On strong solutions to the Cauchy problem of the two-dimensional compressible magnetohydrodynamic equations with vacuum

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Abstract

Two-dimensional barotropic compressible magnetohydrodynamic equations with shear and bulk viscosities being a positive constant and a power function of the density, respectively, are considered. We prove that the Cauchy problem on the whole two-dimensional space with vacuum as the far field density admits a unique local strong solution provided the initial density and magnetic field do not decay very slowly at infinity. In particular, the initial density can have a compact support.

Keywords: compressible magnetohydrodynamic equations, two-dimensional space, vacuum, strong solutions, Cauchy problem

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1. Introduction and main results

We consider the two-dimensional compressible magnetohydrodynamic (MHD) equations which read as follows:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \mu \Delta u + \nabla((\mu + \lambda) \text{div} u) + H \cdot \nabla H - \frac{1}{2} \nabla |H|^2, \\
H_t + u \cdot \nabla H - H \cdot \nabla u + H \text{div} u - \nu \Delta H &= 0, \\
\text{div} H &= 0,
\end{align*}
\]

(1.1)

where \( t \geq 0 \) is the time, \( x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2 \) is the spatial coordinate, \( \rho = \rho(x, t), \ u = (u^1, u^2)(x, t) \) and \( H = (H^1, H^2)(x, t) \) represent, respectively, the fluid density, velocity and magnetic field, and pressure \( P \) is given by

\[ P(\rho) = A\rho^\gamma \quad (A > 0, \gamma > 1). \]

(1.2)

The shear viscosity \( \mu \) and the bulk one \( \lambda \) satisfy the following hypothesis:

\[ 0 < \mu = \text{const}, \quad \lambda(\rho) = b\rho^\beta, \]

(1.3)

where the constants \( b \) and \( \beta \geq 0 \) satisfy

\[
\begin{cases}
  b > 0, & \text{if } \beta > 0, \\
  \mu + b \geq 0, & \text{if } \beta = 0.
\end{cases}
\]

(1.4)

The constant \( \nu > 0 \) is the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields. In what follows, without loss of generality, we set \( \Omega = \mathbb{R}^2 \) and we consider the Cauchy problem with \( (\rho, u, H) \) vanishing at infinity (in some weak sense). For given initial data \( \rho_0, m_0 \) and \( H_0 \), we require that

\[ \rho(x, 0) = \rho_0(x), \quad \rho u(x, 0) = m_0(x), \quad H(x, 0) = H_0(x), \quad x \in \Omega = \mathbb{R}^2. \]

(1.5)

There has been a huge amount of literature on the study of the compressible MHD problem (1.1) by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges; see, e.g., [5, 8, 9, 12, 13, 17, 18, 29, 30] and references therein. In particular, if there is no electromagnetic effect, i.e. \( H = 0 \), then (1.1) reduces to the compressible Navier–Stokes equations for barotropic flows, which have also been discussed in numerous studies; see, e.g., [2–4, 6, 7, 10, 11, 14–16, 19–25, 27, 28, 31] and references therein. Now, we briefly recall some results concerned with the multi-dimensional compressible MHD equations with constant viscosities. The local existence of solutions to the compressible MHD equations was obtained by Vol’pert–Khudiaev [30], Kawashima [17] and Strohmer [29] when the initial density was strictly positive, and by Fan–Yu [9] in \( \mathbb{R}^3 \) with the initial density probably containing vacuum. The global existence of solutions to the compressible MHD equations was obtained in many works: Kawashima [17] firstly obtained the global existence with non-vacuum; Hu–Wang [12, 13] and Fan–Yu [8] proved the global existence of renormalized solutions for large initial data; Li–Xu–Zhang [18] established the global existence of classical solutions in three-dimensional space with large oscillations and vacuum.

However, for the Cauchy problem (1.1)–(1.5) with \( \Omega = \mathbb{R}^2 \), it is still open even for the local existence of strong solutions when the far field density is vacuum; in particular, the initial density may have a compact support. Recently, Li–Liang [19] have obtained the local existence of the strong solutions to the two-dimensional Cauchy problem (1.1)–(1.5) with \( H \equiv 0 \). The aim of this paper is to prove the local existence and uniqueness of strong solutions to the Cauchy problem (1.1)–(1.5) whose definition is as follows:
Definition 1.1. If all derivatives involved in (1.1) for \((\rho, u, H)\) are regular distributions, and equations (1.1) hold almost everywhere in \(\mathbb{R}^2 \times (0, T)\), then \((\rho, u, H)\) is called a strong solution to (1.1).

In this section, for \(1 \leq r \leq \infty\), we denote the standard Lebesgue and Sobolev spaces as follows:

\[
L^r = L^r(\mathbb{R}^2), \quad W^{s,r} = W^{s,r}(\mathbb{R}^2), \quad H^s = H^s(\mathbb{R}^2).
\]

Theorem 1.1. Let \(\eta_0\) be a positive constant and

\[
x_0 \triangleq (e + |x|^2)^{1/2} \log^{1+\eta_0}(e + |x|^2).
\]  \hfill (1.6)

For constants \(q > 2\) and \(a > 1\), assume that the initial data \((\rho_0, m_0, H_0)\) satisfy that

\[
\rho_0 \geq 0, \quad x_0^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \rho_0^{1/2} u_0 \in L^2, \quad x_0^{a/2} H_0 \in L^2, \quad \nabla H_0 \in L^2,
\]  \hfill (1.7)

and that

\[
m_0 = \rho_0 u_0.
\]  \hfill (1.8)

In addition, if \(\beta \in (0, 1)\), suppose that

\[
\lambda(\rho_0) \in L^2, \quad x_0^a \nabla \lambda(\rho_0) \in L^2 \cap L^q,
\]  \hfill (1.9)

for some \(\theta_0 \in (0, \beta)\). Then there exists a positive time \(T_0 > 0\) such that the problem (1.1)–(1.5) has a unique strong solution \((\rho, u, H)\) on \(\mathbb{R}^2 \times (0, T_0]\) satisfying that

\[
\begin{align*}
\rho &\in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), \\
x_0^a \rho &\in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\mu_0}, \nabla u, \sqrt{T} \nabla u, \sqrt{T} \sqrt{\mu_0}, &\in L^\infty(0, T_0; L^2), \\
H, H^2, H x_0^{a/2}, \nabla H, \sqrt{T} H_0, &\in L^\infty(0, T_0; L^2), \\
\nabla u &\in L^2(0, T_0; H^1) \cap L^{(q+1)/2}(0, T_0; W^{1,q}), \\
\nabla H &\in L^2(0, T_0; H^1), \\
\sqrt{T} \nabla u &\in L^2(0, T_0; W^{1,q}), \\
\sqrt{T} \nabla u, \sqrt{T} \nabla H, &\in L^2(\mathbb{R}^2 \times (0, T_0)),
\end{align*}
\]  \hfill (1.10)

and that

\[
\inf_{0 \leq t \leq T_0} \int_{B_4} \rho(x, t) \, dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) \, dx,
\]  \hfill (1.11)

for some constant \(N > 0\) and \(B_N \triangleq \{x \in \mathbb{R}^2 \mid |x| < N\}\).

Remark 1.1. When \(H = 0\), i.e. there is no electromagnetic field effect, (1.1) reduces to the compressible Navier–Stokes equations, and theorem 1.1 is the same as the results of Li–Liang [19]. Roughly speaking, we generalize the results of [19] to the compressible MHD equations.

Remark 1.2. It should be noted here that the only compatibility condition to obtain the local existence and uniqueness of strong solutions in theorem 1.1 is (1.8) which is much weaker than those of [9] where

\[
-\mu \Delta u_0 - \nabla ((\mu + \lambda(\rho_0)) \text{div} u_0) + \nabla P(\rho_0) - H_0 \cdot \nabla H_0 + \frac{1}{2} \nabla |H_0|^2 = \rho_0^{1/2} g
\]

for some \(g \in L^2(\mathbb{R}^2)\) is needed.
Remark 1.3. Similar to [19], if the initial data \((\rho_0, u_0, H_0)\) satisfy some additional regularity and compatibility conditions, the global strong solutions obtained by theorem 1.1 become classical ones.

We now comment on the analysis of this paper. For the two-dimensional case, it seems difficult to bound the \(L^p(\mathbb{R}^2)\)-norm of \(u\) just in terms of \(\|\rho^{1/2}u\|_{L^2(\mathbb{R}^2)}\) and \(\|\nabla u\|_{L^2(\mathbb{R}^2)}\), i.e. the methods used in the three-dimensional case [2–4, 9] cannot be applied directly to our case. Compared with the two-dimensional compressible Navier–Stokes equations [19], for the compressible MHD equations, the strong coupling between the velocity vector field and the magnetic field, such as \(u \cdot \nabla H\) and \(H \cdot \nabla H\), will bring us some new difficulties. In fact, the solutions to overcome these difficulties are as follows: on the one hand, in order to control the terms \(\|u \cdot H\|\) and \(\|u \cdot H\|\), after integration by parts, we obtain a spatial weighted mean estimate of \(H\) and \(|\nabla H|\), which are deduced from the coupled term \(\rho^12u \cdot \nabla H\), after integration by parts, we obtain a spatial weighted mean estimate of \(H\) and \(|\nabla H|\) (i.e. \(\frac{\lambda}{\rho^{1/2}}H\) and \(\frac{\lambda}{\rho^{1/2}}\nabla H\), see (3.27) and (3.64)) and then get the estimates of \(L^2\)-norm of \(H\) and \(t\|\nabla H\|_2^2\); on the other hand, to obtain the estimate of \(\|\nabla u\|_{L^\infty}\), it seems difficult to get the usual \(L^\infty\)-norm (in time) of \(t\|\nabla H\|_2^2\); due to the coupled term \(H \cdot \nabla H\). This is overcome by bounding the \(L^\infty\)-norm of \(t\|H\|_2\|\nabla H\|_2\) instead of \(t\|\nabla H\|_2^2\), (see (3.35)). Next, in order to overcome the difficulties caused by the fact that the bulk viscosity \(\lambda\) depends on \(\rho\), we consider the approximate problems (2.2) in bounded ball \(B_R\) and impose the Navier-slip boundary conditions instead of the usual Dirichlet boundary ones. However, when we extend the approximate solutions by 0 outside the ball, it seems difficult to bound the \(L^2(\mathbb{R}^2)\)-norm of the gradient of the velocity. Motivated by Li–Liang [19], we put an additional term \(-R^{-1}u\) on the right-hand side of (1.1) (see (2.2) for details). Finally, combining all these ideas stated above with those due to [9, 19], we derive some desired bounds which are independent of both the radius of the balls \(B_R\) and the lower bound of the initial density.

