We present additional observations to previous studies on infrared (IR) renormalon in \(SU(N)\) QCD(adj.), the \(SU(N)\) gauge theory with \(n_W\)-flavor adjoint Weyl fermions on \(\mathbb{R}^3 \times S^1\) with the \(\mathbb{Z}_N\) twisted boundary condition. First, we show that, for arbitrary finite \(N\), a logarithmic factor in the vacuum polarization of the “photon” (the gauge boson associated with the Cartan generators of \(SU(N)\)) disappears under the \(S^1\) compactification. Since IR renormalon is attributed to the presence of this logarithmic factor, it is concluded that there is no IR renormalon in this system with finite \(N\). This result generalizes the observation made by Anber and Sulejmanpasic for \(N = 2\) and 3 to arbitrary finite \(N\). Next, we point out that, although renormalon ambiguities do not appear through the Borel procedure in this system, an ambiguity appears in an alternative resummation procedure in which a resumed quantity is given by a momentum integration where the inverse of the vacuum polarization is included as the integrand. Such an ambiguity is caused by a simple zero at non-zero momentum of the vacuum polarization. Under the decompactification \(R \to \infty\), where \(R\) is the radius of the \(S^1\), this ambiguity in the momentum integration smoothly reduces to the IR renormalon ambiguity in \(\mathbb{R}^4\). We term this ambiguity in the momentum integration “renormalon precursor”. The emergence of the IR renormalon ambiguity in \(\mathbb{R}^4\) under the decompactification can be naturally understood with this notion.
1. Introduction

Perturbative expansion of observables typically gives divergent asymptotic series. Such divergent behavior is caused by a factorial growth of perturbative coefficients and often induces intrinsic errors in perturbative predictions. One of the sources of the factorial growth is known as renormalon [1, 2]. This is closely related to renormalization properties, and in asymptotically free theories, infrared (IR) renormalon gives inevitable uncertainties in perturbation theory. The fate of the IR renormalon, for instance how its ambiguity is eliminated, has not been understood well so far.

In Refs. [3–6], the conjecture concerning IR renormalon was proposed: In an $S^1$ compactified spacetime with the $Z_N$ twisted boundary condition, the ambiguity associated with IR renormalon is cancelled against the ambiguity associated with the integration of quasi-collective coordinates of a semi-classical quasi-solution called a bion [7]. This conjecture suggests a semi-classical picture on IR renormalon in an analogous manner to the cancellation of ambiguities between the proliferation of Feynman diagrams and the instanton-anti-instanton pair [8, 9]. The suggested structure would be fascinating to the resurgence program in asymptotically free field theories [10]. To examine this conjecture, the study of IR renormalon in theories on the $S^1$ compactified spacetime was performed [11–15]. See Refs. [16, 17] for detailed analyses from the bion side.

Contrary to the conjecture, however, it was argued in Ref. [11] that the bion ambiguity does not correspond to renormalon ambiguities, because IR renormalon is absent in the $SU(N)$ QCD (adj.), the $SU(N)$ gauge theory with $n_W$-flavor adjoint Weyl fermions with the $Z_N$ twisted boundary condition [7, 18–38] on $\mathbb{R}^3 \times S^1$ with $N = 2$ and 3. Quite recently, the perturbative ambiguity to be cancelled against the bion ambiguity has been identified in Ref. [39]; it has been clarified that such a perturbative ambiguity is not caused by IR renormalon but by the proliferation of Feynman diagrams and enhancement of an amplitude of each diagram, which is specific to the $S^1$ compactification and the twisted boundary condition. In this way, the recent controversial issue whether bion ambiguities truly correspond to renormalon ambiguities or not has been settled to our understanding.

In this paper, nevertheless, we further investigate renormalon ambiguities of a theory on the $S^1$ compactified spacetime. This aims at understanding issues remaining unclear about the renormalon structure in a compactified spacetime itself. A particular purpose of this paper is to understand the relation between two results given in Refs. [11] and [13] on IR renormalon in the $SU(N)$ QCD (adj.) on $\mathbb{R}^3 \times S^1$. In Ref. [11], the vacuum polarization of the “photon” (the gauge boson associated with Cartan generators of $SU(N)$) was analyzed in great detail and it was found that a logarithmic factor in the vacuum polarization, which is responsible for the existence of IR renormalon, disappears as the effect of the $S^1$ compactification. This analysis was performed explicitly for $N = 2$ and 3 and indicates the absence of IR renormalon. On the other hand, in Ref. [13], it was concluded that there exists IR renormalon for $N = \infty$. Therefore, it is of great interest to know how the existence of IR renormalon depends on the value of $N$. In the analyses of the present paper, we entirely rely on the large-$\beta_0$ approximation [1, 10, 42], which is a somewhat ad hoc but widely used approximation in the study of renormalon in asymptotically free theories (see below); this approximation was also adopted in Ref. [13].
In the first part of this paper, we show that the logarithmic factor in the vacuum polarization of the photon disappears for arbitrary finite \( N \), by employing expressions obtained in Ref. 13. This result generalizes the result in Ref. 11, which studied the cases with \( N = 2 \) and 3. We conclude that IR renormalon does not exist for any finite \( N \). We also make remarks on how the point \( N = \infty \) should be regarded singular.

The absence of IR renormalon for arbitrary finite \( N \) is, however, somewhat peculiar because it indicates that IR renormalon does not exist irrelevantly to details of the theory as long as the \( S^1 \) compactification is considered. On the other hand, we know that IR renormalon indeed exists in \( \mathbb{R}^4 \). Then, the question arises how IR renormalon in \( \mathbb{R}^4 \) can emerge in the decompactification limit starting from the theory on the \( S^1 \) compactified spacetime. To gain an insight on this issue, in the second part of this paper, we point out that although renormalon ambiguities do not appear through the Borel procedure, an ambiguity appears in an alternative resummation procedure in which a resumed quantity is given by a momentum integration where the inverse of the vacuum polarization is included as the integrand. Such an ambiguity is caused by a simple zero at non-zero momentum of the vacuum polarization. This ambiguity is generally different from ordinary renormalon ambiguities, which we encounter in the Borel procedure. An advantage to consider such an ambiguity is that we can naturally understand how the IR renormalon ambiguity in \( \mathbb{R}^4 \) emerges under the decompactification limit \( R \to \infty \). We term this ambiguity in the momentum integration “renormalon precursor”. This ambiguity is not IR renormalon in the sense that it is not associated with the factorial growth of the perturbative coefficients; it is nevertheless a “precursor” of IR renormalon in the sense that under the decompactification \( R \to \infty \), the renormalon precursor smoothly reduces to IR renormalon in \( \mathbb{R}^4 \).

This paper is organized as follows. In Sect. 2, we give a review on IR renormalon in \( SU(N) \) QCD(adj.) and collect necessary results obtained in Ref. 13. In Sect. 3, we show the absence of the logarithmic factor in the vacuum polarization at the low momentum limit for arbitrary finite \( N \). In Sect. 4, we introduce “renormalon precursor” and discuss perturbative ambiguities in the decompactification limit. Section 5 is devoted to the conclusion. In Appendix A, we give a rigorous proof on the asymptotic behavior of the vacuum polarization at low momentum limit. In Appendix B, we present some examples to which the notion of the renormalon precursor applies; we see that even the shift of the Borel singularity by \( -1/2 \) under the compactification \( \mathbb{R}^d \to \mathbb{R}^{d-1} \times S^1 \) in some models [12, 14] can be naturally understood by this notion.

2. Preparation: Basics on IR renormalon in QCD(adj.)

Let us start with recalling how IR renormalon arises in QCD(adj.) in the un-compactified spacetime \( \mathbb{R}^4 \). Throughout this paper, we rely on the large-\( \beta_0 \) approximation [1, 40–42], which extracts a certain (gauge invariant) sub-contribution of Feynman diagrams. For this, one first considers the large flavor limit \( n_W \to \infty \) with the combination \( g^2 n_W \) kept fixed, where \( g \) is

\[ \text{Our treatment of the gauge field loop diagrams is somewhat different from that in Ref. [11]; see below.} \]

\[ \text{In the most part of this theory, \( \mathbb{R} \) dependence is controlled by the combination \( NR \) instead of \( \mathbb{R} \) due to the twisted boundary condition. Then, it is naively expected that as \( N \) becomes larger the theory becomes equivalent to that on \( \mathbb{R}^4 \).} \]
the gauge coupling constant. In this limit, the gauge field propagator is dominated by the chain of the fermion one-loop vacuum bubbles. Then, to partially incorporate the effect of the gauge field loops, the number of flavor $n_W$ is replaced by hand with the one-loop coefficient of the beta function of the 't Hooft coupling $\lambda = g^2 N$ as

$$-\frac{2}{3} n_W \rightarrow \beta_0 \equiv \frac{11}{3} - \frac{2}{3} n_W.$$  \tag{2.1}$$

In this large-$\beta_0$ approximation, the gauge field propagator is given by

$$\langle A^a_\mu(x) A^b_\nu(y) \rangle = \frac{\lambda}{N} \delta^{ab} \int \frac{d^4 p}{(2\pi)^4} e^{i p(x-y)} \frac{1}{(p^2)^2} \left\{ 1 - \beta_0 \lambda \frac{1}{16\pi^2} \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) \right\}^{-1} \left( p^2 \delta_{\mu\nu} - p_\mu p_\nu \right) + \frac{1}{\xi p_\mu p_\nu},$$  \tag{2.2}$$

where $\lambda$ is the renormalized 't Hooft coupling in the $\overline{\text{MS}}$ scheme at the renormalization scale $\mu$; $\xi$ is the renormalized gauge parameter. We note that this form is actually consistent with a renormalization group equation. From the geometric series expansion of this expression, the perturbative expansion of a gauge invariant physical quantity $F(\lambda)$ is expected to have the form

$$F(\lambda) \sim \lambda \sum_{k=0}^{\infty} f_k \left( \frac{\beta_0 \lambda}{16\pi^2} \right)^k, \quad f_k = \int \frac{d^4 p}{(2\pi)^4} (p^2)^\alpha \left[ \ln \left( \frac{\mu^2}{p^2} \right) \right]^k.$$  \tag{2.3}$$

Here, we assume that $\alpha + 2 > 0$ so that the perturbative expansion of $F(\lambda)$ does not suffer from IR divergences. For $k \gg 1$, the momentum integral for $f_k$ is dominated by the contribution of the saddle point $p^2 = \mu^2 e^{-k/(\alpha+2)}$ and the large order behavior is given by

$$f_k \big|_{k \gg 1} \sim \frac{\mu^{2\alpha+4}}{16\pi^2} \frac{k!}{(\alpha + 2)^{k+1}}.$$  \tag{2.4}$$

For the Borel transform defined by

$$B[F](u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k,$$  \tag{2.5}$$

the above factorial growth of the perturbative coefficient $f_k$ produces a pole singularity at $u = \alpha + 2$:

$$\frac{\mu^{2\alpha+4}}{16\pi^2} \frac{1}{\alpha + 2 - u}.$$  \tag{2.6}$$

In the Borel procedure, which allows us to resum divergent series, the Borel integral

$$\frac{16\pi^2}{\beta_0} \int_0^\infty du \, B[F](u) e^{-16\pi^2 u/(\beta_0 \lambda)}$$  \tag{2.7}$$

formally gives the original quantity $F(\lambda)$. However, the Borel integral along the positive $u$-axis should be regularized due to the pole singularity of the Borel transform at $u = \alpha + 2 > 0$. The integration contour is often deformed in the complex $u$-plane as $\int_0^\infty \rightarrow \int_{0 \pm i \delta}^\infty$ with a

\footnote{We will explicitly see such an example in Eqs. \ref{4.1}, \ref{4.8}, and \ref{4.9}. Here, we omit the factor $e^{5/3}$, which is not important in this discussion.}
small parameter $\delta$. Accordingly, it possesses an imaginary part, regarded as the ambiguity associated with the pole singularity,

$$
\pm \pi i \frac{1}{\beta_0^2} \Lambda^{2\alpha+4},
$$

(2.8)

where $\Lambda$ is the one-loop dynamical scale

$$
\Lambda^2 \equiv \mu^2 e^{-16\pi^2/(\beta_0 \lambda)}.
$$

(2.9)

Equation (2.8) is the IR renormalon ambiguity. Since the factorial growth of $f_k$ comes from the momentum integration around $p^2 = \mu^2 e^{-k/2(\alpha+2)}$, which goes to 0 as $k \to \infty$ (i.e., for the large order behavior of perturbation theory), the persistent presence of the logarithmic factor $\ln p^2$ in the vacuum polarization in Eq. (2.2) toward $p^2 = 0$ is crucial for the existence of IR renormalon. For example, if the vacuum polarization approaches a constant as $p^2 \to 0$, we do not have the factorial growth of perturbative coefficients.

