The Ma-Qiu index and the Nakanishi index for a fibered knot are equal, and $\omega$-solvability

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Abstract

For a knot $K$ in $S^3$, let $G(K)$ be the knot group of $K$, $a(K)$ the Ma-Qiu index (the MQ index, for short), which is the minimal number of normal generators of the commutator subgroup of $G(K)$, and $m(K)$ the Nakanishi index of $K$, which is the minimal number of generators of the Alexander module of $K$. We generalize the notions for a pair of a group $G$ and its normal subgroup $N$, and we denote them by $a(G, N)$ and $m(G, N)$ respectively. Then it is easy to see $m(G, N) \leq a(G, N)$ in general. We also introduce a notion “$\omega$-solvability” for a group that the intersection of all higher commutator subgroups is trivial. Our main theorem is that if $N$ is $\omega$-solvable, then we have $m(G, N) = a(G, N)$. As corollaries, for a fibered knot $K$, we have $m(K) = a(K)$, and we could determine the MQ indices of prime knots up to 9 crossings completely.

1 Introduction

For a group $G$ and its normal subgroup $N$, $y_1, \ldots, y_d \in N$ (possibly $d$ is infinite) are normal generators of $N$ if any element $x$ in $N$ is a product of conjugates of them:

$$x = \prod_{j=1}^{r} g_j^{-1} y_{i_j}^{\varepsilon_j} g_j \quad (g_j \in G, \varepsilon_j = \pm 1, i_j \in \{1, \ldots, d\}; j = 1, \ldots, r)$$

where $r$ is finite. Then we denote by $N = \langle \langle y_1, \ldots, y_d \rangle \rangle$, and the minimal number of such $d$ is called the Ma-Qiu index (the MQ index, for short) of the group pair $(G, N)$, which is denoted by $a(G, N)$. It is easy to see that $a(G, N) = 0$ if and only if $N$ is trivial (i.e. $N = \{1_G\}$). J. Ma and R. Qiu [MQ] originally defined the MQ index for the case that $G$ is the knot group of a knot $K$ in $S^3$ and $N$ is the first commutator subgroup of $G$ (i.e. $N = [G, G]$), and it is denoted by $a(K)$. Then $a(K) = 0$ if and only if $K$ is the trivial knot by the Dehn lemma as pointed out in [MQ]. Their first main theorem is that the MQ index is a lower bound of the unknotting number (cf.

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minimal size of the Alexander matrix of original Nakanishi index, defined by Y. Nakanishi [Na], is for the case that Kw1 gave an alternative definition that respectively (i.e. [Kw1]). For the proof, they used a modified Wirtinger presentation. Proposition 2.1 (1)). For the Kinoshita-Terasaka knot $K_{KT}$ [KT] and the Conway knot $K_C$ [Co], since they both have the unknotting number one [BM] and they are both non-trivial, we have $a(K_{KT}) = a(K_C) = 1$. Ma and Qiu also asked in [MQ] about additivity of the MQ index under connected sum (cf. Proposition 2.1 (2)). The author and Yang [KY] provided counterexamples for additivity (“1 + 1 = 1”) of the MQ index (cf. Proposition 2.1 (5)). We also showed that the MQ index is a lower bound of both the rank−1 and the tunnel number (cf. Proposition 2.1 (3) and (4)). Note that Ma and Qiu did not use the term “MQ index”, but the author and Yang [KY] firstly used the term.

We regard $N/[N, N]$ as a group homology of $N$ where $[N, N]$ is the first commutator subgroup of $N$. A multiplication in $N/[N, N]$ is an addition in the homology. Moreover in the present case, a conjugation of $n \in N$ by $g \in G$ induces a multiplication to $[n] \in N/[N, N]$ by $\overline{g} \in G/N$ where $\overline{g}$ and $\overline{n}$ are the induced elements from $g$ and $n$ respectively (i.e. $[g^{-1}ng] = [n] \cdot \overline{g}$ ($g \in G$; $n \in N$)). Thus $N/[N, N]$ has a natural right $\mathbb{Z}[G/N]$-module structure. We denote it by

