite direction). The positive cone $\Gamma_+ \subset \Gamma$ is then identified with the cone of dimension vectors of the representations $X$ of $Q$ through $\dim X \equiv \sum_\nu \dim X_\nu e_\nu$.

The emergence of the quiver $Q$ may be understood as follows. Fix a particle with charge $\gamma = \sum_\nu N_\nu e_\nu \in \Gamma_+$; on its word–line we have a one dimensional supersymmetric theory with 4 supercharges, and the BPS particles correspond to states which are SUSY vacua of this 1d theory. The 1d theory turns out to be a quiver theory in the sense that its K"ahler target space is the representation space of $Q$ of dimension $\sum_\nu N_\nu e_\nu$.

\begin{equation}
\prod_{\text{arrows } u \to v} \mathbb{C}^{N_\nu N_\nu} / \prod_{\text{nodes }} \text{GL}(N_\nu, \mathbb{C}) \quad \text{(symplectic quotient).}
\end{equation}

To completely define the 1d theory we need to specify a $\prod_{\text{nodes }} \text{GL}(N_\nu, \mathbb{C})$–invariant superpotential $W$ (and the FI terms implicit in \textsuperscript{(2)}); gauge invariance requires $W$ to be a function of the traces of the products of the bi–fundamental Higgs fields along the closed oriented loops in $Q$. It turns out that this function must be linear (a sum of single–trace operators) and thus canonically identified with a linear combination (with complex coefficients) of the oriented cycles in $Q$. Thus $W$ is a potential for the quiver $Q$ in the sense of DWZ \textsuperscript{[15]}. One shows \textsuperscript{[10]} that a 1d configuration is a classical SUSY vacuum if and only if the bi–fundamental Higgs fields associated to the arrows of $Q$ form a \textit{stable} module $X$ of the Jacobian algebra$\textsuperscript{[4]}$.

\begin{equation}
\mathcal{J}(Q, W) := \mathbb{C}Q/(\partial W),
\end{equation}

and two field configurations are physically equivalent iff the corresponding modules are isomorphic. Stability is defined in terms of the central charge $Z$ of the $\mathcal{N} = 2$ SUSY algebra. Being conserved, $Z$ is a linear combinations of the various charges; hence may be seen as a linear map $Z : \Gamma \to \mathbb{C}$. We assume $\text{Im} Z(\Gamma_+) \geq 0$, so that we have a well–defined function $\arg Z : \Gamma_+ \to [0, \pi]$. Then $X \in \text{mod } \mathcal{J}(Q, W)$ is \textit{stable} (with respect to the given central charge $Z$) iff, for all proper non–zero submodules $Y$, $\arg Z(Y) < \arg Z(X)$. In particular, $X$ is \textit{stable} $\Rightarrow X$ is a \textit{brick}, (a module $X$ of an associative algebra is called a \textit{brick} if End $X = \mathbb{C}$). The isoclasses of stable modules of given dimension $\gamma$ typically form a family parameterized by a K"ahler manifold $M_\gamma$; from the viewpoint of the 1d theory the space $M_\gamma$ corresponds to zero–modes which should be quantized producing $SU(2)_{\text{Spin}} \times SU(2)_R$ quantum numbers. In particular, a $d$–dimensional family corresponds (at least) to a BPS supermultiplet with spin content $(0, \frac{d}{2}) \otimes \frac{d}{2}$ (thus rigid modules corresponds to

\footnote{A module $X \in \text{mod } \mathcal{J}(Q, W)$ of dimension $\sum_\nu N_\nu e_\nu$ is specified by giving, for each arrow $u \to v$, an $N_\nu \times N_\nu$ matrix $X_\nu$ such that the matrices $\{X_\nu\}$ satisfy the relations $\partial X_\beta W(X_\alpha) = 0$ for all arrows $\beta$ in $Q$. Two such representations are \textit{isomorphic} if they are related by a $\prod_\nu \text{GL}(N_\nu, \mathbb{C})$ transformation.}
hypermultiplets, $\mathbb{P}^1$–families to vector supermultiplets, and so on). Notice that the full dependence of the BPS spectrum from the parameters of the theory is encoded in the central charge $Z$, which depends on these parameters as specified by the Seiberg–Witten geometry.

For a given $\mathcal{N} = 2$ theory $(Q, W)$ is not unique; indeed there may be several sets of generators $\{e_i\}$ with the above properties. Two allowed $(Q, W)$ are related by a Seiberg duality, which precisely coincides with the mutations of a quiver with potential in the sense of cluster algebras $\text{[15]}$ (this, in particular, requires $W$ to be non–degenerate in that sense). Therefore, to a QFT we associate a full mutation class of quivers. If the mutation class is finite we say that the corresponding $\mathcal{N} = 2$ QFT is complete $\text{[7]}$ which, in particular, implies that no BPS state has spin larger than 1.

