Abstract. In the present paper, we are interested in developing iterative Krylov subspace methods in tensor structure to solve a class of multilinear systems via Einstein product. In particular, we develop global variants of the GMRES and Golub-Kahan bidiagonalization processes in tensor framework. We further consider the case that mentioned equation may be possibly corresponds to a discrete ill-posed problem. Applications arising from color image and video restoration are included.

Keywords: Arnoldi process, Golub–Kahan, ill-posed problem bidiagonalization, tensor equation, Einstein product, Video processing.

1. Introduction. In this paper, we are interested in approximating the solution of following tensor equation

$$A \ast_N X = C,$$

where $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $C \in \mathbb{R}^{J_1 \times \cdots \times J_M \times I_1 \times \cdots \times I_N}$ are known and $X \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$ is an unknown tensor to be determined. We can also consider the least-squares problem

$$\min \| A \ast_N X - C \|_F.$$
regularization for tensor equation (1.1). Then we propose GMRES and Global Golub–Kahan methods via Einstein in conjunction with Tikhonov regularization. On the basis of Point Spread Function (PSF), in Section 4 we propose a tensor formulation in the form of (1.1) that describes the blurring of color image and video processing. Numerical examples are reported on restoring blurred and noisy color images and videos. Concluding remarks can be found in Section 5.

2. Definitions and Notations. In this section, we briefly review some concepts and notions that are used throughout the paper. A tensor is a multidimensional array of data and a natural extension of scalars, vectors and matrices to a higher order, a scalar is a 0th order tensor, a vector is a 1st order tensor and a matrix is 2nd order tensor. The tensor order is the number of its indices, which is called modes or ways. For a given N-mode tensor \( \mathcal{X} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \), the notation \( x_{i_1 \ldots i_N} \) (with \( 1 \leq i_j \leq I_j, j = 1, \ldots, N \)) stands for element \((i_1, \ldots, i_N)\) of the tensor \( \mathcal{X} \). Corresponding to a given tensor \( \mathcal{X} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \), the notation

\[
\mathcal{X}^j_{i_1 \ldots \cdot j \cdots \cdot i_N}, \quad k = 1, 2, \ldots, I_N,
\]

\((N-1)-\)times
denotes a tensor in \( \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_{N-1}} \) which is obtained by fixing the last index and is called frontal slice; see [20, 21] for more details. Throughout this work, vectors and matrices are respectively denoted by lowercase and capital letters, and tensors of higher order are represented by calligraphic letters.

We first recall the definition of n-mode tensor product with a matrix; see [21].

**Definition 2.1.** The n-mode product of the tensor \( \mathcal{A} = [a_{i_1i_2\ldots i_N}] \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \) and the matrix \( U = [u_{j_1}] \in \mathbb{R}^{j_1 \times i_N} \) is denoted by \( \mathcal{A} \times_n U \) is a tensor of order \( I_1 \times I_2 \times \ldots \times I_{n-1} \times j \times I_{n+1} \times \ldots \times I_N \) and its entries are defined by

\[
(\mathcal{A} \times_n U)_{i_1i_2\ldots i_{n-1}j_1i_{n+1}\ldots i_N} = \sum_{i_N} a_{i_1i_2\ldots i_N} u_{j_1i_N}.
\]

The n-mode product of the tensor \( \mathcal{A} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \) with the vector \( v = [v_{i_N}] \in \mathbb{R}^{i_N} \) is an \((N-1)\)-mode tensor denoted by \( \mathcal{A} \bar{\times} v \) whose elements are given by

\[
(\mathcal{A} \bar{\times} v)_{i_1\ldots i_{n-1}i_N} = \sum_{i_N} a_{i_1\ldots i_N} v_{i_N}.
\]

Next, we recall the definition and some properties of the tensor Einstein product which is an extension of the matrix product; for more details see [3].

**Definition 2.2.** Let \( \mathcal{A} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \), \( \mathcal{B} \in \mathbb{R}^{K_1 \times K_2 \times \ldots \times K_N} \), the Einstein product of tensors \( \mathcal{A} \) and \( \mathcal{B} \) is a tensor of size \( \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N \times K_1 \times K_2 \times \ldots \times K_N} \) whose elements are defined by

\[
(\mathcal{A} \ast_N \mathcal{B})_{i_1\ldots i_Nj_1\ldots j_M} = \sum_{k_1\ldots k_N} a_{i_1\ldots i_Nk_1\ldots k_N} b_{k_1\ldots k_Nj_1\ldots j_M}.
\]

Given a tensor \( \mathcal{A} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \), the tensor \( \mathcal{B} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N \times j} \), and the transpose of \( \mathcal{A} \), if \( b_{i_1\ldots i_Nj} = a_{j_1\ldots j_Ni_1\ldots i_N} \). We denote the transpose of \( \mathcal{A} \) by \( \mathcal{A}^T \). A tensor \( \mathcal{D} = [d_{i_1\ldots i_Mj_1\ldots j_N}] \in \mathbb{R}^{i_1 \times \ldots \times i_M \times j_1 \times \ldots \times j_N} \) is said to be diagonal if all of its entries are equal to zero except for \( d_{i_1\ldots i_Mj_1\ldots j_N} = 1 \). In the case \( d_{i_1\ldots i_Mj_1\ldots j_N} = 1 \), the tensor \( \mathcal{D} \) is called diagonal and denoted by \( \mathcal{I}_N \). We further use the notation \( \mathcal{A}^T \) for a tensor having all its entries equal to zero.

**Definition 2.3.** Let \( \mathcal{A} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \). The tensor \( \mathcal{A} \) is invertible if there exists a tensor \( \mathcal{X} \in \mathbb{R}^{i_1 \times i_2 \times \ldots \times i_N} \) such that \( \mathcal{A} \ast_N \mathcal{X} = \mathcal{X} \ast_N \mathcal{A} = \mathcal{I}_N \).
The trace of an even-order tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ is given by

$$tr(\mathcal{A}) = \sum_{i_1, \ldots, i_N} a_{i_1 \ldots i_N i_1 \ldots i_N}.$$ 

**Definition 2.4.** The inner product of two same size tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ is defined by

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} x_{i_1 i_2 \ldots i_N} y_{i_1 i_2 \ldots i_N}.$$ 

Notice that for even order tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times I_{N+1} \times I_M}$, we have

$$\langle \mathcal{X}, \mathcal{Y} \rangle = tr(\mathcal{X}^T \ast_N \mathcal{Y})$$

where $\mathcal{Y}^T \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_M \times I_{N+1} \times I_N}$ denote de transpose of $\mathcal{Y}$.

