Abstract. In this paper we introduce and study absolute continuity and singularity of positive operators acting on anti-dual pairs. We establish a general theorem that can be considered as a common generalization of various earlier Lebesgue-type decompositions. Different algebraic and topological characterizations of absolute continuity and singularity are supplied and also a complete description of uniqueness of the decomposition is provided. We apply the developed decomposition theory to some concrete objects including Hilbert space operators, Hermitian forms, representable functionals, and additive set functions.

1. Introduction

This paper is part of a unification project aiming to find a common framework and generalization for various results obtained in different branches of functional analysis including extension, dilation and decomposition theory. One important class of such results are decomposition theorems analogous to the well known Lebesgue decomposition of measures. What do we mean about analogous? In several cases, transformations of a given system can be grouped into two extreme classes according to the behavior with respect to their qualitative properties. These particular classes are the so-called regular transformations (i.e., transformations with “nice” properties) and the so-called singular ones (transformations that are hard to deal with). Of course, regularity and singularity may have multiple meanings depending on the context. A decomposition of an object into regular and singular parts is called a Lebesgue-type decomposition.

In order to understand a structure better, it can be effective to characterize its regular and singular elements. This explains why a regular-singular type decomposition theorem may have theoretic importance, especially when the corresponding regular part can be interpreted in a canonical way. The prototype of such results is the celebrated Radon-Nikodym theorem stating that every $\sigma$-finite measure splits uniquely into absolutely continuous and singular parts with respect to any other measure, and the absolutely continuous part has an integral representation. Returning to the previous idea, the Radon-Nikodym theorem can be phrased as follows: if we want to decide whether a set function can be represented as a point function,

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we only need to know if it is absolutely continuous or not. That is to say, in this concrete situation, the appropriate regularity concept is absolute continuity.

In the last 50 years quite a number of authors have made significant contributions to the vast literature of non-commutative Lebesgue-Radon-Nikodym theory – here we mention only Ando [2], Gudder [17], Inoue [22], Kosaki [23] and Simon [30], and from the recent past Di Bella and Trapani [7], Corso [8–10], ter Elst and Sauter [13], Gheondea [16], Hassi et al. [18–21], Sebestyén and Titkos [32], Szűcs [34], Vogt [46].

The purpose of the present paper is to develop and investigate an abstract decomposition theory that can be considered as a common generalization of many of the aforementioned results on Lebesgue-type decompositions. The key observation is that the corresponding absolute continuity and singularity concepts rely only on some topological and algebraic properties of an operator acting between an appropriately chosen vector space and its conjugate dual. So that, the problem of decomposing Hilbert space operators, representable functionals, Hermitian forms and measures can be transformed into the problem of decomposing such an abstract operator.

In this note we are going to investigate Lebesgue decompositions of positive operators on a so-called anti-dual pair. Hence, for the readers sake, we gathered in Section 2 the most important facts about anti-dual pairs and operators between them. We also provide here a variant of the famous Douglas factorization theorem. Section 3 contains the main result of the paper (Theorem 3.3), a direct generalization of Ando’s Lebesgue decomposition theorem [2, Theorem 1] to the anti-dual pair context. It states that every positive operator on a weak-* sequentially complete anti-dual pair splits into a sum of absolutely continuous and singular parts with respect to another positive operator. We also prove that, when decomposing two positive operators with respect to each other, the corresponding absolute continuous parts are always mutually absolutely continuous. In Section 4 we introduce the parallel sum of two positive operators and furnish a different approach to the Lebesgue decomposition in terms of the parallel addition. In Section 5 we establish two characterizations of absolute continuity: Theorem 5.1 is of algebraic nature, as it relies on the order structure of positive operators. Theorem 5.3 is rather topological in character: it states that a positive operator is absolutely continuous with respect to another if and only if it can be uniformly approximated with the other one in a certain sense. Section 6 is devoted to characterizations of singularity, Section 7 deals with the uniqueness of the decomposition. To conclude the paper, in Section 8 we apply the developed decomposition theory to some concrete objects including Hilbert space operators, Hermitian forms, representable functionals, and additive set functions.

2. Preliminaries

The aim of this chapter is to collect all the technical ingredients that are necessary to read the paper.

2.1. Anti-dual pairs. Let $E$ and $F$ be complex vector spaces which are intertwined via a sesquilinear function

$$\langle \cdot, \cdot \rangle : F \times E \to \mathbb{C},$$
which separates the points of $E$ and $F$. We shall refer to $\langle \cdot, \cdot \rangle$ as anti-duality and the triple $(E, F, \langle \cdot, \cdot \rangle)$ will be called an anti-dual pair and shortly denoted by $(F, E)$. In this manner we may speak about symmetric and, first and foremost, positive operators from $E$ to $F$. Namely, we call an operator $A : E \to F$ symmetric, if

$$\langle Ax, y \rangle = \langle Ay, x \rangle, \quad x, y \in E,$$

furthermore, $A$ is said to be positive, if its “quadratic form” is positive semidefinite:

$$\langle Ax, x \rangle \geq 0, \quad x \in E.$$

Clearly, every positive operator is symmetric.

Most natural anti-dual pairs arise in the following way. Let $\hat{E}^*$ denote the conjugate algebraic dual of a complex vector space $E$ and let $F$ be a separating subspace of $\hat{E}^*$. Then

$$\langle f, x \rangle := f(x), \quad x \in E, f \in F$$

defines an anti-duality, and the pair $\langle F, E \rangle$ so obtained is called the natural anti-dual pair. (In fact, every anti-dual pair can be regarded as a natural anti-dual pair when $F$ is identified with $\hat{F}$, the set consisting of the conjugate linear functionals $\langle f, \cdot \rangle$, $f \in F$.) Our prototype of anti-dual pairs is the system $((H, H), \langle \cdot, \cdot \rangle)$ where $H$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Just as in the dual pair case (see e.g. [28]), we may endow $E$ and $F$ with the corresponding weak topologies $\sigma(E, F)$, resp. $\sigma(F, E)$, induced by the families $\{\langle f, \cdot \rangle : f \in F\}$, resp. $\{\langle \cdot, x \rangle : x \in E\}$. Both $\sigma(E, F)$ and $\sigma(F, E)$ are locally convex Hausdorff topologies such that

(2.1) $\hat{E}' = F, \quad F' = E$,

where $F'$ and $\hat{E}'$ refer to the topological dual and anti-dual space of $F$ and $E$, respectively, and the vectors $f \in F$ and $x \in E$ are identified with $\langle f, \cdot \rangle$, and $\langle \cdot, x \rangle$, respectively. We also recall the useful property of weak topologies that, for a topological vector space $(V, \tau)$, a linear operator $T : V \to F$ is $\sigma(F, E)$-continuous if and only if

$$T_x(v) := \langle Tv, x \rangle, \quad v \in V$$

is continuous for every $x \in E$.

This fact and (2.1) enables us to define the adjoint (that is, the topological transpose) of a weakly continuous operator. Let $(F_1, E_1)$ and $(F_2, E_2)$ be anti-dual pairs and $T : E_1 \to F_2$ a weakly continuous linear operator, then the (necessarily weakly continuous) linear operator $T^* : E_2 \to F_1$ satisfying

$$\langle Tx_1, x_2 \rangle = \langle T^* x_2, x_1 \rangle, \quad x_1 \in E_1, x_2 \in E_2$$

is called the adjoint of $T$. In particular, the adjoint of a weakly continuous operator $T : E \to F$ emerges again as an operator of this type. The set of weakly continuous linear operators $T : E \to F$ will be denoted by $\mathcal{L}(E; F)$. An operator $T \in \mathcal{L}(E; F)$ is called self-adjoint if $T^* = T$. It is immediate that every symmetric operator (hence every positive operator) is weakly continuous and self-adjoint.

Finally, we recall that a topological vector space $(V, \tau)$ is called complete if every Cauchy net in $V$ is convergent. Similarly, $V$ is sequentially complete if every Cauchy sequence in $V$ is convergent. We shall call the anti-dual pair $\langle F, E \rangle$ weak-* (sequentially) complete if $(F, \sigma(F, E))$ is (sequentially) complete. It is easy to see that $\langle \hat{E}^*, E \rangle$ is always weak-* complete. It can be deduced from the Banach-Steinhaus...
theorem that, for a Banach space $E$, $\langle E', E \rangle$ is weak-* sequentially complete (but not weak-* complete unless $E$ is finite dimensional).

2.2. Factorization of positive operators. Let $\langle F, E \rangle$ be an anti-dual pair and $A : E \to F$ a positive operator. As we have already mentioned, $A \in \mathcal{L}(E; F)$ and $A = A^*$. Now we give the prototype of positive operators. Let $\mathcal{H}$ be a complex Hilbert space and let $T : E \to \mathcal{H}$ be a weakly continuous (i.e., $\sigma(E, F) \cdot \sigma(\mathcal{H}, \mathcal{H})$ continuous) linear operator, then the adjoint operator $T^* : \mathcal{H} \to F$ is again weakly continuous and the product $T^* T \in \mathcal{L}(E; F)$ is positive:

$$\langle T^* T x, x \rangle = \langle T x, T x \rangle \geq 0, \quad x \in E.$$ 

On a weak-* sequentially complete anti-dual pair $\langle F, E \rangle$, every positive operator $A \in \mathcal{L}(E; F)$ can be written as $A = T^* T$. We sketch here the proof of this fact because we will use the construction continuously; for more details the reader is referred to [11].

Let $\langle F, E \rangle$ be a weak-* sequentially complete anti-dual pair and let $A \in \mathcal{L}(E; F)$ be a positive operator. Endow the range space $\text{ran} \ A$ with the following inner product:

$$(Ax | Ay)_A := \langle Ax, y \rangle, \quad x, y \in E.$$ 

One can show that $(\cdot | \cdot)_A$ is well defined and positive definite, hence $(\text{ran} \ A, (\cdot | \cdot)_A)$ is a pre-Hilbert space. Let $\mathcal{H}_A$ denote its Hilbert completion so that $\text{ran} \ A \subseteq \mathcal{H}_A$ forms a norm dense linear subspace. The canonical embedding operator

$$J_A(Ax) = Ax, \quad x \in E,$$

of $\text{ran} \ A \subseteq \mathcal{H}_A$ into $F$ is weakly continuous, hence $J_A$ extends to an everywhere defined weakly continuous operator because of weak-* sequentially completeness of $F$. We continue to write $J_A \in \mathcal{L}(\mathcal{H}_A, F)$ for this extension. The adjoint operator $J_A^* \in \mathcal{L}(E, \mathcal{H}_A)$ admits the canonical property

$$J_A^* x = Ax \in \mathcal{H}_A, \quad x \in E,$$

that leads to the useful factorization of $A$:

$$A = J_A J_A^*.$$ 

2.3. Range of the adjoint operator. Operators of type $T \in \mathcal{L}(E, \mathcal{H})$ will play a peculiar role in the theory of positive operators, as we have seen, every positive operator $A$ on a weak-* sequentially complete anti-dual pair admits a factorization $A = T^* T$ through a Hilbert space $\mathcal{H}$. In this section we describe the range of the adjoint operator $T^* \in \mathcal{L}(\mathcal{H}, F)$. The key result is a variant to Douglas’ famous range inclusion theorem [12] (for further generalizations to Banach space setting see Barnes [6] and Embry [13]).

