Ladder operators and squeezed coherent states of a three-dimensional generalized isotonic nonlinear oscillator

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Abstract
We explore squeezed coherent states of a three-dimensional generalized isotonic oscillator whose radial part is the newly introduced generalized isotonic oscillator whose bound state solutions have been shown to admit the recently discovered $X_1$-Laguerre polynomials. We construct a complete set of squeezed coherent states of this oscillator by exploring the squeezed coherent states of the radial part and combining the latter with the squeezed coherent states of the angular part. We also prove that the three-mode squeezed coherent states resolve the identity operator. We evaluate Mandel’s $Q$-parameter of the obtained states and demonstrate that these states exhibit sub-Poissonian and super-Poissonian photon statistics. Furthermore, we illustrate the squeezing properties of these states, both in the radial and angular parts, by choosing appropriate observables in the respective parts. We also evaluate the Wigner function of these three-mode squeezed coherent states and demonstrate the squeezing property explicitly.

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(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. New exactly solvable model

Very recently, the exact quantum solvability of the extended radial oscillator/generalized isotonic oscillator potential,

$$V(r) = \omega^2 r^2 + \frac{l(l+1)}{r^2} + \frac{8\omega}{(2\omega r^2 + 2l + 1)} - \frac{16\omega(2l + 1)}{(2\omega r^2 + 2l + 1)^2},$$

(1)

where $\omega$ and $l$ are parameters, has been demonstrated in four different perspectives [1–4]. To begin with, it was shown that the bound state eigenfunctions of the Schrödinger equation associated with the potential (1) can be expressed in terms of the newly found exceptional orthogonal $v$th-degree polynomials, $\hat{L}_k^v(x)$, $v = 0, 1, 2, 3, \ldots$, where $k$ is a positive real number.
parameter, namely $X_1$-Laguerre polynomials [1]. These new polynomials, $\hat{L}_k^j(x)$, are related to the classical Laguerre polynomials $L_k^j(x)$ by the following relation [5]:

$$\hat{L}_k^j(x) = -(x+k+1)L_{k-1}^j(x) + L_{k-2}^j(x).$$

(2)

The bound state solutions of the Schrödinger equation associated with the potential (1) are found to be [1]

$$\phi_{n,l}(r) = N_{n,l} \left( \frac{r^{l+1}}{(2\omega r^2 + 2l + 1)^{1/2}} \right) \hat{L}_n^{l+1}(\omega r^2) e^{-\frac{1}{4} \omega r^2},$$

(3)

with

$$E_{n,l} = 2\omega \left( 2n + l + \frac{3}{2} \right), \quad n = 0, 1, 2, 3 \ldots, l = 0, 1, 2, 3, \ldots$$

(4)

and $N_{n,l} = \left( \frac{8\omega^{l+1} n!}{(n+l+\frac{3}{2})(n+l+\frac{1}{2})} \right)^{1/2}$ is the normalization constant. We note here that the $X_1$-Laguerre polynomials, $\hat{L}_k^j(x)$, are solutions of the following second-order linear ordinary differential equation with rational coefficients, i.e.

$$y'' - \left( \frac{x-k}{x(x+k)} \right)y' + \left( 1 + \frac{k}{x+k} + \nu - 1 \right)y = 0, \quad \nu = \frac{d}{dx},$$

(5)

where $k > 0$ is a real parameter and $\nu = 1, 2, 3, \ldots$. The first few $X_1$-Laguerre polynomials are [5]

$$\hat{L}_0^1(x) = -(x+k+1),$$

(6)

$$\hat{L}_1^1(x) = x^2 - k(k+2),$$

(7)

$$\hat{L}_2^1(x) = -\frac{1}{2} x^3 + \frac{k+3}{2} x^2 + \frac{k(k+3)}{2} x - \frac{k}{2} (3+4k+k^2).$$

(8)

One may observe that the above polynomial sequence starts with a linear polynomial in $x$ instead of a constant which usually the other classical orthogonal polynomials do. However, these new polynomials form a complete set with respect to some possible measure [5]. For more details on the properties of these exceptional orthogonal polynomials, one may refer to [5].

In a different context, while constructing exact analytic solutions of the $d$-dimensional Schrödinger equation associated with the generalized quantum isotonic nonlinear oscillator potential (1), i.e.

$$-\Delta \Psi + \left( B^2 + \frac{2^d}{r^2} + \omega_0^2 r^2 + 2 g \frac{r^2 - a^2}{(r^2 + a^2)^2} \right) \Psi = E \Psi,$$

(9)

where $\Delta$ is the $d$-dimensional ($d \geq 2$) Laplacian operator and $\omega, a, g$ are parameters and $V(r)$ is the central potential given by

$$V(r) = \frac{B^2}{r^2} + \omega^2 r^2 + 2 g \frac{(r^2 - a^2)}{(r^2 + a^2)^2}, \quad B^2 \geq 0,$$

(10)

Hall et al have shown that only for the values $g = 2$ and $\omega^2 a^2 = B^2 + \left( l + \frac{(d-2)}{2} \right)^2$, can the potential (10) be exactly solvable [2]. It is shown that the potential (10) can be regarded as a supersymmetric partner of the Goldmann–Krivchenkov potential whose exact solutions are known [2]. The potential (10) with the parametric restrictions given above exactly matches with the one constructed by Quesne [1]. In a subsequent study [3], Sadd et al have solved the system (10) numerically, for arbitrary values of the system parameters, through an asymptotic iteration method. They have shown that for certain restrictions ($B = 0$ and $d = 1$), the potential (10) can be reduced to the one discussed in [6] (see equation (11) and the paragraph...
that follows). The quasi-polynomial solutions and the energy eigenvalues of (10) are also reported in [3].

In a very recent paper, Agboola and Zhang have considered the potential (1) and transformed the associated Schrödinger equation into the generalized spheroidal wave equation [7]. Using the Bethe ansatz method, they have derived bound state solutions of (1) for certain parametric values [4].

We mention here that the inverse square-type potentials play an important role in physics [8–12]. Solving the Schrödinger equation with non-polynomial potentials [13] is also of interest for physicists at both classical [14] and quantum levels [15].

We also recall here that over the last few years, the one-dimensional version of the potential (1), i.e.

\[ V(x) = a^2 x^2 + 2 g \frac{(x^2 - a^2)}{(x^2 + a^2)^2}, \quad a^2 > 0, \]  

has been studied in different perspectives; see, for example, [3, 6, 16–21]. The generalized isotonic oscillator potential (11) is generalized in such a way that it lies between the harmonic oscillator and the isotonic oscillator potentials [6]. Importantly, this generalization removes the singularity nature of the isotonic oscillator in real space, as the generalized isotonic oscillator has poles only at imaginary points \( x = \pm ia \). The energy values of this generalized isotonic oscillator potential \( g = 2a^2 \omega (1 + 2a^2) \), \( \omega = 1 \), \( a^2 = \frac{1}{2} \) are equidistant [6]. In our earlier studies, we have constructed different types of nonlinear coherent states, nonlinear squeezed states and various non-classical states of the oscillator (11) and analyzed its position-dependent mass Schrödinger equation by fixing the parameters \( g = 2a^2 \omega (1 + 2a^2) \), \( \omega = 1 \) and \( a^2 = \frac{1}{2} \) [20–23]. While constructing nonlinear coherent and squeezed states of the system (11), the deformed ladder operators which we had constructed lead to two non-unitary displacement operators and two squeezing operators [21–23]. While one of the displacement operators produced nonlinear coherent states, the other non-unitary displacement operator failed to produce any new type of nonlinear coherent states (dual pair) [24]. In the case of squeezed states also, while one of the squeezing operators gave the nonlinear squeezed states, the other one failed to produce the dual pair of nonlinear squeezed states which leads us to the conclusion that the dual pair of nonlinear coherent and squeezed states is absent in this system (for more details, one may refer to [22, 23]).

1.2. This work

As a continuation of our earlier studies, in this paper, we focus our attention on the three-dimensional extension of the potential (11). With the restriction \( d = 3 \), \( g = 2 \), \( B = 0 \) and \( a^2 = l + \frac{1}{2} \), the Schrödinger equation (9) turns out to be an exactly solvable three-dimensional isotonic oscillator equation:

\[ -\Delta \Psi + \left( \omega^2 r^2 + \frac{8\omega}{2\omega r^2 + 2l + 1} - \frac{16\omega(2l + 1)}{(2\omega r^2 + 2l + 1)^2} \right) \Psi = E \Psi. \]  

(12)

The isotonic oscillator with the generalized inverse type of potential models (i) the dynamics of harmonic oscillator in the presence of dipoles where it describes the scattering of electrons by polar molecules [25] and (ii) the dynamics of dipoles in cosmic strings [11].

Now considering the three-dimensional Laplacian in polar coordinates with the wavefunction \( \Psi(r, \theta, \phi) = u(r) Y_{l,m}(\theta, \phi) \), where \( Y_{l,m}(\theta, \phi) \) are the spherical harmonics, we have the Schrödinger equation

\[ - \frac{d^2 u}{dr^2} - \frac{2}{r} \frac{du}{dr} + \left( \omega^2 r^2 + \frac{l(l + 1)}{r^2} + \frac{8\omega}{2\omega r^2 + 2l + 1} - \frac{16\omega(2l + 1)}{(2\omega r^2 + 2l + 1)^2} \right) u = Eu, \]  

(13)

where \( l \) is the angular momentum.
Since the bound state solutions of (13) are already known (see equations (3) and (4)), we can unambiguously write the eigenfunctions and energy values of the three-dimensional problem as [1, 2]

\[ \Psi_{n,l}(r, \theta, \phi) = N_{n,l} \frac{r^l}{2\omega r^2 + 2l + 1} e^{-\frac{1}{2}\omega r^2} Y_l m(\theta, \phi), \]  

\[ E_{n,l} = 2\omega \left( 2n + l + \frac{3}{2} \right), \quad n = 0, 1, 2, 3, \ldots, \quad l = 0, 1, 2, 3, \ldots, \quad m = -l \text{ to } l. \]  

From these solutions, we construct a complete set of squeezed coherent states for the three-dimensional system (12). Since the potential under consideration is a spherically symmetric one, we separate the radial part from the angular part and investigate each one of them separately. To construct squeezed coherent states of the radial part, see equation (1), we should know its ladder operators. The ladder operators can be explored through the supersymmetric technique. However, the resultant operators which come out through this formalism do not perform perfect annihilation and creation actions on the number states. To explore the perfect ladder operators, we alternatively examine the shape invariance property of the potential (1). Once the presence of this property is confirmed, we then move on to construct the necessary annihilation and creation operators by adopting the procedure described in [26]. Using these operators, we unambiguously construct squeezed coherent states [27, 28] of the radial part. To explore squeezed coherent states of the angular part, we use the Schwinger representation and define the necessary creation and annihilation operators. In particular, by considering a disentangled form of two-mode squeezing and displacement operators, we derive squeezed coherent states [29]. We obtain a complete set of squeezed coherent states of the three-dimensional oscillator by defining that these states are the tensor product between squeezed coherent states of the radial variable and squeezed states of the angular variables [30]. To make our result more rigorous, we also prove that these three-mode squeezed coherent states satisfy the completeness condition. We also evaluate Mandel’s Q-parameter of the three-mode squeezed coherent states and investigate certain photon statistical properties associated with these states. Our result confirms that these states exhibit sub-Poissonian (non-classical) and super-Poissonian photon statistics for certain parameters. Furthermore, we demonstrate squeezing properties of the states associated with the radial part by considering the generalized position and momentum coordinates with the obtained ladder operators and the angular part by considering the angular momentum quantities \( \hat{L}_x \) and \( \hat{L}_y \), respectively. Finally, we evaluate the Wigner function of the constructed squeezed coherent states by defining a three-mode Wigner function in terms of the coherent states [31, 32]. Our result reveals that these squeezed coherent states exhibit the squeezing property. We intend to carry out all these studies due to the fact that among the non-classical states, squeezed states have attracted much attention because certain observables in this basis show the fluctuations less than that of the vacuum. The squeezed coherent states have found applications not only in quantum optics but also in quantum cryptography [33], quantum teleportation [34] and quantum communication [35], to name a few.

