Completeness of the set of scattering amplitudes *†‡

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Abstract
Let \( f \in L^2(S^2) \) be an arbitrary fixed function on the unit sphere \( S^2 \), with a sufficiently small norm, and \( D \subset \mathbb{R}^3 \) be an arbitrary fixed bounded domain. Let \( k > 0 \) and \( \alpha \in S^2 \) be fixed.

It is proved that there exists a potential \( q \in L^2(D) \) such that the corresponding scattering amplitude \( A(\alpha') = A_q(\alpha') = A_q(\alpha', \alpha, k) \) approximates \( f(\alpha') \) with arbitrary high accuracy: \( \|f(\alpha') - A_q(\alpha')\|_{L^2(S^2)} \leq \varepsilon \), where \( \varepsilon > 0 \) is an arbitrarily small fixed number. The results can be used for constructing nanotechnologically "smart materials".

1 Introduction
Let \( D \subset \mathbb{R}^3 \) be a bounded domain, \( k = \text{const} \), \( \alpha \in S^2 \), \( S^2 \) is the unit sphere. Consider the scattering problem:

\[
[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3,
\]

\[
u = u_0 + A_q(\alpha', \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad u_0 = e^{ik\alpha \cdot x}, \quad r = |x| \to \infty, \quad \alpha' = \frac{x}{r},
\]

The coefficient \( A_q \) is called the scattering amplitude, and \( q \in L^2(D) \) is a potential. The solution to (1)-(2) is called the scattering solution. It solves the equation

\[
u = u_0 - Tu, \quad Tu := \int_D g(x, y)q(y)u(y)dy,
\]

\[
g = g(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad u = u(y) = u(y, \alpha, k).
\]

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The solution $u$ to (3) is unique in $L^2(D)$ for any complex-valued $q$ for which $\|T\| < 1$, i.e., for any sufficiently small $q$.

We are interested in the following problem, which differs from the standard inverse scattering problem with fixed-energy data, studied in [R].

**Question (Problem) P.** Given an arbitrary fixed $f(\alpha') \in L^2(S^2)$, can one find a $q \in L^2(D)$, such that

$$\|A_q(\alpha') - f(\alpha')\|_{L^2(S^2)} \leq \varepsilon,$$

$$A_q(\alpha') = A_q(\alpha', \alpha, k),$$

(4)

where $\varepsilon > 0$ is an arbitrary small number, $\alpha \in S^2$ and $k > 0$ are fixed?

The answer to this question was not known. The scattering problem in (1)–(2) has been studied much (see, e.g., [C], [P], [R]). The inverse scattering problem with fixed-energy data (ISP) was solved in [R1], [R2], [R3], and [R, Chapter 5]. This problem consists of finding $q \in L^2(D)$ from the corresponding scattering amplitude $A_q(\alpha', \alpha, k)$ given for all $\alpha', \alpha \in S^2$ at a fixed $k > 0$. It was proved in [R1] that this problem has a unique solution. In [R3] reconstruction algorithms were proposed for finding $q$ from exact and from noisy data, and stability estimates were established for the solution.

The problem P we have posed in this paper is different from the ISP. The data $A_q(\alpha')$ do not determine $q \in L^2(D)$ uniquely, in general. The potential $q$ for which (4) holds is not unique even if $\varepsilon = 0$ and $f(\alpha') = A_q(\alpha', \alpha)$ for some $\alpha \in S^2$ and $q \in L^2(D)$. We want to know if there is a $q \in L^2(D)$, $q = q_{\varepsilon}(x)$, such that (4) holds with an arbitrarily small fixed $\varepsilon > 0$.

We prove that the answer is yes, provided that $f$ is sufficiently small. The ”smallness” condition will be specified in our proof. The question itself is motivated by the problem $P_1$ studied in [R4]:

**Problem $P_1$.** Can one distribute small acoustically soft particles in a bounded domain so that the resulting domain would have a desired radiation pattern, i.e., a desired scattering amplitude?

The problem P, which is studied here, is of independent interest.

The following two lemmas allow us to give a positive answer to Question P under the ”smallness” assumption.

**Lemma 1.** Let $f \in L^2(S^2)$ be arbitrary and $k > 0$ be fixed. Then

$$\inf_{h \in L^2(D)} \|f(\alpha') + \frac{1}{4\pi} \int_D e^{-ika' \cdot x} h(x) dx\|_{L^2(S^2)} = 0.$$  

(5)

**Lemma 2.** Let $h \in L^2(D)$ be arbitrary, with a sufficiently small norm. Then

$$\inf_{q \in L^2(D)} \|h - qu\|_{L^2(D)} = 0,$$

(6)

where $u = u(x, \alpha, k)$ is the scattering solution corresponding to $q$. Under the above ”smallness” assumption, there exists a potential $q$, such that $qu = h$. 

