Locus of the apices of projectile trajectories under constant drag

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Abstract

Using the hodograph method, we present an analytical solution for projectile coplanar motion under constant drag, parametrised by the velocity angle. We find the locus formed by the apices of the projectile trajectories, and discuss its implementation for the motion of a particle on an inclined plane in presence of Coulomb friction. The range and time of flight are obtained numerically, and we find that the optimal launching angle is smaller than in the drag-free case. This is a good example of a problem with constant dissipation of energy that includes curvature; it is appropriate for intermediate courses of mechanics.

Keywords: projectile motion, dissipation, exact results, non-Newtonian fluid, yield stress, Coulomb friction

(Some figures may appear in colour only in the online journal)

1. Introduction

Projectile trajectory under drag has received much well-deserved attention in the literature, not least because it appears as a common problem in undergraduate physics, and can be related to recent exact analytical results and new analysis [1–5]. The power-law velocity dependent drag

\[ \vec{f} = -\sum_n m g \beta_n v^n \vec{v}, \]  

with \( v = ||\vec{v}|| \), is a series approximation for the real complex problem. The linear, \( n = 1 \), and quadratic, \( n = 2 \), cases are of much used, not only for the analysis of the motion of a particle in midair but also to model other energy dissipation processes. At quantum scales, \( n = 1 \) is the usual model for energy losses [6, 7].

Notwithstanding the usefulness of linear approximation, and that it allows analytical solutions for the projectile motion, equation (1) has another case: \( n = 0 \), the constant drag.
case. It is not trivial if, as is usual, the projectile motion is coplanar, i.e. the vector \( \mathbf{v} \) changes with the orientation of the orbit. This case has received little attention; the only reported works we have found are [8, 9].

There is no evidence that any regime exists where the drag could be considered constant; however, the problem studied here is important for the following reasons:

(i) A series expansion for a retarding force has to have a non-null zeroth term—take, for instance, the integrable Legendre cases

\[
f(v) = \frac{1}{n} (a + bv^n),
\]

where the constant \( a \neq 0 \) appears together with the drag constant \( b \) [10].

(ii) The motion of an object in a non-Newtonian fluid with yield stress could be constant—see, for instance, [11]—i.e. the problem of a particle launched in oil or liquid chocolate contains this constant term. Even more, spheres in loose granular media are another example of an object moving in a fluid with presence of yield stress [12].

(iii) As an undergraduate problem, a constant retarding force could be considered as a point rocket with the thrust pointed against the motion.

(iv) As a simple example of friction that depends on the curvature.

(v) The Coulomb friction, \( f = \mu N \), appearing in the description of sliding bodies, has the same description as that presented here for the coplanar motion of a particle on an inclined plane [13].

In the present paper we analyse such a case, obtaining both the explicit solutions of the problem in the next section and the description of the locus which give title to this work. We discuss the range and the time of flight are given in section 3.3. Conclusions are presented in section 4.

2. The projectile problem with constant drag

The constant drag problem is governed by the following equations in rectangular coordinates:

\[
m \frac{d^2}{dt^2} \mathbf{r} = -mg \mathbf{j} - mgh_0 \frac{v}{v}. 
\]

Notice that the friction is constant in the direction of motion—i.e. it changes with velocity. We choose the drag force in units of weight \( mg \) in order to compare with linear and quadratic drag results. Here, the unit vector \( \mathbf{j} \) points upward.

In order to be clear on what kind of differential equations we are dealing with, we explicitly rewrite the above equations in Cartesian coordinates:

\[
mx = -mgb_0 \frac{\dot{x}}{\sqrt{x^2 + y^2}}. \\
m\ddot{y} = -mg - mgb_0 \frac{\dot{y}}{\sqrt{x^2 + y^2}}.
\]

Here, we use a dot for a time derivative. The above equations are coupled and non-linear. However, an analytical solution parametrised with the velocity angle can be obtained. Some other results require standard numerical methods [14]. The solutions presented here for \( x \) and \( y \) do not require any further numerical integration.

