The linear theory of tidally excited spiral density waves: application to CV and circumplanetary disks

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ABSTRACT
We revisit linear tidal excitation of spiral density waves in the disks of cataclysmic variables (CVs), focusing on scalings with orbital Mach number in order to bridge the gap between numerical simulations and real systems. If an inner Lindblad resonance (ILR) lies within the disk, ingoing waves are robustly excited, and the angular-momentum flux they carry is independent of Mach number. But in most CVs, the ILR lies outside the disk. The wave flux and its scaling with Mach number are then very sensitive to conditions near the disk edge. If the temperature and sound speed vanish there, excitation tends to be exponentially suppressed. If the Mach number remains finite in the outer parts but the radial and vertical density scale lengths become comparable due to subkeplerian rotation, resonance can occur with acoustic-cutoff and stratification frequencies. These previously neglected resonances excite waves, but the Mach-number scaling remains very steep if the radial scale length decreases gradually. The scaling can be less strong—algebraic rather than exponential—if there are sharp changes in surface density at finite sound speed. Shocks excited by streamline-crossing or by the impact of the stream from the companion are unlikely to be important for the angular-momentum budget, at least in quiescence. Our results may also apply to circumplanetary disks, where Mach numbers are likely lower than in CVs.

1 INTRODUCTION
Since Sawada et al. (1986), many numerical simulations of Roche-lobe overflow in semidetached binaries have shown spiral shocks in the disk around the primary (e.g., Schwarzenberg-Czerny & Rozyczka 1988; Sawada & Matsuda 1992; Rozyczka & Spruit 1993; Godon et al. 1998; Stehle 1999; Blondin 2000). Savonije et al. (1994, hereafter SPL) interpret the excitation of these shocks as a linear effect of the tidal field of the secondary and claim reasonable agreement with their own nonlinear two-dimensional simulations as regards the torque exerted on the disk. Most recently, Ju et al. (2016, 2017) report spiral shocks in both hydro and MHD simulations, and they interpret their results partly by reference to the linear framework of SPL.

According to standard linear theory for very thin disks (Goldreich & Tremaine 1979), excitation of density waves by a smooth periodic potential occurs only at resonances where the radial WKBJ wavenumber vanishes. In the two-dimensional approximation adopted by many of the works cited above, the only relevant resonance is the inner Lindblad resonance (hereafter ILR), although others can occur in three dimensions (Lubow 1981). SPL’s linear analysis considers a disk that is truncated well inside the ILR and neglects the impact of the tidal stream (an intrinsically nonlinear process). SPL attribute the excitation of these shocks as a linear effect of the tidal field of the secondary and claim reasonable agreement with their own nonlinear two-dimensional simulations as regards the torque exerted on the disk. Most recently, Ju et al. (2016, 2017) report spiral shocks in both hydro and MHD simulations, and they interpret their results partly by reference to the linear framework of SPL.

Spiral features have been found by doppler tomography of some CV disks, notably IP Peg and U Gem (Steeghs et al. 1997; Neustroev & Borisov 1998; Groot 2001; Baptista et al. 2005). The pitch angles of these observed spirals are too large for density waves at realistic values of \( M \), however, and it has been suggested that the spiral features are due instead to the vertical resonance (Ogilvie 2002), or simply to the
nonaxisymmetry of the tidally induced velocity field in the outer disk; i.e., the compression of closed orbits (neglecting pressure) along the line between the stars (Smak 2001). Accretion may also be driven by spiral shocks in the disks surrounding nascent jovian planets (Zhu et al. 2016). Here the Mach number is expected to be lower than in CVs, making both the excitation (by the stellar tide) and the propagation of these shocks more efficient.

The present paper focuses on the linear excitation of spiral density waves by the tidal potential, especially when there is no ILR within the disk and the Mach number large. We use idealized axisymmetric disk models with adjustable outer radii and Mach numbers, and various density profiles. (In models where $c \rightarrow 0$ at the disk edge, $M$ pertains to somewhat smaller radii where conditions are more characteristic of the disk as a whole.) This reduces the excitation to a well-defined mathematical problem in linear ordinary differential equations, which we treat both analytically and numerically.

This paper is organized as follows. In §2 we review the equations governing linear density-wave perturbations in an axisymmetric disk, their boundary conditions, and methods for calculating their excitation by a tidal potential. Section 3 analyses a model in which the sound speed tends smoothly to zero at the edge, rather like an Emden polytrope with a free boundary. It is shown both numerically and analytically that when such a disk contains no ILR, linear wave excitation tends to be exponentially small at large $M$. §4 examines cases in which there is a finite temperature and sound speed in the outer disk, if only because of irradiation by the white dwarf and its companion, but the radial scale lengths of density and pressure become small due to subkeplerian rotation: the stream through the inner Lagrange point arrives with constant specific angular momentum, while viscous redistribution is weak in quiescence, leading to a small outer region where the orbital angular velocity $\Omega \propto r^{-2}$ rather than $r^{-3/2}$. It is shown that a short radial scale length $(H)$ can lead to resonances where $c/H \sim \Omega$. Such resonances are familiar in asteroseismology but seem to have received little attention so far.

2 METHODS

2.1 Disk models

Let the mass of the primary and the binary companion be $M_1$ and $M_2$ respectively and assume that their orbits are circular. The orbital frequency is therefore $\omega = \sqrt{G(M_1 + M_2)/D^3}$ where $D$ is their separation. Since we work with linear theory, the mass ratio $q \equiv M_2/M_1$ is otherwise treated as infinitesimal; thus the gravitational potential within the disk is simply $-GM_1/r$, the axisymmetric part of the companion’s tide being neglected. We choose units such that $G = M_1 = \omega = 1$. The unit length $R_0 \equiv (\omega^2/GM_1)^{1/3}$ is therefore the radius of corotation with the tidal potential.

We use $\Sigma_d$ to denote the typical surface density of the disk; the exact definition of $\Sigma_d$ differs among models.

The unperturbed equilibrium disk is axisymmetric with outer radius $r_{\text{max}}$ and inner radius $r = 0$. In isothermal models, the surface density cannot completely vanish, but there is a characteristic radius $r_d$ beyond which the surface density $\Sigma_0$ declines swiftly. We presume that the structure of the inner parts of the disk is unimportant for the excitation of waves and for the angular-momentum flux that they carry, as long as the waves are not appreciably reflected by the inner boundary: short wavelengths subject the ingoing waves to dissipation by cooling, and nonlinear steepening into shocks. We assume that $r_{\text{max}} < R_0$: i.e., the disk is always truncated inside the corotation resonance. This is a realistic assumption for CV disks, and probably for circumplanetary ones also. In fact, the specific angular momentum of the stream from the L1 point is approximately conserved until it strikes the disk, so the outer radius of the disk is given by the circularization radius

$$r_c \approx (1 + q)^{-1} (r_{L1}/D)^4.$$  

(1)

Here $r_{L1}$ is the radius of the L1 point, and $q \equiv M_2/M_1$ the mass ratio. The inferred radius of the hot spot in quiescent disks appears to be roughly consistent with eq. (1) (Smak 1971; Stanishev et al. 2004; Baptista et al. 2016), indicating at least partial suppression of the disk viscosity, which would tend to cause the disk to spread outward. When the disk edge coincides with the circularization radius (1), it lies well within the ILR $[r_{L1} \approx 0.63(1 + q)^{-1/2}D]$, let alone the corotation radius, unless the mass ratio is extreme ($q \lesssim 0.004$).

The disk profile can be described by the unperturbed surface density $\Sigma_0$ and enthalpy$^2$ $K_0$ = $\int \Sigma_0^{-1}dP_0$. Force balance relates the latter to the rotation profile:

$$\frac{dK_0}{dr} = \tau \Omega^2 - \frac{GM_1}{r^2}.$$  

(2)

When the disk is not isentropic, radial stratification is measured by the Brunt-Väisälä frequency

$$N_0^2 \equiv \frac{dK_0}{dr} \left( \frac{d\ln \Sigma_0}{dr} - \frac{1}{c^2} \frac{dK_0}{dr} \right).$$  

(3)

2.2 Linearisation

Perturbations can be characterized by the Lagrangian radial displacement $\xi$ and the Eulerian enthalpy perturbation $K = P/\Sigma_0$ where $P$ is the (vertically integrated) Eulerian pressure perturbation. For adiabatic perturbation with azimuthal number $m$, the equations governing their pattern in

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1 in fact reduced by 10-20%, depending on $q$, so that eq. (1) is a mild overestimate (Flannery 1975)

2 Technically, $K_0$ represents the thermodynamic enthalpy only when the disk is isentropic.
an inviscid disk are

\[
\frac{d\xi}{dr} = -\left[\frac{2\Omega\sigma}{r} + \frac{1}{r} \frac{dK_0}{dr}\right] + \frac{m^2}{r^3 \sigma^2} (K + W_m) - \frac{1}{r^2} K.
\]  

(4a)

\[
\frac{dK}{dr} = \frac{2\Omega\sigma}{r} (K + W_m) + \left(\sigma^2 - \kappa^2 - N_0^2\right) \xi - \frac{dW_m}{dr}
\]  

\[ - N_0^2 \left(\frac{dK_0}{dr}\right)^{-1} K.
\]  

(4b)

The isentropic version of these equations can be found in SPL. Here \( \sigma = m(\omega - \Omega) \) is the frequency of the tide in the corotating frame, and \( \kappa \) is the epicyclic frequency, equal to \( \Omega \) when \( \Omega \propto r^{-3/2} \). \( W_m \) is the \( m \)th azimuthal harmonic of the tidal potential, for which we take the \( m = 2 \) quadrupolar approximation

\[
W_2 = -\frac{3GM_2}{4D^2} r^{-2}
\]  

(5)

in all of our numerical calculations. The \( t, \phi \) dependence \( e^{im(\phi - wt)} \) is implicit for all linearised quantities. We ignore the self gravity of the disk; the validity of this approximation is discussed in Appendix A.

