Oblique projections on metric spaces

Matteo Polettini

Complex Systems and Statistical Mechanics, University of Luxembourg,
162a avenue de la Faïencerie, L-1511 Luxembourg (G. D. Luxembourg)

Abstract

It is known that complementary oblique projections \( \hat{P}_0 + \hat{P}_1 = \hat{I} \)
on a Hilbert space \( \mathcal{H} \) have the same standard operator norm \( \|\hat{P}_0\| = \|\hat{P}_1\| \) and the same singular values, but for the multiplicity of 0 and 1.
We generalize these results to Hilbert spaces endowed with a positive-definite metric \( G \) on top of the scalar product. Our main result is that
the volume elements (pseudodeterminants \( \det_+ \)) of the metrics \( L_0, L_1 \)induced by \( G \) on the complementary oblique subspaces \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \),and of those \( I_0, I_1 \) induced on their algebraic duals, obey the relations

\[
\frac{\det_+ L_1}{\det_+ I_0} = \frac{\det_+ L_0}{\det_+ I_1} = \det_+ G.
\]

Furthermore, we break this result down to eigenvalues, proving a “supersymmetry” of the two operators \( \sqrt{L_0 I_0} \) and \( \sqrt{L_1 I_1} \). We connect the former result to a well-known duality property of the weighted-spanning-tree polynomials in graph theory and the latter to the mesh analysis of electrical resistor circuits driven by either voltage or current generators. We conclude with some speculations about an overarching notion of duality that encompasses mathematics and physics.

\[ \text{matteo.polettini@uni.lu} \]
1 Introduction

In our previous work Ref. [1] we used oblique projections to give an algebraic interpretation of the decomposition of the edge space of a graph in the basis of so-called “cycles” and “cocycles”. The results permitted to make statements regarding the spectrum of certain matrices that naturally appear in simple physical theories, e.g. in the analysis of electrical resistor networks when all resistances are identical.

The main objective of this work is to extend those results to arbitrary projection operators, and to graphs that carry positive weights along their edges (e.g. varied resistances). Since such weights can be interpreted as a metric, ultimately the generalization that we present here is to arbitrary oblique projections on metric Hilbert spaces. Some applications to physics and connections to graph theory will be analyzed.

Somewhat to the detriment of simplicity, we opted for abstract definitions in terms of operators and forms rather than matrices, which enable us to switch to the most convenient basis throughout the proofs. However, faithful to Gower’s suggestion “examples first” [2], we will first introduce our main results by simple examples that are self-explicative.
2 Examples first!

Main results

Consider the complementary oblique projections and the positive symmetric matrix:

\[
\hat{P}_0 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}, \quad \hat{P}_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 1 & -1 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad G = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Clearly \(\hat{P}_0^2 = \hat{P}_0\), \(\hat{P}_0 + \hat{P}_1 = I\), and \(\hat{P}_0 \neq \hat{P}_0^\dagger\). Let the induced metrics be given by

\[
L_0 := \hat{P}_0^\dagger G \hat{P}_0 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 4 & 0 & 3 \\
0 & 0 & 0 & 0 \\
0 & 3 & 0 & 3
\end{pmatrix},
\]

\[
\Gamma_0 := \hat{P}_0 G^{-1} \hat{P}_0^\dagger = \begin{pmatrix}
\frac{5}{3} & 0 & \frac{2}{3} & \frac{2}{3} \\
0 & 0 & 0 & 0 \\
\frac{2}{3} & 0 & \frac{5}{3} & \frac{4}{3} \\
\frac{3}{3} & 0 & \frac{3}{3} & \frac{3}{3}
\end{pmatrix},
\]

\[
L_1 := \hat{P}_1^\dagger G \hat{P}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 4 & -1 & 1 \\
0 & -1 & 2 & -2 \\
0 & 1 & -2 & 2
\end{pmatrix},
\]

\[
\Gamma_1 := \hat{P}_1 G^{-1} \hat{P}_1^\dagger = \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} \\
-\frac{2}{3} & -\frac{2}{3} & -1 & -\frac{2}{3} \\
\frac{1}{3} & -\frac{1}{3} & 3 & 1 \\
\frac{2}{3} & -\frac{2}{3} & 1 & \frac{2}{3}
\end{pmatrix}.
\]

Letting \(\det_+\) denote the product of the nonvanishing eigenvalues of a matrix, we obtain

\[
\frac{\det_+ L_1}{\det_+ \Gamma_0} = \frac{\det_+ L_0}{\det_+ \Gamma_1} = \det_+ G = 3.
\]

Furthermore, let us define the following matrices (we won’t give the explicit expressions):

\[
K_0 := \Gamma_0^{1/2} L_0 \Gamma_0^{1/2}, \quad \Sigma_0 := L_0^{1/2} \Gamma_0 L_0^{1/2},
\]

\[
K_1 := \Gamma_1^{1/2} L_1 \Gamma_1^{1/2}, \quad \Sigma_1 := L_1^{1/2} \Gamma_1 L_1^{1/2}.
\]

It can be checked that their spectra coincide, with all positive eigenvalues being larger than 1:

\[
\sigma(K_0) = \sigma(K_1) = \sigma(\Sigma_0) = \sigma(\Sigma_1) = (\sim 5.36, \sim 1.31, 0, 0).
\]
Connection to graph-theoretic polynomials

Consider the following oriented graph, with positive weights $g_1, \ldots, g_5$ on the edges:

There exists a standard procedure (see Appendix of Ref. [1]) to introduce a basis for the linear space of oriented cycles and that of oriented cocycles (a cocycle is a minimal set of edges whose removal disconnects the graph into two components) starting from an arbitrary spanning tree. One such basis of three cocycles and two cycles is

with vector representatives

$$
c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad c_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.
$$

Vectors $c_1$, $c_2$ and $c_3$ span the cocycle space, and vectors $c_4$ and $c_5$ span the cycle space. Letting $e_i$ be the $i$-th Cartesian vector, representing the $i$-th single edge, in Ref. [1] it is shown that $P_0 = \sum_{i=1}^{5} e_i \otimes e_i$ and $P_1 = \sum_{i=1,2,3} e_i \otimes e_i$ are complementary projections.