**Theorem 2.1.** Let $\langle F, E \rangle$ be an anti-dual pair and let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Given two weakly continuous operators $T_j \in \mathcal{L}(\mathcal{H}_j, F)$ ($j = 1, 2$) the following assertions are equivalent:

(i) $\text{ran} \ T_1 \subseteq \text{ran} \ T_2$,

(ii) there is a constant $\alpha \geq 0$ such that

$$\|T_1^* x\|^2 \leq \alpha \|T_2^* x\|^2, \quad x \in E,$$

(iii) for every $h_1 \in \mathcal{H}_1$ there is a constant $\alpha_{h_1} \geq 0$ such that

$$\langle T_1 h_1, x \rangle^2 \leq \alpha_{h_1} \|T_2^* x\|^2, \quad x \in E,$$
(iv) there is a bounded operator \( D : \mathcal{H}_1 \to \mathcal{H}_2 \) such that
\[
T_1 = T_2 D.
\]

Moreover, if any (hence all) of (i)-(iv) is valid, then there is a unique \( D \) such that

\begin{enumerate}[(i)]
  
  \item \( \text{ran } D \subseteq (\ker T_2)^\perp \),
  \item \( \ker T_1 = \ker D \),
  \item \( \|D\|^2 = \inf \{ \alpha \geq 0 : \|T^*_1 x\|^2 \leq \alpha \|T^*_2 x\|^2, (x \in E) \} \).
\end{enumerate}

\[\text{Figure 1. Factorization of } T_1 \text{ along } T_2\]

\textbf{Proof.} Implications (i)\(\Rightarrow\)(iii), (iv)\(\Rightarrow\)(ii)\(\Rightarrow\)(iii) and (iv)\(\Rightarrow\)(i) are immediate. We only prove (iii)\(\Rightarrow\)(iv): fix \( h_1 \) in \( \mathcal{H}_1 \) and define a conjugate linear functional \( \varphi : \text{ran } T^*_2 \to \mathbb{C} \) by
\[
\varphi(T^*_2 x) := \langle T^*_1 h_1, x \rangle, \quad x \in E.
\]

By (iii) one concludes that \( \varphi \) is well defined and continuous by norm bound \( \|\varphi\| \leq \sqrt{\alpha h_1} \). The Riesz representation theorem yields then a unique representing vector \( Dh_1 \in \text{ran } T^*_2 \) such that
\[
\langle T_1 h_1, x \rangle = (Dh_1 | T^*_2 x), \quad x \in E, h_1 \in \mathcal{H}_1.
\]

It is easy to check that \( D : \mathcal{H}_1 \to \mathcal{H}_2 \) is linear and that \( T_2 D = T_1 \). Our only duty is to prove that \( D \) is continuous. Take \( h_2 \in (\text{ran } T^*_2)\perp \) and \( x \in E \), then for any \( h_1 \in \mathcal{H}_1 \)
\[
(Dh_1 | h_2 + T^*_2 x) = (Dh_1 | T^*_2 x) = (h_1 | T^*_1 x).
\]

This means that the domain of \( D^* \) includes the dense set \( \text{ran } T^*_2 + (\text{ran } T^*_2)\perp \), hence \( D \) is closable. By the closed graph theorem we conclude that \( D \) is continuous.

Observe also that \( D \) obtained above fulfills conditions (a)-(c) above. Indeed, (a) and (b) are straightforward, and if (ii) holds for some \( \alpha \geq 0 \) then for \( x \in E \) and \( u \in (\text{ran } T^*_2)\perp \) we have \( D^* u = 0 \) by (a). Consequently,
\[
\|D^*(T^*_2 x + u)\|^2 = \|T^*_1 x\|^2 \leq \alpha \|T^*_2 x\|^2 \leq \alpha \|T^*_2 x + u\|^2,
\]

which implies \( \|D^*\|^2 \leq \alpha \), hence \( D \) satisfies (c). Finally, the uniqueness follows easily from (b).

\textbf{Corollary 2.2.} Let \( \langle F, E \rangle \) be a weak-* sequentially complete anti-dual pair. If \( A, B \in \mathcal{L}(E; F) \) are positive operators such that \( B \leq A \), then there is a unique positive contraction \( C \in \mathcal{B}(\mathcal{H}_A) \) such that \( B = J_A CJ^*_A \).

\textbf{Proof.} Let \( \mathcal{H}_B \) stand for the auxiliary Hilbert space obtained by the procedure of Subsection 2.2 with \( A \) replaced by \( B \). Since \( B \leq A \), we have \( \|J_B x\|^2 \leq \|J_A x\|^2 \) for
every \( x \in E \). By Theorem 2.1 there exists a bounded operator \( D \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_B) \), \( \|D\| \leq 1 \), such that \( J_B = J_A D \), hence \( C := DD^* \) satisfies

\[
J_A C J_A^* = J_B J_B^* = B.
\]

The uniqueness of \( C \) follows easily from the fact that \( \text{ran} J_A^* = \text{ran} A \) is dense in \( \mathcal{H}_A \).

The following range description of the adjoint operator is similar in spirit to [33], cf. also [29].

**Lemma 2.3.** Let \( \langle F, E \rangle \) be an anti-dual pair, \( \mathcal{H} \) a Hilbert space and \( T : E \to \mathcal{H} \) a weakly continuous linear operator. A vector \( y \in F \) belongs to the range of \( T^* \) if and only if there exists \( m_y \geq 0 \) such that

\[
(2.5) \quad |\langle y, x \rangle|^2 \leq m_y \|Tx\|^2, \quad x \in E.
\]

**Proof.** Assume first that \( y = T^* h \) for some \( h \in \mathcal{H} \), then

\[
|\langle y, x \rangle|^2 = |\langle T^* h, x \rangle|^2 = |\langle Tx | h \rangle|^2 \leq \|h\|^2 \|Tx\|^2, \quad x \in E,
\]

hence (i) implies (ii). Conversely, (ii) expresses precisely that the correspondence

\( Tx \mapsto \langle y, x \rangle, \quad x \in E, \)

defines a continuous anti-linear functional from \( \text{ran} T \subseteq \mathcal{H} \) to \( \mathbb{C} \). The Riesz representation theorem yields a vector \( h \in \text{ran} \overline{T} \) such that

\[
\langle y, x \rangle = (h | Tx) = (T^* h, x), \quad x \in E.
\]

Consequently, \( y = T^* h \in \text{ran} T^* \). \( \square \)

### 2.4. Linear relations in Hilbert spaces

If \( A, B \) are positive operators on the anti-dual pair \( \langle F, E \rangle \) then we can associate the auxiliary Hilbert spaces \( \mathcal{H}_A, \mathcal{H}_B \) with them along the procedure given in Subsection 2.2. The vast majority of the results in this paper relies on some topological properties of a “mapping” sending \( Ax \) of \( \mathcal{H}_A \) into \( Bx \) of \( \mathcal{H}_B \). In general, this map is not a function (and thus not a bounded operator). Such “multivalued” operators, i.e., linear subspaces of a product Hilbert space \( \mathcal{H} \times \mathcal{K} \) are called linear relations. In this subsection we gather some basic notions and results of the theory of linear relations. For a comprehensive treatment on linear relations one may refer to [4] and [18].

A linear relation between two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \) is a linear subspace \( T \) of the Cartesian product \( \mathcal{H} \times \mathcal{K} \). Accordingly, \( T \) is called a closed linear relation if it is a closed linear subspace of \( \mathcal{H} \times \mathcal{K} \). If \( T \) is a linear operator from \( \mathcal{H} \) to \( \mathcal{K} \) then the graph of \( T \) is a linear relation. Conversely, a linear relation \( T \) is (the graph of) an operator if and only if \( (0, k) \in T \) implies \( k = 0 \) for every \( k \in \mathcal{K} \). In other words, a linear relation \( T \) is an operator if its multivalued part

\[
\text{mul} T := \{k \in \mathcal{K} : (0, k) \in T\}
\]

is trivial. The domain, kernel and range of a linear relation \( T \), denoted by \( \text{dom} T \), \( \ker T \) and \( \text{ran} T \), respectively, are defined in the obvious manner. A relation \( T \) is called closable if its closure \( \overline{T} \) is an operator, or equivalently, if

\[
\text{mul} \overline{T} = \{k \in \mathcal{K} : \exists (h_n, k_n)_{n \in \mathbb{N}} \subset T, h_n \to 0, k_n \to k\}
\]

is trivial. In the sequel, we shall also need the concept of the adjoint of a linear relation. To this aim let us introduce the unitary operator

\[
V(h, k) := (-k, h), \quad h \in \mathcal{H}, k \in \mathcal{K}
\]
from $\mathcal{H} \times \mathcal{K}$ to $\mathcal{K} \times \mathcal{H}$. The adjoint of a linear relation $T \subseteq \mathcal{H} \times \mathcal{K}$ is given by

$$T^* := [V(T)]^\perp = \{(-k, h) : (h, k) \in T\}^\perp,$$

that agrees with the original concept of the adjoint transformation introduced by J. von Neumann if $T$ is a densely defined operator. Observe immediately that $T^*$ is always closed such that $T^{**} = T$. For a pair of vectors $(g, f) \in \mathcal{K} \times \mathcal{H}$, relation $(g, f) \in T^*$ means that

$$(2.6) \quad (k \mid g) = (h \mid f), \quad \text{for all } (h, k) \in T.$$

In a full analogy with the operator case, the domain of the adjoint relation consists of those vectors $g$ such that

$$(2.7) \quad |(k \mid g)|^2 \leq m_g (h \mid h) \quad \text{for all } (h, k) \in T$$

holds for some $m_g \geq 0$. Furthermore, we have the following useful relations:

$$(2.8) \quad \text{mul} T^* = (\text{dom} T)^\perp \quad \text{and} \quad \overline{\text{dom} T^*} = (\text{mul} T^{**})^\perp.$$

In particular, the adjoint of a densely defined relation is a closed operator and the adjoint of a closable operator is densely defined. Let $P$ denote the orthogonal projection of $\mathcal{K}$ onto $\text{mul} T$. The regular part $T_{\text{reg}}$ of $T$ is defined to be the linear relation

$$(2.9) \quad T_{\text{reg}} := \{(h, (I - P)k) : (h, k) \in T\}.$$

Actually, it can be proved that $T_{\text{reg}}$ is a closable operator and its closure satisfies

$$(2.10) \quad \overline{T_{\text{reg}}} = (T)_{\text{reg}},$$

see [18, Theorem 4.1 and Proposition 4.5]. In particular, the regular part of a closed linear relation is itself closed.

