First passage time and statistical thermodynamics

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Abstract. The first-passage time is proposed as an independent thermodynamic parameter of the statistical distribution that generalizes the Gibbs distribution. The theory does not include the determination of the first passage statistics itself. A random process is set that describes a physical phenomenon. The first passage statistics is determined from this random process. The thermodynamic parameter conjugated to the first-passage time is the same as the Laplace transform parameter of the first-passage time distribution in the partition function. The corresponding partition function is divided into multipliers, one of which is associated with the equilibrium parameters, and the second one - with the parameters of the first-passage time distribution. The thermodynamic parameter conjugated to the first-passage time can be expressed in terms of the deviation of the entropy from the equilibrium value. Thus, all the moments of the distribution of the first passage time are expressed in terms of the deviation of the entropy from its equilibrium value and the external forces acting on the system. By changing the thermodynamic forces, it is possible to change the first passage time. It is shown that the statistical distribution containing the thermodynamic parameter of the first-passage time corresponds to the Zubarev’s non-equilibrium statistical operator.

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1. Introduction

The first-passage time (FPT) is the time during which a stochastic process reaches a certain threshold for the first time. FPT distributions have found numerous applications in physics, biology and finance science [1–3]. Examples of the use of the FPT parameter include frequency-tuning systems [4], problems of crossing a potential barrier, decay of an unstable state [5], conformational changes of proteins, ecological systems, epidemics, diffusion-controlled chemical reaction, autocatalytic reactions, number of individuals in a population, dissociation of a diatomic molecule, potential of a neuron, transition through a potential barrier [6]. In [7] the FPT, the moment of the first reach of a certain threshold by a random process, was used to study the effects of noise on electronic relays, and for other problems of radio engineering. In [8, 9] the possibilities of applying distributions of FPT (referred to as "lifetime" in [7]) to the description of superstatistics and such phenomena as the behavior of systems with multiplicative noise, Van der Pol-Duffing systems, turbulence, Van der Pol generators are noted. The FPT distribution has been studied for a variety of diffusive processes, ranging from ordinary diffusion (Brownian motion) to continuous-time random walks (CTRWs) [10–12], and in many other applications [1–3]. First-passage-time distributions were studied in the context of thermodynamics as well [13, 14–19].

A large variety of problems ranging from noise in vacuum tubes, chemical reactions and nucleation [19] to stochastic resonance [20], behavior of neurons [21], and risk management in finance [1] can be reduced to FPT problems.

A Gaussian approximation was used in [5] in the problem of transition through a potential barrier. For other processes, other expressions are possible for the probability density of
the first passage of a level. In [22], for the probability density of the first passage of the level in the case of Brownian motion, an inverse Gaussian distribution density was obtained depending on the boundary conditions and on the time dependence of the drift and diffusion coefficients. In [23], a generalization of the Eyring – Kramers transition rate formula for irreversible diffusion processes was obtained. In [24] the Weibull distribution was derived for the probability density of the first passage of a level. In [25], the escape probability was obtained for the Feller process. In [26], the limiting exponential distribution for this value was derived. In [27] the authors provided an exact unified framework for studying the comprehensive statistics of first passage time under detailed balance conditions. In [28] an exact determination of the first passage time distribution from the corresponding relaxation spectrum was performed. In [29] the behavior of smooth FPT densities with finite moments was explained.

In [30-33] FPT is determined for the diffusion processes as a time it takes for the fastest “diffusive searcher” to find a target from a large group of searchers. In [34] a framework is set for analyzing extreme value statistics of ergodic reversible Markov processes in confining potentials on the hand of the underlying relaxation eigen spectra. New trends in first-passage methods and overview of FPT problems can be found in [35].

It is of interest to study the possibilities of controls over FPT. In [36, 37, 38], the possibilities of the optimal control problem are considered. In [13] the steady state entropy production rate was estimated using mean first-passage times of suitable physical variables. In the general case, the impact on a physical system are described by the flows into the system from the environment and by the corresponding thermodynamic forces or by the deviation of the entropy of the system from its equilibrium value. This deviation consists of the internal production of entropy, expressed through flows and forces, and the flow of entropy into the system from the outside. The main goal of this work is to establish a relationship between the FPT distribution parameters and the deviation of the entropy of the system from its equilibrium value. Revealing such relations makes it possible to affect FPT by various external controls. The possibility of such controlling depends on the type of FPT distribution. For large times, the limiting distribution of FPT is exponential. Then any impacts (if they act over a sufficiently long period of time) can only decrease the mean FPT. For other FPT distributions, which are possible for shorter periods of time of impact on the system (for example, [39]), there are means to increase the mean FPT. The finite values of FPT are considered, and the distributions of these values should describe finite, and not only infinitely large values of the exposure time.

In this paper, FPT is considered as an independent thermodynamic variable [8, 9, 40, 41, 42]. Similar thermodynamic parameters of flows are considered in the extended irreversible thermodynamics (EIT) [43]. The distribution containing FPT includes the conjugate thermodynamic parameter γ. It is expressed through the deviation of entropy from the equilibrium value Δs. The same parameter γ appears in the FPT Laplace transform. Thus, the mean value and all further moments of FPT are expressed in terms of Δs. Distributions generalizing Gibbs distributions by introducing an additional parameter, generalized or canonical Gibbs reservoirs, which are thermodynamic reservoirs in a field of conservative external forces, are discussed in [44].

There are known relationships between the average time of crossing the barrier and the entropy of the transition in the reaction-rate theory and escape from metastable states (for example, [5, 28]). In [45] an expression was obtained for the transition time scale, which coincides with the expression from [40-41] for the average lifetime (in terms of [7]) and the exponential distribution of the lifetime after changing the sign of the entropy change. Here we consider a more general case.

The relationship between time and entropy has been known for a long time. In [16, 17, 46] FPTs are considered for a certain level of entropy production or for a given flow value. In [46] it is shown that the rate of dissipation $\dot{\gamma}$ in a nonequilibrium stationary process $X$ bounds the moments of FPT of dissipative currents $J$. In this paper we obtain the relation between the FPT, the value directly related to the passage of time, and thermodynamic quantities.
In this paper, an independent thermodynamic variable \( FPT \) is included in the statistical distribution \([8, 9, 42]\). In this case, \( FPT \) is considered not as a temporary variable, but as a quantity that depends on the phase coordinates of the system. Thermodynamics and first passage times both are important fields of study. However, there is a certain gap between them. First passage is a concept from the stochastic theory. The approach proposed in this paper allows us to bring these two important areas closer together, and such a rapprochement would mutually enrich them. Relationships are obtained between the movement, rates of the processes in the system and thermodynamic forces and flows in it. The assumption that \( FPT \) can be an independent thermodynamic parameter makes it possible to relate \( FPT \) to the change in entropy of the system and to combine many stochastic relations for \( FPT \) with the physical results of non-equilibrium and stochastic thermodynamics.

In the second section, the introduction of \( FPT \) as a thermodynamic parameter is substantiated. As an example distribution containing \( FPT \), considered the Zubarev’s non-equilibrium statistical operator (NSO). In the third section the generalizing Gibbs distribution containing the energy and \( FPT \) and their conjugate parameters is introduced. The fourth section deals with generalized thermodynamics of systems with an extra parameter of \( FPT \). In the four section, the entropy and its differentials are determined both for the distribution depending on the energy \( u \) and the \( FPT \) \( T_\gamma \), when the entropy is equal to \( s \) \((23)\), and for the distribution depending only on \( y=u \). The fifth section discusses the \( FPT \) of a certain flow level, \( FPT \) of level \( L \) in the drift-diffusion process and \( FPT \) for Feller process. Explicit dependences of the parameter \( \gamma \), which is the thermodynamically conjugated \( FPT \), through changes of entropy are obtained. The proposed theory does not allow to determine first passage statistics. General relations are obtained that are valid for arbitrary \( FPT \) distributions. The choice and use of specific \( FPT \) distributions depends on the physical situation under study. The choice of a particular distribution from a variety of existing distributions is discussed in section six. Also in the sixth section, an example is considered illustrating the provisions of Sections 3, 4 and based on the experiment. The seventh section contains some concluding remarks of the paper. Appendix A discusses the relationship of superstatistics with \( FPT \).

2. Non-equilibrium statistical operator and First-Passage Time

The initial values of the phase coordinates are unknown. Therefore, statistical ensembles are introduced. The probabilistic interpretation of dynamic processes is carried out. The parameters of the energy \( u \) and \( FPT \) \( T_\gamma \) are considered as random functions in the phase space. Thus, in distributions \((15)-(16)\), \( u \) and \( T_\gamma \) \((1)\) are random variables in the phase space, numerical functions that define a measurable mapping of the phase space \( z \) to the real line. In the reduced description, the kinetic equations for the distribution functions are derived from the Liouville equation \([63]\). The values \( u \) and \( T_\gamma \) are described by direct and inverse conjugate kinetic equations \([4]\).

In the introduction, the objectives of this work are noted. We will clarify and concretize them. The most important step taken in this paper is to include \( FPT \) in the number of thermodynamic variables. This opens up wide possibilities for studying \( FPT \) from the standpoint of statistical thermodynamics, searching for and studying numerous connections between stochastic and thermodynamic quantities and regularities. The introduction of \( FPT \) as a thermodynamic parameter is justified by the dependence of \( FPT \) on the phase variables \([51, 52]\). One more justification can be extended irreversible thermodynamics (\( EIT \)) \([43]\), where fluxes act as additional thermodynamic variables. Fluxes have a lot in common with \( FPT \). In some cases, the flow is inversely proportional to \( FPT \).

\( FPT \) was studied mainly from the point of view of stochastics, an important and necessary direction, to which \( FPT \)'s belong \([1-7, 13-18, 22-39, 45-48]\). This direction is necessary as a mandatory stage for further research, for example, the thermodynamic aspects of
In this paper, the relations of $FPT$ with statistical thermodynamics are established. The obtained results of stochastic studies $FPT$, for example, the Laplace transform of the distribution density $FPT$ are used.

The main goal of this paper is to explicitly relate a random $FPT$ process of a given level to the changes in the entropy of the system that occur during this process. In any real physical process modeled by some random mathematical process, the entropy changes that accompany the process are inevitable. They should be taken into account when correctly describing the $FPT$ process. During the $FPT$ process, the entropy of the system changes. These changes should be taken into account in the $FPT$ moments. $FPT$'s are a special case of stopping times [47, 48]. In this paper, a joint distribution (15), (16) for internal energy and $FPT$ is obtained. The introduced distribution containing $FPT$ as a thermodynamic parameter is compared with the non-equilibrium distribution, Zubarev’s NSO [49, 50]. The case is considered when the distribution of the random $FPT$ does not depend on the random value of the internal energy $u$, but depends on the average values of the internal energy. In the model used in this work and the approximations used in this case, the Laplace transform of the $FPT$ distribution appears, the argument of which is associated with the change in the entropy of the system using thermodynamic relations. Therefore, the value, for example, of the average $FPT$ value, in which the value of the argument of the Laplace transform of the $FPT$ distribution assumed zero, is just an auxiliary value that does not correspond to real events in which the changes in entropy and the values of this argument are not zero. It is necessary to set the value of the argument of the Laplace transform, which corresponds to the change in entropy in the real $FPT$ process of reaching the specified level.

The change in entropy, obtained using the introduced distributions, is expressed in terms of thermodynamic flows and conjugate thermodynamic forces. There is also an inverse relationship: the effect of entropy changes caused by the introduction or change of, for example, thermodynamic forces on the $FPT$. You can set and investigate the task of control of $FPT$. The system can be affected by various kinds of thermodynamic forces that cause thermodynamic flows and entropy changes that affect the $FPT$. Using the found relationship between the average $FPT$ and the change in entropy, the intervals of entropy change are determined, at which the average $FPT$ of reaching the level decreases or increases. Processes in the system slow down or speed up.

The dependence of $FPT$ on the phase coordinates is important, which makes it possible to determine $FPT$ as a thermodynamic parameter [51, 52]. Information about the dynamic properties of the system is also required. You need to know the Laplace transform of the $FPT$ distribution. This value is included in the non-equilibrium part of the partition function. In the proposed approach, the Laplace transform parameter of the $FPT$ distribution coincides with the thermodynamic parameter $\gamma$, the conjugate random thermodynamic parameter $FPT T_\gamma$. The factor $exp\{-T_\gamma\}$ from the statistical distribution (15) for energy and $FPT$ acts as the core of the integral Laplace transform. The relationship (23), (26) for the deviation of entropy from its equilibrium value includes such quantities as the product of the average $FPT \langle T_\gamma \rangle$ and its conjugate thermodynamic parameter $\gamma$, $\gamma \langle T_\gamma \rangle$, and the non-equilibrium part of the internal energy $u_\gamma$. To determine this value $u_\gamma$, it is necessary to know the dependences of the parameters included in the distribution for $FPT$ on the inverse temperature $\beta$. The average value of $FPT \langle T_\gamma \rangle$ depends on the parameter $\gamma$. From the equation for entropy (26), this parameter is determined, which is then substituted into the expressions for $\langle T_\gamma \rangle$ and in the partition function. The results obtained are of a general nature and are applicable, in principle, to arbitrary systems of any nature. The physical approach is based on a general mathematical definition of stopping times. This allows you to control the speed and time to achieve arbitrary goals. Section 5 provide examples of how entropy changes affect the average $FPT$. 

The task of studying the possibilities of influencing and managing the processes of the $FPT$ is relevant. The proposed article suggests solving this problem using the methods of the theory of random processes, non-equilibrium statistical physics and thermodynamics. The time of the first achievement is set as a thermodynamic parameter, a joint distribution is constructed for the energy of the system and the time of the first achievement, and an expression is written for the non-equilibrium entropy of the system. By changing the thermodynamic forces, you can change the entropy of the system and the time of the first achievement.

The system is open, and the parameters of the system under the impact of the environment are random variables that are characterized stochastically. In the present work, as in [8, 9, 41, 42, 43], an additional thermodynamic parameter is introduced to describe a non-equilibrium state.

$FPT$ is the time it takes for a stochastic process $X(t)$ to reach a certain threshold $a$ (1) for the first time. For example, it is the time for a stochastic process that describes a macroscopic parameter $X(t)$ to reach the zero level. The stochastic process $X(t)$ is the order parameter of the system, for example, its energy, number of particles, (in some papers, for example [46], entropy production is considered). The process $X(t)$ can describe many other physical quantities. $FPT$ is, by definition,

$$ T_\gamma = \inf\{t : X(t) = a\}, \quad X(0) = x > 0. \quad (1) $$

The subscript $\gamma$, emphasizing the dependence on the conjugate thermodynamic parameter, is used not to confuse the variable with the temperature $T$. There are other definitions of $FPT$. For example, $T_\gamma = \inf\{t > 0; X(t) \in A\}$, where $A$ is a Borel set on a number line, $T_\gamma$ is the moment of the first achievement of the set $A$ [47]. The moments of the first achievement of the $FPT$ refer to the stopping times and to the Markov moments [47, 48]. Each Markov moment is associated with a set of sets describing a set of events observed over a random time $T_\gamma$ [48]. Physically, this corresponds to a dependency on the system’s history. The events that occur during this time include the change in the entropy of the system. In this way, the system history is taken into account. $FPT$ (1) is a multiplicative functional from a random process $X(t)$ [47]. In distributions (15), (16), as in the NSO method, the dependence on the system’s past is important.

