Partial Resolutions of Orbifold Singularities via Moduli Spaces of HYM-type Bundles

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Abstract. Let $\Gamma$ be a finite group acting linearly on $\mathbb{C}^n$, freely outside the origin, and let $N$ be the number of conjugacy classes of $\Gamma$ minus one.

A construction of Kronheimer \cite{Kro89} of moduli spaces $X_\zeta$ of translation-invariant $\Gamma$-equivariant instantons on $\mathbb{C}^2$ is generalised to $\mathbb{C}^n$.

The moduli spaces $X_\zeta$ depend on a parameter $\zeta \in \mathbb{Q}^N$. The following results are proved: for $\zeta = 0$, $X_0$ is isomorphic to $\mathbb{C}^n/\Gamma$; if $\zeta \neq 0$, the natural maps $X_\zeta \to X_0$ are partial resolutions. The moduli $X_\zeta$ are furthermore shown to admit Kähler metrics which are Asymptotically Locally Euclidean (ALE).

A description of the singularities of $X_\zeta$ using deformation complexes is given, and is applied in particular to the case $\Gamma \subset SU(3)$. It is conjectured that for general $\Gamma$ and generic $\zeta$ that the singularities of $X_\zeta$ are at most quadratic. When $\Gamma \subset SU(3)$ a natural holomorphic 3-form is constructed on the smooth locus of $X_\zeta$, which is conjectured to be non-vanishing. The morphisms $X_\zeta \to X_0$ are expected to be crepant resolutions and $X_\zeta$ to be smooth for generic choices of the parameter $\zeta$. Related open problems in higher-dimensional complex geometry are also mentioned.

The paper has a companion paper \cite{SI96b} which identifies the moduli $X_\zeta$ with representation moduli of McKay quivers, and describes them completely in the case of abelian groups.

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0. Introduction

This paper is concerned with affine orbifold singularities, namely with singularities of the type \( X = \mathbb{C}^n/\Gamma \) for \( \Gamma \) a finite group acting linearly on \( \mathbb{C}^n \).

More precisely, this paper gives a method for constructing partial resolutions of \( X \), namely birational morphisms \( Y \rightarrow X \) which are isomorphisms over the smooth locus of \( X \).

0.1. Background. The method in question was first introduced by Kronheimer [Kro89]. It can be described in various ways, depending on one’s point of view. One description (although maybe not the most straight-forward) is to construct moduli spaces \( X_\zeta \) of instantons on the
trivial bundle \( \mathbb{C}^n \times R \to \mathbb{C}^n \). Here \( R \) denotes the regular representation space for the group \( \Gamma \), and \( \zeta \) is a linearisation of the bundle action. The instantons are required to satisfy Hermitian-Yang-Mills-type equations, as well as additional \( \Gamma \)-equivariance and translation-invariance properties.

In Kronheimer’s case, \( \Gamma \subset SU(2) \), and the moduli spaces \( X_\zeta \) can in fact be viewed as hyper-Kähler quotients. Kronheimer shows that \( X_0 \) is isomorphic to \( \mathbb{C}^n/\Gamma \), that the natural maps \( X_\zeta \to X_0 \) are partial resolutions for \( \zeta \neq 0 \), and that indeed \( X_\zeta \) coincides with the minimal resolution of \( \mathbb{C}^n/\Gamma \) for generic choices of \( \zeta \). Furthermore, \( X_\zeta \) inherit natural hyper-Kähler metrics on their non-singular locus, which are shown to be Asymptotically Locally Euclidean (ALE): they asymptotically approximate the Euclidean metric at infinity (up to terms vanishing with the inverse of the fourth power of the radial coordinate).

0.2. Main Results. In the present case \( n \) is any integer greater than or equal to 2, and \( \Gamma \subset U(n) \) is assumed to act on \( \mathbb{C}^n \) freely outside the origin for any \( n \geq 2 \), which means that \( X = \mathbb{C}^n/\Gamma \) has an isolated singularity. As a result, the moduli \( X_\zeta \) are only Kähler rather than hyper-Kähler quotients (in actual fact they are more conveniently described in term of geometric invariant theory). The main result is

**Theorem** (c.f. Thms. 4.2, 4.7 and 5.1 in the text). Let \( \Gamma \) act linearly on \( \mathbb{C}^n \) and freely outside the origin and let \( X_\zeta \) be the moduli spaces constructed in Section 3.5.

Then \( X_0 \) is isomorphic to \( X = \mathbb{C}^n/\Gamma \), and for \( \zeta \neq 0 \), the natural morphisms \( X_\zeta \to X_0 \) are partial resolutions. Furthermore, the inherited Kähler metrics on the smooth loci of \( X_\zeta \) are Asymptotically Locally Euclidean in the sense of [Kro89].

0.3. Two Conjectures. The final sections of the paper discuss and develop two conjectures.

**Conjecture** (c.f. Conj. 1). The singularities of \( X_\zeta \) (for generic \( \zeta \), say) are at most quadratic algebraic.

This is a common occurrence for moduli spaces of this kind [Nad88, GM87, GM90]. Its proof can be reduced to proving the formality of a certain differential graded Lie algebra (DGLA) by the methods of [GM90]. This is done in Section 3 where the singularities of \( X_\zeta \) are described in terms of deformation complexes [AHS77, DK90]. The concept of formality is explained, and it is suggested that the complex

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1This is for the purpose of simplicity — the method would seem to be applicable to the general case with some modifications
relevant to $X_\zeta$ may be formal, in a way similar to Tian [Tia87] and Todorov’s [Tod89] work. This would imply that the singularities of $X_\zeta$ at most quadratic algebraic. This conjecture is also checked by computer for low order abelian groups in $U(3)$ using the methods in the companion paper [SI96b].

Conjecture (c.f. Conj. 2 in the main text). If $\Gamma \subset SU(3)$, the morphisms $X_\zeta \to X_0$ are crepant, and if $\zeta$ is generic, $X_\zeta$ is smooth and its Euler number is equal to the orbifold Euler number of $X_0$ as defined in [DHVV85].

The fact that $X_\zeta$ has at most quadratic singularities has been verified for the abelian subgroups of order less than 11. The smoothness of $X_\zeta$ has been verified in the abelian cases $\frac{1}{3}(1,1,1)$, $\frac{1}{6}(1,2,3)$, $\frac{1}{7}(1,2,4)$, $\frac{1}{8}(1,2,5)$, $\frac{1}{9}(1,2,6)$, $\frac{1}{10}(1,2,7)$ and $\frac{1}{11}(1,2,8)$. Both these verifications were done by a brute-force listing of singularities of $X_\zeta$ for all possible $\zeta$, using the methods given in the companion paper [SI96b].

The cases $\Gamma \subset SU(n)$ present a particular interest. The problem of constructing a crepant resolution of $C^3/\Gamma$ with the same orbifold Euler number was only recently completed [MOP87, Roa91, Mar93, Roa93, If94, Roa90]. For the case $C^4/\Gamma$, one can obtain some interesting analogous results if one considers terminalisations rather than resolutions [SI96a].

In Section 7, a natural holomorphic 3-form is constructed on the smooth locus of $X_\zeta$: this is conjectured to be non-vanishing. Its norm is shown to be constant if and only if the induced metric on $X_\zeta$ is Ricci-flat (which does not usually turn out to be the case, however).

0.4. Related Questions. This paper has a companion paper [SI96b] in which the moduli $X_\zeta$ are identified with representation moduli of McKay quivers. This allows one to explicitly describe the case of Abelian $\Gamma$ in terms of “flows” on the McKay quiver. Explicit computations are carried out for groups of low order, and the conjectures about the smoothness and the triviality of the canonical bundle are checked (by brute-force computer calculations) for abelian subgroups of SU(3) of order less than or equal to 11.

Many questions are left open by the present work, besides the conjectures already mentioned. For instance, do the birational models $X_\zeta$ of $C^n/\Gamma$ possess any special properties with regards to their singularities (are they terminal?) Are all terminal models for a given 3-fold singularity obtained by this construction? What is the relationship between the different $X_\zeta$? Are they related by flips/flops? Is it possible, by choosing very special values of $\zeta$, to produce blowups $X_\zeta \to X_0$ which
are interesting from the point of view of higher dimensional geometry, for instance, for the construction of flips in dimensions 4 and greater?

0.5. **Methods.** The methods used include geometric invariant theory, Kähler quotients, and elementary theory of the moduli of bundles, the necessary aspects of which are reviewed in Sections 1, 2 and 3. Furthermore, the same construction is presented under different angles with the intention that the reader who is familiar with one of them (or with Kronheimer’s work [Kro89]) will be able to follow the discussion easily.

The later sections devoted to the various conjectures raised by the main results touch on the theory of deformation complexes, and concepts of Kuranishi germs, formality, and so on. Some background is also provided, although not as extensive as to be able to describe it as “self-contained”, given the conjectural nature of the material.

0.6. **Outline.** The outline of this paper is as follows.

Section 1 reviews material regarding geometric invariant theory quotients which is necessary to define the moduli $X_\zeta$. No essentially new material is involved, although the formulation of some of the results may be unfamiliar to non-specialists.

Section 2 review material concerning moduli of Hermitian-Yang-Mills connections. This is not essential to the understanding of the main results, although some familiarity is desirable for the understanding of Section 3.

Section 3 deals with the definition and construction of the moduli $X_\zeta$.

Section 4 gives the proof that $X_0$ is isomorphic to $X$ and that $X_\zeta \to X_0$ are partial resolutions.

Section 5 proves that the induced metrics on $X_\zeta$ are ALE.

Section 6 contains the discussion of the singularities of $X_\zeta$ in the language of deformation complexes. This includes the conjecture that the singularities of $X_\zeta$ are at most quadratic algebraic.

