Twisted bi-symplectic structure on Koszul twisted Calabi-Yau algebras

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Abstract
For a Koszul Artin-Schelter regular algebra (also called twisted Calabi-Yau algebra), we show that it has a “twisted” bi-symplectic structure, which may be viewed as a noncommutative and twisted analog of the shifted symplectic structure introduced by Pantev, Toën, Vaquié and Vezzosi. This structure gives a quasi-isomorphism between the tangent complex and the twisted cotangent complex of the algebra, and may be viewed as a DG enhancement of Van den Bergh’s noncommutative Poincaré duality; it also induces a twisted symplectic structure on its derived representation schemes.

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1 Introduction

Let $A$ be an Artin-Schelter (AS for short) regular algebra of dimension $n$. Then, denote by $\sigma$ the Nakayama automorphism of $A$ and $A^e = A \otimes A^{\text{op}}$ its enveloping algebra. Reyes, Rogalski and Zhang [29] (see also Van den Bergh [33] and Yekutieli and Zhang [38]) proved that there is an isomorphism

$$\text{RHom}_{A^e}(A, A \otimes A) \cong A_\sigma[n]$$

in the derived category $D(A^e)$ of (left) $A^e$-modules, where $A_\sigma$ is the twisted $A^e$-module whose underlying space is $A$, with the action of $A^e$ given by $u \circ x \circ v := \sigma(u)xv$, for all $u, v, x \in A$. Therefore, due to Van den Bergh [34], the noncommutative Poincaré duality of $A$ reads as

$$\text{HH}^\bullet(A) \cong \text{HH}_{n-\bullet}(A, A_\sigma),$$

where $\text{HH}^\bullet(A)$ and $\text{HH}_{n-\bullet}(A, A_\sigma)$ are the Hochschild cohomology and the twisted Hochschild homology of $A$, respectively. This paper aims to understand (2), especially the role of the Nakayama automorphism, from the derived noncommutative geometry point of view.
1.1 Twisted bi-symplectic structure

Before going to the details, let us look at the case where $\sigma$ is the identity map. In this case, the AS-regular algebra is nothing but a Calabi-Yau algebra in the sense of Ginzburg [17] (therefore, an AS-regular algebra is also called a twisted Calabi-Yau algebra in [29]). Then the isomorphism (1) may be interpreted as the isomorphism between the (derived) noncommutative tangent and cotangent complexes of $A$, which gives rise to a shifted noncommutative symplectic structure on $A$. The noncommutative symplectic structure was first introduced by Crawley-Boevey, Etingof and Ginzburg in [13] for associative algebras and was called the bi-symplectic structure, which can easily be generalized to the category of differential graded (DG) algebras and hence their homotopy category (see [10, 27] for some more details); it is also a noncommutative analog of the shifted symplectic structure introduced by Pantev, Toën, Vaquié and Vezzosi in [26].

Now for an arbitrary AS-regular algebra, it is very tempting for us to look for a version, possibly twisted, of the bi-symplectic structure analogous to that of Calabi-Yau algebras; here, by “twisted”, we mean a structure that encodes the Nakayama automorphism.

For simplicity, let us assume $A$ is also Koszul. Denote by $A^!$ the Koszul dual coalgebra of $A$; then its cobar construction $R := \Omega(A^!)$ is a DG free cofibrant resolution of $A$. It is usually more convenient to consider the noncommutative geometric structure on $R$ instead of $A$ in derived algebraic geometry (c.f. [5, 6, 39]). Our first theorem is:

**Theorem 1.1** Let $A$ be a Koszul AS-regular algebra of dimension $n$, and let $R$ be as above. Then there exists a closed 2-form $\omega$ of total degree $n$ in the twisted Karoubi-de Rham complex $\hat{DR}_\omega^2$ of $R$ such that the contraction (see Lemma 2.3 for the definition)

$$\Psi \circ \iota_{(-)} \omega : \widehat{\text{Der}} R \xrightarrow{\sim} \hat{\Omega}^1_{\sigma} R[n - 2]$$

is an isomorphism of DG $R$-bimodules, where $\widehat{\text{Der}} R$ and $\hat{\Omega}^1_{\sigma} R$ are the noncommutative tangent complex and the twisted noncommutative cotangent complex of $R$ respectively.

In Sect.2, we shall give more details about the notations in the above theorem. In the theorem, $\omega$ is the twisted version of the shifted bi-symplectic structure of a Koszul Calabi-Yau algebra and hence is called a twisted bi-symplectic structure. Moreover, the induced isomorphism on the commutator quotient spaces on both sides of (3) is precisely Van den Bergh’s noncommutative Poincaré duality (2). In this sense, we may view (3) as a derived enhancement of (2).

1.2 Derived representation schemes

A guiding principle in the study of noncommutative geometry is via representations due to Kontsevich and Rosenberg [20]. It roughly says that a noncommutative geometric structure on a noncommutative space (in this work, we mean an associative algebra), if it exists, should induce its classical counterpart on the affine schemes of
its representations (called the representation schemes). This principle has achieved much success, for example, in the study of noncommutative Poisson geometry and noncommutative symplectic geometry [12, 13, 35].

Later this principle is generalized to the derived/homotopy setting by Berest, Khachatryan and Ramadoss in [6]. Such a generalization is highly nontrivial, as it solves many issues that appear in the calculations of representation schemes and reveals many new interesting structures. Going back to the Koszul AS-regular algebra case, we have

**Theorem 1.2.** Let $A$ be a Koszul AS-regular algebra. Denote by $\mathcal{D}\text{Rep}_V(A)$ the derived moduli stack of representations of $A$ in a vector space $V$ (see §4.5 below for more details). Then the twisted bi-symplectic structure of $A$ induces a twisted symplectic structure on $\mathcal{D}\text{Rep}_V(A)$.

Here, the **twisted symplectic structure**, parallel to the noncommutative case, gives a quasi-isomorphism, up to degree shifting, between the tangent and twisted cotangent complexes of $\mathcal{D}\text{Rep}_V(A)$. It is the twisted version of the shifted symplectic structure introduced in [26] and also generalizes some previous results obtained in [10, 27, 39] for Calabi-Yau algebras.

The rest of the paper is devoted to proving the above two theorems. It is organized as follows: in §2, we first recall the noncommutative tangent and cotangent complexes as well as their twisted version for DG algebras, and then introduce the notion of twisted bi-symplectic structure; in §3, we first recall the definition of AS-regular algebras and then prove Theorem 1.1; in §4, we briefly recall Berest et al.’s construction of derived representation schemes, and in particular, study the differential forms on them; in the last section, §5, after introducing the twisted differential forms and twisted symplectic structure for DG commutative algebras, we prove Theorem 1.2.

**Convention.** Throughout the paper, $k$ is a field of characteristic zero. All algebras are unital over $k$, and all tensors and homomorphisms are over $k$ unless otherwise specified. Boldface letters such as $\text{Der}(\cdot)$ and $\Omega^\bullet(\cdot)$ mean the corresponding noncommutative structures for DG algebras while the usual letters such as $\text{Der}(\cdot)$ and $\Omega^\bullet(\cdot)$ mean those for DG commutative algebras.

## 2 Noncommutative differential calculus

In this section we study the twisted analogue of the noncommutative bi-symplectic structure; its untwisted version was first introduced by Crawley-Boevey, Etingof and Ginzburg in [13]; see also [10, 27] for the DG/derived case.

### 2.1 Noncommutative differential forms

Suppose $(R, \mu, \partial)$ is a DG $k$-algebra, where $\mu$ is the multiplication and $\partial$ is the differential of degree $-1$. The set of noncommutative 1-forms of $R$, denoted by $\Omega^1 R$, is

$$\Omega^1 R := \ker\{\mu : R \otimes R \to R\}.$$
It is a DG $R$-bimodule generated by $r \otimes 1 - 1 \otimes r \in R \otimes R$ for all $r \in R$, with the differential induced from $R$ and still denoted by $\partial$. Equivalently, $\Omega^1 R$ is the $R$-bimodule generated by $dr$, for all $r \in R$, subject to the following relations: $d(r_1 r_2) = (dr_1)r_2 + r_1(dr_2)$, for all $r_1, r_2 \in R$. The identification of these two $R$-bimodules is given by

$$r \otimes 1 - 1 \otimes r \mapsto dr.$$ 

In what follows, we always raise the degree of 1-forms up by 1; in other words, by the set of noncommutative 1-forms of $R$ we shall mean $\Omega^1 R[−1]$ (also denoted by $\Sigma \Omega^1 R$). Elements like $r \otimes 1 - 1 \otimes r \in \Omega^1 R$ is now identified with $\Sigma (r \otimes 1 - 1 \otimes r) \in \Omega^1 R[−1]$. With this convention, the de Rham differential $d : R \rightarrow \Omega^1 R[−1]$, $r \mapsto \Sigma dr$ has degree 1, and satisfies $d \circ \partial + \partial \circ d = 0$.

**Remark 2.1** It is suggested by Yeung [39] that in practice it is better to take the noncommutative 1-forms to be the cone of the inclusion $i : \Omega^1 R \rightarrow R \otimes R$. Formally, we may view $R \otimes R$ as the set of noncommutative 1-forms generated by $d 1$, where $1$ is the image of $1 \in k$ under the unit map. Taking the degree shifting also into account, we denote this set of augmented noncommutative 1-forms by $\hat{\Omega}^1 R[−1]$; more precisely,

$$\hat{\Omega}^1 R[−1] = \text{cone}\{\Omega^1 R \rightarrow R \otimes R\}[−1].$$

In what follows, we adopt this convention, which will be convenient for us to study the equivariant differential forms on the representation schemes of $R$ (see Theorem 4.9).

The set of noncommutative de Rham forms of $R$, denoted by $\hat{\Omega}^• R$, is the tensor algebra

$$\hat{\Omega}^• R := T_R(\hat{\Omega}^1 R[−1])$$

with $d$ and $\partial$ extended to it by derivation and by letting $d^2 = 0$. The triple $(\hat{\Omega}^• R, \partial, d)$ is a mixed DG algebra.

### 2.2 Double derivations

According to [13, 35], the noncommutative vector fields on a DG associative algebra are given by the double derivations. By definition, the space of double derivations $\text{Der} R$ of $R$ is the set of derivations $\text{Der}(R, R \otimes R)$, where the $R$-bimodule structure on $R \otimes R$ is the outer $R$-bimodule structure, given by

$$u \circ (x \otimes y) \circ v := ux \otimes yv,$$
for any \( x, y, u, v \in R \). Since the map \( R \to \Omega^1 R, x \mapsto dx \) is the universal derivation, meaning that every derivation of \( R \) factors through \( \Omega^1 R \), we have that

\[
\text{Der} R \cong \text{Hom}_{R^e}(\Omega^1 R, R \otimes R).
\]

Observe that \( R \otimes R \) also has an inner \( R \)-bimodule structure, which is given by

\[
u \ast (x \otimes y) \ast v := (-1)^{|u||x|+|v||y|+|u||v|} x v \otimes u y.
\] (5)

With the inner \( R \)-bimodule structure on \( R \otimes R \), \( \text{Der} R \) is a DG \( R \)-bimodule.

Analogously to the 1-forms case (see Remark 2.1), in what follows, we take the space of noncommutative vector fields to be

\[
\hat{\text{Der}} R[1] := \text{Hom}_{R^e}(\hat{\Omega}^1 R[-1], R \otimes R)
\]

and let \( T_R(\hat{\text{Der}} R[1]) \) be the tensor algebra generated by \( \hat{\text{Der}} R[1] \) over \( R \), which is called the space of (noncommutative) polyvector fields of \( R \).

In what follows, we also call \( \hat{\Omega}^1 R[-1] \) and \( \hat{\text{Der}} R[1] \) the noncommutative tangent and cotangent complexes of \( R \) respectively.

