Soft Robots Modeling: a Literature Unwinding

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Abstract—The robotics community has seen an exponential growth in the level of complexity of the theoretical tools presented for the modeling of soft robotics devices. Different solutions have been presented to overcome the difficulties related to the modeling of soft robots, often leveraging on other scientific disciplines, such as continuum mechanics and computer graphics. These theoretical foundations are often taken for granted and this lead to an intricate literature that, consequently, has never been the subject of a complete review. Withing this scenario, the objective of the presented paper is twofold. The common theoretical roots that relate the different families of modeling techniques are highlighted, employing a unifying language that ease the analysis of their main connections and differences. Thus, the listing of the approaches naturally follows and a complete, untangled, review of the main works on the field is finally provided.

I. INTRODUCTION

The term soft robot appeared for the first time in a scientific paper in 2000, describing a McKibben pneumatic artificial muscle \cite{1}, a family of braided pneumatic actuators developed in the 50s to assist polio patients. Even though they were not explicitly called soft robots, McKibben actuators probably represent the first example of a robotic device exploiting its compliance to achieve improved performances with respect to their exclusively rigid counterpart. Since then, soft robotics has been one of the fastest growing research community in the last decades \cite{2, 3}.

Many different soft robotics devices and actuators have been presented, ranging in almost every possible technological field, from biomedical engineering to aerospace and underwater robotics. The increasing interest in soft robotics is demonstrated by the vast number of review papers that have been published to summarize the employed techniques, the achievements and the future prospects of this promising research field \cite{4, 5, 6, 2, 7, 8, 9, 10}. Among the available reviews published with regards to the theoretical modeling components, some have been published for specific groups of approaches, \cite{11, 12, 13, 14, 15}, some for specific application fields \cite{16} and other for specific families of robot \cite{17}. However, a comprehensive review on the modeling tools that have been presented to efficiently describe the static and dynamics of soft robots is still lacking.

An efficient theoretical framework is crucial for the prediction of the robot’s behaviour, the optimization of the design and finally for control. While the kinematic modeling of traditional rigid-link robots is fully defined by the link dimensions and the joint coordinates, the virtually infinite number of degrees of freedom of continuum robots drastically increases the complexity of their modeling. Analytic closed form solutions are available only for few cases characterized by basic loading condition and mostly one dimensional cases. For this reason, one of the biggest effort in the modeling of soft robots is the investigation of simplified assumptions and model reductions that are accurate enough to predict the robot’s behaviour and, at the same time, that are computationally efficient. The inevitable trade-offs between the model complexity, its computational cost and accuracy are the primary principles that should be considered when modeling continuum robots.

Within this scenario, the contribution of this paper is twofold. On one side, we present a complete literature review of the modeling techniques that have been presented so far, spamming from the more general approaches such as Finite Element Method to those that were specifically conceived for soft robotics applications.

On the other side, we present the general background and organize the defining characteristics of these modeling techniques presented in the literature. In this regard we go beyond a simple identification of already published work in the field and attempt to untangle such an intricate research topic. This is to provide the reader with the basic knowledge on the theoretical foundations of the published works.

The remaining parts of the paper are organized as follows. In Section \textsection I some general concepts are drawn and the classification employed in the paper to discuss all the modeling approaches is presented. In Section \textsection II we present the approaches that are based on continuum mechanics theories, that are also briefly presented to the reader. In Section \textsection IV we present the so-called geometrical approaches, which rely on the hypothesis that the deformed shape of the robot follows a specific geometrical curve. Proceeding with the level of reduction, in Section \textsection V the approaches that arise from a specific discretization of the soft body are presented, while in Section \textsection VI surrogate (also called data driven) models are discussed. Finally, Section \textsection VII presents the softwares implementation the proposed approaches, while in Section \textsection VIII the conclusion and the future perspectives are draw.

II. CHARACTERIZATION

The modeling of a mechanical system is the mathematical representation of the considered physical problem. In general,
the model of a mechanical system is defined by the following features \cite{18}, \cite{19}:

- **Configuration**: the set of position of the system’s particles compatible with the internal kinematic constraints (e.g. joints) and external ones (geometric boundary conditions). At any given time, the configuration of a mechanical system defines a subset of the Euclidean space.

- **Generalized Coordinates**: a set of continuous or discrete parameters describing any configuration. They are the coordinates of a unique point in an abstract space or "configuration space". We call them absolute coordinates when they refer to a fixed inertial frame, relative coordinates when they refer to a moving frame covering together with the system.

- **Kinematic map**: it takes as input the generalized coordinates and it returns the configuration of the system. This is also called forward kinematics, while the inverse kinematics represents exactly the opposite process, i.e. the calculation of the coordinates (typically, the actuation ones) required to obtain a specific configuration of the system.

- **Kinetic and Potential Energy**: the functions that determine the fundamental dynamics of the system. In addition, applied non-conservative forces and dissipation internal forces may act on the system through their virtual work. In the context of soft robotics, the potential energy is of particular importance as it can strongly characterize the approach.

It should be noted that these features are not always clearly expressed in the developing of a theoretical model, as sometimes they might be implicitly posed by the assumptions of the model itself. One of the aims of this work is to shed light on the underline structure of the models proposed in literature using a unified language.

Based on these features, in the present paper, the theoretical tools employed for the modeling of soft robots are classified as follows:

1) **Continuum Mechanics models.** They are all characterized by a continuous (infinite-dimensional) configuration space of the system, and on physical considerations about the soft bodies deformations. As such, the models in this category benefit from a physically rigorous definition of the kinetic and potential energy of the system. This family of models can employ both absolute and relative coordinates and they are derived from continuum mechanics theories. When no specific assumptions are made, they are based on the classical three-dimensional continuum mechanics theory. On the other hand, other (reduced) approaches have been presented for surface structures (shells, membranes) or slender structures (beam, rods). In particular, the latter include Cosserat, Kirchhoff and non-linear Euler Bernoulli beam theories, which are frequently employed in soft robotics.

2) **Geometrical models.** These models are all based on some sort of geometrical assumptions on the deformed shape undertaken by the soft body when specific loads are applied. For this group, the central role is taken by the generalized coordinates, on which the system’s kinetic and potential energy are defined. In particular, two main groups emerged. The first one includes the so-called "functional” models, which all rely on the assumption that the deformed shape of the body resembles a theoretical space curve represented by a specific mathematical function. In this case, the generalized coordinates are usually absolute in nature. The second group includes the widely known piecewise-constant-curvature (PCC) models, which are based on the discretization of the continuous soft body in a finite number of sections having circular arc shape, with intrinsically relative coordinates. This is by far the most numerous group of theoretical model proposed in the last decades, even though they proved to be reliable only for specific loading conditions and geometries.

3) **Discrete material models.** As the name itself suggests, these models are based on a discretization of the continuous body in a finite number of discrete material components. As such, they are characterized by an a priori finite-dimensional configuration space of absolute and/or relative coordinates, the relative ones being usually preferred in practice. Once the discretization is defined, different paths can be followed to obtain the kinematics of the system. Some models are based on traditional approaches employed for the modeling of rigid link robots. Other models, on the other hand, are based on the discretization of the soft body in a finite number of discrete elastic rods.

4) **Surrogate models.** In this family of approaches, sometimes also referred to as "data-driven", the configuration of the system is obtained employing sets of data and a learning process. The great majority of the approaches falling in this group use neural networks models and machine learning algorithms, or deep learning.

This classification has been defined to ease the explanation of the models, in an effort to group them based on the different path to discretization that they employ to tackle the presence of the non-linearities intrinsically entailed in the modeling of soft bodies. This could happen from the very first steps of the definition of the kinematics of the body, such as for the Discrete models, or when the involved equations are solved, which is the case of continuum mechanics models. On the other end, surrogate models often work around the problem of the definition of a mechanical model through the employment of machine-learning algorithms. Clearly, this is only one of the classifications that can be conceived and, in some cases, it is possible to note some overlapping between the models, that we will try to underline.
III. Continuum Mechanics Models

Soft robotics is an extremely interdisciplinary field and continuum mechanics represents the most influential community for the theoretical modeling of soft robots. Classical elasticity theories have been used for centuries to precisely model the mechanics of continuum bodies and thus they offer an established and general framework that is already available to the soft robotics community. Thus, one of the main goals when employing these approaches in the robotics community consists in making them computationally efficient for the considered purpose, avoiding any unnecessary redundancy.

Before going into the details of these theories, let us draw some general considerations on the main steps that are required to obtain a model for a continuum body. The first important remark that should be pointed out is that, while traditional rigid robots can be fully represented by some finite discrete set of frames, in soft robotics the medium is a continuum of particles. Even though this consideration is mostly taken for granted, it represents the starting point of all the following theories. In contrast to the Eulerian point of view of fluid mechanics, in the Lagrangian description of solid continuum mechanics, the body’s configuration is parameterized through positional fields, which depend both on the material coordinates and on time. In particular, the material coordinates \( X \) represent the coordinates of the particles of the body when it is in an arbitrary configuration, which is called reference configuration. The motion of a continuum body is defined by a continuous sequence of configurations along time. Differentiating once the positional field with respect to time provides the instantaneous flow velocity of the material particle, while the acceleration is obtained differentiating twice. A change in the configuration of a continuum body results in a displacement, which usually has two components: a rigid-body displacement and a deformation. To describe the internal deformation state of the body, it is necessary to define its strain (time) rate, which is a combination of the gradient of the velocity fields. Definition of strains needs to be objective, i.e., observer independent. Finally, beyond kinematics, the closed formulation describing the time evolution of a continuum medium is given by:

- a principle of the dynamics (Newton’s Law, d’Alembert’s principle, Hamilton’s principle), providing the PDEs relating the stress with the acceleration of the particles and the external forces applied inside and across the boundaries of the medium;
- a set of geometric boundary conditions (BCs);
- a constitutive law that relate the time evolution of the stress to that of the strains.

In general, such closed formulations are extremely nonlinear and one of the biggest challenges is the characterization of the simplifications strategies to solve them. In the following, we will start considering in Section III-B the models that are obtained from the three-dimensional elasticity theory, treating the soft body as a continuum medium. This represents the more general procedure, which however leads to high dimensional, non-linear equations. In this context, Finite Element Method represents by far the most popular approach to solve these equations in a computationally effective way. We will then move in Section III-B to the so-called “director approaches”, which replace the continuum medium to a rod as an oriented body [20]. This is characterized by a flexible curve and a continuous set of rigid cross sections of infinitesimal thickness, represented by a triad of orthonormal basis vectors (directors). These approaches are all based on the Cosserat rod theory which leads, which encompasses the Kirchhoff and large deflections (non-linear) Euler Bernoulli theories. The main advantage of these approaches for the modeling of soft robots is that, in principle, they do not require any hypotheses about the through-the-thickness distributions of displacement and stress fields [21]. For the same reason, however, the formulation of general constitutive laws appears as more complicated with respect to traditional three dimensional theories.

