GLOBAL REGULARITY FOR THE MONGE-AMPÈRE EQUATION WITH NATURAL BOUNDARY CONDITION

SHIBING CHEN, JIAKUN LIU, AND XU-JIA WANG

Abstract. In this paper, we establish the global $C^{2,\alpha}$ and $W^{2,p}$ regularity for the Monge-Ampère equation $\det D^2u = f$ subject to boundary condition $Du(\Omega) = \Omega^*$, where $\Omega$ and $\Omega^*$ are bounded convex domains in the Euclidean space $\mathbb{R}^n$ with $C^{1,1}$ boundaries, and $f$ is a Hölder continuous function. This boundary value problem arises naturally in optimal transportation and many other applications.

1. Introduction

In this paper we establish the global $C^{2,\alpha}$ and $W^{2,p}$ regularity for the Monge-Ampère equation

$$\det D^2u(x) = f(x)$$

subject to the boundary condition

$$Du(\Omega) = \Omega^*,$$

where $\Omega, \Omega^*$ are bounded convex domains in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and $f$ is a positive function. We also assume that $f \in C^0(\Omega)$ for the global $W^{2,p}$ estimate ($p \geq 1$); and $f \in C^\alpha(\Omega)$ for the global $C^{2,\alpha}$ estimate ($\alpha \in (0,1)$).

The boundary value problem (1.1) and (1.2) arises naturally in optimal transportation with the quadratic cost function. It is a fundamental problem in the area and received much attention due to its wide range of applications, such as in fluid mechanics, meteorology, image recognition, reflector design, and also in geometry and probability [14, 16, 26, 33, 34]. In particular it was recently found that the problem (1.1) and (1.2) plays a fundamental role in Wasserstein generative adversarial networks, a fast growing technique in machine learning [22].

The existence and uniqueness of solutions to the problem (1.1) and (1.2) were obtained by Brenier in his pioneering work [3]. Since then the regularity of solutions has been a focus of attention in this area [33, 34], and has been studied in [5, 6, 7, 15, 32]. When $\Omega$ and $\Omega^*$ are bounded convex domains, and $f \geq 0$ satisfies the doubling condition, the global $C^{1,\alpha}$ regularity for the solution was obtained by Caffarelli [5]. In a landmark paper [7], Caffarelli established the global $C^{2,\alpha'}$ regularity for the problem (1.1) and (1.2), assuming that $\Omega$ and $\Omega^*$ are uniformly

Date: March 15, 2022.
2000 Mathematics Subject Classification. 35J96, 35J25, 35B65.
Key words and phrases. Monge-Ampère equation, global regularity.
This work was supported by ARC FL130100118 and ARC DP170100929.
convex with $C^2$ boundary, $f \in C^\alpha(\bar{\Omega})$ and $f > 0$. When $\Omega$ and $\Omega^*$ are uniformly convex and $C^{3,1}$ smooth, and $f \in C^{1,1}(\bar{\Omega})$, the global smooth solution was first obtained by Delanoë [15] in 1991 for dimension two and later extended to high dimensions by Urbas [32]. The results of Caffarelli, Delanoë, and Urbas were used by Brendle and Warren [1, 2] to study the minimal Lagrangian graphs. These results may also be applied to the problem of convex hypersurfaces with prescribed spherical map [25].

The uniform convexity of domains is a natural condition for the regularity of solutions to boundary value problems of the Monge-Ampère equation. In fact, the uniform convexity is a necessary condition for the global regularity of solutions to the Dirichlet problem [9, 27, 31]. It was used extensively and played a critical role in the proof for both the Dirichlet problem and the second boundary value problem (1.1), (1.2) in the above mentioned papers [7, 9, 15, 27, 31, 32], and also in the paper on the Neumann problem [23].

Surprisingly, we found that for the boundary value problem (1.1) and (1.2), the uniform convexity of domains can be dropped. In this paper we obtain the global $C^{2,\alpha}$ regularity for the problem (1.1) and (1.2), assuming both $\Omega$ and $\Omega^*$ are convex only (instead of uniformly convex). From [6], [24, §7.3] it is known that for arbitrary positive and smooth functions $f$, the convexity of domains is necessary for the global $C^1$ regularity.

Not only the uniform convexity of domains can be dropped, we also prove that the boundary smoothness can be reduced to $C^{1,1}$. Note that if the boundaries are $C^2$ and uniformly convex, they will become quadratic polynomials after blowing-up, but if the boundaries are only $C^{1,1}$, we have to deal with the possibility that limit shape is not even $C^1$ smooth after the blowing-up. This is the situation that gives rise to substantial difficulties in our proof (see Remark 4.2). By blowing-up, we mean to normalise a sequence of sub-level sets of the solution.

Under the above assumptions on domains, in this paper we obtain the sharp boundary $C^{2,\alpha}$ regularity when $f \in C^\alpha(\bar{\Omega})$ and $f > 0$; and $C^2$ regularity when $f$ is Dini continuous. For the Dirichlet problem, the sharp boundary $C^{2,\alpha}$ estimate was obtained in [31, 27], and the interior $C^{2,\alpha}$ estimate was obtained in [4].

**Theorem 1.1.** Assume that $\Omega$ and $\Omega^*$ are bounded convex domains in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and assume that $f \in C^\alpha(\bar{\Omega})$ is positive, for some $\alpha \in (0, 1)$. Let $u$ be a convex solution to (1.1) and (1.2). Then we have the estimate

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C,$$

where $C$ is a constant depending only on $n, \alpha, f, \Omega$, and $\Omega^*$.

Our argument also leads to the global $W^{2,p}$ estimate for the solution.

**Theorem 1.2.** Assume that $\Omega$ and $\Omega^*$ are bounded convex domains in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and assume that $f \in C^0(\bar{\Omega})$ is positive. Let $u$ be a convex solution to (1.1) and (1.2). Then we
have the estimate

\[ \|u\|_{W^{2,p}(\Omega)} \leq C \]

for all \( p \geq 1 \), where \( C \) is a constant depending only on \( n, p, f, \Omega, \) and \( \Omega^* \).

The interior \( W^{2,p} \) estimate for the Monge-Ampère equation was proved by Caffarelli [4]. The \( W^{2,p} \) estimate at the boundary was obtained by Savin [28] for the Dirichlet problem, and by Figalli and the first author [10] for the boundary condition (1.2). The proof in [10] relies on the estimates in [7] and thus required the domains to be \( C^2 \) smooth and uniformly convex. In this paper, we assume that the domains are convex with \( C^{1,1} \) boundary, see Remark 6.2 as well.

Relaxing the uniform convexity of domains to convexity does make sense in applications. For example, in Wasserstein generative adversarial networks, a typical case is when the domains \( \Omega \) and \( \Omega^* \) are squares or cubes [22, 30]. Theorems 1.1 and 1.2 imply the regularity of the solution on the faces of the cube. We will prove the \( C^{3,\alpha} \) regularity at the corner in a separate paper. In dimension two, the \( C^{3,\alpha} \) regularity was proved in [20] and it is optimal.

The proof of Theorems 1.1 and 1.2 is based on delicate analysis on sub-level sets of the solution near the boundary and uses various techniques on the Monge-Ampère equation [17, 19], in particular those from Caffarelli’s papers. The uniform density in §2 was introduced by Caffarelli [7] but a different proof is needed for non-uniformly convex domains. The key estimate of the paper is the following uniform obliqueness:

**Lemma 1.1.** Assume that \( \Omega, \Omega^* \) are two convex domains with \( C^{1,1} \) boundaries, and that \( f \) is positive and continuous. Let \( 0 \in \partial \Omega \) and the image \( Du(0) = 0 \in \partial \Omega^* \). Then there exists a positive constant \( \mu \) such that

\[ \langle \nu(0), \nu^*(Du(0)) \rangle \geq \mu > 0, \]

where \( \nu \) and \( \nu^* \) are the unit inner normals of \( \Omega \) and \( \Omega^* \), respectively.

Lemma 1.1 will be proved in §4 and §5. To prove (1.5) for non-uniformly convex domains, we need to introduce a completely different and new idea. We also provide different proof for the boundary \( C^{2,\alpha} \) estimate in §6 for convex domains with \( C^{1,1} \) boundary. These new techniques may apply to other problems related to Monge-Ampère type equations. In particular, we have recently established the \( C^{2,\alpha} \) regularity of free boundaries in optimal transportation [13], thus resolved an open problem raised by Caffarelli and McCann in [8].

This paper is organised as follows. In §2, we introduce some properties on the sub-level sets of solutions to the problem (1.1) and (1.2), and prove the uniform density property. In §3, we obtain the tangential \( C^{1,\alpha} \) regularity for any given \( \alpha \in (0,1) \). In §4 and §5, we prove the uniform obliqueness in dimension two and high dimensions, respectively, which is the key ingredient for the proof of the global \( C^{2,\alpha} \) and \( W^{2,p} \) regularity. Finally in §6, we complete the proof of Theorems 1.1 and 1.2.
2. Uniform density

Consider the optimal transport with density \( f \) in \( \Omega \) and density 1 in \( \Omega^* \). We assume that \( f \) satisfies \( \lambda^{-1} < f < \lambda \) for a constant \( \lambda > 0 \) and \( \int_{\Omega} f(x) \, dx = \int_{\Omega^*} dy \). Let \( u \) and \( v \) be the potential functions in \( \Omega \) and \( \Omega^* \), respectively. Then \( u \) is a solution to (1.1) and (1.2). We extend \( u, v \) to the extended functions \( \tilde{u}, \tilde{v} \) at \( (x_0, \lambda) \) as follows:

\[
\tilde{u}(x) := \sup \{ \ell(x) : \ell \text{ is affine}, \ell \leq u \text{ in } \Omega, \nabla \ell \in \Omega^* \} \quad \text{for } x \in \mathbb{R}^n;
\]

\[
\tilde{v}(y) := \sup \{ \ell(y) : \ell \text{ is affine}, \ell \leq v \text{ in } \Omega^*, \nabla \ell \in \Omega \} \quad \text{for } y \in \mathbb{R}^n.
\]

For simplicity of notations, we denote the extended functions \( \tilde{u}, \tilde{v} \) as \( u, v \). Let \( 0 \in \partial \Omega \) be a boundary point. By subtracting a linear function, we assume that \( u(0) = 0 \) and \( u \geq 0 \). Correspondingly, one has \( 0 \in \partial \Omega^*, v(0) = 0 \) and \( v \geq 0 \) as well.

We introduce two different sub-level sets of \( u \) at \( x_0 \in \Omega \). One is

\[
S_h[u](x_0) = \left\{ x \in \Omega : u(x) < \ell_{x_0}(x) + h \right\},
\]

which may be abbreviated as \( S_h[u] \) or \( S_h(x_0) \) when no confusion arises, where \( \ell_{x_0} \) is a support function of \( u \) at \( x_0 \). The other one is the centred sub-level set

\[
S_h^c[u](x_0) = \left\{ x \in \mathbb{R}^n : u(x) < \hat{\ell}(x) + h \right\}
\]

or simply denoted as \( S_h^c[u] \) or \( S_h^c(x_0) \), where the affine function \( \hat{\ell} \) is chosen such that \( \hat{\ell}(x_0) = u(x_0) \) and \( x_0 \) is the mass centre for \( S_h^c[u](x_0) \). The existence of such a linear function is proved in [5]. Note that \( S_h(x_0) \) is contained in \( \Omega \), but \( S_h^c(x_0) \) may contain both points in and out of \( \Omega \).

The extended function \( u \in C^1(\mathbb{R}^n) \) and satisfies \( \det D^2u = f \chi_\Omega \) in \( \mathbb{R}^n \) if \( \Omega, \Omega^* \) are convex. The following lemma was established by Caffarelli [7, Corollary 2.2].

**Lemma 2.1.** Assume that \( \Omega, \Omega^* \) are convex and bounded. Given a centred sub-level set \( S_h^c(x_0) \) with \( x_0 \in \Omega \), let \( T \) be a linear transform such that \( B_1(0) \subset S^* =: T(S_h^c(x_0)) \subset B_{n}(0) \). Then \( \tilde{u}(x) = h^{-1}|u - \ell|(T^{-1}(x)) \) satisfies

\[
B_r(0) \subset \nabla \tilde{u}(\frac{1}{2} S^*) \subset \tilde{u}(S^*) \subset B_{r^{-1}}(0),
\]

where \( \ell \) is the linear function such that \( u = \ell \) on \( \partial S_h^c(x_0) \). Scaling back, there is an ellipsoid \( E \) centred at \( \nabla \ell \) such that

\[
rE \subset \nabla u(S_h^c(x_0)) \subset r^{-1}E,
\]

where \( \alpha E \) denotes the \( \alpha \)-dilation of \( E \) with respect to its centre, and the constant \( r > 0 \) depends only on \( n, \lambda, \Omega, \Omega^* \), but is independent of \( h \) and \( u \).

Lemma 2.1 implies that \( \nabla \ell \) is a true interior point of \( \nabla u(S_h^c(x_0)) \), namely it has a positive distance from the boundary after normalisation. The first inclusion in (2.1) also follows from the strict convexity of \( u \) [7, Corollary 2.3], namely

\[
u(x) \geq u(x_0) + \nabla u(x_0)(x - x_0) + c_0 h \quad \forall x \in \partial S_h^c(x_0) \cap \Omega,
\]
where $c_0$ is a constant depending on $n, \lambda, \Omega, \Omega^*$ but independent of $h, u$. The last inclusion in (2.1) is due to the doubling condition of $\mu_a$, where $\mu_a$ is the Monge-Ampère measure of $\hat{u}$.

Let $x_0 = 0 \in \partial \Omega$, we shall describe a geometric implication of (2.2). Denote $w = u - \hat{u}$ and assume that $w$ attains its minimum at $p_0$. Let $\phi$ be a convex function whose graph is a convex cone with vertex at $(p_0, w(p_0))$ and satisfies $\phi = w$ on $\partial S_h^c(0)$. Then we have

$$\nabla \phi(x) \cdot (x - p_0) \leq \nabla w(x) \cdot (x - p_0) \leq c \nabla \phi(x) \cdot (x - p_0) \quad \forall x \in \partial S_h^c(0).$$

The first inequality is due to the convexity of $w$ and the second one is due to (2.2).

Let $p \in \partial S_h^c(0)$ such that $p \cdot e_1 = \sup\{x \cdot e_1 : x \in S_h^c(0)\}$, where $e_k$ denotes the unit vector on the $x_k$-axis, for $k = 1, 2, \ldots, n$. Then $\nabla w(p) = |\nabla w(p)| e_1$ and (2.4) implies that

$$(2.5) \quad |\nabla w(p)| \approx \frac{h}{(p - p_0) \cdot e_1}.$$

By the convexity, one sees that (2.5) holds if $p_0$ is replaced by any point in $\frac{1}{2} S_h^c(0)$. In particular it holds when $p_0 = 0$. (2.5) will be used in the proof of Lemma 2.3 below.

In this paper we use the notation $a \gtrsim b$ (resp. $a \lesssim b$) if there exists a constant $C > 0$ depending only on $n, f, \Omega, \Omega^*$ such that $a \gtrsim Cb$ (resp. $a \lesssim Cb$), and $a \approx b$ means that $C^{-1}b \leq a \leq Cb$. For a convex set $A$, we also use the notation $A \sim E$, where $E$ is an ellipsoid, if $C^{-1}E \subset A \subset CE$. For two convex sets $A_1$ and $A_2$, we denote $A_1 \sim A_2$ if there is an ellipsoid $E$ such that $A_1 \sim E$ and $A_2 \sim E$. If $A \sim B$ for a ball $B$, we also say that $A$ has a good shape.

The following lemma shows an equivalence relation between these two sub-level sets.

**Lemma 2.2.** Under the hypotheses of Lemma 2.1 for $h > 0$ small we have

$$S_{b^{-1}h}(0) \cap \Omega \subset S_h(0) \subset S_{bh}(0) \cap \Omega,$$

where the constant $b \geq 1$ depends only on $n, \lambda, \Omega, \Omega^*$, but is independent of $h$ and $u$.

**Proof.** To prove the first inclusion, it suffices to prove that for any $x \in S_h^c(0)$, we have $u(x) \leq Ch$ for a constant $C > 0$ depending only on $n$. Indeed, assume that $\sup\{u(x) : x \in S_h^c(0)\}$ is attained at $p \in \partial S_h^c(0)$. Let $q = -\beta p$, where $\beta > 0$, be a point on $\partial S_h^c(0)$ such that $p, q, 0$ stay on a line segment. Since 0 is the centre of $S_h^c(0)$, we have $c_n^{-1} \leq \beta \leq c_n$ for a constant $c_n$ depending only on $n$. Noting that $u(0) = 0$ and $u = \ell$ on $\partial S_h^c(0)$ for a linear function $\ell$, we have

$$\ell(q) + \beta \ell(p) = (1 + \beta)\ell(0) = (1 + \beta)h.$$

If $\ell(p) = u(p) > Ch$ for a large $C$, we have $u(q) = \ell(q) < 0$, which is a contradiction.

The second inclusion follows readily from the strict convexity, (2.3). \qed

The following uniform density was introduced and proved by Caffarelli in [7], assuming that $\Omega$ is polynomially convex. Here we relax the polynomial convexity to the convexity of domains with $C^{1,1}$ boundary.
Lemma 2.3. Assume that $\Omega, \Omega^*$ are bounded convex domains with $C^{1,1}$ boundary, and that $0 \in \partial \Omega$. Then

$$ \frac{\text{Vol}(\Omega \cap S_h^r(0))}{\text{Vol}(S_h^r(0))} \geq \delta_0 > 0 $$

for some positive constant $\delta_0$ depending on $n, \lambda, \Omega, \Omega^*$, but independent of $u$ and $h$.

Proof. Assume that $\{x_n = 0\}$ is the tangential plane of $\partial \Omega$ at 0 and $\Omega \subset \{x_n > 0\}$. Let $S'_h$ and $S'_{\Omega,h}$ be respectively the projections of $S_h^r$ and $S_h^r \cap \Omega$ on $\{x_n = 0\}$. To prove (2.7), it suffices to prove

$$ |S'_{\Omega,h}| \geq C|S'_h|. $$

In fact, let $\tilde{p} = r_ne_n \in \partial S_h^r$, see Figure 1 below. Then we have $\text{Vol}(S_h^r) \leq C_1r_n|S'_h|$ and $\text{Vol}(\Omega \cap S_h^r) \geq C_2r_n|S'_{\Omega,h}|$, where the constants $C_1, C_2$ only depend on the dimension $n$.

For any unit vector $e \in \{x_n = 0\}$, denote

$$ \lambda_e = \sup\{(x - y) : x, y \in S'_{\Omega,h}\}, $$

$$ r_e = \sup\{t : te \in S'_h\}. $$

Note that $\lambda_e$ is the width of projection of $S'_{\Omega,h}$ in the direction $e$, and $r_e$ is the distance from 0 to the boundary $\partial S'_h$ in the direction $e$. We claim that if there is a positive constant $C$ such that

$$ \frac{\lambda_e}{r_e} \geq C \quad \forall \ e \in \partial B_1(0) \cap \{x_n = 0\}, $$

then (2.8) holds.

To prove this claim, we use induction on dimensions. Let $E$ be the minimum ellipsoid of $S'_h$ with principal radii $r_1 \leq \cdots \leq r_{n-1}$ and principal axes $e_1, \cdots, e_{n-1}$.

Let $p \in \partial S'_{\Omega,h}$ be a point satisfying $|p \cdot e_{n-1}| = \sup\{|x \cdot e_{n-1}| : x \in S'_{\Omega,h}\}$, and $e_p := \frac{p}{|p|}$. By (2.9), $|p \cdot e_{n-1}| \geq C r_{n-1}$. Let $S''_{\Omega,h}$ be the projection of $S'_{\Omega,h}$ on $\{x : x \cdot e_p = 0\}$, and $S''_h := S'_h \cap \{x : x \cdot e_p = 0\}$. Denoting

$$ \lambda'_e = \sup\{(x - y) : x, y \in S''_{\Omega,h}\}, $$

$$ r'_e = \sup\{t : te \in S''_h\}, $$

for any unit vector $e \in \text{span}(e_1, \cdots, e_{n-1})$ and $e \perp e_p$, we still have $\frac{\lambda'_e}{r'_e} \geq C$. Observe that

$$ |S'_{\Omega,h}| \approx |S''_{\Omega,h}| |p| \geq C |S''_{\Omega,h}| r_{n-1} \quad \text{and} \quad |S'_h| \leq C |S''_h| r_{n-1}. $$

Therefore, to prove (2.8) it suffices to prove

$$ |S''_{\Omega,h}| \geq C |S''_h|. $$

By induction we can reduce it to one-dimensional case, in which the claim is trivial.

