JACOB’S LADDERS AND THE NONLOCAL INTERACTION OF THE FUNCTION $|\zeta(1/2 + it)|$ WITH THE FUNCTION $\arg \zeta(1/2 + it)$ ON THE DISTANCE $\sim (1 - c)\pi(t)$

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Abstract. In this paper we obtain a new-type formula - a mixed formula - which connects the functions $|\zeta(1/2 + it)|$ and $\arg \zeta(1/2 + it)$. This formula cannot be obtained in the classical theory of A. Selberg, and, all the less, in the theories of Balasubramanian, Heath-Brown and Ivic.

1. Introduction

Let us remind that in the formula

$\zeta \left( \frac{1}{2} + it \right) = \left| \zeta \left( \frac{1}{2} + it \right) \right| e^{i \arg \zeta \left( \frac{1}{2} + it \right)}$

the $\arg \zeta \left( \frac{1}{2} + it \right)$ is defined as follows. If $t = \gamma$, where the $\beta + i\gamma$ is a zero of $\zeta(s)$, the $\arg \zeta \left( \frac{1}{2} + it \right)$ is obtained by continuous variation along the straight lines joining 2, 2 + it and 1/2 + it, starting from the value $\arg \zeta(2) = 0$. If $t = \gamma$ then

$\arg \zeta \left( \frac{1}{2} + i\gamma \right) = \lim_{t \to \gamma^+} \arg \zeta \left( \frac{1}{2} + it \right)$.

Let

$S(t) = \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + it \right), \quad S_1(T) = \int_0^T S(t)dt$.

First of all, there are the asymptotic formulae for the integrals

$\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt, \quad \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt$,

(Hardy and Littlewood started to study these integrals in 1918, 1922). On the other hand, there are Selberg’s asymptotic formulae (13, 14, in 1944, 1946)

$\int_T^{T+U} \{S(t)\}^{2k}dt \sim \frac{(2k)!}{k!(2\pi)^{2k}} U(\ln \ln T)^k, \quad k \leq 2$

$\int_T^{T+U} \{S_1(t)\}^{2k}dt \sim c_k U, \quad T \to \infty, \quad k = 1, \quad U = T^{1/2+\epsilon}$

where $U = T^{1/2+\epsilon}$ and $k$ is a fixed positive number, (Littlewood and Titchmarsh started to study the integrals (13) and (14), $k = 1$, see 2, 15 in 1925, 1928).

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In this paper we obtain a formula of a new type - a mixed formula - i.e. a formula which connects the functions (see (1.1)) \( |\zeta(1/2 + it)|, \arg \zeta(1/2 + it) \).

This paper is a continuation of the series of papers [3][12].

2. Result

Let \( \varphi_1(T), \ T \geq T_0[\varphi_1] \) stands for the Jacob's ladder. The following theorem holds true.

**Theorem.** For every fixed \( k \in \mathbb{N} \) and for every fixed Jacob's ladder there is the single-valued function of \( T \)

\[ \tau_k = \tau_k(T; \varphi_1) = \tau_k(T), \ T \geq T_1[\varphi_1], \]

for which the following asymptotic formula

\[ \arg \zeta \left( \frac{1}{2} + it \right) dt \sim \pi(c_k) \frac{\arg \zeta(T)}{|\zeta(1/2 + it)|^2} \]

is true, where

(A) \( \tau_k \in (T, T + U), \ \varphi_1(\tau_k) \in (\varphi_1(T), \ \varphi_1(T + U)), \ U = T^{1/2+\varepsilon} \)

(B) \( \varphi_1(T + U) - \varphi_1(T) \sim T, \ \varphi_1(T + U) < T \)

(C) \( \rho([T, T + U]; [\varphi_1(T), \varphi_1(T + U)]) \sim (1 - c)\pi(T) \to \infty \) as \( T \to \infty \) and \( \rho \)

denotes the distance of the corresponding segments, \( c \) is the Euler’s constant and \( \pi(T) \) is the prime-counting function.