The rest of the paper is organized as follows: in section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to the \textit{a priori} estimates which are needed to obtain the local existence and uniqueness of strong solutions. Then finally, the main results, theorem 1.1 is proved in section 4.

2. Preliminaries

First, the following local existence theory on bounded balls, where the initial density is strictly away from vacuum, can be shown by similar arguments as in [9, 17].

Lemma 2.1. For \(R > 0\) and \(B_R = \{x \in \mathbb{R}^2||x| < R\}\), assume that \((\rho_0, u_0, H_0)\) satisfies

\[
(\rho_0, u_0, H_0) \in H^3(B_R), \quad \inf_{x \in B_R} \rho_0(x) > 0,
\]

\[
u_0 \cdot n = 0, \quad \text{rot} u_0 = 0, \quad H_0 = 0, \quad \text{div} H_0 = 0, \quad x \in \partial B_R.
\]

(2.1)

Then there exist a small time \(T_R > 0\) and a unique classical solution \((\rho, u, H)\) to the following initial-boundary-value problem

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P - \mu \Delta u - \nabla ((\mu + \lambda)\text{div} u) &= H \cdot \nabla H - \frac{1}{2} \nabla \|H\|^2 - R^{-1}u, \\
H_t + u \cdot \nabla H - H \cdot \nabla u + H \text{div} u - \nu \Delta H &= 0, \quad \text{div} H = 0 \\
|\rho u| \cdot n &= 0, \quad \text{rot} u = 0, \quad H = 0, \quad x \in \partial B_R, \quad t > 0,
\end{align*}
\]

(2.2)

\[
(\rho, u, H)(x, 0) = (\rho_0, u_0, H_0)(x), \quad x \in B_R.
\]
on $B_R \times (0, T_R)$ such that
\[
\begin{align*}
\rho & \in C\left([0, T_R]; H^3\right), \quad \rho_t \in C\left([0, T_R]; H^2\right),
(a, H) & \in C\left([0, T_R]; H^3\right) \cap L^2\left(0, T_R; H^4\right),
(u_t, H_t) & \in L^\infty\left(0, T_R; H^1\right) \cap L^2\left(0, T_R; H^2\right),
\end{align*}
\tag{2.3}
\]
where we denote $H^k = H^k(B_R)$ for a positive integer $k$.

Next, for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$, the following weighted $L^p$-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) \triangleq \{v \in H^1_{\text{loc}}(\Omega) | \nabla v \in L^2(\Omega)\}$ can be found in [23, theorem B.1].

**Lemma 2.2.** For $m \in [2, \infty)$ and $\theta \in (1 + m/2, \infty)$, there exists a positive constant $C$ such that for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$ and for any $v \in \tilde{D}^{1,2}(\Omega)$,
\[
\left(\int_{\Omega} \left[\frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^\gamma \right]^d x\right)^{1/m} \leq C \|v\|_{L^2(B_1)} + C \|\nabla v\|_{L^2(\Omega)},
\tag{2.4}
\]
A useful consequence of lemma 2.2 is the following weighted bounds for elements of $\tilde{D}^{1,2}(\Omega)$ which in fact will play a crucial role in our analysis.

**Lemma 2.3.** Let $\tilde{x}$ and $\eta_0$ be as in (1.6) and $\Omega$ as in lemma 2.2. For $\gamma > 1$, assume that $\rho \in L^1(\Omega) \cap L^\infty(\Omega)$ is a non-negative function such that
\[
\int_{B_{\gamma}} \rho \, dx \geq M_1, \quad \int_{\Omega} \rho^\gamma \, dx \leq M_2,
\tag{2.5}
\]
for positive constants $M_1, M_2,$ and $N_1 \geq 1$ with $B_{N_1} \subset \Omega$. Then there is a positive constant $C$ depending only on $M_1, M_2, N_1, \gamma,$ and $\eta_0$ such that
\[
\|v \tilde{x}^{-\gamma} \|_{L^2(\Omega)} \leq C \|\rho^{1/2}v\|_{L^2(\Omega)} + C \|\nabla v\|_{L^2(\Omega)},
\tag{2.6}
\]
for $v \in \tilde{D}^{1,2}(\Omega)$. Moreover, for $\varepsilon > 0$ and $\eta > 0$, there is a positive constant $C$ depending only on $\varepsilon, \eta, M_1, M_2, N_1, \gamma,$ and $\eta_0$ such that every $v \in \tilde{D}^{1,2}(\Omega)$ satisfies
\[
\|v \tilde{x}^{-\gamma} \|_{L^{2m/(m+\varepsilon)}(\Omega)} \leq C \|\rho^{1/2}v\|_{L^2(\Omega)} + C \|\nabla v\|_{L^2(\Omega)},
\tag{2.7}
\]
with $\tilde{\eta} = \min\{1, \eta\}$.

**Proof.** It follows from (2.5) and the Poincaré-type inequality [6, lemma 3.2] that there exists a positive constant $C$ depending only on $M_1, M_2, N_1$ and $\gamma$, such that
\[
\|v\|^2_{L^2(B_{N_1})} \leq C \int_{B_{N_1}} \rho v^2 \, dx + C \|\nabla v\|^2_{L^2(B_{N_1})},
\]
which together with (2.4) gives (2.6) and (2.7). The proof of lemma 2.3 is completed.

Next, the following $L^p$-bound for elliptic systems, whose proof is similar to that of [2, lemma 12], is a direct consequence of the combination of a well-known elliptic theory due to Agmon–Douglis–Nirenberg [1] with a standard scaling procedure.

**Lemma 2.4.** For $p > 1$ and $k \geq 0$, there exists a positive constant $C$ depending only on $p$ and $k$ such that
\[
\|\nabla^{k+2} v\|_{L^p(B_R)} \leq C \|\Delta v\|_{W^{k,p}(B_R)},
\tag{2.8}
\]
for every $v \in W^{k+2,p}(B_R)$ satisfying either
\[
v \cdot n = 0, \quad \text{rot} v = 0, \quad \text{on} \, \partial B_R,
\]
or
\[
v = 0, \quad \text{on} \, \partial B_R.
\]
3. A priori estimates

Throughout this section, for \( p \in [1, \infty] \) and \( k \geq 0 \), we denote
\[
\int f \, dx = \int_{B_R} f \, dx, \quad L^p = L^p(B_R), \quad W^{k,p} = W^{k,p}(B_R), \quad H^L = W^{k,2},
\]
and, without loss of generality, we assume that \( \beta > 0 \) since all the estimates obtained in this section and the next hold for the case where \( \beta = 0 \) after some small modifications. Moreover, for \( R > 4N_0 \geq 4 \), assume that \((\rho_0, u_0, H_0)\) satisfies, in addition to (2.1), that
\[
1/2 \leq \frac{1}{B_{R_0}} \rho_0(x) \, dx \leq \frac{1}{B_{R_0}} \rho_0(x) \, dx \leq 3/2.
\]
(3.1)

Lemma 2.1 thus yields that there exists some \( T_R > 0 \) such that the initial-boundary-value problem (2.2) has a unique classical solution \((\rho, u, H)\) on \( B_R \times [0, T_R] \) satisfying (2.3).

For \( \tilde{x}, \eta_0, a, q \) and \( \theta_0 \) as in theorem 1.1, the main aim of this section is to derive the following key a priori estimate on \( \psi \) defined by
\[
\psi(t) = 1 + \|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|R^{-1/2}u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2 + \|\lambda(\rho)\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2.
\]
(3.2)

Proposition 3.1. Assume that \((\rho_0, u_0, H_0)\) satisfies (2.1) and (3.1). Let \((\rho, u, H)\) be the solution to the initial-boundary-value problem (2.2) on \( B_R \times (0, T_R) \) obtained by lemma 2.1. Then there exist positive constants \( T_0 \) and \( M \) both depending only on \( \mu, \nu, \beta, \gamma, b, q, a, \eta_0, \theta_0, N_0 \) and \( E_0 \) such that
\[
\sup_{0 \leq s \leq T_0} \int_0^{T_0} \left( \|\nabla^2 u\|_{L^q}^{q+1/q} + t\|\nabla^2 u\|_{L^q}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \, dt \leq M,
\]
(3.3)

where
\[
E_0 = \|\rho_0^{1/2}u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \|R^{-1/2}u_0\|_{L^2}^2 + \|\nabla H_0\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2 + \|\lambda(\rho_0)\|_{L^2}^2 + \|\tilde{x}\|_{L^2}^2.
\]

To prove proposition 3.1, we begin with the following standard energy estimate for \((\rho, u, H)\), and preliminary \( L^2 \)-bounds for \( \nabla u \) and \( \nabla H \).

Lemma 3.2. Under the conditions of proposition 3.1, let \((\rho, u, H)\) be a smooth solution to the initial-boundary-value problem (2.2). Then there exist a \( T_1 = T_1(N_0, E_0) > 0 \) and a positive constant \( \alpha = \alpha(\gamma, \beta, q) > 1 \) such that for all \( t \in (0, T_1) \),
\[
\sup_{0 \leq s \leq t} \left( \|\rho u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) + \int_0^t \left( \|\rho \frac{1}{2} u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \right) \, ds \leq C + C \int_0^t \psi^\alpha(s) \, ds,
\]
(3.4)

where (and in what follows) \( C \) denotes a generic positive constant depending only on \( \mu, \nu, \beta, b, q, a, \eta_0, \theta_0, N_0 \) and \( E_0 \).

