A theory of Kondo lattices is applied to studying possible magnetic and charge structures of itinerant-electron antiferromagnets. Even helical spin structures can be stabilized when the nesting of the Fermi surface is not sharp and the superexchange interaction, which arises from the virtual exchange of pair excitations across the Mott-Hubbard gap, is mainly responsible for magnetic instability. Sinusoidal spin structures or spin density waves (SDW) are only stabilized when the nesting of the Fermi surface is sharp enough and a novel exchange interaction arising from that of pair excitations of quasi-particles is mainly responsible for magnetic instability. In particular, multiple SDW are stabilized when their incommensurate ordering wave-numbers $\pm Q$ are multiple; magnetizations of different $\pm Q$ components are orthogonal to each other in double and triple SDW when magnetic anisotropy is weak enough. Unless $\pm 2Q$ are commensurate, charge density waves (CDW) with $\pm 2Q$ coexist with SDW with $\pm Q$. Because the quenching of magnetic moments by the Kondo effect depends on local numbers of electrons, the phase of CDW or electron densities is such that magnetic moments are large where the quenching is weak. It is proposed that the so-called stripe order in cuprate-oxide high-temperature superconductors must be the coexisting state of double incommensurate SDW and CDW.

I. INTRODUCTION

It is a long standing and important issue to explain two types of magnetism, local-moment magnetism and itinerant-electron magnetism, in a unified theoretical framework. Some of their physical properties are similar to each other and others are different from each other. Even if they are phenomenologically similar to each other, their microscopic explanations can be totally different from each other. For example, the spin susceptibility obeys the Curie-Weiss law in both types of magnets. The Curie-Weiss law in insulators is due to the temperature dependence of local thermal spin fluctuations. This mechanism is of leading order in $1/d$, with $d$ spatial dimensionality. On the other hand, the Curie-Weiss law in metals is a controversial issue. Two mechanisms were proposed: the temperature dependence of the mode-mode coupling between intersite spin fluctuations and that of Weiss’ magnetic mean fields. The former is of higher order in $1/d$, while the latter is of leading order in $1/d$. It is interesting which is responsible for the Curie-Weiss law in actual $d = 3$ dimensions. There is a sharp peak at the chemical potential in the density of states in typical ferromagnetic metals, and there is sharp nesting of the Fermi surface in typical antiferromagnetic metals. Recent theories showed that in such itinerant-electron magnets the mode-mode coupling plays a totally negative role in the Curie-Weiss law. On the other hand, a novel exchange interaction arising from the virtual exchange of pair excitations of quasi-particles can have a large temperature dependence consistent with the Curie-Weiss law. Weiss’ mean fields are given by magnetic polarizations multiplied by the exchange interaction.

Magnetic structures are different between the two types of antiferromagnets. Helical structures are stabilized in typical local-moment magnets and sinusoidal structures or spin density waves (SDW) in typical itinerant-electron magnets. When ordering wave-numbers are incommensurate, there are two or more than two equivalent wave-numbers depending on lattice symmetry. An interesting issue is which are stabilized, single or multiple SDW, in what conditions.

Itinerant-electron magnets lie in the vicinity of the Mott-Hubbard metal-insulator transition. A combined theory of Hubbard’s and Gutzwiller implies that in metallic phases the density of states is of a three-peak structure, Gutzwiller’s quasi-particle band at the chemical potential between the lower and upper Hubbard bands lying far from the chemical potential. It is unquestionable that the Mott-Hubbard splitting exists even in metallic phases in the vicinity of the transition. Such metals must show a crossover between the two types of magnetism as a function of temperatures. Denote the energy scale of local quantum spin fluctuations by $k_B T_K$. They must behave as local-moment magnets at $T \gg T_K$ because local thermal spin fluctuations are dominant and they behave as itinerant-electron magnets at $T \ll T_K$ because magnetic moments are quenched by local quantum spin fluctuations. When physical phenomena relevant to the Mott-Hubbard transition are examined, first of all, local spin fluctuations should be accurately taken into account.

Local spin fluctuations are rigorously considered in any single-site approximation (SSA) that includes all the single-site terms. Such an SSA is reduced to determining and solving self-consistently the Anderson model, which is one of the effective Hamiltonians for the Kondo problem. Because the Kondo problem was solved, many useful results are available in clarifying single-site properties in lattice systems. The three-peak structure discussed above can be easily understood by the SSA theory or the mapping to the Anderson model. The quenching
of magnetic moments by local quantum spin fluctuations in lattice systems is mapped to that in the Kondo problem, so that $T_K$ discussed above is nothing but the Kondo temperature. A perturbative treatment of intersite terms starting from an unperturbed state constructed in the SSA is nothing but a theory of Kondo lattices. Because the SSA is rigorous for Landau’s normal Fermi-liquid states in infinite dimensions ($d \to +\infty$) it can also be formulated as a 1/$d$ expansion theory.

One of the purposes of this paper is to study which are stabilized in itinerant-electron magnets, single or multi-site terms: $\Sigma_{\sigma}$ and $\Sigma_{\sigma \alpha \beta}$, with $\sigma^{\alpha \beta}$ the Pauli matrix. When only the first and second terms are considered, the model (2.1) is reduced to the Hubbard model, in which the well-known superexchange interaction is involved. Actual systems must be described by a multi-band model. A superexchange interaction arising from multi-band effects is phenomenologically included by the third term. The fourth term is the long range Coulomb interaction: $V^{(c)}_{ij} = e^2/|R_i - R_j|$, with $R_i$ lattice sites. No spin-orbit interaction or no magnetic anisotropy is included in this paper.

Assume Landau’s normal Fermi-liquid state. The single-particle selfenergy is divided into single-site and multi-site terms: $\Sigma_{\sigma}(i\varepsilon_n, \mathbf{k}) = \Sigma_0(i\varepsilon_n) + \Delta \Sigma_{\sigma}(i\varepsilon_n, \mathbf{k})$. The multi-site term, $\Delta \Sigma_{\sigma}(i\varepsilon_n, \mathbf{k})$, is of higher order in 1/$d$, and is ignored here. The single-site term, $\Sigma_0(i\varepsilon_n)$, is of leading order in 1/$d$. In the presence of infinitesimally small external fields, $\mathcal{H}_{ext} = -\sum_{i\sigma}[\Delta \mu + \frac{1}{2} \sigma g_i H_i^\sigma] \hat{n}_{i\sigma}$, it is expanded in such a way that $\Sigma_0(i\varepsilon_n) = \Sigma_0 + (1 - \phi_s)\varepsilon_n + (1 - \phi_c)\frac{1}{2} \sigma g_i H_i^\sigma + (1 - \phi_c)\Delta \mu + \cdots$ for $|\varepsilon_n| \ll k_B T_K$ and $T \ll T_K$, with $T_K$ the Kondo temperature discussed in Introduction. Because $\Sigma_0(i\varepsilon_n)$ is equal to that of the mapped Anderson model, it follows according to Yamada and Yosida’s lemma that $\phi_s = (\phi_s + \phi_c)/2$; $\phi_s$ and $\phi_c$ are approximately given by Eqs. (A11) and (A12) in Appendix. The electron density is given by

$$n_d = \frac{1}{N} \sum_i (\hat{n}_{i\sigma}). \quad (2.2)$$

with $N$ the total number of unit cells. When the system is in the vicinity of the Mott-Hubbard transition, $n_d$ is close to unity so that $\phi_s \gg 1$ and $\phi_c \ll 1$. The single-particle Green function is given by

$$G_{\sigma}(i\varepsilon_n, \mathbf{k}) = \frac{1}{\phi_{\sigma} - i\varepsilon_n - \xi(\mathbf{k})}. \quad (2.3)$$

for $|\varepsilon_n| \ll k_B T_K$ and $T \ll T_K$, with

$$\xi(\mathbf{k}) = \frac{1}{\phi_{\sigma}} \left[ \Sigma_0 - \sum_{i} t_{ij} e^{i\mathbf{k}(R_i - R_j)} - \mu \right]. \quad (2.4)$$

The quasi-particle density of states is given by $\rho^*(\epsilon) = (1/N) \sum \delta(\epsilon - \xi(\mathbf{k}))$. When contributions from local spin fluctuations are only considered, the specific heat at $T \ll T_K$ is given by $C = \gamma T + \cdots$, with $\gamma = (2/3)\pi^2 k_B^2 \rho^*(0)$.

