UNIQUENESS IN INVERSE ELASTIC SCATTERING WITH ONE INCIDENT PLANE WAVE

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Abstract. In this paper, we give a positive answer to a longstanding open problem for determining the shape of an obstacle from the knowledge of the far field pattern for the scattering of time-harmonic elastic wave. We show that the elastic far field pattern by an incoming plane wave with a fixed frequency, a fixed incident direction and a fixed polarization determines the obstacle \( D \) and the boundary condition on \( \partial D \) uniquely. The boundary condition on \( \partial D \) is either the Dirichlet, or the Neumann, or the Robin one.

1. Introduction

Let \( D \) be a bounded domain with boundary \( \partial D \) of class \( C^2 \), and let \( \mathbb{R}^3 \setminus \overline{D} \) be the unbounded connected exterior domain of \( \Omega \). The propagation of time-harmonic elastic waves in a three-dimensional isotropic and homogeneous elastic medium \( \mathbb{R}^3 \setminus \overline{D} \) with Lamé coefficients \( \lambda \) and \( \mu \) satisfying \( \mu > 0 \) and \( \lambda + 2\mu > 0 \) (and density normalized to \( \rho = 1 \)) must obey the Navier equation (or Lamé system)

\[
(\Delta^* + \omega^2) u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \quad \Delta^* := \mu \Delta + (\lambda + \mu) \text{grad} \text{div},
\]

where \( u \) denotes the displacement field, and \( \omega > 0 \) is the frequency. Equivalently, the Navier equation (1.1) can also be written as

\[
-\mu \nabla \times \nabla \times u + (\lambda + 2\mu) \nabla (\nabla \cdot u) + \omega^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}.
\]

As usual, \( \mathbf{a} \cdot \mathbf{b} \) denotes the scalar product and \( \mathbf{a} \times \mathbf{b} \) denotes the vector product of \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \). Denote the linearized strain tensor by

\[
\epsilon(u) := \frac{1}{2}(\nabla u + \nabla u^\top) \in \mathbb{R}^{3 \times 3},
\]

where \( \nabla u \) and \( \nabla u^\top \) stand for the Jacobian matrix of \( u \) and its transpose, respectively. By Hooke’s law the strain tensor is related to the stress tensor via the identity

\[
\tau(u) = \lambda (\nabla \cdot u) \mathbf{I} + 2\mu \epsilon(u) \in \mathbb{R}^{3 \times 3},
\]

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where I denotes the $3 \times 3$ identity matrix. The surface traction (or the stress operator) on $\partial D$ is defined as

$$(1.5) \quad T_\nu u := \nu \cdot \tau(u) = 2\mu\nu \cdot \nabla u + \lambda\nu \cdot \nabla \cdot u + \mu\nu \times \nabla \times u,$$

where the unit normal vector $\nu$ to $\partial D$ always point into $\mathbb{R}^3 \setminus \bar{D}$. For convenience, we denote

$$(1.6) \quad B_1 u := u|_{\partial D}, \quad B_2 u := T_\nu u|_{\partial D}, \quad B_3 u := (T_\nu u + hu)|_{\partial D},$$

where $h=\text{const}$, $\Im h \geq 0$. The obstacle $D$ is supposed to be either a rigid body, or a cavity, or an absorbing body for which $u$ respectively satisfies the Dirichlet boundary condition

$$B_1 u = 0 \quad \text{on} \quad \partial D,$$

or the Neumann boundary condition

$$B_2 u = 0 \quad \text{on} \quad \partial D,$$

or the Robin boundary condition

$$B_3 u = 0 \quad \text{on} \quad \partial D.$$

It follows from Theorem 3.2.5 of [27] that any regular solution $u$ of (1.1) has the form

$$(1.7) \quad u = u^{(p)} + u^{(s)},$$

where $u^{(p)}$ and $u^{(s)}$ satisfy the conditions

$$(1.8) \quad (\Delta + \kappa_p^2) u^{(p)} = 0, \quad \nabla \times u^{(p)} = 0, \quad (\Delta + \kappa_s^2) u^{(s)} = 0, \quad \nabla \cdot u^{(s)} = 0,$$

where

$$(1.9) \quad \kappa_p := \frac{\omega}{\sqrt{\lambda + 2\mu}}, \quad \kappa_s := \frac{\omega}{\sqrt{\mu}}.$$

Here, $u^{(p)}$ denotes the pressure (longitudinal) wave whereas $u^{(s)}$ denotes the shear (transversal) wave, associated with the respective wave number $\kappa_p, \kappa_s$ given by (1.9). In addition, the field $u = u^{(p)} + u^{(s)}$ is required to satisfy the Kupradze radiation condition

$$\lim_{r \to \infty} r \left( \frac{\partial u^{(p)}}{\partial r} - i\kappa_p u^{(p)} \right) = 0, \quad \lim_{r \to \infty} r \left( \frac{\partial u^{(s)}}{\partial r} - i\kappa_s u^{(s)} \right) = 0,$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $\frac{x}{|x|}$. We will refer to solutions of the Navier equation satisfying the Kupradze radiation condition as radiating solutions. For existence and uniqueness of a Kupradze radiating solution to the above boundary value problems via a boundary integral equation approach we refer to Kupradze [27, 28].

In elastic scattering, an important case of incident fields is plane wave defined by

$$(1.10) \quad U^i(x, \alpha, \omega, \eta) := -\frac{1}{\omega^2} \nabla_x (\nabla_x \cdot [e^{i\kappa_p \cdot x/\eta}], \quad x \in \mathbb{R}^3,$$

where $\alpha \in S^2$ is its direction of propagation and $\eta \in \mathbb{R}^3$ controls its amplitude and polarization. The longitudinal and transversal plane waves respectively are

$$P^i(x, \alpha, \kappa_p, \eta) := -\frac{1}{\omega^2} \nabla_x (\nabla_x \cdot [e^{i\kappa_p \cdot x/\eta}] = \frac{1}{\lambda + 2\mu} e^{i\kappa_p \cdot x/\eta} (\alpha \cdot \eta), \quad x \in \mathbb{R}^3,$$

$$S^i(x, \alpha, \kappa_s, \eta) := \frac{1}{\omega^2} \nabla_x \times \nabla_x \times [e^{i\kappa_s \cdot x/\eta}] = -\frac{1}{\mu} e^{i\kappa_s \cdot x/\eta} \times (\alpha \times \eta), \quad x \in \mathbb{R}^3.$$

