Local cohomology modules of bigraded Rees algebras

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Dedicated to Prof. R. C. Cowsik on the occasion of his sixtieth birthday

Abstract. Formulas are obtained in terms of complete reductions for the bigraded components of local cohomology modules of bigraded Rees algebras of 0-dimensional ideals in 2-dimensional Cohen-Macaulay local rings. As a consequence, cohomological expressions for the coefficients of the Bhattacharya polynomial of such ideals are obtained.

1. Introduction

Let \((R, \mathfrak{m})\) be a \(d\)-dimensional local ring. Let \(I\) and \(J\) be \(\mathfrak{m}\)-primary ideals of \(R\). Let \(\lambda\) denote length. The function \(B(r, s) = \lambda(R/I^r J^s)\) is given by a polynomial \(P(r, s)\) for large values of \(r\) and \(s\) \([BH]\). The function \(B(r, s)\) is called the Bhattacharya function of \(I\) and \(J\) and the polynomial \(P(r, s)\) is called the Bhattacharya polynomial of \(I\) and \(J\). The polynomial \(P(r, s)\) has total degree \(d\) \([BH]\), and it can be written as

\[
P(r, s) = \sum_{0 \leq i, j \leq d} e_{ij} \binom{r}{i} \binom{s}{j}.
\]

The integers \(e_{ij}\) for which \(i + j = d\) are called the mixed multiplicities of \(I\) and \(J\).

Associated to \(I\) and \(J\) is the bigraded Rees algebra, \(R = \bigoplus_{r,s \geq 0} I^r J^s R_\mathfrak{m}^r R_\mathfrak{m}^s\). Here \(t_1\) and \(t_2\) are indeterminates. Put \(R_{++} = I t_1 J t_2 R\). Let \(H_{R_{++}}(R)\) denote the local cohomology module of \(R\) with support in \(R_{++}\).

In this paper we study relationships among the Bhattacharya function \(B(r, s)\), the Bhattacharya polynomial \(P(r, s)\) and local cohomology modules \(H_{R_{++}}(R)\) and obtain cohomological expressions for all the coefficients of \(P(r, s)\) when \(R\) is a two-dimensional Cohen-Macaulay local ring.

A relationship between Hilbert coefficients and local cohomology was first observed by Grothendieck and Serre. Their formula expresses the difference between Hilbert polynomial and the Hilbert function of a graded algebra in terms of the lengths of graded components of local cohomology modules with support in the maximal homogeneous ideal \([BH]\), Theorem 4.4.3].

Sally, in \([S1]\), studied the Hilbert function of the maximal ideal \(\mathfrak{m}\) of a Cohen-Macaulay local ring and its relation with the local cohomology modules of the Rees algebra \(R(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n t^n\) with support in \(\mathfrak{m} t R(\mathfrak{m})\). Before we state her results, we recall the concept of reductions of ideals introduced by Northcott and Rees in \([NR]\). An ideal \(J\) of a Noetherian local ring \((R, \mathfrak{m})\) is said to be a reduction of an ideal \(I\) if \(J \subseteq I\) and \(J^m = I^{m+1}\) for \(m \gg 0\). We say \(J\) is a minimal reduction of \(I\) if \(J\) is minimal with respect to inclusion among all reductions of \(I\). If \(R/\mathfrak{m}\) is infinite, then all minimal reductions of \(I\) are generated by same number of elements, \(l(I)\),
called the analytic spread of $I$. The analytic spread of $I$ is equal to the dimension of the fiber cone $F(I) = \oplus_{n \geq 0} I^n/mI^n$. We have $\text{ht}(I) \leq l(I) \leq \min \{ \mu(I), \dim(R) \}$, [R1]. Here $\text{ht}(I)$ denotes the height of $I$ and $\mu(I) = \dim_{R/m} I/mI$ denotes the minimum number of generators for $I$.

Let $x = x_1, \ldots, x_d \in m$ be a minimal reduction of $m$. Let $(x)^{[k]}$ denote the ideal $(x_1^k, \ldots, x_d^k)$. For a graded module $M$ we will use the symbol $M_n$ or $[M]_n$ for the $nth$ graded component of $M$. Sally showed [S1] that

$$[H^d_{\tilde{R}(m)}(R(m))]_n \simeq \lim_{k \to \infty} m^{d^2+n}/(x)^{[k]}m^{d(d-1)k+n}.$$  

Sally showed [S1] that the modules appearing on the right hand side of the above equation are isomorphic for large $k$. The maps in the directed system of modules appearing in the above direct limit are given by multiplication by $x_1x_2 \ldots x_d$. The above results were generalized for $m$-primary ideals, in dimension 2, by Sally [S2], and for Hilbert filtration of ideals in a $d$-dimensional Cohen-Macaulay local ring by Blancafort in her thesis, [D]. By Hilbert filtration of ideals $\{I_n\}$ we mean a descending sequence of ideals of $(R, m)$ such that $I_nI_n \subseteq I_{n+1}$ for all $n$, $I_1I_n = I_{n+1}$ for large $n$, and $I_1$ is $m$-primary.

