A simple topological model with continuous phase transition

F Baroni
Formerly at Dipartimento di Fisica dell’Università di Firenze, Via G Sansone 1, I-50019 Sesto F.no (FI), Italy
E-mail: baronifab@libero.it

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Abstract. In the area of topological and geometric treatment of phase transitions and symmetry breaking in Hamiltonian systems, some general sufficient conditions for these phenomena in $\mathbb{Z}_2$-symmetric systems (i.e. invariant under reflection of coordinates) were found in a recent paper. In this paper we present a simple topological model satisfying the above conditions, hoping to shed light on the mechanism which causes this phenomenon in more general physical models. The symmetry breaking is proved by a continuous magnetization with a nonanalytic point in correspondence with a critical temperature which divides the broken symmetry phase from the unbroken one. A particularity with respect to the common pictures of a phase transition is that the nonanalyticity of the magnetization is not accompanied by a nonanalytic behavior of the free energy.

Keywords: rigorous results in statistical mechanics, classical phase transitions (theory)

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1. Introduction

Phase transitions are sudden changes of the macroscopic behavior of a physical system composed of many interacting parts occurring when an external parameter is smoothly varied, generally the temperature, but, e.g., in a quantum phase transition it is the external magnetic field. The successful description of phase transitions starting from the properties of the microscopic interactions between the components of the system is one of the major achievements of equilibrium statistical mechanics.

From a statistical-mechanical point of view, in the canonical ensemble, phase transitions occur at special values of the temperature $T$ called transition points, where thermodynamic quantities such as pressure, magnetization, or heat capacity are nonanalytic functions of $T$. These points are the boundaries between different phases of the system. Starting from the solution of the two-dimensional Ising model by Onsager [4], these singularities have been found in many models, and later developments like the renormalization group theory [5] have considerably deepened our knowledge of the properties of the transition points.

However, in spite to the success of equilibrium statistical mechanics the issue of the deep origin of a phase transition remains open, and this motivates a study of phase transitions which may also be based on alternative approaches. One such approach, proposed in [6] and developed later [7], is based on simple concepts and tools drawn from differential geometry and topology. The main issue of this new approach is a ‘topological hypothesis’, whose content is that at their deepest level phase transitions in Hamiltonian systems are due to one or more topology changes of suitable submanifolds of configuration space, those where the system ‘lives’ as the number of its degrees of freedom becomes very large.
This idea has been discussed and tested in many recent papers [8, 9]. Moreover, the
topological hypothesis has been given a rigorous background by a theorem [10] which
states that, at least for systems with short-ranged interactions and confining potentials,
topology changes in configuration space submanifolds are a necessary condition for a phase
transition. However, the converse is not trivially true because there are models with
topology changes without phase transitions [11]. The importance of the above theorem
is that it established a strong link between phase transitions in Hamiltonian systems and
the topology of suitable submanifolds of configuration space, and the main issue of the
topological hypothesis is to search for and to test some possible topology-based sufficient
conditions for occurrence of phase transitions.

Indeed, in a recent paper [2] some sufficient topological conditions were been found,
although with the aid of some other conditions of geometric nature. Namely, for $\mathbb{Z}_2$-
symmetric systems a theorem has been shown, according to which if the potential $V$
has two absolute minima separated by a minimum jump proportional to the number of
degrees of freedom $N$, then the system shows at least symmetry breaking [5, 13]. Under
suitable assumptions, the symmetry breaking can also be associated to a phase transition
in the sense of loss of analyticity of the magnetization. In the same paper [2] a very simple
topological toy model, called the ‘hypercubic model’, was also built. This is a model with
a first order symmetry breaking phase transition which shows in a pedagogical way how
the theorem works.

This paper is devoted to the building of another topological toy model, that
we call the ‘hyperspherical model’, showing $\mathbb{Z}_2$-symmetry breaking associated to
continuous magnetization with a second order singularity in correspondence with a critical
temperature. Despite this, the partition function does not show any singularly unlike
what we expected. Indeed, this picture is quite anomalous because a singularity in the
magnetization is generally associated to a singularity also in the other thermodynamic
functions, but this may be not so strange given the extreme simplicity of the model.
We cannot expect it to reproduce all the features of a physical model, e.g. an Ising-like
model. Nevertheless, the success in building up such toy models only by topological
and geometrical ingredients may give some hints to shed light on how the topology and
geometry may affect the behavior of a general Hamiltonian system showing symmetry
breaking and phase transitions.

Since we cannot presuppose an appropriate knowledge of the topological and
geometric approach to symmetry breaking and phase transitions by the average reader, we
devote section 2 to a brief presentation of some basic concepts of it, and we refer to [15] for
a more exhaustive review. In section 3 we recall some results about $\mathbb{Z}_2$-symmetry breaking
obtained in [2], comprised the aforementioned hypercubic model. Finally, in section 4 we
present the original results of this paper, namely the aforementioned hyperspherical model
showing $\mathbb{Z}_2$-symmetry breaking testified by a continuous magnetization.