Now, we explain how the expression (2.2) is modified under the $S^1$ compactification, $\mathbb{R}^4 \to \mathbb{R}^3 \times S^1$ based on Ref. [13]. We compactify the $x^3$-direction and impose the $\mathbb{Z}_N$ twisted boundary condition along $S^1$ (see Eqs. (2.3)–(2.7) in Ref. [13] for the detailed definition). Since the twisted boundary condition is expressed in terms of Cartan generators of $SU(N)$, it is convenient to decompose the field in the Cartan–Weyl basis as

$$
A_\mu(x) = -i \sum_{\ell=1}^{N-1} A^\ell_\mu(x) H_\ell - i \sum_{m \neq n} A^{mn}_\mu(x) E_{mn},
$$

(2.10)

where $H_\ell$ are Cartan generators and $E_{mn}$ are root generators of $SU(N)$. In what follows, we refer the Cartan components $A^\ell_\mu(x)$ to as the “photon”, whereas the root components $A^{mn}_\mu(x)$ the “W-boson”; they have rather different properties.

Gauge field propagators are given in Eq. (2.37) of Ref. [13] in the large-$\beta_0$ approximation as

$$
\left\langle A^\ell_\mu(x) A^\nu_\nu(y) \right\rangle = \frac{1}{N} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} e^{ip(x-y)} \left\{ \left[ (1-L)^{-1} \right]_{tr} p^2 P^L_{\mu\nu} + \left[ (1-T)^{-1} \right]_{tr} p^2 P^T_{\mu\nu} + \delta_{tr} \frac{1}{\xi} p^2 p_{\mu\nu} \right\},
$$

$$
\left\langle A^{mn}_\mu(x) A^{pq}_\nu(y) \right\rangle = \frac{1}{N} \delta_{mq} \delta_{np} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} e^{ip(x-y)} \left\{ \left[ (1-L)^{-1} \right]_{tr} p^2 P^L_{\mu\nu} + (1-T)^{-1} p^2 P^T_{\mu\nu} + \frac{1}{\xi} p^2 p_{\mu\nu} \right\},
$$

(2.11)

Here, we have shown only non-zero propagators. In these expressions, $p_3$ denotes the discrete Kaluza–Klein (KK) momentum along $S^1$,

$$
p_3 = \frac{n}{R}, \quad n \in \mathbb{Z},
$$

(2.12)
and the projection operators $P^T_{\mu \nu}$ and $P^L_{\mu \nu}$ are defined by \[11\]

$$
P^T_{ij} \equiv \delta_{ij} - \frac{p_i p_j}{p^2 - p_3^2}, \quad P^T_{i3} = P^T_{3i} = P^T_{33} = 0,$$

$$
P^L_{\mu \nu} \equiv \delta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} - P^T_{\mu \nu},$$

\begin{equation}
\tag{2.13}
\end{equation}

where the Roman letters $i, j, \ldots$ run only over 0, 1, and 2, the un-compactified directions. The functions $L_{\ell r}, T_{\ell r}, L, T$ in Eq. \eqref{2.11} are given by

$$
L_{\ell r} \equiv \frac{\beta_0 \lambda}{16 \pi^2} \left\{ \delta_{\ell r} \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) + 12 \sum_{j \neq 0} (\sigma_{j,N})_{\ell r} \int_0^1 dx \, e^{i \pi \rho_j} x (1 - x) \left[ K_0(z) - K_2(z) \right] \right\},
$$

$$
T_{\ell r} \equiv \frac{\beta_0 \lambda}{16 \pi^2} \left\{ \delta_{\ell r} \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) + 12 \sum_{j \neq 0} (\sigma_{j,N})_{\ell r} \int_0^1 dx \, e^{i \pi \rho_j} x (1 - x) \left[ K_0(z) - \frac{p_3^2}{p^2} K_2(z) \right] \right\},
$$

$$
L \equiv \frac{\beta_0 \lambda}{16 \pi^2} \left\{ \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) + 12 \sum_{j \neq 0, j = 0 \mod N} \int_0^1 dx \, e^{i \pi \rho_j} x (1 - x) \left[ K_0(z) - K_2(z) \right] \right\},
$$

$$
T \equiv \frac{\beta_0 \lambda}{16 \pi^2} \left\{ \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) + 12 \sum_{j \neq 0, j = 0 \mod N} \int_0^1 dx \, e^{i \pi \rho_j} x (1 - x) \left[ K_0(z) - \frac{p_3^2}{p^2} K_2(z) \right] \right\}, \tag{2.14}
$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind and the variable $z$ is defined by

$$
z \equiv \sqrt{x(1 - x)} \sqrt{p^2 R^2 2\pi |j|}. \tag{2.15}
$$

In the first two expressions in Eq. \eqref{2.11}, $\sigma_{j,N}$ are $(N - 1) \times (N - 1)$ real symmetric matrices whose components are defined by \[13\]

$$
(\sigma_{j,N})_{\ell r} \equiv \frac{1}{N} \sum_{m,n=1}^{N} (\nu^m - \nu^n)_{\ell} (\nu^m - \nu^n)_{r} e^{i(n-m)2\pi j/N}
$$

$$
= \begin{cases}
    \delta_{\ell r}, & \text{for } j = 0 \mod N, \\
    -\frac{1}{N} \frac{1}{\sqrt{(\ell+1)(r+1)}} \Re \left[ \left( e^{-i 2\pi j/N} - 1 \right) - \ell e^{-i 2\pi j/N} \right] \left( e^{i 2\pi j/N} - 1 \right) - \ell e^{i 2\pi j/N} \right], & \text{for } j \neq 0 \mod N.
\end{cases} \tag{2.16}
$$

In this expression, $\nu^m$ is the $SU(N)$ weights, i.e., the diagonal elements of Cartan generators $$(\nu^m)_{\ell} \equiv (H_{\ell})_{mm}$$ (no sum over $m$ is taken here). With the convention in Ref. \[13\] (which we
adopt throughout this paper), we have the relations

$$\sum_{\ell=1}^{N-1} (\nu^m)_{\ell} (\nu^n)_{\ell} = \frac{1}{2} \delta_{mn} - \frac{1}{2N}$$

(2.17)

and

$$\sum_{m=1}^{N} (\nu^m)_{\ell} (\nu^m)_{\ell'} = \frac{1}{2} \delta_{\ell\ell'}, \quad \sum_{m=1}^{N} (\nu^m)_{\ell} = 0.$$  

(2.18)

In the W-boson propagator (the second expression in Eq. (2.11)), the momentum variable $p$ inside the curly brackets is replaced by the twisted momentum,

$$p_{mn,\mu} \equiv p_{\mu} - \delta_{\mu 3} \frac{m-n}{RN}, \quad m \neq n,$$

(2.19)

as a consequence of the twisted boundary condition.

In Eq. (2.14), the terms containing the Bessel functions correspond to the modifications due to the $S^1$ compactification. If we simply discard these terms, then $L_{\ell\ell'} = T_{\ell\ell'}$ and $L = T$ and, Eq. (2.11) reduces to Eq. (2.2) from Eq. (2.13) (under the prescription that $1/(2\pi R) \sum_{p_3} \to \int \frac{dp_3}{2\pi R}$). As we have already noted, for the existence of IR renormalon, the logarithmic factor $\ln p^2$ in the vacuum polarization around $p^2 = 0$ is crucial. Here, we note that $p^2$ can be zero in the vacuum polarization of the photon (the first expression in Eq. (2.11)), whereas it cannot be zero in that of the W-boson (the second expression in Eq. (2.11)). This is because the momentum of the W-boson is replaced by Eq. (2.19) and $p_{mn,3}$ cannot vanish for finite $RN$. Hence, the W-boson vacuum polarization does not give rise to IR renormalon. From these considerations, it is natural to ask how the logarithmic factor $\ln p^2$ in the photon vacuum polarization, which exists in the un-compactified space-time $\mathbb{R}^4$, is affected by the $S^1$ compactification [11]. This question was studied in Ref. [11], and it was shown that the logarithmic factor $\ln p^2$ disappears by the effect of the $S^1$ compactification (with a somewhat different treatment of the gauge field loops to ours) and that there is no IR renormalon; this was shown for $N = 2$ and 3. In the next section, we explicitly generalize this result of Ref. [11] to arbitrary finite $N$. We also comment how the statement of Ref. [13] that IR renormalon exists in the $N = \infty$ system in $\mathbb{R}^3 \times S^1$ should be understood.

3. Asymptotic behavior of the photon vacuum polarization in $\mathbb{R}^3 \times S^1$

In this section, we show that the logarithmic factor $\ln p^2$ at $p^2 \to 0$ disappears in the vacuum polarization of the photon, given by the first two expressions in Eq. (2.11), for arbitrary finite $N$. Since $p^2 = 0$ can be realized only when $p_3 = 0$, we exclusively assume $p_3 = 0$ in the following.

3.1. Properties of $\sigma_{j,N}$

To investigate the finite volume effect parts (terms containing the modified Bessel functions in Eq. (2.11)), we first study properties of the matrix $\sigma_{j,N}$ defined by Eq. (2.16). From the
definition (2.16), \( \sigma_{j,N} \) is periodic in \( j \) with the period \( N \), i.e.,

\[
\sigma_{j+N,N} = \sigma_{j,N}.
\]  

(3.1)

Note also that

\[
\sigma_{N-j,N} = \sigma_{j,N}.
\]  

(3.2)

From Eq. (2.16), we also have

\[
\sigma_{j,N} = \mathbb{1}, \quad \text{for } j = 0 \text{ mod } N,
\]  

(3.3)

where \( \mathbb{1} \) denotes the unit matrix.

For the sum over \( j \), we have

\[
\sum_{j=1}^{N} \sigma_{j,N} = 0
\]  

(3.4)

and

\[
\sum_{j=1}^{N} j \sigma_{j,N} = N \frac{N}{2} \mathbb{1}.
\]  

(3.5)

Equation (3.4) immediately follows from the identity

\[
\sum_{j=1}^{N} e^{i(n-m)2\pi j/N} = N \delta_{n,m}
\]  

(3.6)

and the definition (2.16), because \( n = m \) terms do not contribute in Eq. (2.16). To see Eq. (3.5), we note

\[
\sum_{j=1}^{N} \frac{1}{2} \left[ e^{i(n-m)2\pi j/N} + e^{-i(n-m)2\pi j/N} \right] = \begin{cases} \frac{1}{2}N(N+1), & \text{for } n = m, \\ \frac{1}{2}N, & \text{for } n \neq m, \end{cases}
\]  

(3.7)

and thus from the definition (2.16),

\[
\sum_{j=1}^{N} j (\sigma_{j,N})_{\ell r} = \frac{1}{2} \sum_{m,n=1}^{N} (\nu^m - \nu^n)_{\ell} (\nu^m - \nu^n)_{r} = N \frac{1}{2} \delta_{\ell r},
\]  

(3.8)

where we have used Eq. (2.17).

