PARTIAL GAUSS DECOMPOSITION,
\( U_q(\widehat{\mathfrak{gl}(n-1)}) \in U_q(\widehat{\mathfrak{gl}(n)}) \) AND ZAMOLODCHIKOV ALGEBRA

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Abstract. We use the idea of partial Gauss decomposition to study structures related to \( U_q(\widehat{\mathfrak{gl}(n-1)}) \) inside \( U_q(\widehat{\mathfrak{gl}(n)}) \). This gives a description of \( U_q(\widehat{\mathfrak{gl}(n)}) \) as an extension of \( U_q(\widehat{\mathfrak{gl}(n-1)}) \) with Zamolodchikov algebras. We explain the connection of this new realization with form factors.

1. Introduction

The affine Kac-Moody algebra \( \widehat{\mathfrak{g}} \) associated to a simple Lie algebra \( \mathfrak{g} \) admits a natural realization as a central extension of the corresponding loop algebra \( \mathfrak{g} \otimes \mathbb{C}[\approx, \approx^{-1}] \). Drinfeld gives a similar realization for \( U_q(\mathfrak{g}) \), which is called Drinfeld realization [D1]. Faddeev, Reshetikhin and Takhtajan [FRT] Reshetikhin and Semenov-Tian-Shansky [RS] present a realization of \( U_q(\mathfrak{g}) \) to the quantum loop algebra \( U_q(\mathfrak{g} \otimes [\approx, \approx^{-1}]) \) using a solution of the Yang-Baxter equation depending on a parameter \( z \in \mathbb{C} \).

\[
R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),
\]

where \( R(z) \) is a rational function of \( z \) with values in \( \text{End}(\mathbb{C}^\times \otimes \mathbb{C}^\times) \). An explicit identification between the two realizations of the quantum affine algebra \( U_q(\widehat{\mathfrak{g}}) \) for the case \( \mathfrak{g} = \mathfrak{gl}(n) \) is established [DF] by applying Gauss decomposition to the L-operators for the FRTS realization.

In this paper, we will use the idea of partial Gauss decomposition to study the structures related to \( U_q(\widehat{\mathfrak{gl}(n-1)}) \) inside \( U_q(\widehat{\mathfrak{gl}(n)}) \). We show that \( U_q(\widehat{\mathfrak{gl}(n)}) \) can be described as an extension of \( U_q(\widehat{\mathfrak{gl}(n-1)}) \) with Zamolodchikov algebras, where the Zamolodchikov algebras can be interpreted as intertwiner for \( U_q(\widehat{\mathfrak{gl}(n-1)}) \). The Zamolodchikov algebras are used to derive structures related to form factors and related structures.

This paper is to present the method of the partial Gauss decomposition to find new structures hidden inside the affine quantum groups, which is related to many aspects of the theory of affine quantum groups.
The meaning of this method can be explained using the method of the twisting of Drinfeld [2].

2. **Quantum Algebra** \(U_q(\hat{\mathfrak{gl}}(n-1)) \in U_q(\hat{\mathfrak{gl}}(n))\) and Partial Gauss Decomposition

Let \(V\) be \(\mathbb{C}^\times\) with a fixed basis \(e_i, i=1,\ldots,n\) and \(E_{ij}\) be the standard basis of \(\text{End}(\mathbb{C}^\times)\) dependent on \(e_i\). Let \(R(z)\) be an element of \(\text{End}(\mathbb{C}^\times \otimes \mathbb{C}^\times)\) defined by

\[
R(z) = \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{i \neq j, i,j=1}^{n} E_{ii} \otimes E_{jj} \frac{z-1}{q^{-1}z - q} + \sum_{i < j, i,j=1}^{n} E_{ij} \otimes E_{ji} \frac{z(q^{-1} - q)}{zq^{-1} - q} + \sum_{i > j, i,j=1}^{n} E_{ij} \otimes E_{ji} \frac{z(q^{-1} - q)}{zq^{-1} - q}
\]

where \(q, z\) are formal variables. Then \(R(z)\) satisfies the Yang-Baxter equation and \(R\) is unitary, namely

\[
R_{21}(z)^{-1} = R(z^{-1}),
\]

where \(R_{21}(z) = PR_{12}(z)P\), where \(P\) is the operator permuting the two components \(V \otimes V\). Here \(V = \mathbb{C}^\times\).