A. Classical 3D Models

Defining a (soft) body as a set of material particles \( \Omega \) labelled by their (material, in general curvilinear) coordinates \( X \), the aim of any three dimensional theory is to predict the time-evolution of its configuration \( r \) defined by:

\[
r(\cdot): X \in \Omega \mapsto r(X) \in \mathbb{R}^3 \tag{1}
\]

In order to proceed, the first step consists in obtaining (and solving) the balance equations of the system. Shall we consider a material subpart of the body \( B \subset \Omega \), having frontier \( \partial B \) and outward unit normal \( n \in \mathbb{R}^3 \). In general, volume and surface forces are applied upon \( B \), and modelled by some volume and surface vector densities, \( b \in \mathbb{R}^3 \) and \( t \in \mathbb{R}^3 \) respectively. While the first represents the external forces acting on the volume, the latter represents the forces that \( B \) exchanges with its surroundings (the other parts of \( \Omega \)) across its boundary \( \partial B \), which are also known as internal forces. In particular, \( t \) is called stress vector and we admit the Cauchy theorem, i.e. \( t \) is a linear function of \( n \):

\[
t = \sigma n \tag{2}
\]

where \( \sigma \in \mathbb{R}^3 \times \mathbb{R}^3 \) is the Cauchy’s stress tensor. It is then possible to express the balance equations for any part \( B \) of \( \Omega \), thus the force resultant is null:

\[
\int_B (b - \rho \dot{v}) dV + \int_{\partial B} t dS = 0 \tag{3}
\]

where \( \rho \) is the mass density, while \( v \) is the body velocity. Using Cauchy (2), and divergence theorems yields:

\[
\int_B \left( b_i + \frac{\partial \sigma_{ji}}{\partial x_j} - \rho \ddot{v}_i \right) dV = 0 \tag{4}
\]
This relation should hold for any volume $\mathcal{B}$, and finally the Cauchy equilibrium equations are obtained:

$$\nabla \cdot \sigma + b = \rho \dot{\mathbf{u}}$$  \hspace{1cm} (5)

In order to obtain the configuration of the body from the solution of (5), it is necessary to define a constitutive relation characterizing the response of the body’s material under external forces (or, more generally, any external stimuli) and providing the relationship between the stress tensor $\sigma$ and the strains. There are different ways to define a constitutive relation and the first and most famous example is the Hooke’s law for linear elastic materials, which in the 1D case reads $\sigma = E \epsilon$. One other more general constitutive relation has been proposed by Green (1839, 1842), which employs the concept of strain energy function. More precisely, a Green elastic material (also known as hyperelastic material) is a type of constitutive model for which the stress–strain relationship can be obtained from a strain energy density function (or stored-energy function). This function depends symmetrically on the principal stretches of the deformation $\lambda_1, \lambda_2$ and $\lambda_3$, which, for incompressible materials satisfy the constrain $I_3 = \lambda_1 \lambda_2 \lambda_3 = 1$, where $I_3$ represents the third strain invariant. It is worth to remind that the stretch ratios $\lambda_i$ are defined as the ratio between the stretched length $L_i$ and the undeformed one $L_{0,i}$. Many soft robotics devices and components are realized with rubber-like materials, which, in the case of static deformations, are often treated as hyperelastic (while, in general, they can exhibit other nonlinear behaviors such as hysteresis, viscoelasticity and stress softening). In particular, the Ogden [22] material model represents one of the more general framework for the modeling of hyperelastic materials and it can reduce to the Neo-Hookean and Mooney–Rivlin solids. In the Ogden material the strain energy density is expressed in terms of the principal stretches as:

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{j=1}^{n} \frac{\mu_j}{\alpha_j} (\lambda_1^{\alpha_j} + \lambda_2^{\alpha-j} + \lambda_3^{\alpha_j} - 3)$$  \hspace{1cm} (6)

where $n$, $\mu_j$ and $\alpha_j$ are material constants. In particular, for $n = 1$ and $\alpha = 2$, the Neo-Hookean solid is obtained, while the Mooney-Rivlin material for an incompressible material is obtained for $n = 2$, $\alpha_1 = 2$, $\alpha_2 = -2$. Once the strain energy function is defined, the principal Cauchy stresses $\sigma_1$, $\sigma_2$ and $\sigma_3$ are related to the stretches through the equations:

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p,$$  \hspace{1cm} (7)

where $p$ represents the Lagrangian multiplier associated with the incompressibility constraint. It is worth to remind that the Cauchy principal stresses are related to the corresponding nominal (or engineering) stresses $\sigma_e$ by the relationship $\sigma_{e,i} = \sigma_i \lambda_i^{-1}$. The great majority of the materials that are employed within the soft robotics community can be considered as hyperelastic. However, most soft robotics modeling approaches still rely on a linear-elastic material assumption, as they are mostly focused on the description of the large deformation of the body, rather than the large strains. Many classical theories of rods have been derived from (5) by employing simplifying assumptions on their average solution over the cross section, as in the de Saint Venant’s (1855) principle, which however are not usually employed within the soft robotics community. The most popular numerical approach to solve three dimensional continuum problems is the Finite Element Method.

1) Finite Element Method: The Finite Element Method (FEM) is a well-known and widely-spread numerical technique for finding approximate solutions to Partial Differential Equations (PDEs). It was established by a set of scientific papers in the 40s and it soon became one of the most commonly employed technique for the modeling of a wide range of engineering problems, from structural mechanics to fluid flow and heat transfer. The main property of this technique is the sub-division of the problem’s domain into a set of smaller parts, called finite elements. These are obtained by the construction of a mesh which represents the numerical domain for the solutions, and defines the discrete variables of the system as the value of the position field (and its derivatives) at the edges of the elements, called nodes. Once this discretization is performed, the continuous field $r$ is approximated on each element by a polynomial interpolation of its values at the nodes according to a Ritz reduction approach. Introducing this discretization in the above continuous formulation and projecting it on the same trial (polynomials) basis, changes the PDEs (5) into a set of algebraic discrete equations for steady state problems, or ODEs for transient problems. In the former case, the static equilibrium is obtained as a system of algebraic equations in the form:

$$Q_{int}(q) = Q_{ext}(q)$$  \hspace{1cm} (8)

where $Q_{int}(q)$ and $Q_{ext}(q)$ represent the internal and external generalized forces, respectively, while $q$ are the generalized coordinates, which in the FEM formulation are the nodes displacements. On the other hand, the dynamic equilibrium is given by a system of ODEs:

$$\begin{array}{l}
\dot{q} = v \\
M(q) \ddot{v} + Q_{int}(q, v) = Q_{ext}(q, v)
\end{array}$$  \hspace{1cm} (9)

where $M(q)$ is the generalized mass matrix. Note that the terms appearing in (8) and (9) are computed integrating the distributed variables element-by-element according to the selected shape functions, which play the role of the kinematic map, while the position of each element in the mesh is ensured by the assembly process.

Since nodal coordinates are of absolute nature, the non-linearities appears in $F_{int}(q, v)$. They are named geometric or material, depending whether they come from the large displacements (and namely the rotations) of the elements, or from the constitutive laws. To overcome the geometric non-linearity, corotational approaches are used in three-dimensional FE softwares. One other solution to represent the non-linearities is the employment of an absolute nodal coordinate formulation [23].

The FEM represent a standard for many popular simulation software, such as Abaqus, ANSYS, Comsol. The main advantage of these software is that they provide a powerful and general tool that can be applied to a wide spectrum of physical problems ranging from structural dynamics, fluid-structure
interactions, contact and thermodynamics. They provide a ready made and user-friendly framework that can be easily used for specific studies on multiphysics systems. They have been widely used to model soft robots \(24\), \(25\) (Abaqus), \(26\) ANSYS. At the same time, their generality intrinsically entail an increased computational cost and the simulations, especially in dynamics, can take a lot of time to converge, sometimes even days.

There are also some popular examples of ad-hoc Finite Element approaches (and softwares) that have been specifically proposed for (soft) robotics applications. In 2007 \(27\), a group of researchers from different institutes released Sofa (Simulation Open Framework Architecture) an open-source C++ library which was originally presented as a computational environment for medical simulations \(28\), \(29\). In the following years, Sofa became a comprehensive high-performance library that has been widely implemented for different application fields, and a SoftRobots plugin was also created for the design \(30\), modeling and control \(31\), \(32\), of soft robots, including self-collision scenarios \(33\). Moreover, a model combining the FEM and the discrete Cosserat approach described in Section III-B is presented in \(34\).

A discussion on the architecture of the software is addressed in Section VII while here we mostly focus on the modeling part for the soft bodies. In Sofa, a deformable continuum is modeled using a dynamic or quasi-static system of simulation nodes. The node coordinates are the independent DOFs of the object, and they are typically governed by equations of the type \(5\) and \(9\). Some approaches have also introduced reduced coordinates and these are presented in Section VI-B.

With regards to the actuation, a constraint-based approach is employed. Imposing a linearization of the internal forces \(Q_{\text{int}}(q_i) \approx Q_{\text{int}}(q_{i-1}) + K(q_{i-1})dq_i\) and considering the contribution of the actuation constraints, the static equilibrium at each \(i\)-th step \((8)\) becomes:

\[
Q_{\text{int}}(q_{i-1}) + K(q_{i-1})dq_i = Q_{\text{ext}}(q_{i-1}) + B(q_{i-1})^T\lambda (10)
\]

Above, the term \(B^T\lambda\) models the contributions of the Lagrange multipliers (i.e. the contribution of the actuators), where \(B\) represents the Jacobian of the constraint equations imposed by the actuators. Three steps are then performed. In the first step, a free configuration \(q^{\text{free}}\) is found for \(\lambda = 0\) and, for each constraint, the violation \(\delta^{\text{free}}\) is estimated. In the second step, the solver computes the value of \(\lambda\) through a projection of the mechanics into the constraint space, obtaining the smallest possible projection space \(\delta = BK^{-1}B^T\lambda + \delta^{\text{free}}\). In the last and third step, the final configuration is corrected using the value of the constraint response \(q = q^{\text{free}} + K^{-1}B^T\lambda\).

### B. Directors Approaches

As discussed in Section III-A, in general, in order to determine the stresses in a generic body under the action of given external loads, the solution of the equations of three dimensional elasticity \(5\) appears as the canonical procedure. Sometimes, however, the slenderness of the body (such in the cases often considered in soft robotics) suggests an alternative method, leading to the so-called directors approaches. Indeed the prototype of a slender body is a cylinder whose diameter is much smaller when compared to its length. Thus, a solution to the problem can take the form of a limit of the three dimensional solution when the diameter of the cylinder approaches zero, while the resultants of loads and the global stiffness of the body tend to finite limits \(20\). In order to properly model bending and twisting, the line is endowed by directions that can rotate independently of the deformation of the line itself. The directions and the associated axis are called directors and a material curve combined with a collection of directors associated to each particle is called a Cosserat rod \((20\) Cosserat and \(8\) Cosserat, 1907) \(33\). In other words, the Cosserat model treats the slender body as a deformable curve in which every particle is rigidly connected to a triad of orthonormal vectors (directors) to characterize its orientation.

