The $C^\alpha$ regularity of a class of ultraparabolic equations

ZHANG Liqun *

Institute of Mathematics, AMSS, Academia Sinica, Beijing

Abstract

We prove the $C^\alpha$ regularity for weak solutions to a class of ultraparabolic equation, with measurable coefficients. The results generalized our recent $C^\alpha$ regularity results of Prandtl’s system to high dimensional cases.

keywords: Ultraparabolic equations, Moser iteration, $C^\alpha$ regularity

1 Introduction

The ultraparabolic equation arises in many applications, for example, fluid dynamics, mathematical finance, degenerated diffusion process, etc. There are more and more studies on this problem in recent years. The regularity of this type of equation becomes interesting since it has some special algebraic structures and is degenerated. It is still unclear in general, whether the interior $C^\alpha$ regularity results hold for weak solutions of the ultraparabolic equations with bounded measurable coefficients like the parabolic cases.

*The author currently is working at NSFC.
In the study of boundary layer problem, we obtained the existence global weak solution in the class that Oleinik considered under the assumption that the pressure is favorable [13]. One of the interesting question is whether the weak solution is actually smooth? In particular, in the two dimensional Prandtl’s system with constant pressure, in the Crocco variable, we can deduce the following equation

\[
\frac{\partial u}{\partial t} + y \frac{\partial u}{\partial x} - u^2 \frac{\partial^2 u}{\partial y^2} = 0.
\]

This is of strong degenerated parabolic type equations, more precisely, an ultraparabolic type equation. However, it satisfies the well known Hörmander’s hypoellipticity conditions, which sheds lights on the smoothness of weak solutions. It is proved by Polidoro and Ragusa [12] that the weak solution is in the \(C^\alpha\) class, if the coefficient is in the class of VMO. That is, if the weak solution \(u\) is also in the class of VMO, then the solution \(u\) is \(C^\alpha\) continuous and then the result of Xu [15], or Bramanti, Cerutti and Manfredini [1] implies that the solution is also \(C^\infty\) smooth. In our case, however, we can only prove that the weak solution is in the class of BV space. It is interesting if the weak solution of equation (1.1) is still smooth when the coefficient is only measurable functions.

On the other hand, equation (1.1) has the divergent form if we replace \(u\) by \(\frac{1}{u}\). A recent paper by Pascucci and Polidoro [11] proved that the Moser iterative method still works for a class of ultraparabolic equations with measurable coefficients including equation (1.1). Their results showed that for a non-negative sub-solution \(u\) of (1.1), the \(L^\infty\) norm of \(u\) is bounded by the \(L^p\) norm of \(u\) \((p \geq 1)\). This is a very important step toward the final solution of regularity of the ultraparabolic equations.

We proved in [14] that the weak solution that we obtained in [13] of (1.1) is of \(C^\alpha\) class and then \(u\) is smooth. In this paper, we are concerned with the \(C^\alpha\) regularity of solutions of the ultraparabolic equations. We shall generalize
the result in [13] to high dimensional cases in this paper.

We consider a class of Komogorov-Fokker-Planck type operator on $R^{N+1}$.

(1.2) \[ Lu \equiv \sum_{i,j=1}^{m_0} \partial_{x_j} (a_{ij}(t, x) \partial_{x_i} u) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u = 0, \]

where $(x, t) \in R^{N+1}, 1 \leq m_0 \leq N$, and $b_{ij}$ is constant for every $i, j = 1, \cdots, N$. We make the following assumptions on the coefficients of $L$:

$(H_1)$ $a_{ij} = a_{ji} \in L^\infty(R^{N+1})$ and there exists a $\lambda > 0$ such that

\[ \frac{1}{\lambda} \sum_{i=1}^{m_0} \xi_i^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(t, x) \xi_i \xi_j \leq \lambda \sum_{i=1}^{m_0} \xi_i^2 \]

for every $(t, x) \in R^{N+1}$, and $\xi \in R^{m_0}$.

$(H_2)$ The matrix $B = (b_{ij})_{N \times N}$ has the form

\[
\begin{pmatrix}
0 & B_1 & 0 & \cdots & 0 \\
0 & 0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_d \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

where $B_k$ is a matrix $m_{k-1} \times m_k$ with rank $m_k$ and $m_0 \geq m_1 \geq \cdots \geq m_d$, $m_0 + m_1 + \cdots + m_d = N$. $||B|| \leq \lambda$ where the norm $|| \cdot ||$ is in the sense of matrix norm.

