Abstract: Inspired by general relativity, we suggest an approach for long-range potential scattering. In scattering theory, there is a general theory for short-range potential scattering, but there is no general theory for long-range potential scattering. This is because the scattering boundary conditions for all short-range potentials are the same, but for different long-range potentials are different. In this paper, by introducing tortoise coordinates, we convert long-range potential scattering to short-range potential scattering. This allows us to deal with long-range potential scattering as short-range potential scattering. An explicit expression of the scattering wave function for long-range potential scattering is presented, in which the scattering wave function is represented by the tortoise coordinate and the scattering phase shift. We show that the long-range potential scattering wave function is just the short-range potential scattering wave function with a replacement of a common coordinate by a tortoise coordinate. The approach applies not only to scattering but also applies to bound states. Furthermore, in terms of tortoise coordinates, we suggest a classification scheme for potentials. We also discuss the duality between tortoise coordinates.

Keywords: Long-range potential scattering, Tortoise coordinate, Classification scheme
1. Introduction

Scattering theory is an important issue in both physics and mathematics [1–3]. For short-range potential scattering, a general theory has been established. For long-range potential scattering, however, there is no general treatment. The reason why it is more difficult to establish a general theory for long-range potential scattering than that for short-range potential scattering is that the scattering boundary conditions for all short-range potential scatterings are the same, but for different long-range potentials are different. In this paper, inspired by general relativity, we suggest a general theory for long-range potential scattering by introducing tortoise coordinates. The tortoise coordinate allows us to convert a long-range potential to a short-range potential. Then a general treatment for long-range potentials can also be established similar to that of short-range potentials.

A. Potential scattering

The scattering boundary condition is determined by the large-distance asymptotic solution of the radial equation

\[
\frac{d^2 u_l(r)}{dr^2} + \left[ k^2 - \frac{l(l+1)}{r^2} - V(r) \right] u_l(r) = 0.
\] (1)
For short-range potentials, the large-distance asymptotic behavior is dominated by the centrifugal potential $l(l+1)/r^2$ and the large-distance asymptotics of the radial equation (1) is

$$\frac{d^2 u_l(r)}{dr^2} + \left[ k^2 - \frac{l(l+1)}{r^2} \right] u_l(r) \sim_0 0.$$  (2)

This asymptotic equation is independent of the potential $V(r)$. Consequently, the scattering boundary conditions for all short-range potentials are the same and short-range potential scattering can be treated uniformly [4].

For long-range potentials, however, the large-distance asymptotic behavior is dominated by the potential $V(r)$ and the large-distance asymptotics of the radial equation (1) is

$$\frac{d^2 u_l(r)}{dr^2} + \left( k^2 - V(r) \right) u_l(r) \sim_0 0.$$  (3)

This asymptotic equation depends on the potential $V(r)$. Consequently, the scattering boundary condition depends on the potential and in general different potentials lead to different scattering boundary conditions. This is just the reason why it is difficult to obtain a uniform scattering boundary condition for long-range potential scattering.

B. Tortoise coordinates

Inspect a result in general relativity: in the Schwarzschild spacetime, the tortoise coordinate can convert the large-distance asymptotic equation of a long-range potential to that of a short-range potential.

The Schwarzschild spacetime is described by the metric $ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2$ with $f(r) = 1 - \frac{2M}{r}$. The radial Klein-Gordon equation in the Schwarzschild spacetime is

$$\left( 1 - \frac{2M}{r} \right) \frac{d}{dr} \left( 1 - \frac{2M}{r} \right) \frac{du_l(r)}{dr} + \left\{ \omega^2 - \left( 1 - \frac{2M}{r} \right) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right] \right\} u_l(r) = 0.$$  (4)

The large-distance asymptotics of the radial Klein-Gordon equation is

$$\frac{d^2 u_l(r)}{dr^2} + \left( \omega^2 + \frac{4M\omega^2}{r} \right) u_l(r) \sim_0 0.$$  (5)

This is a radial equation with a long-range potential $\frac{4M\omega^2}{r}$.

By introducing the tortoise coordinate $r_* = \int^r \frac{1}{f(r)} dr = r + 2M \ln(r - 2M)$, the radial-time part of the Schwarzschild spacetime becomes conformally flat, $ds^2 = f(r)(-dt^2 + dr^2_*) + r^2 d\Omega^2$, and the large-distance asymptotics of the radial Klein-Gordon equation (4) under the tortoise coordinate becomes [5,6]

$$\frac{d^2 u_l(r_*)}{dr^2_*} + \left[ \omega^2 - \frac{l(l+1)}{r^2} \right] u_l(r_*) \sim_0 0.$$  (6)

This is just the analogue of the large-distance asymptotic equation of short-range potential scattering, Eq. (2).

In a word, in the Schwarzschild spacetime the tortoise coordinate converts the long-range potential scattering to short-range potential scattering. Similar treatments also applied to the Reissner-Nordström spacetime [7].
C. Inspiration

In general relativity, as shown above, the tortoise coordinate converts a long-range potential to a short-range potential. This inspires us to introduce tortoise coordinates to convert a long-range potential to a short-range potential in the scattering problem. Once a long-range potential is converted to a short-range potential, similar to the treatment for short-range potential scattering, we can also establish a general theory for long-range potential scattering.

In this paper, we establish a general theory for potential scattering by introducing the tortoise coordinate which converts a long-range potential to a short-range potential. Under the tortoise coordinate, the large-distance asymptotic behaviors of all long-range potential scattering are the same. Recalling that the reason why one can establish a general theory for short-range potential scattering is that the large-distance asymptotic behaviors of all short-range potentials are the same, we, in virtue of the tortoise coordinate, can establish a general treatment for long-range potential scattering.

Functions can be classified in terms of their asymptotics [8]. Wave equations with different long-range potentials have different asymptotic wave functions, which allows us to classify long-range potentials by the corresponding asymptotic wave function. Concretely, in the classification scheme suggested in the present paper, a long-range potential is converted to a short-range potential by the tortoise coordinate which is determined by the potential. Different long-range potentials correspond to different tortoise coordinates, so the classification of tortoise coordinates classifies the potentials. Especially, a short-range potential under the tortoise coordinate is just the short-range potential itself and the asymptotic wave functions for all short-range potentials are the same, so in this classification scheme, all short-range potentials are classified into one type.

It is worthy to note here that the result obtained in the present paper not only applies to scattering but also applies to potentials which possesses only bound states.

It is revealed that there is a duality in various physical systems [9]. We show the duality for tortoise coordinates and for asymptotic wave functions. We discuss the relation of the classification of potentials and the duality.

Long-range potential scattering is an important subject in the scattering theory [10]. There are many studies on long-range potential scattering, such as the asymptotic completeness of modified wave operators [11], the low-energy expansion of the phase shift [12], the expansion of the scattering phase shift at \( k = 0 \) [13], the low-energy expansion of the partial-wave Jost function [14], the low-energy scattering theory [15], the scattering length and the effective range [16], the late-time dynamics of the wave equation [17], the Schrödinger operator with long-range electrostatic and magnetic potentials [18], the spectral properties of the corresponding long-range potential scattering matrix of the Schrödinger operator [19], the Gell-Mann-Goldberger formula for long-range potentials [20], and the classical long-range potential scattering [21]. In Ref. [22], the author studies the long-range potential scattering by proposing a certain weakening of the standard criterion. There are also studies on scattering including both long-range and short range potentials, such as the quasi-classical limit of quantum mechanical scattering for short-range potentials and long-range potentials [23,24], the short-range and long-range quantum mechanical scattering [25,26], and the low-energy scattering by a potential consisting of a long-range part and a short-range part [27,28]. For short-range potential scattering, a rigorous treatment without the large-distance asymptotics approximation is proposed [4,29]. The duality discuss in the present paper also exists in various physical systems, such as in the scalar field [9] and in the Gross–Pitaevskii equation [30].

The tortoise coordinate is first introduced in general relativity and is widely used in black hole physics [31]. For the Schwarzschild spacetime, the Eddington–Finkelstein coordinate is established on the tortoise coordinate which is convenient to describe the ingoing
and outgoing waves \[31,32\]. In Ref. \[33\], in order to study the Hawking radiation in anti–de Sitter space, the authors introduce the generalized tortoise coordinate for the AdS black hole. In Ref. \[34\], by using the tortoise coordinate, the radial wave equation in gravitational backgrounds of a constant negative curvature is simplified.