2. Basis of the topological hypothesis

All the particular properties of a Hamiltonian system have to be considered enclosed in
the form of the potential $V$ in a cause–effect relationship, and the topological hypothesis

$^1 \mathbb{Z}_2$ is the group of integers modulo 2 \{0, 1\} which are isomorphic to the symmetry group of the reflection of
coordinates: $q \mapsto -q$. 

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is an attempt to disclose this relationship in respect of phase transitions and symmetry breaking.

In particular, phase transitions show as loss of analyticity of the partition function \( Z \) at some critical temperatures \( T_c \), while symmetry breaking phenomena show by an order parameter which assumes a non-vanishing value in a certain interval of values of temperatures \( T \).

Generally, a phase transition and a symmetry breaking are associated and occur at the same critical temperature \( T_c \). \( T_c \) divides the broken symmetry phase from the unbroken one, although there are cases in which a system shows a phase transition without symmetry breaking. The topological hypothesis attempts to relate these phenomena to the topology of suitable submanifolds of configuration space.

Consider an \( N \) degrees of freedom system with Hamiltonian given by

\[
H(p,q) = \sum_{i=1}^{N} p_i^2 + V(q). \tag{1}
\]

The partition function \( Z \) is

\[
Z(N,T) = \int dp dq e^{-(1/T)H(p,q)} = \int dp e^{-(1/T)\sum_{i=1}^{N} p_i^2} \int dq e^{-(1/T)V(q)} = Z_{\text{kin}} Z_C \tag{2}
\]

where \( Z_{\text{kin}} \) is the kinetic part of \( Z \), and \( Z_C \) is the configurational part. In order to develop what follows we have to assume that the potential is lower bounded, and for convenience the minimum is assumed to be 0. If 0 is not the minimum it is sufficient to add to the potential a non-influential constant term equal to the minimum itself with the sign changed.

Consider \( Z_C \), which can be decomposed as follows:

\[
Z_C(N,T) = \int dq e^{-(1/T)V(q)} = N \int_0^{+\infty} dv e^{-Nv/T} \int_{\Sigma_v} d\Sigma |\nabla V| \tag{3}
\]

where \( v = V/N \) is the potential per degree of freedom, and the \( \Sigma_v^N \)s are the \( v \)-level sets of the potential \( V \) in configuration space

\[
\Sigma_v^N = \{ q \in \mathbb{R}^N : V(q) = Nv \}. \tag{4}
\]

The \( \Sigma_v^N \)s are the boundaries of the \( M_v^N \)s (\( \Sigma_v^N = \partial M_v^N \)) defined as follows:

\[
M_v^N = \{ q \in \mathbb{R}^N : V(q) \leq Nv \}. \tag{5}
\]

Equation (3) shows that configuration space is foliated by the \( \Sigma_v^N \)s by varying \( v \) between 0 and \(+\infty\). The \( \Sigma_v^N \)s are very important submanifolds because in the thermodynamic limit the canonical statistic measure narrows around \( \Sigma_v^\bar{v}(T) \), where \( \bar{v}(T) \) is the average potential per degree of freedom, and thus \( \Sigma_v^\bar{v}(T) \) becomes the unique submanifold accessible to the representative point of the system.

This may have significant consequences on the one hand on the symmetries of the system and thus on the order parameter, and on the other hand on the analyticity of \( Z_C \) in the thermodynamic limit, as the theorem in section 2.1 suggests, owing to the fact that the \( \Sigma_v^N \) have in general a very complex topology which changes by varying \( v \).
The same considerations made about $Z_C$ can be made also for $Z_{\text{kin}}$, but in this case the corresponding submanifolds $\Sigma^N_t$, where $t = T/N$ is the kinetic energy per degree of freedom, are all trivially homeomorphic\(^2\) to the $N$-dim hypersphere. Further, $Z_{\text{kin}}$ is analytic for all values of $T$ in the thermodynamic limit, and thus cannot contribute to any loss of analyticity in $Z$.

2.1. Necessary topological conditions for the occurrence of a phase transition

The main result of the topological hypothesis so far obtained is a theorem which establishes a topological necessary condition for the occurrence of a phase transition in a Hamiltonian system with the potential of the standard form

$$V(q) = \sum_{i=1}^{N} \phi(q_i) + \sum_{i,j=1}^{N} c_{ij} \psi(|q_i - q_j|)$$

which is short range, stable, confining and bounded below. We do not enter into the details of the theorem because it is not essential for the following, so we refer the interested reader to [10,23]. We limit ourselves to reporting the statement and a brief discussion of its main consequences.

**Theorem.** Let $v_0, v_1 \in \mathbb{R}$ such that $v_0 < v_1$, if $\exists N_0 : \forall N > N_0, \forall v, v' \in [v_0, v_1] \Sigma^N_v$ is diffeomorphic to $\Sigma^N_{v'}$ then the limit for large $N$ of Helmholtz free energy $F$ is $C^2$ in the interval $(v_0, v_1)$, hence the system does not have any phase transition in the same interval at least of the second order.

Having a phase transition in the interval $(v_0, v_1)$ means that the critical average potential $v^* = \bar{v}(T_c)$, where $T_c$ is the critical temperature, lies in it.

