APPROXIMATE FUNCTIONAL EQUATION AND UPPER BOUNDS FOR THE BARNES DOUBLE ZETA-FUNCTION

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Abstract. As one of the asymptotic formulas of the zeta-function, Hardy and Littlewood gave asymptotic formulas called the approximate functional equation. In this paper, we prove an approximate functional equation of the Barnes double zeta-function \( \zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha + vm + wn)^{-s} \). Also, applying this approximate functional equation and the van der Corput method, we obtain upper bounds for \( \zeta_2(1/2 + it, \alpha; v, w) \) and \( \zeta_2(3/2 + it, \alpha; v, w) \) with respect to \( t \) as \( t \to \infty \).

1. Introduction and statement of results

Let \( s = \sigma + it \) be a complex variable, and let \( \alpha > 0 \) and \( v, w > 0 \) be real parameters.

In this section, we introduce the approximate functional equations of the Riemann zeta-function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

some results of upper bounds for \( \zeta(1/2 + it) \) with respect to \( t \to \infty \), Barnes multiple zeta-function, and give the main theorems on the approximate functional equation for the Barnes double zeta-function

\[
\zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s}. \tag{1.1}
\]

As a classical asymptotic formula of the Riemann zeta-function, the following formula was proved by Hardy and Littlewood;

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \quad (x \to \infty),
\]

uniformly for \( \sigma \geq \sigma_0 > 0 \), \( |t| < 2\pi x/C \), when \( C > 1 \) is a constant. This formula gives an indication in the discussion in the critical strip of \( \zeta(s) \). Also, Hardy and Littlewood proved the following asymptotic formula (§4 in [12]); suppose that \( 0 \leq \sigma \leq 1 \), \( x \geq 1 \), \( y \geq 1 \) and \( 2\pi xy = |t| \) then

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}), \tag{1.2}
\]

where \( \chi(s) = 2\Gamma(1-s) \sin(\pi s/2)(2\pi)^{s-1} \) and note that the functional equation \( \zeta(s) = \chi(s)\zeta(1-s) \) holds. This formula (1.2) is called the approximate functional equation.
The order of $|\zeta(\sigma + it)|$ with respect to $t$ is an extremely important problem in the deeper theory of the Riemann zeta function $\zeta(s)$. In particular, the order of $|\zeta(1/2 + it)|$ is the most important. For example, Hardy-Littlewood improve to

$$\zeta \left( \frac{1}{2} + it \right) \ll t^{1/6 + \varepsilon}$$

(1.3)

by the van der Corput method, applying to (1.2). In 1988, $\zeta(1/2 + it) \ll t^{9/56 + \varepsilon}$ was proved by Bombieri and Iwaniec, after which many mathematicians gradually improved, and now $\zeta(1/2 + it) \ll t^{13/84 + \varepsilon}$ has been proved by Huxley in 2005. Furthermore in 2017, $\zeta(1/2 + it) \ll t^{32/205 + \varepsilon}$ was proved by Bourgain (in see [4]).

Let $r$ be a positive integer and $w_j > 0$ ($j = 1, \ldots, r$) are complex parameters. The Barnes multiple zeta-function

$$\zeta_r(s, \alpha; w_1, \ldots, w_r) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{1}{(\alpha + w_1 m_1 + \cdots + w_r m_r)^s}$$

(1.4)

where the series on the right hand-side is absolutely convergence for $\text{Re}(s) > r$, and is continued meromorphically to $\mathbb{C}$ and its only singularities are the simple poles located at $s = j$ ($j = 1, \ldots, r$). This series is a multiple version of the Hurwitz zeta-function

$$\zeta_H(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s} \quad (0 < \alpha \leq 1).$$

(1.5)

Also, as a generalization of this series in another direction, the Lerch zeta-function

$$\zeta_L(s, \alpha, \lambda) = \sum_{n=0}^{\infty} \frac{e^{2\pi in\lambda}}{(n + \alpha)^s} \quad (0 < \alpha \leq 1, 0 < \lambda \leq 1)$$

(1.6)

is also an important research subject. These series are absolutely convergent for $\sigma > 1$. Also, if $0 < \lambda < 1$, then the series (1.6) is convergent even for $\sigma > 0$.

**Remark 1.** The Barnes double zeta-function was introduced by E. W. Barnes in the theory of double gamma function, and double series of the form (1.1) is introduced in [2]. Furthermore in [3], in connection with the theory of the multiple gamma function, and multiple series of the form (1.4) was introduced.

**Theorem 1** (Theorem 3 in [11]). Let $0 < \sigma_1 < \sigma_2 < 2$, $x \geq 1$ and $C > 1$. Suppose $s = \sigma + it \in \mathbb{C}$ with $\sigma_1 < \sigma < \sigma_2$ and $|t| \leq 2\pi x/C$. Then

$$\zeta_2(s, \alpha; v, w) = \sum_{0 \leq m, n \leq x} \frac{1}{(\alpha + vm + wn)^s} + \frac{(\alpha + vx)^{2-s} + (\alpha + wx)^{2-s} - (\alpha + vx + wx)^{2-s}}{vw(s-1)(s-2)} + O(x^{1-\sigma})$$

(1.7)

as $x \to \infty$.

We prove an analogue of the approximate functional equation (1.2) for (1.1) (in Theorem 2). In the following theorem, the results when the complex parameter $v, w$ linearly independent are different from the results when $v, w$ are linearly dependent over $\mathbb{Q}$. Also, we consider the upper bounds of $\zeta_2(1/2 + it, \alpha; v, w)$ and $\zeta_2(3/2 + it, \alpha; v, w)$, and the following Theorem 3 and Theorem 4 were obtaind. Proof of Theorem 3 can be obtained by using Theorem 2 and proof of Theorem 4 can be obtained by using Theorem 1.
Theorem 2. Suppose that $0 \leq \sigma \leq 2$, $x = x(t) \geq 1$, $y = y(t) \geq x(t)$ and $2\pi xy = |t|$. Let $L, M, N$ are non-negative integer as satisfying $N = \lfloor x/(v + w) \rfloor$ and $L = \lfloor vy \rfloor, M = \lfloor wy \rfloor$.

(i) If $v, w$ are linearly independent over $\mathbb{Q}$;

$$
\zeta_2(s, \alpha; v, w) = \sum_{0 \leq m, n \leq N} \frac{1}{(\alpha + vm + wn)^s} + \frac{1}{w^s} \sum_{m=0}^{N} \zeta_H^*(s, \alpha_{v,m}) + \frac{1}{v^s} \sum_{n=0}^{N} \zeta_H^*(s, \alpha_{w,n})
$$

$$
- \frac{\Gamma(1 - s)}{(2\pi i)^{1-s}} e^{\pi is} \left\{ \frac{1}{v^s} \sum_{0 < |n| < L} \frac{e^{-2\pi i(n+vN)/v}}{(e^{2\pi i nw/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{0 < |n| < M} \frac{e^{-2\pi i(n+wN)/w}}{(e^{2\pi i nw/w} - 1)n^{1-s}} \right\}
$$

$$
+ O(x^{-\sigma})
$$

(ii) If $v, w$ are linearly dependent over $\mathbb{Q}$, exist $p, q \in \mathbb{N}$ such as $pv = qw$ and $(p, q) = 1$. Then we have

