Field redefinition and renormalisability in scalar field theories

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Abstract

We have addressed the issue of field redefinition in connection with renormalisability. Our study is restricted to theories of interacting scalar fields. We have, in particular, shown that if a theory is renormalisable in the usual power-counting sense then it remains renormalisable in the same sense after a change of variables. This is due to the use of the powerful method of the background field expansion. In the case of a single complex scalar field, it turns out that the determination of the counter-terms is much simpler when polar coordinates are used. We illustrate this by carrying out a one-loop calculation in the latter case.

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1 Introduction

The issue of coordinate transformations (also referred to as field redefinition or reparametrisation invariance) in quantum field theory has been in the past [1–8], and still is [9–17] a subject of renewed interest. The so-called “equivalence theorem” ensures, in principle [18,19], that the elements of the S-matrix remain the same after a field redefinition. However, it is certainly not clear what becomes of the renormalisability of a theory when a coordinate transformation is carried out. We should mention that the reparametrisation invariance of the effective action has been discussed in [20].

For instance, it is well-known that the Lagrangian density for a complex scalar field as given by

\[ L = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 (\Phi^* \Phi) - \lambda (\Phi^* \Phi)^2. \] (1.1)

is a renormalisable theory. In Cartesian coordinates where

\[ \Phi = \frac{1}{\sqrt{2}} (\chi_1 + i \chi_2) \] (1.2)

this model describes two massive real fields \( \chi_1 \) and \( \chi_2 \) with a quartic interaction as seen from the Lagrangian

\[ L = \frac{1}{2} \partial_\mu \chi_1 \partial^\mu \chi_1 + \frac{1}{2} \partial_\mu \chi_2 \partial^\mu \chi_2 - \frac{m^2}{2} (\chi_1^2 + \chi_2^2) - \frac{\lambda}{4} (\chi_1^2 + \chi_2^2)^2. \] (1.3)

Let us now perform a field redefinition and parametrise the complex scalar field \( \Phi \) as

\[ \Phi = \frac{1}{\sqrt{2}} \rho e^{i\theta/v}, \] (1.4)

where \( \rho \) and \( \theta \) are the polar coordinates (real fields) and \( v \) is a dimensionful constant. The Lagrangian density becomes

\[ L = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta - \frac{m^2}{2} \rho^2 - \frac{\lambda}{4} \rho^4. \] (1.5)

The first thing to notice in this parametrisation is that the kinetic term for the real field \( \theta \) is not of the standard form. Furthermore, it is not clear how one can see that the spectrum of the theory contains two massive fields (as in Cartesian coordinates).

If one insists on treating the Lagrangian (1.5) as a conventional field theory, then a change of variables is necessary. For instance, we could change \( \rho \) to

\[ \rho = v e^{\psi/v}, \] (1.6)

where \( \psi \) is the new field. The Lagrangian (1.5) becomes then

\[ L = e^{2\psi/v} \left[ \frac{1}{2} \partial_\mu \psi \partial^\mu \psi + \frac{1}{2} \partial_\mu \theta \partial^\mu \theta \right] - \frac{m^2 v^2}{2} e^{2\psi/v} - \frac{\lambda v^4}{4} e^{4\psi/v}. \] (1.7)
Expanding the exponential, \( e^{2\psi/v} = 1 + 2\psi/v + 2\psi^2/v^2 + \ldots \), leads to a standard kinetic terms for the two real fields \( \psi \) and \( \theta \). In this parametrisation, the field \( \theta \) is massless while all the mass is appropriated by the field \( \psi \).

The important feature of the Lagrangians (1.5) and (1.7) is that their interaction parts are non-polynomial in nature. That is, they involve derivatives of the fields. Therefore, the usual power-counting argument of renormalisability does not apply here.

The aim of this note is to address the issue of the renormalisability of theories like the one in (1.5) when a change of variables is carried out. Our strategy is to treat the type of Lagrangians in (1.5) and (1.7) as a four-dimensional non-linear sigma model supplemented with a potential term and study their renormalisability. We use, for this purpose, the background field method which we will review in the next section.