The balance equations can be derived considering an arbitrary subinterval \([a, b] \subset [0, L]\) of the rod, Figure 3. Across any section splitting the rod in two parts, the action of the material on one side upon the other side is equivalent to the resultant stress vector \(n\) and the resultant couple \(m\) in the global frame. Moreover, we suppose that \(\rho\) is the mass density of the rod, \(I\) is the inertia tensor, while \(\nu\) and \(\omega\) are the linear and angular velocity in the global frame. The rod is also subject to the external forces \(\vec{f}\) and moments \(\vec{l}\) distribution per unit of length \(X\) in the global frame. It is then possible to write the linear and angular balance equation at the equilibrium \(20\):

\[
\begin{align*}
\left\{\begin{array}{l}
n_{ia}^b + \int_a^b (\vec{f} - \rho \vec{A}\dot{\nu}) \, ds = 0, \\
m_{ia}^b + (r \times n)_{ia}^b + \int_a^b (I - \partial_t(\rho I \cdot \omega)) \, ds = 0,
\end{array}\right.
\end{align*}
\]

Considering also the boundary equations, we obtain the dynamic equilibrium:

\[
\begin{align*}
\left\{\begin{array}{l}
n' + \vec{f} = \rho \vec{A}\dot{\nu}, \\
m' + r' \times \vec{f} + \vec{l} = \partial_t(\rho I \cdot \omega)
\end{array}\right.
\end{align*}
\]

\[
\begin{align*}
n(0) = -f_0, \quad n(L) = f_L, \quad m(0) = -l_0, \quad m(L) = l_L,
\end{align*}
\]

Above, \(l_{0,L} f_{0,L}\) are the moments and forces applied in \(X = 0\) and \(X = L\), respectively. Finally, in order to obtain the solution to equations \(12\) and to obtain the configuration of the body, it is necessary to define a constitutive model which relates the internal forces and moments with the strains. A system of six scalar equations is obtained, thus the involved kinematical variables should be six or less.

Cosserat rod theory was introduced in bio-robotics \(36\) to study locomotion dynamics of hyper-redundant swimmers,
in medical robotics for modeling continuum robots, and in soft robotics for modeling soft arms inspired from octopus. The configuration of a Cosserat rod is defined by a vector \( \mathbf{r}(X,t) \in \mathbb{R}^3 \) representing the position of the center-line and a relative reference frame having orientation represented by a rotation matrix \( \mathbf{R}(X,t) \in SO(3) \). Both the mid-line and the orientation matrix are defined as function of a curvilinear abscissa \( X \in [0,L] \), where \( L \) is the length of the robot. Thus, the configuration of a soft body is defined as a curve:

\[
g(\cdot,t) : X \in [0,L] \mapsto g(X,t) \in SE(3)
\]  

which can be expressed in terms of \( \mathbf{r}(X,t) \) and \( \mathbf{R}(X,t) \) by the homogeneous matrix:

\[
g(X,t) = \begin{pmatrix} \mathbf{R}(X,t) & \mathbf{r}(X,t) \\ 0^T & 1 \end{pmatrix}
\]  

Moving to the strain definition, the space variations of the field \( g(X,t) \) can be entirely described by the vector field \( \xi(\cdot,t) : X \in [0,L] \mapsto \xi(X,t) \in se(3) \cong \mathbb{R}^6 \) which is given by the boundary value problem:

\[
g' = g\hat{\xi}, \quad g(0) = g_0
\]  

In particular, it is possible to define the strain field \( (\xi - \xi^*) \), where \( \xi^*(X) \) is given by \([15]\) in the reference configuration \( \xi^*(X) \), while \( \xi^* \) can be expressed as:

\[
\xi = \begin{pmatrix} \mathbf{K}^T, \mathbf{\Gamma}^T \end{pmatrix}^T \in \mathbb{R}^6, \quad \hat{\xi} = \begin{pmatrix} \hat{\mathbf{K}}, \mathbf{\Gamma} \\ 0, 0 \end{pmatrix} \in se(3)
\]  

Above, \( (\mathbf{\Gamma} - \mathbf{\Gamma}^*) \), \( (\mathbf{K} - \mathbf{K}^*) \) \( \in \mathbb{R}^3 \) represent the linear and angular strains, respectively, while \( (\cdot) \) represents the isomorphism between \( \mathbb{R}^3 \) and the algebra of skew-symmetric matrices \( so(3) \).

On the other end, the derivative of the configuration \([14]\) with respect to time yields:

\[
g' = g\hat{\eta},
\]  

providing the velocity twist:

\[
\eta = \begin{pmatrix} \mathbf{w}^T, \mathbf{v}^T \end{pmatrix}^T \in \mathbb{R}^6, \quad \hat{\eta} = \begin{pmatrix} \dot{\mathbf{w}}, \dot{\mathbf{v}} \\ 0, 0 \end{pmatrix} \in se(3)
\]  

where \( \mathbf{w}(X,t), \mathbf{v}(X,t) \in \mathbb{R}^3 \) represent the angular and linear velocity in the local frame, respectively, which are related to their global counter-part through the rotation matrix \( \mathbf{R} \), i.e. \( \mathbf{w} = \mathbf{R}^T \mathbf{w}, \mathbf{v} = \mathbf{R}^T \mathbf{v} \).

Depending on the constraints considered by the definition \([16]\), different non-linear rod theories can be obtained. In the most general case, all six components of the vectors \( \mathbf{\Gamma} \) and \( \mathbf{K} \) can be defined by an arbitrary time evolution which corresponds to the Simo-Reissner beam’s model. On the other end, the Kirchhoff rod theory can be obtained by fixing the first component \( \Gamma_1 \) of \( \mathbf{\Gamma} \) to one, and the two others to zero, which prevents the beam from stretching and shearing respectively.

In \([37]\), equations \([12]\) are expressed in terms of wrenches (i.e. \( 6 \times 1 \) vectors). In particular, for a Cosserat beam subject to a density of state-dependent external wrench \( \mathbf{F} \) on \( [0,1] \) and two tip external wrenches \( \mathbf{F}_0 \) and \( \mathbf{F}_1 \) at \( X = 0 \) and \( X = 1 \) respectively, the dynamic balance equations in the local frames are:

\[
\begin{cases}
M\ddot{\eta} - a\mathbf{d}^T M \eta = \mathbf{N} - a\mathbf{d}^T \mathbf{A} + \ddot{\mathbf{F}}, \\
\mathbf{A}(0) = -\mathbf{F}_0, \quad \mathbf{A}(1) = \mathbf{F}_1
\end{cases}
\]  

where \( \mathbf{a} \) is the adjoint representation of the Lie algebra and \( \mathbf{M} \) is the cross sectional inertia tensor. Above, \( \mathbf{A} \) models the stress field along the beam (it is the dual counterpart of the strain field). This model is directly related to \([12]\), by using the relations between vectors of components in the inertial and local frames: \( \mathbf{A} = \left( (\mathbf{R}^T \mathbf{m})^T, (\mathbf{R}^T \mathbf{n})^T \right)^T \) and \( \dot{\mathbf{F}} = \left( (\mathbf{R}^T \dot{\mathbf{l}})^T, (\mathbf{R}^T \dot{\mathbf{f}})^T \right)^T \). Note that equation \([19]\) is a general framework that can support any non-linear constitutive law beyond the linear Hooke model, although the process of obtaining that constitutive relation is more complicated than in the classical continuum setting.

One of the most common procedure to reduce the number of variables consists in employing Kirchhoff’s hypotheses (1859). The theory is based on the following four constitutive assumptions \([20]\): (1) the axis of the rod is inextensible; (2) the stress couple varies linearly with the curvature and the twist; (3) there are no shear stresses between the cross-section and the axis; (4) the cross-section is undeformable in its own plane. In particular, one of the analytical consequences of these hypotheses is that the constitutive relations between the couple resultant \( \mathbf{m} \) and the deformation of the rod are not simply linear, but they also have the special form \([20]\):

\[
\mathbf{m}(X,t) = \mathbf{R}(X)\text{diag}(A,B,C)\mathbf{K}(X,t)
\]  

where \( A(X), B(X) \) and \( C(X) \) depend on the material and on the geometry of the cross-section. In 1944, the theory was further enhanced by Love \([38]\) to also account for small axial tension and in 1972 \([39]\) and 1981 \([40]\) Reissner completed the theory adding two deformation measures representing the shear of the beam.

A special case of Kirchhoff’s theory is represented by non-linear Euler Bernoulli (EB) rod theory, whose linearized version is still the most used method for the analysis of bending elements in structural mechanics. Let us now consider a rod which is straight in its undeformed configuration and whose bending plane is the one for which the flexural rigidity coincides with the scalar \( B(X) \) in equations \([20]\). As a consequence of this assumption, \( \kappa_1 \) and \( \kappa_3 \) vanish, yielding to the so-called elastica:

\[
B(X)\theta'(X) = M_{\text{ext}}(X)
\]  

where \( B(X) \) is the bending stiffness of the rod , \( \theta' = \kappa_2 \) and \( M_{\text{ext}}(X) \) is the integral of the external bending moment about \( X \) acting on the remaining of the beam.

In the following, we discuss the most important examples of approaches employing the Cosserat rod theory for the modeling of soft robots. In particular, the different approaches are grouped based on the numerical techniques that are employed to solve \([12]\). It should be noted that, while the great majority of these approaches describe the soft robot as a rod, there are also some examples of modeling techniques which have been
presented for 2D shells [41, 42] and combination of Cosserat rods, shells and rigid bodies [43].

1) GE-FEM: One of the most famous and powerful methods to capture in an exact manner the geometric nonlinearities of soft robots is the geometrically-exact FEM (GE-FEM) introduced by J.C. Simo [44, 45, 46]. In this context, the model of rotations is introduced at the same level as that of positions, by replacing the material points of classical medium by the rigid micro-structures of a Cosserat medium. The GE character, then imposes to apply the FEM without resorting to any simplifications on finite rotations, except the unavoidable space and time discretizations required by the numerical resolution. The GE-FEM was particularly developed for shells [41, 42] and rods [47, 48, 49, 50], where micro-structures are rigid transverse fibers, and rigid cross-sections respectively. The method’s tour de force was to generalize all the key operations of the FEM (interpolation, space and time discretizations required by the numerical software through its plugin MECANO. In this software, the model of rods is the full-Cosserat model of Reissner. However, its dynamic resolution is ill-conditioned when the aspect ratio of the rod increases, i.e. for very slender rods. To overcome this limitation, several FEM based on the non-linear Euler-Bernoulli or Kirchhoff rod models, were proposed, such as the Kirchhoff GE-FEM of [49] and [51].

The GE-FEM has been rarely applied to soft robots modelled by Cosserat rods. This is probably due to the fact that it is computationally heavy and difficult of access for researchers not experienced in non-linear structural dynamics. Nevertheless, in [52], a GE-FEM approach for the modeling of inflatable robots is presented. In this case, the rotational dof of a rod are represented by quaternions without unity constraints $q(X) \in \mathbb{H}$. It is then possible to define an additional strain variables $\xi(X) = \Upsilon^{-1}\Upsilon'(X)$ where $\Upsilon(X) = ||q(X)||^2$, which model the inflation of the rod and its space-rate, respectively. These can then be related to the other strain variables to obtain the kinematics, in the quaternion algebra, incorporating bending, twisting, extension, shear and inflation. In this way, the configuration is isophormic to a subspace of 7D functions. Then, the weak form of the static equilibrium is provided and solved employing a off-the-shelf FEM software (COMSOL).