In section 2, we introduce tortoise coordinates to convert long-range potentials to short-range potentials. In section 3, we suggest a classification scheme for potentials in terms of the tortoise coordinate. In section 4, we provide a uniform expression for long-range potential scattering. In section 5, we give an alternative expression of tortoise coordinates. Some examples are given in sections 6. In section 7, we give duality relations for tortoise coordinates and for asymmetric wave functions. The conclusions are summarized in section 8.

2. Tortoise coordinates: Converting long-range potentials to short-range potentials

In this section, inspired by general relativity, we introduce tortoise coordinates and show that a long-range potential can be converted to a short-range potential under the tortoise coordinate.

In the following, potentials are divided into two types to be considered: potentials vanishing at \(r \rightarrow \infty\), which have both scattering states and bound states and potentials nonvanishing at \(r \rightarrow \infty\), which have only bound states. It can be seen that all kinds of potentials are converted to short-range potentials including potentials which have only bound states.

2.1. Potentials vanishing at \(r \rightarrow \infty\)

There are two kinds of potentials vanishing at \(r \rightarrow \infty\): the long-range potential, e.g., the Coulomb potential, and the short-range potential, e.g., the Yukawa potential.

In scattering theory, long-range potential scattering cannot be uniformly treated, because the large-distance asymptotic behaviors are different for different long-range potentials. On the contrary, a general theory has been established for short-range potential scattering, because the large-distance asymptotic behaviors for all short-range potentials are the same. In the following, we convert long-range potential scattering to short-range potential scattering by introducing tortoise coordinates. This allow us to establish a general theory for long-range potential scattering.

For a potential \(V(r)\) which vanishes at \(r \rightarrow \infty\), by introducing the tortoise coordinate

\[
r_s = r - \sum_{\eta=1}^{N} \sigma_{\eta} \int_{r}^{\infty} \left( \frac{V(r)}{k^2} \right)^{\frac{\eta}{2}} dr,
\]

where \(\sigma_{\eta} = \Gamma(\eta - 1/2)/(2\sqrt{\pi} \eta! slow),\) there must exist a non-negative integer \(N\) or \(N \rightarrow \infty\), so that the radial equation (1) with the potential \(V(r)\) can be converted to a large-distance asymptotic radial equation of the potential \(2\sigma_{N+1} \left( \frac{V(r)}{k^2} \right)^{N} V(r)\) which is a short-range potential decreasing faster than \(1/r\):

\[
\frac{d^2u_l(r_s)}{dr_s^2} + \left[ k^2 - \frac{l(l+1)}{r^2} - 2\sigma_{N+1} \left( \frac{V(r)}{k^2} \right)^{N} V(r) \right] u_l(r_s) \sim 0.
\]

The large-distance asymptotics of Eq. (8) under the tortoise coordinate is

\[
\frac{d^2u_l(r_s)}{dr_s^2} + k^2 u_l(r_s) \sim 0.
\]
and the solution of Eq. (9) is

\[ u_l(r_*) \sim e^{\pm ikr_*}. \]  

**Proof.** The radial equation (1) with the potential \( V(r) \) which vanishes at \( r \to \infty \) under the tortoise coordinate (7) becomes

\[
\frac{d^2 u_l(r_*)}{dr_*^2} - \frac{\sum_{\eta=1}^N \eta \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta} V'(r)}{1 - \sum_{\eta=1}^N \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta}} \frac{1}{r^2} u_l(r_*) + \frac{1}{1 - \sum_{\eta=1}^N \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta}} \left( k^2 - \frac{l(l+1)}{r^2} - V(r) \right) u_l(r_*) = 0,
\]

where

\[
dr_* = \left[ 1 - \sum_{\eta=1}^N \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta} \right] dr
\]

is used.

At \( r \to \infty \), the coefficient of \( \frac{d^2 u_l(r_*)}{dr_*^2} \) vanishes, i.e.

\[
\frac{\sum_{\eta=1}^N \eta \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta} V'(r)}{1 - \sum_{\eta=1}^N \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta}} \sim 0.
\]

Note that \( V'(r) \) falls off faster than \( \frac{1}{r} \) and vanishes at \( r \to \infty \).

At \( r \to \infty \), the coefficient of \( u_l(r_*) \) becomes

\[
\frac{1}{1 - \sum_{\eta=1}^N \sigma_\eta \left( \frac{V(r)}{k^2} \right)^{\eta}} \left( k^2 - \frac{l(l+1)}{r^2} - V(r) \right) \sim 0
\]

Clearly, there must exist a value of \( N \) so that \( 2\sigma_{N+1} \left( \frac{V(r)}{k^2} \right)^N \) falls faster than \( 1/r \). This proves Eq. (8).

### 2.2. Potentials nonvanishing at \( r \to \infty \)

Potentials nonvanishing at \( r \to \infty \) are long-range distance potentials which has only bound states. In this section, we show that even potentials which possess only bound states can also be converted to short-range potentials by introducing the tortoise coordinate.

For the potential \( V(r) \) which does not vanish at \( r \to \infty \), by introducing the tortoise coordinate

\[
r_* = \frac{1}{4k} \ln \left( \frac{V(r)}{k^2} \right) - \sum_{\eta=1}^N \sigma_\eta \int_{r_0}^r \frac{k^2}{V(\rho)} \rho^{-1/2} d\rho,
\]

there must exist a non-negative integer \( N \) or \( N \to \infty \), so that the radial equation (1) with the potential \( V(r) \) can be converted to a large-distance asymptotic radial equation of the potential \( \frac{1}{2} \frac{k^2}{V(r)} \frac{V'(r)}{\sqrt{V(r)}} \) which is a short-range potential decreasing faster than \( 1/r \):

\[
\frac{d^2 u_l(r_*)}{dr_*^2} + \left( \frac{1}{k^2} - \frac{k^2}{2} \frac{V(r)}{V(r)} \sqrt{\frac{V(\rho)}{V(r)}} \right) u_l(r_*) \sim 0.
\]
The large-distance asymptotics of Eq. (16) under the tortoise coordinate is

\[ \frac{d^2 u_i(r_\ast)}{dr_\ast^2} - k^2 u_i(r_\ast) \sim 0, \quad (17) \]

and the solution of Eq. (17) is

\[ u_i(r_\ast) \sim e^{-kr_\ast}. \quad (18) \]

Proof. The radial equation (1) with the potential which does not vanish at \( r \to \infty \) under the tortoise coordinate (15) becomes

\[
\frac{d^2 u_i(r_\ast)}{dr_\ast^2} + \sum_{\eta=0}^N (\eta - \frac{1}{2}) \sigma_\eta \left( \frac{k^2}{V(r)} \right)^{\eta-1/2} \frac{V'(r)}{V(r)} + \frac{1}{4k} \left( \frac{V'(r)}{V(r)} - \frac{(V(r))'}{V(r)} \right)^2 \frac{d u_i(r_\ast)}{dr_\ast} \\
+ \left[ - \sum_{\eta=0}^N \sigma_\eta \left( \frac{k^2}{V(r)} \right)^{\eta-1/2} + \frac{1}{4k} \frac{V'(r)}{V(r)} \right] \left( k^2 - \frac{l(l+1)}{r^2} - V(r) \right) u_i(r) = 0, \quad (19) 
\]

where

\[ dr_\ast = \left[ - \sum_{\eta=0}^N \sigma_\eta \left( \frac{k^2}{V(r)} \right)^{\eta-1/2} + \frac{1}{4k} \frac{V'(r)}{V(r)} \right] dr \quad (20) \]

is used.

At \( r \to \infty \), the coefficient of \( \frac{d u_i(r_\ast)}{d r_\ast} \) vanishes, i.e.

\[
\sum_{\eta=0}^N (\eta - \frac{1}{2}) \sigma_\eta \left( \frac{k^2}{V(r)} \right)^{\eta-1/2} \frac{V'(r)}{V(r)} + \frac{1}{4k} \left( \frac{V'(r)}{V(r)} - \frac{(V(r))'}{V(r)} \right)^2 \sim 1 \quad (21) 
\]

Note that \( \frac{1}{2} \sqrt{\frac{k^2}{V(r)}} \frac{V'(r)}{V(r)} \) falls off faster than \( 1/r \) and then vanishes at \( r \to \infty \).

