DEGENERATE FLAG VARIETIES AND THE MEDIAN GENOCCHI NUMBERS

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ABSTRACT. We study the degenerations $\mathcal{F}_\lambda^M$ of the type A flag varieties $\mathcal{F}_\lambda$. We describe these degenerations explicitly as subvarieties in the products of Grassmanians. We construct cell decompositions of $\mathcal{F}_\lambda^M$ and show that for complete flags the number of cells is equal to the normalized median Genocchi numbers $h_n$. This leads to a new combinatorial definition of the numbers $h_n$. We also compute the Poincaré polynomials of the complete degenerate flag varieties via a natural statistics on the set of Dellac’s configurations, similar to the length statistics on the set of permutations. We thus obtain a natural $q$-version of the normalized median Genocchi numbers.

INTRODUCTION

Let $\mathfrak{g} = \mathfrak{sl}_n$, $\mathcal{G} = SL_n$. Fix the Cartan decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{b}$ is a Borel subalgebra, $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. In [Fe3] we considered the degenerate algebra $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian Lie algebra isomorphic to $\mathfrak{n}^-$ as a vector space. The corresponding Lie group is a semi-direct product $\mathcal{G}^a = B \rtimes \mathcal{G}_a^M$, where $\mathcal{G}_a$ is the additive group of the field and $M = \dim \mathfrak{n}$.

For a dominant integral weight $\lambda$ let $V_\lambda$ be the highest weight $\lambda$ irreducible $\mathfrak{g}$-module with a highest weight vector $v_\lambda$. The increasing PBW filtration $F_\bullet$ on $V_\lambda$ is defined as follows:

$$F_0 = \mathbb{C} v_\lambda, \quad F_{s+1} = \text{span}\{ xv : x \in \mathfrak{g}, v \in F_s \}, s \geq 0$$

(see [Fe1], [Fe2], [FFoL1], [FFoL2], [K2]). The associated graded space $V_\lambda^a = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \ldots$ can be naturally endowed with the structure of a $\mathfrak{g}^a$- and $\mathcal{G}_a^M$-module. A degenerate flag variety $\mathcal{F}_\lambda^a$ is a subvariety in $\mathbb{P}(V_\lambda^a)$ defined by $\mathcal{F}_\lambda^a = \mathcal{G}_a^M \cdot \mathbb{C} v_\lambda$. These are the $\mathcal{G}_a^M$-degenerations of the classical (generalized) flag varieties $\mathcal{F}_\lambda$ (see [A], [AS], [Fe3], [HT]). For example, $\mathcal{F}_\lambda^\omega_d \simeq Gr(d,n)$ for all fundamental weights. Recall also that in the classical case (for $\mathfrak{g} = \mathfrak{sl}_n$) the varieties $\mathcal{F}_\lambda = \mathcal{G} \cdot \mathbb{C} v_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ are the usual flag varieties (maybe partial). In particular, if $\lambda$ is regular, i.e. $(\lambda, \omega_d) > 0$ for all $d$, then $\mathcal{F}_\lambda$ is isomorphic to the variety $\mathcal{F}_n$ of complete flags in $n$-dimensional space $V$. Fix a basis $v_1, \ldots, v_n$ of $V$.

For all weights $\lambda$, $\mu$ there exists an embedding of $\mathfrak{g}^a$-modules $V_\lambda^a \hookrightarrow V_\lambda^a \otimes V_\mu^a$ sending $v_{\lambda+\mu}$ to $v_{\lambda} \otimes v_{\mu}$ (see [FFoL1], [FFoL2]). This induces the embedding of varieties $\mathcal{F}_\lambda^{\lambda+\mu} \hookrightarrow \mathcal{F}_\lambda^a \times \mathcal{F}_\mu^a$. Thus for any $\lambda$ we obtain an embedding of $\mathcal{F}_\lambda^a$ into the product of Grassmanians. Our first result is
an explicit description of this embedding. We state the theorem here for complete flag varieties $F_n^a$. For this we need one more piece of notations. Let $pr_d : V \to V$ be the projection along the space $\mathbb{C}v_d$ to the linear span of the vectors $v_i$, $i \neq d$.

**Theorem 0.1.** The image of the embedding of the variety $F_n^a$ in the product $\prod_{d=1}^{n-1} Gr(d,n)$ is equal to the set of chains of subspaces $(V_1, \ldots, V_{n-1})$, $V_d \in Gr(d,n)$ such that

$$pr_{d+1}(V_d) \hookrightarrow V_{d+1}, \quad 1 \leq d \leq n-2.$$  

Our next goal is to compute the Poincaré polynomial of $F_n^a$. Recall that in the classical case the flag variety $F_n$ can be written as a disjoint union of $n!$ cells, each cell being associated with a torus fixed point. The fixed points are labeled by permutations from $S_n$. The length statistics $\sigma \to l(\sigma)$ gives the complex dimension of the cells. Therefore, the Poincaré polynomial $P_{F_n}(t)$ of $F_n$ is equal to

$$P_{F_n}(t) = \sum_{\sigma \in S_n} t^{2l(\sigma)}.$$  

As an immediate corollary of Theorem 0.1 we obtain that the fixed points of the torus $T \subset G^a$ action on $F_n^a$ are labeled by the sequences $I_1, \ldots, I_{n-1}$, $I_d \subset \{1, \ldots, n\}$, $\#I_d = d$, satisfying

$$(0.1) \quad I_d \setminus \{d+1\} \hookrightarrow I_{d+1}, \quad d = 1, \ldots, n-2.$$  

(Note that this set of sequences has a subset with $I_d \hookrightarrow I_{d+1}$, which can be naturally identified with the permutations $S_n$). Our first task is to compute the number of such fixed points. To this end, recall the normalized median Genocchi numbers $h_n$, $n = 1, 2, \ldots$ (also referred to as the normalized Genocchi numbers of second kind). These numbers have several definitions [De], [Du], [DR], [DZ], [G], [Kr], [Vien] (see section 3 for a review). Here we give the Dellac definition, which is the earliest one and which fits our construction in the best way.

Consider a rectangle with $n$ columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs $(l,j)$, with $l = 1, \ldots, n$ and $j = 1, \ldots, 2n$. A Dellac configuration $D$ is a subset of boxes, subject to the following conditions: first, each column contains exactly two boxes from $D$ and each row contains exactly one box from $D$, and, second, if the $(l,j)$-th box is in $D$, then $l \leq j \leq n+l$. Let $DC_n$ be the set of such configurations. Then $h_n$ is the number of elements in $DC_n$. The first several median Genocchi numbers (starting from $h_1$) are as follows: $1, 2, 7, 38, 295$. For instance, the two Dellac configurations for $n=2$ are as follows: (we specify boxes in a configuration by putting fat dots inside)

$$\begin{array}{cccc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} \quad \begin{array}{cccc}
\bullet & \bullet \\
\bullet & \bullet \\
\end{array}$$

We prove the following theorem:

**Theorem 0.2.** The number of sequences $I_1, \ldots, I_{n-1}$ as above, satisfying $$(0.1)$$ is equal to $h_n$. 
We also prove that the Dellac definition \( \text{De} \) is equivalent to the Dumont-Kreweras definition \( \text{Du}, \text{Kr} \) (this fact is known to experts \( \text{G}, \text{S} \) but we were unable to find the proof in the literature).

