On Uniqueness of the Laplace Transform on Time Scales

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Abstract
After introducing the concept of null functions, we shall present a uniqueness result in the sense of the null functions for the Laplace transform on time scales with arbitrary graininess. The result can be regarded as a dynamic extension of the well-known Lerch’s theorem.
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1 Introduction
The Laplace transform is one of the fundamental representatives of integral transformations used in mathematical analysis. This transform is essentially bijective for the majority of practical uses. The Laplace transform has the useful property that many relationships and operations over the originals functions correspond to simpler relationships and operations over the image functions. The discrete analogue of the Laplace transform is called as Z-transform, which is also converts a sequence of real or complex numbers into a complex frequency-domain representation. This transform is also bijective.

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The calculus on time scales has been initiated by Hilger (see [1]) in order to unify the theories of continuous analysis and of discrete analysis. The Laplace transform on time scales was introduced by Hilger in [2], but in a form that tries to unify the (continuous) Laplace transform and the (discrete) Z-transform. For arbitrary time scales, the Laplace transform was introduced by and investigated by Bohner and Peterson in [3] (see also [4, Section 3.10]).

Let \( \sup T = \infty \), for locally \( \Delta \)-integrable function \( f : [s, \infty)_T \to \mathbb{C} \), i.e., \( \Delta \)-integrable over each compact interval of \([s, \infty)_T\), the Laplace transform is defined to be

\[
\mathcal{L}\{f\}(z) := \int_s^\infty f(\eta)e_{\Theta z}(\sigma(\eta), s)\Delta \eta \quad \text{for } z \in D,
\]

where \( D \) consists of such complex numbers for which the improper integral converges. In order to determine an explicit region of convergence, conditions on the class of the determining functions should be provided. This was done by Davis et al. in [5], where some restrictions were imposed on the graininess \( \mu \). In a recent paper [6], Bohner et al. removed the restriction on the graininess of the time scale and considered some fundamental properties of the Laplace transform on time scales. The readers may be referred to [7, 8, 9] for the basic properties of the usual Laplace transform. For other results about the Laplace transform on time scales, see [10, 11, 12, 13].

The uniqueness property of the Laplace transform and of the Z-transform are well-known (see [7, 8, 9]), which is a necessary tool for the inverse Laplace transform. To the best of our knowledge, nothing has been published up to now on the uniqueness of the Laplace transform on arbitrary time scales. In this paper, we shall provide a uniqueness result on time scales with arbitrary graininess for the Laplace transform, which reduces to the well-known the Lerch’s theorem in the continuous case. Our result on time scales with constant graininess (\( \mathbb{R} \) and \( \mathbb{Z} \)), gives a unified proof for the uniqueness of the Laplace transform (the usual Laplace transform and the Z-transform).

The paper is organized as follows: In Section 2, we present some results that are required in the proof of the main result. In Section 3, we state and prove our main results together with some necessary definitions. And in Section 4, as an appendix, we recall a short account concerning the time scale calculus.
2 Auxiliary Lemmas

We define the minimal graininess function \( \mu_* : T \to \mathbb{R}_0^+ \) by

\[
\mu_*(s) := \inf \{ \mu(t) : t \in [s, \infty)_T \} \quad \text{for } s \in T
\]

and the set of positively regressive constants \( \mathcal{R}_c^+(T, \mathbb{C}) \) by

\[
\mathcal{R}_c^+(T, \mathbb{C}) := \{ z \in \mathbb{C} : 1 + z\mu(t) > 0 \text{ for all } t \in T \}.
\]

For \( h \in \mathbb{R}_0^+ \) and \( \lambda \in \mathcal{R}_c^+(T, \mathbb{R}) \), we also define the set \( \mathbb{C}_h(\lambda) \) by

\[
\mathbb{C}_h(\lambda) := \{ z \in \mathbb{C} : \text{Re}(z) > \lambda \}.
\]

Now, we proceed this section with a result quoted from [6].

