CONTINUOUS GUTS POKER AND NUMERICAL OPTIMIZATION OF GENERALIZED RECURSIVE GAMES

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Abstract. We study a type of generalized recursive game introduced by Castronova, Chen, and Zumbrun featuring increasing stakes, with an emphasis on continuous guts poker and 1 v. n coalitions. Our main results are to develop practical numerical algorithms with rigorous underlying theory for the approximation of optimal multiplayer strategies, and to use these to obtain a number of interesting observations about guts. Outcomes are a striking 2-strategy optimum for n-player coalitions, with asymptotic advantage approximately 16%; convergence of Fictitious Play to symmetric Nash equilibrium; and a malevolent interactive n-player “bot” for demonstration. For a mild variant of Guts known as the “Weenie rule” we find the surprising different result that the Nash equilibrium solution is in fact strong; that is, it is optimal against arbitrary coalitions.

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1. Introduction

In this paper, we study a class of single-state generalized recursive games introduced in [CCZ], typified by the popular poker game “Guts,” with an eye toward developing a practical set of theoretical and numerical tools for optimizing multiplayer strategy.

**Generalized recursive games.** Recursive games, introduced by Everett [E], are games like Markov chains consisting of a set of possible states, for which the outcome of a single round is to either terminate with a given fixed payoff or to proceed to another state, with various (fixed) probabilities. This is closely related to the concept of stochastic game introduced by Shapley [Sh1], in which players move from state to state with various probabilities, receiving payoffs at the same time without termination. Indeed, if \( \rho_0 \) is the probability of termination in a one-shot recursive game, \( a_0 \) is the associated payoff, and \( \rho, \ldots, \rho_\ell \) are the probabilities of moving to states 1, \ldots, \( \ell \), then this is equivalent to a *variable-stakes* stochastic game with transition probabilities \( \rho_1, \ldots, \rho_\ell \), one-shot payoff \( \alpha := \rho_0 a_0 \), and new stakes adjusted by multiplication factor \( \beta := (1 - \rho_0) \). That is, the recursive game of Everett may be viewed as a variable-stakes version of the stochastic game of Shapley, where the stakes are multiplied by a factor

\[
0 \leq \beta \leq 1.
\]

The class of games introduced in [CCZ], denoted here as *generalized recursive games* (gRG) have the variable-stake stochastic structure just described, but with the condition (1.1) relaxed to

\[
0 \leq \beta.
\]

That is, in its most general form, in each round a game in state \( i \) moves to a new state \( j \) with transition probability \( \rho_{ij} \), receiving a one-shot payoff \( \alpha_{ij} \) and adjusting the stakes by factor \( \beta_{ij} \geq 0 \), where \( i, j = 1, \ldots, \ell \) enumerate the possible states, and \( \rho_{ij}, \alpha_{ij}, \beta_{ij} \) are functions of the strategies chosen by the players in the game.

Here, as in [CCZ], we will restrict to the simplest case of a *single state*, so that there is a single one-shot payoff \( \alpha \) and a single stakes multiplier \( \beta \) that is nonnegative but not necessarily bounded by one. We will assume further that the game is *zero sum* in the sense that the one-shot game described by \( \alpha \) is zero sum. This does not necessarily mean that total payoff over the infinite course of the game is zero sum, as increasing stakes allow the possibility that some part of the stakes remain “indefinitely in play” and inaccessible to all players, serving as an effective mutual loss. For the same reason, von Neumann’s maximin principle does not necessarily hold in the long run even in the two-player case, though it of course holds for the one-shot payoff function \( \alpha \).

**Discrete and continuous guts poker.** An archetypal example, and our particular target here is the popular poker game “Guts” [S, W1] which can be played with any number of players \( n \geq 2 \) and 2-, 3-, or 5-card hands with standard poker ordering. Players make an initial one unit ante into a pot. Hands are dealt, and on the count of three players either “hold” or “drop” their hands, with no further betting or cards dealt. If only one player holds, they win the pot and the round is terminated. If no players hold, the game is redone, starting over. If \( m \geq 2 \) players hold, the player
with highest hand wins the pot and the remaining \( m - 1 \) players must match it, so that the stakes increase by factor \( m - 1 \). A new hand is then dealt to all players and the game played in the same way but with now higher stakes, this process continuing until the round is terminated. In actual poker there are many identical rounds of play; here, we view each round as an individual game.

Following [CCZ], we will treat a simplified, continuous version of guts, in which hands are replaced by random variables \( p_i \) uniformly distributed on \([0, 1]\), with higher value corresponding to a higher hand. This avoids combinatorial details coming from lack of replacement, while keeping the essential features of the game. We then discretize the values on \([0, 1]\) to obtain an arbitrarily-near game suitable for numerical analysis, returning full circle to a (slightly simpler) finite game.

A “pure” strategy for player \( i \), indexed by \( p^* \in [0, 1] \), is the threshold type strategy to hold for \( p_i > p^*_i \) and otherwise drop. As noted in [CCZ], one might also consider more general type pure strategies to hold for \( p_i \) in a specified subset \( \mathcal{S}_i \subset [0, 1] \); however, these are evidently majorized by the threshold strategies \( p^*_i = |\mathcal{S}_i| \), and so may be ignored. A “mixed,” or “blended” strategy is a random mixture of pure strategies with a given probability weight.

### Coalitions, and value of multiplayer games.

For a symmetric multi-player game such as the one-shot (payoff \( \alpha \)) game associated with guts, there always exists a symmetric Nash Equilibrium \( \{N, W^2\} \), meaning a collection of identical strategies from which deviation by a single player does not improve their payoff hence there is no motivation to leave. The value of this symmetric Nash equilibrium is necessarily zero, by the zero-sum assumption.

If the Nash equilibrium is strong, meaning that deviation of any subset of players does not improve their joint payoff, then this is in fact the value of the game by any reasonable definition. However, typically this is not the case, and indeed it is not the case for one-shot continuous guts [CCZ]. In the latter scenario, following von Neumann and Morganstern [vNM], it is standard to consider the effects of coalitions in assigning a value. Taking the worst-case point of view, we will define the value \( v \) for player one to be the von Neumann minimax value of the two-player game obtained by considering players 2-\( n \) as a single entity consisting of a coalition working together. One might call this a “synchronous” coalition, in that players 2-\( n \) may form mixed strategies blending coordinated or “synchronized” configurations of pure strategies. It is straightforward to see that this value is the maximum one forceable by player one. By the minimax principle, it is equal to the value forceable via synchronous coalition by players 2-\( n \).

An interesting question that we do not address here is what is the minimax value forceable by players 2-\( n \) via asynchronous coalition, that is,

\[
\min_{p_1} \max_{p_2, \ldots, p_n} \psi(p_1, p_2, \ldots, p_n),
\]

where \( \psi \) denotes the expected payoff for blended strategies with probability distributions \( p_j \), with \( j = 1, \ldots, n \). This corresponds to the case that players 2-\( n \) are able to organize a strategy together, but not allowed to communicate during the game, or even to know to which round of play they are responding at a given time. It is clear that the asynchronous minimax is greater than or equal to the minimax, or synchronous coalition value, and strictly less than the Nash equilibrium value, except in the case the the Nash equilibrium is strong, in which case all three values agree.

If the Nash equilibrium is not strong, then (see [CCZ Proposition A.2]) the optimum synchronous coalition strategy contains strategies for which players 2 – \( n \)’s individual strategies do not agree. If this optimum is strict up to symmetry, and has no pure strategy representative, then every asynchronous mixed strategy for players 2-\( n \) gives a larger return to player 1, and so we may deduce that there is a gap between the minimax values forceable by players 2-\( n \) by synchronous vs. asynchronous coalition play. Thus, a gap between synchronous vs. asynchronous minimax values would seem to be the generic situation; based on our numerical experiments, it indeed appears to be the case for continuous Guts (by examination of optimal synchronous strategies found below).
The maximin against asynchronous play on the other hand is the same as the maximin against synchronous play. In the above-described situation that there is a gap between synchronous and asynchronous minimax values, therefore, there is a gap also between minimax and maximin values for asynchronous coalition play. In this situation, for which the minimax is strictly greater than the maximin, one could imagine a negotiation in which both sides agree to some intermediate return, even though neither may force such an outcome.

The method of Fictitious Play. Having framed the one-shot problem as a standard finite two-player zero-sum game, we now discuss practical solution of such a game. Dantzig \[D\] demonstrated an equivalence between such games and the class of linear programming problems. Hence, one approach would be to translate to a linear programming problem and apply the Simplex method or one of the various interior point methods that has been developed in recent years. Alternatively observing that the minimax problem is a convex minimization program, one may solve by subgradient descent or related methods.

An appealing alternative of interest in its own right, involving concepts more directly involved with the game, is the iterative method of Fictitious Play (FP) introduced by Brown \[B\] as a model for learning/dynamics of actual play. In this algorithm, the game is repeated a large number of times, with players at each step following a strategy that is the best response to the empirical probability distribution defined by past play of the opponent player. As shown in a remarkable paper of Robinson \[R\], (FP) converges for finite two-player zero-sum games to the associated Nash equilibrium, in this case equal to the von Neumann minimax. For multiplayer or nonzero-sum games, it may not converge, as shown by a clever counterexample of Shapley \[Sh2, Sh3\] for a class of nonzero-sum 3x3 two-player games, hence, by introduction of a virtual third player with no choice in strategy, also for a class of zero-sum 3x3x1 three-player games.

This algorithm is conveniently supported for finite two-player games in the Python package Nashpy \[NP\], requiring only input of the payoff matrix. We will use it as the basis of our numerical methods, yielding approximate solutions of the one-shot game associated with a (gRG).

Fixed-point iteration: from one-shot game to long-term value. To assign a value to the full, possibly infinite game, we define following \[CCZ\] a termination fee

\[ -t_1, \]

typically \( \leq 0 \) for player one to leave the game, and similarly for player two. For example, in the context of Guts Poker, \( -t_1 = -1 \) corresponds to forfeiting one’s ante, while \( -t_1 = 0 \) corresponds to the convention that players are allowed at any time to collect their share of the pot and leave play without penalty. Here, as in \[CCZ\], we take always the former case \( -t_1 = -1 \).

Evidently, the value \( V_0 \) for a player in zero rounds of play is thus \( V_0 = -t_1 \). Denoting by \( A \) and \( B \) the matrices associated with the one-shot payoff \( \alpha \) and stakes multiplier \( \beta \), so that \( \alpha(x, y) = xAy^T, \beta(x, y) = xBy^T \), we thus have that the value forceable by player one in precisely one round of play is \( V_1 = Value(A + BV_0) \), where \( Value \) of a matrix denotes the standard minimax value. Continuing, we find that the value \( V_n \) forceable in \( n \) round of play is given inductively by

\[ V_{n+1} = Value(A + BV_n). \]

By the assumption \( B_{ij} \geq 0 \), a consequence of \( \beta \geq 0 \), the sequence \( V_n \) is monotone increasing so long as \( v_1 > V_0 \), hence has a well-defined limit

\[ V := \lim_{n \to \infty} V_n. \]

As noted in \[CCZ\], \( V \) is necessarily a fixed point of the value map

\[ T(V) := Value(A + BV), \]

which could be \( V = +\infty \).
For $V_1 > V_0$, we define the value forceable by player one to be the minimum of $V$ and $t_2$, the amount forfeited by player two to terminate the game without play. If $V_1 \leq V_0$ on the other hand, then the value forceable by player one is $V = V_0 = -t$, and player one should not enter into play. Likewise, the infimum $\overline{V}$ of expected returns that can be forced by player two is a fixed point of $T$. Note that both or either could be $\pm \infty$ in general. When $\overline{V} = \overline{V} = V$, with $-t_1 \leq V \leq t_2$, we say that the game has value $V$, similarly as in the nonrecursive case.

