Generalized Fourier transform on Chébli-Trimèche hypergroups

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Abstract In this paper, we prove the Hardy-Littlewood inequality for the generalized Fourier transform on Chébli-Trimèche hypergroups and we study in the particular case of the Jacobi hypergroup the integrability of this transform on Besov-type spaces.

Keywords Chébli-Trimèche hypergroups · Generalized Fourier transform · Jacobi hypergroup · Jacobi function

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1 Introduction

We consider the Chébli-Trimèche hypergroup \((\mathbb{R}^+, \ast (A))\) associated with the function \(A\) which depends on a real parameter \(\alpha > -\frac{1}{2}\) (see next section). We prove the Hardy-Littlewood inequality for the generalized Fourier transform \(\mathcal{F}(f)\) of a function \(f\) in \(L^p(\mathbb{R}^+, A(x)dx)\), \(1 < p \leq 2\). Next, inspired by the definition of usual Besov spaces and Besov-Dunkl spaces (see [2, 5]), we define the Besov-type spaces for Chébli-Trimèche hypergroup denoted by \(\mathcal{B}^{p,q}_{\gamma,\alpha}\), as the subspace of functions \(f \in L^p(\mathbb{R}^+, A(x)dx)\) satisfying

\[
\int_0^{+\infty} \left( \frac{\omega_{A,p}(f,x)}{x^\gamma} \right)^q \frac{dx}{x} < +\infty \quad \text{if } q < +\infty
\]

and

\[
\sup_{x \in [0, +\infty]} \frac{\omega_{A,p}(f,x)}{x^\gamma} < +\infty \quad \text{if } q = +\infty,
\]

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where \( \omega_{A,p}(f,x) = \| \tau_x(f) - f \|_{A,p} \) is the modulus of continuity of first order of \( f \) with \( \tau_x \) the generalized translation operators, \( x \in \mathbb{R}_+ \) (see next section). We establish in the particular case of Jacobi hypergroups further results concerning integrability of the generalized Fourier transform \( \mathcal{F}(f) \) of a function \( f \) when \( f \) belongs to a suitable Besov-type spaces. Analogous results have been obtained for the theory of Dunkl operators in [1, 3, 4].

The contents of this paper are as follows.

In section 2, we collect some results about harmonic analysis on Chébli-Trimèche hypergroups.

In section 3, we prove the Hardy-Littlewood inequality for the generalized Fourier transform on Chébli-Trimèche hypergroups and we study in the particular case of the Jacobi hypergroup the integrability of this transform on Besov-type spaces.

Along this paper we use \( c \) to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- \( \mathbb{C}_{\text{c}}(\mathbb{R}) \) the space of even continuous functions on \( \mathbb{R} \), with compact support.
- \( \mathcal{S}(\mathbb{R}) \) the space of even \( C^\infty \)-functions on \( \mathbb{R} \) with compact support.

## 2 Preliminaries

In this section, we recall some notations and results about harmonic analysis on Chébli-Trimèche hypergroups and we refer for more details to the articles [6, 9, 11, 12].

Let \( A \) be the Chébli-Trimèche function defined on \( \mathbb{R}_+ \) and satisfying the following conditions.

i) \( A(x) = x^{2\alpha+1} B(x) \), with \( \alpha > -\frac{1}{2} \), and \( B \) an even \( C^\infty \)-function on \( \mathbb{R} \) such that \( B(x) \geq 1 \) for all \( x \in \mathbb{R}_+ \).

ii) \( A \) is increasing and unbounded.

iii) \( \frac{A'}{A} \) is decreasing on \( \mathbb{R}_+^\ast = [0, +\infty] \) and \( \lim_{x\to+\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0 \).

iv) There exists a constant \( \eta > 0 \) such that for all \( x \in [x_0, +\infty] \), \( x_0 > 0 \), we have

\[
\frac{A'(x)}{A(x)} = \begin{cases} 
2\rho + e^{-\eta x} F(x), & \text{if } \rho > 0 \\
\frac{\alpha+1}{2\rho} + e^{-\eta x} F(x), & \text{if } \rho = 0,
\end{cases}
\]

where \( F \) is a \( C^\infty \)-function bounded together with its derivatives.

We consider the Chébli-Trimèche hypergroup \( (\mathbb{R}_+, \ast(A)) \) associated with the function \( A \). We note that it is commutative with neutral element 0 and the identity mapping is the involution. The Haar measure \( m \) on \( (\mathbb{R}_+, \ast(A)) \) is absolutely continuous with respect to the Lebesgue measure and can be chosen to have the Lebesgue density \( A \).

