Two-orbital Schwinger Boson Representation of Spin-One: Application to a Non-abelian Spin Liquid with Quaternion Gauge Field

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A non-abelian spin liquid in triangular lattice spin-1 systems was recently formulated in the form of continuum field theory [T. Grover, and T. Senthil, Phys. Rev. Lett. \textbf{107}, 077203 (2011); Cenke Xu, A.W.W. Ludwig, arXiv:1012.5671]. It has spin-1/2 bosonic spinons coupled to emergent quaternion gauge fields, and can be obtained by quantum disordering a non-collinear spin nematic order hypothesized to describe NiGa\textsubscript{2}S\textsubscript{4} [H. Tsunetsugu, and M. Arikawa, J. Phys. Soc. Jpn. \textbf{75}, 083701 (2006)]. However a microscopic lattice description, e.g. the lattice spinon (mean-field) Hamiltonian and the spin wavefunction, has been missing, and it has been noted that the standard Schwinger boson or bosonic triplon representations of spin-1 cannot describe this spin liquid. In this paper a two-orbital Schwinger boson representation for spin-1 systems is developed and used to construct a mean-field description of this quaternion spin liquid. Projecting the mean-field state produces a prototype wavefunction, which is a superposition of close-packed AKLT loop configurations with nontrivial amplitudes. This new formalism and related wavefunctions may be generalized to higher spin systems and can possibly produce spin liquid states with even richer emergent gauge structures.

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Spin liquid states in more than one spatial dimensions were proposed more than three decades ago\textsuperscript{2–6}. They are ground states of Mott insulators with no spontaneous symmetry breaking, thus beyond the symmetry breaking paradigm of phases. Many parent Hamiltonians of spin liquids with\textsuperscript{7–13} and without\textsuperscript{14–16} spin SU(2) symmetry have been constructed. Extensive numerical studies on semi-realistic models have shown evidences of spin liquid ground states in quantum spin models on triangular\textsuperscript{17–19} and kagome lattices\textsuperscript{20–23} and also in electronic Hubbard models\textsuperscript{24,25}. In the last decade several promising candidate materials have also emerged, a review of which is given in Ref.\textsuperscript{19}.

One way to understand some of the spin liquid states is by disordering a spin SU(2) symmetry breaking order without proliferating topological defects\textsuperscript{20–23}. Low energy theory of such description usually contains gapped bosonic spinon and emergent gauge field, and the phase transition from quantum disordered (spin liquid) state to ordered state is the condensation of the bosonic spinon\textsuperscript{20–23}. This approach is believed to work better in the deep Mott insulating limit, where a short-range quantum spin model is appropriate. To get quantum spin liquid ground state, the conventional wisdom suggests that low spin value is important, and spin-1/2 is the best. For spin-1/2 systems the only single-site SU(2)-breaking order parameter is the local magnetic dipole vector $\mathbf{S}$ (which also breaks time-reversal symmetry). Long-range order of $\langle \mathbf{S} \rangle$ is the most commonly used starting point of this bosonic spin liquid approach\textsuperscript{20–23}.

However spin-1 systems may have magnetic quadrupole order (spin nematic order hereafter) that breaks spin SU(2) symmetry with zero local dipole moments and no time-reserval symmetry breaking. The spin nematic order parameter is the real symmetric traceless matrix $Q^{ab} = \langle (S^{a}S^{b} + S^{b}S^{a})/2 \rangle - (2/3)\delta^{ab}$ ($a, b = x, y, z$). In this paper only the uniaxial spin nematic order will be considered, which can be described by the “director” $\mathbf{n}$ as $Q^{ab} = n^{a}n^{b} - (1/3)\delta^{ab}n^{2}$.

Spin nematic orders have been proposed\textsuperscript{24,25} for the spin-1 triangular lattice material NiGa\textsubscript{2}S\textsubscript{4} which had some experimental evidence of a ground state without magnetic dipole order\textsuperscript{26}. In particular Tsunetsugu and Arikawa\textsuperscript{24} proposed an interesting three-sublattice spin nematic order, with the directors on the three sublattices perpendicular to each other (see Fig. 1). This state was also found in a numerical study of spin-1 nearest-neighbor bilinear-biquadratic Heisenberg model on triangular lattice\textsuperscript{26}. Very recently two groups\textsuperscript{27,28} considered possible spin liquid states by disordering this “Tsunetsugu-Arikawa state” (“antiferroquadrupolar order” in Ref.\textsuperscript{27}, “tetrad order” in Ref.\textsuperscript{28}), and found an interesting non-abelian spin liquid with spin-1/2 bosonic spinons coupled to emergent gauge fields in the quaternion group $Q_{8}$, a discrete non-abelian group with eight elements defined as $Q_{8} = \{ \pm 1, \pm i, \pm j, \pm k \}$ with the mul-
The bosonic triplon representation uses a three-component boson \( b \) and writes the spin operator as
\[
S = -ib^\dagger \times b, \tag{2}
\]
with the constraint \( b^\dagger \cdot b = 1 \). Both magnetic order and spin nematic order can be described by a boson condensate in this construction. However the gauge freedom is again \( U(1) \), \( b \to e^{i\theta}b \), with no quanton subgroup. Moreover \( b \) is not a spin-1/2 representation of spin \( SU(2) \) symmetry. What is needed for the quaternion spin liquid is a new representation of spin-1 by spin-1/2 bosons (like the Schwinger boson) with large enough gauge freedom, which can describe the spin nematic order semiclassically by boson condensation (like the bosonic triplon).

The outline of this paper is as follows. In section II the two-orbital Schwinger boson representation for spin-1 will be introduced. Generic mean-field theories of spin liquids in this representation and related gauge structure and generic (projected) mean-field wavefunctions will be presented. In section III a mean-field Hamiltonian for the quaternion spin liquid on triangular lattice will be constructed and analyzed. The Tsunetsugu-Arikawa state will be obtained by boson condensation. A prototype spin wavefunction for the spin liquid will be presented. In section IV remaining issues and possible extensions will be discussed.

### I. TWO-ORBITAL SCHWINGER BOSON REPRESENTATION OF SPIN-ONE AND MEAN-FIELD THEORIES FOR SPIN LIQUIDS

Spin-1 can be viewed as a symmetric combination of two spin-1/2. The Affleck-Kennedy-Lieb-Tasaki (AKLT) state was originally defined in this way. Use this “two-orbital” picture and introduce Schwinger bosons for each spin-1/2, the spin-1 operators in terms of the four-component bosons are
\[
S = \sum_{\alpha=1}^{2} S_\alpha = \sum_{\alpha=1}^{2} \frac{1}{2} \sum_{s,s'=\uparrow,\downarrow} b^\dagger_{\alpha s} \sigma_{ss'} b_{\alpha s'}, \tag{3}
\]
where \( \alpha = 1,2 \) labels orbital, \( s,s' = \uparrow, \downarrow \) label spin. This bosonic representation for \( SU(N) \) spins were briefly mentioned in Ref. [33]. From this alone the gauge freedom seems to be orbital \( U(2) \), namely \( b_{\alpha s} \to \sum_{\beta} u_{\alpha \beta} b_{\beta s} \) with \( \sum_{\beta} u_{\alpha \beta}^* u_{\alpha' \beta'} = \delta_{\alpha \alpha'} \). However the gauge transformations should also leave the constraints invariant.

