UNIPOTENT FACTORIZATION OF VECTOR BUNDLE AUTOMORPHISMS

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ABSTRACT. We provide unipotent factorizations of vector bundle automorphisms of real and complex vector bundles over smooth manifolds. This generalises work of Thurston-Wasserstein and Wasserstein for trivial vector bundles. We also address two symplectic cases and propose a complex geometric analog of the problem in the setting of holomorphic vector bundles over Stein manifolds.

1. Introduction

By elementary linear algebra any matrix in \( \text{SL}_k(\mathbb{R}) \) or \( \text{SL}_k(\mathbb{C}) \) can be written as a product of elementary matrices \( \text{id} + \alpha e_{ij} \), i.e., matrices with ones on the diagonal and at most one non-zero element outside the diagonal. Replacing \( \text{SL}_k(\mathbb{R}) \) or \( \text{SL}_k(\mathbb{C}) \) with \( \text{SL}_k(R) \) where \( R \) is the ring of continuous real or complex valued functions on a topological space \( X \), we arrive at a much more subtle problem. This problem was addressed Thurston and Wasserstein [TW86] in the case where \( X \) is the Euclidean space and more generally by Wasserstein [Was88] for a finite dimensional normal topological space \( X \). In particular, Wasserstein [Was88] proves that for any finite dimensional normal topological space \( X \), and any continuous map \( F : X \to \text{SL}_k(\mathbb{R}) \) for \( k \geq 3 \) (and \( \text{SL}_k(\mathbb{C}) \) for \( k \geq 2 \), respectively) which is homotopic to the constant map \( x \mapsto \text{id} \), there are continuous maps \( E_1, \ldots, E_N \) from \( X \) to the space of elementary real (resp. complex) matrices, such that

\[
F = E_N \circ \cdots \circ E_1.
\]

More recently, the problem has been considered by Doubtsov and Kutzschebauch [DK19].

Recall that a map \( S \) on a vector space is unipotent if \( (S - \text{id})^m = 0 \) for some \( m \). Note that, again by elementary linear algebra, providing a factorisation in terms of elementary matrices is equivalent to providing a factorisation in terms of upper and lower triangular unipotent matrices, i.e., matrices of the form

\[
\text{id} + \sum_{i<j} \alpha_{ij} e_{ij} \quad \text{and} \quad \text{id} + \sum_{i>j} \alpha_{ij} e_{ij},
\]

respectively.

Moreover, note that an element in \( \text{SL}_k(R) \) where \( R \) is the ring of continuous real valued functions on a topological space \( X \) is equivalent to a continuous \( SL_k(\mathbb{R}) \)-valued function on \( X \). Similarly, an element in \( \text{SL}_k(R) \) where \( R \) is the ring of continuous complex valued functions on a topological space \( X \) is equivalent to a continuous \( SL_k(\mathbb{C}) \)-valued function on \( X \).

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In this paper we will consider the following problem: Let \( X \) be a smooth manifold and
\[
\pi : V \to X
\]
a smooth real or complex vector bundle over \( X \) of rank \( k \). Moreover, let \( S \) be a vector bundle automorphism of \( V \) with constant determinant equal to one, in other words \( S \) is a diffeomorphism \( V \to V \) such that \( \pi \circ S = \pi \) and \( S \) restricts to a linear map of determinant one on each fiber \( \pi^{-1}(x), x \in X \). We will call such an object a \textit{special} vector bundle automorphism. Given this data, can \( S \) be factored into a composition of unipotent vector bundle automorphisms, i.e., vector bundle automorphisms that restrict to unipotent maps on the fibers?

Fixing a local trivialization \( \rho : V|_U \to U \times \mathbb{R}^k \) (resp. \( \rho : V|_U \to U \times \mathbb{C}^k \)) of \( V \) over a subset \( U \subset X \) we get that a special vector bundle automorphism \( S \) satisfies
\[
\rho \circ S|_U \circ \rho^{-1}(x,y) = (x,F(x))
\]
where \( F \) is a \( SL_k \)-valued function on \( U \). In other words, special vector bundle automorphisms can be represented locally as \( SL_k \)-valued functions. In this sense the problem considered by Thurston and Wasserstein can be seen as a special case (given by the cases when \( V = X \times \mathbb{R}^k \) and \( V = X \times \mathbb{C}^k \)) of the problem considered in this paper.

We will say that a special vector bundle automorphism \( S \) is \textit{nullhomotopic} if there is a homotopy \( (t,x) \in [0,1] \times X \mapsto S_t(x) \) such that \( S_0 = S, S_1 \) is the identity map on \( V \) and \( S_t \) is a special vector bundle automorphism for each \( t \in [0,1] \). Our first result is the following.

**Theorem 1.** Let \( X \) be a smooth manifold and let \( \pi : V \to X \) be a real (resp. complex) vector bundle of rank \( k \). Assume that \( k \geq 3 \) (resp. \( k \geq 2 \)) and let \( S \) be a nullhomotopic special vector bundle automorphism of \( V \). Then there exist unipotent vector bundle automorphisms \( E_1, \ldots, E_N \) such that
\[
S = E_N \circ \cdots \circ E_1.
\]
Moreover, for all \( j \) we have that \((E_j - \text{id})^2 = 0\) and \( \text{rank}(E_j - \text{id}) \in \{0,1\} \).

**Remark 1.** Note that for a vector space \( V \) and for \( E \in \text{Iso}(V) \), the conditions that \((E - \text{id})^2 = 0\) and \( \text{rank}(E - \text{id}) \in \{0,1\} \) are equivalent to the existence of \( \alpha \in V^* \)
\[
E(u) = u + \alpha(u) \cdot v
\]
with \( \alpha(v) = 0 \). We will call isomorphisms on the form (1) \textit{elementary}.

**Remark 2.** We make a short remark about the assumption in Theorem 1 that \( k \geq 3 \) in the case of a real vector bundle. The fact that sets the case \( k = 2 \) apart is that the fundamental group of \( \text{SO}_2(\mathbb{R}) \) is infinite. In \cite{Was88}, Wasserstein provides an example of a \( \text{SL}_2(\mathbb{R}) \)-valued function on \( \mathbb{R} \) which does not factor into a product of maps into the space of real elementary matrices.

If \( V = \mathbb{R}^k \) (resp. \( \mathbb{C}^k \)) it is customary to require an elementary isomorphism to satisfy the additional requirements that \( v = e_j \) for some \( j \), and \( \alpha(v) = \lambda \cdot \langle v, e_i \rangle \) for some \( i \neq j, \lambda \in \mathbb{R} \) (resp. \( \lambda \in \mathbb{C} \)), where \( \{e_1, \ldots, e_k\} \) denotes the set of standard basis vectors in \( \mathbb{R}^k \). Assuming the necessary additional structure on \( V \), we may improve Theorem 1 to obtain an analogous stronger form of decomposition by elementary matrices.
Theorem 2. Suppose in addition to the data in Theorem 1 that $V$ splits globally into line bundles $L = \{L_1, \ldots, L_k\}$. Then, in addition to the conclusions in Theorem 1 we may achieve that for any point $x \in X$ and for any frame $\{e_1(x), \ldots, e_k(x)\}$ for $V_x$ such that $e_s \in L_s$ for all $s \in \{1, \ldots, k\}$, that $E_j(x)$ is of the form

$$E_j(x)(u) = u + \alpha_j(u) \cdot e_s(x),$$

where $\alpha_j(e_s) \neq 0$ if and only if $s = s_0$ for some fixed $s_0 \neq s_j$.

Remark 3. We will call a vector bundle automorphism satisfying (2) at any point $L$-elementary.

Remark 4. For any vector bundle $V \to X$ we may consider the flag bundle $Y = \text{Fl}(V)$ associated to $V$, with a map $p : Y \to X$ such that $p^*V$ splits into a family $L$ of line bundles (see e.g. Hatcher, [H2003]). Then, by Theorem 2 $p^*S$ factors into a composition of finitely many $L$-elementary vector bundle automorphisms.

