Particle transport in incompressible MHD Kolmogorov flow

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Abstract. In previous work [1], an investigation of Kolmogorov flow for incompressible magnetohydrodynamics (MHD) was performed. It consists of a three-dimensional periodic flow driven by a unidirectional forcing varying on a transverse direction. In practice, the forcing is chosen to be $f_x = A \sin(k_f y)$, and $f_y = f_z = 0$. It was found that vorticity structures with long lifetimes can be formed in the turbulent regime. The presence of such structures may affect the transport of particles interacting with the flow as shown in this preliminary study.

1. Theoretical framework

1.1. Introduction

The general purpose of this work is the description of particle transport in turbulent plasmas. A plasma is generally composed of several species of charged particles as well as neutrals. Its complete description may thus require the use of several coupled kinetic equations. However, in the fluid limit, the large scale features of the plasma can be described using the magnetohydrodynamic equations. Depending on the level of description, single fluid or multi-fluid MHD approaches can be developed. Here, only the simplest fluid description of the plasma will be considered and its evolution is assumed to be driven by the incompressible MHD equations. However, an ensemble of test particles, interacting with the plasma and the electromagnetic fields, will also be considered. These particles are treated as passive. It is a common approximation for impurities in fusion plasmas [2, 3].

A detailed discussion of MHD Kolmogorov flow is proposed in [1]. Such a geometry is used here because it displays large two dimensional structures with long lifetimes. The presence of persistent large-scale structure is reminiscent of zonal flows observed in various fusion plasma situations. Moreover, since particle transport in two-dimensional Navier Stokes Kolmogorov flow displays anomalous transport properties [4], the same situation is expected in three-dimensional flows.

The system of equations describing the evolution of the MHD fields is given by:

\begin{align}
\partial_t u_i &= -\partial_j (u_i u_j - b_i b_j) + \nu \partial_j \partial_j u_i - \partial_i p + f_i \\
\partial_t b_i &= -\partial_j (b_i u_j - u_i b_j) + \eta \partial_j \partial_j b_i \\
\partial_t u_i &= 0 \\
\partial_t b_i &= 0
\end{align}

(1) (2) (3) (4)

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where the flow is assumed to be incompressible and isothermal to simplify the discussion. The only parameters appearing in these equations are the kinematic viscosity of the fluid $\nu$ and the magnetic diffusivity $\eta$. The pressure $p$ represents the sum of an hydrodynamic and a magnetic contributions rescaled by the density. Thanks to the incompressibility condition, no state equation is needed for $p$ which is obtained by imposing the constraint (3) into the equation (1). The flow is driven by the Kolmogorov force, given $f = (A \sin(kfy), 0, 0)$ and appears to be periodic but anisotropic. Indeed, the three directions play different “roles”: the forcing is along the $x$ direction, it varies along $y$, and it should be invariant to the $z$ direction. The structures themselves are in the $xy$ plane, so $z$ is the only direction of homogeneity, in a statistical sense. Indeed, once turbulence develops, the flow exhibits fluctuation in all three directions. Statistical results concerning the turbulent fields are always averaged over $z$, for faster convergence [5].

In [4], fluid particle transport in 2D Kolmogorov flow is investigated. Here, test particles interacting with both the fluid velocity field and the electromagnetic fields generated in 3D MHD Kolmogorov flow are considered. Indeed, the MHD equations do not only provide the fields $u$ and $b$, but also a self-consistent electric field given by:

$$ e = -u \times b + \eta \nabla \times b $$  \hspace{1cm} (5)

All these fields depend implicitly on both the position and time. Several types of test particles can be considered. The simplest example is given by the “Velocity field lines” (VFL) which correspond to fluid particles, or simply tracers:

$$ \frac{d}{dt} r = u $$  \hspace{1cm} (6)

Such particles simply follow the fluid without particular interaction with the electric and magnetic fields. However, if charged particles are considered, they might experience a triple interaction consisting of a friction with the fluid, an acceleration due to the electric field and a locally helical motion due to the magnetic field. The motion can then become extremely complex and very difficult to track numerically. For these reasons, various approximations have been designed. The simplest one, reasonably valid when the magnetic field is very strong, is given by the first guiding center (GC) approximation with the parallel velocity $v_\parallel$:

$$ \frac{d}{dt} r = v_\parallel \frac{b}{|b|} $$  \hspace{1cm} (7)

where $v_\parallel$ is a parameter with the dimensions of a velocity, and its value has to be determined by the specific conditions of the system studied. For instance, a charged particle, whose velocity projected on the local magnetic field is $v_\parallel$, will approximately follow a spiral with an axis that is precisely this guiding center. The guiding center approximation can be improved by adding extra terms to the right hand side of (7) to account for various effects such as the possible curvature of the magnetic field or the interaction with the electric field. Such more sophisticated approximations will be treated in future studies. The approximation of a “large” charged impurity (CP for “charged particle”), for which collisions can be approximated by a drag term is given by:

$$ \frac{d}{dt} r = v $$

$$ \frac{d}{dt} v = \alpha(e + v \times b) + \chi(u - v) $$  \hspace{1cm} (8)

where $\alpha$ corresponds to the charge to mass ratio in the Lorentz force, and $\chi$ is the inverse of a friction timescale. This model is a natural extension of the usual model for particles with mass moving in turbulent Navier-Stokes fluids [6].
1.2. Characterization of the dynamic regimes

