Liouville-type theorems for the forced Euler equations and the Navier-Stokes equations

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Abstract
In this paper we study the Liouville-type properties for solutions to the steady incompressible Euler equations with forces in $\mathbb{R}^N$. If we assume “single signedness condition” on the force, then we can show that a $C^1(\mathbb{R}^N)$ solution $(v,p)$ with $|v|^2 + |p| \in L^q(\mathbb{R}^N)$, $q \in (\frac{3N}{N-1}, \infty)$ is trivial, $v = 0$. For the solution of of the steady Navier-Stokes equations, satisfying $v(x) \to 0$ as $|x| \to \infty$, the condition $\int_{\mathbb{R}^3} |\Delta v|^{\frac{6}{5}} \, dx < \infty$, which is stronger than the important D-condition, $\int_{\mathbb{R}^3} |\nabla v|^2 \, dx < \infty$, but both having the same scaling property, implies that $v = 0$. In the appendix we reprove the Theorem 1.1([1]), using the self-similar Euler equations directly.

AMS Subject Classification Number: 35Q30, 35Q35, 76Dxx
keywords: Euler equations with perturbation, steady solutions, vanishing property

1 Main theorems

1.1 The steady Euler equations with force

Here we are concerned on the steady equations on $\mathbb{R}^N$ with force.

$$\begin{align*}
(v \cdot \nabla)v &= -\nabla p + \Phi, \\
\text{div } v &= 0,
\end{align*}$$

(1.1)
where \( v = v(x) = (v_1(x), \ldots, v_N(x)) \) is the velocity, and \( p = p(x) \) is the pressure. The force function \( \Phi[v] : \mathbb{R}^N \to \mathbb{R}^N \) satisfies the single signedness condition described below. We study Liouville-type property of the solutions to (1.1) under this condition. Let us fix \( N \geq 2, k \geq 0 \). Here we assume that the continuous function

\[
\Phi[v](x) := \Phi_k(x, v(x), Dv(x), \ldots, D^k v(x))
\]

for some \( \Phi_k : \mathbb{R}^M \to \mathbb{R}^N \) for the appropriate \( M(N, k) \), satisfies the condition of single signedness:

either \( \Phi[v](x) \cdot v(x) \geq 0 \) or \( \Phi[v](x) \cdot v(x) \leq 0 \) for all \( x \in \mathbb{R}^N \), (1.2)

and

\[
\Phi[v](x) \cdot v(x) = 0 \text{ if only if } v(x) = 0.
\]

(1.3)

For such given \( \Phi \) we consider the system (1.1). Note that when \( \Phi[v] = -v \) the system (1.1)-(1.3) becomes the usual steady Euler equations with a damping term. We remark that the damped Euler equations corresponds to a special case of the self-similar Euler equations (see Appendix below for more details). More generally \( \Phi[v](x) = G(x, v(x), \ldots, D^k v(x))v(x) \) with a scalar function \( G(x, v(x), \ldots, D^k v(x)) \leq 0 \) satisfies (1.2)-(1.3). We will prove that a Liouville type property for the system (1.1)-(1.3) under quite mild decay conditions at infinity on the solutions. More specifically we will prove the following.

**Theorem 1.1** Let \( k \geq 0 \), and \( v \) be a \( C^k(\mathbb{R}^N) \) solution of (1.1)-(1.3) with \( \Phi = \Phi[v] \). Suppose there exists \( q \in \left( \frac{3N}{N-1}, \infty \right) \) such that

\[
|v|^2 + |p| \in L^q(\mathbb{R}^N).
\]

(1.4)

Then, \( v = 0 \).

**Remark 1.1** If \( \Phi \) satisfies an extra condition \( \text{div } \Phi = 0 \), then the condition \( p \in L^q(\mathbb{R}^N) \) can be replaced by the well-known velocity-pressure relation in the incompressible Euler and the Navier-stokes equations,

\[
p(x) = \sum_{j,k=1}^N R_j R_k(v_j v_k)(x)
\]
with the Riesz transform $R_j$, $j = 1, \cdots, N$, in $\mathbb{R}^N$ (7), which holds under the condition that $p(x) \to 0$ as $|x| \to \infty$. In this case the $L^2$ estimate of the pressure follows from the $L^q$ estimate for the velocity by the Calderon-Zygmund inequality,

$$
\|p\|_{L^2} \leq C \sum_{j,k=1}^{N} \|R_j R_k v_j v_k\|_{L^2} \leq C \|v\|_{L^q}^2 \quad 2 < q < \infty. \quad (1.5)
$$

**Remark 1.2** The theorem implies that $\text{curl}(\Phi[0]) = 0$ is a necessary condition for the well-posedness of the problem, namely $v = 0$ is the unique solution of the equations.

