BRANCHING OF PERIODIC ORBITS IN REVERSIBLE
HAMILTONIAN SYSTEMS

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Abstract. This paper deals with the dynamics of time-reversible Hamiltonian vector fields with 2 and 3 degrees of freedom around an elliptic equilibrium point in presence of symplectic involutions. The main results discuss the existence of one-parameter families of reversible periodic solutions terminating at the equilibrium. The main techniques used are Birkhoff and Belitskii normal forms combined with the Liapunov-Schmidt reduction.

1. Introduction

The resemblance of dynamics between reversible and Hamiltonian contexts, probably first noticed by Poincaré and Birkhoff, has caught much attention since the sixties of the twentieth century. Since then many important results, e.g. KAM theory, Liapunov center theorems, etc, holding in the Hamiltonian context have been carried over to the reversible one (see [13, 20] and reference therein).

The concept of reversibility is linked with an involution $R$, i.e., a map $R : \mathbb{R}^N \to \mathbb{R}^N$ such that $R \circ R = Id$. Let $X$ be a smooth vector field on $\mathbb{R}^N$. The vector field is called $R$–reversible if the following relation is satisfied

$$X(R(x)) = -DR_x.X(x).$$

Reversibility means that $x(t)$ is a solution of $X$ if and only if $Rx(-t)$ is also a solution. The set $Fix(R) = \{ x \in \mathbb{R}^N : R(x) = x \}$ plays an important role in the reversible systems. We say that a singular point $p$ is symmetric if $p \in Fix(R)$, and analogously we say that an orbit $\gamma$ is symmetric if $R(\gamma) = \gamma$.

Many dynamical systems that arise in the context of applications possess robust structural properties, such as for instance symmetries or Hamiltonian structure. In order to understand the typical dynamics of such systems, their structure need to be taken into account, leading one to study phenomena that are generic among dynamical systems with the same structure. In the last decade there has been a surging interest in the study of systems with time-reversal symmetries (see [18] and [11]). Symmetry properties arise naturally and frequently in dynamical systems. In recent years, a lot of

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attention has been devoted to understand and use the interplay between
dynamics and symmetry properties. It is worthwhile to mention that one of
the characteristic properties of Hamiltonian and reversible systems is that
minimal sets appear in one-parameter families. So a number of natural
questions can be formulated, such as: (i) how do branches of such minimal
sets terminate or originate?; (ii) can one branch of minimal sets bifurcate
from another such branch?; (iii) how persistent is such branching process
when the original system is slightly perturbed? Recently, there has been
increased interest in the study of systems with time-reversal symmetries
and we refer [13] for a survey in reversible systems and related problems.

Our main concern, in this article, is to find conditions for the existence
of one-parameter families of periodic orbits terminating at the equilibrium.

We present some relevant historical facts. In 1895 Liapunov published
his celebrated center theorem, see Abraham and Marsden [1] p 498; This
theorem, for analytic Hamiltonians with \( n \) degrees of freedom, states that
if the eigenfrequencies of the linearized Hamiltonian are independent over
\( \mathbb{Z} \), near a stable equilibrium point, then there exists \( n \) families of periodic
solutions filling up smooth 2-dimensional manifolds going through the equi-
librium point. Devaney [6] proved a time-reversible version of the Liapunov
center theorem. Recently this center theorem has been generalized to equi-
variant systems, by Golubitsky, Krupa and Lim [7] in the time-reversible
case, and by Montaldi, Roberts and Stewart [16] in the Hamiltonian case.
We recall that in [7] the Devaney’s theorem was extended and some extra
symmetries were considered. Contrasting Devaney’s geometrical approach,
they used Liapunov-Schmidt reduction, adapting an alternative proof of the
reversible Liapunov center theorem given by Vanderbauwhede [19]. In [16]
the existence of families of periodic orbits around an elliptic semi-simple
equilibrium is analyzed. Systems with symmetry, including time-reversal
symmetry, which are anti-symplectic are studied. Their approach is a con-
tinuation of the work of Vanderbauwhede, in [19], where the families of
periodic solutions correspond bijectively to solutions of a variational prob-
lem.

Recently Buzzi and Teixeira in [3] have analyzed the dynamics of time-
reversible Hamiltonian vector fields with 2 degrees of freedom around an
elliptic equilibrium point in presence of \( 1 : -1 \) resonance. Such systems ap-
pear generically inside a class of Hamiltonian vector fields in which the sym-
plectic structure is assumed to have some symmetric properties. Roughly
speaking, the main result says that under certain conditions the original
Hamiltonian \( H \) is formally equivalent to another Hamiltonian \( \tilde{H} \) such that
the corresponding Hamiltonian vector field \( \mathcal{X}_\tilde{H} \) has two Liapunov families
of symmetric periodic solutions terminating at the equilibrium. It is worth
while to say that all the systems considered there have been derived from
the expression of Birkhoff normal form.
In this paper we address the problem to systems with 2 and 3 degrees of freedom. Physical models of such systems were exhibited in [5, 12]. As usual the main proofs are based on a combined use of normal form theory and the Liapunov-Schmidt Reduction. It is important to mention that our results concerning the existence of Liapunov families generalize those in [3]. As a matter of fact we deal with $C^\infty$ or $C^\omega$.

We begin in Section 2 with an introduction of the terminology and basic concepts for the formulation of our results. In Section 3 the Belitskii normal form is discussed. In Section 4 the Liapunov-Schmidt reduction is presented. In Section 5 the usefulness of Birkhoff normal form in our approach is pointed out. In Section 6 we study the Hamiltonian with 2 degrees of freedom denoted by $\Omega^0$, and we denote by $\Omega^0_B$ the set of vector fields in $\Omega^0$ that satisfy the Birkhoff Condition and by $\Omega^0_\omega$ the vector fields in $\Omega^0$ that are analytic. We generalize some results presented in [3] by proving Theorem A. That result says that there exists an open set $U^0 \subset \Omega^0_B$ (respec. $\Omega^0_\omega$), in the $C^\infty$–topology, such that (a) $U^0$ is determined by the 3–jet of the vector fields; and (b) each $X \in U^0$ possesses two 1–parameter families of periodic solutions terminating at the equilibrium. In Section 7 we study the Hamiltonian with 3 degrees of freedom, and we prove Theorems B and C. In Theorem B we consider the involution associated to the system satisfying $\dim(\text{Fix}(R)) = 2$, and in Theorem C satisfying $\dim(\text{Fix}(R)) = 4$. We denote these spaces of reversible Hamiltonian vector fields by $\Omega^1$ and $\Omega^2$, respectively. Again $\Omega^2_B$ is the set of vector fields in $\Omega^2$ that satisfy the Birkhoff Condition and $\Omega^2_\omega$ is the set of vector fields in $\Omega^2$ that are analytic. The conclusions are the following: In Theorem B there exists an open set $U^1 \subset \Omega^1$, in the $C^\infty$–topology, such that (a) $U^1$ is determined by the 2–jet of the vector fields, and (b) for each $X \in U^1$ there is no periodic orbit arbitrarily close to the equilibrium. In Theorem C there exists an open set $U^2 \subset \Omega^2_B$ (respec. $\Omega^2_\omega$), in the $C^\infty$–topology, such that (a) $U^2$ is determined by the 3–jet of the vector fields, and (b) each $X \in U^2$ has infinitely many one–parameter family of periodic solutions terminating at an equilibrium with the periods tending to $2\pi/\alpha$. In Section 8 we present an example that satisfies the hypotheses of the Theorem A and commented on that is possible to accomplish the vector fields of Theorem C.

2. Preliminaries

Now we introduce some of the terminology and basic concepts for the formulation of our results.

