Explicit solutions of generalized Cauchy-Riemann systems using the transplant operator

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Abstract

In \cite{8} it was shown that the tool introduced there and called the transplant operator transforms solutions of one Vekua equation into solutions of another Vekua equation, related to the first via a Schrödinger equation. In this paper we prove a fundamental property of this operator: it preserves the order of zeros and poles of generalized analytic functions and transforms formal powers of the first Vekua equation into formal powers of the same order for the second Vekua equation. This property allows us to obtain positive formal powers and a generating sequence of a “complicated” Vekua equation from positive formal powers and a generating sequence of a “simpler” Vekua equation. Similar results are obtained regarding the construction of Cauchy kernels. Elliptic and hyperbolic pseudoanalytic function theories are considered and examples are given to illustrate the procedure.

1 Introduction

In the present work a special class of Vekua equations describing generalized analytic or pseudoanalytic functions is considered. It arises naturally in relation with some linear equations of mathematical physics such as the stationary Schrödinger equation, the conductivity equation and others. Vekua equations of this type we call main Vekua equations. They are closely related to another generalization of the Cauchy-Riemann system, the system describing so-called $p$-analytic functions (see the definitions in the next section). The general pseudoanalytic function theory mainly created by L. Bers and his coauthors and presented in \cite{1} among other developments contains deep results on generalizations of the concept of complex differentiability and integrability, Taylor and Laurent series related to generalized analytic functions as well as the generalizations of the Cauchy integral formula and its corollaries. In the core of Bers’ theory there is a concept of a generating sequence related to a Vekua equation. In general a derivative of a generalized analytic function in the sense introduced by Bers is not any more a solution of the same Vekua equation but of another Vekua equation called a successor of the original one. Bers derivatives of solutions of this second Vekua equation will solve another Vekua equation, and in principle this sequence of Vekua equations related to the original one is infinite. If somehow one manages to obtain a pair of solutions in a certain sense independent for each of these Vekua equations then such sequence of pairs is called the generating sequence and it immediately allows one to construct a complete system of positive formal powers related to the original Vekua equation. The formal powers are basic constituents of the Taylor-type series expansions of the pseudoanalytic functions and generalize the usual powers $(z-z_0)^n$ in the sense that being a solution of the Vekua equation a formal power of order $n$ asymptotically behaves like $(z-z_0)^n$ when $z \to z_0$. Moreover, theorems generalizing such facts like the Runge theorem on the completeness of the system of powers in a uniform convergence topology and even stronger results guaranteeing the completeness in the $C$-norm were obtained in the framework of pseudoanalytic function theory.

One of the most significant obstacles for the further development and a broader application of pseudoanalytic function theory is the explicit construction of generating sequences, formal powers and Cauchy kernels corresponding to Vekua...
equations arising in applications. Bers himself and together with Gelbart succeeded in constructing a generating sequence in a very special case (see [1] and [9]). In [7] an algorithm for explicit construction of a generating sequence was proposed for a much more general situation. In application to second-order elliptic equations with the aid of the tools from pseudoanalytic function theory this result allows one to obtain a complete system of solutions of the equation, e.g., of the Schrödinger equation when the equation possesses a particular solution \( f \) in a separable form \( f = U(u)V(v) \) where \( u, v \) are orthogonal solutions of some stationary Schrödinger equation.

In the present paper we substantially extend the class of Vekua equations and of systems describing \( p \)-analytic functions for which a generating sequence and a system of formal powers can be constructed explicitly. For this we use a concept introduced in [8] and called there the transplant operator. In fact, it is an operator transforming solutions of one Vekua equation into solutions of another one related to the first via a Schrödinger equation. Here we prove a crucial property of the transplant operator: it transforms formal powers into formal powers of the same order (see details in Section 3). This means that if we are able to solve a Vekua equation, that is we know its generating sequence then using the transplant operator we can construct positive formal powers and a generating sequence for a related Vekua equation which can be much more complicated. As an example in Section 4 we consider a Vekua equation whose generating sequence is periodic with a period 1, that is it consists of one generating pair only. In this case it is relatively easy to obtain the corresponding positive formal powers. Then using the approach described in the present work, it is possible to obtain systems of positive formal powers and generating sequences for a wide class of Vekua equations related to the first one. The structure of generating sequences and of formal powers for the related Vekua equations are more complicated. We also obtain a similar result regarding the construction of Cauchy kernels as we show in Section 6. If a Cauchy kernel for a Vekua equation is known, it can be used for constructing Cauchy kernels for a wide class of related Vekua equations using the transplant operator.

All the described results have a direct application to linear second-order equations. For example, in the case of the stationary two-dimensional Schrödinger equation \((-\Delta + q(x, y)) u = 0\) with \( q \) being real valued, the existence of one solution \( u \) such that a generating sequence for an associated main Vekua equation can be constructed explicitly leads not only to the construction of a complete system of solutions to this Schrödinger equation but also to the construction of complete systems of solutions to any Schrödinger equation with the potential \( q_f = -q + 2(\nabla f/f)^2 \) where \( f \) is any solution of the original Schrödinger equation with the potential \( q \). Note that the form of the potential \( q_f \) is a precise generalization of the potential obtained after a Darboux transformation in a one-dimensional case (see, e.g., [12]).

2 Some known facts about generalized Cauchy-Riemann systems

Let \( \Omega \) be a domain in \( \mathbb{R}^2 \). Throughout the whole paper we suppose that \( \Omega \) is a simply connected domain.

In the present work we consider two related generalized Cauchy-Riemann systems. The first defines so-called \( p \)-analytic functions [15] (see also [9]) and has the following form

\[
\varphi_x = \frac{1}{p} \psi_y, \quad \varphi_y = -\frac{1}{p} \psi_x
\]

(1)

where \( p \) is a given positive function of two real variables \( x \) and \( y \) which is supposed to be continuously differentiable; \( \varphi \) and \( \psi \) are real-valued continuously differentiable functions. If \( \varphi \) and \( \psi \) are solutions of this system in \( \Omega \), then the complex function \( \omega = \varphi + i\psi \) of a complex variable \( z = x + iy \) is said to be \( p \)-analytic in \( \Omega \).

The second system considered here represents a special case of a general Vekua equation (see, e.g., [17]) and sometimes is referred to as the main Vekua equation [9]. It has the form

\[
Wz = \frac{f}{f} W \quad \text{in } \Omega
\]

(2)

where the subindex \( \overline{z} \) means the application of the operator \( \partial_{\overline{z}} := \frac{i}{2}(\partial_x + i\partial_y) \), \( W \) is a continuously differentiable complex valued function, \( f \) is a positive function of \( x \) and \( y \), twice continuously differentiable, which will be supposed to be a particular solution of some stationary Schrödinger equation

\[
(-\Delta + q) u = 0 \quad \text{in } \Omega,
\]

(3)

that is \( q = \Delta f/f \).
Systems (1) and (2) are equivalent \[8, 9\] in the following sense. Denote 
\[V := \partial z - \frac{fz}{f} C \]
where \(C\) is the operator of complex conjugation. We suppose that \(p = f^2\) and introduce the operator 
\[\Pi := f \partial z P^+ + \frac{1}{f} \partial z P^- , \]
where \(P^\pm := \frac{1}{2}(I \pm C)\) and \(I\) is the identity operator. We have that the equation 
\[\Pi \omega = 0 \quad (4)\]
is equivalent to the system 
\[\varphi_x = \frac{1}{f^2} \psi_y, \quad \varphi_y = -\frac{1}{f^2} \psi_x \quad (5)\]
where \(\varphi = \text{Re} \omega\) and \(\psi = \text{Im} \omega\).

Denote 
\[B := f P^+ + \frac{1}{f} P^- . \]

Then it is easy to see that 
\[B^{-1} = \frac{1}{f} P^+ + f P^- . \]

**Proposition 1** \[8\] 
\[\forall B = \Pi . \]

**Remark 2** From proposition\[7\] we have also that 
\[V = \Pi B^{-1} . \]

Thus, application of the operator \(B\) or \(B^{-1}\) respectively allows us to establish a direct relation between the results corresponding to (1) and (2).

The following factorization of the Schrödinger operator will be used.