In [8, 9, 42] a distribution is introduced that contains $FPT$ (referred to as “lifetime”) as an additional thermodynamic parameter. The choice of $FPT$ as a thermodynamic parameter is possible from the mere definition of a thermodynamic parameter in statistical physics. For example, in [51] and [52] it is stated: "Any function $B(z)$ of dynamic variables $z=(q_1, ... , q_N, p_1, ... , p_N)$ of macroscopic nature is a random inner thermodynamic parameter". The fact that $FPT$ (1) depends on dynamic variables $z$ can be seen from the equations for the distribution of the $FPT$ (or lifetime [7]) in the Markov model [4, 53]. $FPT$ is considered in the statistical ensemble not as a time variable, but as a parameter that depends on the phase coordinates $z=(q_1, ... , q_N, p_1, ... , p_N)$.

The distributions for $FPT$ (1) are described by the Pontryagin equations [4], [53], which are conjugate to the kinetic equations (for example, the Fokker-Planck equations) for the energy distribution of the system, the main variable of statistical thermodynamics and equilibrium statistical physics. Since the Fokker-Planck equation contains functions that depend on phase variables, the same dependences hold for the conjugate equation. An illustration is the method of the NSO. $FPT$ is contained in the NSO method implicitly. In the method of NSO [49]:[50], [55-62] an additional macroscopic parameter in the description of the non-equilibrium system is the time $t-t_0$ elapsed from the birth of the system, the time of the past life, the time until the first crossing of the zero level in the inverse time, the age of the system [54]. In [54] it was shown that averaging over the initial time [50] corresponds to averaging over the distribution of the system lifetime $t-t_0$. We consider the classical case. However, as noted in [49], the term operator is used for both the classical and quantum cases. Below, the NSO method is associated with the distribution introduced in this paper.
The approach of Zubarev to the theory of non-equilibrium processes [49, 50, 55-62] based on a fundamental property of macroscopic systems, which is intimately related to phase space $z=(q(t), p(t))$ of a many-particle system. For time intervals that are not too small in the instability of the classical phase trajectories $Z(t)$, the details of the initial state become inimicant and the number of parameters necessary for the description of the state of the system is reduced.

In [57-62], the steps for constructing the NSO are highlighted. It consists in the basic step in the formalism consisting into introducing a separation of the total Hamiltonian into two parts, namely $\hat{H} = \hat{H}_0 + \hat{H}'$, where $\hat{H}_0$ is the so-called “relevant” (or secular, or quasi-conserving) part, composed by energy operators involving the kinetic energies and a part of the interactions, namely, those strong enough to be responsible for fast dissipative processes with very short relaxation times, meaning those smaller than the characteristic time scale to be used for the description of the system, essentially the resolution time in any given experiment under consideration. The other contribution, $\hat{H}'$ contains the remaining interaction potentials, associated to interactions responsible for processes with long-time relaxations times (meaning larger than the experimental characteristic time), in the dissipative processes that develop in the system.

Fundamental step is to decide on the basic set of dynamical variables appropriate for the description of the macroscopic state of the non-equilibrium system. In [56] it is noted that this problem has not yet been solved, but in [59] algorithms for selecting a set of relevant dynamic parameters are given (Zubarev-Peletminskii symmetry condition). At this point enters the fundamental Bogoliubov’s principle of correlation weakening, and the accompanying hierarchy of relaxation times [63]. According to this view, a series of successively contracted descriptions is possible because the existence, in many cases, of an array of relaxation times, say $\tau_0 << \tau_f << \tau_r << \tau_{eq}$, where $\tau_0$ is the collision time, $\tau_f$ is the time between two successive collisions, $\tau_r$ is the relaxation time, the time for the establishment of local equilibrium in a volume that is macroscopically small but still contains a large number of particles, $\tau_{eq}$ is the time necessary for the relaxation of the system to total equilibrium, such that after each one has elapsed, correlations with lifetimes smaller that each one of these time lengths are damped out (that is, the associated dissipative processes have died down) and can be ignored. Then, an ever shortened set of dynamical variables can be used for a proper description of the macrostate of the system; at a sufficiently long time the equilibrium condition is approached (with all correlations being wiped out), and the most contracted descriptions is to be used.

This leads to the second basic step in the formalism, namely, to introduce a basic set of dynamical variables, call them $\{\hat{P}_m\}$, $m=1,2,...$ (the triangular hat indicating Hermitian operator in Quantum Mechanics, or in Classical Mechanics a dynamical function defined over the phase space). This is done, in analogy with equilibrium, introducing those that are quasi-conserved (quasi-constants of motion) under the dynamics generated by the secular part of the Hamiltonian, namely, $\hat{H}_0$. It is noted in [49] that these dynamic variables do not have to be integrals of motion.

In [50] construct generalized Gibbs ensembles, which are closely related to the thermodynamic description of non-equilibrium systems, when the observable macroscopic quantities depend on time. These generalized Gibbs ensembles are also called the relevant ensembles or quasi-equilibrium ensembles. The relevant statistical distributions serve as auxiliary distributions to select spatial solutions of the Liouville equation that describe irreversible macroscopic processes [57-62].

The non-equilibrium macroscopic state is specified by the set of observables which are the average values $\langle \hat{P}_n \rangle'$ of some relevant dynamical variables $\hat{P}_n$. These variables give the reduced description of the system on the chosen time scale. The existence of different time scales is due to the “hierarchy” of basic relaxation times in macroscopic systems.
Using the Lagrange multipliers method [49], [50] (or other approaches, such as mathematical statistics, section 3), it follows that
\[ \rho_{rel}(t,0) = \exp[-\Phi(t) - \sum_{m} F_{m}(t)\hat{P}_{m}], \]
which will be called the relevant distribution. Here a set of variables \{F_{m}(t)\} play the role of variables thermodynamically conjugated to the macrovariables \( \hat{P}_{m} \). The first \( t \) in the argument of \( \rho_{rel}(t,0) \) stands for the time dependence of the basic thermodynamic variables, and the zero indicates that the dynamical operators are taken in Schrödinger representation,
\[ \rho_{rel}(t',t-t) = \exp[-i(t'-t)L]\rho_{rel}(t',0). \]
Here \( L \) is Liouville operator; \( i\mathcal{L} = [H,\rho] = \sum_{k}\left[ \frac{\partial H}{\partial q_{k}} \frac{\partial \rho}{\partial q_{k}} - \frac{\partial H}{\partial p_{k}} \frac{\partial \rho}{\partial p_{k}} \right], \) \( H \) is Hamilton function, \( p_{k} \) and \( q_{k} \) are pulses and coordinates of particles; \([\ldots]\) is Poisson bracket.

The Massieu-Planck function \( \Phi(t) \) is determined from the normalization condition for the relevant distribution and has the form
\[ \Phi(t) = \ln \text{Tr} (\exp[-\sum_{m} F_{m}(t)\hat{P}_{m}]) \]

The Lagrange multipliers \( F_{m}(t) \) are determined from the self-consistency conditions
\[ \left\langle \hat{P}_{m}' \right\rangle = \left\langle \hat{P}_{m} \right\rangle_{rel} = \text{Tr}(\hat{P}_{m}\rho_{rel}(t,0)), \quad \text{Tr}\rho_{rel}(t,0) = 1. \]

The considerations are concerned with classical and quantum systems if the dynamical variables thermodynamically conjugated to the macrovariables \( \hat{P}_{m} \) are taken in Schrödinger representation,
\[ \rho_{rel} = \exp[-\Phi(t) - \sum_{m} F_{m}(t)\hat{P}_{m}], \]
where \( \Phi(t) \) is determined from
\[ \text{Tr} (\rho(t') = \text{Tr} (\rho(t) \ln \rho(t)) \]
under the constraints
\[ \text{Tr} (\rho(t')) = 1, \quad Q_{m}(r,t) = \text{Tr}(\hat{P}_{m}\rho(t')) \]
for \( t_{0} \leq t' \leq t \), that is, keeping the information on the history of evolution of the system from the initial condition of preparation at time \( t_{0} \) up to the time when the measurement is performed. The result is
\[ \rho(t) = \exp[-\Psi(t) - \sum_{j=1}^{n} \int_{t_{0}}^{t} dt' \int d^{3}r \phi_{j}(r,t,t')\hat{P}_{j}(r)], \]
where
\[ \Psi(t) = \ln \text{Tr} (\exp[-\sum_{j=1}^{n} \int_{t_{0}}^{t} dt' \int d^{3}r \phi_{j}(r,t,t')\hat{P}_{j}(r)]) \]
ensures the normalization of \( \rho(t) \). The Lagrange multipliers \( \phi_{j}(r,t,t') \) are determined by the conditions of Eq. (3), which are redefined in the form
\[ \phi_{j}(r,t,t') = w(t,t')F_{j}(r,t'), \]
introducing the weight function \( w \) in the integration in time [58, 62].

In [49,59] the limit transition \( t_{0} \rightarrow -\infty, t_{0} \rightarrow -\infty, \) is made, after taking the thermodynamic limit. Before this transition is made, the functions in (4) depend on the value \( u = t-t_{0} \), which in [54] is interpreted as the random lifetime of the system, \( \text{FPT} \). The functions \( w(t,t') \) are interpreted in [54] as the densities of the distribution of this lifetime.
It is possible to assume that the system evolves as isolated from the state \( \rho_{\text{rel}}(t_0,0) \) making random transitions with exponent probability \( w(t,t_0) = (1/\langle M \rangle) \exp[-(t-t_0)/\langle M \rangle] \) (where \( \langle M \rangle = 1/e \to \infty \) after \( V \to \infty \)) and to interpret it as influence of a "thermostat" [55]. In [59] functions \( w(t,t_0) \) are considered in a general view. The properties of these "weight functions" [59] are investigated. In [64] \( W(t,t') = 1-(t-t')/\tau \); \( w(t,t') = dW(t,t')/dt' = 1/\tau \); \( t=t_0+\tau \). The value \( \tau \) goes to infinity after the calculations of averages. This function is interpreted as function of the coarse-graining in time [65] (below this function is treated as density of uniform distribution of lifetime). Other (very many) choices of the weight function \( w \) are possible [59].

Zubarev’s approach follows from the choice of the weight function \( w(t,t') \) of Eq. (5) in the form of Abel’s kernel \( \exp\{\varepsilon(t'-t)\} \) and \( t_0 \) taken in the remote past; \( t_0 \) is the initial time of preparation. Hence,

\[
\rho_{\varepsilon}(t) = \exp\{-\varepsilon \int_{-\infty}^{t} dt' \exp\{\varepsilon(t'-t)\} \hat{S}(t',t'-t) \},
\]

which after part integration takes the form

\[
\rho_{\varepsilon}(t) = \exp\{-\hat{S}(t,0) + \hat{\varphi}_{\varepsilon}(t)\}, \quad \hat{\varphi}_{\varepsilon}(t) = \int_{-\infty}^{t} dt' \exp\{\varepsilon(t'-t)\} \frac{d}{dt} \hat{S}(t',t'-t),
\]

where \( \rho_{\text{rel}}(t,0) = \exp\{-\hat{S}(t,0)\} \), \( \hat{S}(t,0) = \Phi(t) \hat{I} + \sum_{n=1}^{\infty} \int d^3r F_n(r,t) \hat{P}_n(r). \)

The entropy production operator [48, 49] is equal

\[
\frac{d \ln \rho_{\text{rel}}(t-y,-y)}{dy} = -\sigma = -\frac{d\hat{S}(t-y,-y)}{dy}; \quad \frac{d\hat{S}(t)}{dt} = \sum_{m} \frac{\partial F_m(t)}{\partial t} \left( \hat{P}_m - \langle \hat{P}_m \rangle \right) + \sum_{m} F_m(t) \hat{P}_m.
\]

The expression for entropy production \( \sigma \) for the local-equilibrium distribution, a special case of the relevant distribution (2), is written in terms of flows in [49].

From the complete group of solutions of Liouville equation (symmetric in time) the subset of retarded "unilateral" in time solutions is selected by means of introducing a source in the Liouville equation (for Zubarev’s \( w(t,t_0) \))

\[
\frac{\partial \rho}{\partial t} + iL \rho(t) = -\varepsilon(\rho(t) - \rho_{\text{rel}}(t,0)),
\]

which tends to zero (value \( \varepsilon \to 0 \)) after thermodynamic limiting transition.

The state \( \rho(t) \) observable in the moment \( t \) is equal to an average on the initial moments of time \( t_0 \) from the solution of an initial value problem for the statistical operator (or function of distribution) \( \exp[-iL(t-t_0)]\rho_{\text{rel}}(t_0,0) \) for an enough large interval of time \( t-t_0 \), necessary for damping the initial nonphysical states.

In [54] other interpretation of functions \( w(t,t') \) (5) and operation (6)-(7) is given. Having done in (6)-(7) the replacement \( t'=t_0, t_1=t_0+t \), rewrite for \( \exp\{-\varepsilon(t-t')\} \to w(t-t') \) (6)-(7) by \( t_0 \to -\infty \), as

\[
\ln \rho(t) = \int_{0}^{\infty} w(y) \ln \rho_{\text{rel}}(t-y,-y) dy = \ln \rho_{\text{rel}}(t,0) - \int_{0}^{\infty} \left( \int w(y) \frac{d \ln \rho_{\text{rel}}(t-y,-y)}{dy} \right) dy,
\]

where in the second equality (9) arguments \( t_1 \) is replaced by \( -y \). Thus \( t_0 \to -y; t_1=t_0+y \), i.e. the current time \( t \) is represented as the sum of the initial moment \( t_0 \) and value \( y \), which represents lifetime of system (random value), (FPT). Note that in the NSO notation in the form (4) there is a random variable \( u=y=t-t_0 \), on which the density of the distribution function depends before the transition to the thermodynamic limit is made. The Liouville operator (in a classical case) affects on dynamic variable \( \hat{P}_m \) leaving values of temporary arguments in \( F_m \) equal \( t_0=t-y \). The expression (9) corresponds to averaging of value \( \exp[-iyL]\rho_{\text{rel}}(t,y,0) \) with probability density \( (1/\langle M \rangle) \exp[-y/\langle M \rangle] \). In the ratio (9) the value \( \langle M \rangle = e^t = \langle t-t_0 \rangle \) are interpreted as the average lifetime of the finite non-equilibrium system, i.e. the average lifetime of the random values \( \hat{P}_m \) at
the description of the system by the non-equilibrium distribution (6), (7). The average lifetime of the system \(<M>\) tending to infinity after thermodynamic limiting transition is explained by the fact that the lifetime of infinite system is also infinite. The relevant (or the quasi-equilibrium) distribution (2) itself, as well as Gibbs distribution, does not contain lifetime. However, the experience shows that all real systems have finite lifetime and are temporary irreversible. In (9) integration by parts in time is carried out at \(\hat{w}(y)dy_{\varepsilon}=0\); \(\hat{w}(y)dy_{\varepsilon}\to \varepsilon=0\); at \(w(y)=\varepsilon \exp\{-\varepsilon y\}; \varepsilon=I/<M>\), the expression (9) passes in NSO (6), (7) [49, 50]. In [66], it is shown, how it is possible to construct from random process \(X(t)\) the set of new processes, introducing the randomized operational time. It is supposed that to each value \(t>0\) there corresponds the random value \(R(t)\) with the distribution \(w(y)\). The random values \(X(R(t))\) form new random process, which, generally speaking, need not to be of Markovian type any more. It is true also initial condition \(\rho(t_0)=\rho_{rel}(t_0,0)\) [50], if in (9) we assume that \(\ln \rho_{rel}(t_0,-y)=0\) at \(y>0\), as at the moment of smaller than \(t_0\), the system does not exist.