Letting $G = \text{diag}(g_i)_{i=1}^5$, in the basis $\{e_i\}_{i=1}^5$ the only nonvanishing blocks of the induced metrics read

$$
L_0 = [c_4, c_5]^T G [c_4, c_5] = \begin{pmatrix} g_2 + g_3 + g_4 & -g_2 \\ -g_2 & g_1 + g_2 + g_5 \end{pmatrix},
$$

$$
\Gamma_0 = [e_4, e_5]^T G^{-1} [e_4, e_5] = \begin{pmatrix} g_4^{-1} & 0 \\ 0 & g_5^{-1} \end{pmatrix},
$$

$$
L_1 = [e_1, e_2, e_3]^T G [e_1, e_2, e_3] = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix},
$$

$$
\Gamma_1 = [c_1, c_2, c_3]^T G^{-1} [c_1, c_2, c_3] = \begin{pmatrix} g_1^{-1} + g_5^{-1} & -g_5^{-1} \\ -g_5^{-1} & g_2^{-1} + g_4^{-1} + g_5^{-1} \\ 0 & g_4^{-1} + g_5^{-1} \end{pmatrix}.
$$
Matrices $L_0$ and $\Gamma_1$ are sometimes called Kirchhoff-Symanzik matrices \cite{3}. We have that
\[
\text{det } L_0 = g_3 g_5 + g_1 g_3 + g_5 g_3 + g_1 g_4 + g_2 g_3 + g_5 g_2 + g_1 g_2 + g_4 g_2
\]
\[
= \sum_{\mathcal{T}} \prod_{e \in \mathcal{T}} g_e
\]
\[
\text{det } \Gamma_1 = \frac{1}{g_1 g_2 g_3} + \frac{1}{g_2 g_4 g_5} + \frac{1}{g_1 g_3 g_4} + \frac{1}{g_2 g_3 g_4} + \frac{1}{g_1 g_4 g_5} + \frac{1}{g_1 g_3 g_5} + \frac{1}{g_1 g_4 g_5}
\]
\[
= \sum_{\mathcal{T}} \prod_{e \in \mathcal{T}} \frac{1}{g_e}
\]
In the second line of both expressions we gave a representation of the determinant expansion in terms of spanning trees (weights are intended to be multiplied over solid edges of the diagram), which we compactly resume in the third line in terms of the spanning-tree polynomials found e.g. in Ref. \cite{3, Th. 3.10}, where $\mathcal{T}$ ranges over spanning trees. Then the identity $\text{det } L_0/\text{det } \Gamma_1 = \text{det } G$ corresponds to a well-known duality of spanning tree polynomials, see e.g. Ref. \cite{4, Eq. (4.11)}.

Furthermore we can specialize this duality to eigenvalues as follows. Let
\[
K_0 = \Gamma_0^{1/2} L_0^{1/2} \Gamma_0^{-1/2} = \begin{pmatrix}
1 + \frac{g_3 + g_5}{g_3} & -\frac{g_3}{\sqrt{g_3 g_5}} \\
-\frac{g_3}{\sqrt{g_3 g_5}} & 1 + \frac{g_3 + g_5}{g_5}
\end{pmatrix}
\]
\[
\Sigma_1 = \Gamma_1^{1/2} L_1^{1/2} \Gamma_1^{-1/2} = \begin{pmatrix}
1 + \frac{g_3}{g_5} & -\frac{\sqrt{g_1 g_5}}{g_5} & 0 \\
-\frac{\sqrt{g_1 g_5}}{g_5} & 1 + \frac{g_2}{g_5} & \frac{g_2}{\sqrt{g_2 g_5}} \\
0 & \frac{g_2}{\sqrt{g_2 g_5}} & 1 + \frac{g_2}{g_4}
\end{pmatrix}
\]
Obviously $\text{det } K_0 = \text{det } \Sigma_1$. It can be checked by direct substitution that the vector $(1/\sqrt{g_1}, 1/\sqrt{g_2}, -1/\sqrt{g_3})^T$ is an eigenvector of $\Sigma_1$ relative to eigenvalue $\lambda = 1$. The characteristic polynomials of the two matrices are given by
\[
\zeta_0(\lambda) = \lambda^2 - (X + 2) \lambda + Y
\]
\[
\zeta_1(\lambda) = \lambda^3 - (X + 3) \lambda^2 + (Y + X + 2) \lambda - Y
\]
where
\[
Y = \frac{g_1 g_2 + g_1 g_3 + g_1 g_4 + g_4 g_2 + g_4 g_5 + g_5 g_2 + g_5 g_3 + g_2 g_3}{g_4 g_5}
\]
\[
X = \frac{g_2 g_5 + g_3 g_5 + g_1 g_4 + g_4 g_2}{g_4 g_5}
\]
Notice that $\zeta_1(\lambda) = (\lambda - 1)\zeta_0(\lambda)$, therefore the two matrices have the same spectrum but for eigenvalue 1. It can be checked that all other eigenvalues are larger than 1.
3 Notation and setup

We consider a finite \( n \)-dimensional Hilbert space \( \mathcal{H} \) with nondegenerate scalar product \( H : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \). Let \( \mathcal{H}^* \) be its algebraic dual. The action of a 1-form \( v^* \) on a vector \( w \) is denoted \( v^*[w] \), and, vice versa, vectors act linearly on 1-forms via \( w[v^*] = v^*[w] \), by virtue of the canonical isomorphism between \( \mathcal{H} \) and the dual’s dual \( \mathcal{H}^{**} \).

