On Backlund transformation of Riccati equation method and its application to nonlinear partial differential equations and differential-difference equations

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Abstract: In this paper, we investigate the equivalence between the Backlund transformation of Riccati equation method and the extended tanh-function method. It is proved that the two methods are equivalent when applying them to partial differential equations and differential-difference equations. Two examples are introduced to justify our results.

Keywords: Extended tanh-function method, Backlund transformation of Riccati equation method, partial differential equations, differential-difference equations.

MSC: 35C07, 35Q51.

1. Introduction

Many physical, biological and chemical phenomena can be modeled using partial differential equations (PDEs) and differential-difference equations (DDEs). So, in the last decades, many researchers have been interested in obtaining exact solutions of PDEs and DDEs. Many methods were proposed for achieving this task. Some of these methods are: the tanh method [1], the extended tanh-function method (ETM) [2], the simplest equation method [3], the integral bifurcation method [4], the extended mapping transformation method [5,6] and the Backlund transformation of Riccati equation method (BTREM) [7–12]. Our objective in this paper is to investigate the equivalence between the BTREM and the ETM.

2. Description of the two methods

In the following two subsections we give a brief description of the two methods.

2.1. The extended tanh-function method [2]

Consider a given partial differential equation with some independent variables, say, x and t and dependent variable u:

\[ H(u, u_t, u_{xx}, u_{xt}, ...) = 0, \]  

(1)

where \( u = u(x, t) \) is an unknown function, \( u_x \) and \( u_t \) are the derivatives of \( u \) with respect to \( x \) and \( t \) respectively. It is assumed that the Equation (1) has the following traveling wave solution:

\[ u = u(z), \quad z = kx + ct + z_0, \]  

(2)

where \( k, c \) and \( z_0 \) are some constants. Substituting Equation (2) into Equation (1), we get the following reduced ordinary differential equation:

\[ H(u, u', u'', ...) = 0, \]  

(3)

where the primes denote the derivative with respect to \( z \). The solution of the Equation (3) can be expressed as:

\[ u = \sum_{i=0}^{n} a_i \phi_i'(z), \]  

(4)
where \(a_i, i = 0, 1, 2, n\) are some constants that will be computed later, \(n\) is a positive integer computed by the balance between the highest-order derivative term and the nonlinear terms in the Equation (3) and \(\phi\) satisfies the following Riccati equation:

\[
\phi'(z) = \sigma + \phi^2(z),
\]

where \(\sigma\) is a constant. The Riccati Equation (5) has the following solutions:

1. If \(\sigma < 0\)
   \[
   \phi(z) = -\sqrt{-\sigma} \tanh(\sqrt{-\sigma}z),
   \]
   \[
   \phi(z) = -\sqrt{-\sigma} \coth(\sqrt{-\sigma}z),
   \]

2. If \(\sigma = 0\)
   \[
   \phi(z) = -\frac{1}{z + \omega}, \quad \omega = \text{const}.
   \]

3. If \(\sigma > 0\)
   \[
   \phi(z) = \sqrt{\sigma} \tan(\sqrt{\sigma}z),
   \]
   \[
   \phi(z) = -\sqrt{\sigma} \cot(\sqrt{\sigma}z).
   \]

Substituting Equation (4) into Equation (3) and making use of Equation (5), then setting the coefficients of \(\phi'(z), i = 0, 1, \ldots\) to zero, we get a set of algebraic equations for \(a_i, i = 0, 1, 2, \ldots, n\). Solving this obtained system will lead to the values of \(a_i, i = 0, 1, 2, \ldots, n\).

2.2. Backlund transformation of Riccati equation method [10]

In this method the solution of the Equation (3) is given in the form:

\[
\Phi(z) = \frac{-\sigma B + D\phi(z)}{D + B\phi(z)},
\]

where \(\Phi(z)\) is given by:

\[
\Phi(z) = \frac{-\sigma B + D\phi(z)}{D + B\phi(z)},
\]

\(b_i, i = 0, 1, 2, n\) are some constants that will be computed later, \(n\) is a positive integer computed by the balance between the highest-order derivative term and nonlinear terms in the Equation (3), \(\phi(z)\) are the known solutions of Ricatti Equation (5), \(B\) and \(D\) are arbitrary constants. Substituting Equation (11) into Equation (3), then setting the coefficients of \(\phi(z)\) to zero, we get some algebraic equations for \(b_i, i = 0, 1, 2, \ldots, n\). Solving this system of algebraic equations will lead to the values of \(b_i, i = 0, 1, 2, \ldots, n\).