This paper is organized as follows. In section 2, we analyze the shape invariance property of the radial part of the three-dimensional generalized isotonic oscillator and construct suitable ladder operators. In section 3, we construct squeezed coherent states of the three-dimensional oscillator. We use these ladder operators to construct squeezed coherent states of the radial part and Schwinger’s operators to construct the angular part. We express the complete form of the squeezed coherent states of the three-dimensional oscillator by suitably combining the radial part with the angular part. We also prove that the three-mode squeezed coherent states satisfy the completeness condition. We investigate certain non-classical properties associated
with these states in section 4. To begin with, we study Mandel’s $Q$-parameter and confirm the non-classical and super-Poissonian nature of these states. We also illustrate the squeezing properties of the constructed three-dimensional squeezed coherent states both in the radial part and in the angular part in this section. In section 5, we evaluate the Wigner function for the three-dimensional generalized isotonic oscillator and demonstrate the squeezing property of the constructed states. Finally, in section 6, we present the conclusion and the outcome of our study. Certain background derivations which are needed to obtain the necessary ladder operators are given in appendix A. The details of evaluating certain multiple sums in the expressions $\langle \hat{n}_i \rangle$ and $\langle \hat{n}_i^2 \rangle$ are given in appendix B. In appendix C we discuss the method of evaluating the multiple sums appearing in the Wigner function.

2. Shape invariance property and ladder operators

2.1. Supersymmetric formalism

To construct annihilation and creation operators, we start with the time-independent Schrödinger equation,

$$\hat{H} \Phi(r) = \left( -\frac{d^2}{dr^2} + V(r) \right) \Phi(r) = E \Phi(r),$$  \hspace{1cm} (16)

with $V(r)$ being given in (1). Let us rewrite the second-order differential operator $\hat{H}$ given in (16) as a product of two first-order differential operators, namely $\hat{A}^+$ and $\hat{A}^-$, such that

$$\hat{A}^+ = \frac{d}{dr} + W(r) \quad \text{and} \quad \hat{A}^- = \frac{d}{dr} + W(r).$$  \hspace{1cm} (17)

In the above equation, $W(r)$ is the superpotential which is found to be [1]

$$W(r) = \omega r - \frac{l + 1}{r} + \frac{4\omega r}{2\omega r^2 + 2l + 1} - \frac{4\omega r}{2\omega r^2 + 2l + 3}. \hspace{1cm} (18)$$

The operators $\hat{A}^+$ and $\hat{A}^-$ factorize the partner Hamiltonian $\hat{H}^{(2)}$ as

$$\hat{H}^{(2)} = \hat{A}^- \hat{A}^+ = \hat{H} - \omega(2l + 1).$$

The Schrödinger equation associated with $\hat{H}^{(1)}$ reads

$$\hat{H}^{(1)} \Phi^{(1)}_{n,l}(r) = E^{(1)}_{n} \Phi^{(1)}_{n,l}(r),$$ \hspace{1cm} (19)

i.e.

$$-\frac{d^2 \Phi^{(1)}_{n,l}(r)}{dr^2} + V_1(r) \Phi^{(1)}_{n,l}(r) = E^{(1)}_{n} \Phi^{(1)}_{n,l}(r),$$  \hspace{1cm} (20)

where the potential $V_1(r)$ is given by

$$V_1(r) = \omega^2 r^2 + \frac{l(l+1)}{r^2} + \frac{8\omega}{2\omega r^2 + 2l + 1} = \frac{16\omega(2l + 1)}{(2\omega r^2 + 2l + 1)^2} - \omega(2l + 3).$$ \hspace{1cm} (21)

Equation (20) shares the same solution as that of the Schrödinger equation associated with the potential (1) admits, i.e.

$$\Phi^{(1)}_{n,l} \equiv \Phi_{n,l} = N_{n,l} \frac{r^{l+1}}{(2\omega r^2 + 2l + 1)^\frac{l+\frac{1}{2}}{2}} L^{(l+\frac{1}{2})}_{n+l+1}(\omega r^2) e^{-\frac{1}{2}\omega^2}$$ \hspace{1cm} (22)

with the energy eigenvalues

$$E^{(1)}_{n} = E_{n,l} - \omega(2l + 3) = 4\pi \omega.$$ \hspace{1cm} (23)
The Schrödinger equation associated with $\hat{H}^{(2)}$ is given by
\[
\hat{H}^{(2)} \Phi_{n,l}^{(2)}(r) = E_n^{(2)} \Phi_{n,l}^{(2)}(r).
\] (24)
Rewriting (24), we obtain
\[
-\frac{d^2\Phi_{n,l}^{(2)}(r)}{dr^2} + V_2(r) \Phi_{n,l}^{(2)}(r) = E_n^{(2)} \Phi_{n,l}^{(2)}(r),
\] (25)
where the partner potential $V_2(r)$ turns out to be
\[
V_2(r) = \omega^2 r^2 + \frac{(l + 1)(l + 2)}{r^2} + \frac{8\omega}{2\omega r^2 + 2l + 3} - \frac{16\omega(2l + 3)}{(2\omega r^2 + 2l + 3)^2} - \omega(2l + 1).
\] (26)

The only difference between equations (21) and (26) is that in the latter the parameter $l$ is modified as $l + 1$ besides the constant term. As a consequence, from the known solutions, (3) and (4), we can generate the solutions of the system (25) straightforwardly, i.e.
\[
\Phi_{n,l}^{(2)}(r) = \tilde{N}_{n,l} \frac{r^{l+2}}{2\omega^2 + 2l + 3} L_{n+1}^{(l+\frac{1}{2})} (\omega r^2) e^{-\frac{1}{2}\omega r^2},
\] (27)
with
\[
E_n^{(2)} = E_{n,l+1} - \omega(2l + 1) = 4\omega(n + 1)
\] (28)
and $\tilde{N}_{n,l} = \left(\frac{8\omega^{1/2} n!}{(n+\frac{1}{2})! (n+\frac{1}{2}+\frac{3}{2})!}\right)^{1/2} = N_{n,l+1}$ is the normalization constant.

Now we investigate the action of SUSY operators, $\hat{A}^\pm$, on the eigenfunctions $\Phi_{n,l}^{(1)}(r)$ with the help of (22). For this purpose, let us calculate the action of $\hat{A}^+$ on $\Phi_{n,l+1}^{(1)}$:
\[
\hat{A}^+ \Phi_{n,l+1}^{(1)} = \left( -\frac{d}{dr} + W(r) \right) N_{n,l+1} \frac{r^{l+2}}{2\omega^2 + 2l + 3} L_{n+1}^{(l+\frac{1}{2})} e^{-\frac{1}{2}\omega r^2}.
\] (29)
Evaluating the right-hand side of (29), we find
\[
\hat{A}^+ \Phi_{n,l+1}^{(1)} = \left[ -\frac{d\tilde{L}_{n+1}^{(l+\frac{1}{2})}}{dr} + \left( \frac{2\omega r - 2l + 3}{r} + \frac{4\omega r}{2\omega r^2 + 2l + 3} \right) \tilde{L}_{n+1}^{(l+\frac{1}{2})} \right] \frac{N_{n,l+1} r^{l+2}}{2\omega^2 + 2l + 3} e^{-\frac{1}{2}\omega r^2}.
\] (30)
By using the properties and recursion relations involving $X_1$-Laguerre polynomials, the terms inside the square bracket can be replaced by $-\frac{1}{r} \left( \frac{2\omega r^2 + 2l + 3}{2\omega^2 + 2l + 3} \right) L_{n+1}^{(l+\frac{1}{2})}$ (see appendix A for details). Substituting this result in (30) and simplifying the resultant expression, we arrive at
\[
\hat{A}^+ \Phi_{n,l+1}^{(1)}(r) = -\sqrt{4\omega(n + 1)} \Phi_{n+1,l}^{(1)}(r).
\] (31)

Now let us evaluate the action of other operator $\hat{A}$ on the eigenfunctions $\Phi_{n,l}^{(1)}$:  
\[
\hat{A}^- \Phi_{n,l}^{(1)} = \left( \frac{d}{dr} + W(r) \right) N_{n,l} \frac{r^{l+1}}{2\omega^2 + 2l + 3} L_{n+1}^{(l+\frac{1}{2})} e^{-\frac{1}{2}\omega r^2}.
\] (32)
Here also while expanding the right-hand side of (32), we find
\[
\hat{A}^- \Phi_{n,l}^{(1)} = \left[ \frac{d\tilde{L}_{n+1}^{(l+\frac{1}{2})}}{dr} - \frac{4\omega r}{2\omega^2 + 2l + 3} \tilde{L}_{n+1}^{(l+\frac{1}{2})} \right] \frac{N_{n,l} r^{l+1}}{2\omega^2 + 2l + 3} e^{-\frac{1}{2}\omega r^2}.
\] (33)
The terms inside the square bracket can be replaced by \(-2\omega r \left( \frac{2\omega r^2 + 2l + 1}{2\omega r^2 + 2l + 1} \right) \hat{L}^{(l+\frac{1}{2})}_n\) (see appendix A for details). With this identification, equation (32) can be simplified to

\[
\hat{A}^{-}\Phi_{n,l}^{(1)}(r) = -\sqrt{4\omega r} \Phi_{n-1,l+1}^{(1)}(r). \tag{34}
\]

Equations (31) and (34) are the intertwining relations that relate the system (19) and (24). These intertwining operators while commuting yield \([\hat{A}^{-}, \hat{A}^{+}] = 2W'(r)\). We recall here that in the case of the harmonic oscillator, one finds that \(W'(r) = 1\) and so the ladder operators associated with the Hamiltonian straightforwardly provide the Heisenberg–Weyl algebra. As a consequence, the ladder operators of the harmonic oscillator perfectly act as annihilation and creation operators. However, in general, the factorization operators are not the ladder operators of the system for other than the harmonic oscillator. The latter situation leads to the (nonlinear) polynomial algebras [36]. In particular, the supersymmetric partners of the harmonic oscillator admit distorted versions of the Heisenberg algebra [37]. These nonlinear algebras can be linearized through the methods given in [38]. However, to the authors’ knowledge, for shape-invariant potentials other than harmonic oscillator, the Balantekin algebraic method [26, 39] is more versatile to derive the necessary annihilation and creation operators. These operators frequently lead to either one of the following algebras: Heisenberg–Weyl algebra, SU(1, 1), SO(2) and \(q\)-deformed algebra depending on the energy spectrum of the underlying shape-invariant potential. As far as the present problem is concerned, on the Fock space, we proceed in the following way.

2.2. Shape invariance property

We attempt to explore the ladder operators by adopting Balantenkin’s method [26]. In this approach, one can derive the ladder operators by analyzing the shape invariance property exhibited by the potential under investigation. The potential (21) can be shown to be shape invariant under the condition

\[
V_2(r; l) = V_1(r; l + 1) + 4\omega. \tag{35}
\]

From this relation, we can define two new operators, namely \(\hat{B}_+\) and \(\hat{B}_-\), which are of the form

\[
\hat{B}_+ = \hat{A}^{+}\hat{T}(l) = \hat{A}^{+} e^{\frac{\hat{J}}{\sqrt{\omega}}}, \quad \hat{B}_- = \hat{T}^{-1}(l)\hat{A}^{-} = e^{-\frac{\hat{J}}{\sqrt{\omega}}}\hat{A}^{-}, \tag{36}
\]

respectively. The commutation relation between these two new operators yields

\[
[\hat{B}_-, \hat{B}_+] = \hat{B}_-\hat{B}_+ - \hat{B}_+\hat{B}_- = e^{-\frac{\hat{J}}{\sqrt{\omega}}}\hat{A}^{-}\hat{A}^{+} e^{\frac{\hat{J}}{\sqrt{\omega}}} - \hat{A}^{-}\hat{A}^{+}. \tag{37}
\]

Recalling the operators identity [40],

\[
e^{\hat{A}^{+}\hat{A}^{-}} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots, \tag{38}
\]

we can evaluate the first term appearing on the right-hand side in equation (37). The result shows

\[
e^{-\frac{\hat{J}}{\sqrt{\omega}}}\hat{A}^{-}\hat{A}^{+} e^{\frac{\hat{J}}{\sqrt{\omega}}} = -\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{l(l+1)}{r^2} + \frac{8\omega}{(2\omega r^2 + 2l + 1)} - \frac{16\omega(2l+1)}{(2\omega r^2 + 2l + 1)}. \tag{39}
\]
On the other hand evaluating $\hat{A}^+ \hat{A}^-$, we find

$$
\hat{A}^+ \hat{A}^- = \frac{d^2}{dr^2} + \omega^2 r^2 + \frac{l(l+1)}{r^2} + \frac{8\omega}{(2\omega r^2 + 2l + 1)} - \frac{16\omega(2l+1)}{(2\omega r^2 + 2l+1)} - \omega(2l+3).
$$

(40)

Subtracting (40) from (39), we obtain

$$
[\hat{B}_-, \hat{B}_+] = 4\omega.
$$

(41)

The relation (41) confirms that the operators $\hat{B}_+$ and $\hat{B}_-$ form the Heisenberg–Weyl algebra, which may also be observed from the energy spectrum of the potential $V_1$, given in (23), which is linear (see equation (23)). We note here that the new operators $\hat{B}_+$ and $\hat{B}_-$ act as the ladder operators of the extended radial oscillator potential as in the case of harmonic oscillator potential [26].

