On the Normal Scalar Curvature Conjecture for Legendrian Submanifolds in Kenmotsu Space Forms

Monia Fouad Naghi\textsuperscript{a}, Mića S. Stanković\textsuperscript{b}, Fatimah Alghamdi\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
\textsuperscript{b}Department of Mathematics, Faculty of Sciences and Mathematics, University of Nis, Serbia

Abstract. In this paper, we prove DDVV conjecture (the generalized Wintgen inequality) for Legendrian submanifolds in Kenmotsu space forms. Further, we derive an inequality for slant submanifolds in Kenmotsu space forms.

1. Introduction

In differential geometry, one of most fundamental research problems is to discover the relationships for intrinsic and extrinsic invariants. In [19], P. Wintgen found a relationship between Gaussian curvature $G$ (an intrinsic invariant), the squared mean curvature $\|H\|^2$ (an extrinsic invariant) and the normal curvature $G^\perp$ of any surface $M^2$ in $\mathbb{E}^4$ always satisfy the inequality

$$G + G^\perp \leq \|H\|^2,$$

and the equality holds if and only if the ellipse of curvature of $M^2$ in $\mathbb{E}^4$ is a circle. The inequality (1) is called Wintgen inequality and the Whitney 2-sphere satisfies the equality case of Wintgen inequality.

Later, the Wintgen inequality was extended for the surfaces $M^2$ of codimension $m$ in a real space form $\tilde{M}^{m+2}(c)$ in [18] and [11] independently as:

$$G + G^\perp \leq \|H\|^2 + c.$$ 

The equality case was also investigated.

In 1999, De Smet, Dillen, Verstraelen, Vrancken [9] developed the generalized Wintgen inequality named as DDVV conjecture for the submanifolds in real space forms as follows:

**Conjecture 1.1.** Let $f : M^n \to \tilde{M}^{n+m}(c)$ be an isometric immersion, where $\tilde{M}^{n+m}(c)$ is a real space form of constant sectional curvature $c$. Then

$$\rho + \rho^\perp \leq \|H\|^2 + c.$$ 

where $\rho$ is the normalised scalar curvature (intrinsic invariant) and $\rho^\perp$ is the normalised scalar normal curvature (extrinsic invariant).
If $K$ and $R^c$ are the sectional curvature and the normal curvature tensor on $M^n$, respectively in $\tilde{M}^{n+m}(c)$, then the normalized scalar curvature tensor $\rho$ is given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{i<j} K(e_i \wedge e_j)$$  \hspace{1cm} (3) $$

where $\tau$ is the scalar curvature, and the normalized scalar normal curvature $\rho^+$ by

$$\rho^+ = \frac{2\tau^+}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \leq i,j \leq n} \sum_{1 \leq r,s \leq m+n} (R^c(e_i, e_j, e_r, e_s))^2$$  \hspace{1cm} (4) $$

The Conjecture 1.1 was proven in [9] for a submanifold $M^n$ of arbitrary dimension $n \geq 2$ and codimension 2 in the real space form $\tilde{M}^{n+2}(c)$ of constant sectional curvature $c$. Later, the DDVV conjecture was proved for general case in [12] and in [10] independently.

For a normally flat submanifold, i.e., $R^c = 0$, this conjecture was proved by B.-Y. Chen in [6]. Hence, the conjecture is true for the hypersurfaces of real space forms.

Recently, I. Mihai proved DDVV conjecture for Lagrangian submanifolds in complex space forms [14] and for Legendrian submanifolds in Sasakian space forms [15]. In this paper, we derive the generalized Wintgen inequality (DDVV conjecture) for Legendrian submanifolds in Kenmotsu space forms.

2. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}^{2m+1}, g)$ is said to be a Kenmotsu manifold if it admits a $(1,1)$ tensor field $\varphi$ of its tangent bundle $TM^{2m+1}$, a vector field $\xi$ and a 1-form $\eta$, satisfying [4]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \tilde{\nabla}_X \xi = X - \eta(X)\xi.$$ 

for all vector fields $X, Y$ on $\tilde{M}^{2m+1}$, where $\tilde{\nabla}$ is the Levi-Civita connection of $g$.