Nevertheless, a geometrically-exact finite element approach for the modeling of soft robots is presented in [53]. Taking the cue from [50], a helical shape function is employed in terms of an exponential map to express the shape of the flexible manipulator, which is given in terms of the initial position and orientation at the base ($X = 0$) and the strain $\xi(X)$. The resulting dynamic equations take the general form [9] and is solved employing a geometric time integration, which has the main benefit of preserving the Lie group structure and guaranteeing the geometrical exactness of the approach.

2) Ritz-Galerking Methods: In numerical analysis, the Ritz method is a direct method to solve a BVP, such as [15] as soon as the rod’s configuration is parameterized by a set of vector fields (e.g. the positional field along the backbone $r$ and the vector field of its 3 cross-sections Euler’s angles $\theta$). Applying the separation of variables, the components of these fields can be approximated on a truncated basis of Ritz functions compatible with the boundary conditions:

$$ r(X, t) \approx \sum_i \Phi_i(X) q_i(t) $$

$$ \theta(X, t) \approx \sum_i \Psi_i(X) q_i(t) $$

where $\Phi(X)$ and $\Psi(X)$ are the spatial shape functions, while $q(t)$ are the time-dependent generalized coordinates.

In this Section we gather the soft robotics models that rely on this approximation, which, in some cases, is directly applied to the configuration variables as in (22), while in others, it is applied to the strain fields of the rod.

In [54], the Ritz and Ritz-Galerkin methods are used to derive a weak-form series solution for a Cosserat rod statics and dynamics. In particular, the kinematics of the rod is obtained expressing equation (15) in a inertial reference system, placed at the base of the manipulator. Then, the backbone kinematics can be approximated as in (22). Applying (22) to this field of angle and introducing all these approximations into the balance equations projected onto the same truncated basis, provides a set of algebraic equations similar to those obtained in the FEM approach which from this view point is no more than a piece-wise Ritz method. In the static case, a numerical solver is employed to find the coefficients $q_i$ that minimize the approximation error, while in the dynamic case the coefficients $q_i(t)$ are found from the time numerical integration of the equations of motion, now in an ODE form [9]. Based on an experimental observation, the system states $q_i$ are assumed to be the coefficients of a series of Lagrange polynomial shape functions. This polynomial, given in the form (22), is finally fitted to three control points (at the base, in the middle and at the tip of the manipulator). Considering that at the base $r(X, t) = [0, 0, 0]$, the final configuration of the continuum robot is reduced to the position of only two physical points.

Alternatively to these reduction methods based on absolute coordinates, the approximated solution in the form (22) can also be obtained using the collocation method. The main idea is to replace the unknown strain field with a polynomial and setting the vanishing of the residual between the two in a finite set of points on the domain, also called collocation points. Considering $m$ collocation points, a total of $m + 1$ equations is obtained, providing the parameters that are required to define an $m$-th order polynomial. In [55], the ODEs describing the kinematics of a Cosserat rod are directly solved in terms of transformations on $SE(3)$, using a combination of orthogonal collocation and forward integration through Magnus Expansion. In particular the unknown $\xi(X)$ in (15) is expressed as a set of three Chebyshev polynomials (which would be six if the shear strains would have been considered). To ensure that the interpolating polynomial satisfies the boundary conditions, the minimization of the residual error is imposed, providing a set of non-linear algebraic equations in the form (8). The
computing of these equations require the integration of \(\mathbf{R}(X)\) at the collocation points and this is finally achieved using a Magnus Expansion.

The Ritz approach has been recently applied to the strain fields of Cosserat rods in the context of the Piece-wise Variable Strain (PVS) approach\(^{[56, \text{57}]}\). In this case, the Ritz coefficients of the strains, stand for the generalized coordinates of a set of homogeneous transformations along the soft robot, similar to the joint transformations for rigid ones\(^{[19]}\). This (relative) strain parametrization, produces a highly reduced set of ODEs in the classical form\(^{[9]}\), where \(q\) is then comparable to the vector of joint angles of a rigid robot. In\(^{[37]}\), the PVS was successfully validated against the GE-FEM with standard benches in statics and dynamics. The results showed that the approach can provide very good results in terms of accuracy with a very few number of generalized coordinates. Moreover, in contrast to absolute Ritz methods, the strain functions do not need to fulfill any boundary conditions. However, the cost to pay for this, is that it handles double space-integrals that are not easy to compute. In\(^{[37]}\), a Newton-Euler inverse dynamic algorithm is used to achieve these computations. Recently, other analytical methods were developed. They exploit the fact that equation\(^{[15]}\) is a system of homogeneous first order ODEs that can be expressed in the matrix form \(Y' = YA(X)\). Then, the exponential representation of the solution can be obtained through the Magnus Expansions, yielding, for a soft manipulator, to the kinematic map in the form:

\[
g(X) = \exp\left(\Omega(q, X)\right)
\]

where \(\Omega(q, X)\) is the Magnus expansion of \(\dot{\xi}(X)\) and \(\exp\) represents the exponential map in \(SE(3)\).

With regards to the velocity twist\(^{[18]}\), the equality of the mixed partial derivative of \(g\) yields\(^{[22]}\):

\[
\dot{\eta}'(X) = \dot{\xi} - \text{ad}_{\xi}\eta,
\]

\[
\dot{\eta}(X) = \dot{\xi} - \text{ad}_{\xi}\eta - \text{ad}_{\xi}\eta,
\]

In\(^{[57]}\), the analytical integration of equations\(^{[24]}\) has been provided. The strain discretization in the form\(^{[25]}\) can then be inserted in the differential kinematics for the system, yielding to the geometric Jacobian of the kinematic map and its time derivative \(\dot{\eta} = J(q)\dot{q} + J'(q)\dot{q}\) in \(se(3)\). The generalized dynamic equation of the system are finally obtained via D’Alembert’s principle. The continuous models is projected onto the space of generalized coordinates through the Jacobians and integrating over the length of the soft links yields to the equation of motion in the classical form\(^{[9]}\). In particular, the actuation forces are now included in the internal forces, and not in the external ones as in other Cosserat rod approaches:

\[
\Lambda = F_{\text{ela}}(X) + \sum_{i=1}^{n_a} \left[ \mathbf{d}_i(X) t_{ci}(X) \right] u_i
\]

where \(n_a\) is the number of actuators, \(\mathbf{d}_i\) is the distance from the mid-line to the actuator, while \(t_{ci}(X)\) represents the tangent unit vector to the actuator path. In\(^{[58]}\), the PVS approach is further extended to the modeling of concentric tube robot systems, including the modeling of the tube’s insertion motion.

Finally, the PVS approach has been implemented in a Matlab Toolbox, \textit{SoRoSim}, for the simulation of soft, rigid and hybrid robots\(^{[57]}\). A special case of the PVS is the Piecewise Constant, Approach (PCS)\(^{[59, \text{60}]}\), where the strains in the sections are assumed to be constant, which was also extended to closed-chain geometries in\(^{[61]}\).

3) \textit{Shooting Methods:} Depending on the considered external forces and initial conditions, the equations describing a Cosserat rod\(^{[19]}\) and its kinematics\(^{[15]}\) represent a boundary value problem (BVP). Here, we gather all the approaches that employ standard numerical techniques, such as the shooting method, to solve this problem. In this way, the rod is treated as a continuous body and the numerical approximation is directly applied to solve the involved equations. The shooting method represents one of the most popular numerical approach to solve a BVP, transforming it into an equivalent initial value problem. The initial value problem is iteratively solved for different values of initial conditions, until a solution that satisfies the desired boundary condition is found. In other words, the solver \textit{shoot} the solution from the starting boundary condition towards the final point and the iterations are continued until the error with respect to the targeted boundary condition is lower than the required precision.

Trivedi\(^{[62]}\) presents the geometrically exact Cosserat model for an internally pressurized soft robotic manipulator. The kinematics of the robot is obtained expressing the directors as functions of quaternions and these are replaced in the definition of the space derivatives of both \(r\) and the directors. In particular, in the planar case, the directors can be expressed as functions of the cross section angle \(\theta\) and an Euler Bernoulli model enhanced with shear is obtained. In the static equilibrium\(^{[12]}\), the internal pressurization is represented as a distributed force acting in the axial direction and subtracted to the internal force \(n\). The other considered external forces are the weight of the robot and the concentrated forces and moments applied by the actuator at the end of each section. Finally, the stress and the moment resultants \(n\) and \(m\) in\(^{[12]}\) are decomposed with respect to the directors to obtain a system of six first order differential equations governing the relationship between the curvatures, the torsion and the stress. Even though this is not stated in the paper, we believe that the shooting method or a finite difference method is finally applied to the obtained BVP.

Rucker et al.\(^{[63]}\) employ the continuum Cosserat theory for the static and the dynamic modeling of continuum robots with tendon actuators. In order to couple the rod and the tendon actuation models, the external forces and moments distributions in\(^{[12]}\) are represented as the sum of the action of the \textit{actual} external loads and the action of the tendons. The tendons are modeled as ideal strings, which can take only a tension \(u_i\) that is assumed to be constant along its length (i.e. there is no friction between the tendon and the soft body). The internal forces acting on the tendon can then be expressed as:

\[
n_i(X) = u_i t_{ci}(X),
\]

where \(t_{ci}\) is the unit vector tangent to the cable path. The distributed force along the tendon can then be obtained imposing the linear equilibrium\(^{[12]}\). Finally, the total distributed force
given by the tendon is equal and opposite to the sum of the individual force distributions on the tendons \( F_i = -n'_i(X) \). Similarly, the distributed moment at the backbone is the sum of the cross products of each moment arm with each force. Expressing each tendon’s parametric space curve as function of the strain variables and through some manipulations, the explicit model equations are obtained as a BVP and the shooting method is applied to solve them numerically. In 64, the numerical solution of the dynamics of Kirchhoff rods is presented. In particular, a time discretization strategy is employed to transform the standard PDEs of a Kirchhoff rod into BVPs in the spatial dimension. This is obtained replacing the time derivative terms with an implicit differentiation formula. The same approach has been employed and extended in [65] for the dynamic modeling of tendon-driven robots.

In [66] and [67], the continuum Cosserat rod theory is applied to the modeling of concentric tubes robots. In both cases, the tubes are treated as Kirchhoff rods. Then, some considerations are drawn based on the assumption that the tubes are, in fact, concentric, which means that they all share the same center-line \( r(X) \) and each tube’s \( R_i(X) \) differs from the others by a rotation about the local tangent axis. In [67] the static equations [12] are modified to take into account the additional concentric tube constraint and then solved employing a shooting method. On the other end, in [66], instead of employing directly equations [12], the Euler-Poincaré equations are applied on \( SE(3) \times S(1) \times \ldots \) to provide the constrained governing equations and, finally, the shooting method is employed to obtain the solutions.

