Lattice simulations of Born-Infeld non-linear QED

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Born-Infeld non-linear electrodynamics was introduced to render the self energy of a point particle finite. It has recently been revived as a field theory for branes and strings. We quantize this theory on a Euclidean space-time lattice, using Metropolis Monte-Carlo simulations to measure the properties of the quantum field theory. Lüscher-Weisz methods are used to measure the electromagnetic fields from a static point charge. The D field from a point charge appears to be identical to that for the normal Maxwell Lagrangian. The E field is enhanced by quantum fluctuations, and shows short distance screening as it does in the classical theory.

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1. Introduction

The \( n + 1 \) dimensional Born-Infeld (non-linear electrodynamics) action \([1, 2]\) is:

\[
S = b^2 \int d^{n+1}x \left[ 1 - \sqrt{-\det \left( g_{\mu \nu} + \frac{1}{b} F_{\mu \nu} \right)} \right].
\] (1.1)

This has seen a revival as a theory of strings and branes \([3, 4, 5, 6, 7]\). Choosing \( n = 9 \), and dimensionally reducing this action from \( 9 + 1 \) to \( p + 1 \) dimensions describes a \( p \)-brane. The \( 9 - p \) additional components of \( A_\mu \) are identified with the transverse components of the string/brane.

Most of the serious work on these theories has dealt with their classical behaviour \([8]\). We are simulating these quantum theories on the lattice. We are starting with the simplest case where \( n = p = 3 \), the original Born-Infeld modification of electrodynamics, designed to make the self energy of a point charge finite.

Section 2 reviews the classical Born-Infeld theory. In section 3 we indicate how this is ported to the lattice allowing Monte-Carlo simulations of the quantum theory. Section 4 details our simulations and preliminary results. A discussion of our results and conclusions are given in section 5.

2. Classical Born-Infeld electrodynamics in Minkowski space-time

This section summarises those results in references \([9, 10, 11, 12]\) which are relevant for our investigations.

Evaluating the determinant in equation (1.1) the Lagrangian in \( 3 + 1 \) dimensions is

\[
\mathcal{L} = b^2 \left[ 1 - \sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(E \cdot B)^2} \right].
\] (2.1)

One can now define \( D \) and \( H \) by

\[
D = \frac{\partial \mathcal{L}}{\partial E} = \frac{E + b^{-2}(E \cdot B)B}{\sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(E \cdot B)^2}}.
\]

\[
H = \frac{\partial \mathcal{L}}{\partial B} = \frac{B - b^{-2}(E \cdot B)E}{\sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(E \cdot B)^2}}.
\] (2.2)

Interaction with charged particles is implemented, as usual, by adding a term \( j_\mu A^\mu \) to the Lagrangian. In terms of \( \mathbf{E}, \mathbf{B}, \mathbf{D} \) and \( \mathbf{H} \), the equations of motion are the standard Maxwell equations. The non-linearity is hidden in equations (2.2).

For a static point charge \( \rho = e \delta^3(\mathbf{r}) \) the electric fields are

\[
\mathbf{D} = \frac{e}{4\pi \mu^2} \hat{\mathbf{r}}
\]

\[
\mathbf{E} = \frac{e}{4\pi} \frac{\hat{\mathbf{r}}}{\sqrt{r^4 + r_0^4}},
\] (2.3)

where \( \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \) and \( r_0 = \sqrt{\frac{e}{4\pi b}} \). Hence the \( \mathbf{D} \) field for a static point charge is identical to the Maxwell solution, while the \( \mathbf{E} \) field is screened at short distances.
3. Lattice Born-Infeld quantum-electrodynamics

The Euclidean space action for Born-Infeld QED

\[ S = b^2 \int d^4x \left[ \sqrt{1 + b^{-2}(E^2 + B^2) + b^{-4}(E \cdot B)^2} - 1 \right] \]  \hspace{1cm} (3.1)

is positive. Hence it can be simulated using Monte-Carlo methods.

On the lattice we use the non-compact formulation:

\[ F_{\mu \nu}(x + \frac{1}{2} \hat{\mu} + \frac{1}{2} \hat{\nu}) = A_{\nu}(x + \hat{\mu}) - A_{\nu}(x) - A_{\mu}(x + \hat{\nu}) + A_{\mu}(x) \]  \hspace{1cm} (3.2)

and average over the 16 choices of 6 plaquettes associated with each lattice site. We also define \( \beta = b^2 a^4 \) where \( a \) is the lattice spacing. Simulations are performed using the Metropolis Monte-Carlo method [13].

We measure the \( E \) and \( D \) fields due to a static point charge. This point charge \( e \) is introduced by including a Wilson Line (Polyakov Loop) \( W(x) \).

\[ W(x) = \exp \left\{ ie \sum_t \left[ A_4(x, t) - \frac{1}{N_t N_y N_z} \sum_y A_4(y, t) \right] \right\} \]  \hspace{1cm} (3.3)

The second ('Jellium') term is needed, since a net charge would be inconsistent with periodic boundary conditions on \( A_\mu \). \( \langle E \rangle \) and \( \langle D \rangle \) in the presence of this charge are given by

\[ i \langle E \rangle_\rho(y - x) = \frac{\langle E(y, t) W(x) \rangle}{\langle W(x) \rangle} \]

\[ i \langle D \rangle_\rho(y - x) = \frac{\langle D(y, t) W(x) \rangle}{\langle W(x) \rangle}. \]  \hspace{1cm} (3.4)

Since \( W \) is complex, there is a sign problem, which causes \( \langle W(x) \rangle \) to fall exponentially with \( N_t \). We use the method of Lüscher and Weisz [14] (Parisi, Petronzio and Rapuano [15]) with thickness 1 and 2 timeslices to overcome this exponential factor.