Another interesting property of \( \sigma_{j,N} \) is that they commute to each other:

\[
[\sigma_{j,N}, \sigma_{k,N}] = 0.
\]  

(3.9)

Therefore, all \( \sigma_{j,N} \) \( (j = 0, 1, \ldots) \) can be diagonalized by making use of an orthogonal transformation on the gauge potential \( A'_{\ell \mu} \) in Eq. (2.10). Equation (3.9) is obvious when \( j = 0 \text{ mod } N \) and/or \( k = 0 \text{ mod } N \), because of Eq. (3.3). For \( j = 1, \ldots, N-1 \) and \( k = 1, \ldots, N-1 \), Eq. (3.9) can be seen from the fact that the matrix product

\[
(\sigma_{j,N} \sigma_{k,N})_{\ell r} = \begin{cases} 0, & \text{when } j + k \neq 0 \text{ mod } N, \\ \frac{1}{N^2} \sum_{m,n} (\nu^m)(\nu^n)_{\ell} [(m-n)2\pi j/N] , & \text{when } j + k = 0 \text{ mod } N, \end{cases}
\]  

(3.10)

which follows from Eq. (2.17), is symmetric under \( j \leftrightarrow k \).
3.2. Asymptotic behavior of the photon vacuum polarization for \( p^2 R^2 \ll 1 \)

We now study the asymptotic behaviors of the functions \( L_{\ell r} \) and \( T_{\ell r} \), which are contained in the photon vacuum polarization. Here, we introduce the function

\[
f_\nu(p^2 R^2)_{\ell r} = 24 \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} \int_0^1 dx x(1-x)K_\nu(\sqrt{x(1-x)} \sqrt{p^2 R^2 2\pi j}), \tag{3.11}
\]

with \( \nu = 0 \) or 2, which corresponds to finite volume corrections. Then, the functions \( L_{\ell r} \) and \( T_{\ell r} \) in Eq. (2.14) with \( p_3 = 0 \) are represented as

\[
L_{\ell r} = \frac{\beta_0 \lambda}{16 \pi^2} \left[ \delta_{\ell r} \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) + f_0(p^2 R^2)_{\ell r} - f_2(p^2 R^2)_{\ell r} \right],
\]

\[
T_{\ell r} = \frac{\beta_0 \lambda}{16 \pi^2} \left[ \delta_{\ell r} \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) + f_0(p^2 R^2)_{\ell r} \right]. \tag{3.12}
\]

We thus study the asymptotic behavior of the function \( f_\nu(p^2 R^2) \) (3.11) with \( \nu = 0 \) and 2 for \( p^2 R^2 \ll 1 \).

For this, we insert \( \lim_{\epsilon \to 0^+} e^{-\epsilon j} = 1 \) into Eq. (3.11):

\[
f_\nu(p^2 R^2)_{\ell r} = 24 \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} \lim_{\epsilon \to 0^+} e^{-\epsilon j} \int_0^1 dx x(1-x)K_\nu(\sqrt{x(1-x)} \sqrt{p^2 R^2 2\pi j}). \tag{3.13}
\]

Since \( |(\sigma_{j,N})_{\ell r}| \) is bounded (it is periodic in \( j \) with the period \( N \); see Eq. (3.1)) and the modified Bessel function decreases rapidly \( K_\nu(z) \sim e^{-z/2} \), the infinite series \( \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \int_0^1 dx x(1-x)K_\nu(\sqrt{x(1-x)} \sqrt{p^2 R^2 2\pi j}) \) converges uniformly in \( \epsilon \geq 0 \).

This allows us to exchange the infinite sum \( \sum_{j=1}^{\infty} \) and the limit \( \lim_{\epsilon \to 0^+} \) as

\[
f_\nu(p^2 R^2)_{\ell r} = \lim_{\epsilon \to 0^+} 24 \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \int_0^1 dx x(1-x)K_\nu(\sqrt{x(1-x)} \sqrt{p^2 R^2 2\pi j}). \tag{3.14}
\]

Next, we use the series expansion of the modified Bessel function

\[
K_\nu(z) = \sum_{k=0}^{\infty} \left[ b_k^{(\nu)} + c_k^{(\nu)} \ln z \right] z^{2k+\nu} + \sum_{k=0}^{\nu-1} d_k^{(\nu)} z^{2k-\nu}, \tag{3.15}
\]

in Eq. (3.14) (for \( \nu = 0 \), the second sum in Eq. (3.15) is set zero). Here, the first few coefficients are given by

\[
b_0^{(0)} = \ln 2 - \gamma, \quad c_0^{(0)} = -1, \tag{3.16}
\]

where \( \gamma \) is the Euler–Mascheroni constant, and

\[
d_0^{(2)} = 2, \quad d_1^{(2)} = -\frac{1}{2}. \tag{3.17}
\]

Note that with the substitution \( z = \sqrt{x(1-x)} \sqrt{p^2 R^2 2\pi j} \), Eq. (3.15) becomes the series expansion in \( p^2 R^2 \) (and \( \log(p^2 R^2) \)). Then, since the damping factor \( e^{-\epsilon j} \) provides a good convergence property for the \( j \)-summation, we intuitively expect that the \( j \)-summation can

---

\(^5\)This can be rigorously proven by an argument similar to that in Appendix B of Ref. [13].
be done for each term in the $k$-summation. This naive exchange of the $j$-summation and the $k$-summation yields

$$f_0(p^2 R^2)_{\ell r} = 2 \lim_{\epsilon \to 0^+} \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \left[ -\ln(p^2 R^2) + \frac{5}{3} - 2 \ln \pi - 2 \gamma - 2 \ln j \right] + O(p^2 R^2 \ln(p^2 R^2)),
$$

$$f_2(p^2 R^2)_{\ell r} = 2 \lim_{\epsilon \to 0^+} \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \left( \frac{6}{\pi^2 j^2 \frac{1}{p^2 R^2}} - 1 \right) + O(p^2 R^2 \ln(p^2 R^2)). \quad (3.18)$$

In fact, it is not easy to give a rigorous justification to the above exchange of the $j$ and $k$ summations (even in the sense of the asymptotic expansion) or, in other words, to show that the last remainder terms in Eq. (3.18) are really $O(p^2 R^2 \ln(p^2 R^2))$. In Appendix A we give a rigorous proof for the leading asymptotic behaviors of $f_\nu(p^2 R^2)_{\ell r}$ with $\nu = 0$ and 2 for $p^2 R^2 \ll 1$ up to $O((p^2 R^2)^0)$ terms; the results are indeed consistent with Eq. (3.18) and consequently also with Eqs. (3.29)–(3.31) below. This proof is sufficient to conclude the disappearance of the logarithmic factor $\ln(p^2)$ in the photon vacuum polarization. For $O((p^2 R^2)^0)$ terms in Eq. (3.18), we do not have a rigorous proof, although it is highly plausible that the above exchange of the $j$ and $k$ summations is legitimate; we also numerically check that Eqs. (3.29)–(3.31) are indeed correct for some small $N$.

Now, in the first expression in Eq. (3.18), the sum of terms not containing $\ln j$ can be computed as

$$\sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} = \sum_{j=1}^{N} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \sum_{b=0}^{\infty} e^{-\epsilon b N} = \sum_{j=1}^{N} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \frac{1}{1 - e^{-\epsilon N}} = \sum_{j=1}^{N} (\sigma_{j,N})_{\ell r} \left[ \frac{1}{N \epsilon} + \frac{1}{2} - \frac{j}{N} + O(\epsilon) \right] = -\frac{1}{2} \delta_{\ell r} + O(\epsilon), \quad (3.19)$$

where in the first equality we have used the fact that $\sigma_{j,N}$ is periodic in $j$ with the period $N$; in the last step, we have used the properties (3.4) and (3.5). Therefore, we obtain

$$\lim_{\epsilon \to 0^+} \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} = -\frac{1}{2} \delta_{\ell r}. \quad (3.20)$$

On the other hand, the sum over terms containing $\ln j$ in the first equation of Eq. (3.18) is evaluated as

$$\sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \ln j = \sum_{j=1}^{N} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \left[ \sum_{b=1}^{\infty} e^{-\epsilon b N} \ln(j + bN) + j \right]. \quad (3.21)$$
The calculation of the first term in the square brackets proceeds as follows:
\[
\sum_{b=1}^{\infty} e^{-\epsilon b N} \ln (j + bN)
\]
\[
= \sum_{b=1}^{\infty} e^{-\epsilon b N} \left\{ \ln \left( 1 + \frac{j}{bN} \right) - \frac{j}{bN} + \ln(bN) \right\}
\]
\[
= -\gamma \frac{j}{N} - \ln(j/N) - \ln \Gamma(j/N) - \frac{j}{N} \ln(\epsilon N) + O(\epsilon) + \sum_{b=1}^{\infty} e^{-\epsilon b N} \ln(bN). \tag{3.22}
\]
We compute the last infinite sum as
\[
\sum_{b=1}^{\infty} e^{-\epsilon b N} \ln(bN) = -\frac{\partial}{\partial s} \left[ \sum_{b=1}^{\infty} e^{-\epsilon b N} (bN)^{-s} \right]_{s=0}
\]
\[
= -\frac{\partial}{\partial s} \left[ N^{-s} \text{Li}_s(e^{-\epsilon N}) \right]_{s=0}, \tag{3.23}
\]
where \( \text{Li}_s(z) \) is the polylogarithm function
\[
\text{Li}_s(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad \text{for } |z| < 1. \tag{3.24}
\]
Then, using the expansion
\[
\text{Li}_s(z) = \Gamma(1-s) (-\ln z)^{s-1} + \sum_{k=0}^{\infty} \frac{\zeta(s-k)}{k!} (\ln z)^k, \tag{3.25}
\]
we have
\[
\sum_{b=1}^{\infty} e^{-\epsilon b N} \ln(bN) = -\frac{1}{\epsilon N} (\ln \epsilon + \gamma) - \frac{1}{2} \ln \frac{N}{2\pi} + O(\epsilon). \tag{3.26}
\]
Using this in Eq. (3.22), Eq. (3.21) is given by
\[
\sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \ln j
\]
\[
= \sum_{j=1}^{N} (\sigma_{j,N})_{\ell r} \left[ -\frac{j}{N} \ln N - \ln \Gamma(j/N) - \frac{1}{\epsilon N} (\ln \epsilon + \gamma) + \ln \sqrt{2\pi N} + O(\epsilon) \right]
\]
\[
= -\frac{1}{2} \ln N \delta_{\ell r} - \sum_{j=1}^{N} (\sigma_{j,N})_{\ell r} \ln \Gamma(j/N) + O(\epsilon), \tag{3.27}
\]
where we have used Eqs. (3.4) and (3.5). In this way, we obtain
\[
\lim_{\epsilon \to 0^+} \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} e^{-\epsilon j} \ln j = -\frac{1}{2} \ln N \delta_{\ell r} - \sum_{j=1}^{N-1} (\sigma_{j,N})_{\ell r} \ln \Gamma(j/N). \tag{3.28}
\]
Finally, by combining Eqs. (3.18), (3.20), and (3.28), we obtain the asymptotic form for \( p^2 R^2 \ll 1 \),
\[
f_0(p^2 R^2)_{\ell r}
\]
\[
= \left[ \ln(p^2 R^2) - \frac{5}{3} + 2 \ln \pi + 2\gamma + 2 \ln N \right] \delta_{\ell r} + 4 \sum_{j=1}^{N-1} (\sigma_{j,N})_{\ell r} \ln \Gamma(j/N) + O(p^2 R^2 \ln(p^2 R^2)), \tag{3.29}
\]
that is,

\[ \delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{\ell r} \]

\[ = (2\gamma + 2 \ln N + 2 \ln \pi) \delta_{\ell r} + 4 \sum_{j=1}^{N-1} (\sigma_{j,N})_{\ell r} \ln \Gamma(j/N) + O(p^2 R^2 \ln(p^2 R^2)), \]

(3.30)

and in a similar way

\[ f_2(p^2 R^2)_{\ell r} = \frac{12}{\pi^2} \sum_{j=1}^{\infty} (\sigma_{j,N})_{\ell r} \frac{1}{j^2 p^2 R^2} + \delta_{\ell r} + O(p^2 R^2 \ln(p^2 R^2)). \]

(3.31)

These are our main results in the first part of this paper. Since the photon vacuum polarization with \( p_3 = 0 \) is given by

\[ L_{\ell r} = \frac{\beta_0 \lambda}{16 \pi^2} \left[ \delta_{\ell r} \ln(\mu^2 R^2) + \delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{\ell r} - f_2(p^2 R^2)_{\ell r} \right], \]

\[ T_{\ell r} = \frac{\beta_0 \lambda}{16 \pi^2} \left[ \delta_{\ell r} \ln(\mu^2 R^2) + \delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{\ell r} \right], \]

(3.32)

it is shown with Eqs. (3.30) and (3.31) that the logarithmic factor \( \ln p^2 \) disappears in the photon vacuum polarization because of the effect of the \( S^1 \) compactification. This generalizes the observation made in Ref. [11] for \( N = 2 \) and 3 to arbitrary finite \( N \). Hence, it is concluded that IR renormalon is absent for arbitrary finite \( N \). (Supplementary explanation for the absence of IR renormalon is given in Sect. 4 with an explicit example of a gauge invariant quantity.)

