Abstract—We consider the problem of perfectly recovering the vertex correspondence between two correlated Erdős-Rényi (ER) graphs on the same vertex set. The correspondence between the vertices can be obscured by randomly permuting the vertex labels of one of the graphs. We determine the information-theoretic threshold for exact recovery, i.e. the conditions under which the entire vertex correspondence can be correctly recovered given unbounded computational resources.

Graph alignment is the problem finding a matching between the vertices of the two graphs that matches, or aligns, many edges of the first graph with edges of the second graph. Alignment is a generalization of graph isomorphism recovery to non-isomorphic graphs. Graph alignment can be applied in the deanonymization of social networks, the analysis of protein interaction networks, and computer vision. Narayanan and Shmatikov successfully deanonymized an anonymized social network dataset by graph alignment with a publicly available network [1]. In order to make privacy guarantees in this setting, it is necessary to understand the conditions under which graph alignment recovery is possible.

We consider graph alignment for a randomized graph-pair model. This generation procedure creates a “planted” alignment: there is a ground-truth relationship between the vertices of two correlated ER graphs. Graph alignment can be applied to non-isomorphic graphs. Graph alignment is the problem of finding a matching between the vertices of two correlated ER graphs. There is a ground-truth relationship between the vertices of the two graphs. When edges of the first graph are matched with meaningless labels, we model this by applying the indicator variable $e$ to the vertices of the anonymized version. We consider the following problem. There are two correlated graphs $G_a$ and $G_b$, both on the vertex set $[n] = \{0, 1, \ldots, n-1\}$. By correlation we mean that for each vertex pair $e$, presence or absence of $e \in E(G_a)$, or equivalently the indicator variable $G_a(e)$, provides some information about $G_b(e)$. The true vertex labels of $G_a$ are removed and replaced with meaningless labels. We model this by applying a uniformly random permutation $\Pi$ to map the vertices of $G_a$ to the vertices of its anonymized version. The anonymized graph is $G_c$, where $G_c(\{\Pi(i), \Pi(j)\}) = G_a(\{i, j\})$ for all $i, j \in [n], i \neq j$. The original vertex labels of $G_b$ are preserved and $G_c$ and $G_b$ are revealed. We would like to know under what conditions it is possible to discover the true correspondence between the vertices of $G_a$ and the vertices of $G_b$. In other words, when can the random permutation $\Pi$ be exactly recovered with high probability?

In this context, an achievability result demonstrates the existence of an algorithm or estimator that exactly recovers $\Pi$ with high probability. A converse result is an upper bound on the probability of exact recovery that applies to any estimator.

B. Correlated Erdős-Rényi graphs

To fully specify this problem, we need to define a joint distribution over $G_a$ and $G_b$. In this paper, we will focus on Erdős-Rényi (ER) graphs. We discuss some of the advantages and drawbacks of this model in Section II-E.

We will generate correlated Erdős-Rényi graphs as follows. Let $G_a$ and $G_b$ be graphs on the vertex set $[n]$. We will think of $(G_a, G_b)$ as a single function with codomain $\{0, 1\}^2$: $(G_a, G_b)(e) = (G_a(e), G_b(e))$. The random variables $(G_a, G_b)(e), e \in \binom{[n]}{2}$, are i.i.d. and

$$(G_a, G_b)(e) = \begin{cases} 
(1, 1) & \text{w.p. } p_{11} \\
(1, 0) & \text{w.p. } p_{10} \\
(0, 1) & \text{w.p. } p_{01} \\
(0, 0) & \text{w.p. } p_{00}.
\end{cases}$$

Call this distribution $ER(n, p)$, where $p = (p_{11}, p_{10}, p_{01}, p_{00})$. Note that the marginal distributions of $G_a$ and $G_b$ are Erdős-Rényi and so is the distribution of the intersection graph $G_a \land G_b$: $G_a \sim ER(n, p_{10} + p_{11}), G_b \sim ER(n, p_{11} + p_{01})$, and $G_a \land G_b \sim ER(n, p_{11})$.

When $p_{11} > (p_{10} + p_{11})(p_{01} + p_{11})$, we say that the graphs $G_a$ and $G_b$ have positive correlation. Observe that

$$p_{11} - (p_{10} + p_{11})(p_{01} + p_{11}) = p_{11}p_{00} - p_{01}p_{10}$$

so $p_{11}p_{00} > p_{10}p_{01}$ is an equivalent, more symmetric condition for positive correlation.

C. Results

All of the results concern the following setting. We have $(G_a, G_b) \sim ER(n, p)$, $\Pi$ is a uniformly random permutation of $[n]$ independent of $(G_a, G_b)$, and $G_c$ is the anonymization of $G_a$ by $\Pi$ as described in Section II-E. Our main result is the following.

I. MODEL

A. The alignment recovery problem

We consider the following problem. There are two correlated graphs $G_a$ and $G_b$, both on the vertex set $[n] = \{0, 1, \ldots, n-1\}$. By correlation we mean that for each vertex pair $e$, presence or absence of $e \in E(G_a)$, or equivalently the indicator variable $G_a(e)$, provides some information about $G_b(e)$. The true vertex labels of $G_a$ are removed and replaced with meaningless labels. We model this by applying a uniformly random permutation $\Pi$ to map the vertices of $G_a$ to the vertices of its anonymized version. The anonymized graph is $G_c$, where $G_c(\{\Pi(i), \Pi(j)\}) = G_a(\{i, j\})$ for all $i, j \in [n], i \neq j$. The original vertex labels of $G_b$ are preserved and $G_c$ and $G_b$ are revealed. We would like to know under what conditions it is possible to discover the true correspondence between the vertices of $G_a$ and the vertices of $G_b$. In other words, when can the random permutation $\Pi$ be exactly recovered with high probability?

In this context, an achievability result demonstrates the existence of an algorithm or estimator that exactly recovers $\Pi$ with high probability. A converse result is an upper bound on the probability of exact recovery that applies to any estimator.

B. Correlated Erdős-Rényi graphs

To fully specify this problem, we need to define a joint distribution over $G_a$ and $G_b$. In this paper, we will focus on Erdős-Rényi (ER) graphs. We discuss some of the advantages and drawbacks of this model in Section II-E.

We will generate correlated Erdős-Rényi graphs as follows. Let $G_a$ and $G_b$ be graphs on the vertex set $[n]$. We will think of $(G_a, G_b)$ as a single function with codomain $\{0, 1\}^2$: $(G_a, G_b)(e) = (G_a(e), G_b(e))$. The random variables $(G_a, G_b)(e), e \in \binom{[n]}{2}$, are i.i.d. and

$$(G_a, G_b)(e) = \begin{cases} 
(1, 1) & \text{w.p. } p_{11} \\
(1, 0) & \text{w.p. } p_{10} \\
(0, 1) & \text{w.p. } p_{01} \\
(0, 0) & \text{w.p. } p_{00}.
\end{cases}$$

Call this distribution $ER(n, p)$, where $p = (p_{11}, p_{10}, p_{01}, p_{00})$. Note that the marginal distributions of $G_a$ and $G_b$ are Erdős-Rényi and so is the distribution of the intersection graph $G_a \land G_b$: $G_a \sim ER(n, p_{10} + p_{11}), G_b \sim ER(n, p_{11} + p_{01})$, and $G_a \land G_b \sim ER(n, p_{11})$.

When $p_{11} > (p_{10} + p_{11})(p_{01} + p_{11})$, we say that the graphs $G_a$ and $G_b$ have positive correlation. Observe that

$$p_{11} - (p_{10} + p_{11})(p_{01} + p_{11}) = p_{11}p_{00} - p_{01}p_{10}$$

so $p_{11}p_{00} > p_{10}p_{01}$ is an equivalent, more symmetric condition for positive correlation.

C. Results

All of the results concern the following setting. We have $(G_a, G_b) \sim ER(n, p)$, $\Pi$ is a uniformly random permutation of $[n]$ independent of $(G_a, G_b)$, and $G_c$ is the anonymization of $G_a$ by $\Pi$ as described in Section II-E. Our main result is the following.
Theorem 1. Let $p$ satisfy the conditions
\begin{align}
    p_{01} + p_{10} &\leq O\left(\frac{1}{\log n}\right) \quad (1) \\
    p_{11} &\leq O\left(\frac{1}{\log n}\right) \quad (2) \\
    \frac{p_{01}p_{10}}{p_{11}p_{00}} &\leq O\left(\frac{1}{(\log n)^2}\right) \quad (3) \\
    p_{11} &\geq \frac{\log n + \omega(1)}{n}. \quad (4)
\end{align}

Then there is an estimator for $\Pi$ given $(G_c, G_b)$ that is correct with probability $1 - o(1)$.

Together, conditions (1) and (2) force $G_a$ and $G_b$ to be mildly sparse. Condition (3) requires $G_a$ and $G_b$ to have nonnegligible positive correlation.

There is a matching converse bound.

Theorem 2 (Converse bound). If $p$ satisfies

$$ p_{11} \leq \frac{\log n - \omega(1)}{n} $$

then any estimator for $\Pi$ given $(G_c, G_b)$ is correct with probability $o(1)$.

Theorem 2 was originally proved by the authors in [3]. The proof is short compared to Theorem 1 and it is included in Section 1.4.

A second achievability theorem applies without conditions (1), (2), and (3). This requires condition (4) to be strengthened.

Theorem 3. If $p$ satisfies

$$ \frac{2\log n + \omega(1)}{n} \leq \left(\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}}\right)^2 $$

then there is an estimator for $\Pi$ given $(G_c, G_b)$ that is correct with probability $1 - o(1)$.

Theorem 3 was also originally proved in [3]. In this paper, it appears as an intermediate step in the proof of Theorem 1.

II. PRELIMINARIES

A. Notation

Throughout, we use capital letters for random objects and lower case letters for fixed objects.

For a graph $g$, let $V(g)$ and $E(g)$ be the node and edge sets respectively. Let $[n]$ denote the set $\{0, \cdots, n-1\}$. All of the $n$-vertex graphs that we consider will have vertex set $[n]$. We will always think of a permutation as a bijective function $[n] \to [n]$. The set of permutations of $[n]$ under the binary operation of function composition forms the group $S_n$.