### 2.3 Twisted differential forms

In the literature, an associative algebra together with an automorphism is called a twisted algebra. Twisted differential forms and twisted derivations for twisted algebras have been studied by many people; see, for example, [1, 19, 22] and references therein. In what follows, we briefly recall these concepts for DG algebras.

Suppose \( R \) is a DG algebra with an automorphism \( \sigma \in \text{Aut} R \). The set of twisted noncommutative differential 1-forms of \( R \) is

\[
\Omega^1_R := \ker(\mu_{\sigma} : R \otimes R \to R, a \otimes b \mapsto a \sigma(b)).
\] (6)

Here \( \mu_{\sigma} \) is called the twisted product of \( R \). There is a close relationship between \( \Omega^1_R \) and \( \Omega^1_{\sigma} R \). To see this, let us first give \( \Omega^1_{\sigma} R \) an \( R \)-bimodule structure as follows:

\[
u \circ (\sum_i x_i \otimes y_i) \circ v := \sum_i \sigma(u) x_i \otimes y_i v, \quad \text{for all } u, v \in R.
\] (7)

Then it is straightforward to see that the following map

\[
\Psi : \Omega^1_R \to \Omega^1_{\sigma} R, \quad \sum_i x_i \otimes y_i \mapsto \sum_i \sigma(x_i) \otimes y_i
\] (8)

is a map of DG \( R \)-bimodules, where \( \Omega^1_{\sigma} R \) is equipped with the above twisted \( R \)-bimodule structure (7).
Parallel to Remark 2.1, in what follows we take the twisted noncommutative 1-forms of \( R \) to be

\[
\hat{\Omega}_\sigma^1 R[-1] := \text{cone}(\Omega_\sigma^1 R \rightarrow R \otimes R)[-1],
\]

where \( R \otimes R \) is equipped with the twisted outer \( R \)-bimodule structure the same as (7). There is a natural map

\[
d_\sigma : R \rightarrow \Omega_\sigma^1 R[-1] \subset \hat{\Omega}_\sigma^1 R[-1], \quad a \mapsto \Sigma(\sigma(a) \otimes 1 - 1 \otimes a),
\]

called the twisted de Rham differential of \( R \). It satisfies the following twisted Leibniz rule

\[
d_\sigma(ab) = (d_\sigma a) \cdot b + (-1)^{|a|} \sigma(a) \cdot (d_\sigma b), \quad \text{for all } a, b \in R.
\]

Moreover, we also have \( d_\sigma \circ \partial + \partial \circ d_\sigma = 0 : R \rightarrow \hat{\Omega}_\sigma^1 R[1] \).

### 2.4 The Karoubi-de Rham complex

Recall that the Karoubi-de Rham complex of \( R \), denoted by \( \hat{\text{DR}}^\bullet R \), is

\[
\hat{\text{DR}}^\bullet R := \hat{\Omega}^\bullet R := \hat{\Omega}^\bullet R / [\hat{\Omega}^\bullet R, \hat{\Omega}^\bullet R],
\]

where \([-, -]\) means the super-commutator subspace. It is equipped with two differentials \( \partial \) and \( d \) induced from \( \hat{\Omega}^\bullet R \) (see, for example, [13]), and hence forms a mixed complex.

Now suppose \((R, \sigma)\) is a twisted algebra. Let

\[
\hat{\text{DR}}_\sigma^\bullet R := \hat{\Omega}^\bullet R / \{xy - (-1)^{|x||y|}y\sigma(x), \text{ where } x, y \in \hat{\Omega}^\bullet R\},
\]

which we would call the twisted Karoubi-de Rham complex of \( R \). Here \( \sigma : \hat{\Omega}^\bullet R \rightarrow \hat{\Omega}^\bullet R \) is the automorphism which extends \( \sigma : R \rightarrow R \). It is related to the so-called twisted Hochschild homology of \( R \). In the literature, the notion of twisted Hochschild cohomology was first introduced by Kustermans, Murphy and Tuset in [21]. Soon after that, Hadfield and Krähmer studied the twisted Hochschild homology and computed them for some twisted algebras in [18]. They were used by Brown and Zhang in the study of some twisted algebras such as the AS-regular algebras in [9]. By definition, the twisted Hochschild homology of a twisted algebra \((A, \sigma)\) is the Hochschild homology of \( A \) with values in \( A_\sigma \). The following theorem gives a relationship between the twisted differential forms and the twisted Hochschild homology.

**Theorem 2.2** Suppose \( R \) is a DG free algebra with \( \sigma \in \text{Aut } R \). Then

\[
\hat{\text{DR}}_\sigma^1 R \simeq \text{CH}_\bullet(R, R_\sigma),
\]
where $\overline{\text{CH}}_\bullet(\mathcal{R}, R_\sigma)$ is the reduced twisted Hochschild chain complex of $R$. In particular, if $R \xrightarrow{\sim} A$ is a DG free resolution of $A$, then

$$\hat{\text{DR}}_\sigma^1 R \simeq \overline{\text{CH}}_\bullet(A, A_\sigma).$$

Observe that in the above theorem, $\sigma$ also induces an automorphism of $A$, so $A_\sigma$ makes sense. Moreover, if $\sigma$ is the identity map, then the above quasi-isomorphism becomes

$$\hat{\text{DR}}_\sigma^1 R = \hat{\text{DR}}^1 R \simeq \overline{\text{CH}}_\bullet(R) \simeq \overline{\text{CH}}_\bullet(A),$$

whose proof can be found in [10, Propositions 3.5]. The proof for the twisted case is similar, so here we only give a sketch of it, just for the sake of completeness; see also Liu [23] for some similar discussions.

**Proof of Theorem 2.2** Suppose $R = k\langle x_1, \cdots, x_n \rangle$ with differential $\partial$. If we denote $V := \text{Span}_k\{x_1, \cdots, x_n\}$, then $V \otimes k$ has a counital, $A_\infty$-coalgebra structure, which is the Koszul dual coalgebra of $R$ (see [8, §1] for more details), and is denoted by $\tilde{V}$. Now equip $R \otimes \tilde{V} \otimes R$ with the following differential $b$: suppose the $A_\infty$-coproduct $\Delta_m(v) = \sum v_1 \otimes \cdots \otimes v_m$, for $v \in \tilde{V}$, then for $r_1 \otimes v \otimes r_2 \in R \otimes \tilde{V} \otimes R$,

$$b(r_1 \otimes v \otimes r_2) = \partial(r_1) \otimes v \otimes r_2 + (-1)^{|r_1|+|v|} r_1 \otimes v \otimes \partial(r_2) + \sum_m \sum_i (-1)^{|r_1|+|r_1|+\cdots+|r_{i-1}|} r_1 \tilde{v}_i \cdots \tilde{v}_{i-1} \otimes v_i \otimes \sigma(\tilde{v}_{i+1} \cdots \tilde{v}_m)r_2,$$

where $\tilde{v}_i$ is the image of $v \in \tilde{V}$ in $V$. It is direct to see

$$\hat{\Omega}_\sigma^1 R[-1] = \text{cone}(\Omega_\sigma^1 R \to R \otimes R)[-1] \cong R \otimes \tilde{V} \otimes R,$$

and therefore we get

$$\hat{\text{DR}}_\sigma^1 R \cong \overline{\text{CH}}_\bullet(\tilde{V}, \tilde{V}_\sigma). \quad (11)$$

By Koszul duality theory for $A_\infty$-algebras ([8, Theorem 1.2]) we have

$$\overline{\text{CH}}_\bullet(\tilde{V}, \tilde{V}_\sigma) \simeq \overline{\text{CH}}_\bullet(R, R_\sigma). \quad (12)$$

(In loc. cit. the authors did not consider the twisted Hochschild homology, but this is straightforward.) Thus combining (11) and (12) we get

$$\hat{\text{DR}}_\sigma^1 R \cong \overline{\text{CH}}_\bullet(R, R_\sigma).$$
Now suppose $R \xrightarrow{\sim} A$ is a DG free resolution. Then it induces a chain map $\overline{\text{CH}}_\bullet(R, R_\sigma) \to \overline{\text{CH}}_\bullet(A, A_\sigma)$. Filtering both complexes with respect to the number of components, we get $\overline{\text{CH}}_\bullet(R, R_\sigma) \simeq \overline{\text{CH}}_\bullet(A, A_\sigma)$ by comparing the associated spectral sequences. Combining it with the above quasi-isomorphism (10), we get the desired result.

In [10, Proposition 3.7] it is also proved that

$$\widehat{\text{Der}} R[1] := \widehat{\text{Der}} R[1]/[R, \widehat{\text{Der}} R[1]] \simeq \overline{\text{CH}}^\bullet(R) \simeq \overline{\text{CH}}^\bullet(A).$$

(13)

2.5 Twisted bi-symplectic structure

From now on, we assume $R$ is a DG free algebra with $\sigma \in \text{Aut} R$. We introduce the twisted noncommutative contraction of double derivations with differential forms, which is a generalization of the noncommutative contraction introduced in [13].

First, for any $\Theta \in \widehat{\text{Der}} R$ and $a \in R$, in what follows we write $\Theta(a)$ in the form $\sum (-1)^{\sigma_n} \Theta(a_i)'' d a_i + \cdots + \sigma(da_n-1)\sigma(\Theta(a_1)''),$

where $(-1)^{\sigma_n}$ is the Koszul sign. We thus get an $R$-bilinear map $\iota_{\Theta}(-) : \widehat{\Omega}^\bullet R \to \widehat{\Omega}^{\bullet-1} R$, $\omega \mapsto \iota_{\Theta}\omega$.

It is straightforward to see that $\iota_{\Theta}(-)$ vanishes on the commutator subspace

$$\{xy - (-1)^{|x||y|} y\sigma(x), \text{ where } x, y \in \widehat{\Omega}^\bullet R\}$$

and hence descends to a well-defined map $\iota_{\Theta}(-) : \overline{\text{DR}}^\bullet R \to \widehat{\Omega}^{\bullet-1} R$.

Now recall that $\overline{\text{DR}}^\bullet R$ is a mixed complex with two differentials $\partial$ and $d$; we say that $\omega \in \overline{\text{DR}}^\bullet R$ extends to be a closed form in the negative cyclic complex associated to $\overline{\text{DR}}^\bullet R$ if there exists a sequence $\omega_0 = \omega, \omega_1, \omega_2, \cdots$ such that

$$\partial \omega_0 = 0, \quad d \omega_i = \partial \omega_{i+1} \quad \text{for all } i = 0, 1, 2, \cdots.$$
Lemma 2.3 If \( \omega \in \hat{\text{DR}}^2_\sigma R \) is \( \sigma \)-invariant and extends to be a closed 2-form, then the following map

\[
\Psi \circ \iota_{(-)} \omega : \text{Der} R \to \hat{\Omega}^1 R[n-2], \Theta \mapsto \Psi(\iota\Theta \omega),
\]

where \( \Psi \) is given by (8), is a map of DG \( \sigma \)-bimodules.