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If Lemmas 1 and 2 are proved, then the positive answer to Question P (under the "smallness" assumption) follows from the well-known formula for the scattering amplitude:

\[ A_q(\alpha') = -\frac{1}{4\pi} \int_D e^{-ika'\cdot x} q(x) u(x, \alpha, k) dx, \]  

in which \( k > 0 \) and \( \alpha \in S^2 \) are fixed. The answer is given in Theorem 1.

**Theorem 1.** Let \( \varepsilon > 0, k > 0, \alpha \in S^2 \) and \( f \in L^2(S^2) \) be arbitrary, fixed, with a sufficiently small norm. Then there is a \( q \in L^2(D) \) such that (4) holds.

There are many potentials \( q \) for which (4) holds. In Section 2 proofs are given.

In Section 3 a method is given for finding \( q \in L^2(D) \) such that (4) holds.

### 2 Proofs

**Proof of Lemma 1.** If (5) is not true, then there is an \( f \in L^2(S^2) \) such that

\[ 0 = \int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta\cdot x} h(x) dx = \int_D dx h(x) \int_{S^2} f(\beta) e^{-ik\beta\cdot x} d\beta \quad \forall h \in L^2(D). \]  

Since \( h(x) \) is arbitrary, relation (8) implies

\[ \int_{S^2} f(\beta) e^{-ik\beta\cdot x} d\beta = 0 \quad \forall x \in D. \]  

The left-hand side of (9) is the Fourier transform of a compactly supported distribution \( \frac{\delta(\lambda - k)}{\lambda^2} f(\beta) \), where \( \delta(\lambda - k) \) is the delta-function. Since the Fourier transform is injective, it follows that \( f(\beta) = 0 \). This proves Lemma 1.

An alternative proof can be given. It is known that

\[ e^{-ik\beta\cdot x} = \sum_{\ell=0, -\ell \leq m \leq \ell} 4\pi i^\ell j_\ell(kr) Y_{\ell,m}(-x^0) Y_{\ell,m}(\beta), \quad r = |x|, \quad x^0 := \frac{x}{r}, \]  

where \( Y_{\ell,m} \) are orthonormal in \( L^2(S^2) \) spherical harmonics, \( Y_{\ell,m}(-x^0) = (-1)^\ell Y_{\ell,m}(x^0) \), \( j_\ell(r) := (\frac{\pi}{2r})^{1/2} J_{\ell + \frac{1}{2}}(r) \), and \( J_\ell(r) \) is the Bessel function. Let

\[ f_{\ell,m} := (f, Y_{\ell,m})_{L^2(S^2)}. \]  

From (9) and (10) it follows that

\[ f_{\ell,m} j_\ell(kr) = 0, \quad \forall x \in D, -\ell \leq m \leq \ell. \]  

If \( k > 0 \) is fixed, one can always find \( r = |x|, x \in D \), such that \( j_\ell(kr) \neq 0 \).

Thus, (12) implies \( f_{\ell,m} = 0 \quad \forall \ell, -\ell \leq m \leq \ell \). Lemma 1 is proved. The "smallness" assumption is not needed in this proof.
Proof of Lemma 2. In this proof we use the "smallness" assumption. If the norm of $f$ is sufficiently small, the the norm of $h$ is small so that condition (23) (see below) is satisfied. If this condition is satisfied, then formula (24) (see below) yields the desired potential $q$, and $h = qu$, where $u$ is the scattering solution, corresponding to $q$. Therefore, the infimum in (6) is attained. Lemma 2 is proved.

Let us give another argument, which shows the role of the "smallness" assumption from a different point of view. Note, that if $||q|| \to 0$, then the set of the functions $qu$ is a linear set. In this case, if one assumes that (6) is not true, then one can claim that there is an $h \in L^2(D), h \neq 0$, such that

$$
\int_D dxh(x)q(x)u(x) = 0 \quad \forall q \in L^2(D), \quad ||q|| << 1.
$$

(13)

Choose

$$
q = c\text{e}^{-ik\alpha \cdot x}, \quad c = \text{const} > 0.
$$

(14)

Let $c$ be so small that $||T|| = O(c) < 1$, where $T : L^2(D) \to L^2(D)$ is defined in (3). Then equation (13) is uniquely solvable and

$$
u = u_0 + O(c) \quad \text{as} \; c \to 0.
$$

(15)

From (13) and (15) one gets

$$
c \int_D |h|^2 dx + O(c^2) = 0 \quad \forall c \in (0, c_0).
$$

(16)

If $c \to 0$, then (16) implies $\int_D |h(x)|^2 dx = 0$. Therefore $h = 0$.