In the subsequent subsections, we will show the asymptotic behaviors explicitly for some small \( N \).

### 3.3 \( N = 2 \)

For \( N = 2 \), \( \ell \) and \( r \) can take only \( \ell = r = 1 \) in Eq. (2.16) and

\[ (\sigma_{j,2})_{11} = \begin{cases} 1, & \text{for } j = 0 \text{ mod } 2, \\ -1, & \text{for } j = 1 \text{ mod } 2. \end{cases} \]

(3.33)

From this, we have

\[ \sum_{j=1}^{1} (\sigma_{j,2})_{11} \ln \Gamma(j/2) = -\ln \sqrt{\pi}, \]

\[ \sum_{j=1}^{\infty} (\sigma_{j,2})_{11} \frac{1}{j^2} = -\frac{\pi^2}{12}. \]

(3.34)

---

\( \delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) \) behavior in \( L_{\ell r} \) implies that naive perturbation theory suffers from IR divergences. To see this, one should note that \( L_{\ell r} \) is \( O(\lambda) \) and thus higher powers of \( L_{\ell r} \) are included in the numerator of the integrand in calculating higher order perturbative coefficients; see the first equation of Eq. (2.11). Hence, we have severer IR divergences at higher orders. This is nothing but the famous IR divergence in finite temperature [44, 45] although here the boundary condition for the adjoint fermion is not anti-periodic. To avoid the IR divergences, the \( 1/(p^2 R^2) \) term should not be expanded and should be kept in the denominator. With this understanding, the absence of IR renormalon is concluded. See also Sect. 4.
and, therefore, from Eqs. (3.30) and (3.31),
\[
\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{11} = 2\gamma + \ln 4 + O(p^2 R^2 \ln(p^2 R^2)),
\]
\[
f_2(p^2 R^2)_{11} = -\frac{1}{p^2 R^2} + 1 + O(p^2 R^2 \ln(p^2 R^2)).
\] (3.35)

In Figs. 1 and 2 we plot the functions appearing in the left-hand side of Eq. (3.35). We numerically compute them directly from the definition (3.11). In Fig. 1, we see that the asymptotic value of \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{11}\) (the blue curve) as \(p^2 R^2 \to 0\) is correctly given by Eq. (3.35). The broken line shows the logarithmic function \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right)\), which diverges as \(p^2 R^2 \to 0\). In Fig. 2, we show the function \(f_2(p^2 R^2)_{11}\) by a solid line (blue), and we can see that it indeed approaches the asymptotic behavior obtained in Eq. (3.35), which is shown by the dashed line of the same color, as \(p^2 R^2 \to 0\).

![Fig. 1: The function \(\delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{\ell r}\). From below to top, \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{11}\) for \(N = 2\), \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{11}\) for \(N = 3\), and \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{11}\) and \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{33}\) for \(N = 4\). The logarithmic function \(\ln \left( \frac{e^{5/3}}{p^2 R^2} \right)\) is also drawn by the broken line.](image-url)

3.4. \(N = 3\)

In this case with \(N = 3\), from Eq. (2.16), we have
\[
(\sigma_{j,3})_{\ell r} = \delta_{\ell r} \begin{cases} 
1, & \text{for } j = 0 \mod 3, \\
-\frac{1}{2}, & \text{for } j = 1, 2 \mod 3,
\end{cases}
\] (3.36)

and then
\[
\sum_{j=1}^{2} (\sigma_{j,3})_{\ell r} \ln \Gamma(j/3) = -\frac{1}{2} \ln \left( \frac{2\pi}{\sqrt{3}} \right) \delta_{\ell r},
\]
\[
\sum_{j=1}^{\infty} (\sigma_{j,3})_{\ell r} \frac{1}{j^2} = -\frac{\pi^2}{18} \delta_{\ell r}.
\] (3.37)
Fig. 2: The function $f_2(p^2 R^2)_{\ell r}$. From below to top, $f_2(p^2 R^2)_{11}$ for $N = 2$, $f_2(p^2 R^2)_{11}$ for $N = 3$, and $f_2(p^2 R^2)_{11}$ and $f_2(p^2 R^2)_{33}$ for $N = 4$. The dashed lines show their asymptotic behaviors obtained from Eqs. (3.35), (3.38), and (3.44).

We thus obtain

$$
\delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{\ell r} = \left[ 2\gamma + \ln \frac{27}{4} + O(p^2 R^2 \ln(p^2 R^2)) \right] \delta_{\ell r},
$$

$$
f_2(p^2 R^2)_{\ell r} = \left[ -\frac{2}{3} \frac{1}{p^2 R^2} + 1 + O(p^2 R^2 \ln(p^2 R^2)) \right] \delta_{\ell r}. \tag{3.38}
$$

In Figs. 1 and 2, we plot the functions appearing in the left-hand side of Eq. (3.38). Again, in Fig. 1 we see that the asymptotic value of $\ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{11}$ (the yellow curve) as $p^2 R^2 \to 0$ is correctly given by Eq. (3.38). Also in Fig. 2, we confirm validity of the asymptotic form of $f_2(p^2 R^2)_{11}$.

We can compare the results in Ref. [11] for $N = 2$ and 3 with our results in Eqs. (3.35) and (3.38). Using $\beta_0 = 11/3 - 2n_W/3$, Eqs. (5.3) and (5.4) of Ref. [11] show that for $p_3 = 0$ (in our notation)

$$
L_{\ell r} = \frac{(\beta_0 - 3)\lambda}{16\pi^2} \left\{ \left[ 1 - 3 \left( \frac{2}{N} - 1 \right)^2 \right] \frac{1}{p^2 R^2} - \frac{2}{3} \right\} \delta_{\ell r} + \frac{\beta_0 \lambda}{16\pi^2} \left[ \ln(\Lambda_0^2 R^2) - \ln 4 - \psi(1/N) - \psi(1 - 1/N) \right] \delta_{\ell r} + O(p^2 R^2 \ln(p^2 R^2)),
$$

$$
T_{\ell r} = \frac{(\beta_0 - 3)\lambda}{16\pi^2} \frac{1}{3} \delta_{\ell r} + \frac{\beta_0 \lambda}{16\pi^2} \left[ \ln(\Lambda_0^2 R^2) - \ln 4 - \psi(1/N) - \psi(1 - 1/N) \right] \delta_{\ell r} + O(p^2 R^2 \ln(p^2 R^2)), \tag{3.39}
$$

where $\Lambda_0$ is the renormalization scale in Ref. [11] and $\psi(z) \equiv (d/dz) \ln \Gamma(z)$. Noting that $\psi(1/2) = -\gamma - \ln 4$ and $\psi(1/3) + \psi(2/3) = -2\gamma - 3 \ln 3$, we see that by choosing

$$
\Lambda_0^2 = e^{-1/3} \mu^2, \tag{3.40}
$$
these expressions perfectly coincide with our results, Eq. (3.32) with Eqs. (3.35) and (3.38), for \( \beta_0 \to \infty \). We note that, because of the difference in the treatment of the gauge field loops, we expect that the results in Ref. [11] coincide with ours only in the limit \( \beta_0 \to \infty \), in which the contribution of the fermion loop diagrams dominates.

3.5. \( N = 4 \)

For this case, Eq. (2.16) gives

\[
(\sigma_{j,4})_{\ell r} = \begin{cases} 
(1 \quad 0 \quad 0), & \text{for } j = 0 \mod 4, \\
0 \quad 1 \quad 0, & \\
0 \quad 0 \quad 1, & \\
\frac{-1}{4} \quad \frac{-1}{4\sqrt{3}} \quad \frac{1}{2\sqrt{6}}, & \text{for } j = 1, 3 \mod 4, \\
\frac{-1}{4\sqrt{3}} \quad -\frac{1}{12} \quad -\frac{1}{6\sqrt{2}}, & \\
\frac{1}{2\sqrt{6}} \quad -\frac{1}{6\sqrt{2}} \quad -\frac{1}{3}, & \\
\frac{-1}{2} \quad \frac{1}{2\sqrt{3}} \quad -\frac{1}{6} \quad -\frac{1}{3\sqrt{2}}, & \text{for } j = 2 \mod 4, \\
\frac{1}{2\sqrt{3}} \quad -\frac{1}{6} \quad \frac{1}{3\sqrt{2}} \quad -\frac{1}{3}, & \\
\frac{1}{\sqrt{6}} \quad \frac{1}{3\sqrt{2}} \quad -\frac{1}{3}, & 
\end{cases}
\]

where the row and the column refer to the indices \( \ell \) and \( r \), respectively. As noted in Eq. (3.9), these matrices can be simultaneously diagonalized by an orthogonal transformation. After this diagonalization, we have

\[
(\sigma_{j,4})_{\ell r} = \begin{cases} 
(1 \quad 0 \quad 0), & \text{for } j = 0 \mod 4, \\
0 \quad 1 \quad 0, & \\
0 \quad 0 \quad 1, & \\
\frac{-1}{2} \quad 0 \quad 0, & \text{for } j = 1, 3 \mod 4, \\
0 \quad -\frac{1}{2} \quad 0, & \\
0 \quad 0 \quad 0, & \\
0 \quad 0 \quad 0, & \text{for } j = 2 \mod 4, \\
0 \quad 0 \quad -1 & 
\end{cases}
\]

In this diagonal basis, we have

\[
\sum_{j=1}^{3} (\sigma_{j,4})_{\ell r} \ln \Gamma(j/4) = \begin{cases} 
-\frac{1}{2} \ln(\sqrt{\pi}), & \text{for } \ell = r = 1, \\
-\frac{1}{2} \ln(\sqrt{2\pi}), & \text{for } \ell = r = 2, \\
-\frac{1}{2} \ln \pi, & \text{for } \ell = r = 3, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\sum_{j=1}^{\infty} (\sigma_{j,4})_{\ell r} \frac{1}{j^2} = \begin{cases} 
-\frac{5\pi^2}{96}, & \text{for } \ell = r = 1, \\
-\frac{5\pi^2}{96}, & \text{for } \ell = r = 2, \\
-\frac{\pi^2}{48}, & \text{for } \ell = r = 3, \\
0, & \text{otherwise.}
\end{cases}
\]
We thus have

\[
\delta_{\ell r} \ln \left( \frac{e^{5/3}}{p^2 R^2} \right) + f_0(p^2 R^2)_{\ell r} = \begin{cases} 
2\gamma + \ln 8 + O(p^2 R^2 \ln(p^2 R^2)), & \text{for } \ell = r = 1, \\
2\gamma + \ln 8 + O(p^2 R^2 \ln(p^2 R^2)), & \text{for } \ell = r = 2, \\
2\gamma + \ln 16 + O(p^2 R^2 \ln(p^2 R^2)), & \text{for } \ell = r = 3, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
f_2(p^2 R^2)_{\ell r} = \begin{cases} 
-\frac{5}{8} \frac{1}{p^2 R^2} + 1 + O(p^2 R^2 \ln(p^2 R^2)), & \text{for } \ell = r = 1, \\
-\frac{5}{8} \frac{1}{p^2 R^2} + 1 + O(p^2 R^2 \ln(p^2 R^2)), & \text{for } \ell = r = 2, \\
-\frac{1}{4} \frac{1}{p^2 R^2} + 1 + O(p^2 R^2 \ln(p^2 R^2)), & \text{for } \ell = r = 3, \\
0, & \text{otherwise.}
\end{cases}
\]

In Figs. 1 and 2 we plot the functions appearing in the left-hand side of Eq. (3.44). In Fig 1 we see that the asymptotic value of \(\delta_{\ell r} \ln[e^{5/3}/(p^2 R^2)] + f_0(p^2 R^2)_{\ell r}\) (the green and orange curves) as \(p^2 R^2 \to 0\) is correctly given by Eq. (3.44). Also in Fig. 2 we can confirm validity of the asymptotic form of \(f_2(p^2 R^2)_{11}\) and \(f_2(p^2 R^2)_{33}\).

