Brownian Bridge Asymptotics for the Subcritical Bernoulli Bond Percolation.

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October 22, 2018

Abstract

For the $d$-dimensional model of a subcritical bond percolation ($p < p_c$) and a point $\vec{a}$ in $\mathbb{Z}^d$, we prove that a cluster conditioned on connecting points $(0, ..., 0)$ and $n\vec{a}$ if scaled by $\frac{1}{n\|\vec{a}\|}$ along $\vec{a}$ and by $\frac{1}{\sqrt{n}}$ in the orthogonal direction converges asymptotically to $\text{Time} \times (d-1)$-dimensional Brownian Bridge.

1 Introduction.

1.1 Percolation and Brownian Bridge.

We begin by briefly stating the notion of a bond percolation based on the material rigorously presented in [8] and [10], and the notion of the Brownian Bridge as well as the word description of the result connecting the two that we have obtained and made the primary objective of this paper.

**Percolation:** For each edge of the $d$-dimensional square lattice $\mathbb{Z}^d$ in turn, we declare the edge open with probability $p$ and closed with probability $1-p$, independently of all other edges. If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^d$. A connected component of the subgraph is called a “cluster”, and the number of edges in a cluster is the “size” of the cluster. The probability $\theta(p)$ that the point $(0,0)$ belongs to a cluster of an infinite size is zero if $p = 0$, and one if $p = 1$. However, there exists a critical probability $0 < p_c < 1$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. In the first case, we say that we are dealing with a subcritical percolation model, and in the second case, we say that we are dealing with a supercritical percolation model.

**Brownian Bridge:** defined as a sample-continuous Gaussian process $B^0_t$ on $[0,1]$ with mean 0 and $E B^0_s B^0_t = s(1-t)$ for $0 \leq s \leq t \leq 1$. So, $B^0_0 = B^0_1 = 0$ a.s. Also, if $B$ is a Brownian motion, then the process $B_t - tB_1$ ($0 \leq t \leq 1$) is a Brownian Bridge. For more details see [2], [3] and [4]. The process $B^0_t \vec{a} \equiv B^0_t + t\vec{a}$ is a Brownian Bridge connecting points zero and $\vec{a}$. 

History of the problem: Below, we consider the $d$-dimensional model of a subcritical bond percolation ($p < p_c$) and a point $\mathbf{a}$ in $\mathbb{Z}^d$, conditioned on the event of zero being connected to $n\mathbf{a}$. We first show that a specifically chosen path connecting points zero and $n\mathbf{a}$ going through some appropriately defined points on the cluster (regeneration points), if scaled $\frac{1}{n\|\mathbf{a}\|}$ times along $\mathbf{a}$ and $\frac{1}{\sqrt{n}}$ times in the direction orthogonal to $\mathbf{a}$, converges to Time $\times (d-1)$-dimensional Brownian Bridge as $n \to +\infty$, where the scaled interval connecting points zero and $n\mathbf{a}$ serves as a $[0,1]$ time interval. In other words, we prove that a scaled “skeleton” going through the regeneration points of the cluster converges to Time $\times (d-1)$-dimensional Brownian Bridge. In a subsequent step, we show that if scaled, then the hitting area of the orthogonal hyper-planes shrinks, implying that for $n$ large enough, all the points of the scaled cluster are within an $\epsilon$-neighborhood of the points in the “skeleton”. One of the major tools used in this research was the renewal technique developed in [1], [3], [4], [5] and [9] as part of the derivation of the Ornstein-Zernike estimate for the subcritical bond percolation model and “similar” processes. A major result related to the study is that for $\mathbf{a} = (1,0,...,0)$, the hitting distribution of the cluster in the intermediate planes, $x_1 = tna$, $0 < t < 1$ obeys a multidimensional local limit theorem (see [3]). Dealing with all other $\mathbf{a} \neq (k,0,...,0)$ became possible only after the corresponding technique further mastering the regeneration structures and equi-decay profiles was developed in [4] and [9]. This technique played a central role in obtaining the research results.

1.2 Asymptotic Convergence.

Here we state a version of a local CLT and a technical result that we later prove.

Local Limit Theorem: In this paper we are going to use the version of the local CLT borrowed from [2]. Let $X_1, X_2, ... \in \mathbb{R}$ be i.i.d. with $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = \sigma^2 \in (0,\infty)$, and having a common lattice distribution with span $h$. If $S_n = X_1 + ... + X_n$ and $P[X_i \in b+h\mathbb{Z}] = 1$ then $P[S_n \in nb+h\mathbb{Z}] = 1$. We put

$$p_n(x) = P[S_n/\sqrt{n} = x]$$

for $x \in \Lambda_n = \{(nb+hz)/\sqrt{n} : z \in \mathbb{Z}\}$ and

$$n(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$$

for $x \in (-\infty, \infty)$.

Local CLT. Under the above hypotheses, $\sup_{x \in \Lambda_n} |\frac{\sqrt{n}}{n} p_n(x) - n(x)| \to 0$ as $n \to \infty$.

Technical Result Concerning Convergence to the Brownian Bridge (to be used in Chapter 2.4, and is proved in Chapter 3.2): The following technical result is going to be proved in the section 3 of this paper, however since we are to use it in the section 2, we will state the result below as part of the introduction. Let $X_1, X_2, ...$ be i.i.d. random variables on $\mathbb{Z}^d$ with the span of the lattice distribution equal to one (see [7], section 2.5), and let there be a $\lambda > 0$ such that the moment-generating function

$$\mathbb{E}(e^{\theta \cdot X_1}) < \infty$$

for all $\theta \in B_{\lambda}$. 
Now, for a given vector \( \vec{a} \in \mathbb{Z}^d \), let \( X_1 + \ldots + X_i = [t_i, Y_i]_f \in \mathbb{Z}^d \) when written in the new orthonormal basis such that \( \vec{a} = [||\vec{a}||, 0]_f \) (in the new basis \( [\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1} \)). Also let \( P[\vec{a} \cdot X_i] > 0 \) = 1. We define the process \( [t, Y_{n,k}(t)]_f \) to be the interpolation of 0 and \( \frac{1}{n} t_i, \frac{1}{\sqrt{n}} Y_i \) when written in the new basis \( [\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1} \). Also let \( P[\vec{a} \cdot X_i] > 0 \) = 1. We define the process \( [t, Y_{n,k}(t)]_f \) to be the interpolation of 0 and \( \frac{1}{n} t_i, \frac{1}{\sqrt{n}} Y_i \) when written in the new basis \( [\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1} \). The process \( \{Y_{n,k}^* \) for some \( k \) such that \( t_k, Y_k \) conditioned on the existence of such \( k \) converges weakly to the Brownian Bridge (of variance that depends only on the law of \( X_1 \)).

## 2 The Main Result in Subcritical Percolation.

In this section we work only with subcritical percolation probabilities \( p < p_c \).

### 2.1 Preliminaries.

Here we briefly go over the definitions that one can find in Section 4 of [4].

We start with the inverse correlation length \( \xi_p(\vec{x}) \):

\[
\xi_p(\vec{x}) \equiv - \lim_{n \to \infty} \frac{1}{n} P_p(0 \leftrightarrow [n\vec{x}]),
\]

where the limit is always defined due to the FKG property of the Bernoulli bond percolation (see [8]). Now, \( \xi_p(\vec{x}) \) is the support function of the compact convex set

\[
K^p \equiv \bigcap_{\vec{n} \in \mathbb{Z}^{d-1}} \{\vec{r} \in \mathbb{R}^d : \vec{r} \cdot \vec{n} \leq \xi_p(\vec{n})\},
\]

with non-empty interior \( \text{int}\{K^p\} \) containing point zero.