Proof Denote by \( \Phi \) the map \( \Theta \mapsto \Psi(\iota\Theta \omega) \). We first show \( \Phi \) is a chain map. In fact, in \( \omega = \sum d\eta' \otimes d\eta'' \), if \( \partial \eta' = \sum \eta'_1 \cdots \eta'_p \) and \( \partial \eta'' = \sum \eta''_1 \cdots \eta''_q \), then

\[
\partial \omega = -\sum (d\partial \eta' \otimes d\eta'' + (-1)^{|d\eta'|} d\eta' \otimes d\partial \eta'')
\]

\[
= -\sum \left( \sum_i (-1)^{|\eta'_{i-1}|} \eta'_1 \cdots d\eta'_i \cdots \eta'_p \otimes d\eta'' 
\right)
\]

\[
+ (-1)^{|d\eta'|} \sum_j (-1)^{|\eta''_{j-1}|} d\eta' \otimes \eta''_1 \cdots d\eta''_j \cdots \eta''_q).
\]

Since \( R \) is free, that \( \partial \omega \) descends to be zero in the quotient space \( \hat{\text{DR}}^2_\sigma R \) means all the summands in the above expression cancel out, which further means, up to a sign difference, the terms in the first summand and the terms in the second summand are equal to each other up to a cyclic permutation followed by a twisting; more precisely, we have, up to a sign, \( \sum_i d\eta'_1 \cdots \eta'_p d\eta'' \sigma(\eta'_1 \cdots \eta'_{i-1}) \) equals \( \sum_j d\eta''_1 \cdots d\eta''_j \cdots \eta''_q \) (see also Lemma 3.3 for an explicit computation when \( p = q = 2 \)). Therefore if we denote by \( \partial \) and \( \delta \) the internal differential (the differential induced from that of \( R \)) of \( \hat{\Omega}^1 R \) and \( \text{Der} R \) respectively, then on one hand,

\[
\partial \circ \Phi(\Theta) = \partial \left( \sum \Theta(\eta'')' \cdot \Psi(d\eta'') \cdot \sigma(\Theta(\eta')) \right)
\]

\[
= \sum (\delta \Theta)(\eta'')' \cdot \Psi(d\eta'') \cdot \sigma((\delta \Theta)(\eta'))
\]

\[
+ \text{those terms involving } \partial \omega
\]

\[
= \sum (\delta \Theta)(\eta'')' \cdot \Psi(d\eta'') \cdot \sigma((\delta \Theta)(\eta')), \tag{15}
\]

where in the above, the third equality holds since those terms involving \( \partial \omega \) vanish due to the twisted cyclicity described above. On the other hand,

\[
\Phi(\delta(\Theta)) = \sum (\delta \Theta)(\eta'')' \cdot \Psi(d\eta'') \cdot \sigma((\delta \Theta)(\eta')) \tag{16}
\]

Thus that \( \Phi \) is a chain map follows from that (15) equals (16).

Next, we show that \( \Phi \) is a map of \( R \)-bimodules. That is, we need to show that, for any \( u, v \in R \),

\[
\Phi(u \ast \Theta \ast v) = u \cdot \Phi(\Theta) \cdot \sigma(v), \quad \text{for all } \Theta \in \text{Der} R.
\]
In fact, if we write $\Theta(\eta) = \sum \Theta'(\eta) \otimes \Theta''(\eta)$, then by (5) we have

$$(u \ast \Theta \ast v)(\eta) = \sum \Theta'(\eta) \cdot v \otimes u \cdot \Theta''(\eta).$$

From this we obtain

$$\Phi(u \ast \Theta \ast v) = t_{u \ast \Theta \ast v}(\sum d\eta' \otimes d\eta'') = \sum (u \cdot \Theta''(\eta')) \cdot \Psi(d\eta'' \cdot \sigma(\Theta'(\eta') \cdot v)) = u \cdot \left( \sum \Theta''(\eta') \cdot \Psi(d\eta'') \cdot \sigma(\Theta'(\eta')) \right) \cdot \sigma(v) = u \cdot \Phi(\Theta) \cdot \sigma(v).$$

(17)

The lemma is now proved.

**Definition 2.4** (Twisted bi-symplectic structure) Let $R$ be a DG free algebra with $\sigma \in \text{Aut} R$. A twisted symmetric 2-form $\omega \in \hat{DR}^2_R \sigma R$ of total degree $n$ is called a $(2 - n)$-shifted twisted bi-symplectic structure if

1. it extends to be a closed 2-form in the negative cyclic complex associated to $\hat{DR}^2_R \sigma R$;
2. it is $\sigma$-invariant and induces a quasi-isomorphism

$$\Psi \circ t_{(-)} : \hat{\text{Der}} R \rightarrow \hat{\text{Omega}}^1 R[n - 2]$$

of DG $R^e$-modules.

We say a DG associative algebra $A$ has a derived twisted bi-symplectic structure if its DG free model has a twisted bi-symplectic structure.

**Remark 2.5** [About the degree shifting] The $n$-shifted symplectic structure for derived stacks was introduced by Pantev et al in [26]. The above definition is the noncommutative and twisted analogue of theirs. In what follows, we prefer to write (18) in the form

$$\Psi \circ t_{(-)} : \hat{\text{Der}} R[1] \rightarrow \hat{\Omega}^1_R[1][n].$$

**Corollary 2.6** Suppose $A$ has a derived twisted $(2 - n)$-shifted symplectic structure. Denote by $\sigma$ the automorphism of $A$. Then

$$\text{HH}^\bullet(A) \cong \text{HH}_{n-\bullet}(A, A_\sigma).$$

**Proof** By (10) and (13) we have

$$[\delta \hat{\text{Der}} R[1]] \cong \cong \text{CH}^\bullet(A) \quad \text{and} \quad [\hat{\Omega}^1_R[-1]] \cong \text{CH}_\bullet(A, A_\sigma)$$

respectively. Thus the corollary follows from Definition 2.4 and Theorem 2.2. □
3 AS-regular algebras

The notion of Artin-Schelter regular algebras was introduced by Artin and Schelter in [2]. They have been extensively studied in the past three decades. One may refer to [37], in particular Chapter 18, for some more backgrounds which are related to our study. In this section, we first recall the definition of AS-regular algebras, discuss their Koszul dual algebra and coalgebra, and then prove Theorem 1.1.

Definition 3.1 [Artin-Schelter [2]] A connected graded algebra $A$ is called Artin-Schelter regular (or AS-regular for short) of dimension $n$ if

1. $A$ has finite global dimension $n$, and
2. $A$ is Gorenstein, that is, $\text{Ext}^i_A(k, A) = 0$ for $i \neq n$ and $\text{Ext}^n_A(k, A) \cong k$.

Every AS-regular algebra has an automorphism, called the Nakayama automorphism. In this paper, we are mostly interested in those AS-regular algebras which are Koszul (c.f. [25, Chapters 1-3] for more details of Koszul algebras). For a Koszul AS-regular algebra, its Nakayama automorphism can be explicitly described; for the Nakayama automorphism of a not-necessarily-Koszul AS regular algebra, see, for example, [29].

Suppose $A$ is a Koszul AS-regular algebra, then the Koszul complex

$$
\cdots \longrightarrow A \otimes A^i_d \longrightarrow A \otimes A^i_{d-1} \longrightarrow \cdots \longrightarrow A \otimes k \longrightarrow k
$$

gives a resolution of $k$ as a left $A$-module, where $A^i$ is the Koszul dual coalgebra of $A$. From this we obtain

$$
\text{RHom}_A(k, A) = \text{Hom}_A(\bigoplus_i A \otimes A^i, A) = A \otimes A^!,$$

where $A^!$ is the Koszul dual algebra of $A$, and thus from $\text{RHom}_A(k, A) \cong k[n]$ we have

$$
A \otimes A^! \cong A \otimes A^i[n].
$$

The compatibility of the differentials gives isomorphism $A^! \cong A^i[n]$ as left $A^!$-modules. Similarly we get $A^! \cong A^i[n]$ as right $A^!$-modules. These two isomorphisms together are equivalently saying that $A^!$ is a Frobenius algebra of degree $n$ (see e.g. [30] for more studies of this result). In what follows, we study the structure of a Frobenius algebra with some more details.

3.1 Frobenius algebras

In the literature, a Frobenius algebra structure of degree $n$ on a finite dimensional graded algebra, say $A^!$, is a non-degenerate bilinear pairing of degree $n$

$$
\langle -, - \rangle : A^! \otimes A^! \rightarrow k
$$

(19)
such that \( \langle a, b \cdot c \rangle = \langle a \cdot b, c \rangle \), for all \( a, b, c \in A^1 \). This is equivalent to saying that \( A^1 \) is isomorphic to its linear dual \( A^1[n] \) as left and right \( A^1 \)-modules. Since \( A^1 \) is finite dimensional, \( A^1 \) is a graded coalgebra, and the adjoint of (19) gives a map

\[
k \rightarrow A^1 \otimes A^1, \quad 1 \mapsto \sum \eta' \otimes \eta''.
\]

(20)

This map factors through \( A^1 \); that is, there is an \( \eta \) in the degree \( n \) component of \( A^1 \) (which is 1-dimensional) such that \( \sum \eta' \otimes \eta'' = \Delta(\eta) \), which \( \Delta(-) \) is the coproduct of \( A^1 \). In this form, the pairing on \( A^1 \) is also given by

\[
(u, v) \mapsto \eta'(v) \cdot \eta''(u), \quad \text{for } u, v \in A^1,
\]

By the non-degeneracy of the pairing on \( A^1 \), we have two isomorphisms of graded vector spaces \( A^1 \rightarrow A^1[n] \) which are given by

\[
\eta' \mapsto \sum \eta''(u) \cdot \eta' \quad \text{and} \quad \eta'' \mapsto \sum \eta'(u) \cdot \eta''.
\]

(21)

Since \( \sum \eta' \otimes \eta'' = \Delta(\eta) \), these two maps are exactly the slant product of \( u \) with \( \eta \) from the left and the right respectively, and therefore by the co-associativity of the co-product, these isomorphisms can be extended to an isomorphism of left and right \( A^1 \)-modules, respectively.

However, on the other hand, these two isomorphisms may not give an \( A^1 \)-bimodule on \( A^1 \), due to the fact that the pairing on \( A^1 \) may not be graded symmetric; that is, \( \langle a, b \rangle \) may not be equal to \( (-1)^{|a||b|} \langle b, a \rangle \). Nevertheless, by the non-degeneracy of the pairing there exists an automorphism \( \sigma^* : A^1 \rightarrow A^1 \) such that \( \langle a, b \rangle = (-1)^{|a||b|} \langle b, \sigma^*(a) \rangle \).

Now, equip \( A^1 \) as an \( A^1 \)-bimodule structure by setting

\[
u \circ x \circ v := \sigma^*(u)xv,
\]

which is denoted by \( A^1_{\sigma} \), then we have \( A^1 \cong A^1_{\sigma}[n] \) as \( A^1 \)-bimodules. Here, \( \sigma^* \) is called the Nakayama automorphism of \( A^1 \), whose adjoint is denoted by \( \sigma : A^1 \rightarrow A^1 \).

Now let \( \Omega(A^1) \) be the cobar construction of \( A^1 \), which is a DG free algebra generated by \( \Sigma^{-1} \tilde{A}^1 \), the augmentation of \( A^1 \) with degree shifted down by 1. Then \( \sigma \) can be lifted to be an automorphism \( \sigma : \Omega(A^1) \rightarrow \Omega(A^1) \). Going back to the Koszul AS-regular algebra case, since \( \Omega(A^1) \rightarrow A \) is a quasi-isomorphisms of DG algebras, \( \sigma \) also induces an automorphism \( \sigma : A \rightarrow A \), which is also called the Nakayama automorphism of \( A \). We have the following result due to Reyes, Rogalski and Zhang in [29, Lemma 1.2]:

**Theorem 3.2** Let \( A \) be an associative algebra. Then \( A \) is AS-regular of dimension \( n \) with Nakayama automorphism \( \sigma \) if and only if \( A \) is twisted Calabi-Yau of dimension \( n \); that is, \( A \) satisfies the following two conditions:

1. \( A \) is homologically smooth, and
2. \( \text{RHom}_{A^e}(A, A \otimes A) \cong A_\sigma[n] \) in \( D(A^e) \).
In the theorem, twisted Calabi-Yau comes from the fact that, if $\sigma$ is the identity map, then $A$ is exactly a Calabi-Yau algebra in the sense of Ginzburg [17]. Observe that the second condition in the above theorem is a derived invariant, and therefore if $R$ is a DG free resolution of $A$, then we also have $\text{RHom}_{R^e}(R, R \otimes R) \cong R_\sigma[n]$ in $D(R^e)$, where $\sigma$ is the lifting of the Nakayama automorphism of $A$. Since now $R$ is DG free, we have a short exact sequence

$$0 \rightarrow \Omega^1 R \rightarrow R \otimes R \rightarrow R \rightarrow 0,$$

where $\Omega^1 R$ is a DG free $R$-bimodule. This means $\widehat{\Omega}^1 R[-1]$ is a DG free resolution of $R$ as $R$-bimodules, and therefore,

$$\text{RHom}_{R^e}(R, R \otimes R) \cong \text{Hom}_{R^e}(\widehat{\Omega}^1 R[-1], R \otimes R) \cong \widehat{\text{Der}} R[1]$$

and similarly $\widehat{\Omega}^1_\sigma R[-1] \cong R_\sigma[n]$ in $D(R^e)$. Thus $\text{RHom}_{R^e}(R, R \otimes R) \cong R_\sigma[n]$ is equivalent to $\widehat{\text{Der}} R[1] \cong \widehat{\Omega}^1_\sigma R[-1][n]$ in $D(R^e)$.