3.6. **Comment on the \(N = \infty\) case**

In Ref. [13], the \(N \to \infty\) limit of the expressions in Eq. (2.14) is considered and it is concluded that IR renormalon exists in this limit. Since we have observed that there is no IR renormalon for arbitrary finite \(N\), we should clarify how these two conclusions are related. The crucial relation that led to the existence of IR renormalon for \(N \to \infty\) is the bounds [13]:

\[
\left| \sum_{j \neq 0} \sigma_{j,N} \int_0^1 dx e^{ixp_j 2\pi R_j} x(1 - x) K_0(z) \right| < \frac{8\zeta(3)}{\pi^3(p^2 R^2)^{3/2}} \left( \frac{1}{N^3} + \frac{4}{N} \right),
\]

and

\[
\left| \sum_{j \neq 0} \sigma_{j,N} \int_0^1 dx e^{ixp_j 2\pi R_j} x(1 - x) K_2(z) \right| < \frac{16\zeta(4)}{\pi^4(p^2 R^2)^2} \left( \frac{1}{N^4} + \frac{4}{N} \right) + \frac{8\zeta(3)}{\pi^3(p^2 R^2)^{3/2}} \left( \frac{1}{N^3} + \frac{4}{N} \right).
\]

Similar bounds hold for the finite volume parts in \(L\) and \(T\) for the W-boson vacuum polarization, i.e., for the expressions where \(\sigma_{j,N}\) is omitted and the sum is replaced by \(\sum_{j \neq 0, j = 0 \mod N}\). From these bounds, for a fixed non-zero momentum \(p\), the terms containing the Bessel functions in Eq. (2.14) (finite volume corrections) vanish as \(N \to \infty\); the vacuum polarizations then become those in \(\mathbb{R}^4\). This is the basic logic in Ref. [13] in concluding IR renormalon (see also Ref. [14]). The problem in this argument is that the bound is not uniform in the momentum \(p\). The situation can be clearly seen in Figs. 1 and 2: for any fixed non-zero \(p^2 R^2\), the functions \(f_0(p^2 R^2)\) and \(f_2(p^2 R^2)\) vanish as \(N \to \infty\). In particular, in Fig. 1 the curves approach the logarithmic function \(\ln[e^{5/3}/(p^2 R^2)]\) (the broken line) as \(N \to \infty\) at each fixed non-zero value of \(p^2 R^2\). However, in Fig. 1 as far as \(N\) is finite, the limiting value as \(p^2 R^2 \to 0\) is finite and does not have the logarithmic behavior \(\ln p^2\) at \(p^2 = 0\). A similar remark applies
to the vacuum polarization of the W-boson, because the logarithmic behavior can appear only when the twisted shift of the momentum vanishes, i.e., when $N = \infty$. In this way, the vacuum polarization may possess the logarithmic factor only at the limiting point $N = \infty$; the existence of IR renormalon is peculiar in this single point. Therefore, $N = \infty$ cannot be used as a starting point for the study of IR renormalon in $SU(N)$ QCD(adj.) with finite $N$ even though it is very large. In this sense, we have to admit that the statement on IR renormalon in Ref. [13] is not wrong but misleading.

4. Decompactification limit $R \to \infty$ and renormalon precursor

We have observed that, under the $S^1$ compactification, the functions $L_{\ell r}$ and $T_{\ell r}$ in Eq. (2.14) appearing in the vacuum polarization of the photon with $p_3 = 0$ lose the logarithmic behavior $\ln p^2$ for $p^2 \to 0$. The vacuum polarization of the W-boson also does not possess $\ln p^2$ behavior around $p^2 = 0$ because its momentum is given by the twisted momentum $p_{mn}$ of Eq. (2.19) and $p_{mn}^2$ cannot take zero as long as $R N$ is finite. According to the discussion in Sect. 2, therefore, there is no ambiguity associated with IR renormalon (i.e., the factorial growth of perturbative coefficients) under the $S^1$ compactification. Then, it is natural to wonder how the ambiguity associated with IR renormalon in $R^4$ can emerge under the decompactification of $S^1$, $R \to \infty$. The purpose of this section is to understand this issue. We are naturally led to introduce the notion of “renormalon precursor” from this consideration.

To illustrate the idea of the renormalon precursor, let us consider the example of the “gluon condensate” in the $N = 2$ theory that is given in the large-$\beta_0$ approximation from Eq. (2.11) by

$$
\langle \text{tr}(F_{\mu \nu}F_{\mu \nu}) \rangle = -\frac{\lambda}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} \left\{ [1 - L_{11}(\mu)]^{-1} + 2 [1 - T_{11}(\mu)]^{-1} \right\} - \frac{\lambda}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} \left\{ [1 - L(\mu)]^{-1} + 2 [1 - T(\mu)]^{-1} \right\}_{p \to p_{12}}
$$

$$
= \frac{\lambda}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} \left[ \frac{1}{L_{11}(\mu = \Lambda)} + \frac{2}{T_{11}(\mu = \Lambda)} \right]_{p \to p_{12, p_{21}}},
$$

(4.1)

where the functions $L_{11}(\mu), T_{11}(\mu)$ (contained in the photon vacuum polarization), $L(\mu)$, and $T(\mu)$ (contained in the W-boson vacuum polarization) are given by Eq. (2.11); here, we have explicitly written the dependence on the renormalization scale $\mu$. To derive the last expression, we have noted

$$
1 - \frac{\beta_0 \lambda}{16\pi^2} \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) = -\frac{\beta_0 \lambda}{16\pi^2} \ln \left( \frac{e^{5/3} \Lambda^2}{p^2} \right),
$$

(4.2)

for the one-loop dynamical scale (2.9). (The renormalization scale of the coupling $\lambda$ is always set to $\mu$ even in $L_{11}(\mu = \Lambda)$ etc.) In these expressions, we mean that only the argument of

---

Recall that for $N = 2$, $l$ and $r$ can take only $l = r = 1$ in $L_{\ell r}$ and $T_{\ell r}$ and these are not matrices but simply numbers.
logarithm is set to $\mu = \Lambda$.) We note that in the last expression of Eq. (4.1), the overall factor $\lambda = \lambda(\mu^2)$ is actually cancelled against the overall factor $\lambda = \lambda(\mu^2)$ of the functions $L_{tr}(\mu = \Lambda)$, $T_{tr}(\mu = \Lambda)$, ... in the denominator, and thus the gluon condensate (4.1) is clearly renormalization group invariant. In Fig. 3, we plot the functions appearing in the denominators of Eq. (4.1), i.e.,

$$
\frac{16\pi^2}{\beta_0 \lambda} \begin{cases}
L_{11}(\mu = \Lambda) |_{p_3=0}, \\
T_{11}(\mu = \Lambda) |_{p_3=0}, \\
L(\mu = \Lambda) |_{p_3=1/(2R)}, \\
T(\mu = \Lambda) |_{p_3=1/(2R)},
\end{cases}
$$

as functions of $p^2 = \sum_{i=0}^{2} p_i^2$ for various values of the compactification radius $R\Lambda$. As already noted, these quantities are renormalization group invariant. Here, $p_3$ is set to the possible smallest values, that is, $p_3 = 0$ for $L_{11}(\mu = \Lambda)$ and $T_{11}(\mu = \Lambda)$, and $p_3 = 1/(NR) = 1/(2R)$ for $L(\mu = \Lambda)$ and $T(\mu = \Lambda)$ as implied in Eq. (4.1).

**Fig. 3:** (a) $16\pi^2/(\beta_0 \lambda)L_{11}(\mu = \Lambda) |_{p_3=0}$ for $R\Lambda = 0.3$, 0.2, and 0.1.

(b) $16\pi^2/(\beta_0 \lambda)T_{11}(\mu = \Lambda) |_{p_3=0}$ for $R\Lambda = 0.3$, 0.2, and 0.1.

(c) $16\pi^2/(\beta_0 \lambda)L(\mu = \Lambda) |_{p_3=1/(2R)}$ for $R\Lambda = 0.3$, 0.2, and 0.1.

(d) $16\pi^2/(\beta_0 \lambda)T(\mu = \Lambda) |_{p_3=1/(2R)}$ for $R\Lambda = 0.3$, 0.2, and 0.1.

Fig. 3: $(16\pi^2/\beta_0 \lambda)L_{11}$, $(16\pi^2/\beta_0 \lambda)T_{11}$, $(16\pi^2/\beta_0 \lambda)L$ and $(16\pi^2/\beta_0 \lambda)T$ at $\mu = \Lambda$ (see Eq. (4.3)) as functions of $p^2 = \sum_{i=0}^{2} p_i^2$ for various values of the compactification radius $R\Lambda$. $p_3$ is set to the possible smallest values as Eq. (4.3). The dashed lines are the infinite volume result, $\log(e^{5/3} \Lambda^2 / p^2)$.
In Fig. 3b, we see that \( T_{11}(\mu = \Lambda)|_{p_3=0} \) goes to a finite value (rather than infinity) as \( p_3 = p^2 \rightarrow 0 \). This is precisely the disappearance of the logarithmic factor we observed in the previous section; the curves in Fig. 3b are nothing but the \( N = 2 \) curve in Fig. 1 up to trivial addition and rescaling. Therefore, according to the argument in Sect. 2, this implies the absence of a factorial growth of perturbative coefficients and the IR renormalon ambiguity in the Borel procedure (i.e., one considers the Borel transform of the perturbative expansion and then performs the Borel integral). This absence of IR renormalon persists as far as the compactification radius \( R \Lambda \) is finite.

However, the momentum integration of \( [T_{11}(\mu = \Lambda)]^{-1} \) (with \( p_3 = 0 \)) itself, as given by Eq. (4.1), becomes ill-defined and ambiguous when \( R \Lambda \) is finite but sufficiently large. This is caused by a simple zero of \( T_{11}(\mu = \Lambda) \) at \( p^2 = p_3 > 0 \), which arises when \( R \Lambda \) is sufficiently large as shown in Fig. 3b. We note that Eq. (4.1) is a resummed quantity of the perturbative series in a different way from the Borel procedure, which is obtained with resummation of a geometric series. Similarly, \( T_{11}(\mu = \Lambda) \) with other discrete values of \( p_3 \) can possess a zero as the function of \( p^2 \) and then the momentum integration becomes ambiguous; these ambiguities are what we term the renormalon precursor. This is not the conventional renormalon because \( T_{11}(\mu = \Lambda)|_{p_3=0} \) has no logarithmic factor \( \sim \ln p_3^2 \) as \( p^2 = p_3 \rightarrow 0 \) and thus the perturbative coefficients do not exhibit factorial growth. However, instead, the momentum integration becomes ambiguous. This is a “precursor” of IR renormalon in the sense that, under the decompactification \( R \Lambda \rightarrow \infty \), the sum of ambiguities arising from each zero of \( T_{11}(\mu = \Lambda) \) (corresponding to different discrete values of \( p_3 \)) smoothly reduces to the IR renormalon ambiguity in \( \mathbb{R}^4 \). (In an example in Appendix B this summation over ambiguities is explicitly calculated.) To see this, we note that, as \( R \Lambda \rightarrow \infty \), the terms containing the modified Bessel function in Eq. (2.14) are suppressed for any finite \( p^2 > 0 \) (see Eqs. (3.45) and (3.46)) and thus

\[
T_{11}(\mu = \Lambda) \xrightarrow{RA \to \infty} \beta_0 \lambda \frac{16}{16 \pi^2} \ln \left( \frac{e^{\frac{5}{3} \Lambda^2}}{p^2} \right), \tag{4.4}
\]

\[\text{We can explain the absence of IR renormalon explicitly also in the following way. For instance for the } T_{11} \text{ part, from the first expression in Eq. (4.1) and Eq. (3.12), the perturbative expansion is given by}
\]

\[
- \lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} \left[ \ln(e^{5/3} \mu^2/p^2) + f_0(p^2 R^2)_{11} \right]^k \left( \frac{\beta_0 \lambda}{16\pi^2} \right)^k
\]

and then the Borel transform \( 2.5 \) is obtained as

\[
B(u) = - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} e^{u[\ln(e^{5/3} \mu^2/p^2) + f_0(p^2 R^2)_{11}]}.
\]

Since \( \ln(e^{5/3} \mu^2/p^2) + f_0(p^2 R^2)_{11} \sim \text{const. in the IR region, the momentum integral (and the sum) giving the Borel transform is not IR divergent for any } u > 0; \text{ the Borel transform does not possess singularities at } u > 0. \text{ This is in contrast with the un-compactified case, where } \ln(e^{5/3} \mu^2/p^2) + f_0(p^2 R^2)_{11} \text{ is replaced by } \ln(e^{5/3} \mu^2/p^2) \text{ (and also } \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} \to \int \frac{d^3p}{(2\pi)^3} \text{) and the Borel transform possesses a singularity at certain } u > 0.\]

19
as shown in Fig. [30], where the dashed line corresponds to the right-hand side of this equation.

