NEW MODIFIED BURR III DISTRIBUTION, PROPERTIES AND APPLICATIONS
Wali Khan, Farrukh Jamal, Muhammad A Nasir, Ali H Abuzaid, M.H. H Tahir, Wali Khan Mashwani

To cite this version:
Wali Khan, Farrukh Jamal, Muhammad A Nasir, Ali H Abuzaid, M.H. H Tahir, et al.. NEW MODIFIED BURR III DISTRIBUTION, PROPERTIES AND APPLICATIONS. 2021. hal-01902854v2

HAL Id: hal-01902854
https://hal.science/hal-01902854v2
Preprint submitted on 7 Jun 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
NEW MODIFIED BURR III DISTRIBUTION, PROPERTIES AND APPLICATIONS

Farrukh Jamal 1, Muhammad A. Nasir 2, Ali H. Abuzaid 3, M.H. Tahir 4, Wali Khan Mashwani 5

1,4 Department of Statistics, the Islamia University of Bahawalpur, Pakistan, dfarrukh1982@gmail.com, mthahir.stats@gmail.com
2 Department of Statistics, Govt. S.E. College Bahawalpur, Pakistan, arslannasir147@gmail.com
3 Department of Mathematics, Al-Azhar University-Gaza, Gaza, Palestine, a.abuzaid@alazhar.edu.ps
4,6 Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, Pakistan, mashwanger8@gmail.com

Abstract

In this paper we introduce a new continuous probability distribution which having increasing, decreasing, bathtub, upside-down bathtub and nearly constant hazard rate shapes is proposed. Some of its properties such as, rth moment, sth incomplete moment, moment generating function, skewness, kurtosis, mode, ith order statistic and stochastic ordering are discussed. The maximum likelihood estimation is employed to estimate the model parameters. Middle-censoring is considered as a modern general scheme of censoring. The usefulness of this model is demonstrated through applications on complete and censored samples.

Keywords: Burr III distribution, Stochastic ordering, Middle-Censoring, Order statistics, Characterization.

1. INTRODUCTION

Burr [1], developed a system of twelve types of distribution functions based on generating the Pearson differential equation. The density function has a range of shapes that is applicable to a wide area of applications. From the system of Burr distributions, the Burr III distribution is widely used model. It is also called the Dagum distribution in studies of income, wage and wealth distribution [2]. In the actuarial literature, it is known as the inverse Burr distribution [3] and the kappa distribution in the meteorological literature [4].

The hazard function of number of distributions have only increasing, decreasing or constant shapes. Thus, they may not be used to model lifetime data with a bathtub shaped hazard function, such as human mortality and machine life cycles. For last few decades, statisticians have been developing various extensions and modified forms of the probability distributions.

In the literature, a mostly modified form of Weibull distribution has been discussed. The two-parameter Weibull extension of [5], has a hazard function that can be increasing, decreasing or bathtub shaped. The reference [6], studied the characteristics and application of the truncated Weibull distribution which has a bathtub shaped hazard function. A three-parameter model, called exponentiated Weibull distribution, was introduced by [7]. Another three- parameter model was introduced by [8] and called extended Weibull distribution. The reference [9], proposed a three-parameter modified Weibull extension with a bathtub shaped hazard function. The reference [10] also introduced a modified version of Weibull distribution having increasing and a bathtub shaped hazard function.

The most popular lifetime distributions including the exponential, Weibull, gamma, Rayleigh, Pareto and Gompert have monotonic hazard rate functions and many others distributions have non-monotonic hazard rate functions. However, certain lifetime data (for example, human mortality, machine life cycles and data from some biological and medical studies) require non-monotonic shapes like the bathtub shape, the unimodal (upside-down bathtub) or modified unimodal shape and many others need monotonic shapes. For many years, using different techniques, many researchers have developed various modified forms of the parent distribution to achieve monotonic as well as non-monotonic shapes. The BIII distribution has monotonic decreasing and unimodal hazard rate function but due to modification NMBIII has monotonic decreasing, increasing, unimodal, bathtub and approximately constant hazard rate shapes.

Figure 1 represents the different shapes of the proposed model i.e. bimodal, reversed-J, right skewed, approximate left skewed and symmetrical shapes for different parameter values. Figure 2 reflects the different shapes of hazard function, which are increasing, decreasing, bathtub, upside-down bathtub and nearly constant for different parameter values.
The summary of the paper is as follows. In Section 1, a new model is defined and in Section 2 expansion of the NMBIII density, rth moment, sth incomplete moment, moment generating function, skewness and kurtosis, mode and order statistics are given, stochastic ordering, density and distribution function of ith order statistic of NMBIII are studied, characterization through lower record values are also included. In Section 3, model parameters are estimated by maximum likelihood method and the Fisher information matrix is derived. In Section 4, middle censoring, simulation for middle censoring is given. In Section 5, four real data sets are analyzed for illustration.

The commutative distribution function (cdf) and probability density function (pdf) of BIII distribution are given below:

\[ F(x; c, k) = (1 + x^{-c})^{-k}, \quad x > 0, \quad \text{with } c, k > 0, \quad \text{(1)} \]

and

\[ f(x; c, k) = c k x^{-c-1} (1 + x^{-c})^{-k-1}. \quad \text{(2)} \]

The reference [11], re-parameterized BIII distribution and named it modified BIII (MBIII), its CDF is given by: \[ F(x) = (1 + \mu x^{-c})^{-\frac{k}{\mu}}. \]

2. MATERIAL AND METHODS

2.1 The New Modified BIII Model

The modified Weibull (MW) distribution of [12] multiplies the Weibull cumulative hazard function \( \alpha x^\beta \) by \( e^{\lambda x} \),

\[ F(x) = \left( 1 + e^{-\alpha x e^{\lambda x}} \right)^{\theta} \quad \text{(3)} \]

which was later generalized to exponentiated form by [13]. Defining a new modified BIII (NMBIII) distribution by modifying equation (1). The cdf and pdf of NMBIII are given below:

\[ F(x; c, k, \lambda) = \left( 1 + x^{-c} e^{-\lambda x} \right)^{-k}, \quad \text{(4)} \]

and

\[ f(x; c, k, \lambda) = \frac{k(1 + \lambda)}{x e^{\lambda x}} \left( 1 + x^{-c} e^{-\lambda x} \right)^{-k-1}. \quad \text{(5)} \]