Let \( \vec{r} \in \partial K^p \), and let \( \vec{e} \) be a basis vector such that \( \vec{e} \cdot \vec{r} \) is maximal. For \( \vec{x}, \vec{y} \in \mathbb{Z}^d \) define

\[
S^r_{\vec{x},\vec{y}} \equiv \{\vec{z} \in \mathbb{R}^d : \vec{r} \cdot \vec{z} \leq \vec{r} \cdot \vec{x} \leq \vec{r} \cdot \vec{y}\}.
\]

Note that \( S^r_{\vec{x},\vec{y}} = \emptyset \) if \( \vec{r} \cdot \vec{y} < \vec{r} \cdot \vec{x} \).

Let \( C^r_{\vec{x},\vec{y}} \) denote the corresponding common open cluster of \( \vec{x} \) and \( \vec{y} \) when we run the percolation process on \( S^r_{\vec{x},\vec{y}} \).

**Definition 1.** For \( \vec{x}, \vec{y} \in \mathbb{Z}^d \) lets define \( h_r \)-connectivity \( \{\vec{x} \leftarrow^{h_r} \vec{y}\} \) of \( \vec{x} \) and \( \vec{y} \) to be the event that

1. \( \vec{x} \) and \( \vec{y} \) are connected in the restriction of the percolation configuration to the slab \( S^r_{\vec{x},\vec{y}} \).
2. If \( \vec{x} \neq \vec{y} \), then \( C^r_{\vec{x},\vec{y}} \cap S^r_{\vec{x},\vec{x}+\vec{e}} = \{\vec{x}, \vec{x}+\vec{e}\} \) and \( C^r_{\vec{x},\vec{y}} \cap S^r_{\vec{y},\vec{y}-\vec{e}} = \{\vec{y}, \vec{y}+\vec{e}\} \).
3. If \( \vec{x} = \vec{y} \) and all the edges adjoined to \( \vec{x} \) and perpendicular to \( \vec{e} \) are closed.

Set

\[
h_r(\vec{x}) \equiv P_p[0 \leftarrow^{h_r} \vec{x}],
\]

Notice that \( h_r(0) = (1 - p)^{2(d-1)} \).
Definition 2. For \( \vec{x}, \vec{y} \in \mathbb{Z}^d \) let's define \( f_r \)-connectivity \( \{ \vec{x} \leftarrow f_r \rightarrow \vec{y} \} \) of \( \vec{x} \) and \( \vec{y} \) to be the event that
1. \( \vec{x} \neq \vec{y} \)
2. \( \vec{x} \leftarrow h_r \rightarrow \vec{y} \).
3. For no \( \vec{z} \in \mathbb{Z}^d \setminus \{ \vec{x}, \vec{y} \} \) both \( \{ \vec{x} \leftarrow h_r \rightarrow \vec{z} \} \) and \( \{ \vec{z} \leftarrow h_r \rightarrow \vec{y} \} \) take place.

Set
\[ f_r(\vec{x}) \equiv P_p[0 \leftarrow f_r \rightarrow \vec{x}]. \]
Notice that \( f_r(0) = 0 \).

Definition 3. Suppose \( 0 \leftarrow h_r \rightarrow \vec{x} \), we say that \( \vec{z} \in \mathbb{Z}^d \) is a regeneration point of \( C^r_{0, \vec{x}} \) if
1. \( \vec{r} \cdot \vec{e} \leq \vec{r} \cdot \vec{z} \leq \vec{r} \cdot (\vec{y} - \vec{e}) \)
2. \( S^r_{\vec{z} - \vec{e}, \vec{z} + \vec{e}} \cap C^r_{0, \vec{x}} \) contains exactly three points: \( \vec{z} - \vec{e}, \vec{z} \) and \( \vec{z} + \vec{e} \), where \( \vec{e} \) is defined as before.

Let also \( \vec{x} \) itself be a regeneration point.

The following Ornstein-Zernike equality is due to be used soon:

Theorem. \( \exists A(\cdot, \cdot) \) on \( (0, p_c) \times S^{d-1} \) s. t.
\[ P_p[0 \leftrightarrow \vec{x}] = \frac{A(p, n(\vec{x}))}{\lVert \vec{x} \rVert^{d-1}} e^{-\xi_p(\vec{x})}(1 + o(1)) \] (1)
uniformly in \( \vec{x} \in \mathbb{Z}^d \), where \( n(\vec{x}) \equiv \frac{\vec{x}}{\lVert \vec{x} \rVert} \).

We refer to [4] for the proof of the theorem.

2.2 Measure \( Q^r_{r_0}(x) \).

It had been proved in section 4 of [4] that for a given \( \vec{r}_0 \in \partial K^p \) there exists \( \bar{\lambda} > 0 \) such that
\[ F_{r_0}(\vec{r}) = \frac{1}{(1-p)^{2(d-1)}} \sum_{x \in \mathbb{Z}^d} f_{r_0}(x)e^{\vec{r} \cdot \vec{x}} = 1 \text{ whenever } \vec{r} \in B_\bar{\lambda}(\vec{r}_0) \cap \partial K^p \]
and therefore
\[ Q^r_{r_0}(\vec{x}) \equiv \frac{1}{(1-p)^{2(d-1)}} f_{r_0}(\vec{x})e^{\vec{r} \cdot \vec{x}} \text{ is a measure on } \mathbb{Z}^d. \]

Also, it was shown that
\[ \mu = \mu_{r_0}(\vec{r}) \equiv E^r_{r_0} X = \sum_{\vec{x} \in \mathbb{Z}^d} \vec{x}Q^r_{r_0}(\vec{x}) = \nabla_r \log F_{r_0}(\vec{r}) \neq 0 \]
and
\[ F_{r_0}(\vec{r}) < \infty \text{ for all } \vec{r} \text{ in } B_\bar{\lambda}(\vec{r}_0). \]

The later implies
\[ F_{r_0}(\vec{r}) = \sum_{\vec{x} \in \mathbb{Z}^d} f_{r_0}(\vec{x})e^{\vec{r} \cdot \vec{x}} = \sum_{\vec{x} \in \mathbb{Z}^d} Q^r_{r_0}(\vec{x})e^{\theta \cdot \vec{x}} < \infty \]
for $\theta = \vec{r} - \vec{r}_0 \in B_{\lambda}(0)$, i.e. the moment generating function $E_{r_0}^\theta(e^{\theta \cdot X_1})$ of the law $Q_{r_0}^\theta$ is finite for all $\theta \in B_{\lambda}(0)$.

Now, there is a renewal relation (see section 1 and section 4 of [4]),

$$h_{r_0}(\vec{x}) = \frac{1}{(1 - p)^{2(d-1)}} \sum_{\vec{z} \in \mathbb{Z}^d} f_{r_0}(\vec{z}) h_{r_0}(\vec{x} - \vec{z}) \text{ with } h_{r_0}(0) = (1 - p)^{2(d-1)}$$

and therefore

$$h_{r_0}([N\mu]) = (1 - p)^{2(d-1)} e^{-r \cdot [N\mu]} \sum_k Q_{r_0}^r (X_1 + ... + X_k = [N\mu]) \text{ for } N > 0,$$

where $X_1, X_2, ...$ is a sequence of i.i.d. random variables distributed according to $Q_{r_0}^r$, as $h_{r_0}$-connection is a chain of $f_{r_0}$-connections with junctions at the regeneration points of $C_{0,x}^{r_0}$.