Recall that the length of a permutation \( \sigma \in S_n \) can be defined as the number of pairs \( 1 \leq l_1 < l_2 \leq n \) satisfying \( \sigma(l_1) > \sigma(l_2) \). We define a length \( l(D) \) of a Dellac configuration \( D \) as the number of squares \( (l_1, j_1), (l_2, j_2) \in D \) such that \( l_1 < l_2 \) and \( j_1 > j_2 \). We prove the following theorem:

**Theorem 0.3.** The Poincaré polynomial \( P_{F_n}(t) \) is given by \( \sum_{D \in DC_n} t^{l(D)} \).

Our paper is organized in the following way:

In Section 1, we recall main definitions and theorems from \( \text{Fe3} \).

In Section 2, we describe explicitly the image of the embedding of the varieties \( F_n \) into the product of Grassmanians and construct the cell decomposition of \( F_n \).

In Section 3, we study the combinatorics of the median Genocchi numbers and compute the Poincaré polynomials of the complete degenerate flag varieties.

1. **PBW Deformation**

1.1. **Definitions.** We first recall basic definitions and constructions from \( \text{FPoL1} \) and \( \text{Fe3} \). Let \( g \) be a simple Lie algebra with the Cartan decomposition \( g = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \). We denote by \( M \) the number of positive roots of \( g \), i.e. \( M = \text{dim} \mathfrak{n} \). Let \( b = \mathfrak{n} \oplus \mathfrak{h} \) be a Borel subalgebra. Then the deformed algebra \( g^a \) is defined as a sum of two subalgebras \( g^a = b \oplus (n^-)^a \), where \((n^-)^a \) is an abelian Lie algebra isomorphic to \( n^- \) as a vector space. The subalgebra \( (n^-)^a \hookrightarrow g^a \) is an abelian ideal and the action of \( b \) on \( (n^-)^a \) is induced from the identification \( (n^-)^a \simeq g/b \).

Let \( G \) be the Lie group of the Lie algebra \( g \). Let \( N, T, N^- \), \( B \) be the Lie groups of the Lie algebras \( n, \mathfrak{h}, n^- \), \( b \). The deformed Lie group \( G^a \) is defined as a semi-direct product of \( B \) and the normal subgroup \( \mathbb{G}_a^M \), where \( \mathbb{G}_a \) is the additive group of the field (thus \( \mathbb{G}_a^M \) is the Lie group of the Lie algebra \( (n^-)^a \)). The Borel group \( B \) acts on the vector space \( (n^-)^a \simeq g/b \) via the restriction of the adjoint action and therefore there exists a natural homomorphism from \( B \) to \( Aut(\mathbb{G}_a^M) \), defining the semi-direct product \( G^a = B \ltimes \mathbb{G}_a^M \).

For a dominant integral weight \( \lambda \) we denote by \( V_\lambda \) the corresponding irreducible highest weight \( g \)-module with a highest weight vector \( v_\lambda \). The Lie algebra \( g^a \) and the Lie group \( G^a \) act on the deformed representations \( V^a_\lambda \), where \( \lambda \) are dominant integral weights of \( g \). The representations \( V^a_\lambda \) are defined as associated graded \( gr^a V_\lambda \) of the representation \( V_\lambda \) with respect to the PBW filtration \( F^a_\lambda \):

\[
F_s = \text{span}\{x_1 \ldots x_l v_\lambda : x_i \in g, l \leq s\}.
\]

So \( V^a_\lambda = \bigoplus_{s \geq 0} V^a_\lambda(s) \), where \( V^a_\lambda(0) = \mathbb{C} v_\lambda \) and \( V^a_\lambda(s) = F^a_\lambda/F^a_{s-1} \) for \( s > 0 \). It is easy to see that the action of \( n^- \) on \( V^a_\lambda \) becomes abelian on \( V^a_\lambda \) (i.e. it
induces the action of \((n^-)^a\) and the action of the Borel subalgebra induces the action of (the same algebra) \(b\). The actions of \((n^-)^a\) and \(b\) glue together to the action of \(g^a\).

**Remark 1.1.** Let \(\tilde{g}^a = g^a \oplus \mathbb{C}p\) be the central extension of \(g^a\) with a single element \(p\) subject to the relations \([p, b] = 0, [p, f_\alpha] = f_\alpha\) for any positive root \(\alpha\) and the corresponding weight element \(f_\alpha \in (n^-)^a\). Thus the Cartan subalgebra of \(\tilde{g}^a\) has one extra dimension. We note that the \(g^a\)-module structure of \(V_\lambda^a\) naturally lifts to the structure of representation of \(\tilde{g}^a\) by setting \(pv_\lambda = 0\) (in general, \(p|_{V_\lambda^a} = s\)). An eigenvalue of the operator \(p\) sometimes referred to as a PBW degree. The character of \(V_\lambda^a\) with respect to \(h\oplus \mathbb{C}p\) was computed in \([FFoL1]\) for \(sl_n\) and in \([FFoL2]\) for symplectic Lie algebras. We denote the Lie group of \(\tilde{g}^a\) by \(\tilde{G}^a\), which differs from \(G^a\) by an additional \(\mathbb{C}^a\).

Consider the action of \(G^a\) on the projective space \(\mathbb{P}(V_\lambda^a)\). Recall that in the classical situation the (generalized) flag varieties are defined as \(F_G = G \cdot C v_\lambda \hookrightarrow \mathbb{P}(V_\lambda)\) (see [K1]). The degenerate flag varieties \(F_G^a = G \cdot C v_\lambda \hookrightarrow \mathbb{P}(V_\lambda^a)\) are defined as the closures of the \(G^a\) orbit (or, equivalently, of the \(C^a\) orbit) of the line \(C v_\lambda\). We note that in the classical case the orbit \(G \cdot C v_\lambda\) already covers the whole flag variety. This is not true in the degenerate case: the orbit \(G^a \cdot C v_\lambda\) is an affine cell, whose closure gives a projective singular variety \(F_G^a\).

**1.2. The type A case.** From now on we assume that \(g = sl_n\) and \(G = SL_n\). Then all positive roots are of the form

\[
\alpha_{i,j} = \alpha_i + \cdots + \alpha_j, \ 1 \leq i \leq j \leq n - 1
\]

(for instance, \(\alpha_{i,i} = \alpha_i\) are the simple roots). We denote by \(f_{i,j} = f_{\alpha_{i,j}} \in n^-\) and \(e_{i,j} = e_{\alpha_{i,j}} \in n\) the corresponding root elements. We have \(F_{\omega_d}^a \simeq F_{\omega_d} \simeq Gr(d, n)\). The reason why the degenerate flag varieties are isomorphic to the non-degenerate ones for fundamental weights is that the radicals in \(sl_n\), corresponding to \(\omega_d\), are abelian. In other words, define the set of positive roots

\[
R_d = \{\alpha_{i,j} : 1 \leq i \leq d \leq j \leq n - 1\}.
\]

Define the subalgebra \(u_d^- = \text{span}\{f_\alpha : \alpha \in R_d\}\). Then \(u_d^-\) is abelian and \(V_{\omega_d} = U(u_d^-) \cdot v_\lambda\).