1 In one dimension, this term only changes the sign of the force, in order to keep it in opposition to the movement.
3. Explicit solution parametrised by $\theta$

In order to obtain a solution of problem (4), we first change the equations for normal, $n$, and tangent, $t$, coordinates to the motion; hence, the corresponding force components are

$$ F_n = -mg \sin \theta - mb_0, $$

and

$$ F_t = -mg \cos \theta. $$

If the mass is constant, we obtain

$$ m\ddot{v} = -mg \sin \theta - mb_0 $$

and

$$ m\frac{\dot{v}^2}{\rho} = -mg \cos \theta. $$

where $\rho = -ds/d\theta$, and $s$ is the arc length. The last equation can be written as

$$ \dot{v} = -g \cos \theta, $$

with the help of the chain rule: $\rho = -ds/d\theta = -(ds/dr)(dr/d\theta)$. Equation (7) for the tangent acceleration can be modified with the same rule; using (9) the result is

$$ \frac{d\dot{v}}{d\theta} = v(\tan \theta + b_0 \sec \theta). $$

This is the hodograph equation, which is widely used in mechanics and hydrodynamics. In particular, the solution for the Legendre cases described in the introduction are obtained using this type of equation—see [10], for instance. We solve this first order differential equation for the initial conditions $v(t = 0) = v_0$ and $\theta(t = 0) = \theta_0$, obtaining

$$ v(\theta) = \frac{v_0 \cos \theta_0}{\cos \theta} \left( \frac{\Delta}{\Delta_0} \right), $$

with

$$ \Delta \equiv (\sec \theta + \tan \theta)^{\Delta_0}, $$

and $\Delta_0 \equiv \Delta(\theta_0)$.

The solution for time is

$$ t(\theta) = -\frac{1}{g} \int_{\theta_0}^{\theta} v(\theta) \sec \theta d\theta $$

$$ = \frac{-v_0 \cos \theta_0}{g \Delta_0} \left( \frac{(b_0 - \sin \theta) \Delta}{(b_0^2 - 1) \eta} - \frac{(b_0 - \sin \theta_0) \Delta_0}{(b_0^2 - 1) \eta_0} \right). $$

where

$$ \eta = (\cos \theta/2 - \sin \theta/2)(\cos \theta/2 + \sin \theta/2), $$

and $\eta_0 \equiv \eta(\theta_0)$. 


Using a similar procedure, we obtain

\[
x(\theta) = -\frac{1}{g} \int_{\theta_0}^{\theta} v(\theta)^2 d\theta = -\frac{1}{g} \left( \frac{v_0 \cos \theta_0}{\Delta_0} \right)^2 \\
\times \left[ \frac{(-2b_0 + \sin \theta) \Delta_0^2}{(2b_0 - 1)(2b_0 + 1)\eta} + \frac{(-2b_0 + \sin \theta_0) \Delta_0^2}{(2b_0 - 1)(2b_0 + 1)\eta_0} \right]
\]

(14)

and

\[
y(\theta) = -\frac{1}{g} \int_{\theta_0}^{\theta} v(\theta)^2 \tan \theta d\theta = -\frac{1}{g} \left( \frac{v_0 \cos \theta_0}{\Delta_0} \right)^2 \\
\times \left[ \frac{\sec^2 \theta(-3 + \cos 2\theta + 4b_0 \sin \theta) \Delta_0^2}{8(b_0^2 - 1)} \right.

\left. - \frac{\sec^2 \theta_0(-3 + \cos 2\theta_0 + 4b_0 \sin \theta_0) \Delta_0^2}{8(b_0^2 - 1)} \right]
\]

(15)

So, (12), (14) and (15) are, formally, the solutions to problem (4). Unfortunately, explicit inversion of \(i(\theta)\) is hard (if not impossible). Notwithstanding, these solutions are analytical, and no additional integration is required. In search of an explicit time dependent solution, the homotopy analysis method could offer a guide, as was the case for quadratic drag [1].

In order to establish that previous expressions are as useful as the time parametrisation we shall use them to plot the usual graph of \(x(t)\) and \(y(t)\), as well as the iconic \(y(x)\) (see 2). For comparison, we rewrite the drag-free solutions as functions of \(\theta\); the results

\[
t(\theta) = -\frac{v_0 \cos \theta_0}{g} (\tan \theta - \tan \theta_0),
\]

(16)

\[
x(\theta) = -\frac{(v_0 \cos \theta_0)^2}{g} (\tan \theta - \tan \theta_0),
\]

(17)

and

\[
y(\theta) = -\frac{(v_0 \cos \theta_0)^2}{2g} (\sec^2 \theta - \sec^2 \theta_0),
\]

(18)

are obtained by solving (7) and (8) for \(b_0 = 0\). It is an exercise to check that the preceding expressions are the familiar solutions of parabolic motion.