### 2.3 The homogeneous problem

It is convenient to represent the dependent variables by a column vector

\[
y = \begin{bmatrix} \sqrt{r \Sigma_0} \xi \\ \sqrt{r \Sigma_0} (K + W_m) \end{bmatrix}
\]  

(6)

so that the system (4) can be written as \( dy/dr = My \) when \( W_m \rightarrow 0 \). The factors of \( \sqrt{r \Sigma_0} \) ensure that the \( 2 \times 2 \) matrix \( M \) is traceless and therefore has eigenvalues \( \pm \sqrt{-k_0^2} \), with

\[
k_0^2 \approx \frac{\sigma^2 - \kappa^2 - N_0^2}{c^2} - \left[ \frac{1}{2\Omega} + N_0^2 \right] \left(\frac{dK_0}{dr}\right)^{-1}.
\]  

(7)

Here \( H \equiv (-d \ln \Sigma_0/dr)^{-1} \) is the density scale length. Exact expressions for \( k_0^2 \) and \( M \) are given in Appendix B. Where \( k_0 \) varies slowly \( (dk_0/dr \ll k_0^2) \), it becomes the WKBJ wavenumber. Homogeneous solutions are then oscillatory for \( k_0^2(r) > 0 \) and evanescent where \( k_0^2(r) < 0 \). Radial at which \( k_0^2(r) = 0 \) are turning points and can be interpreted as resonances (§4). Where \( k_0 \) varies rapidly, however, intuition based on WKBJ may fail utterly. This happens at the edges of our polytropic disks (§3).

Where stratification and density gradients are small, i.e. well away from the edge,

\[
k_0^2 \approx \frac{\sigma^2 - \kappa^2}{c^2}.
\]  

(8)

This form of \( k_0^2(r) \) vanishes at the ILR. Where \( k_0^2 \) is positive and large, WKBJ is applicable and the radial group velocity \( V_g = -m(\partial k_0/\partial \omega) \) is unambiguous. We choose a basis for the homogeneous solutions consisting of an outgoing wave \( y_+ \) and ingoing wave \( y_- \). Interior to the ILR, i.e. where \( \omega < \Omega - \kappa/m \), the ingoing wave has a phase that increases with radius, i.e. \( dy_-/dr \approx \mp ik_0 y_\pm \) for \( k_0 > 0 \). These solutions can be continued outward to the turning point and into the evanescent zone \( k_0^2(r) < 0 \) by solving the homogeneous form of the system (4). The phases and amplitudes of \( y_\pm \) can be scaled so that they are complex conjugates and so that

the solution that decays outward in the evanescent zone is \( y_R \equiv \text{Real}(y_\pm) = (y_+ + y_-)/2 \).

The angular-momentum flux carried by a wave is

\[
F = \pi r^2 \Sigma_0 \text{Real}(v_r^* v_\phi),
\]  

(9)

where \( v_r, v_\phi \) are the velocity perturbation given by (for the homogeneous solution)\(^3\)

\[
v_r = -i \xi, \quad v_\phi = -\frac{\kappa^2}{2\Omega} \xi + \frac{m}{r \sigma} K.
\]  

(10)

For a solution to the forced equation \( (W_m \neq 0) \), \( K \) in the above equation is replaced by \( (K + W_m) \). For homogeneous solutions, \( F \) is independent of \( r \); the angular-momentum flux is conserved. We normalize \( y_\pm \) so that the angular-momentum flux they carry is

\[
F_0 = \Sigma_d R_0^2 \omega^2 \left(\frac{M_2}{M_1}\right)^2 \left(\frac{D}{R_0}\right)^{-6},
\]  

(11)

where \( \Sigma_d \) is a constant characterizing the typical density of the disk. Thus a wave that approaches \( A_m y_\pm \) as \( r \rightarrow 0 \) carries a flux \( F_j = |A_m|^2 F_0 \).

### 2.4 The boundary conditions

For models having a definite edge at \( r_{\text{max}} \), a free outer boundary condition is imposed:

\[
K + \xi \frac{dK_0}{dr} \bigg|_{r_{\text{max}}} = 0.
\]  

(12)

If the disk has a finite surface density at \( r_{\text{max}} \), the free boundary condition requires that the Lagrangian pressure perturbation vanishes there; otherwise, the outer edge is a regular singular point of the linear differential equations, and the free boundary condition selects the regular solution. For models having no definite edge, wavelike part of the solution should decay outward in the evanescent zone. Both conditions express the physical requirement that the angular-momentum fluxes emitted by annuli near the boundary tend to zero with their masses.

Near \( r = 0 \), we require that there is no outgoing wave. This is not always equivalent to saying that the solution approaches a purely ingoing wave: there is always a locally forced non-wave component of the response to the tide, and this dominates even at small \( r \) if \( \Sigma_0 \) increases rapidly inward. (The non-wavelike response is in phase with the tide and hence carries no flux.) For all models in this paper where it is necessary to examine the behavior as \( r \rightarrow 0 \) explicitly, however, it suffices to require that \( y \propto y_- \) as \( r \rightarrow 0 \).

### 2.5 Formal solution for the wave amplitude

The first-order system (4) can be recast as a second-order equation for \( y_1 = \sqrt{r \Sigma_0} \xi \) of the form

\[
\frac{d^2}{dr^2} y_1 + p(r) \frac{d^2}{dr^2} y_1 + k^2(r) y_1 = f(r).
\]  

(13)

\(^3\) When calculating \( (v_r^* v_\phi) \), we directly use the complex \( v_r, v_\phi \) we get from complex \( \xi, K \); if we use the real part instead the result should be smaller by a factor of 2.
Here $p \approx \ln c^2/\Delta r$, $k^2 \approx k_0^2$, and

$$f(r) \approx \frac{1}{c^2} \sqrt{\Delta r} \left( \frac{dW_m}{dr} - \frac{2m\Omega}{r} W_m \right). \quad (14)$$

Exact expressions for $p$, $k^2$ and $f$ are given in Appendix C1. A formal solution of this equation is easy to obtain, and we can use it to calculate the amplitude of the tidally excited density wave.

As always, the solution to the inhomogeneous differential equation can be expressed via integrals over linearly independent homogeneous solutions. For the latter we take $y_{1,-}$, the complex-valued ingoing wave, and $y_{1,1}$, the real-valued homogeneous solution that satisfies the outer boundary condition. Their Wronskian is

$$W(r) \equiv y_{1,1} \frac{dy_{1,-}}{dr} - \frac{dy_{1,-}}{dr} y_{1,1} \propto \exp \int_0^r p(r') dr'. \quad (15)$$

Note that $W(r) \propto 1/c^2$ approximately. The solution of (13) that satisfies our boundary conditions is then

$$y_1(r) = y_{1,-}(r) \int_r^{r_{\text{max}}} \frac{y_{1,1}(r') f(r')}{W(r')} dr' + y_{1,1}(r) \int_0^r \frac{y_{1,-}(r') f(r')}{W(r')} dr'. \quad (16)$$

The amplitude of the ingoing wave at the inner boundary can be read off as

$$A_{\text{in}} \equiv \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{y_{1,1}(r') f(r')}{W(r')} dr.$$ \quad (17)

The angular-momentum flux is then $F_\Omega = |A_{\text{in}}|^2 F_0$, with $F_0$ given by eq. (11).

For a high-Mach-number disk, $y_{1,1}(r)$ oscillates rapidly, on a scale $k^{-1} \sim c/\Omega$, whereas the rest of the integrand of eq. (17) varies slowly. Such integrals tend to be exponentially small (as can be demonstrated formally by distorting the integration contour into the complex $r$ plane) except for possible contributions from resonances where $k_0 \to 0$ or from radii where the background disk is not smooth. A realistic disk might have sharp changes even after azimuthal and temporal averaging due to thermal or ionization fronts, standing shocks at the hot spot, etc. In the idealized models considered here, the only sharp changes occur at the boundary (§3 & §5). Other models have resonances associated with density or entropy gradients, or to the ILR itself if that lies within the disk (§4). In summary, we expect for all high-Mach-number disks that significant wave excitation—by which we mean wave fluxes that scale algebraically rather than exponentially with $M$—should arise locally near such sharp features or resonances.4 The remainder of this paper is dedicated to demonstrating this through examples.

\[4 \] The propagation of the waves, once excited, may be global. This depends on nonlinear and dissipative effects that we do not address here. We simply assume that such effects are unimportant within the first wavelength or so of the excitation region.