1.1. The curvature tensor of a Riemannian manifold. We will now present an interesting special case of Theorem 1. Assume that $X$ is a smooth manifold equipped with a Riemannian metric, let $\pi : TX \to X$ be the tangent bundle of $X$, $\nabla$ its Levi-Civita connection and $R$ its Riemann curvature tensor. A vector bundle endomorphism of $TX$ is a smooth map $V \to V$ such that $\pi \circ S = \pi$ and $S$ restricts to a linear map on each fiber $\pi^{-1}(x), x \in X$. Fixing two smooth vector fields $U, V$ on $X$, we get the Riemann curvature endomorphism of $TX$ given by

$$R(U, V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}.$$

The restriction of this map to any fiber in $TX$ defines a skew-symmetric map with respect to the Riemannian metric. It follows that applying the exponential map to this, we get a vector bundle automorphism of $TX$ which preserves the Riemannian metric, i.e. it can be locally represented by an orthogonal matrix. In particular, it has determinant one. Applying Theorem 1 to this gives the following:

Corollary 5. Let $X$ be a Riemannian manifold, $R(\cdot, \cdot)$ its Riemann curvature tensor, $\nabla$ its Levi-Civita connection, and let $U, V$ be two smooth vector fields on $X$. Then the vector bundle automorphism of $TX$ given by

$$\exp(R(U, V)) = \exp(\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}$$

admits a factorisation in terms of unipotent vector bundle automorphisms.

1.2. Real and complex symplectic vector bundles. In [KL19], Kutzschebauch, Ivarsson, and Løw studies factorization properties of symplectic matrices over various rings of complex valued functions. We will now present two results in this direction, one for complex vector bundles and one for real vector bundles. Recall that a symplectic form on a complex or real vector bundle $V$ is a non-degenerate section $\omega$ of $\Lambda^2 V^*$, i.e., for each point $x \in X$ and any $u \in V_x$ we have that $\omega_x(u, \cdot)$ is not identically zero. We will say that a vector bundle automorphism of $V$ is symplectic if it preserves $\omega$.

We will start with the complex case. Recall that the compact symplectic group is the group of complex $2k \times 2k$-matrices given by

$$\text{sp}_k = \text{Sp}_{2k}(\mathbb{C}) \cap U_{2k}$$

where $\text{Sp}_{2k}(\mathbb{C})$ is the group of symplectic complex $2k \times 2k$-matrices, i.e., maps from $\mathbb{C}^{2k}$ to $\mathbb{C}^{2k}$ preserving the standard complex symplectic form, and $U_{2k}$ is the
group of unitary $2k \times 2k$-matrices. An $sp_k$-bundle is a complex vector bundle of rank $2k$ together with a special family of trivialisations such that the transition functions between any two trivialisations in this family lie in $sp_k$. Note that an $sp_k$-bundle admits a natural complex symplectic form, namely the one given by the standard complex symplectic form in any such trivialisation. We will say that a vector bundle automorphism of an $sp_k$-bundle is symplectic if it preserves this symplectic form or, equivalently, if it is represented by a complex symplectic matrix in each trivialisation.

If in addition, the transition matrices are of the special form

$$
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
$$

then there is a natural splitting $V = L_1 \oplus L_2$ of $V$ by two Lagrangian subspaces, and in this case we will refer to $V$ as a Lagrangian $sp_k$-bundle. (Note that in this case we necessarily have that $B = (A^T)^{-1}$.)

**Theorem 3.** Let $X$ be a smooth manifold and let $\pi : V \to X$ be an $sp_k$-bundle over $X$. Let $S$ be a nullhomotopic symplectic vector bundle automorphism of $V$. Then there exist unipotent symplectic vector bundle automorphisms $E_1, \ldots, E_N$ such that

$$
S = E_N \circ \cdots \circ E_1.
$$

Moreover, for all $j$ we have that $(E_j - \text{id})^2 = 0$ and rank$(E_j - \text{id}) \in \{0, 1, 2\}$ and if $V$ is Lagrangian, we may achieve that the $E_j$’s respect the Lagrangian splitting (see Remark 7 below).

**Remark 6.** In fact, we can achieve that each $E_j$ in Theorem 3 is of the form

$$
E_j(u) = u + \omega(u, v) \cdot w + \omega(u, w) \cdot v
$$

for some $v, w \in V$ such that $\omega(v, w) = 0$.

**Remark 7.** By saying that each $E_j$ respects the Lagrangian splitting, we mean that when expressing each $E_j$ as in (4) we have that either $v, w \in L_1$ or $v, w \in L_2$.

**Remark 8.** The fact that $V$ is an $sp_k$-bundle gives us both a symplectic form and a compatible Hermitian metric. This is crucial in our proof of Theorem 3. It would be interesting to see to what extent Theorem 3 can be extended to the more general setting of $Sp_{2k}(\mathbb{C})$-bundles, where we have a symplectic form but possibly no compatible Hermitian metric.

We now turn to the case of real symplectic vector bundles. We will denote the group of real symplectic $2k \times 2k$-matrices by $Sp_{2k}(\mathbb{R})$.

**Theorem 4.** Let $X$ be a compact smooth manifold and let $\pi : V \to X$ be an $Sp_{2k}(\mathbb{R})$-bundle over $X$. Let $S$ be a nullhomotopic symplectic vector bundle automorphism of $V$. Then there exist unipotent symplectic vector bundle automorphisms $E_1, \ldots, E_N$ such that

$$
S = E_N \circ \cdots \circ E_1.
$$

Moreover, for all $j$ we have that $(E_j - \text{id})^2 = 0$ and rank$(E_j - \text{id}) \in \{0, 1, 2\}$ and if $V$ is Lagrangian, we may achieve that the $E_j$’s respect the Lagrangian splitting.
Remark 9. We make a brief remark about the compactness assumption in Theorem 4. The reason our methods fail in the non-compact case is that the first fundamental group of $U_k = Sp_{2k}(\mathbb{R}) \cap O_{2k}$ is infinite, preventing us from applying Theorem 5. It might be tempting to look for a generalisation of Theorem 4 to the non-compact setting. However, we do not believe the statement of the theorem is true in the non-compact setting. In particular (generalising the construction in [Was88]), we expect that the following null-homotopic maps from $\mathbb{R}$ to $Sp_{2k}(\mathbb{R})$ for any positive $k$ provide examples which can’t be written as a product of unipotent factors:

$$t \mapsto \begin{bmatrix} A(t) & 0 \\ 0 & A(t) \end{bmatrix}$$

where $A(t)$ is the orthogonal matrix

$$\begin{bmatrix} \cos(t) & -\sin(t) & 0 & \ldots & 0 \\ \sin(t) & \cos(t) & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

1.3. Compact Kähler manifolds. Let $X$ be a compact Kähler manifold. Then, since the holonomy of a Kähler manifold is in the symplectic group, we get that $\exp(R(U, V))$ is a symplectic vector bundle automorphism of $TX$ (see Section 4 for details). Applying Theorem 4 to this gives the following corollary:

**Corollary 10.** Let $X$ be a compact Kähler manifold, $R(\cdot, \cdot)$ be its Riemann tensor, $\nabla$ its Levi-Civita connection and $U, V$ be two smooth real vector fields on $X$. Then the symplectic vector bundle automorphism of $TX$ given by

$$\exp(R(U, V)) = \exp(\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}$$

admits a factorisation in terms of unipotent symplectic vector bundle automorphisms.

1.4. Gromov’s Wasserstein problem. Given a holomorphic map $A$ from a Stein manifold $X$ to $SL_k(\mathbb{C})$ it is natural to ask if $A$ can be written as a product of maps into the set of elementary matrices, or (closely related) if $A$ can be written as a product of maps into unipotent subgroups of $SL_k(\mathbb{C})$. This question, sometimes referred to as Gromov’s Wasserstein problem, was suggested by Gromov in [Gro89] as a possible application of his h-principle introduced in the same paper. In 2012 the question was settled by Ivarsson and Kutzschebauch [IK12]. They proved that any holomorphic null-homotopic $SL_k(\mathbb{C})$-valued map on a finite dimensional reduced Stein space can be factorized into a product of holomorphic maps into unipotent subgroups of $SL_k(\mathbb{C})$. The proof was based on Wasserstein’s result in [Was88] together with a refinement of Gromov’s h-principle due to Forstnerič [For10].