### 2.3 Important Observation.

The probability that $0 \xleftarrow{h_{r_0}} x$ with exactly $k$ regeneration points $x_1, x_1 + x_2, ...; \sum_{i=1}^k x_i = x$

$$P_X \equiv P[0 \xleftarrow{h_{r_0}} x; \text{regeneration points: } x_1, x_1 + x_2, ...; \sum_{i=1}^k x_i = x]$$

$$= \frac{1}{(1 - p)^{2(d-1)(k-1)}} P[0 \xleftarrow{f_{r_0}} x_1] P[x_1 \xleftarrow{f_{r_0}} x_1 + x_2] ... P[\sum_{i=1}^{k-1} x_i \xleftarrow{f_{r_0}} \sum_{i=1}^k x_i = x]$$

$$= \frac{1}{(1 - p)^{2(d-1)(k-1)}} f_{r_0}(x_1) f_{r_0}(x_2) ... f_{r_0}(x_k). \quad (2)$$

### 2.4 The Result.

In this section we fix $\vec{a} \in \mathbb{Z}^d$, and let $r = r_0 = \vec{a}\mathbb{R}^+ \cap \partial K^p$. Then we recall that

$$E_{r_0}(e^{\theta \cdot X_1}) < \infty$$

for all $\theta \in B_{\lambda}(0)$. We also denote $h(x) \equiv h_{r_0}(x)$ and $f(x) \equiv f_{r_0}(x)$.

First, we introduce a new basis $\{\vec{f}_1, \vec{f}_2, ..., \vec{f}_d\}$, where $\vec{f}_1 = \frac{\vec{a}}{||\vec{a}||}$. We use $[\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of a vector with respect to the new basis. Obviously $\vec{a} = [||\vec{a}||, 0]_f$.

We want to prove that the process corresponding to the last $d - 1$ coordinates in the new basis of the scaled $(\frac{1}{||\vec{a}||} \cdot \vec{a})$ times along $\vec{a}$ and $\frac{1}{\sqrt{\lambda}}$ times in the orthogonal $d$-1 dimensions) interpolation of regeneration points of $C_{0,n\vec{a}}^{r_0}$ conditioned on $0 \xleftarrow{h} n\vec{a}$ converges weakly to the Brownian Bridge $B^\theta(t)$ (with variance that depends only on measure $Q_{r_0}^\theta$) where $t$ represents the scaled first coordinate in the new basis.
Let $X_1, X_2, \ldots$ be i.i.d. random variables distributed according to $Q_{r_0}^r$ law. We interpolate $0, X_1, (X_1 + X_2), \ldots, (X_1 + \ldots + X_k)$ and scale by $\frac{1}{n||\alpha||} \times \frac{1}{\sqrt{n}}$ along $<\alpha> \times <\alpha>^\perp$ to get the process $[t, Y_{n,k}^*(t)]_f$. The technical theorem (see Chapters (1.2) and (3.2)) implies the following

**Theorem 1.** The process

$$\{Y_{n,k}^* \text{ for some } k \text{ such that } X_1 + \ldots + X_k = n\alpha\}$$

conditioned on the existence of such $k$ converges weakly to the Brownian Bridge (with variance that depends only on measure $Q_{r_0}^r$).

Now, let for $y_1, \ldots, y_k \in \mathbb{Z}^d$ with positive increasing first coordinates $\gamma(y_1, \ldots, y_k)$ be the last $(d - 1)$ coordinates in the new basis of the scaled $(\frac{1}{n||\alpha||} \times \frac{1}{\sqrt{n}})$ interpolation of points 0, $y_1, \ldots, y_k$ (where the first coordinate is time). Notice that $\gamma(y_1, \ldots, y_k) \in C_0[0,1]^{d-1}$ as a function of scaled first coordinate whenever $y_k = n\alpha$.

By the important observation (2) we’ve made before, for any function $F(\cdot)$ on $C[0,1]^{d-1}$,

$$\sum_k \sum_{x_1+\ldots+x_k=n\alpha} F(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i))$$

$$\times P[0 \leftarrow h_{r_0} \rightarrow x ; \text{regeneration points: } x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i = x]$$

$$= \sum_k \sum_{x_1+\ldots+x_k=n\alpha} F(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) \frac{1}{(1 - p)^{2(d-1)(k-1)}} f(x_1) \cdots f(x_k)$$

$$= (1 - p)^{2(d-1)} e^{-r-n\alpha} \sum_k \sum_{x_1+\ldots+x_k=n\alpha} F(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) Q_{r_0}^r(x_1) \cdots Q_{r_0}^r(x_k).$$

Therefore, for any $A \subset C[0,1]^{d-1}$

$$P_p[\gamma(\text{regeneration points}) \in A \mid 0 \leftarrow h \rightarrow n\alpha]$$

$$= \frac{\sum_k \sum_{x_1+\ldots+x_k=n\alpha} I_A(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) \frac{1}{(1 - p)^{2(d-1)(k-1)}} f(x_1) \cdots f(x_k)}{\sum_k \sum_{x_1+\ldots+x_k=n\alpha} \frac{1}{(1 - p)^{2(d-1)(k-1)}} f(x_1) \cdots f(x_k)}$$

$$= \frac{\sum_k \sum_{x_1+\ldots+x_k=n\alpha} I_A(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) Q_{r_0}^r(x_1) \cdots Q_{r_0}^r(x_k)}{\sum_k \sum_{x_1+\ldots+x_k=n\alpha} Q_{r_0}^r(x_1) \cdots Q_{r_0}^r(x_k)}$$

$$= P[Y_{n,k}^* \in A \text{ for the } k \text{ such that } X_1 + \ldots + X_k = n\alpha \mid \exists k \text{ such that } X_1 + \ldots + X_k = n\alpha].$$

Hence, we have proved the following

**Corollary.** The process corresponding to the last $d-1$ coordinates (in the new basis $\{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_d\}$) of the scaled $(\frac{1}{n||\alpha||} \times \frac{1}{\sqrt{n}})$ interpolation of regeneration points of $C_{0,n\alpha}^{r_0}$ (where the first coordinate is time) conditioned on $0 \leftarrow h \rightarrow n\alpha$ converges weakly to the Brownian Bridge (with variance that depends only on measure $Q_{r_0}^r$).
2.5 Shrinking of the Cluster. Main Theorem.

Here for $\vec{a} \in \mathbb{Z}^d$ we let $\vec{r}_0 = \vec{a} \mathbb{R}^+ \cap \partial \mathbb{K}^p$ again. Before we proceed with the proof that the scaled percolation cluster $C^\vec{r}_0_{a,\vec{a}}$ shrinks to the scaled interpolation skeleton of regeneration points, we need to prove the following

**Proposition.** If $\vec{r} = \nabla \xi_p(\vec{r}_0)$ then $Q^r_{\vec{r}_0}$ is a probability measure.

**Proof.** First we notice that $\vec{r}_0 \cdot \vec{r} = \vec{r}_0 \cdot \nabla \xi_p(\vec{r}_0) = D_{\vec{r}_0}(\xi_p(\vec{r}_0)) = \xi_p(\vec{r}_0)$, and thus

$$H_{\vec{r}_0}(\vec{r}) \equiv \frac{1}{(1-p)^{2(d-1)}} \sum_{\vec{x} \in \mathbb{Z}^d} h_{\vec{r}_0}(x)e^{\vec{x} \cdot \vec{r}} \geq \sum_{\vec{x} \in \langle \vec{a} \rangle \cap \mathbb{Z}^d} h_{\vec{r}_0}(x)e^{\vec{x} \cdot \vec{r}} = \sum_{\vec{x} \in \langle \vec{a} \rangle \cap \mathbb{Z}^d} h_{\vec{r}_0}(x)e^{\xi_p(\vec{x})} = +\infty$$

for $d \leq 3$ by Ornstein-Zernike equation (1). For all other $d$ we sum over all $\vec{x}$ inside a small enough cone around $\vec{a}$ to get $H_{\vec{r}_0}(\vec{r}) = +\infty$.

