The isotopy problem for the phase tropical line

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Abstract

Let $H$ be the complex line $1 + z_1 + z_2 = 0$ in $(\mathbb{C}^*)^2$ and $H_{\text{trop}}$ the associated phase tropical line. We show that $H$ and $H_{\text{trop}}$ are isotopic as topological submanifolds, by explicitly constructing the isotopy map.

1 Introduction

Given an $n$-dimensional complex hyperplane $H \subset (\mathbb{C}^*)^n$, one can construct the associated phase tropical hyperplane $H_{\text{trop}} \subset (\mathbb{C}^*)^n$. This is a cell complex whose projection to $\mathbb{R}^n$ is the tropical hyperplane. Recently, in [2] and [3] it has been shown that in any dimension these two objects are homeomorphic. In the present paper we show that for $n = 2$ one can get a stronger result, namely one can continuously deform $H$ into $H_{\text{trop}}$ via homeomorphic objects in $(\mathbb{C}^*)^2$. We prove the following:

Theorem 1.1. Let $H \subset (\mathbb{C}^*)^2$ the complex line $1 + z_1 + z_2 = 0$ and $H_{\text{trop}} \subset (\mathbb{C}^*)^2$ its associated phase tropical line. Let $i_H : H \to (\mathbb{C}^*)^2$ be the canonical embedding. Then there exists a continuous map:

$$
\Psi : H \times [0, 1] \to (\mathbb{C}^*)^2
$$

such that the family of maps:

$$
\Psi_t : H \to (\mathbb{C}^*)^2, \ P \mapsto \Psi(P, t),
$$

for $t \in [0, 1]$, has the following properties:

1. $\Psi_0 = i_H$;
2. $\Psi_1(H) = H_{\text{trop}}$;
3. $\Psi_t$ is a homeomorphism onto the image, for each $t \in [0, 1]$.

Throughout, we identify $(\mathbb{C}^*)^2$ with $\mathbb{R}^2 \times (S^1)^2$. We start in Section 1 by defining the phase tropical line, following [4] and [5]. We describe $H_{\text{trop}}$ fibred over its image in $\mathbb{R}^2$ under the projection map and in Section 2 we do the same thing for the complex line. In Section 3 we prepare $H$ and $H_{\text{trop}}$ for the construction of the isotopy map. To do that, we cut $H$ and $H_{\text{trop}}$ in three parts, see Figure 5, Figure 6, Figure 7 and Definition 4.6. By Lemma 4.7 we get an automorphism $\lambda$ of $\mathbb{R}^2 \times (S^1)^2$ which maps, in a cyclic order, both in $H$ and $H_{\text{trop}}$, each part of the subdivision to another. Therefore, we focus our attention to one of the parts of the subdivision of $H$ and $H_{\text{trop}}$, which we call respectively $H_1$ and $H_{1\text{trop}}$ and we attempt to construct an isotopy between the canonical embedding of $H_1$ in $(\mathbb{C}^*)^2$ and an embedding whose
image is \( H_{1\text{trop}} \). We call this isotopy map \( \Phi_1^t \). In order to do that, we subdivide \( H_1 \) and \( H_{1\text{trop}} \) in two parts, see Figure 8 and Definition 4. In Section 5 we write down at first the isotopy map \( \Phi_1^t \) and then we prove Theorem 1.1.

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2 The phase tropical line

Let \( K \) be the field of the (real-power) Puiseux series:

\[
K := \left\{ \sum_{j \in I} c_j t^j \middle| c_j \in \mathbb{C}, I \subset \mathbb{R} \text{ well ordered} \right\}.
\]

Set \( K^* := K \setminus \{0\} \). We have a map called the complexified valuation:

\[
\text{val} : K^* \to \mathbb{C}^*
\]

\[
|c_{j_0}| e^{i \arg(c_{j_0})} t^{j_0} + \text{higher terms} \mapsto e^{-j_0 + i \arg(c_{j_0})},
\]

This map extends componentwise to a map \( \text{val}_2 : (K^*)^2 \to (\mathbb{C}^*)^2 \). We define the phase tropical line, following [4] and [5]:

Definition 2.1. Let \( H \) be the complex line in \((\mathbb{C}^*)^2\):

\[
H := \left\{ (z_1, z_2) \in (\mathbb{C}^*)^2 \middle| 1 + z_1 + z_2 = 0 \right\}.
\]

Let \( f := 1 + z_1 + z_2 \in K[z_1, z_2] \) and consider its zero set \( Z(f) \subset (K^*)^2 \). The phase tropical line associated to \( H \) is defined by:

\[
H_{\text{trop}} := \overline{\text{val}_2(Z(f))} \subset (\mathbb{C}^*)^2.
\]

Remark 2.2. If we define the map:

\[
\nu : K^* \setminus \{-1\} \mapsto (\mathbb{C}^*)^2, z \mapsto (\text{val}(z), \text{val}(-1 - z)),
\]

then:

\[
H_{\text{trop}} = \nu(K^* \setminus \{-1\}).
\]

Definition 2.3. We identify \((\mathbb{C}^*)^2\) with \(\mathbb{R}^2 \times (S^1)^2\) via the homeomorphism:

\[
h : (\mathbb{C}^*)^2 \to \mathbb{R}^2 \times (S^1)^2, (z_1, z_2) \mapsto (\ln |z_1|, \ln |z_2|, \arg(z_1), \arg(z_2)).
\]

In this case the complexified valuation map becomes:

\[
\text{val} : K^* \to \mathbb{R} \times S^1
\]

\[
|c_{j_0}| e^{i \arg(c_{j_0})} t^{j_0} + \text{higher terms} \mapsto (-j_0, \arg(c_{j_0})).
\]
To describe the phase tropical line in $\mathbb{R}^2 \times (S^1)^2$, let $\nu$ be as in Remark 2.2. We subdivide $K^* \setminus \{-1\}$ into 4 parts. Let $z = r_0 e^{i\varphi} t^{j_0} + \text{higher terms} \in K^* \setminus \{-1\}$:

Part 1: $P_1 := \{ z \in K^* \setminus \{-1\} \mid j_0 > 0 \}$.
In the sum $-1 - z$ the smallest term is $-1$. Thus,

$$\nu(P_1) = \{(x, 0, \varphi, \pi) \in \mathbb{R}^2 \times (S^1)^2 \mid x < 0, \varphi \in S^1 \}.$$ 

Part 2: $P_2 := \{ z \in K^* \setminus \{-1\} \mid j_0 = 0, r_0 e^{i\varphi} = -1 \}$.
In the sum $-1 - z$ the smallest term is $\alpha t^j$, for any $j > 0$ and $\alpha \in \mathbb{C}^*$. Thus,

$$\nu(P_2) = \{(0, y, \pi, \psi) \in \mathbb{R}^2 \times (S^1)^2 \mid y < 0, \psi \in S^1 \}.$$ 

Part 3: $P_3 := \{ z \in K^* \setminus \{-1\} \mid j_0 < 0 \}$.
In the sum $-1 - z$ the smallest term is $-r_0 e^{i\varphi} t^{j_0}$. Thus,

$$\nu(P_3) = \{(x, x, \varphi, \varphi + \pi) \in \mathbb{R}^2 \times (S^1)^2 \mid x > 0, \varphi \in S^1 \}.$$ 

Part 4: $P_4 := \{ z \in K^* \setminus \{-1\} \mid j_0 = 0, r_0 e^{i\varphi} \neq -1 \}$.
In the sum $-1 - z$ the smallest term is $-1 - r_0 e^{i\varphi} t^{j_0}$. Thus,

$$\nu(P_4) = \{(0, 0, \varphi, \text{arg}(-1 - r_0 e^{i\varphi})) \in \mathbb{R}^2 \times (S^1)^2 \mid r_0 > 0, \varphi \in S^1 \}.$$ 

Taking the closure in $\mathbb{R}^2 \times (S^1)^2$ of the union of the subsets $\nu(P_i)$ we get $H_{trop}$.