A bilinear form \( A : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) induces a map \( A : \mathcal{H} \to \mathcal{H}^* \) (denoted by the same symbol) from the Hilbert space to its algebraic dual via \( Av[\cdot] := A(\cdot, v) \). In particular, the scalar product induces a canonical isomorphism between vectors and linear forms, and a scalar product \( H^{-1} : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{C} \) in the dual space, defined by \( H(v, w) =: H^{-1}(Hv, Hw) \).

Operators, i.e. linear maps from the Hilbert space to itself, are adorned by a hat \( \hat{O} : \mathcal{H} \to \mathcal{H} \). Their adjoints \( \hat{O}^\dagger : \mathcal{H} \to \mathcal{H} \) are defined by \( H(v, \hat{O}w) =: H(\hat{O}^\dagger v, w) \). Given a bilinear form \( A \), one can obtain an operator \( \hat{A} := H^{-1} A : \mathcal{H} \to \mathcal{H} \). The pseudodeterminant \( \det_+ A \) of a bilinear form is defined as the product of the nonvanishing eigenvalues of \( \hat{A} \). Furthermore, if \( A \) is self-adjoint positive-semidefinite, one can define the unique map \( \sqrt{A} : \mathcal{H} \to \mathcal{H}^* \) such that \( A(v, w) = H^{-1}(\sqrt{A}v, \sqrt{A}w) \).

Operators \( \hat{O} \) and bilinear forms \( A \) are denoted by italic uppercase symbols, (the nonvanishing blocks of) their matrix representatives in a preferred roman characters \( \hat{O}, A \).

We consider a decomposition \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) into nontrivial complementary subspaces, with \( n_0 = \dim \mathcal{H}_0 \neq 0 \) and \( n_1 = n - n_0 \). Let \( \hat{P}_1 : \mathcal{H} \to \mathcal{H} \) be a projection with range \( \mathcal{H}_1 \) and kernel \( \mathcal{H}_0 \), neither null nor the identity, and \( \hat{P}_0 = \hat{I} - \hat{P}_1 \) its complement with range \( \mathcal{H}_0 \) and kernel \( \mathcal{H}_1 \). In general, \( \hat{P}_0 \) and \( \hat{P}_1 \) are oblique, that is, self-adjoint. Correspondingly, the dual space is decomposed into \( \mathcal{H}^* = \mathcal{H}_0^* \oplus \mathcal{H}_1^* \), where \( \mathcal{H}_0^* \) is the space of all linear forms that vanish on \( \mathcal{H}_0 \) and \( \mathcal{H}_0^* \) that of linear forms that vanish on \( \mathcal{H}_0 \). Obviously \( \dim \mathcal{H}_0^* = n_0 = \dim \mathcal{H}_0 \). We then introduce oblique complementary projections \( \hat{P}_0^* \) and \( \hat{P}_1^* = \hat{I}^* - \hat{P}_1^* \) on \( \mathcal{H}^* \), \( \hat{I}^* \) being identity in \( \mathcal{H}^* \). Obviously \( \hat{P}_0^* = H\hat{P}_0 H^{-1} \).

On top of the scalar product, we introduce another nondegenerate positive-
definite self-adjoint form $G$ that we call the metric, which also induces an inverse metric $G^{-1}$ in the dual space, defined by $G(v, w) =: G^{-1}(Gv, Gw)$.

**Definition 1.** The metrics $L_0, L_1 : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ induced by $G$ respectively on $\mathcal{H}_0$ and on $\mathcal{H}_1$, and those $\Gamma_0, \Gamma_1 : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{C}$ induced by $G^{-1}$ respectively on $\mathcal{H}^*_0$ and on $\mathcal{H}^*_1$, are defined by

$$
L_0(v, w) := G(\hat{P}_0 v, \hat{P}_0 w), \quad \forall v, w \in \mathcal{H},
$$

and by

$$
\Gamma_0(v^*, w^*) := G^{-1}(\hat{P}_0^* v^*, \hat{P}_0^* w^*), \quad \forall v^*, w^* \in \mathcal{H}^*,
$$

$$
\Gamma_1(v^*, w^*) := G^{-1}(\hat{P}_1^* v^*, \hat{P}_1^* w^*), \quad \forall v^*, w^* \in \mathcal{H}^*.
$$

**Definition 2.** The forms $K_0, K_1, \Sigma_0, \Sigma_1 : \mathcal{H}^* \times \mathcal{H}^* \to \mathbb{C}$ are defined as

$$
K_0(v^*, w^*) = H^{-1}(\sqrt{L_0^\dagger} v^*, \sqrt{L_0^\dagger} w^*),
$$

$$
K_1(v^*, w^*) = H^{-1}(\sqrt{L_1^\dagger} v^*, \sqrt{L_1^\dagger} w^*),
$$

$$
\Sigma_1(v^*, w^*) = H^{-1}(\sqrt{\Sigma_0^\dagger} v^*, \sqrt{\Sigma_0^\dagger} w^*), \quad \forall v^*, w^* \in \mathcal{H}^*,
$$

$$
\Sigma_0(v^*, w^*) = H^{-1}(\sqrt{\Sigma_1^\dagger} v^*, \sqrt{\Sigma_1^\dagger} w^*),
$$

(\text{where it is understood that } \sqrt{AB} = \sqrt{A}\sqrt{B}).

