Analysis of a Class of Likelihood Based Continuous Time Stochastic Volatility Models including Ornstein-Uhlenbeck Models in Financial Economics.

LANCELOT F. JAMES

The Hong Kong University of Science and Technology

In a series of recent papers Barndorff-Nielsen and Shephard introduce an attractive class of continuous time stochastic volatility models for financial assets where the volatility processes are functions of positive Ornstein-Uhlenbeck (OU) processes. This models are known to be substantially more flexible than Gaussian based models. One current problem of this approach is the unavailability of a tractable exact analysis of likelihood based stochastic volatility models for the returns of log prices of stocks. With this point in mind, the likelihood models of Barndorff-Nielsen and Shephard are viewed as members of a much larger class of models. That is likelihoods based on n conditionally independent Normal random variables whose mean and variance are representable as linear functionals of a common unobserved Poisson random measure. The analysis of these models is facilitated by applying the methods in James (2005, 2002), in particular an Esscher type transform of Poisson random measures; in conjunction with a special case of the Weber-Sonine formula. It is shown that the marginal likelihood may be expressed in terms of a multidimensional Fourier-cosine transform. This yields tractable forms of the likelihood and also allows a full Bayesian posterior analysis of the integrated volatility process. A general formula for the posterior density of the log price given the observed data is derived, which could potentially have applications to option pricing. We also identify tractable subclasses, where inference can be based on a finite number of independent random variables. We close by obtaining explicit expressions for likelihoods incorporating leverage. It is shown that inference does not necessarily require simulation of random measures. Rather, classical numerical integration can be used in the most general cases.

1 Introduction

Barndorff-Nielsen and Shephard (2001a, b)(BNS) introduce a class of continuous time stochastic volatility (SV) models that allows for more flexibility over Gaussian based models such as the Black-Scholes model[see Black and Scholes (1973) and Merton (1973)]. Their proposed SV model is based on the following differential equation,

\[ dx^*(t) = (\mu + \beta v(t))dt + v^{1/2}(t)dw(t) \]

where \( x^*(t) \) denotes the log-price level, \( w(t) \) is Brownian motion, and independent of \( w(t) \), \( v(t) \) is a stationary Ornstein-Uhlenbeck (OU) process which models the instantaneous volatility. This model is an extension of the Black-Scholes or Samuelson model which arises by replacing \( v \) with a fixed variance, say \( \sigma^2 \). The additional innovation in BNS is that modeling volatility as a random process, \( v(t) \), rather than a random variable, not only allows for heavy-tailed models, but additionally induces serial dependence. This serial dependence is used to account for a clustering affect referred to as volatility persistence. The work of Carr, Geman, Madan, and Yor (2003) discuss this point further.

1 AMS 2000 subject classifications. Primary 62G05; secondary 62F15.
Corresponding authors address. The Hong Kong University of Science and Technology, Department of Information and Systems Management, Clear Water Bay, Kowloon, Hong Kong. lancelot@ust.hk
Keywords and phrases. Bessel Functions, Mixture of Normals, Ornstein-Uhlenbeck Process, Poisson Process, Stochastic Volatility, Weber-Sonine Formula.
See also Duan (1995) and Engle (1982) for different approaches to this type of phenomenon. The model of BNS has gained a great deal of interest with some related works including Carr, Geman, Madan, and Yor (2003), Barndorff-Nielsen and Shephard (2003), Eberlein (2001), Nicolato and Venardos (2001), Benth, Karlsen, and Reikvam (2003). See also the discussion section in Barndorff-Nielsen and Shephard (2001a).

One current drawback of this approach is the unavailability of a tractable analysis of likelihood based stochastic volatility models for the returns of log prices of stocks. These models are based on the integrated volatility process \( \tau(t) = \int_0^t v(u)du \). Several MCMC procedures have been proposed to handle subclasses of these models requiring simulation of points from random processes. See for instance, Roberts, Papaspiliopoulos and Dellaportas (2004) and the discussion section in Barndorff-Nielsen and Shephard (2001a).

In this paper, we shall actually provide a complete analysis of a significantly more complex class of likelihood models. Specifically models where \( \tau \) is expressible as a linear functional of a Poisson random measure. This includes the superposition processes mentioned in Barndorff-Nielsen and Shephard (2001a, b) and much more general spatial models for \( \tau \). Our results are therefore applicable to a wide range of models and applications. We first present a description of these models similar to the framework outlined in Barndorff-Nielsen and Shephard (2001a). We then show how the results in James (2005, 2002) are easily applied to this setting via the usage of a Bessel integral identity involving the cosine function. That is, a special case of what is called the Weber-Sonine formula. This leads to an interesting series of tractable characterizations of such processes, including an identification of simple subclasses of these models. As a byproduct, we derive the posterior predictive density of what could be considered as models for the log price of stocks. This may prove useful to applications in option pricing. Moreover, our methods do not require simulation of random measures and in the most general cases can be handled by more classical numerical integration methods. We point out also that procedures to simulate from random measures often require explicit knowledge of the Lévy density associated with an infinitely divisible random variable. This is an important point as there are some interesting cases where the probability density of a random variable is known explicitly but its corresponding Lévy density is unknown. Our results show that one only requires knowledge of the form of the Lévy exponent or log of the Laplace transform of a corresponding random variable. Inference using the classes of models that we identify in sections 2.5 and 4 can be performed based on at most \( J < \infty \) independent latent random variables. In the case of section 4, \( J = n + 1 \), where \( n \) denotes the number of observations. Section 2.6 discusses another class where exact calculations of a different form are easily obtained. Section 5 describes exact expression for likelihoods of generalized types of leverage effects models.

**Remark 1.** The appearance of integrals involving Bessel functions is certainly not new to applications in finance as can be seen in the case of the important work of Yor (1992) on Asian Options. See also Carr and Schröder (2004). We shall however employ a different, but certainly related, integral identity.

### 1.1 Likelihood model and representation

The model of Barndorff-Nielsen and Shephard (2001a, section 5.4) translates into a likelihood based model as follows. Let \( X_i \) for \( i = 1, \ldots, n \) denote a sequence of aggregate returns of the log price of a stock observed over intervals of length \( \Delta > 0 \). Additionally for each interval \([ (i-1)\Delta, i\Delta] \), let \( \tau_i = \tau(i\Delta) - \tau((i-1)\Delta) \). Now the model in \( \square \) implies that \( X_i | \tau_i, \beta, \mu \) are conditionally independent with

\[
X_i = \mu \Delta + \tau_i \beta + \tau_i^{1/2} \varepsilon_i,
\]
where $\epsilon_i$ are independent standard Normal random variables. Hence if $\tau$ depends on external parameters $\theta$, one is interested in estimating $(\mu, \beta, \theta)$ based on the likelihood

$$\mathcal{L}(X|\mu, \beta, \theta) = \int_{\mathbb{R}^n_+} \left[ \prod_{i=1}^n \phi(X_i|\mu \Delta + \beta \tau_i, \tau_i) \right] f(\tau_1, \ldots, \tau_n|\theta) d\tau_1, \ldots, d\tau_n$$

where, setting $A_i = (X_i - \mu \Delta)$, and $A = n^{-1}\sum_{i=1}^n A_i$,

$$\phi(X_i|\mu \Delta + \beta \tau_i, \tau_i) = e^{A_i} \frac{1}{\sqrt{2\pi}} \tau_i^{-1/2} e^{-A_i^2/(2\tau_i)} e^{-\tau_i \beta^2/2}$$

denotes a Normal density.