**Definition 2.4.** Let $H$ be the complex line and $H_{trop}$ the phase tropical line. Let us consider the projections $\pi : \mathbb{R}^2 \times (S^1)^2 \to \mathbb{R}^2$ and $\text{Arg} : \mathbb{R}^2 \times (S^1)^2 \to (S^1)^2$.

The projections $\pi(H)$ and $\pi(H_{trop})$ are called respectively the amoeba and the tropical amoeba of $H$. Moreover, $\text{Arg}(H)$ and $\text{Arg}(H_{trop})$ are called respectively the coamoeba and tropical coamoeba of $H$.

The following proposition can be found in a general form in [5].

**Proposition 2.5.** Let $H$ be the complex line and $H_{trop}$ the phase tropical line, then the tropical coamoeba equals the closure of the coamoeba.

The tropical amoeba of $H$ is given by the union of three half lines:

$$\pi(H) = \{x \leq 0, y = 0\} \cup \{x = 0, y \leq 0\} \cup \{x = y, y \geq 0\}.$$
The tropical coamoeba, seen in the fundamental domain \([0, 2\pi]^2\), can be pictured as follows:

- Figure 1: The tropical amoeba.
- Figure 2: The tropical coamoeba.

The picture is justified by Proposition 2.5 and Proposition 3.9.

If we consider the projection \(\pi|_{H_{trop}} : H_{trop} \to \pi(H_{trop})\), we see that the fibre of \(\pi|_{H_{trop}}\) is given by \(\{(\varphi, \pi)|\varphi \in S^1\}\) over \(x < 0\), by \(\{(\pi, \psi)|\psi \in S^1\}\) over \(y < 0\), by \(\{((\varphi, \varphi + \pi)|\varphi \in S^1\}\) over \(x = y, y > 0\) and by the whole tropical coamoeba \(\text{Arg}(H_{trop})\) over the point \((0,0)\).

### 3 The complex line

We give a detailed description of the amoeba and coamoeba of the complex line \(H\). In Proposition 3.4 we describe \(H\) by describing the fibres of \(\pi|_H : H \to \pi(H)\).

**Lemma 3.1.** Let \(H\) be the complex line:

\[
H = \{(x, y, \varphi, \psi) \in \mathbb{R}^2 \times (S^1)^2|e^{x+i\varphi} + e^{y+i\psi} + 1 = 0\}.
\]

If \((x, y, \varphi, \psi) \in H\), then:

\[
\begin{align*}
e^{2x} &= 1 + 2e^y \cos \varphi + e^{2y} \\
e^{2y} &= 1 + 2e^x \cos \varphi + e^{2x}
\end{align*}
\]

(1)

Conversely, if \((x, y, \varphi, \psi) \in \mathbb{R}^2 \times (S^1)^2\) satisfies \((1)\), then:

\[
\{(x, y, \varphi, \psi), (x, y, 2\pi - \varphi, 2\pi - \psi)\} \text{ or } \{(x, y, 2\pi - \varphi, \psi), (x, y, \varphi, 2\pi - \psi)\}
\]

is a subset of \(H\).

**Proof.** From the defining equation of \(H\) we get:

\[
\begin{align*}
e^x &= |1 - e^{y+i\psi}| \\
e^y &= |1 - e^{x+i\varphi}|
\end{align*}
\]
which is equivalent to:

\[
\begin{aligned}
(1 + e^y \cos \psi)^2 + (e^y \sin \psi)^2 &= e^{2x} \\
(1 + e^x \cos \varphi)^2 + (e^x \sin \varphi)^2 &= e^{2y}
\end{aligned}
\]  

(2)

which, in turn, is equivalent to (1).

Conversely, let \((x, y, \varphi, \psi) \in \mathbb{R}^2 \times (S^1)^2\) satisfy (1). Then, we immediately see that \((x, y, 2\pi - \varphi, 2\pi - \psi), (x, y, 2\pi - \varphi, \psi)\) and \((x, y, \varphi, 2\pi - \psi)\) are also solutions of (1). Now, straightforward computations show that only one of the two sets:

\[
\{(x, y, \varphi, \psi), (x, y, 2\pi - \varphi, 2\pi - \psi)\}, \{(x, y, 2\pi - \varphi, \psi), (x, y, \varphi, 2\pi - \psi)\}
\]

also satisfies \(e^{x+i\varphi} + e^{y+i\psi} + 1 = 0\).

\[\Box\]

**Proposition 3.2.** Let \(H\) be the complex line and \(\pi(H) \subset \mathbb{R}^2\) its amoeba. Let \((x, y) \in \mathbb{R}^2\). Then, \((x, y) \in \pi(H)\) if and only if:

\[
\begin{cases}
    e^x - e^y \leq 1 \\
    e^y - e^x \leq 1 \\
    e^x + e^y \geq 1
\end{cases}
\]  

(3)

**Proof.** Let \((x, y) \in \mathbb{R}^2\). By Lemma 3.1, \((x, y) \in \pi(H)\) if and only if we can find \((\varphi, \psi) \in (S^1)^2\) for which (1) holds. Now, once we fix \((x, y) \in \mathbb{R}^2\), (1) has a solution in \((S^1)^2\) if and only if:

\[
\begin{cases}
    -1 \leq \frac{e^{2x} - e^{2y} - 1}{2e^y} \leq 1 \\
    -1 \leq \frac{e^{2y} - e^{2xy} - 1}{2e^x} \leq 1
\end{cases}
\]

which is equivalent to:

\[
\begin{cases}
    (e^x - e^y - 1)(e^x + e^y + 1) \leq 0 \\
    (e^x - e^y + 1)(e^x + e^y - 1) \geq 0 \\
    (e^y - e^x - 1)(e^y + e^x + 1) \leq 0 \\
    (e^y - e^x + 1)(e^y + e^x - 1) \geq 0
\end{cases}
\]  

(4)

We easily see that (4) holds if and only if (3) holds.

\[\Box\]
Definition 3.3. We denote by $H^\circ$ the subset of all points $(x, y, \varphi, \psi) \in H$ whose image under $\pi$ lies in the interior of the amoeba, that is $(x, y)$ satisfies:
\[
\begin{align*}
& e^x - e^y < 1 \\
& e^y - e^x < 1 \\
& e^x + e^y > 1
\end{align*}
\]
We denote by $H_\partial$ the subset of all points $(x, y, \varphi, \psi) \in H$ whose image under $\pi$ lies on the boundary of the amoeba, that is $(x, y)$ either satisfies $e^x - e^y = 1$, $e^y - e^x = 1$ or $e^x + e^y = 1$.