The first constraint requires that the total number of bosons is two,
\[
n = \sum_{\alpha} n_\alpha = \sum_{\alpha s} b^\dagger_{\alpha s} b_{\alpha s} = 2. \tag{4}
\]
Define the orbital pseudo-spins \( T \) as
\[
T = \frac{1}{2} \sum_{s,\alpha,\beta} b^\dagger_{\alpha s} \tau_{\alpha \beta} b_{\beta s}, \tag{5}
\]
where \( \tau \) are orbital Pauli matrices. There are ten states for two bosons, one \( S = 0, T = 0 \) spin-orbital singlet \( [2^{-3/2}(b^1 \cdot \tau^y \sigma^y \cdot b^1)][0] \) where \([0]\) is the boson vacuum] and nine \( S = 1, T = 1 \) states. The second constraint is to project out the singlet state,

\[
(b^1 \cdot \tau^y \sigma^y \cdot b^1)(b \cdot \tau^y \sigma^y \cdot b) = 0. 
\]

The \( S = 1, T = 1 \) states can be arranged into a matrix, with row and column indices \( S^2, T^2 = 1, 0, -1 \),

\[
\begin{pmatrix}
\frac{(b^1_1)^2}{\sqrt{2}} & \frac{b^1_1 b^1_2}{\sqrt{2}} & \frac{(b^1_2)^2}{\sqrt{2}} \\
\frac{b^1_1 b^1_2}{\sqrt{2}} & \frac{b^1_1 b^1_3 + b^1_2 b^1_4}{\sqrt{2}} & \frac{b^1_3 b^1_4}{\sqrt{2}} \\
\frac{(b^1_2)^2}{\sqrt{2}} & \frac{b^1_3 b^1_4}{\sqrt{2}} & \frac{(b^1_3)^2}{\sqrt{2}}
\end{pmatrix} | 0 \rangle .
\]

The physical spin-1 states should be a linear combination of the columns in Eq. (7). This final constraint can be formally represented by

\[
N \cdot T = 0, 
\]

with complex unit vector \( N \) (\( N^* \cdot N = 1 \)). The chosen linear combination has coefficients \(- (N^* - iN^y)/\sqrt{2}, N^x, (N^* + iN^y)/\sqrt{2}\) respectively for the three columns in Eq. (7). An equivalent form of Eq. (8) is

\[
\sum_{a,b=\tau,\eta,\zeta} T^a N^{\alpha a} N^{\beta b} T^b = 0. 
\]

The gauge freedom should leave this “vector Higgs condensate” \( N \) invariant up to a complex phase, or equivalently leave the hermitian matrix \( N^{\alpha a} N^{\beta b} \) invariant.

This parton construction unifies the conventional Schwinger boson Eq. (1) and the two-orbital AKLT representation. For example, \( N = (\hat{x} + i\hat{y})/\sqrt{2} \) chooses the first column of Eq. (7) and is the old Schwinger boson representation. In contrast \( N = \hat{z} \) chooses the middle column of Eq. (7) and is the AKLT representation. Three different cases of \( N \) are discussed in the following.

Case 1): \( N^* \times N = 0 \), then a complex phase can be chosen so that \( N \) is a real vector. This represents an orbital nematic ordered state with the director \( N \), and is equivalent to the AKLT representation. The gauge freedom is \( U(1) \times U(1) \times \mathbb{Z}_2 \), generated by

\[
b \rightarrow e^{i\theta} b, \quad b \rightarrow e^{i\theta}(N \cdot \tau) b, \quad b \rightarrow (N^\perp \cdot \tau) b, 
\]

where \( N^\perp \) is a real unit vector perpendicular to \( N \). Note that the last \( \mathbb{Z}_2 \) does not commute with the second \( U(1) \) group, therefore semidirect-product \( \times \) is used.

Case 2): \( N^* \times N \neq 0 \) and \( N \cdot N = 0 \), then the real and imaginary parts of \( N \) are perpendicular to each other and of equal length \( \sqrt{1/2} \). This represents an orbital dipole ordered state with the orbital moment along \( T = - i N^* \times N \), and is equivalent to the single-orbital Schwinger boson representation Eq. (1). The gauge freedom is \( U(1) \),

\[
b \rightarrow e^{i\theta(1 + T \cdot \tau)} b, 
\]

where \( 1 \) is the identity matrix.

Case 3): \( N^* \times N \neq 0 \) and \( N \cdot N \neq 0 \), then a complex phase can be chosen so that the real and imaginary parts of \( N \) are perpendicular to each other but of different length. The orbital dipole moment is nonzero along \( T = - i N^* \times N / |N^* \times N| \). The gauge freedom is \( U(1) \times \mathbb{Z}_2 \), generated by

\[
b \rightarrow e^{i\theta} b, \quad b \rightarrow (T \cdot \tau) \cdot b. 
\]

Consider the Heisenberg antiferromagnetic interaction between two spin-1 at positions \( r \) and \( r' \), namely \( S_r \cdot S_{r'} = \sum_{\alpha, \beta} S_{r \alpha} \cdot S_{r \beta} \). The right-hand-side can be Hubbard-Stratonovich decoupled in the same way as the Schwinger boson mean-field theory. The mean-field Hamiltonian contains spin singlet boson pairing terms, \( A_{rr', \alpha \beta} (b^\dagger_{r \alpha} b^\dagger_{r' \beta} S_{r \alpha} \cdot S_{r \beta}) + h.c. \), or in short form \( b^\dagger_r \cdot A_{rr'} \otimes i\sigma^y \cdot b^\dagger_{r'} + h.c. \) where \( A_{rr'} \) is a generic \( 2 \times 2 \) matrix in the orbital space. Note that \( A_{rr'} = - A^{T}_{rr'} \) where superscript \( T \) stands for matrix transpose. For simplicity the boson hopping terms \( \sum_{r \alpha, \beta} b^\dagger_{r \alpha} b_{r \beta} \) are ignored.