First we explain the solutions in angle parametrisation. To this end we draw equation (12) in figure 2(a), i.e. time as function of \(\theta\) for \(b_0 = 0.25\) (black line), \(b_0 = 0.5\) (red dashed line) and \(b_0 = 0.75\) (blue dotted line) and the drag-free case in blue dashed line, from (16). The launching angle was set to \(\theta_0 = \pi/4\) here; other selections will shift the graphs (not shown). The parameter \(\theta\) goes, asymptotically, to \(-\pi/2\), since the reference frame changes the orientation after the orbit reaches its apex, as it appears in figure 1.

In figure 2(b) solutions (14) for \(x(\theta)\) are presented in the same order as before (graphs diverging to \(\infty\) as \(\theta \to -\pi/2\)). The solutions (14) for \(y(\theta)\) are those that diverge to \(-\infty\) as \(\theta \to -\pi/2\). A close up of these (not shown) could show the angle where \(y = 0\). The numerical solutions to this condition shall be discussed below. Again we draw in blue dashed lines the drag-free solutions from (17) and (18).

In figure 2(c) time solutions are presented for \(x(t)\) (upper graphs) and \(y(t)\) (lower graphs). Using (14), (15) and a simple computational program, we can write the \(x(t(\theta))\) and \(x(t(\theta))\)
data and plot them. We do so, and present the results for the same drag values and color code. We consider only the range of $\theta$, in order to show the $y = 0$ condition. The $y$ results show that the larger the drag, the shorter the maxima. The maxima are reached in shorter times as the drag increases, as well.

Finally, we present the iconic $y(x)$ for projectile motion in figure 2(d). As expected, the larger the $b_0$ value, the shorter the path. Certainly, at first sight the paths are similar to those obtained with a linear drag, but a comparison requires comparing energy losses, not similar values of $b_0$ and $b_1$ [15].

3.1. The locus of the apices

The solution in terms of the angle could be hard to handle, but is straightforward for a particular locus: the locus formed by all the apices for initial launching angle $\theta_0$. The cases for no drag [16–18] and linear drag have been studied previously [19–21].

The apex for each orbit is obtained by setting $\theta = 0$ for $x$ and $y$ in (14) and (15) as can be seen at figure 1. After rearranging factors in these equations, and using $(\cos \theta_0/2 - \sin \theta_0/2)(\cos \theta_0/2 + \sin \theta_0/2) = \cos \theta_0^2$, the locus is written as

$$x(\theta_0) = -\frac{1}{g(4b_0^2 - 1)}\left(\frac{v_0 \cos \theta_0}{\Delta_0}\right)^2 \left[2b_0 + \frac{(-2b_0 + \sin \theta_0)\Delta_0^2}{\cos \theta_0}\right]$$

Figure 1. The normal, $n$, and tangent, $t$, reference frame during the projectile movement (in blue lines) as comparison to the usual reference frame $x - y$ (in magenta lines). The first comparison is made prior to reaching the apex, and the second, afterward. In the upper scale we show the velocity angle scale $\theta$ starting at $\theta_0$ up to the angle when $y = 0$, i.e., $-\theta = \theta^*$. The final, asymptotic, angle is $\theta = -\pi/2$, regardless of the value of the dissipation parameter. As usual, $W = mg$, indicated in red. The plotted trajectory corresponds to $v_0 = 50$, $\theta_0 = \pi/4$ and $b_0 = 0.25$ in SI units.

2 Since the launching angle is always in the first quadrant.
Figure 2. Explicit solution to the projectile motion in presence of a constant drag as function of the angle \( \theta \); in all cases we draw the drag-free solution in blue lines, \( b_0 = 0.25 \) in black lines, \( b_0 = 0.5 \) in red dashed lines, and the blue dotted lines correspond to \( b_0 = 0.75 \). In (a) time is depicted; notice that the angle goes from \( \theta_0 \) to \( -\pi/2 \) as time increases. In (b) we show \( x \) and \( y \) as a function of the angle, functions that grow to \( +\infty \) correspond to \( x \), and those going to \( -\infty \) are for \( y \) solutions. In (c) and (d) we depict the solution in the traditional variables, \( x \) and \( y \) as function of time for the former and \( y \) as function of \( x \) in the latter.

and

\[
y(\theta_0) = \frac{1}{8g(b_0^2 - 1)} \left( \frac{v_0 \cos \theta_0}{\Delta_0} \right)^2 [2 + \sec^2 \theta_0(-3 + \cos 2\theta_0 + 4b_0 \sin \theta_0)\Delta_0^2].
\]

(20)

In figure 3 we show the locus for parameters with values \( v_0 = 50 \) m s\(^{-1} \), \( b_0 = 0.15 \) and \( g = 9.81 \) m s\(^{-2} \). The drag-free solution

\[
x_m = \rho \sin \theta_0 \cos \theta_0,
\]

