Large-time asymptotics to the focusing nonlocal modified Kortweg-de Vries equation with step-like boundary conditions

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Abstract
We investigate the large-time asymptotics of solution for the Cauchy problem of the nonlinear focusing nonlocal modified Kortweg-de Vries (MKdV) equation with step-like initial data, that is, \( u_0(x) \to 0 \) as \( x \to -\infty \), \( u_0(x) \to A \) as \( x \to +\infty \), where \( A \) is an arbitrary positive real number. We first develop the direct scattering theory to establish the basic Riemann-Hilbert (RH) problem associated with step-like initial data. Thanks to the symmetries \( x \to -x \), \( t \to -t \) of nonlocal MKdV equation, we investigate the asymptotics as \( t \to -\infty \) and \( t \to +\infty \), respectively. Our main technique is to use the steepest descent analysis to deform the original matrix-valued RH problem to corresponded regular RH problem, which could be explicitly solved. Finally, we obtain the different large-time asymptotic behaviors of the solution of the Cauchy problem for focusing nonlocal MKdV equation in different space-time sectors \( \mathcal{R}_I \), \( \mathcal{R}_{II} \), \( \mathcal{R}_{III} \), and \( \mathcal{R}_{IV} \) on the whole \((x,t)\)-plane.

Key words
Cauchy problem with step-like initial data, focusing nonlocal MKdV equation, large-time asymptotics, nonlinear steepest method, Riemann–Hilbert problem

JEL classification
35Q51, 35Q15, 35C20, 37K15, 37K40
1 | INTRODUCTION AND STATE OF RESULTS

The pioneering work for the nonlocal integrable systems was introduced by M. Ablowitz and Z. Musslimani to study the nonlinear nonlocal Schrödinger (NNLS) equation with Parity-Time (PT) symmetry.\(^3\) As the first nonlocal integrable system, nonlocal NLS equation is a reduction of a member of the AKNS hierarchy,\(^1\) namely, of the coupled Schrödinger equations

\[ iq_t + q_{xx} + 2q^2r = 0, \quad -ir_t + r_{xx} + 2r^2q = 0, \]

corresponding to \( r(x, t) = \bar{q}(-x, t) \).

Besides the nonlocal NLS equation, researchers apply the PT symmetric reduction to AKNS and the other hierarchies to derive the other types of integrable nonlocal PDEs, which include the space-time Sine-Gordon/Sinh-Gordon equation,\(^2\) the complex/real modified Kortweg-de Vries (MKdV) equation,\(^4,5\) the nonlocal derivative NLS equation,\(^37\) as well as the multidimensional nonlocal Davey–Stewartson equation.\(^12\)

The real nonlocal (also called reverse-space-time) MKdV (NMKdV) equation mentioned in Refs.\(^4,5\) is as follows:

\[ u_t(x, t) + 6\sigma u(x, t)u(-x, -t)u_x(x, t) + u_{xxx}(x, t) = 0, \quad (1) \]

where \( \sigma = \pm 1 \). For \( \sigma = 1 \) and \( \sigma = -1 \), we call the corresponding Equation (1) the focusing NMKdV equation and defocusing NMKdV equation, respectively.

In this paper, we study the following Cauchy problem for the focusing NMKdV equation (which corresponds to \( \sigma = 1 \)) with step-like initial data

\[ u_t(x, t) + 6\sigma u(x, t)u(-x, -t)u_x(x, t) + u_{xxx}(x, t) = 0, \quad (2a) \]

\[ u_0(x) = u(x, t = 0), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (2b) \]

with

\[ u(x, t) = \begin{cases} A + o(1), & x \to +\infty, \\ o(1), & x \to -\infty, \end{cases} \]

where \( u_0(x) \to A \) as \( x \to +\infty \) and \( u_0(x) \to 0 \) as \( x \to -\infty \) sufficiently fast with a real constant \( A > 0 \).

Like the local MKdV equation, the nonlocal MKdV equation also attracted much attention because of its rich physical and mathematical properties. For instance, some exact solutions of the nonlocal MKdV equation including soliton solution, kink solution, rogue-wave, and breathers, which display some new different properties from those of the local MKdV equation, are obtained through Darboux transformation\(^{21}\) as well as the inverse scattering transformation (IST) technique.\(^{22}\) In Ref.\(^{40}\), Zhang and Yan use the Riemann–Hilbert (RH) approach to construct the soliton solutions under the nonzero boundary conditions with symmetric limiting functions like \( |q_\pm| = q_0 \). In physical application, the nonlocal MKdV equation possesses the shifted parity and delayed time reversal symmetry, and thus it is related to the Alice–Bob systems.\(^{23}\) For the long-time asymptotic analysis of NMKdV equation (1), He, Fan, and Xu establish the long-time asymptotics for defocusing nonlocal MKdV equation (corresponding to \( \sigma = -1 \)) with
decaying boundary conditions. Under the nonzero boundary conditions \(|q_\pm| = q_0\), Zhou and Fan use the uniformization technique to investigate the long-time asymptotics for defocusing NMKdV equation via the Dbar steepest method, which was developed by McLaughlin and his collaborators.

Cauchy problems for nonlinear integrable systems with step-like initial data have a long history and origin from the work of Gurevich and Pitaevsky for KdV equation. Since Deift–Zhou nonlinear steepest method became a powerful tool to investigate the long-time asymptotics of Cauchy problem for integrable PDEs, researchers apply it to investigate a large number of integrable equations, in particular, local integrable equations with step-like initial data. The representative works are as follows. Monvel, Lenells, Shepelsky, and their coauthors investigate the long-time asymptotics for focusing NLS equations (corresponding to non–self-adjoint Lax operator). For focusing on the local MKdV equation, refer to Minakov and his collaborators’ work. For defocusing equations (corresponding to self-adjoint Lax operator), please refer to Fromm, Lenells, and Quirchmayr’s work on defocusing local NLS equation, Jenkins’ work on defocusing local NLS with pure step initial functions, as well as Xu and Fan’s work on defocusing local MKdV equation. For the step-like initial-valued problem of nonlocal integrable equations, there is little related work except to some work on nonlocal NLS equations due to Rybalko and Shepelsky. Inspired by the work on NNLS equation, we generate the motivation to investigate the long-time asymptotics for the Cauchy problem of the nonlocal MKdV equation (2), of which the long-time asymptotics has not been studied to the best of our knowledge.

In this work, we assume that the solution \(u(x, t)\) of Cauchy problem (2a)-(2b) satisfies the boundary conditions (consistent with the equation) for all \(t\):

\[
\begin{align*}
    u(x, t) &= A + o(1), \quad x \to +\infty, \\
    u(x, t) &= o(1), \quad x \to -\infty.
\end{align*}
\]

In fact, we will make the sense of \(o(1)\) more precise in the following contents. This choice of initial data and boundary conditions is inspired by some works for the classical (local) MKdV equation (e.g., Ref. 27),

\[
u(x, t) + 6u^2(x, t)u_x(x, t) + u_{xxx}(x, t) = 0
\]

with

\[
\begin{align*}
    u_0(x) &= A + o(1), \quad x \to +\infty, \\
    u_0(x) &= o(1), \quad x \to -\infty,
\end{align*}
\]

where \(A\) indeed is a rarefaction wave solution of local MKdV equation (4).

If we make the initial value \(u_0(x)\) be the pure step function as follows:

\[
u_{0A}(x) := \begin{cases} 
0, & x < 0, \\
A, & x > 0,
\end{cases}
\]

the asymptotic analysis can be simplified. In this paper, we assume that the initial function \(u_0(x)\) is a compact perturbation of the pure step initial function defined by (6), that is, \(u_0(x) - u_{0A}(x) = 0\)
for $|x| > N$ with some $N > 0$, which can make our spectral functions take properties similar to those in the case of the pure step initial functions.

1.1 Main results

We divide the whole $(x, t)$-plane into different space-time zones depicted by Figure 1, in which we obtain the different long-time asymptotic behaviors of the solution for focusing nonlocal MKdV equation (2) with step-like initial data.

As $t \to +\infty$ and $t \to -\infty$, respectively, different asymptotics are presented in different sectors. The asymptotic solution $u(x, t)$ for the Cauchy problem (2) in the regions $\mathcal{R}_{II}$ and $\mathcal{R}_{IV}$ could be called the self-similar solution. In the asymptotic zone $\mathcal{R}_{IV}$, the leading term in the asymptotics is presented by the constant $A$ multiplied by a slowly varying factor, which tends to 1 as $\xi \to -\infty$, in compatible with the boundary condition (3) as $x > 0$. In the asymptotic zone $\mathcal{R}_{II}$, the leading term in the asymptotics is 0, which is also compatible with the boundary condition (3) as $x < 0$. And the asymptotic solution in both $\mathcal{R}_{II}$, $\mathcal{R}_{IV}$ admit the subleading term as the form $\tau^{-1/2}$, which is obtained by parabolic cylinder functions. The subleading terms of the asymptotics in the asymptotic sectors $\mathcal{R}_{II}$ and $\mathcal{R}_{IV}$ admit the typical Zakharov–Manakov form. Additionally, the subleading terms and error terms of the asymptotic formulas in $\mathcal{R}_{II}$ and $\mathcal{R}_{IV}$ depend on the parameter $\text{Im} \nu(-k_0)$. Please refer to Theorem 1 as follows.

Theorem 1. Consider the initial-valued problem (2) and (3), where the initial data $u_0(x)$ is a compact perturbation of the pure step initial function (6): $u_0(x) - u_0A(x) = 0$ for $|x| > N$ with some $N > 0$. Assume that the spectral functions $a_j(k)$, $j = 1, 2$ and $b(k)$ associated to $u_0(x)$ satisfy

Con1. $a_1(k)$ has a single, simple zero in $\mathbb{C}_+$ located $k = ix$, and $a_2(k)$ either has no zeros in $\mathbb{C}_-$ (generic case) or has a single, simple zero at $k = 0$ (nongeneric case);
Con2. $$\Im \nu(-k_0(\xi)) = -\frac{1}{2\pi} d \int_{-\infty}^{-k_0(\xi)} d \arg(1 + r_1(s)r_2(s)) \in (-\frac{1}{2}, \frac{1}{2})$$, where $$r_1(k) = \frac{b(k)}{a_1(k)}$$, $$r_2(k) = \frac{b(k)}{a_2(k)}$$.

Assuming that the solution $$u(x, t)$$ of the Cauchy problem exists, the long-time asymptotics of $$u(x, t)$$ along any line $$\xi := \frac{x}{12t} < 0$$, $$|\xi| = O(1)$$ can be described as follows:

I. For $$x < 0$$, $$t > 0$$ (corresponding to $$\mathcal{R}_{II}$$), as $$t \to +\infty$$

$$u(x, t) = -4\eta(-\tau)^{-\frac{1}{2}-\Im \nu(-k_0(\xi))} \Re \left( \gamma(\xi)e^{t\varphi(\xi,0)}(-\tau)^{i\Re \nu(-k_0(\xi))} \right) + R_1(\xi, -t).$$ (7)

II. For $$x > 0$$, $$t < 0$$ (corresponding to $$\mathcal{R}_{IV}$$), as $$t \to -\infty$$, three types of asymptotic forms are possible, depending on the $$\Im \nu(-k_0)$$, in detail,

(II.a) if $$\Im \nu(-k_0(\xi)) \in (-\frac{1}{2}, -\frac{\alpha}{6})$$,

$$u(x, t) = A\delta^2(0, \xi) - \frac{4c_0^2}{k_0^2(\xi)} \eta\tau^{-\frac{1}{2} - \Im \nu(-k_0(\xi))} \Re \left( i\gamma(\xi)e^{t\varphi(\xi,0)}\tau e^{i\Re \nu(-k_0(\xi))} \right) + R_1(\xi, t),$$ (8a)

(II.b) if $$\Im \nu(-k_0(\xi)) \in (-\frac{\alpha}{6}, \frac{\alpha}{6})$$,

$$u(x, t) = A\delta^2(0, \xi) - \frac{4c_0^2}{k_0^2(\xi)} \eta\tau^{-\frac{1}{2} - \Im \nu(-k_0(\xi))} \Re \left( i\gamma(\xi)e^{t\varphi(\xi,0)}\tau e^{i\Re \nu(-k_0(\xi))} \right) + 4\eta\tau^{-\frac{1}{2} + \Im \nu(-k_0(\xi))} \Re \left( \beta(\xi)e^{-t\varphi(\xi,0)}\tau^{-i\Re \nu(-k_0(\xi))} \right) + R_3(\xi, t),$$ (8b)

(II.c) if $$\Im \nu(-k_0(\xi)) \in [\frac{\alpha}{6}, \frac{1}{2})$$,

$$u(x, t) = A\delta^2(0, \xi) + 4\eta\tau^{-\frac{1}{2} + \Im \nu(-k_0(\xi))} \Re \left( \beta(\xi)e^{-t\varphi(\xi,0)}\tau^{-i\Re \nu(-k_0(\xi))} \right) + R_2(\xi, t).$$ (8c)

Here,

$$\delta(0, \xi) = \exp \left\{ \frac{1}{2\pi i} \int_{(-\infty, k_0(\xi))\cup(k_0(\xi), +\infty)} \frac{\log (1 + r_1(s)r_2(s))}{s} ds \right\},$$

$$k_0(\xi) := \sqrt{-\xi}, \quad \eta := \frac{k_0(\xi)}{2}, \quad \rho = \eta\sqrt{48k_0(\xi)}, \quad \tau := -t\rho^2 = -12tk_0^3(\xi),$$

$$\epsilon := \min \left\{ \eta := \frac{k_0(\xi)}{2}, \frac{1}{2}|i\xi + k_0(\xi)| \right\}, \quad \alpha \in \left(\frac{1}{2}, 1\right),$$

$$\varphi(\xi; \xi') := 2i\theta \left( \xi - k_0(\xi) + \frac{\eta}{\rho} \right) = 16i\rho^3(\xi) - \frac{i}{2}\xi^2 + \frac{i\xi^3}{12\rho},$$
\[
\beta(\xi) = \sqrt{2\pi e^{\frac{\pi i}{4} e^{-\frac{\pi}{2} \nu(-k_0(\xi))}}} q_1(-k_0(\xi)) \Gamma(-\nu(-k_0(\xi))),
\]
\[
q_1(-k_0(\xi)) = e^{-2\chi(\xi,-k_0(\xi))} r_1(-k_0(\xi)) e^{2i\nu(-k_0(\xi))} \log 4,
\]
\[
\gamma(\xi) = \sqrt{2\pi e^{-\frac{\pi i}{4} e^{-\frac{\pi}{2} \nu(-k_0(\xi))}}} q_2(-k_0(\xi)) \Gamma(i\nu(-k_0(\xi))),
\]
\[
q_2(-k_0(\xi)) = e^{2\chi(\xi,-k_0(\xi))} r_2(-k_0(\xi)) e^{-2i\nu(-k_0(\xi))} \log 4.
\]

And the error estimates are as follows:

\[
R_1(\xi, t) = \begin{cases} 
O(\epsilon \tau^{-\frac{\alpha+1}{2}}), & \text{Im } \nu(-k_0(\xi)) \geq 0 \\
O(\epsilon \tau^{-\frac{\alpha+1}{2}+2|\text{Im } \nu(-k_0(\xi))|}), & \text{Im } \nu(-k_0(\xi)) < 0 
\end{cases}
\]

\[
R_2(\xi, t) = \begin{cases} 
O(\epsilon \tau^{-\frac{\alpha+1}{2}+2|\text{Im } \nu(-k_0(\xi))|}), & \text{Im } \nu(-k_0(\xi)) \geq 0 \\
O(\epsilon \tau^{-\frac{\alpha+1}{2}}), & \text{Im } \nu(-k_0(\xi)) < 0 
\end{cases}
\]

and

\[
R_3(\xi, t) := R_1(\xi, t) + R_2(\xi, t) = O(\epsilon \tau^{-\frac{\alpha+1}{2}+2|\text{Im } \nu(-k_0(\xi))|}).
\]

The proof of Theorem 1 is exhibited in Section 4.

Remark 1. For \(x > 0, t < 0\), as \(x \to +\infty\), \(k_0 = \sqrt{-\frac{x}{12t}} \to +\infty\), \(-k_0 = -\sqrt{-\frac{x}{12t}} \to -\infty\), then \(A\delta^2(0, \xi) \to A\), which is compatible to the boundary condition as \(x > 0\). The compatibilities to the boundary condition of the other sectors are obvious.

\(R_I\) and \(R_{III}\) are similar. For \(R_I\), we divide it into three sectors \(R_{I, L}, R_{I, M},\) and \(R_{I, R}\). We name the \(R_{I, L}\) the solitonic region because its asymptotics take the leading term with the one-soliton form. The leading term of asymptotics in \(R_{I, M}\) is the constant \(A\). And the error order of the asymptotics in \(R_{I, M}\) is the same as asymptotics in \(R_{I, L}\). As for \(R_{I, R}\), the asymptotics admit the same leading term as the asymptotics in \(R_{I, M}\), but with the different radiation term, which depends on a small parameter \(\kappa_6 \in (0, \kappa)\). The asymptotics of \(R_{III}\) are obtained by the symmetry of focusing nonlocal MKdV equation \((x \to -x, t \to -t)\). \(R_{III, L}, R_{III, M}, R_{III, R}\) are corresponded to \(R_{I, L}, R_{I, M}, \) and \(R_{I, R}\), respectively. Some similar claims for \(R_I\) are also fitted to \(R_{III}\). For the asymptotics of \(R_I\) and \(R_{III}\), please refer to Theorem 2 as follows.

**Theorem 2.** Under the same conditions of Theorem 1, the long-time asymptotics of \(u(x, t)\) along any line \(\xi := \frac{x}{12t} > 0\), can be described as follows:

I. For \(x > 0, t > 0\) (corresponding to \(R_I\), as \(t \to +\infty\), three asymptotic forms are presented for different \(\xi\) as follows:
(I.a) if $\xi \in (0, \frac{\kappa^2}{3})$ (corresponding to solitonic region $R_{I,E}$)

$$u(x, t) = \frac{A}{1 - C_1(\kappa)e^{-2\kappa x + 8\kappa^3 t}} + O\left(t^{-\frac{1}{2}}e^{-16t\xi^{3/2}}\right)$$

(I.b) if $\xi \in \left(\frac{\kappa^2}{3}, \kappa^2\right)$ (corresponding to region $R_{I,M}$)

$$u(x, t) = A + O\left(t^{-\frac{1}{2}}e^{-16t\xi^{3/2}}\right)$$

(I.c) if $\xi \in (\kappa^2, +\infty)$ (corresponding to region $R_{I,R}$)

$$u(x, t) = A + O\left(t^{-\frac{1}{2}}e^{-8t\kappa_\delta(3\xi - \kappa_\delta^2)}\right).$$

II. For $x < 0$, $t < 0$ (corresponding to $R_{III}$), as $t \to -\infty$, three asymptotic forms are presented for different $\xi$ as follows:

(II.a) if $\xi \in (0, \frac{\kappa^2}{3})$ (corresponding to solitonic region $R_{III,R}$)

$$u(x, t) = \frac{4}{4A\kappa^2 - C_2(\kappa)e^{-2\kappa x + 8\kappa^3 t}} + O\left((-t)^{-\frac{1}{2}}e^{16t\xi^{3/2}}\right),$$

(II.b) if $\xi \in \left(\frac{\kappa^2}{3}, \kappa^2\right)$ (corresponding to region $R_{III,M}$)

$$u(x, t) = O\left((-t)^{-\frac{1}{2}}e^{16t\xi^{3/2}}\right),$$

(II.c) if $\xi \in (\kappa^2, +\infty)$ (corresponding to region $R_{III,E}$)

$$u(x, t) = O\left((-t)^{-\frac{1}{2}}e^{8t\kappa_\delta(3\xi - \kappa_\delta^2)}\right).$$

Here,

$$C_1(\kappa) = \frac{A\gamma_0}{2ia_1'(ix)\kappa^2}, \quad C_2(\kappa) = \frac{2ia_1'(ix)}{\gamma_0}, \quad \kappa_\delta \in (0, \kappa), \quad \gamma_0^2 = 1.$$
**FIGURE 2** Convergence of the leading-term for the asymptotic solution in $R_I$, at (A) $t = 2$, (B) $t = 5$, (C) $t = 10$. Solid blue curves present the exact solution $u(x, t)$ with the parameters $A = 1$, $y_0 = -1$, $\kappa = \frac{1}{2}$, and the dashed red curves present the leading order exhibited in Theorem 2. The figure shows that the leading order is approximated with the one-soliton type solution for $0 < \xi < \frac{\sqrt{3}}{2}$ (numerically, $0 < \xi < 0.083$). And the leading term is compatible with the “background wave” $A$ for $\xi > \frac{\sqrt{3}}{3} (\xi > 0.083)$ when $x > 0$.

**FIGURE 3** Convergence of the leading-term for the asymptotic solution in $R_{III}$, at (A) $t = -2$, (B) $t = -5$, (C) $t = -10$. Solid blue curves present the exact solution $u(x, t)$ with the parameters $A = 1$, $y_0 = -1$, $\kappa = \frac{1}{2}$, and the dashed red curves present the leading order exhibited in Theorem 2. The figure shows that the leading order is approximated with the one-soliton type solution for $0 < \xi < \frac{\sqrt{2}}{3}$ (numerically, $0 < \xi < 0.083$). And the leading term is compatible with the “background wave” $0$ for $\xi > \frac{\sqrt{2}}{3} (\xi > 0.083)$ when $x < 0$.

(i) **The RH problem formalism.** Under the same initial value conditions “$u_0(x) \to A$ as $x \to +\infty$ and $u_0(x) \to 0$ as $x \to -\infty$ sufficiently fast with a real constant $A > 0$,” there exists a cut $(-iA, iA)$ that belongs to imaginary axis for the focusing local MKdV equation, refer to Ref. 27. For the focusing nonlocal MKdV equation, we have a singular point at $k = 0$ as well as a discrete spectrum located at the pure imaginary axis instead of the branch cut.

