ON THE CLASSIFICATION OF QUANDLES OF LOW ORDER

L. VENDRAMIN

Abstract. Using the classification of transitive groups we classify indecomposable quandles of size < 36. This classification is available in Rig, a GAP package for computations related to racks and quandles. As an application, the list of all indecomposable quandles of size < 36 not of type D is computed.

1. Introduction

Racks appeared for the first time in [11] and quandles appeared in [18] and [21]. Racks and quandles are used in modern knot theory because they provide good knot invariants, [18]. They are also useful for the classification problem of pointed Hopf algebras because they provide a powerful tool to understand Yetter-Drinfeld modules over groups, see [5]. Of course, the classification of finite racks (or quandles) is a very difficult problem. Several papers about classifications of different subcategories of racks have appear, see for example [15], [17], [16], [5], [15], [8].

In this paper we use the classification of transitive groups and the program described in [9] to classify indecomposable quandles. With this method, we complete the classification of all non-isomorphic indecomposable quandles of size < 36. This classification is available in Rig, a GAP [1] package designed for computations related to racks and quandles. Rig is a free software and it is available at http://code.google.com/p/rig/.

2. Definitions and examples

We recall basic notions and facts about racks. For additional information we refer for example to [5]. A rack is a pair \((X, \triangleright)\), where \(X\) is a non-empty set and \(\triangleright : X \times X \to X\) is a map (considered as a binary operation on \(X\)) such that

\begin{enumerate}
  \item the map \(\varphi_i : X \to X\), where \(x \mapsto i \triangleright x\), is bijective for all \(i \in X\), and
  \item \(i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)\) for all \(i, j, k \in X\).
\end{enumerate}

A rack \((X, \triangleright)\), or shortly \(X\), is a \textit{quandle} if \(i \triangleright i = i\) for all \(i \in X\). A subrack of a rack \(X\) is a non-empty subset \(Y \subseteq X\) such that \((Y, \triangleright)\) is also a rack.

Example 2.1. A group \(G\) is a quandle with \(x \triangleright y = x y x^{-1}\) for all \(x, y \in G\). If a subset \(X \subseteq G\) is stable under conjugation by \(G\), then it is a subquandle of \(G\).

To construct racks associated to (union of) conjugacy classes of groups use the Rig function \texttt{Rack}. For example, to construct the quandle of three elements associated to the conjugacy class of transpositions in \(S_3\):

\begin{verbatim}
gap> r := Rack(SymmetricGroup(3), (1,2));;
gap> Size(r);
3
\end{verbatim}
Example 2.2. Let $G$ be a group and $s \in \text{Aut}(G)$. Define $x \triangleright y = xs(x^{-1}y)$ for $x, y \in G$. Then $(G, \triangleright)$ is a quandle. Further, let $H \subseteq G$ be a subgroup such that $s(h) = h$ for all $h \in H$. Then $G/H$ is a quandle with $xH \triangleright yH = xs(x^{-1}y)H$. It is called the homogeneous quandle $(G, H, s)$.

Example 2.3. Let $n \geq 2$. The dihedral quandle of order $n$ is $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with $i \triangleright j = 2i - j \pmod{n}$.

The package provides several functions to construct racks and quandles. See the documentation for more information.

Let $X$ be a finite rack. Assume that $X = \{x_1, x_2, \ldots, x_n\}$. With the identification $x_i \equiv i$ the rack $X$ can be presented as a square matrix $M \in \mathbb{N}^{n \times n}$ such that $M_{ij} = i \triangleright j$. This matrix is called the table of the rack. See [10].

Example 2.4. The matrix (or table) of the rack $\mathbb{D}_4$ is
\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4 \\
\end{pmatrix} = \begin{pmatrix}
1 & 4 & 3 & 2 \\
3 & 2 & 1 & 4 \\
1 & 4 & 3 & 2 \\
3 & 2 & 1 & 4 \\
\end{pmatrix}.
\]

The files of the matrix are the permutations of the quandle: $\varphi_1 = \varphi_3 = (2 4)$ and $\varphi_2 = \varphi_4 = (1 3)$.

\texttt{gap> D4 := DihedralQuandle(4);}
\texttt{gap> Permutations(D4);}
\texttt{[ [ 1, 4, 3, 2 ], [ 3, 2, 1, 4 ], [ 1, 4, 3, 2 ], [ 3, 2, 1, 4 ] ]}

Let $(X, \triangleright)$ and $(Y, \triangleright)$ be racks. A map $f : X \to Y$ is a morphism of racks if $f(i \triangleright j) = f(i) \triangleright f(j)$ for all $i, j \in X$.

