EXTREME VALUES OF CLASS NUMBERS OF REAL QUADRATIC
FIELDS

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Abstract. We improve a result of H. L. Montgomery and J. P. Weinberger by establishing the existence of infinitely many fundamental discriminants \( d > 0 \) for which the class number of the real quadratic field \( \mathbb{Q}(\sqrt{d}) \) exceeds \((2e^\gamma + o(1))\sqrt{d}(\log \log d)/\log d\). We believe this bound to be best possible. We also obtain upper and lower bounds of nearly the same order of magnitude, for the number of real quadratic fields with discriminant \( d \leq x \) which have such an extreme class number.

1. Introduction

An important problem in number theory is to understand the size of the class number of an algebraic number field. The case of a quadratic field has a long history going back to Gauss. Let \( d \) be a fundamental discriminant and \( h(d) \) be the class number of the field \( \mathbb{Q}(\sqrt{d}) \). When \( d < 0 \), in which case \( \mathbb{Q}(\sqrt{d}) \) is imaginary quadratic, J. E. Littlewood \cite{l} established, assuming the Generalized Riemann Hypothesis GRH, that

\[
(1.1) \quad h(d) \leq \left( \frac{2e^\gamma}{\pi} + o(1) \right) \sqrt{|d| \log \log |d|},
\]

where \( \gamma \) is the Euler-Mascheroni constant. Littlewood used Dirichlet’s class number formula, which for \( d < -4 \), asserts that

\[
(1.2) \quad h(d) = \frac{\sqrt{|d|}}{\pi} \cdot L(1, \chi_d),
\]

where \( \chi_d = (\frac{d}{\cdot}) \) is the Kronecker symbol. He then deduced \((1.1)\) from the bound

\[
(1.3) \quad L(1, \chi_d) \leq (2e^\gamma + o(1)) \log \log |d|,
\]

which he obtained under GRH for all fundamental discriminants \( d \). Furthermore, under the same hypothesis, Littlewood \cite{l} proved that there exist infinitely many fundamental discriminants \( d \) (both positive and negative) for which

\[
(1.4) \quad L(1, \chi_d) \geq (e^\gamma + o(1)) \log \log |d|,
\]

and hence for those \( d < 0 \), one has

\[
(1.5) \quad h(d) \geq \left( \frac{e^\gamma}{\pi} + o(1) \right) \sqrt{|d| \log \log |d|},
\]
by the class number formula (1.2). The omega result (1.4) was later established unconditionally by S. Chowla [1].

The case of a real quadratic field is notoriously difficult, due to the presence of non-trivial units in $\mathbb{Q}(\sqrt{d})$. For example, Gauss’s conjecture that there are infinitely many positive discriminants $d$ for which $h(d) = 1$ is still open. When $d$ is positive, the class number $h(d)$ is heavily affected by the size of the regulator $R_d$ of $\mathbb{Q}(\sqrt{d})$. In this case, Dirichlet’s class number formula asserts that

$$h(d) = \frac{\sqrt{d}}{R_d} \cdot L(1, \chi_d). \quad (1.6)$$

Recall that $R_d = \log \varepsilon_d$ where $\varepsilon_d$ is the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{d})$, defined as $\varepsilon_d = (a + b\sqrt{d})/2$, where $b > 0$ and $a$ is the smallest positive integer such that $(a, b)$ is a solution to the Pell equations $m^2 - dn^2 = \pm 4$. Since $\varepsilon_d > \sqrt{d}/2$, it follows that when $d$ is large we have

$$R_d \geq \left( \frac{1}{2} + o(1) \right) \log d, \quad (1.7)$$

and hence by Littlewood’s bound (1.3) we deduce that

$$h(d) \leq \left( 4e^\gamma + o(1) \right) \sqrt{d} \cdot \frac{\log \log d}{\log d}. \quad (1.8)$$

for all positive fundamental discriminants $d$, under the assumption of GRH. In 1977, H. L. Montgomery and J. P. Weinberger [8] showed that this bound cannot be improved, apart from the value of the constant. Indeed, they proved that there exist infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$ such that

$$h(d) \gg \sqrt{d} \cdot \frac{\log \log d}{\log d}. \quad (1.9)$$

Recently, W. Duke investigated generalizations of this result to higher degree number fields. In [3], he obtained the corresponding omega result for the class number of abelian cubic fields, while in [2], assuming certain hypotheses (including GRH), he obtained similar results in the case of totally real number fields of a fixed degree whose normal closure has the symmetric group as Galois group.

