BLOW-UP AND SCATTERING FOR THE 1D NLS WITH POINT NONLINEARITY ABOVE THE MASS-ENERGY THRESHOLD

ALEX H. ARDILA

Abstract. In this paper, we study the nonlinear Schrödinger equation with focusing point nonlinearity in dimension one. First, we establish a scattering criterion for the equation based on Kenig-Merle’s compactness-rigidity argument. Then we prove the energy scattering below and above the mass-energy threshold. We also describe the dynamics of solutions with data at the ground state threshold. Finally, we prove a blow-up criteria for the equation with initial data with arbitrarily large energy.

1. Introduction

The nonlinear Schrödinger equations with point interactions have been intensively studied in recent years [11, 12, 14, 18, 21]. This interest is motivated by physical experiments in the theory of Bose-Einstein condensates and in nonlinear optics; see [3–5, 22, 23] and references therein. In this paper, we consider the Cauchy problem for the following nonlinear Schrödinger equation with focusing point nonlinearity

\[
\begin{cases}
i\partial_t u + \partial_x^2 u + \delta(x) |u|^{p-1} u = 0, & x \in \mathbb{R}, t \in \mathbb{R}, \\
u(0) = u_0 \in H^1(\mathbb{R}),
\end{cases}
\]

where \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \), \( \delta(x) \) is the Dirac mass at \( x = 0 \) and the nonlinearity is \( L^2 \)-supercritical, i.e. \( p > 3 \). The Dirac measure is used to model a defect localized at the origin (see, for example, [22]).

The unique local existence of solutions is well known (see [16, Theorem 1.1]): Given \( u_0 \in H^1(\mathbb{R}) \), there exists a unique solution \( u \in C([0,T_+), H^1(\mathbb{R})) \) of the Cauchy problem (1.1) for some maximal existence interval \( [0,T_+] \). We say that the solution \( u \) is global forward in time if \( T_+ = \infty \). Besides, the mass \( M(u(t)) \) and energy \( E(u(t)) \) are independent of \( t \), where

\[
M(f) = \int_{\mathbb{R}} |f(x)|^2 \, dx, \quad E(f) = \frac{1}{2} K(f) - \frac{1}{p+1} N(f),
\]

with

\[
K(f) = \int_{\mathbb{R}} |\partial_x f(x)|^2 \, dx \quad \text{and} \quad N(f) = |f(0)|^{p+1}
\]

for \( f \in H^1(\mathbb{R}) \). The equation (1.1) has a scaling invariance: if \( u(x,t) \) is a solution of the Cauchy problem (1.1), then

\[
\lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \quad \lambda > 0
\]

is also a solution with the rescaled initial data. Thus, by scaling argument it is possible to show that \( \dot{H}^{\gamma_c} \) is the critical Sobolev space to (1.1), with

\[
\gamma_c = \frac{1}{2} - \frac{1}{p-1}.
\]

2010 Mathematics Subject Classification. 35Q55, 37K45, 35P25.

Key words and phrases. NLS with point nonlinearity; Ground state; Scattering; Compactness.
Therefore, the NLS \((1.1)\) is called \(L^2\)-subcritical if \(p < 3\), \(L^2\)-critical if \(p = 3\) and \(L^2\)-supercritical if \(p > 3\). We say that the solution \(u \in C([0, \infty), H^1(\mathbb{R}))\) to \((1.1)\) scatters in \(H^1(\mathbb{R})\) forward in time, if there exists \(\psi^+ \in H^1(\mathbb{R})\) such that
\[
\lim_{t \to \infty} \|u(t) - e^{it\partial_x^2} \psi^+\|_{H^1(\mathbb{R})} = 0.
\]
The equation \((1.1)\) admits a global but nonscattering solution \(u(x, t) = e^{it}Q(x)\), where \(Q \neq 0\) should satisfy the following elliptic equation
\[
-\partial_x^2 Q + Q - \delta(x) |Q|^{p-1}Q = 0, \quad x \in \mathbb{R}. \tag{1.3}
\]
In [16], the authors prove that there exists a unique positive symmetric solution of the stationary problem \((1.3)\), which is given by
\[
Q(x) = 2^{\frac{2}{p-1}} e^{-|x|}. \tag{1.4}
\]
We call \(Q\) the ground state. Recently, Adami-Fukuizumi-Holmer [1] determined the long time dynamics to \((1.1)\) with data below the ground state threshold.

Our purpose in this paper is to discuss the global behavior of the solutions to \((1.1)\) at and above the mass and energy ground states threshold. With this in mind, we give the following unified scattering criterion for the equation \((1.1)\). Using this criterion, we will be able to obtain scattering below, at and above the mass and energy ground states threshold. We set
\[
\sigma_c := \frac{1 - \gamma_c}{\gamma_c}.
\]

**Theorem 1.1** (Scattering criterion). Let \(p > 3\) and \(u_0 \in H^1(\mathbb{R})\). Let \(u(t)\) the corresponding solution of the Cauchy problem \((1.1)\) with initial data \(u_0\) defined on the maximal forward time lifespan \([0, T_+).\) Assume that
\[
\sup_{t \in [0, T_+)} N(u(t))[M(u(t))]^{\sigma_c} < N(Q)[M(Q)]^{\sigma_c}, \tag{1.5}
\]
then the solution \(u(t)\) is global \((T_+ = \infty)\) and scatters in \(H^1(\mathbb{R})\) forward in time.

For the classical NLS, a similar result was originally proven by Duyckaerts-Roudenko [10, Theorem 3.7] through the use concentration-compactness-rigidity argument of Kenig-Merle [19]. Later, in Dinh [7], this result in dimension \(N \geq 3\) was proven using the Dodson-Murphy’ argument [8], which simplifies the process of the proof for the scattering. For other results in this direction see also [13, 24]. Naturally, in this paper, we establish the scattering criterion (Theorem 1.1) for the equation \((1.1)\) based on the argument of Duyckaerts-Roudenko [10].

As a first consequence of the Theorem 1.1, we have the energy scattering below the ground state threshold, which was originally proved by Adami-Fukuizumi-Holmer [1] based on the ideas of [9, 17].

**Theorem 1.2** (Scattering below the threshold, [1]). Let \(p > 3\) and \(u_0 \in H^1(\mathbb{R})\) satisfy
\[
E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c} \tag{1.6}
\]
and
\[
K(u_0)[M(u_0)]^{\sigma_c} < K(Q)[M(Q)]^{\sigma_c}. \tag{1.7}
\]
Then the solution \(u(t)\) of Cauchy problem \((1.1)\) is global and scatters in \(H^1(\mathbb{R})\) in both directions.

**Remark 1.3.** We observe that if \(u_0 \in H^1(\mathbb{R})\) satisfies \((1.6)\) and \(K(u_0)[M(u_0)]^{\sigma_c} > K(Q)[M(Q)]^{\sigma_c}\), then the corresponding solution \(u(t)\) of the Cauchy problem \((1.1)\) blows-up in the positive time direction. An analogous statement holds for negative time; see [16, Theorem 1.4] for more details.
In the following result, we investigate the long time dynamics for the equation (1.1) at mass and energy ground states threshold $E(u_0)[M(u_0)]^\sigma_c = E(Q)[M(Q)]^\sigma_c$. Indeed, using the scattering criterion Theorem 1.1 and the compactness of minimizing sequence for the Gagliardo-Nirenberg inequality (see (2.2) below) we obtain the following result.

**Theorem 1.4 (Dynamics at threshold).** Let $p > 3$ and $u_0 \in H^1(\mathbb{R})$ satisfy

$$E(u_0)[M(u_0)]^\sigma_c = E(Q)[M(Q)]^\sigma_c.$$  \hfill (1.8)

(i) If

$$K(u_0)[M(u_0)]^\sigma_c < K(Q)[M(Q)]^\sigma_c,$$  \hfill (1.9)

then one of the following cases holds.

- The corresponding solution $u(t)$ to (1.1) scatters in $H^1(\mathbb{R})$ forward in time.
- There exist $\theta \in \mathbb{R}$ and a time sequence $t_n \to \infty$ such that

$$u(\cdot, t_n) \to 2^{\frac{1}{p-1}} e^{i\theta} e^{-|\cdot|} \quad \text{in } H^1(\mathbb{R}) \quad \text{as } n \to \infty.$$  \hfill (1.10)

(ii) If

$$K(u_0)[M(u_0)]^\sigma_c = K(Q)[M(Q)]^\sigma_c,$$  \hfill (1.11)

then there exists $\theta \in \mathbb{R}$ such that the corresponding solution $u(t)$ to (1.1) is given by $u(x, t) = 2^{\frac{1}{p-1}} e^{i\theta} e^{-|x|}$.

(iii) If

$$K(u_0)[M(u_0)]^\sigma_c > K(Q)[M(Q)]^\sigma_c,$$  \hfill (1.12)

then one of the following cases holds.

- The corresponding solution $u(t)$ to (1.1) blows-up in finite time.
- The solution $u(t)$ is global and exist $\rho \in \mathbb{R}$ and a time sequence $\tau_n \to \infty$ such that

$$u(\cdot, \tau_n) \to 2^{\frac{1}{p-1}} e^{i\rho} e^{-|\cdot|} \quad \text{in } H^1(\mathbb{R}) \quad \text{as } n \to \infty.$$  \hfill (1.13)

Next, we have interested in a criteria of scattering that includes initial data above the mass-energy threshold. We recall that when $u_0 \in \Sigma := \{f \in H^1(\mathbb{R}) : |x| f \in L^2(\mathbb{R})\}$, the virial quantity of (1.1),

$$V(t) := \int_{\mathbb{R}} x^2 |u(x, t)|^2 dx,$$

is finite for all $t \in [0, T_+]$ and satisfies the virial identity

$$V''(t) = G(u) \quad \text{where } \ G(u) := 8K(u) - 4N(u)$$

is the Pohozaev functional.

As another consequence of Theorem 1.1, we obtain the following scattering result for (1.1) above the mass-energy threshold. An analogous result was proven in [10] for the classic NLS.

**Theorem 1.5 (Scattering above the threshold).** Let $p > 3$ and $u_0 \in \Sigma(\mathbb{R})$. Assume that

$$E(u_0)[M(u_0)]^\sigma_c \geq E(Q)[M(Q)]^\sigma_c,$$  \hfill (1.14)

$$\frac{E(u_0)[M(u_0)]^\sigma_c}{E(Q)[M(Q)]^\sigma_c} \left(1 - \frac{(V'(0))^2}{32 E(u_0)V(0)}\right) \leq 1,$$  \hfill (1.15)

$$N(u_0)[M(u_0)]^\sigma_c < N(Q)[M(Q)]^\sigma_c,$$  \hfill (1.16)

$$V'(0) \geq 0.$$  \hfill (1.17)

Then the solution $u(t)$ with initial data $u_0$ is global and scatters forward in time in $H^1(\mathbb{R})$. 

Complementing the theorem stated above, we extend the scope of blow-up solutions in Remark 1.3 to those with arbitrarily large energy. Indeed, we have the following finite time blow-up result for the equation (1.1).

**Theorem 1.6** (Blow up above the threshold). Let \( u_0 \in \Sigma(\mathbb{R}) \) satisfy (1.15),

\[
N(u_0)[M(u_0)]^{\sigma_c} > N(Q)[M(Q)]^{\sigma_c},
\]

\[
V'(0) \leq 0.
\]

Then the solution \( u(t) \) to Cauchy problem (1.1) with initial data \( u_0 \) blows up forward in finite time.

As a consequence of the Theorems 1.5 and 1.6, we classify the behavior of the ground state modulated by a quadratic phase. More specifically,

**Corollary 1.7.** Let \( \gamma \in \mathbb{R} \) and let \( \psi^\gamma(t) \) be the corresponding solution to (1.1) with initial data

\[
\psi_0^\gamma(x) = 2\pi^{-\frac{1}{2}} e^{i\gamma x^2} e^{-|x|}.
\]

Then, for \( \gamma > 0 \) the solution \( \psi^\gamma(t) \) is globally defined on \( [0, \infty) \), scatters forward in \( H^1(\mathbb{R}) \) and blows up in negative time. Furthermore, for \( \gamma < 0 \) the solution \( \psi^\gamma(t) \) is globally defined on \( (-\infty, 0] \), scatters backward in time in \( H^1(\mathbb{R}) \) and blows up in positive time.

Notice that Corollary 1.7 shows that the results stated in Theorems 1.5 and 1.6 are not symmetric in time. Finally, we have the following result.

**Corollary 1.8.** Let \( \mu \in \mathbb{R} \setminus \{0\}, u_0 \in H^1(\mathbb{R}) \) with finite variance and let \( u_\mu(t) \) be the corresponding solution of (1.1) with initial data

\[
u_{\mu,0}(x) = e^{i\mu x^2} u_0(x).
\]

Assume that

\[
E(u_0)[M(u_0)]^{\sigma_c} \leq E(Q)[M(Q)]^{\sigma_c}.
\]

(i) If \( [M(u_0)]^{\sigma_c} N(u_0) < [M(Q)]^{\sigma_c} N(Q) \), then for any \( \mu > 0 \) the solution \( u_\mu(t) \) scatters in \( H^1(\mathbb{R}) \) forward in the time.

(ii) If \( [M(u_0)]^{\sigma_c} N(u_0) > [M(Q)]^{\sigma_c} N(Q) \), then for any \( \mu < 0 \) the solution \( u_\mu(t) \) blows up in finite positive time.

We observe that under the conditions of Corollary 1.8,

\[
E(u_{\mu,0}) = 2\mu^2 \| Xu_0 \|_{L^2}^2 + 2\mu \operatorname{Im} \int_\mathbb{R} x \bar{u}_0 \psi_0 + E(u_0).
\]

Thus, \( E(u_{\mu,0}) \to \infty \) as \( \mu \to \infty \).

This paper is structured as follows. We fix notations at the end of Section 1. In Section 2, we give some results that are necessary for later sections, including the smoothing properties of the integral equation. In Section 3, we prove the scattering criterion by a concentration-compactness-rigidity argument. Then in Section 4, using the scattering criterion, we prove the Theorems 1.2 and 1.5. In Section 1.4, we establish the long time dynamics at mass and energy ground states threshold (Theorem 1.4). Finally, in Section 6 we show the blow-up result of Theorem 1.6. Moreover, we prove the Corollaries 1.7 and 1.8.