This theorem states a necessary condition for a phase transition, because if $v^* = \bar{v}(T_c)$ exists then the theorem implies that

$$\forall \epsilon > 0, \exists \bar{N} : \forall N > \bar{N} \exists \Sigma^N_{v^*} : |v^* - v_c| < \epsilon$$

where $v_c$ is a value of the potential, generally different from $v^*$, at which a topological change occurs in the $\Sigma^N_v$. Thus it is possible to extract a sequence $\{\Sigma^N_{v^*_i}\}_{i \in N}$ such that $v^*_i \to v^*$ as $i \to \infty$, and in that limit we can say that the presence of a phase transition implies a topological change in the $\Sigma^N_v$s located exactly in correspondence with $v^*$.

In the light of this theorem it is natural to ask whether its converse may hold, that is if a topological change in the $\Sigma^N_v$s necessary causes a phase transition, but the answer is trivially not because it is very easy to find models with a lot of topological changes without phase transitions. Some of these models have already been studied, e.g. the one-dim $XY$ model [11,22].

Further, in some other models the $\Sigma^N_v$s undergoes huge topological changes that increase with $N$ and that are not always in correspondence with a phase transition, e.g. the mean-field $\phi^4$ model [12] and the mean-field $XY$ model [21]. The difference between the last two models is that in the latter $v^*$ corresponds always to a topological change in the $\Sigma^N_v$s for all values of the model’s parameters, while in the former it is possible to find $v^*$ which does not correspond to any topological change in the $\Sigma^N_v$s. There is no contradiction

\(^2\) A homeomorphism is a continuous bijection between manifolds with continuous inverse.
with the above theorem because among its hypothesis a short range potential is requested, while in a mean-field model the interaction range is obviously infinite.

Further, it has been shown that in general no exclusively topological sufficient conditions for phase transitions are possible; we refer the reader to [24,15] for details.

3. Toward sufficient topological and geometric conditions for symmetry breaking phase transitions

In order to search for a sufficient topological and geometric condition for a phase transition it is necessary to study how topological changes in the $\Sigma^N_v$s may affect the analytical properties of the thermodynamic functions. But in this work we will not follow this line of research, and we will shift our attention to the issue of how topology may break the symmetry of a system. In [2] a simple theorem on sufficient conditions for $\mathbb{Z}_2$-symmetry breaking, and an elementary model which illustrates how it works, were found.

Hereafter, we are going to present the trick necessary to perform the thermodynamic limit of $Z$, that is a necessary condition in order for symmetry breaking to occur. We will see how this trick emphasizes the role of $\Sigma^N_v$, and thus of its topology, in that limit.

The second integral on the right hand side of (3) is the $V$-derivative of the measure of the $M^N_v$s induced by the standard metric of $\mathbb{R}^N$ (d$V$ = dx$_1\cdots$dx$_N$)

$$\int_{\Sigma^N_v} \frac{d\Sigma}{|\nabla V|} = \frac{1}{N} d\nu \text{Mis}(M^N_v) = \mu(\Sigma^N_v)$$

that also coincides with the microcanonical volume of the $\Sigma^N_v$s, $\mu(\Sigma^N_v)$. $\text{Mis}(M^N_v)$, being the volume of a subset of $\mathbb{R}^N$, can be rewritten as follows:

$$\text{Mis}(M^N_v) = a_N(v)^N$$

where the function $a_N(v)$ is defined. $a_N(v)$ has the dimension of a length, and is linked to the configurational entropy per degree of freedom $s_N(v)$ by the relation $s_N(v) = \ln a_N(v)$.

The introduction of $a_N(v)$ is useful in order to perform the thermodynamic limit of $Z_C$ by the saddle point trick. $\mu(\Sigma^N_v)$ can be written as a function of $a_N(v)$

$$\mu(\Sigma^N_v) = a'_N(v)a_N(v)^{N-1}$$

where the prime denotes the derivative with respect to $v$, and then we have

$$Z_C = N \int_0^{+\infty} dv \ e^{-Nv/T} \ a'_N(v)a_N(v)^{N-1} = N \int_0^{+\infty} dv \ \frac{a'_N(v)}{a_N(v)} e^{-(N/T)f_N(v,T)}$$

where $f_N(v,T) = v - T \ln a_N(v)$ is the free energy per degree of freedom.

Now we can apply the saddle point trick to evaluate $Z_C$, but before performing the $N = \infty$ limit it is necessary to assume that $\lim_{N\to\infty} a_N(v) = a(v)$ exists. This requirement corresponds to the requirement that microcanonical entropy exists, which from a physical viewpoint seems quite reasonable. Thus

$$Z_C \simeq N \left( \frac{2\pi T}{Nf'(\bar{v},T)} \right)^{1/2} \frac{a'(\bar{v})}{a(\bar{v})} e^{-(N/T)f(\bar{v},T)}$$

where $f = \lim_{N\to\infty} f_N$, and $\bar{v}$ is the $v$-minimum of $f(v,T)$ in the interval $[0, +\infty)$ at fixed $T$. $\bar{v}$ coincides with the average potential per degree of freedom.