1.1. Main results. We now describe briefly our main results.

1.1.1. Analytical. Theoretical centerpiece of [CCZ] was the following “Termination Theorem” giving conditions for existence of a repeated single strategy guaranteeing a winning or neutral outcome for player one. This was used to support rigorously all of the conclusions of that work.

Proposition 1.1 ([CCZ]). Suppose for an arbitrary single-state recursive game with termination constant $-t < 0$, that a certain strategy for player 1 has associated payoffs $\alpha + \beta V$ satisfying

$$\alpha \geq \alpha_0 \geq 0, \beta \geq 0, \text{ and } \alpha \geq t(\beta - 1) + \varepsilon, \text{ for some } 0 < \varepsilon < t.$$  \hspace{1cm} (1.7)

Then, the expected return $V_n$ upon termination of the game at the $n$th step satisfies

$$V_n \geq v_n := \alpha_0 - \alpha_0(1 - \varepsilon/t)^{\max(0,n-1)} - t(1 - \varepsilon/t)^n;$$  \hspace{1cm} (1.8)

that is, the strategy forces a payoff with lower bound exponentially converging to $\alpha_0$. In particular, condition (1.7) gives sharp criteria for existence of a single strategy forcing $\overline{V} > 0$ or $\overline{V} \geq 0$.

Though evidently of practical use, this result has two shortcomings: First, it is designed to sharply differentiate winning from losing games for player one, but not to determine the sharp value of $\overline{V}$ as is clearly of great interest in applications. Second, it concerns only repeated strategies, whereas it could well be that a sequence of different strategies might be necessary in order to go from $V_0 = -t$ to $\overline{V} - \varepsilon$ in the iterative sequence for $\overline{V}$.

Our first new results are the following theorems simplifying and extending the previous one.

Theorem 1.2 (Transition criterion). For a fixed Player 1 strategy $S$, suppose that $\alpha + \beta V > V$ for $V = v_*$ and $\alpha + \beta V \geq V$ for $V = v^*$. Then, for any initial value $V_0 \in [v_*, v^*]$, a limiting value $\overline{V} \geq v^*$ may be achieved by repeated play of that strategy. Moreover, this criterion is sharp.

Theorem 1.3 (Convergence rate). If, above, $\alpha + \beta v_* \geq v_* + \varepsilon$, $\varepsilon > 0$, then, for $V_0 \in [v_*, v^*]$, we have the sharp “geometric series estimate” $(v^* - V_n) \leq (1 - \varepsilon)^n(v^* - v_*)$. Theorems 1.2 and 1.3 recover for the choice $v_* = -t$, $v^* = \alpha_0$ the result of Theorem 1.1. The choice $v_* = -t$, $v^* = \overline{V}$ on the other hand, where $\overline{V}$ is a fixed point of the value map $T$, with $S$ taken to be the optimal fixed-point strategy for player 1, gives a sharp criterion for existence of a repeated strategy forcing $\overline{V}$, remedying the first deficiency mentioned above. Moreover, Theorem 1.2 may be applied repeatedly over a series of interlacing intervals $[v_*, v^*]$ to reach from $-t$ to $\overline{V}$ by finite repetitions of different strategies $S_i$, thus remedying the second as well.

Overshoot. Theorems 1.2, 1.3 are meant to be used in combination with numerical iteration of (1.4). This raises the question of numerical overshoot: is it possible that numerical error could lead to an approximation $\overline{V}_n$ of $V_n$ that is strictly larger than $\overline{V}$, with subsequent approximations $\overline{V}_m$, $m > n$ increasing to a different fixed point $V^* > \overline{V}$ that does not represent the value forceable by player one? In this case, the a priori validation afforded by Theorem 1.2 would be useless, as the obtained value $V^*$ would be incorrect.

To put things another way, what we would like is for $\overline{V}$ to be strictly attracting from above in the sense that for $V > \overline{V}$ and $|V - \overline{V}|$ sufficiently small,

$$T(V) - V \leq -\delta|V - \overline{V}|,$$  \hspace{1cm} (1.9)

for some $\delta > 0$. 
The following theorem, and our final analytical result, shows that this is generically the case, i.e., numerical overshoot is in general not a worry.

**Theorem 1.4 (Attraction from above).** Let the optimal fixed point strategies for players 1 and 2, defined as optimal strategies for game $A + BV$, be unique. Then, $V$ is attracting from above.

1.1.2. **Numerical.** The main conclusions of [CCZ], recorded in Propositions 3.3 and 3.5 below, were to (i) compute explicitly the symmetric Nash equilibrium for $n$-player guts, and (ii) to confirm rigorously that this is an optimal strategy for player 1 against “bloc” coalitions of players 2-$n$, defined as coalitions in which all players play identical strategies, but is not optimal against general, nonbloc coalitions. These results were derived analytically, and were of qualitative (optimal vs. suboptimal) type. Our goal here is by numerical investigation to obtain quantitative results.

As described in more detail later on, our first main numerical result was to develop a workable algorithm for approximating the solution of generalized recursive games, consisting of numerical implementation of the fixed-point iteration (1.4) using the Fictitious Play routine supported in Nashpy [NP3]. Though we did not do it here, this could be used in principle together with the a posteriori verification afforded by Theorem 1.2 to yield rigorous upper and lower bounds to any desired tolerance.

This algorithm worked extremely well in practice, allowing us to treat 3-player continuous guts with 201 mesh point discretization and 4- and 5-player guts at 21 mesh point discretization, all on a standard laptop with little attempt at optimization. Applied to 3-player guts it yielded the result that players 2-3 working in collaboration can win $\approx 1.2\%$ of the ante of player 1, similar to a typical “house edge” in blackjack. This improved on the lower bound of $\approx 0.4\%$ obtained in [CCZ] by explicit construction using a blend of two well-chosen strategies. Strikingly, the approximate optimal fixed-point strategy found here— that is, the optimal strategy for the game $A + BV$ at the fixed-point value— is quite close to the winning strategy constructed in [CCZ], in particular consisting of a blend of just two strategies. See [CCZ] for some intuition behind this choice. Moreover, this strategy yields $\alpha - \beta t_1 > -t_1$, verifying by the condition of Theorem 1.2 that repeated application of this single strategy suffices to yield $V$.

Experiments pitting player 1 versus players 2-$n$ with $n = 4$ and 5 yielded an optimal strategy with a strikingly simple “pseudo-bloc” structure, in the sense that players 3-$n$ play always the same strategy while player 2 sometimes deviates. Imposing this structure by force and solving the resulting reduced strategy-set game allowed us to treat the $1 v. n$ problem up to $n = 15$ with a fairly high discretization of 101 mesh points. As described later on, this appears to yield a smooth curve of payoff with respect to $n$, with asymptotic value $\approx 16\%$ of player 1’s ante as $n \to \infty$.

The optimal pseudo-bloc strategy was then coded into a continuous guts-playing “bot” to form an instructional interactive game, in which players test their skill against a virtual $n$-player coalition armed with the optimal game-theoretic result. The authors and fellow REU students found this game entertaining and even mildly addictive.

A side-experiment was to code up 3-player Fictitious Play for continuous guts with players responding as individual agents. One might imagine some kind of oscillatory behavior with different temporary coalitions forming and dissolving. However, the result was convergence to (the symmetric Nash) equilibrium similarly as in the 2-player case. This gives a sense in which the Nash equilibrium is relevant to the 3-player game.

Last but not least, we considered the “Weenie rule” variant of Guts discussed in [CCZ, Appendix B], extending the investigation of [CCZ] to the nonbloc case. Surprisingly, we found for this slight modification that the Nash equilibrium strategy is actually optimal, not only against bloc strategies, but against arbitrary player 2-$n$ coalitions. That is, continuous Guts with the Weenie rule is a rare example of a realistic multi-player game with a simple and explicit exact solution!
1.2. Discussion and open problems. Our analysis resolves the main questions posed in [CCZ] regarding general single-state generalized recursive games with termination fee, and continuous guts in particular. It might be interesting to consider also the “gambler’s ruin” problem in which play continues until the players resources are depleted. Another important open problem is the analysis of the actual card-game version of Guts Poker. As described in [CCZ], this involves some interesting combinatorial difficulties having to do with lack of replacement of cards within the deck, and associated lack of independence between probabilities of players hands.

The issue of asynchronous vs. synchronous coalition seems very interesting to explore further from a philosophical and practical point of view. The development of an efficient and convenient numerical algorithm for the evaluation of the asynchronous maximin \( \max_x \min_{y,z} \psi(x,y,z) \) we view as an important open problem even for standard one-shot games. In particular, it is a very interesting question whether there exists a winning asynchronous player 2-n coalition strategy for standard continuous guts, in the sense that it forces a negative expected return for player 1.

Likewise, the observed convergence to Nash equilibrium of Fictitious Play for 3-player guts seems quite interesting from the point of view of behavior in multi-player games. An extremely interesting open problem would be to prove convergence of (FP) for 3- and or n-player guts, and identify the property(ies) that guarantee it, particularly if these might extend to other relevant problems.

Related to the last two problems is to investigate Fictitious Play for the modified game in which players 2-n pool their winnings. It is straightforward to see that a symmetric Nash equilibrium for the original game is also a Nash equilibrium for the modified one. But, it may be that other interesting new equilibria exist corresponding to an asynchronous player 2-n coalition. For continuous 3-player guts, (FP) for this modified game is seen again to converge to the symmetric equilibrium; it is an interesting open question whether there exist other equilibria favoring players 2-3.

More generally, one may consider the question for general n-player symmetric zero-sum games whether it is possible for Fictitious Play to select a winning coalition strategy, or, say, oscillate between different such approximate coalitions. We examine this and other issues to do with Fictitious Play in Appendix C.

Finally, it may be interesting to consider the case of \( m \) v. \( n \) coalitions for general \( m, n \). For example, a study of the 2 v. 2 case returned a value of \( V = V^* = 0 \), showing that this game is “fair.” The optimal strategies, though, were different from the symmetric Nash equilibrium for the original 4-player game, being of a “blended, non-bloc” type similar to that of the optimal strategy for players 2-3 in the 1 v. 2 game; see Section 5. There is no inherent obstacle to the application of our numerical scheme to this more general case, the only additional work being to code the associated payoff matrices.