It is widely believed that the true nature of extreme values of $L(1, \chi_d)$ is given by the omega result (1.4) rather than the GRH bound (1.3). A. Granville and K. Soundararajan [5] investigated the distribution of large values of $L(1, \chi_d)$ and their results give strong support to this conjecture. In view of (1.7) and the class number formula (1.6), this leads to the following conjecture

**Conjecture 1.1.** For all large positive fundamental discriminants $d$ we have

$$h(d) \leq \left( 2e^\gamma + o(1) \right) \sqrt{d} \cdot \frac{\log \log d}{\log d}. \quad (1.10)$$
In this paper, we prove the existence of infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$ for which the class number $h(d)$ is as large as the conjectured upper bound $\text{(1.10)}$. We also obtain upper and lower bounds of nearly the same order of magnitude, for the number of real quadratic fields with discriminant $d \leq x$ for which the class number is that large.

**Theorem 1.2.** Let $x$ be large.

(a) There are at least $x^{1/2 - 1/\log \log x}$ real quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminant $d \leq x$, such that

\[ h(d) \geq (2e^\gamma + o(1)) \sqrt{d} \cdot \frac{\log \log d}{\log d}. \tag{1.11} \]

(b) Furthermore, there are at most $x^{1/2 + o(1)}$ real quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminant $d \leq x$, for which $\text{(1.11)}$ holds.

To prove the omega result $\text{(1.9)}$, Montgomery and Weinberger worked over the following special family of fundamental discriminants, first studied by Chowla

\[ D := \{ d \text{ square-free of the form } d = 4n^2 + 1 \text{ for some } n \geq 1 \}. \]

This family has the advantage that the regulator $R_d$ is as small as possible. Indeed, if $d = 4n^2 + 1 \in \mathcal{D}$, then the fundamental unit of $\mathbb{Q}(\sqrt{d})$ is $\epsilon_d = 2n + \sqrt{d} \leq 2\sqrt{d}$, and hence $R_d = (1/2 + o(1)) \log d$. Therefore, the class number formula $\text{(1.6)}$ implies that

\[ h(d) = (2 + o(1)) \frac{\sqrt{d}}{\log d} \cdot L(1, \chi_d) \tag{1.12} \]

for $d \in \mathcal{D}$. Montgomery and Weinberger showed that there exist infinitely many $d \in \mathcal{D}$ for which $L(1, \chi_d) \gg \log \log d$, from which they deduced $\text{(1.9)}$.

Let $\mathcal{D}(x)$ denote the set of $d \in \mathcal{D}$ such that $d \leq x$. To establish the first part of Theorem $\text{1.2}$, we prove that there exist many fundamental discriminants $d \in \mathcal{D}(x)$ for which $L(1, \chi_d)$ is as large as one could hope for, namely $\geq (e^\gamma + o(1)) \log \log d$. Our argument uses elements from the work of Montgomery and Weinberger as well as some new ideas.

The proof of the second part of Theorem $\text{1.2}$ relies on two main ingredients. First, using elementary methods we bound the number of real quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminant $d \leq x$ which have small regulator. Then, we combine Heath-Brown’s quadratic large sieve $\text{[6]}$ with zero-density estimates to show that Littlewood’s GRH bound $L(1, \chi_d) \leq (2e^\gamma + o(1)) \log \log d$ holds unconditionally for all but at most $x^{1/2 + o(1)}$ fundamental discriminants $0 < d < x$. 
2. Real quadratic fields with extreme class number: proof of Theorem 1.2, part (a)

To obtain large values of $L(1, \chi_d)$, a general strategy is to construct fundamental discriminants $d$ for which $\chi_d(p) = 1$ for all the small primes $p$, typically up to $y = \log d$. Montgomery and Weinberger [8] noticed that for square-free $d$ of the form $d = 4n^2 + 1$, one has $\chi_d(p) = 1$ for all the primes $p$ dividing $n$. Hence, this reduces the problem to estimating the number of fundamental discriminants $d \in \mathcal{D}(x)$ for which $(d - 1)/4$ is divisible by all the small primes. To this end they established the following lemma.