**Theorem 3** \[5\] Let \(f\) be a positive in \(\Omega\) particular solution of (3). Then for any real valued function \(\varphi \in C^2(\Omega)\) the following equalities hold
\[\frac{1}{4} (\Delta - q) \varphi = \left( \partial z + \frac{fz}{f} C \right) \left( \partial z - \frac{fz}{f} C \right) \varphi = \left( \partial_x + \frac{f}{f} C \right) \left( \partial_x - \frac{f}{f} C \right) \varphi . \quad (6)\]

An immediate corollary of this theorem is the fact that if \(W\) is a solution of (2) then its real part \(W_1\) is necessarily a solution of (3), meanwhile its imaginary part \(W_2\) is a solution of the following Schrödinger equation 
\[- \Delta W_2 + q_1 W_2 = 0 \quad \text{in} \ \Omega \quad (7)\]
where \(q_1 = 2(\nabla f)^2 / f^2 - q\) and \((\nabla f)^2 = f_x^2 + f_y^2\) (see \[5\] and \[11\]). Moreover, given \(W_1\), the corresponding \(W_2\) can be easily constructed and vice versa. In order to formulate this result we need to introduce the following notation. Note that the operator \(\partial_x\) applied to a real-valued function \(\phi\) can be regarded as a kind of gradient, and if we know that \(\partial_x \phi = \Phi\) in a whole complex plane or in a convex domain, where \(\Phi = \Phi_1 + i \Phi_2\) is a given complex valued function such that its real part \(\Phi_1\) and imaginary part \(\Phi_2\) satisfy the equation 
\[\partial_x \Phi_1 - \partial_y \Phi_2 = 0 , \quad (8)\]
then we can reconstruct \(\phi\) up to an arbitrary real constant \(c\) in the following way 
\[\phi(x, y) = 2 \left( \int_{x_0}^x \Phi_1(\eta, y) d\eta + \int_{y_0}^y \Phi_2(x_0, \xi) d\xi \right) + c \quad (9)\]
where \((x_0, y_0)\) is an arbitrary fixed point in the domain of interest. Note that this formula can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve \(\Gamma\) leading from \((x_0, y_0)\) to \((x, y)\)

\[
\phi(x, y) = 2 \left( \int_{\Gamma} \Phi_1 dx + \Phi_2 dy \right) + c.
\]

By \(\overline{A}\) we denote this integral operator:

\[
\overline{A}[\Phi](x, y) = 2 \left( \int_{x_0}^{x} \Phi_1(\eta, y) d\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi) d\xi \right) + c.
\]

Thus if \(\Phi\) satisfies (8), there exists a family of real valued functions \(\phi\) such that \(\partial z \phi = \Phi\), given by the formula \(\phi = \overline{A}[\Phi]\).

\[\text{Theorem 4}\]

Let \(W_1\) be a real valued solution of (3) in a simply connected domain \(\Omega\). Then the real valued function \(W_2\), solution of (7) such that \(W = W_1 + iW_2\) is a solution of (2), is constructed according to the formula

\[
W_2 = f^{-1} \overline{A}(if^{-2} \partial_z(f^{-1}W_1)).
\]

Given a solution \(W_2\) of (7), the corresponding solution \(W_1\) of (3) such that \(W = W_1 + iW_2\) is a solution of (2), is constructed as follows

\[
W_1 = -f \overline{A}(if^{-2} \partial_z(fW_2)).
\]

\[\text{Remark 5}\]

When in (3) \(q \equiv 0\) and \(f \equiv 1\), equalities (10) and (11) turn into the well known formulas in complex analysis for constructing conjugate harmonic functions.

We will need some definitions and results from Bers’ pseudoanalytic function theory [1] concerning solutions of the general Vekua equation

\[
W_z = a(F,G)W + b(F,G)\overline{W},
\]

where we will suppose that \(a(F,G)\) and \(b(F,G)\) are continuously differentiable complex functions. A couple of solutions of (12) in \(\Omega\), \(F\) and \(G\) satisfying the inequality \(\text{Im}(FG) > 0\) form a so-called generating pair of the Vekua equation. Every complex function \(W\) defined in \(\Omega\) admits the unique representation \(W = \phi F + \psi G\) where the functions \(\phi\) and \(\psi\) are real valued. Sometimes it is convenient to associate with the function \(W\) the function \(\omega = \phi + i\psi\). The correspondence between \(W\) and \(\omega\) is one-to-one.

The following expressions are known as characteristic coefficients of the pair \((F,G)\)

\[
a_{(F,G)} = -\frac{FG_z - Fz \overline{G}}{FG - \overline{FG}}, \quad b_{(F,G)} = \frac{FG_z - Fz \overline{G}}{FG - \overline{FG}},
\]

\[
A_{(F,G)} = -\frac{FG_T - F_T \overline{G}}{FG - \overline{FG}}, \quad B_{(F,G)} = \frac{FG_T - F_T \overline{G}}{FG - \overline{FG}}.
\]

For solutions of (12) the following operation is introduced, called the \((F,G)\)-derivative and denoted as \(W = \frac{d_{(F,G)}W}{dz}\):

\[
W = W_z - A_{(F,G)}W - B_{(F,G)}\overline{W} = \phi_z F + \psi_z G.
\]

The inverse operation is introduced as follows.

\[\text{Definition 6}\]

Let \((F,G)\) be a generating pair. Its adjoint generating pair \((F*,G*)\) is defined by the formulas

\[
F^* = -\frac{2F}{FG - \overline{FG}}, \quad G^* = \frac{2\overline{G}}{FG - \overline{FG}}.
\]
The \((F,G)\)-integral is defined as follows

\[
\int_{\Gamma} W d_{(F,G)} z = F(z_1) \text{Re} \int_{\Gamma} G^* W dz + G(z_1) \text{Re} \int_{\Gamma} F^* W dz
\] (14)

where \(\Gamma\) is a rectifiable curve leading from \(z_0\) to \(z_1\).

If \(W = \phi F + \psi G\) is a solution of (12) where \(\phi\) and \(\psi\) are real valued functions then

\[
\int_{z_0}^{z} W d_{(F,G)} z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z),
\] (15)

and as \(\dot{F} = \dot{G} = 0\), this integral is path-independent and represents the \((F,G)\)-antiderivative of \(W\).

The \((F,G)\)-derivative \(W\) is a solution of another Vekua equation with some other coefficients \((a_1, b_1)\) and possessing another generating pair \((F_1, G_1)\) called a successor of \((F,G)\).

**Definition 7** A sequence of generating pairs \(\{(F_m, G_m)\}\), \(m = 0, \pm 1, \pm 2, \ldots\), is called a generating sequence if \((F_{m+1}, G_{m+1})\) is a successor of \((F_m, G_m)\). If \((F_0, G_0) = (F,G)\), we say that \((F,G)\) is embedded in \(\{(F_m, G_m)\}\).

Let \(W\) be an \((F,G)\)-pseudoanalytic function. Using a generating sequence in which \((F,G)\) is embedded we can define the higher derivatives of \(W\) by the recursion formula

\[
W^{[0]} = W; \quad W^{[m+1]} = \frac{d_{(F_m,G_m)} W^{[m]}}{dz}, \quad m = 1, 2, \ldots
\]

The notion of a generating sequence leads to the concept of formal powers.

**Definition 8** Each formal power \(Z^{(n)}(a, z_0; z)\) corresponding to the Vekua equation (12), with some exponent \(n \in \mathbb{Z}\), \(a\) being a complex number, \(z_0\) a point in \(\Omega\), is a solution of (12) in the whole domain \(\Omega\), such that

\[
\lim_{z \to z_0} \frac{Z^{(n)}(a, z_0; z)}{a(z - z_0)^n} = 1.
\]

That is \(Z^{(n)}(a, z_0; z)\) is a solution of (12) possessing a zero or a pole of order \(n\) depending on the sign of \(n\), and for \(n = 0\) it takes the value \(a\) at \(z_0\).

The nonnegative formal powers \((n \geq 0)\) can be defined also in the following recursive way.

**Definition 9** The formal power \(Z^{(0)}_m(a, z_0; z)\) with center at \(z_0 \in \Omega\), coefficient \(a\) and exponent \(0\) is defined as the linear combination of the generators \(F_m, G_m\) with real constant coefficients \(\lambda, \mu\) chosen so that \(\lambda F_m(z_0) + \mu G_m(z_0) = a\). The formal powers with exponents \(n = 1, 2, \ldots\) are defined by the recursion formula

\[
Z^{(n)}_m(a, z_0; z) = n \int_{z_0}^{z} Z^{(n-1)}_{m+1}(a, z_0; \zeta) d_{(F_m,G_m)} \zeta.
\] (16)

This definition implies the following properties.