At \( r \to \infty \), the coefficients of \( u_i(r_\ast) \) becomes

\[
\frac{1}{\left[ - \sum_{\eta=0}^N \sigma_\eta \left( \frac{k^2}{V(r)} \right)^{\eta-1/2} + \frac{1}{4k} \frac{V'(r)}{V(r)} \right]^2} \left( k^2 - \frac{l(l+1)}{r^2} - V(r) \right) r \sim -k^2 + \frac{k^2}{2V(r)} \frac{V'(r)}{\sqrt{V(r)}} + 2\sigma_{N+1} \left( \frac{k^2}{V(r)} \right)^{N+2} V(r). \quad (22) 
\]

Clearly, there must exist a value of \( N \) so that \( 2\sigma_{N+1} \left( \frac{k^2}{V(r)} \right)^{N+2} V(r) \) falls faster than \( \frac{1}{2} \frac{k^2}{V(r)} \frac{V'(r)}{\sqrt{V(r)}} \) and \( \frac{1}{2} \frac{k^2}{V(r)} \frac{V'(r)}{\sqrt{V(r)}} \) falls faster than \( 1/r \).

It can be seen that the eigenvalue \(-k^2\) in Eq. (16) is less than zero, since the potential considered here has only bound states.

3. Classifying potentials in terms of tortoise coordinates

In this section, we suggest a classification scheme for potentials in terms of tortoise coordinates. Essentially, this scheme classifies potentials by the asymptotic behaviors of the corresponding wave functions. The result given by Eqs. (10) and (18) shows that under the tortoise coordinate the asymptotic wave functions are the same. The difference between asymptotic wave functions is reflected in tortoise coordinates which depend on potentials. Therefore, the classification of tortoise coordinates classifies potentials.
3.1. Potentials vanishing at $r \to \infty$

In the following, we suggest a classification scheme for potentials in terms of the tortoise coordinate according to various values of $N$ in the definition of the tortoise coordinate (7).

The potentials which vanish at $r \to \infty$ can be classified in three types in terms of the value of $N$ in the tortoise coordinate (7):

1. $N = 0$. The potential corresponding to $N = 0$ is a short-range potential satisfying
   \[ \int_a^\infty |V(r)| \, dr < \infty, \quad a > 0, \]
   i.e., the potential $V(r)$ decreases faster than $1/r$ at $r \to \infty$. In this case, the tortoise coordinate is just the radial coordinate itself: $r_* = r$.

2. $N$ is a positive integer. The potential corresponding to a positive integer $N$ is a long-range potential satisfying
   \[ \int_a^\infty |V(r)| r^{A-1} \, dr < \infty, \quad a > 0, \quad A = \frac{1}{N+1}, \]
   \[ \int_b^\infty \frac{1}{|V(r)| r^{B+1+\epsilon}} \, dr < \infty, \quad b > 0, \quad B = \frac{1}{N}, \quad \epsilon \sim 0^+, \]
   i.e., the potential $V(r)$ decreases faster than $1/r^{1/(N+1)}$ and slower than or equally to $1/r^{1/N}$ at $r \to \infty$. Different values of $N$ correspond to different long-range potentials with different potential ranges.

3. $N \to \infty$. The potential corresponding to $N \to \infty$ is another type of long-range potentials satisfying
   \[ \int_a^\infty \frac{|V(r)|}{r} \, dr < \infty, \quad a > 0, \]
   \[ \int_b^\infty \frac{1}{|V(r)| r^{1+\epsilon}} \, dr < \infty, \quad b > 0, \quad \epsilon \sim 0^+. \]

The tortoise coordinate (7) in this case becomes

\[ r_* = \int_r^r \sqrt{1 - \frac{V(r)}{k^2}} \, dr. \]

Under the tortoise coordinate (28), the radial equation (1) becomes

\[ \frac{d^2 u_l(r_*)}{dr_*^2} + \left[ k^2 - \frac{l(l+1)}{r^2} \right] \frac{1}{1 - V(r)/k^2} u_l(r_*) = 0. \]

A special case of the potential corresponding to $N = 0$ is the negative power potential $1/r^a$ with $a > 1$; a special case of the potential corresponding to $N$ equalling a finite positive integer is the negative power potential $1/r^a$ with $0 < a \leq 1$; two special cases of the potentials corresponding to $N \to \infty$ are the constant potential and the potential $V(r) \sim 1/\ln r$.

In the following, we prove the above statements.

Proof. First, we write the radial wave function as

\[ u_l(r) = e^{\pm ikr_*} e^{h(r)}. \]
Substituting Eq. (30) into the radial equation (1) gives an equation of $h(r)$,

$$h''(r) + h'(r)^2 + 2 \left( \frac{e^{\pm i k r_*}}{e^{i k r_*}} \right) h'(r) = -k^2 + \frac{l(l+1)}{r^2} + V(r) - \left( \frac{e^{\pm i k r_*}}{e^{i k r_*}} \right)''. \tag{31}$$

The asymptotics of Eq. (31) at $r \to \infty$ is

$$\pm 2i kh'(r) \sim 2\sigma_{N+1} \left( \frac{V(r)}{k^2} \right)^N V(r). \tag{32}$$

Solving this asymptotic equation gives

$$h(r) \sim \pm \sigma_{N+1} \frac{k}{l} \int_r^r \left( \frac{V(r)}{k^2} \right)^{N+1} dr. \tag{33}$$

To satisfy Eq. (10), we require that

$$h(r) \sim \frac{r}{r^\infty} 0. \tag{34}$$

Eqs. (33) and (34) require that $V(r)$ decreases faster than $1/r^{1+\delta}$ at $r \to \infty$, so

$$\int_a^\infty |V(r)| r^{-1+\delta} dr = \int_a^c |V(r)| r^{-1+\delta} dr + \int_c^\infty |V(r)| r^{-1+\delta} dr \leq \int_a^c |V(r)| r^{-1+\delta} dr + \int_c^\infty 1/r^{1+\delta} dr$$

where $\delta > 0$. Then we have $\int_a^\infty |V(r)| r^{-1+\delta} dr < \infty$ with $a > 0$ and $A = N^{-1}$, since Eq.(35) is finite.

$N = 0$ gives the condition (23), $N$ equaling a nonzero finite integer gives the condition (25), and $N \to \infty$ gives the condition (27), respectively.

Second, we write the radial wave function as

$$u_l(r) = \exp \left( \pm ik \left[ r_* + \sigma_N \int_r^r \left( \frac{V(r)}{k^2} \right)^N dr \right] \right) e^{i g(r)}. \tag{36}$$

Substituting Eq. (36) into the radial equation (1) gives an equation of $g(r)$,

$$g''(r) + g'(r)^2 + 2 \left( \frac{\exp \left( \pm ik \left[ r_* + \sigma_N \int_r^r \left( \frac{V(r)}{k^2} \right)^N dr \right] \right)}{\exp \left( \pm ik \left[ r_* + \sigma_N \int_r^r \left( \frac{V(r)}{k^2} \right)^N dr \right] \right)} \right) g'(r) = -k^2 + \frac{l(l+1)}{r^2} + V(r) - \left( \frac{\exp \left( \pm ik \left[ r_* + \sigma_N \int_r^r \left( \frac{V(r)}{k^2} \right)^N dr \right] \right)}{\exp \left( \pm ik \left[ r_* + \sigma_N \int_r^r \left( \frac{V(r)}{k^2} \right)^N dr \right] \right)} \right)''. \tag{37}$$

The asymptotics of Eq. (37) at $r \to \infty$ reads

$$\pm 2i kg'(r) \sim 2\sigma_N \left( \frac{V(r)}{k^2} \right)^{N-1} V(r). \tag{38}$$

Solving this asymptotic equation gives

$$g(r) \sim \pm \frac{\sigma_N}{ik} \int_r^r \left( \frac{V(r)}{k^2} \right)^N dr. \tag{39}$$
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To satisfy Eq. (10), at $r \to \infty$, the factor $g(r)$ must contribute, i.e.,

$$g(r)^{r \to \infty} \propto 0.$$ (40)

Eqs. (39) and (40) require that $V(r)$ must decrease slower than or equally to $1/r^{1/N}$ at $r \to \infty$, so

$$\int_b^\infty \frac{1}{|V(r)|r^{3/2+\epsilon}} dr = \int_b^c \frac{1}{|V(r)|r^{3/2+\epsilon}} dr + \int_c^\infty \frac{1}{|V(r)|r^{3/2+\epsilon}} dr$$

$$\leq \int_b^c \frac{1}{|V(r)|r^{3/2+\epsilon}} dr + \int_c^\infty \frac{1}{r^{3/2+\epsilon}} dr$$

$$= \int_b^c \frac{1}{|V(r)|r^{3/2+\epsilon}} dr + \int_c^\infty \frac{1}{r^{3/2+\epsilon}} dr,$$ (41)

where $\delta \geq 0$. Then we have $\int_b^\infty \frac{1}{|V(r)|r^{3/2+\epsilon}} dr < \infty$ with $b > 0$, $B = 1/N$, and $c \sim 0^+$, since Eq. (35) is finite.