The quantity $f(\tau_1, \ldots, \tau_n|\theta)$ denotes the joint density of the integrated volatility based on the intervals $[(i - 1)\Delta, i\Delta]$ for $i = 1, \ldots, n$. Barndorff-Nielsen and Shephard (2001a) note that the likelihood is intractable and hence makes exact inference difficult. The apparent intractability is attributed to the complex nature of $f(\tau_1, \ldots, \tau_n|\theta)$ which is derived from a random measure. However, we shall show that in fact it is quite easy to deal with $f(\tau_1, \ldots, \tau_n|\theta)$ for more general $\tau$ by means of the Poisson partition calculus methods outlined in James (2005, 2002). Rather, the stumbling block which currently prevents one from integrating out the infinite-dimensional components in the likelihood, is inherent from the Normal distribution of $X_i|\tau_i, \beta, \mu$. Quite simply the Normal assumption yields exponential terms of the form

$$e^{-A_i^2/(2\tau_i)}$$

rather than $e^{-\tau_i A_i^2}$.

In the next section we apply an integral identity to circumvent this problem.

### 1.2 Bessel integral representation of the likelihood

In order to calculate we first employ a Bessel integral identity which we state in more general terms. Suppose that $J_v(x)$ denotes a Bessel function of the first kind of order $v$. Then for $v > -1$, and numbers $a, p$

$$p^{-2(v+1)} e^{-a^2/4p^2} = 2^{v+1} a^{-v} \int_0^\infty J_v(at) t^{v+1} e^{-t^2} dt.$$  

This is a special case of the Weber-Sonine formula. See for instance Andrews, Askey and Roy (1999, p.222) and Watson (1966, p. 394 eq. (4)) for the identity and also those references for Bessel functions. Taking for each $i$, $p^2 = \tau_i / 2$, $a = |A_i|$ and $v = -1/2$, it follows the marginal likelihood is given by,

$$\mathcal{L}(X|\mu, \beta, \theta) = \frac{e^{n A \beta}}{\pi^n} \int_{\mathbb{R}^n_+} \mathbb{E} \left[ \prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} \right] \prod_{i=1}^n \cos(y_i | A_i) dy_i$$

where

$$\mathbb{E} \left[ \prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} \right] = \int_{\mathbb{R}^n_+} \prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} f(\tau_1, \ldots, \tau_n|\theta) \prod_{i=1}^n d\tau_i.$$  

**Remark 2.** Notice that the expression has nothing to do with the distributional properties of $\tau$. The appearance of the cosine in is due to the identity

$$\sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{\Gamma(k + 1/2)k!} = x^{1/2} J_{-1/2}(x) = \sqrt{\frac{2}{\pi}} \cos(x)$$

where $J_{-1/2}(x)$ is a Bessel function of the first kind of order $-1/2$. See for example Andrews, Askey and Roy (1999, p. 202). In other words we are using the classical Fourier-Cosine Integral

$$\frac{1}{\pi} \int_0^\infty \cos(y | A_i) e^{-y^2/\tau_i} dy = \frac{1}{\sqrt{2\pi}} \tau_i^{-1/2} e^{-A_i^2/2 \tau_i}$$
2 Evaluation of the likelihood for general $\tau$

The representations in (5) and (4) allow us to immediately apply the results in James (2005) to obtain a full analysis for quite general $\tau$, which we now describe. Let $N$ denote a Poisson random measure on some Polish space $\mathcal{V}$ with mean intensity,

$$\mathbb{E}[N(dx)|\nu] = \nu(dx).$$

We denote the Poisson law of $N$ with intensity $\nu$ as $\mathbb{P}(dN|\nu)$. The Laplace functional for $N$ is defined as

$$\mathbb{E}[e^{-N(f)}|\nu] = \int_{\mathcal{M}} e^{-N(f)}\mathbb{P}(dN|\nu) = e^{-\Lambda(f)}$$

where for any positive $f$, $N(f) = \int_{\mathcal{V}} f(x)N(dx)$ and $\Lambda(f) = \int_{\mathcal{V}} (1 - e^{-f(x)})\nu(dx)$. $\mathcal{M}$ denotes the space of boundedly finite measures on $\mathcal{V}$ [see Daley and Vere-Jones (1988)]. We suppose that $\tau_i = N(f_i)$, for $i = 1, \ldots, n$ where $f_1, \ldots, f_n$ are positive measurable functions on $\mathcal{V}$. Notice now that the index $i = 1, \ldots, n$ need not correspond to fixed intervals involving $\Delta$. With this in mind, let $(w_1, \ldots, w_n)$ denote arbitrary non-negative numbers. Define for $i = 1, \ldots, n$, functions $R_i(x) = \sum_{j=1}^{i} w_j f_j(x)$ and $\nu R_i(dx) = e^{-R_i(x)}\nu(dx)$. Then all our results will follow from the following special case of James (2005, Proposition 2.1), which can be viewed as an Esscher-type transform,

$$e^{-N(\sum_{i=1}^{n} w_i f_i)}\mathbb{P}(dN|\nu) = \mathbb{P}(dN|\nu R_n)e^{-\Lambda(\sum_{i=1}^{n} w_i f_i)}.$$

Additionally the following decomposition is sometimes useful

$$\mathbb{E}\left[ e^{-N(\sum_{i=1}^{n} w_i f_i)}|\nu \right] = e^{-\Lambda(\sum_{i=1}^{n} w_i f_i)} = \mathbb{E}\left[ e^{-N(w_i f_i)}|\nu \right] \prod_{i=2}^{n} \mathbb{E}\left[ e^{-N(w_i f_i)}|\nu R_{i-1} \right].$$

This expression appears in James (2002) and may be obtained by repeated application of (6).

Now, throughout, for each $n \geq 1$, define $\Omega_n(x) = \sum_{i=1}^{n} (y_i^2/2 + \beta^2/2)f_i(x)$. This is a special case of $\sum_{i=1}^{n} w_i f_i$, with $w_i = (y_i^2/2 + \beta^2/2)$ for $i = 1, \ldots, n$. The following result is immediate from an application of Fubini’s theorem, (5) and the representation (6).