Section 7 deals with the case $\Gamma \subset SU(3)$, the construction of the holomorphic three-form on the non-singular locus of $X_\zeta$ and conjecture 2.

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2 Minor portions have been rewritten to include references to advances in the field made since then.
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1. Geometric Invariant Theory

This section recalls the geometric invariant theory of affine varieties and proves some results which shall be needed in the sequel. These results have been included here, because, although they are well-known to the experts, no elementary treatment exists.

1.1. Linearisations and GIT quotients. Let $G$ be a reductive group acting linearly on a complex affine variety $X$. In this situation, a $(G)$-linearisation is a lifting of the $G$-action to the trivial line bundle $L \to X$. Such a linearisation is determined completely by the action of $G$ on the fibres of $L$, namely by a character $\zeta: G \to \mathbb{C}^*$. For every character $\zeta$, denote by $L_\zeta$ the trivial bundle endowed with the corresponding linearisation. The space of $G$-invariant sections of $L_\zeta$ is denoted by $H^0(L_\zeta)^G$.

The geometric invariant theory (GIT) quotient of $X$ by $G$ with respect to $\zeta$ is defined by

$$X/\!/\zeta G := \text{Proj} \bigoplus_{k \in \mathbb{N}} H^0(kL_\zeta)^G.$$ 

Example 1.1. If $\zeta = 0$ is the trivial character, then the corresponding quotient $X/\!/0 G$ coincides with the usual affine GIT quotient $X/\!/G$. In fact, suppose that $X = \text{Spec} \, R$ for a finitely generated ring $R$ and let $z_0$ be a coordinate in the fibre of $L_0$. Then $\bigoplus_{k \in \mathbb{N}} H^0(kL_0)^G = R[z_0]^G = R^G[z_0]$, so taking Prox gives:

$$X/\!/0 G = \text{Proj} \, R^G[z_0] = \text{Spec} \, R^G = X/\!/G,$$

which is the usual affine GIT quotient.

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3For the generalisation of these results to arbitrary quasi-projective varieties, see [Tha94, DH94].

4When $X$ is only quasi-projective the definition of a linearisation involves specifying an ample bundle over $X$ as well as a lift of the action to the bundle.
1.2. **Stability and Extended $G$-equivalence.** The GIT quotient can be obtained by first restricting attention to the open set $X^\text{ss}(\zeta) \subseteq X$ of so-called semi-stable points. A point $x$ in $X$ is called semi-stable (with respect to $\zeta$) if there exists a $G$-invariant section of $kL_\zeta$ (for some $k$ in $\mathbb{N}$) which is non-vanishing at $x$. As a set, $X/\!/\zeta G$ is the quotient of $X^\text{ss}(\zeta)$ by the extended $G$-equivalence relation induced by the closure of the $G$-orbits:

$$x \sim y \iff \overline{Gx} \cap \overline{Gy} \neq \emptyset.$$ 

Thus the $G$-invariant quotient map $X^\text{ss}(\zeta) \rightarrow X/\!/\zeta G$ for the equivalence relation can map several $G$-orbits to the same point.

For most of the points, this does not happen, however. This is because the closure of an open orbit is obtained by adding orbits of smaller dimension, so since the dimension of the orbit is a lower semi-continuous function on $X$, it follows that there is an open subset $X^s(\zeta) \subseteq X^\text{ss}(\zeta)$ of points which have full-dimensional closed $G$-orbits — the so-called stable points — and there is a one-one correspondence between the orbits of $G$ in $X^s(\zeta)$ and their images in the GIT quotient. In other words, the GIT quotient contains, as an open set, the geometric quotient $X^s(\zeta)/G$.

1.3. **Quotients for non-trivial linearisations.** Example 1.1 showed that the GIT quotient for the trivial linearisation coincides with the affine GIT quotient. The following theorem uses the notion of stability to show that the quotients for non-trivial $\zeta$ are closely related to the affine quotient.

**Theorem 1.2.** The GIT quotients admit projective morphisms

$$\rho_\zeta : X/\!/\zeta G \rightarrow X/\!/0 G$$

which are isomorphisms over

$$\rho_\zeta^{-1}(X^s(0)) \rightarrow X^s(0).$$

**Proof.** Let $X = \text{Spec } R$ for some finitely generated ring $R$, and let $z_\zeta$ be a complex coordinate in the fibre of $L_\zeta$. The GIT quotient $X/\!/\zeta G$ is given by taking Proj of the $G$-invariant part of $R[z_\zeta]$, where $R[z_\zeta]$ is to be considered as an algebra graded by the powers of $z_\zeta$. The previous example showed that when $\zeta$ is zero, $\text{Proj } R^G[z_0] = \text{Spec } R^G$.

For a general non-zero $\zeta$, $R[z_\zeta]^G \neq R^G[z_\zeta]$. However, the degree-zero part of $R[z_\zeta]^G$ is always $R^G$, and this shows [Har77, Example II.4.8.1 and Cor. II.5.16] that $X/\!/\zeta G$ is projective over $X/\!/G = \text{Spec } R^G$.

Finally, the map $X/\!/\zeta G \rightarrow X/\!/0 G$ comes from the descent of the composition $X^\text{ss}(\zeta) \hookrightarrow X^\text{ss}(0) \rightarrow X/\!/0 G$ and this is one-one whenever $X^\text{ss}(0) \rightarrow X/\!/0 G$ is, so the last statement of the theorem follows. □
Remark 1.3. If \( X \) contains a 0-stable point, then \( X^s(0) \) is open and non-empty, so dense in \( X \). Its image \( X^s(0)/_0 G \) is open and dense in \( X/\!/_0 G \), and therefore, \( \rho_\zeta : X/\!/_\zeta G \to X/\!/_0 G \) is an isomorphism on a dense open subset, i.e. \( \rho_\zeta \) is birational.

2. Moduli of Hermitian-Yang-Mills Connections

This section recalls basic background concerning Hermitian-Yang-Mills connections and the construction of their moduli, following [DK90]. The construction of \( X_\zeta \) will appear as a specialisation of the material in this section. However, the reader wishing to go straight to the point can skip this section and find a self-contained construction of \( X_\zeta \) in section 3.

2.1. Connections over Symplectic Manifolds. Suppose \( X \) is a compact symplectic manifold \( (X, \omega) \) of dimension \( 2n \) and let \( E \) be a complex vector bundle over \( X \). The bundle of infinitesimal automorphisms of \( E \) will be denoted \( \mathfrak{gl}(E) \) or \( \text{End} E \).

2.1.1. Connections. Consider connections on \( E \), namely linear maps \( \nabla : \Omega^0_X(E) \to \Omega^1_X(E) \) which satisfy the Leibniz condition. Any connection on \( E \) can be expressed in a local neighbourhood \( U \subset X \) as

\[
d_\alpha = d + \alpha,
\]

for \( \alpha \in \Omega^1_U(\text{End } E) \). On the other hand, if one considers the difference of two connections, one gets a global one-form with values in \( \text{End } E \).

2.1.2. Hermitian Structure. Let \( h \) be a positive definite Hermitian inner product on the fibres of \( E \) and denote by \( u(E) \subset \mathfrak{gl}(E) \) the real sub-bundle of unitary automorphisms determined by \( h \). The connection \( \nabla \) is said to be compatible with the Hermitian structure if

\[
dh(s, t) = h(\nabla s, t) + h(s, \nabla t).
\]

Such a connection will have local one-form representatives \( \alpha \in \Omega^1_U(u(E)) \); the space \( \mathcal{A} \) of all such connections is an infinite-dimensional affine space modeled on \( \Omega^1_X(u(E)) \).

Remark 2.1. Why does one fix a Hermitian structure on \( E \)? One reason is because fixing a Hermitian structure on \( E \) amounts, by Chern-Weil theory, to fixing the topological invariants of the connections: for any Hermitian connection \( d_\alpha \) on \( E \), the Chern polynomials \( c_1(E) \) and \( c_2(E) - c_1(E)^2 \) are represented respectively by \( \frac{1}{2\pi} \text{trace } F_\alpha \) and \( \text{trace } F_\alpha^2 \).
2.1.3. Symplectic Structure and Gauge Group. The space $\mathcal{A}$ has a symplectic structure defined by

$$\omega(a, b) := \int_X \text{trace}(a \wedge b) \wedge \omega^{n-1},$$

for $a, b \in \Omega^1_X(u(E))$ tangent vectors to $\mathcal{A}$.

The gauge group $\mathcal{G}$ is the group of automorphisms of $E$ which respect the Hermitian structure in the fibres and cover the identity map of $X$. It acts on $\mathcal{A}$ by

$$g \cdot \nabla := g \nabla g^{-1}$$

(the condition that $g$ is unitary ensures that the new connection is compatible with $h$) and preserves the symplectic form $\omega$.

2.1.4. Moment Map. The Lie algebra of the gauge group is $\text{Lie} \mathcal{G} = \Omega^0_X(u(E))$. The moment map for the action of the gauge group is given by \cite[Prop. 6.5.8]{DK90}

$$\mu: \mathcal{A} \rightarrow \Omega^0_X(u(E))^*$$

$$\alpha \mapsto s \mapsto \int_X \text{trace} s F_\alpha \wedge \omega^{n-1}$$

Note that the $\mathcal{G}$-equivariance follows because the curvature transforms as a tensor under gauge transformations.

2.2. Connections over Kähler manifolds, Holomorphic Structures and the Hermitian-Yang-Mills condition. Now suppose that $X$ is in fact a Kähler manifold.