Using Lemma 3.1 one can easily prove the following:

Proposition 3.4. Let $(x, y, \varphi, \psi) \in H$. Then:

$$(\varphi, \psi) = \begin{cases} 
(\pi, 0) & \text{if and only if } e^x - e^y = 1 \\
(0, \pi) & \text{if and only if } e^y - e^x = 1 \\
(\pi, \pi) & \text{if and only if } e^x + e^y = 1
\end{cases}$$

Let $(x, y) \in \pi(H^\circ)$, then:

$\pi^{-1}_H(x, y) = \{(x, y, \varphi, \psi), (x, y, 2\pi - \varphi, 2\pi - \psi)\}$,

for some $(\varphi, \psi) \in (S^1)^2 \setminus \{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$.

Definition 3.5. We define two subsets of $H$, namely $H^+ := \{(x, y, \varphi, \psi) \in H | 0 \leq \varphi \leq \pi \}$ and $H^- := \{(x, y, \varphi, \psi) \in H | \pi \leq \varphi \leq 2\pi \}$.
Proposition 3.6. The maps \( \pi_+ := \pi|_{H^+} \) and \( \pi_- := \pi|_{H^-} \) are homeomorphisms onto their images.

Proof. By Proposition 3.4 the maps are bijective. They are continuous being restrictions of a continuous map. Moreover, since \((S^1)^2\) is compact, the projection \(\pi\) is a closed map, therefore \(\pi_+\) and \(\pi_-\) are also closed maps being restrictions of \(\pi\) to closed subsets.

Lemma 3.7. Let \((x, y, \varphi, \psi) \in H\). Then, \((x, y, \varphi, \psi) \in H_0\) if and only if

\[
\begin{cases}
  e^x = \frac{\sin \varphi}{\sin(\psi - \varphi)} \\
  e^y = \frac{\sin \varphi}{\sin(\psi - \varphi)}
\end{cases}
\]

(5)

Proof. Let \((x, y, \varphi, \psi) \in H\). By Proposition 3.4 \((x, y, \varphi, \psi) \in H_0\) if and only if \((\varphi, \psi) \notin \{(0,0), (0, \pi), (\pi,0), (\pi, \pi)\}\). The defining equation of \(H\) is equivalent to the following system of equations:

\[
\begin{cases}
  e^x \cos \varphi + e^y \cos \psi + 1 = 0 \\
  e^x \sin \varphi + e^y \sin \psi = 0
\end{cases}
\]

(6)

From (6) we see that if \(\sin \psi \neq 0\), then \(\sin \varphi \neq 0\). Now,

\[
\begin{cases}
  \sin \psi \neq 0 \\
  \sin \varphi \neq 0
\end{cases}
\]

if and only if \((\varphi, \psi) \notin \{(0,0), (0, \pi), (\pi,0), (\pi, \pi)\}\). Under this assumption, (6) is equivalent to:

\[
\begin{cases}
  e^x \sin(\psi - \varphi) + \sin \psi = 0 \\
  e^x \sin \varphi + e^y \sin \psi = 0
\end{cases}
\]

(7)

From the first equation in (7), we see that if \(\sin \psi \neq 0\), then also \(\sin(\psi - \varphi) \neq 0\). Therefore, (7) is equivalent to (5).

Definition 3.8. We define two subsets of \((S^1)^2\), namely:

\[
T_1 := \{ (\varphi, \psi) \in (S^1)^2 | 0 < \varphi < \pi, \pi < \psi < \varphi + \pi \}
\]

and

\[
T_2 := \{ (\varphi, \psi) \in (S^1)^2 | \pi < \varphi < 2\pi, \varphi - \pi < \psi < \pi \}
\]

See Figure 4 below.
Proposition 3.9. The coamoeba of $H$ is:

$$\text{Arg}(H) = T_1 \cup T_2 \cup \{(0, \pi), (\pi, 0), (\pi, \pi)\}.$$ 

Moreover, the map $\text{Arg}|_{H_0} : H_0 \rightarrow T_1 \cup T_2$ is a homeomorphism.

Proof. By Proposition 3.4, if $(x, y, \varphi, \psi) \in H_0$, then its image under $\text{Arg}$ is $(0, \pi), (\pi, 0)$ or $(\pi, \pi)$. On the other hand, by Lemma 3.7, if $(x, y, \varphi, \psi) \in H_0$, then $(x, y, \varphi, \psi)$ satisfies (5), hence $(\varphi, \psi)$ satisfies:

\[
\begin{aligned}
\frac{\sin \psi}{\sin (\psi - \varphi)} &> 0 \\
\frac{\sin \varphi}{\sin (\psi - \varphi)} &> 0.
\end{aligned}
\] (8)

Therefore, if $\sin (\psi - \varphi) > 0$, then $\sin \psi < 0$ and $\sin \varphi > 0$. In this case, $\sin \psi < 0$ implies $\pi < \psi < 2\pi$, and $\sin \varphi > 0$ implies $0 < \varphi < \pi$. Moreover, $\sin (\psi - \varphi) > 0$ implies $0 < \psi - \varphi < \pi$. Therefore, we get that $(\psi, \varphi)$ lies in $T_1$. Conversely, if $(\varphi, \psi)$ lies in $T_1$, that is $0 < \varphi < \pi, \pi < \psi < \varphi + \pi$, then we have $\sin \varphi > 0, \sin \psi < 0, \sin (\psi - \varphi) > 0$, hence (8) has a (unique) solution. The symmetry of $H$, $x \leftrightarrow y, \varphi \leftrightarrow \psi$, completes the proof of the first statement. In particular, we have shown that $\text{Arg}(H_0) = T_1 \cup T_2$. Now, $\text{Arg}|_{H_0}$ is continuous being the restriction of a continuous map. Its inverse map is:

$$\text{Arg}|_{H_0}^{-1} : T_1 \cup T_2 \rightarrow H_0$$

$$(\varphi, \psi) \mapsto \left(\ln \left(-\frac{\sin \psi}{\sin (\psi - \varphi)}\right), \ln \left(-\frac{\sin \varphi}{\sin (\psi - \varphi)}\right), \varphi, \psi\right),$$

which is continuous as well.

Figure 4: The coamoeba of $H$.

4 The isotopy preparation procedure

Definition 4.1. We subdivide $H$ in three parts. Namely:
This particular subdivision of the complex line can be found in [1]. The inequalities defining the subdivision of $H$ determine, in particular, a subdivision in three parts of the amoeba $\pi(H)$. This subdivision is represented in the following picture:

![Figure 5](image.png)

Analogously, taking the image under the argument map of $H_1$, $H_2$ and $H_3$ we get an induced subdivision of $\text{Arg}(H)$ in three parts.

