Coarse geometric properties of the Hilbert geometry

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Abstract

We give a necessary and sufficient condition for the natural boundary of a Hilbert geometry to be a corona. In addition, we show that any Hilbert geometry is uniformly contractible and with bounded coarse geometry. As a consequence of these we see that the coarse Novikov conjecture holds for a Hilbert geometry under a mild condition. Also we show that the asymptotic dimension of any two-dimensional Hilbert geometry is just two.

Keywords: Hilbert geometry, corona, coarse Baum-Connes conjecture, coarse Novikov conjecture, asymptotic dimension.

2010MSC: 51F99.

1 Introduction

Any non-empty bounded convex domain $X$ of the $n$-dimensional Euclidean space $\mathbb{R}^n$ is endowed with the Hilbert metric $d$. Then $(X,d)$ is called a Hilbert geometry, which is a proper geodesic metric space. For instance a unit open ball endowed with the Hilbert metric is the projective model of the $n$-dimensional hyperbolic space. On the other hand a Hilbert geometry is not necessarily either CAT(0) or Gromov-hyperbolic. Indeed an investigation of Kelly and Straus [10] implies that $(X,d)$ is CAT(0) if and only if $X$ is an ellipsoid. Also if $(X,d)$ is Gromov-hyperbolic then the boundary $\partial X = \overline{X} \setminus X$ must be of differentiability class $C^1$ where $\overline{X}$ is the closure of $X$ in $\mathbb{R}^n$ (see [9]). In general, it is unclear what kind of coarse geometric properties a Hilbert geometry has. We discuss topics related to the coarse Baum-Connes conjecture for Hilbert geometries in this paper.

The coarse Baum-Connes conjecture ([8], [12] and [14]) is one of the most important themes in coarse geometry. We say that a proper metric space satisfies
the coarse Baum-Connes conjecture (resp. the coarse Novikov conjecture) if the coarse assembly map from the coarse $K$-homology to the $K$-theory of the Roe algebra is an isomorphism (resp. an injection). There are a number of studies concerning these conjectures. For example, any proper geodesic Gromov-hyperbolic space satisfies the coarse Baum-Connes conjecture \cite{8}. Also, some sufficient conditions are known for proper metric spaces to satisfy the coarse Baum-Connes conjecture \cite{15} and \cite{16}. Unfortunately these conjectures are false in general \cite{15} and \cite{7}. Nevertheless, just for uniformly contractible spaces with bounded coarse geometry, there are no counterexample. This enables us to expect that all Hilbert geometries satisfy these conjectures because they are uniformly contractible (Proposition \ref{prop3.4}) and with bounded coarse geometry (Proposition \ref{prop4.3}). Our two main theorems partially give positive results for the expectation.

The following is our first main theorem.

**Theorem 1.1.** Let $X \subset \mathbb{R}^n$ be a non-empty bounded convex domain. Then the boundary $\partial X$ is a corona of the Hilbert geometry $(X, d)$ if and only if the closure $\overline{X}$ is strictly convex in $\mathbb{R}^n$.

By a standard argument, we deduce the following.

**Corollary 1.2.** Let $X \subset \mathbb{R}^n$ be a non-empty bounded convex domain and the closure $\overline{X}$ be strictly convex in $\mathbb{R}^n$. Then the Hilbert geometry $(X, d)$ satisfies the coarse Novikov conjecture.

The following is our second main theorem.

**Theorem 1.3.** The asymptotic dimension of any 2-dimensional Hilbert geometry is equal to 2.

This theorem and Proposition \ref{prop4.3} with Yu’s result \cite{15} imply that any 2-dimensional Hilbert geometry satisfies the coarse Baum-Connes conjecture.

**Notation 1.4.** We collect some notation which will frequently appear in this paper. We always assume that $2 \leq n < \infty$.

- A line through $a, b \in \mathbb{R}^n$ is a set $\{ ta + (1-t)b \in \mathbb{R}^n \mid t \in \mathbb{R} \}$.
- For $x, y \in \mathbb{R}^n$, $[x, y]$ denotes the (directed) segment from $x$ to $y$.
- For $x, y \in \mathbb{R}^n$, $|xy|$ denotes the Euclidean distance between $x$ and $y$.
- We write the closure of an open set $A \subset \mathbb{R}^n$ as $\overline{A}$ and put $\partial A := \overline{A} \setminus A$. For a closed set $B \subset \mathbb{R}^n$, we put $\partial B := B \setminus \text{int}(B)$ where $\text{int}(B)$ is the interior of $B$.
- For a bounded convex domain $X \subset \mathbb{R}^n$ (see \cite{2.1}), a chord $[x', y']$ is a (directed) segment connecting two boundary points $x', y' \in \partial X$.
• For a Hilbert geometry \((X,d)\) (defined in §2.2), \(B(x,r)\) denotes the closed ball of radius \(r > 0\) centered at \(x \in X\) with respect to \(d\). If we do not wish to specify the center, we simply denote by \(B_r\) a closed ball of radius \(r\). When we consider a Euclidean closed \(r\)-ball (resp. a closed \(r\)-ball in a general metric space \((Y,d_Y)\)), we write it as \(B_{euc}(x,r)\) (resp. \(B_Y(x,r)\)).