The polarization vector $\frac{1}{\lambda + 2\mu} (\alpha \cdot \eta)\alpha$ for the longitudinal wave is parallel to $\alpha$ and the polarization vector $-\frac{1}{\mu} \alpha \times (\alpha \times \eta)$ for the transversal wave is orthogonal to $\alpha$. Note that for
Consider the scattering problem:

\[(1.11) \quad (\Delta + \omega^2) \mathbf{U} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus D, \quad \mathcal{B}_i \mathbf{U}|_{\partial D} = 0, \quad \mathbf{U} = \mathbf{U}^i + \hat{\mathbf{U}}, \quad l = 1, 2, 3,\]

where the scattered field \(\hat{\mathbf{U}}(\mathbf{x}, \alpha, \omega, \eta)\) satisfies

\[
\lim_{r \to \infty} r \left( \frac{\partial \hat{U}^{(p)}(\mathbf{x}, \alpha, \omega, \eta)}{\partial r} - i \kappa_p \hat{U}^{(p)}(\mathbf{x}, \alpha, \omega, \eta) \right) = 0,
\]

\[
\lim_{r \to \infty} r \left( \frac{\partial \hat{U}^{(s)}(\mathbf{x}, \alpha, \omega, \eta)}{\partial r} - i \kappa_s \hat{U}^{(s)}(\mathbf{x}, \alpha, \omega, \eta) \right) = 0,
\]

uniformly in all directions, and \(r = |\mathbf{x}|\). Here

\[
\hat{\mathbf{U}} := \hat{\mathbf{U}}^{(p)} + \hat{\mathbf{U}}^{(s)}, \quad \hat{\mathbf{U}}^{(p)} := -\frac{1}{\kappa_p^2} \nabla \nabla \cdot \hat{\mathbf{U}}, \quad \hat{\mathbf{U}}^{(s)} := \frac{1}{\kappa_s} \nabla \times \nabla \times \hat{\mathbf{U}}
\]

satisfying

\[
(\Delta + \kappa_p^2) \hat{\mathbf{U}}^{(p)} = 0, \quad \nabla \times \hat{\mathbf{U}}^{(p)} = 0,
\]

\[
(\Delta + \kappa_s^2) \hat{\mathbf{U}}^{(s)} = 0, \quad \nabla \cdot \hat{\mathbf{U}}^{(s)} = 0.
\]

Moreover, the representation (2.14) of the solution for \(\hat{\mathbf{U}}\) (see Second 2) leads to the asymptotic behavior of the form

\[
\hat{\mathbf{U}}(\mathbf{x}, \alpha, \omega, \eta) = \frac{e^{ik_p|\mathbf{x}|}}{|\mathbf{x}|} \mathbf{U}^{(p, \infty)}(\mathbf{x}, \alpha, \omega, \eta) + \frac{e^{ik_s|\mathbf{x}|}}{|\mathbf{x}|} \mathbf{U}^{(s, \infty)}(\mathbf{x}, \alpha, \omega, \eta) + O(1/|\mathbf{x}|^2), \quad |\mathbf{x}| \to +\infty,
\]

uniformly in all directions \(\mathbf{x} := \frac{\mathbf{x}}{|\mathbf{x}|}\). The fields \(\mathbf{U}^{(p, \infty)}\) and \(\mathbf{U}^{(s, \infty)}\) are defined on the unit sphere \(S^2\) in \(\mathbb{R}^3\) and known as pressure (longitudinal) part and shear (transversal) part of the far-field pattern of \(\hat{\mathbf{U}}\), respectively. They can be represented by (see [3, 14, 15, 16])

\[
\mathbf{U}^{(p, \infty)}(\mathbf{x}, \alpha, \omega, \eta) = \frac{\kappa_p^2}{4\pi\omega^2} \int_{\partial D} \left\{ [T_{\nu}(y)\mathbf{x} \mathbf{x}^\top e^{-i\kappa_p \mathbf{x} \cdot \mathbf{y}}]^\top \cdot \hat{\mathbf{U}}(y, \alpha, \omega, \eta) - (\mathbf{x} \mathbf{x}^\top e^{-i\kappa_p \mathbf{x} \cdot \mathbf{y}}) \cdot T_{\nu}(y) \hat{\mathbf{U}}(y, \alpha, \omega, \eta) \right\} ds(y), \quad \mathbf{x} \in S^2,
\]

and

\[
\mathbf{U}^{(s, \infty)}(\mathbf{x}, \alpha, \omega, \eta) = \frac{\kappa_s^2}{4\pi\omega^2} \int_{\partial D} \left\{ [T_{\nu}(y)(\mathbf{I} - \mathbf{x} \mathbf{x}^\top) e^{-i\kappa_s \mathbf{x} \cdot \mathbf{y}}]^\top \cdot \hat{\mathbf{U}}(y, \alpha, \omega, \eta) - (\mathbf{I} - \mathbf{x} \mathbf{x}^\top) e^{-i\kappa_s \mathbf{x} \cdot \mathbf{y}} \cdot T_{\nu}(y) \hat{\mathbf{U}}(y, \alpha, \omega, \eta) \right\} ds(y), \quad \mathbf{x} \in S^2,
\]

where

\[
\mathbf{x} \mathbf{x}^\top := \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{pmatrix},
\]

\[
\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}.
\]
In view of the linearity of $U^i(x, \alpha, \omega, \eta)$ with respect to $\eta$, we find by the linear Navier equation (1.11) that for fixed $x \in \mathbb{R}^3 \setminus D$, $\hat{x}, \alpha \in S^2$ and $\omega \in \mathbb{R}^1$ the maps

$$\eta \in \mathbb{R}^3 \mapsto \hat{U}(x, \alpha, \omega, \eta) \in \mathbb{C}^3, \quad \eta \in \mathbb{R}^3 \mapsto U^{(p, \infty)}(\hat{x}, \alpha, \omega, \eta) \in C^3,$$

$$\eta \in \mathbb{R}^3 \mapsto U^{(s, \infty)}(\hat{x}, \alpha, \omega, \eta) \in C^3$$

are all linear. Note that for any $\eta \in \mathbb{R}^3$ and $\hat{x} \in S^2$,

$$\eta = (\hat{x} \cdot \eta) \hat{x}, \quad (I - \hat{x} \hat{x}^T) \eta = \hat{x} \times (\hat{x} \times \eta).$$

Thus, we see by (1.15)–(1.16) that the longitudinal part $U^{(p, \infty)}(\hat{x}, \alpha, \omega, \eta)$ is normal to $S^2$ and the transversal part $U^{(s, \infty)}(\hat{x}, \alpha, \omega, \eta)$ is tangential to $S^2$. We define the far-field pattern $U^{(\infty)}(\hat{x}, \alpha, \omega, \eta)$ of the scattered field $\tilde{U}(x, \alpha, \omega, \eta)$ as the sum of $U^{(p, \infty)}(\hat{x}, \alpha, \omega, \eta)$ and $U^{(s, \infty)}(\hat{x}, \alpha, \omega, \eta)$, that is,

$$U^{(\infty)}(\hat{x}, \alpha, \omega, \eta) := U^{(p, \infty)}(\hat{x}, \alpha, \omega, \eta) + U^{(s, \infty)}(\hat{x}, \alpha, \omega, \eta).$$

Clearly, we regard the far field pattern as a matrix-valued map

$$U^{(\infty)} : S^2 \times S^2 \mapsto C^{3 \times 3}$$

satisfying $U^{(\infty)}(\hat{x}, \alpha, \omega, \eta) = U^{(\infty)}(\hat{x}, \alpha, \omega, \eta)$. Especially (see Proof of Lemma 3.1), the knowledge of $U^{\infty}$ allows to compute the longitudinal and transversal parts of the far field pattern. It follows from [22] that for smooth bounded obstacle the far field pattern $U^{(\infty)}(\beta, \alpha, \omega)$ is real analytic matrix of $\beta$ and $\alpha$ on the unit sphere $S^2$. If $U^{(\infty)}(\beta, \alpha, \omega)$ as a matrix of $\beta$ is known on an open subset of $S^2$, it is uniquely extended to all of $S^2$ by analyticity. Physically, a far-field pattern can be obtained by sending a single incident plane wave and then measuring the scattered wave field far away in every possible observation direction (see [3] [14] [15] [16] [23]).