Proof. First, applying a standard energy estimate to (2.2) gives
\[
\sup_{0 \leq s \leq t} \left( \|\rho\|_{L^2}^2 + \|\rho^{1/2}u\|_{L^2}^2 + \|H\|_{L^2}^2 \right) + \int_0^t \left( R^{-1} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \, ds \leq C.
\]
(3.5)

Next, for \( N > 1 \) and \( \varphi_N \in C_0^\infty(B_N) \) such that
\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = 1, \quad \text{if } |x| \leq N/2, \quad |\nabla \varphi_N| \leq CN^{-k}(k = 1, 2),
\]
(3.6)
it follows from (3.5) and (3.1) that
\[
\frac{d}{dt} \int \rho \varphi_{2N_0} \, dx = \int \rho u \cdot \nabla \varphi_{2N_0} \, dx \\
\geq -CN_0^{-1} \left( \int \rho \, dx \right)^{1/2} \left( \int \rho |u|^2 \, dx \right)^{1/2} \geq -\tilde{C}(E_0),
\]
where in the last inequality we have used
\[
\int \rho \, dx = \int \rho_0 \, dx,
\]
due to (2.2)1 and (2.2)4. Integrating (3.7) gives
\[
\inf_{0 \leq t \leq T_1} \int_{B^{2N_0}} \rho \, dx \geq \inf_{0 \leq t \leq T_1} \int \rho \varphi_{2N_0} \, dx \\
\geq \int \rho_0 \varphi_{2N_0} \, dx - \tilde{C}T_1 \geq 1/4,
\]
where \(T_1 = \min\{1, (4\tilde{C})^{-1}\}\). From now on, we will always assume that \(t \leq T_1\). The combination of (3.8), (3.5) and (2.7) yields that for \(\varepsilon > 0\) and \(\eta > 0\), every \(v \in \tilde{D}_{1,2}(BR)\) satisfies
\[
\|v\|_{L^2}^2 \leq C(\varepsilon, \eta) \int \rho |u|^2 \, dx + C(\varepsilon, \eta) \|\nabla v\|_{L^2}^2,
\]
with \(\tilde{\eta} = \min\{1, \eta\}\). In particular, we have
\[
\|\rho \frac{v}{\eta}\|_{L^{2+\varepsilon}/\tilde{\eta}} + \|u \frac{v}{\tilde{\eta}}\|_{L^{2+\varepsilon}/\tilde{\eta}} \leq C(\varepsilon, \eta) \psi^{1+\eta}.
\]
Next, multiplying equations (2.2)2 by \(u_t\) and integration by parts yield
\[
\frac{d}{dt} \int \left( (2\mu + \lambda)(\text{div} u)^2 + \mu \omega^2 + R^{-1} |u|^2 \right) \, dx + \int \rho |u_t|^2 \, dx \\
\leq C \int \rho |u|^2 |\nabla u|^2 \, dx + \int \lambda_t (\text{div} u)^2 \, dx \\
+ 2 \int P \text{div} u, \, dx + \int \left( H \cdot \nabla H - \frac{1}{2} |\nabla H|^2 \right) \cdot u_t \, dx,
\]
where \(\omega \equiv \text{rot} u\) is defined in the following (3.30).

We estimate each term on the right-hand side of (3.11) as follows:

First, the Gagliardo–Nirenberg inequality (see [26]) implies that for all \(p \in (2, +\infty)\),
\[
\|\nabla u\|_{L^p} \leq C(p) \|\nabla u\|_{L^2}^{2/p} \|\nabla u\|_{H^1}^{1-2/p} \\
\leq C(p) \psi + C(p) \psi \|\nabla^2 u\|_{L^2}^{1-2/p},
\]
which together with (3.10) yields that for \(\eta > 0\) and \(\tilde{\eta} = \min\{1, \eta\}\),
\[
\int \rho^\eta |u|^2 |\nabla u|^2 \, dx \leq C \|\rho^\eta/2 u\|_{L^{2+\varepsilon}/\tilde{\eta}}^2 \|\nabla u\|_{L^{2+\varepsilon}/\tilde{\eta}}^2 \\
\leq C(\varepsilon, \eta) \psi^{4+2\eta} \left( 1 + \|\nabla^2 u\|_{L^2}^{3/2} \right) \\
\leq C(\varepsilon, \eta) \psi^{\alpha(\varepsilon)} + \psi^\varepsilon \|\nabla^2 u\|_{L^2}^2.
\]

Then, noticing that \(\lambda = b \rho^\beta\) satisfies
\[
\lambda_t + \text{div}(\lambda u) + (\beta - 1) \lambda \text{div} u = 0,
\]
where
we deduce from (3.10) and the Sobolev inequality that
\[ \int \lambda \|\text{div}(u)\|^2 \, dx \leq C \int \lambda \|\nabla u\|^2 \, dx + C \int \lambda \|\nabla u\|^3 \, dx \]
\[ \leq C(\varepsilon) \psi + C \varepsilon \psi^{-1} \|\nabla^2 u\|^2_{L^2}, \]
where (and in what follows) we use \( \alpha = \alpha(\beta, \gamma, q) > 1 \) to denote a generic constant depending only on \( \beta, \gamma, \) and \( q \), which may be different from line to line.

Next, since \( P_t + \text{div}(Pu) + (\gamma - 1)P\text{div}u = 0 \),
we deduce from (3.10), (3.12) and the Sobolev inequality that
\[ 2 \int P \text{div}u_t \, dx = 2 \frac{d}{dt} \int P \text{div}u \, dx - 2 \int Pu \cdot \nabla \text{div}u \, dx + 2(\gamma - 1) \int P(\text{div}u)^2 \, dx \]
\[ \leq 2 \frac{d}{dt} \int P \text{div}u \, dx + \varepsilon \psi^{-1} \|\nabla^2 u\|^2_{L^2} + C(\varepsilon) \psi^\alpha. \]

Finally, integration by parts together with (3.19) and \( \text{div}H = 0 \) yields
\[ \int \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot u_t \, dx \]
\[ = - \int H \cdot \nabla u_t \cdot H \, dx + \frac{1}{2} \int |H|^2 \text{div}u \, dx \]
\[ = \frac{d}{dt} \left( \frac{1}{2} \int |H|^2 \, dx - \int H \cdot \nabla u \cdot H \, dx \right) - \int H \cdot H_t \text{div}u \, dx \]
\[ + \int H_t \cdot \nabla u \cdot H \, dx + \int H \cdot \nabla u \cdot H_t \, dx \]
\[ \leq \frac{d}{dt} \left( \frac{1}{2} \int |H|^2 \, dx - \int H \cdot \nabla u \cdot H \, dx \right) + C \int |H_t| \|\nabla u\| \, dx \]
\[ \leq \frac{d}{dt} \left( \frac{1}{2} \int |H|^2 \, dx - \int H \cdot \nabla u \cdot H \, dx \right) + \delta \|H_t\|^2_{L^2} + C(\delta) \|H\|^2_{L^2} \|\nabla u\|^2_{L^2}, \]
\[ \leq \frac{d}{dt} \left( \frac{1}{2} \int |H|^2 \, dx - \int H \cdot \nabla u \cdot H \, dx \right) + \delta \|H_t\|^2_{L^2} + C(\delta) \|H\|^2_{L^2} \|\nabla u\|^2_{L^2} \|\nabla u\|_{H^1}, \]
\[ \leq \frac{d}{dt} \left( \frac{1}{2} \int |H|^2 \, dx - \int H \cdot \nabla u \cdot H \, dx \right) + \delta \|H_t\|^2_{L^2} + C(\delta) \|H\|^2_{L^2} \|\nabla u\|^2_{L^2} + C(\varepsilon, \delta) \psi^\alpha, \]
where in the last inequality the following estimate is used:
\[ \sup_{0 \leq t \leq \tau} \|H\|^2_{L^2} + \int_0^\tau \|\nabla H\|_{L^2}^2 \, dx \leq C. \] 
Indeed, multiplying equations (2.2) by \( H |H|^2 \) and integrating by parts yield
\[ \frac{1}{4} \left( \|H\|^2_{L^2} \right)_t + v \|\nabla H\|_{L^2}^2 + \frac{v}{2} \|\nabla^2 H\|^2 \]
\[ \leq C \int |\nabla u| |H|^4 \, dx \leq C \|\nabla u\|_{L^2} \|H\|^2_{L^2}, \]
\[ \leq C \|\nabla u\|_{L^2} \|H\|^2_{L^2} \|\nabla H\|^2_{L^2}, \]
\[ \leq \frac{v}{4} \|\nabla H\|^2_{L^2} + C \|\nabla u\|_{L^2} \|H\|^2_{L^2}, \]
which together with Gronwall’s inequality and (3.5) yields (3.19).
where
\[ B(t) = 2 \int P \text{div} u \, dx + \frac{1}{2} \int |H|^2 \text{div} u \, dx - \int H \cdot \nabla u \cdot H \, dx \]
\[ \leq C\|H\|_{L^2}^4 + C\|P\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u\|_{L^2}^2 \]
\[ \leq C + C\|P\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u\|_{L^2}^2 \]

owing to (3.19).

Moreover, it follows from (2.2) that
\[ \nu \frac{d}{dt} \|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \nu^2 \|\Delta H\|_{L^2}^2 \]
\[ \leq C\|H\|\|\nabla u\|_{L^2}^2 + C\|u\|\|\nabla H\|_{L^2}^2 \]
\[ \leq C\|H\|\|\nabla u\|_{L^2}^2 + C \int |u|^{2\gamma - 1}\gamma \delta \|\nabla H\|\|\nabla H\| \, dx \]
\[ \leq C\|\nabla u\|_{L^2} \|\nabla H\|_{L^2} + C\|\xi^{\alpha/2} \nabla H\|_{L^2} \|\nabla H\|_{L^2} \]
\[ \leq \nu \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C(\epsilon) \psi \|\nabla H\|_{L^2}^2 + C \psi \|\nabla H\|_{L^2}^2 \]
\[ \leq \nu \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + \|\xi^{\alpha/2} \nabla H\|_{L^2}^2 + C(\epsilon) \psi \|\nabla H\|_{L^2}^2 \]
\[ \leq \nu \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + \|\xi^{\alpha/2} \nabla H\|_{L^2}^2 + C(\epsilon) \psi \|\nabla H\|_{L^2}^2 \]

which together with (3.21) and choosing \( \delta \) suitably small yields
\[ \frac{d}{dt} \int ((2\mu + \lambda)(\text{div} u)^2 + \mu \omega^2 + R^{-1}|u|^2) \, dx \]
\[ + \|\rho^{1/2} u\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \]
\[ \leq \frac{d}{dt} B(t) + C\psi \|\nabla H\|_{L^2}^2 + C\|\nabla^{\alpha/2} \nabla H\|_{L^2}^2 + \nu \psi^{-1} \|\nabla^2 u\|_{L^2}^2. \]  
(3.24)

