Generating MHV super-vertices in light-cone gauge

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Abstract: We construct the $\mathcal{N} = 1$ SYM lagrangian in light-cone gauge using chiral superfields instead of the standard vector superfield approach and derive the MHV lagrangian. The canonical transformations of the gauge field and gaugino fields are summarised by the transformation condition of chiral superfields. We show that $\mathcal{N} = 1$ MHV super-vertices can be described by a formula similar to that of the $\mathcal{N} = 4$ MHV super-amplitude. In the discussions we briefly remark on how to derive Nair’s formula for $\mathcal{N} = 4$ SYM theory directly from light-cone lagrangian.

Keywords: Gauge symmetry, QCD, Supersymmetric gauge theory
1. Introduction

Ever since its discovery the CSW prescription originally conjectured by Cachazo, Svrček
and Witten in [1, 3] has been shown to be an efficient method for constructing gluon
scattering amplitudes. In this approach the off-shell continued MHV amplitudes (non-
trivial amplitudes with the largest number of positive helicity gluons) are taken as the new
vertices, which allows numerous varieties of subgraph structures in the standard Feynman
graphs to be represented by Parke-Taylor formula [3]. The CSW rules were successfully
generalised to one-loop level and to include quarks and superpartners [4, 6, 5, 7]. A
lagrangian derivation of the rules was found by Mansfield [8] and independently by Gorsky
and Rosly [9] through canonically transforming the self-dual part of the LCYM lagrangian
into a free field theory

\[ \mathcal{L}^{--} [A] + \mathcal{L}^{--} [\bar{A}] = \mathcal{L}^{--} [B]. \]  

(1.1)

The transverse components \( A \) and \( \bar{A} \) of the gauge field in light-cone coordinates were
assumed to be functionals of the new field variables \( B \) and \( \bar{B} \) so that after performing
the transformation the vertices in the new lagrangian have the same helicity structure as
prescribed by the CSW rules.
\[ A_1 = B_1 + \Upsilon_{123} B_2 B_3 + \cdots \] (1.2)

\[ \bar{\partial} A_1 = \bar{\partial} B_1 + \Xi_{123}^2 \bar{\partial} B_2 B_3 + \Xi_{123}^2 B_2 \bar{\partial} B_3 + \cdots \] (1.3)

In 4-dimensions the MHV vertices were algebraically verified to agree with the Parke-Taylor formula using holomorphy of the translation kernels \( \Upsilon \) and \( \Xi^k \) in [10, 11] and the D-dimensional theory was given in [12] which restored the loop-level amplitudes originally appeared “missing” from the CSW rules. The corresponding MHV lagrangians for QCD and SQCD were derived by Ettle, Morris and Xiao by tranforming the physical field components and their canonical conjugate variables on a pair by pair basis [13, 14].

An alternative strategy was found by Britto, Chachazo, Feng and Witten from analysing singularities of the amplitude when external leg momenta are shifted by a complex value [15, 16]. Using Cauchy’s theorem it was shown that a generic amplitude can be derived from scattering amplitudes of fewer particles whose leg momenta are determined by poles. The method of BCFW recursion has been shown to be a powerful tool for tree-level calculations [18] and was extended to theories containing massive particles and fermions [19, 20]. Combining with generalised unitarity the BCFW recursion relation was also extended to loop-level calculations [21].

Recently, BCFW recursion has been generalised to compute tree-level amplitudes in \( \mathcal{N} = 4 \) supersymmetry Yang-Mills theory [22]. Instead of shifting individual scattering amplitudes labeled by particle species and momenta, in the supersymmetry generalisation of BCFW recursion one considers super-amplitudes whose initial and final states are described by momentum space super-wavefunctions. The super-wavefunction contains a superposition of all the single particle states in the \( \mathcal{N} = 4 \) supermultiplet, each of them being tagged by bookkeeping Grassmann variables \( \eta_A \),

\[
\Phi(p, \eta) = G^+(p) + \eta_A \Gamma^A(p) + \frac{1}{2} \eta_A \eta_B S^{AB}(p) + \frac{1}{3!} \eta_A \eta_B \eta_C \epsilon^{ABCD} \Gamma_D(p) + \frac{1}{4!} \eta_A \eta_B \eta_C \eta_D \epsilon^{ABCD} G^-(p). \tag{1.4}
\]

The conventional scattering amplitude is obtained from differentiating super-amplitude with respect to the appropriate Grassmann variables determined by the species of the particles which participate the scattering event. For example differentiating Nair’s formula for \( \mathcal{N} = 4 \) MHV super-amplitude [24]

\[
A_{\mathcal{N}=4}^{\text{MHV}}(1, 2, \cdots n) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \delta^{(4)} \left( \sum_{i,j=1}^n \langle i \; j \rangle \eta_{Ai} \eta_{Aj} \right) \tag{1.5}
\]

with respect to \( \eta_{Ai} \) and \( \eta_{Aj} \), from \( A = 1 \) to 4, yields the familiar Parke-Taylor formula for gluon scattering, where the \( i \)-th and \( j \)-th legs are associated with negative helicity gluons. The generalised supersymmetry BCFW recursion formula is then obtained by applying Cauchy’s theorem to the super-amplitude where Grassmann variables are shifted along with leg momenta [25]. All tree-level super-amplitudes were computed in [23] by
Drummond and Henn and the super-amplitudes were verified to be superconformal invariant, where the SUSY generators for $\mathcal{N} = 4$ theory were given by (1.5),

$$Q_{\alpha A} = \lambda_{\alpha} \eta_A, \quad \bar{Q}_{\dot{\alpha} A} = \bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta_A}.$$  \hspace{1cm} (1.6)

In this paper we canonically transform the chiral superfields in the usual momentum space description to produce the MHV lagrangian for $\mathcal{N} = 1$ SYM theory. The supersymmetry generalisations to equations (1.2) and (1.3) automatically summarise the relation between the transformation formulae of gluon and gluino fields. We perform a fermionic integral transformation which replaces the super-space Grassmann variables $\theta$ and $\bar{\theta}$ used to label off-shell superfields by a single Grassmann variable $\eta$. The integral transformation allows us to directly derive the $\mathcal{N} = 1$ analogue of (1.6) as representations of the SUSY generators in the new super-space. In section 4.1 we adapt the supersymmetry BCFW recursion to $\mathcal{N} = 1$ theory and calculate the generic n-point “helicity-ordered” MHV super-amplitude formula. The more symmetrical formula derived by Bern, Carrasco, Ita, Johansson and Roiban (7) can be reproduced by superimposing helicity-ordered super-amplitudes with all possible helicity configurations. We compute the translation kernels to all order and show that the MHV super-vertices in the new lagrangian are described by the same formula as that of the MHV super-amplitude

$$V_{\mathcal{N} = 1}^{MHV} (1^+, 2^+ \ldots i^-, j^-, \ldots n^+) = \frac{\langle i, j \rangle^3}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \delta (\sum_{i,j=1}^{n} \langle i, j \rangle \eta_i \eta_j),$$  \hspace{1cm} (1.7)

with the off-shell continuation defined in the light-cone $\hat{p}$ direction which is absent in formula (1.7). In the discussions we remark on how the method is extended to the $\mathcal{N} = 4$ SYM theory to generate the super-space MHV lagrangian that has vertices described by Nair’s formula (1.5). The notation used throughout this paper and, in particular, the definition of spinors for off-shell momenta are summarised in appendix A.