$$
\zeta_2(s, \alpha; v, w) = \sum_{0 \leq m, n \leq N} \frac{1}{(\alpha + vm + wn)^s} + \frac{1}{w^s} \sum_{m=0}^{N} \zeta_H^*(s, \alpha_{v,m}) + \frac{1}{v^s} \sum_{n=0}^{N} \zeta_H^*(s, \alpha_{w,n})
$$

$$
- \frac{\Gamma(1 - s)}{(2\pi i)^{1-s}} e^{\pi is} \left\{ \frac{1}{v^s} \sum_{0 < |n| < L} \frac{e^{-2\pi i(n+vN)/v}}{(e^{2\pi i nw/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{0 < |n| < M} \frac{e^{-2\pi i(n+wN)/w}}{(e^{2\pi i nw/w} - 1)n^{1-s}} \right\}
$$

$$
+ \frac{q^{s-1}}{2\pi ipv^s} (1 - s) \sum_{0 < |n| < M} \frac{e^{-2q\pi in\alpha/v}}{n^{2-s}}
$$

$$
- \left( \frac{\alpha q}{pv^2} + \frac{p + q}{pv} + \frac{vp}{2q} + \frac{v}{2} \right) \left( \frac{q}{v} \right)^{s-1} \sum_{0 < |n| < M} \frac{e^{-2q\pi in\alpha/v}}{n^{1-s}} \right\}
$$

$$
+ O(x^{-\sigma}),
$$

where

$$
\zeta_H^*(s, \alpha_{v,m}) := \zeta_H(s, \alpha_{v,m}) - \sum_{n=0}^{N+n_{v,m}} \frac{1}{(n + \alpha_{v,m})^s},
$$

$$
\alpha_{v,m} := \begin{cases} \left\lfloor \frac{vm + \alpha}{w} \right\rfloor & \left( \frac{vm + \alpha}{w} \notin \mathbb{N} \right), \\ 1 & \left( \frac{vm + \alpha}{w} \in \mathbb{N} \right), \\ \end{cases}
$$

$$
n_{v,m} := \begin{cases} \left\lfloor \frac{vm + \alpha}{w} \right\rfloor - 1 & \left( \frac{vm + \alpha}{w} \geq 1 \right), \\ 0 & \left( 0 < \frac{vm + \alpha}{w} < 1 \right). \\ \end{cases}
$$

The definitions of $\zeta_H^*(s, \alpha_{w,n})$ and $\alpha_{w,n}$ are similar.
Theorem 3. If $v, w$ are linearly independent over $\mathbb{Q}$, then we have
\[
\zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right) \ll \begin{cases} |t|^{1/6} & (\alpha, v, w \text{ are lin. indep. over } \mathbb{Q}), \\ |t|^{1/6} \log |t| & (\alpha, v, w \text{ are lin. dep. over } \mathbb{Q}). \end{cases}
\]
If $v, w$ are linearly dependent over $\mathbb{Q}$, exist $p, q \in \mathbb{N}$ such as $pv = qw$ and $(p, q) = 1$. Then we have
\[
\zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right) = \kappa t + O(|t|^{1/6} \log |t|)
\]
where $\kappa = \kappa(t)$ is a constant with
\[
0 < |\kappa(t)| < \frac{1}{2\pi p\sqrt{qv}} \left| \zeta_L \left( \frac{1}{2} + it, 1, 1 - \frac{q}{v} \alpha \right) \right|.
\]

Theorem 4.
\[
\zeta_2 \left( \frac{3}{2} + it, \alpha; v, w \right) \ll |t|^{1/3}.
\]

2. Proof of Theorem 2

In this section, we give the proof of Theorem 2.

Proof of Theorem 2
Let $N \in \mathbb{N}$ be sufficiently large. Then we consider
\[
\zeta_2(s, \alpha; v, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} = \left( \sum_{m=0}^{N} \sum_{n=0}^{\infty} + \sum_{m=N+1}^{\infty} \sum_{n=0}^{\infty} + \sum_{m=N+1}^{\infty} \sum_{n=N+1}^{\infty} \right) \frac{1}{(\alpha + vm + wn)^s}.
\]
Also, the second term of the above is
\[
\sum_{m=0}^{N} \sum_{n=N+1}^{\infty} \frac{1}{(\alpha + vm + wn)^s} = \frac{1}{w^s} \sum_{m=0}^{N} \left\{ \zeta_H(s, \alpha_{v,m}) - \sum_{n=-n_v,\alpha}^{N} \frac{1}{n + (\alpha + vm)/w} \right\} = \frac{1}{w^s} \sum_{m=0}^{N} \zeta^*_H(s, \alpha_{v,m}).
\]
Similarly, the third term of the above, we have
\[
\sum_{m=N+1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(\alpha + vm + wn)^s} = \frac{1}{w^s} \sum_{n=0}^{N} \zeta^*_H(s, \alpha_{w,n}).
\]
Transform the fourth term on the right hand-side of the above equation to the contour integral to obtain

$$\sum_{m=N+1}^{\infty} \sum_{n=N+1}^{\infty} \frac{1}{(\alpha + \nu m + \nu n)^s} = \frac{\Gamma(1-s)}{2\pi i e^{\pi i s}} \int_{C} z^{s-1} e^{-(\alpha + \nu N + \nu N)z} \frac{dz}{(e^{\nu z} - 1)(e^{\nu z} - 1)}$$

(2.1)

where $C$ is the contour integral path that comes from $+\infty$ to $\varepsilon$ along the real axis, then continues along the circle of radius $\varepsilon$ counter clockwise, and finally goes from $\varepsilon$ to $+\infty$.

Let $\sigma \leq 2, t > 0$ and $1 \leq x < y$, so that $1 \leq x \leq \sqrt{t/2\pi}$. Let $L, M, N$ be non-negative integer satisfying

$$N = \left\lfloor \frac{x}{\nu + \nu} \right\rfloor, \quad L = [\nu y], \quad M = [\nu y]$$

and let $\eta = 2\pi y$. We deform the contour integral path $C$ to the straight lines $C_1, C_2, C_3$ and $C_4$ joining $\infty, c\eta + i\eta(1+c), -c\eta + i(1-c)\eta, -c\eta - (2L+1)\pi i, \infty$ where $c$ is an absolute constant $0 < c \leq 1/2$.

Next we consider the residue of the integrand of (2.1)

$$F(z) = z^{s-1} e^{-(\alpha + \nu N + \nu N)z} \frac{dz}{(e^{\nu z} - 1)(e^{\nu z} - 1)}$$