In order to estimate the third term on the right-hand side of (3.24), multiplying (2.2)_3 by \( H \xi^\alpha \) and integrating by parts yield
\[ \frac{1}{2} \int |H|^2 \xi^\alpha \, dx + \nu \int |\nabla H|^2 \xi^\alpha \, dx \]
\[ = \frac{\nu}{2} \int |H|^2 \Delta \xi^\alpha \, dx + \int H \cdot \nabla u \cdot H \xi^\alpha \, dx \]
\[ - \frac{1}{2} \int \text{div} [H \xi^\alpha] \, dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \xi^\alpha \, dx \doteq \sum_{i=1}^{4} \tilde{I}_i. \]  
(3.25)

Direct calculations yield that
\[ |\tilde{I}_1| \leq C \int |H|^2 \xi^\alpha \|\xi^{-2} \log^{2(1-h_0)}(\epsilon + |x|^2) \, dx \leq C \int |H|^2 \xi^\alpha \, dx, \]
\[ |\tilde{I}_2| + |\tilde{I}_3| \leq C \int |\nabla u| |H|^2 \xi^\alpha \, dx \leq C \|\nabla u\|_{L^2} \|H \xi^{\alpha/2}\|_{L^2}^2 \]
\[ \leq C \|\nabla u\|_{L^2} \|H \xi^{\alpha/2}\|_{L^2} \|H \nabla \xi^{\alpha/2}\|_{L^2} + \|H \nabla \xi^{\alpha/2}\|_{L^2} \]
\[ \leq C(\|\nabla u\|_{L^2}^2 + 1) \|H \xi^{\alpha/2}\|_{L^2}^2 + \frac{\nu}{4} \|H \nabla \xi^{\alpha/2}\|_{L^2}^2, \]  
(3.26)
\[|\tilde{L}| \leq C \int \tilde{x}^a |H|^2 \tilde{x}^{-3/4} |u| \tilde{x}^{-1/4} \log^{(1-n_0)}(e + |x|^2) \, dx \]
\[\leq C \|Hx^{a/2}\|_{L^2} \|Hx^{a/2}\|_{L^2} \|ux^{-3/4}\|_{L^4} \]
\[\leq C \|Hx^{a/2}\|_{L^2}^2 + C \|Hx^{a/2}\|_{L^2}^2 \left(\|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) \]
\[\leq C (1 + \|\nabla u\|_{L^2}) \|Hx^{a/2}\|_{L^2}^2 + \frac{\nu}{4} \|\nabla Hx^{a/2}\|_{L^2}^2.\]

Putting (3.26) into (3.25), after using Gronwall’s inequality and (3.5), we have
\[
\sup_{0 \leq s \leq t} \|Hx^{a/2}\|_{L^2}^2 + \int_0^t \|\nabla Hx^{a/2}\|_{L^2}^2 \, ds \leq C \exp \left\{ C \int_0^t (1 + \|\nabla u\|_{L^2}^2) \, ds \right\} \leq C. \tag{3.27}
\]

To estimate the last term on the right-hand side of (3.24), denoting \( \nabla^\perp \triangleq (\partial_2, -\partial_1) \), together with
\[
(\nabla \times H) \times H = H \cdot \nabla H - \frac{1}{2} \nabla |H|^2
\]
due to div \( H = 0 \), we rewrite the momentum equations (2.2)_2 as
\[
\dot{\rho} + R^{-1}u = \nabla F + \mu \nabla^\perp \omega + H \cdot \nabla H, \tag{3.29}
\]
where
\[
\dot{f} \triangleq f_t + u \cdot \nabla f, \quad F \triangleq (2\mu + \lambda) \text{div} u - P(\rho) - \frac{1}{2} |H|^2, \quad \omega \triangleq \nabla^\perp \cdot u
\]
are the material derivative of \( f \), the effective viscous flux and the vorticity respectively. Thus, (3.29) implies that \( \omega \) satisfies
\[
\begin{aligned}
\mu \Delta \omega &= \nabla^\perp \cdot \left( \rho \dot{u} + R^{-1}u - H \cdot \nabla H \right), & \text{in } B_R, \\
\omega &= 0, & \text{on } \partial B_R. \tag{3.31}
\end{aligned}
\]

Applying the standard \( L^p \) estimate to (3.31) yields that, for \( p \in (1, \infty) \),
\[
\|\nabla \omega\|_{L^p} \leq C(p) \left( \|\rho \dot{u}\|_{L^p} + \|H\|\|\nabla H\|_{L^p} + R^{-1}\|u\|_{L^p}\right),
\]
which together with (3.29) gives
\[
\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C(p) \left( \|\rho \dot{u}\|_{L^p} + \|H\|\|\nabla H\|_{L^p} + R^{-1}\|u\|_{L^p}\right). \tag{3.32}
\]

It follows from (2.8) and (3.32) that for \( p \in [2, q] \),
\[
\|\nabla^2 u\|_{L^p} \leq C \|\nabla \omega\|_{L^p} + C \|\nabla \text{div} u\|_{L^p}
\leq C \left( \|\nabla \omega\|_{L^p} + \|\nabla (2\mu + \lambda) \text{div} u\|_{L^p} + \|\nabla \lambda \| \text{div} u\|_{L^p} \right)
\leq C \left( \|\rho \dot{u}\|_{L^p} + \|H\|\|\nabla H\|_{L^p} + \|\nabla P\|_{L^p} + R^{-1}\|u\|_{L^p} + \|\nabla \lambda \| \text{div} u\|_{L^p} \right),
\]
which together with (3.19), (3.12) and (3.13) leads to
\[
\|\nabla^2 u\|_{L^2} \leq \frac{C}{\psi^{1/2}} \|\sqrt{\rho} u_t\|_{L^2} + \frac{\mu}{2} \|u \cdot \nabla u\|_{L^2} + C \|H\| \|\nabla H\|_{L^2}
\leq C \|\sqrt{\rho} u_t\|_{L^2} + \|H\| \|\nabla H\|_{L^2} + C \psi^{1/2} + \frac{1}{2} \|\nabla^2 u\|_{L^2}. \tag{3.34}
\]

Putting (3.34) into (3.24), integrating the resulting inequality over \((0, t)\) and choosing \( \varepsilon \) suitably small yield
\[
R^{-1}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \int_0^t \left( \|\rho^{1/2}u_t\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \right) \, ds
\leq C + C\|P\|_{L^2}^2 + C \int_0^t \psi \, ds + C \int_0^t \|H\| \|\nabla H\|_{L^2}^2 \, ds + C \int_0^t \|\nabla Hx^{a/2}\|_{L^2}^2 \, ds,
\]
\[ \leq C + C \| P \|_{L^2}^2 + C \int_0^t \psi'' \, ds \]
\[ \leq C + C \int_0^t \psi'' \, ds, \]

where we have used (3.22), (3.19), (3.27) and the following estimate
\[ \| P \|_{L^2}^2 \leq \| P(\rho_0) \|_{L^2}^2 + C \int_0^t \| P \|_{L^2}^{1/2} \| P \|_{L^2}^{1/2} \| \nabla u \|_{L^2} \, ds \leq C + C \int_0^t \psi'' \, ds \]
due to (3.16). The proof of lemma 3.2 is completed.

**Lemma 3.3.** Let \((\rho, u, H)\) and \(T_1\) be as in lemma 3.2. Then, for all \(t \in (0, T_1)\),
\[ \sup_{0 \leq s \leq t} \left( s \| \rho \|_{L^2}^2 + s \| H \|_{L^2}^2 \right) \]
\[ + \int_0^t \left( s R^{-1} \| u \|_{L^2}^2 + s \| \nabla u \|_{L^2} + s \| H \|_{L^2} + s \| \Delta H \|_{L^2} \right) \, ds \]
\[ \leq C \exp \left\{ C \int_0^t \psi'' \, ds \right\}. \]  
(3.35)

**Proof.** Differentiating (2.2) with respect to \(t\) gives
\[ \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u - \mu \nabla^2 \omega_t - \nabla ((2\mu + \lambda) \nabla u) + R^{-1} u, \]
\[ = -\rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u + \nabla (\lambda \partial_t u) - \nabla P + \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right). \]  
(3.36)

Multiplying (3.36) by \(u_t\) and integrating the resulting equation over \(B_R\), we obtain after using (2.2) that
\[ \frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 \, dx + \int (2\mu + \lambda) (\nabla u)_t^2 + \mu \omega_t^2 + R^{-1} |u_t|^2 \, dx \]
\[ \leq C \int \rho |u_t| |\nabla u_t| + |\nabla u_t|^2 + |u_t| \| \nabla u \|_{L^2} \| \nabla u_t \|_{L^2} \, dx \]
\[ + C \int \rho |u_t|^2 |\nabla u| \, dx + C \int |\lambda_t| |\nabla u_t| \, dx \]
\[ + C \int |P_t| |\nabla u_t| \, dx + \int \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot u_t \, dx \]
\[ \leq \sum_{i=1}^6 \hat{I}_i. \]  
(3.37)

Similar to the proof of [19, lemma 3.3], we have that for \(\varepsilon \in (0, 1)\),
\[ \sum_{i=1}^5 \hat{I}_i \leq \varepsilon \| \nabla u_t \|_{L^2}^2 + \frac{1}{2} \int \lambda (\nabla u_t)_t^2 \, dx + C(\varepsilon) \psi'' \left( \| \nabla^2 u \|_{L^2}^2 + \| \rho \|_{L^2}^{1/2} \right). \]  
(3.38)

Using (2.2) and \(\text{div} \, H = 0\), we get
\[ \hat{I}_6 = \int H_t \cdot \nabla H \cdot u_t \, dx + \int H \cdot \nabla H_t \cdot u_t \, dx + \int H \cdot H_t \text{div} \, u_t \, dx \]
\[ = -\int H_t \cdot \nabla u_t \cdot H \, dx - \int H \cdot \nabla u_t \cdot H_t \, dx + \int H \cdot H_t \text{div} \, u_t \, dx \]
\[ \leq C \int |H_t| |H| |\nabla u_t| \, dx \leq C \int (|\Delta H| + |\nabla u| |H| + |\nabla H| |u|) |H| |\nabla u_t| \, dx \]

\[ \leq \epsilon \|\nabla u_t\|_{L^2}^2 + C(\epsilon) \left( \|\Delta H\|_{H^1_{L^2}}^2 + \|\nabla u\|_{H^2_{L^2}}^2 + \|\nabla H\|_{L^2} \|u\|_{H^1_{L^2}} \right) \]

\[ \leq \epsilon \|\nabla u_t\|_{L^2}^2 + C(\epsilon) \left( \|\Delta H\|_{H^1_{L^2}}^2 + \|\nabla u\|_{H^2_{L^2}}^2 + \|\nabla H\|_{L^2} \|u\|_{H^1_{L^2}} \right) \]

\[ + C(\epsilon) \left( \|u\|^2 \tilde{x}^{-a/2} \|H\|_{L^2} \|\nabla H\|^2 \|\nabla H\|_{L^2} \|\nabla H\|_{L^2} \|\nabla H\|^2 \right) \]