We would like to propose the following generalization of Gromov’s Wasserstein problem:

**Problem 11.** Let $X$ be a Stein manifold and $V$ a holomorphic vector bundle over $X$. Assume $S$ is a null-homotopic holomorphic vector bundle automorphism of $V$ of determinant 1. Can $S$ be factored into a composition of unipotent holomorphic vector bundle automorphisms of $V$?
One motivation for the present paper is the possibility that Theorem 1 above together with a similar application of the h-principle as in [IK12] can provide a solution to this problem.

1.5. Overview of proofs. In most of the paper we will consider a vector bundle automorphism $S$ as a section of the endomorphism bundle $V \otimes V^*$. As a first step we will introduce an auxiliary metric on $V$; an inner product in the case when $V$ is a real vector bundle and a Hermitian form in the case when $V$ is a complex vector bundle. This defines a fiberbundle over $X$ consisting of the points in $V \otimes V^*$ that represent orthogonal maps in the case of a real vector bundle and unitary maps in the case of a complex vector bundle. We will then apply a Gram-Schmidt like process to reduce the factorization problem to the case when $S$ takes values in this fiber bundle (Proposition 18). In particular, this means that $S$ can be considered as a section of a fiber bundle with compact fibers. The next ingredient in the proof is a general theorem on homotopies of sections of fiber bundles with compact fibres proved in Section 2 (Theorem 5). While we assume that $S$ is homotopic to the identity, this theorem allows us to conclude that there is a uniform homotopy between $S$ and the identity. Using a simple argument we can then reduce the problem to the case when $S$ is close to the identity (see the first paragraph of the proof of Theorem 1 and Theorem 2 in Section 4). Theorem 1 then follows by applying a Gauss-Jordan type process (Proposition 19).

Theorem 2 will follow by observing that all of the above can be done while choosing local frames that respect the splitting $L$. Theorem 3 and Theorem 4 will follow by a similar argument as in the case of Theorem 1, using the symplectic Gram-Schmidt and Gauss-Jordan processes defined in [IKL19]. However, the compactness assumption in Theorem 4 ensures existence of a uniform homotopy between $S$ and the identity without using Theorem 5.

The main obstacle when using the Gram-Schmidt process and the Gauss-Jordan process is that they both have to be applied locally and they are only well-defined after fixing a local frame of $V$. We ensure that the resulting factorisation is globally well-defined by first introducing a triangulation of the manifold. We then apply Gram-Schmidt and Gauss-Jordan in an inductive manner, first applying them at the vertices of the triangulation, then applying them along the edges of the triangulation and proceeding with higher dimensional faces of the triangulation in the order of their dimension. These induction steps are given by Lemma 20 for the Gram-Schmidt process and Lemma 21 for the Gauss-Jordan process.

Theorem 5 generalises a theorem by Calder and Siegel [CS78, CS80]. The main technical ingredient in the proof is a type of deformations of maps into CW complexes constructed in Proposition 17.

1.6. Organisation of the paper. The paper is organised as follows: Section 2 is devoted to the proof of Theorem 5. The main technical ingredient in the proof, Proposition 17 is formulated and proved in Section 3. Section 4 is devoted to the proof of Theorem 1 - Theorem 4 including the induction steps of Lemma 20 and Lemma 21. Corollary 5 and Corollary 10 are proved at the end of Section 4.

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2. Homotopies and uniform homotopies

In this section we will consider general fiber bundles with compact structure groups over a topological space $X$. We will define a notion of uniform homotopies of sections of such bundles. Moreover, assuming that $X$ is a finite dimensional smooth manifold and that the fibers have finite first fundamental group and the homotopy type of a CW-complex of finite type, we will prove that any two homotopic sections can be joined by a uniform homotopy (see Theorem 5). This generalizes a theorem by Calder and Siegel. [CS78, CS80]

2.1. Uniform homotopies of sections. Let $\pi : W \to X$ be a fiber bundle over a topological space $X$ with fiber $Y$ and compact structure group $G$. A homotopy between two sections $s_0$ and $s_1$ of $W$ is a continuous map $h : X \times I \to W$ such that $H(\cdot, 0) = s_0$, $H(\cdot, 1) = s_1$ and $H(\cdot, t)$ is a section for each $t \in [0, 1]$, i.e., $\pi(h_t(x)) = x$ for all $(x, t) \in X \times I$. We will let $C(I : W)$ denote the space of maps $f : I \to W$ such that $\pi \circ f$ is constant, i.e. such that there is $x_0 \in X$ with $\pi(f(t)) = x_0$ for all $t \in I$. Then a homotopy $H$ of sections of $W$ induces a map $h : X \to C(I : W)$, given by $h(x) = H(x, \cdot)$. In the rest of this section we will use this map, rather than $H$, to refer to a homotopy.

If $(U, \rho)$ is a local trivialization of $W$, then composition with $\rho$ induces an invertible map $\tilde{\rho} : C(I : W|_U) \to C(I : (U \times Y))$. Moreover, letting $\pi_2 : U \times Y \to Y$ be projection on the second factor we get that composition with $\pi_2$ induces a map $\tau : C(I : (U \times Y)) \to C(I, Y)$. Given a homotopy $h : X \to C(I : W)$ and letting $h_U = \tau \circ \tilde{\rho} \circ h|_U$ we get the following commutative diagram

$$
\begin{array}{ccc}
C(I : W|_U) & \xrightarrow{\tilde{\rho}} & C(I : (U \times Y)) \\
& h|_U \downarrow & \tau \downarrow \\
U & \xrightarrow{h_U} & C(I, Y)
\end{array}
$$

The map $h_U$ can be thought of as a local representative of $h$ in the trivialization $(U, \rho)$. Note the structure group $G$ of $W$ acts on $C(I, Y)$ by composition. Given $x \in U$ and $t \in I$ we have that $h(x)(t) \in W$ and $h_U(x)(t) \in Y$ and $h|_U$ can be recovered from $h_U$ by

$$
h(x)(t) = (\tau \circ \tilde{\rho})^{-1}(x, h_U(x)(t)).$$

Moreover, let $(U', \rho')$ be another local trivialization of $W$ and assume $x \in U \cap U'$. Then $\pi_2 \circ \rho' \circ h|_U(x) = g \circ \pi_2 \circ \rho \circ h|_U(x)$ for some element $g$ in the structure group $G$. It follows that

$$
h_{U'}(x) = g \circ h_U(x).
$$

In particular, fixing a $G$ invariant set $K \subset C(I, Y)$, we get that $h_U(x) \in K$ if and only if $h_{U'}(x) \in K$.

**Definition 12.** Let $\pi : W \to X$ be a fiber bundle over a topological space $X$ with fiber $Y$ and compact structure group $G$. Then a homotopy of sections of $W$
is a uniform homotopy if there is a $G$-invariant compact set $K \subset C(I, Y)$ such that for any $x \in X$ and some (hence any) local trivialization $(\rho, U)$ of $W$ over a neighbourhood $U$ of $x$ we have $h_U(x) \in K$.

**Lemma 13.** Let $h$ be a uniform homotopy of a fiber bundle $W$ as above. Assume in addition that the fiber $Y$ is a metric space with distance function $d : Y \times Y \to \mathbb{R}_{\geq 0}$. Then there is, for any $\epsilon > 0$, a finite set of real numbers $0 = t_1 < t_2 < \ldots < t_N = 1$ such that for any $x \in X$ and any trivialization over a neighbourhood $U$ of $x$, we have $d(h_U(x)(t), h_U(x)(t')) < \epsilon$ whenever $t, t' \in [t_i, t_{i+1}]$ for some $i$.

**Proof.** By definition, there is a compact $G$-invariant set $K \subset C(I, Y)$ such that for any $x \in X$ and any local trivialization $(U, \rho)$ of $W$ over a neighbourhood $U$ of $x$, $h_U(x) \in K$. By the Arcela-Ascoli Theorem $K \subset C(I, Y)$ is equicontinuous. This proves the lemma. □

2.2. **Existence of uniform homotopies.** From now on we will assume that $X$ is a smooth manifold of dimension $d < \infty$ and that $Y$ has the homotopy type of a CW-complex of finite type. The main theorem of this section is the following theorem:

**Theorem 5.** Let $X$ be a smooth manifold of finite dimension and $W \to X$ be a fiber bundle over $X$ with compact structure group and fiber $Y$. Assume also that $Y$ has the homotopy type of a CW-complex of finite type and that $\pi_1(Y)$ is finite. Furthermore, let $h : X \times I \to W$ be a homotopy of sections. Then there is a uniform homotopy of sections $\tilde{h} : X \times I \to W$ such that $\tilde{h}(\cdot, 0) = h(\cdot, 0)$ and $\tilde{h}(\cdot, 1) = h(\cdot, 1)$.

