MINIMIZING MOVEMENTS FOR MEAN CURVATURE FLOW OF PARTITIONS

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ABSTRACT. We prove the existence of a weak global in time mean curvature flow of a bounded partition of space using the method of minimizing movements. The result is extended to the case when suitable driving forces are present. We also prove some consistency results for a minimizing movement solution with smooth and viscosity solutions when the evolution starts from a partition made by a union of bounded sets at a positive distance. In addition, the motion starting from the union of convex sets at a positive distance agrees with the classical mean curvature flow and is stable with respect to the Hausdorff convergence of the initial partitions.

1. INTRODUCTION

Mean curvature evolution of partitions became popular in recent years because of its applications in material science and physics, especially evolutions of grain boundaries and motion of immiscible fluid systems, see e.g. [5, 9, 32, 41] and references therein. Behaviour of the motion in the two phase case, i.e. in the case of classical motion by mean curvature of a boundary as a gradient flow of the area functional, is rather well-understood, see for instance [6, 14, 20, 26, 27, 29, 40] and references therein.

Mean curvature evolution of interfaces in the multiphase case in general involves motion of surface junctions in $\mathbb{R}^n$, or triple and multiple points in the plane, an already nontrivial problem. We refer to the survey [41] and references therein for recent results on curvature evolution of planar networks.

Not much seems to be known in higher space dimensions; short time existence of the motion of subgraph-type partitions has been derived in [24, 25] and well-posedness and short time existence of the motion by mean curvature of three surface clusters have been recently shown in [19].

Even in the two phase case, the classical flow describes the motion only up to the appearance of the first singularity. In order to continue the motion through singularities, several notions of generalized solutions have been suggested: Brakke varifold-solution [9], the viscosity solution (see [27] and references therein), the Almgren-Taylor-Wang [1] and Luckhaus-Sturzenhecker [38] solutions, the minimal barrier solution (see [6] and references therein); we also refer to [22, 30] for other types of solutions. At the moment the lack of the comparison principle in the multiphase case results in a lot of difficulties to extend such notions as viscosity and barrier solutions, while besides Brakke solution, some other generalized solutions have been successfully extended to partitions. For example, the authors of [34] have proved the existence of a distributional solution of mean curvature evolution of partitions on the torus using the time thresholding method introduced in [42], see also [21, 43]; furthermore the authors of [31] showed the existence of a Brakke flow.

In [17] De Giorgi generalized the Almgren-Taylor-Wang and Luckhaus-Sturzenhecker approach to what he called the minimizing movements method. In the present paper, we prove the existence of a generalized minimizing movement solution in $\mathbb{P}_b(N + 1)$, the collection of all partitions of $\mathbb{R}^n$, $n \geq 2$, having $N + 1 \geq 2$ components, with the first $N$-components bounded. This is the multiphase generalization of the evolution of a compact boundary in the

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two-phase case \((N = 1)\), for which the generalized minimizing movement solution has been introduced and studied in \([1, 38]\).

Let us recall the definition (see \([17, 18]\), also \([2, 4]\)).

**Definition 1.1 (Generalized minimizing movement for partitions).** Let \(\mathbb{P}_b(N + 1)\) be the set of all bounded \((N + 1)\)-partitions of \(\mathbb{R}^n\) (Definition 3.9) endowed with the \(L^1(\mathbb{R}^n)\)-convergence, and let \(\mathcal{F} : \mathbb{P}_b(N + 1) \times \mathbb{P}_b(N + 1) \times [1, +\infty) \rightarrow [-\infty, +\infty]\) be defined as

\[
\mathcal{F} (\mathcal{A}, \mathcal{B}; \lambda) = \text{Per}(\mathcal{A}) + \frac{\lambda}{2} \sum_{j=1}^{N+1} \int_{A_j \triangle B_j} d(x, \partial B_j) dx, \quad \mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N + 1),
\]

where \(\text{Per}(\mathcal{A}) = \frac{1}{2} \sum_{j=1}^{N+1} P(A_j)\) is the perimeter of the partition \(\mathcal{A} = (A_1, \ldots, A_{N+1})\) and \(d(\cdot, E)\) is the distance function from \(E \subseteq \mathbb{R}^n\). We say that a map \(\mathcal{M} : [0, +\infty) \rightarrow \mathbb{P}_b(N + 1)\) is a generalized minimizing movement (shortly a GMM) associated to \(\mathcal{F}\) starting from \(\mathcal{G} \in \mathbb{P}_b(N + 1)\) and we write \(\mathcal{M} \in \text{GMM}(\mathcal{F}, \mathcal{G})\), if there exist \(\mathcal{L} : [1, +\infty) \times \mathbb{N}_0 \rightarrow \mathbb{P}_b(N + 1)\) and a diverging sequence \(\{\lambda_h\}\) such that

\[
\lim_{h \to +\infty} \mathcal{L}(\lambda_h, [\lambda_h t]) = \mathcal{M}(t) \quad \text{in } L^1(\mathbb{R}^n) \quad \text{for any } t \geq 0,
\]

where the bounded partitions \(\mathcal{L}(\lambda, k), \lambda \geq 1, k \in \mathbb{N}_0,\) are defined inductively as \(\mathcal{L}(\lambda, 0) = \mathcal{G}\) and

\[
\mathcal{F}(\mathcal{L}(\lambda, k + 1), \mathcal{L}(\lambda, k); \lambda) = \min_{\mathcal{A} \in \mathbb{P}_b(N + 1)} \mathcal{F}(\mathcal{A}, \mathcal{L}(\lambda, k); \lambda) \quad \forall k \geq 0.
\]

When \(\text{GMM}(\mathcal{F}, \mathcal{G})\) is a singleton, it is called the minimizing movement starting from \(\mathcal{G}\) and denoted by \(\text{MM}(\mathcal{F}, \mathcal{G})\).

We shall also consider GMM associated to the functional

\[
\mathcal{F}_H(\mathcal{A}, \mathcal{B}; \lambda) = \text{Per}(\mathcal{A}) + \sum_{j=1}^{N+1} \int_{A_j} H_j dx + \frac{\lambda}{2} \sum_{j=1}^{N+1} \int_{A_j \triangle B_j} d(x, \partial B_j) dx, \quad \mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N + 1)
\]

for suitable driving forces \(H_i, \ i = 1, \ldots, N + 1\) (see Section 5).

Our main result is the following (see Theorems 4.9 and 5.1 for the precise statements):

**Theorem 1.2.** For any \(\mathcal{G} \in \mathbb{P}_b(N + 1)\), \(\text{GMM}(\mathcal{F}, \mathcal{G})\) is nonempty, i.e. there exists a generalized minimizing movement starting from \(\mathcal{G}\). Moreover,

1) any such movement \(\mathcal{M}(t) = (M_1(t), \ldots, M_{N+1}(t))\) is locally \(\frac{1}{n+1}\)-Holder continuous in time;

2) \(\bigcup_{j=1}^{N} M_j(t)\) is contained in the closed convex envelope of the union \(\bigcup_{j=1}^{N} G_j\) of the bounded components of \(\mathcal{G}\) for any \(t > 0\).

Finally, similar results are valid for \(\mathcal{F}_H\).

To prove Theorem 1.2 we establish uniform density estimates for minimizers of \(\mathcal{F}\) and \(\mathcal{F}_H\). A lower-type density estimate for minimizers of \(\mathcal{F}\) could be proven using the slicing method for currents as in the thesis \([10]\), or also using the infiltration technique of \([36, \text{Lemma 4.6}]\) (see also \([39, \text{Section 30.2}]\)). In Section 3 we prove that \((\mathcal{A}, r_0)\)-minimizers of \(\text{Per}\) in \(\mathbb{R}^n\) (Definition 3.5) satisfy uniform density estimates using the method of cutting out and filling in with balls, an argument of \([38]\).

Some consistency results of GMM starting from disjoint partitions (Definition 6.7) with other notions of solutions are shown in Section 6. In particular we have:
**Theorem 1.3.** a) Let $G \in \mathbb{P}_b(N+1)$ be a disjoint partition and suppose that for each $i = 1, \ldots, N$ there exists a family of smooth sets $L_i(t), \ t \in [0, t_o)$, whose boundaries evolve smoothly by mean curvature in $[0, t_o)$ such that $L_i(0) = G_i$. Then for any $M \in GMM(\mathcal{G}, G)$ we have

$$M_i(t) = L_i(t), \ t \in [0, t_o).$$

b) Let $G \in \mathbb{P}_b(N+1)$ be a disjoint partition such that for each $i = 1, \ldots, N, \ |\partial G_i| = 0$, and suppose that the viscosity solution $v_i$ [11] of

$$\frac{\partial u}{\partial t} = |\nabla u| \text{ div} \frac{\nabla u}{|\nabla u|}$$

starting from $\chi G_i - \chi G_i$ is unique. Then $GMM(\mathcal{G}, G) = \{M\}$ is a singleton and

$$v_i(x, t) = \chi_{M_i(t)}(x) - \chi_{M_i(t)}(x) \quad \text{for every} \ (x, t) \in \mathbb{R}^n \times [0, +\infty).$$

In Theorem 6.10 we also show the following stability result.

**Theorem 1.4.** Suppose that $C = (C_1, \ldots, C_{N+1}) \in \mathbb{P}_b(N+1)$, where $C_1, \ldots, C_N$ are convex sets whose closures are disjoint. Then the GMM associated to $\mathcal{G}$ and starting from $C$ is the minimizing movement $\{M\} = MM(\mathcal{G}, C)$, and writing

$$M(t) = (M_1(t), \ldots, M_{N+1}(t)),$$

we have that each $M_i(t)$ agrees with the classical mean curvature flow starting from $C_i$, up to the extinction time. Moreover, if a sequence $\{G^{(k)}\} \subset \mathbb{P}_b(N+1)$ converges to $C \in \mathbb{P}_b(N+1)$ in the Hausdorff distance, then any $M^{(k)} \in GMM(\mathcal{G}, G^{(k)})$ converges to $\{M\} = MM(\mathcal{G}, C)$ in the Hausdorff distance at every time $t \geq 0$.

The proof of the consistency with the classical mean curvature flow relies on the results of [7], while for the stability in the Hausdorff distance we employ the comparison results from [8, 12].

Our results do not apply to the case when at least two components of a partition are unbounded, since in this case they have infinite perimeter, and it also may happen that the right hand side of (4.1), which allows to replace $\int_{E_i \Delta F_i} d(x, \partial F_i)dx$ with the signed distance function, is not well-defined.

The plan of the paper is the following.

In Section 2 we set the notation and recall some results from the theory of finite perimeter sets. Section 3 is devoted to the definitions of partitions and density estimates for $(\Lambda, r_0)$-minimizers. In Section 4 we prove the existence of minimizers of $\mathcal{G}$ in $\mathbb{P}_b(N+1)$ (Theorem 4.2), the density estimates (Theorem 4.6), and – one of our main results – the existence of $GMM$ for $\mathcal{G}$ (Theorem 4.9). The existence of $GMM$ for $\mathcal{G}_H$ is shown in Section 5. Finally, in Section 6 we show that any GMM starting from a disjoint partition is also disjoint and prove Theorem 1.3 – the consistency result with smooth mean curvature flow. As a nontrivial application of these facts, we show the consistency and stability results stated in Theorem 1.4.

2. Notation and preliminaries

In this section we introduce the notation and collect some important properties of sets of locally finite perimeter. The standard references for $BV$-functions and sets of finite perimeter are [3, 28].

We use $\mathbb{N}_0$ to denote the set of all nonnegative integers. Given a finite subset $I \subset \mathbb{N}_0$, we write $|I|$ for the number of elements of $I$. The symbol $B_r(x)$ stands for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ of radius $r > 0$. The characteristic function of a Lebesgue measurable set
\( F \) is denoted by \( \chi_F \) and its Lebesgue measure by \( |F| \); we set also \( \omega_n := |B_1(0)| \). We denote by \( E^c \) the complement of \( E \) in \( \mathbb{R}^n \).

\( \text{Op}(\mathbb{R}^n) \) (resp. \( \text{Op}_0(\mathbb{R}^n) \)) is the collection of all open (resp. open and bounded) subsets of \( \mathbb{R}^n \). The set of \( L^1_{\text{loc}}(\mathbb{R}^n) \)-functions having locally bounded total variation in \( \mathbb{R}^n \) is denoted by \( BV_{\text{loc}}(\mathbb{R}^n) \) and the elements of

\[
BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) := \{ E \subseteq \mathbb{R}^n : \chi_E \in BV_{\text{loc}}(\mathbb{R}^n) \}
\]

are called locally finite perimeter sets. Given a \( E \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \) we denote by

a) \( P(E, \Omega) := \int_{\Omega} |D\chi_E| \) the perimeter of \( E \) in \( \Omega \in \text{Op}(\mathbb{R}^n) \);

b) \( \partial E \) the measure-theoretic boundary of \( E \):

\[
\partial E := \{ x \in \mathbb{R}^n : 0 < |B_\rho \cap E| < |B_\rho| \quad \forall \rho > 0 \};
\]

c) \( \partial^* E \) the reduced boundary of \( E \);

d) \( \nu_E \) the outer generalized unit normal to \( \partial^* E \).

For simplicity, we set \( P(E) := P(E, \mathbb{R}^n) \) provided \( E \in BV(\mathbb{R}^n, \{0, 1\}) \). Further, given a Lebesgue measurable set \( E \subseteq \mathbb{R}^n \) and \( \alpha \in [0, 1] \) we define

\[
E^{(\alpha)} := \left\{ x \in \mathbb{R}^n : \lim_{\rho \to 0^+} \frac{|B_\rho(x) \cap E|}{|B_\rho(x)|} = \alpha \right\}.
\]

Unless otherwise stated, we always suppose that any locally finite perimeter set \( E \) we consider coincides with \( E^{(1)} \) (so that by [28, Proposition 3.1] \( \partial E \) coincides with the topological boundary). We recall that \( \partial^* E = \partial E \) and \( D\chi_E = \nu_E d\mathcal{H}^{n-1} \mathcal{L} \partial^* E \), where \( \mathcal{H}^{n-1} \) is the \((n - 1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \) and \( \mathcal{L} \) is the symbol of restriction. Given a nonempty set \( E \subseteq \mathbb{R}^n \), \( d(\cdot, E) \) stands for the distance function from \( E \) and

\[
\tilde{d}(x, \partial E) = d(x, E) - d(x, \mathbb{R}^n \setminus E)
\]

is the signed distance function from \( \partial E \), negative inside \( E \). We also write \( d(A, B) \) to denote the distance between \( A, B \subseteq \mathbb{R}^n \).

**Theorem 2.1.** [16] Let \( E \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \). Then for any \( x \in \partial^* E \)

\[
\lim_{\rho \to 0^+} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = \frac{1}{2}; \quad \lim_{\rho \to 0^+} \frac{P(E, B_\rho(x))}{\omega_{n-1} \rho^{n-1}} = 1.
\]

**Theorem 2.2.** [3, Theorem 3.61] For every \( E \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \)

\[
\mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup E \cup \partial^* E)) = 0.
\]

Moreover, \( \mathcal{H}^{n-1}(E^{(1/2)} \setminus \partial^* E) = 0 \).

**Remark 2.3.** Given \( E \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \) the map \( \Omega \in \text{Op}(\mathbb{R}^n) \mapsto P(E, \Omega) \) extends to a Borel measure in \( \mathbb{R}^n \), so that \( P(E, B) = \mathcal{H}^{n-1}(B \cap \partial^* E) \) for every Borel set \( B \subseteq \mathbb{R}^n \).