3 POLYTROPIC DISK

As a first example, we consider an isentropic disk with a rotation profile that deviates from keplerian by a constant factor $1 - \eta$, with $\eta \ll 1$:

$$\Omega^2 = \frac{1 - \eta}{r^3}.$$ \quad (18)

Assuming a polytropic equation of state with Emden index $n$ and adiabatic exponent $\gamma = 1 + 1/n$, $K_0 = \eta \left( \frac{1}{r} - \frac{1}{r_{\text{max}}} \right)$, $\Sigma_0 = \Sigma_\text{d} \left( \frac{1}{r} - \frac{1}{r_{\text{max}}} \right)^n$, $c^2 = n^{-1} \eta \left( \frac{1}{r} - \frac{1}{r_{\text{max}}} \right)$. \quad (19)

Thus the local Mach number $r\Omega/c$ diverges at the outer edge but approaches the constant $M_0 \equiv \eta^{-1/2}$ at $r \ll r_{\text{max}}$.

For a given tidal perturbation $W_m$, the ingoing-wave amplitude $|A_{\text{in}}|$ depends on three parameters: $\eta$, $r_{\text{max}}$, and $\gamma$. We study the scaling of the result with respect to these, especially the Mach number $M_0 = \eta^{-1/2}$. The only possible resonance in this model is the ILR, and wave excitation depends strongly on whether or not the ILR occurs inside the disk, as we demonstrate below.

3.1 ILR inside the disk

When the ILR lies inside the disk, the wave excitation is dominated by the resonance. The angular-momentum flux at the inner edge of the disk can be analytically estimated (Goldreich & Tremaine 1979, hereafter GT79; see also §4.2):

$$F_{\text{in}} = 2\pi^2 \left\{ \left| \Sigma_0 \frac{d}{r} \left( \sigma^2 - \sigma^2 \right) \right|^{-1} \left( \frac{dW_2}{dr} - \frac{4\Omega}{\sigma} W_2^2 \right) \right\}_{\text{ILR}}.$$ \quad (22)

When the resonance happens at $\Sigma_0 \sim \Sigma_d$, this corresponds to a dimensionless amplitude $|A_{\text{in}}| \sim 1$. According to eq. (22), $F_{\text{in}}$ and $|A_{\text{in}}|$ do not depend on the disk Mach number, but only on the local surface density and orbital frequency near the resonance.

When the ILR is well inside the disk, the numerical results in Figures 1 and 2 differ from the analytic prediction (22) by no more than a few percent.

3.2 ILR outside the disk

When the ILR lies outside the disk, the excitation should mainly take place near the outer edge, since that is where the WKBJ wavelength is longest compared to the scale on which the background disk properties vary. A local analytical approximation predicts that as the Mach number increases, the wave amplitude decreases faster than any power of $M_0$ (Appendix D). We have tested this numerically.

3.2.1 Numerical results

The scaling of $|A_{\text{in}}|$ with respect to $r_{\text{max}}$ for different $\eta$ and $\gamma$ is shown in Figure 1. When the ILR is outside the disk, $|A_{\text{in}}|$ decays approximately exponentially as $r_{\text{max}}$ decreases. The
3.2.2 An empirical formula for the amplitude scaling

The scaling of the amplitude when the ILR is outside the disk can be summarized by the following empirical formula,

\[
\log_{10}|A_{in}| \approx -0.62\eta^{-1/2}\gamma^{-1}\Delta, \quad (23)
\]

where \(\Delta\) is a function of \(r_{\text{max}}\) indicating the deviation of the disk edge from the ILR, and is given by

\[
\Delta \equiv \left[ \frac{\sigma^2 - \kappa^2}{\Omega^2} \frac{\Omega^2}{\omega} \right]^{1/6} r_{\text{max}}^{-1}. \quad (24)
\]

The sign of \(\Delta\) indicates whether the ILR is inside the disk. This empirical formula is only for the ILR outside the disk, i.e. \(\Delta > 0\). Note that \(\Delta \approx (r_{\text{ILR}} - r_{\text{max}})\) when the ILR is relatively close to the disk edge.

We find that the empirical formula (23) gives a very good fit for the \(r_{\text{max}}\) dependence and captures most of the \(\eta\) and \(\gamma\) dependence when \(|A_{in}|\) is small.

3.2.3 The location of wave excitation

We have numerically investigated the radius at which most of the excitation occurs (\(r_{\text{ILR}}\)) by adding a small imaginary part to \(\omega\), which corresponds physically to a tidal perturbation that is growing with time. The wave amplitude near the inner edge is then reduced in proportion to the group delay from the point of excitation. By these means, we find that \(r_{\text{max}} - r_{\text{ILR}} \lesssim \eta\).

3.3 Summary of this section

When the ILR is inside the disk, GT79’s formula (22) accurately predicts the angular-momentum flux, independently of the disk Mach number and other disk properties absent from that formula. When the ILR is outside the disk, the excitation happens near the outer edge and is exponentially weak, as codified by the empirical formula (23). We expect similar behavior from all models where the temperature falls linearly to zero at the disk edge, and no resonance or discontinuity exists within the disk.

4 WAVE EXCITATION BY RESONANCES

Even in two-dimensional disks, other resonances besides the ILR can exist, and in fact should exist near the disk edge provided that the radial scale lengths of density and pressure vary slowly there. These resonances tend to be weaker than the ILR. Although familiar in asteroseismology (e.g., the acoustic cutoff for p-modes), they seem not to have been studied before in this context.

We will start by developing a local model for resonances and obtain a general analytical formula for the angular momentum flux they produce in the limit of high Mach number (short wavelengths). We apply the results to different types of resonances, and discuss the scaling of the angular-momentum flux with Mach number for each case.
4.1 A local model for resonances

We identify resonant radii with zeros of the determinant $k_0^2$ of the matrix $\mathbf{M}$ in the first-order linear differential system (B1). The local analysis of wave excitation at all resonances offered below is justified only if relevant properties of the disk vary slowly near the resonance. The polytropic disks of the previous section formally have an acoustic-cutoff resonance (§4.3), but only very close to the disk edge ($r_{\text{max}}$), where $H = (-d\ln \Sigma_0/dr)^{-1} \approx (r_{\text{max}} - r)/\Sigma$ varies quickly; evidently, the analysis below does not apply to such cases.

Consider the second order equation (13) for $y_1 = \sqrt{r} \Sigma_0 \xi$ near a zero of $k_0^2$. Let $r_{\text{res}}$ be the location of this zero. If $k_0^2$ varies smoothly, the following length scale can be associated with the resonance:

$$\lambda \equiv \left| \frac{dk_0^2}{dr} \right|_{r_{\text{res}}}^{-1/3}$$

(25)

When this is small compared to the radius and to the scales of variation of relevant disk properties—including $H$ but not necessarily $\Sigma_0$ itself—we can approximate (13) by the simplified local model

$$\frac{d^2y_1}{dx^2} - xy_1 = \lambda^2 f(r_{\text{res}}) e^{-\alpha x}$$

(26)

Here $x \equiv (r - r_{\text{res}})/\lambda$ and

$$\alpha \equiv \frac{\lambda}{2H}.$$

The solution for the amplitude and flux of the ingoing wave is given by GT79 for $\alpha = 0$, and can be obtained by the methods of §2.5 with the appropriate replacements. This treatment would extend to $\alpha < 0$ because the integral analogous to eq. (17) for the amplitude of the ingoing wave would then be proportional to

$$\int_{-\infty}^{\infty} Ai(x) e^{-\alpha x} \, dx = e^{-\alpha^3/3} \text{ Real}(\alpha) < 0.$$  

(28)

Here $Ai(x)$ is the Airy function that decays $\propto e^{x^3/2}$ as $x \to +\infty$ and is oscillatory for $x < 0$. We need $\alpha \geq 0$ since $d\ln \Sigma_0/dr < 0$, however. The integral above is then not convergent, so we solve eq. (26) by other means in Appendix C2. It turns out that the amplitude $A_{\text{in}}$ of the ingoing wave is exactly as if eq. (28) held for positive as well as negative $\alpha$. The angular-momentum flux is therefore

$$F_{\text{in}} \approx m^2 \pi \Sigma_0 \frac{dk_0^2}{dr}^{-1} \left| W_m' \right|^2 e^{-2\alpha^3/3},$$

where $W_m' \equiv \frac{dW_m}{dr} - \frac{2m\Omega}{r\sigma} W_m$,  

(29)

all quantities being evaluated at the resonance. Note that the frequent combination hereby abbreviated as $W_m'$ is not simply the radial derivative of the tidal potential $W_m$.

