Dimensional Reduction at High Temperature for Fermions

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Abstract

The concept of dimensional reduction in the high temperature regime is generalized to static Green’s functions of composite operators that contain fermionic fields. The recognition of a natural kinematic region where the lowest Matsubara modes are close to their mass-shell, and the ultraviolet behavior of the running coupling constant of the original theory are crucial for providing the necessary scale hierarchy. The general strategy is illustrated in the asymptotically-free Gross-Neveu model in 1 + 1 dimensions, where we verify that dimensional reduction occurs to the leading order in $g^2(T)$. We also find, in the same model, that the scale parameter characterizing the dependence on temperature of the coupling constant in the reduced theory, $\Lambda_T$, is considerably smaller than $\Lambda_{\overline{\text{MS}}}$. Implications of our results for QCD are also discussed.

Keywords: Finite Temperature Field Theory, Dimensional Reduction, Scale Parameter
I. INTRODUCTION

Physics very often simplifies under extreme situations. For example, specific observables in a $D + 1$-dimensional theory at high temperature ($T$) can sometimes be described by a $D$-dimensional theory: this phenomenon is known as dimensional reduction (DR) \[1\]. The basic concept of DR is that in the high-$T$ limit all temporal excitations are naturally $O(T)$. If, for either kinematic or dynamical reasons, there exist modes of order less than $T$, only these few light modes remain active, while the others decouple. This qualitative expectation can be formalized \[1\] in some theories, such as QED, order by order in perturbation theory, analogously to the usual heavy mass decoupling \[2\].

The approach to DR is easier, when there exists a clear scale separation already at the tree-level. For instance, kinematics makes this scale hierarchy manifest for observables made by elementary bosonic fields. The non-zero Matsubara frequencies act like masses of $O(T)$ compared to the zero-modes, and these non-zero modes, both bosonic and fermionic, can be integrated out. If no other dynamical phenomenon occurs, the result is an effective theory with one less dimension, which can be used to describe static phenomena of the original theory in the high-$T$ limit. So far, the existing literature has exclusively dealt with the dynamics of these bosonic zero modes \[3–7\].

This work will focus on situations where the observables are made explicitly by fermionic fields, the lowest modes are also of $O(T)$, and, therefore, there is no obvious scale separation. If at all, the scale separation must be generated dynamically. We shall systematically study, in the framework of perturbation theory and renormalization group, how the concept of DR can be generalized to include these situations. The purpose of deriving an effective theory for fermions at high-$T$ by integrating out heavy Matsubara modes is to simplify the physics for screening phenomena involving quarks, which are well studied in lattice QCD. Our result provides a formal basis for the recent interpretation \[8–12\] of lattice data \[13–19,10\] related to screening processes at high-$T$, which assumes a picture where only the lowest Matsubara frequencies are important and can be treated non-relativistically.

After stating the precise criterion for DR for observables involving fermionic fields, we illustrate the general principle in the Gross-Neveu model in 1+1 dimensions, and apply it to the calculation of the screening mass.

In the process, we also calculate the scale parameter that fixes the dependence of the effective coupling constant on the temperature.

II. GENERAL STRATEGY AND CRITERIA FOR DIMENSIONAL REDUCTION

Let us first recapitulate the criterion for DR when only static fundamental bosons appear in the external lines. We say that the $D + 1$ dimensional Lagrangian $\mathcal{L}_{D+1}$ undergoes DR to a specific $D$ dimensional Lagrangian $\mathcal{L}_D$ as its effective theory, if the following happens. Static Green’s functions of $\mathcal{L}_{D+1}$ with small external momenta ($|p| \ll T$) are equal to the corresponding Green’s functions of $\mathcal{L}_D$ up to corrections of order $|p|/T$ and $m/T$, where $m$ is any external dimensionful parameter in $\mathcal{L}_{D+1}$, e.g. a mass. In general, the form and parameters of $\mathcal{L}_D$ are determined by the original theory.
As stressed by Landsman [6], naive expectations based on tree-level power countings may fail if there are dynamically generated scales of order $T$. Nevertheless, these dynamically generated scales must be proportional to some power of the coupling constant $M \sim g^n T$, since they are generated by the interaction. Therefore, they induce corrections of order $M/T \sim g^n$. If $g$ is small, the concept is still useful, and we say that the reduction is partial.