It is natural to ask for analogues of these results for the bigraded Rees algebra $R$. We find the analogues in dimension 2. We have studied the case of arbitrary dimension in [JY] where completely different methods are employed. In dimension two we can approach the results in a self contained manner from the first principles. To state these analogues we recall the concept of complete reduction introduced by Rees in [R3]. Let $(R, m)$ be a Noetherian local ring of dimension $d$ and $I$, $J$ be $m$-primary ideals of $R$. Let $x_1, \ldots, x_d \in I$ and $y_1, \ldots, y_d \in J$ and let $z_i = x_iy_i$ for $i = 1, \ldots, d$. Then the system of elements $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ is said to be a complete reduction of $(I, J)$ if $(z_1, \ldots, z_d)$ is a reduction of $IJ$. D. Rees showed that the complete reductions exist if the local ring has infinite residue field.

Let $(R, m)$ be a 2-dimensional local ring and $I$, $J$ be $m$-primary ideals of $R$. Let $(x, y; z, w)$ be a complete reduction of $(I, J)$. We will show in section 4 that

$$[H^2_{B_+}(R)]_{(r, s)} = \lim_{k \to \infty} \frac{J^{2k+r}J^{2k+s}}{(xz, yw)^{[k]}J^{k+r}J^{k+s}}.$$  

We will also show that the modules appearing on the right hand side of the above equation are isomorphic for $k \gg 0$. Sally, [S2] proved,

$$H^2_{B_+}(B)_n = \lim_{k \to \infty} J^{2k+n}/(x)^{[k]}J^{k+n}$$  

where $B = R[It]$ and $B_+$ is the positively graded ideal of $B$. She also showed that [S2, Proposition 2.4] for all $n \geq 0$,

$$\lambda(H^2_{B_+}(B)_n) = P_I(n) - \lambda(R/I^n).$$  

Here $\tilde{I}^n$ denotes the Ratliff-Rush closure of $I^n$ (see section 2) and $P_I(n)$ denotes the Hilbert-Samuel polynomial corresponding to the function $\lambda(R/I^n)$. Following [S2], write $P_I(n)$ as

$$P_I(n) = e(I)\binom{n}{2} + a_1(I)n + a_2(I).$$  

In Section 3 we will provide a simple and short proof, in dimension two, of a theorem of Rees concerning mixed multiplicities and complete reductions [R3]. The purpose of providing this proof is to make this paper self-contained. In section 4 we will show that in dimension two, for $r, s \geq 0$,

$$P(r, s) = \lambda\left([H^2_{R^{++}}(R)]_{(r, s)}\right) + \lambda(R/I^nJ^s).$$
Blancafort obtained an expression for $H^1_{R(I_+)}(R(I))_n$ where $I$ is an $m$-primary ideal of a Cohen-Macaulay local ring of dimension at least 2. She showed that for $n \geq 0$

$$H^1_{R(I_+)}(R(I))_n \cong \tilde{I}/I^n.$$  

We prove, in section 4, that for $r, s \geq 0$,

$$H^1_{R^{++}}(R(r,s)) \cong \tilde{I}^r J^s / I^r J^s.$$  

Sally obtained the following cohomological expressions for the coefficients of the Hilbert-Samuel polynomial $P_m(n)$:

1. $\lambda \left( H^2_{R(m)}(R(m))_0 \right) = a_2(m)$.
2. $\lambda \left( H^2_{R(m)}(R(m))_1 \right) = a_1(m) + a_2(m) - 1$.
3. $\lambda \left( H^2_{R(m)}(R(m))_{-1} \right) = e(m) - a_1(m) + a_2(m)$.

The above formulas have also been deduced in [JoV]. We obtain similar cohomological expressions for the Bhattacharya coefficients as a consequence of the expressions obtained for $H^1_{R^{++}}(R)$ and $H^2_{R^{++}}(R)$. We end the article with an example to show that our results are no longer true if we remove the Cohen-Macaulay hypothesis on the ring.

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2. Ratliff-Rush closure of products of ideals

Since the formulas for the local cohomology modules $H^1_{R^{++}}(R)$ involve Ratliff-Rush closure of products of ideals, we develop their basic properties in this section. Let $R$ be a Noetherian ring and $I$ be an ideal of $R$. The stable value of the sequence $\{I^{n+1} : I^n\}$ is called the Ratliff-Rush closure of $I$, denoted by $\tilde{I}$. An ideal $I$ is said to be Ratliff-Rush if $\tilde{I} = I$. The following proposition summarizes some basic properties of Ratliff-Rush closure found in [RR].

Proposition 2.1. Let $I$ be an ideal containing a regular element in a Noetherian ring $R$. Then

1. $I \subseteq \tilde{I}$ and $\tilde{(I)} = \tilde{I}$.
2. $(\tilde{I})^n = I^n$ for $n \gg 0$. Hence if $I$ is $m$-primary, the Hilbert polynomial of $I$ and $\tilde{I}$ are same.
3. $(\tilde{I}^n) = I^n$ for $n \gg 0$.
4. If $(x_1, \ldots, x_g)$ is a reduction of $I$, then $\tilde{I} = \cup_{n \geq 0} I^{n+1} : (x_1^n, \ldots, x_g^n)$.

In [JoV], the authors studied some of the properties of Ratliff-Rush closure of products of ideals. We recall some of the results proved there. For the sake of completeness, we include the proof of the next result.

