Brauer character degrees and Sylow normalizers

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Abstract

If \(p\) and \(q\) are primes, and \(G\) is a \(p\)-solvable finite group, it is possible to detect that a \(q\)-Sylow normalizer is contained in a \(p\)-Sylow normalizer using the character table of \(G\). This is characterized in terms of the degrees of \(p\)-Brauer characters. Some consequences, which include yet another generalization of the Itô–Michler theorem, are also obtained.

Keywords Character degrees · Brauer characters · Sets of primes

Mathematics Subject Classification 20C15 · 20C20

1 Introduction

If \(G\) is a finite group and \(\pi\) is a set of primes, let \(\text{Irr}_\pi(G)\) be the subset of the irreducible complex characters \(\chi\) of \(G\) such that all the primes dividing the degree \(\chi(1)\) lie in \(\pi\). It is fair to say that the interaction between \(\text{Irr}_\pi(G)\) and the structure of \(G\) is one of the

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recurrent problems in character theory. Recently, in [12], we have asked if it is possible to characterize group-theoretically when \( \text{Irr}_\pi(G) = \text{Irr}_\rho(G) \) for sets of primes \( \pi \) and \( \rho \).

In this paper, we fix a prime \( p \), and we turn our attention to \( p \)-Brauer characters, within the universe of finite \( p \)-solvable groups. (Degrees of modular representations in characteristic \( p \) outside \( p \)-solvable groups are usually deemed an intractable subject.) If \( G \) is a \( p \)-solvable finite group and \( \text{IBr}(G) \) is the set of the irreducible \( p \)-Brauer characters of \( G \), we let \( \text{IBr}_\pi(G) \) be the subset of \( \varphi \in \text{Br}(G) \) such that all the primes dividing \( \varphi(1) \) lie in \( \pi \). As usual, if \( r \) is a prime, \( r' \) denotes the set of primes different from \( r \).

The following is our first main result.

**Theorem A** Let \( G \) be a finite \( p \)-solvable group, and let \( q \) be a prime. Then, \( \text{IBr}_q(G) \subseteq \text{IBr}_{q'}(G) \) if and only if there are \( Q \in \text{Syl}_q(G) \) and \( P \in \text{Syl}_p(G) \) such that \( N_G(Q) \subseteq N_G(P) \).

Theorem A was the main result of [1], assuming that \( G \) is both \( p \)-solvable and \( q \)-solvable. To better understand our this new situation, we invite the reader to consider, for instance, \( G = \text{PSL}_2(3^5) \). If \( q = 2 \), then \( N_G(P) = N_G(Q) \) for some \( p \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \). Of course, \( G \) is \( p \)-solvable, but not \( q \)-solvable. There are families of almost-simple groups like this, and we are able to deal with this new situation thanks to the main result of [6]. In particular, the proof of Theorem A depends on the Classification of Finite Simple Groups.

From Theorem A, and using McKay bijections for Brauer characters in \( p \)-solvable groups together with the recent proof of the divisibility of degrees between Glauberman correspondents by M. Geck in [2], we can prove our second main result.

**Theorem B** Let \( G \) be a finite \( p \)-solvable group, and let \( q \) be a prime different from \( p \). Then \( \text{IBr}_{q'}(G) = \text{IBr}_{q}(G) \) if and only if there are \( P \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \) such that \( N_G(Q) \subseteq N_G(P) \) and \( Q \) is abelian.

As the reader can easily check, in the trivial case where \( p \) does not divide \( |G| \), Theorem B is yet another restatement of the Itô–Michler theorem.

As happens with complex irreducible characters (see [12]), it does not seem easy to group-theoretically characterize when \( \text{IBr}_\pi(G) = \text{IBr}_\rho(G) \) for arbitrary sets of primes \( \pi \) and \( \rho \), even if \( G \) is \( p \)-solvable. In the case, for instance, where \( \pi = \mathbb{P} \) is the set of all primes and \( \rho = q' \), where \( q \) is a prime, this constitutes Problem 3.2 of [9]. (This problem was studied long before in [8] and more recently in [5].) It has now been conjectured that \( \text{IBr}_{q'}(G) = \text{IBr}_p(G) \) if and only if the number of \( p \)-regular classes of \( G \) is the number of \( p \)-regular classes of \( N_G(Q)/Q' \), where \( Q \in \text{Syl}_q(G) \). This is a consequence of the Inductive McKay conjecture, and it seems difficult to obtain a direct proof. (See Conjecture D in [7].)

