Spacelike CMC 1 surfaces 
with elliptic ends in de Sitter 3-Space

(Dedicated to Professor Takeshi Sasaki on the occasion of his sixtieth birthday)

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Abstract. We show that an Osserman-type inequality holds for spacelike surfaces of constant mean curvature 1 with singularities and with elliptic ends in de Sitter 3-space. An immersed end of a constant mean curvature 1 surface is an "elliptic end" if the monodromy representation at the end is diagonalizable with eigenvalues in the unit circle. We also give a necessary and sufficient condition for equality in the inequality to hold, and in the process of doing this we derive a condition for determining when elliptic ends are embedded.

Key words: de Sitter 3-space, spacelike CMC 1 surface, admissible singularities.

Introduction

It is known that there is a representation formula, using holomorphic null immersions into $SL(2, \mathbb{C})$, for spacelike constant mean curvature (CMC) 1 immersions in de Sitter 3-space $S^3_1$ [AA]. Although this formula is very similar to a representation formula for CMC 1 immersions in hyperbolic 3-space $\mathbb{H}^3$ (the so-called Bryant representation formula [B, UY1]), and the global properties of CMC 1 immersions in $\mathbb{H}^3$ have been investigated [CHR, RUY1, RUY2, UY1, UY2, UY3, Yu], global properties and singularities of spacelike CMC 1 immersions in $S^3_1$ are not yet well understood. One of the biggest reasons for this is that the only complete spacelike CMC 1 immersion in $S^3_1$ is the flat totally umbilic immersion [Ak, R]. This situation is somewhat parallel to the relation between minimal immersions in Euclidean 3-space $\mathbb{R}^3$ and spacelike maximal immersions in Lorentz 3-space $\mathbb{R}^3_1$, the $\mathbb{R}^3$ case having been extensively studied while the $\mathbb{R}^3_1$ case still being not well understood. Recently, Umehara and Yamada defined spacelike maximal surfaces with certain kinds of “admissible” singularities, and named them “maxfaces” [UY4]. They then showed that maxfaces are rich objects with respect to global geometry. So in Section 1, we introduce

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admissible singularities of spacelike CMC 1 surfaces in $S^3_1$ and name these surfaces with admissible singularities “CMC 1 faces”.

For CMC 1 immersions in $H^3$, the monodromy representation at each end is always diagonalizable with eigenvalues in $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$, but this is not true for CMC 1 faces in general. However, when the monodromy representation at each end of a CMC 1 face is diagonalizable with eigenvalues in $S^3_1$, we can directly apply many results for CMC 1 immersions in $H^3$ to CMC 1 faces. So in Section 2, we give a definition of “elliptic ends” of CMC 1 faces. In Section 3, we give a necessary and sufficient condition for elliptic ends to be embedded.

The total curvature of complete minimal immersions in $\mathbb{R}^3$ of finite total curvature satisfies the Osserman inequality [O1, Theorem 3.2]. Furthermore, the condition for equality was given in [JM, Theorem 4]. This inequality is a stronger version of the Cohn-Vossen inequality. For a minimal immersion, the total curvature is equal to the degree of its Gauss map, multiplied by $-4\pi$. So the Osserman inequality can be viewed as an inequality about the degree of the Gauss map. In the case of a CMC 1 immersion in $H^3$, the total curvature never satisfies equality of the Cohn-Vossen inequality [UY1, Theorem 4.3], and the Osserman inequality does not hold in general. However, using the hyperbolic Gauss map instead of the total curvature, an Osserman type inequality holds for CMC 1 immersions in $H^3$ [UY2]. Also, Umehara and Yamada showed that the Osserman inequality holds for maxfaces [UY4]. In Section 4, we give the central result of this paper, that the Osserman inequality holds for complete CMC 1 faces of finite type with elliptic ends (Theorem 4.7). We remark that the assumptions of finite type and ellipticity of the ends can actually be removed, because, in fact, any complete CMC 1 face must be of finite type. This deep result will be shown in the forthcoming paper [FRUYY].

Lee and Yang were the first to construct a numerous collection of examples of CMC 1 faces [LY]. Furthermore, they constructed complete irreducible CMC 1 faces with three elliptic ends, by using hypergeometric functions [LY]. We will also give numerous examples here in Section 5, by using a method for transferring CMC 1 immersions in $H^3$ to CMC 1 faces in $S^3_1$. Applying this method to examples in [MU, RUY1, RUY2], we give many examples of complete reducible CMC 1 faces of finite type with elliptic ends.

We remark that the generic singularities of maxfaces in $\mathbb{R}^3_1$ and CMC 1
faces in $S^3_1$ are investigated in [FSUY].

1. CMC 1 faces

Let $\mathbb{R}^4_1$ be the 4-dimensional Lorentz space with the Lorentz metric

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Then de Sitter 3-space is

$$S^3_1 = S^3_1(1) = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\},$$

with metric induced from $\mathbb{R}^4_1$. $S^3_1$ is a simply-connected 3-dimensional Lorentzian manifold with constant sectional curvature 1. We can consider $\mathbb{R}^4_1$ to be the $2 \times 2$ self-adjoint matrices ($X^* = X$, where $X^*$ denotes the transpose of $X$), by the identification

$$\mathbb{R}^4_1 \ni X = (x_0, x_1, x_2, x_3) \leftrightarrow X = \sum_{k=0}^3 x_k e_k = \left(\begin{array}{cc} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{array}\right),$$

where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

Then $S^3_1$ is

$$S^3_1 = \{X \mid X^* = X, \det X = -1\} = \{F e_3 F^* \mid F \in SL(2, \mathbb{C})\}$$

with the metric

$$\langle X, Y \rangle = -\frac{1}{2} \text{trace} (X e_2 (Y^*) e_2).$$

In particular, $\langle X, X \rangle = -\det X$. An immersion in $S^3_1$ is called spacelike if the induced metric on the immersed surface is positive definite.

Aiyama and Akutagawa [AA] showed the following representation formula for simply-connected spacelike CMC 1 immersions.

**Theorem 1.1** The representation of Aiyama-Akutagawa. Let $D$ be a simply-connected domain in $\mathbb{C}$ with a base point $z_0 \in D$. Let $g : D \to (\mathbb{C} \cup \{\infty\}) \setminus \{z \in \mathbb{C} \mid |z| \leq 1\}$ be a meromorphic function and $\omega$ a holomorphic 1-form on $D$ such that

$$ds^2 = (1 + |g|^2)^2 \omega \bar{\omega} \tag{1.1}$$
is a Riemannian metric on $D$. Choose the holomorphic immersion $F = (F_{jk}): D \rightarrow SL(2, \mathbb{C})$ so that $F(z_0) = e_0$ and $F$ satisfies

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.$$  \hspace{1cm} (1.2)

Then $f: D \rightarrow S^3_1$ defined by

$$f = Fe_3F^*$$ \hspace{1cm} (1.3)

is a conformal spacelike CMC 1 immersion. The induced metric $ds^2 = f^*(ds^2_{S^3_1})$ on $D$, the second fundamental form $h$, and the hyperbolic Gauss map $G$ of $f$ are given as follows:

$$ds^2 = (1 - |g|^2)^2 \omega \bar{\omega}, \quad h = Q + \bar{Q} + ds^2, \quad G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}},$$ \hspace{1cm} (1.4)

where $Q = \omega dg$ is the Hopf differential of $f$.

Conversely, any simply-connected spacelike CMC 1 immersion can be represented this way.

**Remark 1.2** We make the following remarks about Theorem 1.1:

1. Following the terminology of Umehara and Yamada, $g$ is called the secondary Gauss map. The pair $(g, \omega)$ is called the Weierstrass data.

2. The oriented unit normal vector field $N$ of $f$ is given as

$$N = \frac{1}{|g|^2 - 1} F \begin{pmatrix} |g|^2 + 1 \\ 2g \\ |g|^2 + 1 \end{pmatrix} F^*,$$

which is a future pointing timelike vector field (See [KY]).

3. The hyperbolic Gauss map has the following geometric meaning: Let $S^2_{\infty}$ be the future pointing ideal boundary of $S^3_1$. Then $S^2_{\infty}$ is identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ in the standard way. Let $\gamma_z$ be the geodesic ray starting at $f(z)$ in $S^3_1$ with the initial velocity vector $N(z)$, the oriented unit normal vector of $f(D)$ at $f(z)$. Then $G(z)$ is the point $S^2_{\infty}$ determined by the asymptotic class of $\gamma_z$. See [B, UY1, UY3].

4. For the holomorphic immersion $F$ satisfying Equation (1.2), $\hat{f} := FF^*: D \rightarrow \mathbb{H}^4$ is a conformal CMC 1 immersion with first fundamental form $\hat{f}^*(ds^2_{\mathbb{H}^4}) = ds^2$ as in Equation (1.1), and with second fundamental form $\hat{h} = -\hat{Q} - \bar{Q} + ds^2$. The Hopf differential $\hat{Q}$ and the hyperbolic Gauss map $\hat{G}$ of $\hat{f}$ are the same as $Q$ and $G$, since $f$ and
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\( \hat{f} \) are both constructed from the same \( F \), and \( F \) determines \( Q \) and \( G \), by Equations (1.2) and (1.4).

