SPHERICAL TWISTS AND THE CENTER OF AUTOEQUIVALENCE GROUPS OF K3 SURFACES

KOHEI KIKUTA

Abstract. Homological mirror symmetry predicts that there is a relation between autoequivalence groups of derived categories of coherent sheaves on Calabi–Yau varieties, and the symplectic mapping class groups of symplectic manifolds. In this paper, as an analogue of Dehn twists for mapping class groups of closed oriented real surfaces, we study spherical twists for derived categories of algebraic varieties. We introduce the intersection number and relate it to group-theoretic properties of spherical twists.

As an application, we compute the center of autoequivalence groups of derived categories of K3 surfaces.

1. Introduction

Let $X$ be a smooth projective variety over a field $K$ and $\mathcal{D}^b(X)$ the bounded derived categories of coherent sheaves on $X$. The autoequivalence group $\text{Aut}(\mathcal{D}^b(X))$ is the group of exact self-equivalences of $\mathcal{D}^b(X)$. There are some attempts to compute $\text{Aut}(\mathcal{D}^b(X))$, but this problem is rather difficult in general. The aim of this paper is to study group-structures of $\text{Aut}(\mathcal{D}^b(X))$ by focusing on spherical twists. The details are explained in the following two subsections.

1.1. Spherical twists and intersection number. Homological mirror symmetry predicts that there is a relation between autoequivalence groups of derived categories of coherent sheaves on Calabi–Yau varieties, and the symplectic mapping class groups of symplectic manifolds. As an analogue of Dehn twists along Lagrangian spheres, Seidel–Thomas introduced spherical objects and some autoequivalence $T_E \in \text{Aut}(\mathcal{D}^b(X))$ called the spherical twist along a spherical object $E \in \mathcal{D}^b(X)$ ([ST]).

To clarify the analogy, we denote the sum of dimensions of all extension groups by $i(M, N)$ i.e. for $M, N \in \mathcal{D}^b(X)$,

$$i(M, N) := \sum_{p \in \mathbb{Z}} \dim_K \text{Hom}(M, N[p]),$$

which we call the intersection number of $M$ and $N$ in this paper. Using the intersection number, we show one-to-one correspondence between spherical twists and spherical objects.
Proposition 1.1 (Proposition 3.6). Let $E_1, E_2 \in \mathcal{D}^b(X)$ be spherical objects. Then the following are equivalent:

(i) We have $T_{E_1} = T_{E_2}$.

(ii) There exists $l \in \mathbb{Z}$ such that $E_1 \simeq E_2[l]$.

In mapping class groups, a mapping class commutes with a Dehn twist along a simple closed curve if and only if it preserves the curve up to isotopy (cf. [FM, Fact 3.8]). In autoequivalence groups, this basic fact corresponds to the following.

Theorem 1.2 (Theorem 3.8). Let $E \in \mathcal{D}^b(X)$ be spherical object. An autoequivalence $\Phi \in \text{Aut}(\mathcal{D}^b(X))$ satisfies $\Phi \circ T_E = T_E \circ \Phi$ if and only if $\Phi(E) = E[l]$ for some $l \in \mathbb{Z}$.

It is well-known that two Dehn twists whose intersection number is greater than one generate the (non-abelian) free group of rank 2 (cf. [FM, Theorem 3.14]). Assuming that a natural inequality (see Conjecture 4.3 (i)), we prove the corresponding result for autoequivalence groups.

Theorem 1.3 (Theorem 4.7). Let $E_1, E_2 \in \mathcal{D}^b(X)$ be spherical objects satisfying $E_1 \not\simeq E_2[l]$ for any $l \in \mathbb{Z}$. Assume that for each $k \in \mathbb{Z}$, $i = 1, 2$ and $M, N \in \mathcal{D}^b(X)$, we have

$$i(E_i, M)i(E_i, N) \leq i(T_k^i M, N) + i(M, N).$$

If $i(E_1, E_2) \geq 2$, then $\langle T_{E_1}, T_{E_2} \rangle$ is isomorphic to the free group $F_2$ of rank 2.

To prove this theorem, we use the ping-pong lemma. This approach is the same as the case of mapping class groups. As a corollary, in lower cases: $i(E_1, E_2) = 0$ or 1, the group-relation of two spherical twists is characterized by the intersection number (Corollary 4.9 and 4.10).

1.2. The center of autoequivalence groups of K3 surfaces. Let $X$ be a complex algebraic K3 surface and $\text{Aut}_{CY}(\mathcal{D}^b(X))$ the subgroup of autoequivalences trivially acting on the transcendental lattice of $X$. The autoequivalence groups are usually studied by using the action on the cohomology. In contrast to the automorphism groups of K3 surfaces, there are non-trivial autoequivalences trivially acting on the cohomology: squares of spherical twists for example. These cohomologically trivial autoequivalences are detected by the action on the space $\text{Stab}(X)$ of stability conditions, which is a Bridgeland’s approach in [Bri2]. Then we naturally consider the subgroups $\text{Aut}^\dagger(\mathcal{D}^b(X))$ and $\text{Aut}_{CY}^\dagger(\mathcal{D}^b(X))$ which preserve the distinguished component of $\text{Stab}(X)$ (see Definition 2.6).

The center (Definition 2.1) measures the commutativity of a given group. The triviality of the center of the mapping class group is proved via the equivalence between
SPHERICAL TWISTS AND THE CENTER OF AUTOEQUIVALENCE GROUPS OF K3 SURFACES

the commutativity with Dehn twists and the fixability of simple closed curves (cf. [FM, Fact 3.8]). Similarly, as an application of Theorem 1.2, we compute the center groups $Z(\text{Aut}^+(D^b(X)))$ and $Z(\text{Aut}_{\text{CY}}^+(D^b(X)))$.

**Theorem 1.4** (Theorem 5.1). Let $X$ be a K3 surface of any Picard rank, $m_X$ the order of the finite cyclic group $\text{Aut}_t(X) := \{f \in \text{Aut}(X) \mid H^2(f)\,|_{\text{NS}(X)} = \text{id}_{\text{NS}(X)}\}$, and $f_t$ a generator of $\text{Aut}_t(X)$. Then we have the following

(i) $Z(\text{Aut}^+(D^b(X))) = \text{Aut}_t(X) \times \mathbb{Z}[1] \simeq (\mathbb{Z}/m_X) \times \mathbb{Z}$.

(ii) $Z(\text{Aut}_{\text{CY}}^+(D^b(X))) = \begin{cases} \langle (f_t^*)^{m_X/2} \circ [1] \rangle & \text{if } m_X \text{ is even} \\ \mathbb{Z}[2] & \text{if } m_X \text{ is odd} \end{cases}
\simeq \mathbb{Z}$.

We also compute the center of the quotient $\text{Aut}_{\text{CY}}^+(D^b(X))/\mathbb{Z}[2]$ (Corollary 5.2), which is closely related to the orbifold fundamental group of the stringy Kähler moduli space of $X$. These results reveal that the number $m_X$ determines the group-structure of the center groups, so we explain some examples of $m_X$ in subsection 5.1.

**Acknowledgements.** The author is supported by JSPS KAKENHI Grant Number 20K22310 and 21K13780.

**Notation and Convention.**

- A *K3 surfaces* $X$ means a complex algebraic K3 surface i.e. a smooth projective surface over the complex number field $\mathbb{C}$ such that $K_X \simeq O_X$ and $H^1(X, O_X) = 0$. Let $\text{Aut}(X)$ be the automorphism group of $X$.
- The category $D^b(X) := D^b(\text{Coh}(X))$ is the derived category of bounded complexes of coherent sheaves on $X$.
- An *autoequivalence* of $D^b(X)$ is an exact self-equivalence $D^b(X) \to D^b(X)$. The *autoequivalence group* $\text{Aut}(D^b(X))$ is the group of (natural isomorphism classes) of autoequivalences of $D^b(X)$.

**2. Preliminaries**

Let $X$ be a K3 surface.
2.1. Hodge structures on the Mukai lattice. The integral cohomology group $H^*(X, \mathbb{Z})$ of $X$ has the lattice structure given by the Mukai pairing

$$((r_1, c_1, s_1), (r_2, c_2, s_2)) := c_1 \cdot c_2 - r_1 s_2 - r_2 s_1$$

for $(r_1, c_1, s_1), (r_2, c_2, s_2) \in H^*(X, \mathbb{Z})$. The lattice $H^*(X, \mathbb{Z})$ called the Mukai lattice of $X$ is an even unimodular lattice of signature $(4, 20)$. The Mukai lattice has a weight two Hodge structure $\tilde{H}(X, \mathbb{Z})$ given by $\tilde{H}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=2} \tilde{H}^{p,q}(X)$ and

$$\tilde{H}^{2,0}(X) := H^{2,0}(X), \tilde{H}^{1,1}(X) := \bigoplus_{p=0}^2 H^{p,p}(X), \tilde{H}^{0,2}(X) := H^{0,2}(X).$$

This Hodge structure contains the ordinary Hodge structure on $H^2(X, \mathbb{Z})$ as a primitive sub-Hodge structure. The algebraic part of $\tilde{H}(X, \mathbb{Z})$ denoted by $\mathbb{N}(X)$, is equal to $H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z})$ and has signature $(2, \rho(X))$.