The requirements of matrix $B$ in $(H_2)$ ensures that the operator $L$ with constant $a_{ij}$ satisfies the well-known Hömander’s hypoellipticity condition.

The Schauder type estimate of (1.2) has been obtained. Besides, the regularity of weak solutions have been studied by Bramanti, Cerutti and Manfredini [1], Manfredini and Polidoro [7], Polidoro and Ragusa [12] assuming a weak continuity on the coefficient $a_{ij}$. It is quite interesting whether the
weak solution has Hölder regularity under the assumption \((H_1)\) on \(a_{ij}\). The first advances is the work of A. Pascucci and S. Polidoro [11] who proved that Moser iteration method still works for equation (1.2). One of the approach to the Hölder estimates is to obtain the Harnack type inequality. In the case of elliptic equation with measurable coefficients, the Harnack inequality is obtained by J. Moser [8] via an estimate of BMO functions due to F. John and L. Nirenberg together with the Moser iteration method. J. Moser [9] also obtained the Harnack inequality for parabolic equations with measurable coefficients by generalizing the John-Nirenberg estimates to the parabolic case. Another approach to the Hölder estimates is given by S. N. Kruzhkov [6], [7] based on the Moser iteration to obtain a local priori estimates, which provides a short proof for the parabolic equations.

We prove a Poincare type inequality for non-negative weak sub-solutions of (1.2). Then we apply it to obtain a local priori estimates which implies the Hölder estimates for ultraparabolic equation (1.2).

Let \(D_{m_0}\) be the gradient with respect to the variables \(x_1, x_2, \ldots, x_{m_0}\). And

\[
Y = \sum_{i,j=1}^{N} b_{ij} x_i \partial x_j - \partial_t.
\]

We say that \(u\) is a weak solution of (1.2) if it satisfies (1.2) in the distribution sense and \(u, D_{m_0}u, Y u \in L^2_{loc}\).

Our main result is the following theorem.

**Theorem 1.1** Under the assumptions \((H_1)\) and \((H_2)\), the weak solution of (1.2) is Hölder continuous.
2 Some Preliminary Results

One of the important feature of equation (1.2) is that the fundamental solution can be written down explicitly if the coefficients $a_{ij}$ is constant, (see [1], [4]). Besides, there are some geometric and algebraic structures in the space $R^{N+1}$ induced by the constant matrix $B$ (see for instance, [1]).

Let $E(\tau) = exp(-\tau B^T)$, where $E(\tau)$ is a polynomial of degree $d$ in $\tau$ with $N \times N$ matrices coefficients. For $(t, x), (\tau, y) \in R^{N+1}$, set

$$(t, x) \circ (\tau, y) = (t + \tau, y + E(\tau)x).$$

Then $(R^{N+1}, \circ)$ is a group with neutral element $(0, 0)$; the inverse of an element $(t, x)$ is $(t, x)^{-1} = (-t, -E(-t)x)$. The left translation by $(\tau, y)$ given by

$$(t, x) \mapsto (\tau, y) \circ (t, x),$$

is a invariant translation to operator $L$ when coefficient $a_{ij}$ is constant.

The associated dilation to operator $L$ with constant coefficient $a_{ij}$ is given by

$$\delta_\lambda = diag(\lambda^2, I_{m_0}, \lambda^3 I_{m_1}, \cdots, \lambda^{2d+1} I_{m_d}),$$

where $I_{m_k}$ denotes the $m_k \times m_k$ identity matrix. Then the operator is homogeneous of degree 2 with respect to the dilation $\delta_\lambda$. Let

$$Q = m_0 + 3m_1 + \cdots + (2d + 1)m_d.$$ 

Then the number $Q + 2$ is usually called the homogeneous dimension of $R^{N+1}$ with respect to the dilation $\delta_\lambda$.