### 2 The covariant background field expansion

The non-linear sigma model is defined as follows: Let \( \Sigma \) denote the four-dimensional space-time with coordinates \( x^\mu \) and derivative \( \partial_\mu \) and let \( \mathcal{M} \) be a Riemannian manifold (the target space) with metric \( g_{ij} \). The field of the sigma model \( \phi^i(x) \) is a map from \( \Sigma \) to \( \mathcal{M} \). The Lagrangian for the non-linear sigma model is

\[
L(\phi) = \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j . 
\]  

(2.1)

The field \( \phi^i(x) \) labels the coordinates of the target space \( \mathcal{M} \).

The background field expansion method [21–26] consists in splitting the field \( \phi^i(x) \) of the non-linear sigma model as \( \phi^i(x) = \varphi^i(x) + \pi^i(x) \), where \( \varphi^i \) is the background field (a classical field) and \( \pi^i \) is the quantum fluctuation around this background field. The quantum field \( \pi^i = \phi^i - \varphi^i \), being a difference between two coordinates of the target space, does not lead to a covariant expansion of the non-linear sigma model. In order to respect the geometric nature of the non-linear sigma model we expand the action in terms of the quantum field \( \xi^i \) instead. This field transforms as a vector on the target space and is defined as follows:

Let \( \sigma^i(x,s) \) be the unique geodesic joining the two target space points \( \varphi^i(x) \) and \( \varphi^i(x) + \pi^i(x) \). The affine parameter \( s \in [0,1] \) parametrises this geodesic and we have the interpolating conditions

\[
\sigma^i(x,s = 0) = \varphi^i(x) \quad \text{and} \quad \sigma^i(x,s = 1) = \phi^i(x) = \varphi^i(x) + \pi^i(x) \ .
\]  

(2.2)

The geodesic equation is given by

\[
\frac{d^2 \sigma^i}{ds^2} + \Gamma_{jk}^i (\sigma) \frac{d\sigma^j}{ds} \frac{d\sigma^k}{ds} = 0 \ ,
\]  

(2.3)

where \( \Gamma_{jk}^i (\sigma) \) are the Christoffel symbols corresponding to the target space metric \( g_{ij}(\sigma) \).

Let \( \xi^i_s(x,s) \) denote the tangent vector to the geodesic \( \sigma^i(x,s) \). In other words,

\[
\xi^i_s(x,s) = \frac{d\sigma^i}{ds} .
\]  

(2.4)
The quantum field $\xi^i$ that will enter in the covariant expansion is defined as the tangent vector to the geodesic at the point $\varphi^i(x)$. That is,

$$\xi^i (x) = \xi^i_s (x, s) \bigg|_{s=0}.$$  \hspace{1cm} (2.5)

Since $\xi^i(x)$ transforms as a vector on the target space, the expansion of the action in terms of this field will be automatically covariant.

In order to obtain the covariant expansion, we start by extending the Lagrangian as

$$L(s) = \frac{1}{2} g_{ij} (\sigma(s)) \partial_\mu \sigma^i (s) \partial^\mu \sigma^j (s)$$  \hspace{1cm} (2.6)

so that $L(\phi) = L(s = 1)$. We then expand $L(s)$ in powers of $s$ around $s = 0$. We obtain

$$L(\varphi + \pi) = L(s = 1) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n L(s) \bigg|_{s=0} = \sum_{n=0}^{\infty} \frac{1}{n!} (\nabla_s)^n L(s) \bigg|_{s=0},$$  \hspace{1cm} (2.7)

where we have used the fact that $L(\sigma(x,s))$ is a scalar to get the last equality. Here $\nabla_s$ is the covariant derivative along the curve $\sigma^i(x,s)$. Its acts on an arbitrary tensor $T_j^i$ as

$$\nabla_s T_j^i = \frac{d}{ds} T_j^i + \Gamma^i_{kl} (\sigma) \xi^k \sigma_j^i - \Gamma^j_{ki} (\sigma) \xi^i \sigma_l^i.$$  \hspace{1cm} (2.8)

Of course, the tensor $T_j^i$ could have more indices. If $T_j^i$ is a tensor function of $\sigma^i(x,s)$ only then

$$\nabla_s T_j^i (\sigma) = \xi^k \nabla_k T_j^i (\sigma) = \xi^k \left[ \frac{\partial}{\partial \xi^k} T_j^i + \Gamma^i_{kl} (\sigma) T_j^l - \Gamma^j_{ki} (\sigma) T_l^i \right].$$  \hspace{1cm} (2.9)

Here $\nabla_i$ is the usual covariant derivative with respect to $\Gamma^i_{jk}$.