Before we prove Theorem 5 we note that by Lemma 13 we have the following corollary:

**Corollary 14.** Assume in addition that the fiber $Y$ of $W$ is a metric space with distance function $d : Y \times Y \to \mathbb{R}_{\geq 0}$. Then there is a homotopy $\tilde{h}$ between $h_0$ and $h_1$, with the property that for any $\epsilon > 0$ there is a finite set of real numbers $0 = t_1 < t_2 < \ldots < t_N = 1$ such that for any $x \in X$ and any trivialization over a neighbourhood $U$ of $x$, we have $d(h_U(x)(t), h_U(x)(t')) < \epsilon$ whenever $t, t' \in [t_i, t_{i+1}]$ for some $i$.

The main technical ingredient in the proof of Theorem 5 is a result on homotopies of maps into CW-complexes proved in Section 3. We will summarize the consequences of this needed to prove Theorem 5 in Lemma 15 below. To state Lemma 15 we will first introduce some notation.

Let $p : C(I, Y) \to Y \times Y$ be the map given by $p(f) = (f(0), f(1))$. Given a homotopy $h : X \to C(I : W)$ and a local trivialization $(U, \rho)$ of $W$, we have the following commutative diagram:

$$
\begin{array}{ccc}
C(I, Y) & \xrightarrow{h_U} & U \\
\downarrow{p} & & \downarrow{p \circ h_U} \\
Y \times Y & \xrightarrow{p \circ h_U} & Y \times Y
\end{array}
$$
Note that if \( h \) is a homotopy between two sections \( s_0, s_1 \) of \( W \), then \( h' : X \to C(I, W) \) is also a homotopy between \( s_0 \) and \( s_1 \) if and only if

\[
p \circ h_U = p \circ h_U'
\]

for any local trivialization \( (U, \rho) \) of \( W \).

We will now fix a point in \( Y \) and let \( \Omega Y \) denote the pointed loop space of \( Y \). By standard theory, there is an open covering \( \{ B_i \}_{i=1}^\infty \) of \( Y \times Y \) such that for each \( i \), the fibration \( p|_{p^{-1}(B_i)} : p^{-1}(\overline{B_i}) \to \overline{B_i} \) is homotopy equivalent to the trivial fibration \( \pi_1 : \overline{B_i} \times \Omega Y \to \overline{B_i} \).

**Lemma 15.** Let \( U \subset X \) and \( p : C(I, Y) \to Y \times Y \) be the fibration above. Assume \( h_U : U \to C(I, Y) \) is a continuous map such that the closure of \( B = p \circ h(U) \subset Y \times Y \) is compact and the fibration \( p \) is trivial over \( \overline{B} \). Let \( A \subset U \) and assume \( h_U(A) \) is contained in a compact set \( K \subset C(I, Y) \). Then there is a compact set \( K' \subset C(I, Y) \) depending only on \( d, K \) and \( B \), and a fiber homotopy \( H : U \times I \to C(I, Y) \) such that \( H(\cdot, 0) = h_U \) and

\[
H(U \times \{1\}) \cup H(A \times I) \subset K'.
\]

**Remark 16.** Lemma [13] above is closely related to Lemma 2 in [CS80], stating that a CW-complex of finite type satisfies the relative compressibility property. However, Lemma [13] above is stronger in that the set \( K' \) only depends on \( d, K \) and \( B \). In other words, given \( d, K \) and \( B \) we have that \( K' \) is independent of \( h_U \) and \( U \). This will be crucial when applying it to prove Theorem [5].

**Proof.** By assumption, the fibration \( p : p^{-1}(\overline{B}) \to \overline{B} \) is homotopy equivalent to the trivial fibration \( \pi_1 : \overline{B} \times \Omega Y \to \overline{B} \).

Let \( \pi_2 : \overline{B} \times \Omega Y \to \Omega Y \) be the projection onto the second factor. Moreover, let

\[
\phi : p^{-1}(\overline{B}) \to \overline{B} \times \Omega Y, \quad \psi : \overline{B} \times \Omega Y \to p^{-1}(\overline{B})
\]

be a homotopy equivalence and \( F : p^{-1}(\overline{B}) \times I \to p^{-1}(\overline{B}) \) be a homotopy between the identity map on \( p^{-1}(\overline{B}) \) and \( \psi \circ \phi \). By [Wal65], \( \Omega Y \) is homotopic to a CW-complex of finite type. Fix such a homotopy equivalence and for positive integers \( m \), let \( (\Omega Y)^m \) be the \( m \)-skeleton of \( \Omega Y \) induced by this homotopy equivalence. By compactness of \( \pi_2 \circ \phi(K) \subset \Omega Y \), we get that \( \pi_2 \circ \phi(K) \) is contained in \( (\Omega Y)^{mK} \) for some \( mK \). Applying Proposition [17] in Section [3] we get a homotopy \( \Theta : U \times I \to \Omega Y \) such that \( \Theta(\cdot, 0) = \pi_2 \circ \phi \circ h_U \) and the following inclusions hold:

\[
\Theta(U \times \{1\}) \subset (\Omega Y)^d, \\
\Theta(A \times I) \subset (\Omega Y)^{mK}.
\]

We will let \( \gamma \) denote the map \( p \circ h_U \). Let \( H \) be given by

\[
H(x, t) = \begin{cases} 
F(h_U(x), 2t), & t \in [0, 1/2] \\
\psi(\gamma(x), \Theta(x, 2t - 1)), & t \in (1/2, 1].
\end{cases}
\]

This means \( H(\cdot, 0) = F(\cdot, 0) \circ h_U = h_U \). Continuity of \( H \) follows from

\[
F(h(x), 1) = \psi \circ \phi \circ h_U(x) = \psi(\gamma(x), \Theta(x, 0)).
\]

Moreover,

\[
H(A, I) \subset F(h_U(A) \times I) \cup \psi(\gamma(A) \times \Theta(A, I)) \\
\subset F(K \times I) \cup \psi(\overline{B} \times (\Omega Y)^{mK})
\]
and

\[ H(U \times \{1\}) \subset \psi(\gamma(U) \times \Theta(U, 1)) \subset \psi(\overline{\mathcal{B}} \times (\Omega Y)^d) \]

which are both compact since the CW structure of \( \Omega Y \) is of finite type. Note that \( F \) and \( \psi \) are determined by \( B \). Letting

\[ K' = F(K \times I) \cup \psi(\overline{\mathcal{B}} \times (\Omega Y)^{\max(d, m, k)}) \]

proves the lemma. \( \square \)

We are now ready to prove Theorem 5.

**Proof of Theorem 5**. Let \( \{B_i\}_{i=1}^m \) be a finite open covering of \( Y \times Y \) such that for each \( i \) we have that \( p^{-1}(\overline{\mathcal{B}}_i) \) is homotopy equivalent to \( \overline{\mathcal{B}}_i \times \Omega Y \).

Moreover, let \( \{U_j\}_{j=1}^\infty \) be an open covering of \( X \) such that the following holds

1. For each \( j \), there is a trivialization of \( W \) over \( U_j \) such that \( p \circ h_{U_j}(U_j) \subset B_{i_j} \) for some \( i_j \in \{1, \ldots, m\} \).
2. There is a family of \( d+1 \) index sets \( I_1, \ldots, I_{d+1} \subset \mathbb{N} \) such that if \( i, j \in I_k \) for some \( k \), then \( U_i \) and \( U_j \) are disjoint, and \( \bigcup_{k=1}^{d+1} I_k = \mathbb{N} \).

The first point above is easily obtained by picking a family of trivializations \( \{(U_j, \rho_j)\}_j \) of \( W \) such that \( \{U_j\}_j \) is a covering of \( X \) and considering the refinement of \( \{U_j\}_j \) given by \( \{U_j \cap (p \circ h_{U_j})^{-1}(B_i)\}_{i,j} \). By possibly passing to a refinement of this second covering, the second point can be obtained by applying Ostrand’s theorem on coloured dimension (see Lemma 3 in [Ost71]).