(5) By Equation (2.6) in [UY1], \( G \) and \( g \) and \( Q \) have the following relation:

\[
2Q = S(g) - S(G),
\]

where \( S(g) = S_z(g)dz^2 \) and

\[
S_z(g) = \left( \frac{g''}{g'} \right)' - \frac{1}{2} \left( \frac{g''}{g'} \right)^2 \quad ('=d/dz)
\]

is the Schwarzian derivative of \( g \).

Our first task is to extend the above theorem to non-simply-connected CMC 1 surfaces with singularities, along the same lines as in [UY4]. To do this, we first define admissible singularities.

**Definition 1.3** Let \( M \) be an oriented 2-manifold. A smooth (that is, \( C^\infty \)) map \( f: M \to S^3_1 \) is called a CMC 1 map if there exists an open dense subset \( W \subset M \) such that \( f|_W \) is a spacelike CMC 1 immersion. A point \( p \in M \) is called a singular point of \( f \) if the induced metric \( ds^2 \) is degenerate at \( p \).

**Definition 1.4** Let \( f: M \to S^3_1 \) be a CMC 1 map and \( W \subset M \) an open dense subset such that \( f|_W \) is a CMC 1 immersion. A singular point \( p \in M \setminus W \) is called an admissible singular point if

(1) there exists a \( C^1 \)-differentiable function \( \beta: U \cap W \to \mathbb{R}^+ \), where \( U \) is a neighborhood of \( p \), such that \( \beta ds^2 \) extends to a \( C^1 \)-differentiable Riemannian metric on \( U \), and

(2) \( df(p) \neq 0 \), that is, \( df \) has rank 1 at \( p \).

We call a CMC 1 map \( f \) a CMC 1 face if each singular point is admissible.

To extend Theorem 1.1 to CMC 1 faces that are not simply-connected, we prepare two propositions. First, we prove the following proposition:

**Proposition 1.5** Let \( M \) be an oriented 2-manifold and \( f: M \to S^3_1 \) a CMC 1 face, where \( W \subset M \) an open dense subset such that \( f|_W \) is a CMC 1 immersion. Then there exists a unique complex structure \( J \) on \( M \) such that

(1) \( f|_W \) is conformal with respect to \( J \), and

(2) there exists an immersion \( F: \tilde{M} \to SL(2, \mathbb{C}) \) which is holomorphic
with respect to $J$ such that

$$\det(dF) = 0 \quad \text{and} \quad f \circ \varrho = Fe_3F^*,$$

where $\varrho: \tilde{M} \to M$ is the universal cover of $M$.

This $F$ is called a holomorphic null lift of $f$.

**Remark 1.6** The holomorphic null lift $F$ of $f$ is unique up to right multiplication by a constant matrix in $SU(1, 1)$. See also Remark 1.11 below.

**Proof of Proposition 1.5.** Since the induced metric $ds^2$ gives a Riemannian metric on $W$, it induces a complex structure $J_0$ on $W$. Let $p$ be an admissible singular point of $f$ and $U$ a local simply-connected neighborhood of $p$. Then by definition, there exists a $C^1$-differentiable function $\beta: U \cap W \to \mathbb{R}^+$ such that $\beta ds^2$ extends to a $C^1$-differentiable Riemannian metric on $U$. Then there exists a positively oriented orthonormal frame field $\{v_1, v_2\}$ with respect to $\beta ds^2$ which is $C^1$-differentiable on $U$. Using this, we can define a $C^1$-differentiable almost complex structure $J$ on $U$ such that

$$J(v_1) = v_2 \quad \text{and} \quad J(v_2) = -v_1. \quad (1.5)$$

Since $ds^2$ is conformal to $\beta ds^2$ on $U \cap W$, $J$ is compatible with $J_0$ on $U \cap W$. There exists a $C^1$-differentiable decomposition

$$\Gamma(T^*M^\mathbb{C} \otimes \mathfrak{sl}(2, \mathbb{C})) = \Gamma(T^*M^{(1,0)} \otimes \mathfrak{sl}(2, \mathbb{C})) \oplus \Gamma(T^*M^{(0,1)} \otimes \mathfrak{sl}(2, \mathbb{C})) \quad (1.6)$$

with respect to $J$, where $\Gamma(E)$ denotes the sections of a vector bundle $E$ on $U$. Since $f$ is smooth, $df \cdot f^{-1}$ is a smooth $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form. We can take the $(1, 0)$-part $\zeta$ of $df \cdot f^{-1}$ with respect to this decomposition. Then $\zeta$ is a $C^1$-differentiable $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form which is holomorphic on $U \cap W$ with respect to the equivalent complex structures $J_0$ and $J$ (which follows from the fact that $f|_W$ is a CMC 1 immersion, so the hyperbolic Gauss map $G$ of $f$ is holomorphic on $W$, which is equivalent to the holomorphicity of $\zeta$ with respect to $J$ and $J_0$ on $U \cap W$). Hence $d\zeta \equiv 0$ on $U \cap W$. Moreover, since $W$ is an open dense subset and $\zeta$ is $C^1$-differentiable on $U$, $d\zeta \equiv 0$ on $U$. Similarly, $\zeta \land \zeta \equiv 0$ on $U$. In particular,

$$d\zeta + \zeta \land \zeta = 0. \quad (1.7)$$

As $U$ is simply-connected, the existence of a $C^1$-differentiable map $F_U = \ldots$
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$$(F_{jk})_{j,k=1,2} : U \rightarrow SL(2, \mathbb{C})$$ such that $dF_U \cdot F_U^{-1} = \zeta$ is equivalent to the condition (1.7). Hence such an $F$ exists.

Note that since $f$ takes Hermitian matrix values, we have $df = \zeta f + (\zeta f)^*$. So $df(p) \neq 0$ (that is, $p$ is an admissible singularity) implies $\zeta \neq 0$. Then at least one entry $dF_{jk}$ of $dF_U$ does not vanish at $p$. Using this $F_{jk}$, we define the function $z = F_{jk} : U \rightarrow \mathbb{C} = \mathbb{R}^2$. Then, $z$ gives a coordinate system on $U$. Since $z = F_{jk}$ is a holomorphic function on $U \cap W$, it gives a complex analytic coordinate around $p$ compatible with respect to that of $U \cap W$. The other entries of $F_U$ are holomorphic functions with respect to $z$ on $U \cap W$ and are $C^1$-differentials on $U$, so each entry of $F_U$ is holomorphic with respect to $z$ on $U$, by the Cauchy-Riemann equations. Since $p$ is an arbitrary fixed admissible singularity, the complex structure of $W$ extends across each singular point $p$.

This complex structure can be seen to be well-defined at singular points as follows: Let $p' \in M \setminus W$ be another singular point and $U'$ a neighborhood of $p'$ so that $U \cap U' \neq \emptyset$. Then by the same argument as above, there exists a $C^1$-differentiable almost complex structure $J'$ on $U'$ and $C^1$-differentiable map $F'_U = (F'_{jk})_{j,k=1,2} : U' \rightarrow SL(2, \mathbb{C})$ such that $dF'_U \cdot F'_U^{-1}$ is the $(1, 0)$-part of $df \cdot f^{-1}$ with respect to Equation (1.6). Define $z' = F'_{jk}$ so that $dF'_{jk} \neq 0$. Then by uniqueness of ordinary differential equations, $F_U = F'_U A$ for some constant matrix $A$. So $z$ and $z'$ are linearly related, and hence they are holomorphically related. Also, because $dz$ and $dz'$ are nonzero, we have $dz/dz' \neq 0$ on $U \cap U'$.

For local coordinates $z$ on $M$ compatible with $J$, $\partial f \cdot f^{-1} := (f_z dz) \cdot f^{-1}$ (which is equal to $\zeta$) is holomorphic on $M$ and there exists a holomorphic map $F : \tilde{M} \rightarrow SL(2, \mathbb{C})$ such that

$$dF \cdot F^{-1} = \partial f \cdot f^{-1}. \quad (1.8)$$

Since $\partial f \cdot f^{-1} \neq 0$, also $dF \neq 0$, and hence $F$ is an immersion. Also, since $f$ is conformal, $0 = \langle \partial f, \partial f \rangle = -\det(\partial f)$. Thus $\det(dF) = 0$.

Finally, we set $\tilde{f} = Fe_3 \tilde{F}^*$, defined on $\tilde{M}$. We consider some simply-connected region $V \subset W$. By Theorem 1.1, there exists a holomorphic null lift $\tilde{F}$ of $f$,

$$f = \tilde{F} e_3 \tilde{F}^*, \quad (1.9)$$
defined on that same $V$. Then by Equations (1.8) and (1.9), we have
\[ d\hat{F} \cdot \hat{F}^{-1} = dF \cdot F^{-1}, \]
and hence \( \hat{F} = FB \) for some constant \( B \in SL(2, \mathbb{C}) \). We are free to choose the solution \( F \) of Equation (1.8) so that \( B = e_0 \), that is, \( \hat{F} = F \), so \( f = \hat{f} \) on \( V \). By the holomorphicity of \( F \), \( \hat{f} \) is real analytic on \( \hat{M} \). Also, \( f \circ \varrho \) is real analytic on \( \tilde{M} \), by Equation (1.8) and the holomorphicity of \( F \). Therefore \( f \circ \varrho = \hat{f} \) on \( \tilde{M} \), proving the proposition. \( \square \)

By Proposition 1.5, the 2-manifold \( M \) on which a CMC 1 face \( f: M \to S^3_1 \) is defined always has a complex structure. So throughout this paper, we will treat \( M \) as a Riemann surface with a complex structure induced as in Proposition 1.5.

