THE CHOW RING OF THE MODULI SPACE
AND ITS RELATED HOMOGENEOUS SPACE
OF BUNDLES ON $\mathbb{P}^2$ WITH CHARGE 1

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Abstract. For an algebraically closed field $K$ with $\text{ch}(K) \neq 2$, let $\mathcal{O}M(1, SO(n, K))$ denote the moduli space of holomorphic bundles on $\mathbb{P}^2$ with the structure group $SO(n, K)$ and half the first Pontryagin index being equal to 1, each of which is trivial on a fixed line $l_\infty$ and has a fixed holomorphic trivialization there. In this paper we determine the Chow ring of $\mathcal{O}M(1, SO(n, K))$.

1. Introduction

Let $G$ be one of the classical groups $SU(n)$, $SO(n)$ or $Sp(n)$, and let $k \geq 0$ be half the first Pontryagin index of a $G$-bundle $P$ over $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Denote by $M(k, G)$ the framed moduli space whose points represent isomorphism classes of pairs:

$$(\text{self-dual } G\text{-connections on } P, \text{isomorphism } P_\infty \simeq G).$$

Let $\mathcal{O}M(k, G^C)$ denote the moduli space of holomorphic bundles on $\mathbb{CP}^2$ for the associated complex group, trivial on a fixed line $l_\infty$ and with a fixed holomorphic trivialization there. Then Donaldson [7] showed a diffeomorphism $M(k, G) \simeq \mathcal{O}M(k, G^C)$.

In [12] the topology of $M(1, SO(n)) \simeq \mathcal{O}M(1, SO(n, \mathbb{C}))$ was studied in detail. The result was used in [11] to prove the fact that the natural homomorphism $J : H._*(M(1, SO(n)), \mathbb{Z}/2) \to H._*(\Omega^3_{\emptyset}\text{Spin}(n), \mathbb{Z}/2)$ is injective. Moreover, the image of $J$ was determined. To prove this, the following description of $\mathcal{O}M(1, SO(n, \mathbb{C}))$ by a homogeneous space was used: We set

$$W_n = SO(n)/(SO(n-4) \times SU(2)).$$

Then there is a diffeomorphism

$$(1.1) \quad \mathcal{O}M(1, SO(n, \mathbb{C})) \simeq \mathbb{R}^5 \times W_n.$$
The purpose of this paper is to generalize the definition of $OM(1, SO(n, \mathbb{C}))$ for any algebraically closed field $K$ with $\text{ch}(K) \neq 2$ and to determine the Chow ring of this. The Chow ring of a classifying space was studied by Totaro [19]. A loop space is considered to be a dual situation of a classifying space in a certain sense. Our result and the result of [11] are the first step for a loop space.

**Definition 1.1.** Let $K$ be an algebraically closed field with $\text{ch}(K) \neq 2$. Let $OM(1, SO(n, K))$ denote the moduli space of holomorphic bundles on $\mathbb{P}^2$ with the structure group $SO(n, K)$ and half the first Pontryagin index being equal to 1, each of which is trivial on a fixed line $l_\infty$ and has a fixed holomorphic trivialization there.

The moduli space $OM(1, SO(n, K))$ is a quasi-projective variety and defines the Chow ring. More explicitly, the diffeomorphism (1.1) is generalized (in the sense of a biregular map) as follows: We set

$$X_n = SO(n, K)/(SO(n - 4, K) \times SL(2, K)) \cdot P_u,$$

where $P_u$ denotes the unipotent radical. (Recall that for a parabolic subgroup $P$ of an algebraic group $G$, $P$ is a semidirect product of a reductive group and its unipotent radical $P_u$.) Then there is a biregular map

$$OM(1, SO(n, K)) \simeq \mathbb{A}^2 \times X_n.$$  

(For the proof of (1.2), see Proposition 2.1.) A formula of Grothendieck [3] shows that

$$CH^\cdot(OM(1, SO(n, K))) \simeq CH^\cdot(X_n).$$

The purpose of this paper is to determine the Chow rings of $X_n$ and its related algebraic variety $Y_n$ explicitly.

The Schubert cell approach of the Chow ring of $Y_n$ by using a Young diagram is done in [13], [15]. However it needs further work to determine the Chow ring of $X_n$ from this. Hence we first calculate the Chow ring of $Y_n$ more explicitly by a different method. Then we calculate the Chow ring from the results. Our results for the Chow ring of $X_n$ are new.

This paper is organized as follows. In Sect. 2 we first prove (1.2). Then we recall basic facts on the Chow ring. In Sect. 3 we determine an integral basis and the ring structure of $CH^\cdot(Y_n)$, where $Y_n$ is an algebraic variety which is related to $X_n$. (See Theorems 3.7 and 3.9.) The ring structure of $CH^\cdot(Y_n)$ proved in Theorem 3.9 is one of our main results. Since the results are long, we give them in tables in Sect. 5. (See 5.2-5.5.) Using the results of Sect. 3, we determine $CH^\cdot(X_n)$ in Sect. 4. (See Theorem 4.1.)
We thank N. Yagita for turning our interest to the Chow ring and explaining the paper [17].

2. Preliminaries

We first prove (1.2):

**Proposition 2.1.** For an algebraically closed field $K$ with $\text{ch}(K) \neq 2$, there is a biregular map

$$\mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times X_n.$$ 

**Proof.** Recall that a monad description of $\mathcal{O}M(k, SO(n, \mathbb{C}))$ was indicated in [7] and given explicitly in [14] and [18]. It is easy to see that the description remains valid for any algebraically closed field $K$. In particular, $\mathcal{O}M(1, SO(n, K))$ is given as follows:

**Lemma 2.2.** Let $C_n$ be the space of $n \times 2$ matrices

$$c = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \\ \vdots & \vdots \\ z_n & w_n \end{pmatrix}$$

with coefficients in $K$ satisfying:

a) $c^T c = O$, that is:

$$\sum_{i=1}^n z_i^2 = 0, \quad \sum_{i=1}^n w_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^n z_i w_i = 0,$$

b) The rank of $c$ over $K$ is 2.

The group $SL(2, K)$ acts on $C_n$ from the right by the multiplication of matrices. Then there is a biregular map

$$\mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times (C_n/SL(2, K)).$$

From the lemma, it suffices to prove $X_n \simeq C_n/SL(2, K)$. We prove this for the case $n = 2m$. (The case $n = 2m + 1$ can be proved similarly.) Recall that in [2], $SO(n, K)$ was defined as follows: Let $q(x)$ be a quadratic form on $\mathbb{A}^n$ defined by $q(x) = \sum_{i=1}^m x_i x_{m+i}$, and let $B(x, y)$ be the associated bilinear form. Then $SO(n, K)$ is defined by

$$SO(n, K) = \{ \sigma \in \text{Aut}(\mathbb{A}^n) : B(\sigma(x), \sigma(y)) = B(x, y) \text{ for } x, y \in \mathbb{A}^n \}.$$ 

We set

$$x_j = z_j + \sqrt{-1}z_j, \quad x_{m+j} = z_j - \sqrt{-1}z_j, \quad y_j = w_j + \sqrt{-1}w_j \quad \text{and} \quad y_{m+j} = w_j - \sqrt{-1}w_j,$$
where \(1 \leq j \leq m\). Then the defining equations of \(C_n\) are given by
\[q(x) = q(y) = 0 \quad \text{and} \quad B(x, y) = 0.\]

Clearly \(SO(n, K)\) acts on \(C_n\). It is easy to prove the following lemma. (See [2, V 23.4].)

