Adaptive meshfree approximation for linear elliptic partial differential equations with PDE-greedy kernel methods

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Abstract

We consider the meshless approximation for solutions of boundary value problems (BVPs) of elliptic Partial Differential Equations (PDEs) via symmetric kernel collocation. We discuss the importance of the choice of the collocation points, in particular by using greedy kernel methods. We introduce a scale of PDE-greedy selection criteria that generalizes existing techniques, such as the PDE-\(P\)-greedy and the PDE-\(f\)-greedy rules for collocation point selection.

For these greedy selection criteria we provide bounds on the approximation error in terms of the number of greedily selected points and analyze the corresponding convergence rates. This is achieved by a novel analysis of Kolmogorov widths of special sets of BVP point-evaluation functionals. Especially, we prove that target-data dependent algorithms that make use of the right hand side functions of the BVP exhibit faster convergence rates than the target-data independent PDE-\(P\)-greedy. The convergence rate of the PDE-\(f\)-greedy possesses a dimension independent rate, which makes it amenable to mitigate the curse of dimensionality.

The advantages of these greedy algorithms are highlighted by numerical examples.

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1 Introduction

Kernel methods \[5,33,38\] are a popular class of techniques in approximation theory, numerical mathematics and machine learning. They revolve around the notion of a kernel \(k\). Given a non-empty set \(\Omega\), a real valued kernel is defined as a symmetric function \(k : \Omega \times \Omega \rightarrow \mathbb{R}\), and here we focus on domains \(\Omega \subset \mathbb{R}^d\).

For given points \(X_n := \{x_1, \ldots, x_n\} \subset \Omega\), the kernel matrix \(K_{X_n} \in \mathbb{R}^{n \times n}\) is defined as \((K_{X_n})_{ij} := (k(x_i, x_j))_{ij}, i, j = 1, \ldots, n\). A kernel is called strictly positive definite (s.p.d.) if the associated kernel matrix is positive definite for any choice of pairwise distinct points \(X_n \subset \Omega, n \in \mathbb{N}\). Those s.p.d. kernels give rise to a unique Hilbert space, the so called Reproducing Kernel Hilbert Space (RKHS) denoted as \(H_k(\Omega)\). The RKHS is a space of real-valued functions from \(\Omega\) to \(\mathbb{R}\) with inner product \((\cdot, \cdot)_{H_k(\Omega)}\), and it is sometimes also called native space. Within the RKHS, the kernel \(k\) acts as a reproducing kernel, i.e. (i) \(k(\cdot, x) \in H_k(\Omega) \forall x \in \Omega\), and (ii) \(f(x) = (f, k(\cdot, x))_{H_k(\Omega)} \forall x \in \Omega, \forall f \in H_k(\Omega)\), which is called the reproducing property. Kernel interpolation can be characterized and analyzed in this framework: For a given closed subspace \(V \subset H_k(\Omega)\) we denote the \(H_k(\Omega)\)-orthogonal projection onto \(V\) with \(\Pi_V : H_k(\Omega) \rightarrow V\). Given \(n\) interpolation points \(X_n\), the unique minimum-norm interpolant \(s_n\) to a function \(f \in H_k(\Omega)\) is equal to the \(H_k(\Omega)\)-orthogonal projection of \(f\) onto the subspace \(V_{X_n} := \text{span} \{k(\cdot, x_i), x_i \in X_n\} \subset H_k(\Omega)\), i.e. it holds

\[
s_n(\cdot) := \Pi_{V_{X_n}}(f)(\cdot) = \sum_{j=1}^{n} \alpha_j k(\cdot, x_j). \tag{1}
\]

The coefficients \(\alpha_j\) can be determined by the interpolation conditions

\[
s_n(x_i) = f(x_i) \quad \forall i = 1, \ldots, n \quad \Leftrightarrow \quad K_{X_n} \alpha = (f(x_j))_{j=1}^{n},
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_n)^{\top} \in \mathbb{R}^n\) is the vector of the coefficients. A standard way to quantify the error between the function \(f\) and the interpolant \(s_n\) in the \(\|\cdot\|_{L^{\infty}(\Omega)}\)-norm is using the so called power function \(P_{X_n}\), which is given as

\[
P_{X_n}(x) := \|k(\cdot, x) - \Pi_{V_{X_n}}(k(\cdot, x))\|_{H_k(\Omega)}
\]

\[
= \sup_{0 \neq f \in H_k(\Omega)} \frac{|f(x) - \Pi_{V_{X_n}}(f)(x)|}{\|f\|_{H_k(\Omega)}}, \tag{2}
\]

and thus allows to estimate the residual \(f - s_n\) as

\[
|(f - s_n)(x)| \leq P_{X_n}(x) \cdot \|f\|_{H_k(\Omega)}. \tag{3}
\]

In order to obtain a good approximation, a suitable choice of the points \(X_n \subset \Omega\) is required, even if it is usually unclear a priori. A viable option is the use of greedy algorithms \[35\], which start with an empty set \(X_0 := \emptyset\) and then add further points step by step as \(X_{n+1} := X_n \cup \{x_{n+1}\}\). The next interpolation point \(x_{n+1}\) is usually chosen according to some optimality criterion, and for greedy kernel interpolation common choices are the so called \(P\)-greedy \[2\], \(f \cdot P\)-greedy (or \(psr\)-greedy) \[4\], \(f\)-greedy \[32\], and \(f/P\)-greedy \[21\] criteria. A unifying analysis of these standard greedy kernel interpolation algorithms was recently provided in \[42\] under the notion of \(\beta\)-greedy algorithms, that we will generalize in the current article from function interpolation to PDE
collocation. Among the greedy kernel interpolation algorithms, the P-greedy scheme does not use the target values of the function approximation problem for selecting the points. Therefore, this variant is called target-data independent. In contrast, the other mentioned selection criteria make use of the target values, hence adapt to the target function, and are denoted as target-data dependent. This terminology will be adopted to the PDE approximation schemes under consideration, where the “target-data” will be the point evaluations of the right hand side of the PDE problem.

The theory of kernel interpolation is not limited to the interpolation of function values. Indeed it is possible to consider arbitrary functionals Λ_n := \{λ_1, ..., λ_n\} ⊂ \mathcal{H}_k(\Omega)', where \mathcal{H}_k(\Omega)' denotes the dual space of \mathcal{H}_k(\Omega), and obtain a function s_n of minimum norm that satisfies

\[ \lambda_i(s_n) = \lambda_i(f) \quad \forall i = 1, \ldots, n. \] (4)

This procedure is called generalized interpolation [38, Chapter 16]. It is possible to show that, similarly to Eq. (1), it holds

\[ s_n(\cdot) = \Pi_V_n (f)(\cdot) = \sum_{j=1}^n \alpha_j v_{\lambda_j}(\cdot), \] (5)

where we now have V_n := span\{v_{\lambda_i} | \lambda_i \in \Lambda_n\} and v_{\lambda_1}, ..., v_{\lambda_n} \in \mathcal{H}_k(\Omega) are the unique Riesz representers of the functionals λ_1, ..., λ_n. Observe that for the special case of standard kernel interpolation, due to the reproducing property, the Riesz representer of the point evaluation functionals δ_{x_j} are simply given by k(\cdot, x_j), such that the generalized interpolant from Eq. (5) again reduces to Eq. (1), and Eq. (5) is indeed a generalized form of interpolation.

This framework of generalized interpolation can especially be applied to approximating solutions of linear PDEs [20, 27]. For this we consider linear elliptic boundary value problems (BVPs)

\[ Lu = f \quad \text{on } \Omega \subset \mathbb{R}^d \]
\[ Bu = g \quad \text{on } \partial \Omega, \] (6)

with an elliptic differential operator L of second order, for example the Laplace operator L = −Δ, and a suitable boundary operator B. In order to keep the focus on symmetric kernel collocation and the distribution of the collocation points, we do not aim for the most general case of BVPs, but rather restrict to problems with Dirichlet boundary conditions, hence we assume B = Id throughout the remaining presentation.

We propose and analyze methods that are based on collocation [1, 8–10], hence we focus on cases where we have well-posedness of the BVP in this strong formulation, i.e. the solution is assumed to be classically differentiable of sufficient order with corresponding continuous derivatives. We comment on more general cases in the outlook.

The constraints imposed by the PDE and the boundary values can be collected in two sets of functionals, i.e. by introducing

\[ \Lambda_L := \{\delta_{x} \circ L : x \in \Omega\} \]
\[ \Lambda_B := \{\delta_{x} : x \in \partial \Omega\} \]
\[ \Lambda := \Lambda_L \cup \Lambda_B. \] (7)
In this paper, we address the problem of designing and analyzing greedy collocation schemes for the approximation of the solution of these kind of problems. First, we extend and unify the framework for the analysis of standard greedy kernel interpolation algorithms from [42] to generalized greedy kernel interpolation. The analysis of these generalized schemes require decay estimates for certain Kolmogorov widths, which we derive here for the first time answering an open question posed in [31]. Combining these new rates with the extension of the framework from [42], we are able to obtain error estimates for the whole family of generalized \( \beta \)-greedy schemes. These estimates are comparable with the rates obtained with optimally placed points, and additionally, when using full adaptivity (\( \beta = 1 \)), the resulting convergence rates break the curse of dimensionality, in the sense of having a dimension-independent decay factor. This indicates that the approach may especially be beneficial for high-dimensional PDEs, where standard mesh-based techniques cannot easily be applied.

The paper is structured as follows. Section 2 starts by giving required background information on PDEs, greedy kernel interpolation, generalized interpolation and pointing to the corresponding literature. Section 3 provides an analysis of Kolmogorov \( n \)-widths related to the functionals of the considered BVP. Section 4 extends the target-data dependent greedy analysis of [42] to generalized kernel interpolation. Section 5 combines the results of the previous sections to derive convergence rates of the error \( \| u - s_n \|_{L^\infty(\Omega)} \) between the approximant and the true solution of the PDE. Section 6 presents numerical experiments on both low-dimensional and high-dimensional BVPs. Section 7 gives a conclusion and formulates an outlook.

2 Background results

In the following, we collect more required background information about partial differential equations (Subsection 2.1), interpolation with translational invariant kernels (Subsection 2.2), greedy and generalized interpolation (Subsection 2.3), as well as other kernel methods to solve PDEs (Subsection 2.4).

2.1 A priori bounds for elliptic PDEs

We make use of the following assumptions, which especially give rise to a well-posed PDE boundary value problem.

**Assumption 1.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain with piecewise smooth boundary \( \partial \Omega \) of dimension \( d - 1 \). Let \( L \) be a uniformly elliptic operator of second order of the form

\[
(Lu)(x) := \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} u(x) + c(x) u(x),
\]

with bounded and smooth\(^1\) coefficient functions \( a_{ij}, b_i, c \) on \( \Omega \) with \( c \leq 0 \). We assume to have Dirichlet boundary values \( g \in \mathcal{C}(\partial \Omega) \), PDE right hand side

\(^1\)Similar to results in literature that we will refer later, e.g. [11], we assume infinite smoothness for the boundary pieces, which then are smooth manifolds. However, we anticipate that by deliberately tracking the exact derivative requirements during the analysis, the same results can be obtained for sufficiently high finite differentiability. The same applies to coefficient functions of the differential operator, which for simplicity we assume to be \( C^\infty(\Omega) \) functions.
$f \in C(\Omega)$ and there exists a unique strong solution $u \in C^2(\Omega) \cap C(\Omega)$ of the BVP (6).

The assumption on the existence of a unique solution can be removed, if one leverages suitable existence and uniqueness theorems for special PDE problems. Kernel methods are intrinsically related to the use of the $\| \cdot \|_{L^\infty(\Omega)}$ norm because of the reproducing property. Therefore we recall an a-priori bound for the $\| \cdot \|_{L^\infty(\Omega)}$ norm of the solution of Eq. (6) in dependence of the data, cf. [27, Theorem 4.11]

$$
\| u \|_{L^\infty(\Omega)} \leq C \| f \|_{L^\infty(\Omega)} + \| g \|_{L^\infty(\partial \Omega)}.
$$

(9)

Given a (kernel) approximant $s_n$, the difference $v := u - s_n$ solves the same BVP but with data $Lv = f - Ls_n$ and $g - s_n$, so that (9) implies that

$$
\| u - s_n \|_{L^\infty(\Omega)} \leq C \left( \| f - Ls_n \|_{L^\infty(\Omega)} + \| g - s_n \|_{L^\infty(\partial \Omega)} \right),
$$

(10)

which allows to conclude convergence rates on $\| u - s_n \|_{L^\infty(\Omega)}$ based on convergence rates on $\| f - Ls_n \|_{L^\infty(\Omega)}$ and $\| g - s_n \|_{L^\infty(\partial \Omega)}$, which will be derived in Section 4. The convergence rate analysis in the current article will be possible by assuming that the target function has some suitable Sobolev regularity, which is satisfied due to the assumed classical differentiability of the solution and the assumptions on the domain.

