LONGEST AND SHORTEST CYCLES IN RANDOM PLANAR GRAPHS

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Abstract. Let $P(n,m)$ be a graph chosen uniformly at random from the class of all planar graphs on vertex set $[1,\ldots,n]$ with $m=m(n)$ edges. We study the cycle and block structure of $P(n,m)$ when $m \sim n/2$. More precisely, we determine the asymptotic order of the length of the longest and shortest cycle in $P(n,m)$ in the critical range when $m = n/2 + o(n)$. In addition, we describe the block structure of $P(n,m)$ in the weakly supercritical regime when $n^{2/3} \ll m \sim n/2 \ll n$.

1. Introduction and results

1.1. Motivation. In their seminal papers [12,13], Erdős and Rényi introduced the uniform random graph $G(n,m)$, also known as the Erdős-Rényi random graph, which is a graph chosen uniformly at random from the class $\mathcal{G}(n,m)$ of all vertex-labelled graphs on vertex set $[n] := \{1,\ldots,n\}$ with $m = m(n)$ edges, denoted by $G(n,m) \in \mathcal{G}(n,m)$. Since then, $G(n,m)$ and its variants, in particular their component structure, were extensively studied (see e.g. [5,6,13,14,19,27]). For example, Erdős and Rényi [13] showed that there is a drastic change of the component structure of $G(n,m)$ when $m \sim n/2$. More precisely, letting $m = dn/2$ for a positive constant $d$ they showed that the following hold in $G(n,m)$ with high probability (meaning with probability tending to 1 as $n$ tends to infinity, whp for short): if $d < 1$, then every component has at most a logarithmic number of vertices; in contrast, if $d > 1$, there is a unique component containing linearly many vertices. These results raised the question whether also the cycle structure of $G(n,m)$ undergoes such a significant change when $d \sim 1$. Ajtai, Komlós, and Szemerédi [11] proved that whp there is a cycle of linear length when $d > 1$, and Bollobás [6, Corollary 5.8] showed that whp every cycle is bounded when $d < 1$. (Throughout the paper, we use the standard Landau notation as well as notations in Definition 2.7 for asymptotic orders.)

Theorem 1.1 ([1,5]). Let $m = dn/2$ for a constant $d > 0$. Then the following hold in $G(n,m) \in \mathcal{G}(n,m)$.

(a) If $d < 1$, then all cycles are of length $O_p(1)$.

(b) If $d > 1$, then whp there is a cycle of length $\Theta(n)$.

Kolchin [25] and later Łuczak [27] took a closer look at the critical range when $m = n/2 + o(n)$ and provided a relation between longest and shortest cycles and the component structure of $G(n,m)$. Their results are strengthened by Łuczak [28] and later by Łuczak, Pittel, and Wierman [29]. Given a graph $H$, we denote by $c(H)$ the length of the longest cycle of $H$ (also known as the circumference) and by $g(H)$ the length of the shortest cycle of $H$ (also known as the girth).

Theorem 1.2 ([26,29]). Let $m = n/2 + s$ for $s = s(n) = o(n)$. Let $G = G(n,m) \in \mathcal{G}(n,m)$, $L_1 = L_1(G)$ be the largest component of $G$, and $R = R(G) = G \setminus L_1$.

(a) If $s^3 n^{-2} \to -\infty$, then whp $L_1$ is a tree. Furthermore, we have $c(R) = \Theta_p\left(n|s|^{-1}\right)$.

(b) If $s = O\left(n^{2/3}\right)$, then the probability that $L_1$ is a tree is bounded away from both 0 and 1. Provided there is a cycle in $L_1$, we have $c(L_1) = \Theta_p\left(n^{1/3}\right)$ and $g(L_1) = \Theta_p\left(n^{1/3}\right)$. Moreover, we have $c(R) = \Theta_p\left(n^{1/3}\right)$.

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Keywords and phrases. Random graphs, planar graphs, cycles, blocks, Pólya urn.

Supported by Austrian Science Fund (FWF): I3747 and W1230.
(c) If $s^3n^{-2} \to \infty$, then whp the longest cycle in $G$ is contained in $L_1$ and $c(L_1) = \Theta(s^2n^{-1})$. In addition, we have $g(L_1) = \Theta_p(ns^{-1})$ and $c(R) = \Theta_p(ns^{-1})$.

Perhaps the most interesting case in Theorem 1.2 is when $s^3n^{-2} \to \infty$ (in the so-called weakly supercritical regime): whp the longest cycle is contained in the largest component $L_1$. Moreover, the length of the shortest cycle in $L_1$ is of the same asymptotic order as the length of the longest cycle outside $L_1$. In other words, there exists a ‘threshold’ function $f(n) := ns^{-1}$ in the sense that whp for all cycles $K$ in $G$ we have

$$
\begin{cases}
    K \text{ is contained in } L_1 & \text{if } |K| = \omega(f(n)); \\
    K \text{ is not contained in } L_1 & \text{if } |K| = o(f(n)),
\end{cases}
$$

(1)

where $|K|$ denotes the length of $K$.

An important structure related to cycles are blocks, because every cycle is contained in a block: a block of a graph $H$ is a maximal 2-connected subgraph of $H$. We emphasise that we do not consider a bridge, i.e. an edge whose deletion increases the number of components, to form a block. Łuczak [28] investigated the block structure of $G(n, m)$ when $m = n/2 + s$ for $s^3n^{-2} \to \infty$, showing that whp there is a unique largest block, while all other blocks are ‘small’. In addition, the largest component contains only a few number of blocks. Given a graph $H$, let $b_i(H)$ denote the number of vertices in the $i$-th largest block $B_i(H)$ of $H$ for each $i \in \mathbb{N}$.

**Theorem 1.3 (28).** Let $m = n/2 + s$ for $s = o(n)$ and $s^3n^{-2} \to \infty$. Let $G = G(n, m) \in \mathcal{G}(n, m)$ and $L_1 = L_1(G)$ be the largest component of $G$. Then the following hold.

(a) $b_1(G) = \Theta(s^2n^{-1})$ whp.

(b) $b_i(G) = O_p(ns^{-1})$ for each $i \in \mathbb{N}$ with $i \geq 2$.

(c) The number of blocks in $L_1$ is $O_p(1)$.

We note that Theorems 1.2(6) and 1.3 imply that whp the longest cycle lies in the largest block $B_1(G)$ and its length grows asymptotically like $b_1(G)$.

In the last few decades various models of random graphs have been introduced by imposing additional constraints to $G(n, m)$, e.g. topological constraints or degree restrictions. Particularly interesting models are random planar graphs and, more generally, random graphs on surfaces which have attained considerable attention [8–10, 15, 21, 23, 32], since the pioneering work of McDiarmid, Steger, and Welsh [33] on random planar graphs and that of McDiarmid [31] on random graphs on surfaces. Many exciting results have been obtained, revealing richer and more complex behaviour. In particular, results are often found to feature thresholds, meaning that the probabilities of various properties change dramatically according to which ‘region’ the edge density falls into.

A natural question is whether random planar graphs satisfy similar properties as in Theorems 1.2 and 1.3. Kang and Łuczak [21] showed that the component structure of a random planar graph $P(n, m)$ changes drastically when $m \sim n/2$, analogously to $G(n, m)$. In contrast, not much is known about the cycle and block structure of $P(n, m)$. In this paper we investigate this open problem, determining the length of the shortest and longest cycle in $P(n, m)$ and the order of blocks in $P(n, m)$, in the light of Theorems 1.2 and 1.3.

### 1.2. Main results

Throughout this section, we let $\mathcal{P}(n, m)$ denote the class of all vertex-labelled planar graphs on vertex set $[n]$ with $m = m(n)$ edges and $P(n, m)$ be a graph chosen uniformly at random from $\mathcal{P}(n, m)$, denoted by $P(n, m) \in R(\mathcal{P}(n, m))$.

Our first main result concerns the distribution of cycles in the random planar graph $P(n, m)$.

**Theorem 1.4.** Let $P = P(n, m) \in R(\mathcal{P}(n, m))$, $L_1 = L_1(P)$ be the largest component of $P$, and $R = R(P) = P \setminus L_1$. Assume $m = n/2 + s$ for $s = o(n)$. Then the following hold.

(a) If $s^3n^{-2} \to -\infty$, then whp $L_1$ is a tree. Furthermore, we have $c(R) = \Theta_p(ns^{-1})$.

(b) If $s = O(n^{2/3})$, then the probability that $L_1$ is a tree is bounded away from both 0 and 1. Provided there is a cycle in $L_1$, we have $c(L_1) = \Theta_p(n^{1/3})$ and $g(L_1) = \Theta_p(n^{1/3})$. Moreover, we have $c(R) = \Theta_p(n^{1/3})$. 


(c) If $s^3n^{-2} \to \infty$, then whp the longest cycle in $P$ is contained in $L_1$ and $c(L_1) = O\left(sn^{-1/3}\right)$. In addition, we have $c(L_1) = \Omega_p\left(n^{1/3}\log\left(sn^{-2/3}\right)\right)$, $g(L_1) = \Theta_p\left(ns^{-1}\right)$, and $c(R) = \Theta_p\left(n^{1/3}\right)$.

As we will see in Theorem 4.5 Theorem 1.3 holds for a more general universal class of graphs, the so-called kernel-stable classes of graphs (see Definition 4.1 for a formal definition), which include the class of series-parallel graphs, the class of planar graphs, and the class of graphs on a surface, to mention a few.

Our second main result deals with the block structure of $P(n, m)$. Due to Theorem 1.4(a) and (b) we focus on the weakly supercritical regime and will show that whp $P(n, m)$ contains a unique largest block $B_1$ which is significantly larger than all other blocks, similarly as in $G(n, m)$. However, the largest component in $P(n, m)$ contains ‘many’ blocks, while the largest component in $G(n, m)$ contains only a bounded number of blocks (cf. Theorem 1.5).

**Theorem 1.5.** Let $P = P(n, m) \in \mathcal{P}(n, m)$ and $L_1 = L_1(P)$ be the largest component of $P$. Assume $m = n/2 + s$ for $s = s(n) = o(n)$ and $s^3n^{-2} \to \infty$. Then the following hold.

(a) $b_1(P) = \Theta_p\left(sn^{-1/3}\right)$.

(b) $b_i(P) = \Theta_p\left(s^2n^{-1/3}\right)$, for each $i \in \mathbb{N}$ with $i \geq 2$.

(c) The number of blocks in $L_1$ is whp $\Theta\left(sn^{-2/3}\right)$.

It is well known that when $s^3n^{-2} \to \infty$ or $s = O\left(n^{2/3}\right)$, the probability that $G(n, m)$ is planar is bounded away from 0 (see e.g. Theorem 4.4(a) and 18 [29, 36]). Hence, each graph property that holds whp in $G(n, m)$ is also true whp in $P(n, m)$. In particular, this implies that the cycle structure of $P(n, m)$ ‘behaves’ similarly like that of $G(n, m)$ (see Theorems 1.4(a), (b) and 1.4(a), (b)). However, when $s^3n^{-2} \to \infty$, whp $G(n, m)$ is not planar (see e.g. [29, 36]) and therefore, $G(n, m)$ and $P(n, m)$ can exhibit different asymptotic behaviours. Theorems 1.4(c) and 1.5 indicate that in view of the cycle and block structure this is indeed the case. For example, a ‘threshold’ function in the sense of (1) does not exist in $P(n, m)$, because $g(L_1) = \Theta_p\left(ns^{-1}\right) \ll c(R) = \Theta_p\left(n^{1/3}\right)$. However, whp the longest cycle in $P(n, m)$ is still contained in the largest component.

Kang and Łuczak [21] proved that in the case of $s^3n^{-2} \to \infty$ the core, i.e. the maximal subgraph of minimum degree at least two (also known as 2-core), is much smaller in $P(n, m)$ compared to $G(n, m)$. More precisely, whp the core of $P(n, m)$ is of order $\Theta\left(sn^{-1/3}\right)$, while that of $G(n, m)$ is of order $\Theta\left(s^2n^{-1}\right)$. This has a natural impact on the longest cycle and largest block in $P(n, m)$ (cf. Theorems 1.4(c) and 1.4(b)) and Theorems 1.4(a) and 1.4(a):

\[ c(P(n, m)) = \Theta\left(sn^{-1/3}\right) \text{ whp } \ll c(G(n, m)) = \Theta\left(s^2n^{-1}\right) \text{ whp} \]

\[ b_1(P(n, m)) = \Theta_p\left(sn^{-1/3}\right) \ll b_1(G(n, m)) = \Theta\left(s^2n^{-1}\right) \text{ whp}. \]

Furthermore, it is known that the ‘edge density’ in the part without the largest component is typically much larger in $P(n, m)$ than in $G(n, m)$ (see e.g. [23 Theorem 1.7]). This affects the order of the longest cycle outside the largest component (cf. Theorems 1.4(c) and 1.4(b)):

\[ c(R(P(n, m))) = \Theta_p\left(n^{1/3}\right) \gg c(R(G(n, m))) = \Theta_p\left(ns^{-1}\right). \]

1.3. Related work. The so-called $n$-vertex model of a random planar graph is a graph chosen uniformly at random from the class of all vertex-labelled planar graphs on vertex set $[n]$, denoted by $P(n) \in \mathcal{P}(n)$. Giménez and Noy [15] showed that whp $P(n)$ has $(1 + o(1)) kn$ edges for a constant $k \approx 2.1$, i.e. $P(n)$ ‘behaves’ like $P(n, m)$ where $m \approx kn$. Many exciting results on the block structure of $P(n)$ were obtained in recent literature, revealing a different behaviour from that of $P(n, m)$ as observed in Theorem 1.5. For example, Panagiotou and Steger [38] proved that whp $b_1(P(n)) = \Theta(n)$. Later, Giménez, Noy, and Rüé [15] established that in fact an Airy-type central limit theorem holds for $b_1(P(n))$. Stufler [41] Theorem 6.20, Corollary 6.42 determined the limiting distribution of $b_i(P(n))$ for any fixed integer $i \geq 2$, showing, among others, that $b_1(P(n)) = O_p\left(n^{2/3}\right)$.

Furthermore, Stufler [40] Remark 9.13 provided a detailed structural description of the graph $P(n) \setminus B_1(P(n))$, i.e. $P(n)$ without its largest block.

Another model related to $P(n, m)$ is the random connected planar graph $C(n, m)$, which is a graph chosen uniformly at random from the class of all connected planar graphs on vertex set $[n]$ having
Panagiotou [37, Theorem 1, Corollary 1] proved that if \( m = \lfloor c n \rfloor \) for a constant \( c \in (1, 3) \), then \( C(n, m) \) has a block of linear order. Moreover, there is a discussion in [16, End of Section 5] sketching how much stronger results on the order of the largest block in \( C(n, m) \) can in principle be obtained.

### 1.4. Key techniques

One of the main proof techniques is the so-called core-kernel approach. We decompose a graph into the simple part (in which each component contains at most one cycle) and the complex part (in which each component contains at least two cycles). Then we decompose the complex part into its core and then into its kernel, a key structure obtained from the core by replacing each path whose internal vertices all have degree exactly two by an edge. Conversely, each graph can be uniquely constructed from the kernel by first subdividing the edges of the kernel, thereby obtaining the core, then replacing vertices of the core with rooted trees and adding the simple part (see Section 2.3).

In order to investigate the cycle and block structure of a random planar graph \( P = P(n, m) \), we begin with the analysis of the structure of its core \( C(P) \), which is itself a random graph. Instead of directly analysing the random core \( C(P) \), we introduce an auxiliary random core model \( \tilde{C} \), in which we split the ‘randomness’ into smaller parts. More precisely, we choose randomly a kernel and then randomly a subdivision number which is a total number of vertices that will be used for a subdivision of the kernel. Given these two random bits (i.e. a random kernel and a random subdivision number) we then randomly construct a core by randomly inserting vertices on the edges of the kernel. A crucial technique to analyse the random core \( \tilde{C} \) is the famous Polya urn model: We derive results on the maximum and minimum number of drawn balls of some colour in order to determine the length of the longest and shortest cycles in the core, respectively.

### 1.5. Outline of the paper

The rest of the paper is organised as follows. After providing the necessary notations, definitions, and concepts in Section 2, we present our proof strategy in Section 3. In Section 4 we define kernel-stable classes of graphs. In Section 5 we provide results on the Polya urn model, which we use in Section 6 to derive the cycle structure of a core randomly built from a fixed kernel and a fixed subdivision number. Section 7 is devoted to a random kernel and Section 8 to the block structure of a random planar graph. In Sections 9 and 10 we provide the proofs of our main and auxiliary results, respectively. Finally in Section 11 we discuss various questions that remain unanswered.

### 2. Preliminaries

#### 2.1. Notations and parameters for graphs

Unless stated otherwise, all considered (simple or multi) graphs are undirected.

**Definition 2.1.** Given a (simple or multi) graph \( H \) we denote by

- \( V(H) \) the vertex set of \( H \) and
- \( v(H) \) the order of \( H \), i.e. the number of vertices in \( H \);
- \( E(H) \) the edge set of \( H \) and
- \( e(H) \) the size of \( H \), i.e. the number of edges in \( H \);
- \( L_1(H) \) the largest component of \( H \);
- \( R(H) \) := \( H \setminus L_1(H) \) the graph outside the largest component;
- \( g(H) \) the girth of \( H \), i.e. the length of the shortest cycle in \( H \);
- \( c(H) \) the circumference of \( H \), i.e. the length of the longest cycle in \( H \);
- \( \lambda(H) \) the number of loops in \( H \);
- \( B_i(H) \) the \( i \)-th largest block of \( H \) and
- \( b_i(H) \) the number of vertices in \( B_i(H) \) for \( i \in \mathbb{N} \).

**Definition 2.2.** Let \( H \) be a (simple or multi) graph and let \( v, w \in V(H) \) be distinct. We denote by

- \( vw \) or \( \{v, w\} \) an edge between \( v \) and \( w \);
- \( vv \) a loop at \( v \);
- \( H + vw \) the graph obtained from \( H \) by adding an additional edge \( vw \);
Definition 2.7. **H − vw** the graph obtained from H by deleting the edge vw.

Definition 2.3. Given a class $\mathcal{A}$ of vertex-labelled graphs (under consideration of certain constraints, e.g. planarity or degree restrictions), we denote by $\mathcal{A}(n)$ the subclass of $\mathcal{A}$ consisting of graphs on vertex set $[n]$ and by $\mathcal{A}(n, m)$ the subclass of $\mathcal{A}$ consisting of graphs on vertex set $[n]$ with $m$ edges, respectively. We denote by $A(n) \in_R \mathcal{A}(n)$ a graph chosen uniformly at random from $\mathcal{A}(n)$ and by $A(n, m) \in_R \mathcal{A}(n, m)$ a graph chosen uniformly at random from $\mathcal{A}(n, m)$, respectively.

Definition 2.4. Let $H$ be a graph and $\mathcal{Q}$ a set of graphs. We call $\mathcal{Q}$ a graph property. And if $H \in \mathcal{Q}$, then we say that $H$ satisfies $\mathcal{Q}$; or that $\mathcal{Q}$ holds in $H$; or that $\mathcal{Q}$ is true in $H$.