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Figure 1. Example of three $f$-level sets of the $\mathbb{Z}_2$-symmetric function $f = -\frac{1}{2}(q_1^2 + q_2^2) + \frac{1}{4}(q_1^4 + q_2^4) - q_1q_2$: the $f$-levels are $-\frac{1}{2}$, 0, and 1 respectively, corresponding to the lines colored blue, magenta, and yellow. The set labeled by the blue line has two connected components which do not conserve the $\mathbb{Z}_2$-symmetry singularly.

The saddle point trick implies that the configuration space accessible to the representative point $q$ of the system reduces to the $\Sigma^N_v$ selected by $\bar{v}(T)$, and thus by $T$, in the limit of large $N$. Now we understand how the topological properties of the $\Sigma^N_v$'s may break the symmetry of a system.

Indeed, suppose that the potential $V(q)$ has some symmetries in configuration space, then the same symmetries have to belong also to the $\Sigma^N_v$'s which in general are composed of a number $n$ of connected components $\Sigma^{N,a}_v$ labeled by the index $a$, so the $\Sigma^N_v$ is the disjointed union of the $\Sigma^{N,a}_v$'s:

$$\Sigma^N_v = \bigcup_{a=1}^n \Sigma^{N,a}_v.$$  \hfill (13)

The crucial observation is that a single $\Sigma^{N,a}_v$ does not need to have the same symmetries as the $\Sigma^N_v$ anymore, as the example in figure 1 illustrates. In the light of this fact, in the thermodynamic limit the representative point $q$ can ‘live’ only on a single c.c. $\Sigma^{N,a}_{\bar{v}(T)}$ because it cannot jump from one to the others anymore. In other words, the ergodicity cannot be assumed on the whole $\Sigma^{N}_{\bar{v}(T)}$, disregarding its topology.

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This fact reflects on the magnetization $m_N$ per degree of freedom defined as

$$m_N = \frac{1}{Z_C} \int dq \frac{1}{N} \sum_{i=1}^{N} q_i e^{-(1/T)V(q)} \quad (14)$$

because in the limit of large $N$ the domain of integration has to be replaced by a single $\Sigma_{\bar{v}(T)}^N$, and thus the expression for $m_N$ has to be replaced with

$$m_N^a \simeq \frac{1}{\mu} \left( \Sigma_{\bar{v}(T)}^{N,a} \right) \frac{1}{\Sigma_{\bar{v}(T)}^N} \sum_{i=1}^{N} q_i \frac{1}{|\nabla V|} \quad (15)$$

In (15) we have assumed the ergodicity on the $\Sigma_{\bar{v}(T)}^{N,a}$. $\{m_N^a\}_{a=1,...,n}$ is the set of all possible values of the magnetization which undergoes a splitting every time that $\Sigma_{\bar{v}(T)}^N$ undergoes a topological change by varying $T$. The temperatures at which the topological changes occur are recognized as critical temperatures.

At this point it must be noted that in the thermodynamic limit other selection mechanisms may intervene to limit the ergodicity of a system on the $\Sigma_{\bar{v}(T)}^N$ besides the topological one here pointed out. For instance, consider the mean-field $\phi^4$ model [12]. For suitable values of the parameters and of the temperature it shows $Z_2$-symmetry breaking with $\Sigma_{\bar{v}(T)}^N$ homeomorphic to the $N$-dim hypersphere, and thus in this case the topology of the $\Sigma_{\bar{v}(T)}^N$ cannot be the acting selection mechanism.

3.1. Sufficient topological and geometric conditions for $Z_2$-symmetry breaking

Hereafter we will deal only with systems having $Z_2$-symmetry, i.e. with potential $V(q)$ invariant under the reflection of coordinates $q \rightarrow -q$. The considerations of section 2 can be applied to these systems and condensed into a straightforward theorem on necessary conditions for $Z_2$-symmetry breaking [2].

Statement:

**Theorem.** Let us consider a Hamiltonian system with $N$ degrees of freedom and a potential $V$ bounded below which is $Z_2$-symmetric. Let the entropy per degree of freedom be well defined in the thermodynamic limit, i.e., the function $a(v)$ defined in equation (9) exist and be continuous and piecewise differentiable. Let $\Sigma_v^N$ be the family of equipotential submanifolds of the configuration space defined as in equation (4). Without loss of generality, let $\min(V) = 0$.

Let $v'' > v' > 0$ be two values of the potential energy per degree of freedom $V/N$ such that $\forall v : v \geq v''$, $\forall N\Sigma_v^N$ is made of a single connected component, and such that $\forall v : v' > v \geq 0$, $\forall N\Sigma_v^N$ is made of more than one connected component which are not $Z_2$-symmetric when considered individually.

Then, in the thermodynamic limit the $Z_2$-symmetry is spontaneously broken for all the temperatures $T < T'$ where $T'$ is such that $v' = \bar{v}(T')$, and is unbroken for all the temperatures $T \geq T''$ where $T''$ is such that $v'' = \bar{v}(T'')$, provided the $\Sigma_v^N$'s remain ergodic in the thermodynamic limit.

