Multi-Spectrally Constrained Transceiver Design against Signal-Dependent Interference

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Abstract—This paper focuses on the joint synthesis of constant envelope transmit signal and receive filter aimed at optimizing radar performance in signal-dependent interference and spectrally contested-congested environments. To ensure the desired Quality of Service (QoS) at each communication system, a precise control of the interference energy injected by the radar in each licensed/shared bandwidth is imposed. Besides, along with an upper bound to the maximum transmitted energy, constant envelope (with either arbitrary or discrete phases) and similarity constraints are forced to ensure compatibility with amplifiers operating in saturation regime and bestow relevant waveform features, respectively. To handle the resulting NP-hard design problems, new iterative procedures (with ensured convergence properties) are devised to account for continuous and discrete phase constraints, capitalizing on the Coordinate Descent (CD) framework. Two heuristic procedures are also proposed to perform valuable initializations. Numerical results are provided to assess the effectiveness of the conceived algorithms in comparison with the existing methods.

Index Terms—Multiple Spectral Compatibility Constraints, Signal-Dependent Interference, Continuous and Discrete Phase-Only Waveform Design, Coordinate Descent (CD) Method.

I. INTRODUCTION

Spectral coexistence among radar and telecommunication systems has drawn flourishing attention due to the conflict between limited Radio Frequency (RF) spectrum resource and the increasing demand of spectrum access [1]–[5]. Waveform diversity and cognitive radar are key candidates to alleviate this problem via on the fly adaptation of the transmitted waveforms to actual spectrally contested and congested environments [6]–[15]. In this respect, a plethora of papers in the open literature have dealt with the problem of designing cognitive radar signals with a suitable frequency allocation so as to induce acceptable interference levels on the frequency-overlaid systems, while improving radar performance in terms of low range-Doppler sidelobes, detection, and tracking abilities [16]–[25].

In [26], a technique to synthesize constant envelope signals sharing a low Integrated Sidelobe Level (ISL) and a sparse spectral allocation is developed, considering as objective function a weighted sum between an ISL-oriented contribution and a term accounting for the waveform energy over the licensed bandwidths. The mentioned approach is generalized to provide a control on the waveform ambiguity function features in [27]. Some interesting algorithms to devise radar waveforms under spectral compatibility requirements, are also proposed in [28]–[30], where different performance criteria, like amplitude dynamic range, Peak Sidelobe Level (PSL), spectral shape features, and distance from a reference code, are considered at the signal design process. [31] introduces the spectral shaping (SHAPE) method to synthesize constant-modulus waveforms aimed at fitting an arbitrary desired spectrum magnitude, whereas in [32] and [33] the Spectral Level Ratio (SLR) is adopted as performance metric assuming, respectively, constant envelope and PAR constraints. In [1], [34]–[36], the maximization of Signal to Interference plus Noise Ratio (SINR) is accomplished in the presence of signal-independent disturbance while controlling the total interference energy on the common band and some desirable features of the transmitted waveform. This framework is extended to incorporate multiple spectral compatibility constraints in [37]. Along this line, to comply with the current amplifier technology, extensions to address optimized synthesis of constant envelop waveforms with the continuous phase [38] and the finite alphabet [39], are developed. However, the studies in [1], [34]–[39] do not account for signal-dependent interference at the design stage, namely they implicitly assume that the radar is pointing toward the sky (with very low antenna sidelobes) or the target range is far enough that ground clutter is substantially absent. Some attempts to design radar transceivers capable of lifting up detection performance in a highly reverberating environment as well as ensuring spectral compatibility have been pursued in the open literature. For instance, [40] and [41] optimize the SINR (in the presence of signal-dependent disturbance) over the transmit signal and receive structure, while controlling the total amount of interference energy injected on the shared frequency bands. Still forcing a constraint on the global spectral interference, [42]–[44] propose waveform design procedures in the context of Multiple-Input Multiple-Output (MIMO) radar systems operating in highly reverberating environments. Besides, [42] also extends the developed framework considering multiple spectral constraints but just an energy constraint is...
forced on each transmitted signal. Nevertheless, the design of constant envelope signals ensuring the appropriate Quality of Service (QoS) to each licensed system still remains an open issue.

Aimed at filling this gap, in this paper, a new radar transceiver design strategy (with phase-only probing signals) is proposed aimed at optimizing surveillance system performance (via SINR maximization in signal-dependent interference) while fully guaranteeing coexistence with the surrounding RF emitters. Specifically, unlike most of the previous works\(^1\), a local control on the interference energy radiated by the constant envelope signal on each reserved frequency bandwidth is performed, so as to enable joint radar and communication activities. Moreover, to comply with the current amplifiers technology (operating in saturation regime) constant envelope waveforms are considered, with either arbitrary or discrete phases. Besides, to fulfill basic radar requirements, in addition to an upper bound to the maximum radiated energy, a similarity constraint is enforced to bestow relevant waveform hallmarks, i.e., a well-shaped ambiguity function. To handle the resulting NP-hard optimization problems, a suitable re-parameterization of the radar code vector is performed; hence, leveraging the Coordinate Descent (CD) method [46], iterative algorithms (monotonically improving the SINR) are proposed (for both continuous and discrete phase constraints), where either a specific entry of the transmitter parameter vector or the receive filter is optimized at a time while keeping fixed the other variables. Specifically, a global optimal solution of each, possibly non-convex, optimization problem (involved in the two developed procedures) is derived in closed form through the evaluation of elementary functions. Regardless of the phase cardinality, the computational complexity is linear with respect to the number of iterations, cubic with reference to the code length, and less than quadratic with respect to the number of spectral constraints. Finally, two heuristic approaches, accounting for spectral compatibility requirements via a penalty term in the objective function, are proposed to initialize the procedures via ad-hoc starting solutions. To shed light on the capability of the devised algorithms to counter signal-dependent interference and ensure coexistence with the overlaid RF emitters, some case studies are provided at the analysis stage. Moreover, appropriate comparisons with some counterparts available in the open literature are presented to prove the effectiveness of the new proposed strategies.

The paper is organized as follows. In Section II, the system model is introduced, followed by the definition and description of the key performance metrics as well as constraints involved into the formulation of radar transceiver design problems under investigation. In Section III, innovative solution methods are developed to handle the NP-hard optimization problems at hand. Section V presents some numerical results to assess the performance. Finally, in Section VI, concluding remarks and some possible future research avenues are provided.

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\(^1\)Preliminary results of continuous phase codes are shown in [45] without technical details.

### Table I

| NOTATIONS |
|-----------|
| $\bullet$ Bold letters, e.g., $a$ (lower case), and $A$ (upper case) denote vector and matrix, respectively. |
| $\bullet$ $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^\dagger$ indicate the transpose, the conjugate, and the conjugate transpose operators, respectively. |
| $\bullet$ $\mathbb{R}^N$, $\mathbb{C}^N$, $\mathbb{H}^N$ are the sets of $N$-dimensional vectors of real and complex numbers, and of $N \times N$-dimensional Hermitian matrices, respectively. |
| $\bullet$ For any $s \in \mathbb{C}^N$, $\|s\|$ and $\|s\|_\infty$ represent the Euclidean and $l$-infinity norm, respectively. |
| $\bullet$ $I_N$ and $0_N$ represent the $N \times N$-dimensional identity matrix and the matrix with zero entries. |
| $\bullet$ $1_N$ is the $N \times 1$-dimensional vector with all entries equal to 1. |
| $\bullet$ $e_n \in \mathbb{R}^N$ is a vector whose $n$-th entry is 1 and other elements are 0. |
| $\bullet$ Letter $j$ represents the imaginary unit (i.e., $j = \sqrt{-1}$). |
| $\bullet$ $\mathbb{R}\{\cdot\}$, $\mathbb{S}\{\cdot\}$ and $|\cdot|$ mean the real, imaginary part, and modulus of a complex number, respectively. |
| $\bullet$ $\arg(x) \in [-\pi, \pi]$ represents the argument of the complex number $x$. |
| $\bullet$ $\text{diag}(a)$ indicates the diagonal matrix formed by the entries of $a$. |
| $\bullet$ $\text{Diag}(A)$ indicates the diagonal matrix whose $i$-th diagonal element is $A(i,i)$. |
| $\bullet$ $J_m \in \mathbb{C}^N$ is the shift matrix with $J_m(i,l) = 1$ if $i - l = m$, else $J_m(i,l) = 0$, $i,l \in \{1, \ldots, N\}$. |
| $\bullet$ $\lambda_{\text{max}}(A)$ is the largest eigenvalue of $A$. |
| $\bullet$ The statistical expectation is indicated as $E\{\cdot\}$. |
| $\bullet$ $[a]$ and $\lfloor a \rfloor (a \in \mathbb{R})$ provide the greatest integer not larger than $a$ and the lowest integer not smaller than $a$, respectively. |
| $\bullet$ $\diamond$ is the Hadamard element-wise product. |
| $\bullet$ $v(P)$ is the optimal value of the optimization Problem $P$. |
| $\bullet$ $\frac{\partial f(x)}{\partial x}$ denote the derivative of $f(x)$ with respect to $x$. |

### II. Problem Formulation

This section is focused on the introduction of the system model accounting for a highly reverberating environment, as well as on the formulation of the constrained optimization problem to jointly design radar transmitter and receiver.

#### A. System Model

Let $s = [s_1, \ldots, s_N]^T \in \mathbb{C}^N$ be the transmitted fast-time radar code with $N$ being the number of coded sub-pulses. The observations from the range-azimuth cell under test (CUT) are collected in the vector $v \in \mathbb{C}^N$, which can be expressed as [40]

$$ v = a_0 s + c + n, \quad (1) $$

where
• $a_0$ accounts for the response of the prospective target within the CUT.

• $c \in \mathbb{C}^N$ is the signal-dependent interference produced by the range cells adjacent to the CUT, namely

$$c = \sum_{m=-N+1}^{N-1} \alpha_m J_m s,$$

with $\alpha_m$ the scattering coefficient associated with the $m$-th range patch. Specifically, $\{\alpha_m\}_{m \neq 0}$ are modeled as independent complex, zero-mean, circularly symmetric, random variables with $\mathbb{E}[|\alpha_m|^2] = \beta_m, m \neq 0$. As a result, the covariance matrix of $c$ can be cast as

$$R^{(c)}_{\delta} = \mathbb{E}[cc^\dagger] = \sum_{m=-N+1}^{N-1} \beta_m J_m sj^\dagger J_m^\dagger,$$

• $n \in \mathbb{C}^N$ denotes the signal-independent interference that comprises thermal noise and other disturbances from interfering emitters. It is modeled as a zero-mean, complex, circularly symmetric, random vector with covariance $\mathbb{E}[nn^\dagger] = R_{\text{ind}}$.

### B. Transmit Code Constraints

In this subsection, some constraints on the transmitted signal are introduced to fulfill the appropriate radar requirements [47], [48].