Lemma 2.2. Let $(R, m)$ be a $d$-dimensional Noetherian local ring with infinite residue field and $I, J$ ideals of $R$. Then

1. $\tilde{IJ} = \bigcup_{r,s \geq 0} I^{r+1} J^{s+1} : I^r J^s$.
2. $\tilde{I^a J^b} = \bigcup_{k \geq 0} I^{a+k} J^{b+k} : I^k J^k$. 

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(3) Let \( I \) and \( J \) are \( \mathfrak{m} \)-primary ideals with a reduction \((\mathbf{z}) = (z_1, \ldots, z_d)\) of \(IJ\) obtained from a complete reduction of \(I\) and \(J\). Put \(\mathbf{z}[^k] = (z_1^k, \ldots, z_d^k)\) then

\[
\widehat{I^a J^b} = \bigcup_{k \geq 0} I^{a+k} J^{b+k} : \mathbf{z}[^k].
\]

**Proof.** (1) Let \( x \in \widehat{IJ} \), then \( xI^n J^m \subseteq I^{n+1} J^{m+1} \) for some \( n \). Conversely if \( xI^n J^m \subseteq I^{n+1} J^{m+1} \) for some \( r, s \geq 0 \) then for \( n = \max\{r, s\} \), \( xI^n J^m \subseteq I^{n+1} J^{m+1} \). Therefore \( x \in \widehat{IJ} \).

(2) Let \( z \in (\widehat{I^a J^b}) \) then for some \( r, s \), by (1), we have \( zI^{ar} J^{bs} \subseteq I^{ar+a} J^{bs+b} \). Set \( k = \max\{ar, bs\} \). Then \( zI^k J^k \subseteq I^{a+k} J^{b+k} \) and hence \( z \in I^{a+k} J^{b+k} : I^k J^k \). Let \( zI^k J^k \subseteq I^{a+k} J^{b+k} \) for some \( k \). We may assume that \( k = nab \) for \( n \gg 0 \). Therefore \( z \in I^{nab+a} J^{nab+b} : I^{nab}. \)

(3) Suppose \( y \in (\widehat{I^a J^b}) \). Then for some \( k, yI^k J^k \subseteq I^{a+k} J^{b+k} \), by (2). Since \( \mathbf{z}[^k] \subseteq I^k J^k \), we have \( y\mathbf{z}[^k] \subseteq I^{a+k} J^{b+k} \). Now, let \( yz_i^k \in I^{a+k} J^{b+k} \) for \( i = 1, \ldots, d \). Write the reduction equation of \(IJ\) with respect to \( \mathbf{z} \): \((IJ)^{m+n} = z^m (IJ)^n \) for all \( m \geq 0 \) and \( n \) large. Then, for \( r \) large, \((IJ)^{r+dk} = (\mathbf{z})^{dk} I^r J^r \). Therefore

\[
yI^{r+dk} J^{r+dk} = y(\mathbf{z})^{dk} I^r J^r = \sum \mathbf{z}_i^{dk} \subseteq I^{a+dk} J^{b+dk} I^r J^r \subseteq I^{a+dk} J^{b+dk} I^r J^r
\]

Hence \( y \in (\widehat{I^a J^b}) \), by (2).

\[\square\]

3. Complete reductions and mixed multiplicities in dimension 2

In the next section, while deriving a formula for the second local cohomology module of the bigraded Rees algebra \( \mathcal{R} \) of two \( \mathfrak{m} \)-primary ideals \( I \) and \( J \) in a two dimensional Cohen-Macaulay local ring \((R, \mathfrak{m})\), we will use a result of Rees linking mixed multiplicities and complete reductions. Recall that a set of elements \((x_1, x_2, \ldots, x_d)\) in a local ring \((R, \mathfrak{m})\) of dimension \( d \) is called a joint reduction of \( I \) if \( x_1 I_2 \cdots \cdots I_d \) is a reduction of \( I_1 I_2 \cdots \cdots I_d \).

Rees showed \([R2]\) that if \((x_1, x_2, \ldots, x_d)\) is a joint reduction of \((I_1, I_2, \ldots, I_d)\) where \( I_1 = I_2 = \cdots = I_i = I \) and \( I_1 = I_2 = \cdots = I_i = I_j = I, i + j = d, \) then \( e_{ij} = e(x_1, x_2, \ldots, x_d) \). The converse is true in quasi-unmixed local rings and it was proved in dimension 2 by Verma \([Sw1]\) and in any dimension by Swanson in \([Sw2]\). We will need the following version for complete reductions.