For an object $E \in \mathcal{D}^b(X)$, the Mukai vector $v(E) \in H^{2*}(X, \mathbb{Q})$ of $E$ is given by

$$v(E) := \text{ch}(E) \sqrt{\text{td}_X} = (\text{rk}(E), c_1(E), \chi(E) - \text{rk}(E)).$$

By the Riemann–Roch formula, we have the isomorphism $v : \mathcal{N}(\mathcal{D}^b(X)) \xrightarrow{\sim} \mathbb{N}(X)$ satisfying $(v(E), v(F)) = -\chi(E, F)$ for any objects $E, F \in \mathcal{D}^b(X)$, where $\mathcal{N}(\mathcal{D}^b(X))$ is the numerical Grothendieck group of $\mathcal{D}^b(X)$ and $\chi$ is the Euler pairing on it.

2.2. Groups. A Hodge isometry $\varphi : \tilde{H}(X, \mathbb{Z}) \rightarrow \tilde{H}(X, \mathbb{Z})$ of $\tilde{H}(X, \mathbb{Z})$ is an isomorphism of the Hodge structure preserving the Mukai pairing. The group of Hodge isometries is denoted by $O(\tilde{H}(X, \mathbb{Z}))$. Let $T(X)$ be the transcendental lattice of $X$ which is the transcendental part of the Hodge structures $\tilde{H}(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$. Restricting the Hodge structure $\tilde{H}(X, \mathbb{Z})$ with the Mukai pairing to sub-Hodge structures $H^2(X, \mathbb{Z}), \text{NS}(X)$ and $T(X)$, we similarly define the groups of Hodge isometries $O(H^2(X, \mathbb{Z})), O(\text{NS}(X))$ and $O(T(X))$, respectively. Then $O(T(X))$ is a finite cyclic group, and faithfully acts on $H^{2,0}(X) \simeq \mathbb{C}$ by a root of unity.

Using the action of $\text{Aut}(X)$ on $H^2(X, \mathbb{Z})$, the following two groups are defined

$$\text{Aut}_s(X) := \left\{ f \in \text{Aut}(X) \mid H^2(f)|_{H^{2,0}(X)} = \text{id}_{H^{2,0}(X)} \right\}$$

$$= \left\{ f \in \text{Aut}(X) \mid H^2(f)|_{T(X)} = \text{id}_{T(X)} \right\}$$

$$\text{Aut}_t(X) := \left\{ f \in \text{Aut}(X) \mid H^2(f)|_{\text{NS}(X)} = \text{id}_{\text{NS}(X)} \right\}.$$

These two subgroups are normal. The group $\text{Aut}_s(X) \subset \text{Aut}(X)$ is of finite index, and its element is called a symplectic automorphism. The natural group homomorphisms

$$\text{Aut}_s(X) \rightarrow O(\text{NS}(X)) \quad \text{and} \quad \text{Aut}_t(X) \rightarrow O(T(X)).$$

are injective, thus $\text{Aut}_t(X)$ is also a finite cyclic group.
For any autoequivalence $\Phi_E \in \text{Aut}(\mathcal{D}^b(X))$, we define the *cohomological Fourier–Mukai transform* $\Phi_E^H : H^*(X, \mathbb{Z}) \sim H^*(X, \mathbb{Z})$ associated to $\Phi_E$ by

$$\Phi_E^H(v) := p_*(q^*(v) \cdot v(\mathcal{E})),$$

which is a Hodge isometry of $\tilde{H}(X, \mathbb{Z})$, thus induces the action

$$\text{Aut}(\mathcal{D}^b(X)) \to O(\tilde{H}(X, \mathbb{Z})).$$

The two subgroups $\text{Aut}_0(\mathcal{D}^b(X))$ and $\text{Aut}_{\text{CY}}(\mathcal{D}^b(X))$ of $\text{Aut}(\mathcal{D}^b(X))$ are defined by

$$\text{Aut}_0(\mathcal{D}^b(X)) := \{ \Phi \in \text{Aut}(\mathcal{D}^b(X)) | \Phi^H = \text{id}_{H^*(X, \mathbb{Z})} \}$$

$$\text{Aut}_{\text{CY}}(\mathcal{D}^b(X)) := \{ \Phi \in \text{Aut}(\mathcal{D}^b(X)) | \Phi^H|_{H^{2,0}(X)} = \text{id}_{H^{2,0}(X)} \} = \{ \Phi \in \text{Aut}(\mathcal{D}^b(X)) | \Phi^H|_{T(X)} = \text{id}_{T(X)} \}.$$

These two subgroups are normal, and clearly $\text{Aut}_0(\mathcal{D}^b(X)) \subset \text{Aut}_{\text{CY}}(\mathcal{D}^b(X))$. The group $\text{Aut}_{\text{CY}}(\mathcal{D}^b(X)) \subset \text{Aut}(\mathcal{D}^b(X))$ is of finite index, and its element is called a *Calabi–Yau autoequivalence*. We note that $\Phi \in \text{Aut}(\mathcal{D}^b(X))$ is Calabi–Yau if and only if it respects the Serre duality pairings

$$\text{Hom}(E, F[i]) \times \text{Hom}(F, E[2-i]) \to \mathbb{C}$$

induced by a choice of holomorphic volume forms in $H^{2,0}(X)$, see [BB, Appendix A].

We recall the definition of centralizer and center groups.

**Definition 2.1.** Let $G$ be a group, and $H$ a subgroup of $G$.

(i) The *centralizer group* $C_G(H)$ of $H$ in $G$ is defined by

$$C_G(H) := \{ g \in G | gh = hg \text{ for any } h \in H \}.$$

(ii) The *center group* $Z(G)$ of $G$ is defined by

$$Z(G) := C_G(G).$$

It is easy to see $\text{Aut}_t(X) \subset Z(\text{Aut}(X))$, but these are not equal in general.

2.3. **Stability conditions on $\mathcal{D}^b(X)$.** We review the space of Bridgeland stability conditions on the derived categories of K3 surfaces and the action on it of the autoequivalence group.

Let $X$ be a K3 surface and fix a norm $|| \cdot ||$ on $\mathcal{N}(\mathcal{D}^b(X)) \otimes \mathbb{R}$.

**Definition 2.2** ([Bri1 Definition 5.1]). A (numerical) stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}^b(X)$ consists of a group homomorphism $Z : \mathcal{N}(\mathcal{D}^b(X)) \to \mathbb{C}$ called central charge and a family $\mathcal{P} = \{ \mathcal{P}(\phi) \}_{\phi \in \mathbb{R}}$ of full additive subcategory of $\mathcal{D}^b(X)$ called slicing, such that

(i) For $0 \neq E \in \mathcal{P}(\phi)$, we have $Z(v(E)) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$. 
(ii) For all \( \phi \in \mathbb{R} \), we have \( \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1] \).

(iii) For \( \phi_1 > \phi_2 \) and \( E_i \in \mathcal{P}(\phi_i) \), we have \( \text{Hom}(E_1, E_2) = 0 \).

(iv) For each \( 0 \neq E \in \mathcal{D}^b(X) \), there is a collection of exact triangles called Harder–Narasimhan filtration of \( E \):

\[
0 = E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{p-1} \rightarrow E_p = E
\]

with \( A_i \in \mathcal{P}(\phi_i) \) and \( \phi_1 > \phi_2 > \cdots > \phi_p \).