### 2 Proof of theorem A

Our notation for Brauer characters follows [9]. The deepest part of the proof of Theorem A comes from the main result in [MN].
Theorem 2.1 Let $G$ be a finite $p$-separable group. Let $H$ be a Hall $p$-subgroup, let $K$ be a $p$-complement of $G$, and let $q$ be a prime. Then, every $\alpha \in \text{Irr}_q(H)$ extends to $G$ if and only if there is $Q \in \text{Syl}_q(H)$ such that $N_G(Q) \subseteq N_G(K)$.

Proof This is Theorem A of [6].

A very useful result to deal with the hypotheses in this paper appears in Suzuki’s book.

Theorem 2.2 Let $G$ be a $p$-solvable group, and let $q$ be a prime. Let $\pi = \{p, q\}$. Then, $G$ has a unique conjugacy class of Hall $\pi$-subgroups, and every $\pi$-subgroup of $G$ is contained in one of them.

Proof This follows from 5.3.13 of [14].

For the reader’s convenience, let us prove the following standard result.

Lemma 2.3 Let $G$ be a finite group. Let $Q$ be a Sylow $q$-subgroup of $G$ and let $N$ be a normal subgroup of $G$. If $\varphi \in \text{IBr}_q(G)$, then $\varphi_N$ has a $Q$-invariant irreducible constituent and any two of them are $N_G(Q)$-conjugate.

Proof Let $\varphi \in \text{IBr}_q(G)$ and let $\theta \in \text{IBr}(N)$ be an irreducible constituent of $\varphi_N$. Let $G_{\theta}$ be the stabilizer of $\theta$ in $G$. By Clifford’s theorem (see Corollary 8.9 of [9], for instance), we have that $|G : G_{\theta}|$ divides $\varphi(1)$, and hence, it is a $q'$-number. It follows that there is an element $g \in G$ such that $Q^g$ is contained in $G_{\theta}$. Hence, $Q \subseteq G_{\theta}^{g^{-1}} = G_{\theta^g}$ and $G_{\theta^g}$ is a $Q$-invariant irreducible constituent of $\varphi_N$.

Now, suppose that $\theta$ and $\mu$ are two $Q$-invariant irreducible constituents of $\varphi_N$. Again by Clifford’s theorem we have that there is an element $g \in G$ such that $\mu = \theta^g$. It follows that $Q$ and $Q^g$ are contained in $G_{\mu}$, and hence, there is an element $x \in G_{\mu}$ such that $Q = (Q^g)^x$. Therefore $gx \in N_G(Q)$ and $\theta^x = (\theta^g)^x = \mu^x = \mu$, as wanted.

The following result implies Theorem A. In its proof, we shall use Fong characters of Brauer characters, and we refer the reader to Chapter 10 of [9] for their main properties. (The term Fong character was coined by I. M. Isaacs after some results of P. Fong.) We also use the fact that if $G$ is $p$-solvable, $H$ is a $p$-complement of $G$, and $\varphi \in \text{IBr}(G)$ has degree not divisible by $p$, then $\varphi_H \in \text{Irr}(H)$. (See Theorem 10.9 of [9].) A complication when dealing with Brauer characters is that we do not have any form of Frobenius reciprocity, even in favorable conditions. For instance, in the previous situation where $H$ is a $p$-complement of a $p$-solvable group $G$, if $\alpha \in \text{Irr}(H)$, and $\varphi$ is an irreducible constituent of $p'$-degree of the induced Brauer character $\alpha^G$, then $\alpha$ needs not be the irreducible character $\varphi_H$; a fact that would simplify our proof below. (If $G = A_4$, $p = 2$, and $H = C_3$, then the Brauer character $(1_H)^G = 21 + \lambda_1 + \lambda_2$, where $\lambda_i$ are distinct linear Brauer characters. Now take $\alpha = 1_H$ and $\varphi = \lambda_i$.)