From the above, we can also establish

$$
\hat{a}_r = -\frac{1}{\sqrt{4\omega}} \hat{B}_-, \quad \hat{a}^+_r = -\frac{1}{\sqrt{4\omega}} \hat{B}_+.
$$

(43)

Furthermore, redefining these two operators, $\hat{B}_-$ and $\hat{B}_+$, in such a way that

$$
\hat{a}_r = -\frac{1}{\sqrt{4\omega}} \Phi_{n-1,l}, \quad \hat{a}^+_r = -\frac{1}{\sqrt{4\omega}} \Phi_{n+1,l}.
$$

(42)

we can show that

$$
\hat{a}_r \Phi_{n,l} = \sqrt{n} \Phi_{n-1,l}, \quad \hat{a}^+_r \Phi_{n,l} = \sqrt{n+1} \Phi_{n+1,l}.
$$

(44)

The above relations confirm that these operators perfectly annihilate and create the eigenstate $\Phi_{n,l}$ by absorbing and emitting one photon.

Using these ladder operators, one can construct coherent, squeezed and other types of states and analyze classical/non-classical properties exhibited by the radial oscillator potential (1). In the following, we carry out these studies.

3. Squeezed coherent states

In this section, using the above ladder operators, we construct squeezed coherent states of the three-dimensional isotonic oscillator. The squeezed coherent states of the three-dimensional spherically symmetric oscillator can be obtained by taking a tensor product of squeezed coherent states of radial excitation with squeezed angular momentum coherent states [30]:

$$
|\xi, \alpha\rangle = |\xi_r, \alpha_r\rangle \otimes |\tilde{\xi}, \tilde{\alpha}\rangle.
$$

(45)

The squeezed coherent states which generalize both coherent and squeezed states are defined to be [27]

$$
|z, \alpha\rangle = \hat{S}(z) \hat{D}(\alpha)|0\rangle = \hat{D}(\alpha_0) \hat{S}(z)|0\rangle,
$$

(46)

where $\hat{S}(z) = \exp \left[ \frac{1}{2} (z^* \hat{a}^2 - z \hat{a}^\dagger z^*) \right]$ is the squeezing operator with $z = R e^{i\phi}$ and $\hat{D}(\alpha) = \exp (\alpha \hat{a}^\dagger - \alpha^* \hat{a})$ in which the operators $\hat{a}$ and $\hat{a}^\dagger$ represent annihilation and creation operators, respectively. These operators satisfy the relations given in (44). Here $\alpha = \alpha_0 \cosh R + \alpha_0^* e^{i\phi} \sinh R$, where $R$ and $\phi$ are the squeezing parameters and $\alpha_0$ and $\alpha_0^*$ are the coherent parameters, respectively.
3.1. Radial part

To begin with, we evaluate the squeezed coherent states of the radial part, i.e. \(|\xi_r, \alpha_r\rangle\), by using the definition

\[ |\xi_r, \alpha_r\rangle = \hat{D}(\alpha_{0,r}) \hat{S}(\xi_r) |0\rangle. \tag{47} \]

The squeezing operator can be disentangled as \([27]\)

\[ \hat{S}(\xi_r) = \frac{1}{\cosh R_r} e^{-\frac{1}{2} \tanh R_r \Delta^2} e^{-\ln \cosh R_r \alpha_r^*} e^{\frac{1}{2} \tanh R_r \Delta^2}. \tag{48} \]

From (48), we find

\[ \hat{S}(\xi_r)|0\rangle = \frac{1}{\cosh R_r} e^{-\frac{1}{2} e^{\phi} \tanh R_r \Delta^2} |0\rangle. \tag{49} \]

Substituting (49) into (47), we obtain

\[ |\xi_r, \alpha_r\rangle = \frac{1}{\cosh R_r} \hat{D}(\alpha_{0,r}) e^{\frac{1}{2} \Delta^2} |0\rangle, \tag{50} \]

where we have defined \(\xi_r = -e^{\phi} \tanh R_r\). Using the identity \(\hat{D}^\dagger(\alpha_{0,r}) \hat{D}(\alpha_{0,r}) = 1\), we can rewrite (50) of the form

\[ |\xi_r, \alpha_r\rangle = \frac{1}{\cosh R_r} \hat{D}(\alpha_{0,r}) e^{\frac{1}{2} \Delta^2} \hat{D}^\dagger(\alpha_{0,r}) |0\rangle. \tag{51} \]

The combined action of \(\hat{D}(\alpha_{0,r}) e^{\frac{1}{2} \Delta^2} \hat{D}^\dagger(\alpha_{0,r})\) yields

\[ \hat{D}(\alpha_{0,r}) e^{\frac{1}{2} \Delta^2} \hat{D}^\dagger(\alpha_{0,r}) = e^{\frac{1}{2} (\alpha_r^* - a_r^*)^2} \approx e^{\frac{1}{2} (\alpha_r^* - a_r^*)^2}. \tag{52} \]

Substituting this result into (51) and simplifying the latter, we obtain

\[ |\xi_r, \alpha_r\rangle = \frac{1}{\cosh R_r} e^{\left(\frac{1}{2} \Delta^2 + (\alpha_r^* - a_r^*)^2\right)} |0\rangle = \frac{1}{\cosh R_r} e^{\left(\frac{1}{2} \Delta^2 + \frac{1}{2} a_r^* \xi_r\right)} |0\rangle, \tag{53} \]

where \(\alpha_r = \cosh R_r \alpha_{0,r} + e^{\phi_r} \sinh R_r \alpha_{0,r}^*\).

We can replace the exponential part in (53) by Hermite functions, \(H_n, n = 0, 1, 2, 3, \ldots\), i.e. \([41]\)

\[ e^{\left(\frac{1}{2} \Delta^2 + \frac{1}{2} a_r^* \xi_r\right)} = \sum_{n=0}^{\infty} \frac{H_n(a^* \sqrt{1-|\xi_r|^2})}{n!} \left(-\frac{\xi_r}{2}\right)^{n/2} \hat{a}_r^n, \tag{54} \]

so that equation (53) now becomes

\[ |\xi_r, \alpha_r\rangle = \sum_{n=0}^{\infty} \frac{H_n(a^* \sqrt{1-|\xi_r|^2})}{n!} \left(-\frac{\xi_r}{2}\right)^{n/2} \hat{a}_r^n |0\rangle, \tag{55} \]

Thus the normalized squeezed coherent states of the radial part are found to be

\[ |\xi_r, \alpha_r\rangle = N_r \sum_{n=0}^{\infty} \frac{H_n(a^* \sqrt{1-|\xi_r|^2})}{\sqrt{n!}} \left(-\frac{\xi_r}{2}\right)^{n/2} |n\rangle, \tag{56} \]

where \(N_r\) is the normalization constant whose exact value is given by

\[ N_r = (1 - |\xi_r|^2)^{1/4} \exp \left[-\frac{1}{2} (\alpha_r^* \xi_r + \alpha_r \xi_r^* + 2|\alpha_r|^2)\right], \quad 0 < |\xi_r| < 1. \tag{57} \]

In the following, we construct squeezed coherent states of the angular part.
3.2. Angular momentum part

The angular momentum part of the Schrödinger equation (9) with \( d = 3 \) admits spherical harmonics \( Y_{l,m}(\theta, \phi) \) as the solution which is defined to be simultaneous eigenstates of the operators \( \hat{L}_2 \) and \( \hat{L}_z \) with the eigenvalues \( l(l+1) \) and \( m \), i.e.

\[
\hat{L}_2 |l, m\rangle = l(l+1) |l, m\rangle,
\]

where \( \langle \theta, \phi | l, m \rangle = Y_{l,m}(\theta, \phi) \) and \( |l, m\rangle \) are nothing but the angular momentum states.

The angular momentum coherent states can be constructed by expressing the angular momentum part, \( \hat{L}_z \), in terms of the new annihilation and creation operators, i.e. \( \hat{a}_\pm \) and \( \hat{a}^\dagger_\pm \) [42, 43]:

\[
\hat{L}_\pm = \hat{a}_\pm \hat{a}_\mp^\dagger.
\]

\[
\hat{L}_z = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_++\hat{a}_-^\dagger \hat{a}_-),
\]

\[
\hat{L}_z^2 = \frac{1}{2}(\hat{n}_++\hat{n}_-)(\frac{1}{2}(\hat{n}_++\hat{n}_-)+1),
\]

where \( \hat{n}_++\hat{n}_- = \hat{n} \) is the number operator. The action of the operators \( \hat{a}_+ \) and \( \hat{a}_- \) on the number states is given by [42]

\[
\hat{a}_+ |n\rangle = \sqrt{n+1} |n+1, n_\mp -1\rangle, \quad \hat{a}_- |n\rangle = \sqrt{n} |n_\mp, n_- -1\rangle,
\]

\[
\hat{a}^\dagger_+ |n\rangle = \sqrt{n+1} |n+1, n_\mp\rangle, \quad \hat{a}^\dagger_- |n\rangle = \sqrt{n} |n_\mp, n_+\rangle.
\]

The operator \( \hat{L}_z \) acts on the states \( |l, m\rangle \) as follows

\[
\hat{L}_z |l, m\rangle = \sqrt{(l+m)(l-m+1)} |l, m - 1\rangle.
\]

The squeezed coherent states associated with the angular momentum part can be obtained by using the definition [29],

\[
|\tilde{\xi}, \tilde{\alpha}\rangle = \hat{D}(\tilde{\alpha}_0) \hat{S}(\tilde{\xi}) |0, 0\rangle.
\]

Since we are dealing two modes, we consider the above equation is of the form

\[
|\tilde{\xi}, \tilde{\alpha}\rangle = \hat{D}_+ (\alpha_{0,+}) \hat{S}_+ (\xi_+) \hat{D}_- (\alpha_{0,-}) \hat{S}_- (\xi_-) |0, 0\rangle,
\]

with \( \xi_\pm = -\tanh R_\pm e^{i\phi_\pm} \) and \( \alpha_\pm = \alpha_{0, \pm} \cosh R_\pm + \alpha_{0, \pm}^* \sinh R_\pm e^{i\phi_\pm} \) are the squeezing parameters and displaced coherent parameters, respectively.

By adopting the calculations given in the radial part, we find the combined action of displacement and squeezed operators to produce (see equations (63) and (64))

\[
\hat{D}_+ (\alpha_{0,+}) \hat{S}_+ (\xi_+) = \exp \left( \frac{\alpha_+}{\cosh R_+} \hat{a}_+^\dagger + \frac{\xi_+}{2} \hat{a}_+^2 \right).
\]

Recalling the identity, \( \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2} \), we can express the operator appearing in (68) as an infinite series in Hermite polynomials:

\[
\hat{D}_+ (\alpha_{0,+}) \hat{S}_+ (\xi_+) = \sum_{n_{\pm}=0}^{\infty} \frac{H_{n_+}(\frac{\alpha_+ \sqrt{1-|\xi_+|^2}}{\sqrt{2R_+}})}{n_+!} \left( -\frac{\xi_+}{2} \right)^{n_+/2} \hat{a}_+^{n_+}.
\]
By induction, we find
\[ \hat{D}_-(\alpha_0, -\hat{\xi}_-) \hat{S}_-(\xi_-) = \sum_{n_-=0}^{\infty} H_{n_-} \frac{(-\xi_-)}{2} \frac{\hat{a}_-^{n_-}}{n_-!}. \] (70)

Substituting (69) and (70) into (67), we obtain
\[ |\tilde{\xi}, \tilde{\alpha}\rangle = N_{\pm} \sum_{n_+, n_-=0}^{\infty} H_{n_+} \frac{(-\xi_+)}{2} \frac{\hat{a}_+^{n_+}}{n_+!} \frac{H_{n_-} \frac{(-\xi_-)}{2} \frac{\hat{a}_-^{n_-}}{n_-!}}{n_+!n_-!}, \] (71)

where \( N_{\pm} \) is the normalization constant which can be fixed as
\[ N_{\pm} = (1 - |\xi_+|^2)^{1/4} (1 - |\xi_-|^2)^{1/4} \exp \left[ -\frac{1}{2} (\alpha_+^2 \xi_+^2 + \alpha_-^2 \xi_-^2 + 2|\alpha_+|^2) \right] \]
\[ \times \exp \left[ -\frac{1}{2} (\alpha_+^2 \xi_+^2 + \alpha_-^2 \xi_-^2 + 2|\alpha_-|^2) \right], \quad 0 < |\xi_\pm| < 1 \] (72)
through the usual procedure.