2. Chiral Construction of the $\mathcal{N} = 1$ SYM lagrangian

In the textbook approach, the supersymmetric non-abelian gauge theory is constructed from the fieldstrength of a vector superfield. To arrive at an MHV lagrangian we can pick a convenient gauge, integrate over unphysical degrees of freedom, and canonically transform the helicity gluon and gluino fields separately. However the underlying supersymmetry implies that we can organise the physical fields that belong to the same supermultiplet into a conceptually simpler structure. In (25) Feng and Huang successfully derived the MHV lagrangian for $\mathcal{N} = 4$ SYM theory in which the chiral superfield has a simple transformation condition in coordinate space. In pursuit of this idea we may be tempted to apply the canonical transformation directly on the $\mathcal{N} = 1$ vector superfield, but the vector superfield depends on a large number of unphysical degrees of freedom and the lagrangian does not have an easily manipulated structure when it is written in terms of vector superfields. As noted by Siegel and Gates (24) alternately the physical field components can be packed into chiral superfields. A systematic method was developed by Ananth, Brink, Lingren,
Nilsson and Ramond to derive the $\mathcal{N} = 4$ light-cone SYM lagrangian dimensionally reduced from 10-dimensions [30, 31]. In light-cone gauge the physical fields are closed under the SUSY subalgebra $Q_1, \bar{Q}_1$.

\[
\begin{align*}
Q_1 A &= i \Lambda, & \bar{Q}_1 \bar{A} &= 0, & \bar{Q}_1 \bar{\Lambda} &= -i \bar{\Lambda}, \\
Q_1 \Lambda &= 0, & \bar{Q}_1 \bar{\Lambda} &= \hat{\partial} A, & Q_1 \bar{\Lambda} &= -\bar{\partial} \bar{A}, & \bar{Q}_1 \bar{A} &= 0,
\end{align*}
\]

where in the above equations we used $\psi_\alpha = (\bar{T} \bar{\Lambda})$, $\bar{\psi}_\dot{\alpha} = (T \bar{\Lambda})$ to denote the gluino field components. Note that with the quadratic terms eliminated by the gauge condition $\hat{A} = 0$, the transformation relations (2.1) and (2.2) remain the same in the on-shell process. The closure of the physical field components under this SUSY subalgebra allows us to define chiral superfields without the need of introducing auxiliary fields [30].

\[
\Phi(x, \theta) = A(y) + i \theta \Lambda(y),
\]

\[
\bar{\Phi}(x, \bar{\theta}) = \bar{A}(\bar{y}) + i \bar{\theta} \bar{\Lambda}(\bar{y}),
\]

where gluons and gluinos having the same helicities are enclosed into the same superfield, $y = (x^+, x^- + \frac{1}{2} i \theta \bar{\theta}, x^z, x^{z\dot{z}})$, and $\bar{y} = (x^+, x^- - \frac{1}{2} i \bar{\theta} \theta, x^z, x^{z\dot{z}})$. We introduce the shorthand notation for representations of the SUSY covariant derivatives and generators $D_1 = d$, $\bar{D}_1 = \bar{d}$, $Q_1 = q$, and $\bar{Q}_1 = \bar{q}$, which stand for

\[
d = \frac{\partial}{\partial \theta} + \frac{i}{2} \bar{\theta} \bar{\partial}, \quad \bar{d} = -\frac{\partial}{\partial \bar{\theta}} - \frac{i}{2} \theta \partial,
\]

\[
q = \frac{\partial}{\partial \theta} - \frac{i}{2} \bar{\theta} \bar{\partial}, \quad \bar{q} = -\frac{\partial}{\partial \bar{\theta}} + \frac{i}{2} \theta \partial.
\]

The superfields defined in (2.3) and (2.4) satisfy the chiral constraints $d \Phi = d \bar{\Phi} = 0$. Using the chiral superfields just defined, one can construct a SUSY invariant lagrangian as a D-term integral [32]

\[
S = -\frac{4i}{g^2} \int d^4 x d\theta d\bar{\theta} \text{tr} \left( \bar{\Phi} \frac{\partial^2}{\partial \bar{\theta}^2} \Phi + [\Phi, \bar{\theta} \bar{\Phi}] \bar{\Phi} + [\bar{\Phi}, \bar{\theta} \Phi] \Phi - i [\Phi, \bar{\Phi}] \frac{1}{\partial^2} [\bar{\Phi}, d \Phi] \right)
\]

It is straightforward to verify that after integrating $\theta, \bar{\theta}$ equation (2.7) is the same as the standard $\mathcal{N} = 1$ SYM lagrangian with $\hat{A}$ eliminated by light-cone gauge condition and all of the unphysical fields integrated out.

### 3. Transforming light-cone gauge SYM into the new representation

In order to interpret coefficients in the interaction terms as vertices in Feynman rules we need to work in an unconstrained description so that the superfields can be taken as functional integral variables. Note that when $(y, \theta, \bar{\theta})$ and $(\bar{y}, \theta, \bar{\theta})$ are regarded as sets of independent variables the chiral conditions demand that the chiral and anti-chiral superfield
to be independent of $\bar{\theta}$ and $\theta$ respectively: $\bar{d}\Phi = \frac{\partial}{\partial \bar{\theta}} \Phi \big|_{y, \theta} = 0$, $d\tilde{\Phi} = \frac{\partial}{\partial \theta} \tilde{\Phi} \big|_{y, \bar{\theta}} = 0$. As pointed out by Mandelstam in [33] for this purpose one can start with a single Grassmann variable representation and reconstruct the light-cone gauge action. Nevertheless in this paper we choose to remove the constraint by mapping the expressions (2.3) and (2.4) through integral transformations. This is possible because when considered individually each superfield has only one Grassmannian dependence. It is the appearance of both chiral and anti-chiral superfields in the action that requires both $\bar{\theta}$ and $\theta$. The mapping generalises to terms that contain multiple superfields and to the MHV lagrangian to be discussed in section 5. For functions of $(\bar{y}, \bar{\theta})$, we define a super-space analogue to the Fourier transform and maps the function to a new super-space labeled by momentum and a Grassmann variable $\eta$. Denoting the inverse of this transform as $T$ we have $T^{-1} : f(\bar{y}, \bar{\theta}) \rightarrow f(p, \eta)$. Since SUSY covariant derivatives anti-commute with themselves, the function $\bar{d}f(\bar{y}, \bar{\theta})$ satisfies anti-chiral constraint and therefore is a function of $(y, \theta)$. This allows us to reuse the integral transformation to bring both functions $f(\bar{y}, \bar{\theta})$ and $f(y, \theta)$ into the same super-space. $(dT)^{-1} : f(y, \theta) \rightarrow f(p, \eta)$. The new super-space $(p, \eta)$ has one Grassmann variable fewer than the original super-space $(x, \theta, \bar{\theta})$ so the superfields are no longer constrained after the transformation. Applying different transformations on chiral and anti-chiral superfields does not generate extra representations to the SUSY generators here. For anti-chiral superfields the generator becomes $T^{-1}qT$, while for chiral superfields we have

$$(dT)^{-1}q(dT) = -(dT)^{-1}d\Phi = -(dT)^{-1}(dT)T^{-1}T^{-1}d\Phi,$$ \hspace{1cm} (3.1)

and the extra minus sign cancels after moving generator in the new representation across the Grassmannian delta function produced from $(dT)^{-1}(dT)$.

We choose the integral transformation for an anti-chiral superfield as

$$\Phi(\bar{y}, \bar{\theta}) = -\int d^{4}\bar{y}d\eta e^{-i\bar{y}p} \delta(\bar{\theta}\bar{p}^{\frac{1}{2}} - \eta) \tilde{\phi}(p, \eta)$$ \hspace{1cm} (3.2)

The transformation for chiral superfields follows from taking covariant derivative on both sides of (3.2). In momentum space the formulae read:¹

$$\tilde{\Phi}(p, \theta) = -\int d\eta e^{-\frac{i}{2}\theta\bar{p}} \delta(\bar{\theta}\bar{p}^{\frac{1}{2}} - \eta) \tilde{\phi}(p, \eta),$$ \hspace{1cm} (3.5)

¹When an analytic continuation for the negative light-cone energy $\bar{p}$ is needed, we use

$$\tilde{\Phi}(p, \theta) = -\int d\eta e^{-\frac{i}{2}\theta\bar{p}} \delta(\bar{\theta}\sqrt{\bar{p}} - \eta) \tilde{\phi}(p, \eta),$$ \hspace{1cm} (3.3)

$$d\tilde{\Phi}(p, \theta) = -\sqrt{\bar{p}}\int d\eta e^{\frac{i}{2}\theta\bar{p}} \delta(1 - \theta\eta\bar{p}/\sqrt{\bar{p}}) \tilde{\phi}(p, \eta)$$ \hspace{1cm} (3.4)

in place of (3.3) and (3.5), where the square roots are determined by the analytic continuation used in the definition of Lorentz invariant bracket $(ij) = (ij)/\sqrt{i^2}$. In the rest of this paper we follow the convention used in [3] and define $\sqrt{\bar{p}} = \text{sgn}(\bar{p}) |\bar{p}|^{\frac{1}{2}}$. Alternatively one can adopt the analytic continuation of [33] and assume $\sqrt{\bar{p}} = i|\bar{p}|^{\frac{1}{2}}$ when $\bar{p}$ is negative, which produces a relative phase factor $i$ in the delta function in propagator (3.13).
\[ \bar{d} \Phi(p, \theta) = -\hat{p}^{\frac{1}{2}} \int d\eta e^{\frac{i}{2} \hat{p} \theta} \delta(1 - \theta \eta \hat{p}^{\frac{1}{2}}) \phi(p, \eta), \] (3.6)

\[ \Phi(p, \theta) = \int d\eta e^{\frac{i}{2} \hat{p} \theta} \delta(1 - \theta \eta \hat{p}^{\frac{1}{2}}) \phi(p, \eta), \] (3.7)

\[ d \Phi(p, \theta) = -\hat{p}^{\frac{1}{2}} \int d\eta e^{-\frac{i}{2} \hat{p} \theta} \delta(\hat{p}^{\frac{1}{2}} - \eta) \phi(p, \eta) \] (3.8)