Theorem 2.4 Let $G$ be a $p$-solvable group, let $\text{IBr}(G)$ be the set of irreducible Brauer characters of $G$, and let $q \not= p$ be a prime. Let $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ such that $U = PQ$ is a subgroup of $G$. Suppose that $H$ is a $p$-complement of $G$ containing $Q$. Then, the following are equivalent.
(a) \( \text{IBr}_{\bar{q}}(G) \subseteq \text{IBr}_{\bar{p}'}(G) \).
(b) \( N_G(Q) \subseteq N_G(P) \).
(c) Every \( \alpha \in \text{Irr}_{\bar{q}}(H) \) extends to \( G \).

**Proof** Set \( \pi = (p, q) \). By Theorem 2.2, notice that we can find \( P \in \text{Syl}_{p}(G) \) and \( Q \in \text{Syl}_{q}(G) \) such that \( U = PQ \) is a Hall \( \pi \)-subgroup of \( G \). (In fact, given a Hall \( \pi \)-subgroup \( U \) of \( G \), then \( U = PQ \) for every \( P \in \text{Syl}_{p}(U) \) and \( Q \in \text{Syl}_{q}(U) \).)

Assume (a). We prove (b) by induction on \(|G|\). If \( K \) is a minimal normal subgroup of \( G \), then we have that \( N_G(Q) \subseteq N_G(P)K \), by using induction in \( G/K \). Since \( G \) is \( p \)-solvable, then \( K \) is either a \( p \)-group or a \( p' \)-group. In the first case, \( N_G(P)K = N_G(P) \) and we are done. So we assume that \( K \) is a \( p' \)-group.

Let \( V \) be any subgroup of \( G \) containing \( KQ \) and let \( \delta \in \text{IBr}_{\bar{q}}(V) \). By Lemma 2.3, there exists a \( Q \)-invariant irreducible constituent \( \tau \in \text{IBr}(K) \) of the restriction \( \delta_K \) (using that \( Q \in \text{Syl}_{q}(V) \)). We claim that \( \tau \) is also \( P \)-invariant. By Corollary 8.7 of [9], we have that \( \delta \) is an irreducible constituent of the induced Brauer character \( \tau^V \). Now consider the Brauer character \( \delta^G \), which has degree \(|G : V| \delta(1)\), which is not divisible by \( q \). Hence, there exists an irreducible constituent \( \varphi \in \text{IBr}(G) \) of \( \delta^G \) of degree not divisible by \( q \). By hypothesis, we have that \( \varphi \) has degree not divisible by \( p \). Now, \( \varphi \) is an irreducible constituent of \( \tau^G \), and therefore, \( \tau \) is an irreducible constituent of the restriction \( \varphi_K \), again by Corollary 8.7 of [9].

If \( I = G_1 \) is the stabilizer of \( \tau \) in \( G \), and \( \mu \in \text{IBr}(I) \) is the Clifford correspondent of \( \varphi \) over \( \tau \), we have that \( IG : I \) is a \( \pi' \)-number. Therefore, using Theorem 2.2, we have that \( U \subseteq I^\varphi \) for some \( g \in G \). Then, \( \tau \) and \( \tau^\varphi \) are \( Q \)-invariant constituents of \( \varphi \), and thus, by Lemma 2.3, we have that \( \tau^\varphi = \tau^\varphi(G) \), for some \( x \in N_G(Q) \subseteq K \). Hence, we may assume that \( \tau^\varphi = \tau^\varphi \) for some \( y \in N_G(P) \). Then, \( I^\varphi = I, U^\varphi = I \), and we conclude that \( P = B^\varphi \subseteq I \). In other words, \( \tau \) is \( P \)-invariant, as claimed.