1) Multi-Spectral Constraints: To assure spectral compatibility with the surrounding licensed emitters, the radar has to control the spectral shape of the probing waveform to manage the amount of interfering energy injected on the shared frequency bandwidths. In this respect, let us denote by $K$ the number of licensed emitters while $f^\dagger_1$ and $f^\dagger_2$ indicate the lower and upper normalized frequencies (with respect to the underlying radar sampling frequency) for the $k$-th system, respectively. Now, denoting by $S_c(f) = \sum_{n=1}^{N} s_n e^{-j2\pi fn}$ the Energy Spectral Density (ESD) of the fast-time code $s$, the energy transmitted by the radar within the $k$-th licensed bandwidth (also denoted by stopband) can be expressed as

$$\int_{f^\dagger_2} f_{^\dagger 1} S_c(f) df = s^\dagger R^k_s s,$$

where for $i, l \in \mathbb{N}$,

$$R^k_s(i, l) = (f^\dagger_2 - f^\dagger_1) e^{j\pi(f^\dagger_2 + f^\dagger_1)(i-l)} \text{sinc}(\pi(f^\dagger_2 - f^\dagger_1)(i-l)).$$

To enable joint radar and communication activities, it is thus demanded that the radar transmitted waveform complies with the constraints

$$s^\dagger R^k_s s \leq E^k_h, k \in \{1, \ldots, K\},$$

where $E^k_h, k \in \{1, \ldots, K\}$, accounts for the acceptable level of disturbance on the $k$-th bandwidth and is tied up to the QoS required by the $k$-th communication system.

2) Constant Envelope and Energy Constraints: To assure compatibility with the amplifier technology and comply with the radar power budget, constant envelope and energy constraints are forced on the sought waveform, which is tantamount to forcing

$$|s_i| = |s_j|, i, j \in \mathbb{N},$$

$$\|s\|^2 \leq 1.$$  

3) Finite Alphabet Constraint: As a limited number of bits are available in digital waveform generators, the finite alphabet constraint is possibly forced, requiring $\arg(s_i) \in \Omega_M$, where $\Omega_M = 2\pi \frac{M}{2} \{-1/2, \ldots, -1/2, \ldots, M-1\}$ denotes the discrete set of $M$ equi-spaced phases.

4) Similarity Constraint: To bestow some desirable attributes (e.g., Doppler tolerance, ISL, PSL and etc.) to the radar probing signal, a similarity constraint is imposed on the transmitted code, i.e.,

$$\|s - s_0\|_\infty \leq \epsilon \sqrt{N},$$

where $0 \leq \epsilon \leq 2$ rules the size of the trust hypervolume, and $s_0$ is a specific constant modulus reference code with $\|s_0\|^2 = 1$.

### C. Transceiver Design Problem Formulation

A transceiver design approach aimed at optimizing target detectability under the probing code requirements of Subsection II-B is now formalized. To this end, supposing the received signal $v$ filtered via $w \in \mathbb{C}^N$, the SINR at the output of the filter, i.e.,

$$\text{SINR}(s, w) = \frac{|a_0|^2 \|w^\dagger s\|^2}{w^\dagger \left(R^{(c)}_{\delta} + R_{\text{ind}}\right) w},$$

is considered as the design metric.

Hence, according to the code limitations introduced in Subsection II-B, the joint transmit-receive pair design assuming either continuous phase or finite alphabet phase codes, respectively, can be formulated as the following non-convex and in general NP-hard optimization problems

$$\max_{s, w} \text{SINR}(s, w) \text{ s.t. } s^\dagger R^k_s s \leq E^k_h, k \in \{1, \ldots, K\},$$

$$\max_{s, w} \text{SINR}(s, w) \text{ s.t. } s^\dagger R^k_s s \leq E^k_h, k \in \{1, \ldots, K\},$$

where $E^k_h, k \in \{1, \ldots, K\}$, accounts for the acceptable level of disturbance on the $k$-th bandwidth and is tied up to the QoS required by the $k$-th communication system.

$$\|s\|^2 \leq 1,$$

$$\|s - s_0\|_\infty \leq \frac{\epsilon}{\sqrt{N}},$$

$$|s_i| = |s_j|, i, j \in \mathbb{N},$$

$$\arg(s_i) \in \Omega_M,$$

and

$$\text{SINR}(s, w) = \text{SINR}(s, w)/|a_0|^2$$

with $\text{SINR}(s, w) = \text{SINR}(s, w)/|a_0|^2$ and $\Omega_M = [-\pi, \pi]$.  

2If $\{\alpha_m\}_{m \neq 0}$ and $n$ are statistically independent and Gaussian distributed, the SINR optimization is tantamount to maximizing the detection probability.
III. CODE AND FILTER SYNTHESIS

In this section, an iterative design procedure is developed to get optimized transmit-receive pairs leveraging the CD paradigm [46]. By invoking elementary function, each non-convex optimization subproblem is solved in closed form, and the monotonic improvement of the SINR is ensured along the iterations. As first step toward this goal, let us re-parameterize the optimization vector \( s \) as

\[
s = \sqrt{P} \hat{s}_\varphi \odot s_0, \tag{12}
\]

where \( \hat{s}_\varphi = [e^{j\varphi_1}, \ldots, e^{j\varphi_N}]^T \in \mathbb{C}^N \), with \( \varphi = [\varphi_1, \ldots, \varphi_N]^T \in \mathbb{R}^N \), and \( P \) \((0 \leq P \leq 1)\) account for code phases and the signal amplitude level, respectively. As a result, the similarity constraint is tantamount to \( \Re\{e^{j\varphi_i}\} \geq 1 - \epsilon^2/2, i \in N \). Hence, denoting by \( \delta = \arccos(1 - \epsilon^2/2) \), for any \( i \in N \)

- \( \varphi_i \in \Psi_\infty = [-\delta, \delta] \) for the continuous case [38];
- \( \varphi_i \in \Psi_M = 2\pi\{\alpha_1, \alpha_1 + 1, \ldots, \alpha_1 + \omega_i - 1\} \) with \( \omega_i = \begin{cases} 1 - 2\alpha_1, & \text{if } i \in [0, 2] \\ M, & \text{if } i = 2 \end{cases} \)

for the finite alphabet case [39].

Additionally, in the transformed domain, the spectral constraints can be cast as

\[
P \hat{s}_\varphi \mathbf{R}_h^k \hat{s}_\varphi \leq E_{\hat{f}}^k, k \in \{1, \ldots, K\}, \tag{13}
\]

where \( \mathbf{R}_h^k = \text{diag}(s_0)R_h^k\text{diag}(s_0) \), \( k = 1, \ldots, K \). Finally, the objective function in (11) can be expressed as

\[
\chi(\varphi, P, w) = \frac{\text{SINR} (\sqrt{P} \hat{s}_\varphi \odot s_0, w)}{P \hat{s}_\varphi M_1^w \hat{s}_\varphi + \phi^w}, \tag{14}
\]

with

\[
M_1^w = \text{diag}(s_0)w^1\text{diag}(s_0),
\]

\[
M_2^w = \text{diag}(s_0)w^2\sum_{m=1}^{N-1} \beta_m J_m^w w^1 J_m^w \text{diag}(s_0),
\]

\[
\phi^w = w^1R_{\text{in}}w.
\]

To proceed further, let us introduce the optimization vector \( y = [\varphi^T, P, w]^T \in \mathbb{R}^{N+1} \times \mathbb{C}^N \) and denote by \( \chi(y) \) the function (14) evaluated at \( ([y_1, \ldots, y_N]^T, y_{N+1}, [y_{N+2}, \ldots, y_{2N+1}]^T) \). Consequently, Problems \( \mathcal{P}_\infty \) and \( \mathcal{P}_M \) can be equivalently cast as

\[
\mathcal{P}_p \left\{ \begin{array}{c}
\max \chi(y) \\
\text{s.t.} \quad \hat{s}_\varphi = [e^{jy_1}, \ldots, e^{jy_N}]^T, \\
y_{N+1} \hat{s}_\varphi \mathbf{R}_h^k \hat{s}_\varphi \leq E_{\hat{f}}^k, k \in \{1, \ldots, K\}, \\
0 \leq y_{N+1} \leq 1, \\
y_i \in \Psi_\infty, i \in N \end{array} \right. \tag{15}
\]

where \( p \) is either an integer number or \( \infty \) and specifies the code alphabet size.

The provided reformulation in (15) paves the way for an effective CD-based optimization process, where the design variables of the vector \( y \) are partitioned into \( N+2 \) blocks given by \( y_1, \ldots, y_{N+1}, y_{N+2} \), with \( y_{N+2} = [y_{N+2}, \ldots, y_{2N+1}]^T \in \mathbb{C}^N \) corresponding to the receive filter. Hence, at the \( h \)-th step, \( h = 1, \ldots, N+2 \), of the \( n \)-th iteration, the \( h \)-th block is optimized with the other blocks fixed at their previous optimized value. Now, denoting by \( y^{(n)} = [y_1^{(n)}, \ldots, y_{N+1}^{(n)}, y_{N+2}^{(n)}]^T \), it follows that

\[
\chi(y^{(n-1)}) \leq \chi\left([y_1^{(n)}, y_2^{(n)}, \ldots, y_{N+1}^{(n)}, y_{N+2}^{(n)}]^T\right) \leq \cdots \leq \chi\left([y_1^{(n)}, \ldots, y_{N+1}^{(n)}, y_{N+2}^{(n)}]^T\right) \leq \chi(y^{(n)})
\]

The above inequalities entail that the objective function values monotonically increase along with the iterations. Thus, being \( \chi(y) \) upper bounded by \( 1/\sigma_0 \) with \( \sigma_0 \) the variance of thermal noise, \( \chi(y^{(n)}) \) converges to a finite value. In the following, the procedures devised to optimize the \( h \)-th block, \( h = 1, \ldots, N+2 \), at the \( n \)-th iteration are developed. This represents the main technical achievement of this paper from an optimization theory standpoint.

A. Code Phase Optimization

Assuming \( h \in N \), the block to optimize is \( y_h \), with the other blocks set to their previous optimized values, i.e., \( y_{-h}^{(n)} = \left[y_1^{(n)}, \ldots, y_{h-1}^{(n)}, y_{h+1}^{(n)}, \ldots, y_{N+1}^{(n)}, y_{N+2}^{(n)}\right]^T \), \( h \in N \). Hence, as shown in Appendix A, the problem to solve is

\[
\mathcal{P}_p^{(n)} \left\{ \begin{array}{c}
\max_{y_h} \chi(y_h; y_{-h}^{(n)}) \\
\text{s.t.} \quad \Re\{e^{jy_h}e^{jy_{-h}}\} \leq c^{(n)}_{k,h}, k \in \{1, \ldots, K\}, y_h \in \Psi_p \end{array} \right. \tag{16}
\]

where

\[
\chi(y_h; y_{-h}^{(n)}) = \frac{\Re\{e^{jy_h}e^{jy_{-h}}\} + b_h^{(n)}}{\Re\{e^{jy_h}e^{jy_{-h}}\} + d_h^{(n)}}, h \in N.
\]

The feasible set \( \mathcal{F}_p \) of Problem \( \mathcal{P}_p^{(n)} \) and the monotonicity properties of \( \chi(y_h; y_{-h}^{(n)}) \) are analyzed in Appendix B with specification of global maximizer and minimizer \( \phi^{(n)}_{y_h} \) and \( \phi^{(n)}_{h-h} \), respectively. In particular, it is shown that \( \mathcal{F}_p \) can be expressed as

\[
\mathcal{F}_p = \begin{cases}
K_p \cup \left\{ \left[\frac{i_{1}^\infty, \hat{u}_i^\infty}{l_i^\infty} \right] \right. & \text{if } p = \infty, \\
K_p \cup \left\{ \left[IM_i^M + \frac{2\pi M}{N}, \hat{u}_M^N \right] \right. & \text{if } p = M,
\end{cases}
\tag{17}
\]

where (see Appendix B for details)

\[\bullet \] \( K_p \) is the number of disjoint intervals that compose \( \mathcal{F}_p \) with \( l_i^\infty, \hat{u}_i^\infty \in \Psi_\infty \), \( i = 1, \ldots, K_p \), depending on the

\[\hat{\epsilon} < 2 \text{ is assumed.}\]
specific instance of Problem $\mathcal{P}_{\phi_0}^{y^{(n)}_h}$ and such that $\hat{t}^\infty_1 < \hat{t}^\infty_{i+1}, i = 1, \ldots, K_\infty - 1$.
- $K_M \leq K_\infty$ is the number of disjoint discrete sets that form $\hat{F}_M$ with $\hat{t}^M_i, \hat{u}^M_i \in \Psi_M, i = 1, \ldots, K_M$, and $\hat{t}^M_i < \hat{u}^M_i < \hat{t}^M_{i+1}, i = 1, \ldots, K_M - 1$.