Let $e_1$ be the direction in which $\inf \\{ \frac{\lambda_e}{r_e} : e \in \{x_n = 0\} \}$ is attained. By the above claim, it suffices to prove that $\frac{\lambda_{e_1}}{r_{e_1}} \geq C$. Let $\ell$ be the linear function such that $u = \ell$ on $\partial S_h^r(0)$. By
subtracting a linear function we assume that $\ell = 0$ (namely we write $u - \ell$ as $u$). Assume $u$ attains its minimum at $p_0$. Let $p_l$ and $p_r$ be the left and right ends of $S_h^c$, namely

$$p_r \cdot e_1 = \sup \{ x \cdot e_1 : x \in S_h^c(0) \},$$
$$p_l \cdot e_1 = \inf \{ x \cdot e_1 : x \in S_h^c(0) \}.$$

Denote $q_l = Du(p_l)$ and $q_r = Du(p_r)$. By definition, $r_e e_1 \in \partial S_h^c$, and there exists $y \in \partial S_h^c(0)$ such that the projection of $y$ on $\{ x_n = 0 \}$ is $r_e e_1$. Since $S_h^c(0)$ is balanced with respect to 0, we may assume $y_n = y \cdot e_n \geq 0$. Observe that $p_r \cdot e_1 \geq y \cdot e_1 = r_e e_1$.

![Figure 1](image)

If the ratio $\lambda e_1 / r e_1$ is sufficiently small, we have

a) $\delta y \notin \Omega$ for some small $\delta > 0$.

b) $\delta p_r, \delta p_l \notin \Omega$.

c) $q_l, q_r \in \partial \Omega^*$.

d) The segment $\overline{q_l q_r}$ is parallel to $e_1$.

e) The point $q_0 := Du(p_0)$ lies on the segment $\overline{q_l q_r}$, and by (2.1), $|q_l - q_0| \approx |q_0 - q_r|$.

f) By the convexity of $S_h^c$, there is a unique number $r_n > 0$ such that $\bar{p} = r_n e_n \in \partial S_h^c$.

The line segment $\overline{pq}$ intersects with $\partial \Omega$ at a point $z = (z_1, \cdots, z_n)$. Since both points $\bar{p}, y \in \partial S_h^c$, we have $z \in S_h^c$. By definition we have $\lambda e_1 \geq |z'|$, where $z' = (z_1, \cdots, z_{n-1})$.

Hence by property a) above and since $y_n \geq 0$, we infer that

$$z_n \geq \frac{r e_1 - |z'|}{r e_1} r_n \geq \frac{1}{2} r_n.$$
Actually, in the triangle vertex at \((0, \tilde{p}, y)\), since \(z \in \tilde{py}\), one has
\[
 z_n \geq \frac{r_{e_1} - |z'|}{r_{e_1}} r_n \\
 \geq \frac{r_{e_1} - \lambda_{e_1}}{r_{e_1}} r_n \geq \frac{1}{2} r_n,
\]
as the ratio \(\lambda_{e_1}/r_{e_1}\) is sufficiently small.

g) By the \(C^{1,\delta}\) regularity of \(u\), we have \(r_n \geq Ch_1^{1+\delta}\). By the \(C^{1,1}\) regularity of \(\partial \Omega\) and property f) above, we then have
\[
r_{e_1} > |z'| \geq Cz_n^{1/2} \geq Ch_1^{1+2\delta}.
\]

Let \(q^* \in \partial \Omega^*\) be the point such that
\[
 |q_0 - q^*| = \inf \{|q - q_0| : q \in \partial \Omega^*\}.
\]

Assume that \(q^* = q_0 + \sigma e^*\) for a unit vector \(e^*\). Note that \(|p_l - p_r|\) is small if \(h\) is small. Hence by the \(C^{1,1}\) smoothness of \(\partial \Omega^*\) and property e) above, we see that
\[
 \sigma = |q^* - q_0| \leq C|q_l - q_r|^2 \quad \text{as } h \to 0.
\]

By (2.5) (note that (2.5) holds when \(p_0\) is replaced by any point in \(\frac{1}{2} S_h^c(0)\)),
\[
 |q_{e_1} - q_l| = |Du(p_l) - Du(p_r)| \leq C \frac{h}{p_r \cdot e_1} \leq C \frac{h}{r_{e_1}} \leq Ch_1^{1+2\delta}.
\]

Hence
\[
 \sigma = |q_0 - q^*| \leq C|q_{e_1} - q_l|^2 \leq Ch_1^{1+2\delta} = Ch_1^{1+\frac{4}{1+\delta}}.
\]

But by (2.5) again, we also have
\[
 \sigma \approx \frac{h}{d_{e^*}}, \quad \text{where } d_{e^*} := \sup \{x \cdot e^* : x \in S_h^c(0)\}.
\]

Hence
\[
 d_{e^*} \approx \frac{h}{\sigma} \geq h^{-\frac{\delta}{1+\delta}} \to \infty \quad \text{as } h \to 0.
\]

This is apparently a contradiction, because \(d_{e^*} \to 0\) as \(h \to 0\), by the strict convexity of the solution. \qed

Remark 2.1. As mentioned before Lemma 2.3, the uniform density was proved by Caffarelli [7, Remark 2, Theorem 3.1], assuming that \(\Omega\) is polynomially convex. In dimension two, a bounded convex domain is polynomially convex. Hence when \(n = 2\), the uniform density holds for any bounded convex domains. No regularity on the boundaries \(\partial \Omega\) and \(\partial \Omega^*\) is needed.

From the uniform density property, we then have [7],
\[
 (2.11) \quad \text{Vol} (S_h^c(0)) \approx \text{Vol} (S_h^c(0) \cap \Omega) \approx h^2
\]
for any \(h > 0\) small. By Lemma 2.2, we also have
\[
 (2.12) \quad \text{Vol} (S_h(0)) \approx h^2.
\]
The following duality result can be found in [7, Corollary 3.2]

**Corollary 2.1 (Duality).** Let $T$ be a unimodular linear transform such that $B_{h^{1/2}} \subset T\{S_h[u](0)\} \subset B_{nh^{1/2}}$. Then we have

$$B_{Ch^{1/2}} \subset T^*\{S_h^c[v](0)\} \subset B_{C^{-1}h^{1/2}},$$

where $T^* = (T')^{-1}$ is the inverse of the transpose of $T$.

**Proof.** As the inner product $x \cdot y$ is invariant under the transforms $T$ and $T^*$, to prove (2.13) one may assume directly that $T$ is the identity mapping. Then (2.13) follows from (2.11) and (2.1).

From Corollary 2.1 we also have the following corollaries, which will be used in §5.1.

**Corollary 2.2.** For any $h > 0$ small, we have

$$|x \cdot y| \leq Ch \quad \forall x \in S_h^c[u](0), \ y \in S_h^c[v](0).$$

Moreover, for any $x \in \partial S_h^c[u](0)$, there exists $y \in \partial S_h^c[v](0)$ such that

$$x \cdot y \geq C^{-1}h,$$

where $C$ is a constant independent of $u$ and $h$.

**Remark 2.2.** Similarly to Corollary 2.2, by Lemma 2.2 we also have the following relation between $S_h[u](0)$ and $S_h[v](0)$:

$$|x \cdot y| \leq Ch \quad \forall x \in S_h[u](0), \ y \in S_h[v](0).$$

**Remark 2.3.** Given any unit vector $e \in \mathbb{R}^n$, let

$$d_1 := \sup \{|x \cdot e| : x \in S_h^c[u](0)\}, \quad d_2 := \sup \{|x \cdot e| : x \in S_h[u](0)\}$$

be the width of $S_h^c[u](0)$ and $S_h[u](0)$, respectively, in $e$ direction. Note that $S_h^c(0) \subset CB S_h^c(0)$ and $S_h(0) \subset CB S_{C^{-1}h}^c(0)$, where $b$ is the constant in Lemma 2.2 and $C$ is a constant independent of $h$ (see [7, Observation b) in Lemma 4.1]). Then by Lemma 2.2 Lemma 2.3 [2.11] and [2.12], we can obtain $d_1 \approx d_2$.

**Remark 2.4.** The estimates in this section are invariant under affine transforms. Let $S_{h_j}[u](x_j)$ be a sequence of sub-level sets and let $T_j$ be a linear transform such that $T_j(S_{h_j}[u](x_j))$ has a good shape and $T_j(x_j) = 0$, where $x_j \in \overline{\Omega}$ and $h_j \to 0$ as $j \to \infty$. Denote

$$u_j(x) := \frac{1}{h_j} u(T_{h_j}^{-1} x) \quad \text{and} \quad \Omega_j := T_j(\Omega).$$

Then the estimates in Lemmas 2.1 2.2 and 2.3 also hold for $u_j, \Omega_j$ with the same constants $r, b, \delta_0$ independent of the sequence $h_j$. Assume that $u_j, \Omega_j$ sub-converge as $j \to \infty$ to limits $u_0, \Omega_0$. One sees that these estimates hold for $u_0$ near 0 as well, again with the same constants $r, b, \delta_0$, which depend only on $n, \lambda, \Omega, \Omega^*$, but are independent of $\Omega_0$. Similarly, by taking limits,
the estimates in Corollaries 2.1–2.2 for the centred sub-level sets $S_{h_j}^c[u], S_{h_j}^c[v]$ also hold for the limits $u_0, v_0$.

Furthermore, by Caffarelli’s geometric decay estimate (see [5, Lemma 4], [7, Lemma 2.2]), one infers the strict convexity and $C^{1,\delta}$ regularity of solutions, namely
\begin{equation}
C^{-1}|x|^{1+\delta-1} \leq u(x) \leq C|x|^{1+\delta} \quad \forall \ x \in S_1[u](0),
\end{equation}
if $u(0) = 0, Du(0) = 0$ and $S_1[u](0)$ is normalised, where $C, \delta > 0$ depend only on $n, \lambda$ (assuming that $\partial \Omega \cap \partial S_1[u](0)$ and $\partial \Omega^*$ are convex). (2.18) also holds for the sequence $u_j$ and its limit $u_0$ near 0, with the same constants.

3. Tangential $C^{1,\alpha}$ regularity

The tangential $C^{1,\alpha}$ regularity of $u$, for any given $\alpha \in (0, 1)$, was established in [7], where $\Omega$ is assumed to be a uniformly convex domain with $C^2$ boundary. But the same strategy applies to convex domains with $C^{1,1}$ boundary. To see this let us outline the proof here.

Let $0 \in \partial \Omega$ be a boundary point. We assume that locally $\partial \Omega$ is given by
\begin{equation}
\{x_n = \rho(x')\}
\end{equation}
for some convex function $\rho \in C^{1,1}$ satisfying $\rho(0) = 0, D\rho(0) = 0, \rho(x') \leq C|x'|^2$, where $x' = (x_1, \ldots, x_{n-1})$. In this section, we assume that $0 < f \in C^0(\overline{\Omega})$ and $f(0) = 1$. To prove the tangential $C^{1,\alpha}$, it suffices to prove

Lemma 3.1. For any given $\alpha \in (0, 1)$, there exists a small constant $C = C_\alpha > 0$ depending only on $n$, the modulus of continuity of $f$ and $\|\partial \Omega\|_{C^{1,1}}$, but independent of $h$, such that for the centred sub-level set $S_h^c(0)$, we have
\begin{equation}
S_h^c(0) \cap \{x_n = 0\} \supset B_{C_\alpha h^{1/(1+\alpha)}}(0) \cap \{x_n = 0\}.
\end{equation}

The idea of the proof is as follows. For each $h > 0$, there is an ellipsoid $E_h$ such that
\begin{equation}
S_h^c(0) \sim E_h = \left\{ \sum_{i=1}^{n-1} \frac{(x_i - k_i x_n)}{a_i}^2 + \frac{x_n}{a_n}^2 \leq 1 \right\},
\end{equation}
where $a_1 \leq \cdots \leq a_{n-1}$, namely $\beta E_h \subset S_h^c(0) \subset \beta^{-1} E_h$ for some constant $\beta$ depending only on $n$. Let $b e_n$ be the intersection of the positive $x_n$-axis and $\partial E_h$.

We first make a linear transform
\begin{equation}
T_1 : \left\{ \begin{array}{ll}
y_i = x_i - k_i x_n & i < n, \\
y_n = x_n.
\end{array} \right.
\end{equation}
This transformation $T_1$ moves the centre of $E_h \cap \{x_n = b\}$ to the point $b e_n$. Hence, the “slope” $k_i$ is bounded by
\begin{equation}
k_i \leq \frac{a_i}{b} \quad \text{for} \ i = 1, \ldots, n - 1.
\end{equation}
If the inclusion (3.1) does not hold, let \( h_0 > 0 \) be the largest constant such that (3.1) holds for \( h > h_0 \) and \( \partial S_{h_0}^c(0) \cap \{ x_n = 0 \} \) touches \( \partial B_{C_0 h_0^{1/(1+\alpha)}} \), where the constant \( C_0 \) is chosen small so that \( h_0 \) is also small. Then
\[
a_1 \leq C_0 h_0^{1/(1+\alpha)}.\tag{3.4}
\]
By the \( C^{1,\delta} \) regularity of \( u \) [5], we have
\[
a_n \geq b \geq C h_0^{1/(1+\delta)}.\tag{3.5}
\]
Next we make the linear transform
\[
(3.6) \quad \mathcal{T}_2: \quad z_i = y_i/a_i \quad i = 1, \ldots , n,
\]
such that the sub-level set \( S_{h_0}^c(0) \) is “normalised”. Denote \( \mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1 \). The next lemma shows that near the origin, the \( \mathcal{T}(\Omega) \) tends to be flat in \( e_1 \) direction as \( h_0 \to 0 \).

**Lemma 3.2.** For any given \( R > 0 \), the limit of \( \mathcal{T}(\partial \Omega) \cap B_R(0) \) (as \( h_0 \to 0 \)) is flat in \( e_1 \) direction.

**Proof.** Let \( p' = (h_0^\gamma, 0, \cdots , 0) \) be a point on the \( x_1 \)-axis, where \( \gamma \) is chosen so that \( \frac{1}{2(1+\delta)} < \gamma < \frac{1}{1+\alpha} \). Denote \( p = (p', \rho(p')) \) and \( q = \mathcal{T}(p) \). By direct computation we have
\[
q_1 = \frac{1}{a_1}(h_0^\gamma - k_1 \rho(p')), \\
q_i = - \frac{1}{a_i}k_i \rho(p'), \quad i = 2, \cdots , n - 1, \\
q_n = \frac{1}{a_n} \rho(p').
\]
Note that \( \rho(p') \leq Ch_0^{2\gamma} \). By (3.3), (3.4) and (3.5) we have
\[
k_1 \rho(p') \leq h_0^{1+\gamma} < h_0^{1/1+\delta} \ll h_0^\gamma,
\]
where the last inequality is due to the choice of \( \gamma \). Hence \( q_1 \to \infty \) as \( h_0 \to 0 \). It is also easy to verify that \( |q_i| \leq \frac{1}{2} h_0^{2\gamma} \to 0 \) (if \( i = 2, \cdots , n - 1 \)), and \( q_n \to 0 \), as \( h_0 \to 0 \). Note that the above computation still works if \( p' = (-h_0^\gamma, 0, \cdots , 0) \). Therefore the limit of \( \mathcal{T}(\partial \Omega) \) (as \( h_0 \to 0 \)) contains the \( x_1 \) axis. By convexity, we see that the limit of \( \mathcal{T}(\Omega) \) is independent of the \( e_1 \) direction. \( \square \)

Since \( \mathcal{T}(S_{h_0}^c(0)) \) is normalised, the domain \( \mathcal{T}(\Omega \cap S_{Mh}^c(0)) \) has a good shape, where \( M > 1 \) is chosen such that \( S_{h}^c(0) \subset \frac{1}{2} S_{Mh}^c(0) \). By the above discussion, the boundary part \( \mathcal{T}(\partial \Omega \cap S_{Mh}^c(0)) \) becomes flat in direction \( e_1 \) as \( h \to 0 \). As in [7], we denote
\[
D_h = \{ z \in \mathcal{T}(S_{Mh}^c(0)) : z = \hat{z} + te_1 \text{ for some } \hat{z} \in \mathcal{T}(S_{Mh}^c(0) \cap \Omega) \text{ and } t \in \mathbb{R} \}
\]
by erasing the dependence on \( x_1 \). Then
\[
\mathcal{T}(S_{Mh}^c(0) \cap \Omega) \subset D_h \subset \mathcal{T}(S_{Mh}^c(0)) \cap \{ x_n > 0 \}
\]
and near the origin, \( \partial D_h \) is flat in the \( x_1 \)-direction.
Let $w$ be the solution to
\begin{equation}
\begin{aligned}
\det D^2 w &= \chi_{D_{h_0}} \quad \text{in } \mathcal{T}(S_{h_0}^c(0)), \\
 w &= \tilde{u} \quad \text{on } \partial\{\mathcal{T}(S_{h_0}^c(0))\},
\end{aligned}
\end{equation}
where $\tilde{u}(z) = |\mathcal{T}|^{2/n}u(|\mathcal{T}^{-1}(z)|)$. A key observation in this proof is that Pogorelov’s interior second derivative estimate applies to $w_{11}$, even though the right hand side of (3.7) is discontinuous in $(z_2, \ldots, z_n)$, and no regularity of $\mathcal{T}(\partial \Omega)$ in $(z_2, \ldots, z_n)$ is assumed. Therefore $w$ is $C^{1,1}$ in $z_1$. By the maximum principle one can give an estimate for $|w - \tilde{u}|$:
\begin{equation}
|w - \tilde{u}| \leq C(\delta_0 + V_{h_0})^{1/n},
\end{equation}
where $\delta_0 = \sup\{|f(x) - 1| : x \in S_{h_0}^c(0) \cap \Omega\}$, and $V_{h_0} = \text{Vol}\{D_{h_0} - \mathcal{T}(S_{h_0}^c(0) \cap \Omega)\} = o(h_0)$.

Changing back one obtains an estimate for $u$ from the estimate $\partial^2 w \leq C$, from which one infers the tangential $C^{1,\alpha}$ for any given $\alpha \in (0,1)$. For details see [7].

**Corollary 3.1.** Assume that the function $f$, defined in $\Omega$, is a positive constant near the origin, and both $\partial \Omega$ and $\partial \Omega^*$ are flat near the origin in a direction $e$. Then near the origin, $u$ is $C^{1,1}$ and uniformly convex in the direction $e$.

**Proof.** From the assumption, one can see that $\delta_0$ and $V_{h_0}$ vanish in estimate (3.8), thus $u \in C^{1,1}$ in the direction $e$. Since $\partial \Omega^*$ is also flat in direction $e$, by Corollary 2.1 we have $B_{C^{1,2}(0)} \subset S_h^c[u](0)$ along $e$ direction. And then by the duality in Corollary 2.2 we have $S_h^c(u^*_0(0)) \subset B_{C^{1,2}(0)}$ along $e$ direction. Hence $u$ is uniformly convex in the direction $e$. \(\square\)

4. Uniform obliqueness in dimension two

The uniform obliqueness (Lemma 1.1) is a key ingredient in proving the boundary $C^{2,\alpha}$ and $W^{2,p}$ estimates. The proof is technically rather complicated. For the reader’s convenience, we divide the proof into two sections. In this section we prove Lemma 1.1 in dimension two. In dimension two, we assume that $\Omega, \Omega^*$ are bounded convex domains with $C^{1,\gamma}$ boundaries for a small $\gamma > 0$, and $f \in C^0(\overline{\Omega})$. In the next section we prove Lemma 1.1 in high dimensions. In dimension two, our proof consists of the following four steps.