In [68], a soft actuator is composed of a pliant hollow body that is internally pneumatically actuated and whose swelling is constrained with rigid hoops placed along its length. On one side, the curvature of the actuator is described using the linear EB rod theory, while a visual tracking system is employed to obtain the actuator’s deformed shape at varying the pressure load.

4) Analytical and Euler Bernoulli: The analytical solution to equation [21], can be obtained only for specific loading conditions (for example, only when the rod is subject to concentrated external loads, and not distributed ones) and it is given in terms of elliptic functions. For example, considering the case of a simply supported beam subjected to a concentrated compressive force \( F \) applied at the tip, the analytical solution of the Euler Bernoulli equation yields:

\[
F = \frac{EI}{L^2} 4 \left[ K \left( \sin \left( \frac{\theta(0)}{2} \right) \right) \right]^2, \tag{27}
\]

where an homogeneous cross section along the length of the beam has been considered (i.e. \( B(X) = EI \), where \( E \) is the material’s Young’s Modulus and \( I \) is the area moment of inertia) and \( K \) is the complete elliptic integral of the first kind. It should be noted that equation [27] provides an implicit expression of the base angle \( \theta(0) \) as a function of the applied load \( F \). Once the angle \( \theta(0) \) is obtained for a given load \( F \), the rotational field \( \theta(X) \) is obtained through integration of [21]. Similarly, the solution can be employed to obtain the displacement field of the rod in terms of other elliptic functions.

Focusing on the application of EB rod theory for soft robotics applications, in [69] it is employed to obtain the static and the dynamic behaviour of a soft arm with a continuously rotating clamp and a dead load applied at the tip. In particular, the equations have been used to analyse the a rise of a snap-back phenomenon and the post buckling behaviour of the system.

In [70], multiple-backbone continuum robots are considered and the kinematic and static modeling is obtained employing EB analytical solutions to express the deflected shapes of the backbones within a sub-segment of the continuum robot. The same research group employed the EB rod theory for the modeling of tendon driven continuum robots (TDCR) [71]. In particular, they can be composed of a series of segments that are connected by spacer disks where the cables can slide inside. Considering a planar bending problem, the static equilibrium of the end disk and the spacer disks are formulated. Finally, the geometric compatibility constraints relate the distances between the cables, the backbone and the disks, which are used in the resolution of the actuation redundancy problem.

In [72], EB rod theory is applied to the modeling of a fiber-reinforced bending actuator. The soft actuator is modeled as a Neo-Hookean material, where the strain energy functional defined in [6] reduces to \( W = \frac{\mu}{2} (I_1 - 3) \), \( \mu \) is the shear modulus and \( I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \) is the first invariant of the principal stretch ratios. The principal nominal stresses can be obtained from the principal Cauchy stresses given by [7], setting \( \sigma_{e,i} = \sigma_i \lambda_i^{-1} \). The bending model is obtained considering the fiber reinforcement on the circumferential direction \( (\lambda_2 = 1) \) and the incompressibility of the material \( (\lambda_1 \lambda_2 \lambda_3 = 1) \), yielding to the definition of the three nominal stress and the Lagrangian multiplier \( p \) in terms of \( \mu \) and \( \lambda_1 \). The bending moment can then be calculated from the integration of the effect of the stresses on the top and bottom layers and, for equilibrium, this is finally set equal to the bending of the the internal air pressure against the distal cap of the actuator. Finally, a relationship between the input air pressure and the bending angle in the free space, is obtained. The actuator force can be obtained imposing the torque equilibrium, under the hypothesis that the actuator is constrained in a flat configuration and that no internal moments were generated under pressurization. This approach is further extended in [73], which also takes into account the pressure on the lateral surface of the inner chamber of the actuator.

In [74] a soft arm made of longitudinal pneumatic actuators is described in terms of its curvature, given by [21], a stretch defining the extension or compression of the base curve, and the bend plane defined by the bending angle at the base. The equilibrium between the internal and the external loads is then formulated, together with the equilibrium on the cross section. The model is finally solved numerically, discretizing the arm into a finite number of sections and driving the residual of the equilibrium equations to zero.
IV. GEOMETRICAL MODELS

In the previous Section we discussed the models which are based on the application of continuum mechanics theories to soft robotics. The presented models are characterized by the assumption of a continuous media and they are all, sometimes implicitly, based on a strain energy assumption. In this Section, we discuss a different family of approaches, which are not based on specific materials considerations. More precisely, the main characteristic of these approaches is that they all rely on the assumption that the deformed shape of the soft body can vary, for instance being discrete or continuous, dynamic equations, these approaches do not rely on the PDEs implicitely, based on a strain energy assumption. In this Section, we discuss a different family of approaches, which are not based on specific materials considerations. More precisely, the constitutive laws modeling the elastic behaviour are expressed as functions of the employed parametrization. All these approaches are based on a representation of the soft body which falls in the definition of a Cosserat rod, thus these approaches are based on a representation of the soft body which falls in the definition of a Cosserat rod, thus these approaches are based on a representation of the soft body which falls in the definition of a Cosserat rod, thus the directors defining the orientation of the cross section. The body which falls in the definition of a Cosserat rod, thus these approaches are based on a representation of the soft body which falls in the definition of a Cosserat rod, thus the directors defining the orientation of the cross section. The main difference from the approaches described in Section III-B is that, when moving from the kinematics to the static and dynamic equations, these approaches do not rely on the PDEs but their are build starting from the generalized coordinates specifically employed to represent the body geometry. We can then conclude that, while the configuration of these models can vary, for instance being discrete or continuous, all these approaches have their roots in some set of shape coordinates mapped to the configuration by:

\[ q \in \mathbb{R}^n \Rightarrow g(q) \]  

(28)

The equilibrium is then given by the Euler-Lagrange equations (or other equivalent principles of dynamics):

\[ \frac{d}{dt} \left( \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \right) - \frac{\partial L(q, \dot{q})}{\partial q} = Q(q, \dot{q}) \]  

(29)

where \( L(q, \dot{q}) = K(q, \dot{q}) - U(q) \) is the Lagrangian of the system, \( K \) and \( U \) are, respectively, the kinetic and potential (including elastic) energy of the system and \( Q \) are the generalized external forces. Note that, depending on the definition of the Lagrangian, the resulting models can be equivalent to the ones derived in the previous section, with particular reference to the Ritz-based approach of section III-B2.

The most popular approach falling in this category is the so called Piecewise Constant Curvature (PCC) model, which discretizes a soft body in a finite number of sections having a circular arc shape. While some works present the Piecewise Constant Curvature (PCC) and the Constant Curvature (CC) as two separate approaches, from the author’s point of view CC models can be seen as a specific case of PCC models, when a single constant curvature section is employed to represent the entire soft body. One other family of approaches based on geometrical assumption on the deformed shape of the soft body are the functional models, which describe the deformed configuration of the soft body through a theoretical geometrical curve.

A. Functional approaches

The functional approaches represent probably one of the first tentative to model soft robotics devices. Their main characteristic is that they all employ a chosen mathematical function to describe the desired space curves representing the geometry of the robot. Thus, for these approaches, the coordinates \( q \) in (28) are within a space of functions.

One of the first example falling in this category is the model developed in [75], which employs a serpinoind curve to describe the kinematics of snake-like robots. Then, the kinematics of the robot is obtained by simple forces and torques equilibrium, considering the action of the ground on the robot and the internal force components [76]. This approach has been employed to model a great variety of snake-like robots, but considering only planar cases, while the 3-dimensional kinematics has not been addressed.

One other important example is the so-called modal approach, that have been originally presented for the modeling of hyper-redundant robots both in statics [77] and in dynamics [78], while a survey on this technique is presented in [12].

In particular, the backbone curve \( r \) is parametrized using the following integral representation:

\[ r(X,t) = \int_0^X \lambda(s,t)t(s,t)ds, \]  

(30)

where \( \lambda(X,t) \) represents a length scaling factor, while \( t(X,t) \) is the unit tangent vector, which can be obtained using any spherical kinematics representations (such as Euler angles or quaternions). In other words, the back-bone curve is represented as a function growing from the base of the robot along its tangent, growing with a magnitude rate \( \lambda(X,t) \) and with direction \( t(X,t) \), implicitly complying with the Kirchhoff assumptions. In order to obtain the complete description of the robot geometry, the backbone curve is finally associated to a set of orthonormal frames and a roll distribution that describes the twist of the robot (or, in other words, the twist of the set of reference frames). In conclusion, the geometry of the robot is defined by a small set of shape functions, describing the backbone curve itself, the chosen reference system and finally the twist. The number of independent shape functions that are required to uniquely define the backbone curve depends on the degrees of freedom of the considered problem. Moreover, the choice of the shape functions is not unique and it depends on the considered task or on the purpose of the modeling (for example, computational efficiency rather than model precision).

In [77], a set of four independent shape functions \( S_i(X,t) = \{ \lambda \theta(X) \phi(X) \psi(X,t) \} \) is employed. In particular, \( \theta(X,t) \) and \( \phi(X,t) \) are the two angles defining \( t(X,t) \) and \( \psi(X,t) \) is the roll distribution function. The inverse kinematic problem then consists in finding the set of shape functions satisfying the task constraints, which, in most cases, are represented by the end-effector positioning. For this purpose, a modal approach is employed and each \( S_i(X,t) \) takes the form:

\[ S_i(X,t) = \sum_{j=1}^{N_{S_i}} \Phi_{ij}(X)q_{ij}(t) \]  

(31)

where \( N_{S_i} \) is the number of modes distributed on the \( i \)-th shape function, \( q_{ij} \) are the modal participation factors and \( \Phi_{ij}(X) \) are the mode functions, which are chosen by the
robot programmer. For example, an extensible planar backbone curve can be fully represented by the two shape functions \( \theta(X, t) \) and \( \lambda(X, t) \), which are restricted to the modal form \( \Phi(X, t) \). Thus, the backbone curve shape is fully described by the modal factors \( q_{ij} \), which are obtained to satisfy the task constraints and which represent the general coordinates of the system.

In [11], it has been proved that, when the curvature of the body is assumed to be constant, this approach provides a transformation which is equivalent to \( \Phi(X, t) \), presented in the following Section. In one other work by the same authors [12], variational approaches are presented in order to compute the optimal shape curves that comply with joint constraints, while, at the same time, satisfying the task constraints. The same kinematical model was also employed in [79] and extended to the dynamical modeling of coiling continuum robots.

One other example of a functional model is the one presented in [80], where a polynomial curvature approach is specifically conceived for control purposes, rather than simulation. The approximated curvature \( k(X, t) \) is defined as an infinite expansion of monomials:

\[
k(X, t) = \sum_{i=0}^{m} X^i q_i(t) ,
\]

where \( q_i \) are the modal components of order \( i \). When \( m = 0 \) the model coincides with the constant curvature one, which is discussed in the following Section IV-B and an analytical solution can be obtained.

