Spin-Nematic and Spin-Density-Wave Orders in Spatially Anisotropic Frustrated Magnets in a Magnetic Field

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We develop a microscopic theory of finite-temperature spin-nematic orderings in three-dimensional spatially anisotropic magnets consisting of weakly-coupled frustrated spin-\(\frac{1}{2}\) chains with nearest-neighbor and next-nearest-neighbor couplings in a magnetic field. Combining a field theoretical technique with density-matrix renormalization group results, we complete finite-temperature phase diagrams in a wide magnetic-field range that possess spin-bond-nematic and incommensurate spin-density-wave ordered phases. The effects of a four-spin interaction are also studied. The relevance of our results to quasi-one-dimensional edge-shared cuprate magnets such as LiCuVO\(_4\) is discussed.

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Introduction.— The quest for novel states of matter has been attracting much attention in condensed-matter physics. Among those states, recently spin-nematic (quadrupolar) phases have been vividly discussed in the field of frustrated magnetism [1,10]. The spin-nematic phase is defined by the presence of a symmetrized rank-2 spin tensor order, such as \(\langle S^x_j S^y_j + H.c. \rangle \neq 0\), and the absence of any spin (dipolar) moment. Geometrical frustrations which generally suppresses spin orders, is an important ingredient for the emergence of spin nematics [1]. In spin-\(\frac{1}{2}\) magnets, the spin nematic operators cannot be defined on a single site because of the commutation relation of spin-\(\frac{1}{2}\) operators. They reside on bonds between different sites [1,3], which is a significant difference from the quadrupolar phases in higher-spin systems [6]. Due to this property, it is generally quite hard to develop theories of spin nematics in spin-\(\frac{1}{2}\) magnets, particularly in two- or three-dimensional (3D) systems. Developing such a theory is a current important issue in magnetism.

Among the existing models predicting spin-nematic phases, the spin-\(\frac{1}{2}\) frustrated chain with a ferromagnetic nearest-neighbor coupling \(J_1 < 0\) and an antiferromagnetic (AF) next-nearest-neighbor one \(J_2 > 0\) would be the most relevant in nature because this system is believed to be an effective model for a series of quasi-1D edge-shared cuprate magnets such as LiCuVO\(_4\) [11–16], Rb\(_2\)Cu\(_2\)MO\(_3\)O\(_{12}\) [17], PbCuSO\(_4\)(OH)\(_2\) [18,19], LiCuSbO\(_4\) [20], and LiCuO\(_2\) [21]. These quasi-1D magnets hence offer a promising playground for spin-nematic phases.

Low-energy properties of the spin-\(\frac{1}{2}\) \(J_1-J_2\) chain have been well understood thanks to recent theoretical efforts [2,3]. The corresponding Hamiltonian is given by

\[
\mathcal{H} = \sum_{n=1,2} \sum_j J_n S_j \cdot S_{j+n} - H \sum_j S_j^z, \tag{1}
\]

where \(S_j\) is the spin-\(\frac{1}{2}\) operator on site \(j\) and \(H\) is an external field. Below the saturation field in the broad parameter range \(-2.7 \lesssim J_1/J_2 < 0\), the nematic operator \(S_j^x S_{j+1}^y\) and the longitudinal spin \(S_j^z\) exhibit quasi-long-range orders, while the transverse spin correlator \(\langle S_j^x S_{j+1}^y \rangle\) decays exponentially due to the formation of two-magnon bound states [3]. This phase is called a spin-nematic Tomonaga-Luttinger (TL) liquid, and it expands down to a low-field regime. The nematic correlation is stronger than the incommensurate longitudinal spin correlation in the high-field regime, while the latter grows stronger in the low-field regime.

From these theoretical results, the quasi-1D cuprates are expected to possess incommensurate longitudinal spin-density-wave (SDW) and spin-nematic long-range orders, respectively, in low- and high-field regimes at sufficiently low temperatures. In fact, recent magnetization measurements of LiCuVO\(_4\) at low temperatures have detected a new phase [12] near saturation, and it is expected to be a 3D spin nematic phase. Some experiments on LiCuVO\(_4\) in an intermediate-field regime find SDW oscillations [13,14] whose wave vectors agree with the result of the nematic TL-liquid theory [2,3,5]. Furthermore, the spin dynamics of LiCuVO\(_4\) observed by NMR [16] seems to be consistent with the prediction from the same theory [3,4]. However, this nematic TL-liquid picture can be applicable only above the 3D ordering temperatures. We have to take into account interchain interactions to explain how 3D spin-nematic and SDW long-range ordered phases are induced with lowering temperature. A mean-field theory for the 3D nematic phase of quasi-1D spin-\(\frac{1}{2}\) magnets [4] has been proposed recently, but it cannot be applied to the SDW phase and does not quantitatively describe finite-temperature effects. It is obscure how both nematic and SDW ordered phases are described in a unified way. A reliable theory for 3D orderings in weakly coupled spin-\(\frac{1}{2}\) \(J_1-J_2\) chains is strongly called for.