$T_2$–duality. The Seiberg duality/DWZ mutation is not the only source of quiver non–uniqueness. The quiver mutations preserve both the number of nodes and 2–acyclicity. There are more general dualities which do not share these properties. As an example consider the Gaiotto theory corresponding to the $A_1$ $(2,0)$ 6d theory on a sphere with 3 regular punctures (the $T_2$ theory) $\text{[16]}$. $T_2$ consists of 4 free hypermultiplets, carrying 4 flavor charges, which corresponds to a disconnected quiver with 4 nodes and no arrows. On the other hand, we may associate to it a quiver with only three nodes, each pair of nodes being connected by a pair of opposite arrows $\rightleftarrows \text{[10]}$. We refer to the equivalence of the two quivers as ‘$T_2$–duality’.

2. The $(Q,W)$ class associated to an $\mathcal{N} = 2$ theory

The BPS states correspond to the stable bricks of the Jacobian algebra. This reduces our problem to a standard problem in Representation Theory provided we know which $(Q,W)$ mutation class is associated to our $\mathcal{N} = 2$ theory. Determining the mutation class for several interesting gauge theories is the main focus of the present note.

For $\mathcal{N} = 2$ models having a corner in their parameter space with a weakly coupled Lagrangian description, we have a very physical criterion to check whether a candidate pair $(Q, W)$ is correct. Simply use the category $\text{mod } \mathcal{F}(Q, W)$ to compute the would–be BPS spectrum in the limit of vanishing YM coupling $g_{YM} \to 0$ and compare the result with the prediction of perturbation theory. The weakly coupled spectrum should consist of

- finitely many mutually–local states with bounded masses as $g_{YM} \to 0$:
  
  1. vector multiplets making one copy of the adjoint representation of the gauge group $G$ (photons and $W$–bosons);
  
  2. hypermultiplets making definite (quaternionic) representations $R_a$ of $G$ (quarks);

- particles non–local relatively to the $W$–bosons with masses $O(1/g_{YM}^2)$ (heavy dyons).

We ask which pairs $(Q, W)$ have such a property (the Ringel property $\text{[11]}$).

2.1. Magnetic charge and weak coupling regime. Consider a quiver $\mathcal{N} = 2$ gauge theory having a weak coupling description with gauge group $G$ (of rank $r$). We pick a particular pair $(Q, W)$ in the corresponding Seiberg mutation–class which is appropriate for the weak coupling regime (along the Coulomb branch).
\textbf{Remarks and Properties}

1. \textit{mod} \mathcal{F}(Q, W) \textit{contains many} ligh subcategories, one for each weakly coupled corner. \textit{E.g.} SU(2) \( N_f = 4 \) has a SL(2, Z) orbit of such subcategories;

2. \( m(\Gamma_+) \not\equiv 0 \Rightarrow \) the light category is \textit{not} the restriction to a subquiver, and its quiver is \textit{not} necessarily 2-acyclic (as in the \( T_2 \) case \textsuperscript{10,11});

3. the category \( \mathcal{L}(Q, W) \) is \textit{tame} (physically: no light BPS state of spin > 1);

4. \textit{universality} of the SYM sector: for given gauge group \( G \)

\[ \mathcal{L}(Q_{\text{SYM}}, W_{\text{SYM}}) \subset \mathcal{L}(Q, W) \]

where \((Q_{\text{SYM}}, W_{\text{SYM}})\) is the pair for pure \( G \) SYM. Only finitely many bricks \( X \in \mathcal{L}(Q, W) \) and \( X \not\in \mathcal{L}(Q_{\text{SYM}}, W_{\text{SYM}}) \), they correspond to ‘quarks’.

\textsuperscript{2} \( h \) stands for the Cartan subalgebra of the complexified Lie algebra of the gauge group \( G \).
As a warm-up we consider four classes of (simple) examples.