Finally, let \( \{V_j\}_{j=1}^\infty \) be an open covering of \( X \) such that \( V_j \subset U_j \) for each \( j \), and for each \( k \in \{1, \ldots, d+1\} \) define

\[ U^k = \bigcup_{i \in I_1 \cup \ldots \cup I_k} U_i, \quad V^k = \bigcup_{i \in I_1 \cup \ldots \cup I_k} V_i. \]

Set \( h^0 := h \) and \( K_0 = V^0 = U^0 = \emptyset \). We will proceed by induction. Assume that \( h^{k-1} \) is a homotopy matching \( h^0 \) at \( \{0, 1\} \), and that \( K_{k-1} \subset C(I, Y) \) is a \( G \)-invariant compact set such that for any \( x \in V^{k-1} \) and any trivialization of \( W \) in a neighbourhood \( U \) of \( x \), we have that

\[ h^{k-1}_{U_i}(x) \in K_{k-1}. \]

We will now find a homotopy \( h^k \) matching \( h^{k-1} \) at \( \{0, 1\} \) and a \( G \)-invariant compact set \( K_k \subset C(I, Y) \) such that for any \( x \in V^k \) and any trivialization of \( W \) in a neighbourhood \( U \) of \( x \), we have that \( h^k_{U_i}(x) \in K_k \).

By our choice of covering \( \{U_j\}_j \), we have that \( W \) is trivial over each \( U_j \) and \( p \circ h_{U_j}(U_j) \subset B_{i_j} \) for some \( i_j \in \{1, \ldots, m\} \). In other words, \( p : C(I, Y) \to Y \times Y \) is trivial over \( p \circ h_{U_j}(U_j) \).

For each \( j \in I_k \) we let

\[ H_j : U_j \times I \to C(I : Y) \]

be the fiber homotopy given by applying Lemma 15 to the map

\[ h^{k-1}_{U_j} : U_j \to C(I : Y) \]

and the set \( A_k = U_j \cap V^{k-1} \).

We get that

\[ H_j(A_k \times I) \cup H_j(U_j \times \{1\}) \]
is contained in a compact set $K^i_k \subset C(I, Y)$ which is determined by $d$, $K_{k-1}$ and $B_i$ (note that $i \notin \{1, \ldots, m\}$).

Let $\eta_k : X \to I$ be a continuous map such that

$$\eta_k^{-1}(1) = \bigcup_{j \in I_k} V_j \quad \eta_k^{-1}(0) = X \setminus \bigcup_{j \in I_k} U_j.$$ 

For any $x \in X \setminus \bigcup_{j \in I_k} U_j$ we put $h^k(x) := h^{k-1}(x)$. As $U_1$ and $U_j$ are disjoint for any distinct $l, j \in I_k$ we get for any $x \in \bigcup_{j \in I_k} U_j$, a unique $j \in I_k$ such that $x \in U_j$.

We define $h^k(x)$ by

$$h^k_{I,j}(x) = H(x, \eta_k(x))$$

It follows that for any $x \in U_j \cap V^{k-1}$, we have that $h^k_{I,j}(x)$ is contained in $H(U_j \cap V^{k-1} \times I)$ and for any $x \in V_j$, we have that $h^k_{I,j}(x)$ is contained in $H(V_j \times \{1\})$ both of which are contained in $K^i_k$. Performing this procedure for all $j \in I_k$ we get that for any $x \in V^k$ and any trivialization $(U, \rho)$ of $W$ in a neighbourhood $U$ of $x$, we have that $h^k_{I,j}(x)$ is in the compact set $K_k := \bigcup_{i=1}^m G K^i_k$,

where $G K^i_k$ denotes the $G$-orbit of $K^i_k \subset C(I, Y)$ which is compact by compactness of $G$. The theorem then follows by induction on $k \in \{1, \ldots, d + 1\}$. In particular, $h^{d+1}$ is the desired uniform homotopy.

\[ \square \]

3. Deformation of maps into CW Complexes

This goal of this section is to state and prove Proposition 17, which is used in the previous section to prove Lemma 15. We begin by recalling the definition of a CW complex.

(1) Start with a discrete set $Y^0$, the zero cells of $Y$.
(2) Inductively, form the $n$-skeleton $Y^n$ from $Y^{n-1}$ by attaching $n$-cells $e^n_\alpha$ via maps $\varphi_\alpha : S^{n-1} \to Y^{n-1}$. This means that $Y^n$ is the quotient space of $X^{n-1} \cup_{\varphi} D^n_\alpha$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. The cell $e^n_\alpha$ is the homeomorphic image of $D^n_\alpha \setminus \partial D^n_\alpha$ under the quotient map.
(3) $Y = \bigcup_{i} Y^n$ is equipped with the weak topology: A set $A \subset Y$ is open (or closed) if and only if $A \cap Y^n$ is open (or closed) in $Y^n$ for each $n$.

The goal of this section is to prove the following proposition. The proposition is used in the proof of Lemma 15 in the previous section and it is closely related to the construction on page 354 of [Dol72].

**Proposition 17.** Let $Y$ be a CW complex, let $X$ be a smooth (connected) manifold of dimension $d$, and let $f : X \to Y$ be a continuous map. Then there exists a homotopy $\Theta : X \times [0, 1] \to Y$ such that the following holds.

(1) $\Theta(x, 0) = f(x)$ for all $x \in X$,
(2) If $f(x) \in e^n_\alpha$ then $\Theta(x, t) \in e^n_\alpha \cup Y^{m-1} \subset Y^m$ for all $t \in [0, 1]$,
(3) $\Theta(x, 1) \in Y^d$ for all $x \in X$.
Proof. We give the proof under the assumption that $X$ is non-compact (the compact case is simpler). Let \{${K}_j$$_{j\in\mathbb{N}}$\} be an exhaustion of $X$ by compact sets. We will construct $\Theta$ inductively, and the following will be our induction hypothesis $I_n$:

Assume that we have constructed a homotopy

$$\Theta_n : X \times [0, 1 - 1/(n + 1)] \to Y$$

such that

1. $\Theta_{n,0} = f$,
2. If $f(x) \in e_{\alpha}^m$ then $\Theta_n(x,t) \in e_{\alpha}^m \cup Y^{m-1}$ for all $t \in [0, 1 - 1/(n + 1)]$,
3. $\Theta_n(t) = f$ on $f^{-1}(Y^d)$ for all $t \in [0, 1 - 1/(n + 1)]$,
4. $\Theta_{n,1-1/(n+1)}(K_n) \subset Y^d$.

We start by constructing $\Theta_1$. Before we start, note that if $X$ is compact then $K_1 = X$. Hence, in the compact case it sufficed to construct $\Theta_1$ to prove the theorem. Now, if $f(K_1) \subset Y^d$ for all $x \in K_1$ we simply set $\Theta_1(x,t) = x$ for all $x \in X$ and all $t$. So from now on we assume that $f(K_1)$ is not contained in the $d$-skeleton of $Y$. Since $K_1$ is compact we have that $f(K_1)$ is contained in a minimal $s_1$-skeleton of $Y$, and $f(K_1)$ intersects only finitely many $s_1$-cells $\{e_{\alpha}^s\}_{j=1}$. We start by constructing for each $j$ a homotopy supported on $e_{\alpha}^s$.