Now, for all $\vec{n} \in \mathbb{S}^{d-1}$, $\vec{n} \cdot \nabla \xi_p(\vec{r}_0) = D_{\vec{r}_0}(\xi_p(\vec{r}_0)) \leq \xi_p(\vec{n})$ by convexity of $\xi_p$, and therefore $\vec{r} = \nabla \xi_p(\vec{r}_0) \in \partial \mathbb{K}^p$. Notice that due to the strict convexity of $\xi_p$ and the way $\mathbb{K}^p$ was defined, $\vec{r} = \nabla \xi_p(\vec{r}_0)$ is the only point on $\partial \mathbb{K}^p$ such that $\vec{r}_0 \cdot \vec{r} = \xi_p(\vec{r}_0)$.

Now, Ornstein-Zernike equation (1) also implies that the sums $H_{\vec{r}_0}(\vec{r})$ and $F_{\vec{r}_0}(\vec{r})$ are finite whenever $\vec{r} \in \alpha \mathbb{K}^p = \bigcap_{\vec{n} \in \mathbb{S}^{d-1}} \{\vec{r} \in \mathbb{R}^d : \vec{n} \cdot \vec{r} \leq \alpha \xi_p(\vec{n})\}$ with $\alpha \in (0, 1)$, and due to the recurrence relation of $f_{\vec{r}_0}$ and $h_{\vec{r}_0}$ connectivity functions, $H_{\vec{r}_0}(\vec{r}) = \frac{1}{1 - F_{\vec{r}_0}(\vec{r})}$ (see [4]). Therefore $F_{\vec{r}_0}(\vec{r}) \equiv \frac{1}{(1-p)^{2(d-1)}} \sum_{\vec{x} \in \mathbb{Z}^d} f_{\vec{r}_0}(x)e^{\vec{x} \cdot \vec{r}} = 1$, where the probability measure $Q^r_{\vec{r}_0}$ has an exponentially decaying tail due to the same reasoning as in chapter 4 of [4] ("mass-gap" property).

With the help of the proposition above we shall show that the consequent regeneration points are situated relatively close to each other:

**Lemma.**

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3}, \text{ x_i-reg. points } | \text{ 0 } \leftrightarrow_{\cdot} \text{ n} \overrightarrow{a}] < \frac{1}{n}$$

for $n$ large enough.

**Proof.** Let $\vec{r} \equiv \nabla \xi_p(\vec{r}_0) = \nabla \xi_p(\vec{a})$. Since $\xi_p(x)$ is strictly convex (see section 4 in [4]),

$$\frac{\xi_p(\vec{a}) - \xi_p(\vec{a} - \vec{x})}{\frac{\|\vec{x}\|}{n}} < \frac{\vec{x}}{\|\vec{x}\|} \cdot \nabla \xi_p(\vec{a})$$

for $\vec{x} \in \mathbb{Z}^d (\vec{x} \neq 0)$, and therefore

$$\xi_p(n\vec{a}) - \xi_p(n\vec{a} - \vec{x}) = \|\vec{x}\| \frac{\xi_p(\vec{a}) - \xi_p(\vec{a} - \vec{x})}{\frac{\|\vec{x}\|}{n}} < \vec{x} \cdot \nabla \xi_p(\vec{a}) = \vec{r} \cdot \vec{x}.$$
Thus, since $Q_{r_0}^r(x)$ decays exponentially and therefore

$$\frac{f(x)}{(1-p)^{2(d-1)}} e^{\xi_p(n\vec{a}) - \xi_p(n\vec{a} - x)} < Q_{r_0}^r(x)$$

and also decays exponentially. Hence by Ornstein-Zernike result \([1]\),

$$P_p[n^{1/3} < |x|, \ x\text{-first reg. point } |0 \leftarrow h \rightarrow n\vec{a}| = \sum_{n^{1/3} < |x|} \frac{f(x)}{(1-p)^{2(d-1)}} \frac{h(n\vec{a} - x)}{h(n\vec{a})} < \frac{1}{n^2}$$

for $n$ large enough. So, since the number of the regeneration points is no greater than $n$,

$$P_p[\max_i |x_i - x_{i-1}| > n^{1/3}, \ x_i\text{- reg. points } |0 \leftarrow h \rightarrow n\vec{a}| < \frac{1}{n}$$

for $n$ large enough.

Now, it is really easy to check that there is a constant $\lambda_f > 0$ such that

$$f(\vec{x}) > e^{-\lambda_f \|\vec{x}\|}$$

for all $\vec{x}$ such that $f(\vec{x}) \neq 0$ (here we only need to connect points $\vec{e}$ and $\vec{x} - \vec{e}$ with two non-intersecting open paths surrounded by the closed edges), and there exists a constant $\lambda_u > 0$ such that

$$P_p[\text{percolation cluster } C(0) \not\subset [\mathbb{R}; B_{R}^{d-1}(0)]_f] < e^{-\lambda_u R}$$

for $R$ large enough due to the exponential decay of the radius distribution for subcritical probabilities (see \([\S]\)). Hence, for a given $\epsilon > 0$

$$P_p[\text{cluster } C_{0,\vec{x}}^{r_0} \not\subset [\mathbb{R}, B_{\epsilon\sqrt{\pi}}(0)]_f | 0 \leftarrow f \rightarrow x] < e^{\lambda_f \|\vec{x}\| - \lambda_u \epsilon \sqrt{\pi}}$$

and therefore, summing over the regeneration points, we get

$$P_p[\text{scaled cluster } C_{0,n\vec{a}}^{r_0} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points } ) | 0 \leftarrow g \rightarrow n\vec{a}]$$

$$< \frac{1}{n} + ne^{\lambda_f n^{1/3} - \lambda_u \epsilon \sqrt{n}}$$

for $n$ large enough.

We can now state the main result of this paper:

**Main Theorem.** The process corresponding to the last $d - 1$ coordinates (in the new basis \(\{\vec{f}_1, \vec{f}_2, ..., \vec{f}_d\}\) of the scaled \(1/n\|\vec{a}\| \times \sqrt{n}\)) interpolation of regeneration points of $C_{0,n\vec{a}}^{r_0}$ (where the first coordinate is time) conditioned on $0 \leftarrow h \rightarrow n\vec{a}$ converges weakly to the Brownian Bridge (with variance that depends only on measure $Q_{r_0}^r$).

Also for a given $\epsilon > 0$

$$P_p[\text{scaled cluster } C_{0,n\vec{a}}^{r_0} \not\subset \epsilon\text{-neighbd. of } [0,1] \times \gamma(\text{ reg. points } ) | 0 \leftarrow h \rightarrow n\vec{a}] \rightarrow 0$$

as $n \rightarrow \infty$.  

8
3 Convergence to Brownian Bridge.

As it was mentioned in the introduction, this chapter is entirely dedicated to proving the Technical Theorem that we have already used in the proof of the main result.

3.1 Simple Case.

Let $Z_1, Z_2, \ldots$ be i.i.d. random variables on $\mathbb{Z}$ with the span of the lattice distribution equal to one (see [3], section 2.5) and mean $\mu = \mathbb{E}Z_1 < \infty$, $\sigma^2 = \text{Var}(Z_1) < \infty$. Also let point zero be inside of the closed convex hull of $\{z : P[Z_1 = z] > 0\}$.