3.1. Example 1: SU(2) SQCD with $N_f \leq 4$. These examples are discussed in detail in [7,10,11]; here we limit ourselves to a description of the resulting categories. One shows [11] that the category $\text{mod} J(Q,W)$ is Seiberg–duality equivalent to the Abelian category $\text{Coh}(\mathbb{P}^1_{N_f})$ of coherent sheaves on $\mathbb{P}^1$ with $N_f$ ‘double points’, that is, the variety in the weighted projective space $\mathbb{W}P(2,2,\ldots,2,1,1)$ of equations

$$X_2^i - \lambda_i X_{N_f+1} - \mu_i X_{N_f+2} = 0, \quad i = 1,2,\ldots,N_f, \quad (\lambda_i : \mu_i) \in \mathbb{P}^1.$$ 

In $\text{Coh}(\mathbb{P}^1_{N_f})$ we have two quantum numbers, degree and rank

$$\text{rank} = \text{magnetic charge}, \quad \text{degree} = 2 \times \text{electric charge}.$$ 

The light subcategory $\text{L}_\text{YM}(G) \supset \mathcal{L} = \{\text{sheaves of finite length}\}$ a.k.a. ‘skyscrapers’, while the dyons correspond to line bundles of various degree.

For $N_f = 4$ the curve $\mathbb{P}^1_4$ is Calabi–Yau, hence an elliptic curve $E$. The moduli space of the degree 1 skyscrapers, which is the curve $E$ itself, is isomorphic to its Jacobian $J(E)$ which parameterizes the line bundles of fixed degree. Quantization of $J(E)$ then produces magnetic charged vector–multiplets. Of course, $E \sim J(E)$ reflects the $S$–duality of the theory. See [11] for more details.

3.2. Example 2: SYM with a simply–laced gauge group $G$. The quiver exchange matrix $B$ is fixed by the Dirac charge quantization [11] (cfr. §2.1). The standard quiver (the square form) corresponds to

$$B = C \otimes S,$$

where \( C \) is the Cartan matrix of $G$, \( S \) is the modular $S$–matrix.

The square quiver is represented (for $G = SU(6)$) in figure 1; it is supplemented by a quartic superpotential $W$ [10,11]. The charge vector of the a–th simple root $W$–boson is equal to \( \delta_a \equiv \alpha_a^{(1)} + \alpha_a^{(2)} \), i.e. the a–th simple–root $W$ bosons corresponds to the $\mathbb{P}^1$–family of bricks associated with the minimal imaginary root of the a–th $\tilde{A}(1,1)$ affine subquiver $\|_a$. The a–th magnetic charge (weight) is (cfr. eqn. (9))

$$m_a(X) = \dim X_{\alpha_a^{(1)}} - \dim X_{\alpha_a^{(2)}}.$$ 

From the discussion around eqn. (7), the light subcategory $\mathcal{L}_\text{YM}(G)$ containing the perturbative BPS spectrum is then given by the modules $X \in \text{mod} J(Q,W)$ with $m_a(X) = 0$ such that all their submodules $Y$ satisfy $m_a(Y) \leq 0, \forall a$. 

![Figure 1. The square form of the quiver for pure SU(6) SYM](image)
We may break $G \to SU(2)_a \times U(1)^{r-1}$ at weak coupling and describe the Higgs mechanism perturbatively; that is, the gauge breaking should respect the light subcategory. Mathematically, this gives the following result at the level of Abelian categories of modules

$$X \in \mathcal{L}^{\text{YM}}(G) \Rightarrow X|_{\mathfrak{m}_a} \in \mathcal{L}^{\text{YM}}(SU(2)) \quad \forall a,$$

which may be checked directly. Then, if $\mathfrak{m}_a$ is indecomposable, in each Kronecker subquiver $\mathfrak{m}_a$ we may set one of the arrows to 1 with the result that the category $\mathcal{L}^{\text{YM}}(G)$ gets identified with the category of modules of a Jacobian algebra

$$\mathcal{L}^{\text{YM}}(G) = \text{mod} \mathcal{I}(Q', \mathcal{W}')$$

where the reduced quiver $Q'$ is the double of the Dynkin graph $G$ with loops $A_v$ attached at the nodes (i.e. the 'N = 2 quiver' of $G$), see figure 2 for the $SU(6)$ example. The reduced quiver $Q'$ is equipped with the superpotential

$$\mathcal{W}' = \sum_{a: \text{edges} \in G} \text{tr}(\tilde{\psi}_a A_{t(a)} \psi_a - \psi_a A_{h(a)} \tilde{\psi}_a).$$

Given a module $X \in \text{mod} \mathcal{I}(Q', \mathcal{W}')$, consider the linear map

$$\ell: (X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_n}) \mapsto (A_1 X_{\alpha_1}, A_2 X_{\alpha_2}, \cdots, A_r X_{\alpha_n}).$$