Notation 2.5. We write $g^G$ for the conjugacy class of $g$ in $G$.

Example 2.6. Let $T_1 = (1 2 3)^{h_1}$ and $T_2 = (1 3 2)^{h_2}$. Then the quandles $T_1$ and $T_2$ are isomorphic.

\texttt{gap> T1 := Rack(AlternatingGroup(4), (1,2,3));}
\texttt{gap> T2 := Rack(AlternatingGroup(4), (1,3,2));}
\texttt{gap> IsomorphismRacks(T1, T2);}
\texttt{(3,4)}

Hence $T_1 \simeq T_2$ and the isomorphism is given by the permutation $\sigma = (3,4)$. More precisely, assume that $T_1 = \{x_1, x_2, x_3, x_4\}$ and $T_2 = \{y_1, y_2, y_3, y_4\}$. Then the map $f : T_1 \to T_2, f(x_i) = y_{\sigma(i)}$, is an isomorphism of racks.

Example 2.7. Let $A$ be an abelian group, and let $T \in \text{Aut}(A)$. We have a quandle structure on $A$ given by
\[ a \triangleright b = (1 - T)a + Tb \]
for $a, b \in A$. The quandle $(A, \triangleright)$ is called affine (or Alexander) quandle and it will be denoted by $\text{Aff}(A, T)$. In particular, let $p$ be a prime number, $q$ a power of $p$ and $\alpha \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. We write $\text{Aff}(\mathbb{F}_q, \alpha)$, or simply $\text{Aff}(q, \alpha)$, for the affine quandle $\text{Aff}(A, g)$, where $A = \mathbb{F}_q$ and $g$ is the automorphism given by $x \mapsto \alpha x$ for all $x \in \mathbb{F}_q$. 

Example 2.8. The tetrahedron quandle is the quandle $T = (1 2 3)^A_4$. It is easy to see that this quandle is isomorphic to an affine quandle over $\mathbb{F}_4$.

The inner group of a rack $X$ is the group generated by the permutations $\varphi_i$ of $X$, where $i \in X$. We write $\text{Inn}(X)$ for the inner group of $X$. A rack is said to be faithful if the map

$$\varphi : X \to \text{Inn}(X), \quad i \mapsto \varphi_i,$$

is injective. We say that a rack $X$ is indecomposable (or connected) if the inner group $\text{Inn}(X)$ acts transitively on $X$. Also, $X$ is decomposable if it is not indecomposable. Any finite rack $X$ is the disjoint union of indecomposable subracks [5, Prop. 1.17] called the components of $X$.

Example 2.9. The dihedral quandle $D_4$ is decomposable: $D_4 = \{1, 3\} \sqcup \{2, 4\}$.

```gap
gap> D4 := DihedralQuandle(4);;
gap> IsIndecomposable(D4);
false
gap> Components(D4);
[ [ 1, 3 ], [ 2, 4 ] ]
```

For any rack $X$, the enveloping group of $X$ is

$$G_X = F(X)/(ijj^{-1} = i \triangleright j, \ i, j \in X),$$

where $F(X)$ denotes the free group generated by $X$. This group is also called the associated group of $X$, see [11]. Let

$$\overline{G_X} = G_X/\langle \text{ord}(\varphi_x) \mid x \in X \rangle.$$

If $X$ is finite then the group $\overline{G_X}$ is finite and it is called the finite enveloping group of $X$, see [14].

Example 2.10. Let $X = T$ be the tetrahedron rack. Then $\text{Inn}(X) \simeq A_4$ and $\overline{G_X} \simeq \text{SL}(2,3)$.

```gap
gap> T := Rack(AlternatingGroup(4), (1,2,3));;
gap> inn := InnerGroup(T);
gap> StructureDescription(inn);
A4
gap> env := FiniteEnvelopingGroup(T);
gap> StructureDescription(env);
SL(2,3)
```

Table 1 contains the inner group and the finite enveloping groups associated to some particular racks. These racks appear in the classification of finite-dimensional Nichols algebras, see for example [2 Table 6].