The constraints must be included in the mean-field theory by introducing on-site terms with Langrange multipliers. The first constraint Eq. (1) may be incorporated by a chemical potential \( \mu \) as \( \mu(r - n) \). The second one Eq. (2) may be included as \(- \lambda(\eta(b^\dagger_r \cdot \sigma^y \cdot b^\dagger_{r'}) - \eta^*(\tau \cdot \sigma^y \cdot b^\dagger_r)) \) with real \( \lambda \), however this is not quadratic in terms of bosons. A non-rigorous Hubbard-Stratonovich decoupling can be performed to reduce this term to \(- \eta(b^\dagger_r \cdot \sigma^y \cdot b^\dagger_{r'}) - \eta^*(\tau \cdot \sigma^y \cdot b^\dagger_r) \) with complex \( \lambda \). This procedure may be made rigorous regardless of the sign of \( \lambda \) by the tricks of Ref. 35. The final constraint Eq. (3) may be included as \(- \mu^* \tau \cdot N^\perp \cdot T + h.c. \) with a complex Langrange multiplier \( \mu^* \). In summary the generic mean-field Hamiltonian for spin liquids is

\[
H_{\text{MF}} = - \sum_{r,r'} [b^\dagger_r \cdot A_{rr'} \otimes i\sigma^y \cdot b^\dagger_{r'} + h.c.] + \sum_{r} \mu_r n_r \\
- \sum_{r} [\eta_r (b^\dagger_r \cdot \tau \cdot \sigma^y \cdot b^\dagger_{r'}) + h.c.] \\
- \sum_{r} [b^\dagger_r \cdot \Re(\mu'_r N^\perp_r) \cdot \tau \cdot b^\dagger_{r'} + \text{const}].
\]

where \( \Re \) means real part. The mean-field constraints are

\[
\langle n_r \rangle_{\text{MF}} = 2, \quad \langle b^\dagger_r \cdot \tau \cdot \sigma^y \cdot b^\dagger_{r'} \rangle_{\text{MF}} = 0, \quad \langle b^\dagger_r \cdot (N^\perp \cdot \tau) \cdot b^\dagger_{r'} \rangle_{\text{MF}} = 0.
\]

The mean-field Hamiltonian Eq. (13) is not gauge invariant. Under a site-dependent gauge transformation \( b_r \rightarrow G(r) \cdot b_r \), the mean-field ansatz \( \{ \mu_r, \eta_r, \mu'_r, N^\perp_r, A_{rr'} \} \) should transform as

\[
A_{rr'} \rightarrow G(r) \cdot A_{rr'} \cdot G^T(r'), \quad \eta_r \rightarrow e^{2i\theta} \eta_r, \\
\mu_r + \Re(\mu'_r N^\perp_r) \cdot \tau \rightarrow G(r) \cdot [\mu_r + \Re(\mu'_r N^\perp_r) \cdot \tau] \cdot G^{-1}(r),
\]

where \( \theta \) is the gauge phase.
where $\theta$ defines the orbital-independent $U(1)$ subgroup in Eqs. (10,12). Gauge-invariant fluxes can be defined in analogy to the Schwinger boson or $Sp(N)$ boson theory. The loop expansion for the mean-field ground state energy can also be performed, and a “flux expulsion” argument for Heisenberg models may be raised as well.

A realistic spin-1 Hamiltonian may contain the bi-quadratic interactions $(S_r \cdot S_r)^2$ and multiple-spin interactions, which cannot be simply decoupled into quadratic terms of bosons. In this situation it is better to view the mean-field theory as a variational approach. The mean-field ground state after projection to physical spin-1 space can be used as a variational wavefunction. This viewpoint will be adopted throughout this paper, so no self-consistent equation of $A$ will be solved, and the overall scale of the ansatz does not matter.

The mean-field ground state $|\text{MF}\rangle$ is generically

$$|\text{MF}\rangle = \exp \left[ \frac{i}{2} \sum_{r,r'} b^\dagger_r f_{rr'} \otimes |\sigma^y b_{r'}^\dagger \rangle \right] |0\rangle,$$

where $f_{rr'}$ are $2 \times 2$ matrices in the orbital space, and have the same symmetry and gauge transformation rule as the mean-field ansatz $A_{rr'}$, e.g. $f_{rr'} = -f_{r'r}^T$. $r = r'$ term with $f_{rr} = -f_{rr}^T \propto \sigma^y$ is allowed but creates only onsite spin singlet and will be projected out. Each $f_{rr',\alpha\beta} = (b_{\alpha r}^\dagger b_{\beta r}^\dagger - b_{\alpha r}^\dagger b_{\beta r}^\dagger)\tau$ term creates a spin singlet from two spin-1/2 on bond $rr'$. The projection onto physical spin-1 space requires two bosons on every site and the onsite symmetrization of the two orbitals. The projected wavefunction $\mathcal{P}|\text{MF}\rangle$ is therefore a superposition of close-packed (every site is covered once) loop configurations $\{\ell\}$, and on each loop $\ell$ the spin-1 form an AKLT state,

$$\mathcal{P}|\text{MF}\rangle = \sum_{\{\ell\}} \prod_{\ell} W_\ell |\text{AKLT on } \ell\rangle.$$ (17)

The “close-packed” loop configurations may involve bonds longer than nearest-neighbor. The amplitude factor $W_\ell$ for a length-$L$ loop $\ell = (r_1 r_2 \ldots r_L)$ is

$$W_\ell = (3/4)^{L/2} \cdot N_L \cdot \text{Tr}[t_{r_1} f_{r_1 r_2} t_{r_2} f_{r_2 r_3} \ldots t_{r_L} f_{r_L r_1}],$$ (18)

where $\text{Tr}$ means matrix trace, $N_L = \sqrt{1 + 3 \cdot (-3)^{-L}}$, and $\tau_r = -i \sigma^y \cdot (N_{r'}^\dagger \cdot \tau)$ comes from the contraint Eq. (5) ($\tau_r = \tau^x$ when $N_{r'} = \zeta$). The factor $(3/4)^{L/2} \cdot N_L$ is the overlap between one spin-1/2 dimer pattern and the AKLT state. $(3/4)^{L/2}$ produces an overall factor for the wavefunction and can be omitted. $N_L \sim 1$ when $L$ is large. $W_\ell$ is gauge invariant up to a global factor, due to the fact that $G^T(r) \cdot (-i \sigma^y) \cdot (N_r^\dagger \cdot \tau) \cdot G(r) = e^{2i\theta} (-i \sigma^y) \cdot G^{-1}(r) \cdot (N_r^\dagger \cdot \tau) \cdot G(r) \sim (-i \sigma^y) \cdot (N_r^\dagger \cdot \tau)$ up to a complex phase, for any $G(r)$ in the gauge group Eqs. (10,12). $L$ can be 2 in which case the AKLT state is the “double-bond” spin singlet state formed by two spin-1.