The Frobenius norm of the tensor $\mathcal{X}$ is given by

$$||\mathcal{X}||_F = \langle \mathcal{X}, \mathcal{Y} \rangle = \sqrt{tr(\mathcal{X}^T \ast_N \mathcal{X})}.$$  \hspace{1cm} (2.1)

The two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times I_{N+1} \times I_M}$ are orthogonal iff $\langle \mathcal{X}, \mathcal{Y} \rangle = 0$.

In [4], the $\otimes^N$ product between $N$-mode tensors $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_{N-1} \times I_N}$ and $\mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_{N-1} \times I_N}$ is defined as an $I_N \times I_N$ matrix whose $(i, j)$-th entry is

$$[\mathcal{X} \otimes^N \mathcal{Y}]_{ij} = \text{tr}(\mathcal{X}^{\ldots i \ldots} \otimes^{N-1} \mathcal{Y}^{\ldots j \ldots}), \quad N = 3, 4, \ldots,$$ 

where

$$\mathcal{X} \otimes^2 \mathcal{Y} = \mathcal{X}^T \mathcal{Y}, \quad \mathcal{X} \in \mathbb{R}^{I_1 \times I_2}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2}.$$ 

Basically, the product $\mathcal{X} \otimes^N \mathcal{Y}$ is the contracted product of $N$-mode tensors $\mathcal{X}$ and $\mathcal{Y}$ along the first $N - 1$ modes.

It is immediate to see that for $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$, we have

$$\langle \mathcal{X}, \mathcal{Y} \rangle = tr(\mathcal{X} \otimes^N \mathcal{Y}), \quad N = 2, 3, \ldots,$$ 

and

$$||\mathcal{X}||^2 = tr(\mathcal{X} \otimes^{N+1} \mathcal{X}) = \mathcal{X} \otimes^{N+1} \mathcal{X},$$

for $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$.

We end the current subsection by recalling the following useful proposition from [4].

**Proposition 2.5.** Suppose that $\mathcal{B} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times m}$ is an $(N + 1)$-mode tensor with the column tensors $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ and $z = (z_1, z_2, \ldots, z_m)^T \in \mathbb{R}^m$. For an arbitrary $(N + 1)$-mode tensor $\mathcal{A}$ with $N$-mode column tensors $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$, the following statement holds

$$\mathcal{A} \otimes^{N+1} (\mathcal{B} \otimes_{N+1} z) = (\mathcal{A} \otimes^{N+1} \mathcal{B}) z.$$  \hspace{1cm} (2.2)
3. Krylov subspace methods via Einstein product. In this section, we recall the tensor global Arnoldi and propose iterative methods based on Global Arnoldi and Global Golub–Kahan bidiagonalization (GGKB) combined with Tikhonov regularization that are applicable to the restoration of a color images and videos from an available blur- and noise-contaminated versions.

3.1. Tikhonov regularization. Many applications require the solution of several ill-conditioning systems of equations of the form (1.1) with a right hand side contaminated by an additive error,

$$A^*N\mathcal{X} = C + E,$$

where $E$ is the matrix of error terms that may stem from measurement and discretization errors. An ill-posed tensor equation may appear in color image restoration, video restoration, and when solving some partial differential equations in several space dimensions. In order to diminish the effect of the noise in the data, we replace the original problem by a stabilized one. One of the most popular regularization methods is due to Tikhonov [30]. Tikhonov regularization problem to solve (3.1) is given by

$$\mathcal{X}_\mu = \text{argmin}_X \left( \| A^*N\mathcal{X} - C \|_F^2 + \mu \| \mathcal{X} \|_F^2 \right),$$

The choice of $\mu$ affects how sensitive $\mathcal{X}_\mu$ is to the error $E$ in the contaminated right-hand side. Many techniques for choosing a suitable value of $\mu$ have been analyzed and illustrated in the literature; see, e.g., [33] and references therein. In this paper we use the discrepancy principle and the Generalized Cross Validation (GCV) techniques.

3.2. Global GMRES method via Einstein product. Let $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times J_1 \times \ldots \times J_M}$ be a square tensor and $\mathcal{V} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times J_1 \times \ldots \times J_M}$. The $m$-th tensor Krylov subspace is defined by

$$K_m(\mathcal{A}, \mathcal{V}) = \text{span} \{ \mathcal{V}, \mathcal{A} \mathcal{V}, \ldots, \mathcal{A}^{m-1} \mathcal{V} \},$$

where $\mathcal{A}^{i} = \mathcal{A}(\mathcal{A}^{i-1})$. The global Arnoldi process for matrix case was proposed in [18]. The algorithm for constructing orthonormal basis of (3.3) can be given as follows: (see [4, 17, 18])

**Algorithm 1** Global Arnoldi process via Einstein product

1. Inputs: A tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times J_1 \times \ldots \times J_M}$ and a tensor $\mathcal{V} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times J_1 \times \ldots \times J_M}$ and the integer $m$.
2. Set $\beta = \| \mathcal{V} \|_F$ and $\mathcal{V}_1 = \mathcal{V} / \beta$.
3. For $j = 1, \ldots, m$
4. \hspace{1em} $\mathcal{W} = \mathcal{A}^*N\mathcal{V}_j$
5. \hspace{1em} for $i = 1, \ldots, j$,\n6. \hspace{2em} $h_{ij} = \langle \mathcal{V}_i, \mathcal{W} \rangle,$
7. \hspace{2em} $\mathcal{W} = \mathcal{W} - h_{ij} \mathcal{V}_i$
8. \hspace{1em} endfor
9. $h_{j+1,j} = \| \mathcal{W} \|_F$. If $h_{j+1,j} = 0$, stop; else
10. $\mathcal{V}_{j+1} = \mathcal{W} / h_{j+1,j}$.
11. EndFor

Let $\tilde{H}_m$ be the upper $(m + 1 \times m)$ Hessenberg matrix whose entries are the $h_{ij}$ from Algorithm 1 and let $H_m$ be the matrix obtained from $\tilde{H}_m$ by deleting the last row. Then, it is not difficult to verify that the
\(\mathcal{V}_i\)'s obtained from Algorithm \([\mathcal{I}]\) form an orthonormal basis of the tensor Krylov subspace \(\mathcal{K}_m(\mathcal{A}, \mathcal{V})\). Analogous to \([4, 18]\), we can prove the following proposition.