**Remark 1** If \( A(x) = 2^\rho (\sinh x)^{2\alpha+1} (\cosh x)^{2\beta+1} \), with \( \alpha \geq \beta \geq -\frac{1}{2}, \alpha \neq -\frac{1}{2} \) and \( \rho = \alpha + \beta + 1 \), \( (\mathbb{R}_+, \ast(A)) \) is called the Jacobi hypergroup.

Let \( \Delta \) be the differential operator on \( \mathbb{R}_+^\ast \) given by

\[
\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.
\]

The solution \( \varphi_{\lambda}, \lambda \in \mathbb{C}, \) of the differential equation

\[
\begin{cases}
\Delta u(x) = -(\lambda^2 + \rho^2) u(x), \\
u(0) = 1, \frac{d}{dx} u(0) = 0,
\end{cases}
\]
is multiplicative on \((\mathbb{R}_+, * (A))\) in the sense that
\[
\forall x, y \in \mathbb{R}_+, \int_{\mathbb{R}_+} \varphi_\lambda(t) \, d(\delta_x * \delta_y)(t) = \varphi_\lambda(x) \varphi_\lambda(y),
\]
where \(\delta_x\) is the point mass at \(x\) and \(\delta_x * \delta_y\) is a probability measure which is absolutely continuous with respect to the measure \(m\) and satisfies
\[
\text{supp} \, \delta_x * \delta_y = [|x - y|, |x + y|].
\]

We list some known properties of the characters \(\varphi_\lambda\) of the hypergroups.

i) For each \(\lambda \in \mathbb{C}\), the function \(x \mapsto \varphi_\lambda(x)\) is an even \(C^\infty\)-function on \(\mathbb{R}\) and for each \(x \in \mathbb{R}_+\), the function \(\lambda \mapsto \varphi_\lambda(x)\) is an entire function on \(\mathbb{C}\).

ii) For every \(\lambda \in \mathbb{C}\), the function \(\varphi_\lambda\) admits the integral representation
\[
\forall x \in \mathbb{R}_+, \quad \varphi_\lambda(x) = \int_0^\infty K(x, y) \cos(\lambda y) \, dy.
\]

Where \(K(x, .)\) is a positive even \(C^\infty\)-function on \([-x, x]\) with support in \([-x, x]\).

**Remark 2** In the Jacobi hypergroup (see Remark 1), we have for all \(x \in \mathbb{R}_+\) and \(\lambda \in \mathbb{C}\),
\[
\varphi_\lambda(x) = \varphi_\lambda^\alpha(x) = 2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1; -\sin^2 x\right),
\]
where \(2F_1\) is the Gauss hypergeometric function (see [9]). The function \(\varphi_\lambda^\alpha(x)\) is the Jacobi function and it satisfies for all \(\lambda \in \mathbb{R}\) and \(t \in \mathbb{R}_+^\ast\)
\[
|1 - \varphi_\lambda^\alpha(t)| \geq c \min\{1, \lambda t^2\}, \quad (1)
\]
where \(c\) is constant which depends only on \(\alpha\) and \(\beta\) (see [7, 8]).

For every \(p \in [1, +\infty]\), we denote by \(L^p_{\alpha}(\mathbb{R}_+)\) the space \(L^p(\mathbb{R}_+, |A(x)| \, dx)\) and by \(L^p_{\lambda}(\mathbb{R}_+)\) the space \(L^p(\mathbb{R}_+, \frac{dx}{|A(x)|^p})\) where \(|c(\lambda)|^{-2}\) is an even continuous function on \(\mathbb{R}\), satisfying the estimates: There exist positive constants \(k, k_1, k_2\) such that

i) If \(\rho = 0\) and \(\alpha > 0\) then
\[
k_1 |\lambda|^{2\alpha + 1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha + 1}, \quad \lambda \in \mathbb{C}. \quad (2)
\]

ii) If \(\rho > 0\) and \(\alpha > -\frac{1}{2}\) then
\[
k_1 |\lambda|^{2\alpha + 1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha + 1}, \quad \lambda \in \mathbb{C}, \, |\lambda| > k, \quad (3)
\]

and
\[
k_1 |\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^2, \quad \lambda \in \mathbb{C}, \, |\lambda| \leq k. \quad (4)
\]

We use \(\|\cdot\|_{\lambda, p}\) and \(\|\cdot\|_{\alpha, p}\) as a shorthand respectively of \(\|\cdot\|_{L^p_{\alpha}(\mathbb{R}_+)}\) and \(\|\cdot\|_{L^p_{\lambda}(\mathbb{R}_+)}\).