The logarithm of NSO (9) is equal the average from the logarithm of relevant (or quasi-equilibrium) distribution on the system lifetime distribution. Introduction of an ad hoc time-smoothing procedure, with properties, which allow to complete the construction of a satisfactory non-equilibrium statistical operator. In particular, the fading-memory characteristic it produces leads to the irreversible evolution of the macrostate of the system, from the initial condition of preparation (the information available to build the formalism) and the description of the system on the basis of such information and its evolution to a final state of equilibrium in the future.

The NSO dependences on \(u=\gamma-t_0\) differ before and after the limit transition \(t_0\to \infty\). So, NSO (4) depends on the random variable \(u=\gamma-t_0\), since \(t_0\) is a random variable. After the limit transition, the dependence on \(u\) becomes integrated over \(u\), averaging occurs, as in (9).

By definition (1) \(FPT\) is the first moment when a random process \(X(t)\) reaches the absorbing set “\(a\)”. For example, in relation (15), the role of the main random process \(X(t)\) is played by the internal energy \(u\). In example (30) in Section 5, this is a random flux \(J\); in examples (35), (41), this is the coordinate of a diffusing particle. \(FPT\) \(T_p\) plays the role of a subordinate process [66]. In NSO (4), (7), (9) the absorbing set “\(a=0\)” is achieved in the reverse time.

If in NSO method we consider the process in reverse time after the replacement \(t \to -t\), then the birth of the system at a random moment \(t_0\) corresponds to its death at this random moment. The process \(X(t)\) from expression (1) in direct time describes the random process of the system’s existence, the time of the system’s past life, from the random moment \(t_0\) to the current moment \(t\). Achievement of a zero value in the reverse time at a random moment \(t_0\) corresponds to the \(FPT\) process reaching the limit \(a=0\) specified in expression (1) by a random process for the number of particles in the system or the energy of the system; the role of a random process \(X(t)\) is played by the parameters \(\tilde{P}_p\) from expression (2). Parameters \(\tilde{P}_p\) start from the current moment \(t\), and at a random moment \(t_0\) in the reverse time reach the absorbing set “\(a=0\)”.

For finite-size systems with finite times of reaching the zero level by the number of particles and energy, lifetime, a special case of \(FTP\), in the Liouville equation (8) there is a source that characterizes the interaction of finite-size system with the environment. For finite lifetimes of the system, the source in the Liouville equation (8) is not zero. For the exponential distribution \(w(t,t_0)=(1/<M>)\exp\{-\gamma(t-t_0)/<M>\}\) [49, 50], this source in the right-hand side of the Liouville equation is (8). It is also possible to choose an expression for \(w(y)\) that is different from the exponential. Examples of such a choice are given in [42, 67]. The lifetime distribution may differ significantly from the limit exponential distribution [42, 67].

### 3. Distributions for energy and \(FPT\)

Section 5 considers an example of \(FPT\) for a diffusion process with an absorbing boundary. In this ensemble, particles reaching the boundary are absorbed there and removed from the
The ensemble is characterized by \( dN \) \( \delta \) \( q \) \( z \) \( s \). It is also possible to use the property of the thermodynamic \( \rho \) \( q - \) \( nt \) probabilities, and cover the entire space of article from a metastable state \( 70 \) \( dN \) \( alization \) of the grand canonical ensemble must be considered, which \( - \) \( \int \). In this case, the \( \rho \) \( e \) \( scales \) are essential.

Reduced description of non-equilibrium systems is a description based on the restriction information about the system. Reduced description was discussed in section 2. In this case, the difference between the spatial and temporal scales of the processes occurring in non-equilibrium systems is used.

There are several different ways to construct the distribution function of thermodynamic parameters \([49, 68, 69]\). To determine the distribution density for energy and \( FPT \), we will proceed from the "objective" school of thought regarding the probability of an event as an objective property of that event, always capable in principle of empirical measurements by observation of frequency ratios in a random experiment \([70]\), based on mathematical statistics. Maximum entropy method \([70], [71]\) is also suitable for the derivation of the distribution density for energy and \( \gamma \) \( \rho \) \( \gamma \) \( \gamma \). To determine the distribution density for energy and \( \gamma \) \( \rho \) \( \gamma \) \( \gamma \), we proceed to a reduced description for the distribution density \( p \) \( q \). We turn from the discussion of a single system to that of a set of similar systems. To this end, let us assume that initially we have \( N \) identical systems. If we subject them to some realistic preparation procedure, the outcome will be a set of \( N \) slightly different systems, which we call an ensemble. Hence it makes sense to describe an ensemble in \( 2Nd \)-dimensional phase space of sufficiently many systems by a continuous function \( \rho(p, q, t) \) defined by

\[
\forall B \subset \Gamma: \int_{B} \rho(p, q, t) d^{2N}p d^{2N}q \propto \text{number of ensemble points in the set } B \text{ and the normalization} \int_{\Gamma} \rho(p, q, t) d^{2N}p d^{2N}q = 1; \Gamma \text{ is phase space of the position and momentum coordinates of the particles in } d \text{ dimensions. It is easy to see that the } \rho \text{ thus defined is just a probability density in phase space } z = q_1, ..., q_N, p_1, ..., p_N. \text{ Function } \rho(p, q, t) = \rho(z, t) \text{ therefore is called a macrostate of classical mechanics. Let } \rho \text{ be a macrostate and } A = A(p, q) \text{ an observable.}

The probability density \( p_{\rho} \) of an observable \( A \) for a macrostate \( \rho \) given by the expression \([68], [69]\):

\[
p_{\rho}(A) \equiv \rho(A) = \int_{\Gamma} \delta(A - A(p, q)) \rho(p, q, t) d^{2N}p d^{2N}q = \langle \delta(A - A(p, q)) \rangle_{\rho}.
\]

where \( \delta(z, t) \) is the density of distribution of coordinates \( q_i \) and momenta \( p_i \) of particles of the system; \( i = 1, 2, ..., N \). The function \( \rho(p, q, t) \) is also determined by the residence time of the system in a certain region of the phase space \([51]\).

As in Section 2, we proceed to an reduced description for the distribution density \( \rho(z, t) \). In this case, as in \([49], [55]\), we seek solutions of the Liouville equation that depend on time only
through the mean values of a certain set of observables \( \langle \hat{H} \rangle \). The distribution density \( \rho(z;t) \) from expression (10) is replaced by a distribution of the form (2) \( \rho_{rel}(z;u,T_{\gamma}) \).

FTP (1) in terms of the stochastic theory [48, 66] is a functional of the main random process \( X(t) \) (for example, energy), and a subordinate random process. FTP is a macroscopic and slowly changing quantity. The distribution for the FPT \( T_{\gamma} \) (1) depends on the macroscopic values of \( X(t) \). Assume that the process \( X(t)=u \) is the energy (or another system order parameter) of the system. The relationship between the distribution density \( p(u,T_{\gamma})=p_{rel}(x,y) \) and the microscopic (coarse-grained) density \( \rho_{rel}(z;u,T_{\gamma}) \) is written as (following the standard procedure, for example, see [68], [69]):

\[
p(u,T_{\gamma}) = \int \delta(u-u(z)) \delta(T_{\gamma}-T_{\gamma}(z)) \rho_{rel}(z;u,T_{\gamma})dz. \tag{11}
\]

In the case of stochastic dynamics and deterministic chaos the function \( \delta(A-A(z)) \) (\( A=u, T_{\gamma} \)) from (11) is replaced by \( \delta(A-f(A(z))) \), where \( f \) is map (a chaotic map is a map(=evolution function) that exhibits some sort of chaotic behavior) [71]. However, in [46] it is written “We say that a random time \( T \in [0, \infty] \cup \{+\infty\} \) is a stopping time if it is a deterministic function defined on the set of trajectories \( X^{+\infty}_0 = \{X(t)\}_{t \in \mathbb{R}} \) that obeys causality; in other words, the value of the stopping time \( T \) is independent of the outcomes of the process \( X \) after the stopping time. If the event does not occur, then \( T = +\infty \).” In this case and for the case of stochastic dynamics in (11) the expression \( \delta(T_{\gamma}-T_{\gamma}(z)) \) is valid. We consider the observable quantity, the thermodynamic parameter of FTP \( T_{\gamma} \), as random variable in the phase space \( z \) (due to the uncertainty of the initial values of the phase coordinates) and deterministic function in the time dynamics. The particles obey the equations of motion of mechanics, \( T_{\gamma} \) is described by the Hamilton equations.

There are different possibilities for determining the function \( \rho_{rel}(z;u,T_{\gamma}) \) from expression (11). For example, in [49] physical quantities characterizing the system \( \xi_{1},...,\xi_{k} \) are specified, but not necessarily the integrals of motion. Average values \( \langle \xi_{i} \rangle \) can characterize some state of incomplete statistical equilibrium. The free energy of a non-equilibrium state, characterized by a given mean value \( \langle \xi_{i} \rangle \), is defined in [51] as the free energy of an equilibrium state in auxiliary fields that make the system in equilibrium at given values \( \langle \xi_{i} \rangle \). In this case, the distribution density is

\[
\rho_{rel} = Q^{-1} \exp[-\beta(H - \mu N - \sum a_{k} \xi_{k})],
\]

where \( \beta \) is the inverse temperature, \( \mu \) is the chemical potential, \( H \) is the energy, \( N \) is the number of particles. The partition function \( Q \) is a function of \( \beta, \mu, a_{k} \) or \( \beta, \mu, \langle \xi_{k} \rangle \) and determines the thermodynamic functions in a state of incomplete statistical equilibrium with a given \( \langle \xi_{k} \rangle \) depending on \( \beta, \mu, \langle \xi_{k} \rangle \). We put \( \xi_{i} = T_{\gamma} \); the auxiliary field will be described by the parameter \( a_{k} = \gamma \) from expression (15). In [49], this approach is also used in the local equilibrium distribution, a particular case of the relevant distribution [50]. This description is valid for not very large deviations from equilibrium. Similar relations were written in [72] for the generalized Gibbs ensemble, which corresponds to a given type of thermodynamic contact of a macroscopic system with the environment.

Let us consider another approach to obtaining the function \( \rho_{rel}(z;u,T_{\gamma}) \) from expression (11). It is based on general results of mathematical statistics and is free from limitations.

It was shown in [73] that for a sequence of independent trials with \( n \) outcomes, each with probability \( p_{i} \), \( \ln P(v_{1},...,v_{n} \in W) \approx -ND(P|R) \), where \( R \) is the maximum point of \( D(P\|Q) \) at \( Q \in W, v_{1},...,v_{n} \) are the frequency of outcomes, the region \( W \) satisfies the requirements of regularity. Information deviation (or relative entropy) \( D(P\|Q) \) of the probability distribution \( Q \)
from the dominated distribution $P, Q \gg P$ is the quantity $D(P\|Q) = \intlog(dP/dQ)dP, \text{ if } P \ll Q.$

The notation $\lambda \gg \mu$ denotes that the measure $\lambda$ dominates $\mu$, $Z_\lambda \subseteq Z_\mu$, where $Z_\mu$ is the ideal of the algebra $S$ of measure $\mu[\cdot]$ on a measurable space $(\Omega, S)$, all sets of zero measure, zero sets, $\mu[\{\omega\}] = 0 \ [74]$. By the Radon-Nikodym theorem, if $Z_\mu \subseteq Z_\gamma$, then on $\Omega$ there is a derivative $\frac{d\nu}{d\mu}(\omega)$ of the measure $\nu$ with respect to measure $\mu$, a measurable finite numerical function $h(\omega)$ such that $\nu[H] = \int h(\omega)\mu(\{\omega\})$ for any $H \subseteq S$. In [75] proved a theorem based on the results of [73]:

Let $(S, \mathcal{B})$ be an arbitrary measurable space, $\Lambda$ is the set of all probability measures on $(S, \mathcal{B})$. If $\Pi \subseteq \Lambda$ is defined by $\Pi = \{ P : \int f_idP \geq 0, \ i = 1,..., k \}$, where $f_1,...,f_k$ are given measurable functions on $(S, \mathcal{B})$, then for a probability measure $Q \in \Lambda$ we have $D = D(\Pi\|Q) < \infty$ if there exists a $P \in \Pi$ with $P \ll Q$. Then the generalized $I$-projection $P'$ of $Q$ on $\Pi$ has $Q$-density of form

$$\frac{dP'}{dQ} = \begin{cases} \exp\{ D + \sum_{i=1}^k \theta_i^f_i \} & \text{on } \{ s : (f_1(s),...,f_k(s)) \in M \} \\ 0 & \text{elsewhere} \end{cases},$$

where $M$ is a linear subspace of $R^k$ and $\theta_i^f = (\theta_1^f,...,\theta_k^f) \in R^k$. The relation (13) holds with $M \in R^k$ i.e. $P'$ belongs to the exponential family $\{ P_\theta : \mathcal{G} \subseteq \Theta \}$ defined by (compare with (2))

$$\frac{dP_\theta}{dQ} = \frac{\exp(\sum_{i=1}^k \theta_i f_i)}{\int \exp(\sum_{i=1}^k \theta_i f_i) dQ}, \quad \Theta = \{ \theta = (\theta_1,...,\theta_k) : \int \exp(\sum_{i=1}^k \theta_i f_i) dQ < \infty \},$$

if there exists a $P \in \Pi$ with $P = Q$, where $P = Q$ designates mutual absolute continuity. Under the last condition

$$D = D(\Pi\|Q) = \max_{\theta \in R^k} [-\log \int \exp(\sum_{i=1}^k \theta_i f_i) dQ],$$

where the maximum is attained if $P_\theta = P'$.

Here $R^k$ is the $k$-dimensional Euclidean space, $R^+_{2k}$ is the half-space in $R^k$, defined by the condition $x^k > 0$. The probability measure $P'$ will be called the $I$-projection of $Q$ on $\Pi$ if $P' \in \Pi$ and $D(P\|Q) = D(\Pi\|Q).$ In our case, probability measure $Q = z = \{ q_1,p_1,...,q_N,p_N \}$ is phase space; $S_q = F_1$ from (3), $f_i = P_i(=u, T_7)$; $k = 2, \theta_1 = \beta, f_1 = u, \theta_2 = y, f_2 = T_7$ (8). The integral $\int f_idP > 0$ in condition (5) is the Lebesgue integral of a given measurable function $f_i$ with respect to the probability measure $P \in \Pi$ with $P \ll Q$. If the integration is carried out over the entire probability space $\Omega$, then this integral is equal to the mathematical expectation of the function $f_i$, if the function $f_i$ is non-negative. The non-negativity condition is satisfied for $f_2 = T_7$.

The results of this theorem can be compared with using of the maximum entropy principle when used only the first moment as information of the density function, when a Gibbs or exponential distribution of the form (14) is written. The extremum of the relative entropy $D(P\|Q)$, the Kullback entropy, is determined. The same operation for finding a non-equilibrium
distribution density similar to (14) is performed in [52]. In [75] it is written “Such results are relevant for statistical physics,...such conditional limit theorems provide a justification of the “maximum entropy principle” in physics”.