**Notations.** We write \( A \lesssim B \) or \( B \gtrsim A \) to signify \( A \leq CB \) for some constant \( C > 0 \). When \( A \lesssim B \lesssim A \) we write \( A \approx B \). For an interval \( I \subset \mathbb{R} \), we use \( L^r(I) \) to denote the Banach space of functions \( f : I : \rightarrow \mathbb{C} \) such that the norm

\[
\|f\|_{L^r(I)} = \left( \int_I |f(x)|^r \, dx \right)^{\frac{1}{r}},
\]
is finite, with the usual adjustments when $r = \infty$.

For $s \in \mathbb{R}$ we introduce the Sobolev space

$$H^s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}), \|f\|_{H^s(\mathbb{R})} := \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R})} \right\}$$

and the homogeneous Sobolev space

$$\dot{H}^s(\mathbb{R}) = \left\{ f \in \mathcal{S}'(\mathbb{R}), \|f\|_{\dot{H}^s(\mathbb{R})} := \|\xi^s \hat{f}(\xi)\|_{L^2(\mathbb{R})} \right\},$$

where $\hat{f}$ denote the Fourier transform. As usual, we denote $\mathbb{R}^+ = (0, +\infty)$. Finally, we define the following indices

$$a = 2(p - 1) \quad \text{and} \quad b = \frac{2(p - 1)}{p}.$$  

2. Preliminaries

In this section, we give preliminary results that will be used later. A function $u : \mathbb{R} \times I \to \mathbb{C}$ on a nonempty interval $[0, T_+]$ is called a solution to the Cauchy problem (1.1) if $u \in C_t^1 H^1(\mathbb{R} \times K) \cap C_t H^{\gamma}(0, T_+) \times \mathbb{R}$ for every compact interval $K \subset (0, T_+)$ and it satisfies the Duhamel formula

$$u(x, t) = e^{i\sqrt{2} t} u_0 + i \int_0^t e^{i \frac{\gamma}{2}(t-s)} \delta(x) |u(x, s)|^{p-1} u(x, s)ds$$

$$= e^{i\sqrt{2} t} u_0 + i \int_0^t \frac{e^{i \frac{\gamma}{2} \tau}}{\sqrt{4\pi i (t - s)}} |u(0, s)|^{p-1} u(0, s)ds,$$

for all $t \in [0, T_+)$. Consider $f \in \dot{H}^{\gamma}$ with $\gamma \in \mathbb{R}$. For $t, s \in \mathbb{R}$ with $t \geq s$ we define the transformation

$$[\mathcal{P}_s f](x, t) := \int_s^t \frac{e^{i \frac{\gamma}{2} \tau}}{\sqrt{4\pi i (t - \tau)}} f(\tau) d\tau.$$

Moreover, for $t \in \mathbb{R}$ we define

$$[\mathcal{S} f](x, t) := \int_{-\infty}^t \frac{e^{i \frac{\gamma}{2} \tau}}{\sqrt{4\pi i (t - \tau)}} f(\tau) d\tau.$$

To prove the scattering result, we will need the following global-in-time estimates (see [1, Proposition 2.1] and [2, Lemma 1]).

**Proposition 2.1.** Let $t, s$ and $\gamma \in \mathbb{R}$. We have

(i) If $f \in \dot{H}^s(\mathbb{R})$, then

$$\|e^{i(t-s)\frac{\gamma}{2}} f(0)\|_{H^s_{\dot{t}}} \lesssim \|f\|_{\dot{H}^s}.$$  

(ii) If $f \in \dot{H}^{\frac{s-1}{2}}$ and $-\frac{1}{2} < \frac{2\gamma - 1}{4} < \frac{1}{2}$, then

$$\|\mathcal{P}_s f(0, \cdot)\|_{H^s_{\dot{t}}} \lesssim \|f\|_{H^{\frac{2\gamma - 1}{4}}},$$

$$\|\mathcal{S}_s f(0, \cdot)\|_{H^s_{\dot{t}}} \lesssim \|f\|_{H^{\frac{2\gamma - 1}{4}}}.$$  

(iii) If $f \in \dot{H}^{\frac{s-1}{2}}$ and $-\frac{1}{2} < \frac{2\gamma - 1}{4} < \frac{1}{2}$, then

$$\|\mathcal{P}_s f\|_{L^p(\mathbb{R}; H^s_{\dot{t}})} \lesssim \|f\|_{H^{\frac{2\gamma - 1}{4}}},$$

$$\|\mathcal{S}_s f\|_{L^p(\mathbb{R}; H^s_{\dot{t}})} \lesssim \|f\|_{H^{\frac{2\gamma - 1}{4}}}.$$
Remark 2.2. We recall that $\gamma_c = \frac{2}{p} - \frac{1}{p-1}$. Notice that if $p > 3$, then
\[
\frac{1}{4} < \frac{2\gamma_c + 1}{4} < \frac{1}{2} \quad \text{and} \quad -\frac{1}{4} < \frac{2\gamma_c - 1}{4} < 0.
\]
Furthermore, by the Sobolev embedding $H^s_x(\mathbb{R}) \hookrightarrow L^{s_q}_x(\mathbb{R})$ with $\frac{1}{s_q} = \frac{1}{p} - s$, we get
\[
\|f\|_{L^2_x} \lesssim \|f\|_{H^s_x}^{\frac{2\gamma_c + 1}{4}} \quad \text{for} \quad a = 2(p-1).
\]
And since $L^{s_q}_x(\mathbb{R}) \hookrightarrow \dot{H}^{-s}_x(\mathbb{R})$, it follows that
\[
\|f\|_{\dot{H}^{s_q - 1}_x} \lesssim \|f\|_{L^2_x} \quad \text{for} \quad b = \frac{2(p-1)}{p}.
\]

Variational Analysis. We recall the following Gagliardo-Nirenberg inequality established by Holmer and Liu [16, Proposition 1.3],
\[
N(u) \leq |K(u)|^{\frac{4}{4p - 2}} |M(u)|^{\frac{4p - 6}{4p - 2}} \quad \text{for all} \quad u \in H^1(\mathbb{R}). \tag{2.2}
\]
The ground state $Q$ optimizes the inequality (2.2), i.e.,
\[
N(Q) = |K(Q)|^{\frac{4}{4p - 2}} |M(Q)|^{\frac{4p - 6}{4p - 2}},
\]
where $Q$ is defined in (1.4). Note also that $Q$ satisfies the Pohozaev identities:
\[
M(Q) = K(Q) = \frac{1}{2} N(Q). \tag{2.3}
\]
Thus,
\[
E(Q) = \frac{(p-3)}{4(p+1)} N(Q) = \frac{(p-3)}{2(p+1)} K(Q). \tag{2.4}
\]
Moreover,
\[
2^{\sigma_c} = (K(Q)|M(Q)|^{\sigma_c})^{\frac{4p - 6}{4p - 2}} = \left(\frac{1}{2} N(Q)|M(Q)|^{\sigma_c}\right)^{\frac{4p - 6}{4p - 2}}. \tag{2.5}
\]
We also have the following coercivity property.

Lemma 2.3. Let $p > 3$ and $f \in H^1(\mathbb{R})$ satisfy
\[
N(f)|M(f)|^{\sigma_c} \leq L < N(Q)|M(Q)|^{\sigma_c}
\]
for some positive constant $L$. Then there exists a positive constant $\eta = \eta(L, Q)$ such that
\[
8K(f) - 4N(f) \geq \eta K(f), \tag{2.6}
\]
\[
E(f) \geq \frac{\eta}{16} K(f). \tag{2.7}
\]

Proof. We can write
\[
L = (1 - r)N(Q)|M(Q)|^{\sigma_c} \tag{2.8}
\]
for some $r > 0$ small. Notice that by Gagliardo-Nirenberg inequality (2.2)
\[
[N(f)]^{\frac{p+1}{p-1}} \leq [K(f)]^{\frac{p+1}{p-1}} (N(f)|M(f)|^{\sigma_c})^{\frac{(p-3)}{p-1}}.
\]
Moreover, using the identities (2.5) we have
\[
N(Q)|M(Q)|^{\sigma_c} = 2^{\sigma_c}.
\]
Combining equations above and (2.8) we get
\[
[N(f)]^{\frac{p+1}{p-1}} \leq 2^{\frac{(p+1)}{p-1}} [K(f)]^{\frac{(p+1)}{p-1}} \left(\frac{N(f)|M(f)|^{\sigma_c}}{N(Q)|M(Q)|^{\sigma_c}}\right)^{\frac{(p-3)}{p-1}} \leq 2^{\frac{(p+1)}{p-1}} [K(f)]^{\frac{(p+1)}{p-1}} (1 - r)^{\frac{(p-3)}{p-1}}.
\]
Then,
\[ 8K(f) - 4N(f) \geq 8K(f) - 8K(f)(1 - r)\frac{r}{2} = \eta K(f), \]
where \( \eta = 8(1 - (1 - r)\frac{r}{2}) > 0 \). This proves the estimate (2.6). Finally, since \( p > 3 \), from (2.6) we obtain
\[ E(f) = \frac{1}{2} \left( K(f) - \frac{1}{2}N(f) \right) + \frac{(p - 3)}{4(p + 1)}N(f) \geq \frac{1}{2} \left( K(f) - \frac{1}{2}N(f) \right) \geq \frac{\eta}{16}K(f). \]
This completes the proof of lemma. \( \Box \)

### 3. Scattering criterion

In this section, we show the Theorem 1.1.

**Cauchy problem.** In the following result we have a sufficiently condition for scattering. See [1, Proposition 2.4] for the proof.

**Proposition 3.1.** Let \( p \geq 3 \), \( u(0) \in H^1(\mathbb{R}) \) and \( u(t) \) be the corresponding solution of Cauchy problem (1.1). If \( u \) is forward global, uniformly bounded in \( H^1(\mathbb{R}) \) and \( \|u(0, \cdot)\|_{L^1(\mathbb{R}^+)} < \infty \), then \( u(t) \) scatters forward in time.

We recall a small data global existence result for the equation 1.1.

**Proposition 3.2.** Let \( p \geq 3 \). There exist \( 0 < \delta_{sd} \leq 1 \) and a positive constant \( C_{sd} \) such that if \( \|u_0\|_{H^\infty(\mathbb{R})} \leq \delta_{sd} \), then the solution of (1.1) with initial data \( u_0 \) is global in \( \dot{H}^\infty(\mathbb{R}) \) and
\[ \|u(0, \cdot)\|_{L^2(\mathbb{R}^+)} + \|u\|_{L^\infty(\mathbb{R}^+, \dot{H}^1(\mathbb{R}))} \leq C_{sd}\|u_0\|_{H^1(\mathbb{R})}. \]

For the proof of Proposition 3.2, see [1, Propositions 2.3 and 2.4].

**Remark 3.3** (Existence of wave operators). Using the same argument developed in the proof of Lemma 4.2 in [1], we can show that given \( \psi \in H^1(\mathbb{R}) \), there exists \( T \in \mathbb{R} \) and a solution \( v \in C([-T, \infty), \dot{H}^1(\mathbb{R})) \) to (1.1) such that
\[ \|v(t) - e^{it\partial^2_x} \psi\|_{H^1(\mathbb{R})} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \]

**Perturbation lemma and linear profile decomposition.** We will use a perturbation result and a lemma for linear profiles.

For the proof of the following result we refer the reader to [1, Proposition 2.5].

**Proposition 3.4.** Let \( p \geq 3 \). For any \( M \gg 1 \), there exist \( \varepsilon = \varepsilon(M) \ll 1 \) and \( C = C(M) > 0 \) such the following holds. If \( u \in C([0, \infty), \dot{H}^1(\mathbb{R})) \) is a solution to (1.1) and if \( \tilde{u} \in C([0, \infty), \dot{H}^1(\mathbb{R})) \) is a solution of the equation with source term \( \varepsilon \):
\[ i\partial_t \tilde{u} + \partial_x^2 \tilde{u} + \delta(x)\tilde{u}|^{p-1}\tilde{u} = \delta(x)e \]
with
\[ \|\varepsilon(0, \cdot)\|_{L^1(\mathbb{R}^+)} < \varepsilon, \quad \|\tilde{u}(0, \cdot)\|_{L^1(\mathbb{R}^+)} \leq M \]
and if
\[ \|e^{i(t-t_0)\partial_x^2}(u(t_0) - \tilde{u}(t_0))\|_{L^2_t((t_0, \infty))} \leq \varepsilon \]
for some \( t_0 \geq 0 \), then
\[ \|u(0, \cdot)\|_{L^2(\mathbb{R}^+)} \leq C(M). \]

Using Proposition 2.1, the proof of the following result is an easy modification of arguments used in [1, Proposition 2.5].
Proposition 3.5. Let \( p \geq 3 \) and \([0, T]\) an compact interval and let \( \tilde{u} : [0, T] \times \mathbb{R} \to \mathbb{C} \) be a solution to
\[
i \partial_t \tilde{u} + \partial_x^2 \tilde{u} + \delta(x)|\tilde{u}|^{p-1} \tilde{u} = \delta \epsilon
\]
for some source term \( \epsilon \). Assume that there exists \( M > 0 \) such that
\[
\| \tilde{u}(0, \cdot) \|_{L^\infty_t([0, T])} \leq M.
\]
Moreover, suppose that \( u : [0, T] \times \mathbb{R} \to \mathbb{C} \) is solution to (1.1). Assume that we have the following smallness conditions
\[
\| [e^{it\phi^2}(u(0) - \tilde{u}(0))] \|_{L^\infty_t([0, T])} \leq \epsilon,
\]
\[
\| e(0, \cdot) \|_{L^\infty_t([0, T])} \leq \epsilon,
\]
for some \( 0 < \epsilon < \epsilon_1 \) with \( \epsilon_1 = \epsilon_1(M) \) small constant. Then
\[
\| ([u - \tilde{u}](0, \cdot)) \|_{L^\infty_t([0, T])} \leq C(M, T).
\]

We need the following linear profile decomposition, which is a key ingredient.

Proposition 3.6 (Linear profile decomposition). Let \( p > 3 \). Let \( \{\phi_n\}_{n \geq 1} \) be a bounded sequence of \( H^1(\mathbb{R}) \). Then for each integer \( J \geq 1 \), there exists a subsequence, which we still denote by \( \{\phi_n\}_{n \geq 1} \), and
(i) for each \( 1 \leq j \leq J \), there exists a fixed profile \( \psi^j \in H^1(\mathbb{R}) \);
(ii) for each \( 1 \leq j \leq J \), there exists a sequence of time shifts \( \{\xi_n\}_{n \geq 1} \subset \mathbb{R} \);
(iii) for each \( 1 \leq j \leq J \), there exits a sequence of remainders \( \{W_n^J\}_{n \geq 1} \subset H^1(\mathbb{R}) \) such that we can write
\[
\phi_n = \sum_{j=1}^{J} e^{-i\xi_n^j \partial_x^2} \psi^j + W_n^J,
\]
and the following hold.