We now discuss some more details on $\widehat{\text{Der}} R[1]$ for $R = \Omega(A^i)$. Since $A^i$ is finite dimensional (and hence so is $A^i$), the product $\mu : A^i \otimes A^i \rightarrow A^i$ induces two maps

$$A^i \rightarrow A^i \otimes A^i, u \mapsto \sum_c \tilde{c}u \otimes c \quad \text{and} \quad A^i \rightarrow A^i \otimes A^i, u \mapsto \sum_c c \otimes u \tilde{c}$$

by adjunction, where $c$ runs over the bases of $A^i$ and $\tilde{c}$ is its linear dual. By the associativity of the product, the above two maps in fact make $A^i$ into a left and right $A^i$-comodules. Moreover, since the product on $A^i$ is an $A^i$-bimodule map, we get that

$$A^i \mapsto A^i \otimes A^i \otimes A^i \subset A^i \otimes A^i \otimes A^i, u \mapsto \sum_c \tilde{c}u \otimes c + c \otimes u \tilde{c}$$

makes $A^i$ into an $A^i$-bi-comodule. The internal differential, say $\delta$, on $\widehat{\text{Der}} R$ is given as follows: since

$$\widehat{\text{Der}} R[1] \cong \text{Hom}_{R^e}(\widehat{\Omega}^1 R[-1], R \otimes R) \cong \text{Hom}_{R^e}(R \otimes A^i \otimes R, R \otimes R) \cong R \otimes A^i \otimes R$$

(22)

as $R^e$-modules, taking a generator $u \in A^i \cong k \otimes A^i \otimes k \subset R \otimes A^i \otimes R$, if we denote by $Du \in \widehat{\text{Der}} R[1]$ its image under the above isomorphism, then

$$\delta(Du) = \sum_c (-1)^{|c|+|u|-1} \tilde{c}u \otimes \tilde{c} + \tilde{c} \otimes u \tilde{c} \in A^i \otimes \Sigma^{-1} \tilde{A}^i \otimes \Sigma^{-1} \tilde{A}^i \otimes A^i \subset R \otimes A^i \otimes R,$$

(23)

where $\tilde{c}$ is image of $c$ under the natural map $A^i \rightarrow \Sigma^{-1} \tilde{A}^i$. Observe that $\delta$ is dual, via (22), to the internal differential on $\widehat{\Omega}^1 R$ induced by $\partial$ on $R$. 

3.2 The twisted bi-symplectic structure

Now suppose \( A \) is a Koszul AS-regular algebra of dimension \( n \) and let \( R := \Omega(A^i) \).
Take \( \sum \eta' \otimes \eta'' = \Delta(\eta) \) as given in \((20)\) and consider the following 2-form
\[
\omega := \frac{1}{2} \sum d\eta' \otimes d\eta'' \in \hat{\Omega}^2 R, \tag{24}
\]
which descends to a 2-form, still denoted by \( \omega \), in \( \hat{\mathrm{Der}}^2 R \). We have the following:

**Lemma 3.3** Suppose \( A \) is Koszul AS-regular. Let \( R = \Omega(A^i) \) be the cobar construction of \( A^i \). Then \( \omega \in \hat{\mathrm{Der}}_R^2 \sigma \) is both \( \partial - \) and \( d - \) closed.

**Proof** (Compare with [10, Lemma 4.5].) It is obvious that \( \omega \) is \( d - \) closed. We next show it is also \( \partial - \) closed. In fact, if we write \( \Delta(\eta) = \sum \eta_1 \otimes \eta_2 \) for any \( \eta \in A^i \), then
\[
\partial \omega = \frac{1}{2} \sum \partial (d\eta') \otimes d\eta'' + (-1)^{||\eta'||} d\eta' \otimes \partial (d\eta'')
\]
\[
= -\frac{1}{2} \sum d(\partial \eta') \otimes d\eta'' + (-1)^{||\eta'||} d\eta' \otimes d(\partial \eta'')
\]
\[
= -\frac{1}{2} \sum (-1)^{||\eta'||} d(\eta_1 \eta_2) \otimes d\eta'' + (-1)^{||\eta'||+||\eta''||} d\eta' \otimes d(\eta_1 \eta_2)
\]
\[
= -\frac{1}{2} \sum \left( (-1)^{||\eta'||} (d\eta_1) \eta_2 \otimes d\eta'' - \eta_1 (d\eta_2) \otimes d\eta''
\right)
\]
\[
+ (-1)^{||\eta'||+||\eta''||} d\eta' \otimes (d\eta_1) \eta_2'' - (-1)^{||\eta'||} d\eta' \otimes \eta_1'' (d\eta_2) \right). \tag{25}
\]

In the right hand side of the last equality, the first and the last summands cancel with each other, due to the co-associativity of \( A^i \). For the second and third summands, let us recall that in \( A^i \), we have
\[
\langle a, b \cdot c \rangle = (-1)^{(|b|+|c|)|a|} \langle b \cdot c, \sigma^*(a) \rangle
\]
\[
= (-1)^{(|b|+|c|)|a|} \langle b, c \cdot \sigma^*(a) \rangle, \quad \text{for all } a, b, c \in A^i.
\]
This means on the linear dual space \( A^i \), suppose \( (id \otimes \Delta) \circ \Delta(\eta) = \sum \eta' \otimes \eta'' \otimes \eta''' \), then
\[
\sum \eta' \otimes \eta'' \otimes \eta''' = \sum (-1)^{||\eta'||(||\eta''||+||\eta'''||)} \eta'' \otimes \eta''' \otimes \sigma(\eta').
\]
Thus when passing to the twisted quotient space, the second and third summands cancel with each other. This proves the lemma.

**Proof of Theorem 1.1** By Definition 2.4 and Lemma 3.3, we only need to show \( \omega \) is \( \sigma - \) invariant and
\[
\Psi \circ \iota(-) \omega : \hat{\mathrm{Der}} R[1] \to \hat{\Omega}_\sigma^1 R[-1][n] \tag{26}
\]
is a quasi-isomorphism.
In fact, since on $A^!$ the pairing satisfies $\langle a, b \rangle = (-1)^{|a||b|} \langle \sigma(b), a \rangle = \langle \sigma(a), \sigma(b) \rangle$, we get that $\Delta(\eta) = \sum \eta' \otimes \eta''$ and hence $\omega$ is $\sigma$-invariant. Also, since

$$\hat{\mathcal{H}}^1_\sigma R[−1] \cong R \otimes A^! \otimes R \quad \text{and} \quad \hat{\text{Der}} R[1] \cong R \otimes A^! \otimes R,$$

the isomorphism (26) on the generators becomes

$$\iota_{(-)} \omega : A^! \to A^! [n], \quad u \mapsto \sum u(\eta') \cdot \eta''[n],$$

which is exactly the isomorphism given by (21). Therefore the theorem follows. \qed

In other words, $\omega$ given by (24) is a (derived) twisted bi-symplectic structure on $A$.

As a by-product, we get the following result due to Van den Bergh [34] in general and Reyes, Rogalski and Zhang [29] in the case of AS-regular algebras, which is in fact the main motivation of the current paper:

**Corollary 3.4** (Van den Bergh [34]; Reyes-Rogalski-Zhang [29]) Suppose $A$ is AS-regular of dimension $n$. Then

$$\text{HH}^\bullet(A) \cong \text{HH}_{n-\bullet}(A, A_\sigma),$$

(27)

which coincides with Van den Bergh’s noncommutative Poincaré duality [34].

**Proof** Follows from Theorem 1.1 and Corollary 2.6. \qed

**Remark 3.5** In the case of Calabi-Yau algebras, de Thanhoffer de Völcsey and Van den Bergh proved in [14,Theorem 5.5] that there exists a volume class $\eta \in \text{HH}_n(A)$ such that the noncommutative Poincaré duality

$$\text{HH}^\bullet(A) \stackrel{\cong}{\longrightarrow} \text{HH}_{n-\bullet}(A)$$

is given by capping with the volume class $\eta$. In the general AS-regular algebra case, this could not be true at first glance, since there is no cap-product between the Hochschild cohomology and the twisted Hochschild homology. Nevertheless, Theorem 1.1 says that after choosing appropriate chain complexes, there does exist a volume class such that capping with it gives the isomorphism (27).

### 3.3 Examples

In this subsection, we study two examples of AS-regular algebras with nontrivial Nakayama automorphism, and give their twisted bi-symplectic structure explicitly.

**Example 3.6** Let $A = k\langle x_1, \cdots, x_n \rangle/(f)$, where

$$f = (x_1, \cdots, x_n) M (x_1, \cdots, x_n)^T, \quad M \in \text{GL}_n(k),$$
and \((f)\) means the ideal generated by \(f\). Dubois-Violette proved in \([15]\) that \(A\) is a Koszul AS-regular algebra of dimension 2. The Koszul dual algebra \(A^!\) is graded Frobenius of global dimension 2 with

\[
A^! = k \oplus A_1^! \oplus A_2^!,
\]

where \(A_1^!\) is the \(k\)-linear dual space of \(\text{Span}_k \{x_1, \ldots, x_n\}\) with the corresponding dual basis \(\{\tilde{x}_1, \ldots, \tilde{x}_n\}\) and \(A_2^! = (A_1^! \otimes A_1^!)/\{a \in V^* \otimes V^* | \alpha(f) = 0\}\), which is 1-dimensional, say, spanned by \(\{z\}\). The Frobenius pair \((-,-)\) on \(A^!\) is given by

\[
\langle a, b \rangle = a M b^T, \quad a = a_1 \tilde{x}_1 + \cdots + a_n \tilde{x}_n, \quad b = b_1 \tilde{x}_1 + \cdots + b_n \tilde{x}_n.
\]

From \(\langle a, b \rangle = \langle b, \sigma^*(a) \rangle\) we get

\[
\sigma^*(a) = (\tilde{x}_1, \ldots, \tilde{x}_n) M^{-1} M^T (a_1, \ldots, a_n)^T, \quad \sigma^*(z) = z.
\]

By Van den Bergh [33,Theorem 9.2], the Nakayama automorphism \(\sigma\) of \(A\) is

\[
\sigma = \epsilon^{2+1}(\sigma^* T)^{-1},
\]

where \(\epsilon(a) = (-1)^m a\) for \(a \in A_m\). On \(A_1\), for \(x = \sum k_i x_i, k_i \in k\),

\[
\sigma(x) = -(x_1, \ldots, x_n) M^T M^{-1} (k_1, \ldots, k_n)^T.
\]

Next we give the explicit twisted bi-symplectic structure on \(A\). The Koszul dual coalgebra \(A^!\) has the form

\[
A^! = k \oplus A_1^! \oplus A_2^!
\]