### 3. Lebesgue Decomposition Theorem for Positive Operators

Modeled by the Lebesgue–Radon–Nikodym theory of positive operators on a Hilbert space (see e.g. [2] or [36]) we can introduce the concepts of absolute continuity and singularity of positive operators on an anti-dual pair. Let $A$ and $B$ be positive operators on an anti-dual pair $(F, E)$. We say that $B$ is absolutely continuous with respect to $A$ (in notation, $A \ll B$) if for any sequence $(x_n)_{n \in \mathbb{N}}$ of $E$,

$$\langle Ax_n, x_n \rangle \to 0 \quad \text{and} \quad \langle B(x_n - x_m), x_n - x_m \rangle \to 0 \quad (n, m \to \infty)$$

imply $\langle Bx_n, x_n \rangle \to 0$. On the other hand, we say that $A$ and $B$ are mutually singular (in notation, $A \perp B$) if $C \leq A$ and $C \leq B$ imply $C = 0$ for any positive operator $C \in \mathcal{L}(E; F)$.

The main purpose of this section is to establish an extension of Ando’s Lebesgue decomposition theorem [2, Theorem 1]. This states that every positive operator $B$ on a weak-* sequentially complete anti-dual pair admits a decomposition $B = B_a + B_s$ where $B_a \ll A$ and $B_s \perp A$. Before passing to the proof, let us make a few remarks.

The following construction is analogous to the one developed in [26]. Let us consider the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and the linear operators $J_A, J_B$, associated with $A$ and $B$, respectively. Introduce the closed linear relation

$$(3.1) \quad \hat{B} := \{(Ax, Bx) \in \mathcal{H}_A \times \mathcal{H}_B : x \in E\}$$
from $\mathcal{H}_A$ to $\mathcal{H}_B$, and denote its multivalued part by $\mathcal{M}$:

$$\mathcal{M} := \{ \xi \in \mathcal{H}_B : (0, \xi) \in \hat{B} \}.$$

According to what has been said in Subsection 2.4, $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}_B$ and one easily verifies that

$$M := \{ \xi \in \mathcal{H}_B : (0, \xi) \in \hat{B} \}.$$

It is easy to check that $B \ll A$ if and only if $\hat{B}$ is a closed operator, or equivalently, if $\mathcal{M} = \{0\}$. Furthermore, since ran $A \subseteq \text{dom } \hat{B}$, the adjoint relation $\hat{B}^*$ is always a single-valued operator from $\mathcal{H}_B$ to $\mathcal{H}_A$ such that

$$\text{(3.3)} (\text{dom } \hat{B}^*)^\perp = \mathcal{M}.$$

The next lemma describes the domain of $\hat{B}^*$:

**Lemma 3.1.** For a vector $\xi \in \mathcal{H}_B$ the following assertions are equivalent:

(i) $\xi \in \text{dom } \hat{B}^*$,

(ii) there exists $m_\xi \geq 0$ such that $|\langle Bx, \xi \rangle_a|^2 \leq m_\xi |\langle Ax, x \rangle|$ for all $x \in E$,

(iii) $J_B \xi \in \text{ran } J_A$.

In any case,

$$\text{(3.4)} J_A \hat{B}^* \xi = J_B \xi, \quad \xi \in \text{dom } \hat{B}^*.$$

**Proof.** The equivalence between (i) and (ii) is clear due to (2.7) and the equivalence between (ii) and (iii) follows from Lemma 2.3. Finally, for $\xi \in \text{dom } \hat{B}^*$ and $x \in E$ we have

$$\langle J_A \hat{B}^* \xi, x \rangle = (\hat{B}^* \xi | Ax)_a = (\xi | Bx)_a = (J_B \xi, x),$$

that proves (3.4). \hfill \Box

Let $P$ stand for the orthogonal projection of $\mathcal{H}_B$ onto $\mathcal{M}$ and set

$$\hat{B}_{\text{reg}} := \{ (\zeta, (I - P) \xi) : (\zeta, \xi) \in \hat{B} \}.$$

Since $\hat{B}_{\text{reg}}$ is the regular part (2.9) of $\hat{B}$, [18, Theorem 1] and identity (2.10) yield the following result:

**Proposition 3.2.** $\hat{B}_{\text{reg}}$ is a densely defined closed linear operator between $\mathcal{H}_A$ and $\mathcal{H}_B$ such that

$$\hat{B}_{\text{reg}} = \{ (Ax, (I - P)(Bx)) : x \in E \}.$$
Proof. It is clear that $B_a, B_s \in \mathcal{L}(E; F)$ are positive operators such that $B = B_a + B_s$. In order to prove absolute continuity of $B_a$, we observe that

$$\langle B_a x, x \rangle = \langle (I - P)J_B x | J_B^* x \rangle_B = \langle (I - P)(Bx) | (I - P)(Bx) \rangle_B$$

$$= \langle \tilde{B}_{\text{reg}}(Ax) | \tilde{B}_{\text{reg}}(Ax) \rangle_B,$$

for $x \in E$, hence $B_a$ is $A$-absolutely continuous according to Proposition 3.2.

Our next claim is to show the maximality of $B_a$. Let us consider a positive operator $C \in \mathcal{L}(E; F)$ such that $C \leq B$ and $C \ll A$. By Corollary 2.2 there is a unique positive operator $\hat{C} \in \mathcal{B}(\mathcal{H}_B)$, $\|C\| \leq 1$, such that $C = J_B \hat{C} J_B^*$. In particular we have

$$\langle Cx, y \rangle = \langle \hat{C}^{1/2}(Bx) | \hat{C}^{1/2}(By) \rangle_B, \quad x, y \in E.$$

We claim that

$$\mathcal{M} \subseteq \ker \hat{C}.$$

For let $\xi \in \mathcal{M}$ and consider a sequence $(x_n)_{n \in \mathbb{N}}$ of $E$ such that

$$\langle Ax_n, Ax_n \rangle_A \to 0 \quad \text{and} \quad Bx_n \to \xi \in \mathcal{H}_B.$$  

By continuity, $\hat{C}^{1/2}(Bx_n) \to \hat{C}^{1/2}\xi$, and by $A$-absolute continuity,

$$\langle \hat{C}^{1/2}(Bx_n) | \hat{C}^{1/2}(Bx_n) \rangle_B = \langle Cx_n, x_n \rangle \to 0,$$

whence $\hat{C}\xi = 0$. This proves (3.7). Let now $x \in E$. By (3.7), $\hat{C}^{1/2} P = 0$. Consequently,

$$\langle Cx, x \rangle = \langle \hat{C}^{1/2}(Bx) | \hat{C}^{1/2}(Bx) \rangle_B = \langle \hat{C}^{1/2}(I - P)(Bx) | \hat{C}^{1/2}(I - P)(Bx) \rangle_B$$

$$\leq \langle (I - P)(Bx) | (I - P)(Bx) \rangle_B = \langle J_B(I - P)J_B^* x, x \rangle$$

$$= \langle B_a x, x \rangle,$$

whence $C \leq B_a$, as it is stated.

Finally, in order to prove that that $B_s$ and $A$ are mutually singular, let $C \in \mathcal{L}(E; F)$ be a positive operator such that $C \leq A$ and $C \ll B_s$. Then $C + B_a \leq B$ so that $C + B_a$ is $A$-absolutely continuous. By the maximality of $B_a$ we conclude that $C + B_a \leq B_a$, i.e., $C = 0$. \hfill \square

Remark 3.4. Observe that

$$\tilde{B}_{\text{reg}} J_A^* x = (I - P)Bx = (I - P)J_B^* x$$

for any $x$ in $E$, whence we obtain yet another useful factorization of the absolute continuous part:

$$B_a = (\tilde{B}_{\text{reg}} J_A^*)^* (\tilde{B}_{\text{reg}} J_A^*).$$
We close the section with an interesting property of the absolute continuous part. Suppose that $A, B$ are positive operators and let $B = B_a + B_s$ be the Lebesgue decomposition of $B$ with respect to $A$ in virtue of Theorem 3.3, i.e.,

$$B_a = J_B(I - P)J_B^*$$

and $B_s = J_B P J_B^*$. Here we have $B_a \ll A$. Interchanging the roles of $A$ and $B$, by the same process we may take the Lebesgue decomposition of $A$ with respect to $B$, namely, $A = A_a + A_s$. We shall prove that the absolutely continuous parts $A_a$ and $B_a$ are absolutely continuous with respect to each other, i.e., $B_a \ll A_a$ and $A_a \ll B_a$.

This surprising property was discovered by T. Titkos in context of nonnegative forms [42] and measures [43]. Theorem 3.6 below is not only a generalization of this fact, it also reproves these results with a completely different technique.

**Lemma 3.5.** Let $H$ and $K$ be Hilbert spaces and let $T$ be a closed linear relation between them and denote by $P_T$ and $Q_T$ the orthogonal projections onto $\text{mul} \hat{T}$ and $\ker \hat{T}$, respectively. Then

$$S := \{((I - Q_T)\xi, (I - P_T)\eta) : (\xi, \eta) \in T\}$$

is (the graph of) a one-to-one closed operator.

**Proof.** It is easy to see that $\ker \hat{T} = \text{mul} \hat{T}^{-1}$ and that $\ker \hat{T} = \ker T_{\text{reg}}$, furthermore the regular part of a closed linear relation is closed itself, hence

$$S = (((T_{\text{reg}})^{-1})_{\text{reg}})^{-1}$$

is a one-to-one closed operator. □

**Theorem 3.6.** Let $(F, E)$ be a weak-* sequentially anti-dual pair and let $A, B \in \mathcal{L}(E; F)$ be positive operators. Then we have

$$A_a \ll B_a \quad \text{and} \quad B_a \ll A_a.$$ 

**Proof.** Let us continue to write $P$ for the orthogonal projection onto $\text{mul} \hat{B}$ and let $Q$ be the orthogonal projection onto $\ker \hat{B}$. Then, according to the preceding lemma,

$$S := \{((I - Q)\xi, (I - P)\eta) : (\xi, \eta) \in \hat{B}\}$$

is (the graph of) a one-to-one closed linear operator from $H_A$ to $H_B$. Since we have $\ker \hat{B} = \text{mul} \hat{B}^{-1}$, it follows that

$$B_a = J_B(I - P)J_B^*, \quad A_a = J_A(I - Q)J_A^*.$$ 

Consider a sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ such that $\langle A_a x_n, x_n \rangle \to 0$ and $\langle B_a (x_n - x_m), x_n - x_m \rangle \to 0$, then $(I - Q)A x_n \to 0$ in $H_A$ and $(I - Q)B x_n \to \eta$ for some $\eta \in H_B$. Since $S$ is closable it follows that $\eta = 0$ and hence that $\langle B_a x_n, x_n \rangle \to 0$, thus $B_a \ll A_a$. A very similar reasoning shows that $A_a \ll B_a$, but this time the closability of $S^{-1}$ is used. □

**Remark 3.7.** We have only proved that the canonical absolute continuous parts $A_a$ and $B_a$ have the property of being mutually absolute continuous. As we shall see, the Lebesgue decomposition is not unique in general, so there might exist other Lebesgue-type decompositions differing from what we have constructed in Theorem 3.3. The statement of Theorem 3.6 is certainly not true for the absolutely continuous parts of such Lebesgue decompositions.
4. The parallel sum

Ando’s key notion in establishing his Lebesgue-type decomposition theorem was the so called parallel sum of two positive operators. Inspired by his treatment, Hassi, Sebestyén, and de Snoo [19] proved an analogous result for nonnegative Hermitian forms by means of the parallel sum as well. Parallel addition may also be defined in various areas of functional analysis, e.g. for measures, representable positive functionals on a $^*$-algebra, and for positive operators from a Banach space to its topological anti-dual, see [25][39][43]. In what follows we provide a common generalization of those concepts.

