Schwinger Pair Production at Finite Temperature in QED

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We use the evolution operator method to find the Schwinger pair-production rate at finite temperature in scalar and spinor QED by counting the vacuum production, the induced production and the stimulated annihilation from the initial ensemble. It is shown that the pair-production rate for each state is factorized into the mean number at zero temperature and the initial thermal distribution for bosons and fermions.

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I. INTRODUCTION

Vacuum polarization and pair production have been issues of continuous concern since the early works by Sauter, Heisenberg and Euler, and Weisskopf [1], and then by Schwinger [2] (for a review and references, see Ref. [3]). The task of directly computing, without relying on the electromagnetic duality, the effective action in electric field backgrounds, however, has been a challenging problem due to the vacuum instability. Dunne and Hall used the resolvent method to directly find the effective action in time-dependent electric fields [4]. In the previous paper [5], employing the evolution operator method, we found the exact one-loop effective actions of scalar and spinor QED at zero temperature in a constant or a pulsed electric field of Sauter-type, which satisfy the exact relation $2\text{Im}L_{\text{eff}} = \pm \sum_n \ln(1 \pm N_n)$ (with + for scalar and − for spinor) between the imaginary part of the effective Lagrangian density $L_{\text{eff}}$ and the mean number of created pairs $N_n$ at state $n$. Even finding the pair production rate by time-dependent or spatially localized electric fields is methodologically nontrivial, which has recently been intensively studied [6, 7, 8].

To calculate the effective action and thereby Schwinger pair production at finite temperature is another challenging problem in QED. In a constant pure magnetic field the QED effective action was studied at finite temperature [9] and at finite temperature and density [10]. However, the presence of an additional electric field raised Schwinger pair production at debate depending on the formalism employed. Some calculations in thermal field theory reported that the effective action in both a constant magnetic and electric field had an imaginary part having dependence on temperature [11, 12, 13]. However, in the real-time formalism, no imaginary part was found in the QED effective action in the presence of both magnetic and electric field [14]. The recent calculation of effective action in the imaginary-time formalism shows an imaginary part only at two-loop but not at one-loop in both a constant electric and magnetic field [15]. In nonequilibrium quantum field theory of scalar QED, a calculation in the real-time formalism shows thermal enhancement of pair production [16].

In this paper, using the evolution operator method, we find the pair-production rate at finite temperature in time-dependent electric fields both in scalar and spinor QED. The evolution operator, unitarily transforming the particle and antiparticle operators from the ingoing vacuum to the outgoing vacuum, carries all the information of quantum evolution. In fact, the evolution operator is completely determined by the Bogoliubov coefficients. The advantage of the evolution operator is the readiness to calculate the probability for transitions among multiparticle states. This allows us to compute the mean number of created pairs at finite temperature in scalar and spinor QED by counting the pairs from the vacuum and the induced production and the stimulated annihilation in a thermal ensemble of bosons and fermions. We find that the mean number of created pairs is factorized into the mean number of created pairs at zero temperature and the initial thermal distribution for bosons and fermions.

The organization of this paper is as follows. In Sec. II, rewriting the Bogoliubov transformation as a unitary transformation by the evolution operator, we find the mean number of created pairs at zero temperature in scalar
and spinor QED. In Sec. III, we calculate the mean number of created pairs at finite temperature and apply it to the Sauter-type electric field.

II. EVOLUTION OPERATOR AND PAIR PRODUCTION AT $T = 0$

We first consider scalar QED for spinless charged bosons under an external electric field with the gauge field $A_\mu$. The electric field is assumed to be acting on for a finite period of time so that the ingoing and the outgoing vacua are well-defined. Thus, at $t_{\text{in}} = -\infty$, before the external electric field being turned on, the scalar field is free and the Hamiltonian takes the usual form

$$H_{\text{in}}^{\text{sc}} = \int \frac{d^3k}{(2\pi)^3} \omega_{k,\text{in}} N_{k,\text{in}} = \int \frac{d^3k}{(2\pi)^3} \omega_{k,\text{in}}(a_{k,\text{in}}^\dagger a_{k,\text{in}} + b_{k,\text{in}}^\dagger b_{k,\text{in}}),$$

where $\omega_{k,\text{in}}$ is the initial frequency at momentum $k$. Here, the gauge is chosen $A_\mu = 0$, so that the ingoing vacuum $|0; t_{\text{in}}\rangle$ is the Minkowski vacuum $|0\rangle_M$, annihilated by $a_{k}(t_{\text{in}})$ and $b_{k}(t_{\text{in}})$ for each momentum $k$. Similarly, the outgoing vacuum at $t_{\text{out}} = \infty$ is defined by $a_{k}(t_{\text{out}})$ and $b_{k}(t_{\text{out}})$. These operators are related through the Bogoliubov transformations $[16]$

$$a_{k,\text{out}} = \mu_k a_{k,\text{in}} + \nu_k b_{k,\text{in}}^\dagger, \quad b_{k,\text{out}} = \mu_k b_{k,\text{in}} + \nu_k a_{k,\text{in}}^\dagger,$$

where $|\mu_k|^2 - |\nu_k|^2 = 1$.