**Remark 1.2.** Let us explain the difference between the structure of \(g\)-module on \(V_{\omega_d}\) and the structure of \(g^a\)-module on \(V_{\omega_d}^a\). The operators \(f_\alpha\) act trivially on \(V_{\omega_d}^a\) unless \(\alpha \in R_d\). Also, \(e_\alpha\) act trivially on \(V_{\omega_d}^a\) if \(\alpha \in R_d\). Therefore, \(g^a\) acts on \(V_{\omega_d}^a\) via the projection to the subalgebra

\[
(1.1) \quad g^a_d = u_d^- \oplus h \oplus \text{span}\{e_\alpha : \alpha \notin R_d\}.
\]

Similarly, the group \(G^a\) acts on \(Gr(d, n)\) via the surjection to the Lie group of \(g^a_d\). In particular, the group \(G^a\) does not act transitively on the deformed flag varieties even in the case of Grassmanians.
Remark 1.3. We note that though \( F_{\omega_d} \cong F_{\omega_d} \cong Gr(d, n) \), the actions of the Borel groups \( B \subset G \) and \( B \subset G^a \) are very different. Let us consider the case \( G = SL_2 \). Then \( g^a \) is spanned by three elements \( e^a \), \( h^a \) and \( f^a \) subject to the relations

\[
[h^a, e^a] = 2e^a, \quad [h^a, f^a] = -2f^a, \quad [e^a, f^a] = 0.
\]

Let \( \lambda \) be a dominant weight of \( sl_2 \), \( \lambda \in \mathbb{Z}_{\geq 0} \). Then \( V^a_\lambda \) is the direct sum of one-dimensional subspaces spanned by vectors \( v_l \), \( l = \lambda, \lambda - 2, \ldots, -\lambda \) such that

\[
h^a v_l = l v_l, \quad f^a v_l = v_{l - 2}, \quad e^a v_l = 0.
\]

Therefore, the Borel subgroup \( B \) acts trivially on \( F_{\lambda} \cong \mathbb{P}^1 \). For instance, there exists one point of \( \mathbb{P}^1 \), which is fixed by the action of the whole group \( G^a \).

Let us now recall the Plücker relations for \( F_\lambda \) [Fu] and the deformed Plücker relations for \( F^a_\lambda \) [Ec3].

Let \( 1 \leq d_1 < \cdots < d_s \leq n - 1 \) be a sequence of increasing numbers. Then for any positive integers \( a_1, \ldots, a_s \) the variety \( F_{a_1 \omega_{d_1} + \cdots + a_s \omega_{d_s}} \) is isomorphic to the partial flag variety

\[
F(d_1, \ldots, d_s) = \{ V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_s \hookrightarrow \mathbb{C}^n : \dim V_i = d_i \}.
\]

In particular, if \( s = 1 \), then \( F(d) \) is the Grassmanian \( Gr(d, n) \) and for \( s = n - 1 \) \( F(1, \ldots, n - 1) \) is the variety of the complete flags. We recall that

\[
V_{\omega_d} = \Lambda^d(V_{\omega_1}) = \Lambda^d(\mathbb{C}^n)
\]

and the embedding \( Gr(d, n) \hookrightarrow \mathbb{P}(\Lambda^d V_{\omega_1}) \) is defined as follows: a subspace with a basis \( w_1, \ldots, w_d \) maps to \( Cw_1 \wedge \cdots \wedge w_d \). For general sequence \( d_1, \ldots, d_s \) one has embeddings:

\[
F(d_1, \ldots, d_s) \hookrightarrow Gr(d_1, n) \times \cdots \times Gr(d_s, n) \hookrightarrow \mathbb{P}(V_{\omega_{d_1}}) \times \cdots \times \mathbb{P}(V_{\omega_{d_s}}).
\]

The composition of these embeddings is called the Plücker embedding. The image is described explicitly in terms of Plücker relations. Namely, let \( v_1, \ldots, v_n \) be a basis of \( \mathbb{C}^n = V_{\omega_1} \). Then one gets a basis \( v_J \) of \( V_{\omega_d} \) \( v_J = v_{j_1} \wedge \cdots \wedge v_{j_d} \) labeled by sequences \( J = (1 \leq j_1 < j_2 < \cdots < j_d \leq n) \). Let \( X_J \in V^{\ast}_{\omega_d} \) be the dual basis. We denote by the same symbols the coordinates of a vector \( v \in V_{\omega_d} \): \( X_J = X_J(v) \). The image of the embedding

\[
F(d_1, \ldots, d_s) \hookrightarrow \prod_{i=1}^s \mathbb{P}(V_{\omega_{d_i}})
\]

is defined by the Plücker relations. These relations are labeled by a pair of numbers \( p \geq q, p, q \in \{d_1, \ldots, d_s\} \), by a number \( k, 1 \leq k \leq q \) and by a pair of sequences \( L = (l_1, \ldots, l_p), J = (j_1, \ldots, j_q), 1 \leq l_\alpha, j_\beta \leq n \). The corresponding relation is denoted by \( R^k_{L, J} \) and is given by

\[
(1.2) \quad R^k_{L, J} = X_L X_J - \sum_{1 \leq r_1 < \cdots < r_k \leq p} X_{L'} X_{J'},
\]
where \( L', J' \) are obtained from \( L, J \) by interchanging \( k \)-tuples \((l_{r_1}, \ldots, l_{r_k})\) and \((j_1, \ldots, j_k)\) in \( L \) and \( J \) respectively, i.e.

\[
J' = (l_{r_1}, \ldots, l_{r_k}, j_{k+1}, \ldots, j_q), \\
L' = (l_1, \ldots, l_{r_1-1}, j_1, l_{r_1+1}, \ldots, l_{r_2-1}, j_2, \ldots, l_p).
\]

We note that for any \( \sigma \in S_d \) the equality

\[
X_{j_{\sigma(1)}, \ldots, j_{\sigma(d)}} = (-1)^\sigma X_{j_1, \ldots, j_d}
\]

is assumed in (1.2). We denote the ideal generated by all \( R_{L,J}^k \) by \( I(d_1, \ldots, d_s) \).

We introduce the notation

\[
\mathcal{F}^a(d_1, \ldots, d_s) = \mathcal{F}_{\omega_{d_1} + \cdots + \omega_{d_s}}, \quad 1 \leq d_1 < \cdots < d_s < n.
\]

**Definition 1.4.** Let \( I^a(d_1, \ldots, d_s) \) be an ideal in the polynomial ring in variables \( X_{j_1, \ldots, j_d} \), \( d = d_1, \ldots, d_s, 1 \leq j_1 < \cdots < j_d < n \), generated by the elements \( R_{L,J}^{ka} \) given below. These elements are labeled by a pair of numbers \( p \geq q, p, q \in \{d_1, \ldots, d_s\} \), by an integer \( k, 1 \leq k \leq q \) and by sequences \( L = (l_1, \ldots, l_p), J = (j_1, \ldots, j_q) \), which are arbitrary subsets of the set \( \{1, \ldots, n\} \). The generating elements are given by the formulas

\[
R_{L,J}^{ka} = X_{l_1, \ldots, l_p}^a X_{j_1, \ldots, j_q}^a - \sum_{1 \leq r_1 < \cdots < r_k \leq p} X_{l_1', \ldots, l_p'}^a X_{j_r', \ldots, j_q'}^a,
\]

where the terms of \( R_{L,J}^{ka} \) are the terms of \( R_{L,J}^k \) (with a superscript \( a \), to be precise) such that

\[
\{l_1, \ldots, l_p\} \cap \{q + 1, \ldots, p\} = \emptyset.
\]

**Remark 1.5.** The initial term \( X_{l_1, \ldots, l_p}^a X_{j_1, \ldots, j_q}^a \) is also subject to the condition (1.2), i.e. it is not present in \( R_{L,J}^{ka} \) if \( \{j_1, \ldots, j_k\} \cap \{q + 1, \ldots, p\} \neq \emptyset \).

