SINGULARITY AND EXISTENCE TO A WAVE SYSTEM OF NEMATIC LIQUID CRYSTALS

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Abstract. In this paper, we prove the global existence and singularity formation for a wave system from modelling nematic liquid crystals in one space dimension. In our model, although the viscous damping term is included, the solution with smooth initial data still has gradient blowup in general, even when the initial energy is arbitrarily small.

Key Words. Liquid crystal; singularity; wave equations.

1. Introduction

The mean orientation of the long molecules in a nematic liquid crystal is described by a director field of unit vectors, \( n \in S^2 \), and the propagation of the orientation waves in the director field could be modelled by below Euler-Lagrangean equations derived from the least action principle [12, 1],

\[
\begin{align*}
\frac{d}{dt}n &= \mu \frac{\partial}{\partial t}n + \frac{\delta W(n, \nabla n)}{\delta n} = \lambda n, \\
\end{align*}
\]

where the well-known Oseen-Franck potential energy density \( W \) is given by

\[
W(n, \nabla n) = \frac{1}{2} \alpha (\nabla \cdot n)^2 + \frac{1}{2} \beta (\nabla \times n)^2 + \frac{1}{2} \gamma |n \times (\nabla \times n)|^2.
\]

The positive constants \( \alpha, \beta, \) and \( \gamma \) are elastic constants of the liquid crystal, corresponding to splay, twist, and bend, respectively. The viscous coefficient \( \mu \) is a non-negative constant and the Lagrangian multiplier \( \lambda \) is determined by the constraint \( n \cdot n = 1 \).

There are many studies on the constrained elliptic system of equations for \( n \) derived through variational principles from the Oseen-Franck potential (1.2), and on the parabolic flow associated with it, see [2, 6, 7, 9, 10, 15].

The global weak existence and singularity formation for the Cauchy problem of the extreme case of (1.1) with \( \mu = 0 \) in one space dimension (1-d) has also been extensively studied [3, 8, 16, 17, 18, 19]. This hyperbolic system describes the model with viscous effects neglected, for which an example with smooth initial data and singularity formation (gradient blowup) in finite time has been provided in [8]. The lack of regularity makes us only be able to consider the existence of weak solutions instead of classical solutions. The existing global existence results in [3, 18, 19] proved by the method of energy-dependent coordinates will be introduced in Section 2.

However, the well-posedness and regularity of the solution for the complete system (1.1) with \( \mu > 0 \) are still wide open.

Date: May 2, 2014.
In this paper, we consider the 1-d system of (1.1) with $\alpha = \beta$ and $\mu \geq 0$, where the derivation will be delayed to Section 2:

$$
\begin{aligned}
\partial_t n_1 + \mu \partial_t n_1 - \partial_x (c^2(n_1) \partial_x n_1) &= (-|n_t|^2 + (2c^2 - \gamma)|n_x|^2)n_1, \\
\partial_t n_2 + \mu \partial_t n_2 - \partial_x (c^2(n_1) \partial_x n_2) &= (-|n_t|^2 + (2c^2 - \alpha)|n_x|^2)n_2, \\
\partial_t n_3 + \mu \partial_t n_3 - \partial_x (c^2(n_1) \partial_x n_3) &= (-|n_t|^2 + (2c^2 - \alpha)|n_x|^2)n_3,
\end{aligned}
$$

where $\mu \geq 0,$

$$
c^2(n_1) \equiv \alpha + (\gamma - \alpha)n_1^2
$$

and

$$
n(t, x) = (n_1(t, x), n_2(t, x), n_3(t, x)) \quad \text{and} \quad |n| = 1, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.
$$

In this paper, we first provide an example with smooth ($C^1$) initial data and singularity formation in finite time. Then we prove the global weak existence of the Cauchy problem for (1.3) with initial data

$$
n_i(0, x) = n_{i0} \in H^1, \quad (n_i)_x(0, x) = n_{i1} \in L^2, \quad i = 1 \sim 3.
$$

Although when $\mu = 0$ the singularity formation and global existence of Cauchy problem for (1.3) has been systematically studied in [8] and [4] respectively, the extension of these results to the case with $\mu > 0$ in this paper is important and non-trivial. First, the model with $\mu = 0$ only describes the extreme case in physics which is much less happening than the one described by the model with $\mu > 0$. Secondly, the comparison between our results and the results when $\mu = 0$ indicates that the regularity of the solutions is basically not impacted by adding or removing the viscous term $\mu n_i$, even when $\mu$ is large, which is to some extend unexpected. Furthermore, now most existing existence and regularity results for the models with $\mu = 0$ can be expected to extend to the models with $\mu > 0$.

We first consider the regularity of solutions of (1.3). A nature question is whether the appearance of the viscous term $\mu n_i$ with $\mu > 0$ can prevent the singularity formation (gradient blowup), which is known existing when $\mu = 0$, or not. However, the singularity formation result in this paper gives a negative answer to this question: For any $\mu \geq 0$, the solution generally includes gradient blowup. In fact, for both the singularity formation example in [8] ($\mu = 0$) and the one in this paper ($\mu > 0$), the initial energy can be arbitrarily small. As a consequence of this surprising result, we need to consider the weak solution instead of the classical solution for the system (1.3).

For simplicity, in order to get a gradient blowup example, we only consider the solution of (1.3) with structure $n = (\cos u(t, x), \sin u(t, x), 0)$ (planar deformation). Clearly $n_3$ is always 0 in the smooth solution if it is vanishing initially. It is easy to get that the system (1.3) is equivalent to

$$
\begin{aligned}
u_{tt} + \mu u_t - c(u)(c(u)u_x)_x &= 0,
\end{aligned}
$$

when $\sin u \neq 0$, where we still use $c$ to denote the wave speed and

$$
c^2(u) = \gamma \cos^2 u + \alpha \sin^2 u.
$$

It is easy to see that there exist positive constants $C_L$, $C_U$ and $C_D$, such that

$$
C_L < c(u) < C_U, \quad |c'(u)| < C_D.
$$

**Theorem 1.1.** We consider the Cauchy problem of (1.6) with initial data satisfying

$$
u(0, x) = u_0 + \varepsilon \phi(\frac{x}{\varepsilon}) + \varepsilon^2 \eta(\frac{x}{\varepsilon^2})
$$

$$
u_x(0, x) = (-c(u(0, x)) + \varepsilon) \ u_x(0, x),
$$

where $\phi$, $\eta$ are smooth functions such that $\phi(0) = 1$, $\eta(1) = 0$, $\phi(1) = 0$, $\phi''(0) = 1$, $\eta'(-1) = 1$. Then for any $\varepsilon > 0$ and $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, the system (1.6) has a solution $\nu$ for $0 < \varepsilon < \varepsilon_0$ which satisfies

$$
\sup_{0 \leq t \leq T} \|\nu(t, \cdot)\|_{H^1} < \infty,
$$

$$
\sup_{0 \leq t \leq T} \|\nu_x(t, \cdot)\|_{L^1} < \infty,
$$

$$
\sup_{0 \leq t \leq T} \|\nu_{xx}(t, \cdot)\|_{L^1} < \infty,
$$

$$
\sup_{0 \leq t \leq T} \|\nu_{xxx}(t, \cdot)\|_{L^1} < \infty.
$$

Furthermore, as $\varepsilon \to 0$, the solution $\nu$ converges to a weak solution $\bar{\nu}$ of the system (1.6) in the sense of distributions. Moreover, the blowup set of $\bar{\nu}$ is the same as the blowup set of the solution $\nu$ when $\varepsilon \to 0$.
where \( u_0 \) is a constant satisfying
\[
c'(u_0) > 0,
\]
and two functions \( \phi(a) \) and \( \eta(a) \) are in \( C^1(\mathbb{R}) \) and satisfy conditions
\[
(1.10) \quad \phi(a), \eta(a) = 0 \text{ when } a \notin (-1,1); \quad \phi'(a) < 0 \text{ when } a \in [0,1),
\]
and
\[
(1.11) \quad -\phi'(0) > \frac{8\mu C_U}{c'(u_0) C_L}.
\]
For any \( \mu \geq 0 \), we can choose \( \varepsilon > 0 \) sufficiently small, such that the \( C^1 \) solution \( u(t,x) \) forms singularity in finite time.

Note, in the solutions we constructed, \( \sin u \neq 0 \) before blowup. Hence, for system \( (1.3) \), for any \( \mu \geq 0 \), we can find examples with \( C^1 \) smooth initial data and singularity formation in finite time. From Theorem 1.2, we know that the gradient blowup we found is not a discontinuity (shock wave).

Then we consider the global existence of weak solution for the initial value problem of \( (1.3) \) \( \sim \) (1.5). Here, the unit vector \( n(t,x) \) with \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \) is a weak solution to the Cauchy problem \( (1.3) \) \( \sim \) (1.5) if it satisfies that

1. The equations \( (1.3) \) hold in distributional sense for test functions \( \phi \in C^1_c(\mathbb{R} \times \mathbb{R}) \). The initial data satisfy \( (1.5) \) in the pointwise sense, and their temporal derivatives hold in \( L^p_{\text{loc}} \) for \( p \in [1,2) \).

2. \( n_i(t,x) \) is locally Hölder continuous with exponent \( 1/2 \), for \( i = 1 \sim 3 \). The map \( t \mapsto (n_1,n_2,n_3)(t,\cdot) \) is continuously differentiable with values in \( L^p_{\text{loc}} \), for all \( 1 \leq p < 2 \). And it is Lipschitz under the \( L^2 \) norm, i.e.
\[
(1.12) \quad \|n_i(t,\cdot) - n_i(s,\cdot)\|_{L^2} \leq L |t-s|, \quad i = 1 \sim 3,
\]
for all \( t,s \in \mathbb{R}^+ \).

The main well-posedness results in this paper are

**Theorem 1.2.** The initial value problem \( (1.3) \) \( \sim \) (1.5) has a global weak solution \( n(t,x) \) for all \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \), where \( n(t,x) \) satisfies that the energy
\[
(1.13) \quad \mathcal{E}(t) \overset{\text{def}}{=} \frac{1}{2} \int \left[ |n_i|^2 + c^2(n_1)|n_x|^2 \right] dx
\]
is less than or equal to \( \mathcal{E}(0) \) for any \( t > 0 \).

**Theorem 1.3.** For the Cauchy problem \( (1.3) \) \( \sim \) (1.4), let a sequence of initial data satisfy
\[
\|(n_{i0}^k)_x - (n_{i0})_x\|_{L^2} \to 0, \quad \|n_{i0}^k - n_{i1}\|_{L^2} \to 0, \quad i = 1 \sim 3,
\]
and \( n_0^k \to n_0 \) uniformly on compact sets, as \( k \to \infty \). Then \( n^k \to n \) uniformly on bounded subsets of the \( (t,x) \)-plane with \( t > 0 \).

In order to prove these theorems, by the method of energy-dependent coordinates, first used in papers [3] and related Camassa-Holm equation, we dilate the singularity then find the energy dissipated solution. In fact, the energy is dissipated when \( \mu < 0 \). When \( \mu = 0 \), the energy is conserved in some sense [4].

Similar existence and continuous dependence results apply to system (1.6), which is a special example of (1.3).

The paper is organized as follows. In Section 2, we derive the system (1.3) and energy equation, then introduce the existing results for the extreme case \( \mu = 0 \). In Section 3, we
prove the singularity formation result: Theorem 1.1. In Section 4, we construct a set of semi-linear equations in the energy-dependent variables by studying the smooth solution of (1.3). Then, in Section 5, we prove the existence of the solutions for the semi-linear system. In Sections 6~8, we transform the solutions back to the original system and prove Theorems 1.2, 1.3.