**Lemma 2.3.**
\[SO(n, K)/SO(n - 4, K) \cdot P_u \simeq C_n,\]
where \(P_u\) is the unipotent radical of a parabolic subgroup with a Levi factor \(SO(n - 4, K) \times GL(2, K)\).

Now Proposition 2.1 follows from Lemma 2.3. This completes the proof of Proposition 2.1. \(\Box\)

Next we recall basic facts on the Chow ring. We suppose that an algebraic variety \(V\) is defined over \(K\). Let \(CH^i(V)\) denote the Chow ring and \(CH^i(V)\) the subgroup of \(CH^i(V)\) generated by the cycles of codimension \(i\).

**Theorem 2.4 ([3]).**
(i) Let \(V\) be a nonsingular variety, \(X\) a nonsingular closed subvariety of \(V\), and \(U = X - V\). Then there exists an exact sequence
\[CH^i(X) \overset{i_*}{\rightarrow} CH^i(V) \overset{j^*}{\rightarrow} CH^i(U) \rightarrow 0,\]
where \(i : X \rightarrow V\) (resp. \(j : U \rightarrow V\)) is a closed immersion (resp. an open immersion).

For the definitions of \(i_*\) and \(j^*\), see also [10].

(ii) Let \(\pi : E \rightarrow V\) be a fiber bundle with an affine space \(A^n\) as a fiber. Then the induced map \(\pi^* : CH^i(V) \rightarrow CH^i(E)\) is an isomorphism.

The Chow ring of the following projective variety is well-known.

**Theorem 2.5 ([1], [6]).** Let \(G\) be a reductive algebraic group and \(P\) a maximal parabolic subgroup. Then
(i) a quotient \(G/P\) is a nonsingular projective variety,
(ii) \(CH^i(G/P)\) is generated by the Schubert varieties.
(iii) \(CH^i(G/P)\) is independent of \(ch(K)\). Moreover, \(CH^i(G/P) \simeq H^i(G/P, \mathbb{Z})\) for \(K = \mathbb{C}\).

3. The ring structure of \(CH^i(Y_n)\)

Before describing the results, we need some notations and results. We set
\[Y_n = SO(n, K)/(SO(n - 4, K) \times GL(2, K)) \cdot P_u.\]
Then we have a principal bundle
\[ G_m \to X_n \overset{\pi}{\to} Y_n. \]

In this section we determine an integral basis and the ring structure of \( CH(Y_n) \). By Theorem 2.5 (ii), (iii), we obtain the following theorem:

**Theorem 3.1 ([12]).** We have an isomorphism as modules:

1. For \( n = 2m \),
   \[ CH(Y_n) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2[c_1, c_2]/(b_{m-1}c_{2m-2}) \otimes \Delta(v_{2m-4}, v_{2m-2}). \]
2. For \( n = 2m + 1 \),
   \[ CH(Y_n) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2[c_1, c_2]/(b_{m-1}c_{2m-2}) \otimes \Delta(v_{2m-2}, v_{2m}), \]
   where \(|c_1| = 2, |c_2| = 4, |b_i| = 2i\) and \(|v_i| = i\).

**Theorem 3.2 ([12]).** Let \( p \) be an odd prime. Then we have a ring isomorphism:

1. For \( n = 2m \),
   \[ CH(Y_n) \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p[c_1, c_2, \chi_{2m-4}]/(c_2\chi_{2m-4}, \chi_{2m-4}^2 - d_m - 1, c_2d_m - 2), \]
   where \( \chi_{2m-4} \in H^{2m-4}(BSO_{2m-4}, \mathbb{Z}/p) \) is the Euler class.
2. For \( n = 2m + 1 \),
   \[ CH(Y_n) \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p[c_1, c_2]/(d_{m-1}, c_2d_m - 2). \]

We recall the definitions of \( b_i, d_i \) and \( v_i \). In a polynomial ring \( \mathbb{Z}[\alpha, \beta]\), we set \( c_1 = \alpha + \beta \) and \( c_2 = \alpha\beta \). Then \( b_k \) and \( d_k \) are defined by

\[ b_k = (-1)^k \sum_{i=0}^{k} \alpha^i \beta^{k-i} \]

and

\[ d_k = (-1)^k \sum_{i=0}^{k} \alpha^{2i} \beta^{2k-2i}. \]

The element \( v_{2r} \in CH^{2r}(Y_n) \) is defined by

1. For \( n = 2m \),
   \[ \begin{cases} 2v_{2m-4} = \chi_{2m-4} - b_{m-2} \\ 2v_{2m-2} = b_{m-1}. \end{cases} \]
2. For \( n = 2m + 1 \),
   \[ \begin{cases} 2v_{2m-2} = b_{m-1} \\ 2v_{2m} = c_2b_{m-2}. \end{cases} \]

The following formulas are easily proved.
**Lemma 3.3.** We have

\[ b_k = (-1)^k \sum_{\mu=0}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^\mu \binom{k-\mu}{\mu} c_1^{k-2\mu} c_2^\mu \]

and

\[ d_k = (-1)^k \sum_{\mu=0}^{k} (-1)^\mu \binom{2k-\mu+1}{\mu} c_1^{2k-2\mu} c_2^\mu. \]

The following lemmas are also easily shown.

**Lemma 3.4.**

\[ \sum_{\mu=0}^{h} (-1)^\mu c_2^{h-\mu} b_{2\mu} = d_h. \]

**Lemma 3.5.** We set \( f_n(x) = (1 + x)^n - (1 + x^n) \) and write \( f_n(x) \) as

\[ f_n(x) = \sum_{\mu=1}^{\left\lfloor \frac{n}{2} \right\rfloor} a_\mu x^{n-2\mu}. \]

Then we have

\[ a_\mu = (-1)^{\mu+1} n \frac{n-1-\mu}{\mu} \left( \frac{n}{\mu} - 1 \right). \]

Especially, the last term is given by

\[ \begin{cases} 
(-1)^{s+1} 2x^s & \text{for } n = 2s \\
(-1)^{s+1} (2s + 1)x^s(1 + x) & \text{for } n = 2s + 1.
\end{cases} \]

For \( n = 2m \) or \( 2m + 1 \), we define a subgroup \( A_n \) of \( CH(Y_n) \) by

\[ (3.4) \quad A_n = \left( \bigoplus_{i=0}^{m-2} \mathbb{Z}[c_1]/(c_1^{m-1-i}) \{ c_2^i \} \right) \otimes B_n, \]

where we set

\[ B_n = \begin{cases} 
\Delta_\mathbb{Z}(v_{2m-4}, v_{2m-2}) & n = 2m \\
\Delta_\mathbb{Z}(v_{2m-2}, v_{2m}) & n = 2m + 1.
\end{cases} \]

The generators \( v_{2i} \) is specified in (3.2) and (3.3).