**Example 1.6.** Let \( s = 1 \). Then \( I^a(d) = I(d) \), since there are no numbers \( l \) such that \( d + 1 \leq l \leq d \) and thus \( R_{L,J}^{ka} = R_{L,J}^k \) (up to a superscript \( a \) in the notations of variables \( X_j \)). Hence \( \mathcal{F}_{\omega_d}^a \simeq \mathcal{F}_{\omega_d} \).

The following theorem is proved in [Fe3].

**Theorem 1.7.** The variety \( \mathcal{F}^a(d_1, \ldots, d_s) \hookrightarrow \times_{i=1}^s \mathbb{P}(\Lambda^d_i \mathbb{C}^n) \) is defined by the ideal \( I^a(d_1, \ldots, d_s) \).

**Example 1.8.** Let \( s = 2, d_1 = 1, d_2 = n - 1 \). Then the classical flag variety \( \mathcal{F}(1, n - 1) \) is a subvariety in \( \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \) defined by a single relation

\[
\sum_{i=1}^n (-1)^{i-1} X_i X_{1, \ldots, i-1, i+1, \ldots, n} = 0.
\]

The degenerate variety \( \mathcal{F}(1, n - 1) \) is also a subvariety in \( \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \), defined by a "degenerate" relation

\[
X_1^a X_2^a, \ldots, n + (-1)^{n-1} X_n^a X_1^a, \ldots, n-1 = 0.
\]
2. CELL DECOMPOSITION

In this section we describe explicitly the image of \( F^\lambda \) inside the product of Grassmanians and construct the cell decomposition of the degenerate flag varieties. We start with the case of \( \lambda = \omega_d \).

2.1. CELL DECOMPOSITION FOR GRASSMANIANS. Recall that \( F^\omega_d \simeq F_\omega_d \simeq Gr(d,n) \). Given an increasing tuple \( L = (l_1 < \cdots < l_d) \) we set

\[
p_L = \text{span}(v_{l_1}, \ldots, v_{l_d}) \in Gr(d,n).
\]

The subspace \( p_L \) is \( T \)-invariant. Let \( k \) be a number such that \( l_k \leq d < l_{k+1} \).

**Proposition 2.1.** The orbit \( G^a \cdot p_L \) is an affine cell and \( Gr(d,n) \) is the disjoint union of all such cells.  

**Proof.** Recall that \( G^a \) acts on \( Gr(d,n) \) via the projection to the Lie group of \( g_d \) (see (1.1)). Therefore the elements of \( G^a \cdot p_L \) are exactly the subspaces of \( V \) having a basis \( e_1, \ldots, e_d \) of the form

\[
e_j = v_j + \sum_{i=1}^{l_j-1} a_{i,j} v_i + \sum_{i=d+1}^{n} a_{i,j} v_i, \quad j = 1, \ldots, k
\]

\[
e_j = v_j + \sum_{i=d+1}^{l_j-1} a_{i,j} v_i, \quad j = k+1, \ldots, d.
\]

Such elements in \( Gr(d,n) \) obviously form an affine cell and one has a decomposition \( Gr(d,n) = \sqcup_L G^a \cdot p_L \). \( \square \)

**Remark 2.2.** Formulas (2.1) and (2.2) can be combined together as follows. Let \( [k]_+ = k \) if \( k > 0 \) and \( [k]_+ = k + n \) if \( k \leq 0 \). Then each element of \( G^a \cdot p_L \) has a basis \( e_1, \ldots, e_d \) of the form

\[
e_j = v_j + \sum_{i=1}^{[l_j-d]_+} a_{i,j} v_{[l_j-i]+}.
\]

**Remark 2.3.** The orbit \( G^a \cdot p_L \) can be identified with a certain cell \( B \cdot p_J \) in the usual cell decomposition of \( Gr(d,n) \). Namely, define \( J \) as follows:

\[
J = (l_{k+1} - d, l_{k+2} - d, \ldots, l_d - d, l_1 - d + n, l_2 - d + n, \ldots, l_k - d + n).
\]

Then the map

\[
\psi : V \to V, \quad \psi(v_i) = v_{[i-d]_+}, \quad i = 1, \ldots, n
\]

sends \( G^a \cdot p_L \) to \( B \cdot p_J \) (this is clear from the explicit description (2.1), (2.2)).

**Example 2.4.** Let \( n = 9, d = 4 \) and \( L = (2, 3, 6, 7) \) (thus \( k = 2 \)). Then the elements of \( G^a \cdot p_L \) can be identified with the following matrices (the
columns of a matrix form a basis of the corresponding subspace):

\[
\begin{pmatrix}
* & * & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
* & * & * & * \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
* & * & 0 & 0 \\
* & * & * & * \\
\end{pmatrix}
\]

Here * denotes arbitrary entries and hence the number of stars coincides with the dimension of the cell.

2.2. Chains of subspaces. In this section we fix the numbers \(d_1, \ldots, d_s\) and write \(\mathcal{F}^a\) for \(\mathcal{F}^a(d_1, \ldots, d_s)\). Let \(v_1, \ldots, v_n\) be some basis of \(V \cong \mathbb{C}^n\). For \(1 \leq i < j \leq n\) we define the projections \(pr_{i+1,j} : V \to V\) by the formula

\[
pr_{i+1,j}(\sum_{l=1}^{n} c_l v_l) = \sum_{l=1}^{i} c_l v_l + \sum_{l=j+1}^{n} c_l v_l.
\]

The goal of this subsection is to prove the following theorem.

**Theorem 2.5.** The variety \(\mathcal{F}^a \hookrightarrow Gr(d_1, n) \times \cdots \times Gr(d_s, n)\) is formed by all sequences \(V_1, \ldots, V_s\), \(V_l \in Gr(d_l, n)\) such that for all \(1 \leq l < m \leq s\)

\[
(2.4) \quad pr_{d_l+1,d_m} V_l \hookrightarrow V_m.
\]

**Remark 2.6.** It is easy to see that the set of conditions \((2.4)\) is equivalent to the subset with \(m = l + 1\), i.e. to the set of conditions

\[
(2.5) \quad pr_{d_l+1,d_{l+1}} V_l \hookrightarrow V_{l+1}, \quad l = 1, \ldots, s - 1.
\]

**Lemma 2.7.** Let \((V_1, \ldots, V_s) \in \mathcal{F}^a\). Then conditions \((2.4)\) are satisfied.

**Proof.** Let us first look at the big cell \(G^aM \cdot \mathbb{C}v_\lambda \subset \mathcal{F}^a\). Note that the line \(\mathbb{C}v_\lambda\) is represented by the point

\[
\times_{i=1}^{s} \text{span}(v_1, \ldots, v_{d_i}) \in \times_{i=1}^{s} Gr(d_i, n).
\]

Take an element \(g = \exp(\sum s_{i,j} f_{i,j}) \in G^aM \subset G^a\). Then one has

\[
g \cdot \text{span}(v_1, \ldots, v_d) = \text{span}(v_1 + \sum_{j=d}^{n-1} s_{1,j} v_{j+1}, \ldots, v_d + \sum_{j=d}^{n-1} s_{d,j} v_{j+1}).
\]

Therefore conditions \((2.4)\) hold for all points from the big cell of the degenerate flag varieties. Since \(\mathcal{F}^a_\lambda\) is the closure of the big cell, the lemma is proved.