The expansion (2.7) is evaluated using (2.9) together with the formulae

$$\nabla_s \partial_\mu \sigma^i = \nabla_\mu \frac{d \sigma^i}{ds} = \nabla_\mu \xi^i_s \equiv \partial_\mu \xi^i_s + \Gamma^i_{jk} (\sigma) \partial_\mu \sigma^j \xi^k_s,$$

$$\nabla_s g_{ij} (\sigma) = 0,$$

$$\nabla_s \frac{d \sigma^i}{ds} = \nabla_s \xi^i_s = 0,$$

$$\nabla_s \nabla_\mu \xi^i_s = R^i_{jkl} \xi^j \xi^k \partial_\mu \sigma^l.$$  \hspace{1cm} (2.10)

Here $R^i_{jkl}$ is the Riemann tensor$^1$.

The first few terms in the expansion of the Lagrangian $L(\phi)$ around the background $\varphi^i$ are

$$L(\phi) = \frac{1}{2} g_{ij} (\varphi) \partial_\mu \varphi^i \partial^\mu \varphi^j + g_{ij} (\varphi) \partial_\mu \varphi^i \nabla_\mu \xi^j + \frac{1}{2} g_{ij} (\varphi) \nabla_\mu \xi^i \nabla^\mu \xi^j + R_{ijkl} (\varphi) \partial_\mu \varphi^i \partial^\mu \varphi^j \xi^k \xi^l + \ldots,$$  \hspace{1cm} (2.11)

where

$$\nabla_\mu \xi^i = \partial_\mu \xi^i + \Gamma^i_{jk} (\varphi) \partial_\mu \varphi^j \xi^k.$$  \hspace{1cm} (2.12)

$^1$Our convention is $R^i_{jkl} = \partial_\kappa \Gamma^i_{jl} + \Gamma^i_{km} \Gamma^m_{jl} - (k \leftrightarrow l).$
If our Lagrangian contains a potential term like
\[ L(\phi) = \frac{1}{2} \eta^{ij} \phi_i \partial_\mu \phi^i \partial_\mu \phi^j - V(\phi) \] (2.13)
then the expansion of the potential is simply
\[ V(\phi) = V(\varphi) + \sum_{n=1}^{\infty} \frac{1}{n!} \nabla_{j_1} \cdots \nabla_{j_n} V(\varphi) \xi_{j_1} \cdots \xi_{j_n}. \] (2.14)

Notice that the term \( g_{ij}(\varphi) \partial_\mu \xi^i \partial^\mu \xi^j \) needed for the determination of the propagator for the quantum field \( \xi^i \) is not of the standard form due to the presence of the non-constant metric \( g_{ij}(\varphi) \). The remedy to this is to define a new quantum field \( \xi^a \) as
\[ \xi^a = e^a_i \xi^i \quad \text{or} \quad \xi^i = E^i_a \xi^a, \] (2.15)
where we have introduced the vielbiens \( e^a_i \) such that
\[ g_{ij} = \eta_{ab} E^a_i e^b_j. \] (2.16)
The constant matrix \( \eta_{ab} \) is invertible and \( E^a_i \) is the inverse of \( e^i_a \). That is,
\[ e^a_i E^i_b = \delta^a_b, \quad E^i_a e^a_j = \delta^i_j. \] (2.17)

