WELL-POSEDNESS OF RIEMANN-LIOUVILLE FRACTIONAL DEGENERATE EQUATIONS WITH FINITE DELAY IN BANACH SPACES

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Abstract. We study the Existence and uniqueness of solutions of the Riemann-Liouville fractional integro-differential degenerate equations

\[ \frac{d}{dt} \left( B^{\frac{1}{\Gamma(1-\alpha)}} \int_{-\infty}^{t} (t-s)^{-\alpha} x(s) ds \right) = Ax(t) + \int_{-\infty}^{t} a(t-s)x(s) ds + L(x(t)) + \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^{t} (t-s)^{\beta-1} x(s) ds + f(t). \]

where \( A \) and \( B \) are a linear closed operators in a Banach space.

Keywords: Riemann-Liouville fractional; integro-differential equations; \( L^p \)-multipliers; UMD-spaces.

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1. INTRODUCTION

Differential equations play an important role in describing many real-world processes. For many years the models are successfully used to study a number of physical, biological. A particular interest is in differential equations with many variables such as partial differential equations and/or integral differential equations in the case when one of the variables is times. In this work, we study the existence of periodic solutions for the following Riemann-Liouville fractional integro-differential degenerate equations.

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\[
\frac{d}{dt} \left( B \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} (t-s)^{-\alpha} x(s)ds \right) = Ax(t) + \int_{-\infty}^{t} a(t-s)x(s)ds + L(x_t) + \frac{1}{\Gamma(\beta)} \int_{-\infty}^{t} (t-s)^{-\beta-1} x(s)ds + f(t); \quad 0 \leq t \leq 2\pi
\]

where \( \Gamma(.) \) is the Euler gamma function, \( \alpha, \beta \in \mathbb{R}^+, 0 \leq \beta \leq \alpha \) and \( A : D(A) \subseteq X \rightarrow X \) and \( B \) are a linear closed operators on Banach space \( (X, \|\cdot\|) \) such that \( D(A) \subseteq D(B) \), \( f \in L^p([-r_{2\pi}, 0], X) \) for all \( p \geq 1 \) and \( r_{2\pi} := 2\pi N \) (some \( N \in \mathbb{N} \)), \( a \in L^1(\mathbb{R}_+) \), \( L \) is a linear operator and \( x_t \) is an element of \( L^p([-r_{2\pi}, 0], X) \) which is defined as follows

\[
x_t(\theta) = x(t + \theta) \text{ for } \theta \in [-r_{2\pi}, 0].
\]

The operator-valued Fourier multiplier Theorems 2.8 have been used by Keyantuo and Lizama in [19] to establish maximal regularity results for an integro-differential equation in Banach space. The authors consider the following problem

\[
x'(t) = Ax(t) + \int_{-\infty}^{t} a(t-s)Ax(s)ds + f(t); \quad x(0) = x(2\pi)
\]

Maximal regularity for the evolution problem in \( L^p \) was treated earlier by Weis [30, 31] (see also [12] for a different proof of the operator-valued Mikhlin multiplier theorem using a transference principle). The study in the \( L^p \) framework (when \( 1 < p < \infty \)) was made possible thanks to the introduction of the concept of randomized boundedness (hereafter \( R \)-boundedness, also known as Riesz-boundedness or Rademacher-boundedness). With this, necessary conditions for operator-valued Fourier multipliers were found in this context. In addition, the space \( X \) must have the \( UMD \) property. This was done initially by L. Weis [30, 31] for the evolutionary problem and then by Arendt-Bu [2] for periodic boundary conditions. For non-degenerate integro-differential equations both in the periodic and non periodic cases, operator-valued Fourier multipliers have been used by various authors to obtain well-posedness in various scales of function spaces: [7, 9, 10, 19, 25, 20, 21, 27] and the corresponding references. The well-posedness or maximal regularity results are important in that they allow for the treatment of nonlinear problems. Earlier results on the application of operator-valued Fourier multiplier theorems to evolutionary integral equations can be found in [12]. More recent examples of second order
integro-differential equations with frictional damping and memory terms have been studied in the paper [11]