**Proposition 2.8.** Let \(V_1, \ldots, V_s\) be a set of subspaces of \(V\) satisfying \((2.4)\) with \(\dim V_l = d_l\). Then \((V_1, \ldots, V_s) \in \mathcal{F}^a\).
Proof. We know that the image of the embedding
\[ \mathcal{F}^q \hookrightarrow \times_{i=1}^s \text{Gr}(d_i, \ell) \hookrightarrow \times_{i=1}^s \mathbb{P}(\Lambda^{d_i} V) \]
is defined by the set of relations \( R^{k:a}_{J,I} = 0 \). Our goal is to prove that (2.4) implies that all the relations \( R^{k:a}_{J,I} \) vanish. Fix a pair \( 1 \leq l \leq m \leq s \). In what follows we denote the projection \( pr_{d_l+1,d_m} \) simply by \( pr \).

Let \( (V_1, \ldots, V_s) \) be a collection of subspaces satisfying (2.4). Fix tuples \( I = (i_1, \ldots, i_l) \) and \( J = (j_1, \ldots, j_m) \) and a number \( k \). We prove that the relation \( R^{k:a}_{J,I} \) vanishes on \( (V_1, \ldots, V_s) \). Without loss of generality we assume that \( i_1, \ldots, i_k \notin [d_l + 1, d_m] \). We also rearrange the entries of \( I \) in such a way that the elements from \( I \cap [d_l + 1, d_m] \) are concentrated at the end of \( I \), i.e. there exists a number \( b \) such that \( i_1, \ldots, i_b \notin [d_l + 1, d_m], \quad i_{b+1}, \ldots, i_l \in [d_l + 1, d_m] \).

Obviously, \( b \geq k \). Let \( l - c = \dim(\ker pr \cap V_l) \). We fix a basis \( e_1, \ldots, e_l \) of \( V_l \) such that \( \text{pre}_{e_1}, \ldots, \text{pre}_{e_c} \) is a basis of \( \ker PR \) and \( e_{c+1}, \ldots, e_l \) form a basis of \( \ker pr \cap V_l \). We denote by \( a_{s,t} \) the coefficients of the expansion of \( e_s \) in terms of \( e_t \):
\[ e_q = \sum_{r=1}^{l} a_{r,q} e_r. \]
The idea of the proof is to use the following decomposition of a Plücker coordinate \( X_I \):
\[ X_I = \sum_{1 \leq \alpha_1 < \cdots < \alpha_{l-b} \leq l} \pm a_{i_{b+1}, \alpha_1} \cdots a_{i_l, \alpha_{l-b}} X_{i_1, \ldots, i_b}. \]
Here \( X_{i_1, \ldots, i_b} \) is the \((i_1, \ldots, i_b)\)-th Plücker coordinate of the vector space \( \text{span}(e_{\beta_1}, \ldots, e_{\beta_b}) \), where the set of \( \beta \)'s is complementary to the set of \( \alpha \)'s, i.e.
\[ \{\beta_1, \ldots, \beta_b\} \cup \{\alpha_1, \ldots, \alpha_{l-b}\} = \{i_1, \ldots, i_l\}. \]
The decomposition (2.6) induces the decomposition of the relation \( R^{k:a}_{J,I} \), such that each term can be shown to vanish. Note that if \( b > c \) then \( X_I \) vanishes on \( V_l \). We thus assume that \( b \leq c \).

Define the subspace \( E_{\beta} = pr(\text{span}(e_{\beta_1}, \ldots, e_{\beta_b})) \).

We know that \( E_{\beta} \hookrightarrow V_m \). In addition, the coordinates \( X_{(i_1, \ldots, i_b)} \) of the space \( \text{span}(e_{\beta_1}, \ldots, e_{\beta_b}) \) coincide with the Plücker coordinates \( Y_{(i_1, \ldots, i_b)} \) of \( E_{\beta} \), because \( i_1, \ldots, i_b \notin [d_l + 1, d_m] \) (we are using the notations \( Y_I \) to distinguish between Plücker coordinated of different spaces). Since \( E_{\beta} \hookrightarrow V_m \), the classical relations \( R^{k:a}_{J,(i_1, \ldots, i_b)} \) vanish on the pair \( (E_{\beta}, V_m) \). Since
\[ E_{\beta} \hookrightarrow \text{span}(v_1, \ldots, v_{d_l}, v_{d_m+1}, \ldots, v_n), \]
a Plücker coordinate \( Y_{q_1, \ldots, q_b} \) of \( E_{\beta} \) vanishes unless non of the indices \( q \) are between \( d_l + 1 \) and \( d_m \). Hence the degenerate Plücker relation \( R^{k:a}_{J,(i_1, \ldots, i_b)} \)
also vanishes on \((E_\beta, V_m)\). Note also that the decomposition (2.6) induces the decomposition
\[
R_{k,l}^{i_1, \ldots, i_b} = \sum_{\alpha_1 < \cdots < \alpha_{l-b}} \pm \alpha_{i_{b+1}, \alpha_1} \cdots a_{i_2, \alpha_{l-b}} R_{k,l}^{i_1, \ldots, i_b}.
\]

But as we have shown above, each of the relations \(R_{k,l}^{i_1, \ldots, i_b}\) vanishes on \((V_i, V_m)\). Hence so does \(R_{k,l}^{i_1, \ldots, i_b}\).

**Example 2.9.** Let \(\lambda = \omega_1 + \omega_{n-1}\), i.e. \(s = 2, d_1 = 1, d_2 = n - 1\). Then the image of \(\mathcal{F}(1, n-1)\) inside \(Gr(1, n) \times Gr(n-1, n)\) is formed by all pairs \(V_1, V_2\) such that \(pr_2, n-1 V_1 \hookrightarrow V_2\). Since \(pr_2, n-1 V_1 \hookrightarrow \text{span}(v_1, v_n)\), the image of the embedding \(\mathcal{F}(1, n-1) \hookrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}\) is defined by a single relation
\[
X_1^a X_2^a \cdots + (-1)^{n-1} X_n^a = 0,
\]
which agrees with Example 2.8.

**Corollary 2.10.** Theorem 2.5 is true.

**Corollary 2.11.** Let \(I^1, \ldots, I^s, I^l \subset \{1, \ldots, n\}\) be a collection of tuples such that the cardinality of \(I^l\) is \(d_l\). Then a point \(p_{l_1} \times \cdots \times p_{l_s}\) belongs to \(\mathcal{F}\) if and only if
\[
I^l \setminus \{d_l + 1, \ldots, d_{l+1}\} \subset I^{l+1}.
\]

**Example 2.12.** Consider the case of the complete flags: \(s = n - 1, d_l = l\). Set \(p_{l_1} = pr_{l_1}\). Then the embedding of \(\mathcal{F}\) into the product of Grassmanians is defined by the conditions
\[
pr_{l+1} V_l \hookrightarrow V_{l+1}, \quad l = 1, \ldots, n - 2
\]
and the conditions (2.7) read as \(I^l \setminus \{l + 1\} \subset I^{l+1}\) for \(l = 1, \ldots, n - 2\).

2.3. **Cells for \(\mathcal{F}_\lambda\).** Recall that the cell decomposition for a Grassmanian is given by the \(G^a\)-orbits of the torus fixed points. However, this is not true for the case of general \(\mathcal{F}_\lambda\). Moreover, the number of \(G^a\)-orbits can be infinite. The simplest example is as follows.