**Theorem 2.4.** [39, Theorem 16.3] If \( E \) and \( F \) are sets of locally finite perimeter, and we let

\[
\{\nu_E = \nu_F\} = \{ x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x) \},
\]

\[
\{\nu_E = -\nu_F\} = \{ x \in \partial^* E \cap \partial^* F : \nu_E(x) = -\nu_F(x) \},
\]

then \( E \cap F, E \setminus F \) and \( E \cup F \) are locally finite perimeter sets with

\[
\partial^*(E \cap F) \approx (F \cap \partial^* E) \cup (E \cap \partial^* F) \cup \{\nu_E = \nu_F\},
\]

\[
\partial^*(E \setminus F) \approx (F^{(0)} \cap \partial^* E) \cup (E \cap \partial^* F) \cup \{\nu_E = -\nu_F\},
\]

\[
\partial^*(E \cup F) \approx (F^{(0)} \cap \partial^* E) \cup (E^{(0)} \cap \partial^* F) \cup \{\nu_E = \nu_F\},
\]

where \( A \approx B \) means \( \mathcal{H}^{n-1}(A \Delta B) = 0 \). Moreover, for every Borel set \( B \subseteq \mathbb{R}^n \)

\[
P(E \cap F, B) = P(E, F \cap B) + P(F, E \cap B) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap B),
\]

(2.4)
Then there exist a partition \( A \in \mathcal{P}(\mathbb{R}^n) \) of \( \Omega \) such that
\[
P(E \setminus F, B) = P(E, F(0) \cap B) + P(F, E \cap B) + H^{n-1}(\{\nu_E = -\nu_F\} \cap B),
\]
\[
P(E \cup F, B) = P(E, F(0) \cap B) + P(F, E(0) \cap B) + H^{n-1}(\{\nu_E = \nu_F\} \cap B).
\]
Finally, recall that for every \( E, F \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \) and \( \Omega \in Op(\mathbb{R}^n) \),
\[
P(E \cap F, \Omega) + P(E \cup F, \Omega) \leq P(E, \Omega) + P(F, \Omega).
\]

3. Partitions

Now we give the notions of partition, \((\Lambda, r_0)\)-minimizer and bounded partition. The main result of this section is represented by the density estimates for \((\Lambda, r_0)\)-minimizers (Theorem 3.6).

**Definition 3.1 (Partition).** Given an integer \( N \geq 2 \), an \( N \)-tuple \( C = (C_1, \ldots, C_N) \) of subsets of \( \mathbb{R}^n \) is called an \( N \)-partition of \( \mathbb{R}^n \) (a partition, for short) if
(a) \( C_i \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \) for every \( i = 1, \ldots, N \),
(b) \( \sum N_{i=1} |C_i \cap K| = |K| \) for each compact \( K \subset \mathbb{R}^n \).

The collection of all \( N \)-partitions of \( \mathbb{R}^n \) is denoted by \( \mathbb{P}(N) \). Our assumptions \( C_i = C_i^{(1)} \) implies \( C_i \cap C_j = \emptyset \) for \( i \neq j \). Notice also that we do not exclude the case \( C_i = \emptyset \).

The elements of \( \mathbb{P}(N) \) are denoted by calligraphic letters \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \) and the components of \( \mathcal{A} \in \mathbb{P}(N) \) by the corresponding roman letters \( (A_1, \ldots, A_N) \). The functional
\[
(\mathcal{A}, \Omega) \in \mathbb{P}(N) \times Op(\mathbb{R}^n) \mapsto Per(\mathcal{A}, \Omega) := \frac{1}{2} \sum N_{j=1} P(A_j, \Omega)
\]
is called the perimeter of the partition \( \mathcal{A} \) in \( \Omega \). For simplicity, we write \( Per(\mathcal{A}) := Per(\mathcal{A}, \mathbb{R}^n) \).

We set
\[
\mathcal{A} \Delta \mathcal{B} := \bigcup_{j=1}^N A_j \Delta B_j
\]
and
\[
|\mathcal{A} \Delta \mathcal{B}| := \sum_{j=1}^N |A_j \Delta B_j|,
\]
where \( \Delta \) is the symmetric difference of sets, i.e. \( E \Delta F = (E \setminus F) \cup (F \setminus E) \).

We say that the sequence \( \{\mathcal{A}^{(k)}\} \subset \mathbb{P}(N) \) converges to \( \mathcal{A} \in \mathbb{P}(N) \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) if
\[
|((\mathcal{A}^{(k)} \Delta \mathcal{A}) \cap K| := \sum_{j=1}^N |(A_j^{(k)} \Delta A_j) \cap K| \to 0 \quad \text{as } k \to +\infty
\]
for every compact set \( K \subset \mathbb{R}^n \). Since \( E \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \mapsto P(E, \Omega) \) is \( L^1_{\text{loc}}(\mathbb{R}^n) \)-lower semicontinuous for any \( \Omega \in Op(\mathbb{R}^n) \), the map \( \mathcal{A} \in \mathbb{P}(N) \mapsto Per(\mathcal{A}, \Omega) \) is \( L^1_{\text{loc}}(\mathbb{R}^n) \)-lower semicontinuous. The following compactness result can be proven using [3, Theorem 3.39] and a diagonal argument.

**Theorem 3.2 (Compactness).** Let \( \{\mathcal{A}^{(l)}\} \subset \mathbb{P}(N) \) be a sequence of partitions such that
\[
\sup_{l \geq 1} Per(\mathcal{A}^{(l)}, \Omega) < +\infty \quad \forall \Omega \in Op_b(\mathbb{R}^n).
\]
Then there exist a partition \( \mathcal{A} \in \mathbb{P}(N) \) and a subsequence \( \{\mathcal{A}^{(l_k)}\} \) converging to \( \mathcal{A} \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( k \to +\infty \).

The next result is proven for the convenience of the reader.
Proposition 3.3 (Boundaries of “neighboring” sets). Let $A \in \mathbb{P}(N)$. Then

$$\mathcal{H}^{n-1}\left(\partial^* A_1 \setminus \bigcup_{j=1, j \neq i}^N \partial^* A_j \right) = 0 \quad \forall i = 1, \ldots, N.$$  

Proof. If $N = 2$, then

$$\partial^* A_1 = \partial^* (\mathbb{R}^n \setminus A_1) = \partial^* A_2,$$

hence we suppose $N \geq 3$. There is no loss of generality in assuming $i = 1$. By virtue of (2.3), there exists an $\mathcal{H}^{n-1}$-negligible set $Z_{2,3} \subset \partial A_2 \cup \partial A_3$ such that

$$\partial^* (A_2 \cup A_3) = Z_{2,3} \cup \left( A_2^{(0)} \cap \partial^* A_3 \right) \cup \left( A_3^{(0)} \cap \partial^* A_2 \right) \cup \{\nu_{A_2} = \nu_{A_3}\}.$$  

Therefore,

$$\partial^* (A_2 \cup A_3) \subseteq Z_{2,3} \cup \left( \partial^* A_2 \cup \partial^* A_3 \right),$$

and by an induction argument, for any $j \in \{3, \ldots, N\}$ there exists an $\mathcal{H}^{n-1}$-negligible set $Z_{2,\ldots,j-1,j} \subset \partial \left( \bigcup_{h=2}^{j-1} A_h \right) \cup \partial A_j$ such that

$$\partial^* \left( \bigcup_{j=2}^N A_j \right) \subseteq \left( \bigcup_{j=3}^N Z_{2,\ldots,j-1,j} \right) \cup \left( \bigcup_{j=2}^N \partial^* A_j \right).$$

Hence

$$\partial^* A_1 \setminus \bigcup_{j=2}^N \partial^* A_j \subseteq \left( \bigcup_{j=3}^N Z_{2,\ldots,j-1,j} \right) \cup \partial^* A_1 \setminus \partial^* \left( \bigcup_{j=2}^N A_j \right).$$

In view of (3.3), we have

$$\partial^* \left( \bigcup_{j=2}^N A_j \right) = \partial^* (\mathbb{R}^n \setminus A_1) = \partial^* A_1,$$

whence from (3.4),

$$\mathcal{H}^{n-1}(\partial^* A_1 \setminus \bigcup_{j=2}^N \partial^* A_j) \leq \sum_{j=3}^N \mathcal{H}^{n-1}(Z_{2,\ldots,j-1,j}) = 0.$$  

Remark 3.4. From Proposition 3.3 it follows that

$$\text{Per}(A, \Omega) = \frac{1}{2} \sum_{j=1}^N \mathcal{H}^{n-1}(\Omega \cap \partial^* A_j) = \sum_{j=2}^N \sum_{i=1}^{j-1} \mathcal{H}^{n-1}(\Omega \cap \partial^* A_j \cap \partial^* A_i).$$

Since $\mathcal{H}^{n-1}(\Omega \cap \partial^* A_j \cap \partial^* A_i)$ is the $(n - 1)$-dimensional area of the interface between the phases $A_i$ and $A_j$, $\text{Per}(A, \Omega)$ measures the total perimeter of the interfaces in $\Omega$.

3.1. $(\Lambda, r_0)$-minimizers. In order to prove Theorem 4.6 it is convenient to give the following definition.

Definition 3.5 ($(\Lambda, r_0)$-minimizers). Given $\Lambda \geq 0$ and $r_0 \in (0, +\infty]$ we say that a partition $A \in \mathbb{P}(N)$ is a $(\Lambda, r_0)$-minimizer of $\text{Per}$ in $\mathbb{R}^n$ (a $(\Lambda, r_0)$-minimizer, for short) if

$$\text{Per}(A, B_r(x)) \leq \text{Per}(B, B_r(x)) + \Lambda |A \Delta B|$$

whenever $x \in \mathbb{R}^n$, $B \in \mathbb{P}(N)$, $A \Delta B \subset \subset B_r(x)$, and $r \in (0, r_0)$.

The crucial technical tool is the following.
Theorem 3.6 (Density estimates for \((\Lambda, r_0)\)-minimizers). Let \(A \in \mathbb{P}(N)\) be a \((\Lambda, r_0)\)-minimizer, \(i \in \{1, \ldots, N\}\) and \(\hat{r}_0 := \min\{r_0, \frac{n}{1+1/2(N-1)}\}\) if \(\Lambda > 0\) and \(\hat{r}_0 := r_0\) if \(\Lambda = 0\). Then for any \(x \in \partial A_i\) and \(r \in (0, \hat{r}_0)\) the following density estimates hold:

\[
\left(\frac{1}{2N}\right)^n \leq \frac{|A_i \cap B_r(x)|}{|B_r(x)|} \leq 1 - \frac{1}{2n} \left(1 - \frac{1}{2(N-1)}\right)^n, \tag{3.5}
\]

\[
c_{n,N} \leq \frac{P(A_i, B_r(x))}{r^{n-1}} \leq \frac{2N-1}{2(N-1)} n \omega_n, \tag{3.6}
\]

where

\[
c_{n,N} := \frac{n \omega^{1/n} (2^{1/n} - 1)}{2^{n+1/n} N^{n-1}}. \tag{3.7}
\]

Moreover,

\[
\sum_{i=1}^N \mathcal{H}^{n-1}(\partial A_i \setminus \partial^* A_i) = 0. \tag{3.8}
\]

Proof. We may suppose \(i = 1\). Moreover, since \(\partial A_1 = \partial A_1\), it suffices to show (3.5)-(3.6) whenever \(x \in \partial A_1\). Writing \(B_\rho := B_\rho(x)\) for \(\rho > 0\), we will show that for a.e. \(r \in (0, \hat{r}_0)\) one has

\[
P(\mathbb{R}^n \setminus A_1, B_r) \leq \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda|(\mathbb{R}^n \setminus A_1) \cap B_r|. \tag{3.9}
\]

Choose \(r \in (0, \hat{r}_0)\) satisfying

\[
\sum_{j=1}^N \mathcal{H}^{n-1}(\partial B_r \cap \partial^* A_j) = 0 \tag{3.10}
\]

and define the competitor \(B \in \mathbb{P}(N)\) as

\[
B := (A_1 \cup B_r, A_2 \setminus B_r, \ldots, A_N \setminus B_r).
\]

Then \(A \Delta B \subset B_s\) for every \(s \in (r, \hat{r}_0)\) and thus, by \((\Lambda, r_0)\)-minimality,

\[
0 \leq 2 \text{Per}(B, B_s) - 2 \text{Per}(A, B_s) + 2\Lambda|A \Delta B| = P(A_1 \cup B_r, B_s) - P(A_1, B_s)
\]

\[
+ \sum_{j=2}^N \left(P(A_j \setminus B_r, B_s) - P(A_j, B_s)\right) + 2\Lambda|B_r \setminus A_1| + 2\Lambda \sum_{j=2}^N |A_j \cap B_r|. \tag{3.11}
\]

By the disjointness of the \(A_j\)'s we have

\[
\sum_{j=2}^N |A_j \cap B_r| = |B_r \setminus A_1|. \tag{3.12}
\]

Moreover, recalling that \(A_j^{(1)} = A_j\) from the relation (2.5), (3.10) and \(\mathcal{H}^{n-1}(B_s \cap \{\nu_{A_j} = -\nu_{B_s}\}) = 0\), we get

\[
P(A_j \setminus B_r, B_s) = P(A_j, B_s \setminus B_r) + \mathcal{H}^{n-1}(A_j \cap \partial B_r) \quad \forall j \in \{2, \ldots, N\}. \tag{3.13}
\]

Thus,

\[
\sum_{j=2}^N P(A_j \setminus B_r, B_s) = \sum_{j=2}^N P(A_j, B_s \setminus B_r) + \sum_{j=2}^N \mathcal{H}^{n-1}(A_j \cap \partial B_r).
\]

By the disjointness of the \(A_j\)'s, Theorem 2.2 and the choice of \(r\) in (3.10),

\[
\sum_{j=2}^N \mathcal{H}^{n-1}(A_j \cap \partial B_r) = \mathcal{H}^{n-1}(A_j^{(1)} \cap \partial B_r) = \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r).
\]
Therefore,
\[ \sum_{j=2}^{N} P(A_j \setminus B_r, B_s) = \sum_{j=2}^{N} P(A_j, B_s \setminus B_r) + \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r). \]  \tag{3.14} 

Finally, since \( \mathcal{H}^{n-1}(B_s \cap \{\nu_{A_1} = \nu_{B_r}\}) = 0 \) by (3.10), from (2.6) we deduce
\[ P(A_1 \cup B_r, B_s) = P(A_1, B_s \setminus B_r) + \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r). \]  \tag{3.15} 

Now inserting (3.12), (3.14) and (3.15) in (3.11) we get
\[ P(A_1, B_r) + \sum_{j=2}^{N} P(A_j, B_r) \leq 2\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 4\Lambda|\mathbb{R}^n \setminus A_1) \cap B_r|. \]  \tag{3.16} 

Applying (2.7) and using the disjointness of the \( A_j \)'s we get
\[ \sum_{j=2}^{N} P(A_j, B_r) \geq P\left( \bigcup_{j=2}^{N} A_j, B_r \right) = P(\mathbb{R}^n \setminus A_1, B_r) = P(A_1, B_r) \]
and thus from (3.16) we establish (3.9).

Adding \( \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) \) to both sides of (3.9) and using (3.10) we get
\[ P((\mathbb{R}^n \setminus A_1) \cap B_r) \leq 2\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda|\mathbb{R}^n \setminus A_1) \cap B_r|. \]  \tag{3.17} 

Now by the isoperimetric inequality [15],
\[ n\omega_n^{1/n}|(\mathbb{R}^n \setminus A_1) \cap B_r|^{\frac{n-1}{n}} \leq 2\mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda|\mathbb{R}^n \setminus A_1) \cap B_r|. \]  \tag{3.18} 

Since \( r < \hat{r}_0 \leq \frac{n\omega_n^{1/n}}{2(N-1)\Lambda} \),
\[ 2\Lambda|\mathbb{R}^n \setminus A_1) \cap B_r|^{\frac{1}{n}} \leq 2\Lambda \omega_n^{1/n}\hat{r}_0 \leq \frac{n\omega_n^{1/n}}{2(N-1)}. \]

As a result, from (3.18) we obtain
\[ \frac{1}{2} \left( 1 - \frac{1}{2(N-1)} \right) n\omega_n^{1/n}|(\mathbb{R}^n \setminus A_1) \cap B_r|^{\frac{n-1}{n}} \leq \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r). \]  \tag{3.19} 

Set \( m(\rho) := |(\mathbb{R}^n \setminus A_1) \cap B_{\rho}|, \rho > 0 \). Since \( x \in \partial A_1 \), one has \( m(\rho) > 0 \) for any \( \rho > 0 \) and by the coarea formula (see e.g. [39, Example 13.4]) \( m(\cdot) \) is absolutely continuous and \( m'(\rho) := \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_{\rho}) \) for a.e. \( \rho > 0 \). Now by (3.19)
\[ \frac{1}{2} \left( 1 - \frac{1}{2(N-1)} \right) n\omega_n^{1/n}m(r)^{\frac{n-1}{n}} \leq m'(r), \text{ for a.e. } r \in (0, \hat{r}_0). \]

Integrating this differential inequality we get
\[ |(\mathbb{R}^n \setminus A_1) \cap B_r| \geq \frac{1}{2^n} \left( 1 - \frac{1}{2(N-1)} \right)^n \omega_n r^n, \]
i.e.
\[ \frac{|A_1 \cap B_r|}{|B_r|} \leq 1 - \frac{1}{2^n} \left( 1 - \frac{1}{2(N-1)} \right)^n, \]
which is the upper volume density estimate in (3.5).