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2. General analysis

We begin by establishing the general results stated in Section 1.1.1.

2.1. A posteriori estimates.

Proof of Theorem 1.2. It is sufficient to establish the result for the the subgame in which player one’s strategy is always \( S \). By linearity with respect to \( V \) of the payoff function \( \alpha + \beta V \), together with the endpoint assumptions \( \alpha + (\beta - 1)v_* > 0 \) and \( \alpha + (\beta - 1)v_* \geq 0 \), we have \( \alpha + (\beta - 1)V > 0 \).
for any $V \in [v_*, v^*)$, hence there is no fixed point of the value map $T$ (for the reduced game) in this half-open interval. In particular, the value $V \geq v_*$ forced by strategy $S$, since it is a fixed point, must satisfy $V \geq v^*$.

**Proof of Theorem 1.3.** Considering the subgame in which player one’s strategy is always $S$, and defining $\theta_n := (V - v_n)/(V - v_0)$, it is equivalent to show that $\theta_n \leq (1 - \varepsilon)^n$. Rearranging, we have

$$v_n = (1 - \theta_n)V + \theta_nv_0,$$

whence, by linearity of the payoff matrix, together with the assumptions,

$$T(v_n) - v_n \geq \theta_n(T(v_0) - v_0)) \geq \theta_n\varepsilon(V - v_0) = \varepsilon(V - v_n).$$

Here, we have used in an important way superlinearity of $\min_j$.

Thus, $v_{n+1} = T(v_n) \geq v_n + \varepsilon(V - v_n)$, and so

$$V - v_{n+1} \leq V - (v_n + \varepsilon(V - v_n)) = (1 - \varepsilon)(V - v_n),$$

giving by induction the asserted rate $(v^* - V_n) \leq (1 - \varepsilon)^n(v^* - v_\alpha).$ 

**Remark 2.1.** Note that sublinearity of $\max_j$ wrecks the above argument for the general case in which player one’s strategy is not necessarily fixed. Indeed, it appears difficult to quantify the convergence rate in the general case, without specifying a particular strategy sequence beforehand. Luckily, for 3-player Guts, the optimum strategies are repeated and so we may readily estimate the convergence rate using Theorem 1.3.

### 2.2. Overshoot.

We next investigate the possibilities for multiple fixed points, and associated phenomena of overshoot, and duality gap, defined as strict inequality between $V$ and $\overline{V}$. We start with the simple example of an $m \times 1$ game, in which player 2 has no choice in strategy, and only player 1’s choices affect the outcome. It is not difficult to see that, among all lines $\alpha_i + \beta_iV$, the maximum over $i$ can intersect the fixed-point line $F(V) \equiv V$ at at most two points, a point $V_\alpha$ with $\beta_i < 1$ and a point $V^*$ with $\beta_i \geq 1$, the first less than the second. Ignoring the degenerate case $\beta_i = 1$, fixed points with $\beta_i < 1$ are attracting from both sides and with $\beta_i > 1$ are repelling from both sides so long as they remain separated. It follows that for $V_\alpha < V^*$ only the point $V_\alpha$ is a candidate for the value $V$ of the game, and it is always attracting from above. Thus, there is no overshoot. However, for player 2 initial values $V^*$ and player 1 initial values $V_\alpha$, there can be a duality gap, with $V = V_\alpha$ and $\overline{V} = +\infty$.

In the degenerate case $V_\alpha = V^*$ that they intersect, the fixed point is attracting from below and repelling from above, hence overshoot to $+\infty$ is possible, even though the correct value of the game is $V = V_\alpha$ for initial values $v_0 < V_\alpha$. An example is the $2 \times 1$ game $(\alpha_1, \alpha_2) = (0, 0)$, $(\beta_1, \beta_2) = (1/2, 2), \overline{V}$, for which any strategy is a fixed point of $\alpha + \beta V$ for $V = 0$. Moreover, for $V < 0$, we see that $Value(\alpha + \beta V) < 0$ as well, so that $V = 0$. This fixed point is attracting from below, but it is repelling from above, so that in this case overshoot can happen. That is, $T(V) := Value(\alpha + \beta V) > V$ for $V < 0$ and for $V > 0$, with $T(V) = V$ only for $V = V$. In the degenerate case that $\beta_i = 1$ at the fixed point, $\alpha_i = 0$ and there is a line of fixed points, all neutral, hence both overshoot and duality gap are possible.

Combining these observations, we may now treat the general $m \times n$ case, completing our analysis.

**Proof of Theorem 1.4.** We have only to note that, when the optimal fixed-point strategies $S_1$ and $S_2$ for players 1 and 2 are unique, then the reduced $m \times 1$ game obtained by fixing player 2’s strategy always as $S_1$ must still have a unique optimal strategy, so that the fixed-point $V$ must coincide with precisely one of the fixed points $V_\alpha$ and $V^*$ described in the discussion above. Thus, $V$ is attracting from above for the reduced $m \times 1$ game, and therefore (since player 2 can always do at least as well in the original game) for the full, unrestricted game.
Further questions. We have seen that \( m \times 1 \) games give rise to \( 1 - 2 \) finite fixed points, in a specific attracting/repelling configuration. It seems a very interesting question how many fixed points are possible for a general \( m \times n \) game, and what configurations of attraction/repulsion.

3. Payoff functions for continuous Guts

We next recall from \[CCZ\] the derivation of the payoff functions for continuous guts. In computing the expected immediate return \( \alpha \), we take the point of view that the ante is to be paid upon termination of the game. Thus, for example, if two players hold, then the immediate return to the winning player is the value \(+ n\) of the entire pot, and to the losing player \(- n\), with the stakes at next round remaining at value 1, the multiplier of the pot. The immediate return to any players that drop in this scenario is 0.

When three players hold, on the other hand, the pot doubles, effectively paying all players one unit of additional ante in the resulting higher-stakes game, which they will in fact never have to pay. So the immediate returns of all players are incremented by one unit and the stakes- and ante-are changed to 2, exactly balancing out. Thus, the winning player receives immediate return \( n + 1 \) and the losing (holding) players receive return \(- n + 1 = -(n - 1)\). Any dropping players receive +1. One may check that the immediate return is in every event zero-sum: if \( r \) players hold, then the stakes are multiplied by \((r - 1)\), giving all players an additional “virtual ante” of \((r - 2)\). Meanwhile, the single winning player receives \(+ n\) return while the \((r - 1)\) losing (holding) players receive \(- n\), for a total immediate return of \(n(r - 2) + n - (r - 1)n = 0\). It follows by summation across events that the expected immediate return \( \alpha \) is zero-sum as well.

For reference, the explicit results for \( n = 2 \) and \( n = 3 \) derived in \[CCZ\] are as follows.

Proposition 3.1 (\[CCZ\]). For continuous 2-player Guts,

\[
\alpha(p_1^*, p_2^*) = \begin{cases} 
(1 - 2p_1^*)(p_1^* - p_2^*) & p_2^* \leq p_1^*, \\
(1 - 2p_2^*)(p_1^* - p_2^*) & p_2^* > p_1^*, 
\end{cases}
\]

\[
\beta(p_1^*, p_2^*) = p_1^* p_2^* + (1 - p_1^*)(1 - p_2^*).
\]

Proposition 3.2 (\[CCZ\]). For 3-player continuous Guts,

\[
\alpha(p_1^*, p_2^*, p_3^*) = \begin{cases} 
2p_1^* - p_2^* - p_3^* + (p_2^*)^3 + 3(p_2^*)^2 p_3^* - 4p_1^* p_2^* p_3^* & p_1^* < p_2^* < p_3^*, \\
2p_1^* - p_3^* - p_2^* + (p_3^*)^2 p_2^* - 4p_1^* p_3^* p_2^* & p_1^* < p_3^* < p_2^*, \\
2p_1^* - p_2^* - p_3^* + (p_1^*)^3 - 3(p_1^*)^2 p_3^* + 2p_1^* p_2^* p_3^* & p_2^* < p_1^* < p_3^*, \\
2p_1^* - p_2^* - p_3^* + (p_1^*)^2 p_2^* + 2p_1^* p_2^* p_3^* & p_3^* < p_1^* < p_2^*, \\
2p_1^* - p_2^* - p_3^* - 2(p_1^*)^3 + 2p_1^* p_2^* p_3^* & p_2^* < p_3^* < p_1^*, \\
2p_1^* - p_2^* - p_3^* + 2(p_1^*)^2 p_3^* & p_3^* < p_2^* < p_1^*, 
\end{cases}
\]

\[
\beta = 2 - p_1^* - p_2^* - p_3^* + 2p_1^* p_2^* p_3^*.
\]

3.1. Bloc strategies and symmetric Nash equilibria. Following \[CCZ\], we define a “bloc” strategy for the \( n \)-player game to be a strategy \((p_1^*, p_2^*, \ldots, p_n^*)\) in which players 2-\( n \) pursue identical strategies. Restricting to bloc strategies reduces the \( n \)-player game to a 2-player game that is a simplified version of player 1 vs. a player 2-\( n \) coalition. It is not difficult to see \[CCZ\] that for a symmetric one-shot game, the optimal bloc strategy corresponds to the symmetric Nash equilibrium guaranteed by \[N\]. Using the explicit payoff functions of Propositions 3.1,3.2 for \( n = 2, 3 \), along with partial information obtained in \[CCZ\] on the bloc payoff function for general \( n \), we have the following explicit description of the symmetric Nash equilibrium, and optimality for the bloc game.
Proposition 3.3 (Cor 8.9). CCZ] For $n$-player continuous Guts, the unique symmetric Nash equilibrium is $(p_1^*, \ldots, p_n^*)$, with

$$p_i^* = \frac{1}{2^{1/(n-1)}}.$$  

Moreover, this choice forces total value $V = 0$ for player 1 in the bloc 1 v. $(n - 1)$ version of Guts.

Remark 3.4. As in [CCZ, Prop. A.2], we note that no coalition employing a bloc strategy (even a mixed strategy) of $n$ players can beat a single player in any zero-sum symmetric game, as player 1 can always switch to the same strategy to give payoff zero.

3.2. Suboptimality in the non-bloc game. Neither the general payoff function nor the bloc version have been computed analytically for $n$-player guts $n \geq 4$, here or in [CCZ]. However, strategically chosen test strategies were used in [CCZ] to obtain the following negative conclusion.

Proposition 3.5 (Cor. 8.16 [CCZ]). For the general (i.e., nonbloc) 1 v. $(n - 1)$ version of continuous Guts, there exists a strategy for players 2-3 forcing a negative return for player 1. In particular, the Nash equilibrium strategy $p_1^*$ is suboptimal for player 1 vs. an $n - 1$ player coalition.

The strategy constructed in Proposition 3.5 was a blend of two pure strategies: one of bloc type $(p_2^*, \ldots, p_n^*)$, and the other of “pseudo-bloc” type $(p_2^*, p_3^*, \ldots, p_n^*)$. As described in Section 1.1, this leaves open the question of the optimal strategy in the full (nonbloc) $n$-player game, $n \geq 3$, along with the values $V$ and $V$ forceable by players 1 and 2-3, respectively, the answers to which are the main focus of our numerical investigations.