**Lemma 2.1** (Lemma 1 of [8]). The number of integers $d \leq x$ such that $d$ is square-free and $d = 4n^2 + 1$ where $q \mid n$ equals

$$\frac{\sqrt{x}}{2q} \prod_{p \mid q} \left(1 - \frac{2}{p^2}\right) + O\left(x^{1/3} \log x\right).$$

Taking $q = \prod_{p \leq y} p$, where $\sqrt{\log x} \leq y \leq (\log x)/8$ is a real number, and noting that $q = e^{y(1+o(1))}$ by the prime number theorem, yields

**Corollary 2.2.** Let $\sqrt{\log x} \leq y \leq (\log x)/8$ be a real number. The number of fundamental discriminants $d \in \mathcal{D}(x)$ such that $\chi_d(p) = 1$ for all primes $p \leq y$ is at least $x^{1/2}e^{-y(1+o(1))}$.

Montgomery and Weinberger then used zero density estimates to prove that for any $0 < \delta < 1$, all but at most $x^{\delta}$ fundamental discriminants $1 \leq d \leq x$ satisfy

(2.1) \[ \log L(1, \chi_d) = \sum_{p \leq y} \frac{\chi_d(p)}{p} + O_\delta(1), \]

where $(\log x)\delta \leq y \leq \log x$ is a real number. Taking $y = (\log x)/9$ and $\delta = 1/4$ in (2.1), and using Corollary 2.2 produces more than $x^{3/8}$ fundamental discriminants $d \leq x$ in $\mathcal{D}$ for which $L(1, \chi_d) \gg \log \log d$.

In order to improve this estimate, we first replace (2.1) with a better approximation to $L(1, \chi_d)$, due to Granville and Soundararajan [5], which is obtained using zero density estimates together with the large sieve.

**Proposition 2.3** (Proposition 2.2 of [5]). Let $A > 2$ be fixed. Then, for all but at most $Q^{2/A+o(1)}$ primitive characters $\chi \pmod{q}$ with $q \leq Q$ we have

$$L(1, \chi) = \prod_{p \leq (\log Q)^A} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\log \log Q}\right)\right).$$

Let $\sqrt{\log x} \leq y \leq (\log x)/8$ be a real number. Then, note that

(2.2) \[ \prod_{p \leq (\log x)^A} \left(1 - \frac{\chi_d(p)}{p}\right)^{-1} = \prod_{p \leq y} \left(1 - \frac{\chi_d(p)}{p}\right)^{-1} \exp \left(\sum_{y < p \leq (\log x)^A} \frac{\chi_d(p)}{p} + O\left(\frac{1}{\sqrt{y \log y}}\right)\right). \]
By Corollary 2.2 there are many fundamental discriminants \( d \in D(x) \) for which the product \( \prod_{p \leq y} (1 - \chi_d(p)/p)^{-1} \) is as large as possible. The key ingredient in the proof of the first part of Theorem 1.2 is the following proposition which gives an upper bound for the \( 2k \)-th moment of \( \sum_{y < p < (\log x)^A} \chi_d(p)/p \) as \( d \) varies in \( D(x) \), uniformly for \( k \) in a large range. In particular, we shall later deduce that with very few exceptions in \( D(x) \), the prime sum \( \sum_{y < p < (\log x)^A} \chi_d(p)/p \) is small.

**Proposition 2.4.** Let \( \sqrt{\log x} < y < \log x \) be a real number, and \( z = (\log x)^A \) where \( A > 2 \) is a constant. Then, for every positive integer \( k \leq \log x/(8A \log \log x) \) we have

\[
\sum_{d \in D(x)} \left( \sum_{y < p < z} \frac{\chi_d(p)}{p} \right)^{2k} \ll \sqrt{x} \left( \frac{ck}{y \log y} \right)^k,
\]

for some absolute constant \( c > 0 \).

To prove this result we first need the following lemma, which gives a non-trivial bound for a certain character sum.

**Lemma 2.5.** Let \( q \) be an odd positive integer, and write \( q = q_2^2 q_0 \) where \( q_0 \) is square-free. If \( x \geq q_2^2 \) then

\[
\sum_{n \leq x} \left( \frac{4n^2 + 1}{q} \right) \ll \frac{x}{q_0},
\]

where \((\frac{\cdot}{q})\) is the Jacobi symbol modulo \( q \).