(21)

\[
y_m = \frac{\rho}{2} \sin^2 \theta_0,
\]

(22)

is shown for comparison. Here \( \rho = v_0^2/g \), and this solution corresponds to an ellipse described from the bottom. In polar coordinates \( r_m \) and \( \theta_m \) has the form

\[
r_m = \frac{2\rho \sin \theta_m}{1 + 3 \sin^2 \theta_m}.
\]

(23)

See [16] and appendix B in [21]. We add to the figure three orbits, those corresponding to launching angles \( \theta_0 = 30^\circ \), \( \theta_0 = 45^\circ \) and \( \theta_0 = 60^\circ \), as is usual in the textbooks.
Additionally, the locus for the linear drag has a closed form in both rectangular and polar coordinates, but it is not shown in the figure. The latter is written using the Lambert-W function, a common function in this problem. A closed form for the quadratic case is not known.

3.2. The locus on an inclined plane with Coulomb friction

In [13] the motion of a particle on an inclined plane under the presence of Coulomb friction is analysed. This friction is constant, and its magnitude is proportional to \( \mu mg \cos \alpha \), \( \mu \) being the kinetic friction constant and \( mg \cos \alpha \) the normal to a plane inclined by an angle \( \alpha \). Except for the coordinates orientation, solutions for \( x \), \( y \) and \( t \) obtained in [13] are equivalent to those obtained here. Notwithstanding that the microscopical nature of the retarding force is different, the equations are equal with the substitutions, of \( ma \rightarrow b \cot 0 \) and \( a \rightarrow g \sin \alpha \) for the physical parameters; the different coordinate orientation and velocity angle give the changes \( q_0 \rightarrow \sin \theta \) to \( \cos \alpha \) and \( q_0 \rightarrow \cos \sin \). Notice that this selection changes the hodograph equation from (10) to

\[
\frac{dv}{d\theta} = v(-\cot \theta + \mu \cot \alpha \csc \theta),
\]

in the cited reference.

For the locus described in the previous subsection, the solution under the influence of Coulomb friction is

\[
x(\theta_0) = -\frac{\eta_0}{g \sin \alpha (4 \mu \cot \alpha^2 - 1)} \left[ 2\mu \cot \alpha + \frac{(-2 \mu \cot \alpha + \sin \theta_0)\Delta_0^2}{\cos \theta_0} \right]
\]

and

\[
y(\theta_0) = \frac{\eta_0}{8 g \sin \alpha (\mu \cot \alpha^2 - 1)} \left[ 2 + \sec^2 \theta_0(-3 + \cos 2\theta_0 + 4 \mu \cot \alpha \sin \theta_0)\Delta_0^2 \right],
\]
with
\[ \eta_0 = \left( \frac{v_0 \cos \theta_0}{\Delta_0} \right)^2. \] (27)

These equations offer a straightforward way to obtain, experimentally, the locus on the plane.

### 3.3. Some important quantities in projectile motion: the range and the flight time

Unfortunately, not all the important quantities are of mathematical significance. While the apex has a mathematical meaning, other loci are important for practical reasons. Such is the case of the range and its maximum. Their value are determined by our choice of the origin and the chord generated. The selection of the origin, and hence the length of the chord, is determined in an arbitrary way. Hence, it is not surprising that we need to solve (15) numerically for \( y = 0 \).

This condition is translated from (15) to solve the equation
\[ \sec^2 \theta (-3 + \cos 2\theta + 4b_0 \sin \theta) \Delta^2 = \sec^2 \theta_0 (-3 + \cos 2\theta_0 + 4b_0 \sin \theta_0) \Delta^2_0. \] (28)

If we call
\[ p(\theta) = \sec^2 \theta (-3 + \cos 2\theta + 4b_0 \sin \theta) \Delta^2, \] (29)

we are looking for solutions such that \( p(\theta) = p(\theta_0) \). For symmetrical functions, the solution would be clear—but this is not the case, as can be seen in figure 4(a). There we plot \( p \) for the indicated values of \( b_0 \) and the drag-free case \( \sec^2 \theta (-4 + 2 \cos^2 \theta) = \sec^2 \theta_0 (-4 + 2 \cos^2 \theta_0) \) with the solution \( \theta^* = \pm \theta_0 \) (in blue dashed line). In this figure, the drag values considered are \( b_0 = 0.05 \) (black line), 0.15 (dashed red line) and 0.25 (dotted blue line). We add the extreme case of \( b_0 = 0.75 \) in order to show how asymmetric the curve \( p(\theta) \) can be. The color code remains in the rest of the graphs.

