THE OPTIMIZED MODEL OF MULTIPLE INVASION PERCOLATION

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We study the optimized version of the multiple invasion percolation model. Some topological aspects as the behavior of the acceptance profile, coordination number and vertex type abundance were investigated and compared to those of the ordinary invasion. Our results indicate that the clusters show a very high degree of connectivity, spoiling the usual nodes-links-blobs geometrical picture.

Keywords: Invasion Percolation, Backbone, Fractal Dimension

1. Introduction

When a nonviscous liquid is injected into a porous medium already filled with a viscous fluid the system can be found in two different regimes: one where the dominant forces are of capillary nature and another where the viscous forces are predominant. The theoretical description of such system is based on two models: diffusion-limited aggregation (DLA) and invasion percolation. DLA describes the fast displacement situation when the viscous forces are dominant. The invasion percolation model is applied to a slow fluid flow, where the leading forces are of capillary nature. The fluid displacement follows minimum resistance paths: the smaller pores are filled or invaded first.

The original invasion percolation model have been modified in order to bring it as close as possible to real world. So, the action of an external gravitational field and the flux with a privileged direction were incorporated into the model. Recently the multiple invasion percolation model was proposed. In this model a certain number of lattice sites can be simultaneously invaded. It is always important to have in mind that invasion percolation is a dynamical model, that is, it intends to explain not only static properties like the clusters fractal dimensions but also how these clusters evolve in time. The lack of a meaning on associating or trying to make a correspondence between one step growth interval (of the algorithm) and one time interval separating two sucessives pore’s invasions (of the real world) is what we were concerned by the time we formulated the multiple invasion percolation model. In our models, one real time interval corresponds exactly to one growth
step and it is, in this sense, more realistic. When water penetrates layered soil, in some cases, a fingering formation is detected. The multiple invasion percolation not only describes successfully the fingers dynamics but also led us to a more ambitious program - to obtain the experimental pores size distribution of the soil. Briefly, the multiple invasion percolation model is useful not only to explain experimental results but also to predict them.

There are two kinds of multiple invasion percolation: the perimeter and optimized models. In the first model the cluster growth is controlled by the flux through the perimeter. The optimized model is governed by a scaling relation between the mass and the gyration radius of the cluster. Both types of invasion were studied in their site versions. For the perimeter model, the acceptance profile, mean coordination number and abundance of vertex type were investigated. In this paper we discuss the acceptance profile and some topological properties of the optimized model clusters.

2. The Optimized Model

We briefly recall the growth mechanism of the optimized model. It was devised to obey exactly the scaling

\[ M \sim (R_g)^D \]  

or as near it as possible \((R_g)\) is the gyration radius and \(D\) is a real positive external parameter that can be tuned).

Basically we use the following strategy: at each growing step we build a list containing all the cluster perimeter sites that can be invaded and we ask for the number of sites that should be invaded in order that equation (2.1) is verified as closely as possible. When \(D \in [1.89, 2]\) these proceedings build a fractal object which is extremely stabilized (in the sense that in any stage or size the scaling is perfectly obeyed and not only in the asymptotic limit). In this interval the parameter \(D\) coincides with the fractal dimension of the cluster. Reference stress that although the ordinary invasion percolation and the optimized model \((D = 1.89)\) have the same fractal dimension the clusters are not the same. While in the ordinary invasion percolation only one site is invaded at each step, in the optimized model several sites can be invaded at same time. The result is that the optimized model generates more massive clusters than those of the ordinary invasion percolation.

In the region \([0, 1.89]\) the system is frustrated from below, that is, the scaling relation recommends an invasion of less than one site which is forbidden by the algorithm (at least one site must be invaded). In the region \(D > 2\), the system is frustrated from above and a very beautiful burst phenomenon takes place. In both cases, \(D\) does not coincide with the real fractal dimension.

3. Numerical Simulations
The Optimized Model of Multiple Invasion Percolation

Figure 1: Some acceptance profiles of the optimized model for fixed $D$ and various lattice sizes. In the inset, we show the behaviour of $a(r)$ for $L = 201$ and different values of $D$.

In this paper we have studied the optimized model only in the region $[1.89, 2]$. To analyze growth mechanisms it is useful to define the acceptance profile $a(r)$ which is the ratio between the number of random numbers in the interval $[r, r+dr]$ accepted into the cluster and the total number of random numbers in that range. In the limit of an infinite lattice, the acceptance profile of the ordinary invasion percolation tends to a step function with the discontinuity located at the critical ordinary percolation threshold $p_c = 0.5928$. For the optimized model we determined the acceptance profile as a function of $D$. Fig. 1 shows the case $D = 1.89$ for many lattice sizes. It does not seem to approach a step function, exhibiting a persistent tail near the inflexion point. This happens because, in order to obey the scaling rule, sites with larger random numbers are invaded. As a consequence, we find that the inflexion point position is bigger than its value in the ordinary invasion. The inset of the Fig. 1 shows that the inflexion point position increases with $D$.