With this field redefinition the expansions (2.11) becomes
\[ L(\phi) = \frac{1}{2} \eta^{ij} e^a_i \phi^i \partial_\mu \phi^j + \eta_{ab} e^a_i \partial_\mu \phi^i D^\mu \xi^b + \frac{1}{2} \eta_{ab} \partial_\mu \xi^a \partial_\mu \xi^b + R_{iabj} (\varphi) \partial_\mu \phi^i \partial^\mu \phi^j \xi^a \xi^b + \cdots, \] (2.18)
where
\[ D^\mu \xi^a = \partial_\mu \xi^a + \omega_{ib}^a \partial_\mu \phi^i \xi^b, \quad \omega_{ib}^a = e^a_j \left( \partial_i E^j_b + \Gamma^j_{ik} E^k_b \right) = e^a_j \nabla_i E^j_b \] (2.19)
and \( R_{iabj} = R_{iklj} E^b_i E^l_j \). The propagator is now computed from the term \( \eta_{ab} \partial_\mu \xi^a \partial_\mu \xi^b \) which has the standard form.

Similarly, the expansion (2.14) of the potential is
\[ V(\phi) = V(\varphi) + \sum_{n=1}^{\infty} \frac{1}{n!} V_{a_1 \cdots a_n}(\varphi) \xi^{a_1} \cdots \xi^{a_n} \]
\[ V_{a_1 \cdots a_n}(\varphi) \equiv E^{a_1}_{j_1} \cdots E^{a_n}_{j_n} \nabla_{j_1} \cdots \nabla_{j_n} V(\varphi). \] (2.20)
It is clear that \( V_{a_1 \cdots a_n} \) is symmetric under the exchange of any two indices. It is convenient to write this as
\[ V_{a_1 \cdots a_n} = D_{a_1} D_{a_2} \cdots D_{a_n} V, \] (2.21)
where the derivative \( D_a \) acts as
\[ D_a X_{bc} = E^a_i \left( \partial_i X_{bc} - \omega_{ib}^d X_{dc} - \omega_{ic}^d X_{bd} \right) \] (2.22)
on an arbitrary tensor \( X_{bc} \).
3 Field redefinition and sigma model

We start with a field theory as described by the Lagrangian

\[ \mathcal{L} = \frac{1}{2} \eta_{ab} \partial_\mu f^a \partial^\mu f^b - V(f) , \]  

(3.1)

where \( \eta_{ab} \) is a constant metric and \( f^a(x) \) is a set of fields. In the special case of the complex scalar theory in (1.3), we have \( f^a = (\chi_1, \chi_2) \) and \( \eta_{ab} = \text{diag}(1, 1) \). In four dimensions, this theory is renormalisable in the usual power-counting sense (Dyson criterion) if the potential \( V(f) \) is at most quartic in the fields \( f^a \).

Let now \( \phi^i \) denote another set of fields which parametrise the same theory as the fields \( f^a \). In other words, we have made a change of variables from the field \( f^a \) to the fields \( \phi^i \). We may therefore write

\[ f^a = f^a(\phi^i) , \quad \phi^i = \phi^i(f^a) , \]  

(3.2)

where we have assumed that the change of variables is invertible. Under this field redefinition, the Lagrangian (3.1) becomes

\[ L = \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi) , \]  

(3.3)

where the metric \( g_{ij} \) is given by

\[ g_{ij} = \eta_{ab} \partial_i f^a \partial_j f^b . \]  

(3.4)

Here \( \partial_a = \frac{\partial}{\partial f^a} \) and \( \partial_i = \frac{\partial}{\partial \phi^i} \) and the range of the indices \( a, b, \ldots \) is the same as the range of the indices \( i, j, \ldots \).

From the relations \( \partial_i \phi^j = \delta^j_i \) and \( \partial_a f^b = \delta^b_a \) together with the chain rule, we deduce that

\[ \partial_i f^a \partial_a \phi^j = \delta^j_i , \quad \partial_a \phi^i \partial_i f^b = \delta^b_a . \]  

(3.5)

Hence we could identify the vielbeins \( e^a_i \) and their inverse \( E^i_a \) with

\[ e^a_i = \partial_i f^a , \quad E^i_a = \partial_a \phi^i . \]  

(3.6)

The inverse of the metric is \( g^{ij} = \eta^{ab} \partial_a \phi^i \partial_b \phi^j \), where \( \eta^{ab} \) is the inverse of \( \eta_{ab} \). The Christoffel connection is then given by

\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) = \partial_j \partial_k f^a \partial_a \phi^i \]  

(3.7)

and all the components of the Riemann tensor \( R^i_{jkl} \) vanish.