The next proposition is the converse to Proposition 1.5:

**Proposition 1.7** Let \( M \) be a Riemann surface and \( F: M \to SL(2, \mathbb{C}) \) a holomorphic null immersion. Assume the symmetric \((0, 2)\)-tensor

\[ \det[d(Fe_3F^*)] \quad (1.10) \]

is not identically zero. Then \( f = Fe_3F^*: M \to S^3_1 \) is a CMC 1 face, and \( p \in M \) is a singular point of \( f \) if and only if \( \det[d(Fe_3F^*)]_p = 0 \). Moreover,

\[ -\det[d(FF^*)] \quad \text{is positive definite} \quad (1.11) \]
on \( M \).

**Proof.** Since (1.10) is not identically zero, the set

\[ W := \{ p \in M \mid \det[d(Fe_3F^*)]_p \neq 0 \} \]
is open and dense in \( M \). Since \( F^{-1}dF \) is a \( \mathfrak{sl}(2, \mathbb{C}) \)-valued 1-form, there exist holomorphic 1-forms \( a_1, a_2, a_3 \) such that

\[ F^{-1}dF = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}. \]

Since \( F \) is a null immersion, that is, \( \text{rank}(dF) = 1 \), \( a_j \ (j = 1, 2, 3) \) satisfy

\[ a_1^2 + a_2a_3 = 0 \quad \text{and} \quad |a_1|^2 + |a_2|^2 + |a_3|^2 > 0. \quad (1.12) \]

Since

\[ d(Fe_3F^*) = F \left( F^{-1}dFe_3 + (F^{-1}dFe_3)^* \right) F^* \]
we have

\[-\det[d(Fe_3F^*)] = -2|a_1|^2 + |a_2|^2 + |a_3|^2\]
\[= -2[a_2a_3] + |a_2|^2 + |a_3|^2\]
\[= |a_2|^2 - 2|a_2||a_3| + |a_3|^2 = (|a_2| - |a_3|)^2 \geq 0.\]

So \(-\det[d(Fe_3F^*)]\) is positive definite on \(W\). Set \(f = Fe_3F^*\). Then \(f|_W: W \to S^3\) determines a conformal immersion with induced metric

\[ds^2 = f^*ds^2_{\mathbb{S}^3} = \langle df, df \rangle = -\det[d(Fe_3F^*)].\]

Furthermore, \(f\) is CMC 1 by Theorem 1.1. Also, by (1.12),

\[-\det[d(FF^*)] = 2|a_1|^2 + |a_2|^2 + |a_3|^2\]

is positive definite on \(M\). Thus if we set

\[\beta = \frac{\det[d(FF^*)]}{\det[d(Fe_3F^*)]}\]

on \(W\), \(\beta\) is a positive function on \(W\) such that

\[\beta ds^2 = -\det[d(FF^*)]\]

extends to a Riemannian metric on \(M\). Also,

\[\partial f \cdot f^{-1} = dF \cdot F^{-1} = F(F^{-1}dF)F^{-1} \neq 0,\]

and so \(df \neq 0\). This completes the proof. \(\square\)

**Remark 1.8** Even if \(F\) is a holomorphic null immersion, \(f = Fe_3F^*\) might not be a CMC 1 face. For example, for the holomorphic null immersion

\[F: \mathbb{C} \ni z \mapsto \begin{pmatrix} z + 1 & -z \\ z & -z + 1 \end{pmatrix} \in SL(2, \mathbb{C}),\]

\(f = Fe_3F^*\) degenerates everywhere on \(\mathbb{C}\). Note that \(\det[d(Fe_3F^*)]\) is identically zero here.

Using Propositions 1.5 and 1.7, we can now extend the representation of Aiyama-Akutagawa for simply-connected CMC 1 immersions to the case of CMC 1 faces with possibly non-simply-connected domains.
Theorem 1.9  Let $M$ be a Riemann surface with a base point $z_0 \in M$. Let $g$ be a meromorphic function and $\omega$ a holomorphic 1-form on the universal cover $\tilde{M}$ such that $ds^2$ in Equation (1.1) is a Riemannian metric on $\tilde{M}$ and $|g|$ is not identically 1. Choose the holomorphic immersion $F = (F_{jk}): \tilde{M} \to SL(2, \mathbb{C})$ so that $F(z_0) = e_0$ and $F$ satisfies Equation (1.2). Then $f: \tilde{M} \to S_3^1$ defined by Equation (1.3) is a CMC 1 face that is conformal away from its singularities. The induced metric $ds^2$ on $M$, the second fundamental form $h$, and the hyperbolic Gauss map $G$ of $f$ are given as in Equation (1.4).

The singularities of the CMC 1 face occur at points where $|g| = 1$.

Conversely, let $M$ be a Riemann surface and $f: M \to S_3^1$ a CMC 1 face. Then there exists a meromorphic function $g$ (so that $|g|$ is not identically 1) and holomorphic 1-form $\omega$ on $\tilde{M}$ such that $ds^2$ is a Riemannian metric on $\tilde{M}$, and such that Equation (1.3) holds, where $F: \tilde{M} \to SL(2, \mathbb{C})$ is an immersion which satisfies Equation (1.2).

Proof. First we prove the first paragraph of the theorem. Since

$$d(Fe_3^*F^*) = F(F^{-1}dF \cdot e_3 + (F^{-1}dF \cdot e_3)^*)F^*, $$

we have

$${\text{det}}[d(Fe_3^*F^*)] = (1 - |g|^2)\omega \bar{\omega}.$$ 

Also, since $ds^2$ gives a Riemannian metric on $\tilde{M}$, $\omega$ has a zero of order $k$ if and only if $g$ has a pole of order $k/2 \in \mathbb{N}$. Therefore $|g| = 1$. Hence by Proposition 1.7, $f = Fe_3^*: \tilde{M} \to S_3^1$ is a CMC 1 face, and $p \in \tilde{M}$ is a singular point of $f$ if and only if $|g(p)| = 1$, proving the first half of the theorem.

We now prove the second paragraph of the theorem. By Proposition 1.5, there exists a holomorphic null lift $F: \tilde{M} \to SL(2, \mathbb{C})$ of the CMC 1 face $f$. Then by the same argument as in the proof of Proposition 1.7, we may set

$$F^{-1}dF = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix},$$

where $a_j (j = 1, 2, 3)$ are holomorphic 1-forms such that (1.12) holds. By changing $F$ into $FB$ for some constant $B \in SU(1, 1)$, if necessary, we may assume that $a_3$ is not identically zero. We set

$$\omega := a_3, \quad g := \frac{a_1}{a_3}$$
Then $\omega$ is a holomorphic 1-form and $g$ is a meromorphic function. Since $a_2/a_3 = a_2a_3/a_3^2 = -(a_1/a_3)^2 = -g^2$, we see that Equation (1.2) holds. Since $|a_2| - |a_3|$ is not identically zero (by the same argument as in the proof of Proposition 1.7), $|g|$ is not identically one. Also, since $g^2\omega = a_2$ is holomorphic, (1.11) implies $-\det[d(FF^*)] = (1 + |g|^2)^2\omega\bar{\omega} = ds^2$ is positive definite, so $ds^2$ gives a Riemannian metric on $\tilde{M}$, proving the converse part of the theorem. 

**Remark 1.10** Since the hyperbolic Gauss map $G$ has the geometric meaning explained in (3) of Remark 1.2, $G$ is single-valued on $M$ itself, although $F$ might not be. We also note that, by Equation (1.4), the Hopf differential $Q$ is single-valued on $M$ as well.

**Remark 1.11** Let $F$ be a holomorphic null lift of a CMC 1 face $f$ with Weierstrass data $(g, \omega)$. For any constant matrix

$$B = \begin{pmatrix} \bar{p} & -q \\ -q & p \end{pmatrix} \in SU(1, 1), \quad p\bar{p} - q\bar{q} = 1,$$

(1.13)

$FB$ is also a holomorphic null lift of $f$. The Weierstrass data $(\hat{g}, \hat{\omega})$ corresponding to $(FB)^{-1}d(FB)$ is given by

$$\hat{g} = \frac{pg + q}{qq + \bar{p}} \quad \text{and} \quad \hat{\omega} = (\bar{q}g + \bar{p})^2\omega.$$

(1.14)

Two Weierstrass data $(g, \omega)$ and $(\hat{g}, \hat{\omega})$ are called equivalent if they satisfy Equation (1.14) for some $B$ as in Equation (1.13). See Equation (1.6) in [UY1]. We shall call the equivalence class of the Weierstrass data $(g, \omega)$ the Weierstrass data associated to $f$. When we wish to emphasis that $(g, \omega)$ is determined by $F$, not $FB$ for some $B$, we call $(g, \omega)$ the Weierstrass data associated to $F$.