4. Numerical algorithms

We now describe the numerical algorithms used to perform our computations.

4.1. Numerical payoff functions.

4.1.1. General $\alpha$-function. In order to study general $n$-player Guts, we developed a function in Python that can compute the expected payoff $\alpha(p_1, p_2, \ldots, p_n)$ for any $n$. The algorithm is as follows.

1) The strategies are first reordered from $(p_1, p_2, \ldots, p_n)$ to $(p_1^*, p_2^*, \ldots, p_n^*)$ such that $p_i^* \leq p_j^*$ for $i \leq j$; essentially, the strategies are sorted in increasing order. An array of $n$ zeroes named value is initialized, with each index $i$ representing the expected payoff of the player playing the strategy $p_i^*$.

2) The number of players who drop is first determined and denoted as $0 \leq d \leq n - 1$. We do not need to account for the case where all players drop, since in that scenario the expected payoff is 0 for all players. The number of players who hold then is simply $n - d$ and is denoted as $h$.

3) A subset $D$ of the players of size $d$ is then chosen from $\binom{n}{d}$ possible subsets to represent the players who drop. The complement $H$ of size $h$ is then the subset of players who hold.

4) Of the players in $H$, there are two possibilities for each player. Let $p^H$ be the highest strategy in $H$. Then either a player receives a hand that is at least $p^H$ in which case the player is in "fair play", or the player receives a hand that is less than $p^H$ in which case the player is guaranteed to lose. A certain number of losers $0 \leq l < h$ is and a subset $L$ that represents the losers from $\binom{h}{l}$ possibilities are chosen. This also determines the number of players in fair play $f$ and a subset $F$ that represents these players so that $F \cup L = H$. 

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5) Now that all players have either dropped, lost, or contended in fair play, we can calculate the probability of this scenario with

\[
\text{probability} = \prod_{p \in D} p \prod_{p \in F} (1 - p^H) \prod_{p \in L} (p^H - p)
\]

We will then multiply this probability with the expected return for each player and add to the value array accordingly. Players who have dropped will have a return of \(-1 + (h - 1) = h - 2\). Players who have lost will have a return of \(-1 - n + (h - 1) = h - n - 2\). Players who contended in fair play will have an expected return of \((-1 + n + h - 1) \cdot \frac{1}{2} + (-1 - n + h - 1) \cdot \frac{L-1}{2}\).

6) Repeat this process with each possible number of players who drop and each possible subset of players who drop. Then, restore the original order of the player strategies given in the value array.

This algorithm has a runtime of \(\Theta(n3^n)\) assuming constant runtime for basic operations.

4.1.2. General \(\beta\)-function. We also developed a function that can compute \(\beta(p_1, \ldots, p_n)\) for any \(n\). The algorithm for this function is as follows.

1) The number of players who drop \(d\) is determined. The number of players who hold is \(h\). Initialize \(\beta = 0\).

2) A subset \(D\) of the players of size \(d\) is then chosen from \(\binom{n}{d}\) possible subsets to represent the players who drop. The complement \(H\) of size \(h\) is then the subset of players who hold.

3) The probability of this scenario is then

\[
\text{probability} = \prod_{p \in D} p \prod_{p \in H} (1 - p)
\]

. Update \(\beta = \beta + \text{probability} \cdot (h - 1)\) unless \(h = 0\), in which case, \(\beta = \beta + \text{probability} \cdot 1\).

4) Repeat this process with each possible number of players who drop and each possible subset of players who drop.

This algorithm has a runtime of \(\Theta(n2^n)\).

4.1.3. Payoff Matrices for \(n\)-player Guts. To make player 1’s payoff matrix for \(n\)-player Guts game where player 1 is against a coalition of players 2-\(n\), we used a special indexing scheme to represent the strategies of the coalition. We created an \(N \times N^{n-1}\) matrix where \(N\) is our discretization of Guts and the column \(j_1 + j_2 \cdot N + j_3 \cdot N^2 + \ldots j_{n-1} \cdot N^{n-2}\) will represent the scenario where player \(i\) in the coalition plays the strategy \(j_i\). The strategy \(i\) played by player \(p_i\) of the coalition can be retrieved from the column \(x\) by taking \(\lfloor (x \mod N^i)/N^{i-1} \rfloor\).

4.1.4. Pseudo-Bloc Modifications. In the pseudo-bloc variation of Guts in which we are concerned with only the payoff of player 1, the \(\alpha\) function algorithm can be simplified as players 3-\(n\) play the same strategies. In a normal player 1 vs players 2-\(n\) Guts game, we have to account for all possible combinations of players dropping and holding. However, in pseudo-bloc Guts, only the number of players dropping and holding in the bloc of players 3-\(n\) is relevant as all of them are playing the same strategies. This allows us to reduce the runtime complexity of the \(\alpha\) function to just \(O(n^2)\).
4.2. Numerical fixed point iteration. To determine the value of the generalized recursive game, we fill each player’s matrix \( \alpha + \beta V_0 \) for each players \( \alpha \), \( \beta \), and \( V_0 \). \( V_0 \) is some anticipated value of the game. A logical starting point is \( -1 \) for a single player and \( -N \) for a coalition of \( N \) players in the \( 1 \) v. \( N \) coalition case. This is a “worst case” that we know a player can guarantee, since any player can play a strategy of 1 (always fold) to guarantee this value. We can instead start each player at \( V_0 = 0 \) to guarantee the game is 0-sum (and thus fictitious play will converge in the 1 v. \( N \) setting).

After generating this first matrix, we can run fictitious play on this matrix to determine each player’s value \( V_1 \). We can repeat this process using each \( V_n \) to compute \( V_{n+1} \), and repeat this process until the player’s \( V_n \) converge to some \( V^* \). This is a fixed point: a value where \( V^* = Value(\alpha + \beta V^*) \). If this fixed point is unique, it is the value of the game. This value iteration will be monotonic in the zero-sum setting, so it will either converge to a fixed point or “overshoot” and diverge to infinity. As we have shown earlier, generically overshoot does not occur.

5. Numerical results

Finally, we describe our numerical results. In the 1 v. 2 and 1 v. 3 player games, we determined the optimal strategy for the coalition, which guarantees advantage against player 1. This strategy has a joint value of approximately 0.013 for players 2 and 3, and -0.013 for player 1. Here, players 2 and 3 play a bloc strategy of 0.68 around 86% of the time. In the remaining 14% of the time, player 2 plays a strategy of 0 while player 3 plays a strategy of 0.86. Player 1’s best response to this mixed strategy, a strategy of 0.64, loses to both. These results were created with a discretization of 101, meaning that there are 101 possible strategies from 0 to 1. The payoffs of the component strategies, denoted A and B, are depicted graphically in Figure 1 and that of the corresponding blended strategy in Figure 2 below.

We next computed the optimal strategies for player 1 vs. a coalition of \( N \) players, with \( N = 3, 4, \) and 5. The discretizations for these experiments was \( M = 21 \). In spite of the low resolution
in the discretization of strategies, a trend emerged where the coalition would randomly select either a “bloc” strategy where every member of the coalition would adopt the same strategy or a “pseudo-bloc” strategy where a single member of the coalition, without loss of generality player 2, would always hold and the rest of the coalition would adopt a conservative bloc strategy with each holding at the same threshold. With this mixed strategy for the coalition, player one cannot force a positive return and the best they can do is to play more conservatively (than for the Nash equilibrium strategy of \( p^*_1 = \frac{1}{2^{1/n}} \)) in order to minimize losses. The reason for the low discretization is that treating each member of the coalition independently costs a significant amount of computational power with asymptotic run time bounded by \( O(M^N N^3) \), where \( I \) denotes the number of iterations in one round of Fictitious Play, with physical run times ranging from minutes for \( N = 2 \) to in excess of an hour for \( N = 5 \). The main reason for this is that modeling each of the \( N \) player’s strategy combinations is quite computationally intensive, requiring a matrix of size \( M^N + 1 \) for a discretization of \( M \).

Motivated by the observed trend that the optimal coalition appears in practice to be of pseudo-bloc type, we reduced complexity by substituting for the full game a reduced game in which the coalition could only use bloc or pseudo-bloc strategies. Computationally, this is significantly cheaper as now the run time is bounded by \( O(M^3 I + M^3 N^2) \). With these computational savings, we were able to explore the strategies for both player one and the coalition of \( N \) players for \( N \) between 2 and 15 with a discretization of \( M = 101 \). We verified that the restrictions we placed on the coalition’s strategies in the more optimized algorithm correctly reproduced the results from the independent actor method for \( N = 2, 3, 4, 5 \) as a crucial test. Using SciPy, we fitted a rational function to the values forceable by the coalition for \( N \) between 2 and 15. The resulting best fit was

\[
V(n) = 0.163 - 0.84/(n - 3.6)
\]

with an excellent \( r^2 \) squared value of \( \approx 0.9991 \), and a predicted asymptote as \( N \to \infty \) of 0.16. Compared to blackjack where the house edge is 0.02; one sees that the coalition in continuous Guts has a much higher expected gain.

Results are recorded in Table 1 showing, for each coalition size \( N \), the value of the game from the coalitions, Player 1’s best response against the optimal strategy, the coalition’s bloc strategy, used a majority of the time, and the strategy played by the pseudo-bloc while player 2 plays a strategy of 0. This pseudo-bloc strategy is played around 10-30 percent of the time, increasing with the player count, however these results were noisy and thus not included. A graphical display, together with the optimized curve fit, is given in Figure 3 below.

An interesting variation was the 2 v. 2 case, which, due to high computational complexity, was carried out with relatively low discretization \( M = 21 \). For this experiment, the optimal values
Table 1. Numerical results for the pseudo-bloc case

| $N$ | Opponent Value | Player 1 Strategy | Bloc Strategy | Pseudo-bloc Strategy |
|-----|----------------|-------------------|---------------|----------------------|
| 1   | 0.0            | 0.5               | 0.5           | -                    |
| 2   | 0.0132         | 0.64              | 0.68          | 0.86                 |
| 3   | 0.0339         | 0.72              | 0.76          | 0.89                 |
| 4   | 0.0516         | 0.77              | 0.81          | 0.91                 |
| 5   | 0.0654         | 0.81              | 0.84          | 0.93                 |
| 6   | 0.0753         | 0.84              | 0.87          | 0.94                 |
| 7   | 0.0847         | 0.86              | 0.88          | 0.94                 |
| 8   | 0.0909         | 0.87              | 0.89          | 0.95                 |
| 9   | 0.0954         | 0.89              | 0.91          | 0.95                 |
| 10  | 0.1007         | 0.89              | 0.91          | 0.96                 |
| 11  | 0.1066         | 0.91              | 0.92          | 0.96                 |
| 12  | 0.1074         | 0.92              | 0.93          | 0.97                 |
| 13  | 0.1110         | 0.92              | 0.93          | 0.97                 |
| 14  | 0.1154         | 0.92              | 0.94          | 0.97                 |
| 15  | 0.1184         | 0.93              | 0.94          | 0.97                 |

Figure 3. Curve fit by rational function of total return for pseudo-bloc strategy, 1 v. $n$.

turned out to be $V = \overline{V} = 0$, so that the game has a value of $V = 0$ in the strong minimax sense of von Neumann [vN]. The optimal strategies for the two coalitions were identical and somewhat similar to the optimal strategies for players 2-3 in the 1 v. 2 game. Namely, players pursue a bloc strategy with thresholds of .75-.8 with probability $\approx 0.95$, and non-bloc strategy with thresholds $(0,0.95)$ with probability 0.05.
5.1. **The marauding guts-bot and performance.** The optimal pseudo-bloc strategies found for a player 2-\((n+1)\) coalition were coded into an interactive game with the user playing the role of player 1 in a 1 v. \(n\) game of continuous guts, discretized at 101 mesh points. The game was tested by authors and some fellow REU students. High scores were 16 (Jay) and 26 (Jacob) for a round starting with 1 unit ante. Interesting informal conjectures were that even though the bot is in the long run unbeatable, certain aggressive or erratic patterns of play seems to increase the chances of a high score in the short term, perhaps by increasing variance in the outcome. This, and other real-world experiments on actual play and play-patterns would be a very interesting direction for further investigation.