We emphasize that our approach can straightforwardly be extended to any linear PDE problem of potential higher order, parabolic or hyperbolic or mixed type, as long as a suitably regular solution exists and a corresponding a priori bound is available. The restriction to second order elliptic problems is mainly for simplifying the presentation and reflecting that our experiments will only cover such elliptic problems.

2.2 RKHS of translational invariant kernels and Sobolev spaces

A translational invariant kernel can be written as

$$
k(x, y) = \Phi(x - y), \quad \text{for all } x, y \in \mathbb{R}^d,
$$

(11)

for some function $\Phi : \mathbb{R}^d \to \mathbb{R}$. Translational invariant kernels can be characterized by the decay of the Fourier transform $\hat{\Phi}$ of $\Phi$, i.e. we assume there exist constants $c, C > 0, \tau > d/2$, such that

$$
c(1 + \| \omega \|^2)^{-\tau} \leq \Phi(\omega) \leq C(1 + \| \omega \|^2)^{-\tau}
$$

(12)

for all $\omega \in \mathbb{R}^d$.

In this case, $\mathcal{H}_k(\mathbb{R}^d)$ is norm equivalent to a Sobolev space, i.e. $\mathcal{H}_k(\mathbb{R}^d) \simeq H^\tau(\mathbb{R}^d) = W^\tau_2(\mathbb{R}^d)$, where the order $\tau > d/2$ is possibly fractional. This norm equivalence can be extended to the RKHS $\mathcal{H}_k(\Omega)$ on bounded domains $\Omega \subset \mathbb{R}^d$ under suitable conditions on the boundary $\partial \Omega$, such as its Lipschitz continuity, i.e. it holds $\mathcal{H}_k(\Omega) \simeq H^\tau(\Omega) = W^\tau_2(\Omega)$.

In order to estimate the error $\| f - s_n \|_{L^\infty(\Omega)}$ between a function $f \in \mathcal{H}_k(\Omega)$ and its kernel interpolant $s_n$, one usually makes use of sampling inequalities and corresponding zero lemmata, which exist for Sobolev spaces or spaces of
analytic functions [22, 30]. They bound the error in terms of the fill distance $h = h_{X, \Omega}$, which describes the largest hole within $\Omega$ where no point from $X$ exists, i.e.

$$h := h_{X, \Omega} := \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|.$$  \hspace{1cm} (13)$$

The same definition of the fill distance can also be used for the boundary $\partial \Omega$ of $\Omega$. Here we recall [13, Theorem 2.2], where $| \cdot |_{W^m_p(\Omega)}$ denotes the Sobolev semi-norm, which uses only the highest derivatives, $\|f\|_\infty := \max_{x \in X} |f(x)|$, and $(x)_+ := \max\{x, 0\}$.

**Theorem 1.** Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain satisfying an interior cone condition and having a Lipschitz boundary. Let $X \subset \Omega$ be a discrete set with sufficiently small fill distance $h \leq h_0$. Let $\tau = k + s$ with $k \in \mathbb{N}, 0 \leq s < 1, 1 \leq p < \infty, 1 \leq q \leq \infty, m \in \mathbb{N}_0$ with $k > m + d/p$ if $p > 1$ or $k \geq m + d/p$ if $p = 1$. Then for each $f \in W^m_p(\Omega)$ we have that

$$|f|_{W^m_p(\Omega)} \leq C \left( h^{\tau - m - d(1/p - 1/q)} |f|_{W^\tau_p(\Omega)} + h^{-m} \|f\|_\infty \right),$$

where $C > 0$ is a constant independent of $f$ and $h$.

Similar statements as in Theorem 1 can also be derived for interpolation on manifolds under some assumptions on the shape of the manifold. In the following we are particularly interested in statements for the boundary of the domain $\Omega$, i.e. the manifolds $\mathbb{M} \subset \partial \Omega$. For this, we recall the result [12, Lemma 10].

**Theorem 2.** Let $\mathbb{M}$ be a smooth manifold of dimension $\tilde{d}$ and let $1 \leq p, q \leq \infty$, $t \in \mathbb{R}$ with $t > \tilde{d}/p$ if $p > 1$ or $t \geq \tilde{d}$ if $p = 1$. Let $\mu \in \mathbb{N}$ satisfy $0 \leq \mu \leq \lfloor t - \tilde{d}(1/p - 1/q)\rfloor - 1$. Also, let $X \subset \mathbb{M}$ be a discrete set with fill distance $h_{X, \mathbb{M}} \leq C_{\mathbb{M}}$. If $f \in W^\mu_p(\mathbb{M})$ satisfies $f|_X = 0$, then

$$|f|_{W^\mu_p(\mathbb{M})} \leq C h^{t - \mu - \tilde{d}(1/p - 1/q)} |f|_{W^\mu_p(\mathbb{M})}.$$ 

Here $h_{X, \mathbb{M}}$ is defined as the fill distance $h_{X, \Omega}$, but using the intrinsic distance on the manifold instead of the Euclidean one, and $W^\mu_p(\mathbb{M})$ is a Sobolev space on the manifold $\mathbb{M}$.

We remark that similar statements as in Theorem 1 and Theorem 2 are also available for interpolation with analytic kernels such as the Gaussian kernel. However, as we focus on Sobolev kernels we do not recall those estimates here.

### 2.3 Generalized kernel interpolation

We consider the given set of functionals $\Lambda_n \equiv \{\lambda_1, \ldots, \lambda_n\} \subset H_k(\Omega)'$, which we assume to be linearly independent and with associated Riesz representers $v_{\lambda_1}, \ldots, v_{\lambda_n} \in H_k(\Omega)$, and we recall that we use the ansatz space $V_n = \text{span}\{v_{\lambda_i}, \lambda_i \in \Lambda_n\}$. In order to bound approximation errors, a generalized power function similar to the standard power function in Eq. (2) can be defined for generalized interpolation [38, Chapter 16] as

$$P_{\Lambda_n}(\lambda) := \|v_\lambda - \Pi_{V_n}(v_\lambda)\|_{H_k(\Omega)} = \sup_{0 \neq u \in H_k(\Omega)} \frac{|\lambda(u - \Pi_{V_n}(u))|}{\|u\|_{H_k(\Omega)}},$$ \hspace{1cm} (14)$$
which immediately gives for all $u \in \mathcal{H}_k(\Omega)$ that
\[
|\lambda(u) - \lambda(\Pi V_n(u))| \leq P_{\Lambda_n}(\lambda) \cdot \|u\|_{\mathcal{H}_k(\Omega)}.
\]

We like to point out again that for $\lambda_1 := \delta_{x_1}, \ldots, \lambda_n := \delta_{x_n}$ the generalized approximant as well as the generalized power function boil down to their standard counterparts by identifying $x$ and $\delta_x$.

In order to avoid a frequent recomputation of the coefficients $\alpha_j$ within the standard kernel expansion from Eq. (1) and Eq. (5), typically the Newton basis $\{v_1, \ldots, v_n\}$ that is obtained by orthonormalizing the Riesz representers $\{v_{\lambda_1}, \ldots, v_{\lambda_n}\}$ is applied. Then the (standard or generalized) interpolant $s_n$ of $u \in \mathcal{H}_k(\Omega)$ can be written as
\[
s_n(\cdot) = \sum_{j=1}^{n} \langle u, v_j \rangle_{\mathcal{H}_k(\Omega)} v_j(\cdot),
\]
and it can be shown that the coefficients of the interpolant expressed in the Newton basis satisfy
\[
\langle u, v_n \rangle_{\mathcal{H}_k(\Omega)} = \frac{\lambda_n(r_{n-1})}{P_{\Lambda_{n-1}}(\lambda_n)} = \frac{\lambda_n(r_{n-1})}{P_{\Lambda_{n-1}}(\lambda_n)}.
\]
(15)

Therefore, it also holds that
\[
\sum_{j=1}^{\infty} \left( \frac{|\lambda_j(u - s_{j-1})|}{P_{\Lambda_{j-1}}(\lambda_j)} \right)^2 \leq \|u\|_{\mathcal{H}_k(\Omega)}^2,
\]
(16)
where the inequality is in fact an equality if and only if $s_n \xrightarrow{n \to \infty} u$ in $\mathcal{H}_k(\Omega)$.

### 2.4 Solving PDEs by collocation

As motivated in the introduction in Section 1, the choice of $\Lambda$ according to Eq. (7) naturally yields an approximation of the PDE solution in the setting of generalized interpolation. This approach is usually called symmetric kernel collocation since it is obtained by a symmetric application of the differential operators to the two arguments of the kernel or, in other words, since the differential operators are applied both to construct the ansatz in Eq. (5), and to apply the generalized interpolation conditions of Eq. (4). Since this method is based on the optimality principles described in Section 1, it can be analyzed by extending in a systematic manner the ideas used for plain function interpolation.

If the approximant $s_n$ in Eq. (5) is instead assumed to be represented as $s_n := \sum_{j=1}^{n} \alpha_j k(\cdot, x_j)$ independently of the specific problem to be solved, the resulting method is usually referred as unsymmetric kernel collocation, or method of Kansa (see [17,18]). This approach has the advantage of requiring less regularity on the kernel and of being usually computationally more flexible, but the corresponding theoretical analysis is more subtle [16,30].

These ideas have been significantly extended by introducing a local approximation scheme known as Radial Basis Function Finite Difference (RBF-FD) [7,36]. The convergence of this method has been extensively studied only very recently [37].
3 Analysis of the Kolmogorov widths for PDE collocation functionals

For the analysis of the generalized greedy kernel approximation algorithms in Section 4 we will extend an analysis introduced in [28,42]. For this, the knowledge about some Kolmogorov $n$-widths [26], that are associated to our approximation problems, are required: The Kolmogorov $n$-widths $d_n(\Lambda)$ of $\Lambda \subset H'$ in the dual Hilbert space $H'$ of a Hilbert space $H$ are defined as

$$d_n(\Lambda) := d_n(\Lambda, H') = \inf_{G_n \subset H'} \sup_{\mu \in \Lambda} \text{dist}(\mu, G_n)_{H'}.$$  \hspace{1cm} (17)

The shorthand notation $d_n(\Lambda)$ will be used whenever the Hilbert space $H$ or its dual $H'$ are clear. We remark that we can rewrite this Kolmogorov width: It is possible to move this definition from the dual space $H'$ to the primal space $H$, since the Riesz representer theorem gives an isometry. Furthermore we calculate the distance in the Hilbert space by using orthogonal projections and obtain that

$$d_n(\Lambda, H') \equiv \inf_{G_n \subset H} \sup_{\mu \in \Lambda} \text{dist}(\mu, H_n)_{H'}$$
$$= \inf_{H_n \subset H} \sup_{\mu \in \Lambda} \|v_{\mu} - \Pi_{H_n}(v_{\mu})\|_H.$$  \hspace{1cm} (18)

For the problem of PDE approximation, the Hilbert space is chosen as the RKHS of the kernel $k$, i.e. $H := \mathcal{H}_k(\Omega)$. The set $\Lambda$ is defined via the PDE equations in [7], i.e. $\Lambda := \Lambda_L \cup \Lambda_B$. With $x_{\lambda} \in \Omega \cup \partial\Omega$ we will denote in the following the point related to a functional $\lambda \in \Lambda \subset \mathcal{H}_k(\Omega)'$, i.e. where the evaluation takes place within $\Omega$ (for evaluations of $\lambda \in \Lambda_L$) or on its boundary $\partial\Omega$ (for evaluations of the boundary functionals $\lambda \in \Lambda_B$). Since we consider kernels that satisfy Eq. (12), we have the norm equivalence of the RKHS $\mathcal{H}_k(\Omega)$ to a Sobolev space $H^\tau(\Omega)$ for some $\tau > d/2$. Therefore we immediately obtain

$$\text{dist}(v_{\mu}, H_n)_{H^\tau(\Omega)} \approx \text{dist}(v_{\mu}, H_n)_{\mathcal{H}_k(\Omega)}$$
$$\Rightarrow \quad d_n(\Lambda, H^\tau(\Omega)) \approx d_n(\Lambda, \mathcal{H}_k(\Omega)),$$

i.e. instead of analyzing the Kolmogorov $n$-widths related to the RKHS $\mathcal{H}_k(\Omega)$ we can analyze the $n$-widths related to the Sobolev space $H^\tau(\Omega)$, because they provide the same asymptotics.