3. Equivalence of the two methods

Case 1: when \(\phi(z) = -\sqrt{-\sigma} \tanh(\sqrt{-\sigma}z)\). In this case, we have

\[
\Phi(z) = -\frac{\sigma B - D\sqrt{-\sigma} \tan(\sqrt{-\sigma}z)}{D - B\sqrt{-\sigma} \tan(\sqrt{-\sigma}z)} = -\sqrt{-\sigma} \frac{\frac{B\sqrt{-\sigma}}{D} - \tanh(\sqrt{-\sigma}z)}{1 - \frac{B\sqrt{-\sigma}}{D} \tanh(\sqrt{-\sigma}z)}.
\]

By assuming that \(\left(-\frac{B\sqrt{-\sigma}}{D}\right) = \tanh(k_1)\), \(k_1\) is a constant, we get \(k_1 = \tanh^{-1}\left(-\frac{B\sqrt{-\sigma}}{D}\right)\). Therefore,

\[
\Phi(z) = -\sqrt{-\sigma} \frac{\tan(\sqrt{-\sigma}z) + \tanh(k_1)}{1 + \tanh(k_1) \tanh(\sqrt{-\sigma}z)} = -\sqrt{-\sigma} \tan(\sqrt{-\sigma}z + k_1).
\]

It is clear that \(\Phi(z)\) and \(\phi(z)\) are only differed by the constant phase shift \(k_1\).

Case 2: when \(\phi(z) = -\sqrt{-\sigma} \coth(\sqrt{-\sigma}z)\). In this case, we get

\[
\Phi(z) = -\frac{\sigma B - D\sqrt{-\sigma} \coth(\sqrt{-\sigma}z)}{D - B\sqrt{-\sigma} \coth(\sqrt{-\sigma}z)} = \sqrt{-\sigma} \frac{1 - \frac{D}{B\sqrt{-\sigma}} \coth(\sqrt{-\sigma}z)}{\frac{D}{B\sqrt{-\sigma}} - \coth(\sqrt{-\sigma}z)}.
\]
By setting \( \left( -\frac{D}{B\sqrt{\sigma}} \right) = \coth(k_2) \), \( k_2 \) is a constant, we obtain
\[ k_2 = \coth^{-1} \left( -\frac{D}{B\sqrt{\sigma}} \right). \]
Therefore,
\[
\Phi(z) = -\sqrt{-\sigma^{2} + \coth(k_2) \coth(\sqrt{-\sigma}z)} \cdot \coth(\sqrt{-\sigma}z + k_2).
\] (16)

It is clear that \( \Phi(z) \) and \( \phi(z) \) are only differed by the constant phase shift \( k_2 \).

**Case 3:** when \( \phi(z) = -\frac{1}{\sqrt{\sigma}}, \sigma = 0 \). In this case, we have
\[
\Phi(z) = \frac{D}{D + B \left( -\frac{1}{z + \omega} \right)} = -\frac{D}{Dz + D\omega - B} = -\frac{1}{z + \omega - \frac{B}{D}}.
\] (17)

It is clear that \( \Phi(z) \) and \( \phi(z) \) are only differed by the constant \( -\frac{B}{D} \).

**Case 4:** when \( \phi(z) = \sqrt{\sigma} \tan(\sqrt{\sigma}z) \). In this case, we get
\[
\Phi(z) = \frac{-\sigma B + D \sqrt{\sigma} \tan(\sqrt{\sigma}z)}{D + B \sqrt{\sigma} \tan(\sqrt{\sigma}z)} = \frac{\sqrt{\sigma} - \frac{B \sqrt{\sigma}}{D} + \tan(\sqrt{\sigma}z)}{1 + \frac{B \sqrt{\sigma}}{D} \tan(\sqrt{\sigma}z)}.
\] (18)

Assuming that \( -\frac{B \sqrt{\sigma}}{D} = \tan(k_3) \), \( k_3 \) is a constant, we get
\[
k_3 = \tan^{-1} \left( -\frac{B \sqrt{\sigma}}{D} \right). \]
Therefore,
\[
\Phi(z) = \sqrt{\sigma} \frac{\tan(\sqrt{\sigma}z) + \tan(k_3)}{1 - \tan(k_3) \tan(\sqrt{\sigma}z)} = \sqrt{\sigma} \tan(\sqrt{\sigma}z + k_3).
\] (19)

It is clear that \( \Phi(z) \) and \( \phi(z) \) are only differed by the constant phase shift \( k_3 \).