In [8] Bu et al studied the well-posedness of the third-order integro-differential equations

\[ \alpha u'''(t) + u''(t) = \beta A u(t) + \beta \int_{-\infty}^{t} a(t-s) A x(s) ds + \gamma B u'(t) + f(t), \]

with periodic boundary conditions \( u(0) = u(2\pi), u'(0) = u'(2\pi), u''(0) = u''(2\pi). \)

In [22], S.Koumla, Kh.Ezzinbi, R.Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay

\[ \frac{d}{dt} x(t) = A x(t) + \int_0^t B(t-s) x(s) ds + f(t, x_t) + h(t, x_t) \]

where \( A : D(A) X \to X \) is the infinitesimal generator of a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on a Banach space \( X \), for \( t \geq 0, B(t) \) is a closed linear operator with domain \( D(B) \supset D(A) \).

This work is organized as follows: In Section 2 we collect some preliminary results and definitions. In section 3, we study the existence and uniqueness of strong \( L^p \)-solution of the Eq. (1.1) solely in terms of a property of R-boundedness for the sequence of operators \( (ik)^\alpha ((ik)^\alpha I - A - L_k - \tilde{a}(ik) - (ik)^{-\beta} I)^{-1}. \) We optain that the following assertion are equivalent in UMD space:

**1:** \( (ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta} I \) is invertible and \( \{ (ik)^\alpha ((ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta} I)^{-1}, k \in \mathbb{Z} \} \) is R-bounded.

**2:** For every \( f \in L^p(\mathbb{T}; X) \) there exist a unique function \( u \in H^{\alpha, p}(\mathbb{T}; X) \) such that \( u \in D(A) \) and equation (1.1) holds for a.e \( t \in [0, 2\pi] \).

## 2. Preliminaries

In this section, we collect some results and definitions that will be used in the sequel. Let \( X \) be a complex Banach space. We denote as usual by \( L^1(0, 2\pi, X) \) the space of Bochner integrable functions with values in \( X \). For a function \( f \in L^1(0, 2\pi; X) \), we denote by \( \hat{f}(k), k \in \mathbb{Z} \) the \( k \)-th Fourier coefficient of \( f \):

\[ \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-kt} f(t) dt, \]
where \( e_k(t) = e^{ikt}, t \in \mathbb{R} \).

**Lemma 2.1.** [24]

Let \( L : L^p(\mathbb{T}, X) \to X \) be a bounded linear operator. Then

\[
\hat{L}(u)(k) = L(e_k \hat{u}(k)) := L_k \hat{u}(k) \quad \text{for all } k \in \mathbb{Z}.
\]

Let \( a \in L^1(\mathbb{R}_+) \). We consider the function

\[
F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds, \quad t \in \mathbb{R}.
\]

Since

\[
F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds = \int_{0}^{\infty} a(s)u(t-s)ds,
\]

we have \( \|F\|_{L^1} \leq \|a\|_1 \|u\|_{L^1} = \|a\|_{L^1(\mathbb{R}_+)} \|u\|_{L^1(0, 2\pi; X)} \) and \( F \) is periodic of period \( T = 2\pi \) as \( u \).

Now using Fubini’s theorem and (2.1) we obtain, for \( k \in \mathbb{Z} \), that

\[
\hat{F}(k) = \check{a}(ik)\hat{u}(k), k \in \mathbb{Z}
\]

where \( \check{a}(\lambda) = \int_{0}^{\infty} e^{-\lambda t}a(t)dt \) denotes the Laplace transform of \( a \). This identity plays a crucial role in the paper.

Let \( X, Y \) be Banach spaces. We denote by \( \mathcal{L}(X, Y) \) the set of all bounded linear operators from \( X \) to \( Y \). When \( X = Y \), we write simply \( \mathcal{L}(X) \).

**Proposition 2.2** ([2, Fejer’s Theorem]). Let \( f \in L^p(0, 2\pi; X) \), then one has

\[
f = \lim_{n \to \infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)
\]

with convergence in \( L^p(0, 2\pi; Y) \).

**R-boundedness-UMD space, \( L^p \)-multiplier and Riemann-Liouville fractional integral.** For \( j \in \mathbb{N} \), denote by \( r_j \) the \( j \)-th Rademacher function on \([0, 1]\), i.e. \( r_j(t) = \text{sgn}(\sin(2^j \pi t)) \). For \( x \in X \) we denote by \( r_j \otimes x \) the vector valued function \( t \to r_j(t)x \).