- Orthogonality of the parameters:
  \[
  \lim_{n \to \infty} |\xi_n^i - \xi_n^j| = \infty, \quad \text{for } 1 \leq i \neq j \leq J. \tag{3.1}
  \]

- Asymptotic smallness of the remainder:
  \[
  \lim_{J \to \infty} \left( \lim_{n \to \infty} \| e^{i\xi_n \phi^2} W_n^J(0) \|_{L^\infty_t(\mathbb{R})} \|_{L^\infty_t(\mathbb{R})} \right) = 0. \tag{3.2}
  \]

- Orthogonality in norms: for fixed \( J \) and any \( \gamma \in [0, 1] \),
  \[
  \|\phi_n\|_{H^\gamma}^2 = \sum_{j=1}^{J} \|\psi^j\|_{H^\gamma}^2 + \|W_n^J\|_{H^\gamma}^2 + o_n(1). \tag{3.3}
  \]

- Asymptotic Pythagorean expansion: for fixed \( J \),
  \[
  E(\phi_n) = \sum_{j=1}^{J} E(e^{-i\xi_n^j \partial_x^2} \psi^j) + E(W_n^J) + o_n(1). \tag{3.4}
  \]

Proof. We follow the same spirit as in the proof of [1, Proposition 3.1]. Let \( A = \limsup_{n \to \infty} \|\phi_n\|_{H^1_\mathbb{R}(\mathbb{R})} \). For \( R > 0 \), we fix a real-valued, symmetric function \( \zeta_R \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \zeta_R \leq 1 \), \( \zeta_R(\xi) = 1 \) for \( \frac{R}{2} \leq |\xi| \leq R \) and supported in \( \frac{R}{4} \leq |\xi| \leq 2R \). Let \( C_1 := \limsup_{n \to \infty} \| e^{i\xi_n \partial_x^2} \phi_n(0) \|_{L^\infty_t(\mathbb{R})} \) and \( B_1 := \limsup_{n \to \infty} \| e^{i\xi_n \partial_x^2} \phi_n(0) \|_{L^\infty_t(\mathbb{R})} \). We have four cases: (i) \( C_1 = 0 \) and \( B_1 = 0 \); (ii) \( C_1 > 0 \) and \( B_1 = 0 \); (iii) \( C_1 = 0 \) and \( B_1 > 0 \) and (iv) \( C_1 > 0 \) and \( B_1 > 0 \). We only deal with the case \( C_1 > 0 \) and \( B_1 > 0 \). The proof in the other cases is similar.
Suppose that $C_1 > 0$ and $B_1 > 0$. Pass to a subsequence, we may assume \( \lim_{n \to \infty} \| e^{it\partial_x^2} \phi_n \|_{L^\infty(x)} = C_1 \). In particular, passing to a subsequence if necessary, we have that

\[
\sup_{\tau \in \mathbb{R}} \| e^{i \tau \partial_x^2} \phi_n \| \geq \frac{C_1}{2}.
\]

It follows that there exists a sequence of times \( \{ t_n^i \}_{n \geq 1} \) such that

\[
\| e^{i t_n^i \partial_x^2} \phi_n \| \geq \frac{C_1}{2},
\]

for all \( n > 1 \). Since \( \| e^{i t_n^i \partial_x^2} \phi_n \|_{H^1} \lesssim 1 \) for all \( n \), there exists \( \psi^1 \) such that \( e^{i t_n^i \partial_x^2} \phi_n \to \psi^1 \) in \( H^1(\mathbb{R}) \) as \( n \to \infty \). By using the compact embedding \( H^1[-1, 1] \hookrightarrow C[-1, 1] \) we infer that \( [e^{i t_n^i \partial_x^2} \phi_n] \to \psi^1(0) \). Then, from (3.5) we get \( |\psi^1(0)| \geq \frac{C_1}{2} \).

In particular, \( \psi^1 \neq 0 \) and by the inequality \( 2|\psi^1(0)|^2 \leq \| \psi^1 \|^2_{H^1} \), we see that

\[
\| \psi^1 \|^2_{H^1} \geq \frac{(C_1)^2}{2}.
\]

We set \( W_n^1 = \phi_n - e^{-it_n^i \partial_x^2} \psi^1 \). Then we obtain that for \( 0 \leq \gamma \leq 1 \),

\[
\lim_{n \to \infty} \| W_n^1 \|_{H^\gamma} = \lim_{n \to \infty} \| \phi_n \|_{H^\gamma} - \| \psi^1 \|_{H^\gamma}.
\]

In particular, \( \sup_{n \to \infty} \| W_n^1 \|_{H^1} < \infty \). Now, by translation invariance of \( L^2(\mathbb{R}) \)-norm, the argument developed in [1, Proposition 3.1] shows that by choosing \( R_1 = \langle 2AB_1^{-1} \rangle^\max\left\{ \frac{\alpha}{\alpha - 2}, \frac{\alpha}{\alpha - 4} \right\} \) (note that \( B_1 > 0 \)), we have

\[
\left( \frac{1}{2} B_1 \right)^\frac{\alpha}{\alpha - 2} = \lim_{n \to \infty} \| e^{i (t + t_n^i) \partial_x^2} \phi_n \|_{L^\infty(\mathbb{R})} \leq [A^2 R_1]^\frac{\alpha}{\alpha - 2} \lim_{n \to \infty} \| [\zeta_{R_1} * e^{i (t + t_n^i) \partial_x^2}] \phi_n \|_{L^\infty(\mathbb{R})}.
\]

Since \( e^{it\partial_x^2} \) commutes with the convolution with \( \zeta_{R_1} \), by the weak convergence we infer that

\[
\lim_{n \to \infty} \| [\zeta_{R_1} * e^{i (t + t_n^i) \partial_x^2}] \phi_n \|_{L^\infty(\mathbb{R})} = \| [\zeta_{R_1} * e^{i t \partial_x^2}] \psi^1 \|_{L^\infty(\mathbb{R})}.
\]

Moreover, by plancherel’s formula we see that

\[
\| [\zeta_{R_1} * e^{i t \partial_x^2}] \psi^1 \|_{L^\infty(\mathbb{R})} \leq C_\gamma R_1^\frac{1 - 2\gamma}{\alpha} \| \psi^1 \|_{H^\gamma}.
\]

Thus, we obtain

\[
\| \psi^1 \|^2_{H^\gamma} \geq [C_{\gamma}]^{-1} \left( \frac{B_1}{2} \right)^\frac{\alpha}{\alpha - 2} A^{-\frac{\alpha}{\alpha - 2}} R_1^{-\theta},
\]

where \( \theta = \frac{\alpha}{\alpha - 2} + 1 - 2\gamma \).

Next, we obtain the functions \( \psi^j \), for all \( j \geq 2 \) inductively (see, for example [17, Lemma 5.2]). Indeed, we construct a sequence \( \{ t_n^j \}_{n \geq 1} \) and a profile \( \psi^j \) such that

\[
\| \psi^j \|^2_{H^1} \geq \frac{(C_j)^2}{2},
\]

\[
\| \psi^j \|^2_{H^\gamma} \geq [C_{\gamma}]^{-1} \left( \frac{B_j}{2} \right)^\frac{\alpha}{\alpha - 2} A^{-\frac{\alpha}{\alpha - 2}} R_j^{-\theta},
\]

where

\[
C_j := \lim_{n \to \infty} \sup_{n \to \infty} \| [e^{i t \partial_x^2} W_n^j] \|_{L^\infty(\mathbb{R})},
\]

\[
B_j := \lim_{n \to \infty} \sup_{n \to \infty} \| [e^{i t \partial_x^2} W_n^j] \|_{L^1(\mathbb{R})}.
\]
But then, by (3.7) and (3.8) we infer

$$\sum_{J=1}^{\infty} \frac{(C_J)^2}{2} \leq \lim_{n \to \infty} \sum_{j=1}^{J} \|\psi^j\|_{L^2}^2 \leq \lim_{n \to \infty} \|\phi_n\|_{L^2}^2 \lesssim 1.$$ 

and

$$\sum_{J=1}^{\infty} B_J \frac{2}{R_J^\theta} \lesssim \lim_{n \to \infty} \sum_{j=1}^{J} \|\psi^j\|_{L^{\infty}} \lesssim 1.$$ 

Therefore $C_J \to 0$ and $B_J \to 0$ as $n \to \infty$. Here we have used that $\theta > 0$.

The remainder of the proof is similar to that of [1, Proposition 3.1]. This completes the proof of proposition. \(\square\)

**Scattering criterion.** Suppose that $u(t)$ is a solution of Cauchy problem (1.1) with initial data $u_0 \in H^1(\mathbb{R})$ satisfying (1.5). Notice that by the conservation of energy and assumption (1.5) we see that

$$\sup_{t \in [0,T_+)} K(u(t)) \leq C(E(u_0),Q),$$

which implies that the solution is global (i.e., $T_+ = \infty$). Note also that to get scattering criterion in Theorem 1.1, from Proposition 3.1, it is enough to get

$$\|u(0,\cdot)\|_{L^2(\mathbb{R}^+)} < \infty.$$ 

Let $L > 0$ and $\delta > 0$. For $u(t)$ satisfying

$$\sup_{t \in [0,\infty)} N(u(t))[M(u(t))]^{\sigma_\varepsilon} \leq L, \quad E(u(t))[M(u(t))]^{\sigma_\varepsilon} \leq \delta,$$

with $L < N(Q)[M(Q)]^{\sigma_\varepsilon}$, we define

$$S(L,\delta) := \sup \{ \|u(0,\cdot)\|_{L^p(\mathbb{R}^+)} : \ u(t) \text{ satisfies (3.9)} \}.$$ 

By (2.7) we infer that $E(u_0) \geq 0$. Thus, by interpolation, (2.7) and (3.9) we obtain

$$\|u_0\|_{H^{\sigma_\varepsilon}} \lesssim K(u_0)[M(u_0)]^{\sigma_\varepsilon} \leq \frac{16}{\eta} \delta.$$ 

By inequality above, Propositions 3.1 and 3.2, we have that $S(L,\delta) < \infty$ for $\delta$ small enough.

Suppose by contradiction that Theorem 1.1 fails. Then $S(L,\delta) = \infty$ for some $\delta < \infty$. As $S(L,\delta) < \infty$ for $\delta \ll 1$, using the monontonicity of $S(L,\delta)$, there exists a critical level $0 < \delta_c(L) < \infty$ such that

$$\delta_c(L) := \delta_c(L) = \sup \{ \delta : S(L,\delta) < \infty \} = \inf \{ \delta : S(L,\delta) = \infty \}.$$ 

Notice that $S(L,\delta_c) = \infty$. By definition, this implies that there exists a sequence of initial data $u_n(0)$ such that

$$\sup_{t \in [0,\infty)} N(u_n(t))[M(u_n(t))]^{\sigma_\varepsilon} \leq L, \quad E(u_n(t))[M(u_n(t))]^{\sigma_\varepsilon} \searrow \delta_c,$$

$$\|u_n(0,\cdot)\|_{L^p(\mathbb{R}^+)} = \infty \quad \text{for all } n,$$

where $u_n(t)$ is a global solution (i.e., $T_+ = \infty$) to (1.1) with initial data $u_n(0)$. Now, our goal is to prove the existence of a critical element $u_c(t)$ such that $E(u_c(t))[M(u_c(t))]^{\sigma_\varepsilon} = \delta_c$, $\sup_{t \in [0,\infty)} N(u_c(t))[M(u_c(t))]^{\sigma_\varepsilon} = L$ and

$$\|u_c(0,\cdot)\|_{L^p(\mathbb{R}^+)} = \infty.$$ 

More precisely, we have the following result.
Proposition 3.7 (Existence and compactness of critical element). There exists $u_c(0) \in H^1(\mathbb{R})$ such that if $u_c(t)$ is the corresponding solution of (1.1) with initial data $u_c(0)$, then $u_c(t)$ satisfies
\begin{align}
M(u_c(t)) &= 1, \quad \sup_{t \in [0, \infty)} N(u_c(t)) = L, \\
E(u_c(t)) &= \delta_c, \quad \|u_c(0, \cdot)\|_{L^1_t L^\infty_x(\mathbb{R}^+)} = \infty
\end{align}
and $\Omega := \{u_c(t) : t \geq 0\}$ is precompact in $H^1(\mathbb{R})$.

Proof. We observe that the quantities $E(u)(M(u))^{1/2}$ and $\sup_{t \geq 0} N(u(t))(M(u(t)))^{1/2}$ are both invariant under the scaling (1.2). Thus, since the equation (1.1) also is invariant under (1.2), we can assume that
\begin{align}
M(u_n(0)) &= 1, \quad \sup_{t \in [0, \infty)} N(u_n(t)) \leq L, \\
E(u_n(0)) &= \delta_c, \quad \|u_n(0, \cdot)\|_{L^1_t L^\infty_x(\mathbb{R}^+)} = \infty,
\end{align}
and we may apply the profile decomposition to $\varphi_n := u_n(0)$. Therefore, by Proposition 3.6 we write
\begin{equation}
\varphi_n = \sum_{j=1}^{J} e^{-it_j \partial_x^2} \psi_j + W_n^J
\end{equation}
for all $n \geq 1$, where the sequences satisfy properties (3.1)-(3.4). Moreover, we may assume that either $t^j_n = 0$ or $|t^j_n| \to \infty$.