1. \(Z^{(n)}(a, z_0; z)\) is an \((F_m, G_m)\)-pseudoanalytic function of \(z\), that is, it is a solution of the Vekua equation \(w_\zeta = a_m w + b_m \bar{w}\) possessing a generating pair \((F_m, G_m)\).

2. If \(a'\) and \(a''\) are real constants, then \(Z^{(n)}_m(a' + ia'', z_0; z) = a' Z^{(n)}_m(1, z_0; z) + a'' Z^{(n)}_m(i, z_0; z)\).

3. The formal powers satisfy the differential relations

\[
\frac{d_{(F_m,G_m)} Z^{(n)}_m(a, z_0; z)}{dz} = n Z^{(n-1)}_{m+1}(a, z_0; z).
\] (17)
4. The asymptotic formulas

\[ Z^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0 \]

hold.

Moreover, the system of all formal powers \( \left\{ Z^{(n)}(a, z_0; z) \right\}_{n=0}^{\infty} \) represents a complete system of solutions of (12) in the following sense. We will omit the subindex 0 when a formal power corresponds to \((F, G)\), that is

\[ Z^{(n)}(a, z_0; z) := Z^{(n)}(a, z_0; z). \]

**Theorem 10** \[2\] A solution of (12) defined in a bounded simply connected domain can be expanded into a normally convergent series of formal polynomials (linear combinations of formal powers with positive exponents).

Moreover, the following stronger result is valid.

**Theorem 11** \[13\] Let \( W \) be a solution of (12) in a domain \( \Omega \) bounded by a Jordan curve and satisfy the Hölder condition on \( \partial \Omega \) with the exponent \( \alpha (0 < \alpha \leq 1) \). Then for any \( \varepsilon > 0 \) and any natural \( n \) there exists a pseudopolynomial of order \( n \) satisfying the inequality

\[ |W(z) - P_n(z)| \leq \frac{\text{Const}}{n^{\alpha - \varepsilon}} \quad \text{for any } z \in \Omega \]

where the constant does not depend on \( n \), but only on \( \varepsilon \).

With the aid of these results concerning pseudoanalytic formal powers and of the relation between solutions of the main Vekua equation to the Schrödinger equation corresponding completeness results were obtained for solutions of the Schrödinger equation as, e.g., the following statement.

**Theorem 12** \[6\] An arbitrary solution of (3) defined in a bounded simply connected domain \( \Omega \) where there exists a positive particular solution \( f \in C^1(\Omega) \) of (3) can be expanded into a normally convergent series of real parts of formal polynomials.

As was mentioned before besides positive formal powers also the negative were defined by L. Bers (see \[1\]). First of all, the existence of the generalized Cauchy kernel was proved, that is the existence of a solution \( w \) of (12) in \( \Omega \setminus \{z_0\} \) which satisfies the relation

\[ \lim_{z \to z_0} \frac{w(z)}{a(z - z_0)^{-1}} = 1 \] (18)

where \( a \) is any complex number. This function is denoted as follows

\[ w(z) = Z^{(-1)}(a, z_0, z). \]

The negative formal powers \( Z^{(-n)} \) for \( n = 2, 3, \ldots \), are constructed using the recursive differential relations like (17).

With the aid of the positive and negative formal powers a whole theory of pseudoanalytic functions was developed including Taylor and Laurent series, and their numerous properties similar to the properties of their special cases corresponding to the usual analytic functions. The generalized Cauchy kernel \( Z^{(-1)}(\alpha, z_0, z) \) makes it possible to prove a generalization of the Cauchy integral formula \[1\], see also \[9\].

Thus, an important problem is to find the way to construct the formal powers explicitly. This is the main subject of this paper.

3 The transplant operator

In this section we define and study the main tool of this paper called the transplant operator. It was introduced in \[9\] and used for constructing Cauchy kernels and Cauchy integral representations for an important subclass of \( p \)-analytic functions, the \( x^k \)-analytic functions. Let us describe the main idea behind this concept.

Let both \( f \) and \( g \) be positive solutions of (3) in \( \Omega \). Together with the main Vekua equation (2) we consider the main Vekua equation corresponding to \( g \):

\[ w_{\bar{z}} = \frac{\bar{g} w}{g} \quad \text{in } \Omega. \] (19)
We have that both \( \text{Re} W \) (where \( W \) is a solution of (2)) and \( \text{Re} w \) satisfy (3) in \( \Omega \), meanwhile \( \text{Im} W \) and \( \text{Im} w \) satisfy in general different Schrödinger equations
\[
(-\Delta + q_1) \text{Im} W = 0 \quad \text{in } \Omega
\] (20)
and
\[
(-\Delta + q_2) \text{Im} w = 0 \quad \text{in } \Omega
\] (21)
where \( q_1 = 2(\nabla f)^2/f^2 - q \) and \( q_2 = 2(\nabla g)^2/g^2 - q \).

Now we introduce an operator which transforms solutions of (2) into solutions of (19) acting in the following way
\[
T_{f,g}[W] = P^+ W + ig^{-1} A [ig^2 \partial_z (g^{-1} P^+ W)].
\] (22)
Its application makes the imaginary part of a solution of (2) drop out and be substituted by an imaginary part constructed according to theorem 4 in such a way that after this “transplant” operation the new complex function \( w = T_{f,g}[W] \) becomes a solution of (19). This is why we call the operator \( T_{f,g} \) the transplant operator.

Assigning a fixed value in a certain point of the domain of interest to the result of application of \( A \) we obtain an invertible one-to-one map establishing a relation between solutions of (2) and (19). The inverse to \( T_{f,g} \) is given by the expression
\[
T_{f,g}^{-1}[w] = T_{g,f}[w] = P^+ w + if^{-1} A [if^2 \partial_z (f^{-1} P^+ w)],
\]

Let us denote the formal powers corresponding to (2) and (19) by \( Z_{f}^{(n)}(a, z_0, z) \) and \( Z_{g}^{(n)}(a, z_0, z) \) respectively. In the following we establish a useful property of the transplant operator. Namely, that it allows one to transform an \( n \)-th formal power to an \( n \)-th formal power. We will consider the case of positive and negative formal powers separately.

Let \( W \) be a solution of (2) such that
\[
\lim_{z \to z_0} \frac{W(z)}{a(z - z_0)^n} = 1
\] (23)
for some \( z_0 \in \Omega, n \in \mathbb{N} \) and a complex number \( a \). That is \( W \) is a formal power \( Z_{f}^{(n)}(a, z_0, z) \) corresponding to (2). As before, we denote \( W_1 = \text{Re} W \) and \( W_2 = \text{Im} W \) and due to theorem 4 we have the equality (10). As \( W \) has a zero at \( z_0 \) it is convenient to write \( A[\Phi] \) where \( \Phi = if^2 \partial_\tau (f^{-1} W_1) \) as follows
\[
A[\Phi](z) = 2 \left( \int_\Gamma \Phi_1 dx + \Phi_2 dy \right)
\] (24)
where \( \Gamma \) is a rectifiable curve leading from \( z_0 \) to \( z \). That is we fix \( z_0 \) as an initial point for integration in \( A \).

