A Gauss–Kuz’min–Lévy theorem for Rényi-type continued fractions

by

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1. Introduction. The present paper continues a series of papers dedicated to Rényi-type continued fraction expansions [LS19, SL19]. Actually, these continued fractions are a particular case of u-backward continued fractions studied by Gröchenig and Haas [GH96]. In 1957, Rényi [R57] showed that every irrational number $x \in [0, 1)$ has an infinite continued fraction expansion of the form

$$x = 1 - \frac{1}{n_1 - 1} - \frac{1}{n_2 - 1} - \frac{1}{n_3 - 1} - \cdots =: [n_1, n_2, n_3, \ldots]_b,$$

where each $n_i$ is an integer greater than 1. We call the expansion in (1.1) a backward continued fraction. The underlying dynamical system is the Rényi map $R$ defined from $[0, 1)$ to $[0, 1)$ by

$$R(x) := \frac{1}{1 - x} - \left\lfloor \frac{1}{1 - x} \right\rfloor,$$

which has a neutral fixed point at 0 and thus is nonuniformly hyperbolic. Rényi showed that the infinite measure $dx/x$ is invariant for $R$. This map does not possess a finite absolutely continuous invariant measure, and the usual inducing trick to study its thermodynamic formalism does not work.

Unlike (1.2), the Gauss map defined from $(0, 1]$ to $(0, 1]$ by

$$G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,$$

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which generates the well-known regular continued fraction expansion, does have a finite absolutely continuous invariant measure, namely the Gauss measure $dx/(x + 1)$. The Gauss map is uniformly expanding, but has infinitely many branches. The graph of $R$ can be obtained from that of $G$ by reflecting the latter in the line $x = 1/2$ (see Figure 1). It is for this reason that the continued fraction (1.1) has been called “backward”.

![Graphs of Gauss (piecewise decreasing, red in the pdf file) and Rényi (piecewise increasing, blue) transformations](image)

Fig. 1. Graphs of Gauss (piecewise decreasing, red in the pdf file) and Rényi (piecewise increasing, blue) transformations

Starting from the expansion in (1.1) and the Rényi transformation $R$, Gröchenig and Haas [GH96] define the family of maps $T_u(x) := \frac{1}{u(1-x)} - \lfloor \frac{1}{u(1-x)} \rfloor$, where $u > 0$, $x \in [0,1)$. Given $u \in (0,4)$ and $x \in [0,1)$, $x$ has the $u$-backward continued fraction expansion

(1.4) \[ x = 1 - \frac{1}{un_1 - \frac{1}{n_2 - \frac{1}{un_3 - \frac{1}{n_4 - \cdots}}}} =: [a_1, a_2, a_3, \ldots]_u, \]

where the integers $n_i = 1 + a_i$ are $\geq 2$ and the coefficient of $n_i$ is 1 or $u$, depending on the parity of $i$. In the particular case $u = 1/N$, for a positive integer $N \geq 2$, they have identified a finite absolutely continuous invariant measure for $T_u$, namely $dx/(x + N - 1)$. For $u = 1/N$, where $N \geq 2$ is an integer, we will call $T_u$ the Rényi-type continued fraction transformation and denote it by $R_N$.

The metrical theory of this algorithm was initiated in [LS19]. The first known metrical problem concerning (regular) continued fractions is due to Gauss. At the start of the 20th century an old discovery of Gauss was shown
to tie the theory of continued fractions to probability theory and ergodic theory. In 1802 and 1812 Gauss found the invariant measure of the transformation \( G \) in (1.3) underlying the regular continued fraction, and asked Lagrange in a letter in 1812 how fast \( \lambda(G^{-n}([0, x])) \) converges to the invariant measure \( \gamma([0, x]) = \log(1 + x)/\log 2 \). Here \( \lambda \) is the Lebesgue measure. In 1928, Kuz’min [K28] answered Gauss’ question by giving an estimate of the remainder. Independently in 1929 Paul Lévy [L29] improved Kuz’min’s result and published another proof. In the 60’s Szüsz [S61] was able to prove the same result by using Kuz’min’s approach.