Consider a one dimensional plane and a walk $X_j$ that starts with $X_0 = 0$ and for a given $X_j$, the $(j+1)$-st step to be $X_{j+1} = X_j + Z_{j+1}$. After interpolation we get

$$X(t) = X[t] + (t - [t])(X_{[t]+1} - X_{[t]})$$

for $0 \leq t < \infty$.

And define $\bar{X}(t) = (t, X(t))$ to be a two dimensional walk.

Now, if for a given integer $n > 0$ we define $X_n(t) \equiv X(nt)\sqrt{n}$ for $0 \leq t \leq 1$, then $X_n(t)$ would belong to $C[0, 1]$ and $X_n(0) = 0$.

Theorem 2. $X_n(t)$ conditioned on $X_n(1) = 0$ converges weakly to the Brownian Bridge.

First we need to prove the theorem when $\mu = 0$. For this we need to prove that

Lemma 1. For $A_0 \subseteq C[0, 1]$, let $P_n(A_0) = P[X_n \in A_0 | X_n(1) = 0]$ to be the law of $X_n$ conditioned on $X_n(1) = 0$. Then

(a) For $\mu = 0$, the finite-dimensional distributions of $P_n$ converge weakly to a Gaussian distributions.

(b) There are positive $\{C_n\}_{n=1,2,\ldots} \to C$ ($C = \sigma^2$ when $\mu = 0$) such that $0 < C < \infty$ and

$$\text{Cov}_{P_n}(X_n(s), X_n(t)) = C_n s(1 - t) + O(\frac{1}{n})$$

for all $0 \leq s \leq t \leq 1$. More precisely: $\text{Cov}_{P_n}(X_n(s), X_n(t)) = C_n s(1 - t)$ if $[ns] < [nt]$ and $\text{Cov}_{P_n}(X_n(s), X_n(t)) = C_n s(1 - t) - C_n \frac{\epsilon_1(1-\epsilon_2)}{n}$ if $[ns] = [nt]$, where $\epsilon_1 = \frac{ns-[ns]}{n} \in [0,1)$ and $\epsilon_2 = \frac{nt-[nt]}{n} \in [0,1)$.

and we need

Lemma 2. For $\mu = 0$, the probability measures $P_n$ induced on the subspace of $X_n(t)$ trajectories in $C[0, 1]$ are tight.
Proof of Lemma 1: (a) Though it is not difficult to show that a finite-dimensional distribution of $P_n$ converges weakly to a gaussian distribution, here we only show the convergence for one and two points on the interval (in case of one point $t \in [0,1]$, we show that the limit variance has to be equal to $t(1-t)\sigma^2$). Take $t \in \frac{1}{n}Z \cap (0,1)$ and let $\alpha = \frac{k}{\sqrt{n}}$, then by the Local CLT,

$$P[X(tn) = k] = \frac{1}{\sqrt{n}} \Phi_\sigma(\sqrt{t}(\alpha) + o(\frac{1}{\sqrt{n}})),$$

where $\Phi_\sigma(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ is the normal density function, and the error term is uniformly bounded by a $o(\frac{1}{\sqrt{n}})$ function independent of $k$.

Therefore, substituting (3),

$$P_n[X_n(t) = \alpha] = \frac{1}{\sqrt{n}} \Phi_\sigma(0) + o(\frac{1}{\sqrt{n}}) = \frac{1}{\sqrt{n}} \Phi_\sigma(t(1-t)(\alpha) + o(\frac{1}{\sqrt{n}})).$$

Thus for a set $A$ in $\mathbb{R}$,

$$P_n[X_n(t) \in A] = \sum_{k \in \sqrt{n}A} \frac{1}{\sqrt{n}} \Phi_\sigma(t(1-t)(\alpha) + o(\frac{1}{\sqrt{n}})) = N[0,t(1-t)\sigma^2](A) + o(1)$$

-here the limit variance is equal to $t(1-t)\sigma^2$. Given that the variance $\sigma^2 < 0$, the convergence follows.

The same method works for more than one point, here we do it for two: Let $\alpha_1 = \frac{k}{\sqrt{n}}$ and $\alpha_2 = \frac{k_2}{\sqrt{n}}$, then as before, for $t_1 < t_2$ in $\frac{1}{n}Z \cap (0,1)$, writing the conditional probability as a ratio of two probabilities, and representing the probabilities according to (3), we get

$$P_n[X_n(s) = \alpha_1, X_n(t) = \alpha_2] = \frac{\sqrt{|A|}}{2\pi \sigma^2} \exp \left\{ -\frac{(\alpha_1, \alpha_2)A(\alpha_1, \alpha_2)^T}{2\sigma^2} \right\} + o(\frac{1}{n}).$$

where

$$A = \begin{pmatrix} \frac{t_2-t_1}{(t_2-t_1)(1-t_2)} & -\frac{1}{(t_2-t_1)} \\ \frac{1}{t_2-t_1} & \frac{t_2-t_1}{(t_2-t_1)(1-t_2)} \end{pmatrix}.$$ 

Thus for sets $A_1$ and $A_2$ in $\mathbb{R}$,

$$P_n[X_n(t_1) \in A_1, X_n(t_2) \in A_2] = \sum_{k_1 \in \sqrt{n}A_1, k_2 \in \sqrt{n}A_2} \frac{\sqrt{|A|}}{2\pi \sigma^2} \exp \left\{ -\frac{(\alpha_1, \alpha_2)A(\alpha_1, \alpha_2)^T}{2\sigma^2} \right\} + o(\frac{1}{n}) = N[0, A^{-1}](A_1 \times A_2) + o(1)$$

10
Observe that \((\sigma^2 A^{-1}) = \begin{pmatrix} t_1(1-t_1)\sigma^2 & t_1(1-t_2)\sigma^2 \\ t_1(1-t_2)\sigma^2 & t_2(1-t_2)\sigma^2 \end{pmatrix}\) is the covariance matrix, and the part (b) of the lemma follows in case \(\mu = 0\).

(b) Though the estimate above produces the needed variance in case when the mean \(\mu = 0\), in general, we need to apply the following approach: We first consider the case when 
\[s < t \quad \text{and both} \quad s, t \in \frac{1}{n} \mathbb{Z} \cap (0, 1)\] where 
\[\mathbf{E}[X_n(s) \mid X_n(t) = y] = \mathbf{E}[Z_1 + ... + Z_{sn} \mid Z_1 + ... + Z_{tn} = y] = \frac{s}{t}y,\]
and therefore
\[\text{Cov}_{P_n}(X_n(s), X_n(t)) = \frac{s}{t}\mathbf{E}[X_n^2(t) \mid X_n(1) = 0]\]
as \{-X_n(1-t) \mid X_n(1) = 0\} and \{X_n(t) \mid X_n(1) = 0\} are identically distributed.