Using the identity,
\[ |l, m\rangle = \frac{(\hat{a}_+^\dagger)^{l+m}}{\sqrt{l!(l+m)!}} |0, 0\rangle, \] (73)
with the restriction \( n_+ = l + m \) and \( n_- = l - m \), we can rewrite the squeezed angular momentum coherent states (71) in the form
\[ |\tilde{\xi}, \tilde{\alpha}\rangle = N_{\pm} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_{l+m} \frac{(-\xi_+)}{2} \frac{\hat{a}_+^l \hat{a}_-^m}{\sqrt{l!(l+m)!}} \left( \frac{-\xi_-}{2} \right)^{-m} \left( \frac{-\xi_-}{2} \right)^{l-m} |l, m\rangle. \] (74)

Finally, substituting the squeezed coherent states of the radial part (see equation (56)) and the angular part (see equation (74)) into (45), we obtain the squeezed coherent states of the three-dimensional system (12) which in turn reads
\[ |\xi, \alpha\rangle = N_{\xi, \alpha} \sum_{n_0=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_{l+m} \frac{(-\xi_+)}{2} \frac{\hat{a}_+^l \hat{a}_-^m}{\sqrt{l!(l+m)!}} \left( \frac{-\xi_-}{2} \right)^{-m} \left( \frac{-\xi_-}{2} \right)^{l-m} |n, l, m\rangle, \quad 0 < |\xi_+| < 1, \quad 0 < |\xi_-| < 1. \] (75)

The normalization constant \( N_{\xi, \alpha} \) is given by
\[ N_{\xi, \alpha} = N_r \times N_{\xi} \]
\[ = \left[ (1 - |\xi_+|^2) (1 - |\xi_-|^2) \right]^{1/4} \exp \left[ -\frac{1}{2} (\alpha_+^2 \xi_+^2 + \alpha_-^2 \xi_-^2 + 2|\alpha_+|^2) \right] \]
\[ \times \exp \left[ -\frac{1}{2} (\alpha_+^2 \xi_+^2 + \alpha_-^2 \xi_-^2 + 2|\alpha_-|^2) \right], \quad 0 < |\xi_+| < 1, \quad 0 < |\xi_-| < 1. \] (76)
Equation (75) can also be written in a more compact form, namely
\[ |\xi, \alpha\rangle = N_{\xi, \alpha} \sum_{n_0=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{n, l, m} |n, l, m\rangle, \] (77)
with

$$c_n = H_n \left( \frac{\alpha \sqrt{1 - |\xi|^2}}{\sqrt{n!}} \right) \left( -\frac{\xi}{2} \right)^n.$$  (78a)

and

$$c_{l,m} = H_{l+m} \left( \frac{\alpha \sqrt{1 - |\xi|^2}}{\sqrt{l + m)!l - m)!} \right) \left( -\frac{\xi}{2} \right)^l \left( -\frac{\xi}{2} \right)^m.$$  (78b)

The squeezed coherent states given in (77) are expressed in terms of both number states ($|n\rangle$) and angular momentum states ($|l, m\rangle$), i.e. the states are expressed in terms of bound state solutions $\psi(r, \theta, \phi)$. In other words, a complete description of the three-mode squeezed coherent states is established now in terms of three independent squeezing parameters $\xi$, $\alpha$, and the range $\xi_r$ $\alpha$ to be

$$\leq 0 < |\xi_j| < 1$$

and the range $\alpha_0^\prime$ to be $-\infty$ to $\infty$ since the normalization constant $N_{\xi, \alpha}$ is defined in the range $0 < |\xi_j| < 1$, where $j = r, +, -$.

In the following section, we analyze the classical/non-classical nature of the obtained squeezed coherent states.

### 3.3. Completeness condition

In this subsection, we demonstrate that the three-mode squeezed coherent states (77) satisfy the completeness condition. The three-mode squeezed coherent states are represented in terms of three coherent parameters ($\alpha$, $\alpha_+$, $\alpha_-$) and three independent displaced coherent parameters ($\xi_r$, $\xi_+$, $\xi_-$). By varying these parameters, we can analyze both the classical and the non-classical nature of these states. We recall here that whenever quantum states exhibit the Poissonian statistics they are said to possess classical nature [44]. Any deviation from this behavior represents the non-classical nature of the states. To proceed further, we restrict the range of values of $\xi_j$ to be $0 < |\xi_j| < 1$ and the range $\alpha_0^\prime$ to be $-\infty$ to $\infty$ since the normalization constant $N_{\xi, \alpha}$ is defined in the range $0 < |\xi_j| < 1$, where $j = r, +, -$.

$$\leq 0 < |\xi_j| < 1$$

where $j = r, +, -$.

In the following section, we analyze the classical/non-classical nature of the obtained squeezed coherent states.

In this subsection, we demonstrate that the three-mode squeezed coherent states (77) satisfy the completeness condition. The three-mode squeezed coherent states are represented in terms of three coherent parameters ($\alpha_0^\prime$, $j = r, +, -$) and three squeezed parameters ($\xi_j$, $j = r, +, -$). Now we prove that the squeezed coherent states resolve the identity operator [45]

$$\left( \frac{i}{\lambda \pi} \right)^3 \int |\xi, \alpha\rangle \langle \xi, \alpha| d^2\alpha = \hat{1},$$  (79)

when the integration is carried over the entire space of $\alpha$ ($\alpha_r$, $\alpha_+$, $\alpha_-$). In this analysis, we consider that $\alpha$ varies only with respect to $\alpha_0^\prime$ and treat $\xi_j$ as arbitrary constants.

For the squeezed coherent parameters,

$$\alpha_j = \alpha_0^\prime - \xi_j \alpha_0^\prime, \quad j = r, \theta, \phi,$$  (80)

we can show that

$$d\alpha_j d\alpha_j^\ast = \left| \frac{\partial \alpha_j}{\partial \alpha_0^\prime}, \frac{\partial \alpha_j}{\partial \alpha_0^\prime} \right| \frac{\partial \alpha_j}{\partial \alpha_0^\prime}, \frac{\partial \alpha_j}{\partial \alpha_0^\prime} \right| d\alpha_0^\prime, d\alpha_0^\prime = d\alpha_0^\prime, d\alpha_0^\prime.$$  (81)

Substituting the expression given in (77) into the integral (79), we obtain

$$G = \int |\xi, \alpha\rangle \langle \xi, \alpha| d^2\alpha = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{n'=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{m'=0}^{l'} |n', l', m'\rangle \langle n, l, m|$$

$$\times \int \int c_{n', l', m'}^{\ast} c_{l, m}^{\ast} c_{c, l, m} N_{\xi, \alpha}^2 d^2\alpha, d^2\alpha_+, d^2\alpha_-.$$  (82)
To start with, we evaluate the radial part (which we call as $G_R$) in (82):

\[
G_R = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} |n'\rangle \langle n| \int c_{n'} c_n N_2 d^2 \alpha_r
\]

\[
= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} |n'\rangle \langle n| \sqrt{n!} \sqrt{n'}! \int \left[ -|\alpha_r|^2 - \frac{1}{2} (\xi_r \alpha_r^* + \overline{\xi_r} \alpha_r) \right] \times H_n \left( \frac{\alpha_r \sqrt{1 - |\xi_r|^2}}{\sqrt{2 \overline{\xi_r}}} \right) H_n \left( \frac{\overline{\alpha_r} \sqrt{1 - |\xi_r|^2}}{\sqrt{2 \xi_r}} \right) d\alpha_r d\alpha_r^*.
\]  

(83)

Substituting (80) and (81) into (83) and separating the integral (which we call as $I_1$) from the summation part, we obtain

\[
I_1 = \int \int \exp \left[ -|\alpha_{0,r}|^2 + \frac{1}{2} (\xi_r \alpha_{0,r}^* + \overline{\xi_r} \alpha_{0,r}) \right] \times H_0 \left( \frac{\alpha_{0,r} \sqrt{1 - |\xi_r|^2}}{\sqrt{2 \overline{\xi_r}}} \right) H_0 \left( \frac{\overline{\alpha_{0,r}} \sqrt{1 - |\xi_r|^2}}{\sqrt{2 \xi_r}} \right) d\alpha_{0,r} d\alpha_{0,r}^*.
\]  

(84)

To evaluate the integral (84), we introduce the transformation

\[
\frac{\alpha_{0,r}}{\sqrt{-2 \overline{\xi}_r}} = \frac{y_j + |\xi_j| y_j^*}{1 - |\xi_j|^2} \quad \text{and} \quad \frac{\alpha_{0,j}^*}{\sqrt{-2 \xi_j}} = \frac{y_j^* + |\xi_j| y_j}{1 - |\xi_j|^2}
\]  

so that

\[
d\alpha_{0,r} d\alpha_{0,r}^* = \left| \frac{\partial \alpha_{0,r}}{\partial y_j} \frac{\partial \alpha_{0,j}^*}{\partial y_j^*} \right| dy_j dy_j^* = \frac{-2|\xi_j|}{1 - |\xi_j|^2} dy_j dy_j^*, \quad j = r, \theta, \phi.
\]  

(85)

(86)

In the new variables, the integral (84) reads

\[
I_1 = \frac{-2|\xi_j|}{1 - |\xi_j|^2} \int \int \exp \left[ \frac{|\xi_j|^2 (y_j^2 + y_j^* y_j^*) + 2|\xi_j| y_j y_j^*}{1 - |\xi_j|^2} \right] H_0(y_j) H_0(y_j^*) dy_j dy_j^*.
\]  

(87)

With another change of variable, $z = i \frac{|\xi_j|}{\sqrt{1 - |\xi_j|^2}} y_j$, the integral (87) can be brought to the form

\[
I_1 = \frac{-2i}{\sqrt{1 - |\xi_j|^2}} \int \exp \left[ \frac{|\xi_j|^2 y_j^2}{1 - |\xi_j|^2} \right] H_0(y_j^*) \int \exp \left[ -z^2 + \frac{-iy_j^*}{\sqrt{1 - |\xi_j|^2}} \right] \times H_0 \left[ -i \frac{1 - |\xi_j|^2}{|\xi_j|} z \right] dz dy_j^*.
\]  

(88)

With the identity [41]

\[
\int e^{-\left(-y^*y\right)} H_n(\alpha x) dx = \sqrt{\pi} (1 - \alpha^2)^{n/2} H_n \left( \frac{\alpha}{\sqrt{1 - \alpha^2}} y \right),
\]  

(89)

the second integral in (88) can be evaluated and given in terms of Hermite polynomials:

\[
I_1 = \frac{-2i}{\sqrt{1 - |\xi_j|^2}} \sqrt{\pi} \left( \frac{1}{|\xi_j|} \right)^n \int \exp \left[ \frac{|\xi_j|^2 (1 - y_j^2)}{1 - |\xi_j|^2} \right] H_n(-y_j^*) H_n(y_j^*) dy_j^*.
\]  

(90)

Now using the orthogonality property of the Hermite polynomials,

\[
\int e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^n n! \delta_{n,m},
\]  

(91)
where $\delta_{n,n'}$ is the Kronecker-delta function, we find

$$I_1 = -2\pi i \frac{2}{|\xi_r|^2} \left( \frac{-2}{|\xi_r|^2} \right)^n n! \delta_{n,n'}.$$ (92)

Substituting this expression, (92), into $G_R$ (see equation (83)) and simplifying the resultant expression, we obtain

$$G_R = -2\pi i \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{|n'|}{\sqrt{n!n'^!}} \left( \frac{-\xi_r^*}{2} \right)^{n'/2} \left( \frac{-\xi_r}{2} \right)^{n/2} \left( \frac{-2}{|\xi_r|^2} \right)^n n! \delta_{n,n'}.$$ (93)

From the above equation, we observe that radial part of the integral provides

$$G_R = -2\pi \sum_{n=0}^{\infty} |n\rangle \langle n|.$$ (94)

In a similar way, we can evaluate the integrals involving the other two variables, namely $\theta$ and $\phi$ (since the procedure is exactly the same as that of radial part, we do not repeat the details here). Our result shows that

$$G_A = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{m'=l}^{l} |l', m'angle \langle l, m| \int \int \int c_n^* c_{l', m'}^* c_n c_{l, m} N_1^2 d^2\alpha_r d^2\alpha_\theta d^2\alpha_\phi$$

$$= -4\pi^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |l, m\rangle \langle l, m|. $$ (95)

The integral (82) is completely evaluated now. The resultant value turns out to be

$$G = G_R \times G_A = 18\pi^3 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |l, m\rangle \langle n| \langle m| |l, m|.$$ (96)

This in turn confirms that the states $|\xi, \alpha\rangle$ resolve the identity operator:

$$\sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |n, l, m\rangle \langle n, l, m| = \hat{I}. $$

The result ensures that the constructed three-mode squeezed coherent states form a complete set.