(ii) **The large-time asymptotics analysis technique.** We shall introduce the related $g$ function to deal with the cut for local MKdV equation with step-like initial data (2). However, in the nonlocal case, our technique is to convert the singularity conditions of $k = 0$ into residue condition so that we can construct a regular RH problem without residue conditions.

(iii) **The asymptotic results.** For focusing local MKdV equation, Minakov and Kotlyrov present a modulated sector between two straight line boundaries $x/t = \text{Const.} 1$ and $x/t = \text{Const.} 2$, where the leading asymptotic term is described in terms of modulated elliptic functions. However, with the Cauchy problem (2), we lack the corresponded modulated sectors and all asymptotics of different sectors are described by genus-0 solution for focusing nonlocal MKdV equation. Besides the difference above, the asymptotic formulas in asymptotic sectors $R_{II}$ and $R_{IV}$ depend on the value $\text{Im}\nu(-k_0)$ (please see Theorem 1), which is the typical feature for the nonlocal MKdV equation.
1.3 | Outline of this paper

The structure of the paper is as follows.

In Section 2, we first perform spectral analysis and set up the direct scattering theory for both focusing and defocusing nonlocal MKdV equation associated to the Cauchy problem (2) in Section 2.1. Then, by inverse scattering theory, we construct the basic RH problem, which is suitable for the asymptotic analysis.

In Section 3, we construct the one-soliton solution under the Assumption 1 for the focusing nonlocal MKdV equation with Cauchy problem (2).

In Section 4, we mainly construct the asymptotics for \( \xi < 0 \) as \( t \to -\infty \). In Section 4.1, we introduce the \( \delta \) function for the first RH transformation. In Section 4.2, we execute the so-called “opening lens” to make the second RH deformation. In Section 4.3, we introduce Blaschke–Potapov factors to reduce the RH problem 4.2 into a regular RH problem without residue conditions. In Section 4.4, we construct the local parametrix for the regular RH problem. In Section 4.5, we do the error analysis via Beals–Coifman theory for regular RH problem. Finally, we review the deformations path and the symmetries of focusing nonlocal MKdV equation to form the asymptotics for \( R_{II} \) and \( R_{IV} \) as described in Theorem 1.

In Section 5, we take the similar technique mentioned in Section 4 to construct the asymptotics for \( \xi > 0 \) as \( t \to +\infty \). According to different asymptotic forms, we divide both \( R_I \) and \( R_{III} \) into three sectors and form the corresponded asymptotics as presented in Theorem 2.

In the last Section 6, we give a brief conclusion for the present work and provide some further discussions for the subsequent work.

2 | INVERSE SCATTERING TRANSFORM AND THE BASIC RH PROBLEM

The concrete ways to deal with RH approach to the step-like problem for classical MKdV equations substantially differ in the focusing case and the defocusing case. For the focusing case of classical MKdV equation, the structure of the spectrum is associated to the non–self-adjoint Lax operator and a part of the spectrum is outside the real axis. For the defocusing case of classical MKdV equation, the structure of the spectrum is associated to the self-adjoint Lax operator and the whole spectrum is located on the real axis. However, we notice that the focusing case and defocusing case for the step-like Cauchy problem of NMKdV are close to each other, in particular, there is a point singularity on the real axis. Owing to this observation, we will present some results of the direct scattering theory for both the cases (focusing and defocusing), see the following Section 2.1.

2.1 | Eigenfunctions and direct scattering

The nonlinear nonlocal MKdV equation admits compatibility condition of the following two linear equations (Lax pairs):

\[
\phi_x + ik_3\phi = U(x,t)\phi, \quad (11a)
\]

\[
\phi_t + 4ik^3\sigma_3\phi = V(x,t,k)\phi, \quad (11b)
\]
where $\phi(x, t, k)$ is a $2 \times 2$ matrix-valued function, $\sigma_3 = \text{diag}(1, -1)$, $k$ is a spectral parameter, and
\[
U(x, t) = \begin{pmatrix} 0 & u(x, t) \\ -\sigma u(-x, -t) & 0 \end{pmatrix}, \quad V(x, t) = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix}
\]
with
\[
V_{11} = 2ik\sigma u(x, t)u(-x, -t) - \sigma u(-x, -t)u_x(x, t) - \sigma u(x, t)u_x(-x, -t),
\]
\[
V_{12} = 4k^2u(x, t) + 2iku_x(x, t) - 2\sigma u^2(x, t)u(-x, -t) - u_{xx}(x, t),
\]
\[
V_{21} = -4k^2\sigma u(-x, -t) - 2ik\sigma u_x(-x, -t) + 2u^2(-x, -t)u(x, t) + \sigma u_{xx}(-x, -t).
\]

Consider the boundary conditions (3), we obtain that the matrices $U(x, t)$ and $V(x, t, k)$ converge to the following matrices:
\[
U(x, t) \to U_{\pm}, \quad V(x, t, k) \to V_{\pm}(k), \quad x \to \pm \infty,
\]
with
\[
U_{\pm} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad U_{-} = \begin{pmatrix} 0 & 0 \\ -\sigma A & 0 \end{pmatrix},
\]
\[
V_{\pm}(k) = \begin{pmatrix} 0 & 4k^2A \\ 0 & 0 \end{pmatrix}, \quad V_{-}(k) = \begin{pmatrix} 0 & 0 \\ -4\sigma k^2A & 0 \end{pmatrix}.
\]

It is easy to notice that the systems
\[
\phi_x + ik\sigma_3\phi = U_{\pm}\phi, \quad \phi_t + 4ik^3\sigma_3\phi = V_{\pm}(k)\phi,
\]
and
\[
\phi_x + ik\sigma_3\phi = U_{-}\phi, \quad \phi_t + 4ik^3\sigma_3\phi = V_{-}(k)\phi,
\]
are still compatible. Then, we could solve the “background solutions” $\phi_{\pm}$ of (15) and (16), respectively, as
\[
\phi_{\pm}(x, t, k) = N_{\pm}(k)e^{-ikx + 4ik^3t}\sigma_3,
\]
where
\[
N_{\pm}(k) = \begin{pmatrix} 1 & A \\ 0 & \frac{2ik}{1} \end{pmatrix}, \quad N_{-}(k) = \begin{pmatrix} 1 & 0 \\ \sigma A & 1 \end{pmatrix}.
\]

We can see that $N_{\pm}(k)$ have singularities at $k = 0$, which plays a significant role in our analysis. Define the Jost solutions as
\[
\psi_j(x, t, k) := \phi_j(x, t, k)e^{ikx + 4ik^3t}\sigma_3, \quad j = 1, 2,
\]
where
\[
\phi_1 \to \phi_{-}, \quad x \to -\infty \quad \text{and} \quad \phi_2 \to \phi_{+}, \quad x \to +\infty.
\]
And $\psi_j, j = 1, 2$ admit the following Lax pairs, respectively:

\begin{align}
(N^{-1}_-\psi_1)_x - ik[N^{-1}_-\psi_1, \sigma_3] &= N^{-1}_-(U - U_-)\psi_1, \quad (21a)
(N^{-1}_-\psi_1)_x - 4ik^3[N^{-1}_-\psi_1, \sigma_3] &= N^{-1}_-(V - V_-)\psi_1, \quad (21b)
\end{align}
as well as

\begin{align}
(N^{-1}_+\psi_2)_x - ik[N^{-1}_+\psi_2, \sigma_3] &= N^{-1}_+(U - U_+)\psi_2, \quad (22a)
(N^{-1}_+\psi_2)_x - 4ik^3[N^{-1}_+\psi_2, \sigma_3] &= N^{-1}_+(V - V_+)\psi_2. \quad (22b)
\end{align}

Now define the $2 \times 2$ matrix-valued functions $\psi_j(x, t, k), j = 1, 2$ as the solutions of the following Volterra integral equations:

\begin{align}
\psi_1(x, t, k) &= N_-(k) + \int_{-\infty}^{x} G_-(x, y, t, k)(U(y, t) - U_-)\psi_1(y, t, k)e^{ik(x-y)\sigma_3} dy, \quad (23a)
\psi_2(x, t, k) &= N_+(k) - \int_{x}^{\infty} G_+(x, y, t, k)(U(y, t) - U_+)\psi_2(y, t, k)e^{ik(x-y)\sigma_3} dy, \quad (23b)
\end{align}

where the $G_\pm(x, y, t, k)$ takes the form

\begin{equation}
G_\pm(x, y, t, k) = \phi_\pm(x, t, k)\phi_\pm^{-1}(y, t, k). \quad (24)
\end{equation}

We summarize some basic properties of $\psi_j, j = 1, 2$ as the following proposition:

**Proposition 1.** The matrix-valued functions $\psi_1(x, t, k)$ and $\psi_2(x, t, k)$ have the following properties:

(i) The columns $\psi_1^{(1)}(x, t, k)$ and $\psi_2^{(2)}(x, t, k)$ are analytic for $k \in \mathbb{C}_+$ and continuous in $\mathbb{C}_+ \setminus \{0\}$. The columns $\psi_1^{(2)}(x, t, k)$ and $\psi_2^{(1)}(x, t, k)$ are analytic for $k \in \mathbb{C}_-$ and continuous in $\mathbb{C}_-$.  

(ii) As $k \to \infty$, $\psi_j(x, t, k) = I + O(k^{-1})$. Moreover,

\begin{equation}
u(x, t) = 2i \lim_{k \to \infty} k\psi_j(x, t, k)_{12}, \quad -\sigma\nu(-x, -t) = 2i \lim_{k \to \infty} k\psi_j(x, t, k)_{21}. \quad (25)\end{equation}

(iii) $\det\psi_j(x, t, k) = 1$ for $x, k \in \mathbb{R}$. 

(iv) The following two symmetric relations hold:

**Symmetry Reduction I:**

\begin{equation}
\Lambda \bar{\psi}_1(-x, -t, -k)\Lambda^{-1} = \psi_2(x, t, k), \quad k \in \left(\mathbb{C}_-, \mathbb{C}_+ \setminus \{0\}\right), \quad (26a)
\end{equation}

where $\Lambda = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$. And $(\mathbb{C}_-, \mathbb{C}_+ \setminus \{0\})$ indicates that $\psi_1^{(1)}$ and $\psi_2^{(2)}$ are defined for $k \in \mathbb{C}_-$ and $\mathbb{C}_+ \setminus \{0\}$, respectively. In particular,

\begin{equation}
\Lambda \bar{\psi}_1(-x, -t, -k)\Lambda^{-1} = \psi_2(x, t, k), \quad k \in \mathbb{R} \setminus \{0\}. \quad (26b)
\end{equation}
Symmetry Reduction II:

\[ \Lambda \psi_1(-x, -t, k) \Lambda^{-1} = \psi_2(x, t, k), \quad k \in \left( \mathbb{C}_-, \mathbb{C}_+ \setminus \{0\} \right). \]  

(v) As \( k \to 0 \), \( \psi_j(x, t, k) \), \( j = 1, 2 \) admit the following singularities:

\[ \psi_1(x, t, k) = \begin{pmatrix}
\frac{1}{k} v_1(x, t) + O(1) & 2iA v_1(x, t) + O(k) \\
\frac{1}{k} v_2(x, t) + O(1) & 2iA v_2(x, t) + O(k)
\end{pmatrix}, \]  
\[ (28a) \]

\[ \psi_2(x, t, k) = \begin{pmatrix}
-\frac{2i}{A} \sigma v_2(-x, -t) + O(k) & -\frac{1}{k} \sigma v_2(-x, -t) + O(1) \\
-\frac{2i}{A} v_1(-x, -t) + O(k) & -\frac{1}{k} v_1(-x, -t) + O(1)
\end{pmatrix}, \]  
\[ (28b) \]

where \( v_j(x, t) \), \( j = 1, 2 \) are some functions and satisfy the following system of integral equations:

\[ \begin{cases}
v_1(x, t) = \int_{-\infty}^{x} u(y, t) v_2(y, t) dy, \\
v_2(x, t) = -iA \frac{1}{2} - \sigma \int_{-\infty}^{x} (u(-y, -t) - A) v_1(y, t) dy,
\end{cases} \]  
\[ (29) \]

Proof. The item (i) directly follows from the representation of \( \psi_j \), \( j = 1, 2 \) in terms of the Neumann series associated with Equations (23).

For item (ii), we expand \( \psi_j \) as

\[ \psi_j = \psi_{j,E_0} + \frac{\psi_{j,E_1}}{k} + O(k^{-2}), \quad k \to \infty. \]  
\[ (30) \]

Notice that \( \psi_j \) satisfies the linear equations

\[ (\psi_j)_x + ik [\sigma_3, \psi_j] = U \psi_j, \]  
\[ (31a) \]

\[ (\psi_j)_t + 4ik^3 [\sigma_3, \psi_j] = V \psi_j, \]  
\[ (31b) \]

we substitute (30) into (31) and compare the order to obtain that \( x \)-part:

\[ O(1) : (\psi_{j,E_0})_x + i[\sigma_3, \psi_{j,E_1}] = U \psi_{j,E_0}, \]  
\[ (32a) \]

\[ O(k) : i[\sigma_3, \psi_{j,E_0}] = 0, \]  
\[ (32b) \]

t-part:

\[ O(k^2) : 4i[\sigma_3, \psi_{j,E_0}] = 4U \psi_{j,E_0}, \]  
\[ (32c) \]

\[ O(k^3) : 4i[\sigma_3, \psi_{j,E_0}] = 0. \]  
\[ (32d) \]
From (32b) and (32d), we know Ψ_{j, E_0} is a diagonal matrix. From (32a) and (32c), we can get that Ψ_{j, E_0} is an independent of parameter x. Then,

$$\psi_{j, E_0} = \lim_{x \to \infty} \lim_{k \to \infty} \psi_j = \lim_{k \to \infty} \lim_{x \to \infty} \psi_j = \lim_{k \to \infty} N_+(k) = I,$$

which implies $\psi_j(x, t, k) = I + O(k^{-1})$ as $k \to \infty$. Furthermore, combining (32a) and (32c), we have

$$u(x, t) = 2i \psi_{j, E_1,12}, \text{ and } -\sigma u(-x, -t) = -2i \psi_{j, E_1,21},$$

which imply (25).

Item (iii) follows from the facts that $U(x, t)$ and $V(x, t)$ are traceless matrices and $\det \psi_j = \det \phi_j$ for $x, k \in \mathbb{R}$.

Item (iv) follows from the symmetries

$$\Lambda U(-x, -t) \Lambda^{-1} = -U(x, t), \quad \Lambda N_(-k) \Lambda^{-1} = N_+(k), \quad \Lambda N_-(k) \Lambda^{-1} = N_+(k).$$

Let us consider item (v). From (23) and the structure of singularity for $N_\pm(k)$ at $k = 0$, it follows that, as $k \to 0$,

$$\psi_{11}^{(1)}(x, t, k) = \frac{1}{k} \left( \begin{array}{c} v_1(x, t) \\ v_2(x, t) \end{array} \right) + O(1), \quad \psi_{11}^{(2)}(x, t, k) = \left( \begin{array}{c} v_1(x, t) \\ v_2(x, t) \end{array} \right) + O(k),$$

$$\psi_{21}^{(1)}(x, t, k) = \left( \begin{array}{c} w_1(x, t) \\ w_2(x, t) \end{array} \right) + O(k), \quad \psi_{21}^{(2)}(x, t, k) = \frac{1}{k} \left( \begin{array}{c} w_1(x, t) \\ w_2(x, t) \end{array} \right) + O(1),$$

with some functions $v_j, \tilde{v}_j, w_j, \tilde{w}_j, j = 1, 2$. The symmetry relation (26b) implies that

$$\left( \begin{array}{c} w_1(x, t) \\ w_2(x, t) \end{array} \right) = \left( \begin{array}{c} -\sigma v_2(-x, -t) \\ -v_1(-x, -t) \end{array} \right), \quad \left( \begin{array}{c} \tilde{w}_1(x, t) \\ \tilde{w}_2(x, t) \end{array} \right) = \left( \begin{array}{c} \tilde{v}_2(-x, -t) \\ \sigma \tilde{v}_1(-x, -t) \end{array} \right).$$

Furthermore, substituting (35a) into (23a), we obtain (29). Similarly, we substitute (35b) into (23b) to obtain

$$\begin{cases} \tilde{v}_1(x, t) = \int_{-\infty}^x u(y, t) \tilde{v}_2(y, t) dy, \\ \tilde{v}_2(x, t) = 1 - \sigma \int_{-\infty}^x (u(-y, -t) - A) \tilde{v}_1(y, t) dy. \end{cases}$$

Comparing (29) with (37), we easily get

$$\left( \begin{array}{c} \tilde{v}_1(x, t) \\ \tilde{v}_2(x, t) \end{array} \right) = \frac{2i \sigma}{A} \left( \begin{array}{c} v_1(x, t) \\ v_2(x, t) \end{array} \right).$$

Summarizing (35), (36), and (38), we complete the proof of item (v). □
Remark 2. Using the second symmetry reduction of item (iv) in Proposition 1, that is, (27), we know that the unprescribed functions \( v_j(x, t), j = 1, 2 \) defined by (28) admit that \( v_j(x, t) + v_j(x, t) = 0 \), which is equivalent to \( v_j(x, t) \), is pure imaginary.

Since \( \phi_1(x, t, k) \) and \( \phi_2(x, t, k) \) are two fundamental matrix-valued solutions of Lax integrable system (11), thus there exists a so-called scattering matrix \( S(k) \) such that

\[
\phi_1(x, t, k) = \phi_2(x, t, k)S(k), \quad k \in \mathbb{R} \setminus \{0\},
\]

equivalently, in terms of \( \psi_j(x, t, k) \)

\[
\psi_1(x, t, k) = \psi_2(x, t, k)e^{-\sigma(ikx + 4i\frac{k^3t}{3})}\sigma S(k)e^{\sigma(ikx + 4i\frac{k^3t}{3})}\sigma, \quad k \in \mathbb{R} \setminus \{0\}.
\]

The following proposition presents the form of scattering matrix \( S(k) \).

**Proposition 2.** The scattering matrix \( S(k) \) defined by (40) admits the form

\[
S(k) = \begin{pmatrix}
a_1(k) & -\sigma b(k) \\
b(k) & a_2(k)
\end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\}.
\]

Moreover,

(i) \( a_1(k) \) is analytic for \( k \in \mathbb{C}_+ \) and continuous in \( \mathbb{C}_+ \setminus \{0\} \), \( a_2(k) \) is analytic for \( k \in \mathbb{C}_- \) and continuous in \( \mathbb{C}_- \).

(ii) \( a_1(k) = a_1(-\bar{k}) \) for \( k \in \mathbb{C}_+ \setminus \{0\} \), \( a_2(k) = a_2(-\bar{k}) \) for \( k \in \mathbb{C}_- \), \( b(k) = \overline{b(-k)} \) for \( k \in \mathbb{R} \). In particular, \( a_1(k) = a_1(-\bar{k}) \) for \( k \in \mathbb{R} \setminus \{0\} \), \( a_2(k) = a_2(-\bar{k}) \) for \( k \in \mathbb{R} \).

(iii) \( a_j(k) = 1 + O(k^{-1}) \) as \( k \to \infty \), \( b(k) = O(k^{-1}) \) as \( k \to \infty \) for \( k \in \mathbb{R} \).

(iv) \( a_1a_2(k) + \sigma b(k)b(-k) = a_1a_2(k) + \sigma b^2(k) = 1 \), for \( k \in \mathbb{R} \setminus \{0\} \).

(v) As \( k \to 0 \), \( a_1(k) = \sigma \frac{A^2a_2(0)}{4k^2} + O(k^{-1}) \) for \( k \in \mathbb{C}_+ \), \( b(k) = \sigma \frac{Aa_2(0)}{2ik} + O(1) \) for \( k \in \mathbb{R} \).

\[\textbf{Proof.}\] Replacing \( x, t \) by \( -x, -t \) in (40) and using the symmetric relation (27), we obtain that

\[
\psi_1(x, t, k) = \psi_2(x, t, k)e^{-\sigma(ikx + 4i\frac{k^3t}{3})}\sigma \Lambda S^{-1}(k)\Lambda e^{\sigma(ikx + 4i\frac{k^3t}{3})}\sigma, \quad k \in \mathbb{R} \setminus \{0\}.
\]

Comparing (42) with (40), we derive the form (41).

Similarly, replacing \( x, t, k \) by \( -x, -t, -\bar{k} \) in (40), then taking conjugation and using the symmetry relation (26b), we obtain that

\[
\psi_1(x, t, k) = \psi_2(x, t, k)e^{-\sigma(ikx + 4i\frac{k^3t}{3})}\sigma \Lambda S^{-1}(-k)\Lambda e^{\sigma(ikx + 4i\frac{k^3t}{3})}\sigma, \quad k \in \mathbb{R} \setminus \{0\}.
\]

Comparing (43) with (40), we obtain the following form:

\[
S(k) = \begin{pmatrix}
a_1(k) & -\sigma b(-k) \\
b(k) & a_2(k)
\end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\}.
\]

Taking into account (41), we know that \( b(k) = \overline{b(-k)} \).

Item (ii) directly follows from the detailed derivation for \( S(k) \).
The spectral functions can be obtained in terms of initial data only

\[ S(k) = \psi_2^{-1}(0,0,k)\psi_1(0,0,k), \]  

(44)

alternatively could be written in terms of determinant representations

\[ a_1(k) = \text{Wr}\left(\psi_1^{(1)}(0,0,k),\psi_2^{(2)}(0,0,k)\right), \quad k \in \mathbb{C}_+ \setminus \{0\}, \]  

(45a)

\[ a_2(k) = \text{Wr}\left(\psi_2^{(1)}(0,0,k),\psi_1^{(2)}(0,0,k)\right), \quad k \in \mathbb{C}_-, \]  

(45b)

\[ b(k) = \text{Wr}\left(\psi_2^{(1)}(0,0,k),\psi_1^{(1)}(0,0,k)\right), \quad k \in \mathbb{R}. \]  

(45c)

Using the properties of \( \psi_j \) (Proposition 1), we easily get item (i) and item (iii). Item (iv) follows from \( \det S(k) = 1 \) for \( k \in \mathbb{R} \).