**Proof.** Let \( f(n) = 4n^2 + 1 \). First we have

(2.3)
\[
\sum_{n \leq x} \left( \frac{f(n)}{q} \right) = \sum_{a=1}^{q} \sum_{n \leq x \mod q} \left( \frac{f(n)}{q} \right) = \sum_{a=1}^{q} \left( \frac{f(a)}{q} \right) \sum_{n \leq x \mod q} 1 = \frac{x}{q} \sum_{a=1}^{q} \left( \frac{f(a)}{q} \right) + O(q).
\]

Note that \( \sum_{a=1}^{q} \left( \frac{f(a)}{q} \right) \) is a complete character sum, and hence if \( q = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) is the prime factorization of \( q \), then

(2.4)
\[
\sum_{a=1}^{q} \left( \frac{f(a)}{q} \right) = \prod_{j=1}^{k} \left( \sum_{\alpha_j=1}^{p_j^{\alpha_j}} \left( \frac{f(a_j)}{p_j^{\alpha_j}} \right) \right) = \prod_{j=1}^{k} \left( \sum_{\alpha_j=1}^{p_j^{\alpha_j}} \left( \frac{f(a_j)}{p_j} \right)^{\alpha_j} \right),
\]

by multiplicativity and the Chinese Remainder theorem. Now, if \( \alpha_j = 2\beta_j \) is even we use the trivial bound

(2.5)
\[
\left| \sum_{a=1}^{p_j^{2\beta_j}} \left( \frac{f(a)}{p_j} \right)^{2\beta_j} \right| \leq p_j^{2\beta_j}.
\]
On the other hand, if $\alpha_j = 2\beta_j + 1$ is odd, then $\left(\frac{f(a_j)}{p_j}\right)^{\alpha_j} = \left(\frac{f(a_j)}{p_j}\right)$, and hence

\begin{equation}
\sum_{a=1}^{p_j^{2\beta_j + 1}} \left(\frac{f(a)}{p_j}\right)^{2\beta_j + 1} = \sum_{b=0}^{p_j^{2\beta_j - 1}} \sum_{c=1}^{p_j} \left(\frac{f(bp_j + c)}{p_j}\right) = p_j^{2\beta_j} \sum_{c=1}^{p_j} \left(\frac{f(c)}{p_j}\right).
\end{equation}

If $p$ is a prime number, then the sum $\sum_{n=1}^{p} \left(\frac{n^2 + b}{p}\right)$ is a Jacobsthal sum, and it is known that (see for example Storer [9])

\begin{equation}
\sum_{n=1}^{p} \left(\frac{n^2 + b}{p}\right) = -1 \text{ if } p \nmid b.
\end{equation}

Therefore, we deduce

\begin{equation}
\sum_{c=1}^{p_j} \left(\frac{4c^2 + 1}{p_j}\right) = - \left(\frac{4}{p_j}\right) = -1.
\end{equation}

Inserting this estimate in (2.6) yields

\begin{equation}
\sum_{a=1}^{p_j^{2\beta_j + 1}} \left(\frac{f(a)}{p_j}\right)^{2\beta_j + 1} = p_j^{2\beta_j}.
\end{equation}

Combining the bounds (2.5) and (2.7) in (2.4) yields

\begin{equation}
\left|\sum_{a=1}^{q} \left(\frac{f(a)}{q}\right)\right| \leq \frac{q}{q_0}.
\end{equation}

The result follows upon inserting this bound in (2.3).

\textbf{Proof of Proposition 2.4.} As before, we let $f(n) = 4n^2 + 1$. By positivity of the summand we have

\begin{equation}
\sum_{d \in \mathcal{D}(x)} \left(\sum_{y < p < z} \frac{\chi_d(p)}{p}\right)^{2k} \leq \sum_{n \leq \sqrt{x}} \left(\sum_{y < p < z} \frac{f(n)}{p}\right)^{2k}.
\end{equation}

Expanding the inner sum we obtain

\begin{equation}
\left(\sum_{y < p < z} \frac{f(n)}{p}\right)^{2k} = \sum_{m} b_{2k}(m; y, z) \left(\frac{f(n)}{m}\right),
\end{equation}

where

\begin{equation}
b_r(m; y, z) := \sum_{y < p_1 < \cdots < p_r < z} 1.
\end{equation}