Wavefunctions for spin-1/2 spinon and gauge flux excitations can be constructed as well. The mean-field state with two spinons at $r, r'$ is given by

$$|B_\ell B_{\ell'}\rangle_{\text{MF}} = \text{Tr}[\hat{B}^\dagger_r \cdot B_{r'} \cdot \text{Tr}[\hat{B}^\dagger_{r'} \cdot B_{r'}]|\text{MF}\rangle$$ (19)

where $B_{r,r'}$ are $2 \times 2$ complex spinon state matrices, and

$$\hat{B} = \begin{pmatrix} b_{1+}^\dagger & b_{2+}^\dagger \\ b_{1-} & b_{2-} \end{pmatrix}.$$ (20)

Projecting this state onto spin-1 space creates a superposition of configurations with one open AKLT chain $\ell_{rr'}$ from $r$ to $r'$ plus close-packed AKLT loops. The direction of the end-spins of open AKLT chain are given by $\text{Tr}[\hat{B}^\dagger \sigma B] = 1$. The amplitude for a length-$(L + 2)$ open chain $\ell_{rr'} = (rr_1 \ldots r_L r')$ with the end-spins at $r, r'$ in $S^2$ eigenstates $s, s' = \uparrow, \downarrow$ is (L can be zero)

$$W_{\ell_{rr',ss'}} = (3/4)^{L/2+1} \cdot N_{L,ss'}$$

$$\times \langle B_r \tau_r f_{rr'} \tau_{r_1} f_{r_1 r_2} \ldots \tau_{r_L} f_{r_L r'} B_{r'}^\dagger \rangle_{ss'},$$ (21)

with $N_{L,ss'} = \sqrt{1 + (-3)^{L-2}(\sigma^z - 1)_{ss'}}$. Multiple (even) number of spinons can be constructed similarly. Gauge flux excitations and topological degeneracy of ground states will be demonstrated in the quantum limit of quaternon spin liquid in subsection II C.

Spin liquids described in this way will have gapped spin-1/2 bosonic spinons. For them to be stable in 2D it is necessary to “Higgs” the continuous compact gauge groups Eqs. (10,12) to a discrete subgroup. Many possibilities exist which can in principle be completely classified by the projective symmetry group (PSG) method. This brute-force approach will not be attempted here, but the PSG language will be used to show that the mean-field theory indeed describes a spin liquid state with no symmetry breaking. This will be achieved by the explicit construction of the PSG elements, $b_r \rightarrow G_X(r) \cdot b_{X(r)}$, for all generators $X$ of the physical symmetry group (space group and time-reversal). The mean-field Hamiltonian shall be invariant under PSG actions.

The case with uniform $N_r = \zeta$ will be considered hereafter only, except subsection II D. The physical spin-1 states $|S^2 = +1, 0, -1\rangle$ are

$$|S^2 = +1\rangle = b_{1+}^\dagger b_{2+}^\dagger |0\rangle,$$ (22a)

$$|S^2 = 0\rangle = \frac{1}{\sqrt{2}} (b_{1+}^\dagger b_{2+}^\dagger + b_{1+}^\dagger b_{2+}^\dagger) |0\rangle,$$ (22b)

$$|S^2 = -1\rangle = b_{1+}^\dagger b_{2+}^\dagger |0\rangle.$$ (22c)

The constraints Eq. (11) and Eq. (5) can be rewritten as

$$a_\alpha = \sum_s b_{\alpha s}^\dagger b_{\alpha s} = 1, \quad \alpha = 1, 2.$$ (23)
The $U(1) \times U(1) \times Z_2$ gauge group is,

$$b \rightarrow e^{i\theta} \cos \phi \cdot \mathbf{1} + \sin \phi \cdot i\tau^z \cdot b, \quad \text{or}$$

$$b \rightarrow e^{i\theta} \cos \phi \cdot i\tau^x + \sin \phi \cdot i\tau^y \cdot b.$$  \hfill (24)

For frustrated (e.g., triangular) lattices the orbital-independent $U(1)$ freedom $e^{i\theta}$ will be removed by boson pairings in mean-field theory. The remaining $U(1) \times Z_2$ [a subgroup of $SU(2)$ by setting $\theta = 0$ in Eq. (24)] will be the starting point of discussions hereafter. It is non-abelian and contains the quaternion group.

A semiclassical picture of the uniaxial nematic spinor order is that the two spin-1/2 have dipole moments antiparallel to each other and along the direction of the director, $\langle S_i^z \rangle = -\langle S_j^z \rangle \propto n$. This can be achieved by a single spin-orbital-entangled condensate, e.g., $\langle (b_{11}, b_{21}, b_{22}, b_{12}) \rangle \propto (1,0,0,1)$ for $n \propto \hat{z}$. However the quadrupole order parameter $Q^{ab}$ is naively zero because $\langle S_i^a \rangle = 0$. This can be remedied by recognizing that $Q^{ab} = (\sum_{\alpha,\beta} (S_{i\alpha}^a S_{i\beta}^b + S_{i\beta}^a S_{i\alpha}^b))/2 - (2/3)\delta^{ab} = (\sum_{\alpha} S_{i\alpha}^a S_{i\alpha}^b) - (1/6)\delta^{ab}$, and the last expression is nontrivial in this condensate state [although not in the form of $n^a n^b - (1/3)\delta^{ab} n^z$]. This can be further justified by projecting the coherent state from this condensate $\exp[w (b_{11}^\dagger + b_{22}^\dagger)]|0\rangle$ onto the physical spin-1 states Eqs. 22a-22c, which gives the uniaxial spin nematic state $(w^2/\sqrt{2}) S_z = 0$ with director along $\hat{z}$ direction. In general the nematic director from a condensate $|b\rangle$ is given by $(b)^* \cdot \tau^z \sigma \cdot (b)$.

II. QUATERNION SPIN LIQUID ON TRIANGULAR LATTICE

For a mean-field theory of quaternion spin liquid, the invariant gauge group IGG must be a representation of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with eight distinct IGG elements $G_q(r) \in U(1) \times Z_2$ such that: 1) the ansatz $\{\mu_r, \eta_r, \mu_r', N_r = \hat{z}, A_r\}$ are invariant under the actions Eq. (15) of $G_q(r)$ for any $q \in Q_8$, and no other element of Eq. (21) can do the same; 2) $G_q(r)$ is a representation of $Q_8$ for any site $r$.

There are three distinct classes (labelled by $A, B, C$) of $Q_8$ representations on a single site. Representatives of each class are given below (g $\in Z$ for all classes),

$$A : \quad G_{\pm 1} = \mp i\tau^x, \quad G_{\pm \pm} = \mp i\tau^y, \quad G_{\pm \pm \pm} = \mp i\tau^z \mp (25a)$$

$$B : \quad G_{\pm 1} = \mp i\tau^y, \quad G_{\pm \pm} = \mp i\tau^x, \quad G_{\pm \pm \pm} = \mp i\tau^z \mp (25b)$$

$$C : \quad G_{\pm 1} = \mp i\tau^x, \quad G_{\pm \pm} = \mp i\tau^y, \quad G_{\pm \pm \pm} = \mp i\tau^z. \mp (25c)$$

Each class is generated by group conjugacy on its representative, $G_q \rightarrow G \cdot G_q \cdot G^{-1}$ for $G \in U(1) \times Z_2$. Therefore by site-dependent gauge transformations all $G_q(r)$ can be reduced to one of those in Eqs. 25a-25c.