2 Preliminaries

2.1 Convexity in the Euclidean space

A metric space \((Y,d_Y)\) is said to be a geodesic space (resp. uniquely geodesic space) if any two points are joined by a geodesic (resp. a unique geodesic). Here, a geodesic is (the image of) an isometric embedding of a closed interval of \(\mathbb{R}\) into \(Y\).

For a uniquely geodesic space \((Y,d_Y)\), we say that a subset \(A\) of \(Y\) is convex (resp. strictly convex) if for every \(x, y \in A\), any point \(z\) distinct from \(x, y\) on the geodesic joining \(x, y\) is contained in \(A\) (resp. the interior of \(A\)). See [11, Definitions 2.5.2, 2.5.6]. If \((Y,d_Y)\) is proper, that is, any bounded closed set is compact, then the closure of a convex set is also convex ([11, Proposition 2.5.3]).

Lemma 2.1. For a bounded domain \(X\) in \(\mathbb{R}^n\), the closure \(\overline{X}\) is strictly convex if and only if \(X\) is convex and its boundary \(\partial X\) does not include any non-trivial segment.

Proof. The necessary condition is obvious. In order to prove the converse, we assume that \([x,y] \cap \partial X\) contains \(z \neq x, y\). Since \(\partial X\) does not include any non-trivial segment, there exist two points \(x' \in [x,z]\) and \(y' \in [z,y]\) with \(x', y' \in X\). Then \(z \in [x', y'] \not\subset X\). This contradicts the convexity of \(X\). \(\square\)

The following is well-known (see [2, 16.3 Proposition, 16.4 Theorem]).

Proposition 2.2. Let \(A\) be a bounded convex domain in \(\mathbb{R}^n\) and let \(o \in A\). Take \(\epsilon > 0\) so that \(B_{euc}(o,\epsilon) \subset A\). Define a map \(\pi: \partial A \to \partial B_{euc}(o,\epsilon)\) as a projection of \(\partial A\) to \(\partial B_{euc}(o,\epsilon)\) toward \(o\). Then \(\pi\) is a homeomorphism and can be extended to a homeomorphism from \(A\) to \(B_{euc}(o,\epsilon)\). Especially \(A\) is contractible.

2.2 The Hilbert geometry

Let \(X \subset \mathbb{R}^n (n \geq 2)\) be a non-empty bounded convex domain. For any different two points \(x, y \in X\) a line passing through \(x\) and \(y\) crosses \(\partial X\) at just two points \(x', y'\) where \(x', x, y, y'\) are arranged in this order. Such a chord \([x', y']\) is uniquely determined for \(x\) and \(y\). The value

\[
\frac{|xy'||yx'|}{|xx'||yy'|}
\]
is called the \textit{cross ratio} of $x$ and $y$. The cross ratio induces a metric $d$ on $X$ by

$$d(x, y) = \begin{cases} 
\log \frac{|xy'| |yy'|}{|xx'| |yx'|} & \text{if } x \neq y, \\
0 & \text{if } x = y,
\end{cases}$$

(for example, [3, 6]). We call $d$ the \textit{Hilbert metric} and $(X, d)$ a \textit{Hilbert geometry}.

We say that a finite set of points in $\mathbb{R}^n$ is \textit{collinear} if it belongs to a single line. We always suppose that elements of a collinear set $\{x_1, \ldots, x_k\}$ are arranged by their indices. The following invariance of the cross ratio under the perspective projection is well-known ([11, Proposition 5.6.4]). See Figure 1.

**Proposition 2.3.** Let $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\} \subset \mathbb{R}^n$ be collinear and consist of distinct points, respectively. If we have four lines $R_i$ passing through $a_i$ and $b_i$ ($i = 1, \ldots, 4$) which meet at a point $p \in \mathbb{R}^n$ or are parallel, then

$$\frac{|a_2a_4||a_3a_1|}{|a_2a_1||a_3a_4|} = \frac{|b_2b_4||b_3b_1|}{|b_2b_1||b_3b_4|}.$$ 

![Figure 1: Proposition 2.3](image)

We recall some basic facts about Hilbert geometries.

**Theorem 2.4 ([3, 6]).** Let $X \subset \mathbb{R}^n$ be a non-empty bounded convex domain.

(i) The Hilbert metric and the restricted Euclidean metric give the same topology on $X$.

(ii) The Hilbert geometry $(X, d)$ is a proper metric space.

(iii) Every segment in $X$ is a geodesic in $(X, d)$.

(iv) The Hilbert geometry $(X, d)$ is uniquely geodesic if and only if there is no pair of non-trivial segments $I, J$ in $\partial X$ such that $I, J$ span an affine plane in $\mathbb{R}^n$. In particular if $X$ is strictly convex in $\mathbb{R}^n$, then $(X, d)$ is uniquely geodesic.
3 Uniform contractibility

We prove that every Hilbert geometry is uniformly contractible.

Definition 3.1. A metric space \((Y, d_Y)\) is uniformly contractible if for any \(R > 0\) there exists \(S \geq R\) such that any closed \(R\)-ball \(B_Y(y, R)\) is contractible to a point in \(B_Y(y, S)\).