4. Non-classical properties

4.1. Mandel’s $Q$-parameter

In this subsection, we study Mandel’s $Q$-parameter for the three-mode squeezed coherent states (77). We evaluate Mandel’s $Q$-parameter associated with each mode [46]

$$Q_j = \frac{\langle n_j^2 \rangle}{\langle \hat{n}_j \rangle} - \langle \hat{n}_j \rangle - 1, \quad j = r, +, -. $$ (97)

Here, $n_j$ denotes the number of particles in the respective mode which can be uniquely determined from their associated number operators $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$. The action of the ladder operators $\hat{a}_j^\dagger$ and $\hat{a}_j$ on the number states $|n_j\rangle$ is given in equations (44), (63) and (64), respectively.

To evaluate Mandel’s $Q$-parameter, we consider the squeezed coherent states to be of the form

$$|\xi, \alpha\rangle = N_{\xi, \alpha} \sum_{n=0}^{\infty} \sum_{n_+ = 0}^{\infty} \sum_{n_- = 0}^{\infty} c_n c_{n_+} c_{n_-} |n, n_+, n_-\rangle,$$ (98)
where the constants $c_b$ and $c_{n_m}$ are the same as given in (78a) and (78b) with $l_\pm m$ being replaced by $n_\pm$. To find the expectation values $\langle \hat{n}_j \rangle$ and $\langle \hat{n}_j^2 \rangle$, as we did earlier, we first calculate the radial part. Doing so, we find

$$
\langle \hat{n}_j \rangle = N_{j,a}^2 \sum_{n=0}^{\infty} \sum_{n_\pm=0}^{\infty} \sum_{n_m=0}^{\infty} c_n^* c_{n_m} c_{n_\pm} n
$$

(99)

and

$$
\langle \hat{n}_j^2 \rangle = N_{j,a}^2 \sum_{n=0}^{\infty} \sum_{n_\pm=0}^{\infty} \sum_{n_m=0}^{\infty} c_n^* c_{n_m} c_{n_\pm} n^2
$$

(101)

The details of evaluating expressions (99) and (101) are given in appendix B.

Since $\hat{n}_\pm |n_+, n_-\rangle = n_\pm |n_+, n_-\rangle$, the expectation values $\langle \hat{n}_\pm \rangle$ and $\langle \hat{n}_\pm^2 \rangle$ associated with the angular part also provide the same expressions given in (100) and (102). Hence, in general, we can write the expectation values $\langle \hat{n}_j \rangle$ and $\langle \hat{n}_j^2 \rangle$ that are of the form

$$
\langle \hat{n}_j \rangle = \frac{1}{(1-|\xi_j|^2)} \left[ |\alpha_j|^2 (1 + |\xi_j|^2) + \xi_j^* \alpha_j^* + \xi_j \alpha_j \right]
$$

(103)

and

$$
\langle \hat{n}_j^2 \rangle = \frac{1}{(1-|\xi_j|^2)} \left[ |\alpha_j|^4 (1 + |\xi_j|^2)^2 + |\xi_j|^2 (2 + |\xi_j|^2) + (\xi_j^* \alpha_j^2 + \xi_j \alpha_j^*)^2
+ (2 (1 + |\alpha_j|^2)(1 + |\xi_j|^2) + 2 |\xi_j|^2)(\xi_j^* \alpha_j^2 + \xi_j \alpha_j^*)
+ |\alpha_j|^2 (1 + 8 |\xi_j|^2 + 3 |\xi_j|^4) \right].
$$

(104)

where $\alpha_j = \frac{\alpha_{0,j} - \xi_j \alpha_{1,j}}{\sqrt{(1-|\xi_j|^2)}}$ and $j = r, +, -$.

From expressions (103) and (104), we can analyze the Poissonian ($Q_j = 0$), sub-Poissonian ($Q_j < 0$) and super-Poissonian ($Q_j > 0$) nature of the states in each mode. We can also calculate the parameter $Q_j$ of the three-mode squeezed coherent states which basically depends on the parameters $\alpha_{0,j}$ and $\xi_j$. The obtained numerical results are depicted in figure 1. The figure shows that when $|\xi_j| < 1$ and $\alpha_{0,j} = 3$, the $Q$-parameter takes positive and negative values which in turn confirm the super-Poissonian ($Q_j > 0$) and sub-Poissonian ($Q_j < 0$) nature of the states. The sub-Poissonian statistics indicates the non-classical nature exhibited by the states.

4.2 Quadrature squeezing

The squeezed coherent states given in (77) are expressed in terms of the variables $r, \theta$ and $\phi$. As far as the radial part is concerned, one can analyze the squeezing in the observables such as the generalized position ($w_r$) and its conjugate momentum ($p_r$). To show the squeezing in the angular variables $\theta$ and $\phi$, we have to consider the angular momentum quantities $L_x, L_y$, and
are defined as \[47\]

In the following, we analyze the squeezing in position and momentum coordinates which are defined as [47]

$$\hat{w}_r = \frac{1}{\sqrt{2}} (\hat{a}_r^\dagger + \hat{a}_r), \quad \hat{p}_r = \frac{i}{\sqrt{2}} (\hat{a}_r^\dagger - \hat{a}_r).$$ \hfill (105)

To analyze the squeezing in the quadratures \(\hat{w}_r\) and \(\hat{p}_r\) in which the Heisenberg uncertainty relation holds, \((\Delta \hat{w}_r)^2 (\Delta \hat{p}_r)^2 \geq \frac{1}{4}\), where \(\Delta \hat{w}_r\) and \(\Delta \hat{p}_r\) denote uncertainties in \(\hat{w}_r\) and \(\hat{p}_r\), respectively, we introduce the following inequalities, i.e.

$$I_1 = |\langle \hat{a}_r \rangle|^2 + |\langle \hat{a}_r^\dagger \rangle|^2 - |\langle \hat{a}_r \rangle|^2 - |\langle \hat{a}_r^\dagger \rangle|^2 - 2|\langle \hat{a}_r \rangle \langle \hat{a}_r^\dagger \rangle| + 2|\langle \hat{a}_r \hat{a}_r^\dagger \rangle| < 0,$$ \hfill (106)

$$I_2 = -|\langle \hat{a}_r \rangle|^2 - |\langle \hat{a}_r^\dagger \rangle|^2 + |\langle \hat{a}_r \rangle|^2 + |\langle \hat{a}_r^\dagger \rangle|^2 - 2|\langle \hat{a}_r \rangle \langle \hat{a}_r^\dagger \rangle| + 2|\langle \hat{a}_r \hat{a}_r^\dagger \rangle| < 0,$$ \hfill (107)

which can be derived from the squeezing condition \((\Delta \hat{r})^2 < \frac{1}{2}\) or \((\Delta \hat{p}_r)^2 < \frac{1}{2}\) by implementing the expressions given in (105). The expectation values should be calculated with respect to the squeezed coherent states \(\xi, \alpha\) in which the squeezing property has to be examined.

For the squeezed coherent states (77), we obtain the following values for the quantities which appear in equations (106) and (107),

$$\langle \hat{a}_r \rangle = \frac{\alpha_r + \xi_r |\alpha_r|}{\sqrt{1 - |\xi_r|^2}}, \quad \langle \hat{a}_r^\dagger \rangle = \frac{\alpha_r^* + \xi_r^* |\alpha_r|}{\sqrt{1 - |\xi_r|^2}},$$ \hfill (108)

$$\langle \hat{a}_r^2 \rangle = \frac{\xi_r + (\alpha_r^* \xi_r + \alpha_r)^2}{1 - |\xi_r|^2}, \quad \langle \hat{a}_r^\dagger^2 \rangle = \frac{\xi_r^* + (\alpha_r \xi_r^* + \alpha_r^*)^2}{1 - |\xi_r|^2},$$ \hfill (109)

$$\langle \hat{a}_r^\dagger \hat{a}_r \rangle = \frac{1}{1 - |\xi_r|^2} (|\alpha_r|^2 (1 + |\xi_r|^2) + |\xi_r|^2 + \alpha_r^* \xi_r + \alpha_r \xi_r^*),$$ \hfill (110)

where \(\alpha_r = \frac{\alpha_0 - \xi_0 \alpha_r^*}{\sqrt{1 - |\xi_r|^2}}\).

With the expressions given in (108)–(110), we evaluate the inequalities (106) and (107) numerically and plot the outcome in figure 2 with \(\alpha_{0,r} = |\alpha_{0,r}|e^{i\theta_{0,r}}\). From figure 2, we observe that the identities given in equations (106) and (107) for the squeezed coherent states \(\xi, \alpha\), satisfying the uncertainty relation, show \(I_2 < 0\) and \(I_1 > 0\). This in turn confirms the squeezing in the quadrature \(\hat{p}_r\), for all values of \(\alpha_{0,r}\).

As far as the angular part is concerned, the squeezing in the quadratures \(\hat{L}_x\) or \(\hat{L}_y\) can be analyzed through either one of the normalized quantities \(S_{L_x} < 0\) or \(S_{L_y} < 0\), where
Figure 2. The plots of (a) \( I_1 \) and (b) \( I_2 \) which are calculated with respect to squeezed coherent states (77) with \( \xi = 0.3 \) for different values of \( \lvert \alpha \rvert \).

\[
S_{\alpha r} = \frac{2\Delta \lvert \alpha \rvert}{\lvert \alpha \rvert} \quad \text{and} \quad S_{\alpha r} = \frac{2\Delta \lvert \alpha \rvert}{\lvert \alpha \rvert} [48].
\]

We can express the operators \( \hat{L}_x \) and \( \hat{L}_y \) in terms of \( \hat{L}_+ \) and \( \hat{L}_- \), namely

\[
\hat{L}_x = \frac{1}{2} (\hat{L}_+ + \hat{L}_-), \quad \hat{L}_y = \frac{i}{2} (\hat{L}_+ - \hat{L}_-)
\]

so that the uncertainty of \( \hat{L}_x \) and \( \hat{L}_y \) can now be expressed in terms of \( \hat{L}_+ \) and \( \hat{L}_- \):

\[
(\Delta \hat{L}_x)^2 = \langle \hat{L}_x^2 \rangle - \langle \hat{L}_x \rangle^2 = \frac{1}{4} ((\hat{L}_+^2) + (\hat{L}_-^2) + (\hat{L}_+ \hat{L}_-) + ((\hat{L}_+ + \hat{L}_-)^2)),
\]

\[
(\Delta \hat{L}_y)^2 = \langle \hat{L}_y^2 \rangle - \langle \hat{L}_y \rangle^2 = -\frac{1}{4} ((\hat{L}_+^2) + (\hat{L}_-^2) - (\hat{L}_+ \hat{L}_-) - ((\hat{L}_+ - \hat{L}_-)^2)).
\]