A Kenmotsu manifold with constant $\varphi$-sectional curvature $c$ is said to be Kenmotsu space form and is denoted by $\tilde{M}^{2m+1}(c)$. Recall that the Riemannian curvature tensor of a Kenmotsu space form $\tilde{M}^{2m+1}(c)$ is given by

$$\tilde{R}(X, Y, Z, W) = \frac{(c-3)}{4} [g(X, W)g(Y, Z) - g(X, Z)g(Y, W)] - \frac{(c + 1)}{4} [\eta(Z)\eta(Y)g(X, W) - \eta(X)g(Y, W)]$$

$$+ \eta(W)g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) - g(\varphi X, W)g(\varphi Y, Z) + g(\varphi X, Z)g(\varphi Y, W)$$

$$+ 2g(\varphi X, Y)g(\varphi Z, W)]$$ \hspace{1cm} (5) $$

for any vector fields $X, Y, Z$ and $W$ tangent to $\tilde{M}^{2m+1}(c)$. As examples of Kenmotsu space forms we mention $\mathbb{R}^{2m+1}$ and $\mathbb{S}^{2m+1}(-1)$, with usual Kenmotsu structures ( for instance, see [4]).

Let $M^n$ be an $n$-dimensional Riemanian manifold isometrically immersed in a Kenmotsu space from $\tilde{M}^{2m+1}(c)$. We denote by $\nabla$ and $h$, the Riemannian connection and the second fundamental form of $M^n$, respectively. Then, the Gauss and Ricci equations are respectively given by

$$R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), $$ \hspace{1cm} (6) $$

$$R^c(X, Y, N_1, N_2) = \tilde{R}(X, Y, N_1, N_2) - g([A_{N_1}, A_{N_2}], X, Y),$$ \hspace{1cm} (7) $$

for all $X, Y, Z, W \in \Gamma(TM^n)$ and $N_1, N_2 \in \Gamma(T^\bot M^n)$, where $\tilde{R}$ is the curvature tensor of $\tilde{M}^{2m+1}$ and $R^c$ is the normal component of $\tilde{R}$, whereas $R$ is the curvature tensor of $M^n$. 
For any orthonormal basis \( \{e_1, \cdots, e_n\} \) of the tangent space \( T_pM^n \), the mean curvature vector \( H(p) \) is given by

\[
H(p) = \frac{1}{h} \sum_{i=1}^{n} h(e_i, e_i), \quad ||H||^2 = \frac{1}{n^2} \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).
\]

A submanifold \( M^n \) is totally geodesic in \( \tilde{M}^{2n+1} \) if \( h = 0 \), and minimal if \( H = 0 \). If \( h(X,Y) = g(X,Y)H \) for all \( X,Y \in \Gamma(TM^n) \), then \( M^n \) is totally umbilical in \( \tilde{M}^{2n+1} \).

A submanifold \( M^n \) normal to the structure vector field \( \xi \) is said to be a \( C\)-totally real submanifold. In this case, it follows that \( \varphi \) maps any tangent space of \( M^n \) into the normal space, that is, \( \varphi(T_pM^n) \subset T_p \mathbb{R}^m \), for each \( p \in M^n \). In particular, if \( n = m \), then \( M^n \) is called a Legendrian submanifold.

For submanifolds tangent to the structure vector field \( \xi \), we mention the following classes of submanifolds.

(i) A submanifold \( M^n \) tangent to \( \xi \) is said to be an invariant submanifold if \( \varphi \) preserves any tangent space of \( M^n \), that is, \( \varphi(T_pM^n) \subset T_pM^n \), for any \( p \in M^n \).

(ii) A submanifold \( M^n \) tangent to \( \xi \) is called an anti-invariant submanifold if \( \varphi \) maps any tangent space of \( M^n \) into the normal space, that is, \( \varphi(T_pM^n) \subset T_p \mathbb{R}^m \), for any \( p \in M^n \).