In [LS19] we started an approach to the metrical theory of Rényi-type continued fraction expansions via dependence with complete connections. More precisely, we obtained a version of the Gauss–Kuz’min theorem for these expansions by applying the theory of random systems with complete connections, due to Iosifescu [IG09]. Once a finite ergodic measure \( \rho_N \) is obtained for the map \( R_N \) (recall that \( R_N \) is ergodic and measure preserving under \( \rho_N \)), classical results of ergodic theory, such as the Birkhoff ergodic theorem, yield precise information on the frequency with which a digit occurs. It should be stressed that the ergodic theorem does not yield any information on the convergence rate in the Gauss–Kuz’min problem that amounts to determining the asymptotic behavior of \( \mu(R_N^{-n}) \) as \( n \to \infty \), where \( \mu \) is an arbitrary probability measure. So that a Gauss–Kuz’min theorem is needed. Using the natural extensions for Rényi-type transformations, we obtained an infinite-order-chain representation \((\bar{a}_\ell)_{\ell \in \mathbb{Z}}\) of the sequence \((a_n)_{n \in \mathbb{N}}\) of incomplete quotients of these expansions. Then we showed that the associated random systems with complete connections are with contraction and their transition operators are regular with respect to the Banach space of Lipschitz functions. This allowed us to solve a variant of the Gauss–Kuz’min problem. In this result the constants involved are far from optimal. So the hunt for the best possible constants started.

In [SL19] we used a Wirsing-type approach [W74] to get close to the optimal convergence rate. Let us notice that Wirsing’s Gauss–Kuz’min–Lévy theorem leads to a good estimation of the convergence rate to 0 as \( n \to \infty \) of the \( \psi \)-mixing coefficients of the incomplete quotients \((a_n)_{n \in \mathbb{N}}\) under \( \rho_N \) or of the extended versions \((\bar{a}_\ell)_{\ell \in \mathbb{Z}}\) under the extended measure \( \bar{\rho}_N \) (see [LS19]). Paralleling the treatment in the case of regular continued fractions, the Gauss–Kuz’min–Lévy problem for the transformation \( R_N \) can be approached in terms of the Perron–Frobenius operator under the invariant measure induced by the limit distribution function. By restricting the domain of the associated Perron–Frobenius operator of \( R_N \) under its invariant measure \( \rho_N \) to the Banach space of functions which have a continuous derivative on \([0, 1]\), we obtained upper and lower bounds of the error which provide a refined estimate of the convergence rate. For example, in case \( N = 100 \), the upper and lower bounds of the convergence rate are respec-
tively $O(w_{100}^n)$ and $O(v_{100}^n)$ as $n \to \infty$, with $v_{100} > 0.00503350150708559$ and $w_{100} < 0.00503358526129032$.

The purpose of this paper is to continue our investigation on the asymptotic behavior of the distribution functions of Rényi-type transformations. In order to prove a Gauss–Kuz’mín–Lévy-type theorem for Rényi-type continued fraction expansions, we apply the method of Szüsz \[S61\]. We mention that using this method, we obtain more information on the convergence rate involved. The main novelty of this paper is the explicit expression in terms of Hurwitz zeta functions of $\eta_N$ that appears in Theorem 3.1. In addition, the estimate we have for $\eta_N$ shows that $\eta_N \to 0$ as $N \to \infty$. Finally, to enable direct comparisons of the results obtained in the last two methods (Wirsing and Szüsz), we give upper and lower bounds of $\eta_N$ for $N = 100:
0.00505050495049505 < \eta_{100} < 0.0050753806723955975$.

2. Rényi-type continued fractions. In this section we briefly present known results about Rényi-type continued fractions.