(i) If the uniform obliqueness does not hold at the origin, we express the boundaries of $\Omega$ and $\Omega^*$ by (4.1), and prove a “balance property” of the sub-level set $S_h[u](0)$ in Lemma 4.1. It implies the decay estimates (4.12) and (4.13).

(ii) We introduce a blow-up argument so that the inhomogeneous term $f$ becomes a positive constant in the limit.

(iii) The blow-up limit $u_0$ may not be smooth. We construct a smooth sequence $\{u_k\}$, which converges to $u_0$ locally uniformly.

(iv) We introduce the auxiliary function $w = \partial_1 u_0 + u_0 - x_1 \partial_1 u_0$. By Steps (ii), (iii) and the maximum principle, the function $w(t) = \inf w(t, \cdot)$ is concave near the origin. The concavity and the decay estimate (4.13) imply that $w \equiv 0$ for $t > 0$ small, which contradicts to the strict convexity of $u_0$. Hence we infer the uniform obliqueness.
4.1. Balance property and decay estimate. Assume that \(0 \in \partial \Omega\) and \(\Omega \subset \{x_2 > 0\}\). To prove the uniform obliqueness, by the global \(C^{1,\delta}\) regularity \([3]\), we may assume to the contrary that \(u(0) = 0\), \(Du(0) = 0\) \(\in \partial \Omega^*\) and \(\Omega^* \subset \{y_1 > 0\}\). Then we have

(i) \(u_1 = u_{x_1} > 0\) in \(\Omega\) and \(v_2 = v_{y_2} > 0\) in \(\Omega^*\); it implies that

(ii) if \(x \in S_h(0)\), then \(x - te_1 \in S_h(0) \ \forall t > 0\), provided \(x - te_1 \in \Omega\),

where \(S_h(0) = S_h[u](0)\) is the sub-level set of \(u\), introduced in \([2]\). Accordingly, the boundaries \(\partial \Omega\) and \(\partial \Omega^*\) near the origin can be expressed as

\[
\partial \Omega = \{x_2 = \rho(x_1)\},
\]

\[
\partial \Omega^* = \{y_1 = \rho^*(y_2)\},
\]

with the following properties:

\(\textbf{(H}_1\textbf{)}\) \(\rho, \rho^* \geq 0\) are convex functions defined in an interval \((-r_0, r_0)\) and satisfying \(\rho(0) = 0\) and \(\rho^*(0) = 0\), where \(r_0 > 0\) is a constant.

\(\textbf{(H}_2\textbf{)}\) Denote \(\sigma(t) = |t|^{1+\gamma}\). By the assumption \(\partial \Omega \in C^{1,\gamma}\), we have

\[
\rho(t) \leq C \sigma(t) \quad \text{for } t \leq 0.
\]

**Remark 4.1.** (i) We will derive a contradiction from \(\textbf{(H}_1\textbf{)}\) and \(\textbf{(H}_2\textbf{)}\). Note that it suffices to assume \(t \leq 0\) in \((4.2)\). By Lemma 4.1 below, we can show that \((4.2)\) holds for \(t > 0\) as well. Hence \(\partial \Omega\) is \(C^1\) at 0.

(ii) As the reader will see, in our argument below we will not use any boundary regularity for \(\partial \Omega^*\). For any point \(p \in \partial \Omega \cap \{x_1 < 0\}\), since the inner product \(\langle \nu(p), \nu^*(Du(p)) \rangle \geq 0\), \((\textbf{H}_2)\) implies that \(\rho^*(t) = o(t)\) for \(t \geq 0\).

(iii) For clarity in this section we will always assume that \(\partial \Omega, \partial \Omega^* \in C^{1,\gamma}\), for a small \(\gamma > 0\). By Remark \([2,4]\) the constants in this section depend on \(n, \lambda, \Omega, \Omega^*\) (inner and outer radii of \(\Omega, \Omega^*\) and \(\gamma\)). In the approximation \(\{u_k\}\) in \([4.3]\) we also allow that the constants depend on \(k\). But all the constants are independent of \(h\) and \(u\) (for \(h > 0\) small). The continuity of \(f\) is used only in the blow-up process, such that the RHS of \((4.18)\) is a constant. In this section we do not use the tangential \(C^{1,\alpha}\) regularity of \([3]\).

Let \(q = (q_1, q_2)\) and \(\xi = (\xi_1, \xi_2)\) be two points on \(\partial S_h(0) \cap \Omega\) such that

\[
\langle q, e_1 \rangle = \sup \{\langle x, e_1 \rangle : x \in S_h(0)\},
\]

\[
\langle \xi, e_1 \rangle = \inf \{\langle x, e_1 \rangle : x \in S_h(0)\}.
\]

Apparently \(q_1 > 0\) and \(\xi_1 < 0\), see Figure 2 below. Note that \(u_2(p) < 0\) for any boundary point \(p \in \partial \Omega \cap \{x_1 > 0\}\). Hence \(q\) is an interior point of \(\Omega\). The following lemma shows that the area of \(S_h[u](0) \cap \{x_1 > 0\}\) can balance that of \(S_h[u](0) \cap \{x_1 < 0\}\).

**Lemma 4.1.** For all \(h > 0\) small, we have the “balance” property

\[
q_1 \geq \delta_0 |\xi_1|,
\]

where \(\delta_0 > 0\) is a constant independent of \(h\).
Proof. To prove (4.4), suppose to the contrary that
\[ q_1 = o(\|\xi_1\|) \] 
for a sequence \( h \to 0 \). Denote \( t_0 = \frac{1}{2}(\xi_1 + q_1) \). There is a unique \( s_0 > \rho(t_0) \) such that \( u(t_0, s_0) = h \). Denote
\[ x^c = (x_1^c, x_2^c) := \left( t_0, \frac{1}{2}(s_0 + \rho(t_0)) \right). \]
The point \( x^c \) can be regarded as the centre of \( S_h(0) \). Denote
\[ \begin{align*}
\lambda_1 &= q_1 - \xi_1, \\
\lambda_2 &= s_0 - \rho(t_0).
\end{align*} \]
Apparently
\[ \text{Vol}(S_h(0)) \approx \lambda_1 \lambda_2. \] 
Moreover, by (4.2) and the property (i), we have
\[ \lambda_2 \leq \xi_2 \leq \sigma(\xi_1) \leq \sigma(\lambda_1). \]
By (4.5), we have \( \frac{1}{2}\xi_1 < x_1^c < \frac{1}{4}\xi_1 \) for \( h > 0 \) small. Let’s make the first change
\[ \begin{align*}
y_1 &= x_1, \\
y_2 &= x_2 - \frac{x_2}{x_1}x_1
\end{align*} \]
such that \( S_h[u] \subset \{ x \in \mathbb{R}^2 : \xi_1 < x_1 < q_1, |x_2| < 4\lambda_2 \} \). Note that such a change does not change the ratio \( \frac{q_1}{\|\xi_1\|} \) in (4.4). We then make the change
\[ \begin{align*}
z_1 &= y_1/\lambda_1, \\
z_2 &= y_2/\lambda_2
\end{align*} \]
and accordingly,
\[ u_h = u/h, \quad \text{where } h = u(\xi). \]
By the volume estimate (4.6), the sub-level set \( S_h(0) \) has a good shape after changes (4.8) and (4.9). By Lemmas 2.2 and 2.3 (in dimension two, the uniform density holds for any bounded
convex domains, see Remark 2.1, the centred sub-level set $S^c_h(0)$ also has a good shape after the change. Hence,

\begin{equation}
\|u_h\|_{L^\infty(B_1(0))} \leq C
\end{equation}

for a constant $C$ independent of $u$ and $h$.

Let $q_h, \xi_h$ be the corresponding points of $q, \xi$ after the above changes. Assume that $q_h \to q_0 = (q_{0,1}, q_{0,2})$, $u_h \to u_0$ as $h \to 0$. By (4.11) we have $q_{0,1} = 0$, namely $q_0$ is on the $x_2$-axis. On the other hand, after the change (4.9), the line \{ $x_1 = q_{h,1}$ \} is tangent to $S_1[u_h]$ at $q_h$, and thus $u_h(x) \geq 1$ for all $x \in \{ x_1 \geq q_{h,1} \}$. Passing to the limit we have $u_0(x) \geq 1$ for $x \in \{ x_1 \geq 0 \}$ and $u_0(0) = \lim_{h \to 0} u_h(0) = 0$, which is a contradiction by (4.11), namely $u_0$ is a limit of a sequence of locally uniformly bounded convex functions $u_h$, $u_0$ must be continuous.

\begin{corollary}
For $t > 0$, denote

$$ u(t) = \inf \{ u(t, x_2) : x_2 \geq \rho(t) \}. $$

We have the asymptotic estimate

\begin{equation}
(4.12) \quad u(t) \leq Ct\sigma(t) \quad \text{for } t > 0 \text{ small.}
\end{equation}

\end{corollary}

\begin{proof}
By the strict convexity of $u$, the sub-level set $S_h(0)$ shrinks to the origin as $h \to 0$. Hence, for any $t > 0$, there exists a unique $h > 0$ such that \{ $x_1 = t$ \} is tangential to $\partial S_h(0)$ at the point $q = (q_1, q_2)$. This implies that $q_1 = t$ and $u(t) = h$.

From (4.4),

$$ \lambda_1 = q_1 - \xi_1 \leq Ct, $$

for some constant $C$ independent of $h$. Then from (4.7) and (4.6) we have

$$ \lambda_2 \leq C\sigma(t) \quad \text{and} \quad \text{Vol}(S_h(0)) \leq C t\sigma(t). $$

Hence by (2.12) we obtain that

\begin{equation}
(4.12) \quad u(t) = h \leq C t\sigma(t).
\end{equation}

\end{proof}

\begin{corollary}
For $t > 0$, denote

$$ \partial_1 u(t) = \inf \{ \partial_1 u(t, x_2) : x_2 \geq \rho(t) \}. $$

Then we have the asymptotic behaviour for $t > 0$ small,

\begin{equation}
(4.13) \quad \partial_1 u(t) \leq C \sigma(t).
\end{equation}

\end{corollary}

\begin{proof}
This is a direct consequence of (4.12). In fact, by the convexity of $u$, for $t > 0$ small

$$ \partial_1 u(t, x_2) \leq \frac{u(2t, x_2) - u(t, x_2)}{t} \leq \frac{u(2t)}{t}. $$

Then taking the infimum in $x_2$, from (4.12), we obtain that

\begin{equation}
(4.12) \quad \partial_1 u(t) = \inf_{x_2} \partial_1 u(t, x_2) \leq \frac{u(2t)}{t} \leq C \sigma(t).
\end{equation}

\end{proof}
4.2. A blow-up sequence. Assume that \( f > 0 \) is continuous. The purpose of blow-up is such that \( f \) becomes a positive constant in the limit.

From the proof of Lemma 4.1, the sub-level set \( S_h[u](0) \) has a good shape under the following normalisation \( T \):

\[
\begin{align*}
y_1 &= x_1/\lambda_1, \quad \text{with} \quad \lambda_1 = q_1 - \xi_1, \\
y_2 &= x_2/\lambda_2, \quad \text{with} \quad \lambda_2 = \rho(\xi_1).
\end{align*}
\]

In fact, as shown in the proof of Lemma 4.1, we have \( \text{Vol}(S_h[u]) \approx \lambda_1 \lambda_2 \). Hence \( T(S_h[u]) \approx 1 \).

Also by the proof of Lemma 4.1, there is an interval \((0, r)\) such that \( \Omega \subset \mathcal{J} \subset [1, r] \times [0, 1] \). Hence \( T(S_h[u]) \) has a good shape.

Accordingly we make the change \( u \to u_h \), where

\[
u_h(x) = u(T^{-1}x)/h.
\]

After the change, the domain \( \Omega \) is changed to \( \Omega_h \), and the boundary \( \{x_2 = \rho(x_1)\} \) is changed to \( \{x_2 = \rho_h(x_1) = \frac{1}{\lambda_2} \rho(\lambda_1 x_1)\} \). By Lemma 2.2 and Lemma 2.3, we also have

\[
B_{\frac{c}{h}}(0) \subset T(S_h^c[u]) \subset B_C(0)
\]

for some constant \( C \) depending only on \( n \), the constants \( b \) in Lemma 2.2 and \( \delta_0 \) in Lemma 2.3, but independent of \( h \). In (4.16), the centred sub-level set \( S_h^c \) can be replaced by the usual sub-level set \( S_h \) if the centre of the concentric ball is properly chosen.

By (4.12), the limit \( \lim_{t \to 0} \frac{u(t)}{\sigma(t)} < \infty \). Hence for any fixed small \( \bar{\varepsilon} > 0 \) (we may fix \( \bar{\varepsilon} = 1 \)), there is a sequence \( t_j \to 0 \) (\( t_j > 0 \)) such that

\[
\frac{u(t)}{\sigma(t)} \leq (1 + \bar{\varepsilon}) \frac{u(t_j)}{\sigma(t_j)} \quad \forall \ t \in (0, t_j).
\]

Denote \( h_j = u(t_j) \). Since \( T(S_h[u]) \) has a good shape, for any \( R > 0 \), \( \Omega_{h_j} \cap B_R(0) \) converges in Hausdorff distance to a limit. Hence by passing to a subsequence, \( \Omega_{h_j} \) converges to a limit \( \Omega_0 \) as \( h_j \to 0 \), which is an unbounded convex domain in \( \mathbb{R}^2 \).

Next we show that \( u_{h_j} \) sub-converges to a limit \( u_0 \) as \( h_j \to 0 \). Indeed, by the geometric decay of sections (Lemma 2.2), for any \( k > 0 \), there exists a constant \( M_k \) such that

\[
kS_{h}^c[u] \subset S_{M_k h}[u] \quad \text{for} \quad h > 0 \text{ small}.
\]

Hence by the convexity of \( u_{h_j} \) and the estimate (4.16), \( u_{h_j} \) is locally uniformly bounded, which implies the sub-convergence.

By the weak convergence of the Monge-Ampère operator, \( u_0 \) satisfies the equation

\[
\det D^2 u_0 = c_1 \chi_{\rho_0} \quad \text{in} \quad \mathbb{R}^2
\]

for a positive constant \( c_1 \). There is no loss of generality in assuming that \( c_1 = 1 \).

By the change (4.14), we have \( 0 \in \partial \Omega_0 \). Let \( \mathcal{J} \) be the projection of \( \Omega_0 \) on the \( x_1 \)-axis. By Lemma 4.1, there is an interval \( (0, r_0) \subset \mathcal{J} \). Hence the lower boundary of \( \Omega_0 \cap \{0 < x_1 < r_0\} \) can be represented by a convex function

\[
x_2 = \rho_0(x_1)
\]
and \( \rho_0 \) is the limit of \( \rho_{h_j} \), passing to a subsequence if necessary.

**Remark 4.2.** (i). In Lemma 4.1, we proved that \( |\xi_1| \leq C q_1 \), but the possibility \( |\xi_1| = o(q_1) \) as \( h \to 0 \) has not been ruled out. Hence, even when \( \partial \Omega \in C^{1,1} \), the limit \( \Omega_0 \) may be contained in the first quadrant, i.e. \( \Omega_0 \subset \{ x_1 > 0, x_2 > 0 \} \). In this case, \( \rho_0 \) is defined in \( \{ x_1 > 0 \} \).

(ii). No matter whether \( \Omega_0 \) is contained in the first quadrant, we point out that the whole positive \( x_2 \) axis is contained in \( \Omega_0 \). To see this, notice that there exists a constant \( \beta_0 \) such that \( \beta e_2 \in \Omega \) \( \forall \beta \in (0, \beta_0) \). By the transform \( T \) in (4.14), we have \( \beta e_2 \in \Omega_h \) \( \forall \beta \in (0, \frac{\beta_0}{\lambda_2}) \). By the strict convexity of \( u \), \( \lambda_2 \to 0 \) and \( \beta_0/\lambda_2 \to \infty \) as \( h \to 0 \). Hence for any \( R > 0 \), \( Re_2 \in \Omega_h \) provided \( h > 0 \) is small enough. Passing to the limit, we have \( Re_2 \in \Omega_0 \).

(iii). In Corollary 4.4 below, we will show that \( \rho_0(t) \leq C \sigma(t) \) for \( t > 0 \) small. But \( \rho_0 \) may not be smooth. In comparison, if \( \partial \Omega \in C^2 \) and is uniformly convex, then \( \rho_0 \) is a quadratic polynomial [7]. The lack of smoothness of \( \rho_0 \) in our case makes the problem much more complicated.

By our choice of the sequences \( t_j \) and \( h_j = u(t_j) \) in (4.17), the asymptotic behaviour (4.12) holds for the limits \( u_0 \) and \( \rho_0 \). Namely, we have the following estimates.

**Corollary 4.3.** Denote

\[
\bar{u}_0(t) = \inf \{ u_0(t, x_2) : x_2 \geq \rho_0(t) \} \quad t > 0.
\]

We have

\[
\bar{u}_0(t) \leq Ct \sigma(t) \quad \text{for } t > 0 \text{ small.}
\]

**Proof.** Let \( q, \xi \) be the points defined in (4.3) with \( h = h_j \), and let \( \lambda_1 = q_1 - \xi_1, \lambda_2 = \rho(\xi_1) \) as in (4.14). From (4.4), \( t_j = q_1 \approx \lambda_1 \).

By (4.15), \( u_{h_j}(x_1, x_2) = \frac{u(\lambda_1 x_1, \lambda_2 x_2)}{u(t_j)} \). Hence, by (4.17)

\[
\inf \{ u_{h_j}(t, x_2) : x_2 \geq \rho_{h_j}(t) \} = \inf \{ u(\lambda_1 t, \lambda_2 x_2) : x_2 \geq \rho(t) \} \frac{u(t_j)}{u(\lambda_1 t)} \leq \frac{(1 + \varepsilon)(\lambda_1 t) \sigma(\lambda_1 t)}{t_j \sigma(t_j)} \leq Ct \sigma(t),
\]

where the constant \( C > 0 \) is independent of \( j \). The above inequality implies that \( u_{h_j}(t) \leq Ct \sigma(t) \). Passing to the limit, we obtain (4.20). \( \square \)

**Corollary 4.4.** We have

\[
\rho_0(t) \leq C \sigma(t) \quad \text{for } t > 0 \text{ small.}
\]
Proof. For any given $h > 0$ small, as in (4.14) we introduce two points $\xi$ and $q$ for the sub-level set $S_h[u_0]$. Let $z = \beta e_2$ be the point on the $x_2$-axis such that $u_0(z) = h$. By Corollary 4.3

\[ h = u(q) \leq Cq_1\sigma(q_1). \]

By (2.12) and Remark 2.4 we have

\[ \frac{1}{2} \beta q_1 \leq \text{Vol}(S_h[u_0]) \leq Ch. \]

Hence $\beta \leq C\sigma(q_1)$. Noting that $\partial_x u_0 \geq 0$, we infer that $q_2 \leq \beta \leq C\sigma(q_1)$, and so (4.21) follows.

Denote $\mathcal{T}^* = \frac{1}{h}(\mathcal{T}')^{-1}$, the dual affine transform for $v$, where $\mathcal{T}'$ is the transpose of $\mathcal{T}$ in (4.14). As in (4.14), we denote $v_0(y) = \frac{1}{h} v((\mathcal{T}^*)^{-1}y)$, and $\Omega_h^* = \mathcal{T}^*(\Omega_h^*)$. The boundary $\partial \Omega_h^*$ near the origin is given by $\{y_1 = \rho^*_h(y_2)\}$. Similarly, $\Omega_h^* \cap \partial \Omega^*_0$ converges to an unbounded convex domain $\Omega_0^*$ and $v_h$ converges to a convex function $v_0$, locally uniformly.

Moreover, for a positive constant $c_2$

\[ (4.22) \quad \det D^2v_0 = c_2 \chi_{\Omega_0^*} \quad \text{in} \quad \mathbb{R}^2 \]

Remark 4.3. As pointed out in Remark 4.2, we need to deal with the case when $\Omega_0^*$ is contained in the first quadrant (note that by Lemma 4.1 it is again the first quadrant, not the fourth quadrant). By Remark 4.2 the whole positive $y_1$-axis is contained in $\Omega_0^*$.