(i) In the case when $\nu, w$ are linear independent over $\mathbb{Q}$, $F(z)$ has simple poles at

$$z = \frac{2\pi i n}{\nu}, \frac{2\pi i n}{\nu} (n = \pm 1, \pm 2, \cdots).$$
Since
\[
\lim_{z \to 2\pi in/v} \left( z - \frac{2\pi in}{v} \right) F(z) = \lim_{z \to 2\pi in/v} \left( z - \frac{2\pi in}{v} \right) \frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz} - 1)(e^{wz} - 1)}
\]
\[
= \lim_{z \to 2\pi in/v} \left( z - \frac{2\pi in}{v} \right) \frac{e^{vz} - e^{2\pi in}}{z - 2\pi in/v}^{-1} z^{s-1}e^{-(\alpha+vN+wN)z} e^{wz} - 1
\]
\[
= \frac{1}{v} \left( \frac{2\pi in}{v} \right)^{s-1} e^{-(\alpha+wN)2\pi in/v} e^{2\pi inw/v - 1},
\]
then we have
\[
\text{Res}_{z=2\pi in/v} F(z) = \left\{ \begin{array}{ll}
\frac{1}{v} (2\pi in)^{s-1} e^{-(\alpha+wN)2\pi in/v} e^{2\pi inw/v - 1} & (n > 0) \\
\frac{1}{v} (2\pi in)^{s-1} e^{2\pi inw/v} (2\pi n)^{s-1} e^{3\pi i(s-1)/2} & (n < 0)
\end{array} \right.
\]
and we obtain
\[
\zeta_2(s, \alpha; v, w) = \sum_{0 \leq m, n \leq N} \frac{1}{(\alpha + vm + wn)^s} + \frac{1}{v^s} \sum_{m=0}^{N} \zeta_H^*(s, \alpha, v, m) + \frac{1}{v^s} \sum_{n=0}^{N} \zeta_H^*(s, \alpha, w, n)
\]
\[-\frac{\Gamma(1-s)}{(2\pi i)^{1-s} \epsilon^{2\pi is}} \left\{ \frac{1}{v^s} \sum_{0 < |n| \leq L} \frac{e^{2\pi in(\alpha+wN)/v}}{(e^{2\pi inw/v} - 1)n^{1-s}} + \frac{1}{w^s} \sum_{0 < |n| \leq M} \frac{e^{-2\pi in(\alpha+vN)/w}}{(e^{2\pi inw/v} - 1)n^{1-s}} \right\}
\]
\[+ \frac{1}{\Gamma(s)(\epsilon^{2\pi is} - 1)} \left( \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz} - 1)(e^{wz} - 1)} \, dz. \quad (2.2)
\]
From here, we consider the order of the integral term on the right-hand side of (2.2). First, we consider it on the integral path $C_4$. Let $z = u + iu' = re^{i\theta}$ then $|z^{s-1}| = r^{s-1}e^{-i\theta}$. Since $\theta \geq (5/4)\pi, r \gg u + cn, |e^{vz} - 1| \gg 1$ and $|e^{wz} - 1| \gg 1$ we have
\[
\int_{C_4} F(z) \, dz = \int_{C_4} \frac{z^{s-1}e^{-(\alpha+vN+wN)z}}{(e^{vz} - 1)(e^{wz} - 1)} \, dz
\]
\[
\ll e^{-(5/4)\pi t} \int_{-cn}^{\infty} (u + cn)^{\sigma-1}e^{-(\alpha+vN+wN)u} \, du
\]
\[
\ll e^{(\alpha+vN+wN)cn-(5/4)\pi t} \int_{0}^{\infty} u^{\sigma-1}e^{-(\alpha+vN+wN)u} \, du
\]
\[
\ll e^{(\alpha+vN+wN)cn-(5/4)\pi t} (\alpha + vN + wN)^{-\sigma} \Gamma(\sigma)
\]
\[
\ll r^{-\sigma} e^{c\sigma y + (c-(5/4))\pi t} \quad (2.3)
\]
Secondly, we consider the order of the integral on $C_3$ of (2.2). Noticing
\[
\arctan \varphi = \int_{0}^{\varphi} \frac{d\mu}{1 + \mu^2} > \int_{0}^{\varphi} \frac{d\mu}{(1 + \mu)^2} = \frac{\varphi}{1 + \varphi},
\]
at $\varphi > 0$, we have
\[
\theta = \arg z = \frac{\pi}{2} + \arctan \frac{c}{1 - c} = \frac{\pi}{2} + c
\]
on $C_3$. Then we have

$$|z^{s-1}e^{-(\alpha+\nu N+wN)z}| \ll \eta^{\sigma-1}e^{-(\pi/2+c)t}e^{(\alpha+\nu N+wN)\eta} \ll \eta^{\sigma-1}e^{-\pi t/2}.$$  

Also, since $|e^{ue^z} - 1| \gg 1$, $|e^{u\eta} - 1| \gg 1$ we have

$$\int_{C_3} F(z)dz = \int_{C_3} \frac{z^{s-1}e^{-(\alpha+\nu N+wN)z}}{(e^{u\eta} - 1)(e^{u^\eta} - 1)}dz \ll \int_{-(2L+1)}^{(1-c)\eta} \eta^{\sigma-1}e^{-\pi t/2}dt \ll \eta^{\sigma}e^{-\pi t/2} \quad (2.4)$$

Thirdly, since $|e^{ue^z} - 1| \gg e^{\nu u}$ and $|e^{u\eta} - 1| \gg e^{u\eta}$ on $C_1$, we have

$$\frac{z^{s-1}e^{-(\alpha+\nu N+wN)z}}{(e^{u\eta} - 1)(e^{u^\eta} - 1)} \ll \eta^{\sigma-1} \exp \left(-t \arctan \left(\frac{1+c\eta}{u} - (\alpha + (N+1)(v + w))u\right)\right).$$

Since $N+1 \geq \frac{\eta}{\nu + w} = t/(v+w)\eta$ are included in the fractional part of the right hand-side of the above $-(\alpha + (N+1)(v + w))u$ may be replaced with $tu/\eta$. Also, since

$$\frac{d}{du} \left( \arctan \frac{1+c\eta}{u} + \frac{u}{\eta} \right) = -\frac{(1+c)\eta}{u^2 + (1+c)^2\eta^2} + \frac{1}{\eta} > 0$$

and

$$\arctan \phi = \int_{\phi}^{\nu} \frac{d\mu}{1 + \mu^2} < \int_{\phi}^{\nu} d\mu = \phi,$$

we have

$$\arctan \frac{1+c\eta}{u} + \frac{u}{\eta} \geq \arctan \frac{1+c}{c} + c = \frac{\pi}{2} - \arctan \frac{c}{1+c} + c$$

$$> \frac{\pi}{2} + A(c)$$

in $c\eta \leq u \leq \pi\eta$, and let $A(c) = c^2/(1+c)^2$. Then we have

$$\frac{z^{s-1}e^{-(\alpha+\nu N+wN)z}}{(e^{u\eta} - 1)(e^{u^\eta} - 1)} \ll \eta^{\sigma-1} \exp \left(\frac{\pi}{2} + A(c)\right) \exp \left(\frac{\pi}{2} + A(c)\right).$$

Also, since

$$\frac{z^{s-1}e^{-(\alpha+\nu N+wN)z}}{(e^{u\eta} - 1)(e^{u^\eta} - 1)} \ll \eta^{\sigma-1} \exp \left(-\alpha + \nu x + wx\right)u$$

in $u \geq \pi\eta$, then we obtain

$$\int_{C_3} \frac{z^{s-1}e^{-(\alpha+\nu N+wN)z}}{(e^{u\eta} - 1)(e^{u^\eta} - 1)}dz$$

$$\ll \eta^{\sigma-1} \left\{ \int_{c\eta}^{\pi\eta} e^{-(\pi/2+A(c))t}dt + \int_{\pi\eta}^{\infty} e^{-(\alpha+\nu x+wx)u}du \right\}$$

$$\ll \eta^{\sigma}e^{-(\pi/2+A(c))t} + \eta^{\sigma-1}e^{-(\alpha+\nu x+wx)\pi\eta}$$