2.2.1. Complex Structure and the Decomposition of Curvature. There is a natural decomposition $\Omega^1_X(\text{End } E) = (\Omega^{1,0}_X \oplus \Omega^{0,1}_X) \otimes \Omega^0(\text{End } E)$, and if a connection is expressed in a local holomorphic frame $\{z_i\}$ according to (2.1), it takes the form

$$\alpha = \sum_i \alpha_i dz_i - \alpha_i^* d\bar{z}_i,$$

for $\alpha_i$ smooth sections of $\text{End } E$.

The local connection $d_\alpha$ splits into a sum of $(1, 0)$ and $(0, 1)$ parts $\partial_\alpha + \bar{\partial}_\alpha$ given by:

$$\partial_\alpha = \partial + \sum_i \alpha_i dz_i$$

$$\bar{\partial}_\alpha = \bar{\partial} - \sum_i \alpha_i^* d\bar{z}_i.$$
are represented by their $\bar{\partial}_\alpha$ operators. The holomorphic tangent space to $\mathcal{A}$ is of course isomorphic to $\Omega^0_X(\text{End } E)$.

The $(1,1)$ and $(2,0)$ parts of the curvature $F_\alpha$ of $d_\alpha$ are given by

$$F_{\alpha}^{1,1} = \sum_{i,j} \left( \frac{\partial \alpha_j}{\partial \bar{z}_i} - \frac{\partial \alpha_j^*}{\partial z_i} - [\alpha_i, \alpha_j^*] \right) dz_i \wedge d\bar{z}_j,$$

(2.5)

and

$$F_{\alpha}^{2,0} = \sum_{i,j} \left( -\frac{\partial \alpha_i}{\partial z_j} + \frac{1}{2} [\alpha_i, \alpha_j] \right) dz_i \wedge dz_j,$$

(2.6)

with $F_{\alpha}^{0,2}$ equal to minus the Hermitian adjoint of $F_{\alpha}^{2,0}$.

2.2.2. The Hermitian-Yang-Mills condition. A connection on $E$ is called Hermitian-Yang-Mills (HYM) if the inner product of its curvature with the Kähler form $\omega$ is a central element of $\Omega^1_X(u(E))$. This is in fact a moment map condition: using the identity

$$F_\alpha \wedge \omega^{n-1} = \frac{1}{n} \langle F_\alpha, \omega \rangle \omega^n =: \frac{1}{n} \langle \Lambda F_\alpha \rangle \omega^n,$$

the map in equation (2.2) becomes

$$\mu(\alpha)(s) = \frac{\text{Vol}(X)}{(n-1)!} \text{trace}(s \Lambda F_\alpha),$$

so, embedding the Lie algebra of $G$ its dual in the usual way, we see that the moment map becomes a constant multiple of

$$\mu^*: \mathcal{A} \rightarrow \Omega^0_X(u(E))$$

$$\alpha \mapsto \Lambda F_\alpha.$$

The moduli space of HYM connections is the Kähler quotient $\mu^*\Gamma(0)/G$.

2.2.3. Holomorphic Bundles. Suppose instead that the Hermitian connection $d_\alpha$ is required to induce a holomorphic structure on $E$. By the Newlander-Nirenberg theorem, prescribing such a structure is exactly equivalent to specifying a connection $d_\alpha$ which is integrable, i.e. whose $(0,1)$-part is such that $\bar{\partial}_\alpha \circ \bar{\partial}_\alpha = 0$. This is equivalent to the condition that $F_\alpha$ is of type $(1,1)$ and gives a Kähler subvariety $\mathcal{A}^{1,1} \subset \mathcal{A}$. This variety parametrises all the possible holomorphic structures which can be put on the $C^\infty$ bundle $E \rightarrow X$.

The action of $\mathcal{G}$ on $\mathcal{A}$ extends to an action of its complexification $\mathcal{G}^c$, which can be thought of naturally as the group of all general linear
automorphisms of $E$ covering the identity map on $X$. Put $\bar{g} := (g^*)^{-1}$ and let

$$g \cdot \bar{\partial}_\alpha := g \bar{\partial}_\alpha g^{-1}, \quad (2.7)$$

$$g \cdot \partial_\alpha := \bar{g} \partial_\alpha \bar{g}^{-1}. \quad (2.8)$$

This action of $G^C$ preserves the space $A^{1,1}$, and its orbits are equivalence classes of holomorphic bundles. To get a nice moduli space (a quasi-projective variety), one must restrict to the so-called semi-stable bundles, or in other words, consider a GIT quotient $A^{1,1} \sslash G^C$. A theorem of Uhlenbeck and Yau [UY86] states that the moduli space of stable bundles with the same topological type as $E \to X$ coincides with the moduli space of Hermitian-Yang-Mills connections on $E$. This is an infinite-dimensional version of the correspondence between symplectic and algebro-geometric quotients.

Remark 2.2. One should really use the quotient of $G^C$ by the scalar automorphisms, since they act trivially on $A$. The resulting group then has a trivial centre so there is only one linearisation of the action; it essentially determined by the degree and rank of $E$.

3. Construction of $X_\zeta$

A construction of $X_\zeta$ from scratch will be given in this section. Let $Q$ be an $n$-dimensional complex representation of a finite group $\Gamma$. Average over the group elements to get a positive definite Hermitian inner product on $Q$ such that $\Gamma \subset U(Q)$. Let $R$ be the regular representation of $\Gamma$, i.e. the free $\Gamma$-module which is generated over $\mathbb{C}$ by a basis $\{e_\gamma | \gamma \in \Gamma\}$, and on which $\Gamma$ acts via the morphism $\varphi : \Gamma \to \text{Aut}_\mathbb{C} R$ defined by:

$$\gamma \cdot e_\delta := \varphi(\gamma)e_\delta := e_{\gamma \delta}. \quad (3.1)$$

3.1. Invariant HYM Connections.

Note. The reader who has not read Section 2 or is not interested in the “moduli of bundles” point of view can jump directly to 3.2.

The construction of $X_\zeta$ is based on a variation on the construction in the previous section. It consists, roughly speaking, in applying the construction to the case where the compact Kähler manifold $X$ is replaced by the germ of the singularity $Q/\Gamma$.

More precisely, start with $Q^*$, the dual vector-space to $Q$, and $E = Q^* \times R \to Q^*$ the trivial vector bundle with fibre $R$. Consider the connections on $E$ which are invariant under all translations in $Q^*$. These connections are determined by their value at one point, and
they form a finite-dimensional vector space which can be identified with \( M = Q \otimes \text{End}_CR \) by choosing an isomorphism \( \Omega_Q^1 \cong Q \). The constructions in the previous section are now valid, because translation invariance eliminates any problems one might have with the non-compactness of base space \( Q^* \). Most aspects are indeed a lot simpler: it suffices to set all the derivatives of the \( \alpha_i \) equal to zero, and to ignore any integrals over the base space and all the formulas remain valid.

An unusual feature of these invariant connections is that there are several moduli spaces: the usual Hermitian-Yang-Mills condition for a connection states that the contraction of the curvature with the Kähler form should be a central element of the Lie algebra of the gauge group. In case of general connections, there is only one gauge-invariant momentum level set because the centre of the gauge group \( \mathcal{G} \) is trivial. In the case of invariant connections the gauge group that is relevant is a much smaller group \( K^G \), consisting of gauge transformations which are invariant with respect to all translations and the action of \( \Gamma \). This group consists of unitary endomorphisms of \( R \) which commute with the action of \( \Gamma \), and has a non-trivial centre (consisting of the traceless \( \Gamma \)-endomorphisms \( \zeta : R \to R \)). The non-triviality of the centre means that there are several gauge invariant momentum level sets, and hence several possible moduli \( X_\zeta \).

For the sake of readers unfamiliar with the material in section 2, the construction of \( X_\zeta \) is given in detail without making any reference to the bundle construction.

### 3.2. The Vector Space \( M^G \)

Let \( M = Q \otimes \text{End}_CR \) and let \( \Gamma \) act on \( \text{End}_CR \) by conjugation via \( \Gamma \): \( \gamma \cdot T := \varphi(\gamma) T \varphi(\gamma)^{-1} \) (3.1)

This makes \( M \) into a \( \Gamma \)-module. Its \( \Gamma \)-invariant part is denoted \( M^G \):

\[
M^G := (Q \otimes \text{End}R)^\Gamma.
\]

The spaces \( M^G \) and \( M \) can be described explicitly by choosing a basis \( \{q_i\}_{l=1}^n \) for \( Q \), and defining the components of \( \alpha \in M \) by:

\[
\alpha = \sum_{l=1}^n q_l \otimes \alpha_l.
\]

In this way the elements \( \alpha \in M \) are identified with \( n \)-tuples of linear maps \( \alpha_i : R \to R \). The elements of \( M^G \) correspond to those \( n \)-tuples which satisfy the following equivariance condition

\[
\sum_i \gamma_{kl}\alpha_l = \varphi(\gamma)\alpha_k\varphi(\gamma)^{-1}, \quad \forall k, \gamma,
\]

(3.4)
where $\gamma = (\gamma_{kl})$ is the matrix corresponding to the action of the element $\gamma$ on $Q$ with respect to the basis $\{q_i\}_{i=1}^n$.