Fix an integer $N \geq 2$. The Rényi-type continued fraction transformation $R_N : [0, 1] \to [0, 1]$ is given by

\begin{equation}
R_N(x) = \frac{N}{1 - x} - \left\lfloor \frac{N}{1 - x} \right\rfloor, \quad x \neq 1; \quad R_N(1) = 0.
\end{equation}

For any irrational $x \in [0, 1]$, $R_N$ generates a new continued fraction expansion of $x$ of the form

\begin{equation}
x = 1 - \frac{N}{1 + a_1 - \frac{N}{1 + a_2 - \frac{N}{1 + a_3 - \cdots}}} =: [a_1, a_2, a_3, \ldots]_{R}.
\end{equation}

Here, $a_n$’s are nonnegative integers greater than or equal to $N$ defined by

\begin{equation}
a_1 := a_1(x) = \left\lfloor \frac{N}{1 - x} \right\rfloor, \quad x \neq 1; \quad a_1(1) = \infty,
\end{equation}

and

\begin{equation}
a_n := a_n(x) = a_1(R_N^{n-1}(x)), \quad n \geq 2,
\end{equation}

with $R_N^0(x) = x$.

The rational approximants to $x$ arise in a manner similar to that in the case of other continued fraction algorithms. In particular we define two integer sequences by $p_0 = 1$, $q_0 = 1$, $p_1 = 1 + a_1 - N$, $q_1 = 1 + a_1$,

\begin{equation}
p_n = (1 + a_n)p_{n-1} - Np_{n-2} \quad \text{and} \quad q_n = (1 + a_n)q_{n-1} - Nq_{n-2}
\end{equation}

for $n \geq 2$. A simple inductive argument gives

\begin{equation}
p_{n-1}q_n - p_nq_{n-1} = N^n, \quad n \in \mathbb{N}_+ := \{1, 2, \ldots\}.
\end{equation}
The rationals \( p_n/q_n, n \in \mathbb{N}_+ \), are the convergents to \( x \) in \([0, 1]\). In [GH96] it was shown that the dynamical system \(([0, 1], \mathcal{B}_{[0,1]}, R_N, \rho_N)\) is measure preserving and ergodic. Here, \( \mathcal{B}_{[0,1]} \) denotes the \( \sigma \)-algebra of all Borel subsets of \([0, 1]\), and the probability measure \( \rho_N \) is defined by

\[
(2.5) \quad \rho_N(A) := \frac{1}{\log \left( \frac{N}{N-1} \right)} \int_A \frac{dx}{x + N - 1}, \quad A \in \mathcal{B}_{[0,1]}.
\]

In [LS19] we investigated the Perron–Frobenius operator of \( R_N \) on the measurable space \(([0, 1], \mathcal{B}_{[0,1]}, \mu)\) such that the probability measure \( \mu \) satisfies \( \mu(R_N^{-1}(A)) = 0 \) whenever \( \mu(A) = 0 \) for \( A \in \mathcal{B}_{[0,1]} \). In particular, we studied the Perron–Frobenius operator \( U \) of \(([0, 1], \mathcal{B}_{[0,1]}, \rho_N, R_N)\), that is, \( U \) is a unique operator on \( L^1([0, 1], \rho_N) := \{ f : [0, 1] \to \mathbb{C} : \int_0^1 |f| \, d\rho_N < \infty \} \) which satisfies

\[
(2.6) \quad \int_A U f \, d\rho_N = \int_{R_N^{-1}(A)} f \, d\rho_N \quad \text{for any } A \in \mathcal{B}_{[0,1]}, f \in L^1([0, 1], \rho_N).
\]

Also, we have found an explicit formula for the Perron–Frobenius operator under the invariant measure \( \rho_N \), namely

\[
(2.7) \quad U f(x) = \sum_{i \geq N} P_{N,i}(x) f(u_{N,i}(x)), \quad f \in L^1([0, 1], \rho_N),
\]

where \( P_{N,i} \) and \( u_{N,i} \) are functions defined on \([0, 1]\) by

\[
(2.8) \quad P_{N,i}(x) := \frac{x + N - 1}{(x + i)(x + i - 1)},
\]

\[
(2.9) \quad u_{N,i}(x) := 1 - \frac{N}{x + i}.
\]

A more thorough account of Rényi-type continued fractions can be found in [GH96, LS19, SL19].