We consider (germs of) smooth functions $H : \mathbb{R}^{2n}, 0 \to \mathbb{R}$ having the origin as an equilibrium point. The corresponding Hamiltonian vector field, to be denoted by $X_H$, has the origin as an equilibrium or singular point. We recall that $dH = \omega(X_H, \cdot)$, where $\omega = dx_1 \land dy_1 + dx_2 \land dy_2 + \cdots + dx_n \land dy_n$
denotes the standard 2-form on $\mathbb{R}^{2n}$. In coordinates $X_H$ is expressed as:

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}; \quad i = 1, \cdots, n.$$  

In $\mathbb{R}^6$ we have

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\vdots \\
\dot{x}_3 \\
\dot{y}_3
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H}{\partial x_1} \\
\frac{\partial H}{\partial y_1} \\
\vdots \\
\frac{\partial H}{\partial x_3} \\
\frac{\partial H}{\partial y_3}
\end{pmatrix}.
\]

Here,

\[J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}\]

is the symplectic structure associated with the 2-form $\omega$ given above.

We say that an involution is \textit{symplectic} when it satisfies the equation $\omega(DR_p(v_p), DR_p(w_p)) = \omega(v_p, w_p)$. If the involution $R$ is linear then this definition is equivalent to $JR = R^TJ$, where $J$ is the symplectic structure and $R^T$ is the transpose matrix of $R$.

The next proposition exhibits normal forms for linear symplectic involutions on $\mathbb{R}^6$.

**Proposition 2.1.** Given the symplectic structure $\omega$ and an involution $R$ there exists a symplectic change of coordinates that transforms $R$ in one of the following normal forms

1. $R_0 = Id$,
2. $R_0(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, x_2, y_2, -x_3, -y_3)$,
3. $R_0(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, -x_2, -y_2, -x_3, -y_3)$,
4. $R_0 = -Id$.

Before giving the proof we observe that the mapping $\psi = (1/2)(R + L)$, where $L = DR(0)$, is a symplectic conjugacy between $R$ and $L$, i. e., $R \circ \psi = \psi \circ L$. So we may and do assume, without loss of generality, that the involution $R$ is linear.

**Lemma 2.2.** If $R$ is a linear symplectic involution, then we have that $\mathbb{R}^6 = \text{Fix}(R) \oplus \text{Fix}(-R)$ and $\omega(\text{Fix}(R), \text{Fix}(-R)) = 0$.

**Proof:** For every $u \in \mathbb{R}^6$, we can write $u = ((u + R(u))/2) + ((u - R(u))/2)$. Notice that $(u + R(u))/2 \in \text{Fix}(R)$ and $(u - R(u))/2 \in \text{Fix}(-R)$. Now, let $u \in \text{Fix}(R)$ and $v \in \text{Fix}(-R)$, so we have that $\omega(u, v) = \omega(R(u), -R(v))$. 


By using that $R$ is symplectic and $R$ is linear, we have that $\omega(R(u), R(v)) = \omega(u, v)$. So $-\omega(u, v) = \omega(u, v)$, and we have proved that $\omega(\operatorname{Fix}(R), \operatorname{Fix}(-R)) = 0$.

A linear subspace $U \subseteq \mathbb{R}^6$ is symplectic if $\omega$ is non-degenerate in $U$, i.e., if $\omega(u, v) = 0$ for all $u \in U$ then $v = 0$.

**Lemma 2.3.** $\operatorname{Fix}(R)$ and $\operatorname{Fix}(-R)$ are symplectic subspaces.

**Proof:** Suppose $u \in \operatorname{Fix}(R)$ and $u \neq 0$ such that $\omega(u, \operatorname{Fix}(R)) = 0$. By using Lemma 2.2, we have $\omega(\operatorname{Fix}(R), \operatorname{Fix}(-R)) = 0$, so $\omega(u, \operatorname{Fix}(-R)) = 0$. Again by Lemma 2.2 $(\mathbb{R}^6 = \operatorname{Fix}(R) \oplus \operatorname{Fix}(-R))$ we have $\omega(u, \mathbb{R}^6) = 0$ and so $\omega$ is degenerate in $\mathbb{R}^6$ which is not true. Then $\operatorname{Fix}(R)$ is a symplectic subspace. The proof for $\operatorname{Fix}(-R)$ is analogous. □

**Proof of Proposition 2.1:** Let $R : \mathbb{R}^6 \to \mathbb{R}^6$ be a linear involution and $\omega$ be a fixed symplectic structure. From Lemma 2.2, $\mathbb{R}^6 = \operatorname{Fix}(R) \oplus \operatorname{Fix}(-R)$ and as $\operatorname{Fix}(R)$ is a symplectic subspace, then $\dim \operatorname{Fix}(R) = 0, 2, 4, \text{ or } 6$.

- if $\dim \operatorname{Fix}(R) = 0$, then we can find a coordinate system, using Darboux Theorem [10], such that $R_0 = -\text{Id}$;
- if $\dim \operatorname{Fix}(R) = 6$, then we can find a coordinate system, using Darboux Theorem [10], such that $R_0 = \text{Id}$;
- if $\dim \operatorname{Fix}(R) = 4$, we consider the bases $\beta_1 = \{e_1, e_2, e_3, e_4\}$ for $\operatorname{Fix}(R)$ and $\beta_2 = \{f_1, f_2\}$ for $\operatorname{Fix}(-R)$. So $\beta = \{e_1, e_2, e_3, e_4, f_1, f_2\}$ is a basis for $\mathbb{R}^6$. Let us show that $\beta$ can be chosen such that $[\omega]_\beta = J$

and $[R]_\beta = R_0 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$. Here $[\omega]_\beta$ means the matrix of $\omega$ with respect to the basis $\beta$.

Note that $\omega(e_i, e_i) = 0$ and $\omega(f_j, f_j) = 0$, $i = 1, 2, 3, 4$ and $j = 1, 2$. By Lemma 2.2, $\omega(e_i, f_j) = 0$, $i = 1, 2, 3, 4$ and $j = 1, 2$. And as $\omega$ is alternating, then $\omega(f_1, f_2) = 1$ and $\omega(f_2, f_1) = -1$.

Define $\omega(e_i, e_j)$ for $i \neq j$. From Darboux’s Theorem there exists a coordinate system around 0 such that $\omega|_{\beta_1}$ in this coordinate system is the symplectic structure $J$.

- if $\dim \operatorname{Fix}(R) = 2$, in the same way as above, we get

$$R_0 = [R]_\beta = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$
Using the previous proposition we consider the following cases:

6 : 2 Case: \( R_1(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, -x_2, -y_2, -x_3, -y_3) \),

6 : 4 Case: \( R_2(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, y_1, x_2, y_2, -x_3, -y_3) \).

2.1. Linear part of a \( R_j \)-reversible Hamiltonian vector field in \( \mathbb{R}^6 \).

Denote by \( \Omega^j \) the space of all \( R_j \)-reversible Hamiltonian vector field, \( X_{H_j} \), in \( \mathbb{R}^6 \) with 3-degrees freedom where \( H_j \) is the associate Hamiltonian and \( j = 1, 2 \). Fix the coordinate system \((x_1, y_1, x_2, y_2, x_3, y_3) \in (\mathbb{R}^6, 0)\). We endow \( \Omega^j \) with the \( C^\infty \)-topology.

The symplectic structure given by \( J \) is:

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}.
\]

Observe that the involution \( R_j \) is symplectic, i.e. \( J R_j - R_j^T J = 0 \), \( j = 1, 2 \).