(iii) A submanifold \( M^n \) tangent to \( \xi \) is said to be a slant submanifold if for each non-zero vector \( X \in T_pM^n \) not proportional to \( \xi_p \), the angle \( \theta(X) \) between \( \varphi X \) and \( T_pM^n \) is constant, which is independent of the choice of \( p \in M^n \) and \( X \in T_pM^n \). The angle \( \theta \) is called slant angle or Wirtinger angle of \( M^n \).

3. DDVV conjecture for Legendrian submanifolds

In this section, we derive the generalized Wintgen inequality for Legendrian submanifolds of Kenmotsu manifolds. An anti-invariant submanifold \( M^n \) normal to the structure vector field \( \xi \) of a Kenmotsu manifold \( \tilde{M}^{2n+1} \) is said to be a Legendrian submanifold, if \( n = m \).

Following [20], we have

\[
K_N = \frac{1}{4} \sum_{r,s=1}^{2m+1-n} \text{Trace}[A_r, A_s]^2,
\]

where \( A_r = A_{(',r)} \), \( r \in \{1, \cdots, 2m + 1 - n\} \), and call it the scalar normal curvature of \( M^n \). The normalized scalar normal curvature is given by \( \rho_N = \frac{2}{m(n-1)} \sqrt{K_N} \). Since \( A_\xi = 0 \), it follows that

\[
K_N = \frac{1}{2} \sum_{1 \leq r < s \leq m-n} \text{Trace}[A_r, A_s]^2 = \sum_{1 \leq r < s \leq m-n} \sum_{1 \leq i} g([A_r, A_s]e_i, e_i)^2.
\]

We denote the second fundamental form \( h_{ij} = g(h(e_i, e_j), e_i) \), \( i, j \in \{1, \cdots, n\} \), \( r \in \{ n+1, \cdots, 2m+1-n\} \). Then, in terms of the components of the second fundamental form, we write

\[
K_N = \sum_{1 \leq i, j \leq m-n} \sum_{1 \leq r, s \leq n} \left( \sum_{k=1}^{n} (h_r^{ij} h_k^{ij} - h_r^{ik} h_k^{ij}) \right)^2.
\]

Lemma 3.1. Let \( M^n \) be an \( n \)-dimensional anti-invariant submanifold normal to \( \xi \) of a \((2m+1)\)-dimensional Kenmotsu space from \( \tilde{M}^{2m+1}(c) \). Then,

\[
\rho + \rho_N \leq ||H||^2 + \frac{c-3}{4}
\]
with equality holding if and only if, with respect to the suitable orthonormal frames \( \{ e_1, \cdots, e_n \} \) and \( \{ e_{m+1}, \cdots, e_{2m+1} = \xi \} \), the shape operator of \( M^n \) in \( M^{2m+1} (c) \) takes the form

\[
A_{\text{ext}} = \begin{pmatrix}
\lambda_1 & \mu & 0 & \cdots & 0 \\
\mu & \lambda_1 & 0 & \cdots & 0 \\
0 & 0 & \lambda_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_1
\end{pmatrix}, \quad A_{\text{ext}2} = \begin{pmatrix}
\lambda_2 + \mu & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 - \mu & 0 & \cdots & 0 \\
0 & 0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_2
\end{pmatrix},
\]

where \( \lambda_1, \lambda_2, \lambda_3 \) and \( \mu \) are real function on \( M^n \),

\[ A_{\text{ext}3} = \cdots = A_{\text{ext}n} = A_{\text{ext}n+1} = 0. \]

**Proof.** Note that in the proof of this lemma, we use the similar arguments and technique used in [12] (see also [14]). From the definition of \( H \), we have

\[
n^2 |H|^2 = \sum_{i=1}^{2m+1-n} \left( \sum_{j=1}^{n} h_{ij}^2 \right)^2.
\]

Since \( h_{ij}^{2m+1} = g(A_{\text{ext}1}, e_i, e_j) = 0 \), then the above expression will be

\[
n^2 |H|^2 = \frac{1}{n!} \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} (h_{ij}^r - h_{ij}^s)^2 + \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} h_{ij}^r h_{ij}^s.
\]