The important concept of \( R \)-bounded for a given family of bounded linear operators is defined as follows.
Definition 2.3. A family $T \subset \mathcal{L}(X,Y)$ is called $R$-bounded if there exists $c_q \geq 0$ such that

$$\| \sum_{j=1}^{n} r_j \otimes T_j x_j \|_{L^q(0,1;X)} \leq c_q \| \sum_{j=1}^{n} r_j \otimes x_j \|_{L^q(0,1;X)}$$

for all $T_1, \ldots, T_n \in T, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. We denote by $R_q(T)$ the smallest constant $c_q$ such that (2.3) holds.

Definition 2.4. Let $\varepsilon \in ]0,1[$ and $1 < p < \infty$. Define the operator $H_{\varepsilon}$ by: for all $f \in L^p(\mathbb{R};X)$

$$(H_{\varepsilon}f)(t) := \frac{1}{\pi} \int_{\varepsilon < |s| < \frac{1}{\varepsilon}} \frac{f(t-s)}{s} ds$$

if $\lim_{\varepsilon \to 0} H_{\varepsilon}f := Hf$ exists in $L^p(\mathbb{R};X)$. Then $Hf$ is called the Hilbert transform of $f$ on $L^p(\mathbb{R},X)$.

Definition 2.5. A Banach space $X$ is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R};X)$ for all $1 < p < \infty$.

Definition 2.6. For $1 \leq p < \infty$, a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X,Y)$ is said to be an $L^p$-multiplier if for each $f \in L^p(\mathbb{T},X)$, there exists $u \in L^p(\mathbb{T},Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proposition 2.7. Let $X$ be a Banach space and $\{M_k\}_{k \in \mathbb{Z}}$ be an $L^p$-multiplier, where $1 \leq p < \infty$. Then the set $\{M_k\}_{k \in \mathbb{Z}}$ is $R$-bounded.

Theorem 2.8. (Marcinkiewicz operator-valued multiplier Theorem). Let $X, Y$ be UMD spaces and $\{M_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k \in \mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ are $R$-bounded, then $\{M_k\}_{k \in \mathbb{Z}}$ is an $L^p$-multiplier for $1 < p < \infty$.

Definition 2.9. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined by

$$\mathcal{I}_{-\infty}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s) ds$$

where $\Gamma(\alpha) = \int_{0}^{\infty} e^{-t} t^{\alpha-1} dt$, is the Euler gamma function.

Definition 2.10. The Riemann-Liouville fractional integral derivative operator of order $\alpha > 0$ is defined by

$$\mathcal{D}_{-\infty}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_{-\infty}^{t} (t-s)^{-\alpha} f(s) ds \right)$$
Those familiar with the Fourier transform know that the Fourier transform of a derivative can be expressed by the following:

\[ \hat{\frac{dx}{dt}}(k) = ik\hat{x}(k), \forall k \in \mathbb{Z} \]

and more generally,

\[ \hat{\frac{d^n x}{dt^n}}(k) = (ik)^n\hat{x}(k), \forall k \in \mathbb{Z} \]

A similar identity holds for anti-derivatives

\[ \mathcal{F}_{-\infty}^s f(k) = (ik)^{-s}\hat{x}(k), \forall k \in \mathbb{Z} \]

\[ \mathcal{F}_{-\infty}^s f(k) = (ik)^s\hat{x}(k), \forall k \in \mathbb{Z} \]

**Remark 2.11.** If we set \( u(x) = e^{ikx} \) for \( k \in \mathbb{Z} \) we have

1) \( \mathcal{D}_{-\infty}^\alpha u(t) = (ik)^\alpha e^{ikx} \)

2) \( \mathcal{D}_{-\infty}^{-\alpha} u(t) = (ik)^{-\alpha} e^{ikx} \).