Depending on which terms in $dk_0^2/dr$ are dominant, the scaling of $F_{\text{in}}$ can be different. Below, we will consider the scaling of $F_{\text{in}}$ for different resonances.

4.2 Lindblad resonance

When stratification and density gradient are both negligible, the resonance is located approximately at $\sigma^2 - \kappa^2 = 0$. In this case,  

$$\frac{dk_0^2}{dr} \approx \frac{1}{c^2} \left| \frac{d(r^2 - \kappa^2)}{dr} \right|^{-1},$$

(30)

and the angular-momentum flux is

$$F_{\text{in}} \approx m^2 \pi \Sigma_0 \frac{d}{dr} \left( r^2 - \kappa^2 \right)^{-1} \left| W_m' \right|^2.$$  

(31)

This is identical to GT79’s result (22), although they also included disk self gravity. The factor $e^{-2\alpha^3/3}$ has been omitted on the presumption that $H \gg \lambda$.

4.3 Acoustic-cutoff resonance (ACR)

In an isentropic disk, or one with negligible radial stratification, (7) at high Mach number becomes

$$k_0^2 \approx \frac{\sigma^2 - \kappa^2}{c^2} - \frac{1}{4H^2}.$$  

(32)

The resonance occurs where this vanishes, i.e. where the acoustic cutoff frequency of the disk $\omega_{\text{ac}} = c/2H$ matches $\sqrt{\sigma^2 - \kappa^2}$. Therefore we refer to this resonance as an acoustic-cutoff resonance (ACR). This requires $H \sim c/\Omega$ (the vertical scale height) unless $\sigma^2 - \kappa^2 \ll \Omega^2$, in which case the resonance reduces essentially to an ILR. The natural location for an ACR proper is in the outermost parts of the disk where $\Sigma_0$ declines rapidly.

4.3.1 Scaling of angular-momentum flux with $M$

The angular-momentum flux has the form (29) but with $k_0^2$ as in eq. (32) rather than eq. (8).

Consider a fixed subkeplerian rotation curve $\Omega(r) < \Omega_K(r)$. For an isothermal equation of state, since the pressure gradient must make up the difference between centrifugal force and gravity, $H = r(\Omega_K^2 - \Omega^2)/c^2 \propto M^{-2}$ at a fixed radius. The ACR will occur where the two terms on the right side of eq. (32) balance, whence $H_{\text{ac}} \propto M^{-1}$ rather than $M^{-2}$. Thus we need $(\Omega_K - \Omega)_{\text{ac}} \propto M^{-1}$.

If the density scale length varies slowly, i.e. $0 < -dH/dr \ll 1$, then the flux excited at an ACR will tend to decrease rapidly with increasing Mach number. Let $r_0 < r_{\text{ac}}$ be some radius interior to the ACR where the surface density is characteristic of the disk as a whole, but close enough so that the scale lengths at the two radii are comparable. Then by the intermediate-value theorem, there exists an intermediate radius $\tilde{r} \in (r_0, r_{\text{ac}})$ such that

$$\ln \left[ \frac{\Sigma_0(r_0)}{\Sigma_0(r_{\text{ac}})} \right] = \int_{r_0}^{r_{\text{ac}}} \frac{dr}{H(r)} = \frac{r_{\text{ac}} - r_0}{H(\tilde{r})}.$$  

If $H$ varies slowly, and since $r/H = O(M)$ at the ACR, the numerator of this last expression can be much larger than the denominator, whence $\Sigma_0(r_{\text{ac}})$, to which the flux is
directly proportional, is small. Furthermore, if the gradient of \( k_0^2 \) near the resonance is dominated by the variation of the scale length, then it follows from eqs. (32), (25), and (27) that \( \alpha^2 = \lambda^2/8H^2 \approx (-(dH/dr))^{-1} \), and so the factor \( e^{-2\alpha^2/3} \) in the flux (29) is also small.

### 4.3.2 Numerical confirmation

Consider an isothermal disk with rotation profile

\[
\Omega^2(r) = \frac{GM_0}{r^4} \left( r^{-\beta} + r_0^{-\beta} \right)^{-1/\beta},
\]

where \( r_0 \) and \( \beta > 1 \) are constants. This corresponds to a disk with a keplerian rotation profile for \( r \ll r_0 \), and constant specific angular momentum for \( r \gg r_0 \). Larger \( \beta \) makes for a sharper transition between the two regimes. To maintain this rotation profile, the density scale height satisfies

\[
1 = \frac{1}{H} = \frac{1}{c^2} \left( \frac{GM_0}{r^2} - \Omega^2 \right) \sim \begin{cases} 0 & (r \ll r_0) \\ \mathcal{M}^2/r & (r \gg r_0) \end{cases}.
\]

The ACR, which requires \(1/H \sim \mathcal{M}/r \), occurs at \( r \sim r_0 \).

As Figure 3 shows, the numerical results broadly agree with the analytical prediction (29). At large Mach number, \( \Sigma_0(r_{ACR}) \) and the angular-momentum flux fall off rapidly—faster than any power law; but the curvature in these log-log plots is less for larger \( \beta \), i.e. for more abrupt transitions between keplerian rotation and constant specific angular momentum.

### 4.4 The effect of stratification: stratification-shifted ACR (SACR)

When the effect of stratification becomes nontrivial (i.e. when the terms with \( N_0^2 \) are among the leading order terms in \( k_0^2 \) near the resonance), the location of the resonance can be shifted. In this case, we refer to this resonance as a stratification-shifted ACR (SACR).

Since SACR depends on too many parameters \((H, N_0^2, dK_0/dr \text{ etc.})\), \( F_{in} \) cannot be simplified unless certain restrictions are imposed on the disk profile. Let us make the following assumptions:

- \( \gamma \equiv (\partial \ln P/\partial \ln \Sigma)_{\text{ad}} \) is \( O(1) \) and slowly varying.
- The disk temperature \( T \propto P_0/\Sigma_0 \) is finite and slowly varying. With the previous condition, the same goes for \( c^2 \).

Under these assumptions, when the ILR lies far outside the disk and \( H \ll r \) (which, as we will show later, is required at the resonance),

\[
N_0^2 \sim \frac{\sigma^2}{H^2}, \quad \left| \frac{dK_0}{dr} \right| \sim \frac{c^2}{H}.
\]

As a result, the SACR corresponds to

\[
\sigma^2 - \kappa^2 \sim N_0^2 \sim \frac{c^2}{H^2} \sim c^2 \left[ \frac{dK_0}{dr} \right]^{-1}.
\]

Thus the \( H \) required for SACR is similar to that required for ACR.

If \( H \) is one of the fastest varying parameters, then

\[
F_{in} \sim \left[ \frac{r\Sigma_0}{\sigma^2 - \kappa^2} \left| \frac{d\ln H}{dr} \right|^{-1} \left| W_{\kappa} \right|^2 \right]_{\text{SACR}}.
\]

In general, one can replace \( d\ln H/dr \) by \( d\ln X/dr \) where \( X \) is the fastest varying parameter. Therefore, the scaling of the flux with \( \mathcal{M} \) should be much the same as for the ACR (§4.3.1).

## 5 WAVE EXCITATION AT DISCONTINUITIES

In this section, we consider another possible mechanism for wave excitation: discontinuities in the equilibrium disk. As an example, we consider a discontinuity in the surface density itself—a disk with finite \( \Sigma_0(r_{\text{max}}) \). The rotation profile is keplerian even at the edge, and \( k_{\text{Or}} \gg 1 \), so that WKB is applicable everywhere. Pressure balance in equilibrium would require a tenuous hot atmosphere to confine the disk edge. This is probably not realistic, but it seems to be what SPL assumed for their analytic estimates. It may also serve as a proxy for sharp changes in surface density within the disk interior, perhaps due to thermal/ionization fronts. The wave excitation is estimated analytically via a local model, and then confirmed numerically.

### 5.1 Discontinuous surface density

The distinguishing feature of this model is that the surface density is nonzero at the edge, as is the sound speed. So to study wave excitation, we adopt a local model in which these are constants, whence \( dK_0/dr = 0 \). For consistency, the rotation curve exactly balances gravity and is exactly keplerian, but we continue to write \( \kappa^2 \) not \( \Omega^2 \) in eq. (8) for \( k_0^2 \) as a reminder of the ILR. We treat \( k_0 \) as locally constant, assuming that the ILR lies a distance \( \gg k_0^{-1} \) beyond \( r_{\text{max}} \). Defining \( x \equiv (r - r_{\text{max}})/k_0 \) and \( f_0 \equiv f(r_{\text{max}}) \), our local approximation to (13) becomes

\[
\frac{d^2y_1}{dx^2} + y_1 = k_0^{-2}f_0.
\]

The Lagrangian pressure perturbations should vanish if the edge is confined by a tenuous atmosphere of constant pressure, so we impose the free boundary condition (12). Since
\[ \frac{dK_0}{dr} = 0, \] this reduces to \( K(r_{\text{max}}) = 0; \) cast in terms of \( y_1 = \sqrt{r_{\Sigma_0}} \), the free boundary condition becomes

\[ x = 0: \quad \frac{dy_1}{dx} + \frac{1}{k_0r^3} \left( \frac{2m\Omega}{\sigma} + \frac{1}{2} \right) y_1 = \frac{m^2\sqrt{r_{\Sigma_0}}}{k_0r^2\sigma^2} W_m. \quad (39) \]

The non-wavelike particular solution to eq. (38) is simply \( y_{1,\text{nv}} = k_0^{-2}f_0. \) Since \( k_0 \propto c^{-1} \) and \( f_0 \propto c^{-2}, \) this solution is independent of \( M, \) but it does not satisfy the boundary condition (39). So we must add to it an incoming wave \( A_{\text{in}} \exp(i\pi) \) that satisfies the homogeneous form of eq. (38).