Now consider
\[
\omega_2 = g^{-1} A (ig^2 \partial_\tau (g^{-1} W_1))
\] (25)
where again \( z_0 \) is used as an initial point for integration. We are interested in the limit
\[
\lim_{z \to z_0} \frac{W_2(z)}{\omega_2(z)} = c \lim_{z \to z_0} \frac{\psi_f(z)}{\psi_g(z)}
\] (26)
where \( c := f^{-1}(z_0)/g^{-1}(z_0) \),
\[
\psi_f := A (if^2 \partial_\tau (f^{-1} W_1))
\] (27)
and
\[
\psi_g := A (ig^2 \partial_\tau (g^{-1} W_1)).
\] (28)
In order to prove its existence and evaluate it let us consider any direction in the plane defined by a vector \( d = (d_1, d_2)^T \), and assume that \( z \) tends to \( z_0 \) along the corresponding path, that is we consider the following limit
\[
\lim_{t \to 0} \frac{\psi_f(x_0 + td_1, y_0 + td_2)}{\psi_g(x_0 + td_1, y_0 + td_2)^2}.
\]
By definition, \( \psi_f(z_0) = \psi_g(z_0) = 0 \), and hence to evaluate this limit we can make use of the l’Hospital rule which here gives us
\[
\lim_{t \to 0} \frac{\psi_f(x_0 + td_1, y_0 + td_2)}{\psi_g(x_0 + td_1, y_0 + td_2)} = \frac{\partial\psi_f(x_0, y_0)}{\partial d} \frac{(\nabla\psi_f(x_0, y_0), d)}{(\nabla\psi_g(x_0, y_0), d)}
\]
where \(\langle \cdot, \cdot \rangle\) denotes the usual scalar product of two vectors. Let us note that the last expression can be written in a complex-analytic form as follows
\[
\frac{\text{Re} (\partial_x \psi_f(z_0) \cdot (d_1 - id_2))}{\text{Re} (\partial_x \psi_g(z_0) \cdot (d_1 - id_2))}.
\]
We recall that \(\psi_f\) and \(\psi_g\) are defined by (27) and (28) respectively. Thus we have
\[
\lim_{t \to 0} \frac{\psi_f(x_0 + td_1, y_0 + td_2)}{\psi_g(x_0 + td_1, y_0 + td_2)} = \frac{\text{Re} (i\psi f^{-1}(f^{-1}W_1) \cdot (d_1 - id_2))}{\text{Re} (i\psi g^{-1}(g^{-1}W_1) \cdot (d_1 - id_2))}.
\]
Now we use the fact that \(W_1(z_0) = 0\) as well as once more that \(f\) and \(g\) are positive and obtain that
\[
\lim_{t \to 0} \frac{\psi_f(x_0 + td_1, y_0 + td_2)}{\psi_g(x_0 + td_1, y_0 + td_2)} = \frac{1}{c}
\]
for any direction \(d\). Thus, the limit (26) exists and \(\lim_{z \to z_0} \frac{W_2(z)}{W_2(z_0)} = 1\). Consequently we obtain that the function \(W_1 + i\omega_2\) satisfies the asymptotic relation (23) as well and represents a formal power \(Z^{(n)}(a, z_0, z)\) corresponding to (19).

Now let us consider negative formal powers. We suppose that \(W\) is a solution of (2) and for this we again consider any direction \(d\) and use the l’Hospital rule as both functions tend to infinity at \(z_0\):
\[
\lim_{t \to 0} \frac{\psi_f(x_0 + td_1, y_0 + td_2)}{\psi_g(x_0 + td_1, y_0 + td_2)} = \frac{1}{c^{2}} \frac{\text{Re} (\partial_x f^{-1}(f^{-1}W_1) \cdot (d_2 + id_1))}{\text{Re} (\partial_x g^{-1}(g^{-1}W_1) \cdot (d_2 + id_1))}.
\]

Here the reasoning we used before, in the case of positive formal powers, is not already applicable. Nevertheless we note that the l’Hospital rule can be applied to the obtained quotient in the opposite direction. Namely, we have
\[
1 \cdot \frac{\text{Re} (\partial_x f^{-1}(f^{-1}W_1) \cdot (d_2 + id_1))}{\text{Re} (\partial_x g^{-1}(g^{-1}W_1) \cdot (d_2 + id_1))} = \frac{1}{c^{2}} \frac{\text{Re} (\partial_x f^{-1}(f^{-1}W_1) \cdot (d_2 + id_1))}{\text{Re} (\partial_x g^{-1}(g^{-1}W_1) \cdot (d_2 + id_1))}.
\]

Thus we proved that with the aid of the transplant operator both positive and negative formal powers corresponding to (2) and (19) can be transformed to each other. We formulate these statements as the following theorems.
Theorem 13 Let $f$ and $g$ be real valued nonvanishing solutions of (3) in a simply connected domain $\Omega \subset \mathbb{R}^2$. Let $z_0 \in \Omega$, $a \in \mathbb{C}$ and $Z^{(n)}_f(a,z_0,z)$, $n \in \mathbb{N}$ be a formal power associated with equation (2). Then the function $Z^{(n)}_g(a,z_0,z) := T_{f,g} [Z^{(n)}_f(a,z_0,z)]$ is a formal power of order $n$, with center at $z_0$ and coefficient $a$, associated with equation (19). Here $T_{f,g}$ is defined by (22) with $\overline{A}$ being defined by (24) where as an initial point of integration is chosen $z_0$.

Theorem 14 Let $f$ and $g$ be real valued nonvanishing solutions of (3) in a simply connected domain $\Omega \subset \mathbb{R}^2$. Let $z_0 \in \Omega$, $a \in \mathbb{C}$ and $Z^{(n)}_f(a,z_0,z)$, $n \in \mathbb{N}$ be a formal power associated with equation (3). Then the function $Z^{(n)}_g(a,z_0,z) := T_{f,g} [Z^{(n)}_f(a,z_0,z)]$ is a formal power of order $-n$, with center at $z_0$ and coefficient $a$, associated with equation (19). Here $T_{f,g}$ is defined by (22) with $\overline{A}$ being defined by (24) where $\Gamma$ is any rectifiable curve belonging to $\Omega$, leading from $z_1$ to $z$ and not passing through $z_0$.

4 Construction of positive formal powers

As we have shown in the previous section the transplant operator allows us to transform positive and negative formal powers of one main Vekua equation, say (2), into formal powers of the same order of another main Vekua equation, say (19), when the coefficients $f$ and $g$ are solutions of the same Schrödinger equation (3). This observation leads to a substantial extension of the class of Vekua equations and of systems of the form (1) for which a generating sequence and a complete system of formal powers can be obtained. Suppose we are interested in solving a Vekua equation of the form (19) or a system describing a corresponding generating sequence.

As an example, let us consider two positive solutions $f = y^2$ and $g = \frac{1 + xy^3}{y}$ of the Schrödinger equation (3) with potential $q = 2/y^2$ in the domain $\Omega = \{(x,y) \mid x > 0 \text{ and } y > 0\}$. In this case, the Vekua equations (2) and (19) take, respectively, the form

$$W_\tau = \frac{i}{y} \nabla \text{ in } \Omega,$$

and

$$w_\tau = \frac{y^4 + i(2xy^3 - 1)}{2y(1 + xy^3)} \nabla \text{ in } \Omega.$$

For the calculations given below we used Maple. Let us first calculate the formal powers $Z^{(n)}_f(a,z_0;z)$ of orders $n = 0, 1, 2$ for the Vekua equation (20) with generating pair $(F,G) = (f,i/f)$ where $z_0 := x_0 + iy_0$ and $(x_0,y_0) \in \Omega$. Using property 2 following definition 9 we are considering $Z^{(n)}_f(1,z_0;z)$ and $Z^{(n)}_f(i,z_0;z)$. By definition 9 we have $Z^{(0)}_f(1,z_0;z) = LF(z) + \mu G(z)$ and $Z^{(0)}_f(i,z_0;z) = LF(z) + \mu' G(z)$ where the constants $(\lambda,\mu),(\lambda',\mu')$ are defined by $\lambda F(z_0) + \mu G(z_0) = 1$ and $\lambda' F(z_0) + \mu' G(z_0) = i$. We find $(\lambda,\mu) = (1/y_0,0)$ and $(\lambda',\mu') = (0,y_0^2)$ such that

$$Z^{(0)}_f(1,z_0;z) = \left(\frac{y}{y_0}\right)^2 \quad \text{and} \quad Z^{(0)}_f(i,z_0;z) = i \left(\frac{y}{y_0}\right)^2.$$

In order to construct $Z^{(1)}_f(1,z_0;z)$ for $\alpha = 1, i$ from formula (16) we need first $Z^{(0)}_{f,1}(\alpha,z_0;z)$. However, for $f$ depending only on $y$ it is shown (see 9) that $(F_m,G_m) = (F,G)$ for $m = 0, \pm 1, \pm 2, \ldots$. Therefore we have

$$Z^{(n)}_{f,m}(\alpha,z_0;z) = Z^{(n)}_f(\alpha,z_0;z) \quad \text{for} \quad \alpha = 1, i \quad \text{and} \quad m = 0, \pm 1, \pm 2, \ldots$$

so that formula (16) gives us

$$Z^{(1)}_f(1,z_0;z) = \int_{z_0}^z Z^{(0)}_f(\alpha,z_0;\zeta)d(F,G)_\zeta, \quad \alpha = 1, i.$$
We calculate these two integrals using (14) where \( F^* = -if \) and \( G^* = 1/f \). Defining \( \zeta := \xi + i\eta \), we obtain

\[
Z_f^{(1)}(1, z_0; z) = y^2 \text{Re} \int_{z_0}^z \frac{d\zeta}{y_0^2} - \frac{i}{y^2} \text{Re} \int_{z_0}^z \frac{i\eta^4}{y_0^2} d\zeta
\]

\[
= (x - x_0) \left( \frac{y}{y_0} \right)^2 + \frac{i}{5} \frac{y^5 - y_0^5}{(y_0y)^2}
\]

and

\[
Z_f^{(1)}(i, z_0; z) = y^2 \text{Re} \int_{z_0}^z \frac{i\eta^2}{\eta^2} d\zeta + \frac{i}{y^2} \text{Re} \int_{z_0}^z \eta^2 d\zeta
\]

\[
= -\frac{1}{3} \frac{y^3 - y_0^3}{y_0y} + i(x - x_0) \left( \frac{y}{y_0} \right)^2.
\]