Next, we introduce some notion for random graphs which have the ‘same’ asymptotic behaviour in the sense that they are indistinguishable in view of properties that hold whp.

Definition 2.5. For each $n \in \mathbb{N}$, let $G_n$ and $H_n$ be random graphs. We say that $G_n$ and $H_n$ are contiguous if for every graph property $\mathcal{Q}$

$$
\lim_{n \to \infty} \mathbb{P}[G_n \in \mathcal{Q}] = 1 \iff \lim_{n \to \infty} \mathbb{P}[H_n \in \mathcal{Q}] = 1.
$$

2.2. **Weighted Multigraphs.** Throughout the paper, we always assume implicitly that multigraphs are weighted by the so-called compensation factor, which was first introduced by Janson, Knuth, Łuczak, and Pittel [18].

Definition 2.6. Given a multigraph $H$ and $i \in \mathbb{N}$, we denote by $m_i(H)$ the number of unordered pairs $(u, w)$ of distinct vertices $v, w \in V(H)$ such that there are precisely $i$ edges between $v$ and $w$. Similarly, let $\lambda_i(H)$ be the number of vertices $v \in V(H)$ such that there are exactly $i$ loops at $v$ and let $\lambda(H) := \sum_{i \in \mathbb{N}} i \lambda_i(H)$ be the total number of loops in $H$. Then the compensation factor (or weight, for short) of $H$ is defined as

$$
w(H) := 2^{-\lambda(H)} \prod_{i \in \mathbb{N}} (i!)^{-\lambda_i(H) - m_i(H)}.
$$

(2)

For a finite class $\mathcal{A}$ of multigraphs we define

$$
|\mathcal{A}| := \sum_{H \in \mathcal{A}} w(H).
$$

2.3. **Complex part, core, and kernel.** We call a component of a graph $H$ complex if it has at least two cycles and define the complex part $Q(H)$ as the union of all complex components of $H$. We decompose the complex part $Q(H)$ further into the core $C(H)$, which is the maximal subgraph of $Q(H)$ of minimum degree at least two. Finally, we extract the kernel $K(H)$ from the core $C(H)$ by considering paths $(v_0, v_1, \ldots, v_l)$ such that $v_0$ and $v_l$ have degree at least three and all internal vertices $v_1, \ldots, v_{l-1}$ have degree two. We allow the case $v_0 = v_l$ in which $(v_0, v_1, \ldots, v_l)$ is a cycle. To obtain the kernel $K(H)$, we replace any such path in the core $C(H)$ by an edge $v_0v_1$. By doing that, loops and multiple edges can be created and therefore in general the kernel $K(H)$ is a multigraph. Finally, we reverse the above decomposition and note that we can construct the core $C(H)$ by subdividing the edges of the kernel $K(H)$ with additional vertices, thereby ensuring that no loops and multiple edges of the kernel survive in the core. The number of additional vertices that are used to subdivide the kernel $K(H)$ to obtain the core $C(H)$ is called the subdivision number, denoted by $S(H)$, i.e. $S(H) := v(C(H)) - v(K(H))$.

2.4. **Asymptotic notation.** We will study asymptotic properties of random graphs on vertex set $[n]$ as $n$ tends to $\infty$, and all asymptotics are taken with respect to $n$. In addition to the standard Landau notation, we will use the notations by Janson [17] to express asymptotic orders of random variables.

Definition 2.7. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables and $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Then, we write

- whp $X_n = O(f)$ if there exists a $c > 0$ such that whp $|X_n| \leq cf(n)$;
- whp $X_n = \Omega(f)$ if there exists a $c > 0$ such that whp $|X_n| \geq cf(n)$;
- whp $X_n = \Theta(f)$ if whp $X_n = O(f)$ and $X_n = \Omega(f)$;
- $X_n = O_p(f)$ if for every $\delta > 0$ there exist $c > 0$ and $N \in \mathbb{N}$ such that $\mathbb{P}[|X_n| \leq cf(n)] \geq 1 - \delta$ for all $n \geq N$;
• $X_n = \Omega_p(f)$ if for every $\delta > 0$ there exist $c > 0$ and $N \in \mathbb{N}$ such that $\mathbb{P} \left( |X_n| \geq cf(n) \right) \geq 1 - \delta$ for all $n \geq N$;
• $X_n = \Theta_p(f)$ if $X_n = O_p(f)$ and $X_n = \Omega_p(f)$.

Moreover, for a sequence $(A_n)_{n \in \mathbb{N}}$ of events we say that the probability that $A_n$ occurs is bounded away from $0$ and $1$ if $\liminf_{n \to \infty} \mathbb{P}[A_n] > 0$ and $\limsup_{n \to \infty} \mathbb{P}[A_n] < 1$, respectively.

We note that $X_n = O_p(f)$ if and only if $\mathbb{P} \left( |X_n| \geq h(n)f(n) \right) = o(1)$ for any function $h = o(1)$. Similarly, we have $X_n = \Omega_p(f)$ if and only if $\mathbb{P} \left( |X_n| \leq f(n)/h(n) \right) = o(1)$ for any function $h = o(1)$. Furthermore, the statement $X_n = O_p(f)$ is slightly weaker than having whp $X_n = O(f)$. The latter says that there exists a constant $c > 0$ such that for every $\delta > 0$ there is a $N \in \mathbb{N}$ satisfying $\mathbb{P} \left( |X_n| \leq cf(n) \right) \geq 1 - \delta$ for all $n \geq N$. In other words, we have a uniform constant $c$ if whp $X_n = O(f)$, while $c$ may depend on $\delta$ in the case of $X_n = O_p(f)$.

### 3. Proof strategy

In order to analyse the cycle and block structure of a random planar graph $P = P(n, m)$, we decompose $P$ into smaller parts: We first decompose $P$ into the complex part $Q(P)$, the core $C(P)$, and the kernel $K(P)$ as in Section 2.3. Moreover, we obtain the core $C(P)$ by subdividing the edges of the kernel $K(P)$ by $S(P)$ many additional vertices. We note that all blocks and cycles that do not lie in unicyclic components are contained in the core $C(P)$. Therefore, it is crucial to understand the cycle and block structure of the core $C(P)$. To this end, we introduce an auxiliary random core $\tilde{C}$ which is created stepwise as follows. We choose randomly a typical kernel $K$ and a typical subdivision number $k$. Then we construct $C$ from $K$ by randomly subdividing the edges of $K$ with $k$ additional vertices. In order to analyse the cycle and block structure of $C(P)$, we ask the following questions. (a) Which properties do a typical kernel $K$ and a typical subdivision number $k$ have, in particular with respect to cycles and blocks? (b) How do these properties translate to $C$ when choosing a random subdivision? (c) How can we relate the random graph $C$ to the original core $C(P)$ where we first choose the random graph $P$ and then extract (deterministically) the core $C(P)$. In the rest of this section we will give an overview how to deal with these questions. We will start by considering question (b) in Section 3.1. The main idea is to find a relationship between a random core $C$ and the Pólya urn model. Next, we will deal with question (a). We note that already lots of results about a typical kernel $K$ and a typical subdivision number $k$ are known, e.g. asymptotic order of $v(K), e(K)$, and $k$ (see Theorem 4.3). In contrast, there are no results on the cycle and block structure of $K$ known. We will deal with these open problems as follows. Firstly, we will show that a typical kernel ‘behaves’ asymptotically like a random cubic (i.e. 3-regular) planar multigraph (see Lemma 7.1). Then we will analyse the cycle (in Section 3.2) and block structure (in Section 3.3) of a random cubic planar multigraph by double counting arguments. Finally, we will introduce the concept of conditional random graphs in Section 3.4 to give an answer to question (c).

#### 3.1. Random core and Pólya urn model

Given a (typical) kernel $K$ and a (typical) subdivision number $k \in \mathbb{N}$ we let $C = C(K, k)$ be a random core chosen uniformly at random from the set of all cores with kernel $K$ and subdivision number $k$. In order to study the cycle and block structure of a random core $C$, we consider an auxiliary random multigraph $\tilde{C}$ that ‘behaves’ similarly like $C$ and is easier to study. We randomly place $k$ additional labelled vertices $\{v_1, \ldots, v_k\}$ one after another on the edges $e_1, \ldots, e_N$ of $K$ to obtain $\tilde{C}$. More precisely, in the $i$-th step of the random process we choose uniformly at random an edge of the current multigraph and place the vertex $v_i$ on that edge. We note that loops and multiple edges are allowed in $\tilde{C}$, although they do not appear in $C$. However, we will show that $C$ and $\tilde{C}$ behave asymptotically quite similarly (see Lemma 6.4 and Corollary 6.5). Thus, it suffices to consider the cycle and block structure of $\tilde{C}$ instead of $C$.

We note that each cycle of $C$ is a cycle in the kernel $K$ together with some additional vertices placed on the edges. Thus, we are interested in the distribution of $(X_1, \ldots, X_N)$, where $X_i$ is the number of vertices placed on edge $e_i$. We observe that we can model the random vector $(X_1, \ldots, X_N)$ with the following Pólya urn model (see e.g. [20, 30] for details of Pólya urns). We start with $N$ balls of $N$ distinct colours $F_1, \ldots, F_N$ in a urn, where colour $F_i$ represents edge $e_i$. Then we draw one ball uniformly at
F  I  G  U  R  E  1. Use of a Pólya urn to construct a random core by sequentially subdividing the edges of a kernel with \( N = 5 \) edges by \( k = 10 \) additional vertices. The left-hand side represents the situation at the beginning, the right-hand side after four drawings.

random from the urn, say \( F_j \), and subdivide the corresponding edge \( e_j \) by vertex \( v_1 \). By doing so the edge \( e_j \) is split into two new edges. Hence, we need in the next step two balls of the colour corresponding to \( e_j \). Therefore, we return the drawn ball to the urn along with an additional ball of the same colour. We repeat that procedure \( k \) times and observe that the number of drawn balls (after \( k \) steps) of colour \( F_i \) is distributed like \( X_i \) for each \( i \in \{N\} \) (see also Figure 1).

The Pólya urn model provides bounds on \( \min_{1 \leq i \leq f} X_i \) and \( \max_{1 \leq i \leq f} X_i \) for various \( f \) satisfying \( 1 \leq f \leq N \) (see Theorem 5.1). Assuming \( e_1, \ldots, e_\lambda \) are the loops of \( K \), we will derive bounds on the length of the shortest and longest cycle in \( \tilde{C} \) (see Lemma 6.7), denoted by \( g(\tilde{C}) \) and \( c(\tilde{C}) \), by applying the following inequalities:

\[
\begin{align*}
g(\tilde{C}) &\leq 1 + \min_{1 \leq i \leq \lambda} X_i; \\
g(\tilde{C}) &\geq \min_{1 \leq i \leq N} X_i; \\
c(\tilde{C}) &\geq \max_{1 \leq i \leq \lambda} X_i.
\end{align*}
\]

Next, we consider how the block structure of \( K \) translates to the block structure of \( \tilde{C} \). We observe that each block in \( \tilde{C} \) is a block or loop in \( K \) together with additional vertices placed on the edges. So in general we can use similar ideas as for the cycle structure above. However, we need to slightly modify our Pólya urn model (see Section 5.2). For the cycle structure, we use many ‘small’ cycles (in fact, loops) simultaneously, but for the block structure we will fix one ‘large’ block \( B \) of \( K \) and consider how many vertices are placed on the edges of \( B \). Therefore, we just need balls of two different colours, one representing edges in \( B \) and the other representing edges outside \( B \). Using standard results on this Pólya urn model, we will show that each ‘large’ block \( B \) of \( K \) translates to a ‘large’ block \( \tilde{B} \) of \( \tilde{C} \) and that \( \nu(\tilde{B}) \) is concentrated around its expectation (see Lemmas 8.10 and 8.11).

3.2. Loops in the kernel. When studying the shortest and longest cycles in the kernel, the number of loops in the kernel plays a crucial role. We will prove in Section 7 that a typical kernel \( K \) on \( N \) edges has \( \Theta(N) \) many loops. Firstly, we will show that the kernel \( K(P) \) ‘behaves’ asymptotically like a random cubic planar multigraph (see Lemma 7.1). Secondly, we estimate the typical number of loops in a random cubic planar multigraph by the second moment method (see Lemma 7.6). To this end, we introduce the so-called loop insertion (see Definition 7.2), which is a natural operation that changes an arbitrary cubic graph with \( 2n \) vertices and an additional loop. By using this loop insertion we can estimate the probability that there is a loop at some fixed vertex in a random cubic planar multigraph, from which we deduce the typical number of loops in a random cubic planar multigraph.

3.3. Blocks of the kernel. As in Section 5.2, we use the fact that the kernel \( K(P) \) behaves asymptotically like a random cubic planar multigraph. In order to analyse the block structure of a random connected cubic planar multigraph \( M \), we assign the bridge number \( \beta(e) \) to a bridge \( e \), defined as the order of the smaller component which we obtain from \( M \) by deleting \( e \) (see Definition 8.1). We
will show that the bridge number $\beta(e)$ is typically quite ‘small’: In fact, we will determine the distribution of a bridge number (see Lemma 8.4), using the so-called bridge insertion operation (see Definition 8.2). Then we combine it with a double counting argument to show that there is one block $B_1$ of linear order (see Lemma 8.10). For the second largest block $B_2$ in $M$ we consider the maximum $A$ of all bridge numbers. As there is a bridge $e$ such that $B_1$ and $B_2$ lie in different components of $M - e$, we will get $v(B_2) \leq A$. On the other hand, if $e$ is a bridge with $\beta(e) = A$, then the smaller component of $M - e$ is distributed similarly as a random connected cubic planar multigraph on $A$ vertices. Hence, this smaller component should contain a block of linear order (in $A$). Thus, the second largest block (and by induction also the $i$-th largest block for every $i \geq 2$) is of the same order as the maximum bridge number $A$.

3.4. Conditional random graphs. In Section 3.1 we considered a random core $C = C(K, k)$ obtained from a (candidate) kernel $K$ by randomly subdividing the edges of $K$ by $k$ additional vertices. In other words, given $K$ and $k$, we considered the ‘conditional’ random core $C$ conditioned on the event that its kernel is equal to $K$ and its subdivision number is equal to $k$. However, we are actually interested in the ‘unconditional’ random core $C(P)$ of a random planar graph $P = P(n, m) \in R \mathcal{P}(n, m)$ for some function $m = m(n)$. In this section we describe a method how to obtain results on an ‘unconditional’ random graph by studying the corresponding ‘conditional’ random graphs (see Lemma 3.2). To do so, we need the following definition.

**Definition 3.1.** Given a class $\mathcal{A}$ of graphs, a set $\mathcal{I}$, and a function $\Phi : \mathcal{A} \to \mathcal{I}$, we call a sequence $s = (s_n)_{n \in \mathbb{N}}$ feasible for $(\mathcal{A}, \Phi)$ if for each $n \in \mathbb{N}$ there exists a graph $H \in \mathcal{A}(n)$ such that $\Phi(H) = s_n$. Moreover, for each $n \in \mathbb{N}$ we denote by $(A | s)(n)$ a graph chosen uniformly at random from the set $\{H \in \mathcal{A}(n) : \Phi(H) = s_n\}$. We will often omit the dependence on $n$ and write just $A | s$ (i.e. ‘$A$ conditioned on $s$’) instead of $(A | s)(n)$.

**Lemma 3.2.** Let $\mathcal{A}$ be a class of graphs, $\mathcal{I}$ a set, $\Phi : \mathcal{A} \to \mathcal{I}$ a function, and $\mathcal{D}$ a graph property. Let $A = A(n) \in R \mathcal{A}(n)$. If for every sequence $s = (s_n)_{n \in \mathbb{N}}$ that is feasible for $(\mathcal{A}, \Phi)$ we have whp $A | s \in \mathcal{D}$, then we have whp $A \in \mathcal{D}$.

The proof of Lemma 3.2 is provided in Section 10. In the following we illustrate how one can use Lemma 3.2 to deduce that a graph property $\mathcal{D}$ holds whp in the core $C(P)$ of $P = P(n, m)$ by studying the random core $C(P)$ of $P = P(n, m)$ by studying the random core $C(P)$ of $P = P(n, m)$ by studying the random core $C(K, k)$, which is obtained by randomly subdividing the edges of a kernel $K$ with $k$ additional vertices. We start with a ‘strong’ property $\mathcal{T}$ that is whp satisfied by the kernel $K(P)$ and the subdivision number $S(P)$, e.g. $v(K(P))$ and $S(P)$ lie in certain intervals (see Theorem 4.8). Then we let $\mathcal{A}(n) \subseteq \mathcal{P}(n, m)$ be the subclass of all graphs in $\mathcal{P}(n, m)$ fulfilling $\mathcal{T}$ and $\mathcal{A}' = \bigcup_{n \in \mathbb{N}} \mathcal{A}(n)$. We define the function $\Phi : \mathcal{A} \to \mathcal{K} \times \mathbb{N}$ by $\Phi(H) := (K(H), S(H))$ and let $s = (k_n, n)_{n \in \mathbb{N}}$ be a sequence that is feasible for $(\mathcal{A}, \Phi)$. The core $C(A | s)$ of the conditional random graph $A | s$ is distributed like $C(K_n, k_n)$ (see Lemma 3.2 for details). Now the main step is to show that whp $C(K_n, k_n) \in \mathcal{D}$. This is usually much easier than proving whp $C(P) \in \mathcal{D}$ directly, as the kernel and subdivision number are not random anymore in $C(K_n, k_n)$ and furthermore, fulfill property $\mathcal{T}$. Knowing that whp $C(K_n, k_n) \in \mathcal{D}$, it follows by Lemma 3.2 that whp also the core $C(A)$ of the random graph $A = A(n) \in R \mathcal{A}(n)$ satisfies $\mathcal{D}$. Finally, this implies that whp $C(P) \in \mathcal{D}$ as desired, because whp $P \in \mathcal{A}$ by definition of $\mathcal{A}$. Applications of Lemma 3.2 can be found e.g. in the proofs of Theorems 4.5 and 8.12.

4. Kernel-stable classes of graphs

In this section we will show that Theorem 1.4 holds for a more general class of graphs, called a kernel-stable class, in which graphs satisfy certain properties that are extracted from the class of planar graphs and are essential for the aforementioned core-kernel approach. Before defining the kernel-stable class, we first recall well-known classes of graphs (see e.g. [2, 24, 33, 34] for details). A class $\mathcal{A}$ of graphs is called

- **weakly addable** (also known as bridge-addable) if it is closed under adding an edge between two components;
- **addable** if $\mathcal{A}$ is weakly addable and fulfills in addition the property that a graph $H$ is in $\mathcal{A}$ if and only if all components of $H$ are in $\mathcal{A}$;
Definition 4.1. A class \( \mathcal{P} \) of graphs is called kernel-stable if it satisfies the following properties.

(P1) [global]. The class \( \mathcal{P} \) is weakly addable and closed under taking minors.

(P2) [kernel]. Let \( \mathcal{K} \) be the class of all kernels of graphs in \( \mathcal{P} \) and \( \mathcal{K}_C \) be the subclass of \( \mathcal{K} \) containing all connected kernels. Then \( \mathcal{K} \) and \( \mathcal{K}_C \) satisfy the following conditions.