The corresponding survival and hazard rate functions are given below

\[ S(x; c, k, \lambda) = 1 - \left( 1 + x^{-c} e^{-\lambda x} \right)^{-k} \]

And

\[ h(x; c, k) = \frac{k(1 + \lambda)}{x e^{\lambda x} \left( 1 + x^{-c} e^{-\lambda x} \right)^{-k-1}} \left( 1 + x^{-c} e^{-\lambda x} \right)^{-k}, \quad x > 0, \quad \text{with } c, k, \lambda > 0. \]
If a new random variable $y$ is defined as $y = \frac{1}{x}$ in equation (4), then obtained the following model named as modified Burr XII distribution with cdf and pdf, respectively, as under

$$G(y) = 1 - \left(1 + \frac{yc}{e^{y/\lambda}}\right)^{-k},$$  \hspace{1cm} (6)

and

$$g(y) = \frac{kyc^{k-1}(c+2)}{e^{y/\lambda}} \left(1 + \frac{yc}{e^{y/\lambda}}\right)^{-k-1}, \hspace{0.5cm} y > 0, \hspace{0.5cm} \text{with} \hspace{0.5cm} c, k, \lambda > 0.$$

In our best knowledge equation (4) and equation (6) are first modification of Burr III distribution and Burr XII distribution respectively. The proposed distribution is more flexible and tractable than its parent BIII distribution and as well as MBIII distributions (See in Table 1).

**Table 1: Sub models of NMBIII distributions**

| Model   | $\lambda$ | $c$ | $k$ | $G(x)$                        | Reference |
|---------|-----------|-----|-----|-------------------------------|-----------|
| Bur III | 0         | -   | -   | $(1 + x^{-c})^{-k}$          | Standard  |
### 2.2. Some Properties of The NMBIII Distribution

In this section, some significant properties of the NMBIII distribution such as rth moment, sth incomplete moment, moment generating function, skewness, kurtosis, mode and order statistics are discussed.

#### Useful Expansion

The generalize binomial theorem or power series is given by:

\[
(1 + z)^{-b-1} = \sum_{i=0}^{\infty} \binom{b + i}{i} (-1)^i z^i .
\]  

Using series expansion in (7), equation (5) becomes

\[
f(x) = \sum_{i=0}^{\infty} \binom{k + i}{i} (-1)^i \frac{k(\lambda + \frac{x}{\gamma})}{x^{c(i+1)}} e^{-\lambda x(i+1)} .
\]  

This expression can be used to obtain the following properties of the NMBIII distribution.

#### Moments

The rth moment of NMBIII distribution is given by:

\[
m'_r = E(X^r) = \int_0^\infty x^r f(x)dx
\]

\[
= \sum_{i=0}^{\infty} \binom{k + i}{i} (-1)^i \int_0^\infty x^{r-c(i+1)} \left(\lambda + \frac{x}{\gamma}\right) e^{-\lambda x(i+1)} dx
\]

\[
= \lambda \sum_{i=0}^{\infty} a_i \int_0^\infty x^{r-c(i+1)} e^{-\lambda x(i+1)} dx + c \sum_{i=0}^{\infty} a_i \int_0^\infty x^{r-c(i+1)-1} e^{-\lambda x(i+1)} dx
\]

\[
= \lambda \sum_{i=0}^{\infty} a_i \Gamma[r - c(i + 1)] \left[\frac{1}{\lambda(i + 1)}\right]^{r-c(i+1)} + c \sum_{i=0}^{\infty} a_i \Gamma[r - c(i + 1) - 1] \left[\frac{1}{\lambda(i + 1)}\right]^{r-c(i+1)-1}
\]

\[
= \lambda \sum_{i=0}^{\infty} a_i \frac{\gamma^{r-c(i+1)-1}}{\lambda(i + 1)^{r-c(i+1)}} \left(\frac{1}{\lambda(i + 1)} + c[r - c(i + 1)] - 1\right).
\]  

\[(9)\]
where \( a_i = \binom{k + i}{i} (-1)^i \) and \( \Gamma(a) b^a = \int_0^\infty x^{a-1} e^{-x} dx \) is a gamma function.

Remark 1. By submitting \( r = 1 \) in equation (9), one can find mean of the NMBIII distribution. The definition of \( s \)th incomplete moment is given by

\[
T'_s(x) = \int_0^x x^s f(x) \, dx.
\]

The \( s \)th incomplete moment of NMBIII distribution is

\[
T'_s(x) = \lambda \sum_{i=0}^\infty a_i \gamma \left( r - c(i + 1) - 1, \frac{x}{\lambda(i+1)} \right) \left( \frac{1}{\lambda(i+1)} \right)^{r-c(i+1)-1} + c \sum_{i=0}^\infty a_i \gamma \left( r - c(i + 1), \frac{x}{\lambda(i+1)} \right) \left( \frac{1}{\lambda(i+1)} \right)^{r-c(i+1)},
\]

(10)

The application of incomplete moment refers to the mean deviations, Bonferroni and Lorenz curves, these curves are usefully economics reliability, demography, insurance and medicine. (See [15])

**Moment Generating Function**

The moment generating function of NMBIII distribution is given by:

\[
M_0(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x) \, dx
\]

\[
= \sum_{i=0}^\infty a_i \int_0^\infty x^{-c(i+1)} \left( \frac{\lambda + c}{\lambda} \right) e^{[t-\lambda(i+1)]x} dx
\]

\[
= \sum_{i=0}^\infty a_i \int_0^\infty x^{-c(i+1)} \left( \frac{\lambda + c}{\lambda} \right) e^{[t-\lambda(i+1)]x} dx + c \int_0^\infty x^{-c(i+1)-1} e^{[t-\lambda(i+1)]x} dx
\]

\[
= \sum_{i=0}^\infty a_i \left( \frac{\lambda - c}{\lambda(i+1)-c} \right)^{1-c(i+1)} + c \frac{r(-c(i+1))}{\lambda(i+1)-c}^{1-c(i+1)}.
\]

(11)

The skewness and kurtosis of the NMBIII distribution can be obtained numerically by the following expression.