For the MHD Kolmogorov flow, the same length and velocity scales defined for the Navier-Stokes case [5] are used:

\[ L = \frac{1}{k_f}, \quad U = \left( \frac{A}{k_f} \right)^{1/2}, \]  

(i.e. based on the characteristics of the forcing, \( f = (A\sin(k_f y), 0, 0) \)). Kinetic and magnetic Reynolds numbers corresponding to these units can then easily be defined:

\[ Re = \frac{UL}{\nu} = \frac{A^{1/2}}{\nu k_f^{3/2}}, \quad Re_m = \frac{UL}{\eta} = \frac{A^{1/2}}{\eta k_f^{3/2}}, \]  

and the magnetic Prandtl number is defined as the ratio of the two Reynolds numbers: \( Pr_m = Re_m/Re \). A timescale can also be introduced:

\[ \tau_f = \frac{L}{U} \equiv \left( \frac{Ak_f}{1} \right)^{1/2} \]  

Throughout this work, Alfven units are used for the fields ([b] = [u] = ms\(^{-1}\) and [e] = \( m^2 s^{-2} \)), so \( \alpha \) (proportional to the charge to mass ratio) is the inverse of a length and \( \chi \) is the inverse of a time. An alternative is then to note \( \ell \equiv 1/\alpha \) and \( \tau \equiv 1/\chi \) — this allows for a straightforward construction of adimensional parameters. The following three adimensional numbers can then be defined:

\[ \beta = \frac{v_\parallel}{U} \equiv \frac{v_\parallel A^{1/2}}{k_f^{1/2}}, \quad \gamma = \frac{\ell}{L} \equiv \frac{k_f}{1}, \quad St = \frac{\tau}{\tau_f} \equiv \left( \frac{Ak_f}{1} \right)^{1/2} \]  

to characterize the behaviour of the particles. The Stokes number \( St \) is well-known in the Navier-Stokes turbulence literature; note that the timescale defined by the forcing has been used here for \( St \) rather than the dissipation timescale. Parameters might be used differently in the literature: in [7] for instance, the classical Stokes number definition is used, and a different adimensional parameter is defined using \( \ell \).

2. Simulations

2.1. Method

The MHD equations are solved using TURBO [8], a pseudo-spectral solver. This code has been extended to study the transport of passive particles interacting with the fields generated by the MHD evolution. Particle trajectories in turbulent flows have been studied numerically for quite some time, but there are a few particularities to charged particles. The electric field is constructed as a product of two turbulent fields summed with derivatives of another turbulent field. Its spectrum might thus be quite different from the velocity field spectrum and, in particular, it might exhibit a slower decrease in \( k \). As a consequence, the interpolation scheme needed to estimate the electric field at the particle position must be well behaved. Furthermore, because particles can have characteristic timescales much smaller than those of the MHD flow, it can be necessary to use high order integration schemes for tracking their position, which in turn, may impose high order spline spatial interpolations for the fields [9].

The particle trajectories are integrated in time using a fourth order implicit Runge Kutta solver [10], coupled with a seventh order spline computed over six grid points (\( S(7,6) \) from [9]). It has been checked that improving the time integration scheme or the interpolation method does not affect the results. Also, double precision floating point arithmetic is used for all the calculations, as this can have a noticeable effect on particle trajectory integration, [11].
2.2. Parameters
A moderately turbulent regime of $Re = Re_m = 100$ has been chosen. The fluid equations are solved with the pseudo-spectral solver, in a periodic box of dimensions $6\pi L \times 2\pi L \times 2\pi L$, using a uniform, rectangular grid with $784 \times 128 \times 128$ nodes. Vorticity structures with very long (possibly infinite) lifetimes, $[1, 5]$ are then observed in the flow. It is worth mentioning that, when these vorticity structures are present, very strong vertical streams are also formed, i.e. in the $y$ direction. They are formed in pairs, so the average fluid flow on the vertical direction is still null. This geometry is reminiscent of stratified turbulence, with alternating vertical sheets of movement in opposite directions.