It is easy to check that $\ell \in \text{End} X$. Hence $X$ a brick $\Rightarrow A_i = \lambda \in \mathbb{C}$ for all $i$ (in fact, $\lambda \in \mathbb{P}^1$). Fixing $\lambda \in \mathbb{P}^1$, the brick $X$ is identified with a brick of the double $\overline{G}$ of the Dynkin graph $G$.

$$\overline{A_5}$$

subjected to relations

$$\sum_{t(a)=v} \psi_a \tilde{\psi}_a - \sum_{h(a)=v} \tilde{\psi}_a \psi_a = 0.$$
The algebra defined by the double quiver $G$ with the relations \((17)\) is known as the Gelfand–Ponomarev preprojective algebra of the graph $G$, written $\mathcal{P}(G)$ \((17)\). There are three basic results on the preprojective algebra of a graph $L$:

- Gelfand and Ponomarev \((17)\): $\dim \mathcal{P}(L) < \infty$ if and only if $L$ is an ADE Dynkin graph;
- Crawley–Boevey \((18)\): Let $C_L = 2 - I_L$ be the Cartan matrix of the graph $L$. Then for all $X \in \text{mod} \mathcal{P}(L)$

\[
2 \dim \text{End} X = (\dim X)^t C_L (\dim X) + \dim \text{Ext}^1(X, X)
\]

- Lusztig \((19)\): Let $X$ be an indecomposable module of $\mathcal{P}(L)$ belonging to a family of non–isomorphic ones parameterized by the (Kähler) moduli space $\mathcal{M}(X)$. Then

\[
\dim \mathcal{M}(X) = \frac{1}{2} \dim \text{Ext}^1(X, X).
\]

If $L$ is an ADE graph $G$, the integral quadratic form $v^t C_G v$ is positive–definite and even; then $X \neq 0$ implies $(\dim X)^t C_L (\dim X) \geq 2$ with equality if and only if $\dim X$ is a positive root of $G$. From eqns.\((18)-(19)\) it follows that if $X$ is a brick of $\mathcal{P}(G)$ it must be rigid with $\dim X$ a positive root of $G$. Going back to $\mathcal{L}^\text{YM}(G)$, we see that a module in the light category is a brick iff $\dim X$ is a positive root of $G$ and $\mathcal{M}(X) = \mathbb{P}^1$. By the dictionary between physics and Representation Theory, this means that the BPS states which are stable and have bounded mass as $\mathcal{L}^\text{YM} \rightarrow 0$ are vector–multiplets in the adjoint of the gauge group $G$. In fact, a more detailed analysis shows \((11)\) that there is precisely one copy of the adjoint in each weakly coupled BPS chamber. This is, clearly, the result expected for pure SYM at weak coupling; in particular, is shows that the identification \((6)\) of $(Q, \mathcal{W})$ is correct.

3.3. Example 3: SQCD with $G$ simply–laced and $N_a$ quarks in the $a$–th fundamental representation. We consider $\mathcal{N} = 2$ SQCD with a simply–laced gauge group $G = ADE$ coupled to $N_a$ full hypermultiplets in the representation $F_a$ with Dynkin label $[0, \cdots, 0, 1, 0, \cdots, 0]$ (1 in the $a$–th position, $a = 1, 2, \ldots, r$). The prescription for the quiver is simple \((10)\): one replaces the $a$–th Kronecker subquiver $\blacktriangledown_a$ of the pure $G$ SYM quiver (cfr. §3.2) as follows

and replaces the pure SYM superpotential $W_{\text{SYM}}$ with

\[
W \rightarrow W_{\text{SYM}} + \sum_{i=1}^{N_i} \text{tr} [(\alpha_i A_a - \beta_i B_a) \phi_i \bar{\phi}_i],
\]

\[
(\alpha_i : \beta_i) \equiv \lambda_a \in \mathbb{P}^1 \text{ pairwise distinct}.
\]

The exchange matrix of the resulting quiver, $B$, has $N_i$ zero eigenvalues corresponding to the $N_a$ flavor charges carried by the quarks. Formally \((10)\), we may extend this construction to the case in which we have quarks in several distinct
fundamental representations, just be applying the substitutions (20)(21) to all the corresponding Kronecker subquivers of the (square) pure SYM quiver.

