SOCLE DEFORMATIONS OF SELFINJECTIVE ORBIT ALGEBRAS OF TILTED TYPE

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Dedicated to José Antonio de la Peña on the occasion of his 60th birthday

Abstract. We survey recent development of the study of finite-dimensional selfinjective algebras over a field which are socle equivalent to selfinjective orbit algebras of tilted type.

1. Introduction

Throughout this article, by an algebra we mean a basic, connected, finite-dimensional associative $K$-algebra with identity over a fixed field $K$. For an algebra $A$, we denote by $\text{mod} \ A$ the category of finite-dimensional right $A$-modules and $\text{ind} \ A$ the full subcategory of $\text{mod} \ A$ of indecomposable modules. We denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$, and by $\tau_A$ and $\tau_A^{-1}$ the Auslander-Reiten translations $D \text{Tr}$ and $\text{Tr} D$, respectively, where $D$ is the standard duality $\text{Hom}_K(-, K)$ on $\text{mod} \ A$ and $\text{Tr}$ is the transpose.

An algebra $A$ is called selfinjective if $A$ is injective in $\text{mod} \ A$, or equivalently the projective modules in $\text{mod} \ A$ are injective. Two selfinjective algebras $A$ and $A'$ are said to be socle equivalent if the quotient algebras $A/\text{soc}(A)$ and $A'/\text{soc}(A')$ are isomorphic. In this case, we also call $A$ a socle deformation of $A'$. Moreover, two selfinjective algebras $A$ and $A'$ are called stably equivalent if their stable categories $\text{mod} \ A$ and $\text{mod} A'$ are equivalent as $K$-categories.

In the representation theory of selfinjective algebras an important role is played by the selfinjective algebras $A$ which admit Galois covering of the form $\hat{B} \to \hat{B}/G = A$, where $\hat{B}$ is the repetitive category of an algebra $B$ and $G$ is an admissible group of automorphisms of $\hat{B}$. In this theory, the selfinjective orbit algebras $\hat{B}/G$ given by algebras $B$ of finite global dimension and infinite cyclic groups $G$ are of special interest. Namely, frequently interesting selfinjective algebras are Morita equivalent to socle deformations of such orbit algebras, and we may reduce their representation theory to that for the corresponding algebras of finite global dimension. For example, it is the case for selfinjective algebras of polynomial growth over an algebraically closed field $K$ (see Section 2 for details). This is also the case for the tame principal blocks of the enveloping algebras of restricted Lie algebras [19], or more general infinitesimal groups [20], over algebraically closed fields of odd characteristic. We also note that the prominent class of special biserial algebras over an algebraically closed field is formed by the orbit algebras of repetitive categories (see [15], [18], [33]).

For an arbitrary field $K$, even the structure of selfinjective algebras of finite representation type over $K$ is far from being understood (see Section 6 for new results toward solution of this problem). In this article, we survey old and new
results concerning selfinjective algebras $A$ over a field which are socle equivalent to orbit algebras $\hat{B}/G$ for an algebra $B$ and $G$ the infinite cyclic group generated by the composition $\varphi \nu B$ of the Nakayama automorphism $\nu B$ and a positive automorphism $\varphi$ of $\hat{B}$. In particular, we present, in Sections 3 and 4 criteria for a selfinjective algebra $A$ to have this property. Sections 5 and 6 are devoted to presentations of interesting selfinjective algebras as socle deformations of orbit algebras of repetitive categories of tilted algebras. In the final Section 7 we show that, for the class of selfinjective algebras of tilted type, discussed in this article, the stable equivalence and socle equivalence coincide.

For basic background on the representation theory discussed in this article we refer to the books [3], [50], [52], and the survey articles [41], [49].

2. Repetitive algebras and orbit algebras

Let $B$ be an algebra and $1_B = e_1 + \cdots + e_n$ a decomposition of the identity of $B$ into a sum of pairwise orthogonal primitive idempotents $e_1, \ldots, e_n$ of $B$. The algebra is regarded as a $K$-category, denoted by $\hat{B}$ again, whose objects are $e_1, \ldots, e_n$ and the morphism space $B(e_i, e_j) = \text{Hom}_B(e_i, e_j)$ is $\text{Hom}_B(e_iB, e_jB)$ for $i, j \in \{1, \ldots, n\}$. We then associate to $B$ a selfinjective locally bounded $K$-category $\hat{B}$, called the repetitive category of $B$ (see [24]). The objects of $\hat{B}$ are $e_{m,i}$ for $m \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$, and the morphism spaces are defined as follows

$$\hat{B}(e_{m,i}, e_{r,j}) = \begin{cases} 
    e_jBe_i, & r = m, \\
    D(e_iBe_j), & r = m + 1, \\
    0, & \text{otherwise}.
\end{cases}$$

Observe that $e_jBe_i = \text{Hom}_B(e_iB, e_jB)$, $D(e_iBe_j) = e_jD(B)e_i$ and

$$\bigoplus_{(m,i) \in \mathbb{Z} \times \{1, \ldots, n\}} \hat{B}(e_{m,i}, e_{r,j}) = e_jB \oplus D(Be_j),$$

for any $r \in \mathbb{Z}$ and $j \in \{1, \ldots, n\}$. The composition of morphisms in $\hat{B}$ is naturally defined by the multiplication in the algebra $B$ and the $B$-bimodule structure of $D(B)$.

An automorphism $\varphi$ of the $K$-category $\hat{B}$ is said to be positive or rigid, respectively, if for each $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$,

$$\varphi(e_{m,i}) = e_{p,j} \text{ for some } p \geq m \text{ and } j \in \{1, \ldots, n\},$$

or

$$(e_{m,i}) = e_{m,j} \text{ for some } j \in \{1, \ldots, n\}.$$

A strictly positive automorphism of $\hat{B}$ is a positive but not rigid automorphism of $\hat{B}$. In particular, the Nakayama automorphism of $\hat{B}$, denoted by $\nu B$, is the strictly positive automorphism defined by

$$\nu B(e_{m,i}) = e_{m+1,i} \text{ for all } (m, i) \in \mathbb{Z} \times \{1, \ldots, n\}.$$

A group $G$ of automorphisms of $\hat{B}$ is said to be admissible if $G$ acts freely on the set of objects of $\hat{B}$ and has finitely many orbits. Following P. Gabriel [21], we may consider the orbit category $\hat{B}/G$ of $\hat{B}$ with respect to $G$, whose objects are the $G$-orbits of objects in $\hat{B}$. The morphism space $(\hat{B}/G)(a, b)$ for objects $a, b$ in $\hat{B}/G$
is the subspace of $\prod_{(x,y) \in a \times b} \hat{B}(x, y)$ consisting of those elements $(f_{y,x})$ satisfying $gf_{y,x} = f_{gy,gx}$ for all $g \in G, (x, y) \in a \times b$.