Lemma 1 (See [6, Theorem 3.4(iii)]). Let \( \sup T = \infty \), \( s \in T \), \( \lambda \in \mathcal{R}_c^+(T, \mathbb{R}) \) and \( z \in \mathbb{C}_{\mu_*(s)}(\lambda) \), then

\[
\lim_{t \to \infty} e_{\lambda \Theta z}(t, s) = 0.
\]

The inclusion \( \mathbb{R}^+ \subset \mathbb{C}_{\mu_*(s)}(0) \) for any \( s \in T \) yields the following corollary.

Corollary 1. Let \( \sup T = \infty \), \( s \in T \) and \( x \in \mathbb{R}^+ \), then

\[
\lim_{t \to \infty} e_{\Theta x}(t, s) = 0 \quad \text{and} \quad \lim_{t \to \infty} e_x(t, s) = \infty.
\]

Next, we present a result on asymptotic property of the time scale exponential.

Lemma 2. Let \( s, t \in T \) and \( \lambda \in \mathbb{R}^+ \), then

\[
\lim_{x \to \infty} \left( x^\lambda e_{\Theta x}(t, s) \right) = \begin{cases} 0, & t > s \\ \infty, & t \leq s. \end{cases}
\]

Proof. As we will be considering the limit as \( x \to \infty \), we may assume that \( x \in \mathbb{R}^+ \). First, we consider the case \( s, t \in T \) with \( t > s \). We may find \( n \in \mathbb{N} \) such that \( n > \lambda \). By the Taylor’s formula, we have

\[
e_x(t, s) = \sum_{\ell=0}^n x^\ell h_{\ell}(t, s) + x^{n+1} \int_s^t h_n(t, \sigma(\eta)) e_x(\eta, s) \Delta \eta \geq x^n h_n(t, s).
\]
Therefore, we see that

\[ 0 \leq x^\lambda e_{\Theta x}(t,s) = \frac{x^\lambda}{e_x(t,s)} \leq \frac{x^\lambda}{x^\mu h_n(t,s)}, \]

which proves \( x^\lambda e_{\Theta x}(t,s) \to 0 \) by letting \( x \to \infty \). Next, let \( s, t \in T \) with \( t \leq s \), then \( e_{\Theta x}(t,s) = e_x(s,t) \geq 1 \) and thus we have \( x^\lambda e_{\Theta x}(t,s) \geq x^\lambda \), which shows that \( x^\lambda e_{\Theta x}(t,s) \to \infty \) as \( x \to \infty \). This completes the proof. \( \square \)

Let us introduce the function \( \Lambda : \mathcal{R}_c(T, \mathbb{R}) \times T \times T \to \mathbb{R} \) defined by

\[
\Lambda(x; t, s) := \exp\left\{-x e_{\Theta x}(t,s)\right\} \quad \text{for} \quad x \in \mathcal{R}_c(T, \mathbb{R}) \text{ and } s, t \in T. \tag{2.1}
\]

Corollary 2. Let \( s, t \in T \), then

\[
\lim_{x \to \infty} \Lambda(x; t, s) = \chi_{(-\infty, t)}(s),
\]

where \( \chi_D : \mathbb{R} \to \{0, 1\} \) is the characteristic function of the set \( D \subseteq \mathbb{R} \).

3 Uniqueness of the Laplace Transform

In this section, we shall always assume that \( \sup T = \infty \). We first start with the definition of the set of null functions.

Definition 1. A function \( f : [s, \infty)_T \to \mathbb{C} \) is called a null function if

\[
\int_s^t f(\eta) \Delta \eta = 0 \quad \text{for all} \quad t \in [s, \infty)_T.
\]

The set of null functions on will be denoted by \( \mathcal{N}([s, \infty)_T, \mathbb{C}) \).

Next, we give some properties of the null functions some of which will be required in the proof of the main result.