Although $\Omega_0$ is unbounded, by Remark 2.4 $u_0$ is locally strictly convex and $u_0 \in C^{1,\delta}(\Omega_0)$. By the convexity of $\Omega_0$ and $\Omega_0^*$, we have $u_0 \in C^1(\mathbb{R}^2)$. By the interior regularity for the Monge-Ampère equation (4.18), $u_0$ is $C^\infty$ smooth inside $\Omega_0$.

4.3. Smooth approximation. In this subsection we shall construct a smooth approximation sequences $\{u_k\}$ converging to $u_0$ in a small neighbourhood of the origin. The $C^{2,\alpha}$-smoothness of $u_k$ is needed for deriving the contradiction in (4.4). Note that we just need the $C^{2,\alpha}$-smoothness of $u_k$ but not a uniform upper bound for the $C^{2,\alpha}$-norm of $u_k$.

Let $V = B_r \cap \Omega_0^*$ for a small constant $r > 0$, and let $U = Dv_0(V)$. By the local strict convexity of $v_0$, there exists $r_0 > 0$ such that $B_{r_0} \cap \Omega_0 \subset U$. Let $U_k, U_k^*$ be $C^\infty$ smooth, bounded domains such that

\begin{align*}
& a) \quad 0 \in \partial U_k, \quad 0 \in \partial U_k^* \quad \forall \quad k \geq 1, \\
& b) \quad U_k \subset \{x_2 > 0\}, \quad U_k^* \subset \{y_1 > 0\} \quad \forall \quad k \geq 1, \\
& c) \quad U_k \to U, \quad U_k^* \to V, \quad \text{as} \quad k \to \infty, \quad \text{in Hausdorff's sense.}
\end{align*}

Moreover, we assume the lower part of the boundary $\partial U_k \cap B_{r_0}$ is the graph of a smooth, uniformly convex function $\rho_k$ in direction $e_2$, that is

\[ (4.23) \quad \Gamma_k = \{x_2 = \rho_k(x_1)\} \cap B_{r_0}, \]

where $\rho_k$ is defined on $\mathcal{J}_k$ that is the projection of $\partial U_k \cap B_{r_0}$ on the $x_1$-axis. In our construction, $0 \in \mathcal{J}_k$ is an interior point of $\mathcal{J}_k$ for all $k \geq 1$. By Corollary 4.4 $(0, \frac{1}{2}r_0) \subset \mathcal{J}_k$. From the above
conditions $a) - c$) and uniform convexity, the function $\rho_k$ satisfies

$$\begin{align*}
\rho_k(0) &= 0; \quad (\rho_k)'(0) = 0; \\
(\rho_k)'(x_1) &> 0 \quad \text{for } x_1 > 0; \quad (\rho_k)'(x_1) < 0 \quad \text{for } x_1 < 0.
\end{align*}$$  
(4.24)

Meanwhile, the left part of the boundary $\partial U^*_k \cap B_{r_0}$ is the graph of a smooth, uniformly convex function $\rho_k^*$ in direction $e_1$, that is

$$\Gamma_k^* =: \{y_1 = \rho_k^*(y_2)\} \cap B_{r_0},$$  
(4.25)

where $\rho_k^*$ is defined on $\mathcal{J}_k^*$ that is the projection of $\partial U^*_k \cap B_{r_0}$ on the $y_2$-axis. In our construction, $0 \in \mathcal{J}_k^*$ is an interior point of $\mathcal{J}_k^*$ for all $k \geq 1$. The function $\rho_k^*$ satisfies

$$\begin{align*}
\rho_k^*(0) &= 0; \quad (\rho_k^*)'(0) = 0; \\
(\rho_k^*)'(y_2) &> 0 \quad \text{for } y_2 > 0; \quad (\rho_k^*)'(y_2) < 0 \quad \text{for } y_2 < 0.
\end{align*}$$  
(4.26)

Let $u_k$ be the potential function for the optimal transport from $(U_k, 1)$ to $(U^*_k, g_k)$, where the density $g_k = \frac{\nu_k}{\det D^2 u_k}$ is a constant. Subtracting a constant we have $u_k(0) = 0$. Since $U_k^*$ is convex, we can extend $u_k$ to $\mathbb{R}^2$ by

$$u_k(x) := \sup\{\ell(x) : \ell \text{ is affine, } \ell \leq u_k \text{ in } U_k, \nabla \ell \in U_k^*\} \quad \text{for } x \in \mathbb{R}^2.$$  
(4.27)

Since $u_0(0) = 0$, by the uniqueness of potential functions we have $u_k \to u_0$ uniformly in $B_{r_0}(0)$ for a different $r_0 > 0$ small, (in this subsection, the constant $r_0$ may change from line to line but they have a uniform positive lower bound independent of $k$). By the uniqueness of optimal mapping $Du_0$, we also have $Du_k \to Du_0$ in $B_{r_0}(0)$.

By (2.18) in Remark 2.4, $u_k$ are uniformly $C^{1,\delta}$ smooth in $\overline{U_k} \cap B_{r_0}(0)$, and the extended $u_k \in C^1(B_{r_0}(0))$, by the convexity of $U_k^*$. Let $v_k$ be the dual of $u_k$, namely the potential function for the optimal transport from $(U_k^*, g_k)$ to $(U_k, 1)$. Similarly to (4.27), let

$$\hat{v}_k(y) := \sup\{\ell(y) : \ell \text{ is affine, } \ell \leq v_k \text{ in } Du_k(B_{r_0}(0)), \nabla \ell \in B_{r_0}(0)\} \quad \text{for } y \in \mathbb{R}^2.$$  
(4.28)

Then $\hat{v}_k = v_k$ in $U^*_k \cap B_{r_0}(0)$, $\hat{v}_k$ is uniformly $C^{1,\delta}$ smooth in $U^*_k \cap B_{r_0}(0)$, and $\hat{v}_k \in C^1(B_{r_0}(0))$, for a different $r_0$.

We have constructed the sequences of functions $\rho_k, \rho^*_k$ defined on $\mathcal{J}_k, \mathcal{J}_k^*$, respectively, with the properties (4.24) and (4.26). Moreover, $[0, r_0/2] \subset \mathcal{J}_k, \mathcal{J}_k^*$. The next lemma shows that $u_k$ satisfies the obliqueness condition and is smooth on $\Gamma_k$ for all $k$.

**Lemma 4.2.** (i) For each $k \geq 1$, we have

$$\nu_k(x) \cdot \nu^*_k(Du_k(x)) > 0 \quad \forall x \in \Gamma_k,$$  
(4.29)

where $\nu_k$ and $\nu^*_k$ are the unit inner normals of the domains $\{x_2 > \rho_k(x_1)\} \cap B_{r_0}$ and $\{y_1 > \rho^*_k(y_2)\} \cap B_{r_0}$, respectively.

(ii) For each $k \geq 1$, $u_k$ is smooth, locally uniformly convex, and $\det D^2 u_k$ is a positive constant in $B_{r_0}(0) \cap \{x_2 \geq \rho_k(x_1)\}$ (up to the boundary $\Gamma_k$).

**Proof.** The proof is as follows (for a fixed $k$):
By definition, the obliqueness holds at $p$ if and only if $\nu(p) \cdot \nu^*(D\nu_k(p)) > 0$.

(a) $(ii)$ actually follows from $(i)$. In fact, if the obliqueness holds, we can apply the argument in §4.2 to obtain the smoothness in $(ii)$. So, it suffices to prove $(i)$.

(b) Suppose to the contrary that the obliqueness fails for $u_k$ at a point $p_0 = (t_0, \rho_k(t_0)) \in \Gamma_k$ with $t_0 < r_0$. We claim that there is an interval $(\tilde{t}, \tilde{t} + \varepsilon)$, where $\tilde{t} \geq t_0$, such that the obliqueness fails for $u_k$ at $\tilde{p} = (\tilde{t}, \rho_k(\tilde{t}))$, but holds at all points on $\Gamma_k \cap \{\tilde{t} < x_1 < \tilde{t} + \varepsilon\}$ (Lemma 4.3). Then by the regularity in §6, $u_k$ is smooth up to the boundary $\Gamma_k \cap \{\tilde{t} < x_1 < \tilde{t} + \varepsilon\}$.

(c) Regard $\tilde{p}$ as the origin. Applying the argument in §4.4 to $u_k$ on $\Gamma_k \cap \{\tilde{t} < x_1 < \tilde{t} + \varepsilon\}$, we reach a contradiction at the point $\tilde{p}$. It implies that the obliqueness must hold for $u_k$ at all points on $\Gamma_k$, namely $(i)$ is proved.

To apply the argument in §4.4 to $u_k$, we do not need to carry out the blow-up for $u_k$ in §4.2 as $\det D^2 u_k$ is already a positive constant. We also point out that the $C^{2,\alpha}$ regularity proved in §6 is localised. That is, let $x_0 \in \partial \Omega$ and $y_0 = Du(x_0) \in \partial \Omega^*$. If $\partial \Omega$ and $\partial \Omega^*$ are $C^{1,1}$ and convex near $x_0$ and $y_0$, respectively, then $u$ is $C^{2,\alpha}$ smooth near $x_0$.

Therefore it remains to verify the claim in (b), which will be proved in Lemma 4.3 below. □

Remark 4.4. Instead of using Lemma 4.3 below and the regularity in §6, we can also use Caffarelli’s localised boundary $C^{2,\alpha}$ regularity to conclude that $u_k$ is smooth up to $\Gamma_k$. However, the proof in [7] is rather involved. Here we use Lemma 4.3 to make our proof self-contained. This lemma will also be used in §5 for high dimensions.

Lemma 4.3. Assume that the obliqueness fails for $u_k$ at a point $p_0 = (t_0, \rho_k(t_0)) \in \Gamma_k$ (for any fixed $k \geq 1$). Then there is a boundary point $\tilde{p} = (\tilde{t}, \rho_k(\tilde{t})) \in \Gamma_k$ with $\tilde{t} \geq t_0$, such that the obliqueness fails at $\tilde{p}$, but holds at all points $p \in \Gamma_k \cap \{\tilde{t} < x_1 < \tilde{t} + \varepsilon\}$ for a constant $\varepsilon > 0$.

Proof. Since the obliqueness fails at $p_0$, by a change of coordinates and subtracting a linear function to $u_k$, we can assume that $p_0 = 0$, $Du_k(0) = 0$, and also

$$0 \in \partial U_k, \quad U_k \subset \{x_2 > 0\}; \quad 0 \in \partial U_k^*, \quad U_k^* \subset \{y_1 > 0\}.$$ 

We still use $\rho_k, \rho_k^*$ to denote the boundary $\partial U_k, \partial U_k^*$ near 0, which are smooth, uniformly convex near 0. Correspondingly, we have $\rho_k(0) = 0, \rho_k \geq 0,$ and $\rho_k^*(0) = 0, \rho_k^* \geq 0$.

For any boundary point $p = (t, \rho_k(t)) \in \Gamma_k$, let $\zeta = \zeta(p)$ denote the unit tangential vector of $\Gamma_k$ at $p$. Denote by $\tau_1$ and $\eta$ the tangential and normal vectors of $\partial S[h[u_k]]$ at $p$, respectively, where $h = u_k(p)$. Let $\alpha$ be the angle between $\eta$ and $\nu$, and let $\beta$ be the angle between $\nu^*$ and $\tau_1$ (see Figure 3). Then,

$$(4.30) \quad \nu(p) \cdot \nu^*(Du_k(p)) = \cos\left(\frac{\pi}{2} - \alpha - \beta\right) \geq \frac{1}{2}(\alpha + \beta).$$

By definition, the obliqueness holds at $p$ if and only if $\nu(p) \cdot \nu^*(Du_k(p)) > 0$. 

We claim that \( \beta > 0 \). Indeed, since \( \rho_k^* \) is smooth, \( (\rho_k^*)'(0) = 0 \). Noticing that \( p^* = Du_k(p) \), we see that \( \overline{op}^* \) is parallel to \( \eta \). Hence by the uniform convexity of \( \partial U_k^* \),

\[
\beta = \arctan \left( \left| (\rho_k^*)'(p^*_2) \right| \right) - \arctan \left( \frac{p_1^*}{|p_2^*|} \right) > 0.
\]

\((\zeta, \zeta^*)\) are tangential vectors, and \( \nu, \nu^* \) are inner unit normals, of \( \partial U_k \), \( \partial U_k^* \) at \( p, p^* = Du_k(p) \), respectively. \( \eta \) and \( \tau_1 \) are the normal and tangential vectors of \( S_h[u_k] \) at \( p \). Hence we have \( \cos \alpha = \langle \eta, \nu \rangle = \langle \tau_1, \zeta \rangle \).

Hence to prove Lemma 4.3, we just need to prove that there is an interval of \( \Gamma_k \) in which \( \alpha \geq 0 \). From Figure 3, it is easy to see that \( \alpha \geq 0 \) if and only if \( \partial \zeta u_k(p) \geq 0 \). Since \( u_k(0) = 0 \) and \( u_k(x) > 0 \ \forall \ x \in \Gamma_k \setminus \{0\} \), there exists a point \( \hat{p} = (\hat{t}, \rho_k(\hat{t})) \in \Gamma_k \) with \( \hat{t} > 0 \) such that \( \partial \zeta u_k(\hat{p}) > 0 \). Hence there exists \( \varepsilon > 0 \) such that

\[
(4.31) \quad \partial \zeta u_k(p) > 0 \quad \text{for} \quad p = (t, \rho_k(t)) \in \Gamma_k, \ t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon).
\]

That means \( \alpha \geq 0 \) and so the obliqueness holds for \( p = (t, \rho_k(t)) \in \Gamma_k \) with \( t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon) \).

Let

\[
(4.32) \quad \tilde{t} = \inf \{ t : \text{ the obliqueness holds at } p = (s, \rho_k(s)) \in \Gamma_k \text{ for all } s \in (t, t + \varepsilon) \}.
\]

By the \( C^{1, \delta} \)-regularity \([5]\), \( \tilde{t} \geq 0 \) is well defined and the obliqueness holds for \( t \in (\tilde{t} - \varepsilon, \tilde{t} + \varepsilon) \) but not at \( \tilde{p} = (\tilde{t}, \rho_k(\tilde{t})) \). This finishes the proof. \( \square \)

4.4. **Contradiction.** Now we derive a contradiction with the help of the approximation sequence \( u_k \). By our construction, \( 0 \in \Gamma_k \) and \( 0 \in \Gamma_k^* \), where \( \Gamma_k \) and \( \Gamma_k^* \) are the curves given in (4.23) and (4.25). First we point out that

\[
(4.33) \quad \partial_{x_2} u_k(p) < 0 \quad \text{for} \quad p \in \Gamma_k \cap \{ x_1 > 0 \}.
\]

Indeed, for any given point \( p \in \Gamma_k \cap \{ x_1 > 0 \} \), the inner normal \( \nu \) of \( U_k \) at \( p \) lies in the second quadrant. Let \( \nu^* \) be the inner normal of \( U_k^* \) at \( p^* = Du_k(p) \in \Gamma_k^* \). By (4.29) we have \( \nu \cdot \nu^* > 0 \). By (4.24), it implies \( p^* \in \Gamma_k^* \cap \{ y_2 < 0 \} \). Hence we have (4.33).

**Lemma 4.4.** We have

\[
(4.34) \quad \partial_{x_1 x_2} u_k(t, \rho_k(t)) < 0 \quad \text{for} \quad t \in (0, r_0).
\]
Proof. By the boundary condition $Du_k(\partial U_k) = \partial U_k^*$, we have 

$$\partial_{x_1} u_k(t, \rho(t)) = \rho_k'(\partial_{x_2} u_k(t, \rho(t))).$$

Differentiating the above equation we have 

$$\partial_{x_1 x_1} u_k + \partial_{x_1 x_2} u_k \rho_k' = (\rho_k')' (\partial_{x_1 x_2} u_k + \partial_{x_2 x_2} u_k \rho_k').$$

Namely, 

$$(4.35) \quad \partial_{x_1 x_1} u_k - \partial_{x_2 x_2} u_k \rho_k' = (\rho_k' - \rho_k') \partial_{x_1 x_2} u_k.$$ 

By the above approximation, $D^2 u_k$ is positive definite and continuous on the boundary. For $t > 0$ small, by (4.33) we have $\rho_k' > 0$ and $(\rho_k')' < 0$. Hence, the LHS of (4.35) is always positive, and the coefficient on the RHS, $(\rho_k' - \rho_k') < 0$. Therefore, we obtain (4.34). □

Introduce the function 

$$(4.36) \quad w_k(x) := \partial_{x_1} u_k + u_k - x_1 \partial_{x_1} u_k.$$ 

By Lemma 4.2 det $D^2 u_k$ is a positive constant. Hence $w_k$ satisfies the equation 

$$(4.37) \quad M^{ij} D_{ij} w_k = 0$$ 

in $B_{r_0} \cap U_k$, where $\{M^{ij}\}$ is the cofactor matrix of $D^2 u_k$.

Corollary 4.5. There exists a constant $\epsilon_0 > 0$ independent of $k$ such that for $t \in (0, \epsilon_0)$, the function $w_k(t, \cdot)$ has an interior local minimum.

Proof. From Lemma 4.4 we have $\partial_{x_1 x_2} u_k(t, \rho(t)) < 0$ for $t \in (0, r_0)$. Hence, by (4.33) 

$$(4.38) \quad \partial_{x_2} w_k(t, \rho(t)) = (1 - t) \partial_{x_1 x_2} u_k(t, \rho(t)) + \partial_{x_2} u_k(t, \rho(t)) < 0$$

for all $t \in (0, r_0)$.

On the other hand, by our assumption $U_k^* \subseteq \{y_1 > 0\}$, $\partial_{x_1} u_k \geq 0$. Hence by the strict convexity of $u_0$ and $u_k \rightarrow u_0$ uniformly in $B_{r_0}$, there exists a constant $\delta_0 > 0$ such that 

$$w_k(x) = u_k(x) + (1 - x_1) \partial_{x_1} u_k > u_k(x) \geq \delta_0$$

when $x \in \partial B_{r_0} \cap U_k$. Hence there is a small $\epsilon_0 > 0$ such that for any $t \in (0, \epsilon_0)$, $w_k(t, \cdot)$ has a local minimum that is smaller than the boundary value $w_k(t, \rho_k(t))$. □

Hence we can define the following function 

$$(4.39) \quad w_k(t) = \inf\{w_k(t, x_2) : x_2 > \rho_k(t), (t, x_2) \in U_k\}, \quad t \in (0, \epsilon_0).$$ 

By Corollary 4.5 the infimum cannot be attained on $\partial U_k \cap \{0 < x_1 < \epsilon_0\}$, and $w_k$ is well defined for all $t \in (0, \epsilon_0)$.

Lemma 4.5. $w_k$ is concave for $t \in (0, \epsilon_0)$. 
Proof. If \( w_k \) is not concave, then there exist constants \( 0 < r_1 < r_2 < c_0 \) and an affine function \( L(t) \) such that \( w_k(r_i) = L(r_i) \) for \( i = 1, 2 \), and the set \( \{ t \in (r_1, r_2) : w_k(t) < L(t) \} \neq \emptyset \). Extend \( L \) to an affine function \( \hat{L} \) defined in \( \mathbb{R}^2 \), such that \( \hat{L}(t,s) = L(t) \). Denote

\[
D_\varepsilon = \{ x \in U_k : x_1 \in (r_1, r_2), \text{ and } w_k(x) < \hat{L}(x) - \varepsilon \}.
\]

By our definition of \( w_k \) and Corollary 4.5 we can choose \( \varepsilon > 0 \) such that

\[
(4.40) \quad \emptyset \neq D_\varepsilon \subseteq U_k.
\]

Indeed, by our choice of \( \hat{L}, D_\varepsilon \varepsilon = \emptyset \). Let \( \varepsilon_1 = \sup \{ \varepsilon : D_\varepsilon \neq \emptyset \} \). Then if \( \varepsilon < \varepsilon_1 \) and sufficiently close to \( \varepsilon_1 \), we have \( D_\varepsilon \neq \emptyset \). By Corollary 4.5 the infimum in (4.39) is attained at an interior point. Hence we also have \( D_\varepsilon \subseteq U_k \).