As remarked in [31], a thorough analysis of those Kolmogorov $n$-widths for PDE collocation functionals does not seem to be available so far in the literature. We refer to Remark 3 below for a discrimination from other types of PDE-related Kolmogorov $n$-widths. Nevertheless [31] conjectured that there should be a $\tau-$ (smoothness of the Hilbert space) and $d-$ (dimensionality of the domain $\Omega \subset \mathbb{R}^d$) dependent decay rate $\kappa(\tau, d)$ in the sense that

$$d_n(\Lambda, H') \leq C n^{-\kappa(\tau, d)} \quad \text{for } n \to \infty.$$

Indeed it will be proven in the following that the Kolmogorov $n$-widths related to our PDE problems exactly behave according to such a relation. We stress that the difficulty in obtaining bounds for $d_n(\Lambda)$ is due to the fact that $\Lambda$ is the union of two sets of different functionals.
Remark 3 (Relation to Kolmogorov n-widths for parametric PDEs). We want to briefly relate and discriminate the present type of Kolmogorov n-widths for PDE collocation functionals to another width in the literature. Especially in the context of parametric PDEs and projection-based model order reduction, solution manifolds of the parametric solutions appear and need to be approximated, e.g. \[15\]. A crucial property for successful approximation in these scenarios is a rapid decay of the Kolmogorov n-widths of those solution manifolds. Especially in transport-dominated cases, the decay can be very slow for the wave equation \[14\] or the convection of discontinuities \[24\]. In contrast, here we are not aiming at approximating solution manifolds of parametric PDE solutions, but addressing the Kolmogorov n-width of the set \(\Lambda\) of the PDE collocation functionals of a single non-parametric PDE.

The next Subsection 3.1 provides results to obtain lower and upper bounds on \(d_n(\bigcup_{j=1}^M \Lambda_j)\) with help of \(d_n(\Lambda_j), j = 1, \ldots, M\). Subsequently we derive in Subsection 3.2 precise decay rates for our special case of Sobolev spaces.

3.1 Abstract setting: Bounds on \(d_n(\Lambda)\)

We call this subsection the abstract setting, because the following Proposition 4 and Proposition 5 provide bounds on the Kolmogorov n-widths of general sets that are given as the union of two or more (not necessarily disjoint) sets. Moreover, although we use the same notation used before for orthogonality, it should be noted that Proposition 4 and Proposition 5 work also for Kolmogorov n-widths in Banach spaces because we only use distance-based arguments and no dot products.

The following Proposition 4 shows that it is possible to lower bound the Kolmogorov n-width \(d_n(\Lambda)\) of \(\Lambda := \bigcup_{j=1}^M \Lambda_j \subset H'\). Then it holds

\[
d_n(\Lambda) \geq \max_{j=1,\ldots,M} d_n(\Lambda_j).
\]

Proposition 4.

Proof. For any \(j = 1, \ldots, M\) we obtain

\[
d_n(\Lambda) \equiv \inf_{G_n \subset H', \dim(G_n)=n} \sup_{\mu \in \Lambda} \text{dist}(\mu, G_n)_{H'} \geq \inf_{G_n \subset H', \dim(G_n)=n} \sup_{\mu \in \Lambda} \text{dist}(\mu, G_n)_{H'} = d_n(\Lambda_j).
\]

Therefore the statement directly follows.

In a similar way it is possible to derive an upper bound on \(d_n(\Lambda)\) via the quantities \(d_n(\Lambda_j), j = 1, \ldots, M\). The proof can be found in Appendix A.

Proposition 5. Let \(\Lambda := \bigcup_{j=1}^M \Lambda_j \subset H'\). Then it holds

\[
d_n(\Lambda) \leq \min_{\sum_{i=1}^M n_j \leq n, j=1,\ldots,M} \max_{1 \leq j \leq M} d_n(\Lambda_j).
\]

Proposition 4 and Proposition 5 enable us to analyze the asymptotics of \(d_n(\Lambda)\) with help of the knowledge of \(d_n(\Lambda_j), 1 \leq j \leq M\). This will be applied in the following subsection using \(M = 2\) and \(\Lambda_L, \Lambda_B\).
3.2 Kernel setting: Bounds on \(d_n(\Lambda)\)

We start by analyzing the Kolmogorov \(n\)-widths \(d_n(\Lambda_L)\) and \(d_n(\Lambda_B)\) as defined in Eq. (7). To the best of our knowledge there are no similar estimates on \(d_n(\Lambda_L)\) and \(d_n(\Lambda_B)\) available in the literature.

For \(d_n(\Lambda_L)\) we obtain the following statement, where the proof can be found in the Appendix.

**Theorem 6** (Upper bound on \(d_n(\Lambda_L)\)). Let \(L\) be a linear differential operator according to Assumption 1, \(\Omega \subset \mathbb{R}^d\) a bounded Lipschitz domain. Let \(k\) be a kernel s.t. \(H_k(\Omega) \simeq H^\tau(\Omega)\) with \(\tau > 2 + d/2\). Then for \(\Lambda_L\) as defined in Eq. (7) and all \(n \geq 1\) it holds

\[
d_n(\Lambda_L, H_k(\Omega)) \leq C_L n^{\frac{1}{2} - \frac{\tau - 2}{d}},
\]

where \(C_L > 0\) is a constant depending on \(L\) but not on \(n\).

Similarly we can state an upper bound of the widths of the set of the boundary functionals. The proof again is postponed to the Appendix.

**Theorem 7** (Upper bound on \(d_n(\Lambda_B)\)). Let \(\Omega \subset \mathbb{R}^d\) be a bounded Lipschitz domain with piecewise smooth boundary \(\partial \Omega\) of dimension \(d - 1\). Let \(k\) be a kernel s.t. \(H_k(\Omega) \simeq H^\tau(\Omega)\) with \(\tau > d/2\). Then for \(\Lambda_B\) as defined in Eq. (7) and all \(n \geq 1\) it holds

\[
d_n(\Lambda_B) \leq C_B n^{\frac{1}{2} - \frac{\tau - 1}{2(d - 1)}},
\]

where \(C_B > 0\) is a constant independent of \(n\).

We remark that \(\frac{1}{2} - \frac{\tau - 1}{2} < \frac{1}{2} - \frac{\tau - 2}{d} < 0\), i.e. the decay in Theorem 7 is faster than the decay in Theorem 6.

We can now leverage the abstract results from Subsection 3.1 and combine them with the concrete estimates for the kernel setting from Theorem 6 and Theorem 7. The basic idea is to carefully balance the number of points which are spent on the interior and on the boundary of the domain. Like this it is possible to derive an upper bound on the Kolmogorov \(n\)-width \(d_n(\Lambda)\).

**Theorem 8** (Upper bound on \(d_n(\Lambda)\)). Let \(\Omega \subset \mathbb{R}^d\) be a bounded Lipschitz domain with piecewise smooth boundary \(\partial \Omega\) of dimension \(d - 1\). Let \(L\) be a linear differential operator according to Assumption 1. Let \(k\) be a kernel s.t. \(H_k(\Omega) \simeq H^\tau(\Omega)\) with \(\tau > 2 + d/2\). Then the Kolmogorov \(n\)-widths of the collocation functionals from (6) satisfy for all \(n \geq 1\)

\[
d_n(\Lambda) \leq C n^{\frac{1}{2} - \frac{\tau - 2}{d}},
\]

where \(C > 0\) depends on \(\tau, d, L\) but not on \(n\).

Applied to the PDE setting of [31] this means that the Kolmogorov \(n\)-widths related to the collocation functionals of the PDE \(Lu = f\) (here given by \(d_n(\Lambda_L) \simeq n^{\frac{1}{2} - \frac{\tau - 2}{d}}\)) decide about the Kolmogorov \(n\)-widths for the whole boundary value problem (here \(d_n(\Lambda)\)), as it was already conjectured in [31]. The additional boundary conditions do not influence the decay of the Kolmogorov \(n\)-widths, because the Kolmogorov \(n\)-widths related to the boundary conditions.
decay faster.

The proposed analysis in this section is in principle not restricted to only two sets of functionals \( \Lambda_L \) and \( \Lambda_B \). This is explained in the following remark.

**Remark 9.** We remark that the results of this section can be extended to more sets of functionals. This was already done for the abstract setting in Subsection 3.1 by considering \( \Lambda = \bigcup_{j=1}^M \Lambda_j \). In the PDE collocation setting this refers e.g. to more types of boundary conditions, i.e. we can incorporate both Dirichlet boundary conditions and Neumann boundary conditions by using \( \Lambda_B \) (for the Dirichlet-related functionals) and \( \Lambda_B \) (for the Neumann-related functionals) instead of only using \( \Lambda_B \).

## 4 Analysis of PDE-greedy schemes

We start by defining the notion of a PDE-greedy kernel algorithm.

**Definition 10.** Consider a BVP (6), a kernel \( k \) and selection criteria \( \{ \eta_j \}_{j=1}^\infty \) with \( \eta_j : \Lambda \to [0, \infty) \) for all \( j \in \mathbb{N} \), and with \( \Lambda \) defined in Eq. (7). An algorithm is called PDE-greedy kernel algorithm if it selects \( \lambda_{n+1} \in \Lambda \) according to

\[
\lambda_{n+1} = \arg \max_{\lambda \in \Lambda} \eta_n(\lambda).
\]

A whole class of PDE-greedy algorithms with specific selection criteria \( \eta_n \) is given in Definition 13. For the moment we continue with this general notion in order to stress the fact that the results of the following Section 4.1 hold for general selection criteria.

For this general class of algorithms some basic analytical bounds on geometric means of power function evaluations can be provided, as presented in Subsection 4.1. Then in Subsection 4.2 we define a scale of \( \beta \)-greedy procedures, which are favourable in the sense that these allow more refined analysis that is presented in Subsection 4.3. Those results will serve for obtaining the main convergence rate statements in Section 5.

### 4.1 Decay of power function quantities

In [42, Section 3] the abstract analysis of greedy algorithms in Hilbert spaces of [3] was extended by considering a broader class of greedy algorithms. Here we will make use of those results and apply them to obtain bounds on

\[
\left[ \prod_{i=n+1}^{2n} P_{\Lambda_i}(\lambda_{i+1}) \right]^{1/n},
\]

see Corollary 12 with \( \Lambda_i \equiv \{ \lambda_1, \ldots, \lambda_i \} \) as introduced in Section 2.3. This will later enable us to derive convergence rates for a range of target-data dependent PDE-greedy kernel algorithms.

For this we briefly recall the notation of [42, Section 3]: Let \( \mathcal{H} \) be a Hilbert space with norm \( \| \cdot \| := \| \cdot \|_\mathcal{H} \) and \( \mathcal{F} \subset \mathcal{H} \) a compact subset. Without loss of generality we assume that \( \| f \| \leq 1 \) for all \( f \in \mathcal{F} \). We consider algorithms that select elements \( f_0, f_1, \ldots \in \mathcal{F} \), without yet specifying any particular selection
criterion. We define $V_n := \text{span}\{f_0, \ldots, f_{n-1}\}$ and the following quantities, where $H_n$ is any $n$-dimensional subspace of $\mathcal{H}$.

$$d_n := d_n(\mathcal{F}) := \inf_{H_n \subset \mathcal{H}} \sup_{f \in \mathcal{F}} \text{dist}(f, H_n),$$

$$\sigma_n := \sigma_n(\mathcal{F}) := \sup_{f \in \mathcal{F}} \text{dist}(f, V_n),$$

$$\nu_n := \text{dist}(f_n, V_n).$$

Now we recall \cite{42} Corollary 2 including its assumptions which were stated before in \cite{42} Theorem 1):

**Corollary 11** (Corollary 2 from \cite{42}). Consider a compact set $\mathcal{F}$ in a Hilbert space $\mathcal{H}$, and a greedy algorithm that selects elements from $\mathcal{F}$ according to any arbitrary selection rule.

i) If $d_n(\mathcal{F}) \leq C_0 n^{-\alpha}, n \geq 1$, then it holds

$$2^n \prod_{i=n+1}^{2n} \nu_i \left( \frac{1}{n} \right) \leq 2^{\alpha+1/2} \tilde{C}_0 e^{\alpha \log(n)} n^{-\alpha}, \ n \geq 3,$$

with $\tilde{C}_0 := \max\{1, C_0\}$.

ii) If $d_n(\mathcal{F}) \leq C_0 e^{-c_0 n^\alpha}, n \geq 1$, then it holds

$$2^n \prod_{i=n+1}^{2n} \nu_i \left( \frac{1}{n} \right) \leq 2 \tilde{C}_0 \cdot e^{-c_1 n^\alpha}, \ n \geq 2,$$

with $\tilde{C}_0 := \max\{1, C_0\}$ and $c_1 = 2^{-(2+\alpha)} c_0 < c_0$.