As the involution is symplectic, then the vector field is \( R_j \)-reversible if and only if the Hamiltonian function \( H_j \) is \( R_j \)-anti-invariant, \( j = 1, 2 \). This is equivalent to say that \( H_j \circ R_j = -H_j \). (See \[3\])

Define the polynomial function with constant coefficients \( a_k \in \mathbb{R} \):

\[
H_j(x_1, y_1, x_2, y_2, x_3, y_3) = a_{01}x_1^2 + a_{02}x_1y_1 + a_{03}x_1x_2 + a_{04}x_1y_2 + a_{05}x_1x_3 + a_{06}x_1y_3 + a_{07}y_1^2 + a_{08}y_1x_2 + a_{09}y_1y_2 + a_{10}y_1x_3 + a_{11}y_1y_3 + a_{12}x_2^2 + a_{13}x_2y_2 + a_{14}x_2x_3 + a_{15}x_2y_3 + a_{16}y_2^2 + a_{17}y_2x_3 + a_{18}y_2y_3 + a_{19}x_3^2 + a_{20}x_3y_3 + a_{21}y_3^2 + h.o.t.
\]

First of all we impose the \( R_j \)-reversibility on our Hamiltonian system, \( j = 1, 2 \). For each case we have:

a) Case 6 : 2

From the reversibility condition, \( H_1 \circ R_1 = -H_1 \), and

\[
R_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix},
\]

we obtain

\[
H_1 = a_{03}x_1x_2 + a_{04}x_1y_2 + a_{05}x_1x_3 + a_{06}x_1y_3 + a_{08}x_2y_1 + a_{09}y_1y_2 + a_{10}x_3y_1 + a_{11}y_1y_3 + h.o.t.
\]
Then, the linear part of Hamiltonian vector field \( X_{H_1} \) is

\[
A_1 = \begin{pmatrix}
0 & 0 & a & b & c & d \\
0 & 0 & e & f & g & h \\
-f & b & 0 & 0 & 0 & 0 \\
e & -a & 0 & 0 & 0 & 0 \\
-h & d & 0 & 0 & 0 & 0 \\
g & -c & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Just to simplify the notation we replace \( a_03, a_04, a_05, a_06, a_08, a_09, a_{10}, a_{11} \) by \( a, b, c, d, -e, -f, -g, -h \), respectively. Note that \( A_1 \) is \( R_1 \)-reversible (i.e., \( R_1.A_1 + A_1.R_1 = 0 \)). The eigenvalues of \( A_1 \) are \( \{0, 0, \pm \sqrt{be - af + dg - ch}, \pm \sqrt{be - af + dg - ch}\} \). We restrict our attention to those systems satisfying the inequality:

\[
(2.1) \quad be - af + dg - ch < 0.
\]

The case when \( be - af + dg - ch > 0 \) will not be considered because the center manifold of the equilibrium has dimension two with double zero eigenvalue. We shall use the Jordan canonical form from \( A_1 \). So we stay, for while, away from the original symplectic structure. We call \( \alpha = \sqrt{-be + af - dg + ch} \), and so the transformation matrix is

\[
P_1 = \begin{pmatrix}
0 & 0 & -\frac{d}{dg-ch}\alpha & 0 & -\frac{c}{dg-ch}\alpha & 0 \\
0 & 0 & \frac{-h}{dg-ch}\alpha & 0 & \frac{-g}{dg-ch}\alpha & 0 \\
\frac{df-bh}{be-af} & \frac{cf-bg}{be-af} & 0 & \frac{-df+bh}{dg-ch} & 0 & \frac{-cf+bg}{dg-ch} \\
\frac{-de+ah}{be-af} & \frac{-ce+ag}{be-af} & 0 & \frac{de-ah}{dg-ch} & 0 & \frac{ce-ag}{dg-ch} \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

So

\[
\hat{A}_1 = P_1^{-1}.A_1.P_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & -\alpha & 0 \\
\end{pmatrix},
\]

where \( P_1^{-1} \) is the inverse matrix of the matrix \( P_1 \). Moreover, in this way, \( \hat{R}_1 = P_1^{-1}.R_1.P \) takes the form

\[
\hat{R}_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]
b) Case 6 : 4

We proceed in the same way as in the previous case. The involution is

\[ R_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

and the Hamiltonian function in this case takes the form:

\[ H_2 = a_{05}x_1 x_3 + a_{14} x_2 x_3 + a_{10} x_3 y_1 + a_{17} x_3 y_2 + a_{06} x_1 y_3 + a_{15} x_2 y_3 + a_{11} y_1 y_3 + a_{18} y_2 y_3 + h.o.t. \]

Then, the linear part of Hamiltonian vector field \( X_{H_2} \) is expressed by:

\[ A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & a & b \\
0 & 0 & 0 & 0 & c & d \\
0 & 0 & 0 & 0 & e & f \\
0 & 0 & 0 & 0 & g & h \\
-c & b & h & f & 0 & 0 \\
-c & a & g & -e & 0 & 0
\end{pmatrix}. \]

Again we change the notation. The eigenvalues of \( A_2 \) are given by \( \{0, 0, \pm \sqrt{bc - ad + fg - eh}, \pm \sqrt{bc - ad + fg - eh}\} \). We consider the case

\[ (2.2) \quad bc - ad + fg - eh < 0. \]

We call \( \alpha = \sqrt{-bc + ad - fg + eh} \) and consider the transformation matrix

\[ P_2 = \begin{pmatrix}
\frac{bc-ad}{bc-ad} & -\frac{bg+ah}{bc-ad} & 0 & -\frac{b}{\alpha} & 0 & -\frac{a}{\alpha} \\
\frac{bc-ad}{bc-ad} & -\frac{dg+eh}{bc-ad} & 0 & -\frac{d}{\alpha} & 0 & -\frac{e}{\alpha} \\
0 & 1 & 0 & -f & 0 & -\frac{e}{\alpha} \\
1 & 0 & 0 & -\frac{h}{\alpha} & 0 & -\frac{g}{\alpha} \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \]

and the Jordan canonical form of \( A_2 \) is:

\[ \widetilde{A}_2 = P_2^{-1} A_2 P_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & -\alpha & 0
\end{pmatrix}. \]
Moreover, in this way, $\hat{R}_2 = P_2^{-1}R_2P_2$ takes the form
\[
\hat{R}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

3. Belitskii normal form

In this section we present the Belitskii Normal Form. When a vector field is in this normal form we can write explicitly the resultant equation of Liapunov–Schmidt reduction.

Consider a formal vector field expressed by
\[
\hat{X}(x) = Ax + \sum_{k \geq 2} X^{(k)}(x)
\]
where $X^{(k)}$ is the homogeneous part of degree $k$. Let us look for a “simple” form of the formal vector field $\hat{Y} = \hat{\phi} \ast \hat{X}$ by means of formal transformation
\[
\hat{\phi} = x + \sum_{k} \phi^{(k)}(x).
\]

The proof of the next theorem is in [2].

**Theorem 3.1.** Given a formal vector field
\[
\hat{X}(x) = Ax + \sum_{k \geq 2} X^{(k)}(x),
\]
there is a formal transformation $\hat{\phi}(x) = x + \ldots$ bringing $\hat{X}$ to the form $(\hat{\phi} \ast X)(x) = Ax + h(x)$ where $h$ is a formal vector field with zero linear part commuting with $A^T$, i.e
\[
A^T h(x) = h'(x) A^T x,
\]
where $A^T$ is the transposed matrix and $h'$ is the derivative of $h$.