Now, let \( V = N_G(P)K \). We claim that \( \text{IBr}_{\bar{q}}(V) \subseteq \text{IBr}_{\bar{p}'}(V) \). By the claim in the previous paragraph, let \( \tau \in \text{IBr}(K) \) be \( PQ \)-invariant under \( \delta \). By Theorem 8.11 of [9], there is a unique \( \hat{\tau} \in \text{IBr}(KP) \) over \( \tau \), extending \( \tau \), and we conclude that \( \delta \) lies over \( \hat{\tau} \). Now, \( KP \triangleleft V, V/KP \) is a \( p' \)-group, and we have that \( \delta(1)/\hat{\tau}(1) \) divides \(|V : KP|h| \) by Theorem 8.30 of [9]. Since \( \tau(1) \) is not divisible by \( p \), we conclude that \( \delta \) has \( p' \)-degree, as claimed. By induction, we may assume that \( V = G \). That is, \( KP \triangleleft G \). We want to use Theorem 2.1.

Next, we show that every \( \alpha \in \text{Irr}_{\bar{q}}(H) \) extends to \( G \). By Lemma 2.3, let \( \tau \in \text{Irr}(K) \) be \( Q \)-invariant under \( \alpha \). By the claim in the third paragraph, we have that \( \tau \) is \( P \)-invariant too. By Corollary 6.28 of [3], we have that \( \tau \) has a canonical extension \( \gamma \in \text{Irr}(KP) \). Using the uniqueness of \( \gamma \), we easily check that \( G_\gamma \cap H = H_\gamma \). By Isaacs restriction theorem (Lemma 6.8(d) of [11]), the Clifford correspondence and Mackey’s theorem (Theorem 1.16 of [10]), we have that \( \alpha \) extends to \( G \). If \( p_0 = p' \) is the set of primes dividing \(|G| \) different from \( p \), by Theorem 2.1 applied to \( p_0 \), we conclude that there is \( Q_1 \in \text{Syl}_{p_0}(H) \) such that \( N_{G_1}(Q_1) \subseteq N_{G_1}(P) \). In particular, \( P \triangleleft PQ_1 \). By Theorem 2.2, \( PQ_1 \) and \( PQ \) are \( G \)-conjugate. Hence \( P \triangleleft PQ \) and therefore \( Q_1 \in \text{Syl}_{p}(N_{G_1}(P)) \). Hence, \( Q_1 = Q \) for some \( z \in N_{G_1}(P) \), and \( N_G(Q) = N_G(Q_1)^\gamma \subseteq N_{G_1}(P) \).

We have that (b) implies (c), by Theorem 2.1 applied to \( p_0 = p' \).

Finally, we prove that (c) implies (a) by using Fong characters. Suppose now that every \( \alpha \in \text{Irr}_{\bar{q}}(H) \) extends to \( G \). We show that \( \text{IBr}_{\bar{q}}(G) \subseteq \text{IBr}_{\bar{p}'}(G) \). Let \( \varphi \in \text{IBr}_{\bar{q}}(G) \), and let \( \alpha \in \text{Irr}(H) \) be an irreducible constituent of \( \varphi_H \) such that \( \alpha(1) = \varphi(1)p' \) (using Theorem 10.18 of [9].) In other words, \( \alpha \) is a Fong character for \( \varphi \). Then, \( \alpha(1) \) is not divisible by \( q \), and by hypothesis, we have that \( \alpha \) extends to some \( \gamma \in \text{Irr}(G) \). Then, the Brauer
character \( \mu = \chi^0 \in \text{IBr}(G) \) extends \( \alpha \). (Indeed, if \( \chi^0 = \varphi_1 + \varphi_2 \) for Brauer characters \( \varphi_i \) of \( G \), then the irreducible character \( \alpha \) would be written as \( (\varphi_1)_H + (\varphi_2)_H \).) By Theorem 10.17 of [9], we have that \( \varphi = \mu \), and \( \varphi(1) = \mu(1) = \alpha(1) \) has degree not divisible by \( p \).

\[ \square \]

3 Proof of theorem B

If \( N \) is a normal subgroup of \( G \) and \( \theta \in \text{IBr}(N) \), we write \( \text{IBr}(G | \theta) \) to denote the set of irreducible Brauer characters \( \varphi \) of \( G \) such that \( \theta \) is an irreducible constituent of \( \varphi_N \). We write \( \text{IBr}_p(G | \theta) = \text{IBr}_p(G) \cap \text{IBr}(G | \theta) \).