Using (14) with \( k=2 \), \( \beta_1=\beta \), \( f_1=u \), \( \beta_2=\gamma \), \( f_2=T_\gamma \), for function \( \rho_{\text{rel}}(z;u,T_\gamma) \) from (11) we obtain

\[
\rho_{\text{rel}}(z;u,T_\gamma) = \frac{e^{-\beta u(z)-\gamma T_\gamma(z)}}{Z(\beta,\gamma)}.
\]  

(15)

Substituting the expression (15) in the relation (11) and replacing the variables, passing from the variables \( z \) to the variables \( u \), \( T_\gamma \), we get

\[
p(u,T_\gamma) = \frac{e^{-\beta u-\gamma T_\gamma}}{Z(\beta,\gamma)},
\]  

(16)

where \( \beta = 1/T \) is the inverse temperature of the reservoir (\( k_B = 1 \), \( k_B \) is Boltzmann constant),

\[
Z(\beta,\gamma) = \int e^{-\beta u-\gamma T_\gamma} dz = \int du dT_\gamma \omega(u,T_\gamma)e^{\beta u+\gamma T_\gamma},
\]  

(17)

is the partition function, \( \beta \) and \( \gamma \) are the Lagrange multipliers satisfying the following expressions for the averages:

\[
\langle u \rangle = -\frac{\partial \ln Z(\beta,\gamma)}{\partial \beta}, \quad \langle T_\gamma \rangle = -\frac{\partial \ln Z(\beta,\gamma)}{\partial \gamma}.
\]  

(18)

The factor \( \omega(u) \) in the case of a distribution for \( u \) is replaced by \( \omega(u,T_\gamma) \), which is the volume of the hypersurface in the phase space containing fixed values of \( u \) and \( T_\gamma \). If \( \mu(u,T_\gamma) \) is the number of states in the phase space with parameter values less than \( u \) and \( T_\gamma \), then \( \omega(u,T_\gamma) = d^2\mu(u,T_\gamma)/dudT_\gamma \). Moreover, \( \int \omega(u,T_\gamma)dT_\gamma = \omega(u) \). The number of phase points with parameters in the interval between \( u \), \( u+du \); \( T_\gamma \), \( T_\gamma+dT_\gamma \), is \( \omega(u,T_\gamma)dudT_\gamma \). Similarly, as the distribution function in [49, 69] is dimensionless and normalized to the minimum size of the phase volume, which in the one-dimensional case is equal to the Planck constant \( h=2\pi\hbar \), let’s make the function \( \omega(u,T_\gamma) \) dimensionless by multiplying it by some unit standard

In [74], a family of distributions

\[
\frac{dp}{d\mu} (\omega) = p(\omega,s) = p_0(\omega)\exp[ \sum_j s^j q_j(\omega) - \Psi(s) ]
\]  

(19)

(compare with (14)) is called an exponent or geodesic family (of a finite number of dimensions) with a canonical affine parameter \( s = (s^1,\ldots,s^g) \), where \( q = (q_1(\omega),\ldots,q_g(\omega)) \) is guide the sufficient statistic, \( \mu(\cdot) \) is a fixed dominant measure, and

\[
\exp[\Psi(s)] = \int \exp[ \sum_j s^j q_j(\omega)] p_0(\omega)\mu(d\omega)
\]

is the normalizing factor. Formula (19) retains its form when replacing a fixed measure \( \mu(\cdot) \), relative to which all densities are taken, by an other measure \( \lambda \gg \mu \). All densities are multiplied by the density variant \( \frac{d\mu}{d\lambda} (\omega) \), going into densities relative to the measure \( \lambda(\cdot) \). This refers to the replacement of the variables in (15), (16). Families (19) belong to a broader class of exponential families with densities of the form

\[
p(\omega;\theta) = p_0(\omega)\exp[ \sum_j s^j(\theta)q_j(\omega) - \Psi(s(\theta))] .
\]  

(20)

where \( (\theta_1,\ldots,\theta_g) \in \theta \), \( s(\theta) = (s^1(\theta),\ldots,s^g(\theta)) \). A probability distribution is called a weighted geodesic mean if the logarithm of its density is

\[
\ln p_\lambda(\omega) = \sum_{i=0}^m \alpha_i \ln p_i(\omega) - H(A)\cdot1(\omega), \quad H(A) = \ln \int \exp[ \sum_{i=0}^m \alpha_i \ln p_i(\omega)]\mu(d\omega),
\]  

(21)
where \( p_0, p_1, \ldots, p_n \) are a finite number of mutually absolutely continuous laws, \( \{ \alpha_0, \alpha_1, \ldots, \alpha_n \} \) is set of weights. In [74], a family of distributions with a logarithm of density
\[
\ln u(\omega) = \int \ln p(\omega; t) \alpha \{ dt \} - H(\alpha)
\]
(22)
the average geodesic of a family of distributions \( u_\alpha \) weighted (with a weight measure \( \alpha \)) is called, where \( H(\alpha) \) is logarithm of the normalizing divisor,
\[
\exp H(\alpha) = \int \exp \left[ \int \ln p(\omega; t) \alpha \{ dt \} \right] R \{ d\omega \},
\]
if the last integral is finite. The distribution (7), (9) of the NSO corresponds to this definition when \( H(\alpha) = 0 \). The relationship between distributions of the form (14) and (19)-(22) shown in [74] corresponds to the relationship between distributions of the form (15) and NSO (4), (9).

In expressions (10), (13)-(14), (19)-(22) the distributions are given in a general way. Thus, expressions (13)-(14) are valid for arbitrary random variables that satisfy condition (12). The explicit form of these distributions for random variables \( u, T \) is given in (11), (15)-(18), and the explicit form FPT distribution of a random process of \( J \) for the flux level \( J_{\text{fr}} \) is in (30), section 5.

In expressions (15)-(18), the values of energy \( u \) and FPT \( T \), chosen as thermodynamic parameters. The thermodynamically conjugated to FPT value \( \gamma \) is associated with the production and flows of entropy, which characterize the non-equilibrium processes in an open statistical system. At \( \gamma = 0 \) and \( \beta = \beta_0 = T_{\text{eq}} \), where \( T_{\text{eq}} \) is the equilibrium temperature, the non-equilibrium distribution (15) converges to equilibrium Gibbs distribution. The distribution (15) over \( u \) and FPT is a generalization of Gibbs distribution to a non-equilibrium situation. The canonical Gibbs distribution is obtained from the microcanonical ensemble in the zero approximation by the interaction of the system with the environment. Using the FPT \( T_\gamma \) (1), an effective account is taken of this interaction (similarly to the methods of McLennan [76] and Zubarev NSO [49, 50], and [72]). The value of \( \gamma \) can be considered as a measure of deviation from equilibrium.

Let us make a remark about the form of the distribution function (15). The value appearing in (15) is \( \exp \{ -\gamma T_\gamma \} \) corresponds to the exponential limit distribution describing infinitely large times. At the same time, the distributions (14), (15) characterizes the guiding statistics [74]. The accuracy of this approximation, based on the theory of rare events, is estimated at [73]. Refinement of the distribution is possible, for example, by taking into account the quadratic terms in \( T_\gamma \) in the exponent (15). In the proposed below approach \( \exp \{ -\gamma T_\gamma \} \) acts as the core of the integral Laplace transform (28). A more detailed distribution of the value \( T_\gamma \) is contained in the distribution function \( f(T_\gamma) \) in \( \alpha(u,T_\gamma) \) (16), (24), which can describe finite arbitrarily small time intervals. A more detailed description is possible using weighted geodesic mean distributions (21). They are applicable to the description of a multi-stage FPT process with different distributions at different stages of the process. For example, in [32] two modes are highlighted, and the transition point between them. For heterogeneity-controlled kinetics in [39] derived exact asymptotic results for the short, intermediate, and long time FPT statistics for an arbitrary degree of heterogeneity.

4. Generalized thermodynamics of systems with parameter of FPT

In the previous section, we found an expression for the FPT distribution. We need to find an expression for the conjugate thermodynamic quantity \( \gamma \). In the general case, the value of FPT \( T_\gamma \) can be selected as a subprocess in a form different from that described above. The equilibrium state then will define some curve in the \( (\beta, \gamma) \) plane, which does not necessarily coincide with the
line $\gamma=0$. Features of the non-analytic behavior of the partition function (or its derivatives) on the $(\beta, \gamma)$ plane indicate non-equilibrium phase transitions.

Assuming that FPT can be instrumentally measured, we introduce the local specific entropy $s\gamma$ corresponding to the distribution (15) ($u$ is specific internal energy) by the relation [49, 50]

$$s\gamma = -\left(\ln\rho(z;u,T\gamma) = \beta(u) + \gamma(T\gamma) + \ln Z(\beta,\gamma)\right); \quad ds\gamma = \beta d\langle u \rangle + \gamma d\langle T\gamma \rangle. \quad (23)$$

Expression (23) is the Legendre transform for the variables $u$ and $T\gamma$. For spatially inhomogeneous systems, the values $\beta$ and $\gamma$ in the general case depend on the spatial coordinate. Distributions (15) can be considered for a small volume element in which the values of $\beta$ and $\gamma$ are replaced by the average values constant over this volume element. In non-equilibrium thermodynamics, the densities of extensive thermodynamic quantities (entropy, internal energy, mass fraction of a component) are considered. We follow this approach, including here the FPT parameter, with $T\gamma \to T\gamma/V$. The assumption about the smallness and homogeneity of the volume elements is similar to the hypothesis of local equilibrium adopted in classical non-equilibrium thermodynamics. However, in EIT [43], where thermodynamic flows variables close to FPT are used, the hypothesis of local equilibrium is not applied.

In the distributions (15)-(16), containing the FPT as a thermodynamic parameter, the joint probability for the quantities $u$ and $T\gamma$ is (16). The distribution (16) corresponds to the general formulas (10), (11). The factor $\omega(u,T\gamma)$ is the joint probability for $u$ and $T\gamma$, considered as the stationary probability of this process. We rewrite the value $\omega(u,T\gamma)$ in the form

$$\omega(u,T\gamma) = \omega(u)\omega_1(u,T\gamma) = \omega(u)\sum_{k=1}^{n} R_k f_{ik}(T\gamma,u). \quad (24)$$

In Eq. (24) it is assumed that there are $n$ classes of states in the system; $R_k$ is the probability that the system is in the $k$-th class of states, $f_{ik}(T\gamma,u)$ is the density of the distribution of FPT $T\gamma$ in this class of (ergodic) states (in the general case, $f_{ik}(T\gamma,u)$ depends on $u$). As a physical example of such a situation (characteristic of metals, glasses, etc.), one can mention the potential of many complex physical systems. Below we restrict ourselves to the case $n=1$.

Such situation is considered in [77, 78]. The minimum points of the potential correspond to metastable phases, disordered structures, etc. The phase space of these systems divided into isolated regions, each of which corresponds to a metastable thermodynamic state, and the number of these regions increases exponentially with increasing total number of (quasi)particles [79].

The form of the function $f_{ik}(T\gamma,u)$ in (24) reflects not only the internal properties of the system, but also the influence of the environment on the open system, and the features of its interaction with the environment. A physical interpretation of the density of the exponential distribution in NSO for the function $f_{ik}(T\gamma,u)$

$$f_{ik}(T\gamma,u) = T_0^{-1}\exp\{-y/T_0\}, \quad T_0 = T|_{\gamma=0}, \quad (25)$$

is given in [49, 55]: the system evolves as an isolated system controlled by the Liouville operator. In addition, the system undergoes random transitions, and the phase point representing the system switches from one trajectory to another with an exponential probability under the influence of a “thermostat”. Exponential distribution describes completely random systems. The impact of the environment on the system can also be included, for example, this applies to systems in a non-equilibrium state with input and output unsteady flows. The nature of the interaction with the environment may be different; therefore, various forms of the function $f_{ik}(T\gamma,u)$ can be used [42]. The exponential distribution is the limiting distribution that is valid in the limit of infinitely large times. Since FPTs are finite, in the general case their behavior is characterized by a variety of distributions, the form of which depends on the interaction of the system with the environment and the random processes governing the system itself.
Note that a value similar to $\gamma$ is determined in [80]–[82] for a fractal repeller object. It is equal to zero for a closed system, and for an open system it is equal to $\Sigma \lambda_i - \lambda_{KS}$, where $\lambda_i$ are Lyapunov exponents, $\lambda_{KS}$ is the Kolmogorov-Sinai entropy. The thermodynamic interpretation of the value $\gamma$ is given below.

We assume that the distribution of the random variable $T_\tau$ does not depend on the random variable $u$. Further, the distribution of the random variable $T_\tau$ does not depend on the random variable $u$, but depends on the mean values.

The initial state is not necessarily an equilibrium, but a stationary non-equilibrium state. For this case, we can write relations similar to (15) - (18), (23). Deviations from the equilibrium state are considered below.

The thermodynamic relations for the entropy (23) are written in [40-41]. We rewrite (23) in the form of the relationship of expressions for equilibrium and non-equilibrium entropy of the form

$$s_{\tau} = s_{eq} - \Delta = \gamma T_\tau + \beta \mu + \ln Z = \beta u_\beta + \ln Z_\beta - \Delta,$$  

(26)

where $s_{eq} = s_{1-\gamma} = \beta u_\beta + \ln Z_\beta$, $Z_\beta = \frac{1}{\beta} \int e^{\beta u} \delta(u) du$. $u_\beta = -\partial \ln Z_\beta / \partial \beta$, $Z_\beta$ is partition function; $u_\beta$ is equilibrium energy; $s_{eq}$ is equilibrium entropy. For extended irreversible thermodynamics (EIT) [43] and for the case of thermal conductivity $\Delta = \lambda q^2 / 2$, $\lambda u = \theta / \rho \lambda \Theta^2$, where $\Theta$ is the non-equilibrium temperature, $\rho$ is the mass density, $\lambda$ is the thermal conductivity, $q$ is the heat flux, $\tau = \tau_q$ is the correlation time of the fluxes from the Maxwell-Cattaneo equation of the form

$$q = -\lambda VT - \tau_q \dot{q} / \partial \lambda.$$

Equating the distributions (4), (9) and (15), we get when $\ln \rho_{eq} (t) = -\beta u - \ln Z_\beta$ and $Z = Z_\beta Z_\gamma$

$$R_1 = \int_{t_0}^{t} e^{-\sigma} \sigma(t - y, y - \gamma) dy = \gamma T_\gamma + \ln Z_\gamma,$$

$$R_2 = \int_{t_0}^{t} e^{\gamma (t - y)} \sigma(t_0 + t_1, t_0 - t + t_1) dt_1 = \gamma T_\gamma + \ln Z_\gamma.$$

(27)

In the right-hand part (27) is a random variable $T_\gamma = u - t_0$, as in (4), (9). If we consider the value (4) before the thermodynamic limit transition, then the left part (27) also depends on the value $u = t - t_0$, which in [54] is interpreted as the random lifetime of the system, FPT. In this case, the corresponding meaning should be given to the expressions on the right-hand side of relation (27). But the identity of distributions (4) and (15) has not been proved. On the left-hand side of (27), the time dependence can be written explicitly, for example, in the form of the Liouville equation with a source (8). The right side describes the stationary distribution. Although, if we use a distribution of the form (20), then the parameter $\sigma$ can be given the meaning of time [83]. Just as in [83] in the family of exponential distributions (20) applied to the NSO, the parameter $\sigma$ is given the meaning of time, so in (15)-(16), by writing these distributions in the form (20), we can give the parameter $\sigma$ the meaning of time and look for the dependencies $\beta(t)$, $\gamma(t)$. The time dependence of the distribution parameters is obtained using the information geometry in [84]. Expression (27) can be considered as generating equality. You can multiply it by random variables, average it, differentiate it by parameters, and so on.