$N$ equaling a nonzero finite integer gives the condition (24) and $N \to \infty$ gives the condition (26), respectively. Especially, for $N \to \infty$, substituting the tortoise coordinate (28) into Eq. (1) gives

$$\frac{d^2u_i(r_*)}{dr_*^2} - \frac{V'(r)}{2k^2\left(1 - \frac{V(r)}{k^2}\right)^{3/2}} \frac{du_i(r_*)}{dr_*} + k^2 \frac{l(l+1)}{r^2} \left(1 - \frac{V(r)}{k^2}\right) u_i(r_*) = 0.$$ (42)

At $r \to \infty$, the coefficient of $du_i(r_*)/dr_*$ falls off faster than $1/r$ and then vanishes; at the meantime, $r^{-1/2}$ is a short-range potential. This proves Eq. (29).

**Remark.** In this classification scheme, the potentials decaying faster than $r^{-2}$ are classified into one class, for their asymptotic scattering wave function are the same.

In literature, there are two classification schemes: (1) in terms of the comparison of the potential and the centrifugal potential $\frac{1}{r^2}$, and (2) in terms of the asymptotic behavior of scattering wave functions. In the scheme (1), the short-range potential decays faster than $\frac{1}{r^2}$, while in the scheme (2), the short-range potential decays faster than $\frac{1}{r^2}$. The scheme (1) corresponds to the requirement

$$\int_a^\infty |V(r)| dr < \infty, \quad a > 0.$$ (43)

The scheme (2) corresponds to the requirement

$$\int_a^\infty |V(r)| dr < \infty, \quad a > 0.$$ (44)

Usually we adopt the scheme (1) for avoiding unnecessary complications [35]. However, the essential classification scheme is the scheme (2) [35,36].

### 3.2. Potentials nonvanishing at $r \to \infty$

In the following, as above, we classify the potential by the tortoise coordinate according to various values of $N$ in the definition of the tortoise coordinate (15).

*The potentials which does not vanish at $r \to \infty$ can be classified in three types in terms of the value of $N$ in the tortoise coordinate (15):*
1. \( N = 0 \). The potential corresponding to \( N = 0 \) satisfies

\[
\int_{b}^{\infty} \frac{1}{|V(r)|} r dr < \infty,
\]

i.e., the potential \( V(r) \) increases faster than \( r^2 \) at \( r \to \infty \). We might call it the superlong-range potential. The tortoise coordinate (7) in this case becomes

\[
r_\ast = \frac{1}{4k} \ln \frac{V(r)}{k^2} + \int_{r}^{\infty} \sqrt{V(r) - \frac{k^2}{r^2}} dr.
\]

2. \( N \) is a positive integer. The potential corresponding to a positive integer \( N \) is a long-range potentials satisfying

\[
\int_{b}^{\infty} \frac{1}{|V(r)|} r^{A-1} dr < \infty, \quad b > 0, \quad A = \frac{1}{N + 1/2},
\]

\[
\int_{a}^{\infty} V(r) \frac{1}{r^{B+1} + \epsilon} dr < \infty, \quad a > 0, \quad B = \frac{1}{N - 1/2}, \quad \epsilon \sim 0^+,
\]

i.e., the potential \( V(r) \) increases faster than \( r^{1/(N+1/2)} \) and slower than or equally to \( r^{1/(N-1/2)} \) at \( r \to \infty \). Different values of \( N \) correspond to different long-range potentials with different potential ranges.

3. \( N \to \infty \). The potential corresponding to \( N \to \infty \) satisfies

\[
\int_{b}^{\infty} \frac{1}{|V(r)|} r^{-1} dr < \infty, \quad b > 0,
\]

\[
\int_{a}^{\infty} V(r) \frac{1}{r^{1+\epsilon} + \epsilon} dr < \infty, \quad a > 0, \quad \epsilon \sim 0^+.
\]

The tortoise coordinate (15) in this case becomes

\[
r_\ast = \frac{1}{4k} \ln \frac{V(r)}{k^2} + \int_{r}^{\infty} dr \sqrt{\frac{V(r)}{k^2} - 1}.
\]

Under the tortoise coordinate (51), the radial equation (1) becomes

\[
\frac{d^2u_l(r_\ast)}{dr_\ast^2} + \left[ -k^2 - \frac{l(l + 1)}{(V(r))^{2/2} - 1} \right] \frac{V'(r)/k}{\sqrt{V(r)/k^2 - 1} - 1} u_l(r_\ast) = 0.
\]

A special case of the potential corresponding to \( N = 0 \) is \( r^\alpha \) with \( \alpha > 2 \); a special case of the potential corresponding to a finite positive integer \( N \) is the positive power potential \( r^\alpha \) with \( 0 < \alpha \leq 2 \); two special cases of the potential corresponding to \( N \to \infty \) are constant potential and the potential \( V(r) \sim \ln r \).

Proof. First, we write the radial wave function as

\[
u_l(r) = e^{-kr_\ast} e^{q(r)}.
\]

Substituting Eq. (53) into the radial equation (1) gives an equation of \( q(r) \),

\[
q''(r) + q'(r)^2 + 2 \frac{e^{-kr_\ast}}{e^{-kr_\ast}} q'(r) = -k^2 + \frac{l(l + 1)}{r^2} + V(r) - \frac{(e^{-kr_\ast})''}{e^{-kr_\ast}}.
\]
The asymptotics of Eq. (54) at \( r \to \infty \) is

\[
2\sqrt{V(r)}q(r) \sim -2\sigma_{N+1} \left( \frac{k^2}{\sqrt{V(r)}} \right)^{N+1} V(r). \tag{55}
\]

Solving this asymptotic equation gives

\[
q(r) \sim -k\sigma_{N+1} \int^r \left( \frac{k^2}{\sqrt{V(r)}} \right)^{N+1/2} dr. \tag{56}
\]

To satisfy Eq. (18), we require that

\[
q(r) \sim r^{-\infty} 0. \tag{57}
\]

Eqs. (56) and (57) require that \( V(r) \) increases faster than \( r^{N+1/2} \), so

\[
\int_b^\infty \frac{1}{|V(r)|} r^{N+1/2-1} dr = \int_b^c \frac{1}{|V(r)|} r^{N+1/2-1} dr + \int_c^\infty \frac{1}{|V(r)|} r^{N+1/2-1} dr
\]

\[
\leq \int_b^c \frac{1}{|V(r)|} r^{N+1/2-1} dr + \int_c^\infty \frac{1}{r^{1+\delta}} r^{N+1/2-1} dr
\]

\[
= \int_b^c \frac{1}{|V(r)|} r^{N+1/2-1} dr + \int_c^\infty \frac{1}{r^{1+\delta}} dr, \tag{58}
\]

where \( \delta > 0 \). Then we have \( \int_b^\infty \frac{1}{|V(r)|} r^{A-1} dr < \infty \) with \( b > 0 \) and \( A = \frac{1}{N+1/2} \), since Eq. (58) is finite.

\( N = 0 \) gives the condition (45); \( N \) equaling a nonzero finite positive integer gives the condition (47); \( N \to \infty \) gives the condition (49), respectively.

Second, we write the radial wave function as

\[
u_l(r) = \exp\left(-kr + k\sigma_N \int^r \left( \frac{k^2}{\sqrt{V(r)}} \right)^{N-1/2} dr\right) e^{p(r)}. \tag{59}
\]

Substituting Eq. (59) into the radial equation Eq. (1) gives an equation of \( p(r) \),

\[
p''(r) + p'(r)^2 + 2\left[ \exp\left(-kr + k\sigma_N \int^r \left( \frac{k^2}{\sqrt{V(r)}} \right)^{N-1/2} dr\right) \right]' p'(r) = -k^2 + \frac{l(l+1)}{r^2} + V(r) - \left[ \exp\left(-kr + k\sigma_N \int^r \left( \frac{k^2}{\sqrt{V(r)}} \right)^{N-1/2} dr\right) \right]'' \tag{60}
\]

The asymptotics of Eq. (60) at \( r \to \infty \) is

\[
2\sqrt{V(r)}p'(r) \sim -2\sigma_N \left( \frac{k^2}{\sqrt{V(r)}} \right)^N V(r). \tag{61}
\]

Solving this asymptotic equation gives

\[
p(r) \sim -k\sigma_N \int^r \left( \frac{k^2}{\sqrt{V(r)}} \right)^{N-1/2} dr. \tag{62}
\]