5.2. **Three-player Fictitious Play.** Finally, we experimented with various aspects of Fictitious Play. We saw that for 3 player free-for-all guts, the Fictitious Play algorithm did not converge to the Nash equilibrium. Curiously, the Fictitious Play algorithm did not converge to the coalition strategy. The reason for this is that in the coalition, the bloc strategy does not beat the equilibrium strategy and the pseudo-bloc strategy does not allow for both players two and three to benefit while simultaneously causing player one to lose. In fact, no strategy allows player two to benefit, player three to be at least neutral and causes player one to lose.

More precisely, in the 1 v. 2 case, the 2 coalition players are incentivized to break and go to the equilibrium strategy. When one player is playing a strategy of 0, the other coalition member is incentivized to play a strategy close to the 3-player equilibrium strategy of \(\frac{1}{\sqrt{2}}\), which will then also trigger the other coalition member to switch to a similar strategy. These players will be incentivized to play a strategy approaching the 3-player equilibrium strategy of \(\frac{1}{\sqrt{2}}\). Through numerical experiments, we determine that no bloc strategy gains advantage against player 1 playing \(\frac{1}{\sqrt{2}}\), if player 1 and player 2 are playing \(\frac{1}{\sqrt{2}}\), no strategy for player 3 gains advantage, and there is no pair of strategies \(S_2, S_3\), where if player 1 plays \(\frac{1}{\sqrt{2}}\), player 2 plays \(S_2\), and player 3 plays \(S_3\), player 1 loses while both players 2 and 3 are at least neutral.

If two non-communicating players attempt to team up against player 1, they will be unable to meaningfully gain advantage, as even if both players play for the team and do not betray each other, players 2 and 3 each playing \(\frac{1}{\sqrt{2}}\) is the only pure equilibrium, so coalitions are unlikely to arrive naturally in 3 player play. This equilibrium of \(\frac{1}{\sqrt{2}}\) was observed in 3 player Fictitious Play, even when players 2 and 3 played for the team and not themselves.

**Appendix A. Computational complexity**

When experimenting with the 1 v. \(n\) case, we found that the most time-consuming portion of the calculation was the construction of the \(A\) matrix. However, we also found that in the cases we could run, the equilibrium was always a “pseudo-bloc” case, as previously described. In order to save computation, we thus restricted the coalition’s allowed strategies to this pseudo-bloc type, in which players 3-\((n+1)\) were required to play a common strategy at all times. This greatly reduced the time necessary to calculate \(A\). Not only did this require populating a much smaller matrix (effectively populating an array of one less dimension), but the defining function for the matrix was also reduced from \(n^3\) to \(n^2\) cases. This allowed us to efficiently run up to coalition size \(n = 15\). To give an idea of the run time, working on a Mac powerbook, with discretization of 101 mesh points, it took approximately 1 minute (real time) to compute the 1 v. 2 solution, and approximately one hour (real time) to compute the 1 v. 15 solution under pseudo-bloc play.

In order to further improve run time, the most efficient means would be to parallelize the definition of the \(A\) matrix. If this were parallelized and run on a GPU, where many cores allow for very fast processing of parallelized operations, the runtime would be cut by a large factor. Rewriting the program in another language could also improve runtime. The results presented were achieved
using python for ease of use. If Fortran or C++ were implemented the results would likely be improved. In short, the applications carried out here did not press much the limits of computation time, leaving plenty of room for improvement in more complicated scenarios.

**Appendix B. The Weenie rule**

A variant on Guts Poker is the addition of the “Weenie rule” penalizing overcautious play. Under this rule, should all players drop, the player with highest hand- the “weenie”- must match the pot, thus doubling the stakes for the next round. It was shown in [CCZ] that the optimal bloc strategy \( p^*_1 = 1/2^{1/(n-2)} \) for standard guts is shifted under the Weenie rule to the more aggressive strategy \( p^*_1 = 1/3^{1/(n-1)} \), as recorded in the following proposition.

**Proposition B.1 ([CCZ]).** For \( n \)-player continuous Guts with the Weenie rule, the unique symmetric Nash equilibrium is \( (1/3^{1/(n-1)}, \ldots, 1/3^{1/(n-1)}) \), and this is optimal for the bloc strategy game. Different from the standard case, it is a strict Nash equilibrium.

Here, we go further, investigating the general (nonbloc) case. The surprising result is that in the case of the Weenie rule, quite differently than in the standard case, the explicit Nash equilibrium strategy appears to be optimal not only against bloc strategies, but against general nonbloc coalition strategies of players \( 2-n \). That is, continuous Guts with the Weenie rule appears to be a rare example of a real-world multi-player game with an exact solution: a simple pure strategy of threshold type.

**B.1. Payoff for \( n = 2, 3 \).** We start by extending the following result of [CCZ] to the case \( n = 3 \).

**Proposition B.2 ([CCZ]).** For 2-player continuous Guts with the Weenie rule,

\[
\alpha(p_1^*, p_2^*) = \begin{cases} 
(1 - 2p_1^* - p_2^*)(p_1^* - p_2^*) & p_2^* \leq p_1^*, \\
(1 - 2p_2^* - p_1^*)(p_2^* - p_1^*) & p_2^* > p_1^*.
\end{cases}
\]

**Proposition B.3.** For 3-player continuous guts with the Weenie rule, \( \beta \) is as for standard Guts, while the payoff for player 1 is \( \alpha(p_1^*, p_2^*, p_3^*) = \)

\[
\begin{align*}
2p_1^* - p_2^* - p_3^* + (p_3^*)^2 + (p_1^*)^3 + 3(p_2^*)^2p_3^* - 3p_1^*p_2^*p_3^*, & \quad p_1^* < p_2^* < p_3^*, \\
2p_1^* - p_2^* - p_3^* + (p_3^*)^3 - (p_1^*)^3 + 3(p_2^*)^2p_3^* - 3p_1^*p_2^*p_3^*, & \quad p_1^* < p_3^* < p_2^*, \\
2p_1^* - p_2^* - p_3^* + (p_2^*)^2/2 + (p_3^*)^3 - 3(p_1^*)^2p_2^* - 2 + 3(p_1^*)^2p_3^* - 3p_1^*p_2^*p_3^*, & \quad p_2^* < p_1^* < p_3^*, \\
2p_1^* - p_2^* - p_3^* + (p_1^*)^3/2 + (p_2^*)^2/2 - 3p_1^*p_2^* - 2 + 3(p_1^*)^2p_3^* - 3p_1^*p_2^*p_3^*, & \quad p_3^* < p_1^* < p_2^*, \\
2p_1^* - p_3^* - p_2^* - 2(p_1^*)^3 + (p_2^*)^2/2 + 3p_2^*(p_3^*)^2/2, & \quad p_2^* < p_3^* < p_1^*, \\
2p_1^* - p_3^* - p_2^* - 2(p_1^*)^3 + (p_3^*)^2/2 + 3p_2^*(p_3^*)^2/2, & \quad p_3^* < p_2^* < p_1^*.
\end{align*}
\]

**Proof.** We first compute the correction to the standard guts payoff in the case that all players drop.

**Case 1:** \( p_1^* < p_2^* < p_3^* \)

Case i) \( p_1, p_2, p_3 \leq p_1^* \), outcome zero.
Case ii) \( p_1, p_3 \leq p_1^*; p_2 \in [p_1^*, p_2^*] \), outcome \( 1 \cdot (p_1^*)^2(p_2^* - p_1^*) \).
Case iii) \( p_1, p_2 \leq p_1^*; p_3 \in [p_1^*, p_2^*] \), outcome \( 1 \cdot (p_1^*)^2(p_3^* - p_1^*) \).
Case iv) \( p_1 \leq p_1^*; p_2, p_3 \in [p_1^*, p_2^*] \), outcome \( 1 \cdot p_1^*(p_2^* - p_1^*)^2 \).
Case v) \( p_1, p_2 \leq p_1^*; p_3 \in [p_2^*, p_3^*] \), outcome \( 1 \cdot (p_1^*)^2(p_3^* - p_2^*) \).
Case vi) \( p_1 \leq p_1^*; p_2 \in [p_1^*, p_2^*]; p_3 \in [p_2^*, p_3^*] \), outcome \( 1 \cdot p_1^*(p_2^* - p_1^*)(p_3^* - p_2^*) \).

Total: \( -(p_1^*)^3 + p_1^*p_2^*p_3^* \)

**Case 2:** \( p_2^* < p_1^* < p_3^* \)

Case i), all fair play, outcome zero
Case ii) \( p_2, p_1 \leq p_2^*; p_3 \in [p_2^*, p_3^*] \), outcome \( 1 \cdot (p_2^*)^2(p_3^* - p_2^*) \).
Case iii) \( p_2, p_1 \leq p_2^*; p_3 \in [p_1^*, p_3^*] \), outcome \( 1 \cdot (p_2^*)^2(p_3^* - p_1^*) \).
Case iv) $p_2, p_3 \leq p_3^*; p_1 \in [p_2^*, p_1^*]$, outcome $-2 \cdot (p_2^*)^2(p_3^* - p_3^*)$

Case v) $p_2 \leq p_2^*; p_1, p_3 \in [p_2^*, p_3^*]$, outcome $-\frac{3}{2} \cdot (p_3^* - p_3^*)^2$

Case vi) $p_2 \leq p_2^*; p_1 \in [p_2^*, p_1^*]; p_3 \in [p_1^*, p_3^*]$, outcome $1 \cdot p_2^*(p_1^* - p_2^*)(p_3^* - p_1^*)$

Total: $(p_2^*)^3/2 - \frac{3}{2}p_2^*(p_1^*)^2 + p_1^*p_2^*p_3^*$

Case 3: $p_2^* < p_3^* < p_1^*$. Case i) fair play, outcome zero

Case ii) $p_2, p_3 \leq p_2^*; p_1 \in [p_2^*, p_3^*]$, outcome $-2 \cdot (p_2^*)^2(p_3^* - p_2^*)$

Case iii) $p_2, p_3 \leq p_2^*; p_1 \in [p_2^*, p_3^*]$, outcome $-2 \cdot (p_3^*)^2(p_1^* - p_2^*)$

Case iv) $p_2, p_1 \leq p_2^*; p_3 \in [p_2^*, p_3^*]$, outcome $1 \cdot (p_2^*)^2(p_3^* - p_2^*)$

Case v) $p_2 \leq p_2^*; p_1 \in [p_2^*, p_3^*]; p_3 \in [p_2^*, p_3^*]$, outcome $-2 \cdot p_2^*(p_3^* - p_2^*)(p_1^* - p_3^*)$

Total: $(p_2^*)^3/2 + \frac{3}{2}(p_2^*)^2 - 2p_1^*p_2^*p_3^*$

Summing with the standard payoffs given in (3.2), we obtain formulas (i), (iii), and (v) of (B.2). The remaining cases follow by invariance under permutation of $p_2^*$ and $p_3^*$.