$$\ll \eta^{\sigma}e^{-(\pi/2+A(c))t} \quad (2.5)$$
Finally, we describe the integral evaluation on $C_2$. Since, it can be rewritten that $z = i\eta + \xi e^{\pi i/4}$ (where $\xi \in \mathbb{R}$ and $|\xi| \leq \sqrt{2}c\eta$), we have
\[ z^{s-1} = \exp \left\{ (s-1) \left( \frac{\pi i}{2} + \log (\eta + \xi e^{-\pi i/4}) \right) \right\} \]
\[ = \exp \left\{ (s-1) \left( \frac{\pi i}{2} + \log \eta + \log \left( 1 + \frac{\xi}{\eta} e^{-\pi i/4} \right) \right) \right\} \]
\[ = \exp \left\{ (s-1) \left( \frac{\pi i}{2} + \log \eta + \frac{\xi}{\eta} e^{-\pi i/4} - \frac{\xi^2}{2\eta^2} e^{-\pi i/2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) \right\} \]
\[ \ll \eta^{s-1} \exp \left\{ \left( -\frac{\pi}{2} + \frac{\xi}{\sqrt{2}\eta} - \frac{\xi^2}{2\eta^2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) t \right\} \quad (\xi \to \infty). \]
as $\eta \to \infty$. Also, since
\[ \frac{e^{-(\alpha+\nu N+wN)z}}{(e^{\nu z} - 1)(e^{wz} - 1)} = \frac{e^{-(\alpha+\nu N+wN)z+(\alpha+\nu x+w x)z}}{(e^{\nu z} - 1)(e^{wz} - 1)} \cdot e^{-(\alpha+\nu x+w x)z} \]
and
\[ \frac{e^{(\nu+w)(x-N)z}}{(e^{\nu z} - 1)(e^{wz} - 1)} \ll \begin{cases} e^{(\nu+w)(x-N-1)u} & (u > \frac{\pi}{2}) \\ e^{(\nu+w)(x-N)u} & (u < -\frac{\pi}{2}) \end{cases}, \]
we have
\[ \frac{e^{-(\alpha+\nu N+wN)z}}{(e^{\nu z} - 1)(e^{wz} - 1)} \ll \left| e^{-(\alpha+\nu x+w x)\eta \sqrt{2}} \right| \quad \left( |u| > \frac{\pi}{2} \right). \]
Hence
\[ \int_{C_2 \cap \{|z|>\pi/2\}} \frac{z^{s-1} e^{-(\alpha+\nu N+wN)z}}{(e^{\nu z} - 1)(e^{wz} - 1)} dz \]
\[ \ll \eta^{s-1} e^{-\pi t/2} \int_{-\pi/2}^{\pi/2} \left| \exp \left\{ \left( \frac{\xi}{\sqrt{2}\eta} (1 - v - w) - \frac{\xi^2}{2\eta^2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) t \right\} \right| d\xi \]
\[ \ll \eta^{s-1} e^{-\pi t/2} \int_{-\infty}^{\infty} \exp \left\{ \frac{-B(c)\xi^2 t}{\eta^2} \right\} d\xi \]
\[ \ll \eta^{s-1} e^{-\pi t/2}, \quad (2.6) \]
where $B(c)$ is a constant depending on $c$. The argument can also be applied to the part $|u| \leq \pi/2$ if $|e^{z-2\pi i\lambda}| > A$. If not, that is the case when the contour goes too near to the pole at $z = 2L\pi i/v$ (or $2M\pi i/w$), we take an arc of the circle $|z-2L\pi i/v| = \varepsilon$ (or $|z-2M\pi i/w| = \varepsilon$). On this arc we can write
\[ z = \frac{2L\pi i}{v} + \varepsilon e^{i\beta} \quad \text{or} \quad z = \frac{2M\pi i}{w} + \varepsilon e^{i\beta}, \]
where $\varepsilon$ is a positive real number less than the distance between any two poles, that is,
\[ 0 < \varepsilon < \min_{k,l} \left\{ \left| \frac{2k\pi i}{v} - \frac{2l\pi i}{w} \right| \left| \frac{2k\pi}{v} - \frac{2l\pi}{w} \right| < \eta \right\}. \]
Then,

$$\log (z^{s-1}) = (s - 1) \log \left( \frac{2L\pi i}{v} + \varepsilon e^{i\beta} \right)$$

$$= (s - 1) \log e^{\pi i/2} \left( \frac{2L\pi}{v} + \frac{\varepsilon e^{i\beta}}{t} \right)$$

$$= (\sigma + it - 1) \left\{ \frac{\pi i}{2} + \log \frac{2L\pi}{v} + \log \left( 1 + \frac{v\varepsilon e^{i\beta}}{2L\pi t} \right) \right\}$$

$$= -\frac{\pi t}{2} + (s - 1) \log \frac{2L\pi}{v} + \frac{v\varepsilon e^{i\beta}}{2L\pi} t + O(1).$$

On the last line of the above calculations, we used $N^2 \gg t$ which follows from the assumption $x \leq y$. Then

$$z^{s-1} e^{-(\alpha + vN + wN)z}$$

$$= \exp \left( -\frac{\pi t}{2} + (s - 1) \log \frac{2L\pi}{v} + \frac{v\varepsilon e^{i\beta}}{2L\pi} t + O(1) \right)$$

$$\times \exp \left( -(\alpha + vN + wN) \left( \frac{2L\pi i}{v} + \varepsilon e^{i\beta} \right) \right)$$

$$= \exp \left( -\frac{\pi t}{2} + (s - 1) \log \frac{2L\pi}{v} + \left( \frac{vt}{2\pi L} - (\alpha + vN + wN) \varepsilon e^{i\beta} + O(1) \right) \right),$$

and since

$$\left( \frac{vt}{2\pi L} - (\alpha + vN + wN) \right) \varepsilon e^{i\beta}$$

$$= \frac{vt - (\alpha + vN + wN)2\pi L}{2\pi L} \varepsilon e^{i\beta}$$

$$= \frac{2\pi xyv - 2\alpha\pi L - (v + w)[x/(v + w)]2\pi L}{2\pi L} \varepsilon e^{i\beta}$$

$$= -\alpha \varepsilon e^{i\beta} + \frac{2\pi xyv - (v + w)[x/(v + w)]2\pi L}{2\pi L} \varepsilon e^{i\beta}$$

$$\times \left\{ x - (v + w) \left[ \frac{x}{v + w} - \alpha \right] \varepsilon e^{i\beta} = O(1) \right\},$$

we have

$$z^{s-1} e^{-(\alpha + vN + wN)z} = \exp \left( -\frac{\pi t}{2} + (s - 1) \log \frac{2L\pi}{v} + O(1) \right)$$

$$\ll \left( \frac{L}{v} \right)^{\sigma - 1} e^{-\pi t/2} = O(\eta^{\sigma - 1} e^{-\pi t/2}).$$

In the case when the path $C_2$ is running around the another poles at $z = 2k\pi i/w + \varepsilon e^{i\beta}$, by using the similar above method to obtain

$$z^{s-1} e^{-(\alpha + vN + wN)z} = O(\eta^{\sigma - 1} e^{-\pi t/2}).$$

Therefore together with (2.6), we have

$$\int_{C_2} \frac{z^{s-1} e^{-(\alpha + vN + wN)z}}{(e^{vz} - 1)(e^{wz} - 1)} dz \ll \eta^{\sigma - 1/2} e^{-\pi t/2} + \eta^{\sigma - 1} e^{-\pi t/2} \quad (2.7)$$
Since, the evaluation of all integrals was obtained, using the evaluation formulas (2.3), (2.4), (2.5), (2.7) and \( \Gamma(1-s) \ll t^{1/2-\sigma}e^{\pi t/2} \), we find that the evaluation of the integral term of (2.2) is

\[
\ll t^{1/2-\sigma}e^{\pi t/2} \{ \eta^\sigma e^{-(\pi/2+A(c))t} + \eta^{-1} t^{-1/2} e^{-\pi t/2} + \eta^{-1} e^{-\pi t/2} + \eta^\sigma t^{-1/2} e^{-(5\pi/4)t} \}
\]

\[
\ll t^{1/2} \left( \frac{\eta}{t} \right)^\sigma e^{-A(c)t} + t^{-1/2} \left( \frac{\eta}{t} \right)^{-1} + t^{1/2-\sigma} e^{-(5\pi/4)t}
\]

\[
\ll e^{-\delta t} + x^{-\sigma} + t^{-1/2} x^{1-\sigma} \ll x^{-\sigma},
\]

where \( \delta \) is a small positive real number. Therefore we obtain (1.8).