Substituting (3.38)–(3.39) into (3.37) and choosing \( \epsilon \) suitably small lead to

\[ \frac{d}{dt} \int \rho |u_t|^2 \, dx + \int (2\mu + \lambda)(\text{div} \ u_t)^2 + \mu \omega^2 + R^{-1} |u|^2 \, dx \]

\[ \leq C \psi^\alpha \left( 1 + \|u_t\|_{H^1_{L^2}}^2 + \|\nabla u\|_{H^2_{L^2}}^2 + \|\nabla H\|_{H^1_{L^2}}^2 \right) \]

\[ + C_1 \|\Delta H\|_{H^1_{L^2}} \|H\|_{H^2_{L^2}}^2 + C \|\nabla H\|_{H^2_{L^2}}^2 . \]

Next, for \( a_1, a_2 \in \{-1, 0, 1\} \), denoting

\[ \tilde{H}(a_1, a_2) = a_1 H + a_2 H^2, \quad \tilde{u}(a_1, a_2) = a_1 u + a_2 u^2 \].

It thus follows from (2.2) that

\[ H_t - \nu \Delta \tilde{H} = H \cdot \nabla \tilde{u} - u \cdot \nabla \tilde{H} - \tilde{H} \text{div} u . \]

Multiplying (3.42) by \( 4v^{-1} \tilde{H} \Delta |\tilde{H}|^2 \) and integrating the resulting equation by parts lead to

\[ \int v^{-1} \left( \|\nabla |\tilde{H}|^2\|_{L^2}^2 \right) + 2 \|\Delta |\tilde{H}|^2\|_{L^2}^2 \]

\[ = 4 \int |\nabla |\tilde{H}|^2\| \Delta |\tilde{H}|^2 \, dx - 4v^{-1} \int H \cdot \nabla \tilde{u} \tilde{H} \Delta |\tilde{H}|^2 \, dx \]

\[ + 4v^{-1} \int \text{div} u \|\tilde{H}|^2\| \Delta |\tilde{H}|^2 \, dx + 2v^{-1} \int u \cdot \nabla \tilde{H} \Delta |\tilde{H}|^2 \, dx \]

\[ \leq C \|\nabla u\|_{L^4}^4 + C \|\nabla H\|_{L^4}^4 + C \|H\|_{H^1_{L^2}}^4 + \|\Delta |\tilde{H}|^2\|_{L^2}^2 \]

\[ \leq C \psi^\alpha \left( 1 + \|\nabla u\|_{L^4}^4 + \|\nabla H\|_{L^4}^4 \right) + C \|\nabla H\|_{L^2}^2 \|\nabla H\|_{H^1_{L^2}}^2 + \|\Delta |\tilde{H}|^2\|_{L^2}^2 . \]

Where in the first inequality we have used the following estimate:

\[ \int u \cdot \nabla |\tilde{H}|^2 \Delta |\tilde{H}|^2 \, dx = - \int u \cdot \nabla |\tilde{H}|^2 \cdot \nabla |\tilde{u}| \, dx + \frac{1}{2} \int \text{div} \nabla |\tilde{u}|^2 \, dx \]

\[ \leq C \|\nabla u\|_{L^4}^4 + C \|\nabla H\|_{L^4}^4 + C \|H\|_{H^1_{L^2}}^4 . \]

Noticing that

\[ \|\nabla H\|_{H^1_{L^2}}^2 \leq \|\nabla H(1, 0)\|_{L^2}^2 + \|\nabla H(0, 1)\|_{L^2}^2 \]

\[ + \|\nabla H(1, 1)\|_{L^2}^2 + \|\nabla |\tilde{H}(1, -1)|\|_{L^2}^2 , \]

and that

\[ \|\Delta |\tilde{H}(1, 0)\|_{L^2}^2 \leq C \|\nabla H\|_{L^4}^4 + \|\nabla H(1, 0)\|_{L^2}^2 + \|\Delta |\tilde{H}(0, 1)|\|_{L^2}^2 \]

\[ + \|\Delta |\tilde{H}(1, 1)|\|_{L^2}^2 + \|\Delta |\tilde{H}(1, -1)|\|_{L^2}^2 . \]

(3.44)
adding (3.43) multiplied by $4(C_1 + 1)$ to (3.40) yields that
\[
\frac{d}{dt} \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) + \|\nabla u_t\|_{L^2}^2 + R^{-1} \|u_t\|_{L^2}^2 + \|\Delta H\|_{L^2}^2 \\
\leq C \psi^\alpha \left( 1 + \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) + \|\nabla H\|_{L^2}^2 + 1 \|\nabla^2 H\|_{L^2}^2 \\
\leq C \psi^\alpha \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) + C \psi^\alpha + C \left( \|\nabla H\|_{L^2}^2 + 1 \right) \|\nabla^2 H\|_{L^2}^2
\] (3.46)
where in the last inequality we have used (3.34). Multiplying (3.46) by $s$, we obtain (3.35) after using Gronwall’s inequality, (3.4) and (3.19). The proof of lemma 3.3 is completed.

**Lemma 3.4.** Let $(\rho, u, H)$ and $T_1$ be as in lemma 3.2. Then, for all $t \in (0, T_1]$,
\[
\sup_{0 \leq t \leq T_1} \left( \|\tilde{x}^\alpha \rho\|_{L^1(\mathbb{R}^N \cap \partial \Omega_0)} + \|\lambda(\rho)\|_{L^2} + \|\nabla (\tilde{x}^\theta \lambda)\|_{L^2} \right) \\
\leq \exp \left( C \int_0^t \psi^\alpha \, ds \right). \tag{3.47}
\]
**Proof.** First, it follows from the Sobolev inequality and (3.10) that for $0 < \delta < 1$,
\[
\|u \tilde{x}^{-\delta}\|_{L^q} \leq C(\delta) \left( \|u \tilde{x}^{-\delta}\|_{L^{q+1/2}} + \|\nabla (u \tilde{x}^{-\delta})\|_{L^2} \right) \leq C(\delta) \left( \|u \tilde{x}^{-\delta}\|_{L^{q+1/2}} + \|\nabla u\|_{L^q} + \|u \tilde{x}^{-\delta}\|_{L^{q+1/2}} \|\tilde{x}^{-1} \nabla \tilde{x}\|_{L^q} \right) \leq C(\delta) \left( \psi^\alpha + \|\nabla^2 u\|_{L^2} \right). \tag{3.48}
\]
With the same arguments as in [19, lemma 3.4], denoting $w \triangleq \rho \tilde{x}^\alpha$, it holds that
\[
\sup_{0 \leq t \leq T_1} \int w \, dx \leq C \exp \left( C \int_0^t \psi^\alpha \, ds \right), \tag{3.49}
\]
and for $p \in [2, q]$ that
\[
\left( \|\nabla w\|_{L^p} \right)^p \leq C \left( \psi^\alpha + \|\nabla^2 u\|_{L^2} \right) \left( 1 + \|\nabla w\|_{L^p} + \|\nabla w\|_{L^q} \right). \tag{3.50}
\]
Moreover, one can obtain from (3.14) that
\[
\left( \|\lambda(\rho)\|_{L^2} + \|\nabla (\tilde{x}^\theta \lambda)\|_{L^2} \right) \leq C \left( \psi^\alpha + \|\nabla^2 u\|_{L^2} \right) \left( 1 + \|\lambda(\rho)\|_{L^2} + \|\nabla (\tilde{x}^\theta \lambda)\|_{L^2} \right). \tag{3.51}
\]
Now, we claim that
\[
\int_0^t \left( \|\nabla^2 u\|_{L^2}^{q+1/2} + s \|\nabla^2 u\|_{L^2}^2 \right) \, ds \leq C \exp \left( C \int_0^t \psi^\alpha(s) \, ds \right), \tag{3.52}
\]
which together with (3.49), (3.50), (3.51) and Gronwall’s inequality yields (3.47). Finally, to complete the proof of lemma 3.4, it only needs to prove (3.52). In fact, on the one hand, it follows from (3.34), (3.19), (3.4) and (3.35) that
\[
\int_0^t \left( \|\nabla^2 u\|_{L^2}^{5/2} + s \|\nabla^2 u\|_{L^2}^2 \right) \, ds \leq C \int_0^t \left( \psi^\alpha + \|\rho^{1/2} u_t\|_{L^2} + \|H\|_{L^2} \right) \, ds \\
+ C \sup_{0 \leq s \leq t} \left( s \rho^{1/2} u_t \|_{L^2} + s \|H\|_{L^2} \right) \int_0^t \psi^\alpha \, ds \leq C \exp \left( C \int_0^t \psi^\alpha(s) \, ds \right). \tag{3.53}
\]
On the other hand, choosing $p = q$ in (3.33), together with (3.9) and the Gagliardo–Nirenberg inequality, gives
\[
\|\nabla^2 u\|_{L^q} \leq C \left( \|\mu u\|_{L^q} + \|H\|_{L^q} + \|\nabla P\|_{L^q} + \|\nabla \lambda\|_{L^q} \right. \\
+ \left. R^{-1} \|u\|_{L^q} + \|\nabla u\|_{L^q} \right)
\]
\[
\leq C \left( \|\mu u\|_{L^q} + \|\mu u\|_{L^{q'}} \|\nabla u\|_{L^{q''}} + \|H\|_{L^{q'}} \|\nabla H\|_{L^{q''}} + \|\nabla \lambda\|_{L^{q'}} \|\nabla \lambda\|_{L^{q''}} \right)
\]
\[
+ C \psi^a \left( 1 + \|\nabla u\|_{L^{q}} \right) + \|\nabla^2 u\|_{L^q}
\]
\[
\leq C \psi^a \left( \|\mu^{1/2} u\|_{L^q} + \|\nabla u\|_{L^q} \right) + \|\nabla^2 u\|_{L^q}
\]
\[
+ C \psi^a \left( 1 + \|\nabla u\|_{L^{q}} \right) + \|\nabla^2 u\|_{L^q}
\]
\[
\leq C \left( \psi^a + \|\mu^{1/2} u\|_{L^q} + \|\nabla u\|_{L^q} \right) + \|\nabla^2 u\|_{L^q}.
\] (3.54)