Lemma 3.2. Let \((X, d)\) be a Hilbert geometry. Then for any \(o, x, y \in X\) and any \(z \in [x, y]\), we have

\[
d(o, z) \leq \max \{d(x, o), d(y, o)\}.
\]

In particular, every open ball in \((X, d)\) is convex with respect to the Euclidean metric.

Proof. If \(o, x, y\) are collinear then the claim is trivial and hence it suffices to consider the case where \(o, x, y\) span a plane \(H\). Let \([x_1, x_2]\) and \([y_1, y_2]\) be two chords through \(o, x\) and \(o, y\) in this order respectively. Note that \([x_1, x_2]\) and \([y_1, y_2]\) belong to the plane \(H\). For any \(z \in [x, y]\) we take a chord \([z_1, z_2]\) passing through \(o, z\) in this order. Since \(X\) is also convex, \([z_1, z_2]\) crosses \([x_1, y_1]\) and \([x_2, y_2]\) at two points \(z'_1, z'_2 \in H \cap X\) respectively. If \([x_1, y_1]\) and \([x_2, y_2]\) are not parallel, let \(p \in H\) be the point on which the line including \([x_1, y_1]\) intersects the line including \([x_2, y_2]\). In the case where \([x_1, y_1]\) and \([x_2, y_2]\) are parallel, we take the point at infinity as \(p\). Considering the line \(L\) through \(p, z\), we see that \(L\) crosses \([x_1, x_2]\) and \([y_1, y_2]\) at \(x_3\) and \(y_3\) respectively. Since \(z\) belongs to the segment \([x, y]\), (i) \(|oz_3| \geq |oz_1|\) and \(|oy_3| \geq |oy_2|\) or (ii) \(|oy_3| \geq |oy_2|\) and \(|oz_3| \geq |ox_3|\) must be satisfied. If (i) happens, by Proposition 2.3 we have

\[
\frac{|ox_2|}{|ox_1|} \geq \frac{|x_3x_1|}{|x_3x_2|} = \frac{|oz'_2|}{|oz'_1|} \geq \frac{|oz_3|}{|oz_1|}.
\]

Thus we conclude that \(d(o, x) \geq d(o, z)\). A similar inequality gives \(d(o, y) \geq d(o, z)\) in the case where (ii) happens.

Remark 3.3. For \(A > 0, B \geq A\) and \(C \geq 0\), we have \(B/A \geq (B + C)/(A + C)\).

Proposition 3.4. Every Hilbert geometry \((X, d)\) is uniformly contractible.

Proof. Every open ball of \((X, d)\) is convex with respect to the Euclidean metric by Lemma 3.2. Thus by Theorem 2.4 (i) and Proposition 2.2 any closed ball with respect to \(d\) is contractible in itself.

4 Bounded coarse geometry

We show that Hilbert geometries are with bounded coarse geometry.

Definition 4.1. A metric space \((Y, d_Y)\) is said to be with bounded coarse geometry if there exists \(\epsilon > 0\) satisfying the following: For any \(R > 0\),

\[
\sup \{ l \mid y \in Y, y_1, \ldots, y_l \in B_Y(y, R), i \neq j, d_Y(y_i, y_j) > \epsilon \} < \infty.
\]
In order to count points in a closed ball we compute the ratio of the volume of closed balls.

**Lemma 4.2.** Let \((X, d)\) be a Hilbert geometry. There exists a constant \(1 > D > 0\) such that for any two closed balls \(B_R\) and \(B(x, r)\) in \((X, d)\) with \(B(x, r) \subset B_R\), the map \(f_x : \mathbb{R}^n \ni y \mapsto x + D(y - x) \in \mathbb{R}^n\) sends \(B_R\) into \(B(x, r)\).

**Proof.** Fixing \(y \in B_R \setminus B(x, r)\) arbitrarily, we take a chord \([x', y']\) passing through \(x, y\) in this order. Let \(z \in \partial B(x, r)\) be the intersection point of the segment \([x, y]\) and \(\partial B(x, r)\). See Figure 2.

Since \(x, y \in B_R\), their distance is at most \(2R\):

\[
\frac{|y'x||x'y'|}{|y'y||x'x'|} = e^{d(x, y)} \leq e^{2R}.
\]  

By expanding \((1)\) with \(|y'x| = |y'y| + |xy|\), \(|x'y| = |x'x| + |xy|\) and \(|y'x'| = |y'y| + |yx| + |xx|\), we obtain

\[
|xy||y'x'| \leq (e^{2R} - 1)|y'y||x'x|.
\]

In a similar way we have

\[
|xz||y'x'| = (e^r - 1)|y'z||x'x|.
\]

Since \(|y'y| \leq |y'z|\) by our assumption, we get

\[
\frac{|xz|}{|xy|} \geq \frac{(e^r - 1)|y'z||x'x|}{(e^{2R} - 1)|y'y||x'x|} \geq \frac{e^r - 1}{e^{2R} - 1}.
\]

Defining \(D = (e^r - 1)/(e^{2R} - 1)\), we have the conclusion. 