(ii) In the case when \( v, w \) are linear dependent over \( \mathbb{Q} \), that is, there exist \( p, q \in \mathbb{N} \) such that \( pv = qw \) and \( (p, q) = 1 \). \( F(z) \) has simple poles at

\[
z = \frac{2\pi in}{v} \text{ (} n \in \mathbb{Z} \setminus \{0\}, \ q \nmid n \), \quad \frac{2\pi in}{w} \text{ (} n \in \mathbb{Z} \setminus \{0\}, \ p \nmid n \).
\]

On the other hand here for \( w = pv/q \),

\[
\lim_{z \to 2q\pi in/v} \frac{d}{dz} \left[ \left( z - \frac{2q\pi in}{v} \right)^2 z^{s-1} e^{-(\alpha+vN+pvN/q)z} \right] = \lim_{z_0 \to 0} \frac{d}{dz_0} \left[ \left( z_0 + \frac{2q\pi in}{v} \right)^{s-1} e^{-(\alpha+vN+pvN/q)z_0} e^{-2q\pi in\alpha/v} \right]
\]

\[
\lim_{z_0 \to 0} \frac{d}{dz_0} \left[ \left( z_0 + \frac{2q\pi in}{v} \right)^{s-1} \times \frac{e^{-(\alpha+vN+pvN/q)z_0} e^{-2q\pi in\alpha/v}}{(vz_0 + \frac{p^2}{q^2} z_0^2 + O(z_0^3)) \left( \frac{pv}{q} \right)^2 z_0^2 + O(z_0^3)} \right)
\]

\[
= \frac{q}{pv^2} (s-1) \left( \frac{2q\pi in}{v} \right)^{s-2} e^{-2q\pi in\alpha/v}
\]

\[
- \left( \frac{\alpha q}{pv} + \frac{(p+q)N}{pv} + \frac{vp^2}{2q} + \frac{v}{2} \right) \left( \frac{2q\pi in}{v} \right)^{s-1} e^{-2q\pi in\alpha/v}
\]

Therefore \( F(z) \) has double poles at

\[
z = \frac{2\pi iqn}{v} = \frac{2\pi iqn}{w} \text{ (} n \in \mathbb{Z} \setminus \{0\} \).
\]

Then, we calculate the following residue sum;

\[
\sum_{0<|n|\leq M} \text{Res} F(z) = \sum_{0<|n|\leq L} \text{Res}_{z=2\pi in/v} F(z) + \sum_{0<|n|\leq K} \text{Res}_{z=2\pi in/w} F(z)
\]

\[
+ \sum_{0<|k|\leq K} \text{Res}_{z=2\pi iqn} F(z)
\]
therefore, we have
\[
\text{Res}_{z=2\pi in/v} F(z) = \frac{1}{v} \left( \frac{2\pi in}{v} \right)^{s-1} e^{-(\alpha+wN)2\pi in/v} e^{2\pi inw/v - 1} = \begin{cases} 
  e^{-2\pi in\alpha} (2\pi n)^{s-1} e^{\pi(s-1)/2} & (n > 0) \\
  e^{2\pi in\alpha} (-2\pi n)^{s-1} e^{3\pi(s-1)/2} & (n < 0)
\end{cases}
\]
and we obtain
\[
\zeta_2(s, \alpha; v, w) = \sum_{0 \leq m, n \leq N} \frac{1}{(\alpha + vm + wn)^s} + \frac{1}{v^s} \sum_{m=0}^{N} \zeta^*_H(s, \alpha_v,m) + \frac{1}{w^s} \sum_{n=0}^{N} \zeta^*_H(s, \alpha_w,n)
\]

\[
- \frac{\Gamma(1-s)}{(2\pi)^{1-s} e^{\pi is}} \left\{ \frac{1}{v^s} \sum_{0 < |n| < L, q \mid n} \frac{e^{-2\pi in(\alpha+wN)/v}}{(e^{2\pi inw/v - 1}n^{1-s})} + \frac{1}{w^s} \sum_{0 < |n| < M, p \mid n} \frac{e^{-2\pi in(\alpha+vN)/w}}{(e^{2\pi invw - 1}n^{1-s})} + \frac{q^{s-1}}{2\pi ipu^s} (1-s) \sum_{0 < |n| < M} \frac{e^{-2q\pi in\alpha/v}}{n^{2-s}} \right\}
\]

\[
- \left( \frac{\alpha q}{pv^2} + \frac{(p+q)N}{pv} + \frac{vp + \nu}{2q} + \frac{\nu}{2} \right) \left( \frac{q}{v} \right)^{s-1} \sum_{0 < |n| < M} \frac{e^{-2q\pi in\alpha/v}}{n^{1-s}}
\]

\[
+ \frac{1}{\Gamma(s)(e^{2\pi is} - 1)} \left( \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) e^{-(\alpha+wN+wN)z} e^{-1} \left( e^{wz} - 1 \right) dz.
\]

Furthermore, four integrals in the last term of the above are evaluated to lead the same result by the similar method as in (i)

Hence proof of Theorem 2 is complete. \(\square\)

3. Some Lemmas

In this section, we will first introduce two basic lemmas (Lemma 5 and Lemma 6) on exponential sum in the van der Corput method. These lemmas are called the second order differential test and the third order differential test, respectively. Also, the Hurwitz zeta-function \(\zeta_H(s, \alpha)\) has similar results to the Hardy-Littlewood’s results (1.2) and (1.3), which are described in Lemma 7 and Lemma 8. There are approximate functional equation for the Lerch zeta-function \(\zeta_L(s, \alpha, \lambda)\) proved by R. Garunkštis, A. Laurinčikas, and J. Steuding (see [5], [7], [8]), Lemma 7 is an analogue of its special case (see [10]).