**Theorem 2.1** Suppose that $\tau_i = N(f_i)$ for $i = 1, \ldots, n$ where $N$ is a Poisson random measure on $\mathcal{V}$ with intensity $\nu$. Then setting $w_i = y_i^2/2 + \beta^2/2$ for $i = 1, \ldots, n$, in (2), the likelihood (3) can be expressed as

$$\mathcal{L}(X|\mu, \beta, \theta) = \frac{e^{n\Lambda(\theta)}}{\pi^n} \int_{\mathbb{R}_+^n} e^{-\Lambda(\Omega_n)} \prod_{i=1}^{n} \cos(y_i | A_i) dy_i.$$

\( \square \)

2.1 Posterior distribution of parameters

Theorem 2.1 shows that Bayesian inference for $(\mu, \beta, \theta)$ may be described as follows.

**Proposition 2.1** Suppose that $\tau$ depends on a $d$-dimensional parameter $\theta$. Then if $q(d\theta)$, $q(d\beta)$, $q(d\mu)$ denote independent prior distributions for $(\beta, \mu, \theta)$, their posterior distribution can be written as,

$$q(d\beta, d\mu, d\theta|X) \propto \int_{\mathbb{R}_+^n} \left[ e^{\Lambda_0(\Omega_n) - n\Lambda(\beta)} q(d\theta)q(d\beta) \right] \prod_{i=1}^{n} \cos(y_i | A_i) dy_i q(d\mu),$$

where $\Lambda_0$ denotes the dependence of $\Lambda$ on $\theta$.\( \square \)
2.2 Posterior distribution of the process

The above results describe the behaviour of the finite-dimensional likelihood and parameters. It is useful to also obtain a description of the underlying random process given the data. This allows one to see directly how the data affects the overall process. Moreover, combined with the results in James (2005), it provides a calculus for more general functionals. Define the measure,

\[ Q_{\mathbf{X}}(dy) = \pi^{-n}e^{-[\Lambda(\Omega_n) - n\tilde{A}\beta]} \prod_{i=1}^{n} \cos(y_i|A_i)dy_i/\mathcal{L}(\mathbf{X}|\mu, \beta, \theta). \]

For notational simplicity we suppose that \((\mu, \beta, \theta)\) are fixed. The next result also follows immediately from an application of Fubini’s theorem, (6) and the representation (5).

**Theorem 2.2** Suppose that the distribution of \(\mathbf{X}\) is given by (3), and that \(\tau\) and \(N\) are defined by the specifications in Theorem 2.1. Let \(\Omega_n(x) = \sum_{i=1}^{n}(y_i^2/2 + \beta^2/2)f_i(x)\). Then the posterior distribution of \(N|\mathbf{X}\) is given by the mixture

\[ \int_{\mathbb{R}_+^n} \mathbb{P}(dN|\nu_{\Omega_n})\mathcal{L}_{\mathbf{X}}(dy) \]

which determines the posterior distribution of \(\tau\) and related quantities. \(\mathbb{P}(dN|\nu_{\Omega_n})\) can be viewed as the posterior distribution of \(N\) given the information in \(\mathbf{Y}, \mathbf{X}\) and corresponds to the law of a Poisson random measures with mean intensity

\[ \nu_{\Omega_n}(dx) := e^{-\Omega_n(x)}\nu(dx) = \nu(dx)e^{-\sum_{i=1}^{n}(y_i^2/2 + \beta^2/2)f_i(x)}. \]

\(\square\)

The next result, which gives an expression for the posterior Laplace functional of \(N\), is an immediate consequence of (6) combined with Theorem 2.2.

**Proposition 2.2** The posterior Laplace functional of \(N|\mathbf{X}\), according to the Theorem 2.2, is given by

\[ \int_{\mathbb{R}_+^n} \left[ \int_{\mathcal{M}} e^{-N(f)}\mathbb{P}(dN|\nu_{\Omega_n}) \right] \mathcal{L}_{\mathbf{X}}(dy) = \int_{\mathbb{R}_+^n} e^{-[\Lambda(f + \Omega_n) - \Lambda(\Omega_n)]} \mathcal{L}_{\mathbf{X}}(dy) \]

for \(f\) such that \(\Lambda(f + \Omega_n) < \infty\). \(\square\)

2.3 A general posterior predictive density for the log price

We now define a random variable similar to (2) which can be thought of as representing the log-price and give an explicit expression for its posterior density given \(\mathbf{X}\). The random variable is defined as,

\[ \tilde{\mathbf{X}} = \mu\tilde{\Delta} + \tilde{\tau}\beta + \tilde{\tau}^{1/2}\tilde{\epsilon} \]

where \(\tilde{\Delta}\) denotes a general positive quantity, \(\tilde{\epsilon}\) is a standard Normal random variable independent of all other variables, and \(\tilde{\tau} = N(\tilde{f})\) for some positive function \(\tilde{f}\) such that Laplace transform of \(\tilde{\tau}\) exists.

**Proposition 2.3** Suppose \(N\) and the data structure of \(\mathbf{X}\) is defined as in Theorem 2.2. Let \(\tilde{\mathbf{X}}\) be defined by (7). Denote its marginal density as \(f_{\tilde{\mathbf{X}}}(|\beta, \mu)\) and its posterior density given the data \(\mathbf{X}\) from (6) as \(f_{\tilde{\mathbf{X}}}(|\beta, \mu, \mathbf{X})\). Then the following results hold

(i) \(f_{\tilde{\mathbf{X}}}(|x, \beta, \mu) = \frac{1}{\pi}e^{-(x-\mu\tilde{\Delta})^2} \int_{0}^{\infty} e^{-\Lambda((y^2/2 + \beta^2/2)f)} \cos(y|x - \mu\tilde{\Delta})dy.\)
The posterior density of the log stock price given \( X \) is, \( f_X(x|\beta, \mu, \mathbf{X}) \), given by

\[
\frac{1}{\pi} e^{-(x-\mu_\Delta)^2} \int_{\mathbb{R}_+^n} \left[ \int_0^\infty e^{-[\Lambda(w_{n+1} \hat{f} + \Omega_n) - \Lambda(\Omega_n)]} \cos(y|x - \mu_\Delta) \right] d\mathbf{X}(dy),
\]

where \( w_{n+1} = (y^2/2 + \beta^2/2) \).

**Proof.** Setting \( \tilde{\Omega}_{n+1}(x) = \Omega_n(x) + w_{n+1} \hat{f}(x) \), the results follow from \( \text{Proposition 2.4} \) and Theorem 2.2, using the fact from \( \text{Proposition 2.4} \) that,

\[ e^{-N(w_{n+1} \hat{f})} \mathbb{P}(dN|\nu_{\tilde{\Omega}_n}) = \mathbb{P}(dN|\nu_{\tilde{\Omega}_{n+1}}) e^{-[\Lambda(w_{n+1} \hat{f} + \Omega_n) - \Lambda(\Omega_n)]}. \]

\[ \square \]

### 2.4 Simplifications for a class of \( \tau \) via an inversion formula

We have shown that for \( \tau \) modeled quite generally that its contribution to the likelihood \( \text{(2)} \) is only through the exponent \( \Lambda(\sum_{i=1}^n w_i f_i) \). That is through the form of \( \sum_{i=1}^n w_i f_i \) and \( \nu \). With a view towards choosing \( \tau \) which are the most tractable we present the following interesting result.