We can leverage this abstract result to obtain convergence statements on the quantity of Eq. (20) by generalizing a convenient link between the abstract setting and the kernel setting, which was first of all established in \cite{28} for function interpolation. We generalize this by choosing $\mathcal{H} := H_k(\Omega), \mathcal{F} := \{v_\lambda \mid \lambda \in \Lambda_L \cup \Lambda_B\}$ and thus $f_0 := v_{\lambda_1}, f_1 := v_{\lambda_2}, \text{ etc.}$ Hereby, we have $V_n = \text{span}\{v_{\lambda_i} \mid i = 1, \ldots, n\}$.

These choices allow us to formulate the abstract definitions of Eq. (20) in the kernel setting for PDE approximation. For this we point again to the definition of the generalized power function from Eq. (14): The quantity $\sigma_n$ is then simply the maximal value $\sup_{\lambda \in \Lambda} \lambda P_{\Lambda_n}(\lambda)$ of the generalized power function $P_{\Lambda_n}$, while $\nu_n$ is the value of the generalized power function $P_{\Lambda_n}$ at the next selected functional $\lambda_{n+1}$ corresponding to the point $x_{\lambda_{n+1}}$:

$$\sigma_n \equiv \sup_{f \in \mathcal{F}} \text{dist}(f, V_n) = \sup_{f \in \mathcal{F}} \|f - \Pi_{V_n}(f)\|_{\mathcal{H}} = \sup_{\lambda \in \Lambda_L \cup \Lambda_B} \|v_\lambda - \Pi_{V_n}(v_\lambda)\|_{\mathcal{H}} = \sup_{\lambda \in \Lambda_L \cup \Lambda_B} P_{\Lambda_n}(\lambda),$$

$$\nu_n \equiv \text{dist}(f_n, V_n) = \|f_n - \Pi_{V_n}(f_n)\|_{\mathcal{H}} = \|v_{\lambda_{n+1}} - \Pi_{V_n}(v_{\lambda_{n+1}})\|_{\mathcal{H}} = P_{\Lambda_n}(\lambda_{n+1}).$$
Furthermore, \( d_n \) can be similarly bounded as
\[
d_n \equiv \inf_{H_n \subset \mathcal{H}} \sup_{f \in F} \text{dist}(f, H_n) = \inf_{H_n \subset \mathcal{H}} \sup_{f \in F} \| f - \Pi_{H_n}(f) \|_{\mathcal{H}} \leq \inf_{H_n \subset \text{span}(F)} \sup_{f \in F} \| f - \Pi_{H_n}(f) \|_{\mathcal{H}} = \inf_{H_n \subset \text{span}(F)} \sup_{\lambda \in \Lambda_L \cup \Lambda_B} \| v_{\lambda} - \Pi_{H_n}(v_{\lambda}) \|_{\mathcal{H}} = \inf_{\Lambda_n \subset \Lambda_L \cup \Lambda_B} \sup_{\lambda \in \Lambda_n} P_{\lambda_n}(\lambda). \tag{22}
\]

Here \( H_n \) always indicates an \( n \)-dimensional subspace. Due to Eq. (22), any bound on \( \sup_{\lambda \in \Lambda_L \cup \Lambda_B} P_{\lambda_n}(\lambda) \) for a given set of functionals \( \Lambda_n \subset F \) (which corresponds to a distribution of points in \( \Omega \) and on the boundary \( \partial \Omega \)) gives a bound on \( d_n \).

Additionally, observe that the assumption \( \| f \|_{\mathcal{H}} \leq 1 \) for \( f \in F \) from the abstract setting is satisfied using the kernel setting as soon as
\[
\sup_{\lambda \in \Lambda_L \cup \Lambda_B} P_0(\lambda) = \sup_{\lambda \in \Lambda_L \cup \Lambda_B} \| v_{\lambda} \|_{\mathcal{H}} \geq \max \left( \sup_{\lambda \in \Lambda_L} \| v_{\lambda} \|_{\mathcal{H}}, \sup_{\lambda \in \Lambda_B} \| v_{\lambda} \|_{\mathcal{H}} \right) \leq \max \left( \sup_{x \in \Omega} \sqrt{L(1)k(x, x)}, \sup_{x \in \partial \Omega} \sqrt{\kappa(x, x)} \right) \leq 1, \tag{23}
\]
where we use the explicit representations of the Riesz representers \( v_{\lambda} = k(x, \cdot) \) if \( \lambda = \delta_x \in \Lambda_B \), and \( v_{\lambda} = L^{(1)}k(x, \cdot) \) if \( \lambda = \delta_x \circ L \in \Lambda_L \). The validity of Eq. (23) can always be ensured by rescaling the kernel \( k \) to \( ek \), \( 0 < e < 1 \) (which does not change the RKHS, but only rescales its norm). Applying the link to Corollary 11 immediately gives the following result:

**Corollary 12.** Consider the set \( F = \{ v_{\lambda} \mid \lambda \in \Lambda_L \cup \Lambda_B \} \subset \mathcal{H}_k(\Omega) \), and a greedy algorithm that selects elements from \( F \) according to any arbitrary selection rule.

1) If \( d_n(F) \leq C_0n^{-\alpha}, n \geq 1 \), then it holds
\[
\left( \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right)^{1/n} \leq 2^{\alpha+1/2}\tilde{C}_0 e^{\alpha} \log(n)^{\alpha}n^{-\alpha}, \ n \geq 3,
\]
with \( \tilde{C}_0 := \max \{1, C_0\} \).

2) If \( d_n(F) \leq C_0 e^{-c_1 n^{\alpha}}, n \geq 1 \), then it holds
\[
\left( \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right)^{1/n} \leq \sqrt{2\tilde{C}_0 \cdot e^{-c_1 n^{\alpha}}}, \ n \geq 2,
\]
with \( \tilde{C}_0 := \max \{1, C_0\} \) and \( c_1 = 2^{-(2+\alpha)C_0} < c_0 \).

**Proof.** This is a direct consequence of Corollary 11 using the choices of \( \mathcal{H}, F \) and \( f_0, f_1, \ldots, f_t \) as introduced before and summarized in Eq. (21).

The assumptions of Corollary 12 about the decay of the Kolmogorov widths \( d_n(F) \) can be ensured via Theorem 8 because of the choice \( F \equiv \{ v_{\lambda}, \lambda \in \Lambda \} \), and the Riesz representor isometry, i.e.
\[
d_n(\Lambda) \equiv d_n(\Lambda, \mathcal{H}_k(\Omega)) = d_n(F, \mathcal{H}_k(\Omega)) \equiv d_n(F).
\]
We note that the results of this subsection simplify to the results of [42, Section 5] if we choose $L = \text{Id}$, i.e. \( \{v_{\lambda} \mid \lambda \in \Lambda_L \cup \Lambda_B\} = \{k(\cdot, x), x \in \Omega\} \).

### 4.2 Definition of PDE-$\beta$-greedy algorithms

We generalize the notion of $\beta$-greedy algorithms as introduced in [42] for standard interpolation to PDE-$\beta$-greedy algorithms via the following definition.

**Definition 13** (Definition 4). A PDE-greedy kernel algorithm is called PDE-$\beta$-greedy algorithm with $\beta \in [0, \infty]$, if the next interpolation functional is chosen

1. for $\beta \in [0, \infty)$ according to
   \[
   \lambda_{n+1} = \arg \max_{0 \neq \lambda \in \Lambda \setminus \Lambda_n} |\lambda(u - \Pi_{V_n}(u))|^\beta \cdot P_{\Lambda_n}(\lambda)^{1-\beta},
   \]
   (24)

2. for $\beta = \infty$ according to
   \[
   \lambda_{n+1} = \arg \max_{0 \neq \lambda \in \Lambda \setminus \Lambda_n} \frac{|\lambda(u - \Pi_{V_n}(u))|}{P_{\Lambda_n}(\lambda)}.
   \]
   (25)

This scale of PDE-greedy algorithms includes several special cases, i.e. $\beta = 0$ (PDE-$P$-greedy), $\beta = 1$ (PDE-$f$-greedy) and $\beta = \infty$ (PDE-$f/P$-greedy) as visualized in Figure 1. Here we introduce the names in brackets in analogy to the existing corresponding greedy kernel schemes for function interpolation.

### 4.3 Analysis of PDE-$\beta$-greedy algorithms

We remark that the following lemma and theorems are generalizations of the corresponding statements within [42], when we pick $\Lambda = \{\delta_x \mid x \in \Omega\}$ or equivalently $\mathcal{F} = \{k(\cdot, x) \mid x \in \Omega\}$. Therefore the proofs are not included in the main text, but can be found in the Appendix B.

**Lemma 14** (Generalization of Lemma 6 from [42]). For any sequence $\{\lambda_i\}_{i \in \mathbb{N}} \subset \Lambda$ and any $u \in \mathcal{H}_k(\Omega)$, using $r_i = u - \Pi_{V_i}(u)$ it holds for all $n \geq 1$ that

\[
\left[\prod_{i=n+1}^{2n} \lambda_{i+1}(r_i)\right]^{1/n} \leq n^{-1/2} \cdot \|u - \Pi_{V_{n+1}}(u)\|_{\mathcal{H}_k(\Omega)} \cdot \left[\prod_{i=n+1}^{2n} P_{\Lambda_i}(\lambda_{i+1})\right]^{1/n}.
\]
   (26)

Now we can adapt [42, Lemma 7].

**Lemma 15** (Generalization of Lemma 7 from [42]). Any PDE-$\beta$-greedy algorithm with $\beta \in [0, \infty]$ applied to a function $u \in \mathcal{H}_k(\Omega)$ satisfies for $i \geq 0$
a) in the case of $\beta \in [0, 1]$

$$\sup_{\lambda \in \Lambda} |\lambda(r_i)| \leq |\lambda_{i+1}(r_i)|^{\beta} \cdot P_\lambda(\lambda_{i+1})^{1-\beta} \cdot \|r_i\|_{H^k(\Omega)}^{1-\beta},$$  \hspace{1cm} (27)

b) in the case of $\beta \in (1, \infty]$ with $1/\infty := 0$

$$\sup_{\lambda \in \Lambda} |\lambda(r_i)| \leq \frac{|\lambda_{i+1}(r_i)|}{P_\lambda(\lambda_{i+1})^{1-1/\beta}} \cdot \sup_{\lambda \in \Lambda} P_\lambda(\lambda)^{1-1/\beta}. \hspace{1cm} (28)$$

Now also the main result of [42, Section 4] can be adapted.

**Theorem 16** (Generalization of Theorem 8 from [42]). Any PDE-$\beta$-greedy algorithm with $\beta \in [0, \infty]$ applied to a function $u \in H^k(\Omega)$ satisfies the following error bound for $n \geq 1$:

a) In the case of $\beta \in [0, 1]$

$$\left[ \prod_{i=n+1}^{2n} \sup_{\lambda \in \Lambda} |\lambda(r_i)| \right]^{1/n} \leq n^{-\beta/2} \cdot \|r_{n+1}\|_{H^k(\Omega)} \cdot \left[ \prod_{i=n+1}^{2n} P_\lambda(\lambda_{i+1}) \right]^{1/n},$$ \hspace{1cm} (29)

b) In the case of $\beta \in (1, \infty]$: \hspace{1cm} (30)

$$\left[ \prod_{i=n+1}^{2n} \sup_{\lambda \in \Lambda} |\lambda(r_i)| \right]^{1/n} \leq n^{-1/2} \cdot \|r_{n+1}\|_{H^k(\Omega)} \cdot \left[ \prod_{i=n+1}^{2n} P_\lambda(\lambda_{i+1})^{1/\beta} \right]^{1/n}.$$ \hspace{1cm} (30)

We remark that also further results from [42] can be extended to this generalized setting, e.g. [42, Corollary 9] which proves an improved standard power function estimate.