The following proposition paves the way to the solution of the non-convex optimization problem $\mathcal{P}_{\phi_0}^{y^{(n)}_h}$.

**Proposition III.1.** An optimal solution $y^{*_h}_h$ to $\mathcal{P}_{\phi_0}^{y^{(n)}_h}$ can be evaluated in closed form via the computation of elementary functions as follows:

- if the unconstrained global optimal solution $\phi_{g,h}^{(n)}$ is feasible, i.e., $\phi_{g,h}^{(n)} \in \hat{F}_\infty$, the optimal solution is $x^*_n = \phi_{g,h}^{(n)}$;
- otherwise, if $\phi_{g,h}^{(n)} \notin \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}$, the optimal solution belongs to $\bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}$, i.e.,
  \[ y^{*_h}_h = \arg \max_{y_n \in \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}} \chi \left( y_n; y^{(n)}_h \right); \quad (19) \]

  where $y^{*_h}_h = \max_{r > \phi_{g,h}^{(n)}} r$, the optimal solution is

  \[ y^{*_h}_h = \arg \max_{y_n \in \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}} \chi \left( y_n; y^{(n)}_h \right); \quad (18) \]

**Proof:** See Appendix C.

According to Proposition III.1, as long as $\phi_{g,h}^{(n)}$ does not belong to one of the closed and bounded intervals $[\hat{t}_i^\infty, \hat{u}_i^\infty], i = 1, \ldots, K_\infty$, the global optimal solution is one of the intervals extremes. As a consequence, embedding $\hat{F}_M$ to an appropriate union of closed intervals the following corollary holds true.

**Corollary III.1.** An optimal solution $y^{*_h}_h$ to $\mathcal{P}_{\phi_0}^{y^{(n)}_h}$ can be derived as follows:

- if there exists an index $q^* \in \{1, \ldots, K_M\}$ satisfying $\hat{t}^M_{q^*} \leq \phi_{g,h}^{(n)} \leq \hat{u}^M_{q^*}$, then
  \[ y^{*_h}_h = \arg \max_{y_n \in \hat{t}^M_{q^*}, \hat{u}^M_{q^*}} \chi \left( y_n; y^{(n)}_h \right); \quad (20) \]

  where

  \[ \phi^M_{q^*} = \frac{\phi_{g,h}^{(n)} \cdot \frac{2\pi}{\hat{M}, \hat{u}^M_{q^*}}}{M}; \quad (21) \]

  otherwise, if $\phi_{g,h}^{(n)} \notin \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}$,

  \[ y^{*_h}_h = \arg \max_{y_n \in \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}} \chi \left( y_n; y^{(n)}_h \right); \quad (22) \]

  else,

  \[ y^{*_h}_h = \arg \max_{y_n \in \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}} \chi \left( y_n; y^{(n)}_h \right); \quad (23) \]

  where $r^* = \max_{\hat{u}^M_{g,h}^{(n)}} r$.

**Proof:** See Appendix D.

Hence, starting from a feasible solution $y^{(n)}_h$, the solution to $\mathcal{P}_{\phi_0}^{y^{(n)}_h}$ for different values of the objective parameters can be easily accomplished following the line of Algorithm 1.

**Algorithm 1** Phase Code Entry Optimization.

**Input:** $y^{(n)}_{h-k, p} \in \{M, \infty\}, h, a_h^{(n)}, b_h^{(n)}$, $\chi, \theta_h^{(n)}, \chi_{k,h}, c_{k,h}$, $k \in \{1, \ldots, K\}$;

**Output:** $y^{*_h}_h$;

1) Compute the feasible set $\hat{F}_p$ as given in (17);

2) If $\phi_{g,h}^{(n)} \notin \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}$, the global optimal solution is one of the intervals extremes. As a consequence, embedding $\hat{F}_M$ to an appropriate union of closed intervals the following corollary holds true.

3) Elseif, $\phi_{g,h}^{(n)} \notin \bigcup_{i=1}^{\infty} \hat{u}^\infty_{K_\infty}$, the optimal solution is

4) Else,

5) Output $y^{*_h}_h$.

**B. Code Amplitude Optimization**

Focusing on $h = N + 1$, $P_p$ w.r.t. $y_{N+1}$ reduces to the following problem,

$$
\mathcal{P}_{\phi_0}^{y^{(n)}_{N+1}} \left\{ \begin{array}{ll}
max & y_{N+1}^{(n)} M_1(y_{N+2}^{(n)}) \bar{s}_{\phi}^{(n)} \\
\text{s.t.} & y_{N+1}^{(n)} \leq E_k, k \in \{1, \ldots, K\} \\
& 0 \leq y_{N+1} \leq 1 \\
\end{array} \right.
$$

where $E_k = \bar{s}_{\phi}^{(n)} \tilde{R}_k^{(n)}$, and $\bar{s}_{\phi}^{(n)} = [e^{s_{r_1}}, \ldots, e^{s_{r_N}}]^T$. The first-order derivative of the objective function satisfies

$$
\mathcal{P}_{\phi_0}^{y^{(n)}_{N+1}} \left\{ \begin{array}{ll}
max & (y_{N+1}^{(n)} M_1(y_{N+2}^{(n)}) \bar{s}_{\phi}^{(n)} \\
\text{s.t.} & y_{N+1} \leq E_k, k \in \{1, \ldots, K\} \\
& 0 \leq y_{N+1} \leq 1 \\
\end{array} \right.
$$

Hence, the objective function monotonically increases over $[0,1]$, and the optimal solution is just the highest feasible value, given by

$$
y_{N+1}^{(n)} = \min \left( \begin{array}{c}
\min \{E_k\} \quad 1
\end{array} \right).
$$

**C. Receive Filter Optimization**

With reference to the $(N+2)$-th variable block, the optimization variable is the receive filter. The update of the receive filter, i.e., $y_{N+2}$, is tantamount to solving

$$
\mathcal{P}_{\phi_0}^{y^{(n)}_{N+2}} \left\{ \begin{array}{ll}
max & \text{SNIR} (s^{(n)}, y_{N+2}) \\
\end{array} \right.
$$
where \( s^{(n)} = \sqrt{\frac{n-1}{N+1}} \phi \circ s_0 \) is the updated transmit signal at the \( n \)-th iteration. The optimal solution [40] is

\[
\begin{align*}
y^{(n)*}_{N+2} = \frac{\left( R_d(s^{(n)}) + R_{\text{ind}} \right)^{-1}}{s^{(n)} \left( R_d(s^{(n)}) + R_{\text{ind}} \right)^{-1}} s^{(n)}.
\end{align*}
\]

(26)

Remark III.1. Supposing \( A = R_d(s^{(n)}) + R_{\text{ind}} \) which is positive definite, the solution to the linear equations \( A \hat{w} = s^{(n)} \) can be derived via the Conjugate Gradient Method (CGM) [49].

D. Optimization Process and Computational Complexity

Starting from a feasible solution \( s^{(0)} \), the overall iterative design procedure is summarized in Algorithm 2. Note that in place of the standard cyclic updating rule, the Maximum Block Improvement (MBI) strategy [50] can be used, too. As to the computational complexity of Algorithm 2, it is linear with the number of iterations. In each iteration, it mainly includes the computation of the code phases (step 3), the code amplitude (step 4), and the receive filter (step 5), whose complexities are now discussed.

Step 3 includes the solution of \( \mathcal{P}_p^{(h)}(n), h \in \mathcal{N} \), and the main actions to perform at each step are:

1. evaluation of the parameter problems;
2. calculation of \( \mathcal{F}_p \);
3. determination of the optimal phase.

As to item 1, the parameters to compute are \( \alpha_h^{(n)}, b_h^{(n)}, c_h^{(n)}, d_h^{(n)}, z_{k,h}^{(n)} \), and \( \hat{c}_{k,h}^{(n)}, k \in \{1,\ldots,K\} \), \( h \in \mathcal{N} \). Leveraging smart recursive schemes based on suitable support variables, it follows that for a complete algorithm iteration the computational complexity of \( \alpha_h^{(n)} \) and \( b_h^{(n)} \) is \( O(N) \) whereas that of \( c_h^{(n)} \) and \( d_h^{(n)} \) is \( O(N^2 \log N) \). With reference to \( z_{k,h}^{(n)} \), it requires \( N \) multiplications for any \( k \in \{1,\ldots,K\} \) and \( h \in \mathcal{N} \). Furthermore, following the same line of reasoning as in [38], for any \( k \in \{1,\ldots,K\} \), \( \hat{c}_{k,h}^{(n)} \), \( h \geq 2 \) can be updated by canceling out the items related to the variable \( y_h^{(n-1)} \) from \( \hat{c}_{k,h-1}^{(n)} \), and adding those involving \( y_h^{(n-1)} \), with a complexity of \( O(N) \); similar considerations hold true with respect to \( h = 1 \). Hence, the overall computational complexity of item 1 at each algorithm iteration is \( O(N^2(\log N + K)) \). As to item 2, note that the determination of \( \mathcal{F}_\infty \) can be performed with a computational complexity of \( O(K \log(K)) \) for any \( h \in \mathcal{N} \) leveraging the results of [38]. Besides, according to Appendix B, \( \mathcal{F}_\infty \) can be obtained resorting to appropriate quantizations of \( \bar{f}_i^{\infty} \) and \( \bar{u}_i^{\infty} \), \( i = 1,\ldots,K_{\infty} \) with an extra computational complexity of \( O(1) \). With reference to item 3, the unconstrained optimal solution \( \hat{o}_{g,h}^{(n)} \) for any \( h \in \mathcal{N} \), can be evaluated via elementary functions using \( a_h^{(n)}, b_h^{(n)}, c_h^{(n)}, d_h^{(n)} \) with a computational burden of \( O(1) \), while the optimal solution to the Problem \( \mathcal{P}_P^{(n)} \) can be accomplished with a complexity at most of \( O(K) \) positioning \( \hat{o}_{g,h}^{(n)} \) within \( \{p_1, p_2, \ldots, p_{K_P}, p_{K_P} \} \), \( p \in \{\infty, M\} \) in the correct sorted location.

With reference to the complexities of step 4 and 5, the former (mainly related to the efficient computation of \( q_{k}^{(n)} \)) is \( O(K) \), the latter requires \( O(N^3) \) operations, with the most demanding task given by the evaluation of \( R_d^{(s^{(n)})} \); in particular, according to the Remark III.1, (26) can be efficiently computed by CGM with a complexity of \( O(N^2Q) \) where \( Q \) is the number of iterations. Therefore, the overall computational complexity of steps 3, 4 and 5 is \( O(N^2(K + N) + KN\log K) \).

Algorithm 2 Transceiver design method.

Input: Reference code \( s_0 \), phase cardinality \( p \), initial feasible code \( s^{(0)} \in \Omega_{\infty} \) (resp. \( \Omega_{M} \)), \( c, \epsilon \), minimum required improvement \( \tilde{c}_1, \tilde{c}_2 \), \( R_{\text{ind}}, J^1_p, J^2_p \) and \( E^1_p, k \in \{1,\ldots,K\} \).