This combined with (3.53), (3.55) and (3.4)
\[
\int_0^t \|\nabla^2 u\|_{L^q}^{(q+1)/q} \, ds
\]
\[
\leq C \int_0^t \psi^a s^{-\frac{q-1}{q+1}} \left( s \|\mu^{1/2} u\|_{L^q} \right)^{\frac{q-1}{2q} - \frac{1}{q+1}} \left( s \|\nabla u\|_{L^q} \right)^{\frac{q-2}{2q} + \frac{1}{q+1}} \psi^a \left( \|\nabla u\|_{L^q} \right) \psi^a \left( 1 + \|\nabla u\|_{L^q} \right) \right) \, ds
\]
\[
\leq C \psi^a \left( \|\mu^{1/2} u\|_{L^q} + \|\nabla u\|_{L^q} \right) + \|\nabla^2 u\|_{L^q}
\]
\[
\leq C \psi^a \left( 1 + \|\nabla u\|_{L^q} \right) + \|\nabla^2 u\|_{L^q}
\]
\[
\leq C \psi^a \left( \|\mu^{1/2} u\|_{L^q} + \|\nabla u\|_{L^q} \right) \left( 1 + \|\nabla^2 u\|_{L^q} \right)
\] (3.55)

and
\[
\int_0^t s \|\nabla^2 u\|_{L^q} \, ds
\]
\[
\leq C \int_0^t \psi^a s \|\mu^{1/2} u\|_{L^q} \psi^a \left( s \|\mu^{1/2} u\|_{L^q} \right)^{2(q-1)(q-2)} \left( s \|\nabla u\|_{L^q} \right)^{q-2q/(q-2)} \, ds
\]
\[
+ C \int_0^t \psi^a \left( 1 + \|\nabla^2 u\|_{L^q} \right)^2 \psi^a \left( 1 + \|\nabla^2 u\|_{L^q} \right) \right) \, ds
\]
\[
\leq C \sup_{0 \leq s \leq t} \left( s \|\mu^{1/2} u\|_{L^q} \right) \int_0^t \psi^a \, ds + C \sup_{0 \leq s \leq t} \left( s \|\mu^{1/2} u\|_{L^q} \right)^{2(q-1)(q-2)} \int_0^t \left( \psi^a + s \|\nabla u\|_{L^q} \right) \, ds
\]
\[
+ C \int_0^t \psi^a \left( 1 + \|\nabla^2 u\|_{L^q} \right)^2 \psi^a \left( 1 + \|\nabla^2 u\|_{L^q} \right) \right) \, ds
\]
\[
\leq C \psi^a \left( \|\mu^{1/2} u\|_{L^q} + \|\nabla u\|_{L^q} \right) \left( 1 + \|\nabla^2 u\|_{L^q} \right) \right) \, ds
\] (3.56)

One thus obtains (3.52) from (3.53)–(3.56) and completes the proof of lemma 3.4.

Now, proposition 3.1 is a direct consequence of lemmas 3.2–3.4.
Proof of proposition 3.1. It follows from (3.5), (3.4), (3.27) and (3.47) that
\[ \psi(t) \leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \right\}. \]
Standard arguments thus yield that for \( M \triangleq C e^{c \varepsilon} \) and \( T_0 \triangleq \min \{ T_1, (CM^\alpha)^{-1} \} \),
\[ \sup_{0 \leq t \leq T_0} \psi(t) \leq M, \]
which together with (3.34), (3.4) and (3.52) gives (3.3). The proof of proposition 3.1 is thus completed.

At the end of this section, we will show some estimates of \( H_t \) and \( \nabla H_t \).

Lemma 3.5. Let \((\rho, u, H)\) be a smooth solution to the initial-boundary-value problem (2.2), and \( T_0 \) is obtained in proposition 3.1, then we have
\[ \sup_{0 \leq s \leq T_0} \left( s\|H_t\|_{L^2}^2 + s\|\Delta H\|_{L^2}^2 \right) + \int_0^{T_0} \left( s\|\nabla H_t\|_{L^2}^2 \right) \, ds \leq C. \]  

Proof. Differentiating (2.2) with respect to \( t \) shows
\[ H_{tt} - H_t + \nabla u - \nabla \cdot H_t + \nabla H_t + H_t \Delta u + H_t \Delta H_t = \nu \Delta H_t. \]  
Multiplying (3.58) by \( H_t \) and integrating the resulting equation over \( B_R \), yields that
\[ \frac{1}{2} \frac{d}{dt} \int |H_t|^2 \, dx + \int |\nabla H_t|^2 \, dx = \int H_t \cdot \nabla u_t \cdot H_t \, dx - \int u_t \cdot \nabla H_t \cdot H_t \, dx - \int H_t \cdot H_t \, dx + \int H_t \cdot \Delta u_t \cdot H_t \, dx + \int u \cdot \nabla H_t \cdot H_t \, dx - \int |H_t|^2 \, dx \triangleq 6 \sum_{i=1}^6 S_i. \]  
For the terms \( S_i (i = 1, \cdots, 6) \) on the right-hand side of (3.59), we have
\[ \sum_{i=1}^6 S_i \leq \int H_t \cdot \nabla u_t \cdot H_t \, dx + \int u_t \cdot \nabla H_t \cdot H_t \, dx \]
\[ \leq C \|\nabla u_t\|_{L^2} \|H_t\|_{L^4} \|H\|_{L^4} + C \|\nabla H_t\|_{L^2} \|u_t\|_{L^2} \|H\|_{L^2} \]
\[ \leq C \|H_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 + \frac{\varepsilon}{2} \|\nabla H_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^2 \]
\[ \leq \varepsilon \|\nabla H_t\|_{L^2}^2 + C(\varepsilon) \|H_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\nabla H_t\|_{L^2}^2 \]
\[ \leq \varepsilon \|\nabla H_t\|_{L^2}^2 + C(\varepsilon) \|H_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u_t\|_{L^2}^2 \]
\[ \leq \varepsilon \|\nabla H_t\|_{L^2}^2 + C(\varepsilon) \|H_t\|_{L^2}^2, \]  
owing to (3.9), (3.19) and (3.27),
\[ \sum_{i=1}^6 S_i \leq C \int |H_t|^2 |\nabla u| \, dx \leq C \|H_t\|_{L^2} \|\nabla H_t\|_{L^2} \|\nabla u\|_{L^2} \]
\[ \leq \varepsilon \|\nabla H_t\|_{L^2}^2 + C(\varepsilon) \|H_t\|_{L^2}^2, \]  
due to (3.3).

Now, putting (3.60) and (3.61) into (3.59), and multiplying the resulting inequality by \( s \), we have after choosing \( \varepsilon \) suitably small that
\[ \frac{d}{dt} \left( s \|H_t\|_{L^2}^2 \right) + \nu s \|\nabla H_t\|_{L^2}^2 \leq C \left( s \|H_t\|_{L^2}^2 \right) + C \left( \|H_t\|_{L^2}^2 + s \rho^{1/2} u_t \|_{L^2}^2 + s \|\nabla u_t\|_{L^2}^2 \right), \]  
(3.62)
which together with Gronwall’s inequality, (3.35), (3.4) and (3.3) yields
\[
\sup_{0 \leq s \leq T_0} (s \| H s \|_{L^2}^2) + \int_0^{T_0} s \| \nabla H s \|_{L^2}^2 \, ds \leq C. \tag{3.63}
\]

Next, we claim that
\[
\sup_{0 \leq s \leq T_0} (s \| \nabla H s a/2 \|_{L^2}^2) + \int_0^{T_0} s \| \Delta H s a/2 \|_{L^2}^2 \, ds \leq C. \tag{3.64}
\]

Moreover, it holds that
\[
\| \Delta H \|_{L^2}^2 \leq C \| H s \|_{L^2}^2 + C \| u s \|_{L^2}^2 + C \| H \|_{L^2}^2 \| u s \|_{L^2}^2,
\]
\[
\leq C \| H s \|_{L^2}^2 + C \| u s \|_{L^2}^2 + C \| H \|_{L^2}^2 \| u s \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 \| \Delta H \|_{L^2}^2 + C \| u \|_{L^2}^2,
\]
\[
\leq C \| H s \|_{L^2}^2 + C \| u \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 \| \Delta H \|_{L^2}^2 + C \| u \|_{L^2}^2,
\]
\[
\leq C \| H s \|_{L^2}^2 + C \| u \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2 \| \Delta H \|_{L^2}^2 + C \| u \|_{L^2}^2,
\]
where in the third inequality one has used (3.9) and (3.3). Multiplying (3.65) by \( s \), we obtain (3.57) directly from (3.63) and (3.64).