To satisfy Eq. (18), the factor \( p(r) \) must contribute at \( r \to \infty \), i.e.,

\[
p(r) \sim r^{-\infty} 0. \tag{63}
\]
Eqs. (62) and (63) require that $V(r)$ must increase slower than or equally to $r^{1/(N-1/2)}$ when $r \to \infty$, so

$$\int_{a}^{\infty} |V(r)| \frac{1}{r^{N/2} + 1 + \varepsilon} dr = \int_{a}^{c} |V(r)| \frac{1}{r^{N/2} + 1 + \varepsilon} dr + \int_{c}^{\infty} |V(r)| \frac{1}{r^{N/2} + 1 + \varepsilon} dr$$

$$\leq \int_{a}^{c} |V(r)| \frac{1}{r^{N/2} + 1 + \varepsilon} dr + \int_{c}^{\infty} r^{-\delta} \frac{1}{r^{N/2} + 1 + \varepsilon} dr$$

$$= \int_{a}^{c} |V(r)| \frac{1}{r^{N/2} + 1 + \varepsilon} dr + \int_{c}^{\infty} \frac{1}{r^{1+\varepsilon+\delta}} dr,$$

(64)

where $\delta \geq 0$. Then we have $\int_{a}^{\infty} |V(r)| \frac{1}{r^{N/2} + 1 + \varepsilon} dr < \infty$ with $a > 0$, $B = \frac{1}{N-1/2}$, and $\varepsilon \to 0^+$, since Eq. (64) is finite.

$N$ equaling a nonzero finite integer gives the condition (48) and $N \to \infty$ gives the condition (50), respectively. Especially, for $N \to \infty$, substituting the tortoise coordinate (51) into Eq. (1) gives

$$\frac{d^2 u_i(r_s)}{dr_s^2} + \frac{1}{4k} \left[ \frac{V''(r)}{V(r)} - \left( \frac{V'(r)}{V(r)} \right)^2 \right] + \frac{V'(r)}{2k^2 \sqrt{\frac{k}{2k^2} - 1}} \frac{du_i(r_s)}{dr_s} + \frac{1}{\left( \frac{V'(r)}{2k^2 \sqrt{\frac{k}{2k^2} - 1}} \right)^2} \left[ k^2 - \frac{l(l+1)}{r^2} - V(r) \right] u_i(r_s) = 0. $$

(65)

At $r \to \infty$, the coefficient of $du_i(r_s)/dr_s$ falls off faster than $1/r$ and then vanishes. At the meantime,

$$\left( \frac{V'(r)}{4kV(r)} + \sqrt{\frac{k}{2k^2} - 1} \right)^2 \left[ k^2 - \frac{l(l+1)}{r^2} - V(r) \right] - k^2 - \frac{l(l+1)}{r^{2k^2}} + \frac{V'(r)/k}{2k^2 \sqrt{\frac{k}{2k^2} - 1}}.$$

(66)

Clearly, $\frac{l(l+1)}{2k^2} - \frac{V'(r)/k}{2k^2 \sqrt{\frac{k}{2k^2} - 1}}$ is a short-range potential. This proves Eq. (52).

4. Scattering wave functions for long-range potential scattering

In scattering theory, a general theory is established for short-range potential scattering, in which a uniform expression of a scattering wave function is given and all the information of scattering is embodied in a scattering phase shift [1,37,38]. Nevertheless, there is no such a general treatment for long-range potential scattering. The reason is that the large-distance behaviors for short-range potential scattering are the same, but for long-range potential scattering are different.

In the above we show that by introducing tortoise coordinates, long-range potential scattering can be converted to short-range potential scattering. This allows us to develop a general theory for long-range potential scattering just like that for short-range potential scattering. After converting long-range potential scattering to short-range potential scattering, we can also give a uniform expression of a scattering wave function under tortoise coordinates and describe long-range-potential scattering by a phase shift. The difference between large-distance asymptotic wave functions of different potentials is reflected in tortoise coordinates.

Potentials vanishing at $r \to \infty$, which are discussed in section 2.1, can have scattering states. In the following we show that the scattering wave function can be uniformly expressed in terms of tortoise coordinates for both long-range potential scattering and short-range potential scattering.
The radial equation with a potential satisfying the conditions \((24)\) and \((25)\) has two linear independent solutions, \(F_i(r)\) and \(G_i(r)\) which satisfy the condition \((10)\). The large-distance asymptotic wave function can be then expressed as

\[
u_i(r) \approx C_i \int_{r}^{\infty} e^{- \sqrt{-1} V(r)} + D_i e^{\sqrt{-1} V(r)}.
\]

By introducing the phase shift

\[
e^{2i\delta_i} = \frac{D_i}{C_i},
\]

the large-distance asymptotic scattering wave function can be then written as

\[
u_i(r) \approx \sin \left( kr - \frac{l\pi}{2} + \delta_i \right).
\]

Comparing with short-range potential scattering in which \(u_i(r) \approx \sin \left( kr - \frac{l\pi}{2} + \delta_i \right)\), we can see that the asymptotic scattering wave function for long-range potential scattering is just to replace coordinate \(r\) with the tortoise coordinate \(r_\ast\) in the asymptotic wave function of short-range potential scattering.

5. Alternative expressions of tortoise coordinates

In this section, we give another expression of tortoise coordinates \((7)\) and \((15)\) in terms of the Gauss hypergeometric function.

5.1. The tortoise coordinate for potentials vanishing at \(r \to \infty\)

For potentials vanishing at \(r \to \infty\), the tortoise coordinate \((7)\) can be expressed as

\[r_\ast = \int^r dr \sqrt{1 - \frac{V(r)}{k^2}} + \frac{\Gamma(N + 1/2)}{2\sqrt{\pi} \Gamma(N + 2)} \int^r dr \frac{V(r)}{k^2} F_1 \left( \frac{1}{2}, N + 1/2, \frac{V(r)}{k^2} \right)^{N+1}.
\]

Here \(F_1\) is the Gauss hypergeometric function \([39]\).

Proof: Expand \(\sqrt{1 - \frac{V(r)}{k^2}}\) as

\[1 - \frac{V(r)}{k^2} = \sum_{\eta=0}^{\infty} (-1)^\eta \frac{\sqrt{\pi}}{2\Gamma(3/2 - \eta)\Gamma(\eta + 1)} \left( \frac{V(r)}{k^2} \right)^\eta
\]

\[= \sum_{\eta=0}^{\infty} (-1)^\eta \frac{\sqrt{\pi}}{2\Gamma(3/2 - \eta)\eta!} \left( \frac{V(r)}{k^2} \right)^\eta
\]

\[+ \sum_{\eta=0}^{\infty} (-1)^{\eta+1+N} \frac{\sqrt{\pi}}{2\Gamma(1/2 - \eta - N)\Gamma(\eta + N + 2)} \left( \frac{V(r)}{k^2} \right)^{\eta+N+1}.
\]

Eq. \((71)\) can be rewritten as

\[1 - \frac{V(r)}{k^2} = \sum_{\eta=0}^{N} \frac{\Gamma(\eta - 1/2)}{2\Gamma(\eta + N + 2)} \left( \frac{V(r)}{k^2} \right)^\eta - \frac{\Gamma(N + 1/2)}{2\Gamma(\eta + N + 2)} \left( \frac{V(r)}{k^2} \right)^{N+1} \sum_{\eta=0}^{\infty} \frac{\Gamma(1 + \eta)\Gamma(\eta + N + 2)}{\Gamma(N + 1/2)\Gamma(\eta + N + 2)\eta!} \left( \frac{V(r)}{k^2} \right)^\eta.
\]
by use of the reflection formula of the gamma, \( \Gamma(-z)\Gamma(z + 1) = -\frac{\pi}{\sin(z\pi)} \). Comparing with the Gauss hypergeometric function gives

\[
\sqrt{1 - \frac{V(r)}{k^2}} = -\sum_{\eta=0}^{N} \frac{\Gamma(\eta - 1/2)}{2\sqrt{\pi}\eta!} \left( \frac{V(r)}{k^2} \right)^{\eta} - \frac{\Gamma(N + 1/2)}{2\sqrt{\pi}\Gamma(N + 2)} \left( \frac{V(r)}{k^2} \right)^{N+1/2} 2F_1 \left( 1, N + 1/2 ; \frac{k^2}{V(r)} \right). \tag{73}
\]

This proves Eq. (70).