Going through the same steps as in § 3.2, one sees that the light category \( L = \text{mod} J(Q', W') \) with \( Q' \) the double of the graph \( G[a, N_a] \) obtained by adding \( N_a \) extra nodes to the Dynkin graph \( G \) connected with a single edge to the \( a \)-th node of \( G \) and having loops only at all ‘old’ nodes of \( G \) (see figure 3 for a typical example) and superpotential

\[
W' = W'_{\text{SYM}} + \sum_i \text{tr}[(\alpha_i A_a - \beta_i)\phi_i \tilde{\phi}_i].
\]

As in § 3.2 X is a brick \( \Rightarrow A_i = \lambda \in \mathbb{P}^1 \). Now we have two distinct cases:

1. \( \lambda \) is generic (i.e. \( \lambda \neq \lambda_i, i = 1, 2, \ldots, N_a \)): the Higgs fields \( \phi_i, \tilde{\phi}_i \) are massive and may be integrated out. Then \( X \) is a brick of \( \mathcal{P}(G) \) and its charge vector \( \text{dim} \, X \) is a positive root of \( G \). These are the same representations as for the light category of pure SYM and they correspond to \( W^{-}\)-bosons in the adjoint of \( G \);

2. \( \lambda = \lambda_a \), then \( X \) is a brick of the preprojective algebra \( \mathcal{P}(G[i, 1]) \). Right properties (finitely many, rigid, in right reprs. of \( G \)) if and only if \( G[i, 1] \) is also a Dynkin graph.

By comparison one gets the following \([11] \):

**Theorem.** (1) Consider \( N = 2 \) SYM with simple simply–laced gauge group \( G \) coupled to a hyper in a representation of the form \( F_a = [0, \ldots, 0, 1, 0, \ldots, 0] \). The resulting QFT is Asymptotically Free if and only if the augmented graph \( G[a, 1] \) obtained by adding to the Dynkin graph of \( G \) an extra node connected by a single edge to the \( a \)-th node of \( G \) is also an ADE Dynkin graph. (2) The model has a Type IIB engineering iff, in addition, the extra node is an extension node in the extended (affine) augmented Dynkin graph \( \hat{G}[a, 1] \).

See figure 4 for the full list of asymptotically free theories of this class. Note that in case (2) the light category automatically contains hypermultiplets in the right representation of \( G \) since, if \( a \) is an extension node in \( \hat{G}[a, 1] \) we have

\[
\text{Ad}(G[a, 1]) = \text{Ad}(G) \oplus [0, \ldots, 0, 1, 0, \ldots, 0] \oplus [0, \ldots, 0, 1, 0, \ldots, 0] \oplus \text{singlets}.
\]
Figure 4. The augmented graphs $G[a, 1]$ corresponding to pairs of gauge group $G = ADE$ and fundamental representation which give an asymptotically free $\mathcal{N} = 2$ gauge theory.
Besides those in figure 4 there is another asymptotically free pair (group, representation), namely $SU(N)$ with the two–index symmetric representation (which is not fundamental) whose augmented graph is identified with the non–simply–laced Dynkin graph of type $B_N$.

3.4. Example 4: $G$ non–simply–laced. The Dynkin graph of a non–simply laced Lie group $G$ arises by folding a parent simply–laced Dynkin graph $G_{\text{parent}}$ along an automorphism group $U$. Specifically, the $G_{\text{parent}} \rightarrow G$ foldings are

\[
\begin{align*}
D_{n+1} & \rightarrow B_n & A_{2n-1} & \rightarrow C_n \\
D_4 & \rightarrow G_2 & E_6 & \rightarrow F_4 .
\end{align*}
\]

$U = \mathbb{Z}_2$ in all cases except for $D_4 \rightarrow G_2$ where it is $\mathbb{Z}_3$. To each node of the folded Dynkin diagrams there is attached an integer $d_a$, namely the number of nodes of the parent graph which were folded into it. This number corresponds to one–half the length–square of the corresponding simple co–root $\alpha_a$

\[
d_a = \frac{1}{2}(\alpha_a^\vee,\alpha_a^\vee) \equiv \frac{2}{(\alpha_a,\alpha_a)} \quad a = 1, 2, \ldots, r .
\]

In general, the light category of a (quiver) $\mathcal{N} = 2$ gauge theory with group $G$ has the structure

\[
\mathcal{L} = \bigvee_{\lambda \in \mathbb{P}^1/U} \mathcal{L}_\lambda
\]

with $U$ acting on the category $\mathcal{L}_\lambda$ through monodromy functors $\mathcal{M}_u$

\[
\mathcal{L}_{u,\lambda} = \mathcal{M}_u(\mathcal{L}_\lambda) \quad u \in U .
\]