The following lemma is proved in the same way as in [12, Lemma 3.8].
Lemma 3.6. For a prime $p$, we abbreviate $\text{CH}^*(Y_n) \otimes \mathbb{Z}/(p)$ as $\text{CH}^*(Y_n)(p)$. If $p$ is odd, we have the following isomorphism of modules:

(i) For $n = 2m$,

$$\text{CH}^*(Y_n)(p) \simeq \mathbb{Z}/(p)[c_1]/(c_1^{2(m-1)})\{1, \chi_{2m-2}\} \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}/(p)[c_i]/(c_i^{2(m-1-i)})\{c_i^{2i-1}, c_i^{2i}\}.$$ 

(ii) For $n = 2m + 1$,

$$\text{CH}^*(Y_n)(p) \simeq \bigoplus_{i=0}^{m-2} \mathbb{Z}/(p)[c_1]/(c_1^{2(m-1-i)})\{c_2^{2i}, c_2^{2i+1}\}.$$ 

Theorem 3.7. An integral basis of $\text{CH}^*(Y_n)$ is constructed from the monomial basis of $A_n$ in (3.4). The results are summed up in Sect. 5, 5.2.

The proof is based on a rather complicated calculation. Its outline is as follows: We construct a set of suitable generators starting from the basis of $A_n$. It is easily verified that it is a basis of $\text{CH}^*(Y_n) \otimes \mathbb{Z}/p$ by using the presentation of Lemma 3.6. Then it is a $\mathbb{Z}$-basis of $\text{CH}^*(Y_n)$. We only prove the case $n = 2m + 1$ and $m$ is even. To simplify the proof, we set:

$$(3.5) \quad C = \text{CH}^*(Y_n)$$

$$b_{m-1} = (-1)^{m-1}b_{m-1} = \sum_{i+j=m-1} \alpha^i \beta^j, \quad e_m = c_2b_{m-2}$$

$$d_{m-1} = (-1)^{m-1}d_{m-1}$$

$$A = (\mathbb{Z}[c_1]/(c_1^{m-1}) \oplus \cdots \mathbb{Z}[c_1]/(c_1^{m-1-i})\{c_2^{m-2}\}) \otimes \mathbb{Z}\{1, b_{m-1}, e_m, b_{m-1}e_m\}.$$ 

Let $\mathcal{A}_n$ the set of a basis

$$\{c_1^i c_2^j b_{m-1}^{e_1} b_{m}^{e_2} : i + j \leq m - 2, e_k = 0 \text{ or } 1 (k = 1, 2)\}.$$ 

Then we have

$$C \subset \mathbb{Q}[c_1, c_2]/(d_{m-1}, c_2^2d_{m-2}), \quad A \subset \mathbb{Q}[c_1, c_2]/(d_{m-1}, c_2^2d_{m-2})$$

and $c_2^2d_{m-1-i} \in (d_{m-1}, c_2^2d_{m-2}), \ i \geq 0.$

Our concern is a homogeneous polynomial algebra $S_\mathbb{Z}(V)$ of $V = \{c_1, c_2\}$. It is identified with an inhomogeneous ring $\mathbb{Z}[x]$ by putting $x = \frac{b}{a}$. Then we have

$$c_1 = 1 + x, \ c_2 = x, \ b_{m-1} = \frac{x^m - 1}{x - 1}, \quad \text{and} \quad d_{m-1} = \frac{x^{2m} - 1}{x^2 - 1}.$$
Using the identification, the next formulas are directly checked: In $\mathbb{Z}[c_1, c_2]$, 
\[ -c_2^{2i+1+j}b_{m-2-2i} = c_2^{j+1}b_{2i-1}b_{m-1} - c_2^{i}b_{2i}e_m \]
and
\[ -c_2^{2i+2+j}b_{m-3-2i} = c_2^{j+1}b_{2i}b_{m-1} - c_2^{i}b_{2i+1}e_m. \]

We set
\[ (3.6) \quad [c_1^{2i-1}c_2^{j+1}b_{m-1}] := c_2^{j+1}b_{2i-1}b_{m-1} - c_2^{i}b_{2i}e_m = -c_2^{2i+j+1}b_{m-2-2i} \]
and
\[ [c_1^{2i}c_2^{j+1}b_{m-1}] := c_2^{j+1}b_{2i}b_{m-1} - c_2^{i}b_{2i+1}e_m = -c_2^{2i+j+2}b_{m-3-2i}. \]

Noting $c_2^{i}d_{m-2} = 0$, we have
\[ c_2^{i}b_{m-2}b_{m-1} - c_2^{i}c_1d_{m-2} = \frac{x^3}{(x-1)^2}(x^{m-2}-1)(x^{m-1}-1) = c_2^{i}b_{m-3}b_{m-2}. \]

Using $c_2^{i}d_{m-1-i} = 0$, $1 \leq i \leq m$, we repeat the argument and get
\[ (3.7) \quad c_2^{i}b_{m-1}e_m = c_2^{2i+1}b_{m-2-i}b_{m-1-i}. \]

In $Y$, we see that $d_{m-1} = 0$. Recall that we set $f_{2i+3}(x) = (1 + x)^{2i+3} - (1 + x^{2i+3})$ (see Lemma 3.5). Then we have
\[ c_1^{2i+1+j}c_2^{i}b_{m-1}e_m - c_2^{i}d_i d_{m-1} \]
\[ = \frac{x^{j+1}(x^m - 1)}{(x^2 - 1)^2}((1 + x)^{2i+3}(x^{m-1} - 1) - (x^{2i+2} - 1)(x^m + 1)) \]
\[ = \frac{x^{j+1}(x^m - 1)}{(x^2 - 1)^2}((1 + x^{2i+3})(x^{m-1} - 1) - (x^{2i+2} - 1)(x^m + 1) + f_{2i+3}(x)(x^{m-1} - 1)) \]
\[ = \frac{x^{2i+j+3}}{(x - 1)(x^2 - 1)}(x^m - 1)(x^{m-2i-3} - 1) + \frac{x^{j+1}}{(x^2 - 1)^2}(x^{m-1} - 1)(x^m - 1)f_{2i+3}(x). \]

We set
\[ I_1 = \frac{x^{2i+j+3}}{(x - 1)(x^2 - 1)}(x^m - 1)(x^{m-2i-3} - 1) \]
and
\[ I_2 = \frac{x^{j+1}}{(x^2 - 1)^2}(x^{m-1} - 1)(x^m - 1)f_{2i+3}(x). \]
Since \( x^{2\lambda}d_{m-\lambda} = 0 \) for \( 0 \leq \lambda \leq m-1 \), we see that \( x^{2i+j+3+\lambda}d_{m-(i+2)-\lambda} = 0 \) for \( 0 \leq \lambda \leq j - 1 \). Hence

\[
I_1 - \sum_{\lambda=0}^{j-1} x^{2i+j+3+\lambda} d_{m-(i+2)-\lambda} = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-j-1}(x^{m-2i-j-3}-1).
\]

From Lemma 3.5,

\[
I_2 = \sum_{\mu=1}^{i} a_{\mu} c_2^{j+\mu} c_1^{2i+1-2\mu} b_{m-1}e_m + (-1)^i k x^{i+j+2} \frac{(x^{m-1} - 1)(x^{m} - 1)}{(x-1)(x^2-1)},
\]

where \( k = 2i + 3 \).