Here we call the normal form $(\hat{\phi} \ast X)(x) = Ax + h(x)$ the Belitskii normal form. By abuse of the terminology, call $X_H = A + h$. 
4. Liapunov–Schmidt reduction

In this section we recall the main features of the Liapunov–Schmidt reduction. As a matter of fact, we adapt the setting presented in [4, 21] to our approach. In this way consider the $R$-reversible system expressed by

\begin{equation}
\dot{x} = X_H(x); \quad x \in \mathbb{R}^6
\end{equation}

satisfying $X_H(Rx) = -RX_H(x)$ with $R$ a linear involution in $\mathbb{R}^6$. Assume that $X_H(0) = 0$ and consider

\begin{equation}
A = D_1X_H(0),
\end{equation}

the Jacobian matrix of $X_H$ in the origin.

In our case the linear part of vector field has the following eigenvalues: 0 with the algebraic and geometric multiplicity 2, and $\pm \alpha i$, also with algebraic and geometric multiplicity 2, $\alpha \in \mathbb{R}$. Performing a time rescaling we may take $\alpha = 1$. We write the real form of the linear part of the vector field $X_J$:

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}.
\]

Let $C^0_{2\pi}$ be the Banach space of $2\pi$-periodic continuous mappings $x : \mathbb{R} \to \mathbb{R}^6$ and $C^1_{2\pi}$ the corresponding $C^1$-subspace. We define an inner product on $C^0_{2\pi}$ by

\[
(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} <x_1(t), x_2(t)> dt
\]

where $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathbb{R}^6$.

The main aim is to find all small periodic solutions of (4.3) with period near $2\pi$.

Define the map $F : C^1_{2\pi} \times \mathbb{R} \to C^0_{2\pi}$ by

\[
F(x, \sigma)(t) = (1 + \sigma)\dot{x}(t) - X_H(x(t)).
\]

Note that if $(x_0, \sigma_0) \in C^1_{2\pi} \times \mathbb{R}$ is such that

\begin{equation}
F(x_0, \sigma_0) = 0,
\end{equation}

then $\tilde{x}(t) := x_0((1 + \sigma_0)t)$ is a $2\pi/(1 + \sigma_0)$-periodic solution of (4.3).

Our task now is to find the zeroes of $F$. Clearly, $(x_0, \sigma_0) = (0, 0)$ is one solution of $F(x_0, \sigma_0) = 0$. Let $L := D_xF(0, 0) : C^1_{2\pi} \to C^0_{2\pi}$; explicitly $L$ is given by

\[
Lx(t) = \dot{x}(t) - Ax(t).
\]
Consider the unique (S-N)-decomposition of $A$, $A = S + N$. Recall that in our case $A$ is semi-simple, i.e., $A = S$. Define the subspace $\mathcal{N}$ of $C^1_{2\pi}$ as

$$\mathcal{N} = \{ q; \dot{q}(t) = Sq(t) \} = \{ q; q(t) = \exp(tS)x; \ x \in \mathbb{R}^6 \}.$$ 

Observe that $\mathcal{N} \subset C^1_{2\pi}$ and a basis for the solutions of $\dot{q} = Sq$ is given by the set $$\{(1, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0), (0, 0, \cos(t), \sin(t), 0, 0), (0, 0, -\sin(t), \cos(t), 0, 0), (0, 0, 0, 0, \cos(t), \sin(t)), (0, 0, 0, 0, -\sin(t), \cos(t))\}.$$ 

In order to study certain properties of the operator $L$ we introduce $\mathcal{N} \subset C^1_{2\pi}$ and the following definitions and notations.

We will put the solution of $F(x_0, \sigma_0) = 0$ in one-to-one correspondence with the solutions of an appropriate equation in $\mathcal{N}$. Define the subspaces

$$X_1 = \{ x \in C^1_{2\pi} : (x, \mathcal{N}) = 0 \}$$

and

$$Y_1 = \{ y \in C^0_{2\pi} : (y, \mathcal{N}) = 0 \}$$

as the orthogonal complements of $\mathcal{N}$ in $C^1_{2\pi}$ and $C^0_{2\pi}$, respectively.

Let $(q_1, q_2, q_3, q_4, q_5, q_6)$ with $q_i = \exp(tS)u_i$ where $u_i$, $i = 1, ..., 6$, is a basis for $\mathbb{R}^6$. Then we define a projection

$$\mathcal{P} : C^0_{2\pi} \to C^0_{2\pi}$$

by

$$\mathcal{P} = \sum_{i=1}^{6} q_i^*(\cdot) q_i \in \mathcal{L}(C^0_{2\pi})$$

with $q_i^*(x) = (q_i, x)$.

We have $\text{Im}(\mathcal{P}) = \mathcal{N}$ and $\text{Ker}(\mathcal{P}) = Y_1$. Hence,

$$C^1_{2\pi} = X_1 \oplus \mathcal{N}, \ C^0_{2\pi} = Y_1 \oplus \mathcal{N}.$$ 

Now we consider

$$F(x, \sigma) = F(q + x_1, \sigma) =: \hat{F}(q, x_1, \sigma); \ q \in \mathcal{N}, \ x_1 \in X_1.$$ 

The proof of next result can be found in [9].

**Lemma 4.1. (Fredholm’s Alternative)** Let $A(t)$ be a matrix in $C^0_T$ and let $f$ be in $C_T$. Here $C^0_T$ is the space of the matrices with entries continuous and $T$-periodic, and $C_T$ is the set of $T$-periodic maps from $\mathbb{R}$ to $\mathbb{R}^n$. Then the equation $\dot{x} = A(t)x + f(t)$ has a solution in $C_T$ if, and only if,

$$\int_0^T \langle y(t), g(t) \rangle dt = 0$$

for all solution $y$ of the adjoint equation

$$\dot{y} = -yA(t)$$

such that $y^T \in C_T$. 
As \( L(\mathcal{N}) \subset \mathcal{N} \) this lemma implies the following:

**Lemma 4.2.** The mapping \( \hat{L} := L|_{X_1} : X_1 \to Y_1 \) is bijective.

Let us study the solutions of \( \hat{F}(q, x_1, \sigma) = 0 \). These solutions are equivalent to the solutions of the system

\[
(I - P) \circ \hat{F}(q, x_1, \sigma) = 0, \\
P \circ \hat{F}(q, x_1, \sigma) = 0.
\]

With Lemma 4.2 and the Implicit Function Theorem we can solve the first equation as \( x_1 = x_1^*(q, \sigma) \). Then, (4.5) is reduced to

\[
\tilde{F}(q, \sigma) := P \circ \hat{F}(q, x_1^*(q, \sigma), \sigma) = 0.
\]

This equation is solved if, and only if,

\[
q_i^*(\hat{F}(q, x_1^*(q, \sigma), \sigma)) = 0, \quad i = 1, \cdots, 6.
\]

Notice that \((u, \sigma)\) is a solution of (4.5) provided that

(4.6) \quad \begin{align*}
B(u, \sigma) &= 0
\end{align*}

with \( B : \mathcal{N} \times \mathbb{R} \to \mathbb{R}^6 \) defined by

\[
B(u, \sigma) := \frac{1}{2\pi} \int_0^{2\pi} \exp(-tS)F(x^*(u, \sigma), \sigma)dt
\]

and

\[
x^*(u, \sigma) := \exp(tS)u + x_1^*(\exp(tS)u, \sigma).
\]

Let us present some properties of the mapping \( B \).

The proof of next lemma can be found in [13].

**Lemma 4.3.** The following relations hold:

i) \( s_\phi B(u, \sigma) = B(s_\phi u, \sigma) \);

ii) \( RB(u, \sigma) = -B(\dot{R}u, \sigma) \), where \( s_\phi \) is the \( S^1 \)–action in \( \mathbb{R}^6 \) defined by \( s_\phi u = \exp(-\phi S_0)u \).