**Definition 4.2.** We set:

\[
H_1 := \{(x, y, \varphi, \psi) \in H | x \leq y, x \leq 0\}, \\
H_2 := \{(x, y, \varphi, \psi) \in H | y \leq x, y \leq 0\}, \\
H_3 := \{(x, y, \varphi, \psi) \in H | 0 \leq x, 0 \leq y\}.
\]

Using Lemma 3.7, straightforward computations prove the following claims:

**Claim 4.3.**

\[
A_1 := \{(\varphi, \psi) \in \text{Arg}(H_1) | 0 \leq \varphi \leq \pi\}, \\
A_2 := \{(\varphi, \psi) \in \text{Arg}(H_1) | \pi \leq \varphi \leq 2\pi\}, \\
B_1 := \{(\varphi, \psi) \in \text{Arg}(H_2) | 0 \leq \varphi \leq \pi\}, \\
B_2 := \{(\varphi, \psi) \in \text{Arg}(H_2) | \pi \leq \varphi \leq 2\pi\}, \\
C_1 := \{(\varphi, \psi) \in \text{Arg}(H_3) | 0 \leq \varphi \leq \pi\}, \\
C_2 := \{(\varphi, \psi) \in \text{Arg}(H_3) | \pi \leq \varphi \leq 2\pi\}.
\]

Using Lemma 3.7, straightforward computations prove the following claims:

**Claim 4.4.**

\[
A_1 = \left\{(\varphi, \psi) \in \text{Arg}(H) | \psi \leq \frac{\varphi}{2} + \pi, \psi \leq -\varphi + 2\pi\right\}, \\
A_2 = \left\{(\varphi, \psi) \in \text{Arg}(H) | \psi \geq \frac{\varphi}{2}, \psi \geq -\varphi + 2\pi\right\}.
\]

**Claim 4.4.**

\[
B_1 = \{(\varphi, \psi) \in \text{Arg}(H) | \psi \leq 2\varphi, \psi \leq -\varphi + 2\pi\}, \\
B_2 = \{(\varphi, \psi) \in \text{Arg}(H) | \psi \geq 2(\varphi - \pi), \psi \leq -\varphi + 2\pi\}.
\]
Claim 4.5.

\[ C_1 = \left\{ (\varphi, \psi) \in \text{Arg}(H) \mid \psi \geq \frac{\varphi}{2} + \pi, \psi \geq 2\varphi \right\}, \]
\[ C_2 = \left\{ (\varphi, \psi) \in \text{Arg}(H) \mid \psi \leq 2(\varphi - \pi), \psi \leq \frac{\varphi}{2} \right\}. \]

The subdivision of \( \text{Arg}(H) \) is represented in the following picture:

![Figure 6](image)

The two points \( O := \left( \frac{2\pi}{3}, \frac{4\pi}{3} \right) \) and \( O' := \left( \frac{4\pi}{3}, \frac{2\pi}{3} \right) \) are given by the intersection of \( \psi = -\varphi + 2\pi \) respectively with \( \psi = \frac{\varphi}{2} \) and \( \psi = 2\varphi \). These two points are the arguments of the fibre of \( \pi|_H \) over \((0,0)\). Indeed, using Lemma [3.1] we get that the fibre of \( \pi|_H \) over \((0,0)\) is given by \( \{(0,0,\varphi,\psi),(0,0,2\pi-\varphi,2\pi-\psi)\} \) or \( \{(0,0,2\pi-\varphi,\psi),(0,0,\varphi,2\pi-\psi)\} \), where \((0,0,\varphi,\psi)\) is a solution of the following system of equations:

\[
\begin{align*}
2 \cos \psi + 1 &= 0 \\
2 \cos \varphi + 1 &= 0
\end{align*}
\]

that is:

\[
\begin{align*}
e^{2x} &= 1 + 2e^y \cos \psi + e^{2y} \\
e^{2y} &= 1 + 2e^x \cos \varphi + e^{2x}
\end{align*}
\]

Take \((\varphi,\psi) = \left( \frac{2\pi}{3}, \frac{4\pi}{3} \right)\), then we get that the two above sets coincide and \((2\pi-\varphi,2\pi-\psi) = \left( \frac{4\pi}{3}, \frac{2\pi}{3} \right)\). We notice that \( O \) and \( O' \) are the barycentres respectively of the triangles \( T_1 \) and \( T_2 \), closure in \((S^1)^2\) of the subsets \( T_1 \) and \( T_2 \).

In a similar way as done for \( H \), we subdivide \( H_{trop} \) in three parts, which we call \( H_{1trop}, H_{2trop}, H_{3trop} \). To do that, we extend the above subdivision of \( \text{Arg}(H) \) to \( \text{Arg}(H) \):
Definition 4.6. We set:

\[ H_{1 \text{ trop}} := \left\{ (x, 0, \varphi, \psi) \in H_{\text{ trop}} | (\varphi, \psi) \in \overline{\text{Arg}}(H_1) \right\}, \]

\[ H_{2 \text{ trop}} := \left\{ (0, y, \varphi, \psi) \in H_{\text{ trop}} | (\varphi, \psi) \in \overline{\text{Arg}}(H_2) \right\}, \]

\[ H_{3 \text{ trop}} := \left\{ (x, x, \varphi, \psi) \in H_{\text{ trop}} | (\varphi, \psi) \in \overline{\text{Arg}}(H_3) \right\}. \]

Lemma 4.7. Let us consider the automorphism of \( \mathbb{R}^2 \times (S^1)^2 \) of order 3:

\[ \lambda : \mathbb{R}^2 \times (S^1)^2 \rightarrow \mathbb{R}^2 \times (S^1)^2 \]

\[ \lambda : (x, y, \varphi, \psi) \mapsto (-y, x - y, -\psi + 2\pi, \varphi - \psi + 2\pi) \]

We have \( \lambda(H_1) = H_2, \lambda(H_2) = H_3, \lambda(H_3) = H_1 \) and \( \lambda(H_{1 \text{ trop}}) = H_{2 \text{ trop}}, \lambda(H_{2 \text{ trop}}) = H_{3 \text{ trop}}, \lambda(H_{3 \text{ trop}}) = H_{1 \text{ trop}}. \)

Proof. We only show \( \lambda(H_1) = H_2 \) and \( \lambda(H_{1 \text{ trop}}) = H_{2 \text{ trop}}. \) Let \( P := (x, y, \varphi, \psi) \in H. \) Then \( P \in H_1 \) if and only if it satisfies:

\[
\begin{align*}
\begin{cases}
e^{x+iy} + e^{y+ix} + 1 = 0 \\
x \leq 0 \\
y \geq x
\end{cases}.
\end{align*}
\] (9)

Similarly, \( P \in H_2 \) if and only if it satisfies:

\[
\begin{align*}
\begin{cases}
e^{x+iy} + e^{y+ix} + 1 = 0 \\
y \leq x \\
y \leq 0
\end{cases}.
\end{align*}
\] (10)
Assume $P \in H_1$. Then $\lambda(P) = (-y, x - y, -\psi + 2\pi, \varphi - \psi + 2\pi)$ satisfies:

$$
\begin{cases}
    e^{-y+i(-\psi)} + e^{x-y+i(\varphi-\psi)} + 1 = 0 \\
    x - y \leq -y \\
    x - y \leq 0
\end{cases}
$$

Indeed, $e^{-y+i(-\psi) + 2\pi} + e^{x-y+i(\varphi-\psi) + 2\pi} + 1 = (1 + e^{x+i\varphi} + e^{y+i\psi})^{-1}$. Using the first equation in (9), we get that the first equation in (11) holds. Moreover, the first and second inequality in (11) are direct consequences respectively of the first and second inequality in (9). Thus, changing coordinates, $\lambda(P) =: (x', y', \varphi', \psi')$, we see that $\lambda(P) \in H_2$. Conversely, let $P' := (x, y, \varphi, \psi)$ satisfy (10). Then, in a similar way as before one shows that $\lambda^{-1}(P') = (y - x, -x, \psi - \varphi + 2\pi, -\psi + 2\pi)$ satisfies (9). To show that $\lambda(H_{1\text{top}}) = H_{2\text{top}}$ we notice that from $\lambda(H_1) = H_2$, it immediately follows that $\lambda$ maps $\pi(H_{1\text{top}})$ onto $\pi(H_{2\text{top}})$ and $\text{Arg}(H_{1\text{top}}) = \text{Arg}(H_1)$ onto $\text{Arg}(H_{2\text{top}}) = \text{Arg}(H_2)$.