For \(f \in L^1_{\lambda}(\mathbb{R}_+)\) the generalized Fourier transform of \(f\) is given by
\[
\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}_+} f(x) \varphi_\lambda(x) A(x) \, dx.
\]
The generalized Fourier transform satisfies the following properties.
i) For $f \in L^1_A(\mathbb{R}+)\), we have

$$\|F(f)\|_{\mathcal{C}} \leq \|f\|_{A,1}$$  \hspace{1cm} (5)$$

ii) For $f$ in $L^1_A(\mathbb{R}+)\) such that $F(f)$ belongs to $L^1_{\mathcal{C}}(\mathbb{R}+)$, we have the following inversion formula for the transform $F$

$$f(x) = \int_{\mathbb{R}+} F(f)(\lambda)\phi_{\lambda}(x) \frac{d\lambda}{|c(\lambda)|^2}, a.e.$$ 

iii) (Plancherel formula) For all $f \in \mathcal{D}_\ast(\mathbb{R})\,

$$\int_{\mathbb{R}+} |f(x)|^2A(x)dx = \int_{\mathbb{R}+} |F(\lambda)|^2 \frac{d\lambda}{|c(\lambda)|^2}. \hspace{1cm} (6)$$

The transform $F$ can be uniquely extended to an isometric isomorphism from $L^2_A(\mathbb{R}+)$ onto $L^2_{\mathcal{C}}(\mathbb{R}+)$. For $1 \leq p \leq 2$, we denote by $p'$ the conjugate of $p$. From (2.5), (2.6) and the Marcinkiewicz interpolation theorem (see [10]), we obtain for $f \in L^p_A(\mathbb{R}+)\)

$$F(f) \in L^{p'}_{\mathcal{C}}(\mathbb{R}+). \hspace{1cm} (7)$$

For $x \in \mathbb{R}+$ and $f \in C_{\ast,\ast}(\mathbb{R})\), the generalized $x$-translate of $f$ is defined by

$$\forall y \in \mathbb{R}+, \quad \tau_x f(y) = \int_{\mathbb{R}+} f(t)d(\delta_x * \delta_y)(t),$$

and we have $\tau_x f(0) = f(x)$. The generalized translation operators $\tau_x, x \in \mathbb{R}+$, satisfy the following properties.

i) For all $x, y \in \mathbb{R}+$ and $\lambda \in \mathbb{C}$, we have the product formula

$$\tau_x \phi_{\lambda}(y) = \phi_{\lambda}(x)\phi_{\lambda}(y).$$

ii) For $f \in \mathcal{D}_\ast(\mathbb{R})$ and $x \in \mathbb{R}+$, the function $y \mapsto \tau_x f(y)$ belongs to $\mathcal{D}_\ast(\mathbb{R})$ and we have

$$\forall \lambda \in \mathbb{R}+, \quad F(\tau_x f)(\lambda) = \phi_{\lambda}(x)F f(\lambda). \hspace{1cm} (8)$$

iii) Let $f \in L^p_{\mathcal{C}}(\mathbb{R}+), \ p \in [1, +\infty]$. For all $x \in \mathbb{R}+$, the function $\tau_x f$ belongs to $L^p_A(\mathbb{R}+), \ p \in [1, +\infty]$, and we have

$$\|\tau_x f\|_{A,p} \leq \|f\|_{A,p}. $$
3 Generalized Fourier transform

Throughout this section, $\kappa$ refers to the constant obtained in (3) and (4) from the estimates of $|c(\lambda)|^{-2}$.

In the following lemma, we prove the Hardy-Littlewood inequality for the Fourier transform.

**Lemma 1** For $f \in L_A^p(\mathbb{R}^+)$, $1 < p \leq 2$, one has
\[
\int_{\mathbb{R}^+} (g(x))^{p-2} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2} \leq c \|f\|_{A,p}^p
\]  
(9)

where
i) $g(x) = x^{2(\alpha+1)}$ if $\rho = 0$ and $\alpha > 0$.
ii) $g(x) = \begin{cases} x^{2(\alpha+1)} & \text{for } x > k \\ x^3 & \text{for } x \leq k \end{cases}$ if $\rho > 0$ and $\alpha > -\frac{1}{2}$

where $k$ refers to the constant obtained from the estimates of $|c(x)|^{-2}$.