The norm in $R^{N+1}$, related to the group of translations and dilation to the equation is defined by

$$\| (t, x) \| = r$$

5
if \( r \) is the unique positive solution to the equation

\[
\frac{x_1^2}{\gamma^{2\alpha_1}} + \frac{x_2^2}{\gamma^{2\alpha_2}} + \cdots + \frac{x_N^2}{\gamma^{2\alpha_N}} + \frac{t^2}{\gamma^4} = 1,
\]

where \((t, x) \in \mathbb{R}^{N+1} \setminus \{0\}\) and

\[
\alpha_1 = \cdots = \alpha_{m_0} = 1, \quad \alpha_{m_0+1} = \cdots = \alpha_{m_0+m_1} = 3, \cdots,
\]

\[
\alpha_{m_0+\cdots+m_{d-1}+1} = \cdots = \alpha_N = 2d + 1.
\]

And \(||(0, 0)|| = 0\). The balls at a point \((t_0, x_0)\) is defined by

\[
\mathcal{B}_r(t_0, x_0) = \{(t, x) | \quad ||(t, x)^{-1} \circ (t_0, x_0)|| \leq r\}.
\]

Let

\[
\mathcal{B}_r^-(t_0, x_0) = \mathcal{B}_r(t_0, x_0) \cap \{t < t_0\}.
\]

For convenience, we sometimes use the cube replace the balls. The cube at point \((0, 0)\) is given by

\[
\mathcal{C}_r(0, 0) = \{(t, x) | \quad |t| \leq r^2, \quad |x_1| \leq r^{\alpha_1}, \cdots, |x_N| \leq r^{\alpha_N}\}.
\]

It is easy to see that there exists a constant \( \Lambda \) such that

\[
\mathcal{C}_{r^\Lambda}(0, 0) \subset \mathcal{B}_r(0, 0) \subset \mathcal{C}_{r^{-\Lambda}}(0, 0),
\]

where \( \Lambda \) only depends on \( B \) and \( N \).

When the matrix \((a_{ij})_{N \times N}\) is of constant matrix, we denoted it by \( A_0 \). Then the operator \( L \) takes the form

\[
L_0 = \text{div}(A_0 D) + Y.
\]

We let \( z = (t, x) \). The fundamental solution \( \Gamma_0(\cdot, \zeta) \) of \( L_0 \) with pole in \( \zeta \in \mathbb{R}^{N+1} \) has been constructed (see [1]) as follows:

\[
\Gamma_0(z, \zeta) = \Gamma_0(\zeta^{-1} \circ z, 0), \quad z, \zeta \in \mathbb{R}^{N+1}, \quad z \neq \zeta,
\]
And $\Gamma_0(z, 0)$ can be written down explicitly. There are some basic estimates for $\Gamma_0$

$$\Gamma_0(z, \zeta) \leq c||\zeta^{-1} \circ z||^{-Q},$$

$$|\partial_{x_i} \Gamma_0(z, \zeta)| \leq c||\zeta^{-1} \circ z||^{-Q-1},$$

where $i = 1, \ldots, m_0$.

A weak sub-solution of (1.2) in a domain $\Omega$ is a function $u$ such that $u$, $D_{m_0}u$, $Yu \in L^2_{\text{loc}}(\Omega)$ and for any $\phi \in C^\infty_0(\Omega)$, $\phi \geq 0$,

(2.1) $$\int_\Omega \phi Yu - (Du)^T AD\phi \geq 0.$$

A result of Pascucci and Polidoro obtained by using the Moser’s iterative method (see [11]) states as follows.

**Lemma 2.1** Let $u$ be a non-negative weak sub-solution of (1.2) in $\Omega$. Let $(t_0, x_0) \in \Omega$ and $\overline{B_r(t_0, x_0)} \subset \Omega$ and let $p \geq 1$. Then there exists a positive constant $c$ which depends only on $\lambda$ and the homogeneous dimension $Q$ such that, for $0 < r \leq 1$

(2.2) $$\sup_{B_{r/2}(t_0, x_0)} u^p \leq \frac{c}{r^{Q+2}} \int_{B_r(t_0, x_0)} u^p,$$

provided that the last integral converges.

We copy a classical potential estimates (see [3]) here to prove the Poincare type inequality.

**Lemma 2.2** Let $\alpha \in (0, Q+2)$ and $G \in C(R^{N+1} \setminus \{0\})$ be a $\delta_\lambda$-homogeneous function of degree $\alpha - Q - 1$. If $f \in L^p(R^{N+1})$ for some $p \in (0, \infty)$, then

$$G_f(z) \equiv \int_{R^{N+1}_+} G(\zeta^{-1} \cdot z) f(\zeta) d\zeta,$$
is defined almost everywhere and there exists a constant $C = C(Q, p)$ such that

$$
\|G f\|_{L^q(R^{N+1})} \leq C \max_{\|z\|=1} |G(z)| \quad \|f\|_{L^p(R^{N+1})},
$$

where $q$ is defined by

$$
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q + 2}.
$$

3 Proof of Main Theorem

We want to obtain a local estimates of solutions of the equation (1.2), for instant, at point $(t_0, x_0)$. Since the equation (1.2) is invariant under the left group translation when $a_{ij}$ is constant, we may consider the estimates at a ball centered at $(0,0)$. We mainly prove the following Lemma 3.4 which is essential in the oscillation estimates in Kruzhkov’s approaches in parabolic case. Then the $C^\alpha$ regularity result follows easily by the standard arguments.