When irreducible two-point polarization functions in the wave-number representation are denoted by $\pi_s(i\omega_l, \mathbf{q})$ and $\pi_c(i\omega_l, \mathbf{q})$, spin and charge channels are given by $\pi_s(i\omega_l, \mathbf{q}) = \pi_{\sigma\sigma}(i\omega_l, \mathbf{q}) - \pi_{\sigma\sigma}(i\omega_l, \mathbf{q})$ and $\pi_c(i\omega_l, \mathbf{q}) = \pi_{\sigma\sigma}(i\omega_l, \mathbf{q}) + \pi_{\sigma\sigma}(i\omega_l, \mathbf{q})$. Spin and charge susceptibilities of the model (2.1) are given by

$$\chi_s(i\omega_l, \mathbf{q}) = \frac{2\pi_s(i\omega_l, \mathbf{q})}{1 - [U + 2J^{(p)}(\mathbf{q})] \pi_s(i\omega_l, \mathbf{q})} \quad (2.5)$$

and

$$\chi_c(i\omega_l, \mathbf{q}) = \frac{2\pi_c(i\omega_l, \mathbf{q})}{1 + [U + 2V^{(c)}(\mathbf{q})] \pi_c(i\omega_l, \mathbf{q})} \quad (2.6)$$

with $J^{(p)}(\mathbf{q}) = \sum_{j(\neq i)} J^{(p)}_{ij} e^{-i\mathbf{q}(\mathbf{R}_i - \mathbf{R}_j)}$ and $V^{(c)}(\mathbf{q}) = \sum_{j(\neq i)} V^{(c)}_{ij} e^{-i\mathbf{q}(\mathbf{R}_i - \mathbf{R}_j)}$. Irreducible polarization functions are also divided into single-site ones, $\pi_s(i\omega_l, \mathbf{q})$ and $\pi_c(i\omega_l, \mathbf{q})$, and multi-site ones, $\Delta \pi_s(i\omega_l, \mathbf{q})$ and $\Delta \pi_c(i\omega_l, \mathbf{q})$: $\pi_s(i\omega_l, \mathbf{q}) = \bar{\pi}_s(i\omega_l, \mathbf{q}) + \Delta \pi_s(i\omega_l, \mathbf{q})$ and $\pi_c(i\omega_l, \mathbf{q}) = \bar{\pi}_c(i\omega_l, \mathbf{q}) + \Delta \pi_c(i\omega_l, \mathbf{q})$. Because the single-site terms are equal to those of the mapped Anderson model, its spin and charge susceptibilities are given by

$$\bar{\chi}_s(i\omega_l) = \frac{2\bar{\pi}_s(i\omega_l)}{1 - \bar{\pi}_s(i\omega_l)} \quad (2.7)$$

and

$$\bar{\chi}_c(i\omega_l) = \frac{2\bar{\pi}_c(i\omega_l)}{1 + \bar{\pi}_c(i\omega_l)} \quad (2.8)$$

The Kondo temperature is defined by $k_B T_K = 1/|\bar{\pi}_s(0)|$. When the Fermi-liquid relation is made use of, it follows that $k_B T_K = (\bar{\phi}_s/\bar{\phi}_c)/\rho^*(0)$ so that $k_B T_K$ is as large as the bandwidth of quasi-particles. In this paper, our study is restricted to the strongly correlated regime defined by

$$k_B T_K/U \ll 1. \quad (2.9)$$

The Kondo temperature should be calculated by determining and solving selfconsistently the mapped Anderson
model. However, it is treated as a phenomenological parameter in this paper.

Eqs. (2.7) and (2.8) are also written in such a way that
\[
\chi_s(i\omega_l, \mathbf{q}) = \frac{\tilde{\chi}_s(i\omega_l)}{1 - \frac{i}{4} I_s(i\omega_l, \mathbf{q}) \tilde{\chi}_s(i\omega_l)} \tag{2.10}
\]
and
\[
\chi_c(i\omega_l, \mathbf{q}) = \frac{\tilde{\chi}_c(i\omega_l)}{1 + V(i\omega_l, \mathbf{q}) \tilde{\chi}_c(i\omega_l)} \tag{2.11}
\]
with
\[
I_s(i\omega_l, \mathbf{q}) = J^{(p)}(\mathbf{q}) + \frac{2\Delta \pi_s(i\omega_l, \mathbf{q})}{\pi_s(i\omega_l) \pi_c(i\omega_l, \mathbf{q})} \tag{2.12}
\]
and
\[
V(i\omega_l, \mathbf{q}) = V^{(c)}(\mathbf{q}) - \frac{\Delta \pi_c(i\omega_l, \mathbf{q})}{2\pi_c(i\omega_l) \pi_c(i\omega_l, \mathbf{q})} \tag{2.13}
\]
It follows from Eqs. (2.3) and (2.7) that
\[
\pi_s(0, \mathbf{q}) = \frac{1}{U} \left[ 1 + O(k_B T_K / U) \right] \tag{2.14}
\]
and
\[
\tilde{\pi}_s(0) = \frac{1}{U} \left[ 1 + O(k_B T_K / U) \right]. \tag{2.15}
\]
Here, \( U \gg |J^{(p)}(\mathbf{q})| \) is assumed. To leading order in \( k_B T_K / U \), Eq. (2.12) becomes
\[
I_s(i\omega_l, \mathbf{q}) = J^{(p)}(\mathbf{q}) + 2U^2 \Delta \pi_s(i\omega_l, \mathbf{q}). \tag{2.16}
\]
It follows from Eqs. (2.9) and (2.8) that
\[
[U + V_c(\mathbf{q})] \pi_c(0, \mathbf{q}) = \frac{1}{2} \left[ U + V_c(\mathbf{q}) \right] \chi_c(0, \mathbf{q}) \tag{2.17}
\]
and \( U \tilde{\pi}_c(0) = \frac{1}{2} U \tilde{\chi}_c(0) / [1 - \frac{1}{4} U \tilde{\chi}_c(0)] \). It is likely that \( [U + V_c(\mathbf{q})] \pi_c(0, \mathbf{q}) \geq 1 \) and \( U \tilde{\pi}_c(0) \geq 1 \) in actual magnets. Note also that \( \tilde{\phi}_n \ll 1 \) for \( n_d \simeq 1 \). Then, the second term of Eq. (2.13) can be ignored. In this paper, Eq. (2.13) is approximated by \( V(i\omega_l, \mathbf{q}) \simeq V^{(c)}(\mathbf{q}) \). In \( d = 3 \) dimensions,
\[
V^{(c)}(\mathbf{q}) = \frac{1}{N} \sum_{\mathbf{G}} \left[ \frac{4\pi e^2}{|\mathbf{q} - \mathbf{G}|^2} - \frac{1}{N} \sum_{\mathbf{p} \neq 0} \frac{4\pi e^2}{|\mathbf{p} - \mathbf{G}|^2} \right], \tag{2.18}
\]
with \( \mathbf{G} \) wave-numbers of reciprocal lattice points, \( p \) the volume of a unit cell, and the summation over \( \mathbf{p} \) restricted to the first Brillouin zone. Because \( \lim_{\mathbf{q} \rightarrow \mathbf{G}} V^{(c)}(\mathbf{q}) / k_B T_K = +\infty \), the charge susceptibility vanishes for commensurate \( \mathbf{q} \):
\[
\lim_{\mathbf{q} \rightarrow \mathbf{G}} \chi_c(i\omega_l, \mathbf{q}) = 0. \tag{2.19}
\]
The susceptibility given by Eq. (2.14) with Eq. (2.16) is consistent with a physical picture for Kondo lattices that local spin fluctuations at different sites interact with each other by intersite exchange interactions. Then, \( I_s(i\omega_l, \mathbf{q}) \) should be called an exchange interaction, which is responsible for magnetic instability. The second term of Eq. (2.10) is mainly composed of three contributions. One is the superexchange interaction, which arises from the virtual exchange of pair excitations across the Mott-Hubbard gap. The phenomenological term \( J^{(p)}(\mathbf{q}) \) should be determined in such a way that the sum of this superexchange interaction and \( J^{(p)}(\mathbf{q}) \) should be equal to the superexchange interaction of actual systems. Another is the novel exchange interaction arising from that of pair excitations of quasi-particles. Whether it is ferromagnetic or antiferromagnetic depends on the dispersion relation of quasi-particles. For example, it is ferromagnetic in the so-called flat-band and band-edge models. On the other hand, it is antiferromagnetic when the nesting of the Fermi surface is sharp or when the chemical potential is at the band center. The other is the mode-mode coupling term among intersite spin fluctuations. Each of the superexchange interaction and the novel exchange interaction can cause magnetic instability, while the mode-mode coupling suppresses magnetic instability. In the following part of this paper, we assume that the sum of the two exchange interactions is antiferromagnetic and is so strong to cause magnetic instability; the Néel temperature \( T_N \) is determined by \( 1/\chi_s(0, \mathbf{Q}) \big|_{T = T_N} = 0 \) or \( 1 - \frac{1}{4} [I_s(0, \mathbf{Q}) \chi_c(0)] \big|_{T = T_N} = 0 \), with \( \mathbf{Q} \) ordering wave-numbers.

