A GENERALIZATION OF THE HASSE-WITT MATRIX
OF A HYPERSURFACE

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Abstract. The Hasse-Witt matrix of a hypersurface in $\mathbb{P}^n$ over a finite field of characteristic $p$ gives essentially complete mod $p$ information about the zeta function of the hypersurface. But if the degree $d$ of the hypersurface is $\leq n$, the zeta function is trivial mod $p$ and the Hasse-Witt matrix is zero-by-zero. We generalize a classical formula for the Hasse-Witt matrix to obtain a matrix that gives a nontrivial congruence for the zeta function for all $d$. We also describe the differential equations satisfied by this matrix and prove that it is generically invertible.

1. Introduction

Let $p$ be a prime number, let $q = p^a$, and let $\mathbb{F}_q$ be the field of $q$ elements. Let

$$\lambda = \sum_{j=1}^{N} \lambda_j x^{a_j} \in \mathbb{F}_q[x_0, \ldots, x_n]$$

be a homogeneous polynomial of degree $d$. We write $a_j = (a_{0j}, \ldots, a_{nj})$ with $\sum_{j=0}^{n} a_{ij} = d$. Let $X_{\lambda} \subseteq \mathbb{P}^n$ be the hypersurface defined by the equation $f_{\lambda}(x) = 0$ and let $Z(X_{\lambda}/\mathbb{F}_q, t)$ be its zeta function. We write

$$Z(X_{\lambda}/\mathbb{F}_q, t) = P_{\lambda}(t) \left(\frac{-1}{(1-t)(1-qt) \cdots (1-q^{n-1}t)}\right)$$

for some rational function $P_{\lambda}(t) \in 1 + t\mathbb{Z}[t]$. If $d = 1$ then $P_{\lambda}(t) = 1$, so we shall always assume that $d \geq 2$.

Define a nonnegative integer $\mu$ by the equation

$$\left\lceil \frac{n+1}{d} \right\rceil = \mu + 1,$$

where $\lceil r \rceil$ denotes the least integer greater than or equal to the real number $r$. By a result of Ax\cite{Ax} (see also Katz\cite[Proposition 2.4]{Katz}) we have

$$P_{\lambda}(q^{-\mu}t) \in 1 + t\mathbb{Z}[t].$$

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Our goal in this paper is to give a mod $p$ congruence for $P_{\lambda}(q^{-\mu}t)$. We do this by defining a generalization of the classical Hasse-Witt matrix, which gives such a congruence for $\mu = 0$. Presumably our matrix is the matrix of a “higher Hasse-Witt” operation as defined by Katz [6, Section 2.3.4], but so far we have not been able to prove this.

It will be convenient to define an augmentation of the vectors $a_j$. Set 
\[ a_j^+ = (a_0j, \ldots, a_nj, 1) \in \mathbb{N}^{n+2}, \quad j = 1, \ldots, N, \]
where $\mathbb{N}$ denotes the nonnegative integers. Note that the vectors $a_j^+$ all lie on the hyperplane $\sum_{i=0}^n u_i = du_{n+1}$ in $\mathbb{R}^{n+2}$. We shall be interested in the lattice points on this hyperplane that lie in $(\mathbb{R}_{>0})^{n+2}$: set 
\[ U = \left\{ u = (u_0, \ldots, u_{n+1}) \in \mathbb{N}^{n+2} \mid \sum_{i=0}^n u_i = du_{n+1} \text{ and } u_i > 0 \text{ for all } i \right\}. \]

Note that $u \in U$ implies that $u_{n+1} \geq \mu + 1$. Let 
\[ U_{\text{min}} = \{ u = (u_0, \ldots, u_{n+1}) \in U \mid u_{n+1} = \mu + 1 \}, \]
a nonempty set by the definition of $\mu$. We define a matrix of polynomials with rows and columns indexed by $U_{\text{min}}$: let $A(\Lambda) = [A_{uv}(\Lambda)]_{u,v \in U_{\text{min}}}$, where 
\[ A_{uv}(\Lambda) = (-1)^{\mu+1} \sum_{\nu \in \mathbb{N}^N} \frac{\Lambda_1^{\nu_1} \cdots \Lambda_N^{\nu_N}}{\nu_1! \cdots \nu_N!} \in \mathbb{Q}[\Lambda_1, \ldots, \Lambda_N]. \]

Note that since the $(n+1)$-st coordinate of each $a_j^+$ equals 1, the condition on the summation implies that 
\[ \sum_{j=1}^N \nu_j = (p-1)(\mu + 1). \]

When $\mu = 0$, it follows that $\nu_j \leq p-1$ for all $j$, hence the matrix $A(\Lambda)$ can be reduced modulo $p$. We denote by $\bar{A}(\Lambda) \in \mathbb{F}_p[\Lambda]$ its reduction modulo $p$. Using the algorithm of Katz [6, Algorithm 2.3.7.14], one then checks that $\bar{A}(\lambda)$ is the Hasse-Witt matrix of the hypersurface $f_{\lambda} = 0$. It is somewhat surprising that even when $\mu > 0$ we still have $\nu_j \leq p-1$ for all $j$.