3. Main result. In this section we show our main theorem. Let \( \mu \) be a nonatomic probability measure on \( \mathcal{B}_{[0,1]} \) and define

\[
(3.1) \quad F_{N,0}(x) := \mu([0, x]), \quad x \in [0, 1],
\]

\[
(3.2) \quad F_{N,n}(x) := \mu(R_N^n \leq x), \quad x \in [0, 1], \quad n \in \mathbb{N}_+.
\]

**Main Theorem 3.1** (A Gauss–Kuz’mín–Lévy-type theorem). Let \( R_N \) and \( F_{N,n} \) be as in (2.1) and (3.2). Then there exists a constant \( 0 < \eta_N < 1 \) such that \( F_n \) can be written as

\[
(3.3) \quad F_{N,n}(x) = \frac{1}{\log \left( \frac{N}{N-1} \right)} \log \left( \frac{x + N - 1}{N - 1} \right) + \mathcal{O}(\eta_N^n)
\]

uniformly with respect to \( x \in [0, 1] \).
Remark 3.2. From (3.3), we see that
\begin{equation}
\lim_{n \to \infty} F_{N,n}(x) = \rho_N([0, x]),
\end{equation}
where $\rho_N$ is the measure defined in (2.5). In fact, Theorem 3.1 estimates the error
\begin{equation}
e_{N,n}(x) = e_{N,n}(x, \mu) = \mu(R_N^n \leq x) - \rho_N([0, x]), \quad x \in [0, 1].
\end{equation}
To prove Theorem 3.1 we need the following results.

Lemma 3.3. For functions $F_{N,n}$ in (3.2), the following Gauss–Kuz’mintype equation holds:
\begin{equation}
F_{N,n+1}(x) = \sum_{i \geq N} \left\{ F_{N,n}\left(1 - \frac{N}{x+i}\right) - F_{N,n}\left(1 - \frac{N}{i}\right) \right\}
\end{equation}
for $x \in [0, 1]$ and $n \in \mathbb{N}$.

Proof. From (2.1) and (2.4), we see that
\begin{equation}
R_N^n(x) = 1 - \frac{N}{a_{n+1} + R_N^{n+1}}, \quad n \in \mathbb{N}_+.
\end{equation}
Now,
\begin{align*}
F_{N,n+1}(x) &= \mu(R_N^{n+1} \leq x) = \sum_{i \geq N} \mu\left(1 - \frac{N}{i} \leq R_N^{n+1} \leq 1 - \frac{N}{i + x}\right) \\
&= \sum_{i \geq N} \left\{ F_{N,n}\left(1 - \frac{N}{x+i}\right) - F_{N,n}\left(1 - \frac{N}{i}\right) \right\}. \quad \blacksquare
\end{align*}

Remark 3.4. Assume that for some $p \in \mathbb{N}$, the derivative $F'_{N,p}$ exists everywhere in $[0, 1]$ and is bounded. Then it is easy to see by induction that $F'_{N,p+n}$ exists and is bounded for all $n \in \mathbb{N}_+$. This allows us to differentiate (3.6) term by term, obtaining
\begin{equation}
F'_{N,n+1}(x) = \sum_{i \in \mathbb{N}} \frac{N}{(x+i)^2} F'_{N,n}\left(1 - \frac{N}{x+i}\right).
\end{equation}

We introduce functions $f_{N,n}$ as follows:
\begin{equation}
f_{N,n}(x) := (x + N - 1)F'_{N,n}(x), \quad x \in [0, 1], \quad n \in \mathbb{N}.
\end{equation}
Then (3.8) is
\begin{equation}
f_{N,n+1}(x) = \sum_{i \geq N} P_{N,i}(x) f_{N,n}(u_{N,i}(x)),
\end{equation}
where $P_{N,i}(x)$ and $u_{N,i}(x)$ are given in (2.8) and (2.9), respectively.
Lemma 3.5. For \( \{f_{N,n}\} \) in (3.9), define \( M_{N,n} := \max_{x \in [0,1]} |f'_{N,n}(x)| \). Then

\[
M_{N,n+1} \leq \eta_N \cdot M_{N,n}
\]

where

\[
\eta_N = \sum_{i \geq N} \left( \frac{1}{i^3} + \frac{N}{i^2(i+1)} \right).
\]