| Quandle | $\text{Inn}(Q)$ | $\overline{G_X}$ |
|---------|----------------|-----------------|
| $D_3$   | $S_3$          | $S_3$           |
| $T$     | $A_4$          | $\text{SL}(2,3)$ |
| Aff(5,2), Aff(5,3) | $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ | $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ |
| $(1 2)^{A_4}$ | $S_4$       | $S_4$           |
| Aff(7,3), Aff(7,5) | $(\mathbb{Z}_7 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ | $(\mathbb{Z}_7 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ |
| $(1 2 3 4)^{A_4}$ | $S_4$       | $\text{SL}(2,3) \rtimes \mathbb{Z}_4$ |
| $(1 2)^{A_5}$ | $S_5$       | $S_5$           |
3. The classification of indecomposable quandles of low order

The main tool for the classification of indecomposable quandles is the following theorem of [9]. Our proof is heavily based on [18, Theorem 7.1]. For completeness we give a proof in the context of this paper.

**Theorem 3.1.** Let $X$ be an indecomposable quandle of $n$ elements. Let $x_0 \in X$, $z = \phi_{x_0}$, $G = \text{Inn}(X)$ and $H = \text{Stab}_G(x_0) = \{ g \in G \mid g \cdot x_0 = x_0 \}$. Then

1. $G$ is a transitive group of degree $n$,
2. $z$ is a central element of $H$,
3. $X$ is isomorphic to the homogeneous quandle $(G, H, I_z)$, where $I_z : G \to G$ is the conjugation $x \mapsto zxz^{-1}$.

**Proof.** The claim (1) follows by definition. The claim (2) follows from [9, Theorem 4.3]. We now prove (3). We consider the quandle structure over $G$ given by $x \bowtie y = xI_z(x^{-1}y)$ for all $x, y \in G$, and let $e : G \to X$, $x \mapsto x \cdot x_0$, be the evaluation map. Since $G$ acts transitively on $X$, the map $e$ is surjective. We claim that $e$ is a rack morphism. Indeed,

$$e(x \bowtie y) = e(xs(x^{-1}y)) = e(xzx^{-1}yz^{-1} \cdot x_0) = xzx^{-1}y \cdot x_0 = x \cdot (x_0 \bowtie (x^{-1}y \cdot x_0)) = e(x) \bowtie e(y)$$

for all $x, y \in G$. Further, $e(x) = e(y)$ if and only if $xH = yH$. Then $e$ induces the isomorphism $G/H \to X$, $xH \mapsto e(x)$. Hence the claim follows. □

**Algorithm 1:** Indecomposable quandles of size $n$

Result: The list $L$ of all non-isomorphic indecomposable quandles

$L \leftarrow \emptyset$;

for all transitive groups $G$ of degree $n$ do

Compute $H = \text{Stab}_G(x_0)$;

Compute $Z(H)$, the center of $H$;

for $z \in Z(H) \setminus \{1\}$ do

Compute the homogeneous quandle $Q = (G, H, I_z)$;

if $Q$ is indecomposable and $Q \not\approx X$ for all $X \in L$ then

Add the quandle $Q$ to $L$;

end

end

end

Recall that all indecomposable quandles of prime order $p$ are affine, see [10]. Let $n \in \mathbb{N}$, $n < 36$, and $n$ not being a prime number. Using Theorem 3.1 and Algorithm 1, the list of all non-isomorphic indecomposable quandles can be constructed. The only requirement is the classification of transitive groups. The complete list of transitive groups up to degree $< 32$ is included in GAP. Hulpke classified several of these transitive groups, see [17]. Further, Hulpke classified transitive groups of degree $33$, $34$ and $35$. Transitive groups of degree $32$ were classified in [6].

For $n \in \mathbb{N}$ let $q(n)$ be the number of non-isomorphic indecomposable quandles of size $n$. In Example 3.2 above, $q(20)$ is computed. Further, Table 2 shows the value of $q(n)$ for $n \in \{1, 2, \ldots, 35\}$.

**Example 3.2.** There are 10 isomorphism classes of indecomposable quandles of order 20.

```
gap> NrSmallQuandles(20);
10
```
Rig contains a huge database with the set of representatives of isomorphism classes of indecomposable quandles of size $< 36$. Let $n \in \{1, 2, \ldots, 35\}$ such that $q(n) \neq 0$, and let $Q_{n, 1}, Q_{n, 2}, \ldots, Q_{n, q(n)}$ be the set of representatives of isomorphism classes of indecomposable quandles of size $n$. In the package, a representative $Q_{n, i}$, $1 \leq i \leq q(n)$, can be obtained with the function $\text{SmallQuandle}$.