On the other hand, the Hopf differential $Q$ and the hyperbolic Gauss map $G$ are independent of the choice of $F$, because

$$\hat{\omega}d\hat{g} = \omega dg \quad \text{and} \quad \frac{\bar{p}dF_{11} + \bar{q}dF_{12}}{\bar{p}dF_{21} + \bar{q}dF_{22}} = \frac{dF_{11}}{dF_{21}},$$

where $F = (F_{jk})$.

This can also be seen from (3) of Remark 1.2, which implies that $G$ is determined just by $f$. Then $S(g) = S(\hat{g})$ and (5) of Remark 1.2 imply $Q$ is independent of the choice of $F$ as well.
2. CMC 1 faces with elliptic ends

It is known that the only complete spacelike CMC 1 immersion is a flat totally umbilic immersion [Ak, R] (see Example 5.1 below). In the case of non-immersed CMC 1 faces in $S^3_1$, we now define the notions of completeness and finiteness of total curvature away from singularities, like in [KUY, UY4].

**Definition 2.1** Let $M$ be a Riemann surface and $f: M \to S^3_1$ a CMC 1 face. Set $ds^2 = f^*(ds^2_{S^3_1})$. $f$ is complete (resp. of finite type) if there exists a compact set $C$ and a symmetric $(0, 2)$-tensor $T$ on $M$ such that $T$ vanishes on $M \setminus C$ and $ds^2 + T$ is a complete (resp. finite total curvature) Riemannian metric.

**Remark 2.2** For CMC 1 immersions in $S^3_1$, the Gauss curvature $K$ is non-negative. So the total curvature is the same as the total absolute curvature. However, for CMC 1 faces with singular points the total curvature is never finite, not even on neighborhoods of singular points, as can be seen from the form $K = 4dg d\bar{g}/(1 - |g|^2)^4 \omega \bar{\omega}$ for the Gaussian curvature, see also [ER]. Hence the phrase “finite type” is more appropriate in Definition 2.1.

**Remark 2.3** The universal covering of a complete (resp. finite type) CMC 1 face might not be complete (resp. finite type), because the singular set might not be compact on the universal cover.

Let $f: M \to S^3_1$ be a complete CMC 1 face of finite type. Then $(M, ds^2 + T)$ is a complete Riemannian manifold of finite total curvature. So by [H, Theorem 13], $M$ has finite topology, where we define a manifold to be of finite topology if it is diffeomorphic to a compact manifold with finitely many points removed. The ends of $f$ correspond to the removed points of that Riemann surface.

Let $g: \tilde{M} \to M$ be the universal cover of $M$, and $F: \tilde{M} \to SL(2, \mathbb{C})$ a holomorphic null lift of a CMC 1 face $f: M \to S^3_1$. We fix a point $z_0 \in M$. Let $\gamma: [0, 1] \to M$ be a loop so that $\gamma(0) = \gamma(1) = z_0$. Then there exists a unique deck transformation $\tau$ of $\tilde{M}$ associated to the homotopy class of $\gamma$. We define the monodromy representation $\Phi_\gamma$ of $F$ as

$$F \circ \tau = F \Phi_\gamma.$$ 

Note that $\Phi_\gamma \in SU(1, 1)$ for any loop $\gamma$, since $f$ is well-defined on $M$. So $\Phi_\gamma$ is conjugate to either
Spacelike CMC 1 surfaces with elliptic ends in $S^3_1$

$E = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ or $H = \pm \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$

or $P = \pm \begin{pmatrix} 1 + i & 1 \\ 1 & 1 - i \end{pmatrix}$ (2.1)

for $\theta \in [0, 2\pi)$, $s \in \mathbb{R} \setminus \{0\}$.

**Definition 2.4**  Let $f: M \to S^3_1$ be a complete CMC 1 face of finite type with holomorphic null lift $F$. An end of $f$ is called an elliptic end or hyperbolic end or parabolic end if its monodromy representation is conjugate to $E$ or $H$ or $P$ in $SU(1, 1)$, respectively.

**Remark 2.5**  A matrix

$$X = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix} \in SU(1, 1)$$

acts on the hyperbolic plane in the Poincaré model $\mathbb{H}^2 = \{w \in \mathbb{C} | |w| < 1\}$, $ds^2_{\mathbb{H}^2} = 4dwd\bar{w}/(1 - |w|^2)^2$ as an isometry:

$$\mathbb{H}^2 \ni w \mapsto \frac{pw + q}{qw + \bar{p}} \in \mathbb{H}^2.$$  

$X$ is called **elliptic** if this action has only one fixed point which is in $\mathbb{H}^2$. $X$ is called **hyperbolic** if there exist two fixed points, both in the ideal boundary $\partial \mathbb{H}^2$. $X$ is called **parabolic** if there exists only one fixed point which is in $\partial \mathbb{H}^2$. This is what motivates the terminology in Definition 2.4.

Since any matrix in $SU(2)$ is conjugate to $E$ in $SU(2)$, CMC 1 immersions in $\mathbb{H}^3$ and CMC 1 faces with elliptic ends in $S^3_1$ share many analogous properties. So in this paper we consider CMC 1 faces with only elliptic ends. We leave the study of hyperbolic ends and parabolic ends for another occasion.

**Proposition 2.6**  Let $V$ be a neighborhood of an end of $f$ and $f|_V$ a spacelike CMC 1 immersion of finite total curvature which is complete at the end. Suppose the end is elliptic. Then there exists a holomorphic null lift $F: \tilde{V} \to SL(2, \mathbb{C})$ of $f$ with Weierstrass data $(g, \omega)$ associated to $F$ such that

$$ds^2|_V = (1 + |g|^2)^2\omega\bar{\omega}$$

is single-valued on $V$. Moreover, $ds^2|_V$ has finite total curvature and is complete at the end.
Proof. Let \( \gamma : [0, 1] \to V \) be a loop around the end and \( \tau \) the deck transformation associated to \( \gamma \). Take a holomorphic null lift \( F_0 : \tilde{V} \to SL(2, \mathbb{C}) \) of \( f \). Then by definition of an elliptic end, there exists a \( \theta \in [0, 2\pi) \) such that

\[
F_0 \circ \tau = F_0 P E_\theta P^{-1},
\]

where \( P \in SU(1, 1) \) and \( E_\theta = \operatorname{diag}(e^{i\theta}, e^{-i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \).

Defining the holomorphic null lift \( F = F_0 P \) of \( f \) and defining \( (g, \omega) \) to be the Weierstrass data associated to \( F \), we have

\[
F \circ \tau = F E_\theta, \quad g \circ \tau = e^{-2i\theta} g \quad \text{and} \quad \omega \circ \tau = e^{2i\theta} \omega.
\]

Thus, \( |g \circ \tau| = |g| \) and \( |\omega \circ \tau| = |\omega| \). This implies \( ds^2|_V \) is single-valued on \( V \). Let \( T \) be a \((0, 2)\)-tensor as in Definition 2.1. Then by Equation (1.4), we have \( ds^2 + T \leq ds^2|_V \) on \( V \setminus C \). Thus, if \( ds^2 + T \) is complete, \( ds^2|_V \) is also complete. We denote the Gaussian curvature of the metric \( ds^2|_V \) by \( K_{ds^2|_V} \) (note that \( K_{ds^2|_V} \) is non-positive). Then we have

\[
(-K_{ds^2|_V}) ds^2|_V = \frac{4dg d\bar{g}}{(1+|g|^2)^2} \leq \frac{4dg d\bar{g}}{(1-|g|^2)^2} = K ds^2
\]
on \( V \setminus C \). Thus, if \( ds^2 + T \) is of finite total curvature, the total curvature of \( ds^2|_V \) is finite, proving the proposition. \( \square \)

**Proposition 2.7** Let \( f : M \to \mathbb{S}^1_1 \) be a complete CMC 1 face of finite type with elliptic ends. Then there exists a compact Riemann surface \( \overline{M} \) and finite number of points \( p_1, \ldots, p_n \in \overline{M} \) so that \( M \) is biholomorphic to \( \overline{M} \setminus \{p_1, \ldots, p_n\} \). Moreover, the Hopf differential \( Q \) of \( f \) extends meromorphically to \( \overline{M} \).

Proof. Since \( f \) is of finite type, \( M \) is finitely connected, by [H, Theorem 13]. Consequently, there exists a compact region \( M_0 \subset M \), bounded by a finite number of regular Jordan curves \( \gamma_1, \ldots, \gamma_n \), such that each component \( M_j \) of \( M \setminus M_0 \) can be conformally mapped onto the annulus \( D_j = \{ z \in \mathbb{C} \mid r_j < |z| < 1 \} \), where \( \gamma_j \) corresponds to the set \( \{|z| = 1\} \). Then by Proposition 2.6, there exists \( ds^2|_{M_j} \), which is single-valued on \( M_j \) and is of finite total curvature and is complete at the end, and so that \( K_{ds^2|_{M_j}} \) is non-
positive. Therefore by [L, Proposition III-16] or [O2, Theorem 9.1], $r_j = 0$, and hence each $M_j$ is biholomorphic to the punctured disk $\{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$. We can, using the biholomorphism from $M_j$ to $D_j$, replace $M_j$ in $M$ with $D_j$ without affecting the conformal structure of $M$. Thus, without loss of generality, $M = \overline{M} \setminus \{ p_1, \ldots, p_n \}$ for some compact Riemann surface $\overline{M}$ and a finite number of points $p_1, \ldots, p_n$ in $\overline{M}$, and each $M_j$ becomes a punctured disk about $p_j$. Hence, by (4) of Remark 1.2, we can apply [B, Proposition 5] to $\hat{f}_j := f|_{M_j}$ to see that $Q = \hat{Q}$ extends meromorphically to $M_j \cup \{ p_j \}$, proving the proposition.