**Theorem 3.1.** Let \((R, \mathfrak{m})\) be a 2-dimensional local ring and \( I \) and \( J \) be \( \mathfrak{m} \)-primary ideals. Let \((x, y; z, w)\) be a complete reduction of \((I, J)\). Then

\[
e(x, w) = e(y, z).
\]

**Proof.** First note that since \((xz, yw)\) is a reduction of \(IJ\), there exists an \( n \) such that \((xz, yw)(IJ)^n \subseteq (IJ)^{n+1} \). Hence

\[
xI^n J^{n+1} + wJ^n I^{n+1} = yI^n J^{n+1} + zJ^n I^{n+1} = (IJ)^{n+1}.
\]

Therefore \((x, w)\) and \((y, z)\) are both joint reductions of \((I, J)\). It is enough to prove that \( e(y, z) = e_{11} \). Consider the \( R \)-module homomorphism

\[
\phi : R/I^n \oplus R/J^n \longrightarrow (y^n, z^n)/(y^n J^n + z^n I^n)
\]

given by \(\phi(\bar{r}, \bar{s}) = rz^n + sy^n + (y^n J^n + z^n I^n)\). Since \(\phi\) is surjective,

\[
\lambda(R/y^n J^n + z^n I^n) \leq \lambda(R/(y^n, z^n)) + \lambda(R/I^n) + \lambda(R/J^n).
\]
Divide by $n^2/2$ and take the limit as $n \to \infty$, to get
\[
\lim_{n \to \infty} \lambda \left( \frac{(R/(y^nJ^n + z^nI^n))}{n^2/2} \right) \leq \lim_{n \to \infty} \lambda \left( \frac{R/(y^n, z^n)}{n^2/2} \right) = e(I) + e(J).
\]

By Lech’s Lemma [M2, Theorem 14.12], the first term on the right hand side of the above inequality is $2e(y,z)$ and the left hand side is bounded below by
\[
\lim_{n \to \infty} \frac{\lambda(R/(yJ + zI^n))}{n^2/2} = e(yJ + zI).
\]

As $yJ + zI$ is a reduction of $IJ$, $e(IJ) = e(yJ + zI)$. Hence $e(IJ) \leq 2e(y,z) + e(I) + e(J)$. Since $e(IJ) = e(I) + 2e_{11} + e(J)$, by the definition of the Bhattacharya polynomial, it follows that $e_{11} \leq e(y,z)$. By repeated application of [L, Section 5], we get
\[
e(x,y) + e(x,w) + e(z,y) + e(z,w) = e(xz,yw) = e(IJ) = 2e_{11} + e(I) + e(J).
\]

Hence $2e_{11} = e(x,w) + e(z,y)$. As $e_{11} \leq e(y,z)$ and $e_{11} \leq e(x,w)$, we get $e_{11} = e(x,w) = e(y,z)$.

\[
\square
\]

4. Local cohomology of bigraded Rees algebras

For the rest of the article, let $(R,m)$ denote a 2-dimensional Cohen-Macaulay local ring with infinite residue field, unless stated otherwise. Let $I$ and $J$ be m-primary ideals of $R$ and $(x,y,z,w)$ be a complete reduction of $I$ and $J$.

Let $R = R[It_1, Jt_2] = \oplus_{r,s \in \mathbb{N}} I^r J^s t_1^r t_2^s$ denote the bigraded Rees algebra of $R$ with respect to $I$ and $J$. Let $R(k,k)$ denote the ring $R$ with the $(r,s)$th graded piece $R(k,k)_{(r,s)} = R_{(r+k,s+k)}$. We explain our method for computing the formulas for the local cohomology modules of the Rees algebra $R$. Consider the complex:
\[
F^k: 0 \to R \xrightarrow{\alpha_k} R(k,k)^2 \xrightarrow{\beta_k} R(2k,2k) \to 0,
\]
where the maps are defined as,
\[
\alpha_k(1) = ((xzt_1 t_2)^k, (yzt_1 t_2)^k), \quad \beta_k(u,v) = (yzt_1 t_2)^k u - (xzt_1 t_2)^k v.
\]
The twists are given so that $\alpha_k$ and $\beta_k$ are degree zero maps. We have the following commutative diagram:
\[
\begin{array}{ccc}
0 & \to & R \\
& \alpha_k \downarrow & \beta_k \\
& g_k \downarrow & h_k \\
0 & \to & R(k,k)^2 \\
& \alpha_k+1 \downarrow & \beta_{k+1} \\
0 & \to & R(k+1,k+1)^2 \\
& \to & R(2k+2,2k+2) \\
& \to & 0
\end{array}
\]

Here the map $f_k$ is the identity map, $g_k(1,0) = (xzt_1 t_2, 0)$, $g_k(0,1) = (0, yzt_1 t_2)$ and $h_k(1) = xz yt w(t_1 t_2)^2$. Thus the cohomology modules $H^i(F_k)$ of the complex $F_k$ form a directed system for each $i$.

Let $C = (xz,yw)$ and $Ct = (xzt_1 t_2, ywt_1 t_2)$. Since both $R_++$ and $Ct$ have same radical and $H^i_C(R) = \lim_k H^i(F_k) = H^i_{R_+} (R)$ for all $i$ by [BS, Theorem 5.2.9].

We begin by deriving an expression for the bigraded components of the first local cohomology module of the bigraded Rees algebra.