Using the chain rule and the relations in (3.5), we find that the spin connection, \( \omega^a_{ib} = e^a_j \left( \partial_i E^j_b + \Gamma^j_{ik} E^k_b \right) \), vanishes. Indeed,

\[ \omega^a_{ib} = \partial_i \partial_j f^a \partial_b \phi^j + \partial_j f^a \partial_i \partial_b \phi^j + \partial_j f^a \partial_i \partial_c \partial_b \phi^j + \partial_j f^a \partial_i \partial_c \partial_b \phi^j = 0 . \]  

(3.8)
We also need the expression of the tensor $V_{a_1...a_n}$ as defined in (2.21). This is found to be

$$V_{ab...c} = \partial_a \partial_b \ldots \partial_c V .$$  \hfill (3.9)

If we assume that $V$ is at most quartic in the fields, then the background field expansion, to all order in the quantum field $\xi^a$, of the Lagrangian yields

$$L = \frac{1}{2} g_{ij}(\varphi) \partial_\mu \varphi^i \partial^\mu \varphi^j - V(\varphi) + \eta_{ab} \epsilon^a_i \partial_\mu \varphi^i \partial^\mu \xi^b + \frac{1}{2} \eta_{ab} \partial_\mu \xi^a \partial^\mu \xi^b$$

$$- \partial_a V \xi^a - \frac{1}{2} \partial_a \partial_b V \xi^a \xi^b - \frac{1}{6} \partial_a \partial_b \partial_c V \xi^a \xi^b \xi^c - \frac{1}{24} \partial_a \partial_b \partial_c \partial_d V \xi^a \xi^b \xi^c \xi^d .$$  \hfill (3.10)

We notice that the resulting expansion is at most quartic in the quantum field $\xi^a$. It is then clear that the theory is renormalisable in the usual power-counting sense. We will illustrate this by an example in the following section.

4 An example: the interacting complex scalar field in polar coordinates

In the notation of the previous sections, we have $f^a = (\chi_1, \chi_2)$ and the constant metric $\eta_{ab} = \text{diag}(1,1)$. The new set of fields are $\phi^i = (\rho, \theta)$ and we have made the change of variables

$$\chi_1 = \rho \cos(\theta/v) \quad , \quad \chi_2 = \rho \sin(\theta/v) .$$  \hfill (4.1)

The Lagrangian (1.5) is a non-linear sigma model with a metric $g_{ij}$ given by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2/v^2 \end{pmatrix} .$$  \hfill (4.2)

The non-vanishing components of its Christoffel symbols are

$$\Gamma^1_{22} = -\frac{\rho}{v^2} \quad , \quad \Gamma^2_{12} = \frac{1}{\rho} .$$  \hfill (4.3)

Using (3.6) and (4.1), the vielbeins $e^a_i$ and their inverses $E^i_a$ are given by

$$e^a_i = \begin{pmatrix} \cos(\theta/v) & -\frac{\rho}{v} \sin(\theta/v) \\ \frac{\rho}{v} \sin(\theta/v) & \cos(\theta/v) \end{pmatrix} , \quad E^i_a = \begin{pmatrix} \cos(\theta/v) & -\frac{\rho}{v} \sin(\theta/v) \\ \frac{\rho}{v} \sin(\theta/v) & \cos(\theta/v) \end{pmatrix} .$$  \hfill (4.4)

According to (3.8), all the components of the spin connection $\omega^a_{ib}$ vanish.\footnote{As matrices, $e^a_i$ and $E^i_a$ should be read as $e_{ia}$ and $E_{ai}$, respectively.}
The background field expansion, to all order in the quantum field $\xi^a$, of the Lagrangian (1.5) yields

$$L(\varphi, \xi) = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \frac{\rho^2}{v^2} \partial_\mu \theta \partial^\mu \theta - \left( \frac{m^2}{2} \rho^2 + \frac{\lambda}{4} \rho^4 \right)$$

$$+ \eta_{ab} \epsilon_i^a \partial_\mu \varphi^i \partial^\mu \xi^b + \frac{1}{2} \eta_{ab} \partial_\mu \xi^a \partial^\mu \xi^b$$

$$- V_a \xi^a - \frac{1}{2} V_{ab} \xi^a \xi^b - \frac{1}{6} V_{abc} \xi^a \xi^b \xi^c - \frac{1}{24} V_{abcd} \xi^a \xi^b \xi^c \xi^d \ . \quad (4.5)$$

Here we have used $\varphi^i = (\rho, \theta)$ to denote also the background fields.

