POISSON–JENSEN FORMULAS AND BALAYAGE OF MEASURES

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Abstract. Our main results are certain developments of the classical Poisson–Jensen formula for subharmonic functions. The basis of the classical Poisson–Jensen formula is the natural duality between harmonic measures and Green's functions. Our generalizations use some duality between the balayage of measures and and their potentials.

subharmonic function, balayage, potential, Riesz measure, Jensen measure

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1. Introduction

1.1. On the classical Poisson-Jensen formula. Let $D$ be a bounded domain in the $d$-dimensional Euclidean space $\mathbb{R}^d$ with the closure $\text{clos} D$ in $\mathbb{R}^d$ and the boundary $\partial D$ in $\mathbb{R}^d$. Then, for any $x \in D$ there are the extended harmonic measure $\omega_D(x, \cdot)$ for $D$ at $x \in D$ as a Borel probability measure on $\mathbb{R}^d$ with support on $\partial D$ and the generalized Green's function $g_D(\cdot, x)$ for $D$ with pole at $x$ extended by zero values on the complement $\mathbb{R}^d \setminus \text{clos} D$ and by the upper semicontinuous regularization on $\partial D$ from $D$ [22], [2], [50], [23], [14], [46] (see also (2.7) and (2.9) in Subsec. 2.3 below).
Let $u \not\equiv -\infty$ be a subharmonic function on $\text{clos} \, D$, i.e., on an open set containing $\text{clos} \, D$, with its Riesz measure $\Delta_u$ on this open set (see in detail §§ 1.2.2–1.2.3 and (1.6)).

**Classical Poisson–Jensen formula** ([22, Theorem 5.27], [50, 4.5]).

(1.1) $u(x) = \int_{\partial D} u \, d \omega_D(x, \cdot) - \int_{\text{clos} \, D} g_D(\cdot, x) \, d \Delta_u$ for each $x \in D$.

For $s \in \mathbb{R}$, we set

(1.2k) $k_s(t) := \begin{cases} 
\ln t & \text{if } s = 0, \\
-s \text{sgn}(s) t^{-s} & \text{if } s \in \mathbb{R} \setminus 0,
\end{cases} \quad t \in \mathbb{R}^+ \setminus 0,$

(1.2K) $K_{d-2}(y, x) := \begin{cases} 
k_{d-2}(|y - x|) & \text{if } y \neq x, \\
-\infty & \text{if } y = x \text{ and } d \geq 2, \\
0 & \text{if } y = x \text{ and } d = 1,
\end{cases} \quad (y, x) \in \mathbb{R}^d \times \mathbb{R}^d.$

The following functions

(1.3) $p : y \mapsto -g_D(y, x) + K_{d-2}(y, x), \quad q : y \mapsto -K_{d-2}(y, x)$

are subharmonic with Riesz probability measures $\Delta_p = \omega_D(x, \cdot)$ and $\Delta_q = \delta_x$, where $\delta_x$ is the Dirac measure at $x \in D$: $\delta_x(\{x\}) = 1$. The following symmetric equivalent form of the classical Poisson–Jensen formula (1.1) immediately follows from the suitable definitions of harmonic measures and Green’s functions and is briefly discussed in Subsec. 2.3.

**Symmetrization of the classical Poisson–Jensen formula.** If we choose $p, q$ as in (1.3) and put $S = \text{clos} \, D$, then (1.1) can be rewritten in the symmetric form

(1.4) $\int_S u \, d \Delta_q + \int_S p \, d \Delta_u = \int_S u \, d \Delta_p + \int_S q \, d \Delta_u.$

Equality (1.4) reflects the fact that the Laplace operator $\triangle$ is self-adjoint for some formal bilinear integral form $(u, \Delta w) := \int u \, \Delta w = \int \Delta u \, w = (\Delta u, w)$, where $w := q - p$.

The following result is a special case of our Main Theorem from Subsec. 2.2, but already significantly develops the classical Poisson–Jensen formula (1.3)–(1.4).

**Theorem 1.** Let $S \subset \mathbb{R}^d$ be a non-empty compact set, and $p \not\equiv -\infty$ and $q \not\equiv -\infty$ be a pair of subharmonic functions on $S$ with Riesz measures $\Delta_p$ and $\Delta_q$, respectively. If $p$ and $q$ are harmonic outside $S$ and $p = q$ outside $S$, then the symmetric Poisson–Jensen formula (1.4) holds for each subharmonic function $u \not\equiv -\infty$ on $S$ with Riesz measure $\Delta_u$.

Our Main Lemma is formulated in Subsec. 2.1 and gives a symmetric Poisson–Jensen formula for measures and their potentials. The Main Lemma is proved in Sec. 3. The proof of the Main Lemma use Theorem 2 on representations for pairs of subharmonic functions. Theorem 2 from Subsec. 3.1 is also of independent interest.
The Main Theorem is formulated in Subsec. 2.2 and gives a full symmetric Poisson–Jensen formula for subharmonic integrands. The proof of the Main Theorem in Sec. 4 essentially uses the Main Lemma. Theorem 1 is deduced from the Main Theorem in Subsec. 2.2. The next Subsec. 2.3 contains a discussion of the classical symmetric Poisson–Jensen formula (1.3)–(1.4) as a consequence of Theorem 1.

Our Duality Theorems 1–3 in Sec. 5 give a complete description of potentials of measures obtained as a certain process of balayage of measures with compact support. In order to prove Duality Theorems 1–3, we use both the Main Lemma and the Main Theorem.

We proceed to precise and detailed definitions and formulations.

1.2. Basic notation, definitions, and conventions. The reader can skip this Subsec. 1.2 and return to it only if necessary.

1.2.1. Sets, topology, order. We denote by \( \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{R} \), and \( \mathbb{R}^+ := \{ x \in \mathbb{R} : x \geq 0 \} \) the sets of natural, of real, and of positive numbers, each endowed with its natural order \((\leq, \sup/\inf)\), algebraic, geometric and topological structure. We denote singleton sets by a symbol without curly brackets. So, \( \mathbb{N}_0 := \{0\} \cup \mathbb{N} =: 0 \cup \mathbb{N} \), and \( \mathbb{R}^+ \setminus 0 := \mathbb{R}^+ \setminus \{0\} \) is the set of strictly positive numbers, etc. The extended real line \( \mathbb{R} := (−\infty, \infty] \) is the order completion of \( \mathbb{R} \) by the disjoint union \( \sqcup \) with \( +\infty := \sup \mathbb{R} \) and \( -\infty := \inf \mathbb{R} \) equipped with the order topology with two ends \( \pm\infty \), \( \mathbb{R}^+ := \mathbb{R}^+ \sqcup +\infty \), \( \inf \emptyset := +\infty \), \( \sup \emptyset := -\infty \) for the empty set \( \emptyset \) etc. The same symbol 0 is also used, depending on the context, to denote zero vector, zero function, zero measure, etc.

We denote by \( \mathbb{R}^d \) the Euclidean space of \( d \in \mathbb{N} \) dimensions with the Euclidean norm \( |x| := \sqrt{x_1^2 + \cdots + x_d^2} \) of \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), and by \( \mathbb{R}^d_\infty := \mathbb{R}^d \sqcup \infty \) we denote the Alexandroff (Alekandrov) one-point compactification of \( \mathbb{R}^d \) obtained by adding one extra point \( \infty \). For a subset \( S \subset \mathbb{R}^d_\infty \) or a subset \( S \subset \mathbb{R}^d \) we let \( \mathbb{C}S := \mathbb{R}^d_\infty \setminus S \), \( \text{clos}S \), \( \text{int}S := \text{clos}(\text{clos}S) \), and \( \partial S := \text{clos}S \setminus \text{int}S \) denote its complement, closure, interior, and boundary always in \( \mathbb{R}^d_\infty \), and \( S \) is equipped with the topology induced from \( \mathbb{R}^d_\infty \). If \( S' \) is a relative compact subset in \( S \), i.e., \( \text{clos}S' \subset S \), then we write \( S' \subset S \). We denote by \( B(x, t) := \{ y \in \mathbb{R}^d : |y - x| < t \} \), \( \overline{B}(x, t) := \{ y \in \mathbb{R}^d : |y - x| \leq t \} \), \( \partial \overline{B}(x, t) := \overline{B}(x, t) \setminus B(x, t) \) an open ball, closed ball, a circle of radius \( t \in \mathbb{R}^+ \) centered at \( x \in \mathbb{R}^d \), respectively.

Let \( T \) be a topological space, and \( S \) be a subset in \( T \). We denote by Conn\( T \) or Conn\( T(S) \) the set of all connected components of \( S \subset T \) in \( T \).

Throughout this paper \( O \neq \emptyset \) will denote an open subset in \( \mathbb{R}^d \), and \( D \neq \emptyset \) is a domain in \( \mathbb{R}^d \), i.e., an open connected subset in \( \mathbb{R}^d \).

1.2.2. Measures and charges. The convex cone over \( \mathbb{R}^+ \) of all Borel, or Radon, positive measures \( \mu \geq 0 \) on the \( \sigma \)-algebra Bor(S) of all Borel subsets of \( S \) is denoted by \( \text{Meas}^+(S) \); \( \text{Meas}_{\text{cmp}}^+(S) \subset \text{Meas}^+(S) \) is the subcone of \( \mu \in \text{Meas}^+(S) \) with compact support \( \text{supp} \mu \) in \( S \), \( \text{Meas}(S) := \text{Meas}^+(S) - \text{Meas}^+(S) \) is the vector lattice
over $\mathbb{R}$ of charges, or signed measures, on $S$, $\text{Meas}^+(S)$ is the convex set of probability measures on $S$, $\text{Meas}_{\text{cmp}}^+(S) := \text{Meas}^+(S) \cap \text{Meas}_{\text{cmp}}(S)$, and $\text{Meas}_{\text{cmp}}(S) := \text{Meas}_{\text{cmp}}^+(S) - \text{Meas}_{\text{cmp}}^-(S)$. For a charge $µ ∈ \text{Meas}(S)$, we let $µ^+ := \text{sup}\{0, µ\}$, $µ^- := (-µ)^+$ and $µ := µ^+ + µ^-$ respectively denote its upper, lower, and total variations, and $µ(x, t) := µ(\overline{B}(x, t))$.