Let $\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$ and $\Delta^* = \Delta \setminus \{ 0 \}$. Let $f : \Delta^* \to S_3^1$ be a conformal spacelike CMC 1 immersion of finite total curvature which has a complete elliptic end at the origin. Then we can take the Weierstrass data associated to $f$ in the following form:

$$\begin{align*}
g &= z^\mu \tilde{g}(z), \\
\omega &= w(z)dz = z^\nu \tilde{w}(z)dz,
\end{align*}$$

where $\tilde{g}$ and $\tilde{w}$ are holomorphic functions on $\Delta$ and $\mu + \nu \in \mathbb{Z}$. (See [UY1] and also [B, Proposition 4] for case that $\mu, \nu \in \mathbb{R}$. Then applying a transformation as in Equation (1.14) if necessary, we may assume $\mu > 0$. Completeness of the end then gives $\nu \leq -1$.)

**Definition 2.8** The end $z = 0$ of $f : \Delta^* \to S_3^1$ is called regular if the hyperbolic Gauss map $G$ extends meromorphically across the end. Otherwise, the end is called irregular.

Since $Q$ extends meromorphically to each end, we have the following proposition, by (4) and (5) of Remark 1.2:

**Proposition 2.9** ([B, Proposition 6]) An end $f : \Delta^* \to S_3^1$ is regular if and only if the order at the end of the Hopf differential of $f$ is at least $-2$.

**Remark 2.10** In [LY], Lee and Yang define normal ends and abnormal ends. Both normal and abnormal ends are biholomorphic to a punctured disk $\Delta^*$, and the Hopf differential has a pole of order 2 at the origin. Normal ends are elliptic ends, and abnormal ends are hyperbolic ends. Moreover, the Lee-Yang catenoids with normal ends are complete in our sense. However, the Lee-Yang catenoids with abnormal ends include incomplete examples, because the singular set of these examples accumulates at the ends. (In fact, CMC 1 face with hyperbolic ends cannot be complete, which is shown
in [FRUYY]).

3. Embeddedness of elliptic ends

In this section we give a criterion for embeddedness of elliptic ends, which is based on results in [UY1]. This criterion will be applied in the next section.

Let \( f : \Delta^* \to S^3 \) be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin. Let \( \gamma : [0, 1] \to \Delta^* \) be a loop around the origin and \( \tau \) the deck transformation of \( \widetilde{\Delta}^* \) associated to the homotopy class of \( \gamma \). Then by the same argument as in the proof of Proposition 2.6, there exists the holomorphic null lift \( F \) of \( f \) such that \( F \circ \tau = F E_\theta \) for some \( \theta \in [0, 2\pi) \), where \( E_\theta = \text{diag}(e^{i\theta}, e^{-i\theta}) \).

Since \( E_\theta \in SU(2) \), \( \hat{f} := FF^* \) in \( H^3 \) is single-valued on \( \Delta^* \).

Let \( (g, \omega) \) be the Weierstrass data associated to \( F \), defined as in (2.2). Then by (4) in Remark 1.2, \( \hat{f} : \Delta^* \to H^3 \) is a conformal CMC 1 immersion with the induced metric \( ds^2 = (1 + |g|^2)^2 \omega \bar{\omega} \). Thus by the final sentence of Proposition 2.6, \( \hat{f} \) has finite total curvature and is complete at the origin.

Since \( \hat{f} \) has the same Hopf differential \( Q \) as \( f \), \( f \) having a regular end immediately implies that \( \hat{f} \) has a regular end. Furthermore we show the following theorem:

**Theorem 3.1** Let \( f : \Delta^* \to S^3 \) be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin. Then there exists a holomorphic null lift \( F : \Delta^* \to SL(2, \mathbb{C}) \) of \( f \) (that is, \( f = Fe_3F^* \)) such that \( \hat{f} = FF^* \) is a conformal CMC 1 finite-total-curvature immersion from \( \Delta^* \) into \( H^3 \) with a complete regular end at the origin. Moreover, \( f \) has an embedded end if and only if \( \hat{f} \) has an embedded end.

**Remark 3.2** The converse of the first part of Theorem 3.1 is also true, that is, the following holds: Let \( \hat{f} : \Delta^* \to H^3 \) be a conformal CMC 1 immersion of finite total curvature with a complete regular end at the origin. Take a holomorphic null lift \( F : \Delta^* \to SL(2, \mathbb{C}) \) of \( \hat{f} \) (that is, \( \hat{f} = FF^* \)) such that the associated Weierstrass data \( (g, \omega) \) is written as in (2.2). Then \( f = Fe_3F^* \) is a conformal spacelike CMC 1 finite-total-curvature immersion from \( \Delta^* \) into \( S^3 \) with a complete regular elliptic end at the origin. Moreover, \( f \) has an embedded end if and only if \( \hat{f} \) has an embedded end. See Proposition 5.5 below.
We already know that such an $F$ exists. So we must prove the equivalency of embeddedness between the ends of $f$ and $\hat{f}$. To prove this we prepare three lemmas.

**Lemma 3.3** [UY1, Lemma 5.3] There exists a $\Lambda \in SL(2, \mathbb{C})$ such that

$$\Lambda F = \begin{pmatrix} z^{\lambda_1} a(z) & z^{\lambda_2} b(z) \\ z^{\lambda_1-m_1} c(z) & z^{\lambda_2-m_2} d(z) \end{pmatrix},$$

(3.1)

where $a, b, c, d$ are holomorphic functions on $\Delta$ that do not vanish at the origin, and $\lambda_j \in \mathbb{R}$ and $m_j \in \mathbb{N}$ ($j = 1, 2$) are defined as follows:

1. If $\text{Ord}_0 Q = \mu + \nu - 1 = -2$, then
   $$m_1 = m_2, \quad \lambda_1 = -\frac{\mu + m_j}{2} \quad \text{and} \quad \lambda_2 = \frac{\mu + m_j}{2}.$$

2. If $\text{Ord}_0 Q = \mu + \nu - 1 \geq -1$, then
   $$m_1 = -(\nu + 1), \quad m_2 = 2\mu + \nu + 1, \quad \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = m_2.$$

(3.2)

(3.3)

Note that in either case we have $\nu < -1$ and

$$\lambda_1 < \lambda_2, \quad \lambda_1 - m_1 < \lambda_2 - m_2 \quad \text{and} \quad \lambda_1 - m_1 < 0.$$

(3.4)

Note also that in the second case we have $m_1 < m_2$.

**Proof of Lemma 3.3.** $F$ satisfies Equation (1.2), which is precisely Equation (1.5) in [UY1]. So we can apply [UY1, Lemma 5.3], since that lemma is based on Equation (1.5) in [UY1]. This gives the result. □

It follows that

$$\Lambda f \Lambda^* = (\Lambda F)e_3 (\Lambda F)^*$$

$$= \begin{pmatrix} |z|^{2\lambda_1} |a|^2 - |z|^{2\lambda_2} |b|^2 & |z|^{2\lambda_1} \bar{z}^{-m_1} a \bar{c} - |z|^{2\lambda_2} \bar{z}^{-m_2} d \bar{d} \\ |z|^{2\lambda_1} \bar{z}^{m_1} a \bar{c} - |z|^{2\lambda_2} \bar{z}^{-m_2} b \bar{d} & |z|^{2(\lambda_1-m_1)} |c|^2 - |z|^{2(\lambda_2-m_2)} |d|^2 \end{pmatrix}.$$  

Note that $\Lambda f \Lambda^*$ is congruent to $f = Fe_3 F^*$, because $(\Lambda F)^{-1}d(\Lambda F) = F^{-1}dF$ determines both the first and second fundamental forms as in Equation (1.4).

To study the behavior of the elliptic end $f$, we present the elliptic end in a 3-ball model as follows: We set

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} = \Lambda f \Lambda^*.$$
Since
\[ x_0 = \frac{1}{2} \text{trace}(\Lambda f \Lambda^*) = \frac{1}{2} \left( |z|^{2\lambda_1} |a|^2 - |z|^{2\lambda_2} |b|^2 + |z|^{2(\lambda_1-m_1)} |c|^2 - |z|^{2(\lambda_2-m_2)} |d|^2 \right), \]

(3.4) implies that \( \lim_{z \to 0} x_0(z) = \infty \). So we may assume that \( x_0(z) > 1 \) for any \( z \in \Delta^* \). So we can define a bijective map
\[
p: \{(x_0, x_1, x_2, x_3) \in S^3_1 \mid x_0 > 1\} \to \mathbb{D}^3 := D^3_1 \setminus \overline{D^3_{1/\sqrt{2}}} \]
as
\[ p(x_0, x_1, x_2, x_3) := \frac{1}{1 + x_0(x_1, x_2, x_3)}, \]
where \( D^3_r \) denotes the open ball of radius \( r \) in \( \mathbb{R}^3 \) and \( \overline{D^3_r} = D^3_r \cup \partial D^3_r \) (See Fig. 1).

