Noncommutative open strings from Dirac quantization

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Abstract

We study Dirac commutators of canonical variables on D-branes with a constant Neveu-Schwarz 2-form field by using the Dirac constraint quantization method, and point out some subtleties appearing in previous works in analyzing constraint structure of the brane system. Overcoming some ad hoc procedures, we obtain desirable noncommutative coordinates exactly compatible with the result of the conformal field theory in recent literatures. Furthermore, we find interesting commutator relations of other canonical variables.

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Noncommutative geometry arises in D-branes on constant antisymmetric tensor fields [1], and gauge theories on the noncommutative space have been studied by Seiberg and Witten [2,3]. The fundamental commutator relations of open string coordinates $x^i$ on the branes are represented by noncommutativity, $[x^i(0, \tau), x^j(0, \tau)] = i\theta^{ij}$ where $\theta^{ij}$ is an antisymmetric constant tensor depending on the background constant fields. Various aspects of the noncommutativity on the branes are extensively studied in Refs. [4–8]. Furthermore, the noncommutativity in the Matrix Model with non-trivial three form field has been observed in Ref. [1]. Especially, the noncommutativity between D-brane coordinates is shown in terms of Green’s function method [3] and string mode expansion method [7,8].

In recent studies [9–11], it has been suggested that noncommutative coordinates on the brane coupled to constant antisymmetric background fields can be derived from the Hamiltonian formulation of the system by treating the mixed boundary condition as a primary constraint. Another intriguing point adopted in Refs. [9,11] for the Hamiltonian formulation is to discretize the string world sheet as $X_n$ ($n = 0, 1, 2\cdots$) where $X_0$ describes the boundary of the open string and the others are for the bulk, and the noncommutative structure of the boundary and the bulk part can be studied respectively. On the other hand, the stability of the primary constraint with respect to time gives infinite number of secondary constraints where the Lagrangian multiplier $u^i$ implemented by the primary constraint was determined as $u^i = 0$. The resulting constraints of the primary and the secondary constraints form second class algebra, and Dirac brackets between the coordinates on the brane yield the noncommutative coordinates. At first sight, however, the noncommutativity seems to appear both at the boundary and partially at the bulk [3,11]. In Ref. [11], the Dirac quantization of this system was performed with the infinite number of secondary constraints and obtained the noncommutative coordinate relations at the boundary, however, it is more or less formal and the regularization of $\delta$-function is needed. In Refs. [3,11], the essential subtlety is due to the fact that the canonical Hamiltonian instead of the primary Hamiltonian has been used in the stability condition of the primary constraint, which it is different from the conventional Dirac quantization procedure [13].
In this paper, we shall reconsider Dirac constraint quantization of the brane on the constant antisymmetric backgrounds and obtain consistent noncommutative commutators of canonical variables including momenta. Compared to the previous works [9–11], the crucial difference comes from the choice of the nonvanishing multiplier in the primary Hamiltonian, which is determined by the stability condition of the primary constraint. In our case, there are no more secondary constraints, instead, the single primary constraint itself forms the second class constraint algebra, which gives various kinds of interesting commutators as well as the expected noncommutative coordinates. In our analysis, the coordinates in the bulk part are definitely commutative, which is different from some earlier works [9,10].

Before proceeding, let us exhibit self-dual constraints appearing in chiral quantum mechanics [12] as a simple illustration of noncommutative coordinates, which is very similar to the constraint system in the open string theory. Then the Lagrangian is given by

$$L = -\frac{e}{2c} B \dot{x}^i \epsilon_{ij} x^j + e\phi(x),$$

where it is obtained from a charged point particle coupled to the static potential $\phi$ on a strong magnetic field background in three dimensions. The canonical momenta are

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = -\frac{e}{2c} B \epsilon_{ij} x^j,$$

which becomes a primary constraint in this theory. This is called simply self-dual constraint in that the momenta are represented by the coordinates

$$\omega_i = p_i + \frac{e}{2c} B \epsilon_{ij} x^j \approx 0.$$  

Considering a primary Hamiltonian [13],

$$H_p = H_c + u^i \omega_i,$$

where $H_c = -e\phi$ and $u^i$ is a multiplier, the stability condition of the primary constraint with respect to time yields

$$\{\omega_i, H_p\} = e \partial_i \phi + \frac{e}{c} B \epsilon_{ij} u^j,$$

$$\approx 0.$$
If the magnetic field is nonvanishing, it is possible to fix the multiplier, whereas for $B \to 0$ the additional constraint called a secondary constraint can be generated, so the primary constraint together with the secondary constraint form second class constraint algebra. However, in our case, the primary constraint itself becomes a second class constraint and the multiplier can be naturally fixed.