**Proposition 3.1.** Let \(\mathcal{V}\) be the \((M+N+1)\)-mode tensor with frontal slices \(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m\) and \(\mathcal{W}_m\) be the \((M+N+1)\)-mode tensor with frontal slices \(\mathcal{A} *_N \mathcal{V}_1, \ldots, \mathcal{A} *_N \mathcal{V}_m\). Then

\[
\mathcal{W}_m = \mathcal{V}_{m+1} \times (M+N+1) \tilde{H}_m^T
= \mathcal{V}_m \times (M+N+1) H_m^T + h_{m+1,m} \mathcal{L} \times (M+N+1) E_m,
\]

where \(E_m = [0, 0, \ldots, 0, e_m]\) with \(e_m\) is the \(m\)-th column of the identity matrix \(I_m\) and \(\mathcal{L}\) is an \((M+N+1)\)-mode whose frontal slices are all zero except that last one being equal to

Let \(\mathcal{A} \in \mathbb{R}^{l_1 \times l_2 \times \cdots \times l_M} \times \mathbb{R}^{l_1 \times l_2 \times \cdots \times l_M} \) and \(\mathcal{V} \in \mathbb{R}^{l_1 \times l_2 \times \cdots \times l_M} \times \mathbb{R}^{l_1 \times l_2 \times \cdots \times l_M} \). Consider now the linear system

\[
\mathcal{A} *_N \mathcal{X} = \mathcal{V}.
\]

Using Algorithm \([\mathcal{I}]\) we can propose the global GMRES method to solve the problem \((3.5)\). As for the global GMRES, we seek for an approximate solution \(\mathcal{X}_m\), starting from \(\mathcal{X}_0\) such that \(\mathcal{X}_m \in \mathcal{X}_0 + \mathcal{K}_m(\mathcal{A}, \mathcal{V})\) and by solving the minimization problem

\[
\|\mathcal{X}_m\|_F = \min_{\mathcal{X} \in \mathcal{X}_0 + \mathcal{K}_m(\mathcal{A}, \mathcal{V})} \|\mathcal{V} - \mathcal{A} *_N \mathcal{X}\|_F.
\]

where \(\mathcal{X}_m = \mathcal{V} - \mathcal{A} *_N \mathcal{X}^*\).

Let \(m\) steps of Algorithm \([\mathcal{I}]\) has been performed. Given an initial guess \(\mathcal{X}_0\), we set

\[
\mathcal{X}_m = \mathcal{X}_0 + \mathcal{V}_m - (M+N+1) y_m,
\]

which results \(\mathcal{X}_m = \mathcal{X}_0 - \mathcal{V}_m (M+N+1) y_m\). Using the relations \((3.4)\), from Proposition \(2.5\) it immediate to observe that

\[
\|\mathcal{V} - \mathcal{A} *_N \mathcal{Y}_m\|_F = \|\mathcal{V}_m \otimes (M+N+1) (\mathcal{V} - \mathcal{A} *_N \mathcal{X}_m)\|_2
= \|\mathcal{V}_m \otimes (M+N+1) (\mathcal{X}_0 - \mathcal{V}_m (M+N+1) y_m)\|_2
= \|\beta e_{1}^{m+1} - \mathcal{V}_m \otimes (M+N+1) (\mathcal{V}_m \otimes (M+N+1) y_m)\|_2.
\]

Therefore, \(y_m\) is determined as follows:

\[
y_m = \arg\min_y \|\beta e_{1}^{m+1} - \mathcal{H}_m y\|_2.
\]

The relations \((3.7)\) and \((3.8)\) define the tensor global GMRES (TG-GMRES). Setting \(\mathcal{X}_0 = 0\) and using the relations \((3.6), (3.7)\) and \((3.8)\) it follows that instead of solving the problem \((3.2)\) we can consider the following low dimensional Tikhonov regularization problem

\[
\|\beta e_{1}^{m+1} - \mathcal{H}_m y\|_2^2 + \mu \|y\|_2^2.
\]

The solution of the problem \((3.9)\) is given by
\[ y_{m, \mu} = \arg\min \left\| \begin{pmatrix} \tilde{H}_m \\ \sqrt{\mu} \mathbf{I} \end{pmatrix} y - \begin{pmatrix} \beta e_{1}^{m+1} \\ 0 \end{pmatrix} \right\|_2. \]  

(3.10)

The minimizer \( y_{m, \mu} \) of the problem (3.10) is computed as the solution of the linear system of equations

\[ \tilde{H}_{m, \mu} y = \tilde{H}_m^T \beta e_{1}^{m+1} \]  

(3.11)

where \( \tilde{H}_{m, \mu} = (\tilde{H}_m^T \tilde{H}_m + \mu \mathbf{I}) \).

Notice that the Tikhonov problem (3.9) is a matrix one with small dimension as \( m \) is generally small. Hence it can be solved by some techniques such as the GCV method [13] or the L-curve criterion \([4, 15, 7, 8]\).

An appropriate selection of the regularization parameter \( \mu \) is important in Tikhonov regularization. Here we can use the generalized cross-validation (GCV) method \([6, 13, 33]\). For this method, the regularization parameter is chosen to minimize the GCV function

\[ \text{GCV}(\mu) = \frac{\| \tilde{H}_m y_{m, \mu} - \beta e_{1}^{m+1} \|_2^2}{\| \text{tr}((I - \tilde{H}_m \tilde{H}_m^T) \beta e_{1}^{m+1}) \|_2^2} = \frac{\| (I - \tilde{H}_m \tilde{H}_m^T) \beta e_{1}^{m+1} \|_2^2}{| \text{tr}((I - \tilde{H}_m \tilde{H}_m^T) \beta e_{1}^{m+1}) |^2} \]

(3.12)

where \( \tilde{H}_{m, \mu} = (\tilde{H}_m^T \tilde{H}_m + \mu \mathbf{I}) \) and \( y_{m, \mu} \) is the solution of (3.11). As the projected problem we are dealing with is of small size, we can use the SVD decomposition of \( \tilde{H}_m \) to obtain a more simple and computable expression of \( \text{GCV}(\mu) \). Consider the SVD decomposition of \( \tilde{H}_m = U \Sigma V^T \). Then the GCV function could be expressed as (see \([33]\))

\[ \text{GCV}(\mu) = \frac{\sum_{i=1}^m \frac{g_i}{\sigma_i^2 + \mu}}{\left( \sum_{i=1}^m \frac{1}{\sigma_i^2 + \mu} \right)^2}, \]

(3.13)

where \( \sigma_i \) is the \( i \)th singular value of the matrix \( \tilde{H}_m \) and \( g_i = \beta_i U^T e_{1}^{m+1} \).