**Proposition 4.1.** $[H^1_{R_+} (R)]_{(r,s)} \cong I^r J^s / I^r J^s$ for all $r,s \geq 0$.

**Proof.** By the above discussion we have:
\[
[H^1_{R_+} (R)]_{(r,s)} = \lim_{k} \frac{(\ker \beta_k)_{(r,s)}}{(\im \alpha_k)_{(r,s)}}.
\]
We show that for large \( k \), \((\ker \beta_k)_{(r,s)} \cong \widetilde{I^rJ^s}\) and \((\im \alpha_k)_{(r,s)} \cong I^rJ^s\). Let 
\([ut_1^{r+k}t_2^{s+k}, vt_1^{r+k}t_2^{s+k}] \in (\ker \beta_k)_{(r,s)}\) for \( k \gg 0 \). Then \((yw)^ku - (xz)v = 0\). Since \((xz, yw)\) is a regular sequence in \( R \), we have \( u = p(xz)^k\) for some \( p \in R \) and \( v = p(yw)^k\). Therefore 
\((ut_1^{r+k}t_2^{s+k}, vt_1^{r+k}t_2^{s+k}) = p((xz)t_1^{r+k}t_2^{s+k}, (yw)t_1^{r+k}t_2^{s+k})\).

Hence \((\ker \beta_k)_{(r,s)} \subseteq \{p((xz)t_1^{r+k}t_2^{s+k}, (yw)t_1^{r+k}t_2^{s+k}) \mid p \in R\}\). The reverse inclusion is clear. Therefore 
\((\ker \beta_k)_{(r,s)} = \{p((xz)t_1^{r+k}t_2^{s+k}, (yw)t_1^{r+k}t_2^{s+k}) \mid p \in R\}\).

Consider the map \( \delta : (\ker \beta_k)_{(r,s)} \to I^{r+k}J^{s+k} : (xz, yw)^k\) defined by 
\(\delta(p((xz)t_1^{r+k}t_2^{s+k}, (yw)t_1^{r+k}t_2^{s+k})) = p\).

It is clear that \( \delta \) is an isomorphism. Hence, by Lemma 2.2(3), \( \ker \beta_k(r,s) \cong \widetilde{I^rJ^s}\).

To see that \((\im \alpha_k)_{(r,s)} \cong I^rJ^s\), consider the map \( \tilde{\phi} : I^rJ^s \to (\im \alpha_k)_{(r,s)}\) defined by \(\tilde{\phi}(p) = (p(xz)t_1^{r+k}t_2^{s+k}, p(yw)t_1^{r+k}t_2^{s+k})\). Since \(xz\) and \(yw\) are regular, \(\tilde{\phi}\) is an isomorphism.

Therefore 
\([H_{R_{++}}^1(R)]_{(r,s)} \cong \widetilde{I^rJ^s}/I^rJ^s\).

Next we obtain an expression for the second local cohomology module of bi-graded Rees algebra in terms of complete reductions. In this result we do not need the usual Cohen-Macaulay hypothesis.

**Proposition 4.2.** Let \((R, \mathfrak{m})\) be a Noetherian local ring of dimension 2 and \(I, J\) be \(\mathfrak{m}\)-primary ideals of \(R\). Let \((x, y, z, w)\) be a complete reduction of \((I, J)\), where \(x, y \in I\) and \(z, w \in J\). Set \(R_{++} = (xzt_1t_2, ywt_1t_2)\). Then 
\([H_{R_{++}}^2(R)]_{(r,s)} \cong \lim \frac{I^{2k+r}J^{2k+s}}{(xz, yw)^k I^{k+r}J^{k+s}}\).

**Proof.** Consider the complex defined above 
\(F^{k^+} : 0 \to R \xrightarrow{\alpha_k} R/(k, k)^2 \xrightarrow{\beta_k} R(2k, 2k) \to 0\).

Thus \(H_{R_{++}}^2(R) = \lim \frac{R(2k, 2k)}{(k, k)}\). Note that the local cohomology modules have a natural \(\mathbb{Z}^2\)-grading which is inherited from the \(\mathbb{N}^2\)-grading of \(R\). Therefore 
\([H_{R_{++}}^2(R)]_{(r,s)} \cong \lim \frac{I^{2k+r}J^{2k+s}t_1^{2k+r}t_2^{2k+s}}{(\im \beta_k)_{(r,s)}}\).

Since \((\im \beta_k)_{(r,s)} = (xzt_1t_2, ywt_1t_2)^k(I_1)^{k+r}(J_2)^{k+s}\), 
\([H_{R_{++}}^2(R)]_{(r,s)} \cong \lim \frac{I^{2k+r}J^{2k+s}}{(xz, yw)^k I^{k+r}J^{k+s}}\).

\(\square\)

For the directed system involved in the above direct limit, the maps 
\(\frac{I^{2k+r}J^{2k+s}}{(xz, yw)^k I^{k+r}J^{k+s}} \xrightarrow{\mu} \frac{I^{2k+r+2}J^{2k+s+2}}{(xz, yw)^{k+1} I^{k+r+1}J^{k+s+1}}\)
are the multiplication by \((xyzw)\). We show that the above map is an isomorphism for \( k \gg 0 \) in the next two lemmas.