**Example 2.13.** Let \(n = 4, \lambda = \omega_1 + \omega_3\). Then \(\mathcal{F}_\lambda^a\) is embedded into \(\mathbb{P}^3 \times \mathbb{P}^3\) (two Grassmanians for \(s_4\)) with the coordinates \((x_1 : x_2 : x_3 : x_4)\) and \((x_{123} : x_{124} : x_{133} : x_{234})\). The variety \(\mathcal{F}_{\omega_1 + \omega_3}\) is defined by a single relation \(x_1 x_{234} - x_4 x_{123} = 0\). Therefore, \(\mathcal{F}_{\omega_1 + \omega_3}\) contains the product \(\mathbb{P}^2 \times \mathbb{P}^2\) defined by \(x_1 = x_{123} = 0\). We note that the subgroup \(G_0^a\) of \(G^a\) acts trivially on this \(\mathbb{P}^2 \times \mathbb{P}^2\) (the PBW-degree in both \(V_{\omega_1}\) and \(V_{\omega_3}\) is at most one). Therefore, we are left with an action of the Borel subgroup. Let \(w_1, w_2, w_3, w_4\) and \(w_{123}, w_{124}, w_{134}, w_{234}\) be the standard bases for \(V_{\omega_1}\) and \(V_{\omega_3}\). The group \(B\) acts on the span of \(w_2, w_3, w_4\) (resp. on the span of \(w_{124}, w_{134}, w_{234}\)) as on the quotient of the vector representation (resp. the dual vector representation) by \(C w_1\) (resp. \(C w_{123}\)). It is easy to see that the corresponding \(B\)-action on \(\mathbb{P}^2 \times \mathbb{P}^2\) has infinitely many orbits.
In the following proposition we describe the cell decomposition for $\mathcal{F}^a = \mathcal{F}^a(d_1, \ldots, d_s)$.

**Proposition 2.14.** Let $I = (I^1, \ldots, I^s)$ be a set of sequences satisfying the condition (2.7). Then there exists a cell decomposition $\mathcal{F}^a = \sqcup_i C_i$, where

$$C_i = (G^a \cdot p_{I^1} \times \cdots \times G^a \cdot p_{I^s}) \cap \mathcal{F}^a.$$  

In other words, a cell is given by the intersection of the degenerate flag variety, embedded into the product of Grassmanians, with the product of the corresponding cells in $Gr(d_i, n)$.

**Proof.** In Theorem 3.6 we compute the dimensions of $C_i$. In the proof we construct explicitly the coordinates on $C_i$ thus showing that $C_i$ is a cell. \(\square\)

### 3. The median Genocchi numbers

#### 3.1. Combinatorics

Let $h_n$ be the normalized Genocchi numbers of the second kind. They are also referred to as the normalized median Genocchi numbers. These numbers have several definitions (see [De], [Du], [Kr], [S]). The first several $h_n$'s are as follows: 1, 2, 7, 38, 295, 3098. We first briefly recall definitions of these numbers.

We start with the Dellac definition (see [De]). Consider a rectangle with $n$ columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs $(l, j)$, where $l = 1, \ldots, n$ is the number of a column and $j = 1, \ldots, 2n$ is the number of a row. A Dellac configuration $D$ is a subset of boxes, subject to the following conditions:

- each column contains exactly two boxes from $D$,
- each row contains exactly one box from $D$,
- if the $(l, j)$-th box is in $D$, then $l \leq j \leq n + l$.

Let $DC_n$ be the set of such configurations. Then the number of elements in $DC_n$ is equal to $h_n$.

We list all Dellac’s configurations for $n = 3$. We specify boxes in a configuration by putting fat dots inside.

(3.1)
The Dellac definition is the earliest one, but the most well-known defi-
nition is via the Seidel triangle. The Seidel triangle is of the form

\[
\begin{array}{cccccc}
1 & 1 & 2 & 8 & 8 & 56 \\
1 & 2 & 6 & 14 & 14 & 48 \\
2 & 3 & 6 & 17 & 18 & 34 \\
8 & 6 & 17 & 17 & & 17 \\
8 & 14 & 18 & & & \\
56 & 48 & 34 & 17 & & \\
56 & 104 & 138 & 155 & 155 & \\
\end{array}
\]

By definition, the triangle is formed by the numbers \( G_{k,n} \) (\( n \) is the number of a row and \( k \) is the number of a column) with \( n = 1, 2, \ldots \) and \( 1 \leq k \leq \frac{n+1}{2} \), subject to the relations \( G_{1,1} = 1 \) and
\[
G_{k,2n} = \sum_{i \geq k} G_{i,2n-1}, \quad G_{k,2n+1} = \sum_{i \leq k} G_{i,2n}.
\]
The numbers \( G_{n,2n-1} \) are called the Genocchi numbers of the first kind and the numbers \( G_{1,2n} \) are called the Genocchi numbers of the second kind (or the median Genocchi numbers). Barsky \cite{Ba} and then Dumont \cite{Du} proved that the number \( G_{1,2n+2} \) is divisible by \( 2^n \). The normalized median Genocchi numbers \( h_n \) are defined as the corresponding ratios: \( h_n = G_{1,2n+2}/2^n \).

In \cite{Kre} Kreweras suggested another description of the numbers \( h_n \). Namely, a permutation \( \sigma \in S_{2n+2} \) is called a normalized Dumont permutation of the second kind if the following conditions are satisfied:

- \( \sigma(k) < k \) if \( k \) is even,
- \( \sigma(k) > k \) if \( k \) is odd,
- \( \sigma^{-1}(2k) < \sigma^{-1}(2k+1) \) for \( k = 1, \ldots, n \).

The set of such permutations is denoted by \( PD2N_n \) (P for permutations, D for Dumont, 2 for the second kind and N for normalized). According to Kreweras, the number of elements of \( PD2N_n \) is equal to \( h_n \). In Proposition \ref{prop:dellac_kreweras_equivalence} we show that the definitions of Dellac and Kreweras are equivalent (this seems to be known to expert – see \cite{G, S}, but we were not able to find a proof in the literature).

In the following proposition we show that the conditions from Example \ref{example:genocchi_numbers} give rise to a new definition of the numbers \( h_n \).

**Proposition 3.1.** The number of tuples \( I^1, \ldots, I^{n-1} \), with \( I^l \subset \{1, \ldots, n\} \), \( \#I^l = l \) subject to the condition
\[
I^{l-1} \setminus \{l\} \subset I^l, \quad l = 2, \ldots, n-1
\]
is equal to \( h_n \).

**Proof.** Let \( \tilde{h}_n \) be the number of tuples as above. We compare \( \tilde{h}_n \) with the Dellac definition of \( h_n \). Given a set \( I^1, \ldots, I^{n-1} \) subject to the condition
we construct the corresponding Dellac’s configuration $D$ and then prove that this map is one-to-one. The rule is as follows. Let us explain what are the boxes of $D$ in the $l$-th column.

First, suppose $l \not\in I_l^1$. Then because of the condition $l \not\in I_l^1$ the difference $I_l^1 \setminus I_l^{l-1}$ contains exactly one number $j$. There are two cases:

- If $j > l$, then $D$ contains boxes $(l, l)$ and $(l, j)$.
- If $j \leq l$, then $D$ contains boxes $(l, l)$ and $(l, j + n)$.

Now, suppose $l \in I_l^1$. Then either $l \in I_l^1$, or $L \not\in I_l^1$. If $l \in I_l^1$, then $I_l^1 \setminus I_l^{l-1}$ contains exactly one number $j$. There are two cases:

- If $j > l$, then $D$ contains boxes $(l, l + n)$ and $(l, j)$.
- If $j \leq l$, then $D$ contains boxes $(l, l + n)$ and $(l, j + n)$.