Substituting (28) into (45), we derive that, as \( k \to 0 \)

\[ a_1(k) = \frac{1}{k^2} (\sigma |v_2(0,0)|^2 - |v_1(0,0)|^2) + O\left(\frac{1}{k}\right), \]  

(46a)

\[ a_2(k) = \frac{4\sigma}{A^2} (\sigma |v_2(0,0)|^2 - |v_1(0,0)|^2) + O(k), \]  

(46b)

\[ b(k) = -\frac{2i}{Ak} (\sigma |v_2(0,0)|^2 - |v_1(0,0)|^2) + O(1), \]  

(46c)

from which item (v) follows.

Remark 3. Since \( \sigma v_2(x,t)v_2(-x,-t) - v_1(x,t)v_1(-x,-t) \) is independent of variables \( x \) and \( t \), we can know that \( \sigma |v_2(0,0)|^2 - |v_1(0,0)|^2 = \sigma v_2(x,t)v_2(-x,-t) - v_1(x,t)v_1(-x,-t) \).

Remark 4. In the case of pure step initial data (6), the scattering matrix \( S(k) \) admits the following specific form:

\[
S(k) = \phi_2^{-1}(0,0,k)\phi_1(0,0,k) = N_+^{-1}(k)N_-(k) = \begin{pmatrix}
1 + \frac{\sigma A^2}{4k^2} & -\frac{A}{2ik} \\
\frac{\sigma A}{2ik} & 1
\end{pmatrix}.
\]

For the focusing case (\( \sigma = 1 \)), \( a_1(k) \) has a single, simple zero at \( k = i\frac{A}{2} \) in the upper half-plane whereas \( a_2(k) \) has no zeros in the lower half-plane. The discussion for the zeros of spectral functions under the pure step initial value could guide us make proper assumptions under the non–pure-step case.

We define the following reflection coefficients as:

\[ r_1(k) := \frac{b(k)}{a_1(k)}, \quad r_2(k) := \frac{b(-k)}{a_2(k)} = \frac{b(k)}{a_2(k)} \text{ (follows from } b(-k) = b(k)). \]  

(47)
We notice that the item (ii) of Proposition 2 implies that
\[ r_1(-k) = r_1(k), \quad r_2(-k) = r_2(k), \quad k \in \mathbb{R} \setminus \{0\}. \tag{48} \]

For \( \sigma = 1 \), via the item (iv) of Proposition 2, we obtain
\[ 1 + r_1(k)r_2(k) = \frac{1}{a_1(k)a_2(k)}, \quad k \in \mathbb{R} \setminus \{0\}. \tag{49} \]

### 2.2 The basic RH problem

The aim of this work is to present the asymptotic analysis of the focusing (\( \sigma = 1 \)) nonlocal MKdV equation with step-like initial data, so we fix \( \sigma = 1 \) in the above results as well as the subsequent contents. The RH problem formalism via the IST technique is based on setting up the following \( 2 \times 2 \) matrix-valued piecewise meromorphic functions \( M(x, t, k) \) with jump condition on the real line:

\[
M(x, t, k) := \begin{cases} 
\left( \frac{\psi^{(1)}_1(x, t, k)}{a_1(k)}, \frac{\psi^{(2)}_2(x, t, k)}{a_2(k)} \right), & k \in \mathbb{C}_+, \\
\left( \frac{\psi^{(1)}_2(x, t, k)}{a_1(k)}, \frac{\psi^{(2)}_1(x, t, k)}{a_2(k)} \right), & k \in \mathbb{C}_-.
\end{cases}
\tag{50}
\]

Then the boundary values \( M_{\pm}(x, t, k) := \lim_{k' \to k, k' \in \mathbb{C}_{\pm}} M(x, t, k), \quad k \in \mathbb{R}, \) satisfy the jump condition
\[ M^+(x, t, k) = M^-(x, t, k)J(x, t, k), \]
where
\[
J(x, t, k) = \begin{pmatrix} 
1 + r_1(k)r_2(k) & r_2(k)e^{-2ikx-8ik^3t} \\
r_1(k)e^{2ikx+8ik^3t} & 1
\end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\}.
\]

Under the view of (46), the behavior of \( M \) near \( k = 0 \) is different in the case \( a_2(0) = 0 \) as well as \( a_2(0) \neq 0 \). The former case contains the case of pure step initial-valued problem, refer to Remark 4, where \( a_1(k) \) has a simple single zero located on the \( i\mathbb{R}_+ \), whereas \( a_2(k) \) has not any zeros in the closure of lower half-plane. Since small perturbation of the pure step initial data preserves these properties, we shall discuss the following two cases in the present work:

**Case I, Generic Case:** The spectral function \( a_1(k) \) has a simple pure imaginary zero at \( k = ix, \quad x > 0 \), and \( a_2(k) \) has no zeros in \( \mathbb{C}_- \).

**Case II, Nongeneric Case:** The spectral function \( a_1(k) \) has a simple pure imaginary zero at \( k = ix, \quad x > 0 \), and \( a_2(k) \) has one simple zero in \( \mathbb{C}_- \) located on \( k = 0 \). Thus, we assume that \( a'_2(0) \neq 0 \), where \( (') = \frac{d}{dk} \). Additionally, we suppose that \( a_1(k) = \frac{a_{11}}{k} + O(1), \quad a_{11} \neq 0 \) as \( k \to 0 \).

**Remark 5.** Under the view of (46), Case I corresponds to the \( |v_2(0, 0)|^2 - |v_1(0, 0)|^2 \neq 0 \) whereas Case II corresponds to the equality \( |v_2(0, 0)|^2 - |v_1(0, 0)|^2 = 0 \).
Now we introduce the following proposition, which shows that the value $\kappa$ could be described in terms of some principle valued integrals with respect to the spectral functions $b(k)$.

**Proposition 3.** The simple zero $k = ix$ of $a_1(k)$, $\kappa > 0$, is determined by the following equalities:

(i) In Case I (generic case),

$$
\kappa = \frac{A}{2} \exp \left\{ -\frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \log \frac{s^2}{s^2 + \alpha^2} \frac{1 - b^2(s)}{s} \, ds \right\},
$$

(ii) In Case II (nongeneric case),

$$
\kappa = A \frac{\sqrt{b^2(0) + I_2^2} - b(0)}{2I_1I_2},
$$

where

$$
I_1 = \exp \left\{ \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{\log(1 - b^2(s))}{s} \, ds \right\}, \quad I_2 = \exp \left\{ \frac{1}{2} \log(1 - b^2(0)) \right\}.
$$

Additionally, $a_{11}a'_2(0) = 1 - b^2(0) \neq 0$.

**Proof.** Following the similar methods, which are used in the Ref. [33, Proposition 3], we can obtain this proposition. More details for the proof are arranged in Appendix A. \qed

Combining the singularities of eigenfunctions $\psi_j(x, t, k)$, $j = 1, 2$ and spectral function $a_1(k)$ at $k = 0$ (see the item (v) in Proposition 1 and item (v) in Proposition 2), we exhibit the behavior of piecewise matrix-valued functions $M(x, t, k)$ defined by (50) at $k = 0$ in the following proposition.

**Proposition 4.** The behavior of $M(x, t, k)$ defined by (50) at $k = 0$ can be presented as follows:

**Case I**

$$
M_+(x, t, k) = \begin{pmatrix}
\frac{4v_1(x, t)}{A^2a_2(0)} & -v_2(-x, -t) & O(k^2) \\
\frac{4v_2(x, t)}{A^2a_2(0)} & -v_1(-x, -t) & O(k^2) \\
\frac{4v_2(x, t)}{A^2a_2(0)} & -v_1(-x, -t) & O(k^2)
\end{pmatrix}
\begin{pmatrix}
k \\
0 \\
1/k
\end{pmatrix}
+ O(k), \quad k \in \mathbb{C}_+, k \to 0,
$$

$$
M_-(x, t, k) = \frac{2i}{A} \begin{pmatrix}
v_2(-x, -t) & \frac{v_1(x, t)}{a_2(0)} \\
v_1(-x, -t) & \frac{v_2(x, t)}{a_2(0)}
\end{pmatrix}
+ O(k), \quad k \in \mathbb{C}_-, k \to 0.
$$
Case II

\[ M_+(x, t, k) = \begin{pmatrix} \frac{v_1(x, t)}{a_{11}} + O(k) & -\frac{1}{k}v_2(-x, -t) + O(1) \\ \frac{v_2(x, t)}{a_{11}} + O(k) & -\frac{1}{k}v_1(-x, -t) + O(1) \end{pmatrix} = \begin{pmatrix} v_1(x, t) & -v_2(-x, -t) \\ v_2(x, t) & -v_1(-x, -t) \end{pmatrix} (I + O(k)) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \quad k \in \mathbb{C}_+, \ k \to 0, \]

\[ M_-(x, t, k) = \frac{2i}{A} \begin{pmatrix} -v_2(-x, -t) + O(k) & v_1(x, t) \\ -v_1(-x, -t) + O(k) & -v_2(x, t) \end{pmatrix} \frac{v_1(x, t)}{ka_2'(0)} + O(1) = \frac{2i}{A} \begin{pmatrix} -v_2(-x, -t) & v_1(x, t) \\ -v_1(-x, -t) & -v_2(x, t) \end{pmatrix} \frac{v_1(x, t)}{ka_2'(0)} + O(1) \]

Proof. Use item (v) in Proposition 1, item (v) in Proposition 2, as well as \(a_{11}\) from the expansion \(a_1(k) = a_{11}k^{-1} + O(1)\) as \(k \to 0\). □

Next, we consider the residue conditions for \(M(x, t, k)\) defined by (50). The results are exhibited as follows:

Proposition 5. Assume that \(ix, \kappa > 0\), is a simple zero of \(a_1(k)\), then \(M(x, t, k)\) satisfies the following residue conditions:

\[ \text{Res}_{k=i\kappa} M^{(1)}(x, t, k) = \frac{\gamma_0}{a_1'(i\kappa)} e^{-2\kappa x + 8\kappa^2 t} M^{(2)}(x, t, i\kappa), \quad \gamma_0^2 = 1, \]

where \(\psi^{(1)}(0, 0, i\kappa) = \gamma_0\psi^{(2)}(0, 0, i\kappa)\).

Proof. The construction for residue conditions is standard. Now we give an explanation for \(\gamma_0^2 = 1\). Taking advantage of the symmetry (26a) and (27), we can obtain that

\[ \begin{pmatrix} \psi_{1, 21}(0, 0, i\kappa) \\ \psi_{1, 11}(0, 0, i\kappa) \end{pmatrix} = \begin{pmatrix} \psi_{2, 12}(0, 0, i\kappa) \\ \psi_{2, 22}(0, 0, i\kappa) \end{pmatrix}, \quad (56a) \]

\[ \begin{pmatrix} \psi_{1, 21}(0, 0, i\kappa) \\ \psi_{1, 11}(0, 0, i\kappa) \end{pmatrix} = \begin{pmatrix} \psi_{2, 12}(0, 0, i\kappa) \\ \psi_{2, 22}(0, 0, i\kappa) \end{pmatrix}, \quad (56b) \]

where \(\psi_{j, lm}\) represents the \((l, m)\)-entry of \(\psi_j\). Combining (56a) and (56b), we can see that \(\psi_{1, 11}(0, 0, i\kappa)\) and \(\psi_{1, 21}(0, 0, i\kappa)\) are real-valued. Taking into account \(\psi^{(1)}_1(0, 0, i\kappa) = \gamma_0\psi^{(2)}_2(0, 0, i\kappa)\),
we have
\[ \begin{align*}
\gamma_0 \psi_{2,22}(0,0,ix) &= \psi_{2,12}(0,0,ix), \\
\gamma_0 \psi_{2,12}(0,0,ix) &= \psi_{2,22}(0,0,ix),
\end{align*} \tag{57} \]
from which, we derive \( \gamma_0^2 = 1 \).

Additionally, we discuss the symmetries of \( M(x,t,k) \) defined by (50) in the following proposition.

**Proposition 6.** The piecewise matrix-valued function \( M(x,t,k) \) satisfies the following symmetry reduction:
\[
M(x,t,k) := \begin{cases} 
\Lambda M(-x,-t,-\bar{k}) \Lambda^{-1} \begin{pmatrix} 1 & 0 \\
0 & a_1(k) \end{pmatrix}, & k \in \mathbb{C}_+ \setminus \{0\} \\
\Lambda M(-x,-t,-\bar{k}) \Lambda^{-1} \begin{pmatrix} a_2(k) & 0 \\
0 & 1/a_2(k) \end{pmatrix}, & k \in \mathbb{C}_- \setminus \{0\}.
\end{cases}
\]

**Proof.** Due to the symmetries of reflection coefficients \( r_1(k) \) and \( r_2(k) \) (see (48)–(49)), we have the symmetry for \( J(x,t,k) \) with
\[
\begin{pmatrix} a_2(k) & 0 \\
0 & 1/a_2(k) \end{pmatrix} J(x,t,k) \begin{pmatrix} a_1(k) & 0 \\
0 & 1/a_1(k) \end{pmatrix} = \Lambda J(-x,-t,-\bar{k}) \Lambda^{-1}, \quad k \in \mathbb{R} \setminus \{0\}.
\]
To match consistency with the behavior of \( M(x,t,k) \) at \( k = 0 \) (see Proposition 4) and residue condition (see Proposition 5), we reach the finale of this proposition.

Summarizing all the contents above in this subsection, we construct the following basic RH problem, which is the basis to construct the soliton solution and long-time asymptotics for the focusing nonlocal MKdV equation.

**RH problem 1 (Basic RH Problem).** Find a \( 2 \times 2 \) matrix-valued function \( M(x,t,k) \) such that
\[
(i) \ M(x,t,k) \text{ is meromorphic for } k \in \mathbb{C} \setminus \mathbb{R} \text{ and has a simple pole located at } k = ix, \ x > 0.
(ii) \ Jump \ conditions. \ The \ nontangential \ limits \ M_{\pm}(x,t,k) = \lim_{k' \to k, k' \in \mathbb{C}_\pm} M(x,t,k') \text{ exist for } k \in \mathbb{R}
\]
and \( M_{\pm}(x,t,k) \) satisfy the jump condition \( M_+(x,t,k) = M_-(x,t,k) J(x,t,k) \) for \( k \in \mathbb{R} \setminus \{0\} \), where
\[
J(x,t,k) = \begin{pmatrix} 1 + r_1(k)r_2(k) & r_2(k)e^{-2ikx-8ik^3t} \\
r_1(k)e^{2ikx+8ik^3t} & 1 \end{pmatrix}, \quad k \in \mathbb{R} \setminus \{0\}.
\tag{58}
\]
with \( r_1 \) and \( r_2 \) given in terms of \( b \) by (47) with (A.4) (Case I) or (A.10) (Case II).
(iii) Normalization condition at $k = \infty$. $M(x, t, k) = I + O(k^{-1})$ uniformly as $k \to \infty$.

(iv) Residue condition at $k = i \lambda$.

$$\text{Res}_{k = i \lambda} M^{(1)}(x, t, k) = \frac{\gamma_0}{a_1'(i \lambda)} e^{-2c x + 8c^3 t} M^{(2)}(x, t, i \lambda), \quad \gamma_0^2 = 1,$$

where $x$ could be exhibited in terms of $b$ using (51) (Case I) or (52) (Case II).

(v) Singularities at the origin. As $k \to 0$, $M(x, t, k)$ satisfies

For Case I,

$$M_+(x, t, k) = \begin{pmatrix} 4v_1(x, t) & -v_2(-x, -t) \\ A^2v_2(x, t) & -v_1(-x, -t) \end{pmatrix} \left( I + O(k) \right) \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \quad k \in \mathbb{C}_+, k \to 0,$$

$$M_-(x, t, k) = \frac{2i}{A} \begin{pmatrix} -v_2(-x, -t) \\ v_1(x, t) \end{pmatrix} \frac{a_2(0)}{a_2'(0)} + O(k), \quad k \in \mathbb{C}_-, k \to 0.$$

For Case II,

$$M_+(x, t, k) = \begin{pmatrix} \frac{v_1(x, t)}{a_1} & -\frac{v_2(-x, -t)}{a_1} \\ \frac{v_2(x, t)}{a_1} & -\frac{v_1(-x, -t)}{a_1} \end{pmatrix} \left( I + O(k) \right) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \quad k \in \mathbb{C}_+, k \to 0,$$

$$M_-(x, t, k) = \frac{2i}{A} \begin{pmatrix} -\frac{v_2(-x, -t)}{a_2} \\ \frac{v_1(x, t)}{a_2} \end{pmatrix} \frac{a_2'(0)}{a_2'(0)} + O(k), \quad k \in \mathbb{C}_-, k \to 0.$$

where $v_j(x, t), j = 1, 2$ are some functions.

(vi) Symmetry. $M(x, t, k)$ satisfies the following symmetry reduction:

$$M(x, t, k) := \begin{cases} \Lambda M(-x, -t, -\bar{k}) \Lambda^{-1} \begin{pmatrix} 1 & 0 \\ 0 & a_1(k) \end{pmatrix}, & k \in \mathbb{C}_+ \setminus \{0\} \\ \Lambda M(-x, -t, -\bar{k}) \Lambda^{-1} \begin{pmatrix} a_2(k) & 0 \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{C}_- \setminus \{0\}. \end{cases}$$

Assume that the RH problem 1 exists a solution $M(x, t, k)$, the solution $u(x, t)$ of the nonlinear nonlocal MKdV equation (2) with initial data $u_0$ can be expressed in terms of the (1,2)-entry and (2,1)-entry of RH problem $M$ via following equalities:

$$u(x, t) = 2i \lim_{k \to \infty} k M_{12}(x, t, k), \quad u(-x, -t) = 2i \lim_{k \to \infty} k M_{21}(x, t, k),$$

which are direct results following from (25).
Remark 6. The solution of the RH problem 1 is unique, if it exists. In fact, if we assume there exist two solutions $M$ and $\tilde{M}$, the behaviors of $M$ at $k = 0$ show that $M\tilde{M}^{-1}$ is bounded at the origin. To reach $M\tilde{M}^{-1} \equiv I$, we just need to obey the standard procedure based on the Liouville theorem.

Remark 7. As long as we derive the long-time $(0 < t \to \infty)$ asymptotics for the nonlocal MKdV equation with $x \in \mathbb{R}$, we can obtain the negative long-time $(0 > t \to -\infty)$ asymptotics and vice versa. This is a typical characteristic of nonlocal MKdV equation.

### 2.3 Jump factorizations

To investigate the long-time asymptotics via nonlinear descent method, factorizations of jump matrix (58) play an important role in our analysis.

Introduce the phase function

$$\theta(k, \xi) := 4k^3 + 12k\xi,$$

(64)

where $\xi := \frac{x}{12t}$. For $\xi < 0$, we define

$$k_0(\xi) = \sqrt{-\xi}.$$

(65)

$\pm k_0$ are the saddle points of phase function $\theta(k, \xi)$ and located on the real axis.

For $\xi > 0$, we solve the saddle points of $\theta(k, \xi)$ as follows:

$$k_0(\xi) = i\sqrt{\xi}.$$

(66)

Since the phase function $\theta(\xi, k)$ is the same as in the case of local MKdV equation, its signature tables (see Figure 4 and Figure 5) suggest us to follow the standard procedure (see Ref. 11) to make factorizations of the jump matrix (58) along the real axis to deform the contours onto those on which the oscillatory jump on the real axis is traded for exponential decay as $t \to +\infty$ or $t \to -\infty$. 

**Figure 4** Signature table of the function $\text{Im} \theta(k, \xi) (\xi < 0)$.
This step is aided by two well-known factorizations of the jump matrix $J(x, t, k)$:

\begin{align}
J(x, t, k) &= \begin{pmatrix}
1 & r_2(k)e^{-2it\theta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
ar_1(k)e^{2it\theta} & 1
\end{pmatrix}, \\
\text{(67a)}
\end{align}

as well as

\begin{align}
J(x, t, k) &= \begin{pmatrix}
1 & 0 \\
\frac{1}{1 + r_1(k)r_2(k)}e^{2it\theta} & 1
\end{pmatrix}
\begin{pmatrix}
1 + r_1(k)r_2(k) & 0 \\
1 + r_1(k)r_2(k) & 1
\end{pmatrix}
\begin{pmatrix}
1 & r_2(k)e^{-2it\theta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{1}{1 + r_1(k)r_2(k)} & 1
\end{pmatrix}.
\text{(67b)}
\end{align}

Additionally, we should present more details for the behavior at the singularity point $k = 0$ when we deal with the RH deformations in the long-time asymptotic analysis.

### 3 | ONE-SOLITON SOLUTION

Before studying the long-time asymptotics for the solution of NMKdV equation, we first construct the one-soliton solution corresponding to the discrete spectrum $k = ix$ under the reflectionless environment via the RH problem 1. Some assumptions are given as follows:

**Assumption 1.** Suppose the spectral functions $a_1(k), a_2(k)$, and $b(k)$ satisfy

- $a_1(k)$ has a single, simple zero $k = ix$ with $x > 0$ in $\mathbb{C}_+$.
- $a_2(k)$ has a single, simple zero located at $k = 0$ in $\mathbb{C}_-$.
- $b(k) = 0$ for $k \in \mathbb{R}$.

Then, the following proposition gives the one-soliton solution associated with the Cauchy problem (2).
FIGURE 6 One-soliton solution with the parameters $\gamma_0 = -1$ and $A = 1$.

FIGURE 7 Singular one-soliton solution with the parameters $\gamma_0 = 1$ and $A = 1$.