**Proof** For $f \in L_A^p(\mathbb{R}^+)$, $1 \leq p \leq 2$, we consider the operator
\[ L(f)(x) = g(x)\mathcal{F}(f)(x), x \in \mathbb{R}^+. \]

For every $f \in L_A^2(\mathbb{R}^+)$, we have from (6)
\[
\left( \int_{\mathbb{R}^+} |L(f)(x)|^2 \frac{dx}{(g(x))^2|c(x)|^2} \right)^{\frac{1}{2}} = \|\mathcal{F}(f)\|_{l^2} = \|f\|_{A,2},
\]

hence $L$ is an operator of strong-type $(2, 2)$ between the spaces $(\mathbb{R}^+, A(x)dx)$ and $(\mathbb{R}^+, \frac{dx}{(g(x))^2|c(x)|^2})$.

i) Assume $\rho = 0$, $\alpha > 0$ and $g(x) = x^{2(\alpha+1)}$. For $\lambda \in [0, +\infty]$, $f \in L_A^1(\mathbb{R}^+)$ and using (2) and (5), we can write
\[
\int_{\{|x| > \lambda\}} \frac{dx}{(g(x))^2|c(x)|^2} = \int_{\{|x| > \lambda\}} \frac{dx}{x^{4(\alpha+1)}|c(x)|^2} \leq c \int_{\{x > \lambda\}} \frac{x^{2\alpha+1}}{x^{4(\alpha+1)}}dx \\
\leq c \frac{\|f\|_{A,1}}{\lambda}
\]

It yields that $L$ is of weak-type $(1, 1)$ between the spaces under consideration.

By the Marcinkiewicz interpolation theorem (see [10]), we can assert that $L$ is an operator of strong-type $(p, p)$, $1 < p \leq 2$ between the spaces $(\mathbb{R}^+, A(x)dx)$ and $(\mathbb{R}^+, \frac{dx}{(g(x))^2|c(x)|^2})$.

We conclude that
\[
\int_{\mathbb{R}^+} |L(f)(x)|^p \frac{dx}{(g(x))^2|c(x)|^2} = \int_{\mathbb{R}^+} |g(x)|^{p-2} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2} \leq c \|f\|_{A,p}^p,
\]

which proves the result.

ii) Suppose now $\rho > 0$, $\alpha > -\frac{1}{2}$ and $g(x) = \begin{cases} x^{2(\alpha+1)} & \text{for } x > k \\ x^3 & \text{for } x \leq k \end{cases}$.
where $k$ is the constant obtained in (3) and (4) from the estimates of $|c(\lambda)|^{-2}$. Let $\lambda \in [0, +\infty[$ and $f \in L^p_\alpha(\mathbb{R}_+)$, by (3), (4) and (5), we have

$$
\int_{\{x \in \mathbb{R}_+: |L(f)(x)| > \lambda\}} \frac{dx}{(g(x))^2|c(x)|^2} \leq \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \frac{dx}{(g(x))^2|c(x)|^2}
$$

$$
\leq \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \frac{dx}{(g(x))^2|c(x)|^2} \leq \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \frac{dx}{(g(x))^2|c(x)|^2}
$$

$$
+ \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \chi_{[k, \infty)} \frac{dx}{(g(x))^2|c(x)|^2}
$$

$$
\leq c \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \frac{dx}{(g(x))^2|c(x)|^2}
$$

$$
\leq c \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \frac{dx}{(g(x))^2|c(x)|^2}
$$

$$
+ \int_{\{x \in \mathbb{R}_+: |g(x)| > \frac{1}{(g_\alpha(x))^2}\}} \frac{dx}{(g(x))^2|c(x)|^2}
$$

Hence $L$ is of weak-type $(1,1)$ between the spaces $(\mathbb{R}_+, A(x)dx)$ and $((\mathbb{R}_+, \frac{dx}{(g(x))^2|c(x)|^2}))$. We conclude by the Marcinkiewicz interpolation theorem that $L$ is of strong-type $(p, p)$, between the spaces under consideration.