Next is the version for Brauer characters and \( p \)-solvable groups of Theorem A of [13], which we shall need to prove Theorem B. It is worth mentioning that its proof uses the recent proof of the divisibility of the degrees of the Glauberman correspondence in [2].

**Theorem 3.1** Let \( G \) be a \( p \)-solvable group, and \( \varphi \in \text{Syl}_p(G) \), then there is a bijection \( f : \text{IBr}_p(G) \to \text{IBr}_p(N_G(P)) \) such that \( f(\varphi)(1) \) divides \( \varphi(1) \) for all \( \varphi \in \text{IBr}_p(G) \). Furthermore, \( \varphi(1)/f(\varphi)(1) \) divides \( |G : N_G(P)| \).

**Proof** We argue by induction on \( |G| \). Since \( O_p(G) \) is in the kernel of every \( \varphi \in \text{IBr}(G) \), by induction we may assume that \( O_p(G) = 1 \) and hence \( K = O_p(G) > 1 \). Let \( S/K = O_p(G/K) \) and notice that \( P_0 = P \cap S \) is a Sylow \( p \)-subgroup of \( S \). By the Frattini argument, we have that \( G = KN_G(P_0) \). Notice also that \( N_G(P_0) < G \) since \( O_p(G) = 1 \).

Let \( \theta_1, \ldots, \theta_s \) be a complete set of representatives of the orbits of the action of \( N_G(P) \) on the \( P \)-invariant irreducible characters of \( K \). By Lemma 2.3, we have that

\[ \text{IBr}_p(G) = \text{IBr}_p(G | \theta_1) \cup \cdots \cup \text{IBr}_p(G | \theta_s) \]

is a disjoint union. Fix \( \theta_i \in \text{Irr}(K) \), \( P \)-invariant, and observe that \( \theta_i \) is also \( P_0 \)-invariant. Let \( \theta_i^* \in \text{Irr}(C_K(P_0)) = \text{IBr}(C_K(P_0)) \) be the Glauberman correspondent of \( \theta_i \) and let \( T_i = G_0 \) be the stabilizer of \( \theta_i \) in \( G \). Since the Glauberman correspondence and the action of \( N_G(P_0) \) commute (see Lemma 2.10 of [10]), it follows that \( N_{T_i}(P_0) = T_i \cap N_G(P_0) \) is the stabilizer of \( \theta_i^* \) in \( N_G(P_0) \). By Dade’s theorem (see Theorem (6.5) of [15]), we have that \( (T_i, K, \theta_i) \) and \( (N_{T_i}(P_0), C_K(P_0), \theta_i^*) \) are isomorphic character triples. In particular, there is a bijection \( \Delta : \text{Irr}(T_i/\theta_i) \to \text{Irr}(N_{T_i}(P_0)/\theta_i^*) \) such that \( \chi(1)/\theta_i(1) = \Delta(\chi)(1)/\theta_i^*(1) \) for all \( \chi \in \text{Irr}(T_i/\theta_i) \). Now using \( \Delta \) and Lemma 3.12 of [4] we can construct a bijection \( \ast : \text{IBr}(T_i/\theta_i) \to \text{IBr}_p(N_{T_i}(P_0)/\theta_i^*) \) such that \( \psi(1)/\theta_i(1) = \psi^*(1)/\theta_i^*(1) \) for all \( \psi \in \text{IBr}(T_i/\theta_i) \), and since \( \theta_i(1) \) and \( \theta_i^*(1) \) are \( p^2 \)-numbers we have that \( \ast : \text{IBr}_p(T_i/\theta_i) \to \text{IBr}_p(N_{T_i}(P_0)/\theta_i^*) \) is a bijection.