In a similar way we construct \( Z_f^{(2)}(\alpha, z_0; z) \) for \( \alpha = 1, i \) where we first need \( Z_f^{(1)}(\alpha, z_0; z) = Z_f^{(1)}(\alpha, z_0; z) \). From formula (16) we obtain

\[
Z_f^{(2)}(1, z_0; z) = 2 \int_{z_0}^z Z_f^{(1)}(1, z_0; z) d_{(F,G)} \zeta
\]

\[
= 2y^2 \text{Re} \int_{z_0}^z \left[ (\zeta - x_0) \left( \frac{\eta}{y_0} \right)^2 + \frac{i}{5} \frac{\eta^5 - y_0^5}{(y_0\eta)^2} \right] \frac{d\zeta}{\eta^2}
\]

\[
- \frac{2i}{y^2} \int_{z_0}^z \left[ (\zeta - x_0) \left( \frac{\eta}{y_0} \right)^2 + \frac{i}{5} \frac{\eta^5 - y_0^5}{(y_0\eta)^2} \right] (i\eta^2) d\zeta
\]

\[
= \frac{1}{15(y_0\eta)^2} \left[ 15(x - x_0)^2 y^4 - 3y^6 + 5y_0^2 y^2 - 2y_0 y \right]
\]

\[
+ 6i(x - x_0)(y^5 - y_0^5)
\]

and

\[
Z_f^{(2)}(i, z_0; z) = 2 \int_{z_0}^z Z_f^{(1)}(i, z_0; z) d_{(F,G)} \zeta
\]

\[
= 2y^2 \text{Re} \int_{z_0}^z \left[ \frac{1}{3} \frac{y_0^3 - \eta^3}{y_0\eta} + i(\zeta - x_0) \left( \frac{y_0}{\eta} \right)^2 \right] \frac{d\zeta}{\eta^2}
\]

\[
- \frac{2i}{y^2} \int_{z_0}^z \left[ \frac{1}{3} \frac{y_0^3 - \eta^3}{y_0\eta} + i(\zeta - x_0) \left( \frac{y_0}{\eta} \right)^2 \right] (i\eta^2) d\zeta
\]

\[
= \frac{1}{15y_0\eta^2} \left[ -10(x - x_0)y(y^3 - y_0^3) \right.
\]

\[
+ i(15y_0^3(x - x_0)^2 + 5y_0^3 y^2 - 2y^5 - 3y_0^5)
\]

We verify easily that \( Z_f^{(n)} \) are indeed solutions of the Vekua equation (30). Moreover, \( \text{Re} Z_f^{(n)} \) are solutions of the Schrödinger equation (3) with \( q = 2/y^2 \) and \( \text{Im} Z_f^{(n)} \) are solutions of the Schrödinger equation (20) with \( q_1 = 6/y^2 \). Finally, we have (see definition 8)

\[
\lim_{z \to z_0} \frac{Z_f^{(n)}(\alpha, z_0; z)}{(z - z_0)^n} = \alpha, \quad \alpha = 1, i.
\]

Now in order to obtain the formal powers \( Z_g^{(n)}(\alpha, z_0; z) = T_{f,g} [Z_f^{(n)}(\alpha, z_0; z)] \) of Vekua equation (31) let us apply the transplant operator to the constructed formal powers \( Z_f^{(n)}(\alpha, z_0; z) \) of (30) for \( n = 1, 2 \).
Since the formal powers $Z_g^{(0)}(\alpha, z_0; z)$ can be easily calculated using definition 9, we are not using the transplant operator in the particular case of formal powers of zero order. Hence, for the generating pair $(F, G) = (g, i/g)$ we find

$$Z_g^{(0)}(1, z_0; z) = k_0 \frac{1 + xy^3}{y} \quad \text{and} \quad Z_g^{(0)}(i, z_0; z) = \frac{i}{k_0} 1 + xy^3,$$

where $k_0 := y_0/(1 + x_0 y_0^3)$.

Now considering application of the transplant operator to $Z_f^{(n)}(\alpha, z_0; z)$ for $n = 1, 2$ we obtain

$$Z_f^{(1)}(1, z_0; z) := T_{f,g} [Z_f^{(1)}(1, z_0; z)] = \Re \{ Z_f^{(1)}(1, z_0; z) + ig^{-1} A \left[ -\frac{3(x - x_0) + i(y + x_0 y^4)}{2y_0^3} \right] \},$$

where

$$A \left[ -\frac{3(x - x_0) + i(y + x_0 y^4)}{2y_0^3} \right] = \int_{x_0}^{x} \frac{3(y - x_0)}{y_0^2} dy + \int_{y_0}^{y} \frac{y + x_0 y^4}{y_0^3} dy + c = \frac{5(y_0^2 - y_0^3)}{10y_0^2} + 2x_0 (y_0^3 - y_0^5) - 15(x - x_0)^2 + c.$$

Therefore, we have

$$Z_f^{(1)}(1, z_0; z) = (x - x_0) \left( \frac{y}{y_0} \right)^2 + i \left[ \frac{5(y_0^2 - y_0^3)}{10y_0^2} + 2x_0 (y_0^3 - y_0^5) - 15(x - x_0)^2 \right] y,$$

where the arbitrary real constant $c$ was chosen equal to zero.

Similar calculations give us:

$$Z_g^{(1)}(i, z_0; z) = -\frac{1}{3} \frac{y_0^3 - y_0^3}{y_0 y} + i \left[ \frac{30(x - x_0) + 15y_0^3(x^2 - x_0^2) - 5y_0^3 y^2 + 2y_0^3 + 3y_0^3}{30y_0(1 + xy_0^3)} \right]$$

$$Z_g^{(2)}(1, z_0; z) = \frac{15(x - x_0)^2 y^4 - 3y_0^6 + 5y_0^3 y^4 - 2y_0^3}{15(y_0 y)^2} + \frac{i y}{105y_0^2(1 + xy_0^3)} \left[ 315x_0(x - x_0) + 105(x - x_0)(y_0^2 - y_0^3) - 105(x_0^3 - x_0^3) + 21(x^2 - x_0^2)(y_0^3 - y_0^3) - 7(y_0 y)^2 y^3 - 9y_0^3 - 3(y_0^3 - y_0^3) \right]$$

$$Z_g^{(2)}(i, z_0; z) = -\frac{1}{3} \frac{y_0^3 - y_0^3}{y_0 y^2} + i \left[ \frac{15(x - x_0)^2 y + 10y_0^3 x^3 y}{15y_0^3(1 + xy_0^3)} \right]$$

$$- 15x_0 y_0^3 x^2 y + 5x_0 y_0^3 y - 2x_0 y_0^3 - 5y_0^3 + 5x_0 y_0^3 y^3 - 10y_0^3 - 3x_0 y_0^3 y + 15y_0^3 y.$$

These formal powers $Z_g^{(n)}$ are solutions of the Vekua equation (31). Moreover, we also have that $\Re Z_g^{(n)}$ are solutions of the Schrödinger equation (3) with $q = 2/y_0^2$ and $\Im Z_g^{(n)}$ are solutions of the Schrödinger equation (21) with potential

$$q_2 = \frac{y_0^5 + 3x_0^2 y^3 - 6x}{(1 + xy_0^3)^2}.$$

Finally, we can verify that $Z_g^{(n)} = T_{f,g} [Z_f^{(n)}]$ satisfy the asymptotics of the formal powers when $z \to z_0$, i.e.

$$\lim_{z \to z_0} \frac{Z_g^{(n)}(\alpha, z_0; z)}{(z - z_0)^n} = \alpha, \quad \alpha = 1, i.$$
5 Construction of a generating sequence