Now, by symmetry (time reversal),
\[\text{Cov}_{P_n}(X_n(s), X_n(t)) = \text{Cov}_{P_n}(X_n(1-t), X_n(1-s)) = \frac{1-t}{1-s}\mathbf{E}[X_n^2(s) \mid X_n(1) = 0],\]
and therefore
\[\frac{\mathbf{E}[X_n^2(s) \mid X_n(1) = 0]}{\mathbf{E}[X_n^2(t) \mid X_n(1) = 0]} = \frac{s(1-s)}{t(1-t)}.\]
Hence, there exists a constant \(C_n\) such that for all \(t \in \frac{1}{n} \mathbb{Z} \cap (0, 1)\)
\[\frac{\mathbf{E}[X_n^2(t) \mid X_n(1) = 0]}{t(1-t)} \equiv C_n.\]
Thus we have shown that for \(s \leq t\) in \(\frac{1}{n} \mathbb{Z} \cap [0, 1]\),
\[\text{Cov}_{P_n}(X_n(s), X_n(t)) = \frac{s}{t}\mathbf{E}[X_n^2(t) \mid X_n(1) = 0] = \frac{s}{t}C_n(1-t) = C_n s(1-t).\]
Now, consider the general case: 
\[s = s_0 + \frac{\epsilon_1}{n} \leq t = t_0 + \frac{\epsilon_2}{n}, \quad \text{where} \quad n s_0, n t_0 \in \mathbb{Z} \quad \text{and} \quad \epsilon_1, \epsilon_2 \in [0, 1).\]
Then the covariance
\[\text{Cov}_{P_n}(X_n(s), X_n(t)) = (1-\epsilon_1)(1-\epsilon_2) \text{Cov}_{P_n}(X_n(s_0), X_n(t_0)) + (1-\epsilon_1)\epsilon_2 \text{Cov}_{P_n}(X_n(s_0), X_n(t_0 + \frac{1}{n})) + \epsilon_1(1-\epsilon_2) \text{Cov}_{P_n}(X_n(s_0 + \frac{1}{n}), X_n(t_0)) + \epsilon_1\epsilon_2 \text{Cov}_{P_n}(X_n(s_0 + \frac{1}{n}), X_n(t_0 + \frac{1}{n}))\]
Therefore
\[ \text{Cov}_{P_n}(X_n(s), X_n(t)) = C_n s(1 - t) \text{ when } s_0 < t_0 \ (\lfloor ns \rfloor < \lfloor nt \rfloor), \]
and
\[ \text{Cov}_{P_n}(X_n(s), X_n(t)) = C_n s(1 - t) - C_n \frac{\epsilon_1(1 - \epsilon_2)}{n} \text{ when } s_0 = t_0 \ (\lfloor ns \rfloor = \lfloor nt \rfloor). \]

Now, plugging in \( s = t = \frac{1}{2} \) we get
\[ C_n = 4 \mathbb{E}[X_{n\left(\frac{1}{2}\right)}]|X_n(1) = 0] \text{ when } n \text{ is even}, \]
and
\[ C_n = 4 \mathbb{E}[X_{n\left(\frac{1}{2}\right)}]|X_n(1) = 0](\frac{n}{n - 1}) \text{ when } n \text{ is odd.} \]

Therefore
\[ C_n = 4 \mathbb{E}[X_{n\left(\frac{1}{2}\right)}]|X_n(1) = 0](1 + O\left(\frac{1}{n}\right)) \rightarrow C = \sigma^2 \]
as \( \{X_{n\left(\frac{1}{2}\right)}, P_n\} \) converges in distribution as \( n \rightarrow +\infty \).

\[ \square \]

**Proof of Lemma 2:** Before we begin the proof of tightness, we notice that the only real obstacle we face is that the process is conditioned on \( X_n = 0 \). The tightness for the case without the conditioning has been proved years ago as part of the Donsker’s Theorem (see Chapter 10 in [2]). With the help of the local CLT we are essentially removing the difference between the two cases.

Given a \( \lambda > 0 \) and let \( m = \lfloor n\delta \rfloor \) for a given \( 0 < \delta \leq 1 \), then for any \( \mu > 0 \),

\[
P_{\lambda} \equiv P\left[ \max_{0 \leq i \leq m} X_i \geq \lambda \sqrt{n} > X_m > -\lambda \sqrt{n} \mid X_n = 0 \right] \]
\[
= \sum_{a = -\lfloor \lambda \sqrt{n} \rfloor}^{\lfloor \lambda \sqrt{n} \rfloor} \frac{P[\max_{0 \leq i \leq m} X_i > \lfloor \lambda \sqrt{n} \rfloor \mid X_m = a \mid X_n = 0]}{P[X_n = 0]} \]
\[
= \sum_{a = -\lfloor \lambda \sqrt{n} \rfloor}^{\lfloor \lambda \sqrt{n} \rfloor} \frac{P[\max_{0 \leq i \leq m} X_i > \lfloor \lambda \sqrt{n} \rfloor \mid X_m = a]P[X_{n-m} = -a]}{P[X_n = 0]} \]
\[
\leq \max_{-\lfloor \lambda \sqrt{n} \rfloor \leq a \leq \lfloor \lambda \sqrt{n} \rfloor} \left( \frac{P[X_{n-m} = -a]}{P[X_n = 0]} \right) \times \sum_{a = -\lfloor \lambda \sqrt{n} \rfloor}^{\lfloor \lambda \sqrt{n} \rfloor} P[\max_{0 \leq i \leq m} X_i > \lfloor \lambda \sqrt{n} \rfloor \mid X_m = a] \]
\[
\leq 2P[\max_{0 \leq i \leq m} X_i \geq \lambda \sqrt{n} > X_m \geq -\lambda \sqrt{n}] \]

for \( n \) large enough, where by the local CLT,
\[
\max_{-\lfloor \lambda \sqrt{n} \rfloor \leq a \leq \lfloor \lambda \sqrt{n} \rfloor} \left( \frac{P[X_{n-m} = -a]}{P[X_n = 0]} \right) \leq 2
\]
for \( n \) large enough as \( n - m \) linearly depends on \( n \).

Therefore, the probability
\[
P[\max_{0 \leq i \leq m} |X_i| \geq \lambda \sqrt{n}|X_n = 0] \leq 2P_{\lambda} + P[|X_m| \geq \lambda \sqrt{n}|X_n = 0],
\]
where
\[
P_{\lambda} \leq 2P[\max_{0 \leq i \leq m} X_i \geq \lambda \sqrt{n}].
\]

Now, due to the point-wise convergence, we can proceed as in Chapter 10 of [2] by bounding the two remaining probabilities:
\[
P[\max_{0 \leq i \leq m} |X_i| \geq \lambda \sqrt{n}] \leq 2P[|X_m| \geq \frac{1}{2}\lambda \sqrt{n}] \rightarrow 2P[\sqrt{\delta}N \geq \frac{\lambda}{2\sigma}] \leq \frac{16\delta^{3/2}\sigma^3}{\lambda^3}\mathbb{E}[|N|^3]
\]
and similarly
\[
P[|X_m| \geq \lambda \sqrt{n}|X_n = 0] \rightarrow P[\sqrt{\delta(1-\delta)}N \geq \frac{\lambda}{\sigma}] \leq \frac{\delta^{3/2}\sigma^3}{\lambda^3}\mathbb{E}[|N|^3].
\]

Thus, for all integer \( k \in [0, n-m] \),
\[
P[\max_{0 \leq i \leq m} |X_{k+i} - X_k| \geq \lambda \sqrt{n}|X_n = 0] = P[\max_{0 \leq i \leq m} |X_i| \geq \lambda \sqrt{n}|X_n = 0] \leq \frac{70\delta^{3/2}\sigma^3}{\lambda^3}\mathbb{E}[|N|^3]
\]
for \( n \) large enough, (see Chapter 10 in [2]). Therefore \( \{P_n\} \) are tight (see Chapter 8 of [2]).