\[
\alpha = \frac{m_3 - 3m_2m_1^2 + m_1^3}{\left[m_2 - (m_1)^2\right]^{3/2}} \quad \text{and} \quad \beta = \frac{m_4 - 4m_3m_2 + 6m_2^2m_1^2 + 3m_1^4}{\left[m_2 - (m_1)^2\right]^2},
\]

where \( m_r \) is the \( r \)th moment which can be obtained from equation (9). Table 2 shows some moments, skewness and kurtosis for the NMBIII distribution for selected values of the parameters. Remark 2. The mode of the NMBIII distribution can be obtained as follows:

Taking log of the equation (5), one obtains

\[
\log f(x) = \log k + \log \left( \lambda + c \right) - c \log x - \lambda x - (k + 1) \log \left( 1 + x^{-c} e^{-\lambda x} \right).
\]

Taking derivative with respect to \( x \), getting

\[
\frac{d}{dx} \log f(x) = \frac{1}{x^2 \lambda + 2} - \frac{c}{x} - \lambda + (k + 1) \frac{x^{-c} e^{-\lambda x} (\lambda + c)}{1 + x^{-c} e^{-\lambda x}}.
\]
by setting the above expression equal to zero and solving for $x$ one can find the mode.

**Table 2:** The numerical values of the first four moments ($m_r'$, $r = 1, 2, 3, 4$), skewness ($\alpha$) and kurtosis ($\beta$) of the NMBIII for some parameter values

| $(c, k, \lambda)$ | $m_1'$ | $m_2'$ | $m_3'$ | $m_4'$ | $\alpha$ | $\beta$ |
|-------------------|--------|--------|--------|--------|----------|--------|
| 0.5, 0.5, 0.5     | 0.6754 | 2.0695 | 10.4250| 72.6365| 3.3418   | 20.5484|
| 1.5, 0.5, 0.5     | 0.6662 | 1.0760 | 3.1293 | 14.2983| 3.1239   | 27.1548|
| 1.5, 1.5, 0.5     | 1.2849 | 2.6939 | 8.7612 | 41.8399| 2.4599   | 23.6830|
| 1.5, 1.5, 1.5     | 0.8024 | 0.8745 | 1.2564 | 2.3394 | 1.6650   | 70.6890|
| 2.0, 0.5, 0.5     | 0.6814 | 0.9031 | 2.0319 | 7.3171 | 2.8155   | 31.7670|
| 2.0, 2.0, 0.5     | 1.3695 | 2.6073 | 7.0943 | 27.8595| 2.4280   | 39.1836|
| 2.0, 2.0, 2.0     | 0.8041 | 0.7682 | 0.8775 | 1.2098 | 1.5101   | 226.2743|

**Order Statistics**

The density function $f_{i:n}(x)$ of the $i$-th order statistic, for $i = 1, \ldots, n$, from i.i.d random variables $X_1, \ldots, X_n$ following MBIII distribution is simply given by:

$$F_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F(x)^{i+j}.$$  \hfill (12)

The corresponding pdf is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x)F(x)^{i+j-1}.$$  \hfill (13)

Using the pdf and cdf of NMBIII in equations (12) and (13), obtaining

$$F_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left[1 + x^{-c} e^{-\lambda x}\right]^{-k(j+i)}. \hfill (14)$$

Using series expansion in equation (14),

$$F_{i:n}(x) = \sum_{j=0}^{n-i} b_j \sum_{l=0}^{\infty} \binom{k(j+i)+l}{l} (-1)^l x^{-cl} e^{-\lambda l x},$$

where $b_j = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \binom{-1}{j}\binom{j+i}{j+i} x^{-cl} e^{-\lambda l x}$. Similarly, following the above algebra, the following result is obtained.

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{l=0}^{\infty} a_j \binom{j+i+1}{l} (-1)^l x^{-c(l+1)} \left(\lambda + \frac{c}{x}\right) e^{-\lambda (l+1) x},$$

where $a_j = \frac{n!}{(i-1)!(n-i)!} \binom{n-i}{j} (-1)^j$.

**Stochastic Ordering**

The concept of stochastic ordering is frequently used to show the ordering mechanism in life time distributions. For more detail about stochastic ordering reader is referred to as [16]. A random variable is
said to be stochastically greater \((X \leq_{st} Y)\) than \(Y\) if \(F_X(x) \leq F_Y(x)\) for all \(x\). In the similar way, \(X\) is said to be stochastically greater \((X \leq_{st} Y)\) than \(Y\) in the

1. **Stochastic order** \((X \leq_{st} Y)\) if \(F_X(x) \geq F_Y(x)\) for all \(x\).

2. **Hazard rate order** \((X \leq_{hr} Y)\) if \(h_X(x) \geq h_Y(x)\) for all \(x\).

3. **Mean residual order** \((X \leq_{mrl} Y)\) if \(m_X(x) \geq m_Y(x)\) for all \(x\).

4. **Likelihood ratio order** \((X \leq_{lr} Y)\) if \(f_X(x) \geq f_Y(x)\) for all \(x\).

5. **Reversed hazard rate order** \((X \leq_{rhr} Y)\) if \(\frac{f_X(x)}{F_Y(x)}\) is decreasing for all \(x\).

The stochastic orders defined above are related to each other, has following implications.

\[
X \leq_{rhr} Y \iff X \leq_{lr} Y \iff X \leq_{hr} Y \iff X \leq_{st} Y \iff X \leq_{mrl} Y.
\]

Let \(X_1 \sim NMBIII(c_1, k_1, \lambda_1)\) and \(X_2 \sim NMBIII(c_2, k_2, \lambda_2)\). Then according to the definition of likelihood ratio ordering \(f(x)\)

\[
f(x) = \frac{k_1 \left(\lambda_1 + \frac{c_1}{x} \right)}{x^{c_1} e^{\lambda_1 x}} \left(1 + x^{-c_1} e^{-\lambda_1 x}\right)^{-k_1 - 1},
\]

\[
g(x) = \frac{k_2 \left(\lambda_2 + \frac{c_2}{x} \right)}{x^{c_2} e^{\lambda_2 x}} \left(1 + x^{-c_2} e^{-\lambda_2 x}\right)^{-k_2 - 1},
\]

And

\[
\frac{d (\log f(x))}{dx} = \frac{c_1}{x (\lambda_1 + \frac{c_1}{x})} - \frac{c_2}{x (\lambda_2 + \frac{c_2}{x})} + \frac{c_2 - c_1}{x} + (\lambda_2 - \lambda_1) + (k_2 + 1) \frac{x^{-c_2} e^{-\lambda_2 x} - c_2 - \lambda_2}{1 + x^{-c_2} e^{-\lambda_2 x}} - (k_1 + 1) \frac{x^{-c_1} e^{-\lambda_1 x} - c_1 - \lambda_1}{1 + x^{-c_1} e^{-\lambda_1 x}}.
\]

If \(\lambda_2 < \lambda_1\), then \(\frac{d (\log f(x))}{dx} < 0\), which implies that \(X \leq_{lr} Y\). The remaining statements follow form the equation \(X \leq_{rhr} Y \iff X \leq_{lr} Y \iff X \leq_{hr} Y \iff X \leq_{st} Y \iff X \leq_{mrl} Y\) becomes true.