**Lemma 5** (Theorem 5.9 in [12]). Let \(a, b\) are real number with \(b \geq a + 1\) and \(c > 1\) is a constant. Suppose that \(f(x)\) be a real two times differentiable function which satisfies

\[
0 < \Lambda \leq f''(x) \leq c\Lambda \quad \text{or} \quad 0 < \Lambda \leq -f''(x) \leq c\Lambda
\]
Lemma 6 (Theorem 5.11 in [12]). Let \(a, b\) be real number with \(b \geq a + 1\), and \(c > 1\) is a constant. Suppose that \(f(x)\) be a real three times differentiable function which satisfies

\[
0 < \Lambda \leq f'''(x) \leq c\Lambda \quad \text{or} \quad 0 < \Lambda \leq -f'''(x) \leq c\Lambda
\]
in \([a, b]\). Then,

\[
\sum_{a < n \leq b} e^{2\pi i f(n)} \ll c(b - a)\Lambda^{1/2} + \Lambda^{-1/2}.
\]

Lemma 7 (Theorem 1(ii) in [10]). Let \(0 < \alpha \leq 1\). Suppose that \(0 \leq \sigma \leq 1\), \(x \geq 1\), \(y \geq 1\) and \(2\pi xy = |t|\). Then

\[
\zeta_H(s, \alpha) = \sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^s} + \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ e^{\pi i (1-\alpha)} \sum_{1 \leq n \leq y} \frac{e^{2\pi i n (1-\alpha)}}{n^{1-s}} + e^{-\pi i \frac{1}{2} (1-\alpha)} \sum_{1 \leq n \leq y} \frac{e^{2\pi i n \alpha}}{n^{1-s}} \right\}
\]

\[
+ O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}). \tag{3.1}
\]

Lemma 8. Let \(x > 0, t > 0\) and \(0 < \alpha \leq 1\). Suppose that \(x \ll t\), then

\[
\sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^{1/2 + it}} \ll t^{1/6} \log x \quad (t \to \infty).
\]

Proof. First, we consider the following single series and by using Lemma 6 with \(f(x) = -t(2\pi)^{-1} \log (x + \alpha)\). Suppose that \(a + 1 \leq b \leq 2a\), then

\[
\sum_{a < n \leq b} (n + \alpha)^{-it} = \sum_{a < n \leq b} e^{2\pi i f(n)} \ll a \left\{ \frac{t}{(a + \alpha)^{3}} \right\}^{1/6} + a^{1/2} \left\{ \frac{t}{(a + \alpha)^{3}} \right\}^{-1/6}
\]

\[
\ll a^{1/2} t^{1/6} + at^{-1/6}. \tag{3.2}
\]

Also by using partial summation formula, and by using (3.2), then

\[
\sum_{a < n \leq b} (n + \alpha)^{-1/2 - it}
\]

\[
= \left\{ \sum_{a < n \leq b} (n + \alpha)^{-it} \right\} (b - a)^{-1/2} + \frac{1}{2} \int_a^b \left\{ \sum_{a < n \leq \xi} (n + \alpha)^{-it} \right\} \xi^{-3/2} d\xi
\]

\[
\ll (a^{1/2} t^{1/6} + at^{-1/6}) a^{-1/2} + \int_a^b (a^{1/2} t^{1/6} + at^{-1/6}) \xi^{-3/2} d\xi
\]

\[
\ll t^{1/6} + a^{1/2} t^{-1/6}.
\]
If \(a \ll t^{2/3}\), the above evaluation formula is
\[
\sum_{a<n \leq b} (n + \alpha)^{-1/2-it} \ll t^{1/6}. \tag{3.3}
\]

Also if \(t^{2/3} \ll a \ll t\), by using Lemma 5 with same \(f(x)\) as above method, then we have
\[
\sum_{a<n \leq b} (n + \alpha)^{-it} \ll a \left\{ \frac{t}{(a + \alpha)^2} \right\}^{1/2} + a^{1/2} \left\{ \frac{t}{(a + \alpha)^2} \right\}^{-1/2}
\ll t^{3/2} + at^{-1/2}.
\]

Similarly, by using partial summation formula to obtain
\[
\sum_{a<n \leq b} (n + \alpha)^{-1/2-it} \ll a^{-1/2} t^{1/2} + a^{1/2} t^{-1/2}.
\]

Therefore, also in the case of \(t^{3/2} \ll a \ll t\), the evaluation formula (3.3) holds. Furthermore, setting \((a, b) = (2^{-j}x, 2^{-j+1}x)\) and taking the sum for \(j = 1, 2, 3, \ldots\), so calculate the sum of \(O(\log x)\) terms, then we obtain
\[
\sum_{0 \leq n \leq x} (n + \alpha)^{-1/2-it} = \sum_{j \geq 1} \sum_{2^{-j}x<n \leq 2^{-j+1}x} (n + \alpha)^{-1/2-it} \ll t^{1/6} \log x.
\]

\[\square\]

4. Proof of Theorem 3 and Theorem 4

Proof of Theorem 3

(i) In the case when \(v, w\) are linear independent over \(\mathbb{Q}\). Setting \(s = 1/2 + it\) on Theorem 2 (i), and note that
\[
-\frac{\Gamma(1-s)}{(2\pi i)^{1-s}e^{\pi is}} \sim 1
\]
so we have following evaluation,
\[
\zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right)
\ll \sum_{0 \leq m, n \leq x/(v+w)} \frac{1}{(\alpha + vm + wn)^{1/2+it}} + \frac{1}{\sqrt{w}} \sum_{0 \leq m \leq x/(v+w)} \zeta^*_H \left( \frac{1}{2} + it, \alpha_{v,m} \right)
+ \frac{1}{\sqrt{v}} \sum_{0 \leq n \leq x/(v+w)} \zeta^*_H \left( \frac{1}{2} + it, \alpha_{w,n} \right)
+ 2\pi \sum_{0 < |n| \leq y_v} \frac{e^{-2\pi in(\alpha+wN)/v}}{e^{2\pi inw/v} - 1} \cdot \frac{1}{n^{1/2-it}}
+ \frac{2\pi}{\sqrt{w}} \sum_{0 < |n| \leq y_w} \frac{e^{-2\pi in(\alpha+vN)/w}}{e^{2\pi inw/w} - 1} \cdot \frac{1}{n^{1/2-it}}. \tag{4.1}
\]

We denote the right-hand side by \(S + T_1 + T_2 + U_1 + U_2\). To evaluation for double series \(S\) on (4.1), we consider the following single series and by using Lemma 6 with
\[ g(x) = -t(2\pi)^{-1} \log(\alpha + vx + wx) \] and \((a, b) = (2^{-j}x/(v + w), 2^{-j+1}x/(v + w))\), we have

\[
\sum_{0 \leq n \leq x/(v+w)} (\alpha + vm + wn)^{-it} = \sum_{j=1}^{\infty} \sum_{a<n\leq b} e^{2\pi \imath g(n)}
\]

\[
\ll \sum_{j=1}^{\infty} \left\{ 2^{-j}x(m+x)^{-1/2}|t|^{1/6} + 2^{-j/2}x^{1/2}(m+x)^{1/2}|t|^{-1/6} \right\}
\]

\[
\ll x(m+x)^{-1/2}|t|^{1/6} + x^{1/2}(m+x)^{1/2}|t|^{-1/6}.
\] (4.2)

Also by using partial summation formula, and by using (4.2), then we have

\[
\sum_{0 \leq n \leq x/(v+w)} (\alpha + vm + wn)^{-1/2-it}
\]

\[
= \left\{ \sum_{0 \leq n \leq x/(v+w)} (\alpha + vm + wn)^{-it} \right\} \left( \frac{x}{v+w} \right)^{-1/2}
\]

\[
-\frac{1}{2}(v+w)^{1/2} \int_{x/(v+w)}^{x/(v+w+1)} \left\{ \sum_{0 \leq n \leq \xi/(v+w)} (\alpha + vm + wn)^{-it} \right\} \xi^{-3/2}d\xi
\]

\[
\ll x^{1/2}(m+x)^{-1/2}|t|^{1/6} + (m+x)^{1/2}|t|^{-1/6}
\]

\[
- \int_{1}^{x/(v+w)} \left\{ \xi^{-1/2}(m+\xi)^{-1/2}|t|^{1/6} + \xi^{-1}(m+\xi)^{1/2}|t|^{-1/6} \right\} d\xi
\]

\[
\ll x^{1/2}(m+x)^{-1/2}|t|^{1/6} + |t|^{-1/6}(m+x)^{1/2} + |t|^{1/6} \log(\sqrt{x} + \sqrt{m+x})
\]

\[
+ |t|^{-1/6} \sqrt{m} \left\{ \log\left(\sqrt{1 + \frac{x}{m} - 1}\right) - \log\left(\sqrt{1 + \frac{x}{m} + 1}\right) \right\}.
\]