In this Letter, we develop a general theory for spin nematic and incommensurate SDW orders in spatially
anisotropic 3D magnets consisting of weakly coupled \( J_1-J_2 \) spin chains with arbitrary interchain couplings in a wide magnetic-field range. Combining field theoretical and numerical results for the \( J_1-J_2 \) spin chain, we obtain finite-temperature phase diagrams, which contain both spin-nematic and SDW phases at sufficiently low temperatures. We thereby reveal characteristic features in the ordering of weakly coupled \( J_1-J_2 \) chains, which cannot be predicted from the theory for the single \( J_1-J_2 \) chain. We also discuss the relevance of our results to real compounds such as LiCuVO4.

**Model.**—Our model of a spatially anisotropic magnet is depicted in Fig. 1. The corresponding Hamiltonian is expressed as

\[
H_{3D} = \sum_r H_r + H_{\text{int}},
\]

where \( r = (r_y, r_z) \) denotes the site index of the square lattice in the \( y-z \) plane, \( H_r \) denotes the Hamiltonian for the \( r \)-th \( J_1-J_2 \) chain along the \( x \) axis in magnetic field \( H \), and \( H_{\text{int}} \) is the inter-chain interaction. In \( H_{\text{int}} \), we introduce weak inter-chain Heisenberg-type exchange interactions with coupling constants \( J_{y_i} \) and \( J_{z_i} \) defined in the \( x-y \) and \( x-z \) planes, respectively.

**Spin-\( \frac{1}{2} \) J1-J2 chain.**—Under the condition \( |J_{y_i,z_i}| \ll |J_{1,2}| \), it is reasonable to choose decoupled \( J_1-J_2 \) spin chains (\( H_r \)) as the starting point for analyzing the 3D model (\( H_{3D} \)). The low-energy effective Hamiltonian for the nematic TL-liquid phase is given by

\[
H^{\text{eff}}_{3D} = \int dx \sum_{\nu = \pm} \frac{v_{\nu}^2}{2} [K_{\nu}(\partial_x \phi_\nu)^2 + K_{\nu}^{-1}(\partial_x \phi_\nu^*)^2] + G_{-} \sin(\pi M) \sin(\sqrt{4\pi\phi_\nu^* + \pi M}),
\]

where \( x = a_{0j} \) (the length \( a_{0j} \) of the \( J_1 \) bond is set equal to unity), \( (\phi_+^\nu(x), \phi_-^\nu(x)) \) is the canonical pair of scalar boson fields, and \( v_+ \) and \( K_+ \) are, respectively, the excitation velocity and the TL-liquid parameter of the \( (\phi_+, \theta_+) \) sector. The sine term makes \( \phi_- \) pinned, inducing an excitation gap in the \( (\phi_-, \theta_-) \) sector. Physically, the gap corresponds to the magnon binding energy \( E_b \). On the other hand, the \( (\phi_+, \theta_+) \) sector describes a massless TL liquid. Vertex operators are renormalized as \( \langle e^{i\alpha \sqrt{\pi} \phi_+} e^{-i\alpha \sqrt{\pi} \phi_-} \rangle_+ = |2/|x||^{K_+/2} \) for \( |x| > 1 \), in which \( \langle \cdots \rangle_\pm \) denotes the average over the \( \langle \phi_\pm, \theta_\pm \rangle \) sector. Spin operators \( S_{x,y} \) are also bosonized as

\[
S_{x,y}^+_r \approx M + \theta_x(\phi_0^y + (-1)^y\phi_0^y)\sqrt{\pi} + (-1)^y A_1 \cos(\sqrt{\pi}\phi_0^y + (-1)^y\phi_0^y) + 2\pi M q_j + \cdots,
\]