The basic inverse problem in elastic scattering theory is to determine the shape of the scatterer $D$ and the boundary condition on $\partial D$ from the knowledge of the elastic far field pattern $U^{(\infty)}(\beta, \alpha, \omega, \eta)$ for one or several incident plane waves with incident directions $\alpha$ and frequencies $\omega$ as well polarizations $\eta$. The study of inverse scattering problem for elastic wave is important in areas such as geophysical exploration, materials characterization and acoustic emission of many important materials and nondestructive testing. There is already a vast literature on inverse elastic scattering problems using the full far-field pattern $U^{(\infty)}$ (see [1], [9], [11], [10], [18], [19], [27], [29], [12], [43] and [44]). We refer to the theoretical uniqueness results proved in [21], [37], [38], [39], [10] and [42], and numerical reconstruction schemes developed in [2], [4], [6], [7], [8], [20] and [22]. In [21], Hähner and Hsiao proved that the far field pattern for a bounded sequence of different frequencies and an incoming plane wave with fixed incident direction and fixed polarization uniquely determine the obstacle. They also proved the same result for a fixed frequency but for all incident directions and a certain set of polarizations. Elschner and Yamamoto [17] derived uniqueness results for polyhedral elastic scatterers with finitely many income plane waves. Hu, Kirsch and Sini in [24] proved that a $C^r$-regular rigid scatter in $\mathbb{R}^3$ can be uniquely identified by the shear part of the far field pattern corresponding to all incident shear plane waves at any fixed frequency (see also [20]). They also showed that uniqueness using the shear part of the far-field pattern corresponding to only one incident plane shear wave holds for a ball or a convex Lipschitz polyhedron.

In the inverse acoustic obstacle scattering (i.e., the Helmholtz equation), the author of this paper in [39] proved that the scattering amplitude for one single incident direction and a fixed wave number uniquely determines the acoustic obstacle. By a similar method, the author in [31] showed the corresponding uniqueness result for the Maxwell equations by incident plane wave with a fixed incident direction, a fixed wave number and a fixed polarization.
However, it is widely known to be a challenging open problem (see, e.g., p. 1 of [17], p. 1 of [23] or [24]) that for a fixed frequency $\omega$, a fixed incident direction $\alpha$ and a fixed polarization direction $\eta$, whether the elastic far field pattern can uniquely determine the general scatterer $D$ and its boundary condition (where scatterer could be rigid or cavity or absorbing which is not required to be known in advance)? Note that the acoustic waves are a type of longitudinal waves that propagate by means of adiabatic compression and decompression; the electric and magnetic fields of an electromagnetic wave are perpendicular to each other and to the direction of the wave; however, the elastic waves are vector waves which have both transverse and longitudinal waves in elastic medium and are coupled by condition on the boundary. It is generally much more difficult to study elastic wave scattering problem than acoustic and electromagnetic ones.

In this paper, by discussing all possible positions of two scatterers and by applying the eigenvalue theory of the Navier operator, we give a positive answer to the above inverse scattering problem for the elastic field. Our main result is the following:

**Theorem 1.1.** Assume that $D_1$ and $D_2$ are two scatterers with boundary conditions $B^{D_1}$ and $B^{D_2}$ such that for a fixed frequency $\omega_0$, a fixed incident direction $\alpha_0$, and a fixed polarization $\eta_0$ the elastic far field patterns of both scatterers coincide (i.e., $U_\infty^\gamma(\beta, \alpha_0, \omega_0)\eta_0 = U_\infty^\gamma(\beta, \alpha_0, \omega_0)\eta_0$ for all $\beta$ in an open subset of $S^2$). Then $D_1 = D_2$ and $B^{D_1} = B^{D_2}$.

Let us remark that our means is completely new and essentially an elementary one. This paper is organized as follows. In Section 2, we introduce some known results for the Navier equation in $\mathbb{R}^3 \setminus \bar{D}$. More important, we prove a new Rellich-type lemma for the Navier operator (Lemma 2.7). In Section 3, by applying Lemma 2.7 we show that when the elastic far field patterns of both scatterers coincide for a fixed incident plane wave, the two scatterers produce the same scattering field (Lemma 3.1). According to these results, in Section 4, we eventually prove the two scatterers and their boundary conditions coincide.

## 2. Preliminaries

Let $g(x)$ be a real-valued function defined in an open set $\Omega$ in $\mathbb{R}^n$. For $y \in \Omega$ we call $g$ real analytic at $y$ if there exist $a_\gamma \in \mathbb{R}^4$ and a neighborhood $V$ of $y$ (all depending on $y$) such that

$$g(x) = \sum_\gamma a_\gamma(x - y)^\gamma$$

for all $x \in V$, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index (a set of non-negative integers), $|\gamma| = \sum_{j=1}^n \gamma_j$, and $(x - y)^\gamma = (x_1 - y_1)^{\gamma_1} \cdots (x_n - y_n)^{\gamma_n}$. We say $g$ is real analytic in $\Omega$, if $g$ is real analytic at each $y \in \Omega$.

**Lemma 2.1** (Unique continuation of real analytic function, see, for example, p. 65 of [25]). *Let $\Omega$ be a connected open set in $\mathbb{R}^n$, and let $g$ be real analytic in $\Omega$. Then $g$ is determined uniquely in $\Omega$ by its values in any nonempty open subset of $\Omega$.*

**Lemma 2.2** (The interior real analyticity of the solutions for real analytic elliptic equations, see [33], [34], [35] or [36]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $L$ be a strongly elliptic
linear differential operator of order $2m$

$$Lu(x) = \sum_{|\gamma| \leq 2m} a_\gamma(x) D^\gamma u(x).$$

If the coefficients $a_\gamma(x)$, $|\gamma| \leq 2m$, and the right-hand side $f(x)$ of the equation $Lu(x) = f(x)$ are real analytic with respect to $x = (x_1, \cdots, x_n)$ in the domain $\Omega$, then any solution $u$ of this equation is also real analytic in $\Omega$.

Consider the following eigenvalue problem of the Navier operator (for the Robin boundary condition, we should assume $h$ is a nonnegative constant) in a bounded domain $\Omega$:

$$\begin{cases}
(\Delta^* + \omega^2)u = 0 \quad \text{in } \Omega, \\
B_j u = 0 \quad \text{on } \partial \Omega, \quad j = 1, 2, 3.
\end{cases}$$

It is well-known (see [41] or [27]) that there exists for the problem (2.1) a countable set of positive wave numbers $\omega^2$ (note that the first Neumann Navier eigenvalue is zero) called eigenvalues, accumulating only at infinity for which the homogeneous problem has nontrivial solutions. We arrange the eigenvalues in non-decreasing order (repeated according to multiplicity):

$$0 \leq \omega_1^2 \leq \omega_2^2 \cdots \leq \cdots \omega_k^2 \leq \cdots.$$

The nontrivial solution $u_k$ of (2.1) corresponding to the eigenvalue $\omega_k^2$ is called the $k$-th Navier eigen-field. Lemma 2.2 leads to the following:

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with piecewise $C^2$-smooth boundary. Assume that $u$ is a Navier eigen-field corresponding to the Navier eigenvalue $\omega^2$. Then $u$ must be a real analytic vector-field in $\Omega$.

**Lemma 2.4** (see [27]). Any continuously differentiable solution to the Navier equation has analytic cartesian components. In particular, the cartesian components of solutions to the Navier equation are automatically two times continuously differentiable.