It yields, that

$$
\int_{\mathbb{R}_+} |L(f)(x)|^p \frac{dx}{(g(x))^2|c(x)|^2} \leq \int_{\mathbb{R}_+} |g(x)|^{p-2} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2}
$$

$$
\leq c \|f\|_{A_p}^p,
$$

thus we obtain the result.

In the following, we study the integrability of the generalized Fourier transform in the Jacobi hypergroup case (see Remarks 1 and 2). For $1 \leq p \leq 2$, we denote by $p'$ the conjugate of $p$.

**Lemma 2** Let $1 \leq p \leq 2$ and $f \in L^p_\alpha(\mathbb{R}_+)$. Then there exists a positive constant $c$ such that for $\delta > 0$, one has

$$
\left( \int_0^{+\infty} \min\{1, (\delta x)^{2p'}\} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2} \right)^{\frac{1}{p'}} \leq c \omega_{\alpha,p}(f)(\delta), \quad \text{if } 1 < p \leq 2
$$

and

$$
\text{ess sup}_{x>0} \left( \min\{1, (\delta x)^2\} |\mathcal{F}(f)(x)| \right) \leq c \omega_{\alpha,1}(f)(\delta), \quad \text{if } p = 1.
$$

**Proof** For $f \in L^p_\alpha(\mathbb{R}_+)$, $1 \leq p \leq 2$, we have by (8)

$$
\mathcal{F}(\tau_\delta(f) - f)(x) = (\phi_\delta - 1)\mathcal{F}(f)(x),
$$

for $\delta > 0$ and a.e $x \in \mathbb{R}_+$. Applying (7), we get

$$
\|\mathcal{F}(\tau_\delta(f) - f)\|_{c,p'} = \left( \int_0^{+\infty} |1 - \phi_\delta(\delta)|^{p'} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|c(x)|^2} \right)^{\frac{1}{p'}}
$$

$$
\leq c \omega_{\alpha,p}(f)(\delta).
$$

From (1), we obtain our results. Here when $p = 1$, we make the usual modification.
Remark 3

i) In the lemma 2, the gauge on the size of the generalized transform in terms of an integral modulus of continuity of \( f \) gives a quantitative form of the Riemann-Lebesgue lemma:

\[
\left( \int_{-\infty}^{+\infty} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|x|^2} \right)^{\frac{1}{p'}} \leq c \omega_{\lambda,p}(f)(\delta), \text{ if } 1 < p \leq 2
\]

and

\[
\text{ess sup}_{x \geq \frac{1}{2}} |\mathcal{F}(f)(x)| \leq c \omega_{\lambda,1}(f)(\delta), \text{ if } p = 1.
\]

ii) We will use the following estimates deduced from lemma 2 to establish the integrability of \( \mathcal{F}(f) \) when \( f \) belongs in \( \mathcal{B}^{p,1}_{\alpha} \) for \( 1 \leq p \leq 2 \):

\[
\delta^{2p} \left( \int_{0}^{\frac{1}{2}} x^{2p'} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|x|^2} \right)^{\frac{1}{p'}} \leq c \omega_{\lambda,p}(f)(\delta), \text{ if } 1 < p \leq 2
\]

and

\[
\text{ess sup}_{0 < x < \frac{1}{2}} \left( (\delta x)^{2p} |\mathcal{F}(f)(x)| \right) \leq c \omega_{\lambda,1}(f)(\delta), \text{ if } p = 1.
\]

Theorem 1

If \( f \in \mathcal{B}^{p,1}_{\alpha} \cap \mathcal{B}^{p,1}_{\alpha} \) for \( 1 < p \leq 2 \), then

\[
\mathcal{F}(f) \in L_{p}^{1}(\mathbb{R}^{+}).
\]

Proof

For \( f \in L_{p}^{1}(\mathbb{R}^{+}) \), \( 1 < p \leq 2 \) and \( \delta > 0 \), we can write from (8) and (9)

\[
\int_{\mathbb{R}^{+}} |1 - \varphi_{1}(\delta)|^{p} |\mathcal{F}(\tau_{\delta}(f))(t)|^{p} |g(t)|^{p-2} \frac{dt}{|x(t)|^2} \leq c (\omega_{\lambda,p}(f)(\delta))^{p},
\]

then by (1), we obtain

\[
\delta^{2p} \int_{0}^{\frac{1}{2}} x^{2p} |\mathcal{F}(f)(t)|^{p} |g(t)|^{p-2} \frac{dt}{|x(t)|^2} \leq c (\omega_{\lambda,p}(f)(\delta))^{p}.
\]  