Finally, we only need to prove (3.64). Indeed, multiplying (2.2)3 by \( \Delta H s a\), integrating the resultant equation by parts yields that
\[
\frac{1}{2} \left( \int |\nabla H|^2 \bar{x} \, dx \right) + v \int |\Delta H|^2 \bar{x} \, dx
\]
\[
\leq C \int |\nabla H| \| H \| \| u \| \| \bar{x} \| \, dx + C \int |\nabla u| \| u \| \| \bar{x} \| \, dx + C \int |\nabla H| \| \Delta H \| \| \bar{x} \| \, dx
\]
\[
+ C \int |\nabla u| \| \Delta H \| \| \bar{x} \| \, dx + C \int |\nabla H| \| \Delta H \| \| \bar{x} \| \, dx \leq \sum_{i=1}^4 J_i. \tag{3.66}
\]

Using the Gagliardo–Nirenberg inequality, (3.27), (3.9) and (3.3), the following holds:
\[
J_1 \leq C \int |\nabla H| \| H \| \| u \| \| \bar{x} \| (\bar{x}^{-1} \| \nabla \bar{x} \|) \, dx
\]
\[
\leq C \| H \bar{x} a/2 \|_{L^2}^4 + C \| \nabla u \|_{L^4}^4 + C \| \nabla H \bar{x} a/2 \|_{L^2}^2,
\]
\[
\leq C \| H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla u \|_{L^4}^4 + C \| \nabla H \bar{x} a/2 \|_{L^2}^2,
\]
\[
\leq C + C \| \nabla u \|_{L^2}^2 + C \| \nabla H \bar{x} a/2 \|_{L^2}^2,
\]
\[
J_2 \leq C \int |\nabla H| \| H \| \| u \| \| \bar{x} \| \| \nabla \bar{x} \| \, dx
\]
\[
\leq C \| H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \bar{x} \| \| \nabla \bar{x} \| \, dx \leq C \| H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2,
\]
\[
\leq C \| H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + \varepsilon \| \Delta H \bar{x} a/2 \|_{L^2}^2,
\]
\[
J_3 + J_4 \leq \varepsilon \| \Delta H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla H \bar{x} a/2 \|_{L^2}^2 + C \| H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4,
\]
\[
\leq \varepsilon \| \Delta H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla H \bar{x} a/2 \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 + C,
\]
\[
J_5 \leq C \| \nabla u \|_{L^2} \| \nabla H \bar{x} a/2 \|_{L^2}^2
\]
\[
\leq C \| \nabla u \|_{L^2}^4 + C \| \nabla \bar{x} a/2 \|_{L^2}^2 + C \| \nabla H \bar{x} a/2 \|_{L^2}^2
\]
\[
\leq C \left( 1 + \| \nabla u \|_{L^2}^{(q+1)/2} \right) \| \nabla H \bar{x} a/2 \|_{L^2}^2. \tag{3.67}
\]
Proof of theorem 1.1.

To prove theorem 1.1, we will only deal with the case that $\beta > 0$, since the same procedure can be applied to the case that $\beta = 0$ after some small modifications.

**Proof of theorem 1.1.** Let $(\rho_0, u_0, H_0)$ be as in theorem 1.1. Without loss of generality, assume that

$$\int_{\mathbb{R}^2} \rho_0 \, dx = 1,$$

which implies that there exists a positive constant $N_0$ such that

$$\int_{B_{R_0}} \rho_0 \, dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 \, dx = \frac{3}{4}. \quad (4.1)$$

We construct $\rho^R_0 = \hat{\rho}^R_0 + R^{-1} e^{-|x|^2}$ where $0 \leq \hat{\rho}^R_0 \in C^\infty_0 (\mathbb{R}^2)$ satisfies that

$$\int_{B_{R_0}} \hat{\rho}^R_0 \, dx \geq 1/2, \quad (4.2)$$

and that

$$\begin{cases}
\hat{x}^a \hat{\rho}^R_0 \to \tilde{x}^a \rho_0 & \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \\
\lambda(\hat{\rho}^R_0) \to \lambda(\rho_0), \quad \tilde{x}^b \nabla \lambda(\hat{\rho}^R_0) \to \tilde{x}^b \nabla \lambda(\rho_0) & \text{in } L^2(\mathbb{R}^2) \cap L^4(\mathbb{R}^2),
\end{cases} \quad (4.3)$$

as $R \to \infty$.

Notice that $H_0 \tilde{x}^a/2 \in L^2(\mathbb{R}^2)$ and $\nabla H_0 \in L^2(\mathbb{R}^2)$, choosing $H^R_0 \in C^\infty_0 (B_R)$ such that

$$H^R_0 \tilde{x}^a/2 \to H_0 \tilde{x}^a/2, \quad \nabla H^R_0 \to \nabla H_0 \quad \text{in } L^2(\mathbb{R}^2), \quad \text{as } R \to \infty. \quad (4.4)$$

Since $\nabla u_0 \in L^2(\mathbb{R}^2)$, choosing $v^R_i \in C^\infty_0 (B_R)(i = 1, 2)$ such that

$$\lim_{R \to \infty} \|v^R_i - \partial_i u_0\|_{L^1(\mathbb{R}^2)} = 0, \quad i = 1, 2, \quad (4.5)$$

we consider the unique smooth solution $u^R_0$ of the following elliptic problem:

$$\begin{cases}
-\Delta u^R_0 + \rho^R_0 u^R_0 + R^{-1} u^R_0 = \sqrt{\rho^R_0} \nabla h^R - \partial_i v^R_i, & \text{in } B_R, \\
\frac{\partial u^R_0}{\partial n} = 0, \quad \text{on } \partial B_R,
\end{cases} \quad (4.6)$$

where $h^R = (\sqrt{\rho^R_0} u_0) * j_{1/R}$, $j_{1/R}$ being the standard mollifying kernel of width $\delta$. Extending $u^R_0$ to $\mathbb{R}^2$ by defining $0$ outside $B_R$ and denoting $u^R \equiv u^R_0 \psi_R$ with $\psi_R$ as in (3.6), with the same arguments as the proof of theorem 1.1 in [19], we obtained that

$$\lim_{R \to \infty} \left( \|\nabla (u^R - u_0)\|_{L^2(\mathbb{R}^2)} + \sqrt{\rho^R_0} \|u^R - \sqrt{\rho^R_0} u_0\|_{L^2(\mathbb{R}^2)} \right) = 0. \quad (4.7)$$

Then, in terms of lemma 2.1, the initial-boundary-value problem (2.2) with the initial data $(\rho^R_0, u^R_0, H^R_0)$ has a classical solution $(\rho^R, u^R, H^R)$ on $B_R \times [0, T_R]$. Moreover, proposition 3.1...
and lemma 3.5 show that there exists a $T_0$ independent of $R$ such that (3.3) and (3.57) hold for $(\rho^R, u^R, H^R)$.

We extend $(\rho^R, u^R, H^R)$ by zero on $\mathbb{R}^2 \setminus B_R$ and denote

$$
\tilde{\rho}^R \triangleq \varphi_R \rho^R, \quad \tilde{u}^R \triangleq \varphi_R u^R, \quad \tilde{H}^R \triangleq \varphi_R H^R, 
$$

with $\varphi_R$ as in (3.6). First, similar to [19], we can also deduce the following estimates of $(\tilde{\rho}^R, \tilde{u}^R)$:

$$
\sup_{0 \leq t \leq T_0} \left( \|\sqrt{\tilde{\rho}^R} u_t^R\|_{L^1(\mathbb{R}^2)} + \|
abla u^R\|_{L^2(\mathbb{R}^2)} \right) \leq C, 
$$

$$
\sup_{0 \leq t \leq T_0} \left( \|\tilde{\rho}^R \tilde{u}^R\|_{L^1(\mathbb{R}^2)} + \|
abla (\tilde{\rho}^R)\|_{L^p(\mathbb{R}^2)} + \|\tilde{u}^R\|_{L^p(\mathbb{R}^2)} \right) \leq C, 
$$

$$
\sup_{0 \leq t \leq T_0} \left( \|\tilde{\lambda}(\tilde{\rho}^R)\|_{L^2(\mathbb{R}^2)} + \|\nabla (\tilde{\lambda})\|_{L^p(\mathbb{R}^2)} + \|\tilde{\lambda}\|_{L^p(\mathbb{R}^2)} \right) \leq C, 
$$

$$
\int_0^{T_0} \left( \|\nabla^2 u^R\|_{L^2(\mathbb{R}^2)}^{2 + \frac{q}{q+1}} + t \|\nabla^2 u^R\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla^2 w^R\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{\lambda}\|_{L^2(\mathbb{R}^2)}^2 \right) dt \leq C, 
$$

$$
\sup_{0 \leq t \leq T_0} t \int_{\mathbb{R}^2} \tilde{\rho}^R |u_t^R|^2 dx + \int_0^{T_0} t \|\nabla u^R\|_{L^2(\mathbb{R}^2)}^2 \leq C, 
$$

where $p \in [2, q]$.

Next, it follows from (3.3) and (3.27) that

$$
\sup_{0 \leq t \leq T_0} \|\nabla \tilde{H}^R\|_{L^1(\mathbb{R}^2)} + \int_0^{T_0} \|\nabla^2 \tilde{H}^R\|_{L^2(\mathbb{R}^2)}^2 dt \leq C \sup_{0 \leq t \leq T_0} \left( \|\nabla H^R\|_{L^2(\mathbb{R}^2)} + R^{-1} \|H^R\|_{L^2(\mathbb{R}^2)} \right)
$$

$$
+ \int_0^{T_0} \left( \|\nabla^2 H^R\|_{L^2(\mathbb{R}^2)}^2 + R^{-2} \|\nabla H^R\|_{L^2(\mathbb{R}^2)}^2 \right) dt \leq C, 
$$

and that

$$
\sup_{0 \leq t \leq T_0} \|\tilde{H}^R \tilde{u}^R\|_{L^2(\mathbb{R}^2)}^2 \leq C \sup_{0 \leq t \leq T_0} \|H^R \tilde{u}^R\|_{L^2(\mathbb{R}^2)}^2 \leq C. 
$$

Moreover, one derives from (3.35), (3.57), (3.64) and (3.3) that

$$
\sup_{0 \leq t \leq T_0} t \left( \int_{\mathbb{R}^2} |\tilde{H}^R|^2 dx + \int_{\mathbb{R}^2} |\Delta \tilde{H}^R|^2 dx \right) + \int_0^{T_0} t \|\nabla \tilde{H}^R\|_{L^2(\mathbb{R}^2)}^2 dt \leq C \sup_{0 \leq t \leq T_0} t \left( \int_{B_R} |H^R|^2 dx + \int_{B_R} |\Delta H^R|^2 dx + R^{-2} \int_{B_R} |\nabla H^R|^2 dx \right)
$$

$$
+ \int_0^{T_0} t \left( \|\nabla H^R\|_{L^2(\mathbb{R}^2)}^2 + R^{-2} \|H^R\|_{L^2(\mathbb{R}^2)}^2 \right) dt \leq C. 
$$

With all these estimates (4.9)–(4.16) at hand, we find that the sequence $(\tilde{\rho}^R, \tilde{u}^R, \tilde{H}^R)$ converges, up to the extraction of subsequences, to some limit $(\rho, u, H)$ in the obvious weak sense, that is, as $R \to \infty$, we have

$$
R^{-1} u^R \to 0, \quad \text{in } L^2(\mathbb{R}^2 \times (0, T_0)),
$$

$$
\tilde{\rho}^R \to \tilde{\rho}, \quad \text{in } C(\overline{B_N} \times [0, T_0]), \quad \text{for any } N > 0, 
$$