**Lemma 2.5** (see Theorem 6.7 of [13]). Assume the bounded domain $D$ is the open complement of an unbounded domain of class $C^2$. Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a solution to the Navier equation

$$\begin{cases}
(\Delta^* + \omega^2)u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}
\end{cases}$$

satisfying the Kupradze radiation condition. Then the radiating solutions $u = u^{(p)} + u^{(s)}$ to the Navier equation automatically satisfy

$$u^{(p)}(x) = O\left(\frac{1}{|x|}\right), \quad u^{(s)}(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

uniformly for all directions $\frac{x}{|x|}$.

**Lemma 2.6** (see [27]). Let $u$ be a solution to the Navier equation in $\mathbb{R}^3$ satisfying the Kupradze radiation condition. Then $u$ must vanish identically in $\mathbb{R}^3$.

The following Rellich-type lemma for the Navier operator $\Delta^* + \omega$ will be needed late:

**Lemma 2.7.** If $u$ satisfies

$$\begin{cases}
(\Delta^* + \omega^2)u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}
\end{cases}$$
and
\[(2.4) \quad \lim_{r \to \infty} \int_{|x|=r} |u(x)|^2 ds = 0,\]
then \(u(x) = 0\) in \(\mathbb{R}^3 \setminus D\).

**Proof.** Recall (see Section 1) that \(u = u^{(p)} + u^{(s)}\), and \(u^{(p)}\) and \(u^{(s)}\) satisfy
\[
(\Delta + \kappa_p^2)u^{(p)} = 0, \quad \nabla \times u^{(p)} = 0,
\]
\[
(\Delta + \kappa_s^2)u^{(s)} = 0, \quad \nabla \cdot u^{(s)} = 0,
\]
where
\[
\kappa_p := \frac{\omega}{\lambda + \sqrt{\lambda \mu}}, \quad \kappa_s := \frac{\omega}{\sqrt{\mu}}.
\]
In fact, one has (see p. 124 of [27])
\[(2.5) \quad u^{(p)} = \frac{1}{\kappa_s^2 - \kappa_p^2} (\Delta + \kappa_p^2)u, \quad u^{(s)} = \frac{1}{\kappa_p^2 - \kappa_s^2} (\Delta + \kappa_s^2)u.
\]

It is well-known that the spherical harmonics
\[(2.6) \quad Y^{m}_n(\theta, \phi) := \sqrt{\frac{2n + 1}{4\pi} \frac{(n - |m|)!}{(n + |m|)!}} P^{|m|}_n(\cos \theta)e^{im\phi}
\]
for \(m = -n, \ldots, n, \ n = 0, 1, 2, \ldots\), form a complete orthonormal system in \(L^2(S^2)\), where \(P^{|m|}_n(t), \ m = 0, 1, \ldots, n,\) is the associated Legendre functions. Thus, for sufficiently large \(|x|\), we have a Fourier expansion
\[(2.7) \quad u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(a^{(p)}_{nm}(|x|) + a^{(s)}_{nm}(|x|)\right) Y^m_n(\hat{x})
\]
with respect to spherical harmonics, where \(\hat{x} = x/|x|\). The coefficients are given by
\[
a^{(p)}_{nm}(r) + a^{(s)}_{nm}(r) = \int_{S^2} u^{(p)}(r\hat{x}) Y^m_n(\hat{x}) d\hat{x} + \int_{S^2} u^{(s)}(r\hat{x}) Y^m_n(\hat{x}) d\hat{x}
\]
\[
= \int_{S^2} \left(\Delta + \kappa_p^2\right)u^{(p)}(r\hat{x}) + \nabla \times u^{(s)}(r\hat{x}) d\hat{x} = \int_{S^2} u^{(p)}(r\hat{x}) Y^m_n(\hat{x}) d\hat{x} = \int_{S^2} u^{(s)}(r\hat{x}) Y^m_n(\hat{x}) d\hat{x}
\]
and satisfy Parseval’s equality
\[
\int_{|x|=r} |u(x)|^2 ds = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |a^{(p)}_{nm}(r) + a^{(s)}_{nm}(r)|^2.
\]

Our assumption \[(2.4)\] implies that
\[(2.8) \quad \lim_{r \to \infty} r^2|a^{(p)}_{nm}(r) + a^{(s)}_{nm}(r)|^2 = 0
\]
for all \(n\) and \(m\). Since \(u^{(p)}, u^{(s)} \in (C^2(\mathbb{R}^3 \setminus \bar{D}))^3\), we can differentiate under the integral and integrate by parts using \(\nabla \times u^{(p)} + \kappa_p u^{(p)} = 0\) and \(\nabla u^{(s)} + \kappa_s u^{(s)} = 0\) to conclude that the \(a^{(p)}_{nm}(r)\) and \(a^{(s)}_{nm}(r)\) are solutions to the spherical Bessel equations
\[
\frac{d^2a^{(p)}_{nm}(r)}{dr^2} + \frac{2}{r} \frac{da^{(p)}_{nm}(r)}{dr} + \left(\kappa_p^2 - \frac{n(n+1)}{r^2}\right) a^{(p)}_{nm}(r) = 0, \quad r \geq R_0,
\]
\[
\frac{d^2a^{(s)}_{nm}(r)}{dr^2} + \frac{2}{r} \frac{da^{(s)}_{nm}(r)}{dr} + \left(\kappa_s^2 - \frac{n(n+1)}{r^2}\right) a^{(s)}_{nm}(r) = 0, \quad r \geq R_0,
\]
respectively. That is,

\[ a_{nm}^{(s)}(r) = \beta_{nm}^{(s)} h_n^{(1)}(\kappa r) + \gamma_{nm}^{(s)} h_n^{(2)}(\kappa r), \]

where \( \beta_{nm}^{(p)} \) and \( \gamma_{nm}^{(s)} \) are constants. Note that (see, e.g., p. 31 of [13]) the spherical Hankel functions have the following asymptotic behavior for large argument

\[ h_n^{(1)}(t) = \frac{1}{t} e^{-i(t - \frac{\pi}{2})} \left( 1 + O\left(\frac{1}{t}\right) \right), \quad t \to \infty, \]

\[ h_n^{(2)}(t) = \frac{1}{t} e^{-i(t - \frac{\pi}{2})} \left( 1 + O\left(\frac{1}{t}\right) \right), \quad t \to \infty. \]

Using the asymptotic formula and substituting (2.9) into (2.8) we obtain \( \beta_{nm}^{(p)} = \gamma_{nm}^{(p)} = \beta_{nm}^{(s)} = \gamma_{nm}^{(s)} = 0 \) for all \( n \) and \( m \). Consequently, \( u = 0 \) outside a sufficiently large ball and hence \( u = 0 \) in \( \mathbb{R}^3 \setminus D \) by analyticity.

Let \( \Delta^* \) and \( T_\nu \) be as in Section 1. If \( u, v : D \to \mathbb{R}^3 \) denote \( C^2(D) \cap C^1(\overline{D}) \) smooth vector fields, then Gauss’ theorem implies

\[ \int_D (u \cdot \Delta^* v - v \cdot \Delta^* u) \, dx = \int_{\partial D} (u \cdot T_\nu v - v \cdot T_\nu u) \, ds(x). \tag{2.10} \]

The above formula is called Betti’s integral formula (see (2.1) of [4] or p. 121 of [27]). If \( v \) and the columns of the matrix \( W = (w_1, w_2, w_3) \) satisfy the Navier equation \( \Delta^* u + \omega^2 u = 0 \) in \( D \), then from (2.10) we deduce that

\[ \int_{\partial D} [(T_\nu W)^T \cdot v - W^T \cdot T_\nu v] \, ds = 0, \tag{2.11} \]

where \( T_\nu W = (T_\nu w_1, T_\nu w_2, T_\nu w_3) \) and \( ^T \) indicates the transpose of a matrix.