### 1.2 The steady Navier-Stokes equations in $\mathbb{R}^3$

Here we study the following system of steady Navier-Stokes equations in $\mathbb{R}^3$.

$$(NS) \left\{ \begin{array}{l}
(v \cdot \nabla)v = -\nabla p + \Delta v, \\
\text{div} v = 0,
\end{array} \right. \quad (1.7)$$

We consider here the generalized solutions of the system (NS), satisfying

$$
\int_{\mathbb{R}^3} |\nabla v|^2 dx < \infty, \quad (1.6)
$$

and

$$
\lim_{|x| \to \infty} v(x) = 0. \quad (1.7)
$$

It is well-known that a generalized solution to (NS) belonging to $W^{1,2}_{\text{loc}}(\mathbb{R}^3)$ implies that $v$ is smooth (see e.g. [4]). Therefore without loss of generality we can assume that our solutions to (NS) satisfying (1.6) are smooth. The uniqueness question, or equivalently the question of Liouville property of solution for the system (NS) under the assumptions (1.6) and (1.7) is a long-standing open problem. On the other hand, it is well-known that the uniqueness of solution holds in the class $L^2(\mathbb{R}^3)$, namely a smooth solution to (NS) satisfying $v \in L^2(\mathbb{R}^3)$ is $v = 0$ (see Theorem 9.7 of [4]). We assume here slightly stronger condition than (1.6), but having the same scaling property, to deduce our Liouville-type result.
Theorem 1.2 Let $v$ be a smooth solution of (NS) satisfying (1.7) and
\[ \int_{\mathbb{R}^3} |\Delta v|^6 \, dx < \infty. \tag{1.8} \]
Then, $v = 0$ on $\mathbb{R}^3$.

Remark 1.3 Under the assumption (1.7) we have the inequalities with the norms of the same scaling properties,
\[ \|v\|_{L^6} \leq C\|\nabla v\|_{L^2} \leq C\|D^2 v\|_{L^6} \leq C\|\Delta v\|_{L^6} < \infty \]
due to the Sobolev and the Calderon-Zygmund inequalities. Thus, (1.8) implies (1.6). There is no, however, mutual implication relation between Theorem 1.2 and the above mentioned $L^2$ result, although our assumption (1.8) corresponds to $L^6(\mathbb{R}^3)$ at the level of scaling.