The above setup greatly simplifies in an orthonormal basis, in which case $\hat{H} = \hat{I} = \text{diag}(1, 1, \ldots, 1)$, $\dagger$ is matrix transposition and complex conjugation, $\hat{P}_0^* = \hat{P}_0^\dagger$, $\hat{P}_1^* = \hat{P}_1^\dagger$. For the induced metrics we have $L_0 = \hat{P}_0^\dagger G \hat{P}_0$, $L_1 = \hat{P}_1^\dagger G \hat{P}_1$, $\Gamma_0 = \hat{P}_0^\dagger G^{-1} \hat{P}_0$, $\Gamma_1 = \hat{P}_1^\dagger G^{-1} \hat{P}_1$, and furthermore $K_0 = \Gamma_0^{1/2} L_0 \Gamma_0^{1/2}$, $K_1 = \Gamma_1^{1/2} L_1 \Gamma_1^{1/2}$, $\Sigma_0 = L_0^{1/2} \Gamma_0 \Gamma_0^{1/2}$ and $\Sigma_1 = L_1^{1/2} \Gamma_1 \Gamma_1^{1/2}$ where the square roots are uniquely defined since Hermitian matrices have unique Hermitian square roots. However, it will be convenient to work in a different basis that gives a handy block-structure of matrices, where the scalar product is not the identity matrix; that’s the reason we maintain this level of formality.
4 Results

Theorem 1 establishes a connection between the volume elements of the metrics induced by $G$ and $G^{-1}$ on the oblique subspaces. Theorem 2 establishes a relation between the forms $K_0$ and $\Sigma_1$, and between $K_1$ and $\Sigma_0$ which, as a consequence, implies that they have the same spectra up to the multiplicity of 0 and 1 (and therefore the same singular values for operators $\sqrt{T_0L_0}$ and $\sqrt{L_1T_1}$, but for 0 and 1).

4.1 Determinants

Theorem 1. The pseudodeterminants of the induced metrics are related by

$$\frac{\det_+ L_1}{\det_+ \Gamma_0} = \frac{\det_+ L_0}{\det_+ \Gamma_1} = \det_+ G. \tag{4}$$

Proof. Let $(v_i, w_j)$, be a basis for $\mathcal{H}$ such that the first $i = 1, \ldots, n_0$ vectors are a basis for $\mathcal{H}_0$ and the last $j = n_0 + 1, \ldots, n$ are a basis for $\mathcal{H}_1$. We denote by $u_i$ a generic vector in this basis. We choose as dual basis the one $(v^*_i, w^*_j)$ defined by $u^*_i[u_j] = \delta_{ij}$, with $u = v, w$. In general, this basis cannot be orthonormal. However, it is possible to choose a basis (that we call natural) such that the first $n_0$ vectors are orthonormal among themselves and the last $n_1$ vectors are orthonormal among themselves, i.e. $H(v_i, v_j) = H(w_i, w_j) = \delta_{ij}$, while in general $\Omega_{ij} := H(v_i, w_j) \neq 0$.

Let us consider $\hat{G} = H^{-1}G$ and $\hat{G}^{-1} = G^{-1}H = G^*H$, which in the natural basis read

$$\hat{G} = \begin{pmatrix} I & \Omega^\dagger \\ \Omega & I \end{pmatrix}^{-1} \begin{pmatrix} L_0 & V^\dagger \\ V & L_1 \end{pmatrix}, \quad \hat{G}^{-1} = \begin{pmatrix} \Gamma_0 & \Lambda^\dagger \\ \Lambda & \Gamma_1 \end{pmatrix} \begin{pmatrix} I & \Omega^\dagger \\ \Omega & I \end{pmatrix} \tag{5}$$

where $L_0, L_1, \Gamma_0, \Gamma_1$ are the nonvanishing self-adjoint square blocks of the matrix representatives of $L_0, L_1, \Gamma_0, \Gamma_1$, and $V_{ij} := G(v_i, w_j), \Lambda_{ij} := G^*(v^*_i, w^*_j)$. From properties of inverses and determinants of partitioned block matrices
we obtain
\[
\begin{align*}
\Gamma_0^{-1} &= L_0 - V^\dagger L_1^{-1} V \\
\Gamma_1^{-1} &= L_1 - V L_0^{-1} V^\dagger \\
L_0^{-1} &= \Gamma_0 - \Lambda \Gamma_1^{-1} \Lambda \\
L_1^{-1} &= \Gamma_1 - \Lambda \Gamma_0^{-1} \Lambda^\dagger
\end{align*}
\] (6)

and the following expressions for the determinants
\[
\begin{align*}
\det \hat{G} &= \frac{\det L_0}{\det (I - \Omega^{\dagger} \Omega) \det \Gamma_1} \\
\det \hat{G}^{-1} &= \frac{\det (I - \Omega^{\dagger} \Omega) \det \Gamma_0}{\det L_1}.
\end{align*}
\] (7)

Finally we want to relate these matrix determinants to operator determinants. Clearly \(\det_{+} G = \det \hat{G}\). As regards \(\det_{+} L_0\), we need to consider the eigenvalues of the operator \(\hat{L}_0 = G^{-1} L_0\) which has matrix form
\[\hat{L}_0 = \begin{pmatrix} I & \Omega^\dagger \\ \Omega & I \end{pmatrix}^{-1} \begin{pmatrix} L_0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (I - \Omega^{\dagger} \Omega)^{-1} L_0 & 0 \\ -(I - \Omega^{\dagger} \Omega)^{-1} \Omega L_0 & 0 \end{pmatrix}.\] (8)

The eigenvalue equation \(\hat{L}_0 v = \lambda v\) in the natural basis reads
\[
\begin{pmatrix} (I - \Omega^{\dagger} \Omega)^{-1} L_0 v_0 \\ -(I - \Omega^{\dagger} \Omega)^{-1} \Omega L_0 v_0 \end{pmatrix} = \lambda \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.
\] (9)

The null eigenvalue corresponds to eigenvectors with \(v_0 = 0\). For nonvanishing eigenvalues, the first equation yields the reduced eigenvalue equation \((I - \Omega^{\dagger} \Omega)^{-1} L_0 v_0 = \lambda v_0\) and the second returns the second part of the eigenvector as \(v_1 = -(I - \Omega^{\dagger} \Omega)^{-1} \Omega (I - \Omega^{\dagger} \Omega) v_0\). Consequently, we obtain \(\det_{+} L_0 = \det L_0 / \det (I - \Omega^{\dagger} \Omega)\). By a similar reasoning, since
\[\hat{\Gamma}_0 = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma_1 \end{pmatrix} \begin{pmatrix} I & \Omega^{\dagger} \\ \Omega & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \Gamma_1 \Omega & \Gamma_1 \end{pmatrix},\] (10)

we obtain \(\det_{+} \Gamma_1 = \det \Gamma_1\), which completes the proof.