By the definition $\hat{B}/G$ has finitely many objects and the morphism spaces are finite-dimensional. It therefore associates naturally the finite-dimensional $K$-algebra $\bigoplus (\hat{B}/G)$ which is the direct sum of all morphism spaces in $\hat{B}/G$, called the orbit algebra of $\hat{B}$ with respect to $G$. A typical example is the trivial extension algebra $B \ltimes D(B)$ of an algebra $B$, which is isomorphic to the orbit algebra $\hat{B}/(\nu_{\hat{B}})$.

More generally, the $r$-fold trivial extension algebra $T(B)^{(r)}$ of $B$ is defined as the orbit algebra $\hat{B}/(\nu_{r\hat{B}})$, for $r \geq 1$. Note that $T(B)^{(1)} = T(B)$.

A selfinjective algebra $A$ is said to be of tilted type if $A$ is isomorphic to an orbit algebra $\hat{B}/G$ for a tilted algebra $B$ and an admissible automorphism group $G$ of $\hat{B}$. An important remark is that in this case any admissible automorphism group of $\hat{B}$ is an infinite cyclic group generated by a strictly positive automorphism of $\hat{B}$ (see [49, Theorem 7.1]). In the paper, by a tilted algebra we mean the endomorphism algebra $B = \text{End}_H(T)$ of a tilting module $T$ in the module category mod $H$ of a basic, connected, hereditary algebra $H$ over a field $K$.

The classification of all representation-finite selfinjective algebras over an algebraically closed field was given in the early 1980's by C. Riedtmann (see [13], [35], [36], [37]) via a combinatorial classification of the Auslander-Reiten quivers of these algebras, based on the fundamental paper [35] linking them to the Dynkin quivers. Equivalently, the Riedtmann's classification can be presented in terms of the orbit algebras as follows (see [41, Section 3]).

**Theorem 2.1.** Let $A$ be a non-simple, basic, connected, selfinjective algebra over an algebraically closed field $K$. Then $A$ is of finite representation type if and only if $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi)$, where $B$ is a tilted algebra of Dynkin type and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

In the theorem, we may replace “socle equivalence” by “isomorphism” if char($K$) $\neq$ 2, but it is not the case in general (see [37] and [41, Section 3]).

We would like to stress that a crucial role in proving the above interpretation of the Riedtmann’s classification theorem is played by the Galois covering techniques introduced by P. Gabriel in [21] and the description of the module categories of repetitive categories of tilted algebras of Dynkin type given by D. Hughes and J. Waschb"{u}sch in [24]. This was the starting point for the study of representation-infinite selfinjective algebras of polynomial growth in [38], where it was shown that all these algebras, having simply connected Galois coverings, are related to tilted algebras of Euclidean type and tubular algebras. Here, the new results on Galois coverings of representation-infinite algebras proved by P. Dowbor and A. Skowroński in [14], [15] are heavily applied.

The classification of arbitrary representation-infinite domestic selfinjective algebras was completed in the series of papers [10], [11], [12], [27] (see [41, Section 4]). In particular, we have the following theorem.

**Theorem 2.2.** Let $A$ be a basic, connected, selfinjective algebra over an algebraically closed field $K$. Then $A$ is representation-infinite domestic if and only if $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi)$, where $B$ is a tilted algebra of Euclidean type and $\varphi$ is a strictly positive automorphism of $\hat{B}$.
We would like to stress that, for any algebraically field $K$, there are representation-infinite domestic selfinjective algebras which are not orbit algebras of repetitive categories of tilted algebras of Euclidean type (see [12] for description of these algebras). We also point that the structure of the module categories mod $A$ of the representation-infinite domestic selfinjective algebras $A$ follows from Theorem 2.2 and the results proved in [2] and [38].

For an algebra $A$, the infinite radical $\text{rad}^\infty_A$ of mod $A$ is the intersection of all powers $\text{rad}^i_A$, $i \geq 1$, of the radical $\text{rad}_A$ of mod $A$. We note that, by a classical result of M. Auslander, $\text{rad}^\infty_A = 0$ if and only if $A$ is representation-finite. The following theorem is a consequence of Theorem 2.2 and the main result proved by O. Kerner and A. Skowroński in [26, Theorem].

**Theorem 2.3.** Let $A$ be a basic, connected, representation-infinite selfinjective algebra over an algebraically closed field $K$. Then the infinite radical $\text{rad}^\infty_A$ of mod $A$ is nilpotent if and only if $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi)$, where $B$ is a tilted algebra of Euclidean type and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

The classification of arbitrary non-domestic selfinjective algebras of polynomial growth was completed in the series of papers [6], [7], [8], [28] (see [41, Section 5] for details). In particular, we have the following theorem.

**Theorem 2.4.** Let $A$ be a basic, connected, selfinjective algebra over an algebraically closed field $K$. Then $A$ is non-domestic of polynomial growth if and only if $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi)$, where $B$ is a tubular algebra and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

In the theorem, we may replace “socle equivalence” by “isomorphism” if $\text{char}(K)$ is different from 2 and 3 (see [8] for details). We also mention that the structure of the module categories mod $A$ of non-domestic selfinjective algebras $A$ of polynomial growth follows from Theorem 2.4 and the results proved in [31] and [38].

Recall that an algebra $A$ is called periodic if it is periodic with respect to action of the syzygy operator $\Omega_A$ in the module category mod $A^e$ over the enveloping algebra $A^e = A^{op} \otimes_K A$. It is known that if $A$ is periodic then $A$ is selfinjective and all nonprojective indecomposable modules in mod $A$ are periodic with respect to action of the syzygy operator $\Omega_A$ in mod $A$. It is known that every non-simple, basic, connected, representation-finite selfinjective algebra $A$ is periodic, by a theorem proved by A. Dugas in [16]. The following theorem is a consequence of Theorem 2.4 and the main result proved by J. Białkowski, K. Erdmann and A. Skowroński in [9, Theorem 1.1].

**Theorem 2.5.** Let $A$ be a basic, connected, representation-infinite selfinjective algebra of polynomial growth over an algebraically closed field $K$. Then $A$ is periodic if and only if $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi)$, where $B$ is a tubular algebra and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

We also note that the algebras $B$ occurring in the above theorems are algebras of global dimension 1 or 2.

Recently, Theorems 2.1, 2.2 and 2.4 have been applied by S. Ariki, R. Kase, K. Miyamoto and K. Wada to provide in [1] a complete classification of selfinjective cellular algebras of polynomial growth, over algebraically closed fields $K$ with $\text{char}(K) \neq 2$. 

We refer also to [17] (respectively, [29]) for the representation theory and homological properties of the orbit algebras \( \hat{B}/G \) of tilted algebras \( B \) of wild type (respectively, quasi-tilted algebras \( B \) of wild canonical type). These algebras \( B \) are also of global dimension 1 or 2.

In general, the following problem arises naturally.

**Problem.** Describe the basic, connected, selfinjective algebras over an arbitrary field \( K \) which are socle equivalent to an orbit algebra \( \hat{B}/G \), where \( B \) is an algebra of finite global dimension and \( G \) is an infinite cyclic group generated by a strictly positive automorphism of \( \hat{B} \).