3.3. **Symplectic and Kähler structure.** Endow $R$ with a fixed positive definite Hermitian inner product $\langle \, , \, \rangle$ which makes the standard basis $e_i$ orthonormal. The inner product on $R$ also defines a real structure on the space $\text{End}_C R$ of $C$-linear endomorphisms of $R$ by the Hermitian adjoint operation in the usual way:

$$\langle T^* x, y \rangle := \langle x, Ty \rangle, \quad x, y \in R.$$ 

Define a positive definite Hermitian inner product $h$ on $M$ by

$$h: \quad M \times M \longrightarrow \mathbb{C} \quad (\alpha, \beta) \longmapsto \sum_i \text{trace}(\alpha_i \beta_i^*)$$

(3.5)

The definition of $h$ is independent of the basis of $Q$ up to a unitary transformations, and restricts to an inner product on $M^\Gamma$. As usual, $h$ induces two forms on the underlying real vector-space to $M$:

- a non-degenerate symmetric bilinear form $g = \Re(h)$ called the **Riemannian metric** associated to $h$

$$g: \quad M \times M \longrightarrow \mathbb{R} \quad (\alpha, \beta) \longmapsto \frac{1}{2} \sum_i \text{trace}(\alpha_i \beta_i^* + \beta_i \alpha_i^*)$$

- a non-degenerate skew-symmetric bilinear form $\omega = \Im(h)$ called the **Kähler** form associated to $h$

$$\omega: \quad M \times M \longrightarrow \mathbb{R} \quad (\alpha, \beta) \longmapsto \frac{1}{2\sqrt{-1}} \sum_i \text{trace}(\alpha_i \beta_i^* - \beta_i \alpha_i^*)$$

This gives a Riemannian metric $g$ and a Kähler form $\omega$ of type $(1, 1)$ on $M$ and $M^\Gamma$, related as usual by

$$\omega(\alpha, \beta) = g(\alpha, i\beta).$$

(3.6)

This makes $M^\Gamma, M$ and all their complex subvarieties into Kähler varieties.

The group $\text{GL}(R)$ of automorphisms of $R$ acts on $M$ by conjugation on $\text{End} R$:

$$\alpha_i \mapsto g\alpha_i g^{-1}, \quad g \in \text{GL}(R).$$

(3.7)

In fact, the scalars act trivially, and the action descends to an action of $G := \text{PGL}(R) := \text{GL}(R)/\text{GL}(1)$.

The subgroup $\text{GL}^\Gamma R$ of endomorphisms which commute with the action of $\Gamma$ acts on $M^\Gamma$ and, in the same way, there is a free action of $G^\Gamma = \text{PGL}(R)^\Gamma$ on $M^\Gamma$. A maximal compact subgroup of $G$ (resp. $G^\Gamma$) is given by $K = \text{PU}(R)$ (resp. $K^\Gamma := \text{PU}^\Gamma(R)$). The compact group $K$ (resp. $K^\Gamma$) leaves the Kähler structure on $M$ (resp. $M^\Gamma$) invariant.
3.4. The Variety $\mathcal{N}^T$ of Commuting Matrices. Define the following natural map:

$$\psi: \mathcal{Q} \otimes \text{End} R \rightarrow \Lambda^2 \mathcal{Q} \otimes \text{End} R$$

$$\sum_k q_k \otimes \alpha_k \mapsto \sum_{k,l} q_k \wedge q_l [\alpha_k, \alpha_l]$$

This definition is independent of the basis of $\mathcal{Q}$ (in terms of connections, it corresponds to calculating the $(0,2)$ part of the curvature). Denote the restriction of $\psi$ to $M^\Gamma$ by the same letter. Define

$$\mathcal{N} := \psi^{-1}(0) \subset M; \quad (3.8)$$

it is a cone (i.e. it is invariant under multiplication by non-zero scalars) which is an intersection of quadrics in $M$ given by the coordinate functions of $\psi$. In the representation of equation (3.3), its points consist of $n$-tuples of commuting $r \times r$ matrices:

$$\mathcal{N} = \{ (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \text{Mat}_r(\mathbb{C}), [\alpha_i, \alpha_j] = 0. \}$$

Its $\Gamma$-invariant part $\mathcal{N}^\Gamma$ consists of those commuting matrices satisfying the equivariance condition (3.4).

3.5. Moment Map. Consider the vector space $M$ with the hermitian inner product $h$ and the action of $K$. The Lie algebra of $K$ is isomorphic to $\mathfrak{su}_R$, which consists of traceless skew-Hermitian endomorphisms of $R$. Using the invariant inner product

$$\langle a, b \rangle := \text{trace}(ab^*) = -\text{trace} ab,$$

identify $\mathfrak{su}_R$ with its dual in the usual way. Then the moment map is for the action of $K$ is

$$\mu: M \rightarrow \mathfrak{su}_R$$

$$\alpha \mapsto \sum_k [\alpha_k^*, \alpha_k]. \quad (3.9)$$

(In the language of of connections, the map $\mu$ corresponds to contracting the curvature of the connection with the Kähler form $\omega = \sum_i dq_i \wedge d\bar{q}_i \in \Omega^1 Q^1$.) The moment map for the action of $K^\Gamma$ on $M^\Gamma$ is obtained simply by restriction.

The Kähler quotients are defined by

$$X_\zeta := \frac{\mu^{-1}(\zeta) \cap \mathcal{N}^T}{K^\Gamma}, \quad \zeta \in \text{Centre}(\mathfrak{su}_R). \quad (3.10)$$

As was remarked in section 1, to make this definition rigorous, one needs to make sense of the Kähler structure on $X_\zeta$. One way to do this is by restricting $\zeta$ to take on integral values. Then, by the correspondence between Kähler and GIT quotients, one has

$$X_\zeta \cong \frac{\mathcal{N}^T}{G^\Gamma}, \quad (3.11)$$
where $\zeta$ on the right-hand side specifies the linearisation of the action of $G^\Gamma$ on the trivial line bundle $\mathcal{N}^\Gamma \times \mathbb{C}$.

**Remark 3.1.** In the case $\Gamma \subset SU(2)$, one has $\Lambda^2 Q \cong R_0$ — the trivial representation. Identifying $\mathbb{C}^2$ with the quaternions, the vector space $M^\Gamma$ becomes a hyper-Kähler manifold with 3 distinct complex structures $I, J, K$ and corresponding associated Kähler forms $\omega_I, \omega_J, \omega_K$. The map $\psi$ is then a moment map for the complex symplectic form $\omega_C = \omega_I + i\omega_K$ which is itself holomorphic with respect to $\omega_I$ and the quotients $X_\zeta$ are quotients of $M^\Gamma$ with respect to the hyper-Kähler moment map given by $(\mu, \psi)$, where the second (complex) variable is set to zero. Kronheimer [Kro86, Kro89] exploits this fact to show that $X_\zeta$ are the minimal resolutions of $\mathbb{C}^2/\Gamma$ for generic values of $\zeta$. Furthermore, by varying the level set of $\psi$, he obtains universal deformations. If $\dim M > 2$ however, $\psi^{-1}(\zeta)$ is not $G^\Gamma$-invariant for non-zero $\zeta$.

**Remark 3.2.** In fact, there is a further action on $M$ by $GL Q$ (acting on $Q$ on the left). If $\rho \in GL Q$, then it is easy to see that this preserves $M^\Gamma$, and indeed $\mathcal{N}^\Gamma$. Furthermore, the centre $Z(\Gamma)$ of $\Gamma$ is a subgroup of $GL Q$ which acts trivially because of (3.4), so there is an action of $G' := GL Q/Z(\Gamma)$ which the quotients $X_\zeta$ inherit. The compact subgroup $K' := U^\Gamma Q/Z(\Gamma)$ acts in a Hamiltonian fashion, and the moment map for this is given by

$$
\mu': \ X_\zeta \longrightarrow \ (\text{Lie } U^\Gamma Q)^* \quad [\alpha] \longrightarrow \ b \mapsto \sum_{ij} b_{ij} \text{trace } \alpha_i \alpha_j^*.
$$

(3.12)

In general this does not provide one with very much information: for instance, if $Q$ is irreducible, $G' = \mathbb{C}^*$ and $\mu'(\alpha)$ is the identity endomorphism of $Q$ times the sum of the norm squared of the $\alpha_i$’s. In the case where $\Gamma$ is abelian, however, $G'$ contains an algebraic torus of dimension $n$ acting freely. The components of $\mu'$ are the norm squared of the matrices $\alpha_i$. This is exploited in the companion paper [S1961] to obtain a complete description of $X_\zeta$ by the methods of toric geometry.

### 4. Variation of Quotients and Partial Resolutions

The zero momentum quotient $X_0$ is better understood if viewed as a two-stage construction: first construct a “universal quotient” $\mathcal{N}_0$ by ignoring the $\Gamma$-equivariance condition (i.e. perform the same construction with $\Gamma$ replaced by the trivial group) and then obtain $X_0$ as its $\Gamma$-invariant part.
4.1. The Universal Quotient. Consider taking symplectic quotients of $\mathcal{N}$ with respect to the action of $K$. Since the centre of its Lie algebra is trivial, there is only one quotient, with momentum zero:

$$\mathcal{N}_0 = \frac{\mathcal{N} \cap \mu^{-1}(0)}{K} = \mathcal{N} / \mathbb{G}.$$ 

**Lemma 4.1.** The reduction $\mathcal{N}_0$ is isomorphic to configuration space of $r = |\Gamma|$ points in $Q$:

$$\mathcal{N}_0 \cong \text{Sym}^r(Q) := Q^r / \Sigma_r,$$

where $\Sigma_r$ denotes the permutation group on $r$ letters acting component-wise on the Cartesian product $Q^r$.