The fundamental solution (or Kupradze’s matrix) to the Navier equation is given by

\[ \Upsilon(x, y, \omega) := \frac{\kappa_s}{4\pi\omega^2} e^{i\kappa_s |x-y|} I + \frac{1}{4\pi\omega^2} \nabla \times \nabla \times \left[ \frac{e^{i\kappa_s |x-y|}}{|x-y|} - \frac{e^{i\kappa_p |x-y|}}{|x-y|}\right], \tag{2.12} \]

where \( I \) denotes the identity matrix. From the definition, we can immediately see that \( \Upsilon(x, y, \omega) \) satisfies \( \Upsilon(x, y, \omega) = [\Upsilon(x, y, \omega)]^T \). By the identity \( \nabla \times \nabla \times F = -\Delta F + \nabla(\nabla \cdot F) \) for any vector field \( F \), we can infer that

\[ \Upsilon(x, y, \omega) = \frac{1}{\omega^2} \nabla \times \nabla \times \left( \frac{e^{i\kappa_s |x-y|}}{4\pi |x-y|} I \right) - \frac{1}{\omega^2} \nabla \times \nabla \times \left( \frac{e^{i\kappa_p |x-y|}}{4\pi |x|} I \right). \tag{2.13} \]

Note that Kupradze’s matrix \( \Upsilon(x, y, \omega) \) for the \( \Delta^* \) operator has the same role as \( 1/(4\pi |x-y|) \) for the \( \Delta \)-operator. It is well-known (see [27] [28]) that \( \Upsilon(x, y, \omega) \) satisfies \( \Delta^* \Upsilon(x, y, \omega) + \omega^2 \Upsilon(x, y, \omega) = -\delta(x-y) I \) for \( x \neq y \). From Betti’s integral formula, for the radiating solution \( u \in (C^2(\mathbb{R}^3 \setminus \overline{D}))^3 \cap (C^1(\overline{\mathbb{R}^3 \setminus \overline{D}}))^3 \) to the Navier equation (1.1), one can derive the integral representation (see p. 131 of [27])

\[ u(x) = \int_{\partial D} \left( [T_\nu \Upsilon(x, y, \omega)]^T \cdot u(y) - \Upsilon(x, y, \omega) \cdot T_\nu u(y) \right) \, ds(y) \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \overline{D}. \tag{2.14} \]

**Lemma 2.8** (Holmgren’s uniqueness theorem for the Navier equation). Let \( D \) be a bounded domain with \( C^2 \)-smooth boundary \( \partial D \) and let \( \Gamma \subset \partial D \) be an open subset with \( \Gamma \cap (\mathbb{R}^3 \setminus D) \neq \emptyset. \)
Assume that \( u \) is a solution of the scattering problem for the Navier equation
\[
\begin{cases}
\Delta^* u + \omega^2 u = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\
u = U^i + u & \text{in } \mathbb{R}^3 \setminus \bar{D},
\end{cases}
\]
where \( U^i \) is as in Section 1, and the scattering solution \( \tilde{u} \) satisfies the Kupradze radiation condition, such that
\[
(2.15) \quad u = T_\nu u = 0 \quad \text{on } \Gamma.
\]
Then \( u \equiv 0 \) in \( \mathbb{R}^3 \setminus \bar{D} \).

*Proof.* We first prove the same conclusion if \( \mathbb{R}^3 \setminus \bar{D} \) is replaced by a bounded domain \( W \) with \( C^2 \)-smooth boundary and \( \Gamma \subset \partial W \). Let \( v \) be a solution of the Navier equation \((\Delta^* + \omega^2)v = 0\) in \( W \). It follows from p. 123 of [27] that
\[
(2.16) \quad \int_{\partial D} \left[ - (T_\nu(y) \Upsilon(x, y, \omega))^T \cdot v(y) + \Upsilon(x, y, \omega) \cdot T_\nu(y) v(y) \right] ds(y) = \begin{cases}
v(x) & \text{for } x \in W, \\
0 & \text{for } x \in \mathbb{R}^3 \setminus W.
\end{cases}
\]
In view of \( v = T_\nu v = 0 \) on \( \Gamma \), we use the formula \((2.16)\) to extend the definition of \( v \) by setting
\[
v(x) := \int_{\partial \bar{W} \setminus \Gamma} \left[ \Upsilon(x, y, \omega) \cdot T_\nu(y) v(y) - (T_\nu(y) \Upsilon(x, y, \omega))^T \cdot v(y) \right] ds(y)
\]
for \( x \in (\mathbb{R}^3 \setminus \bar{W}) \cup \Gamma \). Then, by the representation formula \((2.16)\), we obtain \( v = 0 \) in \( \mathbb{R}^3 \setminus W \).

It is obvious that \( v \) solves the Navier equation in \((\mathbb{R}^3 \setminus \partial W) \cup \Gamma\) and hence \( v = 0 \) in \( W \), because \( \mathbb{R}^3 \setminus W \) and \( W \) are connected through the gap \( \Gamma \subset \partial W \) and \( v \) is analytic in \((\mathbb{R}^3 \setminus \partial W) \cup \Gamma\).

Now, let \( W \) be a bounded domain in \( \mathbb{R}^3 \setminus \bar{D} \) with \( C^2 \)-smooth boundary such that \( W \) and \( \mathbb{R}^3 \setminus \bar{D} \) have the common part boundary surface \( \Gamma \). Since \( u \) still satisfies the Navier equation in \( W \) and \( u = T_\nu u = 0 \) on \( \Gamma \), it follows from the conclusion of the first part that \( u = 0 \) in \( W \), so that \( u = 0 \) in \( \mathbb{R}^3 \setminus \bar{D} \) by the analyticity of \( u \) in \( \mathbb{R}^3 \setminus \bar{D} \). \( \square \)

3. Elastic scattering fields in the exterior of two scatterers

Let \( D_j \) be a bounded domain with a connected \( C^2 \)-smooth boundary \( \partial D_j \), and let \( \mathbb{R}^3 \setminus \bar{D}_j \) be the unbounded exterior domain of \( D_j \) \((j = 1, 2)\). Recall that incident plane wave with incident direction \( \alpha \in \mathbb{S}^2 \) and polarization vector \( \eta \in \mathbb{R}^3 \) is described by \( U^i(x, \alpha, \omega)\eta = P^i(x, \alpha, \kappa_p)\eta + S^i(x, \alpha, \kappa_s)\eta \), where
\[
P^i(x, \alpha, \kappa_p)\eta := \frac{1}{\lambda + 2\mu} e^{i\kappa_p \cdot x} \alpha \cdot (\alpha \cdot \eta) \alpha,
\]
\[
S^i(x, \alpha, \kappa_s)\eta := -\frac{1}{\mu} e^{i\kappa_s \cdot x} \alpha \times (\alpha \times \eta).
\]
Let \( U_j(x, \alpha, \omega)\eta, j = 1, 2, \) be the solution of the scattering problem
\[
(3.1) \quad \begin{cases}
(\Delta^* + \omega^2) U_j = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}_j, \\
U_j(x, \alpha, \omega)\eta = U^i(x, \alpha, \omega)\eta + \tilde{U}_j(x, \alpha, \omega)\eta, \\
\mathcal{B}_l U_j = 0 & \text{on } \partial D_j, \quad l = 1, 2, 3,
\end{cases}
\]
where $\hat{U}_j$ satisfies the Kupradze radiation condition. As pointed out in [1.14] of Section 1, we can write