The volume form \(\eta = (x_1, \ldots, x_n) M (x_1, \ldots, x_n)^T = \sum_{i,j=1}^n m_{ij} x_i x_j\), and therefore the twisted bi-symplectic structure is given by

\[
\omega = d(1) \otimes d\left(\sum_{i,j} m_{ij} x_i x_j\right) + \sum_{i,j} m_{ij} d(x_i) \otimes d(x_j) + d\left(\sum_{i,j} m_{ij} x_i x_j\right) \otimes d(1).
\]

**Example 3.7** (Quantum affine space) Let \(Q = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}\) be an \(n \times n\) matrix over \(k\) with \(q_{ii} = 1, q_{ij} q_{ji} = 1\), for \(1 \leq i, j \leq n\). Let \(A = k \langle x_1, \ldots, x_n \rangle/(x_j x_i - q_{ij} x_i x_j)\). In [24,Proposition 4.1], Liu, Wang and Wu proved that \(A\) is a twisted Calabi-Yau algebra with the Nakayama automorphism \(\sigma\) given by

\[
\sigma(x_i) = (\prod_{j=1}^n q_{ji}) x_i, \quad i = 1, \ldots, n.
\]
The Koszul dual algebra $A^!$ of $A$ is

$$A^! = k\langle \tilde{x}_1, \cdots, \tilde{x}_n \rangle / (q_{ij} \tilde{x}_j \tilde{x}_i + \tilde{x}_i \tilde{x}_j | 1 \leq i, j \leq n).$$

The volume form of $R = \Omega(A^!)$ is

$$\sum_{\sigma = (i_1, i_2) \cdots (i_k, i_{k+1}) \in S_n} \text{sgn}(\sigma)(q_{i_1i_2} \cdots q_{i_{k+1}i_{k+2}})x_{\sigma(1)} \cdots x_{\sigma(n)},$$

from which we can write down the twisted symplectic form explicitly. For example, for $n = 2$, we have $Q = \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}$, then

$$\sigma(x_1) = q_{21}x_1, \quad \sigma(x_2) = q_{12}x_2.$$  

Similarly to Example 3.6, the twisted symplectic structure is

$$\omega = d(1) \otimes d(x_2x_1 - q_{12}x_1x_2) + d(x_2) \otimes d(x_1)$$

$$- q_{12} d(x_1) \otimes d(x_2) + d(x_2x_1 - q_{12}x_1x_2) \otimes d(1).$$

For $n > 2$, the method is the same, and we leave it to the interested reader.

**Remark 3.8** In Theorem 1.1, we have assumed that the AS-regular algebra is Koszul. In fact, by the same technique, the theorem holds for $N$-Koszul AS-regular algebras, too, such as the down-up algebra introduced by Benkart and Roby in [7].

## 4 Derived representation schemes

From now on we study the twisted symplectic structure on the derived representation schemes of a Koszul AS-regular algebra. Derived representation schemes were first introduced by Berest, Khachatryan and Ramadoss in [6] (see also [3–5] for further discussions and applications). The goal of this and the subsequent section is to prove Theorem 1.2. At the same time, these two sections also improve some results presented in [10] if we take the algebra to be Calabi-Yau, and hence serves as supplement to it.

### 4.1 Representation functor of DG algebras

Let $A$ be a DG $k$-algebra. Let $V$ be a chain complex of finite total dimension. Consider the following representation functor

$$\text{Rep}_V(A) : \text{CDGA} \to \text{Sets}, \quad B \mapsto \text{Hom}_{\text{DGA}}(A, \text{End} V \otimes B),$$

where $\text{End} V$ is the DG algebra of endomorphisms of $V$. It is proved in [6, Theorem 2.1] that this functor is representable. That is, there exists a DG commutative algebra $A_V$, which only depends on $A$ and $V$, such that
\[ \text{Hom}_{\text{DGA}}(A, \text{End } V \otimes B) = \text{Hom}_{\text{CDGA}}(A_V, B). \]

In other words, we may view \( \text{Rep}_V(A) \) as a DG affine scheme, and \( A_V = \mathcal{O}(\text{Rep}_V(A)) \).

The rough idea of Berest et al in [6] is as follows. Let \( \text{DGA}_{\text{End}(V)} \) be the category of DG algebras under \( \text{End} V \), that is, the objects of \( \text{DGA}_{\text{End}(V)} \) are DG algebra maps \( \text{End} V \to A \) and the morphisms are commutative triangles

\[
\begin{array}{ccc}
\text{End} V & \xrightarrow{A} & B \\
\end{array}
\]

Then the following functor

\[ G : \text{DGA}_k \to \text{DGA}_{\text{End}(V)}, \quad A \mapsto \text{End} V \otimes A \]

is in fact an equivalence of categories, whose inverse is

\[ G^{-1} : \text{DGA}_{\text{End}(V)} \to \text{DGA}_k, \quad (\text{End} V \to A) \mapsto A^{\text{End}(V)}, \]

where \( A^{\text{End}(V)} \) is the centralizer of \( \text{End} V \) in \( A \); see [6, Lemma 2.1] for more details.

Now introduce the following two functors

\[ V^{\sqrt{-}} : \text{DGA}_k \to \text{DGA}_k, \quad A \mapsto (\text{End} V \ast_k A)^{\text{End} V}, \quad (29) \]

\[ (-)^V : \text{DGA}_k \to \text{CDGA}_k, \quad A \mapsto (\sqrt{A})_{22}, \quad (30) \]

where \( \ast \) is the coproduct of two \( k \)-algebras, and \( (-)^{22} \) is the “commutativization” of the algebra, namely, taking the quotient of the algebra by the two-sided ideal generated by the commutators. The following is proved in [6, Proposition 2.1]:

**Proposition 4.1**  For any \( A, B \in \text{DGA}_k \) and \( C \in \text{CDGA}_k \), there are natural bijections:

1. \( \text{Hom}_{\text{DGA}_k} (\sqrt{A}, B) \cong \text{Hom}_{\text{DGA}_k} (A, \text{End} V \otimes B); \)
2. \( \text{Hom}_{\text{CDGA}_k} (A_V, C) \cong \text{Hom}_{\text{DGA}_k} (A, \text{End} V \otimes C). \)

The second statement of this proposition exactly says that \( A_V \) represents the functor (28). Now fixing \( V \), we get a functor

\[ (-)^V : \text{DGA}_k \to \text{CDGA}_k, \quad A \mapsto A_V. \]

Then [6, Theorem 2.2] says that \( (-)^V \) has a right adjoint

\[ \text{End} V \otimes - : \text{CDGA}_k \to \text{DGA}_k, \quad B \mapsto \text{End} V \otimes B \]
such that

\[ (-)_V : \text{DGA}_k \rightleftharpoons \text{CDGA}_k : \text{End} V \otimes - \tag{31} \]

form a Quillen pair, and hence can be lifted to their homotopy categories.

### 4.2 Quillen’s adjunction theorem

We recommend the survey of Dwyer and Spalinski [16] for an introduction to model categories.

Suppose \( A \) is a model category, then the homotopy category \( \text{Ho}(A) \) of \( A \) is the category where the objects remain the same as those in \( A \), and the morphisms for two objects, say \( A \) and \( B \), are given by

\[
\text{Hom}_{\text{Ho}(A)}(A, B) := \text{Hom}_A(QA, QB)/\text{quasi-isomorphisms},
\]

where \( QA \) and \( QB \) are the cofibrant resolutions of \( A \) and \( B \) respectively (for more details see [16,§5]). Both \( \text{DGA}_k \) and \( \text{CDGA}_k \) are model categories, and their homotopy categories are denoted by \( \text{Ho(DGA}_k) \) and \( \text{Ho(CDGA}_k) \) respectively.

Now suppose \( A, B \) are two model categories, and two functors

\[ F : A \rightleftharpoons B : G \]

form a Quillen pair. Then Quillen’s adjunction theorem says that the derived functors

\[ LF : \text{Ho}(A) \rightleftharpoons \text{Ho}(B) : RG \]

exist, and again form a Quillen pair. The functor \( LF \) is given by \( LF(A) = \gamma F(QA) \), where \( QA \to A \) is a(ny) cofibrant resolution of \( A \), and \( \gamma : A \to \text{Ho}(A) \) is the natural functor from \( A \) to its homotopy category, which is the identity on objects, and sends morphisms to the homotopy equivalence classes of their liftings on cofibrant resolutions.

### 4.3 Derived representation schemes

Since the functors (31) form a Quillen pair, from the above argument they have the associated derived functors.

**Definition 4.2** (Berest, Khachatryan and Ramadoss [6]) For a chain complex \( V \) of finite total dimension, the derived functor

\[ L(-)_V : \text{Ho(DGA)} \to \text{Ho(CDGA)}, A \mapsto (QA)_V \]
is called the derived representation functor in $V$. For $A \in \text{Ho(DGA}_k)$, the DG affine scheme $\text{Spec}(L(A)_V)$ of $L(A)_V$ is called the derived representation scheme\footnote{In [6], the derived representation scheme of $A$ is simply the DG commutative algebra $L(A)_V$, and is also denoted by $\text{DRep}_V(A)$. Here we reserve this terminology to mean its affine scheme, which is consistent with Yeung in [39].} of $A$ in $V$, and is also denoted by $\text{DRep}_V(A)$. The homology $H_\bullet(L(QA)_V)$ is called the representation homology of $A$.

The interested reader may also refer to Ciocan-Fontanine and Kapranov [11] for more details about derived schemes.

Suppose $A$ is an associative algebra, viewed as a DG algebra with zero differential, then any DG free resolution of $A$ is a cofibrant resolution. If we view the cofibrant resolution of $A$ as the best approximation of $A$ by DG free algebras, then $\text{DRep}_V(A)$ best approximates $\text{Rep}_V(A)$. In particular, the zeroth homology $H_0(\text{DRep}_V(A)) = R(\text{Rep}_V(A))$

(see [6,Theorem 2.5]).

\textbf{Example 4.3} Assume $V = k^d$ and $A$ is an object in $\text{Ho(DGA}_k)$. Suppose $(R, \partial) = (k\langle x^\alpha \rangle_{\alpha \in I}, \partial)$ is a cofibrant resolution of $A$. We construct a DG commutative algebra as follows: to each $x^\alpha$ are associated variables $x^\alpha_{ij}, 1 \leq i, j \leq d$, with the same grading as $x^\alpha$. Consider the assignments $x^\alpha \mapsto (x^\alpha_{ij})$, then the differential on these variables are assigned such that

$$\partial(x^\alpha_{ij}) = (\partial x^\alpha)_{ij}.$$ 

$\text{DRep}_V(A)$ is an object in $\text{Ho(CDGA)}$ isomorphic to $(k[x^\alpha_{ij}]_{\alpha \in I}, \partial)$.

From now on, we always assume $V$ is a vector space, that is, $V = k^d$ for some $n \in \mathbb{N}$. $\text{DRep}_V(A)$ is sometimes also denoted by $\text{DRep}_d(A)$.

\section{The GL-invariant subfunctor}

Observe that the general linear group $\text{GL}(V)$, as a subspace of $\text{End } V$, acts from the right on the latter by conjugation $\alpha \mapsto g^{-1}\alpha g$, for all $g \in \text{GL}(V)$. It induces a right action on the functor $\text{End } V \otimes - : \text{CDGA}_k \to \text{DGA}_k$. Through the adjunction in Proposition 4.1(2) we get an action of $\text{GL}(V)$ from the left on the representation functor $(\_)_V : \text{DGA}_k \to \text{CDGA}_k$. Passing to the homotopy category, we get a $\text{GL}(V)$-action on $\text{DRep}_n(A)$.