We denote the collection of all two element subsets of $[n]$ by $\binom{[n]}{2}$. The edge set of a graph $g$ is $E(g) \subseteq \binom{[n]}{2}$.

Represent a labeled graph on the vertex set $[n]$ by its edge indicator function $g : \binom{[n]}{2} \to [2]$. The group $S_n$ has an action on $\binom{[n]}{2}$. We can write the action of the permutation $\pi$ on the graph $g$ as the composition of functions $g \circ l(\pi)$, where $l(\pi)$ is the lifted version of $\pi$:

\[
l(\pi) : \binom{[n]}{2} \to \binom{[n]}{2} \quad \{i, j\} \mapsto \{\pi(i), \pi(j)\}.
\]

Thus $G_c = G_a \circ l(\Pi^{-1})$. Whenever there is only a single permutation under consideration, we will follow the convention $\tau = l(\pi)$.

For a generating function in the formal variable $z$, $[z^j]$ is the coefficient extraction operator:

$$ [z^j]a(z) = [z^j] \sum_i a_i z^i = a_j. $$

When $z$ is a matrix of numbers or formal variables and $k$ is a matrix of numbers, both indexed by $S \times T$, we use the notation

$$ z^k = \prod_{i \in S} \prod_{j \in T} z_{i,j}^{k_{i,j}} $$

for compactness.

B. Graph statistics

Recall that we consider a graph on $[n]$ to be a $[2]$-labeling of the set of vertex pairs $\binom{[n]}{2}$. The following quantities have clear interpretations for graphs, but we define them more generally for reasons that will become apparent shortly.

Definition 1. For a set $S$ and a pair of binary labelings $f_a, f_b : S \to [2]$, define the type

$$ \mu(f_a, f_b) \in [N^{[2]\times[2]}]. $$

The Hamming distance between $f_a$ and $f_b$ is

$$ \Delta(f_a, f_b) = \sum_{e \in S} \mathbf{1}\{f_a(e) \neq f_b(e)\} = \mu(f_a, f_b)_{01} + \mu(f_a, f_b)_{10}. $$

For a permutation $\tau : S \to S$, define

$$ \delta(\tau; f_a, f_b) = \frac{1}{2} \left(\Delta(f_a \circ \tau, f_b) - \Delta(f_a, f_b)\right). $$

In the particular case of graphs (where $S = \binom{[n]}{2}$ and $\tau = l(\pi)$), $\Delta(G, H)$ is the size of the symmetric difference of the edge sets, $|E(G \cup H)| - |E(G \cap H)|$. The quantity $\delta$ is central to both our converse and our achievability arguments (as well as the achievability proof of Pedarsani and Grossglauser [2]). When $f_a$ and $f_b$ are graphs on $[n]$ and $\pi$ is a permutation of $[n]$, $\delta(l(\pi); f_a, f_b)$ is the difference in matching quality between the permutation $\pi$ and the identity permutation.

Lemma 11. Let $f_a, f_b : S \to [2]$. Then there is some $i \in \mathbb{Z}$ such that

$$ \mu(f_a \circ \tau, f_b) - \mu(f_a, f_b) = \begin{pmatrix} -i & i \\ i & -i \end{pmatrix} $$

and $i = \delta(\tau; f_a, f_b)$.

Proof. Let $k = \mu(f_a, f_b)$, and $k' = \mu(f_a \circ \tau, f_b)$. Let $1$ be the vector of all ones. We have $k1 = k'1$ because both vectors give the distribution of symbols in $f_a$. Similarly $1^Tk = 1^Tk'$. 

For a generating function in the formal variable $z$, $[z^j]$ is the coefficient extraction operator:

$$ [z^j]a(z) = [z^j] \sum_i a_i z^i = a_j. $$

When $z$ is a matrix of numbers or formal variables and $k$ is a matrix of numbers, both indexed by $S \times T$, we use the notation

$$ z^k = \prod_{i \in S} \prod_{j \in T} z_{i,j}^{k_{i,j}} $$

for compactness.
The matrix \( k' - k \) has integer entries, so it must have the claimed form for some value of \( i \). Finally,
\[
\begin{align*}
i &= \frac{1}{2}((k_{01} + k_{10}') - (k_{01} + k_{10})) \\
&= \frac{1}{2}(\Delta(G_a \circ \tau, G_b) - \Delta(G_a, G_b)) \\
&= \delta(\tau; f_a, f_b)
\end{align*}
\]

\( C. \) MAP estimation

The maximum a posteriori (MAP) estimator for this problem can be derived as follows. In the following lemma we will be careful to distinguish graph-valued random variables from fixed graphs: we name the former with upper-case letters and the latter with lower-case.

**Lemma II.2.** Let \( (G_a, G_b) \sim ER(n, p) \), let \( \Pi \) be a uniformly random permutation of \([n]\), and let \( G_c = G_a \circ l(\Pi^{-1}) \). Then
\[
P[\Pi = \pi|(G_c, G_b) = (g_c, g_b)] \propto \left( \frac{P_{10}P_{01}}{P_{11}P_{00}} \right)^i
\]
where \( i = \frac{1}{2}\Delta(g_c \circ l(\pi), g_b) \).

**Proof.**
\[
P[\Pi = \pi|(G_c, G_b) = (g_c, g_b)] \\
\begin{align*}
&\propto P[\Pi = \pi, (G_c, G_b) = (g_c, g_b)] \\
&\propto P[\Pi = \pi, (G_a, G_b) = (g_c \circ l(\pi), g_b)] \\
&\propto P[(G_a, G_b) = (g_c \circ l(\pi), g_b)] \\
&\propto P^{(g_c \circ l(\pi), g_b)} \left( \frac{P_{10}P_{10}}{P_{01}P_{01}} \right)^{\frac{1}{2}\Delta(g_c \circ g_b)} \\
&= \left( \frac{P_{01}P_{10}}{P_{11}P_{00}} \right)^{\frac{1}{2}\Delta(g_c \circ g_b)}
\end{align*}
\]

where in (a) we multiply by the constant \( P[(G_c, G_b) = (g_c, g_b)] \), in (b) we apply the relationship \( G_c = G_a \circ l(\Pi^{-1}) \), and in (c) we use the independence of \( (G_a, G_b) \) from \( \Pi \) and the uniformity of \( \Pi \). In (d) we apply the definition of the distribution of \((G_a, G_b)\), in (e), we divide by a constant that does not depend on \( \Pi \), and in (f) we use Lemma II.1.

Thus MAP estimator is
\[
\hat{\Pi} = \arg \max_{\pi} P[\Pi = \pi|(G_c, G_b) = (g_c, g_b)] \\
= \arg \min_{\pi} \Delta(G_c \circ l(\pi), G_b)
\]

The permutation \( \hat{\pi} = \Pi \) achieves an alignment score of \( \Delta(G_a, G_b) \). Although \( \Delta(G_a, G_b) \) is unknown to the estimator, we can analyze its success by considering the distribution of \( \Delta(G_a \circ l(\Pi^{-1} \circ \hat{\pi}), G_b) - \Delta(G_a, G_b) = \delta(l(\Pi^{-1} \circ \hat{\pi}); G_a, G_b) \).

Let
\[
Q = \{\pi \in S_n : \Delta(G_a \circ l(\pi), G_b) \leq \Delta(G_a, G_b)\} \\
= \{\pi \in S_n : \delta(l(\pi); G_a, G_b) \leq 0\},
\]
the set of permutation that give alignments of \( G_a \) and \( G_b \) that are at least as good as the true permutation. The identity permutation \( id \) achieves \( \delta(l(id); G_a, G_b) = 0 \), so it is always in \( Q \) by definition.

Let \( \eta(G_a, G_b) \) be the probability of success of the MAP estimator conditioned on the generation of the graph pair \((G_a, G_b)\). When \( id \) is not minimizer of \( \Delta(G_a \circ l(\pi), G_b) \), i.e. there is some \( \pi \) such that \( \delta(l(\pi); G_a, G_b) < 0 \), \( \eta = 0 \). When \( id \) achieves the minimum, \( \eta = 1/|Q| \).

The converse argument use the fact the overall probability of success is at most \( \mathbb{E}[1/|Q|] \).

The achievability arguments use the fact the overall probability of error is at most
\[
P[|Q| \geq 1] \leq \mathbb{E}[|Q| - 1]
\]
or equivalently
\[
P \left[ \bigvee_{\pi \neq id} (\pi \in Q) \right] \leq \sum_{\pi \neq id} P[\pi \in Q].
\]
Here we have applied linearity of expectation on the indicators for \( \pi \in Q \) or equivalently the union bound on these events.

D. Cycle decomposition and the nontrivial region

The cycle decompositions of the permutations \( \pi \) and \( \tau = l(\pi) \) play a crucial role in the distribution of \( \delta(\tau; G_a, G_b) \).
For a vertex set \([n]\) and a fixed \( \tau \), define \( \bar{S} \), the nontrivial region of the graph, to be the vertex pairs that are not fixed points of \( \tau \), i.e. \( \bar{S} = \{e \in (\binom{n}{2}) : \tau(e) \neq e\} \). We will mark quantities and random variables associated with the nontrivial region with tildes.

Recall that \( n \) is the number of vertices and let \( \tilde{n} \) be the number of vertices that are not fixed points of \( \pi \). Let \( t = \binom{\tilde{n}}{2} \), let \( \tilde{t} = |\bar{S}| \), and let \( t_i \) be the number of vertex pairs in cycles of length \( i \). Then \( \tilde{t} = t - t_i \).

The expected value of \( \delta(\tau; G_a, G_b) \) depends only on the size of the nontrivial region.