Situations where composite operators made by fermions appear in the external lines are of great phenomenological interest: typical examples are the electromagnetic current, mesonic and baryonic interpolating fields in QCD. Observables in screening processes are, for instance, extracted from the spatial correlation functions of these currents at high $T$ and large distances relative to the thermal Compton wavelength $1/T$.

When fermions appear in external lines there are two main differences with the fundamental bosons case. The first difference is that the lowest Matsubara frequencies for fermions, $\omega_\pm = \pm \pi T$, are also of order $T$, and hence it is not obvious that they dominate over the heavier modes. The second difference is that we often need to consider external momenta of order $T$. In fact, if we want the fermions to be close to their mass shell in the reduced theory (this is eventually the physically relevant region), $|p|$ must be of order $T$.

Since $\omega_\pm$ acts in the reduced theory as a large mass, it has been proposed [8–11] that fermions might undergo a non-relativistic dimensional reduction. This motivates us to define $q^2 \equiv p^2 + (\pi T)^2$, and expand the Green’s functions in the dynamical residual momentum $q^2$ (both $p_1$ and $q_1$ must be understood in Minkowski space). In the end, it will be necessary to check the consistency of this expansion by verifying whether $q^2 \ll (\pi T)^2$. Again, we expect corrections of order $M/T$ and $q/T$, and talk of partial reduction if we find that $M$ and/or $|q|$ is proportional to $g^n T$, with $g$ small.

It should be pointed out that there are in fact other singularities which could appear in the fermionic Greens functions at high $T$, corresponding to situations where the external momenta are such that some of the heavy modes are close to their “mass-shell”. However the singularities closest to the origin determine the long distance behavior of spatial correlations. In fact, once the free correlation has been separated out, and therefore the trivial mass contribution to the external momentum has been subtracted, the interesting dynamical behavior of the remaining correlation is characterized by $q$, the off-shellness from the lowest free singularity. This residual-momentum dependence of the correlation function can be measured in lattice QCD simulations, since the free contribution to the correlations is known exactly.

In analogy with the heavy mass decoupling theorem, the decoupling of the heavy modes is manifest only in specific subtraction schemes, such as the BPHZ scheme. A two-step approach better illustrates the need for a judicious choice of the counterterms.

Let us consider a graph schematically in the original theory renormalized in a $T$-independent scheme, e.g. the $\overline{\text{MS}}$ scheme. We can always split it into light and heavy contributions: $G^{D+1}_\text{D}(q, T) = G^D_\text{L}(q, T) + G^{D+1}_\text{H}(q, T)$, where $G^D_\text{L}$ is the contribution of terms where all loop frequencies have their smallest value. Since there are no infinite frequency sums $G^D_\text{L}$ is actually $D$ dimensional. Then, we expand $G^{D+1}_\text{H}$ at $q = 0$, and keep terms that are not suppressed by powers of $T$: $G^{D+1}(q, T) = G^D_\text{L}(q, T) + G^{D+1}_\text{H}(0, T) + O(q/T)$, where we have supposed that only one term in the expansion survives. When DR takes place, the local term $G^{D+1}_\text{H}(0, T)$ contains contributions that either can be eliminated by changing the renormalization prescription, or can be generated in the reduced theory by a finite number
of renormalizable vertices.

The above strategy for the decoupling of the heavy modes is in fact similar to any other decoupling theorem. Our particular case differs only in the relevant kinematic regime where heavy Matsubara modes can possibly decouple, which happens when the lightest Matsubara modes are close to their “mass-shell”.

Because of the $T$-dependent renormalization, the parameters in the reduced graph (and in the effective Lagrangian that generates such graph) necessarily depend on $T$. The effective Lagrangian, and the temperature dependence of its parameters are uniquely determined by the original theory.