3 A method for finding $q$ for which (4) holds

Lemmas 1 and 2 show a method for finding a $q \in L^2(D)$ such that (4) holds. Given $f(\alpha') \in L^2(S^2)$, let us find $h(x)$ such that

$$
||f(\alpha') + \frac{1}{4\pi} \int_D e^{-ik\alpha' \cdot x}h(x)dx ||^2_{L^2(S^2)} < \varepsilon^2.
$$

(17)

This is possible by Lemma 1. It can be done numerically by taking $h = h_n = \sum_{j=1}^{n} c_j \varphi_j(x)$, where $\{\varphi_j\}_{1 \leq j < \infty}$ is a basis of $L^2(D)$, and minimizing the quadratic form on the left-hand side of (17) with respect to $c_j, 1 \leq j < n$. For sufficiently large $n$ the minimum will be less than $\varepsilon^2$ by Lemma 1.

The minimization with the accuracy $\varepsilon^2$ can be done analytically: let $B_b$ be a ball of radius $b$ centered at a point $0 \in D, B_b \subset D, h = 0$ in $D \setminus B_b$, 

$$
h(x) = \sum_{\ell=0,-\ell \leq m \leq \ell}^{\infty} h_{\ell,m}(r)Y_{\ell,m}(x^0) \quad \text{in} \; B_b,
$$

4
\[ f(\alpha') = \sum_{\ell=0,-\ell \leq m \leq \ell}^\infty f_{\ell,m} Y_{\ell,m}(\alpha'), \]

and

\[ \sum_{l>L, -\ell \leq m \leq \ell} |f_{\ell,m}|^2 < \varepsilon^2. \]

For \( 0 \leq \ell \leq L \) let us equate the Fourier coefficients of \( f(\alpha') \) and of the integral from (17). We use the known formula:

\[ e^{-ik\beta \cdot x} = \sum_{\ell=0, -\ell \leq m \leq \ell} 4\pi i^\ell j_\ell(kr)Y_{\ell,m}(-x^0)Y_{\ell,m}(\beta). \]

This leads to the following relations for finding \( h_{\ell,m}(r) \):

\[ f_{\ell,m} = -(-i)^\ell \sqrt{\frac{\pi}{2k}} \int_0^b r^{3/2} J_{\ell+\frac{1}{2}}(kr)h_{\ell,m}(r)dr, \quad 0 \leq \ell \leq L. \] (18)

There are many \( h_{\ell,m}(r) \) satisfying equation (18). Let us take \( b = 1 \) and use [B, Formula 8.5.5],

\[ \int_0^1 x^{\mu+\frac{1}{2}} J_{\nu}(kx)dx = k^{-\mu-\frac{3}{2}} \left[ (\gamma + \mu + \frac{1}{2})k \right] j_{\nu-\frac{1}{2}}(k) - k J_{\nu-1}(k)S_{\nu-\frac{1}{2},\nu-1}(k) + 2^{\mu+\frac{1}{2}} \Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\nu+\nu+\frac{1}{2}\right) \Gamma\left(\nu+\nu+\frac{1}{2}\right) \]

\[ := g_{\mu,\nu}(k), \]

where \( S_{\mu,\nu}(k) \) are Lommel’s functions. Thus, one may take

\[ h_{\ell,m}(r) = \frac{f_{\ell,m}}{-(-i)^\ell \sqrt{2k} g_{1,\ell+\frac{1}{2}}(k)}, \quad 0 \leq \ell \leq L; \quad h_{\ell,m}(r) = 0 \quad \ell > L. \]

Thus, the coefficients \( h_{\ell,m}(r) \) do not depend on \( r \in (0,1) \) in this choice of \( h \): inside the ball \( B_0 \) the function \( h(x)=h(x^0) \) depends only on the angular variables, and outside this ball \( h = 0. \)

With this choice of \( h_{\ell,m}(r) \) the left-hand side of (17) equals to

\[ \sum_{\ell=L+1, -\ell \leq m \leq \ell}^\infty |f_\ell|^2 < \varepsilon^2. \]

The above choice of \( h_{\ell,m}(r) \), which yields an analytical choice of \( h(x) \), is one of the choices for which (17) holds. The function \( g_{1,\ell+\frac{1}{2}}(k) \) in the definition of \( h_{\ell,m}(r) \) decays rapidly when \( \ell \) grows. This makes it difficult numerically to calculate accurately \( h_{\ell,m}(r) \) when \( \ell \) is large. If \( f = 1 \) in a small solid angle and \( f = 0 \) outside of this angle, then one needs
large $L$ to approximate $f$ by formula (17) with small $\varepsilon$. Numerical difficulties arise in this case. This phenomenon is similar to the one known in optics and antenna synthesis as superresolution difficulties (see [R6], [R7], [R8]).