Proposition 7. Let $a_1(k), a_2(k),$ and $b(k)$ be the spectral functions associated with $u_0(x)$ and admit the assumptions in Assumption 1. Then, we have

(i) $\kappa$ is uniquely determined as $\kappa = \frac{A}{2}$.

(ii) The RH problem 1 has a unique solution for all $(x, t)$ with $x \in \mathbb{R}, t \in \mathbb{R}$ except $t = x/A^2$ and $\gamma_0 = 1$.

(iii) The associated exact one-soliton solution $u(x, t)$ of Cauchy problem (2) is exhibited as

$$u(x, t) = \frac{A}{1 - \gamma_0 e^{-Ax + A^3 t}}.$$  \hfill (68)

The one-soliton (68) is depicted by Figure 6 and Figure 7. It is clear that the one-soliton (68) admits the kink type. For $\gamma_0 = 1$, (68) admits the singularity if $t = x/A^2$. We explain it as a singular kink soliton.

Proof. Since $b(k) = 0$ for $k \in \mathbb{R}$, thus $b(0) = 0$. Due to $b(0) = 0$, we are in Case II (nongeneric case) to start our discussion for one-soliton solution. Using (52), we arrive at item (i) immediately. Furthermore, from (A.10) (see Case II in Appendix A), we obtain that

$$a_1(k) = \frac{k - i\frac{A}{2}}{k}, \quad a_2(k) = \frac{k}{k - i\frac{A}{2}},$$  \hfill (69)
and
\[ a_{11} = \lim_{k \to \infty} k a_1(k) = \frac{A}{2i}, \quad a'_2(0) = \frac{2i}{A}. \] (70)

Observing that \( b(k) = 0 \) for \( k \in \mathbb{R} \) (i.e., \( J(x, t, k) = 1 \) for \( k \in \mathbb{R} \)), we find that \( M(x, t, k) \) is a meromorphic function in \( k \) complex plane with the only pole \( k = i \frac{A}{2} \). Due to this observation, we compare (61a) to (61b), and conclude that \( v_1(x, t) = -v_2(-x, -t) \). Thus, we can convert the singularity condition (61) into a formal residue condition
\[ \text{Res}_{k=0} M^{(2)}(x, t, k) = \frac{A}{2i} M^{(1)}(x, t, 0). \] (71)

Moreover, taking into account the normalization condition of \( M(x, t, k) \) at \( k = \infty \) (see item (iii) of RH problem 1), we express the \( M(x, t, k) \) as follows:
\[
M(x, t, k) = \begin{pmatrix}
  k + v_1(x, t) & v_1(x, t) \\
  k - \frac{iA}{2} & \frac{k}{k - \frac{iA}{2}} \\
  -v_1(-x, -t) & k - v_1(-x, -t)
\end{pmatrix}. \] (72)

Using the residue condition (59) at \( k = i \frac{A}{2} \), we obtain that
\[ v_1(x, t) = \frac{A}{2i} \frac{1}{1 - \gamma_0 e^{-A x + A^3 t}}. \] (73)

At the end, by \( u(x, t) = 2i \lim_{k \to \infty} k M_{12} \), the one-soliton solution (68) follows.

Remark 8. Using another potential recovering formulas \( u(-x, -t) = 2i \lim_{k \to \infty} k M_{21} \), we can solve that
\[ u(-x, -t) = -2i v_1(-x, -t) = \frac{A}{1 - \gamma_0 e^{A x - A^3 t}}, \]
which is also the direct result to replace \( x, t \) by \(-x, -t\) in (68).

Remark 9. From the one-soliton solution formulas (68), we can find that \( u(x, t) \to A \) as \( x \to +\infty \), \( u(x, t) \to 0 \) as \( x \to -\infty \), which are consistent to the boundary conditions (3).

4 | ASYMPTOTIC BEHAVIOR FOR \( \xi := x/(12t) < 0, |\xi| = O(1) \)

In this section, we investigate the long-time asymptotics under the condition \( \xi < 0, |\xi| = O(1) \), and the signature table of this case corresponds to Figure 4. Before our analysis, we point out that the analysis for \( t \to -\infty \) instead of \( t \to +\infty \) is more convenient to deal with the singularities of RH problem at \( k = 0 \). See Remark 11 below.
### 4.1  First RH problem transformation: δ function

At the beginning, we introduce the δ function as the solution of the following scalar RH problem:

- \( \delta(k, \xi) \) is holomorphic for \( k \in \mathbb{C} \setminus ((-\infty, -k_0) \cup (k_0, +\infty)) \),
- \( \delta_+(k, \xi) = \delta_-(k, \xi)(1 + r_1(k)r_2(k)), k \in \mathbb{R}\setminus[-k_0, k_0] \),
- normalization condition: \( \delta(k, \xi) \to 1 \) as \( k \to \infty \).

And the solution can be given by the Cauchy-type integral

\[
\delta(k, \xi) = \exp \left\{ \frac{1}{2\pi i} \int_{(−∞,−k_0)∪(k_0,+∞)} \frac{\log(1 + r_1(s)r_2(s))}{s - k} \, ds \right\}. \tag{74}
\]

By (48), \( \delta(k, \xi) \) admits the symmetry

\[
\delta(k, \xi) = \overline{\delta(\bar{k}, \xi)}, \quad k \in \mathbb{C}\setminus\{0\}. \tag{75}
\]

**Remark 10.** In the general case (not the case of pure step initial data), \( 1 + r_1(k)r_2(k) \) is complex-valued, which could make \( \delta \) function be singular at \( k = \pm k_0 \).

**Remark 11.** If we consider the asymptotics as \( t \to +\infty \), we have to use the following \( \delta \) function:

\[
\delta(k, \xi) = \exp \left\{ \frac{1}{2\pi i} \int_{-k_0}^{k_0} \frac{\log(1 + r_1(s)r_2(s))}{s - k} \, ds \right\}.
\]

In this function, we notice that \( 0 \in (-k_0, k_0) \), thus we have to pay more attention to the behavior of \( \delta \) at \( k = 0 \). Luckily, owing to nonlocal term \( u(-x, -t) \) of integrable nonlocal MKdV equation, we can study the large negative \( t \) asymptotics first.

Moreover, integration by parts in formulas (74) yields

\[
\delta(k, \xi) = \frac{(k + k_0)^{i\nu(-k_0)}}{(k - k_0)^{i\nu(-k_0)}} e^{\tilde{\chi}(\xi, k)}, \tag{76}
\]

where \( \tilde{\chi}(\xi, k) \) is a uniformly bounded function

\[
\tilde{\chi}(\xi, k) = -\frac{1}{2\pi i} \int_{(−∞,−k_0)∪(k_0,+∞)} \log (k - s) ds \log(1 + r_1(s)r_2(s)). \tag{77}
\]

And \( \nu(-k_0) \) can be expressed in terms of

\[
\nu(-k_0) := -\frac{1}{2\pi} \log(1 + r_1(-k_0)r_2(-k_0)) = -\frac{1}{2\pi} \log |1 + r_1(-k_0)r_2(-k_0)| - \frac{i}{2\pi} \Delta(-k_0), \tag{78}
\]

with

\[
\Delta(-k_0) := \int_{-\infty}^{-k_0} d \arg (1 + r_1(s)r_2(s)).
\]
Furthermore, we assume that
\[-\pi < \Delta(k) < \pi, \quad k < 0,\] (79)
and thus \(-\frac{1}{2} < \text{Im}\, \nu(k) < \frac{1}{2}\). With this assumption, \(\log(1 + r_1(k)r_2(k))\) is single-valued, consequently the singularities \(k = \pm k_0\) of \(\delta(k, \xi)\) is square integrable.

On the other hand, we can rewrite \(\delta(k, \xi)\) as
\[
\delta(k, \xi) = \left( \frac{k + k_0}{k - k_0} \right)^{i\xi(-k_0)} e^{\chi(\xi, k)},
\] (80)
where
\[
\chi(\xi, k) = \hat{\chi}(\xi, k) + \frac{1}{2\pi i} \log \left( \frac{1 + r_1(-k_0)r_2(-k_0)}{1 + r_1(-k_0)r_2(-k_0)} \right) \log(k - k_0).
\] (81)

**Remark 12.** In the local MKdV equation (see Refs. 11, 24), we can obtain that \(\chi(k, \xi)\) is equivalent to \(\hat{\chi}(k, \xi)\) by \(r_1(k) = r(k), r_2(k) = \overline{r(k)}\).

With the help of \(\delta(k, \xi)\), we define a new RH problem as follows:

\[
\tilde{M}(x, t, k) = M(x, t, k)\delta - \sigma_3(\xi, k).
\] (82)

Then, we have properties for \(\tilde{M}(x, t, k)\) as follows:

**RH problem 2.** Find a \(2 \times 2\) matrix-valued function \(\tilde{M}(x, t, k)\) such that

(i) \(\tilde{M}(x, t, k)\) is meromorphic for \(k \in \mathbb{C} \setminus \mathbb{R}\) and has a simple pole located at \(k = ix, x > 0\).

(ii) Jump conditions. The nontangential limits \(M_{\pm}(x, t, k) = \lim_{k' \to k, k' \in \mathbb{C}_{\pm}} \tilde{M}(x, t, k')\) exist for \(k \in \mathbb{R}\) and \(M_{\pm}(x, t, k)\) satisfies the jump condition \(M_{+}(x, t, k) = \sigma_3 M_{-}(x, t, k)J(x, t, k)\) for \(k \in \mathbb{R} \setminus \{0\}\), where

\[
J(x, t, k) = \begin{cases}
1 & 1 \\
0 & \frac{r_2(k)\delta^2(k, \xi)e^{-2it\theta}}{1 + r_1(k)r_2(k)e^{2it\theta}}
\end{cases}, \quad k \in (-k_0, k_0) \setminus \{0\},
\]

\[
J(x, t, k) = \begin{cases}
1 & 1 \\
0 & 1
\end{cases}, \quad k \in (-\infty, -k_0) \cup (k_0, +\infty).
\] (83)

(iii) Normalization condition at \(k = \infty\). \(\tilde{M}(x, t, k) = I + O(k^{-1})\) uniformly as \(k \to \infty\).

(iv) Residue condition at \(k = ix\).

\[
\text{Res}_{k=ix}^{(1)}(x, t, k) = \frac{\gamma_0}{\alpha'_1(ix)\delta^2(ix, \xi)} e^{-2kx + 8k^2t} \tilde{M}^{(2)}(x, t, ix), \quad \gamma_0^2 = 1.
\] (84)
(v) **Singularities at the origin.** As \( k \to 0 \), \( M(x, t, k) \) satisfies for Case I

\[
\begin{align*}
\tilde{M}_+(x, t, k) &= \begin{pmatrix}
\frac{4v_1(x, t)}{A^2a_2(0)\delta(0, \xi)} & -\delta(0, \xi)v_2(-x, -t) \\
\frac{4v_2(x, t)}{A^2a_2(0)\delta(0, \xi)} & -\delta(0, \xi)v_1(-x, -t)
\end{pmatrix} (I + O(k)) \\
&= \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \quad k \in \mathbb{C}_+, \quad k \to 0, \quad (85a)
\end{align*}
\]

\[
\begin{align*}
\tilde{M}_-(x, t, k) &= 2iA \begin{pmatrix}
\frac{v_2(-x, -t)}{\delta(0, \xi)} & \frac{v_1(x, t)}{a_2(0)} \\
\frac{-v_1(-x, -t)}{\delta(0, \xi)} & \frac{v_2(x, t)}{a_2(0)}
\end{pmatrix} + O(k), \quad k \in \mathbb{C}_-, \quad k \to 0. \quad (85b)
\end{align*}
\]

For Case II

\[
\begin{align*}
\tilde{M}_+(x, t, k) &= \begin{pmatrix}
\frac{v_1(x, t)}{a_1(0)\delta(0, \xi)} & -v_2(-x, -t)\delta(0, \xi) \\
\frac{v_2(x, t)}{a_1(0)\delta(0, \xi)} & -v_1(-x, -t)\delta(0, \xi)
\end{pmatrix} (I + O(k)) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \quad k \in \mathbb{C}_+, \quad k \to 0, \quad (86a)
\end{align*}
\]

\[
\begin{align*}
\tilde{M}_-(x, t, k) &= 2iA \begin{pmatrix}
\frac{-v_2(-x, -t)}{\delta(0, \xi)} & \frac{v_1(x, t)}{a_2'(0)} \\
\frac{-v_1(-x, -t)}{\delta(0, \xi)} & \frac{v_2(x, t)}{a_2'(0)}
\end{pmatrix} (I + O(k)) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \quad k \in \mathbb{C}_-, \quad k \to 0, \quad (86b)
\end{align*}
\]

where \( v_j(x, t), j = 1, 2 \) are some unprescribed functions.

### 4.2 Second RH problem transformation: Opening lens

In this subsection, we execute the standard procedure so-called “opening lens” to make some jump matrices decay to identity \( I \) as the parameter \( t \to -\infty \). In general, the RH deformations for \( \tilde{M}(x, t, k) \) depend on the properties of reflection coefficients \( r_j(k) \). In the classical Deift–Zhou method, \( r_j(k) \) and \( \frac{r_j(k)}{1 + r_1(k)r_2(k)} \), \( j = 1, 2 \) have to be analytically approximated by some rational functions with well-behaved errors (see Ref. 11). Besides the Deift–Zhou method, \( r_j(k) \) and \( \frac{r_j(k)}{1 + r_1(k)r_2(k)} \) could be continuously extended via the \( \tilde{\delta} \) technique, which is developed by McLaughlin and his collaborators (see Refs. 10, 25, 26).

For the sake of clarity, we assume that the initial data \( u_0(x) \) admit a compact perturbation of pure step initial function \( u_{0A}(x) \) defined by (6), which ensures that eigenfunctions \( \psi^{(l)}_j(x, 0, k), j, l = 1, 2 \) and thus \( r_j(k) \) are meromorphic in \( \mathbb{C} \) (see Ref. 33). Then, we open the lens (see Figure 8)
The contours $\Gamma := \Gamma_j \cup \Gamma_j^*$, $j = 1, 2, 3, 4$ and the regions $U_j, U_j^*$, $j = 0, 1, 2, 3$. via defining $\tilde{M}(x, t, k)$ by

$$\tilde{M}(x, t, k) = \begin{cases} 
\tilde{M}(x, t, k) \begin{pmatrix} 1 - \frac{r_2(k)\delta^2(k, \xi)}{1 + r_1(k)r_2(k)}e^{-2it\theta} \\
0 & 1 \end{pmatrix}, & k \in U_1 \cup U_3, \\
\tilde{M}(x, t, k) \begin{pmatrix} 1 & 0 \\
-r_1(k)\delta^{-2}(k, \xi)e^{2it\theta} & 1 \end{pmatrix}, & k \in U_2, \\
\tilde{M}(x, t, k) \begin{pmatrix} 1 & 0 \\
r_1(k)\delta^{-2}(k, \xi)e^{2it\theta} & 1 \end{pmatrix}, & k \in U_1^* \cup U_3^*, \\
\tilde{M}(x, t, k) \begin{pmatrix} 1 & 0 \\
r_2(k)\delta^2(k, \xi)e^{-2it\theta} & 1 \end{pmatrix}, & k \in U_2^*, \\
\tilde{M}(x, t, k), & k \in U_0 \cup U_0^*. 
\end{cases}$$

Here, the angles between the real axis and rays $\Gamma_j = \Gamma_j(\xi)$ are given such that the discrete spectrum $ix$ is located on the sector $U_0$. The RH problem $\tilde{M}(x, t, k)$ have properties as follows:

**RH problem 3.** Find a $2 \times 2$ matrix-valued function $\tilde{M}(x, t, k)$ such that

(i) $\tilde{M}(x, t, k)$ is meromorphic for $k \in \mathbb{C} \setminus \Gamma$ and has a simple pole located at $k = ix, \kappa > 0$.

(ii) Jump conditions. The nontangential limits $\tilde{M}_\pm(x, t, k) = \lim_{k' \to k, k' \in \mathbb{C}_\pm} \tilde{M}(x, t, k')$ exist for $k \in \Gamma := (\cup_{j=1}^4 \Gamma_j) \cup (\cup_{j=1}^4 \Gamma_j^*)$ and $\tilde{M}_\pm(x, t, k)$ satisfy the jump condition
\[ \tilde{M}_+(x, t, k) = \tilde{M}_-(x, t, k) j(x, t, k) \] for \( k \in \Gamma \), where

\[
\tilde{J}(x, t, k) = \begin{cases} 
\begin{pmatrix}
1 & \frac{r_2(k)\delta^2(k, \xi)}{1 + r_1(k)r_2(k)} e^{-2i\theta} \\
0 & 1
\end{pmatrix}, & k \in \Gamma_1 \cup \Gamma_4, \\
\begin{pmatrix}
1 & 0 \\
r_1(k)\delta^{-2}(k, \xi)e^{2i\theta} & 1
\end{pmatrix}, & k \in \Gamma_2 \cup \Gamma_3, \\
\begin{pmatrix}
1 & -r_2(k)\delta^2(k, \xi)e^{-2i\theta} \\
0 & 1
\end{pmatrix}, & k \in \Gamma_2^* \cup \Gamma_3^*, \\
I, & \text{elsewhere.}
\end{cases}
\]

(iii) Normalization condition at \( k = \infty \). \( \tilde{M}(x, t, k) = I + O(k^{-1}) \) as \( k \to \infty \).

(iv) Residue condition at \( k = ix \).

\[
\text{Res}_{k=ix} \tilde{M}^{(1)}(x, t, k) = c_1(x, t) \tilde{M}^{(2)}(x, t, ix),
\]

with \( c_1(x, t) = \frac{\gamma_0}{a'(ix)\delta(\xi, \xi)} e^{(-2ix^2+8ix)} = \frac{\gamma_0}{a'(ix)\delta(\xi, \xi)} e^{(-24ix^3+8ix^2)} \), \( \gamma_0 = 1 \).

(v) Singularities at the origin. In both cases (Cases I and II), as \( k \to 0 \), \( M(x, t, k) \) satisfies

\[
\tilde{M}_+(x, t, k) = \begin{cases} 
\begin{pmatrix}
-2i\bar{v}_2(-x, -t) \\
-2i\bar{v}_1(-x, -t)
\end{pmatrix} + O(k) - \frac{1}{k}\bar{v}_2(-x, -t)\delta(0, \xi) + O(1) \\
\begin{pmatrix}
\bar{v}_2(-x, -t) \\
\bar{v}_1(-x, -t)
\end{pmatrix} + \frac{A}{2ik}\delta(0, \xi)\bar{v}_2(-x, -t) + O(k)
\end{cases}, & k \in \mathbb{C}_+, k \to 0,
\]

\[
\tilde{M}_-(x, t, k) = \begin{cases} 
\begin{pmatrix}
\bar{v}_2(-x, -t) \\
\bar{v}_1(-x, -t)
\end{pmatrix} + O(k) - \frac{A}{2ik}\delta(0, \xi)\bar{v}_2(-x, -t) + O(k) \\
\begin{pmatrix}
\bar{v}_2(-x, -t) \\
\bar{v}_1(-x, -t)
\end{pmatrix} + \frac{A}{2ik}\delta(0, \xi)\bar{v}_1(-x, -t) + O(k)
\end{cases}, & k \in \mathbb{C}_-, k \to 0.
\]

Furthermore, we can see that the singularity conditions at origin can be formally reduced to

\[
\text{Res}_{k=0} \tilde{M}^{(2)}(x, t, k) = c_0(\xi)\tilde{M}^{(1)}(x, t, 0),
\]

with \( c_0(\xi) = \frac{A\delta^2(0, \xi)}{2i} \).
Proof. The items (i)–(iv) are not difficult to check. For the singular behavior of \( \hat{M}(x, t, k) \) at \( k = 0 \), we exhibit more details below.

Notice that \( 0 \in \bar{U}_2 \cup U^*_2 \), thus we should investigate the singular behavior of \( r_j(k) \) at \( k = 0 \), \( j = 1, 2 \).

For Case I, item (v) of Proposition 2, that is, \( a_1(k) = \frac{A^2a_2(0)}{4k^2} + O(k^{-1}) \), \( b(k) = \frac{Aa_2(0)}{2ik} + O(1) \) as \( k \to 0 \) imply that

\[
-r_1(k)\delta^{-2}(k, \xi)e^{2it\theta} = \frac{2i}{A\delta^2(0, \xi)}k + O(k^2), \quad k \in U_2, \ k \to 0, \quad (91a)
\]
\[
r_2(k)\delta^2(k, \xi)e^{-2it\theta} = \frac{A\delta^2(0, \xi)}{2ik} + O(1), \quad k \in U^*_2, \ k \to 0. \quad (91b)
\]

Then, by (87) for \( k \in U_2 \) and \( k \in U^*_2 \), respectively, we derive (90) in Case I.