These $Q_8$ IGGs will constrain allowed ansatzes. For the onsite terms, the $Q_8$ IGGs demand $\mu_r' = 0$ but put no constraint on $\eta_r$. On the converse, however, $\mu_r = 0$ and $\eta_r$ do not reduce the $U(1) \times Z_2$ freedom.

Consider a bond $rr'$ with nonzero $A_{rr'}$. The $Q_8$ IGGs demand $A_{rr'} = G_q(r) \cdot A_{rr'} \cdot G_q(r')^T$ for all $q \in Q_8$. There are three possibilities for $A_{rr'}$ depending on the representation choice combinations $(rr')$, $A_{rr'} \propto \{ \tau^x, (rr') = (AA),(BB),(CC), \}$

$$U, \quad (rr') = (AB),(BC),(CA), \quad U^T, \quad (rr') = (AC),(BA),(CB), \quad \text{where the SU(2) matrix} \ U = (1 - i\tau^x - i\tau^y + i\tau^z)/2 \ \text{will appear frequently. Consider the converse problem, namely whether} A_{rr'} \ \text{can "Higgs" the gauge freedom to} Q_8, \quad A_{rr'} \propto \tau^y \ \text{will not do this job, because all} \ G(r) = G(r') \in U(1) \times Z_2 \ \text{will keep} \ A_{rr'} \propto \tau^y \ \text{invariant. The other two possibilities} A_{rr'} = U \ \text{or} U^T \ \text{will reduce the gauge freedom to} Q_8 \ \text{with the representation choices given above.}$

Therefore consider $G(r') = \cos \phi \cdot \mathbf{1} + \sin \phi \cdot i\tau^z \ \text{and} \ A_{rr'} = U \ \text{the constraint solves for} \ G(r) = \cos \phi \cdot \mathbf{1} + \sin \phi \cdot i\tau^y \ \text{which can be a member of the} U(1) \times Z_2 \ \text{group only if} \ \phi \ \text{is a integral multiple of} \ \pi/2, \ \text{restricting} \ G(r) \ \text{and} \ G(r') \ \text{to be members of} \ Q_8 \ \text{representations.}$

A. Mean-field Theory on Triangular Lattice

With the above general considerations a mean-field Hamiltonian of quaternion spin liquid can be constructed on the triangular lattice. Due to the three-sublattice structure it is natural to assign the three $Q_8$ representations to the three corresponding sublattices. In this paper only the nearest-neighbor ansatz will be considered, with $A_{rr'} = U \ \text{or} U^T$ as shown in Fig. 1. Note that by the variational interpretation the overall scale of $A_{rr'}$ does not matter, and the overall complex phase of $A_{rr'}$ can be removed by a global orbital-independent $U(1)$ phase rotation of bosons. Translation symmetry further requires uniform $\mu_r = \mu$ and $\eta_r = \eta$.

The mean-field Hamiltonian reads (up to a constant),

$$H_{MF} = \sum_r \left[ \mu_r n_r - (\eta_r b_{r \cdot}^\dagger \cdot \tau^y \sigma_y \cdot b^\dagger_{r \cdot} + h.c.) \right. \left. - \sum_{d=1}^3 (b_{r d}^\dagger \cdot U \cdot i\tau^y \cdot b_{r + e_d} + h.c.) \right]. \quad \text{(27)}$$

Physical symmetries are generated by two translations $T_{1,2}$ along $e_{1,2}$, two reflections $\sigma_a$ and $\sigma_d$ (see Fig. 1), and time-reversal $T$. $T_{1,2}$ and $\sigma_a$ are trivial. $\sigma_d$ reverses all bond orientations in Fig. 1. $T$ changes the ansatz to their complex conjugate. Corresponding PSG elements are,

$$T_{1,2} : \quad b_r \rightarrow b_{r \cdot + e_{1,2}}, \quad (28a)$$

$$\sigma_a : \quad b_r \rightarrow b_{\sigma_a(r)}, \quad (28b)$$

$$\sigma_d : \quad b_r \rightarrow \sqrt{1/2}(i\tau^x - i\tau^y) \cdot b_{\sigma_d(r)}, \quad (28c)$$

$$T : \quad b_r \rightarrow (\tau^y \sigma_y \cdot b_r). \quad (28d)$$
The \( Q_8 \) IGG is defined in Eqs. \( 25a, 25c \) for \( A, B, C \) sublattices respectively. Time-reversal symmetry restricts \( \eta \) to be real. With this PSG constructed the mean-field Hamiltonian describes a “symmetric spin liquid”\(^3\) with no broken symmetry.

The mean-field Hamiltonian can be solved in the same way as the Schwinger boson mean-field theories.\(^{31} \)

Do the Fourier transform \( b_k = N_{\text{site}}^{-1/2} \sum_r e^{-i\kappa r} b_r \)
where \( N_{\text{site}} \) is the number of sites, and define \( \Psi_k = (b_{k,1\uparrow}, b_{k,2\uparrow}, b_{k,1\downarrow}, b_{k,2\downarrow})^T \). Eq. (27) becomes

\[
H_{\text{MF}} = \sum_k \left[ \Psi_k \cdot \left( \mu \mathbb{1} - P_k \right) \cdot \Psi_k - 2\mu \right],
\]

where \( P_k = i(1 - i\tau_x + i\tau_z)\xi_k - i\tau_y(\Re\xi_k + \eta) \), \( \Re \) and \( \Im \) are real and imaginary parts, \( \xi_k = e^{ik\cdot e_1} + e^{ik\cdot e_2} + e^{ik\cdot e_3} \).

Do a singular value decomposition

\[
P_k = U_k \cdot \left( \rho_1(k) 0 0 0 \right) \cdot V_k^\dagger
\]

with \( U(2) \) matrices \( U_k, V_k \) given by

\[
U_k = \begin{pmatrix} \sqrt{1+\mu^2} / \sqrt{1+\sqrt{3}} & \sqrt{1-\mu^2} / \sqrt{1+\sqrt{3}} \\ \sqrt{1-\mu^2} / \sqrt{1+\sqrt{3}} & -\sqrt{1+\mu^2} / \sqrt{1+\sqrt{3}} \end{pmatrix},
\]

\[
V_k = \begin{pmatrix} e^{i\pi/4} \sqrt{1+\sqrt{3}} & -e^{-i\pi/4} \sqrt{1+\sqrt{3}} \\ e^{i\pi/4} \sqrt{1-\sqrt{3}} & e^{-i\pi/4} \sqrt{1-\sqrt{3}} \end{pmatrix},
\]

and real singular values \( \rho_{1,2}(k) = \sqrt{3}\Im\xi_k \pm (\Re\xi_k + \eta) \).