(v) (support property) There exists a constant \( C > 0 \) such that for all \( 0 \neq E \in \mathcal{P}(\phi) \), we have

\[
||E|| < C|Z(E)|.
\]

For any interval \( I \subset \mathbb{R} \), define \( \mathcal{P}(I) \) to be the extension-closed subcategory of \( \mathcal{D}^b(X) \) generated by the subcategories \( \mathcal{P}(\phi) \) for \( \phi \in I \). Then \( \mathcal{P}((0, 1]) \) is the heart of a bounded t-structure on \( \mathcal{D}^b(X) \), hence an abelian category. The full subcategory \( \mathcal{P}(\phi) \subset \mathcal{D}^b(X) \) is also shown to be abelian. A non-zero object \( E \in \mathcal{P}(\phi) \) is called \( \sigma \)-semistable of phase \( \phi_\sigma(E) := \phi \), and especially a simple object in \( \mathcal{P}(\phi) \) is called \( \sigma \)-stable. Taking the Harder–Narasimhan filtration \((2.1)\) of \( E \), we define \( \phi_\sigma^+(E) := \phi_\sigma(A_1) \) and \( \phi_\sigma^-(E) := \phi_\sigma(A_p) \). The object \( A_i \) is called \( \sigma \)-semistable factor of \( E \). Define \( \text{Stab}(X) \) to be the set of numerical stability conditions on \( \mathcal{D}^b(X) \).

We prepare some terminologies on the stability on the heart of a t-structure on \( \mathcal{D}^b(X) \).

**Definition 2.3.** Let \( \mathcal{A} \) be the heart of a bounded t-structure on \( \mathcal{D}^b(X) \). A stability function on \( \mathcal{A} \) is a group homomorphism \( Z : \mathcal{N}(\mathcal{D}^b(X)) \rightarrow \mathbb{C} \) such that for all \( 0 \neq E \in \mathcal{A} \subset \mathcal{D}^b(X) \), the complex number \( Z(E) \) lies in the semiclosed upper half plane \( \mathbb{H}_- := \{ re^{i\phi} \in \mathbb{C} \mid r \in \mathbb{R}_{>0}, \phi \in (0, 1] \} \subset \mathbb{C} \).

Given a stability function \( Z : \mathcal{N}(\mathcal{D}^b(X)) \rightarrow \mathbb{C} \) on \( \mathcal{A} \), the phase of an object \( 0 \neq E \in \mathcal{A} \) is defined to be \( \phi(E) := \frac{1}{p} \arg Z(E) \in (0, 1] \). An object \( 0 \neq E \in \mathcal{A} \) is \( Z \)-semistable (resp. \( Z \)-stable) if for all subobjects \( 0 \neq A \subset E \), we have \( \phi(A) \leq \phi(E) \) (resp. \( \phi(A) < \phi(E) \)). We say that a stability function \( Z \) satisfies the Harder–Narasimhan property if each object \( 0 \neq E \in \mathcal{A} \) admits a filtration (called Harder–Narasimhan filtration of \( E \)) \( 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E \) such that \( E_i/E_{i-1} \) is \( Z \)-semistable for \( i = 1, \ldots, m \) with \( \phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_m/E_{m-1}) \), and the support property if there exists a constant \( C > 0 \) such that for all \( Z \)-semistable objects \( E \in \mathcal{A} \), we have \( ||E|| < C|Z(E)| \).
The following proposition shows the relationship between stability conditions and stability functions on the heart of a bounded $t$-structure.

Proposition 2.4 ([Bri1, Proposition 5.3]). To give a stability condition on $\mathcal{D}^b(X)$ is equivalent to giving the heart $\mathcal{A}$ of a bounded $t$-structure on $\mathcal{D}^b(X)$, and a stability function $Z$ on $\mathcal{A}$ with the Harder–Narasimhan property and the support property.

For the proof, we construct the slicing $\mathcal{P}$, from the pair $(Z, \mathcal{A})$, by

$$\mathcal{P}(\phi) := \{E \in \mathcal{A} \mid E \text{ is } Z\text{-semistable with } \phi(E) = \phi \} \text{ for } \phi \in (0, 1),$$

and extend for all $\phi \in \mathbb{R}$ by $\mathcal{P}(\phi + 1) := \mathcal{P}(\phi)[1]$. Conversely, for a stability condition $\sigma = (Z, \mathcal{P})$, the heart $\mathcal{A}$ is given by $\mathcal{A} := \mathcal{P}_\sigma((0, 1])$. We also denote stability conditions by $(Z, \mathcal{A})$.

We recall that the Mukai vector $v : \mathcal{N}(\mathcal{D}^b(X)) \xrightarrow{\sim} \mathbb{N}(X)$ and the Mukai pairing $(-, -)$ on $\mathbb{N}(X) \subset \tilde{H}(X, \mathbb{Z})$, then the central charge of a numerical stability condition takes the form $Z(-) = (\Omega, v(-))$ for some $\Omega \in \mathbb{N}(X) \otimes \mathbb{C}$. Bridgeland constructed a family of stability conditions on $\mathcal{D}^b(X)$ as follows: Let

$$V(X) := \{\beta + i\omega \in \text{NS}(X) \otimes \mathbb{C} \mid \beta, \omega \in \text{NS}(X) \otimes \mathbb{R}, \omega : \mathbb{R}\text{-ample}, \ (\exp(\beta + i\omega), v(E)) \notin \mathbb{R}_{\leq 0} \text{ for all spherical sheaves } E\}.$$ 

For $\beta + i\omega \in V(X)$, we set $Z_{\beta, \omega}(-) := (\exp(\beta + i\omega), v(-))$. The category $\mathcal{A}_{\beta, \omega}$ is the heart of a $t$-structure obtained by tilting the standard $t$-structure with respect to the torsion pair $(\mathcal{T}_{\beta, \omega}, \mathcal{F}_{\beta, \omega})$ on $\text{Coh}(X)$ given by

$$\mathcal{T}_{\beta, \omega} := \{E \in \text{Coh}(X) \mid E \text{ is a torsion sheaf or } \mu^{-}_\omega(E/\text{torsion part}) > \beta.\omega\}$$

$$\mathcal{F}_{\beta, \omega} := \{E \in \text{Coh}(X) \mid E \text{ is torsion free and } \mu^{+}_\omega(E) \leq \beta.\omega\},$$

where, for a torsion free sheaf $E$, $\mu^{+}_\omega(E)$ is the maximal slope of $\mu_\omega$-semistable factors of $E$, and $\mu^{-}_\omega(E)$ is the minimal slope of $\mu_\omega$-semistable factors of $E$. Then $(Z_{\beta, \omega}, \mathcal{A}_{\beta, \omega})$ is a stability condition on $\mathcal{D}^b(X)$ ([Bri2, Lemma 6.2 and Proposition 11.2]).

Let $E \in \mathcal{D}^b(X)$ be a non-zero object of $\mathcal{D}^b(X)$ and $\sigma \in \text{Stab}(X)$ be a stability condition on $\mathcal{D}^b(X)$. The mass $m_\sigma(E) \in \mathbb{R}_{>0}$ of $E$ is defined by

$$m_\sigma(E) := \sum_{i=1}^{p} |Z_\sigma(A_i)|,$$

where $A_1, \cdots, A_p$ are $\sigma$-semistable factors of $E$. The following generalized metric (i.e. with values in $[0, \infty]$) $d_B$ on $\text{Stab}(X)$ is defined by Bridgeland ([Bri1, Proposition 8.1]):

$$d_B(\sigma, \tau) := \sup_{E \neq 0} \left\{ |\phi^+_\sigma(E) - \phi^+_\tau(E)|, |\phi^-_\sigma(E) - \phi^-_\tau(E)|, \left| \log \frac{m_\sigma(E)}{m_\tau(E)} \right| \right\} \in [0, \infty].$$
This generalized metric induces the topology on Stab(X). Then the generalized metric \( d_B \) takes a finite value on each connected component \( \text{Stab}^\circ(X) \) of \( \text{Stab}(X) \), thus \( (\text{Stab}^\circ(X), d_B) \) is a metric space in the strict sense.

**Theorem 2.5** ([Bri1, Theorem 7.1]). The map

\[
\text{Stab}(X) \to N(X) \otimes \mathbb{C}; \quad \sigma = ((\Omega, v(-)), \mathcal{P}) \mapsto \Omega
\]

is a local homeomorphism, where \( N(X) \otimes \mathbb{C} \) is equipped with the natural linear topology.

Therefore the space \( \text{Stab}(X) \) (and each connected component \( \text{Stab}^\circ(X) \)) naturally admits a structure of finite dimensional complex manifolds.

There is a left action of \( \text{Aut}(D^b(X)) \) on \( \text{Stab}(X) \) given by

\[
F.\sigma := (Z_\sigma(F^{-1}(-)), \{F(\mathcal{P}_\sigma(\phi))\}) \quad \text{for } \sigma \in \text{Stab}(X), \ F \in \text{Aut}(D^b(X)).
\]

This action of \( \text{Aut}(D^b(X)) \) is isometric with respect to \( d_B \).