**Example 3.3.** There exists only one (up to isomorphism) indecomposable quandle of order 10. Further, this quandle is isomorphic to the conjugacy class of transpositions in $S_5$.

```gap
gap> NrSmallQuandles(10); 1
gap> Q := SmallQuandle(10, 1);
gap> R := Rack(SymmetricGroup(5), (1,2));
gap> IsomorphismRacks(Q, R);
(3,5,6,10,8,4,9,7)
```

Recall that a crossed set is a quandle $(X, \triangleright)$ which further satisfies $j \triangleright i = j$ whenever $i \triangleright j = i$ for all $i, j \in X$.

**Example 3.4.** It is easy to see that the only indecomposable quandles of size $< 36$ which are not crossed sets are $Q_{30, 4}$ and $Q_{30, 5}$.

**Table 2.** The number of non-isomorphic indecomposable quandles

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|
| $q(n)$ | 1 | 0 | 1 | 1 | 3 | 2 | 5 | 3 | 8 | 1 | 9 | 10 |

| $n$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| $q(n)$ | 11 | 0 | 7 | 9 | 15 | 12 | 17 | 10 | 9 | 0 | 21 | 42 |

| $n$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
|-----|----|----|----|----|----|----|----|----|----|----|----|
| $q(n)$ | 34 | 0 | 65 | 13 | 27 | 24 | 29 | 17 | 11 | 0 | 15 |

**Conjecture 3.5.** Let $p$ be an odd prime number and let $Q$ be an indecomposable quandle of $2p$ elements. Then $p \in \{3, 5\}$.

### 4. Rack homology

Let $X$ be a rack. For $n \geq 0$ let $C_n(X, \mathbb{Z}) = \mathbb{Z}X^n$. Consider $C_*(X, \mathbb{Z})$ as a complex with boundary $\partial_0 = \partial_1 = 0$ and $\partial_{n+1} : C_{n+1}(X, \mathbb{Z}) \to C_n(X, \mathbb{Z})$ defined by

$$
\partial_{n+1}(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}]
$$

for $n \geq 1$. It is straightforward to prove that $\partial^2 = 0$. The homology $H_*(X, \mathbb{Z})$ of $X$ is the homology of the complex $C_*(X, \mathbb{Z})$. See for example [7], [12], [13] for applications to the theory of knots and [5] for applications to the theory of Hopf algebras.

**Example 4.1.** Let $X = D_5$. Then $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$.

```gap
gap> RackHomology(DihedralQuandle(5), 2);
[ 1, [ ] ]
```
Example 4.2. Let $X = (12)(345)^{\overline{5}}$. Then $H_2(X, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_6$.

```gap
gap> r := Rack(SymmetricGroup(5), (1,2)(3,4,5));;
gap> RackHomology(r, 2);
[ 1, [ 6 ] ]
```

Example 4.3. Recall that $T$ is the tetrahedron quandle defined in Example 2.8. Then $H_2(T, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2$ and $H_3(T, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Further, the torsion subgroup of $H_2(T, \mathbb{Z})$ is generated by

$$
\chi = \chi(1,2) + \chi(1,3) + \chi(2,1) + \chi(2,3) + \chi(3,1) + \chi(3,2),
$$

where

$$
\chi_{(i,j)}(a,b) = \begin{cases} 
1 & \text{if } (i,j) = (a,b), \\
0 & \text{otherwise.}
\end{cases}
$$

Indeed,

```gap
gap> T := Rack(AlternatingGroup(3), (1,2,3));;
gap> RackHomology(T, 2);
[ 1, [ 2 ] ]
gap> RackHomology(T, 3);
[ 1, [ 2, 2, 4 ] ]
gap> TorsionGenerators(T, 2);
[ [ 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0 ] ]
```

Table 3 contains the second rack homology group of all the indecomposable quandles of size $\leq 21$. Quandles with a prime number of elements were not included in Table 3 because of the following lemma of [15].

Lemma 4.4. Let $p$ be a prime number. Let $X$ be an indecomposable quandle of $p$ elements. Then $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$.

Proof. It follows from [15, Lemma 5.1] and [20, Theorem 2.2].

5. RACKS OF TYPE D

Recall from [3] that a finite rack $X$ is of type D if there exists an indecomposable subrack $Y = R \sqcup S$ (here $R$ and $S$ are the components of $Y$) such that

$$
r \triangleright (s \triangleright (r \triangleright s)) \neq s
$$

for some $r \in R$ and $s \in S$.