Thus we may consider the invariant subfunctor

$$(\_)_V^{\text{GL}} : \text{DGA}_k \to \text{CDGA}_k, \quad A \mapsto A_V^{\text{GL}}.$$ \hspace{1cm} (32)

It is showed in [6] that such a functor does not seem to have a right adjoint, and therefore the Quillen Adjunction Theorem does not apply. Nevertheless, the authors showed the following:
Theorem 4.4 ([6] Theorem 2.6) (1) The functor (32) has a total left derived functor
\[ L(-)^{GL}_V : \text{Ho}(DGA_k) \to \text{Ho}(CDGA_k). \]

(2) For any \( A \) in \( DGA_k \), there is a natural isomorphism of graded algebras
\[ H_\bullet(L(A)^{GL}_V) \cong H_\bullet(L(A)_V)^{GL}. \]

4.5 Van den Bergh’s functor

For a DG algebra \( A \), in this subsection we study the differential forms on \( \text{DRep}_V(A) \) and on the derived quotient stack \( \text{DRep}_V(A) / \text{GL}(V) \) in the sense of Toën-Vezzosi [32], which we denote by \( \mathcal{D}\text{Rep}_V(A) \) and call the derived moduli stack of representations of \( A \) in \( V \). We shall not go to the general theory of derived stacks; the interested reader may refer to [32]. However, in what follows we shall give enough details which is sufficient for our purpose. The materials are taken from [5, 6, 39, 40].

4.5.1 De Rham model for \( \mathcal{D}\text{Rep}_V(A) \)

For derived quotient stacks, Toën proposed in [31, §5] the following model for their differential forms: Let \( G \) be a reductive smooth group scheme over \( k \) acting on an affine derived scheme \( Y = \text{Spec} A \) and let \( X := [Y/G] \) be the derived quotient stack. The 1-forms of \( Y \), also called the cotangent complex of \( Y \), can be described algebraically (see also §5.1 for some details). The cotangent complex \( L_X \) of \( X \), pulled back to \( Y \), is the fiber \( L \) of the natural morphism \( \rho : L_Y \to \mathcal{O}_Y \otimes_k g^\vee \), dual to the infinitesimal action of \( G \) on \( Y \). In other words, the fiber \( L \) is quasi-isomorphic to the complex:
\[ L \cong \text{cone}(L_Y \xrightarrow{\rho} \mathcal{O}_Y \otimes_k g^\vee)[-1]. \] (33)

The group \( G \) acts on \( Y \) and on the morphism above, and hence on \( L \). The complex of 1-forms on \( X \) is thus given by \( L^G \). In general, the complex of \( p \)-forms \( \Omega^p(X) \) is described as
\[ \Omega^p(X) \cong \left( \bigoplus_{i+j=p} (\wedge^j_A \mathcal{L}_Y) \otimes_k \text{Sym}^i_k(g^\vee)[-j] \right)^G. \] (34)

With Toën’s model described above, a model for the de Rham algebra of \( \mathcal{D}\text{Rep}_V(A) \) is given as follows (see also Yeung [39, 40] for more details). Let \( R \) be a cofibrant resolution of \( A \). Then the 1-forms of \( \mathcal{D}\text{Rep}_V(A) \) is the \( \text{GL}(V) \)-invariant of
\[ L = \text{cone}(L_{\text{DRep}_V(R)} \xrightarrow{\rho} R_V \otimes_k \text{gl}(V)^\vee)[-1]. \] (35)

We now describe the map \( \rho \) in (35). To this end, we go back to see the action
\[ \text{GL}(V) \times \text{Spec} R_V \to \text{Spec} R_V. \]
Since \( \text{Spec } R_V = \text{Hom}_{\text{DGA}}(R, \text{End } V) \), this action is given by

\[
\text{GL}(V) \times \text{Hom}_{\text{DGA}}(R, \text{End } V) \to \text{Hom}_{\text{DGA}}(R, \text{End } V), \quad (g, x) \mapsto g \circ x \circ g^{-1}.
\]

Infinitesimally, the corresponding Lie action is

\[
gl(V) \times \text{Hom}_{\text{DGA}}(R, \text{End } V) \to \text{Hom}_{\text{DGA}}(R, \text{End } V), \quad (u, x) \mapsto \{u \circ x - x \circ u\},
\]

where \( \circ \) is the matrix multiplication. It gives a Lie algebra map \( gl(V) \to \mathcal{X}(R_V) := \text{Der}(R_V) \) as follows. Suppose \( x_{ij}^\alpha \in R_V \) (see Example 4.3 for the notation), which corresponds to a representation of \( R \) in \( V \), namely, \( x : R \to \text{End } V \). Then according to (36) we have a map

\[
gl(V) \to \mathcal{X}(R_V), \quad u \mapsto \left\{ x_{ij}^\alpha \mapsto (u \circ x^\alpha - x^\alpha \circ u)_{ij} \right\}.
\]

(37)

Dually, we get a map

\[
\phi : \Omega^1(R_V) \to gl(V)^\vee \otimes R_V
\]

which is given by

\[
\phi(dx_{ij}^\alpha)(u) = u \circ x_{ij}^\alpha \overset{(37)}{=} \sum_k u_{ik} x_{kj}^\alpha - x_{ik}^\alpha u_{kj}.
\]

(38)

If we identify \( gl(V)^\vee \) with \( gl(V) \) via the canonical pairing \( \langle u, v \rangle = \text{trace}(uv) \), which we denote by \( \text{Tr}^\vee \), then we obtain from (38) that

\[
\rho := \text{Tr}^\vee \circ \phi : \Omega^1(R_V) \to gl(V) \otimes R_V
\]

is given by

\[
\rho : dx_{ij}^\alpha \mapsto \begin{pmatrix} 0 & \cdots & x_{ij}^\alpha & \cdots & 0 \\ 0 & \cdots & x_{2i}^\alpha & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & x_{ni}^\alpha & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ x_{j1}^\alpha & x_{j2}^\alpha & \cdots & x_{jn}^\alpha \\ 0 & 0 & \cdots & 0 \end{pmatrix},
\]

(39)

where in the right of the arrow, the first matrix has zero entries except the \( j \)-th column and the second one has zero entries except the \( i \)-th row. Thus by combining (33)–(39), we obtain the following (see [31, 40]):
Proposition 4.5 With $L$ given by (35) and $\rho$ given by (39), the 1-forms of $\mathcal{D}Rep_V(A)$ is $L_{GL(V)}$, and in general,

$$\Omega^p(\mathcal{D}Rep_V(A)) \cong (\text{Sym}^p L)_{GL(V)},$$

where $\text{Sym}^\bullet(-)$ means the graded symmetric product.

4.5.2 Van den Bergh’s functor

Suppose $A$ is a DG algebra and $V$ is a chain complex. Van den Bergh introduced a functor

$$(-)^{ab}_V: \text{Bimod } A \rightarrow \mathcal{D}\text{GMod } A_V, \quad M \mapsto M \otimes_{A^e} (\text{End } V \otimes A_V). \quad (40)$$

This functor has a derived version as follows: suppose $R \rightarrow A$ is a cofibrant resolution of $A$. Note that $R \otimes_A M$ is a DG $R$-bimodule, and let $F(R, M)$ be its projective (cofibrant) resolution. Berest et al proved the following:

Proposition 4.6 ([6] Corollary 5) Let $A \in \text{Alg}$ and $M$ a complex of bimodules over $A$. The assignment $M \mapsto F(R, M)_V$ induces a well defined functor,

$$L(-)^{ab}_V: \mathcal{D}(\text{Bimod } A) \rightarrow \mathcal{D}(\mathcal{D}\text{GMod } R_V), \quad M \mapsto (F(R, M))^{ab}_V,$$

which is independent of the choice of the resolutions $R \rightarrow A$ and $F(R, M) \rightarrow M$ up to equivalence of $\mathcal{D}(\mathcal{D}\text{GMod } R_V)$ inducing the identity on homology.

In [5, 6], $L(-)^{ab}_V$ is called the derived Van den Bergh functor; sometimes it is also denoted by $L(-)_V$ if the content is clear. Due to this proposition, in what follows we shall consider directly DG bimodules over $R$ instead of those over $A$, which is then given by (40) up to homotopy.

4.5.3 Derived noncommutative tangent and cotangent complexes

Van den Bergh [36] as well as Berest et al [6] proved the following:

Lemma 4.7 Under Van den Bergh’s functor,

$$(\Omega^1 R)_V = \Omega^1(R_V) \quad \text{and} \quad (\text{Der } R)_V = \text{Der}(R_V).$$

Proof See [36,Proposition 3.3.4] for the associative algebra case, and [6,pages 661 and 671] for the derived case. \qed

Now, it is direct to see

$$(R \otimes R)_V = (R \otimes R) \otimes_{R^e} (\text{End } V \otimes R_V) = R_V \otimes \text{End } V.$$
Combining this with Lemma 4.7 we obtain the following identity
\[
\left( \text{cone}\{\Omega^1 R \to R \otimes R\} \right)_V = \text{cone}\{\Omega^1 (R_V) \to R_V \otimes \text{End } V\}. \tag{41}
\]

**Lemma 4.8** We have the following isomorphism:
\[
\text{cone}\{\Omega^1 (R_V) \to R_V \otimes \text{End } V\} \cong \text{cone}\{\mathbb{L}_{\text{DRep}_V(A)} \to R_V \otimes_k \text{gl}(V)^\vee\}. \tag{42}
\]

**Proof** We identify \(\text{gl}(V)^\vee\) with \(\text{gl}(V)\) and notice that
\[
R_V \otimes \text{gl}(V) \cong R_V \otimes \text{End } V.
\]
Therefore, the corresponding components of (42) on both sides are isomorphic, so we only need to check that the maps between two cones are the same.

To this end, let us first describe the map in
\[
\text{cone}\{\Omega^1 (R_V) \to R_V \otimes \text{End } V\},
\]
which we denote by \(\kappa\). By definition, we have the following diagram
\[
\begin{array}{c}
R \otimes A^i \otimes R \\
\downarrow \kappa \downarrow
\end{array}
\begin{array}{c}
R \otimes R
\end{array}
\begin{array}{c}
\Omega^1 (R_V)
\end{array}
\begin{array}{c}
\downarrow \kappa
\end{array}
\begin{array}{c}
\text{End } V \otimes R_V
\end{array}
\]
where \(i\) is the inclusion. That is, \(\kappa = i_V\). From this we have an explicit expression for \(\kappa\):
\[
\kappa \cong i \otimes id : (R \otimes A^i \otimes R) \otimes_{R^e} (\text{End } E \otimes R_V) \to (R \otimes R) \otimes_{R^e} (\text{End } E \otimes R_V).
\]

Pick an element \(dx^\alpha \otimes e_{ij} \in (R \otimes A^i \otimes R) \otimes_{R^e} (\text{End } E \otimes R_V)\), which is identified with \(dx^\alpha_{ij}\), then
\[
\kappa(dx^\alpha_{ij}) = (x^\alpha \otimes 1 - 1 \otimes x^\alpha) \otimes e_{ij}
= x^\alpha \circ e_{ij} - e_{ij} \circ x^\alpha
= \begin{pmatrix}
0 & \cdots & x^\alpha_{1i} & \cdots & 0 \\
0 & \cdots & x^\alpha_{2i} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & x^\alpha_{di} & \cdots & 0
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
x^\alpha_{j1} & x^\alpha_{j2} & \cdots & x^\alpha_{jd} \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}, \tag{43}
\]
where in the second equality, \(\circ\) means the action (representation) of \(x^\alpha\) on the corresponding matrices. Now observe that (43) is exactly the same as \(\rho\) given by (39), from which the lemma follows. \(\square\)
**Theorem 4.9** (See also Yeung [39]) Suppose $A$ is a DG algebra with a cofibrant resolution $R \to A$. Then

$$X(DRep_V(A))[1] \cong ((\widehat{\text{Der}} R[1])_V)^{\text{GL}(V)}$$

and

$$\Omega^1(DRep_V(A))[-1] = (\widehat{\Omega}^1 R[-1])_V^{\text{GL}(V)}.$$

**Proof** By Proposition 4.5, we know that

$$\Omega^1(DRep_V(A)) = (\text{cone}\{\Omega^1(R_V) \to R_V \otimes_k \text{gl}(V)^{\vee}\})^{\text{GL}(V)}$$

while

$$\text{cone}\{\Omega^1(R_V) \to R_V \otimes_k \text{gl}(V)^{\vee}\} \overset{(42)}{=} \text{cone}\{\Omega^1(R_V) \to R_V \otimes \text{End } V\} \overset{(41)}{=} \text{cone}\{\Omega^1 R \to R \otimes R\}_V \overset{(4)}{=} (\widehat{\Omega}^1 R)_V.$$