### 3. Periodic Solutions in UMD space

For \( a \in L^1(\mathbb{R}_+) \), we denote by \( a \ast x \) the function

\[ (a \ast x)(t) := \int_{-\infty}^t a(t-s)x(s)ds \]

with this notation we may rewrite Eq. (1.1) in the following was:

(3.1) \[ \mathcal{D}_{-\infty}^\alpha Bx(t) = Ax(t) + L(x_t) + (a \ast x)(t) + \mathcal{F}_{-\infty}^\beta x(t) + f(t) \text{ for } t \in \mathbb{R}. \]

we have \( \hat{a \ast x}(k) = a(ik)\hat{x}(k) \). We define

\[ \Delta_k = ((ik)^\alpha B - A - L_k - a(ik)I - (ik)^{-\beta}I) \]

and

\[ \sigma_\mathbb{Z}(\Delta) = \{k \in \mathbb{Z} : \Delta_k \text{ is not bijective} \} \]

the periodic vector-valued space is defined by
Definition 3.1. For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an $(L^p, H^{\alpha, p})$-multiplier, if for each $f \in L^p(\mathbb{T}, X)$ there exists $u \in H^{1, p}(\mathbb{T}, Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$ 

Lemma 3.2. Let $1 \leq p < \infty$ and $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{B}(X)$ (B(X) is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:

(i) $(M_k)_{k \in \mathbb{Z}}$ is an $(L^p, H^{\alpha, p})$-multiplier.

(ii) $((ik)^\alpha M_k)_{k \in \mathbb{Z}}$ is an $(L^p, L^p)$-multiplier.

We begin by establishing our concept of strong solution for Eq. (3.1)

Definition 3.3. Let $f \in L^p(\mathbb{T}; X)$. A function $x \in H^{\alpha, p}(\mathbb{T}; X)$ is said to be a $2\pi$-periodic strong $L^p$-solution of Eq. (3.1) if $x(t) \in D(A)$ for all $t \geq 0$ and Eq. (3.1) holds almost everywhere.

Proposition 3.4. Let $A$ be a closed linear operator defined on an UMD space $X$. Suppose that $\sigma_{\alpha}(\Delta) = \emptyset$. Then the following assertions are equivalent:

(i) $\left((ik)^\alpha (i(k)^\alpha B - A - L_k - \bar{a}(ik)I - (ik)^{-\beta} I)^{-1}\right)_{k \in \mathbb{Z}}$ is an $L^p$-multiplier for $1 \leq p < \infty$.

(ii) $\left((ik)^\alpha ((ik)^\alpha B - A - L_k - \bar{a}(ik)I - (ik)^{-\beta} I)^{-1}\right)_{k \in \mathbb{Z}}$ is $R$-bounded.

Proof. (i) $\Rightarrow$ (ii) As a consequence of Proposition (2.7)

(ii) $\Rightarrow$ (i) Let $a_{s,k} = (ik)^{-s}, s \in \mathbb{R}, k \neq 0$

Define $M_k = (ik)^\alpha (C_k - A)^{-1}$, where $C_k := (ik)^\alpha B - L_k - \bar{a}(ik)I - (ik)^{-\beta} I$. By Theorem (2.8) it is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ is $R$-bounded. Since

$$k [M_{k+1} - M_k]$$

$$= k [(i(k+1))^\alpha (C_{k+1} - A)^{-1} - (ik)^\alpha (C_k - A)^{-1}]$$

$$= k (C_{k+1} - A)^{-1} [(i(k+1))^\alpha (C_k - A) - (ik)^\alpha (C_{k+1} - A)] (C_k - A)^{-1}$$

$$= kM_{k+1} \left[a_{\alpha,k}(C_k - A) - a_{\alpha,k+1}(C_{k+1} - A)\right] M_k$$

$$= kM_{k+1} \left[a_{\alpha,k} C_k - a_{\alpha,k+1} C_{k+1} + (a_{\alpha,k+1} - a_{\alpha,k}) A\right] M_k$$
Theorem 3.6. Let $X$ be a Banach space. Suppose that for every $f$ \eqref{3.1} unique strong solution of Eq.

\[ L \]

Lemma 3.7. Suppose that \( \beta \) is a solution of Eq. \( (3.1) \) corresponding to \( f \) and that \( L \)-periodic strong \( L \)-solution \( x \) of Eq. \( (3.1) \). Then \( x \) is the unique \( 2\pi \)-periodic strong \( L \)-solution.