Define the nonlinear profile $v^j$ associated to $\psi_j$ in the following way:
(i) If $t^j_n = 0$, then $v^j$ is the maximal solution to equation (1.1) with initial data $v^j(0) = \psi_j$;
(ii) If $t^j_n \to \infty$, then $v^j$ is the maximal solution to equation (1.1) that scatters backward in time to $e^{it \partial_x^2} \psi_j$, which existence is guaranteed by Remark 3.3. In particular,
\begin{equation}
\lim_{n \to \infty} \|v^j(-t^j_n) - e^{-it \partial_x^2} \psi_j\|_{H^1} = 0.
\end{equation}
(iii) Similarly, if $t^j_n \to -\infty$, then $v^j$ is the maximal solution to equation (1.1) that scatters forward in time to $e^{it \partial_x^2} \psi_j$. In particular,
\begin{equation}
\lim_{n \to \infty} \|v^j(-t^j_n) - e^{-it \partial_x^2} \psi_j\|_{H^1} = 0.
\end{equation}
Let $v^j_n(t) := v^j(t - t^j_n)$. This is still solution of equation (1.1) and satisfies
\begin{equation}
\lim_{n \to \infty} \|v^j_n(0) - e^{-it \partial_x^2} \psi_j\|_{H^1} = 0.
\end{equation}
Now, we rewrite (3.14) as
\begin{equation}
\varphi_n(x) = \sum_{j=1}^{J} v^j_n(x, 0) + \tilde{W}_n^J(x),
\end{equation}
for $n \geq 1$, where
\begin{equation}
\tilde{W}_n^J(x) = \sum_{j=1}^{J} \left| e^{-it \partial_x^2} \psi_j(x) - v^j_n(x, 0) \right| + W_n^J(x).
\end{equation}
From Remark 2.2, Sobolev inequality and Proposition 2.1 (i) we infer that
\[
\|e^{it\partial_x^2}W_n^J(0)\|_{L^p_t(L^q_x)} \leq \sum_{j=1}^J \|e^{-\dot{\theta}_n^T \partial_x^2} \psi_j - v_n^J(0, \cdot)\|_{L^p_t(L^q_x)} + \|e^{it\partial_x^2}W_n^J(0)\|_{L^p_t(L^q_x)}
\]
\[
\leq C \sum_{j=1}^J \|e^{-\dot{\theta}_n^T \partial_x^2} \psi_j - v_n^J(0)\|_{L^p_t(L^q_x)} + \|e^{it\partial_x^2}W_n^J(0)\|_{L^p_t(L^q_x)}.
\]
Using (3.2) and (3.15) we get
\[
\lim_{J \to \infty} \left( \lim_{n \to \infty} \|e^{it\partial_x^2}W_n^J(0)\|_{L^p_t(L^q_x)} \right) = 0.
\]
Note that by (3.3), there exists \(J_* \in \mathbb{N}\) such that \(\|\psi_j\|_{H^1} \leq \delta_{sd}\) for \(j \geq J_*\). Therefore, from (3.15) and using Propositions 3.2 and 3.1 we infer that for \(j \geq J_*\) the solutions \(v_n^J(t)\) to (1.1) are global and
\[
\|v_n^J(0, \cdot)\|_{L^p_t(L^q_x)} + \|v_n^J(0, \cdot)\|_{L^p_t(L^q_x)} \lesssim \|\psi\|_{H^1}
\]
for all \(j \geq J_*\). We have the following

**Claim 1.** There exists \(1 \leq j_0 < J_*\) such that
\[
\limsup_{n \to \infty} \|v_n^{j_0}(0, \cdot)\|_{L^p_t(L^q_x)} = \infty.
\]

Indeed, assume by contradiction that for all \(1 \leq j < J_*\),
\[
\limsup_{n \to \infty} \|v_n^j(0, \cdot)\|_{L^p_t(L^q_x)} < \infty.
\]
Then, there exists \(C > 0\) such that for \(n\) big enough \(\|v_n^j(0, \cdot)\|_{L^p_t(L^q_x)} \leq C\). With the same argument developed in [1, Section 4] and using the Lemma 3.4 we can find that for \(n\) sufficiently large \(\|u_n(0, \cdot)\|_{L^p_t(L^q_x)} < \infty\), which is a contradiction. This proves the Claim 1.

Next, by reordering, we can choose \(1 \leq J_1^* \leq J_*\) such that
\[
\limsup_{n \to \infty} \|v_n^j(0, \cdot)\|_{L^p_t(L^q_x)} = \infty \quad \text{for} \quad 1 \leq j \leq J_1^*;
\]
\[
\limsup_{n \to \infty} \|v_n^j(0, \cdot)\|_{L^p_t(L^q_x)} < \infty \quad \text{for} \quad j > J_1^*.
\]

Following [20, Proposition 5.6], for each \(m, n \in \mathbb{N}\), we define \(j(m, n) \in \{1, 2, \ldots, J_1^*\}\) and a compact interval \(I_m^n\) of the form \([0, T]\) such that
\[
\sup_{1 \leq j \leq J_1^*} \|v_n^j(0, \cdot)\|_{L^p_t(I_m^n)} = \|v_n^{j(m, n)}(0, \cdot)\|_{L^p_t(I_m^n)} = \|M(v_n^{j(m, n)}(0, \cdot))\|_{L^p_t(I_m^n)} = m.
\]
Thus, since \(J_1^* < \infty\), using the pigeonhole principle, there is a \(1 \leq j_1 \leq J_1^*\) such that \(j(m, n) = j_1\) for infinitely many \(m\) and for infinitely many \(n\). There are two scenarios to consider.

**Scenario 1:** More than one \(\psi_j \neq 0\). By (3.3) and (3.4) we infer that
\[
E(v_n^{j_1}(t))[M(v_n^{j_1}(t))]^{\sigma_c} < \delta_c,
\]
where \(j_1\) is defined above. Here we recall that \(\delta_c\) is the critical level. Using (3.21) and (3.10), by definition of \(v_n^{j_1}(t)\) we have
\[
\limsup_{m \to \infty} \limsup_{n \to \infty} \sup_{t \in I_m^n} N(v_n^{j_1}(t))[M(v_n^{j_1}(t))]^{\sigma_c} \geq L.
\]
By reordering we can choose \(j_1 = 1\). Now, we need the following result. We denote NLS \(t\) \(\phi\) the solution to equation (1.1) with initial data \(\phi\). We recall that \(\varphi_n = u_n(0)\).
Lemma 3.8. Let $T > 0$ fixed and assume that the solution $u_n(t) = \text{NLS}(t)\varphi_n$ exists up to time $T$ for all $n \geq 1$, and
\[
\limsup_{n \to \infty} \sup_{t \in [0,T]} K(u_n(t)) < \infty. \tag{3.23}
\]
Then for all $1 \leq j \leq J$, the nonlinear profiles $v_n^j(t)$ exist up to time $T$ and
\[
K(u_n(t)) = \sum_{j=1}^{J} K(v_n^j(t)) + K(\tilde{W}_n^J(t)) + o_n(1), \tag{3.24}
\]
\[
N(u_n(t)) = \sum_{j=1}^{J} N(v_n^j(t)) + N(\tilde{W}_n^J(t)) + o_n(1) \tag{3.25}
\]
for $t \in [0, T]$, where $o_n \to 0$ as $n \to \infty$ uniformly on $0 \leq t \leq T$. Here $\tilde{W}_n^J(t) = \text{NLS}(t)\tilde{W}_n^J$, where $\tilde{W}_n^J$ is the remainder given in (3.16).

Let us assume, for a moment, that Lemma 3.8 is true. We recall that
\[
1 = M(u_n(t)) \geq M(v_n^1(t)).
\]
Since more that one $\psi^j \neq 0$, combining (3.13), (3.22) and (3.25) we obtain
\[
L \geq \limsup_{n \to \infty} \sup_{t \in [0,\infty)} N(u_n(t))[M(u_n(t))]^{\sigma_0}
\]
\[
\geq \lim_{J \to \infty} \limsup_{n \to \infty} \sup_{t \in [0,\infty)} \sum_{j=1}^{J} N(v_n^j(t))[M(v_n^j(t))]^{\sigma_0}
\]
\[
> \limsup_{n \to \infty} \sup_{t \in [0,\infty)} N(v_n^1(t))[M(v_n^1(t))]^{\sigma_0} \geq L,
\]
which is a contradiction.

**Scenario 2:** $\psi^1 \neq 0$ and $\psi^j = 0$ for every $j \geq 2$. In this case we have
\[
\phi_n = e^{-i t_n^j \partial_x^2} v^1 + W_n^1, \quad \lim_{n \to \infty} \|[e^{it\partial_x^2} W_n^1](0)\|_{L_t^\infty(\mathbb{R})} = 0.
\]
If $t_n^1 \to -\infty$, we obtain
\[
\|[e^{it\partial_x^2} \varphi_n](0)\|_{L_t^\infty(\mathbb{R}^+)} \leq \|[e^{it\partial_x^2} \psi^1](0)\|_{L_t^\infty([-t_n^1, \infty))} + \|[e^{it\partial_x^2} W_n^1](0)\|_{L_t^\infty(\mathbb{R})} \to 0
\]
as $n \to \infty$. By inequality above, using a standard continuity argument and Proposition 3.2 we may shows that $\|u_n(0, \cdot)\|_{L_t^\infty(\mathbb{R}^+)} < \infty$ for $n$ big enough, which is a contradiction. A similar claim is valid for $t_n^2 \to \infty$. Thus, by (3.16) we can write
\[
\phi_n(x) = v^1(x, 0) + \tilde{W}_n^1(x).
\]
Notice that by Claim 1 above $\|v^1(0, \cdot)\|_{L_t^\infty(\mathbb{R}^+)} = \infty$. Moreover, from (3.3) and (3.4) we can show that
\[
M(v^1(t)) \leq 1, \quad E(v^1(t)) \leq \delta_c.
\]
Notice also that from (3.22) we have that
\[
\sup_{t \in [0,\infty)} N(v^1(t))[M(v^1(t))]^{\sigma_0} = \limsup_{m \to \infty} \sup_{t \in I_m^n} N(v^1(t))[M(v^1(t))]^{\sigma_0} \geq L,
\]
and, by (3.25),
\[
L \geq \limsup_{n \to \infty} \sup_{t \in [0,\infty)} N(u_n(t))[M(u_n(t))]^{\sigma_0}
\]
\[
\geq \limsup_{m \to \infty} \sup_{t \in I_m^n} N(v^1(t))[M(v^1(t))]^{\sigma_0} \geq L.
\]
Hence, we get
\[
\sup_{t \in [0,\infty)} N(v^1(t))[M(v^1(t))]^{\sigma_0} = L. \tag{3.26}
\]
Next, we claim that
\[ M(v^1(t)) = 1. \]
Suppose, by contradiction that \( M(v^1(t)) < 1 \), then
\[ E(v^1(t)) = E(v^1(t)) < \delta_c. \]
Hence, by (3.26) and definition of \( \delta_c \) (see (3.10)), we obtain \( \|v^j(0, \cdot)\|_{L^p_t(\mathbb{R}^+)} < \infty \), which is a contradiction. Similarly, we can may show that \( M(v^1(t)) \) be the solution to equation (1.1) with initial data \( u_c(0) = v^1(x, 0) \). Then \( u_c(t) \) is global \((T_+ = \infty)\) and satisfies
\[
M(u_c(t)) = M(v^1(t)) = 1, \\
E(u_c(t)) = E(v^1(t)) = \delta_c.
\]
and
\[
\sup_{t \in [0, \infty)} N(u_c(t)) = \sup_{t \in [0, \infty)} N(v^1(t)) = L.
\]
Moreover, we have that \( \|u_c(0, \cdot)\|_{L^p_t(\mathbb{R}^+)} = \infty \). Finally, we consider the precompactness of \( \Omega \). Indeed, notice that for any time sequence \( \{t_n\}_{n \geq 1} \), the sequence \( u_c(t_n) \) is uniformly bound in \( H^1(\mathbb{R}) \) and satisfies
\[
M(u_c(t_n)) = 1, \\
E(u_c(t_n)) = \delta_c \text{ and } \|u_c(0, \cdot)\|_{L^p_t([t_n, \infty))} = \infty.
\]
Hence, regarding \( u_c(t_n) \) as the foregoing \( \phi_n \), and using an argument similar to the above, we can find that there exists sequences \( \{\tau^*_n\}_{n \geq 1} \subset \mathbb{R}, \{W^1_{n}\}_{n \geq 1} \subset H^1(\mathbb{R}) \) and \( \xi^1 \in H^1(\mathbb{R}) \) such that
\[ u_c(t_n) = e^{-ir^1 \partial^2 t} \xi^1 + W^1_n, \tag{3.27} \]
with
\[ M(\xi^1) = 1, \quad \lim_{n \to \infty} M(W^1_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} E(W^1_n) = 0. \]
In particular, by inequality (2.7) we infer that
\[ \lim_{n \to \infty} \|W^1_n\|_{H^1} \leq \lim_{n \to \infty} \left[ \frac{16}{\eta} E(W^1_n) + M(W^1_n) \right] = 0. \tag{3.28} \]
Moreover, arguing as Scenario 2, we obtain that \( \tau^*_n \to \tau_* < \infty \) as \( n \to \infty \). Hence, putting together (3.27) and (3.28) we see that \( u_c(t_n) \) converges in \( H^1(\mathbb{R}) \). This proves the proposition. \( \square \)

**Proof of Lemma 3.8** We follow the ideas of the proof of [15, Lemma 3.9] and [6, Lemma 3.2]. First of all notice that, by definition of the intervals \( I_n^m \) (see (3.21)), there exists \( m_0 \in \mathbb{N} \) sufficiently large such that \( [0, T] \subset I_{m_0}^m \) for infinitely many \( n \). We recall that there exists \( J_+ \in \mathbb{N} \) such that
\[ \|v_n^j(0, \cdot)\|_{L^2_t(\mathbb{R}^+)} + \|v_n^j(0, \cdot)\|_{L^p_t(\mathbb{R}^+)} \lesssim \|\phi^j\|_{H^1} \tag{3.29} \]
for all \( j \geq J_+ \). In particular, it follows that for \( j \geq J_+ \)
\[ \|v_n^j(0, \cdot)\|_{L^2_t([0, T])} + \|v_n^j(0, \cdot)\|_{L^p_t([0, T])} \lesssim \|\phi^j\|_{H^1}. \tag{3.30} \]
Moreover, using (3.20) we infer that
\[ \limsup_{n \to \infty} \|v_n^j(0, \cdot)\|_{L^2_t([0, T])} < \infty, \tag{3.31} \]
for all \( J^+_1 < j < J_* \), where \( J^+_1 \) is given in (3.20). Finally, by (3.21) there exists a constant \( C = C(T) \) (possibly depending on time \( T \)) such that
\[ \|v_n^j(0, \cdot)\|_{L^2_t([0, T])} \leq C(T) \quad \text{for } n \text{ sufficiently large}, \tag{3.32} \]
for all \( 1 \leq j < J^+_1 \).

Now, by reordering, we can choose \( 0 \leq J_1 \leq J_+ \) such that
We treat only the case $J_1 = 1$. The case $J_1 = 0$ is easier. Now we set

$$R := \max \left\{ 1, \limsup_{n \to \infty} \sup_{t \in [0, T]} K(u_n(t)) \right\} < \infty.$$  

Moreover, $T_*$ denote the maximal forward time such that the solution $v^1(t)$ satisfies

$$\sup_{t \in [0, T_1]} K(v^1(t)) \leq 2R.$$  

Notice that we may assume that $[0, T_1] \subset I_{\text{no}}$ for infinitely many $n$.