**Lemma II.3.** \( \mathbb{E}[\delta(\tau; G_a, G_b)] = \tilde{t}(p_{00}p_{11} - p_{01}p_{10}) \).

**Proof.** Let \( S = \binom{\tilde{n}}{2} \). Using the alternative expression for \( \delta(\tau; G_a, G_b) \) from Lemma II.1 we have
\[
\mathbb{E}[\delta(\tau; G_a, G_b)] \\
= \mathbb{E}[\mu(G_a, G_b)_{11} - \mu(G_a \circ \tau, G_b)_{11}] \\
= \sum_{e \in \bar{S}} P(G_a(e) = G_b(e) = 1) - P(G_a(e) = G_b(e) = 1) \\
= \sum_{e \in \bar{S}} p_{11} - (p_{10} + p_{11})(p_{01} + p_{11}) \\
= \tilde{t}(p_{00}p_{11} - p_{01}p_{10}) \]

Let \( M = \mu(G_a, G_b)_{11} \), which is the number of edges in \( G_a \land G_b \). Let \( \tilde{M} \) be the number of edges in \( G_a \land G_b \) in the nontrivial region, i.e. \( |E(G_a \land G_b) \cap S| \). When \((G_a, G_b) \sim ER(p, n)\), the events \((G_a, G_b)(e) = (1, 1)\) for \( e \in \binom{n}{2} \) are independent and occur with probability \( p_{11} \), so both \( M \) and \( \tilde{M} \) have binomial distributions. Conditioned on \( M \), \( \tilde{M} \) has a hypergeometric distribution.
We use the following notation for binomial and hypergeometric distributions. Each of these probability distributions models drawing from a pool of $n$ items, $b$ of which are marked. If we take $a$ samples without replacement, the number of marked items drawn has the hypergeometric distribution $\text{Hyp}(n, a, b)$. If we take $a$ samples with replacement, the number of marked items drawn has a binomial distribution $\text{Bin}(n, a, b)$. Thus

\[ M \sim \text{Bin}(t, p_{11}, 1) \]  
\[ \widetilde{M} \sim \text{Bin}(\tilde{t}, p_{11}, 1) \]  
\[ \widetilde{M} | M = m \sim \text{Hyp}(\tilde{t}, m, t). \]

Hypergeometric and binomial random variables have the following generating functions:

\[ \text{Hyp}(a, b, n; z) \triangleq \frac{[x^a y^b]n^n (1 + x + y + xy z)^n}{[x^a y^b]n^n} \]

\[ \text{Bin}(a, b, n; z) \triangleq \left(1 - \frac{b}{n} + \frac{b}{n} z\right)^a \]

Observe that $\text{Hyp}(a, b, n; z)$ is symmetric in $a$ and $b$. Additionally $\text{Hyp}(a, b, n; z) = z^a \text{Hyp}(n - a, b; n; z^{-1})$ because the number of marked balls that are drawn is equal to the number of draws minus the number of unmarked balls drawn. For the same reason, $\text{Bin}(a, b, n; z) = z^a \text{Bin}(a, n - b, n; z^{-1})$.

\section*{E. Proof outline}
Both of our achievability proofs have the following broad structure.

- Use a union bound over the non-identity permutations and estimate $P[\delta(l(\pi), G_a, G_b) \leq 0]$, where $\pi$ is fixed and $(G_a, G_b)$ are random.
- For a fixed $\pi$, examine the cycle decomposition and relate $\delta(l(\pi))$ to $\delta(l(\pi'))$, where $\pi'$ has the same number of fixed points as $\pi$ but all nontrivial cycles have length two. This is summarized in Theorem 4.
- Use large deviations methods to bound the tail of $\delta(l(\pi'))$. This is done in Theorem 5.

Our first achievability result, Theorem 3, comes from applying Theorem 5 in a direct way. This requires no additional assumptions on $p$ but does not match the converse bound when $G_a \land G_b$ is sparse. If $G_a \land G_b$ has no edges, every permutation is in $Q$ and the union bound is extremely loose. When $p_{11} = \frac{\log n}{n}$, the probability that $G_a \land G_b$ has no edges is

\[ (1 - p_{11})^t \approx \exp(-tp_{11}) = \exp \left(-\frac{c}{2} (n - 1) \log n \right). \]

When $c \leq 2$, this probability is larger than $1/n!$, so the union bound on the error probability becomes larger than one.

To overcome this, in the proof of Theorem 5 we condition on $M = \mu(G_a, G_b)_{11}$ before applying Theorem 5. It is more difficult to apply Theorem 5 to $G_a, G_b|M$. In particular, $\widetilde{M}$, the number of edges of the intersection graph in nontrivial cycles of $\tau$, now has a hypergeometric distribution $\text{Hyp}(\tilde{t}, m, t)$ rather than a binomial distribution $\text{Bin}(\tilde{t}, p_{11}, 1)$. One way to analyze the tail of a hypergeometric random variable is to look at the binomial random variable with the same mean and number of samples. This idea is formalized in Lemma 11.3 Moving from $\text{Hyp}(\tilde{t}, m, t)$ to $\text{Bin}(\tilde{t}, m, t)$ would effectively undo our conditioning on $M$. For the most important values of $t$ and the typical values of $m$, we have $\tilde{t} \ll m$. Thus we exploit the symmetry of the hypergeometric distribution $\text{Hyp}(\tilde{t}, m, t) = \text{Hyp}(m, \tilde{t}, t)$ and replace $\text{Hyp}(m, \tilde{t}, t)$ with $\text{Bin}(m, \tilde{t}, t)$, which is more concentrated than $\text{Bin}(\tilde{t}, p_{11}, 1)$.

\section*{F. Related Work}
In the perfect correlation limit, i.e. $p_{01} = p_{10} = 0$, we have $G_a = G_b$. In this case, the size of the automorphism group of $G_a$ determines whether it is possible to recover the permutation applied to $G_a$. This is because the composition of an automorphism with the true matching gives another matching with no errors. Whenever the automorphism group of $G_a$ is nontrivial, it is impossible to exactly recover the permutation with high probability. We will return to this idea in Section V in the proof of the converse part of Theorem 1.

Wright established that for $1 - o(1)$, the automorphism group of $G \sim \mathcal{P}(n, p)$ is trivial with probability $1 - o(1)$ and that for $p \geq \log n - o(1)$, it is nontrivial with probability $1 - o(1)$ [4]. In fact, he proved a somewhat stronger statement about the growth rate of the number of unlabeled graphs that implies this fact about automorphism groups. Thus our Theorem 1 and Theorem 2 extend Wright’s results. Bollobás later provided a more combinatorial proof of this automorphism group threshold function [5]. The methods we use are closer to those of Bollobás.

Some practical recovery algorithms start by attempting to locate a few seeds. From these seeds, the graph matching is iteratively extended. Algorithms for the latter step can scale efficiently. Narayan and Shmatikov were the first to apply this method [1]. They evaluated their performance empirically on graphs derived from social networks.

More recently, there has been some work evaluating the performance of this type of algorithm on graph inputs from random models. Yartseva and Grossglauser examined a simple percolation algorithm for growing a graph matching [6]. They find a sharp threshold for the number of initial seeds required to ensure that final graph matching includes every vertex. The intersection of the graphs $G_a$ and $G_b$ plays an important role in the analysis of this algorithm. Kazemi et al. extended this work and investigated the performance of a more sophisticated percolation algorithm [7].

If the networks being aligned correspond to two distinct online services, it is unlikely that the user populations of the services are identical. Kazemi et al. investigate alignment recovery of correlated graphs on overlapping but not identical vertex sets [8]. They determine that the information-theoretic penalty for imperfect overlap between the vertex sets of $G_a$ and $G_b$ is relatively mild. This regime is an important test of the robustness of alignment procedures.

\section*{G. Subsampling model}
Pedarsani and Grossglauser [2] introduced the following generative model for correlated Erdős-Rényi (ER) graphs.
Essentially the same model was used in [9, 10]. Let $H$ be an ER graph on $[n]$ with edge probability $r$. Let $G_a$ and $G_b$ be independent random subgraphs of $H$ such that each edge of $H$ appears in $G_a$ and in $G_b$ with probabilities $s_a$ and $s_b$ respectively. We will refer to this as the subsampling model. The $s_a$ and $s_b$ parameters control the level of correlation between the graphs. This model is equivalent to $ER(n, p)$ with

$$
\begin{align*}
p_{11} &= r s_a s_b \\
p_{10} &= r s_a (1 - s_b) \\
p_{01} &= r (1 - s_a) s_b \\
p_{00} &= 1 - r (s_a + s_b - s_a s_b).
\end{align*}
$$

Solving for $r$ from the above definitions, we obtain

$$
r = \frac{(p_{10} + p_{11}) (p_{01} + p_{11})}{p_{11}} = p_{11} + p_{10} + p_{01} + \frac{p_{10} p_{01}}{p_{11}}, \quad (8)
$$

Observe that when $G_a$ and $G_b$ are independent, we have $r = 1$. This reveals that the subsampling model is capable of representing any correlated Erdős-Rényi distribution with nonnegative correlation. From [8], we see that $r = \mathcal{O}\left(\frac{1}{\log n}\right)$ is equivalent to $p_{00} = 1 - \mathcal{O}\left(\frac{1}{\log n}\right)$ and $p_{00} p_{10} p_{01} = \mathcal{O}\left(\frac{1}{\log n}\right)$.

### III. Graphs and cyclic sequences

Let $w$ be a matrix of formal variables indexed by $[2] \times [2]$: 

$$
w = \begin{pmatrix} w_{00} & w_{01} \\
w_{10} & w_{11}\end{pmatrix}
$$

and let $z$ be a single formal variable. For a set $S$ and a permutation $\tau : S \rightarrow S$, define the generating function

$$
A_{S, \tau}(w, z) = \sum_{g \in [2]^S} \sum_{h \in [2]^S} z^{\delta(\tau; g, h)} w^{\mu(g, h)}
$$

When $S = \{1, 2\}$ and $(G_a, G_b) \sim ER(p)$, $A_{S, \tau}$ captures the joint distribution of $\mu(G_a, G_b)$ and $\delta(\tau; G_a, G_b)$:

$$
P[\mu(G_a, G_b) = k, \delta(\tau; G_a, G_b) = i] = [w^k z^i] A_{S, \tau}(p \circ w, z)
$$

where $p \circ w$ is the element-wise product of the matrices $p$ and $w$. This follows immediately from the definition of the $ER(p)$ distribution.

#### A. Generating functions

**Definition 2.** Let $S$ be an finite index set and let $\sigma : S \rightarrow S$ be a permutation consisting of a single cycle of length $|S|$. A cyclic $T$-ary sequence is a pair $(\sigma, f)$ where $f : S \rightarrow T$.