Since the determinant of a metric \(G\) is its volume element (measured in units of the volume element of \(H\)), then the above theorem establishes a fundamental symmetry between volume elements induced by the identification of nonorthogonal subspaces of a metric Hilbert space.
4.2 Eigenvalues

When the metric and the scalar product coincide, \( G = H \), it is well known [6] that for complementary oblique projections

\[
\|\hat{P}_0\| = \|\hat{P}_1\|, \tag{11}
\]

with the standard operator norm induced by the vector norm, and coinciding with the modulus of the largest singular value. This norm identity is actually a corollary of an even stronger result, since it can be proven that \( \hat{P}_0 \) and of \( \hat{P}_1 \) have the same singular values (eigenvalues of \( \hat{P}_0^\dagger \hat{P}_0 \) and \( \hat{P}_1 \hat{P}_1^\dagger \), up to the multiplicity (possibly vanishing) of 0 and 1 [7]. It is the objective of this section to prove an analogous result in our generalized context.

**Theorem 2.** The forms \( K_0, K_1, \Sigma_0 \) and \( \Sigma_1 \) have the same spectrum, up to the multiplicity of eigenvalues 0 and 1. All nonvanishing eigenvalues are not smaller than 1.

**Proof.** In the first part of the proof we will show, as intuitive, that it all boils down to considering the eigenvalues of the matrices \( L_0^{1/2} \Gamma_0 L_0^{1/2} \), \( L_1^{1/2} \Gamma_1 L_1^{1/2} \), \( \Gamma_0^{1/2} L_0 \Gamma_0^{1/2} \), and \( \Gamma_1^{1/2} L_1 \Gamma_1^{1/2} \), in the second part we derive the result.

**First part.** Let us first find the matrix representatives of \( \sqrt{L_0} \), \( \sqrt{T_0} \), \( \sqrt{L_1} \) and \( \sqrt{T_1} \) in the natural basis. We have by definition

\[
\begin{pmatrix}
L_0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
\sqrt{L_0} & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
I & \Omega^\dagger \\
\Omega & I
\end{pmatrix}^{-1} \begin{pmatrix}
\sqrt{L_0} & 0 \\
0 & 0
\end{pmatrix}, \tag{12}
\]

\[
\begin{pmatrix}
0 & 0 \\
0 & L_1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & \sqrt{L_1}
\end{pmatrix} \begin{pmatrix}
I & \Omega^\dagger \\
\Omega & I
\end{pmatrix}^{-1} \begin{pmatrix}
0 & 0 \\
0 & \sqrt{L_1}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\Gamma_0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
\sqrt{\Gamma_0} & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
I & \Omega^\dagger \\
\Omega & I
\end{pmatrix} \begin{pmatrix}
\sqrt{\Gamma_0} & 0 \\
0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 \\
0 & \Gamma_1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & \sqrt{\Gamma_1}
\end{pmatrix} \begin{pmatrix}
I & \Omega^\dagger \\
\Omega & I
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & \sqrt{\Gamma_1}
\end{pmatrix},
\]

yielding

\[
\sqrt{L_0} = (I - \Omega^\dagger \Omega)^{1/2} L_0^{1/2},
\]

\[
\sqrt{L_1} = (I - \Omega \Omega^\dagger)^{1/2} L_1^{1/2},
\]

\[
\sqrt{\Gamma_0} = \Gamma_0^{1/2},
\]

\[
\sqrt{\Gamma_1} = \Gamma_1^{1/2}. \tag{13}
\]
Imposing
\[
\begin{pmatrix} K_0 & 0 \\ 0 & 0 \end{pmatrix} = \left( \begin{pmatrix} \sqrt{L_0^*} & 0 \\ 0 & \sqrt{L_0^*} \end{pmatrix} \begin{pmatrix} \Omega & \Omega^* \\ \Omega^* & \Omega \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} \sqrt{L_0} & 0 \\ 0 & \sqrt{L_0} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)
\]
(14)
\[
\begin{pmatrix} 0 & 0 \\ 0 & K_1 \end{pmatrix} = \left( \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{L_1^*} \end{pmatrix} \begin{pmatrix} \Omega & \Omega^* \\ \Omega^* & \Omega \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{L_1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),
\]
we find the intuitive expressions
\[
K_0 = \Gamma_0^{1/2}L_0\Gamma_0^{1/2}
\]
(15)
\[
K_1 = \Gamma_1^{1/2}L_1\Gamma_1^{1/2}.
\]

Since \(\hat{K}_0\) and \(\hat{K}_1\) are defined by equations analogous to Eq. (10), then their spectra coincide with those of \(K_0\), \(K_1\). We can proceed in an analogous way for \(\hat{\Sigma}_0\) and \(\hat{\Sigma}_1\), finding
\[
\hat{\Sigma}_0 = (1 + \Omega^\dagger\Omega)^{1/2}\Sigma_0(1 + \Omega^\dagger\Omega)^{-1/2}
\]
\[
\hat{\Sigma}_1 = (1 + \Omega^\dagger\Omega)^{1/2}\Sigma_1(1 + \Omega^\dagger\Omega)^{-1/2},
\]
(16)
where
\[
\Sigma_0 := L_0^{1/2}\Gamma_0L_0^{1/2}
\]
\[
\Sigma_1 := L_1^{1/2}\Gamma_1L_1^{1/2}.
\]
(17)
Clearly \(\Sigma_0\) and \(\Sigma_0\) have the same spectrum up to the multiplicity of eigenvalue 0, and so do \(\Sigma_1\) and \(\Sigma_1\).