To express the outgoing vacuum as multiparticle states of the ingoing vacuum, we rewrite the Bogoliubov transformations $[2]$ as a unitary transformation $[5]$

$$a_{k,\text{out}}(A) = U_k(A)a_{k,\text{in}}(0)U_k^\dagger(A), \quad b_{k,\text{out}}(A) = U_k(A)b_{k,\text{in}}(0)U_k^\dagger(A).$$

Here, the evolution operator $U_k$ is factorized into the overall phase factor and the two-mode squeeze operator as $[17, 18]$

$$U_k(A) = e^{i\theta_k}(s_{k,\text{in}}^a s_{k,\text{in}}^b + b_{k,\text{in}}^\dagger b_{k,\text{in}}^\dagger + 1) e^{\xi_k s_{k,\text{in}}^a b_{k,\text{in}}^\dagger} e^{\gamma_k} e^{-\xi_k a_{k,\text{in}}^\dagger b_{k,\text{in}}},$$

where

$$e^{2i\theta_k} = \frac{\mu_k^*}{\mu_k}, \quad \xi_k = \frac{\nu_k^*}{\mu_k}, \quad \gamma_k = -2\ln(\mu_k).$$

From the charge neutrality of the vacuum, equal numbers of particles and antiparticles are produced at zero temperature and they carry the opposite momenta due to the momentum conservation. The multiparticle state of $n$-pairs can be concisely denoted as $|n_k, t\rangle = |n_k; \bar{n}_k; t\rangle$. The probability for $n$-pairs with momentum $k$ to be created from the vacuum is

$$P_n(k) = \langle |n_k, \text{out}; 0, \text{in} \rangle^2 = \langle |n_k, \text{in} |U_k^\dagger(0, \text{in}) |n_k, \text{out} \rangle^2 = e^{2\gamma_k} \xi_k^{2n}.$$

Note that $P_0 = e^{\gamma_k}$ and $P_1 = e^{\gamma_k} |\xi_k|^2$ so that $P_n = P_0(P_1/P_0)^n$ and $\sum_{n=0}^\infty P_n = 1$ for each $k$. Thus, at zero temperature, the mean number of pairs created from the vacuum for each momentum per unit volume is

$$\overline{N}_k^{\text{sc}}(T = 0) = \sum_{n=0}^\infty n P_n(k) = |\nu_k|^2.$$

Next, in spinor QED, before the interaction of an external electric field, the spinor field is free without the gauge potential ($A_\mu = 0$), and has the Hamiltonian given by

$$H_{\text{in}}^{\text{sp}} = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \omega_{n,\text{in}} N_{n,\text{in}} = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \omega_{n,\text{in}}(b_{n,\text{in}}^\dagger b_{n,\text{in}} + d_{n,\text{in}}^\dagger d_{n,\text{in}}),$$

where $b_{n,\text{in}}$ and $d_{n,\text{in}}$ are particle and antiparticle operators in the ingoing vacuum. After the interaction of the electric field, the ingoing vacuum evolves to the outgoing vacuum, whose particle and antiparticle operator are $b_{n,\text{out}}$ and $d_{n,\text{out}}$. 
and \(d_{n,\text{out}}\). The Bogoliubov transformations between the ingoing and the outgoing particle and antiparticle operators, \(b_n, d_n\), are similarly given by

\[
b_{n,\text{out}} = \mu_n b_{n,\text{in}} + i\nu_n^* d_{n,\text{in}}, \quad d_{n,\text{out}} = \mu_n d_{n,\text{in}} - i\nu_n^* b_{n,\text{in}},
\]

where \(|\mu_n|^2 + |\nu_n|^2 = 1\) and \(n = (k, \sigma)\) with \(\sigma = \pm 1/2\). The Bogoliubov transformation can be also written as a unitary transformation

\[
b_{n,\text{out}} = U_n b_{n,\text{in}} U_n^+, \quad d_{n,\text{out}} = U_n d_{n,\text{in}} U_n^+, \quad \text{where}
\]

\[
U_n = e^{\xi_n b_{n,\text{in}}^0 d_{n,\text{in}}^0} e^{i(\omega_n + i\theta_n)(b_{n,\text{in}}^0 b_{n,\text{in}} + d_{n,\text{in}}^0 d_{n,\text{in}} - 1)} e^{i\nu_n^* \xi_n^* b_{n,\text{in}}^0 d_{n,\text{in}}^0}.
\]