**Remark 1.** By (2.1) we have the prediction of the value

\[ \left| \zeta \left( \frac{1}{2} + i\tau_k(T) \right) \right| = |Z(\tau_k(T))|, \ \tau_k(T) \in (T, T + U) \]

generated by the Riemann zeta-function, by means of the value

\[ \left| \arg \zeta \left( \frac{1}{2} + it \right) dt \right|, \ \varphi_1(\tau_k(T)) \in (\varphi_1(T), \varphi_1(T + U)) \]

which descends from very deep past (see (B),(C), comp. [7], Remarks 3,4).

**Remark 2.** It is quite evident that the formula (2.1) cannot be obtained within the classical theory of A. Selberg (see [13],[14]), and, all the less, in the theories of Balasubramanian, Heath-Brown and Ivic, (comp. [1]).

3. The first corollaries

Using the mean-value theorem in (2.1) and putting \( \varphi_1(\tau_k(T)) \sim \tau_k(T), \) (this follows from \( t - \varphi_1(t) \sim (1 - c)\pi(t) \)), we obtain

**Corollary 1.**

\[ \omega[\arg \zeta] \sim \frac{\pi(c_k) \frac{\arg \zeta(T)}{\tau_k(T)}}{|\zeta(1/2 + it)|^2}, \]

where

\[ \omega[\arg \zeta] = \omega[\arg \zeta(1/2 + it); \ t \in [0, \varphi_1(\tau_k(T))]] \]
denotes the mean-value of arg\(\zeta(1/2 + it)\), \(t \in [0, \varphi_1(\tau_k(T))]\) and \(\varphi_1(\tau_k(T)) \in (\varphi_1(T), \varphi_1(T + U))\).

**Remark 3.** By (3.1) the following holds true: the value |\(\zeta(1/2 + it)\)| is asymptotically defined by the mean-value \(\omega[\arg \zeta]\), which descends from the very deep past (comp. Remark 1) and vice-versa.

**Remark 4.** It is evident that the set of zeroes \(t = \gamma\) of the function \(\zeta(1/2 + it)\), \(t \in [T, T + U]\) is the exceptional set for the values of the function \(\tau_k = \tau_k(T)\), i.e. \(\tau_k(T) \neq \gamma\).

In connection with this we have the following additional

**Corollary 2.**

\[
\lim_{T \to \infty} |\omega[\arg \zeta]| \cdot |\zeta(1/2 + it\tau_k(T))|^{1/n} = 0.
\]

**4. The asymptotic formula for the distance \(\tilde{\gamma}' - \tilde{\gamma}\) of some subsequence of consecutive zeroes of \(\zeta(1/2 + it)\)**

4.1. For simplicity, we put \(k = 1\) in (3.1)

\[
|\omega[\arg \zeta]| \cdot \left| \zeta \left( \frac{1}{2} + i\tau_1(T) \right) \right| \sim \pi \sqrt{c_1} \frac{\sqrt{\ln \tau_1(T)}}{\tau_1(T)}.
\]

Let \(t = \tilde{\gamma}, \tilde{\gamma}'\) denotes the consecutive zeroes of the function \(|\zeta(1/2 + it)|\), for which

\[
\tilde{\gamma} < \tau_1(T) < \tilde{\gamma}', \ \tau_1(T) \in (T, T + U)
\]

holds true, and \(n(\tilde{\gamma})\) denotes the order of the zero \(t = \tilde{\gamma}\). Since

\[
\left| \zeta \left( \frac{1}{2} + i\tau_1(T) \right) \right| = |Z[\tau_1(T)]| = \frac{1}{\{n(\tilde{\gamma})\}!} \left| Z^{(n(\tilde{\gamma}))}[\tau_1(T)] \right| \cdot |\tau_1(T) - \tilde{\gamma}|^{n(\tilde{\gamma})}, \ \tilde{\gamma} < \tau_1^1(T) < \tau_1(T)
\]