Since the cylinder $\mathbb{C}^* \subset \mathbb{P}^1$ is identified with the Gaiotto plumbing cylinder associated to the gauge group $G$, this monodromical construction is equivalent to the geometric realization of the non–simply–laced gauge groups in the Gaiotto framework or in F–theory. In the simply–laced case the light category was described in terms of the preprojective algebra of $G$; likewise, to each gauge group $G = BCFG$ we may associate a generalized ‘preprojective’ algebra of the form $\mathcal{J}(Q',W')$. $Q'$ is the same reduced quiver as in the $A_r$ case (see figure 2 for the $r = 5$ example) while the reduced superpotential is

\[
\mathcal{W} = \sum_{\alpha \rightarrow \beta} \left( \alpha A_{n(\alpha)}^{\alpha} \alpha^* - \alpha^* A_{m(\alpha)}^{\alpha} \alpha \right) ,
\]

where the sum is over the edges $a \xrightarrow{\alpha} b$ of $A_r$ and

\[
(n(\alpha), m(\alpha)) = \left( \frac{d_a}{(d_a, d_b)}, \frac{d_b}{(d_a, d_b)} \right) .
\]

One checks that mod $\mathcal{J}(Q',W')$ has the monodromic property and the dimension vectors of its bricks are the positive roots of $G$, so that the light category corresponds to vector multiplets forming a single copy of the adjoint of $G$, as required for pure SYM. From the light subcategory mod $\mathcal{J}(Q',W')$ one reconstructs the full non–perturbative Abelian category mod $\mathcal{J}(Q,W)$, which describes the model in all physical regimes, by using the Dirac integrality conditions described in §2.1. See ref. 13 for details.
4. Half–hypers

4.1. Coupling full hypermultiplets to SYM. The construction of the pairs \((Q_{N_f}, W_{N_f})\) for \(G = ADE\) SQCD coupled to \(N_f\) fundamental full hypermultiplets of refs. \cite{10,11} was relatively easy: each hypermultiplet has a gauge invariant mass \(m_i\), and taking the decoupling limit \(m_i \to \infty\) we make \(N_f \to N_f - 1\). At the level of modules categories this decoupling processes insets

\[
\text{mod } \mathcal{J} (Q_{N_f} - 1, W_{N_f} - 1) \to \text{mod } \mathcal{J} (Q_{N_f}, W_{N_f})
\]

as an extension–closed, exact, full, controlled Abelian subcategory \cite{11}. In general, a control function is a linear map \(\eta: \Gamma \to \mathbb{Z}\), and the controlled subcategory is the full subcategory over the objects \(X\) such that \(\eta(X) = 0\) while for all their subobjects \(\eta(Y) \leq 0\). The light subcategory is an example of controlled one with control function the magnetic charge. All decoupling limits of QFT correspond to controlled subcategories in the RT language.

For the decoupling limit \(m_i \to \infty\) the control function \(f_i: \Gamma \to \mathbb{Z}\) corresponds to the flavor charge dual to \(m_i\). Choosing \(f_i\) so that \(f_i(\Gamma_+) \geq 0\), we realize \(Q_{N_f} - 1\) as a full subquiver of \(Q_{N_f}\) missing one node, the functor \(\text{mod } \mathcal{J} (Q_{N_f} - 1, W_{N_f} - 1) \to \text{mod } \mathcal{J} (Q_{N_f}, W_{N_f})\) being the restriction. This gives a recursion relation in \(N_f\) of the form

\[
Q_{N_f} - 1 = Q_{N_f} Q_{N_f - 1}
\]

where the blue node in the right corresponds to the controlling flavor charge \(f_i\). By repeated use of this relation, we eventually get to pure \(G\) SYM whose quiver is known, see §3.2. The decoupling process may be easily inverted to get a recursive map \(Q_{N_f} - 1 \to Q_{N_f}\). Indeed, to define such a map we have only to determine the red arrows in eqn.\((32)\) which connect \(Q_{N_f} - 1\) to the extra (blue) node in the RHS of \((32)\) which corresponds to an additional massive quark. Given the electric weight (i.e. the \(G\)–representation) of the added quark, \(\omega\), the red arrows are uniquely determined by the Dirac pairing of \(\omega\) with the charges associated with the nodes of \(Q_{N_f} - 1\).

This strategy does not work for SYM coupled to half–hypermultiplets: they carry no flavor symmetry, have no mass parameter. They are tricky theories, always on the verge of inconsistency: most of them are indeed quantum inconsistent, but there are a few consistent models which owe their existence to peculiar ‘miracles’. The typical example being \(G = E_7\) SYM coupled to half a 56.