Define “Bogoliubov quasiparticles”

\[
\Phi_k = \begin{pmatrix} \gamma_{k,1\uparrow} \\ \gamma_{k,2\uparrow} \\ \gamma_{k,1\downarrow} \\ \gamma_{k,2\downarrow} \end{pmatrix} = \begin{pmatrix} C_1 & 0 & S_1 \\ 0 & C_2 & 0 \\ S_1 & 0 & C_1 \\ 0 & S_2 & C_2 \end{pmatrix} \begin{pmatrix} U_k \dagger \\ 0_{2\times2} \\ V_k \dagger \end{pmatrix} \Psi_k,
\]

where \( C_{1,2} = \sqrt{1 + \mu/E_{1,2}(k)} / \sqrt{2} \) and \( S_{1,2} = -\rho_{1,2}(k) / E_{1,2}(k) / 2C_{1,2} \), with the mean-field dispersions

\[
E_{1,2}(k) = \sqrt{\mu^2 - \rho_{1,2}^2(k)}.
\]

Eq. (29) is diagonalized by this \( SU(2,2) \) Bogoliubov transformation,

\[
H_{\text{MF}} = \sum_k \left[ E_1(k) (\gamma_{k,1\uparrow} \gamma_{k,1\uparrow}^\dagger + \gamma_{k,1\downarrow} \gamma_{k,1\downarrow}^\dagger) \\ + E_2(k) (\gamma_{k,2\uparrow} \gamma_{k,2\uparrow}^\dagger + \gamma_{k,2\downarrow} \gamma_{k,2\downarrow}^\dagger) - 2\mu \right].
\]

The mean-field ground state energy per site is

\[
E_{\text{MF}} = N_{\text{site}}^{-1} \sum_k [E_1(k) + E_2(k) - 2\mu].
\]
where $w_{1,2}$ are complex coefficients, and constant $c = \sqrt{2 + \sqrt{3}}$. Note the eigenvectors at $\pm K$ (two rows in the first matrix) form a time-reversal pair. Define

$$
\begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix} = \begin{pmatrix}
  e^{2i\pi/3} & e^{i\pi/12}c \\
  e^{-i\pi/12}c & e^{-2i\pi/3}
\end{pmatrix} \begin{pmatrix}
  w_1 \\
  w_2
\end{pmatrix} \quad (42)
$$

and $SU(2)$ rotor field $\mathcal{Z}$

$$
\mathcal{Z} = \begin{pmatrix}
  z_1 & z_2 \\
  -z_2^* & z_1^*
\end{pmatrix} \quad (43)
$$

The boson condensates $\langle \hat{B}_r \rangle$ ($\hat{B}$ is defined in Eq. (20)) on the three sublattices are

$$
\begin{align*}
\langle \hat{B}_A \rangle &= \mathcal{Z} \cdot (-i\mathbf{r}_y \cdot \mathbf{U})^T, \\
\langle \hat{B}_B \rangle &= \mathcal{Z} \cdot (-i\mathbf{r}_y \cdot \mathbf{U})^T (-i\mathbf{r}_y \cdot \mathbf{U})^T, \\
\langle \hat{B}_C \rangle &= \mathcal{Z}.
\end{align*} \quad (44a)$$

It is easy to see that the total dipole moment is $\langle \mathbf{S} \rangle = (1/2) \text{Tr}[\mathcal{Z}^\dagger \mathbf{S} \mathcal{Z} \mathbf{1}] = 0$. The nematic directors $\text{Tr}[\langle \hat{B}_i \rangle^\dagger \sigma \langle \hat{B}_j \rangle \tau]$ on $A, B, C$ sublattices are given respectively by,

$$
\begin{align*}
\mathbf{n}_A &= -\text{Tr}[\mathcal{Z}^\dagger \sigma \mathcal{Z} \mathbf{r}_y], \\
\mathbf{n}_B &= \text{Tr}[\mathcal{Z}^\dagger \sigma \mathcal{Z} \mathbf{r}_y], \\
\mathbf{n}_C &= \text{Tr}[\mathcal{Z}^\dagger \sigma \mathcal{Z} \mathbf{r}_y],
\end{align*} \quad (45)
$$

which are perpendicular to each other. Therefore this is the Tsunetsugu-Arikawa spin nematic state. For example with $\mathcal{Z} = (1 - i\mathbf{r}_z)/\sqrt{2}$ the three directors are $\hat{x}, \hat{y}, \hat{z}$ on $A, B, C$ sublattices respectively, which is the state depicted in Fig. 1. The left $SU(2)$ transformations of $\mathcal{Z}$ are spin rotations, the right $SU(2)$ are sublattice-dependent orbital rotations [the gauge field is only $U(1) \times U(1) \times \mathbb{Z}_2$]. The PSG transformation rules of $\mathcal{Z}$ can be derived from the PSG of lattice bosons Eqs. (28a, 28d),

$$
\begin{align*}
T_{1,2} : \mathcal{Z} &\rightarrow \mathcal{Z} \cdot \frac{1}{2}(-1 - i\mathbf{r}_x - i\mathbf{r}_y - i\mathbf{r}_z)^T, \\
\sigma_s : \mathcal{Z} &\rightarrow \mathcal{Z}, \\
\sigma_d : \mathcal{Z} &\rightarrow \mathcal{Z} \cdot \frac{1}{\sqrt{2}}(-i\mathbf{r}_y + i\mathbf{r}_z)^T, \\
\mathcal{T} : \mathcal{Z} &\rightarrow \mathcal{Z}^*, \\
i : \mathcal{Z} &\rightarrow \mathcal{Z} \cdot (-i\mathbf{r}_z)^T, \\
\mathbf{j} : \mathcal{Z} &\rightarrow \mathcal{Z} \cdot (-i\mathbf{r}_x)^T, \\
k : \mathcal{Z} &\rightarrow \mathcal{Z} \cdot (-i\mathbf{r}_y)^T, \quad \text{(46a-e)}
\end{align*}
$$

The symmetry allowed form of the low energy action would be ($D$ is covariant derivative),

$$
\int d^3x \left\{ \text{Tr}[(D_\nu \mathcal{Z}^\dagger)(D_\nu \mathcal{Z})] + m^2 \text{Tr}[\mathcal{Z}^\dagger \mathcal{Z}] \\
+ u \text{Tr}[\mathcal{Z}^\dagger \mathcal{Z}]^2 + \ldots \right\},
$$

with $SO(4)$ symmetry. Other aspects of the field theory can be found in Refs. 28, 29 and will not be repeated here.