The parallel sum $A : B$ of two bounded positive operators on a Hilbert space can be introduced in various ways, see eg. [1][15][24][27], cf. also [3][11]. Its quadratic form can be obtained via the formula

$$
(A : B)x = \inf \{(A(x - y) | x - y) + (By | y) : y \in H\},
$$

that uniquely determines the operator $A : B$. Therefore, it seems natural to introduce the parallel sum of two positive operators in the anti-dual pair context as an operator whose quadratic form is (4.1) (the inner product replaced by anti-duality, of course).

The existence of such an operator is established in the following result:

**Theorem 4.1.** Let $(F, E)$ be a weak-* sequentially complete anti-dual and let $A, B \in \mathcal{L}(E; F)$ be positive operators. There exists a unique positive operator $A : B \in \mathcal{L}(E; F)$, called the parallel sum of $A$ and $B$, such that

$$
(A : B)x = \inf \{(A(x - y) | x - y) + (By | y) : y \in E\}, \quad x \in E.
$$

**Proof.** Let us consider the product Hilbert space $H_A \times H_B$ and the weakly continuous operator $V_A : H_A \times H_B \rightarrow F$ arising from the densely defined one

$$
V_A(Ax, By) := Ax, \quad x, y \in E.
$$

A straightforward calculation shows that the adjoint $V_A^* \in \mathcal{L}(E; H_A \times H_B)$ fulfills

$$
V_A^* x = (Ax, 0) \in H_A \times H_B, \quad x \in E.
$$

Consider the orthogonal projection $Q$ of $H_A \times H_B$ onto $\hat{B}^\bot$. The positive operator $V_AQV_A^* \in \mathcal{L}(E; F)$ satisfies then

$$
(V_AQV_A^* x, x) = \|Q(Ax, 0)\|_{H_A \times H_B}^2 = \dist^2((Ax, 0), \hat{B})
$$

$$
= \inf \{\|(Ax, 0) - (Ay, By)\|_{H_A \times H_B}^2 : y \in E\}
$$

$$
= \inf \{(A(x - y) | A(x - y))_A + (By | By)_B : y \in E\}
$$

$$
= \inf \{(A(x - y), x - y) + (By, y) : y \in E\}.
$$

Hence $A : B := V_AQV_A^*$ fulfills (4.2). □

In the next proposition we collected some basic properties of parallel addition:

**Proposition 4.2.** Let $(F, E)$ be a weak-* sequentially complete anti-dual pair and let $A, B \in \mathcal{L}(E; F)$ be positive operators. Then

(a) $A : B = B : A$,

(b) $A : B \leq A$ and $A : B \leq B$,

(c) $A_1 \leq A_2$ and $B_1 \leq B_2$ imply $A_1 : B_1 \leq A_2 : B_2$. 


Proof. Replacing $y$ by $x - y$ in (4.2) yields
\[
\langle (A : B)x, x \rangle = \langle (B : A)x, x \rangle \quad \text{for all } x \in E,
\]
that gives just (a). Properties (b) and (c) are immediate from (4.2). □

Note that the definition $A : B = V_AQV_A^*$ of the parallel sum shows some asymmetry in $A$ and $B$, although we have $A : B = B : A$. If we define $V_B : \mathcal{H}_A \times \mathcal{H}_B \to F$ by
\[
V(Ax, By) = By, \quad x, y \in E,
\]
then we will get $V_BQV_B^* = B : A$ and hence
\[
V_BQV_B^* = V_AQV_A^* = A : B.
\]

**Lemma 4.3.** Let $(F, E)$ be a weak-* sequentially complete anti-dual pair and let $(A_n)_{n \in \mathbb{N}}$ be an increasing sequence of positive operators, bounded by a positive operator $B \in \mathcal{L}(E; F)$:
\[
A_n \leq A_{n+1} \leq B, \quad n = 1, 2, \ldots
\]
Then $(A_n)_{n \in \mathbb{N}}$ converges pointwise to a positive operator $A \in \mathcal{L}(E; F)$, $A \leq B$ (i.e., $A_nx, y \to \langle Ax, y \rangle$ for all $x, y \in E$).

**Proof.** Since we have
\[
\|((A_n - A_m)x, y)\|^2 \leq \langle (A_n - A_m)x, x \rangle \langle (A_n - A_m)y, y \rangle
\leq \langle (A_n - A_m)x, x \rangle \langle By, y \rangle
\]
for every $x, y \in E$ and every integer $n \geq m$, it follows that, for every fixed $x \in E$, $(A_nx)_{n \in \mathbb{N}}$ is a weak Cauchy sequence in $F$. By weak-* sequentially completeness, there is a vector $Ax \in F$ such that $A_nx \to Ax$ weakly. A straightforward calculation shows that the pointwise limit $A : E \to F$ is a positive (hence weakly continuous) operator such that $A \leq B$. □

We are going to use this result in the following situation: let $A, B \in \mathcal{L}(E; F)$ be positive operators on the weak-* sequentially complete anti-dual pair $(F, E)$. Letting $A_n := (nA) : B$, we have
\[
A_n \leq A_{n+1} \leq B, \quad n = 1, 2, \ldots
\]
by Proposition 4.2. Lemma 4.3 tells us that the limit
\[
\langle [A]Bx, y \rangle := \lim_{n \to \infty} \langle ((nA) : B)x, y \rangle, \quad x, y \in E
\]
defines a positive operator $[A]B \in \mathcal{L}(E; F)$ such that $[A]B \leq B$. Our next claim in what follows is to show that $[A]B$ coincides with the $A$-absolutely continuous part $B_0$ of $B$:
\[
[A]B = J_B(I - P)J_B^*.
\]
To establish the last claim let us introduce the following linear subspaces $\mathcal{N}_\alpha$ of $\mathcal{H}_A \times \mathcal{H}_B$ for every positive number $\alpha > 0$ as follows:
\[
\mathcal{N}_\alpha := \{\langle \alpha Ax, Bx \rangle : x \in E \} = \{Ax, \alpha^{-1}Bx \} : x \in E \},
\]
and denote by $Q_\alpha$ the orthogonal projection onto $\mathcal{N}_\alpha^\perp$. With the aid of the $Q_\alpha$’s we provide useful factorizations for $(\alpha A) : B$. 

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Proposition 4.4. Let $A, B \in \mathcal{L}(E; F)$ be positive operators on the weak-* sequentially complete anti-dual pair $\langle F, E \rangle$ and let $V_A, V_B$ and $Q_n$ as above. Then

$$\tag{4.6} (\alpha^2 A) : B = V_B Q_n V_B^* = \alpha^2 V_A Q_n V_A^*.$$ \hfill \Box

Proof. For every $x \in E$ and $\alpha > 0$ we have

$$\langle V_B Q_n V_B^* x, x \rangle = \|Q_n (0, Bx)\|_{H_A \times H_B}^2 = \text{dist}^2((0, Bx), N_{\alpha})$$

$$= \inf\{\|((0, Bx) - (Ay, \alpha^{-1} By))\|_{H_A \times H_B} : y \in E\}$$

$$= \inf\{\langle Ay, y \rangle + \langle B(x - \alpha^{-1} y), x - \alpha^{-1} y \rangle : y \in E\}$$

$$= \inf\{\langle (\alpha^2 A)(x-z), x-z \rangle + \langle Bz, z \rangle : z \in E\}$$

$$= \langle ((\alpha^2 A) : B) x, x \rangle,$$

which proves the first identity. The second one is proved by the same argument:

$$\langle \alpha^2 V_A Q_n V_A^* x, x \rangle = \|Q_n (\alpha Ax, 0)\|_{\mathcal{H}_A \times \mathcal{H}_B}^2$$

$$= \text{dist}^2((\alpha Ax, 0), N_{\alpha})$$

$$= \inf\{\|((\alpha Ax, 0) - (\alpha By, B y))\|_{\mathcal{H}_A \times \mathcal{H}_B} : y \in E\}$$

$$= \inf\{\langle (\alpha^2 A)(x-y), x-y \rangle + \langle By, y \rangle : y \in E\}$$

$$= \langle ((\alpha^2 A) : B) x, x \rangle,$$

as it is claimed. \hfill \Box

The proof of identity $[A]B = B_a$ relies on the ensuing lemma:

Lemma 4.5. Let $A, B \in \mathcal{L}(E; F)$ be positive operators and let $0 \oplus (I - P)$ be the orthogonal projection of $\mathcal{H}_A \times \mathcal{H}_B$ onto $\{0\} \times \mathcal{M}^\perp$. Then $Q_n \to 0 \oplus (I - P)$ in the strong operator topology.

Proof. We proceed in three steps. First we prove that

$$\tag{4.7} Q_n (\xi, 0) \to 0, \quad \xi \in \mathcal{H}_A.$$ \hfill \Box

To this aim take $x \in E$. Since $(n^2 A) : B \leq B$, we obtain by Proposition 4.2 that

$$\|Q_n (Ax, 0)\|^2 = \langle V_A Q_n V_A^* x, x \rangle = \frac{1}{n^2} \langle (n^2 A) : Bx, x \rangle \leq \frac{1}{n^2} \langle Bx, x \rangle \to 0.$$ \hfill \Box

The sequence $(Q_n)_{n \in \mathbb{N}}$ is uniformly bounded, hence (4.7) follows by standard density arguments.