In figure 4(b), we show the solution obtained via Newton–Raphson for the equation \( p(\theta^*) = p(\theta_0) = 0 \), and in figure 4(c), the corresponding case for the range as function of the launching angle.

In the last figure, the maximum range occurs at \( \theta_0 \approx 0.7697, 0.7226 \) and 0.6912 for the indicated values of \( b_0 \). All these values are smaller than \( \theta_0 = \pi/4 \approx 0.7854 \), the corresponding value for the drag-free case (shown in blue broken line). For completeness, we present the time of flight as function of the launching angle in figure 4(d). This time increases with the angle. Notice that the drag-free case and the solution for \( b_0 = 0.05 \) are so close that they appear superimposed.

### 4. Conclusions

We discuss the motion of a projectile under the influence of constant gravitational pull and constant drag. Such a case could be considered as the yield stress in a non-Newtonian fluid and as an example of a simple situation where the retarding force depends on the velocity direction. The two coupled non-linear differential equations in rectangular coordinates can be exactly solved by changing to normal and tangent coordinates, and obtaining the

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3 Calculations for values of \( b_0 \) larger than 0.25 require a better selection of the initial condition, as can be seen for \( b_0 = 0.75 \) in figure 4(a).
The solutions, \((14)\) and \((15)\), are parametrised as functions of the velocity angle. That allows us to obtain the locus of the apices in an explicit way. Other loci or quantities require numerical calculation, as with the range and flight time presented in the previous section.

Since this problem involves the same equations as the problem of a particle moving on an inclined plane under the influence of Coulomb friction, the solutions are the same. We present the solution for the Coulomb case rewritten from our equations. They could offer an easy way to obtain the locus of apices in an experimental way.

This problem serves as a good example to introduce undergraduate students to problems with curvature and retarding forces, beyond the problem of an inclined plane with constant friction.

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**References**

[1] Yabushita K, Yamashita M and Tsuboi K 2007 *J Phys A: Math and Theo* **40** 8403–16

[2] Belgacem C H 2014 *Eur. J. Phys.* **35** 055025
[3] Turkyilmazoglu M 2016 Eur. J. Phys. 37 035001
[4] Morales D A 2016 Acta Mech 227 1593–607
[5] Stewart S M 2012 Eur. J. Phys. 33 149–66
[6] Dittrich T, Hänggi P, Ingold G L, Kramer B, Schön G and Zwerger W 1998 Quantum Transport and Dissipation (Weinheim: Wiley-VCH)
[7] Razavy M 2006 Classical and Quantum Dissipative Systems (London: Imperial College Press)
[8] Jones S E, Caipen T L and Butson G J 1991 International Journal of Applied Engineering Education 7 321–7
[9] Jones S E 1991 The projectile motion problem including the effects of constant thrust and drag Challenges of a changing world (The Organization vol. 1 and 2) ed A S E Educ (Univ. of New Orleans, New Orleans, LA: Amer. Soc. Engineering Education) 1839–41 proceeding paper
[10] MacMillan W 1927 Theoretical Mechanics: Static and the Dynamics of a Particle (New York: McGraw-Hill)
[11] Barnes H A 1999 J. Non-Neut. Fluid Mech. 81 133–78
[12] Bruyn J R and Walsh A M 2004 Can. J. Phys. 82 439?446
[13] Wang X 2014 Trajectory of a particle on a frictional inclined plane Am. J. Phys. 82 764–8
[14] Burden R L, Faires J D and Burden A M 2015 Numerical Analysis (Belmont, CA: Brooks Cole)
[15] Hernández-Saldaña H Energy losses in projectile motion under constant, linear and quadratic drag (work in progress)
[16] Fernández-Chapou J L, Salas-Brito A L and Vargas C A 2004 Am. J. Phys. 72 1109
[17] Thomas G, Weir M, Hass J and Giordano P 2004 Calculus (Boston, MA: Addison-Wesley)
[18] Carrillo-Bernal M A, Mancera-Piña P E, Cerecedo-Núñez H H, Padilla-Sosa P, Núñez Yépez H N and Salas-Brito A L 2014 Am. J. Phys. 82 707
[19] Stewart S 2006 Int. J. Math Educ. Sci. Technol. 37 411–31
[20] Stewart S M 2011 Eur. J. Phys. 32 L7–10
[21] Hernández-Saldaña H 2010 Eur. J. Phys. 31 1319–29