We set \( J = \frac{x^{i+j+2}(x^{m-1}-1)(x^{m}-1)}{(x-1)(x^2-1)} \). Using \( x^{i+j+2+\lambda}d_{m-2-\lambda} = 0 \) for \( 1 \leq \lambda \leq i + j \), we see that

\[
J - \sum_{\lambda=0}^{i+j} x^{i+j+2+\lambda} d_{m-2-\lambda} = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-2-i-j-1}(x^{m-1-i-j}-1).
\]

Afterwards, we introduce the notation

\[
(3.8) \quad [c_1^{2i+1}c_2^{j}b_{m-1}e_m] := c_1^{2i+1}c_2^{j}b_{m-1}e_m - \sum_{\mu=1}^{i} a_{\mu} c_2^{j+\mu} c_1^{2i+1-2\mu} b_{m-1}e_m,
\]

where \( f_{2i+3}(x) = \sum_{\mu=1}^{i+1} a_{\mu} x^\mu (1 + x)^{2i+3-2\mu} \) (see Lemma 3.5).

We sum up these arguments:

\[
(3.9) \quad [c_1^{2i+1}c_2^{j}b_{m-1}e_m] = I_1 + (-1)^i k J,
\]

where

\[
I_1 = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-j-1}(x^{m-2i-j-3}-1),
\]

\[
J = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-2-i-j-1}(x^{m-1-i-j}-1), \quad \text{and} \quad k = 2i + 3.
\]

A direct calculation shows \( J - I_1 = \frac{x^{m+j}}{(x-1)(x^2-1)}(x^{i+1} - 1)(x^{i+2} - 1). \)

Hence we get a key formula

\[
(3.10) \quad [c_1^{2i+1}c_2^{j}b_{m-1}e_m] = ((-1)^i(2i + 3) + 1) I_1 + (-1)^i(2i+3) \frac{x^{m+j}}{(x-1)(x^2-1)}(x^{i+1} - 1)(x^{i+2} - 1).
\]

We note that (3.9) holds for \( m \) even or odd. From now on, we assume that \( m \) is even. When we put \( j = 1 \) in (3.9), we see that

\[
I_1 = c_2^{2i+4}d_{m-i-3}e_m. \quad \text{We set}
\]

\[
\langle c_1^{2i+1}c_2b_{m-1}e_m \rangle := [c_1^{2i+1}c_2b_{m-1}e_m] - ((-1)^i(2i + 3) + 1) c_2^{2i+4}d_{m-i-3}e_m.
\]
Then we have

\begin{equation}
\langle c_1^{2i+1} c_2 b_{m-1} e_m \rangle = \frac{(-1)^i c_2^{m+i} b_i}{c_1}.
\end{equation}

We call a generator $c_1^{m-3-2i} c_2^{2i+1} b_{m-1} e_m$ to be the head of the presentation $A_n$ in (3.4). For the head generator, it is directly shown that $I_1 = 0$ by (3.9). Hence the formula (3.9) implies

\begin{equation}
\frac{c_1^{m-3-2i} c_2^{2i+1} b_{m-1} e_m}{m-1-2i} = (-1)^i \frac{b_{\frac{m}{2}-i} b_{\frac{m}{2}-i-1}}{c_1}.
\end{equation}

Then the formulas (3.10) and (3.11) imply the following relations: We set

\begin{equation}
\langle c_1^{2i+1} b_{m-1} e_m \rangle := [c_1^{2i+1} b_{m-1} e_m] - ((-1)^i(2i + 3) + 1) c_2^{2i+2} b_{\frac{m}{2}-i-2} e_m
\end{equation}

and

\begin{equation}
\langle c_1^{2\alpha+1-2\beta} c_2^{1+\beta} b_{m-1} e_m \rangle := [c_1^{2\alpha+1-2\beta} c_2^{1+\beta} b_{m-1} e_m]
\end{equation}

\begin{equation}
- ((-1)^{\alpha-\beta}(2\alpha - 2\beta + 3) + 1) c_2^{2\alpha+1} b_{\frac{m}{2}-\alpha-3} e_m.
\end{equation}

Then we have

\begin{equation}
\langle c_1^{2i+1} b_{m-1} e_m \rangle = \frac{(-1)^i(2i + 3) + 1}{(-1)^i-1(2i + 1)} \langle c_1^{2i-1} c_2 b_{m-1} e_m \rangle = - \frac{c_2^m}{c_1} b_i b_{i+1}
\end{equation}

\begin{equation}
\langle c_1^{2\alpha+1-2\beta} c_2^{1+\beta} b_{m-1} e_m \rangle = \frac{(-1)^{\beta}(2\alpha - 2\beta + 3) + (-1)^{\alpha}}{2\alpha + 3} \langle c_1^{2\alpha+1} c_2 b_{m-1} e_m \rangle
\end{equation}

\begin{equation}
= - \frac{c_2^{m+\beta+2}}{c_1} b_{\alpha-\beta-1} b_{\alpha-\beta}
\end{equation}

and

\begin{equation}
\left[ c_1^{m-3-2\alpha-2\beta} c_2^{2\alpha+\beta+1} \right] = \frac{(-1)^{\beta}(m-1-2\alpha+2\beta) + (-1)^{\alpha}}{m-1-2\alpha} \left[ c_1^{m-3-2\alpha} c_2^{2\alpha+1} b_{m-1} e_m \right]
\end{equation}

\begin{equation}
= - \frac{c_2^{m+2\alpha+\beta+1}}{c_1} b_{\gamma} b_{\gamma+1}, \text{ where } \gamma = \frac{m}{2} - 2 - \alpha - \beta.
\end{equation}

Last we consider a generator $c_1^{2i+2} c_2^{j} b_{m-1} e_m$. We set

\begin{equation}
\langle c_1^{2i+2} c_2^{j} b_{m-1} e_m \rangle = c_1^{2i+2} c_2^{j} b_{m-1} e_m - \sum_{\mu=1}^{i} a_{\mu} c_2^{j+\mu} c_1^{2i+2-2\mu} b_{m-1} e_m
\end{equation}

\begin{equation}
- ((-1)^{i}(2i + 3) + 1) c_2^{i+j+1} b_{m-1} e_m,
\end{equation}
where \( f_{2i+3}(x) = \sum_{\mu=1}^{i+1} a_{\mu} x^\mu (1 + x)^{2i + 3 - 2\mu} \) (see Lemma 3.5). Then the formula (3.9) implies that
\[
(3.16) \quad \langle c_1^{2i+1} c_2^j b_{m-1} e_m \rangle = -c_2^{m+j} b_i b_{i+1}.
\]

We consider a set
\[
\{ (3.6), (3.10), (3.11), (3.12), (3.13), (3.14) \}
\]
\[
\cup \{ x \in A_n : x = c_1^i c_2^j e_m^k, x = c_1^i b_{m-1}, i + j \leq m - 2, \epsilon_k = 0 \text{ or } 1 \}.
\]

When we reduce it to the mod \( p \) reduction for an odd prime \( p \), the identities from (3.6) to (3.14) show that they are linearly independent from Lemma 3.6. Hence, replacing \( b_{m-1} \) and \( e_m \) by \( v_{2m-2} \) and \( v_{2m} \) respectively, we obtain an integral basis of \( CH(Y_{2m+1}) \) for even \( m \). An integral basis of \( CH(Y_n) \) for other cases is written down in a table of Sect. 5.