Lemma 3. Let \( f \in \mathcal{N}([s, \infty)_T, \mathbb{C}) \) and \( g \in C^1_{id}([s, \infty)_T, \mathbb{C}) \), then \( fg^\sigma \in \mathcal{N}([s, \infty)_T, \mathbb{C}) \).

Proof. Performing an integration by parts, for any \( t \in [s, \infty)_T \), we have

\[
\int_s^t f(\eta)g^\sigma(\eta) \Delta \eta = \left[\int_s^\eta f(\zeta) \Delta \zeta \right]g(\eta)\bigg|_{\eta=t} - \int_s^t \left[\int_s^\eta f(\zeta) \Delta \zeta \right]g^\Delta(\eta) \Delta \eta = 0,
\]

which proves the claim. \( \square \)
Corollary 3. Let $f \in \mathcal{N}([s, \infty)_T, \mathbb{C})$ and $g \in \mathcal{R}([s, \infty)_T, \mathbb{C})$, then $fe_\xi(\sigma(\cdot), s) \in \mathcal{N}([s, \infty)_T, \mathbb{C})$.

Corollary 4. Let $f \in \mathcal{N}([s, \infty)_T, \mathbb{C})$, then

$$\int_{s}^{\infty} f(\eta)e_{\Theta\xi}(\sigma(\eta), s)\Delta \eta = 0 \quad \text{for any } z \in \mathcal{R}_c([s, \infty)_T, \mathbb{C}). \quad (3.1)$$

We have now filled the necessary background for the proof of our main result.

Theorem 1 (Lerch’s theorem). Assume that $f : [s, \infty)_T \to \mathbb{C}$, there exist a divergent sequence $\{c_k\}_{k \in \mathbb{N}_0} \subset \mathbb{R}_0^+$ and $\alpha \in \mathbb{R}_c([s, \infty)_T, \mathbb{C})$ such that

$$\mathcal{L}\{fe_\Theta(n\circ c_k)(\sigma(\cdot), s)|c_k\}(\alpha) = 0 \quad \text{for all } n, k \in \mathbb{N}_0. \quad (3.2)$$

Then $f \in \mathcal{N}([s, \infty)_T, \mathbb{C})$.

Proof. Define the function $g : [s, \infty)_T \to \mathbb{C}$ by $g(t) := f(t)e_{\Theta\alpha}(\sigma(t), s)$ for $t \in [s, \infty)_T$, then we have

$$\int_{s}^{\infty} g(\eta)e_{\Theta(\eta\circ c_k)}(\sigma(\eta), s)\Delta \eta = \int_{s}^{\infty} f(\eta)e_{\Theta(\eta\circ c_k)}(\sigma(\eta), s)e_{\Theta\alpha}(\sigma(\eta), s)\Delta \eta$$

$$= \int_{s}^{\infty} f(\eta)e_{\Theta(\alpha \circ (\eta\circ c_k))}(\sigma(\eta), s)\Delta \eta = 0 \quad (3.3)$$

for all $n, k \in \mathbb{N}_0$. Let $r \in [s, \infty)_T$, and define $h : [s, \infty)_T \to \mathbb{C}$ by

$$h_r(t) := \int_{r}^{t} g(\eta)\Delta \eta \quad \text{for } t \in [s, \infty)_T.$$

It follows from (3.2) with $n = 0$ that

$$\int_{s}^{\infty} g(\eta)\Delta \eta = 0, \quad (3.4)$$

which shows that $\lim_{t \to \infty} h_r(t)$ exists. So we can find $M_r \in \mathbb{R}^+$ such that $|h_r(t)| \leq M_r$ for all $t \in [r, \infty)_T$. We may (and do) assume that $M_r \to 0$ as
$r \to \infty$. Using (3.3), and performing integration by parts, we get