### 5.2. The tortoise coordinate for potentials nonvanishing at \( r \to \infty \)

For potentials nonvanishing at \( r \to \infty \), the tortoise coordinate (15) can be expressed as

\[
r_* = \frac{1}{4k} \ln \frac{V(r)}{k^2} + \int^r dr \sqrt{\frac{V(r)}{k^2} - \frac{k^2}{V(r)}} + \frac{\Gamma(N + 1/2)}{2\sqrt{\pi}\Gamma(N + 2)} \int^r dr 2F_1 \left( 1, N + 1/2 ; \frac{k^2}{V(r)} \right) \left( \frac{k^2}{V(r)} \right)^{N+1/2}. \tag{74}
\]

The proof is similar to the proof of Eq. (70).

Notice that corresponding to \( N \to \infty \), the constant potential is a special case of both the above two kinds of potentials, potentials vanishing at \( r \to \infty \) corresponding to the tortoise coordinate (7) and nonvanishing at \( r \to \infty \) corresponding to the tortoise coordinate (15).

### 6. Examples: Scattering states and bound states

#### 6.1. The Coulomb potential

The Coulomb potential

\[ V(r) = \frac{\alpha}{r} \tag{75} \]

corresponds to \( N = 1 \) in the conditions (24) and (25). Then by Eq. (7), we obtain the tortoise coordinate for the Coulomb potential:

\[ r_* = r - \frac{\alpha}{2k^2} \ln r. \tag{76} \]

Substituting the tortoise coordinate (76) into the large-distance asymptotics (10) gives the large-distance asymptotic wave function of the Coulomb potential

\[ u_l(r_*) \sim e^{\pm ikr_*} = \exp \left[ \pm i \left( kr - \frac{\alpha}{2k} \ln r \right) \right]. \tag{77} \]

This result can also be checked by the exact solution of the Coulomb potential,

\[ u_l(r) = M_{(2l+1/2)}(2ikr), \tag{78} \]

where \( M_{\mu,\nu}(z) \) is the Whittaker hypergeometric function. The large-distance asymptotics of the radial solution Eq. (78) is

\[ u_l(r) \sim \frac{(-i)^l(2ik)^{\mu}}{\Gamma(l + \frac{\mu}{2})} \exp \left( i \left( kr - \frac{\alpha}{2k} \ln r \right) \right) + \frac{i^{l+1}(-2ik)^{\mu}}{\Gamma(l + \frac{\mu}{2} + 1)} \exp \left( -i \left( kr - \frac{\alpha}{2k} \ln r \right) \right). \tag{79} \]

This agrees with the large-distance asymptotics given by the tortoise coordinate, Eq. (77).
Under the tortoise coordinate (76), the long-range Coulomb potential is converted to a short-range potential. Then by Eqs. (69) and (76), the scattering wave function can be represented by a scattering phase shift:

\[ u_l(r) \xrightarrow{r \to \infty} A_l \sin \left( kr_* - \frac{l\pi}{2} + \delta_l \right) = A_l \sin \left( kr - \frac{\alpha}{2k} \ln r - \frac{l\pi}{2} + \delta_l \right), \tag{80} \]

where the phase shift \[ \delta_l = \frac{\Gamma(l + 1 + \frac{ia}{2k})}{\Gamma(l + 1 - \frac{ia}{2k})} \] is the scattering phase shift of the Coulomb potential [40].

6.2. The harmonic-oscillator potential

The harmonic-oscillator potential

\[ V(r) = \omega^2 r^2 \tag{81} \]

corresponds to \( N = 1 \) in the conditions (47) and (48). Then by Eq. (15), we obtain the tortoise coordinate for the harmonic-oscillator potential:

\[ r_* = \frac{1}{2k} \omega r^2 - \frac{k}{2\omega} \ln r + \frac{1}{2k} \ln \omega r. \tag{82} \]

Substituting the tortoise coordinate (82) into the large-distance asymptotics (18) gives the large-distance asymptotic wave function of the harmonic-oscillator potential:

\[ u_l(r_*) \xrightarrow{r_* \to \infty} e^{-kr_*} = \exp \left( - \left( \frac{\omega^2}{2} r^2 - \frac{k^2}{2\omega} \ln r + \frac{1}{2} \ln \omega r \right) \right). \tag{83} \]

This result can also be checked by the exact solution of the harmonic-oscillator potential,

\[ u_l(r) = \frac{1}{\sqrt{\pi}} M_{-k^2(4\omega)(2l+1)/4}(\omega^2 r^2). \tag{84} \]

The large-distance asymptotics of the radial solution (84) is

\[ u_l(r) \xrightarrow{r \to \infty} A_l \omega^{\frac{1}{2}l + \frac{1}{2}} \exp \left( - \left( \frac{\omega^2}{2} r^2 + \frac{k^2}{2\omega} \ln r - \frac{1}{2} \ln \omega r \right) \right). \tag{85} \]

This agrees with the large-distance asymptotics given by the tortoise coordinate, Eq. (83).

6.3. The 1/\( r \)-potential

The 1/\( r \)-potential

\[ V(r) = \frac{\xi}{\sqrt{r}} \tag{86} \]

corresponds to \( N = 2 \) in the conditions (24) and (25). The tortoise coordinate of the 1/\( r \)-potential can be obtained by Eq. (7):

\[ r_* = r - \frac{\xi}{k^2} \sqrt{r} - \frac{\xi^2}{8k^4} \ln r. \tag{87} \]
The large-distance asymptotic radial wave function of the 1/√r-potential can be obtained by substituting the tortoise coordinate (87) into Eq. (10):
\[
    u_l(r^*) \overset{r \to \infty}{=} e^{\pm ik r^*} = \exp \left( \pm i \left( k r - \frac{\zeta}{k} \sqrt{r} - \frac{\zeta^2}{8k^3} \ln r \right) \right), \tag{88}
\]

To check this result, we calculate the large-distance asymptotics of the radial wave function from the exact solution [41]
\[
    u_l(r) = A_l(-2i k r)^{l+1} N \left( 4l + 2, -\frac{2\zeta}{\sqrt{2} k^3}, -\frac{\zeta^2}{2 k^3} 0, \sqrt{-2i k r} \right) \exp \left( i \left( k r - \frac{\zeta}{k} \sqrt{r} - \frac{\zeta^2}{8k^3} \ln r \right) \right), \tag{89}
\]
where \( N(\alpha, \beta, \gamma, \delta, z) \) is the biconfluent Heun function [41–43]. The large-distance asymptotics of the exact solution (89) at \( r \to \infty \) is [41]
\[
    u_l(r) \overset{r \to \infty}{=} A_l K_1 \left( 4l + 2, -\frac{2\zeta}{\sqrt{2} k^3}, -\frac{\zeta^2}{2 k^3} 0 \right) (-2i k)^{-1} \exp \left( i \left( k r - \frac{\zeta}{k} \sqrt{r} - \frac{\zeta^2}{8k^3} \ln r \right) \right) + A_l K_2 \left( 4l + 2, -\frac{2\zeta}{\sqrt{2} k^3}, -\frac{\zeta^2}{2 k^3} 0 \right) (-2i k)^{-1} \exp \left( -i \left( k r - \frac{\zeta}{k} \sqrt{r} - \frac{\zeta^2}{8k^3} \ln r \right) \right), \tag{90}
\]
where \( K_1 \left( 4l + 2, -\frac{2\zeta}{\sqrt{2} k^3}, -\frac{\zeta^2}{2 k^3} 0 \right) \) and \( K_2 \left( 4l + 2, -\frac{2\zeta}{\sqrt{2} k^3}, -\frac{\zeta^2}{2 k^3} 0 \right) \) are linear combination coefficients. This agrees with the result obtained in virtue of the tortoise coordinate, Eq. (88).

Under the tortoise coordinate (87), the long-range 1/√r-potential is converted to a short-range potential. Then by Eqs. (69) and (87), the scattering wave function can be represented by a scattering phase shift:
\[
    u_l(r) \overset{r \to \infty}{=} A_l \sin \left( k r - \frac{l \pi}{2} - \delta_l \right) = A_l \sin \left( k r - \frac{\zeta}{k} \sqrt{r} - \frac{\zeta^2}{8k^3} \ln r - \frac{l \pi}{2} - \delta_l \right), \tag{91}
\]
where \( \delta_l = -\arg K_2 \left[ 4l + 2, -\frac{2\zeta}{\sqrt{2} k^3}, -\frac{\zeta^2}{2 k^3} 0 \right] \) is the scattering phase shift of the 1/√r-potential potential, where the definition of \( K_2(\alpha, \beta, \gamma, z) \) can be found in Ref. [41].