For convenience, in the following discussion, we let $x' = (x_1, \cdots, x_{m_0})$ and $x = (x', \overline{x})$. We consider the estimates in the following cube, instead of $B^{-}_r$,

$\mathcal{C}_r = \{(t, x) | -r^2 \leq t \leq 0, \quad |x'| \leq r, |x_{m_0+1}| \leq \lambda N^2 r^3 \cdots, |x_N| \leq \lambda N^2 r^{2d+1}\}$.

Let

$$
S_r = \{\overline{x} \mid |x_{m_0+1}| \leq \lambda N^2 r^3 \cdots, |x_N| \leq \lambda N^2 r^{2d+1}\}.
$$

Let $0 < \alpha, \beta < 1$ be constant and

$$
K_r = \{x' \mid |x'| \leq r\},
$$

$$
S_{\alpha r} = \{\overline{x} \mid |x_{m_0+1}| \leq \lambda N^2 (\alpha r)^3 \cdots, |x_N| \leq \lambda N^2 (\alpha r)^{2d+1}\},
$$

$$
K_{\alpha r} = \{x' \mid |x'| \leq \alpha r\}.
$$
Now for fixed $t$, let
\[ \mathcal{N}_t = \{(x',x) \in K_{\beta r} \times S_{\beta r}, \quad u \geq h\}. \]

In the following discussions, we sometimes abuse the notations of $\mathcal{B}_r^-$ and $\mathcal{C}_r$, since there are equivalent.

**Lemma 3.1** Suppose that $u(t,x) \geq 0$ be a solution of equation (1.2) in $\mathcal{B}_r^-$ centered at $(0,0)$ and
\[ \text{mes}\{(t,x) \in \mathcal{B}_r^-, \quad u \geq 1\} \geq \frac{1}{2} \text{mes}(\mathcal{B}_r^-). \]

Then there exist constant $\alpha$, $\beta$ and $h$, $0 < \alpha, \beta, h < 1$ which only depend on $\lambda$ and $N$ such that for all $t \in (-\alpha r^2, 0)$,
\[ \text{mes}\{\mathcal{N}_t\} \geq \frac{1}{11} \text{mes}\{K_{\beta r} \times S_{\beta r}\}. \]

**Proof:** Let
\[ v = \ln^+(\frac{1}{u + h^2}), \]
where $h$ is a constant $0 < h < 1$ to be determined later. Then $v$ at points where $v$ is positive, satisfies
\[ \sum_{i,j=1}^{ma} \partial_{x_j} (a_{ij}(t,x) \partial_{x_i} v) - (Dv)^T ADv + x^T BDv - \partial_t v = 0. \]

Let $\eta(x')$ be a smooth cut-off function so that
\[ \eta(x') = 1, \quad \text{for} \quad |x'| < \beta r; \]
\[ \eta(x') = 0, \quad \text{for} \quad |x'| \geq r. \]

Moreover, $0 \leq \eta \leq 1$ and $|D_{ma} \eta| \leq \frac{2ma}{(1-\beta)r}$. 

9
Multiplying $\eta^2(x')$ to (3.1) and integrating by parts

\[
\int_{K_{\beta r}} \int_{S_{\beta r}} v(t, x', \overline{\tau}) d\tau dx' + \frac{1}{2\lambda} \int_{\tau}^{t} \int_{K_{\beta r}} \int_{S_{\beta r}} |D_{w_0} v|^2 d\tau dx' dt
\]

\[
\leq \frac{C}{\beta^2 N(1-\beta)^2} \text{mes}(S_{\beta r}) \text{mes}(K_{\beta r}) + \int_{\tau}^{t} \int_{K_{\beta r}} \int_{S_{\beta r}} \eta^2 x^T BDv d\overline{\tau} dx' dt
\]

\[
+ \int_{K_{\beta r}} \int_{S_{\beta r}} v(\tau, x', \overline{\tau}) d\tau dx',
\]

where $C$ only depends on $\lambda$ and $N$.