In [81], the polynomial curvature approach is further extended to control a soft inverted pendulum, employing an affine curvature function \( k(X, t) = q_0(t) + q_1(t)X \). Thus, the constant and linear terms, \( q_0 \) and \( q_1 \) respectively, represent the configuration variables. Both for the affine and the polynomial case, the local orientation of the robot is obtained by direct integration, providing also the Cartesian space.

One other possible function that can be employed to represent the robot’s geometry consists in the Pythagorean hodograph (PH) curves [82]. The backbone is represented using quadratic polynomials, yielding to

\[
r(X) = \sum_{k=0}^{5} q_k \frac{5}{k} (1 - X)^{5-k} X^k
\]

where \( q_k \) are the chosen control points, which thus represent the generalized coordinates of the system. In order to obtain the optimal quadratic polynomials to compute the control points the minimizing of the potential energy is applied. Finally, a neural network model is employed to predict the optimal quadratic polynomials to compute the control points the minimizing of the potential energy is applied.

procedure is carried out to fit the cspline with the given backbone. To complete the description of the configuration of the robot, the orientation along the curve is finally obtained assuming the minimizing of the torsion of the actuator’s backbone.

In [84], elliptic Fourier descriptors are employed to describe soft deformable morphologies. With respect to the other methods described in this Section, this approach does not model the soft robot through its backbone, but it employs a closed curve to represent the boundary of the 2-dimensional shape (region) that it occupies. This is obtained employing elliptic Fourier descriptor, through a procedure that fits a closed curve to a set of two-dimensional points with arbitrary precision. In particular, the image of the region occupied by the robot is extracted from experimental recordings and its contour is identified through a discrete representation. Finally, the four coefficients of the Fourier series are computed, providing the description of the shape.

B. Constant Curvature Models

Constant curvature is often viewed as a desirable characteristic in continuum robots, due to the simplifications it enables in kinematic modeling as well as in real-time control and other useful computations. This is motivated by the fact that actuators with a path parallel to the mid-line produce a constant curvature shape. For these reasons, the constant-curvature assumption has been successfully applied in a great number of continuum robots modeling approaches. In these models, a soft body is represented as a finite number of circular arcs, each having a curvature that is constant in space (but not in time), Figure 1(b). For these approaches, the coordinates \( q \) in (28) are specifically obtained to describe the constant curvature assumption and, in literature, it is possible to find both continuum and discrete configuration spaces employed to describe soft bodies under the CC assumption.

We can distinguish two main families of approaches: some models are developed from a kinematical relation between the actuator parameters and the arcs parameters (Kinematics-based models), while other approaches are based on the mechanical description of the problem (Mechanics-based models). Finally, a survey on some of the main PCC approaches can be also found in [11] and [13].

1) Kinematics-based models: Once the continuous body is represented as a finite set of constant-curvature segments, each of these can be represented by a finite set of arc parameters and it is possible to obtain a map from them to the task space of the robot. Different parameters can be employed to describe a CC segment, yielding to different kinematics maps. One of the most popular sets of arc parameters that have been proposed consists of triplets of curvature \( \kappa \), the angle of the plane containing the arc \( \Phi \), and arc length \( L \). Different approaches have been proposed to obtain the kinematic map from these arc parameters. In [11] it has been proved that they all provide an identical transformation from the arc base to any
One way to obtain (34) is based on the employment of Denavit-Hartenberg (D-H) parameters, [85], [86], [87]. In [85], the constant curvature assumptions and the Frenet-Serret formulas are employed to obtain the complete evolution in space of the curve. The continuous backbone of the continuum robot is then fitted with a virtual conventional rigid-link manipulator and modified D-H parameters are obtained to consider the coupling posed by the curvature in a continuum system. In this way, the D-H table provides the standard homogenous transformation matrix (34) from the arc parameters to the task position and orientation. In a following work by the same research team [86], a similar, improved, approach is presented. One prismatic and four revolute joints (described by D-H parameters) are employed to provide the parameterization of a single constant curvature section, and geometrical considerations are finally employed to provide the transformation from three parallel actuating tendons to the arc parameters:

\[
L(q) = \frac{l_1 + l_2 + l_3}{3}, \\
\Phi(q) = \tan^{-1} \left( \frac{\sqrt{3} l_2 + l_3 - 2l_1}{3 l_2 - l_3} \right), \\
\kappa(q) = \frac{2 \sqrt{l_1^2 + l_2^2 + l_3^2 - l_1 l_2 - l_1 l_3 - l_2 l_3}}{l_1 + l_2 + l_3}
\]

where \(l_i\) are the actuator’s lengths, while \(d\) is the distance from the center of the section to the actuator. This approach has been applied for the modeling of a great number of continuum robots, such as multi-section bionic manipulators [88], [89], [90], [91], [92] and [93], [94]. Finally, an identical map (34) is also obtained in [95] and [96].

One of the main restrictions of some PCC models is that the employed parametrization and kinematic maps can implicitly provide a numerical singularity which is encountered when the curvature tends to vanish \(k \to 0\), resulting in an infinite or undefined radius of curvature. In order to overcome this numerical limitation, different solutions have been introduced. In [97] and [98], the rotational and position components of the homogeneous transformation (34) are represented in modal form, following a similar idea to the one behind the models presented in Section IV-A. In particular, the entries in (34) are numerically approximated using multivariate Taylor series expansions for the joint variables at \(0\). In this way, the kinematics is computed directly in the joint space without requiring intermediate transformations. Based on this formulation, a Lagrangian approach has been developed in [99] and [100] for the spatial dynamics of a single section continuum arm and this was further extended to multi-section arms in [101].

Other approaches have been mostly focused on the definition of a different parametrization and transformation map to describe the geometry of the soft body. One alternative is provided by the exponential map of [102], [103] and [104], described above. In particular, the transformation (34) can be obtained employing the exponential coordinates from the Lie Group theory, which was introduced in Section III-B. Considering that the transformation for a circular arc can be decomposed in a rotation \(\zeta_{\text{rot}}\) and in-plane \(\zeta_{\text{inp}}\) transformation, applying the product of exponential formula yields:

\[
T(\kappa, \Phi, L) = \exp(\zeta_{\text{rot}} \Phi) \exp(\zeta_{\text{inp}}(\kappa) L)
\]

In [103], this approach is applied to the modeling of concentric tube robots. It is assumed that the tubes directly apply moments to one another which, given the arc assumption, can be considered as constant. The in-plane bending model is obtained employing Euler Bernoulli linear equation and the resultant curvature of two overlapping tubes is obtained through a force balance in analogy with linear springs connected in parallel. The axial tube rotation is obtained considering that two tubes have the same stiffness and assuming torsional rigidity and, finally, the flexural and the torsional elastic energies are obtained. The mapping from the actuator (joint) space to the arc parameters is defined as follows (36).

\[
\Phi(q) = \tan^{-1} \left( \frac{\sqrt{3} l_2 + l_3 - 2l_1}{3 l_2 - l_3} \right), \\
\kappa(q) = \frac{2 \sqrt{l_1^2 + l_2^2 + l_3^2 - l_1 l_2 - l_1 l_3 - l_2 l_3}}{l_1 + l_2 + l_3}
\]

where \(l_i\) are the actuator’s lengths, while \(d\) is the distance from the center of the section to the actuator. This approach has been applied for the modeling of a great number of continuum robots, such as multi-section bionic manipulators [88], [89], [90], [91], [92] and [93], [94]. Finally, an identical map (34) is also obtained in [95] and [96].

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\[
T(\kappa, \Phi, L) = \exp(\zeta_{\text{rot}} \Phi) \exp(\zeta_{\text{inp}}(\kappa) L)
\]
2) Mechanics-based models: In [108] both the forward and the inverse kinematics for a tendon driven manipulator are obtained using the PCC method. More precisely, the manipulator configuration is explicitly given in terms of the measured displacement of the actuation tendons. The soft manipulator is represented as infinite concentric arcs, which can be described in terms of the usual arc parameters, \((\kappa, \Phi, s)\). Considering a single concentric tendon that can experience only a constant tension along its length, force and moment balances are employed to define the internal reaction forces acting on a section of the beam. This model is then extended to consider the presence of a finite number of tendons (i.e. a redundant tendon model), employing an analogy with a system of linear springs acting in parallel. Finally, the generalized coordinates describing the configuration of the beam are the curvature \(\kappa\) and the axial strain field. The static equilibrium leads to the description of the mechanical response of the manipulator in the form

\[
Kq = B^T u
\]

where \(B^T\) represents the tendon moment arms and the axially directed tangent. Linking the tendon tension to the tendon displacements is then possible to obtain the mechanical response of the tendons in terms of their displacements \(y\):

\[
y = C_m u
\]

where \(C_m\) is the compliance matrix, which is function of \(B^T\) and of the manipulator and tendon stiffness matrices and lengths. Finally, combining (37) and (38) yields to

\[
y = A^T q
\]

where \(A = K^{-1} B^T C_m^{-1}\) is the forward kinematics transformation matrix. Equation (39) provides a mechanics-based relationship between the beam and the tendon configuration, analogous to transformations (35) that were obtained from geometric considerations. In a following work by the same authors [116], the 3D static model is also developed with an experimental validation.

In [110] and [111] the dynamics of a braided continuum manipulator is addressed. The model employs the usual arc parameters \((\kappa, \Phi, l)\) and the motion equation are obtained using the principle of virtual powers. A Neo-Hookean model and the Cauchy-Green stretch tensor are employed to define a unit deformation energy, but the Hooke and the Euler-Bernoulli equation are also presented as a simplification. To model the braided thread, a constrained volume is employed, imposing that the volume of the solid is fully constrained by the thread. As a consequence of this, the axial and the radial deformation of the chamber are constrained on an helix, while its tangent deformation of free. In this way, the model defines the deformation of the cross section caused by the constant curvature bending of one manipulator module under general gravitational loads.

In [112] and [113], the method of virtual power, equivalent to the Euler-Lagrange equations [29], is employed to obtain the dynamics of TDCCRs robots. The considered geometry for the robot is the same as in [71], i.e. an elastic backbone connected by rigid disks and three tendon cables. The disks are treated as rigid bodies, while the backbone and the cables sub-segment are treated as circular arcs. The position and orientation of a rigid disk with respect to the adjacent one is given by the two orthogonal curvatures and the twist angle of the sub-segment. With regards to the actuation forces and moments, these can be calculated taking as an input the tension on the cables, once the position of the holes in the disks is defined. For the friction at each disk, assuming a continuous viscous model, an iterative approach is employed, where the contact friction is updated at varying the cable tensions at the entrance and at the exit of the hole.

The modeling of concentric tubes robots using PCC approach was developed in [114], which extends the work presented in [104] to account for the torsional twisting of the tubes. The same results were also obtained in [115], where the calculus of variations was employed. In particular, in [114], the relative twist angle between two tubes is defined as a function of the arc length and then it is employed to write the moment equilibrium and to impose the compatibility equations, enforcing the coincidence of the tube centerlines. The torsional curvature is then obtained considering the Cosserat Rod equilibrium equations (12). In particular, the analytical closed-form solutions are provided for the case of two concentric tubes in terms of Elliptic functions (as seen in Section II-B4), and the presence of multiple solutions, corresponding to instability phenomena, is discussed. The stability analysis is further analyzed in [116] and [117].