In the practical implementation, it’s more convenient to use a restarted version of the global GMRES. As the number of outer iterations increases, it is possible to compute the \( m \)-th residual without forming the solution. This is described in the following theorem.

**Proposition 3.2.** At step \( m \), the residual \( \mathcal{R}_m = \mathcal{C} - \mathcal{A}_N \mathcal{D}_m \) produced by the tensor global GMRES method for solving (1.1) has the following expression

\[ \mathcal{R}_m = \mathcal{V}_{m+1} \times (M+N+1) \left( \gamma_{m+1} Q_{m} e_{m+1} \right), \]

(3.14)

where \( Q_{m} \) is the unitary matrix obtained by QR decomposition of the upper Hessenberg matrix \( \tilde{H}_m \) and \( \gamma_{m+1} \) is the last component of the vector \( \beta Q_{m} e_{m+1} \) in which \( \beta = \| \mathcal{R}_0 \|_F \) and \( e_{\ell} \in \mathbb{R}^\ell \) is the last column of identity matrix. Furthermore,

\[ \| \mathcal{R}_m \|_F = | \gamma_{m+1} | \]

Proof. At step \( m \), the residual \( \mathcal{R}_m = \mathcal{R}_0 - \mathcal{V}_m \tilde{x}_{(M+N+1)} y_{m} \) can be expressed as

\[ \mathcal{R}_m = \mathcal{R}_0 - \left( \mathcal{V}_{m+1} \times (M+N+1) \tilde{H}_m \right) \tilde{x}_{(M+N+1)} y_{m} \]

\[ = \mathcal{R}_0 - \mathcal{V}_{m+1} \tilde{x}_{(M+N+1)} (\tilde{H}_m y_{m}) \]
by considering the QR decomposition \( \tilde{H}_m = Q_m \tilde{U}_m \) of the \((m+1) \times m\) matrix \( \tilde{H}_m \), we get

\[
\mathcal{R}_m = \mathcal{R}_0 - \nabla_{m+1} \tilde{x}_{(M+N+1)} (Q_m \tilde{U}_m y_m).
\]

Straightforward computations show that

\[
|\mathcal{R}_m|^2_F = |\mathcal{R}_0 - \nabla_{m+1} \tilde{x}_{(M+N+1)} (Q_m \tilde{U}_m y_m)|^2_F
\]

\[
= \| \nabla_{m} \tilde{x}_{(M+N+1)} (\mathcal{R}_0 - \nabla_{m+1} \tilde{x}_{(M+N+1)} (Q_m \tilde{U}_m y_m)) \|_2^2
\]

\[
= \| Q_m (Q^T_m \beta e_1^{m+1} - \tilde{U}_m y_m) \|_2^2
\]

\[
= \| Q_m (Q^T_m \beta e_1^{m+1} - \tilde{U}_m y_m) \|_2^2
\]

\[= \| \tilde{z}_m - \tilde{U}_m y_m \|_2^2 + |y_{m+1}|^2 \]

where \( \tilde{z}_m \) denotes vector obtained by deleting the last component of \( Q^T_m \beta e_1^{m+1} \). Since \( y_m \) solves problem (3.8), it follows that \( y_m \) is the solution of \( \tilde{U}_m y_m = \tilde{z}_m \), i.e.,

\[
\| \tilde{z}_m - \tilde{U}_m y_m \|_2 = 0.
\]

Note that \( \mathcal{R}_m \) can be written in the following form

\[
\mathcal{R}_m = 0 \nabla_{m+1} \tilde{x}_{(M+N+1)} e_1^{m+1} - \nabla_{m+1} \tilde{x}_{(M+N+1)} (\tilde{H}_m y_m)
\]

\[
= \nabla_{m+1} \tilde{x}_{(M+N+1)} (\beta e_1^{m+1} - \tilde{H}_m y_m)
\]

\[
= \nabla_{m+1} \tilde{x}_{(M+N+1)} (Q_m (Q^T_m \beta e_1^{m+1} - \tilde{U}_m y_m))
\]

\[
= \nabla_{m+1} \tilde{x}_{(M+N+1)} (Q_m y_m + e_1^{m+1} e_1^{m+1}).
\]

Now the result follows immediately from the above computations.

The tensor form of global GMRES algorithm for solving (1.1) is summarized as follows:

**Algorithm 2** Global GMRES method via Einstein product for Tikhonov regularization

1. **Inputs** The tensors \( \mathcal{A}, \mathcal{B} \), initial guess \( \mathcal{X}_0 \), a tolerance \( \varepsilon \), number of iterations between restarts \( m \) and Maxit: maximum number of outer iterations.
2. Compute \( \mathcal{R}_0 = \mathcal{B} - \mathcal{A} \times \mathcal{X}_0 \), set \( \nu = \mathcal{R}_0 \) and \( k = 0 \)
3. Determine the orthonormal frontal slices \( \nu_1, \ldots, \nu_m \) of \( \nabla_{m} \), and the upper Hessenberg matrix \( \tilde{H}_m \) by applying Algorithm 1 to the pair \( (\mathcal{A}, \nu) \).
4. Determine \( \mu_k \) as the parameter minimizing the GCV function given by (3.12) \( \beta \rightarrow \beta \rightarrow \ldots \)
5. Determine \( y_m \) as the solution of low-dimensional Tikhonov regularization problem (3.9) and set \( \mathcal{X}_m = \mathcal{R}_0 + \nabla_{m} \tilde{x}_{(M+N+1)} y_m \)
6. If \( |y_{m+1}|_F < \varepsilon \) or \( k > \text{Maxit} \): Stop
   else: set \( \mathcal{X}_0 = \mathcal{X}_m \), \( k = k + 1 \), Goto 2

3.3. **Golub–Kahan method via Einstein.** Instead of finding orthonormal basis for the Krylov subspace and using GMRES method, one can apply oblique projection schemes based on biorthogonal bases for \( \mathcal{X}_m (\mathcal{A}, \nu) \) and \( \mathcal{X}_m (\mathcal{A}^T, \nu^T) \); see [19] for instance.