\[
\int_{s}^{r} g(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), s) \Delta \eta = - \int_{r}^{\infty} g(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), s) \Delta \eta \\
= - \left[ e_{\Theta(n \odot c_k)}(\eta, s) h_{r}(\eta) \right]_{\eta=\infty}^{\eta=r} \\
- \int_{r}^{\infty} e_{\Theta(n \odot c_k)}(\eta, s) h_{r}(\eta) \Delta \eta \\
= - \left[ \left( e_{\Theta(c_k)}(\eta, s) \right)^{\eta=\infty} h_{r}(\eta) \right]_{\eta=\infty}^{\eta=r} \\
+ \int_{r}^{\infty} (n \odot c_k)(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), s) h_{r}(\eta) \Delta \eta \\
= - \int_{r}^{\infty} (n \odot c_k)(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), s) h_{r}(\eta) \Delta \eta 
\]

(3.5)

for all $n, k \in \mathbb{N}_0$. Note that above have used Corollary 1 while passing to the last step. Now multiplying both sides of (3.5) with $e_{\Theta(n \odot c_k)}(s, r)$, we have

\[
\int_{s}^{r} g(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), r) \Delta \eta = - \int_{r}^{\infty} (n \odot c_k)(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), r) h_{r}(\eta) \Delta \eta, 
\]

which yields

\[
\left| \int_{s}^{r} g(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), r) \Delta \eta \right| \leq M_{r} \int_{r}^{\infty} (n \odot c_k)(\eta) e_{\Theta(n \odot c_k)}(\sigma(\eta), r) \Delta \eta = M_{r}. 
\]

By the series expansion of the exponential function, we know that

\[
\sum_{\ell \in \mathbb{N}_0} \frac{(-1)^{\ell} c_{k}^{\ell}}{\ell!} e_{\Theta(\ell \odot c_k)}(t, s) = \Lambda(c_{k}; t, s) \quad \text{for} \ s, t \in T \ \text{and} \ k \in \mathbb{N}_0, 
\]

where $\Lambda$ is defined by (2.1). Thus, for all $t \in [s, r]_T$ and all $k \in \mathbb{N}_0$, we can
estimate that 
\[ \left| \int_s^r g(\eta) \Lambda(c_k; \sigma(\eta), t) \Delta \eta \right| = \left| \sum_{\ell \in \mathbb{N}_0} \frac{(-1)^{\ell-1}}{\ell!} \int_s^r g(\eta) e_{\Theta(\ell \circ c_k)}(\sigma(\eta), t) \Delta \eta \right| \]

\[ \leq \left| \sum_{\ell \in \mathbb{N}_0} c_k^\ell e_{\Theta(\ell \circ c_k)}(r, t) \right| \int_s^r g(\eta) \exp(\ell \circ c_k(\sigma(\eta), t)) \Delta \eta \]

\[ \leq M_r \sum_{\ell \in \mathbb{N}_0} \frac{c_k^\ell}{\ell!} \left( e_{\Theta c_k}(r, t) \right)^\ell \rho(c_k(\sigma(\eta), t)) = M \exp\left( c_k e_{\Theta c_k}(r, t) \right). \]

Letting \( r \to \infty \), we have \( M_r \to 0 \) and \( e_{\Theta c_k}(r, t) \to 0 \) by Corollary 1, which yields \( M \exp\left( c_k e_{\Theta c_k}(r, t) \right) \to 0 \) as \( r \to \infty \). We can therefore write

\[ \int_s^\infty g(\eta) \Lambda(c_k; \sigma(\eta), t) \Delta \eta = 0 \quad \text{for all} \quad k \in \mathbb{N}_0. \quad (3.6) \]

By (3.4), the function \( g \) is integrable over \([s, \infty)_T\) and the characteristic function \( \chi \) is piecewise constant. Letting \( k \to \infty \) in (3.6), we get

\[ \int_s^\infty g(\eta) \chi_{(-\infty, t)}(\sigma(\eta)) \Delta \eta = 0 \quad \text{for all} \quad t \in [s, \infty)_T \quad (3.7) \]

by Lebesgue’s dominated convergence theorem and Corollary 2. Now, we are in a position to prove that

\[ \int_s^t g(\eta) \Delta \eta = 0 \quad \text{for all} \quad t \in [s, \infty)_T. \quad (3.8) \]