Since \( 2\Lambda r \leq \frac{n\omega_n^{1/n}}{2(N-1)} \), from (3.9) we obtain also
\[ P(A_1, B_r) \leq \mathcal{H}^{n-1}(\partial B_r) + 2\Lambda|B_r| \leq n\omega_n r^{n-1} + \frac{n\omega_n}{2(N-1)} r^{n-1} = \frac{2N-1}{2(N-1)} n\omega_n r^{n-1} \]
for a.e. \( r \in (0, \hat{r}_0) \). Now the left-continuity of \( \rho \mapsto P(A_1, B_\rho) \) implies the upper perimeter density estimate in (3.6).
Let us prove the lower volume density estimate. As above we may suppose \( i = 1 \) and take \( x \in \partial^* A_1 \). Writing \( B_\rho := B_\rho(x) \) for \( \rho > 0 \), we will show that for a.e. \( r \in (0, \hat{r}_0) \) one has

\[
P(A_1, B_r) \leq (N - 1) \mathcal{H}^{n-1}(A_1 \cap \partial B_r) + 2(N - 1) \Lambda |A_1 \cap B_r|.
\]

(3.20)

Set

\[
I := \{j \in \{2, \ldots, N\} : \mathcal{H}^{n-1}(B_{r_0} \cap \partial^* A_1 \cap \partial^* A_j) > 0\}.
\]

Since \( x \in \partial A_1 \), one has \( I \neq \emptyset \). Let \( r \in (0, \hat{r}_0) \) satisfy (3.10). By virtue of Proposition 3.3 and Remark 3.4,

\[
P(A_1, B_r) \leq \sum_{j=2}^N \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j) = \sum_{j \in I} \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).
\]

(3.21)

For every \( j \in I \) let us define the competitor \( B^{(j)} \in \mathcal{P}(N) \) as

\[
B^{(j)} := (A_1 \setminus B_r, A_2, \ldots, A_{j-1}, A_j \cup (A_1 \cap B_r), A_{j+1}, \ldots, A_N).
\]

By the \((\Lambda, r_0)\)-minimality of \( \mathcal{A} \), for every \( s \in (r, \hat{r}_0) \) one has

\[
P(A_1, B_s) + P(A_j, B_s) \leq P(A_1 \setminus B_r, B_s) + P(A_j \cup (A_1 \cap B_r), B_s) + 4 \Lambda |A_1 \cap B_r|.
\]

(3.22)

From (3.10) and (2.1)

\[
\partial^*(A_1 \cap B_r) \approx (A_1 \cap \partial B_r) \cup (B_r \cap \partial^* A_1).
\]

(3.23)

Observe that

\[
\mathcal{H}^{n-1}(B_s \cap \{\nu_{A_j} = \nu_{A_1} \cap B_r\}) = 0
\]

(3.24)

for any \( j \in I \). Indeed, by (3.23)

\[
B_s \cap \{\nu_{A_j} = \nu_{A_1 \cap B_r}\} \approx (A_1 \cap \{\nu_{A_j} \cap \nu_{B_r}\}) \cup (B_r \cap \{\nu_{A_j} = \nu_{A_1}\}).
\]

By (3.10), \( \mathcal{H}^{n-1}(A_1 \cap \{\nu_{A_j} \cap \nu_{B_r}\}) = 0 \). On the other hand, since \( A_j \cap A_1 = \emptyset \), one has \( \nu_{A_j}(x) = -\nu_{A_1}(x) \) for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial^* A_j \cap \partial^* A_1 \), and hence \( \mathcal{H}^{n-1}(B_r \cap \{\nu_{A_j} = \nu_{A_1}\}) = 0 \).

From (2.6) and (3.24) it follows that

\[
P(A_j \cup (A_1 \cap B_r), B_s) = \mathcal{H}^{n-1}((A_1 \cap B_r)^{(0)} \cap B_s \cap \partial^* A_j) + \mathcal{H}^{n-1}(A_j^{(0)} \cap B_s \cap \partial^* (A_1 \cap B_r)).
\]

(3.25)

By Theorem 2.2

\[
\mathcal{H}^{n-1}(B_s \cap \partial^* A_j) = \mathcal{H}^{n-1}(E^{(0)} \cap B_s \cap \partial^* A_j) + \mathcal{H}^{n-1}(E^{(1)} \cap B_s \cap \partial^* A_j)
\]

\[
+ \mathcal{H}^{n-1}(B_s \cap \partial^* E \cap \partial^* A_j)
\]

(3.26)

for every \( E \in BV_{\text{loc}}(\mathbb{R}^n, \{0, 1\}) \). Hence, applying (3.26) with \( E = A_1 \cap B_r = E^{(1)} \), in view of \( \mathcal{H}^{n-1}(A_1 \cap B_r \cap \partial^* A_j) = 0 \) and \( \mathcal{H}^{n-1}(A_1 \cap \partial B_r \cap \partial^* A_j) = 0 \) (see (3.10) and (3.23) we have

\[
\mathcal{H}^{n-1}((A_1 \cap B_r)^{(0)} \cap B_s \cap \partial^* A_j) = \mathcal{H}^{n-1}(B_s \cap \partial^* A_j) - \mathcal{H}^{n-1}(B_s \cap \partial^* (A_1 \cap B_r) \cap \partial^* A_j).
\]

(3.27)

\[
= P(A_j, B_s) - \mathcal{H}^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).
\]
Similarly, since $A_j \cap A_1 = 0$ and $A_j \cap \partial^* A_1 = \emptyset$ we have $H^{n-1}(A_j \cap \partial B_r \cap \partial^*(A_1 \cap B_r)) = 0$ for any $j \in I$ and hence
\[
H^{n-1}(A_j^{(0)} \cap B_s \cap \partial^*(A_1 \cap B_r)) = H^{n-1}(B_s \cap \partial^*(A_1 \cap B_r)) - H^{n-1}(B_s \cap \partial^*(A_1 \cap B_r) \cap \partial^* A_j)
\]
\[
= H^{n-1}(A_1 \cap \partial B_r) + P(A_1, B_r) - H^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).
\]
Therefore, from (3.25) we get
\[
P(A_j \cup (A_1 \cap B_r), B_s) = P(A_j, B_s) + H^{n-1}(A_1 \cap \partial B_r)
\]
\[
+ P(A_1, B_r) - 2H^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j).
\]
Inserting this and
\[
P(A_1 \setminus B_r, B_s) = P(A_1, B_s \setminus B_r) + H^{n-1}(A_1 \cap \partial B_r)
\]
(whose proof is the same as (3.13)) in (3.22) and using (3.10) once more we get
\[
H^{n-1}(B_r \cap \partial^* A_1 \cap \partial^* A_j) \leq H^{n-1}(A_1 \cap \partial B_r) + 2\Lambda|A_1 \cap B_r|.
\]
(3.28)
Summing these inequalities in $j \in I$ and using (3.21) and $|I| \leq N - 1$, we obtain (3.20). Now adding $H^{n-1}(A_1 \cap \partial B_r)$ to both sides of (3.20) we get
\[
P(A_1 \cap B_r) \leq N H^{n-1}(A_1 \cap \partial B_r) + 2(N - 1)\Lambda|A_1 \cap B_r|.
\]
(3.29)
Since $2(N - 1)\Lambda|A_1 \cap B_r|^{1/n} \leq \frac{\omega_n^{1/n}}{2}$ for any $r \in (0, r_0)$, from the isoperimetric inequality we get
\[
\frac{1}{2N} n \omega_n^{1/n} |A_1 \cap B_r|^{\frac{n-1}{n}} \leq H^{n-1}(A_1 \cap \partial B_r).
\]
Now proceeding as in the proof of the upper volume density estimate we get the lower volume density estimate:
\[
|A_1 \cap B_r| \geq \left( \frac{1}{2N} \right)^n \omega_n r^n.
\]
Now we prove the lower perimeter density estimate in (3.6). Notice that $N \geq 2$, therefore
\[
\frac{1}{2N} \leq \frac{1}{2} \left(1 - \frac{1}{2(N - 1)}\right).
\]
Hence from the volume density estimates (3.5) and [23, Theorem 1]
\[
P(A_1, B_r) \geq \frac{n \omega_n^{1/n} (2^{1/n} - 1) \min \left\{|B_r \cap A_1|^{\frac{n-1}{n}}, |B_r \setminus A_1|^{\frac{n-1}{n}}\right\}}{2^{1+1/n}}
\]
\[
\geq \frac{n \omega_n^{1/n} (2^{1/n} - 1) \min \left\{\frac{1}{2N}, \frac{1}{2} \left(1 - \frac{1}{2(N - 1)}\right)\right\}}{2^{1+1/n}} |B_r|^{\frac{n-1}{n}} = c_n N r^{n-1}.
\]
Finally, (3.8) is a consequence of a standard covering argument. □

**Remark 3.7.** Let $\alpha_1, \alpha_2 > \frac{n-1}{n}$, $\Lambda_1 \geq 0$, $\Lambda_2 > 0$, $r_0 \in (0, +\infty]$. Suppose that $\mathcal{A} \in \mathcal{P}(N)$ satisfies
\[
\text{Per}(\mathcal{A}, B_r(x)) \leq \text{Per}(B, B_r(x)) + \Lambda_1 |\mathcal{A} \Delta B|^{\alpha_1} + \Lambda_2 |\mathcal{A} \Delta B|^{\alpha_2}
\]
whenever $B \in \mathcal{P}(N)$, $\mathcal{A} \Delta B \subset B(x)$ and $r \in (0, r_0)$. Then, repeating the proof of Theorem 3.6, one obtains that (3.17) and (3.29) are replaced by
\[
P((\mathbb{R}^n \setminus A_1) \cap B_r) \leq 2H^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + 2\Lambda_1 |(\mathbb{R}^n \setminus A_1) \cap B_r|^{\alpha_1}
\]
\[
+ 2\Lambda_2 |(\mathbb{R}^n \setminus A_1) \cap B_r|^{\alpha_2}
\]
and
\[ P(A_1 \cap B_r) \leq N \mathcal{H}^{n-1}(A_1 \cap \partial B_r) + 2(N-1)\Lambda_1|A_1 \cap B_r|^{\alpha_1} + 2(N-1)\Lambda_2|A_1 \cap B_r|^{\alpha_2} \]
respectively. Thus, for every \( i \in \{1, \ldots, N\} \), for every \( x \in \partial A_i \) and for any \( r \in (0, \tilde{r}_0) \), the relations (3.5)-(3.8) hold, where
\[
\tilde{r}_0 = \min\{r_0, \omega_n^{-1/n}(\frac{\min\{\omega_n^{1/n}(\frac{1}{4(N-1)\Lambda_2}), \omega_n^{-1/n}(\frac{1}{(N-1)\Lambda_2})\}}{\max\{\Lambda_2-\frac{1}{N}\}})\} \quad \text{if } \Lambda_1 = 0,
\]
\[
\min\{r_0, \omega_n^{-1/n}(\frac{\min\{\omega_n^{1/n}(\frac{1}{4(N-1)\Lambda_2}), \omega_n^{-1/n}(\frac{1}{(N-1)\Lambda_2})\}}{\max\{\Lambda_2-\frac{1}{N}\}})\} \quad \text{if } \Lambda_1 > 0.
\]
This will be used in the proof of Theorem 5.1.

From (3.8) it follows that \( \mathcal{H}^{n-1}(\partial^* A_i) = \mathcal{H}^{n-1}(\partial A_i) \) for every \( i = 1, \ldots, N \).

**Remark 3.8.** Let \( x \in \mathbb{R}^n \) and let \( B_r := B_r(x), \ r \in (0, \tilde{r}_0) \) be any ball such that
\[
\sum_{j=1}^{N} \mathcal{H}^{n-1}(\partial^* A_j \cap \partial^* B_r) = 0
\]
(\( x \) not necessarily lies on \( \bigcup_{j=1}^{N} \partial A_j \)). Then comparing \( \mathcal{A} \) with \( \mathcal{B} := (A_1 \cup B_r, A_2 \setminus B_r, \ldots, A_N \setminus B_r) \) as in the proof of Theorem 3.6 we get
\[
P(A_1, B_r) \leq \mathcal{H}^{n-1}((\mathbb{R}^n \setminus A_1) \cap \partial B_r) + \Lambda|B_r \cap (\mathbb{R}^n \setminus A_1)|
\]
and therefore
\[
\frac{P(A_1, B_r(x))}{r^{n-1}} \leq C(n, N), \quad \forall r \in (0, \tilde{r}_0).
\]
By symmetry (3.30) holds for every \( i = 1, \ldots, N \).

### 3.2. Bounded partitions.

The multiphase analog of a bounded phase in \( \mathbb{R}^n \) is the following.

**Definition 3.9 (Bounded partition).** A partition \( C = (C_1, \ldots, C_{N+1}) \in \mathbb{P}(N+1) \) is called **bounded** if \( C_i \) is bounded for each \( i = 1, \ldots, N \).

Therefore, \( C_{N+1} \) is the only unbounded component of \( C \). We denote by \( \mathbb{P}_b(N+1) \) the collection of all bounded partitions of \( \mathbb{R}^n \).

Given \( \mathcal{A} \in \mathbb{P}_b(N+1) \), we denote by
\[
\text{co}(\mathcal{A})
\]
the closed convex hull of \( \bigcup_{i=1}^{N} A_i \). Since \( \mathcal{A}\Delta \mathcal{B} \subset \subset \mathbb{R}^n \) for every \( \mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1) \),
\[
|\mathcal{A}\Delta \mathcal{B}| = \sum_{j=1}^{N+1} |A_j \Delta B_j|
\]
is the \( L^1(\mathbb{R}^n) \) distance in \( \mathbb{P}_b(N+1) \).

The following compactness result can be proven similarly to Theorem 3.2.

**Theorem 3.10 (Compactness).** Let \( \{\mathcal{A}^{(l)}\} \subset \mathbb{P}_b(N+1) \) and \( \Omega \in \mathbb{O}_p_b(\mathbb{R}^n) \) be such that
\[
\sup_{l \geq 1} \text{Per}(\mathcal{A}^{(l)}) < +\infty, \quad \text{co}(\mathcal{A}^{(l)}) \subseteq \Omega \quad \forall l \geq 1.
\]
Then there exist \( \mathcal{A} \in \mathbb{P}_b(N+1) \) and a subsequence \( \{\mathcal{A}^{(l_k)}\} \) converging to \( \mathcal{A} \) in \( L^1(\mathbb{R}^n) \) as \( k \to +\infty \). Moreover, \( \bigcup_{i=1}^{N} A_j \subset \subset \Omega \).
### 4. Existence of GMM

Given $E, F \subset \mathbb{R}^n$ set

$$\bar{\sigma}(E, F) := \int_{E \Delta F} d(x, \partial F)dx.$$  

Note that $\bar{\sigma}(E, F) = 0$ if $|E \Delta F| = 0$ whereas $\bar{\sigma}(E, F) = +\infty$ if $\partial F = \emptyset$ and $|E \Delta F| > 0$.

Moreover, if $X, Y \subset \mathbb{R}^n$ are measurable and $\partial Y \neq \emptyset$,

$$\int_{X \Delta Y} d(x, \partial Y)dx = \int_{X} \hat{d}(x, \partial Y)dx - \int_{Y} \hat{d}(x, \partial Y)dx \quad \text{if } X \cap Y \text{ is bounded},$$

$$\int_{X \Delta Y} d(x, \partial Y)dx = \int_{Y^c} \hat{d}(x, \partial Y)dx - \int_{X^c} \hat{d}(x, \partial Y)dx \quad \text{if } X^c \cap Y^c \text{ is bounded}. \tag{4.1}$$

Now the nonsymmetric distance between $A, B \in \mathbb{P}_b(N+1)$ is defined as

$$\sigma(A, B) := \sum_{i=1}^{N+1} \bar{\sigma}(A_i, B_i),$$

where $N + 1 \geq 2$. Observe that for every $B \in \mathbb{P}_b(N+1)$ the map $\sigma(\cdot, B)$ is $L^1(\mathbb{R}^n)$-lower semicontinuous.