Now the consistent brackets with the primary constraint (3) are defined as

$$[\mathcal{A}, \mathcal{B}] = i\mathcal{A} \{ \mathcal{A}, \mathcal{B} \} + i\mathcal{A} \omega_i e^{-i\epsilon \epsilon} \omega_j \mathcal{B},$$

where $\mathcal{A}$ and $\mathcal{B}$ are canonical variables and the resulting commutators are

$$[x^i, x^j] = -i \frac{c}{eB} \epsilon^{ij},$$

$$[x^i, p^j] = \frac{i}{2} \mathcal{G}^{ij},$$

$$[p^i, p^j] = -i \frac{c}{4\epsilon} B \epsilon^{ij}.$$

The commutators between the coordinates are noncommuting as well as those of the momenta. In fact, this curious feature is due to the constraint (3) to be imposed on the phase space, which will be studied in the context of the open string theory.

We now consider the open string theory coupled to constant Neveu-Schwarz 2-form fields and $U(1)$ gauge field background given by

$$S = \frac{1}{4\pi\alpha'} \int d\sigma^2 \left[ \partial_a X^i \partial^a X_i + 2\pi\alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j \right]$$

$$+ \int d\tau A_i \partial_\tau X^i|_\pi - \int d\tau A_i \partial_\tau X^i|_0$$

where it has a local $U(1)$ gauge invariance. If both ends of a string attached to the same brane, the last two boundary term in Eq. (8) can be written as $-\frac{1}{2\pi\alpha'} \int d\tau F_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j$, and the action (8) becomes

$$S = \frac{1}{4\pi\alpha'} \int d\sigma^2 \left[ \partial_a X^i \partial^a X_i + 2\pi\alpha' F_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j \right]$$

and the two form field $F = B - F = B - dA$ is invariant under both $U(1)$ and $\Lambda$ transformation defined as $B \to B + d\Lambda$ and $A \to A + \Lambda$. We simply take $F_{ij} = 0$ in Eq. (8) for convenience.
Varying the action \( S \) gives the equation of motion, \( \partial_a \partial^a X^i = 0 \) with the mixed boundary condition

\[
g_{ij} \partial_a X^j + 2 \pi \alpha' B_{ij} \partial_a X^j|_{\sigma=0, \pi} = 0. \tag{10}
\]

Without the constant background fields \( B_{ij} \), the boundary conditions (10) are reduced to Neumann boundary conditions of the open string theory. To distinguish the boundary and the bulk, we discretize our action (8) along the \( \sigma \) parameter \([9]\) and the resulting discretized Lagrangian becomes

\[
L = \frac{1}{4 \pi \alpha'} \sum_n \left[ \epsilon (X^i_n)^2 - \frac{1}{\epsilon} (X^i_{n+1} - X^i_n)^2 + 4 \pi \alpha' B_{ij} \dot{X}^i_n (X^j_{n+1} - X^j_n) \right] \tag{11}
\]

where we take equal spacing, \( \int d\sigma = \epsilon \sum_n \) where the overdot means the derivative with respect to the time-like parameter \( \tau \). And we also discretize the mixed boundary condition (10) as

\[
\frac{g_{ij}}{\epsilon} (X^j_1 - X^j_0) + 2 \pi \alpha' B_{ij} \dot{X}^j_0 = 0 \tag{12}
\]

where it is simply denoted at the one boundary \( \sigma = 0 \) for convenience. From the discretized Lagrangian (11), we obtain canonical momenta given as

\[
2 \pi \alpha' P_{ni} = \left[ \epsilon \dot{X}_{ni} + 2 \pi \alpha' B_{ij} (X^j_{n+1} - X^j_n) \right], \tag{13}
\]

where \( n = 0 \) and \( n = 1, 2, \cdots \) denote the coordinates on the brane and the string bulk, respectively. According to the usual Dirac’s procedure \([13]\), one can define the mixed boundary condition (12) as a primary constraint by using Eqs. (12) and (13),

\[
\Omega_i = \frac{1}{\epsilon} \left[ (2 \pi \alpha')^2 B_{ij} P^j_0 - (2 \pi \alpha')^2 B_{ij} B^{jk} (X_{1k} - X_{0k}) + g_{ij} (X^i_1 - X^i_0) \right] \approx 0. \tag{14}
\]