**Theorem 2.3** Suppose that for arbitrary non-negative \((w_1, \ldots, w_n)\), and an integer \( J \), there is an array of non-negative numbers \((a_{ij})\) such that \( \Lambda(\sum_{i=1}^n w_i f_i) = \sum_{j=1}^J \Lambda([\sum_{i=1}^n w_i a_{ij}] h_j) \). Where \((h_j)\) are non-negative functions on \( \mathcal{V} \) such that \( \Lambda(\omega h_j) < \infty \) for all \( \omega \geq 0 \). Let \((T_1, \ldots, T_J)\) denote \( J \) independent random variables with respective Laplace transforms \( E[e^{-\omega T_j}] = e^{-\Lambda(\omega h_j)} \) for \( j = 1, \ldots, J \). Moreover \( \mathcal{L}(\mathbf{X}|\mu, \beta, \theta) \) denotes the likelihood given in \( \text{(5)} \). Then,

\[(i) \quad \mathbb{E}[e^{-\sum_{i=1}^n w_i \tau_i}] = \prod_{j=1}^J e^{-\Lambda(\sum_{i=1}^n w_i a_{ij}) [h_j]}. \]

\[(ii) \quad \mathcal{L}(\mathbf{X}|\mu, \beta, \theta) = \mathbb{E} \left[ \prod_{i=1}^n \phi(X_i|\mu \Delta + \beta [\sum_{j=1}^J a_{ij} T_j], [\sum_{j=1}^J a_{ij} T_j]) \right], \quad \text{where the expectation is with respect to the distribution of } (T_1, \ldots, T_J). \]

\[ \square \]

**Proof.** Statement (i) is immediate from the specification of \( \Lambda(\sum_{i=1}^n w_i f_i) \). Statement (i) implies that one may replace \( \sum_{i=1}^n w_i \tau_i \) with \( \sum_{i=1}^n w_i [\sum_{j=1}^J a_{ij} T_j] \). Now setting each \( w_i = (y_i^2/2 + \beta^2/2) \) for \( i = 1, \ldots, n \), one uses \( \text{(6)} \) and \( \text{(5)} \) to conclude the result. \( \square \)

Statement (ii) of Theorem 2.3 allows one to approximate the likelihood by the simulation of \( J \) independent random variables. It also demonstrates that it is rather straightforward to conduct parametric Bayesian or frequentist estimation procedures, where \((T_1, \ldots, T_J)\) are viewed as independent latent variables. The next proposition puts this in a Bayesian framework.

**Proposition 2.4** Suppose that \((T_1, \ldots, T_J)\) depend on external parameters, say \( \theta \). Then assuming a joint prior \( q(\theta, \mu, \beta) \), posterior inference may be obtained based on the model derived from augmenting the likelihood in Theorem 2.3. That is, the joint distribution of \((\mathbf{X}, T_1, \ldots, T_J, \theta, \mu, \beta)\) given by

\[
\prod_{i=1}^n \phi(X_i|\mu \Delta + \beta \sum_{j=1}^J a_{ij} T_j), [\sum_{j=1}^J a_{ij} T_j]) \prod_{j=1}^J f_{T_j}(T_j) q(\theta, \mu, \beta, \beta).
\]

\[ \square \]
3 Tractable expressions

It is noted that, at first glance, one may find it difficult to work with the expressions involving cosines. Here, influenced by some arguments in Devroye (1986a), we give a representation of the likelihood that can be numerically evaluated via the simulation of random variables. First let \( p = \{p_1, \ldots, p_n\} \) denote a vector of positive numbers and for each \( i \), let

\[
H(y_i|p_i) = \frac{2}{\sqrt{2\pi p_i}} e^{-\frac{y_i^2}{2p_i}} \quad \text{for} \quad y_i > 0
\]

denote a half normal density. Now, notice that \( 0 \leq 1 - \prod_{i=1}^n \cos(y_i) \leq 2 \), and

\[
\int_{\mathbb{R}_+^n} \left[ 1 - \prod_{i=1}^n \cos(y_i|A_i) \right] H(y_i|p_i) dy_i = 1 - e^{-\sum_{i=1}^n \frac{\pi y_i^2}{2p_i}} = C_n(A, p)
\]

From these facts we describe a joint density

**Proposition 3.1** Augmenting the expression in (8) leads to a joint density of an array of positive random variables \( Y = \{Y_{1,n}, \ldots, Y_{n,n}\} \) given by,

\[
r_n(y|p) = \frac{1 - \prod_{i=1}^n \cos(y_i|A_i)|} {C_n(A, p)} \prod_{i=1}^n H(y_i|p_i)
\]

Equivalently, for \( k = 1, \ldots, n \), the conditional density of \( Y_{k,n}|Y_{1,n}, \ldots, Y_{k-1,n} \) is proportional to \( [1 - \lambda_k \cos(y_k|A_k)]H(y_k|p_k) \), where \( \lambda_k = e^{-\sum_{i=k+1}^n \frac{\pi y_i^2}{2p_i}} \prod_{i=1}^{k-1} \cos(y_i|A_i) \) for \( k = 2, \ldots, n-1 \), \( \lambda_1 = e^{-\sum_{i=2}^n \frac{\pi y_i^2}{2p_i}} \), and \( \lambda_n = \prod_{i=1}^{n-1} \cos(y_i|A_i) \).

Define the function

\[
\Upsilon_n(\beta, \theta) := \frac{1}{\pi^n} \int_{\mathbb{R}_+^n} e^{-\Lambda(\Omega_n)} \prod_{i=1}^n dy_i = E \left[ \prod_{i=1}^n \frac{e^{-\beta \tau_i}}{\sqrt{2\pi \tau_i}} \right] \leq E \left[ \prod_{i=1}^n \frac{1}{\sqrt{\tau_i}} \right]
\]

These points lead to following representation of the likelihood.

**Proposition 3.2** Suppose that for fixed \( n \), \( E \left[ \prod_{i=1}^n \frac{1}{\sqrt{\tau_i}} \right] < \infty \), then the likelihood in Theorem 2.1 may be written as

\[
e^{\hat{A}\beta} \left[ \Upsilon_n(\beta, \theta) - \frac{C_n(A, p)}{\pi^n} E \left[ \prod_{i=1}^n \frac{e^{-\Lambda(\Omega_n)}}{H(Y_{i,n}|p_i)} \right] \right]
\]

where the random vector \( \{Y_{1,n}, \ldots, Y_{n,n}\} \) has its joint distribution described by proposition 3.1, and \( \Omega_n(x) = \sum_{i=1}^n [(Y_{i,n}^2 + \beta^2)/2]|g_i(x) \)

**Remark 3.** Proposition 3.2 shows that one may approximate the likelihood by simulating random variables described in Proposition 3.1. Such an approach should work well with a Bayesian procedure for estimating the parameters \( (\mu, \beta, \theta) \). Methods to easily sample the random variables in proposition 3.1, may be deduced from Devroye (1986a, b). Alternatively one may sample from the densities \( H(y_i|p_i) \). One may also use other densities.