B.2. Optimal strategy for $n = 2, 3$.

Corollary B.4. For 2- and 3-player continuous Guts with the Weenie rule, the Nash equilibrium strategies $p_1^* = 1/3$ and $p_1^* = 1/\sqrt{3}$ are optimal against general (nonbloc) strategies, forcing $V = 0$.

Proof. For $n = 2$, this follows from Proposition B.1 as all strategies are bloc strategies in this case.

We focus therefore on the case $n = 3$.

Case I ($p_1^* \leq p_2^* \leq p_3^*$)

Computing for $p_1^* = 1/\sqrt{3} \leq p_2^* \leq p_3^*$ that

\[
\partial \alpha / \partial p_2^* = -1 + 6p_2^*p_3^* - 3p_1^*p_2^* = (3p_2^*p_3^* - 1) + 3p_3^*(p_2^* - p_1^*) \geq 0,
\]

we find for fixed $p_1^* = 1/\sqrt{3}, p_2^* \geq 1/\sqrt{3}$, that $\alpha$ is minimized at the left endpoint $p_2^* = p_1^* = 1/\sqrt{3}$, or $\alpha(1/\sqrt{3}, 1/\sqrt{3}, p_3^*)$. But, we know from the analysis of the bloc strategy case in [CCZ] (here applied to player 1-2 bloc) that player 3 obtains return $\leq 0$, with equality if and only if $p_3^* = 1/\sqrt{3}$. Thus, player 1 obtains a return $\geq 0$ in this case, with equality if and only if $p_1^* = p_2^* = p_3^* = 1/\sqrt{3}$.

Case II ($p_2^* \leq p_1^* \leq p_3^*$) Fixing $p_1^* = 1/\sqrt{3}$, we obtain

\[
\hat{\alpha}(p_2^*, p_3^*) := \alpha(1/\sqrt{3}, p_2^*, p_3^*) = 2/\sqrt{3} - 3p_2^*/2 - 2p_3^* + (p_2^*)^3/2 + (p_3^*)^3 + \sqrt{3}p_2^*p_3^*.
\]

For fixed $p_3^*$, we find that

\[
\partial \hat{\alpha} / \partial p_2^* = 3(p_2^*)^2 - 2 - 3/2 + \sqrt{3}p_3^*.
\]

This is nonnegative for

\[
2/\sqrt{3} \leq p_2^* \leq 1,
\]

hence is minimized in this case at $p_2^* = 0$, or

\[
\hat{\alpha}(0, p_3^*) = 2/\sqrt{3} - p_3^* + (p_3^*)^3 - p_3 \geq 2/\sqrt{3} + (p_3^*)^3 - 2p_3.
\]

The right-hand side of (B.5) is increasing for $2/\sqrt{3} \leq p_3^* \leq 1$, and at endpoint $p_3^* = 2/\sqrt{3}$ gives value $2/\sqrt{3} + 3p_3^*(p_3^*)^2 - 2 = 2/\sqrt{3}(1 + 4/3 - 2) > 0$. Thus, $\hat{\alpha}(p_2^*, p_3^*) > 0$ in case (B.4).

In the remaining case,

\[
1/\sqrt{3} \leq p_2^* \leq 2/\sqrt{3},
\]

noting that $\hat{\alpha}(. , p_3^*)$ is convex, we find that $\hat{\alpha}$ is minimized at the stationary point $\partial \hat{\alpha} / \partial p_2^* = 0$, or

\[
p_2^* = p_2^*(p_3^*) := \sqrt{1 - 2p_3^*/\sqrt{3}}.
\]
Plugging (B.7) into $\tilde{\alpha}$, we find that the minimum value of $\tilde{\alpha}$ is thus

\begin{equation}
\tag{B.8}
h(p_3) = \tilde{\alpha}(p_2(p_3^*), p_3^*) = 2/\sqrt{3} - 2(1 - 2p_3^*/\sqrt{3})^{1/2} - 2p_3^* + (1 - 2p_3^*/\sqrt{3})^{3/2} + (p_3^*)^3 + \sqrt{3}(1 - 2p_3^*/\sqrt{3})^{1/2}p_3^*.
\end{equation}

We may readily compute that $p_3^*(1/\sqrt{3}) = 1/\sqrt{3}$, hence

\begin{equation}
\tag{B.9}
h(1/\sqrt{3}) = \alpha(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 0, \quad h(2/\sqrt{3}) = \alpha(1/\sqrt{3}, 0, 2/\sqrt{3}) = \tilde{\alpha}(0, 2/\sqrt{3}) > 0
\end{equation}

(the second already computed in case (B.4)). Next, we compute

\[ h'(p_3^*) = (3p_3^*/2 - 1)(1 - 2p_3^*/\sqrt{3})^{-1/2} + (\sqrt{3} + 3/2)(1 - 2p_3^*/\sqrt{3})^{1/2} - 2 + 3p_3^*; \]

and thus

\[ h''(p_3^*) = (1/\sqrt{3})\left(Cp_3^* - 1 - (p_3^*)^2 - \sqrt{3}\right)(1 - 2p_3^*/\sqrt{3})^{-1/2}, \]

for $C = \sqrt{3}/2 + 2/\sqrt{3} = \sqrt{3}/2 + 2/\sqrt{3} > \sqrt{3} \geq 2p_3^*$. Observing for

\[ g(p_3^*) := Cp_3 - 1 - (p_3^*)^2 - \sqrt{3} \]

that $g'(p_3) = C - 2p_3^* > 0$, we find that $g$ is maximized on $p_3^* \in [1/\sqrt{3}, \sqrt{3}/2]$ at the right endpoint

\[ g(2/\sqrt{3}) = 2C/\sqrt{3} - 1 - 4/3 - \sqrt{3} = 14/6 - 6/6 - 8/6 - \sqrt{3} < 0, \]

we find that $h'' < 0$ everywhere on $p_3^* \in [1/\sqrt{3}, \sqrt{3}/2]$. Thus, $h$ lies above its secant line between endpoints $p_2^* = 1/\sqrt{3}$ and $p_2^* = 2/\sqrt{3}$, and so $h(p_3) \geq 0$ by (B.9). Combining with case (B.4), we have that $\alpha \geq 0$ on the whole range $p_3^* \in [1/\sqrt{3}, 1]$, with equality if and only if $p_1^* = p_2^* = p_3^* = 1/\sqrt{3}$.

**Case III** ($p_2^* \leq p_3^* \leq p_1^*$)

Computing $\partial \alpha/\partial p_2^* = -1 + 3(p_2^*)^2/2 + 3(p_3^*)^2/2 \leq -1 + 3(p_1^*)^2 = 0$, we find for fixed $p_1^* = 1/\sqrt{3}$ and $p_3^* \leq 1/\sqrt{3}$ that $\alpha$ is minimized on $[0, p_3^*]$ at $p_2^* = p_3^*$. But, by the results of [CCZ] for the bloc case $\alpha(1/\sqrt{3}, p_2^*, p_2^*)$ is $\geq 0$ with equality if and only if $p_2^* = 1/\sqrt{3}$. Thus, in this case, again, $\alpha \geq 0$ with equality if and only if $p_1^* = p_2^* = p_3^* = 1/\sqrt{3}$.

The result of Corollary [B.4] for continuous Guts with the Weenie rule is in striking contrast to the result for standard continuous Guts, giving an optimal winning strategy for player 1 against a coalition of players 2-n. This is illustrated in Figure 4 by a plot of the surface $\alpha(1/\sqrt{3}, p_2^*, p_3^*)$.

**B.3. The case $n \geq 4$.** For the $n$-player game with $n \geq 4$, we investigated numerically, by checking for discretization $M = 1,001$ the optimality property

\[ \min_{p_2^*, \ldots, p_n^*} \alpha(1/3^{1/(n-1)}, p_2^*, \ldots, p_n^*) \geq 0. \]

For $n = 3, 4, 5$ all yielding (numerical) optimality, indicating optimality of the Nash equilibrium strategy against general nonbloc strategies. We conjecture that this property holds for general $n \geq 2$, in striking contrast to the results for standard continuous Guts.

**B.4. Conclusions.** We see that 3-player continuous Guts with the Weenie rule is a rare instance of a multi-player game with exact minimax solution, or strong Nash equilibrium, namely, the symmetric equilibrium $(p_1^*, p_2^*, p_3^*) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Indeed, it is the still rarer instance of a multi-player game with saddlepoint solution consisting of pure rather than blended strategies, with pure strategies being of simple threshold type.

Note that this does not follow by concave-convexity of the payoff function as in the bloc case [CCZ], but by subtle dynamics of the model. Indeed, for symmetric multi-player ($n \geq 3$) games, considered as a 2-player game between player 1 and a player 2-n coalition, the associated payoff
Figure 4. Plot of $\alpha(1/\sqrt{3}, p_2^*, p_3^*)$ vs. $(p_2^*, p_3^*)$.

function $\alpha$ can never be concave (in $p_1$)-convex (in $(p_2, \ldots, p_n)$), by symmetry, unless $\alpha$ is linear. The fact that standard Guts does not share this property is further evidence of its subtlety.

**Appendix C. Convergence of Fictitious Play**

We include here a sketch of the proof of convergence of Fictitious Play for zero-sum 2-player games given by Jean Robinson [R], the first woman elected to the National Academy of Sciences, as the solution of a RAND corporation prize problem posed by G. Brown. The argument is based on the two ingredients of duality and Hamiltonian structure.

Let $P^n$ and $Q^n$ denote the vector of frequencies of strategies played respectively by players one and two, so that $P^n/n$ and $Q^n/n$ are the associated empirical probability distributions. Define $BR_1(q)$ to be the best response of player one against player two strategy $q$, i.e. $\text{argmax}_p \phi(p, \cdot, q)$. Likewise, define $BR_2(p)$ to be the best response of player two against player one strategy $p$, i.e. $\text{argmax}_q \phi(\cdot, q)$.