**Second part.** Now consider the Eqs. (6), that we rewrite as
\[
\Sigma_0^{-1} = I - B_0^*K_0^{-1}B_0
\]
\[
K_0^{-1} = I - D_0^\dagger\Sigma_0^{-1}D_0
\]
(18)
\[
\Sigma_1^{-1} = I - D_1^*K_1^{-1}D_1
\]
\[
K_1^{-1} = I - B_1^\dagger\Sigma_0^{-1}B_1
\]
where
\[
B_0 := \Gamma_0^{1/2}V\Gamma_0^{-1/2}
\]
\[
B_1 := \Gamma_1^{1/2}L\Gamma_1^{-1/2}
\]
\[
D_0 := \Gamma_0^{1/2}V\Gamma_1^{-1/2}
\]
\[
D_1 := \Gamma_1^{1/2}L\Gamma_0^{-1/2}
\]
(19)
Notice that by taking the product $\hat{G}\hat{G}^{-1} = \hat{I}$ from Eqs. (18), we obtain with a few simple manipulations

\begin{align*}
B_0 &= -B_1^\dagger =: B \\
D_0 &= -D_1^\dagger =: D
\end{align*}

and furthermore

\begin{equation}
D^\dagger D = \sqrt{\Gamma_1 L_1}^{-1} BB^\dagger \sqrt{\Gamma_1 L_1}.
\end{equation}

We then obtain from the second and third of Eqs. (18)

\begin{align*}
K_0^{-1} &= I - D^\dagger \Sigma_1^{-1} D \\
\Sigma_1^{-1} &= I - D K_0^{-1} D^\dagger.
\end{align*}

and similarly for $K_1^{-1}$ and $\Sigma_0^{-1}$ in terms of $B$. Replacing the first equation into the second, we obtain the recursive relations

\begin{align*}
K_0^{-1} &= I - DD^\dagger + DD^\dagger K_0^{-1} DD^\dagger, \\
\Sigma_1^{-1} &= I - D^\dagger D + D^\dagger D \Sigma_1^{-1} D^\dagger D.
\end{align*}

The latter equations are solved by

\begin{align*}
K_0 &= I + DD^\dagger \\
\Sigma_1 &= I + D^\dagger D
\end{align*}

as can be found by direct replacement (notice that $K_0$ and $\Sigma_1$ are uniquely defined by construction). Letting $v$ be an eigenvector of $K_1$ relative to eigenvalue $\lambda$, we find

\begin{equation}
D^\dagger D v = (\lambda - 1)v.
\end{equation}

Since the left-hand side matrix is positive-semidefinite, clearly all eigenvalues of $K_0$ and $\Sigma_1$ must be not smaller than 1. Acting with $D$ on the latter expression we obtain that $Dv$ is an eigenvector of $K_0$ with respect to the same eigenvalue. Let $n_0 \geq n_1$, without loss of generality, and let $r \leq n_1$ be the rank of $D$. We conclude that $K_0$ [resp. $\Sigma_1$] has $r$ positive eigenvalues strictly larger than 1, eigenvalue 1 with multiplicity $n_0 - r$ [resp. $n_1 - r$] and eigenvalue 0 with multiplicity $n_1$ [resp. $n_0$], and that the eigenvalues larger than 1 are the same. Same applies to $K_1$ and $\Sigma_0$ given the similarity Eq. (21).
4.3 Relation to graph polynomials

Here we report on a relationship between the above results and known properties of the spanning-tree polynomial in graph theory. We build upon the results of Ref. [1], to which we refer for further details.

By the spectral theorem we can pick an orthonormal basis \( E = \{ e_i \}_{i=1}^n \) that makes the metric diagonal \( G = \text{diag} \{ g_i \}_{i=1}^n \). We call such basis the edge space. The diagonal entries \( g_i \) can be seen as positive weights associated to each edge. We introduce the vertex set \( X \) of vertices \( x \) and a map \( \delta : E \to X \times X \) associating an ordered pair of vertices to each edge, one (denoted \( e_i \to x \)) being the target and the other (denoted \( e_i \leftarrow x \)) being the origin of the edge. The choice of which \( x \) is the origin and which is the target fixes a completely arbitrary orientation of the graph, on which the results below do not depend. We can make this map into a linear operator \( \delta : H \to H_X \) mapping arbitrary linear combinations of edges into the linear space \( H_X \) generated by vertices. The operator acts on basis vectors according to

\[
\delta_{x,i} = \begin{cases} 
+1, & \text{if } e_i \to x \\
-1, & \text{if } e_i \leftarrow x \\
0, & \text{otherwise}
\end{cases} 
\] (26)

The quadruple \( G = (E, X, \delta, G) \) forms an oriented weighted graph. We assume that the graph is connected, in the sense that for any two complementary subsets of edges \( E_1, E_2 = E \setminus E_1 \), the corresponding sets of boundary vertices intersect (so that there is a path between any two boundary vertices) and their union is \( X \) (so that there are no isolated vertex). Under this hypothesis one has \( |X| \leq |E| + 1 \); the number \( |C| = |E| - |X| + 1 \geq 0 \) is called the cyclomatic number. We assume (as customary) that the graph does not include loops, that is, edges whose boundary vertices coincide.

We stipulate that \( H_0 \) is the kernel of \( \delta \), called the cycle space, with dimension \( n_0 = |C| \), and that \( H_1^* \) is the image of \( \delta \), called the cocycle space, with dimension \( n_1 = |X| - 1 \). A basis for the cycle and for the cocycle spaces is found by the following procedure. We pick an arbitrary spanning tree, i.e. a set of \( |X| - 1 \) edges that connects all vertices. Let the chords \( \{ e_{\alpha} \}_{\alpha=1}^{n_1} \) be the (vector representatives) of the edges that belong to the tree, and the cochords \( \{ e_{\mu} \}_{\mu=1}^{n_1} \) be the (co-vector representatives) of the edges that do not belong to the tree. Adding a chord \( e_{\alpha} \) to the spanning tree identifies a unique basis cycle vector \( c_{\alpha} \). Removing a cochord from the spanning tree identifies...
a unique cocycle co-vector $c_\mu$. One has
\[ c_\mu[c_\alpha] = e_\mu[e_\alpha] = 0, \quad c_\mu[e_{\alpha'}] = \delta_{\mu,\mu'}, \quad e_\alpha[e_{\alpha'}] = \delta_{\alpha,\alpha'}. \]
(27)
The crucial result exposed in Ref. [1] is that chords span $H^0$ and cochords span $H^1$, and more precisely that, letting $\otimes$ be the outer product of a vector and a linear form, the two operators
\[
\hat{P}_0 = \sum_{\alpha=1}^{|C|} c_\alpha \otimes e_\alpha, \quad \hat{P}_1 = \sum_{\mu=|C|+1}^{|E|} e_\mu \otimes c_\mu,
\]
(28)
are complementary projections, typically oblique (except very special cases).