2 Proof of the Main Theorems

Proof of Theorem 1.1 We denote
\[ [f]^+ = \max\{0, f\}, \quad [f]^-= \max\{0, -f\}, \]
and
\[ D_\pm := \left\{ x \in \mathbb{R}^N \mid \left[ p(x) + \frac{1}{2} |v(x)|^2 \right]_\pm > 0 \right\} \]
respectively. We introduce the radial cut-off function $\sigma \in C^\infty_0(\mathbb{R}^N)$ such that
\[ \sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases} \tag{2.1} \]
and $0 \leq \sigma(x) \leq 1$ for $1 < |x| < 2$. Then, for each $R > 0$, we define
\[ \sigma \left( \frac{|x|}{R} \right) := \sigma_R(|x|) \in C^\infty_0(\mathbb{R}^N). \]
We multiply first equations of (1.1) by $v$ to obtain
\[ v \cdot \Phi = v \cdot \nabla \left( p + \frac{1}{2} |v|^2 \right). \tag{2.2} \]
Next, we multiply (2.2) by \([p + \frac{1}{2}|v|^2]_+ \frac{a_{N,q-3N}}{2N} \sigma_R \text{sign}\{v \cdot \Phi\}\) and integrate over \(\mathbb{R}^N\) to have
\[
\int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-3N}}{2N} v \cdot \Phi \sigma_R \, dx
= \text{sign}\{v \cdot \Phi\} \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-3N}}{2N} \sigma_R v \cdot \nabla \left( p + \frac{1}{2}|v|^2 \right) \, dx
:= I \tag{2.3}
\]
We estimate \(I\) as follows.
\[
|I| = \left| \int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-3N}}{2N} \sigma_R v \cdot \nabla \left( p + \frac{1}{2}|v|^2 \right) \, dx \right|
= \left| \int_{D^+} \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-3N}}{2N} \sigma_R v \cdot \nabla \left[ p + \frac{1}{2}|v|^2 \right]_+ \, dx \right|
= \frac{2N}{qN-q-N} \left| \int_{D^+} \sigma_R v \cdot \nabla \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-N}}{2N} \, dx \right|
= \frac{2N}{qN-q-N} \left| \int_{D^+} \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-N}}{2N} v \cdot \nabla \sigma_R \, dx \right|
\leq \frac{C\|\nabla \sigma\|_{L^{\infty}}}{R} \left( \int_{\mathbb{R}^N} (|p| + |v|^2)^{\frac{q}{2}} \, dx \right)^{\frac{N-q-N}{2N}} \|v\|_{L^q(\{R \leq |x| \leq 2R\})} \times \left( \int_{\{R \leq |x| \leq 2R\}} \, dx \right)^{\frac{1}{N}}
\leq C\|\nabla \sigma\|_{L^{\infty}} \left( \|p\|_{L^\frac{q}{2}} + \|v\|^2_{L^q} \right)^{\frac{N-q-N}{2N}} \|v\|_{L^q(\{R \leq |x| \leq 2R\})} \to 0
\]
as \(R \to \infty\). Therefore, passing \(R \to \infty\) in (2.3), we obtain
\[
\int_{\mathbb{R}^N} \left[ p + \frac{1}{2}|v|^2 \right]_+ \frac{a_{N,q-3N}}{2N} |v \cdot \Phi| \, dx = 0 \tag{2.4}
\]
by the Lebesgue Monotone Convergence Theorem. Similarly, multiplying (2.2) by \([p + \frac{1}{2}|v|^2]_- \frac{a_{N,q-3N}}{2N} \sigma_R\), and integrate over \(\mathbb{R}^N\), we deduce by similarly
to the above,
\[
\begin{align*}
\int_{\mathbb{R}^N} & \left[ p + \frac{1}{2} |v|^2 \right]^{qN-q-3N/2N} \sigma \cdot \Phi \, \sigma_R \, dx \\
= & - \int_{\mathbb{R}^N} \left[ p + \frac{1}{2} |v|^2 \right]^{qN-q-3N/2N} \sigma_R v \cdot \nabla \left( p + \frac{1}{2} |v|^2 \right) \, dx \\
= & \int_{\mathbb{R}^N} \left[ p + \frac{1}{2} |v|^2 \right]^{qN-q-3N/2N} \sigma_R v \cdot \nabla \left[ p + \frac{1}{2} |v|^2 \right] \, dx \\
\leq & C \| \nabla \sigma \|_{L^\infty} \left( \| p \|_{L^{qN/2N}} + \| v \|_{L^q}^2 \right)^{aN-q-3N/2N} \| v \|_{L^q(R \leq |x| \leq 2R)} \to 0 
\end{align*}
\]  
(2.5)

as \( R \to \infty \). Hence,
\[
\int_{\mathbb{R}^N} \left[ p + \frac{1}{2} |v|^2 \right]^{qN-q-3N/2N} |v \cdot \Phi| \, \sigma_R \, dx = 0 
\]  
(2.6)

by the Lebesgue Monotone Convergence Theorem again. Let us define
\[
\mathcal{S} = \{ x \in \mathbb{R}^N \mid v(x) \neq 0 \}.
\]

We note that \( \mathcal{S} \) is an open set in \( \mathbb{R}^N \). Suppose \( \mathcal{S} \neq \emptyset \). Then, (2.5) and (2.6) together with (1.2)-(1.3) imply
\[
\left[ p(x) + \frac{1}{2} |v(x)|^2 \right] = \left[ p(x) + \frac{1}{2} |v(x)|^2 \right] = 0 \quad \forall x \in \mathcal{S}.
\]

Namely,
\[
p(x) + \frac{1}{2} |v(x)|^2 = 0 \quad \forall x \in \mathcal{S}.
\]

Since this holds for any open subset of \( \mathcal{S} \), we have also \( \nabla (p + \frac{1}{2} |v|^2)(x) = 0 \) for all \( x \in \mathcal{S} \). From (2.2) this implies
\[
\Phi[v](x) \cdot v(x) = 0 \quad \forall x \in \mathcal{S}. 
\]  
(2.7)