**Lemma 4.3.** The map \(\mu\), defined as above, is surjective for large \( k \).
PROOF. To show that the maps are surjective for large $k$, we need to see that for $k \gg 0$,
\[
I^{2k+r+2}J^{2k+s+2} \subseteq xzywI^{2k+r}J^{2k+s} + (xz, yw)[k+1]I^{k+r+1}J^{k+s+1}
\]
Since $(xz, yw)$ is a reduction of $IJ$, for $k \gg 0$,
\[
I^{2k+r+2}J^{2k+s+2} = (xz, yw)k+1I^{k+r+1}J^{k+s+1}
\]
\[
= (xz, yw)[k+1]I^{k+r+1}J^{k+s+1}
\]
\[
+ \sum_{i=1}^{k} ((xz)^i(yw)^{k+1-i})I^{k+r+1}J^{k+s+1}
\]
\[
\subseteq xzywI^{2k+r}J^{2k+s} + (xz, yw)[k+1]I^{k+r+1}J^{k+s+1}.
\]
Hence $\mu$ is surjective. \qed

**Lemma 4.4.** Fix $r, s$. With the notation as above, the multiplication map $\mu = \mu_{xzyw}$
\[
\frac{I^{2k+r}J^{2k+s}}{(xz, yw)[k]I^{k+r}J^{k+s}} \xrightarrow{\mu} \frac{I^{2k+r+2}J^{2k+s+2}}{(xz, yw)[k+1]I^{k+r+1}J^{k+s+1}}
\]
is an isomorphism for $k \gg 0$.

**Proof.** Let $d \in I^{2k+r}J^{2k+s}$ and suppose that $\tilde{d} \in \ker \mu$. Then
\[
xywd = (xz)^{k+1}p + (yw)^{k+1}q
\]
with $p, q \in I^{k+r+1}J^{k+s+1}$. Then $xz(ywd - (xz)^k p) = (yw)^k q$. Since $(xz, yw)$ is a regular sequence, $q \in (xz)$. Let $q = uxz$ for some $u \in R$. Similarly, $p = vyzw$ for some $v \in R$. Therefore $u \in I^{k+r+1}J^{k+s+1}$ and $v \in I^{k+r+1}J^{k+s+1}$ : $yzw$ and hence $u \in I^{2k+r}J^{2k+s} : (xz)^k$ and $v \in I^{2k+r}J^{2k+s} : (yzw)^k$ for $k \gg 0$. We show that $u, v \in I^{k+r}J^{k+s}$. Substituting the value of $q$ in (4.3) and cancelling $xz$ on both sides we get, $ywd = (xz)^k p + (yw)^{k+1}u$. Therefore $(yw)^{k+1}u = ywd - (xz)^k p \in I^{2k+r+2}J^{2k+s+2}$. Thus $u \in I^{2k+r+1}J^{2k+s+1} : (yzw)^{k+1}$. Hence, for $k \gg 0$,
\[
u \in I^{2k+r+1}J^{2k+s+1} : (xz)^{k+1}, (yzw)^{k+1} = I^{k+r}J^{k+s}, \text{ by Lemma 2.2(3). Since $IJ$}
\]
has a reduction generated by regular elements, we can apply Lemma 3.4 of [JV] to see that for $k \gg 0$, $(I^{k+r}J^{k+s}) = (I^{k+r}J^{k+s})$. Thus the map is injective for large $k$. We have already shown that the map is surjective for large $k$. Therefore $\mu$ is an isomorphism for $k \gg 0$. \qed

Lemma 4.4 shows that to compute $\lambda([H_{R,+}^{2}((R))_{(r,s)}])$ it enough to compute the length of
\[
\frac{I^{2k+r}J^{2k+s}}{(xz, yw)[k]I^{k+r}J^{k+s}} \text{ for } k \gg 0.
\]
We aim to compute $\lambda \left( [H_{R,+}^{2}((R))_{(r,s)}] \right)$ for which we need the following technical result.

**Lemma 4.6.** With notations as before, for $k \gg 0$,
\[
\lambda \left( \frac{(xz, yw)[k]}{(xz, yw)[k]I^{k+r}J^{k+s}} \right) = 2\lambda \left( \frac{R}{I^{k+r}J^{k+s}} \right) - \lambda(R/I^{k}J^{s}).
\]

**Proof.** Consider the exact sequence
\[
0 \rightarrow K \rightarrow \left( \frac{R}{I^{k+r}J^{k+s}} \right)^{2} \xrightarrow{\alpha} \left( \frac{(xz, yw)[k]}{(xz, yw)[k]I^{k+r}J^{k+s}} \right) \rightarrow 0.
\]
where $\alpha(\overline{g}, \overline{h}) = (xz)^k g + (yw)^k h$ and $K = \ker \alpha$. First we compute $K$. Let $(\overline{g}, \overline{h}) \in K$. Then there exist $p, q \in I^{k+r, J^{k+s}}$ such that

$$(xz)^k g + (yw)^k h = (xz)^k p + (yw)^k q \quad \cdots (2).$$

i.e $(xz)^k(p - g) = (yw)^k(h - q)$. Thus $p - g \in ((yw)^k)$, say $p - g = (yw)^k u$ for some $u \in R$. Substituting in (2) and cancelling $(yw)^k$ we get, $g = p - (yw)^k u$ and $h = q + (xz)^k u$. Therefore $(\overline{g}, \overline{h}) = u(\overline{yw}, (xz)^k)$. Therefore $K = \ker \alpha = (- (yw)^k, (xz)^k)$. Clearly $\phi$ is surjective. Also

$$\ker \phi = \{u \in R : u(yw)^k, u(xz)^k \in I^{k+r, J^{k+s}}\} = (I^{r, j^s}).$$