Here, we exploit the tensor Golub–Kahan algorithm via the Einstein product. It should be commented here that the Golub–Kahan algorithm has been already examined for solving ill-posed Sylvester and Lyapunov tensor equations with applications to color image restoration [5].
Algorithm 3 Global Golub–Kahan algorithm via Einstein product

1. Input The tensors $\mathcal{A}$, $\mathcal{C}$, and an integer $\ell$.
2. Set $\sigma_1 = \|\mathcal{C}\|_F$, $\frac{\mathcal{U}_1}{\sigma_1} = \mathcal{C}$ and $\gamma_0 = 0$.
3. For $j = 1, 2, \ldots, \ell$ Do
4. $\tilde{V}_j = \mathcal{A}^T \ast N \mathcal{U}_j - \sigma_j \mathcal{V}_{j-1}$
5. $\rho_j = \|\tilde{V}_j\|_F$ if $\rho_j = 0$ stop, else
6. $\mathcal{V}_j = \tilde{V}_j / \rho_j$
7. $\tilde{U}_j = \mathcal{A} \ast N \mathcal{V}_j - \rho_j \mathcal{U}_j$
8. $\sigma_j + 1 = \|\tilde{U}_j\|_F$
9. if $\rho_j = 0$ stop, else
10. $\mathcal{U}_{j+1} = \tilde{U}_j / \sigma_{j+1}$
11. EndDo

Let tensors $\mathcal{A} \in \mathbb{R}^{I_1 \times \ldots \times I_N \times I_1 \times \ldots \times I_M}$, $\mathcal{V} \in \mathbb{R}^{I_1 \times \ldots \times I_N \times J_1 \times \ldots \times J_M}$ and $\mathcal{U} \in \mathbb{R}^{J_1 \times \ldots \times J_M \times I_1 \times \ldots \times I_N}$ be given.

Then, the global Golub–Kahan bidiagonalization (GGKB) algorithm is summarized in Algorithm 3.

Assume that $\ell$ steps of the GGKB process have been performed, we form the lower bidiagonal matrix $C_{\ell} \in \mathbb{R}^{\ell \times \ell}$

$$C_{\ell} = \begin{bmatrix} \rho_1 & \sigma_2 & \ldots & \sigma_{\ell-1} & \rho_{\ell} \\ \sigma_2 & \rho_2 & \ldots & \rho_{\ell-1} & \sigma_{\ell} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{\ell-1} & \rho_{\ell-1} & \ldots & \rho_{\ell} \\ \sigma_{\ell} & \rho_{\ell} & \ldots & \ldots & \ldots \end{bmatrix}$$

and

$$\tilde{C}_{\ell} = \begin{bmatrix} C_{\ell} \\ \sigma_{\ell+1} \mathbf{e}_T^{\ell} \end{bmatrix} \in \mathbb{R}^{(\ell+1) \times \ell}.$$ 

Proposition 3.3. Assume that $\ell$ have performed and all non-trivial entries of the matrix $\tilde{C}_{\ell}$ are positive. Let $\mathcal{V}_\ell$ and $\mathcal{U}_\ell$ be $(M+N+1)$-mode tensors whose frontal slices are given by $\mathcal{V}_j$ and $\mathcal{U}_j$ for $j = 1, 2, \ldots, \ell$, respectively. Furthermore, suppose that $\mathcal{W}_\ell$ and $\mathcal{W}_\ell^*$ are $(M+N+1)$-mode tensors having frontal slices $\mathcal{A} \ast N \mathcal{V}_j$ and $\mathcal{A}^T \ast N \mathcal{U}_j$ for $j = 1, 2, \ldots, \ell$, respectively. The following relations hold:

$$\mathcal{W}_\ell = \mathcal{V}_{\ell+1} \times_{(M+N+1)} \tilde{C}_{\ell}^T,$$  \hspace{1cm} (3.15)

$$\mathcal{W}_\ell^* = \mathcal{U}_{\ell} \times_{(M+N+1)} \tilde{C}_{\ell}^T.$$  \hspace{1cm} (3.16)

Proof. From Lines 7 and 10 of Algorithm 3 we have

$$\mathcal{A} \ast N \mathcal{V}_j = \rho_j \mathcal{U}_j + \sigma_{j+1} \mathcal{V}_{j+1} \quad j = 1, 2, \ldots, \ell$$

which conclude (3.15) from definition of n-mode product. Similarly, Eq. (3.16) follows from Lines 4 and 6 of Algorithm 3.

Here, we apply the following Tikhonov regularization approach and solve the new problem.
\( \min_{\mathcal{X}} \left( \| \mathcal{A} *_{N} \mathcal{X} - \mathcal{C} \|_{F}^{2} + \mu^{-1} \| \mathcal{C} \|_{F}^{2} \right) , \) \hspace{1cm} (3.17)

We comment on the use of \( \mu^{-1} \) in (3.17) instead of \( \mu \) below. As for the iterative tensor Global GMRES method discussed in the previous subsection, the computation of an accurate approximation \( \mathcal{X}_{\mu} \) requires that a suitable value of the regularization parameter be used. In this subsection, we use the discrepancy principle to determine a suitable regularization parameter assuming that an approximation of the norm of additive error is available, i.e., we have a bound \( \varepsilon \) for \( \| \mathcal{E} \|_{F} \). This priori information suggests that \( \mu \) has to be determined such that,

\[ \| \mathcal{C} - \mathcal{A} *_{N} \mathcal{X}_{\mu} \|_{F} = \eta \varepsilon , \] \hspace{1cm} (3.18)

where \( \eta > 1 \) is the safety factor for the discrepancy principle. A zero-finding method can be used to solve (3.18) in order to find a suitable regularization parameter which also implies that \( \| \mathcal{C} - \mathcal{A} *_{N} \mathcal{X}_{\mu} \|_{F} \) has to be evaluated for several \( \mu \)-values. When the tensor \( \mathcal{A} \) is of moderate size, the quantity \( \| \mathcal{C} - \mathcal{A} *_{N} \mathcal{X}_{\mu} \|_{F} \) can be easily evaluated. This computation becomes expensive when \( \mathcal{A} \) is a large tensor, which means that its evaluation by a zero-finding method can be very difficult and computationally expensive. In what follows, it is shown that this difficulty can be remedied by using a connection between the Golub–Kahan bidiagonalization (GGKB) and Gauss-type quadrature rules. This connection provides approximations of moderate sizes to the quantity \( \| \mathcal{C} - \mathcal{A} *_{N} \mathcal{X}_{\mu} \|_{F} \) and therefore gives a solution method to inexpensively solve (3.18) by evaluating these small quantities; see [1, 2] for discussion on this method.