This theorem also implies the occurrence of a singularity in the order parameter because it has to vanish for $T \geq T''$, and has to be not vanishing for $T \leq T'$, but since
it is not possible to join in an analytical way the null function with one that is not null, necessarily a loss of analyticity must occur for at least a critical temperature $T_c$ such that $T' \leq T_c \leq T''$. $T_c$ corresponds to a critical value of the average potential $v_c = \bar{v}(T_c)$ such that $v' \leq v_c \leq v''$. Further, if we restrict the condition of the theorem in such a way that $v'' = v'$ then the singularity is located exactly in correspondence with the critical potential $v_c = v'' = v'$, and if $T'' = T'$ then $T_c = T'' = T'$.

It must be noted that it might be very hard to show whether the potential $V(q)$ of a general physical model satisfies the assumptions of the theorem, because finding the topology of the $\Sigma^N_v$s is generally a very difficult task. Despite this, it cannot be excluded that future developments of the research in this field could provide suitable tools. For now, we are content to see the theorem at work in two elementary models; one is briefly recalled in section 3.2, and the other is the original part of this paper.

3.2. The hypercubic model with $\mathbb{Z}_2$-symmetry breaking and first order phase transition

Now we describe briefly a topological model given in [2] to shed light, in a pedagogical way, on the content of the theorem of the last subsection. We build a double-hole potential $V$ which is $\mathbb{Z}_2$-symmetric by using $N$-dimensional hypercubes

$$V(q) = \begin{cases} 
0 & \text{if } q \in A^\pm \\
Nv_c & \text{if } q \in B \setminus \{A^+ \cup A^-\} \\
+\infty & \text{if } q \in \mathbb{R}^N \setminus B.
\end{cases} \quad (16)$$

The configuration space is $\mathbb{R}^N$, $A^+$ and $A^-$ are two disjoint hypercubes not centered in the origin and symmetric under $\mathbb{Z}_2$, and $B$ is a hypercube centered in the origin such that $A^+ \cup A^- \subset B$. Figure 2 can help us to understand the arrangement. Note that by construction the minimum jump to pass from one hole to the other is proportional to $N$; this assumption is essential to make this model satisfy the hypothesis of the theorem.

The $\Sigma^N_v$s are as follows:

$$\Sigma^N_v = \begin{cases} 
\emptyset & \text{if } v < 0 \\
A^+ \cup A^- & \text{if } v = 0 \\
\emptyset & \text{if } 0 < v < v_c \\
B \setminus \{A^+ \cup A^-\} & \text{if } v = v_c \\
\emptyset & \text{if } v > v_c
\end{cases} \quad (17)$$

from which we see that there are only two permitted values of the potential: 0 and $v_c$. The partition function $Z_N$ is

$$Z_N(T) = 2a^N + (b^N - 2a^N) e^{-(Nv_c/T)} \quad (18)$$

where $a$ and $b$ are the sides of the hypercubes $A^\pm$ and $B$ respectively.

We called this model the hypercubic model (the name ‘hypercubic model’ is only used by convention, because the hypercubes can be substituted by other geometric figures), which satisfies the hypothesis of the theorem in the last section with $v'' = v_c$, $v' = 0$. Indeed, in the limit $T \to 0$ $\Sigma_0^N$ is selected, and in the limit $T \to \infty$ $\Sigma^N_{v_c}$ is selected. Then we expect the occurrence of $\mathbb{Z}_2$-symmetry breaking associated with a first order singularity in the magnetization for at least a finite critical temperature $T_c$. 

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Indeed, the analytical solution in the thermodynamic limit shows that this is the case with \( T_c = v_c \ln^{-1}(b/a) \). The magnetization per degree of freedom in the broken symmetry phase is simply the coordinate of the center of mass of \( A^+ \) or \( A^- \); the picture is sketched in figure 3. Even though not implicated by the theorem, a singularity at \( T_c \) occurs also in the partition function and thus in the thermodynamic functions, reproducing thus the common picture of a first order phase transition.

It is worth remarking that in building the hypercubic model, instead of hypercubes we can use any other manifolds, provided they are topologically equivalent, e.g. hyperspheres.

Now we make some observations on the relation between symmetry breaking and the singularity in the average potential. The structure of the hypercubic model implies that the former needs the latter, but the converse is not true. Indeed, we can redefine \( \Sigma_0^N \) by only one hypercube \( A \) centered in the origin of coordinates and with the same side \( a \), so that the \( \mathbb{Z}_2 \)-symmetry never breaks, but the solution of the thermodynamic does not change in the thermodynamic limit, and so the critical temperature \( T_c \) stays the same.

4. The hyperspherical model

In this section we build a topological model, called the hyperspherical model, showing \( \mathbb{Z}_2 \)-symmetry breaking with a continuous magnetization which passes by a nonanalytic point. This point separates the broken symmetry phase from the unbroken one. The basic ingredients are \( N \)-dim hyperballs\(^3\) with which we will build the \( M_v^N \)'s of the potential as defined in (5).

