SUBANALYTIC SETS AND COMPLEX ANALYTIC GEOMETRY

YA’ACOV PETERZIL AND SERGEI STARCHENKO

Preliminary Report

In a series of papers ([2], [3], [4]) we considered holomorphic manifolds and maps definable in o-minimal structures, over arbitrary real closed fields. In this short note we translate some of these and more recent results into the subanalytic setting. The theorems are of the following nature: We consider a subanalytic subset $A$ of a complex analytic manifold $M$ (when $M$ is viewed as a real manifold) and formulate conditions under which $A$ is a complex analytic subset of $M$.

Remarks.
1. Even though we formulate the results in the more familiar subanalytic language, they all remain true when the subanalytic category is replaced by any analytic-geometric category over $\mathbb{R}$ (in the sense of van den Dries–Miller [1]); $\mathcal{C}_{\text{an,exp}}$ is an example of such a category.
2. Because we prove most of the theorems in a very general o-minimal setting, most of the results yield certain uniformity in parameters in the appropriate category.

Notation and terminology. For a real analytic manifold $X$ and a subanalytic subset $A$ of $X$ we will denote by $\dim_{\mathbb{R}}(A)$ the usual subanalytic dimension of $A$, i.e. the maximal $d \in \mathbb{N}$ such that $A$ contains a $d$-dimensional $C^1$ submanifold of $X$.

We say that a set $S \subseteq \mathbb{C}^n$ is a subanalytic subset of $\mathbb{C}^n$ if $S$ is a subanalytic subset of $\mathbb{R}^{2n}$ under the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$.

We extend these notions to complex analytic manifolds and their subsets in the obvious way:
For a complex analytic manifold $M$ and $S \subseteq M$ we say that $S$ is a subanalytic subset of $M$ if $S$ is a subanalytic subset of $M$ considered as a real analytic manifold. For such $S$ we will denote by $\dim_{\mathbb{R}}(S)$ its dimension (as a subanalytic subset of $M$).

Date: September 21, 2004.
Recall that for a complex analytic manifold $M$, a subset $S \subseteq M$ is called a locally complex analytic subset of $M$ if for every $p \in S$ there is open $V_p \subseteq M$ containing $p$ such that $S \cap V_p$ is the zero locus of finitely many holomorphic on $V_p$ functions. A locally subanalytic subset $S$ of a complex analytic manifold $M$ is called complex analytic subset of $M$ if, in addition, $S$ is closed in $M$. 
1. Removing singularities from complex analytic sets

**Theorem 1.1.** Let $M$ be a complex analytic manifold, $E \subseteq M$ a closed subanalytic subset of $M$ and $A$ a complex analytic subset of $M \setminus E$ which is also a subanalytic subset of $M$.

Assume that for every open $V \subseteq M$, we have $\text{dim}_\mathbb{R}(V \cap E) \leq \text{dim}_\mathbb{R}(V \cap A) - 2$. Then $\text{Cl}(A)$, the closure of $A$ in $M$, is a complex analytic subset of $M$.

The theorem is an immediate consequence of Shiffman’s Theorem in the case when $A$ is of pure dimension. However, it is false as stated without the subanalyticity assumption:

Take $M = \mathbb{C}^3$, $E = \{(0, y, 1) \in \mathbb{C}^3\}$, and $A = \{(x, e^{1/x}, 1) : x \neq 0\} \cup \{(0, y, z) : z \neq 1\}$.

Using properties of geometric-analytic categories one can derive from the above theorem the following useful result (which again fails without the subanalyticity assumption).

**Theorem 1.2** (Closure Theorem). Let $M$ be a complex manifold and $E \subseteq M$ a complex analytic subset of $M$ (of arbitrary dimension). If $A$ is a complex analytic subset of $M \setminus E$ which is also a subanalytic subset of $M$ then $\text{Cl}(A)$ is a complex analytic subset of $M$.

Another corollary is the following:

**Theorem 1.3** (The Union Theorem). Let $M$ be a complex manifold and $\{A_n : n \in \mathbb{N}\}$ a family of locally complex analytic subsets of $M$. If $A = \bigcup_{n \in \mathbb{N}} A_n$ is a closed subset of $M$ which is also subanalytic in $M$ then $A$ is a complex analytic subset of $M$.

Again, note that the theorem fails without the subanalyticity assumption, even for a union of two locally analytic sets: Take $A_1$ be the graph of the map $e^{1/z}$ in $\mathbb{C} - \{0\} \times \mathbb{C}$ and $A_2 = \{0\} \times \mathbb{C}$.

2. Holomorphic maps and subanalytic sets

2.1. The image under holomorphic maps.

**Theorem 2.1.** Let $f : M \to N$ be a holomorphic map between complex analytic manifolds, $A$ a complex analytic subset of $M$. If $f(A)$ is a closed and subanalytic subset of $N$ then $f(A)$ is a complex analytic subset of $N$.

**Remarks**

1. Notice that Remmert’s Proper Mapping Theorem follows from the above theorem since, in the above notation, if $f : M \to N$ is proper then $f(A)$ is necessarily a subanalytic closed subset of $N$. However,
the theorem still applies to cases where \( f \) is not proper and yet \( f(A) \) is a subanalytic subset of \( N \).

2. The theorem is false without the subanalyticity assumption on \( f(A) \). An example is the projection of the set \( \{(0,0)\} \cup \{(1/n,n) : n \in \mathbb{N}, n \neq 0\} \subseteq \mathbb{C}^2 \) onto the first coordinate.

2.2. **Meromorphic maps.** The following theorem also fails without subanalyticity assumption. (An easy example is the function \( e^{1/z} \).)

**Theorem 2.2.** Let \( M, N \) be complex analytic manifolds and \( E \subseteq M \) a closed subanalytic subset of \( M \) such that \( \dim \mathbb{R} E \leq \dim \mathbb{R} M - 2 \). If \( f : M \setminus E \to N \) is a holomorphic map whose graph is a subanalytic subset of \( M \times N \) then \( f \) is a meromorphic map from \( M \) to \( N \), namely, the closure of the graph of \( f \) is a complex analytic subset of \( M \times N \).

If \( N = \mathbb{C} \) then \( f \) can be written locally, at every point of \( M \), as a quotient of two holomorphic functions.

**Corollary 2.3.** Let \( M, N \) be complex analytic manifolds, \( X \subseteq M \) a complex analytic subset of \( M \), \( Y \subseteq N \) a complex analytic subset of \( N \), and \( f : \text{Reg}(X) \to \text{Reg}(Y) \) a bi-holomorphism. Then \( f \) extends to a bi-meromorphism from \( X \) to \( Y \) if and only if the graph of \( f \) is a subanalytic subset of \( M \times N \).

**References**

[1] Lou van den Dries and Chris Miller, *Geometric categories and o-minimal structures*, Duke Math. J. 84 (1996), 497–540.

[2] Ya’acov Peterzil and Sergei Starchenko, *Expansions of algebraically closed fields in o-minimal structures*, Selecta Math. (N.S.) 7 (2001), 409–445.

[3] ———, *Expansions of algebraically closed fields. II. Functions of several variables*, J. Math. Log. 3 (2003), 1–35.

[4] ———, *Complex manifolds and analytic sets. An o-minimal viewpoint*, In preparation.

Department of Mathematics, University of Haifa, Haifa, ISRAEL

E-mail address: kobi@math.haifa.ac.il

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556

E-mail address: starchenko.1@nd.edu