For a group \( G \) and a subgroup \( H \), let \( [G : H] \) denote the index, i.e. the cardinality of \( G/H \). As a corollary of the above theorem, we have

**Corollary 3.8.** (i) For \( n = 2m \),
\[
[CH(Y_n) : A_n] = \begin{cases} 1^2 \cdot 3^2 \cdot \ldots \cdot (m - 3)^2 \cdot (m - 1) & m: \text{even} \\ 1^2 \cdot 3^2 \cdot \ldots \cdot (m - 2)^2 & m: \text{odd} \end{cases}
\]

(ii) For \( n = 2m + 1 \),
\[
[CH(Y_n) : A_n] = \begin{cases} 1^2 \cdot 3^2 \cdot \ldots \cdot (m - 3)^2 \cdot (m - 1) & m: \text{even} \\ 1^2 \cdot 3^2 \cdot \ldots \cdot (m - 2)^2 \cdot m & m: \text{odd} \end{cases}
\]

The following theorem is proved by using the integral basis of \( CH(Y_n) \) given in Theorem 3.7.

**Theorem 3.9.** The ring structure of \( CH(Y_n) \) is determined. The results are summed up in tables in 5.3 and 5.4 in Sect. 5. (See also 5.5.)

**Proof.** We only show the formula (iii) in Sect. 5, 5.4. We use the notations of the proof of Theorem 3.7. Using \( x^{2i}d_{m-1-i} = 0 \), we have
\[
c_1^{m-2i-1} c_2^{2i} b_{m-1} - x^{2i} d_{m-1-i} = \frac{x^m}{x^2 - 1} (x^{2i} - 1) + \frac{x^{2i}}{x^2 - 1} (x^m - 1) f_{m-2i}(x).
\]

We set \( I_1 = \frac{x^m}{x^2 - 1} (x^{2i} - 1) \) and \( I_2 = \frac{x^{2i}}{x^2 - 1} (x^m - 1) f_{m-2i}(x) \). Then we have
\[
I_2 = \sum_{\mu=1}^{m-2i-1} a_\mu c_1^{m-2i-1-2\mu} c_2^{2\mu+\mu} b_{m-1} + (-1)^{\frac{m-2i+1}{2}} x^{\frac{m}{2} - i + 1} x^{\frac{m}{2} + i} \frac{x^m - 1}{x^2 - 1}.
\]
We set \( J = x^{\frac{m}{2}+i}x^{\frac{m-1}{2}} \). From (3.6), we have

\[
J - c_2^{m+i-1} e_m = \left( \sum_{\mu=0}^{\frac{m}{2}-i} (-1)^\mu \left[ c_1^{1+2\mu} c_2^{m+i-1-\mu} b_{m-1} \right] \right) = (-1)^{\frac{m}{2}-i} x^m \frac{x^{2i} - 1}{x^2 - 1}.
\]

Hence

\[
(3.17)
\]

\[
c_1^{m-2i-1} c_2^{2i} b_{m-1} = \sum_{\mu=1}^{\frac{m}{2}-i-1} a_\mu c_1^{m-2i-1-2\mu} c_2^{2\mu} b_{m-1}
\]

\[
+ (-1)^{\frac{m}{2}-i} 2 \left( c_2^{m+i-1} e_m + \sum_{\mu=0}^{\frac{m}{2}-2-i} (-1)^\mu \left[ c_1^{2\mu+1} c_2^{m+i-1-\mu} b_{m-1} \right] \right)
\]

\[
= -x^m \frac{x^{2i} - 1}{x^2 - 1}.
\]

On the other hand, we see from (3.13) that

\[
(3.18)
\]

\[
x^m \frac{x^{2i} - 1}{x^2 - 1} = \left( c_1^{2i-1} b_{m-1} e_m \right) + \frac{2i + 1}{2i - 1} \left( c_1^{2i-3} c_2 b_{m-1} e_m \right).
\]

We calculate

\[
S := (-1)^{\frac{m}{2}-i} 2 \left( c_2^{m+i-1} e_m + \sum_{\mu=0}^{\frac{m}{2}-2-i} (-1)^\mu \left[ c_1^{2\mu+1} c_2^{m+i-1-\mu} b_{m-1} \right] \right).
\]

By (3.6),

\[
S = (-1)^{\frac{m}{2}-i} 2 (S_1 + S_2),
\]

where

\[
S_1 = \sum_{\mu=0}^{\frac{m}{2}-i} (-1)^\mu c_1^{m+i+1-\mu} b_{2\mu+1} b_{m-1} \quad \text{and} \quad S_2 = c_2^{m+i-1} e_m - \sum_{\mu=0}^{\frac{m}{2}-i} (-1)^\mu c_2^{m+i-2-\mu} b_{2\mu+2} e_m.
\]

We have from Lemma 3.3 that

\[
S_1 = \sum_{\mu=1}^{\frac{m}{2}-1-i} (-1)^{\frac{m}{2}-1-\mu} \binom{m-1-2i-\mu}{\mu-1} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} b_{m-1}.
\]

Using \( \sum_{\mu=0}^{h} (-1)^\mu c_2^{h-\mu} b_{2\mu} = (-1)^h \mathfrak{b}_h \) (see Lemma 3.4),

\[
S_2 = c_2^{\frac{m}{2}+i} \sum_{\mu=0}^{\frac{m}{2}-i} (-1)^\mu c_2^{m-i-1-\mu} b_{2\mu} b_{m-1} = (-1)^{\frac{m}{2}-i} c_2^{2i} \mathfrak{b}_{\frac{m}{2}-1-i} b_{m-1}.
\]
The formula (3.17) is
\[ c_1^{m-2i-1}c_2^{2i} b_{m-1} + \sum_{\mu=1}^{m-i-1} (-1)^\mu \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1}c_2^{2i+\mu} b_{m-1} - 2c_2^{2i} \delta_{m-1-i} c_m \]
\[ = -x^m \frac{x^{2i-1}}{x^2 - 1}. \]
Comparing with (3.18), we obtain
\[ c_1^{m-2i-1}c_2^{2i} b_{m-1} = \sum_{\mu=1}^{m-i-1} (-1)^{\mu+1} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1}c_2^{2i+\mu} b_{m-1} \]
\[ - \frac{2}{2i-1} c_2^{2i} \delta_{m-1-i} c_m + \left[ c_1^{2i-1} b_{m-1} c_m \right] + \frac{2i+1}{2i-1} \left[ c_1^{2i-3} c_2 b_{m-1} c_m \right]. \]
This completes the proof of the formula. \( \square \)

4. The Chow ring of \( X_n \)

**Theorem 4.1.** Let \( T(X_n) \) and \( F(X_n) \) be the torsion part and the free part of \( CH(X_n) \), respectively. Then we have
1. For \( n = 4t \),
   \[ F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t \{ v_{4t-4} \}) \]
   \[ T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t \{ v_{4t-2}, v_{4t-4} v_{4t-2} \}). \]
2. For \( n = 4t + 1 \),
   \[ F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t \{ v_{4t} \}) \oplus \mathbb{Z}[c_2]/(c_2^{t-1}) \{ v_{4t} \} \]
   \[ T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t \{ v_{4t-2}, v_{4t-2} v_{4t} \}) \oplus \mathbb{Z}/(2t) \{ c_2^{t-1} v_{4t} \}. \]
3. For \( n = 4t + 2 \),
   \[ F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t \{ v_{4t} \}) \oplus \mathbb{Z}[v_{4t-2}] \]
   \[ T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^{t-1}) \{ c_2 v_{4t-2}, c_2 v_{4t-2} v_{4t} \} \oplus \mathbb{Z}/4 \{ v_{4t-2} v_{4t} \}. \]
4. For \( n = 4t + 3 \),
   \[ F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t \{ v_{4t} \}) \]
   \[ T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t \{ v_{4t+2}, v_{4t+2} v_{4t+2} \}) \oplus \mathbb{Z}/(2t+1) \{ c_2^t v_{4t} \}. \]

**Proof.** Let \( \tilde{X}_n = X_n \times_G \mathbb{A}^1 \) be the associated bundle of (3.1) and \( s : Y_n \to \tilde{X}_n \) the 0-section. Since \( s^* : CH(\tilde{X}_n) \overset{\sim}{\to} CH(Y_n) \) by Theorem 2.4 (ii), the first assertion of the same theorem for \( V = \tilde{X}_n \) and \( X = s(Y_n) \) gives an exact sequence
\[ CH(Y_n) \overset{s_*}{\to} CH(Y_n) \overset{\pi^*}{\to} CH(X_n) \to 0. \]
Theorem 4.1 follows from this and the ring structure of $CH^i(Y_n)$ in Theorems 3.7 and 3.9.