Let us consider the following functions of \( \mu \),

\[ \Phi(\mu) = \left\| \mathcal{C} - \mathcal{A} *_{N} \mathcal{X}_{\mu} \right\|_{F}^{2} \] \hspace{1cm} (3.19)

\[ \mathcal{G}_{l}f_{\mu} = \| \mathcal{C} \|_{F}^{2} e_{1}^{T} \left( \mu C_{l} C_{l}^{T} + I_{l} \right)^{-2} e_{1} , \] \hspace{1cm} (3.20)

\[ \mathcal{R}_{l+1}f_{\mu} = \| \mathcal{C} \|_{F}^{2} e_{1}^{T} \left( \mu \tilde{C}_{l} \tilde{C}_{l}^{T} + I_{l+1} \right)^{-2} e_{1} . \] \hspace{1cm} (3.21)

\( \mathcal{G}_{l}f \) and \( \mathcal{R}_{l+1}f_{\mu} \) are pairs of Gauss and Gauss-Radau quadrature rules, respectively, and they approximate \( \Phi(\mu) \) as follows

\[ \mathcal{G}_{l}f_{\mu} \leq \Phi(\mu) \leq \mathcal{R}_{l+1}f_{\mu} \] \hspace{1cm} (3.22)

As shown in [1, 2], for a given value of \( l \geq 2 \), we solve for \( \mu \) the nonlinear equation

\[ \mathcal{G}_{l}f_{\mu} = \varepsilon^{2} \] \hspace{1cm} (3.23)

by using Newton’s method.

The use the parameter \( \mu \) in (3.17) instead of \( 1/\mu \), implies that the left-hand side of (3.18) is a decreasing convex function of \( \mu \). Therefore, there is a unique solution, denoted by \( \mu_{\varepsilon} \), of

\[ \Phi(\mu) = \varepsilon^{2} \] \hspace{1cm} (3.24)

for almost all values of \( \varepsilon > 0 \) of practical interest and therefore also of (3.23) for \( \ell \) sufficiently large; see [1, 2] for analyses. We accept \( \mu_{\ell} \) that solve (3.18) as an approximation of \( \mu \), whenever we have

\[ \mathcal{R}_{l+1}f_{\mu} \leq \eta^{2} \varepsilon^{2} . \] \hspace{1cm} (3.24)
If (3.24) does not hold for \( \mu \), we carry out one more GGKB steps, replacing \( \ell \) by \( \ell + 1 \) and solve the nonlinear equation

\[
\mathcal{G}_{\ell + 1} f_{\mu} = \varepsilon^2;
\]

(3.25)

see [1, 2] for more details. Assume now that (3.24) holds for some \( \mu \). The corresponding regularized solution is then computed by

\[
\mathcal{X}_\ell = \mathbb{W}_\ell \tilde{x}_{(M+N+1)\ell},
\]

(3.26)

where \( y_{\mu} \) solves

\[
(\tilde{C}_\ell^T \tilde{C}_\ell + \mu^{-1} I)y = \sigma_1 \tilde{C}_\ell^T e_1, \quad \sigma_1 = \| \varepsilon \|_F.
\]

(3.27)

It is also computed by solving the least-squares problem

\[
\min_{y \in \mathbb{R}^l} \left\| \left[ \mu^{1/2} \tilde{C}_\ell \right] y - \sigma_1 \mu^{1/2} e_1 \right\|_2^2.
\]

(3.28)

The following result shows an important property of the approximate solution (3.26). We include a proof for completeness.

**Proposition 3.4.** Under assumptions of Proposition 3.3, let \( \mu \) solve (3.24) and let \( y_{\mu} \) solve (3.28). Then the associated approximate solution (3.26) of (3.17) satisfies

\[
\| \mathcal{A} *_N \mathcal{X}_{\mu} - \varepsilon \|_F^2 = R_{\ell + 1} f_{\mu}.
\]

**Proof.** By Eq. (3.15) we have

\[
\mathcal{A} *_N \mathcal{X}_{\mu} = \sum_{\ell = 1}^{\ell} (\mathcal{A} *_N \mathcal{Y}_\ell) = \mathbb{W}_\ell \tilde{x}_{(M+N+1)\ell}
\]

\[
= \mathbb{U}_{\ell + 1} \tilde{x}_{(M+N+1)}(\tilde{C}_\ell y_\ell)
\]

Using the above expression gives

\[
\| \mathcal{A} *_N \mathcal{X}_{\mu} - \varepsilon \|_F^2 = \left\| \mathbb{U}_{\ell + 1} \tilde{x}_{(M+N+1)}(\tilde{C}_\ell y_\ell) - \sigma_1 \mathbb{W}_\ell \right\|_F^2
\]

\[
= \left\| \mathbb{U}_{\ell + 1} \tilde{x}_{(M+N+1)}(\tilde{C}_\ell y_\ell) - \mathbb{U}_{\ell + 1} \tilde{x}_{(M+N+1)}(\sigma_1 e_1) \right\|_F^2
\]

\[
= \left\| \mathbb{U}_{\ell + 1} \mathbb{W}(M+N+1) (\mathbb{U}_{\ell + 1} \tilde{x}_{(M+N+1)}(\tilde{C}_\ell y_\ell - \sigma_1 e_1)) \right\|_F^2
\]

\[
= \left\| (\mathbb{U}_{\ell + 1} \tilde{x}_{(M+N+1)}(\tilde{C}_\ell y_\ell - \sigma_1 e_1)) \right\|_2^2
\]

where we recall that \( \sigma_1 = \| \varepsilon \|_F \). We now express \( y_{\mu} \) with the aid of (3.24) and apply the following identity

\[
I - A (A^T A + \mu^{-1} I)^{-1} A^T = (\mu A A^T + I)^{-1}
\]
with $A$ replaced by $	ilde{C}_t$, to obtain

$$
\left\| A \ast_N \mathcal{X}_{\mu, t} - \mathcal{C} \right\|_F^2 = \sigma_t^2 \left\| e_1 - \tilde{C}_t \left( \tilde{C}_t^T \tilde{C}_t + \mu_t I_\ell \right)^{-1} \tilde{C}_t^T e_1 \right\|_F^2
$$

$$
= \sigma_t^2 e_1^T \left( \mu_t \tilde{C}_t \tilde{C}_t^T + I_{\ell+1} \right)^{-2} e_1
$$

$$
= R_{t+1} f_{\mu_t}
$$

which conclude the assertion. □

The following algorithm summarizes the main steps to compute a regularization parameter and a corresponding regularized solution of (1.1) using GGKB and quadrature rules method for Tikhonov regularization.

**Algorithm 4** GGKB and quadrature rules method for Tikhonov regularization via Einstein product

1. **Inputs** Tensors $\mathcal{A}$, $\mathcal{C}$, $\eta \leq 1$ and $\epsilon$.
2. Determine the orthonormal bases $\mathcal{U}_{\ell+1}$ and $\mathcal{V}_l$ of tensors, and the bidiagonal matrices $C_l$ and $C_{\ell}$ by implementing Algorithm 3.
3. Determine $\mu_l$ that satisfies (3.23) with Newton’s method.
4. Determine $y_{\mu_l}$ by solving (3.28) and then compute $X_{\mu_l}$ by (3.26).