The sites of the multiple invasion percolation cluster can be classified according to their number of first nearest neighbors occupied sites. Let $N_k$ be the number of sites surrounded by $k$ ($k = 1, ..., 4$ for the square lattice) occupied sites. We determined the vertex concentrations of kind $k$, i.e., $n_k = \frac{N_k}{\sum_k N_k}$, for sizes $L = 51, 101, 151$ and 201. The result was extrapolated using the BST algorithm.
Table 1: The vertex concentrations $n_k = \frac{N_k}{\sum_k N_k}$ and the coordination number $Z$ of the optimized model for many values of $D$. The data were extrapolated using the BST algorithm.

| $D$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $Z$ |
|-----|-------|-------|-------|-------|-----|
| 1.89 | 0.10  | 0.30  | 0.40  | 0.20  | 2.71|
| 1.92 | 0.07  | 0.26  | 0.41  | 0.26  | 2.78|
| 1.95 | 0.04  | 0.17  | 0.41  | 0.38  | 3.14|
| 1.98 | 0.02  | 0.08  | 0.32  | 0.58  | 3.48|
| 2.00 | 0.01  | 0.01  | 0.10  | 0.88  | 3.88|

and it is presented in the Table 1. The BST is a useful algorithm to extrapolate physical quantities that converge obeying a power law $F(L) = F(\infty) + AL^{-\theta}$. It allows a reliable determination of critical parameters in the thermodynamic limit and its versatility becomes more pronounced if there are only very short sequences available. Our estimated coordination number for the optimized model ($D = 1.89$) is $Z = 2.71$ in contrast with $Z = 2.51$ for the ordinary invasion percolation.

In the table, we see that, as we increase $D$, the cluster becomes more compact favouring vertices of kind 4 in detriment of those of kind 1 which are practically extinct.

The topological properties of the percolating cluster have been investigated for more than two decades. It is well known that the backbone is an important structure of the cluster. It is formed by the union of all self-avoiding walks connecting two points $P_1$ and $P_2$ of the lattice. This means that if we pass a current between these points, the backbone is identified as the set of sites that carries current. The backbone study has possible applications on the conductivity of random systems and on fluids flowing in porous media. The backbone of the critical percolation cluster is non-Euclidean with fractal dimension $1.647 \pm 0.004$.

Many pictures of the percolating cluster backbone were proposed. The most promising model was introduced by Stanley and is known as the nodes-links-blobs model. In this model the backbone consists of a network of nodes connected by one-dimensional links which are often separated by multiconnected pieces or blobs of all length scales. Thus, the backbone may be viewed as a topologically linear string of blobs of all possible sizes. In general, there are sites that, when removed, split the spanning cluster into pieces. These sites are termed red sites. In the nodes-links-blobs model the red sites are also called blobs of size one. At criticality, the number of red sites scales as a power law with the exponent calculated exactly by Coniglio $\nu = 0.75$.

We have already studied the backbone and elastic backbone structures of the multiple invasion percolation model. To determine these structures we employed the burning algorithm. We chose this algorithm because, beyond the backbone and the elastic backbone, it also permits the determination of the red sites number,
the minimum path and the loops number.

For the optimized model, the scaling (equation (2.1)) is also perfectly obeyed by the backbone. At \( D = 1.89 \) we got the fractal dimension \( D_F = 1.74 \pm 0.01 \). This means that although the optimized model at \( D = 1.89 \) and the ordinary invasion have the same fractal dimension, they are intrinsically different since their backbones are not the same. The number of red sites \( N_r \) is very small and random. It does not seem to obey any power law. The red sites number is so small that the probability of disconnecting the cluster by removing any site randomly is practically zero. The optimized algorithm destroys the red sites, increasing the cluster connectivity. The disappearance of the red sites indicates that the nodes-links-blobs model may not be useful to describe the backbones of the optimized model clusters. Indeed, the cluster of the optimized model has a very different form from that of the ordinary invasion percolation as can be seen in the Fig. 2.

4. Conclusions

We studied the optimized model of multiple invasion percolation by comparing its topological properties with those of the ordinary invasion percolation. The high connectivity of the clusters produced by the optimized model together with the almost disappearance of the red sites, led us to conclude that the nodes-links-blobs model is not well suited to describe the backbones of the optimized model.

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