Here, the three parameters \(\xi_n, \gamma_n\) and \(\theta_n\) are determined by the Bogoliubov coefficients as

\[
\xi_n = -i\frac{\nu_n^*}{\mu_n}, \quad \gamma_n = -2\ln(|\mu_n|), \quad e^{2i\theta_n} = \frac{|\mu_n|^2}{\mu_n}.
\]

Note that the pair production on spinor QED is restricted to only one pair of particle and antiparticle for a given quantum number \(n\) due to the Pauli exclusion principle. Thus, the mean number of pairs created from the vacuum for each state \(n\) at zero temperature is calculated as

\[
\overline{N}_n^0 (T = 0) = |\langle 1_n, \text{out} | 0_n, \text{in} \rangle|^2 = |\mu_n|^2.
\]

### III. Pair Production at \(T \neq 0\)

We now calculate the mean number of pairs at finite temperature from the probability for each transition, as in the case of zero temperature. As there is no mode-mixing, we separately calculate the mean number of created pairs for each mode. For an initial thermal ensemble at \(\beta = 1/kT\), which might not be charge neutral, the mean number of produced pairs consists of the vacuum pair production, the induced pair production, and the stimulated pair annihilation as shown in Fig. 1:

\[
\overline{N}_n(T) = \frac{1}{Z_n} \left[ \sum_{n_n > 0} \sum_{p_n > m_n} e^{-\beta E_{p_n,m_n}} \langle n_n, m_n | \sum_{p_n > m_n} P_{n_n, m_n \rightarrow n_n, m_n} (p_n - n_n) P_{p_n, q_n \rightarrow n_n, m_n} \right]
\]

\[
\overline{N}_n(T) = \frac{1}{2Z_n} \sum_{p_n > m_n, q_n > m_n} \sum_{n_n > 0} e^{-\beta E_{p_n,m_n}} \langle n_n, m_n | (N_{n_n,m_n} - U_n^U_n N_{n,n}) n_n, m_n, \in \rangle \langle p_n, q_n, \in | U_n^U_n N_{n_n,m_n} n_n, m_n, \in \rangle \langle p_n, q_n, \in | (N_{n_n,m_n} - U_n^U_n N_{n,n}) n_n, m_n, \in \rangle \langle n_n, m_n | U_n N_{n,n} U_n^U_n n_n, m_n, \in \rangle \langle n_n, m_n | U_n N_{n,n} U_n^U_n n_n, m_n, \in \rangle e^{-\beta H_{n,n}}
\]

where \(N_{n,n} = a_{n,n}^+ a_{n,n} + b_{n,n}^+ b_{n,n}^+ \) for scalar particles and \(N_{n,n} = a_{n,n}^+ a_{n,n} + b_{n,n}^+ b_{n,n}^+ \) for spinor particles and \(H_{n,n} = \omega_{n,n} N_{n,n} \) for both.

From the Bogoliubov transformations (3) and (10), we have \(N_{n,\text{out}} = U_n N_{n,n} U_n^U_n\). Thus, the mean number of pairs created at finite temperature can be written concisely as

\[
\overline{N}_n(T) = \frac{1}{2Z_n} \text{Tr}(N_{n,\text{out}} - N_{n,n}) e^{-\beta H_{n,n}}.
\]
First, for scalar particles, from Eq. (2),

\[ N_{k,\text{out}} - N_{k,\text{in}} = 2|\nu_k|^2(a^\dagger_{k,\text{in}}a_{k,\text{in}} + b^\dagger_{k,\text{in}}b_{k,\text{in}} + 1) + 2\mu_k\nu_k a_{k,\text{in}}b_{k,\text{in}} + 2\mu_k^*\nu_{k,\text{in}}^*b^\dagger_{k,\text{in}}a^\dagger_{k,\text{in}}, \]  

(17)

and \( Z_k = e^{\beta \omega_{k,\text{in}}}(2 \sinh \beta \omega_{k,\text{in}}/2)^{-2} \), the mean number of scalar pairs created in momentum \( k \) at finite temperature is given by

\[ \overline{N}^c_k(T) = \frac{|\nu_k|^2}{Z_k} \sum_{n,m=0}^{\infty} \langle n, m | (n + m + 1)e^{-(n+m)\beta \omega_{\text{in}}} | n, m \rangle = |\nu_k|^2 \coth \frac{\beta \omega_{k,\text{in}}}{2}. \]  

(18)