Output: Optimized solution \( y^* \).

1) \textbf{Initialization.}

- Set \( n := 0 \);
- Compute \( y_{N+2}^{(0)} \) by (26);
- Set \( y_h^{(0)} = arg\left( \frac{s_h^{(0)}}{s_h^{(0)}}, h \in \mathcal{N}; \right) \)
- \( y^{(0)} = \left[ y_1^{(0)}, \ldots, y_N^{(0)}, \|s^{(0)}\|^2, y_{N+2}^{(0)T} \right]^T; \)
- Compute \( \chi(y^{(0)}) \).

2) Set \( n := n + 1; \)

3) For \( h = 1 : N \)

- Update \( y_h^{(n)} \) by Algorithm 1

End

4) Update \( y_{N+1}^{(n)} \) by (24);
5) Update \( y_{N+2}^{(n)} \) by (26);
6) \( y^{(n)} = \left[ y_1^{(n)}, \ldots, y_{N-1}^{(n)}, y_{N+2}^{(n)} \right]^T; \)
7) Compute \( \chi(y^{(n)}) \).
8) If \( |\chi(y^{(n)}) - \chi(y^{(n-1)})| \leq \tilde{c}_1 \), stop. Otherwise, go to step 2;
9) Output \( y^* = y^{(n)} \).

IV. HEURISTIC METHODS FOR ALGORITHM INITIALIZATION

The solution provided by Algorithm 2 depends on the initial feasible sequence \( s^{(0)} \). Thus, the development of a heuristic procedure ensuring high quality starting points is valuable. To this end, an ad-hoc transceiver synthesis strategy is proposed, accounting for the spectral constraints via a penalty term in the objective function. Specifically, the following design problem is considered

\[
\begin{align*}
\max_{s,w} \quad & f(s,w) \\
\text{s.t.} \quad & \|s - s_0\|_\infty \leq \frac{\epsilon}{\sqrt{N}} \\
& \|s_i\| = 1/\sqrt{N}, i \in \mathcal{N} \\
& \arg(s_i) \in \Omega_p \\
\end{align*}
\]

(27)

with

\[
\begin{align*}
f(s,w) &= \text{SINR}(s,w) - \beta s^\dagger Rs \\
\end{align*}
\]

(28)
where $\beta \geq 0$ is a weight factor ruling the relative importance between the two objectives, and $R = \sum_{k=1}^{K} R_{pf}^k$. Thus, denoting by $s^*_i$ an optimized solution to (27), the initial feasible code to \textit{Algorithm 2} can be constructed as

\[
\hat{s}_2 = \frac{s^*_i}{\max \left(1, \max_{k \in \{1, \ldots, K\}} s^*_i R_{pf}^k E_{f_i}^k\right)}.
\]  

Note that Problem (27) is in general NP-hard. As a first step to handle this synthesis, let us re-parameterize the transmit signal as $s = [e^{j\varphi_1}, \ldots, e^{j\varphi_N}]^T \odot s_0$; hence, Problem (27) boils down to

\[
\max_{q \in C_p} \hat{f}(q) \quad \text{s.t.} \quad q \in C_p
\]

where $q = [\bar{q}^T, \tilde{q}^T]^T \in C^{2N}$ is the new optimization vector (with $\bar{q} = [e^{j\varphi_1}, \ldots, e^{j\varphi_N}]^T$ and $\tilde{q} = \bar{q}$), $C_p = \mathbb{D}_p \times C^N \subseteq C^{2N}$ (with $\mathbb{D}_p = \{x = e^{j\varphi} | \varphi \in \Psi_p\}$) is the feasible set of $q$, and $\hat{f}(q) = f(q \odot s_0, \bar{q})$, namely it denotes the objective in (28) evaluated at $(s, w) = (q \odot s_0, \bar{q})$.

To solve Problem (30), the optimization framework in [51] and [50] is used. The main idea of this solution strategy is to partition the variables of the optimization vector $q$ into $I$ decoupled blocks, i.e., avoiding useless complications, $q = [q^T_1, \ldots, q^T_I]$, where $q_i \in C_{p,i} \subseteq C^N$, with $\sum_i = Q_i = 2N$, and update one block at a time. The procedure is reported in \textit{Algorithm 3} assuming an alternating optimization rule and no approximations to the feasible set. Therein, focusing on the optimization of the $i$-th variables block $(i = 1, \ldots, I)$, $\tilde{f}_i(x_i; q)$, for $q \in C_p$ and $\forall p$, represents the surrogate function to maximize, which shares the following properties [50], [51]:

\[
\begin{align*}
(P1) & \tilde{f}_i(x_i; q) \text{ is continuous with respect to } x_i \text{ and } q, \\
(P2) & \tilde{f}_i(x_i; q) \leq \tilde{f}(q^T_1, \ldots, q^T_i-1, x^T_i, q^T_{i+1}, \ldots, q^T_I) \quad \text{for } x_i \in C_{p,i} \text{ and } q^T \in C_p, \\
(P3) & \tilde{f}_i(q_i; q^T_1, \ldots, q^T_i-1, q^T_i, q^T_{i+1}, \ldots, q^T_I) = \\
& \tilde{f}(q^T_1, \ldots, q^T_i-1, q^T_i, q^T_{i+1}, \ldots, q^T_I), \quad \forall q_i \in C_{p,i}, \forall q^T \in C_p.
\end{align*}
\]

According to [50, Proposition 2] and [51, Theorem 1], \textit{Algorithm 3} ensures a monotonically increasing of the objective, and, under some mild technical conditions, the convergence to a stationary point for any limit point of the generated sequence and, under some mild technical conditions, the convergence to an optimized solution to (27), the initial feasible code to (30). The former considers as optimization blocks $\bar{q}$ and $\tilde{q}$. The latter assumes $N + 1$ optimization blocks, given by $q_1, \ldots, q_N, \tilde{q}$.

\[4^\beta \] can be interpreted as a weight which scalarizes the multi-objective optimization problem involving the SINR and the opposite total interference energy on the licensed bands as figure of merits.

\[5^\text{It is assumed that } C_p = C_{p,1} \times \cdots \times C_{p,I}, \text{ with } C_{p,i} \text{ the feasible set of } i\text{-th block of variables, } i = 1, \ldots, I.\]

\[6^\text{Either an alternating or an MBI strategy can be considered.}\]

\begin{algorithm}
\begin{itemize}
\item \textbf{Input:} A feasible starting point $q^{(0)} = [q_1^{(0)T}, \ldots, q_I^{(0)T}]^T$, $p \in \{\infty, M\}, C_{p,i}, i = 1, \ldots, I.$
\item \textbf{Output:} Optimized solution $s^*_i$ to Problem (27):
\begin{enumerate}
\item Set $l = 0$;
\item \textbf{Repeat:}
\item For $i = 1: I, q_i^{(l+1)} = \arg \max_{x_i \in C_{p,i}} \tilde{f}_i(x_i; q^T_i)$;
\item $q^{(l+1)} = [q_1^{(l)T}, \ldots, q_i^{(l+1)}, \tilde{q}_i^{(l+1)}, q_{i+1}^{(l)T}, \ldots, q_I^{(l)T}]^T$;
\item $l := l + 1$;
\item \textbf{End}
\item \textbf{Until convergence:}
\item Output $s^*_i = [q_1^{(I)}, \ldots, q_N^{(I)}] \odot s_0$.
\end{enumerate}
\end{itemize}
\end{algorithm}

\section{A. Heuristic Initialization Via Alternating Optimization with MM (HIVAM)}

The optimization variable $q$ is partitioned into two blocks, i.e., $q_1 = \bar{q} \in C_N$, $q_2 = \tilde{q} \in C_N$. At $(l + 1)$-th step, the surrogate functions involved in the optimization of $q_1$ and $q_2$ are given, respectively, by (see Appendix E in the supplemental material for details)

\[
\tilde{f}_1(x_i; q_1) = \Re \left\{ z(q_1, q_2) x_i \right\} + r(q_1, q_2),
\]

\[
\tilde{f}_2(x_2; q_2) = \Re \left\{ s(q_1, x_2) \right\} - \beta s(q_1, R(s)),
\]

where $s(l) = q_1 \odot s_0$. Moreover, $C_{p,1} = \mathbb{D}_p$, and $C_{p,2} = C_N$. As a consequence, at $(l + 1)$-th step:

1) the first block is updated (the interested reader may refer to Appendix E for technical details) as

\[
q_1^{(l+1)} = e^{j\phi_1}.
\]

\begin{itemize}
\item If $p = \infty$, $\phi_1 = \max(\arg(z_{1,i}), \delta), -\delta$;
\item If $p = M$,
\end{itemize}

\[
\phi_1 = \frac{2\pi}{M} \max \left( \min \left( \frac{\arg(z_{1,i}) M}{2\pi} \right), m_e, M \right), \alpha_e, \omega_e - 1, \text{ else}
\]

2) the second block is updated as

\[
q_2^{(l+1)} = \frac{(R_d(s(l)) + R_{end})^{-1} s(l)}{s(l) l (R_d(s(l)) + R_{end})^{-1} s(l)}.
\]

\section{B. Heuristic Initialization via Alternating Optimization with CD (HIVAC)}

The optimization variable $q$ is partitioned into $N + 1$ blocks, i.e., $q_i = \bar{q}_i, i \in N$. The surrogate functions are obtained restricting the objective function to $q_i, i = 1, \ldots, N + 1$, namely

\[
\tilde{f}_1(x_i; q_1) = \tilde{f}(q_1^{(l)T}, \ldots, q_i^{(l)T}, x_i, q_{i+1}^{(l)T}, \ldots, q_N^{(l)T})^T.
\]
Otherwise stated,
\[
\tilde{f}_i(x_i; q^{(l)}) = \Re\{a_i x_i + b_i q^{(l)} + \Re\{f_i x_i + d_i q^{(l)}, i \in \mathcal{N}\}
\]
(35)
\[
\tilde{f}_i(x_{N+1}; q^{(l)}) = \text{SINR}(s^{(l)}, x_{N+1}) - \beta s^{(l)} R s^{(l)},
\]
(36)
with \(a_i, b_i, c_i, d_i, f_i, g_i\) reported in Appendix F. Besides, \(c_{p,1} = \ldots = c_{p,N} = D_p\) and \(c_{p,N+1} = \mathbb{C}^N\).

As a consequence, at \((l+1)\)-th step,
1) \(q_i, i \in \mathcal{N}\) can be updated (see Appendix F for details) as
\[
q_i^{(l+1)*} = e^{j \phi_i^*},
\]
(37)
with
- \(\phi_i^* = \arg \max_{\phi \in T_{\infty,i}} \tilde{f}_i(e^{j \phi}; q^{(l)})\), if \(p = \infty\), with \(T_{\infty,i}\) the set of at most eight points defined in (66) of the supplemental material;
- \(\phi_i^* = \arg \max_{\phi \in T_{M,i}} \tilde{f}_i(e^{j \phi}; q^{(l)})\), if \(p = M\), where \(T_{M,i}\) is the set of at most fourteen points specified in (68) of the supplemental material;
2) the \((N+1)\)-th block is optimized as
\[
q_{N+1}^{(l+1)*} = \left( \frac{R_d (s^{(l)}) + R_{\text{ind}}} {s^{(l)}} \right)^{-1} s^{(l)},
\]
(38)
where \(s^{(l)} = [q_1^{(l)}, \ldots, q_N^{(l)}]^T \otimes s_0\).