where \( M = \langle S_j^0 \rangle \), \( q_j = \frac{\pi}{2} (\frac{j-1}{j+1}) \) for even (odd) \( j \), and \( A_1 \) and \( B_n \) are nonuniversal constants. Utilizing Eqs. 3 and 4, we can evaluate spin and nematic correlation functions at zero temperature \( (T = 0) \) as follows:

\[
\langle S_j^0 S_{j+1}^- \rangle \approx B_0^2 \cos(\pi j/2)(2/j)^{1/(2K_+)} g_-(x) + \cdots,
\]

\[
\langle S_j^0 S_{j+1}^0 \rangle \approx M^2 + (A_1/2)(e^{i\sqrt{\pi} \phi_-})^2 \cos(\pi j(M - 1/2))(2/j)^{K_+ + 1} + \cdots,
\]

where \( g_-(x) = (e^{i\pm i\sqrt{\pi} \phi_-} e^{-i\pm i\sqrt{\pi} \phi_+})_+ \), \( C_0 \) is a constant and we have omitted the index \( r \). The function \( g_-(x) \) decays exponentially as \( x^{-1/2} e^{-x/\xi_-} \). The parameter \( K_+ \), which is less than 2 in the low magnetization regime, monotonically increases with \( M \) and \( K_+ \to 4 \) at the saturation. Thus, the spin-nematic (SN) correlation is stronger than the incommensurate SDW correlation in the high-field regime with \( K_+ > 2 \) and weaker in the low-field regime with \( K_+ < 2 \).
The correlation length $\xi_-$ is related to $v_-$ via $v_- = \xi_- E_b$, under the assumption that the low-energy theory for the $(\phi_-, \theta_-)$ sector has Lorentz invariance. The velocity $v_+$ has the relation $v_+ = 2K/(\pi \chi_1)$, where $\chi = \partial M/\partial H$ is the uniform susceptibility. Since $K_+, \xi_-, E_b$, and $\chi$ are all determined with reasonable accuracy by using the density-matrix renormalization group (DMRG) method, $v_+$ can be quantitatively evaluated as depicted in Fig. 2. The figure shows that $v_+$ is always larger than $v_-$, in accordance with the perturbative formulas $v_+ \approx v(1 \pm K_1/\pi v \cdots)$ for $|K_1| < K_2$, in which $v$ and $K$ are respectively the spinon velocity and the TL-liquid parameter for the single AF-J$_2$ chain. We also note that $v_+$ approaches zero at $M \rightarrow \frac{1}{2}$.

Analysis of the 3D model. — Let us now analyze the 3D model (2) starting with the effective theory of the J$_1$-J$_2$ chain. We first bosonize all of the inter-chain couplings in $H_{\text{int}}$ through Eq. [1]. To obtain the low-energy effective theory for Eq. [2], we trace out the massive $(\phi^a_-, \theta^a_-)$ sectors in the Euclidean action $S_{\text{int}} = S_0 + S_{\text{int}}$ via the cumulant expansion $S_{\text{eff}} = S_0 + \langle S_{\text{int}} \rangle - \frac{1}{2} \langle (S^2_{\text{int}}) - (S_{\text{int}})^2 \rangle + \cdots$, where $S_0$ and $S_{\text{int}}$ are, respectively, the action for the TL-liquid part of the $(\phi^a_+, \theta^a_+)$ sectors and that for the inter-chain couplings. This corresponds to the series expansion in $J_{y,z}/v_-$. The resultant effective Hamiltonian is expressed as $H_{\text{eff}} = H_{\text{0}} + H_{\text{SDW}} + H_{\text{SN}} + \cdots$. Here, $H_{\text{0}} = \sum_r \int dx \frac{1}{2} [K_+ (\partial_x \phi_+^a)(\partial_x \phi_+^a)' + K_- (\partial_x \theta_+^a)(\partial_x \theta_+^a)']$ is the TL-liquid part and $H_{\text{SDW}}$ and $H_{\text{SN}}$ are, respectively, obtained from the first- and second-order cumulants as follows:

$$H_{\text{SDW}} = G_{\text{SDW}} \int dx \sum_r \sum_{\alpha, \beta = \pm} \sum_{\alpha = \pm} \sum_{\alpha = \pm} \int dx_1 \cos(\sqrt{\pi}(\phi_+^\alpha - \phi_+'^\alpha))$$