Second, for spinor particles, from Eq. (9),

\[ N_{n,\text{out}} - N_{n,\text{in}} = -2|\nu_n|^2(b^\dagger_{n,\text{in}}b_{n,\text{in}} + d^\dagger_{n,\text{in}}d_{n,\text{in}} - 1) + 2\mu_n\nu_n b_{n,\text{in}}d_{n,\text{in}} + 2\mu_n^*\nu_{n,\text{in}}^*d^\dagger_{n,\text{in}}b^\dagger_{n,\text{in}}, \]  

(19)

and \( Z_n = (1 + e^{-\beta \omega_{n,\text{in}}})^2 \), we find the mean number of spinor pairs created in state \( n \) at finite temperature

\[ \overline{N}^p_n(T) = -\frac{|\nu_n|^2}{Z_n} \sum_{n,m=0}^{1} \langle n, m | (n + m - 1)e^{-(n+m)\beta \omega_{\text{in}}} | n, m \rangle = |\nu_n|^2 \tanh \frac{\beta \omega_{n,\text{in}}}{2}. \]  

(20)

As an interesting model for discussions, we consider the Sauter-type electric field \( E(t) = E \text{sech}^2(t/\tau) \) with the gauge choice, \( A_z(t) = -E\tau(1 + \tanh(t/\tau)) \), which allows the exact solution leading to the exact mean number of pairs created, \(|\nu_k|^2\), at zero temperature both in scalar and spinor QED [5, 16, 20, 21]. Also the approximation scheme of the WKB or worldline instanton method has been developed [6, 7, 8]. Because the Sauter-type electric field acts...
effectively for a finite period of time $\tau$, the ingoing thermal states are stable and well defined as required in our formalism. Fig. 2 shows the mean number of created pairs Eqs. 18-20 at finite temperature with the electric field $E$, the duration $\tau$ and temperature $k_B T = 1/\beta$ scaled in terms of the critical strength, the Compton time and the electron mass, respectively. The production of scalar pairs is thermally enhanced, while the production of fermion pairs is thermally suppressed as expected by the Pauli blocking, which is consistent with the calculation by density matrix method 22.

The pure thermal effect on the mean number of created pairs, which is $\Delta \mathcal{N}_k(T) = \mathcal{N}_k(T) - \mathcal{N}_k(0)$, is given by $\Delta \mathcal{N}_k(T) = \pm 2|\nu_k|^2 f_k^B(T)$ with $f_k^B(T)$ being the Bose-Einstein or Fermi-Dirac distribution. For $k_B T \ll \omega_{k, in}$, the distribution $f_k^B(T)$ approximately equal to the Boltzmann factor $f_k \approx e^{-\sqrt{m^2 + k^2}/k_B T}$. Thus, the mean number of created pairs at finite temperature, Eqs. (18,20), is reduced to the zero temperature result, Eqs. (7,13), for a temperature much lower than the rest mass.

Our result Eq. (20) could be compared with the calculation in imaginary-time formalism 15, where the thermal effect appears only at two-loop because thermal one-loop fluctuations are on-shell. On the other hand, Eq. (20), an off-shell calculation, is equivalent to the thermal loop times the vacuum one-loop, in fact, part of two-loops. One interesting comment to be pointed out is that the momentum integral over the distribution function in $\Delta \mathcal{N}_k(T) = \pm 2|\nu_k|^2 f_k^B(T)$ leads to the factor $T^4$ in Ref. 15, though the momentum integral of the distribution is intertwined with the vacuum pair-production rate. To show rigorously the connection between our result and Ref. 15 requires calculating the effective action at finite temperature along the line of Ref. 5, which will be addressed in the future.

IV. CONCLUSION

In this paper, using the evolution operator method, we found that the mean number of created pairs in state $n$ at finite temperature is given by

$$\mathcal{N}_n(T) = \frac{1}{2z_n} \text{Tr}(N_{n, \text{out}} - N_{n, \text{in}}) e^{-\beta H_{n, \text{in}}}. \quad (21)$$

For scalar and spinor QED in external electric fields, the total mean number density of created pairs at finite temperature is given by

$$\mathcal{N}^S(T) = \int \frac{d^3k}{(2\pi)^3} \mathcal{N}_k^S(0) \coth \frac{\beta \omega_{k, in}}{2}, \quad (22)$$

$$\mathcal{N}^P(T) = \sum \sigma \int \frac{d^3k}{(2\pi)^3} \mathcal{N}_n^P(0) \tanh \frac{\beta \omega_{n, in}}{2}, \quad (23)$$

where $\mathcal{N}_k^S(0)$ and $\mathcal{N}_n^P(0)$ are the mean number of created pairs at zero temperature for scalar and spinor QED, respectively.

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