Furthermore by calculating the sum on \(m\), so we can evaluated to \(S\) as follows,

\[
S \ll x^{1/2}|t|^{1/6} \sum_{0 \leq m \leq x} \frac{1}{\sqrt{m+x}} + |t|^{-1/6} \sum_{0 \leq m \leq x} \sqrt{m+x}
\]

\[
+ |t|^{1/6} \sum_{0 \leq m \leq x} \log(\sqrt{x} + \sqrt{m+x}) + |t|^{-1/6} \sum_{0 \leq m \leq x} \sqrt{m} \log\left(1 + \frac{2m}{x}\right)
\]

\[
\ll x|x|^{1/6} + x^{3/2}|t|^{-1/6} + |t|^{1/6}x \log x + |t|^{-1/6}x^{3/2}
\]

\[
\ll |t|^{1/6}x \log x + |t|^{-1/6}x^{3/2}. \] (4.3)
Next, we consider the order of $T_1 + T_2$. By using Lemma 7 we have
\[
\zeta_H^* (\sigma + it, \alpha_{v,m}) = \zeta_H (\sigma + it, \alpha_{v,m}) - \sum_{0 \leq n \leq N+n_{v,m}} \frac{1}{(n + \alpha_{v,m})^{\sigma+it}}
\]
\[
\ll \zeta_H (\sigma + it, \alpha_{v,m}) - \sum_{0 \leq n \leq \varepsilon} \frac{1}{(n + \alpha_{v,m})^{\sigma+it}}
\]
\[
= \Gamma(1-s) \left( \frac{e^{\pi i(1-s)}/2}{(2\pi)^{1-s}} \sum_{1 \leq n \leq y} \frac{e^{2\pi i \alpha_{v,m} n}}{n^{1-s}} + e^{-\pi i(1-s)} \sum_{1 \leq n \leq y} \frac{e^{2\pi i \alpha_{v,m} n}}{n^{1-s}} \right)
\]
\[
+ O(x^{-\sigma}) + O(\|t\|^{1/2-\sigma} y^{\sigma-1})
\]
\[
\ll |t|^{1/2-\sigma} e^{\pi|\tau|t}/2 \left| \sum_{1 \leq n \leq y} \frac{e^{2\pi i \alpha_{v,m} n}}{n^{1-s}} \right| + |t|^{1/2-\sigma} e^{\pi|\tau|t}/2 \left| \sum_{1 \leq n \leq y} \frac{e^{2\pi i \alpha_{v,m} n}}{n^{1-s}} \right|
\]
\[
+ x^{-\sigma} + |t|^{1/2-\sigma} y^{\sigma-1}.
\]
Suppose that $\sigma = 1/2$ and by using Lemma 8
\[
\zeta_H^* \left( \frac{1}{2} + it, \alpha_{v,m} \right) \ll \left| \sum_{1 \leq n \leq y} \frac{e^{2\pi i \alpha_{v,m} n}}{n^{1/2-it}} \right| \ll \begin{cases} 
\begin{align*}
1 & \quad (0 < \alpha_{v,m} < 1), \\
|t|^{1/6} \log y & \quad (\alpha_{v,m} = 1).
\end{align*}
\end{cases}
\]
Here, $\alpha_{v,m} = 1$ is equivalent to $(\alpha + vm)/w \in \mathbb{N}$. We suppose that exists $k \in \mathbb{N}$ such as $(\alpha + vm)/w = k$ that is $\alpha = kw - mv$. Also taking $k', m'$ such that $\alpha = k'w - m'v$, so $(k - k')w - (m' - m)v = 0$ holds. Since $v, w$ are linearly independent over $\mathbb{Q}$, so $(k', m') = (k, m)$. That is, there are at most one pair $(k, m)$ that satisfies $\alpha = kw - mv$. Therefore, we have
\[
T_1 + T_2 \ll \sum_{0 \leq m \leq x/\alpha_{v,m}} \zeta_H^* \left( \frac{1}{2} + it, \alpha_{v,m} \right)
\]
\[
\ll \begin{cases} 
x & \quad (\alpha, v, w \text{ are lin. indep. over } \mathbb{Q}), \\
x + |t|^{1/6} \log y & \quad (\alpha, v, w \text{ are lin. dep. over } \mathbb{Q}).
\end{cases}
\]
Finally, we consider the evaluation of $U_1$ and $U_2$. Let $d(z, \varepsilon)$ be the closed disk whose center is $z \in \mathbb{C}$ with radius $\varepsilon > 0$. We first note that for any $\varepsilon_1 > 0$, then exists $\delta = \delta(\varepsilon_1) > 0$ such that for $z \in \bigcup_{n \in \mathbb{Z}} d(2\pi in, \varepsilon_1)$ the inequality
\[
\left| \frac{1}{e^z - 1} \right| \leq \delta e^{-\max\{\Re(z), 0\}}
\]
holds. Also, since $v, w$ are linearly independent over $\mathbb{Q}$ thus $(\alpha + vN)/v \notin \mathbb{Z}$ and $(\alpha + vN)/w \notin \mathbb{Z}$ for any $N \in \mathbb{N}$, then we have
\[
\sum_{0 < n \leq L} \frac{e^{-2\pi i (\alpha + vN)/v}}{e^{2\pi i vn}/v - 1} \cdot \frac{1}{n^{1/2-it}} \leq \delta \sum_{n=1}^{\infty} \frac{e^{-2\pi i (\alpha + vN)/v}}{n^{1/2-it}},
\]
\[
\sum_{0 < n \leq M} \frac{e^{-2\pi i (\alpha + vN)/w}}{e^{2\pi i vn}/w - 1} \cdot \frac{1}{n^{1/2-it}} \leq \delta \sum_{n=1}^{\infty} \frac{e^{-2\pi i (\alpha + vN)/w}}{n^{1/2-it}}.
\]
and each right-hand side series is convergent, so we have

$$U_1 + U_2 \ll 1.$$  \hfill (4.5)

Since (4.3), (4.4) and (4.5), then we have

$$\zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right) \ll |t|^{1/6} x \log x + |t|^{-1/6} x^{3/2}$$

$$+ \begin{cases} x & (\alpha, v, w \text{ are lin. indep. over } \mathbb{Q}), \\ x + |t|^{1/6} \log y & (\alpha, v, w \text{ are lin. dep. over } \mathbb{Q}). \end{cases}$$

We consider in the case \( \alpha, v, w \) are linearly independent over \( \mathbb{Q} \). Let \( C > 1 \), taking \( x = C, y = |t|/2\pi C \) then \( \zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right) \ll |t|^{1/6} \). On the other hand, in the case when \( \alpha, v, w \) are linearly dependent over \( \mathbb{Q} \), taking \( x = 2\pi (\log t)^{1/2}, y = t(\log t)^{-1/2} \) then, \( \zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right) \ll |t|^{1/6} \log |t| \) so we obtain the result of Theorem 2 (i).