Note that $0 \leq b_r(m; y, z) \leq r!$. Moreover, $b_r(m; y, z) = 0$ unless $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ where $y < p_1 < p_2 < \cdots < p_s < z$ are distinct primes and $\Omega(m) = \alpha_1 + \cdots + \alpha_s = r$ (where
\( \Omega(m) \) is the number of prime divisors of \( m \) counting multiplicities). In this case, we have
\[
(2.10) \quad b_r(m; y, z) = \binom{r}{\alpha_1, \ldots, \alpha_s}.
\]
Using this formula, one can easily deduce that if \( n \) and \( m \) are positive integers with \( \Omega(n) = \ell \) and \( \Omega(m) = r \) then
\[
(2.11) \quad b_{\ell+r}(mn; y, z) \leq \binom{\ell+r}{\ell} b_{\ell}(n; y, z) b_r(m; y, z).
\]
Since \( z^{2k} \leq x^{1/4} \), then it follows from Lemma 2.5 that
\[
(2.12) \quad \sum_{n \leq \sqrt{x}} \left( \sum_{y < p < z} \frac{f(n)}{p} \right)^{2k} \ll x \sum_{m \leq \sqrt{x}} \frac{b_{2k}(m; y, z)}{mm_0},
\]
where \( m_0 \) is the square-free part of \( m \). Let \( m \) be such that \( \Omega(m) = 2k \) and \( p \mid m \Rightarrow y < p < z \), and write \( m = m_1^2 m_0 \), where \( m_0 \) is square-free. Put \( \Omega(m_1) = \ell \). Then by (2.11) we have
\[
b_{2k}(m; y, z) \leq \binom{2k}{2\ell} b_{2\ell}(m_1^2; y, z) b_{2k-2\ell}(m_0; y, z)
\]
\[
\leq \binom{2k}{2\ell} \binom{2\ell}{\ell} (b_{\ell}(m_1; y, z))^2 b_{2k-2\ell}(m_0; y, z)
\]
\[
\leq \binom{2k}{k}! \binom{k}{\ell} b_{\ell}(m_1; y, z) b_{2k-2\ell}(m_0; y, z),
\]
since \( b_{\ell}(m_1; y, z) \leq \ell! \). Therefore, we deduce that
\[
\sum_m \frac{b_{2k}(m; y, z)}{mm_0} \leq \frac{(2k)!}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{m_1} \frac{b_{\ell}(m_1; y, z)}{m_1^2} \sum_{m_0} \frac{b_{2k-2\ell}(m_0; y, z)}{m_0^2}
\]
\[
\leq \frac{2k(2k)!}{k!} \left( \sum_{y < p < z} \frac{1}{p^2} \right)^k \leq \left( \frac{ck}{y \log y} \right)^k,
\]
for some positive constant \( c > 0 \) if \( y \) is large enough, since
\[
(2.13) \quad \sum_n \frac{b_r(n; y, z)}{n^2} = \left( \sum_{y < p < z} \frac{1}{p^2} \right)^r
\]
and \( \sum_{y < p < z} 1/p^2 \ll 1/(y \log y) \) by the prime number theorem. Inserting this bound in (2.12) completes the proof. \( \square \)

We are now ready to prove the first part of Theorem 1.2.