**III. EXPLICIT CALCULATION IN THE GROSS-NEVEU MODEL**

Let us illustrate the general idea of last section in a concrete example of dimensional reduction for static spatial correlations of currents involving fermionic fields. The model we consider is the Gross-Neveu model in 1+1 dimensions \[20\] described by the Lagrangian

$$\mathcal{L} = \bar{\psi} i \gamma \cdot \partial \psi - \bar{\psi}(\sigma + i\pi\gamma_5)\psi - \frac{N}{2g^2}(\sigma^2 + \pi^2),$$  \hspace{1cm} (1)

where $\sigma$ and $\pi$ are the auxiliary scalar and pseudoscalar boson fields, respectively. We study this model in the limit $N \to \infty$ with the coupling constant $g^2$ fixed. In spite of some obvious limitations, this model shares several qualitative features with QCD, and it has been often used to test new concepts that might be relevant for QCD itself. By Fourier transforming the fields

$$\psi(\tau, x) = \sqrt{T} \sum_{n=-\infty}^{\infty} \psi_n(x)e^{i\omega_n \tau}, \quad \sigma(\tau, x) = \sum_{l=-\infty}^{\infty} \sigma_l(x)e^{i\Omega_l \tau},$$  \hspace{1cm} (2)

where $\omega_n = (2n - 1)\pi T$ and $\Omega_l = 2l\pi T$, we can rewrite the action as follows \[4\]

$$\int dx \sum_{n,l} \left\{ -\frac{i}{2g^2T} \left[ \delta_l - \sigma_l(x) + i\gamma_5\pi_l(x) \right] \right\} \psi_n(x) \psi_{n-l}(x) - \frac{N}{2g^2T} \left[ \sigma_l^2(x) + \pi_l^2(x) \right].$$  \hspace{1cm} (3)

Since we are interested in static Green’s functions (zero external frequency), only terms with $l = 0$ are relevant. We are left with a one-dimensional theory with an infinite number of fermions, each with a chirally invariant mass $\omega_n$. The tree-level coupling constant is $g^2T$. As discussed in the preceding section, we say that DR occurs, if the static correlations ($l = 0$) are reproduced by the action \[3\] with only $\omega_n \equiv \pm \pi T$ terms, while the heavier modes give corrections that either are of order $1/T$, or can be absorbed by a proper redefinition of the the parameters of the effective theory, which become dependent on $T$.

In the symmetric phase (high-$T$) and in the large $N$ limit, this model has only one non-trivial irreducible graph: the bubble graph. For static external lines, a calculation along the lines of Ref. \[21,22\] yields

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where we have used dimensional regularization on the spatial integral. We are interested
in the large distance behavior of the spatial correlations, which is determined by the lowest
singularities, i.e. $p_1 \approx \pm 2\pi T i$. Therefore, we analytically continue $p_1$ into Minkowski space,
and subtract the trivial large-mass contribution to the external momentum by defining the reduced
momentum $q_1^2 = p_1^2 + 4\pi^2 T^2$. Then Eq. (5) becomes

$$\frac{2\pi i}{N} \Pi_R(q_1^2) = -\ln \left( \frac{\mu^2 e^{2\gamma_E}}{\pi^2 T^2} \right) + 4 \left[ \left( \frac{q_1}{2\pi T} \right)^2 - 1 \right] \left[ \left( \frac{2\pi T}{q_1} \right)^2 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2 T^2} \right],$$

where the subscript R means that the bubble has been renormalized. For the sake of
concreteness, we used the modified minimal subtraction (\overline{MS}) scheme, which is independent
of $T$. Now we assume that $q_1^2 \ll T^2$ (this assumption must be verified in the final results),
and expand in $q_1^2 / T^2$

$$\frac{2\pi i}{N} \Pi_R(q_1^2) = -\frac{16\pi^2 T^2}{q_1^2} - \ln \left( \frac{\mu^2 e^{2\gamma_E-1}}{16\pi^2 T^2} \right) + \mathcal{O} \left( \frac{q_1^2}{T^2} \right).$$

This result illustrates what has been said in the preceding section. The main momentum
dependence $-16\pi^2 T^2 / q_1^2$ is given by the lightest modes, and can be reproduced by a lower
dimensional theory. The other modes give contributions that are either suppressed by powers
of $q_1^2 / T^2$, or momentum independent, $- \ln (\mu^2 e^{2\gamma_E-1} / 16\pi^2 T^2)$. The large $T$-dependent part of
this local piece can be eliminated by choosing $\mu \propto T$, which corresponds to a $T$-dependent
renormalization scheme. The specific choice of the proportionality constant defines the
 coupling $\tilde{g}^2(T)$, where we use the tilde to stress the fact that in general $\tilde{g}^2(T)$ is a different
 function from $g^2(\mu)$.