**Proposition 4.3.** Every Hilbert geometry is with bounded coarse geometry.

**Proof.** Fix \(\epsilon > 0\) arbitrarily, let \(R > 0\) and take any closed ball \(B(x, R)\). Choose \(\{x_1, \ldots, x_l\} \subset B(x, R)\) so that \(d(x_i, x_j) > 2\epsilon (i \neq j)\) for some \(l \in \mathbb{N}\). This condition is equivalent to \(B(x_i, \epsilon) \cap B(x_j, \epsilon) = \emptyset\) for each \(i \neq j\).
Since \( B(x_i, \epsilon) \subset B(x, R+\epsilon) \), there exists a contracting constant \( D \) and a map \( f_i : x \mapsto x_i + D(x-x_i) \) such that \( f_i(B(x, R+\epsilon)) \subset B(x_i, \epsilon) \) for each \( i \in \{1, \ldots, l\} \) by Lemma 4.2. Note that \( D \) depends only on \( R \) and \( \epsilon \). We denote by \( \mu \) the Lebesgue measure on \( \mathbb{R}^n \). Then we have

\[
\sum_{i=1}^{l} \mu(B(x_i, \epsilon)) = \mu \left( \bigcup_{i=1}^{l} B(x_i, \epsilon) \right) \leq \mu(B(x, R+\epsilon)).
\] (2)

On the other hand, we also have

\[
\sum_{i=1}^{l} \mu(B(x_i, \epsilon)) \geq \sum_{i=1}^{l} \mu(f_i(B(x, R+\epsilon))) = lD^n \mu(B(x, R+\epsilon)).
\] (3)

Combining (2) and (3), we obtain

\[ l \leq \frac{1}{D^n}. \]

Thus we conclude

\[ \sup \{ l \mid x \in X, x_1, \ldots, x_l \in B(x, R), i \neq j, d(x_i, x_j) > 2\epsilon \} \leq \frac{1}{D^n}. \]

This completes the proof. \( \square \)

5 Corona

The natural boundary of a bounded convex domain \( X \) gives a compactification of the Hilbert geometry \((X, d)\). This is a consequence of Theorem 2.4 (i).

Corollary 5.1. Let \((X, d)\) be a Hilbert geometry. Then the closure \( \overline{X} \) of \( X \) in \( \mathbb{R}^n \) gives a compactification of \((X, d)\).

We discuss when the natural boundary of a Hilbert geometry is a corona. Let \((Y, d_Y)\) be a proper metric space. A bounded continuous function \( f : Y \to \mathbb{C} \) is a Higson function on \( Y \) if for any \( \epsilon > 0 \) and any \( C > 0 \), there exists a bounded set \( B \subset Y \) such that for \( x, y \in Y \) with \( d_Y(x, y) \leq C \) and \( x \notin B \) we have \( |f(x) - f(y)| < \epsilon \).

Definition 5.2. A metrizable compactification \( \overline{Y} \) of a proper metric space \( Y \) is a coarse compactification if the restriction of every continuous function on \( \overline{Y} \) is a Higson function on \( Y \). We call the boundary \( \overline{Y} \setminus Y \) a corona of \( Y \).

The following is self-evident by the definition of the Hilbert metric.

Lemma 5.3. Let \((X, d)\) be a Hilbert geometry. If a subset \( A \subset X \) satisfies \( d_{\text{cut}}(A, \partial X) > \epsilon \) for some \( \epsilon > 0 \) then it is bounded, i.e., the diameter of \( A \) with respect to \( d \) is finite.
Lemma 5.4. For two converging sequences \( x_i \to x \) and \( y_i \to y \) in \( \mathbb{R}^n \), any sequence \( \{z_i\}_i \) consisting of \( z_i \in [x_i, y_i] \) (\( i \in \mathbb{N} \)) has a converging subsequence. Moreover the limit point lies in \([x, y]\).

Proof. From the assumption, there exists \( t_i \in [0, 1] \) such that \( z_i = t_i x_i + (1-t_i) y_i \) for each \( i \in \mathbb{N} \). By taking a subsequence we may assume that \( \{t_i\}_i \) converges to some point \( t \in [0, 1] \). Set \( z = tx + (1-t)y \). Then we have

\[
||z_i - z|| \leq ||t_i(x_i - x)|| + ||(t_i - t)||x|| + ||(1 - t_i)(y_i - y)|| + ||(t - t_i)y|| \to 0
\]

by the triangle inequality. □

The next lemma is the key to proving Theorem 1.1.

Lemma 5.5. Let \((X, d)\) be a Hilbert geometry with a base point \( o \in X \). Suppose that \( \overline{X} \) is strictly convex in \( \mathbb{R}^n \). Then for any \( C > 0 \) and \( \delta > 0 \) there exists a bounded subset \( M_{\delta,C} \) satisfying the following: For any \( x, y \in X \) if \( x \notin M_{\delta,C} \) and \( d(x, y) \leq C \) then \( |xy| < \delta \).