For an extended numerical function $f: S → \mathbb{R}$ we allow values $±\infty$ for Lebesgue integrals [22, Ch. 3, Definition 3.3.2] (see also [9])

$$\int_S f \, dµ ∈ \mathbb{R}, \quad µ ∈ \text{Meas}^+(S),$$

and we say that $f$ is $µ$-integrable on $S$ if the integral in (1.5) is finite.

1.2.3. Subharmonic functions. We denote by $\text{sbh}(O)$ the convex cone over $\mathbb{R}^+$ of all subharmonic (locally convex if $d = 1$) functions on $O$, including functions that are identically equal to $-\infty$ on some components $C ∈ \text{Conn}_{\mathbb{R}^d}(O)$. Thus, $\text{har}(O) := \text{sbh}(O) \cap (- \text{sbh}(O))$ is the vector space over $\mathbb{R}$ of all harmonic (locally affine if $d = 1$) functions on $O$. Each function $u ∈ \text{sbh}_*(O) := \{u ∈ \text{sbh}(O): u \neq -\infty \text{ on each } C ∈ \text{Conn}_{\mathbb{R}^d}(O)\}$ is associated with its Riesz measure

$$(1.6) \quad \Delta_u := c_d \Delta u ∈ \text{Meas}^+(O), \quad c_d := \frac{Γ(d/2)}{2\pi^{d/2}\max\{1, d - 2\}},$$

where $\Delta$ is the Laplace operator acting in the sense of the theory of distribution or generalized functions, and $Γ$ is the gamma function. If $u ≡ -\infty$ on $C ∈ \text{Conn}_{\mathbb{R}^d}(O)$, then we set $\Delta_{-\infty}(S) := +\infty$ for each $S ⊂ C$. Given $S ⊂ \mathbb{R}^d$, we set

$$\text{Sbh}(S) := ∪\{\text{sbh}(O'): S ⊂ O' \overset{\text{open}}{=} \text{int } O' ⊂ \mathbb{R}^d\},$$

$$\text{Sbh}_*(S) := ∪\{\text{sbh}_*(O'): S ⊂ O' \overset{\text{open}}{=} \text{int } O' ⊂ \mathbb{R}^d\},$$

$$\text{Har}_*(S) := ∪\{\text{har}(O'): S ⊂ O' \overset{\text{open}}{=} \text{int } O' ⊂ \mathbb{R}^d\}.$$

Consider a binary relation $≃ ⊂ \text{Sbh}(\mathbb{R}^d) × \text{Sbh}(\mathbb{R}^d)$ on $\text{Sbh}(\mathbb{R}^d)$ defined by the rule: $U ≃ V$ if there is an open set $O' ⊃ S$ in $\mathbb{R}^d$ such that $U ∈ \text{sbh}(O')$, $V ∈ \text{sbh}(O')$, and $U(x) = V(x)$ for each $x ∈ O'$. This relation $≃$ is an equivalence relation on $\text{Sbh}(\mathbb{R}^d)$, $\text{Sbh}_*(\mathbb{R}^d)$, and on $\text{Har}(\mathbb{R}^d)$. The quotient sets of $\text{Sbh}(\mathbb{R}^d)$, of $\text{Sbh}_*(\mathbb{R}^d)$, and of $\text{Har}(\mathbb{R}^d)$ by $≃$ are denoted below by $\text{sbh}(S) := \text{Sbh}(\mathbb{R}^d)/≃$, $\text{sbh}_*(S) := \text{Sbh}_*(\mathbb{R}^d)/≃$, and $\text{har}(S) := \text{Har}(\mathbb{R}^d)/≃$, respectively. The equivalence class $[u]$ of $u$ is denoted without square brackets as simply $u$, and we do not distinguish between the equivalence class $[u]$ and the function $u$ when possible. So, for $u, v ∈ \text{sbh}(\mathbb{R}^d)$, we write “$u = v \text{ on } S$” if $[u] = [v]$ in $\text{sbh}(S) := \text{Sbh}(\mathbb{R}^d)/≃$, or, equivalently, $u ≃ v$ on $\text{Sbh}(\mathbb{R}^d)$, and we write $u ≃ -\infty$ if $u ∈ \text{sbh}_*(S)$. The concept of the Riesz measure $\Delta_u$ of $u ∈ \text{sbh}(\mathbb{R}^d)$ is correctly and uniquely defined by the restriction $\Delta_u \big|_S$ of the Riesz measure $\Delta_u$ to $S$. For $u ∈ \text{sbh}(S)$ and $v ∈ \text{sbh}(S)$, the concepts “$u ≤ v \text{ on } S$”, and “$u = v \text{ outside } S$”, “$u ≤ v \text{ outside } S$”.


“u is harmonic outside S”, etc. defined naturally: $u(x) \leq v(x)$ for each $x \in S$, and there exists an open set $O' \supset S$ such that $u(x) = v(x)$ for each $x \in O' \setminus S$, $u(x) \leq v(x)$ for each $x \in O' \setminus S$, the restriction $u \big|_{O' \setminus S}$ is harmonic on $O' \setminus S$, respectively. So, Theorem 1 from Introduction is formulated precisely in this interpretation.

1.2.4. Balayage. Let $S \in \text{Bor}(\mathbb{R}^d)$ and $H$ be a set of upper semicontinuous functions $f: S \to \mathbb{R} \setminus +\infty$. A measure $\omega \in \text{Meas}_{\text{cmp}}^+(S)$ is called the balayage of a measure $\Delta \in \text{Meas}_{\text{cmp}}^+(S)$ for $S$ with respect to $H$ [49], [8], [41, Definition 5.2], or, briefly, $\omega$ is a $H$-balayage of $\Delta$, and we write $\Delta \preceq_H \omega$ or $\omega \succeq_H \Delta$ if

$$
\int_{S} h \, d\Delta \leq \int_{S} h \, d\omega \quad \text{for each } h \in H \text{ in accordance with } (1.5).
$$

If $\Delta \preceq_H \omega$ and $\omega \preceq_H \Delta$, then we write $\Delta \simeq_H \omega$. The following properties are obvious:

1. The binary relation $\preceq_H$ (respectively $\simeq_H$) on $\text{Meas}_{\text{cmp}}^+(S)$ is a preorder, i.e., a reflexive and transitive relation, (respectively, an equivalence) on $\text{Meas}_{\text{cmp}}^+(S)$.

2. If $H$ contains a strictly positive (respectively, negative) constant, then $\Delta(S) \preceq \omega(S)$ ($\Delta(S) \succeq \omega(S)$, respectively).

3. If $H = -H$, then the order $\preceq_H$ is the equivalence $\simeq_H$. So, if $H = \text{har}(S)$, then $\omega$ is a har$(S)$-balayage of $\Delta$ if and only if $\Delta \simeq_{\text{har}(S)} \omega$, i.e.,

$$
\int_{S} h \, d\Delta = \int_{S} h \, d\omega \quad \text{for each } h \in \text{har}(S) \quad \text{and} \quad \Delta(S) = \omega(S).
$$

4. If $\Delta \preceq_{\text{sbh}(S)} \omega$, then $\Delta \preceq_{\text{har}(S)} \omega$. The converse is not true [43, XIB2], [48, Example].

5. If $\omega \in \text{Meas}_{\text{cmp}}^+(O)$ is a $(\text{sbh}(O) \cap C^\infty(O))$-balayage of $\Delta \in \text{Meas}_{\text{cmp}}^+(O)$, where $C^\infty(O)$ is the class of all infinitely differentiable functions on $O$, then $\Delta \preceq_{\text{sbh}(O)} \omega$, since for each function $u \in \text{sbh}(O)$ there exists a sequence of functions $u_j \in \text{sbh}(O)$ decreasing to it [14, Ch. 4, 10, Approximation Theorem].

Remark 1. Balayage of charges and measures with a non-compact support is also occur frequently and are used in Analysis. So, a bounded domain $D \subset \mathbb{R}^d$ is called a quadrature domain (for harmonic functions) if there is a charge $\Delta \in \text{Meas}_{\text{cmp}}(D)$ such that the restriction of the Lebesgue measure $\lambda$ to $D$ is a balayage of $\Delta$ with respect to the class of all harmonic $\lambda$-integrable functions on $D$. In connection with the quadrature domains, see very informative overview [19, 3] and bibliography in it.

1.2.5. Potentials. For a charge $\mu \in \text{Meas}_{\text{cmp}}(O)$ its potential

$$
\text{pt}_\mu: \mathbb{R}^d_\infty \to \mathbb{R}, \quad \text{pt}_\mu(y) := \int_{O} K_{d-2}(x,y) \, d\mu(x),
$$

is uniquely determined on $[3]$, [40, 3.1]

$$
\text{Dom} \text{pt}_\mu := \left\{ y \in \mathbb{R}^d : \inf \left\{ \int_{0}^{1} \frac{\mu^-(y,t)}{t^{d-1}} \, dt, \int_{0}^{1} \frac{\mu^+(y,t)}{t^{d-1}} \, dt \right\} < +\infty \right\}
$$
by values in $\mathbb{R}$, and the set $E := (\bar{\text{Dom} \, pt_\mu}) \setminus \infty$ is polar with zero outer capacity

$$\text{Cap}^*(E) := \inf_{E \subset O^\text{open}} \sup_{\nu \in \text{Meas}^+_c(O)} \left( \int \int K_{d-2}(x, y) \, d\nu(x) \, d\nu(y) \right).$$

Evidently $pt_\mu \in \text{har}(\mathbb{R}^d \setminus \text{supp} |\mu|)$, and if $\mu \in \text{Meas}^+_c(\mathbb{R}^d)$, then $pt_\mu \in \text{sbh}_s(\mathbb{R}^d)$.