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\[ \tilde{x}^a \tilde{\rho}^R \to \tilde{x}^a \rho, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; \mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \]  
(4.19)

\[ \nabla (\tilde{x}^a \lambda(\tilde{\rho}^R)) \to \nabla (\tilde{x}^a \lambda(\rho)), \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; L^2(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)), \]  
(4.20)

\[ \tilde{H}^R \to \tilde{H}, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; L^2(\mathbb{R}^2)), \]  
(4.21)

\[ \sqrt{\tilde{\rho}} w^R \to \sqrt{\rho} u, \quad \nabla w^R \to \nabla u, \quad \nabla \tilde{H}^R \to \nabla H, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; L^2(\mathbb{R}^2)), \]  
(4.22)

\[ \nabla^2 w^R \to \nabla^2 u, \quad \text{weakly in } L^{(q+1)/2}(0, T_0; L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^2 \times (0, T_0)), \]  
(4.23)

\[ \nabla \tilde{H}^R \to \nabla H, \quad \text{weakly in } L^2(\mathbb{R}^2 \times (0, T_0)), \]  
(4.24)

\[ t^{1/2} \nabla^2 w^R \to t^{1/2} \nabla^2 u, \quad \text{weakly in } L^2(0, T_0; L^2(\mathbb{R}^2)), \]  
(4.25)

\[ \tilde{\nabla} \sqrt{\tilde{\rho}} w^R \to \tilde{\nabla} \sqrt{\rho} u, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; L^2(\mathbb{R}^2)), \]  
(4.26)

\[ \tilde{\nabla} \tilde{H}^R \to \tilde{\nabla} H, \quad \tilde{\nabla} \Delta \tilde{H}^R \to \tilde{\nabla} \Delta H, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; L^2(\mathbb{R}^2)), \]  
(4.27)

\[ \tilde{\nabla} \tilde{\nabla} w^R \to \tilde{\nabla} \tilde{\nabla} u, \quad \tilde{\nabla} \nabla \tilde{H}^R \to \tilde{\nabla} \nabla H, \quad \text{weakly in } L^2(\mathbb{R}^2 \times (0, T_0)), \]  
(4.28)

with

\[ \inf_{0 \leq t \leq T_0} \int_{\mathbb{R}^{2d}} \rho(x, t) \, dx \geq \frac{1}{4}. \]  
(4.29)

Next, for any function \( \phi \in C_0^\infty(\mathbb{R}^2 \times (0, T_0)) \) and suitably large enough \( \tilde{\beta} \), we take \( \phi(\varphi) \tilde{\beta} \) as the test function in the initial-boundary-value problem (2.2) with the initial data \( (\rho^R_0, u^R_0, H^R_0) \). Then letting \( R \to \infty \), it follows from (4.17)–(4.29) that \( (\rho, u, H) \) is a strong solution of (1.1)–(1.5) on \( \mathbb{R}^2 \times (0, T_0] \) satisfying (1.10) and (1.11). The proof of the existence part of theorem 1.1 is completed.

It only remains to prove the uniqueness of the strong solutions satisfying (1.10) and (1.11). We only treat the case \( \beta > 0 \), since the procedure can be adapted to the case \( \beta = 0 \) after some small modifications. Let \( (\rho, u, H) \) and \( (\tilde{\rho}, \tilde{u}, \tilde{H}) \) be two strong solutions satisfying (1.10) and (1.11) with the same initial data.

First, subtracting the mass equation for \((\rho, u, H)\) and \((\tilde{\rho}, \tilde{u}, \tilde{H})\) gives

\[ \Theta, \tilde{\rho} - \tilde{u} \cdot \nabla \Theta + \Theta \text{div} \tilde{u} + \rho \text{div} U + U \cdot \nabla \rho = 0, \]  
(4.30)

where \( \Theta \triangleq \rho - \tilde{\rho} \) and \( U \triangleq u - \tilde{u} \). Multiplying (4.30) by \( 2 \Theta \tilde{x}^{2r} \) and integrating by parts lead to

\[ \left\| \Theta \tilde{x}^{2r} \right\|_{L^2} \lessapprox C \left( \left\| \tilde{x}^{2r} \right\|_{L^2} + \left\| \tilde{x}^{-1/2} \right\|_{L^\infty} \right) \left\| \Theta \tilde{x}^{2r} \right\|_{L^2}^2 + C \left\| \rho \tilde{x}^{2r} \right\|_{L^\infty} \left\| \nabla U \right\|_{L^2} \left\| \Theta \tilde{x}^{2r} \right\|_{L^2}, \]  
(4.31)

where in the second inequality we have used (1.11), (3.9) and (3.48). This combined with Gronwall’s inequality yields that for all \( 0 \leq t \leq T_0 \)

\[ \left\| \Theta \tilde{x}^{2r} \right\|_{L^2} \lessapprox C \int_0^t \left( \left\| \nabla U \right\|_{L^2} + \left\| \tilde{\rho} \right\|_{L^2} \right) \, ds. \]  
(4.32)
and
\[ \Phi_1 - \nu \Delta \Phi = H \cdot \nabla U + \Phi \cdot \nabla \bar{u} - u \cdot \nabla \Phi - U \cdot \nabla \bar{H} - H \text{div} U - \Phi \text{div} \bar{u} \quad (4.33) \]
with \( \Phi = H - \bar{H} \). Since \( \mu + \lambda > 0 \), multiplying (4.32) and (4.33) by \( U \) and \( \Phi \), respectively, and adding the resulting equations together, and then integrating by parts lead to
\[
\frac{d}{dt} \int (\rho |U|^2 + |\Phi|^2) \, dx + 2 \int (\mu |\nabla U|^2 + |\Phi|^2) \, dx \\
\leq C \left( \|\nabla \bar{u} \|_{L^\infty} + \|\nabla u \|_{L^\infty} \right) \int (\rho |U|^2 + |\Phi|^2) \, dx + C \int |\Theta| |U| (|\bar{u}| + |\bar{u}| |\nabla \bar{u}|) \, dx \\
+ C (\|P(\rho) - P(\bar{\rho})\|_{L^1} + \|\nabla \bar{u} \|_{L^\infty} + \|\lambda(\rho) - \lambda(\bar{\rho})\|_{L^1}) \|\text{div} U\|_{L^2} \\
+ \frac{1}{2} \int (|H|^2 - |ar{H}|^2) \, dx - \int \Phi \cdot \nabla U \cdot \bar{H} \, dx - \int H \cdot \Phi \text{div} U \, dx - \int U \cdot \nabla \bar{H} \cdot \Phi \, dx \\
\leq C \left( \|\nabla \bar{u} \|_{L^\infty} + \|\nabla u \|_{L^\infty} \right) \int (\rho |U|^2 + |\Phi|^2) \, dx + \sum_{i=1}^{6} K_i. \quad (4.34)
\]

For the terms \( K_1 \) and \( K_2 \), we have by the same arguments as \([19]\) that
\[
K_1 + K_2 \leq C(\varepsilon) \left( 1 + \varepsilon t \|\nabla \bar{u}\|_{L^2}^2 + t \|\nabla \bar{u}\|_{L^2}^2 \right) \int_0^t \left( \|\nabla U\|_{L^2}^2 + \|\sqrt{\rho} U\|_{L^2}^2 \right) \, ds \\
+ \varepsilon \left( \|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \right). \quad (4.35)
\]

For the term \( K_3 \), we have
\[
K_3 = \frac{1}{2} \int (H \cdot \Phi + \bar{H} \cdot \Phi) \, dx \\
\leq C \left( \|H\|_{L^4} + \|\bar{H}\|_{L^4} \right) \|\Phi\|_{L^4 \cdot \|\nabla U\|_{L^2}} \\
\leq \varepsilon \|\nabla U\|_{L^2}^2 + \varepsilon \|\nabla \Phi\|_{L^2}^2 + C(\varepsilon) \|\Phi\|_{L^2}^2. \quad \text{(4.36)}
\]

Similarly, it holds that
\[
K_4 + K_5 \leq C \left( \|H\|_{L^2} + \|\bar{H}\|_{L^2} \right) \|\Phi\|_{L^2 \cdot \|\nabla U\|_{L^2}} \\
\leq \varepsilon \|\nabla U\|_{L^2}^2 + \varepsilon \|\nabla \Phi\|_{L^2}^2 + C(\varepsilon) \|\Phi\|_{L^2}^2. \quad \text{(4.37)}
\]

The last term \( K_6 \) can be estimated as follows:
\[
K_6 \leq C \left( \|U\|_{L^2}^2 + \|\nabla \bar{H}\|_{L^2} \|\nabla \bar{H}\|_{L^2} \|\nabla U\|_{L^2} \|\nabla \Phi\|_{L^1} \right) \\
\leq C \left( \|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \right) \|\nabla \bar{H}\|_{L^2} \|\bar{H}\|_{L^2} \|\Phi\|_{L^2}^2 \\
\leq \varepsilon \left( \|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \right) + C(\varepsilon) \|\nabla \bar{H}\|_{L^2} \|\bar{H}\|_{L^2} \|\Phi\|_{L^2}^2. \quad \text{(4.38)}
\]

Denoting
\[
G(t) \triangleq \|\sqrt{\rho} U\|_{L^2}^2 + \|\Phi\|_{L^2}^2 + \int_0^t \left( \|\nabla U\|_{L^2}^2 + \|\sqrt{\rho} U\|_{L^2}^2 \right) \, ds,
\]
putting (4.35)–(4.38) into (4.34), and choosing \( \varepsilon \) suitably small lead to
\[
G(t) \leq C \left( 1 + \|\nabla \bar{u}\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla \bar{H}\|_{L^2}^2 + t \|\nabla \bar{u}\|_{L^2}^2 + t \|\nabla^2 u\|_{L^2}^2 \right) G(t),
\]
which together with Gronwall’s inequality, (3.27) and (1.10) yields \( G(t) = 0 \). Hence, \( U(x, t) = 0 \) and \( \Phi(x, t) = 0 \) for almost everywhere \((x, t) \in \mathbb{R}^2 \times (0, T)\). Then, one can deduce from (4.31) that \( \Theta = 0 \) for almost everywhere \((x, t) \in \mathbb{R}^2 \times (0, T)\). The proof of theorem 1.1 is completed.
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