V. PERFORMANCE ANALYSIS

This section is devoted to the performance assessment of Algorithm 2 in terms of achievable target detectability, spectral shape of the synthesized transmit waveform and receive filter, and cross-correlation features. In this respect, a radar with a two-sided bandwidth of 2 MHz and a pulse length of 100 \(\mu\)s (leading to \(N = 200\)) is considered. As to the reference code \(s_0\), a unitary energy Linear Frequency Modulated (LFM) pulse of 100 \(\mu\)s and a chirp rate \(K_s = (1950 \times 10^3)/(100 \times 10^{-6})\) Hz/s is employed for the continuous phase. The \(M\)-quantized version of the above chirp is instead considered for the finite alphabet case, with cardinality \(M^7\).

The covariance matrix of signal-independent interference is
\[
R_{\text{ind}} = \sigma_0 I + \sum_{k=1}^{2} \frac{\sigma_{f,k}}{\Delta f_k} R_k + \sum_{k=1}^{2} \sigma_{J,k} R_{J,k},
\]
(39)
where \(\sigma_0 = 0 \text{ dB}\) is the thermal noise level; \(\sigma_{f,k} = 10 \text{ dB}\), \(k = 1, 2\), accounts for the energy of the \(k\)-th licensed emitter operating over the normalized frequency interval \(\Delta f_k = [f_{k,1}, f_{k,2}]\), with \(\Delta f_k = f_{k,2} - f_{k,1}\) the related bandwidth extent \((\Omega_1 = [0.2112, 0.2534], \Omega_2 = [0.5856, 0.6112])\); \(R_k, k = 1, 2\) is the normalized covariance matrix of the \(k\)-th non-licensed active source, whose normalized carrier frequency, bandwidth, and power are denoted by \(f_{J,k}, \Delta f_k\) and \(\sigma_{J,k}\), respectively (\(\sigma_{J,1} = 35 \text{ dB}, f_{J,1} = 0.823, \sigma_{J,2} = 40 \text{ dB}, f_{J,2} = 0.925, \Delta f_{J,k} = 0.001, k = 1, 2\)). For the signal-dependent interference, \(\beta_m = 8 \text{ dB}\), \(m \in \{\pm 1, \ldots, \pm (N-1)\}\), is assumed. Besides, the radar probing waveform is required to fulfill the spectral compatibility constraints corresponding to \(E_{I, \text{dB}} = -30 \text{ dB}\) and \(E_{J, \text{dB}} = -35 \text{ dB}\), respectively.

A. Target Detectability Assessment

Fig. 1 depicts the normalized SINR achieved by the proposed transceiver design strategy versus \(\epsilon\) for continuous and discrete phase codes \((M = 2, 4, 8, 16, 32, 64)\). As expected, a larger similarity parameter leads to a higher SINR value regardless of phase cardinality being available more degrees of freedom (DOF) at the design stage. Besides, the finer the phase discretization, the better the performance, with SINR curves of the synthesized discrete phase codes closer and closer to that of the continuous phase benchmark. Finally, it is worth pointing out that the designed sequence coincides with a scaled version of reference code, with an energy modulation implemented to comply with the forced spectral constraints if \(\epsilon = 0\) (for the continuous phase) or \(\alpha_n = 0\) (for the finite alphabet). For instance, for the binary case \(\alpha_n = 0\) as long as \(\epsilon \neq 2\), and thus a SINR performance improvement just appears at \(\epsilon = 2\). Consequently, the similarity parameter should be carefully selected to balance the detection performance and waveform characteristics.

Now, assuming \(\{\alpha_0\}_{\alpha \neq 0}\) and \(n\) statistically independent and Gaussian distributed as well as a Swerling 0 target, the probability of detection \((P_d)\) versus \(|\alpha_0|^2\), as function of \(\epsilon\) and the phase discretization step, is shown in Fig. 2. Therein, the same environmental characterization as in Fig. 1 is considered and the false alarm probability \((P_{fa})\) is set to \(10^{-4}\). As expected, increasing of similarity parameter, the cardinality of the code alphabet, and \(|\alpha_0|^2\) provide \(P_d\) improvements.
To the best of the Authors’ knowledge, the waveform design methods currently available in the open literature are not able to solve Problem $\mathcal{P}_p, p \in \{\infty, M\}$ in its general form. Aimed at providing a complete assessment of Algorithm 2, in Subsection V-C some specific instances of Problem $\mathcal{P}_p, p \in \{\infty, M\}$ are analyzed, reporting the comparison of our new method with other suitable approaches already devised in the open literature [42], [52].

B. Transceiver Characteristics

In Fig. 3, the spectral behavior of the signals synthesized through Algorithm 2 (in terms of ESD versus the normalized frequency) is provided as function of the similarity parameter. Specifically, Fig. 3 (a) refers to the continuous phase design, whereas Fig. 3 (b) is related to $M = 64$. Therein, the stopbands $\Omega_i, i = 1, 2$ are shaded in light gray. The curves highlight the capability of the devised techniques to suitably control the amount of interference energy produced over the shared frequency bandwidths, as required by the imposed spectral compatibility constraints. Furthermore, inspection of the figures reveals that, regardless of the code cardinality, the ESD curve corresponding to $\epsilon = 0.001$ almost coincides with a scaled version of the considered reference code, as a result of the energy modulation performed to ensure cohabitation, along with the similarity requirement. Finally, an improvement in the “useful” energy distribution is achieved as $\epsilon$ increases, with deeper and deeper spectral notches in correspondence of the jammed frequencies, especially for the continuous alphabet code.

To shed light on the capabilities of the devised transceiver to mitigate signal-dependent disturbance, the PSL and ISL of the Cross-Correlation Function (CCF) normalized to $|w^* s^*|$ between the transmit waveform and the receive filter versus the iteration index, are reported in Table II for the continuous phase and $M = 64$, assuming $\epsilon = 1$ and HIVAM as the initialization method. The results show that lower and lower PSL and ISL values are achieved as the iteration step $n$ grows up for both continuous and discrete phase codes. Otherwise stated, the devised strategy is able to iteratively improve the rejection of the signal-dependent interference.

| $n$ | 0 | 2 | 4 | 8 | 15 |
|-----|---|---|---|---|----|
| PSL(dB) | -18.25 | -19.63 | -20.39 | -21.19 | -21.22 |
| ISL(dB) | -6.95 | -7.78 | -8.12 | -8.47 | -8.52 |

(b) $M = 64$

(a) Continuous phase

| $n$ | 0 | 2 | 5 | 10 | 20 |
|-----|---|---|---|----|----|
| PSL(dB) | -18.39 | -20.24 | -20.96 | -21.08 | -21.34 |
| ISL(dB) | -6.49 | -7.89 | -8.18 | -8.32 | -8.39 |

Stated, the devised strategy is able to iteratively improve the rejection of the signal-dependent interference.

TABLE II

PSL AND ISL OF THE NORMALIZED CCFs BETWEEN THE TRANSMIT WAVEFORM AND THE RECEIVE FILTER, AT DIFFERENT ITERATIONS, ASSUMING $\epsilon = 1$.

(a) Continuous phase (the number of iterations up to convergence is 15)

(b) $M = 64$ (the number of iterations up to convergence is 20)
C. Comparison with Available Algorithms

If spectral coexistence requirements are not considered, the resulting problem can be solved by Dinkelbach-Type Algorithm (DTA) [52] and Majorized Iterative Algorithm with the Constant Modulus and Similarity Constraint (MIA-CMSC) [42], which can be further distinguished into MIA-CMSC1 and MIA-CMSC2 depending on different majorization methods\(^8\). Figs. 4(a)-(b) depict the SINR achieved by Algorithm 2, DTA, MIA-CMSC1, and MIA-CMSC2 assuming at the design stage a continuous phase and \(M = 32\), respectively\(^9\). Looking over the figures unveils that the proposed CD framework outperforms the counterparts. The superior performance with respect to DTA can be attributed to the slight phase-search sub-optimality of the DTA, which is intrinsic in the Dinkelbach iterative method. Instead, the performance loss incurred by the MIA-CMSC methods reasonably results from the approximation of the objective function performed in the procedure. Finally, in the design of discrete phase codes MIA-CMSC procedures experience a higher performance gap with respect to Algorithm 2 than that observed for the continuous phase instance.

To assess the convergence property and computational complexity of the different methods, Figs. 5(a)-(b) depict the SINR versus CPU time with \(\epsilon = 0.8\) for the continuous phase and \(M = 32\), respectively. The reference code is used for this case study to initialize all the algorithms. As expected, the objective function of all the methods monotonically increases. Inspection of the curves shows that Algorithm 2 is substantially capable of obtaining larger SINR values than the counterparts for any given time budgets, which provides a practical proof of the CD framework efficiency. Specifically, Algorithm 2 outperforms the counterparts, but for the continuous phase code synthesis and up to 0.38s where MIA-CMSC1 is better.

To proceed further, let us observe that, neglecting the constant envelope as well as the similarity requirements, \(\mathcal{P}_c\) with \(K = 2\) can be solved via Majorized Iterative Algorithm with the Spectrum Compatibility Constraint for Local design (MIA-SCCL) [42], as well as a variant of the algorithm in [40], denoted as SemiDefinite Programming-based Design (SDPD)\(^8\). The SINR, the computational time, and the PAR\(^9\) of the sequences synthesized via SDPD, MIA-SCCL and Algorithm 2 are summarized in the Table III. The sequence devised via HIVAC is adopted as the initialization for all the algorithms, which costs 1.8724s for computation. The proposed CD algorithm is capable of devising the radar transceiver with a shorter running time than the counterparts. However, as expected, it experiences a SINR loss with respect to SDPD and MIA-SCCL. Indeed, SDPD and MIA-SCCL may capitalize on code amplitude variation to boost the radar performance, at the price of larger PAR values, which in turn demand for more sophisticated amplifiers.

VI. Conclusion

The synthesis of radar transceivers in signal-dependent interference and spectrally contested-congested environments has been addressed in this paper. Specifically, assuming constant modulus signals with either continuous or finite alphabet phases, the design has aimed at maximizing the SINR at the output of the receive filter while ensuring cohabitation with surrounding RF systems via multiple spectral constraints. Furthermore, a similarity constraint has been forced on the

\(^8\)The parameters specified in [52] and [42] are used to implement these algorithms; specifically, \(\kappa = 10^{-5}\) and \(\kappa = 10^{-4}\) for DTA and \(\epsilon_{obj} = 10^{-4}\) for MIA-CMSC1 and MIA-CMSC2.

\(^9\)For all the algorithms at each \(\epsilon\) both the reference sequence \(s_0\) and the code optimized at the previous \(\epsilon\) are adopted as the initializations. The one providing the highest SINR between the two synthesized sequences is picked up. Although DTA and MIA-CMSC are not provided in [52] and [42] with reference to discrete phase codes, their extension to encompass also this design constraint is straightforward.

\(^10\)The parameters specified in [42] and [40] are used for implementing these algorithms; specifically, \(\epsilon_{alc} = 10^{-8}\) and \(\epsilon_{obj} = 10^{-4}\) for MIA-SCCL and \(\zeta = 10^{-4}\) for SDPD.
may induce other degrees of freedom which can be optimized to improve the performance of the overall method.

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APPENDIX

A. Derivation of Problem (16)

Before proceeding further, let us introduce the following lemma.