2. The Derivation of System

In this section, we derive (1.3) from (1.1). In one space dimension, (1.1) is

\[ \partial_{tt}c + \mu \partial_t c + \partial_x W(c, \partial_x c) - \partial_x [\partial_{c x} W(c, \partial_x c)] = \lambda c, \quad \text{for} \quad i = 1, 2, 3. \]

Using \(|n| = 1\), we get by multiplying \(n_i\) to (2.1) and summing up \(i\) from 1 to 3

\[ \lambda = \sum_{i=1}^{3} \{ -[\partial_t n_i]^2 + n_i \partial_t W(n, \partial_x n) - n_i \partial_x [\partial_{c x} W(n, \partial_x n)] \}. \]

It is easy to calculate that (2.1) has the following energy equations:

\[ \partial_t \left[ \frac{1}{2} |n|^2 + W(n, \partial_x n) \right] - \partial_x \left[ \sum_{i=1}^{3} \partial_t n_i \partial_{c x} W(n, \partial_x n) \right] = -\mu |n|^2 \leq 0. \]

In 1-d case, the Oseen-Franck potential energy density (1.2) is

\[ W(n, \partial_x n) = \frac{\alpha}{2} [\partial_x n_1]^2 + \frac{\beta}{2} [\partial_x n_2]^2 + \left[ (\partial_x n_2)^2 + (\partial_x n_3)^2 \right] + \frac{1}{2} (\gamma - \beta) n_1^2 |\partial_x n|^2, \]

from which and (2.2), we infer

\[ \lambda = -|n|^2 + (\beta + 2(\gamma - \beta)n_1^2) |\partial_x n|^2 + (\beta - \alpha) n_1 \partial_x^2 n_1. \]

Then by (2.1), (2.4) and (2.5), we have

\[
\begin{cases}
\partial_{tt}n_1 + \mu \partial_t n_1 - \partial_x [c_1^3(n_1) \partial_x n_1] = \{-|n|^2 + (2c_2^2 - \gamma)|n_x|^2 + 2(\alpha - \beta)(\partial_x n_1)^2\} n_1 \\
\partial_{tt}n_2 + \mu \partial_t n_2 - \partial_x [c_2^3(n_1) \partial_x n_2] = \{-|n|^2 + (2c_2^2 - \beta)|n_x|^2 + (\beta - \alpha)n_1 \partial_x n_1\} n_2 \\
\partial_{tt}n_3 + \mu \partial_t n_3 - \partial_x [c_3^3(n_1) \partial_x n_3] = \{-|n|^2 + (2c_2^2 - \beta)|n_x|^2 + (\beta - \alpha)n_1 \partial_x n_1\} n_3
\end{cases}
\]

with

\[ c_1^3(n_1) \overset{\text{def}}{=} \alpha + (\gamma - \alpha)n_1^2 \quad \text{and} \quad c_2^3(n_1) \overset{\text{def}}{=} \beta + (\gamma - \beta)n_1^2. \]

In particular, taking \(\alpha = \beta\) in (2.6), we get (1.3) and

\[ c^2(n_1) = c_1^3(n_1) = c_2^3(n_1) = \alpha + (\gamma - \alpha)n_1^2, \]

and it is easy to check that there are positive numbers \(C_L < C_U\) such that

\[ 0 < C_L < c(n_1) < C_U < \infty; \quad |c'(n_1)| < C_N, \quad \text{for any} \quad |n_1| \leq 1. \]

The energy equation (2.3) is

\[ \frac{1}{2} \partial_t [n^2 + c^2(n_1)|n_x|^2] - \partial_x [c^2(n_1)n_t \cdot n_x] = -\mu |n|^2 \leq 0, \]

where the energy density \(W\) in (2.4) is

\[ W(n, \partial_x n) = \frac{1}{2} c^2(n_1) |\partial_x n|^2. \]

Finally, we introduce the existing existence results for weak solutions of 1-d systems of (1.1) with \(\mu = 0\), proved by the method of energy-dependent coordinates. This method was first applied to equation (1.6) with \(\mu = 0\) for an existence proof by [3]. When \(n\) is arbitrary
in $S^2$ and $\mu = 0$, under the a priori assumption that $n(x,t)$ is uniformly away from $(1,0,0)$, the existences of 1-d solutions for (1.3) and (2.6) with $\beta < \alpha$ have been provided by [18] and [19], respectively. In a recent paper [4], the existence of 1-d solution for system (1.3) and $\mu = 0$ has been established, without any a priori assumption.

3. Singularity formation: Proof of Theorem 1.1

We introduce Riemann variables $R$ and $S$ for equation (1.6):

$$ R \overset{\text{def}}{=} u_t + c(u)u_x, \quad S \overset{\text{def}}{=} u_t - c(u)u_x. $$

By (1.6), the $C^1$ solution satisfies

$$ R_t - cR_x = \frac{c'(u)}{4c(u)}(R^2 - S^2) - \frac{\mu}{2}(R + S), $$
$$ S_t + cS_x = \frac{c'(u)}{4c(u)}(S^2 - R^2) - \frac{\mu}{2}(R + S), $$

where the left hand sides are two directional derivatives along characteristics.

Before we give the proof, we first explain the basic idea. The singularity is essentially caused by the quadratic increase of the Riemann variables. In order to find it out, we need to get a decoupled Riccati type inequality on some characteristics. First, we show that the energy is in $O(\varepsilon)$ for our example. Using this estimate, we can make sure that $c'(u) > \frac{1}{2}c'(u_0)$ before $t = \sigma/\varepsilon$ for some positive constant $\sigma$, when $\varepsilon > 0$ is sufficiently small. This means that the coefficients of the quadratic terms in (3.2)(3.3) have fixed sign. Then the key step of this proof is to prove the domain: $0 \geq R > -M_2\varepsilon^2, \quad S > 0$ is invariant, when $(t,x)$ is taken on some characteristic trapezoid. Collecting all these informations, we can find the decoupled Riccati type inequality then prove the blowup.

We will prove the Theorem 1.1 in several steps.

1. It is easy to get from the initial condition that

$$ u_x(0,x) = \phi'(\frac{x}{\varepsilon}) + \varepsilon^{\frac{3}{2}} \eta'(\varepsilon^{\frac{5}{2}}x), $$
$$ R(0,x) = \varepsilon u_x(0,x), $$
$$ S(0,x) = (-2c(u(0,x)) + \varepsilon) u_x(0,x). $$

Choosing $\varepsilon < C_L$ and $0 < \varepsilon \ll 1$, we have, when $x \in [0, \varepsilon^{-\frac{2}{5}})$

$$ u_x(0,x) < 0, \quad R(0,x) < 0, \quad S(0,x) > 0, $$

because $\phi'(a)$ and $\eta'(a)$ are both negative when $a \in [0,1]$.

2. By (3.2) and (3.3), we have the energy equation

$$ (R^2 + S^2)_t + (c(S^2 - R^2))_x = -\mu(S + R)^2 \leq 0. $$

This equation agrees with the energy equations (2.3) and (2.9). We define the energy function $E$ as

$$ E(t) \equiv E(u(\cdot,t)) = \int_{-\infty}^{\infty} u_t^2(x,t) + c^2(u(x,t))u_x^2(x,t) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} R(t,x) + S(t,x) \, dx. $$
Here we use $E$ to denote energy for equation (1.6) instead of $\mathcal{E}$ for (1.3) to avoid confusion. Integrate (3.8) with respect to $x$, then we have

\[
E(t) \leq E(0) = \frac{1}{2} \int_{-\infty}^{\infty} R^2(0, x) + S^2(0, x) \, dx \\
\leq \frac{1}{2} \int_{-\infty}^{\infty} [(-2c(u(0, x))) + \varepsilon)^2 + \varepsilon^2] (\phi'(\frac{x}{\varepsilon}) + \varepsilon^\frac{2}{3} \eta'(\varepsilon^\frac{2}{3} x))^2 \, dx \\
\leq M\varepsilon,
\]

for some positive constant $M$, where we use $0 < \varepsilon \ll 1$.

**Figure 1.** Characteristic triangle

3. We consider any characteristic triangle in Figure 1 with the characteristic boundaries $t_\pm$ denoted by

\[
\frac{dt_\pm(x)}{dx} = \pm \frac{1}{c(u)}.
\]

Integrating (3.8) on this characteristic triangle, and by the divergence theorem, we have

\[
(3.10) \quad \int_{x_1}^{x_0} R^2(t_+(x), x) \, dx + \int_{x_0}^{x_2} S^2(t_-(x), x) \, dx \leq \frac{1}{2} \int_{x_1}^{x_2} [R^2(0, x) + S^2(0, x)] \, dx.
\]

4. Then we control the sign of $c'(u)$ by choosing $\varepsilon$ small enough. Since equation (1.6) has finite propagation speed, clearly seen from (3.10), we get that

\[
|u(t, x) - u_0| = \left| \int_{-\infty}^{x} u_x(t, x) \, dx \right| \\
\leq \int_{-\varepsilon^{-\frac{2}{3}} + C_U t}^{\varepsilon^{-\frac{2}{3}} + C_U t} |u_x(t, x)| \, dx \\
\leq \|u_x\|_{L^2} \sqrt{2\varepsilon^{-\frac{2}{3}} + 2C_U t} \\
\leq \sqrt{\frac{M}{C_L}} \sqrt{2\varepsilon^{\frac{1}{3}} + 2C_U t \varepsilon},
\]

(3.11)
where we use the bound on $E(t)$ in part 2. Hence, we can find a small positive number $\sigma$ independent of $\varepsilon$ then another small positive number $\varepsilon_0$, such that,

\[(3.12) \quad c'(u(x, t)) > \frac{c'(u_0)}{2} > 0,\]

if

\[0 \leq t < \sigma/\varepsilon, \quad 0 < \varepsilon < \varepsilon_0.\]

We only consider the problem in this time interval, in which (3.12) is satisfied.

5. We next prove an a priori estimate under the assumption that $R \leq 0$ and $S \geq 0$ in the characteristic triangle in Figure 1, with $x_2 - x_1 < \varepsilon^{-\frac{2}{3}}$.

Under this assumption, we have

\[R_t - cR_x \geq -\frac{C_D}{4C_L}S^2 - \frac{\mu}{2} S.\]

Integrating it along the backward characteristic by (3.10), we have

\[R(t_0, x_0) \geq -\frac{C_D}{4C_L} \int_{t_0}^{t} S^2(t, x_-(t))dt - \frac{\mu}{2} \int_{t_0}^{t} S(t, x_-(t))dt + R(0, x_1)\]

\[\geq -\frac{C_D}{4C_L} \int_{x_0}^{x_2} S^2(t_-(x), x) \frac{c(t_-(x), x)}{c(t_-(x), x)} dx - \frac{\mu}{2} \int_{x_0}^{x_2} S(t_-(x), x) \frac{c(t_-(x), x)}{c(t_-(x), x)} dx + R(0, x_1)\]

\[\geq -\frac{M_2}{4C_L} E(0) - \frac{\mu}{2CL} \sqrt{E(0)} \sqrt{|x_2 - x_0|} + R(0, x_1)\]

for some positive constant $M_2$. Choosing $\varepsilon < \left(\frac{2\mu C_L}{M_2 C_N}\right)^6$, we have

\[-\frac{c'(u)}{4c(u)} R^2 - \frac{\mu}{2} R \geq 0\]

because $R \leq 0$ under the a priori assumption we assumed. So

\[S_t + cS_x \geq \frac{c'(u)}{4c(u)} S^2 - \frac{\mu}{2} S.\]

6. Then we use this a priori estimate to prove that $R < 0$ and $S > 0$ in the closed region bounded by the forward characteristic starting from the origin on $(t, x)$-plane, $t = \sigma/\varepsilon, \ t = 0$ and the backward characteristic starting form the point $(0, x_1) = 0$ in the characteristic triangle in Figure 1, with $x_2 - x_1 < \varepsilon^{-\frac{2}{3}}$. We choose $\varepsilon^{-\frac{2}{3}} - \varepsilon^2$ instead of $\varepsilon^{-\frac{2}{3}}$ to exclude the backward characteristic starting from the point $(0, \varepsilon^{-\frac{2}{3}})$.