Proof of Theorem 1.2, part (a). Let \( z = (\log x)^6 \), and \( \sqrt{\log x} \leq y \leq (\log x)/8 \) be a real number to be chosen later. Then, by Proposition 2.3 and equation (2.2) it follows that
for all but at most $x^{2/5}$ fundamental discriminants $d \in \mathcal{D}(x)$, we have
\begin{equation}
L(1, \chi_d) = \prod_{p \leq y} \left(1 - \frac{\chi_d(p)}{p}\right)^{-1} \exp \left(\sum_{y < p < z} \frac{\chi_d(p)}{p} + O\left(\frac{1}{\sqrt{y \log y}}\right)\right).
\end{equation}
Furthermore, taking $k = \lceil \log x/(50 \log \log x) \rceil$ in Proposition 2.4 implies that the number of fundamental discriminants $d \in \mathcal{D}(x)$ such that
\begin{equation}
\left|\sum_{y < p < z} \frac{\chi_d(p)}{p}\right| > \frac{1}{(\log \log x)^{1/4}}
\end{equation}
is
\begin{equation}
\ll \sqrt{x} \left(\frac{\log x}{y \log y (\log \log x)^{1/3}}\right)^k.
\end{equation}
On the other hand, it follows from Corollary 2.2 that there are at least $\sqrt{x}e^{-y(1+o(1))}$ fundamental discriminants $d \in \mathcal{D}(x)$ for which $\chi_d(p) = 1$ for all primes $p \leq y$. Therefore, choosing $y = \log x/(2 \log \log x)$ we deduce from (2.14) and (2.15) that there are at least $x^{1/2 - 1/\log \log x}$ fundamental discriminants $d \in \mathcal{D}(x)$ such that $\chi_d(p) = 1$ for all primes $p \leq y$, (2.14) holds and
\begin{equation}
\left|\sum_{y < p < z} \frac{\chi_d(p)}{p}\right| \leq \frac{1}{(\log \log x)^{1/4}}.
\end{equation}
For these $d$, we have by (2.14) that
\begin{equation}
L(1, \chi_d) = e^{\gamma} \log \log x \left(1 + O\left(\frac{1}{(\log \log x)^{1/4}}\right)\right).
\end{equation}
Inserting this estimate in (1.12) completes the proof.

3. AN UPPER BOUND FOR THE NUMBER OF REAL QUADRATIC FIELDS WITH EXTREME CLASS NUMBER: PROOF OF THEOREM 1.2, PART (B)

In order to bound the number of real quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminant $d \leq x$ for which the class number $h(d)$ is extremely large (that is, $h(d)$ satisfies (1.11)), we shall first bound the number of small solutions to the Pell equations $m^2 - dn^2 = \pm 4$. We prove the following lemma.

Lemma 3.1. Let $x$ be large, and let $d \leq x$ be a positive integer. For a real number $\theta \in (1/2, 3/2)$ denote by $S_\theta(d)$ the set of positive solutions $(m, n)$ to the Pell equations
\begin{equation}
m^2 - dn^2 = \pm 4,
\end{equation}
such that $m \leq d^\theta$. Then, we have
\begin{equation}
\sum_{d \leq x} |S_\theta(d)| \ll (x^{1/2} + x^{\theta - 1/2})(\log x)^2.
\end{equation}
Proof. Let $P_\theta(d)$ be the set of solutions $(m, n)$ to the positive Pell equation $m^2 - dn^2 = 4$, such that $m \leq d^\theta$. Similarly, let $N_\theta(d)$ be the set of solutions $(m, n)$ to the negative Pell equation $m^2 - dn^2 = -4$, such that $m \leq d^\theta$. We shall only bound $\sum_{d \leq x} |P_\theta(d)|$ since the treatment for $\sum_{d \leq x} |N_\theta(d)|$ is similar. Let $(m, n) \in P_\theta(d)$. Then, note that $dn^2 \leq m^2 \leq d^2 \theta$, and hence $n \leq d^\theta - 1/2$. Furthermore, for a fixed $n \leq x^\theta - 1/2$, if $(m, n) \in P_\theta(d)$ for some $d \leq x$ then $m \leq n\sqrt{x} + 2$ and $m^2 \equiv 4 \pmod{n^2}$. Therefore, we deduce that

$$\sum_{d \leq x} |P_\theta(d)| \leq \sum_{n \leq x^\theta - 1/2} |\{m \leq n\sqrt{x} + 2, \text{ such that } m^2 \equiv 4 \pmod{n^2}\}|.$$ 

Let $\ell(q)$ be the number of solutions $m \pmod{q}$ of the congruence $m^2 \equiv 4 \pmod{q}$. Then, $\ell(q)$ is a multiplicative function, and

$$\ell(p^k) \leq \begin{cases} 2 & \text{if } p > 2, \\ 4 & \text{if } p = 2. \end{cases}$$

Hence, we derive

$$\sum_{n \leq x^\theta - 1/2} |\{m \leq n\sqrt{x} + 2, \text{ such that } m^2 \equiv 4 \pmod{n^2}\}| \ll \sum_{n \leq x^\theta - 1/2} \ell(n^2) \left(\frac{\sqrt{x}}{n} + 1\right) \ll \left(x^{1/2} + x^{\theta - 1/2}\right) \sum_{n \leq x^\theta - 1/2} \frac{\ell(n^2)}{n}.$$