$$- J_{\alpha 2} \sin(\sqrt{\pi}(\phi_+^\alpha - \phi_+'^\alpha) + \pi M)$$

$$+ J_{\alpha 3} \sin(\sqrt{\pi}(\phi_+^\alpha - \phi_+'^\alpha) + \pi M)$$

$$H_{\text{SN}} = G_{\text{SN}} \int dx \sum_r \sum_{\alpha, \beta = \pm} \sum_{\alpha = \pm} \sum_{\alpha = \pm} \int dx_1 [J_{\alpha 1}^2 - (J_{\alpha 2} - J_{\alpha 3})^2]$$

$$\times \cos(\sqrt{4\pi}(\theta_+^\alpha - \theta_+'^\alpha))$$

with coupling constants $G_{\text{SDW}} = A_2^1 |e^{i \sqrt{\pi} \phi_+}|^2$ and $G_{\text{SN}} = -\frac{K_2}{\pi} \int dx \partial_x \phi_.(x, \tau)^2$ (and imaginary time). The summations run over all nearest neighbor pairs of chains, where $r' = r + e_\alpha$ ($\alpha = y, z$), $e_\alpha$ denotes the unit vector along the $\alpha$-axis, and we have assumed that the field $\phi_+$ smoothly varies in $x$. The first-order term $H_{\text{SDW}}$ contains an inter-chain interaction between the operators $e^{\pm i \sqrt{\pi} \phi_+}$, which essentially induces a 3D spin longitudinal order. Similarly, the term $H_{\text{SN}}$ contains an inter-chain interaction between the spin-nematic operators $S_{2}^{\pm} \phi_+^{\pm,1,r} \approx (-1)^{\tau} e^{\pm i \sqrt{\pi} \theta_+^\alpha}$, which enhances a 3D spin nematic correlation. We should notice that the effective theory $H_{\text{eff}}$ is reliable under the condition that temperature $T$ is sufficiently smaller than the binding energy $E_b$ and the velocities $v_\pm$.

Both the couplings $G_{\text{SDW,SN}}$ can be numerically evaluated from the DMRG data of correlation functions [1, 23]: $G_{\text{SDW}}$ corresponds to the amplitude of the leading term of the longitudinal correlator $\langle S^\alpha_+ S^-_0 \rangle$ given in Eq. [3] and $G_{\text{SN}}$ can be evaluated as $G_{\text{SN}} = \pi^{-1} \sum_{\alpha = \pm} \langle (J/2)^{1/4} r \alpha j(S^\alpha_+ S^-_0)^{1/2} \rangle$. We have checked that the finite-size correction to the sum is small enough when the cutoff $L$ is larger than the $\xi_-$. We emphasize that there is no free parameter in $H_{\text{eff}}$.

To obtain the finite-temperature phase diagram, we apply the inter-chain mean-field (ICMF) approximation [25, 26] to the effective Hamiltonian $H_{\text{eff}}$. To this end, we introduce the "effective" SDW operator $O_{\text{SDW}} = e^{i \tau \alpha} e^{i \sqrt{\pi} \phi_+}$ and the spin-nematic operator $O_{\text{SN}} = (-1)^{\tau} e^{i \sqrt{\pi} \theta_+^\alpha}$. Within the ICMF approach, the finite-temperature dynamical susceptibilities of $O_{\text{A}}$ ($A =$SDW or SN) above 3D ordering temperatures are calculated as

$$\chi_A(k_x, k_y, k_z) = \frac{\chi^{\text{SDW}}_{\text{eff}}(k_x, \omega)}{1 + J_{\text{eff}}^x(k_x)}$$

where $k = (k_x, k_y, k_z)$ is the wave vector in the $y-z$ plane, $\omega$ is the frequency, and the effective coupling constants $J_{\text{eff}}$ are given by

$$J_{\text{eff}}^x(k_x) = G_{\text{SDW}} \sum_{\alpha = y, z} [J_{\alpha 1} \cos(k_x - \omega - \pi M)$$

$$+ J_{\alpha 3} \sin(k_x + \pi M)],$$

$$J_{\text{eff}}^x(k_x) = G_{\text{SN}} \sum_{\alpha = y, z} [J_{\alpha 1}^2 - (J_{\alpha 2} - J_{\alpha 3})^2] \cos(k_x).$$