The inverse of the integral transformations above are given by

\[ \phi(p, \eta) = -\hat{p}^{-\frac{1}{2}} \int d\theta e^{-\frac{i}{2} \hat{p} \theta} \delta(1 + \theta \eta \hat{p}^{\frac{1}{2}}) \Phi(p, \theta), \] (3.9)

\[ \tilde{\phi}(p, \eta) = -\frac{1}{\hat{p}} \int d\theta e^{-\frac{i}{2} \hat{p} \theta} \delta(1 + \theta \eta \hat{p}^{\frac{1}{2}}) \bar{d} \Phi(p, \theta), \] (3.10)

and the momentum space superfields in the new representation are

\[ \phi(p, \eta) = i \bar{\Lambda}(p) \hat{p}^{\frac{1}{2}} + \eta A(p), \] (3.11)

\[ \tilde{\phi}(p, \eta) = \bar{A}(p) + \eta i \Lambda(p) \hat{p}^{-\frac{1}{2}}. \] (3.12)

In the same way that the symmetry generators of the Poincare group can be converted to momentum space, the representation of the SUSY generators in the light-cone coordinates \( (2.6) \) can be expressed in terms of \( \eta \).

\[ q = \hat{p}^{\frac{1}{2}} \eta, \quad \bar{q} = \hat{p}^{\frac{1}{2}} \frac{\partial}{\partial \eta}. \] (3.13)

It is straightforward to verify that after neglecting all of the commutators appearing in the SUSY transformation relations, for asymptotic states we have

\[ Q_\alpha = \lambda_\alpha \eta, \quad \bar{Q}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta}, \] (3.14)

which are the \( \mathcal{N} = 1 \) versions of the on-shell SUSY generators introduced by Witten in [2]. Note however, that the representation (3.13) holds even when off-shell.

### 3.1 3-point MHV and \( \overline{\text{MHV}} \) vertices

The light-cone gauge SYM lagrangian can be quickly rewritten in terms of the new representation by applying (3.5) to (3.8) to its field contents. In addition to a momentum conservation delta function, the free field part contains a Grassmanian delta function, which forbids interchanges between gluons and gluinos in the absence of an interaction.

\[ S_{\text{free}} = \int d^4x \, d\theta d\bar{\theta} \, L^{-+} = \int d^4p_1 \, d^4p_2 \, d\theta \, d\bar{\theta} \, \bar{\Phi} \frac{b_2^2}{p_2} \delta^{(4)}(p_1 + p_2) \Phi \]

\[ = \int d^4p_1 \, d^4p_2 \, d\eta_1 \, d\eta_2 \, \bar{\phi}(p_1, \eta_1) \delta(p_1 + p_2) \delta(\eta_1 + \eta_2) \frac{p_2^2}{p_2} \phi(p_2, \eta_2) \] (3.15)

For the \((-+-+)\) interaction term we apply the same transformation again to have
\begin{equation}
\int d^4x \, d\theta \tilde{\theta} \mathcal{L}^{++-} = tr \int d^4x \, d\theta \tilde{\theta} [\Phi, \frac{\partial}{\partial \tilde{\Phi}}] \Phi
= tr \int d^4p_1 \ldots d\eta_1 \ldots \frac{(12)^3}{12} 3^\frac{i}{2} (1^\frac{i}{2} \eta_1 + 2^\frac{i}{2} \eta_2 + 3^\frac{i}{2} \eta_3) \bar{\phi}_1 \bar{\phi}_2 \bar{\phi}_3
\end{equation}

(3.16)

Using momentum conservation condition we can combine all of the hat-components and the round bracket into spinor brackets. Note that the vertex factor resembles the super-wavefunction (1.4) and the super-amplitude formula given by Nair for $\mathcal{N} = 4$ SYM theory.

\begin{equation}
tr \int d^4p_1 \ldots d\eta_1 \ldots \frac{(12)^3}{12} (12) \eta_1 \eta_2 + (23) \eta_2 \eta_3 + (31) \eta_3 \eta_1 \bar{\phi}_1 \bar{\phi}_2 \bar{\phi}_3
\end{equation}

(3.17)

We then compute the $(+ + -)$ term to give

\begin{equation}
\int d^4x \, d\theta \tilde{\theta} \mathcal{L}^{+++} = tr \int d^4x \, d\theta \tilde{\theta} [\Phi, \frac{\partial}{\partial \tilde{\Phi}}] \Phi
= tr \int d^4p_1 \ldots d\eta_1 \ldots \frac{[12]^3}{[12] [23] [31]} (23) \eta_1 + [31] \eta_2 + [12] \eta_3) \phi_1 \phi_2 \phi_3
\end{equation}

(3.18)

In the defining equations for integral transformations (3.5) to (3.8) we chose to substitute the $\theta \phi$ dependence in $\Phi$ and $d \Phi$ by the newly introduced Grassmann variable $\eta$. As an alternative we can choose our definition to substitute the $\theta \phi$ in $\Phi$ and $d \Phi$. In that case we will have the same spinor bracket factors for the 3-point MHV vertex (3.17) but with the sum $(12) \eta_1 \eta_2 + (23) \eta_2 \eta_3 + (31) \eta_3 \eta_1$ replaced by $(12) \eta_3 + (23) \eta_1 + (31) \eta_2$, and similarly for the MHV term. The same correspondence between the $\eta$ assignment of the super-wavefunction (1.4) and the $\eta$ dependence in the super-amplitudes for $\mathcal{N} = 4$ SYM was found by Drummond, Henn, Korchemsky and Sokatchev in [23, 35, 36].

The remaining two 4-point vertices can be calculated following the same procedure.

\begin{equation}
V_{1234}^2 = \frac{1}{(2 + 3)^2} \left( (13 + 24) \eta_1 \eta_2 - 1^\frac{i}{2} 3^\frac{i}{2} (2 + 3) \eta_2 \eta_3 + 2^\frac{i}{2} 3^\frac{i}{2} (1 + 4) \eta_1 \eta_3 \\
+ 2 1^\frac{i}{2} 2^\frac{i}{2} 3^\frac{i}{2} 4^\frac{i}{2} \eta_3 \eta_4 - 2^\frac{i}{2} 4^\frac{i}{2} (1 + 4) \eta_4 \eta_1 + 1^\frac{i}{2} 4^\frac{i}{2} (2 + 3) \eta_2 \eta_4 \right),
\end{equation}

(3.19)

and

\begin{align}
V_{1234}^3 = & - \left( \frac{3^\frac{i}{2}}{(1 + 4)^2} + \frac{1^\frac{i}{2}}{(3 + 4)^2} \right) \eta_1 \eta_3 - \frac{1^\frac{i}{2} 2^\frac{i}{2} 4^\frac{i}{2}}{(3 + 4)^2} \eta_2 \eta_3 - \frac{2^\frac{i}{2} 3^\frac{i}{2} 4^\frac{i}{2}}{(1 + 4)^2} \eta_1 \eta_2 \\
+ & 3^\frac{i}{2} 4^\frac{i}{2} \left( \frac{1}{(1 + 4)} - \frac{1}{(3 + 4)^2} \right) \eta_4 \eta_1 + 4^\frac{i}{2} 1^\frac{i}{2} \left( \frac{1}{(3 + 4)} - \frac{3}{(1 + 4)^2} \right) \eta_3 \eta_4.
\end{align}
\[ -\frac{1}{2} + \frac{3}{2} + \frac{1}{2} \left( \frac{1}{1+4} + \frac{1}{3+4} \right) \eta_2 \eta_4, \]  

(3.20)

where we used \( V_{1234}^2 \) and \( V_{1234}^3 \) to denote the vertices that have adjacent and next-to-adjacent negative helicity legs.

\[
\int d^4x d\theta d\bar{\theta} L^{--++} = \frac{4}{g^2} \int d^4x d\theta d\Phi \frac{1}{(i\dot{\phi})^2} \left[ \bar{\Phi}, \Phi \right]
\]

\[
= \frac{4}{g^2} tr \int V_{1234}^2 \bar{\phi}_1 \phi_2 \phi_3 \phi_4 + V_{1234}^3 \bar{\phi}_1 \phi_2 \bar{\phi}_3 \phi_4
\]