By equation (4.37) and the boundary condition \( w_k = \hat{L} - \varepsilon \) on \( \partial D_\varepsilon \), we apply the maximum principle to \( w_k \) in \( D_\varepsilon \) and conclude that \( w_k = \hat{L} - \varepsilon \) in \( D_\varepsilon \). However, by our definition of \( D_\varepsilon \), we have \( w_k < \hat{L} - \varepsilon \) in \( D_\varepsilon \). We reach a contradiction. \( \square \)

Note that for any fixed \( t \in (0, \varepsilon_0) \), the minimum point in Corollary 4.5 may not be unique. In this case, the domain \( D_\varepsilon \) in the above proof may contain more than one connected component. But each component is compactly contained in \( U_k \). Hence we can still use the maximum principle.

We have now established Lemmas 4.4 and 4.5 for the approximation sequence \( u_k \). Denote

\[
w_0 = \partial x_1 u_0 + u_0 - x_1 \partial x_1 u_0 = \lim_{k \to \infty} w_k.
\]

Let

\[
\underline{w_0}(t) = \inf \{ w_0(t,x_2) : x_2 > \rho_0(t), (t,x_2) \in \Omega_0 \}, \quad t \in (0,\varepsilon_0).
\]

From (4.20), we have \( \underline{w_0}(t) \to 0 \) as \( t \to 0 \). More precisely,

\[
\underline{w_0}(t) \leq C \sigma(2t) \quad \text{for } t > 0 \text{ small}.
\]

To see this, let \( q = (2t,q_2) \) such that \( \underline{w_0}(2t) = u_0(q) \), where \( \underline{w_0} \) is defined in Corollary 4.3. By (4.20), we have \( u_0(q) \leq C t \sigma(2t) \). Let \( \hat{q} = (t,q_2) \). Since \( \partial x_1 u_0 \geq 0 \) (i.e. \( \Omega_0 \subset \{ y_1 > 0 \} \)), we have

\[
 u_0(\hat{q}) \leq u_0(q) \leq C t \sigma(2t).
\]

By the convexity of \( u_0 \), one also has

\[
 \partial x_1 u_0(\hat{q}) \leq \frac{u_0(q) - u_0(\hat{q})}{t} \leq C \sigma(2t).
\]

Hence,

\[
(4.41) \quad \underline{w_0}(t) = \inf \underline{w_0}(t, \cdot) \leq \underline{w_0}(\hat{q}) = (1-t) \partial x_1 u_0(\hat{q}) + u_0(\hat{q}) \leq C \sigma(2t).
\]

Recall that \( \sigma(t) = |t|^{1+\gamma} \). Hence by the concavity of \( w_0 \) (Lemma 4.5 taking the limit \( k \to \infty \)), we conclude that \( \underline{w_0}(t) \leq 0 \) for all \( t \in (0,\varepsilon_0) \). On the other hand, since \( \partial x_1 u_0 \geq 0 \), by the strictly convexity of \( u_0 \), we have

\[
w_0(x) = u_0(x) + (1-x_1) \partial x_1 u_0(x) \geq u_0(x) > 0 \quad \text{if } x \in \Omega_0, \ x_1 \in (0,\varepsilon_0).
\]
It implies that $w_0(t) > 0$ when $t > 0$. We reach a contradiction. Therefore the uniform obliqueness in dimension two is proved. 

5. Uniform obliqueness in high dimensions

In this section we prove the uniform obliqueness in high dimensions. Suppose the domains $\Omega$ and $\Omega^*$ are bounded, convex, with $C^{1,1}$ boundaries, and $f \in C^0(\overline{\Omega})$. The proof of obliqueness uses the ideas in §4 and also the following:

(i) Suppose the obliqueness fails at the point 0 $\in \partial \Omega$ and 0 = $Du(0)$ $\in \partial \Omega^*$. To understand the geometry of the sub-level set $S_h[u](0)$, we prove that $S_h[u](0)$ is contained in a cuboid $Q$ with volume $|Q| \leq C|S_h[u](0)|$. Then by a rescaling of the coordinates such that $Q$ becomes a cube, $S_h[u](0)$ changes accordingly to a convex set with good shape. In particular, we show that the boundary $\partial \Omega$ becomes flat in directions orthogonal to the inner normals $\nu$ and $\nu^*$ as $h \to 0$. This property enables us to employ the techniques in §4.

(ii) As in dimension two, we need to construct a smooth approximation sequence to derive the contradiction. The construction in high dimensions is more complicated.

Similarly as in Remark 4.1 (iii), the constants in this section depends on $n, \lambda, \Omega, \Omega^*$ (diameters and $C^{1,1}$ norm of $\Omega, \Omega^*$). They also depend on $k$ in the argument on the approximation sequence $\{u_k\}$, but are independent of $h$ and $u$ for small $h > 0$. The continuity of $f$ is used only in the blow-up process. The tangential $C^{1,\alpha}$ regularity of §3 is used only in §5.1 for $u$ and in the approximation sequence $\{u_k\}$, where the domains are $C^{1,1}$ smooth. We do not need the tangential $C^{1,\alpha}$ regularity for the limit $u_0$.

5.1. The limit profile. To prove the uniform obliqueness, by the $C^{1,\delta}$ regularity [3], we may suppose to the contrary that 0 $\in \partial \Omega$, $u(0) = 0$, $Du(0) = 0 \in \partial \Omega^*$, and locally

\begin{align}
\{x_n > C|x'|^2\} & \subset \Omega \subset \{x_n > 0\}, \quad \text{where } x' = (x_1, \cdots, x_{n-1}), \\
\{y_1 > C|\tilde{y}|^2\} & \subset \Omega^* \subset \{y_1 > 0\}, \quad \text{where } \tilde{y} = (y_2, \cdots, y_n).
\end{align}

Corresponding to properties (i) and (ii) in §4.1 similarly we have

(i) $u_1 > 0$ in $\Omega$ and $v_n > 0$ in $\Omega^*$;

(ii) if $x \in S_h[u]$, then $x - te_1 \in S_h[u] \ \forall \ t > 0$, provided $x - te_1 \in \Omega$.

First we prove a lemma that strengthens Lemma 4.1.

Lemma 5.1. For any given point $p \in \partial S_h[u] \cap \Omega$, let $\mathcal{H}$ be the tangential plane of $S_h[u]$ at $p$. Assume $\mathcal{H} = \{x \in \mathbb{R}^n : x \cdot \gamma = a\}$ for a unit vector $\gamma$, where $a$ is a positive constant. Then

\begin{equation}
 x \cdot \gamma \geq -Ca \quad \forall \ x \in S_h[u],
\end{equation}

where $C > 0$ is a constant independent of $h$ ($h > 0$ small) and $u$. 
Proof. Denote \( b = \inf \{ x \cdot \gamma : x \in S_h[u] \} \) and denote \( \mathcal{H}_1 = \{ x \in \mathbb{R}^n : x \cdot \gamma = b \} \). Suppose to the contrary that the ratio \( \frac{a}{|b|} \to 0 \) as \( h \to 0 \). Let \( \mathcal{A}_h \) be an affine transformation such that \( \mathcal{A}_h(S_h[u]) \sim B_1(z) \) for some point \( z \). Note that the transform does not change the ratio \( \frac{a}{|b|} \).

Accordingly let \( u_h(x) = u(A_h^{-1} x)/h \). By Caffarelli’s geometric decay estimate, similarly as in (4.11), \( u_h \) is locally uniformly bounded and sub-converges to a limit \( u_0 \) as \( h \to 0 \), and \( u_0(0) = \lim_{h \to 0} u_h(0) = 0 \).

On the other hand, by passing to a subsequence, we have
\[
\mathcal{A}_h(\mathcal{H}) \to \mathcal{H}^*, \quad \mathcal{A}_h(\mathcal{H}_1) \to \mathcal{H}_1^*
\]
as \( h \to 0 \). Observe that
\[
\frac{\text{dist}(0, \mathcal{H}^*)}{\text{dist}(0, \mathcal{H}_1^*)} = \lim_{h \to 0} \frac{a}{|b|} = 0.
\]
Hence \( \mathcal{H}^* \) passes through the origin. It implies that \( u_0 \geq 1 \) on one side of \( \mathcal{H}^* \), which is a contradiction since \( u_0 \) is continuous and \( u_0(0) = 0 \). \( \square \)

Similarly as in (4.3), let \( q, \xi \in \partial S_h(0) \) such that
\[
q_1 = \langle q, e_1 \rangle = \sup \{ \langle x, e_1 \rangle : x \in S_h(0) \},
\]
\[
\xi_1 = \langle \xi, e_1 \rangle = \inf \{ \langle x, e_1 \rangle : x \in S_h(0) \}.
\]
Obviously \( q_1 > 0 \) and \( \xi_1 < 0 \). We point out that \( q \) is an interior point of \( \Omega \). To see this, let \( \ell^* = \{ t e_1 : t \in (0, t_0) \} \) be a line segment in \( \Omega^* \). Let \( \ell = (D_u)^{-1}(\ell^*) \subset \Omega \) be the pre-image of \( \ell^* \). Then \( q \) is the intersection of \( \ell \) with \( \partial S_h[u](0) \). From Lemma 5.1 we have

Corollary 5.1. We have
\[
q_1 \geq \delta_0 |\xi_1|
\]
for some constant \( \delta_0 > 0 \) independent of \( h \) and \( u \).

Having obtained the balance property (5.5), one would expect a decay estimate like Corollary 4.1 as in dimension two. But to obtain the decay estimate in high dimensions, one needs a bound of \( S_h[u] \) along \( x_2, \ldots, x_{n-1} \) directions.

Denote \( S_{h,1}^c[u] = S_h^c[u] \cap \{ x_1 = 0 \} \) and denote \( B'_r(0) \) the ball of radius \( r \) in \( \mathbb{R}^{n-2} = \text{span}(e_2, \ldots, e_{n-1}) \). We have the following estimates for \( S_h^c[u] \) in \( x_2, \ldots, x_{n-1} \) directions.

Lemma 5.2. For any given \( \varepsilon > 0 \) small, we have
\[
B'_{C^{-1} h^{1/2+\varepsilon}}(0) \subset S_{h,1}^c[u] \cap \{ x_n = 0 \} \subset B'_{Ch^{2-\varepsilon}}(0).
\]
In particular, for any \( i = 2, \ldots, n-1 \) and any \( x \in S_h^c[u] \),
\[
|x_i| = |x \cdot e_i| \leq C h^{\frac{1}{2} - \varepsilon},
\]
provided \( h > 0 \) is sufficiently small, where the constant \( C = C(\varepsilon) \) is independent of \( h \) and \( u \).
Proof. By the tangential $C^{1,1-\varepsilon}$ regularity for $u$, we have $S_{h,1}^{c}[u] \cap \{x_{n} = 0\} \supset B'_{C^{-1}h^{1/2+\varepsilon}}(0)$, which is the first inclusion of (5.6).

For any points $x \in S_{h}^{c}[u]$ and $y \in S_{h}^{c}[v]$, by (2.14) we have $|x \cdot y| \leq Ch$. By the tangential $C^{1,1-\varepsilon}$ regularity for $v$, we have $C^{-1}h^{1/2+\varepsilon}e_{i} \in S_{h}^{c}[v], (i = 2, \cdots, n-1)$. Hence $(h^{1/2+\varepsilon}e_{i}) \cdot x \leq Ch \forall x \in S_{h}^{c}[u]$, namely

\begin{equation}
|x_{i}| = |x \cdot e_{i}| \leq Ch^{1/2-\varepsilon} \forall x \in S_{h}^{c}[u].
\end{equation}

We obtain (5.7) and the second inclusion of (5.6).

From (5.7) and Remark 2.3, we also obtain

\begin{equation}
|x \cdot e_{i}| \leq Ch^{1/2-\varepsilon} \forall x \in S_{h}^{c}[u] \text{ and } i = 2, \cdots, n-1.
\end{equation}

The same estimate is true for $S_{h}^{c}[v], S_{h}^{c}[v]$ as well. From (5.9) we will derive two consequences: one is a decomposition (5.12), the other one is the decay estimate in Corollary 5.3.

We shall first derive the decomposition (5.12). Note that in high dimensions, there may be a small portion of $S_{h}^{c}[u] \cap \{x_{1} > 0\}$, whose projection on the plane $\{x_{1} = 0\}$ is not contained in $S_{h,1} := S_{h}^{c}[u] \cap \{x_{1} = 0\}$. Nevertheless, we have the following inclusion.

**Corollary 5.2.** Let $S_{h,1}[u]$ be the projection of $S_{h}^{c}[u] \cap \{x_{1} > 0\}$ on the plane $\{x_{1} = 0\}$. Then

\begin{equation}
S_{h,1}[u] \subset (1 + o(1))S_{h,1}[u]
\end{equation}

as $h \to 0$, where the dilation is with respect to $z$, the centre of $S_{h,1}[u]$.

**Proof.** Let $\tilde{x} = (0, x''', x_{n}) \in S_{h,1}^{c}$, where $x''' = (x_{2}, \cdots, x_{m-1})$. By definition of $S_{h,1}^{c}[u]$, there is $t > 0$ such that $x = (t, x''', x_{n}) \in S_{h}^{c}[u]$ and $u(x) < h$. If $\tilde{x} \in \Omega$, since $u_{1} > 0$, one must have $u(\tilde{x}) < h$, and thus $\tilde{x} \in S_{h,1}[u]$.

If $\tilde{x} \notin \Omega$, let $z$ be the centre of $S_{h,1}[u]$. By the $C^{1,\delta}$ regularity of $u$, we have $z_{n} = z \cdot e_{n} \geq Ch^{1/1+\delta}$. From (5.9), $|x'''| \leq Ch^{1/2-\varepsilon}$. Since $\tilde{x} \notin \Omega$ and $\partial \Omega \in C^{1,1}$,

\begin{equation}
x_{n} \leq C|x'''|^{2} \leq Ch^{1-2\varepsilon} = o(h^{1+\delta}) = o(z_{n}).
\end{equation}

Let $\ell$ be the segment connecting $z$ and $\tilde{x}$, and let $y$ be the intersection of $\ell$ and $\partial \Omega$. Since $u(z) < h$, $u(\tilde{x}) < h$, we have $u(y) < h$ and thus $y \in S_{h,1}[u]$. Write $y = (0, y'', y_{n})$. By (5.9) again, we have $|y''| \leq Ch^{1/2-\varepsilon}$. Then since $y \in \partial \Omega$, one has $y_{n} < Ch^{1-2\varepsilon} \ll z_{n}$. Therefore,

\[
\lim_{h \to 0} \frac{|z\tilde{x}|}{|zy|} = \lim_{h \to 0} \frac{|z_{n} - x_{n}|}{|z_{n} - y_{n}|} = 1,
\]

from which one easily obtains (5.10). \qed

Next we estimate the size of $S_{h}^{c}[u] \cap \{x_{1} < 0\}$. We introduce a cone with vertex $q$ (see (5.3)) and passing through $S_{h,1}[u]$, namely

\[
V = \{q + t(x - q) : x \in S_{h,1}[u], t \geq 0\}.
\]
By the convexity of $S_h[u]$, we have $S_h[u] \cap \{x_1 < 0\} \subset \mathcal{V}$. Hence by Corollaries 5.1 and 5.2
\begin{equation}
(5.12)
S_h[u] \subset [\xi_1, q_1] \times \beta S_{h,1}[u]
\end{equation}
for a constant $\beta > 0$ independent of $h$, where $\beta S_{h,1}[u]$ denotes the $\beta$-dilation with respect to the centre of $S_{h,1}[u]$. Indeed, by performing an affine transform in the $e_2, \cdots, e_n$ directions, we may assume $S_{h,1}[u]$ is normalised. Then by Corollary 5.2 we have that $q' = (0, q_2, \cdots, q_n) \in (1 + o(1)) S_{h,1}[u]$. Hence, by Corollary 5.1 and using the fact that $S_h[u] \cap \{x_1 < 0\} \subset \mathcal{V}$ we have
\begin{equation}
(5.12)
\end{equation}

**Remark 5.1.** Replacing the $e_1$-direction by the $e_n$-direction, the same argument for (5.12) also applies to $S_h[v]$ and yields
\begin{equation}
(5.13)
S_h[v] \subset [\xi^*_n, q^*_n] \times \beta^* S_{h,n}[v]
\end{equation}
for a constant $\beta^* > 0$ independent of $h$, where $\xi^*_n, q^* \in \partial S_h[v]$ is defined analogously to (5.3) (where $e_1$ is replaced by $e_n$) and $S_{h,n}[v] := S_h[v] \cap \{y_n = 0\}$.

As another consequence of (5.9), we next derive a decay estimate analogous to Corollary 4.1.

**Lemma 5.3.** For any given $\varepsilon > 0$ small, we have $q_1 \geq h^{\frac{1}{n} + \varepsilon}$, provided $h$ is sufficiently small.

**Proof.** For any $x \in S_h[u]$, by (5.9) we have $|x_i| \leq Ch^{\frac{1}{n} - \varepsilon}$ for $i = 2, \cdots, n - 1$. By Corollary 5.1 we also have $q_1 \geq C|x_1|$. Since $u_1 > 0$, we see that sup\{\(e_n \cdot x : x \in S_h[u]\)\} must be attained on the boundary $\partial \Omega$. Since $\partial \Omega \in C^{1,1}$, we have
\[
 x_n \leq C \sum_{i=1}^{n-1} x_i^2 \leq C(q_1^2 + h^{1-2\varepsilon}) \quad \forall x \in S_h[u] \cap \partial \Omega.
\]
From (2.12), the volume $|S_h[u]| \approx h^\frac{n}{2}$. Hence
\[
h^{\frac{n}{2}} \approx |S_h[u]| \leq Cq_1(q_1^2 + h^{1-2\varepsilon}) h^\frac{n-2}{2} - (n-2)\varepsilon.
\]
Therefore $q_1 \geq h^{\frac{1}{n} + \varepsilon}$ for any given $\varepsilon > 0$ small. \hfill \Box

From Lemma 5.3 similarly to (4.12), we have the following corollary.

**Corollary 5.3.** For $t > 0$ small, denote
\[
 u(t) = \inf\{u(t, x_2, \cdots, x_n) : (t, x_2, \cdots, x_n) \in \Omega\},
\]
\[
\partial_1 u(t) = \inf\{\partial_1 u(t, x_2, \cdots, x_n) : (t, x_2, \cdots, x_n) \in \Omega\}.
\]
We have the asymptotic behaviour
\begin{equation}
(5.14)
\frac{u(t)}{t^3} \leq Ct^{3-\varepsilon},
\end{equation}
\begin{equation}
\frac{\partial_1 u(t)}{t^{2-\varepsilon}} \leq C t^{2-\varepsilon}
\end{equation}
for $t > 0$ small, where $\varepsilon > 0$ is any given small constant.
Remark 5.2. By Lemma 2.2 and Lemma 5.3 we have
\[ s \geq q_1 \geq h^{\frac{1}{2} + \epsilon}, \quad \text{where } s := \sup \{ x \cdot e_1 : x \in S_{bh}^c[u] \}. \]

Let \( d = \sup \{ x \cdot e_n : x \in S_{bh}^c[u], x_1 = 0 \} \). Then by Lemma 5.2 and (2.11), we have
\[ h^{-\frac{1}{2}} \approx |S_{bh}^c[u]| \lesssim h^{-\frac{1}{2} + 2(1+\epsilon) s_d}, \]
which implies \( d \lesssim h^{\frac{1}{2} - \epsilon} \). By Lemma 2.2 again, we obtain that
\[ (5.15) \quad \sup \{ x \cdot e_n : x \in S_{h,1}[u] \} \lesssim h^{\frac{1}{2} - \epsilon}. \]

In order to bound the sub-level set \( S_h[u] \) by a cuboid, we need to further decompose \( S_{h,1}[u] \) in (5.12) along \( e_n \) direction. Denote
\[ S_{h,1,0}^c = S_{h,1}^c[u] \cap \{ x_n = 0 \}, \]
where \( S_{h,1}^c[u] = S_h^c[u] \cap \{ x_1 = 0 \} \) was introduced above.

Lemma 5.4. Let \( P_h \) be the projection of \( S_{h,1}[u] \) on \( \{ x_n = 0 \} \). Then we have
\[ (5.16) \quad P_h \subset \beta S_{h,1,0} \]
for a constant \( \beta \) independent of \( h \) and \( u \).