Let \( \text{Stab}^\dagger(X) \) be the connected component of \( \text{Stab}(X) \) containing the set of geometric stability conditions i.e. one for which all structure sheaves of points are stable of the same phase. It is easy to check that the above stability condition \( (Z_\beta, \omega, A_\beta, \omega) \) for some \( \beta + i\omega \in V(X) \) is geometric.

**Definition 2.6.** The group \( \text{Aut}^\dagger(D^b(X)) \) is defined as the subgroup of \( \text{Aut}(D^b(X)) \) which preserves the connected component \( \text{Stab}^\dagger(X) \).

**Proposition 2.7** ([Har, proof of Proposition 7.9]). The following autoequivalences preserve the distinguished component \( \text{Stab}^\dagger(X) \) i.e. are elements in \( \text{Aut}^\dagger(D^b(X)) \):

- **shift** \([n]\) for \( n \in \mathbb{Z} \)
- **line bundle tensor** \(- \otimes \mathcal{L} \) for \( \mathcal{L} \in \text{Pic}(X) \)
- **pullback** \( f^* \) for \( f \in \text{Aut}(X) \)
- **The composition**

\[
g^* \circ \Phi^{-1}_E : D^b(X) \to D^b(M) \to D^b(X),
\]

where \( M \) is a 2-dimensional fine compact moduli space of Gieseker-stable torsion free sheaves on \( X \) with the universal family \( \mathcal{E} \) such that \( g : X \cong M \).

- **spherical twist** \( T_{\mathcal{O}_X} \) along \( \mathcal{O}_X \)
- **spherical twist** \( T_{\mathcal{O}_C} \) along \( \mathcal{O}_C \) for any \((-2)\)-curve \( C \) on \( X \)

We additionally define the following groups

\[
\text{Aut}^\dagger_{\text{CY}}(D^b(X)) := \text{Aut}_{\text{CY}}(D^b(X)) \cap \text{Aut}^\dagger(D^b(X))
\]

\[
\text{Aut}^\dagger_0(D^b(X)) := \text{Aut}_0(D^b(X)) \cap \text{Aut}^\dagger(D^b(X))
\]
Define the open subset $\mathcal{P}(X) \subset \mathbb{N}(X) \otimes \mathbb{C}$ consisting of vectors $\Omega \in \mathbb{N}(X) \otimes \mathbb{C}$ whose real and imaginary parts span a positive definite 2-plane $((\text{Re}\Omega, \text{Im}\Omega)_\mathbb{R}, (-, -))$ in $\mathbb{N}(X) \otimes \mathbb{R}$. This subset has two connected components, distinguished by the orientation induced on this 2-plane; let $\mathcal{P}^+(X)$ to be the component containing vectors of the form $(1, i\omega, -\frac{1}{2}\omega^2)$ for an ample class $\omega \in \text{NS}(X) \otimes \mathbb{R}$. Consider the root system

$$\Delta(X) := \{ \delta \in \mathbb{N}(X) \mid (\delta, \delta) = -2 \}$$

consisting of $(-2)$-classes in $\mathbb{N}(X)$, and the corresponding hyperplane complement

$$\mathcal{P}^+_0(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

**Theorem 2.8 ([Bri2, Theorem 1.1]).** The map

$$\pi : \text{Stab}^\dagger(X) \rightarrow \mathcal{P}^+_0(X); \ (\Omega, v(-)), \mathcal{P}) \mapsto \Omega$$

is a normal covering, and the group $\text{Aut}^\dagger_0(D^b(X))$ is identified with the group of deck transformations of $\pi$.

The following is a Bridgeland conjecture on the space of stability conditions on $D^b(X)$ and the action on it of $\text{Aut}(D^b(X))$.

**Conjecture 2.9 ([Bri2, Conjecture 1.2]).** Let $X$ be a K3 surface. Then

(i) $\text{Stab}^\dagger(X)$ is simply-connected.

(ii) any autoequivalence of $D^b(X)$ preserves the distinguished component $\text{Stab}^\dagger(X)$, equivalently we have

$$\text{Aut}^\dagger(D^b(X)) = \text{Aut}(D^b(X)).$$

This conjecture clearly implies an isomorphism

$$\pi_1(\mathcal{P}^+_0(X)) \simeq \text{Aut}_0(D^b(X)).$$

**Theorem 2.10 ([BB, Theorem 1.3]).** Conjecture 2.9 holds in the case of Picard rank one.

More strongly, Bayer–Bridgeland proved the contractibility of $\text{Stab}^\dagger(X)$ in [BB].

In Section 5, we consider the center groups $Z(\text{Aut}^\dagger(D^b(X)))$, $Z(\text{Aut}^\dagger_{\text{CY}}(D^b(X)))$, and the centralizer group $C_{\text{Aut}^\dagger(D^b(X))}(\text{Aut}^\dagger_{\text{CY}}(D^b(X)))$.

3. **Spherical twists**

We firstly recall the definition of spherical twists, and then consider their group-theoretic properties via the intersection number.

Throughout this section, $X$ is a smooth projective variety over a field $K$ of dimension $d > 1$. The assumption of $d > 1$ is needed in Proposition 3.6 and after that.
3.1. Basics.

Definition 3.1. An object $E \in \mathcal{D}^b(X)$ is $(d)$-spherical if it satisfies

$$E \otimes \omega_X \simeq E \quad \text{and} \quad \text{Hom}(E, E[i]) = \begin{cases} k & \text{if } i = 0, \dim X \\ 0 & \text{otherwise.} \end{cases}$$

For a spherical object $E \in \mathcal{D}^b(X)$, the spherical twist $T_E \in \text{Aut}(\mathcal{D}^b(X))$ along $E$ is defined as a Fourier–Mukai transform $T_E := \Phi_{P_E}$ whose kernel is the cone of the composition of the restriction to the diagonal $\Delta$ with the trace $P_E := C \left( E^c \otimes E \to (E^c \otimes E)|_\Delta \xrightarrow{\text{tr}} \mathcal{O}_\Delta \right) \in \mathcal{D}^b(X \times X)$, see [Huy1, Section 8.1] for details. For any $M \in \mathcal{D}^b(X)$, the object $T_E(M)$ fits into an exact triangle

$$\text{Hom}^\bullet(E, M) \otimes E \xrightarrow{ev} M \to T_E(M), \quad (3.1)$$

where $\text{Hom}^\bullet(M, N) := \bigoplus_{p \in \mathbb{Z}} \text{Hom}^p(M, N)[-p]$ and $\text{Hom}^p(M, N) := \text{Hom}(M, N[p])$.

Lemma 3.2 (cf. [Huy1] Chapter 8). Let $E \in \mathcal{D}^b(X)$ be a spherical object. We have the following.

(i) $T_E(E) = E[1 - d]$.

(ii) $\Phi \circ T_E \circ \Phi^{-1} = T_{\Phi(E)}$ for any autoequivalence $\Phi \in \text{Aut}(\mathcal{D}^b(X))$.

3.2. Intersections numbers. For any two objects $M, N \in \mathcal{D}^b(X)$, the intersection number $i(M, N)$ of $M$ and $N$ is defined by

$$i(M, N) := \sum_{p \in \mathbb{Z}} \text{hom}^p(M, N) \in \mathbb{Z}_{\geq 0},$$

where $\text{hom}^p(M, N) := \dim_K \text{Hom}^p(M, N)$. For a spherical object $E \in \mathcal{D}^b(X)$, the condition $E \otimes \omega_X \simeq E$ implies $i(E, M) = i(M, E)$ for any $M \in \mathcal{D}^b(X)$.

Lemma 3.3. For a spherical object $E \in \mathcal{D}^b(X)$, objects $M, N \in \mathcal{D}^b(X)$ and $k = \pm 1$, we have

$$|i(T^k_E M, N) - |k|i(E, M)i(E, N)| \leq i(M, N). \quad (3.2)$$

Proof. We prove only the case of $k = 1$. Applying $\text{Hom}(\cdot, N)$ to the exact triangle (3.1), we have $i(T_E M, N) \leq i(M, N) + i(\text{Hom}^\bullet(E, M) \otimes E, N)$. It is easy to check that $i(\text{Hom}^\bullet(E, M) \otimes E, N) = i(E, M)i(E, N)$. The remaining inequality is similarly proved.

Proposition 3.4 ([Kim] Lemma B.3 and [Kea] Section 4.4]). Let $E_1, E_2 \in \mathcal{D}^b(X)$ be spherical objects. Then the following are equivalent:
Lemma 3.5. Let $E_1 \simeq E_2[l]$. 

(i) There is no integer $l \in \mathbb{Z}$ such that $E_1 \simeq E_2[l]$.