Considering the conditions on \( \Phi \) in (1.2)-(1.3), we have a contradiction, and therefore we need \( \mathcal{S} = \emptyset \), namely \( v = 0 \) on \( \mathbb{R}^N \). \( \Box \)

Next in order to prove Theorem 1.2 we recall the following result proved by Galdi (see Theorem X.5.1 of [4] for more general version).
Theorem 2.1 Let $v(x)$ be a generalized solution of (NS) satisfying (1.6) and (1.7) and $p(x)$ be the associated pressure, then there exists $p_1 \in \mathbb{R}$ such that

$$\lim_{|x| \to \infty} |D^\alpha v(x)| + \lim_{|x| \to \infty} |D^\alpha (p(x) - p_1)| = 0$$

uniformly for all multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in [\mathbb{N} \cup \{0\}]^3$.

Proof of Theorem 1.2 Under the assumption (1.8) and Remark 1.1, Theorem IX.6.1 of [4] implies that

$$\lim_{|x| \to \infty} |p(x) - p_1| = 0.$$ (2.8)

for a constant $p_1$. Therefore, if we set

$$Q(x) := \frac{1}{2} |v(x)|^2 + p(x) - p_1,$$

then

$$\lim_{|x| \to \infty} |Q(x)| = 0.$$ (2.9)

As before we denote $[f]_+ = \max\{0, f\}$, $[f]_- = \max\{0, -f\}$. Given $\varepsilon > 0$, we define

$$D^\varepsilon_+ = \left\{ x \in \mathbb{R}^3 \left| [Q(x) - \varepsilon]_+ > 0 \right. \right\},$$

$$D^\varepsilon_- = \left\{ x \in \mathbb{R}^3 \left| [Q(x) + \varepsilon]_- > 0 \right. \right\},$$

respectively. Note that (2.9) implies that $D^\varepsilon_\pm$ are bounded sets in $\mathbb{R}^3$. Moreover,

$$Q \mp \varepsilon = 0 \quad \text{on} \quad \partial D^\varepsilon_\pm$$ (2.10)

respectively. Also, thanks to the Sard theorem combined with the implicit function theorem $\partial D^\varepsilon_\pm$‘s are smooth level surfaces in $\mathbb{R}^3$ except the values of $\varepsilon > 0$, having the zero Lebesgue measure, which corresponds to the critical values of $z = Q(x)$. It is understood that our values of $\varepsilon$ below avoids these exceptional ones. We write the system (NS) in the form,

$$-v \times \text{curl} v = -\nabla Q + \Delta v.$$ (2.11)
Let us multiply (2.11) by $v [Q - \varepsilon ]_+$, and integrate it over $\mathbb{R}^3$. Then, since $v \times \text{curl} v \cdot v = 0$, we have

$$
0 = - \int_{\mathbb{R}^3} [Q - \varepsilon ]_+ v \cdot \nabla (Q - \varepsilon ) \, dx + \int_{\mathbb{R}^3} [Q - \varepsilon ]_+ v \cdot \Delta v \, dx := I_1 + I_2. \tag{2.12}
$$

Integrating by parts, using (2.10), we obtain

$$
I_1 = - \int_{D_+^\varepsilon} (Q - \varepsilon ) v \cdot \nabla (Q - \varepsilon ) \, dx = - \frac{1}{2} \int_{D_+^\varepsilon} v \cdot \nabla (Q - \varepsilon )^2 \, dx. = 0
$$

Using

$$
v \cdot \Delta v = \Delta \left( \frac{1}{2} |v|^2 \right) - |\nabla v|^2, \tag{2.13}
$$

and the well-known formula for the Navier-Stokes equations,

$$
\Delta p = |\omega|^2 - |\nabla v|^2, \tag{2.14}
$$

we have

$$
I_2 = - \int_{\mathbb{R}^3} |\nabla v|^2 [Q - \varepsilon ]_+ \, dx + \int_{\mathbb{R}^3} \Delta \left( \frac{1}{2} |v|^2 \right) [Q - \varepsilon ]_+ \, dx
\quad = - \int_{\mathbb{R}^3} |\omega|^2 [Q - \varepsilon ]_+ \, dx + \int_{\mathbb{R}^3} \Delta (Q - \varepsilon ) [Q - \varepsilon ]_+ \, dx
\quad := J_1 + J_2. \tag{2.15}
$$