Observe that under the condition \( i) \) the mapping \( B \) is \( S^1 \)–equivariant whereas condition \( ii) \) states that the mapping \( B \) is \( R \)–anti-equivariant, i.e., \( B \) inherits the anti-symmetric properties of \( X_H \).

Assume that (4.3) is in Belitskii normal form truncated at the order \( p \). So \( X_H(x) = Ax + h(x) + r(x) \) where \( r(x) = O(\|x\|^{p+1}) \). The proof of next result is in [21].

**Theorem 4.4.** The following relations hold:

i) \( x^*(u, \sigma) = \exp(tS)u + O(\|x\|^{p+1}) \),

ii) \( B(u, \sigma) = (1 + \sigma)Su - Au - h(u) + O(\|x\|^{p+1}) \) for \( \sigma \) near the origin.
If \((u, \sigma)\) is a solution of (4.6) then \(x = x^*(u, \sigma)\) corresponds to a \(2\pi/(1+\sigma)\)-periodic solution of (4.5).

Recall that the periodic solution of (4.6) is \(R\)-symmetric if and only if it intersects \(\text{Fix}(R)\) in exactly two points. In conclusion, we obtain all small symmetric periodic solutions of (4.6) by solving the equation

\[
G(u, \sigma) = B(u, \sigma) |_{\text{Fix}(R)} = 0.
\]

5. Birkhoff normal form

In this section we briefly discuss some points concerning the Birkhoff normal form that will be useful in the sequel. The Birkhoff normal form is useful because it preserves the symplectic structure. In our cases if the vector field is in the Birkhoff normal form then it is in the Belitskii normal form, and so we can apply Theorem 4.4.

The function \(\{f, g\} = \omega(X_f, X_g)\) is called the Poisson bracket of the smooth functions \(f\) and \(g\). Let \(H_n\) be the set of all homogeneous polynomials of degree \(n\). The adjoint map \(Ad_{H_2} : H_n \rightarrow H_n\) is defined by

\[
Ad_{H_2}(H) = \{H_2, H\} = \omega(X_{H_2}, X_H) = -X_{H_2} \cdot \nabla H.
\]

The Birkhoff Normal Form Theorem (cf. [17, 8, 22]) states that if we have a Hamiltonian \(H = H_2 + H_3 + H_4 + \cdots\), where \(H_i \in H_i\) is the homogeneous part of degree \(i\), and \(G_i \subset H_i\) satisfies \(G_i \oplus \text{Range}(Ad_{H_2}) = H_i\), then there exists a formal symplectic power series transformation \(\Phi\) such that \(H \circ \Phi = H_2 + \tilde{H}_3 + \tilde{H}_4 + \cdots\) where \(\tilde{H}_i \in G_i\) \((i = 3, 4, \ldots)\). In particular, if \(Ad_{H_2}\) is semi-simple, as in our case, then \(\text{Ker}(Ad_{H_2})\) is the complement of \(\text{Range}(Ad_{H_2})\).

As \(R_j\) is symplectic, the change of coordinates \(\Phi\) can be chosen in such a way that \(H \circ \Phi\) satisfies \(H \circ \Phi \circ R_j = -H \circ \Phi\). In order to see this, we can split \(H_i = H_i^+ \oplus H_i^-\), where \(H_i^+ = \{H \in H_i : H \circ R_j = H\}\) and \(H_i^- = \{H \in H_i : H \circ R_j = -H\}\). If \(R_j\) is symplectic, then \(Ad_{H_2}(H_i^+) = H_i^+\) and \(Ad_{H_2}(H_i^-) = H_i^-\). In this case, if \(H_i = G_i \oplus Ad_{H_2}(H_i)\), then \(H_i^- = (G_i \cap H_i^-) \oplus Ad_{H_2}(H_i^+)\).

Now we can perform the change of coordinates restricted to \(H_i^-\). It implies that all monomial terms in the image of the adjoint restricted to \(H_i^-\) can be removed and it will remain only monomials in the kernel of the adjoint restricted to \(H_i^-\). And so, the normal form is also \(R_j\)-reversible.

**Definition 5.1.** We say that a Hamiltonian vector field \(X_H\) satisfies the Birkhoff Condition (BC) if \(Ad_{H_2}(H) = 0\).

**Remark 5.2.** By the equalities (5.8), the condition of the Definition 5.1 is equivalent to \(\omega(X_{H_2}, X_H) = 0\) or \(\{H_2, H\} = 0\).
6. Two degrees of freedom

In [3] a Birkhoff normal form for each \( X \in \Omega^0 \) is derived and the following result is obtained:

**Theorem 6.1.** Assume \( H \) is a Hamiltonian that is anti-invariant with respect to the involution and the associated vector field \( X_H \) has an elliptical equilibrium point. Then there exists another Hamiltonian \( \tilde{H} \), formally \( C^k \)-equivalent to \( H \), such that the vector field \( X_{\tilde{H}} \) has two one- parameter families of symmetric periodic solutions, with period near \( 2\pi/\sqrt{ad-bc} \), as in the Liapunov’s Theorem, going through the equilibrium point.

Let \( \Omega^0 \) be the space of the \( C^\infty \) \( R_0 \)-reversible Hamiltonian vector fields with two degrees of freedom in \( \mathbb{R}^4 \) and fix the coordinate system \((x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \). We endow \( \Omega^0 \) with the \( C^\infty \)-topology. Let \( \Omega^0_B \subset \Omega^0 \) be the space of the vector fields that satisfy the Birkhoff condition and \( \Omega^0_\omega \subset \Omega^0 \) be the space of the analytic ones. We prove the following result, which generalizes the previous one.

**Theorem A:** There exists an open set \( U^0 \subset \Omega^0_B \) (resp. \( \Omega^0_\omega \)) such that

(a) \( U^0 \) is determined by the 3-jet of the vector fields.

(b) each \( X \in U^0 \) possesses two 1-parameter families of symmetric periodic solutions terminating at the equilibrium point.

**Proof:** Fix on \( \mathbb{R}^4 \) a symplectic structure as in the Proposition 2.1. So the normal form of an involution has one of the following form: \( Id_{\mathbb{R}^4} \), or \( -Id_{\mathbb{R}^4} \), or \( R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \). We just work with \( R_0 \)-reversible vector fields.

As in the cases in \( \mathbb{R}^6 \) we have that by the hypothesis the Hamiltonian \( H \) satisfies \( H \circ R_0 = -H \), so the linear part of the vector field \( X_H \) is given by

\[
A = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & c & d \\
-d & b & 0 & 0 \\
c & -a & 0 & 0
\end{pmatrix}.
\]

and their eigenvalues are \( \{ \pm \sqrt{bc-ad}, \pm \sqrt{bc-ad} \} \). We are interested in the case with \( bc-ad < 0 \). We call \( \alpha = \sqrt{ad-bc} \) and in order to obtain the Jordan canonical form of the matrix \( A \) we consider the transformation matrix

\[
P = \begin{pmatrix}
0 & -\frac{b}{\alpha} & 0 & -\frac{a}{\alpha} \\
0 & \frac{b}{\alpha} & 0 & \frac{a}{\alpha} \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
After this transformation we obtain

\[ \hat{A} = P^{-1}AP = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}, \]

and

\[ \hat{R}_0 = P^{-1}R_0P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

where \( P^{-1} \) is the inverse matrix of \( P \).