**Definition 4.1 (The functional $\mathfrak{F}$).** We let $\mathfrak{F} : \mathbb{P}_b(N+1) \times \mathbb{P}_b(N+1) \times [1, +\infty) \rightarrow [0, +\infty]$ be the functional defined as

$$\mathfrak{F}(B, A; \lambda) = \text{Per}(B) + \frac{\lambda}{2} \sigma(B, A) = \frac{1}{2} \sum_{j=1}^{N+1} P(B_j) + \frac{\lambda}{2} \sum_{j=1}^{N+1} \int_{B_j \Delta A_j} d(x, \partial A_j)dx.$$  

The domain of $\mathfrak{F}$ is independent of $Z$, and $\mathfrak{F}$ is the natural generalization of the Almgren-Taylor-Wang functional [1] to the case of partitions [10, 18]. One can readily check that the map $B \in \mathbb{P}_b(N+1) \mapsto \mathfrak{F}(B, A; \lambda)$ is $L^1(\mathbb{R}^n)$-lower semicontinuous.

**Theorem 4.2 (Existence of minimizers of $\mathfrak{F}$).** Given $A \in \mathbb{P}_b(N+1)$ and $\lambda \geq 1$ the problem

$$\inf_{B \in \mathbb{P}_b(N+1)} \mathfrak{F}(B, A; \lambda) \tag{4.2}$$

has a solution. Moreover, every minimizer $A(\lambda) = (A_1(\lambda), \ldots, A_{N+1}(\lambda))$ satisfies the bound

$$\bigcup_{i=1}^{N} A_i(\lambda) \subseteq \text{co}(A).$$

**Proof.** Given a partition $B \in \mathbb{P}_b(N+1)$ define the competitor $B' \in \mathbb{P}_b(N+1)$ as

$$B' := \left( B_1 \cap \text{co}(A), \ldots, B_N \cap \text{co}(A), \mathbb{R}^n \setminus \bigcup_{i=1}^{N} (B_i \cap \text{co}(A)) \right). \tag{4.3}$$

Since $\text{co}(A)$ is convex and closed, by the comparison theorem of [2, page 152] we have $P(B_i) \geq P(B_i \cap \text{co}(A))$ for $i = 1, \ldots, N$, and

$$P(B_{N+1}) = P\left( \bigcup_{i=1}^{N} B_i \right) \geq P\left( \bigcup_{i=1}^{N} B_i \cap \text{co}(A) \right) \geq P\left( \bigcup_{i=1}^{N} (B_i \cap \text{co}(A)) \right) = P\left( \bigcup_{i=1}^{N} (B_i \cap \text{co}(A)) \right) = \left( \mathbb{R}^n \setminus \bigcup_{i=1}^{N} (B_i \cap \text{co}(A)) \right).$$
with equality if and only if \( \left| \bigcup_{i=1}^{N} B_i \setminus \text{co}(A) \right| = 0 \). In addition, for \( i = 1, \ldots, N \),
\[
\int_{B_i \Delta A_i} d(x, \partial A_i) dx = \int_{B_i \setminus A_i} d(x, \partial A_i) dx + \int_{A_i \setminus B_i} d(x, \partial A_i) dx
\]
\[
\geq \int_{(B_i \cap \text{co}(A)) \setminus A_i} d(x, \partial A_i) dx + \int_{A_i \setminus (B_i \cap \text{co}(A))} d(x, \partial A_i) dx
\]
\[
= \int_{(B_i \cap \text{co}(A)) \Delta A_i} d(x, \partial A_i) dx,
\]
where we used the nonnegativity of the distance function and \( A_i \setminus B_i = A_i \setminus (B_i \cap \text{co}(A)) \).
The equality in (4.4) holds if and only if \( \left| \bigcup_{i=1}^{N} B_i \setminus \text{co}(A) \right| = 0 \). For the same reason, since \( A_{N+1}^{c} = \bigcup_{i=1}^{N} A_i \subseteq \text{co}(A) \),
\[
\int_{B_{N+1} \Delta A_{N+1}} d(x, \partial A_{N+1}) dx = \int_{B_{N+1}^{c} \Delta A_{N+1}^{c}} d(x, \partial A_{N+1}) dx
\]
\[
\geq \int_{(B_{N+1} \cap \text{co}(A)) \Delta A_{N+1}^{c}} d(x, \partial A_{N+1}) dx.
\]
So we have
\[
\mathfrak{F}(B, A; \lambda) \geq \mathfrak{F}(B', A; \lambda) \quad \forall B \in \mathbb{P}_b(N+1)
\]
and the inequality is strict whenever \( \left| \bigcup_{i=1}^{N} B_i \setminus \text{co}(A) \right| > 0 \).

Let \( \{B^{(k)}\} \subseteq \mathbb{P}_b(N+1) \) be a minimizing sequence, which can be supposed so that \( \text{co}(B^{(k)}) \subseteq \text{co}(A) \) and \( \mathfrak{F}(B^{(k)}, A; \lambda) \leq \mathfrak{F}(T, A; \lambda) \), \( T := (\emptyset, \ldots, \emptyset, \mathbb{R}^n) \) being the trivial partition, so that
\[
\text{Per}(B^{(k)}) \leq \lambda \int A_j \left( d(x, \partial A_j) + d(x, \partial A_{N+1}) \right) dx \quad \forall k \geq 1.
\]
By Theorem 3.10 there exists \( A(\lambda) \in \mathbb{P}_b(N+1) \) such that (passing to a not relabelled subsequence) \( B^{(k)} \to A(\lambda) \) in \( L^1(\mathbb{R}^n) \) as \( k \to +\infty \). Then the \( L^1(\mathbb{R}^n) \)-lower semicontinuity of \( \mathfrak{F}(\cdot, A; \lambda) \) implies that \( A(\lambda) \) is a solution to (4.2).

Now let \( A(\lambda) \) be a minimizer of \( \mathfrak{F}(\cdot, A; \lambda) \). If \( \left| \bigcup_{j=1}^{N} A_j(\lambda) \setminus \text{co}(A) \right| > 0 \) then, as shown above, \( \mathfrak{F}(A(\lambda), A; \lambda) > \mathfrak{F}(A(\lambda)', A; \lambda) \), where \( A(\lambda)' \) is defined as in (4.3), which contradicts the minimality of \( A(\lambda) \).

**Remark 4.3.** Let \( C \subseteq \mathbb{R}^n \) be a compact convex set. Suppose that \( G \in \mathbb{P}_b(N+1) \) satisfies \( \bigcup_{j=1}^{N} G_j \subseteq C \); from Theorem 4.2 it follows that every minimizer \( A(\lambda) \in \mathbb{P}_b(N+1) \) of \( \mathfrak{F}(\cdot, G; \lambda) \) satisfies \( \text{co}(A(\lambda)) \subseteq C \). This gives an a priori bound for minimizers of \( \mathfrak{F}(\cdot, G; \lambda) \) just from a bound for the initial partition and will be used in the proofs of Theorems 4.9 and 5.1.

**Remark 4.4.** Suppose that \( G \in \mathbb{P}_b(N+1) \) and \( G_i = \emptyset \) for some \( i \in \{1, \ldots, N\} \). Then by definition of \( \bar{\sigma} \) every minimizer \( A(\lambda) \in \mathbb{P}_b(N+1) \) of \( \mathfrak{F}(\cdot, G; \lambda) \) satisfies \( A_i(\lambda) = \emptyset \). In particular, for \( G = (G, \emptyset, \ldots, \emptyset, \mathbb{R}^n \setminus G) \), the GMM problem for \( \mathfrak{F}(\cdot, G; \lambda) \) agrees with the
\textit{GMM} problem for the Almgren-Taylor-Wang functional

\begin{equation}
E \in BV(\mathbb{R}^n) \mapsto \mathfrak{A}(E, G; \lambda) := P(E) + \lambda \int_{E \Delta G} d(x, \partial G) dx.
\end{equation}

**Proposition 4.5 (Behaviour of $\mathcal{A}(\lambda)$ as time goes to 0).** Let $\mathcal{A} \in \mathbb{P}(N + 1)$ be such that
\[
\sum_{j=1}^{N+1} |A_j \setminus A_j| = 0, \quad \text{and } \mathcal{A}(\lambda) \text{ be a minimizer of } \mathfrak{F}(\cdot, \mathcal{A}; \lambda). \text{ Then:}
\]

\begin{enumerate}[a)]
\item \(\lim_{\lambda \to +\infty} |\mathcal{A}(\lambda) \Delta \mathcal{A}| = 0,\)
\item \(\lim_{\lambda \to +\infty} \text{Per}(\mathcal{A}(\lambda)) = \text{Per}(\mathcal{A}),\)
\item \(\lim_{\lambda \to +\infty} \lambda \sigma(\mathcal{A}(\lambda), \mathcal{A}) = 0.\)
\end{enumerate}

**Proof.** a) Choose any sequence \(\lambda_k \to +\infty.\) Since \(\mathfrak{F}(\mathcal{A}(\lambda_k), \mathcal{A}; \lambda_k) \leq \mathfrak{F}(\mathcal{A}, \mathcal{A}; \lambda_k) = \text{Per}(\mathcal{A}),\) we have \(\text{Per}(\mathcal{A}(\lambda)) \leq \text{Per}(\mathcal{A})\) and
\[
\lim_{k \to +\infty} \sigma(\mathcal{A}(\lambda_k), \mathcal{A}) = 0.
\]

Moreover, by Theorem 4.2 \(\text{co}(\mathcal{A}(\lambda)) \subseteq \text{co}(\mathcal{A}),\) therefore Proposition 3.10 yields the existence of a subsequence \(\{\lambda_k\}_l\) and of \(B \in \mathbb{P}(N + 1)\) such that \(\mathcal{A}(\lambda_k) \to B\) in \(L^1(\mathbb{R}^n)\) as \(l \to +\infty.\) Now the lower semicontinuity of \(\sigma(\cdot, \mathcal{A})\) and (4.6) imply \(\sigma(B, \mathcal{A}) = 0.\) Then from the assumption on \(\mathcal{A}\) we get \(\mathcal{A} = B.\) Since \(\lambda_k\) is arbitrary, a) follows.

b) Since \(\text{Per}(\mathcal{A}(\lambda)) \leq \text{Per}(\mathcal{A}),\) from a) we obtain
\[
\text{Per}(\mathcal{A}) \leq \liminf_{\lambda \to +\infty} \text{Per}(\mathcal{A}(\lambda)) \leq \limsup_{\lambda \to +\infty} \text{Per}(\mathcal{A}(\lambda)) \leq \text{Per}(\mathcal{A}).
\]

c) From b) we have
\[
\limsup_{\lambda \to +\infty} \lambda \sigma(\mathcal{A}(\lambda), \mathcal{A}) \leq 2 \limsup_{\lambda \to +\infty} (\text{Per}(\mathcal{A}) - \text{Per}(\mathcal{A}(\lambda))) = 0.
\]

**Theorem 4.6 (Density estimates).** Suppose that \(\mathcal{A} \in \mathbb{P}(N + 1)\) and let \(\mathcal{A}(\lambda) \in \mathbb{P}(N + 1)\) be a minimizer of \(\mathfrak{F}(\cdot, \mathcal{A}; \lambda).\) Then for every \(i \in \{1, \ldots, N + 1\}\)
\[
\left(\frac{1}{2(N+1)}\right)^n \leq \frac{|A_i(\lambda) \cap B_r(x)|}{|B_r(x)|} \leq 1 - \frac{1}{2^n} \left(1 - \frac{1}{2N}\right)^n, \quad \text{(4.7)}
\]
\[
c_{n,N+1} \leq \frac{P(A_i(\lambda), B_r(x))}{r^{n-1}} \leq \frac{2N + 1}{2N} n \omega_n \quad \text{(4.8)}
\]

for any \(x \in \partial A_i(\lambda)\) and \(r \in (0, \min\{1, \frac{n}{2(N + 1)}\})\), where \(c_{n,N+1}\) is defined in (3.7) (with \(N + 1\) in place of \(N\)). Moreover
\[
\sum_{j=1}^{N+1} \mathcal{H}^{n-1}(\partial A_j(\lambda) \setminus \partial^* A_j(\lambda)) = 0.
\]

**Proof.** Without loss of generality, we may suppose \(\partial A_i \neq \emptyset\) for every \(i = 1, \ldots, N + 1.\) Fix \(r_0 > 0.\) Then for every \(x \in \mathbb{R}^n\) and \(\mathcal{C} \in \mathbb{P}(N + 1)\) such that \(\mathcal{C} \Delta \mathcal{A}(\lambda) \subset \subset B_{r_0}(x)\) with \(\rho \in (0, r_0),\) by Theorem 4.2 one has
\[
d(z, \partial A_i) \leq \text{diam co}(\mathcal{A}) + 2\rho \quad \forall i = 1, \ldots, N + 1, \quad z \in \mathcal{C} \Delta \mathcal{A}(\lambda).
\]

Therefore the minimality of \(\mathcal{A}(\lambda)\) implies
\[
\text{Per}(\mathcal{A}(\lambda), B_\rho(x)) \leq \text{Per}(\mathcal{C}, B_\rho(x)) + \frac{\lambda}{2} \left(\text{diam co}(\mathcal{A}) + 2r_0\right)|\mathcal{C} \Delta \mathcal{A}(\lambda)|,
\]
\( \mathcal{A}(\lambda) \) is a \((\Lambda, r_0)\)-minimizer with \( \Lambda = \frac{\lambda}{2} \left( \text{diam}\ co(\mathcal{A}) + 2r_0 \right) \).

Now application of Theorem 3.6 to \( \mathcal{A}(\lambda) \) with \( r_0 = 1 \) finishes the proof. \( \square \)

**Remark 4.7.** The density estimates show that the components of \( \mathcal{A}(\lambda) \) are Lebesgue-equivalent to open sets. Indeed, since using \( E \setminus E \subset \partial E \), and \( E \setminus E \subset \partial E \) (\( G \) being the interior of \( G \)), we have
\[
\sum_{j=1}^{N+1} |A_j(\lambda) \Delta \mathcal{A}_j(\lambda)| \leq \sum_{j=1}^{N+1} |A_j(\lambda) \setminus A_j| + \sum_{j=1}^{N+1} |A_j(\lambda) \setminus \mathcal{A}_j(\lambda)| \leq 2 \sum_{j=1}^{N+1} |\partial A_j(\lambda)|.
\]

Now by the density estimates \( \sum_{j=1}^{N+1} |\partial A_j(\lambda)| = 0 \), and therefore \( \sum_{j=1}^{N+1} |A_j(\lambda) \Delta \mathcal{A}_j(\lambda)| = 0 \).

To prove the existence of \( \text{GMM} \), we need the following corollary of Theorem 4.6.