The non-vanishing components of the symmetric tensor $V_{a_1...a_n}$ are

$$V_1 = \rho \left( m^2 + \lambda \rho^2 \right) \cos(\theta/v) \ ,$$

$$V_2 = \rho \left( m^2 + \lambda \rho^2 \right) \sin(\theta/v) \ ,$$

$$V_{11} = m^2 + \lambda \rho^2 \left( 1 + 2 \cos(\theta/v)^2 \right) \ ,$$

$$V_{12} = 2 \lambda \rho^2 \cos(\theta/v) \sin(\theta/v) \ ,$$

$$V_{22} = m^2 + \lambda \rho^2 \left( 1 + 2 \sin(\theta/v)^2 \right) \ ,$$

$$V_{111} = 6 \lambda \rho \cos(\theta/v) \ ,$$

$$V_{112} = 2 \lambda \rho \sin(\theta/v) \ ,$$

$$V_{122} = 2 \lambda \rho \cos(\theta/v) \ ,$$

$$V_{222} = 6 \lambda \rho \sin(\theta/v) \ ,$$

$$V_{1111} = 6 \lambda \ ,$$

$$V_{1112} = 2 \lambda \ ,$$

$$V_{2222} = 6 \lambda \ . \quad (4.6)$$

Let us now see what happens at the one-loop level in perturbation theory.

**One-loop renormalisation:**

The generating functional for connected Green’s functions, $W[J]$, is defined in the usual way by

$$Z[J] = e^{iW[J]} = N \int [d\xi] e^{i[S(\varphi, \xi) + J_a \xi^a]} \ , \quad (4.7)$$

where $N$ is normalising factor and

$$[d\xi] = \prod_x \prod_a \sqrt{g(x)} \, d\xi^a \quad (4.8)$$

is the coordinate independent measure.
The Feynman propagator is computed from the free part of the action
\[ L_{\text{free}}(\xi) = \frac{1}{2} \eta_{ab} \partial_\mu \xi^a \partial^\mu \xi^b - \frac{1}{2} m^2 \eta_{ab} \xi^a \xi^b. \] (4.9)

The mass term comes from \( V_{11} \) and \( V_{22} \). This propagator is given by
\[ \Delta^{ab}_F(x - y) = \eta^{ab} \frac{1}{(2\pi)^4} \int \frac{e^{-ik.(x-y)}}{k^2 - m^2} d^4k \] (4.10)
and satisfies
\[ \eta_{ac} (\partial_\mu \partial^\mu + m^2) \Delta^{cb}_F(x) = -\delta^a_b \delta^4(x). \] (4.11)

The loop expansion in terms of Feynman graphs is generated using the Dyson-Wick perturbation theory. This is obtained from
\[ e^{iW(\varphi)} = \langle 0 | \exp \left[ i \int d^4x \ L_{\text{int}}(\varphi, \xi) \right] | 0 \rangle, \] (4.12)
where
\[ L_{\text{int}}(\varphi, \xi) = L(\varphi, \xi) - L_{\text{free}}(\xi). \] (4.13)

We then expand \( \exp \left[ i \int d^4x \ L_{\text{int}}(\varphi, \xi) \right] \) and calculate the vacuum graphs by considering all the possible Wick contractions involving the quantum field \( \xi^a(x) \). The background field \( \varphi \) is treated as an external field.