Proof. Suppose that \( x_1 \) and \( x_2 \) two strong \( L \)-solution of Eq. \( (3.1) \) then \( x = x_1 - x_2 \) is a strong \( L \)-solution of Eq. \( (3.1) \) corresponding to \( f = 0 \). Taking Fourier transform in \( (3.1) \), we obtain that

\[ (ik)^{\alpha} B \hat{x}(k) = A \hat{x}(k) + L_k \hat{x}(k) + \alpha(ik) \hat{x}(k) + (ik)^{-\beta} \hat{x}(k), k \in \mathbb{Z}. \]

Then

\[ (ik)^{\alpha} B - A - L_k - \alpha(ik) I - (ik)^{-\beta} I) \hat{x}(k) = 0 \]

It follows that \( \hat{x}(k) = 0 \) for every \( k \in \mathbb{Z} \) and therefore \( x = 0 \). Then \( x_1 = x_2 \).

Theorem 3.6. Let $X$ be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists a \( 2\pi \)-periodic strong \( L^p \)-solution of Eq. \( (3.1) \) for \( 1 \leq p < \infty \). Then

1. for every \( k \in \mathbb{Z} \) the operator \( \Delta_k = ((ik)^{\alpha} B - A - L_k - \alpha(ik) I - (ik)^{-\beta} I) \) has bounded inverse
2. \( \{(ik)^{\alpha} \Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is \( R \)-bounded.

Before to give the proof of Theorem 3.6, we need the following Lemma.

Lemma 3.7. If \((ik)^{\alpha} B - A - L_k - \alpha(ik) I - (ik)^{-\beta} I)(x) = 0 \) for all \( k \in \mathbb{Z} \), then \( u(t) = e^{ikt} x \) is a \( 2\pi \)-periodic strong \( L^p \)-solution of the following equation

\[ \mathcal{D}^{\alpha}_{-\infty}(Bu)(t) = Au(t) + (a * u)(t) + \mathcal{D}^\beta_{-\infty}(u)(t). \]
Proof. We have \((ik)^\alpha B - A - L_k - \tilde{a}(ik)I - (ik)^{-\beta}I)x = 0\).

Then
\[
(ik)^\alpha Bx = Ax + L_k x + \tilde{a}(ik)x + (ik)^{-\beta}x
\]
We have \(u(t) = e^{ikt}x\). In fact, since \(u_t(\theta) = e^{ik\theta}u(t)\) we obtain \(u_t = e_k u(t)\). By Remark \(2.11(2),\)
\[
\mathcal{D}_{-\infty}(Bu)(t) = (ik)^\alpha Be^{ikt}x = e^{ikt}((ik)^\alpha Bx)
\]
\[
= e^{ikt}[Ax + L_k x + \tilde{a}(ik)x + (ik)^{-\beta}x]
\]
\[
= Ae^{ikt}x + L_k(e^{ikt}x) + \tilde{a}(ik)e^{ikt}x + (ik)^{-\beta}e^{ikt}x
\]
\[
= Au(t) + L(e_k u(t)) + \tilde{a}(ik)u(t) + (ik)^{-\beta}u(t)
\]
\[
= Au(t) + L(u_t) + (a*u)(t) + \mathcal{D}_{-\infty}u(t)
\]

Proof of Theorem 3.6: 1) Let \(k \in \mathbb{Z}\) and \(y \in X\). Then for \(f(t) = e^{ikt}y\), there exists \(x \in H^{\alpha,p}(\mathbb{T};X)\) such that:
\[
\mathcal{D}_{-\infty}(Bx)(t) = Ax(t) + L(x_t) + (a*x)(t) + \mathcal{D}_{-\infty}^\beta(x)(t) + f(t)
\]
Taking Fourier transform. We have \(\mathcal{D}_{-\infty}Bx(k) = (ik)^\alpha B\hat{x}(k)\) and \(\mathcal{D}_{-\infty}^\beta x(k) = (ik)^{-\beta}\hat{x}(k)\)

Consequently, we have
\[
(ik)^\alpha B\hat{x}(k) = A\hat{x}(k) + L_k\hat{x}(k) + \tilde{a}(ik)\hat{x}(k) + (ik)^{-\beta}\hat{x}(k) + \hat{f}(k)
\]
\[
[(ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta}]\hat{x}(k) = \hat{f}(k) = y \Rightarrow ((ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})\] is surjective.

if \([(ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})(u) = 0\), then by Lemma \(3.7\), \(x(t) = e^{ikt}u\) is a \(2\pi\)-periodic strong \(L^p\)-solution of Eq.\(3.1\) corresponing to the function \(f(t) = 0\) Hence \(x(t) = 0\) and \(u = 0\) then \((ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})\) is injective.