Fictitious play (FP) is thus given by the difference system

\begin{equation}
P^{n+1} - P^n = BR_1(Q^n/n), \quad Q^{n+1} - Q^n = BR_2(P^n/n).
\end{equation}

For simplicity, we consider instead Continuous Fictitious Play (cFP)

\begin{equation}
\dot{P}(t) = BR_1(Q(t)/t), \quad \dot{Q}(t) = BR_2(P(t)/t).
\end{equation}

for $t \geq 1$ (substituting $t$ for $n$ and $\dot{\cdot}$ for difference quotients).

**Duality.** Both the value $V$ of the game and $\phi(P(t)/t, Q(t)/t)$ lie between $\max_p \phi(p, Q(t)/t)$ and $\min_q \phi(P(t)/t, q)$. Thus, the duality gap

$$g := \max_p \phi(p, Q(t)/t) - \min_q \phi(P(t)/t, q)$$

converges to zero as $t \to \infty$ if and only if $\phi(P(t)/t, Q(t)/t) \to V$. 

Hamiltonian structure: Where \( BR_j(\cdot) \) are smooth, \( \mathcal{G} := \phi(BR(Q/t),Q) - \phi(P, BR(P/t)) = tg \) is stationary with respect to the BR variables, and otherwise decreasing. Thus, applying the chain rule, we find computing variations of \( \mathcal{G} \) that

\[
(C.3) \quad J\dot{U} \geq \nabla\mathcal{G}, \quad U = \begin{pmatrix} P \\ Q \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -A \\ A^T & 0 \end{pmatrix} \text{ skew},
\]

where the inequality \( \geq \) in the first equation is understood to hold in the sense of subdifferentials, i.e., with respect to perturbations in the arguments of \( \mathcal{G} \). That is, (cFP) has the form of a Generalized Hamiltonian system.

Convergence for 2-player games. Computing, we thus have \( \mathcal{G} \leq (J\dot{U}) \cdot \dot{U} = 0 \), and therefore \( \mathcal{G} \leq \) constant and \( g = \mathcal{G}/t \leq \) constant/\( t \to 0 \), establishing convergence of (cFP).

C.1. The \( n \)-player case. For the \( n \)-player zero-sum case, defining \( p_1(t), \ldots, p_n(t) \) to be the empirical probability distributions for players 1-\( n \), we point out that one may similarly define a duality gap

\[
\begin{align*}
g := & \max_z \phi_1(z, p_2(t), \ldots, p_n(t)) + \max_z \phi_2(p_1(t), z, p_3(t), \ldots, p_n(t)) + \ldots \\
& + \max_z \phi_n(p_1(t), \ldots, p_{n-1}, z),
\end{align*}
\]

where \( \phi_j \) denotes the payoff function for player \( j \), satisfying \( g \geq \sum_{j=1}^n \phi_j(p_1(t), \ldots, p_n(t)) = 0 \) with equality if and only if \( (p_1(t), \ldots, p_n(t)) \) is a Nash equilibrium. However, a computation for the \( n \)-player (cFP) analogous to (C.3) does not appear to yield anything useful, as there is no cancellation in general for this case.

Recall from the introduction, however, that for continuous 3-player (standard) Guts, Fictitious play does converge, to the unique symmetric Nash equilibrium \( (p_1^*, p_2^*, p_3^*) = (1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}) \). We plot the associated evolution of the gap \( g(n) \) in Figure 5(ii). For a discretization of 501 mesh points; convergence to Nash equilibrium \( p_* = 1/\sqrt{2} \) of the \( n \)th plays \( p_j(n) \) for players \( j = 1, 2, 3 \) (implying convergence of the associated empirical probability distributions to a delta-function centered at \( 1/\sqrt{2} \)) is illustrated in Figure 5(iii). Note the eventual monotone decrease of both the gap \( g(n) \) and the rescaled gap \( G(n) := ng(n) \), similarly as in the 2-player case.\(^1\) This may give some insight toward a proof of convergence in this case. Namely, though the straightforward calculation showing decrease of \( G \) in the 2-player case breaks down in the general 3-player case, the specifics of 3-player Guts, accounted more carefully, may yield the same result by a more circuitous route.

We see also that \( G \sim \) constant, suggesting a rate of decay \( g(n) \sim 1/n \) similarly as in the 2-player case. We conjecture, but have not made a corresponding study, that Fictitious play converges for \( m \)-player guts for all \( m \geq 2 \), with rate \( g(n) \sim 1/n \).

C.2. The Jacob game. Using the 3 player fictitious play algorithm, we also study numerically a simple 3-player symmetric game, referred to here as the Jacob Game. Each player chooses a number: 1 or 2. If two players choose the same number, and the remaining player chooses the other, the players choosing the same number each pay a value equal to their number to the player who chose the other number. We can define a player’s strategy \( s_n \) to be the proportion of the time they choose 1. We can then characterize any equilibrium as the triplet \( (s_1, s_2, s_3) \) of the 3 player’s strategies. The Jacob Game has 3 distinct Nash equilibria, of the form \( (0, 1, 1) \), \( (2/3, 2/3, 2/3) \) and \( (1/3, 1/3, 1/3) \). Since the Jacob Game is symmetric, any permutation of the strategies in any one equilibrium is itself an equilibrium.

This game is particularly interesting, as it features multiple equilibria in which one player wins while the others lose in a symmetric game, additionally the only pure and only non-weak equilibrium

\(^1\)Monotone decrease for (cFP) implies eventual monotone decrease for (FP), as may be seen by the effective step size of \( 1/(n+1) \) for (FP) considered as a discretization of (cFP).
C.3. Convergence to coalition strategies: the Jacob game II. One may ask whether or when $n$-player Fictitious Play might converge to a coalition-type strategy instead of a symmetric Nash equilibrium, in particular why it does not do so for 3-player Guts. As to the latter question, we note first that any limit point must be some Nash equilibrium, even if not a symmetric one. The winning bloc strategy for players 2-3 on the other hand is not a Nash equilibrium, since examination shows that one of players 2, 3 can always benefit by departing from the bloc strategy. Second, we note that limiting strategies must necessarily be of asynchronous type, meaning that all players independently choose randomly from their individual blended strategies. As the winning strategy found above is, rather, of synchronous type, meaning that player 2’s and player 3’s strategies are not chosen independently, it is disqualified for this second reason as well.

These considerations suggest the question whether there exists a symmetric 3-player game with an asymmetric Nash equilibrium consisting of a winning coalition strategy for two players vs. the third, and if so what is its behavior under Fictitious Play. An interesting example in this regard is a symmetric, zero-sum, 3-player game that we refer to as the Jacob game II. In this game, each player chooses from strategies $(1, 2, 3)$, with payoff schedule

$$
\begin{align*}
1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}; & \quad 2 : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -4 \\ 1 & -4 & 16 \end{pmatrix}; & \quad 3 : \begin{pmatrix} 0 & 1 & 0 \\ 1 & 8 & -8 \\ 0 & -8 & 0 \end{pmatrix}
\end{align*}
$$

displayed in the form $s_j : A_j$, where the payoff $\alpha(s_1, s_2, s_3)$ to player 1 for pure strategy choices $s_1, s_2, s_3 = 1, 2, 3$ is the $s_2$-$s_3$ entry of the matrix $A_j$. Payoffs to players 2 and 3 are then determined by symmetry, as is possible in a consistent way thanks to symmetry of the matrices $A_j$. One may check by hand the zero sum property $\alpha(s_1, s_2, s_3) + \alpha(s_2, s_1, s_3) + \alpha(s_3, s_1, s_2) = 0$ for symmetric games.

This game possesses a pure nonstrict symmetric Nash equilibrium in which each player chooses strategy 1, and also a blended nonstrict symmetric Nash equilibria in which each player chooses strategy $(s_1, s_2, s_3)$ with probabilities $(0, 2/3, 1/3)$. It possesses also a pure asymmetric “coalition-type” strong Nash equilibria in which each player $j$ chooses $s_j = j$, with player 1 losing $-2$ and players 2 and 3 each gaining $+1$, along with its various permutations: six in total. There may be other equilibria we have not found. In experiments using Fictitious Play with randomly generated initial strategy ensembles for players 1-3, we found that the process always converged to a coalition-type equilibrium of one type or the other, in striking contrast to the situation for Guts poker.
standard Fictitious play protocol of choosing best response randomly in case of a tie explains the absence of “intermediate behavior” of separatrix type.)

One may ask also whether there exist symmetric games with winning coalitions that are non-strong Nash equilibria, and what would be the result for Fictitious Play. The expanded modification (C.6)
\[
\begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & -2 & 2 \\
0 & 2 & 0 & 2 \\
2 & 2 & 2 & 0
\end{bmatrix}; \quad \begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 0 & -8 & 2 \\
1 & -8 & 32 & 2 \\
2 & 2 & 2 & -4
\end{bmatrix};
\]
\[\text{(C.6)}\]
\[
\begin{bmatrix}
0 & 1 & 0 & 2 \\
1 & 16 & -16 & 2 \\
0 & -16 & 0 & 2 \\
2 & 2 & 2 & 2
\end{bmatrix}; \quad \begin{bmatrix}
-4 & -4 & -4 & -4 & 0 \\
-4 & -4 & -4 & -4 & 2 \\
-4 & -4 & -4 & -4 & -1 \\
0 & 2 & -1 & 0
\end{bmatrix}
\]
of (C.5) features a winning coalition-type Nash equilibrium corresponding for players 1-3 to the one for (C.5); however, it is no longer a strong equilibrium, due to the possibility of “traitorous” play in which one partner in the 2-player coalition colludes with the remaining player to improve their joint returns. Experiment shows that Fictitious Play converges for this game also to this coalition-type equilibrium, or a symmetric permutation thereof.