4.2. Coupling half hypermultiplets. We use yet another decoupling limit: extreme Higgs. Given a \(\mathcal{N} = 2\) gauge theory with group \(G_r\), of rank \(r\), we take a
v.e.v. of the adjoint field $\langle \Phi \rangle \in \mathfrak{h}$ such that

$$
\alpha_b(\langle \Phi \rangle) = \begin{cases} 
t e^{i\phi}, & t \to +\infty, 
b = a 
O(1) & \text{otherwise}
\end{cases}
$$

(33)

States having electric weight $\rho$ such that $\rho(\langle \Phi \rangle) = O(t)$ decouple, and we remain with a gauge theory with a gauge group $G_{r-1}$ whose Dynkin diagram is obtained by deleting the $a$–th node from that of $G_r$ (coupled to specific matter). E.g. starting from $G_7 = E_7$ coupled to $\frac{1}{2} 56$ and choosing $a = 1$ we get $G_6 = \text{Spin}(12)$ coupled to $\frac{1}{2} 32$ corresponding to deleting the black node in the Dynkin graph.

(34)

Again, the decoupling limit should correspond to a controlled Abelian subcategory of the representations of $(Q_{G_r}, W_{G_r})$. One can choose $(Q_{G_r}, W_{G_r})$ in its mutation–class and the phase $\phi$ in (33) so that the control function $\lambda(\cdot)$ is non–negative on the positive–cone $\Gamma_+$. Then $Q_{G_{r-1}}$ is a full subquiver of $Q_{G_r}$ and $W_{G_{r-1}}$ is just the restriction of $W_{G_r}$. It is easy to see that the complementary full subquiver is a two–nodes Kronecker one $\uparrow \downarrow$. Putting everything together, we get a recursion of the quiver with respect to the rank $r$ of $G_r$ of the form

(35)

If we know the simpler quiver $Q_{G_{r-1}}$, to get $Q_{G_r}$ we need just the fix the red arrows connecting the Kronecker to $Q_{G_{r-1}}$ in the above figure. Just as in §4.1, the red arrows are uniquely fixed by Dirac charge quantization. Indeed, by the recursion assumption, we know the representations $X_{\alpha}$ associated to all simple–root $W$–bosons of $G_r$; under the maximal torus $U(1)^r \subset G$ the simple–root $W$–bosons have charges $q_a(X_{\alpha}) = C_{ab}$ (Cartan matrix), while the dual magnetic charges are given by eqn.(6) which explicitly depends on the red arrows. It turns out that $m_a(X) \in \Gamma_{\text{root}}$ for all $X$ for a unique choice of the arrows which are then fixed. Then $Q_{G_r}$ is uniquely determined if we know $Q_{G_{r-1}}$. $W_{G_r}$ is also essentially determined, up to some higher–order ambiguity $\uparrow \downarrow$. Taking a suitable chain of such Higgs decouplings/symmetry breakings

(36)

$$
G_r \to G_{r-1} \to G_{r-2} \to \cdots \to G_k,
$$

we eventually end up with a complete $\mathcal{N} = 2$ with gauge group $G_k = SU(2)^k$. The complete $\mathcal{N} = 2$ quivers are known by classification $\uparrow \downarrow$. Inverting the Higgs procedure, we may construct the pair $(Q_{G_r}, W_{G_r})$ for the theory of interest by ‘pulling back’ through the chain $\uparrow \downarrow$ the pair $(Q_{\text{max comp}}, W_{\text{max comp}})$ of their maximal
complete subsector. For the models of interest the ‘pull back’ chain is presented in figure 5. The bottom model $SU(2)^3$ with $\frac{1}{2}(2, 2, 2)$ is complete \[7\] \[11\].

The pair $(Q_{E_7}, W_{E_7})$ for the model $G = E_7$ coupled to $\frac{1}{2}56$ is given in figure 6; the other models in figure 5 correspond to the restriction to suitable subquivers of $(Q_{E_7}, W_{E_7})$ \[12\]. The light category deduced from these pairs contains light vectors forming one copy of the adjoint of $G$ plus light hypermultiplets in the $G$–representation $\frac{1}{2} R$, with $R$ irreducible quaternionic \[12\]. Indeed, the light category has again the form $\text{mod } J(Q', W')$ for a reduced pair $(Q', W')$. See figure 7 for the the reduced pair for $G = E_7$ coupled to $\frac{1}{2}56$; the other models are obtained by restriction of this one. Note that $Q'_{E_7}$ (and hence all reduced quivers $Q'_{G_r}$ in the Higgs chain) contains as a full subquiver the quiver of the Gaiotto $A_1$ theory on $S^2$ with 3 punctures (the $T_2$ theory) described in \[10\]. Hence for all these models the ‘$T_2$–duality’ of §1.1 is operative; this duality is crucial — together with special properties of the relevant Dynkin graphs — to check the above claims on the BPS spectrum at weak coupling. Details may be found in \[12\].
Figure 6. Quiver and superpotential for the $\mathcal{N} = 2$ $E_7$ SYM coupled to $\frac{1}{2}56$ quark.