4. Simulations and Results

We have performed preliminary simulations of 500,000 10-hit Metropolis sweeps of the lattice at \( \beta = 100, 1.0, 0.01, 0.0001 \) and 100,000 sweeps at \( \beta = 5, 2, 0.5, 0.2, 0.1 \), making measurements every 100 sweeps. We measured the \( E \) and \( D \) fields for on axis separations from the point charge. Figure 1 shows the expectation value of the Wilson lines (Polyakov loops) obtained from these simulations. Note that the value falls rapidly with increasing \( e \). The rate of falloff also increases with increasing non-linearity (decreasing \( \beta \)). Note also the small relative errors, even when the magnitude has fallen 8 orders of magnitude, which shows the effectiveness of the Lüscher-Weisz method.

Figure 2 shows the ratio of the \( E \) field in the direction of the separation from the charge to the \( D = E \) field for the free field (Maxwell) theory, at the minimum separation (\( Z = 0.5 \)), in the limit of zero charge. At large \( \beta \), where the Born-Infeld theory asymptotes to the Maxwell theory, this ratio
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Figure 1: Wilson Lines as functions of charge $e$, for a range of $\beta = a^2 b^2$.

Figure 2: $E/D$ ratio at minimum separation for $e \to 0$ as a function of $\beta$.

Figure 3: a) $D$ fields at distance $Z$ from the point charge $e$, scaled by the free field (Maxwell) values, for $\beta = 0.0001$. b) $E$ fields at distance $Z$ from the point charge $e$, scaled by the free field (Maxwell) values, for $\beta = 0.0001$.

approaches 1. Classically, this ratio is 1 for all values of $b$. Quantum fluctuations cause this ratio to increase with increasing non-linearity (decreasing $\beta$), approaching a value close to 3 for small $\beta$.

In figure 3a, we plot the $D$ fields scaled by their free field values at each separation, as a function of charge, for $\beta = 0.0001$, where the non-linearity is large, and the $D$ field comes almost entirely from the $(E \cdot B)B$ term in its definition. The fact that this ratio is still 1 for all $e$ at minimal separation is because $D$ still obeys $\nabla \cdot D = \rho$, combined with cubic symmetry. Because we do not have full rotational symmetry on the lattice $\nabla \cdot D = \rho$ is insufficient to make this ratio 1 at other
separations. The fact that this ratio is never more than 15% from 1 suggests that it would be 1 if we had rotational symmetry. However, we would expect rotational symmetry to be restored at large distances, which is why the ratio is closer to 1 for larger separations. Figure 3b is a similar graph for the $E$ field. As well as showing the effects of quantum fluctuations as in figure 3a, the $E$ field is clearly screened at short distances as $e$ increases, similar to what is seen in the classical theory.

What is different from the classical theory is that classically the screening length continues to increase with decreasing $b$. The quantum theory approaches a limit as $\beta \rightarrow 0$.

5. Discussion and conclusions

We have succeeded in using lattice Monte-Carlo methods to extract non-perturbative physics from Born-Infeld electrodynamics, quantized using the Euclidean-time functional integral approach. The on-axis (quantum) electrostatic fields of a point charge are measured as functions of the charge $e$ introduced as a Wilson Line. The approach of Lüscher and Weisz, which reduces these measurements from an exponential- to a polynomial-time problem, was essential for extracting these quantities.

In the classical field-theory $E/D \rightarrow 1$ as $e \rightarrow 0$. For the quantum theory $E/D$ increases from 1 as the nonlinearity is increased indicating that the dielectric constant $\varepsilon < 1$.

As $|e|$ is increased, the $E$ field is screened at short distances. Screening increases with $|e|$ and with increasing nonlinearity ($\beta$ or $b$). The screening length $r_0$ appears to increase as $\sqrt{|e|}$ as for the classical theory. $D$ shows no such screening and appears to independent of the non-linearity, while $D/e$ appears independent of $e$.

Unlike the classical theory, where the screening length diverges as $b \rightarrow 0$, the quantum theory approaches a fixed-point field theory as $\beta = b^2 a^4 \rightarrow 0$. This conformal field theory has Euclidean Lagrangian $\mathcal{L}_E = \frac{1}{4} |E \cdot B|$ and Hamiltonian $\mathcal{H} = |D \times B|$ [9, 16].

Normally, Born-Infeld QED is considered as an effective field theory with a momentum cutoff. However, as this cutoff $\rightarrow \infty$ it approaches the above fixed-point theory. If this fixed-point field theory is non-trivial, it would serve to define Born-Infeld QED without a cutoff.

These first simulations were performed on $8^4$ lattices. We are extending these simulations to larger lattices. We then plan to study those $p$-brane theories obtained by dimensional reduction of $n+1$ dimensional Born-Infeld theories to determine if the quantized theories continue to show string/brane dynamics.

Acknowledgements

Our simulations are currently running on the Rachael supercomputer at the Pittsburgh Supercomputer Center.

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