(10)

We use the following inequality given in [12],

\[
\sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} (h_{ij}^r - h_{ij}^s)^2 + 2n \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} h_{ij}^r h_{ij}^s \geq 2n \left[ \sum_{1 \leq r < s \leq 2m-n} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{n} (h_{ik}^r - h_{ik}^s)^2 \right) \right]^{1/2}.
\]

(11)

Then, with the help of above inequality, (10) takes the form

\[
n^2 |H|^2 \geq \frac{2n}{n-1} \left[ \sum_{1 \leq r < s \leq 2m-n} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{n} (h_{ik}^r - h_{ik}^s)^2 \right) \right]^{1/2} + \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} \left[ h_{ij}^r h_{ij}^s - (h_{ij}^s)^2 \right].
\]

Thus, from the definition of normalized scalar curvature and (9), we derive

\[
n^2 |H|^2 \geq n^2 \rho_n + \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} \left[ h_{ij}^r h_{ij}^s - (h_{ij}^s)^2 \right].
\]

(12)

On the other hand, from the Gauss equation we have

\[
2 \tau = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_i, e_j) + 2 \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} \left[ h_{ij}^r h_{ij}^s - (h_{ij}^s)^2 \right].
\]
where for orthonormal vector fields from (5), we derive
\[
\sum_{1 \leq i < j \leq n} R(e_i, e_j, e_j, e_i) = \frac{(c - 3)}{4} n(n - 1).
\]

Then, we find
\[
\tau = \frac{n(n - 1)(c - 3)}{8} + \frac{\sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} [h''_{ij} - (h''_{ij})^2]}{12},
\]

which is the required inequality. The equality case holds identically if and only if the shape operators takes the given form with respect to the suitable frames (we use the similar arguments given in [12] see also [14]).

We have the following corollary as a consequence of Lemma 3.1.

**Corollary 3.2.** Let \( M^n \) be an \( n \)-dimensional anti-invariant submanifold normal to \( \xi \) of \( \mathbb{H}^{2m+1}(-1) \). Then, we have
\[
\rho + \rho_N \leq \|H\|^2 - 1.
\]

Now, we derive the generalized Wintgen inequality (DDVV conjecture) for Legendrian submanifolds in a Kenmotsu space form.

**Theorem 3.3.** Let \( M^n \) be a Legendrian submanifold of a Kenmotsu space form \( \tilde{M}^{2m+1}(c) \). Then, we have
\[
(\rho^2) \leq \left( \|H\|^2 - \rho + \frac{c - 3}{4} \right)^2 + \frac{c + 1}{n(n - 1)} \left( \rho - \frac{c - 3}{4} \right) + \frac{(c + 1)^2}{8n(n - 1)}.
\]

**Proof.** Consider the orthonormal frame fields on \( M^n \) as \( \{e_1, \ldots, e_n\} \); then \( \{e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n, e_{2n+1} = \xi \} \) is an orthonormal frame in the normal bundle \( T^\perp M^n \). From (5) and (7), we have
\[
R^2(e_i, e_j, e_{n+}, e_{n+}) = -\frac{c + 1}{4} \left[ g(\varphi e_i, e_{n+})g(\varphi e_j, e_{n+}) + g(\varphi e_i, e_{n+})g(\varphi e_j, e_{n+}) \right] - g([A_r, A_s]e_i, e_j).
\]

Using the considered frame field, we derive
\[
\sum_{1 \leq r < s \leq n} \sum_{1 \leq i \leq n} R^2(e_i, e_j, e_{n+}, e_{n+}) = \frac{c + 1}{4} \left( g(\varphi e_i, e_{n+})g(\varphi e_j, e_{n+}) + g(\varphi e_i, e_{n+})g(\varphi e_j, e_{n+}) \right) - g([A_r, A_s]e_i, e_j)
\]

for all \( i, j \in \{1, \ldots, n\}, \ r, s \in \{1, \ldots, n\} \). Now, we find
\[
(\tau^2) = \sum_{1 \leq r < s \leq n} \sum_{1 \leq i < j \leq n} R^2(e_i, e_j, e_{n+}, e_{n+})^2.
\]