In relation (26), the function $Z_\gamma$ (28) is assumed to be known. From the expressions (18), (28), the values $u_\gamma$ and $\langle T_\gamma \rangle$ are determined. They are substituted into the relations (26), (29) which is considered as an equation for determining the value $\gamma$. The second Lagrange multiplier $\gamma$ is determined not from the a maximum entropy ansatz, but from the algebraic equation (26), as a function of the deviation of entropy from the equilibrium value $\Delta$. In equation (26), for the value
\(\gamma(A)\), we set the value \(A\) in accordance with the random process that occurs. If you set a different value of \(A\), then, accordingly, the values of the quantities \(\gamma, T_\gamma, \langle T_\gamma \rangle, u_\gamma\) will change. By changing or including some thermodynamic forces, we change the fluxes in the system, \(FPT \ T_\gamma, \langle T_\gamma \rangle\), and non-equilibrium internal energy \(u_\gamma\).

It is shown in [40-42] that the parameter \(T_0\) of the exponential distribution (25) of the \(FPT\), which is equal to the average unperturbed \(FPT\), does not depend on a random value of energy, but depends on the average value of equilibrium energy and inverse equilibrium temperature. In this case, the distribution \(f_{th}(T_\gamma, u_\gamma)\) is independent of random energy \(u\), the integration variables are separated, and the partition function (17) is written as the product of the equilibrium and non-equilibrium factors, \(Z(\beta, \gamma) = Z_\beta Z_\gamma\). Let us assume that the distribution \(f_{th}(T_\gamma, u_\gamma)\) is independent of the random variable \(u\) and the validity of this independency for distributions other than (25). The non-equilibrium part of the partition function in this case is the Laplace transform of the distribution of the \(FPT\). The average \(FPT\) (and other moments) depends on the non-equilibrium parameter \(\gamma\), which is expressed in terms of the difference between the equilibrium and non-equilibrium entropy from Eq. (26), (29). When the non-equilibrium parameter \(\gamma\) and the entropy difference tends to zero, the average \(FPT\) of the system assumes an equilibrium value \(T_0\), which also assumed to finite due to the openness of the system and the presence of fluctuations. The internal energy is equal to the sum of the equilibrium and non-equilibrium parts; non-equilibrium internal energy \(u_\gamma\) is equal to zero in equilibrium:

\[
\bar{u} = -\frac{\partial \ln Z}{\partial \beta} = u_\beta + u_\gamma, \quad u_\beta = -\frac{\partial \ln Z_\beta}{\partial \beta}, \quad u_\gamma = -\frac{\partial \ln Z_\gamma}{\partial \beta} = \int_0^\infty e^{-\gamma\beta} \left(\frac{\partial f(T_\gamma)}{\partial \beta}\right) dT_\gamma \frac{1}{Z_\gamma},
\]

where \(f(T_\gamma)\) is the probability density of the distribution of the \(FPT\).

We also note that the quantity \(Z_\gamma = \int_0^\infty e^{-\gamma T_\gamma} f(T_\gamma) dT_\gamma\) (28) has probabilistic meaning:

\[
Z_\gamma = \int_0^\infty e^{-\gamma T_\gamma} f(T_\gamma) dT_\gamma = P[T_\gamma \leq \tau]; \quad P[\tau > t] = \exp[-\gamma t]. \, \text{At} \quad \gamma \sim \sigma_s, \quad \sigma_s \, \text{is entropy production},
\]

\(P[\tau > t] \sim 1, \text{when} \, \sigma_s \rightarrow 0, \, \text{and} \, P[\tau > t] \sim 0, \text{when} \, \sigma_s \rightarrow \infty; \, P[T_\gamma \leq \tau] \sim 1, \, \tau \rightarrow 0\).

If we use a distribution of the form (20), then the parameter \(\theta\) can be given the meaning of time [83]. Just as in [83] in the family of exponential distributions (20) applied to the \(NSO\), the parameter \(\theta\) is given the meaning of time, so in (15)-(16), by writing these distributions in the form (20), we can give the parameter \(\theta\) the meaning of time and look for the dependencies \(\beta(t), \gamma(t)\).

5. Examples. \(FPT\) for flow values, for level \(L\) in drift-diffusion process and for the Feller process

Let us illustrate the application of the above approach by examples. In [16] and [17] \(FPTs\) are considered for achieving a certain level of entropy production and flux values. The above possibilities of connecting the \(FPT\) distribution with the thermodynamic characteristics of the system are applicable to this situation, as well as to others cases. Consider reaching a certain value of the flow level. As an independent thermodynamic parameter, we choose a random value \(T_{th}\) of the time of the first achievement of a given flow value \(J_{th}\). Adding this parameter, along with the energy, to the exponential distribution, as in (15), we write the relation for the entropy of the form (23), (26) taking into account (28).
\[ s_{\beta} = \beta u + \gamma T_{\beta} + \ln Z = \beta u_\beta + \ln Z_\beta - \Delta, \quad \bar{u} = u_\beta + u_\gamma; \]
\[- \Delta = \beta u_\gamma + \gamma T_{\beta} + \ln Z_\gamma, \quad \Delta = s_{eq} - s_{\beta}. \quad (29)\]

As in [17], for the FPT distribution density of the flux level \( J_{thr} \), we use the inverse Gaussian distribution (Wald's distribution) with a density of the form
\[
F_{\text{inv}}(t) = \left[ \frac{J_{th} \sigma (\bar{t})}{4 \pi t^2} \right] \exp [- \frac{J_{th} \sigma (t - \bar{t})^2}{4 t}], \quad (30)
\]
where \( \sigma \) is entropy production rate (with Boltzmann's constant \( k_B = 1 \)), \( \bar{t} \) is average (with \( \gamma = 0 \)) value of FPT. The quantity \( X \) from (1) in this case is equal to the flow \( J \). The Laplace transform of distribution (30) is
\[
Z_{Jy} = \int_0^\infty e^{-\gamma F_{\text{inv}}(t)} dt = \exp \left\{ \frac{a T_{J0}}{2} (1 - \sqrt{1 + 4 \gamma / a}) \right\}, \quad a = J_{th} \sigma, \quad (31)
\]
where \( \bar{t} = T_{J0} \) is the value of \( T_{\beta} \) in the absence of disturbances, at \( \gamma = 0 \). The expression obtained from relation (31) for the average time to reach the level \( J_{th} \) in (6) is
\[
\bar{T}_{\beta} = - \frac{\partial \ln Z_{Jy}}{\partial \gamma} \bigg|_{\beta} = \frac{T_{J0}}{\sqrt{1 + 4 \gamma / a}} , \quad T_{J0} = \langle \bar{t} \rangle. \quad (32)
\]

In expression (29) it is necessary to use the value \( u_\gamma = - \frac{\partial \ln Z_{Jy}}{\partial \beta} \bigg|_{\gamma} \) (18), (28). Considering
\[
\frac{\partial a}{\partial \beta}, \frac{\partial T_{J0}}{\partial \beta}, \quad (33)
\]
we obtain
\[
- \beta u_\gamma = \frac{1}{2} (1 - \sqrt{1 + 4 \gamma / a}) \beta(T_{J0} \beta \frac{\partial a}{\partial \beta} + a(T_{J0} \beta \frac{\partial a}{\partial \beta} + a \gamma(T_{J0} \beta \frac{\partial a}{\partial \beta} + \beta a \gamma(T_{J0} \beta \frac{\partial a}{\partial \beta})}. \quad (33)
\]
Substituting (32), (33) into (29), we obtain, taking into account (28), a quadratic equation to determine the dependence \( \gamma(\Delta) \) form \( a_\gamma(\gamma)^2 + b_\gamma(\gamma) + c_\gamma = 0 \) with the solution
\[
\gamma = \frac{b_\gamma}{2a_\gamma} \left[ \sqrt{1 - 4a_\gamma c_\gamma / b_\gamma^2} - 1 \right]. \quad (34)
\]
The “+” sign is selected to match the condition \( \gamma_{|a=0} = 0 \).

In (34) \( a_\gamma = a_4^2, \quad a_4 = \frac{\beta}{a} \frac{\partial a}{\partial \beta} T_{J0} + 2 \beta \frac{\partial a}{\partial \beta} T_{J0}, \quad b_\gamma = -k_3 \beta \frac{\partial a}{\partial \beta} T_{J0} + \frac{4}{a}, \quad c_\gamma = k_3 \Delta - \Delta^2, \quad k_3 = -\beta \frac{\partial (a T_{J0})}{\partial \beta} + a T_{J0}. \) The value \( \Delta \) depends on the type of flows \( J \). For example, for heat fluxes in EIT [43] \( J = q, \quad \Delta = a_\beta q^2 / 2, \quad a_\beta = \tau / \rho \lambda \theta^2 \).

Substitution of (34) into (32) gives the relation
\[
\bar{T}_{\beta} = \frac{T_{J0}}{\sqrt{1 + \frac{4[4 \Delta(\Delta + k_3) + ak_3 \beta \beta T_{J0} / \partial \beta]}{(k_3 - \beta \beta T_{J0} / \partial \beta)^2} \left[ 1 - \frac{4 \Delta(\Delta + k_3) - ak_3 \beta \beta T_{J0} / \partial \beta)^2}{[4 \Delta(\Delta + k_3) + ak_3 \beta \beta T_{J0} / \partial \beta]^2} \right]}}. \quad (35)
\]
Knowledge of the value \( \gamma \) (34) also makes it possible to use the probabilistic interpretation of the Laplace transform \( Z_{Jy} = \int e^{-\gamma Z_{Jy}} P(T_{Jy})dT_{Jy} = P(T_{Jy} \leq \tau), \quad P(\tau > t) = e^{-\tau}. \)

The same distribution (30) [85] is written in [22] and is applied in [16] to various problems, including for FPT level \( L \) in drift-diffusion process. The Langevin equation for drift-diffusion process
\[ \frac{dX(t)}{dt} = v + \zeta(t), \]

where \( X(t) \) from (1) is the coordinate, \( v = F/\zeta \) is the drift velocity, \( F \) is an external force, \( \zeta \) is a friction coefficient, and \( \zeta(t) \) is a Gaussian white noise with zero mean \( \langle \zeta(t) \rangle = 0 \) and with autocorrelation \( \langle \zeta(t)\zeta(t') \rangle = D \delta(t-t') \), \( D = k_B T / \zeta \) is the diffusion coefficient. We consider a system containing \( N \) different constituents labeled \( k=1,2,\ldots,N \) with mass density \( \rho_k \) and velocity \( v_k \);
\[
\rho = \sum_{k=1}^{N} \rho_k \]
is total density and the center-of-mass velocity is \( \rho v = \sum_{k=1}^{N} \rho_k v_k \), \( J_k \) is diffusion flux. In [16] the FPT distribution for \( X \) to pass at time \( T_\gamma = t \) for the FPT the threshold \( L>0 \), starting from the initial condition \( X(0) = 0 \), is given by Wald's distribution [85, 16]
\[
F_{\gamma}(t) = \sqrt{\frac{|V|}{4\pi Dt}} \exp[-\frac{(L-vt)^2}{4Dt}].
\]

Distributions (30) and (35) coincide for \( \langle \gamma \rangle = L/v \), \( a = v^2 / D \) in (35) (in (30)-(31) \( a = J_{\text{tot}} / \sigma \)). For the deviation of entropy from the equilibrium value during diffusion in the time \( T_\gamma \), can write the expression from EIT [43]
\[
\Delta = s_{\text{eq}} - s = \dot{s} T_\gamma / \rho, \quad \dot{s} = \sum_{k=1}^{N} \frac{\mu_k}{T} \nabla J_k, \quad J_k = \rho_k (v_k - v),
\]
\( \mu_k \) is the chemical potential. Expression for \( T_\gamma \) we write from (31) - (32), (35). From the ratio
\[
\beta u_\gamma + \gamma T_\gamma + \ln Z_\gamma = -\Delta = -\dot{s} T_\gamma / \rho
\]
we obtain a quadratic equation for \( \gamma \) with the solution
\[
\gamma = \frac{1}{M_1} \left[ c_1 \beta (\partial T_\gamma / \partial \beta) \right] (1 - \frac{\dot{s}}{\rho c_1 \beta^2 (\partial T_\gamma / \partial \beta)^2} M_1 (M_1 + \beta (\partial T_\gamma / \partial \beta)) + \frac{\dot{s}}{\rho} T_\gamma),
\]
\[
T_\gamma = T_\gamma_{\gamma=0}, \quad c_1 = \frac{1}{2} \left[ \frac{a T_0}{2} - \beta (\frac{a T_0}{2}) \right], \quad M_1 = T_\gamma (1 - \frac{\beta a}{a} - 2 \beta \frac{(\partial T_\gamma)}{\partial \beta}).
\]

At small \( \dot{s} \), when
\[
\frac{1}{1 + \frac{\dot{s}}{\rho c_1 \beta^2 (\partial T_\gamma / \partial \beta)^2}} M_1 (M_1 + \beta (\partial T_\gamma / \partial \beta)) \approx 1 + \frac{\dot{s}}{\rho c_1 \beta^2 (\partial T_\gamma / \partial \beta)^2} M_1 (M_1 + \beta (\partial T_\gamma / \partial \beta)),
\]
\[
\gamma \approx -\frac{\dot{s}}{\rho \beta (\partial T_\gamma / \partial \beta)}.
\]

If we assume that the energy of the diffusing particle is equal \( u = mv^2 / 2 = u_\rho \), then \( u_\gamma = 0 \). Derivatives of \( v \) and \( a \) are equal
\[
\frac{\partial T_\gamma}{\partial \beta} = -\frac{T_0}{v} \frac{\partial v}{\partial \beta}, \quad \frac{\partial a}{\partial \beta} = a (\frac{2}{v} \frac{\partial v}{\partial \beta} - \frac{1}{D} \frac{\partial D}{\partial \beta}), \quad \frac{\partial D}{\partial \beta} \approx -E_a D, \quad E_a \text{ is the activation energy.}
\]
From the condition \( u_\gamma = 0 \) we find
\[
\frac{1}{v} \frac{\partial v}{\partial \beta} = -E_a \frac{\sqrt{1 + 4 \gamma / a} - 1 - 2 \gamma / a}{\sqrt{1 + 4 \gamma / a} - 1}.
\]

By \( \gamma \to 0 \), \( \partial v / \partial \beta \to 0 \);
\[
\frac{1}{T_0} \frac{\partial T_0}{\partial \beta} = E_a \frac{\sqrt{1 + 4 \gamma / a} - 1 - 2 \gamma / a}{\sqrt{1 + 4 \gamma / a} - 1}, \quad \frac{1}{a} \frac{\partial a}{\partial \beta} = -E_a \frac{\sqrt{1 + 4 \gamma / a} - 1 - 4 \gamma / a}{\sqrt{1 + 4 \gamma / a} - 1}, \quad M_1 = T_0 (1 - \beta E_a).
\]

For small \( \gamma \),
\[
\frac{1}{T_0} \frac{\partial T_0}{\partial \beta} \approx -E_a \frac{2 \gamma}{a}, \quad \gamma^2 \approx \frac{\dot{s}}{\rho} \frac{a}{2 \beta E_a}.
\]