Lemma 1. For any known matrix $R \in \mathbb{H}^N$, $g(a) = a^H Ra$ with $a \in \mathbb{C}^N$ can be recast as a function of a specific entry $a_h, h \in N$, i.e.,

$$g(a_h; a_{-h}) = (a_{-h} + a_h e_h)^H R (a_{-h} + a_h e_h) = e_h^H R e_h |a_h|^2 + 2 \Re \{ \hat{a}_{-h}^H R e_h a_h \}$$

where $\hat{a}_{-h} = a - a_h e_h \in \mathbb{C}^N$, and $a_{-h} = [a_1, \ldots, a_{h-1}, a_{h+1}, \ldots, a_N]^T \in \mathbb{C}^{N-1}$.

According to Lemma 1 and exploiting $|\hat{s}_k|^2 = |e^{jy_i}|^2 = 1$, $i \in N$ as well as $y_{N+1} \geq 0$, $\chi(y_h; y_{-h})$ w.r.t. $y_h$ can be expressed as

$$\chi(y_h; y_{-h}) = \Re \{ a_h^H e_h y_{-h} \} + b_h^H + d_h^H, h \in N,$$

where

- $a_h^H = 2 s_{-h}^H M_1^H y_{N+2}^{(n-1)} e_h$
- $b_h^H = s_{-h}^H M_1^H e_h$
- $c_h^H = 2 \tilde{s}_{-h}^H M_2^H y_{N+2}^{(n-1)} e_h$
- $d_h^H = \tilde{s}_{-h}^H M_2^H e_h + \tilde{e}_h^H M_2^H y_{N+2}^{(n-1)} e_h + \tilde{g}(y_{N+1}^{(n-1)})/y_{N+1}^{(n-1)}$

with $\tilde{s}_{-h}^H = [e_{y_1}^{(n-1)}, \ldots, e_{y_{N-1}}^{(n-1)}, e_{y_{N+1}}^{(n-1)}, \ldots, e_{y_{N+1}}^{(n-1)}]^T$.

Similarly, the spectral constraints can be transformed as

$$y_{N+1}^{(n-1)} \left( \tilde{z}_{k,h}^{(n)} + \Re \{ \tilde{z}_{k,h}^{(n)} e^{jy_{k,h}} \} \right) \leq E_k, k = 1, \ldots, K,$$

where $\tilde{z}_{k,h}^{(n)} = s_{-h}^{(n)} R_k^H e_{y_{k,h}}^H + e_h^H R_k^H e_{y_{k,h}}^H$. As a result, being $y_{N+1}^{(n-1)} \geq 0$, the inequalities in (42) are tantamount to $\Re \{ \tilde{z}_{k,h}^{(n)} e^{jy_{k,h}} \} \leq c_{k,h}^{(n)}, k = 1, \ldots, K$, where $c_{k,h}^{(n)} = \frac{E_k^{(n-1)}}{y_{N+1}^{(n-1)}} - \tilde{z}_{k,h}^{(n)} \in \mathbb{R}$.
B. Monotonicity Study of $\chi \left( y_h; y_{-h}^{(n)} \right)$ and Evaluation of $\mathcal{F}_p$

Before proceeding further, let us observe that both the characterization of the objective function monotonicities and the feasible set derivation can be performed by means of a change of variable, that defines a one-to-one monotonically increasing mapping. To this end, let us consider $t = \tan (y_h / 2)$. In the transformed domain, the objective function of Problem (16) can be rewritten as

$$R_h^{(n)}(t) = \frac{a_{1,h}^{(n)} t^2 + b_{1,h}^{(n)} t + c_{1,h}^{(n)}}{a_{2,h}^{(n)} t^2 + b_{2,h}^{(n)} t + c_{2,h}^{(n)}},$$

where

$$a_{1,h}^{(n)} = b_{1,h}^{(n)} - \Re \{ a_{1,h}^{(n)} \}, a_{2,h}^{(n)} = d_t^{(n)} - \Re \{ c_t^{(n)} \},$$

$$b_{1,h}^{(n)} = -2 \Re \{ a_{1,h}^{(n)} \}, b_{2,h}^{(n)} = -2 \Re \{ c_{1,h}^{(n)} \},$$

$$c_{1,h}^{(n)} = b_{1,h}^{(n)} + \Re \{ a_{2,h}^{(n)} \}, c_{2,h}^{(n)} = d_t^{(n)} + \Re \{ c_{1,h}^{(n)} \}. $$

Similarly, $R_h^{(n)}(t) \leq z_{k,h}^{(n)}$ can be cast as

$$a_{k,h}^{(n)} t^2 + b_{k,h}^{(n)} t + c_{k,h}^{(n)} \leq 0,$$

where

$$a_{k,h}^{(n)} = -\Re \{ z_{k,h}^{(n)} \}, a_{k,h}^{(n)} = c_{k,h}^{(n)},$$

$$b_{k,h}^{(n)} = -2 \Re \{ z_{k,h}^{(n)} \}, b_{k,h}^{(n)} = -2 \Re \{ c_{k,h}^{(n)} \},$$

$$c_{k,h}^{(n)} = R \{ z_{k,h}^{(n)} \} - c_{k,h}^{(n)}.$$

Let us first characterize the behavior of the objective function $R_h^{(n)}(t)$. To this end, note that $a_{2,h}^{(n)} t^2 + b_{2,h}^{(n)} t + c_{2,h}^{(n)} > 0$, and $\forall t$, which implies that either $a_{2,h}^{(n)} > 0$ and $b_{2,h}^{(n)} - 4a_{2,h}^{(n)} c_{2,h}^{(n)} \leq 0$, or $a_{2,h}^{(n)} = b_{2,h}^{(n)} = 0$ and $c_{2,h}^{(n)} > 0$. As to the first-order derivation of $R_h^{(n)}(t)$, it is given by

$$R_h^{(n)}(t) = \frac{\hat{\gamma}_h^{(n)} t^2 + \hat{e}_h^{(n)} t + \hat{f}_h^{(n)}}{a_{2,h}^{(n)} t^2 + b_{2,h}^{(n)} t + c_{2,h}^{(n)}},$$

where $\hat{\gamma}_h^{(n)} = a_{1,h}^{(n)} b_{2,h}^{(n)} - a_{1,h}^{(n)} b_{1,h}^{(n)}$, $\hat{e}_h^{(n)} = 2(a_{1,h}^{(n)} b_{2,h}^{(n)} - a_{2,h}^{(n)} c_{1,h}^{(n)})$, and $\hat{f}_h^{(n)} = b_{1,h}^{(n)} c_{2,h}^{(n)} - b_{2,h}^{(n)} c_{1,h}^{(n)}$. According to Lemma 2 (reported below), if $\hat{\gamma}_h^{(n)} \neq 0$, $R_h^{(n)}(t)$ admits two stationary points. In particular, $R_h^{(n)}(t)$ exhibits the following behavior:

- if $\hat{\gamma}_h^{(n)} = 0$, it follows that
  - if $\hat{e}_h^{(n)} = 0$, it follows that $\hat{f}_h^{(n)} = 0$, implying that $R_h^{(n)}(t)$ is a constant function;
  - if $\hat{e}_h^{(n)} > 0$ (see Fig. 6 (a) for a notional example), $R_h^{(n)}(t)$ is strictly decreasing over $t < -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$, and strictly increasing over $t > -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$. Thus, the minimum point is $t_{s,h} = -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$, and the finite supremum is achieved at $\pm \infty$;
  - if $\hat{e}_h^{(n)} < 0$ (see Fig. 6 (b) for a notional example), $R_h^{(n)}(t)$ is strictly increasing over $t < -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$, and strictly decreasing over $t > -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$. Thus, the maximum point is $t_{s,h} = -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$, and the finite infimum is achieved at $\pm \infty$;
- if $\hat{\gamma}_h^{(n)} \neq 0$, the roots of (47) are ruled by the discriminant
  $$\Delta_h = \hat{e}_h^{(n)2} - 4 \hat{f}_h^{(n)} \hat{\gamma}_h^{(n)},$$
  which are given by $t_{s,h} = -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$, and $t_{s,h} = -\hat{f}_h^{(n)}/\hat{e}_h^{(n)}$;
  - if $\hat{\gamma}_h^{(n)} > 0$ (see Fig. 6 (c) for a notional example), $R_h^{(n)}(t)$ is strictly increasing over $t < t_{s,h}$, strictly decreasing over $t_{s,h} < t < t_{s,h}$, and strictly increasing over $t > t_{s,h}$;
  - if $\hat{\gamma}_h^{(n)} < 0$ (see Fig. 6 (d) for a notional example), $R_h^{(n)}(t)$ is strictly decreasing over $t < t_{s,h}$, strictly increasing over $t_{s,h} < t < t_{s,h}$, and strictly decreasing over $t > t_{s,h}$.

As already pointed out, since $y_h \in [-\pi, \pi]$ is mapped to $t \in [-\infty, \infty]$ by a strictly increasing function, the monotonicities of $\chi \left( y_h; y_{-h}^{(n)} \right)$ are directly derived starting from those of $R_h^{(n)}(t)$. For instance, suppose $\hat{\gamma}_h^{(n)} > 0$, $\chi \left( y_h; y_{-h}^{(n)} \right)$ is strictly increasing over $-\pi < y_h < \phi_{s,h}$ with $\phi_{s,h} = 2 \arctan(t_{s,h})$, then decreasing over $\phi_{s,h} < y_h < \phi_{s,h}$ with $\phi_{s,h} = 2 \arctan(t_{s,h})$, and increasing over $\phi_{s,h} < y_h < \pi$. As a result, the maximum and minimum points $\phi_{s,h}, \psi_{s,h}$ of the objective function $\chi \left( y_h; y_{-h}^{(n)} \right)$ are

$$\phi_{s,h} = \begin{cases} 2 \arctan(\hat{f}_h^{(n)}/\hat{e}_h^{(n)}), & \text{if } \hat{\gamma}_h^{(n)} = 0, \hat{e}_h^{(n)} < 0, \\ -\pi, & \text{if } \hat{\gamma}_h^{(n)} = 0, \hat{e}_h^{(n)} > 0, \\ 2 \arctan(-\hat{e}_h^{(n)}/\sqrt{\Delta_h}/2d_{\hat{\gamma}_h}^{(n)}), & \text{if } \hat{\gamma}_h^{(n)} \neq 0, \end{cases}$$

and

$$\psi_{s,h} = \begin{cases} -\pi, & \text{if } \hat{\gamma}_h^{(n)} = 0, \hat{e}_h^{(n)} < 0, \\ 2 \arctan(\hat{f}_h^{(n)}/\hat{e}_h^{(n)}), & \text{if } \hat{\gamma}_h^{(n)} = 0, \hat{e}_h^{(n)} > 0, \\ 2 \arctan(-\hat{e}_h^{(n)}/\sqrt{\Delta_h}/2d_{\hat{\gamma}_h}^{(n)}), & \text{if } \hat{\gamma}_h^{(n)} \neq 0. \end{cases}$$

Let us now focus on the feasible set evaluation, which is not empty being $y^{(n)}$ a feasible solution to $\mathcal{P}_p$. Precisely, omitting the dependence on $d$ and $n$ for notational simplicity and considering the continuous phase case, the feasible set in the transformed domain is given by

$$\mathcal{F}_\infty = (\cap_{k=1}^K S_k) \cap \mathcal{P}_\infty,$$

where

$$\mathcal{S}_k = \left\{ t : a_{k,h}^{(n)} t^2 + b_{k,h}^{(n)} t + c_{k,h}^{(n)} \leq 0 \right\},$$

and $\mathcal{P}_\infty = [\tan(\delta/2), \tan(\delta/2)]$. As shown in [38], $\mathcal{F}_\infty$ can be cast as

$$\mathcal{F}_\infty = \{ K \}_{i=1}^K \left[ l_i, u_i \right],$$

with $l_i \leq u_i < l_{i+1}, i = 1, \ldots, K$, $K = 1 \leq K$, whose

11Note that if $\hat{\gamma}_h^{(n)} = 0$ and $\hat{e}_h^{(n)} = 0$, $\chi \left( y_h; y_{-h}^{(n)} \right)$ is a constant function, thus the maximum and minimum points are assigned as $\phi_{s,h} = -\pi$ and $\psi_{s,h} = 0$. 