**Case 5:** when \( \phi(z) = -\sqrt{\sigma} \cot(\sqrt{\sigma}z) \). In this case, we get
\[
\Phi(z) = \frac{-\sigma B - D \sqrt{\sigma} \cot(\sqrt{\sigma}z)}{D - B \sqrt{\sigma} \cot(\sqrt{\sigma}z)} = -\sqrt{\sigma} \frac{1 + \frac{D}{B \sqrt{\sigma}} \cot(\sqrt{\sigma}z)}{\frac{B \sqrt{\sigma}}{D} \cot(\sqrt{\sigma}z)}.
\] (20)

By setting \( -\frac{D}{B \sqrt{\sigma}} = \cot(k_4) \), \( k_4 \) is a constant, we get
\[
k_4 = \cot^{-1} \left( -\frac{D}{B \sqrt{\sigma}} \right). \]
Therefore,
\[
\Phi(z) = -\sqrt{\sigma} \frac{\cot(k_4) \cot(\sqrt{\sigma}z) - 1}{\cot(\sqrt{\sigma}z) + \cot(k_4)} = -\sqrt{\sigma} \cot(\sqrt{\sigma}z + k_4).
\] (21)

It is clear that \( \Phi(z) \) and \( \phi(z) \) are only differed by the constant phase shift \( k_4 \).

4. The Drinfeld-Sokolov-Wilson equation

The Drinfeld-Sokolov-Wilson equation is given by [10]
\[
\begin{align*}
u_t + P\nu_x &= 0, \\
v_t + ruv_x + su_xv + qv_{xxx} &= 0,
\end{align*}
\] (22)
where \( p, q, r, \) and \( s \) are some nonzero constants. The authors in [10] have introduced the traveling wave transformation:
\[
\begin{align*}u(x,t) &= U(z), \\
v(x,t) &= V(z), \\
z &= k(x - ct),
\end{align*}
\] (23)
where \( k \) and \( c \) are constants. Substituting Equation (23) into Equation (22), we obtain the following ordinary differential equations:
\[
-kcU' + pkVV' = 0, \\
-kcV' + rkUU' + skU'V + qk^2V''' = 0.
\] (24)

After applying the BTREM in [10], the authors have obtained four solutions for the Equation (22). These four solutions are the same solutions obtained in [13] as will be shown in the following discussion.

The first solution is given by:
\[ u_1(x,t) = \frac{6c}{r+2s} \left( \frac{\sqrt{-\frac{2}{2qk^2}} B - D \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{D - \sqrt{-\frac{2}{2qk^2}} B \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2 \]

\[ = \frac{6c}{r+2s} \left( \frac{-\sqrt{-\frac{2}{2qk^2}} B + \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{1 - \sqrt{-\frac{2}{2qk^2}} B \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2. \quad (25) \]

Assume that \( -\sqrt{-\frac{2}{2qk^2}} B \) = \( \tanh(k_5) \), \( k_5 \) is a constant. Therefore,

\[ u_1(x,t) = \frac{6c}{r+2s} \left( \frac{\tanh(k_5) + \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{1 + \tanh(k_5) \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2 \]

\[ = \frac{6c}{r+2s} \left( \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct) + k_5) \right) \right)^2, \quad (26) \]

\[ v_1(x,t) = \pm \frac{\sqrt{12c^2}}{p(r+2s)} \left( \frac{\sqrt{-\frac{2}{2qk^2}} B - D \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{D - \sqrt{-\frac{2}{2qk^2}} B \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right) \]

\[ = \pm \frac{\sqrt{12c^2}}{p(r+2s)} \left( \tanh \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct) + k_5) \right) \right), \quad (27) \]

which is the same solution given in [13]. They are only differed by the phase shift constant \( k_5 \).