$$U_j(x, \alpha, \omega)\eta = \left( \frac{1}{\lambda + 2\mu} e^{i\kappa_x \cdot x}(\alpha \cdot \eta)\alpha - \frac{1}{\mu} e^{i\kappa_x \cdot x} \times (\alpha \times \eta) \right)$$

$$+ \frac{e^{i\kappa_x |x|}}{|x|} \left( U_j^{(p, \infty)}(\hat{x}, \alpha, \omega)\eta + \frac{e^{i\kappa_x |x|}}{|x|} \left( U_j^{(s, \infty)}(\hat{x}, \alpha, \omega)\right) \eta \right) + O\left( \frac{1}{|x|^2} \right), \text{ as } |x| \to \infty,$$

uniformly in all directions $\hat{x} := \frac{x}{|x|}$. Clearly,

$$U_j^{(\infty)}(\hat{x}, \alpha, \omega)\eta := U_j^{(p, \infty)}(\hat{x}, \alpha, \omega)\eta + U_j^{(s, \infty)}(\hat{x}, \alpha, \omega)\eta.$$

**Lemma 3.1.** Let $U_j(x, \alpha_0, \omega_0)\eta_0$ be the total elastic scattering field corresponding to the incident plane wave $U_j(x, \alpha_0, \omega_0)\eta_0$ in $\mathbb{R}^3 \setminus \hat{D}_j$ $(j = 1, 2)$. If $U_j^{(\infty)}(\beta, \alpha_0, \omega_0)\eta_0 = U_2^{(\infty)}(\beta, \alpha_0, \omega_0)\eta_0$ for all $\beta \in S^2$, a fixed $\alpha_0$, a fixed $\omega_0$ and a fixed $\eta_0$, then

$$U_1(x, \alpha_0, \omega_0)\eta_0 = U_2(x, \alpha_0, \omega_0)\eta_0 \quad \text{for all } x \in D_{12},$$

where $D_{12} := \mathbb{R}^3 \setminus (D_1 \cup D_2)$.

**Proof.** We can write $U_j(x, \alpha_0, \omega_0)\eta_0 = P_j(x, \alpha_0, \omega_0)\eta_0 + S_j(x, \alpha_0, \omega_0)\eta_0$ for $x \in \mathbb{R}^3 \setminus \hat{D}_j$, where $P_j(x, \alpha_0, \omega_0)\eta_0$ and $S_j(x, \alpha_0, \omega_0)\eta_0$ are as in [1.12]–[1.13] for $x \in \mathbb{R}^3 \setminus \hat{D}_j$. Let $A(\hat{x}) \in \mathbb{R}^{3 \times 3}$ denote the matrix $\hat{x}^T \hat{x}$ having the entry $\hat{x}_j \hat{x}_k$ in the $j$th row and $k$th column (see [1.17]) where $\hat{x} \in S^2$ is a unit vector. Then [1.12]–[1.13] and the relation

$$P_j(x, \alpha_0, \omega_0)\eta_0 = U_j^{(p, \infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 + U_j^{(s, \infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0$$

yield $P_j^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 = U_j^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 = U_j^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0$. Similarly, we can compute $S_j^{(x)}(\hat{x}, \alpha_0, \omega_0)\eta_0 = U_j^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0$. Here $\kappa^0_0 = \frac{\omega}{\sqrt{\lambda + 2\mu}}$, $\kappa^0_0 = \frac{\omega}{\sqrt{\mu}}$. Thus, if $U_j^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 = U_2^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0$ for all $\hat{x} \in S^2$, a fixed $\alpha_0 \in S^2$, a fixed $\omega_0 \in \mathbb{R}^3$ and a fixed $\eta_0$, then, for all $\hat{x} \in S^2$,

$$P_1^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 = \left( U_1^{(\infty)}(\hat{x}, \alpha_0, \omega_0, A(\alpha_0))\eta_0 \right)$$

$$= \left( U_2^{(\infty)}(\hat{x}, \alpha_0, \omega_0, A(\alpha_0))\eta_0 \right) = P_2^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0,$$

$$S_1^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 = \left( U_1^{(\infty)}(\hat{x}, \alpha_0, \omega_0, (I - A(\alpha_0))\eta_0) \right)$$

$$= \left( U_2^{(\infty)}(\hat{x}, \alpha_0, \omega_0, (I - A(\alpha_0))\eta_0) \right) = S_2^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0.$$

By [3.2] we get

$$U_j(x, \alpha_0, \omega_0)\eta_0 = \left( \frac{1}{\lambda + 2\mu} e^{i\kappa_x \alpha_0 \cdot \eta_0} \alpha_0 - \frac{1}{\mu} e^{i\kappa_x \cdot \alpha_0 \times \eta_0} \right)$$

$$+ \frac{e^{i\kappa_x |x|}}{|x|} \left( P_j^{(\infty)}(\hat{x}, \alpha_0, \omega_0)\eta_0 + \frac{e^{i\kappa_x |x|}}{|x|} \left( S_j^{\infty}(\hat{x}, \alpha_0, \omega_0)\eta_0 \right) \right)$$

$$+ O\left( \frac{1}{|x|^2} \right), \text{ as } |x| \to \infty, \quad j = 1, 2,$$
so that

\[(3.6) \quad U_1(x, \alpha_0, \omega_0) \eta_0 - U_2(x, \alpha_0, \omega_0) \eta_0 = \frac{e^{i\kappa_0|x|}}{|x|} \left[ P_1^{(\infty)}(x, \alpha_0, \omega_0) \eta_0 - P_2^{(\infty)}(x, \alpha_0, \omega_0) \eta_0 \right] + \frac{e^{i\kappa_0|x|}}{|x|} \left[ S_1^{(\infty)}(x, \alpha_0, \omega_0) \eta_0 - S_2^{(\infty)}(x, \alpha_0, \omega_0) \eta_0 \right] + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty.\]

Therefore, we find by (3.4) and (3.6) that

\[(3.7) \quad U_1(x, \alpha_0, \omega_0) \eta_0 - U_2(x, \alpha_0, \omega_0) \eta_0 = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,\]

Obviously, \(U_1(x, \alpha_0, \omega_0) \eta_0 - U_2(x, \alpha_0, \omega_0) \eta_0\) satisfies the Navier equation:

\[(3.8) \quad \Delta^* (U_1 - U_1) + \omega^2 (U_1 - U_2) = 0 \text{ in } D_{12}.\]

Combining (3.8), (3.7) and Rellich-type’s lemma for Navier operator (i.e., Lemma 2.7), we find that

\[U_1(x, \alpha_0, \omega_0) \eta_0 = U_2(x, \alpha_0, \omega_0) \eta_0 \quad \text{for all } x \in D_{12}.\]

\[\Box\]

4. Proof of main result

Proof of theorem 1.1. For convenience, we assume that the obstacles are rigid bodies (i.e., the Dirichlet boundary conditions), but our proof is valid for the cavity or the absorbing obstacle (i.e., the Neumann or the Robin) boundary condition as well. It is an obvious fact that if two bounded domains \(D_1\) and \(D_2\) of class \(C^2\) satisfying \(D_1 \neq D_2\), then either \(D_1 \neq D_2\) and \(D_1 \cap D_2 = \emptyset\), or \(D_1 \neq D_2\) and \(D_1 \cap D_2 \neq \emptyset\). We will show that the above two cases can never occur.