\(^3\) By hyperball we mean a hypersphere with its interior.
Figure 3. Magnetization per degree of freedom versus $T$ as $N \to \infty$ of the hypercubic model for $a = 1$, $b = 2a$, $v_c = 1$, and so $T_c = 1/\ln 2$. From [2].

Figure 4. Average potential of the hypercubic model versus $T$. The different smooth curves are the finite-$N$ results with $N = 10, 20, 50$, while the piecewise constant curve is the $N \to \infty$ limit. The numerical values are as in figure 3. From [2].

Generally, when a potential is defined the starting point is its explicit expression $V(q)$, but here we define directly the $M_v^N$s because we are interested in how their topology and measure match to entail a phase transition, and not in the explicit expression of $V$ itself. This is simply possible under the condition that if $v' < v''$ then $M_{v'} \subseteq M_{v''}$. Nevertheless, it is not excluded that the potential we are going to define may also be given by an explicit expression, but it may be very complicated and it would be useless for our present purposes.

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Figure 5. Some $\Sigma^N_v$s at $N = 2$ in the configuration space $(q_1, q_2)$ for the hyperspherical model.

Figure 6. Scheme of the topological mechanism of the symmetry breaking in the hyperspherical model. On the left $\Sigma^N_v(T)$ is schematically sketched at $N = 2$ at $T > T_c$, in the center at $T = T_c$, and on the right at $T < T_c$.

Let us fix a $v_c > 0$. For $v \geq v_c$ let $M^N_v$ be an $N$-dim hyperball $B_v$ centered in the origin of coordinates such that $\text{Mis}(B_v) = v^N$, while for $v_c > v \geq 0$ let $M^N_v$ be the disjointed union of two $N$-dim semi-hyperballs $B^+_v$, $B^-_v$. $B^+_v$, $B^-_v$ are obtained by dividing one $N$-dim hyperball centered in the origin by the hyperplane $\sum_{i=1}^N q_i = 0$, and by widening the $B^+_v$ and $B^-_v$ thus obtained. Thus constructed, $B^+_v$ and $B^-_v$ are the images of each other under $Z_2$-symmetry. Further, we assume $\text{Mis}(B^+_v \cup B^-_v) = v^N$.

Resuming, at the varying of $v$ from $+\infty$ to 0 $M^N_v$ is an $N$-dim hyperball with measure $v^N$ which disconnects into two connected components for a critical value $v_c$:

$$
M^N_v = \begin{cases} 
B_v & \text{if } v \geq v_c \\
B^+_v \cup B^-_v & \text{if } v_c > v \geq 0.
\end{cases}
$$

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Figure 7. Magnetization per degree of freedom versus $T$ for the hyperspherical model (red line). $v_c = 1$ is assumed, and thus $T_c = v_c = 1$. The dashed line labels another pattern of the magnetization among the other infinite possible ones in the broken symmetry phase corresponding to a different choice of arranging the $M_v$s as $v < v_c$, as explained in the text.

The last step in building the model is to choose how to widen $B_v^+$ and $B_v^-$ by translating them parallel to their axes of symmetry. We can choose between two extreme cases: to not widen $B_v^+$ and $B_v^-$ at all, letting them touch, or to widen $B_v^+$ and $B_v^-$ maximally in such a way that all the $B_v^+$'s have their poles in common for $v_c \geq v \geq 0$, and the same for the $B_v^-$'s.

These choices reflect on the shape of the magnetization, which will be vanishing without symmetry breaking at all for the first choice, or which will reach its maximum value for the second choice. In the following we will consider the second choice. Obviously, infinite intermediate possible arrangements exist, dictated only by the requirement that the potential $V$ is a single value function of the coordinate $q$s. In these all intermediate cases the only constraint on the shape of the magnetization is such that its slope is bounded below by the tangent of the maximum magnetization sketched in figure 7 and upper bounded by 0.

Now we have a look at what the $\Sigma^N_v$s (4) are. We recall that they are the boundaries of the $M^N_v$: $\Sigma^N_v = \partial M^N_v$. For $v > v_c$ $\Sigma^N_v$ is the boundary of $B_v$, called $S_v$, that is an $N$-dim hypersphere of the same radius. For $v_c > v \geq 0$ $\Sigma^N_v$ is the boundary of $B_v^+ \cup B_v^-$, called $S^+_v \cup S^-_v$, that is two $N$-dim semi-hyperspheres each closed by one $(N-1)$-dim hypersphere of the same radius. Finally, for $v = v_c$ $\Sigma^N_v$, called $S_{v_c}$, is an $N$-dim hypersphere jointed with an $(N-1)$-dim hypersphere of the same radius. Summarizing

$$
\Sigma^N_v = \begin{cases} 
S_v & \text{if } v > v_c \\
S_{v_c} & \text{if } v = v_c \\
S^+_v \cup S^-_v & \text{if } v_c > v \geq 0.
\end{cases}
$$

(20)

$S_{v_c}$ plays the role of the critical $v$-level by which the disconnection from one connected component of the $\Sigma^N_v$s to two occurs.