**Corollary 4.8.** Let \( \varepsilon > 0 \) and suppose that the components of \( \mathcal{A} \in \mathbb{P}_b(N + 1) \) satisfy the density estimates (4.7)-(4.8) for all \( r \in (0, \varepsilon) \). Then for every minimizer \( \mathcal{A}(\lambda) \) of \( \mathcal{F}(\cdot, \mathcal{A}; \lambda) \) in \( \mathbb{P}_b(N + 1) \) one has
\[
|\mathcal{A}(\lambda) \Delta \mathcal{A}| \leq \frac{5^n \omega_n}{c_{n,N+1}} \left( \frac{\ell}{\varepsilon} \right)^{n-1} \text{Per}(\mathcal{A}) \ell + \frac{1}{\ell} \sigma(\mathcal{A}(\lambda), \mathcal{A}), \quad \ell \geq \varepsilon. \tag{4.9}
\]

**Proof.** Fix \( \ell \geq \varepsilon \) and \( i \in \{1, \ldots, N + 1\} \) and set
\[
E := \{ x \in A_i(\lambda) \Delta A_i : d(x, \partial A_i) \geq \ell \}, \quad F := \{ x \in A_i(\lambda) \Delta A_i : d(x, \partial A_i) < \ell \}.
\]

By the Chebyshev inequality,
\[
|E| \leq \frac{1}{\ell} \int_E d(x, \partial A_i) dx \leq \frac{1}{\ell} \int_{A_i(\lambda) \Delta A_i} d(x, \partial A_i) dx.
\]

We cover the set \( F \) with a family \( \{ \overline{B_\ell(x)} : x \in \partial A_i \} \) of closed balls. By the Vitali lemma, there exists a finite subset \( \{ \overline{B_\ell(x_j)} \}_{j=1}^j \) of the covering, consisting of disjoint balls, such that \( F \subset \bigcup_{j=1}^j \overline{B_\ell(x_j)} \). Since by assumption \( A_j \) satisfies the lower perimeter density estimate in (4.8) with \( r = \varepsilon \),
\[
|F| \leq \sum_j 5^n \omega_n \ell^n \leq \frac{5^n \omega_n}{c_{n,N+1}} \left( \frac{\ell}{\varepsilon} \right)^n \varepsilon \sum_j P(A_i, B_\ell(x_j)) \leq \frac{5^n \omega_n}{c_{n,N+1}} \left( \frac{\ell}{\varepsilon} \right)^{n-1} P(A_i) \ell.
\]

Thus,
\[
|A_i(\lambda) \Delta A_i| \leq |E| + |F| \leq \frac{1}{\ell} \int_{A_i(\lambda) \Delta A_i} d(x, \partial A_i) dx + \frac{5^n \omega_n}{c_{n,N+1}} \left( \frac{\ell}{\varepsilon} \right)^{n-1} P(A_i) \ell. \tag{4.10}
\]

Now (4.9) follows summing (4.10) with respect to \( i \). \( \square \)

One of the main results of the present paper reads as follows.

**Theorem 4.9 (Existence of \( \text{GMM} \)).** Let \( \mathcal{G} \in \mathbb{P}_b(N + 1) \). Then \( \text{GMM}(\mathcal{F}, \mathcal{G}) \) is non empty. Moreover, there exists a constant \( \hat{c} = \hat{c}(N, n, \mathcal{G}) > 0 \) such that for any \( \mathcal{M} \in \text{GMM}(\mathcal{F}, \mathcal{G}) \),
\[
|\mathcal{M}(t) \Delta \mathcal{M}(t')| \leq \hat{c} |t - t'|^{\frac{1}{n-1}} \quad \forall t, t' > 0, \ |t - t'| < 1 \tag{4.11}
\]

and
\[
\bigcup_{j=1}^N M_j(t) \subseteq co(\mathcal{G}) \quad \forall t \geq 0. \tag{4.12}
\]
In addition, if \( \sum_{j=1}^{N+1} |G_j \setminus G_j| = 0 \), then (4.11) holds for any \( t, t' \geq 0 \) and \( |t - t'| < 1 \).

**Proof.** Set \( 2R := \text{diam} \, \text{co}(\mathcal{G}) \). Let \( \mathcal{L}(\lambda, k) = (L_1(\lambda, k), \ldots, L_{N+1}(\lambda, k)) \), \( \lambda \geq 1, \ k \in \mathbb{N}_0 \) be defined as follows: \( \mathcal{L}(\lambda, 0) := \mathcal{G} \), and for \( k \geq 1 \)

\[
\mathfrak{F}(\lambda, k, \mathcal{L}(\lambda, k-1); \lambda) = \min_{A \in \mathfrak{P}_k(N+1)} \mathfrak{F}(A, \mathcal{L}(\lambda, k-1); \lambda);
\]

recall that the existence of minimizers follows from Theorem 4.2 and also

\[
\bigcup_{j=1}^{N} L_j(\lambda, k) \subseteq \text{co}(\mathcal{G}) \quad \forall \lambda \geq 1, \ k \in \mathbb{N}_0.
\]

Clearly, \( \mathfrak{F}(\lambda, k, \mathcal{L}(\lambda, k-1); \lambda) \leq \mathfrak{F}(\lambda, k-1, \mathcal{L}(\lambda, k-1); \lambda) \), hence

\[
\lambda \sigma(\mathcal{L}(\lambda, k), \mathcal{L}(\lambda, k-1)) \leq 2 \left( \text{Per}(\mathcal{L}(\lambda, k-1)) - \text{Per}(\mathcal{L}(\lambda, k)) \right) \quad \forall k \geq 1.
\]

Therefore, the sequence \( k \in \mathbb{N}_0 \mapsto \text{Per}(\mathcal{L}(\lambda, k)) \) is nonincreasing, and \( \text{Per}(\mathcal{L}(\lambda, k)) \leq \text{Per}(\mathcal{G}) \) for all \( k \in \mathbb{N}_0 \) and \( \lambda \geq 1 \), since \( \mathcal{L}(\lambda, 0) = \mathcal{G} \).

For every \( t, t' > 0 \), \( 0 < t - t' < 1 \) let us prove

\[
|\mathcal{L}(\lambda, [\lambda t]) \mathcal{L}(\lambda, [\lambda t'])| \leq \hat{c} |t - t'|^{\frac{1}{n+1}}
\]

provided that \( \lambda \) is sufficiently large depending on \( |t - t'| \), \( n \), \( N \) and \( R \), where

\[
\hat{c} := \hat{c}(N, n, \mathcal{G}) = \left( \frac{5^n \omega_n}{c_{n,N+1}} + \frac{8N(R+1)}{n} \right) \text{Per}(\mathcal{G}).
\]

Set \( k_0 := [\lambda t'] \), \( m_0 := [\lambda t] \). Let \( \lambda \geq \max\{ \frac{n}{4(N+1)N}, \frac{1}{|t-t'|} \} \) be so large that \( m_0 \geq k_0 + 3 \geq 4 \) and \( \frac{4N(R+1)n}{4N(N+1)|t-t'|^\alpha} < 1 \), \( \alpha := \frac{1}{n+1} \). Since each \( \mathcal{L}(\lambda, k) \), \( k \geq 1 \), satisfies the density estimates (4.7)-(4.8) (Theorem 4.6) for \( r \in (0, \frac{n}{4N(N+1)}) \), we may apply Corollary 4.8 with \( \ell = \frac{n}{4N(N+1)|t-t'|^\alpha} \) and \( \varepsilon = \frac{n}{4N(N+1)} \), the inequality \( \text{Per}(\mathcal{L}(\lambda, k)) \leq \text{Per}(\mathcal{G}) \) and (4.14) to get

\[
|\mathcal{L}(\lambda, m_0) \mathcal{L}(\lambda, k_0)| \leq \sum_{k=k_0+1}^{m_0} |\mathcal{L}(\lambda, k) \mathcal{L}(\lambda, k-1)|
\]

\[
\leq \sum_{k=k_0+1}^{m_0} \frac{5^n \omega_n}{c_{n,N+1}} \frac{n}{4N(N+1)} |t - t'|^{-\alpha} \text{Per}(\mathcal{L}(\lambda, k-1))
\]

\[
+ \frac{4N(R+1)}{n} \lambda |t - t'|^{\alpha} \sigma(\mathcal{L}(\lambda, k), \mathcal{L}(\lambda, k-1))
\]

\[
\leq \sum_{k=k_0+1}^{m_0} \frac{5^n \omega_n}{c_{n,N+1}} \frac{n}{4N(N+1)} \text{Per}(\mathcal{G}) |t - t'|^{-\alpha} \frac{m_0 - k_0}{\lambda}
\]

\[
+ \frac{8N(R+1)}{n} |t - t'|^{\alpha} \sum_{k=k_0+1}^{m_0} (\text{Per}(\mathcal{L}(\lambda, k-1)) - \text{Per}(\mathcal{L}(\lambda, k))).
\]

Since

\[
m_0 - k_0 \leq \lambda |t - t'| + 1 \leq 2\lambda |t - t'|,
\]

from the choice of \( \alpha \) and the bound \( \text{Per}(\mathcal{L}(\lambda, k)) \leq \text{Per}(\mathcal{G}) \), we establish

\[
|\mathcal{L}(\lambda, m_0) \mathcal{L}(\lambda, k_0)| \leq \left( \frac{5^n \omega_n}{c_{n,N+1}} \frac{n}{2N(N+1)} + \frac{8N(R+1)}{n} \right) \text{Per}(\mathcal{G}) |t - t'|^{\frac{1}{n+1}},
\]

which is (4.15).
Now we prove the assertions of the theorem. Using the inclusion (4.13), the inequality
\[ \text{Per}(\mathcal{L}(\lambda, k)) \leq \text{Per}(\mathcal{G}) \], Proposition 3.10 and a diagonal argument we obtain the existence
of a diverging sequence \( \{\lambda_h\} \) and an estimate of the form
\[ \lim_{h \to +\infty} |\mathcal{L}(\lambda_h, [\lambda_h t])\Delta \mathcal{M}(t)| = 0 \] (4.17)
for every rational \( t > 0 \) and also (4.12) holds. By (4.15) \( \mathcal{M}(t) \) satisfies
\[ |\mathcal{M}(t)\Delta \mathcal{M}(t')| \leq c |t - t'|^{\frac{1}{\pi + 1}} \quad \forall t', t \in \mathbb{Q} \cap (0, +\infty), \ |t - t'| < 1. \]
Hence this map extends uniquely to a map \( \{\mathcal{M}(t) : t > 0\} \subseteq \mathbb{P}_b(N + 1) \) satisfying (4.11) and
(4.12).

To show that \( \mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G}) \) it remains only to prove (4.17) for any \( t \geq 0 \). Case \( t = 0 \)
is trivial: \( \mathcal{M}(0) = \mathcal{G} \). Fix \( t > 0 \). For every \( \varepsilon \in (0, 1) \) take \( t_\varepsilon \in \mathbb{Q} \cap (0, +\infty) \) such that
\[ |t - t_\varepsilon| < \varepsilon^{n+1} \] (recall that (4.17) holds with \( t_\varepsilon \)). Since \( \mathcal{M} \) satisfies (4.11), from (4.15) and
(4.17) (applied with \( t_\varepsilon \)) we deduce
\[ \limsup_{h \to +\infty} |\mathcal{L}(\lambda_h, [\lambda_h t])\Delta \mathcal{M}(t)| \leq \limsup_{h \to +\infty} |\mathcal{L}(\lambda_h, [\lambda_h t_{\varepsilon}])\Delta \mathcal{L}(\lambda_h, [\lambda_h t_{\varepsilon}])| \]
\[ + \limsup_{h \to +\infty} |\mathcal{L}(\lambda_h, [\lambda_h t_{\varepsilon}])\Delta \mathcal{M}(t_{\varepsilon})| + |\mathcal{M}(t_{\varepsilon})\Delta \mathcal{M}(t)| \]
\[ \leq 2\widehat{c} |t - t_{\varepsilon}|^{\frac{1}{\pi + 1}} \leq 2\widehat{c} \varepsilon \]
and the assertion is obtained letting \( \varepsilon \to 0^+ \).

Finally, let \( \sum_{j=1}^{N+1} |G_j \setminus G_j| = 0 \). Given \( t \in (0, 1) \), choosing \( \lambda \) sufficiently large, from (4.15)
we get
\[ |\mathcal{L}(\lambda, [\lambda t])\Delta \mathcal{L}(\lambda, 0)| \leq |\mathcal{L}(\lambda, [\lambda t])\Delta \mathcal{L}(\lambda, 1)| + |\mathcal{L}(\lambda, 1)\Delta \mathcal{G}| \]
\[ \leq \widehat{c} \left| t - \frac{1}{\lambda} \right|^{\frac{1}{\pi + 1}} + |\mathcal{L}(\lambda, 1)\Delta \mathcal{G}|. \]
Now letting \( \lambda \to +\infty \) and using Proposition 4.5 a) we establish
\[ |\mathcal{M}(t)\Delta \mathcal{M}(0)| \leq \widehat{c} t^{\frac{1}{\pi + 1}}. \]
\[ \square \]

In order to improve the Hölder exponent \( \frac{1}{\pi + 1} \) to the value \( \frac{1}{2} \) in (4.11) we expect to be useful,
for minimizers \( \mathcal{A}(\lambda) \) of \( \mathfrak{F}(\cdot, \mathcal{A}; \lambda) \), an estimate of the form
\[ \sum_{i=1}^{N+1} \sup_{x \in A_i(\lambda) \Delta A_i} d(x, \partial A_i) \leq O(\lambda^{-1/2}). \]
We miss the proof of such an estimate; however, a partial result in this direction is given in
Lemma 6.9.

5. EXISTENCE OF GMM IN THE PRESENCE OF EXTERNAL FORCES

In this section we consider the problem of the mean curvature evolution of bounded partitions
with forcing terms. Given \( \mathcal{A} \in \mathbb{P}_b(N + 1) \) and measurable functions \( H_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \ldots, N + 1 \), consider
the functional
\[ \mathfrak{F}_H(\mathcal{B}, \mathcal{A}; \lambda) = \mathfrak{F}(\mathcal{B}, \mathcal{A}; \lambda) + \sum_{i=1}^{N+1} \int_{B_i} H_i dx, \quad \mathcal{B} \in \mathbb{P}_b(N + 1). \]
When \( N = 1 \) and \( H_2 = 0 \), we get the Almgren-Taylor-Wang functional with an external force \( H_1 \). We suppose:

\[
\begin{align*}
H_i &\in L^p_{\text{loc}}(\mathbb{R}^n), \quad i = 1, \ldots, N + 1, \text{ for some } p > n \text{ and } H_{N+1} \in L^1(\mathbb{R}^n); \\
\text{there exists } R > 0 \text{ such that } H_i \geq H_{N+1} \text{ a.e. in } \mathbb{R}^n \setminus B_R(0) \text{ for any } i = 1, \ldots, N; \\
\end{align*}
\]  

(5.1)

in particular \( \mathcal{F}_H(\cdot, \mathcal{A}; \lambda) \) is well-defined and \( L^1(\mathbb{R}^n) \)-lower semicontinuous.

In the two-phase case (\( N = 1 \)), evolutions with a forcing term \( H \) depending on both position and time have been studied for example in [35, 38] (with \( H \in C^\infty(\Omega \times [0, T]) \) and \( \Omega \subset \mathbb{R}^n \) bounded), in [13] (with discontinuous \( H \) and \( \int_0^t H(x, s)ds \) locally Lipschitz in \( x \) and continuous in \( t \)); see also references therein.

The aim of this section is to prove the following result, generalizing Theorem 4.9.

**Theorem 5.1.** Suppose that \( H_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \ldots, N + 1, \) satisfy (5.1) and let \( G \in \mathcal{P}_b(N + 1) \). Then \( GMM(\mathcal{F}_H, G) \) is non empty. Moreover, there exists a constant \( C = C(N, n, G, p, H_1, \ldots, H_{N+1}) > 0 \) such that for any \( \mathcal{M} \in GMM(\mathcal{F}_H, G) \)

\[
|M(t) - M(t')| \leq C|t - t'|^{\frac{1}{n+1}}, \quad \forall t, t' > 0, \quad |t - t'| < 1
\]

and

\[
\bigcup_{j=1}^N M_j(t) \subseteq \text{closed convex hull of } \text{co}(G) \cup B_R(0) \quad \forall t \geq 0.
\]

(5.3)

In addition, if \( \sum_{j=1}^{N+1} |G_j \setminus G_j| = 0 \), then (5.2) holds for any \( t, t' \geq 0 \) and \( |t - t'| < 1 \).