Using the conservation of mass and (3.3),

$$M(v^2_n(t)) = M(v^2(-t_0^n)) \lesssim 1 + M(\psi^j).$$

By inequality above (for $j = 1$) and the Gagliardo-Nirenberg inequality (2.2) we get

$$\|v^1(0, \cdot)\|_{L^\infty([0, T_1])} \leq \sup_{t \in [0, T_1]} [M(v^1(t))K(u^1(t))]^{\frac{1}{p}} \lesssim R^\frac{1}{p}. \quad (3.33)$$

Here we recall that $t_1^n = 0$ for all $n$. On the other hand, if $t_0^n \to -\infty$, we obtain

$$\|v^2_0(0, \cdot)\|_{L^\infty([0, T_1])} \lesssim \|v^2_n(t) - e^{i(t-t_0^n)\partial^2_x} \psi^j\|_{L^\infty([0, T_1], H^1(\mathbb{R}))} + \|e^{i(t-t_0^n)\partial^2_x} \psi^j\|_{L^1(0, T_1), H^1(\mathbb{R})} \lesssim \|v^2_n(t) - e^{i(t-t_0^n)\partial_x} \psi\|_{L^1(0, T_1), H^1(\mathbb{R})} + 1 \lesssim 1,$$

for $n$ sufficiently large. A similar statement is valid for $t_0^n \to \infty$. Therefore, for each $2 \leq j < J_*$ we get the estimate

$$\limsup_{n \to \infty} \|v^2_n(0, \cdot)\|_{L^\infty([0, T_1])} < \infty. \quad (3.34)$$

Combining (3.31), (3.32), (3.33) and (3.34) we have that for each $1 \leq j < J_*$

$$\|v^j(0, \cdot)\|_{L^2([0, T_1])} \to L^\infty(0, T_1]) \leq C(R, T_*) \quad (3.35)$$

for $n$ sufficiently large.

**Step 1.** We show that for all $t \in [0, T_1]$

$$N(u_n(t)) = \sum_{j=1}^J N(v^j_n(t)) + N(\tilde{W}^j_n(t)) + o_{J, n}(1) \quad (3.36)$$

where $o_{J, n}(1) \to as n \to \infty$. Indeed, we write

$$u^j_n(x, t) = \sum_{j=1}^J v^j_n(x, t) + \tilde{W}^j_n(x, t).$$

Notice that $u^j_n$ satisfies (3.16)

$$\begin{cases}
  i\partial_t u^j + \partial_x^2 u^j + \delta(x)|u|^p u^j = \delta(x)e^j_n, & x \in \mathbb{R}, \ t \in \mathbb{R} \\
  u^j_n(0, x) = \varphi_n,
\end{cases}$$

where

$$e^j_n = \left[ \sum_{j=1}^J F(v^j_n) - F \left( \sum_{j=1}^J v^j_n \right) \right] + [(F(u^j_n) - F(\tilde{W}^j_n)) - F(u^j_n - \tilde{W}^j_n)].$$
with $F(z) := |z|^{p-1}z$. We want to apply Proposition 3.5. We begin by estimating $\|u_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))}$. Indeed, by Proposition 2.1, Duhamel formula and (3.18) we infer that

$$\lim_{J \to \infty} \lim_{n \to \infty} \|\tilde{W}_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))} = 0. \tag{3.37}$$

Then, by (3.30) and (3.35) we obtain that there exists a constant $C > 0$ such that for $J < J_*$,

$$\limsup_{n \to \infty} \|u_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))} \lesssim 1.$$ 

Moreover, for $J \geq J_*$, and using (3.30) and (3.3) we infer that

$$\limsup_{n \to \infty} \|u_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))} \lesssim 1, \tag{3.38}$$

uniformly in $J$. On the other hand, we claim that

$$\lim_{J \to \infty} \lim_{n \to \infty} \|v_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))} = 0. \tag{3.39}$$

Indeed, by (3.37) and (3.38) we have

$$\lim_{J \to \infty} \lim_{n \to \infty} \|F(u_n^J - F(\tilde{W}_n^J)) - F(u_n^J - \tilde{W}_n^J)(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))}$$

$$\lesssim \lim_{J \to \infty} \lim_{n \to \infty} \|\tilde{W}_n^{J-1} + |\tilde{W}_n^J|^p(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))}$$

$$\lesssim \lim_{J \to \infty} \lim_{n \to \infty} \|\tilde{W}_n^{J-1} + |\tilde{W}_n^J|^p(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))}$$

$$= 0.$$ 

Here we have used that

$$\|\tilde{W}_n^{J-1} + |\tilde{W}_n^J|^p(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))} \leq \|u_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))}^{p-1} \|\tilde{W}_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))}.$$ 

In addition, using the inequality

$$\left\| \sum_{j=1}^J F(v_n^J) - F\left(\sum_{j=1}^J v_n^J\right) \right\|_{L^p([0,T]\cap L^\infty([0,T]))}$$

$$\lesssim \sum_{j \neq k} \|v_n^J\|_{L^p([0,T]\cap L^\infty([0,T]))}$$

together with the orthogonality (3.1), one can easily show (see proof of Theorem 1.4 in [1]) that

$$\lim_{J \to \infty} \lim_{n \to \infty} \left\| \sum_{j=1}^J F(v_n^J) - F\left(\sum_{j=1}^J v_n^J\right) \right\|_{L^p([0,T]\cap L^\infty([0,T]))} = 0.$$ 

This proves the claim (3.39). Thus, Lemma 3.4 implies

$$\lim_{J \to \infty} \lim_{n \to \infty} \|u_n(0, \cdot) - u_n^J(0, \cdot)\|_{L^p([0,T]\cap L^\infty([0,T]))} = 0. \tag{3.40}$$

Then, from (3.40), we can repeat the same argument developed in [1, Proposition 3.1] to obtain

$$N(u_n(t)) = \sum_{j=1}^J N(v_n^j(t)) + N(\tilde{W}_n^J(t)) + o_{n,J}(1), \quad \text{for all } t \in [0, T_*]. \tag{3.41}$$

**Step 2.** Conclusion. First, notice that putting together (3.4) and (3.15), by the conservation the energy we get

$$E(u_n(t)) = \sum_{j=1}^J E(v_n^j(t)) + E(\tilde{W}_n^J(t)) + o_{n,J}(1) \quad \text{for all } t \in [0, T_*]. \tag{3.42}$$
Then, combining (3.41) and (3.42) we get (3.24). Finally, we also note that $T \leq T_*$. Indeed, by contradiction assume that $T_* < T$. Since $t_n^1 = 0$ for all $n$, by (3.24) it follows that
\[
\sup_{t \in [0,T_1]} K(u^1(t)) \leq \sup_{t \in [0,T_2]} K(u_n(t)) \leq \sup_{t \in [0,T]} K(u_n(t)) \leq R,
\]
which is a contradiction with the choice of $T_*$. This proves that $T \leq T_*$. \hfill \Box

**Proof of Theorem 1.1.** Assume that $\delta_e < \infty$, then the Proposition 3.7 implies that there exists a critical element $u_{e,0} \neq 0$ such that the corresponding solution to equation (1.1) verifies that the set $\{u(t) : t \geq 0\} \subset H^1(\mathbb{R})$ is precompact in $H^1(\mathbb{R})$. Now, by inequality (2.6) and using the same argument as in [17, Theorem 6.1] we obtain $u_{e,0} = 0$, which is a contradiction. This proves that $\delta_e = +\infty$, which implies Theorem 1.1. \hfill \Box

4. SCATTERING RESULTS BELOW AND ABOVE THE GROUND STATES THRESHOLD

In this section we show the Theorems 1.2 and 1.5.

**Proof of Theorem 1.2.** We proceed in two steps.

*Step 1.* We show that there exists $\theta = \theta(Q,u_0) > 0$ small such that
\[
K(u(t))[M(u(t))]^{\sigma_e} \leq (1 - \theta)K(Q)[M(Q)]^{\sigma_e} \quad \text{for all} \ t \in [0,T_+). \tag{4.1}
\]
Indeed, from the Gagliardo-Nirenberg inequality (2.2) we obtain
\[
E(u(t))[M(u(t))]^{\sigma_e} = \frac{1}{2}K(u(t))[M(u(t))]^{\sigma_e} - \frac{1}{p+1}N(u(t))[M(u(t))]^{\sigma_e} \\
\geq \frac{1}{2}K(u(t))[M(u(t))]^{\sigma_e} - \frac{1}{p+1}[K(u(t))]^{\sigma_e + \frac{p+1}{\sigma_e}}[M(u(t))]^{\sigma_e + \frac{p+1}{\sigma_e}} \\
= \frac{1}{2}K(u(t))[M(u(t))]^{\sigma_e} - \frac{1}{p+1}(K(u(t))[M(u(t))]^{\sigma_e})^{\frac{p+1}{\sigma_e}} \\
= \Phi(K(u(t))[M(u(t))]^{\sigma_e}), \tag{4.2}
\]
where
\[
\Phi(x) := \frac{1}{2}x - \frac{1}{p+1}x^{\frac{p+1}{\sigma_e}}.
\]
Combining the identities (2.3) and (2.5) we see that
\[
\Phi(K(Q)[M(Q)]^{\sigma_e}) = \frac{1}{2}K(Q)[M(Q)]^{\sigma_e} - \frac{2}{p+1}N(Q)[M(Q)]^{\sigma_e} \\
= \frac{(p-3)}{2(p+1)}K(Q)[M(Q)]^{\sigma_e} = E(Q)[M(Q)]^{\sigma_e}.
\]
Then, by assumption (1.6) and conservation laws, \[
\Phi(K(u(t))[M(u(t))]^{\sigma_e}) \leq E(u_0)[M(u_0)]^{\sigma_e} < E(Q)[M(Q)]^{\sigma_e} = \Phi(K(Q)[M(Q)]^{\sigma_e}),
\]
for every $t \in [0,T_+)$. Thus, by continuity and assumption (1.7) we infer that
\[
K(u(t))[M(u(t))]^{\sigma_e} < K(Q)[M(Q)]^{\sigma_e} \quad \text{for all} \ t \in [0,T_+). \tag{4.3}
\]
Again, by assumption (1.6) we have that there exists $a > 0$ small such that
\[
E(u_0)[M(u_0)]^{\sigma_e} < (1-a)E(Q)[M(Q)]^{\sigma_e}. \tag{4.4}
\]
In addition, from identities (2.4) and (2.5) we get
\[
E(Q)[M(Q)]^{\sigma_e} = \frac{(p-3)}{2(p+1)}K(Q)[M(Q)]^{\sigma_e} = \frac{(p-3)}{4(p+1)}(K(Q)[M(Q)]^{\sigma_e})^{\frac{p+1}{4}}.
\]
Thus, by equation above and inequality (4.2) we obtain
\[
\left( \frac{p + 1}{p - 3} \right) \left( \frac{K(u(t))[M(u(t))]^{\sigma_e}}{K(Q)[M(Q)]^{\sigma_e}} \right) - \frac{4}{p - 3} \left( \frac{K(u(t))[M(u(t))]^{\sigma_e}}{K(Q)[M(Q)]^{\sigma_e}} \right)^{\frac{p+1}{p-3}} \leq 1 - a .
\]
Now we set
\[
\Psi(x) = \left( \frac{p + 1}{p - 3} \right)^\gamma - \frac{4}{p - 3} x^{\frac{p+1}{p-3}} \text{ for all } x \in (0, 1).
\]
Notice that \( \Psi(0) = 0, \Psi(1) = 1 \) and \( \Psi'(x) > 0 \) for all \( x \in (0, 1) \). Since
\[
\Psi \left( \frac{K(u(t))[M(u(t))]^{\sigma_e}}{K(Q)[M(Q)]^{\sigma_e}} \right) \leq 1 - a ,
\]
it follows from (4.3) that there exists \( \theta = \theta(a) > 0 \) such that \( x < 1 - \theta \), which implies (4.1).

**Step 2.** Conclusion. By Gagliardo-Nirenberg inequality (2.2) and (4.1) we deduce
\[
N(u(t))[M(u(t))]^{\sigma_e} \leq [K(u(t))]^{\frac{p+1}{p-1}}[M(u(t))]^{\frac{p+1}{p-1}}[M(u(t))]^{\sigma_e}
\]
\[
= (K(u(t))[M(u(t))]^{\sigma_e})^{\frac{p+1}{p-1}}
\]
\[
< (1 - \theta)^{\frac{p+1}{p-1}}(K(Q)[M(Q)]^{\sigma_e})^{\frac{p+1}{p-1}} \quad \text{for all } t \in [0, T) .
\]
Thus, since
\[
(K(Q)[M(Q)]^{\sigma_e})^{\frac{p+1}{p-1}} = (K(Q)[M(Q)])^{\frac{p+1}{p-1}}[M(Q)]^{\sigma_e} = N(Q)[M(Q)]^{\sigma_e} ,
\]
we find that
\[
N(u(t))[M(u(t))]^{\sigma_e} < (1 - \theta)^{\frac{p+1}{p-1}}N(Q)[M(Q)]^{\sigma_e} \quad \text{for all } t \in [0, T) ,
\]
which implies (1.5). So the scattering criterion, Theorem 1.1, tell us that \( u(t) \) scatters. This completes the proof of Theorem 1.2. \( \square \)

Before establishing the following result, we recall that the virial quantity satisfies the following identities
\[
V'(t) = 4 \text{Im} \int_R x \partial_x u(x, t) \pi(x, t) dx
\]
and
\[
V''(t) = 8(K(u(t)) - \frac{1}{2} N(u(t)) \quad (4.5)
\]
\[
= 4(p + 1)E(u(t)) - 2(p - 3)K(u(t)) \quad (4.6)
\]
\[
= 16E(u(t)) + 4 \frac{3 - p}{p + 1} N(u(t)). \quad (4.7)
\]
We use the following Cauchy-Schwartz inequality.