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Lemma 2. If $\tilde{\bar{d}}_h^{(n)} \neq 0$, Eq. (47) admits two roots.

Proof: Let us proceed by contradiction and assume $\Delta_h^{(n)} \leq 0$. If $\tilde{\bar{d}}_h^{(n)} \neq 0$, which entails that $\phi_{g,h}^{(n)} > 0$, two situations may occur:

1) $\tilde{\bar{d}}_h^{(n)} > 0$, which implies that $R_h^{(n)}(t) \geq 0, \forall t$; thus, $R_h^{(n)}(t)$ is a non-decreasing function and in particular, $\lim_{t \to -\infty} R_h^{(n)}(t) \leq R_h^{(n)}(0) \leq \lim_{t \to +\infty} R_h^{(n)}(t)$.

2) $\tilde{\bar{d}}_h^{(n)} < 0$, leading to $R_h^{(n)}(t) \leq 0, \forall t$; hence, $R_h^{(n)}(t)$ is a non increasing function implying that $\lim_{t \to -\infty} R_h^{(n)}(t) \geq R_h^{(n)}(0) \geq \lim_{t \to +\infty} R_h^{(n)}(t)$.

Based on (43), $\lim_{t \to +\infty} R_h^{(n)}(t) = \lim_{t \to -\infty} R_h^{(n)}(t) = \frac{\alpha_1^{(n)}}{\alpha_2^{(n)}}$, that according to the aforementioned monotonicities leads to $R_h^{(n)}(t) = \frac{\alpha_1^{(n)}}{\alpha_2^{(n)}} \forall t$. Hence, $R_h^{(n)}(t) = 0, \forall t$, namely, $\tilde{\bar{d}}_h = \hat{\epsilon}_h = \tilde{\bar{d}}_h = 0$, which contradicts the assumption $\tilde{\bar{d}}_h^{(n)} \neq 0$.

C. Proof of Proposition III.1

Proof: To solve Problem $\mathcal{P}_{\beta,\gamma}^{(n)}$, it is convenient to distinguish among different situations according to the specific instances of $\mathcal{F}_\infty$ and $\phi_{g,h}^{(n)}$. Evidently, if $\phi_{g,h}^{(n)}$ is feasible, i.e., $\phi_{g,h}^{(n)} \in \mathcal{F}_\infty$, of course $y_h^{(n)} = \phi_{g,h}^{(n)}$.

Let us now suppose that $\phi_{g,h}^{(n)}$ is outside of $[l_1^{\infty}, u_1^{\infty}]$, of course $y_h^{(n)} = \phi_{g,h}^{(n)}$.

- if $-\pi < \phi_{g,h}^{(n)} \leq l_1^{\infty}$, the optimal solution depends on the actual monotonicities of the objective function:
  - if $\tilde{\bar{d}}_h^{(n)} = 0$ and $\bar{e}_h^{(n)} < 0$, or $\tilde{\bar{d}}_h^{(n)} > 0$, $\chi \left(y_h; y_h^{(n)} \right)$ monotonically decreases over $\phi_{g,h}^{(n)} \leq y_h < \pi$, which implies $y_h^* = \bar{u}_1^{\infty}$;
  - if $\tilde{\bar{d}}_h^{(n)} > 0$, $\chi \left(y_h; y_h^{(n)} \right)$ monotonically decreases over $\phi_{g,h}^{(n)} \leq y_h \leq 0$, and then strictly increases over $y_h < \phi_{g,h}^{(n)} < y_h < 0$, then $y_h^* \in [l_1^{\infty}, u_1^{\infty}]$;

- if $\phi_{g,h}^{(n)} \geq u_1^{\infty}$, a situation dual to $\phi_{g,h}^{(n)} \leq l_1^{\infty}$ occurs:
  - if $\tilde{\bar{d}}_h^{(n)} = 0$ and $\bar{e}_h^{(n)} < 0$, or $\tilde{\bar{d}}_h^{(n)} > 0$, $\chi \left(y_h; y_h^{(n)} \right)$ monotonically increases over $-\pi \leq y_h \leq \phi_{g,h}^{(n)}$, which implies $y_h^* = \bar{u}_1^{\infty}$;
  - if $\tilde{\bar{d}}_h^{(n)} < 0$, $\chi \left(y_h; y_h^{(n)} \right)$ monotonically decreases over $-\pi \leq y_h \leq \phi_{g,h}^{(n)}$, and then decreases over $\phi_{g,h}^{(n)} < y_h \leq \phi_{g,h}^{(n)}$, thus $y_h^* \in [l_1^{\infty}, u_1^{\infty}]$.

- if $\phi_{g,h}^{(n)} = -\pi$, two situations may occur:
  - if $\tilde{\bar{d}}_h^{(n)} = 0$ and $\bar{e}_h^{(n)} = 0$, $\chi \left(y_h; y_h^{(n)} \right)$ is a constant for all $y_h$, thus $y_h^* = \bar{u}_1^{\infty}$;
  - if $\tilde{\bar{d}}_h^{(n)} = 0$ and $\bar{e}_h^{(n)} > 0$, $\chi \left(y_h; y_h^{(n)} \right)$ monotonically decreases over $\phi_{g,h}^{(n)} \leq y_h \leq \phi_{s,h}^{(n)}$, and then increases over $\phi_{s,h}^{(n)} < y_h < \pi$, then $y_h^* \in [l_1^{\infty}, u_1^{\infty}]$.
Hence in this case, the optimal solution to Problem $P^h_\infty$ is
\[ y^*_h = \arg \max_{y_h \in \{\hat{l}_1^\infty, \tilde{u}_h^\infty\}} \chi \left( y_h; y_{-h}^* \right). \]

Finally, if $\phi_{g,h} \in [\hat{l}_1^\infty, \tilde{u}_h^\infty]$ but $\phi^{(n)}_{g,h} \notin \mathcal{F}_\infty$, $\phi^{(n)}_{g,h}$ belongs to $[\hat{l}_1^\infty, \tilde{u}_h^\infty + 1]$ with $r^*$ the highest index such that $\hat{u}_{r^*}^\infty < \phi^{(n)}_{g,h}$.

It follows that
- if $\hat{l}_1^{(n)} h < 0$ and $\epsilon^{(n)} h < 0$, $\chi \left( y_h; y_{-h}^* \right)$ monotonically increases over $-\pi \leq y_h \leq \phi^{(n)}_{g,h}$, then decreases over $\phi^{(n)}_{g,h} < y_h < \pi$, thus $y_h \in \{\hat{l}_1^\infty, \tilde{u}_h^\infty + 1\};$
- if $\hat{l}_1^{(n)} h > 0$, $\chi \left( y_h; y_{-h}^* \right)$ monotonically increases over $\hat{l}_1^\infty \leq y_h \leq \phi^{(n)}_{g,h}$; and it is quasi-convex over $[\phi^{(n)}_{s,h}, \pi]$, thus $y_h \in \{\hat{l}_1^\infty, \tilde{u}_h^\infty, \hat{l}_{r^*+1}^\infty, \tilde{u}_{r^*+1}^\infty\};$
- if $\hat{u}_{r^*}^\infty < \phi^{(n)}_{g,h}$, and it is quasi-convex over $[-\pi, \phi^{(n)}_{g,h}]$, thus $y_h \in \{\hat{l}_1^\infty, \tilde{u}_h^\infty, \hat{l}_{r^*+1}^\infty, \tilde{u}_{r^*+1}^\infty\}.$

As a result, in general terms, $y^*_h \in \{[\hat{l}_1^\infty, \tilde{u}_h^\infty, \hat{l}_{r^*+1}^\infty, \tilde{u}_{r^*+1}^\infty]\}$.

\section*{D. Proof of Corollary III.1}

\textbf{Proof:} Two situations may occur: there exists an index $q^{*} \in \{1, \ldots, K_M\}$ satisfying $\hat{l}_1^{q^*} \leq \phi^{(n)}_{g,h} \leq \tilde{u}_{q^*}^\infty$ or such an index does not exist.

As to the former case, $\phi^{(n)}_M = \left[ \frac{\phi^{(n)}_M}{2\pi} \right] \times \frac{2\pi}{\hat{l}_1^\infty}$ and $\phi^{(n)}_u = \left[ \frac{\phi^{(n)}_M}{2\pi} \right] \times \frac{2\pi}{\hat{l}_1^\infty}$ represent the closest feasible points from below and from above, respectively, to $\phi^{(n)}_{g,h}$. To proceed further, let us consider the optimal solution to the relaxed problem

\[
\max_{y_h} \chi \left( y_h; y_{-h}^* \right)
\text{ s.t. } y_h \in \{\hat{l}_1^M, \phi^{(n)}_M\} \bigcup \{\phi^{(n)}_u, \tilde{u}_h^M, \hat{l}_1^{K_M+1}\}.
\] (56)

It follows that
- if $\phi^{(n)}_{g,h} > \phi^{(n)}_{s,d}$ (i.e., either $\hat{l}_1^{(n)} h = 0$ and $\epsilon^{(n)} h < 0$, or $\hat{l}_1^{(n)} h > 0$), $\chi \left( y_h; y_{-h}^* \right)$ monotonically decreases over $\phi^{(n)}_{s,d} \leq y_h < \pi$, and $\phi^{(n)}_M$ is the maximum point over $[\phi^{(n)}_u, \pi];$
- if $\phi^{(n)}_{s,d} \leq \hat{l}_1^{M}$, $\chi \left( y_h; y_{-h}^* \right)$ monotonically increases over $\hat{l}_1^{M} \leq y_h \leq \phi^{(n)}_M$, and $\phi^{(n)}_M$ is the maximizer over $[\hat{l}_1^{M}, \phi^{(n)}_M]$, implying that the optimal solution to (56) is $y^*_h = \arg \max_{y_h \in \{\phi^{(n)}_M, \phi^{(n)}_u\}} \chi \left( y_h; y_{-h}^* \right);$
- instead if $-\pi < \hat{l}_1^{M} < \phi^{(n)}_{s,d}$, $\chi \left( y_h; y_{-h}^* \right)$ monotonically decreases over $-\pi \leq y_h \leq \hat{l}_1^{M}$, thus $\chi \left( y_h; y_{-h}^* \right) = \chi \left( \hat{l}_1^{M}; y_{-h}^* \right),$ implying that the optimal solution is $y^*_h = \arg \max_{y_h \in \{\phi^{(n)}_M, \phi^{(n)}_u\}} \chi \left( y_h; y_{-h}^* \right);$
- if $\phi^{(n)}_{s,d} < \phi^{(n)}_{s,d}$, following a line of reasoning similar to that for $\phi^{(n)}_{g,h} > \phi^{(n)}_{s,d}$, it can be concluded that $y^*_h = \arg \max_{y_h \in \{\phi^{(n)}_M, \phi^{(n)}_u\}} \chi \left( y_h; y_{-h}^* \right).$

Let us now focus on the latter situation. In this case let us consider the following relaxed problem,

\[
\begin{aligned}
\max_{y_h} & \chi \left( y_h; y_{-h}^* \right)
\text{ s.t. } y_h \in \{\hat{l}_1^{M}, \tilde{u}_h^M\},
\end{aligned}
\] (57)

Now, according to Proposition III.1,
- if $\phi^{(n)}_{g,h} \notin [\hat{l}_1^{M}, \tilde{u}_h^M]$, the optimal solution to (57) is $y^*_h = \arg \max_{y_h \in \{\hat{l}_1^{M}, \tilde{u}_h^M\}} \chi \left( y_h; y_{-h}^* \right);$ and
- otherwise, denoting by $r^* = \max r$, the optimal solution to (57) is $y^*_h = \arg \max_{y_h \in \{\hat{l}_1^{M}, \tilde{u}_h^M\}} \chi \left( y_h; y_{-h}^* \right).$

Being the optimal solution to the relaxed versions of $P^h_\infty$ in Problems (56) and (57) feasible to Problem $P^h_\infty$, the proof is concluded.