Inserting the sum of these two into the boundary condition yields, to leading order in \((k_0r)^{-1}, \)

\[ A_{\text{in}} \approx k_0^{-1}\sqrt{r_{\Sigma_0}} W_m \bigg|_{r_{\text{max}}}, \quad (40) \]

in which the dimensionless quantity

\[ \widetilde{W}_m \approx \frac{W_m}{r^2\sigma^2} \left[ m^2 - \frac{\sigma^2 + 4m\Omega\sigma}{2(\sigma^2 - \kappa^2)} \left( \frac{d\ln W_m}{dr} - \frac{2m\Omega}{\sigma} \right) \right] \]

involves the tidal potential but not \( c \) or \( \Sigma_0. \)

### 5.1.1 Mach-number scaling

The angular-momentum flux \( \pi mc^2 k_0|y_1|^2 \) carried by the ingoing wave in this model is

\[ F_{\text{in}} = \pi m|W_m|^2 \frac{c^2\Sigma_0}{\sqrt{\sigma^2 - \kappa^2}} \bigg|_{r_{\text{max}}}, \quad (42) \]

which clearly scales as \( M^{-3}. \) The scaling with \( r_{\text{max}} \) at fixed \( M = (r\Omega/c)_{\text{max}} \) is \( F_{\text{in}} \propto r_{\text{max}}^2 \Sigma_0(r_{\text{max}}) \) if \( W_m \propto r^m \) and \( r_{\text{max}} \ll r_{\text{ILR}}. \)

SPL seem to have used a model similar to this one for their analytical estimates, though they are not explicit about the equilibrium conditions at the disk edge. They discuss both free and rigid (vanishing radial displacement) boundary conditions but do not say which they used for their estimates of the angular-momentum flux. It is easy to see from the analysis above that the two boundary conditions lead to different scalings with Mach number: For suppose that \( y_1 = 0 \) at the edge. Then since the non-wave particular solution does not vanish, \( y_{1,\text{nv}} = k_0^{-2}f_0 \propto M^0, \) an equal and opposite wavelike part would have to be added to it, leading to \( F_{\text{in}} \propto M^{-1} \) rather than \( F_{\text{in}} \propto M^{-3}. \)

### 5.1.2 Numerical confirmation

To test the scaling relations predicted above, we have made numerical calculations for a disk with uniform \( c \) and \( \Sigma_0 = \Sigma_\infty. \) These results confirm that when the ILR lies well outside the disk, \( F_{\text{in}} \propto M^{-3}, \) and that when the ILR lies inside the disk, \( F_{\text{in}} \) is independent of \( M \) (Fig. 4).

Note that there is an intermediate regime at \( r_{\text{max}} \sim r_{\text{ILR}} \) which connects the two limiting cases (ILR deep inside / far outside the disk). At large \( M, \) \( F_{\text{in}} \) has a steep dependence on \( r_{\text{max}} \) in this intermediate regime.

### 6 NONLINEAR EFFECTS

In this section, we discuss several nonlinear effects which are relevant to the problem. These nonlinear effects include the truncation of the disk by the companion which tends to keep the disk edge inside the ILR and the mass inflow from the companion which causes angular momentum transport by itself.

#### 6.1 Disk truncation by the binary companion

In our discussion above we have effectively assumed that the companion mass ratio \( q \equiv M_2/M_1 \) is infinitesimal. This means that the ILR can be inside the disk. However, this is not the case in the systems we are interested in. For CV disks, \( q \) is typically of order 0.1 to 1, while for circumplanetary disks, \( q \gtrsim 10^3. \) Having a finite \( q \) affects the maximum possible size of the disk; especially, the edge of the largest possible disk may still be far from the ILR, in which case the excitation is bound to be weak.

We investigate the maximum possible size of the disk using the method in Paczynski (1977). The edge of the largest possible disk is defined as the maximum stable periodic streamline around the primary when the pressure and gravity of the disk is neglected. We survey a large range of \( q \) from \( 10^{-2} \) to \( 10^4, \) and \( \sigma^2 - \kappa^2 \) at the disk edge is always positive and comparable to \( \omega^2 \) (the binary’s orbital frequency).\(^7\)

The size of the disk is also limited by the angular momentum of the accretion stream. In this case, the circularization radius \( r_c \) (1) gives an estimate of the size of the disk. \( r_c \) is usually smaller than the maximum possible size of the disk defined above.

One caveat is that we only consider an infinitely thin disk. Disk pressure, self gravity and viscosity may help increase the stability of the disk and make the disk edge closer to the ILR.

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\(^7\) Here we assume \( \kappa = \Omega \) for simplicity. Using a more realistic assumption, that \( \kappa^2 = 2\Omega R^{-1}d(R^2\omega)/dR \) where \( 2\Omega \) is the diameter of a periodic streamline and \( 2\pi/\Omega \) the period of it, gives the same qualitative result.
6.2 Shocks launched by the accretion stream

The impact on the disk edge of the stream of material from the L1 point can also launch shocks. In a time-averaged sense at least, these shocks (like the tidally excited ones) have angular pattern speed \( \omega \), the orbital angular velocity of the binary; since this is less than the orbital angular velocity of the disk, these shocks also deposit negative angular momentum in the disk. Here we make a rough estimate of the torque on the disk due to these shocks.

Suppose that the outer edge is located at the circularization radius (1). The relative velocity \( v_{\text{rel}} \) between the stream and the outer edge of the disk where they collide is then approximately radial, since the stream and the disk edge have similar specific angular momentum. The associated torque \( \dot{M} \) delivers radial momentum to the disk edge at rate \( \dot{M} v_{\text{rel}} \) and the disk edge

\[
\dot{M} v_{\text{rel}} \approx \frac{\dot{M} \Omega_{\text{edge}}}{\Omega_{\text{disk}}} \]

\[
\times \left[ \frac{\sqrt{\alpha}}{\Omega_{\text{disk}}} - 1 \right] \frac{c}{\Omega_{\text{disk}} r_{\text{c}}}.
\]

The last inequality presumes that the total velocity of the stream is approximately the local escape velocity \( \sqrt{2} \Omega_{\text{disk}} r_{\text{c}} \).

At high Mach number \( c \ll \Omega_{\text{disk}} r_{\text{c}} \), the torque exerted by the impact of the stream is removes only a small fraction of the angular momentum that the stream itself adds to the disk.

Apart from its dependence on Mach number, the torque exerted by the stream scales with the rate of addition of mass, whereas the torque exerted by the tide scales with the mass already present in the disk. Therefore, if the outer disk were to become sufficiently massive, an equilibrium could in principle be reached between the rate of addition of angular momentum by the stream and the rate of its removal by the tide (though the required mass may be unreasonably large if the disk has high Mach number and ends at the circularization radius, rather than the streamline-crossing or ILR radius). The inequality (43) argues that the shocks excited by the stream itself cannot achieve such an equilibrium at any disk mass.

7 APPLICATIONS

7.1 CV disks in quiescence

For CV disks in quiescence, observational results (Menou 2000; Cannizzo et al. 2012) suggest that tidal perturbations may be important for the accretion to the primary and the evolution of the disk. Here we discuss whether the claim that accretion during quiescence is mainly due to the tidal excitation of density waves is consistent with our theory.

Consider the effective \( \alpha \) of the disk, which in the context of a forced and accreting disk can be defined as

\[
\alpha = \frac{\dot{M}}{3\pi \Sigma c h}
\]

where \( h = c/\Omega \) is the disk scale height. Assuming that accretion conserves the mean specific angular momentum of the disk, this effective \( \alpha \) is related to our dimensionless amplitude \( |A_{\text{in}}| = (F_{\text{in}}/F_{\text{out}})^{1/2} \) by (for mass ratio \( q < 1 \))

\[
\alpha \sim q^3 |A_{\text{in}}|^2 \dot{M}^{-2},
\]

where \( \dot{M} \) is the typical Mach number of the disk. For CV disks in quiescence, observations of disk temperatures suggest a \( \dot{M} \approx 100 \) to 200 at quiescence (see, e.g. Rutkowski et al. 2016), and the typical mass ratio is \( q \approx 0.1 \). The effective \( \alpha \) during outburst and quiescence are \( \alpha_{\text{hot}} \approx 0.1 \) (Smak 1984) and \( \alpha_{\text{cold}} \approx 10^{-5} \) (Cannizzo et al. 2012), although with considerable variation from one system to another.