(3.21)

4. Calculating super-amplitudes using functional methods

One of the advantages of rewriting light-cone \( \mathcal{N} = 1 \) SYM lagrangian in terms of chiral superfields is that it allows us to compute super-amplitudes from a set of manifestly supersymmetric Feynman rules. In a system where supersymmetry is effectively unbroken it is reasonable that particles in the same same supermultiplet can be treated as a single entity. Nevertheless, in the standard functional integral approach to Green function calculations there is a fundamental distinction between a field and the field of its superpartner. The gluon fields are bosonic while the gluino fields are taken as fermionic, both fields are regarded as independent variables to be integrated over in the functional integral. A naive attempt to combine these two fields by a change of variables reduces the degrees of freedom and does not make sense mathematically. So instead of on integration variables we focus on the generating functional that generates Green functions.

\[
Z[J] = \int \mathcal{D}A \mathcal{D}\bar{A} \mathcal{D}\bar{\Lambda} \mathcal{D}\Lambda e^{iS+i \int j^{(A)} A + \bar{A} j^{(A)} + j^{(\Lambda)} \Lambda + \bar{\Lambda} j^{(\Lambda)}},
\]

(4.1)

where \( S \) is the \( \mathcal{N} = 1 \) SYM action (2.7) in light-cone gauge, and we introduce generating currents \( j^{(A)} \), \( j^{(\bar{A})} \), \( j^{(\Lambda)} \) and \( j^{(\bar{\Lambda})} \) for every physical fields. We note that the current term integral can be simplify by the introduction of super-currents

\[
\int d^4p j^{(A)} A + \bar{A} j^{(\bar{A})} + j^{(\Lambda)} \Lambda + \bar{\Lambda} j^{(\bar{\Lambda})} = \int d^4p d\eta J \phi + \bar{\phi} \bar{J},
\]

(4.2)

where we defined

\[
J = j^{(A)} - i \hat{p}^\perp \eta j^{(A)}, \quad \bar{J} = \eta j^{(A)} - i \hat{p}^\perp \bar{J}.
\]

(4.3)

As in the standard calculation we extract the interaction terms as variation operators of the generating currents. From equations (3.16) to (3.20) we saw the interactions in the lagrangian can be written as functionals of the momentum space superfields. This means that using the chain rules the variations with respect to the currents associated with physical fields can be combined as \( -i \frac{\delta}{\delta J} \) and \( i \frac{\delta}{\delta \bar{J}} \), and the vertices are simply given by
equations (3.16) to (3.20), where the variation with respect to a function of both bosonic and fermionic variables is defined as in [37]

$$\frac{\delta J(p', \eta')}{\delta J(p, \eta)} = \delta^4(p' - p) \delta(\eta' - \eta) \quad (4.4)$$

Therefore we have

$$Z[J] = e^{iS_{int}[J]} Z_0[J] \quad (4.5)$$

The free generating functional is calculated from integrating over all field variables.

$$Z_0[J] = e^{i\int j^{(\bar{A})} \Delta_{(A)} j^{(A)} + j^{(\bar{\Lambda})} \Delta_{(\Lambda)} j^{(\Lambda)}} = e^{i\int J \Delta J} \quad (4.6)$$

where the light-cone gauge gluon and gluino propagators are given by

$$\Delta_{(A)}(p_1, p_2) = \frac{1}{p_2^2} \delta^4(p_1 + p_2), \quad \Delta_{(\Lambda)}(p_1, p_2) = \frac{2}{p_2^2} \delta^4(p_1 + p_2).$$

We find that the currents associated with gluons and gluinos can be again organised into the super-currents (4.3), and the propagators of two different particle species are replaced by

$$\Delta(p_1, \eta_1; p_2, \eta_2) = \frac{1}{p_2^2} \delta^4(p_1 + p_2) \delta(\eta_1 + \eta_2) \quad (4.7)$$

Note that despite the free generating functional (4.6) was derived without treating the chiral superfields as field variables, the propagator (4.7) takes the form as the inverse of the free superfield lagrangian (3.15), allowing us to introduce superfields $\phi$ and $\bar{\phi}$ as auxiliary field variables, where we generalised the functional integral to fields labeled by both bosonic and fermionic indices $p$ and $\eta$ in the same way as in [37, 38] so that the integration over $\phi$ and $\bar{\phi}$ has the same properties as over ordinary fields. The interaction part of the action extracted as a functional of variation operators can be applied back to the $Z_0[J]$ to restore the lagrangian as a functional of $\phi$ and $\bar{\phi}$. It is easy to see that the lagrangian has the same propagator and vertices given in (3.15) to (3.20).

$$Z_0[J] = \int D\phi D\bar{\phi} e^{iS_{free} + \int J \phi + \bar{\phi} J}, \quad Z[J] = \int D\phi D\bar{\phi} e^{iS + \int J \Delta J} \quad (4.8)$$

For the purpose of computing the generating functional and the Green function it makes no different whether the functional integral was defined from the physical fields or from the superfield viewpoint.

Using the standard Wick contraction and the LSZ reduction on (4.5) and (4.6), it is straightforward to derive a set of supersymmetric Feynman rules for $\mathcal{N} = 1$ SYM theory based on the super-momentum space lagrangian, and the method naturally leads to a combination of scattering amplitudes related to each other by supersymmetry transformation. Therefore we define the “helicity-ordered” super-amplitude as the LSZ reduction of the superfield Green function

$$A(p_i, \sigma_i, \eta_i) = \lim_{p_i^2 \to m^2} \prod_i p_i^2 \left\langle \cdots \phi \cdot \cdots \bar{\phi} \cdot \cdots \right\rangle, \quad (4.9)$$
at the expense of manifest CPT symmetry comparing with the definition used in [7]. To convert the super-amplitude into the physical scattering amplitudes of gluons and gluinos we extract terms with the Grassmann variables corresponding to the particle species participating the event. From the definitions of superfields (3.11), (3.12) we see a Grassmannian momentum $\eta_i$ is present whenever there is a positive helicity gluon or a negative helicity gluino. The appropriate polarisations factors for the LSZ reduction formula are automatically included from the definition of a super-amplitude (4.9).

$$\langle 1^+ \ldots 2^+_\Lambda \ldots 3^- \ldots 4^-_\Lambda \rangle = \frac{\partial}{\partial \eta_4} \ldots \frac{\partial}{\partial \eta_1} \prod_i p_i^2 \langle A_1 \ldots \Lambda_{i+1} \ldots \Lambda_3 \ldots \Lambda_4 \rangle$$

(4.10)

Note that the superfields $\phi$ and $\bar{\phi}$ here are regarded as auxiliary fields introduced in the functional integral (4.8) which do not contain physical gluon or gluino fields as components. The expansion from the definition of a super-amplitude (4.9) into a series of physical scattering amplitudes relies on current algebra. However since the chain rule of variations does not distinguish whether the current $j^A(p) - i p^\alpha \eta_j \Lambda j^A(p)$ is multiplied by the combination $i \Lambda(p) p^{\frac{1}{2}} + \eta \Lambda(p)$ or the newly introduced integration variable $\phi(p, \eta)$, the scattering amplitude calculated from integrating over gluon and gluino fields is the same as the amplitude calculated from integrating over $\phi(p, \eta)$.

### 4.1 Applying BCFW to calculate $\mathcal{N}=1$ MHV super-amplitudes

In [22, 25, 35, 36] the BCFW recursion method is generalised to $\mathcal{N}=4$ SYM theory to compute super-amplitudes that have super-wavefunctions as end states. We adapt the argument provided by Brandhuber, Heslop and Travaglini originally designed to apply on super-amplitudes having two positive helicity gluon lines shifted in the $\mathcal{N}=4$ theory [25] to super-amplitudes with positive and one negative leg shifted in the $\mathcal{N}=1$ theory and derive the formula for 4-point MHV super-amplitude.