**Lemma 4.1.** Let \( f \in H^1(\mathbb{R}) \) such that \(|x|f \in L^2(\mathbb{R}) \). Then
\[
\left( \text{Im} \int_R x \partial_x \overline{f} dx \right)^2 \leq \int_R |x|^2 |f|^2 dx \left[ K(f) - \frac{[N(f)]^{\frac{p+1}{p-1}}}{M(f)} \right]. \quad (4.8)
\]

**Proof.** We follow a similar argument as in [10, Lemma 2.1]. Given \( \lambda \in \mathbb{R} \), a simple calculation shows that
\[
K(e^{\lambda |x|^2} f) = 4\lambda^2 \int_R |x|^2 |f|^2 dx + 4\lambda \text{Im} \int_R x \partial_x \overline{f} dx + K(f).
\]
Moreover, \( M(e^{i\lambda|x|^2}f) = M(f) \) and \( N(e^{i\lambda|x|^2}f) = N(f) \). Now we define the quadratic polynomial in \( \lambda \),

\[
\Phi(\lambda) = K(e^{i\lambda|x|^2}) - \frac{[N(e^{i\lambda|x|^2})]^{\frac{1}{p+1}}}{M(e^{i\lambda|x|^2})} \\
= 4\lambda^2 \int_\mathbb{R} |x|^2 |f|^2 dx + 4\lambda \Im \int_\mathbb{R} x \partial_x \bar{f} f dx + \left( K(f) - \frac{[N(f)]^{\frac{1}{p+1}}}{M(f)} \right).
\]

By using the Gagliardo-Nirenberg inequality (2.2) we infer that \( \Phi(\lambda) \geq 0 \), this implies that the discriminant of \( \Phi \) is non-positive, which implies the inequality (4.8).

We are now ready to give the proof of Theorem 1.5.

**Proof of Theorem 1.5.** We adapt here a proof given in [10, Theorem 1.4]. From (4.6) and (4.7) we have that

\[
K(u(t)) = \frac{1}{2(p-3)}(4(p+1)E(u(t)) - V''(t)), \quad N(u(t)) = \frac{(p+1)}{4(p-3)}(16E(u(t)) - V''(t)).
\]

Notice that as \( N(u(t)) \geq 0 \) it follows \( V''(t) \leq 16E(u(t)) \). Moreover, by inequality (4.8) and identities above we get

\[
(V'(t))^2 \leq 16V(t) \left( \frac{1}{2(p-3)}(4(p+1)E(u(t)) - V''(t)) - [M(u(t))]^{-1} \left( \frac{(p+1)}{4(p-3)}(16E(u(t)) - V''(t)) \right)^{\frac{1}{p+1}} \right).
\]

We set

\[
\Phi(x) = \frac{1}{2(p-3)}(4(p+1)E(u_0) - x) - [M(u_0)]^{-1} \left( \frac{(p+1)}{4(p-3)}(16E(u_0) - x) \right)^{\frac{1}{p+1}}, \tag{4.9}
\]

for all \( x \leq 16E(u_0) \). Thus we have that

\[
(z'(t))^2 \leq 4\Phi(V''(t)), \quad \text{where} \quad z(t) := \sqrt{V(t)}. \tag{4.10}
\]

Now notice that

\[
\Phi'(x) = - \frac{1}{2(p-3)} + \frac{[M(u_0)]^{-1}}{(p-3)} \left( \frac{(p+1)}{4(p-3)}(16E(u_0) - x) \right)^{-\frac{p-3}{p+1}}.
\]

Since \( p > 3 \), we deduce that exists a unique point \( x_0 \) such that \( \Phi'(x_0) = 0 \), where \( x_0 \) satisfies

\[
\frac{1}{2(p-3)} = \frac{[M(u_0)]^{-1}}{(p-3)} \left( \frac{(p+1)}{4(p-3)}(16E(u_0) - x_0) \right)^{\frac{p-3}{p+1}}.
\]

At the same time, as \( p > 3 \), we have that \( \Phi(x) \) is decreasing on \( (-\infty, x_0) \) and increasing on the interval \( (x_0, 16E(u_0)) \). Moreover, by equation above we infer that

\[
[M(u_0)]^{-1} = \frac{1}{2} \left( \frac{(p+1)}{4(p-3)}(16E(u_0) - x_0) \right)^{\frac{p-3}{p+1}}. \tag{4.11}
\]
which implies

\[
\Phi(x_0) = \frac{1}{2(p-3)}(4(p+1)E(u_0) - x_0)
- \frac{1}{2} \left( \frac{(p+1)}{4(p-3)}(16E(u_0) - x_0) \right)^{\frac{p-3}{p+1}} \left( \frac{(p+1)}{4(p-3)}(16E(u_0) - x_0) \right)^{\frac{p-3}{p+1}}
= \frac{x_0}{8}.
\]

By (4.11) we obtain

\[
\left( \frac{p+1}{4(p-3)} \right) [M(u_0)]^{\sigma_c} (16E(u_0) - x_0) = 2^{\sigma_c}.
\]

Similarly, from identities (2.4) and (2.5) we get

\[
2 \left( \frac{2(p+1)}{p-3} \right) E(Q)[M(Q)]^{\sigma_c} = 2^{\sigma_c}.
\]

Therefore, combining (4.12) and (4.13) we deduce

\[
16E(Q)[M(Q)]^{\sigma_c} = 1,
\]
equivalently,

\[
\frac{E(u_0)[M(u_0)]^{\sigma_c}}{E(Q)[M(Q)]^{\sigma_c}} \left( 1 - \frac{x_0}{16E(u_0)} \right) = 1.
\]

In particular,

\[
x_0 = 16E(u_0) \left( 1 - \frac{E(Q)[M(Q)]^{\sigma_c}}{E(u_0)[M(u_0)]^{\sigma_c}} \right).
\]

Notice that by assumption (1.14), we infer that \(x_0 \geq 0\). On the other hand, in view of assumption (1.16), (2.4) and (4.7) we get

\[
V''(0) = 16E(u_0) - \frac{4(3-p)}{(p+1)} N(u_0)
> 16E(u_0) - \frac{4(p-3)N(Q)[M(Q)]^{\sigma_c}}{(p+1)[M(u_0)]^{\sigma_c}}
= 16E(u_0) \left( 1 - \frac{E(Q)[M(Q)]^{\sigma_c}}{E(u_0)[M(u_0)]^{\sigma_c}} \right)
= x_0.
\]

Furthermore, the assumption (1.17) means

\[
z'(0) \geq 0
\]

and assumption (1.15) implies that

\[
(z'(0))^2 \geq \frac{x_0}{2} = 4\Phi(x_0).
\]

We prove the theorem in two steps as follows.

**Step 1.** We show that there exists \(\delta_0 > 0\) such that

\[
V''(t) \geq x_0 + \delta_0, \quad \text{for all } t \in [0, T_+).
\]

Indeed, by continuity and (4.16), we infer that there exists \(\delta_1 > 0\) and \(t_0 > 0\) such that

\[
V''(t) > x_0 + \delta_1 \quad \text{for all } t \in [0, t_0).
\]

On the other hand, taking \(t_0\) more smaller if necessary, we may assume that

\[
z'(t_0) > 2\sqrt{\Phi(x_0)}.
\]
Indeed, if \( z'(0) > 2\sqrt{\Phi(x_0)} \), then by continuity we have the result. If \( z'(0) = 2\sqrt{\Phi(x_0)} \), it follows from (4.16) that
\[
z''(0) = \frac{1}{z(0)} \left( \frac{V''(0)}{2} - (z(0))^2 \right) > \frac{1}{z(0)} \left( \frac{x_0}{2} - \frac{x_0}{2} \right) = 0,
\]
therefore, (4.21) holds for some \( t_0 > 0 \). We define \( \varepsilon_0 > 0 \) such that
\[
z'(t_0) = 2\sqrt{\Phi(x_0)} + 2\varepsilon_0. \tag{4.22}
\]
We claim that
\[
z'(t) > 2\sqrt{\Phi(x_0)} + \varepsilon_0 \quad \text{for all } t \geq t_0. \tag{4.23}
\]
Indeed, suppose that the claim (4.23) is false and let
\[t_1 := \inf \left\{ t \geq t_0; z'(t) \leq 2\sqrt{\Phi(x_0)} + \varepsilon_0 \right\}.
\]
By using the continuity of \( z'(t) \), we get
\[
z''(t) \geq 2\sqrt{\Phi(x_0)} + \varepsilon_0, \quad \text{for all } t \in [t_0, t_1] \tag{4.24}
\]
with
\[
z'(t_1) = 2\sqrt{\Phi(x_0)} + \varepsilon_0. \tag{4.25}
\]
By inequality (4.10) and (4.24) we see that
\[
(2\sqrt{\Phi(x_0)} + \varepsilon_0)^2 \leq (z'(t))^2 \leq 4\Phi(V''(t)) \quad \text{for all } t \in [t_0, t_1]. \tag{4.26}
\]
Thus, we deduce \( \Phi(V''(t)) > \Phi(x_0) \) for all \( t \in [t_0, t_1] \), that is, \( V''(t) \neq x_0 \) and by continuity \( V''(t) > x_0 \) for all \( t \in [t_0, t_1] \). Next we show that there exists a universal positive constant \( L \) such that
\[
V''(t) \geq x_0 + \frac{\sqrt{\varepsilon_0}}{L} \quad \text{for all } t \in [t_0, t_1]. \tag{4.27}
\]
We consider two cases:
(i) If \( V''(t) \geq x_0 + 1 \), then for \( L > 0 \) sufficiently large (4.27) holds.
(ii) Assume that \( x_0 < V''(t) < x_0 + 1 \). By the Taylor expansion of \( \Phi \) around of \( x_0 \), we infer that there exists a universal positive constant \( L \) such that
\[
\Phi(x) \leq \Phi(x_0) + r(x - x_0)^2 \quad \text{when } |x - x_0| \leq 1,
\]
which implies, by (4.26),
\[
4\sqrt{\Phi(x_0)}\varepsilon_0 < 4a(V''(t) - x_0)^2,
\]
and taking \( L = \sqrt{\Phi(x_0)} \varepsilon_0 < 4a(V''(t) - x_0)^2 \), (4.27) holds.

Now, we may use (4.23) and (4.27) to obtain
\[
z''(t_1) = \frac{1}{z(t_1)} \left( \frac{V''(t)}{2} - (z(t_1))^2 \right)
\geq \frac{1}{z(t_1)} \left( \frac{x_0}{2} + \frac{\sqrt{\varepsilon_0}}{L} - (2\sqrt{\Phi(x_0)} + \varepsilon_0)^2 \right)
\geq \frac{1}{z(t_1)} \left( \frac{\sqrt{\varepsilon_0}}{L} - 4\varepsilon_0\sqrt{\Phi(x_0)} - \varepsilon_0^2 \right) > 0
\]
when \( \varepsilon_0 \) is small enough, which is a contradiction with (4.25). Thus we obtain the claim (4.23). Consequently, we have
\[
V''(t) \geq x_0 + \frac{\sqrt{\varepsilon_0}}{L}, \quad \text{for all } t \in [t_0, T_+). \tag{4.28}
\]
Hence, using (4.20) and (4.28) we obtain (4.19) with \( \delta_0 = \min \left\{ \frac{\sqrt{\varepsilon_0}}{L}, \delta_1 \right\} \).
Step 2. Conclusion. Combining (4.19), (4.14) and (2.4) we have
\[ N(u(t))[M(u(t))]^{\sigma_c} = \frac{(p+1)}{4(p-3)}(16E(u_0) - V''(t))[M(u_0)]^{\sigma_c} \]
\[ \leq \frac{(p+1)}{4(p-3)}(16E(u_0) - x_0 - \delta_0)[M(u_0)]^{\sigma_c} \]
\[ = \frac{4(p+1)}{(p-3)}E(Q)[M(Q)]^{\sigma_c} - \delta_0 \frac{(p+1)}{4(p-3)}[M(u_0)]^{\sigma_c} \]
\[ = N(Q)[M(Q)]^{\sigma_c} - \delta_0 \frac{(p+1)}{4(p-3)}[M(u_0)]^{\sigma_c} \]
\[ = (1 - \theta)N(Q)[M(Q)]^{\sigma_c} \text{ for all } t \in [0, T_+), \]
with \( \theta = \delta_0 \frac{(p+1)}{4(p-3)}M(u_0)^{\sigma_c} \). Thus, by the scattering criterion in Theorem 1.1, we infer that the solution \( u(t) \) scatters in \( H^1(\mathbb{R}) \) forward in time. \( \square \)

5. Long time dynamics at threshold

In this section we show the Theorem 1.4. We start with the following result.

Lemma 5.1. If \( \{u_n\}_{n \geq 1} \) is a sequence of \( H^1(\mathbb{R}) \) such that
\[ E(u_n) = E(Q), \quad M(u_n) = M(Q) \text{ and } \lim_{n \to \infty} K(u_n) = K(Q), \]
then there exists \( \theta \in \mathbb{R} \) such that, possibly for a subsequence only,
\[ u_n \to e^{i\theta}Q \quad \text{strongly in } H^1(\mathbb{R}) \text{ as } n \to \infty. \]

Proof. Let \( \{u_n\}_{n \geq 1} \subset H^1(\mathbb{R}) \) be a sequence such that \( E(u_n) = E(Q), M(u_n) = M(Q) \) and \( \lim_{n \to \infty} K(u_n) = K(Q) \). We observe that
\[ \lim_{n \to \infty} N(u_n) = N(Q). \quad (5.1) \]
Notice also that the sequence \( \{u_n\}_{n \geq 1} \) is bounded in \( H^1(\mathbb{R}) \). Thus, as \( H^1(\mathbb{R}) \) is reflexive, we infer that there exists \( v \in H^1(\mathbb{R}) \) such that, possibly for a subsequence only, \( u_n \rightharpoonup v \) weakly in \( H^1(\mathbb{R}) \) and \( u_n(x) \to v(x) \) a.e. \( x \in \mathbb{R} \). Moreover, since \( H^1[-1, 1] \hookrightarrow C[-1, 1] \) is compact, it follows that
\[ u_n(0) \to v(0) \quad \text{as } n \to \infty. \quad (5.2) \]
Putting together (5.1) and (5.2) we get
\[ N(v) = \lim_{n \to \infty} N(u_n) = N(Q). \quad (5.3) \]
Thus, from the weak convergent, (5.3) and the Gagliardo-Nirenberg inequality (2.2) we have
\[ N\text{--}
\[ N\text{--}Q = N\text{--}\pi\left(\pi\right)\]
\[ \leq M(v)K(v) \]
\[ \leq \left[ \lim_{n \to \infty} M(u_n) \right] \left[ \lim_{n \to \infty} K(u_n) \right] \]
\[ \leq M(Q)K(Q). \quad (5.4) \]
Therefore, \( N\text{--}Q = M(v)K(v) \), which implies that \( v(x) = e^{i\theta}Q(rx) \) for some \( \theta \), \( \lambda, r \in \mathbb{R} \) (see [16, Proposition 1.3]). Since \( N(Q) = N(v) \) and \( M(Q) = M(v) \) we see that \( \lambda = 1 \) and \( r = 1 \). Finally, by using the fact that \( M(v) = M(Q) = M(u_n) \) and \( K(v) = K(Q) = \lim_{n \to \infty} K(u_n) \), we infer that \( u_n \to v = e^{i\theta}Q \) strongly in \( H^1(\mathbb{R}) \) as \( n \to \infty \). This completes the proof of lemma. \( \square \)
Lemma 5.2. Let $u_0 \in H^1(\mathbb{R})$. Let $u(t)$ be the solution to Cauchy problem \eqref{1.1} defined on the maximal forward time $[0,T_+)$. If
\begin{equation}
\sup_{t \in [0,T_+)} G(u(t)) \leq -\eta,
\end{equation}
for some $\eta > 0$, then the solution $u(t)$ blows up in finite time, i.e., $T_+ < \infty$.