Performing a time rescaling we can assume that \( \alpha = 1 \). We write the canonical real Jordan form of \( A \) as

\[ \hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \]

First we obtain the Belitskii normal form of \( X_H \), by considering \( h : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) up to 3rd order, which is given by \( X_H(x_1, y_1, x_2, y_2) = A[x_1, y_1, x_2, y_2] + h(x_1, y_1, x_2, y_2) \); and after we require the condition that the Belitskii normal form is \( \hat{R}_0 \)-reversible, i.e., \( X_H \hat{R}_0 = -\hat{R}_0 X_H \). Then the system obtained is given by

(6.10)

\[
\begin{align*}
x_1' &= y_1 + (e_{21}y_1 + e_{23}y_2)(x_1^2 + y_1^2) + e_{30}y_2(x_2^2 + y_2^2) \\
&\quad + (e_{16}x_1 + e_{24}x_2)(y_1x_2 - x_1y_2) + e_{26}y_2(x_1x_2 + y_1y_2), \\
y_1 &= -x_1 + (-e_{21}x_1 - e_{23}x_2)(x_1^2 + y_1^2) - e_{30}x_2(x_2^2 + y_2^2) \\
&\quad + (e_{16}y_1 + e_{24}y_2)(y_1x_2 - x_1y_2) - e_{26}x_2(x_1x_2 + y_1y_2), \\
x_2' &= y_2 + (-d_{15}y_1 - d_{22}y_2)(x_1^2 + y_1^2) - (d_{20}y_1 + d_{29}y_2)(x_2^2 + y_2^2) \\
&\quad - (d_{17}y_1 + d_{25}y_2)(x_1x_2 + y_1y_2), \\
y_2 &= -x_2 + (d_{15}x_1 + d_{22}x_2)(x_1^2 + y_1^2) + (d_{20}x_1 + d_{29}x_2)(x_2^2 + y_2^2) \\
&\quad + (d_{17}x_1 + d_{25}x_2)(x_1x_2 + y_1y_2).
\end{align*}
\]

Now we use the fact that the vector field satisfies the Birkhoff Condition. First of all we observe that the canonical symplectic matrix

\[ J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

after the linear change of coordinates \( P \), is transformed into

\[ \hat{J} = P^{-1}JP = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \]
We take a general Hamiltonian function \( H : \mathbb{R}^4 \to \mathbb{R} \) of 4\textsuperscript{th} order, compute the kernel of \( Ad_{H_2} \) defined on (5.2), where \( H_2 \) is the homogeneous part of degree 2 of \( H \), and require that \( H \) satisfies \( H \circ \hat{R}_0 = -H \). The terms up to 3\textsuperscript{rd} order is given by \( h_b(x) = \hat{J} \cdot \nabla H(x) \); its expression is

\[
\begin{align*}
\dot{x}_1 &= y_1 + a_1 y_1 (x_1^2 + y_1^2) + a_2 (2x_1 x_2 y_1 - x_1^2 y_2 + y_1^2 y_2) \\
&\quad + a_3 (3x_2^2 y_1 - 2x_1 x_2 y_2 + y_1 y_2^2),
\dot{y}_1 &= -x_1 + (a_2 y_1 + 2a_3 y_2) (x_2 y_1 - x_1 y_2) - x_1 (a_1 x_1^2 + y_1^2) \\
&\quad + a_2 (x_1 x_2 + y_1 y_2) + a_3 (x_2^2 + y_2^2)),
\dot{x}_2 &= y_2 + (a_1 x_1 + a_2 x_2) (-x_2 y_1 + x_1 y_2) + y_2 (a_2 x_1^2 + y_1^2) \\
&\quad + a_2 (x_1 x_2 + y_1 y_2) + a_3 (x_2^2 + y_2^2)),
\dot{y}_2 &= -x_2 + (a_1 y_1 + a_2 y_2) (-x_2 y_1 + x_1 y_2) - x_2 (a_1 x_1^2 + y_1^2) \\
&\quad + a_2 (x_1 x_2 + y_1 y_2) + a_3 (x_2^2 + y_2^2)).
\end{align*}
\] (6.11)

**Remark 6.2.** We observe here that we can apply Theorem 4.4 when the vector field is in the Belitskii normal form. This is not a restriction because if the vector field satisfies the Birkhoff Condition then it is in the Belitskii Normal Form. It is easy to see that if \{\( H_2, H \)\} = 0 then \( D(\{H_2, H\}) = 0 \), and so \( A_0^T X_H - DX_H A_0^T (x) = 0 \). For example, in our case we have

\[
\hat{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\text{ and } X_{H_2} = \begin{pmatrix} y_1 \\ -x_1 \\ y_2 \\ -x_2 \end{pmatrix}.
\]

The Birkhoff condition implies \(-y_1 H_{x_1} + x_1 H_{y_1} - y_2 H_{x_2} + x_2 H_{y_2} = 0 \). So

\[
\begin{align*}
H_{y_1} - y_1 H_{x_1 x_1} + x_1 H_{y_1 x_1} - y_2 H_{x_2 x_1} + x_2 H_{y_2 x_1} &= 0, \\
-H_{x_1} - y_1 H_{x_1 y_1} + x_1 H_{y_1 y_1} - y_2 H_{x_2 y_1} + x_2 H_{y_2 y_1} &= 0, \\
H_{y_2} - y_1 H_{x_1 x_2} + x_1 H_{y_1 x_2} - y_2 H_{x_2 x_2} + x_2 H_{y_2 x_2} &= 0, \\
-H_{x_2} - y_1 H_{x_1 y_2} + x_1 H_{y_1 y_2} - y_2 H_{x_2 y_2} + x_2 H_{y_2 y_2} &= 0.
\end{align*}
\] (6.12)

On the other hand if we compute \( A_0^T X_H - DX_H A_0^T (x) \), we obtain

\[
-\begin{pmatrix} -H_{y_2} \\ H_{x_2} \\ H_{y_1} \\ -H_{x_1} \end{pmatrix} + \begin{pmatrix} -H_{x_2 x_1} & -H_{x_2 y_1} & -H_{x_2 x_2} & -H_{x_2 y_2} \\ -H_{y_2 x_1} & -H_{y_2 y_1} & -H_{y_2 x_2} & -H_{y_2 y_2} \\ H_{x_1 x_1} & H_{x_1 y_1} & H_{x_1 x_2} & H_{x_1 y_2} \\ H_{y_1 x_1} & H_{y_1 y_1} & H_{y_1 x_2} & H_{y_1 y_2} \end{pmatrix} \begin{pmatrix} y_1 \\ -x_1 \\ y_2 \\ -x_2 \end{pmatrix},
\]

and by (6.12) we have that \( A_0^T X_H - DX_H A_0^T (x) = 0 \), i.e. the system is in the Belitskii Normal Form.

The Liapunov-Schmidt reduction gives us all small \( \hat{R}_0 \)-symmetric periodic solutions by solving the equation

\[
B(x, \sigma) |_{x \in \text{Fix}(\hat{R}_0)} = 0,
\]

with

\[
B(x, \sigma) = (1 + \sigma) Sx - \hat{A}x - h_b(x), \quad x \in \mathbb{R}^4,
\] (6.13)
where $S$ is the semi-simple part of (unique) $S - N-$decomposition of $\hat{A}$.
(See [15]).

In our case, $\hat{A}$ is semi-simple and $\text{Fix}(\hat{H}_0) = \{(0, y_1, 0, y_2); y_1, y_2 \in \mathbb{R}\}$. Recall that the reduced equation, $B(x, \sigma) = 0$, is defined in $\mathcal{N} \times \mathbb{R}$, where
$$\mathcal{N} = \{\exp(\hat{A}t)x; x \in V\} \subset C_{2\pi}^1$$ and $V = \text{span}\{e_1, e_2, e_3, e_4\}$.