5 Convergence rates for PDE-greedy algorithms

This section serves as an amalgamation of the results from the previous sections: We apply the analysis of the PDE-$\beta$-greedy algorithms from Section 4 to concrete kernels and PDEs. By using estimates on the corresponding Kolmogorov $n$-widths from Section 3 we obtain convergence rates for PDE-greedy algorithms in terms of the error in the PDE functionals. Using the maximum principle for elliptic operators (see Eq. (10)) we can transfer the obtained convergence rates directly to convergence rates of the approximation of the true solution.

**Theorem 17.** We consider the BVP as in Eq. (6) under Assumption 1 and there exists a solution in $H^r(\Omega)$, and we assume that $k$ is a Sobolev kernel of smoothness $\tau > d/2 + 2$, i.e. $H_k(\Omega) \approx H^r(\Omega)$.

Then any PDE-$\beta$-greedy algorithm with $\beta \in [0, \infty]$ satisfies the following error bound for $n \geq 3$ (using $1/\infty = 0$):

$$\min_{n+1 \leq i \leq 2n} \|u - s_i\|_{L^\infty(\Omega)} \leq C \cdot n^{-\min\{\frac{\min\{1, \beta\}}{2}, 1\}} \cdot (\log(n) \cdot n^{-1})^{\frac{\min\{1, \beta\}}{2}} \cdot \|u - s_{n+1}\|_{H^k(\Omega)}.$$

$$\hspace{1cm} (31)$$
with \( C := (2^{\alpha+1/2} \max \{1, C_0\} e^\alpha)^{\max(1, \sigma)} \) and \( \alpha := \frac{\tau-2}{\tau} - \frac{1}{2} > 0 \). In particular
\[
\min_{n+1 \leq i \leq 2n} \| u - s_i \|_{L^\infty(\Omega)} \leq C \cdot \log(n)^{\alpha} \cdot \| u - s_{n+1} \|_{\mathcal{H}_k(\Omega)} \cdot \begin{cases} 
\min \left\{ \begin{array}{ll}
\max n^{-\alpha - 1/2} & \text{PDE} - f - \text{greedy} \\
\max n^{-\alpha - 1/4} & \text{PDE} - f \cdot P - \text{greedy} \\
\max n^{-\alpha} & \text{PDE} - P - \text{greedy}
\end{array} \right. 
\end{cases}
\]

Note that the theorem also could be formulated for \( n \geq 1 \) by enlarging the constant \( C_0 \).

**Proof.** The proof is structured into several steps:

- Since \( k \) is a kernel of finite smoothness such that \( \mathcal{H}_k(\Omega) \supset H^\tau(\Omega) \), Theorem \( \text{[16]} \) gives a decay of the corresponding Kolmogorov \( n \)-width \( d_n(\Lambda) \) of
\[
d_n(\Lambda) \leq C \cdot n^{\frac{1}{2} - \frac{1}{2\tau}} \equiv C \cdot n^{-\alpha}.
\]
This bound satisfies the assumptions of Corollary \( \text{[12]} \) such that it can be applied to derive a decay of the geometric mean of subsequent power function values as
\[
\left( \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right)^{1/n} \leq 2^{n+1/2} \tilde{C}_0 e^\alpha \log(n)^{\alpha} n^{-\alpha}, \quad n \geq 3
\]
for any choice of functionals, thus especially for the \( \beta \)-greedy choices.

- Now we apply Theorem \( \text{[16]} \) especially Eq. \( \text{[29]} \) for \( \beta \in [0,1] \) and Eq. \( \text{[30]} \) for \( \beta \in (1, \infty] \) and insert the bound on the geometric mean of the generalized power function values from the previous step. This gives for \( \beta \in [0,1], n \geq 3 \)
\[
\left( \prod_{i=n+1}^{2n} \max_{\lambda \in \Lambda} |\lambda(r_i)| \right)^{1/n} \leq n^{-\beta/2} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)} \cdot \left( \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right)^{1/n} \leq n^{-\beta/2} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)} \cdot 2^{n+1/2} \tilde{C}_0 e^\alpha \log(n)^{\alpha} n^{-\alpha},
\]
and for \( \beta \in (1, \infty) \) (using \( 1/\infty = 0 \)) and again \( n \geq 3 \)
\[
\left( \prod_{i=n+1}^{2n} \max_{\lambda \in \Lambda} |\lambda(r_i)| \right)^{1/n} \leq n^{-1/2} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)} \cdot \left( \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1})^{1/\beta} \right)^{1/n} \leq n^{-1/2} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)} \cdot \left( 2^{n+1/2} \tilde{C}_0 e^\alpha \right)^{1/\beta} \log(n)^{\alpha/\beta} n^{-\alpha/\beta}.
\]

In both cases, the left hand side can be estimated with help of
\[
\min_{n+1 \leq j \leq 2n} \max_{\lambda \in \Lambda} |\lambda(r_j)| \leq \max_{\lambda \in \Lambda} |\lambda(r_i)|
\]
for all \( i = n + 1, \ldots, 2n \), which finally yields
\[
\min_{n+1 \leq j \leq 2n} \max_{\lambda \in \Lambda} |\lambda(r_j)| \leq C' \cdot n^{-\min\left(\frac{1}{2}, \frac{1}{\beta}\right)} \log(n) \cdot n^{-1} \cdot \max \{1, \sigma\} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)}
\]
with \( C' := (2^{\alpha+1/2} \tilde{C}_0 e^\alpha)^{\max(1, \sigma)} \).
Now we make use of the maximum principle from Eq. (10) which is available due to our assumptions on the BVP. Therefore it holds
\[ \|u - s_n\|_{L^\infty(\Omega)} \leq C \left( \|u - s_n\|_{L^\infty(\partial\Omega)} + \|Lu - Ls_n\|_{L^\infty(\Omega)} \right) \]
\[ = C \cdot \sup_{\lambda \in \Lambda_L \cup \Lambda_B} |\lambda(u - s_n)|, \]
\[ \Rightarrow \min_{j=n+1, \ldots, 2n} \|u - s_j\|_{L^\infty(\Omega)} \leq C \cdot \max_{j=n+1, \ldots, 2n} \sup_{\lambda \in \Lambda_L \cup \Lambda_B} |\lambda(u - s_j)|. \]

Since \( r_j = u - s_j \) we can directly combine this last equation with the result of the previous step to obtain
\[ \min_{j=n+1, \ldots, 2n} \|u - s_j\|_{L^\infty(\Omega)} \leq CC' \cdot n^{-\min\{1, \beta\}} \cdot \max_{j=1, \ldots, n} \|L_{r_{n+1}}\|_{\mathcal{H}_k(\Omega)}, \]
and this concludes the proof.

For \( \beta \in (0, 1] \) it can be observed that we obtain a faster rate of convergence in the number of selected functionals \( \lambda_i \in \Lambda_L \cup \Lambda_B \) than for \( \beta = 0 \) and therefore in the number of selected collocation points \( x_i \in \Omega \cup \partial\Omega \). This additional decay of \( n^{-\beta/2} \) increases with increasing \( \beta \in (0, 1] \), which corresponds to a more “target-data dependent” choice of the next interpolation point according to the definition of PDE-\( \beta \)-greedy algorithms, see Definition 13. In particular the proven decay for PDE-\( f \)-greedy (\( \beta = 1 \)) is better than the proven decay for the PDE-\( P \)-greedy. This makes intuitively sense, since the PDE-\( f \)-greedy choice is adapted to the right hand side of the BVP, while the PDE-\( P \)-greedy is independent of the right hand side and only worst-case optimal [31]. It is furthermore optimal in the sense of realizing the same decay rate as the Kolmogorov \( n \)-width (which is based on the abstract result of [3]). A visualization of the different point distributions can be found in Section 6.1. Moreover, for \( \beta \in (0, 1] \) the new estimate [31] breaks the curse of dimensionality. Indeed, since \( \tau > d/2 + 2 \) by assumption, we can only guarantee \( \alpha = \frac{1}{2} - \frac{\tau - 2}{d} < 0 \) and thus the term \( (\log(n) \cdot n^{-1})^{\alpha} \) may yield arbitrarily slow convergence, unless additional smoothness is taken into consideration. On the contrary, the term \( n^{-\beta/2} \) is converging to zero at a speed that is independent on the dimension \( d \) of the space. This breaking of the curse of dimensionality is theoretically and practically appealing for high dimensional PDE problems.

As already remarked in [42], we conjecture that the analysis so far is suboptimal. In particular in view of [29] we do not expect that the additional \( \log(n)^\alpha \) factor is really required, which occurs for \( \beta > 0 \).

Finally we want to remark that the whole analysis also works for PDEs on manifolds, e.g. Theorem 16 holds for both \( \Omega \) a domain or a manifold. The analysis of target-data dependent greedy kernel algorithms works in very general terms, because it simply leveraged the abstract analysis of [3] and [42, Section 3] for Hilbert spaces and connects it to kernel approximation in RKHS.

In order to derive practically usable convergence rates, the Kolmogorov \( n \)-widths need to be analyzed (Section 3), and to derive those for manifolds, sampling inequalities and corresponding zero lemmas for manifolds can be leveraged, as e.g. in Theorem 2. In order to transfer the resulting convergence rates on
$\sup_{\lambda \in \Lambda} |\lambda(r_i)|$ to convergence rates on $r_i \to 0$ in some norm, PDE solution theory is required, e.g. some well-posedness estimate or a maximum principle.

**Remark 18.** Similar statements as in Theorem 17 can be derived for the case of analytic kernels such as the Gaussian kernel, but we do not consider them here since their RKHSs are difficult to describe [34], and especially they are usually subsets of Sobolev spaces of arbitrary orders. They are thus not particularly well-suited for our purposes.

However, it should be noted that already in the case $\beta = 0$ no converge rates of the various PDE-greedy methods with these kernels are known in the literature. Moreover, even if interpolation with these kernels usually gives spectral convergence that scales as $\exp(-n^{1/4})$, for $\beta > 0$ the additional $\beta$-dependent term is still mitigating the curse of dimensionality, and in particular the dimension-independent rate of $-1/2$ is in line with the rates of interpolation obtained in [6] for the Gaussian kernel.

6 Numerical experiments

The implementation of the PDE greedy algorithms is coined PDE-VKOGA, in analogy to the Vectorial Kernel Orthogonal Greedy Algorithm (VKOGA) for interpolation, see [43], and is provided via a corresponding git repository. The code for reproducing the subsequent numerical experiments is also freely available.

In order to better balance between the importance of the functionals in the interior $\Omega$ and on the boundary $\partial \Omega$, we introduce a corresponding weighting into the PDE-greedy selection criteria and thus use

$$\lambda_{n+1} = \arg \max_{0 \neq \lambda \in \Lambda \setminus \Lambda_n} \left\{ \frac{|\lambda(u - \Pi_{V_n}(u))|^{\beta} \cdot P_{\Lambda_n}(\lambda)^{1-\beta}}{w \cdot |\lambda(u - \Pi_{V_n}(u))|^{\beta} \cdot P_{\Lambda_n}(\lambda)^{1-\beta}} \biggm| \lambda \in \Lambda_L \right\} \biggm| \lambda \in \Lambda_B.$$

As the greedy approach can be interpreted as a pivoted partial Cholesky decomposition of the kernel collocation matrix (see [25]), the choice of the pivoting elements is influenced by this weighting.

As a first experiment we choose a low-dimensional case of $d = 2$ with a smooth solution which allows comparisons of the different greedy variants with the finite element method. The second experiment adresses the case of solutions in $d = 2$ with singularities, which are more challenging for the present approach. As final setting we demonstrate the potential of the greedy procedures for high-dimensional problems where $d = 12$.