Let \( \chi \in \text{IBr}_p(G) \) and let \( \theta_i \in \text{IBr}(K) = \text{Irr}(K) \) be such that \( \chi \in \text{IBr}_p(G | \theta_i) \). By the Clifford correspondence (Theorem 8.9 of [9]), we have that there is \( \psi \in \text{IBr}_p(T_i/\theta_i) \) such that \( \psi^G = \chi \). Since \( N_{T_i}(P_0) \) is the stabilizer of \( \theta_i^* \) in \( N_G(P_0) \), again by the Clifford correspondence we have that \( \psi^*N_{T_i}(P_0) \in \text{Irr}(N_G(P_0)) \). Furthermore, since \( P \subseteq N_{T_i}(P_0) \), we have that

\[ (\psi^*)^*N_{T_i}(P_0)(1) = |N_G(P_0) : N_{T_i}(P_0)| \psi^*(1) \]

is a \( p' \)-number. Hence, we define
g : IBr_{p'}(G) \rightarrow IBr_{p'}(N_G(P_0))

by \( g(\chi) = (\psi^*)^{N_G(P_0)} \).

Since \( \theta_1, \ldots, \theta_s \) is a complete set of representatives of the action of \( N_G(P) \) on the 
\( P \)-invariant characters of \( K \) and the Glauberman correspondence commutes with the action of 
\( N_G(P) \subseteq N_G(P_0) \), we have that \( \theta_1, \ldots, \theta_s \) is a complete set of representatives of the 
action of \( N_G(P) \) on the \( P \)-invariant irreducible characters of \( Irr(C_K(P_0)) = IBr(C_K(P_0)) \).

By Lemma 2.3, we have that

\[
IBr_{p'}(N_G(P_0)) = IBr_{p'}(N_G(P_0) | \theta_1^*) \cup \cdots \cup IBr_{p'}(N_G(P_0) | \theta_s^*)
\]

is a disjoint union and it follows that \( g \) is a bijection.

Let \( \chi \in IBr_{p'}(G|\theta_i) \) and let \( \psi \in IBr_{p'}(T_i|\theta_i) \) with \( \psi^G = \chi \). Then \( \psi(1) = \psi^*(1)\theta_i(1)/\theta_i^*(1) \) and, since \( \theta_i^*(1) \) divides \( \theta_i(1) \) (by the recent main theorem of [2]), we have that \( \psi^*(1) \) divides \( \psi(1) \). Since \( G = N_G(P_0)K \) we have that \( g(\chi)(1) = |N_G(P_0) : N_{T_i}(P_0)|\psi^*(1) \) divides 
\( |G : T_i|\psi(1) = \chi(1) \).

Finally, since \( N_G(P_0) < G \), we apply induction to obtain a bijection

\[
h : IBr_{p'}(N_G(P_0)) \rightarrow IBr_{p'}(N_G(P))
\]

such that \( h(\psi)(1) \) divides \( \psi(1) \) and \( \chi(1)/h(\chi)(1) \) divides \( |N_G(P_0) : N_G(P)| \) for all 
\( \psi \in IBr_{p'}(N_G(P_0)) \). Let \( f = gh = h \circ g \). Clearly, \( f \) is a bijection and \( f(\chi)(1) \) divides \( \chi(1) \) 
for all \( \chi \in IBr_{p'}(G) \). Now, since \( g(\chi)(1)/h(g(\chi))(1) \) divides \( |N_G(P_0) : N_G(P)| \), we have that 
\( \chi(1)/f(\chi)(1) \) divides \( |N_G(P_0) : N_G(P)|\psi(1)/\psi^*(1) = |N_G(P_0) : N_G(P)|\theta_i(1)/\theta_i^*(1) \).

By Problem 13.2 of [3], \( \theta_i(1)/\theta_i^*(1) \) divides \( |K : C_K(P_0)| = |G : N_G(P_0)| \). Hence, 
\( \chi(1)/f(\chi)(1) \) divides \( |G : N_G(P_0)| \).

\[ \square \]

The following is Theorem B.

**Theorem 3.2** Suppose that \( G \) is a \( p \)-solvable finite group and let \( q \) be a prime different from 
\( p \). Then

\[
IBr_{p'}(G) = IBr_{q'}(G)
\]

if and only if there is a Sylow \( p \)-subgroup \( P \) of \( G \) and a Sylow \( q \)-subgroup \( Q \) of \( G \), such that 
\( N_G(P) = PN_G(Q) \) and \( Q \) is abelian.