Proof. The proof is essentially given in \cite{16}. For the convenience of the reader, we give the details. Consider the local virial identity,
\begin{equation*}
I(t) = \int_{\mathbb{R}} a(x)|u(x,t)|^2 \, dx,
\end{equation*}
then a calculation leads to
\begin{equation*}
I''(t) = 4 \int_{\mathbb{R}} a''(x)|\partial_x u(x,t)|^2 \, dx - 2a''(0)N(u(t)) - \int_{\mathbb{R}} a''''(x)|u(x,t)|^2 \, dx,
\end{equation*}
for any $a \in C^4(\mathbb{R})$ with $a(0) = a'(0) = a''(0) = 0$. Moreover, we can choose $a \in C^4(\mathbb{R})$ such that $0 \leq a(x) \leq C\varepsilon^{-2}$, $a''(x) \leq 2$, $a''(0) = 0$ and $|a''''(x)| \leq C\varepsilon^2$ for all $x \in \mathbb{R}$; see proof of Theorem 1.4 in \cite{16}. Thus, we obtain that
\begin{equation*}
I''(t) \leq 8K(u(t)) - 4N(u(t)) + C\varepsilon^2 M(u_0) = G(u(t)) + C\varepsilon^2 M(u_0),
\end{equation*}
for $t \in [0,T_+)$. Therefore, the hypothesis \eqref{5.5} implies that
\begin{equation*}
I''(t) \leq -\eta + C\varepsilon^2 M(u_0) \quad \text{for all } t \in [0,T_+).
\end{equation*}
Suppose that $T_+ = \infty$. Choosing $\varepsilon$ sufficiently small, we find $t_0 > 0$ such that $I(t_0) < 0$, which is a contradiction because $I(t) \geq 0$. Thus, we conclude the proof of blow-up. \hfill \Box

We are now ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4. First of all note that since \eqref{1.8} is scale-invariant under the scaling
\begin{equation}
\lambda \overset{\text{def}}{=} \frac{u_0(\lambda x)}{M(u_0)},
\end{equation}
taking $\lambda \overset{\text{def}}{=} \frac{M(Q)}{M(u_0)}$, we may assume that
\begin{equation}
M(u_0) = M(Q), \quad E(u_0) = E(Q).
\end{equation}

(i) If $u_0$ satisfies \eqref{1.9}, then again by the scaling above we have
\begin{equation}
K(u_0) < K(Q).
\end{equation}
We claim that
\begin{equation}
K(u(t)) < K(Q) \quad \text{for all } t \in [0,T_+),
\end{equation}
where $u(t)$ is the corresponding solution to Cauchy problem \eqref{1.1} with initial data $u_0$ defined on the maximal forward time lifespan $[0,T_+)$. Indeed, suppose by contradiction that there exists $t^* \in [0,T_+)$ such that $K(u(t^*)) = K(Q)$. By \eqref{5.7} we infer that
\begin{equation*}
K(u(t^*)) = K(Q), \quad M(u(t^*)) = M(Q), \quad N(u(t^*)) = N(Q).
\end{equation*}
By the same argument as in proof of Lemma 5.1 (see \eqref{5.4}) we obtain that $u(t^*) = e^{it}Q$ for some $\theta \in \mathbb{R}$. Thus, by the uniqueness of the solution for the Cauchy problem \eqref{1.1} we see that $u(x,t) = e^{it}e^{i(t-t')}Q(x)$, which is a contradiction with \eqref{5.8}. This implies that \eqref{5.9} holds. In particular, the solution $u(t)$ is global, i.e., $T_+ = \infty$.

Next we consider two cases:

Case 1. Suppose
\begin{equation*}
\sup_{t \in [0,\infty)} K(u(t)) < K(Q).
\end{equation*}
In this case, we have that there exists \( a > 0 \) small such that \( K(u(t)) < (1 - a)K(Q) \) for all \( t \geq 0 \). Then by the same argument as in proof of Theorem 1.2-Step 2, we can show that there exists \( b > 0 \) small such that

\[
N(u(t))[M(u(t))]^{\sigma_e} < (1 - b)N(Q)[M(Q)]^{\sigma_e}
\]

for all \( t \in [0, T_+] \).

Therefore, by using the scattering criterion, Theorem 1.1, we see that \( u(t) \) scatters in \( H^1(\mathbb{R}) \) forward in the time.

**Case 2.** Suppose

\[
\sup_{t \in [0, \infty)} K(u(t)) = K(Q).
\]

Then, by definition, there exists a sequence of positive times \( t_n \) such that

\[
E(u(t_n)) = E(Q), \quad M(u(t_n)) = M(Q), \quad \lim_{n \to \infty} K(u(t_n)) = K(Q).
\]

Notice that \( t_n \to \infty \) as \( n \to \infty \). Indeed, if \( t_n \to t^* < \infty \), then, by continuity of the flow we have that \( u(t_n) \to u(t^*) \) in \( H^1(\mathbb{R}) \), which implies that \( K(u(t^*)) = K(Q) \), \( M(u(t^*)) = M(Q) \) and \( N(u(t^*)) = N(Q) \). By the same argument as above, there exists \( \theta \in \mathbb{R} \) such that \( u(x,t) = e^{i\theta}e^{i(t-t^*)Q(x)} \), which is impossible because (5.9).

This proves that \( t^* = \infty \). Finally, we may use Lemma 5.1 to obtain, up subsequence,

\[
u(t_n, \cdot) \to e^{i\theta}Q(\cdot) \quad \text{in} \quad H^1(\mathbb{R}) \quad \text{as} \quad n \to \infty,
\]

for some \( \theta \in \mathbb{R} \).

(iii) If \( u_0 \) satisfies (1.11), then by the scaling (5.6) we may suppose

\[
M(u_0) = M(Q), \quad E(u_0) = E(Q), \quad K(u_0) = K(Q).
\]

In particular, it follows that \( N(u_0) = N(Q) \). From (5.4) we obtain that \( N(u_0) = M_{\infty}^{-1}(u_0)K_{\infty}(u_0) \). Therefore, by uniqueness of minimizer we infer that \( u_0(x) = e^{i\theta}Q(x) \) for some \( \theta \in \mathbb{R} \). Finally, by using the uniqueness of the solution for (1.1) we have \( u(x,t) = e^{it\theta}e^{i\theta}Q(x) \). This proves the statement (ii).

(iii) If \( u_0 \) satisfies (1.12), again by the scaling (5.6) we get

\[
K(u_0) > K(Q). \tag{5.10}
\]

By the same argument as in proof of Claim (5.9) we can show that

\[
K(u(t)) > K(Q) \quad \text{for all} \quad t \in [0, T_+). \tag{5.11}
\]

Next we consider two cases:

**Case 1.** Suppose

\[
\sup_{t \in [0, T_+)} K(u(t)) > K(Q).
\]

Then there exists \( \delta > 0 \) such that

\[
K(u(t)) \geq (1 + \delta)K(Q) \quad \text{for all} \quad t \in [0, T_+).
\]

As a consequence, inequality above and (2.3) yield

\[
N(u(t)) = (p + 1) \left( \frac{1}{2}K(u(t)) - E(u_0) \right)
\geq (p + 1) \left( \frac{1 + \delta}{2}K(Q) - E(Q) \right)
= N(Q) + \frac{\delta(p + 1)}{2}K(Q)
= \left( 1 + \frac{\delta(p + 1)}{4} \right)N(Q) \quad \text{for all} \quad t \in [0, T_+).
\]
Consequently, by inequality above and (2.4) we see that

\[ G(M(t))M(u(t)) = 16E(u_0)M(u_0) - \frac{4(p-3)}{(p+1)}N(u(t))M(u_0) \leq 16E(Q)M(Q) - \left(1 + \frac{\delta(p+1)}{4}\right)\left(\frac{4(p-3)}{(p+1)}\right)N(Q)M(Q) = -\frac{\delta(p+1)}{4}N(Q)M(Q), \quad \text{for all } t \in [0,T_+). \]

By using Lemma 5.2 we then obtain that \(T_+ < \infty\) and therefore the solution \(u(t)\) blows-up in finite time.

**Case 2.** Suppose

\[ \sup_{t \in [0,T_+)} K(u(t)) = K(Q). \]

Then there exists a sequence \(t_n \in [0,T_+]\) such that \(K(u(t)) \to K(Q)\) as \(n \to \infty\). We may assume that, possibly for a subsequence only, \(t_n \to t^* \in [0,T_+]\). Now, suppose that \(t^* < T_+ < \infty\), then, by the same argument as above we infer that \(u_0(x) = e^{i\theta}Q(x)\) for some \(\theta \in \mathbb{R}\), which is a contradiction with (5.10). On the other hand, if \(t^* = T_+ < \infty\), then

\[ E(u(t_n)) = E(Q), \quad M(u(t_n)) = M(Q), \quad \underset{n \to \infty}{\lim} K(u(t_n)) = K(Q). \]

Thus, by Lemma 5.1 there exists \(\theta \in \mathbb{R}\) such that, up to a subsequence, \(u(t_n) \to e^{i\theta}Q\) strongly in \(H^1(\mathbb{R})\) as \(n \to \infty\). However, by the blow-up alternative, this is impossible because \(T_+ < \infty\). In conclusion we have that either \(t^* = T_+ = \infty\) or \(t^* = T_+ = \infty\). Again, if \(t^* < T_+ = \infty\), then \(u_0(x) = e^{i\theta}Q(x)\), which is impossible. Therefore, \(t^* = T_+ = \infty\). Hence (1.13) follows from Lemma 5.1. The proof of the theorem is complete.

\[ \square \]

**6. Blow-up and Applications**

In this section we show the Theorem 1.6 and Corollaries 1.7 and 1.8.

**Proof of Theorem 1.6.** First we recall the definition of \(z(t)\), \(\Phi(x)\) and \(x_0\) in (4.10), (4.9) and (4.14), respectively. By hypothesis (1.19), we have

\[ z'(0) = \frac{V''(0)}{2\sqrt{V(0)}} \leq 0. \]

(6.1)

Moreover, the assumption (1.18) is equivalent to

\[ \frac{(p-3)}{4(p+1)}\left(\frac{M(u_0)}{M(Q)}\right)^{\sigma_c}\left(\frac{N(u_0)}{E(Q)}\right) > 1, \]

where we have used (2.4). Thus, combining inequality above with identity (4.14) and (4.7) implies

\[ V''(0) < x_0. \]

(6.2)

Now, by using the assumption (1.15) we get

\[ (z'(0))^2 \geq \frac{x_0}{2} = 4\Phi(x_0). \]

(6.3)

Therefore, using (6.2) and (6.3) we obtain the inequality

\[ z''(0) = \frac{1}{z(0)}\left(\frac{V''(0)}{2} - (z'(0))^2\right) < \frac{1}{z(0)}\left(\frac{x_0}{2} - \frac{x_0}{2}\right) = 0. \]

Next, we show that

\[ z''(t) < 0, \quad \text{for all } t \in [0,T_+). \]

(6.4)
Indeed, suppose by contradiction that for some \( t^* \in [0, T_+) \) we have \( z''(t^*) \geq 0 \). Since \( z''(0) < 0 \), the intermediate value theorem implies that there exists \( t_0 \in (0, T_+) \) such that
\[
z''(t_0) = 0 \quad \text{and} \quad z''(t) < 0, \quad \text{for all } t \in [0, t_0).
\]
Therefore, as \( z'(0) \leq 0 \), by (6.3) we obtain
\[
z'(t) < z'(0) \leq -\sqrt{4\Phi(x_0)}, \quad \text{for all } t \in (0, t_0],
\]
which implies
\[
(z'(t))^2 > 4\Phi(x_0) \quad \text{for all } t \in (0, t_0].
\]
Then using (4.10) we obtain
\[
4\Phi(V''(t)) > 4\Phi(x_0), \quad \text{for all } t \in (0, t_0].
\]
The last inequality combined with (6.2) yields
\[
V''(t_0) < x_0 \quad \text{for all } t \in [0, t_0].
\]
Finally, by using the inequality above and (6.3) we get
\[
z''(t_0) = \frac{1}{z(t_0)} \left( \frac{V''(t_0)}{2} - (z'(t_0))^2 \right) < \frac{1}{z(t_0)} \left( \frac{x_0}{2} - \frac{x_0}{2} \right) = 0,
\]
which is impossible by definition of \( t_0 \). Thus, (6.4) holds. Now, we proceed by contradiction. Suppose that \( T_+ = \infty \). By \( z'(0) \leq 0 \) and (6.4) we obtain
\[
z(t) = z(1) + \int_1^t z'(t)dt < z(1) + z'(1)(t-1) < 0
\]
for \( t \) large, which is impossible because \( z(t) \) is nonnegative. This completes the proof of theorem.

\[\square\]