As discussed in Introduction, local spin fluctuation are quite different between the two temperature regions, \( T \gg T_K \) and \( T \ll T_K \). Then, local-moment magnetism is characterized by \( T_N \gg T_K \) while itinerant-electron magnetism by \( T_N \ll T_K \). In this paper, our study is restricted to itinerant-electron magnetism or \( T_N \ll T_K \).

\section*{B. Landau’s free energy}

Magnetizations appear below \( T_N \). In this subsection, Landau’s free energy in the presence of external fields
\[
\mathcal{H}_{ext} = -g \mu_B \sum_i (\mathbf{H}_i \cdot \hat{s}_i) - \sum_i \Delta \mu_i \hat{n}_i, \tag{2.20}
\]
is studied. For this purpose, it is convenient to consider
\[
\mathcal{H}_Q = \mathcal{H} + \mathcal{H}_{ext} + \mathcal{H}_L, \tag{2.21}
\]
instead of \( \mathcal{H} + \mathcal{H}_{ext} \), with
\[
\mathcal{H}_L = -\sum_i \lambda_i^{(s)} (2\hat{s}_i - m) - \sum_i \lambda_i^{(c)} (\hat{n}_i - n_i) \tag{2.22}
\]
including no operator and
\[
\mathcal{H}_Q = -\sum_i \lambda_i^{(s)} (2\hat{s}_i - m) - \sum_i \lambda_i^{(c)} (\hat{n}_i - n_i) \tag{2.23}
\]
including Lagrange’s multipliers, \( \lambda_i^{(s)} \) and \( \lambda_i^{(c)} \). The thermodynamic potential \( \Omega \) for \( \mathcal{H}_Q \) is given by
\[
e^{-\Omega / k_B T} = \text{Tr} \exp \left[ -\frac{1}{k_B T} (\mathcal{H}_Q - \mu \sum_i \hat{n}_i) \right], \tag{2.24}
\]
with $\mu$ the chemical potential. Lagrange’s multipliers should satisfy
\[
\frac{\partial \Omega}{\partial \lambda^{(s)}_{i\nu}} = m_{i\nu} - 2\langle \hat{\sigma}_{i\nu} \rangle_{\Omega} = 0
\]  
(2.25)
and
\[
\frac{\partial \Omega}{\partial n_i} = n_i - \langle \hat{n}_i \rangle_{\Omega} = 0,
\]  
(2.26)
with $\langle \cdots \rangle_{\Omega}$ standing for thermal averages for $\mathcal{H}_\Omega$. Then, magnetizations and charge densities are given by sets of $\mathbf{m}$ and $n_i$, $\{\mathbf{m}\}$ and $\{n\}$. It follows that
\[
\frac{\partial \Omega}{\partial m_{i\nu}} = \lambda^{(s)}_{i\nu} (\{\mathbf{m}\}, \{n\}) - \frac{1}{2} g\mu_B H_{i\nu}
\]  
(2.27)
and
\[
\frac{\partial \Omega}{\partial n_i} = \lambda^{(c)}_i (\{\mathbf{m}\}, \{n\}) - \Delta \mu_i.
\]  
(2.28)
Here, it is not necessary to consider the dependence of $\Omega$ on $\{\mathbf{m}\}$ and $\{n\}$ through $\lambda^{(s)}_{i\nu}$ and $\lambda^{(c)}_i$, because of Eqs. (2.25) and (2.26). When $\Omega$ is minimized as a function of $\{\mathbf{m}\}$ and $\{n\}$, it follows from Eqs. (2.27) and (2.28) that $\lambda^{(s)}_{i\nu} = \frac{1}{2} g\mu_B H_{i\nu}$ and $\lambda^{(c)}_i = \Delta \mu_i$, so that $\mathcal{H}_\Omega$ becomes $\mathcal{H} + \mathcal{H}_{\text{ext}}$. The set of Eqs. (2.27) and (2.28) is nothing but the selfconsistency condition to determine magnetizations and charge densities for $\mathcal{H} + \mathcal{H}_{\text{ext}}$, and $\Omega$ defined this way is Landau’s free energy for $\mathcal{H} + \mathcal{H}_{\text{ext}}$.