Substitution of (38) into (32) gives the relation
Substitution of expressions (39) - (40) into relation (38) leads to an equation for
\[ \gamma = 2y / a(\sqrt{1+4y/a} - 1), \quad \gamma = ay(y-1), \] of the form
\[ \frac{\dot{\gamma}}{\rho} a[1-y(2-\beta E_a)] + y^2(1-\beta E_a)] - \left(\frac{\dot{\gamma}}{\rho}\right)^2 = a^2 y(y-1)^2[(1-\beta E_a)y + \beta E_a]. \quad (42) \]
When \( \gamma \to 0, \quad y \to 1 \). The solution of equation (42) after substitution in (41) leads to an expression for the diffusion coefficient of the form
\[ D = \frac{L^2}{T_\gamma} = D_0 \frac{T_0}{T_\gamma} = D_0 \sqrt{1 + \frac{4}{aM_1} \left[ C_1 \frac{T_0}{M_1} \frac{\partial T_0}{\partial \gamma} \right] \left( 1 + \frac{\dot{\gamma}}{\rho} \frac{2T_0M_1(\beta T_0/\partial \gamma) + \dot{\gamma} T_0)}{C_1(\beta T_0/\partial \gamma)^2} \right]}, \quad D_0 = \frac{L^2}{T_0}, \quad C_1 = \frac{1}{2} \left( aT_0 - \beta \frac{\partial (aT_0)}{\partial \gamma} \right), \quad M_1 = T_0(1-\beta E_a). \quad (43) \]
For small \( \gamma \), when \( y \approx 1 \), we write in (42) \( y = 1 + \epsilon \gamma \), where \( \epsilon \) is a small value. Neglecting powers \( \epsilon^2 \) higher than two in the resulting equation, we obtain a quadratic equation for \( \epsilon \gamma \) with the solution
\[ \epsilon = \frac{1}{2(1-\beta E_a)(1-2\dot{\gamma} / \rho a)} \left( 1 - 3\beta E_a \right)^2 + \sqrt{R_0} \right), \]
\[ R_0 = 4 \frac{\dot{\gamma}}{\rho a} (1-\beta E_a)^2 + \left( \frac{\dot{\gamma}}{\rho a} \right)^2 (8 + \beta E_a + (\beta E_a)^2) + 8 \left( \frac{\dot{\gamma}}{\rho a} \right)^2 (1-\beta E_a), \]
\[ \gamma = aM_1(1 + aM_3), \quad M_3 = \epsilon \gamma. \quad (44) \]
The + sign is selected to \( \epsilon > 0, \quad \gamma > 0 \). Then expressions (41) and (43) take the form
\[ T_\gamma = \frac{T_0}{\sqrt{1 + 4M_3(1 + M_3)}}, \quad D = D_0 \sqrt{1 + 4M_3(1 + M_3)}. \quad (45) \]
For molecular diffusion in perfect fluid mixtures in [43] the expressions for the Helmholtz’s free energy \( \Psi = u - T_s \) were obtained \( \Psi_{eq} - \Psi = -\frac{1}{2} \frac{T^2}{\rho} \), where \( J = J \) is the diffusion flux of the first component \( \tau \) is the relaxation time of \( J \). Using expressions (28), (29), we find \( \Psi_{eq} - \Psi = u_{eq} - u - T(s_{eq} - s) = -u_{eq} - T\Delta = -K, \quad \gamma T_\gamma + \ln Z_{\gamma} + \beta K = 0 \). Solving this equation for \( \gamma \), we find
\[ \gamma = R \left( \frac{2R}{aT_0} + \frac{1}{T_0} \sqrt{1 + \frac{4R^2}{(aT_0)^2}} \right), \quad R = \sqrt{K(K + aT_0)}, \quad T_\gamma = \frac{T_0}{\sqrt{1 + 4R^2/aT_0^2 + 1 + 4R^2/(aT_0)^2}} \quad (46) \]
Relation (46) can be rewritten as
\[ T_\gamma = \frac{T_0}{\sqrt{1 + 2p(p + 1 + p^2)}}, \quad D = D_0 \sqrt{1 + 2p(p + 1 + p^2)}, \quad p = 2R/aT_0. \quad (47) \]
The described approach is applicable to multiple problems with FPT concept. For example, in [86], using FPTs, problems such as the thermal motion of a small tracer in a viscous medium, adhesion bond dissociation under mechanical stress, algorithmic trading, first crossing of a moving boundary by Brownian motion, quadratic double-well potential are considered. In [86]
studied the random variable \( \tau = \inf \{ t > a : X(t) > L \} \), i.e. the first exit time of the process \( X(t) \) from an interval \([L, L]\) when started from \( x_0 \) at \( t_0 = 0 \). The expression for the Laplace transform of the FPT distribution used in [86] is more complicated than (31) (the same expression (48) was obtained in [25] for Feller processes), and has the form

\[
Z_{\gamma}(r_0, \gamma) = \frac{U(\frac{L^2}{4\kappa D}, \frac{r_0^2}{2}, \frac{r_0}{L})}{U(0, \frac{d}{2}, \kappa)} U(0, \frac{d}{2}, \frac{r_0}{L}),
\]

where

\[
U(a, b; z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b; z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b; z)
\]

is the confluent hypergeometric function of the second kind [87] (also known as Tricomi function), \( M(a, b; z) = e^t \Gamma(a) \). In [86] a diffusing particle of mass \( m \) trapped by a harmonic potential of strength \( k \) and pulled by a constant force \( F_0 \) is considered; the force in the Langevin equation \( F(X(t)) = -kX(t) + F_0 \) includes the externally applied Hooke term and constant force \( F_0, D = k_bT/\zeta \) is the diffusion coefficient, \( \zeta \) is the drag constant.

An equation of the form (29) for \( \gamma(\Delta) \) and Laplace transform (48) takes on a more complex form and is difficult for an analytical solution. But in some approximations [86] a closed analytical expression to the problem can be found.

Consider the Feller process, which is a diffusion process with linear drift and linear diffusion coefficient vanishing at the origin of coordinates. In [25], an expression was obtained for the Laplace transform of the probability density of the first reaching by the Feller process \( Y(t) \) of the boundary \( x = x_c \) for \( x > x_c \) (\( x \) is the initial value of the process at the initial moment of time) in the form

\[
Z_{\gamma} = \frac{U(\gamma, b; x_c) U(0, b; x_c)}{U(\gamma, b; x_c) U(0, b; x)}.
\]

This expression (50) differs from (48) only in the values of the parameters. In (50), expression (49) with parameters \( \alpha = \gamma, \beta = 2\beta_1k^2 \), where \( \beta_1, k > 0 \) are drift and diffusion parameters of the Feller process. The time evolution of the process is thus governed by

\[
dY(t) = [-\alpha Y(t) + \beta_1] dt + k\sqrt{Y(t)} dW(t),
\]

where \( \alpha > 0, \beta_1, k \) are constant parameters, \( W(t) \) is the Wiener process. In the approximation of small values of \( \alpha = \gamma \) in (49), (50), one can use small \( \gamma \) expansion of the Kummer function [25]

\[
U(a, b, x) = 1 + aU_1(x) + O(a^2), \quad U_1(x) = -\Psi(1-b) - \frac{1}{2} U(1, 1+b, z) dz, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad 0 < b < 1.
\]

In this approximation at \( x = 0 \)

\[
Z_{\gamma} = \frac{1+\gamma U_1(x)}{1+\gamma U_1(0)}, \quad \overline{\gamma} = \frac{U_1(0)}{1+\gamma U_1(0)} - \frac{U_1(x)}{1+\gamma U_1(0)}, \quad T_0 = U_1(0) - U_1(x) = \frac{1}{2} U(1, 1+b, z) dz.
\]

An increase of the mean FPT, when \( \overline{\gamma} > T_0 \), in this case is possible at

\[
U_1(0) < T_0 < 2U_1(0), \quad U_1(0) = -\Psi(1-b) > 0, \quad 0 < b < 1.
\]
We assume that $\gamma>0$. Consider two cases. In the first case $1+\gamma U_1(x)=1-\gamma(T_0-U_1(0))>0$, $T_0-U_1(0)>0$, $U_1(x)=U_1(0)-T_0<0$. Then the values of the parameter $\gamma$ for which the inequality $\bar{T}_\gamma>T_0$ for the quantities from (51) is satisfied lie in the interval

$$\frac{2U_1(0)-T_0}{U_1(0)[T_0-U_1(0)]} < \gamma < \frac{1}{T_0-U_1(0)}.$$  

(53)

For the case $1+\gamma U_1(x)=1-\gamma(T_0-U_1(0))<0$, the opposite inequality

$$\frac{2U_1(0)-T_0}{U_1(0)[T_0-U_1(0)]} > \gamma > \frac{1}{T_0-U_1(0)}$$

should hold, which will not hold, since it follows from (52) that $0<2-T_0/U_1(0)<1$.

The fulfillment of inequality (43) depends on the parameters $b$ and $x$. Equation (29) for determining the dependence of $\gamma$ on $A$ taking into account the term of the form (17), (18), (28) in this case takes the form

$$-\Delta = \ln(1+\gamma U_1(x)) - \ln(1+\gamma U_1(0)) + \frac{\gamma[U(1)(0)+\beta \partial U_1(x)]}{1+\gamma U_1(0)} \partial \beta.$$ 

Expanding the right-hand side of this equation in a series in $\gamma$, and taking into account the terms up to $\gamma^2$, we obtain the quadratic equation

$$\gamma^2 a_2 + \gamma b_2 - \Delta = 0, \quad a_2 = (U_1^2(0)-U_1^2(x))/2 + U_1(0)/\partial \beta - U_1(x)\beta \partial U_1(x)/\partial \beta,$$ 

(54)

$$b_2 = -\beta(\partial U_1(x)/\partial \beta) - \partial U_1(0)/\partial \beta = -\beta \partial T_0/\partial \beta + \partial U_1(0)/\partial \beta = \partial U_1(x)/\partial \beta.$$ 

Expanding the right-hand side of (54) in a series in $\gamma$, and taking into account the terms up to $\gamma^2$, we obtain the quadratic equation

$$\gamma^2 a_2 + \gamma b_2 - \Delta = 0, \quad a_2 = (U_1^2(0)-U_1^2(x))/2 + U_1(0)/\partial \beta - U_1(x)\beta \partial U_1(x)/\partial \beta,$$ 

(54)

$$b_2 = -\beta(\partial U_1(x)/\partial \beta) - \partial U_1(0)/\partial \beta = -\beta \partial T_0/\partial \beta + \partial U_1(0)/\partial \beta = \partial U_1(x)/\partial \beta.$$ 

To solution the quadratic equation (54), condition (53) takes the form

$$\frac{[2U_1(0)-T_0]}{U_1^2(0)[T_0-U_1(0)]} \left[ T_0[2U_1(0)-T_0][1/2](2U_1(0)-T_0)+\beta \partial U_1(0)/\partial \beta \right] + [U_1(0)-T_0][3U_1(0)-T_0]\beta \partial T_0/\partial \beta < \Delta <$$

$$< \frac{1}{T_0-U_1(0)} \left[ T_0[2U_1(0)-T_0]+\beta \partial U_1(0)/\partial \beta \right] + [2U_1(0)-T_0]\beta \partial T_0/\partial \beta.$$ 

(55)

In (52)-(55) do not include explicit expressions for $\partial U_1(0)/\partial \beta$, $\partial T_0/\partial \beta$. Let’s find them. Suppose the parameter $\alpha$ in the stochastic equation for Feller process is $\alpha = 1/T_0$. For the Feller process, the drift parameter $f = -\alpha Y + b_1$. As above, in (30) - (31), we set the parameter $f$ to be $f = L/T_0$, where $L = x-x_$. For $x_0=0$, $L=x$, where $x$ is the initial value of the process at the initial moment of time. Then $\beta_i = f + \alpha Y = (x+y)/T_0$. The diffusion coefficient is $D = k^2 Y$, and the parameter $b$ of (49)-(52) is equal to $b = 2\beta_2/k^2 = 2(x+y)Y/T_0D$. At $\partial D/\partial \beta = -E_a D$

$$\frac{\partial b}{\partial \beta} = b(E_a - \frac{1}{T_0} \partial T_0/\partial \beta).$$ 

(56)

From (51) - (52) we obtain

$$\frac{\partial T_0}{\partial \beta} = \frac{\partial U_1(0)}{\partial \beta} - \frac{\partial U_1(x)}{\partial \beta}, \quad \frac{\partial U_1(0)}{\partial \beta} = \frac{\partial U_1(x)}{\partial \beta}.$$ 

From (18), (51) we find

$$U_\gamma = \frac{\gamma \partial U_1(x)}{1+\gamma U_1(0)}.$$ 

As above, we assume $U_\gamma = 0$. From here...
\[
\frac{\partial T_0}{\partial \beta} = \frac{\partial \Psi T_0 b E_a}{1 + \gamma (U_1(0) + b \partial \Psi)}, \quad \frac{\partial \Psi}{\partial \beta} = \frac{\partial \Psi(1 - b)}{\partial (1 - b)}. \tag{57}
\]
\[
\frac{\partial U_1(0)}{\partial \beta} = \frac{\partial \Psi(1 + \gamma U_1(0)) b E_a}{1 + \gamma (U_1(0) + b \partial \Psi)}, \quad \frac{\partial U_1(x)}{\partial \beta} = \frac{\partial \Psi(1 + \gamma U_1(x)) b E_a}{1 + \gamma (U_1(0) + b \partial \Psi)}. \tag{58}
\]
Substituting (57), (58) into (54), we obtain
\[
a_2 = \frac{1}{2} (U^2_1(0) - U^2_1(x)) + \frac{b b E_a \partial \Psi}{1 + \gamma (U_1(0) + b \partial \Psi)} [T_0 + \gamma (U^2_1(0) - U^2_1(x))],
\]
\[
b_2 = -\beta \frac{\partial T_0}{\partial \beta} = \frac{\gamma \partial \Psi T_0 b b E_a}{1 + \gamma (U_1(0) + b \partial \Psi)}.
\]
Equation (54) for determining \(\gamma\) in terms of \(\Delta\) takes the form
\[
\gamma^2 (U^2_1(0) - U^2_1(x)) [b b E_a \partial \Psi + (U_1(0) + b \partial \Psi) / 2] + \gamma^2 (U^2_1(0) - U^2_1(x)) / 2 - \gamma \Delta (U_1(0) + b \partial \Psi) - \Delta = 0.
\]
Neglecting the terms with \(\gamma^3\), we find a positive solution to the quadratic equation
\[
\gamma \approx \frac{1}{(U^2_1(0) - U^2_1(x))} [\Delta (U_1(0) + b \partial \Psi) + \sqrt{\Delta^2 (U_1(0) + b \partial \Psi)^2 + 2 \Delta (U^2_1(0) - U^2_1(x))}]. \tag{59}
\]
At small values of \(\Delta\), when
\[
\sqrt{2 \Delta (U^2_1(0) - U^2_1(x))} \left[1 + \Delta (U_1(0) + b \partial \Psi)^2 \right] \frac{1}{2(U^2_1(0) - U^2_1(x))} \approx \left[1 + \Delta (U_1(0) + b \partial \Psi)^2 \right] \frac{1}{4(U^2_1(0) - U^2_1(x))} + \ldots
\]
expression (59) is rewritten as
\[
\gamma \approx \frac{1}{(U^2_1(0) - U^2_1(x))} [\sqrt{2 \Delta (U^2_1(0) - U^2_1(x))} + \Delta (U_1(0) + b \partial \Psi)(1 + \sqrt{2 \Delta (U^2_1(0) - U^2_1(x)) (U_1(0) + b \partial \Psi)}) + \ldots]
\]
or
\[
\gamma = \gamma_0 \sqrt{\Delta} + \gamma_1 \Delta + \gamma_2 \Delta^{3/2}, \quad \gamma_0 = \frac{2}{(U^2_1(0) - U^2_1(x))}, \quad \gamma_1 = \frac{U_1(0) + b \partial \Psi}{(U^2_1(0) - U^2_1(x))}, \quad \gamma_2 = \frac{\gamma_0 (U^2_1(0) + b \partial \Psi)^2 / 4}{4(U^2_1(0) - U^2_1(x))}. \tag{61}
\]
We rewrite the condition \(T_\gamma > T_0\) in the form \(T_0 > T_0 [1 + \gamma (U_1(x) + U_1(0)) + \gamma^2 U_1(x) U_1(0)]\),
\[
(U_1(0) + U_1(x)) + \gamma U_1(0) U_1(x) < 0, \quad \gamma (-U_1(x) U_1(0)) > U_1(0) + U_1(x).
\]
This condition can be fulfilled for \(U_1(x) < 0, \quad U_1(x) + U_1(0) > 0\), which is possible at certain values \(x\). Substituting expression (61) into (62), ignoring the term with \(\Delta^{3/2}\), we find the condition that \(T_\gamma > T_0\) in the form
\[
\sqrt{\Delta} > \gamma_+, \quad \gamma_+ = \frac{\sqrt{U^2_1(0) - U^2_1(x))}}{2(U_1(0) + b \partial \Psi)} \left[1 - \frac{2(U_1(0) + U_1(x)) (U_1(0) + b \partial \Psi)}{(U_1(0) U_1(x))} - 1\right], \tag{63}
\]
where \(\gamma_+\) is the positive solution to the quadratic equation \(\gamma_+ \Delta + \gamma_0 \sqrt{\Delta} + \frac{U_1(0) + U_1(x)}{U_1(0) U_1(x)} = 0\).