From (3.7), for all \( t \in [s, \infty)_T \), we have

\[ \int_s^\infty g(\eta) \chi_{(-\infty, t)}(\sigma(\eta)) \Delta \eta = \int_s^t g(\eta) \chi_{[s, t)}^\Lambda(\sigma(\eta)) \Delta \eta \]

\[ = \int_s^t g(\eta) \left[ \chi^\Lambda_{[s, t)}(\eta) + \mu(\eta) \chi^{\Lambda_{[s, t)}}_{[s, t)}(\eta) \right] \Delta \eta \]

\[ = \int_s^t g(\eta) \chi_{[s, t)}(\eta) \Delta \eta, \]

which together with the definition of the characteristic function \( \chi \) and (3.7) gives (3.8). Therefore, we learn that \( g \) is a null function. An application of Corollary 3 shows that \( f = g e_\alpha(\sigma(\cdot), s) \) is a null function too. This completes the proof. \( \square \)
Corollary 5. Assume that \( f, g : [s, \infty)_T \to \mathbb{C} \), there exist an increasing divergent sequence \( \{c_k\}_{k \in \mathbb{N}_0} \subset \mathbb{R}^+ \) and \( \alpha \in \mathcal{R}_C^+([s, \infty)_T, \mathbb{C}) \) such that
\[
\mathcal{L}\{f e^{(n \circ c_k)}(\sigma(\cdot), s)\}(\alpha) = \mathcal{L}\{g e^{(n \circ c_k)}(\sigma(\cdot), s)\}(\alpha) \quad \text{for all } n, k \in \mathbb{N}_0.
\]
Then \( f - g \in \mathcal{N}([s, \infty)_T, \mathbb{C}) \).

Corollary 6. Assume that the graininess function \( \mu \) is constant and there exists \( \alpha \in \mathcal{R}_C^+([s, \infty)_T, \mathbb{C}) \) such that
\[
\mathcal{L}\{f \} (z) = 0 \quad \text{for all } z \in \mathcal{C}_{\mu,(s)}(\alpha).
\]
Then \( f \in \mathcal{N}([s, \infty)_T, \mathbb{C}) \).

Proof. In this case, for any fixed \( \beta \in \mathcal{R}_{\mu,(s)}(\alpha) \subset \mathcal{C}_{\mu,(s)}(\alpha) \), we have
\[
\mathcal{L}\{f\}(\beta) = 0,
\]
which yields \( \beta \oplus ((nk) \circ c) \in \mathcal{R}_{\mu,(s)}(\alpha) \subset \mathcal{C}_{\mu,(s)}(\alpha) \) for all \( n, k \in \mathbb{N}_0 \) and all \( t \in [s, \infty)_T \), where \( c \in \mathbb{R}^+ \), i.e.,
\[
\mathcal{L}\{f\}(\beta \oplus ((nk) \circ c)) = \mathcal{L}\{f e^{((nk) \circ c)}(\sigma(\cdot), s)\}(\beta) = 0 \quad \text{for all } n, k \in \mathbb{N}_0.
\]
This shows that the conditions of Theorem 1 hold with \( c_k := k \circ c \) for \( k \in \mathbb{N}_0 \).