1.2.6. Inward filling of sets with respect to an open set. Let $O$ be an open set in $\mathbb{R}^d$. The union of $S \subset O$ with all components $C \subset \text{Conn}_O(O \setminus S)$ such that $C \subset O$ will be called the inward filling of $S$ with respect to $O$, and we denote this union by in-fill$_O S$ or in-fill$_O(S)$, although in [16, 1.7], [6], [17, Sec. 12], [35, § 1] another notation $\tilde{S}$ was used. Denote by $O_\infty$ the Alexandroff one-point compactification of $O$ with underlying set $O \cup \infty$, where extra point $\infty \notin O$ can be identified with the boundary $\partial O$ or the complement $\mathbb{C} O$, considered as a single point. The following elementary properties of the inward filling will often be used without mentioning them.

Proposition 1 ([16, 6.3], [17], [6], [18]). Let $S$ be a compact set in an open set $O \subset \mathbb{R}^d$. Then

1. in-fill$_O S$ is a compact subset in $O$, and in-fill$_O(\text{in-fill}_O S) = \text{in-fill}_O S$;
2. the set $O_\infty \setminus \text{in-fill}_O S$ is connected and locally connected subset in $O_\infty$;
3. the inward filling of $S$ with respect to $O$ coincides with the complement in $O_\infty$ of connected component of $O_\infty \setminus S$ containing the point $\infty$, i.e., in-fill$_O S = O_\infty \setminus C_\infty$, where $\infty \in C_\infty \subset \text{Conn}_O(O_\infty \setminus S)$;
4. if $O' \subset \mathbb{R}^d$ is an open subset and $O \subset O'$, then in-fill$_O S \subset \text{in-fill}_{O'} S$;
5. if $S' \subset S$ is a compact subset in $O$, then in-fill$_O S' \subset \text{in-fill}_O S$;
6. $\mathbb{R}^d \setminus \text{in-fill}_O S$ has only finitely many components, i.e., $\# \text{Conn}_{\mathbb{R}^d}(\mathbb{R}^d \setminus \text{in-fill}_O S) < \infty$.

2. Poisson–Jensen formulas

2.1. Main result for measures and their potentials.

Main Lemma. Let $\Delta \in \text{Meas}^+_c(O)$, $\omega \in \text{Meas}^+_c(O)$, and

$$(2.1) \quad S_O := \text{in-fill}_O(\text{supp} \Delta \cup \text{supp} \omega).$$

The following seven statements are equivalent:

1. $\Delta \preceq_{\text{har}(O)} \omega$.
2. $\Delta \preceq_{\text{har}(S_O)} \omega$.
3. $pt_\Delta = pt_\omega$ on $\mathbb{R}^d \setminus S_O$.
4. There are a compact subset $S$ in $O$, a function $q \in \text{sbh}_s(S)$ with Riesz measure $\Delta_q = \Delta$, and a function $p \in \text{sbh}_s(S)$ with Riesz measure $\Delta_p = \omega$ such that $q$ and $p$ are harmonic outside $S$, and $q = p$ outside $S$. 

V. The symmetric Poisson–Jensen formula for measures and their potentials is valid:

\[(2.2f) \quad \int u \, d\Delta + \int_B p \, d\Delta_u = \int u \, d\omega + \int_B p \, d\Delta_u\]

\[(2.2B) \quad \text{for each } B \in \text{Bor}(\mathbb{R}^d) \text{ such that } S_O \subset B \subset O \text{ and for each } u \in \text{sbh}_*(\text{clos } B)\].

VI. For each \( q \in \text{sbh}_*(S_O) \) with \( \Delta_q = \Delta \) there is \( p \in \text{sbh}_*(S_O) \) with \( \Delta_p = \omega \) such that

\[(2.3) \quad \int u \, d\Delta + \int_{S_O} p \, d\Delta_u = \int u \, d\omega + \int_{S_O} q \, d\Delta_u \quad \text{for each } u \in \text{sbh}_*(O)\].

VII. There are a compact subset \( S \subset S_O \) in \( O \) and a pair of functions \( q \in \text{sbh}_*(S_O) \) and \( p \in \text{sbh}_*(S_O) \) with Riesz measures \( \Delta_q = \Delta \) and \( \Delta_p = \omega \), respectively, such that the equality in \( (2.3) \) is fulfilled for each special subharmonic function \( u_x : y \mapsto K_{d-2}(y, x) \) with \( x \in O \setminus S \) instead of all functions \( u \in \text{sbh}_*(O) \) in \( (2.3) \).

The proof of the Main Lemma will be given only after some preparation in Section 3.

2.2. Full version of the Poisson–Jensen formula for subharmonic integrands. The starting point of the Main Lemma is a pair of measures \( \Delta, \omega \in \text{Meas}_\text{cmp}(O) \). Our Main Theorem is a functional counterpart of the Main Lemma. The starting point in it is now a pair of subharmonic functions from \( (2.4s) \).

**Main Theorem.** Let

\[(2.4S) \quad \emptyset \neq S^\text{closed} \subset \text{clos } S^\text{compact} \subset O^\text{open} \equiv \text{int } O \subset \mathbb{R}^d, \quad S_O := \text{in-fill}_O S,\]

\[(2.4s) \quad q \in \text{sbh}_*(O) \cap \text{har}(O \setminus S), \quad p \in \text{sbh}_*(O) \cap \text{har}(O \setminus S),\]

\[(2.4\neq) \quad S_{\neq} := \{ x \in O : q(x) \neq p(x) \}.

The following four statements are equivalent:

[I] \( S_{\neq} \subset S_O \), i.e., \( q = p \) on \( O \setminus S_O \).

[II] There is a function \( h \in \text{har}(O) \) such that

\[(2.5) \quad \begin{cases} q = \text{pt}_{\Delta_q} + h & \text{on } O \text{ and } \text{pt}_{\Delta_q} = \text{pt}_{\Delta_p} \text{ on } \mathbb{R}^d \setminus S_O, \\ p = \text{pt}_{\Delta_p} + h & \end{cases}\]

where \( \Delta_q \in \text{Meas}^+(S) \) and \( \Delta_p \in \text{Meas}^+(S) \) are the Riesz measures of \( q \) and \( p \).

[III] The full symmetric Poisson–Jensen formula is valid:

\[(2.6f) \quad \int_S u \, d\Delta_q + \int_B p \, d\Delta_u = \int_S u \, d\Delta_p + \int_B q \, d\Delta_u \quad \text{for each } B \in \text{Bor}(\mathbb{R}^d)\]

\[(2.6B) \quad \text{under } S_O \cap S_{\neq} \subset B \subset O \text{ and for each } u \in \text{sbh}_*(S_O \cup \text{clos } B)\].

[IV] \( (2.6) \) holds for a sequence of sets \( B_j \subset B \in \text{Bor}(O) \) such that \( B_0 := S_O \subset \bigcup_{j \in \mathbb{N}} B_j = O \) instead of all \( B \in \text{Bor}(\mathbb{R}^d) \) with \( S_O \cap S_{\neq} \subset B \subset O \).
in (2.6) and for each special subharmonic function \( u_x : y \mapsto K_{d-2}(y, x) \) with \( x \in O \backslash B_0 = O \backslash S_O \) instead of all functions \( u \in \text{sbh}_*(S_O \cup \text{clos} \, B) \) in (2.6B).

We can now prove Theorem 1 of the Introduction.

of Theorem 1. There is an open set \( O \subseteq \mathbb{R}^d \) such that \( u \in \text{sbh}_*(O) \), \( q \in \text{sbh}_*(O) \) and \( p \in \text{sbh}_*(O) \) are harmonic on \( O \backslash S \), and also \( q = p \) on \( O \backslash S \). Evidently, in the notation (2.4), we have \( S_O \cap S_\neq \subset S \subseteq S_O \subseteq O \) and \( u \in \text{sbh}_*(S_O) \). Theorem 1 with (1.4) follows from implication [I] \( \Rightarrow \) [III] of the Main Theorem, since we can choose \( B := S \) in (2.6).