The second solution is given by:

\[ u_2(x,t) = \frac{6c}{r+2s} \left( \frac{\sqrt{-\frac{2}{2qk^2}} B - D \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{D - \sqrt{-\frac{2}{2qk^2}} B \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2 \]

\[ = \frac{6c}{r+2s} \left( \frac{1 - \sqrt{-\frac{2}{2qk^2}} B \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{-\sqrt{-\frac{2}{2qk^2}} B + \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2. \quad (28) \]

Let \( -\sqrt{-\frac{2}{2qk^2}} B \) = \( \coth(k_6) \), \( k_6 \) is a constant. Therefore,

\[ u_2(x,t) = \frac{6c}{r+2s} \left( \frac{1 + \coth(k_6) \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)}{\coth(k_6) + \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2 \]

\[ = \frac{6c}{r+2s} \left( \coth \left( \sqrt{-\frac{c}{2qk^2}} (k(x-ct) + k_6) \right) \right)^2, \quad (29) \]
\[ v_2(x, t) = \pm \frac{12c^2}{p(r+2s)} \left( \frac{\sqrt{\frac{c}{2qk^2}} B - D \coth \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}{D - \sqrt{\frac{c}{2qk^2}} B \coth \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)} \right) \]
\[ = \pm \frac{12c^2}{p(r+2s)} \left( \coth \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct) + k_6) \right) \right), \quad (30) \]

which is the same solution given in [13]. They are only differed by the phase shift constant \( k_6 \).

The third solution is given by:

\[ u_3(x, t) = \frac{-6c}{r+2s} \left( -\frac{\sqrt{\frac{c}{2qk^2}} B + D \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}{D + \sqrt{\frac{c}{2qk^2}} B \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2 \]
\[ = \frac{-6c}{r+2s} \left( \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct) + k_7) \right) \right)^2, \quad (31) \]

Assume that \(-\sqrt{\frac{c}{2qk^2}} B\) = \( \tan(k_7) \), \( k_7 \) is a constant. Therefore,

\[ u_3(x, t) = \frac{-6c}{r+2s} \left( \tan(k_7) + \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right) \right)^2 \]
\[ = \frac{-6c}{r+2s} \left( \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct) + k_7) \right) \right)^2, \quad (32) \]

\[ v_3(x, t) = \pm \frac{-12c^2}{p(r+2s)} \left( -\frac{\sqrt{\frac{c}{2qk^2}} B + D \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}{D + \sqrt{\frac{c}{2qk^2}} B \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)} \right) \]
\[ = \pm \frac{-12c^2}{p(r+2s)} \left( \tan \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct) + k_7) \right) \right), \quad (33) \]

which is the same solution given in [13]. They are only differed by the phase shift constant \( k_7 \).

The forth solution is given by:

\[ u_4(x, t) = \frac{-6c}{r+2s} \left( \sqrt{\frac{c}{2qk^2}} B + D \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right) \right)^2 \]
\[ - \frac{-6c}{r+2s} \left( \frac{\sqrt{\frac{c}{2qk^2}} B + D \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}{D + \sqrt{\frac{c}{2qk^2}} B \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)} \right)^2 \]
\[ = \frac{-6c}{r+2s} \left( -1 - \sqrt{\frac{c}{2qk^2}} \frac{D}{\pi} \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right) \right)^2 \]
\[ = \frac{-1 - \sqrt{\frac{c}{2qk^2}} \frac{D}{\pi} \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}{\cot(k_8) + \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}, \quad (34) \]

Let \(-\sqrt{\frac{2qk^2}{\pi}} D\) = \( \cot(k_8) \), \( k_8 \) is a constant. Therefore,

\[ u_4(x, t) = \frac{-6c}{r+2s} \left( -1 + \cot(k_8) \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right) \right)^2 \]
\[ = \frac{-1 + \cot(k_8) \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)}{\cot(k_8) + \cot \left( \sqrt{\frac{c}{2qk^2}} (k(x-ct)) \right)} \]
which is a solution in the form of the cot function only with a constant phase shift

The second solution is given by:

\[ v_2(x, t) = \pm \sqrt{\frac{-6c}{r + 2s}} \left( \cot \left( \frac{c}{2qk^2}(k(x - ct) + k_8) \right) \right)^2, \]  

which is the same solution given in [13]. They are only differed by the phase shift constant \( k_8 \).