This means

$$\Omega^1(DRep_V(A))[-1] = (\widehat{\Omega}^1 R[-1])_V^{\text{GL}(V)}.$$

Similarly,

$$X(DRep_V(A))[1] = ((\widehat{\text{Der}} R[1])_V)^{\text{GL}(V)},$$

whose proof is left to the reader. This proves the theorem. 

\[\square\]

**4.5.4 The derived Procesi trace map**

Prior to Van den Bergh, Procesi gave a map in [28]

$$\text{Tr} : A \to (A)^{\text{GL}}_V, \, \tilde{a} \mapsto \{\rho \mapsto \text{Trace}(\rho(a))\},$$

where $\tilde{a}$ means the equivalence class represented by $a \in A$. This map was latter generalized by Berest et al in [5, 6] to $A$-bimodules as follows: first, recall that Van den Bergh’s functor (40) is

$$(\text{-})_V^{\text{ab}} : \text{Bimod } A \to \text{DGMod } A_V, \, M \mapsto M_V^{\text{ab}} := M \otimes_{A^e} (\text{End } V \otimes A_V).$$

There is an analogue of Procesi’s map for bimodules (see [6,(5.11)]) or [5,(87)])

$$\text{Tr}_V(M) : M \to \text{End } V \otimes M_V^{\text{ab}} \to M_V^{\text{ab}},$$
which is a trace map, that is, it factors through $M := M/[M, A]$. It therefore gives a morphism of functors

$$\text{Tr}_V : (-)_\natural \rightarrow (-)_V.$$ 

Now, the derived Procesi map is given by

$$\text{Tr}_V : L(-)_\natural \rightarrow L(-)_V. \quad (44)$$

The difference between $L(M)^{ab}_V$ and $\text{Tr}_V(L(M))$ is that, we would view the former as sheaves on $\text{DRep}_V(A)$ and the latter as their global sections. We learned this point of view from Yeung [39, 40].

**Example 4.10** Suppose $R \rightarrow A$ is a DG free resolution of $A$, then $\Omega^1(R)[-1]$ is a DG free $R$-bimodule. By Lemma 4.8, we have

$$\Omega^1(\text{DRep}_V(A))[-1] = (\Omega^1 R[-1])^\text{GL}(V).$$

Recall that $\Omega^1 R[-1]_\natural = \Omega^1 \text{DR}_1(R)$. Then by (44), the morphism $\text{Tr}_V$ is given by

$$\text{Tr}_V : \Omega^1 \text{DR}_1(R) \rightarrow ((\Omega^1 R[-1])^\text{GL}(V), \ dx^\alpha \mapsto \text{Tr}(dx^\alpha \otimes e_{ij}) = \sum_{i=1}^{d} dx^\alpha_{ii}.$$

Similarly, for 2-forms we have $\text{Tr}_V(dx^\alpha \otimes dx^\beta) = \sum_{i,j=1}^{d} dx^\alpha_{ij} \otimes dx^\beta_{ji}.$

5 Derived representation schemes of AS-regular algebras

Now suppose $A$ is a Kosul AS-regular algebra of dimension $n$. In this section we study the twisted symplectic structure on $\text{DRep}_V(A)$, and prove Theorem 1.2.

5.1 Twisted differential forms

The twisted differential forms of DG commutative algebras are defined similarly to those of DG associative algebras. Some other studies of twisted differential forms can be found in, for example, [1, 19, 22].

Let $(A, \mu, \partial)$ be DG commutative algebra over $k$. Recall that the set of Kähler differential forms of $A$ is $\Omega^1_A := I/I^2$, where $I = \ker{\mu : A \otimes A \rightarrow A, x \otimes y \mapsto xy}$. Now suppose $A$ has an automorphism $\sigma \in \text{Aut} A$. Let

$$J := \ker{\mu_\sigma : A \otimes A \rightarrow A, x \otimes y \mapsto x \cdot \sigma(y)}.$$ 

and let $\Omega^1_A := J/J^2$, which is called the set of twisted Kähler differential forms of $A$. We have the following:
Lemma 5.1 Let $\Omega^1_\sigma A$ be as above. Then

(1) For any $a \in A$, the map $d_\sigma : A \to \Omega^1_\sigma A, a \mapsto \sigma(a) \otimes 1 - 1 \otimes a$

satisfies $d_\sigma(ab) = d_\sigma(a) \cdot b + \sigma(a) \cdot d_\sigma(b)$, for all $a, b \in A$;

(2) For any $u \in \Omega^1_\sigma A$ and $a \in A$, we have

$$\sigma(a) \cdot u - (-1)^{|a||u|} u \cdot a = 0 \in \Omega^1_\sigma A.$$  

Proof (1) For any $a, b \in A$, we have

$$d_\sigma(a) \cdot b + \sigma(a) \cdot d_\sigma(b)$$

$$= (\sigma(a) \otimes 1 - 1 \otimes a) \cdot b + \sigma(a) \cdot (\sigma(b) \otimes 1 - 1 \otimes b)$$

$$= (\sigma(a) \otimes b - 1 \otimes ab) + (\sigma(ab) \otimes 1 - \sigma(a) \otimes b)$$

$$= \sigma(ab) \otimes 1 - 1 \otimes ab$$

$$= d_\sigma(ab).$$

(2) We only need to show that, for any $u \in J/J^2$ and $a \in A$, if $u$ is represented by $\sum_i u_i \otimes v_i \in J$, then

$$\sigma(a) \cdot (\sum_i u_i \otimes v_i) - (-1)^{|u_i||v_i|}|a| (\sum_i u_i \otimes v_i) \cdot a \in J^2.$$  

In fact,

$$\sigma(a) \cdot (\sum_i u_i \otimes v_i) - (-1)^{|u_i||v_i|}|a| (\sum_i u_i \otimes v_i) \cdot a$$

$$= (\sigma(a) \otimes 1 \cdot (\sum_i u_i \otimes v_i) - (1 \otimes a) \cdot (\sum_i u_i \otimes v_i))$$

$$= (\sigma(a) \otimes 1 - 1 \otimes a) \cdot (\sum_i u_i \otimes v_i) \in J^2.$$  

By this lemma, if we define the left and right action of $A$ on $\Omega^1_\sigma A$ by

$$x \circ u \circ y := \sigma(x)uy, \quad \text{for all } u \in \Omega^1_\sigma A \text{ and } x, y \in A,$$

then $\Omega^1_\sigma A$ is a DG $A$-module. Moreover, it is direct to see the following map

$$\Psi : \Omega^1 A \mapsto \Omega^1_\sigma A, \quad dx \mapsto d_\sigma(x)$$

(45)

is an isomorphism of $A$-modules.

Parallel to the noncommutative case, in what follows we prefer to call $\Omega^1 A[-1]$ and $\Omega^1_\sigma A[-1]$ the sets of 1-forms and twisted 1-forms of $A$ respectively. In this case,
we have \( d \circ \partial + \partial \circ d = 0 \) and \( d_\sigma \circ \partial + \partial \circ d_\sigma = 0 \), and moreover, \( d : x \mapsto dx \) and \( d_\sigma : x \mapsto d_\sigma x \) satisfies the Leibniz and twisted Leibniz rule respectively.

The twisted differential forms of \( A \), denoted by \( \Omega^*_A \), is given as follows (see [19]): let \( T_A(\Omega^1 A[-1]) \) be the tensor algebra generated by \( \Omega^1 A[-1] \) over \( A \), with \( \partial \) and \( d \) extending to it by derivation and by letting \( d^2 = 0 \), then \( \Omega^*_A \) is the quotient of \( T_A(\Omega^1 A[-1]) \) by the two-sided DG ideal generated by \( x \cdot y - (-1)^{|x||y|} y \cdot \sigma(x) \), for all \( x \in A \) and \( y \in \Omega^1 A[-1] \), or both \( x, y \in \Omega^1 A[-1] \). Observe that the degree one part of \( \Omega^*_A \) is exactly \( \Omega^1_\sigma A[-1] \) given above.

### 5.2 Twisted symplectic structure

Now suppose \( \omega \in \Omega^2_\sigma A \), similarly to the noncommutative case, we say it is \( \sigma \)-invariant if \( \sigma(\omega) = \omega \), where \( \sigma \) is induced from \( \sigma : A \to A \). For such an \( \omega \), suppose it is represented by \( \sum da_1 \otimes da_2 \); then we may define the following contraction

\[
\Psi \circ \iota(-) \omega : \mathcal{X}(A) \to \Omega^1_\sigma A[n-2], \ X \mapsto \sum X(da_1)\Psi(da_2) + X(da_2)\Psi(\sigma(da_1)),
\]

(46)

where \( \mathcal{X}(A) \) is identified with \( \text{Hom}_A(\Omega^1 A, A) \) and hence \( \mathcal{X}(da_1) \) is the evaluation, and \( \Psi \) is given by (45).

**Lemma 5.2** If \( \omega \) is \( \sigma \)-invariant and extends to be closed, then (46) is a map of DG \( A \)-modules.

**Proof** This is completely analogous to Lemma 2.3. \( \square \)

The following definition is completely parallel to Definition 2.4; if \( \sigma \) is the identity map, then it coincides with the one of Pantev et. al. [26] in the DG affine case.

**Definition 5.3** (Twisted symplectic structure) Let \( A \) be a DG free commutative algebra with \( \sigma \in \text{Aut} \ A \). A twisted 2-form \( \omega \in \Omega^2_\sigma A \) of total degree \( n \) is called a twisted \( (2-n) \)-shifted symplectic structure if

1. it extends to be closed in the negative cyclic complex associated to \( \Omega^2_\sigma A \);
2. it is \( \sigma \)-invariant and induces a quasi-isomorphism

\[
\Psi \circ \iota(-) \omega : \mathcal{X}(A)[1] \to \Omega^1_\sigma A[-1][n].
\]

of DG \( A \)-modules.

If \( A \) is a DG free resolution of a twisted DG algebra \( B \), then we say \( B \) has a derived twisted symplectic structure if \( A \) has a twisted symplectic structure.

The above definition can be generalized to the global quotients of derived affine schemes, viewed as a derived stack, without any difficulty.

Now suppose \( A \) is a Koszul AS-regular algebra of dimension \( n \). Let \( R = \Omega(A^i) \) be the cobar construction of its Koszul dual coalgebra \( A^i \). Let \( x^0 = 1, x^1, \cdots, x^\ell \) be a basis of \( A^i \). By Example 4.3, for \( V = k^d \), \( \text{DRep}_V(A) \) is isomorphic to

\[
R_V = k[x^\alpha_{ij} | 1 \leq \alpha \leq \ell, 1 \leq i, j \leq d, |x^\alpha_{ij}| = |x^\alpha| - 1].
\]
Proof of Theorem 1.2  Denote $\mathbb{T} = (\hat{\text{Der}} R[1])_V$ and $\mathbb{L} = (\Omega^1 R[-1])_V$. Then by Theorem 4.9, $\mathbb{T}^{\text{GL}(V)}$ and $\mathbb{L}^{\text{GL}(V)}$ are the tangent and cotangent complexes of $\mathcal{D}\text{Rep}_V(A)$ respectively. By the same argument, it is also direct to see that the $\text{GL}(V)$-invariant of $\mathbb{L}_\sigma := (\hat{\text{Der}} R[-1])_V$, denoted by $\mathbb{L}^{\text{GL}(V)}_\sigma$, is the twisted cotangent complex of $\mathcal{D}\text{Rep}_V(A)$. Thus by applying Van den Bergh’s functor to Theorem 1.1, we get Theorem 1.2. Since the computations are quite involved, we here give some details of the proof.