Integrating by parts, we have for any $i, j$

\[
\int_{K_{\beta r}} \int_{S_{\beta r}} \eta^2 x_i b_{ij} \partial_x v d\overline{\tau} dx' \leq \frac{r^{-2}}{4N^2} \beta^{-2Q} \ln\left(\frac{1}{h^2}\right) \text{mes}(S_{\beta r}) \text{mes}(K_{\beta r}).
\]

Then

\[
\int_{\tau}^{t} \int_{K_{\beta r}} \int_{S_{\beta r}} \eta^2 x^T BDv d\overline{\tau} dx' dt \leq \frac{1}{4} \beta^{-2Q} \ln\left(\frac{1}{h^2}\right) \text{mes}(S_{\beta r}) \text{mes}(K_{\beta r}).
\]

We shall estimate the measure of the set $\mathcal{N}_1$. Let

\[
\mu(t) = \text{mes}\{(x', \overline{\tau})|\ x' \in K_r, \ \overline{\tau} \in S_r, \ u \geq 1\}.
\]

By our assumption, (for convenience, we may let $B_r^{-}$ be replaced by $C_r^{-}$), for $0 < \alpha < \frac{1}{2}$

\[
\frac{1}{2} r^2 \text{mes}(S_r) \text{mes}(K_r) \leq \int_{-\tau^2}^{0} \mu(t) dt = \int_{-\tau^2}^{-\alpha r^2} \mu(t) dt + \int_{-\alpha r^2}^{0} \mu(t) dt.
\]

That is

\[
\int_{-\tau^2}^{-\alpha r^2} \mu(t) dt \geq \left(\frac{1}{2} - \alpha\right) r^2 \text{mes}(S_r) \text{mes}(K_r).
\]

Then there exists a $\tau \in (-\tau^2, -\alpha r^2)$, such that

\[
\mu(\tau) \geq \left(\frac{1}{2} - \alpha\right)(1 - \alpha)^{-1} \text{mes}(S_r) \text{mes}(K_r).
\]
From (3.2) and (3.6), we have by noticing $v = 0$ when $u \geq 1$

\begin{equation}
\int_{K_r} \int_{S_{\beta r}} v(\tau, x', x) dx' dx \leq \frac{1}{2} (1 - \alpha)^{-1} mes(S_r) mes(K_r) \ln \left( \frac{1}{h^{\frac{a}{b}}} \right).
\end{equation}

Now we choose $\alpha$ (near zero) and $\beta$ (near one), so that

\begin{equation}
\frac{1}{4\beta^{2q}} + \frac{1}{2\beta^{2q}(1 - \alpha)} \leq \frac{4}{5}.
\end{equation}

By (3.2), (3.4), (3.7) and (3.8), we deduce

\begin{equation}
\ln \left( \frac{1}{2h} \right) mes(K_{\beta r} \times S_{\beta r} \setminus \mathcal{N}_t) \leq \left[ C(1 - \beta)^{-2} \beta^{-q} + \frac{4}{5} \ln \left( \frac{1}{h^{\frac{a}{b}}} \right) \right] mes(K_{\beta r} \times S_{\beta r}).
\end{equation}

Since

\begin{equation}
\frac{\ln(h^{-\frac{a}{b}})}{\ln(h^{-1})} \to \frac{9}{8}, \quad \text{as} \quad h \to 0,
\end{equation}

then there exists constant $h_1$ such that for $0 < h < h_1$ and $t \in [-\alpha r^2, 0]$

\begin{equation}
mes(K_{\beta r} \times S_{\beta r} \setminus \mathcal{N}_t) \leq \frac{10}{11} mes(K_{\beta r} \times S_{\beta r}).
\end{equation}

Then we proved our lemma.

**Corollary 3.1** Under the assumptions of Lemma 3.1, we can choose $\theta$, $0 < \theta < \alpha$ and $\theta < \beta$ small enough so that

\[ mes \{ B_{\beta r} \setminus B_{\theta r} \cap \{(t, x) \mid u \geq h\} \} \geq C_0(\alpha, \beta, \Lambda) mes \{ B_{\beta r} \}, \]

where $0 < C_0(\alpha, \beta, \Lambda) < 1$.