The lemma follows upon noting that

$$\sum_{n \leq x^\theta - 1/2} \frac{\ell(n^2)}{n} \ll \prod_{2 < p \leq x} \left(1 - \frac{1}{p}\right)^{-2} \ll (\log x)^2.$$ 

□

The second ingredient in the proof of part (b) of Theorem 1.2 is to show that $L(1, \chi_d) \leq (2e^\gamma + o(1)) \log \log d$ for all but at most $x^{1/2 + o(1)}$ fundamental discriminants $0 < d < x$. To this end, we shall use Heath-Brown’s quadratic large sieve to show that Proposition 2.3 can be improved if we restrict our attention to quadratic characters. More precisely, we prove that $L(1, \chi_d)$ can be approximated by an Euler product over the primes $p \leq (\log x)^A$, for all but at most $x^{1/A + o(1)}$ fundamental discriminants $0 < d < x$.

Proposition 3.2. Let $A > 1$ be fixed. Then for all but at most $x^{1/A + o(1)}$ fundamental discriminants $0 < d < x$ we have

$$L(1, \chi_d) = \prod_{p \leq (\log x)^A} \left(1 - \frac{\chi_d(p)}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\log \log x}\right)\right).$$
Proof. First, it follows from Proposition 2.3 that

$$L(1, \chi_d) = \prod_{p \leq (\log x)^{2A}} \left( 1 - \frac{\chi_d(p)}{p} \right)^{-1} \left( 1 + O \left( \frac{1}{\log \log x} \right) \right),$$

for all except at most $x^{1/A + o(1)}$ fundamental discriminants $0 < d < x$. To prove the result, we are going to show that

$$\sum_{(\log x)^A < p < (\log x)^{2A}} \frac{\chi_d(p)}{p} = O \left( \frac{1}{\log \log x} \right),$$

for all but at most $x^{1/A + o(1)}$ fundamental discriminants $0 < d < x$. To this end, we will exploit Heath-Brown’s quadratic large sieve (see Corollary 2 of [6]) which asserts that

$$\sum_{0 < d < x} \left| \sum_{n \leq N} a(n) \chi_d(n) \right|^2 \ll \epsilon (xN)^{\epsilon} (x + N) \sum_{n_1 n_2 \leq N} |a(n_1)a(n_2)|,$$

where the $\sum^b$ is taken over fundamental discriminants, and the $a(n)$ are arbitrary complex numbers.

For $0 \leq j \leq J := [A \log \log x / \log 2]$, we define $z_j = 2^j (\log x)^A$, and put $z_{j+1} = (\log x)^{2A}$. Also, we let $k = [(\log x) / (A \log x)] + 1$ so that $z_j^k \geq x$ for all $0 \leq j \leq J + 1$. Now, similarly to (2.8) we have

$$\left( \sum_{z_j^k < p < z_{j+1}^k} \frac{\chi_d(p)}{p} \right)^k = \sum_{z_j^k < m < z_{j+1}^k} b_k(m; z_j, z_{j+1}) \chi_d(m),$$

where the coefficient $b_k(m; z_j, z_{j+1})$ is defined in (2.9). Then, by (3.3) we obtain

$$\sum_{0 < d < x} \left( \sum_{z_j^k < p < z_{j+1}^k} \frac{\chi_d(p)}{p} \right)^{2k} = \sum_{0 < d < x} \left( \sum_{z_j^k < n < z_{j+1}^k} \frac{b_k(n; z_j, z_{j+1}) \chi_d(n)}{n} \right)^2 \ll_{\epsilon} \epsilon (z_{j+1}^{1+\epsilon})^{k(1+\epsilon)} \sum_{z_j^k < m < z_{j+1}^k} \frac{b_k(m; z_j, z_{j+1}) b_k(n; z_j, z_{j+1})}{mn}.$$