6.4. The \( r^{2/3} \)-potential

The \( r^{2/3} \)-potential
\[
    V(r) = \zeta r^{2/3} \tag{92}
\]
corresponds to \( N = 2 \) in the conditions (47) and (48). The tortoise coordinate of the \( r^{2/3} \)-potential can be obtained by Eq. (15):
\[
    r_* = \frac{3\zeta^{1/2}}{4k} r^{4/3} - \frac{3k}{4\zeta^{1/2}} r^{2/3} - \frac{k^3}{8\zeta^{3/2}} \ln r + \frac{1}{4k} \ln(\zeta r^{2/3}). \tag{93}
\]
The large-distance radial wave function of the \( r^{2/3} \)-potential can be obtained by substituting the tortoise coordinate (93) into Eq. (18):
\[
    u_l(r^*) \overset{r \to \infty}{=} e^{-k r^*} = \exp \left( - \left( \frac{3\zeta^{1/2}}{4} r^{4/3} - \frac{3k^2}{4\zeta^{1/2}} r^{2/3} - \frac{k^4}{8\zeta^{3/2}} \ln r + \frac{1}{4} \ln(\zeta r^{2/3}) \right) \right). \tag{94}
\]
To check the above result, we calculate the large-distance asymptotics of the radial wave function from the exact solution

$$u_l(r) = A_l \exp \left( -\frac{3}{4} r^{1/2} + \frac{3k^2}{4r^{1/2}} r^{2/3} \right) \left( \frac{\sqrt{6}}{2} \xi^{1/4} r^{1/3} \right)^{3l+1/2} r^{l+1} N \left( 3l + \frac{3}{2}, \frac{\sqrt{6} k^2}{2\xi^{3/4}}, \frac{3k^4}{8\xi^{3/2}}, 0, \frac{\sqrt{6}}{2} \xi^{1/4} r^{2/3} \right).$$

(95)

The large-distance asymptotics of the exact solution (95) at $r \to \infty$ is

$$u_l(r) \sim \exp \left( -\frac{3}{4} r^{1/2} + \frac{3k^2}{4r^{1/2}} r^{2/3} + \frac{k^4}{8\xi^{3/2}} \ln r + \frac{1}{6} \ln r \right).$$

This agrees with the result obtained in virtue of the tortoise coordinate, Eq. (94).

### 6.5. The $1/r^{3/2}$-potential

The tortoise coordinate of the $1/r^{3/2}$-potential

$$V(r) = \frac{\xi}{r^{3/2}}$$

by Eq. (7) is

$$r_* = r,$$

(97)

i.e., the tortoise coordinate of the $1/r^{3/2}$-potential is the radial coordinate $r$ itself. This means that the $1/r^{3/2}$-potential is in fact a short-range potential. Then the large-distance asymptotic radial wave function of the $1/r^{3/2}$-potential is just the large-distance asymptotic radial wave function of all short-range potentials:

$$u_l(r_*) \sim e^{\pm ikr_*} = e^{\pm ikr}.$$

(98)

The large-distance asymptotic radial wave function of the $1/r^{3/2}$-potential can be obtained from the exact solution

$$u_l(r) = A_l e^{ikr} (-2ik r^{1/2})^{l+1} N \left( 4l + 2, 0, 0, -\frac{8\xi}{\sqrt{-2ik}}, -\sqrt{-2ik} \right).$$

(99)

The large-distance asymptotics of the exact solution (99) at $r \to \infty$ is

$$u_l(r) \sim A_l K_1 \left( 4l + 2, 0, 0, -\frac{8\xi}{\sqrt{-2ik}} \right) e^{-ikr} + A_l K_2 \left( 4l + 2, 0, 0, -\frac{8\xi}{\sqrt{-2ik}} \right) e^{-ikr}.$$

(100)

This agrees with the result given by the tortoise coordinate, Eq. (98).

The $1/r^{3/2}$-potential is indeed a short-range potential. By Eqs. (69) and (97), the scattering wave function can be represented by a scattering phase shift:

$$u_l(r) \sim A_l \sin \left( kr_* - \frac{l\pi}{2} + \delta_l \right)$$

$$= A_l \sin \left( kr - \frac{l\pi}{2} + \delta_l \right),$$

(101)

where $\delta_l = -\arg K_2 \left( 4l + 2, 0, 0, -\frac{8\xi}{\sqrt{-2ik}} \right)$ is the scattering phase shift of the $1/r^{3/2}$-potential.
6.6. The $r^6$-potential

The $r^6$-potential

$$V(r) = \zeta r^6$$

has only bound states. The tortoise coordinate of the $r^6$-potential by Eq. (7) reads

$$r_* = \frac{1}{4k} \zeta^{1/2} r^4 + \frac{1}{4k} \ln(\zeta r^6).$$

(103)

The large-distance asymptotic radial wave function of the $r^6$-potential can be obtained by substituting the tortoise coordinate (103) into Eq. (18):

$$u_l(r) \to \infty \sim e^{-kr_*} = \exp \left( -\frac{1}{4} \zeta^{1/2} r^4 - \frac{1}{4} \ln(\zeta r^6) \right).$$

(104)

To check the above result, we calculate the large-distance asymptotic radial wave function from the exact solution of the $r^6$-potential

$$u_l(r) = A_l \left( \frac{\zeta^{1/2}}{2} \right)^{l+1/4} \exp \left( -\frac{1}{4} \zeta^{1/2} r^4 \right) r^{l+1} N \left( l + \frac{1}{2}, 0, 0, \frac{k^2}{\sqrt{2} \zeta^{1/4}}, 1 \right).$$

(105)

The large-distance asymptotics of the exact solution (105) at $r \to \infty$ is

$$u_l(r) \to \infty \sim A_l \exp \left( -\frac{1}{4} \zeta^{1/2} r^4 - \frac{3}{2} \ln r \right).$$

(106)

This agrees with the result given by the tortoise coordinate, Eq. (104).

7. The duality: tortoise coordinates and asymmetric wave functions

7.1. The duality relation

In section 3, we classify potentials in terms of tortoise coordinates. Newton discovered a duality in classical mechanics, called the Newton-Hooke duality [44]. Kasner and Arnol’d generalized this duality to arbitrary power potentials in classical mechanics [45–49]. Ref. [9] gives the general result of this duality, including the duality of arbitrary potentials in classical and quantum mechanics and in scalar fields. In the following, we discuss the relation between the duality and the classification of potentials.

Consider two central potentials $V(r)$ and $U(\rho)$. Their radial equations are

$$\frac{d^2 u_l(r)}{dr^2} + \left[ k^2 - \frac{l(l+1)}{r^2} - V(r) \right] u_l(r) = 0,$$

$$\frac{d^2 v_\ell(\rho)}{d\rho^2} + \left[ \kappa^2 - \frac{\ell(\ell+1)}{\rho^2} - U(\rho) \right] v_\ell(\rho) = 0.$$

(107)

(108)

Suppose the potentials $V(r)$ and $U(\rho)$ can be expanded as a general polynomial which is a polynomial with arbitrary real number powers:

$$V(r) = \zeta r^a + \sum_{b_n} \mu_n r^{b_n}, \quad b_n < a,$$

(109)

$$U(\rho) = \zeta \rho^A + \sum_{B_n} \lambda_n \rho^{B_n}, \quad B_n < A.$$

(110)
According to the duality relation between \( V(r) \) and \( U(\rho) \) given in Ref. [9], the duality relation between the expansions (109) and (110) is

\[
\frac{a+2}{2} = \frac{2}{A+2},
\]

(111)

\[
\sqrt{\frac{2}{a+2}(b_n+2)} = \sqrt{\frac{2}{A+2}(B_n+2)}.
\]

(112)

The wave functions of these two systems can be transformed into each other by the duality transform

\[
r \rightarrow \rho^{(A+2)/2},
\]

(113)

\[
u_l(r) \rightarrow \rho^{A/4} v_l(\rho),
\]

(114)

and

\[
k^2 \rightarrow -\zeta \left( \frac{2}{A+2} \right)^2,
\]

(115)

\[
\zeta \rightarrow -k^2 \left( \frac{2}{A+2} \right)^2,
\]

(116)

\[
l + \frac{1}{2} \rightarrow \frac{2}{A+2} \left( \ell + \frac{1}{2} \right).
\]

(117)

By the above duality relation, we obtain the following duality relation between tortoise coordinates and asymptotic wave functions.

If the central potential \( V(r) \) satisfies the conditions (24) and (25) at infinity, then its dual potential \( U(\rho) \) satisfies the conditions (47) and (48) at infinity. The dual potentials \( V(r) \) and \( U(\rho) \) correspond to the same value of the positive integer \( N \), and their tortoise coordinates and asymptotic wave functions are related by the duality transforms (113)-(117).