Case 1. Suppose by contradiction that \(D_1 \neq D_2\) and \(D_2 \cap D_1 = \emptyset\). Since \(U_1^\infty(\beta, \alpha_0, \omega_0) \eta_0 = U_2^\infty(\beta, \alpha_0, \omega_0) \eta_0\) for all \(\beta\) in an open subset of \(S^2\), we immediately get that the above relation is still true for all \(\beta \in S^2\) by analyticity. Therefore we find from Lemma 3.1 that

\[U_1(x, \alpha_0, \omega_0) \eta_0 = U_2(x, \alpha_0, \omega_0) \eta_0 \quad \text{for all } x \in D_{12},\]

where \(U_j(x, \alpha_0, \omega_0) \eta_0\) is the solution of scattering problem for the Navier equation in \(\mathbb{R}^3 \setminus \hat{D}_j\) \((j = 1, 2)\), and \(D_{12}\) is the unbounded connected component of \(\mathbb{R}^3 \setminus (\hat{D}_1 \cup \hat{D}_2)\). Note that the real and imaginary parts of cartesian components of \(U_j\) are both real analytic in \(\mathbb{R}^3 \setminus \hat{D}_j\) \((j = 1, 2)\) by Lemma 2.4. Since \(U_1(x, \alpha_0, \omega_0) \eta_0\) is defined in \(D_2\) and satisfies there the Navier equation, the unique continuation property implies that \(U_2(x, \alpha_0, \omega_0) \eta_0\) can be defined in \(D_2\) and satisfies there the Navier equation. Consequently, \(U_2(x, \alpha_0, \omega_0) \eta_0\) is defined in \(\mathbb{R}^3\), it is a smooth vector-valued function that satisfies the Navier equation in \(\mathbb{R}^3\), and the same is true for \(U_1(x, \alpha_0, \omega_0) \eta_0\). Therefore the scattered fields \(U_1(x, \alpha_0, \omega_0) \eta_0\) and \(U_2(x, \alpha_0, \omega_0) \eta_0\) of the total fields \(U_1(x, \alpha_0, \omega_0) \eta_0\) and \(U_2(x, \alpha_0, \omega_0) \eta_0\) satisfy the Navier equation \((\Delta^* + \omega^2 = 0)\) in \(\mathbb{R}^3\) for \(\eta \neq 0\).
\( \omega^2 u = 0 \) in \( \mathbb{R}^3 \) and have the Kupradze radiation conditions. It follows from Lemma 2.6 that
\[
U_1(x, \alpha_0, \omega_0) \eta_0 = U_1(x, \alpha_0, \omega_0) \eta_0 = 0 \quad \text{in} \quad \mathbb{R}^3 \quad \text{and hence}
\]
\[
U_1(x, \alpha_0, \omega_0) \eta_0 = U_2(x, \alpha_0, \omega_0) \eta_0
\]
\[
= \frac{1}{\lambda + 2\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 - \frac{1}{\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 - \frac{1}{\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 = 0 \quad \text{in} \quad \mathbb{R}^3.
\]
This is impossible since \( U_j(x, \alpha_0, \omega_0) \eta_0 = 0 \) on \( \partial D_j, \) \( j = 1, 2, \) while \( \frac{1}{\lambda + 2\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 - \frac{1}{\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 \) can not vanish identically for all \( x \in \partial D_j. \) Thus, we must have \( \tilde{D}_1 = D_2. \)

Case 2. Suppose by contradiction that \( D_2 \neq D_2 \) and \( D_1 \cap D_2 \neq \emptyset. \) Then either \((\mathbb{R}^3 \setminus \tilde{D}_1) \cap (\mathbb{R}^3 \setminus D_2)\) or \((\mathbb{R}^3 \setminus D_2) \cap (\mathbb{R}^3 \setminus D_1)\) has only finitely many connected components, and each of them adjoins the unbounded domain \( D_{12} \) by sharing a common \( C^2 \)-smooth surface, where \( D_{12} \) is the unbounded connected component of \( \mathbb{R}^3 \setminus (\tilde{D}_1 \cup D_2). \) Let us assume that \( \Omega \) be any one of the above connected components. Clearly, \( \Omega \) is a bounded domain with piecewise \( C^2 \)-smooth boundary. Without loss of generality, we let \( \Omega \subset \mathbb{R}^3 \setminus \tilde{D}_1. \) Since \( U^\infty_\Omega(\beta, \alpha_0, \omega_0) \eta_0 = U^\infty_\Omega(\beta, \alpha_0, \omega_0) \eta_0 \) for all \( \beta \in \mathbb{S}^2 \) by analyticity, applying Lemma 3.1 once more we find that
\[
U_1(x, \alpha_0, \omega_0) \eta_0 = U_2(x, \alpha_0, \omega_0) \eta_0 \quad \text{for all} \quad x \in D_{12},
\]
where \( U_j(x, \alpha_0, \omega_0) \eta_0 \) is the total solution of the scattering problem for the Navier equation in \( \mathbb{R}^3 \setminus D_j \) \( (j = 1, 2). \) Note that \( U_j |_{\partial D_j} = 0, \) \( j = 1, 2, \) and \( U_1|_{\partial D_{12}} = U_2|_{\partial D_{12}} = 0. \) It is easy to see from this and the definition of \( \Omega \) that the restriction of \( U_1(x, \alpha_0, \omega_0) \eta_0 \) to \( \Omega \) satisfies
\[
\begin{cases}
\Delta^* u + \omega_0^2 u = 0 & \text{in} \ \Omega, \\
u = 0 & \text{on} \ \partial \Omega,
\end{cases}
\]
i.e., the restriction of \( U_1(x, \alpha_0, \omega_0) \eta_0 \) is a Navier eigen-field corresponding to the Navier eigen-value \( \omega_0^2. \) We find by Lemma 2.4 that \( \text{Re} \ U_1(x, \alpha_0, \omega_0) \eta_0 \) and \( \text{Im} \ U_1(x, \alpha_0, \omega_0) \eta_0 \) are both real analytic vector-valued function in \( \mathbb{R}^3 \setminus \tilde{D}_1, \) where \( \text{Re} \ U_1(x, \alpha_0, \omega_0) \eta_0 \) and \( \text{Im} \ U_1(x, \alpha_0, \omega_0) \eta_0 \) are the real part and imaginary part of the full scattering field \( U_1(x, \alpha_0, \omega_0) \eta_0 \), i.e., \( U_1(x, \alpha_0, \omega_0) \eta_0 = \text{Re} \ U_1(x, \alpha_0, \omega_0) \eta_0 + i \text{Im} \ U_1(x, \alpha_0, \omega_0) \eta_0. \) By the definition of the total scattering field \( U_1(x, \alpha_0, \omega_0) \eta_0, \) we have that for all \( x \in \mathbb{R}^3 \setminus \tilde{D}_1, \)
\[
U_1(x, \alpha_0, \omega_0) \eta_0 = \frac{1}{\lambda + 2\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0
\]
\[
- \frac{1}{\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 - \frac{1}{\mu} e^{i \kappa \cdot x} (\alpha_0 \cdot \eta_0) \alpha_0 = 0 \quad \text{in} \quad \mathbb{R}^3.
\]
Note that the Navier equation is real analytic in \( \Omega \) and the Dirichlet boundary condition is real boundary condition. It follows from Lemma 2.3 that the Navier eigen-field \( U_1(x, \alpha_0, \omega_0) \eta_0 \)
must be a real analytic vector-valued function in \( \Omega \). From this and \[(1.2), \] we get that
\[
\frac{1}{\lambda + 2\mu}(\sin(k_p^0 \alpha_0 \cdot x))(\alpha_0 \cdot \eta_0)\alpha_0 - \frac{1}{\mu}(\sin(k_s^0 \alpha_0 \cdot x))\alpha_0 \times (\alpha_0 \times \eta_0) + \Im \hat{U}_1(x, \alpha_0, \omega_0)\eta_0
\]
must vanish identically for all \( x \in \Omega \), i.e.,
\[
(4.3) \quad \Im \hat{U}_1(x, \alpha_0, \omega_0)\eta_0 = -\frac{1}{\lambda + 2\mu}(\sin(k_p^0 \alpha_0 \cdot x))(\alpha_0 \cdot \eta_0)\alpha_0 + \frac{1}{\mu}(\sin(k_s^0 \alpha_0 \cdot x))\alpha_0 \times (\alpha_0 \times \eta_0)
\]
defined for \( x \in \Omega \) has just a unique real analytic extension to \( (\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12}))^c) \), that is,
\[
(4.4) \quad -\frac{1}{\lambda + 2\mu}(\sin(k_p^0 \alpha_0 \cdot x))(\alpha_0 \cdot \eta_0)\alpha_0 + \frac{1}{\mu}(\sin(k_s^0 \alpha_0 \cdot x))\alpha_0 \times (\alpha_0 \times \eta_0)
\]
for \( x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12}))^c) \).