Let $\sigma$ be a permutation of $[\ell]$ with a single cycle. For any such choices of $\sigma$, the sets of cyclic sequences obtained are isomorphic, so we can define the generating function

$$
a_{\ell}(w, z) = A_{[\ell], \sigma}(w, z).
$$

**Lemma III.1.** Let $\tau : S \rightarrow S$ be a permutation. Let $t_\ell$ be the number of cycles of length $\ell$ in $\tau$. Then $\sum_\ell \ell t_\ell = |S|$ and

$$
A_{S, \tau}(w, z) = \prod_{\ell \in \mathbb{N}} a_{\ell}(w, z)^{t_\ell}.
$$

**Proof.** Let $\gamma(a, b, c) = \frac{1}{2}(1\{a \neq c\} - 1\{b \neq c\})$, so

$$
\delta(\tau; g, h) = \sum_{\ell \in \mathbb{N}} \gamma(g(e), g(\tau(e)), h(e)).
$$

Let $T$ be the partition of $S$ from the cycle decomposition of $\tau$. Then we have an alternate expression for $A_{S, \tau}$:

$$
A_{S, \tau}(w, z) = \sum_{\ell \in \mathbb{N}} \sum_{S \in T} \prod_{\ell \in \mathbb{N}} z^{\gamma(g(e), g(\tau(e)), h(e))} w^{\mu(g, h)}
$$

In (a), we use the fact that $e \in S_i$ implies $\tau(e) \in S_i$.

For $l = 1$, the generating function $a_1(x, y)$ is very simple. There are 4 possible pairs of cyclic $[2]$-ary sequences of length one. A cycle of length one in a permutation is a fixed point, so these cyclic sequences are unchanged by the application of $\sigma$ and $\delta(\sigma; g, h)$ is zero for each of them. Thus $a_1(w, z) = (w_{00} + w_{01} + w_{10} + w_{11})$.

We define

$$
\tilde{A}_{S, \tau}(w, z) = \prod_{\ell \geq 2} a_{\ell}(w, z)^{t_\ell}.
$$

Just as $A_{S, \tau}$ captures the joint distribution of $M$ and $\delta(\tau)$, $\tilde{A}_{S, \tau}$ captures the joint distribution of $\tilde{M}$ and $\tilde{\delta}(\tau)$. Because $z$ does not appear in $a_1(w, z)$, we have

$$
[z^i] A_{S, \tau}(w, z) = a_1(w, z)^{t_1} z^i \tilde{A}_{S, \tau}(w, z).
$$

This implies that $\tilde{\delta}(\tau; G_a, G_b)$ and $M$ are conditionally independent given $\tilde{M}$.

#### B. Nontrivial cycles

For $l = 2$, there are 16 possible pairs of sequences. There are only 4 pairs for which $\delta(\sigma; g, h) \neq 0$: the cases where $g$ and $h$ are each either $(0, 1)$ or $(1, 0)$. In the two cases where $g = h$, $\mu(g, h) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\mu(g \circ \sigma, h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\delta(\sigma; g, h) = 1$. In the two cases where $g \neq h$, $\delta(\sigma; g, h) = -1$. Thus

$$
a_2(w, z) = (w_{00} + w_{01} + w_{10} + w_{11})^2 + 2w_{00}w_{11}(z - 1) + 2w_{01}w_{10}(z^{-1} - 1). \quad (9)
$$

The following theorem relates longer cycles to cycles of length two.

**Theorem 4.** Let $w \in \mathbb{R}_{\geq 0}^{[2] \times [2]}$, and $z \in \mathbb{R}$. Then for all $\ell \geq 2$,

$$
a_{\ell}(w, z) \leq a_2(w, z)^{\ell/2}.
$$

The proofs of Theorem 4 and several intermediate lemmas are in Appendix A.
C. Tail bounds from generating functions

The following lemma is a well known inequality that we will apply in the proof of Theorem \[\text{Theorem 4}\] and in several other places.

**Lemma III.2.** For a generating function \(g(z) = \sum_i g_i z^i\) where \(g_i \geq 0\) for all \(i \in \mathbb{Z}\) and a real number \(0 \leq z_* \leq 1\),

\[\sum_{i \leq j} [z^i]g(z) \leq z_*^j g(z_*)\.

**Proof.**

\[\sum_{i \leq j} [z^i]g(z) = \sum_i g_i \leq \sum_i g_i z_*^{i-j} = z_*^j g(z_*)\]. \(\square\)

**Theorem 5.** For \(w \in \mathbb{R}^{2|2}_0\),

\[
\sum_{i \leq 0} [z^i]A_{S,r}(w, z) \leq \left(\left(\sum_{i=0}^n w_{01} w_{01} + w_{10} + w_{11}\right)^2 - 2 \left(\sqrt{w_{00} w_{11}} - \sqrt{w_{01} w_{10}}\right)^2\right)^{\frac{1}{2}}.
\]

**Proof.** For all \(0 \leq z_1 \leq 1\), we have

\[
\sum_{i \leq 0} [z^i]A_{S,r}(x, z) = \sum_{i \leq 0} [z^i] \prod_{\ell \geq 2} a_{\ell}(w, z)_i^{\ell} \leq \prod_{\ell \geq 2} a_{\ell}(w, z_1)_i^{\ell} \leq a_2(w, z_1)^{i/2}.
\]

where (a) follows from by Lemma \[\text{Lemma III.2}\] and (b) follows from Theorem \[\text{Theorem 4}\] and \(\sum_{j=2}^{\infty} \ell \ell = \ell\).

From (10), \(a_2(w, z) = u^2 + 2v\) where

\[
u = \frac{u_{00} w_{11} (z - 1) + w_{01} w_{10} (z - 1)}{u_{00} w_{11} w_{01} w_{10}}
\]

We would like to choose \(z\) to minimize \(v\). Substituting the optimal choice, \(z_1 = \left(\frac{u_{00} w_{11} w_{01} w_{10}}{u_{00} w_{11}}\right)^{1/2}\), into the expression for \(v\), we obtain

\[
\min \frac{u_{00} w_{11} z - u_{00} w_{11} - w_{01} w_{10} + w_{01} w_{10} z^{-1}}{u_{00} w_{11} w_{01} w_{10}} = 2\sqrt{w_{00} w_{11} w_{10} w_{11} z - w_{00} w_{11} - w_{01} w_{10} + w_{01} w_{10} w_{11}} = -\left(\sqrt{w_{00} w_{11}} - \sqrt{w_{01} w_{10}}\right)^2.
\]

Combining this with \(a_2(w, z) = u^2 + 2v\) and (10) gives the claimed bound. \(\square\)

D. Hypergeometric and binomial g.f.

Chvátal provided an upper bound on the tail probabilities of a hypergeometric random variable \(\text{Hyp}(a, b, n; z)\). The following lemma is essentially a translation of that bound into the language of generating functions.

**Lemma III.3.** For all \(a, b, n \in \mathbb{N}\), \(a \leq n\) and \(b \leq n\), and all \(z \in \mathbb{R}, z > 0\)

\[
\text{Hyp}(a, b, n; z) \leq \text{Bin}(a, b, n; z).
\]

**Proof.** First, we have

\[
\binom{n}{a} \binom{n}{b} \text{Hyp}(a, b, n; z) = [a^n b^n (1 + x + y + z)^n]
\]

\[
= [a^n b^n ((1 + x)(1 + y) + xy(z-1))^n]
\]

\[
= \sum_{\ell} \binom{n}{\ell} [a^{\ell} b^n ((1 + x)(1 + y) + xy(z-1))^n] z_{\ell}^{n-\ell}
\]

\[
= \sum_{\ell} \binom{n}{\ell} [a^{\ell} b^n ((1 + x)(1 + y) + xy(z-1))^n] z_{\ell}^{n-\ell}
\]

\[
= \sum_{\ell} \binom{n}{\ell} \binom{n - \ell}{\ell} \binom{a}{b}(n - \ell) z_{\ell}^{n-\ell}
\]

\[
= \binom{n}{a} \binom{n}{b} \sum_{\ell} \binom{n}{\ell} [a^{\ell} b^n] z_{\ell}^{n-\ell}.
\]

Observe that

\[
\frac{\binom{n}{a} \binom{n}{b}}{\binom{n}{a} \binom{n}{b}} = \prod_{i=0}^{\ell-1} (\frac{b-i}{n-i}) b \prod_{i=0}^{\ell-1} (\frac{b-i}{n-i}) b \leq \prod_{i=0}^{\ell-1} \frac{b-i}{n-i} \leq 1.
\]

Thus for \(z \geq 1\),

\[
\text{Hyp}(a, b, n; z) \leq \sum_{\ell} \binom{n}{\ell} [a^{\ell} b^n] z_{\ell}^{n-\ell}
\]

\[
= \binom{a}{b}(n - \ell) z_{\ell}^{n-\ell}
\]

\[
= \text{Bin}(a, b, n; z).
\]

If \(0 < z \leq 1\), then \(z^{-1} > 1\) and

\[
\text{Hyp}(a, b, n; z) = z^n \text{Hyp}(a, n-b, n; z^{-1})
\]

\[
\leq z^n \text{Bin}(a, n-b, n; z^{-1})
\]

\[
= \text{Bin}(a, b, n; z).
\]

\(\square\)

IV. Achievability Theorems

In this section we will prove Theorem \[\text{Theorem 1}\]. This is the combination of two results for different values of \(p_{11}\). Corollary \[\text{Corollary 1}\] in Section IV-B handles \(2 \log n + o(1) \leq p_{11}\) and Theorem \[\text{Theorem 6}\] in Section IV-C handles \(\log n + o(1) \leq p_{11} \leq O\left(\frac{\log n}{n}\right)\).