### 2.3. Maximum Likelihood Estimation

In this section, the maximum likelihood method to estimate the unknown parameters of the proposed model from complete samples only is discussed. Let \(x_1, x_2, \ldots, x_n\) be a random sample of size \(n\) from the NMBIII family given in equation (5) distribution. The log-likelihood function for the vector of parameter \(\Theta = (c, k, \lambda)\) can be expressed as.

\[
l(\Theta) = n \log k - c \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log \left(\lambda + \frac{c}{x_i}\right) - (k + 1) \sum_{i=1}^{n} \log \left(1 + x^{-c} e^{-\lambda x_i}\right).
\]
Taking derivative with respective to $\lambda$, $c$, $k$, getting

$$U_k = \frac{\partial l(\theta)}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n} \log(1 + x^{-c}e^{-\lambda x_i})$$

$$U_\lambda = \frac{\partial l(\theta)}{\partial \lambda} = -\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \left(\frac{\lambda + c}{x_i}\right)^{-1} + (k + 1) \sum_{i=1}^{n} \left(\frac{x_i^{-c}e^{-\lambda x_i}}{1 + x^{-c}e^{-\lambda x_i}}\right)$$

$$U_c = \frac{\partial l(\theta)}{\partial c} = -\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \frac{1}{x_i} \left(\frac{\lambda + c}{x_i}\right)^{-1} + (k + 1) \sum_{i=1}^{n} \left(\frac{x_i^{-c}e^{-\lambda x_i} \log x_i}{1 + x^{-c}e^{-\lambda x_i}}\right).$$

Setting $U_k$, $U_\lambda$ and $U_c$ equal to zero and solving these equations simultaneously yields the maximum likelihood estimates.

The observed information matrix for the parameter vector is given by

$$\begin{pmatrix}
U_{kk} & U_{k\lambda} & U_{k, c} \\
U_{k\lambda} & U_{\lambda\lambda} & U_{\lambda, c} \\
U_{k, c} & U_{\lambda, c} & U_{c, c}
\end{pmatrix}$$

whose elements are given below

$$U_{kk} = -\frac{n}{k^2}$$

$$U_{k\lambda} = -\sum_{i=1}^{n} \left(\frac{x_i^{-c}e^{-\lambda x_i}}{1 + x^{-c}e^{-\lambda x_i}}\right)$$

$$U_{k, c} = -\sum_{i=1}^{n} \left(\frac{x_i^{-c}e^{-\lambda x_i} \log x_i}{1 + x^{-c}e^{-\lambda x_i}}\right)$$

$$U_{\lambda\lambda} = -\sum_{i=1}^{n} \left(\frac{\lambda + c}{x_i}\right)^{-2} - (k + 1) \sum_{i=1}^{n} x_i^2 \frac{x_i^{-c}e^{-\lambda x_i}}{(1 + x^{-c}e^{-\lambda x_i})^2}$$

$$U_{\lambda, c} = -\sum_{i=1}^{n} \frac{1}{x_i} \left(\frac{\lambda + c}{x_i}\right)^{-2} - (k + 1) \sum_{i=1}^{n} \frac{x_i^{-c}e^{-\lambda x_i} \log x_i}{(1 + x^{-c}e^{-\lambda x_i})^2}$$

$$U_{c, c} = -\sum_{i=1}^{n} \frac{1}{x_i^2} \left(\frac{\lambda + c}{x_i}\right)^{-2} - (k + 1) \sum_{i=1}^{n} \frac{x_i^{-c}e^{-\lambda x_i} (\log x_i)^2}{(1 + x^{-c}e^{-\lambda x_i})^2}.$$ 

Equating $U_k$, $U_\lambda$ and $U_c$ with zero and solving these equations simultaneously yield the maximum likelihood estimates of vector $\Theta$ that can be obtained by solving the non-linear equations above numerically for $\lambda$, $c$ and $k$ using statistical softwares such as R and Mathematica packages.

2.4. Middle- Censoring
The middle-censoring scheme is a non-parametric general censoring mechanism was proposed by [18], where other censoring schemes can be obtained as special cases of this middle-censoring scheme [19]. For $n$ identically distributed lifetimes $T_1, \ldots, T_n$ with a random censoring interval $(L_i, R_i)$ at the $i$th item with some unknown bivariate distribution. Then, the exact value of $T_i$ is observable only if $T_i \notin (L_i, R_i)$, otherwise the interval $(L_i, R_i)$ is observed. Middle-censoring has been applied to exponential and Burr-XII lifetime distributions [18, 20]. Furthermore, it was extended to parametric models with covariates, [21] and then its robustness was investigated by [22].

In this section, it is analyzed that the $X_i'$s NMBIII lifetime data when they are middle-censored. Assume that $T_1, \ldots, T_n$ are i.i.d NMBIII $(c, \lambda, k)$ random variable and let $Z_i = R_i - L_i$, $i = 1, \ldots, n$ be another random variable defining the length of the censoring interval with exponential distribution with mean $\gamma^{-1}$, where the left-censoring point for each individual $L_i$ is assumed to be also an exponential random variable with mean $\theta^{-1}$. Moreover, the $T_i'$s, $L_i'$s and $Z_i'$s are all independent of each others and the observed data, $X_i'$s are given by

$$X_i = \begin{cases} T_i & \text{if } T_i \notin (L_i, R_i) \\ (L_i, R_i) & \text{otherwise} \end{cases}$$

**Estimation and Simulation**

For $n$ randomly selected units from NMBIII $(c, \lambda, k)$ population, where $c$, $\lambda$ and $k$ are unknown, and were tested under middle-censoring scheme. In such setting, there are $n_1 > 0$ uncensored observations and $n_2 > 0$ censored observations. Then, by re-ordering the observed data into the uncensored and censored observations. Therefore, having the following data

$$\{T_1, \ldots, T_{n_1}, (L_{n_1+1}, R_{n_1+2}), \ldots, (L_{n_1+n_2}, R_{n_1+n_2})\},$$

where $n_1 + n_2 = n$.