This seems to be a very hard problem in general. In the article, we present recent results concerning the above problem in the following case (coming naturally in several considerations):

- \( B \) is a tilted algebra and \( G \) is generated by \( \varphi \nu \hat{B} \) for a positive automorphism \( \varphi \) of \( \hat{B} \).

To illustrate the situation, let us consider the following simple example.

**Example 2.6.** Let \( Q_n \) be the cyclic quiver

\[
1 \overset{\alpha_1}{\rightarrow} 2 \overset{\alpha_2}{\rightarrow} \cdots \overset{\alpha_{n-1}}{\rightarrow} n \overset{\alpha_n}{\rightarrow} 1, \quad n > 1,
\]

and \( A_n = KQ_n/N \), where \( KQ_n \) is the path algebra and \( N \) is the ideal generated by the compositions of all consecutive three arrows. Let \( A = A_n \) for simplicity. Then \( A \) is a selfinjective Nakayama algebra with \( \text{rad}^3(A) = 0 \). Take the right module \( M = e_1A/\text{rad}^2(e_1A) \oplus e_2A/\text{rad}^2(e_2A) \), and the right annihilator \( I = r_A(M) \) of \( M \) in \( A \). Then

\[
I = \text{rad}^2(e_1A) \oplus \text{rad}^2(e_2A) \oplus e_3A \oplus \cdots \oplus e_nA.
\]

The factor algebra \( B = A/I \) is isomorphic to the path algebra of the quiver \( \Delta \) of the form

\[
1 \overset{\alpha_1}{\rightarrow} 2
\]

of Dynkin type \( A_2 \). It is easily seen that \( A \cong \hat{B}/(\varphi \nu \hat{B}) \) for a positive automorphism \( \varphi \) of \( \hat{B} \). For example, in the case \( n = 5 \), we have \( \varphi = \sigma \nu \hat{B} \) where \( \sigma \) is an automorphism satisfying \( \sigma^2 = \nu \hat{B} \).

Thus the selfinjective algebra \( A_n \) with arbitrarily \( n \) many simple modules is always recovered from the path algebra \( B = K\Delta \) and two indecomposable modules \( M_1, M_2 \) lying on a slice of \( \Gamma_{A_n} \).

### 3. Criterion Theorems

We recall ring theoretical criterion theorems for a selfinjective algebra \( A \) to be socle equivalent to an orbit algebra of tilted type. Throughout this section, \( A \) denotes a selfinjective algebra and \( \{ e_1, \ldots, e_r \} \) is a set of pairwise orthogonal primitive idempotents of \( A \) with \( 1_A = e_1 + \cdots + e_r \).

For an algebra \( \Lambda \) and a subset \( X \) of a right or left \( \Lambda \)-module \( M \), by we denote the right or left annihilator of \( X \) in \( \Lambda \) respectively,

\[
r_\Lambda(X) = \{ \lambda \in \Lambda \mid X\lambda = 0 \}, \quad l_\Lambda(X) = \{ \lambda \in \Lambda \mid \lambda X = 0 \}.
\]

Let \( I \) be an ideal of \( A \) and \( B = A/I \) the factor algebra. We can take an idempotent \( e \) of \( A \) such that \( e + I \) is the identity of \( B \), \( e = e_1 + \cdots + e_n \) (\( n \leq r \))
and \( \{e_1, \ldots, e_n\} \) is the set of all idempotents in \( \{e_i \mid 1 \leq i \leq r\} \) not contained in \( I \). Thus \( e_1 + I, \ldots, e_n + I \) are pairwise orthogonal primitive idempotents of \( B \) and \( 1_B = \bar{e}_1 + \cdots + \bar{e}_n \), where \( \bar{e}_i = e_i + I \in B \) for \( i \in \{1, \ldots, n\} \). By Krull-Schmidt theorem, such an idempotent \( e \) is uniquely determined up to inner automorphism of \( A \), and is called a \textit{residual identity} of \( B = A/I \).

By a Nakayama's theorem \cite[Theorem IV.6.10]{50} the annihilator operations \( l_A \) and \( r_A \) induce a Galois correspondence between the lattice \( R_A \) of right ideals and the lattice \( L_A \) of left ideals of \( A \): \( l_A : R_A \to L_A, \ r_A : L_A \to R_A \), each of which is the inverse to the other. An important consequence of this Galois correspondence is the following statement on the residual identity (see \cite[Proposition 2.3]{42} and \cite[Lemma 5.1]{47}).

\textbf{Lemma 3.1.} Let \( A \) be a selfinjective algebra and \( I \) an ideal of \( A \). Then an idempotent \( e \) of \( A \) is a residual identity of \( B = A/I \) if \( l_A(I) = eI \) or \( r_A(I) = Ie \). Moreover, in this case, \( \text{soc} A \subseteq I \) and \( l_{eAe}(I) = eI = r_{eAe}(I) \).

Observe that the canonical correspondence \( B \to eAe/eIe \), \( a + I \mapsto eae + eIe \), is an algebra isomorphism.

In the following first criterion theorem, the implication (ii) \( \Rightarrow \) (i) was proved in \cite{44} and the converse in \cite{47}.

\textbf{Theorem 3.2.} For a basic, connected, selfinjective algebra \( A \) over a field \( K \), the following statements are equivalent:

\begin{enumerate}[(i)]  
  \item \( A \) is isomorphic to an orbit algebra \( \hat{B}/(\varphi \nu_{\hat{B}}) \) for an algebra \( B \) and a positive automorphism \( \varphi \) of \( \hat{B} \).
  \item There is an ideal of \( I \) of \( A \) and an idempotent \( e \) of \( A \) such that
    \begin{enumerate}[(a)]      \item \( r_A(I) = eI \),
      \item The canonical algebra homomorphism \( \rho : eAe \to eAe/eIe \), \( eae \mapsto eae + eIe \), is a retraction, that is, there is an algebra homomorphism \( \eta : eAe/eIe \to eAe \) with \( \rho \eta = \text{id}_{eAe/eIe} \),
    \end{enumerate}
\end{enumerate}

It should be noticed that the statement (ii)(b) holds always in case \( K \) is algebraically closed. Under the statement (ii), \( B \) in (i) may be chosen as \( A/I \), and the idempotent \( e \) in (ii) is determined by \( I \) as a residual identity of \( A/I \), see Lemma 3.1.

It was observed that in many important situations we may replace a selfinjective algebra \( A \) by its socle deformation satisfying the statement (ii)(b). In order to explain this we need the notion of deforming ideal from \cite{42}.

\textbf{Definition 3.3.} Let \( A \) be a selfinjective algebra, \( I \) an ideal of \( A \), and \( e \) a residual identity of \( A/I \). Then \( I \) is said to be a \textit{deforming ideal} of \( A \) if the following conditions are satisfied:

\begin{enumerate}[(D1)]  
  \item \( l_{eAe}(I) = eIe = r_{eAe}(I) \),
  \item the valued quiver \( Q_{A/I} \) of \( A/I \) is acyclic.
\end{enumerate}

The following statement is an immediate consequence of Lemma 3.1 and Definition 3.3 and is important for further considerations.