Integrating by parts, we transform $J_2$ into

$$
J_2 = \int_{D_+^\varepsilon} \Delta (Q - \varepsilon ) (Q - \varepsilon ) \, dx = - \int_{D_+^\varepsilon} |\nabla (Q - \varepsilon )|^2 \, dx. \tag{2.16}
$$

Thus, the derivations (2.12)-(2.16) lead us to

$$
0 = \int_{D_+^\varepsilon} |\omega|^2 |Q - \varepsilon | \, dx + \int_{D_+^\varepsilon} |\nabla (Q - \varepsilon )|^2 \, dx \tag{2.17}
$$

for all $\varepsilon > 0$. The vanishing of the second term of (2.17) implies

$$
[Q - \varepsilon ]_+ = C_0 \quad \text{on} \quad D_+^\varepsilon
$$
for a constant $C_0$. From the fact (2.10) we have $C_0 = 0$, and $[Q - \varepsilon]_+ = 0$ on $\mathbb{R}^3$, which holds for all $\varepsilon > 0$. Hence,

$$[Q]_+ = 0 \quad \text{on } \mathbb{R}^3. \quad (2.18)$$

This shows that $Q \leq 0$ on $\mathbb{R}^3$. Suppose $Q = 0$ on $\mathbb{R}^3$. Then, from (2.11), we have $v \cdot \Delta v = 0$ on $\mathbb{R}^3$. Hence,

$$\Delta p = -\frac{1}{2} \Delta |v|^2 = -v \cdot \Delta v - |\nabla v|^2 = -|\nabla v|^2.$$ 

Comparing this with (2.14), we have $\omega = 0$. Combining this with $\text{div } v = 0$, we find that $v$ is a harmonic function in $\mathbb{R}^3$. Thus, by (1.7) and the Liouville theorem for the harmonic function, we have $v = 0$, and we are done. Hence, without loss of generality, we may assume

$$0 > \inf_{x \in \mathbb{R}^3} Q(x).$$

Given $\delta > 0$, we multiply (2.11) by $v [Q + \varepsilon]_\delta^-$, and integrate it over $\mathbb{R}^3$. Then, similarly to the above we have

$$0 = -\int_{\mathbb{R}^3} [Q + \varepsilon]_\delta^- v \cdot \nabla (Q + \varepsilon) \, dx + \int_{\mathbb{R}^3} [Q + \varepsilon]_\delta^- v \cdot \Delta v \, dx := I_1' + I_2'.$$ 

(2.19)

Observing $Q(x) + \varepsilon = -[Q(x) + \varepsilon]_-$ for all $x \in D_\varepsilon^-$, integrating by part, we obtain

$$I_1' = \int_{D_\varepsilon^-} [Q + \varepsilon]_\delta^- v \cdot \nabla [Q + \varepsilon]_- \, dx$$

$$= \frac{1}{\delta + 1} \int_{D_\varepsilon^-} v \cdot \nabla [Q + \varepsilon]_{\delta + 1}^- \, dx = 0.$$

Thus, using (2.13), we have

$$0 = -\int_{D_\varepsilon^-} |\nabla v|^2 [Q + \varepsilon]_\delta^- \, dx + \frac{1}{2} \int_{D_\varepsilon^-} [Q + \varepsilon]_\delta^- \Delta |v|^2 \, dx \quad (2.20)$$

Now, we have the point-wise convergence

$$[Q(x) + \varepsilon]_\delta^- \to 1 \quad \forall x \in D_\varepsilon^-.$$
as $\delta \downarrow 0$. Since
\[
\int_{\mathbb{R}^3} |v \cdot \Delta v| \, dx \leq \|v\|_{L^6} \|\Delta v\|_{L^\infty} \leq C \|\nabla v\|_{L^2} \|\Delta v\|_{L^\infty},
\]
\[
\leq C \|\Delta v\|^2_{L^\infty} < \infty,
\]
we have
\[
\Delta |v|^2 = 2v \cdot \Delta v + 2|\nabla v|^2 \in L^1(\mathbb{R}^2).
\] (2.21)