2.3. In detail on the classical Poisson–Jensen formula. If \( x \in D \subseteq O \), then the extended harmonic measure \( \omega_D(x, \cdot) \in \text{Meas}^1(\partial D) \subseteq \text{Meas}_{\text{cmp}}^1(\mathbb{R}^d) \) (for \( D \) at \( x \)) defined on sets \( B \in \text{Bor}(\mathbb{R}^d) \) by

\[
\omega_D(x, B) := \sup \left\{ u(x) : u \in \text{sbh}_D \right\}, \quad \limsup_{D \ni y \to \partial D} u(y') \leq \begin{cases} 1 & \text{for } y \in B \cap \partial D \\ 0 & \text{for } y \notin B \cap \partial D \end{cases}
\]

is a \( \text{har}(O) \)-balayage of \( \delta_x \) with obvious equalities

\[
\text{in-fill}(\sup \delta_x \cup \sup \omega_D(x, \cdot)) = \text{in-fill}(x \cup \partial D) = \text{clos} \, D,
\]

the potential (see [46, Ch. 4, §1,2])

\[
\text{pt}_{\omega_D(x, \cdot)}(y) = \text{pt}_{\omega_D(x, \cdot)}(y) - \text{pt}_{\delta_x}(y)
\]

\[
= \int_{\partial D} K_{d-2}(y, x') \, d_{x'} \omega_D(x, x') - K_{d-2}(y, x) = g_D(y, x), \quad y \in \mathbb{R}^d, \quad x \in D,
\]

is equal to the generalized Green’s function \( g_D(\cdot, x) : \mathbb{R}^d_x \to \mathbb{R}^+ \) (for \( D \) with pole at \( x \) and \( g_D(x, x) := +\infty \) defined on \( \mathbb{R}^d \setminus \{x\} \) by upper semicontinuous regularization

\[
g_D(y, x) := \tilde{g}^*(y, x) := \limsup_{x \to x} \tilde{g}(y', x) \in \mathbb{R}^+ \quad \text{for each } y \in \mathbb{R}^d \setminus x, \text{ where}
\]

\[
\tilde{g}(y, x) := \sup \left\{ u(y) : u \in \text{sbh}(\mathbb{R}^d \setminus x), \quad \begin{cases} u(y') \leq 0 & \text{for each } y \notin \text{clos} \, D, \\ \limsup_{x \to x} \frac{u(y)}{K_{d-2}(x, y)} \leq 1 \end{cases} \right\}.
\]

The equalities (2.8) give (1.3) with \( \Delta_p = \omega_D(x, \cdot) \) and \( \Delta_q = \delta_x \). Thus, Theorem 1 implies the symmetric Poisson–Jensen formula (1.4) which can be written in detail as

\[
\int_{\text{clos} \, D} u \, d\delta_x + \int_{\text{clos} \, D} \left( g_D(\cdot, x) + K_{d-2}(\cdot, x) \right) \, d\Delta_u
\]

\[
= \int_{\text{clos} \, D} u \, d\omega_D(x, \cdot) + \int_{\text{clos} \, D} K_{d-2}(\cdot, x) \, d\Delta_u.
\]

The latter coincides with the classical Poisson–Jensen formula (1.1).
2.4. The Poisson–Jensen formula for the Arens–Singer, and Jensen measures and potentials. Our results presented in this Subsec. 2.4 are intermediate between the classical Poisson–Jensen formula (1.1) and the symmetric Poisson–Jensen formula (1.4) of Theorem 1.

If \( x \in O \) and \( \delta_x \leq \text{har}(O) \) \( \omega \in \text{Meas}^+(O) \), then \( \omega \) is called an Arens–Singer measure on \( O \) at \( x \) [15, Ch. 3], [52], [30], [35, Definition 1], [34], [36], [45], or representing measure. We denote by \( AS_x(O) \subseteq \text{Meas}^+(O) \) the class of all Arens–Singer measure on \( O \) at \( x \). If \( \omega \in AS_x(O) \), then the potential [50, 3.1], [35, Definition 2], [40, 3.1, 3.2], [11]

\[
pt_{\omega - \delta_x}(y) \overset{(1.9)}{=} pt_{\omega} - pt_{\delta_x}(y) \overset{(1.2)}{=} pt_{\omega}(y) - K_d-2(y,x), \quad y \in \mathbb{R}_x^d \backslash x,
\]

satisfies conditions [35, § 1] (see also Duality Theorem 3 in Sec. 5 below)

\[
pt_{\omega - \delta_x} \in \text{sbl}(\mathbb{R}_x^d \backslash x), \quad pt_{\omega - \delta_x}(\infty) := 0,
\]

\[(2.11) \quad pt_{\omega - \delta_x} \equiv 0 \quad \text{on } \mathbb{R}_x^d \backslash \text{in-fill}_O(x \cup \text{supp } \omega),
\]

\[
pt_{\omega - \delta_x}(y) \leq -K_d-2(x,y) + O(1) \quad \text{as } x \neq y \to x.
\]

If \( x \in O \) and \( \delta_x \leq \text{sbl}(O) \) \( \omega \), then this measure \( \omega \) is called a Jensen measure on \( O \) at \( x \) [15, 3], [43], [44], [12], [13], [51], [10], [20], [21], [24], [42], [7], [38]. The class of these measures is denoted by \( J_x(O) \), and properties (2.11) are supplemented by the positivity property \( pt_{\omega - \delta_x} \geq 0 \) on \( \mathbb{R}_x^d \backslash x \) for all measures \( \omega \in J_x(O) \subseteq AS_x(O) \). These measures can be considered as generalizations of the extended harmonic measures (2.7).

By the implication \( I \Rightarrow V \) of the Main Lemma with \( \Delta := \delta_x \), we obtain the following our version [35, Proposition 1.2, (1.3)] of the Poisson-Jensen formula for Arens–Singer measures \( \omega \in AS_x(O) \), generalizing the classical Poisson-Jensen formula (1.1).

**Poisson–Jensen formula for Arens–Singer and Jensen measures.** If \( \omega \in AS_x(O) \), then

\[
(2.12) \quad u(x) = \int u \, d\omega - \int pt_{\omega - \delta_x} \, d\Delta_u \quad \text{for each } u \in \text{sbh}_x(O) \text{ with } u(x) \neq -\infty.
\]

If \( \omega \in J_x(O) \) and \( \omega \neq \delta_x \), then the restriction \( u(x) \neq -\infty \) in (2.12) can be removed.

For \( x \in \mathbb{R}_x^d \), a function \( V \in \text{sbh}_x(\mathbb{R}_x^d \backslash x) \) is called an Arens–Singer potential on \( O \) with pole at \( x \in O \) [15, 3], [52], [34], [35], [36, Definition 6], [45, § 4] if there is \( S_V \in O \) such that

\[
(2.13) \quad V \equiv 0 \quad \text{on } \mathbb{C}S_V \quad \text{and } \quad \limsup_{x \neq y \to x} \frac{V(y)}{-K_d-2(x,y)} \leq 1.
\]

The class of all Arens–Singer potentials on \( O \) with pole at \( x \in O \) denote by \( ASP_x(O) \).

A positive Arens–Singer potential is called a Jensen potential on \( O \) with pole at \( x \in O \) [15, 3], [1], [35], [47], [36, Definition 8], [44, IIIIC], [37], [42], [7]. We denote by \( JP_x(O) \) the class of all Jensen potentials on \( O \) with pole at \( x \in O \). These potentials can be considered as generalizations of the Green’s functions (2.9). For \( V \in ASP_x(O) \), we choose (cf. (1.3))

\[
(2.14) \quad p: y \mapsto V(y) + K_d-2(y,x), \quad q: y \mapsto K_d-2(y,x) \quad \text{for } y \in \mathbb{R}_x^d.
\]
Then these subharmonic functions on $\mathbb{R}^d$ are harmonic and coincide outside $\text{clos}\, S_V$ by (2.13), and the implication $[\text{I}] \Rightarrow [\text{III}]$ of the Main Theorem give the equality (cf. (2.10))

$$
\int_O u \, d\delta_x + \int_{S_O} (V(y) + K_{d-2}(\cdot, x)) \, d\Delta_u = \int_O u \, d\Delta_V + \int_{S_O} K_{d-2}(\cdot, x) \, d\Delta_u,
$$

where $S_O = \text{in-fill} \left( x \cup \left( \bigcup \{ y \in O : V \not\in \text{har}(y) \} \right) \right)$. Hence we obtain

**Poisson–Jensen formula for Arens–Singer and Jensen potentials.** If $V \in \text{ASP}_x(O)$, then

$$(2.15) \quad u(x) = \int_O u \, d\Delta_V - \int O \, d\Delta_u \quad \text{for each } u \in \text{sbh}_x(O) \text{ with } u(x) \neq -\infty.
$$

If $V \in \text{JP}_x(O)$ and $V \not\equiv 0$ on $\mathbb{C}x$, then the restriction $u(x) \neq -\infty$ in (2.15) can be removed.

### 3. PROOF OF THE MAIN LEMMA

#### 3.1. Representations for pairs of subharmonic functions.

**Proposition 2.** If $\mu \in \text{Meas}_{\text{cmp}}(\mathbb{R}^d)$, then

$$(3.1h) \quad \text{pt}_\mu \in \text{sbh}(\mathbb{R}^d) \bigcap \text{har}(\mathbb{R}^d \setminus \text{supp} \mu),$$

$$(3.1\infty) \quad \text{pt}_\mu(x) \overset{(1.2k)}{=} \mu(\mathbb{R}^d)k_{d-2}(|x|) + O(1/|x|^{d-1}), \quad x \to \infty.
$$

**Proof.** For $d = 1$, we have

$$|\text{pt}_\mu(x) - \mu(\mathbb{R})| \leq \int |x - y| \, d|\mu|(y) \leq \int |y| \, d|\mu|(y) = O(1), \quad |x| \to +\infty.$$  

See [50, Theorem 3.1.2] for $d = 2$.