In this paper we have discussed three main cases of wave excitation when the ILR does not lie inside the disk:

- When the disk contains no resonance or discontinuity, the amplitude scales exponentially with respect to the Mach number. For quiescent CV disks, the Mach number is large and the excitation trivial.
- Resonances other than the ILR (i.e. ACR and SACR) may exist in the disk. It is possible for them to produce sufficient angular-momentum flux to be the main accretion mechanism. However, in realistic disks, \( \Sigma_0 \) at the resonance tends to be much smaller than the typical surface density of the disk, reducing the wave excitation. Therefore, it remains uncertain whether resonant excitation of density waves can produce enough accretion.
- The disk profile may contain discontinuities. In the context of quiescent CV disks, the accretion stream and inefficient angular momentum transport may cause the disk to have a sharp edge. If the disk edge is sharp enough (i.e. the density drops from some large value to trivially small within a length scale \( \ll r/\dot{M} \)), this mechanism is very promising. For instance, for a disk with uniform density and sound speed, with \( r_{\text{max}} = 0.5 \) the effective \( \alpha \) is given by

\[
\alpha \approx 0.0015 \left( \frac{\dot{M}(r_{\text{max}})}{100} \right)^{-1}.
\]

This is comparable to the observed \( \alpha_{\text{cold}} \).

The above discussion suggests that the accretion driven by tidally excited density waves can indeed produce the observed \( \alpha_{\text{cold}} \) when the disk contains a resonance (ACR or SACR) that happens at a sufficiently large density, or when the disk has a sharp edge.

Our theory is also consistent with the observation that \( \alpha_{\text{cold}} \) has a steep dependence on the disk size (Cannizzo et al. 2012). At high Mach number, the excitation at an ACR or SACR is much weaker than at an ILR. Yet the net excitation is a continuous function of disk radius. Necessarily therefore, the wave torque varies rapidly with \( r_{\text{max}} \). This effect is illustrated in Fig. 4 for discontinuous disks but occurs more generally.

Overall, our result is in broad agreement with observation, and it is possible that the accretion during quiescence is mainly driven by tidally excited density waves. However, since the amplitude can vary by orders of magnitude across different disk models, whether tidally excited density waves
are the main cause of accretion remains an open question until the disk profile and evolution can be determined with high accuracy. Our results also suggest that extra caution has to be taken when modelling the disk in simulations and extrapolating low-Mach-number results to higher Mach number.

7.2 Circumplanetary disks
Circumplanetary disks are different from quiescent CV disks mainly in that their Mach number tends to be much smaller, especially at their outer edges, by a factor $\sim r_H/a = (M_{\text{planet}}/3M_{\text{star}})^{1/3}$, in which $a$ is the planet’s orbital semi-major axis and $r_H$ its Hill radius. As a result, the excitation can still be relatively large even if the ILR is far outside the disk.

It is shown in all our models that increasing the disk size or density tends to increase the rate of accretion and angular momentum extraction. Therefore, there always exists an equilibrium state where the accretion onto the planet and angular momentum extraction balance the mass and angular momentum input from the accretion onto the disk.\(^8\) This suggests that accretion is mainly limited by the mass input from the accretion stream, rather than the ability to remove excessive angular momentum from the disk.

8 SUMMARY AND CONCLUSION
In this paper we investigate the linear excitation of spiral density waves by tidal perturbation in CV and circumplanetary disks, focusing on the excitation mechanism and the amplitude scaling with respect to the Mach number $M$ of the disk when the inner Lindblad resonance (ILR) lies outside the disk. We summarize our main results below, and discuss their implications.

8.1 Wave excitation mechanisms and amplitude scaling
The strength of wave excitation can be characterized by the dimensionless wave amplitude $|A_{in}| = (F_{in}/F_0)^{1/2}$, with the unit angular-momentum flux $F_0$ given in (11).

When the ILR lies inside the disk, wave excitation is dominated by the ILR and the dimensionless wave amplitude is $|A_{in}| \sim 1$, which is independent of $M$ and insensitive to different disk models (§3.1 and §4.2). This agrees with the result of Goldreich & Tremaine (1979).

When the ILR lies far outside the disk, there are several possible linear excitation mechanisms, and the scaling of $|A_{in}|$ with respect to $M$ depends sensitively on the disk profile.

- When the disk contains no discontinuity or resonance, the excitation tends to be exponentially small, with $\log |A_{in}| \sim -M$ (§3.2).
- The disk may contain resonances other than the ILR, namely acoustic-cutoff resonance (ACR, §4.3) and stratification-shifted acoustic-cutoff resonance (SACR, §4.4). Both ACR and SACR require $|d\ln \Sigma_a/dr| \sim M/r$.

8 For CV disks, such equilibrium in principle also exists, but outbursts are triggered before this equilibrium is reached.

suggesting that they should be located near the disk edge. The angular momentum flux and amplitude of the excited wave can be analytically estimated by a local model. The strength of excitation by ACR and SACR depends sensitively on the disk profile. The amplitude tends to be small, mainly limited by the surface density at the resonance.

- Near-discontinuities in the disk profile (significant change over a radial distance $\lesssim r_H/M$) can also produce non-trivial excitation, with $|A_{in}| \propto M^{-1/2}$ (§5).
- Excitation by the impact of the stream on the disk edge scales with the rate at which mass is added to the disk rather than the mass already there, and is therefore likely less important than tidal excitation for driving accretion.
- Excitation of shocks at the streamline-crossing radius may truncate CV disks in outburst but is likely less important than the linear mechanisms discussed here for exciting waves in quiescent disks if these extend only to the circularization radius of the incoming stream.

8.2 Applications and discussion
We discuss the applications of our results in the context of CV disks and circumplanetary disks in §7.

- For quiescent CV disks in which MRI is suppressed, density waves might drive the accretion rate if the resonance (ACR or SACR) happens at sufficiently large density, or if the disk has a sharp edge at which the sound speed does not vanish.\(^9\) Linear theory also predicts a steep dependence on the disk size that is qualitatively consistent with observations.
- For circumplanetary disks, these results suggest that there may exist an equilibrium state where the angular momentum added by accretion onto the disk is balanced by the tidal torque. As a result, accretion of the planet is not limited by the ability to remove excessive angular momentum from the disk.

More quantitative statements cannot be made with confidence for the lack of knowledge of the disk structure, especially in its outer parts. Our results suggest that discontinuities or ACR/SACR resonances near the disk edge are necessary for significant excitation, at least in quiescent disks that lie well inside their ILR and streamline-crossing radii. But observationally determining the properties of the disk edge is difficult. Nevertheless, if density waves are at all important for the angular momentum and accretion of quiescent CV disks, the waves must be driven predominantly by the tidal potential of the companion rather than the impact of the stream from the L1 point.

Our results also suggest that one has to be cautious when simulating the problem, since prescribing a disk model that is not fully realistic may cause the result to deviate by\(^9\) This presumes that the waves reach the inner parts of the disk before completely damping, as the three-dimensional nonlinear simulations of Ju et al. (2016, 2017) suggest; however, they ignore vertical stratification of density and temperature. Density waves will not go far in disks that are hottest near their midplanes, as might be expected of MRI-turbulent disks with high optical depth (Lubow & Ogilvie 1998). But passively illuminated quiescent disks may have the opposite thermal stratification, which favors long-range propagation (Ogilvie & Lubow 1999).
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orders of magnitude, especially when the disk Mach number is large. Moreover, extrapolating from simulations performed at Mach numbers $M \sim 10^{-20}$ to real disks at $M \sim 100$ is problematic, since the Mach number scaling is not likely to be a simple power law, and can be correctly determined only when both the equilibrium disk model and the main wave excitation mechanism are known. Because of the sensitivity to Mach number the temperature structure of the outer disk is just as important as its surface-density profile (unless the disk extends to the ILR). Simplified (e.g. isothermal) equations of state may give misleading results.

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APPENDIX A: VALIDITY OF IGNORING DISK SELF GRAVITY

Consider the gravitational potential perturbation $\varphi$ due to the density perturbation in the disk. Assume that the disk is 2D (i.e. zero thickness in $z$), we have

$$\nabla^2 \varphi = 4\pi G \Sigma(z).$$

(A1)

For $z \neq 0$, the WKB solution of $\varphi$ is Goldreich & Tremaine (1979)

$$\varphi(r, z) = \Phi(r) \exp \left( -|h(r)z| + i \int h(s) ds \right),$$

(A2)

with $\Phi(r)$ and $h(r)$ being real functions. Physically, $h(r)$ is the WKBJ frequency. Note that we have dropped the $\phi$ and $t$ dependence $\exp[i(m\phi - \omega t)]$ as we do for other variables. Now (A1) at $z = 0$ becomes

$$-2|h| \Phi \exp \left( i \int h(s) ds \right) = 4\pi G \Sigma.$$  

(A3)

To match the phase, for the most part of the disk we have

$$h(r) \sim k(r) \sim c^{-1}. $$

(A4)

Ignoring disk self gravity is a good approximation when $|\varphi(r, z = 0)/K(r)| \ll 1$. Since

$$\left| \frac{\varphi(r, z = 0)}{K(r)} \right| \sim \frac{c \Sigma}{c^2 \Sigma/\Sigma_0} \sim \Sigma_d M,$$

the condition under which our approximation holds become

$$\Sigma_d M \ll 1.$$  

(A6)

Note that $\Sigma_d$, when the disk is not too small compared to the Roche lobe, is of the order of the disk to primary mass ratio. This condition is satisfied for typical CV disks in quiescence and circumplanetary disks, therefore ignoring disk self gravity is a valid approximation for most part of the disk.