4 Appendix: Time Scales Essentials

A time scale, which inherits the standard topology on \( \mathbb{R} \), is a nonempty closed subset of reals. Throughout this paper, the time scale is assumed to be unbounded above and will be denoted by the symbol \( T \), and the intervals with a subscript \( T \) are used to denote the intersection of the usual interval with \( T \). For \( t \in T \), we define the forward jump operator \( \sigma : T \to T \) by \( \sigma(t) := \inf(t, \infty)_T \) while the graininess function \( \mu : T \to \mathbb{R}_0^+ \) is defined to be \( \mu(t) := \sigma(t) - t \). A point \( t \in T \) is called right-dense if \( \sigma(t) = t \); otherwise, it is called right-scattered, and similarly left-dense and left-scattered points are defined in terms of the so-called backward jump operator. A function \( f : T \to \mathbb{C} \) is said to be Hilger differentiable (or \( \Delta \)-differentiable) at the point \( t \in T \) if there exists \( \ell \in \mathbb{C} \) such that for any \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) such that
\[
||f(\sigma(t)) - f(s) - \ell(\sigma(t) - s)|| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U,
\]
and in this case we denote \( f^\Delta(t) = \ell \). A function \( f \) is called rd-continuous provided that it is continuous at right-dense points in \( T \), and has finite limits at left-dense points, and the set of rd-continuous functions is denoted by \( C_{rd}(T, \mathbb{C}) \). The set of functions \( C^1_{rd}(T, \mathbb{C}) \) includes the functions whose derivative is in \( C_{rd}(T, \mathbb{C}) \) too. For \( f \in C^1_{rd}(T, \mathbb{C}) \), we have

\[
f^\sigma = f + \mu f^\Delta \quad \text{on } T^\kappa,
\]

where \( f^\sigma := f \circ \sigma \) and \( T^\kappa := T \setminus \{\sup T\} \) if \( \sup T = \max T \) and satisfies \( \rho(\max T) \neq \max T \); otherwise, \( T^\kappa := T \). For \( s, t \in T \) and a function \( f \in C_{rd}(T, \mathbb{C}) \), the \( \Delta \)-integral of \( f \) is defined by

\[
\int_s^t f(\eta) \Delta \eta = F(t) - F(s) \quad \text{for } s, t \in T,
\]

where \( F : T \to \mathbb{C} \) is an antiderivative of \( f \), i.e., \( F^\Delta = f \) on \( T^\kappa \).

A function \( f \in C_{rd}(T, \mathbb{C}) \) is called regressive if \( 1 + \mu f \neq 0 \) on \( T \), and positively regressive if it is real valued and \( 1 + \mu f > 0 \) on \( T \). The set of regressive functions and the set of positively regressive functions are denoted by \( R(T, \mathbb{C}) \) and \( R^+(T, \mathbb{R}) \), respectively, and \( R^-(T, \mathbb{R}) \) is defined similarly. For simplicity, we denote by \( R_c(T, \mathbb{C}) \) the set of complex regressive constants, and similarly, we define the sets \( R^+_c(T, \mathbb{R}) \) and \( R^-_c(T, \mathbb{R}) \).

Let \( f \in R(T, \mathbb{C}) \). Then the exponential function \( e_f(\cdot, s) \) is defined to be the unique solution of the initial value problem

\[
\begin{cases}
x^\Delta = f x & \text{on } T^\kappa \\
x(s) = 1
\end{cases}
\]

for some fixed \( s \in T \). For \( h > 0 \), set

\[
\mathbb{C}_h := \{ z \in \mathbb{C} : z = -\frac{1}{h} \} \quad \text{and} \quad \mathbb{Z}_h := \{ z \in \mathbb{C} : \pi/h < \text{Im}(z) \leq \pi/h \},
\]

and \( \mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C} \). For \( h \in \mathbb{R}^+_0 \), the Hilger real part and imaginary part of a complex number are given by

\[
\text{Re}_h(z) := \lim_{\nu \to h} \frac{1}{\nu} (|1 + \nu z| - 1) \quad \text{and} \quad \text{Im}_h(z) := \lim_{\nu \to h} \frac{1}{\nu} \text{Arg}(1 + \nu z),
\]