(see (4.2)) then from (4.1), \((\tau_1(T) \sim \tilde{\gamma})\), the asymptotic formula

\[
\tau_1(T) - \tilde{\gamma} \sim \left( \frac{\pi \sqrt{c_1} \{n(\tilde{\gamma})\}! \sqrt{\ln \gamma}}{\gamma |\omega[\arg \zeta]| \cdot |Z^{(n(\tilde{\gamma}))}[\tau_1^1(T)]|^n} \right)^{1/n(\tilde{\gamma})}
\]

follows, and similarly, we obtain

\[
|\tau_1(T) - \tilde{\gamma}'| = \tilde{\gamma}' - \tau_1(T) \sim \left( \frac{\pi \sqrt{c_1} \{n(\tilde{\gamma}')\}! \sqrt{\ln \gamma}}{\gamma |\omega[\arg \zeta]| \cdot |Z^{(n(\tilde{\gamma}'))}[\tau_1^1(T)]|^n} \right)^{1/n(\tilde{\gamma}')}, \ \tau_1(T) < \tau_1^2(T) < \tilde{\gamma}'.
\]

Then we obtain from (4.4), (4.5)

**Corollary 3.** If the pair of consecutive zeroes \(\tilde{\gamma}, \tilde{\gamma}'\) fulfils (4.2) then the asymptotic formula

\[
\tilde{\gamma}' - \tilde{\gamma} \sim \left( \frac{\pi \sqrt{c_1} \{n(\tilde{\gamma}')\}! \sqrt{\ln \gamma}}{\gamma |\omega[\arg \zeta]| \cdot |Z^{(n(\tilde{\gamma}'))}[\tau_1^1(T)]|^n} \right)^{1/n(\tilde{\gamma}')}
\]

\[
+ \left( \frac{\pi \sqrt{c_1} \{n(\tilde{\gamma}')\}! \sqrt{\ln \gamma}}{\gamma |\omega[\arg \zeta]| \cdot |Z^{(n(\tilde{\gamma}'))}[\tau_1^1(T)]|^n} \right)^{1/n(\tilde{\gamma})}, \ \tilde{\gamma} < \tau_1^1(T) < \tau_1(T) < \tau_1^2(T) < \tilde{\gamma}'
\]

holds true.
5. LEMMAS

5.1. Let us remind that

\[ \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \quad \varphi_1(t) = \frac{1}{2} \varphi(t), \]

where

\[ \tilde{Z}^2(t) = \frac{Z^2(t)}{2\psi |\varphi(t)|} = \frac{Z^2(t)}{\left\{ 1 + \mathcal{O}\left( \frac{\ln \ln t}{\ln t} \right) \right\} \ln t}. \]

(see [4], (3.9); [5], (1.3); [9], (1.1), (3.1), (3.2)). The following Lemma holds true (see [8], (2.5); [9], (3.3)).

Lemma 1. For every integrable function (in the Lebesgue sense) \( f(x), x \in [\varphi_1(T), \varphi_1(T + U)] \) the following is true

\[ \int_T^{T+U} f[\varphi_1(t)]\tilde{Z}^2(t)dt = \int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x)dx, \quad U \in \left(0, \frac{T}{\ln T}\right). \]

where \( t - \varphi_1(t) \sim (1 - c)\pi(t). \)

Remark 8. The formula \eqref{5.3} remains true also in the case when the integral on the right-hand side of \eqref{5.3} is only relatively convergent improper integral of the second kind (in the Riemann sense).
5.2. Next, the following $\mathcal{Z}^2$-transformation of the formulae of A. Selberg (1.3), (1.4) holds true (comp. [7], Concluding remarks).