The normal Hartree term is given by $\lambda_0 = -\left(\frac{1}{2} U n_d + \sum_j V_{ij} n_d\right)$, with $n_d$ the average electron density defined by Eq. (2.23). All of its effects are included in the unperturbed state considered in Sec. IIA. Anomalous Hartree and Fock terms, which are due to spin or charge polarizations, are given by
\[
\lambda^{(s)}_{i\nu} = \lambda^{(s)}_{i\nu} + \left(\frac{1}{2} U m_{i\nu} + \frac{1}{4} \sum_{j(\neq i)} J_{ij}(p) m_{j\nu}\right)
\]  
(2.29)
for spin channels, and
\[
\lambda^{(c)}_i = \lambda^{(c)}_i - \left(\frac{1}{2} U \Delta n_i + \sum_{j(\neq i)} V_{ij}(c) \Delta n_j\right),
\]  
(2.30)
for the charge channel, with
\[
\Delta n_i = n_i - n_d.
\]  
(2.31)
Here, Lagrange’s multipliers, $\lambda^{(s)}_{i\nu}$ and $\lambda^{(c)}_i$, are included, and thermal averages $\langle \hat{\sigma}_{i\nu} \rangle_{\Omega}$ and $\langle \hat{n}_i \rangle_{\Omega}$ are replaced by $m_{i\nu}/2$ and $n_i$, because Eqs. (2.25) and (2.26) are eventually satisfied. When $\lambda^{(s)}_{i\nu}$ and $\lambda^{(c)}_i$ are perturbatively considered starting from the Fermi-liquid state considered in Sec. IIA, it follows from the selfconsistency condition of Eqs. (2.25) and (2.26) that
\[
m_{i\nu} = 2 \sum_j \pi_{\nu\nu'}(i, j) \lambda^{(s)}_{j\nu'} + 2 \sum_{j, j'} \pi_{\nu\nu'}(i, j, j') \lambda^{(s)}_{j\nu'} \lambda^{(c)}_{j'\nu'} + \sum_{j, j', j''} \pi_{\nu\nu'}(i, j, j') \lambda^{(c)}_{j\nu'} \lambda^{(c)}_{j'\nu''} \lambda^{(c)}_{j''\nu''}
\]  
(2.32)
and
\[
\Delta n_i = 2 \sum_j \pi_{\nu\nu'}(i, j) \lambda^{(c)}_{j\nu'}
\]  
(2.33)
with $\pi_{\nu\nu'}(i, j)$ static two-point polarization functions in spin channels, $\pi_{\nu\nu'}(i, j)$ those in charge channels, $\pi_{\nu\nu'}(i, j, j')$, $\pi_{\nu\nu'}(i, j, j', j'')$ and $\pi_{\nu\nu'}(i, j, j', j'', j''')$ static three-point ones with two spin vertexes and a charge vertex, and $\pi_{\nu\nu'}(i, j, j', j'', j''')$ static four-point ones with four spin vertexes.

Note that $\pi_s(0, q) = \sum_j \pi_{\nu\nu'}(i, j) e^{-i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)}$ for $\nu = x, y$ and $z$, and $\pi_c(0, q) = \sum_j \pi_{\nu\nu'}(i, j) e^{-i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)}$. It follows from Eqs. (2.32) and (2.33) that in the wavefunction representation
\[
\lambda^{(s)}_{i\nu} = \frac{m_{i\nu}}{\chi_s(0, q)} - \frac{2}{\pi_c(0, q)} \sum_{q_1, q_2} \pi_{\nu\nu'}(q, q_1, q_2) \lambda^{(s)}_{q_1, q_2} \lambda^{(c)}_{q_3},
\]  
\[
- \frac{1}{\pi_c(0, q)} \sum_{q_1, q_2, q_3} \pi_{\nu\nu'}(q, q_1, q_2, q_3) \lambda^{(s)}_{q_1, q_2} \lambda^{(s)}_{q_3, q_4} + \cdots
\]  
(2.34)
and
\[
\lambda^{(c)}_i = \frac{\Delta n_i}{\chi_c(0, q)} - \frac{1}{\pi_c(0, q)} \sum_{q_1, q_2, q_3} \pi_{\nu\nu'}(q, q_1, q_2) \lambda^{(c)}_{q_3, q_4} + \cdots,
\]  
(2.35)
with
\[
m_{i\nu} = \frac{1}{N} \sum_i m_{i\nu} e^{-i\mathbf{q} \cdot \mathbf{R}_i},
\]  
(2.36)
\[
\Delta n_i = \frac{1}{N} \sum_i \Delta n_i e^{-i\mathbf{q} \cdot \mathbf{R}_i},
\]  
(2.37)
\[
\lambda^{(s)}_{i\nu} = \frac{1}{N} \sum_i \lambda^{(s)}_{i\nu} e^{-i\mathbf{q} \cdot \mathbf{R}_i},
\]  
(2.38)
\[
\lambda^{(c)}_i = \frac{1}{N} \sum_i \lambda^{(c)}_i e^{-i\mathbf{q} \cdot \mathbf{R}_i},
\]  
(2.39)
\[
\lambda^{(s)}_{i\nu} = \frac{1}{N} \sum_i \lambda^{(s)}_{i\nu} e^{-i\mathbf{q} \cdot \mathbf{R}_i},
\]  
(2.40)
\[
\lambda^{(c)}_i = \frac{1}{N} \sum_i \lambda^{(c)}_i e^{-i\mathbf{q} \cdot \mathbf{R}_i},
\]  
(2.41)
\[ \pi_{\nu_1 \nu_2 \nu_3}(q_1, q_2, q_3) = \frac{1}{N} \sum_{i_1,i_2,i_3} \pi_{\nu_1 \nu_2 \nu_3}(i_1, i_2, i_3) \times e^{-i[(q_1 \cdot R_{i_1})+(q_2 \cdot R_{i_2})+(q_3 \cdot R_{i_3})]}, \]  
and

\[ \pi_{\nu_1 \nu_2 \nu_3 \nu_4}(q_1, q_2, q_3, q_4) = \frac{1}{N} \sum_{i_1,i_2,i_3,i_4} \pi_{\nu_1 \nu_2 \nu_3 \nu_4}(i_1, i_2, i_3, i_4) \times e^{-i[(q_1 \cdot R_{i_1})+(q_2 \cdot R_{i_2})+(q_3 \cdot R_{i_3})+(q_4 \cdot R_{i_4})]}, \]

Because of the translational symmetry,

\[ \pi_{\nu_1 \nu_2 \nu_3}(q_1, q_2, q_3) \propto \delta_{q_1+q_2+q_3}, \]

and

\[ \pi_{\nu_1 \nu_2 \nu_3 \nu_4}(q_1, q_2, q_3, q_4) \propto \delta_{q_1+q_2+q_3+q_4}, \]

with

\[ \delta_q = \begin{cases} 1, & q = \mathbf{G} \\ 0, & q \neq \mathbf{G} \end{cases}. \]

Eqs. (2.34) and (2.35) can be solved in an iterative manner. When \( |m_{q\nu}| \ll 1 \) and \( |\Delta n_{q\nu}| \ll 1 \), Eqs. (2.34) and (2.35) give trivial relations, \( \lambda_{q\nu}^{(s)} = m_{q\nu}/\chi_s(0, q) \) and \( \lambda_{q\nu}^{(c)} = \Delta n_{q\nu}/\chi_c(0, q) \). When these approximate relations are made use of, it follows that

\[ \lambda_{q\nu}^{(s)} = \frac{m_{q\nu}}{2\pi_s(0, q)} \approx \frac{1}{2} U m_{q\nu} \]  
and

\[ \lambda_{q}^{(c)} = \frac{\Delta n_{q\nu}}{2\pi_c(0, q)}, \]

In Eq. (2.47), Eq. (2.14) is also made use of. When these are used in Eqs. (2.34) and (2.35), it follows that

\[ \lambda_{q\nu}^{(s)} = \frac{m_{q\nu}}{\chi_s(0, q)} - 2 \sum_{q_1,q_2} g(q, q_1, q_2)m_{q_1 \nu} \Delta n_{q_2} \]

\[ + \sum_{\nu_1 \nu_2 \nu_3 q_1 q_2 q_3} b_{\nu_1 \nu_2 \nu_3}(q_1, q_2, q_3) \times m_{q_1 \nu_1} m_{q_2 \nu_2} m_{q_3 \nu_3} + \cdots \]

and

\[ \lambda_{q}^{(c)} = \frac{\Delta n_{q\nu}}{\chi_c(0, q)} - \sum_{\nu_1 q_1 q_2} g(q, q_1, q_2)m_{q_1 \nu_1} m_{q_2 \nu_2 \nu_3} + \cdots, \]

with

\[ g(q_1, q_2, q_3) = \frac{U^2 \pi_s^2(q_1, q_2, q_3)}{2^2 \pi_s(0, q_1)} \]

and

\[ b_{\nu_1 \nu_2 \nu_3 \nu_4}(q_1, q_2, q_3, q_4) = -\frac{U^4}{2^3 \pi_{\nu_1 \nu_2 \nu_3 \nu_4}}(q_1, q_2, q_3, q_4). \]