\textbf{Proposition 3.4.} Let \( A \) be a selfinjective algebra, \( I \) an ideal of \( A \) and \( B = A/I \). Assume that
(i) \( r_A(I) = eI \) for an idempotent \( e \) of \( A \),
(ii) the valued quiver \( Q_B \) of \( B \) is acyclic.

Then \( I \) is a deforming ideal of \( A \) and \( e \) is a residual identity of \( B \).

Now assume that \( I \) is a deforming ideal of a selfinjective algebra \( A \). Then the canonical correspondence \( eAe/eIe \to A/I, eae + I \to eae + I \), is an algebra isomorphism, which makes \( I \) as an \( eAe/eIe \)-bimodule and allows us to define a new algebra, denoted by \( A[I] \), as follows. Let \( A[I] \) be the direct sum of \( K \)-vector spaces \( (eAe/eIe) \oplus I \) and define the multiplication in \( A[I] \) by the rule

\[
(b, x) \cdot (c, y) = (bc, by + xc + xy)
\]

for \( b, c \in eAe/eIe \) and \( x, y \in I \). Then \( A[I] \) is actually a \( K \)-algebra with the identity \( (e + eIe, 1_A - e) \) and the ideal \( \{(0, x) \mid x \in I\} \). By identifying \( x \in I \) with \( (0, x) \in A[I] \), we may regard \( I \) as an ideal of \( A[I] \), so that \( A[I]/I \) denotes the factor algebra of \( A[I] \) by \( I \) with the residual identity \( e = (e + eIe, 0) \). Thus, by identifying \( e \in A \) with \( (e, 0) \in A[I] \), we have

\[
A[I]/I = eAe/eIe \cong A/I, \quad eA[I]e = (eAe/eIe) \oplus eIe.
\]

Moreover, the canonical algebra epimorphism \( eA[I]e \to eA[I]e/eIe \) is a retraction, which is obvious from the definition but the main reason why the algebra \( A[I] \) has been introduced. In fact, \( A[I] \) keeps other important properties of \( A \) as shown in the next theorem established in [42, Theorem 4.1], [43, Theorem 3] and [48, Lemma 3.1].

**Theorem 3.5.** Let \( A \) be a selfinjective algebra over a field \( K \) and \( I \) a deforming ideal of \( A \). Then the following statements hold.

(i) \( A[I] \) is a selfinjective algebra with the same Nakayama permutation as \( A \) and \( I \) is a deforming ideal of \( A[I] \).
(ii) \( A \) and \( A[I] \) are socle equivalent.
(iii) \( A \) and \( A[I] \) are stably equivalent.
(iv) \( A[I] \) is a symmetric algebra if \( A \) is a symmetric algebra.

We note here that a socle equivalence does not imply a stably equivalence in general, as pointed out by J. Rickard (see [32, 50]). Moreover, a selfinjective algebra \( A \) with a deforming ideal \( I \) is not always isomorphic to \( A[I] \) (see [44, Example 4.2]), but this is the case when \( K \) is an algebraically closed field (32, Theorem 3.2)).

The following theorem proved in [44, Theorem 4.1] shows the importance of the algebras \( A[I] \).

**Theorem 3.6.** Let \( A \) be a selfinjective algebra, \( I \) an ideal of \( A \), \( B = A/I \) and \( e \) an idempotent of \( A \). Assume that \( r_A(I) = eI \) and \( Q_B \) is acyclic. Then the following statements are true:

(i) \( A[I] \) is isomorphic to an orbit algebra \( \hat{B}/(\varphi \nu_{\hat{B}}) \) for some positive automorphism \( \varphi \) of \( \hat{B} \).
(ii) \( A \) is a socle deformation of \( \hat{B}/(\varphi \nu_{\hat{B}}) \) for some positive automorphism \( \varphi \) of \( \hat{B} \).

**Proof.** First note that, by Proposition [3.3] the ideal \( I \) is a deforming ideal of \( A \), so \( A[I] \) is well defined.

(ii) follows from (i) and Theorem [3.5](ii). To show (i), we apply Theorem [3.2] to \( A[I] \) and its ideal \( I \). In fact, it is seen that \( r_{A[I]}(I) = eI \) in \( A[I] \), and the canonical algebra homomorphism \( eA[I]e \to eA[I]/eIe \) is a retraction. As a result,
the conditions (a) and (b) in (ii) of Theorem 3.2 are satisfied, which ensures the existence of an algebra isomorphism $A[I] \to \hat{B}/(\varphi \nu_{\hat{B}})$ for a positive automorphism $\varphi$ of $\hat{B}$.

In the theorem, the algebra $A$ is not necessarily isomorphic to an orbit algebra $\hat{B}/(\varphi \nu_{\hat{B}})$, where $B$ is an algebra and $\varphi$ is a positive automorphism of $\hat{B}$ (see [45, Proposition 4]).

The following result proved in [46, Proposition 3.2] describes a situation when the algebras $A$ and $A[I]$ are isomorphic.

**Theorem 3.7.** Let $A$ be a selfinjective algebra over a field $K$, having a deforming ideal $I$, $B = A/I$, $e$ be a residual identity of $B$, and $\nu$ the Nakayama permutation of $A$. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand $e_i$ of $e$. Then the algebras $A$ and $A[I]$ are isomorphic. In particular, $A$ is isomorphic to an orbit algebra $\hat{B}/(\varphi \nu_{\hat{B}})$ for some positive automorphism $\varphi$ of $\hat{B}$.

4. Hereditary stable slice

In this section we explain a new characterization from [53] of the socle deformations of selfinjective orbit algebras $\hat{B}/G$ of tilted type where $G$ is an admissible group generated by $\varphi \nu_{\hat{B}}$ for a positive automorphism $\varphi$ of $\hat{B}$.

Let $A$ be a selfinjective algebra over a field $K$ and $\Gamma_A$ the stable Auslander-Reiten quiver of $A$, obtained from $\Gamma_A$ by removing the projective modules and the arrows attached to them. By $\text{ind}\mathcal{P}(A)$ we understand the family of indecomposable projective modules in $\text{mod}\,A$.

Following [51], a full valued subquiver $\Delta$ of $\Gamma_A$ is said to be a stable slice if the following conditions are satisfied:

1. $\Delta$ is connected, acyclic, and without projective modules.
2. For any valued arrow $V \xrightarrow{(a,a')} U$ in $\Gamma_A$ with $U$ in $\Delta$ and $V$ non-projective, $V$ belongs to $\Delta$ or to $\tau_A \Delta$.
3. For any valued arrow $U \xrightarrow{(b,b')} V$ in $\Gamma_A$ with $U$ in $\Delta$ and $V$ non-projective, $V$ belongs to $\Delta$ or to $\tau_A^{-1} \Delta$.