First of all, recall that in (41)–(42) we have identified

$$(R \otimes R)_V \cong (R \otimes d1 \otimes R)_V \cong R \otimes \text{End } V \cong R \otimes \mathfrak{gl}(V).$$

As before, we write $\{x_{ij}^0\}$ to be the set of generators of $R \otimes \text{End } V$ (the corresponding elementary matrices). Let

$$\omega := \frac{1}{2} \sum_{i,j=1}^{d} d\eta'_{ij} \otimes d\eta''_{ij},$$

which is exactly the image of $\omega = \frac{1}{2} \sum d\eta' \otimes d\eta'' \in \text{DR}_2 R$ under the derived Procesi map $\text{Tr}_V$ described in Example 4.10. It is straightforward to see that $\omega$ is: 1) $\sigma$-invariant of total degree $n$, and 2) closed under both $d$ and $\partial$.

Now by (46) the contraction with $\omega$ is given by

$$\Psi \circ \iota(-)\omega : \mathbb{T} \to \mathbb{L}_\sigma[n], \quad D\tilde{x}_{ij}^\alpha \mapsto \sum_{\beta=0}^{\ell} (\tilde{x}^\alpha, \tilde{x}^\beta) d_\sigma(x_{ij}^\beta), \quad (47)$$

which extends to a map of $R_V$-modules. Here $D\tilde{x}_{ij}^\alpha$ is $(D\tilde{x}^\alpha)_{ij}$, whose linear dual is $dx_{ij}^\alpha$. We claim that (47) is a quasi-isomorphism of chain complexes. First, denote $\delta : \mathbb{T} \to \mathbb{T}$ to be the internal differential on $\mathbb{T}$ via $\delta(D\tilde{x}_{ij}^\alpha) := (\delta(D\tilde{x}^\alpha))_{ij}$, where $\delta$ in the right hand side is given by (23). More precisely, we have

$$\delta(D\tilde{x}_{ij}^\alpha) = - \sum_{\gamma} \sum_{k} (-1)^{|\alpha|+|\gamma|-1} D(\tilde{x}^\gamma \tilde{x}^\alpha)_{ik} \cdot x_{kj}^\gamma - x_{ik}^\gamma \cdot D(\tilde{x}^\alpha \tilde{x}^\gamma)_{kj}.$$ 

Here, for simplicity, we write $(-1)^{|\alpha|}$ as $(-1)^{|\alpha|}$. Denote $\Phi := \Psi \circ \iota(-)\omega$; we have

$$\Phi(\delta(D\tilde{x}_{ij}^\alpha))$$

$$= \Phi\left(- \sum_{\gamma} \sum_{k} (-1)^{|\alpha|+|\gamma|-1} D(\tilde{x}^\gamma \tilde{x}^\alpha)_{ik} \cdot x_{kj}^\gamma - x_{ik}^\gamma \cdot D(\tilde{x}^\alpha \tilde{x}^\gamma)_{kj}\right)$$

$$= - \sum_{\gamma} \sum_{k} \sum_{\beta} (-1)^{|\alpha|+|\gamma|+|\beta|+|\gamma|-1} (\tilde{x}^\gamma \tilde{x}^\alpha, \tilde{x}^\beta) d_\sigma(x_{ik}^\beta) \cdot \sigma(x_{kj}^\gamma)$$

$$- (\tilde{x}^\alpha, \tilde{x}^\beta) x_{ik}^\gamma \cdot d_\sigma(x_{kj}^\beta).$$
\[ \begin{align*}
&= - \sum_{\gamma} \sum_{\beta} \sum_{k} (-1)^{|\beta|} (\tilde{x}^\alpha, \tilde{x}^\beta \sigma^* (\tilde{x}^\gamma)) d_\sigma (x_{ik}^\beta) \cdot \sigma (x_{kj}^\gamma) \\
&\quad - (\tilde{x}^\alpha, \tilde{x}^\gamma \tilde{x}^\beta) x_{ik}^\gamma \cdot d_\sigma (x_{kj}^\beta) \\
&= - \sum_{\zeta} \sum_{k} (\tilde{x}^\alpha, \tilde{x}^\zeta) \left( (-1)^{|\zeta'|} d_\sigma (x_{ik}^{\zeta'}) \cdot \sigma (x_{kj}^{\zeta''}) - x_{ik}^{\zeta'} \cdot d_\sigma (x_{kj}^{\zeta''}) \right) \\
&= - \sum_{\zeta} \sum_{k} (\tilde{x}^\alpha, \tilde{x}^\zeta) \left( (-1)^{|\zeta'|} x_{kj}^{\zeta''} d_\sigma (x_{ik}^{\zeta'}) - x_{ik}^{\zeta'} \cdot d_\sigma (x_{kj}^{\zeta''}) \right), \quad (48)
\end{align*} \]

while

\[
\partial (\Phi (D\tilde{x}_{ij}^\alpha)) = \sum_{\beta} (\tilde{x}^\alpha, \tilde{x}^\beta) \cdot \partial (d_\sigma (x_{ij}^\beta)) = - \sum_{\beta} (\tilde{x}^\alpha, \tilde{x}^\beta) \cdot d_\sigma \circ \partial (x_{ij}^\beta) \\
= - \sum_{\beta} (\tilde{x}^\alpha, \tilde{x}^\beta) \cdot d_\sigma \left( \sum_{k} (-1)^{|\beta'|} x_{ik}^{\beta'} x_{kj}^{\beta''} \right) \\
= - \sum_{\beta} \sum_{k} (\tilde{x}^\alpha, \tilde{x}^\beta) \left( (-1)^{|\beta'|} x_{kj}^{\beta''} d_\sigma (x_{ik}^{\beta'}) - x_{ik}^{\beta'} \cdot d_\sigma (x_{kj}^{\beta''}) \right). \quad (49)
\]

Observe that (48) and (49) are equal to each other, from which we obtain that \( \Phi \) is a chain map.

We now show \( \Phi \) is non-degenerate, that is, it is a quasi-isomorphism of DG \( R_V \)-modules. In fact, this is a direct corollary of the non-degeneracy of the pairing on \( A \) and the non-degeneracy of the canonical pairing on matrices (namely, the trace of the matrix product).

Taking the GL-invariants of (47) gives the desired quasi-isomorphism from the tangent complex to the twisted cotangent complex on \( DRep_V (A) \). This means \( \omega \) given above is a twisted symplectic form, from which the theorem follows.

\[ \square \]

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**References**

1. André, Y.: Différentielles non commutatives et théorie de Galois différentielle ou aux différences. Ann. Sci. École Norm. Sup. **4**(345), 685–739 (2001)
2. Artin, M., Schelter, W.F.: Graded algebras of global dimension 3. Adv. Math. **66**, 171–216 (1987)
3. Berest, Y., Felder, G., Patotski, S., Ramadoss, A., Willwacher, T.: Representation homology, Lie algebra cohomology and the derived Harish-Chandra homomorphism. J. Eur. Math. Soc. **19**(9), 2811–2893 (2017)
4. Berest, Y., Felder, G., Patotski, S., Ramadoss, A., Willwacher, T.: Chern-Simons forms and higher character maps of Lie representations, Int. Math. Res. Not. IMRN (2017) no. 1, 158–212
5. Berest, Y., Felder, G., Ramadoss, A.: Derived representation schemes and noncommutative geometry. Contemp. Math. **607**, 113–162 (2014)
6. Berest, Y., Khachatryan, G., Ramadoss, A.: Derived representation schemes and cyclic homology. Adv. Math. **245**, 625–689 (2013)
7. Benkart, G., Roby, T.: Down-up algebras. J. Algebra 206, 305–344 (1998)
8. Berglund, A., Börjeson, K.: Koszul $A_{\infty}$-algebras and free loop space homology. Proc. Edinburgh Math. Soc. 63(1), 37–65 (2020)
9. Brown, K.A., Zhang, J.J.: Dualizing complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras. J. Algebra 320, 1814–1850 (2008)
10. Chen, X., Eshmatov, F.: Calabi-Yau algebras and the shifted non-commutative symplectic structure. Adv. Math. 367, 107126 (2020)
11. Ciocan-Fontanine, I., Kapranov, M.: Derived Quot schemes. Ann. Sci. ENS 34, 403–440 (2011)
12. Crawley-Boevey, W.: Poisson structures on moduli spaces of representations. J. Algebra 325, 205–215 (2011)
13. Crawley-Boevey, W., Etingof, P., Ginzburg, V.: Noncommutative geometry and quiver algebras. Adv. Math. 209(1), 274–336 (2007)
14. de Thanhoffer de Volcsey, L., Van den Bergh, M.: Calabi-Yau deformations and negative cyclic homology. J. Noncommut. Geom. 12(4), 1255–1291 (2018)
15. Dubois-Violette, M.: Multilinear forms and graded algebras. J. Algebra 317, 198–225 (2007)
16. Dwyer, W.G., Spalinski, J.: Twisted homology of the quantum $SL_2$, K-Theory, 34, 327–360
17. Hadfield, T., Krähmer, U.: Twisted homology of the quantum $SL(2)$. K-Theory, 34, 327–360
18. Karoubi, M., Suarez-Alvarez, M.: Twisted Kähler differential forms. J. Pure Appl. Algebra 118, 279–289 (2003)
19. Kontsevich, M., Rosenberg, A.: Noncommutative smooth spaces. The Gelfand Mathematical Seminars 1996–1999, 85–108, Birkhäuser Boston, Boston, MA, (2000)
20. Kustermans, J., Murphy, G., Tušet, L.: Differential calculi over quantum groups and twisted cyclic cocycles. J. Geom. Phys. 44(4), 570–594 (2003)
21. Le Stum, E., Quirós, A.: Formal confluence of quantum differential operators. Pacific J. Math. 292(2), 427–478 (2018)
22. Liu, L.: Koszul duality and the Hochschild cohomology of Artin-Schelter algebras. Homol. Homotopy Applpl. 22(2), 181–202 (2020)
23. Liu, L.Y., Wang, S.Q., Wu, Q.S.: Twisted Calabi-Yau property of Ore extensions. J. Noncommut. Geom. 8, 587–609 (2014)
24. Pridham, J.P.: Non-commutative derived moduli prestacks, arXiv:2008.11684
25. Procesi, C.: The invariant theory of $n \times n$ matrices. Adv. Math. 19, 306–381 (1976)
26. Reyes, M., Rogalski, D., Zhang, J.J.: Skew Calabi-Yau algebras and homological identities. Adv. Math. 264, 308–354 (2014)
27. Smith, S.P.: Some finite dimensional algebras related to elliptic curves, Representation theory of algebras and related topics (Mexico City, 1994), 315–348. In: CMS Conf. Proc. 19, Amer. Math. Soc., Providence, RI, 1996
28. Toën, B.: Derived algebraic geometry. EMS Surv. Math. Sci. 1(2), 153–240 (2014)
29. Toën, B., Vaquié, M., Vezzosi, G.: Shifted symplectic structures. Publ. Math. Inst. Hautes Études Sci. 117, 271–328 (2013)
30. Vérbökás, M., Rogalski, D., Zhang, J.J.: Skew Calabi-Yau algebras and homological identities. Adv. Math. 264, 308–354 (2014)
31. Yekutieli, A.: Derived categories. Cambridge Studies in Advanced Mathematics 183. Cambridge University Press, Cambridge, 2020
32. Yekutieli, A., Zhang, J.J.: Homological transcendence degree. Proc. London Math. Soc. 93, 105–137 (2006)
39. Yeung, W.-K.: Pre-Calabi-Yau structures and moduli of representations. arXiv:1802.05398
40. Yeung, W.-K.: Shifted symplectic and Poisson structures on global quotients. arXiv:2103.09491

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