A. Permutations

Let \(S_{a, n}\) be the set of permutations of \([n]\) with exactly \(n-\tilde{n}\) fixed points. If \(\pi \in S_{a, n}\), then \(e = \{i, j\}\) is a fixed point of \(\pi\) if either \(i\) and \(j\) are both fixed points of \(\pi\) or \(i\) and \(j\) form a cycle of length 2 in \(\pi\). Thus \(t_1\), the number of fixed points of \(\pi\), satisfies

\[
\binom{n-\tilde{n}}{2} \leq t_1 \leq \binom{n-\tilde{n}}{2} + \frac{\tilde{n}}{2}.
\]
We will need a number of variations on these inequalities and we collect them in this section. Let \( \nu = \hat{n}/n \). Then
\[
\frac{t_1}{t} \leq \frac{(n - \hat{n})(n - 1 - \hat{n}) + \hat{n}}{n(n - 1)} \\
= (1 - \nu) \left(1 - \frac{n}{n - 1}\right) + \frac{\nu}{n - 1} \\
= (1 - \nu)^2 + \frac{\nu}{n - 1}(1 - (1 - \nu)) \\
= (1 - \nu)^2 + \frac{\nu^2}{n - 1}.
\]
(12)
For \( 0 \leq \nu \leq 1 \), the best linear upper bound on \( t_1 \) is
\[
\frac{t_1}{t} \leq 1 - \frac{\nu(n - 2)}{n - 1}.
\]
Equivalently,
\[
\hat{t} = t - t_1 \geq \frac{\hat{n}(n - 2)}{2}.
\]
(13)
In the other direction, we have
\[
\hat{t} \leq \frac{n}{2} - \frac{n - \hat{n}}{2} \\
= \frac{n\hat{n} + (n - 1)\hat{n} - \hat{n}^2}{2} \\
\leq \hat{n}\hat{n}.
\]
(14)

Lemma IV.1. For \( z \in \mathbb{R} \) such that \( 0 \leq z < n^{-1} \),
\[
\sum_{\hat{n}} |S_{\hat{n}}\bar{n}|z^{\hat{n}} \leq 1 + \frac{n^2z^2}{1 - nz}.
\]
Proof. We have \( |S_{\hat{n}}\bar{n}| = \binom{n}{\hat{n}}! / \hat{n}! \), where \( !/\hat{n} \) is the number of derangements of \( \hat{n} \), i.e., the permutations of \( \hat{n} \) with no fixed points. Note that \( !0 = 1 \) and \( !1 = 0 \), so \( |S_{\hat{n},0}| = 1 \) and \( |S_{\hat{n},1}| = 0 \). For \( n \geq 2 \), we use \( |S_{\hat{n},\bar{n}}| \leq \binom{n}{\hat{n}}! \hat{n}! \leq n^{\hat{n}} \). For \( 0 \leq x < 1 \),
\[
\sum_{\hat{n} \geq 2} x^{\hat{n}} \leq \sum_{\hat{n} \geq 2} x^{\hat{n}} = \frac{x^2}{1 - x}.
\]
\[\Box\]

B. Achievability theorem for dense graphs

We have developed enough tools to prove the first achievability theorem.

Lemma IV.2. For all \( \pi \in S_{\hat{n},\bar{n}} \), with \( \tau = l(\pi) \),
\[
P[\delta(\pi) \leq 0] \leq z_2\bar{z}
\]
where
\[
z_2 = \exp \left( -\frac{(n - 2)(\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}})^2}{2} \right)
\]
Proof. For all \( \pi \in S_{\hat{n},\bar{n}} \),
\[
P[\delta(\pi) \leq 0] = \sum_{z \leq 0} |z|\bar{A}_r(p, z) \\
\leq (1 - 2(\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}})^2)\bar{z} \\
\leq \exp(-\hat{t}(\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}})^2) \\
\leq \exp \left( -\frac{\hat{n}(n - 2)(\sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}})^2}{2} \right)
\]
where \((a)\) follows from Theorem\[3\] \((b)\) uses \( 1 + x \leq e^x \), and \((c)\) uses \((13)\). \[\Box\]

Theorem 3. If \( p \) satisfies
\[
\frac{2\log n + \omega(1)}{n} \leq \left( \sqrt{p_{11}p_{00}} - \sqrt{p_{01}p_{10}} \right)^2
\]
then there is an estimator for \( \hat{m} \) given \( G_v, G_b \) that is correct with probability \( 1 - o(1) \).

Proof. FromLemma IV.2 for all \( \pi \in S_{\hat{n},\bar{n}} \), \( P[\delta(\pi) \leq 0] \leq z_2^\hat{n} \). Applying the union bound over permutations and Lemma IV.1
\[
P[\hat{\tau}_{\hat{n}} \neq l(\pi) \leq 0] \leq \sum_{\hat{n} \geq 2} |S_{\hat{n},\bar{n}}|z_2^\hat{n} \leq \frac{n^2z_2^2}{1 - nz_2^2}
\]
which is \( o(1) \) whenever \( nz_2 \) is \( o(1) \). \[\Box\]

Specializing Theorem 3 to the conditions of Theorem 1 gives the following.

Corollary 1. Let \( p \) satisfy the conditions \((1), (2), \) and \((3)\) and let
\[
\frac{2\log n + \omega(1)}{n} \leq p_{11}.
\]
Then the MAP estimator is correct with probability \( 1 - o(1) \).

Proof. In this regime, both \( p_{00} = 1 - p_{01} - p_{10} - p_{11} \) and \( 1 - \sqrt{p_{11}p_{00}} \) are \( 1 - O \left( \frac{1}{\log n} \right) \), so the condition of Theorem 3 is equivalent to the claimed condition.

Note that \((3)\) is slightly stronger than what we require.

C. Achievability theorem for sparse graphs

Throughout this section, we will replace \((2)\) with a much sparser constraint on \( G_v \cap G_b \):
\[
p_{11} \leq O \left( \frac{\log n}{n} \right).
\]
(15)

Lemma IV.3. Let \( p \) satisfy the conditions \((1), (3), \) and \((15)\) and let \( \tilde{m} \leq O \left( \frac{\log n}{n} \right) \). Then for all \( \hat{n} \) and all \( \pi \in S_{\hat{n},\bar{n}} \),
\[
P[\delta(\pi) \leq 0|\tilde{M} = \tilde{m}] \leq z_4^\hat{n}z_5^\hat{n},
\]
where \( z_4 = O \left( \frac{1}{\log n} \right) \) and \( z_5 = O(1) \).

We have \( \tilde{M} \sim \text{Bin}(\tilde{t}, p_{11}, 1) \), so \( \mathbb{E}[\tilde{M}] = \tilde{t}p_{11} \leq O \left( \frac{\log n}{n} \right) \). Thus the assumption \( \tilde{m} \leq O \left( \frac{\log n}{n} \right) \) excludes the possibility that \( \tilde{M} \) is more than a constant factor larger than its expected value. In this regime, each additional appearance of \((1, 1)\) in the nontrivial region significantly reduces the probability that \( \delta(\tau) \leq 0 \). If \( \tilde{m} \) is large enough some 2-cycles contain two appearances of \((1, 1)\). These collisions contribute nothing to \( \delta(\tau) \) and diminish the return we receive for increasing \( \tilde{m} \). Thus the condition \( \tilde{m} \leq O \left( \frac{\log n}{n} \right) \) is required.

Similarly, condition \((1)\), \( p_{01} + p_{10} \leq O((\log n)^{-1}) \), is required to ensure that the number of 2-cycles containing a \((1, 1)\) and a \((1, 0)\) is negligible. \[\Box\]
Proof.

\begin{align*}
P[\delta(\tau) \leq 0, \tilde{M} = \tilde{m}] &= \sum_{d \leq 0} \left[ \tilde{d} w_{11}^{d}\tilde{A}_S,\tau \right] \left( \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} w_{11} \end{pmatrix}, z \right) \\
&\leq \sum_{d \leq 0} \left[ \tilde{d} w_{*}^{\tilde{m}} \tilde{A}_S,\tau \right] \left( \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} w_{*} \end{pmatrix}, z \right) \\
&\leq w_{*}^{\tilde{m}} \alpha(p, w_{*}) \tilde{t} \tag{16}
\end{align*}

where (a) follows from Lemma III.2, (b) follows from Theorem 5, and \( \alpha(p, w_{*}) \) is

\[[1 - p_{11} + p_{11} w_{*}]^2 - 2(\sqrt{p_{00} p_{11} w_{*}} - \sqrt{p_{01} p_{10}})^2.\]

We have

\[P[\tilde{M} = \tilde{m}] = [w_{11}^{\tilde{m}}] \tilde{A}_S,\tau(p \circ w, 1) = \left( \frac{\tilde{t}}{\tilde{m}} \right) p_{11}^{\tilde{m}} (1 - p_{11})^{\tilde{t} - \tilde{m}} \leq \left( \frac{\tilde{t}}{\tilde{m}} \right) p_{11}^{\tilde{m}} (1 - p_{11})^{\tilde{t} - \tilde{m}} \leq \left( \frac{\tilde{t}}{\tilde{m}} \right) \tilde{t}^{\tilde{m}} (1 - p_{11})^{\tilde{t}}. \tag{17}\]

Combining (16) and (17), we get

\[P[\delta(\tau) \leq 0 | \tilde{M} = \tilde{m}] \leq \left( \frac{\tilde{m} (1 - p_{11})}{\tilde{t} p_{11} w_{*}} \right) \left( \frac{\alpha(p, w_{*}) (1 - p_{11})^2}{ (1 - p_{11})^{2}} \right) \leq O \left( \frac{\tilde{m} \log n}{\tilde{t} p_{11} w_{*}} \right). \tag{18}\]

Let \( p_{i,j} = p_{ij} / (1 - p_{11}) \), so

\[\alpha(p, w_{*}) (1 - p_{11})^{-2} = (1 + p_{11} w_{*})^2 - 2(\sqrt{p_{00} p_{11} w_{*}} - \sqrt{p_{01} p_{10}})^2 \]

\[= 1 + (p_{11} w_{*})^2 + 2p_{11}^2 w_{*} - 2p_{00} p_{11} w_{*} - 2p_{01} p_{10} + 4 \sqrt{p_{00} p_{11} p_{10} p_{11} w_{*}} \]

\[\leq 1 + (p_{11} w_{*})^2 + 2(p_{01} + p_{10}) p_{11} w_{*} + 4 \sqrt{p_{00} p_{11} p_{10} p_{11} w_{*}} \]

where we used \( 1 - p_{00} = p_{01}' + p_{10}' \). Now let \( w_{*} = \tilde{w}_{*} = \frac{\tilde{m} \log n}{\tilde{t} p_{11} w_{*}} \).