Proof. Take \( x, y \in X \) arbitrarily and suppose that a chord \([\xi, \eta]\) pass through \( x, y \) in this order. By the definition of the Hilbert metric we have

\[
e^{d(x,y)} = \frac{(|xy|)(|x\xi| + |y\eta|)}{|x\xi||y\eta|}.
\]

If \( d(x, y) \leq C \) and \( \delta \leq |xy| \), then we obtain

\[
e^C \geq \frac{(|xy|)(|x\xi| + |y\eta|)}{|x\xi||y\eta|} \geq \frac{\delta^2}{|x\xi||y\eta|}.
\]

Let \( \text{diam}_{\text{euc}}(X) \) be the diameter of \( \overline{X} \) with respect to the Euclidean metric. By putting \( E = \delta^2/(\text{e}^C \cdot \text{diam}_{\text{euc}}(X)) \) we see that \( |x\xi|, |y\eta| \) should satisfy \( |x\xi|, |y\eta| \geq E \) since \( |x\xi|, |y\eta| \leq |\xi| \leq \text{diam}_{\text{euc}}(X) \). We also have \( |x\xi| = |xy| + |y\eta| \geq E \).

With this in mind, we consider a set

\[
M_{\delta,C} = \{ x \in X \mid \exists y \in X, |xy| \geq \delta \text{ and } d(x, y) \leq C \}.
\]

We claim that \( M_{\delta,C} \) is bounded. Then the assertion follows. To verify this claim, we assume that \( M_{\delta,C} \) is not bounded. Then by Lemma 5.3 for a fixed decreasing sequence \( \epsilon_i \to 0 \) there exists a sequence \( \{x_i\}_i \) in \( M_{\delta,C} \) satisfying \( \inf_{\xi \in \partial X} |x_i\xi| \leq \epsilon_i \) for each \( i \). By the definition of \( M_{\delta,C} \), we have a sequence \( \{y_i\}_i \) in \( X \) such that \( |x_i y_i| \geq \delta \) and \( d(x_i, y_i) \leq C \). Applying the above argument to \( x_i \) and \( y_i \), we get \( \xi_i, \eta_i \in \partial X \) so that \( x_i \in [\xi_i, \eta_i] \) and \( |x_i \xi_i|, |x_i \eta_i| \geq E \) for each \( i \). By taking subsequences we may assume \( \xi_i, \eta_i \to \xi_\infty, \eta_\infty \) (\( i \to \infty \)) for some \( \xi_\infty, \eta_\infty \in \partial X \). Lemma 5.4 shows that the corresponding subsequence \( \{x_i\}_i \) has an accumulation point \( x_\infty \in [\xi_\infty, \eta_\infty] \). Furthermore, \( x_\infty \neq \xi_\infty, \eta_\infty \), because \( |x_i \xi_i|, |x_i \eta_i| \geq E \) for all \( i \in \mathbb{N} \). However since \( \inf_{\xi \in \partial X} |x_\infty \xi| \leq \epsilon_i \to 0 \), we have \( x_\infty \in \partial X \). This contradicts Lemma 2.1 □
proof of Theorem 1.1. Fix $o \in X$ to be a base point. Suppose that $\partial X$ includes a non-trivial segment $[\alpha, \beta]$. We take two distinct points $\xi, \eta$ in $[\alpha, \beta] \setminus \{\alpha, \beta\}$ and a continuous function $f$ on $X$ which separates $\xi$ and $\eta$, i.e., $f(\xi) \neq f(\eta)$. We show that the Hausdorff distance with respect to $d$ between $[o, \xi] \setminus \{\xi\}$ and $[o, \eta] \setminus \{\eta\}$ is finite. Then it follows that $f$ is not a Higson function and hence $\partial X$ is not a corona. Suppose that $|\alpha \xi| < |\alpha \eta|$. For any $x \in [o, \xi] \setminus \{\xi\}$ let $y \in [o, \eta] \setminus \{\eta\}$ be the point such that the line $L$ through $x$ and $y$ is parallel to the line through $\alpha$ and $\beta$. Let $x', y' \in X$ be the intersection points of $L$ and $[o, \alpha], [o, \beta]$ respectively. Then by Proposition 2.3 and Remark 3.3 we have

$$\frac{|\xi \beta||\eta \alpha|}{|\xi \alpha||\eta \beta|} \geq e^{d(x,y)}.$$  

By symmetry, we have the conclusion.

To show the converse, we note that $X$ is a compact metric space with respect to the restriction of the Euclidean metric. Since any continuous function on $X$ is uniformly continuous, the assertion follows from Lemma 5.5.

proof of Corollary 1.2. Since $X$ is a coarse compactification, the transgression map $T_{\partial X}$ and the Higson-Roe map $b_{\partial X}$ are well-defined and the following diagram is commutative:

Here $K_\bullet(X)$ is the $K$-homology of $X$, $C^*X$ is the Roe algebra of $X$, $K_\bullet(C^*X)$ is the $K$-theory of $C^*X$, $A(X)$ is the assembly map for $X$, $c(X)$ is the coarsening map for $X$, $\mu(X)$ is the coarse assembly map for $X$, $T_{\partial X}$ is the reduced $K$-homology of $\partial X$ and $\partial_{\partial X}$ is a connecting map of the $K$-homology (see [8, Section 6] and also [5, Section 1]).

By Proposition 2.2, $\partial_{\partial X}$ is an isomorphism. Combining Propositions 3.4 and 4.3 we see that $c(X)$ is also an isomorphism (see [4, Proof of Theorem 4.8] and also [5, Section 3.2]). By tracing the diagram, we have the conclusion.