Let $h \in L^2(D)$ be a function for which (17) holds. Consider the equation

$$h = qu,$$  \hspace{1cm} (19)

where $u = u(x; q)$ is the scattering solution, i.e., the solution to equation (3). Let

$$w := e^{-ik\alpha x}u, \quad G(x, y) := g(x, y)e^{-ik\alpha(x-y)}, \quad H := he^{-ik\alpha x}, \quad \psi := qw.$$  \hspace{1cm} (20)

Then equation (3) is equivalent to the equation

$$w = 1 - \int_D G(x, y)\psi(y)dy,$$  \hspace{1cm} (21)

and $\psi(y) = H(y)$ by (19). Multiply (21) by $q$ and get

$$\psi(x) = q(x) - q(x) \int_D G(x, y)\psi(y)dy.$$  

Since $\psi(x) = H(x)$, we get

$$q(x) = H(x) \left[1 - \int_D G(x, y)H(y)dy\right]^{-1}. \hspace{1cm} (22)$$

If $H(x)$ is such that

$$\inf_{x \in D} \left|1 - \int_D G(x, y)H(y)dy\right| > 0,$$  \hspace{1cm} (23)

then $q(x)$, defined in (22), belongs to $L^2(D)$ and (21) holds. If $H$ is sufficiently small then (23) holds. If $D$ is small and $H$ is fixed, then (23) holds because

$$\left|\int_D G(x, y)Hdy\right| \leq \int_D \frac{|H|dy}{4\pi|x-y|} \leq \frac{1}{4\pi}\|H\|_{L^2(D)} \left(\int_D \frac{dy}{|x-y|^2}\right)^{1/2} = O(a^{1/2})\|H\|,$$

where $a$ is the radius of a smallest ball containing $D$.

Formula (22) can be written as

$$q(x) = h(x) \left[u_0 - \int_D g(x, y)h(y)dy\right]^{-1},$$  \hspace{1cm} (24)

where $u_0$ and $g$ are defined in (3). Numerically equation (24) worked for $f(\beta)$ which were large in absolute value.
4 An idea of a method for making a material with the desired radiation pattern

Here we describe an idea of a method for calculation of a distribution of small particles, embedded in a medium, so that the resulting medium would have a desired radiation pattern for the plane wave scattering by this medium. This idea is described in more detail in [R4]. The results of this paper complement the results in [R4], but are completely independent of [R4], and are of independent interest.

Suppose that a bounded domain $D$ is filled in by a homogeneous material. Assume that we embed many small particles into $D$. Smallness means that $k_0a << 1$, where $a$ is the characteristic dimension of a particle and $k_0$ is the wavenumber in the region $D$ before the small particles were embedded in $D$. The question is:

Can the density of the distribution of these particles in $D$ be chosen so that the resulting medium would have the desired radiation pattern for scattering of a plane wave by this medium?

For example, if the direction $\alpha$ of the incident plane wave is fixed, and the wave number $k$ of the incident plane wave $e^{ik\alpha \cdot x}$ in the free space outside $D$ is fixed, then can the particles be distributed in $D$ with such a density that the scattering amplitude $A(\alpha', \alpha, k)$ of the resulting medium would approximate with the desired accuracy an a priori given arbitrary function $f(\alpha') \in L^2(S^2)$ in the sense (4)?

Assume that the particles are acoustically soft, that is, the Dirichlet boundary condition is assumed on their boundary, that $k_0a << 1$ and $a << 1$, where $d > 0$ is the minimal distance between two distinct particles, and that the number $N$ of the small particles tends to infinity in such a way that the limit $C(x) := \lim_{N \to \infty} \lim_{r \to 0} \frac{\sum_{D_j \subset B(x,r)} C_j}{|B(x,r)|}$ exists. Under these assumptions, it is proved in [R4] that the scattered field can be described by equation (1) with $q(x)$ that can be expressed analytically via $C(x)$. Here $D_j$ is the region occupied by $j$-th acoustically soft particle, $C_j$ is the electrical capacitance of the perfect conductor with the shape $D_j$, $B(x,r)$ is the ball of radius $r$ centered at the point $x$, and $|B(x,r)| = \frac{4\pi r^3}{3}$ is the volume of this ball. If the small particles are identical, and $C$ is the electrical capacitance of one small particle, then $C(x) = N(x)C$, where $N(x)$ is the number of small particles per unit volume around the point $x$. Formulas for the electrical capacitances for conductors of arbitrary shapes are derived in [R5].

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