2) Let \(f \in L^p(\mathbb{T};X)\). By hypothesis, there exists a unique \(x \in H^{\alpha,p}(\mathbb{T},X)\) such that the Eq. \(3.1\) is valid. Taking Fourier transforms, we deduce that
\[
\hat{x}(k) = ((ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.
\]
Hence
\[
(ik)^\alpha \hat{x}(k) = (ik)^\alpha ((ik)^\alpha B - A - L_k - \tilde{a}(ik) - (ik)^{-\beta})^{-1}\hat{f}(k) \quad \text{for all } k \in \mathbb{Z}
\]
Since \( x \in H^{\alpha,p}(\mathbb{T}; X) \), then there exists \( v \in L^p(\mathbb{T}; X) \) such that
\[
\hat{v}(k) = (ik)^\alpha \hat{x}(k) = (ik)^\alpha ((ik)^\alpha B - A - L_k - \bar{a}(ik) - (ik)^{-\beta})^{-1} \hat{f}(k).
\]
Then \( \{(ik)^\alpha \Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier and \( \{(ik)^\alpha \Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is \( R \)-bounded. \( \square \)

4. Main Result

Our main result in this work is to establish that the converse of Theorem 3.6, are true, provided \( X \) is an UMD space.

**Theorem 4.1.** Let \( X \) be an UMD space and \( A : D(A) \subset X \to X \) be an closed linear operator. Then the following assertions are equivalent for \( 1 < p < \infty \).

1. for every \( f \in L^p(\mathbb{T}; X) \) there exists a unique \( 2\pi \)-periodic strong \( L^p \)-solution of Eq. (3.1).
2. \( \sigma_{\mathbb{Z}}(\Delta) = \phi \) and \( \{(ik)^\alpha \Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is \( R \)-bounded.

**Lemma 4.2.** [2]. Let \( f, g \in L^p(\mathbb{T}; X) \). If \( \hat{f}(k) \in D(A) \) and \( A \hat{f}(k) = \hat{g}(k) \) for all \( k \in \mathbb{Z} \) Then
\[
f(t) \in D(A) \quad \text{and} \quad Af(t) = g(t) \quad \text{for all} \quad t \in [0, 2\pi].
\]

**Proof.** 1) \( \Rightarrow \) 2) see Theorem 3.6

1) \( \Leftarrow \) 2) Let \( f \in L^p(\mathbb{T}; X) \). Define
\[
\Delta_k = ((ik)^\alpha B - A - L_k - \bar{a}(ik)I - (ik)^{-\beta}I)
\]
By Lemma 3.2, the family \( \{(ik)^ \alpha \Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier it is equivalent to the family \( \{\Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier that maps \( L^p(\mathbb{T}; X) \) into \( H^{\alpha,p}(\mathbb{T}; X) \), namely there exists \( x \in H^{1,p}(\mathbb{T}, X) \) such that

\[
\hat{x}(k) = \Delta_k^{-1} \hat{f}(k) = ((ik)^\alpha B - A - L_k - \bar{a}(ik)I - (ik)^{-\beta}I)^{-1} \hat{f}(k)
\]
In particular, \( x \in L^p(\mathbb{T}; X) \) and there exists \( v \in L^p(\mathbb{T}; X) \) such that \( \hat{v}(k) = (ik)^\alpha \hat{x}(k) \)

\[
\mathcal{D}_{\alpha,\infty}Bx(k) := \hat{v}(k) = (ik)^\alpha B\hat{x}(k)
\]
Using now (4.1) and (4.2) we have:
\[
\mathcal{D}_{\alpha,\infty}Bx(k) = (ik)^\alpha B\hat{x}(k) = A\hat{x}(k) + L(x)(k) + \hat{a} \ast \hat{x}(k) + \mathcal{J}_{\alpha,\infty}x(k) + \hat{f}(k)
\]
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for all $k \in \mathbb{Z}$. Since $A$ is closed, then $x(t) \in D(A)$ [Lemma 4.2]

and from the uniqueness theorem of Fourier coefficients, that Eq. (3.1) is valid.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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