**Proof** Suppose that \( IBr_{q'}(G) = IBr_{p'}(G) \). By Theorem 2.4, we have that there is a Sylow 
\( p \)-subgroup of \( G \) and a Sylow \( q \)-subgroup of \( G \) such that \( N_G(Q) \subseteq N_G(P) \). We claim that

\[
IBr_{p'}(N_G(P)) = IBr_{q'}(N_G(P))
\]

First, we notice that Theorem 2.4 applied to \( N_G(P) \) shows \( IBr_{q'}(N_G(P)) \subseteq IBr_{p'}(N_G(P)) \). Let 
\( \mu \in IBr_{p'}(N_G(P)) \). By Theorem 3.1, there is \( \varphi \in IBr_{p'}(G) \) such that \( \mu(1) \) divides \( \varphi(1) \) which 
is a \( q' \)-number. Hence, we conclude \( \mu \in IBr_{q'}(N_G(P)) \). Then \( IBr_{q'}(N_G(P)) = IBr_{p'}(N_G(P)) \), 
as claimed.

Therefore, we may assume, arguing by induction that \( P \triangleleft G \). Then, we have that

\[
Irr_{q'}(G/P) = IBr_{q'}(G) = IBr_{p'}(G) = IBr_{p'}(G/P) = Irr(G/P).
\]
By the Ito–Michler theorem, we know that $G/P$ has a normal and abelian Sylow $q$-subgroup $PQ/Q$. Hence, $Q$ is abelian and $PQ \triangleleft G$. Then $G = P\text{N}_G(Q)$ by the Frattini’s argument.

Conversely, suppose that there is $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ such that $\text{N}_G(P) = P\text{N}_G(Q)$ and $Q$ is abelian. Notice that since $\text{N}_G(Q) \subseteq \text{N}_G(P)$ by Theorem 2.4 we have that $\text{IBr}_{q'}(G) \subseteq \text{IBr}_{q'}(G)$. We only need to proved the reverse containment. If $\text{N}_G(P) < G$, arguing by induction, we have that $\text{IBr}_{q'}(\text{N}_G(P)) = \text{IBr}_{q'}(\text{N}_G(P))$. Let $\varphi \in \text{IBr}_{q'}(G)$ and let $f : \text{IBr}_{p'}(G) \rightarrow \text{IBr}_{p'}(\text{N}_G(P))$ be the bijection given in Theorem 3.1. Hence $\varphi(1)/f(\varphi)(1)$ divides $|G : \text{N}_G(P)|$ which is not divisible by $q$. Since $\text{IBr}_{q'}(\text{N}_G(P)) = \text{IBr}_{q'}(\text{N}_G(P))$, we have that $f(\varphi)(1)$ is not divisible by $q$ and thus $\varphi \in \text{IBr}_{q'}(G)$.

Hence, we may assume that $P$ is a normal subgroup of $G$ and then $\text{IBr}(G) = \text{Irr}(G/P)$. Also $G = \text{N}_G(Q)P$ and hence $PQ$ is normal in $G$. Then, $PQ/P$ is an abelian normal Sylow $q$-subgroup of $G/P$. By Ito’s theorem, we have that $\text{Irr}(G/P) = \text{Irr}_{q'}(G/P)$. It follows that

$$\text{IBr}_{q'}(G) = \text{Irr}_{q'}(G/P) = \text{Irr}(G/P) = \text{IBr}_{q'}(G),$$

and we are done.

To prove the assertion in the Abstract, it is enough to notice that in $p$-solvable groups, the ordinary character table of $G$ uniquely determines the Brauer characters of $G$, by using the Fong–Swan theorem. (See Theorem 10.1 and Corollary 10.4 of [9].)

Finally, using Isaacs $\pi$-characters, the Glauberman–Isaacs correspondence, and some ad hoc arguments, it is possible to replace in Theorems A and B of this paper $p'$ by $\pi$, $p$-solvable groups by $\pi$-separable groups, and Sylow $p$-subgroups by Hall $\pi$-complements.

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