For Case II, review that \( a_1(k) = \frac{a_{11}}{k} + O(1) \) and \( a_2(k) = ka'_2(0) + O(k^2) \) as \( k \to 0 \), we obtain

\[
-r_1(k)\delta^{-2}(k, \xi)e^{2it\theta} = \frac{b(0)}{a_{11}\delta^2(0, \xi)}k + O(k^2), \quad k \in U_2, \ k \to 0, \quad (92a)
\]
\[
r_2(k)\delta^2(k, \xi)e^{-2it\theta} = \frac{b(0)}{ka'_2(0)}\delta(0, \xi) + O(1), \quad k \in U^*_2, \ k \to 0. \quad (92b)
\]

Then, by (87) for \( k \in U_2 \) and \( k \in U^*_2 \), respectively, we derive

\[
\hat{M}_+(x, t, k) = \begin{pmatrix}
v_1(x, t) + b(0)v_2(-x, -t) + O(k) & -v_2(-x, -t)k\delta(0, \xi) + O(1) \\
v_2(x, t) + b(0)v_1(-x, -t) + O(k) & -v_1(-x, -t)k\delta(0, \xi) + O(1)
\end{pmatrix}, \quad k \in \mathbb{C}_+, \ k \to 0, \quad (93a)
\]
\[
\hat{M}_-(x, t, k) = \begin{pmatrix}
-\frac{v_2(-x, -t)}{\delta(0, \xi)} + O(k) & \frac{v_1(x, t) - b(0)v_2(-x, -t)}{ka'_2(0)}\delta(0, \xi) + O(1) \\
-\frac{v_1(-x, -t)}{\delta(0, \xi)} + O(k) & \frac{v_2(x, t) - b(0)v_1(-x, -t)}{ka'_2(0)}\delta(0, \xi) + O(1)
\end{pmatrix}, \quad k \in \mathbb{C}_-, \ k \to 0. \quad (93b)
\]

Comparing (93a) with (93b) (cf. (71)), we have two equations as follows:

\[
\tilde{v}_1(x, t) = -\left(\frac{2i}{A} + b(0)\right)\tilde{v}_2(-x, -t), \quad (94a)
\]
\[
\tilde{v}_1(x, t) = \left(b(0) - \frac{Aa'_2(0)}{2i}\right)\tilde{v}_2(-x, -t), \quad (94b)
\]

Indeed, we can find that the two equations in (94) are the same one by taking into account (A.14), from which, we derive (90) in Case II.  \( \square \)
4.3 | Regular RH problem: Leading term of the asymptotics

In this subsection, we convert the RH problem \(3\) with two formal residue conditions \((89)\) and \((90c)\) into a regular RH problem without residue conditions via using the Blaschke–Potapov factors (e.g., Refs. [33, Proposition 6] and [15]).

Make the transformation

\[
\tilde{M} = B(x, t, k) \tilde{M}'(x, t, k) \begin{pmatrix} 1 & 0 \\ 0 & \frac{k - ix}{k} \end{pmatrix}, \quad k \in \mathbb{C},
\]

where \(B(x, t, k) = I + \frac{ix}{k - ix} P(x, t)\). And \(B(x, t, k), P(x, t)\) are the Blaschke–Potapov factors. Next RH problem reveals the relation between the regular RH problem \(\tilde{M}'\) and the Blaschke–Potapov factors.

**RH problem 4.** Find a \(2 \times 2\) matrix-valued function \(\tilde{M}'(x, t, k)\) such that

\(\begin{enumerate}
\item \(\tilde{M}'(x, t, k)\) is analytic for \(k \in \mathbb{C} \setminus \Gamma\).
\item \(\tilde{M}'_+(x, t, k) = \tilde{M}'_-(x, t, k) \tilde{J}'(x, t, k)\) with
\end{enumerate}\)

\[
\tilde{J}'(x, t, k) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{k - ix}{k} \end{pmatrix} J(x, t, k) \begin{pmatrix} 1 & 0 \\ 0 & \frac{k}{k - ix} \end{pmatrix}.
\]

More specifically,

\[
\tilde{J}'(x, t, k) = \begin{cases}
\begin{pmatrix} 1 & \frac{r'_2(k)\delta^2(k, \xi)}{1 + r'_1(k)r'_2(k)} e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \Gamma_1 \cup \Gamma_4, \\
\begin{pmatrix} 1 & 0 \\ \frac{r'_1(k)\delta^{-2}(k, \xi)}{1 + r'_1(k)r'_2(k)} e^{2it\theta} & 1 \end{pmatrix}, & k \in \Gamma^*_1 \cup \Gamma^*_4,
\end{cases}
\]

\[
\tilde{J}'(x, t, k) = \begin{cases}
\begin{pmatrix} 1 & 0 \\ r'_1(k)\delta^{-2}(\xi, k) e^{2it\theta} & 1 \end{pmatrix}, & k \in \Gamma_2 \cup \Gamma_3, \\
\begin{pmatrix} 1 & -r'_2(k)\delta^2(\xi, k) e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \Gamma^*_2 \cup \Gamma^*_3,
\end{cases}
\]

with

\[
r'_1(k) = \frac{k - ix}{k} r'_1(k), \quad r'_2(k) = \frac{k}{k - ix} r'_2(k).
\]
(iii) Normalization at $k = \infty$. $\tilde{M}^r(x, t, k) \to I$ as $k \to \infty$.

(iv) Matrix-valued factor $P(x, t)$ is determined in terms of $\tilde{M}^r(x, t, k)$:

$$P_{12}(x, t) = \frac{g_1(x, t)h_1(x, t)}{g_1(x, t)h_2(x, t) - g_2(x, t)h_1(x, t)},$$

$$P_{21}(x, t) = -\frac{g_2(x, t)h_2(x, t)}{g_1(x, t)h_2(x, t) - g_2(x, t)h_1(x, t)},$$

$$P_{11}(x, t) = -\frac{P_{12}(x, t)g_2(x, t)}{g_1(x, t)}, \quad P_{22}(x, t) = -\frac{P_{21}(x, t)g_1(x, t)}{g_2(x, t)},$$

where $g(x, t) = (g_1(x, t), g_2(x, t))^T$ and $h(x, t) = (h_1(x, t), h_2(x, t))^T$ are given by

$$g(x, t) = i\kappa \tilde{M}^r(1)(x, t, i\kappa) - c_1(x, t)\tilde{M}^r(2)(x, t, i\kappa),$$

$$h(x, t) = i\kappa \tilde{M}^r(2)(x, t, 0) + c_0(\xi)\tilde{M}^r(1)(x, t, 0),$$

where $\tilde{M}^r(j)$, $j = 1, 2$ represent the $j$-th column of $\tilde{M}^r$.

**Proof.** The proof of this RH problem is similar to [33, Proposition 6].

Moreover, by (48), we can derive

$$r_1'(k) = r_1'(-\bar{k}), \quad r_2'(k) = r_2'(-\bar{k}).$$

Combine the symmetry of $\delta(k, \xi)$, we obtain that

$$\tilde{I}^r(x, t, k) = \tilde{I}^r(x, t, -\bar{k}).$$

With these symmetries, we conclude that the regular RH problem $\bar{\tilde{M}}^r(x, t, k)$ admits the symmetry

$$\bar{\tilde{M}}^r(x, t, k) = \tilde{M}^r(x, t, -\bar{k}).$$

Next, we introduce the following proposition, which establishes the relation between the solution $u(x, t)$ of (2), (3) and the regular RH problem $\tilde{M}^r$. From the relation, we can obtain a rough result of large-time asymptotics for $u(x, t)$.

**Proposition 8.** The solution $u(x, t)$ of the Cauchy problem (2), (3) can be expressed in terms of

$$u(x, t) = -2\kappa P_{12}(x, t) + 2i \lim_{k \to \infty} k\tilde{M}^r_{12}(x, t, k), \quad x > 0, \ t < 0,$$

$$u(x, t) = -2\kappa P_{21}(-x, -t) + 2i \lim_{k \to \infty} k\tilde{M}^r_{21}(-x, -t, k), \quad x < 0, \ t > 0.$$ 

Moreover, as $t \to -\infty$, we have a rough estimation

$$u(x, t) = A\delta^2(0, \xi) + o(1), \quad x > 0, \ t < 0.$$
along any ray $\xi = \frac{x}{12t} = \text{const} < 0$. And as $t \to +\infty$,
\[ u(x, t) = o(1), \quad x < 0, \ t > 0 \] (105)
along any ray $\xi = \frac{x}{12t} = \text{const} < 0$.

Proof. First, we use (95) to obtain the large-$k$ behavior for $\tilde{M}(x, t, k)$:
\[
\tilde{M}(x, t, k) = \begin{pmatrix} 1 & 0 \\ 0 & k - i\kappa \end{pmatrix} + \frac{\tilde{M}'(x, t)}{k} + \frac{i\kappa}{k - i\kappa} P(x, t) + O(k^{-2}), \quad \text{as } k \to \infty, \tag{106}
\]
where $\tilde{M}'(x, t)$ comes from the expansion $\tilde{M}'(x, t, k) = I + \frac{\tilde{M}'(x, t)}{k} + O(\frac{1}{k^2}), \ k \to \infty$. Then, we combine (63) and recall a series of transformations for the basic RH problem $M(x, t, k)$ to obtain (103).

Because the regular RH problem $\tilde{M}'$ has the same form as in the case of zero background, refer to Ref. 18, $\tilde{M}' \approx I$ as $t \to -\infty$. Owing to this, we can give the leading term of $u(x, t)$ as $t \to -\infty$ before formulating the more precise asymptotics. Indeed, $\tilde{M}' \approx I$ as $t \to -\infty$ reveals $(g_1(x, t), g_2(x, t))^T \approx (i\kappa, -c_1(x, t))^T \approx (i\kappa, 0)^T$ as well as $(h_1(x, t), h_2(x, t))^T \approx (c_0(\xi), i\kappa)^T$. Thus, for $x > 0$, we have
\[ u(x, t) \approx -2\kappa P_{12}(x, t) \approx \frac{2i\kappa^2 c_0(\xi)}{\kappa^2 - c_1(x, t)c_0(\xi)} \approx \frac{2ic_0(\xi)}{t \to -\infty} A\delta^2(0, \xi), \tag{107} \]
which is the leading term of $u(x, t)$ as $t \to -\infty, x > 0$.

Similarly, for $x < 0, t \to +\infty$, we have $(g_1(-x, -t), g_2(-x, -t))^T \approx (i\kappa, -c_1(-x, -t))^T \approx (i\kappa, 0)^T$ as well as $(h_1(-x, -t), h_2(-x, -t))^T \approx (c_0(\xi), i\kappa)^T$
\[ u(x, t) \approx -2\kappa P_{21}(-x, -t) \approx \frac{-c_1(-x, -t)(-i\kappa)}{-\kappa^2 + c_0(\xi)c_1(-x, -t)} \approx 0, \tag{108} \]
which is the leading term of $u(x, t)$ as $t \to +\infty, x < 0$.

\]

4.4 Local parametrix near saddle points

In this subsection, our goal is to use the parabolic cylinder functions to present a good approximation of $\tilde{M}'(x, t, k)$ locally around the saddle points $-k_0$ and $k_0$. We use $\tilde{M}_{-k_0}'$ and $\tilde{M}_{k_0}'$ to express the local parametrix near $-k_0$ and $k_0$, respectively. To be brief, we only exhibit the details for $\tilde{M}_{-k_0}'$, then use symmetry relation to obtain the information of the local neighborhood of $\tilde{M}_{k_0}'$. The methods we use in this subsection mainly refer to some works for zero boundary conditions.18,24,31

Denote $U_\varepsilon(-k_0)$ and $U_\varepsilon(k_0)$ as the open disc of radius $\varepsilon$ around $-k_0$ and $k_0$, respectively. Define $\varepsilon$
\[ \varepsilon := \min \left\{ \eta := \frac{k_0}{2}, \quad \frac{1}{2}|i\kappa + k_0| \right\}, \tag{109} \]
which make the 0 and $i\kappa$ do not belong to the neighborhood of $U_\varepsilon(-k_0)$ and $U_\varepsilon(k_0)$. 

\[ \frac{\kappa^2 - c_1(x, t)c_0(\xi)}{k - i\kappa} \approx \frac{2ic_0(\xi)}{t \to -\infty} A\delta^2(0, \xi), \tag{107} \]
And define the contours as follows (see Figure 9):

\[ \Gamma_{k_0, \epsilon} := \Gamma \cap U_{\epsilon}(k_0) = \Gamma_{j, \epsilon} \cup \Gamma_{j, \epsilon}^*, \quad j = 1, 2, \quad (110a) \]

\[ \Gamma_{-k_0, \epsilon} := \Gamma \cap U_{\epsilon}(-k_0) = \Gamma_{j, \epsilon} \cup \Gamma_{j, \epsilon}^*, \quad j = 3, 4, \quad (110b) \]

\[ \Gamma_{\epsilon} := \Gamma_{k_0, \epsilon} \cup \Gamma_{-k_0, \epsilon}. \quad (110c) \]

Here, we consider \( r_j(-k_0) \neq 0, j = 1, 2 \). If one of the \( r_j(-k_0), j = 1, 2 \) equals to 0 or both \( r_j(-k_0), j = 1, 2 \) equal to 0, we have \( \nu(-k_0) = 0 \). For this case, we suggest the interested readers to refer [14], section 1.6, chapter 2]. Additionally, we notice that \( \tilde{M}'(x, t, k) \) is similar to the RH problem [18] under zero boundary condition, so we follow the idea of Ref. 18 to complete the rest asymptotic analysis.

At first, we define some quantities, which are convenient for our expressions in this subsection.

\[ \eta := \frac{k_0}{2}, \quad \rho = \eta \sqrt{48 k_0}, \quad \tau := -t \rho^2 = -12 t k_0^3 > 0, \quad \nu := \nu(-k_0). \quad (111) \]

And

\[ \varphi(\xi; \zeta) := 2i \theta \left( \xi, -k_0 + \frac{\eta}{\rho} \right) = 16 i k_0 k^3 - \frac{i}{2} \rho^2 + \frac{i k^3}{12 \rho}. \quad (112) \]

Remark 13. We can see that: when \( t \to -\infty, \tau \to +\infty \). However, in Ref. 18, we have \( \tau := t \rho^2 = 12 t k_0^3 \) for \( t > 0, \tau \to +\infty \) as \( t \to +\infty \).

For \( k \) near \(-k_0\), we have

\[ \theta(k, \xi) = 8k_0^3 - 12k_0(k + k_0)^2 + 4(k + k_0)^3. \quad (113) \]
Thus, for \( k \in U_\varepsilon(-k_0) \), we define the rescaled variable \( \zeta \) by

\[
k = \frac{\zeta}{\sqrt{-48tk_0}} - k_0 = \frac{\eta}{\sqrt{\varepsilon}} \zeta - k_0
\]

(correspondingly, for \( k \in U_\varepsilon(k_0) \), \( k = \frac{\zeta}{\sqrt{-48tk_0}} + k_0 \)). And the scaling operator \( N_{-k_0} \) admits the mapping

\[
f(\zeta) \mapsto (N_{-k_0}f)(\zeta) = f\left(\frac{\zeta}{\sqrt{-48tk_0}} - k_0\right).
\]

Now we match the \( q_j^r(-k_0) \), \( j = 1, 2 \) that appeared in (B.2). For \( k \in \Gamma \cap U_\varepsilon(-k_0) \), for instance, \( k \in \Gamma_{3,\varepsilon} \), using formulas (80) and (114), we have

\[
\delta^{-2}r^r_1(k)e^{2it\theta} \big|_{k=-k_0} = e^{16itk_0^3 - \frac{\Gamma(\zeta)}{4}(144\tau)^{\frac{1}{2}}} e^{-2\chi(\zeta, k) \tau} e^{2i\gamma(\zeta, -k_0)\tau} e^{\frac{i}{2}\zeta}\zeta^{2i\nu(-k_0)}.
\]

Comparing (116) to the first jump of (B.2), we obtain

\[
q^r_1(-k_0) = e^{-2\chi(\zeta, -k_0)} r^r_1(-k_0) e^{2i\nu \log 4}.
\]

Similarly, we derive

\[
q^r_2(-k_0) = e^{2\chi(\zeta, -k_0)} r^r_2(-k_0) e^{-2i\nu \log 4}.
\]

The local parametrix \( \tilde{M}^r_{-k_0} \) is determined by

\[
\tilde{M}^r_{-k_0}(x, t, k) = \Xi(\zeta, t) m_{-k_0}^{pc}(\zeta, \zeta(k) = \sqrt{-48tk_0(k + k_0)}) \Xi^{-1}(x, t),
\]

where

\[
\Xi(\zeta, t) = e^{-\frac{1}{2}i\zeta\theta(\zeta, 0)\sigma_3} \cdot \tau - \frac{iv(-k_0)\sigma_3}{2}.
\]

The RH problem \( m_{-k_0}^{pc}(\zeta, \zeta) \) in (119) can be solved explicitly, in terms of parabolic cylinder functions (see Appendix B) and admits the following asymptotic behavior as \( \zeta \to \infty \) :

\[
m_{-k_0}^{pc}(\zeta, \zeta(k)) = I + \frac{i}{\zeta} \begin{pmatrix} 0 & -\beta'(\zeta) \\ -\gamma'(\zeta) & 0 \end{pmatrix} + O\left(\frac{1}{\zeta^2}\right), \quad \zeta \to \infty,
\]

where

\[
\beta'(\zeta) = \frac{\sqrt{2\pi e^{\frac{i\pi}{4}} e^{-\frac{\pi\nu(-k_0)}{2}}}}{q^r_1(-k_0)\Gamma(-iv(-k_0))}, \quad \gamma'(\zeta) = \frac{\sqrt{2\pi e^{\frac{i\pi}{4}} e^{-\frac{\pi\nu(-k_0)}{2}}}}{q^r_2(-k_0)\Gamma(iv(-k_0))}.
\]

By the symmetry condition (102), we extend the local parametrix to a neighborhood of \( k_0 \) by

\[
\tilde{M}^r_{k_0}(x, t, k) = \tilde{M}^r_{-k_0}(x, t, -k), \quad |k - k_0| < \varepsilon.
\]
4.5 Error analysis

Having constructed the local parametrix $\tilde{M}_{\pm k_0}$, then we define the so-called error matrix as follows:

$$E(x, t, k) = \begin{cases} 
\tilde{M}_r(x, t, k) \left( \tilde{M}_{-k_0}^{-1} \right)(x, t, k), & k \in U_e(-k_0) = \{ k : |k + k_0| < \varepsilon \}, \\
\tilde{M}_r(x, t, k) \left( \tilde{M}_{k_0}^{-1} \right)(x, t, k), & k \in U_e(k_0) = \{ k : |k - k_0| < \varepsilon \}, \\
\tilde{M}_r(x, t, k), & \text{elsewhere.}
\end{cases} \quad (124)$$

The jump contours of $E(x, t, k)$ is depicted as Figure 10.

And $E(x, t, k)$ satisfies the RH problem as follows:

**RH problem 5.** Find a $2 \times 2$ matrix-valued function $E(x, t, k)$ such that

(i) $E(x, t, k)$ is analytic for $k \in \mathbb{C} \setminus \Gamma_E$, where $\Gamma_E : = \Gamma \cup \partial U_e(-k_0) \cup \partial U_e(k_0)$.
(ii) $E(x, t, k)$ takes continuous boundary conditions $E_+(x, t, k) = E_-(x, t, k)J_E(x, t, k)$ with

$$J_E(x, t, k) = \begin{cases} 
\tilde{M}_{-k_0}^r(x, t, k)J_r(x, t, k) \left( \tilde{M}_{-k_0}^r \right)^{-1}(x, t, k), & k \in \Gamma \cap U_e(-k_0) = \Gamma_{-k_0, e}, \\
\tilde{M}_{k_0}^r(x, t, k)J_r(x, t, k) \left( \tilde{M}_{k_0}^r \right)^{-1}(x, t, k), & k \in \Gamma \cap U_e(k_0) = \Gamma_{k_0, e}, \\
\left( \tilde{M}_{-k_0}^r \right)^{-1}(x, t, k), & k \in \partial U_e(-k_0), \\
\left( \tilde{M}_{k_0}^r \right)^{-1}(x, t, k), & k \in \partial U_e(k_0), \\
J_r(x, t, k), & \Gamma \setminus \Gamma_e. 
\end{cases} \quad (125)$$

(iii) $E(x, t, k) = I + O(k^{-1})$, as $k \to \infty$.

Define $w(x, t, k) := J_E(x, t, k) - I$. By (101) and (102), we know that $w(x, t, k)$ admits the symmetry

$$w(x, t, k) = w(x, t, -\bar{k}). \quad (126)$$

The next lemma shows that some estimates for $k \in \Gamma \setminus \Gamma_e$, $k \in \Gamma_e$, and $k \in \partial U_e(\pm k_0)$, respectively.
Lemma 1. As $\tau \to \infty$, $w(x, t, k)$ admits some estimates as follows:

(i) For $k \in \Gamma \setminus \Gamma_{\epsilon}$, as $\tau \to \infty$

\[
\|w(x, t, \cdot)\|_{L^p(\Gamma \setminus \Gamma_{\epsilon})} = O(\epsilon^{p/2} \tau^{-1}), \quad p = 1, 2,
\]

(ii) For $k \in \Gamma_{\epsilon}$, as $\tau \to \infty$

\[
\|w(x, t, \cdot)\|_{L^{\infty}(\Gamma \setminus \Gamma_{\epsilon})} = O(\tau^{-1}).
\]

(iii) For $k \in \partial U_{\epsilon}(-k_0)$, as $\tau \to \infty$

\[
\frac{w(x, t, k)}{(M^r_{-k_0})^{-1}(x, t, k) - I} = \mathcal{B}_0(\xi, t) + O\left(\frac{1}{\tau} + \frac{1}{\tau^{1/2}}\right),
\]

where $\alpha \in (1/2, 1)$.

Proof. For (127) and (128), we can follow the proof of [18, Lemma 2] and [18, Lemma 4], respectively. Now we turn to the proof of (129).

Recalling the $\Xi(\xi, t)$ defined by (120), we obtain that

\[
\Xi(\xi, t) = O\left(\frac{\tau^{-1}}{\tau^{1/2}}\right), \quad \tau \to \infty.
\]

For $k \in \partial U_{\epsilon}(-k_0)$, as $\tau \to \infty$

\[
w = \left(M^r_{-k_0}\right)^{-1}(x, t, k) - I
\]

\[
\Xi(\xi, t)\left(m_{-k_0}\right)^{-1}\left(\frac{\sqrt{\tau}}{\eta}(k + k_0) - I\right)\Xi^{-1}(\xi, t)
\]

\[
\Xi(\xi, t)\left(I - \frac{i\eta}{\sqrt{\tau}(k + k_0)}\right)\left(\frac{\beta'(\xi)}{\eta^r(\xi)}\right) + O(\tau^{-1}) - I\Xi^{-1}(\xi, t).
\]

Using (120) and (131), we arrive at (129).