Then the primary hamiltonian can be constructed by performing the Legendre transformation of Eq.(11) and by introducing the primary constraint implemented by the multiplier \( u^i(\tau) \), which is given as

\[
H_p = H_c + u^i \Omega_i \]
\[
= \frac{1}{4 \pi \alpha' \epsilon} \sum_n \left[ (2 \pi \alpha')^2 \left( P^i_n - B^{ij} (X^j_{n+1} - X^j_n) \right)^2 + (X^i_{n+1} - X^i_n)^2 \right] + u^i \Omega_i, \tag{15}
\]
where $H_c$ is a canonical Hamiltonian.

Using Poisson brackets defined by

$$
\{X^i_n, X^j_m\} = 0 = \{P^i_n, P^j_m\},
$$
$$
\{X^i_n, P^j_m\} = g^{ij} \delta_{nm},
$$
(16)

the time evolution of the primary constraint (14) yields

$$
\{\Omega_i, H_p\} = -\frac{2\pi\alpha'}{\epsilon^2} \left[ B_{ij} \left( (2\pi\alpha')^2 P_0 B - (2\pi\alpha')^2 B^2 (X_1 - X_0) - (X_1 - X_0) \right)^j
- \left( g - (2\pi\alpha')^2 B^2 \right)_{ij} (P_0 - B(X_1 - X_0))^j
+ \left( g - (2\pi\alpha')^2 B^2 \right)_{ij} (P_1 - B(X_2 - X_1))^j
+ 2(2\pi\alpha') B_{ij} (g - (2\pi\alpha')^2 B^2)^k u^k(\tau) \right]
$$
(17)

where this condition determines the multiplier rather than it generates additional constraints since $B_{ij}$ is invertible [3]. In Refs. [9–11], however, infinite chain of constraints appeared as secondary constraints and it is different from the usual Dirac quantization procedure. The essential reason why the multiplier $u^i$ can be fixed is due to the Poisson algebra of the primary constraint (14) which yields second class constraint algebra as

$$
\{\Omega_i, \Omega_j\} = \frac{2}{\epsilon^2} (2\pi\alpha')^2 \left[ (g - 2\pi\alpha' B) B (g + 2\pi\alpha' B) \right]_{ij}.
$$
(18)

It would be interesting to note that for $B_{ij} \to 0$ the primary constraint becomes first class constraint, in that case, the time evolution of the primary constraint by using the primary Hamiltonian gives secondary constraint. These constraints form the second class constraint algebra and the Lagrangian multiplier can be fixed.

Returning to our analysis, let us now construct the Dirac matrix defined by

$$
C_{ij} = \{\Omega_i, \Omega_j\},
$$
(19)

and the inverse matrix can be obtained as

$$
C^{-1}_{ij} = \frac{\epsilon^2}{2(2\pi\alpha')^2} \left[ \frac{1}{(g + (2\pi\alpha') B)} \frac{1}{B} \frac{1}{(g - (2\pi\alpha') B)} \right]_{ij}.
$$
(20)
Then we can calculate Dirac commutators straightforwardly by means of the following definition

\[ [A, B] = i\{A, B\} - i\{A, \Omega_i\}C^{ij}^{-1}\{\Omega_j, B\}. \] (21)

The small constant parameter \(\epsilon\) can be canceled out in Dirac commutators and commutator relations are valid for the prescription \(\epsilon \to 0\) after finishing all Dirac procedures. The resulting commutators at the boundary \((\sigma = 0)\) of the open string are

\[
[X_0^i, X_0^j] = -\frac{i}{2}(2\pi\alpha')^2 \left[ \frac{1}{(g + (2\pi\alpha')B)} B \frac{1}{(g - (2\pi\alpha')B)} \right]^{ij},
\] (22)

\[
[X_0^i, P_0^j] = \frac{i}{2}g^{ij},
\] (23)

\[
[P_0^i, P_0^j] = \frac{i}{2(2\pi\alpha')^2} \left[ (g + (2\pi\alpha')B) \frac{1}{B} (g - (2\pi\alpha')B) \right]^{ij}.
\] (24)

If we redefine the phase space variables as \(X_0^i \to \frac{1}{\sqrt{2}} X_0^i\), then the Dirac commutator of the coordinates (22) is equivalent to the result derived from the propagator by Seiberg and Witten in Ref. [3]. The commutator relations between the canonical momenta are nontrivial, which is similar to the form of the point particle case on the presence of the magnetic field.