Let $\chi(s)$ be a smooth function given by

\[ \chi(s) = \begin{cases} 1 & \text{if} \ s \leq \sqrt{\theta}r, \\ 0 & \text{if} \ s > \beta r, \end{cases} \]
where $\sqrt{\theta} < \frac{\beta}{2}$ is a constant. Moreover, we assume that

$$0 \leq -\chi'(s) \leq \frac{2}{(\beta - \sqrt{\theta})r},$$

and $\chi'(s) < 0$ if $\sqrt{\theta}r < s < \beta r$. We set

$$\phi_0(t, x) = \chi(\|t\|Q(C^{-1}(\|t\|)e^{tB^T}x, e^{tB^T}x) + \sum_{i=m_0}^N \frac{x_i^2}{r^{2q_i-2q}} - c_1r^{2q-2})^{\frac{1}{2}},$$

$$\phi_1(x) = \chi(\theta|x'|),$$

(3.10) $$\phi(t, x) = \phi_0(t, x)\phi_1(x),$$

where $c_1 > 1$ is chosen so that

$$|2 \sum x_i b_{ij} \frac{x_j^2}{r^{2q_i-2q}}| + |t|^Q(A_0e^{tB}C^{-1}e^{tB^T}x, e^{tB}C^{-1}e^{tB^T}x) < c_1r^{2q-2},$$

for $-r^2 \leq t \leq 0$ and $x \in K_r \times S_r$.

We now have the following Poincaré’s type inequality.

**Lemma 3.2** Let $w$ be a non-negative weak sub-solution of (1.2) in $B_1$. Then there exists a constant $C$, only depends on $\lambda$ and $N$, such that for $r < \theta < 1$

(3.11) $$\int_{B^-_{\theta r}} (w(z) - I_0)^2 \leq C \frac{r^2}{(1 - \theta)^2} \int_{B^-_{\theta r}} |D_{m_0} w|^2,$$

where $I_0$ is given by

(3.12) $$I_0 = \max_{B^-_{\theta r}} [I_1(z) + C_2(z)],$$

and

(3.13) $$I_1(z) = \int_{B^-_{\theta r}} \langle (\phi_1 A_0 D\phi_0, D\Gamma(z, \cdot))w - \Gamma_0(z, \cdot) w\rangle \phi(\zeta) d\zeta,$$

$$C_2(z) = \int_{B^-_{\theta r}} \langle (\phi_0 A_0 D\phi_1, D\Gamma_0(z, \cdot))w \rangle(\zeta) d\zeta,$$

where $\Gamma_0$ is the fundamental solution, and $\phi$ is given by (3.10).
Proof: We represent $w$ in terms of the fundamental solution of $\Gamma_0$. For $z \in \mathcal{B}_{\theta r}$, we have

$$
\begin{align*}
    w(z) &= \int_{\mathcal{B}_{\theta r}} \left[ \langle A_0 D(w\phi), D\Gamma_0(z, \cdot) \rangle - \Gamma_0(z, \cdot) Y(w\phi) \right](\zeta) d\zeta \\
    &= I_1(z) + I_2(z) + I_3(z) + C_2(z),
\end{align*}
$$

where $I_1(z)$ is given by (3.13) and

$$
I_2(z) = \int_{\mathcal{B}_{\theta r}} \left[ \langle (A_0 - A) Dw, D\Gamma_0(z, \cdot) \phi - \Gamma_0(z, \cdot) \langle ADw, D\phi \rangle \right](\zeta) d\zeta,
$$

$$
I_3(z) = \int_{\mathcal{B}_{\theta r}} \left[ \langle ADw, D(\Gamma_0(z, \cdot) \phi) \rangle - \Gamma_0(z, \cdot) \phi Y w \right](\zeta) d\zeta,
$$

And $C_2(z)$ denotes the remaining parts of the integral of $I_1$.

From our assumption that $w$ is a weak sub-solution of (1.2), then $I_3(z) \leq 0$ (see[11]). Then in $\mathcal{B}_{\theta r}$,

$$
0 \leq (w(z) - I_0)_+ \leq I_2(z).
$$

By Lemma 2.2 we have

$$
\|I_2\|_{L^2(\mathcal{B}_{\theta r})} \leq \theta r \|I_2\|_{L^2(\mathcal{B}_{\theta r})} \leq \frac{C \theta^2 r}{1 - \theta} \|D_{m_0}w\|_{L^2(\mathcal{B}_{\theta r})}.
$$

Then we proved our lemma.

Now we apply Lemma 3.2. to the function

$$
w = \ln^+ \frac{h}{u + h^\frac{\alpha}{\Lambda}}.
$$

We estimate the value of $I_0$ given by (3.12) and (3.13) in Lemma 3.2.