Remark 5.6. On the above setting, the transgression map $T_{\partial X}$ is an isomorphism and the assembly map $A(X)$ is injective. We note that the coarsening map $c(X)$ is an isomorphism for any Hilbert geometry because Proposition 4.3 and 3.4 are generally correct.
6 Asymptotic dimension

We begin with the definition of the asymptotic dimension of a metric space. For a family \( U \) of subsets of a metric space, the \( r \)-multiplicity of \( U \) is the smallest number \( n \) such that the every closed \( r \)-ball intersects at most \( n \) elements of \( U \). There are several equivalent ways to define the asymptotic dimension (see, for example, [1, §3]). In this paper, we adopt the following definition.

**Definition 6.1.** Let \( Y \) be a metric space. We say that the asymptotic dimension \( \text{asdim}(Y) \) of \( Y \) does not exceed \( m \) if for each \( r > 0 \) there exists a uniformly bounded cover \( U \) with \( r \)-multiplicity \( \leq m + 1 \). In this case we write \( \text{asdim}(Y) \leq m \). If \( \text{asdim}(Y) \leq m \) but \( \text{asdim}(Y) \not\leq m - 1 \), then we say that the asymptotic dimension of \( Y \) is \( m \).

6.1 Lower bound

To get the lower bound we use the coarse cohomology. For a proper metric space \( Y \), \( H^m_X(Y) \) and \( H^m_c(Y) \) denote the \( m \)-dimensional coarse cohomology and Alexander-Spanier (or equivalently, Čech) cohomology with compact supports of \( Y \) respectively. It is known that there naturally exists a character map \( c^m(Y) : H^m_X(Y) \to H^m_c(Y) \). See [12] and [13] for details.

**Lemma 6.2.** Let \( Y \) be a proper metric space. If \( H^m_X(Y) \) is not trivial and the character map \( c^m(Y) \) is injective, then we have \( m \leq \text{asdim}(Y) \).

**Proof.** We can assume that \( m' = \text{asdim}(Y) < \infty \). Then we have an anti-Čech system \( \{U_k\}_k \) such that each nerve complex \( |U_k| \) satisfies \( H^m_c(|U_k|) = 0 \) for any \( m > m' \) by [13, Theorem 9.9(c)]. We take a partition of unity \( \rho \) subordinate to the cover \( U_k \) of \( Y \) for some \( k \). Then it defines a proper continuous map \( \kappa : Y \to |U_k| \). Since \( |U_k| \) admits a proper metric which is coarsely equivalent to \( Y \) by \( \kappa \) (see, [17]) and the coarse cohomology is coarsely invariant, the character map \( c^m(Y) : H^m_X(Y) \to H^m_c(Y) \) factors through \( H^m_c(|U_k|) \). The map must be a 0-map if \( m > m' \). If the character map \( c^m(Y) \) is injective, then we have \( H^m_X(Y) = 0 \).

**Proposition 6.3.** The asymptotic dimension of any \( m \)-dimensional Hilbert geometry is at least \( m \).

**Proof.** Let \( (X,d) \) be an \( m \)-dimensional Hilbert geometry. Since \( X \) is uniformly contractible (Proposition 3.4) we see that the character map is an isomorphism ([12, (3.33) Proposition]). On the other hand we have \( H^m_c(X) = \mathbb{R} \) because \( X \) is homeomorphic to \( \mathbb{R}^m \). By Lemma 6.2 we have \( \text{asdim}(X) \geq m \).

6.2 Lemmas

Let \( X \subset \mathbb{R}^n \) be a non-empty bounded convex domain and \( (X,d) \) the Hilbert geometry. Fix \( o \in X \) to be a base point. We define a ray \( \ell \) as (the image of) an isometric embedding from \( [0, \infty) \) into \( X \) such that its image is included in a line of \( \mathbb{R}^n \) and \( \ell(0) = o \).
Lemma 6.4. Let \((X, d)\) be a Hilbert geometry with a base point \(o\). For \(a_2, b_2 \in X\), take two chords \([a_1, a_3]\) and \([b_1, b_3]\) passing through \(o, a_2\) and \(o, b_2\) in this order respectively. Let \(L_i\) \((i = 1, 2, 3)\) be the line through \(a_i\) and \(b_i\). If \(o, a_2, b_2\) are not collinear and satisfy \(d(o, a_2) = d(o, b_2)\), then three lines \(L_1, L_2, L_3\) meet at one point in \(\mathbb{R}^n \setminus X\) or are parallel.

Proof. Note that seven points \(a_1, a_2, a_3, b_1, b_2, b_3, o\) are on the same plane in \(\mathbb{R}^n\). We assume that \(L_1\) and \(L_3\) are not parallel. Let \(p\) be the point where \(L_1\) and \(L_3\) intersect. By the choice of \(a_1, a_3, b_1, b_3\), two chords \([a_1, b_1]\) and \([a_3, b_3]\) do not intersect each other in \(X\). Since \(X\) is convex, \(p\) is not contained in \(X\). Consider a line \(L'\) through \(b_2\) and \(p\). Then \(L'\) crosses the segment \([o, a_3]\) at a point \(q\) (Figure 3). By Proposition 2.3, we see that \(d(o, b_2) = d(o, q)\) and thus \(d(o, a_2) = d(o, q)\). This shows that \(a_2 = q\) because \(o, a_2, q\) are on the same segment \([o, a_3]\). For the case where \(L_1\) and \(L_3\) are parallel, we can show the lemma in a similar way. 