Substitution of expression (61) into the expression for \(T_\gamma\) leads to the expression
\[
T_\gamma = \frac{T_0}{R_2}, \quad R_2 = 1 + d_{1/2} \Delta^{1/2} + d_1 \Delta^1 + d_{3/2} \Delta^{3/2} + d_2 \Delta^2 + d_{5/2} \Delta^{5/2} + d_3 \Delta^3, \tag{64}
\]
\[ d_{1/2} = \sqrt{\frac{2(U_1(0) + U_1(x))}{T_0}}, \quad d_1 = \frac{1}{T_0} [U_1(0) + b \partial \Psi + \frac{2}{T_0} \frac{U_1(0)U_1(x)}{(U_1(0) + U_1(x))^{3/2}}], \]

\[ d_{3/2} = \frac{2}{T_0} (U_1(0) + b \partial \Psi)^{1/2} \frac{1}{4} \sqrt{(U_1(0) + U_1(x))(U_1(0) + b \partial \Psi) + \frac{2}{T_0} \frac{U_1(0)U_1(x)}{(U_1(0) + U_1(x))^{3/2}}}, \]

\[ d_2 = U_1(0)U_1(x) \frac{\partial^2}{4} (U_1(0) + b \partial \Psi)^2 (2 + \gamma_0^2), \quad d_{5/2} = U_1(0)U_1(x) \frac{\partial^2}{4} (U_1(0) + b \partial \Psi)^2 \gamma_0, \]

\[ d_3 = U_1(0)U_1(x) \frac{\partial^2}{4} (U_1(0) + b \partial \Psi)^4 \frac{1}{4}. \]

6. Choice of distribution for FPT. An example with electrons transfer.

In this work, the theory does not allow to determined first passage statistics. In this case, the "incorporate" of first passage statistics naturally arises as a choice of the random process that describes the phenomenon under study and the appropriate FPT.

There are many papers exploring FPT statistics [1-7, 10-35, 37-39, 46-48, 86, 89, 107-109]. As a general mathematical plan [1-6, 10-12, 15, 23-35, 46-48, 86], and in connection with a variety of applications [4-7, 13-14, 16-22, 37-39, 89, 107-109], with a variety of types of FPT statistics and in connection with the required tasks. All of them can be used for the purposes of this article.

The main problem is to adequately and as completely as possible establish a correspondence between a physical phenomenon and the stochastic process with which we describe this phenomenon. With respect to FPT, the variety of existing stochastic processes, in each of which a change in the boundary conditions can significantly affect the properties of the process (for example [4, 22]), makes it possible to single out, for example, random processes for which the average FPT can only decrease under various influences on the system, and a more general, but also a more difficult amenable to analytical calculations class of random processes, for which an increase in the average FPT is possible under certain influences.

What are the criteria for choosing a random process and the corresponding FPT? Better match with a random process. At least qualitative agreement with the experimental results. Some kind of internal criteria. In general, this is the subject of a separate study. The behavior of the same phenomenon under different conditions can be different and can be described by different random processes, different distributions and their Laplace transforms. Examples of this kind are given in [88]. We are not interested FPT distributions, but their Laplace transforms. This makes the task some what easier.

FPT statistics is determined by the random process that characterizes the phenomenon under study. In cases where the exact solution cannot be found, one can use general results for FPT, for example, the first few terms of the expansions obtained in [27-28] for Laplace transform of FPT, determining the quantities included in them depending on the process under consideration. There are other options for selecting and setting FPT statistics models. The use of various kinds of approximations also substantially depends on the stage of evolution at which the system under consideration is [32, 39].

The range of applicability of the results obtained is extensive and, apparently, corresponds to the range of applicability of Gibbs statistics. This generality does not allow us to restrict ourselves to any one FPT distributions, but includes the most arbitrary distributions.

Let's give one more example. Unlike the examples in Section 5, it is based on experimental results. This example clearly shows the importance of taking into account the change in entropy and the effect of external influences on the average FPT.
In [89, 90] was investigated the fluctuations of the time elapsed until the electric charge transferred through a conductor reaches a given threshold value. The distribution of the first-passage times for the net number of electrons transferred between two metallic islands in Coulomb blockade regime is considered.

In [89] was derived a simple analytical approximation for the first-passage-time (FPT) distribution, which takes into account the non-Gaussian statistics of the electron transport, and showed that it describes the experimental distributions with high accuracy.

In the context of mesoscopic electron transport, the interest in the distributions of first passage times and waiting times [91, 92] has been inspired by the tremendous progress in nanotechnology allowing very precise single-electron counting experiments [93, 94].

In [89, 90] is to studied the first-passage-time probability distribution $P_N(t)$ for a stochastic process to first reach or surpass a given value $N$ at time $t$, the first-passage-time statistics for electrons transferred through a metallic double dot in the Coulomb-blockade regime, the distribution of the first passage times for electrons tunneling between two islands in the Coulomb blockade regime employing single electron counting technique. For this purpose, in [89, 90] was obtained the full time-record of electron tunneling events between its two metallic islands (see Ref. [89, 90] for details).

The distribution of FPT was obtained in [89, 90] widely used in various tasks (e.g. in queue theory [95]). This distribution has the form

$$P_N(t) = \frac{1}{t} \left| N^{\star} \right| e^{-\frac{t}{t_{\text{FPT}}}} \left( \frac{C_2 + \sqrt{C_1 C_3}}{C_2 - \sqrt{C_1 C_3}} \right)^{N^{\star}/2} I_0 \left( \frac{C_1 \sqrt{C_2^2 - C_1 C_3}}{C_3} t \right),$$

where $N^{\star} = [N \sqrt{C_1/C_3}]$, $C_i, i=1,2,3$ are cumulants of the distribution $P_d(N)$ for the process to take the value $N$ at a fixed time $t$, $I_d(x)$ is the modified Bessel function of the first kind, and $N$ is the threshold value for the number of virtual particles such that the charge transmitted by them, $e' N^{\star}$, $e$ is the electron charge, $e' = e \sqrt{C_2/C_1}$ gets as close as possible to the net charge of real electrons $eN$.

The cumulants $C_1$ and $C_2$ are related to the average electric current $\langle I \rangle$ and the current noise $\langle I(t) I(0) - \langle I \rangle^2 \rangle = 2e^2 C_2$. The values $C_3$ and $C_4$ are the third and the fourth cumulants of the distribution of the number of transmitted electrons $P_d(N)$. In [89, 90] have also assumed that $C_1, C_3 > 0$ and $C_2^2 > C_1 C_3$. Distribution (56) was obtained in [89] from a general approach with certain approximations (for sufficiently long times, setting the approximate form of the cumulant generating function, setting the transition speeds, the rates of jumping forward and backward).

We consider the connection between this process of the first-passage times for the net number of electrons transferred between two metallic islands and the accompanying change in the entropy of the system.

In [90] experimentally studied negative fluctuations of stochastic entropy production in an electronic double dot operating in nonequilibrium steady-state conditions. In this paper, we consider the thermodynamic aspects of FPT moments, in particular, the relationship between the entropy change accompanying the first moment of FPT and the first moment of FPT.

The value of the average FPT value, in which the value of the argument of the Laplace transform of the FPT distribution is assumed to be zero, is just an auxiliary value that does not correspond to real events in which the changes in entropy and the values of this argument are not zero. It is necessary to set the value of the argument of the Laplace transform, which corresponds to the change in entropy in the real FPT process of reaching the specified level.

An external DC voltage $V_b$ is considered as external thermodynamic forces. An external DC voltage $V_b$ controls the net current through the double-dot. In [89, 90] measured nonequilibrium charge-state fluctuations in a hybrid normal metal–superconductor double dot in the strong Coulomb blockade regime subject to a time-independent bias voltage. An external DC
bias voltage $V_b$ is applied between the two leads and brings the system into a nonequilibrium steady state.

The approximate expression for the first-passage-time distribution in [89, 90] is based on the exact result for the one dimensional biased random walk [1, 66],

$$ P_N(t) = \frac{1}{t} |N| e^{-\left(\frac{\Gamma_+ + \Gamma_-}{2}\right) N/2} I_N(2\sqrt{\Gamma_+ \Gamma_-} t) , $$

where expression (65) is obtained from (66) at $\Gamma_\pm = \frac{C_1}{2C_3} (C_2 \pm \sqrt{C_2 C_3})$ are, respectively, the rates of jumping forward and backward. This model also describes the transport of charged particles through a voltage biased tunnel junction [96]. At short times the expression (65) behaves as $t^{-1}$.

In [89] was compared the experimental results with the predictions of the exact theory [14, 15] and observed perfect agreement.

The Laplace transform of the distribution (65) has the form

$$ Z_{\gamma}(s) = \frac{(2\Gamma_\gamma)^N}{(s + \Gamma_\gamma + \sqrt{(s + \Gamma_\gamma)^2 - 4\Gamma_\gamma \Gamma_-})^N}, \quad \Gamma_\gamma = \Gamma_+ + \Gamma_- = \frac{C_1 C_2}{C_3}, \quad \Gamma_+ - \Gamma_- = \frac{C_1^{3/2}}{C_3^{1/2}} . $$

The mean value of $\text{FPT}$ determined from expressions (18), (26), (28), (67) is equal to

$$ \langle T_\gamma \rangle = \frac{N^\gamma}{\sqrt{(s + \Gamma_\gamma)^2 - 4\Gamma_\gamma \Gamma_-}} = \frac{T_0}{\sqrt{1 + \frac{s(s + 2\Gamma_\gamma)}{(\Gamma_+ - \Gamma_-)^2}}} , \quad T_0 = \langle T_\gamma \rangle = \frac{N^\gamma}{\Gamma_+ - \Gamma_-} = \frac{N}{C_1} . $$

By $\gamma \geq 0$, $\langle T_\gamma \rangle \leq T_0$.

The values of $\Delta e$ change in entropy at different voltages are taken from Fig. 4, 12 and Table II [90]. Figure 1 shows the dependences $T_0(N) = N/C_1$ (dashed line) and $\langle T_\gamma \rangle (\Delta e) = \langle T_\gamma \rangle (\Delta e)$ for different values of $N$ and $\Delta e$ - solid line. Let us consider external influences using the example of the applied bias voltage $V_b$. Figure 2 shows the dependence of $\langle T(\Delta e) \rangle$ at $T_0(N=10) = 2.174$ on the applied voltage $V_b$. 

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Fig. 1. Dependencies $T_0(N) = \bar{T}_{\gamma_0} = N / C_1$ (dashed line) and $T_1 = \bar{T}_{\gamma_{\lambda_1}}$ (solid line approximating the calculated points) for the value $N$ of the process $N(t)$ to take at a fixed time $t$. As to stochastic process to first reach or surpass a given value $N$, $N(t)$ is the net number of transmitted electrons \[88, 89\]. Bias voltage $V_b = 90 \mu V$. 
Pattern symmetric about the y-axis for negative $V_b$ values.

The average $FPT$ is expressed in terms of the entropy change accompanying this process. In addition to the changes that are necessarily present in the course of a random process, any other changes in entropy can also be taken into account, which correspond to other possible processes occurring in the system. In this paper, the effect of such changes on the average $FPT$ is illustrated by taking into account for different values of the bias voltage. The mean $FPT$ values in accordance with expression (68) decrease when the system is influenced. This is illustrated in Fig. 2, where the applied voltage acts as an effect on the system. It is shown that the average $FPT$ with a zero value of the argument of the Laplace transform of the $FPT$ distribution density, which is associated with changes in the entropy of the system, does not reflect the effect of real processes on the average $FPT$. It is necessary to take into account those changes in entropy that accompany the random process of reaching a certain boundary.

7 Conclusion

Let us formulate, perhaps repeating itself, the significance of this work.

First, the random variable $FPT$ is considered as a thermodynamic parameter. A generalized Gibbs distribution is introduced containing $FPT$ as a thermodynamic parameter. This distribution of the form (15) - (16) is important in many areas. For example, in [97] from a distribution of the form (16) containing $FPT$ in the form lifetime, the distribution of neutron energy in a nuclear reactor was obtained. This theoretically obtained distribution coincides with the experimentally observed distribution of neutrons in energy in a nuclear reactor: the Maxwell distribution for thermal neutrons and the Fermi distribution for fast neutrons with a transition region between them. The use of a distribution of the form (16) is justified by the finite lifetime of neutrons, due to which the equilibrium state of the system of neutrons with moderator nuclei is not achieved. The finite lifetime and size are characteristic of many systems: atomic nuclei, liquid droplets [98], micelles [99], formation in biological membranes [100-102], etc.

Second, the new physical interpretation of $FPT$ makes it possible to relate $FPT$ to thermodynamic quantities, in particular, to the change in the entropy of the system. Knowledge of such patterns allows you to purposefully influence the processes occurring in the system. In many cases, the pace of events is important and often decisive. The events themselves refer to the moments of stopping, of which $FPT$ is a special case. The change in entropy includes various kinds of effects on the system, which can speed up or slow down the course of events in the system. Knowing the relationship between the change in entropy and $FPT$, one can purposefully control the $FPT$ by choosing such effects on the system that, depending on the need, will decrease or increase the average $FPT$. Thus, it is possible to investigate the possibility of influencing the rate of micelle formation [99], the rate of formation of aerosols [103], evaporation and condensation of droplets [98], the formation of nuclei of a new phase, etc. In [99-103] to the description of the processes of micelle formation and the behavior of aerosols and rafts (special areas (microdomains) of the plasma membrane, enriched with glycosphingolipids and cholesterol) in biological membranes, a stochastic storage model was used. Accordingly, $FPT$ for these systems can be obtained from the stochastic storage model.

The third important point is to obtain numerous connections between the stochastic theory $FPT$ - and these theories are diverse, including the theory of queues, the stochastic storage theory, etc. - and statistical physics and nonequilibrium thermodynamics. Such connections will create many applications, both among those where $FPT$ has already been used, but at a new level, taking into account the obtained patterns, and in new directions. For example, application
to the Kramers problem and related applications, to the behavior of various physicochemical systems (emulsions, foams, aerosols, micelles, etc.).

General relationships are important, the implementation of which is difficult to predict. Generalizing ideas, if correct, should lead to a large number of specific applications. The results obtained in this work are also more general, perhaps the most important. Since \( FPT \) characterizes the behavior of a system in time, it can be hoped that, perhaps, further studies of the connections between \( FPT \) and nonequilibrium statistical physics and nonequilibrium thermodynamics will make it possible to advance in understanding such still largely mysterious phenomena, such as the lifetime and time in general.