Once the quasi-stationary regime is reached, particles are placed uniformly in the field, on a $15 \times 5 \times 5$ grid. The initial velocity for CP is random and of the same order as the fluid velocity $|\mathbf{v}| \sim U$. Because of the drag term, it is expected that the particle’s velocities will be close to the local fluid velocity after a relaxation time. Two values of $\beta$ (1 and 10) are investigated, and four pairs $(\gamma, St)$: $(1/2, 1), (1/2, 1/10), (1/4, 1)$ and $(1/4, 1/10)$.

2.3. Mean squared displacements
Preliminary results show that particles are very much influenced by the large scale structures in the Kolmogorov flow. Figure 1 shows that, after the ballistic regime, VFL along the $y$ direction seem to be super-diffusive. The number of particles appears to be small, as there are significant irregularities in the curves; it must however be recognised that in evolving turbulence bursts in the fluid flow can affect the short time behavior of particles as well. The black dash-dot line corresponding to normal diffusion is shown for comparisons.

![Figure 1](image1.png)

**Figure 1.** Mean squared displacements for both the velocity field lines and the guiding center approximation.

Charged particles (see Figure 2) also exhibit anisotropic diffusion properties. Again, the mean-square displacement is very close to classical diffusion in the forcing ($x$) and transverse ($z$) directions. However, due to the large-scale structures that correspond to strong flows, a super-diffusive regime is observed in the $y$ direction.

2.4. Velocity PDF
The velocity distribution functions for VFL and CP are shown in figure 3. It seems that, for these values of the parameters, the drag term controls CP behaviour (because CP results are very close to VFL results). It is noted that the distribution function for $v_y$ is apparently the sum of two Gaussian distributions. This further confirms the presence of two “streams” along the $y$ axis. Superdiffusive transport for VFL is explained by these two maxima — particles may be
Figure 2. Mean squared displacements for charged particles submitted to both the Stokes and the Lorentz forces.

caught in one vertical stream or the other, and they can experience long flights. Superdiffusive transport was observed for stratified turbulence in [12, 13], but it is not obvious if the generating mechanism is similar (for instance the forcing is different here).

Figure 3. Velocity PDF for VFL and CP

Another interesting result is obtained when looking at the GC approximation. Since these particles are moving at a constant velocity $v_\parallel$ (in amplitude) along the local magnetic field, a Dirac delta distribution has to be observed for the norm of their velocity $|v|$. However, the distributions for individual coordinates do show interesting properties, as seen in figure 4.
Because these particles have a fixed velocity amplitude, all three PDFs have finite support. Some correlation between the magnetic field and the velocity field is to be expected, but it is interesting that both the \( x \) and the \( y \) components of the velocity have distributions with two maxima.

### 2.5. Acceleration PDF

![Figure 4. Velocity PDF for GC.](image)

![Figure 5. Acceleration PDF for VFL and CP.](image)

The probability distribution functions of the accelerations for VFL and CP are shown in figure 5. It is apparent that even though the drag term is present, there are important differences...
between charged particles and velocity field lines. The case $\gamma = 1/2$, $St = 1$ is the one that is closest to the behaviour of fluid particles. Note that the $x$ and $y$ components of the acceleration are sums of two different distributions. The long tailed distribution is generated in the initial relaxation phase (this is shown by results to be discussed elsewhere), whereas the central peak is generated in the quasistationary regime. It is however quite interesting that the second distribution with a longer tail is not observed for the $z$ component of the acceleration.

3. Conclusions

The transport of charged test-particles in a fluid interacting with both the fluid velocity and the electromagnetic fields has been considered. The fluid flow is generated by the so-called Kolmogorov forcing which is known to induce persistent large-scale structures in the system. The fluid is assumed to be described by the incompressible MHD formalism.

Three types of trajectories have been considered. Test particles that follow the guiding center approximation and charged particles subject to a combination of the Stokes and the Lorentz forces have been compared to velocity field lines. Strong similarities between velocity field lines and charged particles can be observed. Also, the distributions of velocities for the guiding center trajectories are strongly affected by the constraint that the velocity is always directed along the local magnetic field with a constant amplitude. In all cases, the interaction of the particle trajectories with large-scale structures in the Kolmogorov flow generates anisotropic transport properties.

This preliminary study can be extended in various ways. First, more sophisticated guiding center approximations taken into account the presence of the electric field or the topology of the magnetic field could be investigated. The comparison with the charge particle trajectories would be a very valuable assessment of the guiding center approximations. Secondly, a more systematic parameter study could be initiated. It would definitively be interesting to evaluate the respective influence of the Stokes and Lorentz force on the diffusive properties of charged particles in the study of impurity transport in fusion plasmas. Finally, the influence of an imposed external magnetic field could also be investigated. Such a constant field would be an additional source of anisotropy that could compete with the large-scale structure influence.

3.1. Acknowledgements

D.C. is research director of the FRS-FNRS (Belgium). This work has been partly supported by the contract of association EURATOM-Belgian state. D.C. and C.C.L. would like to acknowledge support from COST Action MP0806.