The 1D susceptibilities $\chi^{\text{SDW}}_{\text{eff}}(k_x, \omega) = \frac{1}{2} \sum_{\alpha = y, z} e^{-i k_x \alpha} \int_0^\beta d\tau e^{i \omega \tau} (O_{\text{A}}(j, \tau) O_{\text{A}}^\dagger(0, 0))_{\omega = \omega + i \epsilon}$ are analytically computed by using the field theoretical technique ($\beta = 1/T$ and $\epsilon \rightarrow +0$) [27]. Those for SDW and spin-nematic operators respectively take the maximum at $k^\text{max}_{x} = \frac{1}{4} (\epsilon - M) \pi$ and $\chi^{\text{SDW}}_{\text{eff}}(k_x, 0) = \frac{2}{\pi} (\frac{\pi}{\sqrt{2}}) K_2^{-1/2} \sin(\frac{\pi}{4} B K_2^{-1} - \frac{K_2}{4})^2$ and $\chi^{\text{SN}}_{\text{eff}}(\pi, 0) = \frac{2}{\pi} (\frac{\pi}{\sqrt{2}}) K_2^{-1/2} \sin(\frac{\pi}{4} B K_2^{-1} - \frac{K_2}{4})^2$, where $B(x, y)$ is the beta function.

The transition temperature of each order is obtained from the divergent point of its susceptibility at $\omega \rightarrow 0$, which is given by

$$1 + \text{Min}_k |J_{\text{eff}}^x(k)| \chi^{\text{SDW}}_{\text{eff}}(k^\text{max}, 0) = 0.$$
This is because the effective couplings respectively generated from the first- and second-order ordered phase, we find the commensurate ordering vector compared to the crossover line (\(T/2J\)) for \(J_{1-2}\) chains in the \(M-T\) plane, which are derived from the ICMF approach. The temperatures \(T_{\text{SDW(SN)}}\) denote the 3D SDW (nematic) transition points. The vertical dashed lines denote the crossover lines between nematic dominant and SDW dominant TL liquids in the 1D \(J_1-J_2\) chain.

framework becomes less reliable and we need to consider subleading terms in \(\mathcal{H}_{\text{SD}}\).

From Eqs. (2) and (4), we find that the ordering wave numbers \(k_{y,z}\) tend to be a commensurate value \(k_{y,z} = 0\) or \(\pi\) (see also Ref. [24]). Thus the SDW ordered phase has the wave vector \(k_x = (\frac{1}{2} - M)\pi\) and \(k_{y,z} = 0\) or \(\pi\). This agrees with the experimental result in the intermediate-field phase of LiCuVO\(_4\) [13, 14]. For the spin-nematic ordered phase, we find the commensurate ordering vector \((k_x, k_{y,z}) = (\pi, 0)\) for \(|J_{y_1(z_1)}| > |J_{y_2(z_2)} - J_{y_3(z_3)}|\) and \((k_x, k_{y,z}) = (\pi, \pi)\) for \(|J_{y_1(z_1)}| < |J_{y_2(z_2)} - J_{y_3(z_3)}|\). This commensurate nature of \(k_{x,y,z}\) in the nematic phase is consistent with Ref. [8].

We show typical examples of obtained phase diagrams in Fig. 3. When interchain couplings are not frustrated as the \(J_{y_1,z}\) dominant cases of Figs. 3(a) and 3(b), the SDW ordered phase is largely enhanced and the nematic ordered phase is reduced to a higher-field regime compared to the crossover line \((K_+ = 2)\) in the \(J_1-J_2\) chain. This is because the effective couplings \(J_{\text{eff}}^{\text{SDW}}\) and \(J_{\text{eff}}^{\text{SN}}\) are respectively generated from the first- and second-order cumulants, and therefore \(J_{\text{eff}}^{\text{SDW}}\) is generally larger than \(J_{\text{eff}}^{\text{SN}}\) in non-frustrated systems with weak interchain couplings. When both the couplings \(J_{y_1,y_2}\) are dominant, we find a similar tendency. We note that a model with dominant \(J_{y_1,y_2}\) has been proposed for LiCuVO\(_4\) [11], where a new phase expected to be a 3D nematic phase has been observed only near the saturation [12]. From the calculations for the cases of \(|J_1|/|J_2| = 0.5, 1.0,\) and 2.0, we find that the nematic phase region in the \(M-T\) phase diagram generally becomes smaller with increase in \(|J_1|/|J_2|\) since the value \(g(x)\) in \(\mathcal{G}_{\text{SN}}\) decreases. When there is a certain frustration in interchain couplings, however, the nematic phase region can expand, as shown in Fig. 3(c). When the signs of \(J_{y_1}\) and \(J_{y_2}\) are opposite, \(J_{\text{eff}}^{\text{SDW}}\) becomes small, and the 3D nematic phase expands down to a relatively lower-field regime. We emphasize that our theory succeeds in quantitatively analyzing the competition between SDW and nematic ordered phases in quasi-1D magnets.