We thus have

\[- \lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{p_3} \frac{1}{T_{11}(\mu = \Lambda)} \xrightarrow{RA \to \infty} - \lambda \int \frac{d^4p}{(2\pi)^4} \frac{1}{\beta_0 \lambda} \frac{1}{16\pi^2} \ln \left( \frac{e^{5/3} \Lambda^2}{p^2} \right). \tag{4.5}\]

The integrand of the momentum integration in Eq. (4.5) possesses the pole singularity at \( p^2 = e^{5/3} \Lambda^2 \) as

\[\sim \frac{16\pi^2}{\beta_0} \int \frac{d^4p}{(2\pi)^4} \frac{e^{5/3} \Lambda^2}{p^2 - e^{5/3} \Lambda^2} = \frac{1}{\beta_0} \int_0^\infty d(p^2) \frac{e^{5/3} \Lambda^2}{p^2 - e^{5/3} \Lambda^2}. \tag{4.6}\]

The ambiguity of this momentum integration (the renormalon precursor) in the \( R \to \infty \) limit gives rise to an ambiguity

\[\pm i\pi \frac{e^{10/3}}{\beta_0} \Lambda^4. \tag{4.7}\]

We have defined the ambiguity by the imaginary part that appears when the integration contour is deformed in the complex \( p^2 \)-plane such that it avoids the pole.

This ambiguity of the renormalon precursor in the \( R \to \infty \) limit is exactly the same as the renormalon ambiguity in \( \mathbb{R}^4 \). In \( \mathbb{R}^4 \), the corresponding part and its perturbative expansion are given by

\[- \lambda \int \frac{d^4p}{(2\pi)^4} \frac{1}{\beta_0 \lambda} \frac{1}{16\pi^2} \ln \left( \frac{e^{5/3} \Lambda^2}{p^2} \right) \text{ expansion in } \lambda \xrightarrow{\lambda \to \infty} \lambda \sum_{k=0}^{\infty} f_k \left( \frac{\beta_0 \lambda}{16\pi^2} \right)^k \tag{4.8}\]

with

\[f_k = \int \frac{d^4p}{(2\pi)^4} \left[ \ln \left( \frac{e^{5/3} \mu^2}{p^2} \right) \right]^k \xrightarrow{k \to \infty} \frac{e^{10/3}}{16\pi^2} \mu^4 \frac{k!}{2^{k+1}}, \tag{4.9}\]

where we have used Eq. (4.2) in the perturbative expansion. This factorial growth of the perturbative coefficients produces the pole in the Borel transform \( (2.5) - e^{10/3} \mu^4/(16\pi^2) 1/(u - 2) \) and, through the Borel integral (2.7), the IR renormalon ambiguity

\[\pm i\pi \frac{e^{10/3}}{\beta_0} \Lambda^4. \tag{4.10}\]

This is the same as Eq. (4.7).

The situation is similar for \( L(\mu = \Lambda) \) and \( T(\mu = \Lambda) \) in Eq. (1.1). As far as \( RA \) is finite, \( L(\mu = \Lambda) \) and \( T(\mu = \Lambda) \) in Eq. (1.1) do not diverge as \( p^2 \to 0 \), because of the twisted momentum, \( p_3 = n/R + 1/(2R) \neq 0 \) for \( n \in \mathbb{Z} \). Thus, there is neither logarithmic factor nor IR renormalon. On the other hand, it can be shown that the terms containing the modified Bessel function in Eq. (2.14) are suppressed for any finite \( p \) as \( RA \to \infty \) and

\[L(\mu = \Lambda), T(\mu = \Lambda) \xrightarrow{RA \to \infty} \beta_0 \lambda \frac{e^{5/3} \Lambda^2}{16\pi^2} \ln \left( \frac{e^{5/3} \Lambda^2}{p^2} \right). \tag{4.11}\]

According to these behaviors, \( L(\mu = \Lambda) \) and \( T(\mu = \Lambda) \) acquire zeros as \( RA \) becomes larger (but still finite) and the momentum integrations of \( L(\mu = \Lambda)^{-1} \) and \( T(\mu = \Lambda)^{-1} \) become

\[\text{In } \mathbb{R}^4, \text{ the IR renormalon ambiguity can be viewed as the ambiguity arising from the momentum integration [40] as well as the ambiguity in the Borel integral. The renormalon precursor is thus analogous to the former picture; the renormalon precursor however does not always coincide with the ambiguity in the Borel integral.}\]
ambiguous; the presence of these zeroes can be seen in Figs. 3c and 3d. The ambiguity caused by these zeroes (i.e., the renormalon precursor) coincides with the IR renormalon ambiguity as $R\Lambda \to \infty$ as in the above case of $T_{11}(\mu = \Lambda)$.

The situation is slightly different for $L_{11}(\mu = \Lambda)$. As shown in Eqs. (3.32) and (3.35), $L_{11}(\mu = \Lambda)|_{p_3=0}$ has a more singular behavior for $p^2 = p^2 \to 0$ as

$$L_{11}(\mu = \Lambda)|_{p_3=0} = \frac{\beta_0 \lambda}{16\pi^2} \left[ \frac{1}{p^2 R^2} + \ln(\Lambda^2 R^2) + 2\gamma + \ln 4 - 1 + O(p^2 R^2 \ln(p^2 R^2)) \right]$$

and $L_{11}(\mu = \Lambda)|_{p_3=0}$ diverges as $p^2 = p^2 \to 0$ as clearly seen in Fig. 3a. Here, the screening mass $m_{sc}$ has been introduced by $m_{sc}^2 \equiv -\beta_0 \lambda/(16\pi^2 R^2)$. As noted in footnote 6, in perturbative expansion, we do not regard $m_{sc}$ as an $O(\lambda)$ quantity and treat it as if it was an $O(\lambda^0)$ quantity, to avoid IR divergences in fixed order perturbation theory. In this treatment, the situation becomes similar to the above cases, and it is concluded that there is no IR renormalon. On the other hand, again in $L_{11}(\mu = \Lambda)$ the terms containing the modified Bessel function in Eq. (2.14) are suppressed for fixed $p^2 > 0$ as $R\Lambda \to \infty$,

$$L_{11}(\mu = \Lambda)|_{p_3=0} \to R\Lambda \to \infty \frac{\beta_0 \lambda}{16\pi^2} \ln \left( \frac{e^{5/3} \Lambda^2}{p^2} \right).$$

Thus, the ambiguity in the momentum integration of $L_{11}(\mu = \Lambda)|_{p_3=0}$, i.e., the renormalon precursor, caused by the zero in Fig. 3a is smoothly reduced to the IR renormalon ambiguity in $\mathbb{R}^4$ as $R\Lambda \to \infty$.

Although in this section we demonstrated the presence of the renormalon precursor only in the $N = 2$ theory, this notion must be quite general being applicable to QCD(adj.) with any $N$. See also Appendix B for other examples where this notion applies.

5. Conclusion

In this paper, we made some remarks on the issue of possible existence of IR renormalon in the $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ with the $Z_N$ twisted boundary condition, by making use of the large-$\beta_0$ approximation. In the first part of this paper, we showed that for any finite $N$ the photon vacuum polarization loses the logarithmic factor $\ln p^2$ as $p^2 \to 0$ and there is no IR renormalon in the compactified spacetime $\mathbb{R}^3 \times S^1$. In the second part, we presented the notion of the renormalon precursor, i.e., the ambiguity in the momentum integration, which smoothly reduces to the IR renormalon ambiguity in $\mathbb{R}^4$ under the decompactification $\mathbb{R}^3 \times S^1 \to \mathbb{R}^4$. On the first issue, although our demonstration of the disappearance of the logarithmic factor required very detailed calculations, there might be a more direct and simpler way to understand the absence of the logarithmic factor. On the second issue, the renormalon precursor is a quite general notion as an object which smoothly complements the difference between the absence and existence of IR renormalon under the removal of an IR cutoff (such as the compactification radius, the mass, etc.).

Acknowledgements

This work was supported by JSPS Grant-in-Aid for Scientific Research Grant Numbers JP18J20935 (O.M.), JP16H03982, JP20H01903 (H.S.), and JP19K14711 (H.T.).
A. Rigorous proof of Eqs. (3.29)–(3.31) up to $O((p^2 R^2)^0)$

In this appendix, we give a rigorous proof for the asymptotic expansion in Eqs. (3.29)–(3.31) up to $O((p^2 R^2)^0)$ terms. This is sufficient to conclude the disappearance of the logarithmic factor $\ln p^2$ as $p^2 R^2 \to 0$ in the photon vacuum polarization and the absence of IR renormalon.

We study the function (3.11),

$$ f_\nu(p^2 R^2)_{tr} = 24 \sum_{j=1}^{\infty} (\sigma_{j,N})_{tr} \int_0^1 dx \, x(1-x) K_\nu(2 j \hat{p}(x)) \tag{A1} $$

for $\nu = 0$ and 2, where we have set $\hat{p}(x) \equiv \pi \sqrt{x(1-x)} \, p^2 R^2$. First, we note that $(\sigma_{j,N})_{tr}$ in Eq. (2.16) can be represented as

$$ (\sigma_{j,N})_{tr} = \sum_{m,n=1}^{N} C_{tr}^{mn} e^{i(n-m)2\pi j/N} \tag{A2} $$

with

$$ C_{tr}^{mn} \equiv \frac{1}{N} (\nu^m - \nu^n) e^{(\nu^m - \nu^n)_{tr}}. \tag{A3} $$

Then, using integral representations of the modified Bessel functions,

$$ K_0(z) = \int_0^1 \frac{dt}{t} e^{-z(t+\frac{1}{t})}, \quad K_2(z) = \frac{1}{2} \int_0^1 \frac{dt}{t} \left( t^2 + \frac{1}{t^2} \right) e^{-z(t+\frac{1}{t})}, \tag{A4} $$

we obtain

$$ f_\nu(p^2 R^2)_{tr} = 24 \sum_{m,n=1}^{N} C_{tr}^{mn} \lim_{j_{max} \to \infty} \sum_{j=1}^{j_{max}} e^{i(n-m)2\pi j/N} \int_0^1 dx \, x(1-x) \int_0^1 \frac{dt}{t} e^{-j \hat{p}(x)(t+\frac{1}{t})} \tag{A5} $$

$$ = 24 \sum_{m,n=1}^{N} C_{tr}^{mn} \lim_{j_{max} \to \infty} \int_0^1 dx \, x(1-x) \int_0^1 \frac{dt}{t} \sum_{j=1}^{j_{max}} \left[ e^{i(n-m)2\pi j/N} e^{-\hat{p}(x)(t+\frac{1}{t})} \right]^j $$

$$ = 24 \sum_{m,n=1}^{N} C_{tr}^{mn} e^{i(n-m)2\pi/N} \int_0^1 dx \, x(1-x) \int_0^1 \frac{dt}{t} \frac{1}{e^{\hat{p}(x)(t+\frac{1}{t})} - e^{i(n-m)2\pi/N}} $$

$$ - 24 \sum_{m,n=1}^{N} C_{tr}^{mn} e^{i(n-m)2\pi/N} \lim_{j_{max} \to \infty} \int_0^1 \frac{dt}{t} \left[ e^{i(n-m)2\pi j/N} e^{-\hat{p}(x)(t+\frac{1}{t})} \right]^{j_{max}}. \tag{A5} $$