For $d > 2$ and $|x| \geq 2 \sup\{|y| : y \in \text{supp} \mu\}$, we have

$$|\text{pt}_\mu(x) - \mu(\mathbb{R}^d)k_{d-2}(|x|)| = \left| \int \left( \frac{1}{|x|^{d-2}} - \frac{1}{|x - y|^{d-2}} \right) \, d\mu(y) \right|$$

$$\leq \int \left| \frac{1}{|x|^{d-2}} - \frac{1}{|x - y|^{d-2}} \right| \, d|\mu|(y) \leq \frac{2^{d-2}}{|x|^{2d-4}} \int \left| |x - y|^{d-2} - |x|^{d-2} \right| \, d|\mu|(y)$$

$$\leq \frac{2^{d-2}}{|x|^{2d-4}} \int |y| |x|^{-d-3} \sum_{k=0}^{d-3} \left( \frac{3}{2} \right)^k \, d|\mu|(y) \leq 2 \frac{3^{d-2}}{|x|^{d-1}} \int |y| \, d|\mu|(y) = O\left( \frac{1}{|x|^{d-1}} \right).$$

$\square$

**Theorem 2.** Let $O \subset \mathbb{R}^d$ be an open set, and let $p \in \text{sbh}_+(O)$ and $q \in \text{sbh}_+(O)$ be pair of functions such that $p$ and $q$ are harmonic outside a compact subset in $O$. If there is a compact set $S \Subset O$ such that $p = q$ on $O \setminus S$, then, for Riesz measures $\Delta_p \in \text{Meas}_{\text{cmp}}(O)$ of $p$ and $\Delta_q \in \text{Meas}_{\text{cmp}}(O)$ of $q$, we have

$$(3.2) \quad \Delta_p(O) = \Delta_q(O), \quad \text{pt}_\Delta = \text{pt}_\Delta \text{ on } \mathbb{R}^d \setminus S,$$
and there is a harmonic function $H$ on $O$ such that

$$
(3.3) \begin{cases}
p = pt_{\Delta_p} + H \\
q = pt_{\Delta_q} + H
\end{cases} \quad \text{on } O, \quad H \in \text{har}(O).
$$

**Proof.** By Weyl's lemma on the Laplace equation, we have

$$
\begin{align*}
\triangle(p - pt_{\Delta_p}) &= \frac{1}{c_d}(\Delta_p - \Delta_p) = 0 \\
\triangle(q - pt_{\Delta_q}) &= \frac{1}{c_d}(\Delta_q - \Delta_q) = 0
\end{align*}
\implies \begin{cases}
h_p := p - pt_{\Delta_p} \in \text{har}(O) \\
h_q := q - pt_{\Delta_q} \in \text{har}(O)
\end{cases}
$$

and obtain representations

$$
(3.4) \begin{cases}
p = pt_{\Delta_p} + h_p \\
q = pt_{\Delta_q} + h_q
\end{cases} \quad \text{on } O \text{ with } h_p \in \text{har}(O) \text{ and } h_q \in \text{har}(O).
$$

Let us first consider separately

**The case** $O := \mathbb{R}^d$ in the notation $P := p$ and $Q := q$. Put

$$
(3.5) \quad h := h_P - h_Q \in \text{har}(\mathbb{R}^d).
$$

By the conditions of Theorem 2 and Proposition 2, we have

$$
(3.6) \quad h(x) = h_P(x) - h_Q(x) = -pt_{\Delta_P}(x) + pt_{\Delta_Q}(x) + (P(x) - Q(x)) \equiv bk_{d-2}(|x|) + O(|x|^{1-d}), \quad |x| \to +\infty,
$$

where $b := \Delta_Q(\mathbb{R}^d) - \Delta_P(\mathbb{R}^d)$.

The case $d > 2$. If $d \geq 3$, then, in view of (3.6), this harmonic function $h$ bounded on $\mathbb{R}^d$. By Liouville's Theorem [5, Ch. 3], $h$ is constant, and $h_P - h_Q = h \equiv 0$ on $\mathbb{R}^d$. In particular, $|b| = |b + |x|^{d-2}h(x)| \equiv O(1/|x|)$ as $x \to \infty$, i.e., $b = 0$. Thus, for $H := h_P = h_Q$, by (3.4), we obtain representations (3.3) together with $pt_{\Delta_P} = pt_{\Delta_Q}$ on $\mathbb{R}^d \setminus S$, as required.

The case $d = 2$. Using (3.6) we obtain $|h(x) - b \log |x|| \equiv O(1/|x|)$ as $x \to \infty$. Hence, this harmonic function $h$ is bounded from below if $b \geq 0$ or bounded from above if $b < 0$. Therefore, by Liouville’s Theorem, $h$ is constant, $b = 0$, i.e., $\Delta_P(\mathbb{R}^2) \equiv \Delta_Q(\mathbb{R}^2)$, and $h \equiv 0$ on $\mathbb{R}^2$. Thus, we obtain (3.3) together with (3.2).

The case $d = 1$. Using (3.6) we obtain $|h(x) - b|x|| \equiv O(1)$ as $x \to \infty$. Hence, this affine function $h$ on $\mathbb{R}$ is bounded from below if $b \geq 0$ or bounded from above if $b < 0$. Therefore, $h$ is constant, $b = 0$, i.e., $\Delta_P(\mathbb{R}) \equiv \Delta_Q(\mathbb{R})$, and $h \equiv C$ on $\mathbb{R}$ for a constant $C \in \mathbb{R}$. Thus,

$$
(3.7) \quad \begin{cases}
P(x) = pt_{\Delta_P}(x) + ax + b + C \\
Q(x) = pt_{\Delta_Q}(x) + ax + b
\end{cases} \quad \text{for } x \in \mathbb{R} \text{ with } h_Q(x) \equiv ax + b.
$$

The definition (1.9) of potentials in the case $d = 1$ immediately implies
Lemma 1. Let $\Delta \in \text{Meas}_{\text{cmp}}^+(\mathbb{R})$, and $s_l := \inf \supp \Delta$, $s_r := \sup \supp \Delta$. Then

$$\text{pt}_\Delta(x) = \begin{cases} \Delta(\mathbb{R})x - \int y \, d\Delta(y) & \text{if } x \geq s_r, \\ -\Delta(\mathbb{R})x + \int y \, d\Delta(y) & \text{if } x \leq s_l. \end{cases}$$

We set

$$\begin{cases} t := \Delta_P(\mathbb{R}) = \Delta_Q(\mathbb{R}) \in \mathbb{R}^+, \\ S_l := \inf(S \cup \supp \Delta_P \cup \supp \Delta_Q) \in \mathbb{R}, \\ S_r := \sup(S \cup \supp \Delta_P \cup \supp \Delta_Q) \geq S_l. \end{cases}$$

In view of $P(x) \equiv Q(x)$ for $x \in \mathbb{R} \setminus S$, by Lemma 1, we have

$$\begin{cases} tx - \int y \, d\Delta_P(y) + ax + b + C = tx - \int y \, d\Delta_Q(y) + ax + b & \text{if } x \geq S_r, \\ -tx + \int y \, d\Delta_P(y) + ax + b + C = -tx + \int y \, d\Delta_Q(y) + ax + b & \text{if } x \leq S_l, \end{cases}$$

whence

$$\begin{cases} -\int y \, d\Delta_P(y) + C = -\int y \, d\Delta_Q(y), \\ \int y \, d\Delta_P(y) + C = \int y \, d\Delta_Q(y). \end{cases}$$

Adding these equalities, we obtain $C = 0$. Thus, we get (3.3) together with (3.2).

The general case of an open set $O \subset \mathbb{R}^d$. Let’s start again with the representations (3.4). We set

$$S := \text{supp} \Delta_q \cup \text{supp} \Delta_p \subset O,$$

(3.8S) $w := p - q, \Delta_w(1.6) := c_d \Delta w = \Delta_p - \Delta_q \in \text{Meas}(S) \subset \text{Meas}_{\text{cmp}}(O).$

This difference $w \in \text{sbh}_*(O) - \text{sbh}_*(O)$ of subharmonic functions, i.e., a $\delta$-subharmonic function [3], [4], [40, 3.1], is uniquely defined on $O$ outside a polar set (cf. (1.10))

$$\text{Dom } w := \left\{ x \in O : \inf \left\{ \int_0^1 \frac{\Delta^-(w,x,t)}{t^{d-1}} \, dt, \int_0^1 \frac{\Delta^+(w,x,t)}{t^{d-1}} \, dt \right\} < +\infty \right\} \subset S,$$

(3.9) and $w \equiv 0$ on $O \setminus S$ since $p = q$ outside $S \subset S$ in (3.8w), and $p, q \in \text{har}(O \setminus S)$. The Riesz charge $\Delta_w \in \text{Meas}_{\text{cmp}}(O)$ of this $\delta$-subharmonic function $w$ on $O$ is also uniquely determined on $O$ with $\supp |\Delta_w| \subset S$ [3, Theorem 2]. The function $w : O \setminus \text{Dom } w \to \mathbb{R}$ can be extended from $O$ to the whole of $\mathbb{R}^d \setminus \text{Dom } w$ by zero values:

$$w \equiv 0 \text{ on } \mathbb{R}^d \setminus S \subset \mathbb{R}^d \setminus O, \Delta_w = \Delta_p - \Delta_q \in \text{Meas}(S).$$

(3.10) This function $w$ on $\mathbb{R}^d \setminus \text{Dom } w$ is still a $\delta$-subharmonic function, but already on $\mathbb{R}^d$, since $\delta$-subharmonic functions are defined locally [3, Theorem 3]. The Riesz charge of this $\delta$-subharmonic function $w : \mathbb{R}^d \setminus \text{Dom } d \to \mathbb{R}$ on $\mathbb{R}^d$ is the same charge $\Delta_d \in \text{Meas}(S)$. There is a canonical representation [3, Definition 5] of $w$ such that [3, Theorem 5]

$$w = P - Q \text{ on } \mathbb{R}^d \setminus \text{Dom } w, \text{ where } P, Q \in \text{sbh}_*(\mathbb{R}^d) \cap \text{har}(\mathbb{R}^d \setminus S)$$

(3.11d)
are functions with Riesz measures

\[(3.11\Delta)\]

\[\Delta_p \overset{(1.6)}{=} c_d \Delta P = \Delta^+_{w} \overset{(3.11d)}{\in} \text{Meas}^+(S), \quad \Delta_Q \overset{(1.6)}{=} c_d \Delta Q = \Delta^-_{w} \overset{(3.11d)}{\in} \text{Meas}^+(S),\]

\[(3.11\equiv)\]

\[P \overset{(3.10),(3.11d)}{=} Q \quad \text{on } \mathbb{R}^d \setminus S,\]

and there is a function \(s \in \text{sbh}_s(O)\) with Riesz measure

\[(3.11s)\]

\[\Delta_s = \Delta_p - \Delta^+_{w} \overset{(3.10),(3.11\Delta)}{=} \Delta_q - \Delta^-_{w} \in \text{Meas}^+(S)\]

\[(3.11r)\]

such that \(p = P + s, \quad q = Q + s\) on \(O\).