Meanwhile in the example considered in the previous section the generating sequence for the equation (30) is very simple and consists of one generating pair only \((y^2, i/y^2)\), the generating sequence for the related equation (31) is more complicated. However the procedure based on the application of the transplant operator gives us to obtain a generating sequence for a “more complicated” main Vekua equation from a generating sequence corresponding to a “simpler” main Vekua equation. Here the algorithm is following. First, using a generating sequence for equation (2), which is assumed to be known, one can construct the complete system of positive formal powers corresponding to (2). Next, as was explained in the preceding two sections, application of the transplant operator gives a complete system of positive formal powers for equation (19) where \(g\) is related to \(f\) via the Schrödinger equation (3). Finally, to obtain a generating sequence for (19) one can use property 3 of formal powers. We illustrate this by the following scheme.

\[
\begin{align*}
(Z^{(3)}(1,z_0;z), Z^{(3)}(i,z_0;z)) &= \frac{d(g,i/g)}{dz} \cdot Z^{(2)}(1,z_0;z), Z^{(2)}(i,z_0;z)) \\
(Z^{(2)}(1,z_0;z), Z^{(2)}(i,z_0;z)) &= \frac{d(F_2,G_1)}{dz} \cdot Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z)) \\
(Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z)) &= \frac{d(F_1,G_1)}{dz} \cdot Z^{(0)}(1,z_0;z), Z^{(0)}(i,z_0;z)) \\
(Z^{(0)}(1,z_0;z), Z^{(0)}(i,z_0;z)) &= \frac{d(F_1,G_1)}{dz} \cdot Z^{(0)}(1,z_0;z), Z^{(0)}(i,z_0;z)) \\
&\vdots
\end{align*}
\]

In order to obtain the successor \((F_1,G_1)\) one can apply the differential operator \(\frac{d(g,i/g)}{dz}\) to the pair of formal powers \((Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z))\) obtaining \((Z^{(0)}(1,z_0;z), Z^{(0)}(i,z_0;z))\) which can be chosen as \((F_1,G_1)\). Then this newly obtained generating pair serves for obtaining \((F_2,G_2)\) (differentiating \((Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z))\) in the sense of Bers with respect to \((F_1,G_1)\) and positive formal powers of subindex 2, and in this way the whole generating sequence corresponding to (19) can be constructed.

As an illustration of the algorithm, we consider the example from the preceding section. The generating pair \((y^2, i/y^2)\) was used to obtain formal powers of order \(n = 0, 1, 2\) for \(Z_f^{(n)}(\alpha, z_0; z)\) of the Vekua equation (30). Then, using the transplant operator, the corresponding formal powers \(Z_g^{(n)}(\alpha, z_0; z)\) of the Vekua equation (31) were obtained. Looking now for a generating sequence corresponding to the Vekua equation (31) we already have \((F,G) = (g,i/g)\), where we recall that \(g = \frac{1 + xy^3}{y}\). To obtain other elements of the generating sequence for the Vekua equation (31), we follow the algorithm presented above. We have

\[
(F_1, G_1) = \frac{d(g,i/g)}{dz} \left( Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z) \right)
= \frac{d}{dz} \left( Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z) \right) - A(g,i/g) \left( Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z) \right) - B(g,i/g) \left( Z^{(1)}(1,z_0;z), Z^{(1)}(i,z_0;z) \right)
\]

where we used equation (13) for the \((g,i/g)\)-derivative in the sense of Bers. As \(A(g,i/g) = 0\) and

\[
B(g,i/g) = \frac{1}{2} \frac{y^4 - 2ixy^3 + i}{y(1 + xy^3)}
\]

we obtain

\[
F_1 = \frac{y}{y^2(1 + xy^3)} \left[ y(1 + x_0 y^3) - 3i(x - x_0) \right] \quad \text{and} \quad G_1 = \frac{y}{3y^2(1 + xy^3)} \left[ 3y (y^3 - y_0^3) + 3i(1 + xy_0^3) \right].
\]
One can verify that \((F_1, G_1)\) satisfies the required property for a generating pair on the considered domain \(\Omega = \{(x, y) \mid x, y > 0\}:

\[
\text{Im}(F_1 G_1) = \frac{1}{2} \left( \frac{y}{y_0} \right)^2 \frac{g(z_0)}{g(z)} > 0 \quad \text{in } \Omega.
\]

Looking now for \((F_2, G_2)\) we have first to calculate \(Z_{g,1}^{(1)}(\alpha, z_0; z)\):

\[
\begin{aligned}
Z_{g,1}^{(1)}(1, z_0; z) &= Z_{g,1}^{(2)}(1, z_0; z) = \frac{d(g, i/g)}{dz} 
Z_g^{(2)}(1, z_0; z) = \frac{d}{dz} 
Z_g^{(2)}(1, z_0; z) - B_{(g, i/g)}(Z_g^{(2)}(1, z_0; z), Z_g^{(2)}(i, z_0; z)).
\end{aligned}
\]

We obtain

\[
Z_{g,1}^{(1)}(1, z_0; z) = \frac{y}{15y_0^2(1 + xy^3)} \left[ y \left( 15y^3(x^2 - x_0^2) + 30(x - x_0) + 3y^5 - 5y_0^2y^3 + 2y_0^4 \right) 
+ i \left( 15(y_0^2 - y_0^3) + 6x(y^3 - y_0^3) - 45(x - x_0)^2 \right) \right]
\]
and

\[
Z_{i,g}^{(1)}(1, z_0; z) = \frac{2}{3y_0y(1 + xy^3)} \left[ -(y^3 - y_0^3)(1 + x_0y^3) + 3i(x - x_0)y^2(1 + xy^3) \right].
\]

The generating pair \((F_2, G_2)\) is then given by

\[
(F_2, G_2) = \frac{d(F_1, G_1)}{dz} \left( Z_{g,1}^{(1)}(1, z_0; z), Z_{g,1}^{(1)}(i, z_0; z) \right)
= \frac{d}{dz} \left( Z_{g,1}^{(1)}(1, z_0; z), Z_{g,1}^{(1)}(i, z_0; z) \right) - A_{(F_1, G_1)} \left( Z_{g,1}^{(1)}(1, z_0; z), Z_{g,1}^{(1)}(i, z_0; z) \right) - B_{(F_1, G_1)} \left( Z_{g,1}^{(1)}(1, z_0; z), Z_{g,1}^{(1)}(i, z_0; z) \right)
\]
where

\[
A_{(F_1, G_1)} = - \frac{y^4 + 3i}{2y(1 + xy^3)} \quad \text{and} \quad B_{(F_1, G_1)} = - \frac{i}{y}.
\]

Combining these results we find

\[
F_2 = 2 \left( \frac{y}{y_0} \right)^2 \quad \text{and} \quad G_2 = 2i \left( \frac{y_0}{y} \right)^2,
\]

which obviously satisfies the required property that \(\text{Im}(F_2 G_2) > 0\). Notice that in this special case the obtained pair \((F_2, G_2)\) is equivalent (in the sense introduced in [1], see also [9]) to the generating pair \((y^2, i/y^2)\) which was used for starting the proposed procedure. As it depends on the variable \(y\) only the succeeding generating pairs can be chosen again equal to it. Thus, in this example we constructed a complete generating sequence into which the generating pair \((F, G) = (g, i/g)\) is embedded. Namely, \((F_0, G_0) = \left( \frac{1 + x_0y^3}{y}, \frac{iy}{1 + xy^3} \right), (F_1, G_1) = \left( \frac{y}{y_0^3(1 + xy^3)} \left[ y(1 + x_0y^3) - 3i(x - x_0) \right], \frac{y}{y_0^3(1 + xy^3)} \left[ y(y^3 - y_0^3) + 3i(1 + xy_0^3) \right] \right), (F_n, G_n) = (y^2, i/y^2)\) for \(n = 2, 3, \ldots\).

### 6 Construction of Cauchy kernels

In Section 5 we showed that the transplant operator \(T_{f,g}\) transforms a Cauchy kernel \(Z_{f}^{(1)}(\alpha, z_0; z)\) corresponding to equation (2) into a Cauchy kernel \(Z_{g}^{(1)}(\alpha, z_0; z)\) corresponding to equation (19) when \(f\) and \(g\) are solutions of a same Schrödinger equation (3). Here we give an example of the application of this procedure.