Our next claim is to show

$$\tag{4.8} Q_n (0, \xi) = 0, \quad \xi \in \mathcal{M}$$

for every integer $n$. First observe that $(n^2 A) : B \leq n^2 A$ and hence $(n^2 A) : B$ is $A$-absolutely continuous. We have $(n^2 A) : B \leq B$ on the other hand, so $(n^2 A) : B \leq B_a$ by Theorem 3.3. That yields

$$\|Q_n (0, Bx)\|^2 = \langle (n^2 A) : Bx, x \rangle \leq \langle B_a x, x \rangle$$

$$= \langle (I - P)(Bx) : (I - P)(Bx) \rangle_a = \|((0 \oplus (I - P))(0, Bx))\|^2.$$ \hfill \Box

Hence, by density, $\|Q_n (0, \xi)\|^2 \leq \|((0 \oplus (I - P))(0, \xi))\|^2$ for all $\xi \in \mathcal{H}_B$. This implies (4.8).

In the final step of the proof we show that

$$\tag{4.9} Q_n (0, \xi) \to (0, \xi), \quad \xi \in \mathcal{M}^\perp.$$ \hfill \Box
Since $M^\perp = \text{dom } \hat{B}^*$, it suffices to prove (14.9) for $\xi \in \text{dom } \hat{B}^*$ because of uniform boundedness. Consider $\xi \in \text{dom } \hat{B}^*$. According to Lemma 3.1 there is $m_\xi \geq 0$ such that
\[
|\langle Bx, \xi \rangle|_B|^2 \leq m_\xi \langle Ax, x \rangle
\]
for all $x \in E$. Consequently,
\[
\|\left(I - Q_n\right)0,\xi\|_B^2 = \sup\left\{|\langle (nAx, Bx), (0, \xi) \rangle|_B|^2 : x \in E, \|nAx, Bx\| \leq 1\right\}
\leq \sup\left\{|\langle Bx, \xi \rangle|_B|^2 : x \in E, n^2 \langle Ax, x \rangle + \langle Bx, x \rangle \leq 1\right\}
\leq \frac{m_\xi}{n^2} \to 0.
\]
The proof is complete. □

We can now prove the following result:

**Theorem 4.6.** Let $A, B \in \mathcal{L}(E; F)$ be positive operators on the weak-* sequentially complete anti-dual pair $\langle F, E \rangle$, then
\[
\lim_{n \to \infty} \langle (nA) : B \rangle x, y = \langle JB(I - P)J_B^* x, y \rangle, \quad x, y \in E.
\]
In other words, $[A]B$ is identical with the $A$-absolutely continuous part $B_a$ of $B$.

**Proof.** By Lemma 4.5 we infer that
\[
\langle [A]B x, y \rangle = \lim_{n \to \infty} \langle (n^2 A) : B \rangle x, y = \lim_{n \to \infty} \langle V_B Q_n V_B^* x, y \rangle
\leq \sup\left\{\|Bx, \xi\|_B^2 : x \in E, n^2 \langle Ax, x \rangle \leq 1\right\}
\leq \frac{m_\xi}{n^2} \to 0.
\]
The proof is complete. □

5. Characterizations of absolute continuity

It is clear from Theorem 3.3 that a positive operator $B$ is absolutely continuous with respect to the positive operator $A$ if and only if $B$ is identical with its $A$-absolutely continuous part $B_a$. In the light of this, Theorem 4.6 yields yet another characterization absolute continuity, namely, $B \ll A$ if and only if
\[
B = [A]B.
\]
In particular, every $A$-absolutely continuous operator $B$ on a weak-* sequentially complete anti-dual pair can be obtained as the pointwise limit of a monotone increasing sequence $(B_n)_{n \in \mathbb{N}}$ such that $B_n \leq \alpha_n A$ for some nonnegative sequence $(\alpha_n)_{n \in \mathbb{N}}$. For positive operators on a Hilbert space, Ando [2] introduced the concept of being absolutely continuous just by this property. Adopting the rather expressive terminology of [19], such a positive operator $B$ will be called “almost dominated” by $A$.

In the first result of this section we are going to show that almost dominated operators are just the absolutely continuous ones:

**Theorem 5.1.** Let $A, B$ be positive operators on the weak-* sequentially anti-dual pair $\langle F, E \rangle$. The following conditions are equivalent:

(i) $B$ is absolutely continuous with respect to $A$. 
(ii) $B$ is almost dominated by $A$, that is, there exists a monotone increasing sequence $(B_n)_{n \in \mathbb{N}}$ of positive operators in $\mathcal{L}(E;F)$ and $(\alpha_n)_{n \in \mathbb{N}}$ of positive numbers such that $B_n \leq \alpha_n A$ and $B_n \to B$ pointwise on $E$.

Proof. Implication (i)$\Rightarrow$(ii) is clear from Theorem 1.6 and from what has been said above. For the converse implication let $(B_n)_{n \in \mathbb{N}}$ be a sequence satisfying (ii). By Corollary 2.2, for any integer $n$ there is a positive operator $C_n \in \mathcal{B}(\mathcal{H}_B)$, $\|C_n\| \leq 1$, such that $B_n = J_B^* C_n J_B^*$. We claim that

\begin{equation}
\text{ran } C_n \subseteq \text{dom } \hat{B}^*, \quad n = 1, 2, \ldots.
\end{equation}

For let $\xi \in \mathcal{H}_B$, then for every $x \in E$ we have

\begin{align*}
|\langle J_B C_n \xi, x \rangle|^2 &= |\langle \xi, C_n J_B^* x \rangle|^2 \\
&\leq \|\xi\|^2_2 \|C_n J_B^* x\|^2_{B^*} \\
&\leq \|\xi\|^2_2 \|C_n^{1/2} J_B^* x\|^2_{B^*} \\
&= \|\xi\|^2_{B^*} (B_n x, x) \\
&= \alpha_n \|\xi\|^2_{B^*} (Ax, x) \\
&= \alpha_n \|\xi\|^2_{B^*} \|J_B^* x\|^2_A,
\end{align*}

whence we conclude that $J_B C_n \xi \in \text{ran } J_A$ by Lemma 2.3. By Lemma 3.1, $C_n \xi \in \text{dom } \hat{B}^*$, that proves (5.1). Since $B$ is absolutely continuous precisely when $\hat{B}^*$ is densely defined, it suffices to prove that the union of ranges of the $C_n$’s is dense in $\mathcal{H}_B$, or equivalently,

\begin{equation}
\bigcap_{n=1}^{\infty} \ker C_n = \{0\}.
\end{equation}

To check this identity take $x \in E$, then

\begin{align*}
\|Bx - C_n(Bx)\|^2_B &= \langle Bx, x \rangle - 2 \langle C_n J_B^* x \mid J_B^* x \rangle_B + \|C_n(Bx)\|^2_B \\
&\leq 2 \langle Bx, x \rangle - 2 \langle B_n x, x \rangle \to 0,
\end{align*}

from which we deduce that $C_n$ converges strongly to the identity operator of $\mathcal{H}_B$, and this clearly implies (5.2).

We remark that the existence of a Lebesgue-type decomposition can be proved easily by means of (ii) with an elementary iteration involving parallel addition (see [5] and [15]). However, the iteration itself does not guarantee the maximality of the resulted absolutely continuous part.

In the rest of the section, our goal is to give a Radon–Nikodym-type characterization of absolute continuity. In order to formulate our main result we need some preliminaries.

Let $(F, E)$ be a weak-* sequentially complete dual pair and consider two positive operators $A, B \in \mathcal{L}(E;F)$ on it. Denote by $\mathcal{H}_{A+B}$ the corresponding auxiliary Hilbert space, and by $J := J_{A+B}$ the natural embedding operator of $\mathcal{H}_{A+B}$ into $F$. A straightforward application of Lemma 2.2 gives then two positive contractions $C_A, C_B \in \mathcal{B}(\mathcal{H}_{A+B})$ such that

\begin{align*}
\langle C_A J^* x \mid J^* y \rangle_{A+B} &= \langle Ax, y \rangle, \quad \langle C_B J^* x \mid J^* y \rangle_{A+B} = \langle Bx, y \rangle
\end{align*}

for every $x, y \in E$. Our first technical lemma gives some characterizations of the absolute continuity in terms of $C_A$ and $C_B$:
Lemma 5.2. The following assertions are equivalent:

(i) \( B \) is absolutely continuous with respect to \( A \),
(ii) \( \ker C_A = \{0\} \),
(iii) \( \text{ran} \ C_B \subseteq (\ker C_A)^\perp \).

Proof. Assume first that \( B \) is \( A \)-absolutely continuous. If \( \xi \in \ker C_A \), then there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( E \) so that \( J^* x_n \to \xi \) in \( \mathcal{H}_{A+B} \) and \( \langle Ax_n, x_n \rangle \to 0 \). It is clear that \( \langle B(x_n - x_m), x_n - x_m \rangle \to 0 \), hence

\[
\|\xi\|^2_{A+B} = \lim_{n \to \infty} \langle (A+B)x_n, x_n \rangle = \lim_{n \to \infty} \langle Bx_n, x_n \rangle = 0,
\]

because of absolute continuity. Thus (i) implies (ii). The converse implication goes similar: assume that \( \ker C_A = \{0\} \) and consider a sequence \( (x_n)_{n \in \mathbb{N}} \) so that \( \langle Ax_n, x_n \rangle \to 0 \) and that \( \langle B(x_n - x_m), x_n - x_m \rangle \to 0 \). Clearly, \( (A+B)x_n \) is a Cauchy sequence in \( \mathcal{H}_{A+B} \) and its limit \( \xi \) belongs to \( \ker C_A = \{0\} \). This yields

\[
\langle Bx_n, x_n \rangle \leq \langle (A+B)x_n, x_n \rangle \to \|\xi\|^2_{A+B} = 0,
\]

thus \( B \ll A \). It remains to show that (iii) implies (ii) (the backward implication being trivial). That will follow apparently if we show that

\[
\ker C_B \subseteq (\ker C_A)^\perp
\]

holds for arbitrary \( A \) and \( B \). To this end, take \( \xi \in \ker C_A \), \( \zeta \in \ker C_B \), and choose a sequence \( (x_n)_{n \in \mathbb{N}} \) such that

\[
J^* x_n \to \xi, \quad \text{and} \quad \langle Ax_n, x_n \rangle \to 0,
\]

and choose another sequence \( (y_n)_{n \in \mathbb{N}} \) such that

\[
J^* y_n \to \zeta, \quad \text{and} \quad \langle By_n, y_n \rangle \to 0.
\]

Both the sequences \( \langle Bx_n, x_n \rangle \) and \( \langle Ay_n, y_n \rangle \) are bounded, hence

\[
|\langle \xi, \zeta \rangle_{A+B}| = \lim_{n \to \infty} |\langle J^* x_n, J^* y_n \rangle_{A+B}|
= \lim_{n \to \infty} |\langle Ax_n, y_n \rangle + \langle Bx_n, y_n \rangle|
\leq \limsup_{n \to \infty} \langle Ax_n, x_n \rangle^{1/2} \langle Ay_n, y_n \rangle^{1/2}
\quad + \limsup_{n \to \infty} \langle Bx_n, x_n \rangle^{1/2} \langle By_n, y_n \rangle^{1/2} = 0,
\]

which proves (5.3). \( \square \)