**Proof.** Step 1. Given \( \mathcal{A} \in \mathcal{P}_b(N + 1) \), the problem

\[
\inf_{B \in \mathcal{P}_b(N + 1)} \mathcal{F}_H(\cdot, \mathcal{A}; \lambda)
\]

has a solution. Let \( D \) stand for the closed convex hull of \( \text{co}(\mathcal{A}) \cup B_R(0) \) and for every \( B \in \mathcal{P}_b(N + 1) \) define the competitor \( B' \in \mathcal{P}_b(N + 1) \) as

\[
B' := \left( B_1 \cap D, \ldots, B_N \cap D, \mathbb{R}^n \setminus \bigcup_{i=1}^N (B_i \cap D) \right).
\]

Observe that

\[
\mathcal{F}_H(\mathcal{B}, \mathcal{A}; \lambda) = \mathcal{F}(\mathcal{B}, \mathcal{A}; \lambda) + \sum_{j=1}^N \int_{B_j} (H_j - H_{N+1})dx + \int_{\mathbb{R}^n} H_{N+1}dx.
\]

(5.4)

By Remark 4.3 we have \( \mathcal{F}(\mathcal{B}, \mathcal{A}; \lambda) \geq \mathcal{F}(\mathcal{B}', \mathcal{A}; \lambda) \), with the equality if and only if \( \bigcup_{j=1}^N B_j \setminus D \) = 0. Since \( H_j \geq H_{N+1} \) a.e. in \( \mathbb{R}^n \setminus D \), one has also

\[
\sum_{j=1}^N \int_{B_j} (H_j - H_{N+1})dx \geq \sum_{j=1}^N \int_{B_j \cap D} (H_j - H_{N+1})dx.
\]

Therefore, (5.4) implies \( \mathcal{F}_H(\mathcal{B}, \mathcal{A}; \lambda) \geq \mathcal{F}_H(\mathcal{B}', \mathcal{A}; \lambda) \) with the strict inequality when \( \bigcup_{j=1}^N B_j \setminus D \) > 0. Now proceeding as in the proof of Theorem 4.2 we can show that there exists a minimizer \( \mathcal{A}(\lambda) \) of \( \mathcal{F}_H(\cdot, \mathcal{A}; \lambda) \). Moreover, every minimizer \( \mathcal{A}(\lambda) \) satisfies

\[
\text{co}(\mathcal{A}(\lambda)) \subseteq D.
\]

(5.5)

Now we prove the density estimates for \( \mathcal{A}(\lambda) \).
Step 2. Let us fix \( r_0 \in (0, R) \) and take any \( B \in \mathbb{P}_b(N + 1) \) with \( \mathcal{A}(\lambda) \Delta B \subset B_r \), \( r \in (0, r_0) \). We show

\[
\text{Per}(\mathcal{A}(\lambda), B_r) \leq \text{Per}(B, B_r) + \Lambda_1|\mathcal{A}(\lambda) \Delta B|^{1-1/p} + \Lambda_2|\mathcal{A}(\lambda) \Delta B|,
\]

(5.6)

where \( p \) is given in (5.1) and

\[
\Lambda_1 := N^{1/p} \max_{i \leq N} \|H_i - H_{N+1}\|_{L^p(D)}, \quad \Lambda_2 := \frac{\lambda}{2} (\text{diam } D + 2r_0).
\]

Indeed, from (5.5) one has

\[
d(z, \partial A_j) \leq \text{diam } D + 2r, \quad j = 1, \ldots, N + 1, \quad z \in \mathcal{A}(\lambda) \Delta B,
\]

hence using (4.1)

\[
\left| \sigma(B, A) - \sigma(\mathcal{A}(\lambda), A) \right| \leq \sum_{j=1}^{N+1} \int_{B_j \Delta A_j(\lambda)} d(z, \partial A_j) dz \leq (\text{diam } D + 2r_0)|B \Delta \mathcal{A}(\lambda)|,
\]

since \( B \Delta \mathcal{A}(\lambda) \subset B_{r_0} \). Moreover, from the Hölder inequality

\[
\left| \int_{A_i(\lambda)} (H_i - H_{N+1}) dx - \int_{B_i} (H_i - H_{N+1}) dx \right| \leq \int_{A_i(\lambda) \Delta B_i} |H_i - H_{N+1}| dx
\]

\[
\leq |A_i(\lambda) \Delta B_i|^{1-1/p} \left( \int_{A_i(\lambda) \Delta B_i} |H_i - H_{N+1}|^p dx \right)^{1/p}
\]

\[
\leq \|H_i - H_{N+1}\|_{L^p(D)} |A_i(\lambda) \Delta B_i|^{1-1/p}.
\]

Then the concavity of the function \( t \in (0, +\infty) \mapsto t^{1-1/p} \) implies that

\[
\left| \sum_{i=1}^N \int_{A_i(\lambda)} (H_i - H_{N+1}) dx - \int_{B_i} (H_i - H_{N+1}) dx \right|
\]

\[
\leq N^{1/p} \max_{i \leq N} \|H_i - H_{N+1}\|_{L^p(D)} |\mathcal{A}(\lambda) \Delta B|^{1-1/p}.
\]

Now minimality of \( \mathcal{A}(\lambda) \) (Step 1) yields (5.6).

Thus we can apply Remark 3.7 with \( \alpha_1 = 1 - 1/p > 1 - 1/n, \ \alpha_2 = 1, \ r_0 \in (0, R) \) and

\[
\tilde{r}_0 = \begin{cases} r_0, & \text{if } \Lambda_1 = 0, \\ \min \left\{ r_0, \omega_n^{-1/n} \left( \frac{n}{8(1+\Lambda_2 N)} \right)^{1/p}, \frac{n}{8\Lambda_2 N} \right\} & \text{if } \Lambda_1 > 0, \end{cases}
\]

to get that for every \( i \in \{1, \ldots, N + 1\}, \) (4.7)-(4.8) hold for any \( x \in \partial A_i(\lambda) \) and \( r \in (0, \tilde{r}_0) \).

In particular, \( \sum_{j=1}^{N+1} H^{n-1}(\partial A_j(\lambda) \setminus \partial^* A_j(\lambda)) = 0 \).

Step 3. Given \( \mathcal{G} \in \mathbb{P}_b(N + 1) \) let \( K \) denote the closed convex hull of \( \text{co}(\mathcal{G}) \cup B_R(0) \). Let \( \mathcal{L}(\lambda, 0) := \mathcal{G} \) and \( \mathcal{L}(\lambda, k) \) be defined as

\[
\mathcal{F}_H(\mathcal{L}(\lambda, k), \mathcal{L}(\lambda, k - 1); \lambda) = \min_{\mathcal{A} \in \mathbb{P}_b(N + 1)} \mathcal{F}_H(\mathcal{A}, \mathcal{L}(\lambda, k - 1); \lambda), \quad k \geq 1.
\]

Notice that by (5.5), \( \text{co}(\mathcal{L}(\lambda, k)) \subset K \) for any \( \lambda \geq 1 \) and \( k \geq 0 \). Observe that for any \( \lambda \geq 1 \) the map

\[
k \in \mathbb{N}_0 \mapsto \Psi(\lambda, k) := \text{Per}(\mathcal{L}(\lambda, k)) + \sum_{j=1}^N \int_{L_j(\lambda,k)} (H_j - H_{N+1}) dx
\]
is nonincreasing. Indeed, since $\mathcal{F}_H(\mathcal{L}(\lambda, k), \mathcal{L}(\lambda, k - 1); \lambda) \leq \mathcal{F}_H(\mathcal{L}(\lambda, k - 1), \mathcal{L}(\lambda, k - 1); \lambda)$, recalling (5.4) one has

$$\lambda \sigma(\mathcal{L}(\lambda, k), \mathcal{L}(\lambda, k - 1)) \leq 2(\Psi(\lambda, k - 1) - \Psi(\lambda, k)).$$

In particular, from $\Psi(\lambda, k) \leq \Psi(\lambda, 0)$ it follows that

$$\text{Per}(\mathcal{L}(\lambda, k)) \leq \text{Per}(\mathcal{G}) + \sum_{j=1}^{N} \int_{L_{j}(\lambda,k) \Delta G_{j}} |H_{j} - H_{N+1}| \, dx$$

(5.8)

$$\leq \text{Per}(\mathcal{G}) + N \max_{j \leq N} \|H_{j} - H_{N+1}\|_{L^{1}(K)} =: \kappa.$$}

We claim that for every $t, t' > 0$, $0 < t - t' < 1$,

$$|\mathcal{L}(\lambda, [\lambda t]) \Delta \mathcal{L}(\lambda, [\lambda t'])| \leq C |t - t'|^{\frac{1}{4} + \frac{1}{n}}$$

(5.9)

provided that $\lambda \geq \max\{4/t', 4/(t - t')\}$ is sufficiently large so that the density estimates (4.7)-(4.8) hold for $r \in (0, \delta)$, $\delta = \frac{\lambda}{4N(diam K + 2r_{0})}$, where

$$C := C(N, n, \mathcal{G}) = \left(\frac{5^{n} \omega_{n}}{c_{n,N+1}} \frac{n}{2 \lambda N(diam K + 2r_{0})} + \frac{8N(diam K + 2r_{0})}{n} \right) \text{Per}(\mathcal{G}),$$

and $c_{n,N+1}$ is defined in (3.7) (with $N + 1$ in place of $N$).

Set $k_{0} := [\lambda t']$, $m_{0} := [\lambda t]$. By the choice of $\lambda$ we have $m_{0} \geq k_{0} + 3 \geq 4$. Note that

$$\sum_{k=k_{0}+1}^{m_{0}} \left(\Psi(\lambda, k - 1) - \Psi(\lambda, k)\right) \leq \Psi(\lambda, 0) - \Psi(\lambda, m_{0}) \leq \text{Per}(\mathcal{G}) - \text{Per}(\mathcal{L}(\lambda, m_{0}))$$

$$\leq \text{Per}(\mathcal{G}) + N \max_{j \leq N} \|H_{j} - H_{N+1}\|_{L^{1}(K)} = \kappa$$

Since $\mathcal{L}(\lambda, k)$, $k \geq 1$, satisfies the density estimates (4.7)-(4.8) according to Step 2, applying Corollary 4.8 with $\ell = \delta |t - t'|^{-\frac{1}{n+1}}$, we get

$$|\mathcal{L}(\lambda, [\lambda t]) \Delta \mathcal{L}(\lambda, [\lambda t'])| \leq \sum_{k=k_{0}+1}^{m_{0}} |\mathcal{L}(\lambda, k) \Delta \mathcal{L}(\lambda, k - 1)|$$

$$\leq \frac{5^{n} \omega_{n}}{c_{n,N+1}} \frac{n}{4 \lambda N(diam K + 2r_{0})} |t - t'|^{\frac{n}{n+1}} \sum_{k=k_{0}+1}^{m_{0}} \text{Per}(\mathcal{L}(\lambda, k - 1))$$

$$+ \frac{4N(diam K + 2r_{0})}{n} \lambda |t - t'|^{\frac{1}{n+1}} \sum_{k=k_{0}+1}^{m_{0}} \sigma(\mathcal{L}(\lambda, k), \mathcal{L}(\lambda, k - 1))$$

$$\leq \frac{5^{n} \omega_{n}}{c_{n,N+1}} \frac{k_{0} n}{4 \lambda N(diam K + 2r_{0})} |t - t'|^{\frac{n}{n+1}} \sum_{k=k_{0}+1}^{m_{0}} \left(\Psi(\lambda, k - 1) - \Psi(\lambda, k)\right)$$

$$+ \frac{4N(diam K + 2r_{0})}{n} |t - t'|^{\frac{1}{n+1}} \sum_{k=k_{0}+1}^{m_{0}} \left(\Psi(\lambda, k - 1) - \Psi(\lambda, k)\right).$$

Then (5.9) follows using (4.16).

Now the proofs of (5.2) and (5.3) are exactly the same as in the proof of Theorem 4.9.
Step 4. Finally, let us show that if $\sum_{j=1}^{N+1} |G_j \setminus G_j| = 0$, then (5.2) holds for any $t, t' \geq 0$, $|t - t'| < 1$. We need just to show that $|L(\lambda, 1)\Delta G| \to 0$ as $\lambda \to +\infty$, and then we proceed as in the proof of the final assertion of Theorem 4.9.

Using the minimality of $L(\lambda, 1)$ we have $\bar{\mathcal{F}}_H(L(\lambda, 1), G; \lambda) \leq \bar{\mathcal{F}}_H(G, G; \lambda)$, i.e.

$$\frac{\lambda}{2} \sigma(L(\lambda), G) \leq \text{Per}(G) - \text{Per}(L(\lambda, 1)) + N \max_{j \leq N} \|H_j - H_{N+1}\|_{L^1(K)} \leq \kappa. \quad (5.10)$$

Choose an arbitrary diverging sequence $\{\lambda_k\}$. By (5.8) it follows $\text{Per}(L(\lambda_k, 1)) \leq \kappa$ for any $k \geq 1$ and since $\bigcup_{j=1}^{N} L_j(\lambda_k, 1) \subseteq K$, by Theorem 3.10 there exists a (not relabelled) subsequence and $A \in \mathbb{P}_b(N + 1)$ such that $L(\lambda_k, 1) \to A$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$. Then the $L^1(\mathbb{R}^n)$-lower semicontinuity of $\sigma$ and (5.10) yield

$$\sigma(A, G) \leq \liminf_{k \to +\infty} \sigma(L(\lambda_k, 1), G) \leq \liminf_{k \to +\infty} \frac{2\kappa}{\lambda_k} = 0.$$

Hence $\sigma(A, G) = 0$ and by the assumption of $G$ we have $A = G$. Since $\{\lambda_k\}$ is arbitrary, $L(\lambda, 1) \to G$ in $L^1(\mathbb{R}^n)$ as $\lambda \to +\infty$. \hfill $\square$

6. Evolution of Disjoint Partitions

In this section we study the evolution of disjoint partitions and the compatibility results of GMM starting from the disjoint initial partition with other notions of solution.

6.1. Some comparison results for the 2-phase case ($N = 1$). Let us start with recalling some comparison arguments for the Almgren-Taylor-Wang functional $\mathcal{A}(\cdot, \cdot; \lambda)$ in (4.5) from [8, Section 6] and [12, Section 6].

Define

$$\mathcal{M}_b := \{ E \in BV(\mathbb{R}^n, \{0, 1\}) : E \text{ is bounded} \},$$

$$\mathcal{M}_u := \{ E \in BV(\mathbb{R}^n, \{0, 1\}) : E^c \text{ is bounded} \}.$$

Notice that $\mathcal{A}(\cdot, \cdot; \lambda)$ is well-defined for both $\mathcal{M}_b$ and $\mathcal{M}_u$. The following result is well-known, and is a particular case of Theorem 4.2 (applied with $N = 1$).