Hence, passing $\delta \downarrow 0$ in (2.20), by the dominated convergence theorem, we obtain
\[
\int_{D_{\varepsilon}} |\nabla v|^2 \, dx = \frac{1}{2} \int_{D_{\varepsilon}} \Delta |v|^2 \, dx,
\] (2.22)
which holds for all $\varepsilon > 0$. For a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$, we observe
\[
D_{\varepsilon_n} \uparrow \bigcup_{n=1}^\infty D_{\varepsilon_n} = D_- := \{x \in \mathbb{R}^3 \mid Q(x) < 0\}.
\]
Thus, observing (2.21) again, we can apply the dominated convergence theorem in passing $\varepsilon \downarrow 0$ in (2.22) to deduce
\[
\int_{D_-} |\nabla v|^2 \, dx = \frac{1}{2} \int_{D_-} \Delta |v|^2 \, dx.
\] (2.23)

Now, thanks to (2.18) the set
\[
S = \{x \in \mathbb{R}^3 \mid Q(x) = 0\}
\]
consists of critical(maximum) points of $Q$, and hence $\nabla Q(x) = 0$ for all $x \in S$, and the system (2.11) reduces to
\[
- v \times \omega = \Delta v \quad \text{on} \quad S.
\] (2.24)

Multiplying (2.24) by $v$, we have that
\[
0 = v \cdot \Delta v = \frac{1}{2} \Delta |v|^2 - |\nabla v|^2 \quad \text{on} \quad S.
\]
Therefore, one can extend the domain of integration in (2.23) from $D_-$ to $D_- \cup S = \mathbb{R}^3$, and therefore
\[
\int_{\mathbb{R}^3} |\nabla v|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^3} \Delta |v|^2 \, dx.
\] (2.25)
We now claim the right hand side of (2.25) vanishes. Indeed, since \(\Delta |v|^2 \in L^1(\mathbb{R}^3)\) from (2.21), applying the dominated convergence theorem, we have

\[
\left| \int_{\mathbb{R}^3} \Delta |v|^2 \, dx \right| = \lim_{R \to \infty} \left| \int_{\mathbb{R}^3} \Delta |v|^2 \sigma_R \, dx \right| = \lim_{R \to \infty} \left| \int_{\mathbb{R}^3} |v|^2 \Delta \sigma_R \, dx \right|
\]

\[
\leq \lim_{R \to \infty} \int_{\mathbb{R}^3} |v|^2 |\Delta \sigma_R| \, dx
\]

\[
\leq \lim_{R \to \infty} \frac{\|D^2 \sigma\|_{L^\infty}}{R^2} \|v\|^2_{L^6(R \leq |x| \leq 2R)} \left( \int_{\{R \leq |x| \leq 2R\}} \, dx \right)^\frac{1}{2}
\]

\[
\leq C \|D^2 \sigma\|_{L^\infty} \lim_{R \to \infty} \|v\|^2_{L^6(R \leq |x| \leq 2R)} = 0
\]

as claimed. Thus (2.25) implies that

\[\nabla v = 0 \quad \text{on} \quad \mathbb{R}^3,\]

and \(v = \text{constant}\). By (1.7) we have \(v = 0\). □

**Remark after the proof of Theorem 1.2:** The first part of the above proof, showing \([Q]_+ = 0\) can be also done by applying the maximum principle, which follows from the following identity for \(Q\),

\[-\Delta Q + v \cdot \nabla Q = -|\omega|^2 \leq 0\]

I do not think, however, the maximum principle can also be applied to the proof of the second part, showing \([Q]_- = 0\), which is more subtle than the first part. The above proof overall shows that the argument of the proof I used for this second part can also be adapted for the first part without using the maximum principle, which exhibits consistency.

**Appendix**

### A Remarks on the self-similar Euler equations

Let \(a, b\) are given constants with \(b \neq 0\). We study here the system in \(\mathbb{R}^3\).

\[
\begin{cases}
(v \cdot \nabla)v = -\nabla p + av + b(x \cdot \nabla)v, \\
\text{div} v = 0.
\end{cases}
\]  

(A.1)
In the special case of $a = -\frac{\alpha}{\alpha + 1}, b = -\frac{1}{\alpha + 1}$ the system \((A.1)\) reduces to the self-similar Euler equations.