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Now we resolve the thermodynamic of the model. The function $a_N(v)$, given by definition (9), is independent of $N$ and is

$$a(v) = v \quad \text{if } v \geq 0.\quad (21)$$

The configurational partition function $Z_C$ is given by (11)

$$Z_C = N \int_0^{+\infty} dv e^{-Nv/T} v^{N-1} = \frac{N!}{N^N} T^N\quad (22)$$

Disregarding terms depending on $N$ only, the free energy per degree of freedom is

$$f(T) = -\frac{T}{N} \ln Z_C = -T \ln T.\quad (23)$$

The average potential per degree of freedom and the specific heat are respectively

$$\bar{v}(T) = -T^2 \frac{\partial}{\partial T} \left( \frac{f}{T} \right) = T\quad (24)$$

$$c_v(T) = \frac{\partial \bar{v}}{\partial T} = 1.\quad (25)$$

Note that they do not depend on $N$, and therefore it is not necessary to perform the thermodynamic limit. At $T = v_c$ the disconnection of the $\Sigma^N_v$ occurs, and thus we define $T_c = v_c$ as the critical temperature.

Now we pass to study of the magnetization per degree of freedom $m_N$ defined in (15). We start by finding the average of the representative point $\langle q \rangle$ on the $\Sigma^N_v$'s at finite $N$, and then performing the limit $N \to \infty$. The radii of the hyperspheres that constitute the $M^N_v$'s are fixed by the formula of the volume of the $N$-dim hypersphere

$$a(v)^N = \frac{\pi^{N/2}}{(N/2)!} R_N(v)^N\quad (26)$$

and by the definition (21) of $a(v)$. As $v \geq v_c$, $\langle q \rangle$ vanishes on the $S_v$'s for reasons of symmetry. As $v_c > v \geq 0$ we have to calculate $\langle q \rangle$ on the $S^+_v$'s or on the $S^-_v$'s, thus we need the knowledge of $((\nabla V))^{-1}$ on them, but owing to their properties of symmetry we can perform the same calculation without the detailed knowledge of $((\nabla V))^{-1}$.

Indeed, because of our choice of arranging the $S^+_v$'s and the $S^-_v$'s, $((\nabla V))^{-1}$ vanishes at the pole, and it is constant on the parallels by an increasing value as the parallel goes from the pole to the equator, where it reaches its maximum. This maximum is the same on the $(N-1)$-dim hyperball which closes the $N$-dim semi-hypersphere. From this we can deduce that $\langle q \rangle$ is a point lying on the axis of symmetry of the $S^+_v$'s or $S^-_v$'s, and lying between the center of mass of the $N$-dim semi-hypersphere and the center of the corresponding whole $N$-dim hypersphere.

Now we focus our attention on the center of mass of the $N$-dim semi-hypersphere. We fix a coordinate system with the origin coinciding with the center of the corresponding hypersphere. The axis $x_N$ coincides with the axis of symmetry of the $N$-dim semi-hypersphere, and the other axes are orthonormal to $x_N$. Obviously, for reasons of symmetry only the component along $x_N$ is not vanishing, and in the appendix we show
that its value is

\[ B_N = \frac{2R}{\sqrt{\pi N}} \]  

where \( R \) is the radius.

Returning to \( \langle q \rangle = (\langle q_1 \rangle, \ldots, \langle q_N \rangle) \), it is linked to the magnetization per degree of freedom \( m_N \) by the relation

\[ m_N = \frac{1}{N} \sum_{i=1}^{N} \langle q_i \rangle. \]  

(28)

However, since \( \forall i, j \langle q_i \rangle = \langle q_j \rangle \) for reasons of symmetry, it follows that

\[ |\langle q \rangle|^2 = \sum_{i=1}^{N} \langle q_i \rangle^2 = Nm_N^2. \]  

(29)

As \( 0 \leq v < v_c \)

\[ R_N(v_c) - R_N(v) < |\langle q \rangle| < R_N(v_c) - R_N(v) + B_N(v) \]  

(30)

where \( R_N(v) \) is the radius of \( B^{N^+}_v \) or \( B^{N^-}_v \). Then, by using (26), (29) and (30) we obtain

\[ \frac{1}{\sqrt{N}} \left( \frac{N! (N/2)^{1/N} 2^{N/2} \pi^{1/2}}{\sqrt{N}} \right)^{1/N} (a(v_c) - a(v)) \]  

\[ < \frac{1}{\sqrt{N}} \left( \frac{N! (N/2)^{1/N} 2^{N/2} \pi^{1/2}}{\sqrt{N}} \right)^{1/N} \left( a(v_c) - a(v) \left( 1 - \frac{2}{\sqrt{\pi N}} \right) \right). \]  

(31)

By performing the limit \( N \to \infty \), and by using the relations \( \lim_{N \to \infty} N!^{1/N} = 1, \lim_{N \to \infty} (N!^{1/N}/N) = (1/e) \) we have

\[ m(v) = \lim_{N \to \infty} m_N(v) = \frac{1}{\sqrt{2\pi e}} (a(v_c) - a(v)). \]  

(32)

Finally, by using (21) and (24), we reconstruct the link with the temperature \( T \)

\[ m(T) = \begin{cases} 
0 & \text{if } T \geq T_c \\
\frac{1}{\sqrt{2\pi e}} (v_c - T) & \text{if } T \leq T_c 
\end{cases} \]  

(33)

where, we recall, \( T_c = v_c \).