The likelihood function of the observed data is given by

$$L(c, \lambda, k|x) = \omega(k)^{n_1} \prod_{i=1}^{n_1} \left( \lambda + \frac{c}{x_i} \right) \prod_{i=1}^{n_1} \left( x_i^{-c} e^{-\lambda x_i} \right) \prod_{i=1}^{n_1} \left( 1 + x_i^{-c} e^{-\lambda x_i} \right)^{k-1} \prod_{i=n_1+1}^{n_1+n_2} \left[ \left( 1 + r_i^{-c} e^{-\lambda r_i} \right)^{-k} - \left( 1 + l_i^{-c} e^{-\lambda l_i} \right)^{-k} \right],$$

where $\omega$ is a normalizing constant depending on $\gamma$ and $\theta$ and the estimation of them is not of interest, thus it is left as a constant. The log-likelihood function is given by

$$l(c, \lambda, k|x) = \log \omega + n_1 \log k + \sum_{i=1}^{n_1} \log \left( \lambda + \frac{c}{x_i} \right) + \sum_{i=1}^{n_1} \left( x_i^{-c} e^{-\lambda x_i} \right) - (k + 1) \sum_{i=1}^{n_1} \left( 1 + x_i^{-c} e^{-\lambda x_i} \right) + \sum_{i=n_1+1}^{n_1+n_2} \log \left[ \left( 1 + r_i^{-c} e^{-\lambda r_i} \right)^{-k} - \left( 1 + l_i^{-c} e^{-\lambda l_i} \right)^{-k} \right].$$

The maximum likelihood estimation (MLE) of $c$, $\lambda$ and $k$ denoted by $c_M$, $\lambda_M$ and $k_M$ can be derived by solving the following equations:

$$\frac{\partial l(c, \lambda, k|x)}{\partial c} = \sum_{i=1}^{n_1} \left( \lambda x_i + c \right)^{-1} - \sum_{i=1}^{n_1} \log x_i + (k + 1) \sum_{i=1}^{n_1} \frac{x_i^{-c} e^{-\lambda x_i} \log x_i}{1 + x_i^{-c} e^{-\lambda x_i}}$$
It is obvious that the MLE of \( \theta \) and \( \lambda \) cannot be solved explicitly. Therefore, the solutions could be obtained by using Newton-Raphson method, or numerically by using the solve systems of nonlinear equations "nleqslv" package in R.

Since the MLE is asymptotically normal, thus the approximate confidence intervals for the parameters \( \theta \) and \( \lambda \) can be computed as follows:

\[
\hat{\theta}_M = \lambda \pm z_{\alpha} \sqrt{\frac{\hat{\sigma}_\theta^2}{2}}, \quad \hat{\lambda}_M = \lambda \pm z_{\alpha} \sqrt{\frac{\hat{\sigma}_\lambda^2}{2}}
\]

and \( z_{\alpha} \) the value of the standard normal curve and \( \alpha \) is the level of significance. 5.2 Simulation Result

Monte Carlo simulation studies are conducted to assess the finite sample behavior of the MLEs of the parameters \( c, k \) and \( \lambda \) based on two settings, the first is the random variable were generated from the NMBIII distribution, while the other is considered the case when the NMBIII lifetime data were middle-censored. The random samples for both settings were generated from distribution NMBIII\((c, k, \lambda)\) based on accept-reject approach. Without loss of generality, random samples with five different sizes viz n=10, 30, 50, 70 and 100 from NMBIII\((1,2,0.5)\), NMBIII\((0.5,2,0.5)\) and NMBIII\((2,2,2)\) distributions. The middle censoring settings considered three combinations of the censoring schemes \((y^{-1}, \theta^{-1}) = (0.25,0.25), (1,0.75)\) and \((1.25,0.5)\).

The results are obtained from 1000 Monte Carlo replications from simulations carried out using the software R, and the average estimates, the mean squared error (MSE) are obtained and reported in Table 3. Results in Table 3 shows that the ML estimates for both settings behave almost similarly. In general, there is a decreasing function between the sample size and the mean squared error, which verifies the consistency property of the derived estimators. The average estimates are insignificantly affected by the censoring status.

### Table 3: Average MLE estimates and the corresponding MSE (within brackets)

| Distribution | n   | \( c \) | \( k \) | \( \lambda \) | \( c \) | \( k \) | \( \lambda \) | \( c \) | \( k \) | \( \lambda \) | \( c \) | \( k \) | \( \lambda \) |
|--------------|-----|--------|--------|------------|--------|--------|------------|--------|--------|------------|--------|--------|------------|
| NMBIII \( c = 0.5 \)| 10  | 1.114  | 2.079  | 0.397      | 1.123  | 2.233  | 0.447      | 1.087  | 2.130  | 0.524      | 1.196  | 2.088  | 0.561      |
|               |     | (0.130)| (0.102)| (0.122)   | (0.141)| (0.163)| (0.096)   | (0.111)| (0.159)| (0.108)   | (0.121)| (0.099)| (0.125)   |
| NMBIII \( k = 2, \lambda = 0.5 \)| 30  | 1.039  | 2.036  | 0.464      | 1.082  | 2.170  | 0.452      | 1.072  | 2.080  | 0.519      | 1.127  | 2.080  | 0.547      |
|               |     | (0.034)| (0.039)| (0.080)   | (0.095)| (0.072)| (0.043)   | (0.036)| (0.082)| (0.046)   | (0.052)| (0.093)| (0.057)   |
|               | 50  | 1.036  | 2.032  | 0.484      | 1.071  | 2.096  | 0.536      | 1.066  | 2.071  | 0.508      | 1.103  | 2.022  | 0.529      |
|               |     | (0.031)| (0.029)| (0.053)   | (0.033)| (0.031)| (0.032)   | (0.028)| (0.032)| (0.028)   | (0.022)| (0.025)| (0.027)   |
|               | 70  | 1.015  | 1.984  | 0.511      | 1.035  | 2.018  | 0.510      | 1.042  | 2.053  | 0.496      | 1.042  | 1.985  | 0.476      |
In this section, application of the NMBIII model to censored data set by comparing NMBIII with MBIII and BIII distributions are provided. Noting that goodness-of-fit statistics computations have not been developed for censored data but the quality of fit can be checked by Akaike and Bayesian information criteria (AIC and BIC), see [26]. The considered data are the times to failure of 20 aluminum reduction cells. Failure times, in units of 1000, quoted in [27].