We prove it by contradiction. Assume that $R = 0$ or $S = 0$ at some point in the region we consider. By (3.17), $R(0, x) < 0$ and $S(0, x) > 0$ when $x \in [0, \varepsilon^{-\frac{2}{3}}]$, so we can find the lowest line $t = t_1 > 0$ such that $R = 0$ or $S = 0$ on some point of this line, while $R < 0$ and $S > 0$ when $0 \leq t < t_1$. So we can find a closed characteristic triangle with $R = 0$ or $S = 0$ on the upper vertex while $R < 0$ and $S > 0$ elsewhere. Then we derive a contradiction by proving that $R$ and $S$ cannot be zero on the upper vertex.

Since $S \geq 0$ in the characteristic triangle,

\[R_t - cR_x \leq \frac{c'(u)}{4c(u)} R^2 - \frac{\mu}{2} R.\]
Then by standard ODE comparison theorem (see [11] for reference), (1.8) and $R < 0$ except the upper vertex, we have $R < 0$ on the vertex. By $S \geq 0$ and $R \leq 0$ in the characteristic triangle, the a priori estimate in the previous part shows that

$$S_t + cS_x \geq \frac{c'(u)}{4c(u)}S^2 - \frac{\mu}{2}S.$$  

Then by ODE comparison theorem, (1.8) and $S > 0$ except the upper vertex, we have $S > 0$ on the vertex. Hence, we get a contradiction.

7. Finally, we prove the singularity formation by considering the forward characteristic starting from the origin. We have already proved that along this characteristic, before $t = \frac{\sigma}{\epsilon}$ and $t = \frac{\epsilon^2}{2CU}$,

$$S_t + cS_x \geq \frac{c'(u)}{4c(u)}S^2 - \frac{\mu}{2}S \geq \frac{c'(u_0)}{8CU}S^2 - \frac{\mu}{2}S \geq \frac{c'(u_0)}{16CU}S^2 - \frac{\mu}{2}S.$$

When $\epsilon < C_L$, (3.6) and the initial conditions (1.10) and (1.11) give that

$$-C_U\phi'(0) > S(0,0) = -c(u(0,0))\phi'(0) > \frac{8\mu C_U}{c'(u_0)}$$

which shows

$$\frac{c'(u_0)}{16CU}S^2(0,0) > \frac{\mu}{2}S(0,0).$$

By the ODE comparison theorem, on the characteristic, (3.13) gives

$$S(t, x) > \frac{8\mu C_U}{c'(u_0)},$$

i.e.

$$\frac{c'(u_0)}{16CU}S^2(t, x) > \frac{\mu}{2}S(t, x).$$

Hence, by (3.13),

$$S_t + cS_x \geq \frac{c'(u_0)}{16CU}S^2.$$  

So $S$ blowups before $t = \frac{16CU}{c'(u_0)S(0,0)}$ which is in $O(1)$ by (3.14). We complete the proof of Theorem 1.1.

4. SYSTEMS IN ENERGY-DEPENDENT COORDINATES

In this section, by restricting our consideration on smooth solutions, we derive a semi-linear system on new coordinates. In the next two sections, we construct the solution for the new semi-linear system, then after the reverse transformation we show that this solution is also a weak solution of the original system.
4.1. Energy-dependent coordinates. We denote

\[ \vec{R} = (R_1, R_2, R_3) \overset{\text{def}}{=} \mathbf{n}_t + c(n_1) \mathbf{n}_x, \quad \vec{S} = (S_1, S_2, S_3) \overset{\text{def}}{=} \mathbf{n}_t - c(n_1) \mathbf{n}_x. \]

Without confusion, we still use the letters \( R \) and \( S \) to denote Riemann variables as in the previous section. Then (1.3) can be reformulated as:

\[
\begin{align*}
\partial_t R_1 - c(n_1)\partial_x R_1 &= \frac{1}{4c(n_1)} \left\{ (c^2(n_1) - \gamma)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \gamma)\vec{R} \cdot \vec{S} \right\} n_1 \\
\partial_t S_1 + c(n_1)\partial_x S_1 &= \frac{1}{4c(n_1)} \left\{ (c^2(n_1) - \gamma)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \gamma)\vec{R} \cdot \vec{S} \right\} n_1 \\
\partial_t R_2 - c(n_1)\partial_x R_2 &= \frac{1}{4c(n_1)} \left\{ (c^2(n_1) - \alpha)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \alpha)\vec{R} \cdot \vec{S} \right\} n_2 \\
\partial_t S_2 + c(n_1)\partial_x S_2 &= \frac{1}{4c(n_1)} \left\{ (c^2(n_1) - \alpha)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \alpha)\vec{R} \cdot \vec{S} \right\} n_2 \\
\partial_t R_3 - c(n_1)\partial_x R_3 &= \frac{1}{4c(n_1)} \left\{ (c^2(n_1) - \alpha)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \alpha)\vec{R} \cdot \vec{S} \right\} n_3 \\
\partial_t S_3 + c(n_1)\partial_x S_3 &= \frac{1}{4c(n_1)} \left\{ (c^2(n_1) - \alpha)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \alpha)\vec{R} \cdot \vec{S} \right\} n_3 \\
\mathbf{n}_x = \frac{\vec{R} - \vec{S}}{2c(n_1)} \quad \text{or} \quad \mathbf{n}_t = \frac{\vec{R} + \vec{S}}{2}.
\end{align*}
\]

The energy equation (2.9) equals to

\[
\frac{1}{4} \partial_t(|\vec{R}|^2 + |\vec{S}|^2) - \frac{1}{4} \partial_x [c(n_1)(|\vec{R}|^2 - |\vec{S}|^2)] = -\frac{n}{4} |\vec{R} + \vec{S}|^2 \leq 0.
\]

We define the forward and backward characteristics as follows

\[
\begin{cases}
\frac{dx^\pm}{ds}(s, t, x) = \pm c(n_1)(s, x^\pm(s, t, x)), \\
x^\pm|_s = t = x.
\end{cases}
\]

Then we define the coordinate transformation:

\[
X \overset{\text{def}}{=} \int_0^{x^-(0, t, x)} [1 + |\vec{R}|^2(0, y)] dy, \quad \text{and} \quad Y \overset{\text{def}}{=} \int_{x^+(0, t, x)}^0 [1 + |\vec{S}|^2(0, y)] dy.
\]

This implies

\[
X_t - c(n_1)X_x = 0, \quad Y_t + c(n_1)Y_x = 0.
\]

Furthermore, for any smooth function \( f \), we get by using (4.5) that

\[
\begin{align*}
f_t + c(n_1)f_x &= (X_t + c(n_1)X_x)f_X = 2c(n_1)X_xf_X \\
f_t - c(n_1)f_x &= (Y_t - c(n_1)Y_x)f_Y = -2c(n_1)Y_xf_Y.
\end{align*}
\]

Using (4.6), we can transform the directional derivatives \( \partial_t \pm c\partial_x \) on the \((t, x)\)-coordinates into \( \partial_X \) and \( \partial_Y \) on the \((X, Y)\)-coordinates, hence get a new semi-linear system on the \((X, Y)\)-coordinates. We start the calculation from \( u_X \) and \( u_Y \). To complete the system, we introduce several new variables:

\[
p \overset{\text{def}}{=} \frac{1 + |\vec{R}|^2}{X_x}, \quad q \overset{\text{def}}{=} \frac{1 + |\vec{S}|^2}{-Y_x},
\]
For smooth solutions, $|\vec{\ell}|$, $|\vec{m}|$, $|h_1|$ and $|h_2|$ are all less than 1. However, $|p|$ and $|q|$ might go to infinity when the possible gradient blowup happens. In the next section, in order to estimating them, we will consider the energy equation, where the energy equation can help us estimate $p$ and $q$ since $X$ and $Y$ are energy-dependent coordinates.

We derive the new system in several steps:

1. Using (4.6) and (4.8), we have

   \[
   \partial_t \mathbf{n} = \frac{1}{2c(n_1)(-Y_x)}(\partial_t \mathbf{n} - c(n_1)\partial_x \mathbf{n}) = \frac{q}{2c(n_1)}\vec{m},
   \]

   (4.9)

   \[
   \partial_x \mathbf{n} = \frac{1}{2c(n_1)X_x}(\partial_t \mathbf{n} + c(n_1)\partial_x \mathbf{n}) = \frac{p}{2c(n_1)}\vec{\ell}.
   \]

2. Then

   \[
   \partial_t p - c(n_1)\partial_x p = 2(X_x)^{-1}[\vec{R} \cdot (\partial_t \vec{R} - c(n_1)\partial_t \vec{R})]
   \]

   \[
   - (X_x)^{-2}[\partial_x X_x - c(n_1)\partial_x X_x](1 + |\vec{R}|^2).
   \]

While using (4.2), we have

\[
\vec{R} \cdot (\partial_t \vec{R} - c(n_1)\partial_x \vec{R}) = \frac{1}{4c^2(n_1)} \left\{ (c^2(n_1) - \alpha)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \alpha)\vec{R} \cdot \vec{S} \right\} \vec{R} \cdot \mathbf{n}
\]

\[
+ \frac{\alpha - \gamma}{4c^2(n_1)}(|\vec{R}|^2 + |\vec{S}|^2 - 2\vec{R} \cdot \vec{S})R_1n_1 + \frac{c'(n_1)}{2c(n_1)}R_1(|\vec{R}|^2 - \vec{R} \cdot \vec{S}) - \frac{\mu}{2}(|\vec{R}|^2 + \vec{R} \cdot \vec{S}),
\]

while notice that $|\mathbf{n}| = 1$ and $c'(n_1) = \frac{(\gamma - \alpha)n_1}{c(n_1)}$, so that $\vec{R} \cdot \mathbf{n} = 0$, and there holds

(4.10)

\[
\vec{R} \cdot (\partial_t \vec{R} - c(n_1)\partial_x \vec{R}) = \frac{c'(n_1)}{4c(n_1)}R_1(|\vec{R}|^2 - |\vec{S}|^2) - \frac{\mu}{2}(|\vec{R}|^2 + \vec{R} \cdot \vec{S}),
\]

which together with (4.5) applied gives

\[
\partial_t p - c(n_1)\partial_x p = \frac{p}{1 + |\vec{R}|^2}\left[ \frac{c'(n_1)}{2c(n_1)}(-R_1(1 + |\vec{S}|^2) + S_1(1 + |\vec{R}|^2)) - \mu(|\vec{R}|^2 + \vec{R} \cdot \vec{S}) \right],
\]

from which, and (4.6, 4.7), we infer

\[
p_Y = \frac{1}{2c(n_1)(-Y_x)}(p_t - c(n_1)p_x).
\]