Then, with the help of (14), the above relation expresses as
\[
(\tau^2) = \frac{(c + 1)^2}{16} \sum_{1 \leq r < s \leq n} \sum_{1 \leq i < j \leq n} (\delta_{ij} \delta_{jp} - \delta_{ip} \delta_{jp})^2 + K_N + \frac{c + 1}{2} \sum_{1 \leq r < s \leq n} \sum_{1 \leq i < j \leq n} \sum_{1 \leq i < j \leq n} (\delta_{ij} \delta_{jp} - \delta_{ip} \delta_{jp}) g([A_r, A_s]e_i, e_j)
\]

\[
= \frac{n(n - 1)(c + 1)^2}{32} + \frac{n^2(n - 1)^2}{4} \rho_N^2 - \frac{c + 1}{4} \|H\|^2 + \frac{c + 1}{4} n^2 \|H\|^2.
\]

(15)
On the other hand, by Gauss equation and (5), we have
\[ 2\tau = n(n - 1)\frac{c - 3}{4} + n^2\|H\|^2 - \|h\|^2, \]
equivalently,
\[ n^2\|H\|^2 - \|h\|^2 = n(n - 1)\left(\rho - \frac{c - 3}{4}\right). \]
Then, with the help of above relation, (15) takes the form
\[ (\rho^+)^2 = \rho_N + \frac{c + 1}{8(n - 1)} \left(\rho - \frac{c - 3}{4}\right) + \frac{(c + 1)^2}{8n(n - 1)}. \]
Then, by Lemma 3.1, we derive
\[ (\rho^+)^2 \leq \left(\|H\|^2 - \rho + \frac{c - 3}{4}\right)^2 + \frac{c + 1}{n(n - 1)} \left(\rho - \frac{c - 3}{4}\right) + \frac{(c + 1)^2}{8n(n - 1)}, \]
which is required inequality. □

4. Another inequality

In this section we derive an inequality for normalized scalar curvature \( \rho \) and normalized normal scalar curvature \( \rho_N \) for a slant submanifold \( M \) in Kenmotsu space forms. We consider the structure vector field \( \xi \) tangent to \( M \).

**Theorem 4.1.** Let \( M^n \) be an \( n \)-dimensional slant submanifold of a Kenmotsu space form \( M^{2m+1}(c) \). Then, we have
\[ \rho + \rho_N \leq \|H\|^2 + \frac{c - 3}{4} + \frac{(3\cos^2 \theta - 2)(c + 1)}{4n}. \]

**Proof.** Consider the orthonormal frame field on \( M^n \) as follows: \( \{e_1, e_2 = \sec \theta Te_1, \cdots, e_{n-2}, e_{n-1} = \sec \theta Te_{n-2}, e_n = \xi\} \). Then, we have \( g(e_i, e_{i+1}) = -g(\nu e_i, \sec \theta e_1) = -\cos \theta \). Consequently, \( g^2(e_i, qe_{i+1}) = \cos^2 \theta \). Using this fact in Gauss equation with (2), we derive
\[ 2\tau = n(n - 1)\frac{c - 3}{4} - \frac{c + 1}{4} (n - 1) (3\cos^2 \theta - 2) + 2 \sum_{i=1}^{2m+1-n} \sum_{1 \leq i < j \leq n} \left[ h_i^c h_j^c - (h_i^c)^2 \right]. \tag{16} \]

On the other hand, by similar argument as in proof of Lemma 3.1 (relation (12)), we have
\[ n^2\|H\|^2 \geq \frac{2n}{n - 1} \sum_{i=1}^{2m+1-n} \sum_{1 \leq i < j \leq n} \left[ h_i^c h_j^c - (h_i^c)^2 \right]. \tag{17} \]
Then, from (16) and (17), we derive
\[ \rho + \rho_N \leq \|H\|^2 + \frac{c - 3}{4} + \frac{(3\cos^2 \theta - 2)(c + 1)}{4n}. \]
Hence, we achieve the result. □