**Proof.** The proof simply adapts Kronheimer’s [Kro86, Lemma 5.2.1]. It is shown that the $K$-orbits in $\mathcal{N} \cap \mu^{-1}(0)$ can be identified in a one-one way with the $\Sigma_r$-orbits in $Q^r$. Let $\alpha \in \mathcal{N} \cap \mu^{-1}(0)$ have components $\alpha_i$ with respect to a basis $q_i$. The conditions $\psi(\alpha) = \mu(\alpha) = 0$ give

$$[\alpha_i, \alpha_j] = 0, \quad \text{for all } i, j$$

$$\sum_i [\alpha_i^*, \alpha_i] = 0.$$

If one denotes by $A_i$ the operator $\text{ad}(\alpha_i)$, one has, using the Jacobi identity and the above equations:

$$\sum_i A_i A_i^* (\alpha_j^*) = \sum_i [[\alpha_i^*, \alpha_i], \alpha_j^*] + [[\alpha_j^*, \alpha_i^*], \alpha_i] = 0.$$

The positivity of $A_i A_i^*$ implies $A_i A_i (\alpha_j^*) = 0$ for all $i$ and $j$, and hence $A_j (\alpha_j^*) = [\alpha_j^*, \alpha_j] = 0$. Thus $\mathcal{N}_0$ is the variety of $n$-tuples of normal commuting endomorphisms of $R$ modulo simultaneous conjugation. Any such $n$-tuple can simultaneously diagonalised by conjugation by a unitary matrix. This means that the orbit of an $n$-tuple is determined by the eigenvalues of its components; more precisely, there are orthonormal vectors $v_\gamma \in R$ indexed by the elements $\gamma \in \Gamma$ and corresponding eigenvalues $\lambda^i_\gamma \in \mathbb{C}$ such that:

$$\alpha_i(v_\gamma) = \lambda^i_\gamma v_\gamma, \quad \text{for all } i, \gamma.$$  \hspace{1cm} (4.1)

This gives $r$ elements

$$\lambda_\gamma := \sum_i \lambda^i_\gamma q_i \in Q,$$

which could be called the *eigenvalues* of $\alpha$. These are defined up to a permutation, because one can always conjugate by an elementary matrix which permutes the rows of the $\alpha_i$’s.
In geometrical language, denote by $\Delta \subset M$ the subspace of $n$-tuples of matrices which are diagonal with respect to the standard basis $e_\gamma$ of $R$. The unitary automorphism of $R$ which maps $e_\gamma$ to $v_\gamma$ moves $\alpha$ into $\Delta$. The slice $\Delta$ can be identified with $Q^r$ by mapping $\alpha \in \Delta$ to its $r$ eigenvalues (listed in some specified order). In this way $\Delta$ inherits an action of $\Sigma_r$, and the $U(R)$-orbit of $\alpha \in \mu^{-1}(0) \cap \mathcal{N}$ intersects $\Delta$ in a single $\Sigma_r$-orbit. \hfill $\square$

4.2. The Zero-Momentum Quotient.

Theorem 4.2. If $\Gamma$ acts freely outside the origin, then $X_0 \cong Q/\Gamma$ as varieties.

Proof. The proof consists in showing that the $K^\Gamma$-orbits in $\mathcal{N}^T \cap \mu^{-1}(0)$ can be identified in a one-one way with the $\Gamma$-orbits in $Q$. Let $\alpha \in \mathcal{N}^T \cap \mu^{-1}(0)$ and let $v_1$ be an eigenvector of $\alpha$ with eigenvalue $\lambda_1 := \sum_i \lambda_i q_i \in Q$:

$$\alpha_i(v_1) = \lambda_i v_1, \quad i = 1, \ldots, n.$$  

By equivariance of $\alpha$,

$$\alpha_i(R(\gamma)v) = (Q(\gamma)\lambda_1)^i(R(\gamma)v), \text{ for all } \gamma, \text{ and all } i,$$

so the eigenvectors and eigenvalues of $\alpha$ are given by

$$\lambda_\gamma := Q(\gamma)\lambda_1 \text{ and } v_\gamma := R(\gamma)v_1, \quad \text{for all } \gamma$$

i.e. they lie in orbits of $\Gamma$. Since $\Gamma$ acts freely outside the origin, the eigenvalues are either all zero or all distinct and non-zero. In the latter case, the eigenvectors therefore form a basis of $R$. The unitary automorphism of $R$ defined by $e_\gamma \mapsto v_\gamma$ commutes with the action of $\Gamma$, so defines an element of $K^\Gamma$. If $\Delta^\Gamma \subset M^\Gamma$ denotes the $n$-tuples of endomorphisms of $R$ which are diagonal with respect to the standard basis $\{e_\gamma\}$ then the automorphism carries $\alpha$ into $\Delta^\Gamma$. The map $\alpha \mapsto \sum_i \lambda_i q_i$ identifies $\Delta^\Gamma$ with $Q$ in a manner that is compatible with the $\Gamma$-action on both sides. Furthermore, the $K^\Gamma$-orbit of $\alpha$ intersects $\Delta^\Gamma$ in precisely one $\Gamma$-orbit. Thus $X_0 \cong \Delta^\Gamma/\Gamma \cong Q/\Gamma$. \hfill $\square$

The following lemma will be useful in the section on ALE metrics (cf. [Kro86]).

Lemma 4.3. If $\Gamma$ acts freely outside the origin the map

$$\mu^{-1}(0) \cap \mathcal{N}^T/K^\Gamma \to \Delta^\Gamma/\Gamma$$

is an isometry when $\Delta^\Gamma$ is given the metric it inherits as a subspace of $M^\Gamma$, namely the Euclidean metric. Furthermore, the bundle $\mu^{-1}(0) \cap \mathcal{N}^\Gamma \to X_0$ is flat.
Proof. The key point is that the subspace $\Delta^\Gamma$ is everywhere orthogonal to the orbits of $K^\Gamma$: a tangent vector to the orbits consists of an $n$-tuple of matrices of the form $[\xi, \alpha_i]$ for some $\xi \in \mathfrak{su}^\Gamma(R)$, and these matrices are always zero on the diagonal, so orthogonal to $\Delta^\Gamma$.

This shows that the bundle $\mu^{-1}(0) \cap \mathcal{N}^\Gamma \to X_0$ is flat, and the definition of the quotient metric on $X_0$ implies that the map $X_0 = \mu^{-1}(0) \cap \mathcal{N}^\Gamma / K^\Gamma \to \Delta^\Gamma / \Gamma$ is an isometry.

4.2.1. Case when $\Gamma$ does not act freely outside the origin. Let $\alpha \in \mathcal{N}^\Gamma \cap \mu^{-1}(0)$ have an eigenvalue $\lambda \in Q$. If the stabiliser $\Gamma_{\lambda}$ of $\lambda$ is trivial, then $\alpha$ has $r$ distinct eigenvalues, corresponding to the elements of the orbit $\Gamma \lambda$. This determines the components $\alpha_i$ completely on the whole of $R$.

On the other hand, if $\lambda$ has a non-trivial stabiliser $\Gamma_{\lambda}$, then this determines $\alpha$ on the sub-representation $W_{\lambda} := \text{span} \Gamma \cdot E_{\lambda} \subset R$, where $E_{\lambda}$ is the eigenspace corresponding to $\lambda$. In fact, $E_{\lambda}$ is a representation of the stabiliser subgroup $\Gamma_{\lambda}$ and $W_{\lambda}$ is simply the representation of $\Gamma$ induced by $E_{\lambda}$:

$$W_{\lambda} = \text{Ind}_{\Gamma_{\lambda}}^{\Gamma} E_{\lambda}.$$  

If $\dim E_{\lambda} < |\Gamma_{\lambda}|$, then $W_{\lambda} \neq R$ and $\alpha$ restricts to an endomorphism of $W_{\lambda}^\perp$. Let $\lambda'$ be an eigenvalue of the restriction; the equivariance condition then determines $\alpha$ on the factor $W_{\lambda'}$. Continuing in this way, one obtains a decomposition of $R$:

$$R = W_{\lambda} \oplus W_{\lambda'} \oplus \ldots.$$  

From this discussion, one obtains the following description of the quotient $X_0$:

**Theorem 4.4.** There is an inclusion $Q / \Gamma \hookrightarrow X_0$; this inclusion is an isomorphism if and only if $\Gamma$ acts freely on $Q$ outside the origin.

Proof. The first statement follows because, for any orbit $\Gamma \lambda$ in $Q$, one can construct an $n$-tuple $\alpha$ of diagonal matrices whose $\lambda$-eigenspace has dimension equal to the stabiliser $\Gamma_{\lambda}$. The orbit of such an $\alpha$ under $K^\Gamma$ consists of commuting matrices with eigenvalue $\gamma \lambda$ with multiplicity $|\Gamma_{\lambda}|$, for all $\gamma \in \Gamma$.

For the second statement, note that the if direction is theorem 4.2. The only if direction follows because if $\lambda$ is an eigenvalue of $\alpha$ with non-trivial stabiliser and with multiplicity one, one can set $\alpha$ to be zero on $W_{\lambda}^\perp$ (since 0 is a fixed point of $\Gamma$, the equivariance condition (3.4) does not imply the existence of other eigenvalues).

In general, $X_0$ corresponds configurations of $r = |\Gamma|$ points of $Q$ which are unions of orbits of $\Gamma$, and hence give rise to a decomposition
of $R$ into induced representations

$$R = \bigoplus_i \text{Ind}^R_{\Gamma \lambda_i} E_{\lambda_i},$$

where $E_{\lambda_i}$ denote the $\lambda$-eigenspace of an element $\alpha$.  

**Remark 4.5.** When $\Gamma$ doesn't act freely outside the origin, the quotient $X_0$ may end up containing all sorts of things. For instance, for the group action $\frac{1}{5}(0,1,-1)$, the quotient $X_0$ contains a copy of $\mathbb{C}^3 / \mathbb{Z}_5$, a copy of $\mathbb{C}^5$, eight copies of $\mathbb{C}^2$, etc...

4.3. **Non-zero Momentum and Partial Resolutions.** By theorem [1.2] in the case where $\zeta$ is integral, there are projective morphisms $\rho_\zeta: X_\zeta \to X_0$ which are isomorphisms over the set of points which have finite $K^\Gamma$-stabilisers.