Choose non-empty open sets $V_{1,j} \subset V_2 \subset e_{\alpha}^s$, and let $\chi_j \in C^1(e_{\alpha}^s)$ (identifying $e_{\alpha}^s$ with $\text{int}(D_{\alpha}^s)$) with $\chi_j \equiv 1$ near $\partial V_1$, and with $\text{Supp}(\chi_j) \subset V_2$. Our first step is to deform $f$ to a smooth map on a neighbourhood of $f^{-1}(\partial V_1)$. To this end, we may approximate $f$ on $f^{-1}(V_2)$ to arbitrary precision by a $C^1$-smooth map $g_j : f^{-1}(V_2) \to e_{\alpha}^s$, and if the approximation is good enough we may define $\tilde{g}_j := (\chi_j \circ f) \cdot g_j + (1 - \chi_j \circ f) \cdot f$, and obtain another map $\tilde{g}_j : f^{-1}(V_2) \to e_{\alpha}^s$. Note that $\tilde{g}_j$ is smooth on $f^{-1}(\text{int}\{\chi_j = 1\})$ and that $\tilde{g}_j = f$ outside $f^{-1}(\text{Supp}(\chi_j)) \subset f^{-1}(V_2)$. And, assuming that the approximation of $f$ by $g_j$ was close enough, we may assume that $\tilde{g}_j$ is smooth on $\tilde{g}_j^{-1}(V_2)$. Then we may define an homotopy of maps $\varphi^s_1 : f^{-1}(V_2) \to e_{\alpha}^s$, $t \in [0, 1]$, by

$$\varphi^s_1(x) = \varphi^s(x,t) := f(x) + t \cdot (\tilde{g}_j(x) - f(x))$$

for $t \in [0, 1]$, and extend the homotopy to be identically equal to $f$ on $X \setminus f^{-1}(V_2)$.

Now $\varphi^s_1(\cdot)$ is smooth on $(\varphi^s_1)^{-1}(V_1)$ and so it follows from Sard’s Theorem that $\varphi^s_1(X) \cap V_1$ has measure zero since $\dim X = d < s_1$. In particular, there exists a point $a \in e_{\alpha}^s$ with $a \notin \varphi^s_1(X)$, and without loss of generality we may assume that $a = 0$. We now define a second homotopy $\psi^s_1 : e_{\alpha}^s \to e_{\alpha}^s$ by setting

$$\psi^s_1(x) := \frac{x}{\|x\|}(\|x\| + t(1 - \|x\|)),$$

and then we extend $\psi^s_1$ to the rest of $X$ by setting $\psi^s_1(x) = x$ for all $t$ for all $x \in X \setminus e_{\alpha}^s$.

We now define a third homotopy by concatenating $\varphi^s_1$ and $\psi^s_1 \circ \varphi^s_1$, i.e., define $\gamma^s_{1,t} : X \to Y$ by setting $\gamma^s_{1,t}(x) := \varphi^s_2(x)$ for $0 \leq t \leq 1/2$, and $\gamma^s_{1,t}(x) := \psi^s_2 \circ \varphi^s_1(x)$ for $1/2 \leq t \leq 1$. We have thus obtained an homotopy $\gamma^s_1 : X \times [0, 1] \to Y$ with the following properties

(a) $\gamma^s_{1,0} = f$. 

\[ \Box \]
of isomorphisms of $\gamma$ are determined by maps $\psi$ in an inductive construction on above for 4.1.

Vector bundle automorphisms as sections of fiber bundles.

Let $W_{GL}$ be the fiber bundle over $X$ consisting of the points in the endomorphism bundle $V \otimes V^*$ that represent invertible maps on the fibers of $V$. Alternatively, $W_{GL}$ can be constructed in the following manner:

There is an open cover $\{U_j\}_{j \in \mathbb{N}}$ of $X$ and trivialisations $\psi_j : V_j := \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$ (resp. $\mathbb{C}^k$), and the transition maps $\psi_{ij}$ from $V_j$ to $V_i$ over $U_{ij} := U_i \cap U_j$ are determined by maps $\psi_{ij} : U_{ij} \to G$.

Then $W_{GL}$ is the fiber bundle whose fiber $W_{x, GL}$ over a point $x \in X$ is the group of isomorphisms of $V_x$. The transition map from $(W_{GL})^1$ to $(W_{GL})^j$ over $U_{ij}$ is given by $B_{ji}(x)(A_i(x)) = \psi_{ji}(x) \circ A_i(x) \circ \psi_{ij}(x)$.

We also get the following fiber bundles over $X$:

1. $W_{SL}$ denoting the fiber bundle over $X$ consisting of the elements $W_{GL}$ that has determinant one.

2. $W_n$ denoting the fiber bundle over $X$ consisting of the unipotent elements $W_{GL}$.

If $U \subset X$ is an open set and $W$ any of the fiber bundles above we denote the space of continuous sections of $W|_U$ by $\Gamma(U, W)$.. Note that a vector bundle automorphism $S$ of $V$ defines a section in $W_{GL}$. In the rest of the paper we will identify $S$ with this section and hence consider $S$ an element in $\Gamma(X, W_{GL})$.

4.2. Auxiliary metrics and local frames for Theorem 1 and Theorem 2

In the case when $V$ is a real vector bundle we fix an inner product on $V$. We may then choose the local trivialisations $\psi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$ such that the pullback of the inner product under $\psi_j$ is the standard Euclidean on $\mathbb{R}^k$ for all $x \in U_j$. In that case the transition functions $\psi_{ij}$ all take values in the orthogonal group $O_k$. Similarly, if $V$ is a complex vector bundle we fix a Hermitian form on $V$ and pick trivialisations such that the transition functions take values in the unitary group $U_k$. 

(b) $\gamma_{ij}^t(x) = f(x)$ for all $x \notin f^{-1}(e_{\alpha_j}^t)$ for all $t \in [0, 1]$,

(c) $\gamma_{ij}^t(f^{-1}(e_{\alpha_j}^t)) \subset e_{\alpha_j}^t$ for all $t \in [0, 1]$, and

(d) $\gamma_{ij}^t(X) \cap e_{\alpha_j}^t = \emptyset$.

Finally, because of (b) we may drop the superscript $j$ and regard the collection of $\gamma_{ij}^t$'s as a single homotopy $\gamma_1 : X \times [0, 1] \to Y$ such that

\begin{enumerate}
  \item $\gamma_{1,0} = f$,
  \item $\gamma_{1,t}(x) = f(x)$ for all $x \notin f^{-1}(e_{\alpha_j}^t)$ for some $j$, for all $t \in [0, 1]$,
  \item $\gamma_{1,t}(f^{-1}(e_{\alpha_j}^t)) \subset e_{\alpha_j}^t$ for all $j$ and for all $t \in [0, 1]$, and
  \item $\gamma_{1,1}(X) \cap e_{\alpha_j}^t = \emptyset$ for all $j$.
\end{enumerate}

Note that it now follows from (b')–(d') that $\gamma_{1,1}(K_1) \subset Y^{s_1-1}$. We may therefore repeat the construction $s_1 - d$ times to obtain homotopies $\gamma_{i,t}$ such that, letting $\Theta_1 : X \times [0, 1/2] \to Y$ denote the concatenation of the homotopies, we have (1)–(4) above for $n = 1$.

Finally, by replacing $f$ by $\Theta_{n,1-1/n}$, it is now also clear that we may invoke this procedure in an inductive construction on $n$ to pass from $I_n$ to $I_{n+1}$. 

4. Proofs of the main theorems

4.1. Vector bundle automorphisms as sections of fiber bundles. Let $W_{GL}$ be the fiber bundle over $X$ consisting of the points in the endomorphism bundle $V \otimes V^*$ that represent invertible maps on the fibers of $V$. Alternatively, $W_{GL}$ can be constructed in the following manner:

Theorem 1 and Theorem 2.
These are both compact groups, in other words $W^{\text{GL}}$ is a fiber bundle with compact structure group. In the proof of Theorem 1 we will always assume that $V$ is equipped with this structure. This will be crucial when applying Theorem 5. Moreover, the auxiliary metric above also give us two fiber bundles over $X$. In the case when $V$ is a real vector bundle:

$(3) \quad W^{\text{SO}}$, denoting the fiber bundle over $X$ consisting of the elements in $W^{\text{SL}}$ that are orthogonal with respect to the inner product on $V$

and in the case when $V$ is a complex vector bundle:

$(3) \quad W^{\text{SU}}$, denoting the fiber bundle over $X$ consisting of the elements in $W^{\text{SL}}$ that are unitary with respect to the Hermitian form on $V$

The significance of these fiber bundles is that their fibers are compact. This will also be crucial when applying Theorem 5.