Effects of four-spin term.— Finally, we study effects of an interchain four-spin interaction. The Hamiltonian we consider is

\[
\mathcal{H}_4 = -J_4 \sum_{j(r,r')} S^+_{j,r} S^+_{j+1,r} S^-_{j-r} S^-_{j+1,r} + \text{H.c.} \tag{10}
\]

This interaction is a part of the spin-phonon coupling \(\mathcal{H}_{\text{sp}} = -J_{\text{sp}} \sum_{j(r,r')} (S_{j,r} \cdot S_{j+1,r})(S_{j+1,r} \cdot S_{j+1,r'})\) and therefore it really exists in some compounds. One easily finds that Eq. (10) enhances the spin-nematic ordering. Applying the field theoretical strategy to the system \(\mathcal{H}_{\text{3D}} + \mathcal{H}_4\), we find that \(J_{\text{eff}}^{\text{SN}}\) is replaced with \(J_{\text{eff}}^{\text{SN}} - 4J_4 C_0 \cos k_y + \cos k_z\). We thus obtain the phase diagram for \(\mathcal{H}_{\text{3D}} + \mathcal{H}_4\), as shown in Fig. 4. Comparing Figs. 3(a) and 4 we see that an inter-chain four-spin interaction definitely enhances the 3D nematic phase even if its coupling constant \(J_4\) is small. Since \(J_4\) is usually positive, it favors ferrotopy nematic ordering along the \(y\) and \(z\) axes; i.e., \(k_{y,z} = 0\).

Conclusion.— We have constructed finite-temperature phase diagrams for 3D spatially anisotropic magnets, which consist of weakly coupled spin-\(\frac{1}{2}\) \(J_1-J_2\) chains, in an applied magnetic field. Incommensurate SDW and

![FIG. 3: (color online) Phase diagrams of the weakly coupled \(J_1-J_2\) chains in the \(M-T\) plane, which are derived from the ICMF approach. The temperatures \(T_{\text{SDW(SN)}}\) denote the 3D SDW (nematic) transition points. The vertical dashed lines denote the crossover lines between nematic dominant and SDW dominant TL liquids in the 1D \(J_1-J_2\) chain.](image)

![FIG. 4: (color online) Phase diagram of the weakly coupled \(J_1-J_2\) spin chains with a four-spin interaction \(\mathcal{H}_4\).](image)
spin-nematic ordered phases appear at sufficiently low temperatures, triggered by the nematic TL-liquid properties in the $J_1$-$J_2$ spin chains. We reveal several nature
s of orderings in the coupled $J_1$-$J_2$ chains: The 3D nematic ordered phase is generally smaller than the 1D nematic dominant region, while it can be larger if we somewhat tune the inter-chain couplings. The ordering wave numbers $k_{y,z}$ tend to be 0 or $\pi$, and a small four-spin interaction $\mathcal{H}_4$ efficiently helps the 3D nematic ordering. We finally note that our theory can also be applied to AF-$J_1$ systems.