**Proposition 4.6.** If $\Gamma$ acts freely outside the origin, The stabilisers of $K^\Gamma$ on $N^\Gamma \cap \mu^{-1}(0)$ are trivial everywhere except at $\alpha = 0$.

**Proof.** An automorphism $T$ of $R$ which fixes $\alpha \in N^\Gamma \cap \mu^{-1}(0)$ must preserve the (simultaneous) eigenspaces of $\alpha$. If $T$ also commutes with the action of $\Gamma$, its action on an eigenvector $v \in R$ determines its action on the linear span of the $\Gamma$-orbit of $v$. In the case where $\Gamma$ acts freely and $\alpha$ is non-zero this means that $T$ is only allowed to multiply each eigenvector by the same non-zero constant — and this constant must be of modulus one if $T$ is unitary. Such a $T$ thus corresponds to the identity element in the quotient group $K^\Gamma = PU^\Gamma(R)$.

Applying the theorem about Kähler quotients, one gets the following theorem:

**Theorem 4.7.** If $\Gamma$ acts on $Q$ freely outside the origin, and $\zeta$ is integral, there are projective morphisms $\rho_\zeta: X_\zeta \to X_0 = Q/\Gamma$ which are isomorphisms outside the set $\rho_\zeta^{-1}(0)$.

**Remark 4.8.** Even in the case that $\Gamma$ does not act freely outside the origin, it is likely that there are still birational maps from $X_\zeta$ to the component of $X_0$ which is isomorphic to $Q/\Gamma$ and which are isomorphisms outside the singular set.

5. **ALE Metrics**

The quotients $X_\zeta$ inherit a metric $g_\zeta$ from the metric $g$ on the ambient space $M^\Gamma$. This section shows that these are ALE metrics.

A metric $g$ on a real $m$-dimensional Riemannian manifold $X$ is called *asymptotically locally Euclidean (ALE)* if there exists a compact subset
$C \subset X$ whose complement $X \setminus C$ has a finite covering $\widetilde{X \setminus C}$ which is diffeomorphic to the complement of a ball in $\mathbb{R}^m$, and such that, in the pulled-back coordinates $x_1, \ldots, x_m$ on $\widetilde{X \setminus C}$, $g$ takes the form

$$g_{ij} = \delta_{ij} + a_{ij},$$  \hspace{1cm} (5.1)

where $|\partial^p a_{ij}| = O(r^{-4-p})$ for $p \geq 0$, where $r = \sqrt{\sum_i x_i^2}$ denotes the radial distance in $\mathbb{R}^m$ and $\partial$ denotes the differentiation with respect to the coordinates $x_1, \ldots, x_m$.

**Theorem 5.1.** The metrics on $X_\zeta$ are ALE: for any $\zeta$, there is an expansion in powers of $r$

$$g_\zeta = \delta + \sum_{k\geq 2} h_k(\theta)r^{-2k},$$  \hspace{1cm} (5.2)

where $(r, \theta)$ denote polar coordinates in $\mathbb{R}^{2n} \cong Q$. This expansion is analytic and may be differentiated term by term.

**Proof.** Kronheimer’s proof [Kro86, Prop.5.5.1] goes through with the appropriate modifications. The metric $g_\zeta$ restricted to the unit ball $r = 1$ is an analytic function of $\zeta$, so admits an expansion

$$g_\zeta|_{r=1} = \sum_\nu f_\nu \zeta^\nu$$

where $\nu$ are multi-indices in the coordinates of $\zeta$. The moment map being quadratic homogeneous implies that

$$g_\zeta(r, \theta) = g_{r^{-2}\zeta}(1, \theta).$$

Hence the expansion for $g_\zeta$ takes the form

$$\sum_{k\geq 0} h_k(\theta)r^{-2k},$$

where the $h_k = \sum_{|\nu|=k} f_\nu \zeta^\nu$ are analytic functions of the radial coordinates.

It remains to show that $h_0 = \delta$ and that $h_1 = 0$. The first statement is equivalent to showing that the identification $X_0 \to Q/\Gamma$ is an isometry. This was done in Lemma 4.3.

For the second statement, one must show that the variation of $g_\zeta$ with $\zeta$ is zero at $\zeta = 0$ in every direction $\lambda \in (\text{Lie } K^*)^K$. The metric $g_\zeta$ is determined entirely by the Kähler form $\omega_\zeta$ and the induced complex structure $J_\zeta$. Since the latter is the same for all $\zeta$, it is sufficient to prove that

$$\partial_\lambda\omega_\zeta|_{\zeta=0} = 0$$
for all \( \lambda \). A general formula for the variation of the induced symplectic form is given by Duistermaat and Heckman in [DH82]. Away from the singularities, the projection \( \mu^{-1}(\zeta) \cap N^T \to X_\zeta \) is a principal \( R^T \)-bundle whose connection is given by the Levi-Civita connection for the induced metric on \( X_\zeta \). If \( \Omega_\zeta \) denotes the curvature, regarded locally as an element of \( \Omega^2_{X_\zeta} \otimes \mathfrak{su}^T(R) \), then the formula for the variation of \( \omega_\zeta \) is given by

\[
\partial_\lambda \omega_\zeta = \langle \lambda, \Omega_\zeta \rangle.
\]

In the present case, lemma 4.3 tells us that \( \Omega_0 = 0 \), so the variation is zero for \( \zeta = 0 \), and this concludes the proof. \( \square \)

6. Deformation Complexes

The question of the local geometry of the moduli spaces \( X_\zeta \) can be studied using the tools of deformation complexes.

6.1. Differential Forms and Graded Lie Algebras. Define the vectorspaces

\[ M^{p,q} := \Omega^{p,q}_{\zeta} \otimes \text{End } R, \]

for \( p, q \in \mathbb{N} \), whose typical element \( \beta \) is of the form

\[ \beta = \beta_I^J dq_I \wedge d\bar{q}^J, \]

where the summation convention is used for the multi-indices \( I, J \) and where, as usual, \( dq_I = \wedge_{i \in I} dq_i \), and \( d\bar{q}^J = \wedge_{j \in J} d\bar{q}_j \). Define the degree of \( \beta \) to be \( \deg \beta := |I| + |J| \) and write \( (-1)^{\deg \beta} \) for \( (-1)^{\deg \beta} \). The product of two elements \( \alpha, \beta \) is defined to be

\[ \alpha \beta := \alpha_I^J \beta_I'^{J'} dq_I \wedge d\bar{q}^J \wedge dq_{I'} \wedge d\bar{q}^{J'}, \]

and the bracket of any two elements \( \alpha \in M^{p,q} \) and \( \beta \in M^{p',q'} \) is defined by

\[ [\alpha, \beta] := [\alpha_I^J, \beta_I'^{J'}] dq_I \wedge d\bar{q}^J \wedge dq_{I'} \wedge d\bar{q}^{J'} = \alpha \beta - (-1)^{\alpha \beta} \beta \alpha. \quad (6.1) \]

In the equation above and elsewhere, \( (-1)^{\alpha \beta} \) means \( (-1)^{\deg \alpha \deg \beta} \) and not \( (-1)^{\deg \alpha} (-1)^{\deg \beta} \). Writing \( M^r := \sum_{p+q=r} M^{p,q} \), the algebra \( M^r \) inherits the structure of a graded Lie algebra, namely a graded algebra with a bracket satisfying

\[ [M^r, M^s] \subset M^{r+s}, \]

(graded) skew-commutativity

\[ [\alpha, \beta] = -(-1)^{\alpha \beta} [\beta, \alpha] \]

and the (graded) Jacobi identity:

\[ (-1)^{\alpha \gamma} [\alpha, [\beta, \gamma]] + (-1)^{\beta \alpha} [\beta, [\gamma, \alpha]] + (-1)^{\gamma \beta} [\gamma, [\alpha, \beta]] = 0. \]
There are two sub-algebras $M^{*,0}$ and $M^{0,*}$ of $M^*$. Defining the adjoint of $\alpha = \alpha^I dq_I \wedge d\bar{q}^J$ to be

$$\alpha^* := (\alpha^J)^* dq^I \wedge dq_J,$$

then $(\alpha \beta)^* = (-1)^{\alpha \beta} \beta^* \alpha^*$ and $[\alpha, \beta]^* = (-1)^{\alpha \beta} [\beta^*, \alpha^*]$.

The Jacobi identity implies that if $\alpha$ or $\beta$ has odd degree then

$$[\alpha, [\alpha, \beta]] = 1$$

is satisfied.