For $k \gg 0$, $I^{k+r, J^{k+s}} : (xz, yw)^k = (I^{r, j^s})$. Thus $K \cong R/(I^{r, j^s})$ for $k \gg 0$. Hence for large $k$,

$$\lambda\left(\frac{(xz, yw)^k}{(xz, yw)^k I^{k+r, J^{k+s}}}\right) = 2\lambda(R/I^{k+r, J^{k+s}}) - \lambda(R/(I^{r, j^s})).$$

\[ \square \]

**Theorem 4.7.** Let $r, s \geq 0$. With notations as before

$$\lambda\left(H^2_{R+}(R)/(r, s)\right) = P(r, s) - \lambda(R/I^{r, j^s}).$$

**Proof.** By Proposition 4.2 and Lemma 4.4

$$\lambda\left(H^2_{R+}(R)/(r, s)\right) = \lambda\left(\frac{I^{2k+r, J^{2k+s}}}{(xz, yw)^k I^{k+r, J^{k+s}}}\right)$$

$$= \lambda\left(\frac{(xz, yw)^k}{(xz, yw)^k I^{k+r, J^{k+s}}}\right) + \lambda\left(\frac{R}{(xz, yw)^k}\right)$$

$$- \lambda\left(\frac{R}{I^{2k+r, J^{2k+s}}}\right)$$

$$= 2\lambda\left(\frac{R}{I^{k+r, J^{k+s}}}\right) - \lambda\left(\frac{R}{I^{r, J^s}}\right) - \lambda\left(\frac{R}{I^{2k+r, J^{2k+s}}}\right)$$

$$+ \lambda\left(\frac{R}{(xz, yw)^k}\right).$$

Hence for $k \gg 0$, we have

$$\lambda\left(H^2_{R+}(R)/(r, s)\right) = 2P(k + r, k + s) - P(2k + r, 2k + s)$$

$$+ \lambda\left(\frac{R}{(xz, yw)^k}\right) - \lambda\left(\frac{R}{I^{r, J^s}}\right) \cdots (s)$$

Since $(x, y), (x, w), (z, y)$ and $(z, w)$ are regular sequences in $R$, we get

$$\lambda\left(\frac{R}{(xz, yw)^k}\right) = k^2 \lambda(R/(xz, yw))$$

$$= k^2[\lambda(R/(x, y)) + \lambda(R/(x, w)) + \lambda(R/(z, y)) + \lambda(R/(z, w))]$$

so that, by Theorem 3.3

$$\lambda\left(H^2_{R+}(R)/(r, s)\right) = 2P(k + r, k + s) - P(2k + r, 2k + s)$$

$$+ k^2(e(I) + 2e_{11} + e(J)) - \lambda\left(\frac{R}{I^{r, J^s}}\right).$$
Similarly one can get the required expressions for $e_{r, s}$

Substituting the values of $S = (3)$ now put ($r, s$) make it possible to find formulas for the Bhattacharya coefficients.

The formulas obtained for $H_{R_{++}}^1 (R)$ and $H_{R_{++}}^2 (R)$ enable us to express the difference $P(r, s) - B(r, s)$ in terms of the Euler characteristic of local cohomology of $R$. The formula proved below has been generalized to arbitrary dimension in [JV]. The proof in arbitrary dimension is quite different from the proof given here. The proof of the general result uses analysis of cohomology of modified Koszul complex.

Set $h^i(r, s) = \lambda (H_{R_{++}}^i (R)|_ {(r, s)})$.

**Theorem 4.9.** $P(r, s) - B(r, s) = \sum_{i=0}^2 (-1)^i h^i(r, s)$ for all $r, s \geq 0$.

**Proof.** Since $R_{++}$ contains a regular element, $H_{R_{++}}^0 (R) = 0$ so that $h^0(r, s) = 0$. By Theorem 4.1 we have $h^1(r, s) = \lambda (\overline{I^rJ_s/I^rJ_s})$ and by Theorem 4.7 we have $h^2(r, s) = P(r, s) - \lambda (R/\overline{I^rJ_s})$. Therefore

\[
\sum_{i=0}^2 (-1)^i h^i(r, s) = P(r, s) - \lambda (R/\overline{I^rJ_s}) - \lambda (\overline{I^rJ_s/I^rJ_s}) = P(r, s) - B(r, s).
\]

One does not know *apriori* the stage at which the Bhattacharya function equals the Bhattacharya polynomial. Due to this, it is difficult to calculate the coefficients of the Bhattacharya polynomials. Therefore it is desirable to have effective methods for computing these coefficients since they contain information about the bigraded Rees algebra, for example, its depth [JV]. The formulas we have obtained for the bigraded components of the local cohomology modules of bigraded Rees algebra make it possible to find formulas for the Bhattacharya coefficients.

**Corollary 4.10.** With notations as before

1. $e_{00} = h^2(0, 0)$.
2. $e_{10} = h^2(1, 0) - h^2(0, 0) + \lambda (R/\overline{I})$.
3. $e_{01} = h^2(0, 1) - h^2(0, 0) + \lambda (R/\overline{J})$.
4. $e_{20} = h^2(2, 0) - 2h^2(1, 0) + h^2(0, 0) - 2\lambda (R/\overline{I}) + \lambda (R/\overline{I^2})$.
5. $e_{02} = h^2(0, 2) - 2h^2(0, 1) + h^2(0, 0) - 2\lambda (R/\overline{J}) + \lambda (R/\overline{J^2})$.