For the squeezed coherent states (77), the expectation values in (112) and (113) are found to be

\[
\langle \hat{L}_+ \rangle = \left( \frac{\alpha^*_+ - \xi^*_+ \alpha_+}{\sqrt{1 - \lvert \xi_+ \rvert^2}} \right) \left( \frac{\alpha_- - \xi_- \alpha^*_+}{\sqrt{1 - \lvert \xi_- \rvert^2}} \right),
\]

\[
\langle \hat{L}_- \rangle = \left( \frac{\alpha^*_+ - \xi^*_+ \alpha_+}{\sqrt{1 - \lvert \xi_+ \rvert^2}} \right) \left( \frac{\alpha_-^* - \xi_-^* \alpha_+}{\sqrt{1 - \lvert \xi_- \rvert^2}} \right),
\]

\[
\langle \hat{L}_+^2 \rangle = \frac{\xi^*_+ \xi_-}{\sqrt{1 - \lvert \xi_+ \rvert^2 \sqrt{1 - \lvert \xi_- \rvert^2}}} \left( \frac{(\alpha^*_+ - \xi^*_+ \alpha_+)^2}{\xi_+} + 1 \right) \left( \frac{(\alpha_-^* - \xi_-^* \alpha_+)^2}{\xi_-} + 1 \right),
\]

\[
\langle \hat{L}_-^2 \rangle = \frac{\xi^*_+ \xi_-}{\sqrt{1 - \lvert \xi_+ \rvert^2 \sqrt{1 - \lvert \xi_- \rvert^2}}} \left( \frac{(\alpha^*_+ - \xi^*_+ \alpha_+)^2}{\xi_+} + 1 \right) \left( \frac{(\alpha_-^* - \xi_-^* \alpha_+)^2}{\xi_-} + 1 \right),
\]

\[
\langle \hat{L}_+ \hat{L}_- \rangle = \left( \frac{|\alpha^*_+ - \xi^*_+ \alpha_+|^2}{\sqrt{|\xi_+|^2 - 1}} + \frac{|\xi_+|^2}{|\xi_+|^2 - 1} \right) \left( 1 + \frac{|\alpha_-^* - \xi_-^* \alpha_+|^2}{\sqrt{|\xi_-|^2 - 1}} + \frac{|\xi_-|^2}{|\xi_-|^2 - 1} \right),
\]

\[
|\alpha_0| = 0.3 \quad \text{for different values of } |\alpha|.
\]
Figure 3. The plot of normalized quantities (a) $S_{L_{x}}$ and (b) $S_{L_{y}}$ with $\xi_{+} = \xi_{-} = 0.1$ and $\alpha_{0,+} = 1.3$.

\[
\langle \hat{L}_{-}, \hat{L}_{+} \rangle = \left( \frac{\alpha_{+} - \xi_{+} \alpha_{+}}{\sqrt{\xi_{+}}^2 - 1} + \frac{\xi_{+}}{\sqrt{\xi_{+}}^2} + 1 \right) \left( \frac{\alpha_{-} - \xi_{-} \alpha_{-}}{\sqrt{\xi_{-}}^2 - 1} + \frac{\xi_{-}}{\sqrt{\xi_{-}}^2} - 1 \right), \quad (119)
\]

\[
\langle \hat{L}_{c} \rangle = \frac{1}{2} \left( \frac{1}{\sqrt{\xi_{+}}^2} - 1 \right) \left( (\alpha_{+} - \xi_{+} \alpha^{*}_{+})(\alpha^{*}_{+} - \xi_{+} \alpha_{+}) + |\xi_{+}| \right) \]

\[- \frac{1}{2} \left( \frac{1}{\sqrt{\xi_{-}}^2} - 1 \right) \left( (\alpha_{-} - \xi_{-} \alpha^{*}_{-})(\alpha^{*}_{-} - \xi_{-} \alpha_{-}) + |\xi_{-}| \right). \quad (120)
\]

where $\alpha_{\pm} = \alpha_{0,\pm} e^{\pm s \sqrt{\xi_{\pm}}}$.

We evaluate the normalized quantities $S_{L_{x}}$ and $S_{L_{y}}$ numerically by substituting the expectation values (114)–(120) into the uncertainties (112) and (113). We plot the numerical results in figure 3 where we have considered $\alpha_{0,+} = x_{+} + iy_{+}$ with $\xi_{+} = \xi_{-} = 0.1$ and $\alpha_{0,-} = 1.3$. For this choice of parameters, figure 3 shows that $S_{L_{x}} > 0$ and $S_{L_{y}} < 0$ which explicitly demonstrate the squeezing in $\hat{L}_{c}$.

5. Wigner function for the three-dimensional isotonic oscillator

In this subsection, we evaluate the Wigner function ($W(x, p)$) of the squeezed coherent states (75). The Wigner function (a quasi-probability distribution function), which was introduced as quantum corrections in classical statistical mechanics, normally takes negative values in certain domains of phase space so that it cannot be interpreted as a classical distribution function which is non-negative by necessity [49].

The Wigner function for the single mode state is described by the density operator $\hat{\rho}$ which can be written as [31]

\[
W(\zeta) = 2 \text{Tr}[\hat{\rho} \hat{T}(\zeta, 0)], \quad \hat{T}(\zeta, 0) = \hat{D}(\zeta, \zeta^{*}) e^{i \pi \hat{a}^{\dagger} \hat{a}} \hat{D}^{-1}(\zeta, \zeta^{*}), \quad (121)
\]

where $\hat{D}(\zeta, \zeta^{*})$ and $\hat{\rho}$ are the displacement and density operators, respectively, and $\hat{T}$ is the complex Fourier transform of the $s$-parameterized displacement operator $\hat{D}(\eta, s)$ in which $\eta$ is the coherent eigenvalue with $s = 0$ corresponding to the Wigner distribution function.

Since $\hat{\rho} = |\xi, \alpha\rangle \langle \xi, \alpha|$, the Wigner function for a single mode can be explicitly calculated from

\[
W(\zeta) = 2\langle \xi, \alpha | \hat{T}(\zeta, 0) | \alpha, \xi \rangle. \quad (122)
\]

Prolonging the definition (122) to three orthogonal modes [32], we obtain

\[
W(\langle \zeta_{1}, \zeta_{2}, \zeta_{3} \rangle) = 8 \text{Tr} \left[ \hat{\rho} \prod_{i=1}^{3} \hat{T}_{i}(\zeta_{i}, 0) \right], \quad \hat{T}_{i}(\zeta_{i}, 0) = \hat{D}(\zeta_{i}) e^{i \pi \hat{a}^{\dagger}_{i} \hat{a}_{i}} \hat{D}^{-1}(\zeta_{i}), \quad (123)
\]
where $\xi_1 = \zeta_r$, $\xi_2 = \zeta_s$, and $\xi_3 = \zeta_-$. By applying this definition, (123), to the three-mode squeezed coherent states (75), we obtain

$$W((\xi_r, \xi_s, \xi_-)) = N_{\xi, \alpha}^2 \sum_{n, n'} \sum_{\ell, \ell'} \sum_{m, m'} \sum_{l, l'} c_n^\ast c_{n'} c_{m'} c_{l'} c_{l, m}(n')\tilde{T}(\xi_r, 0)\tilde{T}(\xi_s, 0)\tilde{T}(\xi_-, 0)\langle n|\tilde{T}(\xi_r, 0)|n\rangle,$$

(124)

where $c_n$ and $c_{m, l}$ are given in (78a) and (78b). The above equation can be evaluated by separating the radial part from the angular part:

$$W((\xi_r, \xi_s, \xi_-)) = N_{\xi, \alpha}^2 \sum_{n, n'} \sum_{\ell, \ell'} \sum_{m, m'} \sum_{l, l'} c_n^\ast c_{n'} c_{m'} c_{l'} c_{l, m}(n')\tilde{T}(\xi_r, 0)|n\rangle$$

(125)

The operation of the dual state $|n'\rangle$ on the state $\tilde{T}(\xi_r, 0)|n\rangle$ may be known as transition probability, expressed in terms of $n$, has already been reported in [31]. The result shows that

$$|n'|\tilde{T}(\xi_r, 0)|n\rangle = e^{-2|\xi|} \left(\frac{n!}{n'}\right)^{1/2} 2^{n-n'+1} (-1)^{n'} (\xi_+)^n (\xi_-)^{n'} L_{n'}^n (4|\xi|^2),$$

(126)

where $L_{n'}^n$ is the associated Laguerre polynomial of degree $n$. In a similar way, the quantities corresponding to the operators $\tilde{T}(\xi_+, 0)$ and $\tilde{T}(\xi_-, 0)$ can also be found in terms of the number states $n_{\pm}$. The resultant expressions turn out to be

$$|n'_+\tilde{T}(\xi_+, 0)|n_+\rangle = e^{-2|\xi|} \left(\frac{n'_+!}{n_+!}\right)^{1/2} 2^{n_+-n'_++1} (-1)^{n'_+} (\xi_+)^n (\xi_-)^{n'} L_{n'_+}^{n_+} (4|\xi|^2),$$

(127)

and

$$|n'_-\tilde{T}(\xi_-, 0)|n_-\rangle = e^{-2|\xi|} \left(\frac{n'_-!}{n_-!}\right)^{1/2} 2^{n_--n'_-+1} (-1)^{n'_-} (\xi_+)^{n'} (\xi_-)^n L_{n'_-}^{n_-} (4|\xi|^2).$$

(128)

Substituting (126)–(128) into (125), we can obtain the Wigner function for the squeezed coherent states (75) with $n_+ = l + m$, $n'_+ = l' + m'$ and $n_- = l - m$, $n'_- = l' - m'$ in the form

$$W((\xi_r, \xi_s, \xi_-)) = e^{-2|\xi|^2-2|\xi|^2-2|\xi|^2} N_{\xi, \alpha}^2 \sum_{n, n'} \sum_{\ell, \ell'} \sum_{m, m'} \sum_{l, l'} c_n^\ast c_{n'} c_{m'} c_{l'} c_{l, m}(n')\tilde{T}(\xi_r, 0)|n\rangle$$

$$\times \frac{n'_+!}{n!} (l + m)! (l' + m')! \left(\frac{n!}{n'} (l + m)! (l - m)!\right)^{1/2} (\xi_+)^{n-n'} (\xi_-)^{n'-n} L_{n'_+}^{n_+} (4|\xi|^2) L_{n'_-}^{n_-} (4|\xi|^2),$$

(129)

where $\xi_j = x_j + ip_j$, $\xi = \frac{1}{2} \tanh R_j \phi$ and $\alpha_j = \alpha_{0, \phi} \cosh R_j + \alpha_{0, \phi}^\ast \sinh R_j$, $j = r, s$.

On evaluating the right-hand side (the details are given in appendix C), we find

$$W((\xi_r, \xi_s, \xi_-)) = \Pi_{j=1}^3 \exp \left[\frac{-2|\xi|^2}{1 - |\xi|^2} + \frac{2(\xi_\ast \xi + \xi_\ast \xi)}{\sqrt{1 - |\xi|^2}} - 2|\alpha_j|^2 \right]$$

$$+ \frac{2(\alpha_+ \xi_\ast + \alpha_- \xi_\ast)}{\sqrt{1 - |\xi|^2}} - \frac{2(\alpha_+ \xi + \xi_\ast \alpha_-)}{\sqrt{1 - |\xi|^2}}.$$
6. Conclusion

In this paper, we have constructed squeezed coherent states and studied their non-classical properties associated with the three-dimensional generalized isotonic oscillator. Since the potential under consideration is a spherically symmetric one, we dealt with the radial and angular momentum parts separately. As far as the radial part is concerned, we observed that it is nothing but the newly found extended radial oscillator whose eigenfunctions are expressed in terms of the recently discovered $X_1$-Laguerre polynomials. We have shown that this radial part exhibits the shape invariance property. Using this property and implementing Balantekin’s method, we have obtained two new operators which in turn constitute the intertwining supersymmetric operators. We have demonstrated that these operators produce the Heisenberg–Weyl algebra and perfectly annihilate and create the eigenstates of the radial part. Using these ladder operators, we have constructed squeezed coherent states of the radial part.

As far as the angular part is concerned, we have used Schwinger’s representation to define the creation and annihilation operators. We have derived the associated squeezed coherent states by considering the disentangled form of two-mode squeezing and displacement operators with certain restrictions on the allowed values of angular momentum variables $l$ and $m$. Finally, we have expressed the three-mode squeezed coherent states of the generalized isotonic oscillator as a tensor product of the squeezed coherent states of the radial part and squeezed coherent states of the angular momentum part. To make our results more rigorous, we have proved that these three-mode squeezed coherent states resolve the identity operator. We have explicitly illustrated the squeezing properties of the constructed three-dimensional squeezed states both in the radial part and in the angular part. As far as the radial part is concerned, we have considered the conjugate variables, the generalized position $w_r$ and the momentum $p_r$ and proved the quadrature squeezing. Similarly, to show the squeezing in the angular momentum, we have considered two normalized quantities, namely $S_{L_x} = \frac{2(\Delta L_x)^2 - |\langle \hat{L}_x \rangle|}{|\langle \hat{L}_x \rangle|}$ and $S_{L_y} = \frac{2(\Delta L_y)^2 - |\langle \hat{L}_y \rangle|}{|\langle \hat{L}_y \rangle|}$ and demonstrated explicitly the squeezing properties possessed by these states in the angular variables. In addition to the above, we have evaluated the Wigner function of these squeezed coherent states and shown that these states exhibit the squeezing property. Since the ladder operators of this three-dimensional system act on the eigenstates only linearly, one can deform these operators nonlinearly and construct certain non-classical states such as nonlinear coherent states, nonlinear squeezed states, generalized intelligent states and so on. The details will be presented elsewhere.