Thus, we have that for all \( x \in (\Omega \cup D_{12} \cup ((\partial \Omega) \cap (\partial D_{12}))^c) \),
\[
(4.5) \quad \Im \hat{U}_1(x, \alpha_0, \omega_0)\eta_0 = -\frac{1}{\lambda + 2\mu}(\sin(k_p^0 \alpha_0 \cdot x))(\alpha_0 \cdot \eta_0)\alpha_0 + \frac{1}{\mu}(\sin(k_s^0 \alpha_0 \cdot x))\alpha_0 \times (\alpha_0 \times \eta_0).
\]

Since \( \hat{U}_1(x, \alpha_0, \omega_0)\eta_0 \) is the scattering solution of the Navier equation in \( \mathbb{R}^3 \setminus \bar{D}_1 \) satisfying the Kupradze radiation condition, by \[(2.2) \] of Lemma 2.5 we get \( \lim_{|x| \to \infty} |\hat{U}_1(x, \alpha_0, \omega_0)\eta_0| = 0 \) uniformly for all directions. On the other hand, from \[(4.5) \] and the orthogonality of the vectors \( \alpha_0 \cdot \eta_0 \alpha_0 \) and \( \alpha_0 \times (\alpha_0 \times \eta_0) \), we see that
\[
|\hat{U}_1(x, \alpha_0, \omega_0)\eta_0| = \left[ |\Re \hat{U}_1(x, \alpha_0, \omega_0)\eta_0|^2 + |\Im \hat{U}_1(x, \alpha_0, \omega_0)\eta_0|^2 \right]^{1/2}
\]
\[
= \left[ |\Re \hat{U}_1(x, \alpha_0, \omega_0)\eta_0|^2 + \frac{1}{\lambda + 2\mu}(\sin(k_p^0 \alpha_0 \cdot x))(\alpha_0 \cdot \eta_0)\alpha_0 \right.
\]
\[
- \frac{1}{\mu}(\sin(k_s^0 \alpha_0 \cdot x))\alpha_0 \times (\alpha_0 \times \eta_0) \bigg]^2 \left[ \right]^{1/2}
\]
\[
\geq \left[ \frac{1}{\lambda + 2\mu}(\sin(k_p^0 \alpha_0 \cdot x))(\alpha_0 \cdot \eta_0)\alpha_0 \right]^2 + \left[ \frac{1}{\mu}(\sin(k_s^0 \alpha_0 \cdot x))\alpha_0 \times (\alpha_0 \times \eta_0) \right]^2 \bigg]^{1/2}
\]
for all \( x \in ((\Omega \cup D_{12}) \cup ((\partial \Omega) \cap (\partial D_{12}))^c) \), and so \( |\hat{U}_1(x, \alpha_0, \omega_0)\eta_0| \) can't tend to zero as \( |x| \to \infty \) uniformly for all directions \( \frac{b}{|x|} \). Here \( |b| \) denotes the Euclidean norm of a vector \( b \) in \( \mathbb{R}^3 \). This is a contradiction, which implies that any domain \( \Omega \) mentioned above can never appear. Therefore we must have \( D_1 = D_2 \).
Finally, denoting $D = D_1 = D_2$, $U = U_1 = U_2$, we assume that we have different boundary condition $B^{D_1} \neq B^{D_2}$. For the sake of generality, consider the case where we have the Robin boundary conditions with two different continuous elastic impedance functions $h_1 \neq h_2$. Then, from $T_\nu U + h_j U = 0$ on $\partial D$ for $j = 1, 2$ we observe that $(h_1(x) - h_2(x))U(x) = 0$ for $x \in \partial D$. Therefore for the open set $\Gamma := \{x \in \partial D \mid h_1(x) \neq h_2(x)\}$ we have that $U = 0$ on $\Gamma$. Consequently, we further obtain $T_\nu U = 0$ on $\Gamma$ by the given boundary condition. Hence, it follows from Holmgren’s uniqueness theorem for the Navier equation (see Lemma 2.8) that $U = 0$ in $\mathbb{R}^3 \setminus D$, which implies that the scattered field $\tilde{U} = -U^i$ in $\mathbb{R}^3 \setminus \bar{D}$ and $\tilde{U}$ satisfies the Kupradze radiation condition. But the incident field $U^i$ doesn’t satisfy the Kupradze radiation condition. This is a contradiction. Hence $h_1 = h_2$. The case where one of the boundary conditions is the Dirichlet (or Neumann) boundary conditions can be treated analogously. □

Remark 4.1. Let us remark that in the case 2 of the proof of theorem 1.1, if we choose $\Omega \subset \mathbb{R}^3 \setminus \bar{D}_2$, then we should correspondingly discuss $U_2(x, \alpha_0, \omega_0)\eta_0$.

Remark 4.2. Our result as well as method is still valid for more general domain. For example, we may assume that the bounded set $D$ is the open complement of an unbounded domain of class $C^2$, that is, we include scattering from more than one obstacle in our analysis noting that the $C^2$ smoothness implies that $D$ has only a finite number of components.

Remark 4.3. i) In elastic theory, for the Navier equation there are six kinds of boundary conditions (see [27]). Our result is still true for other boundary value problems.

ii) We can also prove the same result by only considering pure shear part of the elastic far field pattern according to our new technique and a method from [22].

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