The induced metrics then read
\[
L_0 = \sum_{\alpha,\alpha'=1}^{|C|} (L_0)_{\alpha,\alpha'} c_\alpha \otimes e_{\alpha'}, \quad L_1 = \sum_{\mu,\mu'=1}^{|V|-1} (L_0)_{\mu,\mu'} e_\mu \otimes c_{\mu'},
\]
\[
\Gamma_0 = \sum_{\alpha,\alpha'=1}^{|C|} (\Gamma_0)_{\alpha,\alpha'} e_\alpha \otimes c_{\alpha'}, \quad \Gamma_1 = \sum_{\mu,\mu'=1}^{|V|-1} (\Gamma_1)_{\mu,\mu'} e_\mu \otimes e_{\mu'},
\]
(29)
where $(L_0)_{\alpha,\alpha'} = G[c_\alpha, c_{\alpha'}]$, $(L_0)_{\mu,\mu'} = G[e_\mu, e_{\mu'}]$, $(\Gamma_0)_{\alpha,\alpha'} = G^{-1}[e_\alpha, e_{\alpha'}]$, and $(\Gamma_1)_{\mu,\mu'} = G_{\mu,\mu'}^{-1}[e_\mu, c_{\mu'}]$. That $\det L_1 / \det \Gamma_0 = \det G = \prod_i g_i$ is obvious from the fact that they are diagonal matrices covering all the edges. Instead, the determinants of the cycle and cocycle overlap matrices $L_0$ and $\Gamma_1$ are well-known to give the spanning tree/cotree polynomials described in the opening example, see Th. 3.10 in Ref. [3] (see also [8]), and $\det L_0 / \det \Gamma_1 = \det G = \prod_i g_i$ corresponds to Eq. (4.11) in the review paper Ref. [4], that relates a the Tutte polynomial of a planar graph and its dual $G^*$. Now let $e_i^*[e_j] = \delta_{i,j}$ and let us introduce the matrix
\[
D_{\mu,\alpha} = \sqrt{\frac{g_\mu}{g_\alpha}} c_\mu[e_\alpha^*] = -\sqrt{\frac{g_\mu}{g_\alpha}} e_\mu^*[c_\alpha].
\]
(30)
The second identity follows from [1, Theorem 3]. Finally, we have a particularly simple representation for $K_0$ and $\Sigma_1$:
\[
(K_0)_{\alpha,\alpha'} = \frac{(L_0)_{\alpha,\alpha'}}{\sqrt{g_\alpha g_{\alpha'}}} = \delta_{\alpha,\alpha'} + \sum_{\mu=1}^{n_1} D_{\mu,\alpha} D_{\mu,\alpha'}
\]
\[
(\Sigma_1)_{\mu,\mu'} = \sqrt{g_\mu g_{\mu'}} (\Gamma_1)_{\mu,\mu'} = \delta_{\mu,\mu'} + \sum_{\alpha=n_1+1}^n D_{\mu,\alpha} D_{\mu,\alpha'}.
\]
(31)
These latter relations follow from the fact that the only chord belonging to cycle $c_\alpha$ is $e_\alpha$, which is accounted for by the Kroenecker delta, while cochords are enough to span the rest of a cycle and the intersections between two cycles $c_\alpha$ and $c_{\alpha'}$. Similarly for cocycles. We have thus reproduced by direct computation the result Eq. (24).

5 Application: electrical resistor networks

In this section we describe a physical context where the two matrices $K_0$ and $\Sigma_1$ appear in a natural way. We consider here an electrical resistor network with resistors $R_i$ (the connection between the cohomology of graphs and electrical circuits dates back to Weyl, who also introduced a method to solve circuits in terms of orthogonal projections [9]). Steady currents $I_i$ can be driven either by electromotive forces that energize charges, or by current generators that input/output charge from outside. The flow of a current along a resistor causes a drop in potential energy $V_i$. The relation between currents and potential drops is prescribed by Ohm’s law

$$V_i = R_i I_i, \quad (32)$$

while the dissipated power is given by Joule’s law

$$P = \sum_i I_i V_i = \sum_i R_i I_i^2. \quad (33)$$

Clearly this quadratic form can be seen as a metric on a weighted graph. The problem to be solved is the determination of the network’s currents and potential drops given a sufficient number of electromotive forces or input currents. The analysis is typically conducted via Kirchhoff’s mesh analysis.