A stable slice $\Delta$ of $\Gamma_A$ is said to be right regular (respectively, left regular) if $\Delta$ does not contain the radical $P$ (respectively, the socle factor $P/\text{soc}\,P$) of any $P$ from $\text{ind}\mathcal{P}(A)$. A stable slice $\Delta$ is said to be almost right regular if for any $P$ from $\text{ind}\mathcal{P}(A)$ with $P$ lying on $\Delta$, $P$ is a sink of $\Delta$, and almost left regular if for any $P \in \text{ind}\mathcal{P}(A)$ with $P/\text{soc}\,P$ lying on $\Delta$, $P/\text{soc}\,P$ is a source of $\Delta$. Moreover, for a finite stable slice $\Delta$, we denote by $M(\Delta)$ the direct sum of all modules lying on $\Delta$ and $H(\Delta) = \text{End}_A(M(\Delta))$ the endomorphism algebra of $M(\Delta)$. Then a finite stable slice $\Delta$ of $\Gamma_A$ is said to be hereditary if the endomorphism algebra $H(\Delta) = \text{End}_A(M(\Delta))$ of $M(\Delta)$ is a hereditary algebra and its valued quiver $Q_{H(\Delta)}$ is the opposite quiver $\Delta^\text{op}$ of $\Delta$.

The following theorem is proved in [53, Theorem 1.1] and extends results established in [44] and [51] to a general case, as explained in the next section.

**Theorem 4.1.** Let $A$ be a selfinjective algebra over a field $K$. The following statements are equivalent:

1. $\Gamma_A$ admits a hereditary almost right regular stable slice.
2. $\Gamma_A$ admits a hereditary almost left regular stable slice.
(iii) A is socle equivalent to an orbit algebra $\hat{B}/(\varphi \nu_{\hat{B}})$, where $B$ is a tilted algebra and $\varphi$ is a positive automorphism of $\hat{B}$.

It should be stressed that, if $K$ is algebraically closed, we may replace in (iii) “socle equivalent” by “isomorphic”, but the replacement is not possible in general.

Idea of the proof. The equivalence of (i) and (ii) is easy, because a selfinjective algebra $A$ satisfies (i) if and only if $A^\text{op}$ satisfies (ii). Similarly, $A$ satisfies (iii) if and only if $A^\text{op}$ satisfies (iii) in view of the canonical algebra isomorphism

$$(\hat{B}/(\varphi \nu_{\hat{B}}))^{\text{op}} \cong \hat{B}^{\text{op}}/(\psi \nu_{\hat{B}}^{\text{op}})$$

for a positive automorphism $\psi$ of $\hat{B}^{\text{op}}$.

Now we explain how to define the algebra $B$ in (iii) under the assumption (i) and conversely how to find an almost right regular stable slice being hereditary in (i) under the condition (iii).

Assume that a selfinjective algebra $A$ has a hereditary and almost right regular stable slice $\Delta$ in the Auslander-Reiten quiver $\Gamma_A$. Take the direct sum $M$ of all indecomposable modules in $\text{mod } A$ lying on $\Delta$, and let $I$ be the right annihilator $r_A(M) = \{a \in A \mid Ma = 0\}$ of $M$, $B = A/I$ the factor algebra of $A$, and $H = \text{End}_A(M)$. By the definition of hereditary stable slice, $H$ is a hereditary algebra and the valued quiver $Q_H$ of $H$ is the opposite quiver $\Delta^{\text{op}}$ of $\Delta$. Then it is shown in [53, Section 3] that $B$ is a desired tilted algebra and $I$ a deforming ideal of $A$ satisfying $r_A(I) = eI$, so that Theorem 3.6 implies the assertion (iii).

Conversely, assume that a selfinjective algebra $A$ satisfies the assertion (iii). We shall show how we find a hereditary and almost right regular stable slice $\Delta$ in $\Gamma_A$. Let $A$ be socle equivalent to $\hat{B}/(\varphi \nu_{\hat{B}})$ as in (iii), and $\Lambda = \hat{B}/(\varphi \nu_{\hat{B}})$. We observe that, if $\Lambda$ has a stable slice $\Delta$ in $\Gamma_\Lambda$ being hereditary and almost right regular, then $A$ has a hereditary almost right regular stable slice, which corresponds to $\Delta$ under the given isomorphism $A/\text{soc}(A) \rightarrow \Lambda/\text{soc}(\Lambda)$, so $A$ may be identified with $\Lambda$, that is, $A = \hat{B}/(\varphi \nu_{\hat{B}})$. We consider two cases:

1. Assume $A$ is of infinite representation type. Since $B$ is not of finite representation type, by [24], [23], the tilted algebra $B$ is not of Dynkin type. It follows from general theory developed in [2], [17], [44] and [45] that $\Gamma_A$ admits an acyclic component $C$ containing a right stable full translation subquiver $D$ which is closed under successors in $C$ and generalized standard (see below for definition). Then any stable slice $\Delta$ in $D$ is a hereditary and right regular stable slice.

2. Assume $A$ is of finite representation type. (a) First consider the case when $A$ is a selfinjective Nakayama algebra. Then, for any indecomposable projective module $P$, the module $P/\text{soc } P$ is always the radical of a projective module. For an arbitrarily chosen indecomposable projective module $P$, consider the sectional path

$$\Delta : \text{soc } P = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = \text{rad } P$$

of canonical irreducible monomorphisms. Then $\Delta$ is a hereditary and almost right regular stable slice $\Gamma_A$.

(b) Next assume that $A$ is of finite representation type but not a Nakayama algebra. Observe that then there exists an indecomposable projective module $P$ such that $P/\text{soc } P$ is not the radical of a projective module, which allows us to take
the full subquiver $\Delta_P$ of $\Gamma_A$ whose vertices are $\tau_A^{-1}(P/\operatorname{soc}P)$ and the indecomposable modules $X$ such that there is a non-trivial sectional path in $\Gamma_A$ from $P/\operatorname{soc}P$ to $X$. It is shown in [53, Proposition 4.4] that $\Delta = \tau_A(\Delta_P)$ is a hereditary and right regular stable slice [25, Theorem 3.1].

We recall some definitions from [51]. Let $A$ be a selfinjective $K$-algebra. A stable slice $\Delta$ of $\Gamma_A$ is said to be semi-regular if $\Delta$ is left or right regular, and regular if $\Delta$ is left and right regular. Moreover, a stable slice $\Delta$ is double $\tau_A$-rigid if $\operatorname{Hom}_A(X,\tau_AY) = 0$ and $\operatorname{Hom}_A(\tau_A^{-1}X,Y) = 0$ for all indecomposable modules $X$ and $Y$ from $\Delta$. Note that a double $\tau_A$-rigid stable slice $\Delta$ is finite ([40], [3, Lemma VIII.5.3]) and a full valued subquiver of a connected component $C$ of $\Gamma_A$ intersecting every $\tau_A$-orbit in $C$ exactly once. The following theorem proved in [51, Theorem 1] is a special case of Theorem 4.1.

**Theorem 4.2.** Let $A$ be a basic, connected, selfinjective algebra over a field $K$. The following statements are equivalent:

(i) $\Gamma_A$ admits a semi-regular double $\tau_A$-rigid stable slice.