**Remark 4.8.** The automorphism $\lambda$ of $\mathbb{R}^2 \times (S^1)^2$ descends from the following automorphism of $(\mathbb{C}^*)^2$ of order 3:

$$
\lambda' : (z, w) \mapsto (w^{-1}, zw^{-1})
$$

via the identification

$$
h : (\mathbb{C}^*)^2 \rightarrow \mathbb{R}^2 \times (S^1)^2
$$

as in Definition 2.3.

We further subdivide $H_1$ and $H_{1\text{top}}$ in two parts.

**Definition 4.9.** We set:

$$
H_{1L} := \{(x, y, \varphi, \psi) \in H_1 | y \geq 2x + \ln 2\}
$$

and

$$
H_{1T} := \{(x, y, \varphi, \psi) \in H_1 | y \leq 2x + \ln 2\}.
$$

Taking the image under $\pi$ of $H_{1L}$ and $H_{1T}$ we get an induced subdivision of $\pi(H_1)$.

**Definition 4.10.** We set:

$$
\Gamma_1 := \{(x, y, \varphi, \psi) \in H_1 | y = 2x + \ln 2\}.
$$

The subdivision of $\pi(H_1)$ is represented in the following picture:
The picture justifies the names given two the two parts of the subdivision. Namely, ‘L’ stays for ‘Leg’ and ‘T’ for ‘Triangle’.

Taking the image under $\text{Arg} \ H_1^L$ and $H_1^T$ we get an induced subdivision of $\text{Arg}(H_1)$. Using Proposition 3.4 and Lemma 3.7 one can immediately prove:

**Proposition 4.11.** $\text{Arg}(\Gamma_1)$ equals:

$$\left\{ (\varphi, \psi) \in \text{Arg}(H) \mid \sin^2 \psi = \frac{1}{2} \sin \varphi \sin(\psi - \varphi), \frac{\sin \psi}{\sin(\psi - \varphi)} > 0 \right\} \cup \{(0, \pi), (\pi, \pi)\}.$$ 

The subdivision of $\text{Arg}(H_1)$ is represented in the following picture:

**Lemma 4.12.** The map:

$$\text{Arg}|_{H_1^T} : H_1^T \longrightarrow \text{Arg}(H_1^T)$$

is a homeomorphism.
Proof. By Proposition 3.9 we get that $Arg|_{H_{1T}}$ is bijective on $H_{1T} \cap H_{o}$. By Proposition 3.4 we see that if $(\varphi, \psi) \in \{(0, \pi), (\pi, \pi)\}$, then there exists only one point in $H_{1T}$ which is mapped to $(\varphi, \psi)$ under $Arg$. Moreover, it is continuous being the restriction of a continuous map. We easily see that $H_{1T}$ is compact and since $Arg(H_{1T})$ is Hausdorff, we get that $Arg|_{H_{1T}}$ is a homeomorphism. 

Definition 4.13. We set:

$$H_{1\text{trop}} := \{(x,0,\varphi,\pi) \in H_{1\text{trop}} | x \leq 0\}, H_{1\text{trop}}^{T} := \{(0,0,\varphi,\psi) \in H_{1\text{trop}}\}.$$ 

Lemma 4.14. Let $c \geq \ln 2$. The straight line $r \subset \mathbb{R}^{2}$ of equation $y = 2x + c$ intersects the boundary of the amoeba of $H$ in two points. Indeed, it has no intersection points with the curve $e^{x} - e^{y} = 1$. The intersection point between $r$ and the curve $e^{y} - e^{x} = 1$ is unique and it is given by $x = \ln\left(\frac{1+\sqrt{1+4e^{c}}}{2e^{c}}\right)$. Moreover, $r$ intersects the curve $e^{y} + e^{x} = 1$ in a unique point given by $x = \ln\left(\frac{1-\sqrt{1+4e^{c}}}{2e^{c}}\right)$.

Lemma 4.15. Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable strictly concave function. Let $P \in \mathbb{R}^{2}$ and $c \in [a, b]$. Let $r := \{(x, y) \in \mathbb{R}^{2} | y = mx + q\}$ the straight line passing through $P$ and through $(c, f(c))$. If $m \geq f'(a)$ or $m \leq f'(b)$, then $(c, f(c)) \in \mathbb{R}^{2}$ is the unique point of intersection between $r$ and the graph of $f$.

Proof. Assume $m \geq f'(a)$. By contradiction, let $(d, f(d)) \in \mathbb{R}^{2}$ another point of intersection, with $d \in [a, b]$ and assume $d > c$. Then, by the Mean value theorem there exists $c' \in (c,d)$ such that $f'(c') = m$. Therefore, we get $f'(c') \geq f'(a)$. On the other hand, since the derivative of a strictly concave function is monotonically strictly decreasing, being $a < c'$, we get $f'(a) > f'(c')$. A similar contradiction we get assuming $m \leq f'(b)$. 

Proposition 4.16. Let $c \geq \ln 2$ and $W_{c} := \{(x, y, \varphi, \psi) \in H | y = 2x + c\}$. Let us consider the point $(x_{0}, y_{0}, \varphi_{0}, \psi_{0}) \in W_{c}$, with $0 \leq \varphi_{0} \leq \pi$. Let $s$ be the straight line in $(S^{1})^{2}$ passing through $(\varphi_{0}, \psi_{0})$ and $O$. Then, for $0 \leq \varphi \leq \pi$, $(\varphi_{0}, \psi_{0})$ is the unique point of intersection between $s$ and $Arg(W_{c})$. Similarly, let $(x_{0}, y_{0}, \varphi_{0}, \psi_{0}) \in W_{c}$, with $\pi \leq \varphi_{0} \leq 2\pi$. Let $s'$ be the straight line in $(S^{1})^{2}$ passing through $(\varphi_{0}, \psi_{0})$ and $O'$. Then, for $\pi \leq \varphi \leq 2\pi$, $(\varphi_{0}, \psi_{0})$ is the unique point of intersection between $s'$ and $Arg(W_{c})$.