From (3) and (4), we have

\[
\int_{0}^{\frac{1}{2}} t |\mathcal{F}(f)(t)| \frac{dt}{|x(t)|^2}
\]

\[
= \int_{0}^{\frac{1}{2}} t |\mathcal{F}(f)(t)| \chi_{[0,\delta]}(t) \frac{dt}{|x(t)|^2} + \int_{0}^{\frac{1}{2}} t |\mathcal{F}(f)(t)| \chi_{\delta,\infty}(t) \frac{dt}{|x(t)|^2}
\]

\[
\leq c \int_{0}^{\frac{1}{2}} t |\mathcal{F}(f)(t)| \chi_{[0,\delta]}(t) |t|^{2} dt + c \int_{0}^{\frac{1}{2}} t |\mathcal{F}(f)(t)| \chi_{\delta,\infty}(t) |t|^{2+1} dt,
\]
by Hölder’s inequality and (12), we have
\[\int_0^\frac{t}{t} |F(f)(t)| \frac{dt}{|c(t)|^2} \]
\[\leq c \left( \int_0^\frac{t}{t} t^{3(p-2)+2p} |F(f)(t)|^p |\chi_{[0,t]}(t)|^2 dt \right)^\frac{1}{p} \left( \int_0^\frac{t}{t} t^{(p-2)} \chi_{[0,t]}(t) dt \right)^\frac{1}{p} \]
\[+ c \left( \int_0^\frac{t}{t} t^{2(\alpha+1)(p-2)+2p} |F(f)(t)|^p |\chi_{[t,\infty]}(t)|^2 dt \right)^\frac{1}{p} \]
\[\times \left( \int_0^\frac{t}{t} t^{(2\alpha+1)(p-2)+2\alpha-1} \chi_{[t,\infty]}(t) dt \right)^\frac{1}{p} \]
\[\leq c \left( \int_0^\frac{t}{t} t^{3(p-2)} |F(f)(t)|^p (g(t))^{p-2} \frac{dt}{|c(t)|^2} \right)^\frac{1}{p} \]
\[\times \left\{ \left( \int_0^\frac{t}{t} t^{2(p-2)} dt \right)^\frac{1}{p} + \left( \int_0^\frac{t}{t} t^{(2\alpha+1)(p-2)+2\alpha-1} dt \right)^\frac{1}{p} \right\} \]
\[\leq c \delta^{-2} \omega_{1,p}(f)(\delta) \left( \frac{1}{\delta^{p-1}} + \frac{1}{\delta^{2(p-2)+1}} \right) \leq c \left( \frac{\omega_{1,p}(f)(\delta)}{\delta^{p-1}} + \frac{\omega_{1,p}(f)(\delta)}{\delta^{2(p-2)+1}} \right).\]

Integrating with respect to \( \delta \) over \( \mathbb{R}_+ \) for \( f \in \mathcal{R}^p_{\gamma,\alpha+1} \cap \mathcal{R}^p_{\gamma,\alpha} \), the double integral is evaluated by interchanging the orders of integration, it yields
\[\int_0^{+\infty} |F(f)(t)| \frac{dt}{|c(t)|^2} < +\infty.\]

This complete the proof.

**Theorem 2** Let \( \gamma > 0, 1 \leq p \leq 2 \) and \( f \in \mathcal{R}^p_{\gamma,\alpha}, \) then

i) For \( p \neq 1 \) and \( 0 < \gamma \leq \frac{2(\alpha+1)}{p} \), one has
\[F(f) \in L^2_{\gamma}(\mathbb{R}_+) \] provided that \( \frac{2(\alpha+1)p}{2p+2(\alpha+1)(p-1)} < s \leq p' \).

ii) For \( p \neq 1 \) and \( \gamma > \frac{2(\alpha+1)}{p} \), one has
\[F(f) \in L^1_{\gamma}(\mathbb{R}_+).\]

iii) For \( p = 1 \) and \( \gamma > \sup(3,2(\alpha+1)), \) one has
\[F(f) \in L^1_{\gamma}(\mathbb{R}_+).\]