4 Analysis of the BNS-OU model

In this section we will show how our results apply to the basic integrated volatility model of Barndorff-Nielsen and Shephard (2001a, b). We shall refer to this model as the BNS-OU model. First suppose that \( N \) is a Poisson random measure on \((0, \infty) \times (-\infty, \infty)\) with intensity

\[
\nu(du, dy) = \rho(du)dy
\]
where $\rho$ is the Lévy density of an infinite-divisible random variable, say $T$, with Laplace transform for $\omega \geq 0$,

$$E[e^{-\omega T}] = e^{-\psi(\omega)} \text{ where } \psi(\omega) = \int_0^\infty (1 - e^{-u\omega})\rho(du).$$

Now we model the background driving Lévy process (BDLP), say $z$, as a completely random measure which is expressible in distribution as $z(dt) = \int_0^\infty uN(du, dt)$. Note that for any non-negative function $g$ on $(-\infty, \infty)$, it follows that

$$z(g) = \int_{-\infty}^\infty g(y)z(dy) = N(f_g)$$

where $f_g(u, y) = ug(y)$ on $(0, \infty) \times (-\infty, \infty)$. Additionally,

$$E\left[e^{-z(g)}\right] = \int_\mathbb{M} e^{-N(f_g)}\mathbb{P}(dN|\nu) = e^{-\int_{-\infty}^{\infty} \psi(g(y))dy} = e^{-\Lambda(f_g)}.$$

One may express the Barndorff-Nielsen and Shephard (2001 a, b) integrated OU process as

$$\tau(t) = \lambda^{-1}[1 - e^{-\lambda t}] \int_{-\infty}^0 e^y z(dy) + \int_0^t (1 - e^{-\lambda(t-s)})z(dy)$$

where $v(0) := v_0 = \int_0^\infty e^y z(dy)$. The form in (9) is taken from Carr, Geman, Madan and Yor (2003, p. 365). It follows that for any $s < t$, $\tau(t) - \tau(s) = z(g_{s,t}) = N(f_{s,t})$ where $f_{s,t}(u, y) = ug_{s,t}(y)$ and $\lambda g_{t,s}(y)$ equals,

$$e^{-\lambda s}(1 - e^{-\lambda(t-s)})e^y I_{y \leq 0} + (1 - e^{-\lambda(t-y)})I_{s < y \leq t} + e^{-\lambda s}(1 - e^{-\lambda(t-s)})e^y I_{0 < y \leq s}.$$

The first component in (10) represents the contribution from $v_0$. Specializing this to $s = (i-1)\Delta$ and $t = i\Delta$ one has $\tau_i = z(g_{i_1, i_2}) = N(f_i)$ where $f_i(u, y) = u[g_{i_1, i_2}(y)]$ and

$$g_{i_1, y}(y) = \lambda^{-1}[(1 - e^{-\lambda(i\Delta - y)})I_{(i-1)\Delta < y \leq i\Delta}) + e^{-\lambda(i-1)\Delta}(1 - e^{-\lambda \Delta})e^y I_{y \leq 0}]$$

and

$$g_{i_2, y}(y) = \lambda^{-1}e^{-\lambda(i-1)\Delta}(1 - e^{-\lambda \Delta})e^y I_{0 < y \leq (i-1)\Delta}.$$

Now for $i = 1, \ldots, n$, set $r_i = \lambda^{-1}[\sum_{k=1}^n w_k e^{-\lambda(k-1)\Delta}] (1 - e^{-\lambda \Delta})$. Now notice that for any sequence of numbers, the simplest expression will be obtained by utilizing the following facts.

$$\sum_{j=1}^n w_j [g_{j_1, y}(y) + g_{j_2, y}(y)] = r_1 e^y \text{ for } y \leq 0$$

and for $i = 1, \ldots, n$

$$\sum_{j=1}^n w_j [g_{j_1, y}(y) + g_{j_2, y}(y)] = \zeta(y|w_i, r_{i+1}) \text{ for } (i-1)\Delta < y \leq i\Delta.$$

Where for each $i$, $\zeta(y|w_i, r_{i+1}) = [\lambda^{-1} w_i (1 - e^{-\lambda(i\Delta - y)})] + r_{i+1} e^y$.

**Proposition 4.1** For $0 \leq s < t$, let $\tau(t) - \tau(s)$ be defined by (4) and (14). Then the results of Theorem 2.1 and 2.2 hold with $f_i(u, y) = u[g_{i_1, y}(y) + g_{i_2, y}(y)]$, $w_i = (y^2/2 + \beta^2/2)$, as described in (10) and (12). In particular, using a change of variable,
Specializing this to (15), one has the following distributional equivalence of marginal distributions, 

\( L \) = \( \text{Lancelot F. James} \)

\( \Phi(w_n|\tau_{i+1}) = \int_{1-\lambda \Delta}^{1} \lambda^{-1} \psi(\tau_{i+1} e^{\lambda \Delta}(1-u) + \lambda^{-1}w_{n}u) \frac{du}{1-u}, \text{for } i = 1, \ldots, n-1 \)

\( \Phi(w_n) = \int_{1-\lambda \Delta}^{1} \lambda^{-1} \psi(\lambda^{-1}w_{n}u) \frac{du}{1-u} \)

\( \Phi_{0}(r_{1}) = \int_{0}^{1} \psi(r_{1}u) \frac{du}{u}, \text{where } e^{-\Phi_{0}(r_{1})} = E[e^{-r_{1}v_{0}}] \)

Remark 4. Expressions of the form in [(iii)] of Proposition 3.1 are known to be a key component in option pricing using the BNS-OU model. However explicit calculations have only been given for a few cases. See Barndorff-Nielsen and Shephard (2003), Nicolato and Venardos (2003) and Carr, Geman, Madan and Yor (2003). Note that if for \( y > 0 \), we change the Lebesque measure, \( du \), to \( e^{\lambda u} du \), the calculations for \( \Phi(w_n|\tau_{i+1}) \) for \( i = 1, \ldots, n \), where \( \Phi(w_n|\tau_{n+1}) = \Phi(w_n) \), are greatly simplified. See section 4.2 for a closely related discussion.