(4.11)

\[
= \frac{pq}{2c(n_1)(2c(n_1)} \left[ \frac{c'(n_1)}{2c(n_1)}(1 - \frac{R_1}{1 + |\vec{R}|^2} + \frac{S_1}{1 + |\vec{S}|^2}) - \mu\frac{|\vec{R}|^2 + \vec{R} \cdot \vec{S}}{(1 + |\vec{S}|^2)(1 + |\vec{R}|^2)} \right].
\]

Similarly,

\[
\partial_t q + c(n_1)\partial_x q = 2(-Y_x)^{-1}[\vec{S} \cdot (\partial_t \vec{S} + c(n_1)\partial_x \vec{S})]
\]

\[
+ (Y_x)^{-2}(\partial_x Y_x + c(n_1)\partial_x Y_x)(1 + |\vec{S}|^2),
\]
then by (4.2), we have
\[ S \cdot (\partial_t S + c(n_1)\partial_x S) = \frac{1}{4c^2(n_1)} \left\{ (c^2(n_1) - \alpha)(|\vec{R}|^2 + |\vec{S}|^2) - 2(3c^2(n_1) - \alpha)\vec{R} \cdot \vec{S} \right\} S \cdot \mathbf{n} \]
\[ + \frac{\alpha - \gamma}{4c^2(n_1)} (|\vec{R}|^2 + |\vec{S}|^2 - 2\vec{R} \cdot \vec{S}) S_1 n_1 + \frac{c'(n_1)}{2c(n_1)} S_1 (|\vec{S}|^2 - \vec{R} \cdot \vec{S}) - \frac{\mu}{2} (|\vec{S}|^2 + \vec{R} \cdot \vec{S}), \]
which along with the fact that \( S \cdot \mathbf{n} = 0 \) leads to
\[ S \cdot (\partial_t S + c(n_1)\partial_x S) = -\frac{c'(n_1)}{4c(n_1)} S_1 (|\vec{R}|^2 - |\vec{S}|^2) - \frac{\mu}{2} (|\vec{S}|^2 + \vec{R} \cdot \vec{S}), \]
from which and (4.5), we infer
\[ \partial_t q + c(n_1)\partial_x q = \frac{q}{1 + |\vec{S}|^2} \left[ \frac{c'(n_1)}{2c(n_1)} R_1 (1 + |\vec{S}|^2) - S_1 (1 + |\vec{R}|^2) \right] - \mu (|\vec{S}|^2 + \vec{R} \cdot \vec{S}). \]

Next by (4.6, 4.7), we have
\[ q_x = \frac{1}{2c(n_1)X_x} (q_t + c(n_1)q_x) \]
\[ = \frac{pq}{2c(n_1)} \left[ \frac{c'(n_1)}{2c(n_1)} \left( \frac{R_1}{1 + |\vec{R}|^2} - \frac{S_1}{1 + |\vec{S}|^2} \right) - \mu \frac{|\vec{S}|^2 + \vec{R} \cdot \vec{S}}{(1 + |\vec{S}|^2)(1 + |\vec{R}|^2)} \right]. \]

3. Next,
\[ \partial_t \ell_1 - c(n_1)\partial_x \ell_1 = (1 + |\vec{R}|^2)^{-1} (\partial_t R_1 - c(n_1)\partial_x R_1) \]
\[ - 2(1 + |\vec{R}|^2)^{-2} R_1 \left[ \vec{R} \cdot (\partial_t \vec{R} - c(n_1)\partial_x \vec{R}) \right], \]
which together with (4.2) and (4.10) implies
\[ \partial_t \ell_1 - c(n_1)\partial_x \ell_1 = (1 + |\vec{R}|^2)^{-1} \left[ \frac{c^2(n_1) - \gamma}{2c^2(n_1)} \right] \]
\[ + (1 + |\vec{R}|^2)^{-2} R_1 \left[ (1 + |\vec{S}|^2) - S_1 (1 + |\vec{R}|^2) \right] \]
\[ + (1 + |\vec{R}|^2)^{-2} R_1 \mu (|\vec{R}|^2 + \vec{R} \cdot \vec{S}) - (1 + |\vec{R}|^2)^{-1} \frac{\mu}{2} R_1 + S_1. \]

Using (4.6, 4.7), we obtain
\[ \partial_y \ell_1 = \frac{q}{8c^3(n_1)} \left[ (c^2(n_1) - \gamma) h_1 + h_2 - 2h_1 + 2(3c^2(n_1) - \gamma) \ell \cdot \vec{m} \right] n_1 \]
\[ + \frac{c'(n_1)}{4c^2(n_1)} \ell_1 q (\ell_1 - m_1) + \frac{\mu q}{4c(n_1)} \left[ 2\ell_1 (h_2 - h_1 + \ell \cdot \vec{m}) - \ell_1 h_2 - m_1 h_1 \right]. \]

Similarly by (4.8), we have
\[ \partial_t m_1 + c(n_1)\partial_x m_1 = (1 + |\vec{S}|^2)^{-1} (\partial_t S_1 + c(n_1)\partial_x S_1) \]
\[ - 2(1 + |\vec{S}|^2)^{-2} S_1 \left[ \vec{S} \cdot (\partial_t \vec{S} + c(n_1)\partial_x \vec{S}) \right], \]
which together with (4.6) and (4.12) ensures that
\[
\partial_t m_1 + c(n_1) \partial_x m_1 = (1 + |\vec{S}|^2)^{-1} \left[ \frac{c^2(n_1) - \gamma}{4c^2(n_1)} (|\vec{R}|^2 + |\vec{S}|^2) - \frac{3c^2(n_1) - \gamma}{2c^2(n_1)} \vec{R} \cdot \vec{S} \right] n_1 \\
-(1 + |\vec{S}|^2)^{-2} S_1 \frac{c'(n_1)}{2c(n_1)} \left[ (R_1(1 + |\vec{S}|^2) - S_1(1 + |\vec{R}|^2) \right] \\
+(1 + |\vec{S}|^2)^{-2} S_1 \mu (|\vec{S}|^2 + \vec{R} \cdot \vec{S}) - (1 + |\vec{S}|^2)^{-1} \frac{\mu}{2} (R_1 + S_1),
\]
from which and (4.6, 4.7), we infer
\[
\partial_X m_1 = \frac{p}{8c^3(n_1)} \left[ (c^2(n_1) - \gamma) (h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \gamma) \vec{e} \cdot \vec{m} \right] n_1 \\
(4.15) \\
- \frac{c'(n_1)}{4c^2(n_1)} m_1 p (\ell_1 - m_1) + \frac{\mu p}{4c(n_1)} \left[ 2m_1 (h_1 - h_2) + \vec{e} \cdot \vec{m} - \ell_1 h_2 - m_1 h_1 \right].
\]
Following the same line, we deduce from (4.2) and (4.10) that
\[
\partial_t \ell_2 - c(n_1) \partial_x \ell_2 = (1 + |\vec{R}|^2)^{-1} \left[ \frac{c^2(n_1) - \alpha}{4c^2(n_1)} (|\vec{R}|^2 + |\vec{S}|^2) - \frac{3c^2(n_1) - \alpha \vec{R} \cdot \vec{S}}{2c^2(n_1)} \right] n_2 \\
+(1 + |\vec{R}|^2)^{-2} R_1 \frac{c'(n_1)}{2c(n_1)} \left[ R_2(1 + |\vec{S}|^2) - S_2(1 + |\vec{R}|^2) \right] \\
+(1 + |\vec{R}|^2)^{-2} R_2 \mu (|\vec{R}|^2 + \vec{R} \cdot \vec{S}) - (1 + |\vec{R}|^2)^{-1} \frac{\mu}{2} (R_2 + S_2),
\]
and
\[
\partial_t \ell_3 - c(n_1) \partial_x \ell_3 = (1 + |\vec{R}|^2)^{-1} \left[ \frac{c^2(n_1) - \alpha}{4c^2(n_1)} (|\vec{R}|^2 + |\vec{S}|^2) - \frac{3c^2(n_1) - \alpha \vec{R} \cdot \vec{S}}{2c^2(n_1)} \right] n_3 \\
+(1 + |\vec{R}|^2)^{-2} R_1 \frac{c'(n_1)}{2c(n_1)} \left[ R_3(1 + |\vec{S}|^2) - S_3(1 + |\vec{R}|^2) \right] \\
+(1 + |\vec{R}|^2)^{-2} R_3 \mu (|\vec{R}|^2 + \vec{R} \cdot \vec{S}) - (1 + |\vec{R}|^2)^{-1} \frac{\mu}{2} (R_3 + S_3),
\]
which together with (4.6, 4.7) derive that
\[
\partial_Y \ell_2 = \frac{q}{8c^3(n_1)} \left[ (c^2(n_1) - \alpha) (h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha) \vec{e} \cdot \vec{m} \right] n_2 \\
+ \frac{c'(n_1)}{4c^2(n_1)} \ell_1 q (\ell_2 - m_2) + \frac{\mu q}{4c(n_1)} \left[ 2\ell_2 (h_2 - h_1h_2 + \vec{e} \cdot \vec{m}) - \ell_2 h_2 - m_2 h_1 \right], \\
(4.16) \\
\partial_Y \ell_3 = \frac{q}{8c^3(n_1)} \left[ (c^2(n_1) - \alpha) (h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha) \vec{e} \cdot \vec{m} \right] n_3 \\
+ \frac{c'(n_1)}{4c^2(n_1)} \ell_1 q (\ell_3 - m_3) + \frac{\mu q}{4c(n_1)} \left[ 2\ell_3 (h_2 - h_1h_2 + \vec{e} \cdot \vec{m}) - \ell_3 h_2 - m_3 h_1 \right].
\]
While we deduce from (4.2) and (4.12) that
\[
\partial_t m_2 + c(n_1) \partial_x m_2 = (1 + |\vec{S}|^2)^{-1} \left[ \frac{c^2(n_1) - \alpha}{4c^2(n_1)} (|\vec{R}|^2 + |\vec{S}|^2) - \frac{3c^2(n_1) - \alpha}{2c^2(n_1)} \vec{R} \cdot \vec{S} \right] n_2 \\
-(1 + |\vec{S}|^2)^{-2} S_1 \frac{c'(n_1)}{2c(n_1)} \left[ R_2(1 + |\vec{S}|^2) - S_2(1 + |\vec{R}|^2) \right] \\
+(1 + |\vec{S}|^2)^{-2} S_2 \mu (|\vec{S}|^2 + \vec{R} \cdot \vec{S}) - (1 + |\vec{S}|^2)^{-1} \frac{\mu}{2} (R_2 + S_2),
\]
which together with (4.6-4.7) implies that

\[
\partial_t m_3 + c(n_1)\partial_x m_3 = (1 + |\vec{S}|^2)^{-1}\left[\frac{c^2(n_1) - \alpha}{4c^2(n_1)}(|\vec{R}|^2 + |\vec{S}|^2) - \frac{3c^2(n_1) - \alpha}{2c^2(n_1)}\vec{R} \cdot \vec{S}\right]n_3 \\
-(1 + |\vec{S}|^2)^{-2}S_1\frac{c'(n_1)}{2c(n_1)}\left[R_3(1 + |\vec{S}|^2) - S_3(1 + |\vec{R}|^2)\right] \\
+(1 + |\vec{S}|^2)^{-2}S_3\mu(|\vec{S}|^2 + \vec{R} \cdot \vec{S}) - (1 + |\vec{S}|^2)^{-1}\frac{\mu}{2}(R_3 + S_3),
\]

which together with (4.6-4.7) implies that

\[
\partial_t m_2 = \frac{p}{8c^3(n_1)}\left[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\vec{\ell} \cdot \vec{m}\right]n_2 \\
- \frac{c'(n_1)}{4c^2(n_1)}m_1p(\ell_2 - m_2) + \frac{\mu p}{4c(n_1)}\left[2m_2(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m}) - \ell_2h_2 - m_2h_1\right],
\]

\[
\partial_t m_3 = \frac{p}{8c^3(n_1)}\left[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\vec{\ell} \cdot \vec{m}\right]n_3 \\
- \frac{c'(n_1)}{4c^2(n_1)}m_1p(\ell_3 - m_3) + \frac{\mu p}{4c(n_1)}\left[2m_3(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m}) - \ell_3h_2 - m_3h_1\right].
\]