Faddeev, Reshetikhin and Takhtajan defined a Hopf algebra using \(R(z)\), which satisfies Yang-Baxter equation. Reshetikhin and Semenov-Tian-Shansky obtained a central extension of this algebra. The algebra defined with the \(R(z)\) above is isomorphic \(U_q(\hat{\mathfrak{gl}}(n))\). The central extension is incorporated in shifts of the parameter \(z\) in \(R(z)\).

**Definition 2.1.** \(U_q(\hat{\mathfrak{gl}}(n))\) is an associative algebra with generators \(\{ l_{ij}^\pm, m \in \mathbb{Z}_+, 0 \leq l_{ij}^0, l_{ji}^0, 1 \leq j < i \leq n \}\). Let \(l_{ij}^\pm(z) = \sum_{m=0}^{\infty} l_{ij}^m z^\pm m\), where \(l_{ij}^0 = l_{ji}^0 = 0\), for \(1 \leq j > i \leq n\). Let \(L^\pm(z) = (l_{ij}^\pm(z))_{i,j=1}^n\). Then the defining relations are the following:

\[
l_{ii}^0 l_{ii}^0 = l_{ii}^0 l_{ii}^0 = 1,
\]

\[
R(z_w) L_{1}^\pm (z) L_{2}^\pm (w) = L_{2}^\pm (w) L_{1}^\pm (z) R(z_w),
\]

where \(z_{\pm} = z q^{\pm \frac{\epsilon}{2}}\). The expansion direction of \(R(z_w)\) are chosen to be in \(\frac{z}{w}\) or \(\frac{w}{z}\) respectively. [DF]
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The Hopf algebra is given by:

$$\Delta L^\pm(z) = L^\pm(zq^{\pm(1 \otimes z)}) \otimes L^\pm(zq^{(z \otimes 1)})$$

or

$$\Delta(\hat{l}_ij^\pm(z)) = \sum_{k=1}^n \hat{l}_{ik}^\pm(zq^{\pm(1 \otimes z)}) \otimes \hat{l}_{kj}^\pm(zq^{(z \otimes 1)}),$$

and its antipode is

$$S(L^\pm(z)) = L^\pm(z)^{-1}.$$  

The invertibility of $L^\pm(z)$ follows from the properties that $l_{ii}^\pm$ are invertible and $L^\pm(0)$ are upper triangular and lower triangular, respectively.

$L^\pm(z)$ have the following unique decompositions:

$$L^\pm(z) = \begin{pmatrix} 
1 & 0 & \cdots & 0 \\
\hat{e}_{2,1}^\pm(z) & \cdots & \cdots & \cdots \\
\hat{e}_{3,1}^\pm(z) & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\hat{e}_{n,1}^\pm(z) & \cdots & \hat{e}_{n,n-1}^\pm(z) & \hat{e}_{n,n}^\pm(z)
\end{pmatrix}
\begin{pmatrix} 
\hat{k}_1^\pm(z) & 0 & \cdots & 0 \\
\hat{k}_2^\pm(z) & \cdots & \cdots & \cdots \\
\hat{k}_3^\pm(z) & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \hat{k}_n^\pm(z)
\end{pmatrix}
\times .22
\frac{22}{(3)}$$

which is used to establish the isomorphism between Drinfeld realizations of $U_q(\widehat{\mathfrak{gl}(n)})$ and its FRTS realization.