6.1 Smooth case for $d = 2$

As domain we consider the sector of the 2D unit ball with opening angle $\alpha \pi$ for $\alpha = \frac{2}{3}$ as visualized in Figure 3. We consider the boundary value problem

$$-\Delta u = f, \quad \text{on } \Omega,$$
$$u = g, \quad \text{on } \partial \Omega,$$
which has a smooth solution \( u(x) = \|x\|^2 \) for source function \( f(x) = -4 \) and Dirichlet-boundary values \( g(x) = u(x) \).

As a kernel, we use the cubic Matérn kernel, which is a translational invariant kernel with \( \Phi(x-y) = \varphi(\|x-y\|), \varphi(r) = (15 + 15r + 6r^2 + r^3) \cdot \exp(-r) \). The sets \( \Omega \) and \( \partial \Omega \) were uniformly randomly sampled using 50000 and 1000 points, respectively. The \( L^2(\Omega) \) and \( L^\infty(\Omega) \) errors were numerically approximated using a uniform grid of mesh width \( 2 \cdot 10^{-3} \) (approximately 250000 points).

We consider PDE-\( P \)-, PDE-\( f \cdot P \)- and PDE-\( f \)-greedy, using weight factors of \( w \in \{10^0, 10^1, \ldots, 10^8\} \). The results are displayed in Table 1, Figure 2 and Figure 3. First we compare the approximation of the three PDE-greedy methods to a classical P1 finite element method (FEM), which uses a triangular grid with polygonal approximation of the sector and different global refinement levels.

Table 1 lists the \( L^2(\Omega) \) and the \( L^\infty(\Omega) \) approximation error for \( u - s_n \) for several expansion sizes, which match the number of degrees of freedom for the FEM solution. Note that we do not compute the kernel approximants for more than 5000 degrees of freedom, as our current implementation is most efficient for smaller expansion sizes. In any case, the final FEM accuracy using 20193 DOFs is already satisfied by the kernel models using 341 collocation points. This clearly demonstrates that both PDE-\( P \)-greedy and PDE-\( f \)-greedy outperform the FEM approximation. Figure 2 gives an overview on the influence of the weighting parameter \( w \) for PDE-\( f \)-, PDE-\( f \cdot P \)- and PDE-\( P \)-greedy. For all the cases, the plots show the decay of the errors \( \sup_{\lambda \in \Lambda} |\lambda(u - s_n)| \) and \( \sup_{x \in \Omega} |(u - s_n)(x)| \) in the number of collocation points. Given a suitable weighting (of e.g. \( w = 10^6 \)), all three PDE-greedy methods provide approximately the same rate of decay for the \( \sup_{x \in \Omega} |(u - s_n)(x)| \) error (right column). However in terms of the \( \sup_{x \in \Lambda} |\lambda(u - s_n)| \) decay, the PDE-\( f \)-greedy method provides the faster rate of decay (left column). Figure 3 visualizes the selected collocation points for both PDE-\( f \)-greed and PDE-\( P \)-greedy for using a weighting of \( w = 10^6 \).

One can observe that PDE-\( f \)-greedy chooses more collocation points close to the boundary, while the PDE-\( P \)-greedy collocation points are rather uniformly distributed, which can be even proved in the case of interpolation (instead of collocation), see [41].
Table 2: Numerical results regarding Section 6.2: Errors for singular solution on sector example

| # DOFs | \( L^2(\Omega) \) | \( L^\infty(\Omega) \) | \( L^2(\Omega) \) | \( L^\infty(\Omega) \) |
|--------|-----------------|-----------------|-----------------|-----------------|
| FEM    | 2.50e-4         | 2.39e-3         | 6.31e-5         | 8.45e-4         |
| PDE-P-greedy | 9.04e-5       | 1.23e-3         | 5.86e-5         | 1.04e-3         |
| PDE-f-greedy | 2.39e-3       | 5.11e-3         | 8.10e-5         | 2.32e-4         |

6.2 Singular case for \( d = 2 \)

As next example we consider the identical geometry and PDE as in the previous section, but now with data-functions that result in a solution with singularity in its derivatives, i.e. the solution is

\[
u(x) = \|x\|^{1/\alpha} \sin(\phi(x)/\alpha)
\]

with \(\phi(x) = \arctan(x_2/x_1)\), which is the solution of the PDE for \( f(x) = 0 \) and \( g(x) = u(x) \). We apply the same kernel, sampling and greedy procedures as in the previous section.

Instead of showing the results for all the weightings, we focus on suitable weightings which are given as \( w = 10^3 \) for the PDE-f-greedy and \( w = 10^6 \) for the PDE-P-greedy. The results are displayed in Table 2, Figure 4 and Figure 5. As before, in Table 2 we show errors in both the \( L^2(\Omega) \) and \( L^\infty(\Omega) \) norms, including a comparison to a classical P1 FEM. In this case of a singular solution, the PDE-greedy methods do no longer outperform the FEM, though still match the accuracy of the FEM. We remark that the performance of the PDE-greedy method deteriorate further, when the singularity is intensified, e.g. by increasing the opening angle \( \alpha \) towards 2. In Figure 4 we observe that we still obtain convergence in the residual error and a bit slower in the collocation functional values. This convergence actually is a nice result, as it demonstrates that we can even approximate some target functions which are lying outside of the RKHS of the chosen kernel. But we observe a clear quantitative difference compared to the smooth example, which is that the errors only decay about half the orders of magnitude. Finally, Figure 5 visualizes the selected collocation points for PDE-f-greedy and PDE-P-greedy. One can clearly observe a clustering of the PDE-f-greedy collocation points towards the singularity, while the PDE-P-greedy collocation points are again uniformly distributed.

6.3 High-dimensional PDE for \( d = 12 \)

In order to show the applicability of the proposed PDE-greedy methods in high dimensional problems, we consider the linear Poisson problem \(-\Delta u(x) = -2d\) on the scaled unit hypercube \( \Omega = d^{-1/2} \cdot [0,1]^d \) in dimension \( d = 12 \) with Dirichlet data on \( \partial \Omega \) given by the function \( g(x) = 1 + \sum_{i=1}^d x_i^2 \). Like this, the exact solution to the corresponding BVP is given by \( u(x) = 1 + \sum_{i=1}^d x_i^2 \).

We approximate the solution by approximants \( s_n \) using PDE-\( \beta \)-greedy with \( \beta = 1 \), i.e. PDE-f-greedy and the cubic Matérn kernel.
For the numerical experiment, the boundary $\partial \Omega$ was discretized using 9600 uniformly randomly sampled training points, while $10^5$ randomly sampled training points were used for the interior $\Omega$. To evaluate the error in $\Omega$ independently of these training points, we used further $10^5$ randomly sampled test points.

Figure 6 visualizes the resulting decay of the error $\|u - s_n\|_{L^\infty(\Omega)}$ as well as $\sup_{\lambda \in \Lambda} |\lambda(u - s_n)|$ over the expansion size $n$. Overall the weighting of interior vs. boundary points seems to play an essential role in the quality of the approximate solution. In principle this weighting could as well be interpreted as an additional hyperparameter of the greedy schemes, which could be chosen based on suitable model selection procedures in order to further optimize the results. Independent of this dependence of the weighting, the error decay curves clearly indicate that the scheme is applicable to higher-dimensional geometrical input domains, where classical methods such as FEM are not applicable.

7 Conclusion and outlook

This paper considered the approximation of solutions of linear PDEs via symmetric kernel collocation with help of PDE-$\beta$-greedy strategies. After deriving estimates on the Kolmogorov $n$-widths of the sets of linear functionals of those BVP, an abstract analysis of greedy kernel methods was applied to derive worst-case optimal approximation rates. Extending an analysis of target-data dependent greedy kernel algorithms for standard interpolation to symmetric kernel collocation, it was possible to derive convergence rates for the full scale of values of $\beta$, with increasingly fast rates going from target-data-independent ($\beta = 0$) to for target-data dependent ($\beta > 0$) algorithms. These algorithms yield point sets which are adapted to both the domain $\Omega$ and the right hand side of the considered BVP.

The experiments demonstrated that for smooth target functions the PDE-greedy algorithms behave very favourably. This even holds in comparison to FEM, for low dimensional problem where the FEM can be applied.

An important insight for the PDE-greedy schemes compared to pure function approximation is revealed by our study: In the PDE context there is a crucial difference between the functional error $\sup_{\lambda \in \Lambda} |\lambda(r_n)|$ and the residual error $\|r_n\|_{L^\infty(\Omega)}$ (which coincide in the case of plain function approximation, i.e. $L = Id$). Especially the schemes for $\beta > 0$ directly aim at driving the functional error to zero. But even if the theoretical convergence rates guarantee an identical decay for both quantities, this decay, however, may differ by a large constant. In order to drive the residual error down, additional preference to choosing boundary functionals via weighting of the boundary selection indicator turned out to be essential.

Overall we conclude that kernel PDE-greedy collocation algorithms are useful approaches for approximation of linear PDE problems which admit sufficiently smooth solutions. In this scope the method’s benefits are the use of meshless points and the ease of implementation in a dimension-independent fashion. This renders those methods particularly useful in high-dimensional cases, where traditional mesh-based schemes such as finite volumes/differences/elements can not easily be applied due to the increasingly complex grid management and curse of dimensionality. This curse of dimensionality is provably overcome with the kernel PDE-greedy approaches. Clear limitations can, however, be seen in
approximation of target functions that have singularities.

We want to comment on some directions for future research: An extension to nonlinear PDE problems seems possible by using the presented scheme for the linearized problems within corresponding fix-point schemes of nonlinear solvers. Further work may as well consider “escaping the native space”, i.e. problems where the solution of the PDE problem is not in the RKHS of the chosen kernel. We expect to be able to derive convergence orders, which are however reduced in accordance with the limited smoothness of the solution \([23,40]\). Future work will also address the analysis of greedy algorithms for other methods of solving PDEs via kernel methods, such as unsymmetric kernel collocation or RBF-FD methods. Further open questions include the optimality of the proven decay rates as well as approximation in different norms and work on analytic kernels such as the Gaussian kernel.

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Figure 2: Numerical results regarding Section 6.1. Visualization of the decay of the errors $\sup_{\lambda \in \Lambda} |\lambda(u - s_n)|$ (left) and $\sup_{x \in \Omega} |(u - s_n)(x)|$ (right) for PDE-P-greedy (top), PDE-$f \cdot P$-greedy (middle), PDE-$f$-greedy (bottom) for different values of the weighting parameter $w$. For each of the three PDE-greedy methods, the number of selected collocation points $n$ which are selected on the boundary is denoted by $n_{\partial \Omega}$. 
Figure 3: Numerical results regarding Section 6.1: Visualization of the distribution of 341 collocation points selected by PDE-$f$-greedy (left) and PDE-$P$-greedy (right) for the smooth solution. Collocation points on the boundary are visualized as crosses. One can observe that the PDE-$f$-greedy selected centers cluster adaptively next to the boundary, while the PDE-$P$-greedy centers are rather uniformly distributed.

Figure 4: Numerical results regarding Section 6.2: Visualization of the decay of the errors $\sup_{\lambda \in \Lambda} |\lambda(u - s_n)|$ (left) and $\sup_{x \in \Omega} |(u - s_n)(x)|$ (right) for PDE-$P$-greedy (top) using $w = 10^6$ and PDE-$f$-greedy (bottom) using $w = 10^3$. 

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Figure 5: Numerical results regarding Section 6.2: Visualization of the distribution of 1305 collocation points selected by PDE-$f$-greedy (left) and PDE-$P$-greedy (right) for the singular solution. Collocation points on the boundary are visualized as crosses. One can observe that the PDE-$f$-greedy selected centers cluster adaptively next to the singularity, while the PDE-$P$-greedy centers are rather uniformly distributed.

\[ \sup_{\lambda \in \Lambda} |\lambda(u - s_n)| \]

\[ \sup_{x \in \Omega} |(u - s_n)(x)| \]

Figure 6: Numerical results regarding Section 6.3: Visualization of the approximation error $\sup_{\lambda \in \Lambda} |\lambda(u - s_n)|$ and $\|u - s_n\|_{L^\infty(\Omega)}$ (evaluated on $10^5$ randomly sampled test points) over the expansion size $n$ (x-axis) for PDE-$f$-greedy on a Poisson problem in $\Omega \subset \mathbb{R}^{12}$ for different weightings $w$ of boundary vs. interior values.