Now we can prove the main result of this section:

Theorem 5.3. Let \( (F,E) \) be a weak-\( * \) sequentially complete anti-dual pair let \( A,B \in \mathcal{L}(E;F) \) be positive operators. The following are equivalent:

(i) \( B \) is absolutely continuous with respect to \( A \),
(ii) for every \( y \in E \) there exists a sequence \( (y_n)_{n \in \mathbb{N}} \) in \( E \) such that

\[
\langle Bx, y \rangle = \lim_{n \to \infty} \langle Ax, y_n \rangle, \quad x \in E,
\]

and the convergence is uniform on the set \( \{x \in E: \langle (A+B)x, x \rangle \leq 1\} \).
Proof. Recall that $B$ is $A$-absolutely continuous precisely if $\hat{B}$ is (the graph of) a closable operator, or equivalently, if $\text{dom}\, \hat{B}^*$ is dense in $\mathcal{H}_B$. Hence, if $B \ll A$ then for every vector $\xi \in \mathcal{H}_B$ there is a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $\text{dom}\, \hat{B}^*$ such that $\|\xi - \xi_n\|_B \leq 1/n, ~n = 1, 2, \ldots$ Furthermore, by density, we can find a sequence $(y_n)_{n \in \mathbb{N}}$ in $E$ such that $\|\hat{B}^*\xi_n - A y_n\|_A \leq 1/n$ for each $n$. Hence
\[
\|Bx \xi_n - (Ax, y_n)\| \leq \|Bx \xi - \xi_n\|_B + \|Bx \xi_n - (Ax, y_n)\| \\
= \|Bx \xi - \xi_n\|_B + \|Ax \hat{B}^*\xi_n - (Ax, y_n)\|_A \\
\leq \frac{1}{n}(\|Ax\|_A + \|Bx\|_B) \leq \frac{\sqrt{2}}{n}\sqrt{(A + B)x, x}.
\]
The choice $\xi := By \in \mathcal{H}_B$ with an arbitrary $y \in E$ gives that (i) implies (ii). Let us prove now the backward implication: let $y \in E$ and choose $(y_n)_{n \in \mathbb{N}}$ according to (ii). Since $\text{ran} \, J^*$ is dense in $\mathcal{H}_{A+B}$, it follows that
\[
\|C_B J^* y - C_A J^* y_n\|_{A+B} \\
= \sup_{x \in E, \|J^* x\|_{A+B} \leq 1} |(J^* x, C_B J^* y - C_A J^* y_n)|_{A+B} \\
= \sup_{x \in E, \|(A+B)x, x\| \leq 1} |(Bx, y) - (Ax, y_n)| \to 0.
\]
We see therefore that $\text{ran} \, C_B J^* \subseteq \text{ran} \, C_A$, and hence $\text{ran} \, C_B \subseteq (\ker \, C_A)^\perp$. Lemma 5.2 completes the proof. \qed

6. Characterizations of singularity

This section is devoted to some characterizations of singularity. Note that the original definition of singularity is rather algebraic as depending on the ordering induced by positivity. Below we are going to provide some further equivalent characterizations which reflect some geometric and metric features of singularity. For analogous results see [2][18][30].

Theorem 6.1. Let $(F, E)$ be a weak-* sequentially complete anti-dual pair and let $A, B \in \mathcal{L}(E; F)$ be positive operators on it. The following assertions are equivalent:

(i) $A$ and $B$ are mutually singular,

(ii) $A : B = 0$,

(iii) the set $\{(Ax, Bx) : x \in E\}$ is dense in $\mathcal{H}_A \times \mathcal{H}_B$,

(iv) $\xi = 0$ is the only vector in $\mathcal{H}_B$ such that $|\langle Bx, \xi \rangle|^2 \leq M_\xi \langle Ax, x \rangle$ for every $x \in E$,

(v) $\mathcal{M} = \mathcal{H}_B$,

(vi) $\text{ran} \, J_A \cap \text{ran} \, J_B = \{0\}$,

(vii) for every $x \in E$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $\langle Ax_n, x_n \rangle \to 0$ and $\langle B(x - x_n), x - x_n \rangle \to 0$.

Proof. Since $A : B \leq A$ and $A : B \leq B$, (i) implies (ii). Assume (ii), then
\[
A : B = V_A Q V_A^* = V_B Q V_B^* = 0,
\]
where $Q$ is the orthogonal projection of $\mathcal{H}_A \times \mathcal{H}_B$ onto $\hat{B}^\perp$. That gives $Q(Ax, 0) = 0 = Q(0, By)$ for every $x, y \in E$. Since $\text{ran} \, A \times \text{ran} \, B$ is dense in $\mathcal{H}_A \times \mathcal{H}_B$ it follows that
\[
\{0\} = \text{ran} \, Q = \{(Ax, Bx) : x \in E\}^\perp.
\]
hence (ii) implies (iii). That (iii) implies (ii) is clear from identity \( A : B = V_A Q V_A \).

Observe furthermore that

\[
A : B \leq (nA) : B \leq (nA) : (nB) = n(A : B),
\]

hence \( A : B = 0 \) implies \( (nA) : B = 0 \) for each \( n \), and we have therefore \( B_n = [A]B = 0 \) by Theorem 4.6. This means that \( B = B_n \) in the view of Theorem 3.3.

\[
n(A) : B \leq 0.
\]

This proves that (ii) implies (i). The equivalence between (iv), (v), (vi) is clear from Lemma 3.1 and identity (3.3). Supposing (v) we have \( B_n = 0 \) and hence \( B_n = B \) and hence \( B \perp A \) by Theorem 3.3. Conversely, if \( B \) and \( A \) are mutually singular then, as it has been shown above, \( (nA) : B = 0 \) for each \( n \) and hence \( J_n \) is the unique positive operator which is simultaneously \( A \)-absolutely continuous and \( A \)-singular. In other words, \( B \ll A \) and \( B \perp A \) imply \( B = 0 \).

As an immediate consequence we conclude that absolute continuity and singularity are complementary notions in some sense:

**Corollary 6.2.** Let \( A \) be a positive operator on the weak-* sequentially complete anti-dual pair \( (F, E) \). Then \( B = 0 \) is the unique positive operator which is simultaneously \( A \)-absolutely continuous and \( A \)-singular. In other words, \( B \ll A \) and \( B \perp A \) imply \( B = 0 \).

### 7. Uniqueness of the decomposition

It was pointed out by Ando [2] that the Lebesgue decomposition among positive operators on an infinite dimensional Hilbert space is not unique. Since anti-dual pairs are even more general, we expect the same in our case. The reason why non-uniqueness occurs in the non-commutative integration theory is that absolute continuity is not hereditary: \( B \ll A \) and \( C \leq B \) do not imply \( C \ll A \). In fact, it may even happen that \( C \neq 0 \) and \( C \perp A \). More explicitly, we have the following result:

**Proposition 7.1.** Let \( A, B \) be positive operators on the weak-* sequentially anti-dual pair \( (F, E) \). Suppose that \( B \) is \( A \)-absolutely continuous but not \( A \)-dominated, i.e., there is no \( \alpha \geq 0 \) such that \( B \leq \alpha A \). Then there is a non-zero positive operator \( B' \leq B \) such that \( B' \perp A \).

**Proof.** By assumption, \( \tilde{B} \) is a densely defined closed and unbounded operator between \( H_A \) and \( H_B \), hence \( \text{dom} \, \tilde{B}^* \) is a proper dense subspace of \( H_B \). Choose a vector \( \zeta \in H_B \setminus \text{dom} \, \tilde{B}^* \) and denote by \( Q_\zeta \) the orthogonal projection onto the one-dimensional subspace \( H_\zeta \) generated by \( \zeta \). Set \( B' := J_B Q_\zeta J_B^* \), then clearly \( B' \leq B \).

We claim that \( B' \perp A \), which is equivalent to \( \text{ran} \, J_A \cap \text{ran} \, J_{B'} = \{0\} \) by Theorem 6.1. To see this we observe first that \( \text{ran} \, J_{B'} = \text{ran} \, J_B Q_\zeta \) because of Theorem 2.1,

\[
\|J_{B'}^* x\|_{B'}^2 = \langle B' x, x \rangle = \langle J_B Q_\zeta J_B^* x, x \rangle = \|Q_\zeta J_B^* x\|_B^2, \quad x \in E.
\]
Suppose \( f = J_B Q_\xi \xi \) belongs to \( \text{ran} \ J_A \), which means that \( Q_\xi \xi \in \text{dom} \ \hat{B}^* \), according to Lemma 3.1. Since we have \( H_\xi \cap \text{dom} \ \hat{B}^* = \{0\} \), it follows that \( Q_\xi \xi = 0 \) and \( f = 0 \), accordingly.

The next result gives a complete characterization of uniqueness of the Lebesgue decomposition. We mention that this is a direct generalization of Ando’s uniqueness result [2, Theorem 6]. We also refer the reader to [20, Theorem 7.8] and [19, Theorem 4.6].

**Theorem 7.2.** Let \( \langle F, E \rangle \) be a weak-* sequentially complete anti-dual pair and let \( A, B \in \mathcal{L}(E; F) \) be positive operators. The following statements are equivalent:

(i) the Lebesgue-decomposition of \( B \) into \( A \)-absolutely continuous and \( A \)-singular parts is unique,

(ii) \( \text{dom} \ \hat{B}^* \subseteq \mathcal{H}_B \) is closed,

(iii) \( \hat{B}_{\text{reg}} \) is norm continuous between \( \mathcal{H}_A \) and \( \mathcal{H}_B \),

(iv) \( B_\alpha \leq \alpha A \) for some \( \alpha \geq 0 \),

(v) \( J_B (\mathcal{M}^\perp) \subseteq \text{ran} \ J_A \).