By \((3.11d)\) and \((3.11\equiv)\), all conditions of Theorem 2 are fulfilled for functions \(P, Q\) from \((3.11)\) instead of \(p, q\), but in the case \(\mathbb{R}^d\) instead of \(O\) and \(S\) instead of \(S\). Thus, we have \((3.2)\) in the form

\[(3.12\Delta)\]

\[\Delta^+_{w}(O) \overset{(3.11\Delta)}{=} \Delta_p(\mathbb{R}^d) \overset{(3.2)}{=} \Delta_Q(\mathbb{R}^d) \overset{(3.11\Delta)}{=} \Delta^-_{w}(O),\]

\[](3.12p)

\[\text{pt}_{\Delta^+_{w}} \overset{(3.11\Delta)}{=} \text{pt}_{\Delta_p} = \text{pt}_{\Delta_Q} \overset{(3.11\Delta)}{=} \text{pt}_{\Delta^-_{w}} \quad \text{on } \mathbb{R}^d \setminus S,\]

and the representations \((3.3)\) in the form

\[(3.13)\]

\[
\begin{cases}
P \overset{(3.3)}{=} \text{pt}_{\Delta_p} + h \overset{(3.12p)}{=} \text{pt}_{\Delta^+_{w}} + h \quad \text{on } \mathbb{R}^d, \\
Q \overset{(3.3)}{=} \text{pt}_{\Delta_Q} + h \overset{(3.12p)}{=} \text{pt}_{\Delta^-_{w}} + h \quad \text{on } \mathbb{R}^d, \\
h \in \text{har}(\mathbb{R}^d),
\end{cases}
\]

Hence, by representation \((3.11r)\), we obtain the following representations

\[(3.14)\]

\[
\begin{cases}
p \overset{(3.11r),(3.13)}{=} \text{pt}_{\Delta^+_{w}} + h + s, \quad \text{on } O, \\
q \overset{(3.11r),(3.13)}{=} \text{pt}_{\Delta^-_{w}} + h + s, \quad \text{on } O,
\end{cases}
\]

\(h \in \text{har}(\mathbb{R}^d), \quad \text{pt}_{\Delta^+_{w}} \overset{(3.12p)}{=} \text{pt}_{\Delta^-_{w}} \quad \text{on } \mathbb{R}^d \setminus S, \quad \Delta^+_{w}(O) \overset{(3.12\Delta)}{=} \Delta^-_{w}(O).\)

Besides, the function \(l \overset{(3.11s)}{=} s - \text{pt}_{\Delta_s}\) is harmonic on \(O\) by Weyl’s lemma on the Laplace equation \(\Delta (s - \text{pt}_{\Delta_s}) \overset{(3.11s)}{=} \Delta_s - \Delta_s = 0\). Hence

\[(3.15)\]

\[
\begin{cases}
p \overset{(3.14)}{=} \text{pt}_{\Delta^+_{w}} + \text{pt}_{\Delta_s} + h + l, \quad \text{on } O, \quad \text{where } h \in \text{har}(\mathbb{R}^d) \quad \text{and } l \in \text{har}(O), \\
q \overset{(3.14)}{=} \text{pt}_{\Delta^-_{w}} + \text{pt}_{\Delta_s} + h + l
\end{cases}
\]

\(\text{pt}_{\Delta^+_{w}} + \text{pt}_{\Delta_s} \overset{(3.14)}{=} \text{pt}_{\Delta^-_{w}} + \text{pt}_{\Delta_s} \quad \text{on } \mathbb{R}^d \setminus S, \quad \Delta^+_{w}(O) \overset{(3.14)}{=} \Delta^-_{w}(O).\)

By construction, we have

\[
\begin{cases}
\text{pt}_{\Delta^+_{w}} + \text{pt}_{\Delta_s} = \text{pt}_{\Delta^+_{w} + \Delta_s} \overset{(3.11s)}{=} \text{pt}_{\Delta_p}, \\
\text{pt}_{\Delta^-_{w}} + \text{pt}_{\Delta_s} = \text{pt}_{\Delta^-_{w} + \Delta_s} \overset{(3.11s)}{=} \text{pt}_{\Delta_q}, \\
\Delta_p(O) = (\Delta^+_{w} + \Delta_s)(O) \overset{(3.11s)}{=} (\Delta^-_{w} + \Delta_s)(O) = \Delta_p(O).
\end{cases}
\]
Hence, if we set $H := h + l \in \text{har}(O)$, then, by (3.15), we obtain exactly (3.3), as well as (3.2), with the only difference being that in (3.2) we have $S \supset (3.8S)$ instead of $S$. Moreover, it immediately follows from the representation (3.3) and the condition $p = q$ on $S \setminus S \supset O \setminus S$ that $\text{pt}_\Delta = \text{pt}_\omega$ on $R^d \setminus S = (R^d \setminus S) \cup (S \setminus S)$. Theorem 2 is proved.

3.2. Duality between balayage of measures and their potentials. In this Subsec. 3.2, the equivalence of the first four statements of the Main Lemma according to the scheme

\[
\begin{array}{cccc}
I & \rightarrow & II \\
& \leftarrow & \downarrow \\
IV & \leftrightarrow & III
\end{array}
\]  

will be established. We write $A \overset{\text{proof}}{\rightarrow} B$ if the implication $A \Rightarrow B$ is proved or discussed below.

$I \overset{\text{proof}}{\rightarrow} II$. By Proposition 1(i-ii) and [16, Theorem 1.7], if $h \in \text{har}(S_O)$ is harmonic on the inward filling $S_O \supset \text{in-fill(supp } \Delta \cup \text{ supp } \omega) = \text{in-fill } S_O \subset O$ of $S$, then there are functions $h_k \in \text{har}(O)$ such that the sequence $(h_k)_{k \in \mathbb{N}}$ converges to $h$ in the space $C(S_O)$ of all continuous functions on the compact set $S_O \subset O$ with sup-norm.

Hence,

\[
\int_{S_O} h \, d\Delta = \int_{S_O} \lim_{k \to \infty} h_k \, d\Delta = \lim_{k \to \infty} \int_{S_O} h_k \, d\Delta = \lim_{k \to \infty} \int_{O} h_k \, d\Delta = \int_{O} \lim_{k \to \infty} h_k \, d\omega = \int_{O} h \, d\Delta.
\]

The statement II of the Main Lemma is established.

$II \overset{\text{proof}}{\rightarrow} III$. If $x \notin S_O$, then the subharmonic function

\[
(3.17) \quad u_x : y \mapsto K_{d-2}(y, x)
\]

is harmonic on $S_O$. Hence, for $x \notin S_O$,

\[
\text{pt}_\Delta(x) = \int_{\text{supp } \Delta} K_{d-2}(\cdot, x) \, d\Delta = \int_{S_O} K_{d-2}(\cdot, x) \, d\Delta \overset{\text{II}(1.8)}{=} \int_{S_O} u_x \, d\Delta \overset{\text{II}(1.8)}{=} \int_{S_O} u_x \, d\omega \overset{(3.17)}{=} \int_{S_O} K_{d-2}(\cdot, x) \, d\omega = \int_{\text{supp } \omega} K_{d-2}(\cdot, x) \, d\Delta = \text{pt}_\omega(x).
\]

The statement III of the Main Lemma is established.

$III \overset{\text{proof}}{\rightarrow} IV$. This implication is obvious if we choose $p := \text{pt}_\omega$ and $q := \text{pt}_\Delta$.

$IV \overset{\text{proof}}{\rightarrow} III$. This implication is a special case of Theorem 2 with the conclusion (3.2).

$III \overset{\text{proof}}{\rightarrow} I$. We use the following
Lemma 2 ([16, Lemma 1.8]). Let $F$ be a compact subset in $\mathbb{R}^d$, $h \in \text{har}(F)$, and $b \in \mathbb{R}^+\setminus\{0\}$. Then there are points $y_1, y_2, \ldots, y_m$ in $\mathbb{R}^d\setminus F$ and constants $a_1, a_2, \ldots, a_m \in \mathbb{R}$ such that

\[
(3.18) \quad |h(x) - \sum_{j=1}^{m} a_j k_{d-2}(|x - y_j|)| < b \quad \text{for all } x \in F.
\]

Applying Lemma 2 to the compact set $F := S_O \subseteq O$ and a function $h \in \text{har}(O)$, we obtain

\[
\left| \int_{O} h \, d(\omega - \Delta) \right| = \left| \int_{S_O} h \, d(\omega - \Delta) \right| \overset{\text{(1.9)}}{=} \left| \int_{S_O} h \, d(\omega - \Delta) - \sum_{j=1}^{m} a_j \left( \int_{S_O} K_{d-2}(y, y_j) \, d\omega(y) - \int_{S_O} K_{d-2}(y, y_j) \, d\Delta(y) \right) \right|
\]

\[
\overset{\text{(1.2K)}}{=} \int_{S_O} h(y) \, d(\omega - \Delta)(y) - \int_{S_O} \sum_{j=1}^{m} a_j k_{d-2}(|y - y_j|) \, d(\omega - \Delta)(y)
\]

\[
\leq \sup_{y \in S_O} \left| h(y) - \sum_{j=1}^{m} a_j k_{d-2}(|y - y_j|) \right| \left( \omega(O) + \Delta(O) \right) \overset{(3.18)}{\leq} b(\omega(O) + \Delta(O))
\]

for each $b \in \mathbb{R}^+\setminus\{0\}$. Hence $\Delta \leq_{\text{har}(O)} \omega$. Thus, we obtain I and complete (3.16).