Quandles of type D are very important for the classification of finite-dimensional pointed Hopf algebras, see for example the program described in [2, §2.6]. For some interesting applications we refer to [3, 4].

Proposition 5.1. Let $Q$ be an indecomposable quandle of size $< 36$. Then $Q$ is of type D if and only if $Q$ is isomorphic to one of the following quandles:

1. $Q_{12,1}$,
2. $Q_{18,i}$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
3. $Q_{20,3}$,
4. $Q_{24,i}$ for $i \in \{1, 2, 3, 4, 5, 6, 8, 10, 11, 16, 17, 21, 22, 23, 26, 27, 28, 32\}$,
5. $Q_{27,i}$ for $i \in \{1, 14\}$,
6. $Q_{30,i}$ for $i \in \{1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15, 16\}$,
7. $Q_{32,i}$ for $i \in \{1, 2, 3, 5, 6, 7, 8, 9\}$.
Indecomposable quandle $Q$ & $H_3(Q, \mathbb{Z})$ & \\
$Q_{4.1}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{6.1}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{6.2}$ & $\mathbb{Z} \times \mathbb{Z}_4$ & \\
$Q_{8.1}, Q_{8.2}, Q_{8.3}$ & $\mathbb{Z}$ & \\
$Q_{9.4}, Q_{9.5}, Q_{9.7}, Q_{9.8}$ & $\mathbb{Z} \times \mathbb{Z}_3$ & \\
$Q_{9.2}, Q_{9.3}, Q_{9.6}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{10.1}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{12.1}, Q_{12.2}, Q_{12.4}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{12.3}$ & $\mathbb{Z} \times \mathbb{Z}_{10}$ & \\
$Q_{12.1}, Q_{12.2}, Q_{12.4}$ & $\mathbb{Z} \times \mathbb{Z}_4$ & \\
$Q_{12.7}$ & $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$ & \\
$Q_{12.8}$ & $\mathbb{Z} \times \mathbb{Z}_2^2$ & \\
$Q_{12.9}$ & $\mathbb{Z} \times \mathbb{Z}_4^2$ & \\
$Q_{12.10}$ & $\mathbb{Z} \times \mathbb{Z}_6$ & \\
$Q_{15.1}, Q_{15.3}, Q_{15.4}$ & $\mathbb{Z}$ & \\
$Q_{15.2}$ & $\mathbb{Z} \times \mathbb{Z}_2^2$ & \\
$Q_{15.5}, Q_{15.6}$ & $\mathbb{Z} \times \mathbb{Z}_5$ & \\
$Q_{15.7}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{16.1}, Q_{16.7}$ & $\mathbb{Z} \times \mathbb{Z}_4$ & \\
$Q_{16.2}$ & $\mathbb{Z} \times \mathbb{Z}_2^2$ & \\
$Q_{16.3}, Q_{16.4}$ & $\mathbb{Z} \times \mathbb{Z}_4^2$ & \\
$Q_{16.5}, Q_{16.6}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{16.8}, Q_{16.9}$ & $\mathbb{Z}$ & \\
$Q_{18.1}, Q_{18.8}, Q_{18.11}, Q_{18.12}$ & $\mathbb{Z} \times \mathbb{Z}_6$ & \\
$Q_{18.2}, Q_{18.9}, Q_{18.10}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{18.3}, Q_{18.6}, Q_{18.7}$ & $\mathbb{Z} \times \mathbb{Z}_4$ & \\
$Q_{18.4}, Q_{18.5}$ & $\mathbb{Z} \times \mathbb{Z}_{12}$ & \\
$Q_{20.1}, Q_{20.2}, Q_{20.3}$ & $\mathbb{Z} \times \mathbb{Z}_6$ & \\
$Q_{20.4}, Q_{20.7}, Q_{20.8}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\
$Q_{20.5}, Q_{20.9}$ & $\mathbb{Z} \times \mathbb{Z}_2^2$ & \\
$Q_{20.6}$ & $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$ & \\
$Q_{20.10}$ & $\mathbb{Z} \times \mathbb{Z}_4$ & \\
$Q_{21.1}, Q_{21.2}, Q_{21.3}, Q_{21.4}, Q_{21.5}$ & $\mathbb{Z}$ & \\
$Q_{21.6}$ & $\mathbb{Z} \times \mathbb{Z}_2^3$ & \\
$Q_{21.7}, Q_{21.8}$ & $\mathbb{Z} \times \mathbb{Z}_7$ & \\
$Q_{21.9}$ & $\mathbb{Z} \times \mathbb{Z}_2$ & \\

**Proof.** By [10], indecomposable quandles of size $p$ are affine. Further, [2] Prop. 4.2 implies that affine quandles with $p$ elements are not of type D. Therefore we may assume that the size of $Q$ is not a prime number. Now the claim follows from a straightforward computer calculation. 