One caveat is that in the analysis above we are considering only regions in the disk where the WKBJ approximation is applicable. However, as we show in the main text, density waves are mainly excited where the WKBJ approximation does not hold. Still, based on the agreement between our result (ignoring self gravity) and that of Goldreich & Tremaine (1979) (including self gravity) for wave excitation by ILR, it is likely that self gravity in the wave excitation region will not significantly affect any nontrivial excitation.

APPENDIX B: EIGENVALUES OF THE SYSTEM

When written in the scale variables $y = [\sqrt{r\Sigma_0} \xi, \sqrt{r\Sigma_0} (K + W_m)]^T$ the first order equations (4a) and (4b) become

$$\frac{dy}{dr} = M \cdot y + w,$$

(B1)

where

$$M = \begin{bmatrix}
-\left[ \frac{2m\Omega}{r\sigma} + \frac{1}{2} \frac{d \ln (r \Sigma_0)}{dr} - N_0^2 \left( \frac{dK_0}{dr} \right)^{-1} \right] & m^2 \left( \frac{\sigma^2}{r^2 \sigma^2} - \frac{1}{c^2} \right) \\
\sigma^2 - \kappa^2 - N_0^2 & + \left[ \frac{2m\Omega}{r\sigma} + \frac{1}{2} \frac{d \ln (r \Sigma_0)}{dr} - N_0^2 \left( \frac{dK_0}{dr} \right)^{-1} \right]
\end{bmatrix},$$

(B2)

$$w = \begin{bmatrix}
\frac{1}{c^2} \sqrt{r\Sigma_0} W_m \\
N_0^2 \left( \frac{dK_0}{dr} \right)^{-1} \sqrt{r\Sigma_0} W_m
\end{bmatrix}. $$

(B3)

Note that $M$ is traceless. The eigenvalues of the system, therefore, are both purely real or both purely imaginary. Note that this is not the unique choice of $y$ that gives a traceless $M$. However, any different scaling will make $M$ directly depend on $\Sigma_0$, while here $M$ depends only on $1/H \equiv -d \ln \Sigma_0/dr$.

The eigenvalue is $\pm ik_0$ with

$$k_0^2 = \frac{\sigma^2 - \kappa^2 - N_0^2}{c^2} \left( 1 - \frac{m^2 c^2}{r^2 \sigma^2} \right) - \left[ \frac{2m\Omega}{r\sigma} + \frac{1}{2} \frac{d \ln (r \Sigma_0)}{dr} - N_0^2 \left( \frac{dK_0}{dr} \right)^{-1} \right]^2$$

(B4)

$$\approx \frac{\sigma^2 - \kappa^2 - N_0^2}{c^2} - \left( \frac{1}{2H} + N_0^2 \left( \frac{dK_0}{dr} \right)^{-1} \right)^2.$$
In the second line, all terms that are always $\lesssim O(1)$ are dropped. This gives the eigenvalue in (7).

**APPENDIX C: THE SECOND ORDER EQUATION AND THE LOCAL MODEL FOR RESONANCES**

**C1 The second order equation**

The first order equation (B1) can be rewritten as a first order equation in $y_1$ in the form of (13), with

$$p(r) = -\frac{d\ln|M_{12}|}{dr} \approx \frac{d\ln c^2}{dr}, \quad (C1a)$$

$$k^2(r) = M_{11} \left( \frac{d\ln M_{12}}{dr} - \frac{d\ln M_{11}}{dr} \right) + k_0^2 \approx k_0^2, \quad (C1b)$$

$$f(r) = M_{12} \frac{d(M_{12}w_1)}{dr} + M_{11}w_1 + M_{12}w_2$$

$$= \frac{1}{c^2} \sqrt{\Sigma_0} \left( \frac{dW_2}{dr} - \frac{2m\Omega}{r\sigma} W_2 \right) - \frac{d\ln(1 - m^2 c^2/r^2 \sigma^2)}{dr} \frac{1}{c^2} \sqrt{\Sigma_0} W_2 + \frac{m^2}{r^2 \sigma^2} N_0^2 \left( \frac{dK_0}{dr} \right)^{-1} \sqrt{\Sigma_0} W_2$$

$$\approx \frac{1}{c^2} \sqrt{\Sigma_0} \left( \frac{dW_2}{dr} - \frac{2m\Omega}{r\sigma} W_2 \right). \quad (C1c)$$

The approximate final equalities in each of eqs. (C1), and the final form of $k_0^2$ in eq. (8), are accurate when $\mathcal{M} \gg 1$ and $\mathcal{M}^2/r \gg 1/H, N_0^2(dK_0/dr)^{-1}$.

**C2 Obtaining the local model for resonances**

Near a resonance (i.e. a zero of $k_0^2$), the second-order equation can be further approximated as

$$\frac{d^2y_1}{dx^2} - xy_1 = \lambda^2 f(r), \quad (C2)$$

with $x \equiv (r - r_{\text{res}})/\lambda$ and $\lambda$ the lengthscale (25), provided that

$$\lambda \ll r, \quad \left| \frac{d\ln M_{12}}{dr} \right| \ll \lambda^{-1}, \quad \left| \frac{d}{dr} \left[ M_{11} \left( \frac{d\ln M_{12}}{dr} - \frac{d\ln M_{11}}{dr} \right) \right] \right| \ll \lambda^{-3}. \quad (C3)$$

A sufficient condition for (C3) to hold is that $c^2, H, N_0^2, dK_0/dr$ (and $\sigma^2 - \kappa^2$ when the resonance is not the ILR) are effectively constant over distances $\sim \lambda \ll r$.

Note, however, that we do not require $\Sigma_0$ to be slowly varying; indeed, its scale length $H$ may be comparable to $\lambda$ at an ACR or SACR, provided only that $H$ itself varies slowly. The variation of $f(r)$ in eq. (C2) is then dominated by the factor $\sqrt{\Sigma_0}$ that it contains. Defining $\alpha \equiv \lambda/2H$, we approximate eq. (C2) still further by putting

$$f(x) \approx f(r_{\text{res}}) e^{-\alpha x}, \quad \alpha \equiv \lambda/2H, \quad (C4)$$

which gives eq. (26).

**C3 Amplitude of the ingoing wave**

At large negative $x$, the solution to eq. (26) should consist of an ingoing wave that satisfies the homogeneous Airy equation—plus a non-wavelike part that is in phase with the local forcing term

$$y_1(x) \approx A_{in}[\text{Ai}(x) + i\text{Bi}(x)] + (-x)^{-1}\lambda^2 f(r_{\text{res}}) e^{-\alpha x} \quad x \rightarrow -\infty. \quad (C5)$$

The angular-momentum flux is proportional to the square of the first (wavelike) term, but the second term is much larger because of the exponential factor. In the forbidden zone, $y_1 \rightarrow 0$ as $x \rightarrow +\infty$.

To solve (26) for the coefficient $A_{in}$, we first adopt a new dependent variable $u(x) = e^{\alpha x} y(x)$, which satisfies

$$\frac{d^2u}{dx^2} - 2\alpha \frac{du}{dx} + (\alpha^2 - x)u(x) = \lambda^2 f(r_{\text{res}}). \quad (C6)$$

The solution for $u(x)$ should tend to zero in both directions, so it will have a Fourier transform $\tilde{u}(q)$,

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqx} \tilde{u}(q) \, dq. \quad (C7)$$

The transform $\tilde{u}(q)$ satisfies the transform of eq. (6),

$$\frac{d\tilde{u}}{dq} + (-\alpha^2 + 2\alpha q + i\alpha^2)\tilde{u} = 2\pi i \tilde{\delta}(q) \lambda^2 f(r_{\text{res}}), \quad (C8)$$

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with initial condition $\tilde{u}(q) = 0$ at $q < 0$, since the phase of $u(x)$ should increase with $x$ to have negative radial group velocity:

$$u(x) = i\lambda^2 f(r_{\text{res}}) \int_0^\infty \exp \left[ \frac{i}{2} q x^3 - \alpha q^2 + i q (x - \alpha^2) \right] dq.$$  \hfill (C9)

This strongly resembles the standard integral representations for the Airy functions, with convergence for all $x$ thanks to the factor $\exp(-\alpha q^2)$ in the integrand. For large negative $x$, two contributions dominate the integral. One of these comes from the lower endpoint $q \approx 0$:

$$u_{\text{nonwave}}(x) \approx (\alpha^2 - x)^{-1} \lambda^2 f(r_{\text{res}}) + O(x^{-3}).$$  \hfill (C10)