**Proof.** We start by proving that (i) implies (ii). Suppose therefore that \( \text{dom} \ \hat{B}^* \) is not closed and consider a unit vector \( \zeta \in \mathcal{M}^\perp \setminus \text{dom} \ \hat{B}^* \). Denote by \( Q_\zeta \) the orthogonal projection onto the one dimensional subspace \( H_\zeta \) spanned by \( \zeta \). Then \( P_1 := P + Q_\zeta \) is a projection and

\[
B_1 := J_B (I - P_1) J_B^* , \quad B_2 := J_B P_1 J_B^* 
\]

are positive operators from \( E \) to \( F \) such that \( B_1 + B_2 = B \). Clearly, \( B_1 \neq B_2 \) and \( B_2 \neq B_\zeta \). We claim that \( B_1 \ll A \) and \( B_2 \perp A \). Since the map

\[
T(Ax) := (I - P)Bx, \quad x \in E
\]

defines a closable operator between \( \mathcal{H}_A \) and \( \mathcal{H}_B \), it follows that \( (I - Q_\zeta)T \) is closable too. Indeed,

\[
\text{dom}(T^*(I - Q_\zeta)) = H_\zeta \oplus [\text{dom} T^* \cap H_\zeta^\perp],
\]

and it is known that a dense subspace of a Hilbert space is dense in every finite co-dimensional subspace. Consequently, \( ((I - Q_\zeta)T)^* \) is densely defined and hence \( (I - Q_\zeta)T \) is closable. A straightforward calculation shows that

\[
\| (I - Q_\zeta)T(Ax) \|_{B}^2 = \langle B_1 x, x \rangle , \quad x \in E,
\]

whence it follows that \( B_1 \ll A \). To check that \( B_2 \perp A \) we argue as in the proof of Proposition [7,1]. First we observe that \( \text{ran} \ J_{B_2} = \text{ran} \ J_B (P + Q_\zeta) \) by Theorem 2.1. Furthermore, if \( f = J_B (P + Q_\zeta) \xi \in \text{ran} \ J_A \), then \( (P + Q_\zeta) \xi \in \text{dom} \ \hat{B}^* \), according to Lemma 3.1. But we have \( \text{ran} \ (P + Q_\zeta) \cap \text{dom} \ \hat{B}^* = \{0\} \), hence \( f = 0 \). Consequently, \( \text{ran} \ J_A \cap \text{ran} \ J_{B_2} = \{0\} \), and therefore \( B_2 \perp A \). Summing up, \( B = B_1 + B_2 \) is a Lebesgue decomposition of \( B \) with respect to \( A \) that differs from the canonical Lebesgue decomposition \( B = B_\zeta + B_\zeta \), i.e., the Lebesgue decomposition is not unique.

To prove that (ii) implies (iii) assume that \( \text{dom} \ \hat{B}^* \) is a closed subspace of \( \mathcal{H}_B \), then \( \hat{B}^* \) is bounded by the closed graph theorem. The same holds true for \( \hat{B}_{\text{reg}} \). If \( \hat{B}_{\text{reg}} \) is bounded, then from (3.8) we conclude that \( B_\alpha = J_A \hat{B}_{\text{reg}} \hat{B}_{\text{reg}} J_A \), and therefore \( B_\alpha \leq \alpha A \) with \( \alpha := \| \hat{B}_{\text{reg}} \|^2 \). Hence (iii) implies (iv). Note that (iv) is equivalent to \( \text{ran} \ J_B (I - P) \subseteq \text{ran} \ J_A \) in virtue of Theorem 2.1, hence (iv) and (v).
are equivalent. Assume finally (iv) and let $B_1 + B_2$ be any Lebesgue decomposition of $B$ with respect to $A$, where $B_1 \ll A$ and $B_2 \perp A$. By Theorem 7.3, $B_2 \leq B_0$, hence $0 \leq B_0 - B_1 \leq \alpha A$. Consequently, $0 \leq B_2 - B_1 = (B_0 - B_1) \leq \alpha A$ and $B_2 - B_1 \leq B_2$, and therefore $B_2 - B_1 = 0$ by singularity. This means that $B_2 = B_1$ and $B_1 = B_0$, proving that the Lebesgue decomposition is unique. The proof is complete. □

Below we give a sufficient condition on an operator $A$ such that the $A$-Lebesgue decomposition of every operator $B$ be unique.

**Lemma 7.3.** Let $A$ be a positive operator on a weak-* sequentially complete antidual pair $(F, E)$. The following assertions are equivalent:

(i) $\text{ran } A$ is weak-* sequentially closed in $F$,

(ii) $\text{ran } A$ is a Hilbert space under the inner product $(\cdot | \cdot)_A$.

**Proof.** Assume first that $\text{ran } A$ is weak-* sequentially closed in $F$. We are going to show that $\mathcal{H}_A = \text{ran } A$. It suffices to show that $J_A$ coincides with its restriction to $\text{ran } A$. That will be obtained by showing that $\ker J_A \subseteq \ker (J_A|_{\text{ran } A})$ and $\text{ran } J_A \subseteq \text{ran } (J_A|_{\text{ran } A})$. The first inclusion is clear because $J_A$ is injective:

$$\ker J_A = \text{ran } J_A^\perp = (\text{ran } A)\perp = \{0\}.$$ 

The second range inclusion follows from the fact that $\text{ran } J_A$ is contained in the weak-* sequential closure of the range of $J_A|_{\text{ran } A}$ in $F$, that is identical with $\text{ran } A$ by (i). This proves that (i) implies (ii). Assume conversely that $\text{ran } A = \mathcal{H}_A$ and let $f$ belong to the weak-* sequential closure of $\text{ran } A$ in $F$. Choose a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$f(x) = \lim_{n \to \infty} \langle Ax_n, x \rangle, \quad x \in E.$$ 

For every $n$ let us define a continuous conjugate linear functional $\varphi_n : \mathcal{H}_A \to \mathbb{C}$ by

$$\varphi_n(Ax) := (Ax_n | Ax)_A = (Ax_n, x), \quad x \in E.$$ 

Then $(\varphi_n)_{n \in \mathbb{N}}$ converges pointwise to some bounded conjugate linear functional $\varphi : \mathcal{H}_A \to \mathbb{C}$, because of the Banach–Steinhaus theorem. By the Riesz representation theorem, there exists $z \in E$ such that $\varphi(Ax) = (Az | Ax)_A, x \in E,$ and therefore

$$f(x) = (Az | Ax)_A = (Az, x), \quad x \in E.$$ 

Consequently, $f = Az \in \text{ran } A$. □

**Theorem 7.4.** Let $A$ be a positive operator on a weak-* sequentially complete antidual pair $(F, E)$. If the range of $A$ is weak-* sequentially closed then every positive operator $B \in \mathcal{L}(E; F)$ admits a unique Lebesgue decomposition with respect to $A$.

**Proof.** According to the preceding lemma, $\text{ran } A$ is complete under the inner product $(\cdot | \cdot)_A$, i.e., $\text{ran } A = \mathcal{H}_A$. The closed operator $\hat{B}_{\text{reg}}$ is everywhere defined on $\mathcal{H}_A$ and thus bounded by the closed graph theorem. The Lebesgue decomposition of $B$ with respect to $A$ is unique by Theorem 7.3. □

8. Applications

To conclude the paper we apply the developed decomposition theory to some concrete objects including Hilbert space operators, Hermitian forms, representable functionals, and additive set functions.
8.1. Positive operators on Hilbert spaces. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then $\langle \mathcal{H}, \mathcal{H} \rangle$ forms a anti-dual pair with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle$. An immediate application of the Banach–Steinhaus theorem shows that $\langle \mathcal{H}, \mathcal{H} \rangle$ is weak-* sequentially complete, thus everything what has been said so far remains valid for $\langle \mathcal{H}, \mathcal{H} \rangle$ and the positive operators on it.

We shortly summarize Ando’s main results [2] Theorem 2 and 6] in a statement. The proof follows immediately from Theorem 3.3, 5.1 and 7.2

**Theorem 8.1.** Let $A, B$ be bounded positive operators on a complex Hilbert space $\mathcal{H}$ and let $B_n := \lim_{n \to \infty} (nA) : B$ where the limit is taken in the strong operator topology and let $B_s := B - B_a$. Then
\[
B = B_a + B_s
\]
is a Lebesgue-type decomposition, i.e., $B_a$ is $A$-absolutely continuous and $B_s$ is $A$-singular. $B_a$ is maximal among those positive operators $C \geq 0$ such that $C \leq B$ and $C \ll A$. The Lebesgue decomposition (8.1) is unique if and only if $B_a \leq \alpha A$ for some constant $\alpha \geq 0$.

8.2. Nonnegative forms. Let $\mathfrak{D}$ be a complex vector space and let $t, \omega$ be nonnegative Hermitian forms on it. Let us denote by $\mathfrak{D}^*$ the algebraic dual space of $\mathfrak{D}$, then $\langle \mathfrak{D}^*, \mathfrak{D} \rangle$ forms a weak-* sequentially complete anti-dual pair and
\[
(Tx, y) := t(x, y), \quad (Wx, y) := \omega(x, y), \quad x, y \in \mathfrak{D}
\]
define two positive operators $T, W : \mathfrak{D} \to \mathfrak{D}^*$. We recall that the form $t$ is called $\omega$-almost dominated if there is a monotonically nondecreasing sequence of forms $t_n$ such that $t_n \leq \alpha_n \omega$ for some $\alpha_n \geq 0$ and $t_n \to t$ pointwise. Similarly, $t$ is called $\omega$-closable if for every sequence $(x_n)_{n \in \mathbb{N}}$ of $\mathfrak{D}$ such that $\omega(x_n, x_n) \to 0$ and $t(x_n, x_n, x_n - x_m) \to 0$ it follows that $t(x_n, x_n) \to 0$.

It is immediate to conclude that the form $t$ is $\omega$-closable if and only if the operator $T$ is $W$-absolutely continuous. Similarly, $t$ is $\omega$-almost dominated precisely when $T$ is $W$-almost dominated. Consequently, from Theorem 5.1 it follows that the notions of closability and almost dominatedness are equivalent (cf. also [19, Theorem 3.8]). The map $t \mapsto T$ between nonnegative hermitian forms and positive operators on $\mathfrak{D}$ is a bijection, so from Theorem 3.3 and 7.2 we conclude the following result (see [19, Theorem 2.11 and 4.6]):

**Theorem 8.2.** Let $t, \omega$ be nonnegative Hermitian forms on a complex vector space $\mathfrak{D}$ and let $t_n(x, x) := \lim_{n \to \infty} ((n t) : \mathfrak{s})(x, x), x \in \mathfrak{D}$ and $t_s := t - t_n$. Then
\[
t = t_n + t_s
\]
is a Lebesgue-type decomposition of $t$ with respect to $\omega$, i.e., $t_n$ is $\omega$-absolutely continuous and $t_s$ is $\omega$-singular. Furthermore, $t_n$ is maximal among those forms $s$ such that $s \leq t$ and $s \ll \omega$. The Lebesgue decomposition (8.2) is unique if and only if $t_n \leq \alpha \omega$ for some constant $\alpha \geq 0$.