![Figure 1: Shifting the helicity-ordered super-amplitude $A(1^-, 2^-, 3^+, 4^+)$](image)

Consider shifting the leg 1 and 4 of the super-amplitude $A(1^-, 2^-, 3^+, 4^+)$ (Fig.1). The momenta $p_1$ and $p_4$ are shifted in the same way as in the pure Yang-Mills theory,

$$\begin{align*}
\hat{P}_1^{\alpha a}(z) &= \lambda_{1a} \bar{\lambda}_{1\dot{a}} - z \lambda_{1a} \bar{\lambda}_{4\dot{a}}, \\
\hat{P}_4^{\alpha a}(z) &= \lambda_{4a} \bar{\lambda}_{4\dot{a}} + z \lambda_{1a} \bar{\lambda}_{4\dot{a}}.
\end{align*}$$

(4.11)

In addition to momenta we also shift the Grassmann variable associated with the negative helicity leg as
while all other momenta and Grassmann variables are unchanged. A super-amplitude defined in (4.9) generically contains a series of physical amplitudes, each of them being multiplied by the corresponding Grassmann variables. The ratios between scattering amplitudes of different particle species are fixed by the SUSY Ward identities. Because the shiftings given by equations (4.11) and (4.12) leave the SUSY generators invariant,

\[
\bar{Q}_\alpha' = \sum_{i=1}^{4} \bar{\lambda}_{i\alpha} \frac{\partial}{\partial \eta_i} = \bar{Q}_\alpha, \\
Q_\alpha' = \sum_{i=1}^{4} \lambda_{i\alpha} ' \eta_i = Q_\alpha,
\]

(4.13)

these ratios can be shown to be independent of the complex variable \(z\). The physical amplitudes contained in the super-amplitude therefore have the same \(z\) dependence as the pure gluon scattering amplitude and the super-amplitude vanishes asymptotically as \(z \to \infty\). For example for the MHV super-amplitude \(A(1^-, 2^-, 3^+, 4^+)\), we solve the ratios explicitly by repeatedly applying SUSY Ward identity with different SUSY transformation parameters, and the super-amplitude is proportional to

\[
\langle 12 \rangle \eta_1 \eta_2 + \langle 23 \rangle \eta_2 \eta_3 + \langle 34 \rangle \eta_3 \eta_4 + \langle 41 \rangle \eta_4 \eta_1 + \langle 13 \rangle \eta_1 \eta_3 + \langle 24 \rangle \eta_2 \eta_4,
\]

(4.14)

which is invariant under the shifting (4.11) and (4.12). From the above argument we also see that the generalisation to \(\mathcal{N} = 1\) SYM does not introduce new singularities, therefore we have, from the BCFW recursion,

\[
A_4(0) = \int d\eta_q d\eta_{q'} A_L(z) \left. \frac{\delta(\eta_q + \eta_{q'})}{q^2} \right|_{z = -\langle 34 \rangle/\langle 13 \rangle} A_R(z)
\]

(4.15)

\[
= \frac{(\langle 12 \rangle)^2}{\langle 2q' \rangle \langle 13 \rangle} \frac{1}{\langle 34 \rangle \langle 4q' \rangle \langle q'3 \rangle} \times \left( \langle 12 \rangle \langle 34 \rangle \eta_1 \eta_2 - \langle 2q' \rangle \langle 4q' \rangle \eta_2 \eta_3 - \langle 2q' \rangle \langle q'3 \rangle \eta_2 \eta_4 + \langle q'1 \rangle \langle 4q' \rangle \eta_1 \eta_3 + \langle q'1 \rangle \langle q'3 \rangle \eta_1 \eta_4 \right),
\]

(4.16)

where \(q = p_3 + p_4\). Simplifying the above expression gives the 4-point super-amplitude

\[
A(1^-, 2^-, 3^+, 4^+) = \frac{(\langle 12 \rangle)^3}{(\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle)} \left( \sum_{i,j=1}^{4} \langle i, j \rangle \eta_i \eta_j \right).
\]

(4.17)

Since the argument for asymptotic behavior and the algebraic derivation we used do not depend on the number of legs, we can replace the amplitude \(A_L(z)\) on the left hand side of the propagator by an \((n-1)\)-point MHV super-amplitude. By induction an \(n\)-point MHV super-amplitude is given by the formula (1.7). The BCFW recursion also extends beyond MHV super-amplitudes because the argument for asymptotic behaviour only rely on the fact that the SUSY generators are invariant under the shifting (4.11) and (4.12).
5. Super-space canonical transformation

So far we have derived a supersymmetry equivalent to the LCYM theory. A natural next step is to perform a canonical transformation on the field variables as in the pure Yang-Mills theory \[8\] to absorb the unwanted MHV term so that in terms of the new variables the lagrangian automatically generates CSW rules for $N = 1$ SYM theory. In \[14\] Morris and Xiao applied the canonical transformation pair by pair. In the gauge field sector the gluon, gluino fields and their canonical conjugate momenta $\{A, \bar{\partial}A\}$ and $\{\Lambda, \bar{\Lambda}\}$ were transformed into the corresponding new fields $\{B, \hat{\partial}\bar{B}\}$ and $\{\Pi, \bar{\Pi}\}$ according to the following expansions

\[
\begin{align*}
A_1 &= B_1 + \Upsilon_{123}B_2B_3 + \cdots, \\
\hat{\partial}A_1 &= \hat{\partial}B_1 + \Xi^2_{123}\hat{\partial}B_2B_3 + \Xi^3_{123}B_2\hat{\partial}B_3 + \cdots \tag{5.1} \\
+&\Xi^2_{123}\bar{\Pi}_2\Pi_3 + \Xi^3_{123}\bar{\Pi}_2\Pi_3 + \Xi^3_{1234}\bar{\Pi}_2\Pi_3B_4 + \cdots, \tag{5.2} \\
\Lambda_1 &= \Pi_1 + \Upsilon_{123}\Pi_2B_3 + \Upsilon_{123}B_2\Pi_3 + \cdots, \tag{5.3} \\
\bar{\Lambda}_1 &= \bar{\Pi}_1 + \Xi^2_{123}\bar{\Pi}_2\Pi_3 + \Xi^3_{123}\bar{\Pi}_2\Pi_3 + \cdots. \tag{5.4}
\end{align*}
\]

In order to keep the notation simple we neglected the momentum conservation delta functions in the higher power terms. The coefficients $\Upsilon$ and $\Xi^k$ in the expansion are the translation kernels originally defined for the pure Yang-Mills theory \[10\].

\[
\begin{align*}
\Upsilon_{12\cdots n} &= \frac{\hat{1}\hat{3}\cdots\hat{n}}{(23)(34)\cdots(n-1,n)}, \\
\Xi^k_{12\cdots n} &= \frac{\hat{k}\hat{3}\cdots\hat{n}}{(23)(34)\cdots(n-1,n)}. \tag{5.6}
\end{align*}
\]

The transformation expansions \((5.1)\) to \((5.5)\) were verified to generate a unit Jacobian and have the effect of absorbing the $L^{++}_A$, $L^{++}_{\bar{A}}$ terms into the new lagrangian. In this paper we take a different approach and apply the transformation on superfields directly. As noted in section \[9\] the Green function can be computed from functional integral over superfields labeled by super-space momenta $p$ and $\eta$. Similarly a generating functional of currents in coordinate space originally derived from integrating over physical filed components $A, \bar{A}, \Lambda, \bar{\Lambda}$ can be reorganised as a functional of super-currents

\[
Z_0[J] = \int DAD\bar{A}D\bar{D}A\bar{\Delta}
\]

\[
\exp\left\{iS \int d^4x d\theta d\bar{\theta} \mathcal{L}_{\text{free}} + i \int d^4x j^{(A)}A + j^{(\Lambda)}\Lambda + \bar{A}j^{(\bar{A})} + \bar{\Lambda}j^{(\bar{\Lambda})}\right\} \tag{5.7}
\]

\[
= \exp\left\{ \int d^4x j^{(\bar{A})}\Delta^{(A)}j^{(A)} + j^{(\Lambda)}\Delta^{(\Lambda)}j^{(\Lambda)}\right\} = \exp\left\{ \int d^4x d\theta d\bar{\theta} \bar{J}\Delta J\right\} \tag{5.8}
\]
where we defined the super-currents in coordinate space as

\[ J(x, \theta) = \theta \bar{\theta} j^{(A)}(x) - i \bar{\theta} j^{(A)}, \]  

(5.9)

\[ \bar{J}(x, \theta) = \frac{1}{i \delta} j^{(A)} - i \theta \bar{\theta} j^{(A)}, \]  

(5.10)

and we have extracted the interaction part of the lagrangian as variation operators with respect to the supercurrents. As in the momentum space we introduce superfields as auxiliary field variables and restore the full action in chiral and anti-chiral superfields by operating the variation operators back onto the free generating functional.

\[ Z[J] = \int D\Phi(x, \theta) D\bar{\Phi}(x, \theta) \exp \left\{ i S[\Phi] + i \int J \Phi + \bar{\Phi} \bar{J} \right\}, \]  

(5.11)

where \( S[\Phi] \) is the \( \mathcal{N} = 1 \) SYM action in light-cone gauge introduced in (2.7). Inspired by the canonical transformation originally applied on pure Yang-Mills to derive an MHV lagrangian \[8\] we make a change of variables. The superfield \( \Phi(\tau, x, \theta) \) at light-cone time \( \tau \) is assumed to be a functional of \( \chi(\tau, y, \xi) \) defined through power expansion. As in the pure Yang-Mills theory the expansion for the anti-chiral superfield \( \bar{\Phi}(\tau, x, \theta) \) is assumed to contain only one \( \bar{\chi}(\tau, y, \theta) \) in each term, while the power of \( \chi(\tau, y, \xi) \) increases term by term. This arrangement ensures that the new lagrangian will have exactly two anti-chiral superfields in every vertex as demanded by the CSW rules. We assume the transformation is given by

\[ d\bar{\Phi}^a(x, \theta) = \int d^3 y d\xi d\xi' \frac{\delta}{\delta \Phi^a(x, \theta)} \bar{\chi}^b(y, \xi) \]  

(5.12)

and after the transformation the unwanted super-vertex is absorbed into the new free field lagrangian \( \mathcal{L}^{++}[\Phi] + \mathcal{L}^{--}[\Phi] = \mathcal{L}^{++}[\chi] \).