Using Lemma 1 and closely following [24, Lemma 2.6] or [18, eq. (99)], we have the proposition as follows:

Proposition 9. As $\tau \to \infty$,
\[
\|w(x, t, \cdot)\|_{L^2(\Gamma_E)} = O\left(\frac{1}{\varepsilon^2 \tau} - \frac{\alpha}{2} + |\text{Im} \nu(-k_0)|\right),
\]
(133a)

\[
\|w(x, t, \cdot)\|_{L^\infty(\Gamma_E)} = O\left(\tau^{-\frac{\alpha}{2}} + |\text{Im} \nu(-k_0)|\right),
\]
(133b)

\[
\|w(x, t, \cdot)\|_{L^p(\Gamma_\varepsilon)} = O\left(\frac{1}{\varepsilon^2 \tau^{\frac{1}{2p}}} - \frac{\alpha}{2} + |\text{Im} \nu(-k_0)|\right),
\]
(133c)

where \( p \in [1, \infty) \). Moreover, recall that \( w^{(j)} \) denotes the \( j \)-th of matrix \( w \), we have

\[
\|w^{(j)}(x, t, \cdot)\|_{L^2(\Gamma_E)} = O\left(\frac{1}{\varepsilon^2 \tau} - \frac{\alpha}{2} + (-1)^j \text{Im} \nu(-k_0)\right),
\]
(134a)

\[
\|w^{(j)}(x, t, \cdot)\|_{L^\infty(\Gamma_E)} = O\left(\tau^{-\frac{\alpha}{2}} + (-1)^j \text{Im} \nu(-k_0)\right),
\]
(134b)

\[
\|w^{(j)}(x, t, \cdot)\|_{L^p(\Gamma_\varepsilon)} = O\left(\frac{1}{\varepsilon^2 \tau^{\frac{1}{2p}}} - \frac{\alpha}{2} + (-1)^j \text{Im} \nu(-k_0)\right),
\]
(134c)

for \( j = 1, 2 \).

Define the Cauchy-type operator \( C_w : L^2(\Gamma_E) + L^\infty(\Gamma_E) \to L^2(\Gamma_E) \) by \( C_w f = C_-(f w) \), where \( (C_- f)(k), k \in \Gamma_E \) is the negative (according to the orientation of \( \Gamma_E \)) nontangential boundary value of

\[
(C f)(k') := \frac{1}{2\pi i} \int_{\Gamma_E} \frac{f(s)}{s-k'} ds, \quad k' \in C \setminus \Gamma_E.
\]
(135)

Since \( C_- \) is an operator of \( L^2(\Gamma_E) \to L^2(\Gamma_E) \), we obtain that

\[
\|C_w\| \leq \text{Const.} \|w\|_{L^\infty(\Gamma_E)} \quad \text{(133b)} = O\left(\tau^{-\frac{\alpha}{2}} + |\text{Im} \nu(-k_0)|\right), \quad \tau \to \infty.
\]
(136)

Then, by \( \alpha \in (1/2, 1) \), \( \|C_w\| \) decays to zero as \( \tau \to \infty \). Hence we conclude that \( I - C_w \) is invertible for large \( \tau \).

By classical Beals–Coifman theory,\(^6\) the \( E(x, t, k) \) could be expressed in terms of a singular integral equation, which is dependent on \( w := J_E - I \) and the standard normalization condition \( E(x, t, k) \to I \) as \( k \to \infty \), that is,

\[
E(x, t, k) = I + C(\mu w) = I + \frac{1}{2\pi i} \int_{\Gamma_E} \mu(x, t, s)w(x, t, s) \frac{ds}{s-k'},
\]
(137)

where \( \mu \) is the solution of the Fredholm-type equation \( (I - C_w)\mu = I \). Moreover, by (133a) and \( \mu - I = (I - C_w)^{-1}C_w I \), we have

\[
\|\mu - I\|_{L^2(\Gamma_E)} = O\left(\frac{1}{\varepsilon^2 \tau} - \frac{\alpha}{2} + |\text{Im} \nu(-k_0)|\right), \quad \tau \to \infty.
\]
(138)
Lemma 2. As $\tau \to \infty$,

$$
\lim_{k \to \infty} k(E(x, t, k) - I) = \mathcal{B}^r(\xi, t) - \overline{\mathcal{B}^r(\xi, t)} + R(\xi, t),
$$

(139)

where

$$
\mathcal{B}^r(\xi, t) = \begin{pmatrix}
0 & i\eta \beta^r(\xi)e^{-i\varphi(\xi, 0)}e^{-i\nu(-k_0)\frac{1}{2}} \\
-i\eta \gamma^r(\xi)e^{i\varphi(\xi, 0)}e^{i\nu(-k_0)\frac{1}{2}} & 0
\end{pmatrix},
$$

(140)

and $R(\xi, t) = \hat{R}_1(\xi, t) + \hat{R}_2(\xi, t) + \hat{R}_3(\xi, t)$,

$$
\hat{R}_1(\xi, t) = \left( O(\tau^{-1} - \text{Im }\nu(-k_0)) \quad O(\tau^{-1} + \text{Im }\nu(-k_0)) \right), \quad \tau \to \infty,
$$

(141a)

$$
\hat{R}_2(\xi, t) = \left( O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + |\text{Im }\nu(-k_0)| - |\text{Im }\nu(-k_0)|\right) \quad O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + 2|\text{Im }\nu(-k_0)|\right) \right), \quad \tau \to \infty.
$$

(141b)

$$
\hat{R}_3(\xi, t) = \left( O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + |\text{Im }\nu(-k_0)| - |\text{Im }\nu(-k_0)|\right) \quad O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + |\text{Im }\nu(-k_0)| + |\text{Im }\nu(-k_0)|\right) \right), \quad \tau \to \infty.
$$

(141c)

Moreover, recalling $\alpha < 1$, we have

$$
R(\xi, t) := \left( O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + |\text{Im }\nu(-k_0)| - |\text{Im }\nu(-k_0)|\right) \quad O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + |\text{Im }\nu(-k_0)| + |\text{Im }\nu(-k_0)|\right) \right), \quad \tau \to \infty.
$$

(142)

In detail, we set $R(\xi, t) = \begin{pmatrix} R_1(\xi, t) & R_2(\xi, t) \\ R_1(\xi, t) & R_2(\xi, t) \end{pmatrix}$, and

$$
R_1(\xi, t) = \begin{cases} O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}}\right), & \text{Im }\nu(-k_0) \geq 0 \\
O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + 2|\text{Im }\nu(-k_0)|\right), & \text{Im }\nu(-k_0) < 0
\end{cases}
$$

(143a)

$$
R_2(\xi, t) = \begin{cases} O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}} + 2|\text{Im }\nu(-k_0)|\right), & \text{Im }\nu(-k_0) \geq 0 \\
O\left(\varepsilon \tau^{-\frac{1+\alpha}{2}}\right), & \text{Im }\nu(-k_0) < 0
\end{cases}
$$

(143b)

which also hold true for $\hat{R}_2^{(j)}(\xi, t)$ and $\hat{R}_3^{(j)}(\xi, t)$, $j = 1, 2$.

Proof. Start from (137)

$$
\lim_{k \to \infty} k(E(x, t, k) - I) = -\frac{1}{2\pi i} \int_{\Gamma_E} \mu(x, t, s)w(x, t, s)ds
$$

$$
= -\frac{1}{2\pi i} \left( \oint_{|s+k_0|=\varepsilon} + \oint_{|s-k_0|=\varepsilon} \right) \mu(x, t, s)w(x, t, s)ds - \frac{1}{2\pi i} \int_{\Gamma} \mu(x, t, s)w(x, t, s)ds.
$$

(144)
For the first term of (144), as \( \tau \to \infty \),
\[
\oint_{|s+k_0|=\varepsilon} \mu(x, t, s)w(x, t, s)ds = \oint_{|s+k_0|=\varepsilon} w(x, t, s)ds + \oint_{|s+k_0|=\varepsilon} (\mu(x, t, s) - I)w(x, t, s)ds
\]
\[
= \frac{\mathfrak{B}_0(\xi, t)}{\sqrt{\tau}} \oint_{|s+k_0|=\varepsilon} \frac{1}{s+k_0}ds + \hat{R}_1(\xi, t) + \hat{R}_2(\xi, t)
\]
\[
= -2\pi i \mathfrak{B}'(\xi, t) + \hat{R}_1(\xi, t) + \hat{R}_2(\xi, t)
\]
(145)
and
\[
\oint_{|s-k_0|=\varepsilon} \mu(x, t, s)w(x, t, s)ds = \oint_{|s-k_0|=\varepsilon} w(x, t, s)ds + \oint_{|s-k_0|=\varepsilon} (\mu(x, t, s) - I)w(x, t, s)ds
\]
\[
= \oint_{|s+k_0|=\varepsilon} w(x, t, s)ds + \oint_{|s-k_0|=\varepsilon} (\mu(x, t, s) - I)w(x, t, s)ds
\]
\[
= \frac{\mathfrak{B}_0(\xi, t)}{\sqrt{\tau}} \oint_{|s+k_0|=\varepsilon} \frac{1}{s+k_0}ds + \hat{R}_1(\xi, t) + \hat{R}_2(\xi, t)
\]
\[
= 2\pi i \mathfrak{B}'(\xi, t) + \hat{R}_1(\xi, t) + \hat{R}_2(\xi, t),
\]
(146)
where
\[
\mathfrak{B}'(\xi, t) = -\frac{\mathfrak{B}_0}{\sqrt{\tau}} = \begin{pmatrix} 0 & -i\eta \beta^\prime(\xi)e^{-t\varphi(\xi, 0)}e^{-1/2} \\ i\eta \gamma^\prime(\xi)e^{t\varphi(\xi, 0)}e^{-1/2} & 0 \end{pmatrix}.
\]
Moreover, we obtain
\[
\hat{R}_2(\xi, t) = \|\mu - I\|_{L^2(\partial U_x(-k_0))} O(|\mathfrak{B}'(\xi, t)|)
\]
\[
= O\left(\frac{1}{\tau^{3/4} + |\text{Im} \nu(-k_0)|}\right) \cdot O\left(\frac{1}{\tau^{3/4} - 1/2 + \text{Im} \nu(-k_0)}\right)
\]
\[
= O\left(\tau^{-1/2} + |\text{Im} \nu(-k_0)| - 1/2 + \text{Im} \nu(-k_0)\right).
\]
(147)
For the \( j \)-th column of the second term in (144), for instance, we consider the second column
\[
\left| \left( \int_{\Gamma} \mu(x, t, s)w(x, t, s) \right)^{(2)} ds \right| \leq \left| \int_{\Gamma} (\mu(x, t, s) - I)w^{(2)}(x, t, s) ds \right| + \left| \int_{\Gamma} w^{(2)}(x, t, s) ds \right|
\]
\[
\leq \|\mu - I\|_{L^2(\Gamma)} \|w^{(2)}\|_{L^2(\Gamma)} + \|w^{(2)}\|_{L^1(\Gamma)}.
\]
(148)
Recalling (127a), (134c), we have
\[
\|w^{(2)}\|_{L^1(\Gamma)} = O\left(\tau^{-1} + \tau^{-1/2} + \text{Im} \nu(-k_0)\right),
\]
(149a)
\[
\|w^{(2)}\|_{L^2(\Gamma)} = O\left(\varepsilon^\frac{1}{2} \tau^{-1} + \varepsilon^\frac{1}{2} \tau^{-\frac{1}{4}} - \frac{\alpha}{2} + \text{Im} \nu(-k_0)\right),
\]

then take into account \(138\) and obtain
\[
\left| \int_\Gamma (\mu(x, t, s)w(x, t, s))^{(2)} ds \right| = O\left(\varepsilon \tau^{-\frac{1}{4}} + \frac{\alpha}{2} + \text{Im} \nu(-k_0)\right).
\]

Similarly, we have
\[
\left| \int_\Gamma (\mu(x, t, s)w(x, t, s))^{(1)} ds \right| = O\left(\varepsilon \tau^{-\frac{1}{4}} + \frac{\alpha}{2} - \text{Im} \nu(-k_0)\right).
\]

As a consequence, we know that
\[
\int_\Gamma \mu(x, t, s)w(x, t, s) ds = O\left(\varepsilon \tau^{-\frac{1}{4}} + \frac{\alpha}{2} + \text{Im} \nu(-k_0)\right) = : \hat{R}_3(\xi, t), \quad \tau \to \infty.
\]

Summarizing \((144)–(151)\), we complete the proof of this lemma.

\[\square\]

4.6 | Long-time asymptotics

To recover the potential \(u(x, t)\), we mainly refer to [33, p. 722]. From Lemma 2, we see that the main term in the large negative \(t\) development of \(E\) in \((137)\) is given by the integral along the circle \(|s + k_0| = \varepsilon\) and \(|s - k_0| = \varepsilon\), which in turn gives

\[
E(x, t, k) = I - \frac{1}{2\pi i} \int_{|s + k_0| = \varepsilon} \frac{\mathcal{B}'(\xi, t)}{(s + k_0)(s - k)} ds + \frac{1}{2\pi i} \int_{|s - k_0| = \varepsilon} \frac{\overline{\mathcal{B}'}(\xi, t)}{(s - k_0)(s - k)} ds + R(\xi, t),
\]

\(|k \pm k_0| > \varepsilon, \quad \tau \to \infty. \quad (152)\)

By \((124)\), \(E(x, t, k) = \tilde{M}'(x, t, k)\) for all \(k : |k \pm k_0| > \varepsilon\), thus we have
\[
\lim_{k \to \infty} k(M'(x, t, k) - I) = \lim_{k \to \infty} k(E(x, t, k) - I) = \mathcal{B}'(\xi, t) - \overline{\mathcal{B}'}(\xi, t) + R(\xi, t). \quad (153)
\]

Using residue theorem, we additionally have
\[
\tilde{M}'(x, t, 0) = I + \frac{\mathcal{B}'(\xi, t)}{k_0} + \frac{\overline{\mathcal{B}'}(\xi, t)}{k_0} + R(\xi, t), \quad (154a)
\]

\[
\tilde{M}'(x, t, i\kappa) = I + \frac{\mathcal{B}'(\xi, t)}{k_0 + i\kappa} + \frac{\overline{\mathcal{B}'}(\xi, t)}{k_0 - i\kappa} + R(\xi, t). \quad (154b)
\]
Now we turn to evaluate $P_{12}(x, t)$ as well as $P_{21}(x, t)$ in (103). First, we consider $g(x, t) = (g_1(x, t), g_2(x, t))^T$ and $h(x, t) = (h_1(x, t), h_2(x, t))^T$ defined by (100). Using (154), we obtain that

$$\begin{align*}
g_1(x, t) &= i\kappa + R_1(\xi, t), \\
g_2(x, t) &= i\kappa \left( \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 + i\kappa} + \frac{\mathcal{Y}_{12}(\xi, t)}{k_0 - i\kappa} \right) + R_1(\xi, t),
\end{align*}$$

and

$$\begin{align*}
h_1(x, t) &= c_0(\xi) + \frac{i\kappa}{k_0} \left( \mathcal{Y}_{12}(\xi, t) + \mathcal{Y}_{21}(\xi, t) \right) + R_3(\xi, t), \\
h_2(x, t) &= i\kappa \left( \frac{\mathcal{Y}_{12}(\xi, t)}{k_0 + i\kappa} + \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 - i\kappa} \right) + R_3(\xi, t),
\end{align*}$$

where we have used the facts that $\mathcal{Y}_{11}(\xi, t) = \mathcal{Y}_{22}(\xi, t) = 0$, the algebraic decay term taking up more dominant than exponential decay term when $\tau \to \infty$, as well as $R_3(\xi, t) = R_1(\xi, t) + R_2(\xi, t)$ with

$$R_3(\xi, t) = O(\varepsilon \tau^{-\frac{1+\alpha}{2}} + 2|\text{Im} \nu| \tau).$$

Furthermore, we obtain

$$\begin{align*}
g_1 h_1 &= i\kappa c_0(\xi) - \kappa^2 \frac{c_0(\xi)}{k_0} \left( \mathcal{Y}_{12}(\xi, t) + \mathcal{Y}_{21}(\xi, t) \right) + R_3(\xi, t), \\
g_1 h_2 &= -\kappa^2 + i\kappa c_0(\xi) \left( \mathcal{Y}_{12}(\xi, t) + \mathcal{Y}_{21}(\xi, t) \right) + R_3(\xi, t), \\
g_2 h_1 &= i\kappa c_0(\xi) \left( \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 + i\kappa} + \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 - i\kappa} \right) + R_1(\xi, t), \\
g_2 h_2 &= -\kappa^2 \left( \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 + i\kappa} + \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 - i\kappa} \right) + R_1(\xi, t).
\end{align*}$$

Substituting (157) into (99), we have

$$\begin{align*}
P_{12}(x, t) &= -\frac{i\kappa}{\kappa} + \frac{1}{k_0} \left( \mathcal{Y}_{12}(\xi, t) + \mathcal{Y}_{21}(\xi, t) \right) + \frac{ic_0^2(\xi)}{k_0 \kappa} \left( \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 + i\kappa} + \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 - i\kappa} \right) + R_3(\xi, t),
\end{align*}$$

and

$$\begin{align*}
P_{21}(x, t) &= -\mathcal{Y}_{21}(\xi, t) + \frac{\mathcal{Y}_{21}(\xi, t)}{k_0 + i\kappa} + R_1(\xi, t).
\end{align*}$$

We can see that (158) involves $\kappa$, then we should introduce the following matrix $\mathfrak{B}$, which can help us make those terms which are dependent on $\kappa$ vanish in the main asymptotic terms.
Introduce
\[
\mathfrak{B}(\xi, t) = \begin{pmatrix}
0 & -\eta \beta(\xi) e^{-t\varphi(\xi, 0)} r_i(0) e_{\varphi(-k_0) - \frac{1}{2}} \\
\eta \gamma(\xi) e^{i\varphi(\xi, 0)} e^{i\varphi(-k_0) - \frac{1}{2}} & \frac{i}{2} \gamma(\xi) e^{2i\varphi(-k_0)}
\end{pmatrix},
\]
and we can see that \(\mathfrak{B}(\xi, t)\) is defined similarly to \(\mathfrak{B}_r(\xi, t)\), see (140), with \(q_j(0)\) being replaced by \(q_j(-k_0)\) (i.e., \(r_j(0)\) is replaced by \(r_j(-k_0)\)). In detail,
\[
\beta(\xi) = \sqrt{2\pi e^{-\frac{\pi^2}{4} \xi}} e^{-2\varphi(\xi, -k_0)} r_1(0) e^{2i\varphi(-k_0)} \log^4,
\]
\[
\gamma(\xi) = \sqrt{2\pi e^{-\frac{\pi^2}{4} \xi}} e^{2\varphi(\xi, -k_0)} r_2(0) e^{-2i\varphi(-k_0)} \log^4.
\]
Comparing to (122), and taking into account (98), we obtain
\[
\beta_r(\xi) = \frac{k_0}{k_0 + i\kappa} \beta(\xi), \quad \gamma_r(\xi) = \frac{k_0 + i\kappa}{k_0} \gamma(\xi),
\]
which implies that
\[
\mathfrak{B}_{12}(\xi, t) = \frac{k_0}{k_0 + i\kappa} \mathfrak{B}_{12}(\xi, t), \quad \mathfrak{B}_{21}(\xi, t) = \frac{k_0 + i\kappa}{k_0} \mathfrak{B}_{21}(\xi, t).
\]
Substituting (158), (162), and (153) into (103), we finally obtain the asymptotics, which are described by Theorem 1.

5 \quad ASYMPTOTIC BEHAVIOR FOR \(\xi := x/(12t) > 0\)

In this section, we investigate the long-time asymptotics under the condition \(\xi > 0\), and the signature table of this case corresponds to Figure 5 with
\[
\text{Im} \vartheta(k, \xi) = 4 \text{Im} k \left(3(\text{Re} k)^2 - (\text{Im} k)^2 + 3\xi^2\right).
\]
Comparing to Section 4, we point out that the analysis for \(t \to +\infty\) is more convenient to deal with the singularities of the RH problem at \(k = 0\) in this case of \(\xi > 0\).

5.1 \quad Asymptotic analysis for \(0 < \xi < \kappa^2\)

In this case, \(i\sqrt{\xi} \leq i\sqrt{3\xi} < i\kappa\) for \(\xi \in (0, \frac{\kappa^2}{3})\) and \(i\sqrt{\xi} < i\kappa < i\sqrt{3\xi}\) for \(\xi \in (\frac{\kappa^2}{3}, \kappa^2)\). Our technique is to deform the information of jump contour on \(\mathbb{R}\) to other contours off real axis. Define two contours \(\Gamma_1 = \{k : k = k_1 + i\sqrt{\xi}\}, \Gamma_1^* = \{k : k = k_1 - i\sqrt{\xi}\}\), which are parallel to \(\mathbb{R}\) and the domains \(U_1 = \{k : 0 < \text{Im} k < i\sqrt{\xi}\}, U_1^* = \{k : -i\sqrt{\xi} < \text{Im} k < 0\}\) (see Figure 11, Figure 12).
Next, we deform contours (see Figure 11 and Figure 12) via defining $\tilde{N}(x, t, k)$ by

$$
\tilde{N}(x, t, k) = \begin{cases} 
M(x, t, k) \begin{pmatrix} 1 & 0 \\ -r_1(k)e^{2i\theta} & 1 \end{pmatrix}, & k \in U_1, \\
M(x, t, k) \begin{pmatrix} 1 & r_2(k)e^{-2i\theta} \\ 0 & 1 \end{pmatrix}, & k \in U^{*}_1, \\
M(x, t, k), & \text{elsewhere.}
\end{cases}
$$

(164)

$\tilde{N}(x, t, k)$ satisfies the following RH problem:

**RH problem 6.** Find a $2 \times 2$ matrix-valued function $\tilde{N}(x, t, k)$ such that

(i) $\tilde{N}(x, t, k)$ is meromorphic for $k \in \mathbb{C} \setminus \{0\} \cup \Gamma_1 \cup \Gamma^{*}_1$ and has a simple pole located at $k = ix$, $\kappa > 0$. 