Note that for a point particle limit of \(\alpha' \to 0\), the commutation relation for the coordinates (22) is reduced to \([x^i(\tau), x^j(\tau)] = i(B^{-1})^{ij}\) whose form is the same with that of Eq.(7) with a constant redefinition.

On the other hand, at the nearest point from the boundary for \(n = 1\), the Dirac commutators are

\[
[X_1^i, X_1^j] = 0,
\] (25)

\[
[X_1^i, P_1^j] = ig^{ij},
\] (26)

\[
[P_1^i, P_1^j] = \frac{i}{2(2\pi\alpha')^2} \left[ (g + (2\pi\alpha')B) \frac{1}{B} (g - (2\pi\alpha')B) \right]^{ij}.
\] (27)

Note that commutator relations between the coordinates \(X_1^i\) in Eq.(25) imply that they are commuting each other, which is in contrasted with the result of the noncommutative coordinates [3]. Next, the cross commutation relations are also given as
\[ [X^i_0, X^j_1] = 0, \]
\[ [X^i_0, P^j_1] = \frac{i}{2} g^{ij}, \]
\[ [X^i_1, P^j_0] = 0, \]
\[ [P^i_0, P^j_1] = -[P^i_0, P^j_0]. \quad (28) \]

Furthermore, the other Dirac commutators for the string bulk, i.e. \( n = 2, 3, \ldots \), are the same with the usual Poisson brackets,
\[ [X^i_n, X^j_m] = 0 = [P^i_n, P^j_m], \]
\[ [X^i_n, P^j_m] = i\delta_{nm} g^{ij}. \quad (29) \]

From Eqs. (22)-(29), we find that the boundary coordinate \( X^i_0 \) is nontrivially correlated with the other canonical variables for \( n = 0, 1 \), whereas the coordinate \( X^i_1 \) is commuting with other canonical variables except its conjugate momentum \( P^i_1 \).

The similar commutators for the \( \sigma = \pi \) boundary of the D-brane are given as
\[ [X^i_\pi, X^j_\pi] = \frac{i}{2} (2\pi\alpha')^2 \left[ \frac{1}{(g + (2\pi\alpha')B)}B \frac{1}{(g - (2\pi\alpha')B)} \right]^{ij}, \]
\[ [X^i_\pi, P^j_\pi] = \frac{i}{2} g^{ij}, \]
\[ [X^i_{\pi-\epsilon}, X^j_{\pi-\epsilon}] = 0, \]
\[ [P^i_\pi, P^j_\pi] = -\frac{i}{2(2\pi\alpha')^2} \left[ (g + (2\pi\alpha')B) \frac{1}{B} (g - (2\pi\alpha')B) \right]^{ij} \]
\[ = [P^i_{\pi-\epsilon}, P^j_{\pi-\epsilon}] \]
\[ = -[P^i_{\pi-\epsilon}, P^j_{\pi-\epsilon}], \]
\[ [X^i_\pi, P^j_{\pi-\epsilon}] = \frac{i}{2} g^{ij}, \]
\[ [X^i_{\pi-\epsilon}, P^j_{\pi-\epsilon}] = ig^{ij}, \]
\[ [X^i_{\pi-\epsilon}, P^j_\pi] = 0 = [X^i_\pi, X^j_{\pi-\epsilon}]. \quad (30) \]

As a result, the noncommutativity of the coordinates emerges only on the branes not in the string bulk.

On the other hand, commutation relation of the center of mass(CM) coordinate is calculated as
\[ [X^i_{\text{CM}}, X^j_{\text{CM}}] = \left( \frac{\epsilon}{\pi} \right)^2 \sum_n \sum_m [X^i_n, X^j_m] \]
\[ = \left( \frac{\epsilon}{\pi} \right)^2 \left( [X^i_0, X^j_0] + [X^i_\pi, X^j_\pi] \right) \]
\[ = i \left( \frac{\epsilon}{\pi} \right)^2 (2\pi\alpha')^2 \left[ \left( (\tilde{g} - (2\pi\alpha')^2\tilde{B}^2)^{-1}\tilde{B} \right)^{ij} - \left( (g - (2\pi\alpha')^2B^2)^{-1}B \right)^{ij} \right] \] (31)