This paper discussed the change in the entropy of a non-equilibrium statistical system during the processes occurring in it, when a random process describing its certain physical quantity reaches some given boundary, is associated with the Laplace transform parameter of the \( FPT \) distribution and with the thermodynamic parameter conjugate to \( FPT \). The Laplace transform of the \( FPT \) distribution and its derivatives, \( FPT \) moments are expressed in terms of the change in the entropy and other thermodynamic quantities. This is achieved by choosing \( FPT \) as an additional thermodynamic parameter. Besides (30), (48), (56) many other \( FPT \) distributions for various processes [27-29] and physical, biological and other systems can be considered [1-3].

The proposed approach combines mathematical \( FPT \) research with non-equilibrium statistical physics and statistical thermodynamics. In this case, a generalization of the well-known Gibbs ensemble is used. This allows us to consider a large number of problems related to both the Gibbs distribution and the set of \( FPT \) problems noted in the introduction. However, knowledge of the mathematical results of the \( FPT \) of the stochastic study is necessary to determine the Laplace transform of the \( FPT \) distribution. A new look opens up new possibilities for both various aspects of statistical thermodynamics and conventional first passage theory. Practical applications of the proposed approach are wide and varied. This is both obtaining different distributions for many phenomena associated with superstatistics [8, 9], and non-equilibrium thermodynamics [104, 105], and the behavior of neutrons in nuclear reactors [97, 106], and the behavior of aerosols [103] and rafts in biological membranes [100-102] (storage models were used for them, but the approach of this work is also applicable). The proposed method is applicable, for example, to radiation-enhanced diffusion, where this method appears in integral form, without concentrating on the details of the description, to the distribution of the first-passage times for the net number of electrons transferred between two metallic islands in Coulomb blockade regime [89], etc.

The results for \( FPT \) of various stochastic processes, for different values and different limits, different levels of their achievement, can be described by the proposed algorithm. The corresponding \( FPT \) is selected as an additional thermodynamic variable. The conjugate thermodynamic force associated with the change in entropy in the process of reaching various limits by a random process is determined.

The relevant distribution (15) obtained in Section 3 is similar to the relevant \( NSO \) distribution (2). However, distribution (2) does not contain the thermodynamic variable \( FPT \). Distribution (15) is written as a special case of distribution (14) obtained using the results of mathematical statistics.

For exponential distribution (25), arbitrary impact on the mean \( FPT \), expressed in terms of the deviation of entropy from the equilibrium value \( \Delta = s_{eq} - s \) (29), can only decrease the mean \( FPT \). This also applies to mean \( FPT \) (32), (68) at \( \gamma > 0 \). However, there are distributions in which external impacts can increase the average \( FPT \). These are, for example, the distribution (48), the Weibull distribution, which is valid in the limiting case for a two-dimensional continuous-time dynamical system, with an attracting fixed point [24], and many other distributions. In general, for the growth of the mean \( FPT \) \( \overline{T}_\gamma \) the condition must be met \( \overline{T}_\gamma > \overline{T}_{\gamma 0} \), \( \overline{T}_\gamma = - \frac{\partial \ln Z_\gamma}{\partial \gamma} \mid_{\gamma} \).
\[ Z_\gamma = \int_0^\infty e^{-\gamma t} f(x)dx, \quad \int_0^\infty x f(x)dx = \overline{T}_\gamma, \quad \int_0^\infty x e^{-\gamma t} f(x)dx = \int_0^\infty e^{-\gamma t} f(x)dx \int_0^\infty xf(x)dx, \quad Z_\gamma < e^{-\overline{T}_\gamma}, \]

where \( f(x) \) is the probability density of the distribution of the FPT.

The paper provides examples of the application of this algorithm. FPT of a certain flow level value, FPT level \( L \) in drift-diffusion process and FPT Feller process are considered. Explicit dependences of the parameter \( \gamma \), which is the thermodynamically conjugated of FPT, through changes of the entropy are obtained. For distribution (48), an analytical solution is possible only for special cases, in contrast to distributions (30), (35), (65). The proposed approach makes it possible to determine the intervals of entropy deviation from the equilibrium value \( \Delta \approx \Delta \), for which the average FPT increases, as in (62), (64).

Let us indicate some open questions, limitations and approximations.

- It is not determined how far from equilibrium the proposed approach is fair. One of the possibilities for estimating the accuracy of expression (14) is to consider the approximations made in obtaining (14) in [73, 75] for each individual case.

- Why \( u_\gamma = -\frac{\partial \ln Z_\gamma}{\partial \beta} \mid_{\beta} = \int_0^\infty e^{-\gamma t} \left( \frac{\partial f(T_\gamma)}{\partial \beta} \right) dt \frac{1}{Z_\gamma} \) (28), but the symmetric, as in (18), expression \( \overline{T}_\gamma = -\frac{\partial \ln Z_\gamma}{\partial \gamma} \mid_{\gamma} = \int_0^\infty e^{-\gamma t} \left( \frac{\partial f(T_\gamma)}{\partial \gamma} \right) dt \frac{1}{Z_\gamma} \) is not taken into account?

- The limitations of the proposed approach are the assumption that the distribution for FPT is independent of the random value of the internal energy; the need to know \( Z_\gamma \), the Laplace transform of FPT.

- Chaotic regimes, conditionally Lévy processes, are not considered.

- An important role in equations (23), (26) for determining the parameter \( \gamma \) is played by the quantity \( \Delta \), the deviation of entropy from its equilibrium value. In the general case, this value is unknown. Various approximations are used to determine it.

- Not all first passage processes (more precisely trajectories of said process) can be reduced to a one-dimensional support. For example, bivariate Wiener process [107, 108]; in [39, 109] and in other papers was considered the first passage times which not correspond to a one-dimensional (deriving from a projection in full phase space) process. In this work, only one-dimensional processes \( X(t) \) are considered.

Appendix A: Superstatistics and FPT

In the introduction, it was mentioned that the distribution with FPT is related to superstatistics [110]. Let's dwell on this issue in more detail.

In [8, 9], a model of a statistical system with \( n \) classes of states is considered. We use distribution (16) with a function \( \omega(u, T_\gamma) \) of the form (24). Such physical systems were considered in [77–79], and in [79] it was shown that the number of classes of states, regions corresponding to metastable thermodynamic states, increases exponentially with an increase in the total number of particles. The FPT distribution function \( f_k(x) \) from (24) is selected in the form of a gamma distribution \( f_k(x) = x^{\alpha_k-1} e^{-x/b_k} / \Gamma(\alpha_k)b_k^{\alpha_k} \). The distribution parameters are chosen equal \( \alpha_k = \gamma_k / \lambda_k \), \( b_k = T_{\gamma0k} \lambda_k / \gamma_k \). Integration over \( T_\gamma \) in (16) gives \( \int_0^\infty e^{-\gamma t} f_k(x)dx = (1+\gamma_k b_k)^{-\alpha_k} \).

A stochastic storage model [111] is used in which \( 1+\gamma_k T_{\gamma0k} = Q_k = \exp[\beta_k P_k V_k] \). We denote \( Q_k \) is the grand statistical sum of the grand canonical ensemble of the part of system in \( k \)-th
metastable state, pressure $P_k$, volume $V_k$, temperature $T_k$ in $k$-th metastable state, $V$ is the full system volume, $\alpha_k = \gamma_k / \lambda_k$, $\lambda_k$ is the intensity of energy flow in the system (subsystem), equal in dynamical equilibrium of an output intensity. For realistic metastable systems the detailed balance principle is satisfied and $\gamma_k \neq 0, \gamma_k = \gamma$ from (16) in $k$-th metastable area. From (16), (24), after integration over $T_\gamma$, the distribution for $u$ is written in the form

$$p(u) = e^{-\beta_0} \omega(u) Z^{-1} \sum_{k=1}^{n} R_k e^{-\gamma_k R_k / \mu} ,$$

(69)

where $R_k$ is the probability that system will in $k$-th class of ergodic states, $\beta_0 = \sum_{k=1}^{n} R_k \beta_k$, $u = \sum_{k=1}^{n} R_k u_k$. We multiply and divide the value $(P_k V_k \alpha_k)$ by a random value $u$, and in the denominator we replace $u$ with the average value $\bar{u}$. The value conjugate to the random variable $u$ in the exponent (a) will have the form $y_k \beta_0 r_0$, where, $r_k = (P_k V_k \alpha_k) / \bar{u}$, $u=U/V$, $\bar{u} = \bar{U} / \bar{V}$, $r_0 = \sum_{k=1}^{n} R_k r_k$, $y_k = \beta_k r_k / \beta_0 r_0$, $U$ is the energy of the full system. The error introduced by this operation is compensated by the form of the obtained approximate distribution. Parameter $r_0$ is related to the controlling parameter of a problem (for example, with the feedback coefficient in the Van der Pol generator [88], the birth parameter in the Maltus-Ferhuelst process etc). Distribution (60) has the form of the product of the Gibbs distribution $\exp[-\beta u]$ and a multiplier in the form of superstatistics. The prevalence of some of the factors depends on the control parameter $r_0$. If we go from the discrete case (69) to a continuous energy relief, then for $r_k = (P'(x) V'(x) \alpha'(x)) / \bar{u}$, $\alpha_k \rightarrow \alpha'(x)$, $R_k \rightarrow R'(x)$, $T_k \rightarrow T'(x)$, $f'(x) = dR'(x) / dx$, $\beta_1 = 1 / k \beta T'(x)$, $y(x) = \beta(x) r(x) / \beta_0 r_0$, $y_1 = \beta_1 r_1 / \beta_0 r_0$, and from the discrete distribution (69) we go over to a continuous distribution of the form

$$p(u) = e^{-\beta_0} \omega(u) Z^{-1} \int \int f(y_1) e^{-\gamma_i \beta_i / \mu} dy_1, \quad \int \int f(y) dy = 1, \quad f(y_1 = \beta_1 r_1 / \beta_0 r_0) = \frac{dR'(x) / dx}{dy_1 / dx},$$

(70)

where $r_0 = (P_0 V_0 \alpha_0) / \bar{u} = \langle r \rangle = \int r f(r) dr = \langle \alpha P V \rangle / \bar{u}$, $\alpha = \gamma / \lambda$, $\alpha_0 = \int \alpha f(\beta r_1) d(\beta r_1)$. In the open system with complex potential an extra factor $\beta \alpha PV$ appears, taking into account the exchange with surrounding (values $\lambda, \gamma$) and the fluctuations of the inverse temperature $\beta$ and values of $V, P, \alpha$. If one performs the replacement of variables $\beta' = \beta + r_1 \beta_1$ and assumes that the situation $\beta = 0$ is possible, instead of (70) we shall obtain

$$p(u) = \omega(u) Z^{-1} \int \int f_i(\beta') e^{-\beta' \mu} d\beta', \quad f_i(\beta') = f(\beta, r_1) = f(\beta - \beta_0 (1 - r_0)) ,$$

(71)

which coincides with superstatistics [110]. But the correlation (70) describes a more general situation and superstatistics forms here a special case (61). Right-hand side (69) using the thermodynamic relation $\Omega = \overline{U} - TS - \mu \overline{N}$ where $\Omega$ is the thermodynamic potential of the grand canonical ensemble, $S$ is the entropy, and $\mu$ is the chemical potential, $\overline{N}$ is the average number of particles, written in the form

$$p(u) = e^{-\beta_0} \omega(u) Z^{-1} \sum_{k=1}^{n} R_k m_k e^{-\gamma_k R_k / \mu} , \quad m_k = \exp(\beta_k (T_k S_k + \mu \overline{N}_k)), \quad r_{\text{f}} = (U_k \alpha_k) / \bar{u} , \quad y_{\text{f}} = (r_{\text{f}} k \beta_k) / (r_{\text{f}} \beta_0 r_0), \quad r_{\text{0}} = \sum_{k=1}^{n} R_k r_{\text{f}} k , \quad \beta_k \alpha_k u_k = \beta_k r_{\text{f}} k u = y_{\text{f}} (r_{\text{f}} \beta_0) \beta u .$$

For $r_0 \rightarrow 0$ from (70) we obtain the Gibbs distribution; $r_0=0$ for $\alpha = 0, \gamma = 0$ when there is no connection with the environment; for $r_0 \rightarrow 1$ from (70) and (71) we obtain superstatistics. The
obtained distribution of a kind (70) gives the wide opportunities of the description of various physical situations.

Distributions (69) - (71) can be obtained in another way, if we assume that the intensity of energy flow in the system $\lambda_k$ depends on the density of the total internal energy of the system $u$ as $\lambda \rightarrow \lambda_0 / (n_0 + u)$. The value $x_0$ depends as $\bar{x} \rightarrow \bar{x}_0 (n_0 + u)$. For constant release rate equal to one of the storage model [111] the partition function $Q_k$ wherein not changes; $Q_k^{-1} = P_{ok} = 1 - \rho$; $\rho = \lambda \bar{x}$, $\bar{x} = \int x b(x) dx$, $b(x)$ is distribution function one jumps input with average value $\bar{x}$. Such a replacement can be substantiated by physical considerations that with an increase in the energy of the system and an increase in the number of particles in it, it is more difficult for flows of energy and the number of particles to enter the system. Then $\alpha_k = \gamma_k / \lambda_0 k \rightarrow \gamma_k (n_0 + u) / \lambda_0 k = \alpha_0 k (n_0 + u)$, $\alpha_0 k = \gamma_k / \lambda_0 k$. The exponent in (69) will be equal to $-\alpha_k \ln Q_k = -\alpha_k \beta_k P_k V_k = -\alpha_0 k n_0 k \beta_k P_k V_k - \alpha_0 k \beta_k P_k V_k u$. The first term is included in the probabilities $R_k$; the normalization of this distribution changes accordingly. Allowance for the second term leads to distribution (69).

In our work we have obtained the superstatistics in a way different from that of [110]. This way is free from the shortcomings pointed out in [112]. The suggested approach can be considered as broadening and detailing the superstatistics theory. The obtained distribution contains the new parameter related to a thermodynamic state of the system, and also with distribution of a lifetime of a metastable states and interaction of this states with an environment.

The Gibbs distribution does not describe the dissipative processes that develop in the system. Superstatistics describe systems by constantly putting energy into the system, which is dissipated. The value $\alpha = \gamma \lambda$ is connected with dissipative processes in the system (through $\gamma$). She defines a correlation between Gibbs and superstatistics multipliers in distribution (70). In addition to (69) - (71), other forms of writing distributions of the form of superstatistics are possible, obtained from a distribution of the form (15) - (16) [113].

The original distribution (16) contains two distribution functions: $f$ and $R$, in contrast to superstatistics, which contain only one distribution density for the reciprocal temperature. Therefore, distribution (16) and the distributions (60)-(62) obtained from it have greater capabilities than superstatistics. When obtaining expressions (69)-(71) from (16), only the gamma distribution function for $f$ was used. You can choose another type for this distribution function.

In [83], superstatistics were obtained using distributions of the form (20), (22). The parameter $\theta$ in (20) represents some extensive thermodynamic parameter corresponding internal thermodynamic parameter, describing nonequilibrium of the system. It can be coordinate of the center of weights in a field of weight, the electric moment of dielectric in an external electric field, etc. As $\vec{\theta}$ is vector value expressions of a kind (20), describe also superstatistics with several fluctuating thermodynamic parameters. Such expressions are received in [8, 9, 113].

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