Proof. Since

\[
P_{N,i}(x) = \frac{i + 1 - N}{x + i} - \frac{i - N}{x + i - 1},
\]

we have

\[
f'_{N,n+1}(x) = \sum_{i \geq N} \left\{ P'_{N,i}(x) f_{N,n}(u_{N,i}(x)) + P_{N,i}(x) f'_{N,n}(u_{N,i}(x)) \right\}
\]

\[
= \sum_{i \geq N} \left\{ \left( \frac{i - N}{(x + i - 1)^2} - \frac{i + 1 - N}{(x + i)^2} \right) f_{N,n}(u_{N,i}(x))\right.
\]

\[
+ P_{N,i}(x) f'_{N,n}(u_{N,i}(x)) \frac{N}{(x + i)^2} \right\}
\]

\[
= \sum_{i \geq N} \left\{ \frac{i + 1 - N}{(x + i)^2} \left[ f_{N,n}(u_{N,i+1}(x)) - f_{N,n}(u_{N,i}(x)) \right] \right.
\]

\[
+ P_{N,i}(x) f'_{N,n}(u_{N,i}(x)) \frac{N}{(x + i)^2} \right\}
\]

\[
= \sum_{i \geq N} \left\{ \frac{i + 1 - N}{(x + i)^3(x + i + 1)} f'_{N,n}(\theta_i) \right.
\]

\[
+ f'_{N,n}(u_{N,i}(x)) \frac{NP_{N,i}(x)}{(x + i)^2} \right\}
\]

where \( u_{N,i+1}(x) < \theta_i < u_{N,i}(x) \). Now (3.13) implies

\[
M_{N,n+1} \leq M_{N,n} \cdot \max_{x \in [0,1]} \left( \sum_{i \geq N} \frac{i + 1 - N}{(x + i)^3(x + i + 1)} \right)
\]

\[
+ N \sum_{i \geq N} \frac{x + N - 1}{(x + i)^3(x + i - 1)}.
\]

We must now calculate the maximum value of the sums in this expression. Since \( x \in [0,1] \) and \( i \geq N \), we get

\[
\frac{i + 1 - N}{(x + i)^3(x + i + 1)} \leq \frac{i + 1 - N}{i^3(i+1)}
\]

and

\[
\frac{x + N - 1}{(x + i)^3(x + i - 1)} \leq \frac{1}{(x + i)^3} \leq \frac{1}{i^3}.
\]
Thus,

\[ M_{N,n+1} \leq M_{N,n} \cdot \sum_{i \geq N} \left( \frac{1}{i^3} + \frac{N}{i^2(i+1)} \right) \]

and the proof is complete. ■

**Proof of Theorem 3.1** Let \( \{F_{N,n}\} \) and \( \{f_{N,n}\} \) be as in (3.2) and (3.9), respectively. Then

\[ F'_{N,n}(x) = \frac{1}{x+N-1} f_{N,n}(x), \quad x \in [0,1], n \in \mathbb{N}. \]

If we can show the existence of a constant \( 0 < \eta_N < 1 \) such that

\[ f_{N,n}(x) = \frac{1}{\log(\frac{N}{N-1})} + \mathcal{O}(\eta_N^n), \]

then integrating (3.16) we will establish (3.3). To demonstrate that \( f_{N,n}(x) \) has the desired form, it suffices to establish that \( f'_{N,n}(x) = \mathcal{O}(\eta_N^n) \), as the \( 1/\log(\frac{N}{N-1}) \) constant in (3.17) will follow from the normalization requirement that \( F_{N,n}(0) = 0 \) and \( F_{N,n}(1) = 1 \).

It remains to prove the following lemma.