**FIGURE 11** Contours of RH problem $\tilde{N}(x, t, k)$ for $0 < \xi < \frac{\kappa^2}{3}$.

**FIGURE 12** Contours of RH problem $\tilde{N}(x, t, k)$ for $\frac{\kappa^2}{3} < \xi < \kappa^2$. 

\[ i \kappa, \quad i \sqrt{3} \xi, \quad k_0 = i \sqrt{\xi}, \quad \Gamma_1 \]

\[ U_1 \]

\[ U^{*}_1 \]

\[ -k_0 = -i \sqrt{\xi}, \quad -i \sqrt{3} \xi, \quad -i \kappa, \quad i \sqrt{3} \xi, \quad i \kappa, \quad k_0 = i \sqrt{\xi}, \quad \Gamma_1 \]

\[ U_1 \]

\[ U^{*}_1 \]

\[ -k_0 = -i \sqrt{\xi}, \quad -i \kappa, \quad -i \sqrt{3} \xi. \]
(ii) Jump conditions. The nontangential limits $\tilde{N}_\pm(x, t, k) = \lim_{{k' \to k, k' \in \mathbb{C}_\pm}} \tilde{N}(x, t, k')$ exist for $k \in \Gamma_1 \cup \Gamma_1^*$ and $\tilde{N}_\pm(x, t, k)$ satisfy the jump condition $\tilde{N}_+(x, t, k) = \tilde{N}_-(x, t, k) J_{\tilde{N}}(x, t, k)$ for $k \in \Gamma_1 \cup \Gamma_1^*$, where

$$J_{\tilde{N}}(x, t, k) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ r_1(k) e^{2i\theta} & 1 \end{pmatrix}, & k \in \Gamma_1, \\
\begin{pmatrix} 1 & r_2(k) e^{-2i\theta} \\ 0 & 1 \end{pmatrix}, & k \in \Gamma_1^*, \\
I, & k \in \mathbb{R} \setminus \{0\}.
\end{cases} \tag{165}$$

(iii) Normalization condition at $k = \infty$. $\tilde{N}(x, t, k) = I + O(k^{-1})$ as $k \to \infty$.

(iv) Residue condition at $k = ix$.

$$\text{Res}_{k=ix} \tilde{N}(x, t, k) = c_2(x, t) \tilde{N}^{(2)}(x, t, ix), \tag{166}$$

with $c_2(x, t) = \frac{\gamma_0}{a'(ix)} e^{-2x^3 + 8x^3 t} = \frac{\gamma_0}{a'(ix)} e^{(-24x^3 + 8x^3)}$, $\gamma_0^2 = 1$.

(v) Singularities at the origin. In both cases (Cases I and II), as $k \to 0$, $\tilde{N}(x, t, k)$ satisfies

$$\tilde{N}_+(x, t, k) = \begin{pmatrix} -2i \bar{v}_2(-x, -t) + O(k) - \frac{1}{k} \bar{v}_2(-x, -t) + O(1) \\
-2i \bar{v}_1(-x, -t) + O(k) - \frac{1}{k} \bar{v}_1(-x, -t) + O(1) \end{pmatrix}, \quad k \in \mathbb{C}_+, k \to 0, \tag{167a}$$

$$\tilde{N}_-(x, t, k) = \begin{pmatrix} -\bar{v}_2(-x, -t) + O(k) - \frac{A}{2ik} \bar{v}_2(-x, -t) + O(k) \\
-\bar{v}_1(-x, -t) + O(k) - \frac{A}{2ik} \bar{v}_1(-x, -t) + O(k) \end{pmatrix}, \quad k \in \mathbb{C}_-, k \to 0. \tag{167b}$$

Furthermore, we can see that the singularity conditions at the origin can be reduced to

$$\text{Res}_{k=0} \tilde{N}^{(2)}(x, t, k) = \frac{A}{2i} \tilde{N}^{(1)}(x, t, 0), \tag{167c}$$

Proof. The proof is similar to RH problem 3.

Still using the so-called Blaschke–Potapov factors and following the similar steps of Section 4.3, we can reduce $\tilde{N}(x, t, k)$ to a regular RH problem $\tilde{N}^r(x, t, k)$ as follows:

**RH problem 7.** Find a $2 \times 2$ matrix-valued function $\tilde{N}^r(x, t, k)$ such that

(i) $\tilde{N}^r(x, t, k)$ is analytic for $k \in \mathbb{C} \setminus (\Gamma_1 \cup \Gamma_1^*)$. 
(ii) $\tilde{N}_r^r(x, t, k) = \tilde{N}_r^r(x, t, k)J_{N_r^r}(x, t, k)$ with

$$J_{N_r^r}(x, t, k) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{k - ik}{k} \end{pmatrix} J_{N_r^r}(x, t, k) \begin{pmatrix} 1 & 0 \\ 0 & \frac{k}{k - ik} \end{pmatrix}. \quad (168)$$

More specifically,

$$J_{N_r^r}(x, t, k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_1'(k)e^{2it\theta} & 1 \end{pmatrix}, & k \in \Gamma_1, \\ \begin{pmatrix} 1 & 0 \\ r_2'(k)e^{-2it\theta} & 1 \end{pmatrix}, & k \in \Gamma_1^* \end{cases} \quad (169)$$

with

$$r_1'(k) = \frac{k - ik}{k} r_1(k), \quad r_2'(k) = \frac{k}{k - ik} r_2(k).$$

(iii) Normalization at $k = \infty$. $\tilde{N}_r^r(x, t, k) \to I$ as $k \to \infty$.

(iv) Matrix-valued factor $P(x, t)$ is determined in terms of $\tilde{N}_r^r(x, t, k)$ as (99) and (100) via replacing $c_1(x, t)$, $c_0(\xi)$ by $c_2(x, t)$, $A_{2i}$, respectively.

Then we have the potential recovering formulas:

$$u(x, t) = -2\kappa P_{12}(x, t) + 2i \lim_{k \to \infty} k\tilde{N}_{12}^r(x, t, k), \quad x > 0, \ t > 0, \quad (170a)$$

$$u(x, t) = -2\kappa P_{21}(-x, -t) + 2i \lim_{k \to \infty} k\tilde{N}_{21}^r(-x, -t, k), \quad x < 0, \ t < 0, \quad (170b)$$

where $P_{12}(x, t)$, $P_{21}(x, t)$ defined by the item (iv) of RH problem 4. For $0 < \xi < \frac{\kappa^2}{3}$, and $\frac{\kappa^2}{3} < \xi < \kappa^2$, different asymptotics are presented.

5.1.1 Asymptotics for $\frac{\kappa^2}{3} < \xi < \kappa^2$

Consider $\tilde{N}_r^r(x, t, k) \approx I$ for $t \to \infty$, we have

$$u(x, t) \approx -2\kappa P_{12}(x, t) \approx \frac{2i\kappa^2 \cdot A_{2i} \cdot \frac{A_{2i}}{2i}}{\kappa^2 - c_2(x, t) \cdot \frac{A_{2i}}{2i}} t \to +\infty A, \quad (171)$$

where we used the fact that $c_2(x, t) = \frac{\gamma_0}{a_1'(ix)} e^{-24\kappa^2 \xi + 8\kappa^3} \to 0$ as $t \to \infty$ for $x > 0$, $\xi \in (\frac{\kappa^2}{3}, \kappa^2)$. And $A$ is the leading term of $u(x, t)$ for $x > 0$, $t \to +\infty$. 


Similarly, we also have
\[
    u(x, t) \approx -2xP^{12}_{21}(-x, -t) \approx 2x \frac{-c_2(-x, -t)(-ix)}{-\kappa^2 + \frac{A}{2i}c_2(-x, -t)} \approx 0, \quad \kappa \to -\infty.
\]  
which is the leading term for \(x < 0, t \to -\infty\).

Using (163), we easily find that as \(t \to +\infty\),
\[
    J_N(x, t, k) = I + O(e^{-16t\xi^{3/2}}), \quad k \in \Gamma_1 \cup \Gamma^*_{\text{1}}, \quad t \to \infty.
\]  
Obeying the error analysis similar to Section 4.5 and using (170), we obtain the following asymptotics:
\[
    u(x, t) = A + O\left(\left(-t\right)^{-1/2}e^{-16t\xi^{3/2}}\right), \quad x > 0, \quad t > 0, \quad \frac{\kappa^2}{3} < \xi < \kappa^2.
\]
\[
    u(x, t) = O\left(\left(-t\right)^{-1/2}e^{16t\xi^{3/2}}\right), \quad x < 0, \quad t < 0, \quad \frac{\kappa^2}{3} < \xi < \kappa^2,
\]
which are the (9b), (10b) in Theorem 2, respectively.

### 5.1.2 Asymptotics for \(0 < \xi < \frac{\kappa^2}{3}\)

In this case, \(c_2(x, t)\) cannot decay 0 as \(t \to \infty\). For this reason, we make our asymptotics well-defined in the so-called solitonic region. Still noticing \(\overline{N}(x, t, k) \approx I\) for \(t \to \infty\), we have
\[
    u(x, t) \approx -2xP^{12}_{12}(x, t) \approx \frac{2ix^2 \cdot \frac{A}{2i}}{\kappa^2 - c_2(x, t) \cdot \frac{A}{2i}} \approx \frac{A}{1 - C_1(\kappa)e^{-2\kappa x + 8\kappa^3 t}},
\]
where \(C_1(\kappa) = \frac{A\gamma_0}{2ia'_1(\xi)c^2}\) with \(\gamma_0^2 = 1\).

Similarly, we also have
\[
    u(x, t) \approx -2xP^{12}_{21}(-x, -t) \approx (-2x) \frac{(-c_2(-x, -t)(ix))}{-\kappa^2 + \frac{A}{2i}c_2(-x, -t)} \approx \frac{4}{Ax^{-2} - C_2(\kappa)e^{-2\kappa x + 8\kappa^3 t}},
\]
where \(C_2(\kappa) = \frac{2ia'_1(\xi)}{\gamma_0}\) with \(\gamma_0^2 = 1\). And the radiation term is similar to the previous Section 5.1.1. Final asymptotics for this case are described by (9a) and (10a) in Theorem 2, respectively.

### 5.2 Asymptotic analysis for \(\xi > \kappa^2\)

In this case, \(ix < i\sqrt{\xi} < i\sqrt{\kappa^2}\), we introduce a new variable \(\kappa_6 \in (0, \kappa)\) and redefine contours \(\Gamma_1 = \{k : k = k_1 + i\kappa_6\},\ \Gamma^*_{\text{1}} = \{k : k = k_1 - i\kappa_6\}\) and two domains \(U_1 = \{k : 0 < \text{Im} \, k < i\kappa_6\},\ U^*_1 = \{k : -i\kappa_6 < \text{Im} \, k < 0\}\) (see Figure 13).
FIGURE 13  Contours of RH problem $\tilde{N}(x,t,k)$ for $\xi > \kappa^2$.

As the considerations in Section 5.1, using (163), we can find

$$ J_{\tilde{N}}(x,t,k) = I + O \left( e^{-8t\kappa_0(3\xi - \kappa^2)} \right), \quad k \in \Gamma_1 \cup \Gamma_1^*, \quad t \to \infty. \quad (177) $$

At last, we obtain the following asymptotics:

$$ u(x,t) = A + O \left( t^{-1} e^{-8t\kappa_0(3\xi - \kappa^2)} \right), \quad x > 0, \ t > 0, \ \xi > \kappa^2. \quad (178a) $$

$$ u(x,t) = O \left( (-t)^{-1} e^{8t\kappa_0(3\xi - \kappa^2)} \right), \quad x < 0, \ t < 0, \ \xi > \kappa^2, \quad (178b) $$

which are presented as (9c) and (10c) of Theorem 2, respectively.

6  CONCLUSIONS AND FURTHER DISCUSSIONS

In the present work, we mainly investigate the large-time asymptotics of solution for the Cauchy problem of the nonlinear focusing nonlocal MKdV equation with step-like initial data, that is, $u_0(x) \to 0$ as $x \to -\infty$, $u_0(x) \to A$ as $x \to +\infty$, where $A$ is an arbitrary positive real number. We first develop the direct scattering theory to establish the basic RH problem associated with step-like initial data. Thanks to the symmetries $x \to -x$, $t \to -t$ of nonlocal integrable systems, we investigate the asymptotics for $t \to -\infty$ and $t \to +\infty$, respectively. Our main technique is to use the steepest descent analysis to deform the original matrix-valued RH problem to a corresponded regular RH problem, which could be explicitly solved. Finally, we obtain the different long-time asymptotic behaviors of the solution of the Cauchy problem (2) for focusing nonlocal MKdV equation in different space-time regions $\mathcal{R}_I$, $\mathcal{R}_{II}$, $\mathcal{R}_{III}$, and $\mathcal{R}_{IV}$ on the whole $(x,t)$-plane. The subleading terms of the asymptotics in the sectors $\mathcal{R}_{II}$ and $\mathcal{R}_{IV}$ admit typical Zakharov–Manakov form. The subleading terms and error terms of the asymptotic formulas in $\mathcal{R}_{II}$ and $\mathcal{R}_{IV}$ depend on the value of $\text{Im} \nu(-k_0)$ (see Theorem 1). $\mathcal{R}_I$ can be divided into three regions, which are $\mathcal{R}_{I,R}$, $\mathcal{R}_{I,M}$, and $\mathcal{R}_{I,L}$. In particular, the $\mathcal{R}_{I,L}$ is the solitonic region. The asymptotics in $\mathcal{R}_{I,L}$ admit the exponential decay error term. Correspondingly, $\mathcal{R}_{III}$ can be also divided into similar three regions,
which are $R_{III,R}$, $R_{III,M}$, $R_{III,C}$. And they admit similar asymptotic types as the three regions in $R_I$ correspondingly (see Theorem 2).

In the present work, the nonlocal term $u(-x,-t)$ of integrable nonlocal MKdV is the great help to us. When we face the obstacles as $t \to +\infty$, we turn to analyze the asymptotics as $t \to -\infty$, see Section 4. In the same way, analysis as $t \to +\infty$ is more convenient than $t \to -\infty$ in Section 5.

Finally, we list some problems, which remain to be discussed in our forthcoming paper.

(i) How to directly analyze the asymptotics as $t \to +\infty$ in Section 4? In this case, main difficulties have been described in Remark 11. Besides, after opening lens, we have to deal with the singularities at $k = 0$ under the second jump factorization with $\frac{r_j}{1 + r_1 r_2}$ instead of $r_j$ when we directly consider $t \to +\infty$. Through our analysis, we can see that it is not easy to convert the singularity conditions of $k = 0$ into the residue condition (90c) in this case.

(ii) Asymptotics of transitions regions between $x > 0$ and $x < 0$ are still open. We hold the view that the main difficult comes from the origin point. For the case of $x > 0, \xi < 0$, we see that the leading term $A\delta^2(\xi,0)$ increasing oscillations as $-k_0 \to 0^-, k_0 \to 0^+$, respectively. We notice that in Ref. 35, Rybalko and Shepelsky give a good solvable result along the curved wedges in the long-time asymptotics for the integrable NNSL equation for this problem. We look forward to extending similar results to our forthcoming work for nonlocal MKdV equation.

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Data sharing is not applicable to this article as no data sets were generated or analyzed during this study.

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REFERENCES
1. Ablowitz MJ, Kaup DJ, Newell AC, Segur H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud Appl Math.* 1974;53:249-315.
2. Ablowitz MJ, Feng BF, Luo XD, Musslimani ZH. Reverse pace-time nonlocal Sine-Gordon/Sinh-Gordon equations with nonzero boundary conditions. *Stud Appl Math.* 2018;141:267-307.
3. Ablowitz M, Musslimani Z. Integrable nonlocal nonlinear Schrödinger equation. *Phys Rev Lett.* 2013;110:064105.
4. Ablowitz M, Musslimani Z. Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. *Nonlinearity.* 2016;29:915-946.
5. Ablowitz M, Musslimani Z. Integrable nonlocal nonlinear equations. *Stud Appl Math.* 2017;139:7-59.
6. Beals R, Coifman RR. Scattering and inverse scattering for first order systems. *Commun Pur Appl Math.* 1984;37:39-90.
7. Boutet de Monvel A, Lenells J, Shepelsky D. The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics. *Commun Math Phys.* 2021;383:893-952.
8. Boutet de Monvel A, Lenells J, Shepelsky D. The focusing NLS equation with step-like oscillating background: the genus 3 sector. *Commun Math Phys.* 2022;390:1081-1148.
9. Boutet de Monvel A, Kotlyrov VP, Shepelsky D. Focusing NLS equation: long-time dynamics of step-like initial data. *Int Math Res Notices.* 7:1613-1653. https://doi.org/10.1093/imrn/rnq129
10. Dieng M, McLaughlin KTR. Dispersive asymptotics for linear and integrable equations by the D-bar steepest descent method. In: Miller PD, Perry PA, Saut JC, Sulem C, eds. Nonlinear Dispersive Partial Differential Equations and Inverse Scattering Vol 83. Fields Inst. Commun. Springer; 2019:253-291.

11. Deift P, Zhou X. A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the mKdV equation. *Ann Math.* 1993;137:295-368.

12. Fokas AS. Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation. *Nonlinearity.* 2016;29:319-324.

13. Fromm S, Lenells J, Quirchmayr R. The defocusing nonlinear Schrödinger equation with step-like oscillatory initial data. arXiv: 2104.03714v1, 2021.

14. Fokas AS, Its AR, Kapaev AA, Yu NV. Painlevé Transcendents: The Riemann-Hilbert Approach. AMS; 2006.

15. Faddeev LD, Takhtajan LA. *Hamiltonian Methods in the Theory of Solitons.* Springer Series in Soviet Mathematics. Springer-Verlag; 1987.

16. Gurevich AV, Pitaevskii LP. Decay of initial discontinuity in the Kortweg-de Vries equation. *JETP Lett.* 1973;17/5:193.

17. Grava T, Minakov A. On the long-time asymptotic behavior of the modified Kortweg-de Vries equation with step-like initial data. *SIAM J Math Anal.* 2020;52:5892-5993.

18. He FJ, Fan EG, Xu J. Long-time asymptotics for the nonlocal mKdV equation. *Commun Theor Phys.* 2019;71:475-488.

19. Its AR. Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations. *Dokl Akad Nauk SSSR.* 1981;261(1):14-18.

20. Jenkins R. Regularization of a sharp shock by the defocusing nonlinear Schrödinger equation. *Nonlinearity.* 2015;28:2131-2180.

21. Ji JL, ZhuZN. On a nonlocal modified Korteweg-de Vries equation: integrability, Darboux transformation and soliton solutions. *Stud Appl Math.* 2017;42:699-708.

22. Ji JL, ZhuZN. Soliton solutions of an integrable nonlocal modified Korteweg-de Vries equation through inverse scattering transform. *J Math Anal Appl.* 2017;453:973-984.

23. Lou SY, Huang F. Alice-Bob physics: coherent solutions of nonlocal KdV systems. *Sci Rep.* 2017;7:869.

24. Lenells J. The nonlinear steepest descent method for Riemann-Hilbert problems of low regularity. *Indiana U Math J.* 2017;66:1287-1332.

25. McLaughlin KTR, Miller PD. The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying non-analytic weights. *Int Math Res Not.* 2006; Art. ID 48673.

26. McLaughlin KTR, Miller PD. The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights. *Int Math Res Not.* 2008; Art. ID 075.

27. Kotlyrov VP, Minakov A. Riemann-Hilbert problem to the modified Korteveg-de Vries equation: long-time dynamics of the steplike initial data. *J Math Phys.* 2010;51:093506.

28. Kotlyrov VP, Minakov A. Asymptotics of rarefaction wave solution to the mKdV equation. *J Math Phys Anal Geo.* 2011;17:59-86.

29. Minakov A. Long-time behavior of the solution to the mKdV equation with step-like initial data. *J Phys A: Math Theor.* 2011;44:085206.

30. Kotlyrov VP, Minakov A. Step-initial function to the MKdV equation: hyper-elliptic long time asymptotics of the solution. *J Math Phys Anal Geo.* 2012;18:38-62.

31. Rybalko Y, Shepelsky D. Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation. *J Math Phys.* 2019;60:031504.

32. Rybalko Y, Shepelsky D. Defocusing nonlocal nonlinear Schrödinger equation with step-like boundary conditions: long-time behavior for shifted initial data. *J Math Phys Anal Geom.* 2020;16(4):418-453.

33. Rybalko Y, Shepelsky D. Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation with step-like initial data. *J Diff Equ.* 2021;270:694-724.

34. Rybalko Y, Shepelsky D. Long-time asymptotics for the integrable nonlocal focusing nonlinear Schrödinger equation for a family of step-like initial data. *Comm Math Phys.* 2021;382:87-121.

35. Rybalko Y, Shepelsky D. Curved wedges in the long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation. *Stud Appl Math.* 2021;147:872-903.
36. Xu TY, Fan EG. On the Cauchy problem of defocusing mKdV equation: long-time asymptotics with step-like initial data. arXiv:2204.01299v1, 2022.

37. Zhou ZX. Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation. *Commun Nonlinear Sci Numer Simul.* 2018;62:480-488.

38. Zhou X, Fan EG. Long time asymptotics for the nonlocal mKdV equation with finite density initial data. *Phys D.* 2022;440:133458.

39. Zhou X, Fan EG. Long time asymptotic behavior for the nonlocal mKdV equation in space-time solitonic regions. *Math Phys Anal Geom.* 2023;26. https://doi.org/10.1007/s11040-023-09445-w

40. Zhang G, Yan Z. Inverse scattering transforms and soliton solutions of focusing and defocusing nonlocal mKdV equations with non-zero boundary conditions. *Phys D.* 2020;402:132170.