We use dimensional regularisation to isolate the divergences in Feynman integrals. The dimension of spacetime is assumed to be \( d = 4 - \epsilon \). The original Lagrangian (1.5) is extended to \( d \) dimensions as
\[ L = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta - \left( \frac{m^2}{2} \rho^2 + \frac{\mu^{4-d} \lambda}{4 \rho^4} \right), \] (4.14)
where \( \mu \) is an arbitrary mass parameter. In \( d \) dimensions \( \rho \) has a mass dimension equal to \( \frac{1}{2} d - 1 \) while \( \theta \) has a mass dimension equal to 1. The coupling constant \( \lambda \) is kept dimensionless in \( d \) dimensions.

The first divergent diagram is shown in figure 1 and contributes
\[ I_1 = (i)^2 \times \left( -\frac{1}{2} \right) \tilde{V}_{ab} \eta^{ab} \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2} \] (4.15)
where \( \tilde{V}_{11} = V_{11} - m^2, \tilde{V}_{22} = V_{22} - m^2 \) and \( \tilde{V}_{12} = V_{12} \). The divergent part of the integral is
(see for example [27])
\[ I_1 = -2\lambda \rho^2 \left( \frac{im^2}{8\pi^2 \epsilon} + \text{finite} \right). \] (4.16)

The second divergent graph is drawn in figure 2 and gives
\[ I_2 = \frac{(i)^4}{2} \times \left( -\frac{1}{2} \right)^2 \tilde{V}_{ab} \tilde{V}_{cd} \left( \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc} \right) \times \left( \mu^2 \right)^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2) \left[ (p - q)^2 - m^2 \right]} \] (4.17)
Figure 1: One loop divergent diagram with one vertex $\tilde{V}_{ab}$.

Figure 2: One loop divergent diagram with two vertices $\tilde{V}_{ab}$ and $\tilde{V}_{cd}$.
The infinite part of the integral is extracted (see for example [27]) and we find
\[ I_2 = \frac{5}{2} \lambda^2 \rho^4 \left( \frac{i \mu^\epsilon}{8\pi^2 \epsilon} + \text{finite} \right). \tag{4.18} \]

Green’s functions are then rendered finite by adding to the Lagrangian (4.14) the counter-term Lagrangian
\[ L_{CT} = \frac{A}{2} \partial_\mu \rho \partial^\mu \rho + \frac{B}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta - \left( \frac{C}{2} \rho^2 + \frac{D \mu^\epsilon \lambda}{4} \rho^4 \right). \tag{4.19} \]

This leads to the bare Lagrangian
\[ L_B = L + L_{CT} = \frac{1}{2} \partial_\mu \rho_B \partial^\mu \rho_B + \frac{1}{2} \rho_B^2 \partial_\mu \theta_B \partial^\mu \theta_B - \left( \frac{m_B^2}{2} \rho_B^2 + \frac{\lambda_B}{4} \rho_B^4 \right). \tag{4.20} \]

The bare quantities are defined as
\[
\begin{align*}
\rho_B &= \sqrt{Z_\rho} \rho, & Z_\rho &= 1 + A, \\
m_B &= Z_m m, & Z_m &= \frac{m^2 + C}{m^2(1 + A)}, \\
\lambda_B &= \mu^\epsilon Z_\lambda \lambda, & Z_\lambda &= \frac{1 + D}{(1 + A)^2}, \\
\theta_B &= \sqrt{Z_\theta} \theta, & v_B &= Z_v v, & \frac{Z_\rho Z_\theta}{Z_v^2} &= 1 + B. \tag{4.21}
\end{align*}
\]

At the one loop level, we have
\[ C = -\frac{\lambda m^2}{2\pi^2 \epsilon}, \quad D = \frac{5\lambda}{4\pi^2 \epsilon}, \quad A = B = 0. \tag{4.22} \]

This leads to \( Z_\rho = Z_\theta = Z_v = 1 \). The one-loop result is precisely the one found in the literature (see for example [28]).

In conclusion, we have shown that a power-counting renormalisable scalar field theory maintains this property in another reparametrisation of the fields. The question of field redefinitions is much more important in theories involving gauge fields. This is due to the fact that in some cases (like the Higgs model) one could work with gauge invariant variables which would eliminate the Faddeev-Popov ghosts. It is then crucial to see what happens to the theory under such change of variables. Some progress has already been made in this direction [29–32].

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