Similarly we have the following partial Gauss decomposition:

**Proposition 2.1.** The operator $L^\pm(z)$ can be uniquely decomposed as

$$L^\pm(z) = \begin{pmatrix} 
I & \hat{f}_{1,2}^\pm(z) & \cdots & \hat{f}_{1,n}^\pm(z) \\
e^{\pm}(z) & I & \cdots & \cdots \\
\hat{e}_{3,1}^\pm(z) & \hat{e}_{3,2}^\pm(z) & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
\hat{e}_{n,1}^\pm(z) & \cdots & \hat{e}_{n,n-1}^\pm(z) & I
\end{pmatrix}
\begin{pmatrix} 
\hat{K}_1^\pm(z) & 0 & \cdots & 0 \\
0 & \hat{K}_2^\pm(z) & \cdots & \cdots \\
0 & 0 & \hat{K}_3^\pm(z) & \cdots \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \hat{K}_n^\pm(z)
\end{pmatrix},$$

where $K^\pm(z)$ and $k^\pm(z)$ are $n-1 \times n-1$ invertible matrix operators, $\hat{e}(z)$ is a size $n-1$ column and $\hat{f}(z)$ is a size $n-1$ arrow.

Because $K^\pm(z)$ are invertible, the elements $\hat{e}^\pm(z)$, $\hat{f}^\pm(z)$ and $k^\pm(z)$ are uniquely expressed in terms of the matrix coefficients of $L^\pm(z)$.

We have:

**Proposition 2.2.** The algebra generated by entries of operator matrices $K^\pm(z)$ is $U_q(\widehat{\mathfrak{gl}(n-1)})$. 
We will follow the steps as in the case of [DF] to find out the complete commutation relations for the operators in the above decomposition. For the calculation, we need the following formulas:

\[
L^\pm(z) = \begin{pmatrix}
    K^\pm(z) & K^\pm(z)f^\pm(z) \\
    e^\pm(z)K^\pm(z) & (k^\pm(z) + e^\pm(z)K^\pm(z)f^\pm(z))
\end{pmatrix},
\]

\[
L_1(z)L_2(w) =
\begin{pmatrix}
    K(z)K(w) & K(z)K(w)f(w) & K(z)f(z)K(w) & K(z)f(z)K(w)f(w) \\
    K(z)e(w)K(w) & K(z)D(z) & K(z)f(z)e(w)K(w) & K(z)f(z)D(w) \\
    e(z)K(z)K(w) & e(z)K(z)K(w)f(w) & D(z)K(w) & D(z)(K(w)f(w)) \\
    e(z)K(z)e(w)K(w) & e(z)K(z)D(w) & D(z)e(w)K(w) & D(z)(D(w))K(w)f(w)
\end{pmatrix}
\]

\[
(L^\pm(z))^{-1} = \begin{pmatrix}
    (K^\pm(z))^{-1} + f^\pm(z)k^\pm(z)^{-1}e^\pm(z) & -f^\pm(z)k^\pm(z)^{-1} \\
    -k^\pm(z)^{-1}e^\pm(z) & k^\pm(z)^{-1}
\end{pmatrix},
\]

\[
R_{12}\left(\frac{z}{w}\right) = \begin{pmatrix}
    \tilde{R}(z/w) & 0 & 0 & 0 \\
    0 & \frac{z-w}{zq^{-1}-wq}A & -\frac{z(q-q^{-1})}{wq^{-1}-zq}B & 0 \\
    0 & -\frac{w(q-q^{-1})}{zq^{-1}-wq}C & \frac{z-w}{zq^{-1}-wq}D & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
R_{12}\left(\frac{z}{w}\right)^{-1} = R_{21}\left(\frac{w}{z}\right) = \begin{pmatrix}
    \tilde{R}_{21}(w/z) & 0 & 0 & 0 \\
    0 & \frac{z-w}{zq^{-1}-wq}A & -\frac{z(q-q^{-1})}{wq^{-1}-zq}B & 0 \\
    0 & -\frac{w(q-q^{-1})}{zq^{-1}-wq}C & \frac{z-w}{zq^{-1}-wq}D & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
(L_1(w))^{-1} =
\begin{pmatrix}
    * & 0 & -f(w)k(w)^{-1} & 0 \\
    0 & * & 0 & -f(w)k(w)^{-1} \\
    -k(w)^{-1}e(w) & 0 & k(w)^{-1} & 0 \\
    0 & -k(w)^{-1}e(w) & 0 & k(w)^{-1}
\end{pmatrix},
\]
where \( A = \sum_{i \neq n} E_{ii} \otimes E_{nn}, \) \( D = \sum_{i \neq n} E_{nn} \otimes E_{ii} \) \( C = \sum_{j<n+1} E_{nj} \otimes E_{jn}, \) \( B = \sum_{j<n+1} E_{jn} \otimes E_{nj}, \) \( \hat{R}(z) \) is the \( R \)-matrix restricted to the subspace \( V' \otimes V', \) \( V' \) is generated on the subspace generated by \( e_i, i = 1, \ldots, n - 1, \) \( D^\pm(z) = (k^\pm(z) + e^\pm(z)K^\pm(z)f^\pm(z)), \) and