(K1) [stability]. A graph is in \( \mathcal{P} \) if and only if its kernel is in \( \mathcal{K} \).

(K2) [asymptotic behaviour]. Let \( \mathcal{K}(2n,3n) \) be the subclass of \( \mathcal{K} \) consisting of all kernels on vertex set \([2n]\) having \(3n\) edges. Then there exist constants \( \gamma > 0 \), \( c \geq c_1 > 0 \), and \( \alpha \in \mathbb{R} \) such that

\[
|\mathcal{K}(2n,3n)| = (1 + o(1))cn^{-\alpha}\gamma^n(2n)!
\]

and

\[
|\mathcal{K}_C(2n,3n)| = (1 + o(1))cn^{-\alpha}\gamma^n(2n)!.
\]

(K3) [giant component]. Let \( K(2n,3n) \in \mathbb{R} \mathcal{K}(2n,3n) \). Then \( v(L_1(K(2n,3n))) = 2n - O(p) \).

In addition, for each \( i \in \mathbb{N} \), the asymptotic probability that \( v(L_1(K(2n,3n))) = 2n - 2i \) is bounded away from both 0 and 1.

The constant \( \alpha \) in (K2) is called the critical exponent. For short, we say \( \mathcal{P} \) is kernel-stable with critical exponent \( \alpha \) if it satisfies (P1) and (P2).

We can show that condition (K3) in Definition 4.1 can be deduced from conditions (K1) and (K2) if \( \mathcal{P} \) is addable. In addition, each graph without complex components is in any kernel-stable class. The proofs of Lemmas 4.2 and 4.3 can be found in Section 10.

Lemma 4.2. Let \( \mathcal{P} \) be a class of graphs satisfying (K1) and (K2). If, in addition, \( \mathcal{P} \) is addable, then \( \mathcal{P} \) satisfies (K3).

Lemma 4.3. Let \( \mathcal{P} \) be a kernel-stable class of graphs and \( H \) a graph without complex components. Then \( H \in \mathcal{P} \).

The next lemma indicates that a kernel-stable class is quite universal and rich, because it includes the class of cactus graphs (a cactus graph is a graph in which every edge belongs to at most one cycle), the class of series-parallel graphs, the class of planar graphs, and the class of graphs on a surface.

Lemma 4.4 (\([21,22,35]\)). The following classes of graphs are kernel-stable with critical exponent \( \alpha \):

(a) the class of cactus graphs with \( \alpha = 5/2 \);
(b) the class of series-parallel graphs with \( \alpha = 5/2 \);
(c) the class of planar graphs with \( \alpha = 7/2 \);
(d) the class of graphs embeddable on an orientable surface of genus \( g \in \mathbb{N} \) with \( \alpha = 5g/2 + 7/2 \).

Proof of Lemma 4.4. As shown in \([21,22,35]\), the class of planar graphs \([21]\), the class of series-parallel graphs \([35]\), and the class of cactus graphs \([22]\) satisfy (K2).

Obviously, these classes also fulfill (P1), (K1) and are addable. Thus, they are kernel-stable classes due to Lemma 4.2. Moreover, in \([23]\) it was shown that the class of graphs that are embeddable on an orientable surface of genus \( g \in \mathbb{N} \cup \{0\} \) satisfies (P2). Thus, they also form a kernel-stable class of graphs, since they trivially fulfill (P1).

Instead of proving Theorem 1.4 only for the class of planar graphs, we will show the following generalisation to kernel-stable classes of graphs in Section 9.

Theorem 4.5. Theorem 1.4 is true for any kernel-stable class of graphs.

This immediately implies that Theorem 1.4 is also true for the graph classes in Lemma 4.4.

Corollary 4.6. Theorem 1.4 is true for the class of cactus graphs, the class of series-parallel graphs, and the class of graphs embeddable on an orientable surface of genus \( g \in \mathbb{N} \cup \{0\} \).
In contrast to the classes of graphs in Corollary 4.6, the class of outerplanar graphs is not kernel-stable, since subdividing an edge in an outerplanar graph can lead to a non-outerplanar graph. Hence, a non-outerplanar graph can have an outerplanar kernel, and thus (K1) is violated. Nevertheless, we can prove that Theorem 1.4 is also true for outerplanar graphs by using some results from [22]: (i) for the cases $s^3n^{-2} \to -\infty$ and $s = O(n^{2/3})$ we use that the asymptotic probability that the uniform random graph $G(n, m)$ is outerplanar is bounded away from 0 (see Theorem 4.8(a)); (ii) if $s^3n^{-2} \to -\infty$, we use the fact that a random outerplanar graph is whp a cactus graph [22, Theorem 4].

**Corollary 4.7.** Theorem 1.4 is true for the class of outerplanar graphs.

In order to prove Theorem 4.8 in Section 5 we will need the following two known facts. The first statement was shown in [23] by applying the core-kernel approach and provides useful information about a typical core and a typical kernel when $s$ statement was shown in [23] by applying the core-kernel approach and provides useful information about a typical core and a typical kernel when $s$ statement was shown in [23] by applying the core-kernel approach and provides useful information about a typical core and a typical kernel when $s$ statement was shown in [23] by applying the core-kernel approach and provides useful information about a typical core and a typical kernel when $s$

**Theorem 4.8.** Let $\mathcal{P}$ be a kernel-stable class of graphs, $P = P(n, m) \in \mathcal{P}(n, m)$, $L_1 = L_1(P)$ the largest component of $P$, and $R = R(P) = P \setminus L_1$. Assume $m = n/2 + s$ for $s = o(n)$ and $s^3n^{-2} \to -\infty$. Then the following hold:

(a) whp $v(C(L_1)) = \Theta(sn^{-1/3})$;

(b) whp $v(K(L_1)) = \Theta(sn^{-2/3})$;

(c) $v(C(R)) = O_p(n^{1/3})$;

(d) $v(K(R)) = O_p(1)$;

(e) whp $K(P)$ is cubic (i.e. 3-regular);

(f) $v(L_1) = 2s + O_p(n^{2/3})$;

(g) $e(L_1) = 2s + O_p(n^{2/3})$.

We note that the results in Theorem 4.8 were not explicitly proven in [23], but they immediately follow by combining Theorems 1.4(iii) and 5.4(iii), (v), (vi) and Corollaries 5.3 and 5.5 from [23]. Strictly speaking, the authors of [23] proved Theorem 4.8 only for the class of graphs embeddable on an orientable surface of genus $g \in \mathbb{N} \cup \{0\}$, but they pointed out that Theorem 4.8 generalises to the more general setting of kernel-stable graph classes (see [23, Remark 8.3]).

**Theorem 4.9.** Let $m = n/2 + s$, where $s \leq Mn^{2/3}$ for some constant $M \in \mathbb{R}$ and let $G = G(n, m) \in \mathcal{P}(n, m)$ be the uniform random graph. Then the following hold:

(a) $\liminf_{n \to \infty} P[G \text{ has no complex component}] > 0$;

(b) $v(L_1(G)) = O_p(n^{2/3})$.

We note that if $\mathcal{P}$ is a kernel-stable class of graphs, then each graph without a complex component lies in $\mathcal{P}$ (see Lemma 4.3). Thus, Theorem 4.9(a) implies $\liminf_{n \to \infty} P[G(n, m) \in \mathcal{P}] > 0$ in the case of $s \leq Mn^{2/3}$. That will be useful in the proof of Theorem 4.5.

5. Pólya urn model

In this section we present several useful results on the Pólya urn model introduced in Section 3.1.

5.1. Model with $N$ colours. Given $N, k \in \mathbb{N}$, there are initially $N$ balls of $N$ distinct colours $F_1, \ldots, F_N$ in a urn. In each step we draw a ball uniformly at random from the urn. Then the drawn ball is returned to the urn along with an additional ball of the same colour. We repeat that procedure $k$ times. For each $i \in [N]$ we denote by $X_i$ the number of drawn balls of colour $F_i$ at the end of the procedure (i.e. after $k$ steps).

To derive bounds on the length of the shortest and longest cycle in the core (see Lemma 5.7), we need the following bounds on the minimum and maximum values of the total numbers $X_1, \ldots, X_f$ of drawn balls of the first $f$ colours when $N, k, f$ are functions in $n$. Although we believe such results should be known, we could not find them in literature and therefore we include their proofs in Appendix A for completeness.
Theorem 5.1. For every \( n \in \mathbb{N} \), we let \( N = N(n), k = k(n) \in \mathbb{N} \), and \( f = f(n) \in \mathbb{N} \) with \( 1 \leq f \leq N \). We assume that \( N = o(1) \) and that \( f = O(1) \) or \( f = o(1) \). Then the following hold.

(a)

\[
X_* := \min_{i \leq i \leq f} X_i = \begin{cases} 
\Theta(n) \left( \frac{k}{N} \right) & \text{if } k = \omega(Nf), \\
O_p(1) & \text{if } k = O(Nf).
\end{cases}
\]

(b)

\[
X^* := \max_{i \leq i \leq f} X_i = \begin{cases} 
\Theta(n) \left( 1 + \log f \right) & \text{if } k = \omega(N), \\
O_p(1 + \log f) & \text{if } k = O(N).
\end{cases}
\]

Another useful fact about the Pólya urn model (which will be used in the proof of Lemma 6.4(a)) is the following result on the distribution of \( X_i \), whose proof can be found in Appendix A.

Proposition 5.2. Let \( N, k \in \mathbb{N} \) be given. Then we have

\[
\sum_{i=1}^{N} \mathbb{P} |X_i| \leq 1 \leq \frac{2N^2}{k}.
\]

5.2. Model with two colours. Given \( b, w, k \in \mathbb{N} \), there are initially \( b \) black and \( w \) white balls in a urn. Then we draw \( k \) times uniformly at random a ball from the urn. In each step we return the drawn ball together with an additional ball of the same colour. Then we denote by \( X \) the number of drawn black balls at the end of the procedure, i.e. after \( k \) steps.

Theorem 5.3 (11). Let \( b, w, k \in \mathbb{N} \) and \( X \) be the number of drawn black balls after \( k \) steps in the Pólya urn model with initially \( b \) black and \( w \) white balls. Then we have

\[
\mathbb{E}[X] = \frac{bk}{b+w} \quad \text{and} \quad \mathbb{V}[X] = \frac{bwk(b+w+k)}{(b+w)^2(b+w+1)}.
\]

6. Random core

In this section we study the process of obtaining a random core \( C = C(K, k) \) from a fixed kernel \( K \) by randomly subdividing the edges of \( K \) with \( k \) additional vertices for given \( (K, k) \). Because it is hard to directly analyse \( C \), we circumvent this difficulty by introducing an auxiliary random multigraph \( \tilde{C} \) which behaves asymptotically like \( C \) and fits into the scheme of the Pólya urn model.

Definition 6.1. Given a pair \((K, k)\) of a multigraph \( K \) on vertex set \([v(K)]\) of minimum degree at least three and \( k \in \mathbb{N} \), we denote by \( \mathcal{C}(K, k) \) the set of all simple graphs on vertex set \([v(K) + k]\) obtained from \( K \) by subdividing the edges of \( K \) by the vertices \( v(K) + 1, \ldots, v(K) + k \). In other words, \( \mathcal{C}(K, k) \) is the set of all cores on vertex set \([v(K) + k]\) whose kernel is equal to \( K \). Let \( C = C(K, k) \in_R \mathcal{C}(K, k) \). In addition, we define a random multigraph \( \tilde{C} = \tilde{C}(K, k) \) by the following random experiment: we start with \( G_0 = K \). Given the multigraph \( G_{i-1} \) we construct \( G_i \) as follows (for \( i = 1, \ldots, k \)). We choose uniformly at random an edge \( e \) of \( G_{i-1} \) and subdivide \( e \) by one additional vertex, which obtains the label \( v(K) + i \). We note that \( E(G_{i-1}) \) is a multiset, i.e. if there are \( r \) edges between vertices \( v \) and \( w \), we choose one of these edges with probability \( r/e(G_{i-1}) \). Then we let \( \tilde{C} = G_k \) be the resulting multigraph after \( k \) steps.

Later we will study a random kernel-stable graph \( P \) conditioned on the event that \( K(P) = K \) and \( S(P) = k \) for some fixed kernel \( K \) and fixed \( k \in \mathbb{N} \). The next lemma says that the core of this conditional random graph is distributed like \( C(K, k) \) from Definition 6.1. That fact will be quite useful when we apply Lemma 5.2.

Lemma 6.2. Let \( \mathcal{P} \) be a kernel-stable class of graphs and \( \mathcal{K} \) the class of all kernels of graphs in \( \mathcal{P} \). Given a pair \((K, k)\) of a multigraph \( K \in \mathcal{K} \) on vertex set \([v(K)]\) and \( k \in \mathbb{N} \), we let \( \mathcal{P}_{(K,k)}(n,m) \) be the subclass of \( \mathcal{P}(n,m) \) consisting of all graphs \( F \) whose kernel \( K(F) \) is equal to \( K \) and whose subdivision number \( S(F) \) is equal to \( k \), i.e.

\[
\mathcal{P}_{(K,k)}(n,m) := \{ F \in \mathcal{P}(n,m) \mid K(F) = K \quad \text{and} \quad S(F) = k \}.
\]
Let $P | (K, k) \in R \mathcal{K}_K(n, m)$ and $C(K, k)$ be the random core as defined in Definition 6.7. Then the core of $P | (K, k)$ is distributed like $C(K, k)$: for each fixed graph $H$, we have

$$P[|C(P | (K, k)) = H|] = P[C(K, k) = H].$$

In the next step, we provide a relation between the two random (multi-) graphs $C$ and $\tilde{C}$ (see Lemma 6.4). In particular, we show that if $k = \omega(N^2)$, they are contiguous in the sense of Definition 2.5 (see Corollary 6.5), where $N = e(K)$ is the number of edges in $K$. In our applications we will have $k = \Theta(sn^{-1/3})$ and $N = \Theta(sn^{-2/3})$ (see Theorem 4.3) and we note that $sn^{-1/3} = \omega\left((sn^{-2/3})^2\right)$. In order to state this result, we need the following definition.

**Definition 6.3** (2-simple). Given a graph $H$ we consider the construction of obtaining the core $C(H)$ from the kernel $K(H)$ (described in Section 2.3). We call $H$ 2-simple if each edge of $K(H)$ is subdivided by at least two vertices in that construction.

**Lemma 6.4.** Let $K$ be a multigraph on vertex set $\{v(K)\}$ of minimum degree at least three with $N$ edges and let $k \in \mathbb{N}$. In addition, let the random multigraphs $C = C(K, k)$ and $\tilde{C} = \tilde{C}(K, k)$ be defined as in Definition 6.7.

1. We have
   $$P[\tilde{C} \text{ is simple}] \geq P[\tilde{C} \text{ is 2-simple}] \geq 1 - \frac{2N^2}{k}.$$

2. Conditioning on the event that $\tilde{C}$ is simple, the distributions of $\tilde{C}$ and $C$ are the same: for each graph $H$, we have
   $$P[\tilde{C} = H | \tilde{C} \text{ is simple}] = P[C = H].$$

**Corollary 6.5.** For every $n \in \mathbb{N}$, let $K = K(n)$ be a multigraph on vertex set $\{v(K)\}$ of minimum degree at least three with $N = N(n)$ edges and let $k = k(n) \in \mathbb{N}$. Let $C = C(n) = C(K(n), k(n))$ and $\tilde{C} = \tilde{C}(K(n), k(n))$ be as in Definition 6.7. If $k = \omega(N^2)$, then $C$ and $\tilde{C}$ are contiguous.

Amongst others, Lemma 6.4(a) states that whp $\tilde{C}$ is 2-simple if $k = \omega(N^2)$. Using that we can deduce the following result, which we will use later in the proof of Lemma 7.1.

**Corollary 6.6.** Let $\mathcal{P}$ be a kernel-stable class of graphs and $P = P(n, m) \in R \mathcal{P}(n, m)$. Assume $m = n/2 + s$ for $s = s(n) = o(n)$ and $s^3n^{-2} \to \infty$. Then whp $P$ is 2-simple.

Our next results provide bounds on the lengths $g(\tilde{C})$ and $c(\tilde{C})$ of the shortest and longest cycle in $\tilde{C}$. We note that if $k = \omega(N^2)$, these results also hold for $C$ due to Corollary 6.5.

**Lemma 6.7.** For every $n \in \mathbb{N}$, let $K = K(n)$ be a multigraph on vertex set $\{v(K)\}$ of minimum degree at least three with $N = N(n)$ edges and $\lambda = \lambda(n)$ loops. In addition, let $k = k(n) \in \mathbb{N}$ and the random multigraph $\tilde{C} = \tilde{C}(n) = \tilde{C}(K(n), k(n))$ be as in Definition 6.7. We assume that $N = \omega(1)$.

1. If $\lambda = O(1)$ or $\lambda = \omega(1)$, then
   $$g(\tilde{C}) = \begin{cases} O_p\left(\frac{k}{N^2}\right) & \text{if } k = \omega(N\lambda) \text{ and } \lambda \neq 0, \\ O_p(1) & \text{if } k = O(N\lambda), \end{cases}$$

   and
   $$g(\tilde{C}) = \Omega_p\left(\frac{k}{N^2}\right) \text{ if } k = \omega(N^2).$$

   In particular, if $\lambda = \Theta(N)$, then
   $$g(\tilde{C}) = \begin{cases} \Theta_p\left(\frac{k}{N^2}\right) & \text{if } k = \omega(N^2), \\ O_p(1) & \text{if } k = O(N^2). \end{cases}$$

2. If $\lambda > 0$ and $k = \omega(N)$, then we have
   $$c(\tilde{C}) = \Omega_p\left(\frac{k}{N} \left(1 + \log \lambda\right)\right).$$

In order to use Lemma 6.7 in the proof of Theorem 4.5, we will study the number of loops in a typical kernel of a random kernel-stable graph in Section 7 (see Corollary 7.1).

We conclude this section with a result on the block structure of $\tilde{C}$, which provides also an insight into the block structure of $C$ due to Corollary 6.5. Roughly speaking, the next lemma says that the $i$-th
largest block of a cubic kernel $K$ translates (during the process of constructing $C$) to the $i$-th largest block of $C$, provided the block is not too ‘small’.

**Lemma 6.8.** For every $n \in \mathbb{N}$, let $k = k(n) \in \mathbb{N}$, $K = K(n)$ be a cubic multigraph on vertex set $[v(K)]$, and the random multigraph $\tilde{C} = \tilde{C}(n) = \tilde{C}(K(n), k(n))$ be defined as in Definition 6.1. We assume that $k = \omega(v(K))$ and let $i \in \mathbb{N}$ be fixed such that $b_i(K) = \omega\left(\left(v(K)\right)^{1/2}\right)$. Then whp

$$b_i(\tilde{C}) = \Theta\left(\frac{k b_i(K)}{v(K)}\right).$$

**Remark 6.9.** The condition in Lemma 6.8 that $b_i(K)$ is not too small, i.e. $b_i(K) = \omega\left(\left(v(K)\right)^{1/2}\right)$, can be weakened by using stronger concentration results on the Pólya urn model than the results in Theorem 5.3. We believe that using similar results as in Theorem 5.1 one can show that the statement of Lemma 6.8 is true even under the condition that $b_i(K) = \omega\left(\left(v(K)\right)^{2}\right)$ for some $\varepsilon > 0$.