**Corollary 5.2.** Let $Q$ be an indecomposable simple quandle of size $< 36$. Assume that $Q$ is of type D. Then $Q \simeq Q_{30.3}$. 

**Acknowledgement.** I am grateful to M. Graña for writing several functions for the package. I would like to thank N. Andruskiewitsch, E. Clark, F. Fantino, M. Farinati, J. A. Guccione, J. J Guccione, I. Heckenberger and A. Lochmann for several conversations related to racks and quandles. I also thank J. A. Hulpke for the list.
of transitive groups of degree 33, 34 and 35 and D. Holt for the list of transitive
groups of degree 32.

References

[1] The GAP Group, 2006. GAP – Groups, Algorithms, and Programming, Version 4.4.12. Available at http://www.gap-system.org.
[2] N. Andruskiewitsch, F. Fantino, G. A. Garcia, and L. Vendramin. On Nichols algebras associated to simple racks. Contemp. Math. 537 (2011) 31-56.
[3] N. Andruskiewitsch, F. Fantino, M. Graña, and L. Vendramin. Finite-dimensional pointed Hopf algebras with alternating groups are trivial. Ann. Mat. Pura Appl. (4) 190 (2011), no. 2, 225-245.
[4] N. Andruskiewitsch, F. Fantino, M. Graña, and L. Vendramin. Pointed Hopf algebras over the sporadic simple groups. J. Algebra, 325:305-320, 2011.
[5] N. Andruskiewitsch and M. Graña. From racks to pointed Hopf algebras. Adv. Math., 178(2):177–243, 2003.
[6] J. J. Cannon and D. F. Holt. The transitive permutation groups of degree 32. Experiment. Math., 17(3):307–314, 2008.
[7] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces. Trans. Amer. Math. Soc., 355(10):3947–3989, 2003.
[8] F. J. B. J. Clauwens. Small connected quandles. Preprint: arXiv:1011.2456.
[9] G. Ehrman, A. Gurpinar, M. Thibault, and D. N. Yetter. Toward a classification of finite quandles. J. Knot Theory Ramifications, 17(4):511–520, 2008.
[10] P. Etingof, A. Soloviev, and R. Guralnick. Indecomposable set-theoretical solutions to the quantum Yang-Baxter equation on a set with a prime number of elements. J. Algebra, 242(2):709–719, 2001.
[11] R. Fenn and C. Rourke. Racks and links in codimension two. J. Knot Theory Ramifications, 1(4):343–406, 1992.
[12] R. Fenn, C. Rourke, and B. Sanderson. James bundles. Proc. London Math. Soc. (3), 89(1):217–240, 2004.
[13] R. Fenn, C. Rourke, and B. Sanderson. The rack space. Trans. Amer. Math. Soc., 359(2):701–740 (electronic), 2007.
[14] M. Graña, I. Heckenberger, and L. Vendramin. Nichols algebras of group type with many quadratic relations. Adv. Math., 227(5):1956–1989, 2011.
[15] M. Graña. Indecomposable racks of order $p^2$. Beiträge Algebra Geom., 45(2):665–676, 2004.
[16] B. Ho and S. Nelson. Matrices and finite quandles. Homology Homotopy Appl., 7(1):197–208, 2005.
[17] A. Hulpke. Constructing transitive permutation groups. J. Symbolic Comput., 39(1):1–30, 2005.
[18] D. Joyce. A classifying invariant of knots, the knot quandle. J. Pure Appl. Algebra, 23(1):37–65, 1982.
[19] D. Joyce. Simple quandles. J. Algebra, 79(2):307–318, 1982.
[20] R. A. Litherland and S. Nelson. The Betti numbers of some finite racks. J. Pure Appl. Algebra, 178(2):187–202, 2003.
[21] S. V. Matveev. Distributive groupoids in knot theory. Mat. Sb. (N.S.), 119(161)(1):78–88, 160, 1982.