**Proof of Theorem 2:** The lemmas above imply the convergence when the mean \( \mu = 0 \). Now, for \( \mu \neq 0 \), there exists a \( \rho \in \mathbb{R} \) such that
\[
\sum_{z \in \mathbb{Z}} z e^{\rho z} P[Z_1 = z] = 0.
\]
Then we let \( \hat{Z}_1, \hat{Z}_2, \ldots \) be i.i.d. random variables with their distribution defined in the following fashion:
\[
P[\hat{Z}_j = z] = \frac{e^{\rho z}}{C_{\rho}} P[Z_j = z]
\]
for all \( j \) and \( z \in \mathbb{R} \), where \( C_{\rho} \equiv \sum_{z \in \mathbb{Z}} P[\hat{Z}_1 = z] = \sum_{z \in \mathbb{Z}} e^{\rho z} P[Z_1 = z] \). Then the law of \( Z_1, \ldots, Z_n \) conditioned on \( Z_1 + \ldots + Z_n = 0 \) is the same as that of \( \hat{Z}_1, \ldots, \hat{Z}_n \) conditioned on \( \hat{Z}_1 + \ldots + \hat{Z}_n = 0 \), and the case is reduced to that of \( \mu = 0 \) as \( \mathbb{E}[\hat{Z}_j] = 0 \). We also estimate the covariance equal to \( \hat{C} s(1-t) \) for all \( 0 \leq s \leq t \leq 1 \), where as before
\[
\hat{C} = \lim_{n \to +\infty} \mathbb{E}[Z_1^2 | Z_1 + \ldots + Z_n = 0].
\]
Observe that the result can be modified for $X_1, X_2, ...$ defined on a multidimensional lattice $L \subset \mathbb{R}^d, d > 1$, if we condition on $X_n(1) = a(n) = a + o(1) \in \{ z \sqrt{n} : z \in \bigoplus_1^n L \}$. We again let point zero be inside the closed convex hull of $\{ z : P[Z_1 = z] > 0 \}$. In this case the process $X_n(t) = X_n(t) + (a - a(n))t$ converges to the Brownian Bridge $B_{0, a}$, and convergence is uniform whenever $a(n)$ uniformly converges to zero thanks to the Local CLT.

**Theorem 3.** $\hat{X}_n(t)$ conditioned on $X_n(1) = a(n) = a + o(1)$ converges weakly to the Brownian Bridge.

Here, as before, if we take $t \in \frac{1}{n} \mathbb{Z} \cap [0, 1]$ and let $\alpha = \frac{t}{\sqrt{n}}$, then

\[
P[X_n(t) = \alpha \mid X_n(1) = a(n)] = \frac{\left( \frac{1}{\sqrt{n}} \Phi_{\sigma \sqrt{t}}(\alpha) + o\left(\frac{1}{\sqrt{n}}\right) \right) - \left( \frac{1}{\sqrt{n}} \Phi_{\sigma \sqrt{1-t}}(a(n) - \alpha) + o\left(\frac{1}{\sqrt{n}}\right) \right)}{\sqrt{n} \Phi_{\sigma}(a(n)) + o\left(\frac{1}{\sqrt{n}}\right)}
\]

\[
= \frac{1}{\sqrt{n}} \Phi_{\sigma \sqrt{1-t}}(\alpha - a(n)t) + o\left(\frac{1}{\sqrt{n}}\right).
\]

### 3.2 General Case.

As before, for a given non-zero vector $\vec{a} \in \mathbb{Z}^d$, we let $X_1, X_2, ...$ be i.i.d. random variables on $\mathbb{Z}^d$ with the span of the lattice distribution equal to one (see [4]) such that the probability $P[\vec{a} \cdot X_1 > 0] = 1$, the mean $\mu = E X_1 < \infty$ and there is a constant $\lambda > 0$ such that the moment-generating function

\[E(e^{\theta X_1}) < \infty\]

for all $\theta \in B_\lambda$. Also we let $P_{\vec{a}}$ denote the projection map on $< \vec{a} >$ and $P_{\vec{a}}^\perp$ denote the orthogonal projection on $< \vec{a} >^\perp$. Now we can decompose the mean $\mu = \mu_a \times \mu_{or}$, where $\mu_a \equiv P_{\vec{a}} \mu$ and $\mu_{or} \equiv P_{\vec{a}}^\perp \mu$.

As before we introduce a new basis $\{ \vec{f}_1, \vec{f}_2, ..., \vec{f}_d \}$, where $\vec{f}_1 = \frac{\vec{a}}{||\vec{a}||}$. We again use $[\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of a vector with respect to the new basis. We denote $X_i = [T_i, Z_i]_f \in \mathbb{Z} \times \mathbb{Z}^{d-1}$, where $[T_i, 0]_f = P_{\vec{a}} X_i$ and $[0, Z_i]_f = P_{\vec{a}}^\perp X_i$, and we let $X_1 + ... + X_i = [t_i, Y_i]_f \in \mathbb{Z} \times \mathbb{Z}^{d-1}$. Note: $T_i$ and $Z_i$ don’t have to be independent. Interpolating $Y_i$, we get

\[Y(t) = Y[t] + (t - [t])(Y[t+1] - Y[t])\]

for $0 \leq t \leq \infty$ and if we now define $Y_n(t) \equiv \frac{Y(n t)}{\sqrt{n}}$ for $0 \leq t \leq 1$, then the following theorem easily follows from the previous result:

**Corollary.** $Y_n(t)$ conditioned on $Y_n(1) = 0$ converges weakly to the Brownian Bridge.
want the same result for \( ET_i = \| \mu_a \| \) and \( \text{Var} T_i < \infty \).

We first let \( \bar{X} \equiv X_i - \mu_a \), then \( \text{E} \bar{X}_i = \mu_{ov} \) and \( \text{Var} \bar{X}_i < \infty \). We again interpolate:

\[
\bar{X}(t) = \bar{X}[t] + (t - [t])(\bar{X}[t+1] - \bar{X}[t])
\]

for \( 0 \leq t \leq \infty \), and scale \( \bar{X}_k(t) \equiv \frac{\bar{X}(kt)}{\sqrt{k}} \). Note: the last \( d - 1 \) coordinates of \( \bar{X}_k(t) \) w.r.t. the new basis are \( Y_k(t) \) (e.g. \( \mathbb{P}_{\bar{a}} \bar{X}_k(t) = [0, Y_k(t)] \)).

From here on we denote \( S_j \equiv [t_j, Y_j] = X_1 + \ldots + X_j \) and \( \bar{S}_j \equiv \bar{X}_1 + \ldots + \bar{X}_j = S_j - j\mu_a \) for any positive integer \( j \). As a first important step, we state another important

**Corollary.** For \( k = k(n) = \lfloor \frac{n\|a\|}{\mu_a} \rfloor + k_0 \sqrt{n} \), \( \{ \bar{X}_k(t) - (k_0 \sqrt{\frac{\mu_a}{\|a\|}} \mu_a + \frac{n\|a\| - k\mu_a}{\sqrt{k}}) t \} \) conditioned on \( \bar{X}_k(1) = n\|a\| - k\mu_a \) (e.g. \( [t_k, Y_k] = n\|a\| \)) converges weakly to the Brownian Bridge \( B^{0, -k_0 \sqrt{\frac{\mu_a}{\|a\|}} \mu_a} \).

Observe that \( n\|a\| - k\mu_a = -k_0 \sqrt{n}\mu_a + o(\sqrt{n}) \) and that the convergence is uniform for all \( k_0 \) in a compact set. Now, looking only at the last \( d - 1 \) coordinates of \( \bar{X}_k(t) \), w.r.t. the new basis the last Corollary implies:

**Lemma 3.** For \( k = k(n) = \lfloor \frac{n\|a\|}{\mu_a} \rfloor + k_0 \sqrt{n} \), \( Y_k(t) \) conditioned on \( t_k = n\|a\| \) and \( Y_k(1) = 0 \) converges weakly to the Brownian Bridge.