### 4. Numerical results.

This section provides some numerical results to show the performance of Algorithms 2 and Algorithm 4 when applied to the restoration of blurred and noisy color images and videos. For clarity and definiteness, we first focus on the formulation of a tensor model, describing the blurring that is taking place in the process of going from the exact to the blurred RGB image (or video). Notwithstanding what has just been said, recovering RGB (or video) from their blurry and noisy observations can be seen as a tensor problem of the form (1.1). Therefore, its very important to understand how the model (1.1) can be constructed for RGB images and color video deblurring problems. In what follows, we will concentrate only on the formulation of the tensor model for RGB image deblurring problems and will comment at the end of this section how a similar one can be formulated for color video deblurring problems. We recall that an RGB image is just multidimensional array of dimension $M \times N \times 3$ whose entries are the light intensity. Throughout this paper, we assume that the original RGB image has the same dimensions as the blurred one, and we refer to it as $N \times N \times 3$ tensor. Let $\mathcal{C}$ represent the available blurred RGB image, let $\mathcal{X}$ denote the desired unknown blurred RGB, and let $\mathcal{A}$ be the tensor describing the blurring that is taking place in the process of going from $\mathcal{X}$ to $\mathcal{C}$. It is well known in the literature of image processing that all the blurring operators can be characterized by a Point Spread Function (PSF) describing the blurring process and the boundary conditions outside the image, see [16]. Once the two-dimensional PSF array, $P$, is specified, we can as well build the blurring tensor $\mathcal{A}$. By using the fact that the blurring process of an RGB image is simply a multi-dimensional convolution operation of the PSF array $P$ and the original three-dimensional image $\mathcal{X}$, the blurring tensor $\mathcal{A}$ can be easily constructed by placing the elements of $P$ in the appropriate positions. Note that the PSF is a two-dimensional array $P$ describing the image of a single white pixel, which makes its dimensions much smaller than $N$. Therefore, $P$ contains all the required information about the blurring throughout the RGB image $\mathcal{C}$. To illustrate this, the discrete operation for multi-dimensional convolution using a $3 \times 3$ local and spatially invariant PSF array $P$ with $p_{22}$ is its center, and assuming zero boundary conditions, is given by:

$$
\begin{align*}
\mathcal{A}_{i j k} &= p_{33} \mathcal{X}_{i-1, j-1, k} + p_{32} \mathcal{X}_{i-1, j, k} + p_{31} \mathcal{X}_{i-1, j+1, k} + p_{33} \mathcal{X}_{i-1, j-1, k} + p_{22} \mathcal{X}_{i j k} \\
&+ p_{21} \mathcal{X}_{i j+1 k} + p_{13} \mathcal{X}_{i+1 j-1, k} + p_{12} \mathcal{X}_{i+1 j+1 k} + p_{11} \mathcal{X}_{i+1 j+1 k},
\end{align*}
$$

(4.1)

$$
\begin{align*}
\mathcal{C}_{i j k} &= p_{33} \mathcal{X}_{i-1, j-1, k} + p_{32} \mathcal{X}_{i-1, j, k} + p_{31} \mathcal{X}_{i-1, j+1, k} + p_{33} \mathcal{X}_{i-1, j-1, k} + p_{22} \mathcal{X}_{i j k} \\
&+ p_{21} \mathcal{X}_{i j+1 k} + p_{13} \mathcal{X}_{i+1 j-1, k} + p_{12} \mathcal{X}_{i+1 j+1 k} + p_{11} \mathcal{X}_{i+1 j+1 k}.
\end{align*}
$$

(4.2)
for $i, j = 1, ..., N$ and $k = 1, 2, 3$. Here the zero boundary conditions are imposed so the values of $\mathcal{D}$ are zero outside the RGB image, i.e., $\mathcal{D}_{ijk} = \mathcal{D}_{iN+1k} = \mathcal{D}_{0jk} = \mathcal{D}_{N+i1k} = 0$ for $0 < i, j < N + 1$ and $k = 1, 2, 3$. By using Definition ?? and Definition ?? a fourth order tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times N \times N}$ associated with $\mathcal{A}_{i}^{k}$ with partition $(1, N, 1, N)$, can be partitioned into matrix blocks of size $N \times N$. Each block is denoted by $\mathcal{A}_{i}^{(2,4)}_{jk} = \mathcal{A}(i, i_{2}, i_{3}, i_{4}) \in \mathbb{R}^{N \times N}$ with $i_{2}, i_{3}, i_{4} = 1, ..., N$ and $i = 1, ..., N$. The nonzero entries of the matrix block $\mathcal{A}_{a,b}^{(2,4)} \in \mathbb{R}^{N \times N}$ are given by

\[
\begin{align*}
(\mathcal{A}_{a,b}^{(2,4)})_{a-1b-1} &= p_{33}; \\
(\mathcal{A}_{a,b}^{(2,4)})_{a-1b} &= p_{32}; \\
(\mathcal{A}_{a,b}^{(2,4)})_{a+1b-1} &= p_{13}; \\
(\mathcal{A}_{a,b}^{(2,4)})_{a+1b} &= p_{12}; \\
(\mathcal{A}_{a,b}^{(2,4)})_{ab-1} &= p_{23}; \\
(\mathcal{A}_{a,b}^{(2,4)})_{ab} &= p_{22};
\end{align*}
\]

for $a, b = 2, ..., N - 1$.

The first following examples applies Algorithms ?? and ?? to the restoration of blurred color image and video that have been contaminated by Gaussian blur and by additive zero-mean white Gaussian noise. We consider the blurring to be local and spatially invariant. In this case the entries of the Gaussian PSF array $P$ are given by

\[
p_{ij} = \exp \left( -\frac{1}{2} \left( \frac{(i-k)}{\sigma} \right)^2 - \frac{1}{2} \left( \frac{(j-\ell)}{\sigma} \right)^2 \right),
\]

where $\sigma$ controls the width of the Gaussian PSF and $(k, \ell)$ is its center, see [16]. Note that $\sigma$ controls the amount of smoothing, i.e., the larger the $\sigma$, the more ill posed the problem. The original tensor image is denoted by $\hat{\mathcal{X}}$ in each example and $\mathcal{A}$ represents the blurring tensor. The tensor $\mathcal{G} = \mathcal{A} \ast_{N} \hat{\mathcal{X}}$ represents the associated blurred and noise-free multichannel image. We generated a blurred and noisy tensor image $\mathcal{G} = \mathcal{G} + \mathcal{N}$, where $\mathcal{N}$ is a noise tensor with normally distributed random entries with zero mean and with variance chosen to correspond to a specific noise level $v := \|\mathcal{N}\|_{F}/\|\mathcal{G}\|_{F}$. To determine the effectiveness of our solution methods, we evaluate