4. It follows from (4.10) and (4.8) that

\[
\partial_t h_1 - c(n_1)\partial_x h_1 = -2(1 + |\vec{R}|^2)^{-2}\left[\vec{R} \cdot (\partial_t \vec{R} - c(n_1)\partial_x \vec{R})\right] \\
- (1 + |\vec{R}|^2)^{-2}\left[\frac{c'(n_1)}{2c(n_1)}\vec{R}_3(|\vec{R}|^2 - |\vec{S}|^2) - \mu(|\vec{R}|^2 + \vec{R} \cdot \vec{S})\right].
\]

Then we get by using (4.6-4.7) that

\[
\partial_Y h_1 = \frac{c'(n_1)}{4c^2(n_1)}q\ell_1(h_1 - h_2) + \frac{\mu}{2c(n_1)}qh_1(h_2 - h_1h_2 + \vec{\ell} \cdot \vec{m}).
\]

Similar calculations together with (4.12) gives

\[
\partial_t h_2 + c(n_1)\partial_x h_2 = (1 + |\vec{S}|^2)^{-2}\left[\frac{c'(n_1)}{2c(n_1)}S_1(|\vec{R}|^2 - |\vec{S}|^2) + \mu(|\vec{S}|^2 + \vec{R} \cdot \vec{S})\right],
\]

which together with (4.6-4.7) implies that

\[
\partial_X h_2 = \frac{c'(n_1)}{4c^2(n_1)}pm_1(h_2 - h_1) + \frac{\mu}{2c(n_1)}ph_2(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m}).
\]
In summary, we obtain

\begin{align}
\partial_Y \ell_1 &= \frac{q}{8c(n_1)}[(c^2(n_1) - \gamma)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \gamma)\vec{\ell} \cdot \vec{m}]n_1 \\
\partial_Y q_1 &= \frac{c(n_1)}{4c^2(n_1)}[q\ell_1 - m_1] + \frac{pq}{4c(n_1)}[2\ell_1(h_2 - h_1h_2 + \vec{\ell} \cdot \vec{m}) - \ell_1h_2 - m_1h_1], \\
\partial_X m_1 &= \frac{p}{8c(n_1)}[\ell_1q_1 - m_1] + \frac{pq}{4c(n_1)}[2m_1(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m}) - \ell_1h_2 - m_1h_1], \\
\partial_Y \ell_2 &= \frac{q}{8c^2(n_1)}[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\vec{\ell} \cdot \vec{m}]n_1 \\
\partial_Y \ell_2 &= \frac{c(n_1)}{4c^2(n_1)}[\ell_1q_2 - m_2] + \frac{pq}{4c(n_1)}[2\ell_2(h_2 - h_1h_1 + \vec{\ell} \cdot \vec{m}) - \ell_2h_2 - m_2h_1], \\
\partial_Y m_2 &= \frac{p}{8c^2(n_1)}[\ell_1q_2 - m_2] + \frac{pq}{4c(n_1)}[2m_2(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m}) - \ell_2h_2 - m_2h_1], \\
\partial_Y \ell_3 &= \frac{q}{8c^3(n_1)}[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\vec{\ell} \cdot \vec{m}]n_3 \\
\partial_Y \ell_3 &= \frac{c(n_1)}{4c^3(n_1)}[\ell_1q_3 - m_3] + \frac{pq}{4c(n_1)}[2\ell_3(h_2 - h_1h_2 + \vec{\ell} \cdot \vec{m}) - \ell_3h_2 - m_3h_1], \\
\partial_X \ell_1 &= \frac{q}{2c(n_1)}\vec{m}, \quad \text{or} \quad \partial_X m_1 = \frac{p}{2c(n_1)}\vec{\ell}, \\
\partial_X h_1 &= \frac{c(n_1)}{4c^2(n_1)}q\ell_1 - m_1 + \frac{pq}{2c(n_1)}qh_1(h_2 - h_1h_2 + \vec{\ell} \cdot \vec{m}), \\
\partial_X h_2 &= \frac{c(n_1)}{4c^2(n_1)}pm_1(h_2 - h_1) + \frac{pq}{2c(n_1)}ph_2(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m}), \\
p_Y &= \frac{pq}{2c(n_1)}[-c(n_1)(\ell_1 - m_1) - \mu(h_2 - h_1h_2 + \vec{\ell} \cdot \vec{m})], \\
q_X &= \frac{pq}{2c(n_1)}[c(n_1)(\ell_1 - m_1) - \mu(h_1 - h_1h_2 + \vec{\ell} \cdot \vec{m})].
\end{align}

4.2. Consistency of variables. Before we prove the global existence of (4.20), we first show that the various variables introduced for (4.20) are consistent, following from the proposition below. Note we only use (4.20) in the proof of this proposition, while the original equation (1.3) and definitions (4.7) and (4.8) are not used.

**Proposition 4.1.** For smooth enough data, the following conservative quantities hold:

\begin{align}
\vec{\ell} \cdot \vec{n}(X,Y) &= \vec{m} \cdot \vec{n}(X,Y) = 0 \quad |\vec{n}(X,Y)| = 1 \quad \text{and} \\
|\vec{\ell}(X,Y)|^2 + h_1^2(X,Y) &= h_1(X,Y), \quad |\vec{m}(X,Y)|^2 + h_2^2(X,Y) = h_2(X,Y) \quad \forall X,Y,
\end{align}

as long as

\begin{align*}
\vec{\ell} \cdot \vec{n}(X,\varphi(X)) &= \vec{m} \cdot \vec{n}(X,\varphi(X)) = 0 \quad |\vec{n}(X,\varphi(X))| = 1 \quad \text{and} \\
|\vec{\ell}(X,\varphi(X))|^2 + h_1^2(X,\varphi(X)) &= h_1(X,\varphi(X)), \\
|\vec{m}(X,\varphi(X))|^2 + h_2^2(X,\varphi(X)) &= h_2(X,\varphi(X)).
\end{align*}

Here $\varphi$ represents the initial curve, see (5.1) from the next section.
Proof. We first deduce from (4.20) that
\[\partial_Y [\ell^2 + h_1^2] = \frac{q}{4c^3(n_1)}[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar{m}] \ell \cdot \mathbf{n} + \frac{(\alpha - \gamma)n_1}{4c^3(n_1)}q[(h_1 + h_2 - 2h_1h_2) - 2\ell \cdot \bar{m}] \ell_1 + c'(n_1)q\ell_1 [2(\ell^2 - \ell \cdot \bar{m}) + 2h_1(h_1 - h_2) - (h_1 - h_2)] + \frac{\mu q}{2c(n_1)}\ell \cdot [2h_2(h_1h_2 + \ell \cdot \bar{m}) - \ell h_2 + \ell h_1] + \frac{\mu q}{2c(n_1)}(2h_1^2 - h_1)(h_1 - h_2 + \ell \cdot \bar{m}),\]
from which and the fact that $c'(n_1) = \frac{(\gamma - \alpha)n_1}{c(n_1)}$, we deduce that
\[\partial_Y [\ell^2 + h_1^2] = \frac{q}{4c^3(n_1)}[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar{m}] \ell \cdot \mathbf{n} + c'(n_1)q\ell_1 + \frac{\mu q}{2c(n_1)}[h_2 - 2h_1h_2 + 2\ell \cdot \bar{m})][\ell^2 + h_1^2] = \ell \cdot \mathbf{n} + \ell \cdot \partial_Y \mathbf{n}.
\]
Similarly it follows from (4.20) that
\[\partial_Y [\ell \cdot \mathbf{n}] = \partial_Y \ell \cdot \mathbf{n} + \ell \cdot \partial_Y \mathbf{n}
\]
which along with the fact that $c^2(n_1) = \alpha + (\gamma - \alpha)n_1^2$ leads to
\[\partial_Y [\ell \cdot \mathbf{n}] = \frac{q}{8c^3(n_1)}[(c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar{m}] (|\mathbf{n}|^2 - 1)
\]
On the other hand, observe that $\partial_Y \mathbf{n} = \frac{q}{2c(n_1)}\bar{m}$ is consistent with $\partial_X \mathbf{n} = \frac{p}{2c(n_1)}\ell$. In fact, again thanks to (4.20), one has on the one hand
\[\partial_X[\partial_Y \mathbf{n}] = -\frac{c'(n_1)}{8c^3(n_1)}pq(\ell_1 + m_1)\bar{m} - \frac{\mu q}{4c^2(n_1)}pq(h_1 - h_1h_2 + \ell \cdot \bar{m})\ell + \frac{q}{2c(n_1)}\partial_X \bar{m},\]
and on the other hand
\[\partial_Y[\partial_X \mathbf{n}] = -\frac{c'(n_1)}{8c^3(n_1)}pq\ell_1 + m_1)\ell - \frac{\mu q}{4c^2(n_1)}pq(h_2 - h_1h_2 + \ell \cdot \bar{m})\ell + \frac{p}{2c(n_1)}\partial_Y \ell,\]
which along with the $\ell$ equations and $m$ equations of (4.20) shows that $\partial_X[\partial_Y n] = \partial_Y[\partial_X n]$. So we can also use the equation $\partial_X n = \frac{p}{2c(n_1)}\ell$, from which and the $\bar m$ equation of (4.20), we get

\[
\partial_X[\bar m \cdot n] = \partial_X\bar m \cdot n + \bar m \cdot \partial_X n \\
= \frac{p}{8c^3(n_1)} \left[ (c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1 h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar m) \right] |n|^2 - 1) \\
+ \frac{p}{8c^3(n_1)} \left[ (c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1 h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar m) \right] \\
+ \frac{\alpha - \gamma}{4c(n_1)} \bar m^2 [h_1 + h_2 - 2h_1 h_2) - 2\ell \cdot \bar m] \\
+ \frac{c'(n_1)}{4c^2(n_1)} \bar m \cdot [\bar m - \ell \cdot n] + \frac{p}{2c(n_1)} \ell \cdot \bar m, \\
+ \frac{\mu p}{4c(n_1)} [\bar m(h_1 - 2h_1 h_2 + 2\ell \cdot \bar m) - \ell h_2) \cdot n],
\]

which gives rise to

(4.24)

\[
\partial_X[\bar m \cdot n] = \frac{p}{8c^3(n_1)} \left[ (c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1 h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar m) \right] |n|^2 - 1) \\
- \frac{c'(n_1)}{4c^2(n_1)} \bar m \cdot [\bar m - \ell \cdot n] + \frac{\mu p}{4c(n_1)} (h_1 - 2h_1 h_2 + 2\ell \cdot \bar m) \bar m + \frac{\mu p h_2}{4c(n_1)} \ell \cdot n),
\]