Eqs. (2.27) and (2.28) can also be written in the momentum representation, and Lagrange’s multipliers are given by Eqs. (2.44) and (2.50). By integrating them we obtain

\[ \Omega = \Omega_0 + \Omega_{\text{ext}} + \Omega_{s^2} + \Omega_{s^2} + \Omega_{s^2} + \cdots, \]

with \( \Omega_0 \) the term including no spin nor charge polarizations,

\[ \Omega_{\text{ext}} = -N \left[ \frac{1}{2} g \mu B \sum_q (H_q - m_{-q}) + \sum_q \Delta \mu q \Delta n_{-q} \right], \]

\[ \Omega_{s^2} = N \sum_q \frac{|m_{q\nu}|^2}{2\chi_s(0, q)}, \]

\[ \Omega_{s^2} = N \sum_q \frac{|\Delta n_{q\nu}|^2}{2\chi_c(0, q)}, \]

\[ \Omega_{s^2} = -N \sum_{q_1,q_2=q_3} g(q_1, q_2, q_3)(m_{q_1 \nu} \cdot m_{q_2 \nu}) \Delta n_{q_3}, \]

and

\[ \Omega_{s^2} = \frac{N}{4} \sum_{q_1,q_2,q_3,q_4} b_{\nu_1}(q_1, q_2, q_3, q_4) \]

\[ \times \left[ (m_{q_1 \nu} \cdot m_{q_2 \nu}) (m_{q_3 \nu} \cdot m_{q_4 \nu}) - (m_{q_1 \nu} \cdot m_{q_3 \nu}) (m_{q_2 \nu} \cdot m_{q_4 \nu}) \right. \]

\[ \left. + (m_{q_1 \nu} \cdot m_{q_4 \nu}) (m_{q_2 \nu} \cdot m_{q_3 \nu}) \right]. \]

In Eq. (2.58), the following relations of

\[ b_{\nu_1}(q_1, q_2, q_3, q_4) = b_{\nu_2}(q_1, q_2, q_3, q_4) \]

\[ = - \left[ b_{\nu_1 \nu_2 \nu_3}(q_1, q_2, q_3, q_1) \right]_{\nu_1 \neq \nu_3} \]

are made use of; \( b_{\nu_1 \nu_2 \nu_3 \nu_4}(q_1, q_2, q_3, q_4) = 0 \) for other combinations of \( \nu_i \)'s.

When ordering wave-numbers are incommensurate, there can be several equivalent ones to denote. Denote the \( i \)th pair by \( \pm Q_i \), and assume that not only \( \pm Q_i \) but also \( \pm 2Q_i \) and \( \pm 4Q_i \) are incommensurate; \( \pm 2Q_i \neq \mathbf{G} \) or \( \pm 4Q_i \neq \mathbf{G} \). When SDW with \( \pm Q_i \) are stabilized, CDW with \( \pm 2Q_i \) appear because of the coupling term between SDW and CDW given by Eq. (2.58) or

\[ \Omega_{s^2} = -N \sum_i g(Q_i, Q_i, -2Q_i) \]

\[ \times \left[ (m_{Q_i})^2 \Delta n_{-2Q_i} + (m_{-Q_i})^2 \Delta n_{2Q_i} \right]. \]

Here and in the following part, the summation over \( Q_i \) is made over pairs or it is made in such a way that one of \( \pm Q_i \) is considered for each \( i \). When only SDW
or helical structures with \( Q \) and CDW with \( 2Q \) are considered,

\[
\Omega_{\text{ext}} = -N \sum_i \left\{ \frac{1}{2} g \mu_B \left[ (H_{Q_i} \cdot m_{-Q_i}) + (H_{-Q_i} \cdot m_{Q_i}) \right] + \left[ \Delta \mu_{2Q_i} \Delta n_{-2Q_i} + \Delta \mu_{-2Q_i} \Delta n_{2Q_i} \right] \right\},
\]

(2.61)

\[
\Omega_{\text{s}^2} = N \sum_i \frac{|m_{Q_i}|^2}{\chi_s(0, Q_i)},
\]

(2.62)

\[
\Omega_{\text{c}^2} = N \sum_i \frac{|\Delta n_{2Q_i}|^2}{\chi_c(0, 2Q_i)}
\]

(2.63)

and

\[
\Delta \Omega_{\text{s}^4} = \frac{N}{2} \sum_i \left\{ B_1 \left[ 2|m_{Q_i}|^4 - |(m_{Q_i} \cdot m_{Q_i})|^2 \right] + 2B_2 |(m_{Q_i} \cdot m_{Q_i})|^2 \right\} \\
+ \frac{N}{2} \sum_{i \neq j} \left\{ 2B_3|m_{Q_i}||m_{Q_j}|^2 \right\}
\]

\[
+ 2B_4 \left[ |(m_{Q_i} \cdot m_{Q_j})|^2 + |(m_{Q_i} \cdot m_{-Q_j})|^2 \right] \\
+ B_5 \left[ |(m_{Q_i} \cdot m_{Q_j})|^2 - |(m_{Q_i} \cdot m_{-Q_j})|^2 \right],
\]

(2.64)

with

\[
B_1 = b_{\nu,4}(Q, -Q, Q, -Q),
\]

(2.65)

\[
B_2 = b_{\nu,4}(Q, Q, -Q, -Q),
\]

(2.66)

\[
B_3 = b_{\nu,4}(Q, -Q, Q', -Q') + b_{\nu,4}(Q, -Q, -Q', Q') - \frac{1}{2} b_{\nu,4}(Q, Q', -Q', -Q') - \frac{1}{2} b_{\nu,4}(Q, -Q', -Q', Q'),
\]

(2.67)

\[
B_4 = \frac{1}{2} b_{\nu,4}(Q, Q', -Q, -Q') + \frac{1}{2} b_{\nu,4}(Q, -Q', -Q', Q'),
\]

(2.68)

and

\[
B_5 = 2b_{\nu,4}(Q, -Q, Q', -Q') - 2b_{\nu,4}(Q, -Q, -Q', Q').
\]

(2.69)

Here, \( Q \) are \( Q' \) (\( Q \neq \pm Q' \)) are two different wave-numbers among considered \( Q \)'s, and symmetrical relations such as \( b_{\nu,4}(q_1, q_2, q_3, q_4) = b_{\nu,4}(q_1, q_2, q_3, q_4) \) and \( b_{\nu,4}(-q_1, -q_2, -q_3, -q_4) \) and so on, are made use of.