### 7. Random kernel

Throughout this section, let $\mathcal{P}$ be a kernel-stable class of graphs and $\mathcal{K}$ the class of all kernels of graphs in $\mathcal{P}$. We let $P = P(n, m) \in \mathbb{R} \setminus \mathcal{P} (n, m)$ and consider the weakly supercritical regime when $m = n/2 + s$ for $s = s(n) = o(n)$ and $s^3 n^2 \to \infty$.

Due to Theorem 4.8 we know that whp $K(P)$ is cubic and $v(K(P)) = \Theta\left(s n^{-2/3}\right)$. Hence, we might expect that $K(P)$ behaves asymptotically like a graph chosen uniformly at random from all cubic multigraphs in $\mathcal{K}$ with $\Theta\left(s n^{-2/3}\right)$ many vertices. In the next lemma we show that this is indeed true. We note, however, that this result is not straightforward, since $K(P)$ is not equally distributed on the set of all possible cubic kernels in $\mathcal{K}$ with $\Theta\left(s n^{-2/3}\right)$ many vertices.

**Lemma 7.1.** Let $\mathcal{P}$ be a kernel-stable class of graphs and $\mathcal{K}$ the class of all kernels of graphs in $\mathcal{P}$. Let $F : \mathcal{P} \to \mathbb{N}$ be a graph theoretic function and $g_1, g_2 : \mathbb{N} \to \mathbb{N}$ non-decreasing functions. We assume that for $K(2n, 3n) \in \mathcal{K}(2n, 3n)$, we have whp $g_1(n) \leq F(K(2n, 3n)) \leq g_2(n)$. In addition, assume $m = n/2 + s$ for $s = s(n) = o(n)$ and $s^3 n^{-2} \to \infty$. Then there exist constants $c_2 \geq c_1 > 0$ such that for $P = P(n, m) \in \mathcal{P}(n, m)$ we have whp

$$g_1\left(c_1 s n^{-2/3}\right) \leq F(K(P)) \leq g_2\left(c_2 s n^{-2/3}\right).$$

We recall that $\lambda(H)$ denotes the number of loops in a multigraph $H$. Next, we will show that for $K(2n, 3n) \in \mathcal{K}(2n, 3n)$, we have whp $\lambda(K(2n, 3n)) = \Theta(n)$, which implies whp $\lambda(K(P)) = \Theta\left(sn^{-2/3}\right)$ by Lemma 7.1. Our proof of that result will be based on the following observation. Let $H$ be a cubic multigraph with a single loop at $w$. Then $w$ has precisely one neighbour $x \neq w$. Assuming that there is no loop at $x$, there are two (not necessarily distinct) additional neighbours $y$ and $z$ of $x$. Then we obtain again a cubic multigraph if we delete $w$ and $x$ in $H$ and add an edge $yz$. We note that by reversing this operation we can create a multigraph with a loop at $w$. This reverse operation leads to the following definition (see also Figure 2).

**Definition 7.2** (Loop insertion). Let $H_1, H_2$ be multigraphs, $yz \in E(H_1)$, and $(w, x) \in (V(H_2))^2$ with $w \neq x$. We say that $H_2$ can be obtained from $H_1$ by a loop insertion at edge $yz$ with vertex pair $(w, x)$ if

$$V(H_2) = V(H_1) \cup \{w, x\}$$

and

$$E(H_2) = E(H_1) - yz + xy + xz + wx + w w.$$
Remark 7.4. Let $H$ be a multigraph on vertex set $[n]$ and $\mathcal{H}$ be the set of all multigraphs which can be obtained by subdividing precisely one edge of $H$ by additional vertex $n + 1$. Then we have $|\mathcal{H}| = w(H) \cdot e(H)$, where $w(H)$ is the weight of $H$ defined in (2) and $|\mathcal{H}|$ is the total weight of all multigraphs in $\mathcal{H}$.

As a consequence of Remark 7.4 we obtain the number of ways to construct multigraphs by a loop insertion.

Remark 7.5. Let $H$ be a multigraph on vertex set $[n]$ and $\mathcal{H}$ be the set of all multigraphs which can be obtained by performing a loop insertion (cf. Definition 7.2) at some edge $e \in E(H)$ with vertex pair $(n + 1, n + 2)$. Then we have $|\mathcal{H}| = w(H) \cdot e(H) / 2$, where $w(H)$ is the weight of $H$ defined in (2).

In order to see that Remark 7.5 is true, we imagine that we perform a loop insertion in two steps. Firstly, we choose the edge $e \in E(H)$ and subdivide it by the vertex $n + 1$. Secondly, we add the vertex $n + 2$ together with an edge $\{n + 1, n + 2\}$ and a loop at $n + 2$. Then Remark 7.5 follows from Remark 7.4 and the fact that inserting a loop halves the weight of a graph.

Next, we show that typically the number of loops in the kernel is linear (in the number of edges in the kernel). Due to Lemma 7.1 it suffices to prove this result only for a random cubic kernel chosen from $\mathcal{K}(2n, 3n)$. We use a loop insertion to construct all graphs in $\mathcal{K}(2n, 3n)$ with a loop at a fixed vertex $v$. By doing that we can estimate the expected number of loops. Then we use the second moment method to show concentration around the mean. Recall that the number of loops in a graph $H$ is denoted by $\lambda(H)$.

Lemma 7.6. Let $\mathcal{P}$ be a kernel-stable class of graphs, $\mathcal{K}$ the class of all kernels of graphs in $\mathcal{P}$, and $K(2n, 3n) \in_R \mathcal{K}(2n, 3n)$. Then whp

$$\lambda(K(2n, 3n)) = (1 + o(1)) \frac{3}{2\gamma} n,$$

where $\gamma > 0$ is as in Definition 4.1.

Corollary 7.7. Let $\mathcal{P}$ be a kernel-stable class of graphs and $P = P(n, m) \in_R \mathcal{P}(n, m)$. Assume $m = n/2 + s$ for $s = s(n) = o(n)$ and $s^3 n^{-2} \to \infty$. Then whp

$$\lambda(K(P)) = \Theta \left( sn^{-2/3} \right).$$

Furthermore, the largest component $L_1 = L_1(P)$ of $P$ satisfies whp

$$\lambda(K(L_1)) = \Theta \left( sn^{-2/3} \right).$$

8. Block structure

In this section we present several results which lead to the conclusion that Theorem 1.5 is also true for all kernel-stable classes of graphs $\mathcal{P}$ that are addable and have a critical exponent $3 < \alpha < 4$ (see Theorem 8.12). First we will analyse the block structure of a random cubic multigraph chosen from an appropriate class of multigraphs $\mathcal{M}$ (the so-called bridge-stable class), which will be later chosen as the class of connected kernels of graphs in $\mathcal{P}$. Then we will deduce the block structure of the random kernel-stable graph in Theorem 8.12 by using Lemmas 6.8 and 7.1.

We recall that, as in the case of a simple graph, a block of a multigraph $H$ is a maximal 2-connected subgraph of $H$. Here we insist that a vertex with a loop forms a block. In order to understand the
block structure of a random cubic multigraph chosen from the class $\mathcal{M}$, we will first analyse bridge numbers defined below (see also Figure 3).

**Definition 8.1** (Bridge number). Given a connected multigraph $H$, an edge $e \in E(H)$ is called a *bridge* if $H - e$ is disconnected. For two distinct vertices $w, x \in V(H)$ we define the bridge number $\beta(wx)$ as follows. If $wx$ is a bridge in $H$, then we set $\beta(wx) = \beta_H(wx) := \min\{v(C_1), v(C_2)\}$, where $C_1$ and $C_2$ are the two components of $H - wx$. Otherwise, we define $\beta(wx) := 0$.

We observe that if $wx \notin E(H)$, then we have $\beta(wx) = 0$. We will determine the distribution of the bridge number $\beta(wx)$ for fixed vertices $w$ and $x$. Intuitively, if $\beta(wx)$ is typically ‘small’, then we should have a unique largest block which is significantly larger than all other blocks. We will show that this is indeed the case (see Lemmas 8.10 and 8.11). To do so, we introduce the bridge insertion, which is an operation that connects two components of a graph via a bridge (see also Figure 3).

**Definition 8.2** (Bridge insertion). Let $H_1, H_2, H_3$ be connected multigraphs, $e_1 = y_1z_1 \in E(H_1), e_2 = y_2z_2 \in E(H_2)$, and $w, x \in V(H_3)$ be distinct. Then we say that $H_3$ can be obtained from $H_1$ and $H_2$ by a *bridge insertion* at edges $e_1$ and $e_2$ with vertices $w$ and $x$ if

$$V(H_3) = V(H_1) \cup V(H_2) \cup \{w, x\}$$

and

$$E(H_3) = E(H_1) \cup E(H_2) - y_1z_1 - y_2z_2 + wy_1 + wz_1 + wx + xy_2 + xz_2.$$

In order to successfully apply bridge insertions in some graph class $\mathcal{M}$, we require two natural properties of $\mathcal{M}$.

**Definition 8.3** (Bridge-stable). A class $\mathcal{M}$ of connected multigraphs is called *bridge-stable* if the following two condition hold.

1. (B1) $\mathcal{M}$ is stable under bridge insertions: if $H_3$ can be obtained by a bridge insertion in $H_1$ and $H_2$, then we have $H_1, H_2 \in \mathcal{M}$ if and only if $H_3 \in \mathcal{M}$.

2. (B2) Let $\mathcal{M}(2n, 3n)$ be the subclass of $\mathcal{M}$ consisting of all connected multigraphs on vertex set $[2n]$ having $3n$ edges. Then there exist constants $\gamma > 0, c > 0$, and $\alpha \in \mathbb{R}$ such that

$$|\mathcal{M}(2n, 3n)| = (1 + o(1))cn^{-\alpha}\gamma^n(2n)!.$$  \hspace{1cm} (6)

The constant $\alpha$ in (B2) is called the *critical exponent*.

We note that the class of all connected kernels of an addable kernel-stable class is bridge-stable. Next, let $\mathcal{M}$ be some bridge-stable class of multigraphs. We study the asymptotic distribution of the bridge number $\beta_M(vw)$ for fixed vertices $v \neq w$ and $M = M(2n, 3n) \in \mathcal{M}(2n, 3n)$. More precisely, we estimate the probability $\mathbb{P} [\beta_M(wx) \geq 2f(n) + 1]$ for an arbitrary function $f$.

**Lemma 8.4.** Let $\mathcal{M}$ be a bridge-stable class of multigraphs with critical exponent $\alpha > 2$ and $f$ a function with $f(n) = o(1)$, but $f(n) = o(n)$. In addition, let $M = M(2n, 3n) \in \mathcal{M}(2n, 3n)$ and let $c$ and $\gamma$ be as in (6). Then for any pair of distinct vertices $w, x \in [2n]$ we have

$$\mathbb{P} [\beta_M(wx) \geq 2f(n) + 1] = (1 + o(1)) \frac{9c}{2\gamma(\alpha - 2)} \cdot \frac{1}{f(n)^{\alpha - 2}}.$$

Using the second moment method, we will show that this number is concentrated around its expectation.
Secondly, we show that the dominant block has linearly many vertices (see Lemma 8.10).

**Definition 8.7.** Let $B$ be a bridge-stable class of multigraphs with critical exponent $\alpha > 3$ and $f$ be a function with $f(n) = \Theta(1), f(n) = o(n)$, and $f(n) = o(n^{1/(\alpha - 2)})$. Let $M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$. In addition, let $w_1, w_2, w_3, w_4 \in [2n]$ be pairwise distinct and let $c$ and $\gamma$ be as in \[.\] Then we have

$$P \{ \beta_M(w_1 w_2), \beta_M(w_3 w_4) \geq 2f(n) + 1 \} = (1 + o(1)) \left( \frac{9c}{2\gamma (\alpha - 2)} \cdot \frac{1}{f(n)^{\alpha-2} n} \right)^2.$$  

Now we can combine Lemmas 8.4 and 8.5 to determine the number of pairs $(w, x)$ with $\beta_M(w x) \geq 2f(n) + 1$.

**Lemma 8.8.** Let $\mathcal{M}$ be a bridge-stable class of multigraphs with critical exponent $\alpha > 3$ and $f$ be a function with $f(n) = \Theta(1), f(n) = o(n)$, and $f(n) = o(n^{1/(\alpha - 2)})$. In addition, let $M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$ and let $c$ and $\gamma$ be as in \[.\] Then $\beta_M$ has many unordered pairs of vertices $w, x \in [2n]$ with $\beta_M(w x) \geq 2f(n) + 1$.

Next, we show in two steps that $\beta_M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$ has a block with linearly many vertices. Firstly, we prove that $\beta_M$ has a dominant block (see Definition 8.7) and Lemma 8.8. Secondly, we show that $\beta_M$ this dominant block has linearly many vertices (see Lemma 8.10).

**Definition 8.7.** Let $H$ be a connected multigraph. A block $B$ of $H$ is called dominant if every bridge $e$ that shares a vertex with $B$ satisfies $v(C_e) > v(R_e)$, where $C_e$ is the component of $H - e$ containing $B$ and $R_e$ the other component of $H - e$.

**Lemma 8.7.** Let $H$ be a connected multigraph. A block $B$ of $H$ is called dominant if every bridge $e$ that shares a vertex with $B$ satisfies $v(C_e) > v(R_e)$, where $C_e$ is the component of $H - e$ containing $B$ and $R_e$ the other component of $H - e$.

**Lemma 8.8.** Let $H$ be a connected cubic multigraph on vertex set $[2n]$ such that $\beta_H(w x) \leq (2n - 2)/3$ for every pair of distinct vertices $w, x \in [2n]$. Then $H$ contains a dominant block.

Due to Lemmas 8.4 and 8.8 we know that $\beta_M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$ has a dominant block $B$ (in the case $\alpha > 3$). Let $e_1, \ldots, e_r$ be the bridges that share a vertex with $B$. Then we get $2n = v(M) = \sum_{i=1}^r \beta_M(e_i)$. Assuming that the bridge numbers $\beta_M(e_i)$ are 'small', we get that $v(B) or r needs to be 'large'. We note that in the latter case we again obtain that $v(B)$ is 'large', since each vertex in $B$ can lie in at most one bridge. In Lemma 8.10 we will make this idea more precise. We already saw in Lemma 8.4 that the bridge numbers are typically 'small'. However, we need the following stronger result in the proof of Lemma 8.10.

**Lemma 8.9.** Let $\mathcal{M}$ be a bridge-stable class of multigraphs with critical exponent $\alpha > 2, 0 < \mu < \alpha - 2$, and $M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$. Then for any pair of distinct vertices $w, x \in [2n]$ we have

$$\mathbb{E} [ (\beta_M(w x))^\mu] = \Theta(1/n).$$

Recall that given a graph $H$ and $i \in \mathbb{N}$, we denote by $b_i(H)$ the number of vertices in the $i$-th largest block $B_i(H)$ of $H$.

**Lemma 8.10.** Let $\mathcal{M}$ be a bridge-stable class of multigraphs with critical exponent $\alpha > 3$. For $M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$, we have $b_i(M) = \Theta_p(n^{1/(\alpha - 2)})$.

Next, we consider the $i$-th largest block of $M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$ for $i \geq 2$. We note that $b_i(M) \leq \max \beta_M(e)$ and thus $b_i(M) = O_p\left(n^{1/(\alpha - 2)}\right)$ by Lemma 8.4. On the other hand, we know by Lemma 8.6 that there is a bridge $e$ in $M$ with $\beta_M(e) = \Omega_p\left(n^{1/(\alpha - 2)}\right)$. Due to Lemma 8.10 we intuitively expect that both components of $M - e$ have again a block whose number of vertices is linear in the order of the component. By induction this would imply $b_i(M) = \Omega_p\left(n^{1/(\alpha - 2)}\right)$. In the following lemma we show that is indeed the case.

**Lemma 8.11.** Let $\mathcal{M}$ be a bridge-stable class of multigraphs with critical exponent $\alpha > 3$ and $M = M(2n, 3n) \in R \mathcal{M}(2n, 3n)$. Then for each $i \geq 2$, we have $b_i(M) = \Theta_p\left(n^{1/(\alpha - 2)}\right)$.  

16
Lemmas 8.10 and 8.11 together with Lemma 7.1 give us the block structure of a random kernel 
$K(P)$. Now we combine this information with Lemma 6.8 to obtain the block structure of a random 
kernel-stable graph.

**Theorem 8.12.** Let $\mathcal{P}$ be a kernel-stable class of graphs which is addable and has a critical exponent 
$3 < \alpha < 4$. In addition, let $P = P(n, m) \in_R \mathcal{P}(n, m)$ and $L_1 = L_1(P)$ denote the largest component of $P$. 
Assume $m = n/2 + s$ for $s = s(n) = o(n)$ and $s^3 n^{-2} \to \infty$. Then the following hold.

(a) $b_1(P) = \Theta_p\left(sn^{-1/3}\right)$.

(b) $b_1(P) = \Theta_p\left(s^{1/(\alpha-2)}n^{-\frac{1}{2\alpha-2}}\right)$ for each $i \in \mathbb{N}$ with $i \geq 2$.

(c) The number of blocks in $L_1$ is $\omega\left(sn^{-2/3}\right)$.

The proof of Theorem 8.12 can be found in Section 9.5

**Remark 8.13.** We believe that Theorem 8.12 is also true when $\alpha \geq 4$. In that case we would need an 
improved version of Lemma 6.8 where we can weaken the condition on $b_1(K) = \omega((v(K))^{1/2})$ to 
$b_1(K) = \omega((v(K))^\epsilon)$ for some $\epsilon > 0$. Using the ideas presented in Section 5 one may deduce such an 
improved statement (see also Remark 6.9). Nevertheless, we omit details, since we expect that the proofs become rather technical but do not provide any new insights.

9. PROOFS OF MAIN RESULTS

Throughout this section, let $\mathcal{P}$ be a kernel-stable class of graphs and $\mathcal{K}$ the class of all kernels of 
graphs in $\mathcal{P}$. Let $\mathcal{P}(n, m)$ be the subclass of $\mathcal{P}$ consisting of all graphs on vertex set $[n]$ with 
m = m(n) edges and $P = P(n, m) \in_R \mathcal{P}(n, m)$. Assume $m = n/2 + s$ for $s = s(n) = o(n)$. Given a graph $H$ we let 
$L_1(H)$ denote the largest component of $H$ and $R(H) := H \setminus L_1(H)$, i.e. the graph obtained from $H$ 
by deleting the largest component $L_1(H)$.

9.1. **Proof of Theorem 4.5.** We first consider the cases when $s^3 n^{-2} \to -\infty$ or $s = O\left(n^{2/3}\right)$. Due to 
Lemma 4.3 each graph without complex components lies in $\mathcal{P}$. Hence, we obtain by Theorem 4.3(a) that 
$\liminf_{n \to \infty} |P(G(n, m) \in \mathcal{P}(n, m))| > 0$. Thus, each property that holds whp in $G(n, m)$ is also true 
whp in $P(n, m)$ and statements (a) and (b) follow from Theorem 1.2.