Next we consider the cycle map. The cohomology groups mean an etale cohomology [9, 13]. All varieties are defined over $K'$, which is a subfield of an algebraically closed field $K$. Let $l$ be a prime with $(l, \text{ch}(K)) = 1$. We denote a locally constant sheaf $\mu_l^{\otimes i}$ by $\mathbb{Z}/l^{(i)}$.

**Corollary 4.2.** The homomorphism $cl : CH^i(X_n) \to H^{2i}(X_n, \mathbb{Z}/l^{(i)})$ is injective.

**Proof.** Since $(\tilde{X}_n, Y_n)$ is a smooth pair, we have the Gysin sequence as in [5, Appendice 1.3.3] and [13, VI Remark 5.4]. Since the cycle map and the Gysin map are commutative, we have the following commutative diagram, where each row is exact:

\[
\begin{array}{cccccc}
CH^i(Y_n) & \xrightarrow{\cdot c_1} & CH^{i+1}(Y_n) & \xrightarrow{\pi^*} & CH^{i+1}(X_n) & \longrightarrow 0 \\
\downarrow cl & & \downarrow cl & & \downarrow cl & \\
H^{2i}(Y_n, \mathbb{Z}/l^{(i)}) & \xrightarrow{\cdot c_1} & H^{2(i+1)}(Y_n, \mathbb{Z}/l^{(i+1)}) & \xrightarrow{\pi^*} & H^{2(i+1)}(X_n, \mathbb{Z}/l^{(i+1)}) & \\
\end{array}
\]

Corollary 4.2 follows from Theorem 4.1 and this diagram.

**Remark 4.3.** Assume that we have a $K'$-isomorphism $Y_n \simeq SO(n, K)/(SO(n-4, K) \times GL(2, K))$, where $SO(n, K)$ and $SO(n-4, K)$ are split over $K'$, and that $\pi : Y_n \to X_n$ is a $K'$-map. Then the Galois actions $\overline{G} = \text{Gal}(\overline{K}/K')$ on $H^i(Y_n, \mathbb{Z}/l^{(i)})$ and $H^i(X_n, \mathbb{Z}/l^{(i)})$ are described by the character group $X(T)$ of a $K'$-split maximal torus $T$ of $SO(n, K)$. It follows from a result of [5] 8-2.

5. Tables of the ring structure of $CH^i(Y_n)$

5.1. Notations. (i) For $k \in \mathbb{N} \cup \{0\}$, we define $b_k$ and $d_k \in \mathbb{Z}[c_1, c_2]$ as follows:

\[
b_k = (-1)^k \sum_{\mu=0}^{[\frac{k}{2}]} (-1)^\mu \binom{k-\mu}{\mu} c_1^{k-2\mu} c_2^\mu
\]

and

\[
d_k = (-1)^k \sum_{\mu=0}^{k} (-1)^\mu \binom{2k-\mu+1}{\mu} c_1^{2k-2\mu} c_2^\mu.
\]
(ii) For \( g \in \mathbb{N} \) and \( \mu \in \mathbb{N} \cup \{0, -1\} \), we define \( a_{g, \mu} \in \mathbb{Z} \) by

\[
a_{g, \mu} = \begin{cases} 
(-1)^{1+\mu} \frac{g-1-\mu}{\mu-1} & \mu \geq 1 \\
-1 & \mu = 0 \\
0 & \mu = -1.
\end{cases}
\]

Then the integers \( a_{g, \mu} \) are characterized by

\[
(1 + x)^g = 1 + x^g + \sum_{\mu=1}^{[\frac{g}{2}]} a_{g, \mu} x^\mu (1 + x)^{g-2\mu}.
\]

(iii) The generators \( v_{2i} \) are given by (3.2) and (3.3).

5.2. An integral basis of \( CH(Y_n) \). In the following (I) and (II), we give an integral basis of \( CH(Y_n) \). The notations are explained as follows: Let \( S_n \) be the set of the monomial basis of \( A_n \) in (3.4). Let \( T \) be a subset of \( S_n \). Then for an element \( \xi \in T \), \( \langle \xi \rangle \) (resp. \( \langle \xi \rangle' \)) is defined to be the right-hand side of an equation (1)-(8) below. We consider a set

\[
\left\{ \langle \xi \rangle_{l_\xi} : \xi \in T \right\} \cup \{ \eta : \eta \in S_n - T \},
\]

where \( l_\xi \in \mathbb{N} \). Following this procedure, we obtain an integral basis of \( CH(Y_n) \). We abbreviate this basis as \( \left\{ \langle \xi \rangle_{l_\xi} : \xi \in T \right\} \).

(I) The case \( n = 2m \).

(i) For even \( m \),

\[
\left\{ \frac{c_1^{2i+1} c_2 v_{2m-4} v_{2m-2}}{2i+3}, \frac{c_1^{m-2j-3} c_2^{2j+1} v_{2m-4} v_{2m-2}}{m-2j-1} : 0 \leq i \leq \frac{m}{2} - 2, 1 \leq j \leq \frac{m}{2} - 2 \right\}.
\]

(ii) For odd \( m \),

\[
\left\{ \frac{c_1^{2i+1} c_2 v_{2m-4} v_{2m-2}}{2i+3}, \frac{c_1^{m-2j-2} c_2^{2j} v_{2m-4} v_{2m-2}}{m-2j} : 0 \leq i \leq \frac{m-5}{2}, 1 \leq j \leq \frac{m-3}{2} \right\}.
\]

(II) The case \( n = 2m + 1 \).