We note that the nilpotency of the Grassmann variables \( \theta \bar{\theta} \) allows us to replace the anti-chiral superfield by \( \bar{\theta} d \bar{\Phi} \),

\[ tr \int d^4 x d\theta d\bar{\theta} \bar{\Phi} \frac{\partial^2}{\partial \Phi} + [\Phi, \frac{\partial}{\partial \Phi}] = tr \int d^4 x d\theta d\bar{\theta} \bar{\Phi} \left( \frac{\partial^2}{\partial \Phi} + [\Phi, \frac{\partial}{\partial \Phi}] \right) \]  

(5.13)

Using the condition (5.12) and stripping off an \( \bar{\theta} d \bar{\chi} \) from both sides of the equation, we have

\[ \frac{\partial^2}{\partial \Phi(x, \theta) + [\Phi, \frac{\partial}{\partial \Phi}](x, \theta) = \int d^3 y d\xi d\xi' \frac{\partial^2}{\partial \chi(y, \xi)} \frac{\delta \Phi(x, \theta)}{\delta \chi(y, \xi)} \]  

(5.14)

From (5.14) we determine the translation kernels in the expansions of \( \Phi \) and \( \bar{\Phi} \). Since the above condition is the same as the condition used to solve for translation kernels in the pure Yang-Mills theory \[8\], we see that the kernels are simply given by the same formulae as in (5.6).

\[ \Phi(x_1, \theta) = \chi(x_1, \theta) + \int \Upsilon_{123} \chi(x_2, \theta) \chi(x_3, \theta) + \cdots \]  

(5.15)
\[ \tilde{\Phi}(x_1, \theta) = \tilde{\chi}(x_1, \theta) + \int \Xi^2_{123} (\tilde{\chi}(x_2, \theta)) \chi(x_3, \theta) \]
\[ + \int \Xi^3_{123} \chi(x_2, \theta) (\tilde{\chi}(x_3, \theta)) \cdots \] (5.16)

We Fourier transform the chiral superfields into momentum space and then apply the integrals defined in equations (3.5) to (3.8) to obtain the superfields in the new representation. The expansion formulae in momentum space are

\[ \phi(p_1, \eta_1) = \chi(p_1, \eta_1) \]
\[ + \int Y_{123} \frac{-1}{1^2} (-\eta_1 \hat{1}^\frac{1}{2} + \eta_2 \hat{2}^\frac{1}{2} + \eta_3 \hat{3}^\frac{1}{2}) \chi(p_2, \eta_2) \chi(p_3, \eta_3) + \cdots \]
\[ + \int Y_{12\ldots n} \frac{-1}{1^2} (-\eta_1 \hat{1}^\frac{1}{2} + \eta_2 \hat{2}^\frac{1}{2} + \cdots \eta_n \hat{n}^\frac{1}{2}) \chi(p_2, \eta_2) \cdots \chi(p_n, \eta_n) + \cdots, \] (5.17)
and

\[ \tilde{\phi}(p_1, \eta_1) = \tilde{\chi}(p_1, \eta_1') \]
\[ + \int \Xi^2_{123} \frac{-2^\frac{1}{2}}{1} (-\eta_1 \hat{1}^\frac{1}{2} + \eta_2 \hat{2}^\frac{1}{2} + \eta_3 \hat{3}^\frac{1}{2}) \tilde{\chi}(p_2, \eta_2) \chi(p_3, \eta_3) + \cdots \]
\[ + \int \Xi^k_{12\ldots n} \frac{-k^\frac{1}{2}}{1} (-\eta_1 \hat{1}^\frac{1}{2} + \eta_2 \hat{2}^\frac{1}{2} + \cdots \eta_n \hat{n}^\frac{1}{2}) \chi(p_2, \eta_2) \cdots \tilde{\chi}(p_k, \eta_k) \cdots \chi(p_n, \eta_n) + \cdots. \] (5.18)

In order to avoid introducing too many symbols we slightly abuse the notation and use \( \chi \) both for superfields before and after the integral transformations (3.5) to (3.8), in the same spirit as the same symbol is used for wave functions before and after the Fourier transformation in the standard notation. The distinction between these two types of fields should be clear judging from the labels \((x, \theta)\) or \((p, \eta)\) attached to the superfields. In equations (5.17) and (5.18) we neglected the overall momentum conservation deltafunction and the integrations are understood to be performed over momenta \( p_2 \) to \( p_n \) as well as superspace momenta \( \eta_2 \) to \( \eta_n \).

The above expansion formulae can be conveniently summarised if we generalise the graphical notation introduced for the pure Yang-Mills in [12]. When an \( n \)-th order term in (5.17) contribute to the calculation we use a blank circle follow by \( (n+1) \) lines to represent the translation kernel, where one of the lines comes from the superfield \( \phi \) being translated. For the \( \tilde{\phi} \) translation, we use a similar graph with the blank circle replaced by a gray circle.

**Figure 2:** Graphical representations of superfield expansions
5.1 Generating MHV super-vertices

Following the same steps as for the pure Yang-Mills theory the super-amplitude given by applying LSZ reduction to the Green function is generically transformed into a series, each of the term contains a number of translation kernels

\[
\prod_{l} p_{l}^{2} \langle \ldots \phi_{i} \ldots \bar{\phi}_{j} \ldots \rangle
\]

\[
= \sum_{m,n} \prod_{l} p_{l}^{2} \langle \ldots (\Upsilon_{i_{2} \ldots i_{m}} x_{i_{2}} x_{i_{3}} \ldots x_{i_{m}}) \ldots (\Xi_{j_{2} \ldots j_{n}} x_{j_{2}} \ldots x_{j_{n}}) \ldots \rangle \ . \tag{5.19}
\]

At tree-level these kernels are suppressed by LSZ factors \( p_{l}^{2} \), so an MHV super-amplitude simply equals the on-shell limit of the corresponding MHV super-vertex. In section (4.1) we derived the formula for a generic n-point \( \mathcal{N} = 1 \) MHV super-amplitude from supersymmetry BCFW recursion, which can only differ from the MHV super-vertices by squares of leg momenta. Since the expansion coefficients in (5.17) and (5.18) are holomorphic, such difference is absent, so an n-point MHV super-vertex is provided by the same formula as for the super-amplitude.

Alternatively, we can directly compute the n-point MHV super-vertex. An MHV super-vertex in the new lagrangian receives contributions from the original 3-point (3.16) and the 4-point vertices (3.19), (3.20), and the superfields attached to the legs of the vertices branch into trees of new field variables according to the expansion formulae (5.17) and (5.18). In the appendix B we prove that the formula for an n-point MHV super-vertex is the same as the super-amplitude (5.20) by matching their coefficients under partial fraction expansion.

\[
V(1^{+}, 2^{+} \ldots i^{-}, j^{-}, \ldots n^{+}) = \frac{\langle i, j \rangle^{3}}{(12) \langle 23 \rangle \ldots \langle n1 \rangle} \left( \sum_{a,b=1}^{n} \langle a \ b \rangle \eta_{a} \eta_{b} \right) . \tag{5.20}
\]

As noted in section (4), although \( \phi \) and \( \bar{\phi} \) are introduced as integration variables, from current algebra the superfields appear in the Green function can be interchanged by the corresponding combinations of gluon and gluino fields. Substituting (3.11) and (3.12) into the canonical transformation formulae of superfields (5.15) and (5.16) suggests that the MHV lagrangian can be as well derived by transforming physical field components separately. Writing the new field variables as

\[
\chi(x, \theta) = B(y) + i \theta \Pi(y), \tag{5.21}
\]

\[
\bar{\chi}(x, \theta) = \bar{B}(y) + i \bar{\theta} \bar{\Pi}(y), \tag{5.22}
\]

and integrating over Grassmann variables \( \theta \) and \( \bar{\theta} \), we find the same transformation relations as equations (5.1) to (5.5) originally given by Morris and Xiao in [14].