**Lemma 3.3** Under the assumptions of Lemma 3.2, there exist constants $\lambda_0$, $r_0$ and $r_0 < \theta$ only depend on constants $\alpha$, $\beta$, $\Lambda$, $\lambda$, $\phi$ and $\theta$, $0 < \lambda_0 < 1$, such that for $r < r_0$

$$
|I_0| \leq \lambda_0 \ln(\frac{1}{h^\frac{\alpha}{\Lambda}}).
$$

13
Proof: We first note that the support of the function $\phi_0 D\phi_1$ is contained in the set $B_{\theta r}^- \cap \{(t, x', \tau) | \sqrt{\theta} < |x'| < \frac{r}{\theta}\}$. Then it is easy to check that there exists a constant $C$ which only depends on $\lambda$ and $N$ such that, for $z \in B_{\theta r}^-$

$$|C_2(z)| \leq C\theta \ln\left(\frac{1}{h} \right).$$

Therefore $C_2(z) \to 0$ as $\theta \to 0$.

Now we let $w \equiv 1$ then (3.14) gives, for $z \in B_{\theta r}^-$,

$$1 = \int_{B_{\theta r}^-} \left[ \langle \phi_1 A_0 D\phi_0, D\Gamma_0(z, \cdot) \rangle - \Gamma_0(z, \cdot)Y\phi \rangle(\zeta) d\zeta + C_2(z) \right]_{w=1},$$

where $\phi$ is given by (3.10). Since the matrix $C^{-1}(t)$ is positive definite for $t > 0$ and by the assumption of matrix $B$, one can check

$$Y \langle C^{-1}(|t|)x, x \rangle = -\langle A_0 C^{-1}(|t|)x, C^{-1}(|t|)x \rangle,$$

then by the choosing of $c_1$, it is easy to see that

$$Y \phi \leq 0.$$

For $z = 0$, by our construction of $\phi$, we have

$$\langle \phi_1 A_0 D\phi_0, D\Gamma_0(z, \cdot) \rangle \geq 0,$$

therefore

$$\langle \phi_1 A_0 D\phi_0, D\Gamma_0(z, \cdot) \rangle - \Gamma_0(z, \cdot)Y\phi \geq 0.$$

In fact,

$$\langle A_0 D\langle C^{-1}(|t|)e^{tB^T} x, e^{tB^T} x \rangle, D\langle C^{-1}(|t|)e^{tB^T} x, e^{tB^T} x \rangle \rangle \geq 0.$$
We note that the support of $\chi'(s)$ is in the region $\sqrt{\theta}r < s < \beta r$. Thus for some $\beta' < \beta$, the set $B_{\beta'} \setminus B_{\sqrt{\theta}r}$ with $|t| > \theta^2 r^2$ is contained in the support of $\phi'$ and then the inequality holds in (3.20). By the choosing of $c_1$, we know that (3.20) is positive in $B_{\beta'} \setminus B_{\sqrt{\theta}r}$ with $|t| > \theta^2 r^2$. Then the integral of (3.20) on the domain $B_{\beta'} \setminus B_{\sqrt{\theta}r}$ with $|t| > \theta^2 r^2$ is lower bounded by a positive constant which independent of small $r$ and $\theta$.

Then we may choose $\theta$ small so that (3.20) still holds for $z \in B_{\beta'}$ and the inequality holds in $B_{\beta'} \setminus B_{\sqrt{\theta}r}$ with $|t| > \theta^2 r^2$. Therefore the integral function in (3.17) is nonnegative for $z \in B_{\beta'}$ and positive in $B_{\beta'} \setminus B_{\sqrt{\theta}r}$ with $|t| > \theta^2 r^2$. Since $w = 0$ when $u \geq h$ and $w \leq \ln(h^{-1/8})$, then by choosing $\theta$ small enough, our lemma follows from Corollary 3.1 and (3.17).