To prove (c), we assume $s^3 n^{-2} \to \infty$. By Theorem 4.3(c) i.e. whp $v(C(L_1(P))) = \Theta\left(sn^{-1/3}\right)$, 
and the simple observation that $c(L_1(P)) \leq v(C(L_1(P)))$ we obtain that whp 
$c(L_1(P)) = O\left(sn^{-1/3}\right)$.

In order to prove the two other results on the girth and circumference of the largest component 
$L_1(P)$, i.e. $g(L_1(P)) = \Theta_p\left(n s^{-2}\right)$ and $c(L_1(P)) = \Omega_p\left(n^{1/3} \log\left(sn^{-2/3}\right)\right)$, we will use typical properties of the core and kernel of $P$. More precisely, let $\mathcal{A}(n)$ be the subclass of $\mathcal{P}(n, m)$ consisting of those graphs $H$ with largest component $L_1(H)$ satisfying the following properties 
$v(C(L_1(H))) = \Theta\left(sn^{-1/3}\right)$,

$e(K(L_1(H))) = \Theta\left(sn^{-2/3}\right)$,

and $\lambda(K(L_1(H))) = \Theta\left(sn^{-2/3}\right)$.

From Theorem 4.8(a) and (b) we obtain that whp $v(C(L_1(P))) = \Theta\left(sn^{-1/3}\right)$ and $e(K(L_1(P))) = 
3/2 \cdot v(K(L_1(P))) = \Theta\left(s n^{-2/3}\right)$. Furthermore, Corollary 7.7 says that whp $\lambda(K(L_1(P))) = \Theta\left(s n^{-2/3}\right)$. Therefore, we have whp $P \in \mathcal{A}(n)$.

Next, we will apply Lemma 3.2 to the class $\mathcal{A} := \bigcup_{n \in \mathbb{N}} \mathcal{A}(n)$. So, we define the function 
$\Phi : \mathcal{A} \to \mathcal{K} \times \mathbb{N}$ which maps a graph $H \in \mathcal{A}$ to the pair of kernel $K(L_1(H))$ and subdivision number $S(L_1(H))$, i.e.

$\Phi(H) := (K(L_1(H)), S(L_1(H)))$.

Let $s = (k_n)_{n \in \mathbb{N}}$ be a sequence that is feasible for $(\mathcal{A}, \Phi)$ (cf. Definition 3.1) and let $A = A(n) \in_R \mathcal{A}(n)$. Due to the definition of $\Phi$ all possible realisations of $L_1(A \mid s)$ have the same kernel $K_n$ and 
the same subdivision number $k_n$. Hence, by Lemma 6.2 we have that $C(L_1(A \mid s))$ is distributed like
C(K_n, R_n), a graph chosen uniformly at random from the class of all cores with kernel K_n and subdivision number k_n. From the definition of ω(n) we have k_n = Θ(sn^{-1/3}), e(K_n) = Θ(sn^{-2/3}), and λ(K_n) = Θ(sn^{-1/3}). In particular, this yields k_n = ω(e(K_n)^2) and λ(K_n) = Θ(e(K_n)). Hence, by combining Lemma 6.7 and Corollary 6.5 we obtain

\[ g(L_1(A|s)) = g(C(K_n, k_n)) = \Theta_p \left( \frac{k_n}{e(K_n)^2} \right) = \Theta_p \left( \frac{sn^{-1/3}}{(sn^{-2/3})^2} \right) = \Theta_p \left( \frac{n}{s} \right) \]

and

\[ c(L_1(A|s)) = c(C(K_n, k_n)) = \Omega_p \left( \frac{k_n}{e(K_n)^2} \log \lambda(K_n) \right) = \Omega_p \left( n^{1/3} \log \left( sn^{-2/3} \right) \right) \]

As the sequence s was arbitrary, Lemma 3.2 implies that the results above hold also for the (unconditional) random graph A. Since whp P ∈ A(n), the same is true for P, i.e.

\[ g(L_1(P)) = \Theta_p \left( ns^{-1} \right) \quad \text{and} \quad c(L_1(P)) = \Omega_p \left( n^{1/3} \log \left( sn^{-2/3} \right) \right), \]

as desired.

Next, we study the graph \( R(P) := P \setminus L_1(P) \) and prove that its circumference satisfies \( c(R(P)) = \Theta_p \left( n^{1/3} \right) \). To this end, we will show that \( R(P) \) behaves similarly like the uniform random graph \( G(n - v(L_1), m - e(L_1)) \) and then apply Theorem 1.2(b). More precisely, given some constant \( M > 0 \) we denote by \( \mathcal{A}(n) \) the subclass of \( \mathcal{P}(n, m) \) consisting of those graphs \( H \) such that

\[ v(L_1(H)), e(L_1(H)) \in [2s - Mn^{2/3}, 2s + Mn^{2/3}]. \]

We set \( \mathcal{A} := \bigcup_{n \in \mathbb{N}} \mathcal{A}(n) \) and let \( \delta > 0 \). By Theorem 4.11 and (g) we can choose the constant \( M \) such that

\[ \mathbb{P} \left[ P \in \mathcal{A}(n) \right] \geq 1 - \delta \]

for all \( n \) large enough. Let \( \mathcal{H} = \mathcal{H}(n) \in \mathcal{A}(n) \) and \( \Phi \) be the function that assigns each graph \( H \in \mathcal{A} \) its largest component, i.e. \( \Phi(H) := L_1(H) \). Moreover, let \( \mathcal{H} = (H_n) \in \mathcal{A}(n) \) be a sequence that is feasible for \( (\mathcal{A}, \Phi) \). We denote by \( n_U = n_U(n) := n - v(H_n) \) and \( m_U = m_U(n) := m - e(H_n) \) the number of vertices and edges in \( R(\mathcal{A} \setminus \mathcal{H}) \), respectively. Next, we will study relations between the distributions of \( \mathcal{H}_n := R(\mathcal{A} | \mathcal{H}) \) and the uniform random graph \( R_n := G(n_U, m_U) \in \mathcal{G}(n_U, m_U) \). To this end, let \( \mathcal{R}(n) \) be the subclass of \( \mathcal{G}(n_U, m_U) \) consisting of all possible realisations of \( \mathcal{H}_n \), i.e. the set of graphs \( R \) such that

\[ R \cup H_n \in \mathcal{A} \quad \text{and} \quad L_1(R \cup H_n) = H_n. \]

We claim that each graph \( R \in \mathcal{G}(n_U, m_U) \) having no complex components and satisfying \( L_1(R) < v(H_n) \) is in \( \mathcal{R}(n) \). Indeed such a graph \( R \) satisfies \( L_1(R \cup H_n) = H_n \). Moreover, we have \( K(R \cup H_n) = K(H_n) \). Thus, by the stability property (K1) of kernel-stable classes (cf. Definition 4.1) we get \( R \cup H_n \in \mathcal{P} \) and therefore \( R \cup H_n \in \mathcal{A} \). This implies \( R \in \mathcal{R}(n) \) due to (g). Next, we will show that \( |\mathcal{R}(n)| \) is ‘large’ in the sense that \( |\mathcal{R}(n)| \geq |\mathcal{G}(n_U, m_U)| \) is bounded away from 0. To this end, we use (7) to obtain

\[ m_U = m - e(H_n) \leq m - 2s + Mn^{2/3} = \frac{n}{2} - s + Mn^{2/3} \leq \frac{n}{2} - \frac{v(H_n)}{2} + \frac{3Mn^{2/3}}{2} = \frac{n_U}{2} + \frac{3Mn^{2/3}}{2}. \]

Using (7) and the fact that \( s = o(n) \) we get \( n_U = (1 + o(1))n \). Combining this with (10) yields that for \( n \) large enough

\[ m_U \leq \frac{n_U}{2} + 2Mn^{2/3}. \]

Together with Theorem 4.9(a) this implies

\[ \liminf_{n \to \infty} \mathbb{P} \left[ R_n \text{ has no complex component} \right] > 0. \]

As \( v(H_n) = \Theta(s) \), we obtain by Theorem 4.9(b) that whp

\[ v(L_1(R_n)) < v(H_n). \]

Combining (12) and (13) with the claim shown above yields

\[ \liminf_{n \to \infty} \mathbb{P} \left[ R_n \in \mathcal{R}(n) \right] > 0. \]
Similarly as in (10) we use (7) to get
\[
  m_U = m - e(H_n) \geq m - 2s - Mn^{2/3} = \frac{n}{2} - v(H_n) - 3Mn^{2/3} = \frac{n_U}{2} - 3Mn^{2/3},
\]
which yields \( m_U \geq n_U/2 - 2Mn^{2/3} \) for large \( n \). Combining this with (11) we obtain \( m_U = n_U/2 + O(n^{2/3}) \). Hence, we get by Theorem 4.8(b) that \( c(R_n) = \Theta_p(n^{2/3}) = \Theta_p(n^{1/3}). \) By (13) each property that holds whp in \( R_n \) is also true whp in \( R_{\theta} \in B(R(n)). \) Thus, we have \( c(R_n) = \Theta_p(n^{1/3}) \). From the definition of \( R_n \) we get \( c(R(\tilde{A} | H)) = \Theta_p(n^{1/3}). \) Hence, by Lemma 4.2 we have \( c(R(\tilde{A})) = \Theta_p(n^{1/3}). \)

Finally, using (9) and observing that the choice of \( \delta > 0 \) was arbitrary we have
\[
  c(R(P)) = \Theta_p(n^{1/3}),
\]
which completes the proof. □

9.2. **Proof of Theorem 4.4** The statement follows by combining Lemma 4.4(c) and Theorem 4.5 □

9.3. **Proof of Corollary 4.6** The result follows directly from Lemma 4.4 and Theorem 4.5 □

9.4. **Proof of Corollary 4.7** The results in the regimes \( s^3n^{-2} \rightarrow -\infty \) and \( s = O(n^{2/3}) \) follow analogously as for kernel-stable classes (see Section 9.1). In [22, Theorem 4] it is proven that in the case \( s^3n^{-2} \rightarrow 0 \) whp a random outerplanar graph is a cactus graph. Thus, the statements in that regime follow directly from Corollary 4.6 □

9.5. **Proof of Theorem 4.12** We start by considering the blocks of the kernel \( K(P) \). By Lemma 7.1 we have that \( K(P) \) ‘behaves’ like a random cubic multigraph chosen from \( \mathcal{A} \). Furthermore, by Theorem 4.8(b) and (d) we know that whp \( v(K(P)) = \Theta_p(s^{n-2/3}). \) Combining that with Lemma 8.10 implies
\[
  b_1(K(P)) = \Theta_p(s^{n-2/3}).
\]

Similarly, by Lemma 8.11 we have that for each \( i \geq 2 \)
\[
  b_i(K(P)) = \Theta_p(v(K(P))\{1/(a-2)\} = \Theta_p(s^{1/(a-2)} n^{-2/(3(a-2))}).
\]

Next, we determine the orders of the blocks in the core \( C(P) \). To this end, we will use Lemmas 3.2 and 8.8 and we fix \( i \geq 2 \). Let \( \mathcal{A}(n) \) be the subclass of \( \mathcal{P}(n, m) \) consisting of those graphs \( H \) satisfying the following properties
\[
  v(C(H)) = \Theta(s^{n-1/3}),
\]
\[
  e(K(H)) = \Theta(s^{n-2/3}),
\]
\[
  b_1(K(H)) = \Theta(s^{n-2/3}),
\]
\[
  b_i(K(H)) = \Theta(s^{1/(a-2)} n^{-2/(3(a-2))}),
\]
and
\( K(H) \) is cubic.

Due to Theorem 4.8(a) and (15), and (16) we can choose the implicit hidden constants in the above equations such that \( P \{ P \in \mathcal{A}(n) > 1 - \delta, \) for a fixed constant \( \delta > 0 \). Let \( \mathcal{A} := \bigcup_{n \in N} \mathcal{A}(n), \tilde{A} = \tilde{A}(n) \in R \mathcal{A}(n) \), and define the function \( \Phi : \mathcal{A} \rightarrow \mathcal{A} \times \mathcal{N} \) which assigns each graph \( H \in \mathcal{A} \) to the pair of its kernel \( K(H) \) and subdivision number \( S(H), i.e.
\[
  \Phi(H) := (K(H), S(H)).
\]

Let \( s = (K_n, k_n)_{n \in N} \) be a sequence feasible for \( \mathcal{A}, \Phi \) and \( C_n := C(K_n, k_n) \) as in Definition 6.1 i.e. a graph chosen uniformly at random from all cores with kernel \( K_n \) and subdivision number \( k_n \). By the definition of \( \mathcal{A}(n) \) we have \( k_n = \omega(e(K_n)^2) \). Thus, combining Corollary 6.3 and Lemma 8.8 yields
\[
  b_1(C_n) = \Theta\left(\frac{k_n b_1(K_n)}{v(K_n)}\right) = \Theta\left(s^{n-1/3}\right)
\]
and
\[
  b_i(C_n) = \Theta\left(\frac{k_n b_1(K_n)}{v(K_n)}\right) = \Theta\left(s^{1/(a-2)} n^{-2/(3(a-2))}\right).
\]
Due to Lemma[6.2] we have that \(C(A|s)\) is distributed like \(G_n\). Hence, by Lemma[3.2] we obtain that whp

\[
b_1(C(A)) = \Theta(sn^{-1/3}) \quad \text{and} \quad b_1(C(A)) = \Theta\left(s^{1/(a-2)}n^{\frac{1}{3a-2}}\right).
\]

As \(\delta > 0\) was arbitrary and \(P [P \in \mathcal{A}(n)] > 1 - \delta\), this implies

\[
b_1(C(P)) = \Theta_p(sn^{-1/3}) \quad \text{and} \quad b_1(C(P)) = \Theta_p\left(s^{1/(a-2)}n^{\frac{1}{3a-2}}\right).
\]

(17)

We note that each block outside the core \(C(P)\) is a cycle. Due to Theorem[4.5], the length of such a cycle is of order \(O_P(n^{1/3})\). This together with (17) and the observations that \(n^{1/3} = o\left(s^{1/(a-2)}n^{\frac{1}{3a-2}}\right)\) implies statements (a) and (b).

For statement (c) we first observe that the number of blocks in \(C(L_1(P))\) is at most \(\nu(K(L_1(P)))\).

Thus, by Theorem[4.3(a)] we have that whp \(C(L_1(P))\), and therefore also \(L_1(P)\), has \(\Omega(sn^{-2/3})\) many blocks. On the other hand, whp \(K(L_1(P))\) has \(\Theta(sn^{-2/3})\) many loops due to Corollary[7.7]. All these loops ‘translate’ to different blocks in the core \(C(L_1(P))\). Thus, whp \(L_1(P)\) has \(\Omega(sn^{-2/3})\) many blocks. Summing up, whp \(L_1(P)\) contains \(\Theta(sn^{-2/3})\) many blocks.

9.6. **Proof of Theorem[1.5]**. The statement follows by combining Lemma[5.2] and Theorem[3.12].

10. **Proofs of auxiliary results**

10.1. **Proof of Lemma[3.2]**. For each \(n \in \mathbb{N}\) let \(s_n^* \in \mathcal{F}\) be such that \(P[A(n) \in \mathcal{P} | \Phi(A(n)) = s]\) is minimised for \(s = s_n^*\) (among all \(s\) for which there exists a graph \(H \in \mathcal{A}(n)\) with \(\Phi(H) = s\)). Note that the sequence \(s^* = (s_n^*)_{n \in \mathbb{N}}\) is feasible for \((\mathcal{A}, \Phi)\) and therefore we obtain

\[
P[A(n) \in \mathcal{P}] = \sum_{s \in \mathcal{F}} P[\Phi(A(n)) = s] \cdot P[A(n) \in \mathcal{P} | \Phi(A(n)) = s] 
\geq \sum_{s \in \mathcal{F}} P[\Phi(A(n)) = s] \cdot P[A(n) \in \mathcal{P} | \Phi(A(n)) = s^*_n] 
= P[A(s^*)^*(n) \in \mathcal{P}] 
= 1 - o(1),
\]

where the sums are taken over all \(s \in \mathcal{F}\) for which there exists some \(H \in \mathcal{A}(n)\) with \(\Phi(H) = s\).

10.2. **Proof of Lemma[4.2]**. The assertion follows along the lines of the proof of Lemma 2 in [21].

10.3. **Proof of Lemma[4.3]**. We use the properties of \(\mathcal{P}\) from Definition[4.1]. By (K2) there exists some \(H_1 \in \mathcal{P}\). Due to (K1) this yields \(H \cup H_1 \in \mathcal{P}\), as \(K(H \cup H_1) = K(H)\). Finally, (P1) implies \(H \in \mathcal{P}\).

10.4. **Proof of Lemma[6.2]**. The assertion is equivalent to the statement that \(P[C(P | (K,k)) = H]\) is independent of the choice of \(H \in \mathcal{H}(K,k)\). Hence, it suffices to prove that for each \(H \in \mathcal{H}(K,k)\) the set \(\{F \in \mathcal{P}(n,m) \mid C(F) = H, K(F) = K, S(F) = k\}\) has the same number of elements. We observe that \(\{F \in \mathcal{P}(n,m) \mid C(F) = H, K(F) = K, S(F) = k\} = \{F \in \mathcal{P}(n,m) \mid C(F) = H\}\) and each graph \(F \in \mathcal{P}(n,m)\) having \(H\) as its core can be constructed as follows. First we replace each vertex in \(H\) by a rooted tree and then we attach a graph without complex components such that we obtain a graph on \(n\) vertices and \(m\) edges (see [23, Section 3] for details). The number of different possibilities of that construction depends only on \(n, m, v(H),\) and \(e(H)\). We have \(v(H) = v(K) + k\) and \(e(H) = e(K) + k\), i.e. \(v(H)\) and \(e(H)\) are independent of the choice of \(H \in \mathcal{H}(K,k)\). Hence, the size of the set \(\{F \in \mathcal{P}(n,m) \mid C(F) = H\}\) is the same for all \(H \in \mathcal{H}(K,k)\), which implies the statement.
10.5. **Proof of Lemma 6.4.** Let \( e_1, \ldots, e_N \) be the edges of \( K \) and we denote by \( X_i \) the number of vertices that are placed on edge \( e_i \) when we subdivide \( K \) to obtain \( \tilde{C} \). To prove \([a]\) we observe that

\[
\mathbb{P} [\tilde{C} \text{ is simple}] \geq \mathbb{P} [\tilde{C} \text{ is 2-simple}] = \mathbb{P} \left[ \bigwedge_{i=1}^{N} (X_i \geq 2) \right] \geq 1 - \sum_{i=1}^{N} \mathbb{P} [X_i \leq 1] \geq 1 - \frac{2N^2}{k},
\]

where the last inequality follows from Proposition 5.2.