Let us consider the following Vekua equation

\[
w_z = \frac{1}{2} \left( \frac{1}{x} + \frac{i}{y} \right) \overline{w}
\]  
(32)
in the domain $\Omega = \{(x, y) \mid x > 0 \text{ and } y > 0\}$. Note that the coefficient in the equation admits a representation in the form of a logarithmic derivative of a real-valued function:

$$\frac{1}{2} \left( \frac{1}{x} + \frac{i}{y} \right) = \frac{g_x}{g}$$

where moreover, $g = xy$ is a harmonic function. The simplest nontrivial harmonic function is, of course, $f \equiv 1$. The corresponding Vekua equation \[7\] is just the Cauchy-Riemann system for which the Cauchy kernel is well known. In order to obtain $Z_g^{(-1)}(\alpha, z_0; z)$ for any $\alpha \in \mathbb{C}$ we need to calculate $Z_g^{(-1)}(1, z_0; z)$ and $Z_g^{(-1)}(i, z_0; z)$ applying the transplant operator $T_{1,xy}$ to $1/(z - z_0)$ and $i/(z - z_0)$, respectively. We will show the result for $Z_g^{(-1)}(1, z_0; z)$ (an expression for $Z_g^{(-1)}(i, z_0; z)$ can be obtained analogously) calculated with the aid of Maple.

Thus we consider $W = 1/(z - z_0)$ for $(x_0, y_0) \in \Omega$ and find

$$Z_g^{(-1)}(1, z_0; z) = T_{1,xy}[W] = \frac{x - x_0}{|z - z_0|^2} + \frac{i}{xy} \left[ i(xy)^2 \partial_z \left( \frac{x - x_0}{xy|z - z_0|^2} \right) \right],$$

where

$$i(xy)^2 \partial_z \left( \frac{x - x_0}{xy|z - z_0|^2} \right) = \frac{1}{2|z - z_0|^2} \left\{ \left( y_0^2 x^2 - x_0^3 x + 3x_0^2 y^2 - 3x_0 x^3 + 3x_0^2 x^2 - 4x_0 y_0 xy \right) \right.$$  

$$\left. - 4y_0 x^2 y - 3x_0 xy^2 - x_0 y_0^2 x + x^4 \right) - i \left( -x_0 y_0^2 y + 4x_0^2 xy \right. 
$$

$$\left. + 2x_0 y_0^2 y - 5x_0 x^2 y + 2x^3 y - x_0^3 y - x_0 y^3 \right).$$

Applying the $\mathcal{A}$ operator (integrating from $(0,0)$ to $(x,y)$) we obtain

$$\mathcal{A} \left[ i(xy)^2 \partial_z \left( \frac{x - x_0}{xy|z - z_0|^2} \right) \right] = \frac{1}{2|z - z_0|^2} \left\{ \left( 2x_0^2 y_0 - 4y_0^2 y - 4x_0 y_0 x + 2y_0 y^2 + 2y_0^2 + 2y_0 x^2 \right) \right.$$  

$$\left. \arctan \left( \frac{y_0}{x_0} \right) \right.$$  

$$\left. + \left( -2x_0^2 x + x_0 x^2 + x_0 ^3 + x_0 y^2 + x_0 y_0^2 - 4x_0 y_0 y \right) \ln \left| \frac{x - z_0}{x} \right| \right.$$  

$$\left. + \left( -2x_0^2 y_0 + 4y_0^2 y + 4x_0 y_0 x - 2y_0 y^2 - 2y_0^3 - 2y_0 x^2 \right) \arctan \left( \frac{y - y_0}{x - x_0} \right) \right.$$  

$$\left. + \left( -2y_0 x + 2x_0 x + x_0 y^2 + x_0 x^2 - 4x_0 x^2 \right) \right\} + c.$$  

Choosing $c = 0$ we find

$$Z_g^{(-1)}(1, z_0; z) = T_{1,xy}[W]$$

$$= \frac{x - x_0}{|z - z_0|^2} + \frac{i}{xy|z - z_0|^2} \left\{ \left( 2x_0^2 y_0 - 4y_0^2 y - 4x_0 y_0 x + 2y_0 y^2 + 2y_0^2 + 2y_0 x^2 \right) \right.$$  

$$\left. \arctan \left( \frac{y_0}{x_0} \right) \right.$$  

$$\left. + \left( -2x_0^2 x + x_0 x^2 + x_0 ^3 + x_0 y^2 + x_0 y_0^2 - 4x_0 y_0 y \right) \ln \left| \frac{x - z_0}{x} \right| \right.$$  

$$\left. + \left( -2x_0^2 y_0 + 4y_0^2 y + 4x_0 y_0 x - 2y_0 y^2 - 2y_0^3 - 2y_0 x^2 \right) \arctan \left( \frac{y - y_0}{x - x_0} \right) \right.$$  

$$\left. + \left( -2y_0 x + 2x_0 x + x_0 y^2 + x_0 x^2 - 4x_0 x^2 \right) \right\}. \quad (33)$$

Property \[18\] for $Z_g^{(-1)}(1, z_0; z)$ is illustrated in Figure 1 by the graph of the function $H(x, y) = \frac{Z_g^{(-1)}(1, z_0; z)}{(z - z_0)^{-1}}$ for $(x_0, y_0) = (1, 1) \in \Omega$.  

14
7 Hyperbolic pseudoanalytic function theory

In [10] and [4] “hyperbolic pseudoanalytic function theory” was studied where hyperbolic numbers \( \mathbb{D} \) (also called duplex numbers) [10] defined by
\[
\mathbb{D} := \{ x + tj : j^2 = 1, \ x, t \in \mathbb{R} \} \cong \mathbb{Cl}_2(0,1)
\]
are considered instead of (elliptic) complex numbers. Here we show that the concept of the transplant operator can be introduced in this context as well and it allows one to solve hyperbolic main Vekua equations related to the Klein-Gordon equation.

As in the case of complex numbers, we denote the real and imaginary parts of \( z = x + tj \in \mathbb{D} \) by \( x = \text{Re} \ z \) and \( t = \text{Im} \ z \). Now, by defining the conjugate as \( \bar{z} = Cz := x - tj \) and the hyperbolic modulus as \( |z|^2 := z\bar{z} = x^2 - t^2 \), we can verify that the inverse of \( z \) whenever exists is given by
\[
z^{-1} = \frac{z}{|z|^2}.
\]
The set \( \mathcal{NC} \) of zero divisors for hyperbolic numbers \( \mathbb{D} \), called the null-cone, is given by \( \mathcal{NC} = \{ x + tj : |x| = |t| \} \).

**Definition 15** Let \( U \) be an open set in \( \mathbb{D} \) and \( z_0 \in U \). Then \( w : U \subseteq \mathbb{D} \longrightarrow \mathbb{D} \) is said to be \( \mathbb{D} \)-differentiable at \( z_0 \) with derivative equal to \( w'(z_0) \in \mathbb{D} \) if
\[
\lim_{z \to z_0} \frac{w(z) - w(z_0)}{z - z_0} = w'(z_0).
\]
Here \( z \) tends to \( z_0 \) following the invertible trajectories. We also say that the function \( w \) is \( \mathbb{D} \)-holomorphic on an open set \( U \) if and only if \( w \) is \( \mathbb{D} \)-differentiable at each point of \( U \).

**Theorem 16** Let \( U \) be an open set and \( w : U \subseteq \mathbb{D} \longrightarrow \mathbb{D} \) such that \( w \in C^1(U) \). Let also \( w(x + tj) = w_1(x,t) + w_2(x,t)j \). Then \( w \) is \( \mathbb{D} \)-holomorphic on \( U \) if and only if
\[
\frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial t} \quad \text{and} \quad \frac{\partial w_2}{\partial x} = \frac{\partial w_1}{\partial t}.
\]
Moreover \( w' = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x} j \) and \( w'(z) \) is invertible if and only if \( \det \mathcal{J}_w(z) \neq 0 \), where \( \mathcal{J}_w(z) \) is the Jacobian matrix of \( w \) at \( z \).

System of equations (34) is called “hyperbolic Cauchy-Riemann” equations. It was considered in [3, 14, 11].

For \( z = x + tj \) where \( x, t \) are real variables, we define the operators \( \partial_x \) and \( \partial_t \) in the hyperbolic function theory as
\[
\partial_z = \frac{1}{2} \left( \partial_x + j\partial_t \right) \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2} \left( \partial_x - j\partial_t \right),
\]
such that \( w_\partial(z) = 0 \) if and only if hyperbolic Cauchy-Riemann equations (34) are satisfied.