\[
\begin{aligned}
(SSE) & \begin{cases}
\frac{\alpha}{\alpha + 1} v + \frac{1}{\alpha + 1} (x \cdot \nabla)v + (v \cdot \nabla)v = -\nabla p, \\
\text{div} v = 0,
\end{cases}
\end{aligned}
\] (A.2)

which is obtained from the time dependent Euler equations,

\[
\begin{aligned}
\begin{cases}
u_t + (u \cdot \nabla)u = -\nabla \pi \\
\text{div} u = 0,
\end{cases}
\end{aligned}
\]

by the self-similar ansatz,

\[
\begin{aligned}
u(x, t) &= \frac{1}{(T-t)^{\frac{1}{\alpha+1}}} v \left( \frac{x - x^*}{(T-t)^{\frac{1}{\alpha+1}}} \right), \\
\pi(x, t) &= \frac{1}{(T-t)^{\frac{\omega}{\alpha}}} p \left( \frac{x - x^*}{(T-t)^{\frac{\omega}{\alpha}}} \right).
\end{aligned}
\]

Note that the damped Euler equation, which is a trivial case of \((1.1)\) is the case when $\alpha = \infty$ in \((A.2)\). In \cite{1}, in particular, a Liouville-type result for the system \((A.2)\) was derived, using the time dependent Euler equations, where we need to use existence result of a back-to-label map due to Constantin\cite{3}. In the following we prove similar result for the general system \((A.2)\).

**Theorem A.1** Let $v$ be a $C^2(\mathbb{R}^3)$ solution to \((A.1)\) with $b \neq 0$, satisfying

\[
\|\nabla v\|_{L^\infty} < \infty \quad \text{and} \quad \omega \in \bigcup_{r>0} \bigcap_{0<q<r} L^q(\mathbb{R}^3). \tag{A.3}
\]

Then, $v = \nabla h$ for a harmonic scalar function $h$ on $\mathbb{R}^3$. Thus, if we impose further the condition $\lim_{|x| \to \infty} |v(x)| = 0$, then $v = 0$.

**Proof** We first observe that from the calculus identity

\[
v(x) = v(0) + \int_0^1 \partial_s v(sx)ds = v(0) + \int_0^1 x \cdot \nabla v(sx)ds,
\]

we have $|v(x)| \leq |v(0)| + |x|\|\nabla v\|_{L^\infty} \leq C(1 + |x|)(\|\nabla v\|_{L^\infty} + |v(0)|)$, and

\[
\sup_{x \in \mathbb{R}^3} |v(x)| \leq C(\|\nabla v\|_{L^\infty} + |v(0)|). \tag{A.4}
\]
We consider the vorticity equation of (A.1),
\[-(a + b)\omega - b(x \cdot \nabla)\omega + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v.\]  
(A.5)

Let $\delta > 0$, and take $L^2(\mathbb{R}^3)$ inner product (A.5) by $\omega(\delta + |\omega|^2)^{\frac{q}{2}} - 1 \sigma_R$, and integrate over $\mathbb{R}^3$ to obtain
\[-(a + b) \int_{\mathbb{R}^3} |\omega|^2(\delta + |\omega|^2)^{\frac{q}{2}} - 1 \sigma_R dx - \int_{\mathbb{R}^3} [(\omega \cdot \nabla)v] \cdot \omega(\delta + |\omega|^2)^{\frac{q}{2}} - 1 \sigma_R dx = \frac{1}{q} \int_{\mathbb{R}^3} ((bx - v) \cdot \nabla)(\delta + |\omega|^2)^{\frac{q}{2}} \sigma_R dx.\]  
(A.6)

For fixed $\delta > 0$ and $R > 0$ the integrands in the right hand side of (A.6) are sufficiently smooth functions having the compact support, and one can integrate by part them to obtain
\[-(a + b) \int_{\mathbb{R}^3} |\omega|^2(\delta + |\omega|^2)^{\frac{q}{2}} - 1 \sigma_R dx - \int_{\mathbb{R}^3} [(\omega \cdot \nabla)v] \cdot \omega(\delta + |\omega|^2)^{\frac{q}{2}} - 1 \sigma_R dx = -(a + b) \int_{\mathbb{R}^3} |\omega|^2(\delta + |\omega|^2)^{\frac{q}{2}} - 1 \sigma_R dx - \frac{1}{q} \int_{\mathbb{R}^3} (\delta + |\omega|^2)^{\frac{q}{2}} ((bx - v) \cdot \nabla) \sigma_R dx.\]  
(A.7)

Passing $\delta \downarrow 0$ in (A.7), using the dominated convergence theorem, we have
\[
\left(-a - b + \frac{3b}{q}\right) \int_{\mathbb{R}^3} |\omega|^q \sigma_R dx - \int_{\mathbb{R}^3} (\omega \cdot \nabla)v \cdot \omega|\omega|^q - 2 \sigma_R dx = -\frac{b}{q} \int_{\mathbb{R}^3} |\omega|^q (x \cdot \nabla) \sigma_R dx + \frac{1}{q} \int_{\mathbb{R}^3} |\omega|^q (v \cdot \nabla) \sigma_R dx
\]
\[:= I + J.\]  
(A.8)