**Proof of Corollary 1.7.** Let \( \gamma > 0 \). First we show that there exists \( t_0 \) such that \( \psi(\gamma(0)) \) satisfies the assumptions (1.14)-(1.17) in Theorem 1.5. We recall that \( \psi(0) = e^{\gamma x^2}Q(x) \) with \( Q(x) = 2\pi^{-\frac{1}{2}}e^{-|x|} \). Notice that a direct calculation shows that
\[
\partial_x \psi(0) = e^{\gamma x^2}(2i\gamma xQ + \partial_x Q),
\]
and therefore
\[
\text{Im} \int_\mathbb{R} x \partial_x \psi(0) \overline{\psi(0)}dx = 2\gamma \int_\mathbb{R} x^2Q^2(x)dx > 0.
\]
Then, by continuity we get
\[
\text{Im} \int_\mathbb{R} x \partial_x \psi(\gamma(t_0)) \overline{\psi(\gamma(t_0))} > 0
\]
for \( t_0 \) sufficiently small, i.e., the assumption (1.17) holds when \( t_0 \) is sufficiently small. Moreover, since
\[
E(\psi(\gamma(t_0))) = E(Q) + 2\gamma \text{Im} \int_\mathbb{R} x \partial_x \psi(\gamma(t_0)) \overline{\psi(\gamma(t_0))} + 2\gamma^2 \int_\mathbb{R} x^2|\psi(\gamma(t_0))|^2dx,
\]
(6.7) it follows that
\[
E(\psi(\gamma(t_0)))|\mathcal{M}(\psi(\gamma(t_0)))|^\sigma_c \geq E(Q)|\mathcal{M}(Q)|^\sigma_c,
\]
which implies that the assumption (1.14) holds. On the other hand, from equation (1.1) we obtain
\[
\partial_t N(\psi(t)) = (p + 1)|\psi(0, t)|^{p-1} \text{Re}[\overline{\psi(0, t)} \partial_t \psi(0, t)]
\]
\[
= (p + 1)|\psi(0, t)|^{p-1} \text{Re}[\overline{\psi(0, t)}(-i\partial^2_\psi \psi(0, t) + i\psi(0, t)\overline{\psi(0, t)})]
\]
\[
= -(p + 1)|\psi(0, t)|^{p-1} \text{Im}[\overline{\psi(0, t)} \partial^2_\psi \psi(0, t)].
\]
By using the fact
\[ \partial_t^2 \psi(x, 0) = e^{i\gamma x^2} (2i \gamma Q - 4i \gamma 2\pi \sqrt{|x|} e^{-|x|} - 4 \gamma^2 x^2 Q + \partial_x^2 Q) \]
we have that
\[ \partial_t N(\psi^\gamma(t))|_{t=0} = -2\gamma(p+1)N(Q) < 0. \]  
(6.8)
Thus, by using the fact
\[ [M(\psi^\gamma(0))]^{\sigma_e} N(\psi^\gamma(0)) = M^{\sigma_e}(Q)N(Q) \]
we conclude that assumption (1.16) holds for \( t_0 \) small. Now we set
\[ H(t) = [M(\psi^\gamma)]^{\sigma_e} \left( E(\psi^\gamma) - \frac{1}{8}(z'(t))^2 \right) - [M(Q)]^{\sigma_e} E(Q) \]
or, equivalently,
\[ H(t) = [M(\psi^\gamma)]^{\sigma_e} \left( E(\psi^\gamma) - \frac{1}{8}(z'(t))^2 \right) - [M(Q)]^{\sigma_e} E(Q) \]  
(6.9)
where (see proof of Theorem 1.5)
\[ z(t) = \sqrt{V(t)}, \quad V(t) = \int_{\mathbb{R}} x^2 |\psi^\gamma(x, t)|^2 dx. \]
Notice that putting together (6.6) and (6.7) we deduce
\[ E(\psi^\gamma) - \frac{1}{2} \int_{\mathbb{R}} x^2 |\partial_x \psi^\gamma(0)|^2 dx = E(Q), \]
which implies that \( H(0) = 0 \). From (6.9), we see that
\[ H'(t) = -\frac{1}{4} [M(\psi^\gamma)]^{\sigma_e} z'(t) z''(t). \]  
(6.10)
Now, as a consequence of (6.6) we get
\[ V'(0) = 8\gamma V(0). \]  
(6.11)
Moreover, since
\[ K(e^{i\gamma x^2} Q) = 4\gamma^2 V(0) + K(Q), \]
it follows from (2.3) and (4.5),
\[ V''(0) = 8K(e^{i\gamma x^2} Q) - 4N(e^{i\gamma x^2} Q) \]
\[ = 32\gamma^2 V(0) + 8K(Q) - 4N(Q) \]  
(6.12)
\[ = 32\gamma^2 V(0). \]
Then combining (6.11) and (6.12) we infer that
\[ (z'(0))^2 = \frac{1}{2} V''(0), \]
which implies
\[ z''(0) = \frac{1}{2} z(0) \left( \frac{V''(0)}{2} - (z'(0))^2 \right) = 0. \]
As a consequence \( H'(0) = 0 \),
\[ H''(0) = -\frac{1}{4} [M(\psi^\gamma)]^{\sigma_e} z'(0) z'''(0). \]
and
\[ V'''(0) = 2z(0) z''''(0). \]
Therefore, \( H''(0) = -\frac{1}{8} [M(\psi^\gamma)]^{\sigma_e} V'''(0). \) Finally, putting together (4.7) and (6.8) we have
\[ V'''(0) = \frac{4(p-3)}{(p+1)} \partial_t N(\psi^\gamma(t))|_{t=0} > 0, \]
and thus, $H''(0) < 0$. In particular, this implies that $H(t)$ is negative when $t > 0$ is small. Therefore, the assumption (1.15) holds for $t$ sufficiently small. Hence an application of Theorem 1.5 shows that the solution $\psi(t)$ scatters in $H^1(\mathbb{R})$ forward in time.

Next, by the same argument as above we may show that the solution $\overline{\psi(t)}$ to (1.1) satisfies the assumptions (1.15), (1.18) and (1.19) in Theorem 1.6. Hence by Theorem 1.5 we infer that the solution $\overline{\psi(t)}$ blow-up in positive time, i.e., $\psi(t)$ blow up in negative time. This proves the corollary when $\gamma > 0$. The second part of the corollary, when $\gamma < 0$, can be proved in a similar way. $\square$

**Proof of Corollary 1.8.** Consider the solution $u_\mu(t)$ of (1.1) with initial data $u_{\mu,0} = e^{i\mu x^2} u_0$. We will assume that $\mu > 0$ and $[M(u_0)]^{\sigma_\epsilon} N(u_0) < [M(Q)]^{\sigma_\epsilon} N(Q)$; the proof of statement (ii) is similar.

We consider two cases:

**Case 1.** We first assume that

$$E(u_{\mu,0})[M(u_{\mu,0})]^{\sigma_\epsilon} \geq E(Q)[M(Q)]^{\sigma_\epsilon}. \quad (6.13)$$

Then we will show that the initial data $u_{\mu,0}$ satisfies the hypotheses (1.15)-(1.17) in Theorem 1.5. Indeed, a direct calculation shows that

$$E(u_{\mu,0}) = E(u_0) + 2i \mu \text{Im} \int_\mathbb{R} x \cdot \partial_x u_0 \overline{u_0} dx + 2\mu^2 \int_\mathbb{R} |x|^2 |u_0|^2 dx$$

and

$$\text{Im} \int_\mathbb{R} x \cdot \partial_x u_{\mu,0} \overline{u_{\mu,0}} dx = \text{Im} \int_\mathbb{R} x \cdot \partial_x u_0 \overline{u_0} dx + 2\mu \int_\mathbb{R} |x|^2 |u_0|^2 dx. \quad (6.14)$$

Combining the equations above, we obtain that

$$E(u_{\mu,0}) - \frac{\left(\text{Im} \int_\mathbb{R} x \cdot \partial_x u_{\mu,0} \overline{u_{\mu,0}} dx\right)^2}{2 \int_\mathbb{R} |x|^2 |u_{\mu,0}|^2 dx} = E(u_0) - \frac{\left(\text{Im} \int_\mathbb{R} x \cdot \partial_x u_0 \overline{u_0} dx\right)^2}{2 \int_\mathbb{R} |x|^2 |u_0|^2 dx} \leq E(u_0),$$

or, equivalently

$$\frac{E(u_{\mu,0})[M(u_{\mu,0})]^{\sigma_\epsilon}}{E(Q)[M(Q)]^{\sigma_\epsilon}} \left(1 - \frac{\left(\text{Im} \int_\mathbb{R} x \cdot \partial_x u_{\mu,0} \overline{u_{\mu,0}} dx\right)^2}{2E(u_0) \int_\mathbb{R} |x|^2 |u_0|^2 dx}\right)\leq 1,$$

(6.15)

where we have used (1.20). Notice that (6.15) implies the assumption (1.15). Moreover, as $[M(u_{\mu,0})]^{\sigma_\epsilon} N(u_{\mu,0}) = [M(u_0)]^{\sigma_\epsilon} N(u_0)$, it is clear that the assumption (1.16) of Theorem 1.5 is fulfilled.

Next we consider the quadratic polynomial in $r$,

$$P(r) := E(u_0) + 2r \text{ Im} \int_\mathbb{R} x \cdot \partial_x u_0 \overline{u_0} dx + 2r^2 \int_\mathbb{R} |x|^2 |u_0|^2 dx - \frac{E(Q)[M(Q)]^{\sigma_\epsilon}}{[M(u_0)]^{\sigma_\epsilon}}.$$ 

Notice that

$$[M(u_{\mu,0})]^{\sigma_\epsilon} P(\mu) = [M(u_{\mu,0})]^{\sigma_\epsilon} E(u_{\mu,0}) \left(1 - \frac{E(Q)[M(Q)]^{\sigma_\epsilon}}{E(u_{\mu,0})[M(u_{\mu,0})]^{\sigma_\epsilon}}\right).$$

Thus, using (6.13) we obtain that $P(\mu) \geq 0$. Moreover, assumption (1.20) is equivalent to $P(0) < 1$. Therefore, there exists $r_0 \geq 0$ such that $\mu \geq r_0$, where $r_0$ satisfies $P(r_0) = 0$. Since $P(r_0) = 0$, by (1.20) we obtain the inequality

$$\text{Im} \int_\mathbb{R} x \cdot \partial_x u_0 \overline{u_0} dx + r_0 \int_\mathbb{R} |x|^2 |u_0|^2 dx \geq 0.$$
Thus, by (6.14) we conclude
\[
\text{Im} \int_{\mathbb{R}} x \cdot \partial_x u_{\mu,0} \overline{u_{\mu,0}} \, dx \geq \mu \int_{\mathbb{R}} |x|^2 |u_0|^2 \, dx,
\]
which yields (1.17). In view of Theorem 1.5, this implies that solution scatters forward in time.

**Case 2.** Now we suppose that \( M \). We show that condition \( [M(u_0)]^{\sigma_c} N(u_0) < [M(Q)]^{\sigma_c} N(Q) \) implies (1.7). Indeed, from (6.16) we have that
\[
E(Q)[M(Q)]^{\sigma_c} > E(u_{\mu,0})[M(u_{\mu,0})]^{\sigma_c} > \frac{1}{2} K(u_{\mu,0})[M(u_{\mu,0})]^{\sigma_c} - \frac{1}{p+1} K(Q)[M(Q)]^{\sigma_c},
\]
combining the first and last term, we infer that \( u_{\mu,0} \) satisfies (1.7). Hence, by (6.16) and an application of Theorem 1.2 completes the proof. \( \square \)

**Acknowledgments**

The author would like to express their sincere thanks to the referees for many helpful comments.

**References**

[1] R. Adami, R. Fukuizumi, and J. Holmer, Scattering for the \( L^2 \) supercritical point NLS, Trans. Amer. Math. Soc., 374 (2021), pp. 35–60.

[2] R. Adami and A. Teta, A class of nonlinear Schrödinger equations with concentrated nonlinearity, J. of Funct. Anal., 180 (2001), pp. 148–175.

[3] B. Bellazi and M. Mintchev, Quantum field theory on star graphs, Phys. A: Math. Theor., 39 (2006), pp. 1101–1117.

[4] G. Berkolaiko, C. Carlson, S. Fulling, and P. Kuchment, Quantum Graphs and Their Applications, vol. 415 of Contemporary Math., American Math. Society, Providence, RI, 2006.

[5] V. Caudrelier, M. Mintchev, and E. Ragoucy, Solving the quantum nonlinear Schrödinger equation with \( \delta \)-type impurity, J. Math. Phys., 4 (2005), pp. 1–24.

[6] V. Dinh, L. Forcella, and H. Hajaiej, Mass-Energy threshold dynamics for dipolar Quantum Gases, Preprint arXiv:2009.05933, (2020), p. 31 pages.

[7] V. D. Dinh, A unified approach for energy scattering for focusing nonlinear Schrödinger equations, Discrete Contin. Dyn. Syst., 40 (2020), pp. 6441–6471.

[8] B. Dodson and J. Murphy, A new proof of scattering below the ground state for the non-radial focusing NLS, Math. Res. Lett., 25 (2018), pp. 1805–1825.

[9] T. Duyckaerts, J. Holmer, and S. Roudenko, Scattering for the non-radial 3D cubic nonlinear Schrödinger equation, Math. Res. Lett., 15 (2008), pp. 1233–1250.

[10] T. Duyckaerts and S. Roudenko, Going beyond the threshold: scattering and blow-up in the focusing NLS equation, Comm. Math. Phys., 334 (2015), pp. 1573–1615.

[11] R. Fukuizumi and L. Jeanjean, Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential, Discrete Contin. Dyn. Syst., 21 (2008), pp. 121–136.

[12] R. Fukuizumi, M. Ohta, and T. Ozawa, Nonlinear Schrödinger equation with a point defect, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), pp. 837–845.

[13] Y. Gao and Z. Wang, Below and beyond the mass–energy threshold: scattering for the Hartree equation with radial data in \( d \geq 5 \), Angew. Math. Phys., 71 (2020).

[14] H. Goodman, P. Holmes, and M. Weinstein, Strong NLS soliton–defect interactions, Physica D, 192 (2004), pp. 215–248.

[15] C. D. Guerra, Global behavior of finite energy solutions to the \( d \)-dimensional focusing nonlinear Schrödinger equation, Appl. Math. Res. Express, 2014 (2013), pp. 177–243.

[16] J. Holmer and C. Liu, Blow-up for the 1D nonlinear Schrödinger equation with point nonlinearity I: Basic theory, J. Math. Anal. Appl., 483 (2020), p. 123522.

[17] J. Holmer and S. Roudenko, A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation, Comm. Math. Phys., 282 (2008), pp. 435–467.
[18] M. Ikeda and T. Inui, Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential, Anal. PDE, 10 (2017), pp. 481–512.

[19] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math, 166 (2006), pp. 645–675.

[20] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity, in Lecture notes of the 2008 Clay summer school ”Evolution Equations”, 2008.

[21] S. Le Coz, R. Fukuzumi, G. Fibich, B. Ksherim, and Y. Sivan, Instability of bound states of a nonlinear Schrödinger equation with a dirac potential, Phys. D, 237 (2008), pp. 1103–1128.

[22] W. C. K. Mak, B. A. Malomed, and P. L. Chu, Interaction of a soliton with a local defect in a fiber Bragg grating, J. Opt. Soc. Am. B, 20 (2003), pp. 725–735.

[23] F. A. Mehmeti, J. von Below, and S. Nicaise, eds., Partial Differential equations on multistructures, no. 219 in Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2001.

[24] T. Saanouni, Scattering versus blow-up beyond the threshold for the focusing choquard equation, J. Math. Anal. Appl., 492 (2020).

Universidade Federal de Minas Gerais, ICEx-UFMG, CEP 30123-970, MG, Brazil
Email address: ardila@impa.br