This matches the expected second term on the right side of eq. (C5) to leading order in $(-x)^{-1}$. The remaining contribution comes from the vicinity of the steepest-descent point, i.e. the point $q = q_0$ such that $\partial \psi(q, x)/\partial q = 0$ if $\psi(q, x)$ represents the argument of the exponential in the integral (C9).\textsuperscript{10} The result of the steepest-descent calculation matches the asymptotic expansion of $\text{Ai}(x) + i\text{Bi}(x)$ to leading order, and the coefficient implies

$$A_{\text{in}} = i\pi \lambda^2 f(r_{\text{res}}) e^{-\alpha^3/3}.$$  \hfill (C11)

**APPENDIX D: WAVE EXCITATION IN THE POLYTROPIC DISKS: ANALYTIC RESULTS**

When ILR lies outside the disk, the integrand of (17) (and hence the torque density on the disk) oscillates rapidly. Such integrals tend to be exponentially small when the envelope of the oscillation tapers slowly and smoothly to zero at both limits of integration. The integration (17) ends abruptly at $r_{\text{max}}$, however, and furthermore has a branch point there if $n = (\gamma - 1)^{-1}$ is not an integer, because $W^{-1}(r) \propto \Sigma_0(r) \propto (r_{\text{max}} - r)^{\gamma-1}$. This raises the possibility that the disk edge makes a contribution to eq. (17) that decreases as some power of $M_0$ rather than exponentially. To address this possibility, we analyse eq. (13) in a simpler local approximation.

When written in terms of $\tilde{K} \equiv K + W_m$ as the dependent variable instead of $y_1$, eq. (13) becomes

$$\frac{d^2}{dr^2} \tilde{K} + \frac{d}{dr} \tilde{K} + \tilde{K} = \tilde{f},$$  \hfill (D1)

with

$$\tilde{p}(r) = \frac{d}{dr} \ln \left( \frac{\Sigma_0 r}{|\sigma^2 - \kappa^2|} \right),$$  \hfill (D2)

$$\tilde{k}^2(r) = -\frac{2m \Omega}{r \sigma} \left[ \frac{d}{dr} \ln \left( \frac{\Sigma_0 \Omega}{|\sigma^2 - \kappa^2|} \right) - \frac{m^2}{r^2} + \frac{\sigma^2 - \kappa^2}{c^2} \right],$$  \hfill (D3)

$$\tilde{f}(r) = \frac{\sigma^2 - \kappa^2}{c^2} W_m.$$  \hfill (D4)

At the outer edge of the disk, $\Sigma_0$ and $c^2$ both go to zero and $\tilde{p}, \tilde{k}^2, \tilde{f}$ all have simple poles. The residue of $\tilde{k}^2$ defines a length scale $\delta$,

$$\delta^{-1} \equiv n \left[ \frac{\sigma^2 - \kappa^2}{\eta G M_1/r^2} + \frac{2m \Omega}{r \sigma} \right]_{r = r_{\text{max}}}.$$  \hfill (D5)

Note that $\delta/r \sim \mathcal{O}(\eta) \sim \mathcal{O}(M_0^{-2})$. So far, the purpose of studying wave excitation at the disk edge when $M_0 \gg 1$, it is reasonable to discard all but the leading-order behaviors (poles) of the functions (D2), (D3), and (D4), and to adopt a scaled independent variable

$$x \equiv \frac{r - r_{\text{max}}}{\delta}.$$  \hfill (D6)

The resulting simplified equation is

$$x^2 \frac{d^2}{dx^2} \tilde{K} + \frac{d}{dx} \tilde{K} - \tilde{K} = \mathcal{F}_0.$$  \hfill (D7)

The constant $\mathcal{F}_0 = \delta n [r^2 (\sigma^2 - \kappa^2) W_m/\eta G M_1]_{r = r_{\text{max}}}$ differs from $W_m(r_{\text{max}})$ by a small $O(\eta)$ correction.

Equation (D7) has a regular singular point at $x = 0$. The change of variable $x = -z^2/4$ makes it

$$\frac{d^2}{dz^2} \frac{K}{z} + \frac{2n - 1}{z} \frac{d}{dz} \frac{K}{z} - \frac{K}{z} = -\mathcal{F}_0.$$  \hfill (D8)

Clearly $n = 1/2$ is a particularly simple case because the homogeneous solutions are then sinusoids. The regular solution of eq. (D7) has a convergent power series in integral powers of $x$ and hence even powers of $z$, so we take $\cos z$ as the “regular”

\textsuperscript{10} There are two roots, $q_0 = -i\alpha \pm \sqrt{-x}$, but only one of these is close to the positive real axis if $x \ll -1$, and neither if $x \gg +1$.  

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solution of eq. (D8). The ingoing wave is \( \exp(-iz) \), and the coefficient of this wave as \( z \to \infty \) is formally \( -i \int_{0}^{\infty} F_0 \cos z \, dz \) [compare eq. (17)]. As it stands, this integral is not convergent. But \( F_0 \) is a proxy for \( W_m(r) \), which (since \( m \neq 0 \)) tends to zero as \( r \to 0 \), corresponding to \( z \to \infty \). So we should think of \( F_0 \) as a smooth slowly decreasing function \( F(z/M_0) \); furthermore \( F(t) = F(-t) \) because \( W_m \) is a regular function of \( r - r_{\text{max}} = -z^2 \delta/4 \). So the amplitude of the ingoing wave is 

\[
(-i/2)M_0 \int_{-\infty}^{\infty} F(t) \cos(M_0 t) \, dt, 
\]

which is exponentially small at large \( M_0 \) since \( F(t) \) is smooth.

Returning now to the general case, let \( \nu \equiv n - 1 \) and set \( \tilde{K} = z^{-\nu} u(z) \), so that eq. (D8) becomes

\[
\frac{d^2 u}{dz^2} + \frac{1}{2} \frac{du}{dz} + \left( 1 - \frac{\nu^2}{z^2} \right) u = -z^{\nu} F(z/M_0). 
\]

The homogeneous solutions of this last equation are Bessel functions of order \( \nu \). The solution corresponding to the regular homogeneous solution of (D7) is \( J_\nu(z) \), the ingoing wave is \( H_\nu^{(2)}(z) = J_\nu(z) - i Y_\nu(z) \), and the amplitude of this wave in the particular solution as \( z \to \infty \) is

\[
A_{\text{in}}(n) \approx \frac{i \pi}{2} \int_{0}^{\infty} z^{\nu+1} J_\nu(z) F(z/M_0) \, dz, 
\]

which has made use of the Wronskian \( W[J_\nu, H_\nu^{(2)}] = -i W[J_\nu, Y_\nu] = 2i/\pi z \). We write \( \approx \) rather than \( = \) because eq. (D10) is based on the local approximation (D7) rather than the exact IWE. This last integral is again exponentially small, which can be seen as follows. Set \( F(t) = g(t^2) \) (since \( F \) is even in its argument) and suppose that \( g(u) \) has an inverse Laplace transform \( \hat{g}(s) \), so that \( F(t) = \int_{0}^{\infty} \hat{g}(s) e^{-s t^2} \, ds \). Putting this into eq. (D10), reversing the order of integration, and invoking Abramowitz & Stegun (1972, §11.4.29) yields

\[
A_{\text{in}}(n) \approx \frac{i \pi}{2} \int_{0}^{\infty} \left( \frac{M_0^2}{2s} \right)^{\nu+1} \exp \left( -\frac{M_0^2 s}{4} \right) \hat{g}(s) \, ds
\]

\[
= \frac{i \pi}{2} M_0^2 \int_{0}^{\infty} (2\sigma)^{-\nu-1} e^{-1/(4\sigma)} \hat{g}(M_0^2 \sigma) \, d\sigma. 
\]

Note finally that \( \hat{g}(s) \) decreases faster than any power of \( s \) as \( s \to \infty \) because \( g(u) = F(\sqrt{u}) \) has a convergent Taylor series at \( u = 0 \) \( (r = r_{\text{max}}) \), so that

\[
\frac{d^k \hat{g}}{du^k}(0) = (-1)^k \int_{0}^{\infty} \hat{g}(s) s^k \, ds
\]

must exist. Therefore the integral (D11) decreases faster than any power of \( M_0^{-2} \) as \( M_0 \to \infty \).

To sum up, by inspecting the integral (17) we find that the excitation will be exponentially small except when the thin region near outer edge of the disk \( (\text{with characteristic size } \delta \sim \eta) \) makes a nontrivial contribution. Then, we illustrated (using the local model discussed above) that the contribution from this region decreases faster than any power of \( M_0^{-2} \) as \( M_0 \to \infty \). As the numerical result in §3.2 shows, this is indeed the case and the amplitude decreases exponentially as \( M_0 \) increases.

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11 Interior to the ILR, negative radial group velocity corresponds to a phase that decreases inward from the disk edge, and therefore with increasing \( z \).