In order to prove \([b]\) it suffices to show that \( \mathbb{P} [\tilde{C} = H] \) is independent of the choice of \( H \in \mathcal{C}(K, k) \). To that end, we count the number of ways our random process ends up with \( \tilde{C} = H \). We observe that there is a unique sequence \( (G_0, \ldots, G_k) \) that leads to \( G_k = H \). Thus, in each step \( i \) there is a unique unordered pair of vertices \( \{u_i, v_i\} \) such that subdividing an edge between \( u_i \) and \( v_i \) in \( G_{i-1} \) leads to \( G_i \). We denote by \( q_i \) the number of edges in \( G_{i-1} \) between \( u_i \) and \( v_i \). Then, there are \( \prod_{i=1}^{k} q_i \) many ways of creating \( H \). We note that the only way a multiple edge can be created during the process is by subdividing a loop and that all loops and multiple edges are destroyed in the end. Thus, we obtain \( \prod_{i=1}^{k} q_i = \frac{1}{w(K)} \), where \( w(K) \) is defined as in \((4)\). This shows \([b]\) since \( w(K) \) is independent of the choice of \( H \in \mathcal{C}(K, k) \).

\[ \square \]

10.6. **Proof of Corollary 6.5.** We observe that by Lemma 6.4 we have that whp \( \tilde{C} \) is simple. For each graph property \( \mathcal{D} \) we obtain by Lemma 6.4 \([b]\)

\[
\mathbb{P} [\tilde{C} \in \mathcal{D}] = \mathbb{P} [\tilde{C} \text{ is simple}] \mathbb{P} [\tilde{C} \in \mathcal{D} | \text{\tilde{C} is simple}] + \mathbb{P} [\tilde{C} \text{ is not simple}] \mathbb{P} [\tilde{C} \in \mathcal{D} | \text{\tilde{C} is not simple}] \\
= (1 + o(1)) \mathbb{P} [\tilde{C} \in \mathcal{D}] + o(1).
\]

This implies that whp \( C \in \mathcal{D} \) if and only if whp \( \tilde{C} \in \mathcal{D} \).

\[ \square \]

10.7. **Proof of Corollary 6.6.** We will use Lemma 6.2. Let \( \mathcal{D} \) be the graph property of being 2-simple. In addition, let \( \mathcal{A}(n) \) be the subclass of \( \mathcal{P}(n, m) \) consisting of all graphs \( H \) with \( v(C(H)) = \Theta \left( sn^{-1/3} \right) \) and \( e(C(H)) = \Theta \left( sn^{-2/3} \right) \). Due to Theorem 4.8 \([a]\) and \([c]\) we have that whp \( v(C(P)) = \Theta \left( sn^{-1/3} \right) \). In addition, by Theorem 4.8 \([b]\), \([d]\) and \([e]\) we have that whp

\[
e(C(P)) = 3/2 \cdot v(C(P)) = \Theta \left( sn^{-2/3} \right).
\]

Hence, we obtain that whp \( P \in \mathcal{A}(n) \). Let \( \mathcal{A} := \bigcup_{n \in \mathbb{N}} \mathcal{A}(n) \) and define the function \( \Phi \) for a graph \( H \in \mathcal{A} \) by \( \Phi(H) := (K(H), S(H)) \). Let \( s = (K_n, k_n) \) be a sequence feasible for \( \langle \mathcal{A}, \Phi \rangle \) and let \( C(K_n, k_n) \) and \( \tilde{C}(K_n, k_n) \) be as in Definition 6.1. By definition of \( \mathcal{A}(n) \) we have \( k_n = \omega \left( e(K_n)^2 \right) \). Thus, by Lemma 6.4 \([a]\) we have that whp \( \tilde{C}(K_n, k_n) \) is 2-simple. By Corollary 6.5 this is also true for \( C(K_n, k_n) \). Let \( A = A(n) \in_R \mathcal{A}(n) \). We note that \( C(A|s) \) is distributed like \( C(K_n, k_n) \) due to Lemma 6.2. Thus, by Lemma 6.2 we have that whp \( C(A) \) is 2-simple. Since whp \( P \in \mathcal{A}(n) \), it is also true that whp \( C(P) \) is 2-simple. Finally, the statement follows, because \( P \) is 2-simple if and only if \( C(P) \) is.

\[ \square \]

10.8. **Proof of Lemma 6.7.** Let \( e_1, \ldots, e_N \) be the edges of \( K \) and \( X_i \) the number of vertices that are placed on edge \( e_i \) if we subdivide \( K \) to obtain \( \tilde{C} \). Without loss of generality we may assume that \( e_1, \ldots, e_3 \) are the loops of \( K \). Then the upper bounds on \( g(\tilde{C}) \) follow by Theorem 5.1 \([a]\) and inequality \( \[3\] \). For the lower bound on \( g(\tilde{C}) \) we use Theorem 5.3 \([a]\) and \([4]\). The ‘in particular’ statements follow immediately by combining the lower and upper bounds on \( g(\tilde{C}) \). Finally, we note that \([b]\) follows by Theorem 5.3 \([b]\) and \([5]\).

\[ \square \]

10.9. **Proof of Lemma 6.8.** We note that the blocks of \( \tilde{C} \) are the blocks of \( K \) with additional vertices placed on the edges of \( K \). For \( j \in \mathbb{N} \) let \( X_j \) be the total number of vertices that are placed on edges of the \( j \)-th largest block \( B_j(K) \) of \( K \). The minimum degree of each block is at least two and together with the fact that \( K \) is cubic, this implies

\[
b_j(K) \leq e(B_j(k)) \leq 3/2b_j(K).
\]

By Theorem 5.3 we have

\[
\mathbb{E}[X_j] = \frac{k e(B_j(k))}{e(K)} \quad \text{and} \quad \forall X_j = O \left( 1 \right) \frac{k^2 e(B_j(k))}{e(K)^2}.
\]
Thus, by Chebyshev’s inequality, (13), and (19) we obtain
\[
\mathbb{P} \left[ X_j \leq \frac{kb_j(K)}{3v(K)} \right] \leq \mathbb{P} \left[ X_j \leq \frac{\mathbb{E}[X_j]}{2} \right] \leq \frac{4\mathbb{V}[X_j]}{\mathbb{E}[X_j]^2} = O(1) \frac{k^2 e(B_j(k))}{e(K)^2} = O(1) b_j(K)^{-1}.
\]

Hence, whp for all \( j \leq i \) we have \( X_j \geq \frac{kb_j(K)}{3v(K)} \geq \frac{kb_j(K)}{3v(K)} \), which shows whp \( b_i(\tilde{C}) = \Omega \left( \frac{kb_j(K)}{v(K)} \right) \). By again applying Chebyshev’s inequality, (13), and (19), we have uniformly over all \( j \geq i \)
\[
\mathbb{P} \left[ X_j \geq \frac{2k b_j(K)}{v(K)} \right] \leq \mathbb{P} \left[ X_j \geq \frac{kb_j(K)}{v(K)} + \frac{v(K)^2 \mathbb{V}[X_j]}{k^2 b_j(K)^2} \right] = O(1) b_j(K)^{-2}.
\]

Thus, by a standard union bound we obtain
\[
\mathbb{P} \left[ \exists j \geq i : X_j \geq \frac{2k b_j(K)}{v(K)} \right] = O(1) b_j(K)^{-2} \sum_{j \geq i} b_j(K) = O(1) b_j(K)^{-2} v(K) = o(1).
\]

That yields whp \( b_i(\tilde{C}) = O \left( \frac{kb_j(K)}{v(K)} \right) \), which completes the proof.

10.10. **Proof of Lemma 7.6.** We will use Lemma 3.2. To this end, let \( c_1, c_2 > 0 \) and \( \mathcal{A}(n) \) be the subclass of \( \mathcal{P}(n, m) \) consisting of all 2-simple graphs \( H \) with a cubic kernel \( K(H) \) and satisfying \( c_1 s n^{-2/3} \leq v(K(H)) / 2 \leq c_2 s n^{-2/3} \). Due to Corollary 6.6 we know that whp \( P \) is 2-simple. Moreover, by Theorem 4.8(b), (d), and (e) we have that whp \( K \) is cubic and \( v(K(P)) = O(s^{-2/3}) \). Thus, we can choose \( c_1, c_2 \) such that whp \( P \in \mathcal{A}(n) \). Let \( \mathcal{A} := \bigcup_{n \in \mathbb{N}} \mathcal{A}(n) \) and define the function \( \Phi \) for a graph \( H \in \mathcal{A} \) by
\[
\Phi(H) := v(K(H)) / 2.
\]

Let \( s = (\ell_n)_{n \in \mathbb{N}} \) be a sequence feasible for \( (\mathcal{A}, \Phi) \) and \( A = (A(n))_{n \in \mathbb{R}} \mathcal{A}(n) \). We note that for a fixed kernel \( K \in \mathcal{K}(2\ell, 3\ell) \) and a fixed \( k \in \mathbb{N} \) there are \( w(K) \left( \frac{k - 3\ell}{3\ell - 1} \right)! \) many ways to construct a 2-simple core with kernel \( K \) and subdivision number \( k \). Thus, \( K(A | s) \) is distributed like \( K(2\ell_n, 3\ell_n) \) by Lemma 6.2.

Hence, we obtain
\[
\mathbb{P} \left[ g_1(c_1 s n^{-2/3}) \leq F(K(A | s)) \leq g_2 \left( c_2 s n^{-2/3} \right) \right] \geq \mathbb{P} \left[ g_1(\ell_n) \leq F(K(2\ell_n, 3\ell_n)) \leq g_2(\ell_n) \right] = 1 - o(1),
\]
as \( \ell \to c_1 s n^{-2/3} \to \infty \). Thus, the statement follows by Lemma 5.2.

10.11. **Proof of Lemma 7.8.** We recall that \( \lambda(K) \) is the number of loops in \( K = K(2n, 3n) \) and observe that \( \lambda(K) = \sum_{w \in [2n]} Z_w \), where \( Z_w \) is the indicator random variable for the event that there is a loop at vertex \( w \). In order to apply the second moment method, we estimate the probabilities \( \mathbb{P} [Z_w = 1] \) and \( \mathbb{P} [Z_u = Z_w = 1] \) for \( u \neq w \). To this end, we will use loop insertions (cf. Definition 7.2). We fix a vertex \( w \in [2n] \) and consider all multigraphs in \( \mathcal{K}(2n, 3n) \) with a loop at \( w \). We note that in all these multigraphs \( w \) has precisely one neighbour \( x \neq w \). We distinguish two cases depending on whether there is a loop at \( x \) or not. Due to Remark 7.5 we can enumerate all these multigraphs with no loop at \( x \) as follows:

- choose a vertex \( x \in [2n] \setminus \{ w \} \);
- choose \( H \in \mathcal{K}(2(n - 1), 3(n - 1)) \) and relabel the vertices with the labels \( [2n] \setminus \{ w, x \} \);
- choose an edge \( e \in E(H) \) and perform a loop insertion at edge \( e \) with vertex pair \( (w, x) \).

For simplicity we set \( a_n := | \mathcal{K}(2n, 3n) | \). If \( x \) and \( H \) are fixed, the total weight of all multigraphs that can be built by choosing an edge \( e \in E(H) \) and performing a loop insertion at edge \( e \) with vertex pair \( (w, x) \) is \( w(H) 3(n - 1) / 2 \) by Remark 7.5. Hence, the total weight of all multigraphs which can be obtained by the above construction is
\[
(2n - 1) \cdot a_n \cdot 3(n - 1) / 2.
\]
(20) On the other hand, if there is a loop at \( x \), the vertices \( w \) and \( x \) form a component with weight 1/4. Thus, all such multigraphs can be enumerated as follows:
• choose a vertex \( x \in [2n] \setminus \{w\} \);
• choose \( H \in \mathcal{K}(2(n-1), 3(n-1)) \) and relabel the vertices with the labels \([2n] \setminus \{w, x\}\);
• add the component \( C \) with \( V(C) = \{w, x\} \) and \( E(C) = \{ww, xx, wx\} \) to \( H \).

The total weight of all multigraphs constructed in that way is

\[
(2n-1)a_{n-1}/4. \tag{21}
\]

We observe that each multigraph can be obtained at most once by one of the two above constructions. Thus, by combining (20) and (21) we obtain

\[
\mathbb{P}[Z_v = 1] = \frac{(2n-1)a_{n-1}3(n-1)/2 + (2n-1)a_{n-1}/4}{a_n} = (1 + o(1))3n^2a_{n-1}/a_n. \tag{22}
\]

By using (K2) in Definition 4.1 we obtain

\[
\frac{a_{n-1}}{a_n} = \frac{(1 + o(1))}{4n^2 \gamma}. \tag{23}
\]

Plugging in (23) in (22) yields

\[
\mathbb{P}[Z_v = 1] = (1 + o(1))\frac{3}{4 \gamma}. \tag{24}
\]

Similarly, we estimate the number of multigraphs with loops at \( u \) and \( w \). We observe that all such multigraphs in which \( u \) and \( w \) are not adjacent can be construct as follows:

• choose a vertex \( x \in [2n] \setminus \{u, w\} \);
• choose \( H \in \mathcal{K}(2(n-1), 3(n-1)) \) and relabel the vertices with the labels \([2n] \setminus \{w, x\}\) such that we obtain a multigraph with a loop at \( u \);
• choose an edge \( e \neq uu \) in \( H \) and perform a loop insertion at edge \( e \) with vertex pair \((w, x)\).

In the above construction we have \((2n-2)\) possible choices for \( x \) and by (24) the weight of all multigraphs that can be chosen for \( H \) is

\[
(1 + o(1))\frac{3}{4 \gamma}a_{n-1}.
\]

For fixed \( x \) and \( H \), the total weight of all multigraphs obtained by choosing an edge \( e \) and performing the loop insertion is \( w(H)(3n-4)/2 \) due to Remark 7.5. On the other hand, if \( u \) and \( w \) are adjacent and there are loops at \( u \) and \( w \), then \( u \) and \( w \) form an own component with weight \( 1/4 \). Combining these two cases we get

\[
\mathbb{P}[Z_u = Z_w = 1] = \left(\frac{3}{4 \gamma}a_{n-1}\frac{3n-4}{2} + \frac{1}{4}a_{n-1}\right) / a_n = (1 + o(1))\frac{9n^2a_{n-1}}{4 \gamma a_n}.\]

Finally, from this together with (23) we obtain

\[
\mathbb{P}[Z_u = Z_w = 1] = (1 + o(1))\left(\frac{3}{4 \gamma}\right)^2 = (1 + o(1))\mathbb{P}[Z_u = 1]\mathbb{P}[Z_w = 1].
\]

Hence, the statement follows by the second moment method. \(\square\)

10.12. **Proof of Corollary 7.7.** The first statement follows directly from Lemmas 7.1 and 7.6. For the second statement we note that \( \lambda(K(R)) = o(K(R)) = O_p(1) \) by Theorem 4.8(d). Thus, by combining that with the first statement we obtain whp

\[
\lambda(K(L)) = \lambda(K(P)) - \lambda(K(R)) = \Theta(sn^{-2/3}).
\]

\(\square\)

In order to prove Lemmas 8.4 and 8.5, we need the following two results, whose proofs are elementary and can be found in Appendix B.
Claim 1. Let $a > 2$ and $f$ be a function such that $f(n) = o(n)$ and $f(n) = o(n)$. We set $I(n) := \{(j, k) \in \mathbb{N}^2 \mid j + k = n - 1, j, k \geq f(n)\}$ and $m_i := \|\mathcal{M}(2i, 3i)\|$ for $i \in \mathbb{N}$. Then we have
\[
\frac{1}{m_i} \sum_{(j, k) \in I(n)} \binom{2n - 2}{2j} m_j m_k j k = (1 + o(1)) \frac{c}{2^{\gamma(a - 2)}} \cdot \frac{1}{f(n)^{a - 2 n}}.
\]

Claim 2. Let $a > 3$ and $f$ be a function with $f(n) = o(n)$, $f(n) = o(n)$, and $f(n) = o(n^{1/(a-2)})$. We set $I(n) := \{(j, k, l) \in \mathbb{N}^3 \mid j + k + l = n - 2, j, k \geq f(n)\}$ and $m_i := \|\mathcal{M}(2i, 3i)\|$ for $i \in \mathbb{N}$. Then we have
\[
\frac{324}{m_n} \sum_{(j, k, l) \in I(n)} \binom{2n - 4}{2j, 2k, 2l} m_j m_k m_l j k l = (1 + o(1)) \left( \frac{9c}{2^{\gamma(a - 2)}} \cdot \frac{1}{f(n)^{a - 2 n}} \right)^2.
\]

10.13. Proof of Lemma 8.4. Each multigraph $H \in \mathcal{M}(2n, 3n)$ with $\beta_H(w, x) \geq 2f(n) + 1$ can be constructed in the following way:
• choose $j, k \in \mathbb{N}$ such that $j + k = n - 1$ and $j, k \geq f(n)$;
• choose $2j$ labels from $[2n] \setminus \{w, x\}$ and denote them by $L$;
• choose $H_1 \in \mathcal{M}(2j, 3j)$ and relabel the vertices with $L$;
• choose $H_2 \in \mathcal{M}(2k, 3k)$ and relabel the vertices with $[2n] \setminus (L \cup \{w, x\})$;
• choose edges $e_1 \in E(H_1), e_2 \in E(H_2)$ and perform a bridge insertion (cf. Definition 8.2) at edges $e_1$ and $e_2$ with vertices $w$ and $x$.

By Remark 7.4 this construction gives multigraphs with a total weight of
\[
\sum_{(j, k) \in I(n)} \binom{2n - 2}{2j} m_j m_k \cdot (3j) \cdot (3k),
\]
where $I(n) := \{(j, k) \in \mathbb{N}^2 \mid j + k = n - 1, j, k \geq f(n)\}$ and $m_i := \|\mathcal{M}(2i, 3i)\|$ for $i \in \mathbb{N}$. Hence, using Claim 1 yields
\[
P[\beta_M(w, x) \geq 2f(n) + 1] = \frac{1}{m_n} \sum_{(j, k) \in I(n)} \binom{2n - 2}{2j} m_j m_k \cdot (3j) \cdot (3k) = (1 + o(1)) \left( \frac{9c}{2^{\gamma(a - 2)}} \cdot \frac{1}{f(n)^{a - 2 n}} \right)^2
\]
as desired.