Proof. Let \( e \in \text{span}\{e_2, \ldots, e_{n-1}\} \) be a unit vector, and denote \( r_e := \sup \{ t : te \in S_{h,1,0}^c[u] \} \). To prove (5.16), it suffices to show that
\[ (5.17) \quad |x \cdot e| \leq \beta r_e \quad \forall x \in S_h[u] \cap \text{span}\{e_n, e\} \]
for all unit vectors \( e \in \text{span}\{e_2, \ldots, e_{n-1}\} \). By Lemma 5.2 we have \( r_e \geq C^{-1} h^{\frac{1}{2} + \epsilon} \).

Given a unit vector \( e \in \text{span}\{e_2, \ldots, e_{n-1}\} \), and a point \( p \in S_h[u] \cap \text{span}\{e_n, e\} \), up to a rotation of coordinates, we assume \( e = e_2 \) and \( p = (0, p_2, 0, \ldots, 0, p_n) \) with \( p_2 > 0 \). By Remark 5.2, we have
\[ (5.18) \quad p_n \leq Ch^{\frac{1}{2} - \epsilon} \]
for \( \epsilon \) as small as we want. In order to prove (5.17), it suffices to show that \( p_2 \leq \beta r_{e_2} \). If \( p_2 \ll h^{\frac{1}{2} + \epsilon} \), we readily have \( p_2 = p \cdot e_2 \leq r_{e_2} \). Hence it suffices to consider the case
\[ (5.19) \quad p_2 \geq Ch^{\frac{1}{2} + \epsilon} \gg p_n \quad \text{(and thus } p_2 \approx |p| \text{ for } h \text{ small}). \]

By Remark 2.2 we have
\[ (5.20) \quad |y \cdot p| \leq Ch \quad \forall y \in S_h[v]. \]
In particular, when \( y \in S_{h,n}[v] := S_h[v] \cap \{ y_n = 0 \} \), \( y \cdot p = y_2 p_2 \). Thus we obtain
\[ \sup \{ |y_2| : y \in S_{h,n}[v] \} \leq C \frac{h}{p_2}. \]
By Remark 5.1, \( \sup \{ |y_2| : y \in S_h[v] \} \leq \beta^* \sup \{ |y_2| : y \in S_{h,n}[v] \} \). Therefore, we obtain
\[ (5.21) \quad \sup \{ |y_2| : y \in S_h[v] \} \leq C \beta^* \frac{h}{p_2}. \]
By the definition of \( r_{e_2} \), we have \( r_{e_2} e_2 \in \partial S_h^c[u] \). Hence by (2.15), there exists \( z^* \in \partial S_h^c[u] \) such that
\[
(5.22) \quad z^* \cdot (r_{e_2} e_2) \geq C^{-1} h.
\]
By (5.21) and Remark 2.3 we have
\[
(5.23) \quad z^* \cdot e_2 \leq \sup \{ y \cdot e_2 : y \in S_h^c[v] \} \leq C^\alpha \frac{h}{p_2}.
\]
Hence from (5.22) and (5.23), we obtain the desired inequality
\[
(5.24) \quad r_{e_2} \geq C^{-1} h z^* \cdot e_2 \geq \frac{1}{C^\beta p_2},
\]
for a different constant \( C > 0 \). This finishes the proof with \( \beta = C^\beta \).
\[\square\]

Thanks to (5.12) and Lemma 5.4 we can now show that \( S_h[u] \) is contained in a cuboid as follows. Denote
\[
(5.25) \quad d_n = \sup \{ e_n \cdot x : x \in S_{h,1}^c[u] \}
\]
to be the height of \( S_h^c[u] \) on the section \( \{ x_1 = 0 \} \). We have
\[
(5.26) \quad d_n \geq \sup \{ e_n \cdot x : x \in S_{b^{-1}h,1}^c[u] \} \geq \sup \{ e_n \cdot x : x \in S_{b^{-1}h}^c[u] \} \geq \sup \{ e_n \cdot x : x \in S_h[u] \},
\]
where the first inequality is due to Lemma 2.2, the second inequalities follows from (5.12), and the last inequality is due to the convexity of \( u \), which implies that \( S_h[u] \subset bS_{b^{-1}h}^c[u] \), \( (b > 1) \).

Let \( \tilde{q} \in \partial S_h^c[u] \) be the point such that
\[
(5.27) \quad \tilde{q}_1 = \tilde{q} \cdot e_1 = \sup \{ x \cdot e_1 : x \in S_h^c[u] \}.
\]
By Remark 2.3 we have that
\[
(5.28) \quad \sup \{ |x \cdot e_1| : x \in S_h[u] \} \lesssim \tilde{q}_1.
\]
Let
\[
(5.29) \quad \mathcal{R}_h = [-\tilde{q}_1, \tilde{q}_1] \times E_h' \times [-d_n, d_n]
\]
be a cuboid, where \( E_h' \subset \mathbb{R}^{n-2} \) is an ellipsoid centred at 0 such that \( E_h' \sim S_{h,1,0}^c[u] \cap \{ x_n = 0 \} \). By Lemma 5.4 (5.12) and (5.28), we have
\[
(5.30) \quad S_h[u] \subset C \mathcal{R}_h
\]
for some constant \( C \) independent of \( h \). Moreover, by (5.25) and (5.27) the volume
\[
|S_h^c[u]| \gtrsim |E_h'| d_n \tilde{q}_1 \gtrsim |\mathcal{R}_h|.
\]
Hence,
\[
(5.31) \quad C^{-1} |\mathcal{R}_h| \leq |S_h^c[u]| \approx |S_h[u]| \leq C |\mathcal{R}_h|.
\]
Now we make a linear transform \( T = T_2 \circ T_1 \) such that the sub-level set \( S_h[u] \) has a good shape, where \( T_1 \) is a linear transform normalising \( E'_h \) to the unit ball in \( \mathbb{R}^{n-2} = \text{span}(e_2, \cdots, e_{n-1}) \) while leaving \( x_1 \) and \( x_n \) unchanged; and \( T_2 \) is given by
\[
T_2 : \begin{cases}
    \bar{x}_1 = x_1/\bar{q}_1, & \bar{x}_n = x_n/d_n, \\
    \bar{x}_i = x_i & \text{for } 2 \leq i \leq n - 1.
\end{cases}
\]
It is easy to see that \( T_2 \circ T_1 = T_1 \circ T_2 \).

After the transform \( T \), the set \( T(S_h[u]) \) is contained in the cube \( D = [-C, C]^n \), and the volume \( |T(S_h[u])| \geq \delta_0 \) for a positive constant \( \delta_0 \) independent of \( h \) and \( u \). Hence \( T(S_h[u]) \) has a good shape. By Lemma 2.2 and Lemma 2.3 we see that \( T(S'_h[u]) \) also has a good shape. By rescaling back and using Lemma 2.2 again, we have
\[
C^{-1} \mathcal{R}_n \cap \Omega \subset S_h[u] \subset C \mathcal{R}_n
\]
for a constant \( C \) independent of \( h \).

Having made the transform \( T \) (note that \( T = T_h \) depends on \( h \)), accordingly we also make the change \( u_h(x) = u(T^{-1}x)/h \).

Let \( u(t) \) be the function introduced in Corollary 5.3. Similarly to (4.17), we choose a sequence \( \{t_j\} \to 0 \) such that
\[
u(t) \leq 2 \left( \frac{t}{t_j} \right)^{3-\varepsilon} \quad \forall t \in (0, t_j),
\]
where \( \varepsilon > 0 \) is the small constant in (5.14). Denote \( T_j = T_{h_j}, u_j = u_{h_j} \), where \( h_j = u(t_j) \).

Similarly as in §4 by passing to a subsequence, \( \Omega_{h_j} := T_j(\Omega) \) converges to a limit \( \Omega_0 \) as \( j \to \infty \), and \( \Omega_0 \) is an unbounded convex domain in \( \mathbb{R}^n \). Also, \( u_j \) converges to a limit \( u_0 \) as \( j \to \infty \), which satisfies the Monge-Ampère equation (4.18) in \( \mathbb{R}^n \).

By the proof of Corollary 4.3, \( u_0 \) satisfies the asymptotic behaviours (5.14). Moreover, \( u_0 \) is strictly convex and \( C^{1,\alpha} \) regular in \( B_k \cap \overline{\Omega}_0 \) for any \( k > 0 \), and \( u_0 \in C^1(\mathbb{R}^n) \), (see Remark 4.3).

Thanks to the above cuboid decomposition (5.29), we can prove that the boundary \( \partial \Omega_0 \) is flat in \( x_2, \cdots, x_{n-1} \) directions.

**Lemma 5.5.** Assume \( \Omega_h \) sub-converges as \( h \to 0 \) to a convex domain \( \Omega_0 \), locally in the sense of Hausdorff. Then \( \Omega_0 = \mathbb{R}^{n-2} \times \omega_0 \), where \( \omega_0 \) is a convex set in the 2-dim space \( \text{span}\{e_1, e_n\} \).

**Proof.** By the global \( C^{1,\delta} \) regularity [5], we have \( d_n \geq C h^{1+\delta} \) for some \( \delta > 0 \), where \( d_n = d_n(h) \) is given in (5.25) and (5.26). By (5.6), we have
\[
S'_{h,1,0}[u] = S'_{h,1}[u] \cap \{x_n = 0\} \subset B'_{Ch^{1/2-\varepsilon}}(0).
\]
Hence by the \( C^{1,1} \) regularity of the boundary \( \partial \Omega \), the height of \( S'_{h,1}[u] \cap \partial \Omega \) satisfies
\[
d_{n,h}(h) := \sup \{e_n : x \in S'_{h,1}[u] \cap \partial \Omega\} \leq Ch^{1-2\varepsilon},
\]
where \( \varepsilon > 0 \) is fixed but can be as small as we want. Hence,
\[
\frac{d_{n,h}(h^{1-\delta/2})}{d_{n}(h)} \to 0 \quad \text{as } h \to 0.
\]
Note that by (5.6),
\[
\text{diam}(S_{h^1,0}^{c}[u]) = o(1)\text{diam}(S_{h^{1-\delta/2};1,0}^{c}[u]) \quad \text{when } h \to 0.
\]
The above formula implies that \( T_h(\partial \Omega \cap S_{h^{1-\delta/2};1}^{c}[u]) \) is becoming flat and so its limit is the plane \( \text{span}(e_2, \ldots, e_{n-1}) \). Namely \( T_h(\partial \Omega \cap B_R(0) \cap \{x_1 = 0\}) \) becomes flat as \( h \to 0, \forall R > 0 \).

It is well known that if a convex set \( G \) contains a straight line \( \ell \), then \( G \) can be expressed as a product \( G = G' \times \ell \). The lemma is proved. \( \square \)

Denote \( T^* = \frac{1}{\ell}(T')^{-1} \), the dual affine transformation for \( v \), where \( T' \) is the transpose of \( T \). Similarly, we denote \( v_h(y) = \frac{1}{\ell}v((T^*)^{-1}y) \), and \( \Omega_h^* = T^*(\Omega^*) \). Applying Lemma 5.5 to \( v \), we see that \( \Omega_h^* \) converges to an unbounded convex domain \( \Omega_0^* = \mathbb{R}^{n-2} \times \omega_0^* \), where \( \omega_0^* \subset \text{span}\{e_1, e_n\} \) is a convex set. And \( v_{h_j} \) converges to a convex function \( v_0 \) locally uniformly, which satisfies the equation (4.22) correspondingly.

5.2. Smooth approximation. First we construct a smooth approximation sequence \( \{u_k\} \) converging to \( u_0 \) in a small neighbourhood of the origin similarly as in §4.3.

Let \( V = B_r \cap \Omega_0^* \) for a small constant \( r \), and let \( U = Dv_0(V) \). Then \( B_{r_0} \cap \Omega_0 \subset U \) for a small constant \( r_0 > 0 \). By Lemma 5.5 we approximate \( U, V \) by a sequence of bounded smooth sets \( U_k, U_k^* \) respectively such that
\[
\begin{align*}
& a) \ 0 \in \partial U_k, \ 0 \in \partial U_k^* \ \forall \ k \geq 1, \\
& b) \ U_k \subset \{x_n > 0\}, \ U_k^* \subset \{y_1 > 0\} \ \forall \ k \geq 1, \\
& c) \ U_k \to U, \ U_k^* \to V, \ \text{as } k \to \infty, \ \text{in Hausdorff’s sense}, \\
& d) \ \exists \text{smooth, uniformly convex sets } \hat{\omega}_k, \hat{\omega}_k^* \subset \text{span}\{e_1, e_n\} \text{ such that} \\
& \quad U_k \cap B_{r_0} = (\hat{\omega}_k \times \mathbb{R}^{n-2}) \cap B_{r_0}, \\
& \quad U_k^* \cap B_{r_0} = (\hat{\omega}_k^* \times \mathbb{R}^{n-2}) \cap B_{r_0}
\end{align*}
\]
for a different, smaller constant \( r_0 > 0 \).

In this subsection, the constant \( r_0 \) may change from line to line but they have a uniform positive lower bound independent of \( k \).

Moreover, we assume the lower part of the boundary \( \partial U_k \cap B_{r_0} \) is the graph of a smooth, uniformly convex function \( \rho_k \) in direction \( e_n \), that is
\[
(5.35) \quad \Gamma_k =: \{x \in \mathbb{R}^n : x_n = \rho_k(x_1)\} \cap B_{r_0},
\]
where \( \rho_k \) is defined on \( J_k \) that is the projection of \( \partial U_k \cap B_{r_0} \) on the \( x_1 \)-axis with \([0, \frac{1}{2}r_0] \subset J_k \). The function \( \rho_k \) satisfies (4.24) as in dimension two.
Similarly, the left part of the boundary \( \partial U_k^* \cap B_{r_0} \) is the graph of a smooth, uniformly convex function \( \rho_k^* \) in direction \( e_1 \), that is

\[
\Gamma_k^* = \{ y \in \mathbb{R}^n : y_1 = \rho_k^*(y_n) \} \cap B_{r_0},
\]

where \( \rho_k^* \) is defined on \( J_k^* \) that is the projection of \( \partial U_k^* \cap B_{r_0} \) on the \( y_n \)-axis containing the origin inside. The function \( \rho_k^* \) satisfies (4.26) as in dimension two.

Let \( u_k \) be the potential function for the optimal transport from \( (U_k, g_k) \) to \( (U_k^*, g_k) \), where the density \( g_k = \frac{\lvert \nabla u \rvert}{\lvert \nabla u_k \rvert} \) is a constant. Subtracting a constant we have \( u_k(0) = 0 \). Since \( U_k^* \) is convex, as before we can extend \( u_k \) to \( \mathbb{R}^n \) by

\[
u_k(x) := \sup \{ \ell(x) : \ell \text{ affine, } \ell \leq u_k \text{ in } U_k, \nabla \ell \in U_k^* \} \quad \text{for } x \in \mathbb{R}^n.
\]

Since \( u_0 \in C^1(\mathbb{R}^n) \) and \( u_0(0) = 0 \), by the uniqueness of potential functions, \( u_k \to u_0 \) uniformly in \( B_{r_0}(0) \) for a different \( r_0 > 0 \) small. In addition we have \( \| u_k - u_0 \|_{C^1(B_{r_0}/2)} \to 0 \) as \( k \to \infty \).

**Lemma 5.6.**

(i) For each \( k \geq 1 \), we have

\[
u_k(x) \cdot \nu_k^*(Du_k(x)) > 0 \quad \forall x \in \Gamma_k,
\]

where \( \nu_k \) and \( \nu_k^* \) are the unit inner normals of the domains \( \{ x \in \mathbb{R}^n : x_n > \rho_k(x_1) \} \cap B_{r_0} \) and \( \{ y \in \mathbb{R}^n : y_1 > \rho_k^*(y_n) \} \cap B_{r_0} \), respectively.

(ii) For each \( k \geq 1 \), \( u_k \) is smooth, locally uniformly convex, and \( \det D^2 u_k \) is a positive constant in \( B_{r_0}(0) \cap \{ x_n > \rho_k(x_1) \} \) (up to the boundary \( \Gamma_k \)).

**Proof.** Similarly as in Lemma 4.2 (ii) follows from (i). That is, if the obliqueness \( 5.37 \) holds, by \( \xi \) we have the smoothness of \( u_k \) in (ii). The proof of (i) will be given in the following two lemmas. \( \square \)

**Lemma 5.7.** For any fixed \( k \geq 1 \), assume that \( u_k(0) = 0 \) and \( Du_k(0) = 0 \). Then for any \( x = (t, x'', \rho_k(t)) \in \Gamma_k \) with \( t \leq |x''|^2/3 \), we have

\[
u_k(x) \approx |x''|^2.
\]

**Proof.** Since the boundaries \( \Gamma_k, \Gamma_k^* \) are flat in \( x'' = (x_2, \ldots, x_{n-1}) \), and \( \rho_k, \rho_k^* \) are smooth and uniformly convex, we can choose \( \varepsilon = 0 \) in Lemmas 5.2 and 5.3 (similarly as in Corollary 3.1). From (5.33), we have

\[
C^{-1}Q \cap U_k \subset S_{h[u_k]} \subset CQ \quad \text{with} \quad Q := [-\tilde{q}_1, \tilde{q}_1] \times B_{h^{1/2}}(0) \times [-d_n, d_n],
\]

where \( \tilde{q}, d_n \) are defined in (5.25), (5.27) respectively. Similarly to (5.3), let \( q, \xi \in \partial S_{h[u_k]} \) be the points on \( \partial S_{h[u_k]} \) such that

\[
q_1 = \langle q, e_1 \rangle = \sup \{ \langle x, e_1 \rangle : x \in S_{h[u_k]} \},
\]

\[
\xi_n = \langle \xi, e_n \rangle = \sup \{ \langle x, e_n \rangle : x \in S_{h[u_k]} \}.
\]

Since \( Du_k(U_k) \subset U_k^* \subset \{ y_1 \geq 0 \} \), \( u_k \) is increasing in \( e_1 \) direction. Hence \( \xi \) can be chosen on \( \Gamma_k \). Then by (5.39), \( C^{-1}q_1 \leq q_1 \leq C\tilde{q}_1 \) (see also Remark 2.3) and \( C^{-1}d_n < \xi_n \leq Cd_n \).
Since $\rho_k \in C^2$, we have $\xi_n \leq C_1 \xi_1^2$. By the uniformly convexity of $\rho_k$, we have $q_n \geq C_2 q_1^2$. By Corollary 5.1, we then obtain

$$\tilde{C}_1 q_1^2 \geq \xi_n \geq q_n \geq \tilde{C}_2 q_1^2,$$

thus $d_n \approx \tilde{q}_1^2$. By the fact that $|S_h[u_k]| \approx h^{n/2}$, we then have $\tilde{q}_1 \approx h^{1/3}$. Hence, when $x = (t, x'', \rho_k(t)) \in \Gamma_k$ with $t \leq |x''|^2/3$, we obtain (5.38).

\[\square\]

Remark 5.3. In dimension two, we can use Caffarelli’s regularity to conclude that $u_k$ is $C^{2,\alpha}$ smooth up to $\Gamma_k$ (Remark 4.4). In high dimensions, for the proof of Lemma 1.1 in [5,3], we have to choose the domains $U_k, U_k^*$ which are flat in $e_2, \cdots, e_{n-1}$ directions. Hence we cannot use Caffarelli’s boundary $C^{2,\alpha}$ regularity [4] directly. But with the help of Lemma 5.7, one can modify Caffarelli’s argument to prove that $u_k$ is smooth up to $\Gamma_k$. In fact, Lemma 5.7 implies that the solution $u_k$ (for any fixed $k$) behaves nicely in $x''$, and so the directions $x''$ wouldn’t cause us new troubles. Here we will not use the argument in [4] but provide an independent proof of (5.37), based on Lemma 4.3.

Lemma 5.8. For any fixed $k \geq 1$, (5.37) holds.

Proof. Suppose to the contrary that (5.37) fails at a point $\hat{x} \in \Gamma_k$, that is

$$\nu_k(\hat{x}) \cdot \nu_k^*(Du_k(\hat{x})) = 0.$$

By a change of coordinates and subtracting a linear function, we can assume $\hat{x} = 0$, $u_k(0) = 0$ and $Du_k(0) = 0$ such that the hypotheses of Lemma 5.7 are satisfied.