The symplectic structure $J$ give us that $X_H$ is written in the following form $h_b(x) = h_b(x_1, y_1, x_2, y_2) = (-H_{x_1}(x_1, y_1, x_2, y_2), -H_{y_1}(x_1, y_1, x_2, y_2), H_{x_1}(x_1, y_1, x_2, y_2), H_{y_1}(x_1, y_1, x_2, y_2))$. Using the fact that $h_b$ satisfies the Birkhoff Condition we have that
$$y_1H_{x_1}(x_1, y_1, x_2, y_2) - x_1H_{y_1}(x_1, y_1, x_2, y_2) + y_2H_{x_2}(x_1, y_1, x_2, y_2) - x_2H_{y_2}(x_1, y_1, x_2, y_2) = 0, \forall (x_1, x_2, y_2) \in \mathbb{R}^4.$$
Hence at the points $(0, 0, 0, y_2)$ we have $y_2H_{x_2}(0, 0, 0, y_2) = 0$. It implies that $H_{x_2}(0, y_1, 0, y_2) = y_1f(y_1, y_2)$. Analogously we have that $H_{x_1}(0, y_1, 0, y_2) = y_2g(y_1, y_2)$. So

(6.14)
$$G(y_1, y_2, \sigma) = B(x, \sigma)|_{x \in \text{Fix}(\hat{H}_0)} = \begin{pmatrix} -y_1(a_1y_1^2 + a_2y_1y_2 + a_3y_2^2 - \sigma + \cdots) \\ -y_2(a_1y_1 + a_2y_1y_2 + a_3y_2^2 - \sigma + \cdots) \end{pmatrix}.$$ 

For the analytic case we have that the equation
$$G(y_1, y_2, \sigma) = (0, 0)$$
is given by
$$-y_1(a_1y_1^2 + a_2y_1y_2 + a_3y_2^2 - \sigma) + H_1(y_1, y_2) = 0,$$
$$-y_2(a_1y_1 + a_2y_1y_2 + a_3y_2^2 - \sigma) + H_2(y_1, y_2) = 0,$$
and multiplying the first equation by $-y_2$ and the second by $y_1$ we get $y_2H_1 = y_1H_2$. Using the fact that $H_1$ and $H_2$ are analytic we have that there exists $\bar{H}$ such that $H_1 = y_1\bar{H}$ and $H_2 = y_2\bar{H}$ for all $(y_1, y_2)$.

If $a_1a_3 \neq 0$ in (6.14), then we have two solutions for the equation $G(y_1, y_2, \sigma) = 0$. One solution is $y_1 = 0$ and $y_2(\sigma) = \pm \sqrt{\frac{a_2}{a_3}} + \ldots$. And the second solution is $y_2 = 0$ and $y_1(\sigma) = \pm \sqrt{\frac{a_2}{a_3}} + \ldots$.

We define $\mathcal{U}^0 = \mathcal{U}^0_1 \cap \mathcal{U}^0_2$ where
$$\mathcal{U}^0_1 = \{ X \in \Omega^0_B; \text{ the canonical form of } DX(0) \text{ satisfies } ad - bc > 0 \}$$
and
$$\mathcal{U}^0_2 = \{ X \in \Omega^0_B; \text{ the coefficients of } (6.11) \text{ satisfies } a_1a_3 \neq 0 \}.$$ 

In $\mathcal{U}^0 = \mathcal{U}^0_1 \cap \mathcal{U}^0_2 \subset \Omega^0_B$ for each $\sigma$ the equation $G(y_1, y_2, \sigma) = 0$ has two nonzero solutions terminating at the origin when $\sigma$ is tending to $0$. So, in the original problem we have two one parameter families of periodic solutions terminating the origin (when $\sigma \to 0$).
7. Three degrees of freedom

As in the previous section, let $\Omega^1$ (resp. $\Omega^2$) be the space of the $C^\infty$ $R_1$-reversible (resp. $R_2$-reversible) Hamiltonian vector fields with three degrees of freedom in $\mathbb{R}^6$ and fix a coordinate system $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$. We endow $\Omega^1$ and $\Omega^2$ with the $C^\infty$-topology. Let $\Omega^2_B$ (resp. $\Omega^2_G$) be the space of vector fields in $\Omega^2$ that satisfy the Birkhoff Condition (resp. that are analytic).

7.1. Case 6:2.

**Theorem B:** There exists an open set $U^1 \subset \Omega^1$ such that

(a) $U^1$ is determined by the 2–jet of the vector fields.

(b) for each $X \in U^1$ there is no symmetric periodic orbit arbitrarily close to the equilibrium point.

**Proof:** First we obtain the Belitskii normal form of $X_H$, by considering $h : \mathbb{R}^6 \to \mathbb{R}^6$ up to $2^{\text{nd}}$ order, and then we require that the Belitskii normal form is $\tilde{R}_1$–reversible, i. e, $X_H \tilde{R}_1 = -\tilde{R}_1 X_H$. After that we take the Birkhoff normal form. The new symplectic structure is $\tilde{J} = P_1^t J P_1$, where $P_1$ is the linear matrix that brings the linear part of the vector field to the Jordan canonical form. The Birkhoff normal form is obtained by taking a general Hamiltonian function $H : \mathbb{R}^6 \to \mathbb{R}$ of $3^{\text{rd}}$ order, computing the kernel of $Ad H_2$ and requiring that $H$ satisfies $H \circ \tilde{R}_1 = -H$. The Birkhoff normal form up to $2^{\text{nd}}$ order is given by $h_b(x) = \tilde{J} \cdot \nabla H(x)$. Finally, the Liapunov-Schmidt reduction gives us all small $\tilde{R}_1$–symmetric periodic solutions by solving the equation

$$B(x, \sigma)|_{x \in \text{Fix}(\tilde{R}_1)} = 0,$$

with

$$B(x, \sigma) = (1 + \sigma) S x - \tilde{A}_1 x - h_b(x), \ x \in \mathbb{R}^6,$$

and $S$ is the semi-simple part of (unique) $S - N$–decomposition of $\tilde{A}_1$. (See [L5]). In our case, $\tilde{A}_1$ is semi-simple and $\text{Fix}(\tilde{R}_1) = \{(0, 0, x_2, 0, x_3, 0) ; \ x_2, x_3 \in \mathbb{R}\}$. We recall that the reduced equation of the Liapunov-Schmidt, $B(x, \sigma) = 0$, is defined in $\mathcal{N} \times \mathbb{R}$, where $\mathcal{N} = \{\exp(\tilde{A}_1 t) x ; x \in V\} \in C^1_{2\pi}$ and $V = \text{ger}\{e_1, e_2, e_3, e_4, e_5, e_6\}$.

We derive the following expression

$$G(x_2, x_3, \sigma) = B(x, \sigma)|_{x \in \text{Fix}(\tilde{R}_1)} =

\begin{bmatrix}
  b_1 x_2^2 + x_3 (b_2 x_2 + b_3 x_3) + \cdots \\
  b_4 x_2^2 + x_3 (b_5 x_2 + b_6 x_3) + \cdots \\
  x_2 (-\sigma + \delta) + \cdots \\
  x_3 (-\sigma + \delta) + \cdots
\end{bmatrix}.

(7.15)

Observe that the equation $b_1 x_2^2 + b_2 x_2 x_3 + b_3 x_3^2 = 0$, generically, either has the solution $(x_2, x_3) = (0, 0)$, or has a pair of straight lines solutions given
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by \((c_1 x_2 + d_1 x_3)(c_2 x_2 + d_2 x_3) = 0\). The equation \(b_1 x_2^2 + b_5 x_2 x_3 + b_6 x_3^2 = 0\) is analogous. We can conclude that if the two first components of (7.13) have no common factor of the form \(c x_2 + d x_3\) then we have just the solution \((x_2, x_3) = (0, 0)\) for the two previous equations.