If $\alpha = \alpha_I dq_I \in M^{0,1}$, and one defines

$$\bar{\partial}_\alpha : M^{p,q} \rightarrow M^{p,q+1},$$

then, using (6.2), one sees that the sequences

$$M^{p,*} : M^{p,0} \xrightarrow{\bar{\partial}_0} M^{p,1} \xrightarrow{\bar{\partial}_1} M^{p,2} \rightarrow \ldots$$

are complexes precisely when $[\alpha, \alpha] = 0$, i.e. when $\alpha \in \mathcal{N}$. Write $H^p_{\alpha,q}$ for the cohomology groups $H^q(M^{p,*}, \bar{\partial}_\alpha)$. If one introduces a metric on $M^{p,q}$ by using the standard inner product on $\text{End} R$ and making

$$\frac{1}{\sqrt{2^{p+q}}} \{ dq_I \wedge d\bar{q}^J \}_{|I|=p, |J|=q}$$

orthonormal [GH78, p.80], one can define the adjoint operator $\bar{\partial}_\alpha^*$, and the Laplacian $\square = \bar{\partial}_\alpha^* \bar{\partial}_\alpha + \bar{\partial}_\alpha \bar{\partial}_\alpha^*$. Their kernels give harmonic representatives for the cohomology groups in the usual way

$$H^p_{\alpha,q} := \ker \square_{\alpha,q} \subset M^{p,q}.$$

If $\Lambda : M^{p,q} \rightarrow M^{p-1,q-1}$ denotes the operation of contraction with the Kähler form $\omega = dq_i \wedge d\bar{q}^j$, then the definition of the adjoint and the invariance of the trace under cyclic permutations give

$$\bar{\partial}_\alpha^* \beta = -\Lambda [\alpha^*, \beta],$$

or, in coordinates,

$$(\bar{\partial}_\alpha^* \beta)^I_j = [\alpha^*, \beta^I_j].$$

Writing $\kappa := dq_1 \wedge \cdots \wedge dq_n$, the $n$-th power of $\omega$ is

$$\omega^n = n!(-1)^{(n-1)(n-2)/2} \frac{i^n}{2^n} \kappa \wedge \kappa^*.$$
6.2. **Local Description of** $X_\zeta$. Under the identification $M = M^{0,1}$, the derivative of the action (3.7) of $\text{GL}(R)$ on $M$ is given by

$$\tilde{\partial}_\alpha^0 : M^{0,0} \to M^{0,1}.$$ 

On the other hand, the derivative of $\alpha \mapsto F^{0,2}_\alpha$ is (twice)

$$\tilde{\partial}_\alpha^1 : M^{0,1} \to M^{0,2},$$

so the Zariski tangent space to $\mathcal{N} \sslash\text{GL}(R)$ at an element $\alpha$ is given by the first cohomology group of the Atiyah-Hitchin-Singer [AHS77] deformation complex

$$M^{0,0} \xrightarrow{\tilde{\partial}_\alpha^0} M^{0,1} \xrightarrow{\tilde{\partial}_\alpha^1} M^{0,2},$$

i.e. by $H^{0,1}_\alpha$. From the point of view of the Kähler quotient, one can see this as follows: the Zariski tangent space to $X_\zeta$ at $[\alpha]$ is given by $\ker d\mu(\alpha) \cap \ker d\psi(\alpha)$. By definition, the derivative of $\mu$ is dual to the action of $K^\Gamma$, so $d\mu(\alpha) = -2\tilde{\partial}_\alpha^* = 2\Lambda[\alpha^*, ]$, as can be verified by remarking that $\mu(\alpha) = \Lambda[\alpha^*, \alpha]$. Hence $d\mu(\alpha) = \text{Im} \tilde{\partial}_\alpha^*$, so the tangent spaces indeed coincide.

A local model for $\mathcal{N} \sslash\text{GL}(R)$ in a neighbourhood of a point $[\alpha]$ where $\alpha \in \mathcal{N}$ has trivial stabiliser is given by solving the equation $F^{0,2}_{\alpha + \beta} = 0$ for $\beta$ in the slice

$$\{ \beta \in M^{0,1} : \tilde{\partial}_\alpha^* \beta = 0, \| \beta \| \text{ small} \}.$$ 

This comes down to solving the system of equations

$$\tilde{\partial}_\alpha^* \beta = 0 \quad (6.7)$$

$$\tilde{\partial}_\alpha \beta + \frac{1}{2}[\beta, \beta] = 0, \quad (6.8)$$

in a neighbourhood of the origin. Kuranishi’s argument [Kur62, Kur65] shows that the solution set is given by the zero set of a map $\Phi : \mathcal{H}^{0,1}_\alpha \to \mathcal{H}^{0,2}_\alpha$ whose two-jet at the origin is given by

$$\Phi(2) : \mathcal{H}^{0,1}_\alpha \xrightarrow{\beta} \mathcal{H}^{0,2}_\alpha \xrightarrow{\mathcal{H}_\alpha([\beta, \beta])}.$$

where

$$\mathcal{H}_\alpha : M^{0,2} \to \mathcal{H}^{0,2}_\alpha$$

denotes the orthogonal projection to the harmonic subspace. Similar statements hold for $X_\zeta$ and $M^{0,*,\Gamma}$. 
6.3. Kuranishi Germs and Formality. This whole discussion can be phrased in more abstract language of deformation functors and differential graded Lie algebras. Additional details and background can be found in [GM90, GM88] and [DGMS75]. The algebra $M^{p,*}$ is actually a differential graded Lie algebra (DGLA) when endowed with the differential $\bar{\partial}_\alpha$. The metric on $M^{p,q}$ makes it into an analytic DGLA, namely a DGLA which possesses a norm compatible with its differential and bracket, and which induces what is essentially a Hodge decomposition of its graded pieces with finite dimensional topological summands $H^i$ which are the analogues of the harmonic forms. When $\alpha$ has trivial stabiliser, $X_\zeta$ is locally analytically isomorphic in the neighbourhood of $[\alpha]$ to the Kuranishi germ $K_{M_0^{*,\Gamma}}$ associated to $M^{0,*}$, $\Gamma$. The results so far are stated in the following theorem.

**Theorem 6.1.** The sequence of vector spaces $(M^{0,*}, \bar{\partial}_\alpha)$ is a complex (and therefore a differential graded Lie algebra) if and only if $\alpha \in N^\Gamma$. Furthermore, if $\alpha \in N^\Gamma$, the Zariski tangent space to $X_\zeta$ at $[\alpha]$ is isomorphic to the first cohomology group of its $\Gamma$-invariant part

$$H^{0,1,\Gamma}_\alpha := H^1(M^{0,*}, \bar{\partial}_\alpha)$$

and if $\alpha$ has trivial $K^\Gamma$-stabilisers, then $X_\zeta$ is locally isomorphic to its Kuranishi germ

$$K_{M^{0,*},\Gamma} = \{ \beta \in M^{0,1,\Gamma} | \bar{\partial}_\alpha^* \beta = \bar{\partial}_\alpha \beta + \frac{1}{2} [\beta, \beta] = 0 \}.$$

In general the Kuranishi germ $K_L$ of an analytic DGLA $(L,d)$ is (Banach analytically) isomorphic to the germ at 0 of

$$\{ \beta \in (\text{Im } d)^+ \subset L^1 | d\beta + \frac{1}{2} [\beta, \beta] = 0 \},$$

where $(\text{Im } d)^+$ is a fixed complement of the image of $d$ in $L^1$. Goldman and Millson [GM90] prove that $K_L$ is invariant under quasi-isomorphisms, namely chains of homomorphisms of DGLAs

$$L \rightarrow L' \leftarrow L'' \rightarrow \cdots \leftarrow L'''$$

which induce isomorphisms in cohomology. When $L$ is quasi-isomorphic to its cohomology (which is a DGLA when endowed with the zero differential), $L$ is called formal and it follows that $K_L$ is analytically isomorphic to the quadratic cone

$$Q_L := \{ \beta \in H^1 : [\beta, \beta] = 0 \}.$$
One way in which this can happen is if the bracket of two harmonic elements of degree 1 is harmonic. This is the case, for instance for the moduli space of flat Hermitian-Yang-Mills connections over a compact Kähler manifold \[ \text{[Nad88, GM87, GM90]} \]. If in addition, the cup-product on \( H^1 \) is zero, then \( K_L \cong H^1 \) and the deformation space is a smooth manifold (even if \( H^2 \neq 0 \)). This is the case, for instance for the moduli space of complex structures over a Calabi-Yau \( n \)-fold, namely a compact Kähler manifold with a nowhere vanishing holomorphic \((n,0)\)-form \[ \text{[GM90]} \]. These moduli were studied by F. Bogomolov.

The key fact which implies the formality of the DGLA and the vanishing of the cup-product in this case was proved by Tian \[ \text{[Tia87]} \] and Todorov \[ \text{[Tod89]} \].

In the case of the algebra \( M^{0,*} \), formula \( (6.2) \) with \( \alpha \) and \( \beta \) interchanged shows that the bracket of two harmonic elements in \( H^0,1 \) is \( \bar{\partial} \alpha \)-closed. However, it does not follow that \( \bar{\partial}_\alpha (\{\beta,\beta\}) = 0 \); indeed this is easily seen to be false, since \( \{\beta,\beta\} = 2\beta \beta \). Nevertheless, it does not seem unreasonable to expect that \( M^{0,*} \Gamma \) can also be proved to be formal for generic \( \zeta \), maybe by imitating Tian and Todorov’s method.

**Conjecture 1.** The differential graded Lie algebra \((M^{0,*}\Gamma, \bar{\partial}_\alpha)\) is formal for all \( \alpha \in \mathcal{N}^\Gamma \cap \mu^{-1}(\zeta) \) and generic \( \zeta \), and therefore \( X_\zeta \) has, for these \( \zeta \), at worst quadratic algebraic singularities.

Another conjecture is the following:

**Conjecture 2.** If \( \Gamma \subset \text{SU}(3) \), can one imitate the Tian-Todorov proof and show that the Kuranishi germ of \((M^{0,*}\Gamma, \bar{\partial}_\alpha)\) is isomorphic to \( H^{0,1}_{\alpha} \) for generic \( \zeta \), i.e. that \( X_\zeta \) is smooth?

The fact that \( X_\zeta \) has at most quadratic singularities has been verified for the abelian subgroups of order less than 11. The smoothness of \( X_\zeta \) has been verified in the abelian cases \( \frac{1}{3}(1,1,1), \frac{1}{6}(1,2,3), \frac{1}{7}(1,2,4), \frac{1}{8}(1,2,5), \frac{1}{9}(1,2,6), \frac{1}{10}(1,2,7) \) and \( \frac{1}{11}(1,2,8) \). Both these verifications were done by exhaustive listing of singularities of \( X_\zeta \) for all possible \( \zeta \), using the methods given in the companion paper \[ \text{[SI96]} \].