![Diagram](image)

Fig. 1. The projection \( p \).

We set \((X_1, X_2, X_3) = p \circ (\Lambda f \Lambda^*)\). Then we have
\[
X_1 + iX_2 = \frac{2a\bar{c}|z|^{2(\lambda_1-m_1)} z^{m_1} - 2b\bar{d}|z|^{2(\lambda_2-m_2)} z^{m_2}}{2 + |a|^2|z|^{2\lambda_1} + |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} - |d|^2|z|^{2(\lambda_2-m_2)}}, \]
(3.5)
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We now define a function $U: \Delta^* \rightarrow \mathbb{C}$ that will be useful for proving Theorem 3.1:

$$U(z) = z^{-m_1}(X_1 + iX_2).$$ (3.7)

Then by making just a few sign changes to the argument in [UY1, Lemma 5.4], we have the following lemma:

**Lemma 3.4** The function $U$ is a $C^\infty$ function on $\Delta^*$ that extends continuously to $\Delta$ with $U(0) \neq 0$. Moreover,

$$\lim_{z \to 0} z \frac{\partial U}{\partial \bar{z}} = 0 \quad \text{and} \quad \lim_{z \to 0} \frac{\partial U}{\partial z} = 0.$$ (3.8)

**Proof.** If $\text{Ord}_0 Q = -2$, then Equation (3.7) is reduced to

$$U(z) = \frac{2ac - 2bdz}{2|z|\mu + m_1 + |a|^2|z|^{2m_1} + |c|^2 - |b|^2|z|^{2(\mu + m_1)} - |d|^2|z|^{2\mu}}.$$ (3.9)

Then we have $U(0) = 2a(0)/c(0) \neq 0$, because $\mu > 0$ and $m_1 \geq 1$. Also, by a straightforward calculation, we see that Equation (3.8) holds.

If $\text{Ord}_0 Q \geq -1$, then Equation (3.7) is reduced to

$$U(z) = \frac{2ac - 2bdz^{m_2-z^{m_1}}}{(2 + |a|^2 - |d|^2)|z|^{2m_1} - |b|^2|z|^{2(m_1 + m_2)} + |c|^2}.$$ (3.10)

Then we have $U(0) = 2a(0)/c(0) \neq 0$, because $m_1, m_2 \geq 1$. Also, since $U$ is $C^1$-differentiable at the origin, we see that Equation (3.8) holds. □

Also, we have the following lemma, analogous to Remark 5.5 in [UY1]:

**Lemma 3.5** $p \circ (Af\Lambda^*) = (X_1, X_2, X_3)$ converges to the single point $(0, 0, -1) \in \partial D^3_1$. Moreover, $p \circ (Af\Lambda^*)$ is tangent to $\partial D^3_1$ at the end $z = 0$.

**Proof.** By (3.2)–(3.6), we see that

$$\lim_{z \to 0} (X_1, X_2, X_3) = (0, 0, -1).$$

Define a function $V: \Delta^* \rightarrow \mathbb{C}$ by $V := z^{-m_1}(X_3 + 1)$. Then from either
case of the proof of Lemma 3.4, we see that
\[ \lim_{z \to 0} V = 0 \quad \text{and} \quad \lim_{z \to 0} z \frac{\partial V}{\partial z} = 0. \]
Therefore
\[ 0 = \lim_{z \to 0} z \frac{\partial V}{\partial z} = \lim_{z \to 0} \left( z^{-m_1+1} \frac{\partial X_3}{\partial z} \right) - m_1 V(0), \]
\[ 0 = \lim_{z \to 0} z \frac{\partial U}{\partial z} = \lim_{z \to 0} \left( z^{-m_1+1} \frac{\partial (X_1 + iX_2)}{\partial z} \right) - m_1 U(0) \]
imply that
\[ \lim_{X_1 + iX_2 \to 0} \frac{\partial X_3}{\partial (X_1 + iX_2)} = \frac{V(0)}{U(0)} = 0, \]
proving the lemma.

**Proof of Theorem 3.1.** [UY1, Theorem 5.2] shows that $\hat{f}$ has an embedded end if and only if $m_1 = 1$, so the theorem will be proven by showing that $f$ also has an embedded end if and only if $m_1 = 1$. We now show this:

Since $U(0) \neq 0$, by taking a suitable branch we can define the function $u : \Delta \to \mathbb{C}$ by
\[ u(z) = z (U(z))^{1/m_1}. \]
Then $u$ is a $C^1$ function such that
\[ \frac{\partial u}{\partial z}(0) \neq 0 \quad \text{and} \quad \frac{\partial u}{\partial \bar{z}}(0) = 0. \tag{3.11} \]
Assume that $m_1 = 1$. By Equation (3.7), $X_1 + iX_2 = u$ holds. Then by Equation (3.11), $X_1 + iX_2$ is one-to-one on some neighborhood of the origin $z = 0$. Hence $f$ has an embedded end.

Conversely, assume that $f$ has an embedded end. Let $p_3 : \mathbb{D}^3 \to \Delta$ be the projection defined by $p_3(X_1, X_2, X_3) = X_1 + iX_2$. By Equation (3.7) we have
\[ p_3 \circ \rho \circ (Af \Lambda^*) = u^{m_1}. \tag{3.12} \]
By Equations (3.11) and (3.12) the map $p_3 \circ \rho \circ (Af \Lambda^*)$ is an $m_1$-fold cover of $\Delta^*_1 = \{ z \in \mathbb{C} \mid 0 < |z| < \epsilon \}$ for a sufficiently small $\epsilon > 0$. Thus, by Lemma 3.5, $m_1$ must be 1, by the same topological arguments as at the end of the proof of Theorem 5.2 in [UY1].
Therefore, we have that $f$ has an embedded end if and only if $m_1 = 1$.

\[\square\]

4. The Osserman-type inequality

Let $f: M = \overline{M} \setminus \{p_1, \ldots, p_n\} \to S^3_1$ be a complete CMC 1 face of finite type with hyperbolic Gauss map $G$ and Hopf differential $Q$.

**Definition 4.1** We set

\[d\hat{s}^2 := (1 + |G|^2)^2 \frac{Q}{dG} \left( \frac{Q}{dG} \right)\]  \hspace{1cm} (4.1)

and call it the *lift-metric* of the CMC 1 face $f$. We also set

\[d\hat{\sigma}^2 := (-K_{d\hat{s}^2})d\hat{s}^2 = \frac{4dGd\bar{G}}{(1 + |G|^2)^2}.\]

**Remark 4.2** Since $G$ and $Q$ are defined on $M$, both $d\hat{s}^2$ and $d\hat{\sigma}^2$ are also defined on $M$.

We define the order of pseudometrics as in [UY3, Definition 2.1], that is:

**Definition 4.3** A pseudometric $d\varsigma^2$ on $\overline{M}$ is of order $m_j$ at $p_j$ if $d\varsigma^2$ has a local expression

\[d\varsigma^2 = e^{2u_j}dzd\bar{z}\]

around $p_j$ such that $u_j - m_j \log |z - z(p_j)|$ is continuous at $p_j$. We denote $m_j$ by $\text{Ord}_{p_j}(d\varsigma^2)$. In particular, if $d\varsigma^2$ gives a Riemannian metric around $p_j$, then $\text{Ord}_{p_j}(d\varsigma^2) = 0$.

We now apply [UY2, Lemma 3] for regular ends in $\mathbb{H}^3$ to regular elliptic ends in $S^3_1$, that is, we show the following proposition:

**Proposition 4.4** Let $f: \Delta^* \to S^3_1$ be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin $z = 0$. Then the following inequality holds:

\[\text{Ord}_0(d\hat{\sigma}^2) - \text{Ord}_0(Q) \geq 2.\]  \hspace{1cm} (4.2)

Moreover, equality holds if and only if the end is embedded.
Proof. By Theorem 3.1, there exists a holomorphic null lift $F: \tilde{\Delta}^* \to SL(2, \mathbb{C})$ of $f$ such that $\tilde{f} = FF^*: \Delta^* \to \mathbb{H}^3$ is a conformal CMC 1 immersion of finite total curvature with a complete regular end at the origin. Then by [UY2, Lemma 3], we have (4.2). Moreover, equality holds if and only if the end of $\tilde{f}$ is embedded, by [UY2, Lemma 3]. This is equivalent to the end of $f$ being embedded, by Theorem 3.1, proving the proposition.

The following lemma is a variant on known results in [Yu, KTUY]. In fact, [Yu] showed that $d\tilde{s}^2$ is complete if and only if $d\tilde{s}^\#$ is complete, see also [KTUY, Lemma 4.1].

**Lemma 4.5** Let $f: M \to S^3_1$ be a CMC 1 face. Assume that each end of $f$ is regular and elliptic. If $f$ is complete and of finite type, then the lift-metric $d\tilde{s}^2$ is complete and of finite total curvature on $M$. In particular,

$$\operatorname{Ord}_{p_j}(d\tilde{s}^2) \leq -2$$

(4.3)

holds at each end $p_j$ ($j = 1, \ldots, n$).