Figure 7. Reduced pair of the light category $\mathcal{Z}_{E_7} \equiv \text{mod } J(Q', \mathcal{W}')$ for $E_7$ SYM with $\frac{1}{2}56$ quark.
References

[1] D. Gaiotto, G. W. Moore, and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” arXiv:0907.3987 [hep-th].
[2] D. Gaiotto, G. W. Moore, and A. Neitzke, “Framed BPS States,” arXiv:1006.0146 [hep-th].
[3] D. Gaiotto, G. W. Moore, and A. Neitzke, “Wall–Crossing in Coupled 2d–4d Systems,” arXiv:1103.2598 [hep-th].
[4] D. Gaiotto, G. W. Moore, and A. Neitzke, “Spectral networks,” arXiv:1204.4824 [hep-th].
[5] D. Gaiotto, G. W. Moore, and A. Neitzke, “Spectral networks and Snakes,” arXiv:1209.0866 [hep-th].
[6] S. Cecotti, A. Neitzke, and C. Vafa, “R-Twisting and 4d/2d Correspondences,” arXiv:1006.3435 [hep-th].
[7] S. Cecotti and C. Vafa, “Classification of complete N=2 supersymmetric theories in 4 dimensions,” in SURVEYS IN DIFFERENTIAL GEOMETRY volume 18 (2013) pages 19–101, arXiv:1103.5832 [hep-th].
[8] S. Cecotti and M. Del Zotto, On Arnold’s 14 ‘exceptional’ N = 2 superconformal gauge theories, JHEP 1110 (2011) 099, arXiv:1107.5747 [hep-th].
[9] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, “BPS Quivers and Spectra of Complete N=2 Quantum Field Theories,” Comm. Math. Phys. 323 (2013) 1185–1227, arXiv:1109.4941 [hep-th].
[10] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, “N = 2 Quantum Field Theories and their BPS Quivers,” Advances in Theor. Math. Phys. 18 (2014) 27–127, arXiv:1112.3984 [hep-th].
[11] S. Cecotti, “Categorical tinkertoys for N = 2 gauge theories”, Int. J. Mod. Phys. A 28 (2013) 1330006, arXiv:1203.6743 [hep-th].
[12] S. Cecotti and M. Del Zotto, “Half–Hypers and Quivers,” JHEP 09 (2012) 135, arXiv:1207.2275 [hep-th].
[13] S. Cecotti and M. Del Zotto, “4d N=2 Gauge Theories and Quivers: the Non-Simply Laced Case,” JHEP 10 (2012) 190, arXiv:1207.7205 [hep-th].
[14] S. Cecotti and M. Del Zotto, “Infinitely many N=2 SCFT with ADE flavor symmetry,” JHEP 01 (2013) 191, arXiv:1210.2886 [hep-th].
[15] H. Derksen, J. Wyman, and A. Zelevinsky, “Quivers with potentials and their representations I. Mutations,” Selecta Mathematica 14 (2008) 59–119.
[16] D. Gaiotto, “N=2 dualities,” arXiv:0904.2715 [hep-th].
[17] I.M. Gelfand and V.A. Ponomarev, “Model algebras and representations of graphs,” Funktsional. Anal. i Prilozhen. 13 (1979) 1–12.
[18] W. Crawley–Boevey, “On the exceptional fibres of Kleinian singularities,” Amer. J. Math. 122 (2000) 1027–1037.
[19] G. Lusztig, “Quivers, perverse sheaves, and quantized enveloping algebras,” J. Amer. Soc. 4 (1991) 313–324;
[20] Y. Tachikawa, “N=2 S-duality via outer–automorphism twists”, arXiv:1009.0339 [hep-th].
[21] M. Bershadsky, K.A. Intriligator, S. Kachru, D.R. Morrison, V. Sadov, and C. Vafa, “Geometric singularities and enhanced gauge symmetries,” Nucl. Phys. B481 (1996) 215 arXiv:hep-th/9605200 [hep-th].

Scuola Internazionale di Studi Avanzati, via Bonomea 265, I-34100 Trieste, ITALY
E-mail address: cecotti@sissa.it