Proof. By Proposition 3.4 and Lemma 3.7 we have that $(\varphi, \psi) \in Arg(W_{c})$ if and only if either $(\varphi, \psi) \in \{(0, \pi), (\pi, \pi)\}$ or it satisfies:

$$\begin{align*}
e^{x} &= \frac{\sin \psi}{\sin(\psi - \varphi)} \\
e^{y} &= \frac{\sin \varphi}{\sin(\psi - \varphi)} \\
y &= 2x + c
\end{align*}$$

which is equivalent to:

$$\begin{align*}
sin^{2}(\psi) &= k \sin(\varphi) \sin(\psi - \varphi) \\
\frac{-\sin \psi}{\sin(\psi - \varphi)} &> 0 \\
\frac{-\sin \psi}{\sin(\psi - \varphi)} &> 0
\end{align*}$$
where \( k := \frac{1}{e} \). We show that \( \text{Arg}(W_c) \), seen as a function, \( \psi = f(\varphi) \), is a differentiable strictly concave function on the interval \( 0 \leq \varphi \leq \pi \). By implicit differentiation, we get that its first derivative is:

\[
\frac{d\psi}{d\varphi} = \frac{2k \sin(\psi - 2\varphi)}{2 \sin(2\psi) - k \sin \psi + k \sin(\psi - 2\varphi)}.
\] (14)

We have that \( \frac{d\psi}{d\varphi} \) is continuous on \( 0 < \varphi < \pi \) and it can be naturally extended to obtain continuity on the closed interval. Moreover, we have that \( \frac{d\psi}{d\varphi} \) is strictly decreasing. To show that, let \( \pi_+ \) be as in Proposition 3.6, we notice that the curve (13) corresponds in \( \pi(H) \) to the path given by the line segment \( V := \{ y = 2x + c \} \cap \pi(H) \) via the diffeomorphism \( \text{Arg} \circ \pi_+^{-1} \), which maps \( (0,\pi), (\pi, \pi) \in \text{Arg}(W_c) \) to the points in \( V \) respectively with coordinate \( x_0 \) and \( x_1 \), where:

\[
e^{-x_0} = \frac{k}{2} (-1 + \sqrt{1 + \frac{4}{k}}) \quad \text{and} \quad e^{x_1} = \frac{k}{2} (1 + \sqrt{1 + \frac{4}{k}}),
\]
as stated in Lemma 4.14. Therefore, we want to rewrite (14) in the \((x,y)\) coordinates. We have that (14) equals:

\[
\frac{2k(\cos(2\varphi) - 2 \cos \varphi \cos \psi \frac{\sin \varphi}{\sin \psi})}{4 \cos \psi - k + k(\cos(2\varphi) - 2 \cos \varphi \cos \psi \frac{\sin \varphi}{\sin \psi})}.
\]

Using (12), the identity \( \cos(2\varphi) = -1 + 2 \cos^2 \varphi \) and the expressions for \( \cos \varphi, \cos \psi \) given in (1), we can rewrite the last expression as:

\[
\frac{2k(-e^{2y} + 1)}{2(e^{4x-y} - e^{2x+y} - e^{2x-y} - k(e^{2x} + e^{2y} - 1))},
\]

which, using the relations \( y = 2x + c \) and \( k = \frac{1}{e} \), can be rewritten as:

\[
\frac{2e^{4x} - 2k^2}{3e^{4x} - k^2 e^{2x} + k^2}.
\] (15)

Now,

\[
\frac{d}{dx} \left( \frac{2e^{4x} - 2k^2}{3e^{4x} - k^2 e^{2x} + k^2} \right) = 4k^2 e^{2x} - e^{4x} + 8e^{2x} - k^2 \cdot \frac{3e^{4x} - k^2 e^{2x} + k^2}{(3e^{4x} - k^2 e^{2x} + k^2)^2}.
\] (16)

We have that (16) is positive if and only if \( 4 - \sqrt{16 - k^2} < e^{2x} < 4 + \sqrt{16 - k^2} \). For \( x \in [x_0,x_1] \), we have that (16) is always positive. Moreover, by (1) we have:

\[
\cos \varphi = \frac{e^{2x} - e^{2x} - 1}{2e^x},
\]
hence for \( 0 \leq \varphi \leq \pi \), if \( \varphi \) strictly increases, then \( x \) strictly decreases. Thus, \( \frac{d\psi}{d\varphi} \) is strictly decreasing. Now, we have that the straight line passing through \( O \) and \( (0,\pi) \) has angular coefficient equal to \( \frac{1}{2} \), while the straight line passing through \( O \) and \( (\pi, \pi) \) has angular coefficient equal to \( -1 \). Therefore, let \( (\varphi_0, \psi_0) \in \text{Arg}(W_c) \), with \( 0 \leq \varphi_0 \leq \pi \), and \( s \) be as in the statement. One easily sees that the angular
coefficient $m$ of $s$ satisfies $m \geq \frac{1}{2}$ or $m \leq -1$. On the other hand, we notice that if $c \geq \ln 2$, then $k \leq \frac{1}{2}$, which, in turn, implies $\frac{d\psi}{d\phi}((0, \pi)) \leq \frac{1}{2}$. Similarly, $k \leq \frac{1}{2}$ implies $\frac{d\psi}{d\phi}((\pi, \pi)) \geq -1$. Therefore, since by assumption $c \geq \ln 2$, we have $m \geq \frac{d\psi}{d\phi}((0, \pi))$ or $m \leq \frac{d\psi}{d\phi}((\pi, \pi))$. Thus, by Lemma 4.15, we have that it does not exist another point of intersection $(\varphi, \psi)$ between $s$ and $\text{Arg}(W_c)$, with $0 \leq \varphi \leq \pi$. The second part of the statement follows from the fact that $H$ is symmetric with respect to the transformation $(x, y, \varphi, \psi) \mapsto (x, y, 2\pi - \varphi, 2\pi - \psi)$.

\[ \text{Figure 10: In red } \text{Arg}(W_c) \text{ for } k = \frac{1}{2}, \text{ in green for } k = \frac{1}{8}. \]

5 The isotopy map

**Definition 5.1.** Let $A_1, A_2 \subset \text{Arg}(H_1)$ as in Definition 4.2. Let $(\varphi, \psi) \in A_1$. Let $\text{Arg}(\Gamma_1)$ as in Proposition 4.11 and $r \subset (S^1)^2$ the straight line passing through $O$ and through $(\varphi, \psi)$. Let $Q' \in \text{Arg}(H_1)$ the point of intersection between $r$ and the straight line in $(S^1)^2$ of equation $\psi = \pi$. Let $d : (S^1)^2 \times (S^1)^2 \to \mathbb{R}_{\geq 0}$ be the flat metric on $(S^1)^2$. We set:

\[ b := d(O, Q'). \]

If $(\varphi, \psi) \in A_1 \cap \text{Arg}(H_{1T})$, let $Q$ be the intersection point between $r$ and $\text{Arg}(\Gamma_1)$, we set:

\[ a := d(O, Q). \]

If $(\varphi, \psi) \in A_1 \cap \text{Arg}(H_{1L})$, then we set:

\[ a' := d(O, (\varphi, \psi)). \]

See Figure 11 below:
Now, for any \( t \in [0, 1] \) we define a map:

\[
(\varphi_t, \psi_t) : \text{Arg}(H_1) \longrightarrow \overline{\text{Arg}(H_1)},
\]

by:

\[
(\varphi_t, \psi_t)|_{A_1} : (\varphi, \psi) \mapsto \begin{cases} 
O + \left(\frac{b}{a}\right)^t((\varphi, \psi) - O), & \text{if } (\varphi, \psi) \in A_1 \cap \text{Arg}(H_{1T}) \setminus \{O\}; \\
O + \left(\frac{b}{a}\right)^t((\varphi, \psi) - O), & \text{if } (\varphi, \psi) \in A_1 \cap \text{Arg}(H_{1L}) \setminus \{O\}; \\
O, & \text{if } (\varphi, \psi) = O.
\end{cases}
\]

The definition of \((\varphi_t, \psi_t)|_{A_2}\) is obtained by replacing \(O\) with \(O'\) and \(A_1\) with \(A_2\) in the above definitions of \(a, a', b\) and \((\varphi_t, \psi_t)|_{A_1}\).