**Lemma 3.6.** For every integer \( N \geq 2 \) there exists a constant \( 0 < \eta_N < 1 \) such that

\[ f'_{N,n}(x) = \mathcal{O}(\eta_N^n), \quad x \in [0,1], n \in \mathbb{N}. \]

Moreover, for any integer \( N \geq 2 \),

\[ \frac{1}{N^3} + \frac{1}{2N(N+1)} + \frac{1}{2N} < \eta_N < \frac{1}{2N(N-1)} + \frac{1}{N} - \frac{1}{2N+1}. \]

**Proof.** Let \( \eta_N \) be as in Lemma 3.5. Using that lemma, to show (3.18) it is enough to prove that \( \eta_N < 1 \). First, we will write \( \eta_N \) in terms of Hurwitz zeta functions:

\[ \eta_N = \sum_{i \geq N} \left( \frac{1}{i^3} + \frac{N}{i^2(i+1)} \right) = \sum_{i \geq N} \left( \frac{1}{i^3} + \frac{N}{i^2} \right) - 1 = \zeta(3, N) + N\zeta(2, N) - 1. \]

For \( i \geq N \) and \( a := \frac{1}{2}(\sqrt{4N^2 + 1} - (2N+1)) > -\frac{1}{2} \), we have

\[ a^2 + (2N+1)a + N = 0 \]

and

\[ a^2 + (2N+1)a + N + (2a+1)(i - N) \geq 0, \]

i.e.,

\[ i^2 \leq (i + a)(i + 1 + a). \]
Hence,
\[ \zeta(2, N) \geq \sum_{i \geq N} \left( \frac{1}{i + a} - \frac{1}{i + 1 + a} \right) = \frac{1}{N + a} = \frac{2}{\sqrt{4N^2 + 1} - 1}. \]

Also we have
\[ \zeta(2, N) < \frac{1}{N^2} + \sum_{i \geq N+1} \frac{1}{(i - 1/2)(i + 1/2)} = \frac{1}{N^2} + \frac{2}{2N + 1}. \]

For \( i \geq N \) and \( b := N(\sqrt{N^2 + 1} - N) < \frac{1}{2} \), we have
\[ b^2 + 2N^2b - N^2 = 0 \]
and
\[ b^2 + 2N^2b - N^2 + (2b - 1)(i^2 - N^2) \leq 0, \]
i.e.,
\[ i^4 \geq (i^2 - i + b)(i^2 + i + b). \]

Hence
\[ \zeta(3, N) < \frac{1}{2} \sum_{i \geq N} \left( \frac{1}{(i - 1)i + b} - \frac{1}{i(i + 1) + b} \right) = \frac{1}{2(N^2 - N + b)} \]
\[ = \frac{1}{2N(\sqrt{N^2 + 1} - 1)}. \]

Also, we have
\[ \zeta(3, N) > \frac{1}{N^3} + \frac{1}{2} \sum_{i \geq N+1} \left( \frac{1}{i^2 - i + 1/2} + \frac{1}{i^2 + i + 1/2} \right) \]
\[ = \frac{1}{N^3} + \frac{1}{2(N^2 + N + 1/2)}. \]

Therefore,
\[ \eta_N < \frac{1}{2N(\sqrt{N^2 + 1} - 1)} + N \left( \frac{1}{N^2} + \frac{2}{2N + 1} \right) - 1 \]
\[ < \frac{1}{2N(N - 1)} + \frac{1}{N} - \frac{1}{2N + 1}, \]
and
\[ \eta_N > \frac{1}{N^3} + \frac{1}{2(N^2 + N + 1/2)} + \frac{2N}{\sqrt{4N^2 + 1} - 1} - 1 \]
\[ > \frac{1}{N^3} + \frac{1}{2(N^2 + N + 1/2)} + \frac{2N}{2N + 1} - 1 \]
\[ = \frac{1}{N^3} + \frac{1}{2(N^2 + N + 1/2)} + \frac{1}{2N} > \frac{1}{N^3} + \frac{1}{2N(N + 1)} + \frac{1}{2N}. \]
For example, we have

| N     | Lower bound of $\eta_N$ | Upper bound of $\eta_N$ |
|-------|-------------------------|-------------------------|
| 2     | 0.4583333333333333      | 0.55                    |
| 10    | 0.055545454545454544    | 0.05793650793650794     |
| 100   | 0.00505050495049505     | 0.0050753806723955975   |
| 500   | 0.001002004007984032    | 0.001003003009015033    |
| 1000  | 0.0005005005004995005   | 0.0005007503755629693   |
| 5000  | 0.00010002000400079984  | 0.00010003000300090015  |
| 10000 | 0.00005000500050004999  | 0.000050007500375056254 |

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