5 Analysis of a simple class of models

This last section, which is based on a class of models from section 2.4, examines models which are the most tractable and we believe still flexible enough to be applied to general classes of problems. Implicitly, we are taking a Bayesian nonparametrics viewpoint of seeking random measures as priors which are both flexible in a modeling sense and easily manipulated. For concreteness, we start out with a variation of the Barndorff-Nielsen and Shephard (2001a,b, 2003) integrated OU process \( \tau \). Here we set,

\[ \tau(t) = \lambda^{-1}[(1-e^{-\lambda t}) \int_{-\infty}^{0} e^{\lambda u} z(dy) + \int_{0}^{\lambda t} (1-e^{-y})z(dy)] \]

where again \( v(0) = v_{0} = \int_{-\infty}^{0} e^{\lambda u} z(dy) \). Interestingly from Barndorff-Nielsen and Shephard (2003, p. 282), one has the following distributional equivalence of marginal distributions,

\[ \lambda^{-1} \int_{0}^{\lambda t} (1-e^{-y})z(dy) = \lambda^{-1} \int_{0}^{\lambda t} (1-e^{-\lambda(t-y)})z(d\lambda y) \]

where the right hand side equates with the model in section 3. However, now for \( s < t \),

\[ \lambda[\tau(t) - \tau(s)] = e^{-\lambda s}(1-e^{-\lambda(t-s)}) \int_{-\infty}^{0} e^{\lambda u} z(dy) + \int_{0}^{\lambda t} (1-e^{-y})I_{\{\lambda s < y \leq \lambda t\}} z(dy) \]

Specializing to \( t = \Delta \) and \( s = (i-1)\Delta \) yields

\[ \tau_{i} = \int_{-\infty}^{\infty} g_{i}(y)z(dy) = z(g_{i}) = N(f_{i}) \]

where \( f_{i}(u, y) = u g_{i}(y) \) with

\[ g_{i}(y) = \lambda^{-1}[(1-e^{-y})I_{\{\lambda(i-1)\Delta < y \leq \lambda i \Delta\}} + e^{-\lambda(i-1)\Delta}(1-e^{-\lambda \Delta})e^{\lambda y}I_{y < 0}] \].

Now notice that for any sequence of numbers, the simplest expression will be obtained by utilizing the following facts.

\[ \sum_{j=1}^{n} w_{j} g_{j}(y) = \lambda^{-1}[\sum_{i=1}^{n} w_{i} e^{-\lambda(i-1)\Delta}(1-e^{-\lambda \Delta})e^{\lambda y} \text{ for } y \leq 0} \]
and for $i = 1, \ldots, n$

$$
\sum_{j=1}^{n} w_jg_j(y) = \lambda^{-1}w_i(1 - e^{-\eta_i}) \text{ for } \lambda(i-1)\Delta < y \leq \lambda i \Delta.
$$

More generally suppose that for $t > s$, $\tau(t) - \tau(s) = z(g_{s,t}) = N(f_{s,t})$ where $f_{s,t}(u,y) = ug_{s,t}(y)$ and

$$
g_{s,t}(y) = h_{1,s,t}(y)I_{\{\lambda s < y \leq \lambda t\}} + h_{2,s,t}(y)I_{\{y \leq 0\}},
$$

for $h_{1,s,t}, h_{2,s,t}$ and $F(y)$ non-negative functions satisfying suitable integrability conditions and $h_{2,s,t}, h_{2,s,t}$ a positive quantity not depending on $y$. Hence, one could choose for each $i$, $g_i(y) = h_{1,i}(y)I_{\{(i-1)\Delta < y \leq \lambda i \Delta\}} + h_{2,i}(y)I_{\{y \leq 0\}},$

for arbitrary positive functions $h_{1,i}, h_{2,i}$ whose form is determined by the general difference $\tau(t) - \tau(s)$ for $t > s$. These models all exhibit behavior similar to (16) and (17). That is for any sequence of numbers $(w_1, \ldots, w_n)$, it follows that

$$
\sum_{j=1}^{n} w_jg_j(y) = w_ih_{1,i,y}(y) \text{ for } \lambda(i-1)\Delta < y \leq \lambda i \Delta.
$$

Now let $\nu$ denote a Poisson random measure on $(0, \infty) \times (-\infty, \infty)$ with

$$
\nu(dw, dy) = \rho_1(dw)\eta_1(dy)I_{\{y > 0\}} + \rho_2(dw)\eta_2(dy)I_{\{y \leq 0\}},
$$

where $\rho_1$ and $\rho_2$ are Lévy densities generating Lévy exponents $\psi_1$ and $\psi_2$, and $\eta_1, \eta_2$ are non-negative sigma-finite measures. It follows from (20) that

$$
\int_{0}^{\infty} \psi_1(\sum_{i=1}^{n} w_i g_i(y))\eta_1(dy) + \int_{-\infty}^{0} \psi_2(\sum_{i=1}^{n} w_i g_i(y))\eta_2(dy) = \Phi_0(s_n) + \sum_{i=1}^{n} \Phi_i(w_i)
$$

where

$$
\Phi_i(w_i) = \int_{\lambda(i-1)\Delta}^{\lambda i \Delta} \psi_1(w_i h_{1,i,y}(y))\eta_1(dy) \text{ and } \Phi_0(s_n) = \int_{-\infty}^{0} \psi_2(\sum_{i=1}^{n} w_i h_{2,i,y}(y))F(y)\eta_2(dy)
$$

for $i = 1, \ldots, n$, where $s_n = \sum_{i=1}^{n} w_i h_{2,i,y}(y)$. In the case of (16) and (17), for $\rho_1 = \rho_2$, and $\eta_1(dy) = \eta_2(dy) = dy$, $s_n = [\sum_{i=1}^{n} w_i e^{-\lambda(i-1)\Delta}] (1 - e^{-\lambda \Delta})$. One has for $i = 1, \ldots, n$,

$$
\Phi_i(w_i) = \int_{(1-e^{-\lambda i \Delta})}^{(1-e^{-\lambda(i-1)\Delta})} \psi(\lambda^{-1} w_i u)\frac{du}{1-u} = \int_{\lambda(i-1)\Delta}^{\lambda i \Delta} \psi(\lambda^{-1} w_i(1 - e^{-\eta_i}))dy
$$

and $\Phi_0(s_n) = \int_{0}^{1} \psi(\lambda^{-1} s_n u)\frac{du}{u}$ is the Lévy exponent corresponding to the prior distribution of $v_0$ evaluated at $s_n$.

**Theorem 5.1** Let $N$ denote a Poisson random measure on $(0, \infty) \times (-\infty, \infty)$ with intensity $\nu$ defined in (21). Define $\tau$ by (16) and (17) and the general specification of $\nu$ above. Let $\Phi_i$ for $j = 0, 1, \ldots, n$ denote the quantities defined by (22). Let $(v_0, T_1, \ldots, T_n)$ denote $n + 1$ independent random variables with respective Laplace transforms $e^{-\Phi_i(\omega)}$ for $i = 0, 1, \ldots, n$. Then,
where $X_{s,t}$ is an independent standard Normal distribution. Our results yield an explicit tractable expression of a predictive density of $X_{s,t}$ given previously observed data $X$.