For $f \in \Gamma(X, W^{\text{GL}})$, the inner product on $V$ (resp. the Hermitian form) induce a fiberwise operator norm

$$\|f\|_x := \sup_{v \in V_x \setminus \{0\}} \frac{|f(x)(v)|}{|v|},$$

where $|\cdot|$ is the norm induced by the inner produce (resp. Hermitian form), and a distance function $d_W : \Gamma(X, W^{\text{GL}}) \times \Gamma(X, W^{\text{GL}}) \to [0, \infty]$ by

$$d_W(f, g) := \sup_{x \in X} \|f - g\|_x.$$  

The space of bounded sections, i.e., $f \in \Gamma(X, W^{\text{GL}})$ such that

$$\|f\|_W := d_W(f, 0) < \infty,$$

is thus given the structure of a normed space which we denote by $\Gamma_b(X, W^{\text{GL}})$. In particular, $\Gamma(X, W^{\text{SO}})$ (and $\Gamma(X, W^{\text{SU}})$ respectively) is naturally included in $\Gamma_b(X, W^{\text{GL}})$.

If in addition $V$ admits a splitting $\mathcal{L} = \{L_1, ..., L_k\}$ into line bundles, we may choose the inner product (or Hermitian form, respectively) such that the $L_i$'s are mutually orthogonal. We may also achieve that each $\psi_j$ sends each $L_i$ to the line generated by the $i$'th standard basis vector in $\mathbb{R}^n$ ($\mathbb{C}^n$, respectively). Then the transition maps $\psi_{ij}(x)$ are diagonal matrices whose entries in the real case take values in $\{-1, 1\}$, and in the complex case $\{z \in \mathbb{C} : |z| = 1\}$. This will be used in the proof of Theorem 2.

4.3. Gauss-Jordan, Gram-Schmidt and the proofs of Theorem 1 and Theorem 2

The main ingredients in the proofs of Theorem 1 and Theorem 2 are Theorem 5 on uniform homotopies together with Proposition 18 and Proposition 19. Proposition 18 is related to the Gram-Schmidt process and says that a section in $W^{\text{SL}}$, or more generally a continuous family of sections $S_t, t \in [0, 1]$ of $W^{\text{SL}}$ can after composing with a number of unipotent sections be assumed to take values in $W^{\text{SO}}$ (or $W^{\text{SU}}$, respectively).

**Proposition 18** (The Gram-Schmidt process). Let $S_t, t \in [0, 1]$ be a homotopy of sections in $\Gamma(X, W^{\text{SL}})$. Then there exist homotopies of unipotent sections $E_t^j, j = 1, ..., N$ in $\Gamma(X, W^n)$ such that in the case when $V$ is a real vector bundle

$$E_N^1 \circ \cdots \circ E_1^1 \circ S_t \in \Gamma(X, W^{\text{SO}})$$
and in the case when $V$ is a complex vector bundle
\[ E_N^1 \circ \cdots \circ E_1^1 \circ S_t \in \Gamma(X, W^{SU}) \]
for all $t \in [0, 1]$. Moreover, if $S_0 = \text{id}$ then $E_j^0 = \text{id}$ for $j = 1, ..., N$.

Proposition 19 is related to the Gauss-Jordan process and says that any $S \in \Gamma_b(X, W^{SL})$ which is sufficiently close to the identity can be factorised into a product of unipotent sections.

**Proposition 19 (The Gauss-Jordan process).** There exists an $\epsilon > 0$ such that if $S \in \Gamma_b(X, W)$ satisfies $d_W(S, \text{id}) < \epsilon$, then there exist unipotent sections $E_j \in \Gamma^u(X, W)$ for $j = 1, ..., N$, such that
\[ E_N \circ \cdots \circ E_1 \circ S = \text{id}. \]

For the proofs of Proposition 18 and Proposition 19 we will introduce a triangulation $\mathcal{T}$ of $X$ subordinate to $\{U_j\}$, and use it to prove two key lemmata: Lemma 20 and Lemma 21. The proofs of Proposition 18 and Proposition 19 are carried out by straightforward repeated use of Lemma 20 and Lemma 21 respectively. Given Proposition 18 and Proposition 19, Theorem 1 and Theorem 2 is proved in the following manner:

**Proof of Theorems 1 and Theorem 2:** Let $S_t, t \in [0, 1]$, be a homotopy between $\text{id}$ and $S$. By Proposition 18 we may assume that $S_t \in \Gamma(X, W^{SO})$ for all $t \in [0, 1]$ if $V$ is a real vector bundle and $S_t \in \Gamma(X, W^{SU})$ for all $t \in [0, 1]$ if $V$ is a complex vector bundle. By the discussion in Section 4.2, $W^{SL}$ has compact structure group.

To apply Theorem 2 it suffices to verify that $W^{SO}$ and $W^{SU}$ have compact fibers with finite first fundamental group. But this is immediate since the fibers are isomorphic to $SO_k$ and $SU_k$ respectively, and $\pi_1(SO_k)$ is finite for $k \geq 3$ and $\pi_1(SU_k)$ is finite for any $k$.

After applying Theorem 5 (or rather its corollary) we may assume that the homotopy $S_t$ is uniform. Hence, there is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\tilde{S}_j := S_{t_{j+1}} \circ S_{t_j}^{-1}$ satisfies $d_W(\tilde{S}_j, \text{id}) < \epsilon$ for $j = 0, ..., m - 1$, where $\epsilon$ is the constant from Proposition 19. Since
\[ S = \tilde{S}_{m-1} \circ \tilde{S}_{m-2} \circ \cdots \circ \tilde{S}_0 \]
the proof of Theorem 1 is completed by an application of Proposition 19 to each $\tilde{S}_j$, and then composing. Finally, the proof of Theorem 2 is immediate, since in that case, all the local coordinates used in the proofs of Lemma 20 and Lemma 21 below respect the splitting $L = \{L_1, ..., L_k\}$ (see the last paragraph in Section 4.2).

4.4. The triangulation an proofs of Proposition 18 and Proposition 19. Letting $\{e_0, ..., e_q\}$ denote the standard basis vectors on $\mathbb{R}^{q+1}$, recall that the (standard) affine $q$-simplex is defined by
\[ F^q := \{x = \sum_{j=0}^q t_j e_j : \sum_{j=0}^q t_j = 1, t_j \geq 0\}, \]

For $q' \leq q$, the convex hull of any $q'+1$ distinct points in $\{e_0, ..., e_q\}$ is called a $q'$-face of $F^q$, and will be identified with $F^{q'}$.

The triangulation $\mathcal{T}$ is given by a family $\{g_j : F^n \to X\}$ of embeddings such that the following holds
(1) For each $j$ we have that $F^n_j := g_j(F^n_i) \subset U_i$ for some $i$,
(2) $X = \cup_j F^n_j$,
(3) $\text{int}(F^n_j) \cap \text{int}(F^n_i) = \emptyset$ for $i \neq j$,
(4) If a $q$-face $F^n_j$ intersects a $q'$-face $F^n_i$, it is along a common subface,
(5) Each $n$-face $F^n_j$ coincides with precisely one distinct $n$-face $F^n_i$,
(6) The set $\{F^n_j\}$ is locally finite.

Lemma 20. Let $S_t \in \Gamma(X, W^{SL})$, $t \in [0, 1]$ be a homotopy of sections and assume that for any $q$-face $F^n_j$, there exists an open neighbourhood $U$ of $F^n_j$ such that $S_t|_U \in \Gamma(U, W^{SO})$ (resp. $\Gamma(U, W^{SU})$) for all $t \in [0, 1]$. Then there exist homotopies of unipotent sections $E^n_1, \ldots, E^n_m \in \Gamma(X, W^n)$ such that for any $(q + 1)$-face $F^{q+1}_k$, there exists an open neighbourhood $V$ of $F^{q+1}_k$ such that

$$\left( E^n_1 \circ \cdots \circ E^n_{m-1} \circ E^n_i \circ S_t \right) |_{V} \in \Gamma(V, W^{SO})$$

(resp. $\Gamma(V, W^{SU})$) for all $t \in [0, 1]$.