(ii) The composition map $\text{Hom}^\bullet(E_i, E_j) \otimes \text{Hom}^\bullet(E_j, E_i) \to \text{Hom}^\bullet(E_i, E_i)$ does not hit the identity $\text{id}_{E_i}$ for every $i \neq j$.

(iii) The composition maps $\text{Hom}^{d}(E_i, E_i)[-d] \otimes \text{Hom}^\bullet(E_i, E_j) \to \text{Hom}^\bullet(E_i, E_j)$ and $\text{Hom}^\bullet(E_i, E_j) \otimes \text{Hom}^{d}(E_j, E_j)[-d] \to \text{Hom}^\bullet(E_i, E_j)$ vanish for all $i \neq j$.

Two objects $M, N \in \mathcal{D}^b(X)$ are called distinct if there is no integer $l \in \mathbb{Z}$ such that $M \simeq N[l]$ (cf. Proposition 3.4(i)).

Lemma 3.5. Let $E_1, E_2 \in \mathcal{D}^b(X)$ be distinct spherical objects. Then there exist objects $S_1, S_2 \in \mathcal{D}^b(X)$ such that

$$i(E_1, S_1) > i(E_2, S_1) \text{ and } i(E_2, S_2) > i(E_1, S_2).$$

Proof. When $i(E_1, E_2) \leq 1$ (resp. $i(E_1, E_2) \geq 3$), it suffices to set $S_1 := E_1$ and $S_2 := E_2$ (resp. $S_1 := E_2$ and $S_2 := E_1$).

Let us consider the case of $i(E_1, E_2) = 2$. By shifting, we may assume that $i(E_1, E_2) = \text{hom}^0(E_1, E_2) + \text{hom}^p(E_1, E_2)$ for some $p \geq 0$. Following the arguments in the proof of [Kim] Proposition 5.1, we define

$$Z := \text{Hom}^d(E_1, E_1)[-d] \otimes E_1 \oplus \text{Hom}^\bullet(E_2, E_1) \otimes E_2 \in \mathcal{D}^b(X),$$

and let $E'_1$ be the cone of the natural evaluation map i.e. $Z \to E_1 \to E'_1$ is an exact triangle in $\mathcal{D}^b(X)$. Applying $\text{Hom}(E_j, -)$ $(j = 1, 2)$ to this triangle, we see that $\text{Hom}^d(E_1, Z) \to \text{Hom}^d(E_1, E_1)$ is surjective, and $\text{Hom}^i(E_2, Z) \to \text{Hom}^i(E_2, E_1)$ are surjective for all $i \in \mathbb{Z}$. Moreover by Proposition 3.4 (ii), $\text{Hom}^0(E_1, Z) \to \text{Hom}^0(E_1, E_1)$ is zero. For $p = 0$, we have

$$\text{hom}^i(E_1, Z) = \begin{cases} 5 & i = d \\ 1 & i = 2d \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \text{hom}^i(E_2, Z) = \begin{cases} 2 & i = d \\ 4 & i = 2d \\ 0 & \text{otherwise} \end{cases},$$

and for $p > 0$,

$$\text{hom}^i(E_1, Z) = \begin{cases} 1 & i = d - p \\ 3 & i = d \\ 1 & i = d + p \end{cases} \quad \text{and} \quad \text{hom}^i(E_2, Z) = \begin{cases} 1 & i = d - p \\ 1 & i = d \\ 2 & i = 2d - p \end{cases}.$$
where we have \( \text{hom}^{d+p}(E_1, Z) = 2 \) (resp. \( \text{hom}^d(E_2, Z) = 3 \)) if \( d + p = 2d \) (resp. \( d = 2d - p \)). Direct computations give \( i(E_1, E'_1) = 6 > 4 = i(E_2, E'_1) \). It therefore suffices to set \( S_1 := E'_1 \), and \( S_2 \) is obtained by the same construction.

**Proposition 3.6.** Let \( E_1, E_2 \in \mathcal{D}^b(X) \) be spherical objects. Then the following are equivalent:

(i) We have \( T_{E_1} = T_{E_2} \).

(ii) There exists \( l \in \mathbb{Z} \) such that \( E_1 \simeq E_2[l] \), i.e. they are not distinct.

*Proof.* The direction from (ii) to (i) follows from [ST, Proposition 2.6] and \( T_{E[l]} = T_E \).

We now consider the converse. Suppose that \( E_1 \) and \( E_2 \) are distinct. When \( i(E_1, E_2) = 0 \), we have \( T_{E_1}(E_1) = E_1[1 - d] \neq E_1 = T_{E_2}(E_1) \) by \( d > 1 \), hence \( T_{E_1} \neq T_{E_2} \). When \( i(E_1, E_2) = 1 \), there exists \( p_0 \in \mathbb{Z} \) such that \( i(E_1, E_2) = \text{hom}^{p_0}(E_1, E_2) = 1 \).

By \( T_{E[l]} = T_E \), we may assume that \( p_0 = 1 \). Then, by [Kim] Proposition 5.1 and Remark 5.3 and \( d > 1 \), there exists an object \( S \in \mathcal{D}(\text{QCoh}(X)) \) satisfying \( i(E_1, S) = 1 \) and \( i(E_2, S) = 0 \). The inequality (3.2) implies that

\[
\begin{align*}
i(T_{E_1}E_2, S) & \geq i(E_1, E_2)i(E_1, S) - i(E_2, S) \\
& = i(E_1, E_2) \\
& > 0 = i(T_{E_2}E_2, S).
\end{align*}
\]

We here note that the inequality (3.2) also holds for the unbounded complex \( S \). We thus have \( T_{E_1} \neq T_{E_2} \).

When \( i(E_1, E_2) \geq 2 \), there exists \( S \in \mathcal{D}^b(X) \) satisfying \( i(E_1, S) > i(E_2, S) \) by Lemma 3.5. The inequality (3.2) implies that

\[
\begin{align*}
i(T_{E_1}E_2, S) & \geq i(E_1, E_2)i(E_1, S) - i(E_2, S) \\
& > i(E_1, E_2)i(E_2, S) - i(E_2, S) = (i(E_1, E_2) - 1)i(E_2, S) \\
& \geq i(E_2, S) = i(T_{E_2}E_2, S).
\end{align*}
\]

Hence we have \( T_{E_1} \neq T_{E_2} \). \( \square \)

**Proposition 3.7.** Suppose that \( \mathcal{D}^b(X) \) has at least two distinct spherical objects. Then each spherical twist \( T_E \in \text{Aut}(\mathcal{D}^b(X)) \) is of infinite order, and \( T_E \neq [k] \) for any \( k \in \mathbb{Z} \).

*Proof.* Let \( E \in \mathcal{D}^b(X) \) be a spherical object. Then the spherical twist \( T_E \in \text{Aut}(\mathcal{D}^b(X)) \) is clearly of infinite order by Lemma 3.2 (i) and \( d > 1 \). The claim of \( T_E \neq [k] \) follows from the same argument of Proposition 3.6 and \( i(T_E(E), M) = i(E, M) = i(E[k], M) \). \( \square \)

The following is a main result of this section.
Theorem 3.8. Let $E \in D^b(X)$ be spherical object. An autoequivalence $\Phi \in \text{Aut}(D^b(X))$ satisfies $\Phi \circ T_E = T_E \circ \Phi$ if and only if $\Phi(E) = E[l]$ for some $l \in \mathbb{Z}$.

Proof. Suppose that $\Phi \in \text{Aut}(D^b(X))$ satisfies $\Phi \circ T_E = T_E \circ \Phi$. By Lemma 3.2 (ii) and Proposition 3.6, we have $\Phi(E) = E[l]$ for some $l \in \mathbb{Z}$. The other direction follows from Lemma 3.2 (ii) and $T_E[1] = T_E$. \qed

Similarly we can prove the following.

Corollary 3.9. Let $E_1, E_2 \in D^b(X)$ be spherical objects. $T_{E_1}$ and $T_{E_2}$ are conjugate in $\text{Aut}(D^b(X))$ if and only if $\Phi(E_1) = E_2$ for some $\Phi \in \text{Aut}(D^b(X))$.

4. Group relations of two spherical twists

We relate the intersection number to group relations of two spherical twists. As an analogue of spherical twists, we firstly introduce some known results for (non-abelian) free groups generated by two Dehn twists in the mapping class groups of real surfaces.