The first term of (18) becomes \( \frac{1}{\log n} \tilde{m} \). The logarithm of the second term is

\[\frac{1}{2} \log \left( \alpha(p, w_{*}) (1 - p_{11})^{-2} \right) \leq \frac{1}{2} \left( p_{11} w_{*} \right)^2 + \frac{1}{2} (p_{01} + p_{10}) p_{11} w_{*} + \frac{1}{2} \sqrt{p_{00} p_{11} p_{10} p_{11} w_{*}} \]

\[= \frac{\tilde{m}^2 (\log n)^2}{2} + \frac{p_{00} + p_{01} p_{10} + p_{11} w_{*}}{\tilde{m} \log n} + 2 \sqrt{p_{00} p_{11} p_{10} p_{11} w_{*}} \]

\[\leq O \left( \frac{\tilde{m} (\log n)^2}{n} \right) + O(\tilde{m}) + O \left( \frac{\tilde{t}}{n} \right) \leq O(\tilde{m}) + O(\tilde{n}). \]

In (a), we used \( \log(1 + x) \leq x \). Observe that (3) and (15) imply \( p_{01} p_{10} \leq O \left( \frac{1}{n (\log n)^2} \right) \). In (b), we used this along with

\[\tilde{m} \leq O \left( \frac{\tilde{t} \log n}{n} \right) \text{ and } p_{01} + p_{10} \leq O \left( \frac{1}{\log n} \right) \text{ (condition (1)).} \]

In (c) we used \( \tilde{t} \leq \tilde{n} \) (14). Overall, we have

\[P[\delta(\tau) \leq 0 | \tilde{M} = \tilde{m}] \leq \left( \frac{1}{\log n} \right) \tilde{m} (\log n)^2 + O(\tilde{m}) \tag{19}. \]

\textbf{Lemma IV.4.} Let \( \textbf{p} \) satisfy the conditions (1), (3), and (15) and let \( m \leq O(n \log n) \). Then for all \( \tilde{n} \) and all \( \pi \in S_{n,\tilde{n}} \),

\[P[\delta(\tau) \leq 0 | |M| = m] \leq \tilde{z}_\tau^2. \]

where \( z_\tau \leq \exp \left( - \frac{2m}{\tau} + O(1) \right) \).

\textbf{Proof.} For a fixed \( \pi \), and thus a fixed \( \tau \), we condition on \( \tilde{M} \), the number of edges of \( G_a \land G_b \) that are in nontrivial cycles of \( \tau \). We do this because we need some upper bound on \( \tilde{m} \) to apply Lemma IV.3. Recall that \( |\tilde{M}| = m \) has a hypergeometric distribution and that \( E[|\tilde{M}| = m] = \frac{m}{\tilde{t}} \).

Define \( m^* = e^2 \tilde{m} \) and write

\[P[\delta(\tau) \leq 0 | |M| = m] = P[\delta(\tau) \leq 0, \tilde{M} \leq \tilde{m}^* | |M| = m] + P[\delta(\tau) \leq 0, \tilde{M} > \tilde{m}^* | |M| = m]. \]

The first error term contains the typical values of \( \tilde{M} \).

\[\epsilon_1 \triangleq P[\delta(\tau) \leq 0, \tilde{M} \leq \tilde{m}^* | |M| = m] \]

\[= \sum_{\tilde{m} \leq \tilde{m}^*} P[\tilde{M} = \tilde{m}, |M| = m] P[\delta(\tau) \leq 0 | \tilde{M} = \tilde{m}] \]

\[\leq \sum_{\tilde{m} \leq \tilde{m}^*} P[\tilde{M} = \tilde{m}, |M| = m] \tilde{z}_5^{\tilde{m}^* \tilde{m}} \]

\[= \tilde{z}_5^{\tilde{m}} \text{Hyp}(m, \tilde{t}, \tilde{t}; z_4) \]

\[\leq \tilde{z}_5^{\tilde{m}} \text{Bin}(m, \tilde{t}, \tilde{t}; z_4) \]

\[= \tilde{z}_5^{\tilde{m}} \left( 1 + \frac{\tilde{t}}{\tilde{t}} (z_4 - 1) \right)^m \]

\[= \tilde{z}_5^{\tilde{m}} \left( 1 + \left( \frac{1}{\tilde{t}} - 1 \right) (1 - z_4) \right)^m \]

where (a) uses the conditionally independence of \( \delta(\tau) \) and \( \tilde{M} \) given \( M \), (b) follows from Lemma IV.3 (c) follows from (7), and (d) follows from Lemma III.3.

Let \( \nu = \frac{\tilde{t}}{\tilde{t}} - 1 \). For sufficiently large \( n \), we have \( z_4 < 1 \) so we can apply (12):

\[1 + \left( \frac{1}{\tilde{t}} - 1 \right) (1 - z_4) \]

\[\leq 1 + \left( 1 - \nu^2 \right) + \left( \frac{\nu^2}{n - 1} - 1 \right) (1 - z_4) \]

\[= (1 - \nu)^2 + \frac{\nu^2}{n - 1} (1 - z_4^2) + \nu(2 - \nu) z_4 \]

\[\leq (1 - \nu)^2 + \frac{\nu^2}{n - 1} + 2\nu z_4. \]
Now let $z_6 = (n-1)^{-1} + 2z_4$. We will handle small and large values of $\bar{n}$ separately. In the region $2 \leq \bar{n} \leq n(1-e^{-1})$, or equivalently $e^{-1} \leq 1-\nu \leq 1$, we have
\[
(1-\nu)^2 + \nu z_6 \\
= (1-\nu) \left[ 1 - \frac{z_6}{1-\nu} \right] \\
\leq (1-\nu)(1-\nu(1-ez_6)) \\
\leq e^{-\nu} \exp(-\nu(1-ez_6)) \\
= \exp(-\nu(2-ez_6))
\]

Thus
\[
\log \epsilon_1 \leq \bar{n} \log z_5 - \frac{m\bar{n}}{n} (2-ez_6) \\
= \bar{n} \left[ -\frac{2m}{n} + \frac{e\nu z_6}{n} + \log z_5 \right] \\
\leq \bar{n} \left[ -\frac{2m}{n} + O(1) \right]
\]
because $z_6 = 2z_4 + \frac{1}{n-1}$, $z_4 = O\left(\frac{1}{\log n}\right)$, $\frac{m}{n} = O(\log n)$, and $\log z_5 = O(1)$.

The second error term is small compared to the first, so we do not need to obtain the best possible exponent. We have
\[
\epsilon_2 \overset{\Delta}{=} P[\delta(\tau) \leq 0, \bar{M} > m^*|M = m] \\
\leq P[\bar{M} > m^*|M = m] \\
\overset{(a)}{=} \text{Hyp}(m, \bar{t}, \bar{z}; \bar{z}_3)^{-e^m \bar{t}/t} \\
\overset{(b)}{=} \text{Bin}(m, \bar{t}, \bar{z}; \bar{z}_3)^{-e^m \bar{t}/t} \\
= \left(1 + \frac{\bar{t}}{\bar{z}_3} + \frac{t}{\bar{z}_3} \right)^{-e^m \bar{t}/t} \\
\leq \exp \left( -e^{z_3-1} \left( \frac{m\bar{t}}{t} - \frac{2e^m \bar{m}}{t} \right) \right) \\
\overset{(c)}{=} \exp \left( -(e^2 + 1) \frac{m\bar{n}(n-2)}{n(n-1)} \right) \\
\overset{(d)}{=} \exp \left( -(e^2 + 1) \frac{m\bar{n}(n-2)}{n(n-1)} \right)
\]
where (a) follows from Lemma III.2, (b) follows from Lemma III.3, (c) follows from letting $z_3 = e^2$, and (d) follows from (13).

Thus $\epsilon_2$ is exponentially smaller than $\epsilon_1$ and we have the claimed bound.

**Lemma IV.5.** Let $p$ satisfy the conditions (1), (3), and (15), and let $m \leq \mathcal{O}(n \log n)$. Then
\[
P[\bar{\pi}(\tau) \leq 0 | M = m] \leq \mathcal{O}(n^2 z_8^m)
\]
where $z_8 \leq \frac{n}{n+4}$.

**Proof.** Recall from Section IV.A that $S_{n, \bar{n}}$ is the set of permutations of $[n]$ with $n - \bar{n}$ fixed points. Apply the union bound over all of the non-identity permutations:
\[
P[\bar{\pi}(\tau) \leq 0 | M = m] \\
\leq \sum_{\pi \neq \text{id}} P[\delta(\tau) \leq 0 | M = m] \\
= \sum_{\bar{n} = 2}^n \sum_{\pi \in S_{n, \bar{n}}} P[\delta(\tau) \leq 0 | M = m] \\
\leq \sum_{\bar{n} = 2}^n |S_{n, \bar{n}}| \max_{\pi \in S_{n, \bar{n}}} P[\delta(\tau) \leq 0 | M = m] \\
\overset{(a)}{=} \sum_{\bar{n} = 2}^n |S_{n, \bar{n}}| z_7^{\bar{n}},
\]

where (a) follows from Lemma IV.4. Now we would like to apply Lemma IV.1 but for that we need $z_7 < \frac{1}{n}$. There is some $C$ such that $z_7 \leq \exp \left( -\frac{2m}{n} + C \right)$ for sufficiently large $n$. If $\exp \left( -\frac{2m}{n} + C \right) \leq \frac{1}{2n}$, then
\[
\sum_{\bar{n} = 2}^n |S_{n, \bar{n}}| z_7^{\bar{n}} \leq \frac{n^2 z_7^2}{1 - n z_7} \leq 2n^2 z_7^2 \leq 2n^2 \exp \left( -\frac{4m}{n} + 2C \right).
\]

If instead $\exp \left( -\frac{2m}{n} + C \right) \geq \frac{1}{2n}$, then
\[
P[\bar{\pi}(\tau) \leq 0 | M = m] \leq 1 \leq 4n^2 \exp \left( -\frac{4m}{n} + 2C \right).
\]

Thus $P[\bar{\pi}(\tau) \leq 0 | M = m] \leq 4n^2 e^{2C} z_8^m$ where $z_8 = \exp \left( -\frac{4m}{n} \right) = \frac{1}{\exp \left( \frac{4m}{n} \right)} \leq 1 + \frac{m}{n} = \frac{n}{n+4}$.  