5.2 SUSY Ward identity

In [2] Witten introduced an on-shell representation of the SUSY generators for \( \mathcal{N} = 4 \) SYM theory and verified that the MHV super-amplitude given by Nair [24] is superconformal
invariant. We find both the on-shell SUSY generators and the super-amplitude for \( \mathcal{N} = 4 \) theory resemble the formulae we derived in (3.14) and (5.20). It is then straightforward to verify that the n-point \( \mathcal{N} = 1 \) MHV super-amplitude is SUSY invariant. The on-shell SUSY transformation operator \( Q(\xi) \) is given by

\[
Q(\xi) = \sum_i \langle i \xi \rangle \eta_i + [\xi] \frac{\partial}{\partial \eta_i}
\]  

(5.23)

The transformation operator consists of a multiplication part and a differentiation part. When operating on formula (5.21) we find the two parts are separately zero. Collecting terms having the same Grassmann numbers, the multiplication part vanishes because from Jacobi identity

\[
\langle 1 \rangle \langle 23 \rangle \eta_1 \eta_2 \eta_3 + \langle 3 \rangle \langle 12 \rangle \eta_3 \eta_1 \eta_2 + \langle 2 \rangle \langle 31 \rangle \eta_2 \eta_3 \eta_1 = 0
\]  

(5.24)

for any three of the momenta carried by external lines. The differentiation part is proportional to

\[
\sum_i [\xi i] \langle ij \rangle \eta_j = 0
\]  

(5.25)

which vanishes from conservation of momentum. We note that supersymmetry is taken as a build-in property of the super-amplitude. Since the functional integral is invariant under SUSY transformation

\[
\prod \rho_i^2 \langle Q(\xi) \phi_1 \phi_2 \phi_3 \cdots \phi_n \rangle = 0.
\]  

(5.26)

Differentiating both sides of the identity (5.26) with respect to \( \eta_1 \eta_2 \eta_j \) for example, gives the familiar SUSY Ward identity relating the amplitude that consists of a pair of gluino and the amplitude of all gluons.

\[
\langle 21 \rangle \langle 1^+, 2^+, 3^-, \cdots j^- \cdots n^- \rangle + \langle 2j \rangle \langle 1^+, 2^+, 3^-, \cdots n^- \rangle = 0
\]  

(5.27)

6. Conclusion and discussions

In this paper we performed a fermionic integral transformation followed by a supersymmetric canonical transformation on the superfield variables to directly generate Nair’s like \( \mathcal{N} = 1 \) MHV super-vertices (5.20). We showed that in such a single Grassmann variable representation the superfields are unconstrained and act as functional integration variables. The supersymmetric Feynman rules derived from the functional integral provide the \( \mathcal{N} = 1 \) version CSW rules, in which the reference spinors used to extract spinors of internal lines are all chosen as \( (0,1)^T \), as in the case of pure Yang-Mills theory [8]. (Other options correspond to using different null-vectors to define light-cone coordinates.) We applied BCFW recursion to compute the generic n-point helicity-ordered MHV super-amplitude defined through superfield Green functions. Symmetrising with respect to all helicity configurations this reproduces the super-amplitude formula derived in [4].
As an alternative approach we note that the MHV lagrangian can be derived entirely from the spinor language. In this approach the gauge field and gaugino field components are spanned by the bispinors and spinors consisting of various combinations of $\mu_\alpha$ and $\lambda_\alpha$, where

$$\mu_\alpha = \begin{pmatrix} \sqrt{\hat{p}} - p\hat{p}/\hat{p} & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mu}_{\dot{\alpha}} = \begin{pmatrix} \sqrt{\hat{p}} - p\hat{p}/\hat{p} & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.1)$$

and

$$\lambda_\alpha = \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}, \quad \bar{\lambda}_{\dot{\alpha}} = \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}, \quad (6.2)$$

and a generic off-shell momentum is written as $P_{\alpha\dot{\alpha}} = \mu_\alpha \bar{\mu}_{\dot{\alpha}} + \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$. We identify the positive, negative helicity physical field components as the coefficients of the terms $\eta_\alpha \bar{\lambda}_{\dot{\alpha}}$, $\lambda_\alpha \bar{\eta}_{\dot{\alpha}}$, $\lambda_\alpha$, $\bar{\lambda}_{\dot{\alpha}}$ respectively. Analogously to deriving the light-cone gauge Yang-Mills lagrangian we eliminate the $\lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$ component of the gluon field by fixing gauge condition and integrate over all unphysical components and Grassmann variables $\theta^\mu$, $\bar{\theta}^\mu$. The resulting lagrangian is the same as the one built from chiral superfields which are composed of helicity field components and Grassmann variables $\theta^\lambda$, $\bar{\theta}^\lambda$. From this point of view the fermionic integral transformation defined through equations (3.5) to (3.8) can be regarded as the off-shell version of identifying $\eta_A = [\lambda \theta_A]$ in the momentum twistor space $[27, 39]$ and the chiral prescription described above is the momentum space analogue to the fields developed in $[26]$ by Boels, Mason and Skinner in the ordinary twistor space.

With a few modifications the method can be extended to $\mathcal{N} = 4$ SYM theory. It has long been known that the $\mathcal{N} = 4$ SYM lagrangian in light-cone gauge can be constructed by chiral superfields $[31]$. The corresponding fermionic integral transformation which gives rise to the Grassmann variables $\eta_A$ in $\mathcal{N} = 4$ SYM theory (and notably in $\mathcal{N} = 8$ supergravity) was found by Kallosh in $[40]$. The generic n-point vertices in the $\mathcal{N} = 4$ MHV lagrangian can be shown to assume the form described by Nair’s formula $[24]$ either from holomrphy or from explicit computation.

Acknowledgements

We would like to thank Paul Mansfield for comments and for bringing our attention to recent developments in twistor field theory. We are also grateful for the discussions with Paolo Benincasa, James Ettle and Zhiguang Xiao.

A. Notation

Here we summarise the notation used throughout this paper.

As a shorthand notation we use $(\hat{p}, \hat{p}, p, \bar{p})$ to describe covariant vectors in the light-cone coordinates, which is related to the Minkowski coordinates by

$$\hat{p} = (p_0 - p_3), \quad \hat{p} = (p_0 + p_3), \quad p = (p_1 - ip_2), \quad \bar{p} = (p_1 + ip_2). \quad (A.1)$$
In the light-cone coordinates the metric becomes off-diagonal. The Lorentz invariant product of two vectors is given by

\[ p \cdot q = \frac{(\hat{p} \hat{q} + \tilde{p} \tilde{q} - p \tilde{q} - \tilde{p} q)}{2}. \] (A.2)

To keep the notation as simple as possible, the momentum components \( p_n \mu \) of the \( n \)th external leg are denoted by number \( n \) with the appropriate decoration \( (\hat{n}, \tilde{n}, \bar{n}, \check{n}) \). Note that a tilde is used for the \( p = p_1 - i p_2 \) component to avoid possible confusions with numerical factors.

A 4-vector can be written in the form of a bispinor

\[ P_{\alpha \check{\alpha}} = p^\mu \sigma_{\mu \alpha} = \begin{pmatrix} \hat{p} & -p \\ \tilde{p} & \check{p} \end{pmatrix}, \] (A.3)

by contracting with \( \sigma_\mu = (I_2, \vec{\sigma}) \), where \( I_2 \) is the \( 2 \times 2 \) identity matrix and \( \vec{\sigma} \) stands for Pauli matrices.

If \( p^\mu \) is lightlike, \( \hat{p} = \tilde{p}/\sqrt{\hat{p}} \) and the bispinor factorises \( p_{\alpha \check{\alpha}} = \lambda_\alpha \bar{\lambda}_{\check{\alpha}} \), where

\[ \lambda_\alpha = \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}, \bar{\lambda}_{\check{\alpha}} = \begin{pmatrix} -\tilde{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}. \] (A.4)

Spinors \( \lambda_{i\alpha} \) associated with different massless particles can be contracted to given a Lorentz invariant angle bracket

\[ \langle 12 \rangle = \epsilon^{\alpha \beta} \lambda_{1\alpha} \lambda_{2\beta} = \frac{(12)}{\sqrt{\langle 12 \rangle}}, \] (A.5)

and we define a round bracket as

\[ (12) = \hat{1} \hat{2} - \tilde{2} \tilde{1}. \] (A.6)

B. \( \mathcal{N} = 1 \) MHV super-vertices

In this appendix we prove that when the superfields \( \phi \) and \( \bar{\phi} \) are canonically transformed into new fields \( \chi \) and \( \bar{\chi} \) the original 3-point and 4-point vertices in the light-cone gauge SYM generate MHV super-vertices of the form (5.20). For simplicity we show this is true when the two negative helicity particles are adjacent. The method outlined here generalise to arbitrary configurations.