\[
\text{RE} = \frac{\|\hat{\mathcal{X}} - \mathcal{X}_{\text{restored}}\|_{F}}{\|\hat{\mathcal{X}}\|_{F}}
\]

and the Signal-to-Noise Ratio (SNR) defined by

\[
\text{SNR}(\mathcal{X}_{\text{restored}}) = 10\log_{10} \frac{\|\hat{\mathcal{X}} - E(\hat{\mathcal{X}})\|_{F}^{2}}{\|\mathcal{X}_{\text{restored}} - \hat{\mathcal{X}}\|_{F}^{2}}
\]

where $E(\hat{\mathcal{X}})$ denotes the mean gray-level of the uncontaminated image $\hat{\mathcal{X}}$. All computations were carried out using the MATLAB environment on an Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz (8 CPUs) computer with 12 GB of RAM. The computations were done with approximately 15 decimal digits of relative accuracy.

4.1. Example 1. This example illustrates the performance of Algorithms ?? and ?? when applied to the restoration of 3-channel RGB color image that have been contaminated by Gaussian blur and additive noise. The original (unknown) RGB image $\hat{\mathcal{X}} \in \mathbb{R}^{256 \times 256 \times 3}$ is the papav256 image from
MATLAB. It is shown on the left-hand side of Figure 4.1. For the blurring tensor $\mathcal{A}$, we consider a PSF array $P$ with $\sigma = 2$ under zero boundary conditions. The associated blurred and noisy RGB image $\hat{C} = \mathcal{A} \ast_N \hat{X}$ is shown on the right-hand side of Figure 4.1. The noise level is $\nu = 10^{-3}$. Given the contaminated RGB image $\mathcal{C}$, we would like to recover an approximation of the original RGB image $\hat{X}$. Table 4.1 compares the computing time (in seconds), the relative errors and the PSNR of the computed restorations. Note that in this table, the allowed maximum number of outer iterations for Algorithm 2 with noise level $\nu = 10^{-2}$ was 4. The restoration for noise level $\nu = 10^{-3}$ is shown on the left-hand side of Figure 4.2 and it is obtained by applying Einstein tensor global GMRES method (Algorithm 2) with input $\mathcal{A}, \mathcal{C}, X_0 = \hat{C}, \varepsilon = 10^{-6}, m = 10$ and Maxit = 10. Using GCV, the computed optimal value for the projected problem in Algorithm 2 was $\mu_5 = 9.44 \times 10^{-4}$. The restoration obtained with Algorithm 4 is shown on the right-hand side of Figure 4.2. The discrepancy principle with $\eta = 1.1$ is satisfied when $\ell = 61$ steps of the Einstein tensor GGKB method have been carried out, producing a regularization parameter given by $\mu_\ell = 2.95 \times 10^{-4}$.

| Noise level | Method | PSNR      | RE        | CPU-time (seconds) |
|-------------|--------|-----------|-----------|--------------------|
| $10^{-3}$   | Algorithm 2 | 21.76     | $6.09 \times 10^{-2}$ | 8.28               |
|             | Algorithm 4 | 24.37     | $4.51 \times 10^{-2}$ | 7.29               |
| $10^{-2}$   | Algorithm 2 | 20.60     | $6.96 \times 10^{-2}$ | 3.31               |
|             | Algorithm 4 | 20.97     | $6.67 \times 10^{-2}$ | 1.58               |

4.2. Example 2. In this example, we evaluate the effectiveness of Algorithms 2 and 3 when applied to the restoration of a color video defined by a sequence of RGB images. Video restoration is the problem of restoring a sequence of $k$ color images (frames). Each frame is represented by a tensor of $N \times N \times 3$ pixels. In the present example, we are interested in restoring 10 consecutive frames of a contaminated video. We consider the xylophone video from MATLAB. The video clip is in MP4 format with each frame having 240 $\times$ 240 pixels. The (unknown) blur- and noise-free frames are stored in the tensor $\hat{C} \in \mathbb{R}^{N \times N \times 3 \times 10}$. These frames are blurred by a blurring tensor $\mathcal{A}$ of the same kind and with the same parameters as in the previous example. Figure 4.3 shows the 5th exact (original) frame and the contaminated version, which is to be restored. Blurred and noisy frames are generated by
\[ \hat{G} = \mathcal{A} \ast_N \hat{X} \]
where the tensor $\mathcal{E}$ represents white Gaussian noise of levels $\nu = 10^{-3}$ or $\nu = 10^{-2}$. Table 4.2 displays the performance of algorithms. For Algorithm 2, we have used as an input $\mathcal{A}$, $\mathcal{E}$, $\mathcal{X}_0 = \mathcal{O}$, $\varepsilon = 10^{-6}$, $m = 10$ and Maxit = 10. For the ten outer iterations, minimizing the GCV function produces $\mu_{10} = 9.44 \times 10^{-4}$. Using Algorithm 4, the discrepancy principle with $\eta = 1.1$ have been satisfied after $\ell = 59$ steps of the Einstein tensor GGKB method, producing a regularization parameter given by $\mu_{\ell} = 1.06 \times 10^{-4}$. The restorations obtained with Algorithms 2 and 4 are shown on the left-hand and right-hand sides of Figure 4.4 respectively.

**Table 4.2**

*Results for Example 2.*

| Noise level | Method    | PSNR  | Relative error | CPU-time (second) |
|------------|-----------|-------|----------------|-------------------|
| $10^{-3}$  | Algorithm 2 | 15.48 | $6.84 \times 10^{-2}$ | 38.93            |
|            | Algorithm 4 | 19.24 | $4.43 \times 10^{-2}$ | 27.37            |
| $10^{-2}$  | Algorithm 2 | 14.50 | $7.65 \times 10^{-2}$ | 15.55            |
|            | Algorithm 4 | 15.13 | $7.11 \times 10^{-2}$ | 4.40             |
5. Conclusion. We extended the GMRES and Golub–Kahan bidiagonalization in conjunction of Tikhonov regularization for solving (possibly) ill-conditioned multilinear systems via Einstein product with perturbed right-hand side. Numerical experiments were disclosed for image and video processing to demonstrate the feasibility of proposed iterative algorithms.

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