Let $m$ and $n$ be positive integers such that $\Omega(m) = \Omega(n) = k$ and put $d = (m, n)$. Also, put $n = d n_0$ and $m = d m_0$ where $(m_1, n_1) = 1$. Since $mn = \square$ then both $m_1$ and $n_1$ are squares. Let $n_1 = \ell_1^2$ and $m_1 = \ell_2^2$, and put $s = \Omega(\ell_1)$. Since $\Omega(n) = \Omega(m) = k$,
then $\Omega(\ell_2) = s$ and $\Omega(d) = k - 2s$. Therefore, using (2.11) we obtain

$$b_k(n; z_j, z_{j+1})b_k(m; z_j, z_{j+1})$$

$$\leq \left( \frac{k}{2s} \right)^2 b_{2s}(\ell_1; z_j, z_{j+1})b_{2s}(\ell_2; z_j, z_{j+1}) \left( b_{k-2s}(d; z_j, z_{j+1}) \right)^2$$

$$\leq \left( \frac{k}{2s} \right)^2 \left( \frac{2s}{s} \right)^2 b_s(\ell_1; z_j, z_{j+1})b_s(\ell_2; z_j, z_{j+1})b_{k-2s}(d; z_j, z_{j+1})$$

$$\leq k! \left( \frac{k}{s, s, k - 2s} \right) b_s(\ell_1; z_j, z_{j+1})b_s(\ell_2; z_j, z_{j+1})b_{k-2s}(d; z_j, z_{j+1}),$$

since $b_r(e; z_j, z_{j+1}) \leq r!$ for any positive integers $r$ and $e$. Thus, we deduce

$$\sum_{z_j^b < n, m < z_{j+1}^b} b_k(n; z_j, z_{j+1})b_k(m; z_j, z_{j+1}) \sum_{mn}$$

$$\leq k! \sum_{0 \leq k \leq k/2} \left( \frac{k}{s, s, k - 2s} \right) \left( \sum_d b_{k-2s}(d; z_j, z_{j+1}) \right) \left( \sum_\ell b_\ell(\ell; z_j, z_{j+1}) \right)^2$$

$$\leq 3^k k! \left( \sum_{z_j < p < z_{j+1}} \frac{1}{p^2} \right)^k,$$

by (2.13). Inserting this bound in (3.4) and using that $\sum_{p > z_j} 1/p^2 \ll 1/(z_j \log z_j)$ yields

$$\sum_{0 < d < x} \left( \sum_{z_j < p < z_{j+1}} \frac{\chi_d(p)}{p} \right) \ll_{\epsilon} \left( z_{j+1}^\epsilon k \right) \ll_{\epsilon} \left( (\log x)^{2\epsilon A} k \right)^k.$$

Therefore, the number of fundamental discriminants $0 < d < x$ such that

$$\sum_{z_j < p < z_{j+1}} \frac{\chi_d(p)}{p} > \frac{1}{A(\log \log x)^2},$$

is

$$\ll_{\epsilon} \left( A^2 (\log \log x)^4 (\log x)^{2\epsilon A} k \right)^k \ll_{\epsilon} x^{1/A + 3\epsilon}.$$

Thus, we deduce that (3.2) holds for all but at most $O_{\epsilon}(x^{1/A + 3\epsilon} \log \log x)$ fundamental discriminants $0 < d < x$, as desired.

\[\square\]

**Proof of Theorem 1.2, part (b).** Taking $A = 2$ in Proposition 3.2, we deduce that for all but at most $x^{1/2 + o(1)}$ fundamental discriminants $x^{1/4} < d < x$, we have

$$L(1, \chi_d) \leq (2e^\gamma + o(1)) \log \log d.$$

Now, the class number formula (1.6) implies that if $d$ satisfies (3.5) then

$$h(d) \leq (2e^\gamma + o(1)) \sqrt{d} \cdot \frac{\log \log d}{\log \varepsilon_d}.$$
Therefore, if \( d \) is a fundamental discriminant such that \( x^{1/4} < d < x \), \( L(1, \chi_d) \) satisfies (3.3) and \( h(d) \) satisfies the bound (1.11), then \( \varepsilon_d \leq d^{1+o(1)} \). Now, recall that \( \varepsilon_d = (m + n\sqrt{d})/2 > m/2 \), where \((m, n)\) is a solution to the Pell equations (3.1). Thus, we deduce from Lemma 3.1 that the number of these fundamental discriminants is at most \( x^{1/2+o(1)} \), which completes the proof. \( \square \)

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