**Corollary 4.2.** Let \( M^n \) be a slant submanifold of \( H^{2m+1} \). Then, we have
\[ \rho + \rho_N \leq \|H\|^2 - 1. \]
Notice that the inequality for the normal scalar curvature and normalized normal scalar curvature in terms of mean curvature does not change for the different submanifolds in \( H^{2m+1} \). For example; in Corollary 3.2, the inequality is obtained for anti-invariant submanifolds, while; in Corollary 4.2, it is for slant slant submanifold but in both cases the inequality is same.
Acknowledgement

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-12-130-39). The authors, therefore, acknowledge with thanks DSR technical and financial support.

The second author acknowledge the grant of the Ministry of Education, Science and Technological Development of Serbia 451-03-68/2020-14/200124 for carrying out this research.

References

[1] M. Aquib, M.H. Shahid, Generalized Wintgen inequality for submanifolds in Kenmotsu space forms, Tamkang J. Math. 50 (2019), no. 2, 155-164.
[2] M. E. Aydin, I. Mihai, Wintgen inequality for statistical surfaces, (2015).arXiv:1511.04987[math.DG].
[3] M.E. Aydin, A. Mihai, I. Mihai, Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature, Bull. Math. Sci. 7 (2017),155–166.
[4] D. E. Blair, Contact manifolds in Riemannian Geometry, Lecture Notes in Math, 509, Springer-Verlag, Berlin, 1976.
[5] B.-Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math. (Basel) 60 (1993), 568-578.
[6] B.-Y. Chen, Mean curvature and shape operator of isometric immersions in real space forms, Glasgow Math. J. 38 (1996), 87–97.
[7] B.-Y. Chen, Classification of Wintgen ideal surfaces in Euclidean 4-space with equal Gauss and normal curvature, Ann. Global Anal. Geom. 38 (2010), 145–160.
[8] S. Decu, S. Haesen, L. Verstraelen, G. E. Vilcu, Curvature Invariants of Statistical Submanifolds in Kenmotsu Statistical Manifolds of Constant ϕ-Sectional Curvature, Entropy, 20 (2018), 529.
[9] P. J. De Smet, F. Dillen, L. Verstraelen, L. Vrancken, A pointwise inequality in submanifold theory, Arch. Math. (Brno) 35 (1999), 115–128.
[10] J. Ge, Z. Tang, A proof of the DDVV conjecture and its equality case, Pacific. J. Math. 237 (2008), 87–95.
[11] I.V. Guadalupe and L. Rodriguez, Normal curvature of surfaces in space forms, Pacific J. Math. 106 (1983), 95–103.
[12] Z. Lu, Normal scalar curvature conjecture and its applications, J. Funct. Anal. 261 (2011), 1284–1308.
[13] A. Mihai, I. Mihai, Curvature Invariants for Statistical Submanifolds of Hessian Manifolds of Constant Hessian Curvature, Mathematics 6(3) (2018), 44. doi:10.3390/math6030044
[14] I. Mihai, On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms, Nonlinear Anal. 95 (2014), 714-720.
[15] I. Mihai, On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms, Tohoku Math. J. 69 (2017), 43–53.
[16] C. Murathan, B. Sahin, A study of Wintgen like inequality for submanifolds in statistical warped product manifolds, J. Geom., 109 (2018), 30.
[17] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J., 24 (1972), 93-103.
[18] B. Rouxel, Sur une famille de A-surfaces d’un espace euclidien E^4, Proc. X Österreichischer Mathematiker Kongress, Innsbruck, (1981), pp. 185.
[19] P. Wintgen, Sur l’inégalité de Chen-Willmore, C. R. Acad. Sci. Paris Sér. A-B 288 (1979), A993–A995.
[20] K. Yano, M. Kon, Anti-invariant submanifolds, M. Dekker, New York, 1976.