**Lemma 2.**

\begin{equation}
\int_T^{T+U} \{S[\varphi_1(t)]\}^{2k} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim \frac{(2k)!}{k!(2\pi)^{2k}} U \ln T (\ln \ln T)^k,
\end{equation}

\begin{equation}
\int_T^{T+U} \{S_1[\varphi_1(t)]\}^{2k} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim c_k U \ln T.
\end{equation}

**Proof.** From (1.3) and by Lemma 1 we obtain

\begin{equation}
\int_T^{T+U} \{S[\varphi_1(t)]\}^{2k} \mathcal{Z}^2(t) dt = \int_{\varphi_1(T)}^{\varphi_1(T+U)} \{S[\varphi_1(t)]\}^{2k} dt \sim \\
\frac{(2k)!}{k!(2\pi)^{2k}} \{\varphi_1(T+U) - \varphi_1(T)\} \ln \varphi_1(T) = \\
\frac{(2k)!}{k!(2\pi)^{2k}} \frac{\varphi_1(T+U) - \varphi_1(T)}{U} \ln \varphi_1(T) = \\
\frac{(2k)!}{k!(2\pi)^{2k}} U \tan[\alpha(T,U)] \ln \varphi_1(T),
\end{equation}

where $\alpha(T,U)$ is the angle of the chord of the curve $y = \varphi_1(t)$ that binds the points $[T, \varphi_1(T)]$ and $[T+U, \varphi_1(T+U)]$. Let us remind that

\begin{equation}
\tan[\alpha(T,U)] \sim 1, \quad U \in \left[ T^{1/3+\epsilon}, \frac{T}{\ln T} \right],
\end{equation}

(comp. [3], (8.3), [7], (3.9)) and

\begin{equation}
t - \varphi_1(t) \sim (1-c)\pi(t),
\end{equation}

(see (5.1), [3], (6.2)). Since by (5.8) we have

\begin{equation}
\varphi_1(T) \sim T, \quad \ln \varphi_1(T) \sim \ln \ln T,
\end{equation}

then from (5.6) by (5.7), (5.9) this formula

\begin{equation}
\int_T^{T+U} \{S[\varphi_1(t)]\}^{2k} \mathcal{Z}^2(t) dt \sim \frac{(2k)!}{k!(2\pi)^{2k}} U (\ln \ln T)^k
\end{equation}

follows. Using the mean-value theorem in (6.10) we obtain (5.10) by (5.2). Similarly we obtain (5.3).

6. **PROOF OF THE THEOREM**

Using the mean-value theorem in (6.10) we obtain

\begin{equation}
|S_1[\varphi_1(\tau)]|^{2k} \mathcal{Z}^2(\tau) \sim c_k \ln T \sim c_k \ln \tau,
\end{equation}

$\tau = \tau(T,U,k) \in (T, T+U)$, $\varphi_1(\tau) \in (\varphi_1(T), \varphi_1(T+U))$.

Since $U = T^{1/2+\epsilon}$ then $\tau = \tau(T,k,\epsilon)$, $T \geq T_0[\varphi_1]$. Next, if for every $T \geq T_0[\varphi_1]$ we take one mean-value $\tau_k(T) \in \{\tau(T,k,\epsilon)\}$ then we have the single-valued function $\tau_k = \tau_k(T)$ for fixed $k, \epsilon$ (comp. [7], Remark 2). Hence by (5.11) we have

\begin{equation}
|S_1[\varphi_1(\tau_k(T))]|^{2k} \sim c_k \frac{\ln \tau_k(T)}{\left| \zeta \left( \frac{1}{2} + i\tau_k(T) \right) \right|^2},
\end{equation}

$\tau_k(T) \in (T, T+U)$, $\varphi_1(\tau_k(T)) \in (\varphi_1(T), \varphi_1(T+U))$.
Then from (6.2) the asymptotic formula (2.1) and (A) follows. The expressions (B) and (C) are identical with (C), (D), \(k=1\) of Theorem in [9].

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