3.3. The symmetric Poisson–Jensen formula for measures and their potentials. In this Subsec. 3.3, we complete the proof of the Main Lemma by the scheme

\[
(3.19) \quad (\text{II } \cap \text{ III}) \quad \longrightarrow \quad V \quad \longrightarrow \quad VI \quad \longrightarrow \quad VII \quad \longrightarrow \quad IV,
\]

where $\text{II } \cap \text{ III}$ means that statements II and III are simultaneously satisfied, and the equivalence $\text{(II } \cap \text{ III)} \Leftrightarrow \text{IV}$ of the extreme statements (II $\cap$ III) and IV of (3.19) has already been proved in the previous Subsec. 3.2 by the scheme (3.16).

\[\text{(II } \cap \text{ III)} \quad \overset{\text{proof}}{\longrightarrow} \quad V.\]

Let $u \overset{\text{(2.2B)}}{\in} \text{sbh}_*(\text{clos } B)$, where $S_O \overset{\text{(2.2B)}}{\subseteq} B \subseteq O$. We can choose an open set $O'$ such that $B \subseteq O' \subseteq O$ and $u \in \text{sbh}_*(\text{clos } O')$. Consider first the case

\[
(3.20) \quad -\infty < \int u \, d\Delta, \quad \text{where supp } \Delta \overset{(2.1)}{\subseteq} S_O \subseteq O'.
\]

Let

\[
(3.21) \quad \mu' := \Delta_u \bigg|_{\text{clos } O'}
\]

be the restriction of Riesz measure of $u \in \text{sbh}_*(\text{clos } O')$ to $\text{clos } O' \subseteq O$. By the Riesz Decomposition Theorem [50, Theorem 3.7.1], [22, Theorem 3.9], [2, Theorem 4.4.1], [23, Theorem 6.18] we obtain a representation

\[
(3.22) \quad u = \text{pt}_{\mu'} + h \quad \text{on } O', \quad \text{where } h \in \text{har}(O') \text{ is continuous and bounded on } S_O.
\]
Integrating this representation with respect to \( d\omega \) and \( d\Delta \), we obtain

\[ (3.23 \omega) \quad \int u \, d\omega = \int pt_{\mu'} \, d\omega + \int h \, d\omega, \quad \text{supp} \, \omega \subset S_O, \]

\[ (3.23 \Delta) \quad \int u \, d\Delta = \int pt_{\mu'} \, d\Delta + \int h \, d\Delta, \quad \text{supp} \, \Delta \subset S_O, \]

where the three integrals in \( (3.23 \Delta) \) are finite, although in the equality \( (3.23 \omega) \) the first two integrals can take simultaneously the value of \(-\infty\), but the last integral in \( (3.23 \omega) \) is finite. Therefore, the difference \( (3.23 \omega) - (3.23 \Delta) \) of these two equalities is well defined:

\[ (3.24) \quad \int u \, d\omega - \int u \, d\Delta = \int pt_{\mu'} \, d\omega - \int pt_{\mu'} \, d\Delta + \int_{\text{supp} \, \omega} h \, d(\omega - \Delta), \]

where the first and third integrals can simultaneously take the value of \(-\infty\), and the remaining integrals are finite. By the statement II we have \( \Delta \simeq_{\text{har}} (S_O) \, \omega \). Hence the last integral in \( (3.24) \) vanishes according to \( (1.8) \). Using Fubini's Theorem on repeated integrals, in view of the symmetry property of kernel \( K_{d-2} \) in \( (1.2K) \), we have

\[ (3.25) \quad \int pt_{\mu'} \, d\Delta = \int \int K_{d-2}(y, x) \, d\mu'(y) \, d\Delta(x) \]

\[ = \int \int K_{d-2}(x, y) \, d\Delta(x) \, d\mu'(y) = \int_{\text{clos} \, O'} pt_{\Delta} \, d\Delta_u, \]

and the same way

\[ (3.26) \quad \int pt_{\mu'} \, d\omega = \int \int K_{d-2}(y, x) \, d\mu'(y) \, d\omega(x) \]

\[ = \int \int K_{d-2}(x, y) \, d\omega(x) \, d\mu'(y) = \int_{\text{clos} \, O'} pt_{\omega} \, d\Delta_u \]

even if the integral on the left side of equalities \( (3.26) \) takes the value \(-\infty\) because the integrand \( K_{d-2}(\cdot, \cdot) \) is bounded from above on the compact set \( \text{clos} \, O' \times \text{clos} \, O' \) [22, Theorem 3.5]. Hence equality \( (3.24) \) can be rewritten as

\[ \int u \, d\omega - \int u \, d\Delta = \int_{\text{clos} \, O'} pt_{\omega} \, d\Delta_u - \int_{\text{clos} \, O'} pt_{\Delta} \, d\Delta_u \]

or in the form

\[ (3.27) \quad \int u \, d\omega + \int_{\text{clos} \, O'} pt_{\Delta} \, d\Delta_u = \int u \, d\Delta + \int_{\text{clos} \, O'} pt_{\omega} \, d\Delta_u. \]

But by the statement III, we have

\[ pt_{\omega} \equiv pt_{\Delta} \quad \text{on} \; \mathbb{R}^d \setminus S_O \supset \text{clos} \, O' \setminus B. \]

Hence, by equality \( (3.27) \), we obtain equality \( (2.2f) \) in the case \( (3.20) \).

If condition \( (3.20) \) is not fulfilled, then from the representation \( (3.23 \Delta) \) it follows that the integral on the left-hand side of \( (3.25) \) also takes the value \(-\infty\). The equalities
(3.25) is still true [22, Theorem 3.5]. Hence, the second integral on the right side of the formula (2.2f) also takes the value \(-\infty\) and this formula (2.2f) remains true.

\[ V \overset{\text{proof}}{\Rightarrow} VI. \] Let \( q \in \text{sbh}_+(S_O) \) be a function with Riesz measure \( \Delta_q = \Delta \). Then there is a function \( h \in \text{har}(O) \) such that \( q = \text{pt}_\Delta + h \) on \( O \). By the statement \( V \), we have (2.2f) for \( B = S_O \). If we set \( p := \text{pt}_\omega + h \), then \( \Delta_p = \omega \), and (2.3) follows from (2.2f) with \( B = S_O \).

\[ VI \overset{\text{proof}}{\Rightarrow} VII. \] We set \( q := \text{pt}_\Delta \in \text{sbh}_+(\mathbb{R}^d) \) with \( \Delta_q = \Delta \). By statement VI, there is a function \( p \in \text{sbh}_+(\mathbb{R}^d) \) with \( \Delta_p = \omega \) such that we have (2.3). In particular, the equality in (2.3) is true for each special subharmonic function \( u_x : y \mapsto K_{d-2}(y, x) \), \( x \in \mathbb{R}^d \), and we obtain VII.

\[ VII \overset{\text{proof}}{\Rightarrow} IV. \] Each special function \( u_x \) in VII is subharmonic on \( \mathbb{R}^d \) with Riesz measure \( \delta_x \). If \( x \in O \setminus S \), where \( S_O \subset S \subset O \), then \( S_O \cap \text{supp} \delta_x = \emptyset \). Thus,

\[ \int_{S_O} p \, d\delta_x = \int_{S_O} q \, d\delta_x = 0 \quad \text{for each } x \in O \setminus S. \]

Hence, by (2.3) with \( u_x \) instead of \( u \), we obtain

\[ \text{pt}_\Delta(x) = \int_{S_O} K(y, x) \, d\Delta(y) = \int_{S_O} u_x \, d\Delta \overset{(2.3), (3.28)}{=} \int_{S_O} u_x \, d\omega = \int_{S_O} K(y, x) \, d\omega(y) = \text{pt}_\omega(x) \]

for each \( x \in O \setminus S \). Thus, we obtain the statement IV for \( q := \text{pt}_\Delta \) and \( p := \text{pt}_\omega \).

The Main Lemma is proved.

4. Proof of the Main Theorem

\[ [I] \overset{\text{proof}}{\Rightarrow} [II]. \] Without loss of generality, we can assume that \( S = S_O \) in (2.4S). Then the statement \([I]\) with (2.5) follows from Theorem 2 with (3.2)–(3.3).

\[ [II] \overset{\text{proof}}{\Rightarrow} [III]. \] By the equality \( \text{pt}_\Delta \overset{(2.5)}{=} \text{pt}_\Delta \) on \( \mathbb{R}^d \setminus S_O \), we have the statement III of the Main Lemma for \( \Delta := \Delta_q \in \text{Meas}^+(S) \) and \( \omega := \Delta_p \in \text{Meas}^+(S) \). By implication III \( \Rightarrow V \) of the Main Lemma, we obtain

\[ (4.1) \quad \int_{\text{supp} \Delta_q} u \, d\Delta_q + \int_B \text{pt}_\Delta d\Delta_u \overset{(2.2)}{=} \int_{\text{supp} \Delta_p} u \, d\Delta_p + \int_B \text{pt}_\Delta d\Delta_u \]

for each \( B \in \text{Bor}(\mathbb{R}^d) \) under \( S_O \subset B \subset O \) and for each \( u \in \text{sbh}_+(\text{clos} B) \), where we returned to the separate notation \( S \subset S_O := \text{in-fill} \ S \). Obviously,

\[ (4.2) \quad \int_B h \, d\Delta_u = \int_B h \, d\Delta_u \quad \text{for each } u \in \text{sbh}_+(\text{clos} B) \text{ and } h \in \text{har}(O). \]