We define the following open sets:

\[ U_1 = \{ X \in \Omega; \text{the canonical form of } DX(0) \text{ satisfies (2.1)} \}, \]
\[ U_2 = \{ X \in \Omega; \text{the 2–jet of the two first equations of (7.15)} \} \text{ have no common factor} \]

Then \( U_1 = U_1 \cap U_2 \) is an open set in \( \Omega \).

The pair \((x_2, x_3) = (0, 0)\) is the unique solution of the equation \( G = 0 \). So, near the origin there are no symmetric periodic orbits for this case. \(\square\)

7.2. Case 6:4.

Theorem C: There exists an open set \( U_2 \subset \Omega_2^\beta \) (resp. \( \Omega_2^\omega \)) such that

(a) \( U_2 \) is determined by the 3–jet of the vector fields.

(b) each \( X \in U_2 \) has two 2–parameter families of periodic solutions \( \gamma^1 \sigma,\lambda \) and \( \gamma^2 \sigma,\lambda \) with \( \sigma \in (-\epsilon, \epsilon) \) and \( \lambda \in [0, 2\pi] \), such that, for each \( \lambda_0 \),

\[ \lim_{\sigma \to 0} \gamma^j \sigma,\lambda_0 = 0, \text{ for } j = 1, 2, \text{ and the periods tend to } 2\pi/\alpha \text{ when } \sigma \to 0. \]

Proof: First of all we derive the reversible Belitskii normal form of \( X_H \) up to 2\(^{nd}\) order. We observe that it coincides with the reversible Birkhoff normal form and is given by:

\[
X_{bh} = \begin{bmatrix} \frac{-b(x_3 y_2 - x_2 y_3)\alpha^2}{\beta} \\ \frac{a(x_3 y_2 - x_2 y_3)\alpha^2}{\beta} \\ (-ax_1 - by_1)y_2 + \alpha y_2 \\ x_2(ax_1 + by_1) - \alpha x_2 \\ (-ax_1 - by_1)y_3 + \alpha y_3 \\ x_3(ax_1 + by_1) - \alpha x_3 \end{bmatrix},
\]

(7.16)

where \( a = b_{65}/\alpha, b = b_{71}/\alpha \) and \( \alpha = \sqrt{-a_{06}a_{10} + a_{05}a_{11} - a_{15}a_{17} + a_{14}a_{18}} \).

As in the other cases, the Liapunov-Schmidt reduction gives us all small \( \widehat{R}_2 \)-symmetric periodic solutions by solving the equation

\[ B(x, \sigma)|_{x \in \text{Fix}(\widehat{R}_2)} = 0, \]
with
\[ B(x, \sigma) = (1 + \sigma)Sx - \hat{A}_2x - h_b(x), \quad x \in \mathbb{R}^6. \]

As before \( S \) is the semi-simple part of (the unique) \( S - N \)-decomposition of \( \hat{A}_2 \). (See [15]). In our case, \( \hat{A}_2 \) is semi-simple and Fix(\( \hat{R}_2 \)) = \{(x_1, y_1, 0, y_2, 0, y_3); \quad x_1, y_1, y_2, y_3 \in \mathbb{R} \}. We recall that the reduced equation of the Liapunov-Schmidt, \( B(x, \sigma) = 0 \), is defined on \( N \times \mathbb{R} \), where \( N = \{ \exp(\hat{A}_2 t) x; \quad x \in V \} \subset C_{2\pi}^1 \) and \( V = \text{ger}\{e_1, e_2, e_3, e_4, e_5, e_6\}. \)

Like in the proof of Theorem A, we derive the following expression

\[ (7.17) \quad G(x_1, y_1, y_2, y_3, \sigma) = B(x, \sigma)|_{x \in \text{Fix}(\hat{R}_2)} = \]
\[ = \left[ \begin{array}{c} y_2(\sigma + a_1 x_1 + a_2 y_1 + a_3 x_2^2 + a_4 y_1^2 + a_5 y_2^2 + a_6 y_2 y_3 + a_7 y_3^2 + \cdots ) \\ y_3(\sigma + a_1 x_1 + a_2 y_1 + a_3 x_2^2 + a_4 y_1^2 + a_5 y_2^2 + a_6 y_2 y_3 + a_7 y_3^2 + \cdots ) \end{array} \right]. \]

If \( a_5 a_7 \neq 0 \) in (7.17), then for each \((x_1, y_1)\) close to \((0, 0)\) we have two solutions for the equation \( G(x_1, y_1, y_2, y_3, \sigma) = 0 \). One solution is \( y_2 = 0 \) and \( y_3(x_1, y_1, \sigma) = \pm \sqrt{\frac{\sigma + a_1 x_1 + a_2 y_1}{a_5}} + \cdots \) and the second solution is \( y_3 = 0 \) and \( y_2(x_1, y_1, \sigma) = \pm \sqrt{\frac{\sigma + a_1 x_1 + a_2 y_1}{a_5}} + \cdots \).

We define the following open sets:
\[ U_2^1 = \{ X \in \Omega_B^2; \quad \text{the canonical form of } DX(0) \text{ satisfies (2.22)} \}, \]
\[ U_2^2 = \{ X \in \Omega_B^2; \quad \text{the coefficients of (7.17) satisfies } a_5 a_7 \neq 0 \}. \]

Then \( U_2 = U_2^1 \cap U_2^2 \) is an open set in \( \Omega_B^2 \). For each \( X \in U_2 \) and \( \sigma \) we consider \( \gamma_1^1 : (x_1, y_1) \mapsto (x_1, y_1, 0, y_3(x_1, y_1, \sigma)) \) and \( \gamma_2^2 : (x_1, y_1) \mapsto (x_1, y_1, y_2(x_1, y_1, \sigma), 0) \). Now we take the parametrization \((x_1, y_1) \mapsto (a \sigma, b \sigma)\). We have \( \gamma_1^1_{\sigma \lambda_0} : (a \sigma, b \sigma) \mapsto (a \sigma, b \sigma, 0, y_3(a \sigma, b \sigma, \sigma)) \) and \( \gamma_2^2_{\sigma \lambda_0} : (a \sigma, b \sigma) \mapsto (a \sigma, b \sigma, y_2(a \sigma, b \sigma, \sigma), 0) \) where \( \lambda_0 = a/b \). Then, there exists two \( 2 \)-parameter family of periodic orbits \( \gamma_1^{\lambda} \) and \( \gamma_2^{\lambda} \) such that for each \( \lambda_0 \in \mathbb{R} \), the families of periodic orbits \( \gamma_j^{\lambda_0} \), for \( j = 1, 2 \), are Liapunov families; i.e, \( \lim_{\sigma \to 0} \gamma_j^{\lambda_0} = 0 \) and the period tends to \( 2\pi/\alpha \).

8. Examples

This section is devoted to present a mechanical example for the Case \( 4 : 2 \).

We consider two objects \( m_1 \) and \( m_2 \) with charge \( q \) and \(-q\). They are at the position \((a, b) \in \mathbb{R}^2 \) and \((-a, -b) \in \mathbb{R}^2 \), respectively. We assume that the system does not have kinetic energy. So the total energy, i.e the Hamiltonian function is:
\[ H(x, u, y, v) = \frac{-q}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{q}{\sqrt{(x+a)^2 + (y+b)^2}}. \]
Note that this Hamiltonian function satisfies the condition
\[ H(\tilde{R}_0 \cdot (x, u, y, v)) = -H(x, u, y, v), \]
where
\[
\tilde{R}_0 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

In another words, our system is a Hamiltonian \( \tilde{R}_0 \)-reversible vector field.

**Remark 8.1.** It is worth to say that the system (7.16) (case 6 : 4 ) can be considered , in a similar way as [23], a mathematical model of a theoretical electrical circuit diagram.

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