Four-point polarization functions are also divided into single and multisite terms so that

\[
b_{\nu,4}(q_1, q_2, q_3, q_4) = \tilde{b}_{\nu,4} + \Delta b_{\nu,4}(q_1, q_2, q_3, q_4).
\]

(2.70)

The single-site term is approximately given by Eq. (2.67): \( \tilde{b}_{\nu,4} = 2 \bar{\chi}_s(0) = 2k_B T_K \). When a diagram corresponding to Fig. 2(b) of the previous paper\( \footnote{\text{[10]}} \) is considered and Eq. (2.3) is approximately used, the multi-site term is calculated so that for \( T \ll T_K \)

\[
\Delta b_{\nu,4}(q_1, q_2, q_3, q_4) = \delta_{q_1+q_2+q_3+q_4} \left( \frac{\phi_{q_1}}{\phi_{q_1} \bar{\chi}_s(0)} \right)^4 \times
\]

\[
\times k_B T_K \sum_n \frac{1}{n} \sum_{\xi_n} \frac{1}{\xi_n - \xi(k)} \frac{1}{\xi_n - \xi(k + q_1)} \times \frac{1}{\xi_n - \xi(k + q_2)} \frac{1}{\xi_n - \xi(k + q_1 + q_2 + q_3)}. \]

(2.71)

Here, the Ward relation for the three-point vertex function in spin channels is made use of, as it was in the previous paper. In general,

\[
|B_5| \ll \text{Min}(B_1, B_2, B_3, B_4).
\]

(2.72)

The nesting of the Fermi surface is characterized by \( \xi(k) \approx -\xi(k + Q) \) in a wide \( k \) region in the vicinity of the Fermi surface. When the nesting of the Fermi surface is sharp, \( B_1 \) is much larger than other \( B_i \)'s so that

\[
B_1 \gg \text{Max}(B_2, B_3, B_4) \gg |B_5|. \]

(2.73)

### III. INCOMMENSURATE SDW AND CDW

#### A. Sinusoidal or helical structures

In this subsection, it is examined which are stabilized, sinusoidal or helical structures. The second order term in magnetizations, \( \Delta \Omega_{\text{s}^2} \), does not depend on magnetic structures. When magnetizations are small, magnetic structures are determined by the fourth order term \( \Delta \Omega_{\text{s}^4} \).

When single \( Q \) structures are assumed, the fourth order term is written as

\[
\Delta \Omega_{\text{s}^4} = \frac{N}{2} \left\{ B_1 \left[ 2|m_{Q}|^4 - |(m_{Q} \cdot m_{Q})|^2 \right] + 2B_2 |(m_{Q} \cdot m_{Q})|^2 \right\}.
\]

(3.1)

When magnetizations are written as \( m_Q = m_Q' + i m_Q'' \), with \( m_Q' \) and \( m_Q'' \) being real, it follows that

\[
(|m_Q \cdot m_Q'|)^2 = (m_Q')^2 - (m_Q'')^2 + i2(m_Q' \cdot m_Q'').
\]

(3.2)

It is easy to see that \(|(m_Q \cdot m_Q'|)^2| \) becomes the smallest for helical structures, which are characterized by \( m_Q' \parallel m_Q'' \), such as

\[
|(|m_Q \cdot m_Q'|)^2| = 0.
\]

(3.3)

On the other hand, it becomes the largest for sinusoidal structures, which are characterized by \( m_Q' \parallel m_Q'' \), such as
Therefore, helical structures are stabilized when
\[ B_1 - 2B_2 < 0, \]  
and sinusoidal structures are stabilized when
\[ B_1 - 2B_2 > 0. \]  

When no multi-site terms are included, \( B_1 = B_2 = B_3 = B_4 = b_3 = \) and \( B_5 = 0. \) When there is no sharp nesting of the Fermi surface, therefore, it is likely that Eq. (3.5) is satisfied. The novel exchange interaction can never be strongly antiferromagnetic in this case. However, antiferromagnetic instability can still occur when the superexchange interaction is strongly antiferromagnetic. Even helical structures are stabilized in such itinerant-electron antiferromagnets. When the nesting of the Fermi surface is so sharp for incommensurate \( Q \) that Eq. (3.6) might be satisfied, sinusoidal structures are stabilized as is expected.

B. Single or multiple \( Q \) structures

In this subsection, it is examined whether multiple \( Q \) structures are stabilized or not. First, examine helical structures under an assumption that Eq. (3.2) is satisfied. Because \( (\mathbf{m}_Q \cdot \mathbf{m}_Q) = 0 \) for helical structures, \( \Delta \Omega_s \) is reduced to
\[
\Delta \Omega_s = \frac{N}{2} \sum_i B_1 |\mathbf{m}_Q|^2 + \frac{N}{2} \sum_{i \neq j} \left( 2B_3 |\mathbf{m}_Q|^2 |\mathbf{m}_Q|^2 + 2B_4 |(\mathbf{m}_Q \cdot \mathbf{m}_Q)|^2 + 2B_5 |(\mathbf{m}_Q \cdot \mathbf{m}_Q)|^2 - |(\mathbf{m}_Q \cdot \mathbf{m}_Q)|^2 \right). \]  

When Eq. (3.5) is satisfied, it is likely that \( B_1 < 2B_3, B_1 < 2B_4, \) and \( B_5 \) is much smaller than other \( B_i. \) In such a case, multiple helical structures are never stabilized. When helical structures appear at temperatures much lower than \( T_K, \) they must be of single \( Q. \)

Next, examine sinusoidal structures under an assumption that Eq. (3.6) is satisfied. In this case, we can put \( \mathbf{m}_Q = e^{i\theta} \mathbf{m}_i \) and \( \mathbf{m}_- \mathbf{Q}_i = e^{-i\theta} \mathbf{m}_i, \) with \( \mathbf{m}_i \) being real. The fourth order term is given by
\[
\Omega_s = \frac{N}{2} \sum_i (B_1 + 2B_2) |\mathbf{m}_i|^4 + N \sum_{i \neq j} \left[ B_3 |\mathbf{m}_i|^2 |\mathbf{m}_j|^2 + 2B_4 (\mathbf{m}_i \cdot \mathbf{m}_j)^2 \right]. \]  

When the multiplicity of \( Q \) is two or three, \( \Omega_s \) becomes the smallest when
\[
(\mathbf{m}_i \cdot \mathbf{m}_j) = 0 \]  
for any pair of \( i \neq j. \) When Eq. (3.6) is satisfied, it is likely that
\[
B_1 + 2B_2 > 2B_3 \]  
is also satisfied. Then, multiple sinusoidal structures are stabilized and their polarization vectors are orthogonal to each other. It follows that
\[
\Omega_{s_2} + \Omega_{s_4} = N \lambda \left( \frac{m^2}{\chi_s(0, Q)} + \frac{1}{2} B_\perp(\lambda)m^4 \right), \]  
with
\[
B_\perp(\lambda) = B_1 + 2B_2 + 2(\lambda-1)B_3. \]  
Here, \( |\mathbf{m}_i| \) is simply denoted by \( m \) because all of \( |\mathbf{m}_i| \) are the same as each other, and \( \lambda \) is the multiplicity of ordering wave-numbers, \( \lambda = 1, 2 \) or 3. Below \( T < T_N, \)
\[
m^2 = -\frac{1}{2B_\perp(\lambda)|\chi_s(0, Q)|^2} \]  
and
\[
\Omega_{s_2} + \Omega_{s_4} = -\frac{\lambda N}{2B_\perp(\lambda)|\chi_s(0, Q)|^2}. \]  
Because Eq. (3.14) is an increasing function of \( \lambda, \) a symmetry broken ordered state is never stabilized as long as \( \lambda \leq 3; \) if there are three equivalent \( Q \)’s, for example, single or double \( Q \) structures are never stabilized.