**Proposition 5.2** For $t > s > n\Delta$, let $X_{s,t}$ be defined according to (24). Let $\tau$ be defined by the general specifications in Theorem 5.1 Let $f_{T_1}, \ldots, f_{T_n}$ denote the densities of the corresponding independent random variables. Then the joint distribution of $X_{s,t} \mid X_1, \ldots, X_n$ is given by the formula,

$$
\int_0^\infty \left[ \int_0^\infty \phi(x) \mu(t-s) + \beta z_{s,t}, z_{s,t} \right] f_{T_1}(z_{s,t} - b_{s,t}v) \right] \right] \right] r(X\mid v) f_{v_0}(v) dv
$$

where, $r(X\mid v) = \left[ \prod_{i=1}^n \int_0^\infty \phi(x_i) \mu + \beta z_{i,t} \right] f_{T_i}(z_i - b_{i,t}v) dZ_{i,t}$ / $\mathcal{L}(X\mid \mu, \beta, \theta)$ and $b_{s,t} = h_{2,s,t,\lambda}$. The quantity, $f_{T_1}$, denotes the density of an independent random variable $T_{s,t}$ with law determined by the Lévy exponent $\Phi_{s,t}(w) = \int_{\lambda s}^\Lambda \psi_1(w_{1,i,\lambda}(y)) \eta_1(dy)$. $\square$

**5.2 Example**

The expressions in Theorem 4.1 suggests that an easily analyzed model would arise if $(\eta_0, \tau_1, \ldots, \tau_n)$ were all from GIG class of densities. Here, going back to the variant of the Barndorff-Nielsen and Shephard model characterized by (24), we shall show that the choice of a stable law yields very nice results. Recall that the Lévy exponent of a stable law of index $0 < \alpha < 1$, is such that $\psi(\omega) = \omega^\alpha / \alpha$. Recall also that the case of $\alpha = 1/2$ leads to the Inverse Gamma distribution of index $1/2$, and that the Inverse Gaussian arises from an exponential tilting of this law. Using this fact we arrive at the following result.
Proposition 5.3 Suppose that $\tau$ is specified by \((15)\) with $\nu(ds,dy) = s^{-\alpha-1}/[\Gamma(1 - \alpha)]dsdy$ for $0 < \alpha < 1$. Then the random variables $(v_0, T_1, \ldots, T_n)$, appearing in Theorem 5.1, are independent stable random variables of index $\alpha$. The respective Lévy exponents are,

\begin{enumerate}
  \item \(\Phi_0(\omega) = \omega^\alpha \lambda^{-\alpha} (1 - e^{-\lambda \Delta})^\alpha / \alpha\) and
  \item \(\Phi_i(\omega) = \omega^\alpha \lambda^{-\alpha} \int_{1-i}^{i} e^{-\lambda(1-1)\Delta} \frac{\omega^\alpha}{\alpha(1-\alpha)} du\) for $i = 1, \ldots, n$
  \item If one instead uses $\eta_1(dy) = e^{-y}dy$, then
    \[\Phi_i(\omega) = \frac{\omega^\alpha}{\alpha(\alpha + 1)} \lambda^{-\alpha} [(1 - e^{-\lambda \Delta})^{\alpha+1} - (1 - e^{-\lambda(i-1)\Delta})^{\alpha+1}],\]
    for $i = 1, \ldots, n$.
\end{enumerate}

Notice that the proposition above shows that a change from Lebesque measure to $\eta_1(dy)$ only affects the constants in the Laplace transform and generally preserves the distributional property of the $(T_i)$. This fits into what has been evidenced in Bayesian nonparametric problems where the choice of quantities such as $\eta_1, \eta_2$ are done mainly for computational convenience. The rationale is that viewing the specifications for $\tau$ as a prior model, experience from the Bayesian nonparametric literature suggests that many such choices of $\tau$ will eventually lead to the similar conclusions in the presence of enough data $X$. Note however that we do not advocate removing the dependence of the Lévy exponents on $(i, \Delta)$, as this is related to the data. We close by noting it is always possible to arrange for the random variables $(v_0, T_1, \ldots, T_n)$ to be self-decomposable by choosing $\psi_1$ and $\psi_2, \eta_1$ and $\eta_2$ such that the random variables are Generalized Gamma Convolutions (GGC). See Thorin (1977) and Bondesson (1979, 1992) for this rich class of models. That is $\Phi_i(\omega) = \int_{\alpha}^{c_i} \ln(1 + \omega/y)\mathcal{U}(dy)$, for $a_i, c_i$ depending on $\Delta, \mathcal{U}$, with $\mathcal{U}(0) = 0$, is called a Thorin function or measure. This applies more generally to the models described in section 2.4. We shall leave it to the reader to investigate which of the general models discussed in section 2 are most suitable to their particular application.

6 Extension: SV Likelihood models with correlated jumps in price, leverage effects models.

Recall that the Barndorff-Nielsen and Shephard (2001a,b) OU process $v(t)$, which models the instantaneous volatility, satisfies the differential equation

$$dv(t) = -\lambda v(t) + dz(\lambda t),$$

where the process $z$ is defined in section 3, and hence the volatility possesses jumps. An important extension of the model in \((14)\) and hence to our general framework described in section 2, is where one includes jumps in the log-price model which are correlated with the the volatility $v$. These types of continuous time models, wherein Duffie, Pan and Singleton (2000) is an early reference, serve to incorporate the leverage effect discussed in for instance Black (1976) and Nelson (1991). We shall be rather brief on this growing literature and refer the reader to the works of Eraker, Johannes and Polson (2003), Duan, Ritchken and Sun (2004) and Duffie, Singleton and Pan (2000), for more extensive background and rationale for these types of models and its parametric variations.
6.1 BNS-OU SV model for leverage effects

Barndorff-Nielsen and Shephard (2001a, eq. 8) describe this type of extension as follows,

\[ dx^\ast(t) = (\mu + \beta v(t))dt + \nu^{1/2}(t)dw(t) + \rho d[z(\lambda t) - \mathbb{E}[d\lambda(t)]] \]

assuming of course that \( \mathbb{E}[z(\lambda t)] < \infty \). One can incorporate modifications to relax this condition. It follows that obviously the log price and the volatility are negatively correlated if \( \rho < 0 \). Thus modeling the leverage effect that a fall in price results in an increase in future volatility. Barndorff-Nielsen and Shephard (2001a, section 4) discuss further details of this model. The likelihood model based on \( \text{(25)} \) was not explicitly discussed in that paper, and moreover is considered even more challenging. However as we shall show, this extension and a variety of natural extensions of the models described in section 2, incorporating a leverage type effect, are easily handled by the type of methodology we have presented so far.

First, assuming a similar framework as in section 1.1, and using the BNS-OU model described in section 3, note that

\[ z_i := z(g_{i,3}) = z(i\Delta \lambda) - z(\lambda(i - 1)\Delta) = \int_0^\infty \int_0^\infty I_{\{\lambda(y) \leq \lambda(i)\Delta\}} uN(du, dy), \]

where \( N \) is a Poisson random measure with intensity \( \nu(du, dy) = \rho(du)dy \) on \((0, \infty) \times (-\infty, \infty)\), and \( g_{i,3}(y) = I_{\{\lambda(y) \leq \lambda(i)\Delta\}} \). Assuming a finite first moment, one has \( \mathbb{E}[z_i] = \Delta \int_0^\infty u\rho(du) \). Hence the model \( \text{(25)} \) implies that \( X_i | \tau_i, z_i, \beta, \mu \) are conditionally independent with

\[ X_i = \mu\Delta + \tau_i\beta + z(g_i,3) + \rho(z_i - \mathbb{E}[z_i]), \]

which may be rewritten for each \( i \), as \( X_i = (\mu\Delta - \rho\mathbb{E}[z_i]) + (\tau_i\beta + \rho z_i) + z(g_i,3) + \sqrt{z(g_i,1 + g_i,2)}\epsilon_i \). Hence on may write the expression in \( \text{(26)} \) as,

\[ X_i = (\mu\Delta - \rho\mathbb{E}[z_i]) + (z(g_i,1 + g_i,2)\beta + \rho z(g_i,3)) + \sqrt{z(g_i,1 + g_i,2)}\epsilon_i, \]

which obviously may be further expressed in terms of a common Poisson random measure.