where we used the definition of the center of mass coordinate as
\[ X^i_{\text{CM}} = \frac{1}{\pi} \int_0^\pi d\sigma X^i = \frac{\epsilon}{\pi} \sum_n X^i_n. \] The only boundary commutators at \( \sigma = 0 \) and \( \sigma = \pi \) contribute to the center of mass commutation relation. The tilde fields are defined at the \( \sigma = \pi \) boundary of the brane. As discussed in [9], for the \( MM \) brane system, the center of mass commutator is vanishing while for the \( MM' \) brane system it is vanishing in this discretized version. In Ref. [9], noncommutative coordinate relations of the boundary and the bulk contribute to the calculation of the commutation relation of the center of mass coordinates. In our case, if \( \epsilon \to 0 \), then the commutator relation of the center of mass coordinates (31) is always vanishing. This fact seems to be plausible in that the CM coordinate belongs the bulk part which is commuting.

It seems to be appropriate to comment on the nontrivial commutator relations between coordinates and momenta. At first sight, one might wonder why the coordinate \( X^i_0 \) is related to the momentum \( P^i_1 \) through the commutation relation as seen in Eq.(28). As a matter of fact, this is due to the structure of the primary constraint (14) since it contains the canonical variables \( X^i_1 \) and affects the Dirac bracket. So the constraint is not only defined at the boundary point \( X^i_0 \) but also it is related to the next point \( X^i_1 \) through the derivative at \( \sigma = 0 \) in the mixed boundary condition (14). Strictly speaking, the coordinates are noncommutative only at the boundary; however, the canonical variables including momenta at the boundary \( \sigma = 0 \) are correlated up to the next canonical variables. As for the canonical momenta, they are noncommutative for \( n = 0, 1 \), whereas for \( n = 2, 3, \cdots \) they are commuting.

In conclusions, we have shown that the noncommutative coordinates can be obtained at the boundary of open strings on the constant antisymmetric fields, while the bulk coordinates as expected follow the usual commuting relations, which has been studied in the context of
the Dirac constraint quantization method compatible with the propagator method used in
the conformal field theory.

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REFERENCES

[1] A. Connes, M. R. Douglas, and A. Schwarz, *Noncommutative Geometry and Matrix Theory: Compactification on Tori*, JHEP 02 (1998) 003, [hep-th/9711162](http://arxiv.org/abs/hep-th/9711162).

[2] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. B460 (1996) 335, [hep-th/9510135](http://arxiv.org/abs/hep-th/9510135).

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP 09 (1999) 032, [hep-th/9908142](http://arxiv.org/abs/hep-th/9908142).

[4] M. R. Douglas and C. Hull, *D-branes and the noncommutative torus*, JHEP 02 (1998) 008, [hep-th/9711165](http://arxiv.org/abs/hep-th/9711165).

[5] M. M. Sheikh-Jabbari, *More on mixed boundary conditions and D-branes bound states*, Phys. Lett. B425 (1998) 48, [hep-th/9712199](http://arxiv.org/abs/hep-th/9712199).

[6] F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, JHEP 02 (1999) 016, [hep-th/9810072](http://arxiv.org/abs/hep-th/9810072).

[7] H. Arfaei and M. M. Sheikh-Jabbari, *Mixed boundary conditions and brane-string bound states*, Nucl. Phys. B526 (1998) 278, [hep-th/9709054](http://arxiv.org/abs/hep-th/9709054).

[8] C.-S. Chu and P.-M. Ho, *Non-commutative open string and D-brane*, Nucl. Phys. B550 (1999) 151, [hep-th/9812219](http://arxiv.org/abs/hep-th/9812219).

[9] F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, *Dirac Quantization of Open Strings and Noncommutativity in Branes*, [hep-th/9906161](http://arxiv.org/abs/hep-th/9906161).

[10] M. M. Sheikh-Jabbari and A. Shirzad, *Boundary Conditions as Dirac Constraints*, [hep-th/9907055](http://arxiv.org/abs/hep-th/9907055).

[11] C.-S. Chu and P.-M. Ho, *Constrained Quantization of Open String in Background B Field and Noncommutative D-brane*, [hep-th/9906192](http://arxiv.org/abs/hep-th/9906192).
[12] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, [hep-th/9908056](http://arxiv.org/abs/hep-th/9908056).

[13] P. A. M. Dirac, *Lectures on quantum mechanics* (Belfer Graduate School of Science, Yeshiva U.P., New York, 1964).