**Claim 5.2.** The map \((\varphi_t, \psi_t)\) is well defined.

**Proof.** If \((\varphi, \psi) \in A_1 \cap \text{Arg}(H_{1T})\), then \(a\) is well defined as the intersection point \(Q\) between \(r\) and \(\text{Arg}(\Gamma_1)\) is unique by Proposition 4.16. We immediately see that on the intersection between \(\text{Arg}(H_{1T})\) and \(\text{Arg}(H_{1L})\), i.e. on the curve \(\text{Arg}(\Gamma_1)\), the definitions of \(a\) and \(a'\) coincide. Moreover, on \(A_1 \cap A_2\), \((\varphi_t, \psi_t)|_{A_1} = (\varphi_t, \psi_t)|_{A_2}\). □

**Lemma 5.3.** The map \((\varphi_t, \psi_t)\) is continuous. Moreover, it is injective on \(\text{Arg}(H_{1T})\) and \((\varphi_1, \psi_1)(\text{Arg}(H_{1T})) = \overline{\text{Arg}(H_1)}\).

**Proof.** The map is clearly continuous. From the definition of \((\varphi_t, \psi_t)\) one easily sees that \(O\) and \(O'\) are the unique points respectively sent to \(O\) and \(O'\). Now, let \((\varphi, \psi) \neq O, O'\) and \((\varphi', \psi') \neq O, O'\) in \(\text{Arg}(H_{1T})\). By construction, we have that if \((\varphi_t, \psi_t)(\varphi, \psi) = (\varphi_t, \psi_t)(\varphi', \psi')\), then \((\varphi, \psi)\) and \((\varphi', \psi')\) lie on the same straight
line passing through $O$ or through $O'$. Since this straight line uniquely determines \( a \) and \( b \), we get \( (\varphi, \psi) = (\varphi', \psi') \). Finally, the last part of the statement follows immediately from the definition of \( (\varphi_1, \psi_1) \).

**Definition 5.4.** Let \( P = (x, y, \varphi, \psi) \in H_1 \). For each \( t \in [0, 1] \), we define maps:

\[
\Phi^t_{1T} : H_{1T} \rightarrow \mathbb{R}^2 \times (S^1)^2 \\
(x, y, \varphi, \psi) \mapsto (x(1-t), y(1-t), \varphi_t, \psi_t)
\]

and

\[
\Phi^t_{1L} : H_{1L} \rightarrow \mathbb{R}^2 \times (S^1)^2 \\
(x, y, \varphi, \psi) \mapsto (x - t\frac{y - \ln 2}{2}, y(1-t), \varphi_t, \psi_t).
\]

Now, we define:

\[
\Phi^t_1 : H_1 \rightarrow \mathbb{R}^2 \times (S^1)^2
\]

by:

\[
\Phi^t_1 : P \mapsto \begin{cases} 
\Phi^t_{1T}(P) & \text{if } P \in H_{1T}; \\
\Phi^t_{1L}(P) & \text{if } P \in H_{1L}.
\end{cases}
\]  

(18)

**Claim 5.5.** The map \( \Phi^t_1 \) is well defined.

**Proof.** On the subset \( S := H_{1T} \cap H_{1L} = \{(x, y, \varphi, \psi) \in H_1 | y = 2x + \ln 2\} \), the two equations defining \( \Phi^t_1 \) agree. Indeed, they clearly agree on the first two components, for all \( (x, y, \varphi, \psi) \in S \). They also agree on the last two components as, by Claim \( 5.2 \), the definitions of \( a \) and \( a' \) agree on \( S \).

**Proposition 5.6.** Let \( i_{H_1} : H_1 \hookrightarrow (\mathbb{C}^*)^2 \) be the canonical inclusion. We have \( \Phi^0_1 = i_{H_1} \) and \( \Phi^1_1(H_1) = H_{1trop} \).

**Proof.** The first part of the statement follows immediately by substituting \( t = 0 \) in the definition of \( \Phi^t_1 \). From the definition of \( \Phi^t_{1T} \) and from Lemma \( 5.3 \) we get \( \pi(\Phi^t_{1T}(H_{1T})) = (0, 0) \) and \( Arg(\Phi^t_{1T}(H_{1T})) = (\varphi_1, \psi_1)(H_{1T}) = Arg(H_1) \), thus \( \Phi^t_{1T}(H_{1T}) = H_{1trop} \). Now, let \( c \geq \ln 2 \). Consider the subset of \( H_{1L} \):

\[
W_c := \{(x, y, \varphi, \psi) \in H_{1L} | y = 2x + c\}.
\]

From the definition of \( \Phi^t_{1L} \), we get:

\[
\Phi^t_{1L}(W_c) = \left\{(x - \frac{2x + c - \ln 2}{2}, 0, \varphi, \pi) | \varphi \in S^1 \right\}
\]

that is:

\[
\Phi^t_{1L}(W_c) = \left\{(\frac{-c + \ln 2}{2}, 0, \varphi, \pi) | \varphi \in S^1 \right\}.
\]
Since we have:

\[ H_{1L} = \bigcup_{c \geq \ln 2} W_c, \]

we get:

\[ \Phi_{1L}(H_{1L}) = \left\{ \left( \frac{-c + \ln 2}{2}, 0, \varphi, \pi \right) | \varphi \in S^1, c \geq \ln 2 \right\}. \]

From \( c \geq \ln 2 \), we get \( \frac{-c + \ln 2}{2} \leq 0 \). Therefore, \( \Phi_{1L}(H_{1L}) = H_{1\text{trop}L} \).

**Proposition 5.7.** The map \( \Phi_1 \) is injective.

**Proof.** Let \( P, P' \in H_{1T} \) and assume \( \Phi_{1T}(P) = \Phi_{1T}(P') \). By Lemma 5.3, the map \((\varphi_t, \psi_t)\) is injective on \( \text{Arg}(H_{1T}) \), hence \( \varphi = \varphi' \) and \( \psi = \psi' \) and using Lemma 4.12 we get \( P = P' \). Now, let \( P, P' \in H_{1L} \) and assume \( \Phi_{1L}(P) = \Phi_{1L}(P') \). If \( t \neq 1 \), then \( x = x' \) and \( y = y' \). Therefore, by Proposition 3.4 \((\varphi, \psi) = (\varphi', \psi') \) or \((\varphi, \psi) = (2\pi - \varphi', 2\pi - \psi') \). If both hold, then we get \((\varphi, \psi) = (\varphi', \psi') = (\pi, \pi) \) and hence \( P = P' \). It cannot happen that only the second equality holds as by definition of \((\varphi_t, \psi_t)\), \( \Phi_{1L}(P) = \Phi_{1L}(P') \) implies that \((\varphi, \psi) \) and \((\varphi', \psi') \) are either both in \( A_1 \) or both in \( A_2 \). If \( t = 1 \), then \( \Phi_{1L}(P) = \Phi_{1L}(P') \) implies that \((\varphi, \psi) \) and \((\varphi', \psi') \) lie both on the same straight line passing through \( O \) or through \( O' \). On the other hand, we also get \( 2(x - x') = y - y' \), which means that \( P, P' \in W_c \), where \( c \geq \ln 2 \). Thus, by Proposition 4.16 we get \((\varphi, \psi) = (\varphi', \psi') \), and hence \( P = P' \) as \( \text{Arg}|W_c \) is injective.

