A Problem in Categories

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Abstract

The problem is posed to find out for arbitrary nonvoid sets $X$ which are all the mappings $T : X \rightarrow X$ that can be defined and each separately identified through means of categories alone. As argued, this problem may have a certain foundational relevance.

1. The Problem

Find out which mappings $T : X \rightarrow X$, with arbitrary nonvoid sets $X$, can be defined and identified each separately by means of categories only.

2. Examples

1) If $T = id_X$ is the identity mapping of $X$, then for every two mappings $f, g : X \rightarrow Y$, where $Y$ is an arbitrary set, one has the cancellation property

$$f \circ T = g \circ T \implies f = g$$

while for every two mappings $f, g : Y \rightarrow X$, where $Y$ is an arbitrary set, one has the dual cancellation property
(2) \( T \circ f = T \circ g \implies f = g \)

In terms of categories, the identity mapping \( T = id_X \) has of course the axiomatic property

(3) \( f \circ T = f, \ T \circ g = g, \ f : X \rightarrow Y, \ g : Y \rightarrow X \)

from which (1) and (2) result immediately. However, the question remains to what extent is the identity mapping \( T = id_X \) characterized by (3), or for that matter, (1) and (2), in terms of categories only.

2) If \( T \) is a constant mapping, that is, for a certain \( c \in X \), we have \( T(x) = c \), with \( x \in X \), then for every two mappings \( f, g : Y \rightarrow X \), where \( Y \) is an arbitrary set, one has the coequalizer property

(4) \( T \circ f = T \circ g \)

We note however that, while (4) may happen to define the set of constant mappings \( T : X \rightarrow X \) as a whole, it certainly does not in general identify them individually as well.

Therefore, the Problem above has in fact two subproblems:

I) Define by means of categories the largest class of mappings \( T : X \rightarrow X \), where \( X \) is an arbitrary set.

II) Identify individually by means of categories the largest class of mappings \( T : X \rightarrow X \), where \( X \) is an arbitrary set.

3. On the Relevance of the Problem

As far as the author is concerned, he has not seen the above Problem formulated, let alone solved anywhere in the literature. The relevance of the Problem, in case it has indeed not been considered before, may be foundational, as argued in what follows.
Category Theory, as introduced in [1], and typically presented ever since in the respective literature, starts from Set Theory which is assumed to be given, and then follows with the definition of categories through certain axioms formulated in set theoretic terms. However, from foundational point of view, this approach is not the only one which has been considered in the literature, [3, pp. 235-250]. In particular, the position of sets, versus categories, when seen in a foundational perspective, can be changed, with categories being considered as given, and sets being introduced in terms of categories.

In that latter case, however, the question arises to what extent can one recover, purely in terms of categories, the structural richness involved in each and every specific set, as inherent in it, when considered with Set Theory?

And obviously, for any given set $X$, one of the immediate and naturally associated structures is that of the set $X^X$ of all mappings $T : X \rightarrow X$.

In this way, the above Problem does indeed address the foundational issue of whether there exists the possibility of recovering the specific structural richness of Set Theory, and recovering it in terms of Category Theory alone.

References

[1] Eilenberg S, Mac Lane S : General theory of natural equivalences. Trans. AMS, 1945, Vol. 58, 231-294

[2] Herrlich H, Strecker G E : Category Theory. Allyn & Bacon, 1973

[3] Lawvere F W, Rosebrugh R : Sets for Mathematics. Cambridge Univ. Press, 2003