The second term in the last line vanishes because

$$ \left| \int_0^1 dx \, x(1-x) \int_0^1 \frac{dt}{t} \left| e^{i(n-m)2\pi/N} e^{-\hat{p}(x)(t+\frac{1}{t})} \right| \right| $$

$$ \leq \int_0^1 dx \, x(1-x) \int_0^1 \frac{dt}{t} \left| e^{-j_{max} \hat{p}(x)(t+\frac{1}{t})} \right| $$

$$ \leq \frac{1}{|\text{Im}(e^{i(n-m)2\pi/N})|} \int_0^1 dx \, x(1-x) \int_0^1 \frac{dt}{t} e^{-j_{max} \hat{p}(x)(t+\frac{1}{t})} $$

$$ < \frac{1}{j_{max} |\text{Im}(e^{i(n-m)2\pi/N})|} \int_0^1 dx \, x(1-x) \frac{1}{\hat{p}(x)}. \tag{A6} $$
where, in the last step, we have used
\[
\int_0^1 \frac{dt}{t} e^{-j_{\text{max}} \hat{p}(x)(t+\frac{1}{2})} = K_0(2j_{\text{max}} \hat{p}(x)) < \frac{1}{j_{\text{max}} \hat{p}(x)} e^{-j_{\text{max}} \hat{p}(x)} < \frac{1}{j_{\text{max}} \hat{p}(x)}
\] (A7)
for \( \hat{p}(x) > 0 \), as shown in Eq. (B3) of Ref. [12]. Hence, we obtain
\[
f_0(\hat{p}^2 R^2)_{tr} = 24 \sum_{m,n=1}^N C_{tr}^{mn} e^{i(n-m)2\pi/N} \int_0^1 \frac{dt}{t} x(1-x) \int_0^1 \frac{dt}{t} e^{i\hat{p}(x)(t+\frac{1}{2})} e^{i(n-m)2\pi/N},
\] (A8)
and, in a similar manner\(^{10}\)
\[
f_2(\hat{p}^2 R^2)_{tr} = 12 \sum_{m,n=1}^N C_{tr}^{mn} e^{i(n-m)2\pi/N} \int_0^1 \frac{dt}{t} x(1-x) \int_0^1 \frac{dt}{t} \left( t^2 + \frac{1}{t^2} \right) e^{i\hat{p}(x)(t+\frac{1}{2})} e^{i(n-m)2\pi/N}.
\] (A9)
To obtain the asymptotic behavior of \( f_0(\hat{p}^2 R^2)_{tr} \) from Eq. (A8), we show
\[
\int_0^1 \frac{dt}{t} e^{\hat{p}(x)(t+\frac{1}{2})} - c \left[ - \frac{1}{1-c} \ln \hat{p}(x) \right] = \int_0^{\hat{p}(x)} \frac{dt}{t} e^{\hat{p}(x)(t+\frac{1}{2})} - c + \int_0^{\hat{p}(x)} \frac{dt}{t} \left[ \frac{1}{e^{\hat{p}(x)(t+\frac{1}{2})} - c} - \frac{1}{1-c} \right] = O(\hat{p}(x)^0)
\] (A10)
for small \( \hat{p}(x) > 0 \)\(^{11}\) here and hereafter, we set \( c \equiv e^{i(n-m)2\pi/N} \neq 1 \). Then, Eq. (A10) tells us that
\[
\int_0^1 \frac{dt}{t} e^{\hat{p}(x)(t+\frac{1}{2})} - c = - \frac{1}{1-c} \ln \hat{p}(x) + O(\hat{p}(x)^0).
\] (A11)
Now, for the first term in the second line of Eq. (A10), we have
\[
\left| \int_0^{\hat{p}(x)} \frac{dt}{t} e^{\hat{p}(x)(t+\frac{1}{2})} - c \right| \leq \int_0^{\hat{p}(x)} \frac{dt}{t} e^{\hat{p}(x)(t+\frac{1}{2})} - c \leq \int_0^{\hat{p}(x)} \frac{dt}{t} e^{\hat{p}(x)(t+\frac{1}{2})} - c - \int_0^1 \frac{dt}{t} e^{-1/t} = O(\hat{p}(x)^0).
\] (A12)
\(^{10}\)For \( f_2(\hat{p}^2 R^2)_{tr} \), one can use \( K_2(2z) < [(1/z) + (1/z)^2]e^{-z} \), which is shown in Eq. (B5) of Ref. [13].
\(^{11}\)We note that
\[
\int_0^1 dx x(1-x) \int_0^1 \frac{dt}{t} e^{i\hat{p}(x)(t+\frac{1}{2})} - e^{i(n-m)2\pi/N} = \lim_{\delta_1, \delta_2 \to 0} \int_{\delta_1}^{1-\delta_2} dx x(1-x) \int_0^1 \frac{dt}{t} e^{i\hat{p}(x)(t+\frac{1}{2})} - e^{i(n-m)2\pi/N},
\]
and \( \hat{p}(x) \) can be assumed to be positive.
Here, we have used \( e^{\hat{p}(x)(t+1/t)} - 1 \geq e^{\hat{p}(x)/t} - 1 \). The second term in the second line of Eq. (A10) can be bounded as

\[
\left| \int_{\hat{p}(x)}^{1} \frac{dt}{t} \frac{e^{\hat{p}(x)(t+1)} - 1}{[e^{\hat{p}(x)(t+1/t)} - c](1 - c)} \right| \leq \frac{1}{|\text{Im}c|^{2}} \int_{\hat{p}(x)}^{1} \frac{dt}{t} \left[ e^{\hat{p}(x)(t+1/t)} - 1 \right].
\]

(A13)

Using, for instance, \( s \) we can show

\[
e^{\hat{p}(x)(t+1/t)} - 1 \leq 8\hat{p}(x) \left( t + \frac{1}{t} \right), \quad \text{for } \hat{p}(x) \leq t \leq 1,
\]

(A14)

we can show

\[
\left| \int_{\hat{p}(x)}^{1} \frac{dt}{t} \frac{e^{\hat{p}(x)(t+1)} - 1}{[e^{\hat{p}(x)(t+1/t)} - c](1 - c)} \right| = O(\hat{p}(x)^{0}).
\]

(A15)

Equations (A12) and (A15) show Eq. (A10) and thus Eq. (A11).

We now study \( f_{2}(\rho R^{2})_{tr} \) in Eq. (A9). We first note

\[
\int_{0}^{1} \frac{dt}{t} \left( s^{2} + \frac{1}{t^{2}} \right) \frac{1}{e^{\hat{p}(x)(t+1/t)} - c} = \int_{1}^{\infty} \frac{ds}{\sqrt{s^{2} - 1}} (4s^{2} - 2) e^{2\hat{p}(x)s} - c
\]

under the change of variable, \( 2s = t + 1/t \). In the following we prove for the right-hand side,

\[
\int_{1}^{\infty} \frac{ds}{\sqrt{s^{2} - 1}} (4s^{2} - 2) e^{2\hat{p}(x)s} - c - \int_{0}^{\infty} ds \frac{4s}{e^{2\hat{p}(x)s} - c} = \int_{1}^{\infty} \frac{ds}{\sqrt{s^{2} - 1}} (4s^{2} - 2) e^{2\hat{p}(x)s} - c - \int_{0}^{1} ds \frac{4s}{e^{2\hat{p}(x)s} - c} = O(\hat{p}(x)^{0}).
\]

(A16)

Then, we obtain

\[
\int_{1}^{\infty} \frac{ds}{\sqrt{s^{2} - 1}} (4s^{2} - 2) e^{2\hat{p}(x)s} - c = \frac{\text{Li}_{2}(c)}{c} \frac{1}{\hat{p}(x)^{2}} + O(\hat{p}(x)^{0}),
\]

(A18)

by noting that the polylogarithm function \( \text{Li}_{2} \) can be represented as

\[
\text{Li}_{2}(c) = c\hat{p}(x)^{2} \int_{0}^{\infty} ds \frac{4s}{e^{2\hat{p}(x)s} - c}.
\]

(A19)

Now we show Eq. (A17). For the first term in Eq. (A17), using

\[
\frac{4s^{2} - 2}{\sqrt{s^{2} - 1}} - 4s \leq \frac{4}{\sqrt{s^{2} - 1}} \frac{1}{s^{2} - 1},
\]

(A20)

we obtain

\[
\left| \int_{1}^{\infty} ds \frac{4s^{2} - 2}{\sqrt{s^{2} - 1}} - 4s \right| \frac{1}{e^{2\hat{p}(x)s} - c} \leq \int_{1}^{\infty} ds \frac{4}{\sqrt{s^{2} - 1}} \frac{1}{e^{2\hat{p}(x)s} - c} = \frac{1}{|\text{Im}c|} \int_{1}^{\infty} ds \frac{4}{\sqrt{s^{2} - 1}} = O(\hat{p}(x)^{0}).
\]

(A21)

For the second term in Eq. (A17), we immediately obtain

\[
\left| \int_{0}^{1} ds \frac{4s}{e^{2\hat{p}(x)s} - c} \right| \leq \int_{0}^{1} ds \frac{4s}{e^{2\hat{p}(x)s} - c} \leq \frac{1}{|\text{Im}c|} \int_{0}^{1} ds = O(\hat{p}(x)^{0}).
\]

(A22)

Hence, Eq. (A17) and thus Eq. (A18) have been shown.

\footnote{We assume that \( p^{2}R^{2} \) is small enough such that \( \hat{p}(x) \leq 1 \) is satisfied for \( 0 \leq x \leq 1 \).}
Finally, from Eqs. (A8) and (A11), we obtain

\[
 f_0(p^2 R^2)_{tr} = -2 \sum_{m,n=1}^{N} C_{mn}^{tr} \frac{e^{i(n-m)2\pi/N}}{1 - e^{i(n-m)2\pi/N}} \ln(p^2 R^2) + O((p^2 R^2)^0)
\]

\[
 = \frac{2}{N} \sum_{j=1}^{N} \frac{1}{j} \ln(p^2 R^2) + O((p^2 R^2)^0)
\]

\[
 = \delta_\varepsilon \ln(p^2 R^2) + O((p^2 R^2)^0),
\]

(A23)

where we have used \(e^{i(n-m)2\pi/N}/[1 - e^{i(n-m)2\pi/N}] = -\frac{1}{N} \sum_{j=1}^{N} je^{i(n-m)2\pi j/N}\) and Eq. (3.8).

Also, from Eqs. (A9), (A16), and (A18), we obtain

\[
 f_2(p^2 R^2)_{tr} = \frac{12}{\pi^2} \sum_{m,n=1}^{N} C_{mn}^{tr} \text{Li}_2(e^{i(n-m)2\pi/N}) \frac{1}{p^2 R^2} + O((p^2 R^2)^0)
\]

\[
 = \frac{12}{\pi^2} \sum_{j=1}^{\infty} (\sigma_j,N)_{tr} \frac{1}{j^2 p^2 R^2} + O((p^2 R^2)^0).
\]

(A24)

These results [Eqs. (A23) and (A24)] prove the asymptotic expansion in Eqs. (3.29)–(3.31) up to \(O((p^2 R^2)^0)\).

**B. Renormalon precursor in wider context**

In this appendix, we present some other examples to which the notion of the renormalon precursor applies.