![Figure 3: Proof of Lemma 6.4](image)

Lemma 6.5. For a pointed Hilbert geometry \((X, d, o)\), given two distinct rays \(\ell_1\) and \(\ell_2\), if \(0 < s < t\) then

\[
d(\ell_1(s), \ell_2(s)) \leq d(\ell_1(t), \ell_2(t)).
\]

Proof. Suppose that two chords \([a_1, b_1]\) and \([a_s, b_s]\) pass through \(\ell_1(t), \ell_2(t)\) and \(\ell_1(s), \ell_2(s)\) in this order respectively. Applying Lemma 6.4 to points on \(\ell_1\) and \(\ell_2\) twice, we notice that \([a_1, b_1]\) and \([a_s, b_s]\) do not intersect each other in \(X\). Putting this and the convexity of \(X\) together we see that \([a_t, o]\) and \([b_t, o]\) cross \([a_s, b_s]\). We denote individual intersection points by \(a'_t, b'_t\) so that \(a_s, a'_t, b'_t, b_s\) are arranged in this order. The segment \([a'_t, b'_t]\) contains \(\ell_1(s)\) and \(\ell_2(s)\). By Proposition 2.3 we have

\[
e^{d(\ell_1(t), \ell_2(t))} = \frac{\|\ell_1(t)b_1\|\|\ell_2(t)a_t\|}{\|\ell_1(t)a_t\|\|\ell_2(t)b_1\|} = \frac{\|\ell_1(s)b'_t\|\|\ell_2(s)a'_t\|}{\|\ell_1(s)a'_t\|\|\ell_2(s)b'_t\|}.
\]

On the other hand by Remark 3.3 we also have

\[
\frac{\|\ell_1(s)b'_t\|\|\ell_2(s)a'_t\|}{\|\ell_1(s)a'_t\|\|\ell_2(s)b'_t\|} \geq \frac{\|\ell_1(s)b_s\|\|\ell_2(s)a_s\|}{\|\ell_1(s)a_s\|\|\ell_2(s)b_s\|} = e^{d(\ell_1(s), \ell_2(s))}.
\]
This shows the lemma.  

The next lemma follows from the triangle inequality and Lemma \[\text{Lemma 6.5}\]

**Lemma 6.6.** Let \((X, d, o)\) be a pointed Hilbert geometry. For \(x, y \in X\) let \(\ell_x\) and \(\ell_y\) be rays passing through \(x\) and \(y\) respectively. If \(d(x, y) \leq r\) and \(d(o, x) \leq d(o, y)\) then \(d(\ell_x(s), \ell_y(s)) \leq 2r\) for \(s \leq d(o, y)\).

**Proof.** Note that for any \(d(o, x) \leq t \leq d(o, y)\), we have

\[
d(\ell_x(t), x) + d(y, \ell_y(t)) = d(o, y) - d(o, x) \leq d(x, y) \leq r,
\]

by the triangle inequality. Hence it holds

\[
d(\ell_x(t), \ell_y(t)) \leq d(\ell_x(t), x) + d(x, y) + d(y, \ell_y(t)) \leq 2r.
\]

By Lemma \[\text{Lemma 6.5}\] we have \(d(\ell_x(s), \ell_y(s)) \leq 2r\) for any \(s \leq d(o, y)\).  

### 6.3 Upper bound

Henceforth, we concentrate on 2-dimensional Hilbert geometries. In such a case, the boundary of a ball with respect to the Hilbert metric is homeomorphic to the circle \(S^1\) by Lemmas \[\text{2.2}\] and \[\text{3.2}\]. Because of this we may simply call the boundary of a ball a *circle* and assume that each circle is endowed with the counterclockwise (CCW) direction. For distinct two points \(a, b\) on a circle, the *arc* \(ab\) from \(a\) to \(b\) stands for the closed subpath on the circle from \(a\) to \(b\) in the CCW direction.

Let \(R > 0\). We consider two conditions for an arc \(\widehat{ac}\):

\[
\begin{align*}
\text{★1} & \quad \text{There exists a point } b \in \widehat{ac} \text{ such that } d(a, b) \geq R. \\
\text{★2} & \quad \text{The diameter } \diam(\widehat{ac}) = \max_{x, y \in \widehat{ac}} d(x, y) \text{ is not larger than } 4R.
\end{align*}
\]

**Lemma 6.7.** Let \(R > 0\). If an arc \(\widehat{ac}\) satisfies ★1 for \(R\) then \(\widehat{ac}\) can be decomposed into arcs \(\widehat{a_0a_1}, \widehat{a_1a_2}, \ldots, \widehat{a_{k-1}a_k}\) \((k \in 2\Z + 1)\), each of which satisfies ★1 and ★2 for \(R\).