While it is easy to observe that

(4.25)

\[
\partial_Y [|n|^2 - 1] = \frac{q}{c(n_1)} \bar m \cdot n.
\]

Finally to control the evolution of $|\bar m|^2 + h_2^2 - h_2$, we get by applying (4.20) that

\[
\partial_X \left[ |\bar m|^2 + h_2^2 - h_2 \right] = \frac{p}{4c^3(n_1)} \left[ (c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1 h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar m) \right] \bar m \cdot n \\
+ \frac{\alpha - \gamma}{4c^3(n_1)} \bar m^2 [h_1 + h_2 - 2h_1 h_2) - 2\ell \cdot \bar m] m_1 m_1 \\
+ \frac{c'(n_1)}{2c^2(n_1)} \bar m \cdot \bar m \cdot [|\bar m|^2 - \ell \cdot \bar m] + (h_2 - 1/2)(h_2 - h_1)] \\
+ \frac{\mu p}{2c(n_1)} \bar m \cdot \left[ 2\bar m(h_1 - h_1 h_2 + \ell \cdot \bar m) - \ell h_2 - \bar m h_1 \right] \\
+ \frac{\mu p}{2c(n_1)} (2h_2^2 - h_2) (h_1 - h_1 h_2 + \ell \cdot \bar m)
\]

which leads to

(4.26)

\[
\partial_X \left[ |\bar m|^2 + h_2^2 - h_2 \right] = \frac{p}{4c^3(n_1)} \left[ (c^2(n_1) - \alpha)(h_1 + h_2 - 2h_1 h_2) - 2(3c^2(n_1) - \alpha)\ell \cdot \bar m) \right] \bar m \cdot n \\
+ \left( \frac{c'(n_1)}{2c^2(n_1)} pm_1 + \frac{\mu p}{2c(n_1)} (h_1 - 2h_1 h_2 + 2\ell \cdot \bar m) \right) \left[ |\bar m|^2 + h_2^2 - h_2 \right]
\]

Summing up (4.22) to (4.26) gives rise to (4.21). This completes the proof of the proposition. $\square$
5. Solutions in the energy coordinates

In this section, we prove the existence of the solution for (4.20) with boundary data converted from (1.5). To avoid the confusion, the reader should aware that in this section we solve variables \( u, p, q, \vec{\ell}, \vec{m}, h_1, h_2 \) by system (4.20) instead of (1.3) and we do not use the definitions (4.7) and (4.8) except when we assign the boundary data.

The initial line \( t = 0 \) in the \((t,x)\)-plane is transformed to a parametric curve

\[
\gamma : \quad Y = \varphi(X)
\]

in the \((X,Y)\) plane, where \( Y = \varphi(X) \) if and only if there is an \( x \) such that

\[
\begin{align*}
X &= \int_0^x [1 + |\vec{R}|^2(0,y)] \, dy, \\
Y &= \int_0^x [1 + |\vec{S}|^2(0,y)] \, dy.
\end{align*}
\]

The curve is non-characteristic. We introduce

\[
E_0 \overset{\text{def}}{=} \frac{1}{4} \int [ |\vec{R}|^2(0,y) + |\vec{S}|^2(0,y) ] \, dy < \infty.
\]

It equals to the number in (1.13) with \( t = 0 \). The two functions \( X = X(x), Y = Y(x) \) from (5.2) are well-defined and absolutely continuous, provided that (1.5) is satisfied. So \( \varphi(X) \) is continuous and strictly decreasing on \( X \) since \( X(x) \) is strictly increasing while \( Y(x) \) is strictly decreasing. From (5.3) it follows

\[
|X + \varphi(X)| \leq 4E_0.
\]

As \((t, x)\) ranges over the domain \( \mathbb{R}^+ \times \mathbb{R} \), the corresponding variables \((X, Y)\) range over the set

\[
\Omega^+ := \{(X,Y) ; \ Y \geq \varphi(X)\}.
\]

Along the curve

\[
\gamma := \{(X,Y) ; \ Y = \varphi(X)\} \subset \mathbb{R}^2
\]

parametrized by \( x \mapsto (X(x), Y(x)) \), we can thus assign the boundary data \((\vec{\ell}, \vec{m}, h_1, h_2, \bar{p}, \bar{q}, \bar{n}) \in L^\infty\) defined by their definition evaluated at the initial data (1.5), i.e.,

\[
\bar{n} = n_0(x), \quad \bar{p} = 1, \quad \bar{q} = 1, \\
\vec{\ell} = \vec{R}(0,x)\bar{h}_1, \quad \vec{m} = \vec{S}(0,x)\bar{h}_2,
\]

\[
\bar{h}_1 = \frac{1}{1 + |\vec{R}|^2(0,x)}, \quad \text{and} \\
\bar{h}_2 = \frac{1}{1 + |\vec{S}|^2(0,x)}.
\]

where

\[
\vec{R}(0,x) = \mathbf{n}_1(x) + c(n_{10}(x))\mathbf{n}_0'(x), \quad \vec{S}(0,x) = \mathbf{n}_1(x) - c(n_{10}(x))\mathbf{n}_0'(x).
\]

We consider solutions to the boundary value problem (4.20) (5.6) (1.4).

**Theorem 5.1.** The problem (4.20) (5.6) (1.5) (1.4) has a unique global solution defined for all \((X,Y) \in \Omega^+\).
Sketch of Proof. Noticing that all equations in (4.20) have a locally Lipschitz continuous right hand side, the construction of a local solution as fixed point of a suitable integral transformation is straightforward. Note the consistency condition \( \partial_X [\partial_Y n] = \partial_Y [\partial_X n] \) proved in the previous section shows that we can use either the equation for \( \partial_Y n \) or the one for \( \partial_X n \) to find the solution.

To make sure that this solution is actually defined on \( \Omega^+ \), one must establish a priori bounds, showing that the solution remains bounded on bounded subset of \( \Omega^+ \).

By (4.21), we have
\[
(5.7) \quad h_1 (1 - h_1) \geq 0, \quad h_2 (1 - h_2) \geq 0.
\]
Thus \( h_1, h_2 \) are bounded between zero and one, and \( |\vec{\ell}| \) and \( |\vec{m}| \) are both uniformly bounded by (4.21).

By the \( p \) and \( q \) equations in (4.20) and (4.21), we have
\[
(5.8) \quad p_Y + q_X = -\frac{\mu pq h_1 h_2}{2c(n_1)} \left| \frac{\vec{\ell}}{h_1} + \frac{\vec{m}}{h_2} \right|^2 \leq 0
\]
which implies that
\[
\int_{\phi^{-1}(Y)}^X p(X', Y) dX' + \int_{\phi(X)}^Y q(X, Y') dY' \leq X - \phi^{-1}(Y) + Y - \phi(X)
\]
where \( \phi^{-1} \) denotes the inverse of \( \phi \), following an integration over the characteristic triangle with vertex \((X, Y)\). Thus, by the energy assumption (5.3), we find
\[
(5.9) \quad \int_{\phi^{-1}(Y)}^X p(X, Y') dX' + \int_{\phi(X)}^Y q(X, Y') dY' \leq 2(|X| + |Y| + 4\mathcal{E}_0).
\]
Integrating the \( p \) equation in (4.20) vertically and use the bound on \( q \) from (5.9), we find
\begin{align*}
(5.10) \quad p(X, Y) & = \exp \left\{ \int_{\phi(X)}^Y q(X, Y') \left[ -\frac{c(n_1)}{2c(n_1)} \left( \ell_1 - m_1 \right) - \mu (h_2 - h_1 h_2 + \vec{\ell} \cdot \vec{m}) \right] dY' \right\} \\
& \leq \exp \left\{ C_0 \int_{\phi(X)}^Y q(X, Y') dY' \right\} \\
& \leq \exp \{2C_0 (|X| + |Y| + 4\mathcal{E}_0)\}.
\end{align*}
Here \( C_0 \) represents a finite number. Similarly, we have
\[
(5.11) \quad q(X, Y) \leq \exp\{2C_0 (|X| + |Y| + 4\mathcal{E}_0)\}.
\]

Relying on the local bounds (5.10), (5.11), the local solution to (4.20) (5.6) (1.5) (1.4) can be extended to \( \Omega^+ \). One may consult paper [3] for details. This completes the sketch of the proof.

**Corollary 5.1.** If the initial data \((n_0, n_1)\) are smooth, the solution \( U := (n, p, q, \vec{\ell}, \vec{m}, h_1, h_2)\) of (4.20) (5.6) (1.5) (1.4) is a smooth function of the variables \((X, Y)\). Moreover, assume that a sequence of smooth functions \((n_0^i, n_1^i)_{i \geq 1}\) satisfies
\[
n_0^i \to n_0, \quad (n_0^i)_x \to (n_0)_x, \quad n_1^i \to n_1
\]
uniformly on compact subsets of \( \mathbb{R} \). Then one has the convergence of the corresponding solutions:
\[
(n^i, p^i, q^i, \vec{\ell}^i, \vec{m}^i, h_1^i, h_2^i) \to U
\]
uniformly on bounded subsets of \( \Omega^+ \).
6. Inverse Transformation

By expressing the solution $\mathbf{n}(X,Y)$ in terms of the original variables $(t,x)$, we shall recover a solution of the Cauchy problem (1.3)~(1.4). This will prove Theorem 1.2 except the estimate $\mathcal{E}(t) \leq \mathcal{E}_0$ which will be proved in the next section.

Using (4.6) by letting $f = t$ or $x$, we have equations:

\begin{equation}
(6.1) \quad t_X = \frac{ph_1}{2c}, \quad t_Y = \frac{qh_2}{2c}, \quad x_X = \frac{ph_1}{2}, \quad x_Y = -\frac{qh_2}{2}.
\end{equation}

In fact, we only need one of the two equations of $t_X$ and $t_Y$, which are consistent since $t_{XY} = t_{YX}$. The same is true for $x$. We integrate (6.1) with data $t = 0, x = x$ on $\gamma$ to find $t = t(X,Y), x = x(X,Y)$, which exist for all $(X,Y)$ in $\Omega^+$. We need the inverse functions $X = X(t,x), Y = Y(t,x)$. The inverse functions do not exist as a one-to-one correspondence between $(t,x)$ in $\mathbb{R}^+ \times \mathbb{R}$ and $(X,Y)$ in $\Omega^+$. There may be a nontrivial set of points in $\Omega^+$ that maps to a single point $(t,x)$. To investigate it, we find the partial derivatives of the inverse mapping, valid at points where $h_1 \neq 0, h_2 \neq 0$,

\begin{equation}
(6.2) \quad X_t = \frac{c}{ph_1}, \quad Y_t = \frac{c}{qh_2}, \quad X_x = \frac{1}{ph_1}, \quad Y_x = -\frac{1}{qh_2}.
\end{equation}

Thus (4.5) holds and so does (4.6) for our solution.

We first examine the regularity of the solution constructed in the previous Section. Since the initial data $(\mathbf{m}_0)_x, \mathbf{m}_1$ etc. are only assumed to be in $L^2$, the functions $U$ may well be discontinuous. More precisely, on bounded subsets of the $\Omega^+$, the solutions satisfy the following:

- The functions $\ell_1, \ell_2, \ell_3, h_1, p$ are Lipschitz continuous w.r.t. $Y$, measurable w.r.t. $X$.
- The functions $m_1, m_2, m_3, h_2, q$ are Lipschitz continuous w.r.t. $X$, measurable w.r.t. $Y$.
- The vector field $\mathbf{n}$ is Lipschitz continuous w.r.t. both $X$ and $Y$.