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A Proofs for Section 3

Proof of Proposition 5. We calculate:

\[ d_n(\Lambda) \equiv \inf_{G_n \subseteq H'} \sup_{\mu \in \Lambda} \dim(G_n) \sup_{H_n \subseteq H_k(\Omega)} \| \mu(u - \Pi_{H_n}(u)) \|_{H_k(\Omega)} \]

\[ = \inf_{G_n \subseteq H'} \max_{j=1, \ldots, M} \left( \sup_{\mu \in \Lambda_j} \sup_{H_n \subseteq H_k(\Omega)} \| \mu(u - \Pi_{H_n}(u)) \|_{H_k(\Omega)} \right) \]

\[ \leq \inf_{G_n \subseteq H'} \max_{j=1, \ldots, M} \left( \sup_{\mu \in \Lambda_j} \sup_{H_n \subseteq H_k(\Omega)} \| \mu(u - \Pi_{H_n}(u)) \|_{H_k(\Omega)} \right) \]

\[ = \min_{\sum_{j=1}^{M} \eta_j \leq n} \inf_{G_n \subseteq H'} \ldots \inf_{H_n \subseteq H_k(\Omega)} \max_{j=1, \ldots, M} \left( \sup_{\mu \in \Lambda_j} \sup_{H_n \subseteq H_k(\Omega)} \| \mu(u - \Pi_{H_n}(u)) \|_{H_k(\Omega)} \right) \]

\[ = \min_{\sum_{j=1}^{M} \eta_j \leq n} \max_{j=1, \ldots, M} \left( \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in \Lambda_j} \sup_{H_n \subseteq H_k(\Omega)} \| \mu(u - \Pi_{H_n}(u)) \|_{H_k(\Omega)} \right) \]

\[ = \min_{\sum_{j=1}^{M} \eta_j \leq n} \max_{j=1, \ldots, M} \left( d_n(\Lambda_j) \right). \]

\[ \square \]

Proof of Theorem 6. Based on Eq. (18) and Eq. (14) we have the following representation of the Kolmogorov n-width \( d_n(\Lambda_L) \) (using the Riesz representer \( v_\mu \) of a functional \( \mu \)):

\[ d_n(\Lambda_L, H_k(\Omega)) = \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in \Lambda_L} \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in \Lambda_L} \| u \|_{H_k(\Omega)} \]

\[ = \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in \Lambda_L} \| L(u - \Pi_{H_n}(u)) \|_{L^\infty(\Omega)} \]

\[ \leq \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in H_k(\Omega) \setminus \{0\}} \| L(u - \Pi_{H_n}(u)) \|_{L^\infty(\Omega)} \]

For the third equality \( \Lambda_L = \{ \delta_x \circ L \mid x \in \Omega \} \) was used and in the final step the infimum was upper bounded by the infimum computed over spaces spanned by any \( n \) elements of \( \Lambda_L \).

Since \( u, \Pi_{H_n}(u) \in H_k(\Omega) \approx H^r(\Omega) \) and the differential operator \( L \) is of degree 2, it follows that \( Lu, L\Pi_{H_n}(u) \in H^{r-2}(\Omega) \). Recall that \( x_\lambda \in \Omega \cup \partial \Omega \)
is the point related to the functional $\lambda \in \Lambda_L \cup \Lambda_B \subset \mathcal{H}_k(\Omega)'$, i.e. where the evaluation takes place within $\Omega$ (for evaluations of $\lambda \in \Lambda_L$) or on its boundary $\partial \Omega$ (for evaluations of the boundary functionals $\lambda \in \Lambda_B$). Then (because of the construction of $\Pi_{H_n}(u)$) it holds

$$(\Pi_{H_n}(u))(x_i) = (Lu)(x_i) \quad \forall i = 1, \ldots, n$$

$$(Lu - \Pi_{H_n}(u))(x_i) = 0 \quad \forall i = 1, \ldots, n,$$

i.e. $Lu - \Pi_{H_n}(u) \in H^{r-2}(\Omega)$ is a function which is zero at least on the set $X_n := \{x_i\}_{i=1}^n$. Therefore the sampling inequalities from Theorem 1 can be applied, provided $h \equiv h_{X_n, \Omega} \leq h_0$ (using $m = 0, q = \infty, p = 2, r = 2$ instead of $\tau$ and $Lu - \Pi_{H_n}(u)$ instead of $u$)

$$\|Lu - \Pi_{H_n}(u)\|_{L^\infty(\Omega)} \leq C h^{r-2-d/2} \cdot |Lu - \Pi_{H_n}(u)_{H^{r-2}(\Omega)}.$$

To bound the infimum over all $H_n \subset \mathcal{H}_k(\Omega)$ that are induced by $\Lambda_L$-functionals, one can now consider functionals related to asymptotically uniformly distributed points. For such a choice of points the fill distance $h_{X_n, \Omega}$ decays according to $h_{X_n, \Omega} \leq cn^{-1/d}$, such that we obtain

$$\|Lu - \Pi_{H_n}(u)\|_{L^\infty(\Omega)} \leq C n^{\frac{1}{2} - \frac{r}{d}} \cdot |Lu - \Pi_{H_n}(u)_{H^{r-2}(\Omega)}$$

$$\leq C n^{\frac{1}{2} - \frac{r}{d}} \cdot \|Lu - \Pi_{H_n}(u)\|_{H^{r-2}(\Omega)}.$$

This result can be plugged into the (in)equality chain above for the Kolmogorov $n$-width $d_n(\Lambda_L)$, such that we obtain

$$d_n(\Lambda_L, \mathcal{H}_k(\Omega)') \leq C n^{\frac{1}{2} - \frac{r}{d}} \cdot \inf_{H_n \subset \mathcal{H}_k(\Omega)} \sup_{\dim(H_n) = n} \frac{\|Lu - \Pi_{H_n}(u)\|_{H^{r-2}(\Omega)}}{\|u\|_{\mathcal{H}_k(\Omega)}} \leq C' n^{\frac{1}{2} - \frac{r}{d}} \cdot \inf_{H_n \subset \mathcal{H}_k(\Omega)} \sup_{\dim(H_n) = n} \frac{\|u - \Pi_{H_n}(u)\|_{H^r(\Omega)}}{\|u\|_{\mathcal{H}_k(\Omega)}} \leq C'' n^{\frac{1}{2} - \frac{r}{d}} \cdot \inf_{H_n \subset \mathcal{H}_k(\Omega)} \sup_{\dim(H_n) = n} \frac{\|u\|_{\mathcal{H}_k(\Omega)}}{\|u\|_{\mathcal{H}_k(\Omega)}}$$

The second inequality is due to [19] Chapter 1: Lemma 7.2+Remark 8.1] and the triangle inequality.

**Proof of Theorem** For this proof we introduce $\alpha := -\left(\frac{1}{2} - \frac{r}{d}\right) > 0$ and $\beta := -\left(\frac{1}{2} - \frac{r-1/2}{d-1/2}\right) > 0$, i.e. it holds $\beta > \alpha > 0$. We choose

$$n_2(n) = \lceil n^{\alpha/\beta} \rceil$$

$$n_1(n) = \lfloor n - n^{\beta/\alpha} \rfloor = \lfloor n \cdot (1 - n^{\alpha/\beta-1}) \rfloor.$$
For \( n \geq 2 \) we have \( 0 < 1 - 2^{\alpha/\beta - 1} \leq 1 - n^{\alpha/\beta - 1} \leq 1 \), therefore we have \( n_1(n) > n \).

In more details, there are constants \( c_1, c_2 > 0 \) such that 
\[
 c_1 n \leq n_1(n) \leq c_2 n \quad \text{for} \quad n \geq 2, \quad c_1, c_2 \text{ are independent of } n, \quad \text{and} \quad c_1 < 1.
\]

Thus, using Theorem 5, 6 and 7 gives
\[
d_n(A) \leq \inf_{n_1 + n_2 \leq n} \max (d_{n_1}(A_L), d_{n_2}(A_B)) \leq \max(C_{L}, C_{B}) \cdot \inf_{n_1 + n_2 \leq n} \max \left( n_1^{-\alpha}, n_2^{-\beta} \right) \leq \max(C_{L}, C_{B}) \cdot \max \left( n_1(n)^{-\alpha}, n_2(n)^{-\beta} \right).
\]

A tiny extra calculation helps to resolve the expressions \( n_1(n)^{-\alpha} \) and \( n_2(n)^{-\beta} \):
\[
 n_1(n)^{-\alpha} \leq (c_1 n)^{-\alpha} = c_1^{-\alpha} n^{-\alpha}
\]
\[
 n_2(n)^{-\beta} \leq [n^{\alpha/\beta}]^{-\beta} \leq (n^{\alpha/\beta})^{-\beta} = n^{-\alpha}.
\]

Therefore we finally obtain
\[
d_n(A) \leq \max(C_{L}, C_{B}) \cdot \max(c_1^{-\alpha}, 1) \cdot n^{-\alpha} \leq \max(C_{L}, C_{B}) \cdot c_1^{-\alpha} \cdot n^{-\alpha}.
\]

Recalling the abbreviation \(-\alpha = \frac{1}{2} - \frac{\beta}{\alpha}\) we obtain the statement.

**Proof of Theorem 7**

The proof follows the same strategy as the one of Theorem 6. We start by decomposing the boundary \( \partial \Omega \) into smooth manifolds \( M_i \), i.e. \( \partial \Omega = \cup_{i=1}^{N} M_i \).

Based on Eq. (18) and Eq. (14) we have analogously to the calculation in Eq. (33) the following representation of the Kolmogorov n-width \( d_n(A_B) \) (using the Riesz representer \( v_\mu \) of a functional \( \mu \)):

\[
d_n(A_B, H_k(\Omega)) = \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in \Lambda_n} \| v_\mu - \Pi_{H_n}(v_\mu) \|_{H_k(\Omega)} \leq \inf_{H_n \subseteq H_k(\Omega)} \sup_{\mu \in \Lambda_n} \| u - \Pi_{H_n}(u) \|_{L^\infty(\partial \Omega)} \| v_\mu \|_{H_k(\Omega)}.
\]

As all involved function are continuous, we waived to include the trace operator. In order to bound \( \| u - \Pi_{H_n}(u) \|_{L^\infty(\partial \Omega)} = \max_{x \in \Omega} (u - \Pi_{H_n}(u)) \| L^\infty(\Omega) \), we leverage standard restriction theorems of RKHS to conclude that \( u - \Pi_{H_n}(u) \| L^\infty(\Omega) \| H_k(\Omega_i) = H^{\tau - 1/2}(\Omega_i) \). We recall that the functionals \( \lambda_1, \ldots, \lambda_n \) are associated to the interpolation points \( x_{\lambda_1}, \ldots, x_{\lambda_n} \), i.e. \( u - \Pi_{H_n}(u) \) is a function with zeros in these interpolation points \( x_{\lambda_1}, \ldots, x_{\lambda_n} \). Therefore the sampling inequalities from Theorem 2 can be applied to any smooth manifold \( M_i \), \( i = 1, \ldots, N \), provided \( h_{X_n, \Omega_i} \leq C_{\| \cdot \|} \) (using \( k = d - 1, \mu = 0, q = \infty, p = 2, t = \tau - 1/2 \) and \( u - \Pi_{H_n}(u) \) instead of \( u \), which suffice the assumptions):

\[
 \| u - \Pi_{H_n}(u) \|_{L^\infty(\Omega_i)} \leq C_{\| \cdot \|} h_{X_n, \Omega_i}^{-1/2 - (d-1)/2} \cdot | u - \Pi_{H_n}(u) |_{W^\tau - 1/2(\Omega_i)}^{1/2},
\]

where \( h_{X_n, \Omega_i} \) is the fill distance related to \( \{ x_{\lambda_1}, \ldots, x_{\lambda_n} \} \cap \Omega_i \subseteq \Omega_i \subseteq \partial \Omega \). Observing that \( H_k(\Omega_i) \) is norm-equivalent to \( H^{\tau - 1/2}(\Omega_i) \) for all \( i = 1, \ldots, N \) (see
we can estimate the semi-norm \(|u - \Pi_{H_n}(u)|_{W^{-1/2}_{\infty}(\partial \Omega)}\) as follows:

\[
|u - \Pi_{H_n}(u)|_{W^{-1/2}_{\infty}(\partial \Omega)} \leq \|u - \Pi_{H_n}(u)\|_{W^{1/2}_2(\Omega)} \\
\leq C_n\|u - \Pi_{H_n}(u)\|_{H_k}\left(\frac{d}{2}\right) \\
\leq C_n\|u\|_{H_k}\left(\frac{d}{2}\right)
\]