The general strategy we adopt to define the temperature dependent couplings is that the
 expansion in $q^2 / T^2$ of the divergent one-particle irreducible (1PI) graphs calculated at one-
loop level should have no finite corrections in the high-$T$ limit. This is always possible
in a renormalizable theory, since we have a renormalization constant for each divergent 1PI
graph (non-divergent graphs are already suppressed by powers of $q/T$). The advantages of
this definition are that the reduced theory with these “optimal” couplings reproduces exactly
all the divergent 1PI graphs of the original theory to one loop up to power corrections in
$q^2 / T^2$, and that it can be easily implemented in any renormalizable theory, since it only
requires perturbative calculations to one loop. It will become clear later that this choice
also minimizes on average the sub-leading corrections in $\tilde{g}^2(T)$ to physical quantities in the
original theory.

In this particular model the only divergent 1PI graph is the bubble graph. The requirement
that the finite terms vanish implies $\mu^2 = 16\pi^2 T^2 / e^{2\gamma_E-1}$, which, recalling that
$2\pi / g^2(\mu) \equiv \ln(\mu^2/\Lambda^2_{\overline{MS}})$, defines
\[
\frac{2\pi}{\tilde{g}^2(T)} = \ln\left(\frac{16\pi^2 T^2}{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E-1}}\right) \equiv \ln\left(\frac{T^2}{\Lambda_T^2}\right),
\] (8)

where we have defined \(\Lambda_T^2 = \Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E-1}/(4\pi)^2\). Using the fact that the critical temperature at which the chiral symmetry gets restored in this model is \(T_c = \Lambda_{\overline{\text{MS}}}^2 \exp(\gamma_E)/\pi\), \(\Lambda_T^2 = T_c^2/(16e)\). It can be easily checked that the 1-dimensional Lagrangian

\[
\int dx \left\{ \sum_{n=\pm} \bar{\psi}_n(x) \left[ -\omega_n \gamma_0 - i\gamma_1 \partial_1 - \sigma(x) + i\gamma_5 \pi(x) \right] \psi_n(x) - \frac{N}{2g_1^2} \left[ \sigma^2(x) + \pi^2(x) \right] \right\},
\] (9)

with \(g_1^2 = T\tilde{g}^2(T)\), reproduces the bubble graph of the 2-dimensional Lagrangian up to power-suppressed terms.

Let us now calculate and compare a measurable quantity in both the original and the reduced theory. The screening mass \(\tilde{m}\) is calculated in both theories by solving the equation

\[
\frac{2\pi}{\tilde{g}^2(T)} + 2\pi i \Pi/N = 0,
\]

with \(p_1^2 = -\tilde{m}^2\) (\(q_1^2 = (2\pi T)^2 - \tilde{m}^2\)). In the original theory, by using Eqs. (6) and (8), we find

\[
\frac{2\pi}{\tilde{g}^2(T)} - \frac{16\pi^2 T^2}{q_1^2} + 4 - \ln(16) + \sum_{n=1}^{\infty} \frac{(q_1^2/2\pi T)^2 - 1}{(2n+1)[4n(n+1) + (q_1^2/2\pi T)^2]} = 0,
\] (10)

whose solution in powers of \(\tilde{g}^2(T)\) is

\[
\tilde{m}_{1+1} = 2\pi T \left[ 1 - \frac{\tilde{g}^2(T)}{\pi} + 3[\ln(16) - 3] \left(\frac{\tilde{g}^2(T)}{\pi}\right)^3 + \mathcal{O}(\tilde{g}^8(T)) \right],
\] (11)

while in the reduced theory we easily find:

\[
\tilde{m}_1 = 2\pi T \left[ 1 - \frac{\tilde{g}^2(T)}{\pi} \right].
\] (12)