10.14. Proof of Lemma 8.5. We denote by $E_i$ the event that $\beta(w_{2i-1} w_{2i}) \geq 2f(n) + 1$, where $i \in \{1, 2\}$. In addition, let $E_3$ be the event that there is an edge in $M$ with one endpoint in $\{w_1, w_2\}$ and the other in $\{w_3, w_4\}$. We start by estimating the probability $P[E_1 \land E_2 \land E_3]$. We observe that if $E_3$ is true, then at least one of the four events $w_1 w_3 \in E(M), w_1 w_4 \in E(M), w_2 w_3 \in E(M), w_2 w_4 \in E(M)$ is true. Thus, by symmetry reasons we obtain
\[
P[E_1 \land E_2 \land E_3] \leq 4P[E_1 \land E_2 \land (w_2 w_3 \in E(M))].
\]

Next, we note that the event $E_2$ implies $w_3 w_4 \in E(M)$. Using that in (26) yields
\[
P[E_1 \land E_2 \land E_3] \leq 4P[E_1 \land (w_2 w_3, w_3 w_4 \in E(M))]
= 4P[E_1] \cdot P[w_2 w_3, w_3 w_4 \in E(M) \mid E_1].
\]

Using Lemma 8.4 for an estimate of $P[E_1]$ and the fact that $P[w_2 w_3, w_3 w_4 \in E(M) \mid E_1] = \Theta(n^{-2})$ we get
\[
P[E_1 \land E_2 \land E_3] = O(1) \cdot \frac{1}{f(n)^{a - 2 n}} \cdot \frac{1}{n^2} = o(1) \cdot \frac{1}{f(n)^{a - 4 n^2}}.
\]

Next, we estimate the probability $P[E_1 \land E_2 \land \neg E_3]$, where $\neg E_3$ is the event that $E_3$ is not true. We observe that we can enumerate all multigraphs $H \in \mathcal{M}(2n, 3n)$ satisfying $\beta_H(w_1 w_2), \beta_H(w_3 w_4) \geq 2f(n) + 1$, and $w_1 w_3, w_1 w_4, w_2 w_3, w_2 w_4 \in E(H)$ by the following construction:
• choose $j, k, l \in \mathbb{N}$ with $j + k + l = n - 2$ and $j, k \geq f(n)$;
• choose a partition $J \cup K \cup L = [2n] \setminus \{w_1, \ldots, w_4\}$ with $|J| = 2j, |K| = 2k, |L| = 2l$;
• choose $M_1 \in \mathcal{M}(2j, 3j), M_2 \in \mathcal{M}(2k, 3k), M_3 \in \mathcal{M}(2l, 3l)$ and relabel these multigraphs with the labels $J, K$ and $L$, respectively;
• choose $i \in \{1, 2\}$ and $i' \in \{3, 4\}$;
Thus, we obtain (6) and (25) we have that uniformly over all 1 \leq j \leq n, \delta.

Proof of Lemma 8.9. This implies that each vertex of \( \tilde{H} \) corresponds to either a vertex or a block in \( H \). We note that the edges of \( \tilde{H} \) are oriented away from \( v \), and thus, has a vertex \( v \) with no incoming edge. If \( v \) is a vertex in \( H \), then \( v \) has three neighbours \( w_1, w_2, w_3 \) in \( H \) and the edges \( vw_1, vw_2, vw_3 \) are all bridges and oriented away from \( v \). Hence, we obtain

\[ 2n = \nu(H) = 1 + \beta_H(vw_1) + \beta_H(vw_2) + \beta_H(vw_3) \leq 1 + 3 \cdot (2n - 2)/3 = 2n - 1, \]

where we used in the last inequality the assumption that each bridge number is at most \( (2n - 2)/3 \).

10.16. Proof of Lemma 8.8 We construct an auxiliary directed graph \( \tilde{H} \) as follows: First we orient each bridge \( wx \) in \( H \) from \( w \) to \( x \) such that \( \nu(H_w) > \nu(H_x) \), where \( H_w \) and \( H_x \) are the components of \( H - wx \) containing \( w \) and \( x \), respectively. Then we contract each block to a single vertex and denote by \( \tilde{H} \) the obtained directed graph. We note that the edges of \( \tilde{H} \) are exactly the bridges of \( H \) and that each vertex of \( \tilde{H} \) corresponds to either a vertex or a block in \( H \). Furthermore, \( \tilde{H} \) contains no cycle and therefore, has a vertex \( v \) with no incoming edge. If \( v \) is a vertex in \( H \), then \( v \) has three neighbours \( w_1, w_2, w_3 \) in \( H \) and the edges \( vw_1, vw_2, vw_3 \) are all bridges and oriented away from \( v \). Hence, we obtain

\[ 2n = \nu(H) = 1 + \beta_H(vw_1) + \beta_H(vw_2) + \beta_H(vw_3) \leq 1 + 3 \cdot (2n - 2)/3 = 2n - 1, \]

where we used in the last inequality the assumption that each bridge number is at most \( (2n - 2)/3 \).

This implies that \( v \) corresponds to a block in \( H \). As all bridges are oriented away from \( v \), this block is dominant.

10.17. Proof of Lemma 8.9 We note that \( \mathbb{P} \left[ \beta(wx) = 1 \right] \leq \mathbb{P} \left[ wx \in E(M) \right] = \Theta(1/n) \). In addition, by (6) and (25) we have that uniformly over all \( 1 \leq j \leq (n - 1)/2 \)

\[ \mathbb{P} \left[ \beta(wx) = 2j + 1 \right] = \Theta(1) j^{-a+1} / n. \]

Thus, we obtain

\[ \mathbb{E} \left[ \beta_M(wx)^\mu \right] = \mathbb{P} \left[ \beta(wx) = 1 \right] + \sum_{j=1}^{(n-1)/2} (2j + 1)^{\mu} \mathbb{P} \left[ \beta(wx) = 2j + 1 \right] \]

\[ = O(1/n) + \Theta(1/n) \sum_{j=1}^{(n-1)/2} j^{-a+\mu+1} = \Theta(1/n), \]

since \( -a + \mu + 1 < -1 \). This completes the proof. 

\[ \square \]
10.18. **Proof of Lemma 8.10.** Let \( h(n) = o(1) \) be a function and \( 1 < \mu \leq \alpha - 2 \) a constant. Using Lemma 8.9 we have that whp \( M \) satisfies \( \sum_{w \neq x} \beta_M(x)^2 \leq n h(n)^{\alpha - \mu} \). Moreover, due to Lemma 8.4 whp all pairs of vertices \( u \neq v \in [2n] \) satisfy \( \beta_M(u,v) \leq (2n-2)/3 \). Hence, \( M \) contains whp a dominant block by Lemma 8.8. Now it suffices to show that each \( H \in \mathcal{M}(2n,3n) \) with a dominant block and satisfying \( \sum_{w \neq x} \beta_H(x)^2 \leq n h(n)^{\alpha - \mu} \) has a block with at least \( n h(n) \) many vertices. To that end, let \( B \) be a dominant block in \( H \) and \( e_1, \ldots, e_r \) the bridges that share a vertex with \( B \). For \( i \in [r] \) we denote by \( C_i \) the component of \( H - e_i \) not containing \( B \). We note that \( v(B) + \sum_{i=1}^r v(C_i) = 2n \) and \( v(C_i) = \beta_H(e_i) \). Hence, we get \( \sum_{i=1}^r v(C_i) \leq n h(n)^{\alpha - \mu} \). Now we assume that \( v(B) < n h(n) \). Then by using Jensen's inequality for the convex function \( x \mapsto x^\alpha \) and the simple fact \( r \leq v(B) \), we obtain that for \( n \) large enough

\[
n h(n)^{\alpha - \mu} \geq \sum_{i=1}^r v(C_i)^\mu \geq \left( \sum_{i=1}^r v(C_i) \right)^\mu / r^{\alpha - \mu} \\
\geq (2n - n h(n))^\mu / (n h(n))^{\alpha - \mu} > n h(n)^{\alpha - \mu - 1},
\]

a contradiction. Hence, we obtain \( v(B) \geq n h(n) \), which completes the proof. \( \square \)

**Remark 10.1.** In the proof of Lemma 8.10 we actually showed the following stronger statement. If there are \( \Omega_p(n) \) many vertices outside the largest block of \( M \), then \( M \) contains a block which shares a vertex with \( \Omega_p(n) \) many bridges. We note that in most of our applications this assumption is satisfied, for example, when \( M \) contains linearly many loops (see Lemma 7.6).

10.19. **Proof of Lemma 8.11.** We note that for every pair of two different blocks \( B \neq B' \) there is a bridge \( e \) such that \( B \) and \( B' \) lie in different components of \( M - e \). Hence, we obtain by Lemma 8.4 that \( b_1(M) \leq \max_{w \neq x} \beta_M(x) = O_p \left( n^{1/(\alpha - 2)} \right) \). Next, we show \( b_1(M) = \Omega_p \left( n^{1/(\alpha - 2)} \right) \) by induction on \( i \). To that end, let \( L > 0 \) be a constant and \( L' = L'(n) = L n^{1/(\alpha - 2)} \). In addition, let \( \mathcal{M}(n) \) be the class of pairs \( (H,e) \), where \( H \in \mathcal{M}(2n,3n) \) and \( \beta_H(e) \geq L'(n) \). Moreover, let \( (M',e') \) be a pair chosen uniformly at random from \( \mathcal{M}(n) \). Next, we show that the distributions of \( M \) and \( M' \) are 'similar'. More precisely, let \( \mathcal{D} \) be some graph property. We claim that

\[
(\forall L > 0 : M' \in \mathcal{D} \text{ whp}) \implies (M \in \mathcal{D} \text{ whp}). \tag{27}
\]

To prove it, we define \( \beta(H,j) := |\{ e \in E(H) \mid \beta_H(e) \geq j \} | \) for a multigraph \( H \) and \( j \in \mathbb{N} \). With this notation we obtain by Lemma 8.4

\[
|\mathcal{M}(n)| = |\mathcal{M}(2n,3n)| \cdot E \left[ \beta \left( M, L' \right) \right] = \Theta(1) \cdot |\mathcal{M}(2n,3n)|. \tag{28}
\]

Next, we observe that for each \( H \in \mathcal{M}(2n,3n) \) we have

\[
P \left[ M' = H \right] = P \left[ M = H \right] \frac{|\mathcal{M}(2n,3n)| / |\mathcal{M}(n)|}. \tag{29}
\]

Combining 28 and 29 yields

\[
P \left[ M \in \mathcal{D} \right] \leq P \left[ \beta(M,L') = 0 \right] + P \left[ M \in \mathcal{D} \land \beta(M,L') \geq 1 \right]
= P \left[ \beta(M,L') = 0 \right] + O(1) P \left[ M' \notin \mathcal{D} \right]
= P \left[ \beta(M,L') = 0 \right] + o(1),
\]

where we assumed that whp \( M' \in \mathcal{D} \). Finally, we observe that for each \( \delta > 0 \), we can choose \( L > 0 \) such that \( P \left[ \beta(M,L') = 0 \right] \leq \delta \) by Lemma 8.4. This shows 27.

Next, we prove \( b_1(M') = \Omega_p \left( n^{1/(\alpha - 2)} \right) \), which implies \( b_1(M) = \Omega_p \left( n^{1/(\alpha - 2)} \right) \) by 27. We note that we can enumerate all pairs \( (H,e) \in \mathcal{M}(n) \) as follows:

- choose an unordered pair \([w,x]\) with \( w \neq x \in [2n] \) and set \( e = wx \);
- choose \( j,k \in \mathbb{N} \) such that \( j + k = n - 1 \) and \( j, k \geq L'/2 \);
- choose a partition \( J \cup K = [2n] \setminus \{w,x\} \) with \( |J| = 2j \) and \( |K| = 2k \);
- choose \( H_1 \in \mathcal{M}(2j,3j) \) and \( H_2 \in \mathcal{M}(2k,3k) \) and relabel the vertices with \( J \) and \( K \), respectively;
- choose edges \( e_1 \in E(H_1) \) and \( e_2 \in E(H_2) \) and perform a bridge insertion (cf. Definition 8.2) at edges \( e_1 \) and \( e_2 \) with vertices \( w \) and \( x \).
Next, we will use Lemma 5.2, which we only formulated for graph classes, but it is straightforward that it is also true for classes of graphs where one edge is marked. Let $\mathcal{A} (n) = \mathcal{A}' (n)$ and define the function $\Phi$ for a pair $(H, e) \in \mathcal{A}$ by $\Phi (H, e) := (e, \beta_H (e))$. Let $(e_n, j_n)_{n \in \mathbb{N}}$ be a sequence chosen uniformly at random from all elements in $\mathcal{A} (n)$ that evaluate to $(e_n, j_n)$ under $\Phi$. Now let $k_n = n - 1 - j_n$ and let $C_1$ and $C_2$ be the two graphs obtained by reversing the bridge insertion of $e_n$ in $F_n$. By the above construction, $C_1$ and $C_2$ are distributed like $M (2 j_n, 3 j_n)$ and $M (2 k_n, 3 k_n)$, respectively. Without loss of generality we may assume $j_n \geq (n - 1) / 2$. Now by Lemma 8.10 and induction hypothesis we obtain $b_{i - 1} (C_1) = \Omega_p (n^{1/ (a - 2)})$ and $b_1 (C_2) = \Omega_p (k_n) = \Omega_p (n^{1/ (a - 2)})$. Hence, we have $b_1 (F_n) = \Omega_p (n^{1/ (a - 2)})$. Now we get $b_1 (M') = \Omega_p (n^{1/ (a - 2)})$ by Lemma 5.2. This together with [27] implies that $b_1 (M) = \Omega_p (n^{1/ (a - 2)})$.

\[ \square \]

11. DISCUSSION

Theorem 1.4(c) raises the question about the precise asymptotic order of the circumference $c (L)$ of the largest component $L$ of a uniform random planar graph in the weakly supercritical regime. The reason why we provided only a lower and an upper bound for $c (L)$ is partly because we could not determine the precise order of the circumference $c (K)$ of a random cubic planar multigraph $K$. If there were a function $f = f (n)$ such that $c (K) = \Theta_p (f)$, then our proof would imply that $c (L) = \Omega_p (n^{1/3} f (sn^{-2/3}))$ and $c (L) = O_p (n^{1/3} f (sn^{-2/3}) \log n)$. That closes the gap up to a factor of $\log n$. Moreover, if $f (n) = o (n^{1/2})$, then our methods lead even to $c (L) = \Theta_p (n^{1/3} f (sn^{-2/3}))$.

We note that Robinson and Wormald [39] showed that whp a random cubic (general, not necessarily planar) graph has a Hamiltonian cycle. We know that this is not the case for random cubic planar multigraphs, since the longest cycle misses all vertices which have a loop attached. By Lemma 7.6 there are linearly many such vertices. Nevertheless, we believe that whp there is a cycle of linear length.

Conjecture 11.1. Let $K = K (2n, 3n)$ be a graph chosen uniformly at random from the class of all cubic planar multigraphs on vertex set $[2n]$. Then, we have $c (K) = \Theta_p (n)$.

If the above conjecture were true, we would immediately obtain the following result.

Conjecture 11.2. Let $\mathcal{P}$ be the class of planar graphs, $P = P (n, m) \in \mathcal{P} (n, m)$, and $L = L (P)$ the largest component of $P$. Assume $m = n / 2 + s$ for $s = s (n) = o (n)$ and $s^3 n^{-2} \rightarrow \infty$. Then $c (L) = \Theta_p (sn^{-1/3})$.

In Theorem 8.12 we determined the block structure for kernel-stable classes of graphs which are addable and have a critical exponent $3 < \alpha < 4$. We already pointed out that it should be straightforward to generalise these results to the case $\alpha \geq 4$ (see Remark 8.13). However, we believe that this is not the case any more if $\alpha < 3$. Panagiotou and Steger [39] showed that in random $n$-vertex graphs, there is a drastic change in the block structure when the critical exponent $\alpha$ is around $3$. For example, they showed that a random planar graph on $n$ vertices (whose critical exponent is known to be $\alpha = 7 / 2$) has whp a block of order linear in $n$, while the largest block of a random outerplanar graph or in a random series-parallel graph on $n$ vertices (whose critical exponent is known to be $\alpha = 5 / 2$) is of order $O (\log n)$. This leads to the following conjecture (see Lemmas 8.10 and 8.11 for comparable results for the case $\alpha > 3$).

Conjecture 11.3. Let $\mathcal{M}$ be a bridge-stable class of multigraphs with critical exponent $\alpha < 3$ and $M = M (2n, 3n) \in \mathcal{M} (2n, 3n)$. Then for each $i \in \mathbb{N}$, we have $b_i (M) = O (\log n)$.

If this were true, we would obtain the following result (see Theorem 8.12 for a comparable statement for $\alpha > 3$).

Conjecture 11.4. Let $\mathcal{P}$ be a kernel-stable class of graphs which is addable and has a critical exponent $\alpha < 3$ and $P = P (n, m) \in \mathcal{P} (n, m)$. Assume $m = n / 2 + s$ for $s = s (n) = o (n)$ and $s^3 n^{-2} \rightarrow \infty$. Then for each $i \in \mathbb{N}$, we have $b_i (P) = \Omega_p (n^{1/3})$ and $b_i (P) = O_p (n^{1/3} (\log n)^2)$. 27
ACKNOWLEDGEMENT

The authors thank the anonymous referees for many helpful remarks to improve the presentation of this paper.

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APPENDIX A. PROOFS OF THEOREM 5.1 AND PROPOSITION 5.2

Before providing the proof of Theorem 5.1, we briefly illustrate the proof strategy. As for an upper bound for \( X_i \), we will find a ‘small’ function \( g_1 = g_1(n) \) such that \( \mathbb{P}[X_i \geq g_1] = o(1) \). For \( 1 \leq i \leq f \) we denote by \( A_i \) the event that \( X_i \geq g_1 \) and observe that \( X_i \geq g_1 \) if and only if \( A_i \) is true for all \( 1 \leq i \leq f \). Moreover, we intuitively expect that

\[
\mathbb{P}
\left[
\bigwedge_{i=1}^{f} A_i
\right]
\leq
\prod_{i=1}^{f} \mathbb{P}[A_i],
\]  

(30)

because given that \( X_1, \ldots, X_{i-1} \) are ‘large’ (for some \( 1 \leq i \leq f \)), the probability that \( X_i \) is also ‘large’ might decrease. If (30) holds (see Proposition A.2), then we obtain

\[
\mathbb{P}[X_i \geq g_1] = \mathbb{P}
\left[
\bigwedge_{i=1}^{f} A_i
\right]
\leq
\prod_{i=1}^{f} \mathbb{P}[A_i] = \prod_{i=1}^{f} \mathbb{P}[X_i \geq g_1].
\]

In order to derive a lower bound for \( X_i \), we will determine a ‘large’ function \( g_2 = g_2(n) \) such that \( \mathbb{P}[X_i \leq g_2] = o(1) \). To that end, we observe that if \( X_i \leq g_2 \), then there is at least one \( 1 \leq i \leq f \) such that \( X_i \leq g_2 \). Thus, we obtain

\[
\mathbb{P}[X_i \leq g_2] \leq \sum_{i=1}^{f} \mathbb{P}[X_i \leq g_2].
\]

Therefore, in both cases, it is enough to find good bounds for \( \mathbb{P}[X_i \geq g_1] \) and \( \mathbb{P}[X_i \leq g_2] \). Such bounds are obtained in Proposition A.3.

In order to make the aforementioned idea more precise, we need two known facts about the Pólya urn model. The first one is about the marginal distribution of \( X_i \) and will be our starting point for deducing bounds on \( \mathbb{P}[X_i \geq g_1] \) and \( \mathbb{P}[X_i \leq g_2] \).

**Proposition A.1 (30 Theorem 3.1).** Let \( N, k \in \mathbb{N} \) be given. For \( i \in [N] \) and \( x \in \{0, \ldots, k\} \), we have

\[
\mathbb{P}[X_i = x] = \frac{(k+N-x-2)\cdots(k-2)(N-1)}{(k+N-1)\cdots(k+1)},
\]

and in particular, \( \mathbb{P}[X_i = x] \leq \mathbb{P}[X_i = 0] \leq \frac{N}{k+N} \).