4.1. Mapping class groups. Let $\Sigma_g$ be a connected oriented closed surface of genus $g \geq 2$, and $\text{MCG}(\Sigma_g)$ be the mapping class groups of $\Sigma_g$. For an isotopy class of a simple closed curve in $\Sigma_g$, the Dehn twist along $a$ is denoted by $T_a \in \text{MCG}(\Sigma_g)$.

Proposition 4.1 (cf. [FM, Proposition 3.4]). Let $a, b, c$ be isotopy classes of simple closed curves in $\Sigma_g$. Then for any $k \in \mathbb{Z}$,

$$|i(T_a^k(c), b) - |k|i(a, b)i(a, c)| \leq i(b, c).$$

Theorem 4.2 ([Ish, Theorem 1.2]). Let $a, b$ be two isotopy classes of simple closed curves in $\Sigma_g$. If the intersection number $i(a, b) \geq 2$, then $\langle T_a, T_b \rangle \simeq F_2$, where $F_2$ is the free group of rank 2.

4.2. Autoequivalence groups. Throughout this subsection, $X$ is a $d$-dimensional smooth projective variety over a field $K$ of dimension $d > 1$.

We conjecture a generalization of Lemma 3.2.

Conjecture 4.3. Let $E \in D^b(X)$ be a $(d)$-spherical object and $M, N \in D^b(X)$ objects.

(i) For any $k \in \mathbb{Z}$, we have

$$i(E, M)i(E, N) \leq i(T_E^k M, N) + i(M, N). \quad (4.1)$$

(ii) Suppose that $E$ and $M$ (resp. $N$) are distinct. For any $k \in \mathbb{Z}$, we have

$$|i(T_E^k M, N) - |k|i(E, M)i(E, N)| \leq i(M, N). \quad (4.2)$$
Remark 4.4.  
(i) The inequality (4.1) is a weak form of the inequality (4.2).
(ii) Conjecture 4.3 (ii) is a complete analogue of Proposition 4.1.
(iii) A part of the inequality (4.2)
\[ i(T^k \mathcal{M}, \mathcal{N}) - |k|i(E, M)i(E, N) \leq i(M, N) \]
always true by the similar arguments of Lemma 3.2 and the induction.
(iv) When \( E \) and \( M \) (resp. \( N \)) are not distinct, the inequality (4.2) is clearly false.
(v) For the \( \mathbb{Z}/2 \)-graded Fukaya category of exact symplectic manifolds, Keating proved the inequality (4.1) [Kea, Proposition 7.4].

We can prove the following under the conjecture, see also Proposition 3.6.

Proposition 4.5. Let \( E_1, E_2 \in \mathcal{D}^b(X) \) be distinct spherical objects. Assume Conjecture 4.3 (i) and that there exists an object \( S \in \mathcal{D}^b(X) \) such that
\[ i(E_1, S) > 0 = i(E_2, S). \]
Then we have \( T_{E_1}^{k_1} \neq T_{E_2}^{k_2} \) for all \( k_1, k_2 \in \mathbb{Z}\{0\} \).

Proof. When \( i(E_1, E_2) = 0 \), we have
\[ T_{E_1}^{k_1}(E_1) = E_1[k_1(1-d)] \neq E_1 = T_{E_2}^{k_2}(E_2) \]
by \( d > 1 \) and \( k_1 \neq 0 \), hence \( T_{E_1}^{k_1} \neq T_{E_2}^{k_2} \).

When \( i(E_1, E_2) > 0 \), it follows from the assumptions that
\[ i(T_{E_1}^{k_1}(E_2), S) \geq i(E_1, E_2)i(E_1, S) - i(E_2, S) \]
\[ = i(E_1, E_2)i(E_1, S) \]
\[ > 0 = i(T_{E_2}^{k_1}(E_2), S), \]
which implies \( T_{E_1}^{k_1} \neq T_{E_2}^{k_2} \). \( \square \)

We then reveal the relationship between the intersection number and group relations of two spherical twists.

Lemma 4.6 (Ping-pong lemma). Let \( G \) be a group acting on a set \( W \), and \( g_1, g_2 \) elements of \( G \). Suppose that there are non-empty, disjoint subsets \( W_1, W_2 \) of \( W \) with the property that, for each \( i, j \) (\( i \neq j \)), we have \( g_i^k(W_j) \subset W_i \) for every nonzero integer \( k \). Then the subgroup \( \langle g_1, g_2 \rangle \) generated by \( g_1 \) and \( g_2 \) is isomorphic to \( F_2 \).

Using ping-pong lemma, we prove the complete analogue of Theorem 4.2.

Theorem 4.7. Let \( E_1, E_2 \in \mathcal{D}^b(X) \) be distinct spherical objects. Assume Conjecture 4.3 (i). If \( i(E_1, E_2) \geq 2 \), then \( \langle T_{E_1}, T_{E_2} \rangle \simeq F_2 \).
Proof. To apply the ping-pong lemma, we define the subsets $W_1, W_2$ of the set of isomorphism classes of objects in $\mathcal{D}^b(X)$ as follows:

\begin{align*}
W_1 &:= \{ [S_2] \mid S_2 \in \mathcal{D}^b(X) \text{ such that } i(S_2, E_2) > i(S_2, E_1) \} \\
W_2 &:= \{ [S_1] \mid S_1 \in \mathcal{D}^b(X) \text{ such that } i(S_1, E_1) > i(S_1, E_2) \}.
\end{align*}

These are obviously disjoint, and non-empty by Lemma 3.5. By the ping-pong lemma, it suffices to check that $T_{E_1}^k(W_2) \subset W_1$ and $T_{E_1}^k(W_2) \subset W_1$ for each $k \in \mathbb{Z}\{0\}$. We only show the former inclusion.

For each $S \in W_2$ and $k \in \mathbb{Z}\{0\}$, Conjecture 4.3 (i) gives

\[i(T_{E_1}^k(S), E_2) \geq i(E_1, S)i(E_1, E_2) - i(S, E_2)\]
\[\geq 2i(E_1, S) - i(S, E_2)\]
\[> 2i(E_1, S) - i(S, E_1)\]
\[= i(E_1, S) = i(E_1, T_{E_1}^k(S)) = i(T_{E_1}^k(S), E_1).
\]

Thus $T_{E_1}^k(S) \in W_1$ as desired. \hfill \Box

Remark 4.8. We add some remarks on known results.

(i) Keating proved similar results for Dehn twists along Lagrangian spheres of exact symplectic manifolds in the same approach ([Kea, Theorem 1.1]).

(ii) In his thesis [Kim, Theorem 4.4], Kim proved that general $n$ spherical twists whose intersection number is greater than one respectively, generate the free group of rank $n$ under the formality assumption for some dg-algebra obtained by spherical objects.

Corollary 4.9. Assume Conjecture 4.3 (i). Let $E_1, E_2 \in \mathcal{D}^b(X)$ be distinct spherical objects. Then the following are equivalent:

(i) $T_{E_1} \circ T_{E_2} \circ T_{E_1} = T_{E_2} \circ T_{E_1} \circ T_{E_2}$ (braid relation)

(ii) $T_{E_1} T_{E_2}(E_1) = E_2[l]$ for some $l \in \mathbb{Z}$

(iii) $i(E_1, E_2) = 1$.

Proof. The assertions (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are shown by Seidel–Thomas ([ST Proposition 2.13], see also [Huy1 Proposition 8.22]). The Braid relation gives $(T_{E_1} T_{E_2}) T_{E_1} (T_{E_1} T_{E_2})^{-1} = T_{E_2}$. Lemma 3.2 (ii) and Proposition 3.6 then imply $T_{E_1} T_{E_2}(E_1) = E_2[l]$ for some $l \in \mathbb{Z}$.

The inequality

\[|i(T_{E_1} E_2, E_2) - i(E_1, E_2)^2| \leq i(E_2, E_2) = 2.
\]
follows from (3.2). Applying $T_{E_1}T_{E_2}(E_1) = E_2[l]$, we have

$$i(T_{E_1}E_2, E_2) = i(E_2, T_{E_1}^{-1}E_2) = i(E_2, T_{E_2}^{-1}T_{E_1}^{-1}E_2) = i(E_2, E_1),$$

hence this inequality holds only in the case of $i(E_1, E_2) = 0, 1$ or 2. When $i(E_1, E_2) = 0$, we have $T_{E_1} = T_{E_2}$ by the braid relation and $T_{E_1} \circ T_{E_2} = T_{E_2} \circ T_{E_1}$, which contradicts the distinctness. Assume that $i(E_1, E_2) = 2$. Then the subgroup $\langle T_{E_1}, T_{E_2} \rangle$ is isomorphic to the rank 2 free group by Theorem 4.7, which contradicts the braid relation. We therefore have $i(E_1, E_2) = 1$. □

**Corollary 4.10.** Assume Conjecture 4.3 (i). Let $E_1, E_2 \in \mathcal{D}^b(X)$ be distinct spherical objects. Then the following are equivalent:

(i) $T_{E_1} \circ T_{E_2} = T_{E_2} \circ T_{E_1}$
(ii) $T_{E_1}(E_2) = E_2[l]$ for some $l \in \mathbb{Z}$
(iii) $i(E_1, E_2) = 0$.