In order to define $\mathbf{n}$ as a vector field of the original variables $t, x$, we should formally invert the map $(X,Y) \mapsto (t,x)$ and write $\mathbf{n}(t,x) = \mathbf{n}(X(t,x), Y(t,x))$. The fact that the above map may not be one-to-one does not cause any real difficulty. Indeed, given $(t^*, x^*)$, we can choose an arbitrary point $(X^*, Y^*)$ such that $t(X^*, Y^*) = t^*, x(X^*, Y^*) = x^*$, and define $\mathbf{n}(t^*, x^*) = \mathbf{n}(X^*, Y^*)$. To prove that the values of $\mathbf{n}$ do not depend on the choice of $(X^*, Y^*)$, we proceed as follows. Assume that there are two distinct points such that $t(X_1, Y_1) = t(X_2, Y_2) = t^*, x(X_1, Y_1) = x(X_2, Y_2) = x^*$. We consider two cases:

**Case 1:** $X_1 \leq X_2, Y_1 \leq Y_2$. Consider the set

$$
\Gamma_{x^*} := \left\{ (X,Y) ; x(X,Y) \leq x^* \right\}
$$

and call $\partial \Gamma_{x^*}$ its boundary. By (6.1), $x$ is increasing with $X$ and decreasing with $Y$. Hence, this boundary can be represented as the graph of a Lipschitz continuous function: $X - Y = \phi(X + Y)$. We now construct the Lipschitz continuous curve $\gamma$ consisting of

- a horizontal segment joining $(X_1, Y_1)$ with a point $A = (X_A, Y_A)$ on $\partial \Gamma_{x^*}$, with $Y_A = Y_1$,
- a portion of the boundary $\partial \Gamma_{x^*}$,
- a vertical segment joining $(X_2, Y_2)$ to a point $B = (X_B, Y_B)$ on $\partial \Gamma_{x^*}$, with $X_B = X_2$.

Observe that the map $(X,Y) \mapsto (t,x)$ is constant along $\gamma$. By (4.6) this implies $h_1 = 0$ on the horizontal segment, $h_2 = 0$ on the vertical segment, and $h_1 = h_2 = 0$ on the portion of the boundary $\partial \Gamma_{x^*}$. When either $h_1 = 0$ or $h_2 = 0$ or both, we have from the conserved quantities (4.22) that $\ell = 0$ or $\bar{m} = 0$ or both, correspondingly. Upon examining the derivatives of $\mathbf{n}$ in
we have the same pattern of vanishing property. Thus, along the path from $A$ to $B$, the values of the components of $n$ remain constant, proving our claim.

Case 2: $X_1 \leq X_2$, $Y_1 \geq Y_2$. In this case, we consider the set

$$\Gamma_t := \{(X,Y); t(X,Y) \leq t^*\},$$

and construct a curve $\gamma$ connecting $(X_1,Y_1)$ with $(X_2,Y_2)$ similarly as in case 1. Details are entirely similar to Case 1.

We now prove that the function $n(t,x) = n(X(t,x),Y(t,x))$ thus obtained are Hölder continuous on bounded sets. Toward this goal, consider any characteristic curve, say $t \mapsto x^+(t)$, with $dx^+/dt = c(n_1)$. By construction, this is parametrized by the function $X \mapsto (t(X,Y),x(X,Y))$, for some fixed $Y$. Using the chain rule and the inverse mapping formulas

$$\n_t + cn_x = n_X(X_t + cX_t) + n_Y(Y_t + cY_t) = 2cX_t n_X.$$

Thus we have

$$\int_0^\tau |n_t + cn_x|^2 \, dt = \int_{X_0}^{X_t} (2cX_t|n_X|^2)(2X_t)^{-1} \, dX$$

(6.3)

$$= \int_{X_0}^{X_t} (2c \frac{1}{ph_1} \frac{|\ell|}{2c})^2 \frac{ph_1}{2c} \, dX$$

$$= \int_{X_0}^{X_t} \frac{p}{2c} \frac{|\ell|^2}{h_1} \, dX \leq \int_{X_0}^{X_t} \frac{p}{2c} \, dX \leq C_\tau,$$

for some constant $C_\tau$ depending only on $\tau$. Notice we have used $|\ell|^2 \leq h_1$, which follows from (4.21). Similarly, integrating along any backward characteristics $t \mapsto x^-(t)$ we obtain

(6.4)

$$\int_0^\tau |n_t - cn_x|^2 \, dt \leq C_\tau.$$

Since the speed of characteristics is $\pm c(n_1)$, and $c(n_1)$ is uniformly positive and bounded, the bounds (6.3)–(6.4) imply that the function $n = n(t,x)$ is Hölder continuous with exponent 1/2. In turn, this implies that all characteristic curves are $C^1$ with Hölder continuous derivative. In addition, from (6.3)–(6.4) it follows that $\tilde{R}, \tilde{S}$ at (4.1) are square integrable on bounded subsets of the $t$-$x$ plane. However, we should check the consistency that $\tilde{R}, \tilde{S}$ at (4.1) are indeed the same as recovered from (4.8). Let us check only one of them, $\tilde{R} = \frac{\ell}{h_1}$. We find

$$n_t + cn_x = 2cX_t n_X = 2c \frac{1}{ph_1} \frac{p\ell}{2c} = \frac{\ell}{h_1}.$$

Finally, we prove that $n$ satisfies the equations of system (1.3) in distributional sense, according to (iii) of Definition 1.1. We note that

$$\iint [\phi_t n_t - \phi_x c^2 n_x] + 2\mu \phi n \, dxdt$$

$$= \iint [\phi_t [(n_t + cn_x) + (n_t - cn_x)]$$

$$- c\phi_x [(n_t + cn_x) - (n_t - cn_x)] + 2\mu \phi n \, dxdt$$

(6.5)

$$= \iint [\phi_t - c\phi_x] (n_t + cn_x)$$

$$+ \iint [\phi_t + c\phi_x] (n_t - cn_x) + 2\mu \phi n \, dxdt$$

$$= \iint [\phi_t - c\phi_x] \tilde{R} \, dxdt + \iint [\phi_t + c\phi_x] \tilde{S} \, dxdt + \iint 2\mu \phi n \, dxdt.$$
By (4.6), this is equal to
\[
\int \int \left[ -2cY_x \phi_Y \vec{R} + 2cX_x \phi_X \vec{S} + 2\mu c(X_x \phi_X - Y_x \phi_Y) \mathbf{n} \right] dx dt.
\]

Using the Jacobian
\[
\frac{\partial (x,t)}{\partial (X,Y)} = \frac{pqh_1h_2}{2c}
\]
derived by (6.1), and the inverse (6.2), it is equal to
\[
\int \int \left[ \frac{2c}{qh_2} \vec{R} \phi_Y + \frac{2c}{ph_1} \vec{S} \phi_X + \frac{2\mu c}{qh_2} \phi_X + \frac{2\mu c}{ph_1} \phi_Y \right] \frac{pqh_1h_2}{2c} dX dY
\]
\[
= \int \int \left[ ph_1 \vec{R} \phi_Y + qh_2 \vec{S} \phi_X + \mu qh_2 \mathbf{n} \phi_X + \mu ph_1 \mathbf{n} \phi_Y \right] dX dY
\]
\[
= \int \int \left[ p\vec{R} \phi_Y + q\vec{S} \phi_X + \mu qh_2 \mathbf{n} \phi_X + \mu ph_1 \mathbf{n} \phi_Y \right] dX dY
\]
\[
= \int \int \left[ - (p\vec{R})_Y - (q\vec{S})_X - \mu (qh_2 \mathbf{n})_X - \mu (ph_1 \mathbf{n})_Y \right] \phi dX dY.
\]

Thanks to (4.1), (4.8) and (4.20), we have that the first component of the integrand equals to
\[
\phi \left[ -(p\vec{R})_Y - (q m)_X - \mu (qh_2 \mathbf{n})_X - \mu (ph_1 \mathbf{n})_Y \right]
\]
\[
= -\phi \frac{pq}{4c^3} \left[ (c^2 - \gamma)(h_1 + h_2 - 2h_1h_2) - 2(3c^2 - \gamma)\vec{\ell} \cdot \vec{m} \right] n_1
\]
\[
= -\phi \frac{pq h_1h_2}{2c} \left[ c^2 - \gamma \left( \frac{1}{h_2} + \frac{1}{h_1} + 2 \right) - \frac{3c^2 - \gamma}{h_1 h_2} \right] n_1
\]
\[
= -\phi \frac{pq h_1h_2}{2c} \left[ c^2 - \gamma (\vec{R}^2 + |\vec{S}|^2) - \frac{3c^2 - \gamma}{c^2} \vec{R} \cdot \vec{S} \right] n_1
\]
\[
= -\phi \frac{pq h_1h_2}{2c} \left[ c^2 - \gamma (|\mathbf{n}|^2 + c^2 |\mathbf{n}_x|^2) - \frac{3c^2 - \gamma}{c^2} (|\mathbf{n}|^2 - c^2 |\mathbf{n}_x|^2) \right] n_1.
\]
\[
= -2\phi \frac{pq h_1h_2}{2c} (|\mathbf{n}|^2 + (2c^2 - \gamma) |\mathbf{n}_x|^2) n_1,
\]
which implies that the first equation in (1.3) holds in integral form. Similarly, \( \mathbf{n} \) satisfies the second and third equations of system (1.3) in distributional sense.

7. Upper bound on energy

We convert the energy equation (2.9) formally to the \((X,Y)\)-plane to look for the upper bound of energy.

Recall that the energy equation (2.9) can be written as (4.3) in terms of \(|\vec{R}|\) and \(|\vec{S}|\). By the variables (4.8), we rewrite the equation (4.3) as
\[
\left( \frac{1}{4h_1} + \frac{1}{4h_2} - \frac{1}{2} \right) t - \left[ \frac{c}{4} \left( \frac{1}{h_1} - \frac{1}{h_2} \right) \right] x \leq 0.
\]

We can write the 1-form
\[
\left( \frac{1}{4h_1} + \frac{1}{4h_2} - \frac{1}{2} \right) dx + \left[ \frac{c}{4} \left( \frac{1}{h_1} - \frac{1}{h_2} \right) \right] dt
\]
as
\begin{equation}
\frac{p(1-h_1)}{4}dX - \frac{q(1-h_2)}{4}dY,
\end{equation}
by the formula
\begin{align}
dt &= t_X dX + t_Y dY = \frac{pb_1}{2c}dX + \frac{qb_2}{2c}dY \\
dx &= x_X dX + x_Y dY = \frac{pb_1}{2}dX - \frac{qb_2}{2}dY.
\end{align}
By (4.20) then by (4.21), we have
\begin{align}
\frac{1}{4}[(p(1-h_1))_Y + (q(1-h_2))_X] &= -\frac{pq\mu}{2c}(h_1 + h_2 - 2h_1h_2 + 2\vec{\ell} \cdot \vec{m}) \\
&= -\frac{pq\mu}{2c}(|\vec{\ell} + \vec{m}|^2 + (h_1 - h_2)^2) \\
&\leq 0.
\end{align}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{energy_conserv.pdf}
\caption{Energy conservation}
\end{figure}