(ii) $A$ has one of the following forms:

(a) $A \cong \hat{B}/(\varphi \nu_{\hat{B}})$ as algebras, where $B = \operatorname{End}_H(T)$ for a hereditary algebra $H$ and a tilting module $T$ in $\operatorname{mod}H$ either without nonzero projective direct summand or without nonzero injective direct summand, and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

(b) $A$ is socle equivalent to $\hat{B}/(\varphi \nu_{\hat{B}})$, where $B = \operatorname{End}_H(T)$ for a hereditary algebra $H$ and a tilting module $T$ in $\operatorname{mod}H$ without nonzero projective or injective direct summands, and $\varphi$ is a rigid automorphism of $\hat{B}$.

Moreover, if $K$ is an algebraically closed field, we may replace in (ii)(b) “socle equivalent” by “isomorphic”.

5. Selfinjective orbit algebras of infinite representation type

The first study of the socle deformations of a selfinjective orbit algebra over an arbitrary field $K$ was given in [42], [44], where representation-infinite selfinjective algebras are mainly considered. In this section, we present two theorems on representation-infinite socle deformations of a selfinjective orbit algebra of a tilted type.

Following [39], a full translation subquiver $\Sigma$ of the Auslander-Reiten quiver $\Gamma_A$ of an algebra $A$ is said to be generalized standard if $\bigcap_{m>0} \operatorname{rad}^m\Sigma = 0$. The following theorem is from [42] and [44].

**Theorem 5.1.** Let $A$ be a basic, connected, selfinjective algebra over a field $K$. The following statements are equivalent:

(i) $\Gamma_A$ admits an acyclic generalized standard right stable full translation subquiver $\Sigma$ which is closed under successors in $\Gamma_A$.

(ii) $\Gamma_A$ admits an acyclic generalized standard left stable full translation subquiver $\Omega$ which is closed under predecessors in $\Gamma_A$.

(iii) $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi \nu_{\hat{B}})$, where $B$ is a tilted algebra not of Dynkin type and $\varphi$ is a positive automorphism of $\hat{B}$.

Moreover, if $K$ is an algebraically closed field, we may replace in (iii) “socle equivalent” by “isomorphic”.
In order to apply Theorem 4.1, let $\Sigma$ be the generalized standard translation subquiver given in the statement (i), and let $I$ be the annihilator $\text{ann}_A \Sigma$ of $\Sigma$ in $A$ and $B = A/I$. Then it is shown in [42] that $I$ is a deforming ideal of $A$ such that
\[ r_A(I) = eI \quad \text{and} \quad l_A(I) = Ie, \]
for a residual identity $e$ of $B$. It should be noted that $\text{ann}_A(\Sigma)$ is the same as the annihilator in $A$ of any section in $\Sigma$.

The following theorem from [51, Theorem 2] shows another characterization of the socle deformations of a representation-infinite orbit algebra $\hat{B}/(\varphi \nu_{\hat{B}})$ of tilted type, and is a consequence of Theorem 4.2.

**Theorem 5.2.** Let $A$ be a basic, connected, selfinjective algebra of infinite representation type over a field $K$. The following statements are equivalent:

(i) $\Gamma_A$ admits a regular double $\tau_A$-rigid stable slice.

(ii) $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi \nu_{\hat{B}})$, where $B$ is a tilted algebra not of Dynkin type and $\varphi$ is a positive automorphism of $\hat{B}$.

Moreover, if $K$ is an algebraically closed field, we may replace in (ii) “socle equivalent” by “isomorphic”.

6. **Selfinjective orbit algebras of finite representation type**

In this section, we present recent results concerning the structure of selfinjective algebras of finite representation type, which are socle equivalent to selfinjective orbit algebras of tilted algebras of Dynkin type.

We first recall the following old result proved by C. Riedtmann [34] and G. Todorov [54] describing the structure of the stable Auslander-Reiten quiver of a selfinjective algebra of finite representation type (see [50, Theorem IV.15.6] for a proof).

**Theorem 6.1.** Let $A$ be a non-simple, basic, connected, selfinjective algebra of finite representation type over a field $K$. Then the stable Auslander-Reiten quiver $\Gamma_A$ of $A$ is isomorphic to the orbit valued translation quiver $\mathbb{Z} \Delta / G$, where $\Delta$ is a Dynkin quiver of type $A_n(n \geq 1)$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_n(n \geq 4)$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$, and $G$ is an admissible infinite cyclic group of automorphisms of $\mathbb{Z} \Delta$.

We note that if $A = \hat{B}/(\varphi)$ is an orbit algebra with $B$ a tilted algebra of Dynkin type $\Delta$ over a field $K$ and $\varphi$ is a strictly positive automorphism of $\hat{B}$ then $\Gamma_A \cong \mathbb{Z} \Delta / G$, where $G$ is the infinite cyclic group of automorphisms of $\mathbb{Z} \Delta$ induced by $\varphi$. It would be interesting to know when a selfinjective algebra $A'$ over an arbitrary field $K$ is socle equivalent to a selfinjective orbit algebra $A = \hat{B}/(\varphi)$ of Dynkin type. It follows from the classification result of C. Riedtmann, presented in Section 2 that all representation-finite selfinjective algebras over an algebraically closed field have this property. But for an arbitrary field $K$, this seems to be a hard problem.

We consider first selfinjective Nakayama algebras. A module $X$ in a module category $\text{mod} A$ is said to be composition free if all simple composition factors of $X$ occur with multiplicity one. The following result is a consequence of Theorem 4.1.

**Theorem 6.2.** Let $A$ be a non-simple, basic, connected, selfinjective Nakayama algebra over a field $K$. Then the following statements are equivalent:

(i) Any indecomposable projective module $P$ in $\text{mod} A$ has composition free radical $\text{rad} P$. 

(ii) Any indecomposable projective module $P$ in $\text{mod } A$ has composition free socle factor $P/\text{soc } P$.

(iii) $A$ is socle equivalent to an orbit algebra $\tilde{H}/(\varphi \nu_{\tilde{H}})$, where $H$ is a hereditary Nakayama algebra and $\varphi$ is a positive automorphism of $\tilde{H}$.

(iv) $A$ is socle equivalent to an orbit algebra $\tilde{B}/(\varphi \nu_{\tilde{B}})$, where $B$ is an algebra and $\varphi$ is a positive automorphism of $\tilde{B}$.

Proof. The equivalence (i) $\iff$ (ii) and the implication (iii) $\Rightarrow$ (iv) are trivial.

Assume that (i) holds. Let $P$ be an indecomposable projective module in $\text{mod } A$. Since $A$ is Nakayama, we have in $\Gamma_A$ the sectional path

$$\Delta : \text{soc } P = X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n = \text{rad } P,$$

given by the irreducible inclusion monomorphisms, which is an almost right regular stable slice of $\Gamma_A$. Moreover, the slice $\Delta$ is hereditary, because $\text{rad } P$ is composition free. Therefore, applying arguments from the proof of Theorem 4.1 we conclude that $A$ is socle equivalent to an orbit algebra $\tilde{H}/(\varphi \nu_{\tilde{H}})$ for a hereditary Nakayama algebra $H$ with the Gabriel quiver $Q_H = \Delta = \Delta^{\text{op}}$ and a positive automorphism $\varphi$ of $\tilde{H}$. Hence (iii) is satisfied.