\section*{E. Derivation of the Surrogate Function $\tilde{f}_1(x_1; q^{(l)})$}

Let us observe that the objective function of Problem (30) restricted to $q_1 = x_1$ is given by

\[
\tilde{f}(x_1^T, q_2^T) = x_1^T M_1(q_1) x_1 - \beta x_1^T R x_1
\]

where $M_1(q_1) = M_1(q_1) + q g(q_1) I_{N}/N$, and $R = diag \{s_0\}^T R diag \{s_0\}$, which can be expressed as $f(x_1^T, q_2^T) = g(x_1, x_1, x_1, q_2) - \beta x_1^T R x_1$, with

\[
g(x_1, x_1, x_1, q_2) = x_1^T M_1(q_1) x_1 - \operatorname{tr} \left( M_2(q_1) x_1 \right).
\]

Being $g(x_1, X_1, q_2)$ jointly convex with respect to $x_1$ and $X_1$, where $M_1(q_1), M_2(q_2), X_1$ are positive definite matrices, a tight expansion is provided by its first-order approximation around any given point $(q_1, Q_1)$, which yields [42]

\[
u \left( x_1, X_1; q_1, Q_1, q_2 \right) = \frac{1}{2} \| \left[ \hat{q}_1^T M_1(q_2) x_1 \right] - \hat{q}_1^T M_1(q_1) q_1 - \operatorname{tr} \left( M_2(q_1) Q_1 \right) \|_2
\]

Denoting by $\hat{u}(x_1; q_1, q_2) = u(x_1, x_1, x_1, q_1, q_1, q_1, q_1, q_1, q_2) + R \nu (x_1; q_1, q_2)$, it follows that

- $f(x_1^T, q_2^T) \geq \hat{u}(x_1; q_1, q_2) - \beta x_1^T R x_1$,
- $f(x_1^T, q_2^T) \geq \hat{u}(x_1, x_1, q_2) - \beta x_1^T R x_1$.

Now, letting $v(x_1; q_1, q_2) = \hat{u}(x_1; q_1, q_2) - \beta x_1^T R x_1$, after some algebraic manipulations it follows that

\[
v(x_1; q_1, q_2) = - \beta x_1^T P(q_1, q_2) x_1 + \Re \left\{ v(q_1, q_2)^{i} x_1 \right\},
\]

where

\[
P(q_1, q_2) = \frac{\hat{q}_1^T M_1(q_1) q_1}{\left( \hat{q}_1^T M_1(q_2) q_1 \right)^2} M_2(q_2) + \beta R.
\]
which is a valid surrogate function of \( \hat{\xi}^\ast \) optimality condition, i.e., nulling the derivative of the objective function.

Note that \( u \), \( \hat{\xi} = \{ \hat{x}_1, (\xi_2 + \phi) \} \) and

\[
\begin{align*}
q_1 &= \hat{z}^T (q_1, q_2) = 2 \left( \lambda I - \hat{p}(q_1, q_2) \right) \hat{q}_1 + w_1 (q_1, q_2), \\
r_1 (q_1, q_2) &= \hat{q}_1^T \left( \hat{p}(q_1, q_2) - \lambda I \right) \hat{q}_1 - \lambda N,
\end{align*}
\]

which is a valid surrogate function of \( \hat{f}([q_1^T, q_2^T]^T) \), since it satisfies the properties (P1)-(P3).

**F. The Solution of \( q_i^{(i+1)*} \) in HIVAC**

As to the continuous phase codes, candidate optimal solutions are the boundary points satisfying the first order optimality condition, i.e., nulling the derivative of the objective function.

To proceed further, let us observe that

\[
\begin{align*}
\max_{x_i} & \quad \tilde{f}_1 (x_i; q^{(i)}) \\
\text{s.t.} & \quad x_i \in \mathcal{D}_p
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\max_{\phi_i} & \quad h(\phi_i) \\
\text{s.t.} & \quad \phi_i \in \Psi_p
\end{align*}
\]

with \( h(\phi_i) = \tilde{f}_1 (e^{j\phi_i}; q^{(i)}) \). Indeed, denoting by \( \phi_i^* \) the optimal solution to \( (62) \), \( e^{j\phi_i^*} \) is an optimal solution to \( (61) \).

Let us now observe that

\[
\frac{dh(\phi_i)}{d\phi_i} = \frac{dh(2 \arctan(\tan(\phi_i/2)))}{d\phi_i} = \frac{d\tilde{f}(\xi)}{d\xi} \frac{1}{2 \cos^2(\phi_i/2)},
\]

with \( \tilde{f}(\xi) = h(2 \arctan(\xi)) \) and \( \xi = \tan(\phi_i/2) \), whose closed form expression is

\[
\tilde{f}(\xi) = \frac{u^{(2)}_1 \xi^2 + v^{(2)}_1 \xi + w^{(2)}_1}{u^{(2)}_1 \xi^2 + v^{(2)}_1 \xi + w^{(2)}_1} + \frac{u^{(3)}_1 \xi^2 + v^{(3)}_1 \xi - u^{(3)}_1}{u^{(3)}_1 \xi^2 + v^{(3)}_1 \xi - u^{(3)}_1} + \frac{v^{(1)}_1 \xi^2 - v^{(1)}_1 \xi - v^{(1)}_1}{v^{(1)}_1 \xi^2 - v^{(1)}_1 \xi - v^{(1)}_1} + \frac{1}{1 + \xi^2},
\]

with

\[
\begin{align*}
u^{(2)}_1 &= b_1 + \Re \{ a_1 \}, \\
u^{(3)}_1 &= d_1 - \Re \{ c_1 \}, \\
u^{(2)}_1 &= -\Re \{ b_1 \}, \\
u^{(3)}_1 &= -23 \{ a_1 \}, \\
u^{(2)}_1 &= -23 \{ c_1 \}, \\
u^{(3)}_1 &= -23 \{ c_1 \},
\end{align*}
\]

As a result, the stationary points, belonging to \([-\pi, \pi]\), of the objective in \( (62) \) can be computed solving the equation

\[
\frac{d\tilde{f}(\xi)}{d\xi} = \frac{\xi^2 + \xi + 1}{(u^{(2)}_1 \xi^2 + v^{(2)}_1 \xi + w^{(2)}_1)^2} - \frac{\xi^2 - \xi - 1}{(u^{(3)}_1 \xi^2 - \xi - 1)^2} = 0.
\]

where \( \tilde{f}(\xi) = u^{(1)}_1 v^{(1)}_1 - u^{(2)}_1 v^{(2)}_1, \xi(\xi) = 2(u^{(1)}_1 w^{(1)}_1 - u^{(2)}_1 w^{(2)}_1) \)

After some algebraic manipulation, it is not difficult to show that \( (64) \) is tantamount to solving

\[
\begin{align*}
\zeta_0 \xi^6 + \zeta_0 \xi^4 + \zeta_0 \xi^2 + \zeta_0 = 0,
\end{align*}
\]

where \( \zeta_0 = -u^{(2)}_1 v^{(2)}_1 + \tilde{d}(\xi) \), \( \zeta_5 = \xi(\xi) + 4u^{(2)}_1 v^{(2)}_1 - 2u^{(2)}_1 v^{(2)}_1, \zeta_4 = \xi(\xi) + 2u^{(2)}_1 v^{(2)}_1 + 8u^{(2)}_1 \xi v^{(2)}_1 - v^{(1)}_1 (2u^{(2)}_1 w^{(2)}_1 + v^{(2)}_1), \zeta_1 = \xi(\xi) + 2u^{(2)}_1 v^{(2)}_1 + 4\xi(\xi) + 4v^{(2)}_1 (2u^{(2)}_1 w^{(2)}_1 + v^{(2)}_1) - 2u^{(2)}_1 v^{(2)}_1, \zeta_1 = \xi(\xi) + d(\xi) + v^{(1)}_1 (2u^{(2)}_1 w^{(2)}_1 + v^{(2)}_1) + 8\xi(\xi) + 8u^{(2)}_1 w^{(2)}_1 - v^{(1)}_1 w^{(2)}_1, \zeta_1 = \xi(\xi) + 2u^{(2)}_1 w^{(2)}_1 + v^{(2)}_1 + 4u^{(2)}_1 w^{(2)}_1, \zeta_1 = \xi(\xi) + u^{(2)}_1 w^{(2)}_1, \zeta_0 = \xi(\xi) + u^{(2)}_1 w^{(2)}_1.
\]

The real roots of \( (65) \) are at most six and can be obtained by Matlab "roots" function.

Hence, denoting by \( T_{\infty} = \{ \xi_1, \ldots, \xi_{M} \}, T_{\infty} \leq 6 \), the set of real roots of \( (65) \) belonging to \([-\tan(\delta/2), \tan(\delta/2)]\) and

\[
\tilde{T}_{\infty,i} = \{ -\delta, \delta, 2 \arctan(\xi_1), \ldots, 2 \arctan(\xi_{M}) \},
\]

the optimal solution to \( (62) \) is given by

\[
\phi_i^* = \arg \max_{\phi_i \in \tilde{T}_{\infty,i}} \tilde{f}_1 (e^{j\phi_i}; q^{(i)}).
\]

For the finite alphabet case, candidate optimal solutions are the feasible points closest to the stationary points from below and from above, respectively.

Denoting by \( T_M = \{ m_1, \ldots, m_{MT} \}, T_M \leq 6 \), the real roots of \( (65) \) with \( m_i \in [\tan(\alpha_i \pi), \tan(\pi \omega_i - 1)] \), \( i = 1, \ldots, M \), the stationary points of \( h(\phi_i) \) are \( \{ \phi_1, \ldots, \phi_{MT} \} \) with \( \phi_i = 2 \arctan(m_i), i = 1, \ldots, M \). Then, the sets of feasible points in \( (62) \) closest to \( \phi_i, i = 1, \ldots, M \), from below and from above are

\[
\begin{align*}
\tilde{T}_{1,i} &= \{ \theta_i, \theta_i, \ldots, \theta_i \} \cap T_{\infty}, \\
\tilde{T}_{2,i} &= \{ \theta_i, \theta_i, \ldots, \theta_i \} \cap T_{\infty},
\end{align*}
\]

Hence, letting

\[
\tilde{T}_{M,i} = \{ 2\pi \alpha_i, \pi(\alpha_i + \omega_i - 1), \frac{1}{M} \},
\]

the optimal solution to \( (62) \) for the finite alphabet case is given by

\[
\phi_i^* = \arg \max_{\phi_i \in \tilde{T}_{M,i}} \tilde{f}_1 (e^{j\phi_i}; q^{(i)}).
\]

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Note that if \( \omega_i \leq 14 \), the direct search may require a lower computational complexity.
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