**Proposition 6.1.** Given $G \in \mathcal{M}_b$ (resp. $G \in \mathcal{M}_u$) and $\lambda \geq 1$ the problem

$$\inf_{E \in \mathcal{M}_b} \mathcal{A}(E, G; \lambda) \quad \text{(resp.} \inf_{E \in \mathcal{M}_u} \mathcal{A}(E, G; \lambda)\text{)}$$

has a solution. Moreover, any minimizer $G(\lambda)$ satisfies the inclusion

$$G(\lambda) \subseteq \text{co}(G) \quad \text{(resp.} \mathbb{R}^n \setminus G(\lambda) \subseteq \text{co}(\mathbb{R}^n \setminus G)\text{)}.$$

**Proposition 6.2 (Maximal and minimal minimizers [8, 12]).** Given $E \in \mathcal{M}_b$ (resp. $E \in \mathcal{M}_u$) and $\lambda \geq 1$ there exist the maximal and the minimal minimizer $E(\lambda)^* \text{ and } E(\lambda)^*$ of $\mathcal{A}(\cdot, E; \lambda)$, in the sense that any other minimizer $E(\lambda)$ satisfies

$$E(\lambda)^* \subseteq E(\lambda) \subseteq E(\lambda)^*.$$

Given a set $E \subset \mathbb{R}^n$ and $\varepsilon > 0$ we write

$$E^+_\varepsilon = \{ x \in \mathbb{R}^n : d(x, E) < \varepsilon \}. \quad (6.1)$$

We recall the following comparison principles for the minimizers of $\mathcal{A}$ from [12, section 6], see also [8, Section 6].
Theorem 6.3 (Comparison principles). Let $\varepsilon > 0$, $E, F \in \mathcal{M}_b$ (or $E, F \in \mathcal{M}_u$ or $E \in \mathcal{M}_b$ and $F \in \mathcal{M}_u$) be such that
\[
E_\varepsilon^+ \subseteq F.
\] (6.2)
Then
\[(E(\lambda))_\varepsilon^+ \subseteq F(\lambda), \quad \delta < \varepsilon,
\] (6.3)
for every $\lambda \geq 1$ and every minimizer $E(\lambda)$ and $F(\lambda)$ of $\mathfrak{A}(\cdot, E; \lambda)$ and $\mathfrak{A}(\cdot, F; \lambda)$, respectively. Moreover,
\[(E(\lambda)_\varepsilon)^+ \subseteq F(\lambda)_\varepsilon, \quad (E(\lambda)^*)_\varepsilon \subseteq F(\lambda)^*.
\] (6.4)

Corollary 6.4. Suppose that $E, F \in \mathcal{M}_b$ are such that
\[
d(E, F) > 0.
\] Then for any $\lambda \geq 1$, every minimizer $E(\lambda)$ (resp. $F(\lambda)$) of $\mathfrak{A}(\cdot, E; \lambda)$ (resp. $\mathfrak{A}(\cdot, F; \lambda)$) satisfies
\[
d(E(\lambda), F(\lambda)) \geq d(E, F).
\] (6.5)

Definition 6.5 (Minimal and maximal GMM associated with a sequence). For $E \in \mathcal{M}_b$, \{\(E_i(t)\}\} $\in$ GMM(\(\mathfrak{A}, E\)) (resp. \{\(E_i^*(t)\}\} $\in$ GMM(\(\mathfrak{A}, E\)) \(\mathfrak{A}\)) is called the minimal (resp. the maximal) GMM associated with a sequence \{\(\lambda_k\)\} if
\[
E(\lambda_k, [\lambda_k t])_\ast \rightarrow E_\ast(t) \quad (\text{resp. } E(\lambda_k, [\lambda_k t])^*_\ast \rightarrow E^*_\ast(t)) \quad \text{as } k \rightarrow +\infty \text{ in } L^1(\mathbb{R}^n),
\] where $E(\lambda, 0)_\ast = E(\lambda, 0)^*_\ast = E$, and $E(\lambda, l)_\ast$ (resp. $E(\lambda, l)^*_\ast$), $\lambda \geq 1$ and $l \in \mathbb{N}$, is the minimal (resp. maximal) minimizer of $\mathfrak{A}(\cdot, (E(\lambda, l)_\ast); \lambda)$ (resp. $\mathfrak{A}(\cdot, (E(\lambda, l)_\ast); \lambda)$).

The minimal and maximal GMM satisfy the following comparison theorem [8, Theorem 7.3].

Theorem 6.6 (Comparison for minimal and maximal GMM). Let $E, F \in \mathcal{M}_b$, $E \subseteq F$ and let \{\(E(t)_\ast\)\} (resp. \{\(E(t)^*_\ast\)\}) be the minimal (resp. maximal) GMM associated with a same sequence \{\(\lambda_k\)\}. Then
\[
E(t)_\ast \subseteq F(t)_\ast \quad (\text{resp. } E(t)^*_\ast \subseteq F(t)^*_\ast) \quad \text{for all } t \geq 0. \tag{6.6}
\]

6.2. Evolution of disjoint partitions. Now we study the evolution of disjoint partitions.

Definition 6.7 (Disjoint partitions). A partition $\mathcal{A} \in \mathcal{P}_b(N + 1)$ is called disjoint provided
\[
\min_{1 \leq i < j \leq N} d(A_i, A_j) > 0.
\]

Notice that if $\mathcal{A} \in \mathcal{P}_b(N + 1)$ is disjoint, then $\text{Per}(\mathcal{A}) = \sum_{j=1}^{N} P(A_j)$. Moreover, if $\mathcal{A}$ and $\mathcal{G}$ are disjoint and satisfy
\[
\bigcup_{j=1}^{N} (A_j \Delta G_j) = \left( \bigcup_{j=1}^{N} A_j \right) \Delta \left( \bigcup_{j=1}^{N} G_j \right),
\] (6.7)
then $\sigma(\mathcal{A}, \mathcal{G}) = 2 \sum_{j=1}^{N} \int_{A_j \Delta G_j} d(x, \partial G_j) dx$ and
\[
\mathfrak{F}(\mathcal{A}, \mathcal{G}; \lambda) = \sum_{j=1}^{N} \left( P(A_j) + \lambda \int_{A_j \Delta G_j} d(x, \partial G_j) dx \right) = \sum_{j=1}^{N} \mathfrak{F}(A_j, G_j; \lambda). \tag{6.8}
\]

In the next two lemmas, no disjointness hypothesis is assumed. The proof of the following lemma is an adaptation of the proof of Theorem 3.6.
Lemma 6.8. Given \( \mathcal{G} \in \mathbb{P}_b(N+1) \), let \( \mathcal{G}(\lambda) \in \mathbb{P}_b(N+1) \) be a minimizer of \( \mathcal{F}(\cdot, \mathcal{G}; \lambda) \). Fix \( i \in \{1, \ldots, N+1\} \). If \( x \in G_i(\lambda)^c \cap G_i \) and \( d(x, \partial G_i) \geq \rho > 0 \), then
\[
\frac{1}{2^n} \leq \frac{|B_{\rho}(x) \cap G_i(\lambda)^c|}{|B_{\rho}(x)|}. \tag{6.9}
\]

Proof. Without loss of generality we suppose \( i = 1 \). As usual, write \( B_r := B_r(x) \) and set
\[
I := \{ j \in \{2, \ldots, N+1\} : \mathcal{H}^{n-1}(B_{\rho} \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) > 0 \}.
\]
Clearly, if \( I = \emptyset \), then \( B_{\rho} \subseteq G_1(\lambda)^c \) and (6.9) is satisfied, hence we can suppose \( I \neq \emptyset \). Fix any \( r \in (0, \rho) \) such that
\[
\sum_{j=1}^{N+1} \mathcal{H}^{n-1}(\partial B_r \cap \partial^* G_j(\lambda)) = 0. \tag{6.10}
\]
For each \( j \in I \) define the competitor \( C^{(j)} \in \mathbb{P}_b(N+1) \) as
\[
C^{(j)} := (G_1(\lambda) \cup (G_j(\lambda) \cap B_r), G_2(\lambda), \ldots, G_{j-1}(\lambda), G_j(\lambda) \setminus B_r, G_{j+1}(\lambda), \ldots, G_{N+1}(\lambda)). \tag{6.11}
\]
Fix \( s \in (r, \rho) \). Arguing as in the proofs of (3.27) and (3.13),
\[
P(G_1(\lambda) \cup (G_j(\lambda) \cap B_r), B_s) = P(G_1(\lambda), B_s) + \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) + P(G_j(\lambda), B_r)
\]
\[- 2\mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda))\]
\[
P(G_j(\lambda) \setminus B_r, B_s) = P(G_j(\lambda), B_s \setminus \overline{B_r}) + \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r).
\]
Therefore from (6.10)
\[
\lim_{s \to \rho^-} \left( P(G_1(\lambda) \cup (G_j(\lambda) \cap B_r), B_s)ight.
\]
\[
+ P(G_j(\lambda) \setminus B_r, B_s) - P(G_j(\lambda), B_s) - P(G_j(\lambda), B_s))
\]
\[- 2\mathcal{H}^{n-1}(G_1(\lambda) \cap \partial B_r) - 2\mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)).
\]
Hence the inequality \( \mathcal{F}(\mathcal{G}(\lambda), \mathcal{G}; \lambda) \leq \mathcal{F}(C^{(j)}, \mathcal{G}; \lambda) \) due to the minimality of \( \mathcal{G}(\lambda) \) and (4.1) imply
\[
\mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) - \mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda))
\]
\[\geq \frac{\lambda}{2} \int_{G_j(\lambda) \cap B_r} (\tilde{d}(y, \partial G_j) - \tilde{d}(y, \partial G_1)) dy. \tag{6.12}\]
Since by assumption \( B_{\rho} \subseteq G_1 \) (and hence \( B_{\rho} \cap G_j = \emptyset \)) we have
\[
\tilde{d}(y, \partial G_j) - \tilde{d}(y, \partial G_1) = d(y, \partial G_j) + d(y, \partial G_1) \geq 0 \quad \forall y \in G_j(\lambda) \cap B_r, \tag{6.13}\]
and therefore
\[
\mathcal{H}^{n-1}(B_r \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) \leq \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r). \tag{6.14}\]
Then summation of (6.14) over \( j \in I \) and the use of Remark 3.4 yield
\[
P(G_1(\lambda)^c, B_r) \leq \sum_{j \in I} \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r) \leq \sum_{j=2}^{N+1} \mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_r)
\]
\[= \mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r).
\]
Now adding $\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r)$ to both sides we get
\[
P(G_1(\lambda)^c \cap B_r) \leq 2 \mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r).
\]
From the isoperimetric inequality, for a.e. $r \in (0, \rho)$ we obtain
\[
n \omega_n^{1/n} |G_1(\lambda)^c \cap \partial B_r| \frac{\rho}{n} \leq 2 \mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_r).
\]
(6.15)
Since $x \in G_1(\lambda)^c$, one has $|G_1(\lambda)^c \cap B_r| > 0$ for any $r > 0$, therefore integrating (6.15) in $(0, \rho)$, we get (6.9).

Lemma 6.9. Given $\mathcal{G} \in \mathcal{P}_{b}(N + 1)$ let $\mathcal{G}(\lambda) \in \mathcal{P}_{b}(N + 1)$ be a minimizer of $\mathcal{F}(\cdot, \mathcal{G}; \lambda)$. Then for any $i \in \{1, \ldots, N + 1\}$,
\[
\sup_{x \in G_1(\lambda)^c \cap \partial G_i} d(x, \partial G_i) \leq \frac{\sqrt{2n+2}}{\sqrt{\lambda}}.
\]
Proof. Without loss of generality we suppose $i = 1$. By contradiction, let $x \in G_1(\lambda)^c \cap G_1$ be such that $d(x, \partial G_1) \geq \rho := \frac{\sqrt{2n+2} \pm \varepsilon}{\sqrt{\lambda}}$ for some $\varepsilon > 0$. We may suppose that $x \in \partial G_1(\lambda)$ and $\varepsilon$ are such that
\[
\mathcal{H}^{n-1}(\partial^* G_1(\lambda) \cap \partial B_\rho) = 0,
\]
where $B_\rho := B_\rho(x)$. Then the set
\[
J := \{j \in \{2, \ldots, N + 1\} : |B_{\rho/2} \cap G_j(\lambda)| > 0\}
\]
is nonempty. By assumption on $x$ and $\rho$, $B_{\rho/2}(y) \subset G_1$ for every $y \in B_{\rho/2}$, and hence
\[
d(y, \partial G_j) \geq d(y, \partial G_1) \geq \rho/2 \forall j \in J.
\]
Therefore, for each $j \in J$, defining the competitor as in (6.11) with $r = \rho/2$, from the minimality of $\mathcal{G}(\lambda)$, (4.1) and (6.12) we get
\[
\mathcal{H}^{n-1}(G_j(\lambda) \cap \partial B_{\rho/2}) - \mathcal{H}^{n-1}(B_{\rho/2} \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) \geq \frac{\lambda}{2} \int_{G_j(\lambda) \cap \partial B_{\rho/2}} (\tilde{d}(y, \partial G_j) - \tilde{d}(y, \partial G_1)) dy \geq \frac{\lambda \rho}{2} |G_j(\lambda) \cap B_{\rho/2}|,
\]
since $\tilde{d}(y, \partial G_j) = d(y, \partial G_j)$ and $\tilde{d}(y, \partial G_1) = -d(y, \partial G_1)$ for any $y \in B_{\rho/2}$. Summing these inequalities over $j \in J$ and using \(\bigcup_{j=1}^{N+1} (G_j(\lambda) \cap B_{\rho/2}) = \bigcup_{j \in J} (G_j(\lambda) \cap B_{\rho/2}) = G_1(\lambda)^c \cap B_{\rho/2}\) (up to a negligible set), we get
\[
\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_{\rho/2}) \geq \sum_{j \in J} \mathcal{H}^{n-1}(B_{\rho/2} \cap \partial^* G_1(\lambda) \cap \partial^* G_j(\lambda)) + \frac{\lambda \rho}{2} |G_1(\lambda)^c \cap B_{\rho/2}|.
\]
Now Lemma 6.8 yields
\[
\left(\frac{1}{2}\right)^{n+1} \lambda \rho \omega_n \left(\frac{\rho}{2}\right)^n \leq \mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_{\rho/2}),
\]
and clearly, $\mathcal{H}^{n-1}(G_1(\lambda)^c \cap \partial B_{\rho/2}) \leq n \omega_n \left(\frac{\rho}{2}\right)^{-1}$. Therefore, $\rho = \frac{\sqrt{2n+2} \pm \varepsilon}{\sqrt{\lambda}} \leq \frac{\sqrt{2n+2}}{\sqrt{\lambda}}$, a contradiction, since $\varepsilon > 0$.

The following theorem shows that if the components of the initial partition $\mathcal{G}$ are far from each other, then so are the components of minimizers of $\mathcal{F}(\cdot, \mathcal{G}; \lambda)$, provided $\lambda$ is large enough.
Theorem 6.10 (Minimizers of $\mathfrak{F}$ for a disjoint initial partition). Suppose that $G \in \mathbb{P}_b(N+1)$ is disjoint and set
\[
\min_{1 \leq i < j \leq N} d(G_i, G_j) =: \varepsilon_0 > 0. \tag{6.16}
\]
Then for $\lambda > 2^{n+6} n_0^{-2}$ any minimizer $G(\lambda)$ of $\mathfrak{F}(\cdot, G; \lambda)$ satisfies
\[
G_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}, \quad j = 1, \ldots, N. \tag{6.17}
\]

Proof. We claim that the choice of $\lambda$ implies
\[
G_{N+1}(\lambda)^c \subseteq (G_{N+1})^+_{\varepsilon_0/4}. \tag{6.18}
\]
Indeed, obviously $G_{N+1}(\lambda)^c \cap G_{N+1}^c \subseteq (G_{N+1})^+_{\varepsilon_0/4}$. Now if $x \in G_{N+1}(\lambda)^c \cap G_{N+1}^c$, then $d(x, G_{N+1}) = d(x, \partial G_{N+1})$ and therefore by Lemma 6.9
\[
d(x, G_{N+1}) \leq \sup_{y \in G_{N+1}(\lambda)^c \cap G_{N+1}} d(y, \partial G_{N+1}) \leq \frac{\sqrt{2 n+2} n_0}{\lambda} < \frac{\varepsilon_0}{4}.
\]
Hence $x \in (G_{N+1})^+_{\varepsilon_0/4}$ and (6.18) follows.