\[
\begin{align*}
L_1^\pm(w)^{-1}R_{21}(\frac{z}{w})L_2^\pm(z) &= L_2^\pm(z)R_{21}(\frac{z}{w})L_1^\pm(w)^{-1} \\
L_1^\pm(w)^{-1}R_{21}(\frac{z^+}{w^-})L_2^\pm(z) &= L_2^\pm(z)R_{21}(\frac{z^-}{w_1^+})L_1^\pm(w)^{-1} \\
R_{21}(\frac{z^-}{w_1^+})L_2^-(z)L_1^+(w) &= L_2^+(w)L_2^-(z)R_{21}(\frac{z^-}{w_1^+}) \\
L_1^-(w)^{-1}R_{21}(\frac{z^-}{w_1^-})L_2^-(z) &= L_2^-(z)R_{21}(\frac{z^+}{w_1^-})L_1^-(w) \\
L_2^\pm(\pm)^{-1}(L_1^\pm(w))^{-1}R(\frac{z}{w}) &= R_{21}(\frac{z^-}{w_1^-})(L_1^\pm(w))^{-1}(L_2^\pm(z))^{-1} \\
L_2^\pm(z)^{-1}L_1^-(w)^{-1}R_{21}(\frac{z^-}{w_1^-}) &= R_{21}(\frac{z^-}{w_1^-})(L_1^-(w))^{-1}(L_2^\pm(z))^{-1}
\end{align*}
\]

Using the same calculation technique as in [DF], we have:

**Lemma 2.3.**

\[
\begin{align*}
\hat{R}(z/w)K_1^\pm(z)K_2^\pm(w) &= K_2^\pm(w)K_1^\pm(z)\hat{R}(z/w) \\
k^\pm(z)k^\pm(w) &= k^\pm(w)k^\pm(z) \\
\hat{R}(z+/w-)K_1^\pm(z)K_2^-(w) &= K_2^-(w)K_1^\pm(z)\hat{R}(z+/w-) \\
k^\pm(z)k^-(w) &= k^-(w)k^\pm(w) \\
k^\pm(z)k^\pm(w) &= k^\mp(w)k^\pm(z) \\
\frac{z_\mp q^{-1} - w_\pm q}{z_\mp - w_\pm}k^\mp(w)^{-1}K^\pm(z) &= K^\pm(z)k^\mp(w)\frac{z_\mp q^{-1} - w_\pm q}{z_\mp - w_\pm}, \\
K_1^\pm(z)E_2(w) &= \frac{zq^{\frac{\mp}{\mp}} - w q}{zq^{\frac{\mp}{\mp}} - w}E_2(w)\hat{R}(zq^{\frac{\mp}{\mp}}w)K_1^\pm(z), \\
K_1^\pm(z)E_2(w) &= \frac{zq^{\frac{\mp}{\mp}} - w q}{zq^{\frac{\mp}{\mp}} - w}E_2(w)\hat{R}(zq^{\frac{\mp}{\mp}}w)K_1^\pm(z), \\
k^\pm(z)E(w) &= \frac{zq^{\frac{\mp}{\mp} + 1} - w q^{-1}}{zq^{\frac{\mp}{\mp}} - w}E(w)k^\pm(z), \\
k^\pm(z)F(w) &= \frac{zq^{\frac{\mp}{\mp}} - w}{zq^{\frac{\mp}{\mp} + 1} - w q^{-1}}F(w)k^\pm(z), \\
(z - wq^2)E_1(z)E_2(w)R(z/w) &= (zq^2 - w)E_2(w)E_1(z), \\
(zq^2 - w)F(z)F(w) &= R(z/w)(z - wq^2)F(w)F(z),
\end{align*}
\]
\[ E_2(z)(F_1(w)) - F_1(w)E_2(z) = (q - q^{-1}) \left( \delta \left( \frac{w}{z}q^c \right) k^- (wq^\hat{x})K^- (wq^\hat{x})^{-1} - \delta \left( \frac{w}{z}q^{-c} \right) k^+ (wq^\hat{x})K^+ (wq^\hat{x})^{-1} \right), \]