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[1] N. Shannon, T. Momoi, and P. Sindzingre, Phys. Rev. Lett. 96, 027213 (2006).
[2] T. Vekua, A. Honecker, H.-J. Mikeska, and F. Heidrich-Meisner, Phys. Rev. B 76, 174420 (2007).
[3] T. Hikihara, L. Kecke, T. Momoi, and A. Furusaki, Phys. Rev. B 78, 144404 (2008).
[4] J. Sudan, A. Luscher, and A.M. Läuchli, Phys. Rev. B 80, 140402(R) (2009).
[5] M. Sato, T. Momoi, and A. Furusaki, Phys. Rev. B 79, 060406(R) (2009).
[6] M. Sato, T. Hikihara, and T. Momoi, Phys. Rev. B 83, 064405 (2011).
[7] See, for example, K. Penc, and A. M. Läuchli, in Introduction to Frustrated Magnetism, edited by C. Lacroix, P. Mendels, and F. Mila (Springer-Verlag, Berlin, 2011), p.331.
[8] R. Shindou and T. Momoi, Phys. Rev. B 80, 064410 (2009).
[9] M. E. Zhitomirsky and H. Tsunetsugu, Europhys. Lett. 92, 37001 (2010).
[10] S. Nishimoto, S.-L. Drechsler, R. O. Kuzian, J. van den Brink, J. Richter, W. E. A. Lorenz, Y. Skourski, R. Klingeler, and B. Büchner, Phys. Rev. Lett. 107, 097201 (2011).
[11] M. Enderle, C. Mukherjee, B. Fäk, R. K. Kremer, J.-M. Broto, H. Rosner, S.-L. Drechsler, J. Richter, J. Malek, A. Prokofiev, W. Assmus, S. Pujol, J.-L. Raggazzoni, H. Rakoto, M. Rheinstadter, and H. M. Ronnow, Europhys. Lett. 70, 237 (2005).
[12] L. E. Svistov, T. Fujita, H. Yamaguchi, S. Kimura, K. Omura, A. Prokofiev, A. I. Smirnov, Z. Honda, and M. Hagiwara, J. Exp. Theor. Phys. Lett. 93, 24 (2011).
[13] T. Masuda, M. Hagiwara, Y. Kondoh, K. Kaneko, and N. Metoki, J. Phys. Soc. Jpn. 80, 113705 (2011).
[14] M. Mourigal, M. Enderle, B. Fäk, R. K. Kremer, J. M. Law, A. Schneidewind, A. Hiess, and A. Prokofiev, Phys. Rev. Lett. 109, 027203 (2012).
[15] N. Bütten, P. Kuhns, A. Prokofiev, A. P. Reyes, and L. E. Svistov, Phys. Rev. B 85, 214421 (2012).
[16] K. Nawa, K. Yoshimura, M. Yoshida, and M. Takigawa, private communication.
[17] M. Hase, H. Kuroe, K. Otsawa, O. Suzuki, H. Kitazawa, G. Kido, and T. Sekine, Phys. Rev. B 70, 104426 (2004).
[18] Y. Yasui, M. Sato, and I. Terasaki, J. Phys. Soc. Jpn. 80, 033707 (2011).
[19] A.U.B. Wolter, F. Lipps, M. Schäpers, S.-L. Drechsler, S. Nishimoto, R. Vogel, V. Katal, B. Büchner, H. Rosner, M. Schmitt, M. Uhlarz, Y. Skourski, J. Wosnitza, S. Süllow, and K. C. Rule, Phys. Rev. B 85, 014407 (2012).
[20] S.E. Dutton, M. Kumar, M. Mourigal, Z.G. Soos, J.-J. Wen, C.L. Broholm, N.H. Andersen, Q. Huang, M. Zbiri, R. Toft-Petersen, and R.J. Cava, Phys. Rev. Lett. 108, 187206 (2012).
[21] T. Masuda, A. Zheludev, B. Roessli, A. Bush, M. Markina, and A. Vasiliev, Phys. Rev. B 72, 014405 (2005).
[22] One can easily take account of other different inter-chain couplings using our theoretical framework.
[23] T. Hikihara and A. Furusaki, Phys. Rev. B 69, 064427 (2004).
[24] Correctly speaking, $G_{SDW}$ can take both positive and negative values $\pm A_1|\langle e^{-\sqrt{4}\pi\phi_-}\rangle_0|^2$ due to degenerate pinning positions $\sqrt{4}\pi\phi_- + \pi M = (2n + \frac{1}{2})\pi$ (n: integer) in each $J_1$-$J_2$ chain. This ambiguity is lifted by subleading inter-chain terms omitted in $\mathcal{H}_{\text{eff}}$. We note that the 3D SDW ordering temperature does not depend on the sign of $G_{SDW}$ in the weak-coupling regime $|J_{y,z}| \ll |J_1,2|$.
[25] D. J. Scalapino, Y. Inui, and P. Pincus, Phys. Rev. B 11, 2042 (1975).
[26] M. Bocquet, F. H. L. Essler, A. M. Tsvelik, and A. O. Gogolin, Phys. Rev. B 64, 094425 (2001).
[27] See, for example, T. Giamarchi, Quantum Physics in One Dimension (Oxford University Press, New York, 2004).