**Theorem 6.** Let $p$ satisfy the conditions (1), (3), (4), and (15). Then the MAP estimator is correct with probability $1 - o(1)$.

**Proof.** We have $M \sim \text{Bin}(t, p_{11}, 1)$, so $\mathbb{E}[M] = tp_{11} \geq n \log n$. Thus the probability that $M \leq (1+\epsilon)(tp_{11})$ is $o(1)$ for any $\epsilon > 0$. We have
\[
P[\bar{\pi}(\tau) \leq 0] \\
\leq o(1) + P[\bar{\pi}(\tau) \leq 0 | M \leq (1+\epsilon)tp_{11}].
\]
Now we analyze the main term:

\[
P[\land_{\pi \neq id} \delta(\tau) \leq 0 | M \leq (1 + \epsilon) t_p 11] \leq \sum_{m \leq (1 + \epsilon) t_p 11} P[\land_{\pi \neq id} \delta(\tau) \leq 0 | M = m] P[M = m]
\]

This follows from Lemma V.1. The inequality \( \delta(\tau) \leq 0 \) follows from (4). In (b), we used \( \delta(\tau; g, a, b) = \delta(g_a, g_b) = \delta(g_a, g_b) - \delta(g_a, g_b) \).

Proof. Let \( \tau = l(\pi) \) and recall that

\[
\Delta(g_a, g_b) = \sum_{e \in [n]} |g_a(e) - g_b(e)|
\]

\[
\delta(\tau; g, a, b) = \Delta(g_a, g_b) - \Delta(g_a, g_b).
\]

The inequality (a) follows from Lemma IV.5. (b) follows from the value of \( \delta(\tau; g, a, b) \) in Lemma IV.5. (c) follows from \( e^\tau \geq 1 + x \), and (d) follows from \( l(x) \). In (e), we used

\[
\log n + o(1) \left( 1 - \Theta \left( \frac{1}{\log n} \right) \right) = \log n - \Theta(1) + o(1).
\]

V. PROOF OF CONVERSE

The converse statement depends on the following lemma.

**Lemma V.1.** Let \( g_a \) and \( g_b \) be graphs on the vertex set \([n]\). For all \( \pi \in \text{Aut}(g_a \land g_b) \), \( \delta(\tau; g_a, g_b) \leq 0 \).

Proof. Let \( \tau = l(\pi) \) and recall that

\[
\Delta(g_a, g_b) = \sum_{e \in [n]} |g_a(e) - g_b(e)|
\]

\[
\delta(\tau; g_a, g_b) = \Delta(g_a, g_b) - \Delta(g_a, g_b).
\]

Let \( e \in [n] \). Suppose that \( (g_a, g_b)(e) = (1, 1) \), so \( (g_a \land g_b)(e) = 0 \). Then the cycle of \( \tau \) containing \( e \) is \( S = \{ e^i : i \in \mathbb{N} \} \). For all \( e' \in S, (g_a \land g_b)(e') = 0 \) or \( (g_a, g_b)(e') = (0, 1) \). Thus the contribution of \( S \) to \( \Delta(g_a, g_b) \) is equal to total number of edges in \( g_a \) and \( g_b \) in \( S \). The contribution of \( S \) to \( \Delta(g_a \land \tau; g_b) \) cannot be larger.

It is well-known that Erdős-Rényi graphs with average degree less than \( n \) have many automorphisms. The following lemma is precise version of this fact that is suitable for our purposes.

**Lemma V.2.** Let \( G \sim \text{ER}(n, p) \). If \( p \leq \frac{\log n - \omega(1)}{n} \), then there is some sequence \( \epsilon_n \to 0 \) such that \( P[|\text{Aut}(G)|] \leq \epsilon_n^{-1} \leq \epsilon_n \).

This follows easily from the second moment method. Full details can be found in [3].

**Theorem 2.** If \( p \) satisfies

\[
p_{11} \leq \frac{\log n - \omega(1)}{n}
\]

then any estimator for \( \Pi \) given \((G_c, G_b)\) is correct with probability \( o(1) \).

Proof. For all sufficiently large \( n \), we have \( \frac{n_{11}}{n_{10} n_{01}} > 1 \), so from Lemma IV.2 if \( |\Delta(G_a, G_b)| \geq |\Delta(G_a, G_b)| \), then the posterior probability of \( \pi \) is at least as large as the true permutation. From Lemma V.1 there are at least \( |\text{Aut}(G_a \land G_b)| \) such permutations. Thus any estimator for \( \Pi \) succeeds with probability at most \( |\text{Aut}(G_a \land G_b)|^{-1} \). The graph \( G_a \land G_b \) is distributed as \( \text{ER}(n, p_{11}) \). With high probability, the size of the automorphism group of an \( \text{ER}(n, p_{11}) \) graph goes to infinity with \( n \). More precisely, if \( p_{11} \leq \frac{\log n - \omega(1)}{n} \), then from Lemma V.2 there is some sequence \( \epsilon_n \to 0 \) such that

\[
P[|\text{Aut}(G_a \land G_b)|^{-1} \geq \epsilon_n] \leq \epsilon_n.
\]

Any estimator succeeds with probability at most \( 2\epsilon_n \). □

APPENDIX A

GENERATING FUNCTION PROOFS

Let \( \sigma \) be a permutation of \([\ell]\) with a single cycle and let \( x \) and \( y \) be matrices of formal variables indexed by \([2]^\ell \times [2]^\ell \). For \( \ell \in \mathbb{N} \), define the generating function

\[
b_\ell(x, y) = \sum_{g \in [2]^\ell} \sum_{h \in [2]^\ell} x^\mu(g, h) y^\mu(g_\sigma, h).
\]

**Lemma A.1.**

\[
a_\ell(x \circ y, \frac{y_{01} y_{10}}{y_{10} y_{01}}) = b_\ell(x, y)
\]

Proof. Define

\[
a_{\ell, g, h}(w, z) = \mu^{\delta(\sigma; g, h)} b_{\ell, g, h}(x, y) = x^{\mu(g, h)} y^{\mu(g_\sigma, h)}.
\]

For each \( g, h \in [2]^\ell \),

\[
a_{\ell, g, h}(x \circ y, \frac{y_{01} y_{10}}{y_{10} y_{01}}) = (x \circ y)^{\mu(g, h)} \left( \frac{y_{01} y_{10}}{y_{10} y_{01}} \right)^{\delta(\sigma; g, h)}
\]

where (a) follows from Lemma I.1. We have

\[
a_{\ell}(w, z) = \sum_{g, h \in [2]^\ell} a_{\ell, g, h}(w, z)
\]

\[
b_\ell(z, y) = \sum_{g, h \in [2]^\ell} b_{\ell, g, h}(x, y)
\]
so the claimed identity follows. □

Let \( x \) be a matrix of formal variables indexed by \( [2] \times [2] \). For \( \ell \in \mathbb{N} \), define the generating function

\[
    c_\ell(x) = \sum_{f \in [2]^S} x^{\mu(f, f_0 \sigma)}.
\]

\[ \text{(19)} \]

Lemma A.2. For all \( \ell \in \mathbb{N} \),

\[
    b_\ell(x, y) = c_\ell(xy^\top)
\]

where \( y^\top \) is the transpose of \( y \).

Proof. Consider a cyclic \(( [2] \times [2] )\)-ary sequence \(( g, h )\) indexed by \( S \), where \(| S | = \ell \), with the cyclic permutation \( \sigma \). We have

\[
    b_\ell(x, y) = \sum_{g \in [2]^S} \sum_{h \in [2]^S} \prod_{e \in S} x_{g(e), h(e), g(\sigma(e)), h(e)}
\]

\[
    = \sum_{g \in [2]^S} \prod_{e \in S} x_{g(e), h(e), g(\sigma(e)), h(e)}
\]

\[
    = \sum_{g \in [2]^S} (xy^\top)_{g(e), g(\sigma(e))}
\]

\[
    = c_\ell(xy^\top)
\]

where (a) follows because \( h(e) \) appears in only one term of the product over \( S \). (In contrast \( g(e) \) appears in both the \( e \) term and the \( \sigma^{-1}(e) \) term.) □

A. Cyclic sequence bijections

Let \( f \) be a cyclic \([2] \)-ary sequence indexed by \( S \) with no consecutive ones (i.e. there is no \( e \in S \) such that \( f(e) = 1 \) and \( f(\sigma(e)) = 1 \)). Each such \( f \) corresponds to a partition of \( S \) into blocks of size one and two: \( e \in S \) is in the same block as \( \sigma(e) \) when \( f(e) = 1 \). Thus each block either contains a one followed by a zero or just a zero. Call this a cyclic partition of \( S \).

There are \( \mu(f, f \circ \sigma)_{00} \) blocks of size one and \( \mu(f, f \circ \sigma)_{01} = \mu(f, f \circ \sigma)_{10} \) blocks of size two. Define the following generating function for these restricted cyclic sequences:

\[
    d_\ell(u, v) = \sum_{f \in [2]^S: \mu(f, f \circ \sigma)_{11} = 0} u^{\mu(f, f \circ \sigma)_{00}} v^{\mu(f, f \circ \sigma)_{01}}.
\]

Lemma A.3.

\[
    c_\ell(x) = d_\ell(x_00 + x_{11}, x_01x_{10} - x_00x_{11})
\]

Proof. The left side of the equation enumerates cyclic \([2] \)-ary sequences as described in (19). From each such sequence, we can obtain a cyclic partition of \( S \) as follows. If \( f(e) = 0 \) and \( f(\sigma(e)) = 1 \), put \( e \) and \( \sigma(e) \) in a block of size two and tag the block with the formal variables \( x_{01}x_{10} \). If \( f(e) = 0 \) and \( f(\sigma(e)) = 0 \), put \( e \) in a block of size one and tag the block with \( x_{00} \). If \( f(e) = 1 \) and \( f(\sigma(e)) = 1 \), put \( \sigma(e) \) in a block of size one and tag the block with \( x_{11} \).