This model satisfies the assumptions of the theorem in section 3.1 with \( v'' = v' = v_c \), \( T'' = T' = T_c \), thus we had to expect exactly what the analytical solution shows: \( \mathbb{Z}_2 \)-symmetry breaking associated with a singularity in the magnetization exactly located at \( T = T_c \). But this does not reproduce the usual picture of a symmetry breaking phase transition because it is not accompanied by any singularity in the partition function \( Z \), and thus in the thermodynamic functions. This is quite surprising, and up to now we have not been able to understand why this is the case.
5. Concluding remarks and outlook

In the book [1] the author, among a lot of other things, points out the strong relation between phase transitions and symmetry breaking and the topology of the \( v \)-level sets of the potential \( \Sigma^N_v \) (4) in \( N \) degrees of freedom Hamiltonian systems. However, the question whether topology might be involved in the deep origin of these phenomena remains substantially open, although some recent results [24] show the impossibility of purely topological sufficient conditions.

In [2] an attempt to address this issue was made by showing a straightforward theorem, reported in section 3.1, on topological and geometric sufficient conditions for \( \mathbb{Z}_2 \)-symmetry breaking, which points out the importance of the topology of the \( \Sigma^N_{v(T)} \) in the thermodynamic limit selected by the temperature \( T \).

Indeed, in that limit the canonical measure narrows more and more around \( \Sigma^N_{v(T)} \), and, since the latter is generally made by more than one connected component which do not need to be \( \mathbb{Z}_2 \)-symmetric, the representative point has to choose among them and thus the \( \mathbb{Z}_2 \)-symmetry can be broken.

The original part of this work consisted of the construction of a topological model, called the ‘hyperspherical model’, which illustrates how the above mentioned theorem works. The \( \Sigma^N_v \)'s are directly defined in terms of hyperspheres (hence the name of the model) which disconnect into two connected components below a critical temperature \( T_c \).

The magnetization is continuous and shows a second order singularity in correspondence with \( T_c \). Despite this, the partition function shows no singularity, and thus neither do the thermodynamic functions. This is quite surprising for a model with symmetry breaking, but its extreme abstractness does not guarantee a realistic reproduction of the properties of a physical model, e.g. an Ising-like model.

Anyway, the aim of this toy model, as well as the hypercubic one, is to highlight how the topological mechanism of selection among distinct connected components of the \( \Sigma^N_v \)’s works, although it is not the only possible selection mechanism able to induce symmetry breaking. For example, in the mean-field \( \phi^4 \) model [12] we have found that, for some values of the parameters and the temperature, the \( \Sigma^N_v \)'s are homeomorphic to a hypersphere also in the broken symmetry phase, and thus another selection mechanism has to act.

However, this fact does not exclude that the topological mechanism may be at the origin of symmetry breaking in any case, because it might be applied to other subsets of configuration space more constrained with respect to the \( \Sigma^N_v \)'s. Indeed, there is no reason to assume the ergodicity on the \( \Sigma^N_{v(T)} \) in the limit of large \( N \); we have simply assumed it both in the theorem and in the hyperspherical model.

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Appendix. The center of mass of the \( N \)-dim semi-hypersphere

Let a coordinate system have its origin coinciding with the center of the \( N \)-dim hypersphere of radius \( R \). The axis \( x_N \) coincides with its axis of symmetry, and the other axes are orthogonal to \( x_N \). Because of the symmetry of the semi-hypersphere, only
the component of the center of mass along \( x_N \), \( B_N \), is not vanishing, and thus we limit ourselves to calculating only this.

We start with the \( N = 3 \) case, and then we will generalize the result by induction to a general \( N \):

\[
B_3 = \frac{2R}{4\pi} \int_0^\pi d\theta_2 \sin^2 \theta_2 \int_0^\pi d\theta_1 \sin \theta_1
\]

where \((\theta_1, \theta_2)\) are standard polar coordinates and \( R \) is the radius. The generalization is straightforward:

\[
B_N = \frac{2R}{\text{Mis}(S^{N-1})} \int_0^\pi d\theta_{N-1} \sin^{N-1} \theta_{N-1} \cdots \int_0^\pi d\theta_1 \sin \theta_1
\]

where \( S^N \) denotes the \( N \)-dim hypersphere of unitary radius. By using the following formulas:

\[
\text{Mis}(S^{N-1}) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \tag{A.2}
\]

\[
\int_0^\pi d\theta_1 \sin^N \theta = \frac{\sqrt{\pi} \Gamma((N/2) + (1/2))}{\Gamma(1/2)} \tag{A.3}
\]

\[
\Gamma(1) = 0! = 1 \tag{A.4}
\]

\[
\Gamma(x + 1) = x\Gamma(x) \tag{A.5}
\]

and by some trivial algebraic manipulation the result we are searching for is

\[
B_N = \frac{2R}{\sqrt{\pi N}}. \tag{A.6}
\]

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