Adding (4.1) and (4.2), according to representations (2.5) of \( q \) and \( p \), we obtain

\[ (4.3) \quad \int_S u \, d\Delta_q + \int_B p \, d\Delta_u \overset{(2.2)}{=} \int_S u \, d\Delta_p + \int_B q \, d\Delta_u, \]

where \( B \) can be replaced with \( B \cap S_q \). This proves (2.6f) already for a set \( B \) and functions \( u \) of the form (2.6B). Thus, we obtain statement [III].

\[ [III] \overset{\text{proof}}{\Rightarrow} [IV]. \] All functions \( u_x \) in [IV] are subharmonic on \( \mathbb{R}^d \supset O \).
The Riesz measure of $u_x$ is the Dirac measure $\delta_x$, and, by [IV],
\[ \int_S u_x \, d\Delta_q + \int_{B_j} p \, d\delta_x = \int_S u_x \, d\Delta_p + \int_{B_j} q \, d\delta_x \quad \text{for each } j \in \mathbb{N} \text{ and } x \in O. \]

If $j = 0$ and $x \notin S_O = B_0$, then $\text{supp} \, \delta_x = x \notin S_O$ and
\[ \int_{S_O} p \, d\delta_x = \int_{S_O} q \, d\delta_x = 0. \]

These equalities do not depend on $j \in \mathbb{N}_0$ for points $x \notin S_O$. Hence
\[ \int_S u_x \, d\Delta_q (4.4) = \int_S u_x \, d\Delta_p \quad \text{for each } j \in \mathbb{N}_0 \text{ and } x \notin S_O \supset S. \]

Therefore, it follows from (4.4) that
\[ \int_{B_j} p \, d\delta_x (4.4) = \int_{B_j} q \, d\delta_x \quad \text{for each } j \in \mathbb{N}_0 \text{ and } x \notin S_O, \]
i.e., $p(x) = q(x)$ for each $j \in \mathbb{N}_0$ and for every $x \in B_j \setminus S_O$. Thus, $p(x) = q(x)$ for each point $x \in \bigcup_{j \in \mathbb{N}_0} B_j \setminus S_O = O \setminus S_O$, and statement [I] is established.

5. **Duality Theorems for Balayage**

Part of some equivalences of the Main Lemma and the Main Theorem allows us to give an internal dual description for the potentials of measures obtained through the balayage processes. Such descriptions in particular cases of Arens–Singer and Jensen measures and their potentials have already found important applications in the study of various problems of function theory [15, Ch. 3 etc.], [1], [24], [25], [26], [27], [28], [29], [44], [30], [31], [33], [34], [51], [32], [35], [36], [37], [7], [41], [42], [39].

**Duality Theorem 1** (for $\text{har}(O)$-balayage). Let $\Delta \in \text{Meas}_{\text{cmp}}^+(O)$.

If a measure $\omega \in \text{Meas}_{\text{cmp}}^+(O)$ is a $\text{har}(O)$-balayage of $\Delta$, then (cf. (2.11))
\[ \text{pt}_\omega \in \text{sbh}_*(\mathbb{R}^d) \cap \text{har}(\mathbb{R}^d \setminus \text{supp } \omega), \]
\[ \text{pt}_\omega = \text{pt}_\Delta \text{ on } \mathbb{R}^d \setminus \text{in-fill}_O(\text{supp } \Delta \cup \text{supp } \omega). \]

Conversely, suppose that there are a compact subset $S \subseteq O$ and a function $p$ such that
\[ p \in \text{cf.}(5.1) \subseteq \text{sbh}(O) \cap \text{har}(O \setminus S), \]
\[ p \in \text{cf.}(5.1) = \text{pt}_\Delta \text{ on } O \setminus S. \]

Then the Riesz measure
\[ \omega := c_d \Delta p \in \text{Meas}^+(S) \subseteq \text{Meas}_{\text{cmp}}^+(O) \]
of this function $p$ is a $\text{har}(O)$-balayage of the measure $\Delta$. 

Proof. Properties (5.1) for $\omega \geq_{\text{har}(O)} \Delta$ directly follow from the implication $\text{I} \Rightarrow \text{III}$ of the Main Lemma. In the opposite direction, we can use the implication $\text{IV} \Rightarrow \text{I}$ of the Main Lemma with $p$ from (5.2) and $q := \text{pt}_{\Delta}$. □

**Duality Theorem 2** (for $\text{sbh}(O)$-balayage). Let $\Delta \in \text{Meas}_{\text{cmp}}^{+}(O)$. If $\omega \geq_{\text{sbh}(O)} \Delta$, then we have (5.1) and $\text{pt}_{\omega} \geq \text{pt}_{\Delta}$ on $\mathbb{R}^{d}$. Conversely, suppose that there are a compact subset $S$ in $O$ containing $\text{supp} \Delta$, and a function $p$ satisfying (5.2) such that

(5.4) \[ p \geq \text{pt}_{\Delta} \quad \text{on } S_{O} := \text{in-fill}(S). \]

Then the Riesz measure (5.3) of this function $p$ is a $\text{sbh}(O)$-balayage of the measure $\Delta$.

Proof. If $\Delta \leq_{\text{sbh}(O)} \omega$, then $\Delta \leq_{\text{har}(O)} \omega$, which was noted earlier in §1.2.4(4), and, by Duality Theorem 1, we obtain (5.1). Besides, functions $y \mapsto K_{d-2}(y, x)$ are subharmonic on $\mathbb{R}^{d}$ for each $x \in \mathbb{R}^{d}$, and

$$\text{pt}_{\Delta}(x) = \int K_{d-2}(y, x) \, d\Delta(y) \leq \int K_{d-2}(y, x) \, d\omega(y) = \text{pt}_{\omega}(x) \quad \text{for each } x \in \mathbb{R}^{d}. \tag{1.7}$$

In the opposite direction, we set $q := \text{pt}_{\Delta} \in \text{sbh}_{\ast}(\mathbb{R}^{d}) \cap \text{har}(O \setminus S)$. By Duality Theorem 1, the Riesz measure $\Delta_{p}$ (5.3) $\omega \in \text{Meas}_{\text{cmp}}^{+}(O)$ of the function $p$ is a $\text{har}(O)$-balayage of $\Delta$. By condition (5.2) in the notation (5.4), we have the equality $p = q$ on $O \setminus S_{O} \subset O \setminus S$, and, by condition (5.2p), the functions $p$ and $q$ are harmonic on $O \setminus S$. Thus, the statement [I] of the Main Theorem is fulfilled. By the implication [I] $\Rightarrow$ [III] of the Main Theorem, using the full symmetric Poisson–Jensen formula (2.6f) with $B \subseteq S_{O}$, we get

(5.5) \[ \int_{S} u \, d\Delta_{q} + \int_{S_{O}} p \, d\Delta_{u} \overset{(2.6f)}{=} \int_{S} u \, d\Delta_{p} + \int_{S_{O}} q \, d\Delta_{u} \quad \text{for each } u \in \text{sbh}_{\ast}(O). \tag{2.6B} \]

Hence, by the condition $p \geq \text{pt}_{\Delta} = q$ on $S_{O}$, we obtain

(5.6) \[ \int_{O} u \, d\Delta + \int_{S_{O}} q \, d\Delta_{u} = \int_{S} u \, d\Delta_{q} + \int_{S_{O}} q \, d\Delta_{u} \leq \int_{S} u \, d\Delta_{q} + \int_{S_{O}} p \, d\Delta_{u} \overset{(5.5)}{=} \int_{S} u \, d\Delta_{p} + \int_{S_{O}} q \, d\Delta_{u} = \int_{O} u \, d\omega + \int_{S_{O}} q \, d\Delta_{u} \quad \text{for each } u \in \text{sbh}_{\ast}(O). \]

In particular, if $u \in \text{sbh}_{\ast}(O) \cap C^{\infty}(O)$, then the function $q$ is $\Delta_{u}$-integrable on $S_{O}$, and it is follows from (5.6) that

$$\int_{O} u \, d\Delta \overset{(5.6)}{\leq} \int_{O} u \, d\omega \quad \text{for each } u \in \text{sbh}_{\ast}(O) \cap C^{\infty}(O).$$

Hence, by §1.2.4(5), we obtain $\Delta \leq_{\text{sbh}(O)} \omega$. □

The following long-known result for Arens–Singer and Jensen measures and their potentials on domains in $\mathbb{R}^{d}$ with $d \geq 2$ has found numerous applications in the theory of functions of one and several complex variables, which is partially reflected in the
bibliographic sources listed at the beginning of Sec. 5. The proof of this result immediately follows from Duality Theorems 1 and 2, but already for open sets $O \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$.

**Duality Theorem 3 ([35, Proposition 1.4, Duality Theorem]).** Let $x \in O \subset \mathbb{R}^d$. The map

$$(5.7) \quad \mathcal{P}_x : \omega \mapsto \text{pt}_x - \delta_x$$

defines an affine bijection from $\text{AS}_x(O)$ onto $\text{ASP}_x(O)$, as well as from $J_x(O)$ onto $\text{JP}_x(O)$ (see also, in addition, (2.11)) with the inverse map

$$(5.8) \quad \mathcal{P}_x^{-1} : V \mapsto c_d \Delta V \mid_{\mathbb{R}^d \setminus x} + \left( 1 - \limsup_{x \not\to y \atop x \not\to x} \frac{V(y)}{-K_{d-2}(x, y)} \right) \cdot \delta_x.$$  

**Remark 2.** Theorems 1 and 2 can also be formulated in a form close to Theorem 3, using some affine bijection of type (5.7)–(5.8) and definitions of the generalized Arens–Singer and Jensen potentials. But such formulations require some development of the theory of $\delta$-subharmonic functions [3], [4], [40, 3.1] and a delicate approach to upper/lower integrals (1.5) with values in $\mathbb{R}$. We will not discuss similar interpretations of Theorems 1 and 2 here.

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