$$H(G, N) = H_1(N; \mathbb{Z}[G/N]),$$

and call it the group homology of $(G, N)$. The Nakanishi index of the group pair $(G, N)$, denoted by $m(G, N)$, as the minimal number of generators of $H(G, N)$ as a right $\mathbb{Z}[G/N]$-module. It is easy to see that $m(G, N) = 0$ if and only if $H(G, N) = 0$, and $m(G, N) = 1$ if and only if $H(G, N)$ is a non-trivial cyclic $\mathbb{Z}[G/N]$-module. The original Nakanishi index, defined by Y. Nakanishi [Na], is for the case that $G$ is the knot group of a knot $K$ in $S^3$ and $N$ is the first commutator subgroup of $G$, the index is the minimal size of the Alexander matrix of $K$, and it is denoted by $m(K)$. A. Kawauchi [Kw1] gave an alternative definition that $m(K)$ is minimal number of generators of the Alexander module of $K$. Then $m(K) = 0$ if and only if the Alexander polynomial of $K$, denoted by $\Delta_K(t)$, is trivial (i.e. $\Delta_K(t) = 1$) (cf. [Ne, Theorem 4.9.1]). It is easy to see that

$$m(G, N) \leq a(G, N)$$

(1.1) because a set of normal generators realizing $a(G, N)$ also generates $H(G, N)$ via the projections $G \to G/N$ and $N \to N/[N, N]$. The equality of (1.1) does not hold in general. For the Kinoshita-Terasaka knot $K_{KT}$ [KT] and the Conway knot $K_C$ [Co], we have $m(K_{KT}) = m(K_C) = 0$ and $a(K_{KT}) = a(K_C) = 1$.

For a group $H$, we denote the $n$-th commutator subgroup of $H$ by $D^{(n)}(H)$. In particular, $D^{(0)}(H) = H$ and $D^{(1)}(H) = [H, H]$. Then $H$ is solvable if there exists a finite $n$ such that $D^{(n)}(H) = \{1_H\}$. We define that $H$ is $\omega$-solvable if the intersection of all $D^{(n)}(H)$ is $\{1_H\}$. It is easy to see that a solvable group is $\omega$-solvable. A free group with rank greater than one is not solvable, but $\omega$-solvable (see Section 2).

The following is our main theorem:

**Theorem 1.1** For a group pair $(G, N)$, if $N$ is $\omega$-solvable, then we have $m(G, N) = a(G, N)$. 

For a fibered knot $K$, since the commutator subgroup of $G(K)$ is a free group with rank greater than one, which is the fundamental group of the fiber surface, $G(K)$ is $\omega$-solvable. Therefore we have:

**Corollary 1.2** Let $K$ be a fibered knot in $S^3$. Then we have $m(K) = a(K)$.

This is the affirmative answer for Question 5.7 in [KY]. As corollaries, (1) one of the main theorems in [KY], Theorem 1.3, is obtained immediately (cf. Proposition 2.1 (5)), and (2) for prime knots up to 9 crossings, the MQ index is determined completely (see Section 3).

## 2 Preliminaries

For a knot $K$ in $S^3$, let $u(K)$ be the unknotting number of $K$, $t(K)$ the tunnel number of $K$, and $r(K)$ the rank (of the knot group) of $K$. Ma and Qiu [MQ], and the author and Z. Yang [KY] showed the following:

**Proposition 2.1** Let $K$, $K_1$ and $K_2$ be knots in $S^3$.

1. [MQ] $m(K) \leq a(K) \leq u(K)$.
2. [MQ] $\max\{a(K_1), a(K_2)\} \leq a(K_1 \sharp K_2) \leq a(K_1) + a(K_2)$.
3. [KY] Theorem 1.1 $a(K) \leq r(K) - 1$.
4. [KY] Corollary 1.2 $m(K) \leq a(K) \leq \min\{r(K) - 1, u(K)\} \leq \min\{t(K), u(K)\}$.
5. [KY] Theorem 1.3 For odd integers $p$ and $q$ with $|p|, |q| \geq 3$, let $K_{p,q}$ be the connected sum of the $(2, p)$-torus knot and the $(2, q)$-torus knot. Then $m(K_{p,q}) = a(K_{p,q}) = 1$ or 2. Moreover they are 1 if and only if $\gcd(p, q) = 1$.

Here we give the definition of “$\omega$-solvability”. For a group $H$, the $n$-th commutator subgroup of $H$, denoted by $D^{(n)}(H)$, is defined inductively by (C1) and (C2) as follows:

- (C1) : $D^{(0)}(H) = H$ (and $D^{(1)}(H) = [H, H]$),
- (C2) : $D^{(n+1)}(H) = [D^{(n)}(H), D^{(n)}(H)]$,

where for subgroups $H_1$ and $H_2$ of $H$, $[H_1, H_2]$ is the commutator subgroup generated by $H_1$ and $H_2$. We note that $D^{(n)}(H)$ is a characteristic subgroup (i.e. a special case of a normal subgroup, which is stable under any automorphism of $H$), and

$$D^{(0)}(H) = H \supset D^{(1)}(H) = [H, H] \supset \cdots \supset D^{(n)}(H) \supset D^{(n+1)}(H) \supset \cdots \quad (2.1)$$