Proof. (1) Suppose that at \( r \to \infty \), the potential \( V(r) \) satisfies the conditions (24) and (25), i.e., \( V(r) \) falls off faster than \( \frac{1}{r^{N+1}} \) and slower than or equally to \( \frac{1}{r^N} \); or, equivalently,

\[
-\frac{1}{N} \leq a < -\frac{1}{N+1}, \quad N \in \mathbb{Z}^+.
\]

(118)

By the duality relation (111) and the expansion of the dual potentials \( V(r) \) and \( U(\rho) \), we have

\[
A = -\frac{2a}{2+a}.
\]

(119)

Substituting Eq. (119) into Eq. (118) gives

\[
\frac{2}{2N+1} < A \leq \frac{2}{2N-1}.
\]

(120)

Moreover, by the duality relation (112) we have

\[
B_n < A.
\]

(121)
This proves that \( U(\rho) \) satisfies the conditions (47) and (48), i.e., \( U(\rho) \) falls off faster than \( r^{2N+1} \) and and slower than or equally to \( r^{2N} \). Therefore, the tortoise coordinate and asymptotic wave function of \( U(\rho) \) are given by (15) and (18).

(2) Next, we show that the asymptotic wave function of \( U(\rho) \) can be obtained by performing the duality transforms (113)-(116).

Because at \( r \to \infty \), \( V(r) \) falls off as \( \frac{1}{r^{2N+1}} < \frac{1}{r^N} \leq \frac{1}{r^{2N}} \), by Eqs. (7) and (10), the asymptotic wave function of \( V(r) \) at \( r \to \infty \) is

\[
u_l(r) \sim r^{\infty} \exp \left\{ \pm i k \left[ r - \sum_{\eta=1}^{N} \frac{\Gamma(\eta - \frac{1}{2})}{2\sqrt{\pi} \eta! k^{2\eta}} \int V(r)^{\eta} \frac{d r}{r} \right] \right\}.
\] (122)

Taking only the leading contribution into account gives

\[
u_l(r) \sim r^{\infty} \exp \left\{ \pm i k \left[ r - \sum_{\eta=1}^{N} \frac{\Gamma(\eta - \frac{1}{2})}{2\sqrt{\pi} \eta! k^{2\eta}} \int \left( \frac{\xi}{r} \right)^{\eta} dr \right] \right\}.
\] (123)

Substituting the duality relations (113)-(116) into Eq. (123) gives

\[
\rho^{A/4} \nu_l(p) \sim \exp \left\{ \pm i \sqrt{-\zeta \left( \frac{2}{A+2} \right)^2 [\rho^{(A+2)/2} - \frac{\zeta \left( \frac{2}{A+2} \right)^2}{(\rho^{(A+2)/2})}]^{2/2}} \right\}.
\] (124)

Note that by Eq. (113) we can see that \( \rho \to \infty \) when \( r \to \infty \).

Rewriting Eq. (124) as

\[
u_l(p) \rho^{\infty} \exp \left\{ \pm \int \sqrt{U(p)} dp - \sum_{\eta=1}^{N} \frac{\Gamma(\eta - \frac{1}{2})}{2\sqrt{\pi} \eta!} \int \left( \frac{1}{U(p)} \right)^{(2\eta-1)/2} \frac{d p}{p} \right\}.
\] (125)

For potentials satisfying \( V(r) \sim r^\beta \) (\( \beta > 0 \)), their asymptotic wave function should tend to zero.

The asymptotic wave function of the potential satisfying \( V(r) \sim r^\beta \) (\( \beta > 0 \)) should vanish at \( r \to \infty \), so we choose minus sign:

\[
u_l(p) \rho^{\infty} \exp \left\{ \pm \int \sqrt{U(p)} dp - \sum_{\eta=1}^{M} \frac{\Gamma(\eta - \frac{1}{2})}{2\sqrt{\pi} \eta!} \int \left( \frac{1}{U(p)} \right)^{(2\eta-1)/2} \frac{d p}{p} \right\}.
\] (126)

This is consistent with the asymptotics given by Eqs. (15) and (18).

### 7.2. Classification of potentials

In section 3, we classify potentials in terms of tortoise coordinates. In this section, we show the relation between the classification and the duality.

For convenience, we consider the power potential. Since the long-distance behavior of potentials can be analyzed by their power expansion, the conclusion drawn from the power potential is not limited to the power potential.

According to the duality relation (111), for power potential \( V(r) \sim r^\alpha \), we have the following conclusion.
### Case 1. The dual potentials with the powers

\[-1 \leq a \leq 0\]  \hspace{1cm} (127)

are the potentials with the powers

\[0 \leq a \leq 2.\]  \hspace{1cm} (128)

Especially, \(V(r) \sim r^0\), i.e., \(a = 0\), is self-dual.

### Case 2. The dual potentials with the powers

\[-2 < a < -1\]  \hspace{1cm} (129)

are the potentials with the powers

\[a > 2.\]  \hspace{1cm} (130)

### Case 3. The dual potentials with the powers

\[a < -2\]  \hspace{1cm} (131)

are the potentials with the powers

\[a < -2.\]  \hspace{1cm} (132)

In case 1, the potentials satisfying (127) are long-range potentials, and their dual potentials, satisfying (128), are also long-range potential.

It is worth noting that in case 2, the potential satisfying (129) is a short-range potential, because their tortoise coordinates are the same as those satisfying (131). However, from the viewpoint of duality, the dual potential of the potentials satisfying (129) is also a positive power long-range potential, but the dual potential of the short-range potential satisfying (131), which satisfies (131), is still a short-range potential. That is, the short-range potential satisfies (129) is somewhat special.

The above classification can also be used to discuss the existence of bound states.

If two potentials are dual, their wave functions can be obtained from each other through the dual transform. A positive-power potential has only bound states. After the dual transform, the bound-state wave function is still a bound-state wave function. Therefore, the dual potential of a positive-power potential must have bound states, although the dual potential can also have scattering states at the same time.

More concretely, the positive-power potential \(U(r) = \lambda r^a\) with \(\lambda > 0\) and \(a > 0\) has only bound states. Its dual potential \(V(\rho) = \eta \rho^b\) whose \(\eta < 0\) and \(-2 < b < 0\) must also has bound-states. The duality transform transforms the bound state of \(U(r)\) to the bound state of its dual potential \(V(\rho)\). But \(V(\rho)\) also has scattering states besides bound states. The scattering state of \(V(\rho)\) is dually related to the scattering state of \(U(r) = \lambda r^a\) with \(\lambda < 0\) and \(a > 0\), which, however, is not lower bounded for \(\lambda < 0\).

Detailed discussions on the existence condition of bound states can be found in Refs [50–52].

### 8. Conclusion

Inspired by general relativity, we suggest a general treatment for long-range potential scattering by introducing tortoise coordinates. While in common scattering theory, only the short-range potential can be treated generally.

The key treatment in our scheme is to introduce the tortoise coordinate. The tortoise coordinate in general relativity is introduced to convert a curved spacetime to a partially...
conformally flat spacetime. In our scheme, the tortoise coordinate is introduced to convert a long-range potential to a short-range potential.

Starting from the tortoise coordinate, we suggest a classification scheme for potentials. Newton and Euler classify functions in virtue of their asymptotics [8]. The asymptotic behavior of wave functions is reflected in the tortoise coordinate. In the paper, we classify potentials by the corresponding tortoise coordinates. Moreover, we also show a relation between classification of potentials and a duality in quantum mechanics.

This classification scheme is indeed a classification of the Schrödinger operator. The Schrödinger operator is one of the Laplacian type operator which consists of a Laplacian operator and a potential function. Replacing the Laplacian operator with the Laplace-Beltrami operator, a Laplacian operator in curved space, we can use a similar approach to classify spacetime manifolds on which the Laplace-Beltrami operator is defined.

By tortoise coordinates, a problem of long-range potentials is converted to a problem of short-range potentials. A short-range potential scattering can be fully described by the scattering phase shift [4,29]. The scattering phase shift can be expressed by tortoise coordinates in gravity related scattering problems, which are the long-range scattering [5–7]. This inspires us to describe long-range scattering in virtue of the tortoise coordinate by a phase shift. Therefore, the method for the calculation of scattering phase shifts for short-range potential scattering can be then applied to long-range potential scattering.

Furthermore, the method suggested in the present paper can also be applied to scattering in curved space. We will also consider the application of the tortoise coordinate in gauge field theory.

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