3. RESULTS AND DISCUSSION

3.1. Applications

This section provides four applications, three of them are for complete (uncensored) data sets and the other for censored data set, to show how the NMBIII distribution can be applied in practice. NMBIII distribution is compare to MBIII and BIII distributions. In these applications, the model parameters are estimated by the method of maximum likelihood. The Akaike information criterion (AIC), Bayesian information criterion (BIC), A* (Anderson Darling), W* (Cramer-von Mises) are computed to compare the fitted models. In general, the smaller the values of these statistics, better the fit to the data. The plots of the fitted PDFs and CDFs of some distributions are displayed for visual comparison. The required computations are carried out in the R software.

### Uncensored/Complete Data Sets

The first data set consists of 119 observations on fracture toughness of Alumina (Al2O3) (in the units of MPa m1/2). This data was studied by [23]. The second data set refers to the 50 observations with hole and sheet thickness are 12 mm and 3.15 mm reported by [24]. The third data set was first analyzed by [25], and it represents the survival time, in weeks, of 33 patients suffering from Acute Myelogenous Leukemia. Tables 4–6 lists the MLEs, standard errors, AIC, BIC, A* and W* of the model for the data sets 1–3. The results in Table 4–6 indicates that the NMBIII model provides the best fit as compared to the other models. Figures 3–5 also support the results of Table 4–6.

### Censored Data Set

In this section, application of the NMBIII model to censored data set by comparing NMBIII with MBIII and BIII distributions are provided. Noting that goodness-of-fit statistics computations have not been developed for censored data but the quality of fit can be checked by Akaike and Bayesian information criteria (AIC and BIC), see [26]. The considered data are the times to failure of 20 aluminum reduction cells. Failure times, in units of 1000, quoted in [27].

Consider a data set \( D = (x, r) \), where \( x = (x_1, x_2, \ldots, x_n)^T \) are the observed failure times and \( r_i = (r_{i1}, r_{i2}, \ldots, r_{in})^T \) are the censored failure times. The \( r_i \) is equal to 1 if a failure is observed and 0 otherwise.

| Data Set | MBIII | NMBIII | BIII |
|----------|-------|--------|------|
|          |       |        |      |
|   10     | 0.621 | 0.012  | 0.613|
|          | 2.074 | 0.040  | 2.057|
|          | 0.427 | 0.151  | 0.464|
|          | 0.582 | 0.127  | 0.531|
|          | 2.325 | 0.086  | 2.264|
|          | 0.522 | 0.063  | 0.513|
|          | 0.534 | 0.056  | 0.529|
|          | 2.135 | 0.084  | 2.104|
|          | 0.524 | 0.088  | 0.516|
|          | 1.196 | 0.080  | 1.127|
|          | 2.098 | 0.105  | 2.087|
|          | 0.530 | 0.096  | 0.521|
| 30       | 0.538 | 0.026  | 0.611|
|          | 2.101 | 0.044  | 2.057|
|          | 0.384 | 0.094  | 0.464|
|          | 0.519 | 0.067  | 0.531|
|          | 2.125 | 0.032  | 2.264|
|          | 0.489 | 0.019  | 0.513|
|          | 0.518 | 0.064  | 0.529|
|          | 2.014 | 0.037  | 2.104|
|          | 0.505 | 0.031  | 1.103|
|          | 2.054 | 0.016  | 2.087|
|          | 0.518 | 0.031  | 0.521|
| 50       | 0.501 | 0.002  | 0.492|
|          | 1.928 | 0.001  | 2.003|
|          | 0.507 | 0.027  | 0.504|
|          | 0.492 | 0.006  | 2.003|
|          | 2.044 | 0.011  | 1.923|
|          | 0.499 | 0.010  | 0.495|
| 70       | 0.501 | 0.001  | 2.212|
|          | 1.928 | 0.027  | 2.452|
|          | 0.492 | 0.006  | 2.517|
|          | 0.492 | 0.007  | 2.298|
|          | 0.232 | 0.006  | 2.571|
|          | 0.280 | 0.026  | 2.322|
|          | 2.371 | 0.088  | 2.331|
|          | 1.102 | 0.082  | 2.493|
|          | 2.508 | 0.084  | 2.256|
| 100      | 0.501 | 0.011  | 2.144|
|          | 2.144 | 0.012  | 2.452|
|          | 0.492 | 0.012  | 2.517|
|          | 0.492 | 0.010  | 2.298|
|          | 0.232 | 0.006  | 2.571|
|          | 0.280 | 0.027  | 2.322|
|          | 2.371 | 0.088  | 2.331|
|          | 1.102 | 0.082  | 2.493|
|          | 2.508 | 0.084  | 2.256|

### Table 6

| Experiment | MBIII | NMBIII | BIII |
|------------|-------|--------|------|
|           |       |        |      |
|           | 1.001 | 0.016  | 1.001|
|           | 1.991 | 0.015  | 1.991|
|           | 0.502 | 0.019  | 0.502|
|           | 1.019 | 0.017  | 1.019|
|           | 1.998 | 0.021  | 1.998|
|           | 0.495 | 0.022  | 0.495|
|           | 0.980 | 0.023  | 0.980|
|           | 2.020 | 0.024  | 2.020|
|           | 0.498 | 0.026  | 0.498|
|           | 0.981 | 0.021  | 0.981|
|           | 1.907 | 0.016  | 1.907|
|           | 0.491 | 0.017  | 0.491|
|           | 0.013  | 0.013  |      |
Suppose that the data are independently and identically distributed and come from a distribution with PDF given by equation (5). Let \( \Theta = (c, k, \lambda)^T \) denote the vector of parameters.