Let \( \Omega \) be a domain in \( \mathbb{R}^2 \) without zero divisors. We consider now the hyperbolic Vekua equation
\[
W_\varphi = \frac{f_\varphi}{f} \quad \text{in} \ \Omega,
\]
where \( f \) is a positive function of \( x \) and \( t \), twice continuously differentiable, which will be supposed to be a particular solution of the following \((1 + 1)\)-dimensional Klein-Gordon equation
\[
(\Box - q)\varphi = 0 \quad \text{in} \ \Omega.
\]
Here \( \Box := \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \), the potential \( q \) is a real valued function and \( \varphi \) is a twice continuously differentiable real valued function of \( x \) and \( t \).
Theorem 17 [10] Let \( f \) be a positive particular solution of (36) in \( \Omega \). Then for any real valued function \( \varphi \in C^{2}(\Omega) \) the following equalities hold

\[
\frac{1}{4}(\Box - q)\varphi = (\partial_{\overline{z}} + f_{\overline{z}} C) (\partial_{z} - f_{z} C) \varphi = (\partial_{\overline{z}} + f_{\overline{z}} C) (\partial_{z} - f_{z} C) \varphi.
\]

In a similar way as in the elliptic case, an immediate consequence of this theorem is the fact that if \( W \) is a solution of (35) then its real part \( W_{1} \) is a solution of (36), meanwhile its imaginary part \( W_{2} \) is a solution of the following Klein-Gordon equation

\[
(\Box - q_{1})W_{2} = 0 \quad \text{in} \ \Omega,
\]

where \( q_{1} = -q + 8 \frac{|f_{z}|^{2}}{f^{2}} \) (see [10], [9]).

Note that in the hyperbolic case the family of real valued functions \( \phi \) such that \( \partial_{z} \phi = \Phi \), and \( \Phi = \Phi_{1} + j \Phi_{2} \), can be constructed as

\[
\mathcal{A}_{h}[\Phi](x, t) = 2 \left( \int_{x_{0}}^{x} \Phi_{1}(\eta, t) d\eta - \int_{t_{0}}^{t} \Phi_{2}(x_{0}, \xi) d\xi \right) + c,
\]

when \( \Phi \) satisfies the compatibility condition

\[
\partial_{\overline{z}} \Phi_{1} + \partial_{z} \Phi_{2} = 0.
\]

Theorem 18 [10] Given a solution \( W_{1} \) of the Klein-Gordon equation (37), the corresponding \( W_{2} \) such that \( W = W_{1} + j W_{2} \) is a solution of the hyperbolic Vekua equation (38) can be constructed according to the formula

\[
W_{2} = -f^{-1} \mathcal{A}_{h} [j f^{2} \partial_{\overline{z}}(f^{-1} W_{1})].
\]

Vice versa, given a solution \( W_{2} \) of the Klein-Gordon equation (37), the corresponding \( W_{1} \) such that \( W = W_{1} + j W_{2} \) is a solution of the hyperbolic Vekua equation (38) can be constructed according to the formula

\[
W_{1} = -f \mathcal{A}_{h} [j f^{-2} \partial_{\overline{z}}(f W_{2})].
\]

Now, let \( g \) be another positive solution of (36) associated with the following hyperbolic Vekua equation

\[
w_{z} = \frac{g_{z}}{g} \quad \text{in} \ \Omega.
\]

For \( W \) and \( w \) solutions of hyperbolic Vekua equations (35) and (38), respectively, we have that both \( \text{Re} W \) and \( \text{Re} w \) satisfy (36), meanwhile \( \text{Im} W \) and \( \text{Im} w \) satisfy two different Klein-Gordon equations

\[
(\Box - q_{1}) \text{Im} W = 0 \quad \text{in} \ \Omega
\]

\[
(\Box - q_{2}) \text{Im} w = 0 \quad \text{in} \ \Omega,
\]

respectively, where \( q_{1} = -q + 8 \frac{|f_{z}|^{2}}{f^{2}} \) and \( q_{2} = -q + 8 \frac{|g_{z}|^{2}}{g^{2}} \).

In a similar way as in the elliptic case, we introduce a hyperbolic transplant operator \( \tilde{T}_{f,g} \) which transforms solutions of (35) into solutions of (38) in the following way:

\[
\tilde{T}_{f,g}[W] = P^{+} W - j g^{-1} \mathcal{A}_{h} [j g^{2} \partial_{z} (g^{-1} P^{+} W)]
\]

where \( P^{+} = \frac{1}{2} (I + C) \).

Again by assigning a fixed value in a certain point of \( \Omega \) to the result of application of \( \mathcal{A}_{h} \), we obtain an invertible one-to-one map establishing a relation between solutions of (35) and (38). The inverse of \( \tilde{T}_{f,g} \) is given by the expression

\[
\tilde{T}_{f,g}^{-1}[w] = \tilde{T}_{g,f}[w] = P^{+} w - j f^{-1} \mathcal{A}_{h} [j f^{2} \partial_{z} (f^{-1} P^{+} w)].
\]

The nonnegative formal powers \( Z_{a,0}^{m}(a, z_{0}, z) \), where \( a, z_{0} \) and \( z \) hyperbolic numbers, are defined in hyperbolic pseudoanalytic function theory as in definition [3] for usual (elliptic) pseudoanalytic theory. These formal powers have same properties replacing \( i \) by \( j \) everywhere [10]. Therefore, using hyperbolic transplant operator as in the elliptic case, nonnegative formal powers and generating sequence of the hyperbolic Vekua equation (38) can be obtained from nonnegative formal powers and generating sequence of (35) in a domain \( \Omega \).
References

[1] L. Bers, Theory of pseudo-analytic functions, New York University, 1952.
[2] L. Bers, Formal powers and power series, Communications on Pure and Applied Mathematics 9 (1956) 693-711.
[3] G.C. Wen, Linear and Quasilinear Complex Equations of Hyperbolic and Mixed Type, Taylor & Francis London, 2003.
[4] V.G. Kravchenko, V. V. Kravchenko, S. Tremblay, Zakharov-Shabat system and hyperbolic pseudoanalytic function theory. Mathematical Methods in the Applied Sciences, Published Online, DOI: 10.1002/mma.1206.
[5] V.V. Kravchenko, On a relation of pseudoanalytic function theory to the two-dimensional stationary Schrödinger equation and Taylor series in formal powers for its solutions, Journal of Physics A: Mathematical and General 38 No. 18 (2005) 3947-3964.
[6] V.V. Kravchenko, On a factorization of second order elliptic operators and applications, Journal of Physics A: Mathematical and General 39 No. 40 (2006) 12407-12425.
[7] V.V. Kravchenko, Recent developments in applied pseudomnalytic function theory. In “Some topics on value distribution and differentiability in complex and p-adic analysis”, eds. A. Escassut, W. Tutschke and C. C. Yang, Science Press 293-328, 2008.
[8] V.V. Kravchenko, On a transplant operator and explicit construction of Cauchy-type integral representations for $p$-analytic functions, Journal of Mathematical Analysis and Applications v. 339 issue 2 (2008) 1103-1111.
[9] V.V. Kravchenko, Applied pseudoanalytic function theory, Basel: Birkhäuser, Series: Frontiers in Mathematics, 2009.
[10] V.V. Kravchenko, D. Rochon and S. Tremblay, On the Klein–Gordon equation and hyperbolic pseudoanalytic function theory, Journal of Physics A: Mathematical and General 41 No. 6 (2008) 65205-65222.
[11] M.A. Lavrentyev and B.V. Shabat, Hydrodynamics problems and their mathematical models, Nauka Moscow (in Russian), 1977.
[12] V. Matveev and M. Salle, Darboux transformations and solitons, N.Y. Springer, 1991.
[13] K. Menke, Zur Approximation pseudoanalytischer Funktionen durch Pseudopolynome, Manuscripta Math. 11 (1974) 111-125.
[14] A.F. Motter and M.A.F. Rosa, Hyperbolic calculus, Adv. Appl. Clifford Algebras 8 No 1 (1998) 109-128.
[15] G.N. Polozhy, Generalization of the theory of analytic functions of complex variables: $p$-analytic and $(p, q)$-analytic functions and some applications, Kiev University Publishers (in Russian), 1965.
[16] G. Sobczyk, The Hyperbolic Number Plane, The College Mathematics Journal 26 No 4 (1995) 268-280.
[17] I.N. Vekua, Generalized analytic functions, Pergamon Press Oxford, 1962.
Figure 1: Graph of the function $H(x, y) = |(z - z_0) Z_g^{(-1)}(1, z_0; z)|$ for the function $Z_g^{(-1)}(1, z_0; z)$ given by (33) and $(x_0, y_0) = (1, 5)$. 