When the multiplicity of \( Q \) is four or larger than four (\( \lambda \geq 4), \) it is impossible that Eq. (3.3) is satisfied for any pair of \( Q_i \) and \( Q_j. \) Only two possible magnetic structures are examined in this paper; triple or quartet \( Q \) structures. Triple structures are symmetry broken states, where only three \( Q \)’s among \( \lambda \) wave-numbers are ordered and their magnetizations are orthogonal to each other. Their free energy is given by Eq. (3.14) with \( \lambda = 3. \) In quartet structures, four \( Q \) components are ordered and each pair of magnetizations make the same angle, \( \cos^{-1}(-1/3) = 121.63 \cdots \) degree; for example, \( \mathbf{m}_1 = m (0, 0, 1), \) \( \mathbf{m}_2 = m (0, \sqrt{8}/3, -1/3), \) \( \mathbf{m}_3 = m (\sqrt{6}/3, -\sqrt{2}/3, -1/3), \) and \( \mathbf{m}_4 = m (-\sqrt{6}/3, -\sqrt{2}/3, -1/3). \) Then, it follows that
\[
\Omega_{s_2} + \Omega_{s_4} = 4N \left( \frac{m^2}{\chi_s(0, Q)} + \frac{1}{2} B_Q m^4 \right) \]  
with
\[
B_Q = B_1 + 2B_2 + 6B_3 + \frac{4}{3} B_4. \]  
When the free energy is minimized,
\[
\Omega_{s_2} + \Omega_{s_4} = -\frac{4N}{2B_Q|\chi_s(0, Q)|^2} \]  
below \( T_N. \) When \( 4/B_Q > 3/B_\perp(3) \) or
\[
B_1 > 4B_4 - 2B_2 + 2B_3, \]  
is satisfied, quartet \( Q \) structures are stabilized. Otherwise, triple \( Q \) structures are stabilized. One can conclude that quartet \( Q \) structures can be stabilized only when the nesting of the Fermi surface is extraordinarily sharp.
C. Coexistence of SDW and CDW

The coupling between helical structures and CDW vanishes according to Eqs. (2.60) and (3.3), and that between SDW and CDW exists according to Eqs. (2.60) and (3.4). Once SDW with \( \pm Q \) appear, therefore, CDW with \( \pm 2Q \) appear. The amplitude of CDW is given by

\[
\Delta n_{-2Q} = g(Q, Q, -2Q) \chi_c(0, 2Q) m^2. \tag{3.19}
\]

When multi-site terms are ignored, \( g(Q, Q, -2Q) \) is approximately given by Eq. (3.3):

\[
g(Q, Q, -2Q) \simeq \begin{cases} 
\Delta/\pi, & n_d < 1 \\
-\Delta/\pi, & n_d > 1 
\end{cases}, \tag{3.20}
\]

with \( \Delta \) the hybridization energy of the mapped Anderson model. According to the mapping condition, \( 1/\Delta \simeq -\text{Im}(1/N) \sum_{\mathbf{k}} G_{\sigma} (+i0, \mathbf{k}) \), so that the coupling constant is rather large. However, the charge susceptibility is small. Then, the amplitude of CDW must be small. In particular, it vanishes when \( 2Q \) is commensurate because of Eq. (3.19).

Denote polarizations as \( \mathbf{m}_Q = \frac{1}{4} \mathbf{m} \text{e}^{i\theta} \) and \( \Delta n_{2Q} = \frac{1}{4} |\Delta n| \text{e}^{i\theta} \), with \( \mathbf{m} \) being real. Then, the coupling term is given by

\[
\Delta \Omega_{s c} = -\frac{1}{2} N g(Q, Q, -2Q) \chi_c(0, Q) \times |\mathbf{m}|^2 |\Delta n| \cos(2\theta_s - \theta_c). \tag{3.21}
\]

The free energy takes its minimum when

\[
2\theta_s - \theta_c = \begin{cases} 
2l\pi, & n_d < 1 \\
(2l + 1)\pi, & n_d > 1
\end{cases}, \tag{3.22}
\]

with \( l \) being integers. Then, it follows that

\[
\mathbf{m}_l = \mathbf{m} \cos(\mathbf{Q} \cdot \mathbf{R}_i + \theta_s) \tag{3.23}
\]

and

\[
\Delta n_l = \begin{cases} 
+|\Delta n| \cos[2(Q \cdot \mathbf{R}_i + \theta_s)], & n_d < 1 \\
-|\Delta n| \cos[2(Q \cdot \mathbf{R}_i + \theta_s)], & n_d > 1
\end{cases}. \tag{3.24}
\]

The quenching of magnetic moments by local quantum spin fluctuations, which is one of the most essential effects in Kondo lattices, sensitively depends on local number of electrons. According to Eq. (A.2) in Appendix, the Kondo temperature at the \( i \)th site is approximately given by

\[
[k_B T_K]_i = \frac{2\Delta}{\pi} |1 - (n_d + \Delta n_i)|. \tag{3.25}
\]

Eqs. (3.23), (3.24) and (3.25) show that electron numbers \( \Delta n_i \) are modulated in such a way that magnetic moments are much quenched where \( T_K \) is high. In the less than half-filled case \( (n_d < 1) \), for example, doped holes go mainly into sites where magnetic moments are small.

IV. DISCUSSION

Because no magnetic anisotropy is taken into account, directions of magnetic anisotropy are totally independent of those of wave-numbers \( Q \) in this paper. In actual magnets, absolute directions of magnetizations are mainly determined by magnetic anisotropy. However, the relative angles of magnetizations between different \( Q \) components must be \( 90^\circ \) or at least close to \( 90^\circ \) in double or triple \( Q \) structures even if magnetic anisotropy is taken into account; the magnetic part of the free energy is lower when the relative angles are closer to \( 90^\circ \).

In Kondo lattices, the coupling between SDW and CDW is strong and the phases of SDW and CDW are never independent of each other. Then, even nonmagnetic impurities have a large pinning effect on coexisting SDW and CDW.

Incommensurate SDW were observed in many metallic magnets, some of which are of high enough symmetry such as cubic CeAl\(_2\) \( \text{Ce}_2\text{Al}_2 \) cubic \( \text{Ce}_2\text{Al}_3 \) cuprate-oxide high-temperature superconductors, \( \text{Ce}_2\text{Al}_3 \) which are approximately regarded as orthorhombic lattices, and so on. In this paper, Landau’s free energy is obtained up to the fourth order in magnetizations. Because saturated magnetic moments are large in CeAl\(_2\) and Cr, higher than the fourth order terms are required to discuss their physical properties at \( T \ll T_N \). However, it is still interesting to examine whether or not a multiple \( Q \) structure is actually stabilized in CeAl\(_2\) because magnetic moments are small just below \( T_N \). It should be mentioned that a triple \( Q \) structure was actually proposed for CeAl\(_2\) \( \text{Ce}_2\text{Al}_2 \) \( \text{Ce}_2\text{Al}_3 \). The transition at \( T_N \approx 38^\circ \text{C} \) in Cr is of first order, and it is not certain if magnetizations are small enough even just below \( T_N \). Because a multiple \( Q \) structure was also suggested \( \text{Ce}_2\text{Al}_2 \) \( \text{Ce}_2\text{Al}_3 \) it is interesting to reexamine SDW in cubic Cr. It has been claimed that the so called stripe order must be stabilized in the cuprate oxides \( \text{Ce}_2\text{Al}_2 \) \( \text{Ce}_2\text{Al}_3 \). Because magnetizations are small, the theory of this paper is applicable to magnetic states in the cuprate oxides. Within the theoretical framework of this paper, the stripe order must be nothing but the coexisting state of incommensurate SDW and CDW. It is interesting which is actually responsible for two equivalent satellites in neutron diffraction, two equivalent magnetic domains whose volumes are accidentally almost the same as each other or a double \( Q \) structure of SDW and CDW.