Consider the restriction of $\partial U_k$ in span$\{e_1, e_n\}$. For a boundary point $p = (t, 0, \cdots, 0, \rho_k(t)) \in \Gamma_k$, let $h = u_k(p)$. Denote by $\hat{\eta}$ the unit inner normal of $S_h[u_k]$ at $p$, and $\eta$ the projection of $\hat{\eta}$ on span$\{e_1, e_n\}$. Denote by $\nu$ the unit inner normal of $\partial U_k$ at $p$, and $\alpha$ the angle between $\eta$ and $\nu$ (see Figure 3). Note that by $d)$ in our domain constructions, $\partial U_k, \partial U_k^*$ are flat along $e_2, \cdots, e_{n-1}$ directions near the origin. Hence the normal vectors $\nu(p), \nu^*(p^*)$ and the tangential vectors $\zeta(p), \zeta^*(p^*)$ are all in the 2-dim plane span$\{e_1, e_n\}$, where $p^* = Du_k(p)$. By the strict convexity of $u_k$ and the proof of Lemma 4.3, there exists a small $t_0 > 0$ such that $\alpha \geq 0$ at $p_0 = (t_0, 0, \rho_k(t_0))$, which implies the obliqueness holds at $p_0$. Hence by the $C^{1,\delta}$ regularity, there is a small constant $\epsilon_0 > 0$ such that

$$\nu_k(p) \cdot \nu_k^*(Du_k(p)) > 0, \quad \forall p = (t, p'', \rho_k(t)) \quad \text{with} \quad t \in (t_0 - \epsilon_0, t_0) \quad \text{and} \quad |p''| \leq \epsilon_0.$$

For any $t \in (0, t_0)$, denote

$$C_t = \{(x_1, x'', 0) : t < x_1 < t_0, \; |x''| < \epsilon_0(x_1 - t)\},$$

which is an $(n-1)$-dimensional round open cone in the hyperplane $\{x_n = 0\}$ with vertex at $(t, 0, 0)$ and base on the disk $\{(t_0, x'', 0) : |x''| \leq \epsilon_0(t_0 - t)\}$.

Let

$$\tilde{t} = \inf\{t : \quad \text{the obliqueness holds} \; \forall \; p \in \Gamma_k, \; \text{provided} \; (p - p_ne_n) \in C_t\},$$
where \( p - p_n e_n \) is the projection of \( p \) on the plane \( \{ x_n = 0 \} \). Obviously \( \tilde{t} \geq 0 \), and there is a point \( (\tilde{x}_1, \tilde{x}''_1, 0) \in \partial \mathcal{C}_\tilde{t} \), with \( \tilde{x}_1 < t_0 \), such that the obliqueness fails at \( \tilde{x} = (\tilde{x}_1, \tilde{x}''_1, \rho_k(\tilde{x}_1)) \) but it holds in \( \{ (x_1, x'', \rho_k(x_1)) : (x_1, x'', 0) \in \mathcal{C}_t \} \).

Therefore by a change of coordinates, we can assume that the obliqueness fails at the origin but it holds for all \( x \in \Gamma_k \) whose projection \( (x - x_n e_n) \in \mathcal{C}_0 \), where \( \mathcal{C}_0 = \mathcal{C}_t |_{t=0} \) was the cone defined above. By subtracting a linear function, we again have \( u_k(0) = 0 \), \( Du_k(0) = 0 \).

Now, we introduce the auxiliary function
\[
(5.40) \quad w = \partial_1 u_k + K(u_k - \frac{n}{2} x_1 \partial_1 u_k),
\]
where \( K \) is a large constant to be determined. Let \( w \) be the function given by
\[
(5.41) \quad w(t) = \inf\{ w(t, x_2, \cdots, x_n) : (t, x_2, \cdots, x_n) \in U_k \cap B_{r_0} \}
\]
for \( t > 0 \) small.

We claim that the infimum in (5.41) cannot be attained on the boundary \( \partial(U_k \cap B_{r_0}) \) for \( t > 0 \) small. Indeed, as in the proof of Corollary 4.5 there exists a small constant \( \tau_0 > 0 \) such that for \( t \in (0, \tau_0) \), the infimum in (5.41) cannot be attained on \( U_k \cap \partial B_{r_0} \). Hence, it suffices to prove the claim over the part \( \Gamma_k = \partial U_k \cap B_{r_0} \). For any given \( 0 < t < \min\{\tau_0, t_0\} \), denote
\[
\Gamma_k \cap \{ x_1 = t \} = \partial_{in}(t) \cup \partial_{out}(t),
\]
where \( \partial_{in}(t) \) denotes the boundary points \( x \in \Gamma_k \) whose projection \( (x - x_n e_n) \in \mathcal{C}_0 \), while \( \partial_{out}(t) = \Gamma_k \cap \{ x_1 = t \} \) - \( \partial_{in}(t) \).

By our choice of the cone \( \mathcal{C}_0 \), the obliqueness holds at all points \( x \in \partial_{in}(t) \). Hence \( u_k \) is smooth up to the boundary \( \partial_{in}(t) \). Similarly to Lemma 4.4, we then infer that \( \partial_{in} u_k < 0 \) and \( \partial_{in} w < 0 \) on \( \partial_{in}(t) \). Hence the infimum in (5.41) cannot be attained on \( \partial_{in}(t) \).

Next we show that the infimum cannot be attained on \( \partial_{out}(t) \) either. On the one hand, since \( U_k, U^*_k \) satisfy the condition (5.1) of (5.1) by Corollary 5.3 and (5.38), when \( t > 0 \) sufficiently small, we have
\[
u_k(t) = \inf\{ u_k(t, x_2, \cdots, x_n) : (t, x_2, \cdots, x_n) \in U_k \cap B_{r_0} \} \leq C t^3.
\]
Note that due to the flatness of \( \partial U_k, \partial U^*_k \) in \( e_2, \cdots, e_{n-1} \) directions, we can choose \( \varepsilon = 0 \) in (5.14). Hence similarly to (4.41) we obtain
\[
(5.42) \quad w(t) \leq C t^2 + CK^3.
\]
On the other hand, for any point \( x = (t, x'', \rho_k(t)) \in \partial_{out}(t) \), we have \( |x''| > \varepsilon_0 t \). Hence by (5.38), we have \( u_k(x) \geq c_0 |x''|^2 > c_0 \varepsilon_0^2 t^2 \). Since \( \partial_1 u_k \geq 0 \), we then obtain that for \( t < 2/(nK)^{-1} \) small,
\[
(5.43) \quad w(x) > Ku_k(x) > Kc_0 \varepsilon_0^2 t^2 \quad \forall x \in \partial_{out}(t).
\]
Therefore, by choosing \( K \) sufficiently large, from (5.42) and (5.43) one can see that \( w(x) > w(t) \) for all \( x \in \partial_{out}(t) \), namely the infimum in (5.41) cannot be attained on \( \partial_{out}(t) \).
Once the claim is proved, we can show that $w$ is concave and reach a contradiction by a similar argument as in §4.4. The proof of Lemma 5.8 is finished. □

With the preparations in §5.1 and §5.2, we are now in position to prove Lemma 1.1.

5.3. Proof of Lemma 1.1. By our construction, $0 \in \Gamma_k$, $0 \in \Gamma^*_k$, and (4.24), (4.26) hold for $\rho_k, \rho_k^*$, respectively. Similarly to (4.33) we have

$\partial_{x^n} u_k(t, x'' \rho_k(t)) < 0$ for $(t, x'' \rho_k(t)) \in \Gamma_k \cap \{ x_1 > 0 \}$ near the origin.

Now, we can prove Lemma 1.1 in a similar way as in §4, which is outlined as follows:

(i) By the computation as in Lemma 4.4, we have

$\partial_{x^n} u_k(x) < 0 \quad \forall \quad x \in \partial U_k \cap \{ x \in B_{r_0} : x_1 > 0 \}.$

(ii) Define the auxiliary function

$w_k(x) := \partial_{x_1} u_k + u_k - \frac{n}{2} x_1 \partial_{x_1} u_k$

that satisfies

$M^{ij} D_{ij} w_k = 0$

in $B_{r_0} \cap U_k$, where $\{ M^{ij} \}$ is the cofactor matrix of $D^2 u_k$.

(iii) By (5.45), similarly to Corollary 4.5, we see that there exists a constant $\epsilon_0$ independent of $k$, such that for any given $t \in (0, \epsilon_0)$, the function $w_k(t, \cdot)$ has an interior local minimum. Hence we can define

$\bar{w}_k(t) = \inf \{ w_k(t, x_2, \ldots, x_n) : (t, x_2, \ldots, x_n) \in U_k \}$

for $t \in (0, \epsilon_0)$. Note that by (5.45), the infimum cannot be attained on $\partial U_k \cap B_{\epsilon_0}$.

(iv) Similarly to Lemma 4.5, we can prove that $\bar{w}_k$ is concave in $(0, \epsilon_0)$.

(v) By letting $k \to \infty$, we have now obtained the function $\bar{w}$ that satisfies

a) $\bar{w} \geq 0$, and $\bar{w}(t) \to 0$ as $t \to 0$.

b) $\bar{w}$ is concave.

c) $\bar{w}(t) \leq C t^{2-\epsilon}$ for $t > 0$ small (by (5.14) that also holds for $u_0$ with the same constants).

Therefore $\bar{w} \equiv 0$, and we reach a contradiction analogous to that of dimension two.

This completes the proof of Lemma 1.1.

□

6. Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2 namely the global $C^{2,\alpha}$ and $W^{2,p}$ estimates for the problem (1.1), (1.2). In [7], Caffarelli established the global $C^{2,\alpha'}$ estimate for some $\alpha' \in (0, \alpha)$. The exponent $\alpha'$ can be improved to $\alpha$, using the global $C^{2,\alpha}$ estimate for the
Dirichlet problem in [31, 27]. Here we give a direct proof. We also obtain the continuity of $D^2 u$ for Dini continuous and positive $f$.

Assume that $0 \in \partial \Omega$, $u(0) = 0$ and $Du(0) = 0 \in \partial \Omega^*$. By the uniform obliqueness (1.5) and a linear transform of the coordinates, we may assume that locally

$$\partial \Omega = \{x_n = \rho(x')\},$$
$$\partial \Omega^* = \{y_n = \rho^*(y')\},$$

where $\rho, \rho^* \in C^{1,1}$ satisfying $\rho, \rho^* \geq 0$ and $\rho(0) = \rho^*(0) = 0$. Note that this expression implies that $u_{x_n} > 0$ in $\Omega$.

Extending $u$ to $\mathbb{R}^n$ as at the beginning of §2. Denote

$$D_{h,a}^+ = \{x \in \mathbb{R}^n : u(x) < h\} \cap \{x_n > a\},$$

where $a \geq 0$ is a small constant. Let $a_h$ be the smallest number such that $D_{h,a_h}^+ \subset \Omega$, but $D_{h,a_h-\varepsilon}^+ \nsubseteq \Omega$ for any $\varepsilon > 0$. One can see that $a_h \to 0$ as $h \to 0$. For simplicity, we denote $D_{h,a_h}^+$ by $D_h^+$.

Let $D_h^-$ be the reflection of $D_h^+$ with respect to the plane $\{x_n = a_h\}$, and $D_h := D_h^+ \cup D_h^-$. Since $D_n u \geq 0$, the domain $D_h$ is convex. Moreover, $D_h$ shrinks to the origin as $h \to 0$.

**Lemma 6.1.** The shape of $D_h$ is close to a ball of radius $h^{1/2}$, in the sense that

$$B_{C^{-1}h^{1/2}+\varepsilon}(x_h) \subset D_h \subset B_{Ch^{1/2}-\varepsilon}(x_h)$$

for any given small $\varepsilon > 0$, where the centre $x_h = a_h e_n$.

**Proof.** First we show the centred sub-level set $S_{h}^c[u]$ is close to a ball of radius $h^{1/2}$, namely

$$B_{C^{-1}h^{1/2}+\varepsilon}(0) \subset S_{h}^c[u] \subset B_{Ch^{1/2}-\varepsilon}(0)$$

for any small $\varepsilon > 0$. Indeed, from Lemma 3.1

$$B_{C^{-1}h^{1/2}+\varepsilon}(0) \cap \{x_n = 0\} \subset S_{h}^c[u] \cap \{x_n = 0\}$$

for any small $\varepsilon > 0$. Similarly, this also holds for centred sub-level sets $S_{h}^c[v]$, for the dual potential $v$.

Let $e'$ be a unit tangential vector with $e' \perp e_n$, and let $t > 0$ such that $te' \in \partial S_{h}^c[u]$. Applying (6.4) to $v$, we have $t \geq C^{-1}h^{1/2}+\varepsilon$. For any $x \in S_{h}^c[u]$, from (2.14)

$$|x \cdot e'| \leq C \frac{h}{t} \leq Ch^{1/2}-\varepsilon,$$

which implies that $S_{h}^c[u]$ is contained in a vertical cylinder centred at the origin with radius $r' \leq Ch^{1/2}-\varepsilon$, for any given small $\varepsilon > 0$. Hence we have proved that

$$B_{C^{-1}h^{1/2}+\varepsilon}(0) \cap \{x_n = 0\} \subset S_{h}^c[u] \cap \{x_n = 0\} \subset B_{Ch^{1/2}-\varepsilon}(0).$$
Let \( r_n e_n \in \partial S_h^c[u] \) and let \( S_h^c[u] \) be the projection of \( S^c_h[u] \) on \( \{ x_n = 0 \} \). By the convexity of \( u \) and noticing that \( u_n \geq 0 \),

\[
    r_n |S_h^c[u]| = C \text{Vol}(S^c_h[u]) = Ch^{2 \frac{n}{2}},
\]

where for the last equality we use \((2.11)\). By \((6.5)\), \( |S_h^c[u]| \leq Ch^{(n-1)\left(\frac{1}{2} - \epsilon\right)} \), and thus we obtain \((6.6)\)

\[
    r_n \geq Ch^{\frac{1}{2} + \epsilon}
\]

for a different small \( \epsilon > 0 \). \((6.6)\) is also true for the dual centred sub-level set \( S^c_h[v] \). By \((6.4)\), we have, similarly to \((6.5)\),

\[
    |x \cdot e_n| \leq Ch^{\frac{1}{2} - \epsilon} \quad \forall \; x \in S^c_h[u].
\]

Combining \((6.4)\)–\((6.7)\) we obtain \((6.3)\).

Next we show that there exist two constants \( b_1, b_2 \), independent of \( u \) and \( h \), such that

\[
    S_{b_1 h}^c \subset D_{h,0}^c \subset S_{b_2 h}^c,
\]

where \( S_{b_1 h}^c = S^c_h[u] \cap \{ x_n > 0 \} \), and \( D_{h,0}^c \) is given in \((6.1)\). The first inclusion can be proved similarly as that of \((2.6)\). Indeed, for any \( x \in \partial S^c_h \cap \{ x_n = 0 \} \), by \((6.3)\) and since \( \partial \Omega \in C^{1,1} \), we have \( \text{dist}(x, \partial S_h^c \cap \Omega) \leq Ch^{1-\epsilon} \). By \((6.3)\) we also have \( |Du| \leq Ch^{\frac{1}{2} - \epsilon} \) in \( S^c_h \). Hence

\[
    u(x) \geq Ch \quad \forall \; x \in \partial S^c_h \cap \{ x_n = 0 \}
\]

and \((6.8)\) follows. The second inclusion of \((6.8)\) also follows from \((2.6)\).

We are ready to prove \((6.2)\). Combining \((6.3)\) and \((6.8)\), there exists a constant \( C \) independent of \( u \) and \( h \) such that

\[
    B_{C^{-1}h^{\frac{1}{2} + \epsilon}}(0) \cap \{ x_n > 0 \} \subset D_{h,0}^c \subset B_{C h^{\frac{1}{2} - \epsilon}}(0) \cap \{ x_n > 0 \}
\]

for any given small \( \epsilon > 0 \). Since \( \partial \Omega \in C^{1,1} \), by the definition of \( a_h \) (after \((6.1)\)), one has \( a_h < Ch^{1-\epsilon} \) for some \( \epsilon > 0 \) as small as we want. Recall that \( D_h = D_{h,a_h}^+ \cup D_{h,a_h}^- \) and \( D_{h,a_h}^+ \) is an even extension of \( D_{h,a_h}^+ \) with respect to \( \{ x_n = a_h \} \). We obtain \((6.2)\) from \((6.9)\). \( \square \)

By \((6.2)\), we infer that

**Corollary 6.1.** For any given small \( \epsilon > 0 \), \( u \in C^{1,1-\epsilon}(\Omega) \).

From the above \( C^{1,\alpha} \) regularity for all \( \alpha < 1 \), we can prove the global \( W^{2,p} \) regularity (Theorem 1.2). As mentioned in the introduction, the global \( W^{2,p} \) estimate for the problem \((1.1)-(1.2)\) was obtained in [10], using the estimates of Caffarelli in [7]. Hence the domains are the uniform convexity with \( C^2 \) boundaries in [10]. By our estimates above, we can remove the uniform convexity condition and reduce the smoothness assumption on domains.

**Proof of Theorem 1.2.** The proof is based on the estimate \((6.2)\) and uses the argument of Savin [28], see also [10]. For completeness, let us outline the main steps. Given \( x \in \Omega \), let \( \bar{h}_x \) be the maximal value of \( h \) such that \( S_h[u](x) \subset \Omega \), i.e.

\[
    \bar{h}_x := \max\{ h \geq 0 : S_h(x) \subset \Omega \}.
\]
Let $T$ be a unimodular linear transform such that $T(S_{h_x}[u](x)) \sim B_{\tilde{h}^{1/2}_x}$. By (6.2), one has $\|T\|, \|T^{-1}\| \lesssim \tilde{h}^{-\varepsilon}_x$, for any small $\varepsilon > 0$. Hence, when $\tilde{h}_x$ is small

$$S_{\tilde{h}_x}[u](x) \subset D_{C\tilde{h}^{1/2}_x} := \{ z \in \overline{\Omega} : \text{dist}(z, \partial \Omega) \leq C\tilde{h}^{1/2}_x \}$$

for any small $\varepsilon > 0$.

By subtracting a linear function we may assume that $x = 0$, $u(0) = 0$ and $Du(0) = 0$. Let

$$\tilde{u}(x) = \frac{1}{h}u(h^{1/2}T^{-1}x),$$

where $x \in \tilde{S}_1(0) = \tilde{h}^{-1/2}T(S_h(0))$. The interior $W^{2,p}$ estimate for $\tilde{u}$ in $\tilde{S}_1(0)$ gives $\int_{\tilde{S}_1/2(0)} \|D^2\tilde{u}\|^p dx \leq C$; hence by rescaling

$$\int_{\tilde{S}_{h}/2(0)} \|D^2u\|^p dx = \int_{\tilde{S}_{h}/2(0)} \|T'D^2\tilde{u}T\|^p \tilde{h}^{n/2} dx \leq C\tilde{h}^{n-2\varepsilon p}.$$  

From Vitali covering lemma, there exists a sequence of disjoint sub-level sets $\{S_{\delta\tilde{h}_i}(x_i)\}$, $\tilde{h}_i = \tilde{h}_{x_i}$ such that $\Omega \subset \bigcup_{i=1}^{\infty} S_{\tilde{h}_i/2}(x_i)$, where $\delta > 0$ is a small constant, (see [28 Lemma 2.3]). Then

$$\int_{\Omega} \|D^2u\|^p dx \leq \sum_i \int_{S_{\tilde{h}/2}(x_i)} \|D^2u\|^p dx.$$

Note that it suffices to consider those $\tilde{h}_i \leq c_1$ for a small constant $c_1 > 0$. We can adopt the argument of Savin [28]: Consider the family $\mathcal{F}_d$ of those $S_{\tilde{h}_i/2}(x_i)$ satisfying

$$d/2 < \tilde{h}_i \leq d$$

for a constant $d \leq c_1$. By (6.11) and (2.12)

$$\int_{S_{\tilde{h}/2}(x_i)} \|D^2u\|^p dx \leq Cd^{-2\varepsilon p}|S_{\delta h_i}(x_i)|.$$  

By (6.10) and Vitali covering lemma, $S_{\delta h_i}(x_i) \subset D_{Cd^{1/2} - \varepsilon}$ and are disjoint. Hence, we have

$$\sum_{i \in \mathcal{F}_d} \int_{S_{\tilde{h}_i/2}(x_i)} \|D^2u\|^p dx \leq C d^{2\varepsilon - 2\varepsilon p} \leq C d^{2\varepsilon - 3\varepsilon p}.$$  

Let $d = c_1 2^{-k}$, $k = 0, 1, 2, \ldots$, and by adding the sequence of inequalities, we obtain

$$\int_{\Omega} \|D^2u\|^p dx \leq C + C_1 \sum_{k=0}^{\infty} 2^{-k(\frac{1}{2} - 3\varepsilon p)}.$$  

For any $p \geq 1$, as $\varepsilon$ is arbitrarily small so that $3\varepsilon p < \frac{1}{4}$, therefore the series is convergent. \hfill $\square$

Now we continue with the proof of global $C^{2,\alpha}$ estimate. Let $w$ be the solution of

$$\det D^2w = 1 \quad \text{in } D_h,$$

$$w = h \quad \text{on } \partial D_h.$$  

Denote by $\tilde{u}$ the even extension of $u$ with respect to $\{x_n = 0\}$, namely

$$\tilde{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{if } x_n \geq 0, \\ u(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

...
For simplicity, we still denote \( \tilde{u} \) by \( u \). The following lemma gives an estimate on the difference between the “original” solution \( u \) and the “good” solution \( w \).