6.2 A General class of likelihoods which incorporate leverage type effects

We note that from our point of view a model such as \( \text{(24)} \) poses no additional complications. Similar to section 2, we will obtain exact expressions for likelihoods of quite general extensions of models with correlated jumps in price and volatility. As before, for a general Poisson random measure on \( \mathcal{Y} \), with intensity \( \nu \), let \( \tau_i = N(f_i) \) for \( i = 1, \ldots, n \). Additionally, for real-valued functions \( \varphi_1, \ldots, \varphi_n \) on \( \mathcal{Y} \), each satisfying the condition \( \Lambda(\varphi_i) < \infty \), for \( i = 1, \ldots, n \), define \( \gamma_i := N(\varphi_i) \). Now a general version of \( \text{(27)} \) is given by the case of conditionally independent

\[ X_i = \mu\Delta + \rho\gamma_i + \tau_i\beta + z(g_i,3) + \sqrt{z(g_i,1 + g_i,2)}\epsilon_i, \]

Let \( \mathcal{Y} = \mathbb{R} \times \mathbb{R}_+ \), and assume that \( N \) depends on a parameter \( \theta \). Then the likelihood of \( X | \mu, \beta, \theta, \rho \) determined by \( \text{(28)} \) can be expressed as

\[ \mathcal{L}(X | \mu, \beta, \theta, \rho) = \left( \prod_{i=1}^n \phi(X_i | \mu\Delta + \rho_1\gamma_i + \beta \tau_1, \tau_i) \right) f((\tau_1, \gamma_1), \ldots, (\tau_n, \gamma_n)) \theta \prod_{i=1}^n d\gamma_i d\tau_i \]

where,

\[ \phi(X_i | \mu\Delta + \rho_1\gamma_i + \beta \tau_i, \tau_i) = e^{(A_1 - \rho_1\gamma_i)\beta} \frac{1}{\sqrt{2\pi}} \tau_i^{-1/2} e^{-(A_1 - \rho_1\gamma_i)^2/(2\tau_i)} e^{-\tau_i \beta^2/2} \]
The difficulty in evaluating this likelihood now manifests itself in the term in brackets where both $\gamma_i$ and $\tau_i$ are functionals of a common Poisson random measure, and moreover are not pairwise independent across $i$. Clearly one could apply (4) however the cosine representation, now involving random terms, would generally lead to expressions which are less aesthetically pleasing. Here we will simply use an identity deduced from the characteristic function of a Normal random variable. This is equivalent to (4). We close by describing the explicit likelihood.

**Theorem 6.1** Suppose that $N$ is a Poisson random measure with intensity $\nu$ on $\mathcal{V}$. Then define the complex valued function $\Upsilon_n(x) = \sum_{i=1}^{n} [\rho(\beta + \xi y_i)](x)$ on $\mathcal{V}$, where $\xi$ denotes the imaginary number. Then defining $\Omega_n(x)$ as in Theorem 2.1 with $w_i = y_i^2/2 + \beta^2/2$ for $i = 1, \ldots, n$, the likelihood (29) can be expressed as

$$
\mathcal{L}(\mathbf{X}|\mu, \beta, \theta, \rho) = e^{-nA\beta/(2\pi)} \int_{\mathbb{R}^n} e^{-\Lambda(\Omega_n + \Upsilon_n)} \prod_{i=1}^{n} e^{x_1 y_i} dy_i.
$$

where $(\varphi_i)$ and $(f_i)$ are chosen such that $\Lambda(\Omega_n + \Upsilon_n) < \infty$. □

**Proof.** Here we use the fact that for each $i$ one has the identity deduced from the characteristic function of a Normal distribution, with mean 0 and variance $2/\tau_i$, evaluated at $\varpi_i = A_i - \rho \gamma_i$. That is,

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{x_1 y_i - \tau_i y_i^2/2} dy_i = \frac{1}{\sqrt{\tau_i}} e^{-(\varpi_i)^2/2\tau_i}
$$

Now similar to the results in section 2, apply an appropriate substitution, Fubini’s theorem and the fact that $\rho \gamma_i(\beta + \xi y_i) + \tau_i w_i = N(\rho(\beta + \xi y_i)\varphi_i + w_i f_i)$. That is after rearranging terms it remains to calculate the expectation of $e^{-N(\Omega_n + \Upsilon_n)}$ □

**Remark 5.** Again we note that similar to the likelihood in Theorem 2.1, the likelihood incorporating a generalized notion of leverage effects in Theorem 5.1 can be easily evaluated by classical numerical integration. Additionally although we have concentrated on the Barndorff-Nielsen and Shephard (2001a, b) models, our framework covers a large class of popular models in the literature, which can now be be analyzed in a likelihood framework with leverage effects. For some examples, see Carr, Geman, Madan, and Yor (2003). We note further that since we used an identity that does not depend on the distributional features of the Poisson linear functionals the results can be easily adapted to other processes with for instance possible additional Gaussian components.

**Acknowledgements** This paper was heavily influenced by my interactions with John W. Lau, to whom I extend my thanks. Thanks also to Sam Wong for his support and stimulating conversation related to this topic. Thanks to Albert Lo for pointing out the literature on models incorporating leverage effects.

**References**

Andrews, G., Askey, R. and Roy, R. (1999). *Special functions. Encyclopedia of Mathematics and its Applications*, 71. Cambridge University Press, Cambridge.

Barndorff-Nielsen, O.E. and Shephard, N. (2001a). Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. Royal Statist. Soc., Series B* 63 167-241.

Barndorff-Nielsen, O.E. and Shephard, N. (2001b). Modelling by Lévy processes for financial econometrics. In Lévy processes. Theory and applications. Edited by Ole E. Barndorff-Nielsen, Thomas Mikosch and Sidney I. Resnick. p. 283-318. Birkhäuser Boston, Inc., Boston, MA.
THORIN, O. (1977). On the infinite divisibility of the lognormal distribution. *Scand. Actuar. J.* 3 121-148.

WATSON, G. N. (1966). *A treatise on the theory of Bessel functions. Paperback Edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge.*

YOR, M. (1992). On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.* 24 509-531.

LANCELOT F. JAMES  
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY  
DEPARTMENT OF INFORMATION AND SYSTEMS MANAGEMENT  
CLEAR WATER BAY, KOWLOON  
HONG KONG  
lancelot@ust.hk