Proof. Since $f$ is complete and of finite type, each end is complete and has finite total curvature. Then by (4.2) and the relation

$$\operatorname{Ord}_{p_j}(d\tilde{s}^2) + \operatorname{Ord}_{p_j}(d\tilde{\sigma}^2) = \operatorname{Ord}_{p_j}(Q)$$

(4.4)

(which follows from the Gauss equation $d\tilde{\sigma}^2 d\tilde{s}^2 = 4Q\tilde{Q}$), we have (4.3) at each end $p_j$. Hence $d\tilde{s}^2$ is a complete metric. Also, again by (4.2), we have

$$\operatorname{Ord}_{p_j}(d\tilde{\sigma}^2) \geq 2 + \operatorname{Ord}_{p_j}(Q) \geq 0,$$

because $p_j$ is regular (that is, $\operatorname{Ord}_{p_j}(Q) \geq -2$, by Proposition 2.9). This implies that the total curvature of $d\tilde{s}^2$ is finite.

**Remark 4.6** Consider a CMC 1 face with regular elliptic ends. If it is complete and of finite type, then, by Lemma 4.5, the lift-metric is complete and of finite total curvature. But the converse is not true. See [FRUYY].

**Theorem 4.7** (Osserman-type inequality) Let $f: M \to S^3_1$ be a complete CMC 1 face of finite type with $n$ elliptic ends and no other ends. Let $G$ be its hyperbolic Gauss map. Then the following inequality holds:

$$2 \deg(G) \geq -\chi(M) + n,$$

(4.5)
where $\deg(G)$ is the mapping degree of $G$ (if $G$ has essential singularities, then we define $\deg(G) = \infty$). Furthermore, equality holds if and only if each end is regular and embedded.

**Remark 4.8** As we remarked in the introduction, the completeness of a CMC 1 face $f$ implies that $f$ must be of finite type. See the forthcoming paper [FRUY].

*Proof of Theorem 4.7.* Recall that we can set $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$, where $\overline{M}$ is a compact Riemann surface and $p_1, \ldots, p_n$ is a set of points in $\overline{M}$, by Proposition 2.7. If $f$ has irregular ends, then $G$ has essential singularities at those ends. So $\deg(G) = \infty$ and then (4.5) automatically holds. Hence we may assume $f$ has only regular ends. Using the Riemann-Hurwitz formula and the Gauss equation $d\hat{s}^2 d\hat{\sigma}^2 = 4Q\overline{Q}$, we have

\[
2 \deg(G) = \chi(\overline{M}) + \sum_{p \in \overline{M}} \text{Ord}_p d\hat{s}^2
\]

\[
= \chi(\overline{M}) + \sum_{p \in \overline{M}} (\text{Ord}_p Q - \text{Ord}_p d\hat{s}^2)
\]

\[
= \chi(\overline{M}) + \sum_{p \in \overline{M}} \text{Ord}_p Q - \sum_{p \in M} \text{Ord}_p d\hat{s}^2 - \sum_{j=1}^n \text{Ord}_{p_j} d\hat{s}^2
\]

\[
= - \chi(\overline{M}) + 2n \quad \text{(because $d\hat{s}^2$ is complete, by (4.3))}
\]

\[
= - \chi(M) + n.
\]

Equality in (4.5) holds if and only if equality in (4.3) holds at each end, which is equivalent to equality in (4.2) holding at each end, by Equation (4.4). Thus by Proposition 4.4, we have the conclusion. □

### 5. Examples

To visualize CMC 1 faces, we use the hollow ball model of $S^3_1$, as in [KY, LY, Y]. For any point

\[
\begin{pmatrix}
x_0 + x_3 & x_1 + ix_2 \\
x_1 - ix_2 & x_0 - x_3
\end{pmatrix} \leftrightarrow (x_0, x_1, x_2, x_3) \in S^3_1,
\]

where $x_0, x_1, x_2, x_3$ are real numbers and $i$ is the imaginary unit.
define
\[ y_k = \frac{e^{\arctan x_0}}{\sqrt{1 + x_0^2}} x_k, \quad k = 1, 2, 3. \]

Then \( e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^\pi \). The identification \((x_0, x_1, x_2, x_3) \leftrightarrow (y_1, y_2, y_3)\) is then a bijection from \( \mathbb{S}_3 \) to the hollow ball
\[ \mathcal{H} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^\pi \}. \]

So \( \mathbb{S}_3 \) is identified with the hollow ball \( \mathcal{H} \), and we show the graphics here using this identification to \( \mathcal{H} \).

We shall first introduce four basic examples, using the same Weierstrass data as for CMC 1 immersions in \( \mathbb{H}^3 \).

**Example 5.1** The CMC 1 face associated to horosphere in \( \mathbb{H}^3 \) is given by the Weierstrass data \((g, \omega) = (c_1, c_2dz)\) with \( c_1 \in \mathbb{C}\setminus\mathbb{S}^1, c_2 \in \mathbb{C}\setminus\{0\} \) on the Riemann surface \( \mathbb{C} \). This CMC 1 face has no singularities. So this example is indeed a complete spacelike CMC 1 immersion. [Ak] and [R, Theorem 7] independently showed that the only complete spacelike CMC 1 immersion in \( \mathbb{S}_3 \) is a flat totally umbilic immersion.

**Example 5.2** The CMC 1 face associated to the Enneper cousin in \( \mathbb{H}^3 \) is given by the Weierstrass data \((g, \omega) = (z, cdz)\) with \( c \in \mathbb{R} \setminus \{0\} \) on the Riemann surface \( \mathbb{C} \). The induced metric \( ds^2 = c^2(1 - |z|^2)^2dzd\bar{z} \) degenerates

![Fig. 2. Pictures of Example 5.1. The left-hand side is drawn with \( c_1 = 1.2, c_2 = 1 \) and the right-hand side is drawn with \( c_1 = 0, c_2 = 1 \).]
where \(|z| = 1\). Take a \(p \in \mathbb{C}\) which satisfies \(|p| = 1\). Define

\[
\beta := \left( \frac{1 + |z|^2}{1 - |z|^2} \right)^2.
\]

Then

\[
\lim_{z \to p} \beta ds^2 = 4c^2 dz d\bar{z}.
\]

So all singularities are admissible and hence this is a CMC 1 face. Moreover, it is easily seen that this CMC 1 face is complete and of finite type. Since this CMC 1 face is simply-connected, the end of this CMC 1 face is an elliptic end. Since \(\text{Ord}_{\infty} Q = -4 < -2\), the end of this CMC 1 face is irregular. Hence this CMC 1 face does not satisfy equality in the inequality (4.5).

\[
\{ z \in \mathbb{C} ; |z| < 1.3 \}. \quad \{ z \in \mathbb{C} ; 0.8 < |z| < 1.3 \}.
\]

Fig. 3. Pictures of Example 5.2, where \(c = 1\).

**Example 5.3** The CMC 1 face associated to the helicoid cousin in \(\mathbb{H}^3\) is given by the Weierstrass data \((g, \omega) = (e^z, ice^{-z}dz)\) with \(c \in \mathbb{R} \setminus \{0\}\) on the Riemann surface \(\mathbb{C}\). Set \(z = x + iy\). The induced metric \(ds^2 = 4c^2 \sinh^2 x (dx^2 + dy^2)\) degenerates where \(x = 0\). Take a \(p \in \mathbb{C}\) which satisfies \(\text{Re}(p) = 0\). Define \(\beta := \tanh^{-2} x\). Then

\[
\lim_{z \to p} \beta ds^2 = 4c^2 (dx^2 + dy^2).
\]
So all singularities are admissible and hence this is a CMC 1 face. Since the singular set is non-compact, this CMC 1 face is neither complete nor of finite type.

\[ \{ z \in \mathbb{C} : -0.9 < \text{Re} z < 0.9, -4\pi < \text{Im} z < 4\pi \} \]

\[ \{ z \in \mathbb{C} : -0.8 < \text{Re} z < 0.8, -0.3 < \text{Im} z < 0.3 \} \]

Fig. 4. Pictures of Example 5.3, where \( c = 1 \).