Finally, let $l \in I_l^{l-1}$ and $l \not\in I_l^1$. Then $I_l^1 \setminus I_l^{l-1}$ contains exactly two numbers $j_1$ and $j_2$. There are four variants:

- If $j_1 > l$ and $j_2 > l$, then $D$ contains boxes $(l, j_1)$ and $(l, j_2)$.
- If $j_1 > l$ and $j_2 \leq l$, then $D$ contains boxes $(l, j_1)$ and $(l, n + j_2)$.
- If $j_1 \leq l$ and $j_2 > l$, then $D$ contains boxes $(l, j_1 + n)$ and $(l, j_2)$.
- If $j_1 \leq l$ and $j_2 \leq l$, then $D$ contains boxes $(l, j_1 + n)$ and $(l, j_2 + n)$.

This rule explains how to pick boxes in columns from 1 to $n - 1$. To complete the configuration we simply pick two boxes in the last column in the unique way to make $D$ a configuration.

In order to prove that this map is a bijection, we construct the inverse map. Let $D$ be a Dellac configuration. We define $I_l^n$ inductively. First, let $l = 1$. Then the box $(1, 1)$ necessarily belongs to $D$. Let $j > 1$ and $D$ contains $(1, j)$. Then if $j = n + 1$, then $I_l^1 = (1)$. Otherwise $I_l^1 = (j)$.

Now assume that $I_l^{l-1}$ is already defined. First, suppose that the $(l, l)$-th box belongs to $D$. Then there exists one more box $(l, j)$ in $D$ with $n + l \geq j > l$. If $j \leq n$ we set $I_l^1 = I_l^{l-1} \cup \{j\}$. Otherwise, we set $I_l^1 = I_l^{l-1} \cup \{j - n\}$. Second, suppose that the $(l, l)$-th box does not belong to $D$. Since the $l$-th row of $D$ contains exactly one box, there exists $l_1 < l$ such that the $(l_1, l)$-th box belongs to $D$. Therefore, $l \subset I_l^{l-1}$. There exist exactly two boxes $(l, j_1)$ and $(l, j_2)$ in $D$ in the $l$-th column. Then we set $I_l^1 = I_l^{l-1} \setminus \{l\} \cup \{j_1, j_2\}$, where $j = j$, if $j \leq n$ and $j = j - n$ otherwise.

Example 3.2. Let $n = 3$. The pairs $I_1^1, I_2^1$, corresponding to the Dellac configurations $3.1$ are as follows (the order is the same as on picture $3.1$):

- $\{(2), (13)\}$, $\{(2), (23)\}$, $\{(2), (12)\}$, $\{(3), (13)\}$,
- $\{(3), (23)\}$, $\{(1), (13)\}$, $\{(1), (12)\}$.

We now compare the definitions by Dellac and by Kreweras.

**Proposition 3.3.** The number of elements in $PD2N_n$ is equal to the number of elements in $DC_n$.

**Proof.** We construct a bijection $A : PD2N_n \rightarrow DC_n$. Let $\sigma \in PD2N_n$. We determine the boxes in the $k$-th column of $A(\sigma)$ using the values of $\sigma^{-1}(2k)$ and $\sigma^{-1}(2k + 1)$.
Let us start with \( k = 1 \). We note that \( \sigma(2) = 1 \), \( \sigma(4) \) is equal to 2 or to 3. In addition, \( \sigma^{-1}(2) = 1 \) or 4 and the possible values of \( \sigma^{-1}(3) \) are 4, 6, \ldots, 2n + 2. Therefore, all possible values of the pair \((\sigma^{-1}(2), \sigma^{-1}(3))\) are as follows:

\[
(1, 4), (4, 6), (4, 8), \ldots, (4, 2n + 2).
\]

If the first possibility occurs, then by definition the first column of \( A(\sigma) \) contains boxes \((1, 1)\) (as any Dellac’s configuration) and \((1, n + 1)\). If \( \sigma^{-1}(2) = 4 \) and \( \sigma^{-1}(3) = 2l + 2 \), then the first column of \( A(\sigma) \) contains boxes \((1, 1)\) and \((1, l)\).

Now let us consider the case \( k = n \). We note that \( \sigma(2n + 1) = 2n \), \( \sigma(2n - 1) \) is equal to 2n or to 2n + 1. In addition, \( \sigma^{-1}(2n + 1) = 2n + 2 \) or 2n - 1 and the possible values of \( \sigma^{-1}(2n) \) are 1, 3, \ldots, 2n - 1. Therefore, all possible values of the pair \((\sigma^{-1}(2n), \sigma^{-1}(2n + 1))\) are as follows:

\[
(2n - 1, 2n + 2), (1, 2n - 1), (3, 2n - 1), \ldots, (2n - 3, 2n - 1).
\]

If the first possibility occurs, then by definition the \( n \)-th column of \( A(\sigma) \) contains boxes \((n, 2n)\) (as any Dellac’s configuration) and \((n, n)\). If

\[
(\sigma^{-1}(2n), \sigma^{-1}(2n + 1)) = (2l - 1, 2n - 1),
\]

then the first column of \( A(\sigma) \) contains boxes \((n, 2n)\) and \((n, n + l)\).

Finally, take \( k = 2, \ldots, n - 1 \). We note that the possible values of \( \sigma^{-1}(2k) \) are 1, 3, \ldots, 2k - 1, 2k + 2, \ldots, 2n. Also, the possible values of \( \sigma^{-1}(2k + 1) \) are 3, 5, \ldots, 2k - 1, 2k + 2, \ldots, 2n, 2n + 2. We now define the \( k \)-th column of \( A(\sigma) \) as follows:

(i) If the pair \((\sigma^{-1}(2k), \sigma^{-1}(2k + 1))\) contains \( 2l - 1, l = 1, \ldots, k \), then the \( k \)-th column of \( A(\sigma) \) contains a box \((k, n + l)\).

(ii) If the pair \((\sigma^{-1}(2k), \sigma^{-1}(2k + 1))\) contains \( 2l + 2, l = k, \ldots, n \), then the \( k \)-th column of \( A(\sigma) \) contains a box \((k, l)\).

We note that \( A(\sigma) \in DC_n \). In fact, by definition any column of \( A(\sigma) \) contains exactly two boxes and every row contains exactly one box (this follows from the definition above and because \( \sigma \) is one-to-one). In order to prove that \( A \) is a bijection it suffices to note that formulas (i) and (ii) allow to construct explicitly the map \( A^{-1} \). \[\square\]

**Example 3.4.** Let \( n = 3 \). The elements of \( PD2N_3 \) corresponding to the Dellac configurations on picture (3.1) are as follows (the order is the same as on picture (3.1)):

\[
(41627385), (61427385), (41526387), (41627583), (61427583), (21637485), (21436587).
\]
We recall that the main ingredient for the Kreweras construction of $PD2N_n$ is the following triangle:

\begin{align*}
1 \\
1 & 1 \\
2 & 3 & 2 \\
7 & 12 & 12 & 7 \\
38 & 69 & 81 & 69 & 38 \\
295 & 552 & 702 & 702 & 552 & 295 \\
\end{align*}

The rule is as follows: denote the numbers in the $n$-th line by $h_{n,1}, \ldots, h_{n,n}$. For example, $h_{4,2} = 12$. Then the Kreweras triangle is defined by

\begin{align*}
h_{n,1} &= h_{n-1,1} + \cdots + h_{n-1,n-1}, \\
h_{n,2} &= 2h_{n,1} - h_{n-1,1}, \\
h_{n,k} &= 2h_{n,k-1} - h_{n,k-2} - h_{n-1,k-2} - h_{n-1,k-1}, \quad k \geq 3.
\end{align*}

Kreweras proved that $h_{n+1,1}$ is the $n$-th Genocchi number $h_n$ and in general $h_{n+1,k}$ is the number of the normalized Dumont permutations $\sigma \in S_{2n+2}$ of the second kind such that $\sigma(1) = 2k$. The following is an immediate corollary from the explicit bijections above.