(iii) For even \( m \),

\[
\left\{ \frac{c_1^{2i+1} c_2 v_{2m-2} v_{2m}}{2i+3}, \frac{c_1^{m-2j-3} c_2^{2j+1} v_{2m-2} v_{2m}}{m-2j-1} : 0 \leq i \leq \frac{m}{2} - 2, 1 \leq j \leq \frac{m}{2} - 2 \right\}.
\]
(iv) For odd \( m \),

\[
\left\{ \frac{c_1^{2i+1}c_{2m-2}v_{2m}}{2i+3}, \frac{c_1^{m-2j-2}c_2^{2j}v_{2m-2}v_{2m}}{m-2j} : 0 \leq i \leq \frac{m-3}{2}, 1 \leq j \leq \frac{m-3}{2} \right\}.
\]

Here \( \langle \, \rangle \) and \( \langle \, \rangle' \) are defined as follows:

(1)

\[
\langle c_1^{2i+1}c_2v_{2m-4}v_{2m-2} \rangle = c_1^{2i+1}c_2v_{2m-4}v_{2m-2} + (-1)^{\frac{m+2i+3}{2}} \frac{(-1)^i(2i+3) + 1}{2} c_2^{2i+4}d_{m-2i-6}v_{2m-4}

- \sum_{\mu=1}^{i} a_{2i+3,\mu} c_1^{2i+1-2\mu} c_2^{1+\mu} v_{2m-4}v_{2m-2}.
\]

(2)

\[
\langle c_1^{m-2j-3}c_2^{2j+1}v_{2m-4}v_{2m-2} \rangle = c_1^{m-2j-3}c_2^{2j+1}v_{2m-4}v_{2m-2}

- \sum_{\mu=1}^{m-2j-4} a_{m-2j-1,\mu} c_1^{m-2j-3-2\mu} c_2^{2j+1+\mu} v_{2m-4}v_{2m-2}.
\]

(3)

\[
\langle c_1^{2i+1}c_2v_{2m-4}v_{2m-2} \rangle' = c_1^{2i+1}c_2v_{2m-4}v_{2m-2} + (-1)^{\frac{m+2i+1}{2}} \frac{(-1)^i(2i+3) + 1}{2} c_2^{2i+3}d_{m-2i-5}v_{2m-2}

- \sum_{\mu=1}^{i} a_{2i+3,\mu} c_1^{2i+1-2\mu} c_2^{1+\mu} v_{2m-4}v_{2m-2}.
\]

(4)

\[
\langle c_1^{m-2j-2}c_2^{2j}v_{2m-4}v_{2m-2} \rangle' = c_1^{m-2j-2}c_2^{2j}v_{2m-4}v_{2m-2}

- \sum_{\mu=1}^{m-2j-3} a_{m-2j,\mu} c_1^{m-2j-2-2\mu} c_2^{2j+\mu} v_{2m-4}v_{2m-2}.
\]

(5)

\[
\langle c_1^{2i+1}c_2v_{2m-2}v_{2m} \rangle = c_1^{2i+1}c_2v_{2m-2}v_{2m} + (-1)^{\frac{m+2i+2}{2}} \frac{(-1)^i(2i+3) + 1}{2} c_2^{2i+4}d_{m-2i-6}v_{2m}

- \sum_{\mu=1}^{i} a_{2i+3,\mu} c_1^{2i+1-2\mu} c_2^{1+\mu} v_{2m-2}v_{2m}.
\]

(6)

\[
\langle c_1^{m-2j-3}c_2^{2j+1}v_{2m-2}v_{2m} \rangle = c_1^{m-2j-3}c_2^{2j+1}v_{2m-2}v_{2m}
\]
5.3. The ring structure of $CH(Y_n)_{(2)}$ for $n = 2m$.

Here

\[ (1) = \left\{ \sum_{\mu=1}^{[m-1]} (-1)^{1+\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^\mu \right\} + (-1)^{m+1} 2v_{2m-2}. \]

\[ (2) = \left\{ \sum_{\mu=1}^{[m-k-1]} (-1)^{1+\mu} \binom{m-k-1-\mu}{\mu} c_1^{m-k-1-2\mu} c_2^{k+\mu} \right\} \]
\[ + \left\{ (-1)^{m+k} 2c_2 b_{k-1} \right\} v_{2m-4} + \left\{ (-1)^{m+k} 2c_2 b_{k-2} \right\} v_{2m-2}. \]

\[ (3) = \left\{ \sum_{\mu=1}^{m-1} (-1)^{1+\mu} \left( \frac{m-1-\mu}{\mu} \right) c_1^{m-1-2\mu} c_2^\mu \right\} v_{2m-4} + (-1)^{m+2} v_{2m-4} v_{2m-2}. \]

\[ (4) = \left\{ \sum_{\mu=1}^{m-2i-1} (-1)^{1+\mu} \left( \frac{m-2i-1-\mu}{\mu} \right) c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-4} \]

\[ + \left\{ (-1)^{m+2} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-4}{2} \right\} v_{2m-4} \]

\[ (5) = \left\{ \sum_{\mu=1}^{m-2i-3} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-4}{2} \right\} v_{2m-4} \]

\[ (6) = \left\{ \sum_{\mu=1}^{m-2i-3} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-4}{2} \right\} v_{2m-4} \]

\[ (7) = \left\{ \sum_{\mu=1}^{m-2i-2} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-4}{2} \right\} v_{2m-4} \]

\[ (8) = \left\{ \sum_{\mu=1}^{m-2i-3} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-3}{2} \right\} v_{2m-4} \]

\[ + \left\{ (-1)^{m+2} \sum_{\mu=1}^{m-2i-1} (-1)^{1+\mu} \left( \frac{m-2i-1-\mu}{\mu} \right) c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-4} \]

\[ + \left\{ \sum_{\mu=1}^{m-2i-1} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-3}{2} \right\} v_{2m-4} \]

\[ + \left\{ \sum_{\mu=1}^{m-2i-1} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-3}{2} \right\} v_{2m-4} \]

\[ + \left\{ (-1)^{m+2} \sum_{\mu=1}^{m-2i-1} (-1)^{1+\mu} \left( \frac{m-2i-1-\mu}{\mu} \right) c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-4} \]

\[ + \left\{ \sum_{\mu=1}^{m-2i-1} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-3}{2} \right\} v_{2m-4} \]

\[ + \left\{ \sum_{\mu=1}^{m-2i-1} \frac{2}{2i+1} c_2^{2i+1} d \frac{m-2i-3}{2} \right\} v_{2m-4} \]
\[-\left\{ \sum_{\mu=0}^{i-2} \left( a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{2+\mu} \right\} v_{2m-4} v_{2m-2}. \]

(9) = \left\{ \sum_{\mu=1}^{\frac{m-2i-1}{2}} (-1)^{1+\mu} \left( m-2i-2-\mu \right) c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2}

+ \left\{ (-1)^{m-2} \sum_{\mu=0}^{i} a_{2i+1,\mu} c_1^{2i-2\mu} c_2^{1+\mu} \right\} v_{2m-4} v_{2m-2}.

(10) = \left\{ \sum_{\mu=0}^{m-2i-1} \left( \frac{m-2i+1}{m-2i-1} a_{m-2i-1,\mu} + a_{m-2i+1,1+\mu} \right) c_1^{m-2i-3-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-4} v_{2m-2}.

(11) = \left\{ \sum_{\mu=1}^{m-2i-2} a_{m-2i+1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-4} v_{2m-2}.

(12) = \left\{ \sum_{\mu=0}^{m-2i-5} a_{m-2i-2,\mu} a_{m-2i+1,1+\mu} c_1^{m-2i-4-2\mu} c_2^{2i+2+\mu} \right\} v_{2m-4} v_{2m-2}.

(13) = (-1)^{m+2} \frac{d_{m-2}}{2} v_{2m-4}.

(14) = (-1)^{m+3} \frac{d_{m-2}}{2} v_{2m-4}.

(15) = - b_{m-2} v_{2m-4} + (-1)^{m+3} \frac{d_{m-2}}{2} v_{2m-2}.

(16) = (-1)^{m+2} \frac{c_2 d_{m-4}}{2} v_{2m-4}.