(ii) In the case when \( v, w \) are linear dependent over \( \mathbb{Q} \). Setting \( s = 1/2 + it \) on Theorem 2 (ii), we have

$$\zeta_2 \left( \frac{1}{2} + it, \alpha; v, w \right)$$

$$= \sum_{0 \leq m, n \leq N} \frac{1}{(\alpha + vm + wn)^{1/2 + it}} + \frac{1}{w^{1/2 + it}} \sum_{m=0}^{N} \sum_{n=0}^{N} \zeta_H^* \left( \frac{1}{2} + it, \alpha_{v,m} \right)$$

$$+ \frac{1}{w^{1/2 + it}} \sum_{n=0}^{N} \zeta_H^* \left( \frac{1}{2} + it, \alpha_{w,n} \right)$$

$$- \frac{\Gamma(1/2 - it)}{(2\pi i)^{1/2 - it} e^{\pi i(1/2 + it)}} \left\{ \frac{2\pi}{w^{1/2 + it}} \sum_{0 < |n| < L \atop q \mid n} e^{-2\pi \alpha n N / \nu} e^{2\pi in \nu / w - 1} n^{-1/2 + it} \right\}$$

$$+ \frac{2\pi}{w^{1/2 + it}} \sum_{0 < |n| < M \atop p \mid n} e^{-2\pi \alpha n N / \nu} e^{2\pi in \nu / w - 1} n^{-1/2 + it}$$

$$- \frac{\Gamma(1/2 - it)}{(2\pi i)^{1/2 - it} e^{\pi i(1/2 + it)}} \cdot \frac{q^{1/2 + it}}{2\pi pq^{1/2 + it}} \left( \frac{1}{2} - it \right)$$

$$\sum_{0 < |n| < M \atop 2q \mid n} \frac{e^{-2q \pi \alpha n / \nu}}{n^{3/2 - it}}$$

$$+ \frac{\Gamma(1/2 - it)}{(2\pi i)^{1/2 - it} e^{\pi i(1/2 + it)}} \left( \frac{\alpha q + (p + q)N}{pv} + \frac{vp + v}{2q} \right)$$

$$\times \left( \frac{q}{v} \right)^{-1/2 + it} \sum_{0 < |n| < M \atop 2q \mid n} \frac{e^{-2q \pi \alpha n / \nu}}{n^{1/2 - it}}$$

$$+ O(x^{-1/2}).$$  \hfill (4.6)

The sum of the first term to fourth term on the right-hand side is evaluated as \( \ll |t|^{1/6} \log |t| \) by using the same method as in (i). We denote the fifth term and
the sixth term on the right-hand side by $V_1$ and $V_2$, respectively. By using the Stirling formula, we obtain
\[
\frac{\Gamma(1/2 - it)}{(2\pi i)^{1/2 - it} e^{\pi i (1/2 + it)}} = 1 + O(t^{-1})
\]
so
\[
|V_1| = \frac{1}{2\pi p \sqrt{qv}} \left| \sum_{0 < |n| < M} \frac{e^{2\pi i n/\alpha}}{n^{3/2 - it}} \right| + O(1).
\]
Also, since $N \approx x$ and
\[
\sum_{0 < |n| < M} \frac{e^{-2\pi i n/\alpha}}{n^{1/2 - it}} \ll x|t|^{1/6} \log |t|
\]
is established, then $V_2 \ll x|t|^{1/6} \log |t|$. Therefore, we have
\[
\zeta_2\left(\frac{1}{2} + it, \alpha; v, w\right) = \kappa t + O(|t|^{1/6} \log |t|)
\]
where $\kappa = \kappa(t)$ is constant that depends on $t$ with
\[
0 < |\kappa| < \frac{1}{2\pi p \sqrt{qv}} \left| \zeta_L\left(\frac{3}{2} - it, 1, 1 - \frac{q}{v} \alpha\right)\right|.
\]

**Remark 2.** The order of $\zeta_2(1/2 + it, \alpha; v, w)$ greatly different between when $v, w$ are linearly dependent over $\mathbb{Q}$ and it is not so. For example, considering the special case of $v = w = 1$, since
\[
\zeta_2(s, \alpha; 1, 1) = (1 - \alpha)\zeta_H(s, \alpha) + \zeta_H(s - 1, \alpha)
\]
holes (See in [9], p. 86) and $\zeta_H(\sigma + it, \alpha) = O(t^{1/2 - \sigma})$ ($\sigma < 0$) is well-known, in the above equation, let $s = 1/2 + it$ then the second term on the right-hand side is $\zeta_H(-1/2 + it, \alpha) = O(t)$. From this, we can see that $\zeta_2(1/2 + it, \alpha; 1, 1)$ is linear expression of $t$.

**Proof of Theorem 4.** Setting $s = 3/2 + it$ on Theorem 1 then
\[
\zeta_2\left(\frac{3}{2} + it, \alpha; v, w\right) = \sum_{0 \leq m, n \leq x} (\alpha + vm + wn)^{-3/2 - it}
+ \frac{(\alpha + vx)^{1/2 - it} + (\alpha + wx)^{1/2 - it} - (\alpha + vx + wx)^{1/2 - it}}{-vw(1/2 + it)(1/2 - it)} + O(x^{-1/2}).
\]
To evaluation for double series on the right-hand side of the above, we consider the following single series and by using Lemma 6 with same $g(x)$ and $(a, b)$ in the proof of Theorem 3(i), then we have
\[
\sum_{0 \leq n \leq x} (\alpha + vm + wn)^{-it} = \sum_{j=1}^{\infty} \sum_{a < n \leq b} \zeta^{2\pi ig(n)}
\ll x(m + x)^{-1/2}|t|^{1/6} + x^{1/2}(m + x)^{1/2}|t|^{-1/6}.
\]
Similarly, by using partial summation formula and by using (4.7), then
\[
\sum_{0 \leq n \leq x/(v+w)} (\alpha + v m + w n)^{-1/2-it} \ll x^{-1/2}(m + x)^{-1/2}|t|^{1/6} + x^{-1}|t|^{-1/6}(m + x)^{1/2} + x^{-1/2}|t|^{1/6}(m + x)^{1/2}m^{-1}
\]
\[
-|t|^{-1/6}\left(\frac{\sqrt{m + x}}{x}\frac{1}{\sqrt{m}} + \left\{\log\left(\sqrt{1 + \frac{x}{m}} - 1\right) - \log\left(\sqrt{1 + \frac{x}{m}} + 1\right)\right\}\right).
\]
Furthermore, by calculating the sum on \(m\), so we can evaluated this series as follows,
\[
\sum_{0 \leq m,n \leq x}(\alpha + w m + w n)^{-3/2-it} \ll |t|^{1/6} + x^{1/2}|t|^{-1/6}.
\]
Therefore, we have
\[
\zeta_2\left(\frac{3}{2} + it, \alpha; v, w\right) \ll |t|^{1/6} + x^{1/2}|t|^{-1/6} + x^{1/2}|t|^{-2} + x^{-1/2}.
\]
Taking \(x \asymp |t|\) then \(\zeta_2\left(3/2 + it, \alpha; v, w\right) \ll |t|^{1/3}\), so proof of Theorem 4 is complete. \(\square\)

**Remark 3.** In the above proof by used Theorem 1 but if we use Theorem 2, we have only weak results of \(\zeta_2\left(3/2 + it, \alpha; v, w\right) \ll |t|^{2/3}\). From this, Theorem 1 is more effective when \(s = 3/2 + it\).

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