Assume that (iii) holds. Let $H$ be a hereditary Nakayama algebra and $\varphi$ is a positive automorphism of $\tilde{H}$ such that the orbit algebra $\Lambda = \tilde{H}/(\varphi \nu_{\tilde{H}})$ is socle equivalent to $A$. Without loss of generality, we may assume that the quotient algebras $\Lambda/\text{soc}(\Lambda)$ and $A/\text{soc}(A)$ are equal. Let $1_H = e_1 + \cdots + e_n$ be a decomposition of the identity of $H$ into a sum of pairwise orthogonal primitive idempotents. Then it follows from the definition of $\tilde{H}$ that every indecomposable projective $\tilde{H}$-module $e_{m,i}\tilde{H}$, $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, is uniserial. Moreover, for any $(m, i)$ and $(r, j)$ in $\mathbb{Z} \times \{1, \ldots, n\}$ with $m \neq r$, the socle factors $e_{m,i}\tilde{H}/\text{soc}(e_{m,i}\tilde{H})$ and $e_{r,j}\tilde{H}/\text{soc}(e_{r,j}\tilde{H})$ have disjoint simple composition factors. Then it follows that any indecomposable projective module $P$ in $\text{mod } \Lambda$ has composition free socle factor $P/\text{soc } P$. Since the socle factors of indecomposable projective modules in $\text{mod } \Lambda$ and $\text{mod } A$ are exactly the indecomposable projective modules in $\text{mod } \Lambda/\text{soc}(\Lambda) = \text{mod } A/\text{soc}(A)$, we conclude that $A$ satisfies (ii).

Assume now that (iv) holds, that is, $A$ is socle equivalent to an orbit algebra $\Lambda = \tilde{B}/(\varphi \nu_{\tilde{B}})$, where $B$ is an algebra and $\varphi$ is a positive automorphism of $\tilde{B}$. Clearly, such an algebra $B$ is basic and connected. We claim that $B$ is a hereditary Nakayama algebra, and hence $A$ satisfies (iii). Since $A$ is a Nakayama algebra and $A/\text{soc}(A) \cong \Lambda/\text{soc}(\Lambda)$, we conclude that $\Lambda$ is a non-simple selfinjective Nakayama algebra. We also note that $B$ is a quotient algebra of $\Lambda$, because $\Lambda$ is an orbit algebra of $\tilde{B}$ with respect to $(\varphi \nu_{\tilde{B}})$ with $\varphi$ positive. Therefore, we obtain that $B$ is a Nakayama algebra. Hence, we have only to show that any non-simple projective module in $\text{mod } B$ has projective socle. Let $1_B = e_1 + \cdots + e_n$ be a decomposition of the identity of $\tilde{B}$ into a sum of pairwise orthogonal primitive idempotents. Remember that $e_{m,i}$, $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, are pairwise orthogonal primitive idempotents of $\tilde{B}$ and $e_{m,i} \tilde{B} = e_i B \oplus D(Be_i)$. For convenience, we write $B_m$ for $1_m \tilde{B}$, where $1_m = e_{m,1} + \cdots + e_{m,n}$, and $D(\tilde{B}) = \bigoplus_{m \in \mathbb{Z}} D(B_m)$. Note that $B_m$ is a copy of $B$, and observe that

$$\text{rad } \tilde{B} = \left( \bigoplus_{m \in \mathbb{Z}} \text{rad } B_m \right) \oplus D(\tilde{B}).$$
Since $\Lambda = \hat{B}/(\varphi \nu_B)$ is a Nakayama algebra, each $e_{m,i}\hat{B}$ and so $e_iB$ is uniserial for all $m \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$. Now let $P$ be a non-simple indecomposable projective module in $\mod B$, and assume $P = e_1B$, without loss of generality. Let

$$\soc P \cong \top(e_1B)$$

for some $t \in \{1, \ldots, n\}$. Take the $\hat{B}$-submodule $M = \soc(e_1B) \oplus D(Be_1)$ of $e_{0,1}\hat{B} = e_1B \oplus D(Be_1)$. Then $M(\rad \hat{B}) = D(Be_1)$, because $M/D(Be_1)$ is a simple $\hat{B}$-module and $M$ is uniserial. Let $f : e_{0,t}\hat{B} \rightarrow e_{0,1}\hat{B}$ be a $\hat{B}$-homomorphism with $f(e_{0,t}\hat{B}) = M$. It is shown that

$$M \cdot \mathbb{D}(\hat{B}) = D(Be_1)$$

taking into account the uniseriality of $e_{0,t}\hat{B}$. On the other hand, $f(e_{0,t}\hat{B} \cdot \rad \hat{B}) = M(\rad \hat{B})$ and $f(e_{0,t}\hat{B} \cdot \mathbb{D}(\hat{B})) = M \cdot \mathbb{D}(\hat{B})$. Consequently, we have $f(e_{0,t}\hat{B} \cdot \rad \hat{B}) = f(e_{0,t}\hat{B} \cdot \mathbb{D}(\hat{B}))$, which implies

$$e_{0,t}\hat{B} \cdot \rad \hat{B} = e_{0,t}\hat{B} \cdot \mathbb{D}(\hat{B}) + \ker f = e_{0,t}\hat{B} \cdot \mathbb{D}(\hat{B}).$$

Indeed, since $f(e_{0,1}\hat{B} \cdot \mathbb{D}(\hat{B})) = D(Be_1) \neq 0$ and $e_{0,t}\hat{B}$ is uniserial, $\ker f$ is contained in $e_{0,t}\hat{B} \cdot \mathbb{D}(\hat{B})$. As a result, we conclude that

$$\top(e_1B) = e_{0,1}\hat{B}/(e_{0,1}\hat{B} \cdot \rad \hat{B}) = e_{0,1}\hat{B}/(e_{0,1}\hat{B} \cdot \mathbb{D}(\hat{B})) = e_1B,$$

and hence $\soc P$ is projective in $\mod B$, as desired. \hfill $\Box$

We obtain also the following consequence of Theorems 3.7 and 3.2

**Corollary 6.3.** Let $A$ be a non-simple, basic, connected, selfinjective Nakayama algebra over a field $K$. Then the following statements are equivalent:

(i) Any indecomposable projective module $P$ in $\mod A$ is composition free.

(ii) $A$ is isomorphic to an orbit algebra $\hat{H}/(\varphi \nu_B)$, where $H$ is a hereditary Nakayama algebra and $\varphi$ is a strictly positive automorphism of $\hat{H}$.

(iii) $A$ is isomorphic to an orbit algebra $\hat{B}/(\varphi \nu_B)$, where $B$ is an algebra and $\varphi$ is a strictly positive automorphism of $\hat{B}$.