(17) = (-1)^{m+1} \frac{c_2 d_{m-3}}{2} v_{2m-2}.
5.4. The ring structure of \( CH(Y_n) \) for \( n = 2m + 1 \).

|                | even m | odd m |
|----------------|--------|-------|
| \( c_1^{m-1} \) |        | (i)   |
| \( c_1^{m-k-1} k_2^k \) \( (k \geq 1) \) |        | (ii)  |
| \( c_1^{m-2i-1} c_2^i v_{2m-2} \) \( (i \geq 0) \) | (iii)  | (iv)  |
| \( c_1^{m-2i-2} c_2^{2i+1} v_{2m-2} \) \( (i \geq 0) \) | (v)    |       |
| \( c_1^{m-2i-1} v_{2m} \) | (vi)   |       |
| \( c_1^{m-2i-2} c_2^{2i} v_{2m} \) \( (i \geq 1) \) | (vii)  |       |
| \( c_1^{m-2i-1} c_2^{2i+1} v_{2m} \) \( (i \geq 0) \) | (viii) | (ix)  |
| \( c_1^{m-2i-2} c_2^{2i+1} v_{2m-2} v_{2m} \) \( (i \geq 0) \) | (x)    | (xi)  |
| \( v_{2m-2} \) | (xii)  | (xiii)|
| \( v_{2m} \) | (xiv)  | (xv)  |

Here

(i) \[ (i) = \left\{ \sum_{\mu=1}^{m+1-k} \frac{1}{\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^\mu \right\} + (-1)^{m+1} v_{2m-2}. \]

(ii) \[ (ii) = \left\{ \sum_{\mu=1}^{m+k-2} \frac{1}{\mu} \binom{m-k-1-\mu}{\mu} c_1^{m-k-1-2\mu} c_2^\mu \right\} + \left\{ (-1)^{m+k+2} c_2 b_{k-1} \right\} v_{2m-2} + \left\{ (-1)^{m+k+1} 2 b_{k-1} \right\} v_{2m}. \]

(iii) \[ (iii) = \left\{ \sum_{\mu=1}^{m-2i-2} \frac{1}{\mu} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1-2\mu} c_2^\mu \right\} v_{2m-2} + \left\{ (-1) \frac{m-2i}{2i-1} c_2^{2i} d_{m-2i-2} \right\} v_{2m} - \left\{ \sum_{\mu=0}^{i} \frac{2i+1}{2i-1} a_{2i-1,1+\mu} + a_{2i+1,\mu} c_2^{2i-1-2\mu} c_2^\mu \right\} v_{2m-2} v_{2m}. \]

(iv) \[ (iv) = \left\{ (-1) \frac{m+2i+3}{2i+1} c_2^{2i+1} d_{m-2i-3} + \sum_{\mu=1}^{m-2i-1} a_{m-2i,\mu} c_1^{m-2i-1-2\mu} c_2^\mu \right\} v_{2m-2} + \left\{ \sum_{\mu=-1}^{i-2} \frac{2i-1}{2i+1} a_{2i+1,1+\mu} c_2^{2i-3-2\mu} c_2^\mu \right\} v_{2m-2} v_{2m}. \]
\[(v) = \left\{ \sum_{\mu=1}^{m-2i-2} (-1)^{1+\mu} \binom{m-2i-2-\mu}{\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} + \left\{ (-1)^{m+2} \sum_{\mu=0}^{i} a_{2i+1,\mu} c_1^{2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_2 m.\]

\[(vi) = \left\{ \sum_{\mu=1}^{m-2i} (-1)^{1+\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^{2i+1+\mu} \right\} v_2 m + (-1)^{m+1} v_{2m-2} v_2 m.\]

\[(vii) = \left\{ \sum_{\mu=1}^{m-2i-1} (-1)^{1+\mu} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1-2\mu} c_2^{2i+1+\mu} \right\} v_2 m + (-1)^{m+2} \sum_{\mu=0}^{i-1} a_{2i-1,\mu} c_1^{2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_2 m.\]

\[(viii) = \left\{ (-1)^{m+2i+4} \frac{4i}{2i+1} c_2^{2i+2} d_{m-2i-4} + \sum_{\mu=1}^{m-2i-3} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_2 m + \left\{ 2 \sum_{\mu=1}^{2i-2} \left( a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_2 m.\]

\[(ix) = \left\{ (-1)^{m+2i+3} \frac{2}{2i+3} c_2^{2i+3} d_{m-2i-5} \right\} v_{2m-2} + \left\{ \sum_{\mu=1}^{m-2i-3} (-1)^{1+\mu} \binom{m-2i-2-\mu}{\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_2 m + \left\{ 2 \sum_{\mu=1}^{2i-1} \left( a_{2i+1,\mu} - a_{2i+1,1+\mu} + \frac{2i+1}{2i+3} a_{2i+3,1+\mu} \right) c_1^{2i-1-2\mu} c_2^{1+\mu} \right\} v_{2m-2} v_2 m.\]

\[(x) = \left\{ \sum_{\mu=0}^{m-2i-4} \frac{m-2i+1}{m-2i-1} a_{m-2i-1,\mu} + a_{m-2i+1,1+\mu} \right\} c_1^{m-2i-3-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_2 m.\]

\[(xi) = \left\{ \sum_{\mu=1}^{m-2i-1} a_{m-2i,\mu} c_1^{m-2i-1-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_2 m.\]
\[(xii) = \left\{ \sum_{\mu=1}^{m-2i-2} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_{2m}. \]

\[(xiii) = \left\{ \sum_{\mu=0}^{m-2i-5} \left( \frac{m-2i}{m-2i-2} a_{m-2i-2,\mu} + a_{m-2i,1+\mu} \right) c_1^{m-2i-4-2\mu} c_2^{2i+2+\mu} \right\} v_{2m-2} v_{2m}. \]

\[(xiv) = (-1)^{\frac{m+1}{2}} d_{\frac{m-2}{2}} v_{2m}. \]

\[(xv) = (-1)^{\frac{m+1}{2}} c_2 d_{\frac{m-3}{2}} v_{2m-2}. \]

\[(xvi) = (-1)^{\frac{m}{2}} c_2 d_{\frac{m-4}{2}} v_{2m}. \]

\[(xvii) = (-1)^{\frac{m+1}{3}} \frac{1}{3} c_2 d_{\frac{m-5}{2}} v_{2m-2} - \frac{2}{3} c_1 v_{2m-2} v_{2m}. \]

5.5. **A remark on the ring structure of** \(CH(Y_n)\). We have given the ring structure of \(CH(Y_n)\) in 5.3 and 5.4. But actually, it is easy to determine the ring structure of \(CH(Y_n)\) from 5.2, 5.3 and 5.4. For example, by the basis in 5.2, the formula 5.3 (5) is rewritten as follows:

\[(5)' \quad c_1^{m-2i-2} c_2^{2i+1} v_{2m-4} = \left\{ (-1)^{\frac{m+2i+2}{2}} \left( (-1)^i (2i-1) + 1 \right) c_2^{2i+2} d_{\frac{m-2i-4}{2}} \right. \]

\[\left. + \sum_{\mu=1}^{m-2i-2} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-4} \]

\[+ \left( 2 \sum_{\mu=-1}^{i-2} a_{2i-1,\mu} c_1^{2i-3-2\mu} c_2^{2i+1+\mu} \right) v_{2m-4} v_{2m-2} - \frac{4i-2}{2i+1} (c_1^{2i-1} c_2 v_{2m-4} v_{2m-2}). \]

The other cases can be calculated similarly.

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