The spectral weight of Gutzwiller’s quasi-particle band is small in strongly correlated electron systems in the vicinity of the Mott transition. A large part of the spectral weight exists in the lower and upper Hubbard bands, which are far from the chemical potential. The formation of SDW and CDW causes not only gaps in quasi-particle spectra but also pseudogaps in the lower and upper Hubbard bands. Their pseudogaps are as large as \( U|\mathbf{m}(Q)| \), although gaps in quasi-particle spectra are of the order of \( k_B T_N \). It is interesting to observe changes in the density of states caused by the formation of SDW and CDW not only in the vicinity of the chemical potential but also far from the chemical potential.
V. CONCLUSION

Magnetic and charge structures in strongly correlated electron liquids are studied within the theoretical framework of Kondo lattices. When there is no sharp nesting in the Fermi surface and the superexchange interaction is strongly antiferromagnetic, helical structures are stabilized even in itinerant-electron magnets. When the nesting of the Fermi surface is sharp enough, sinusoidal structures are stabilized. When incommensurate ordering wave-numbers are multiple, in particular, multiple sinusoidal structures are stabilized. Their magnetic polarizations are orthogonal to each other in double or triple sinusoidal structures, when magnetic anisotropy is small enough.

Because the quenching of magnetic moments by local quantum spin fluctuations sensitively depends on local numbers of electrons in Kondo lattices, incommensurate CDW is inevitably driven by incommensurate SDW. In hole or electron doped systems at the vicinity of the Mott transition, therefore, the phase of CDW is adjusted in such a way that the density of doped carriers, holes or electrons, is large at sites where magnetic moments are small.

ACKNOWLEDGMENTS

The author is thankful to K. Fujiwara for showing him the references. This work was supported by a Grant-in-Aid for Scientific Research (C) No. 13640342 from the Ministry of Education, Cultures, Sports, Science and Technology of Japan.

APPENDIX A: NON-MAPPED SINGLE-IMPURITY ANDERSON MODEL

Consider the Anderson model with infinitely large $U$, constant hybridization energy $\Delta$, and conduction bandwidth $2D$. According to a previous paper the ground state energy of such an Anderson model is given by

$$ E_G = \epsilon_d - \mu - \tilde{\lambda}_c - \sqrt{(k_B T_K)^2 e^{-\pi \tilde{\lambda}_c / \Delta} + \sum_\nu \tilde{\lambda}_\nu^2}, \quad (A1) $$

with $k_B T_K = \sqrt{D \Delta} \exp[\pi (\epsilon_d - \mu) / 2 \Delta]$. Here, $\epsilon_d$ is the energy level of strongly correlated electrons and $\mu$ is the chemical potential; $\lambda_c$ and $\lambda_\nu$ ($\nu = x, y$ and $z$) are external fields or Lagrange multipliers, which are determined in such a way that magnetic moments and electron numbers are $m = (m_x, m_y, m_z)$ and $n$.

In the absence of fields, the electron number is given by $n_d = -[\partial E_G / \partial \lambda_c]_{\lambda_c = 0} = 1 - (\pi T_K / 2 \Delta)$, so that

$$ k_B T_K = \frac{2 \Delta}{\pi} (1 - n_d). \quad (A2) $$

In the presence of fields, magnetic moments and the electron number are given by

$$ m_\nu = -\frac{\partial E_G}{\partial \lambda_\nu} = \frac{\tilde{\lambda}_\nu}{\sqrt{T_K^2 e^{-\pi \tilde{\lambda}_c / \Delta} + \sum_\nu \lambda_\nu^2}}, \quad (A3) $$

and

$$ n = -\frac{\partial E_G}{\partial \lambda_c} = 1 - \frac{\pi}{2 \Delta} \frac{T_K^2 e^{-\pi \tilde{\lambda}_c / \Delta}}{\sqrt{T_K^2 e^{-\pi \tilde{\lambda}_c / \Delta} + \sum_\nu \lambda_\nu^2}}. \quad (A4) $$

It follows from these equations that

$$ \tilde{\lambda}_\nu(m, n) = \frac{2 \Delta m_\nu (1 - n)}{1 - m^2}, \quad (A5) $$

and

$$ \tilde{\lambda}_c(m, n) = -\frac{2 \Delta}{\pi} \ln \frac{1 - n}{(1 - n_d) \sqrt{1 - m^2}}. \quad (A6) $$

Define a thermodynamic potential by

$$ \tilde{\omega}(m, n) \equiv E_G(m, n) + n \tilde{\lambda}_c + \sum_\nu m_\nu \tilde{\lambda}_\nu $$

$$ = \epsilon_d - \mu - \frac{2 \Delta}{\pi} (1 - n) $$

$$ + (1 - n) \frac{2 \Delta}{\pi} \ln \frac{1 - n}{(1 - n_d) \sqrt{1 - m^2}}. \quad (A7) $$

It is easy to confirm that $\partial \tilde{\omega}(m, n) / \partial m_\nu = \tilde{\lambda}_\nu(m, n)$ and $\partial \tilde{\omega}(m, n) / \partial n = \tilde{\lambda}_c(m, n)$.

The case when the electron number is smaller than unity is examined so far. The case where the electron number is larger than unity can also be treated within this theoretical framework if the hole picture is taken.

In the presence of actual external fields, $\Delta \mu$ and $\mathbf{H}$, the thermodynamic potential to be minimized as a function of $\mathbf{m}$ and $n$ is given by

$$ \tilde{\Omega}_A = \tilde{\omega}(m, n) - n \Delta \mu - \frac{1}{2} g \mu_B (\mathbf{m} \cdot \mathbf{H}) $$

$$ = \epsilon_d - \mu - T_K n \Delta \mu - \frac{1}{2} g \mu_B (\mathbf{m} \cdot \mathbf{H}) $$

$$ + \frac{1}{2 \tilde{\chi}_s(0)} \left[ m^2 + \frac{1}{2} m^4 + \cdots \right] $$

$$ + \frac{1}{2 \tilde{\chi}_c(0)} \Delta n^2 - \tilde{g} m^2 n + \cdots, \quad (A8) $$

with $\Delta n = n - n_d$, $\tilde{\chi}_s(0) = (\pi / 2 \Delta) / [1 - n_d]$, $\tilde{\chi}_c(0) = (\pi / 2 \Delta) / [1 - n_d]$, and

$$ \tilde{g} = \begin{cases} \Delta / \pi, & n_d < 1 \\ -\Delta / \pi, & n_d > 1. \end{cases} \quad (A9) $$

Here, the results for $n_d > 1$ in the hole picture are interpreted into those in the electron picture. Then, it follows that

$$ \tilde{b}_{\nu,s} = \frac{2}{\tilde{\chi}_s(0)} = \frac{4 \Delta}{\pi} \left[ 1 - n_d \right] = 2 k_B T_K. \quad (A10) $$

When the Friedel sum rule is made use of, the density of states at the chemical potential is given by
\[ \rho(0) = \sin^2\left(\frac{\pi n_d}{2}\right) / \pi \Delta. \]  

When the Fermi-liquid relations, \( \tilde{\chi}_s(0) = \tilde{\phi}_s \rho(0) \) and \( \tilde{\chi}_c(0) = \tilde{\phi}_c \rho(0) \), are made use of, it follows that

\[ \tilde{\phi}_s = \frac{1}{\sqrt{1 - n_d}} \left( \frac{\pi n_d}{2} \right)^2 \sin^2\left(\frac{\pi n_d}{2}\right), \tag{A11} \]

and

\[ \tilde{\phi}_c = |1 - n_d| \left( \frac{\pi n_d}{2} \right)^2 \sin^2\left(\frac{\pi n_d}{2}\right). \tag{A12} \]