We use (7.3) to prove that the
\begin{equation}
\mathcal{E}(t) \leq \mathcal{E}_0 \quad \text{for any } \quad t > 0.
\end{equation}
Fix \( \tau > 0 \), and \( r >> 1 \). Define the set
\begin{equation}
\Omega_r := \left\{(X,Y) \mid 0 \leq t(X,Y) \leq \tau, \quad X \leq r, \quad Y \leq r\right\}.
\end{equation}
See Figure 2, where segment \( A_\tau A_0 \) is where \( Y = r \) while segment \( B_\tau B_0 \) is where \( X = r \). By construction, the map \((X,Y) \mapsto (t,x)\) will act as follows:
\begin{align*}
A_\tau \mapsto (\tau,a), \quad B_\tau \mapsto (\tau,b), \quad B_0 \mapsto (0,c), \quad A_0 \mapsto (0,d),
\end{align*}
for some \( a < b \) and \( d < c \). By the divergence theorem and (7.2), the integral of the form (7.3) along the curve \( A_\tau \to B_\tau \to B_0 \to A_0 \) is less than or equal to zero. Integrating the 1-form
along the boundary of $\Omega^+_\tau$ we obtain

\[
\int_{A_r B_r} \frac{p(1-h_1)}{4} dX - \frac{q(1-h_2)}{4} dY \\
\leq \int_{A_0 B_0} \frac{p(1-h_1)}{4} dX - \frac{q(1-h_2)}{4} dY - \int_{A_0 A_r} \frac{p(1-h_1)}{4} dX - \int_{B_0 B_r} \frac{q(1-h_2)}{4} dY
\]

(7.8)

We claim that

\[
\int_{a}^{b} \frac{1}{2} \left[ |u|^{2}(\tau, x) + c^2(n_1(\tau, x)) |u_x|^{2}(\tau, x) \right] dx
\]

(7.9)

On the other hand, we use (7.4) to compute

\[
\int_{a}^{b} \frac{1}{2} \left[ |u|^{2}(\tau, x) + c^2(n_1(\tau, x)) |u_x|^{2}(\tau, x) \right] dx
\]

Letting $r \to +\infty$ in (7.7), one has $a \to -\infty$, $b \to +\infty$. Therefore (7.8) and (7.9) together imply (7.6).

8. Regularity of trajectories

8.1. Lipschitz continuity. In this part, we first prove the Lipschitz continuity of the map $t \mapsto (n_1, n_2, n_3)(t, \cdot)$ in the $L^2$ distance, stated in (1.12). For any $h > 0$, we have

\[
n(t + h, x) - n(t, x) = h \int_{0}^{1} n_t(t + \tau h, x) d\tau.
\]

Thus

(8.1) \[\|n_i(t + h, x) - n_i(t, x)\|_{L^2} \leq h \int_{0}^{1} \|n_{it}(t + \tau h, \cdot)\|_{L^2} d\tau \leq h \sqrt{\mathcal{E}_0}, \quad i = 1 \sim 3.\]

8.2. Continuity of derivatives. We prove the continuity of functions $t \mapsto (n_{1t}, n_{2t}, n_{3t})(t, \cdot)$ and $t \mapsto (n_{1x}, n_{1y}, n_{1z})(t, \cdot)$, as functions with values in $L^p$, $1 \leq p < 2$. (To keep tradition, we use the exponent $p$ here at the expense of repeating one of our primary variables.) This will complete the proof of Theorem 1.2.

We first consider the case where the initial data $(n_0)_x$ and $n_1$ are smooth with compact support. In this case, the solution $n = n(X, Y)$ remains smooth in $\Omega^+$. Fix a time $\tau > 0$. We claim that

(8.2) \[\frac{d}{dt} n(t, \cdot) \bigg|_{t=\tau} = n_t(\tau, \cdot)\]

where

(8.3) \[n_t(\tau, x) := n_X X_t + n_Y Y_t = \frac{\tilde{\ell}}{2c} \frac{c}{p h_1} + \frac{\tilde{m}}{2c} \frac{c}{q h_2} = \frac{\tilde{\ell}}{2h_1} + \frac{\tilde{m}}{2h_2}.\]

Notice that (8.3) defines the values of $n_t(\tau, \cdot)$ at almost every point $x \in \mathbb{R}$. By the inequality (7.6), we obtain

(8.4) \[\int_{\mathbb{R}} |n_t(\tau, x)|^2 dx \leq 2 \mathcal{E}(\tau) \leq 2 \mathcal{E}_0.\]
To prove (8.2), we consider the set
\[
\Gamma_\tau := \{(X,Y) \mid t(X,Y) \leq \tau\},
\]
and let $\gamma_\tau$ be its boundary. Let $\varepsilon > 0$ be given. There exist finitely many disjoint intervals $[a_i, b_i] \subset \mathbb{R}$, $i = 1, \ldots, N$, with the following property. Call $\mathcal{A}_i, \mathcal{B}_i$ the points on $\gamma_\tau$ such that $x(\mathcal{A}_i) = a_i$, $x(\mathcal{B}_i) = b_i$. Then one has
\[
\min \{h_1(P), h_2(P)\} < 2\varepsilon
\]
at every point $P$ on $\gamma_\tau$ contained in one of the arcs $\mathcal{A}_i \mathcal{B}_i$, while
\[
h_1(P) > \varepsilon, \quad h_2(P) > \varepsilon,
\]
for every point $P$ along $\gamma_\tau$, not contained in any of the arcs $\mathcal{A}_i \mathcal{B}_i$. Call $J := \bigcup_{1 \leq i \leq N} [a_i, b_i]$, $J' = \mathbb{R} \setminus J$, and notice that, as a function of the original variables, $n = n(t, x)$ is smooth in a neighborhood of the set $\{\tau\} \times J'$. Using Minkowski's inequality and the differentiability of $n$ on $J'$, we can write, for $i = 1 \sim 3$,
\[
\lim_{h \to 0} \frac{1}{h} \left( \int_{\mathcal{A}_i \mathcal{B}_i} \left| n_i(\tau + h, x) - n_i(\tau, x) - h n_{ix}(\tau, x) \right|^p dx \right)^{1/p}
\leq \lim_{h \to 0} \frac{1}{h} \left( \int_{J} \left| n_i(\tau + h, x) - n_i(\tau, x) \right|^p dx \right)^{1/p} + \left( \int_{J} |n_{ix}(\tau, x)|^p dx \right)^{1/p}.
\]
We now provide an estimate on the measure of the “bad” set $J$:
\[
\operatorname{meas} (J) = \int_J dx = \sum_i \int_{\mathcal{A}_i \mathcal{B}_i} \frac{p h_i}{2} dX - \frac{q h_2}{4} dY
\leq \frac{2\varepsilon}{1 - 2\varepsilon} \sum_i \int_{\mathcal{A}_i \mathcal{B}_i} \frac{p(1-h_1)}{2} dX - \frac{q(1-h_2)}{4} dY
\leq \frac{4\varepsilon}{1 - 2\varepsilon} \int_{\gamma_\tau} \frac{p(1-h_1)}{4} dX - \frac{q(1-h_2)}{4} dY \leq \frac{4\varepsilon}{1 - 2\varepsilon} \mathcal{E}_0.
\]
Notice that $dt = 0$ on $\gamma_\tau$, so the two parts of the integral are actually equal. Now choose $q = 2/(2 - p)$ so that $\frac{p}{2} + \frac{1}{q} = 1$. Using Hölder’s inequality with conjugate exponents $2/p$ and $q$, and recalling (8.1), we obtain for any $i = 1 \sim 3$
\[
\int_J \left| n_i(\tau + h, x) - n_i(\tau, x) \right|^p dx
\leq \operatorname{meas} (J)^{1/q} \cdot \left( \int_J \left| n_i(\tau + h, x) - n_i(\tau, x) \right|^2 dx \right)^{p/2}
\leq \operatorname{meas} (J)^{1/q} \cdot \left( \left\| n_i(\tau + h, \cdot) - n_i(\tau, \cdot) \right\|_{L^2}^2 \right)^{p/2}
\leq \operatorname{meas} (J)^{1/q} \cdot \left( h^2 \left[ 2\mathcal{E}_0 \right] \right)^{p/2}.
\]
Therefore,
\[
\limsup_{h \to 0} \frac{1}{h} \left( \int_J \left| n_i(\tau + h, x) - n_i(\tau, x) \right|^p dx \right)^{1/p}
\leq \left[ \frac{4\varepsilon}{1 - 2\varepsilon} \mathcal{E}_0 \right]^{1/p} \cdot \left[ 2\mathcal{E}_0 \right]^{1/2}.
\]
In a similar way we estimate

\[ \int_J |n_{it}(\tau, x)|^p \, dx \leq \left[ \text{meas}(J) \right]^{1/q} \cdot \left( \int_J |n_{it}(\tau, x)|^{2} \, dx \right)^{p/2}, \]

(8.11)

\[ \left( \int_J |n_{it}(\tau, x)|^p \, dx \right)^{1/p} \leq \text{meas}(J)^{1/pq} \cdot [2\mathcal{E}_0]^{p/2}. \]

Since \( \varepsilon > 0 \) is arbitrary, from (8.8), (8.10) and (8.11) we conclude

\[ \lim_{h \to 0} \frac{1}{h} \left( \int_R |n_i(\tau + h, x) - n_i(\tau, x) - h n_{it}(\tau, \cdot, x)|^p \, dx \right)^{1/p} = 0. \]

(8.12)

The proof of continuity of the map \( t \mapsto n_{it} \) is similar. Fix \( \varepsilon > 0 \). Consider the intervals \([a_i, b_i]\) as before. Since \( n \) is smooth on a neighborhood of \( \{\tau\} \times J' \), it suffices to estimate

\[ \limsup_{h \to 0} \int |n_{it}(\tau + h, x) - n_i(\tau, x)|^p \, dx \]

\[ \leq \limsup_{h \to 0} \int |n_{it}(\tau + h, x) - n_{it}(\tau, x)|^p \, dx \]

\[ \leq \limsup_{h \to 0} \left[ \text{meas}(J) \right]^{1/q} \cdot \left( \int_J |n_{it}(\tau + h, x) - n_{it}(\tau, x)|^2 \, dx \right)^{p/2} \]

\[ \leq \limsup_{h \to 0} \left[ \frac{4\varepsilon}{1 - 2^q} \mathcal{E}_0 \right]^{1/q} \cdot \left( \|n_{it}(\tau + h, \cdot)\|_{L^2} + \|n_{it}(\tau, \cdot)\|_{L^2} \right)^{p} \]

\[ \leq \left[ \frac{4\varepsilon}{1 - 2^q} \mathcal{E}_0 \right]^{1/q} \cdot [4\mathcal{E}_0]^p. \]

Since \( \varepsilon > 0 \) is arbitrary, this proves continuity.

To extend the result to general initial data, such that \((n_0)_x, \ n_1 = n_{i|t=0} \in L^2\), we use Corollary 5.1 and consider a sequence of smooth initial data, with \((n_{0}^\nu)_x, n_{1}^\nu \in C_c^\infty\), with \( n_{0}^\nu \to n_0 \) uniformly, \((n_{1}^\nu)_x \to (n_0)_x\) almost everywhere and in \( L^2 \), \( n_{1}^\nu \to n_1 \) almost everywhere and in \( L^2 \).

The continuity of the function \( t \mapsto n_x(t, \cdot) \) as maps with values in \( L^p \), \( 1 \leq p < 2 \), is proved in an entirely similar way.

Acknowledgments. Yuxi Zheng is partially supported by NSF DMS 0908207.

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