The infimum within Eq. (34) over all \(H_n \subset \mathcal{H}\) that are induced by \(\Lambda_B\)-functionals can be upper bounded by considering asymptotically uniformly distributed (wrt. the intrinsic distance on the manifold \(\partial \Omega\)) points \(\{x_{\lambda_1}, ..., x_{\lambda_n}\} \subset \partial \Omega\). For such a choice of points the fill distance \(h_{X_n,\partial \Omega}\) decays according to \(h_{X_n,\partial \Omega} \leq cn^{-1/(d-1)}\) (where the constant \(c\) also depends on the number \(N\) of manifolds \(\mathcal{H}\)), such that Eq. (35) turns into (due to \(-\frac{1}{d-1}(r-1/2 - (d-1)/2) = \frac{r-1/2}{d-1}\)):

\[
\|u - \Pi_{H_n}(u)\|_{L^\infty(\Omega)} \leq C_n^2 n^{-\frac{r-1/2}{d-1}} \cdot \|u - \Pi_{H_n}(u)\|_{W^{-1/2}_{\infty}(\partial \Omega)} \\
\leq \tilde{C}_n \|u\|_{H_k(\Omega)} \\
\Rightarrow \max_{i=1,...,N} \|u - \Pi_{H_n}(u)\|_{L^\infty(\Omega)} \leq \left( \max_{i=1,...,N} \tilde{C}_i \right) n^{-\frac{r-1/2}{d-1}} \cdot \|u\|_{H_k(\Omega)}.
\]

This result can be plugged into the (in-)equality chain for the Kolmogorov \(n\)-width \(d_n(\Lambda_B)\) of Eq. (34), such that we obtain directly

\[
d_n(\Lambda_B, H_k(\Omega)) \leq \left( \max_{i=1,...,N} \tilde{C}_i \right) n^{-\frac{r-1/2}{d-1}}.
\]

\[\square\]

**B Proofs for Section 4**

*Proof of Lemma 14* Let

\[
R_n^2 := \left[ \prod_{i=n+1}^{2n} \left( \frac{\lambda_{i+1}(r_i)}{P_{H_i}(\lambda_{i+1})} \right)^{2/n} \right]^{1/n}
\]

The geometric arithmetic mean inequality gives

\[
R_n^2 \leq \frac{1}{n} \sum_{i=n+1}^{2n} \left( \frac{\lambda_{i+1}(r_i)}{P_{H_i}(\lambda_{i+1})} \right)^2 = \frac{1}{n} \sum_{i=0}^{2n} \left( \frac{\lambda_{i+1}(r_i)}{P_{H_i}(\lambda_{i+1})} \right)^2 - \frac{1}{n} \sum_{i=0}^{n} \left( \frac{\lambda_{i+1}(r_i)}{P_{H_i}(\lambda_{i+1})} \right)^2
\]

We now use Eq. (13) applied to \(s_{2n+1} + s_{n+1}\), and the properties of orthogonal projections to obtain

\[
R_n^2 \leq \frac{1}{n} \left( \|s_{2n+1}\|_{H_k(\Omega)}^2 - \|s_{n+1}\|_{H_k(\Omega)}^2 \right) \leq \frac{1}{n} \left( \|f\|_{H_k(\Omega)}^2 - \|s_{n+1}\|_{H_k(\Omega)}^2 \right) \\
= \frac{1}{n} \|f - s_{n+1}\|_{H_k(\Omega)}^2 = \frac{1}{n} \|r_{n+1}\|_{H_k(\Omega)}^2.
\]
It follows that \( R_n \leq n^{-1/2} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)} \), and thus
\[
\left[ \prod_{j=n+1}^{2n} |\lambda_{i+1}(r_i)| \right]^{1/n} \leq n^{-1/2} \cdot \| r_{n+1} \|_{\mathcal{H}_k(\Omega)} \cdot \left[ \prod_{j=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right]^{1/n} .
\]

**Proof of Lemma 15** We prove the two cases separately:

a) For \( \beta = 0 \), i.e. the PDE-P-greedy algorithm, this is the standard power function estimate in conjunction with the PDE-P-greedy selection criterion \( P_{\Lambda_i}(\lambda_{n+1}) = \sup_{\lambda \in \Lambda} P_{\lambda_i}(\lambda) \). For \( \beta = 1 \) this holds with equality as it is simply the selection criterion of PDE-f-greedy since we have here \( \lambda_{n+1}(r_n) = \sup_{\lambda \in \Lambda} |r_n| \). We thus consider \( \beta \in (0, 1) \) and let \( \tilde{\lambda}_{i+1} \in \Lambda \) be such that \( |\tilde{\lambda}_{i+1}(r_i)| = \sup_{\lambda \in \Lambda} |\lambda(r_i)| \). Then the selection criterion from Eq. (24) gives
\[
|\lambda(r_i)|^\beta \cdot P_{\lambda_i}(\lambda)^{1-\beta} \leq |\lambda_{i+1}(r_i)|^\beta \cdot P_{\lambda_i}(\lambda_{i+1})^{1-\beta} \quad \forall \lambda \in \Lambda ,
\]
and in particular
\[
P_{\lambda_i}(\tilde{\lambda}_{i+1}) \leq \frac{|\lambda_{i+1}(r_i)|^\beta}{|\tilde{\lambda}_{i+1}(r_i)|^\beta} \cdot P_{\lambda_i}(\lambda_{i+1}).
\]

Using this bound with the standard power function estimate gives
\[
\sup_{\lambda \in \Lambda} |\lambda(r_i)| = |\tilde{\lambda}_{i+1}(r_i)| \leq P_{\lambda_i}(\tilde{\lambda}_{i+1}) \cdot \| f - s_i \|_{\mathcal{H}_k(\Omega)}
\]
\[
\leq \frac{|\lambda_{i+1}(r_i)|^\beta}{|\tilde{\lambda}_{i+1}(r_i)|^\beta} \cdot P_{\lambda_i}(\lambda_{i+1}) \cdot \| f - s_i \|_{\mathcal{H}_k(\Omega)}
\]
\[
= \frac{|\lambda_{i+1}(r_i)|^\beta}{\sup_{\lambda \in \Lambda} |\lambda(r_i)|^\beta} \cdot P_{\lambda_i}(\lambda_{i+1}) \cdot \| f - s_i \|_{\mathcal{H}_k(\Omega)}.
\]

This can be rearranged for \( \sup_{\lambda \in \Lambda} |\lambda(r_i)| \) to yield the final result.

b) For \( \beta \in (1, \infty) \), the selection criterion from Eq. (24) can be rearranged to
\[
|\lambda(r_i)|^\beta \leq \frac{|\lambda_{i+1}(r_i)|^\beta}{P_{\lambda_i}(\lambda_{i+1})^\beta - 1} \cdot P_{\lambda_i}(\lambda)^{\beta - 1} \quad \forall \lambda \in \mathbb{A} \setminus \Lambda_i,
\]
and taking the supremum \( \sup_{\lambda \in \mathbb{A} \setminus \Lambda_i} \) gives
\[
\sup_{\lambda \in \mathbb{A}} |\lambda(r_i)| \leq \frac{|\lambda_{i+1}(r_i)|^\beta}{P_{\lambda_i}(\lambda_{i+1})^\beta - 1} \cdot \sup_{\lambda \in \mathbb{A}} P_{\lambda_i}(\lambda)^{\beta - 1} \quad \forall \lambda \in \mathbb{A} \setminus \Lambda_i.
\]

For \( \beta = \infty \), the selection criterion of the PDE-f/P-greedy algorithm can be directly rearranged to yield the statement (when using the notation \( 1/\infty = 0 \)).

\[\square\]
Proof of Theorem 16. We prove the two cases separately:

a) For $\beta = 0$, i.e. PDE-P-greedy, Eq. (27) gives $\sup_{\lambda \in \Lambda} |\lambda(r_i)| \leq P_{\lambda_i}(\lambda_{i+1}) \cdot ||r_i||_{\mathcal{H}_k(\Omega)}$. Taking the product $\prod_{i=n+1}^{2n}$ and the $n$-th root in conjunction with the estimate $||r_i||_{\mathcal{H}_k(\Omega)} \leq ||r_{n+1}||_{\mathcal{H}_k(\Omega)}$ for $i = n + 1, \ldots, 2n$, we get

$$n^{-1/2} ||r_{n+1}||_{\mathcal{H}_k(\Omega)} \cdot \left[ \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right]^{1/n} \geq \left[ \prod_{i=n+1}^{2n} |\lambda_{i+1}(r_i)| \right]^{1/n}.$$ 

For $\beta \in (0,1]$, we start by reorganizing the estimate (27) of Lemma 15 to get

$$|\lambda_{i+1}(r_i)| \geq \left( \sup_{\lambda \in \Lambda} \lambda(r_i)^{1/\beta} \right) / \left( P_{\lambda_i}(\lambda_{i+1})^{1-\beta} \cdot ||r_i||_{\mathcal{H}_k(\Omega)}^{1-\beta} \right),$$

and we use this to bound the left hand side of Eq. (26) as

$$n^{-1/2} ||r_{n+1}||_{\mathcal{H}_k(\Omega)} \cdot \left[ \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1}) \right]^{1/n} \geq \left[ \prod_{i=n+1}^{2n} |\lambda_{i+1}(r_i)| \right]^{1/n}.$$

Rearranging the factors, and using again the fact that $||r_i||_{\mathcal{H}_k(\Omega)} \leq ||r_{n+1}||_{\mathcal{H}_k(\Omega)}$ for $i = n + 1, \ldots, 2n$, gives

$$\left[ \prod_{i=n+1}^{2n} \left( \sup_{\lambda \in \Lambda} \lambda(r_i)^{1/\beta} \right)^{1/n} \right] \leq n^{-1/2} \cdot ||r_{n+1}||_{\mathcal{H}_k(\Omega)} \cdot \left[ \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_{i+1})^{1/\beta} \right]^{1/n} \cdot \left[ \prod_{i=n+1}^{2n} ||r_i||_{\mathcal{H}_k(\Omega)}^{1-\beta} \right]^{1/n}.$$

Now, the inequality can be raised to the exponent $\beta$ to give the final statement.

b) For $\beta \in (1,\infty]$ we proceed similarly by first rewriting Eq. (28) of Lemma 15 as

$$|r_i(x+i)| \geq \left( ||r_i||_{L^{\infty}(\Omega)} \cdot P_i(x+i)^{1-1/\beta} \right) / \left( ||P_i||_{L^{1/\beta}(\Omega)}^{1-1/\beta} \right),$$

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and we lower bound the left hand side of Equation (26) as

\begin{equation}
n^{-1/2} \sup_{\lambda \in \Lambda} |r_{n+1}| \cdot \left[ \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_i+1) \right]^{1/n} \geq \left[ \prod_{i=n+1}^{2n} |\lambda_i+1(r_i)| \right]^{1/n} \leq \left[ \prod_{i=n+1}^{2n} \left( \sup_{\lambda \in \Lambda} |r_i| \cdot P_{\lambda_i}(\lambda_i+1)^{1-1/\beta} \right) / \left( \sup_{\lambda \in \Lambda} P_{\lambda_i}(\lambda)^{1-1/\beta} \right) \right]^{1/n}.
\end{equation}

Rearranging for \( \left[ \prod_{i=n+1}^{2n} \sup_{\lambda \in \Lambda} |\lambda(r_i)| \right]^{1/n} \) yields

\begin{equation}
\left[ \prod_{i=n+1}^{2n} \sup_{\lambda \in \Lambda} |\lambda(r_i)| \right]^{1/n} \leq n^{-1/2} \cdot \|r_{n+1}\|_{H_\alpha(\Omega)} \cdot \left[ \prod_{i=n+1}^{2n} \sup_{\lambda \in \Lambda} P_{\lambda_i}(\lambda)^{1-1/\beta} \right]^{1/n} \cdot \left[ \prod_{i=n+1}^{2n} P_{\lambda_i}(\lambda_i+1)^{1/\beta} \right]^{1/n},
\end{equation}

which gives the final result due to \( \sup_{\lambda \in \Lambda} P_i(\lambda) \leq 1 \) for all \( i = 0, 1, ... \).