We estimate $I$ and $J$ easily as follows.
\[|I| \leq \frac{|b|}{qR} \int_{\{R \leq |x| \leq 2R\}} |\omega|^q |x| |\nabla \sigma| dx \leq \frac{2|b|}{q} \|\nabla \sigma\|_{L^\infty} \|\omega\|_{L^p(R \leq |x| \leq 2R)}^q \rightarrow 0\]
as $R \rightarrow \infty$.
\[|J| \leq \frac{1}{qR} \int_{\{R \leq |x| \leq 2R\}} |\omega|^q |v| |\nabla \sigma| dx \leq \frac{1 + 2R}{qR} \int_{\{R \leq |x| \leq 2R\}} \frac{|v(x)|}{1 + |x|} |\omega|^q |\nabla \sigma| dx \leq \frac{C(1 + 2R)}{qR} \|\nabla \sigma\|_{L^\infty} (\|\nabla v\|_{L^\infty} + |v(0)|) \|\omega\|_{L^p(R \leq |x| \leq 2R)}^q \rightarrow 0\]
as \( R \to \infty \), where we used (A.4). Therefore, passing \( R \to \infty \) in (A.8), and using the dominated convergence theorem for the left hand side, we obtain,

\[
\left( -a - b + \frac{3b}{q} \right) \int_{\mathbb{R}^3} |\omega|^q dx = \int_{\mathbb{R}^3} (\omega \cdot \nabla)v \cdot \omega |\omega|^{q-2} dx,
\]
from which we deduce easily

\[
- \|\nabla v\|_{L^\infty} \|\omega\|_{L^q}^q \leq \left( -a - b + \frac{3b}{q} \right) \|\omega\|_{L^q}^q \leq \|\nabla v\|_{L^\infty} \|\omega\|_{L^q}^q.
\]

Suppose there exists \( x_0 \in \mathbb{R}^3 \) such that \( \omega(x_0) \neq 0 \), then since \( \omega \) is a continuous function, one has \( \|\omega\|_{L^q} > 0 \), and we can divide (A.9) by \( \|\omega\|_{L^q}^q \) to have

\[
- \|\nabla v\|_{L^\infty} \leq \left( -a - b + \frac{3b}{q} \right) \leq \|\nabla v\|_{L^\infty},
\]
which holds for all \( q \in (0, r) \) and for some \( r > 0 \). Since \( b \neq 0 \), passing \( q \downarrow 0 \) in (A.10), we obtain desired contradiction. Therefore \( \omega = \text{curl} v = 0 \). This, together with \( \text{div} v = 0 \), provides us with the fact that \( v = \nabla h \) for a scalar harmonic function \( h \) on \( \mathbb{R}^3 \). □

**Acknowledgements**

The author would like to thank deeply to the anonymous referee for careful reading and constructive criticism. This work was supported partially by the NRF grant. no. 2006-0093854 and also by Chung-Ang University Research Grants in 2012.

**References**

[1] D. Chae, *Nonexistence of self-similar singularities for the 3D incompressible Euler equations*, Comm. Math. Phys., 273, no. 1, (2007), pp. 203-215.

[2] D. Chae, *Nonexistence of asymptotically self-similar singularities in the Euler and the Navier-Stokes equations*, Math. Ann., 338, no. 2, (2007), pp. 435-449.

[3] P. Constantin, *An Eulerian-Lagrangian approach for incompressible fluids: local theory*, J. Amer. Math. Soc. 14(2), (2001), pp. 263-278.
[4] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. II*, Springer, (1994).

[5] J. Leray, *Essai sur le mouvement d’un fluide visqueux emplissant l’espace*, Acta Math. 63 (1934), pp. 193-248.

[6] J. Nečas, M. Ružička and V. Šverák, *On Leray’s self-similar solutions of the Navier-Stokes equations*, Acta Math., 176, no. 2, (1996), pp. 283-294.

[7] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, (1970).

[8] T-P. Tsai, *On Leray’s self-similar solutions of the Navier-Stokes equations satisfying local energy estimates*, Arch. Rat. Mech. Anal., 143, no. 1, (1998), pp. 29-51.