The difference between Eqs. (11) and (12) is a clear indication that DR is only partial. This result also explains the physical reason why a partial DR takes place in this theory, and why this DR implies a non-relativistic reduction. The screening state is a bound state of a quark and an antiquark of mass \(\pi T\) in one dimension, with a binding energy \(\approx 2\tilde{g}^2(T)T\). The binding energy in units of the quark mass decreases logarithmically, therefore our original assumption \(q_1^2 \ll T^2\) is verified. In addition, we expect that the screening mass can be solved from an appropriate Schrödinger equation in the high-\(T\) limit. The point here is that there exists a simplified physical picture, similar to the non-relativistic reduction assumed by several authors \[8-11\], because of the fact that asymptotically free theories possess an additional scale in the large \(T\) limit. This new scale, \(T/\ln(T/\Lambda_T)\), makes partial DR possible. On the contrary, theories with only finite ultraviolet fixed-points lack this scale hierarchy, and should not undergo even this partial DR, as we have explicitly verified in the 2+1 Gross-Neveu model \[23\].

We point out that our “optimal” choice of \(\tilde{g}^2(T)\), i.e. of \(\Lambda_T\), makes corrections of order \(\tilde{g}^4(T)\) in Eq. (11) disappear. Had we chosen a different \(\Lambda_T\), these corrections would have been there. This result cannot be universal, since choosing the renormalization scale only
reorganizes the perturbative expansion, and other quantities with different expansions cannot be “fixed” by a single parameter. Nevertheless, our perturbative choice of absorbing all finite contributions to the divergent 1PI graphs in the couplings should at least make the corrections of order $\tilde{g}^4(T)$ small in most typical quantities.

At last, we draw the attention on the amount of the change of scale: $\Lambda_T$ is about $0.15T_c$, which implies that the perturbative regime sets in much earlier for $T$ than for $\mu$. This large change of scale can partially be explained by using $2\pi T$ as the typical temperature energy scale instead of $T$.

IV. CONCLUSIONS

We have generalized the concept of dimensional reduction to static correlations of operators made by fermionic fields (e.g. electromagnetic current, mesonic currents, etc.).

The basic idea has been illustrated in the Gross-Neveu model, and applied to the calculation of the screening mass in the high-$T$ limit.

The experience gained in the model study suggests that complete DR (corrections are suppressed by powers of $T$) for operators made by fermionic fields is not possible, since the overall scale is set by $T$. However, in asymptotically-free theories there appears a new scale, $\tilde{g}^2(T)T$, which makes a partial DR possible (corrections are suppressed by powers of $\tilde{g}^2(T)$).

Nevertheless, this partial DR still yields a simplified physical picture. The relevant static correlations can be described by the reduced theory, which, in addition, becomes non-relativistic in the high-$T$ limit. This lower dimensional, non-relativistic theory can, for instance, be used to interpret lattice data at high-$T$.

We are in the process of explicitly verifying whether this non-relativistic dimensional reduction also applies to currents made by fermions in QED and QCD, as the present model calculation strongly suggests.

We also show that the dependence on the temperature $T$ of the coupling constant $\tilde{g}^2(T)$ in the reduced theory is $\tilde{g}^2(T) = g^2(T\Lambda_{\overline{MS}}/\Lambda_T)$, where $g^2(\mu)$ is the coupling constant in the original theory in the $\overline{MS}$ scheme. An explicit calculation shows that the change of scale $\Lambda_{\overline{MS}}/\Lambda_T$ is quite large. This result is not typical of the Gross-Neveu model. In fact we have verified that a change of scale of about the same magnitude also appears in QCD.

A rather remarkable consequence of this large reduction of the scale-parameter going from $\mu$ to $T$ is that the high-$T$ regime in the reduced theory sets in for temperatures much lower than naively expected by directly comparing temperatures to typical perturbative momenta at zero temperature. For instance, the effective Lagrangian of finite temperature QCD, which is derived perturbatively (nevertheless its solution can be non-perturbative), appears to become reliable at temperatures as low as about 0.5 GeV, to be compared to momenta of about 5 GeV necessary at zero temperature for perturbation to become reliable.
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