The right side enumerates cyclically-partitioned cyclic \([2] \)-ary sequences built of the following blocks: \((0, x_{00}), (1, x_{11}), (01, x_{01}x_{10}) \), and \((01, -x_{00}x_{11}) \). Let \( f \) be a cyclic \([2] \)-ary sequence with \( k = \mu(f, f \circ \sigma) \). Then \( f \) can be partitioned in \( 3^{k_0} \) ways: each appearance of \( 01 \) can be produced by \((0, x_{00})\) followed by \((1, x_{11})\), by \((01, x_{01}x_{10})\), and by \((01, -x_{00}x_{11})\).

Thus the total contribution of the partitions of \( f \) to the right hand side is

\[
    (x_{00}x_{11} + x_{01}x_{10} - x_{00}x_{11})^{k_{00}}x_{00}^{k_{10}}x_{11}^{k_{11}} = x_{01}^{k_{00}}x_{10}^{k_{10}}x_{00}^{k_{00}}x_{11}^{k_{11}}.
\]

That is, only the partition that is counted on the left side is not canceled by some other partition. Figure 1 illustrates the partitions compatible with one example of \( f \). □

\[ \text{Fig. 1. An illustration of the cancellations that occur on the right hand side} \]

\[ \text{of the identity in Lemma A.3. There are nine cyclic partitions compatible} \]

\[ \text{with the labeling 110100. Blocks containing 01 are tagged with •} \]

\[ \text{and are tagged with x_{01}x_{10} otherwise. The first} \]

\[ \text{row contains the only partition that produces the correct monomial for the} \]

\[ \text{labeling}. \]

\[ \text{Lemma A.4.} \]

\[
    d_\ell(2u, v) = 2 \sum_{i} \binom{\ell}{2i} u^{\ell-2i} (u^2 + v)^i
\]

Proof. Both sides enumerates cyclically-partitioned cyclic \([2] \)-ary sequences built of the following blocks: \((0, u), (1, u), (01, v)\). On the left side, the \([2] \)-ary labels only serve to distinguish the two types of blocks of size one. The right side enumerates each cyclic \([2] \)-ary sequence \( f \) via its cyclic sequence of differences, \( g : e \mapsto f(\sigma(e)) - f(e) \). Each \( g \) has the same number of ones as negative ones and there are \( 2\binom{\ell}{2i} \) sequences with \( i \) ones and \( i \) negative ones. The ones in \( g \) correspond to appearances of \( 01 \) in \( f \), which can be produced either by the block \((0, u)\) followed by the block \((1, u)\) or by the block \((01, v)\). The other \( \ell-2i \) positions in \( f \) are produced by either \((0, u)\) or \((1, u)\).

\[ \text{□} \]

All of the generating function identities of this section combine to give the following theorem.

Lemma A.5. Let \( \tau \) be a permutation of \( S \) with \( t_\ell \) cycles of length \( \ell \) for each \( \ell \in \mathbb{N} \). Then

\[
    A_{S, \tau}(w, z) = \prod_{\ell \in \mathbb{N}} \left( 2 \sum_{i} \binom{\ell}{2i} \left( \frac{u}{2} \right)^{\ell-2i} \left( \frac{u^2 + v}{4} \right)^i \right)^{t_\ell}
\]

where

\[
    u = w_{00} + w_{01} + w_{10} + w_{11}
\]

\[
    v = w_{00}w_{11}(z_1 - 1) + w_{01}w_{10}(z_1^{-1} - 1)
\]
Proof. For $i,j \in [2]$, let $w_{ij} = x_i y_j$. Let $z = \frac{w_{01} y_{11} - w_{01} y_{10}}{y_{00} y_{11}}$. Then from Lemma A.1 we have $A_{S,\tau}(w, z) = B_{S,\tau}(x, y)$. Combining Lemmas III.1, A.2, and A.3, we have

$$B_{S,\tau}(x, y) = \prod_{\ell \in \mathbb{N}} d_{\ell}(\text{tr}(xy^T), -\text{det}(xy^T))^{\ell}.$$ 

Then $\text{tr}(xy^T) = u$ and

$$\text{det}(xy^T) = \left(z_{00} x_{01}^2 - x_{01} x_{10} y_{01} y_{10}ight) = w_{00} w_{11} + w_{01} w_{10} - x_{01} x_{10} y_{01} y_{10} - x_{00} x_{11} y_{01} y_{10} = w_{00} w_{11} + w_{01} w_{10} - w_{01} w_{10} z_{10} - w_{00} w_{11}.$$

Lemma A.4 completes the proof. \qed

Lemma A.6. Let $u, v \in \mathbb{R}$ such that $u \geq 0$ and $u^2 + 4v \geq 0$. Then for all $\ell \geq 2$, $d_{\ell}(u, v) \leq d_2(u, v)^{\ell/2}$.

Proof. We have

$$d_{\ell}(u, v) = \sum_i \left(\frac{\ell}{2}\right) \left(\frac{u^2}{2}\right)^{\ell/2} \left(\frac{u^2}{4} + v\right)^i \leq \sum_{j} \left(\frac{\ell}{2}\right) \left(1 + (-1)^j\right) \left(\frac{u^2}{2}\right)^{\ell-j} \left(\frac{u^2}{4} + v\right)^j = \sum_{j} \left(\frac{\ell}{2}\right) \left(\frac{u^2}{2}\right)^{\ell-j} \left(\sqrt{\frac{u^2}{4} + v}\right)^j + \sum_{j} \left(\frac{\ell}{2}\right) \left(\frac{u^2}{2}\right)^{\ell-j} \left(-\sqrt{\frac{u^2}{4} + v}\right)^j = \left(\frac{u^2}{2} + \sqrt{\frac{u^2}{4} + v}\right)^\ell + \left(\frac{u^2}{2} - \sqrt{\frac{u^2}{4} + v}\right)^\ell \leq \left(\left(\frac{u^2}{2} + \sqrt{\frac{u^2}{4} + v}\right)^2 + \left(\frac{u^2}{2} - \sqrt{\frac{u^2}{4} + v}\right)^2\right)^{\ell/2} = (u^2 + 2v)^{\ell/2} = d_2(u, v)^{\ell/2}.$$ 

Here (a) uses Lemma A.4 and (b) uses the binomial theorem. In (b), we note that $i$ only appears as in the expression as $2i$, so we switch to a sum over $j$ with a factor of $(1 + (-1)^j)$ to eliminate the odd terms. In (c), we apply the binomial theorem to each term. In (d), we have used a standard $p$-norm inequality: for a vector $x$, $\|x\|_p \leq \|x\|_2$ when $\ell \geq 2$. \qed

Theorem 4. Let $w \in \mathbb{R}^{[2] \times [2]}_{>0}$, and $z \in \mathbb{R}$. Then for all $\ell \geq 2$, $\alpha_\ell(w, z) \leq \alpha_2(w, z)^{\ell/2}$.

Proof. We have

$$\alpha_\ell(w, z) = d_{\ell}(u, v) \geq \frac{w_{01} w_{10} + w_{01} w_{10}}{w_{00} w_{11}} \geq \sqrt{w_{00} w_{11}}.$$ 

Note that for all $w \in \mathbb{R}^{[2] \times [2]}$, we have $\frac{w_{00} + w_{11}}{2} \geq \frac{\sqrt{w_{00} w_{11}}}{2}$. Thus

$$\frac{u^2}{4} \geq \left(\frac{w_{00} + w_{11}}{2}\right)^2 \geq \left(\sqrt{w_{00} w_{11} - w_{01} w_{10}}\right)^2 \geq -v.$$ 

where (a) follows from (11) in Section III-C. \qed

REFERENCES

[1] A. Narayanan and V. Shmatikov, “De-anonymizing social networks,” in Security and Privacy, 2009 30th IEEE Symposium on. IEEE, 2009, pp. 173–187.

[2] P. Pedarsani and M. Grossglauser, “On the privacy of anonymized networks,” in Proceedings of the 17th ACM SIGKDD international conference on Knowledge discovery and data mining. ACM, 2011, pp. 1235–1243.

[3] D. Cullina and N. Kiyavash, “Improved achievable and converse bounds for Erdos-Renyi graph matching,” in Proceedings of the 2016 ACM SIGMETRICS International Conference on Measurement and Modeling of Computer Science. ACM, 2016, pp. 63–72.

[4] E. M. Wright, “Graphs on unlabelled nodes with a given number of edges,” Acta Mathematica, vol. 126, no. 1, pp. 1–9. 1971.

[5] B. Bollobas, “Distinguishing Vertices of Random Graphs,” North-Holland Mathematics Studies, vol. 62, pp. 33–49, Jan. 1982.

[6] L. Yartseva and M. Grossglauser, “On the performance of percolation graph matching,” in Proceedings of the first ACM conference on Online social networks. ACM, 2013, pp. 119–130.

[7] E. Kazemi, H. S Hamed, and M. Grossglauser, “Growing a Graph Matching from a Handful of Seeds,” in Proceedings of the VLDB Endowment International Conference on Very Large Data Bases, vol. 8, 2015.

[8] E. Kazemi, L. Yartseva, and M. Grossglauser, “When Can Two Unlabeled Networks Be Aligned Under Partial Overlap?” in Proceedings of the 53rd Annual Allerton Conference on Communication, Control, and Computing, 2015.

[9] S. Ji, W. Li, M. Srivatsa, and R. Beyah, “Structural Data De-anonymization: Quantification, Practice, and Implications,” in Proceedings of the 2014 ACM SIGSAC Conference on Computer and Communications Security. ACM, 2014, pp. 1040–1053.

[10] S. Ji, W. Li, N. Z. Gong, P. Mittal, and R. Beyah, “On Your Social Network De-anonymizability: Quantification and Large Scale Evaluation with Seed Knowledge,” 2015.

[11] V. Chv´atal, “The tail of the hypergeometric distribution,” Discrete Mathematics, vol. 25, no. 3, pp. 285–287, 1979.

[12] B. Bollobas, Random graphs. Springer, 1998.