**Proof.** (1) Putting $(r, s) = (0, 0)$ in (4.8) we get, $e_{00} = h^2(0, 0)$.

(2) Put $(r, s) = (1, 0)$ in (4.8) to get $e_{10} + e_{00} = h^2(1, 0) + \lambda (R/\overline{I})$. This yields (2).

(3) Now put $(r, s) = (2, 0)$ in (4.8) to get $e_{20} + 2e_{10} + e_{00} = h^2(2, 0) + \lambda (R/\overline{I^2})$. Substituting the values of $e_{10}$ and $e_{00}$, we get

\[e_{20} = h^2(2, 0) - 2h^2(1, 0) + h^2(0, 0) - 2\lambda (R/\overline{I}) + \lambda (R/\overline{I^2}).\]

Similarly one can get the required expressions for $e_{01}$ and $e_{02}$. □

We conclude the paper with an example to show that our results need the Cohen-Macaulay hypothesis.

**Example 4.11.** Let $k$ be a field and $T = k[[X, Y, Z]]$ be a power series ring over $k$. The ring $R = T/(X^2, XY)T$ is a two-dimensional local ring of depth one. Let $x$ (resp. $y$ and $z$) be images of $X$ (resp. $Y$ and $Z$) in $R$. Then $x^2 = xy = 0$. Put $S = k[X, Y, Z]$ and $L = (X^2, XY)S$. Then $L = (X)S \cap (X^2, Y)S$. Put $I = (X)S$ and
J = (X^2, Y)S. Let m denote the unique maximal ideal of R. Then the associated graded ring $G := G(m) \simeq S/L$. Indeed, let $R^*$ denote the m-adic completion of R. Then $R^* \simeq T^*/(X^2, XY) = S/L$ by [M]. Theorems 54 and 55. Since $G(m) \simeq G(m^*)$, and $R^*$ is homogeneous, it follows that $G \simeq S/L$. To find the Hilbert series of $G$, consider the exact sequence:

$$0 \rightarrow G \xrightarrow{\alpha} S/I \oplus S/J \xrightarrow{\beta} S/(I + J) \rightarrow 0.$$ 

Here $\alpha(r') = (r', r')$ and $\beta((a', b')) = (a - b)'$. Hence,

$$H(G, t) := \sum_{n=0}^{\infty} \lambda \left( \frac{m^n}{m^{n+1}} t^n \right) = H(S/I, t) + H(S/J, t) - H(S/(I + J), t).$$

Since $H(S/I, t) = 1/(1 - t)^2$, $H(S/J, t) = (1 + t)/(1 - t)$ and $H(S/(I + J), t) = 1/(1 - t)$, we see that

$$H(G, t) = \frac{1 + t - t^2}{(1 - t)^2}.$$ 

Write the Hilbert polynomial $P(n)$ corresponding to the Hilbert function $\lambda(R/m^n)$ in the following way:

$$P(n) = e(m) \left( \frac{n + 1}{2} \right) - e_1(m)n + e_2(m).$$

Then, by [BH], Proposition 4.1.9, $e(m) = 1$, $e_1(m) = -1$ and $e_2(m) = -1$. Thus $P(n)$ is given by

$$P(n) = \left( \frac{n + 1}{2} \right) + n - 1.$$ 

Therefore

$$\lambda(R/m^r m^s) = \left( \frac{r + s + 1}{2} \right) + r + s - 1$$

$$= \left( \frac{r}{2} \right) + rs + \left( \frac{s}{2} \right) + 2(r + s) - 1.$$ 

This shows that $e_{00} = -1$ which is not equal to the length of any module. Therefore Theorem 4.7 (and hence Corollary 4.10) does not hold if the assumption of Cohen-Macaulayness on the ring is removed.

We now show that for $I = J = m$, $H^2_{\mathfrak{R}_{++}}(\mathcal{R}) = 0$. Since $\dim G(m) = 2$, any reduction minimally generated by 2 elements is a minimal reduction [NR]. Since $x^2 = 0$, the ideal $(y, z)$ is a reduction of $m$. Hence it is a minimal reduction of $m$. By Proposition [I],

$$[H^2_{\mathfrak{R}_{++}}(\mathcal{R})]_{(r, s)} = \lim_{k \to \infty} \frac{m^{4k+r+s}}{(y^2, z^2)^k m^{2k+r+s}}.$$ 

For $n \geq 2$

$$m^n = (y^n, y^{n-1} z, y^{n-2} z^2, \ldots, y^2 z^{n-2}, x y z^{n-1}, z^n).$$ 

It is easy to see, by using the above formula for $m^n$, that $(y^{2k}, z^{2k})m^{2k} = m^{4k}$. Therefore $(y^{2k}, z^{2k})$ is a minimal reduction of $m^{2k}$. Hence by the above formula for $H^2_{\mathfrak{R}_{++}}(\mathcal{R})$, we conclude that $H^2_{\mathfrak{R}_{++}}(\mathcal{R}) = 0$. Since $\dim G(m) = 1$, $m^n = m^n$ for all $n \geq 0$. This shows that none of the formulas hold true in Corollary 4.10.
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