The “mega”-version (C.7)
\[
\begin{bmatrix}
0 & 0 & 0 & 2 & 2 & 2 \\
0 & 0 & -2 & 2 & 2 & 2 \\
0 & -2 & 0 & 2 & 2 & 2 \\
2 & 2 & 2 & -4 & -4 & -4 \\
2 & 2 & 2 & -4 & -4 & -4 \\
2 & 2 & 2 & -4 & -4 & -4 \\
\end{bmatrix}; \quad \begin{bmatrix}
0 & 0 & 1 & 2 & 2 & 2 \\
0 & 0 & -8 & 2 & 2 & 2 \\
-8 & 32 & 2 & 2 & 2 \\
2 & 2 & 2 & -4 & -4 & -4 \\
2 & 2 & 2 & -4 & -4 & -4 \\
2 & 2 & 2 & -4 & -4 & -4 \\
\end{bmatrix};
\]
\[\text{(C.7)}\]
\[
\begin{bmatrix}
0 & 1 & 0 & 2 & 2 & 2 \\
1 & 16 & -16 & 2 & 2 & 2 \\
0 & -16 & 0 & 2 & 2 & 2 \\
2 & 2 & 2 & -4 & -4 & -4 \\
2 & 2 & 2 & -4 & -4 & -4 \\
2 & 2 & 2 & -4 & -4 & -4 \\
\end{bmatrix}; \quad \begin{bmatrix}
-4 & -4 & -4 & 2 & 2 & 2 \\
-4 & -4 & -4 & 2 & 2 & 2 \\
-4 & -4 & -4 & 2 & 2 & 2 \\
2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
\end{bmatrix};
\]
\[\text{of (C.6)}\], meanwhile, allows improvement not only through traitorous departure from coalition, but also through jointly favorable modification of the coalition strategies. Thus, though the described coalitions are optimal among synchronous strategies, they cannot be optimal with respect to “synchronous” coalition strategies where players strategies may (as for continuous guts) be linked. Running fictitious play for this game yields convergence to a coalition of either a player playing 2 and a player playing 3 against a player playing 1, or a player playing 5 and a player playing 6 against a player playing 4. This gives each member of the coalition +1, however if the players could play a synchronous strategy in which they synchronously decide to either play [2,3] or [5,6], they would on average get +1.5 each against the opposing player’s best response. Thus, it appears
that fictitious play can converge to an asynchronous coalition strategy, even when it is not optimal among the larger class of synchronous coalition strategies.

More generally, could there be examples in which approximate coalitions spontaneously form and dissolve? Or would there occur some more chaotic possibility? This question seems worthy of further investigation.

**Appendix D. Synchronous vs. Asynchronous Coalition: Two Examples**

We conclude with two examples demonstrating the possibility of a gap between the values forceable by synchronous and by asynchronous coalitions, as mentioned in the introduction.

D.1. **Odd man in.** Consider the following symmetric, zero-sum 3-player game. Each player chooses a value 1, 2, or 3. If all choices are the same, or all are different, there is no payoff. If two players choose a common number, however, and the third player a different one, then the first two each pay a value of 1 to the third, i.e., the first two receive payoff -1 and the third +2. Clearly, the strategy distribution \( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) for player 1 gives average return of +2/3 if the other two players play the same number, and -2/3 if they play different numbers. Thus, player 1 can force \( \geq -2/3 \). On the other hand, if players 2-3 choose with equal probability between pairs of choices \( (1, 2), (1, 3), \) and \( (2, 3) \), then the average payoff to player 1 is independent of player 1’s choice of strategy, and equal to \( (2/3) \times (-1) + (1/3) \times (0) = -2/3 \). Thus, players 2-3 can force a return of \( \leq -2/3 \) to player 1 by synchronous coalition play, and the value of the player 1 vs. players 2-3 game is -2/3.

As described earlier, this is the maximum value that player 1 can force against either synchronous or asynchronous play; however, the value forceable by synchronous play of players 2-3 may in principle be larger. In fact, it is larger, as we now show.

Let \( y := (y_1, y_2, y_2) \) and \( z := (z_1, z_2, z_3) \) denote probability distributions describing mixed strategies for players 2 and 3. Then, it is readily computed that the payoff to player 1 is

\[
\begin{align*}
\Psi_1(y, z) &:= 2y \cdot v - (y_1 + z_1) \\
\Psi_2(y, z) &:= 2y \cdot v - (y_2 + z_2) \\
\Psi_3(y, z) &:= 2y \cdot v - (y_3 + z_3)
\end{align*}
\]

The value \( \Psi(y, z) := \max_j \Psi_j(y, z) \) is thus the minimum value forceable by choice \( (y, z) \), and

\[
\nabla = \min_{y, z} \Psi(y, z)
\]

is the minimum value forceable by players 2-3 via asynchronous play, and by continuity of \( \Psi \) is achieved for some feasible pair of strategies \( (y_*, z_*) \).

Noting that the average of \( \Psi_j \) is

\[
2y \cdot z - (1/3) \sum_j (y_j + z_j) = 2y \cdot z - (2/3) \geq -2/3,
\]

we have that \( \Psi(y, z) \geq -2/3 \), with equality if and only if simultaneously \( y \cdot z = 0 \) and \( y_j + z_j = 2/3 \) for all \( j \). But, these together imply that one of each pair \( y_j, z_j \) has value zero and the other value \( 2/3 \), which is impossible to reconcile with \( \sum_j y_j = \sum_j z_j = 1 \). Thus, evaluating at \( (y, z) = (y_*, z_*) \), we obtain \( \nabla = \Psi(y_*, z_*) > -2/3 \), verifying that there is indeed a gap between this value and the value -2/3 forceable by synchronous coalition play. Indeed, the optimum asynchronous strategy can be shown to be \( y_* = (1, 0, 0), z_* = (0, 1/2, 1/2) \), forcing an expected payoff to player 1 of \( \leq -1/2 \): thus, a gap of \( -1/2 - (-2/3) = 1/6 \).

To complete the picture, recall from [CCZ, Prop. A.2] that for a symmetric zero-sum game, the symmetric Nash equilibrium is equal to the optimum strategy for player 1 in the modified “bloc
strategy” game of player 1 vs. players 2-3, where the latter are required to play the same strategy, in this case the game with payoff

\[(D.2) \quad \Psi(y, y) = \min_j 2(|y|^2 - y_j).\]

By Jensen’s inequality, \(|y|^2/3 \geq [(\sum_j y_j)/3]^2 = 1/9\), or \(|y|^2 \geq 1/3\), while \(\min_j y_j \leq (\sum_j y_j)/3\), both with equality if and only if \(y_j \equiv 1/3\). Thus, \(\Psi(y, y) \geq 0\) with equality only if \(y = (1/3, 1/3, 1/3)\), and so \(x, y, z = (1/3, 1/3, 1/3)\) is the unique strict symmetric Nash equilibrium, returning value 0, as it must for a symmetric zero-sum game.

D.2. Odd man out. Next, consider the same game, but with payoff function multiplied by \(-1\): that is, the “reverse” game, in which the odd player is penalized instead of rewarded. Again, the symmetric Nash equilibrium is \(x = y = z = (1/3, 1/3, 1/3)\), returning payoff zero to all players. An optimum synchronized strategy is a blend of pure strategy pairs 1-1, 2-2, 3-3, each chosen with probability 1/3, yielding again value \((1/3)(-2) = -2/3\).

However, the optimum value forceable by asynchronous coalition by players 2-3 is now \(\tilde{V} = \min_{y,z} \tilde{\Psi}(y, z)\), where

\[\tilde{\Psi}(y, z) := \min -\psi_j(y, z) = -2y \cdot z + \max_j (y_j + z_j),\]

and this, by Cauchy-Schwarz and Jensen’s inequalities \(y \cdot z \leq |y||z|\) and \(|y| \leq (\sum_j y_j)/\sqrt{3} = 1/\sqrt{3}\), together with the fact that maximum is greater than or equal to average, is greater than or equal to \(-2/3 + 2/3 = 0\), with equality if and only if \(y = z = (1/3, 1/3, 1/3)\), the Nash equilibrium values. Thus, in the reverse game as in the forward one, player one can force at most \(-2/3\), the value forceable by players 2-3 by synchronized coalition play; hence, there is again a gap. However, players 2-3 can force at most 0 by asynchronous coalition play; that is, they cannot guarantee by such strategy a joint profit for themselves.

D.3. Discussion. We note, as described in the introduction, that a similar situation appears to hold for (standard) 3-player continuous guts. That is, the optimum synchronous coalition strategy appears to be strict up to symmetry and genuinely mixed-type in the sense that it has no pure-strategy representative, so that the optimum asynchronous strategy must return a strictly larger maximum expected value to player 1 and there is a gap between values forceable by players 2-3 via synchronous vs. asynchronous coalition play. A very interesting open question is whether for Guts, as in the forward example, there is also a gap between the value forceable by asynchronous coalition and the symmetric Nash equilibrium return, so that players 2-3 can guarantee by asynchronous play a joint profit to themselves, or whether as in the reverse example they can guarantee at most a zero outcome by asynchronous play. We conjecture that for Guts, the latter holds; to determine whether or when this is true for general games is an extremely interesting problem worthy of further investigation.

For games with a gap between values forceable by synchronous vs. asynchronous coalition values, as seem from the above discussion to be somewhat typical, it is a further very interesting philosophical question what are the implications for player 1. For, following the classical worst-case scenario analysis, player 1 cannot guarantee better than the value \(V\) against synchronous play. However, suppose that players 2 are prevented from communicating or even knowing which round is being played, so that their play is truly asynchronous. Then, they cannot in their worst-case scenario force less than the value \(\tilde{V} > V\) obtainable by asynchronous play. But, is there any way for player 1 to take advantage of this gap, other than negotiation outside the game? This seems to be a subtle side-issue for symmetric zero sum games, apart from the standard one of determining value/optimal play when there exist successful asynchronous coalition strategies.
REFERENCES

[B] G.W. Brown, *Iterative Solutions of Games by Fictitious Play*. In Activity Analysis of Production and Allocation, T. C. Koopmans (Ed.), New York: Wiley (1951).

[CCZ] L. Castronova, Y. Chen, and K. Zumbrun, *Game-theoretic analysis of Guts Poker*, Preprint; arXiv:2108.06556

[D] G.B. Dantzig, *A proof of the equivalence of the programming problem and the game problem*. In: Koopmans TC (ed) Activity analysis of production and allocation. Wiley, New York (1951) pp 330–335.

[E] H. Everett, *Recursive games*, Contributions to the theory of games, vol. 3, pp. 47–78. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957.

[G] Github depository: https://github.com/pangyjay/Guts.

[N] J.F. Nash, *Non-Cooperative Games*, Annals of Mathematics Second Series, Vol. 54, No. 2 (Sep., 1951), pp. 286–295.

[NPy] Nashpy documentation, https://nashpy.readthedocs.io/

[O] G. Owen, *Game theory*, Third edition. Academic Press, Inc., San Diego, CA, 1995. xii+447 pp. ISBN: 0-12-531151-6.

[R] J. Robinson, *An Iterative Method of Solving a Game*, Annals of Mathematics 54 (1959) 296–301.

[S] M. Shackleford, *Guts poker*, (internet article); https://wizardofodds.com/games/guts-poker/

[Sh1] L.S. Shapley, *Stochastic games*, PNAS 39 (1953) no. 10, 1095–1100.

[Sh2] L.S. Shapley, *Nonconvergence of fictitious play*, RAND memo RM-3026-PS (1962).

[Sh3] L.S. Shapley, *Some Topics in Two-Person Games*. In Advances in Game Theory M. Dresher, L.S. Shapley, and A.W. Tucker (Eds.), Princeton: Princeton University Press (1968).

[vN] J. Von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. 100 (1928) 295–320.

[vNM] J. Von Neumann and O. Morganstern, *Theory of Games and Economic Behavior*, Princeton and Woodstock: Princeton University Press (1944).

[W1] Wikipedia, https://en.wikipedia.org/wiki/Guts_(card_game).

[W2] Wikipedia, https://en.wikipedia.org/wiki/Strong_Nash_equilibrium.

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