\section{C. Prototype Wavefunction in Quantum Limit}

The “quantum limit” is achieved by relaxing the total boson density constraint Eq. (38) and going to the low density limit (with $\langle n \rangle \ll 1$ and $\mu \gg 1$). The mean-field constraint equations Eqs. (39-40) can be solved in power series of $\mu^{-1}$,

$$
\langle n \rangle_{MF} = 6\mu^{-2} + \frac{135}{2}\mu^{-4} + O(\mu^{-6}),
$$

$$
\eta = 6\mu^{-2} + 81\mu^{-4} + O(\mu^{-6}).
$$

By inverting the first equation every quantity can also be expressed in terms of $\langle n \rangle_{MF}$.

The mean-field bond amplitudes $f_{rr'}$ in Eq. (35) can also be expanded in power series of $\mu^{-1}$ and will decay exponentially as $\mu^{-|r-r'|}$ with respect to the distance $|r-r'|$. For example, bond amplitudes $f_{rr'}$ on the nearest- and second- and third-neighbor bonds are given by

$$
\begin{align*}
f_{r, r+e_1} &= \mathcal{U} \cdot \frac{\mu^{-1}}{4} + \frac{15\mu^{-3}}{16} + O(\mu^{-5}), \\
f_{r, r+e_2 - e_3} &= i\mathbf{r}_y \cdot \frac{3\mu^{-3}}{4} + O(\mu^{-5}), \\
f_{r, r+2e_1} &= -\mathcal{U}^T \cdot \frac{3\mu^{-3}}{8} + O(\mu^{-5}),
\end{align*} \quad (49a-c)
$$

and those related trivially by cyclic permutations of $e_{1,2,3}$ (three-fold rotations), and by $f_{r, r'} = -f_{r', r}^T$.

The wavefunction in the extreme quantum limit $\mu \rightarrow \infty$ simplifies to the extreme “short-range resonating valence bonds” state, with nonzero amplitudes only on nearest-neighbor bonds. The overall factor of $f_{rr'}$ does

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Mean-field dispersions Eq. (44) along high symmetry lines at the critical point $\eta_c \approx 0.5287$ and $\mu_c = 6 - \eta_c \approx 5.4713$. Inset shows the Brillouin zone and high symmetry points and lines. Note that at $K$ point there is a critical mode ($\mathcal{Z}$ in text) and a gapped mode ($\mathcal{W}$ in text).}
\end{figure}
\end{center}
The loop weight $W_l$ in Eq. (18) becomes

$$W_l = N_L \cdot \text{Tr}[(−iπ^x)A_{r_i r_{i+1}} \cdots (−iπ^x)A_{r_L r_1}], \quad \text{(51)}$$

with nearest-neighbor bonds $< r_1 r_2 >, \ldots, < r_L r_1 >$, and $(3/4)L/2$ factor ignored. The matrix product inside the trace symbol is a $SU(2)$ matrix because every factor belongs to $SU(2)$, so the trace must be real.

In fact the trace can only take three values, ±2 or 0. The proof is the following. Denote the number of bonds with orientation along the loop direction ($\chi_{r_{i+1} r_i} = U$) by $N_L$, and the number of those opposite to the loop direction ($\chi_{r_i r_{i+1}} = (−U^T)$) by $N_\perp$. Due to the three-sublattice structure $N_\perp + 2N_\parallel \equiv 0 \mod 3$. This can be formally proved by assigning $Z_3$ numbers 0, 1, 2 to $A, B, C$ sublattices respectively, and noting that travelling along (or against) a bond increase this number by unity (or two) modulo three. Use the identity $−UU^T = (−iπ^x)\bar{U}$ to replace the $N_\perp$ factors of $U^T$ by $2N_\perp$ factors of $\bar{U}$, the matrix product becomes $qU \bar{q} U \bar{q} \cdots qU \bar{q} U \bar{q} U \bar{q}$, where the $q$s belong to the quaternion group $\{±1, ±iπ^x, ±iπ^y, ±iπ^z\}$. Use the commutation relations, $UU(±iπ^x) = (±iπ^x)U$, $U(±iπ^y) = (±iπ^y)U$, and $U(±iπ^z) = (±iπ^z)U$, to move all the $q_i$ factors in front of all $U$ factors, the matrix product becomes $q \cdot U^{N_\parallel + 2N_\perp}$ where $q$ is some quaternion group element. Finally use $U^3 = −1$ and the fact that $N_\parallel + N_\perp \equiv 0 \mod 3$, the trace becomes $\text{Tr}[±q]$ which can only be 0 (if $q$ is not ±1) or ±2.

The AKLT normalization factor $N_L$ approaches unity in the long length $L \to \infty$ limit, so may be omitted without changing the long distance behavior especially the topological order. $W_l$ for several short loops are presented in Fig. 3. Interestingly the weight of “double-bond” singlet vanishes, so the wavefunction is made purely by AKLT loops of length $L \geq 3$.

The prototype wavefunctions for gauge flux excitations and nontrivial topological sectors can be constructed in this quantum limit by the standard method (see e.g. Refs. 28, 29). Gauge flux on a length-even loop ($r_1 \ldots r_{2L}$) can be defined as $A_{r_1 r_2}(−A^*_r r_{2L−1}) \cdots A_{r_{2L−1} r_1}$. In the prototype wavefunction, the flux in each rhombus (unit cell) is ±. For a $6n \times 6n$ lattice ($m, n$ are integers) with periodic boundary condition the fluxes on the noncontractible (NC) loops are also ±. Creation operators of a pair of local fluxes are defined on the string on dual lattice connecting them. Creation operators of flux in a NC loop are defined on a NC loop of the dual lattice traversing it. Creation of gauge flux of class $q \in Q_S$ amounts to $A_{r_{i+1} r_i} \to G_q(r) \cdot A_{r_{i+1} r_i}$ for all bonds $r'r'$ cut by the string or NC loop on dual lattice. Examples on a $6 \times 6$ lattice are shown in Fig. 4. Two fluxes on NC loops along $e_d$ ($d = 1, 2$) direction can be explicitly defined as $q_d = A_{0,5e_d}(−A^*_{3e_d} 4e_d)A_{4e_d, 3e_d}(−A^*_{4e_d} 2e_d)A_{2e_d, e_d}(−A^*_{e_d} 0)$. The 22 topological sectors are given by the conjugacy classes of the pair $(q_{1, q_2})$ with the condition $q_1^{-1} q_2^{-1} q_2 q_1 = 1$, and are explicitly $±\{1, ±1\}$, $±\{1, ±i\}$, $(−1, −1)$, $(−i, 1)$, $(i, 1)$, $(i, ±i)$, $iσ^x$, $iσ^y$, $iσ^z$, $−iσ^x$, $−iσ^y$, $−iσ^z$ with $a = x, y, z$.