Proof. Let $\{F^{q+1}_k\}$ be the collection of all $(q + 1)$-faces, where each face appears only once (so each one may be a subface of many other $(q + 1 + m)$-faces). For each $k$, fix $j_k$ such that $F^{q+1}_k \subset U_{j_k}$, and let $V_k \subset U_{j_k}$ be open neighborhoods of the $F^{q+1}_k$’s such that if $l \neq k$ then

$$S|_{V_l \cap V_k} \in \Gamma(V_l \cap V_k, W^{SO})$$

(resp. $\Gamma(V_l \cap V_k, W^{SU})$). For each $V_k$ we perform the Gram-Schmidt process on $S$ with respect to the local coordinates given by $\phi_{j_k}$, to obtain elementary matrices $E^{k,t}_1, \ldots, E^{k,t}_m$ such that

$$\left( E^{k,t}_m \circ \cdots \circ E^{k,t}_1 \circ S_t \right) |_{V_k} \in \Gamma(V_k, W^{SO})$$

(resp. $\Gamma(V_k, W^{SU})$).

Noting that the Gram-Schmidt process leaves any orthogonal matrix (resp. unitary matrix) unchanged, we see that by (3) for any $V_l$ with $V_l \cap V_k \neq \emptyset$ we have that $E^{k,t}_j = \text{Id}$ on $V_l \cap V_k$ for all $j$. Now each $E^{k,t}_j$ is determined by a continuous function $h^{k,t}_j$ that appears off the diagonal, and clearly $h^{k,t}_j = 0$ on $V_l \cap V_k$ for any $l$. Choose cutoff functions $\chi_k$ which are one near the $F^{q+1}_k$’s and are compactly supported in $V_k$ for all $k$. For each $k$ let $\hat{E}^{k,t}_j$ be the elementary matrix obtained by replacing $h^{k,t}_j$ by $\chi_k \cdot h^{k,t}_j$. Then $\hat{E}^{k,t}_j \in \Gamma(X, W^n)$, $\hat{E}^{k,t}_j = E^{k,t}_j$ near $F^{q+1}_k$, and $\hat{E}^{k,t}_j = \text{Id}$ on each $V_l \cap V_k$. Thus, for each fixed $j$ the collection $\{\hat{E}^{k,t}_j\}$ for $k \in \mathbb{N}$, defines a global section $\hat{E}^{k,t}_j \in \Gamma(X, W^n)$. And we have that

$$\hat{E}^{k,t}_m \circ \cdots \circ \hat{E}^{k,t}_1 \circ S_t \in \Gamma(\hat{V}_k, W^{SO}),$$

(resp. $\Gamma(\hat{V}_k, W^{SU})$) for all $t \in [0, 1]$ for some open set $\hat{V}_k$ that contains $F^{q+1}_k$ for each $k$. \qed
Lemma 21. There exist an $\epsilon > 0$ such that the following holds for any $S \in \Gamma_b(X, W^{SL})$ with $d_W(S, \text{Id}) < \epsilon$. If for any $q$-face $F_j$, there exists an open neighbourhood $U$ of $F_j$ such that $S|_U = \text{Id}$, then there exist $E_1, \ldots, E_m \in \Gamma(X, W^{SU})$ such that for any $(q+1)$-face $F_{j+1}$, there exists an open neighbourhood $V$ of $F_{j+1}$ such that $(E_m \circ E_{m-1} \circ \cdots \circ E_1)|_V = S|_V$. Moreover, we have that the dependence of $E_1, \ldots, E_m$ on $S$ can be described in the following manner: $(E_1, \ldots, E_m) = \Lambda(S)$ where

$$
\Lambda : \Gamma_b, \epsilon(X, W^{SL}) \rightarrow \Gamma_b(X, W^{SU}) \times \cdots \times \Gamma_b(X, W^{SU})
$$

is a continuous operator satisfying $\Lambda(\text{Id}) = (\text{Id}, \ldots, \text{Id})$.

Proof. The proof follows the same line of reasoning as the previous one, but uses Gauss-Jordan instead of Gram-Schmidt. □

We may now prove Proposition 18 and Proposition 19.

Proof of Proposition 18 and Proposition 19. The proof of Proposition 18 is carried out by repeated use of Lemma 20. Starting with $q = -1$ (for which the assumption in Lemma 20 on neighbourhoods of $q$-faces is vacuous) we may, after applying Lemma 20, replace $S_t$ by $E_t^m \circ E_t^{m-1} \circ \cdots \circ E_t^1 \circ S_t$ to get a homotopy of sections which satisfies the assumptions of Lemma 20 for $q = 0$. Applying Lemma 20 again for $q = 0, 1, \ldots, n$ where $n$ is the dimension of $X$ finishes the proof. Proposition 19 is proved in the same way using Lemma 21. □

4.5. Proofs of Theorems 3 and 4. The proof of Theorem 3 is parallel to that of Theorem 1. It is shown in [IKL19] that the Gram-Schmidt process can be modified in such a way that it maps symplectic matrices to symplectic matrices. Using this in local coordinates we get analogues of Lemma 20 and Proposition 18 in the symplectic case. This means we have reduced the proof of Theorem 3 to the case where $S$ takes values in $sp_k = Sp_{2k}(\mathbb{C}) \cap U_{2k}$. Next, we have that $sp_k$ is compact and $\pi_1(sp_k)$ is trivial, so we may apply Theorem 5. Since it is also shown in [IKL19] that the Gauss-Jordan process admits a symplectic version similarly as the Gram-Schmidt process, we get an analogue for Lemma 21 and Proposition 19 in the case at hand.

Finally, the proof of Theorem 4 is analogous, except that in this case it is not necessary to apply Theorem 5 since $X$ is assumed to be compact.
4.6. **Proofs of Corollary 5 and Corollary 10.**

**Proof of Corollary 5.** By Theorem 1 it suffices to verify that \( \exp(R(U, V)) \) is a null-homotopic special vector bundle automorphism. First of all, noting that \( \exp(R(0, 0)) = \exp(0) = \text{Id} \), we see that a homotopy from the identity map to \( \exp(R(U, V)) \) is given by

\[
t \mapsto S_t = \exp(R(tU, tV)).
\]

Moreover, to see that \( \exp(R(U, V)) \) has determinant one we need to show that \( R(U, V) \) lies in the Lie algebra of \( \text{SL}_k(\mathbb{R}) \), in other words that the trace of \( R(U, V) \) is zero. A conceptual proof of this is given by the fact that \( R(U, V) \) arise as the variation of the holonomy of small quadrilaterals on \( X \) (the restricted holonomy of a Riemannian manifold lies in \( \text{SO}_k \)). For a more direct proof, let \( \langle \cdot, \cdot \rangle \) be the inner product on \( TX \) given by the Riemannian metric. One of the basic symmetries of the Riemann tensor is that

\[
\langle R(U, V)Y, Z \rangle = -\langle Y, R(U, V)Z \rangle.
\]

Writing this in a frame which is orthogonal with respect to the Riemannian metric we get that the matrix representing \( R(U, V) \) is skew-symmetric. In particular its trace is zero. \( \square \)

**Proof of Corollary 10.** By the same argument as in the proof of Corollary 5 it suffices to verify that \( \exp(R(U, V)) \) is a symplectic vector bundle automorphism, in other words that \( R(U, V) \) lies in the Lie algebra of the symplectic group. As in the proof of Corollary 5, since the holonomy of a Kähler manifold lies in \( U_k = \text{Sp}_{2k}(\mathbb{R}) \cap \text{O}_{2k} \), a conceptual proof is given by the fact that \( R(U, V) \) arise as the variation of the holonomy of small quadrilaterals on \( X \). For a more direct proof, note that the Lie algebra of \( \text{Sp}_{2k}(\mathbb{R}) \) is given by the group of matrices \( A \) satisfying

\[
\Omega A + A^T \Omega = 0
\]

where \( \Omega \) is the \( 2k \times 2k \)-matrix given by

\[
\Omega = \begin{bmatrix}
0 & Id_k \\
-Id_k & 0
\end{bmatrix}.
\]

Now, since the complex structure \( J \) on \( X \) is parallel with respect to the Levi-Civita connection of the Riemannian metric, we get that

\[
R(U, V)J = (\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}) J = J(\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}) = JR(U, V).
\]

Writing this identity in a frame which is orthogonal with respect to the Riemannian metric and in which \( J \) is represented by the standard matrix \( \Omega \) gives \( \Omega A = A\Omega \) where \( A \) is the matrix representing \( R(U, V) \). Since \( A \) is also skew-symmetric by \( (6) \), we get

\[
\Omega A = A\Omega = -A^T \Omega
\]

and \( (7) \) follows. \( \square \)
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