**Corollary 3.5.** The number of the Dellac configurations $D \in DC_n$ such that $\min\{i : (i, n+1) \in D\} = k$ is equal to $h_{n,k}$. The number of tuples $I^1, \ldots, I^{n-1}$ subject to the condition $I^{l-1} \setminus \{l\} \subset I^l$ with an extra condition $\min\{j : 1 \in I^j\} = k$ is equal to $h_{n,k}$.

### 3.2. The Poincaré polynomials.

For a tuple $I = (I^1, \ldots, I^{n-1})$ subject to the relation $I^{l-1} \setminus \{l\} \subset I^l$ we denote by $D_I$ the corresponding Dellac configuration. For a Dellac configuration $D \in DC_n$ we define the length $l(D)$ of $D$ as the number of pairs $(l_1, j_1), (l_2, j_2)$ such that the boxes $(l_1, j_1)$ and $(l_2, j_2)$ are both in $D$ and $l_1 < l_2$, $j_1 > j_2$. We call such a pair of boxes $(l_1, j_1), (l_2, j_2)$ a disorder. This definition resembles the definition of the length of a permutation. We note that in the classical case the dimension of a cell attached to a permutation $\sigma$ in a flag variety is equal to the number of pairs $j_1 < j_2$ such that $\sigma(j_1) > \sigma(j_2)$ (which equals to the length of $\sigma$).

**Theorem 3.6.** The dimension of a cell $C_I$ is equal to $l(D_I)$.

**Proof.** We prove the dimension formula by constructing explicitly the coordinates on the cell $C_I$. Let

\[ I = (I^1, \ldots, I^{n-1}), \quad I^d = (i_1^d < \cdots < i_d^d). \]

Recall the description of the cells $C_{I^d} \subset Gr(d, n)$ from Proposition 2.1. Using this description we construct the coordinates on $C_I$ inductively on $d$. Let $(V_1, \ldots, V_{n-1}) \in C_I$. For a number $k$ we set $[k]_+ = k$ if $k > 0$ and $[k]_+ = k + n$ if $k \leq 0$.

We start with $d = 1$. An element $V_I \in C_I$ is of the form $C e_I$ with

\[ e_I = v_i^1 + a_1 v_{[i-1]}^1 + \cdots + a_1^{[i-1]} v_{[i-1]}^1 \]
We state that \([i_1^1 - 1]_+ - 1\) (which is exactly the number of the degrees of freedom we have so far) is exactly the number of boxes \((l, j) \in D_1\) such that \(l > 1\) and \(j < i_1^1\) (note that the box \((1, 1)\) is necessarily in \(D_1\), but it does not add anything to the length of \(D_1\), since for any \((l, j) \in D_1\) with \(l > 1\) we have \(j > 1\)). In fact, the first column of \(D_1\) contains boxes in the first row and in the \((i_1^1 - 1)_+ + 1\)-st row (see the proof of Proposition 3.1). Since any row of \(D_1\) contains exactly one box, the rows number 2, \ldots, \([i_1^1 - 1]_+\) are occupied by boxes in the columns from 2 to \(n\). Therefore, the box \((1, [i_1^1 - 1]_+ + 1)\) produces exactly \([i_1^1 - 1]_+ - 1\) disorders.

The second step is to construct the coordinates on those subspaces from \(C_{I^2}\) which contain \(pr_2V_1\). There are two possibilities: either \(i_1^1 = 2\) or \(i_1^1 \neq 2\). In the first case the condition \(pr_2V_1 \hookrightarrow V_2\) is empty. Therefore, we have to choose two basis vectors \(e_2^1, e_2^2\) of \(V_2 \subset C_{I^2}\), with the coordinates

\[
eq 1)
\begin{align*}
eq 2) + 1) \text{ contains one box in the columns 3, 4, \ldots, } n\). Hence adding appropriately normalized vector \(e_2^1\) one can vanish the coefficient of \(e_2^2\) in front of \(v_{i_1^1}\). We note that since \(i_1^1 = 2\), the second column of \(D_1\) contains boxes in the rows \((i_1^2 - 2)_+ + 2)\) and \((i_1^2 - 2)_+ + 2)\) (see the proof of Proposition 3.1). We state that \([i_1^2 - 2]_+ + 1 + [i_1^2 - 2]_+ - 2\) (the number of degrees of freedom we have fixing the vectors \(e_2^1\) and \(e_2^2\)) is exactly the number of boxes in the columns 3, 4, \ldots, \(n\), having disorders with boxes in the second column. In fact, each row from 3 to \([i_1^2 - 2]_+ + 1\) contains one box in the columns 3 and greater (recall \(i_1^1 = 2\)). This produces \([i_1^2 - 2]_+ - 1\) disorders with the box \((2, [i_1^2 - 2]_+ + 1)\). Similarly, we obtain \([i_1^2 - 2]_+ - 2\) disorders with the second box in the second column.

Now assume \(i_1^2 = 2\). Then the space \(pr_2V_1\) is nontrivial and spanned by a single vector \(e_2^2 = pr_2e_1^1\). Therefore in order to specify \(V_2\) we need to fix one more vector \(e_2^3\) such that \(\text{span}(e_2^1, e_2^2) \subset C_{I^2}\). Recall that since \(i_1^1 \neq 2\) we have \(I^2 \setminus I^1 = \{j\}\). Also, the second column of \(D_1\) contains boxes in the second row and in the row number \([j - 2]_+ + 2\) (see the proof of Proposition 3.1). The box \((2, 2)\) does not produce any disorder with boxes in the columns greater than 2. As for the box \((2, [j - 2]_+ + 2)\), the number of disorders it produces is equal to the number of degrees of freedom of choosing the vector \(e_2^2\) (the argument is very similar to the ones above in the case \(i_1^1 = 2\)).

Now let us consider the general induction step. Assume that we have already computed the number of degrees of freedom while fixing the subspaces \(V_1, \ldots, V_{d-1}\). Our goal is to show that the number of degrees of freedom of \(V_d\) is equal to the number of disorders produced by the boxes in the \(d\)-th column with the boxes in columns \(l\) with \(l > d\). As in the previous case, one has to consider two cases: \(d \in I^{d-1}\) and \(d \notin I^{d-1}\). The proof is very similar to the one in the case \(d = 2\) and we omit it. \(\square\)
Corollary 3.7. The Poincaré polynomial $P_n(t) = P_{F_n}(t)$ is given by

$$P_n(t) = \sum_{D \in DC_n} t^{2l(D)}.$$ 

Let $q = t^2$. Then $P_n$ are polynomials in $q$ with $P_n(1) = h_n$. Thus the Poincaré polynomials of the degenerate flag varieties provide a natural $q$-version of the normalized median Genocchi numbers (it would be interesting to compare our $q$-version with the one in [HZ]).

Example 3.8. The first four polynomials $P_n(q)$ are as follows:

\begin{align*}
P_1(q) &= 1, \\
P_2(q) &= 1 + q, \\
P_3(q) &= 1 + 2q + 3q^2 + q^3, \\
P_4(q) &= 1 + 3q + 7q^2 + 10q^3 + 10q^4 + 6q^5 + q^6.
\end{align*}

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