We prove (6.17) arguing by contradiction. Suppose for example $j = 1$ and $G_1(\lambda)$ is not contained in $(G_1)^+_{\varepsilon_0/4}$. In view of (6.18) and (6.16)
\[
G_1(\lambda) \subseteq \bigcup_{j=1}^N G_j(\lambda) \subseteq \left( \bigcup_{j=1}^N (G_j)^+_{\varepsilon_0/4} \right) = \bigcup_{j=1}^N (G_j)^+_{\varepsilon_0/4}. \tag{6.19}
\]
Since $G_1(\lambda) \setminus (G_1)^+_{\varepsilon_0/4} \neq \emptyset$, (6.19) implies $G_1(\lambda) \cap (G_j)^+_{\varepsilon_0/4} \neq \emptyset$ for some $j \in \{2, \ldots, N\}$. By virtue of Remark 4.7 the set $G_1(\lambda)$ can be supposed to be open so that there exists a ball $B_r$ of radius $r > 0$ whose closure is contained in $G_1(\lambda) \cap (G_j)^+_{\varepsilon_0/4}$. For shortness, let $j = 2$. Thus setting $B := (G_1(\lambda) \setminus B_r, G_2(\lambda) \cup B_r, G_3(\lambda), \ldots, G_{N+1}(\lambda))$, and using $P(G_1(\lambda)) - P(G_1(\lambda) \setminus B_r) = P(B_r)$, we obtain
\[
2\mathfrak{F}(G(\lambda), G; \lambda) - 2\mathfrak{F}(B, G; \lambda) = P(B_r) + P(G_2(\lambda)) - P(G_2(\lambda) \cup B_r)
\]
\[\quad + \lambda \int_{B_r} (\tilde{d}(x, \partial G_1) - \tilde{d}(x, \partial G_2)) dx.
\]
Now,
\[
P(B_r) + P(G_2(\lambda)) - P(B_r \cup G_2(\lambda)) \geq 0.
\]
In addition, by the definition of $\varepsilon_0$, $d(\cdot, G_1) \geq \frac{3\varepsilon_0}{4}$ in $B_r$ (thus $\tilde{d}(\cdot, \partial G_1) = d(\cdot, \partial G_1)$ in $B_r$); moreover, since $B_r \subseteq (G_2)^+_{\varepsilon_0/4}$, one has
\[
\tilde{d}(x, \partial G_1) - \tilde{d}(x, \partial G_2) \geq \frac{\varepsilon_0}{4} \quad \forall x \in B_r
\]
and therefore
\[
\mathfrak{F}(G(\lambda), G; \lambda) - \mathfrak{F}(B, G; \lambda) \geq \frac{\lambda\varepsilon_0}{8} |B_r| > 0.
\]
This implies that $G(\lambda)$ is not a minimizer of $\mathfrak{F}(\cdot, G; \lambda)$. \hfill \Box

Corollary 6.11. Suppose that $G \in \mathbb{P}_b(N+1)$ is disjoint and let $\varepsilon_0$ be as in (6.16). Then for $\lambda$ sufficiently large (depending only on $\varepsilon_0$ and $n$), $G(\lambda)$ is a minimizer of $\mathfrak{F}(\cdot, G; \lambda)$ if and only if each bounded component $G_j(\lambda)$, $j = 1, \ldots, N$, of $G(\lambda)$ is a minimizer of $\mathfrak{A}(\cdot, G_j; \lambda)$. Moreover, every minimizer $G(\lambda)$ satisfies
\[
\min_{1 \leq i < j \leq N} d(G_i(\lambda), G_j(\lambda)) \geq \varepsilon_0. \tag{6.20}
\]
Proof. By [38, Lemma 2.1] there exists $c(n) > 0$ (depending only on $n$) such that for every $\lambda \geq 1$ and every minimizer $A_j(\lambda)$, $j = 1, \ldots, N$, of $\mathfrak{A}(\cdot, G_j; \lambda)$ one has

$$
\sup_{x \in A_j(\lambda) \Delta G_j} d(x, \partial G_j) \leq \sqrt{\frac{c(n)}{\lambda}}.
$$

Therefore, taking

$$
\lambda > \bar{c}(n) \varepsilon_0^{-2}, \quad \bar{c}(n) := \max\{2^{n+6} n, 16c(n)\}, \quad (6.21)
$$

we deduce $A_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}$, $j = 1, \ldots, N$.

Set $\mathcal{A}(\lambda) = (A_1(\lambda), \ldots, A_N(\lambda), \mathbb{R}^n \setminus \bigcup_{j=1}^N A_j(\lambda))$. Let us show that for $\lambda$ as in (6.21), $\mathcal{A}(\lambda)$ minimizes $\mathfrak{F}(\cdot, G; \lambda)$. Indeed, take any minimizer $G(\lambda)$ of $\mathfrak{F}(\cdot, G; \lambda)$. By Theorem 6.10 we have $G_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}$, therefore both $(\mathcal{A}(\lambda), G)$ and $(G(\lambda), G)$ satisfy (6.7). Hence, (6.8) and the minimality of $A_j(\lambda)$ yield

$$
\mathfrak{F}(G(\lambda), G; \lambda) = \sum_{j=1}^N \left( P(G_j(\lambda)) + \lambda \int_{G_j(\lambda) \Delta G_j} d(x, \partial G_j) dx \right)
$$

$$
\geq \sum_{j=1}^N \left( P(A_j(\lambda)) + \lambda \int_{A_j(\lambda) \Delta G_j} d(x, \partial G_j) dx \right) = \mathfrak{F}(\mathcal{A}(\lambda), G; \lambda).
$$

This implies that $\mathcal{A}(\lambda)$ is also a minimizer $\mathfrak{F}(\cdot, G; \lambda)$.

Conversely, suppose that $\lambda$ satisfies (6.21) and $G(\lambda)$ minimizes $\mathfrak{F}(\cdot, G; \lambda)$ and let $A_j(\lambda)$, $j = 1, \ldots, N$, be a minimizer of $\mathfrak{A}(\cdot, G_j; \lambda)$. By (6.17), $A_j(\lambda) \subseteq (G_j)^+_{\varepsilon_0/4}$, $j = 1, \ldots, N$. Set $\mathcal{A}(\lambda) = (A_1(\lambda), \ldots, A_N(\lambda), \mathbb{R}^n \setminus \bigcup_{j=1}^N A_j(\lambda))$. Then from the minimality of $A_j(\lambda)$ and $G(\lambda)$, as well as (6.8), we deduce

$$
\mathfrak{F}(G(\lambda), G; \lambda) \leq \mathfrak{F}(\mathcal{A}(\lambda), G; \lambda) = \sum_{j=1}^N \left( P(A_j(\lambda)) + \lambda \int_{A_j(\lambda) \Delta G_j} d(x, \partial G_j) dx \right)
$$

$$
\leq \sum_{j=1}^N \left( P(G_j(\lambda)) + \lambda \int_{G_j(\lambda) \Delta G_j} d(x, \partial G_j) dx \right) = \mathfrak{F}(G(\lambda), G; \lambda).
$$

Thus all inequalities are in fact equalities, which is possible if and only if

$$
P(G_j(\lambda)) + \lambda \int_{G_j(\lambda) \Delta G_j} d(x, \partial G_j) = P(A_j(\lambda)) + \lambda \int_{A_j(\lambda) \Delta G_j} d(x, \partial G_j) dx
$$

for any $j = 1, \ldots, N$. Hence, $G_j(\lambda)$ is a minimizer of $\mathfrak{A}(\cdot, G_j; \lambda)$.

Finally, (6.20) directly follows from Corollary 6.4. \hfill \Box

Theorem 6.12 (Evolution of disjoint partitions). Assume that $G \in \mathbb{P}_k(N+1)$ is disjoint, and $\{M\} = \{(M_1, \ldots, M_{N+1}) \in GMM(\mathfrak{F}, G)$. Then $M_i \in GMM(\mathfrak{F}, G_i)$ for any $i = 1, \ldots, N$. In particular, there exists $C(n) > 0$ such that

$$
|M(t) \Delta M(t')| \leq C(n) \operatorname{Per}(G) |t - t'|^{1/2} \quad \forall t, t' > 0, \quad |t - t'| < 1. \quad (6.22)
$$

Proof. Let $\varepsilon_o > 0$ be defined as in (6.16) and take $c_o := c_o(n, \varepsilon_o)$ so that Corollary 6.11 holds for $\lambda > c_o$. 26
Let $M \in GMM(\mathcal{F}, \mathcal{G})$ and let
\[
\lim_{t \to +\infty} |M(t) \Delta L_i(\lambda, [\lambda t])| = 0, \quad t \geq 0,
\] (6.23)
where $\mathcal{L}(\lambda, k)$ is defined as $\mathcal{L}(\lambda, 0) := \mathcal{G}$, and $\mathcal{L}(\lambda, k)$, $k \geq 1$, is a solution of
\[
\min_{A \in \mathcal{P}_b(N+1)} \mathcal{F}(A, \mathcal{L}(\lambda, k-1); \lambda),
\]
and \{\{\lambda_i\}_{i \in \mathbb{N}}\} is a diverging sequence. By induction on $k \geq 1$, and by Corollary 6.11, one can show that
\[
\min_{1 \leq i < j \leq N} d(L_i(\lambda, k), L_j(\lambda, k)) \geq \varepsilon_0
\] (6.24)
for all $\lambda > c_0$ and $k \geq 1$. Therefore, by virtue of Corollary 6.11, for every $k \geq 1$ and $\lambda > c_0$, each $L_i(\lambda, k)$, $i = 1, \ldots, N$, minimizes $\mathcal{A}(\cdot, L_i(\lambda, k-1); \lambda)$. Moreover, from (6.23) we obtain
\[
\lim_{t \to +\infty} |M_i(t) \Delta L_i(\lambda, [\lambda t])| = 0, \quad t \geq 0.
\]
Since $L_i(\lambda, 0) = G_i$, from Definition 1.1 we obtain $M_i \in GMM(\mathcal{A}, G_i)$.

Finally, by [8, 38], there exists $C(n) > 0$ such that each $M_i \in GMM(\mathcal{A}, G_i)$, $i = 1, \ldots, N$, satisfies
\[
|M_i(t) \Delta M_i(t)| \leq C(n) P(G_i) |t - t'|^{1/2}, \quad \forall t, t' > 0, \ |t - t'| < 1.
\] (6.25)
Now (6.22) follows summing (6.25) in $i = 1, \ldots, N$, and using $|A \Delta B| = 2 \sum_{i=1}^N |A_i \Delta B_i|$. \hfill $\square$

**Remark 6.13.** Let $M_i \in GMM(\mathcal{A}, G_i)$, $i = 1, \ldots, N$, and \{\{\lambda_i\}\} be a diverging sequence such that
\[
\lim_{t \to +\infty} |M_i(t) \Delta L_i(\lambda, [\lambda t])| = 0, \quad t \geq 0,
\] (6.26)
where $L_i(\lambda, k)$ is defined as $L_i(\lambda, 0) := G_i$ and $L_i(\lambda, k)$, $k \geq 1$, is a solution of
\[
\min_{A \in \mathcal{B}(\mathbb{R}^n \times \{0, 1\})} \mathcal{A}(A, L_i(\lambda, k-1); \lambda).
\]
Applying an induction argument on $k$ and Corollary 6.4, we establish (6.24) for all $\lambda \geq c_0$ and $k \geq 1$. Therefore, again an induction argument on $k \geq 1$ and Corollary 6.11 imply that the partition $\mathcal{L}(\lambda, k)$ defined for such $\lambda$ and $k$ as
\[
\mathcal{L}(\lambda, k) := \left( L_1(\lambda, k), \ldots, L_N(\lambda, k), \mathbb{R}^n \setminus \bigcup_{i=1}^N L_i(\lambda, k) \right)
\]
minimizes $\mathcal{F}(\cdot, \mathcal{L}(\lambda, k); \lambda)$. Finally, if we denote by $M$ the partition whose bounded components are $M_i$, $i = 1, \ldots, N$, then by (6.26),
\[
\limsup_{t \to +\infty} |\mathcal{L}(\lambda, [\lambda t]) \Delta M(t)| \leq 2 \sum_{i=1}^N \lim_{t \to +\infty} |M_i(t) \Delta L_i(\lambda, [\lambda t])| = 0, \quad t \geq 0,
\]
and hence $M \in GMM(\mathcal{F}, \mathcal{G})$.

Now we are in a position to prove Theorem 1.3.

*Proof.* a) follows combining [1, Theorem 7.4] and Theorem 6.12, whereas b) is a consequence of Theorem 6.12 and [11, Theorem 4]. \hfill $\square$

One can say more about the evolution of convex disjoint partitions.

**Definition 6.14 (Convex disjoint partitions).** A disjoint partition $A \in \mathcal{P}_b(N + 1)$ is called convex if the bounded components of $A$ are convex.
We define the Hausdorff distance between two partitions \( \mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N + 1) \) as
\[
\text{HHD}(\mathcal{A}, \mathcal{B}) := \sum_{i=1}^{N} \text{HHD}(A_i, B_i),
\]
where \( \text{HHD}(A_i, B_i) \) denotes the Hausdorff distance between \( A_i \) and \( B_i \).

**Theorem 6.15 (Evolution and stability of convex disjoint partitions).** Assume that \( C \in \mathbb{P}_b(N + 1) \) is disjoint and convex. Then
\[
GMM(\mathcal{G}, C) = \{\mathcal{M}\} = \{(M_1, \ldots, M_{N+1})\}
\]
is a singleton. In particular, for any \( i, j \in \{1, \ldots, N\}, i \neq j \), the function
\[
t \in [0, \min\{t_i^1, t_j^1\}] \mapsto d(M_i(t), M_j(t))
\]
is nondecreasing. Moreover, for any \( i = 1, \ldots, N \), \( M_i(\cdot) \) agrees with the classical mean curvature flow starting from \( C_t \) up to its extinction time \( t_i^1 \) [29]. Finally, if the sequence \( \{G_t^{(h)}\} \subset \mathbb{P}_b(N + 1) \) converges to \( C \) in the Hausdorff distance \( \text{HHD} \) as \( h \to +\infty \), then for any \( \mathcal{M}^{(h)} \in GMM(\mathcal{G}, C^{(h)}) \),
\[
\lim_{h \to +\infty} \text{HHD}(\mathcal{M}^{(h)}(t), \mathcal{M}(t)) = 0 \quad \forall t \in [0, \min_{i \leq N} t_i^1).
\]

**Proof.** The first part of the theorem follows from Theorem 6.12 and [7, Corollary 5]. Before proving the second part of the theorem, we show the following stability property of convex sets. **Claim.** Let \( C \subset \mathbb{R}^n \) be a nonempty bounded convex set and let \( \{G^{(h)}\} \) be a sequence of sets of finite perimeter converging to \( C \) in the Hausdorff distance as \( h \to +\infty \). Then
\[
G^{(h)}(t) \xrightarrow{\text{HHD}} C(t), \quad t \in [0, t_C^1),
\]
where \( G^{(h)}(t) \) and \( C(t) \) are Almgren-Taylor-Wang solutions starting from \( G^{(h)} \) and \( C \) respectively (recall that \( C(\cdot) \) is unique [7, Corollary 5]), and \( t_C^1 \) is the extinction time of \( C \).

Indeed, consider arbitrary sequences \( \{A^{(l)}\}, \{B^{(l)}\} \) of nonempty bounded convex sets such that \( A^{(l)} \subset C \subset C \subset B^{(l)}, \ l \geq 1, \) and \( A^{(l)}, B^{(l)} \xrightarrow{\text{HHD}} C \) as \( l \to +\infty \). Then for any \( l \geq 1 \), there exists \( h_l \in \mathbb{N} \) such that \( A^{(l)} \subseteq G^{(h)} \subseteq B^{(l)} \) for any \( h > h_l \). We may suppose that \( h_l \to +\infty \) as \( l \to +\infty \). Let \( A^{(l)}(t) \) (resp. \( B^{(l)}(t) \)) be the minimizing movement starting from \( A^{(l)} \) (resp. \( B^{(l)} \)) for the Almgren-Taylor-Wang functional (4.5) and \( G^{(h)}(t)^* \) and \( G^{(h)}(t)^\ast \), be the maximal and minimal \( GMMs \) (Definition 6.5) for (4.5) starting from \( G^{(h)} \), so that \( G^{(h)}(t)^\ast \subseteq G^{(h)}(t) \subseteq G^{(h)}(t)^* \) for all \( t \geq 0 \). By Theorem 6.6, \( A^{(l)}(t) \subseteq G^{(h)}(t)^\ast \) and \( \{G^{(h)}(t)^\ast \} \subseteq B^{(l)}(t) \) for any \( t \geq 0 \) and \( h > h_l \). Moreover, from [7, Theorem 12] we have \( A^{(l)}(t), B^{(l)}(t) \xrightarrow{\text{HHD}} C(t) \) as \( l \to +\infty \) for any \( t \in [0, t_C^1] \), and since \( h_l \to +\infty \), (6.28) follows.

Now we prove the second part of Theorem 6.15. Since the partition \( C \) is disjoint and \( \text{HHD}(G^{(h)}, C) \to 0 \) as \( h \to +\infty \), one has that \( G^{(h)} \) is also disjoint provided \( h \) is large enough. Let \( \mathcal{M}^{(h)} \in GMM(\mathcal{G}, G^{(h)}) \); by Theorem 6.12 \( \mathcal{M}^{(h)}_i \in GMM(\mathcal{G}_i, G^{(h)}_i), \ i = 1, \ldots, N, \) and therefore by virtue of \( G^{(h)} \xrightarrow{\text{HHD}} C \) and the previous claim, \( \mathcal{M}^{(h)}_i(t) \xrightarrow{\text{HHD}} M_i(t), \ i = 1, \ldots, N, \) as \( h \to +\infty \) for any \( t \in [0, t_i^1) \). \( \square \)

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