Then we denote by

$$D^{(\omega)}(H) = \bigcap_{n=0}^{\infty} D^{(n)}(H) \quad (2.2)$$

and we define that $H$ is $\omega$-solvable if $D^{(\omega)}(H) = \{1_H\}$. It is easy to see that a $\omega$-solvable group is a characteristic subgroup, and a solvable group is $\omega$-solvable. A free group with rank greater than one is not solvable, but $\omega$-solvable.
3 Proof of Theorem 1.1

We take \( z_1, \ldots, z_m \in H(G, N) \) such that \( z_1, \ldots, z_m \) generate \( H(G, N) \) as a right \( \mathbb{Z}[G/H] \)-module realizing \( m = m(G, N) \), and \( y_1, \ldots, y_m \in N \) such that \( y_j \) is a lift of \( z_j \) \((j = 1, \ldots, m)\) by the natural projection \( p : N \to N/[N, N] \) (i.e. \( p(y_j) = z_j \) \((j = 1, \ldots, m)\)). Then we set \( Y = \{y_1, \ldots, y_m\} \) and \( \langle \langle y_1, \ldots, y_m \rangle \rangle = \langle \langle Y \rangle \rangle \subset N \). By (1.1), we will only show \( \langle \langle Y \rangle \rangle = N \). Let \( q_i : D^{(i)}(N) \to N/\langle \langle Y \rangle \rangle \) \((i = 0, 1, \ldots)\) be a homomorphism induced by the natural inclusion. Let \( \text{Im}(q_i) \) be the image of \( q_i \) in \( N/\langle \langle Y \rangle \rangle \). Then it is easy to see \( \text{Im}(q_i) \supset \text{Im}(q_{i+1}) \) by (2.1). We will show \( \text{Im}(q_i) \subset \text{Im}(q_{i+1}) \) (as a result, \( \text{Im}(q_i) = \text{Im}(q_{i+1}) \)) for all \( i \) \((i = 0, 1, \ldots)\).

We show the following key lemma:

Lemma 3.1 For any \( a_i, b_i \in D^{(i)}(N) \), there exist \( a_{i+1}, b_{i+1} \in D^{(i+1)}(N) \) and \( c_i, d_i \in \langle \langle Y \rangle \rangle \) such that \( a_i = c_i a_{i+1}, b_i = d_i b_{i+1} \) and

\[
[a_i, b_i] \equiv [a_{i+1}, b_{i+1}] \pmod{\langle \langle Y \rangle \rangle}
\]

for all \( i \) \((i = 0, 1, \ldots)\).

**Proof** Suppose that there exist \( c_i, d_i \in \langle \langle Y \rangle \rangle \) such that \( a_i = c_i a_{i+1}, b_i = d_i b_{i+1} \). Then we have:

\[
[a_i, b_i] = a_i^{-1}b_i^{-1}a_i b_i \\
= (c_i a_{i+1})^{-1}(d_i b_{i+1})^{-1}(c_i a_{i+1})(d_i b_{i+1}) \\
= a_i^{-1}c_i^{-1}b_i^{-1}d_i^{-1}c_i a_{i+1} d_i b_{i+1} \\
= a_i^{-1}c_i^{-1}b_i^{-1}d_i^{-1}c_i (a_i d_i a_i^{-1}) a_{i+1} b_{i+1} \\
= a_i^{-1}c_i^{-1}(b_i^{-1}c_i (a_i d_i a_i^{-1}) b_{i+1}) b_i^{-1} a_{i+1} b_{i+1} \\
= (a_i^{-1} c_i^{-1} (b_i^{-1} c_i (a_i d_i a_i^{-1}) b_{i+1})) a_{i+1} b_{i+1} \\
\equiv [a_{i+1}, b_{i+1}] \pmod{\langle \langle Y \rangle \rangle}.
\]

We show the lemma by induction on \( i \).

1. The case \( i = 0 \).

By the assumption on \( p : N \to N/[N, N] \) and \( Y = \{y_1, \ldots, y_m\} \), for any \( a_0, b_0 \in D^{(0)}(N) = N \), there exist \( a_1, b_1 \in D^{(1)}(N) = [N, N] \) and \( c_0, d_0 \in \langle \langle Y \rangle \rangle \) such that \( a_0 = c_0 a_1, b_0 = d_0 b_1 \) (i.e. \( p(a_0) = p(c_0), p(b_0) = p(d_0) \)). By the calculation above, we have \( [a_0, b_0] \equiv [a_1, b_1] \pmod{\langle \langle Y \rangle \rangle} \).