**Proof.** Note that \(\widehat{ac}\) is homeomorphic to a closed interval. Since the distance function \(d(a, -)\) is continuous, there exists a unique point \(a_1 \in \widehat{ac}\) satisfying \(d(a, a_1) = R\) and \(d(a, b) < R\) for any \(b \in \overline{aa_1} \setminus \{a_1\}\). We have three cases: (i) \(a_1 = c\). (ii) \(a_1 \neq c\) and \(\overrightarrow{ao_1c}\) does not satisfy ★1. (iii) \(a_1 \neq c\) and \(\overrightarrow{ao_1c}\) satisfies ★1. In the case (ii), we put \(a_2 = c\). For the case (iii), take \(a_2 \in \overrightarrow{ao_1c}\) satisfying \(d(a_1, a_2) = R\) and \(d(a_1, b) < R\) for any \(b \in \overline{aa_1a_2} \setminus \{a_2\}\).

By repeating this procedure, we have a sequence \(a = a_0, a_1, \ldots, a_k = c\) on \(\widehat{ac}\) arranged in the CCW direction. For each \(i (i \neq k)\), the arc \(\widehat{a_{i-1}a_i}\) satisfies ★1 and ★2. If \(\overrightarrow{a_{k-1}a_k}\) does not satisfy ★1 then \(\overrightarrow{a_{k-2}a_k}\) has ★1 and ★2. In such a case, we erase \(a_{k-1}\) and rename \(a_k\) to \(a_{k-1}\).

Finally if the number of resulting arcs is even then by erasing \(a_1\) and renaming \(a_i\) to \(a_{i-1}\) for \(i > 1\) we obtain the required arcs.  

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**Proposition 6.8.** Let \((X,d)\) be a 2-dimensional Hilbert geometry. Then the asymptotic dimension of \((X,d)\) is at most 2.

**Proof.** We fix \(R > 4r > 0\) arbitrarily and \(o\) to be a base point of \(X\). We set \(A_0 = B(o,R)\). For each \(i \in \mathbb{N}\), we define

\[
A_i = \{ x \in X \mid iR \leq d(o,x) \leq (i+1)R \}, \quad S_i = \{ x \in X \mid d(o,x) = iR \}.
\]

For \(i > 1\), set a map \(\pi_i : S_i \to S_{i-1}\) to be the projection toward \(o\), which is a homeomorphism (see Lemma 2.2).

To construct a cover of \(X\) we put markers on each \(S_i\) inductively. For each step \(i\), we would like to decompose \(S_i\) into an even number of arcs

\[
\overline{x_0y_0}, \overline{y_0x_1}, \ldots, \overline{x_ky_k}, \overline{y_kx_0},
\]

each of which satisfies \(\star 1\) and \(\star 2\) for \(R\). For \(i > 1\) we require the following: for any \(j \in \mathbb{Z}/(k_i + 1)\mathbb{Z}\) there exist \(p, q \in \mathbb{Z}/(k_i + 1)\mathbb{Z}\) such that \(\pi_i(x_j^i) = y_p^{i-1}\) and \(\pi_i(y_j^i) = x_q^i\). We say that such a decomposition of \(S_i\) is admissible.

**Step 1:** Since \(S_1\) is decomposed into two arcs with the half length of \(S_1\), we can construct an admissible decomposition of \(S_1\) by Lemma 6.7. Note that the number of resulting arcs is even. Label individual end points of the arcs as \(x_0^{i+1} := w_0^i\), \(x_1^{i+1}, x_2^{i+1}, \ldots, x_k^{i+1}, y_k^{i+1}\), in the CCW direction on \(S_{i+1}\). Then we have an admissible decomposition of \(S_{i+1}\).

Put \(U_{0,0} = A_0\) and define \(U_{i,j}\) as a bounded closed set enclosed by

\[
\overline{x_j^ix_{j+1}^i} \cup [x_{j+1}^i, z_{j+1}^i] \cup [z_{j+1}^i, z_j^i] \cup [z_j^i, x_j^i].
\]

See Figure 4. Then \(
\text{diam}(U_{i,j}) \leq 10R\) by \(\star 2\) and \(U = \{ U_{i,j} \}_{i,j}\) is a cover of \(X\).

We check that the \(r\)-multiplicity of \(U\) is at most 3. Take a closed \(r\)-ball \(B_r\) which is not included in the interior of \(A_0\). Then there exists the smallest number \(i\) so that \(B_r\) is included in the interior of \(A_{i-1} \cup A_i\) since \(2r < R\). Consider the set

\[
\Pi_i(B_r) := S_i \cap \{ \ell_x(t) \mid 0 < t < \infty, \ell_x \text{ is a ray through } x \in B_r \}.
\]

Then we see that \(\text{diam}(\Pi_i(B_r)) \leq 4r\) by Lemma 6.6. Since \(\Pi_i(B_r)\) is an arc on \(S_i\), the inequality \(4r < R\) implies that at most one of \(x_0^i, y_0^i, \ldots, x_k^i, y_k^i\) is contained in \(\Pi_i(B_r)\). Consequently, we have the following: (i) If \(B_r \cap S_i = \emptyset\) then at most two elements of \(U\) intersect \(B_r\). (ii) If \(B_r \cap S_i \neq \emptyset\) then at most three elements of \(U\) intersect \(B_r\).

**proof of Theorem 1.3** The assertion follows from Proposition 6.3 and 6.8.
Figure 4: A piece $U_{i,j}$ of the cover $U$.

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