**Proof.** The assertions (iii) ⇒ (ii) ⇒ (i) follow from Lemma 3.2 (ii). Clearly, (i) implies (ii) by Theorem 3.8.

The inequality

$$|i(T_{E_1}E_2, E_2) - i(E_1, E_2)| \leq i(E_2, E_2) = 2.$$

follows from (3.2). Applying $T_{E_1}(E_2) = E_2[l]$, we have $i(T_{E_1}E_2, E_2) = i(E_2[l], E_2) = 2$, hence this inequality holds only in the case of $i(E_1, E_2) = 0, 1$ or 2.

When $i(E_1, E_2) = 1$, we have $T_{E_1} = T_{E_2}$ by the commutative relation and Corollary 4.9, which contradicts the distinctness. Assume that $i(E_1, E_2) = 2$. Then the subgroup $\langle T_{E_1}, T_{E_2} \rangle$ is isomorphic to the rank 2 free group by Theorem 4.7, which contradicts the commutative relation. We therefore have $i(E_1, E_2) = 0$. □

**Corollary 4.11.** Assume Conjecture 4.3 (i). Let $E_1, E_2 \in \mathcal{D}^b(X)$ be distinct spherical objects. Then the following are equivalent:

(i) $\langle T_{E_1}, T_{E_2} \rangle \simeq F_2$
(ii) $i(E_1, E_2) \geq 2$.

**Proof.** The assertion (ii) ⇒ (i) follows from Theorem 4.7. The converse is given by Corollary 4.9 and Corollary 4.10. □

5. The center groups

Let $X$ be a K3 surface of any Picard rank. We compute the center groups of $\text{Aut}^\dagger(\mathcal{D}^b(X))$ and $\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))$ and the centralizer groups $C_{\text{Aut}^\dagger(\mathcal{D}^b(X))}(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X)))$, which is a main result of this section.
Theorem 5.1. Let $X$ be a K3 surface, $m_X$ the order of $\text{Aut}_t(X)$, and $f_t$ a generator of $\text{Aut}_t(X)$. Then we have the following

(i) $Z(\text{Aut}^\dagger(\mathcal{D}^b(X))) = C_{\text{Aut}^\dagger(\mathcal{D}^b(X))}(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))) = \text{Aut}_t(X) \times \mathbb{Z}[1] \cong (\mathbb{Z}/m_X) \times \mathbb{Z}$.

(ii) $Z(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))) = \begin{cases} \langle (f_t^*)^{mx/2} \circ [1] \rangle & \text{if } m_X \text{ is even} \\ \mathbb{Z}[2] & \text{if } m_X \text{ is odd} \end{cases}$

Proof. (i) We note that $\text{Aut}(X) \ltimes \text{Pic}(X) \subset \text{Aut}^\dagger(\mathcal{D}^b(X))$ and $\mathcal{T}_X \in \text{Aut}^\dagger(\mathcal{D}^b(X))$, see Proposition 2.7. Fix any $\Phi \in Z(\text{Aut}^\dagger(\mathcal{D}^b(X)))$. By Theorem 3.8, the relation $\Phi \circ T_{\mathcal{T}_X} = T_{\mathcal{T}_X} \circ \Phi$ implies $\Phi(\mathcal{O}_X) = \mathcal{O}_X[i]$ for some $i \in \mathbb{Z}$. For any line bundle $\mathcal{L} \in \text{Pic}(X)$ on $X$, we have $(\Phi \circ [-i])(\mathcal{L}) = \mathcal{L}$ by $\Phi \circ (-\otimes \mathcal{L}) = (-\otimes \mathcal{L}) \circ \Phi$. By the proof of [Huy2, Lemma A.2], $\Phi \circ [-i]$ is in $\text{Aut}_t(X)$, hence $\Phi \in \langle \text{Aut}_t(X), [1] \rangle \cong \text{Aut}_t(X) \times \mathbb{Z}[1]$. It remains to show that $\text{Aut}_t(X) \subset Z(\text{Aut}^\dagger(\mathcal{D}^b(X)))$. Fix any $\Psi \in \text{Aut}^\dagger(\mathcal{D}^b(X))$ and set $\Psi_t := \Psi \circ f_t^* \circ \Psi^{-1} \circ (f_t^*)^{-1} \in \text{Aut}^\dagger(\mathcal{D}^b(X))$.

Since the transcendental part of the Hodge structure $\tilde{H}(X, \mathbb{Z})$ is equal to $T(X)$, $\mathbb{N}(X) \oplus T(X) \subset \tilde{H}(X, \mathbb{Z})$ is of finite index, so that $\Psi_t \in \text{Aut}_0^\dagger(\mathcal{D}^b(X))$. By [Huy2, Lemma A.3], $f_t^*$ acts on $\text{Stab}^\dagger(X)$ trivially, hence $\Psi_t$ also does. We thus have $\Psi_t = \text{id}_{\mathcal{D}^b(X)}$ since $\text{Aut}_0^\dagger(\mathcal{D}^b(X))$ is isomorphic to the group of deck transformations of the normal cover $\text{Stab}^\dagger(X) \to \mathcal{P}_0^\dagger(X)$, see Theorem 2.8. We therefore have $f_t^* \in Z(\text{Aut}^\dagger(\mathcal{D}^b(X)))$, thus $Z(\text{Aut}^\dagger(\mathcal{D}^b(X))) = \text{Aut}_t(X) \times \mathbb{Z}[1]$.

By $T_{\mathcal{T}_X} - \otimes \mathcal{L} \in \text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))$, similar arguments give

$$C_{\text{Aut}^\dagger(\mathcal{D}^b(X))}(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))) \subset \text{Aut}_t(X) \times \mathbb{Z}[1].$$

The other inclusion follows from

$$\text{Aut}_t(X) \times \mathbb{Z}[1] = Z(\text{Aut}^\dagger(\mathcal{D}^b(X))) \subset C_{\text{Aut}^\dagger(\mathcal{D}^b(X))}(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))).$$

(ii) By (i), we have

$$Z(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))) = C_{\text{Aut}^\dagger(\mathcal{D}^b(X))}(\text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X))) \cap \text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X)) = (\text{Aut}_t(X) \times \mathbb{Z}[1]) \cap \text{Aut}_{\text{CY}}^\dagger(\mathcal{D}^b(X)).$$

Each element of $\text{Aut}_t(X) \times \mathbb{Z}[1]$ is of the form $(f_t^*)^l \circ [i]$ for some $l \in \mathbb{Z}/m_X$ and $i \in \mathbb{Z}$. Then $((f_t^*)^l \circ [i])^H|_{\mathcal{H}^{2,0}(X)} = (-1)^i(f_t^*)^{lH}|_{\mathcal{H}^{2,0}(X)}$ is equal to $\text{id}_{\mathcal{H}^{2,0}(X)}$ if and only if (a) $i$ is odd and $(f_t^*)^{lH}|_{\mathcal{H}^{2,0}(X)} = -\text{id}_{\mathcal{H}^{2,0}(X)}$, or (b) $i$ is even and $(f_t^*)^{lH}|_{\mathcal{H}^{2,0}(X)} = \text{id}_{\mathcal{H}^{2,0}(X)}$ i.e. $l = 0$. By the faithfulness of the action of $\text{Aut}_t(X)$
on $H^{2,0}(X)$, the case (a) is realized only when $m_X$ is even and $l = \frac{m_X}{2}$. We therefore have
\[
(Aut_t(X) \times \mathbb{Z}[1]) \cap Aut^\dagger_{\text{CY}}(D^b(X)) = \begin{cases}
\langle (f^*_t)^{m_X/2} \circ [1] \rangle & \text{if } m_X \text{ is even} \\
\mathbb{Z}[2] & \text{if } m_X \text{ is odd},
\end{cases}
\]
which completes the proof. \hfill \Box

When Conjecture 2.9 (i) and (ii) are true, the group $Aut^\dagger_{\text{CY}}(D^b(X))/\mathbb{Z}[2]$ is naturally isomorphic to the orbifold fundamental group of the stringy Kähler moduli space of $X$ ([BB Section 7] and [Huy3 Conjecture 3.14]), so this quotient group is important in the context of mirror symmetry.