2. Suppose the case \( i \) \((i \geq 0)\) is true. We show the case \((i + 1)\).

By the assumption on \( i \), for any \( a_i, b_i \in D^{(i)}(N) \), there exist \( a_{i+1}, b_{i+1} \in D^{(i+1)}(N) \) and \( c_i, d_i \in \langle \langle Y \rangle \rangle \) such that \( a_i = c_i a_{i+1}, b_i = d_i b_{i+1} \). By the calculation above, we have
\[ [a_i, b_i] \equiv [a_{i+1}, b_{i+1}] \pmod{\langle Y \rangle}. \] Since any element of \( D^{(i+1)}(N) \) is a product of the form \([a_i, b_i] a_i, b_i \in D^{(i)}(N)\), we have the result.  

Now we go back the proof of Theorem \[1.1\]. Lemma \[3.1\] shows \( \text{Im}(q_i) \subset \text{Im}(q_{i+1}) \). Since \( \text{Im}(q_i) \subset \text{Im}(q_{i+1}) \) for all \( i = 0, 1, \ldots \), \( q_0 \) is surjective, and \( D^{(\omega)}(N) = \{1_N\} \) (cf. \[2.2\]) by the assumption, we have that \( N/\langle Y \rangle \) is trivial. Hence we have \( N = \langle Y \rangle \). This completes the proof.  

As corollaries of Theorem \[1.1\] we have Corollary \[1.2\], which is the affirmative answer for Question 5.7 in \[KY\], and Proposition \[2.1\] (5) (\[KY\] Theorem 1.3) immediately.

4. **The MQ indices for prime knots up to 10 crossings**

By Proposition \[2.1\] (4) and \[Kw2, Kw3, KW, MSY\], the MQ index for a prime knot up to 10 crossings is 1 or 2. By Corollary \[1.2\], we could determine that the MQ indices of the following 26 knots are all one:

\[
\begin{align*}
8_{16}, 9_{29}, 9_{32}, 10_{62}, 10_{64}, 10_{79}, 10_{81}, 10_{85}, 10_{89}, 10_{94}, 10_{96}, 10_{100}, 10_{105}, 10_{106}, \\
10_{109}, 10_{110}, 10_{112}, 10_{116}, 10_{148}, 10_{149}, 10_{150}, 10_{151}, 10_{152}, 10_{153}, 10_{154}, 10_{158}.
\end{align*}
\]

On the other hand, the MQ indices of the following 19 knots are still open:

\[
\begin{align*}
10_{65}, 10_{66}, 10_{67}, 10_{68}, 10_{80}, 10_{83}, 10_{86}, 10_{87}, 10_{90}, 10_{92}, 10_{93}, \\
10_{97}, 10_{108}, 10_{111}, 10_{117}, 10_{120}, 10_{121}, 10_{163}, 10_{166}.
\end{align*}
\]

**Corollary 4.1** Let \( K \) be a prime knot up to 10 crossings other than 19 knots above. Then we have \( m(K) = a(K) \).

5. **Final remarks**

In \[JKRS\], the MQ index is discussed as a lower bound of the unknotting number of a 2-knot. In the present paper, we clarified that a group theoretic property \( \omega \)-solvability induces the equality of the MQ index and the Nakanishi index. Now we ask the opposite direction:

**Question 5.1** For a group pair \((G, N)\), if \( m(G, N) = a(G, N) \), then is \( N \) always \( \omega \)-solvable?

A contrast notion of \( \omega \)-solvability may be perfectivity. Actually, the Alexander polynomial of a non-trivial knot \( K \) is equal to 1 if and only if \([G(K), G(K)]\) is perfect, and then we have \( m(K) = 0 \). On the other hand, a knot \( K \) is non-trivial if and only if \( a(K) \geq 1 \). A group \( H \) is finite order perfect/higher order perfect/order \( k \) perfect if \( D^{(k)}(H) = D^{(k+1)}(H) \neq \{1_H\} \) for finite \( k \). We raise the following question:
Question 5.2 For a group pair \((G, N)\), \(m(G, N) < a(G, N)\) if and only if \(N\) is finite order perfect?

As the final remark of the paper, since we do not ask that \(N\) should include \([G, G]\), Theorem 1.1 can be applied for the case that \(G/N\) is non-commutative.

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