where \( E(z) = e^+(zq^\hat{x}) - e^-(zq^\hat{x}) \), \( F(z) = f^+(zq^\hat{x}) - f^-(zq^\hat{x}) \) and
\[ \delta(x) = \sum_{m \in \mathbb{Z}} x^m. \]

The algebra generated by \( E(z) \) or \( F(z) \) gives a realization of the Zamolodchikov algebra. On the other hand, we can reformulate the definition of \( ZUR(n) \) using the similar formulas. The important point is that from the definition we can see that \( E(z) \) and \( F(z) \) is nothing but intertwiner for the affine algebra \( \hat{U_q}(gl(n)) \) generated by the operators \( K^\pm(z)(k^\pm(z))^{-1} \). The last formula of the commutation relations implies the constructions like in \( [M] \) \( [Sm] \).

Let \( E(z) = E(z)K^- (zq^{\ell/2})k^- (zq^{\ell/2}) \), then

**Theorem 2.4.** \( ZUR(n) \) is isomorphic to \( \hat{U_q}(gl(n)) \).

The proof follows form the above lemma and the similar argument in \([DF]\). From the point of view of \([DF]\), we can similarly to give a new Hopf algebra structure to this formulation using the similar formulas. The important point is that from the definition we can see that \( E(z) \) and \( F(z) \) is nothing but intertwiner for the affine algebra \( \hat{U_q}(gl(n - 1)) \) generated by the operators \( K^\pm(z)(k^\pm(z))^{-1} \). Then

**Proposition 2.5.**

\[
E_2(z) \frac{(1 - z/wq^{-c})R(z/w)(z - wq^2)}{zq^2 - w}(F_1(w)) - F_1(w)E_2(z) = (q - q^{-1})(1 - q^{-2c})\delta \left( \frac{w}{z}q^{-c} \right) k^+ (wq^\hat{x})K^+ (wq^\hat{x})^{-1}k^- (wq^\hat{x})^{-1}K^- (wq^\hat{x}),
\]

\[ (zq^2 - w)E_1(z)E_2(w) = (z - wq^2)E_2(w)E_1(z)R(z/w), \]

\[ (zq^2 - w)F(z)F_1(w) = R(z/w)(z - wq^2)F_2(w)F_1(z), \]

\[ (1 - z/wq^c)E_2(z)R(z/w)\frac{z - wq^2}{zq^2 - w}(F_1(w)) - F_1(w)E_2(z) = (q - q^{-1})(1 - q^{2c})\delta \left( \frac{w}{z}q^c \right). \]
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The first one of the formulas above coincides with the spinor constructions of affine quantum groups in [Di2].

This last three formula above says that these operators generate an algebra almost the same as the Zamolodchikov-Faddeev algebra used to describe the theory of form factors [Sm]. Similarly we can also define a new operator \( \hat{F}(z) = (k^+(wq^{-\frac{z}{2}}))^{-1}K^+(wq^{-\frac{z}{2}})F(z) \). This operator with \( E(z) \) generate another algebra similar to the definition above. From the point view of intertwiners as in [MJ], those operator can give a complete theory of form factors, where one copy of the algebra is explained as the the Zamolodchikov-Faddeev algebra to define the model and the other one is explained as local operators, which commutes with the first algebra up to certain functions. In a subsequent paper, we will apply the same method to Yangian, and the elliptic algebra [LKP] [F]

We will give the complete details to the descriptions of more general Zamolodchikov-Faddeev type of algebras, whose degeneration gives us the corresponding results in this paper.

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