respectively, where \( \text{Arg} \) denotes the principle argument function, i.e., \( \text{Arg} : \mathbb{C} \to (-\pi, \pi]_{\mathbb{R}} \). For \( h \in \mathbb{R}^+_0 \) and any fixed \( z \in \mathbb{C}_h \), the Hilger real part \( \text{Re}_h(z) \)
is a nondecreasing function of \( h \in \mathbb{R}_0^+ \), i.e., \( \text{Re}_{h_1}(z) \geq \text{Re}_{h_2}(z) \) for \( h_1, h_2 \in \mathbb{R}_0^+ \) with \( h_1 \geq h_2 \). For \( h \in \mathbb{R}_0^+ \), we define the cylinder transformation \( \xi_h : \mathbb{C}_h \to \mathbb{Z}_h \) by

\[
\xi_h(z) := \lim_{\nu \to h} \frac{1}{\nu} \log(1 + \nu z) \quad \text{for} \quad z \in \mathbb{C}_h.
\]

Then the exponential function can also be written in the form

\[
e_f(t,s) := \exp \left\{ \int_s^t \xi_{\mu(\eta)}(f(\eta)) \Delta \eta \right\} \quad \text{for} \quad s,t \in \mathbb{T}.
\]

It is known that the exponential function \( e_f(\cdot,s) \) is strictly positive on \([s,\infty)_T\) provided that \( f \in \mathcal{R}^+(\mathbb{T},\mathbb{R})\), while \( e_f(\cdot,s) \) alternates in sign at right-scattered points of the interval \([s,\infty)_T\) provided that \( f \in \mathcal{R}^-(\mathbb{T},\mathbb{R})\). For \( h \in \mathbb{R}_0^+ \) and \( w,z \in \mathbb{C}_h \), the circle plus and the circle minus are defined by

\[
z \oplus_h w := z + w + hw \quad \text{and} \quad z \ominus \mu w := \frac{z - w}{1 + hw},
\]

respectively. It is known that \((\mathcal{R}(\mathbb{T},\mathbb{C}),\oplus_{\mu})\) is a group, and the inverse of \( f \in \mathcal{R}(\mathbb{T},\mathbb{C}) \) is \( \ominus_{\mu} f := 0 \ominus_{\mu} f \). Moreover, \( \mathcal{R}_{c}^+(\mathbb{T},\mathbb{C}) \) is a subgroup of \( \mathcal{R}_{c}(\mathbb{T},\mathbb{C}) \). For \( \lambda \in \mathbb{C} \) and \( z \in \mathbb{C}_h \), the circle dot is defined by

\[
\lambda \odot_h z := \lim_{\nu \to h} \frac{1}{\nu} \left( (1 + \nu z)\lambda^\nu - 1 \right).
\]

With this multiplication, \((\mathcal{R}(\mathbb{T},\mathbb{C}),\oplus_{\mu},\otimes_{\mu})\) becomes a complex vector space. It should be noted that

\[
e_{\lambda \otimes f}(t,s) = \left( e_f(t,s) \right)^\lambda \quad \text{for} \quad s,t \in \mathbb{T},
\]

where \( \lambda \in \mathbb{C} \) and \( f \in \mathcal{R}(\mathbb{T},\mathbb{C}) \). For simplicity in the notation, we shall use \( \oplus,\otimes \) and \( \odot \) instead of \( \oplus_{\mu},\otimes_{\mu} \) and \( \odot_{\mu} \), respectively.

The definition of the generalized monomials on time scales (see \[\text{[4, § 1.6]}\]) \( h_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R} \) is given as

\[
h_n(t,s) := \begin{cases} 1, & n = 0, \\ \int_s^t h_{n-1}(\eta,s) \Delta \eta, & n \in \mathbb{N} \end{cases} \quad \text{for} \quad s,t \in \mathbb{T}.
\]

Using induction, it is easy to see that \( h_n(t,s) \geq 0 \) holds for all \( n \in \mathbb{N}_0 \) and all \( s,t \in \mathbb{T} \) with \( t \geq s \), and \((-1)^n h_n(t,s) \geq 0 \) holds for all \( n \in \mathbb{N} \) and all \( s,t \in \mathbb{T} \) with \( t \leq s \).

The readers are referred to \[\text{[4]}\] for fundamentals of time scale theory.
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