8.3. Representable functionals. Let $\mathcal{A}$ be a *-algebra (with or without unit), i.e., an algebra endowed with an involution. A functional $f : \mathcal{A} \to \mathbb{C}$ is called representable if there is a triple $(\mathcal{H}_f, \pi_f, \zeta_f)$ such that $\mathcal{H}_f$ is a Hilbert space, $\zeta_f \in \mathcal{H}_f$ and $\pi_f : \mathcal{A} \to \mathfrak{B}(\mathcal{H}_f)$ is a *-algebra homomorphism such that
\[
f(a) = \langle \pi_f(a) \zeta_f | \zeta_f \rangle_f, \quad a \in \mathcal{A}.
\]
A straightforward verification shows that every representable functional \( f \) is positive hence the map \( \mathcal{A} \to \mathcal{A}^* \) defined by
\[
\langle Aa, b \rangle := f(b^*a), \quad a, b \in \mathcal{A}
\]
is a positive operator. (Note however that not every positive operator \( A \) arises from a representable functional \( f \) in the above way.) Denote by \( \mathcal{H}_A \) the corresponding auxiliary Hilbert space. It is easy to show that \( \pi_f : \mathcal{A} \to \mathcal{B}(\mathcal{H}_A), \ a \mapsto \pi_f(a) \) is a \( * \)-homomorphism, where the bounded operator \( \pi_f(a) \) arises from the densely defined one given by
\[
\pi_f(a)(Ab) := A(ab), \quad b \in \mathcal{A}.
\]
It follows from the representability of \( f \) that \( |f(a)|^2 \leq Cf(a^*a), \ a \in \mathcal{A}, \) for some constant \( C \geq 0 \) and hence
\[
Aa \mapsto f(a), \quad a \in \mathcal{A}
\]
defines a continuous linear functional from ran \( A \subseteq \mathcal{H}_A \) to \( \mathbb{C} \). The corresponding representing functional \( \zeta_f \) satisfies
\[
\langle Aa | \zeta_f \rangle_A = f(a), \quad a \in \mathcal{A},
\]
and admits the useful property \( \pi_f(a)\zeta_f = Aa \). It follows therefore that
\[
f(a) = \langle \pi_f(a)\zeta_f | \zeta_f \rangle_A, \quad a \in \mathcal{A}.
\]
Let \( g \) be another representable functional on \( \mathcal{A} \). We say that \( g \) is \( f \)-absolutely continuous if for every sequence \( (a_n)_{n \in \mathbb{N}} \) of \( \mathcal{A} \) such that \( f(a_n^*a_n) \to 0 \) and \( g((a_n - a_m)^*(a_n - a_m)) \to 0 \) it follows that \( g(a_n^*a_n) \to 0 \). Furthermore, \( g \) and \( f \) are singular with respect to each other if \( h = 0 \) is the only representable functional such that \( h \leq f \) and \( h \leq g \).

Denote by \( B : \mathcal{A} \to \mathcal{A}^* \) be the positive operator associated with \( g \) and let \( (\mathcal{H}_B, \pi_g, \zeta_g) \) the corresponding GNS-triplet obtained along the above procedure. Let us introduce \( \mathcal{M} \subseteq \mathcal{H}_B \) and \( P \) as in Section 3. Then \( \mathcal{M} \) and \( \mathcal{M}^\perp \) are both \( \pi_g \)-invariant, so
\[
g_a(a) := (\pi_g(a)P\zeta_g | P\zeta_g)_a, \quad g_a(a) := (\pi_g(a)(I - P)\zeta_g | (I - P)\zeta_g)_a
\]
are representable functionals on \( \mathcal{A} \) such that
\[
\langle B_a a, b \rangle = g_a(b^*a), \quad \langle B_a a, b \rangle = g_a(b^*a).
\]
It is clear therefore that \( g_a \ll f \) and \( g_a \perp f \). If \( \mathcal{A} \) has a unit element \( 1 \) then the absolutely continuous and singular parts can be written in a much simpler form:
\[
g_a(a) = \langle B_1, a \rangle, \quad g_s(a) = \langle B_1, a \rangle, \quad a \in \mathcal{A}.
\]
After these observations we can state the corresponding Lebesgue decomposition theorem of representable functionals [38, Theorem 3.3]; cf. also [17, Corollary 3] and [10, Theorem 3.3]:

**Theorem 8.3.** Let \( f, g \) be representable functionals on the \( * \)-algebra \( \mathcal{A} \), then \( g_a \) and \( g_s \) are representable functionals such that \( g = g_a + g_s \), where \( g_a \) is \( f \)-absolutely continuous and \( g_s \) is \( f \)-singular. Furthermore, \( g_a \) is is maximal among those representable functionals \( h \) such that \( h \leq g \) and \( h \ll f \).

Finally we note that not every positive operator \( A : \mathcal{A} \to \mathcal{A}^* \) arises from a representable functional, hence the question of uniqueness of the Lebesgue decomposition cannot be answered via Theorem 7.2. For a detailed discussion of this delicate problem we refer the reader to [40].
8.4. **Finitely additive and \( \sigma \)-additive set functions.** Let \( X \) be a non-empty set and \( \mathcal{A} \) be an algebra of sets on \( X \). Let \( \alpha \) be a non-negative finitely additive measure and denote by \( \mathcal{F} \) the unital *-algebra of \( \mathcal{A} \)-measurable functions, then \( \alpha \) induces a positive operator \( A : \mathcal{F} \to \mathcal{F}^* \) by

\[
\langle A\varphi, \psi \rangle := \int \varphi \psi \, d\alpha, \quad \varphi, \psi \in \mathcal{F}.
\]

We notice that we can easily recover \( \alpha \) from \( A \), namely

\[
\alpha(R) = \langle A\chi_R, \chi_R \rangle, \quad R \in \mathcal{A}.
\]

However, not every positive operator \( A : \mathcal{F} \to \mathcal{F}^* \) induces a finitely additive measure, as it turns out from the next statement.

**Proposition 8.4.** If \( A : \mathcal{F} \to \mathcal{F}^* \) is a positive operator then \( \alpha \) defined by \( \alpha(R) = \langle A\chi_R, \chi_R \rangle, \quad R \in \mathcal{A} \) is an additive set function if and only if

\[
\langle A[\varphi], [\varphi] \rangle = \langle A\varphi, \varphi \rangle, \quad \varphi \in \mathcal{F}.
\]

**Proof.** The “only if” part of the statement is clear. For the converse suppose that \( A \) satisfies \( \alpha \). For every two disjoint sets \( R_1, R_2 \in \mathcal{A} \) we have

\[
\alpha(R_1 \cup R_2) = \frac{1}{2} \left\{ \langle A(\chi_{R_1} + \chi_{R_2}), \chi_{R_1} + \chi_{R_2} \rangle + \langle A(\chi_{R_1} - \chi_{R_2}), \chi_{R_1} - \chi_{R_2} \rangle \right\}
\]

\[
= \langle A\chi_{R_1}, \chi_{R_1} \rangle + \langle A\chi_{R_2}, \chi_{R_2} \rangle = \alpha(R_1) + \alpha(R_2),
\]

proving the additivity of \( \alpha \). \( \square \)

Assume that we are given another nonnegative additive set function \( \beta \) on \( \mathcal{A} \), then \( \beta \) is called absolutely continuous with respect to \( \alpha \) if for each \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that \( R \in \mathcal{A} \) and \( \alpha(R) < \delta \) imply \( \beta(R) < \varepsilon \). Furthermore, \( \alpha \) and \( \beta \) are mutually singular if \( \gamma = 0 \) is the only nonnegative additive set function such that \( \gamma \leq \alpha \) and \( \gamma \leq \beta \).

Our claim is to prove that the Lebesgue decomposition of \( \beta \) with respect to \( \alpha \) can also be derived from that of the induced positive operators. To this aim we first note that singularity of \( A \) and \( B \) obviously implies the singularity of \( \alpha \) and \( \beta \).

It is less obvious that \( A \)-absolute continuity of \( B \) implies the \( \alpha \)-absolute continuity of \( \beta \) (cf. also [37, Lemma 3.1]). To see this consider a sequence \( (R_n)_{n \in \mathbb{N}} \) of \( \mathcal{A} \) such that \( \alpha(R_n) \to 0 \). Clearly,

\[
(J_A^* \chi_{R_n} | J_B^* \chi_{R_n})_\lambda = \langle A\chi_{R_n}, \chi_{R_n} \rangle \to 0.
\]

Since \( (J_B^* \chi_{R_n} | J_B^* \chi_{R_n})_\lambda \leq \beta(X) \), the sequence \( (J_B^* \chi_{R_n})_{n \in \mathbb{N}} \) is bounded in \( \mathcal{H}_B \), and for every \( \xi \in \text{dom } B^* \),

\[
(J_B^* \chi_{R_n} | \xi)_\lambda = (J_A^* \chi_{R_n} | B^* \xi)_\lambda \to 0.
\]

Consequently, \( J_B^* \chi_{R_n} \to 0 \) weakly in \( \mathcal{H}_B \), and hence \( B\chi_{R_n} \to 0 \) in \( \mathcal{F}^* \) with respect to the weak-* topology \( \sigma(\mathcal{F}^*, \mathcal{F}) \). This implies that

\[
\beta(R_n) = \langle B\chi_{R_n}, 1 \rangle \to 0,
\]

hence \( \beta \ll \alpha \).

**Theorem 8.5.** Let \( \alpha, \beta : \mathcal{A} \to \mathbb{R}_+ \) be nonnegative additive set functions. There exist two nonnegative additive set functions \( \beta_a, \beta_s \) such that \( \beta = \beta_a + \beta_s \), where \( \beta_a \) is \( \alpha \)-absolutely continuous and \( \beta_s \) is \( \alpha \)-singular.
Proof. Consider the $A$-Lebesgue-decomposition $B = B_a + B_s$ of the corresponding induced operators. According to the above observation it suffices to show that $B_a$ (and hence also $B_s$) is induced by an additive set function $\beta_a$ (respectively, $\beta_s$). By Proposition 8.4, this will be done if we prove that
\[
\langle B_a \varphi, \varphi \rangle = \langle B_a |\varphi|, |\varphi| \rangle, \quad \varphi \in \mathcal{F}.
\]
Set
\[
f(\varphi) := \int \varphi \, d\alpha, \quad g(\varphi) := \int \varphi \, d\beta, \quad \varphi \in \mathcal{F},
\]
so that $f, g$ are representable functionals on $\mathcal{F}$. By Theorem 8.3, $g$ splits into $f$-absolutely continuous and $f$-singular parts $g_a, g_s$ respectively. By (8.4),
\[
\langle B_a \varphi, \varphi \rangle = g_a (|\varphi|^2), \quad \varphi \in \mathcal{F}.
\]
This obviously gives (8.7). □

Finally, assume that $A$ is a $\sigma$-algebra and $\mu, \nu$ are finite measures on $A$. By Theorem 8.5 there exist two nonnegative (finitely) additive set functions $\nu_a, \nu_s$ such that $\nu = \nu_a + \nu_s$ where $\nu_a \ll \mu$ and $\nu_s \perp \mu$. Note that both functions are dominated by the measure $\nu$, hence $\nu_a, \nu_s$ are forced to be $\sigma$-additive. This fact leads us the Lebesgue decomposition of measures:

**Corollary 8.6.** If $\mu, \nu$ are finite measures on a $\sigma$-algebra $A$ then there exist two measures $\nu_a, \nu_s$ such that $\nu = \nu_a + \nu_s$, where $\nu_a$ is $\mu$-absolutely continuous and $\nu_s$ is $\mu$-singular.

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