**Proof** Let \( f \in \mathcal{R}^p_{\gamma,\alpha}, 1 \leq p \leq 2, \)
i) Suppose that \( p \neq 1 \) and \( 0 < \gamma \leq \frac{2(\alpha+1)}{p} \). Let \( \frac{2(\alpha+1)p}{2p+2(\alpha+1)(p-1)} < s \leq p' \), we define the function
\[g(t) = \int_k^t |F(f)(x)|^s \frac{dx}{|c(x)|^2}, \quad t > k.\]
By Hölder’s inequality, (4) and (10) we have
\[g(t) \leq \left( \int_k^t |F(f)(x)|^s \frac{dx}{|c(x)|^2} \right)^{\frac{1}{s}} \left( \int_k^t \frac{dx}{|c(x)|^2} \right)^{1-\frac{1}{s}} \]
\[\leq ct^{2s} \left( \omega_{1,p}(f)(t)^{\frac{1}{s}} \right) \left( \int_k^t \frac{dx}{|c(x)|^2} \right)^{1-\frac{1}{s}} \]
\[\leq ct^{2(\gamma+1)} \left( \int_k^t x^{2(\alpha+1)} dx \right)^{1-\frac{1}{s}} \leq ct^{2(\gamma+1)(2(\alpha+1)(1-\frac{1}{s})).\]
Then we get
\[
\int \frac{|\mathcal{F}(f)(x)|^2}{|c(x)|^2} \, dx = \int \frac{x^{-2s}g'(t)}{|c(x)|^2} \, dx
\]
\[
= t^{-2s}g(t) + 2s \int x^{-2s-1}g(x) \, dx
\]
\[
\leq c \left( t^{-\gamma + 2(\alpha + 1)(1-\frac{1}{p})} + \int x^{-\gamma + 2(\alpha + 1)(1-\frac{1}{p})-1} \, dx \right)
\]
\[
\leq c \left( t^{-\gamma + 2(\alpha + 1)(1-\frac{1}{p})} + 1 \right),
\]
it yields that \( \mathcal{F}(f) \in L^p([-k, +\infty[ \times \{ \frac{dx}{|c(x)|^2} \} ). \) Since \( \mathcal{F}(f) \in L^p([-k, +\infty[ \times \{ \frac{dx}{|c(x)|^2} \} ) \subset L^1([-k, +\infty[ \times \{ \frac{dx}{|c(x)|^2} \} ), \) \( \) we deduce that \( \mathcal{F}(f) \) is in \( L^p_1(\mathbb{R}_+). \)

ii) Assume now \( \gamma > \frac{2(\alpha + 1)}{p}. \) For \( p \neq 1, \) by proceeding in the same manner as the proof of i) with \( s = 1, \) we obtain the desired result.

iii) For \( p = 1 \) and \( \gamma > \sup(3, 2(\alpha + 1)). \) By Hölder’s inequality, (3), (4) and (11), we have for \( t > 0 \)
\[
\int \frac{1}{|\mathcal{F}(f)(x)|^2} \, dx \leq \text{ess sup}_{0 < t \leq 1} \frac{1}{x} |\mathcal{F}(f)(x)| \int \frac{1}{x} \frac{dx}{|c(x)|^2}
\]
\[
\leq c t^{-2} \left( \int \frac{1}{x} x \chi_{\{0 \leq x \leq k\}} \frac{dx}{|c(x)|^2} + \int \frac{1}{x} x \chi_{\{x > k\}} \frac{dx}{|c(x)|^2} \right)
\]
\[
\leq c t^{-2} \left[ t^{-2} + t^{-(2\alpha + 1)} \right] \leq c \left[ t^{-1} + t^{-(\alpha + 1)} \right].
\]

Integration with respect to \( t \) over \( (0, 1) \) and applying Fubini’s theorem we obtain
\[
\int_1^{+\infty} \frac{1}{|\mathcal{F}(f)(x)|^2} \, dx \leq c \left( \int_0^1 t^{-(\alpha + 1) - 1} \, dt + \int_1^{+\infty} t^{-2(\alpha + 1) - 1} \, dt \right) < \infty.
\]
Since \( L^\infty(\mathbb{R}_+, \{ \frac{dx}{|c(x)|^2} \}) \subset L^1(\mathbb{R}_+, \{ \frac{dx}{|c(x)|^2} \} ), \) \( \) then \( \mathcal{F}(f) \in L^1_1(\mathbb{R}_+). \)

Remark 4 For \( \gamma > \sup(3, 2(\alpha + 1)), \) we can assert from the theorem 2, iii) that \( \mathcal{F}_{\gamma, \alpha}^1 \) is an example of space where we can apply the inversion formula.

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