We recall the following well-known results in general topology:

**Lemma 5.8.** Let \( f : X \rightarrow Y \) be a continuous map between locally compact Hausdorff spaces. If \( f \) is proper, then it is closed.

**Lemma 5.9.** Let \( f : X \rightarrow Y \) be a bijective continuous map. If \( f \) is closed, then it is bicontinuous.

**Lemma 5.10.** Let \( p : X \times Y \rightarrow X \) be the projection onto the first factor. If \( Y \) is compact, then \( p \) is proper.

Now, we prove:

**Lemma 5.11.** For any \( t \in [0, 1] \), the map \( \Phi_{1L} : H_{1L} \rightarrow \Phi_{1L}(H_{1L}) \) is a proper map between locally compact Hausdorff spaces.

**Proof.** For any \( t \in [0, 1] \), \( \Phi_{1L}(H_{1L}) \) is clearly a closed subset of \( \mathbb{R}^2 \times (S^1)^2 \), hence it is locally compact and Hausdorff. Now, we show that \( \Phi_{1L} \) is proper.

Let \( \pi : \mathbb{R}^2 \times (S^1)^2 \rightarrow \mathbb{R}^2 \) be the projection and consider the family of maps:

\[ g_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]
\[(x, y) \mapsto (x - t \frac{y - \ln 2}{2}, y(1 - t)),\]

for \(t \in [0, 1]\). We have the following commutative diagram:

\[
\begin{array}{ccc}
H_{1L} & \xrightarrow{\Phi^t_{1L}} & \Phi^t_{1L}(H_{1L}) \\
\pi|_{H_{1L}} & & \pi|_{\Phi^t_{1L}(H_{1L})} \\
\pi(H_{1L}) & \xrightarrow{g|_{\pi(H_{1L})}} & \pi(\Phi^t_{1L}(H_{1L}))
\end{array}
\]

For \(t \neq 1\), the map \(g_t\) is a homeomorphism, hence \(g_t|_{\pi(H_{1L})}\) is proper. If \(t = 1\), then we see that \(g_1\) projects the points on the straight line \(y = 2x + c\), for \(c \in \mathbb{R}\), to the point \((\frac{-c + \ln 2}{2}, 0)\). Thus, \(g_1|_{\pi(H_{1L})}\) is also proper. Now, by Lemma 5.10, \(\pi\) is a proper map and since \(\Phi^t_{1L}(H_{1L})\) is a closed subset of \(\mathbb{R}^2 \times (S^1)^2\), we have that \(\pi|_{\Phi^t_{1L}(H_{1L})}\) is still a proper map for all \(t \in [0, 1]\). Thus, \(g_t|_{\pi(H_{1L})} \circ \pi|_{H_{1L}}\) is a proper map. But, \(g_t|_{\pi(H_{1L})} \circ \pi|_{H_{1L}} = \pi|_{\Phi^t_{1L}(H_{1L})} \circ \Phi^t_{1L}\), one easily sees that \(\Phi^t_{1L}\) is a proper map too. \(\square\)

**Theorem 5.12.** The map \(\Phi^t_1 : H_1 \rightarrow \Phi^t_1(H_1)\) is a homeomorphism.

**Proof.** The map is bijective by Proposition 5.7 and it is clearly continuous. The map
\(\Phi^t_{1T} : H_{1T} \rightarrow \Phi^t_{1T}(H_{1T})\) is bicontinuous as it is a bijective continuous map from a compact space onto a Hausdorff space. We also have that \(\Phi^t_{1L} : H_{1L} \rightarrow \Phi^t_{1L}(H_{1L})\) is bicontinuous. Indeed, by Lemma 5.11, \(\Phi^t_{1L}\) is a proper map between locally compact Hausdorff spaces. Therefore, by Lemma 5.8, \(\Phi^t_{1L}\) is a closed map and hence, by Lemma 5.9, it is bicontinuous. \(\square\)

**Theorem 5.13.** Let \(H \subset (\mathbb{C}^*)^2\) the complex line \(1 + z_1 + z_2 = 0\) and \(H_{\text{trop}} \subset (\mathbb{C}^*)^2\) its associated phase tropical line. Let \(i_H : H \hookrightarrow (\mathbb{C}^*)^2\) be the canonical embedding. Then there exists a continuous map:

\[\Psi : H \times [0, 1] \rightarrow (\mathbb{C}^*)^2\]

such that the family of maps:

\[\Psi_t : H \rightarrow (\mathbb{C}^*)^2, P \mapsto \Psi(P, t),\]

for \(t \in [0, 1]\), has the following properties:

1. \(\Psi_0 = i_H;\)
2. \(\Psi_1(H) = H_{\text{trop}};\)
3. \(\Psi_t\) is a homeomorphism onto the image, for each \(t \in [0, 1]\).

**Proof.** Let \(t \in [0, 1]\). \(\Phi^t_1\) as in Definition 5.4 and Theorem 5.12, let \(\lambda\) be as in Lemma 4.7 and \(H_2, H_3 \subset H\) as in Definition 4.1. We can define two maps:

\[\Phi^t_2 : H_2 \rightarrow \mathbb{R}^2 \times (S^1)^2, \Phi^t := \lambda \circ \Phi^t_1 \circ \lambda^{-1}\]
and
\[ \Phi_t^3 : H_3 \longrightarrow \mathbb{R}^2 \times (S^1)^2, \Phi_t^3 := \lambda^{-1} \circ \Phi_t^1 \circ \lambda. \]

Using Lemma 4.7 we easily see that \( \Phi_0^2 = i_{H_2}, \Phi_0^3 = i_{H_3} \) and \( \Phi_1^2(H_2) = H_{2\text{trop}}, \)
\( \Phi_3^3(H_3) = H_{3\text{trop}}. \) Moreover, \( \Phi_t^2 \) and \( \Phi_t^3 \) are homeomorphisms onto their images being compositions of homeomorphisms. We define a map:
\[ \Psi_t' : H \longrightarrow \mathbb{R}^2 \times (S^1)^2 \]
defined by:
\[ \Psi_t' : P \mapsto \begin{cases} 
\Phi_t^1(P), & \text{if } P \in H_1 \\
\Phi_t^2(P), & \text{if } P \in H_2 \\
\Phi_t^3(P), & \text{if } P \in H_3
\end{cases} \tag{19} \]

It is clearly well defined. Let us consider now the homeomorphism:
\[ h : (\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2 \times (S^1)^2 \]
as in Definition 2.3. The map
\[ \Psi_t := h^{-1} \circ \Psi_t' \circ h \]
satisfies properties 1.-3. in the statement of the theorem. Now,
\[ \Psi : H \times [0, 1] \longrightarrow (\mathbb{C}^*)^2, (P, t) \mapsto \Psi_t(P) \]
is the claimed map.

\[ \square \]

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