The likelihood of \( \Theta \) can be written as

\[
l(D; \Theta) = \prod_{i=1}^{n}[f(x_i; \Theta)]^{r_i}[1 - F(x_i; \Theta)]^{1-r_i},
\]

then the log likelihood reduces

\[
l(D; \Theta) = r_i \sum_{i=1}^{n} \log[f(x_i; \Theta)] + (1 - r_i) \sum_{i=1}^{n} [1 - F(x_i; \Theta)]. \tag{20}
\]

Using the equations (4) and (5) into (20) getting

\[
l(D; \Theta) = r_i \sum_{i=1}^{n} \left[ \log k + \log \left( \lambda + \frac{c}{x_i} \right) - c \log x_i + \lambda x_i - (k + 1) \log (1 + x^{-c} e^{-\lambda x_i}) \right] + (1 - r_i) \sum_{i=1}^{n} \log \left[ 1 - \left[ 1 + x^{-c} e^{-\lambda x_i} \right]^{-k} \right].
\]

The log likelihood function can be maximized numerically to obtain the MLEs. There are various routines available for numerical maximization of \( l \), the routine optimum in the R software is used. Table 7 lists the MLEs, standard errors, AIC and BIC of the model for the censored data sets. The results in Table 7 indicate that the NMBIII model provides the best fit as compared to the other models. Figure 6 also supports the results of Table 7.

For further illustration of the middle censoring, reconsidering the second real data set “hole and sheet thickness” which was analyzed in the previous subsection, where NMBIII model cannot be rejected to fit the data. The hole and sheet thickness data were artificially middle-censored by considering that the left end was an exponential random variable with mean 0.05 and the width was exponential with mean 0.1. Then the data were rearranged and given below:

Data set: 0.02, 0.02, 0.04, 0.04, 0.04, 0.06, 0.06, 0.06, 0.06, 0.08, 0.12, 0.12, 0.14, 0.14, 0.14, 0.14, 0.14, 0.14, 0.16, 0.16, 0.16, 0.16, 0.26, 0.32, 0.22, 0.24, 0.24, 0.18, 0.18, 0.18, 0.18, 0.18, 0.22, 0.22, 0.22, 0.22, 0.22, 0.24, 0.24, 0.32, 0.26, (0.01,0.21), (0.02,0.25), (0.05,0.31), (0.07, 0.33), (0.09, 0.41) There are five middle-censored observations that are listed at the end of the data set and with censoring percentage 10.0%. The MLE of parameters are \( a=2.615, k=0.294 \) and \( \lambda = 14.682 \).

| Model | Parameters | MLE    | Standard error | AIC    | BIC    | A*    | W*    |
|-------|------------|--------|----------------|--------|--------|-------|-------|
| NMBIII | c          | 2.5425 | 0.5067         | 362.1594 | 370.4968 | 1.8880 | 0.2959 |
|       | k          | 25.2429 | 5.1851         |        |        |       |       |
|       | \( \lambda \) | 1.7030 | 0.1793         |        |        |       |       |
| MBIII | c          | 1111.2300 | 461.8195 | 379.3803 | 387.7177 | 3.5150 | 0.5834 |
|       | k          | 4.94320 | 0.2806         |        |        |       |       |
|       | \( \mu \)  | 770.0503 | 398.9633 |        |        |       |       |
| BIII  | c          | 3.0579 | 0.1799         | 423.5352 | 429.0935 | 7.6579 | 1.3649 |
Table 5: MLE, Standard error, AIC, BIC A*, and W* for Data set 2

| Model | Parameters | MLE   | Standard error | AIC     | BIC     | A*     | W*     |
|-------|------------|-------|----------------|---------|---------|--------|--------|
| NMBIII| c          | 2.8018| 1.6204         | -106.3580| -100.6219| 0.5240 | 0.0897 |
|       | k          | 0.3168| 0.2191         |         |         |        |        |
|       | \(\lambda\) | 17.2742| 5.6051         |         |         |        |        |
| MBIII | c          | 0.0020| 0.0002         | -99.77807| -94.042 | 0.9880 | 0.1587 |
|       | k          | 3.4663| 0.2049         |         |         |        |        |
|       | \(\mu\)   | 0.0039| 0.0007         |         |         |        |        |
| BIII  | c          | 7.78772313| 26.5717     | -26.0265| -22.2024| 1.0562 | 0.1766 |
|       | k          | 0.0646| 0.2206         |         |         |        |        |

Table 6: MLE, Standard error, AIC, BIC A*, and W* for Data set 3.

| Model | Parameters | MLE   | Standard error | AIC     | BIC     | A*     | W*     |
|-------|------------|-------|----------------|---------|---------|--------|--------|
| NMBIII| c          | 0.5211| 0.1207         | 303.7034| 308.1006| 0.44047| 0.0643 |
|       | k          | 4.7342| 1.0650         |         |         |        |        |
|       | \(\lambda\) | 0.0123| 0.0053         |         |         |        |        |
| MBIII | c          | 153.5921| 319.6150     | 309.4653| 313.8625| 0.6719 | 0.0982 |
|       | k          | 1.4938| 0.4639         |         |         |        |        |
|       | \(\mu\)   | .2008 | 796.0167       |         |         |        |        |
| BIII  | c          | 0.7553| 0.0924         | 309.7136| 312.6450| 0.9192 | 0.1514 |
|       | k          | 5.7047| 1.2277         |         |         |        |        |

Table 7: MLE, Standard error, AIC, BIC and for Censored data set

| Model | Parameters | MLE   | Standard error | AIC     | BIC     |
|-------|------------|-------|----------------|---------|---------|
| NMBIII| c          | 0.5330| 1.4661         | 43.1694 | 47.15664|
|       | k          | 7.0572| 8.0104         |         |         |
|       | \(\lambda\) | 1.4156| 1.1435         |         |         |
| MBIII | c          | 4.1838| 2.7028         | 44.5785 | 47.9657 |
|       | k          | 6.1127| 12.4016        |         |         |
|       | \(\mu\)   | 8.8005| 28.2002        |         |         |
| BIII  | c          | 2.7027| 0.4920         | 45.9750 | 48.9663 |
|       | k          | 2.0785| 0.4649         |         |         |
Figure 3: The NMBIII, MBIII and BIII estimated pdf's superimposed on the histogram of first data (left panel); the estimated cdf's and empirical cdf (right panel)