Our first example is the understanding of the shift of the Borel singularity by \(-1/2\) under the compactification \(\mathbb{R}^d \to \mathbb{R}^{d-1} \times S^1\) [14] and its relation to the decompactification limit. We start with the integral in \(\mathbb{R}^d\)

\[
 \mathcal{I}(\alpha; d) \equiv \int \frac{d^d p}{(2\pi)^d} (p^2)^\alpha \lambda(p^2),
\]

(B1)

which provides a typical example where we have IR renormalon; here, \(\lambda(p^2)\) is the one-loop running coupling

\[
 \lambda(p^2) = \frac{(4\pi)^{d/2}}{\beta_0} \frac{1}{\ln(p^2/\Lambda^2)}, \quad \Lambda^2 \equiv \mu^2 e^{-(4\pi)^{d/2}/[\beta_0 \lambda(\mu^2)]}.
\]

(B2)

Then by noting

\[
 \lambda(p^2) = \lambda(\mu^2) \sum_{k=0}^{\infty} \left[ \ln \left( \frac{\mu^2}{p^2} \right) \right]^k \left[ \frac{\beta_0 \lambda(\mu^2)}{(4\pi)^{d/2}} \right]^k,
\]

the perturbative expansion of Eq. (11) is given by

\[
 \mathcal{I}(\alpha; d) \sim \lambda(\mu^2) \sum_{k=0}^{\infty} f_k \left[ \frac{\beta_0 \lambda(\mu^2)}{(4\pi)^{d/2}} \right]^k, \quad f_k = \int \frac{d^d p}{(2\pi)^d} (p^2)\alpha \left[ \ln \left( \frac{\mu^2}{p^2} \right) \right]^k.
\]

(B4)

The corresponding Borel transform (see Eq. (2.5)) is

\[
 B[\mathcal{I}(\alpha; d)](u) = \int \frac{d^d p}{(2\pi)^d} (p^2)\alpha \left( \frac{\mu^2}{p^2} \right)^u
\]

\[
 = \mu^{2u} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)} \int_0^1 q^{2\alpha u - d - 2u} \frac{d u}{\alpha + d/2 - u},
\]

(B5)
where we have introduced an ultraviolet cutoff $q$, that is $p^2 < q^2$. This Borel transform possesses a simple pole at $u = \alpha + d/2$ and thus the Borel integral

$$\frac{(4\pi)^{d/2}}{\beta_0} \int_0^\infty du B[I(\alpha, d)](u) e^{-(4\pi)^{d/2} u/[\beta_0 \lambda(\mu^2)]}$$

(B6)

has the ambiguity (the IR renormalon ambiguity),

$$\pm i\pi \frac{1}{\beta_0} \frac{1}{\Gamma(d/2)} \Lambda^{2\alpha + d}.$$ 

(B7)

Now, let us consider the $S^1$ compactification, $\mathbb{R}^d \to \mathbb{R}^{d-1} \times S^1$ and suppose that the integrand does not change under this compactification. Moreover, let us suppose that the KK momentum $p_{d-1}$ is simply given by $p_{d-1} = n/R$ with $n \in \mathbb{Z}$ (rather than the twisted momentum). This kind of situation can occur for instance in the large $N$ limit of the two-dimensional $\mathbb{C}P^{N-1}$ models defined on the compactified spacetime $\mathbb{R}^2 \to \mathbb{R} \times S^1$. Under this situation, the integral (B1) is replaced by

$$I_C(\alpha; d) \equiv \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{2\pi R} \sum_{p_{d-1}} (p^2)^\alpha \lambda(p^2) \left|_{p_{d-1}=0} \right.$$  

(B8)

As noted in Sect. 2 for the factorial growth of perturbative coefficients, the presence of the logarithmic behavior of the integrand as $p^2 = p^2 + p_{d-1}^2$ is crucial. Then, it is sufficient to focus on the contribution with $p_{d-1} = 0$

$$\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{2\pi R} (p^2)^\alpha \lambda(p^2) \left|_{p_{d-1}=0} \right.$$  

(B9)

to detect IR renormalon. Since this is $I(\alpha, d-1)/(2\pi R)$, the Borel transform associated with this $p_{d-1} = 0$ contribution is given, from Eq. (B5), by

$$\mu^{2u} \frac{1}{2\pi R} \frac{1}{(4\pi)^{(d-1)/2}} \frac{1}{\Gamma((d-1)/2)} \frac{1}{\alpha + (d-1)/2 - u}.$$  

(B10)

Thus, the location of the Borel singularity is shifted by $-1/2$ under the compactification and the associated IR renormalon ambiguity in Eq. (B6) is, instead of Eq. (B7),

$$\pm i\pi \frac{1}{\beta_0} \frac{\sqrt{4\pi}}{\Gamma((d-1)/2)} \frac{1}{2\pi R}.$$

(B11)

This is the IR renormalon ambiguity in $I_C(\alpha; d)$ (B8). Thus, under the decompactification $R \to \infty$, the behavior of the IR renormalon ambiguity suddenly changes; Eq. (B11) vanishes as $R \to \infty$ and Eq. (B7) emerges suddenly at $R = \infty$. The renormalon precursor fills this gap and provides a smooth change under the decompactification.
To see this, we note that when the KK momentum in Eq. (B8) satisfies
\[ |p_{d-1}| < \Lambda \]
the integrand of the $d-1$ dimensional momentum integral possesses a simple pole at
\[ p^2 = \Lambda^2 - p_{d-1}^2 > 0. \]

The sum of the contributions of these poles reads (by noting $1/\ln(p^2/\Lambda^2) \sim \Lambda^2/(p^2 - \Lambda^2)$ around the pole)
\[
\frac{1}{2\pi R} \frac{(4\pi)^{d/2}}{\beta_0} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_{|p_{d-1}|<\Lambda} \frac{(p^2 + p_{d-1}^2)^\alpha}{p^2 + p_{d-1}^2 - \Lambda^2} \lambda^2 = \frac{1}{\beta_0} \frac{\sqrt{4\pi}}{\Gamma((d-1)/2)} \frac{1}{2\pi R} \int_0^{\infty} dp^2 \frac{d(p^2)}{\Lambda^2} \sum_{|p_{d-1}|<\Lambda} \frac{(p^2 + p_{d-1}^2)^\alpha}{p^2 + p_{d-1}^2 - \Lambda^2}. \tag{B14}
\]

The sum of ambiguities arising from the momentum integrals is thus
\[
\pm i \int \frac{1}{\beta_0} \frac{\sqrt{4\pi}}{\Gamma((d-1)/2)} \frac{1}{2\pi R \Lambda} \sum_{|n|<R\Lambda} \left[ 1 - \frac{n^2}{(R\Lambda)^2} \right]^{(d-3)/2}. \tag{B15}
\]

In this sum, the $n = 0$ term is the IR renormalon ambiguity in the compactified theory, Eq. (B11). Other terms in the sum are not the renormalon in the compactified theory; their sum is what we call the renormalon precursor, the ambiguity of the momentum integral, which does not correspond to IR renormalon. The total ambiguity in the compactified theory is given by Eq. (B15) and in the decompactified limit $R\Lambda \to \infty$, it becomes
\[
R\Lambda \to \infty \quad \pm i \int \frac{1}{\beta_0} \frac{\sqrt{4\pi}}{\Gamma((d-1)/2)} \frac{\Lambda^{2\alpha+d}}{2\pi} \int_{-1}^{1} dx \left( 1 - x^2 \right)^{(d-3)/2} = \pm i \int \frac{1}{\beta_0} \frac{1}{\Gamma(d/2)} \Lambda^{2\alpha+d}, \tag{B16}
\]

which precisely coincides with the IR renormalon ambiguity in $\mathbb{R}^d$, Eq. (B17). In this way, by introducing the renormalon precursor, we have a smooth transition of the ambiguity under the decompactification.

Our next example is the $SU(N)$ gauge theory in $\mathbb{R}^4$ with massive fermions with a degenerate mass $m$; the mass of the fermions acts as the IR cutoff and is analogous to the inverse of the compactification radius, $1/R$, in the above examples. In the large $\beta_0$ approximation\(^{13}\) the propagator of the gauge field reads
\[
\langle A^a_\mu(x) A^b_\nu(y) \rangle = \frac{\lambda(\mu^2)}{N} \delta^{ab} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{(p^2)^2} \left\{ \left[ 1 - \Pi(p^2, m^2) \right]^{-1} \left[ p^2 \delta_{\mu\nu} - p_\mu p_\nu \right] + \frac{1}{\xi} p_\mu p_\nu \right\}, \tag{B17}
\]

where
\[
\Pi(p^2, m^2) = \frac{3\beta_0 \lambda(\mu^2)}{8\pi^2} \int_0^1 dx \frac{x(1-x)p^2 + m^2}{\mu^2}. \tag{B18}
\]

\(^{13}\) Here, we naively carry out the replacement (2.1) and do not properly consider the fact that the gluons are massless. In this regard, it is more appropriate to regard that the present analysis is done in Abelian gauge theories (with suitable modifications).
We then consider a gauge invariant quantity $\mathcal{F}$ which is given by
\begin{equation}
\mathcal{F} = \lambda(\mu^2) \int \frac{d^4 p}{(2\pi)^4} (p^2)^\alpha \frac{1}{1 - \Pi(p^2, m^2)}.
\end{equation}
See Eq. (23). Now, for finite $m^2$, $\Pi(p^2, m^2) \to \text{const.}$ as $p^2 \to 0$ and this does not possess the logarithmic factor. Therefore, the perturbative expansion of Eq. (B19) does not produce a factorially divergent series or the IR renormalon ambiguity.

Nevertheless, the momentum integral in Eq. (B19) can be ill-defined and ambiguous when $m^2$ is sufficiently small. To see this, we first note
\begin{equation}
\frac{1}{2} \frac{\partial}{\partial p^2} [1 - \Pi(p^2, m^2)] = \frac{3\beta_0 \lambda(\mu^2)}{8\pi^2} \int_0^1 dx \frac{[x(1-x)]^2}{x(1-x)p^2 + m^2} > 0,
\end{equation}
where we have assumed the asymptotic freedom $\beta_0 > 0$. Therefore, the function $1 - \Pi(p^2, m^2)$ is a monotonically increasing function of $p^2$. On the other hand, at the end points,
\begin{align}
1 - \Pi(p^2 = 0, m^2) &= 1 + \frac{3\beta_0 \lambda(\mu^2)}{16\pi^2} \ln \left( \frac{m^2}{\mu^2} \right) = \frac{\beta_0 \lambda(\mu^2)}{16\pi^2} \ln \left( \frac{m^2}{\Lambda^2} \right), \\
1 - \Pi(p^2 = \infty, m^2) &= +\infty,
\end{align}
where we have used the dynamical scale $\Lambda$ given by Eq. (B2) with $d = 4$.

Now, if $m^2 > \Lambda^2$, Eq. (B21) shows that $1 - \Pi(p^2, m^2)$ is positive definite and the function $[1 - \Pi(p^2, m^2)]^{-1}$ in the integrand of Eq. (B19) does not possess any singularity; Eq. (B19) is well-defined. On the other hand, if the mass is small enough as $0 < m^2 < \Lambda^2$, then $1 - \Pi(p^2, m^2)$ develops a simple zero in $p^2$ and the momentum integral (B19) becomes ill-defined and ambiguous; this is the renormalon precursor in the present example. Finally, in the massless limit $m^2 \to 0$, the integral (B19) reduces to Eq. (B1) with Eq. (B2) (with $d = 4$ and rescaling the renormalization scale of the coupling $p^2 \to e^{-5/3} p^2$) and the ambiguity of the renormalon precursor coincides with the IR renormalon ambiguity.

References
[1] G. ’t Hooft, Subnucl. Ser. 15, 943 (1979) doi:10.1007/978-1-4684-0991-8_17 PRINT-77-0723 (UTRECHT).
[2] M. Beneke, Phys. Rept. 317, 1-142 (1999) doi:10.1016/S0370-1573(98)00130-6 [arXiv:hep-ph/9807443 [hep-ph]].
[3] P. Argyres and M. Únsal, Phys. Rev. Lett. 109, 121601 (2012) doi:10.1103/PhysRevLett.109.121601 [arXiv:1204.1661 [hep-th]].
[4] P. C. Argyres and M. Únsal, JHEP 08, 063 (2012) doi:10.1007/JHEP08(2012)063 [arXiv:1206.1890 [hep-th]].
[5] G. V. Dunne and M. Únsal, JHEP 11, 170 (2012) doi:10.1007/JHEP11(2012)170 [arXiv:1210.2423 [hep-th]].
[6] G. V. Dunne and M. Únsal, Phys. Rev. D 87, 025015 (2013) doi:10.1103/PhysRevD.87.025015 [arXiv:1210.3646 [hep-th]].
[7] M. Únsal, Phys. Rev. D 80, 065001 (2009) doi:10.1103/PhysRevD.80.065001 [arXiv:0907.3260 [hep-th]].
[8] E. Bogomolny, Phys. Lett. B 91, 431-435 (1980) doi:10.1016/0370-2693(80)91014-X.
[9] J. Zinn-Justin, Nucl. Phys. B 192, 125-140 (1981) doi:10.1016/0550-3213(81)90197-8.
[10] G. V. Dunne and M. Únsal, PoS LATTICE2015, 010 (2016) doi:10.22323/1.251.0010 [arXiv:1511.05977 [hep-lat]].
[11] M. M. Anber and T. Sulejmanpasic, JHEP 01, 139 (2015) doi:10.1007/JHEP01(2015)139 [arXiv:1410.0121 [hep-th]].
[12] K. Ishikawa, O. Morikawa, A. Nakayama, K. Shibata, H. Suzuki and H. Takaoka, PTEP 2020, no.2, 023B10 (2020) doi:10.1093/ptep/ptaa002 [arXiv:1908.00373 [hep-th]].
[13] M. Ashie, O. Morikawa, H. Suzuki, H. Takaoka and K. Takeuchi, PTEP 2020, no.2, 023B01 (2020) doi:10.1093/ptep/ptz157 [arXiv:1909.05489 [hep-th]].