Labeling the negative helicity leg momenta as leg 1 and 2 the formula (5.20) reads

\[ V(1^{-}, 2^{-}, 3^{+} \ldots n^{+}) = \frac{(12)^3}{(12) \langle 23 \rangle \ldots \langle n1 \rangle} \left( \sum_{a,b=1}^{n} \langle a \, b \rangle \eta_a \eta_b \right) \] (B.1)

\[ = \frac{(12)^2}{(23) \ldots \langle n1 \rangle} \frac{\hat{3} \ldots \hat{n}}{1^{1/2} 2^{1/2}} \left( \sum_{a,b=1}^{n} \langle a \, b \rangle \frac{\eta_a \eta_b}{a^{1/2} b^{1/2}} \right) \] (B.2)
From (5.6) we saw the factors Υ and Ξ appearing in the translation formula (5.17) and (5.18) contain in the denominators a sequential product of round brackets. After the canonical transformation the 3-point and 4-point vertices constitute three and four groups of products of round brackets (Fig. 3 and 4). Therefore when regarded as functions of tilde component variables \( \tilde{p} \) which are contained in the round brackets, both formula (B.2) and the translated 3-point and 4-point vertices can be spanned by terms of the form

\[
\frac{1}{(23)(34) \cdots (k-1,k)} \times \frac{1}{(l+1,l+2) \cdots (n,1)}
\]

(B.3)

together with terms of the form

\[
\frac{1}{(23)(34) \cdots (k-1,k)} \times \frac{1}{(k+1,k+2) \cdots (m-1,m)} \times \frac{1}{(l+1,l+2) \cdots (n,1)}
\]

(B.4)

where the expansion coefficients depend on pairs of Grassmann numbers \( \eta_i \eta_j \) with \( i \) \( j \) running through all possible combinations of external legs and on hat component momenta \( \hat{p} \).

To obtain the coefficients we use the method of partial fractions. For terms of the form (B.3) we adjust tilde components to set

\[
(12) = 0
\]

(B.5)

\[
(23) = \cdots = (k-1,k) = 0
\]

(B.6)

\[
(k+1,k+2) = \cdots = (m-1,m) = 0
\]

(B.7)

\[
(m+1,m+2) = \cdots = (l-1,l) = 0
\]

(B.8)

\[
(l+1,l+2) = \cdots = (n,1) = 0
\]

(B.9)
Similarly, the coefficient of term (B.4) can be obtained by applying the conditions

\[(23) = \cdots = (k - 1, k) = 0 \quad (B.10)\]
\[(k + 1, k + 2) = \cdots = (l - 1, l) = 0 \quad (B.11)\]
\[(l + 1, l + 2) = \cdots = (n, 1) = 0 \quad (B.12)\]

We shall prove that the translated 3-point and 4-point vertices contribute to give the formula (B.2) by showing their expansion coefficients agree with each other.

First we check the coefficients for four sequential products of brackets (B.3). Applying conditions (B.9) on formula (B.2) gives zero because of the \((12)\) dependence in the numerator. The 3-point vertex contribution to the coefficient of (B.3) can be from the following three cases:

\[(a)\] \[a + b c 1 \alpha l + 1 l \delta m + 1 k \beta m \gamma \delta \eta \]
\[(b)\] \[b + c \alpha l + 1 l \delta m + 1 k \beta m \gamma \delta \eta\]
\[(c)\] \[1 \alpha l + 1 l \delta m + 1 k \beta m \gamma \delta \eta\]

In each graph one of the sequential products of brackets splits into two. Summing over contributions from these three graphs gives

\[
\sum_{\alpha, \beta, \gamma, \delta} \frac{1}{c^2 a^2 (b + c)^2} \left( (ac + bd) \hat{\alpha} \hat{\beta} \hat{\gamma} \hat{\delta} \eta_{\alpha} \eta_{\beta} - a (b + c) \hat{\beta} \hat{\gamma} \hat{\delta} \eta_{\beta} \eta_{\gamma} - b (a + d) \hat{\delta} \hat{\gamma} \hat{\alpha} \eta_{\delta} \eta_{\alpha} - b (a + c) \hat{\delta} \hat{\beta} \hat{\alpha} \eta_{\delta} \eta_{\alpha} + 2 ab \gamma \hat{\delta} \eta_{\gamma} \eta_{\delta} + b (a + d) \hat{\alpha} \hat{\gamma} \hat{\delta} \eta_{\alpha} \eta_{\gamma} + a (b + c) \hat{\beta} \hat{\gamma} \hat{\delta} \eta_{\beta} \eta_{\delta} \right) \quad (B.13)
\]

where we used \(\eta_{\alpha}, \eta_{\beta}, \eta_{\gamma}, \eta_{\delta}\) to denote Grassmann variables associated with legs from each of the four branches emerging from the original 4-point vertex. The index \(\alpha\) is to be summed over from \((l + 1)\) to 1, \(\beta\) from 2 to \(k\), \(\gamma\) from \((k + 1)\) to \(m\), and finally \(\delta\) from \((m + 1)\) to \(l\). For simplicity we denote the four internal lines of (Fig.5) by \(a, b, c\) and \(d\).

\[a = \hat{l} + 1 + \cdots + \hat{n} + \hat{1} \quad (B.14)\]
\[b = \hat{2} + \hat{3} + \cdots + \hat{k} \quad (B.15)\]
\[c = \hat{k} + 1 + \cdots + \hat{m} \quad (B.16)\]
\[d = \hat{m} + 1 + \cdots + \hat{l} \quad (B.17)\]
The contribution from the original 4-point vertex can be readily derived by translating the superfields $\phi$ and $\bar{\phi}$ attached to (3.19), which cancels (B.13). The expansion coefficient of terms (B.3) vanishes, therefore agrees with the coefficients obtained by expanding the formula (B.2).

The coefficients of term (B.4) can be calculated using the same method. However we note that since in the pure YM case the LCYM 3-point vertex was verified to give the same expansion coefficients as the Parke-Taylor formula, which corresponds to the $\eta_i\eta_j$ term of the formula (B.2), the proof is complete as long as the ratio between the expansion coefficients of each $\eta_i\eta_j$ term from the original 3-point vertex is the same as ratio of coefficients of $\eta_i\eta_j$ from the formula (B.2). Using $\alpha$, $\beta$ and $\gamma$ to denote external legs from each of the three branches of (Fig.5), we find the translated 3-point vertex contribute to the expansion coefficient of (B.4) as

$$\sum_{\alpha,\beta,\gamma} c \hat{\eta}_\alpha \eta_\beta + a \hat{\eta}_\beta \eta_\gamma + b \hat{\eta}_\gamma \eta_\alpha$$

where $a$, $b$ and $c$ here stand for

$$a = \hat{l} + 1 + \cdots + \hat{\eta} + \hat{1}$$
$$b = \hat{2} + \hat{3} + \cdots + \hat{k}$$
$$c = \hat{k} + 1 + \cdots + \hat{l}$$

The numerator of the formula (B.2) can accordingly be written as

$$\sum_{\alpha,\beta,\gamma} \langle \alpha \beta \rangle \eta_\alpha \eta_\beta + \langle \beta \gamma \rangle \eta_\beta \eta_\gamma + \langle \gamma \delta \rangle \eta_\gamma \eta_\alpha$$
Applying (B.12) this becomes

$$
\frac{12}{12} \cdot c \left( \sum_{\alpha, \beta, \gamma} c \hat{\alpha} \hat{\beta} \hat{\gamma} \eta_{\alpha} \eta_{\beta} + a \hat{\beta} \hat{\gamma} \hat{\alpha} \eta_{\beta} \eta_{\gamma} + b \hat{\gamma} \hat{\alpha} \hat{\beta} \eta_{\gamma} \eta_{\alpha} \right)
$$

(B.23)

The ratio between the coefficients of the $\eta_{\alpha} \eta_{\beta}$ term, $\eta_{\beta} \eta_{\gamma}$ term and $\eta_{\gamma} \eta_{\alpha}$ term are the same as the ratio in (B.18).

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