V. DISCRETE MODELS

In this Section we discuss the approaches where the configuration of the system is discrete from the very beginning (i.e. the system is not discretized at the equation level or through some assumptions on its geometry). In particular, we can distinguish three main groups of models: the discrete rods, the pseudo-rigid and the lumped-mass models.

A. Discrete rods

The computer graphics research community represents an important source of inspiration for soft robotics modeling. This is the case of the approaches that are described in this Section, which are all based, with different extent, on the discrete elastic rod (DER) approach, originally introduced in [118], [119] and [120] and sometimes also referred to as discrete differential geometry DDG [121]. In this formulation, the material curve of the rod is approximated by a discrete set of lines connected at vertices. Thus, the configuration space takes the form:

\[
Q \subset SE(3) \times SE(3) \times SE(3) \cdots \times SE(3)
\]

In particular, in the DER formulation, the inextensible Kirchhoff rod assumptions [27] are considered and, similarly to the Cosserat rod model, the configuration \(g\) of an elastic rod is given by a curve \(r(X)\), representing the centerline, and a material frame, where the first axis is always tangent to the curve. The Bishop (natural) frame \(\{t, v, w\}\) is then introduced, providing, for a given centerline, the most relaxed frame, i.e.
with zero twist. Finally, the parallel transport is employed to define the evolution of the Bishop frame along the mid-line.

In this way, the model provides a simple parametrization of the material frame, which is given by the curve \( r(X) \) and an angle \( \theta(X) \) between the Bishop and the material frame. This parametrization represents the basis of the discrete theory employed for the kinematical description of the continuum body. In particular, the rod is discretized into a series of finite elements. Each node is characterized by a position vector \( r_i \), while each segment is associated with the edge vector \( e_i = r_{i+1} - r_i \) and its tangent unit vector \( t_i \). Figure 4 shows the representation of a rod in the DER approach.

In general, the configuration space of the soft body is given in the form:

\[
Q \subset SE(3) \times SE(3) \times SE(3) \cdots \times SE(3)
\]

where \( Q \) represents the configuration space, and \( SE(3) \) represents the special Euclidean group. The configuration space is then introduced in the system dynamics using a modified mass-based method.

### B. Pseudo-rigid Models

When dealing with soft robotics modeling, the chance of exploiting well known and established rigid robotics theories is clearly intriguing. This is the idea that motivated the family of models that are described in this Section. Here, the soft bodies are represented as series of rigid links which are connected by revolute, universal or spherical joints, as seen in Figure 1(c). Thus, in general, the configuration space of the soft body is given in the form:

\[
Q \subset SE(3) \times SE(3) \times SE(3) \cdots \times SE(3)
\]

While these approaches can provide satisfactory results for the modeling of hyper-redundant or snake-like robots, when dealing with continuous elastic structures they provide a low order of spatial approximation accuracy.

In [128], a dexterous catheter manipulator is represented as a series of pin joints which are connected by a set of revolute and universal joints. Proceeding with approaches for biomedical applications, in [127], a pseudo-rigid 3D approach is applied for the modeling of a steerable ablation catheter. The catheter is treated as a cantilever beam, which is represented as four rigid links connected by three revolute joints and three torsional springs. [128] presents a 6 dof pseudo-rigid model for continuum manipulators subject to multiple external loads. In this case, a flexible manipulator is represented as four rigid links of given length which are connected by three three joints having 2dof each. In [129], the dynamical modeling of compliant mechanism is addressed and a flexible beam is represented as a mass-less rigid body with a torsion spring attached at one end.

In [130], the modeling of hyper-flexible manipulators is addressed using a serial rigid chain, where the number of kinematic dof goes to infinity. The backbone curve is first described as a continuous curve with extended Frenet frames attached to each point along its length. Then, for the numerical simulations, the backbone curve is approximated by a serial chain of rigid bodies. In particular, the backbone is described in terms of a variable vector that corresponds to the joint angle vector of a serial-chain manipulator.

In [131], a steering catheter is represented as a combination of three sections: the virtual base of the distal shaft, which is represented as a prismatic joint; the bending section of the distal shaft is represented as two revolute joints, one prismatic

![Fig. 4: Representation of a rod in the DER approach](Image)
joint and two revolute joints; the distal end of the catheter, which is treated as a rigid body. D-H parameters are then employed to obtain the forward kinematics of the model.

In [132], a pseudo-rigid approach is employed for the modeling of a 8 arms swimmer inspired by the octopus. In particular, each arm is modeled as a kinematic chain of cylindrical rigid segments which are connected by planar rotatory joints. The first joints in the chain, connecting the arms of the octopus to the main body, are modeled as actuated rotatory joints, while the other joints are modeled as (unactuated) rotatory linear spring and damper elements.

C. Lumped-mass Models

One of the most straightforward approach to model a continuum body is to represent it as an array of discrete masses, dampers and springs, Figure 5. In this way, the governing equations can be obtained by energy methods or applying classical Newton equations to describe the propagation of the forces from one particle to the other. Thus, the configurations space is given in the form:

\[ Q \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \cdots \times \mathbb{R}^3 \]  

(43)

The main advantage of lumped-parameter mechanics models is that they are characterized by a simple structure, which is easy to adapt to include complex phenomena such as nonlinear friction, material hysteresis, and inertial dynamics. On the other end, to reach the same fidelity of the continuum mechanics models or the FEM approach, they required a data-expensive system identification procedure.

In [133] and [134], mass-spring-damper arrays are used to model the large deformation of a continuum surface (LDCS) actuated by continuum arms.

While the vast majority of the soft robotics devices is made out of parts that can be treated as rods, LDCS consists in a flexible surface which is actuated into multiple curvature profiles by integrated continuum arms. In particular, the above mentioned works consider the 3D problem of a system composed of two interconnected layers of LDCSs, where bending and shear deformations are allowed. The surface is modeled as a lattice where the masses are defined as the nodes of the system and they are connected through linear spring and dampers. In particular, the presence of springs in the diagonal directions around each node, allows the modeling of the shear effects. Assuming a uniformly stiff surface, the elastic constant of the springs and the damping coefficients are assumed to be constant. Considering an isotropic linear elastic material and assuming a straight undeformed configuration, the equation of motion for each mass is obtained by combining direct application of Newton’s second law and Hooke’s law.

In [135], a discrete lumped-mass approach is presented for the modeling of tendon driven medical robotic catheters. The axial and the bending stiffness of the model are adjusted through the selection of the spring stiffness and their radial location, while the smooth motion of the catheter is modeled through the dampers. Torsion is not considered, even though it could be implemented adding torsional springs to the model. Moreover, a modified Dahl friction formulation, is incorporated at the intersection of the control tendons and the mass elements.

Because of their simplicity, lumped mass models are particularly suited for large robotics simulations library, such as the one presented in [136]. Titan is a GPU-accelerated C++ software library for the modeling of soft bodies and multi-agent robots at massive scales. The soft robotic body structure is represented through spring-mass lattices and an interactive approach is employed. In particular, the first iteration is applied to all the springs to compute, with Hooke’s Law, the forces acting on each corresponding mass and then a second iteration is performed over all masses to update their accelerations and positions. The main benefit of this approach is the possibility to perform parallel computing, resulting in great computational performances.

VI. SURROGATE MODELS

As extensively discussed in all the previous Sections, soft robotics present challenging problems when dealing with modeling and they in general require great efforts and some expertise in the continuum mechanics field. In order to overcome some of these difficulties, some authors presented a complete different solution to tackle the problem, where the modeling of the system is obtained from large sets of data (inputs and outputs) that are derived from various forms of external sources. These approaches are often referred to as surrogate or data-driven models and a survey on some of these approaches is presented in [14] and [15]. While one of the main benefit of these solutions is that they do not require a physical model, on the other end they rely on large amounts of representative data, that are sometimes difficult to collect. Most of data-driven approaches presented for soft robotics modeling are based on machine learning algorithms, mostly involving neural networks models.

A. Neural Networks

Neural networks (NN) have been proved to be an effective tool to solve many kinds of non linear problems in different application fields, including robotics. In general, the goal is to find some sort of underlying pattern in a set of given data. As
the name suggests, they are inspired from the biological neural networks that operates animal brains: the artificial neurons, which represent the elementary units of NNs, can transmit a signals to the other neurons. The signal is usually a real number and the output of each neuron is computed as a non-linear combination of all the inputs. Neurons are connected by edges and both neurons and edges are characterized by weights that are adjusted during the learning process. Usually, neurons are gathered in layers: the first layer is also called input layer, the last one is called output layer and the intermediate layers are called hidden layers, Figure 6. NN results in a high dimensional set of nested functions:

$$y = f_M(A_M, \ldots f_2(A_2, f_1(A_1, x)) \ldots)$$ (44)

where $x$ are the values of the input nodes, $A_i$ are the edges weights, $f_i$ are the activation functions and $y$ are the values of the output nodes. In most soft robotics applications, the input and the output layers represent the actuation variables $u$ and the shape parameters $q$. The learning process results in an optimization over the network weights, which is performed through back-propagation, a form of chain rule. The idea is that, after each forward step through the network, a backward pass is performed that computes the network Jacobian and adjusts the model’s weights accordingly.

With regards to the modeling of soft robots, neural networks have been used to approximate the kinematics, the statics and also the dynamics description of the considered systems. Typically, Feed-forward neural network (FNN) are employed for kinematics and static modeling. They represent the first and simplest example of neural networks where, as the name suggests, the signals travel in only one direction (forward). On the other end, Recurrent Neural Networks (RNN) are usually preferred for dynamical modeling. They use time series data and they can use their memory to process sequences of inputs, which makes them particularly suitable to model temporal problems.

The first examples of the employment of neural networks algorithms for the control of continuum robots were presented in [137] 138. In particular, the authors present a controller for continuum robots which uses a FNN component to compensate the dynamic uncertainties of the system, in an attempt to reduce the uncertainty bound. In [139] the forward kinematic modeling of a bionic assistant trunk is obtained through Multilayer Perceptron (MLP) and Radial Basis Function (RBF) neural networks, which are both class of FNNs. The input neurons propagate the input variables to the following layers, while each neuron in the hidden layer is associated with a RBF kernel (usually Gaussian). The data are obtained experimentally, measuring the arm’s tip position at varying the actuation pressure, using a trilateration algorithm. In one other paper by the same authors [140], the same handling assistant trunk is modeled, both with a data-driven and a pseudo-rigid modeling approach. With regards to the latter, the trunk is modelled as a series of rigid vertabrae connected by a total of four prismatic joints. For the data-driven approach, a modified Elman neural network is employed. This network is also trained using the experimental results based on a trilateration system.