Note that convergence is uniform for \( k_0 \) in a compact set.

What the Lemma above says is the following: the interpolation of \( \left[ \frac{t}{k}, \frac{1}{\sqrt{k}} Y_i \right] \) conditioned on \( [t_k, Y_k] = n\|a\| \) converges to the Brownian Bridge. Now, define the process \( [t, Y_{n,k}(t)] \) to be the interpolation of \( \left[ \frac{t}{n\|a\|}, \frac{1}{\sqrt{n}} Y_i \right] \) for \( i = 0, 1, \ldots k \), then

**Theorem 4.** For \( k = k(n) = \lfloor \frac{n\|a\|}{\mu_a} \rfloor + k_0 \sqrt{n} \), \( \sqrt{n} Y_{n,k}(t) \) conditioned on \( t_k = n\|a\| \) and \( Y_k(t) = 0 \) converges weakly to the Brownian Bridge.

**Proof:** Here we observe that the mean \( \text{E}[\frac{t_j}{\|a\|} - \frac{j-1}{\|a\|}] \) is actually equal to \( \frac{\|a\|}{n\|a\|} = \frac{1}{k_0 \sqrt{n}} + o(\frac{1}{n}) \), and that for a given \( \epsilon > 0 \), the probability of the \( \| \left( \frac{t}{n\|a\|}, \frac{1}{\sqrt{n}} Y_i \right) - \left( \frac{t}{k_0 \sqrt{n}}, \frac{1}{\sqrt{k}} Y_i \right) \| = \| \frac{t}{n\|a\|} - \frac{t}{k_0 \sqrt{n}} \| \) exceeding \( \epsilon \) for some \( j \leq k \),

\[
P( \max_{0 \leq j \leq k} \| t_j - \frac{n\|a\|}{k} \| j \geq n\epsilon \mid S_n = n\|a\| ) \leq P( \max_{0 \leq j \leq k} \| S_j - \frac{n\|a\|}{k} \| j \geq n\epsilon \mid S_k = n\|a\| ) \leq P( \max_{0 \leq j \leq k} \| \bar{S}_j \| \geq n\epsilon \mid \bar{S}_k = \left[ n\|a\| - k\|\mu_a\|, 0 \right] ) \to 0
\]

as \( n \to +\infty \) since \( n\|a\| - k\|\mu_a\| = -\|\mu_a\|k_0 \sqrt{n} + o(\sqrt{n}) \).

Now, the next step is to prove that the process

\( \{ Y_{n,k} \} \) for some \( k \) such that \( [t_k, Y_k] = n\|a\| \)

conditioned on the existence of such \( k \) converges weakly to the Brownian Bridge.

First of all the last theorem implies
Lemma 4. For given \( k = k(n) = \left[ \frac{n\|a\|}{\|\mu_a\|} + k_0\sqrt{n} \right], \) \( Y^*_{n,k}(t) \) conditioned on \( t = n\|\bar{a}\| \) and \( Y_k(1) = 0 \) converges weakly to the Brownian Bridge.

For a fixed \( M > 0, \) convergence is also uniform on \( k \in \left[ \frac{n\|a\|}{\|\mu_a\|} - M\sqrt{n}, \frac{n\|a\|}{\|\mu_a\|} + M\sqrt{n} \right]. \) For the future purposes we denote \( \kappa \equiv \frac{\|\mu_a\|}{\|a\|} \) and \( I_M \equiv \left[ \frac{n\kappa}{\kappa} - M\sqrt{n}, \frac{n\kappa}{\kappa} + M\sqrt{n} \right] \cap \mathbb{Z}. \)

Finally, we want to prove the following technical result, in which we use the uniformity of convergence for all \( k = k(n) \in I_M \) and the truncation techniques to show the convergence of \( Y^*_{n,k} \) to the Brownian Bridge in case when we condition only on the existence of such \( k. \)

Technical Theorem. The process

\[ \{ Y^*_{n,k} \text{ for some } k \text{ such that } [t, Y_k]_f = n\vec{a} \} \]

conditioned on the existence of such \( k \) converges weakly to the Brownian Bridge.

Proof: Take \( M \) large, notice that for \( A \subset C^{d-1}[0,1], \)

\[ \max_{k \in I_M} \left| P[Y^*_{k} \in A \mid [t, Y_k]_f = n\vec{a}] - P[B^o \in A] \right| = o(1), \]

where the Brownian Bridge \( B^o \) is scaled up to the same constant for all those \( k. \)

Hence,

\[ \lim_{n \to +\infty} \frac{\sum_{k \in I_M} P[S_k = n\vec{a}] P[Y^*_{n,k} \in A \mid S_k = n\vec{a}]}{\sum_{k \in I_M} P[S_k = n\vec{a}]} = P[B^o \in A]. \]

Therefore we are only left to prove the truncation argument as \( M \to +\infty. \) Now, for any \( \epsilon > 0 \) there exists \( M > 0 \) such that

\[ (1 + \epsilon) \sum_{k \in I_M} P[S_k = n\vec{a}] \leq \sum_{k \in I_M} P[S_k = n\vec{a}] \leq (1 + 2\epsilon) \sum_{k \in I_M} P[S_k = n\vec{a}] \]

for \( n \) large enough, as by the large deviation upper bound, there is a constant \( \bar{C}_{LD} > 0 \) such that

\[ P[S_k = n\vec{a}] \leq e^{-C_{LD} \frac{(n-k\kappa)^2}{k} \wedge |n-k\kappa|}, \]

and therefore \( \exists C_{LD} > 0 \) such that

\[ \sum_{|n-k\kappa| > n^{2/3}} P[S_k = n\vec{a}] < e^{-C_{LD}n^{1/3}}. \]

Also, by the local CLT,

\[ P[S_k = n\vec{a}] = P[\bar{S}_k = (n-k\kappa)\vec{a}] = \frac{1}{k^{d/2} \sqrt{Var X_1(2\pi)^d}} e^{-\frac{1}{2Var X_1} \frac{(n-k\kappa)^2}{k}} + o\left( \frac{1}{k^{d/2}} \right) \]
implying
\[
\sum_{|n-k\ell| \leq n^{2/3}} P[S_k = n\vec{a}] = \frac{1}{n^{d-1}} \left[ \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\text{Var} X_1(2\pi)^d}} e^{-x^2/2\text{Var} X_1} dx + o(1) \right]
\]
where
\[
\sum_{k \in I_M} P[S_k = n\vec{a}] = \frac{1}{n^{d-1}} \left[ \int_{-M}^{M} \frac{1}{\sqrt{\text{Var} X_1(2\pi)^d}} e^{-x^2/2\text{Var} X_1} dx + o(1) \right].
\]
Therefore
\[
1 + 2\epsilon \sum_{k \in I_M} P[S_k = n\vec{a}] P[Y_{n,k}^* \in A | S_k = n\vec{a}] \leq \frac{\sum_k P[S_k = n\vec{a}] P[Y_{n,k}^* \in A | S_k = n\vec{a}]}{\sum_k P[S_k = n\vec{a}]} \leq \frac{1}{1 + \epsilon} \frac{\sum_{k \in I_M} P[S_k = n\vec{a}] P[Y_{n,k}^* \in A | S_k = n\vec{a}]}{\sum_{k \in I_M} P[S_k = n\vec{a}]}
\]
for all \( A \subset C^{d-1}[0,1] \). Taking the lim inf and lim sup of the fraction in the middle completes the proof.

\[\square\]

Acknowledgements

The author wishes to thank D.Ioffe, who posed the problem, and A.Dembo for providing him with valuable and insightful comments and suggestions concerning the matter of this research.

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