D. A Different Perspective

Previous discussions are based on uniform $\mathcal{N}_r = \frac{3}{4}$ in constraint Eq. (5). A different perspective by allowing non-uniform $\mathcal{N}_r$ will dramatically simplify the picture.
and results. Do a sublattice-dependent orbital rotation
\[ b_A \rightarrow (U^T \cdot i \tau^y)^{-1} \cdot b_A, \quad b_B \rightarrow (U^T \cdot i \tau^y)^{-2} \cdot b_B, \quad b_C \rightarrow b_C. \]  
(52)
The constraints Eq. \ref{eq:N} become
\[ \mathcal{N}_A = \hat{y}, \quad \mathcal{N}_B = \hat{x}, \quad \mathcal{N}_C = \hat{z}. \]  
(53)
The mean-field Hamiltonian Eq. \ref{eq:HMF} becomes
\[ H_{\text{MF}} = \sum_i \left[ \mu n_{\tau} - (\eta b_{\tau}^i \cdot y^y b_{\tau}^i + h.c.) \right. \\
- \left. 3 \sum_{d=1}^3 (b_{\tau}^i \cdot i \tau^y \otimes i \sigma^y \cdot b_{\tau}^{i+e_d} + h.c.) \right], \]  
(54)
and is a spin-orbital singlet. In this gauge choice the quaternion spin liquid state looks like “Tsunetsugu-Arikawa orbital nematic state”, with \( T^y = 0, T^x = 0, T^z = 0 \) on sublattices \( A, B, C \) respectively. This “orbital order” reduces the orbital \( SU(2) \) gauge freedom to quaternion group. However it does not break any physical symmetry in the spin liquid phase. The three-sublattice structure becomes physical only upon a spin-orbital-entangled condensation of \( Z \) at \( k = 0 \), with \( \langle \hat{B}_r \rangle = Z \). The PSG of bosons under this gauge is similar to the PSG of \( Z \). In this picture it is clear that the low energy theory Eq. \ref{eq:HMF} contains the coupling of spinon \( Z \) to the PSG of \( Z \).

III. DISCUSSIONS

In Ref. \cite{28} it was argued that no gauge invariant bilinears of the low energy spinon field can be constructed to carry spin-1 quantum number. Therefore it was suggested the standard projective construction by rewriting spin operators into spinon bilinears cannot describe the quaternion spin liquid. The argument is indeed true here, and there is no gauge invariant spin-1 bilinears of the low energy field \( Z \). However there is a high energy branch of spinons \( W \) (see Fig. \ref{fig:2}) which remains gapped across the spin nematic ordering transition [see Eq. \ref{eq:Zwf} and related discussions]. They together can make gauge invariant spin-1 bilinears \( \langle \hat{W}[Z^i] \sigma W \rangle + c.c. \sim S \) Eq. \ref{eq:Zwf}). This situation was overlooked in the analysis of the low energy theory in Ref. \cite{28}. So there is no real contradiction.

An important issue is what spin-1 Hamiltonian may favor this quaternion spin liquid as the ground state. In a numerical study of nearest-neighbor bilinear-biquadratic Heisenberg Hamiltonian on triangular lattice \cite{32},

\[ H = \sum_{\langle ij \rangle} J S_i \cdot S_j + K(S_i \cdot S_j)^2, \]  
(58)
it was found that a three-sublattice “antiferroquadrupolar” state, same as the Tsunetsugu-Arikawa proposal \cite{23}, is the ground state if \( K > J > 0 \). This may serve as the starting point. Farther neighbor and multiple-spin interactions can then be added to destroy the long-range order, the sign of these terms may be hinted by looking at the loop-products of spinon pairings related to the gauge invariant flux \cite{44}. For example the term \(-\langle b_i \cdot A_{ij} \otimes i \sigma^y \cdot b_j \rangle \langle b_{ij}^* \cdot A_{ik} \otimes i \sigma^y \cdot b_{kj}^* \rangle \langle b_{jk} \cdot A_{kl} \otimes i \sigma^y \cdot b_{lj} \rangle \), defined on a rhombus \( ij \) that may favor the quaternion spin liquid. After projection to the physical spin-1 space it contains a term \(-\langle S_i \cdot S_j \rangle \langle S_j \cdot S_k \rangle \langle S_k \cdot S_l \rangle = (S_i \cdot S_j) \langle S_j \cdot S_k \rangle - (1/2) \langle S_l + S_j + S_j + S_j \rangle^2 \) similar to the 4-site ring exchange of spin-1/2, but with opposite sign to that would naturally arise in a Hubbard model \cite{24}.

Several possible extensions of the current work exist. The two-orbital formalism may also be used to describe spin liquids in proximity to other spin nematic or dipole orders. One interesting example would be the \( Z_4 \) spin liquid proposed in Ref. \cite{28}. A complete classification of PSG in this formalism may be a useful guide along this direction.

The projected spin-1/2 Schwoinger boson wavefunctions have been numerically studied on small lattices by brute-force evaluation of permanents \cite{42, 28}. Generalization to the current case is likely very hard, because the overlap between a \( S^z \) basis state and the projected wavefunction is not a single but many \( (2^{N_{\text{spin}}} \) permanents.

The two-orbital AKLT representation can be directly generalized to higher spin systems. Spin-\( S \) can be represented by \( 2S \) orbitals of spin-1/2 Schwoinger bosons, with \( 2S \) single occupancy constraints generalizing Eq. \ref{eq:2} on each orbital and \( S(2S-1) \) symmetrization constraints generalizing Eq. \ref{eq:7} between each pair of two orbitals.
The gauge freedom is $[U(1)]^{2S} \times S_{2S}$ where $S_{2S}$ is the symmetric group of degree $2S$. This formalism can describe higher degree multipole orders by boson condensation, and spin liquids with even richer gauge structures (thus richer topological orders) may be obtained via the projective construction.

Multiple-orbital fermionic representation has been considered for general $SU(N)$ spins, and used in the context of alkaline-earth cold atom systems. Large-$N$ generalization of the multiple-orbital bosonic representation may also be useful in theoretical studies. More recently the two-orbital fermionic representation of spin-1 was employed in hope of describing the experimental evidence of gapless spin liquid in $\text{Ba}_3\text{NiSb}_2\text{O}_{8.58}$.

The prototype wavefunction defined by Eqs. (17-18) may be of some interest by itself. It is not clear how to check directly the quaternion structure without reference to the mean-field theory. It is also possible that confinement happens due to the projection of the mean-field wavefunction. The confined phase may have nontrivial quantum numbers of the space group. More insight on the amplitude (matrix trace) are much needed for these purposes. And it will be very interesting if the matrix trace form of the loop amplitudes Eq. (18) can be used to represent other nontrivial phases.

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