**Example 5.4** The CMC 1 face associated to the catenoid cousin in \( \mathbb{H}^3 \) is given by the Weierstrass data \( (g, \omega) = (z^\mu, (1 - \mu^2)dz/4\mu z^{\mu+1}) \) with \( \mu \in \mathbb{R}^+ \setminus \{1\} \) on the Riemann surface \( \mathbb{C} \setminus \{0\} \). The induced metric

\[
 ds^2 = \left( \frac{|z|^\mu - |z|^{-\mu}}{|z|} \cdot \frac{1 - \mu^2}{4\mu} \right)^2 dz d\bar{z}
\]

degenerates where \( |z| = 1 \). Take a \( p \in \mathbb{C} \) which satisfies \( |p| = 1 \). Define

\[
 \beta := \left( \frac{|z|^\mu + |z|^{-\mu}}{|z|^\mu - |z|^{-\mu}} \right)^2.
\]

Then

\[
 \lim_{z \to p} \beta ds^2 = 4 \left( \frac{1 - \mu^2}{4\mu} \right)^2 dz d\bar{z}.
\]

So all singularities are admissible and hence this is a CMC 1 face. Moreover, it is easily seen that this CMC 1 face is complete and of finite type. Since the eigenvalues of the monodromy representation at each end are \(-e^{\mu \pi i}, -e^{-\mu \pi i} \in S^1\), each end of this CMC 1 face is an elliptic end. Since \( \text{Ord}_0 Q = \text{Ord}_{\infty} Q = -2 \), each end of this CMC 1 face is regular. Also, each end is of
this CMC 1 face is embedded, so this CMC 1 face satisfies equality in the inequality (4.5).

\[ \{ z \in \mathbb{C}; e^{-5} < |z| < e^5 \} \]
\[ \{ z \in \mathbb{C}; e^{-5} < |z| < e^5 \} \]
\[ \{ z \in \mathbb{C}; e^{-5} < |z| < e^5 \} \]

Fig. 5. Pictures of Example 5.4, where \( \mu = 0.8 \).

\[ \{ z \in \mathbb{C}; e^{-5} < |z| < e^5 \} \]
\[ \{ z \in \mathbb{C}; e^{-5} < |z| < e^5 \} \]
\[ \{ z \in \mathbb{C}; e^{-5} < |z| < e^5 \} \]

Fig. 6. Pictures of Example 5.4, where \( \mu = 1.2 \).

To produce further examples, we consider a relationship between CMC 1 faces and CMC 1 immersions in the hyperbolic space \( \mathbb{H}^3 \), and shall give a method for transferring from CMC 1 immersions in \( \mathbb{H}^3 \) to CMC 1 faces in \( \mathbb{S}^3 \).

Let \( \hat{f}: M = \overline{M} \setminus \{ p_1, \ldots, p_n \} \to \mathbb{H}^3 \) be a reducible CMC 1 immersion, whose first fundamental form \( ds^2 \) has finite total curvature and is complete, where we define \( \hat{f} \) to be reducible if there exists a holomorphic null lift \( F \) of \( \hat{f} \) such that the image of the monodromy representation is in

\[ U(1) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}. \]
Let \((g, \omega)\) be the Weierstrass data associated to \(F\), that is, \((g, \omega)\) satisfies
\[
F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.
\]
Then \(|g|\) and \(|\omega|\) are single-valued on \(M\), as seen in the proof of Proposition 2.6. Assume that the absolute value of the secondary Gauss map is not equal to 1 at all ends \(p_1, \ldots, p_n\). Define \(f := F_3 F^*\). Then \(f\) is defined on \(M\) as well. Furthermore, we have the following proposition:

**Proposition 5.5** The CMC 1 face \(f: M \to S^3_1\) defined as above, using \(\hat{f}\) and its lift \(F\) with monodromy in \(U(1)\), is complete and of finite type with only elliptic ends. Moreover, an end of \(\hat{f}\) is embedded if and only if the corresponding end of \(f\) is embedded.

**Proof.** Fix an end \(p_j\) and assume \(|g(p_j)| < 1\). Then we can take a neighborhood \(U_j\) such that \(|g| < 1 - \epsilon\) holds on \(U_j\), where \(\epsilon \in (0, 1)\) is a constant. In this case,
\[
ds^2 = (1 - |g|^2)^2 \omega \bar{\omega} \geq \frac{\epsilon^2}{4} (1 + |g|^2)^2 \omega \bar{\omega} = \frac{\epsilon^2}{4} d\hat{s}^2
\]
holds on \(U_j\). Since \(d\hat{s}^2\) is complete at \(p_j\), \(ds^2\) is also complete. Moreover, the Gaussian curvatures \(K\) and \(K_{ds^2}\) satisfy
\[
K ds^2 = \frac{4dgd\bar{g}}{(1 - |g|^2)^2} \leq \left( \frac{2}{\epsilon - 1} \right)^2 \frac{4dgd\bar{g}}{(1 + |g|^2)^2} = \left( \frac{2}{\epsilon - 1} \right)^2 (-K_{ds^2}) d\hat{s}^2.
\]
Hence \(ds^2\) is of finite total curvature at the end \(p_j\).

On the other hand, if \(|g(p_j)| > 1\), we can choose the neighborhood \(U_j\) such that \(|g|^{-2} < 1 - \epsilon\) holds on \(U_j\). Then
\[
ds^2 = (1 - |g|^{-2})^2 |g|^4 \omega \bar{\omega} \geq \frac{\epsilon^2}{5} |g|^4 \omega \bar{\omega} \geq \frac{\epsilon^2}{5} (1 + |g|^2)^2 \omega \bar{\omega} = \frac{\epsilon^2}{5} d\hat{s}^2.
\]
Hence \(ds^2\) is complete at \(p_j\). Moreover, since
\[
K ds^2 = \frac{4dgd\bar{g}}{(1 - |g|^2)^2} \leq \left( \frac{2}{\epsilon - 1} \right)^2 \frac{4dgd\bar{g}}{(1 + |g|^2)^2} = \left( \frac{2}{\epsilon - 1} \right)^2 (-K_{ds^2}) d\hat{s}^2,
\]
$ds^2$ is of finite total curvature. The proof of the final sentence of the proposition follows from the proof of Theorem 3.1, by showing $m_1 = 1$ for both $f$ and $\hat{f}$.

Moreover, [UY1, Theorem 3.3] shows that for each $\lambda \in \mathbb{R} \setminus \{0\}$, $(\lambda g, \lambda^{-1} \omega)$ induces a CMC 1 immersion $f_\lambda: M \to \mathbb{H}^3$, where $(g, \omega)$ is stated in Proposition 5.5. Thus we have the following theorem:

**Theorem 5.6** Let $\hat{f}: M \to \mathbb{H}^3$ be a reducible complete CMC 1 immersion of finite total curvature with $n$ ends. Then there exists the holomorphic null lift $F$ so that the image of the monodromy representation is in $U(1)$. Let $(g, \omega)$ be the Weierstrass data associated to $F$. Then there exist $m$ ($0 \leq m \leq n$) positive real numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^+$ such that $f_\lambda: M \to \mathbb{S}^3_1$ induced from the Weierstrass data $(\lambda g, \lambda^{-1} \omega)$, is a complete CMC 1 face of finite type with only elliptic ends for any $\lambda \in \mathbb{R} \setminus \{0, \pm \lambda_1, \ldots, \pm \lambda_m\}$.

**Proof.** Let $\hat{f}$, $F$ and $(g, \omega)$ be as above and set $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$. Then by [UY1, Theorem 3.3], there exists a 1-parameter family of reducible complete CMC 1 immersions $\hat{f}_\lambda: M \to \mathbb{H}^3$ of finite total curvature with $n$ ends. Define $\lambda_j \in \mathbb{R}^+ \cup \{0, \infty\}$ ($j = 1, \ldots, m$) as

$$
\lambda_j = \begin{cases} 
0 & \text{if } |g(p_j)| = \infty, \\
\infty & \text{if } |g(p_j)| = 0, \\
|g(p_j)|^{-1} & \text{otherwise.}
\end{cases}
$$

Then for any $\lambda \in (\mathbb{R} \cup \{\infty\}) \setminus \{0, \pm \lambda_1, \ldots, \pm \lambda_m\}$, $|\lambda g(p_j)| \neq 1$ for all $p_j$, and hence $f_\lambda: M \to \mathbb{S}^3_1$ induced from the Weierstrass data $(\lambda g, \lambda^{-1} \omega)$ is a complete CMC 1 face of finite type with only elliptic ends.

Complete CMC 1 immersions with low total curvature and low dual total curvature in $\mathbb{H}^3$ were classified in [RUY1, RUY2]. Applying Theorem 5.6 to the reducible examples in their classification, we have the following:

**Corollary 5.7** There exist the following twelve types of complete CMC 1 faces $f: M \to \mathbb{S}^3_1$ of finite type with elliptic ends:

- $O(0)$, $O(-5)$, $O(-2, -3)$, $O(-1, -1, -2)$,
- $O(-4)$, $O(-6)$, $O(-2, -4)$, $O(-1, -2, -2)$,
- $O(-2, -2)$, $O(-1, -4)$, $O(-3, -3)$, $O(-2, -2, -2)$,

where $f$ is of type $O(d_1, \ldots, d_n)$ when $M = (\mathbb{C} \cup \{\infty\}) \setminus \{p_1, \ldots, p_n\}$ and $Q$ has order $d_j$ at each end $p_j$. 

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Space-like CMC 1 surfaces with elliptic ends in $\mathbb{S}^3_1$
Furthermore, reducible complete CMC 1 immersions of genus zero with an arbitrary number of regular ends and one irregular end and finite total curvature are constructed in [MU], using an analogue of the so-called UP-iteration. Applying Theorem 5.6 to their results, we have the following:

**Corollary 5.8** Set $M = \mathbb{C} \setminus \{p_1, \ldots, p_n\}$ for arbitrary $n \in \mathbb{N}$. Then there exist choices for $p_1, \ldots, p_n$ so that there exist complete CMC 1 faces $f : M \to S^3_1$ of finite type with $n$ regular elliptic ends and one irregular elliptic end.

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