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Appendix A. Evaluation of (30) and (33)

To evaluate the terms inside the square bracket in (30), we recall the following property associated with the \( X_l \)-Laguerre polynomials [5]. The differential equation (5) can be factorized by two operators [5], namely

\[
A_k(y) = -\frac{(x + k + 1)^2}{(x + k)} \frac{d}{dx} \left( \frac{y}{x + k + 1} \right),
\]

and

\[
B_k(y) = \frac{x(x + k)}{(x + k + 1)} \left( y' - y \right) + ky.
\]

One can unambiguously check that \( A_k B_k(y) \) factorizes equation (5) as \( A_k B_k(y) = \lambda(y) \).

Rescaling the variable \( x = \omega r^2 \) and the constant \( k = l + \frac{1}{2} \) in (A.2), we obtain

\[
B_l(y) = -\frac{r}{2} \left( \frac{2\omega r^2 + 2l + 1}{2\omega r^2 + 2l + 3} \right) \left[ -\frac{dy}{dr} + \left( \frac{2\omega r^2 - 2l + 3}{r} - \frac{4\omega r}{(2\omega r^2 + 2l + 1)} \right) y \right].
\]

Using the identity [5] \( B_l \tilde{I}_m(l + \frac{1}{2}) = n \tilde{I}_{n+1}(l + \frac{1}{2}) \) with \( y = \tilde{I}_{n+1} \), equation (A.3) can be brought to the form

\[
-\frac{d\tilde{I}_{n+1}(l + \frac{1}{2})}{dr} + \left( \frac{2\omega r^2 - 2l + 3}{2\omega r^2 + 2l + 3} \right) \tilde{I}_{n+1}(l + \frac{1}{2}) = -\frac{2r}{2\omega r^2 + 2l + 3} (n + 1) \tilde{I}_{n+2}(l + \frac{1}{2}).
\]

We use this expression to simplify equation (30).

To evaluate (33), we again rescale the relation (A.1) in such a way that \( x = \omega r^2 \) and \( k = l + \frac{1}{2} \) so that the identity (A.1) now becomes

\[
A_l(y) = -\frac{1}{2\omega r} \left( \frac{2\omega r^2 + 2l + 3}{2\omega r^2 + 2l + 1} \right) \left[ \frac{dy}{dr} - \frac{4\omega r}{2\omega r^2 + 2l + 3} y \right].
\]

Recalling another identity \( A_l \tilde{I}_{n+1}(l + \frac{1}{2}) = \tilde{I}_{n+1}(l + \frac{1}{2}) \) with \( y = \tilde{I}_{n+1} \), equation (A.5) can be rewritten as

\[
-\frac{d\tilde{I}_{n+1}(l + \frac{1}{2})}{dr} - \frac{4\omega r}{2\omega r^2 + 2l + 3} \tilde{I}_{n+1}(l + \frac{1}{2}) = -2\omega r \left( \frac{2\omega r^2 + 2l + 1}{2\omega r^2 + 2l + 3} \right) \tilde{I}_{n+1}(l + \frac{1}{2}).
\]

This relation is used to simplify the intertwining relation (33).

Appendix B. Evaluation of (99) and (101)

In the following, we discuss the method of evaluating the series \( \langle \hat{n}_r \rangle \) given in (99):

\[
\langle \hat{n}_r \rangle = N^2_{\pm} \sum_{n=0}^{\infty} \sum_{n_{r}, n_{-r}, n_{-r}, n} c_n^* c_{n_{r}} c_{n_{r}} c_{n_{-r}}, \quad n_{-r}, n_{-r}, \ldots
\]

With the definition \( N^2_{\pm} = \sum_{n_{-r}, n_{-r}, n_{-r}, n_{-r}}^\infty c_{n_{-r}} c_{n_{-r}, n_{-r}, n_{-r}} \), where \( N^2_{\pm} \) is given in (72), the above equation can be written in the following compact form:

\[
\langle \hat{n}_r \rangle = N^2_{\pm} \sum_{n=0}^{\infty} c_n^* c_{n} n.
\]
Substituting (78a) into (B.1) and redefining the summation appropriately, we obtain
\[
\langle \hat{\eta}_r \rangle = N_r^2 \sum_{s=0}^{\infty} \frac{H_{s+1}(x_r^*) H_{s+1}(x_r)}{s!} \left( \frac{-|\xi_r|}{2} \right)^{s+1}.
\]  
(B.2)

Recalling the integral representation associated with the Hermite polynomials, namely
\[
H_q(z) = \frac{2^q}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} (z + it)^q \, dt \quad [41],
\]
the above equation can be brought to the form
\[
\langle \hat{\eta}_r \rangle = \frac{-2|\xi_r|N_r^2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-z_1^2 - z_2^2} (x_r + iz_1)(x_r^* + iz_2) 
\times \sum_{s=0}^{\infty} \frac{(-2\xi_r(x_r + iz_1)(x_r^* + iz_2))^s}{s!} \, dz_1 \, dz_2.
\]  
(B.3)

We observe that the summation inside the integral can also be written as an exponential function. The non-exponential terms left over inside the integrals, i.e. \((x_r + iz_1)(x_r^* + iz_2)\), can also be brought into an exponential form through the identity
\[
(x_r + iz_1)(x_r^* + iz_2) = \left\{ \frac{d}{dp} \left[ p(x_r + iz_1)(x_r^* + iz_2) \right] \right\}_{p=0}.
\]
As a result, equation (B.3) now becomes
\[
\langle \hat{\eta}_r \rangle = \frac{-2|\xi_r|N_r^2}{\pi} \left\{ \frac{d}{dp} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -z_1^2 - z_2^2 \right. \right.ight.
\left. \left. \left. + \, (p - 2|\xi_r|)(x_r + iz_1)(x_r^* + iz_2) \right] \, dz_1 \, dz_2 \right] \right\}_{p=-2|\xi_r|}.
\]  
(B.4)

To evaluate (B.4), we introduce the transformation \( \eta = p - 2|\xi_r| \). In the new variable, equation (B.4) reads
\[
\langle \hat{\eta}_r \rangle = \frac{-2|\xi_r|N_r^2}{\pi} \left\{ \frac{d}{d\eta} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -z_1^2 - z_2^2 + \eta(x_r + iz_1)(x_r^* + iz_2) \right] \, dz_1 \, dz_2 \right] \right\}_{\eta=-2|\xi_r|}.
\]  
(B.5)

To begin with, we evaluate the following integral in (B.5):
\[
I_1 = \int_{-\infty}^{\infty} \exp \left[ -z_1^2 + \eta(x_r + iz_1)(x_r^* + iz_2) \right] \, dz_1.
\]  
(B.6)

We noted that this integral is nothing but the Fourier transform of an exponential function. With this identification, we find
\[
I_1 = \sqrt{\frac{\pi}{1 - \frac{\eta^2}{4}}} \exp \left[ \frac{\eta^2}{4} - \frac{x_r^* \eta^2}{4} - \frac{i\eta^2}{2} x_r^* z_2 + \eta |x_r|^2 + i\eta x_r z_2 \right].
\]  
(B.7)

Substituting (B.7) into the second integral in (B.5), we obtain
\[
I_2 = \int_{-\infty}^{\infty} \exp \left[ -z_2^2 \right] I_1 \, dz_2.
\]  
(B.8)

The above integral also turns out to be the Fourier transform of an exponential function. The explicit integration leads us to
\[
I_2 = \sqrt{\frac{\pi}{1 - \frac{\eta^2}{4}}} \exp \left[ \eta |x_r|^2 - \frac{x_r^* \eta^2}{4} - \frac{\eta^2}{4} \left( x_r - \frac{\eta}{2} x_r^* \right)^2 \right].
\]  
(B.9)

Substituting (B.9) into (B.5) and simplifying the resultant expression, we arrive at
\[
\langle \hat{\eta}_r \rangle = -2|\xi_r|N_r^2 \left\{ \frac{d}{d\eta} \left[ \frac{1}{\sqrt{1 - \frac{\eta^2}{4}}} \exp \left[ \eta |x_r|^2 - \frac{\eta^2}{4} \left( x_r^* + x_r^* \right) \right] \right] \right\}_{\eta=-2|\xi_r|}.
\]  
(B.10)
Now carrying out the differentiation and substituting the expressions \( x_r(= \frac{\mu \sqrt{1 - |\xi_r|^2}}{\sqrt{2} \epsilon \omega_r}) \) and its complex conjugate in the resultant equation, we obtain the following expression for \( \langle \hat{n}_r \rangle \):

\[
\langle \hat{n}_r \rangle = \frac{1}{(1 - |\xi_r|^2)} \left[ |\alpha_r|^2 (1 + |\xi_r|^2) + \alpha_r^* \xi + |\xi_r|^2 \right].
\]

(100)

We use this expression to evaluate Mandel’s \( Q \)-parameter. Now we evaluate the expectation value \( \langle \hat{n}_r^2 \rangle \) given in (101):

\[
\langle \hat{n}_r^2 \rangle = N_{\xi,\alpha}^2 \sum_{n=0}^{\infty} \sum_{n_r=0}^{\infty} c_n^* c_{n_r} c_n c_{n_r} n^2.
\]

(B.11)

Separating the radial part from the angular part by recalling \( N_{\xi,\alpha}^2 = \sum_{n=0}^{\infty} \sum_{n_r=0}^{\infty} c_n^* c_{n_r} c_n c_{n_r} \), with \( N_{\xi}^2 \) being defined in (72), the triple sum in (B.11) can be reduced to the form

\[
\langle \hat{n}_r^2 \rangle = N_{\xi}^2 \sum_{n=0}^{\infty} c_n^* c_n n^2.
\]

(B.12)

Substituting (78a) into (B.12) and redefining the summation appropriately, we find

\[
\langle \hat{n}_r^2 \rangle = N_{\xi}^2 \sum_{j=0}^{\infty} \frac{H_{s+j+1}(x_r)}{s!} \left( \frac{-|\xi_r|^2}{2} \right)^{s+j+1} + \sum_{j=0}^{\infty} \frac{H_{s+j}(x_r) H_{s+j+1}(x_r)}{j!} \left( \frac{-|\xi_r|^2}{2} \right)^{j+1}.
\]

(B.13)

The second term in (B.13) is nothing but \( \langle \hat{n}_r \rangle \) (see equation (B.2)) which has already been evaluated (see equation (100)). As a consequence, we confine our attention only to the first term in equation (B.13) (which we call as \( G \)), namely

\[
G = N_{\xi}^2 \sum_{s=0}^{\infty} \frac{H_{s+1}(x_r)}{s!} \left( \frac{-|\xi_r|^2}{2} \right)^{s+1}.
\]

(B.14)

Using the integral representation, \( H_s(z) = \frac{\sqrt{\pi}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (z + iT)^s \, dt \), we can rewrite (B.14) as

\[
G = \frac{4N_{\xi}^2|\xi_r|^2}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\xi^2} (x_r + i\xi_1)^2 (x_r + i\xi_2)^2 \\
\times \sum_{j=0}^{\infty} \left( |\xi_r|^2 (x_r + i\xi_1) (x_r + i\xi_2) \right)^j \, d\xi_1 \, d\xi_2.
\]

(B.15)

By recognizing that the sum appearing in (B.15) is nothing but an exponential function and the terms \((x_r + i\xi_1)^2 (x_r + i\xi_2)^2\) can also be rewritten as an exponential function, we can bring equation to the form

\[
G = \frac{4N_{\xi}^2|\xi_r|^2}{\pi} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -z_1^2 - z_2^2 \\
+ (p - 2|\xi_r|^2)(x_r + i\xi_1)(x_r + i\xi_2) \right] \, dz_1 \, dz_2 \right\}_{p=0}.
\]

(B.16)

To evaluate the integrals, we introduce the transformation \( \eta = p - 2|\xi_r|^2 \) so that equation (B.16) in the new variable reads

\[
G = \frac{4N_{\xi}^2|\xi_r|^2}{\pi} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -z_1^2 - z_2^2 + \eta (x_r + i\xi_1)(x_r + i\xi_2) \right] \, dz_1 \, dz_2 \right\}_{\eta=-2|\xi_r|^2}.
\]