Following [34] a short cycle in a module category $\mod A$ is a sequence $X \rightarrow Y \rightarrow X$ of non-isomorphisms between two indecomposable modules $X$ and $Y$ in $\mod A$. It was shown in [34 Corollary 2.2] that if an indecomposable module $M$ does not lie on a short cycle in $\mod A$ then $M$ is uniquely determined (up to isomorphism) by its composition factors, that is, by its image $[M]$ in the Grothendieck group $K_0(A)$. Moreover, it is known that if $\mod A$ has no short cycles, then $A$ is of finite representation type [22]. We have also the following fact proved in [25 Lemma 3.2].

**Lemma 6.4.** Let $A$ be a selfinjective algebra which does not admit a short cycle in $\mod A$. Then all stable slices in $\Gamma_A$ are double $\tau_A$-rigid.

We also recall the following fact proved in [25 Theorem 3.1].

**Theorem 6.5.** Let $A$ be a non-simple, connected, selfinjective algebra of finite representation type over a field $K$. The following statements are equivalent:

(i) $\Gamma_A$ admits a semi-regular stable slice.

(ii) $A$ is not a Nakayama algebra.
The following theorem was proved by A. Jaworska-Pastuszak and A. Skowroński in [25].

**Theorem 6.6.** Let $A$ be a non-simple, basic, connected, selfinjective algebra over a field $K$. The following statements are equivalent:

(i) $\mod A$ has no short cycles.

(ii) $A$ is isomorphic to an orbit algebra $\tilde{B}/(\psi \nu \tilde{B})$, where $B$ is a tilted algebra of Dynkin type and $\psi$ is a strictly positive automorphism of $\tilde{B}$.

It was shown in [25, Theorem 3.7] that the equivalence of (i) and (ii) holds if $A = \tilde{B}/G$ for a tilted algebra $B$ of Dynkin type and an admissible group $G$ of automorphisms of $\tilde{B}$, applying the fact that the push-down functor $F_\lambda: \mod \tilde{B} \to \mod A$, induced by the canonical Galois covering functor $F: \tilde{B} \to \tilde{B}/G = A$, is a Galois covering of module categories. We explain now why (i) implies (ii).

Therefore, assume that $\mod A$ has no short cycles. By the above comments, it is enough to show that $A$ is isomorphic to an orbit algebra $\tilde{B}/(\phi \psi \tilde{B})$ for a tilted algebra $B$ of Dynkin type and a strictly positive automorphism $\phi$ of $\tilde{B}$. If $A$ is not a Nakayama algebra, this follows from Theorems 4.2 and 6.5, and Lemma 6.4. Assume $A$ is a Nakayama algebra. Since every indecomposable projective module $P$ in $\mod A$ does not lie on a short cycle, we easily conclude that $P$ is composition free, and then $A$ has a required form $\tilde{B}/(\phi \psi \tilde{B})$ by Theorem 6.2.

In [9] M. Blaszkievicz and A. Skowroński investigated the structure of selfinjective algebras of finite representation type whose module category admits maximal almost split sequences. For an algebra $A$ and an almost split sequence

$$0 \to \tau_A X \to Y \to X \to 0$$

in $\mod A$, we may consider the numerical invariant $\alpha(X)$ of $X$ being the number of summands in a decomposition $Y = Y_1 \oplus \cdots \oplus Y_r$ of $Y$ into a direct sum of indecomposable modules. Then $\alpha(X)$ measures the complexity of homomorphisms in $\mod A$ with domain $\tau_A X$ and codomain $X$. It has been proved by R. Bautista and S. Brenner in [4] (see also [30] for an alternative proof) that if $A$ is of finite representation type and $X$ is an indecomposable nonprojective module in $\mod A$, then $\alpha(X) \leq 4$, and if $\alpha(X) = 4$, then the middle term $Y$ of an almost split sequence in $\mod A$ with the right term $X$ admits an indecomposable projective-injective direct summand. Recall that (by general theory) if $P$ is an indecomposable projective-injective module in $\mod A$, then there is in $\mod A$ an almost split sequence of the form

$$0 \to \rad P \to (\rad P/ \soc P) \oplus P \to P/ \soc P \to 0.$$

If $A$ is of finite representation type, then such an almost split sequence with $\alpha(P/ \soc P) = 4$ is called a maximal almost split sequence in $\mod A$.

Let $A$ be a selfinjective algebra of finite representation type, and assume that $\mod A$ admits an indecomposable projective-injective module $P$ with $\alpha(P/ \soc P) = 4$. We denote by $\Delta_P$ the full valued subquiver of $\Gamma_A$ given by the module $\tau_A^{-1}(P/ \soc P)$ and all indecomposable modules in $\mod A$ such that there is a non-trivial sectional path in $\Gamma_A$ from $P/ \soc P$ to $X$. It was shown in [9, Theorem 5.2] that $\Delta_P$ is a Dynkin quiver such that $\Gamma_A = \mathbb{Z}\Delta_P/G$ for an admissible automorphism group $G$ of the translation quiver $\mathbb{Z}\Delta_P$. We denote by $M_P$ the direct sum of all modules lying on $\Delta_P$. Then the main theorem in [9, Theorems 1 and 2] can be formulated as follows.
Theorem 6.7. Let $A$ be a basic, connected, selfinjective algebra over a field $K$. Then the following statements are equivalent:

(i) $A$ is of finite representation type and admits an indecomposable projective module $P$ with $\alpha(P/\text{soc}P) = 4$ and $\text{Hom}_A(M_P, \tau_A M_P) = 0$.

(ii) $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi \nu_{B}^m)$ for some positive integer $m$, a rigid automorphism $\varphi$ of $\hat{B}$, and a tilted algebra $B$ of Dynkin type having an indecomposable projective module $Q$ whose top is injective and the radical is a direct sum of three indecomposable projective modules.

We note that, if $K$ is algebraically closed, then the condition $\text{Hom}_A(M_P, \tau_A M_P) = 0$ in (i) is superfluous, and we may replace in (ii) “socle equivalent” by “isomorphic”. We refer to [9, Section 3] for a complete description of the tilted algebras $B$ of Dynkin type occurring in (ii).

7. Stable equivalences of selfinjective orbit algebras

We end this article with the following combination of [43, Theorem 1] and [48, Theorem 1], and its consequence.

Theorem 7.1. Let $A$ be a non-simple, basic, connected, selfinjective algebra over a field $K$. Then the following statements are equivalent:

(i) $A$ is stably equivalent to an orbit algebra $\hat{B}/(\varphi \nu_{B})$, where $B$ is a tilted algebra and $\varphi$ is a positive automorphism of $\hat{B}$.

(ii) $A$ is socle equivalent to an orbit algebra $\hat{B}/(\varphi \nu_{B})$, where $B$ is a tilted algebra and $\varphi$ is a positive automorphism of $\hat{B}$.

Moreover, if $K$ is an algebraically closed field, we may replace in (ii) “socle equivalent” by “isomorphic”.

Corollary 7.2. Let $K$ be an algebraically closed field. Then the class of orbit algebras $\hat{B}/(\varphi \nu_{B})$, where $B$ is a tilted algebra over $K$ and $\varphi$ is a positive automorphism of $\hat{B}$, is closed under stable equivalences.

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