FNNs are also employed in [141] and [142], which deal with the inverse kinetics of a cable driven soft manipulator. While the kinematical model describes the configuration of the robot without considering the involved loads, the kinetics model relates the motion of the robot with its actuation forces. In the above mentioned papers, the direct kinetics model is obtained using a geometrically-exact model that follows the approach described in [143], which uses the continuous Cosserat rod model [12]. Thus, a FFN, taking the tip position as an input and giving the cable tension as an output, is employed. After fabricating the considered 3-cable driven manipulator, a data collection phase is carried out, using a set of cable tensions and obtaining the tip position with an infrared vision system. These data were used for the optimization and training of the FNN. After the training phase, the performance of the FNN are measured on test sets, using the output of the FNN as the input of the direct kinetics model.

One other example of a machine learning formulation for the global inverse kinematics of continuum manipulators was presented in [144].

The data samples are generated by continuous motor babbling and a single hidden-layer artificial neural network is employed to learn directly the mapping $(x_{i+1}, q_i) \mapsto q_{i+1}$, where $x$ is the pose of the end effector. In one other work by the same authors [145], a dynamic model for open-loop control of soft robotics manipulators is presented. The PCS model described in [11–12] is employed to obtain the dynamics of a cable driven robot operating underwater. Considering the case where the robot and the task space have the same number of DoF, the forward dynamics of the system (7) is formulated using only the task space variables and the direct mapping between the states of the task space variables and the control inputs can be obtained. A nonlinear auto-regressive network with exogenous inputs (NARX) is used. The samples for the network learning are obtained from the PCS simulations and, then, the training is carried out in two steps. The first step consists in an open loop training of the network, by unfolding the recurrent network and training by Bayesian Regularization. Then, the network is closed and the training is continued using the same network weights. Once the training is completed, an open loop predictive controller is developed through a trajectory optimization that is carried out with an iterative sequential quadratic programming. In [146], the closed-loop controller is implemented.

The inverse kinematic modeling of a bionic trunk is addressed in [147]. In particular, a learning phase is carried...
out considering a volume of desired cartesian position for the robot’s tip, defining a finite set of target vertices. The inverse model is asked to estimate a posture that allows to move the effector to each vertex, and the training is carried out until the distance between the target and the actual position for each vertex is minimized. The inverse kinematics is learned using an online goal babbling scheme. Finally, for the learning of the system, a regression mechanics is defined in the form of a local linear map and the estimated inverse model is obtained as a linear combination of local linear functions weighted by a Gaussian responsibility function.

B. Data-driven Order Reduction

Here we introduce the modeling tools that employ some forms of data-driven order reduction to efficiently approximate the physical model. In [148], a Koopman Operator Theory is employed for a data driven controller of soft robots. In the data-driven Koopman representation, a dynamical system is represented in a infinite function space \( F \), which is composed of real-valued functions inside the state of the system domain \( t \). The elements \( q \in F \) are called observables. The Koopman operators, denoted by \( U_t \), are defined as the linear transformation \( U_t q = q \circ T_t \), where \( \circ \) indicates the composition operation, while \( T_t \) is the flow (or dynamic) map of the system. In other words, the Koopman operator lifts the observables in the state space to the space of the observables, describing the evolution of the observables \( q \) along the trajectories of the system. Its main advantage is that it provides a linear representation of the flow of a non-linear system, but in the infinite-dimensional space of the observables. The discrete approximate of the Koopman operator can be obtained from a set of experimental measured state, given in the form of snapshot pairs.

Some approaches have been focused on the model reduction of FEM that were presented in Section III-A1. Model reduction methods are based on the projection of the FE equations of motion to attractive sub-spaces of smaller dimensions. In this way, the size and the computational time of the simulation are drastically lowered, allowing the application of FE methods for control purposes. In [32], an implicit time integration is applied to \( \dot{q} = \tau \), where the internal forces are linearized applying a Taylor series expansion. A snapshot proper orthogonal decomposition (POD) is then employed to generate relevant vectors that are fabricable, and one other optimization problem is defined (motion-tracking, ease of actuation, resilience to failure) and the optimization is performed using a sensitivity analysis.

Development and use of modeling software and toolboxes for soft robotics is essential to avoid the reinvention of the wheel. They could potentially avoid the need to write a custom code script in a programming language (C/C++, Python, MATLAB, etc.) for each robot design, optimize the robot design for a particular task, or be used for the model-based control of a specific robot. The modeling framework should be user-friendly (necessary for the multi-disciplinary soft robotic community), accurate, fast, and reliable. There are several industrial as well as open-source soft robotic software/toolboxes available. Here we briefly summarize the most popular ones: COMSOL, ANSYS, ABAQUS, SOFA (available at [155]), PyElastica [156], SoRoSim [157], TMTDyn [158], SoMo [159], ChainQueen [160], and IPC-Sim [161].

The SoftRobots plugin of SOFA, one of the earliest open-source platforms for physics-based simulation, uses FEM to model, simulate, and control soft robots [28]. SOFA employs a multi-model representation of a deformable body, allowing the presence of several representations (mechanical, thermal...
and visual) of the same object. The representations are then connected together through a mechanism called "the Mappings". This multi-modal representation allows the simulation of scenarios where different bodies, such as different rigid and/or deformable bodies or fluids, are interacting together. In particular, deformable bodies are modeled employing FEM with different constitutive laws. In general, a simulation in SOFA is described as a scene which is composed by different nodes organized as a tree or as a Directed Acylic Graph (DAG).

PyElastica, which is the python implementation of Elastica, employs Discrete Geometry and the Cosserat rod theory of section [16]A to model soft bodies with a finite number of lumped degrees of freedom [123]. SoRoSim and TMTDyn are MATLAB toolboxes that use geometrically exact piecewise variable strain models [57] and discretized lumped systems with reduced-order models [162], respectively. SoMo (Soft Motion) toolbox couples the pseudo-rigid model discretization of section [17]-B with a rigid body physics engine of python to model soft robots [163].

ChainQueen, and IPC-Sim (Incremental Potential Contact Simulator) are simulators oriented towards computer graphics rather than mechanical systems. ChainQueen is a python toolbox that uses the Moving Least Square Material Point Method, a hybrid Eulerian/Lagrangian FEM which uses both particles and grids to simulate soft bodies [164]. IPC-Sim solves extreme non-linear volumetric elastodynamic models using FEM (section [18]-A) [165]. Particular toolboxes are developed for a specific soft robotic problem, such as the DiffAqua, an optimization toolbox for soft underwater swimming robots [166], and Vine Simulator, a toolbox dedicated to growing soft robots [167].

VIII. DISCUSSION AND PERSPECTIVE

Except for the few sporadic examples that were discussed in Section [19]B-4 all the above approaches, at some level, require a form of numerical discretization. It should be noted that discretization and reduction are two separate concepts that should not be mistaken. A discretization is employed to obtain a numerical solution to the non-linear problem and, for example, the finite difference methods, such the one developed in [168], fall in this category. On the other side, a reduction consists in the depiction of the smallest basis of functions allowing a proper description of the kinematic fields of the body. For example, reduction methods include the mode parametrizations that were discussed in Section [20]B-2 and [21]V but a reduction of the model was also presented in some works discussed in Section [22]V-B. Within this scenario, Finite Element Methods seen in Sections [23]A-1 and [24]B-1 can be seen as a middle ground between the two, since they involve an explicit discretization of the material domain in finite elements, over which the kinematic fields are reduced on some polynomials, or other shape functions.

We can conclude that there are three main levels of reduction that can be pursued:

1) **Material geometry.** When one or two dimensions are much smaller in comparison with the third one, the three dimensional position fields can be approximated through Taylor expansions with respect to the material dimensions. In this way, two main continuous reduced approaches are obtained: rod (1D) and shell (2D) theory. One other option to reduce the material geometry is to represent the material as a Cosserat media, i.e. a continuous collection of rigid microstructures that are staked along some dominant material direction.

2) **Rheology.** i.e. constitutive law. In the more general case, the stress field is a functional of the history of strains (and possibly also of thermo-dynamical variables). In soft robotics, the material is usually treated as elastic, thus the stress is a function (and not a functional) of the strain. Moreover, assuming that the strains are small, a linear function (Hooke’s law) is also typically employed. Both these assumptions can appear as physically questionable, since the materials usually employed in soft robotics are not elastic, let alone linear. There are also some examples of approaches using hyper-elastic constitutive models, such as [22] and [169].

3) **Geometry of deformations.** Even when the strains are small, soft robots are subject to finite deformations. Thus, the equations describing their motions are fundamentally nonlinear. With the assumption of linear elasticity and small strains, the non-linearities are geometric nonlinearities of rigid-body type, that can be expressed in different reduced ways, depending on the considered material geometry and the parametrization used.

One other important characteristic of each approach is whether they rely on absolute or relative coordinates. In general, a relative parametrization eases the modeling of any internal actuator and sensing devise, while an absolute one appears as more suitable for the modeling of the interaction with the external environment.

Despite the classification that is used in this paper, is it possible to find similarities between the different families of models. In the Ritz-Galerking approaches (Section [II]B-2), the configuration of the backbone of the rod is given as the sum of a finite number of shape and time functions, similarly to the modal approach that was described in Section [IV]A. One other observation that should be highlighted is that the GE-FEM (Section [III]B-1), the Piecewise Constant Strain (which is a specific case of the PVS approach discussed in Section [II]B-2) and the Piecewise Constant Curvature (Section [IV]B models are all based on the same backbone geometry (helices or arcs that are a special case of helices) and the continuum rod is reduced in a finite number of sections where this geometry is repeated. Finally, in the FEM representation (Section [III]A), the continuous domain is discretized through a mesh of nodes and a similar idea is followed by the lumped mass approaches presented in Section [V]-C.

For what concerns the actuation, while for the DEM these were included as constraints, in the Directors approaches the thread-like actuators actions were simply added to the linear and angular balances as external distributed wrenches [63] and, sometimes, internal wrenches [60]. This simple adjustment is one of the main benefit of these approaches, which on the other end struggle in the modeling of other forms of actuation, which involve 3D problems (such as ballooning actuators and
electroactive polymers). In this case, FEM appears as a more suitable choice. On the other end, some PCC approaches, specialize to a specific actuators configuration and provide a kinematic or mechanical transformation from the actuating tendons to the parameters describing the configuration of the rod. However, this is valid only when the constant curvature assumption is actually preserved in the real world problem.

Soft robotics was developed in the last two decades and, legitimately, the main efforts so far have been put into the simulation of the robot’s configuration and in the static and dynamic representation of its behaviour. Now the direction of this topic and the efforts of many researchers are mostly towards the extension of these approaches for control and for their integration with optimization tools. Once more, the main source of inspiration towards the achievements of these goals are more likely to come from other research disciplines. Indeed, while going through the papers that have been presented so far for the modeling of soft robots, we tackled a great number of disciplines that represent their theoretical foundations: continuum and solid mechanics, computational mechanics, machine learning, computer graphics, just to name a few. This interdisciplinary is probably what makes soft robotics so interesting, attracting scientists from different research fields. On the other side, being the topic so widespread, it is not easy to grasp it and this constitutes the main motivation behind this manuscript. In this way, we were able to recognize more clearly the uniqueness and the commonalities between the different techniques that have been presented so far, in the effort to untangle such a vast research topic.

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