Figure 4: The NMBIII, MBIII and BIII estimated pdf's superimposed on the histogram of second data (left panel); the estimated cdf's and empirical cdf (right panel)

Figure 5: The NMBIII, MBIII and BIII estimated pdf's superimposed on the histogram of third data (left panel); the estimated cdf's and empirical cdf (right panel)
Figure 6: The NMBIII, MBIII and BIII estimated cdf’s superimposed on the empirical cdf of censored data (left panel); the estimated survival functions and empirical survival functions (right panel)

4. CONCLUSION

In this manuscript, new modified form of Burr III (BIII) distribution has been introduced having increasing, decreasing, bathtub, upside-down bathtub and nearly constant hazard rate shapes. Some of its statistical properties such as, rth moment, sth incomplete moment, moment generating function, skewness, kurtosis, mode, ith order statistic and stochastic ordering have been derived. The maximum likelihood estimation is employed to estimate the model parameters. Middle-censoring is considered as a modern general scheme of censoring and characterization is also given through lower record values. The usefulness of this model is demonstrated through applications on complete and censored samples. Simulation study is also performed. To save the idea the archive version also uploaded on hal.archives-ouvertes.fr, [28].

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

Authors’ Contributions

Farrukh Jamal, gave the initial idea and proposed new model. Muhammad A. Nasir, investigated mathematical properties. Ali H. Abuzaid, add the Middle-Censoring part of the paper. Wali Khan Mashwani and M.H.Tahir mutually wrote and structured the manuscript, they also investigate numerical calculations applied to the dataset.

REFERENCES

[1] Burr, I.W., “Cumulative frequency functions”, The Annals of Mathematical Statistics, 13(2):215-32, (1942).
[2] Dagum, C., “New model of personal income-distribution-specification and estimation”, Economie appliquée, 30(3):413-37, (1977).
[3] Kleiber, C., Kotz, S., “Statistical size distributions in economics and actuarial sciences” John Wiley and Sons; (2003).
[4] Mielke, Jr, PW., “Another family of distributions for describing and analyzing precipitation data” Journal of Applied Meteorology, 12(2):275-80, (1973).
[5] Bebbington, M., Lai, CD, Zitikis R., “A flexible Weibull extension” Reliability Engineering & System Safety, 92(6):719-26, (2007).
[6] Zhang, T., Xie, M., “On the upper truncated Weibull distribution and its reliability implications” Reliability Engineering & System Safety, 96(1):194-200, (2011).
[7] Mudholkar, G.S., Srivastava, D.K., “Exponentiated Weibull family for analyzing bathtub failure-rate data” IEEE transactions on reliability, 42(2):299-302, (1993).
[8] Marshall, AW, Olkin I., “A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families” Biometrika, 84(3):641-52, (1997).
[9] Xie, M., Tang, Y., Goh TN., “A modified Weibull extension with bathtub-shaped failure rate function” Reliability Engineering & System Safety, 76(3):279-85, (2002).
[10] Almalki, SJ., Yuan, J., “A new modified Weibull distribution” Reliability Engineering & System Safety, 111:164-70, (2013).
[11] Ali, A., Hasnain, SA., Ahmad, M., “Modified Burr III Distribution, Properties and Applications” Pak. J. Statist, 31(6):697-708, (2015).
[12] Lai CD, Xie M, Murthy DN. A modified Weibull distribution. IEEE Transactions on reliability, (2003).
[13] Carrasco JM, Ortega EM, Cordeiro GM. A generalized modified Weibull distribution for lifetime modeling. Computational Statistics & Data Analysis, (2008).
[14] Aljouiee, A., Elbatal, I., Al-Mofleh, H., “A New Five-Parameter Lifetime Model: Theory and Applications” Pakistan Journal of Statistics and Operation Research, (2018).
[15] Johnson, N.L., Kotz, S., Balakrishnan, N., “Continuous Univariate Distributions, Volume 2” Wiley, (1995).
[16] Shaked, M., Shanthikumar, J.G., “Stochastic Orders and Their Applications” New York, Wiley. (1994).
[17] Gradshteyn, I.S., Ryzhik, I.M., “Table of Integrals, Series, and Products” Sixth ed. Academic Press, San Diego, (2007).
[18] Jammalamadaka, SR., Mangalam, V., “Nonparametric estimation for middle-censored data” Journal of nonparametric statistics,15(2):253-65, (2003).
[19] Iyer, SK., Jammalamadaka, SR., Kundu D., “Analysis of middle-censored data with exponential lifetime distributions” Journal of Statistical Planning and Inference, 138(11):3550-60, (2008).
[20] Abuzaid, AH., “The estimation of the Burr-XII parameters with middle-censored data” SpringerPlus, (2015).
[21] Bennett, NA., “Some contributions to middle-censoring” University of California, Santa Barbara, (2011).
[22] Abuzaid, AH., El-Qumsan M., El-Habil, AM., “On the robustness of right and middle censoring schemes in parametric survival models” Communications in Statistics-Simulation and Computation, (2017).
[23] Nadarajah, S., Kotz, S., “On the alternative to the Weibull function. Engineering fracture mechanics” (2007).
[24] Dasgupta, “R. On the distribution of burr with applications” Sankhya B, (2011).
[25] Feigl, P., Zelen, M., “Estimation of exponential survival probabilities with concomitant information” Biometrics, 826-38, (1965).
[26] Delignette-Muller, ML., Dutang, C., “fitdistrplus: An R package for fitting distributions” Journal of statistical software, 64(4):1-34, (2015).
[27] Lawless JF., “Statistical Models and Methods for Lifetime Data” John Wiley & Sons, Inc., 3:577, (2003).
[28] Jamal, Farrukh, Ali Abuzaid, Muhammad Nasir, Abdus Saboor, and Muhammad Khan. "NEW MODIFIED BURR III DISTRIBUTION, PROPERTIES AND APPLICATIONS." (2018). https://hal.archives-ouvertes.fr/hal-01902854