**Lemma 3.4** Suppose that $u(t, x) \geq 0$ be a solution of equation (1.2) in $B_r$ centered at $(0, 0)$ and

$$\text{mes}\{(t, x) \in B_r, \quad u \geq 1\} \geq \frac{1}{2}\text{mes}(B_r).$$

Then there exist constant $\theta$ and $h_0$, $0 < \theta, h_0 < 1$ which only depend on $\lambda$, $\lambda_0$ and $N$ such that

$$u(t, x) \geq h_0 \quad \text{in} \quad B_{\beta r} \setminus B_{\sqrt{\theta}r}.$$  

**Proof:** We consider

$$w = \ln^+(\frac{h}{u + h^{\frac{1}{2}}}),$$

for $t \in [-\alpha r^2, 0]$. By applying Lemma 3.2 to $w$ and as in the proof of Lemma 3.1, we have

$$(3.21) \quad \int_{B_{\sqrt{\theta}r}} (w - I_0)^2 \leq C(\ln(h^{-\frac{1}{8}})).$$

By Lemma 2.1, there exists a constant, still denoted by $\theta$, such that for $z \in B_{\beta r}$,

$$(3.22) \quad w - I_0 \leq C(\ln(h^{-\frac{1}{8}}))^{\frac{1}{2}}.$$
Therefore we may choose $h_0$ small enough, so that
\[ C(\ln(\frac{1}{h_0^2}))^\frac{1}{2} \leq \ln(\frac{1}{2h_0^2}) - \lambda_0 \ln(\frac{1}{h_0^2}). \]

Then (3.16) and (3.22) implies
\[ \max_{B_{\frac{h_0}{2}}} \frac{h_0}{u + h_0} \leq \frac{1}{2h_0}, \]
which implies $\min_{B_{\frac{h_0}{2}}} u \geq h_0$. Then we finished the proof of Lemma.

**Proof of Theorem 1.1.** We may assume that $M = \max_{B_r} \pm u$, otherwise we replace $u$ by $u - c$. Then either $1 + \frac{u}{M}$ or $1 - \frac{u}{M}$ satisfies the assumption of Lemma 3.4, thus Lemma 3.4 implies
\[ \text{Osc}_{B_{\frac{h_0}{2}}} u \leq (1 - \frac{h_0}{2})\text{Osc}_{B_{\frac{h_0}{2}}} u, \]
which implies the $C^\alpha$ regularity of $u$ near point $(0, 0)$ by the standard iteration arguments. By the left invariant translation group action, we know that $u$ is $C^\alpha$ in the interior.

**Acknowledgements:** The research is partially supported by the Chinese NSF under grant 10325104 and the innovation program at CAS. The author thanks Prof. Xin Zhouping for many valuable discussions on this subject.

**References**

[1] M. Bramanti, M. C. Cerutti and M. Manfredini. $L^p$ estimates for some ultraparabolic equations, J. Math. Anal. Appl., 200 (2) 332-354 (1996).

[2] M. Bramanti, and M. C., Cerutti, Commutators of singular integrals on homogeneous spaces, Boll. Un. Mat. Ital. B (7) 10 no. 4, 843–883 (1996).
[3] G. B. Folland, *Subellitic estimates and function space on nilpotent Lie groups*, Ark. Math., 13 (2): 161-207, (1975).

[4] L. P. Kupcov, *The fundamental solutions of a certain class of elliptic-parabolic second order equations*, Differencial’nye Uravnenija, 8: 1649-1660, 1716, (1972).

[5] S. N. Kruzhkov, *A priori bounds and some properties of solutions of elliptic and parabolic equations*, Math. Sb. (N.S.) 65 (109) 522-570, (1964).

[6] S.N. Kruzhkov, *A priori bounds for generalized solutions of second-order elliptic and parabolic equations*, (Russian) Dokl. Akad. Nauk SSSR 150 748–751, (1963).

[7] M. Manfredini and S. Polidoro, *Interior regularity for weak solutions of ultraparabolic equations in the divergence form with discontinuous coefficients*, Boll Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), 1 (3) 651-675, (1998).

[8] J. Moser, *On Harnack’s theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14 577–591 (1961).

[9] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. 17 101–134 (1964).

[10] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80, 931-954, (1958).

[11] A. Pascucci and S., Polidoro, *The moser’s iterative method for a class of ultraparabolic equations*, Commun. Contemp. Math., 6,2 (2004), 1-23.
[12] S. Polidoro and M. A., Ragusa, Hölder regularity for solutions of ultraparabolic equations in divergence form, Potential Anal. 14 no. 4, 341–350 (2001).

[13] Z.P. Xin and L. Zhang On the global existence of solutions to the Prandtl’s system, Adv. in Math. 181 88-133 (2004).

[14] Z.P. Xin, L. Zhang and J. N Zhao, Global well-posedness for the two dimensional Prandtl’s boundary layer equations, (preprint).

[15] C. J. Xu, Regularity for quasilinear second-order subelliptic equations, Comm. Pure Appl. Math. 45, no. 1, 77–96 (1992).