Before moving on to mesh analysis, though, in order to make things adimensional and eigenvalues comparable, we choose to redefine quantities in such a way that both scaled currents and scaled voltages have the same units. In view of Eq. (32), we define

$$J_i := \frac{V_i}{\sqrt{R_i}} = \sqrt{R_i} I_i \quad (34)$$

and $J = (J_i)_i$. The dissipated power then reads

$$P = \sum_i J_i^2. \quad (35)$$
Current-driven circuit

We consider the following circuit driven by current generators:

We conventionally stipulate that currents/potentials are positive when they flow/drop left-to-right or top-to-bottom. Then $I_1$, $I_2$ and $I_3$ are the input currents. Notice that current generators are disposed along a cotree (the complement of a tree), to avoid short-circuits that would burn the resistances. Kirchhoff’s Current Law states that conservation of charge at the three points denoted by bullets in the diagram yields $I_1 + I_5 + I_3 = 0$, $I_1 - I_2 - I_4 = 0$, $I_2 + I_6 + I_3 = 0$, so we can solve for $I_4, I_5, I_6$. Plugging into the generated power we obtain

$$P_{\ominus} = I_1 (V_1 + V_4 - V_5) + I_2 (V_2 - V_6 - V_4) + I_3 (V_3 - V_5 - V_6). \quad (36)$$

We can now use Ohm’s relation, use Kirchhoff’s current law again and, as motivated above, move to the $J_i$. Letting $J_{\ominus} = (J_4, J_5, J_6)$ we obtain

$$P_{\ominus} = J_{\ominus} \cdot K_0 J_{\ominus} \quad (37)$$

where

$$K_0 = \begin{pmatrix}
1 + \frac{R_4}{R_1} + \frac{R_5}{R_1} & -\frac{R_1}{\sqrt{R_1 R_2}} & \frac{R_6}{\sqrt{R_2 R_3}} \\
-\frac{R_1}{\sqrt{R_1 R_2}} & 1 + \frac{R_5}{R_2} + \frac{R_6}{R_2} & \frac{R_6}{\sqrt{R_2 R_3}} \\
\frac{R_5}{\sqrt{R_1 R_2}} & \frac{R_6}{\sqrt{R_2 R_3}} & 1 + \frac{R_5}{R_3} + \frac{R_6}{R_3}
\end{pmatrix}. \quad (38)$$
**Voltage-driven circuit**

We consider the exact same resistor network, but this time it is driven by electromotive forces $V_4$, $V_5$ and $V_6$. They are placed in parallel to the resistances belonging to the spanning tree complementary to the cotree chosen above, again to avoid a short circuit:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{R_1}{R_2}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{R_4}{R_5}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{V_4}{V_5}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{V_6}{R_3}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{R_6}{R_2}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Kirchhoff’s Loop Law states that by energy conservation $V_1 + V_5 - V_4 = 0$, $V_4 + V_6 - V_2 = 0$, $V_5 + V_6 - V_3 = 0$ and we can solve for $V_1, V_2, V_3$. Plugging into the generated power and performing the same kind of manipulations as above, and letting $J_⊖ = (J_1, J_2, J_3)$, we obtain

\[
P_⊖ = V_4 (I_1 + I_2 + I_4) + V_5 (-I_1 + I_3 + I_5) + V_6 (I_2 + I_3 + I_6)
= J_⊖ \cdot \Sigma_1 J_⊖
\] (39)

where

\[
\Sigma_1 = \left( \begin{array}{ccc}
1 + \frac{R_3}{R_1} + \frac{R_1}{R_2} & -\sqrt{\frac{R_4 R_5}{R_1}} & \sqrt{\frac{R_4 R_6}{R_5}} \\
-\sqrt{\frac{R_4 R_5}{R_2}} & 1 + \frac{R_5}{R_3} + \frac{R_3}{R_4} & \sqrt{\frac{R_5 R_6}{R_4}} \\
\sqrt{\frac{R_4 R_6}{R_3}} & -\sqrt{\frac{R_4 R_6}{R_3}} & 1 + \frac{R_6}{R_2} + \frac{R_2}{R_3}
\end{array} \right).
\] (40)

That the characteristic polynomials of $K_0$ and $\sigma_1$ coincide can be checked by hand calculation. The above network is self-dual in the sense that the two matrices interchange upon the transformation $R_1 \leftrightarrow 1/R_6$, $R_2 \leftrightarrow 1/R_5$ and $R_3 \leftrightarrow 1/R_4$, which corresponds to planar-graph duality with inversion of the weights.
6 Conclusion and digression

Apart from electrical networks and their variants [10], our results may be applied to Quantum Mechanics in all those cases where it is necessary or convenient to consider projections onto physically meaningful states that are not orthogonal, such as molecular orbitals [11, 12] or coherent states, or to discern between the measurements of two non-commuting observers [13]. The Hamiltonian may play the role of the metric, and therefore the theory might make statements on dual spectra of physical systems. These matrices also appear in the study of Feynman path integrals [3].

The concept of “duality” pervades mathematics and physics [14]. Several different concepts of duality are lightly touched by our theory: Algebraic duality $\mathcal{H} \to \mathcal{H}^*$; Projection duality $\hat{P}_0 \to \hat{P}_1$; Planar graph duality $\mathcal{G} \to \mathcal{G}^*$, and more generally Hodge duality; Strong/weak coupling duality $g \to 1/g$; Electromagnetic duality; Supersymmetry. The connection to electromagnetic duality was analyzed by the Author in Ref. [15]. Supersymmetry emerges as follows. Defining

$$\mathcal{H} = \begin{pmatrix} K_1 - I & 0 \\ 0 & \Sigma_1 - I \end{pmatrix}$$
$$Q = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$$

we can interpret $\mathcal{H}$ as a Hamiltonian and $Q$ as a supercharge, satisfying the usual SUSY algebra $0 = [\mathcal{H}, Q] = [\mathcal{H}, Q^\dagger] = \{Q, Q\} = \{Q^\dagger, Q^\dagger\}$, and $\{Q, Q^\dagger\} = \mathcal{H}$ [16, Ch. 3]. One can further define the Hermitian Dirac operators $Q_+ = Q + Q^\dagger$ and $Q_- = i(Q - Q^\dagger)$ such that $\mathcal{H} = Q_+^2 = Q_-^2$. Witten has highlighted the connection between supersymmetry and cohomology [17]. In our case, the ground states of the Hamiltonian correspond to the orthogonal subspaces. Since the machinery of spanning trees, cycles and cocycles can be generalized to higher-dimensional cellular complexes, in principle all of the above results can be extended to arbitrary (discretized) manifolds.

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