A different approach is available in the specific case of \( \text{SU}(3) \); this is presented next.

### 7. Subgroups of SU(3) and Cubic Forms

Suppose that \( \Gamma \subset \text{SU}(3) \). If \( \alpha \in \mu^{-1}(\zeta) \) and \( \beta, \delta \in H^{0,1}_{\alpha,\Gamma} \), then, as remarked in the previous section,

\[
\bar{\partial}_\alpha [\beta, \delta] = 0,
\]
but \([\beta, \delta]\) is not in \(H^{0,2,\Gamma}_\alpha\). However, considerations of type show that it differs from its harmonic projection by a term \(\bar{\partial}_\alpha \epsilon\), for some \(\epsilon \in M^{0,1,\Gamma}\). For \(\eta \in H^{0,1,\Gamma}_\alpha\)

\[
\text{trace}(\eta[\beta, \delta]) - \text{trace}(\eta H_\alpha([\beta, \delta])) = \text{trace}(\eta[\alpha, \epsilon]) \\
= \text{trace}(\epsilon[\eta, \alpha]) \\
= 0, \quad \text{since } \eta \in H^{0,1,\Gamma}_\alpha. \quad (7.1)
\]

This shows that the tensor

\[
H^{0,1,\Gamma}_\alpha \otimes H^{0,1,\Gamma}_\alpha \otimes H^{0,1,\Gamma}_\alpha \twoheadrightarrow \mathbb{C} \\
(\eta, \beta, \delta) \mapsto \kappa \text{trace}(\eta H_\alpha([\beta, \delta])),
\]

is totally symmetric on \(H^{0,1,\Gamma}_\alpha\) (the isomorphism \(\Omega^{3,0}_Q \cong \mathbb{C}\) has been used). An easy polarisation argument shows that it is completely determined by the corresponding cubic form

\[
C: H^{0,1,\Gamma}_\alpha \rightarrow \mathbb{C} \\
\beta \mapsto \kappa \text{trace}(\beta([\beta, \beta])).
\]

**Proposition 7.1.** The singularity of \(X_\zeta\) has no quadratic part if and only if \(C(\beta) = 0\) for all \(\beta \in H^{0,1,\Gamma}_\alpha\) and all \(\alpha \in \mu^{-1}(\zeta)\).

**Proof.** Suppose \(X_\zeta\) has no quadratic part at \([\alpha]\). Then \(\Phi^{(2)}(\beta) = H_\alpha([\beta, \beta]) = 0\). But this implies that \(C(\beta) = 0\) by equation (7.1).

Conversely, if \(C(\beta) = 0\) for all \(\beta \in H^{0,1,\Gamma}_\alpha\) then the corresponding totally symmetric tensor vanishes on all triples \((\eta, \beta, \beta)\) for all \(\eta, \beta \in H^{0,1,\Gamma}_\alpha\). Since this is true for all \(\eta\), it must be that \(H_\alpha([\beta, \beta]) \in \text{Im } \bar{\partial}_\alpha\), i.e. \(\Phi^{(2)}(\beta) = 0\) in \(H^{0,2,\Gamma}_\alpha\). \(\square\)

There is a natural 3-vector \(\Omega\) whose value on three elements of \(H^{0,1,\Gamma}_\alpha\) of is given by

\[
\Omega(\eta, \beta, \delta) := \kappa \text{trace}(\eta \beta \delta). \quad (7.2)
\]

This is symmetric under cyclic permutations of the entries, so decomposes into a totally skew-symmetric part \(\Omega_{\text{skew}}\) and a totally symmetric part, which is nothing but the totally symmetric tensor corresponding to \(C\). The proposition above implies the

**Corollary 7.2.** If \(X_\zeta\) is smooth, then \(\Omega\) defines an element of \(\Omega^{3,0}(X_\zeta)\).

**Conjecture 3.** The canonical sheaf \(\mathcal{O}_{X_\zeta}(K_{X_\zeta})\) is locally free, and is generated by the non-vanishing \((3, 0)\)-form \(\Omega\) when \(X_\zeta\) is smooth.
Taking the wedge of $\Omega$ with its complex conjugate gives
\[
\Omega \wedge \Omega^*(\eta, \beta, \delta, \eta^*, \beta^*, \delta^*) = (\epsilon_{ijk} \text{trace} \eta_i \beta_j \delta_k)(\epsilon_{ijk} \text{trace} \eta_i \beta_j \delta_k) \kappa \wedge \kappa^* ,
\]
(7.3)
and
\[
= \left| \sum_{ijk} \text{trace} \eta_i \beta_j \delta_k \right|^2 \kappa \wedge \kappa^* ,
\]
(7.4)
where $\sum_{ijk}$ denotes the sum over distinct $i, j$ and $k$. On the other hand, the symplectic form $\omega_\zeta$ on $X_\zeta$ is simply the restriction of the symplectic form $\omega$ defined in (3.5), and equation (6.6) gives
\[
\omega_\zeta \wedge \omega_\zeta \wedge \omega_\zeta = 3 \frac{i}{4} \kappa \wedge \kappa^* .
\]
(7.5)
Suppose that $(\eta, \beta, \delta)$ are an orthonormal triple in $T_{\alpha}^{1,0} X_\zeta$. Then the value of the coefficient of $\kappa \wedge \kappa^*$ in (7.4) is equal to $\|\Omega\|^2 \|\kappa \wedge \kappa^* \|^{-2}$.

Hence $X_\zeta$ has trivial canonical bundle if this coefficient is never zero for all $\alpha \in \mu^{-1}(\zeta)$.

**Lemma 7.3.** The Kähler manifold $X_\zeta$ is Ricci-flat if and only if the coefficient of $\kappa \wedge \kappa^*$ in (7.4) is constant for all orthonormal triples $(\eta, \beta, \delta)$ in $H_{\alpha}^{0,1,\Gamma}$ and all $\alpha \in \mu^{-1}(\zeta) \cap N^T$.

**Proof.** This follows because if $X_\zeta$ is Ricci-flat, there exists a holomorphic $(3, 0)$-form $\Omega'$ which is covariant constant on $X_\zeta$. Hence $\Omega$ will differ from $\Omega'$ by a holomorphic function $f$. Now Liouville’s theorem implies that $f$ is either constant or unbounded. Since $\Omega$ is clearly bounded (by $6 \kappa \wedge \kappa^*$), $f$ must be constant. \qed

**7.1. Example.** Let us work out a specific example. Consider the group $\Gamma = \mu_3$ of order 3 acting on $\mathbb{C}^3$ with weights $(1, 1, 1)$. The following configuration of matrices is easily seen to define a point of $\mu^{-1}(\zeta) \cap N^T$, where $\zeta = (-|A|^2, |A|^2 - |B|^2, |B|^2)$, $(A, B \in \mathbb{R})$:
\[
\alpha_1 = \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} d\bar{q}^1, \quad \alpha_2 = \alpha_3 = 0 .
\]
(7.6)
The tangent space is three-dimensional and is generated by the following orthonormal elements (recall that $\|d\bar{q}\|^2 = 2$)
\[
\beta_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} d\bar{q}^1, \quad \beta_i = \frac{1}{\sqrt{2(A^2 + B^2)}} \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} d\bar{q}^i ,
\]
(7.7)
for \( i = 2, 3 \), so this defines a smooth point of \( X_\zeta \). The value of \( \| \Omega \|^2 \) at this point is

\[
\left| \frac{1.A.B + 1.B.A}{2\sqrt{2}(A^2 + B^2)} \right|^2 \kappa \wedge \kappa^* = \frac{1}{2} \left( \frac{AB}{A^2 + B^2} \right)^2 \kappa \wedge \kappa^*,
\]

(7.8)

and so this is non-zero away from \( AB = 0 \) (which correspond to non-generic values of \( \zeta \)).

At the point

\[
\alpha_1 = \begin{pmatrix} 0 & A + C & 0 \\ 0 & 0 & B + C \\ C & 0 & 0 \end{pmatrix} dq^1, \quad \alpha_2 = \alpha_3 = 0,
\]

(7.9)

in \( \mu^{-1}(\zeta) \cap \mathcal{N}^\Gamma \), the tangent space is still three-dimensional, with orthonormal generators

\[
\beta_i = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} dq^i, \quad i = 1, 2, 3,
\]

(7.10)

The value of \( \| \Omega \|^2 \) however is now

\[
\left| 6 \left( \frac{1}{\sqrt{6}} \right)^3 \right|^2 \kappa \wedge \kappa^* = \frac{1}{6} \kappa \wedge \kappa^*.
\]

(7.11)

In fact, all the points of \( \mu^{-1}(\zeta) \cap \mathcal{N}^\Gamma \) are of the form (7.6) or (7.9) (modulo permutations of the indices 1, 2, 3). Thus it has been shown, in a rather laborious way, that away from certain degenerate values of \( \zeta \), \( \Omega \) is non-vanishing on \( X_\zeta \) and \( K_{X_\zeta} \) is therefore trivial. In fact, \( X_\zeta = \mathcal{O}_{\mathbb{P}^2}(-3) \).

Since the coefficient of \( \kappa \wedge \kappa^* \) in (7.8) is always smaller than 1/8, one also deduces that \( \Omega \wedge \Omega^i \) is not a constant multiple of \( \omega_\zeta \wedge \omega_\zeta \wedge \omega_\zeta \) on any of the quotients \( X_\zeta \), and therefore by lemma 7.3 that the induced metric is never Ricci-flat.

Remark 7.4. The space \( \mathcal{O}_{\mathbb{P}^2}(-3) \) does have a standard Ricci-flat metric, as was first noted by Calabi [Cal79].

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