**Proposition A.2 (425).** Let \( N \in \mathbb{N} \) be given. For \( i \in [N] \) and \( x_1, \ldots, x_i \in \mathbb{N}_0 \), we have

\[
\mathbb{P}
\left[
\bigwedge_{j=1}^{i} (X_j \geq x_j)
\right]
\leq
\prod_{j=1}^{i} \mathbb{P}[X_j \geq x_j],
\]  

(31)

and

\[
\mathbb{P}
\left[
\bigwedge_{j=1}^{i} (X_j \leq x_j)
\right]
\leq
\prod_{j=1}^{i} \mathbb{P}[X_j \leq x_j].
\]  

(32)

We note that a more general version of Proposition A.2 was proven in 4 Example 5.5) by using a fact from 25 (1.8). A random vector \( (X_1, \ldots, X_N) \) satisfying (31) and (32) is also called **negatively dependent** (see e.g. 3 for details). Next, we derive some bounds for \( \mathbb{P}[X_i \leq x] \) and \( \mathbb{P}[X_i \geq x] \) by using Proposition A.1.

**Proposition A.3.** Let \( N, k \in \mathbb{N} \) be given.

(a) For \( i \in [N] \) and \( x \in \{0, \ldots, k\} \), we have

\[
\mathbb{P}[X_i \leq x] \leq (x+1)\frac{N}{k+N}
\]

and

\[
\mathbb{P}[X_i \geq x] \leq 2 \exp\left(-\frac{(N-2)}{k+N}x\right).
\]

(b) If in addition \( x \leq \frac{N}{2} \), then we have

\[
\mathbb{P}[X_i \geq x] \leq 1 - \frac{(N-1)}{k+N} \left[1 - \exp\left(\frac{2N}{k+x}\right)\right].
\]
(c) If in addition \(k \geq 8N\) and \(x \leq \frac{k}{2}\), then we have

\[
P[X_i \leq x] \leq \exp\left(\frac{1}{64} \exp\left(\frac{-2N}{k}x\right)\right).
\]

A.1. **Proof of Proposition A.3.** Throughout the proof, we use Proposition A.1 without stating explicitly. Then the first inequality in \([a]\) follows by

\[
P[X_i \leq x] = \sum_{y=0}^{x} P[X_i = y] \leq \sum_{y=0}^{x} P[X_i = 0] \leq (x+1) \frac{N}{k+N}.
\]

For the second inequality in \([a]\) we may assume \(N \geq 3\), since otherwise the statement is trivially fulfilled. We get by using \(1 + z \leq \exp(z)\) for \(z \in \mathbb{R}\)

\[
P[X_i = x] = \frac{N-1}{k+N-1} \prod_{a=2}^{N-1} \frac{k+N-x-a}{k+N-a} \leq \frac{N-1}{k+N-1} \left(\frac{k+N-x-2}{k+N-2}\right)^{N-2} \leq \frac{N-1}{k+N-1} \exp\left(-\frac{(N-2)x}{k+N}\right).
\]

Next, we observe that for \(y \in \{0, \ldots, k-1\}\)

\[
\frac{P[X_i = y+1]}{P[X_i = y]} = \frac{k-y}{k+N-y-2} \leq 1 - \frac{N-2}{k+N-2}.
\]

Hence, we obtain

\[
P[X_i \geq x] = \sum_{y=x}^{k} P[X_i = y] \\
\leq P[X_i = x] \sum_{y=x}^{k} \left(1 - \frac{N-2}{k+N-2}\right)^{y-x} \\
\leq \frac{N-1}{k+N-1} \exp\left(-\frac{(N-2)x}{k+N}\right) \frac{k+N-2}{N-2} \\
\leq 2 \exp\left(-\frac{(N-2)x}{k+N}\right),
\]

which proves \([a]\). Next, we assume \(x \leq \frac{k}{2}\) and show \([b]\). To that end, we use \(1 - z \geq \exp(-2z)\) for \(z \in \left[0, \frac{1}{2}\right]\) to obtain

\[
P[X_i = x] = \frac{N-1}{k+N-1} \prod_{a=2}^{N-1} \frac{k+N-x-a}{k+N-a} \geq \frac{N-1}{k+N} \left(\frac{k-x}{k}\right)^{N} \geq \frac{N-1}{k+N} \exp\left(-\frac{2Nk}{k}\right).
\]

Using that yields

\[
P[X_i < x] = \sum_{y=0}^{x-1} P[X_i = y] \geq xP[X_i = x] \geq x \frac{N-1}{k+N} \exp\left(-\frac{2Nk}{k}\right).
\]

This shows \([b]\) Finally, we assume \(k \geq 8N\) and \(x \leq \frac{k}{2}\). Then, we have for \(y \leq \frac{3k}{4}\)

\[
\frac{P[X_i = y+1]}{P[X_i = y]} = \frac{k-y}{k+N-y-2} \geq 1 - \frac{N}{k-k-y} \geq 1 - \frac{4N}{k}.
\]
Thus, for \( x \leq \frac{k}{2} \) we obtain by using (33)

\[
\mathbb{P} \{ X_i > x \} \geq \sum_{y=x+1}^{\frac{k}{2}} \mathbb{P} \{ X_i = y \} \geq \mathbb{P} \{ X_i = x \} \sum_{y=x+1}^{\frac{k}{2}} \left( 1 - \frac{4N}{k} \right)^{y-x} \\
\geq \mathbb{P} \{ X_i = x \} \frac{k}{8N} \left( 1 - \left( 1 - \frac{4N}{k} \right)^{\frac{k}{2}} \right) \\
\geq \mathbb{P} \{ X_i = x \} \frac{k}{8N} \left( 1 - \exp(-N) \right) \\
\geq \frac{N-1}{k+N} \exp\left( -\frac{2N x}{k} \right) \frac{k}{16N} \\
\geq \frac{1}{64} \exp\left( -\frac{2N x}{k} \right).
\]

Hence, we conclude the proof with

\[
\mathbb{P} \{ X_i \leq x \} \leq 1 - \frac{1}{64} \exp\left( -\frac{2N x}{k} \right) \leq \exp\left( -\frac{1}{64} \exp\left( -\frac{2N x}{k} \right) \right).
\]

\[\square\]

A.2. Proof of Theorem 5.1

Throughout the proof, we let \( n \) be large and \( h = h(n) = o(1) \). To obtain the claimed bounds on \( X_* = \min_{1 \leq i \leq f} X_i \), it suffices to show

(a) if \( k = o\left( N f \right) \), then \( X_* = \Omega_p \left( \frac{k}{N f} \right) \);

(b) if \( k = \Omega(N) \) and \( f = o(1) \), then \( X_* = O_p \left( \frac{k}{N f} \right) \);

(c) \( X_* = O_p \left( \frac{k}{N f} \right) \).

To prove (a), it is enough to show \( \mathbb{P} \{ X_* \leq \frac{k}{N f} \} = o(1) \) for any \( h = o \left( \frac{k}{N f} \right) \). To this end, let \( x = \frac{k}{N f} \).

If \( X_* \leq x \), then \( X_i \leq x \) for some \( 1 \leq i \leq f \). Thus, by Proposition A.3(a) we obtain

\[
\mathbb{P} \{ X_* \leq x \} \leq \sum_{i=1}^{f} \mathbb{P} \{ X_i \leq x \} \leq \sum_{i=1}^{f} (x+1) \frac{N}{k+N} \leq 2f \frac{N}{k+N} = 2f \frac{k}{h N f} \frac{N}{k+N} = \Theta(1) \frac{1}{h} = o(1),
\]
as desired.

To prove (b), it suffices to show \( \mathbb{P} \{ X_* \geq \frac{h k}{N f} \} = o(1) \) for any \( h = o \left( \frac{k}{N f} \right) \). Now let \( x = \frac{h k}{N f} \) and for each \( 1 \leq i \leq f \) we denote by \( A_i \) the event that \( X_i \geq x \). If \( X_* \geq x \), then \( X_i \geq x \) for all \( 1 \leq i \leq f \). Thus, by Proposition A.2 we have

\[
\mathbb{P} \{ X_* \geq x \} = \mathbb{P} \left[ \bigwedge_{i=1}^{f} A_i \right] \leq \prod_{i=1}^{f} \mathbb{P} \{ A_i \}.
\]

(34)

By Proposition A.4(b) uniformly over all \( 1 \leq i \leq f \), we have

\[
\mathbb{P} \{ A_i \} \leq 1 - \frac{(N-1)x}{k+N} \exp\left( -\frac{2N x}{k} \right) = 1 - \frac{N-1}{k+N} \frac{h k}{N f} \exp\left( -\frac{2N h k}{k N f} \right) \\
\leq 1 - \Theta(1) \frac{h}{f} \\
\leq \exp\left( -\Theta(1) \frac{h}{f} \right).
\]

This together with (34) yields the desired result

\[
\mathbb{P} \{ X_* \geq x \} \leq \exp\left( -\Theta(1) \frac{h}{f} \right)^f = \exp\left( -\Theta(1) h \right) = \exp\left( -o(1) \right) = o(1).
\]

Finally, (c) follows by Markov’s inequality and the fact that \( \mathbb{E} \{ X_* \} \leq \mathbb{E} \{ X_1 \} = \frac{k}{N f} \).
In order to derive the claimed bounds on $X^* = \max_{1 \leq i \leq f} X_i$, we prove the following assertions.

(d) If $k = \omega(N)$ and $f = \omega(1)$, then $X^* = \Omega_p \left( \frac{k}{N} (1 + \log f) \right)$;

(e) if $k = \omega(N)$, then $X^* = \Omega_p \left( \frac{k}{N} \right)$;

(f) if $k = \omega(N)$, then $X^* = O_p \left( \frac{k}{N} (1 + \log f) \right)$;

(g) if $k = O(N)$, then $X^* = O_p (1 + \log f)$.

To show (d), we assume $k = \omega(N)$ and let $x = \frac{k}{hN} (1 + \log f)$. If $X^* \leq x$, then $X_i \leq x$ for all $1 \leq i \leq f$. For each $1 \leq i \leq f$ we denote by $B_i$ the event that $X_i \leq x$. Using Proposition A.3(b) (for the first inequality) and Proposition A.3(c) (for the second inequality) yields for large $n$

$$\mathbb{P} [X^* \leq x] = \mathbb{P} \left[ \bigwedge_{i=1}^f B_i \right] \leq \prod_{i=1}^f \mathbb{P} [B_i] = \prod_{i=1}^f \exp \left( -\frac{1}{64} \exp \left( -\frac{2N}{k} x \right) \right) = \exp \left( -\frac{f}{64} \exp \left( -\frac{2N}{k} \frac{k}{hN} (1 + \log f) \right) \right) \leq \exp \left( -\frac{1}{64} \exp \left( \log f - \frac{4}{h} \log f \right) \right) = \exp \left( -\frac{1}{64} \exp \left( \left( 1 - \frac{4}{h} \right) \log f \right) \right) = o(1).$$

In order to prove (e) we let $x = \frac{k}{hN}$ and use Proposition A.3(a) to get

$$\mathbb{P} [X^* \leq x] \leq \mathbb{P} [X_1 \leq x] \leq (x + 1) \frac{N}{k + N} = \left( \frac{k}{hN} + 1 \right) \frac{N}{k + N} \leq \frac{1}{h} + \frac{N}{k} = o(1),$$

where we used in the last equality that $h = \omega(1)$ and $k = \omega(N)$.

To show (f) we assume $k = \omega(N)$ and let $x = \frac{k}{hN} (1 + \log f)$. If $X^* \leq x$, then $X_i \leq x$ for some $1 \leq i \leq f$.

Therefore, by Proposition A.3(a) we obtain

$$\mathbb{P} [X^* \geq x] \leq \sum_{i=1}^f \mathbb{P} [X_i \geq x] \leq \sum_{i=1}^f 2 \exp \left( -\frac{(N-2)}{k + N} x \right) = 2f \exp \left( -\frac{(N-2)h}{k + N} (1 + \log f) \right) = 2 \exp \left( \log f - \omega(1) (1 + \log f) \right) = o(1).$$

To prove (g) we assume $k = O(N)$ and let $x = h(1 + \log f)$. Using Proposition A.3(a) we get

$$\mathbb{P} [X^* \geq x] \leq \sum_{i=1}^f \mathbb{P} [X_i \geq x] \leq 2f \exp \left( -\frac{(N-2)}{k + N} x \right) = 2f \exp \left( -\frac{(N-2)h}{k + N} (1 + \log f) \right) = 2 \exp \left( \log f - \omega(1) (1 + \log f) \right) = o(1).$$

This concludes the proof. □

A.3. Proof of Proposition 5.2. It follows directly from the first inequality of Proposition A.3(a). □
APPENDIX B. PROOFS OF CLAIMS 1 AND 2

B.1. Proof of Claim 1 Using the formula for \( m_l \) from (3) yields

\[
\frac{1}{m_n} \sum_{(j,k) \in I(n)} \binom{2n-2}{2j} m_j m_k j k = (1 + o(1)) \frac{c n^{a-2}}{4\gamma} \sum_{(j,k) \in I(n)} j^{-a+1} k^{-a+1}.
\]

Thus, it suffices to show

\[
\sum_{(j,k) \in I(n)} j^{-a+1} k^{-a+1} = (1 + o(1)) \frac{2}{\alpha-2} \frac{1}{f(n)^{a-2} n^{a-1}}.
\]  

(35)

To that end, let \( h = \omega(1) \) be such that \( h(n) = \omega(f(n)) \) and \( h(n) = o(n) \). We obtain

\[
\sum_{(j,k) \in I(n)} j^{-a+1} k^{-a+1} \leq 2 \sum_{j=f(n)}^{h(n)} j^{-a+1} (n-1-j)^{-a+1} + 2 \sum_{j=h(n)+1}^{[n/2]} j^{-a+1} (n-1-j)^{-a+1} \\
\leq (2 + o(1)) n^{-a+1} \int_{f(n)-1}^{h(n)} x^{-a+1} dx + \Theta(1) n^{-a+1} \int_{h(n)-1}^{\infty} x^{-a+1} dx \\
= (2 + o(1)) n^{-a+1} \frac{f(n)^{-a+2}}{\alpha-2}.
\]

Similarly, we have that for \( n \to \infty \)

\[
\sum_{(j,k) \in I(n)} j^{-a+1} k^{-a+1} \geq 2 \sum_{j=f(n)}^{h(n)} j^{-a+1} (n-1-j)^{-a+1} \\
\geq 2 n^{-a+1} \int_{f(n)}^{h(n)} x^{-a+1} dx \\
= (2 + o(1)) n^{-a+1} \frac{f(n)^{-a+2}}{\alpha-2}.
\]

This shows (35), which finishes the proof. □

B.2. Proof of Claim 2 We note that the asymptotic formula for \( m_l \) in (3) holds only for ‘large’ \( l \).

Thus, we split the given sum into two parts, one for the terms where \( l \) is ‘small’ and the other for terms where \( l \) is ‘big’. To make that more precise, we set \( I_1(n) := \{ j, k, l \in I(n) \mid l \leq n/2 \} \). Due to (3) there exists a constant \( A > 0 \) such that \( m_l \leq \frac{A}{l^{1/2}} \) for all \( I \in \mathbb{N} \). Combining that with the asymptotic formulas for \( m_j, m_k, \) and \( m_n \) from (3) yields

\[
\sum_{(j,k,l) \in I_1(n)} \frac{324}{m_n} \binom{2n-2}{2j,2k,2l} m_j m_k j k l^2 = O(1) n^{-4} S_1,
\]

where

\[
S_1 = \sum_{(j,k,l) \in I_1(n)} j^{-a+1} k^{-a+1} l^{-a+2}.
\]

We define \( f(l) := \{ (j, k) \in \mathbb{N}^2 \mid j + k = n - 2 - l, j, k \geq f(n) \} \). Analogous to the proof of (35) we obtain

\[
S_1 = \sum_{l=1}^{[n/2]} l^{-a+2} \sum_{(j,k) \in f(l)} j^{-a+1} k^{-a+1} \\
= \sum_{l=1}^{[n/2]} l^{-a+2} \sum_{(j,k) \in f(l)} \Theta(1) (n-l)^{-a+1} f(n)^{-a+2} \\
= \Theta(1) f(n)^{-a+2} n^{-a+1} \sum_{l=1}^{[n/2]} l^{-a+2} \\
= \Theta(1) f(n)^{-a+2} n^{-a+1} \\
= o(1) f(n)^{-2a+4} n^{-a+2}.
\]

34
This in (36) implies
\[
\frac{324}{m_n} \sum_{(j,k,l) \in \bar{I}_1(n)} (2n-4)_{j,k,l}^2 m_j m_k m_l = o(1) f(n)^{-2a+4} n^{-2}. \tag{37}
\]

Next, we consider those terms where \( l \) is ‘big’, i.e. terms with indices in \( \bar{I}_1(n) : = I(n) \setminus I_1(n) \). Using (6) we obtain
\[
\frac{324}{m_n} \sum_{(j,k,l) \in \bar{I}_1(n)} (2n-4)_{j,k,l}^2 m_j m_k m_l = (1 + o(1)) \frac{81 c^2}{4 \gamma^2} n^{a-4} \bar{S}_1, \tag{38}
\]
where
\[
\bar{S}_1 = \sum_{(j,k,l) \in \bar{I}_1(n)} j^{-a+1} k^{-a+1} l^{-a+2}.
\]

Now we partition \( \bar{I}_1(n) \) into smaller parts. More precisely, let \( h \) be a function such that \( h(n) = \omega(f(n)) \), but \( h(n) = o(n) \). We define
\[
I_2(n) := \{(j, k, l) \in \bar{I}_1(n) \mid j \geq h(n) \}; \\
I_3(n) := \{(j, k, l) \in \bar{I}_1(n) \mid j < h(n), k \geq h(n) \}; \\
I_4(n) := \{(j, k, l) \in \bar{I}_1(n) \mid j, k < h(n) \}.
\]
In addition, for \( i \in \{2, 3, 4\} \) we set
\[
S_i := \sum_{(j,k,l) \in I_i(n)} j^{-a+1} k^{-a+1} l^{-a+2}.
\]
Similarly as in the proof of (35), we have
\[
S_2 \leq (n/2)^{-a+2} \sum_{j=h(n)}^{\infty} j^{-a+1} \sum_{k=h(n)}^{\infty} k^{-a+1} = \Theta(1) n^{-a+2} h(n)^{-a+2} f(n)^{-a+2} = o(1) f(n)^{-2a+4} n^{-a+2}.
\]
Next, we observe \( S_3 \leq S_2 \) and
\[
S_4 = (1 + o(1)) n^{-a+2} \sum_{j=h(n)}^{h(n)-1} j^{-a+1} \sum_{k=h(n)}^{h(n)-1} k^{-a+1} = (1 + o(1)) n^{-a+2} f(n)^{-2a+4} \frac{n^{-a+2}}{(-a+2)^2}.
\]
Thus, we have
\[
\bar{S}_1 = S_2 + S_3 + S_4 = \frac{(1 + o(1)) f(n)^{-2a+4} n^{-a+2}}{(a-2)^2}.
\]
Plugging this in (38) yields
\[
\frac{324}{m_n} \sum_{(j,k,l) \in \bar{I}_1(n)} (2n-4)_{j,k,l}^2 m_j m_k m_l = (1 + o(1)) \frac{81 c^2}{4 \gamma^2 (a-2)^2} f(n)^{-2a+4} n^{-2}.
\]
Combining that with (37) yields the statement.