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**APPENDIX A: PROOF OF PROPOSITION 3**

**Case I: generic case**

Define functions $\tilde{a}_1(k)$ and $\tilde{a}_2(k)$ by

$$\tilde{a}_1(k) = a_1(k) \frac{k^2}{(k - i\xi)(k + i)}, \quad \tilde{a}_2(k) = a_2(k) \frac{k - i\xi}{k - i}. \quad (A.1)$$

Via the determinant relation (item (iv) in Proposition 2) and asymptotic of $a_j(k)$ as $k \to \infty$ (item (iii) in Proposition 2), we can construct the following scalar RH problem with respect to $\tilde{a}_j(k)$, $j = 1, 2$.

- $\tilde{a}_1(k)$ and $\tilde{a}_2(k)$ are analytic and have no zeros in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.
- $\tilde{a}_j(k)$, $j = 1, 2$ admit the jump condition

$$\tilde{a}_1(k)\tilde{a}_2(k) = \frac{k^2}{1 + k^2}(1 - b^2(k)), \quad k \in \mathbb{R}. \quad (A.2)$$

- $\tilde{a}_j(k) \to 1$ as $k \to \infty$.

The unique solution of the scalar RH problem is given by

$$\tilde{a}_1(k) = e^{\chi(k)}, \quad \tilde{a}_2(k) = e^{-\chi(k)},$$

where

$$\chi(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \frac{s^2}{1 + s^2}(1 - b^2(s)) \frac{ds}{s - k}. \quad (A.3)$$

Moreover, we have

$$a_1(k) = \frac{(k + i)(k - i\xi)}{k^2}e^{\chi(k)}, \quad a_2(k) = \frac{k - i\xi}{k - i}e^{-\chi(k)}. \quad (A.4)$$
Near \( k = 0 \), we obtain
\[
a_1(k) = \frac{\kappa}{k^2} e^{\chi(+i0)}(1 + o(k)), \quad a_2(0) = \frac{1}{k} e^{-\chi(-i0)},
\]
where \( \pm i0 \) represent that \( k \) tends to 0 from the positive side and negative side, respectively. By Sokhotski–Plemelj formulas, we have
\[
\chi(+i0) + \chi(-i0) = \frac{1}{i\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\log \frac{s^2}{1+s^2} (1 - b^2(s))}{s} ds.
\]
Recall the item (v) in Proposition 2,
\[
a_1(k) = \frac{A^2}{4k^2} a_2(0)(1 + o(k)), \quad k \to 0.
\]
Comparing (A.5) to (A.7), also using (A.6), we reach (51).

**Case II: Nongeneric case**
Define functions \( \tilde{a}_1(k) \) and \( \tilde{a}_2(k) \) by
\[
\tilde{a}_1(k) = a_1(k) \frac{k}{k - i\kappa}, \quad \tilde{a}_2(k) = a_2(k) \frac{k - i\kappa}{k}.
\]
Similar to the claims in Case I, \( \tilde{a}_j(k), j = 1, 2 \) satisfy the following scalar RH problem
\[
\begin{align*}
\text{• } \tilde{a}_1(k) \text{ and } \tilde{a}_2(k) \text{ are analytic and have no zeros in } \mathbb{C}_+ \text{ and } \mathbb{C}_-, \text{ respectively.} \\
\text{• } \tilde{a}_j(k), j = 1, 2 \text{ admit the jump condition} \\
\tilde{a}_1(k)\tilde{a}_2(k) = 1 - b^2(k), \quad k \in \mathbb{R}. \\
\text{• } \tilde{a}_j(k) \to 1 \text{ as } k \to \infty.
\end{align*}
\]
We can solve that
\[
\begin{align*}
a_1(k) &= \frac{k - i\kappa}{k} \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \frac{1 - b^2(s)}{s - k} ds \right\}, \\
a_2(k) &= \frac{k}{k - i\kappa} \exp \left\{ -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \frac{1 - b^2(s)}{s - k} ds \right\}.
\end{align*}
\]
Applying Sokhotski–Plemelj formulas to (A.10), we find
\[
\begin{align*}
a_{11} &= -i\kappa I_1 I_2, \quad a'_2(0) = i\kappa^{-1} I_1^{-1} I_2,
\end{align*}
\]
where \( I_1, I_2 \) are defined by (53). Additionally, we can see that \( a_{11} a'_2(0) = I_2^2 = 1 - b^2(0) \neq 0 \).
On the other hand, due to the symmetry relation (26b) and the item (v) of Proposition 1, the behavior of \( \psi_j(x, t, k), j = 1, 2 \) as \( k \to 0 \) could be exhibited as follows:
\[
\psi^{(1)}_1(x, t, k) = \frac{1}{k} \begin{pmatrix} v_1(x, t) \\ v_2(x, t) \end{pmatrix} + \begin{pmatrix} s_1(x, t) \\ s_2(x, t) \end{pmatrix} + O(k), \quad k \to 0,
\]
\begin{align}
\psi_1^{(2)}(x, t, k) &= \frac{2i}{A} \left( \begin{array}{c} v_1(x, t) \\ v_2(x, t) \end{array} \right) + k \left( \begin{array}{c} h_1(x, t) \\ h_2(x, t) \end{array} \right) + O(k^2), \quad k \to 0, \\
\psi_2^{(1)}(x, t, k) &= -\frac{2i}{A} \left( \begin{array}{c} v_2(-x, -t) \\ v_1(-x, -t) \end{array} \right) - k \left( \begin{array}{c} h_2(-x, -t) \\ h_1(-x, -t) \end{array} \right) + O(k^2), \quad k \to 0, \\
\psi_2^{(2)}(x, t, k) &= -\frac{1}{k} \left( \begin{array}{c} v_2(-x, -t) \\ v_1(-x, -t) \end{array} \right) + \left( \begin{array}{c} s_2(-x, -t) \\ s_1(-x, -t) \end{array} \right) + O(k), \quad k \to 0.
\end{align}

with some functions \( v_j, h_j \) and \( s_j \), \( j = 1, 2 \). Using determinant representation for spectral functions (see (45)) as \( k \to 0 \) and taking into account that \( |v_2(0, 0)|^2 - |v_1(0, 0)|^2 = 0 \) under the Case II (see Remark 5), we have

\begin{align}
a_1(k) &= \frac{1}{k} (v_1 \bar{s}_1 - v_2 \bar{s}_2 + \bar{v}_1 s_1 + \bar{v}_2 s_2)|_{x=0, t=0} + O(1), \quad k \to 0, \\
a_2(k) &= k \frac{2i}{A} (v_1 \bar{h}_1 + v_2 \bar{h}_2 - \bar{v}_1 h_1 - \bar{v}_2 h_2)|_{x=0, t=0} + O(k^2), \quad k \to 0, \\
b(k) &= v_1 \bar{h}_1 - v_2 \bar{h}_2 + \frac{2i}{A} (v_1 \bar{s}_1 - v_2 \bar{s}_2)|_{x=0, t=0} + O(k), \quad k \to 0.
\end{align}

From (A.13), we find (recall that \( a_{11} = \lim_{k \to 0} k a_1(k) \))

\[ b(0) + \overline{b(0)} = \frac{A}{2i} a'_2(0) - \frac{2i}{A} a_{11}, \]

which is equivalent to (notice \( b(k) = \overline{b(-k)} \))

\[ a_{11} = i Ab(0) - \frac{A^2}{4} a'_2(0). \]

Using (A.11), we arrive at (52).

**APPENDIX B: PARABOLIC CYLINDER MODEL**

The Appendix B is based on the fundamental work of A.R.\textsuperscript{19} Find a matrix-valued function

\[ m_{-k_0}^{pc}(\zeta) := m_{-k_0}^{pc}(\zeta; \xi) \]

with the following properties:

* \( m_{-k_0}^{pc}(\zeta) \) is analytical in \( \mathbb{C} \setminus \Sigma_{pc} \) with \( \Sigma_{pc} \) shown in Figure B.1;
* \( m_{-k_0}^{pc} \) has continuous boundary values \( m_{-k_0, \pm}^{pc} \) on \( \Sigma_{pc} \) and

\[ m_{-k_0, +}^{pc}(\zeta) = m_{-k_0, -}^{pc}(\zeta) J_{pc}(\zeta), \quad \zeta \in \Sigma_{pc}, \]

\[ (B.1) \]
where

\[
J^{PC}(\zeta) = \begin{cases} 
\zeta i \nu \frac{\sigma_3}{4} e^{-\frac{\zeta^2}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ q^*_1(-k_0) & 1 \end{pmatrix}, & \zeta \in \Sigma^{PC}_3, \\
\zeta i \nu \frac{\sigma_3}{4} e^{-\frac{\zeta^2}{4} \sigma_3} \begin{pmatrix} 1 & -q^*_2(-k_0) \\ 0 & 1 \end{pmatrix}, & \zeta \in \Sigma^{PC*}_3, \\
\zeta i \nu \frac{\sigma_3}{4} e^{-\frac{\zeta^2}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ q^*_1(-k_0) & 1+q^*_1(-k_0)q^*_2(-k_0) \end{pmatrix}, & \zeta \in \Sigma^{PC}_4 \\
\zeta i \nu \frac{\sigma_3}{4} e^{-\frac{\zeta^2}{4} \sigma_3} \begin{pmatrix} 1 & 0 \\ -q^*_1(-k_0) & 1+q^*_1(-k_0)q^*_2(-k_0) \end{pmatrix}, & \zeta \in \Sigma^{PC*}_4.
\end{cases}
\tag{B.2}
\]

and \( \nu = \nu(-k_0) \).

* \( m^{PC}_{-k_0}(\zeta) = I + m^{PC}_{-k_0,1}\zeta^{-1} + O(\zeta^{-2}), \zeta \to \infty \).

The RH problem \( m^{PC}_{-k_0}(\zeta) \) has an explicit solution, which can be expressed in terms of Webber equation \( \left( \frac{\partial^2}{\partial z^2} + \left( \frac{1}{2} - \frac{z^2}{2} + \alpha \right) \right) D_\alpha(z) = 0 \).

Taking the transformation

\[
m^{PC}_{-k_0} = m_0(\zeta)P^{\nu\sigma_3} e^{i\sigma_3\zeta^2}, \tag{B.3}
\]
where

\[
P(\xi) = \begin{cases}
    \begin{pmatrix}
        1 & 0 \\
        -q_1^r(-k_0) & 1 \\
    \end{pmatrix}, & \xi \in U_2^{pc}, \\
    \begin{pmatrix}
        1 & q_2^r(-k_0) \\
        0 & 1 \\
    \end{pmatrix}, & \xi \in U_2^{pc*}, \\
    \begin{pmatrix}
        1 & -q_2^r(-k_0) \\
        1 + q_1^r(-k_0)q_2^r(-k_0) & 1 \\
    \end{pmatrix}, & \xi \in U_3^{pc}, \\
    \begin{pmatrix}
        q_1^r(-k_0) \\
        1 + q_1^r(-k_0)q_2^r(-k_0) & 1 \\
    \end{pmatrix}, & \xi \in U_3^{pc*}, \\
    I, & \text{elsewhere}.
\end{cases}
\]  

The matrix-valued function \(m_0(\xi)\) satisfies the following properties:

* \(m_0(\xi)\) is analytical in \(\mathbb{C}\setminus\mathbb{R}\);
* \(m_0(\xi)\) takes continuous boundary values \(m_{0,\pm}(\xi)\) on \(\mathbb{R}\) and

\[
m_{0,+}(\xi) = m_{0,-}(\xi)J_0, \quad \xi \in \mathbb{R},
\]  

where

\[
J_0(\xi) = \begin{pmatrix}
    1 + q_1^r(-k_0)q_2^r(-k_0) & q_2^r(-k_0) \\
    q_1^r(-k_0) & 1
\end{pmatrix}.
\]  

* Asymptotic behavior:

\[
m_0(\xi) = \left( I + m_{-k_0,1}^{pc} \xi^{-1} + O(\xi^{-2}) \right)^{\frac{i}{2} \sigma_3} e^{-\frac{i}{2} \xi^2 \sigma_3}, \quad \text{as} \quad \xi \to \infty.
\]  

Differentiating (B.5) with respect to \(\xi\), and combining \(\frac{i\xi}{2} \sigma_3 m_{0,+} = \frac{i\xi}{2} \sigma_3 m_{0,-} - J_0\), we obtain

\[
\left( \frac{dm_0}{d\xi} + \frac{i\xi}{2} \sigma_3 m_0 \right)_+ = \left( \frac{dm_0}{d\xi} + \frac{i\xi}{2} \sigma_3 m_0 \right)_- J_0.
\]  

It is not difficult to verify the matrix function \(\frac{dm_0}{d\xi} + \frac{i\xi}{2} \sigma_3 m_0 \) has no jump along the real axis and is an entire function with respect to \(\xi\). Combine (B.3), we can directly calculate that

\[
\left( \frac{dm_0}{d\xi} + \frac{i\xi}{2} \sigma_3 m_0 \right)^{-1} m_0^{-1} = \left[ \frac{dm_{-k_0}^{pc}}{d\xi} + m_{-k_0}^{pc} \frac{iv}{\xi} \sigma_3 \right] \left( m_{-k_0}^{pc} \right)^{-1} + \frac{i\xi}{2} \left[ \sigma_3, m_{-k_0}^{pc} \right] \left( m_{-k_0}^{pc} \right)^{-1},
\]  

(B.9)
The first term in the R.H.S of (B.9) tends to zero as \( \zeta \to \infty \). We use \( m_{-k_0}^{pc} = I + m_{-k_0,1}^{pc} \zeta^{-1} + O(\zeta^{-2}) \) as well as Liouville theorem to obtain that there exists a constant matrix \( \beta^{mat} \) such that
\[
\beta^{mat} := \begin{pmatrix} 0 & \beta^r(\zeta) \\ \gamma^r(\zeta) & 0 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} \sigma_3, m_{-k_0,1}^{pc} \end{pmatrix} = \begin{pmatrix} 0 & i[m_{-k_0,1}^{pc}]_{12} \\ -i[m_{-k_0,1}^{pc}]_{21} & 0 \end{pmatrix},
\]
which implies that \( [m_{-k_0,1}^{pc}]_{12} = -i\beta^r(\zeta), [m_{-k_0,1}^{pc}]_{21} = i\gamma^r(\zeta) \). Using Liouville theorem again, we have
\[
\begin{pmatrix} d m_0 \over d \zeta + i \zeta \over 2 \sigma_3 m_0 \end{pmatrix} = \beta^{mat} m_0.
\]
We rewrite the above equality to the following ODE systems:
\[
\begin{align*}
\frac{d m_{0,11}}{d \zeta} + \frac{i \zeta}{2} m_{0,11} &= \beta^r(\zeta) m_{0,21}, \\
\frac{d m_{0,21}}{d \zeta} - \frac{i \zeta}{2} m_{0,21} &= \gamma^r(\zeta) m_{0,11},
\end{align*}
\]
as well as
\[
\begin{align*}
\frac{d m_{0,12}}{d \zeta} + \frac{i \zeta}{2} m_{0,12} &= \beta^r(\zeta) m_{0,22}, \\
\frac{d m_{0,22}}{d \zeta} - \frac{i \zeta}{2} m_{0,22} &= \gamma^r(\zeta) m_{0,12}.
\end{align*}
\]
From (B.12) to (B.15), we can obtain
\[
\begin{align*}
\frac{d^2 m_{0,11}}{d \zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta^r(\zeta) \gamma^r(\zeta) \right) m_{0,11} &= 0, \\
\frac{d^2 m_{0,21}}{d \zeta^2} + \left( -\frac{i}{2} + \frac{\zeta^2}{4} - \beta^r(\zeta) \gamma^r(\zeta) \right) m_{0,21} &= 0, \\
\frac{d^2 m_{0,12}}{d \zeta^2} + \left( \frac{i}{2} + \frac{\zeta^2}{4} - \beta^r(\zeta) \gamma^r(\zeta) \right) m_{0,12} &= 0, \\
\frac{d^2 m_{0,22}}{d \zeta^2} + \left( -\frac{i}{2} + \frac{\zeta^2}{4} - \beta^r(\zeta) \gamma^r(\zeta) \right) m_{0,22} &= 0.
\end{align*}
\]
Set \( \nu = \beta^r(\zeta) \gamma^r(\zeta) \). For \( m_{0,11}, \text{Im} \zeta > 0 \), we introduce the new variable \( \tilde{\eta} = \zeta e^{-\frac{3\pi}{4}} \), and the first equation of (B.16) becomes
\[
\frac{d^2 m_{0,11}}{d \tilde{\eta}^2} + \left( \frac{1}{2} - \frac{\tilde{\eta}^2}{4} + i \nu \right) m_{0,11} = 0.
\]
For \( \zeta \in \mathbb{C}^+, \text{Arg} \zeta < \pi, -\frac{3\pi}{4} < \text{Arg} \tilde{\eta} < \frac{\pi}{4} \). We have \( m_{0,11} = e^{\frac{1+\nu}{4}(-k_0)} D_{iv(-k_0)} (e^{\frac{1+\nu}{4} \zeta}) \sim \zeta^i e^{-\frac{i}{4} \zeta^2} \) corresponding to the (1,1)-entry of (B.7). To save the space, we present the other results for \( m_0 \) below.
The unique solution to $m_0(\zeta)$ is when $\zeta \in \mathbb{C}^+$,
$$m_0(\zeta) = \begin{cases} e^{-\frac{3\pi}{4} y(-k_0)} D_{iy(-k_0)}(e^{-\frac{3\pi}{4} i \zeta}) & -\frac{i y(-k_0)}{\gamma'(\zeta)} e^{\frac{\pi}{4} y(-k_0)-i} D_{-iy(-k_0)-1}(e^{-\frac{\pi i}{4} \zeta}) \\ \frac{i y(-k_0)}{\beta'(\zeta)} e^{\frac{3\pi}{4} y(-k_0)+i} D_{iy(-k_0)-1}(e^{-\frac{3\pi i}{4} \zeta}) & \frac{\pi y(-k_0)}{\beta'(\zeta)} e^{-\frac{3\pi}{4} y(-k_0)} D_{-iy(-k_0)}(e^{\frac{3\pi i}{4} \zeta}) \end{cases}.$$ (B.19)

When $\zeta \in \mathbb{C}^-$,
$$m_0(\zeta) = \begin{cases} e^{\frac{\pi}{4} y(-k_0)} D_{iy(-k_0)}(e^{\frac{\pi i}{4} \zeta}) & -\frac{i y(-k_0)}{\gamma'(\zeta)} e^{-\frac{3\pi}{4} y(-k_0)-i} D_{-iy(-k_0)-1}(e^{\frac{\pi i}{4} \zeta}) \\ \frac{i y(-k_0)}{\beta'(\zeta)} e^{\frac{3\pi}{4} y(-k_0)+i} D_{iy(-k_0)-1}(e^{\frac{3\pi i}{4} \zeta}) & \frac{\pi y(-k_0)}{\beta'(\zeta)} e^{-\frac{3\pi}{4} y(-k_0)} D_{-iy(-k_0)}(e^{\frac{3\pi i}{4} \zeta}) \end{cases}.$$ (B.20)

Which is derived in Ref. 24.

From (B.5), we know that $(m_{0,-})^{-1}m_{0,+} = J_0$ and
$$q_1^r(-k_0) = m_{0,-11}m_{0,+21} - m_{0,-21}m_{0,+11}$$
$$= e^{\frac{\pi}{4} y(-k_0)} D_{iy(-k_0)}(e^{\frac{\pi i}{4} \zeta}) \cdot \frac{e^{-\frac{3\pi}{4} y(-k_0)}}{\beta'(\zeta)} \left[ \partial_\zeta \left( D_{iy(-k_0)}(e^{-\frac{3\pi i}{4} \zeta}) \right) + \frac{i \zeta}{2} D_{iy(-k_0)}(e^{-\frac{3\pi i}{4} \zeta}) \right]$$
$$- e^{-\frac{3\pi}{4} y(-k_0)} D_{iy(-k_0)}(e^{\frac{\pi i}{4} \zeta}) \cdot \frac{e^{\frac{\pi}{4} y(-k_0)}}{\beta'(\zeta)} \left[ \partial_\zeta \left( D_{iy(-k_0)}(e^{\frac{3\pi i}{4} \zeta}) \right) + \frac{i \zeta}{2} D_{iy(-k_0)}(e^{\frac{3\pi i}{4} \zeta}) \right]$$
$$= \frac{e^{-\frac{\pi}{2} y(-k_0)}}{\beta'(\zeta)} \cdot \sqrt{\frac{2\pi e^{\frac{\pi}{2} i}}{\Gamma(-i y(-k_0))}}\cdot D_{iy(-k_0)}(e^{\frac{\pi i}{4} \zeta}), D_{iy(-k_0)}(e^{-\frac{3\pi i}{4} \zeta})$$
$$= \frac{e^{-\frac{\pi}{2} y(-k_0)}}{\beta'(\zeta)} \cdot \frac{\sqrt{2\pi e^{\frac{\pi}{2} i}}}{\Gamma(-i y(-k_0))}.$$ (B.21)

And,
$$\beta'(\zeta) = \frac{\sqrt{2\pi e^{4} e^{-\frac{\pi}{2}}}}{q_1^r(-k_0)\Gamma(-i y(-k_0))}, \quad \gamma'(\zeta) = \frac{\sqrt{2\pi e^{4} e^{-\frac{\pi}{2}}}}{q_2^r(-k_0)\Gamma(i y(-k_0))}.$$ (B.22)

Finally, we have
$$m_{0,-k_0}^{pc} = I + \frac{i}{\zeta} \begin{pmatrix} 0 & -\beta'(\zeta) \\ \gamma'(\zeta) & 0 \end{pmatrix} + O(\zeta^{-2}).$$ (B.23)