**Corollary 5.2.** Let $X$ be a K3 surface. Then we have
\[
Z(Aut^\dagger_{\text{CY}}(D^b(X))/\mathbb{Z}[2]) \cong Z(Aut^\dagger_{\text{CY}}(D^b(X)))/\mathbb{Z}[2] \cong \begin{cases}
\mathbb{Z}/2 & \text{if } m_X \text{ is even} \\
\text{trivial} & \text{if } m_X \text{ is odd}.
\end{cases}
\]

**Proof.** Let $\varphi : Z(Aut^\dagger_{\text{CY}}(D^b(X))) \to Z(Aut^\dagger_{\text{CY}}(D^b(X))/\mathbb{Z}[2])$ be a natural group homomorphism induced by the quotient. By Theorem 5.1(ii), $\ker \varphi = \mathbb{Z}[2]$ whether $m_X$ is even or not. We note that $\Phi \in Z(Aut^\dagger_{\text{CY}}(D^b(X))/\mathbb{Z}[2])$ if and only if any $\Psi \in Aut^\dagger_{\text{CY}}(D^b(X))$ satisfies $\Psi \circ \Phi \circ \Psi^{-1} \circ \Phi^{-1} = [2l]$ for some $l \in \mathbb{Z}$.

Let
\[
h_t(-) : \mathbb{R} \times \text{Aut}(D^b(X)) \to \mathbb{R}; \quad (t, \Psi') \mapsto h_t(\Psi')
\]
be the categorical entropy ([DHKK Definition 2.5]). Then by [KST Lemma 2.7(v) and Corollary 2.10], we have
\[
h_t(\Psi \circ \Phi \circ \Psi^{-1}) = h_t(\Phi) \\
h_t(\Phi[2l]) = h_t(\Phi) + 2lt
\]
for all $t \in \mathbb{R}$, which concludes that $l = 0$. The morphism $\varphi$ is therefore surjective, which completes the proof. \hfill \Box

**5.1. Examples.** We here collect several examples of the order $m_X$ of the finite cyclic group $Aut_t(X)$, which determines the center groups by Theorem 5.1 and Corollary 5.2.

**Example 5.3.** Let $X$ be a K3 surface of odd Picard rank. Then $O(T(X))$ is isomorphic to $\mathbb{Z}/2$ (cf [Huy3 Cor 3.3.5]). Therefore $m_X = 2$ (resp. $m_X = 1$) if and only if $Aut_t(X) = O(T(X))$ (resp. $Aut_t(X) = \{\text{id}_X\}$).
Example 5.4. Let $X$ be a K3 surface of Picard rank 1, and $H$ the ample generator of $\text{NS}(X)$. As a special case of Example 5.3, we have the following:

(i) If $H^2 = 2$, then $\text{Aut}(X) = \text{Aut}_t(X) = \langle i^* \rangle = \text{O}(T(X))$, where $i \in \text{Aut}_t(X)$ is the covering involution of the double cover $X \rightarrow \mathbb{P}^2$ branched along a smooth curve of degree six.

(ii) If $H^2 > 2$, then $\text{Aut}(X) = \text{Aut}_t(X) = \{ \text{id}_X \} \neq \text{O}(T(X))$.

Moreover as mentioned in Theorem 2.10, we have $\text{Aut}^\dagger(\mathcal{D}^b(X)) = \text{Aut}(\mathcal{D}^b(X))$.

Example 5.5. Let $X$ be a K3 surface of Picard rank 2, with infinite automorphism group. Then by [GLP, Corollary 1], $\text{Aut}(X)$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/2 \ast \mathbb{Z}/2$. Since $\text{Aut}_t(X)$ is finite and $\text{Aut}_t(X) \subset \mathbb{Z}(\text{Aut}(X))$, one has $m_X = 1$.

Example 5.6. Let $X_3$ and $X_4$ be the K3 surfaces of Picard rank 20 whose transcendental lattices are of the form

$$T(X_3) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad T(X_4) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

respectively, see [SI] and [Huy3, Corollary 14.3.21]. By Vinberg [Vin, Theorem in 2.4 and Theorem in 3.3], one has $\text{Aut}_t(X) \simeq U_0$ in his notation, hence $m_{X_3} = 3$ and $m_{X_4} = 2$.

Example 5.7. Let $X$ be a K3 surface with unimodular or 2-elementary transcendental lattice. Then there exists an involution $\sigma \in \text{Aut}(X)$ satisfying $H^2(\sigma)|_{\text{NS}(X)} = \text{id}_{\text{NS}(X)}$ and $H^2(\sigma)|_{T(X)} = -\text{id}_{T(X)}$, hence $m_X$ is even.

Example 5.8. Let $X$ be a K3 surface with $\varphi(m_X) = \text{rk} \ T(X)$, where $\varphi$ is the Euler function. Then $m_X$ is in the set

$$\{12, 28, 36, 42, 44, 66, 3^k(1 \leq k \leq 3), 5^l(l = 1, 2), 7, 11, 13, 17, 19\}$$

and $X$ is uniquely determined by $m_X$ due to Kondo [Kon, Main Theorem], Vorontsov [Vor], Machida–Oguiso [MO, Theorem 3] and Oguiso–Zhang [OZ, Theorem 2]. Especially, $m_X$ is even (resp. odd) if and only if $T(X)$ is unimodular (resp. non-unimodular).

REFERENCES

[BB] A. Bayer, T. Bridgeland, Derived automorphism groups of K3 surfaces of Picard rank 1, Duke Math. J., 166 (2017), no. 1, 75–124.

[Bri1] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math., 166 (2007), 317–345.

[Bri2] T. Bridgeland, Stability conditions on K3 surfaces, Duke Math. J., 141 (2008) no. 2, 241–291.

[DHKK] G. Dimitrov, F. Haiden, L. Katzarkov and M. Kontsevich, Dynamical systems and categories, Contemporary Mathematics, 621 (2014), 133–170, DOI: 10.1090/conm/621.
[FM] B. Farb and D. Margalit, *A primer on mapping class groups*, volume 49 of Princeton Mathematical Series, Princeton University Press, Princeton (2012).

[GLP] F. Galluzzi, G. Lombardo and C. Peters, *Automorphs of indefinite binary quadratic forms and K3-surfaces with Picard number 2*, Rend. Sem. Mat. Univ. Politec. Torino, **68**(1) (2010), 57–77.

[Har] H. Hartmann, *Cusps of the Kähler moduli space and stability conditions on K3 surfaces*, Math. Ann., **354**(1) (2012), 1–42.

[Huy1] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford (2006).

[Huy2] D. Huybrechts, *Stability conditions via spherical objects*, Math. Z., **271**(3-4) (2012), 1253–1270.

[Huy3] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2016).

[Ish] A. Ishida, *The structure of subgroup of mapping class groups generated by two Dehn twists*, Proc. Japan Acad. Ser. A Math. Sci. **72** no. 10 (1996), 240–241.

[Kea] A. Keating, *Dehn twists and free subgroups of symplectic mapping class groups*, J. Topology **7**(2) (2014), 436–474.

[KST] K. Kikuta, Y. Shiraishi, A. Takahashi, *A note on entropy of auto-equivalences: lower bound and the case of orbifold projective lines*, Nagoya Math. J., **238** (2020), 86–103.

[Kim] J. Kim, *A freeness criterion for spherical twists*, PhD thesis, Nagoya University (2018).

[Kon] S. Kondo, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan., **44** (1992), 75–98.

[MO] N. Machida and K. Oguiso, *On K3 surfaces admitting finite non-symplectic group actions*, J. Math. Sci. Univ. Tokyo, **5** (1998), no. 2, 273–297.

[OZ] K. Oguiso and D.-Q. Zhang, *On Vorontsov’s theorem on K3 surfaces with non-symplectic group actions*, Proc. Amer. Math. Soc., **128**(6) (2000), 1571–1580.

[ST] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108.

[SI] T. Shioda and H. Inose, *On singular K3 surfaces*, In Complex analysis and algebraic geometry, pages 119–136, Iwanami Shoten, Tokyo (1977).

[Vin] È. Vinberg, *The two most algebraic K3 surfaces*, Math. Ann., **265**(1) (1983), 1–21.

[Vor] S. P. Vorontsov, *Automorphisms of even lattices that arise in connection with automorphisms of algebraic K3 surfaces*, Vestnik Mosk. Univ. Math. **38** (1983), 19–21.

Department of Mathematics, Chuo University, Tokyo, 112-0003, Japan

Email address: kikuta@math.chuo-u.ac.jp