**Lemma 6.2.** Assume \( |f - 1| \leq h^\delta \) in \( D_h \cap \Omega \) for some \( \delta \in (0, 1/2) \). We have

\[
|u - w| \leq C h^{1+\delta} \quad \text{in } D_h \cap \Omega, \tag{6.12}
\]

where the constant \( C \) is independent of \( h, \delta \).

**Proof.** Divide \( \partial D^+_h = C_1 \cup C_2 \) into two parts, where \( C_1 \subset \{x_n > a_h\} \) and \( C_2 \subset \{x_n = a_h\} \). On \( C_1 \) we have \( u = w \). On \( C_2 \), by symmetry we have \( D_n w = 0 \). As \( a_h < Ch^{1-\epsilon} \), by Corollary 6.1 we have \( 0 \leq D_n u \leq C_1 h^{1-\epsilon} \) on \( C_2 \), for any given small \( \epsilon > 0 \).

Let 

\[
\hat{w} = (1 - h^\delta)^{1/n} w - (1 - h^\delta)^{1/n} h + h.
\]

Then

\[
\det D^2 \hat{w} \leq \det D^2 u \quad \text{in } D^+_h,
\]

\[
\hat{w} = u = h \quad \text{on } C_1,
\]

\[
D_n \hat{w} = 0 < D_n u \quad \text{on } C_2.
\]

By comparison principle we have \( \hat{w} \geq u \) in \( D^+_h \).

On the other hand, let

\[
\hat{w} = (1 + h^\delta)^{1/n} w - (1 + h^\delta)^{1/n} h + h + C_1 (x_n - Ch^{1/2-\epsilon}) h^{1-\epsilon}.
\]

Then

\[
\det D^2 \hat{w} \geq \det D^2 u \quad \text{in } D^+_h,
\]

\[
\hat{w} \leq u = h \quad \text{on } C_1,
\]

\[
D_n \hat{w} = C_1 h^{1-\epsilon} > D_n u \quad \text{on } C_2.
\]

Hence by comparison principle, \( \hat{w} \leq u \) in \( D^+_h \).

Since \( h > 0 \) is small, \( \delta < 1/2 \), and \( \epsilon > 0 \) is small, we obtain

\[
|u - w| \leq C h^{1+\delta} \quad \text{in } D^+_h. \tag{6.13}
\]

Next, we estimate \( |u - w| \) in \( D_h^c \cap \Omega \). For \( x = (x', x_n) \in D_h^c \cap \Omega \), let \( z = (x', 2a_h - x_n) \in D_h^c \). Then \( |x - z| \leq Ch^{1-\epsilon} \). From (6.13), \( |u(z) - w(z)| \leq C h^{1+\delta} \). Since \( w \) is symmetric with respect to \( \{x_n = a_h\} \), we have \( w(x) = w(z) \). Since \( u \in C^{1,1-\epsilon}(\bar{\Omega}) \), we obtain

\[
|u(x) - u(z)| \leq \|Du\|_{L^\infty(D_h)} |x - z| \leq Ch^{3/2-\epsilon}.
\]

Therefore, as \( \delta < 1/2 \) is a given constant

\[
|u(x) - w(x)| \leq |u(x) - u(z)| + |u(z) - w(z)| \leq Ch^{1+\delta}.
\]

Combining with (6.13) we thus obtain the desired \( L^\infty \) estimate

\[
|u - w| \leq C h^{1+\delta} \quad \text{in } D_h \cap \Omega. \tag{6.14}
\]

\( \square \)
We are now in position to prove the global $C^{2,\alpha}$ estimate. We will adopt the argument in [21]. Note that when $f$ is Hölder continuous with exponent $\alpha \in (0,1)$ and $f(0) = 1$, from Lemma 6.1 the oscillation

(6.15) \[ \omega_f(h) := \sup_{D_h \cap \Omega} |f - 1| \leq Ch^\delta \]

for some

(6.16) \[ \delta \geq \alpha/2 - \varepsilon, \]

where $\varepsilon > 0$ is a small constant arising in (6.2). We point out that if $\varepsilon = 0$ in (6.2), then $\delta = \alpha/2$. We first quote two lemmas from [21].

**Lemma 6.3.** Let $u \in C^2$ be a convex solution of $\det D^2u = 1$ in $D$, vanishing on $\partial D$. Suppose $u$ attains its minimum at the origin, and $|D^2u(0)| \leq C_0$ for some constant $C_0 > 0$. Then the domain $D$ is of good shape.

**Lemma 6.4.** Let $u_i, i = 1,2$, be two convex solutions of $\det D^2u = 1$ in $B_1(0)$. Suppose $\|u_i\|_{C^4} \leq C_0$. Then if $|u_1 - u_2| \leq \delta$ in $B_1(0)$ for some constant $\delta > 0$, we have, for $1 \leq k \leq 3$,

\[ |D^k(u_1 - u_2)| \leq C\delta \quad \text{in } B_{1/2}. \]

**Proof of Theorem 1.1.** We sketch the proof here as it is similar to that in [21]. Choose a sufficiently small initial height $h_0 > 0$, and normalise $D_{h_0}$ by a transformation $T$ such that $T(D_{h_0}) \sim B_1(0)$.

After the change, $D_1$ has a good shape. Denote

(6.17) \[ \omega(h) = \omega_f(h), \quad \omega_k := \omega(4^{-k}) \]

for $k = 0,1,2,\cdots$, where $\omega_f(h)$ is given in (6.15). Define

(6.18) \[ D_k := D_{4^{-k}}, \quad f_k := \inf_{D_k \cap \Omega} f > 0. \]

We claim that $D_k \sim D_{k+1}$, namely there is a constant $C$ depending only on $n$ such that

(6.19) \[ C^{-1}D_k \subset D_{k+1} \subset CD_k. \]

To see this, note that (before the change $T$) by (6.2) and $a_{h_0} < Ch_0^{1-\varepsilon}$, one has $|D_{h_0}| \approx |S_{h_0}u| \approx h_0^{n/2}$ and $|D_{h_0/4} \cap \{x_n > a_{h_0}\}| \approx h_0^{n/2}$. Since $|\det T| \approx h_0^{-n/2}$, we have $|T(D_{h_0/4} \cap \{x_n > a_{h_0}\})| \approx 1$. By definition $D_{h_0/4} \cap \{x_n > a_{h_0}\} \subset D_{h_0}$, thus $T(D_{h_0/4} \cap \{x_n > a_{h_0}\})$ is bounded. Therefore, by symmetry and the fact $a_{h_0} < Ch_0^{1-\varepsilon} \ll h_0^{3/2}$ that is the width of $D_{h_0/4}$ in $e_n$ direction, we obtain $D_1 \sim D_0$. Similarly we can obtain (6.19) for all $k = 0,1,2,\cdots$.

Let $u_k, k = 0,1,2,\cdots$, be the convex solution of

(6.20) \[ \det D^2u_k = f_k \quad \text{in } D_k, \quad u_k = 4^{-k} \quad \text{on } \partial D_k. \]
When \( k = 0 \), since initially \( \mathcal{D}_0 \) has a good shape, by interior regularity \([18]\),
\[
\|u_0\|_{C^4(D_{3/4})} \leq C.
\]
From Lemma 6.2,
\[
\sup_{\mathcal{D}_0 \cap \Omega} |u - u_0| \leq C_{\omega_0},
\]
which implies that \( \mathcal{D}_1 \) has a good shape (also shown in (6.19)), and thus
\[
\|u_1\|_{C^4(D_{3/16})} \leq C.
\]
Hence, from Lemma 6.4 and (6.19)
\[
|D^2u_0(x) - D^2u_1(x)| \leq C_{\omega_0}
\]
for \( x \in C^{-1}\mathcal{D}_2 \), where \( 1 \leq k \leq 3 \). By Lemma 6.3, this estimate then implies that \( \mathcal{D}_2 \) has a good shape.

By induction and (6.12), (6.19), we obtain
\[
|D^2u_k(x) - D^2u_{k+1}(x)| \leq C_{\omega_k}
\]
for \( x \in C^{-1}\mathcal{D}_{k+2} \).

Therefore, for any given point \( z \in \overline{\Omega} \) near the origin such that \( 4^{-k-4} \leq u(z) \leq 4^{-k-3} \),
\[
|D^2u(z) - D^2u(0)| \leq I_1 + I_2 + I_3 :=
\]
\[
|D^2u_k(z) - D^2u_k(0)| + |D^2u_k(0) - D^2u(0)| + |D^2u(z) - D^2u_k(z)|.
\]
By (6.22),
\[
I_2 \leq C \sum_{j=k}^{\infty} \omega_j \leq C \int_0^{\frac{|z|}{r}} \frac{\omega(r)}{r}.
\]
Similarly to (6.24), as in [21] one can derive that
\[
I_3 \leq C \int_0^{\frac{|z|}{r}} \frac{\omega(r)}{r}.
\]
To estimate \( I_1 \), denote \( h_j = u_j - u_{j-1} \). By Lemma 6.4
\[
|D^2h_j(z) - D^2h_j(0)| \leq C2^j\omega_j|z|.
\]
Hence
\[
I_1 \leq |D^2u_0(z) - D^2u_0(0)| + \sum_{j=1}^{k} |D^2h_j(z) - D^2h_j(0)|
\]
\[
\leq C|z| \left( 1 + \int_{|z|}^{1} \frac{\omega(r)}{r^2} \right).
\]
Since \( f \) is Hölder continuous with exponent \( \alpha \), inserting (6.24)–(6.26) into (6.23) we obtain the Hölder continuity at the origin, namely for any point \( z \) near the origin,
\[
|D^2u(z) - D^2u(0)| \leq C|z|^\alpha.
\]
In (6.27), we obtained the Hölder continuity of $D^2u$ at the boundary. If two points $x, y \in \Omega$ are both interior points, let $\hat{x}, \hat{y} \in \partial \Omega$ be the closest points to $x, y$, respectively. In the case $|x - y| \geq \delta_0 (\text{dist}(x, \partial \Omega) + \text{dist}(y, \partial \Omega))$ for some constant $\delta_0 > 0$, by (6.27) we have
\[
|D^2u(x) - D^2u(y)| \leq |D^2u(x) - D^2u(\hat{x})| + |D^2u(\hat{x}) - D^2u(\hat{y})| + |D^2u(\hat{y}) - D^2u(y)| \leq C|x - y|^\alpha.
\]
Otherwise, the estimate for $|D^2u(x) - D^2u(y)|$ has been established in [4, 21] for the interior $C^{2,\alpha}$ regularity.

We have proved the global $C^{2,\alpha-\varepsilon}$ regularity for problem (1.1), (1.2). To remove the small constant $\varepsilon$, observe that once the second derivatives are uniformly bounded, the inclusions (6.2) holds for $\varepsilon = 0$. Therefore (6.16) can be improved to $\delta = \alpha/2$. Repeating the above argument, we then obtain the global $C^{2,\alpha}$ regularity for problem (1.1), (1.2). \qed

**Remark 6.1.** The above argument also implies that if $f$ is Dini continuous, that is if
\[
\int_0^1 \frac{\omega_f(t)}{t} \, dt < \infty,
\]
where $\omega(t) = \sup\{|f(x) - f(y)| : |x - y| < t\}$, then the integrals in (6.24) and (6.26) are convergent. Hence $D^2u$ is positive definite and continuous up to the boundary. Therefore we have proved the following result.

**Theorem 6.1.** Assume that $\Omega$ and $\Omega^*$ are bounded convex domains in $\mathbb{R}^n$ with $C^{1,1}$ boundaries, and assume that $f$ is positive and Dini continuous. Then the second derivatives of the solution $u$ to the problem (1.1) and (1.2) are continuous in $\Omega$.

**Remark 6.2.** Checking the proof of the uniform density (Lemma 2.3), the tangential $C^{1,\alpha}$ regularity (Lemma 3.1), the uniform obliqueness (Lemma 1.1), we find that the $C^{1,1}$ regularity of the boundaries $\partial \Omega$ and $\partial \Omega^*$ can be weakened to $C^{1,1-\theta}$ for some $\theta > 0$ depending on the constant $\delta$, provided $u$ is globally $C^{1,\delta}$ smooth [5]. Therefore our main result, Theorem 1.1, holds for $C^{1,1-\theta}$ convex domains $\Omega, \Omega^*$. In particular, we prove that it suffices to assume $\partial \Omega, \partial \Omega^* \in C^{1,\alpha}$ in dimension two [11]. When $f \equiv 1$, very recently Savin and Yu [29] obtained the global $W^{2,p}$ estimate for arbitrary bounded convex domains $\Omega, \Omega^* \subset \mathbb{R}^2$. In general dimensions, it may be possible to relax the $C^{1,1}$ regularity of the boundaries to $C^{1,\alpha}$. Indeed, if one can manage this relaxation for the uniform density estimate and the tangential $C^{1,1-\varepsilon}$ estimate (for all $\varepsilon > 0$), then our method for the uniform obliqueness can be applied.

**Remark 6.3.** From [24, §7.3] it is known that for arbitrary positive and smooth functions $f$, the convexity of domains is necessary for the global $C^1$ regularity. However, for a fixed function $f > 0$, by Theorem 1.1 and a perturbation argument, we can prove that the solution is smooth up to the boundary, if the domains $\Omega$ and $\Omega^*$ are smooth perturbations of convex ones, even though they are not convex themselves.
References

[1] Brendle, S., Minimal Lagrangian diffeomorphisms between domains in hyperbolic planes, J. Diff. Geom., 80 (2008), 1–22.
[2] Brendle, S. and Warren, M., A boundary value problem for minimal Lagrangian graphs, J. Diff. Geom., 84 (2010), 267–287.
[3] Brenier, Y., Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math., 44 (1991), 375–417.
[4] Caffarelli, L.A., Interior $W^{2,p}$ estimates for solutions of Monge-Ampère equations, Ann. Math., 131 (1990), 135–150.
[5] Caffarelli, L.A., Boundary regularity of maps with convex potentials, Comm. Pure Appl. Math., 45 (1992), 1141–1151.
[6] Caffarelli, L.A., The regularity of mappings with a convex potential, J. Amer. Math. Soc., 5 (1992), 99–104.
[7] Caffarelli, L.A., Boundary regularity of maps with convex potentials – II, Ann. Math. (2), 144 (1996), 453–496.
[8] Caffarelli, L.A. and McCann, R.J., Free boundaries in optimal transport and Monge-Ampère obstacle problems, Ann. of Math. (2), 171 (2010), 673–730.
[9] Caffarelli, L.A.; Nirenberg, L. and Spruck, J., Dirichlet problem for nonlinear second order elliptic equations I, Monge-Ampère equations, Comm. Pure Appl. Math., 37 (1984), 369–402.
[10] Chen, S. and Figalli, A., Partial $W^{2,p}$ regularity for optimal transport maps, J. Funct. Anal., 272 (2017), 4588–4605.
[11] Chen, S.; Liu, J. and Wang, X.-J., Boundary regularity for the second boundary-value problem of Monge-Ampère equations in dimension two, arXiv:1806.09482.
[12] Chen, S.; Liu, J. and Wang, X.-J., Global regularity of optimal mappings in non-convex domains, Sci. China Mathematics, 62 (2019), 2057–2072.
[13] Chen, S.; Liu, J. and Wang, X.-J., $C^{2,\alpha}$ regularity of free boundaries in optimal transportation, arXiv:1911.10503.
[14] De Philippis, G. and Figalli, A., The Monge-Ampère equation and its link to optimal transportation, Bull. Amer. Math. Soc. (N.S.), 51 (2014), 527–580.
[15] Delanoë, Ph., Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator, Ann. Inst. Henri Poincaré, Analyse Non Linéaire, 8 (1991), 443–457.
[16] Evans, L.C., Partial differential equations and Monge-Kantorovich mass transfer, In: Current development in mathematics, Int. Press, Boston, 65–126, 1999.
[17] Figalli, A., The Monge-Ampère equation and its applications, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), 2017.
[18] Gilbarg, D. and Trudinger, N. S., Elliptic partial differential equations of second order. Springer-Verlag, Berlin, 1983.
[19] Gutiérrez, C.E., The Monge-Ampère equation, Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA, 2001.
[20] Jhaveri, Y., On the (in)stability of the identity map in optimal transportation, Calc. Var. PDE, 58, 96 (2019).
[21] Jian, H.Y. and Wang, X.-J., Continuity estimates for the Monge-Ampère equation, SIAM J. Math. Anal., 39 (2007), 608–626.
[22] Lei, N.; Guo, Y.; An, D.; Qi, X.; Luo, Z.; Yau, S.-T. and Gu, X., Mode collapse and regularity of optimal transportation maps, arXiv:1902.02934.
[23] Lions, P.L.; Trudinger, N.S. and Urbas, J.I.E., The Neumann problem for equations of Monge-Ampère type, Comm. Pure Appl. Math., 39 (1986), 539–563.
[24] Ma, X.N.; Trudinger, N.S. and Wang, X.-J., Regularity of potential functions of the optimal transportation problem, Arch. Rat. Mech. Anal., 177 (2005), 151–183.
[25] Pogorelov, A.V., The extrinsic geometry of convex surfaces, Moscow 1969, 759 pp.
[26] Rachev, S.T. and Ruschendorf, L., Mass transportation problems, Vol. I. Theory; Vol II. Applications, Springer-Verlag, 1998.
[27] Savin, O., Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation, J. Amer. Math. Soc., 26 (2013), 63–99.
[28] Savin, O., Global $W^{2,p}$ estimates for the Monge-Ampère equations, Proc. Amer. Math. Soc. 141 (2013), 3573–3578.
[29] Savin, O. and Yu, H., Regularity of optimal transport between planar convex domains, *Duke Math. J.*, 169 (2020), 1305–1327.

[30] Solomon, J.; de Goes, F.; Peyré, G.; Cuturi, M.; Butscher, A.; Nguyen, A.; Du, T. and Guibas, L., Convolutional Wasserstein Distances: Efficient Optimal Transportation on Geometric Domains, *ACM Transactions on Graphics*, 34 (2015), pp.66:1–66:11.

[31] Trudinger, N.S. and Wang, X.-J., Boundary regularity of the Monge-Ampère and affine maximal surface equations, *Ann. Math.*, 167 (2008), 993–1028.

[32] Urbas, J., On the second boundary value problem of Monge-Ampère type, *J. Reine Angew. Math.*, 487 (1997), 115–124.

[33] Villani, C., *Topics in optimal transportation*, Grad. Stud. Math. 58, Amer. Math. Soc., 2003.

[34] Villani, C., *Optimal transport, Old and new*, Springer, Berlin, 2006.

Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, AUSTRALIA

Email address: chenshibing1982@hotmail.com

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, AUSTRALIA

Email address: jiakunl@uow.edu.au

Centre for Mathematics and Its Applications, The Australian National University, Canberra, ACT 0200, AUSTRALIA

Email address: Xu-Jia.Wang@anu.edu.au