5. The equivalence between the two methods when solving differential-difference equations

In this section, we also prove that the BTREM is equivalent to the ETM when applied to differential-difference equations. To achieve this task we choose the following example.

5.1. The discrete mKdV equation

The discrete mKdV equation is given by [14]:

\[ \frac{\partial u_n(t)}{\partial t} = (\theta - u_n^2)(u_{n+1} - u_{n-1}), \]  

where \( \theta \) is a constant. To get the traveling wave solutions for Equation (37), the following transformation was introduced [14]:

\[ u_n(t) = u(\xi_n), \quad \xi_n = dn + c_1 t + c_0, \]  

to transform Equation (37) into:

\[ c_1 u'(\xi_n) = (\theta - u_n^2(\xi_n))(u_{n+1}(\xi_n) - u_{n-1}(\xi_n)) \]  

where \( d, c_1 \) and \( c_0 \) are constants. After using the BTREM, the authors in [14] have obtained the following solutions:

The first solution is given by:

\[ u_1(\xi_n) = a_0 + a_1 \frac{-rb + a \sqrt{r} \tan(\sqrt{r}\xi_n)}{a + b \sqrt{r} \tan(\sqrt{r}\xi_n)} = a_0 + a_1 \sqrt{r} \frac{-rb + \tan(\sqrt{r}\xi_n)}{1 + \frac{bv}{a} \tan(\sqrt{r}\xi_n)}, \]  

where \( a, b, r, a_0 \) and \( a_1 \) are constants. Let \( \left( \frac{-rb}{a \sqrt{r}} \right) = \tan(m_1) \), \( m_1 \) is a constant. Therefore,

\[ u_1(\xi_n) = a_0 + a_1 \sqrt{r} \frac{\tan(m_1) + \tan(\sqrt{r}\xi_n)}{1 - \tan(m_1) \tan(\sqrt{r}\xi_n)} = a_0 + a_1 \sqrt{r} \tan(\sqrt{r}\xi_n + m_1), \]  

which is a solution in the form of the tan function only with a constant phase shift \( m_1 \).

The second solution is given by:

\[ u_2(\xi_n) = a_0 + a_1 \frac{-rb - a \sqrt{r} \cot(\sqrt{r}\xi_n)}{a - b \sqrt{r} \cot(\sqrt{r}\xi_n)} = a_0 - a_1 \sqrt{r} \frac{-a \cot(\sqrt{r}\xi_n) - 1}{\cot(\sqrt{r}\xi_n) - \frac{b}{\sqrt{r}}}. \]  

Let \( \left( \frac{-a}{b \sqrt{r}} \right) = \cot(m_2) \), \( m_2 \) is a constant. Therefore,

\[ u_2(\xi_n) = a_0 - a_1 \sqrt{r} \frac{\cot(m_2) \cot(\sqrt{r}\xi_n) - 1}{\cot(\sqrt{r}\xi_n) + \cot(m_2)} = a_0 - a_1 \sqrt{r} \cot(\sqrt{r}\xi_n + m_2), \]  

which is a solution in the form of the cot function only with a constant phase shift \( m_2 \).
The third solution is given by:

\[
    u_3(\xi_n) = a_0 + a_1 \frac{-rb - a\sqrt{-r} \coth(\sqrt{-r}\xi_n)}{a - b\sqrt{-r} \coth(\sqrt{-r}\xi_n)} = a_0 - a_1 \sqrt{-r} \frac{-\frac{a}{\sqrt{-r}} \coth(\sqrt{-r}\xi_n) + 1}{\coth(\sqrt{-r}\xi_n) - \frac{a}{b\sqrt{-r}}},
\]

(44)

Let \( \left( \frac{-\frac{a}{\sqrt{-r}}}{b\sqrt{-r}} \right) = \coth(m_3) \), \( m_3 \) is a constant. Therefore,

\[
    u_3(\xi_n) = a_0 - a_1 \sqrt{-r} \coth(m_3) \frac{(-r\xi_n) + 1}{\coth(-r\xi_n) + \coth(m_3)} = a_0 - a_1 \sqrt{-r} \coth(\sqrt{-r}\xi_n + m_3),
\]

(45)

which is a solution in the form of the coth function only with a constant phase shift \( m_3 \).

6. Conclusion

We have proved that the the BTREM is equivalent to the ETM. We demonstrated this fact using two examples from partial differential equations and differential-difference equations.

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