(B.17)
Evaluating the integrals with the help of Fourier transform, we find
\[ G = \frac{4N_r^2|\xi_r|^2}{\pi} \left\{ \frac{d^2}{dn^2} \left[ \frac{1}{\sqrt{1 - \frac{n^2}{4}}} \exp \left[ \eta|\chi_r|^2 - \frac{\eta}{2} (\chi_r^2 + \chi_r^\ast)^2 \right] \right] \right\} _{n=-2L_r}. \]  

(B.18)

Now differentiating the terms inside the square bracket two times with respect to \( n \) and then substituting the expression for \( N_r \) into the resultant equation and simplifying the latter, we arrive at
\[ G = \frac{1}{(1 - |\xi_r|^2)^2} \left[ |\alpha_r|^4 (1 + |\xi_r|^2)^2 + (\alpha^2_\xi^* + \alpha^*_\xi \xi_r)^2 + 4|\alpha_r|^2|\xi_r|^2 (2 + |\xi_r|^2) \\
+ |\xi_r|^2 (1 + 2|\xi_r|^2) + (2|\alpha_r|^2 (1 + |\xi_r|^2) + 1 + 5|\xi_r|^2) (\alpha^2_\xi^* + \alpha^*_\xi \xi_r) \right]. \]  

(B.19)

Substituting equations (B.19) and (100) into (B.13), we obtain the following expression for \( \langle \hat{n}^2_r \rangle \):
\[ \langle \hat{n}^2_r \rangle = \frac{1}{(1 - |\xi_r|^2)^2} \left[ |\alpha_r|^4 (1 + |\xi_r|^2)^2 + |\xi_r|^2 (2 + |\xi_r|^2) + (\xi_r^* \alpha^2_r + \xi_r \alpha^*_r)^2 \\
+ (2 (1 + |\alpha_r|^2) (1 + |\xi_r|^2) + 2 |\xi_r|^2) (\xi_r^* \alpha^2_r + \xi_r \alpha^*_r) + |\alpha_r|^2 (1 + 8 |\xi_r|^2 + 3 |\xi_r|^4) \right]. \]

(102)

To numerically evaluate Mandel’s \( Q \)-parameter, we use expression (102).

**Appendix C. Evaluation of Wigner function (129)**

Let us first evaluate the radial part in (129). Separating the radial part from the angular part, we obtain
\[ W_r(\zeta_r) = e^{-2|\zeta|^2} N_r^2 \sum_{n=0}^{\infty} \sum_{\eta=0}^{\infty} \frac{H_n(x_r)H_n(x_r^\ast)}{n!} (2\zeta_r^*)^n \left( -\frac{\xi_r}{2} \right)^{n/2} \left( \frac{\xi_r^2}{2} \right)^{n/2} \left( -1 \right)^n. \]

(C.1)

where \( x_r = \frac{\alpha_r}{\sqrt{1 - |\xi_r|^2}} \).

To start with, we evaluate the following sum appearing in (C.1):
\[ G_1 = \sum_{n=0}^{\infty} \frac{H_n(x_r)}{n!} (2\zeta_r^*)^n \left( -\frac{\xi_r}{2} \right)^{n/2} L_{n/2}^{2/2} (4|\xi_r|^2). \]

(C.2)

Recalling the integral and summation representations of the Hermite and the associated Laguerre polynomials, namely \( H_n(z) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} (z + it)^n dt \) and \( \sum_{n=0}^{\infty} \frac{L_{n}^{\mu}(z)}{n!} = e^z L_{\mu}^{\mu} (z - w) \) [41], we can reduce equation (C.2) to the form
\[ G_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -t^2 + 2 (x_r + it) \zeta_r \sqrt{-2 \xi_r} L_{n/2}^{2/2} (4|\xi_r|^2 - 2 \sqrt{-2 |\xi_r|^2} \zeta_r (x_r + it)) \right] dt. \]

(C.3)

This expression can further be simplified by recalling yet another identity, i.e. \( L_{n/2}^{2/2} (x) = \frac{(-1)^n}{n! \sqrt{\pi}} \). With this simplification, equation (C.3) reshapes into
\[ G_1 = \frac{(-1)^n}{n! \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -t^2 + 2 (x_r + it) \zeta_r \sqrt{-2 \xi_r} (4|\xi_r|^2 - 2 \sqrt{-2 |\xi_r|^2} \zeta_r (x_r + it)) \right] dt. \]

(C.4)
The second term in (C.4) can also be written as an exponential function through the relation 
\[(4|\xi_r|^2 - 2\sqrt{-2\xi_r\xi_r^{*} (x_r + i\eta)})^\prime = \left\{ \frac{d}{d\eta} \left[ \exp \left[ p(4|\xi_r|^2 - 2(x_r + i\eta)\sqrt{-2\xi_r\xi_r^{*}}) \right] \right] \right\}_{p=0}.\]
By doing so, we find
\[
G_1 = \frac{(-1)^n}{n!\sqrt{\pi}} \left\{ \frac{d^n}{dp^n} \left[ \exp \left[4p|\xi_r|^2 + 2(1 - p)\sqrt{-2\xi_r\xi_r^{*}x_r} \right] \right] \right\}_{p=0} \quad (C.5).
\]
Now evaluating the integral using the relation \(\int_{-\infty}^{\infty} e^{ix}\ e^{-x^2} \, dx = \sqrt{\pi} e^{-\frac{x^2}{4}},\) we obtain
\[
G_1 = \frac{(-1)^n}{n!} \left\{ \frac{d^n}{dp^n} \left[ \exp \left[4p|\xi_r|^2 + 2(1 - p)\sqrt{-2\xi_r\xi_r^{*}x_r} + 2(1 - p)^2\xi_r\xi_r^{*} \right] \right] \right\}_{p=0} \quad (C.6).
\]
To proceed further, we introduce the transformation, \(\theta = \sqrt{-2\xi_r\xi_r^{*} (1 - p)},\) in (C.6). In this new variable \(\theta,\) equation (C.6) reads
\[
G_1 = \frac{(-\sqrt{-2\xi_r\xi_r^{*}})^n}{n!} \left\{ \exp \left[4|\xi_r|^2 \left( x_r + \sqrt{\frac{2}{\xi_r}} \xi_r^{*} \right) \right] \right\}_{\theta = \sqrt{-2\xi_r\xi_r^{*}}} \quad (C.7).
\]
Expressing the differential part appearing in (C.7) in terms of Hermite polynomials through the Rodrigues formula, \(H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} [41],\) we find
\[
G_1 = \frac{(-\sqrt{-2\xi_r\xi_r^{*}})^n}{n!} \exp \left[2\xi_r\xi_r^{*} + 2\sqrt{-2\xi_r\xi_r^{*}x_r} \right] H_n \left[ \sqrt{-2\xi_r\xi_r^{*}} - \sqrt{\frac{2}{\xi_r}} \xi_r - x_r \right] \quad (C.8).
\]
Substituting (C.8) into the Wigner function (C.1), we obtain
\[
W = N^2 \exp \left[ -2|\xi_r|^2 + 2\sqrt{-2\xi_r\xi_r^{*}x_r} \right] \sum_{n=0}^{\infty} \frac{H_n (x_r^{*}) H_n (\bar{\theta})}{n!} \left( \frac{|\xi_r|^2}{2} \right)^n \quad (C.9),
\]
where \(\bar{\theta} = \sqrt{-2\xi_r\xi_r^{*}} - \sqrt{\frac{2}{\xi_r}} \xi_r - x_r.\)

The summation in (C.9) can be evaluated through the identity [41]
\[
\sum_{n=0}^{\infty} \frac{H_n (x) H_n (y) \nu^n}{n!} = \frac{1}{\sqrt{1 - 4\nu}} \exp \left[ \frac{4\nu^2 (x^2 + y^2) - 4\nu xy}{4\nu^2 - 1} \right], \quad |\nu| < \frac{1}{2}. \quad (C.10)
\]
On replacing this result in (C.9), we obtain
\[
W = N^2 \exp \left[ -2\xi_r \xi_r^{*} + 2\sqrt{-2\xi_r\xi_r^{*}x_r} - 2|\xi_r|^2 \right.
+ \frac{|\xi_r|^2}{|\xi_r|^2 - 1} \left( x_r^2 + x_r^{*2} + 4|\xi_r|^2 - 2\sqrt{-2\xi_r\xi_r^{*}x_r} - 2|\xi_r|^2 \right.
+ \frac{2|\xi_r|^2}{|\xi_r|^2 - 1} \left( - |x_r|^2 - 2x_r x_r^{*} + \sqrt{-2\xi_r\xi_r^{*}x_r} \right). \quad (C.11)
\]
The final task is to substitute the normalization constant, (57), in (C.11) and simplify the resultant expression with \( x_r = \frac{a_r}{\sqrt{1 - |\xi_r|^2}} \). On completing this task, we arrive at the following expression for the radial part:

\[
W_r = \exp \left[ -\frac{2|x_r|^2(|\xi_r|^2 + 1) + 2(\xi_r^*\xi_r^2 + \xi_r^2\xi_r^2)}{1 - |\xi_r|^2} - 2|\alpha_r|^2 + \frac{2(\alpha_r\xi_r^* + \alpha_r^*\xi_r)}{\sqrt{1 - |\xi_r|^2}} \right].
\]

(C.12)

In a similar way, one can also evaluate the other two modes present in the Wigner function. Our result shows that (since the procedure is repetitive, in the following, we give only the final form of the expression)

\[
W_{\pm} = \exp \left[ -\frac{2|x_{\pm}|^2(|\xi_{\pm}|^2 + 1) + 2(\xi_{\pm}^*\xi_{\pm}^2 + \xi_{\pm}^2\xi_{\pm}^2)}{1 - |\xi_{\pm}|^2} - 2|\alpha_{\pm}|^2 + \frac{2(\alpha_{\pm}\xi_{\pm}^* + \alpha_{\pm}^*\xi_{\pm})}{\sqrt{1 - |\xi_{\pm}|^2}} \right].
\]

(C.13)

As a result, we obtain the following form for the Wigner function for the three-dimensional generalized isotonic oscillator:

\[
W((\xi_r, \xi_+, \xi_-)) = W_r \times W_{\pm}
\]

\[
= \Pi_{j=1}^3 \exp \left[ -\frac{2|x_j|^2(|\xi_j|^2 + 1) + 2(\xi_j^*\xi_j^2 + \xi_j^2\xi_j^2)}{1 - |\xi_j|^2} - 2|\alpha_j|^2 + \frac{2(\alpha_j\xi_j^* + \alpha_j^*\xi_j)}{\sqrt{1 - |\xi_j|^2}} \right].
\]

(C.14)

We use this result to investigate the Wigner function numerically.

References

[1] Quesne C 2008 J. Phys. A: Math. Theor. 41 392001
[2] Hall R L, Saad N and Yesiltas O 2010 J. Phys. A: Math. Theor. 43 465304
[3] Sadd N, Hall R L, Ciftci H and Yesiltas O 2011 Adv. Math. Phys. 2011 750168
[4] Agboola D and Zhang Y-Z 2012 J. Math. Phys. 53 042101
[5] Gomez-Ullate D, Kamran N and Milson R 2009 J. Math. Anal. Appl. 359 352
Gomez-Ullate D, Kamran N and Milson R 2012 J. Math. Anal. Appl. 387 410
Gomez-Ullate D, Kamran N and Milson R 2012 Found. Comput. Math. at press (doi:10.1007/s10208-012-9128-6)
[6] Carinena J F, Perelomov A M, Rañada M F and Santander M 2008 J. Phys. A: Math. Theor. 41 085301
[7] Leaver E W 1986 J. Math. Phys. 27 1238
[8] Lidsey J E 2004 Class. Quantum Grav. 21 777
Haas F 2002 Phys. Rev. A 65 033603
[9] Wright E M, Arlt J and Dholakia K 2000 Phys. Rev. A 63 013608
[10] DelloStritto M and De Silve T N 2012 Phys. Lett. A 376 2298
[11] Hawkins R M and Lidsey J E 2002 Phys. Rev. D 66 023523
Bouaziz D and Basw M 2008 Phys. Rev. A 78 032110
[12] Fernandez Guasti M and Moya-Cessa H 2003 Phys. Rev. A 67 063803
Gauthier S 1984 J. Phys. A: Math. Gen. 17 2633
Ray J R and Reid J L 1979 Phys. Lett. 71 317
[13] Calogero F 1969 J. Math. Phys. 10 2191
[14] Chalykh O A and Veselov A P 2005 J. Nonlinear Math. Phys. 12 179