**Lemma 1.3.** If $u, v \in U_{\text{min}}$, $\nu \in \mathbb{N}^N$, and $\sum_{j=1}^N \nu_j a_j^+ = pu - v$, then $\nu_j \leq p - 1$ for all $j$. In particular, $A_{uv}(\Lambda) \in (\mathbb{Q} \cap \mathbb{Z}_p)[\Lambda]$, so $A_{uv}(\Lambda)$ can be reduced modulo $p$. 
The proof of Lemma 1.3 will be given in Section 2. By the results of [1, Theorem 2.7 or Theorem 3.1], which will be recalled in Section 2, Lemma 1.3 implies immediately that each $A_{uv}(\Lambda)$ is a mod $p$ solution of an $A$-hypergeometric system of differential equations.

Write the rational function $P_\lambda(t)$ of (1.2) as

$$P_\lambda(t) = \frac{Q_\lambda(t)}{R_\lambda(t)},$$

where $Q_\lambda(t)$ and $R_\lambda(t)$ are relatively prime polynomials with

$$Q_\lambda(q^{-\mu}t). R_\lambda(q^{-\mu}t) \in 1 + t\mathbb{Z}[t].$$

If $X_\lambda$ is smooth, it is known that $P_\lambda(t)$ is a polynomial, i.e., $R_\lambda(t) = 1$. Our main result is the following, which does not require any smoothness assumption.

**Theorem 1.4.** If $n$ is not divisible by $d$, then $R_\lambda(q^{-\mu}t) \equiv 1 \pmod{q}$ and

$$Q_\lambda(q^{-\mu}t) \equiv \det(I - t\bar{A}(\lambda^{p^a-1})\bar{A}(\lambda^{p^a-2}) \cdots \bar{A}(\lambda)) \pmod{p}.$$

Note that even in the classical case of the Hasse-Witt matrix ($\mu = 0$), this result contains something new, as we do not assume that $X_\lambda$ is a smooth hypersurface.

The proof of Theorem 1.4 will occupy Sections 3–5. To describe the zeta function, we apply the $p$-adic cohomology theory of Dwork, as in Katz [7, Sections 4–6]. Indeed, Equation (3.5) below is a refined version of [7, Equation (4.5.33)]. We discuss the case $d | n$ in Section 6. If $d | n$, the conclusion of Theorem 1.4 need not hold, and the rational function $P_\lambda(q^{-\mu}t) \pmod{p}$ is instead described by Theorem 6.2. We prove the generic invertibility of the matrix $\bar{A}(\Lambda)$ in Section 7.

2. Proof of Lemma 1.3

It will be convenient for later applications to prove a more general version of Lemma 1.3. Put $S = \{0, 1, \ldots, n\}$ and let $I \subseteq S$. Define an integer $\mu_I$ by the equation

$$\left\lfloor \frac{|I|}{d} \right\rfloor = \mu_I + 1.$$

Note that $\mu_I \geq 0$ if $I \neq \emptyset$, $\mu_\emptyset = -1$, and, in the notation of the Introduction, $\mu_S = \mu$. Set

$$U^I = \left\{ u = (u_0, \ldots, u_{n+1}) \in \mathbb{N}^{n+2} \mid \sum_{i=0}^{n} u_i = du_{n+1} \text{ and } u_i > 0 \text{ for all } i \in I \right\}.$$
Note that $u \in U^I$ implies that $u_{n+1} \geq \mu_I + 1$. Let

$$U^I_{\min} = \{u = (u_0, \ldots, u_{n+1}) \in U^I \mid u_{n+1} = \mu_I + 1\},$$

a nonempty set by the definition of $\mu_I$. Lemma 1.3 is the special case $I = S$ of the following result.

**Lemma 2.1.** If $u, v \in U^I_{\min}$, $\nu \in \mathbb{N}^N$, and $\sum_{j=1}^N \nu_j a_j^+ = pu - v$, then $\nu_j \leq p - 1$ for all $j$.

**Proof.** The result is trivial when $I = \emptyset$ since $U^\emptyset_{\min} = \{(0, \ldots, 0)\}$, so assume $I \neq \emptyset$. Let $u = (u_0, \ldots, u_n, \mu_I + 1), v = (v_0, \ldots, v_n, \mu_I + 1) \in U^I_{\min}$. Fix $k \in \{1, \ldots, N\}$. We claim there exists an index $i_0 \in \{0, \ldots, n\}$ such that

$$a_{i_0 k} \geq \begin{cases} u_{i_0} & \text{if } i_0 \in I, \\ u_{i_0} + 1 & \text{if } i_0 \notin I. \end{cases}$$

(2.2)

For if (2.2) fails for all $i_0 \in \{0, \ldots, n\}$, then

$$u - a_k^+ = (u_0 - a_{0k}, \ldots, u_n - a_{nk}, \mu_I) \in U^I,$$

contradicting the definition of $\mu_I$.

If $\nu_k \geq p$, then

$$\nu_k a_{i_0 k} \geq \begin{cases} pu_{i_0} & \text{if } i_0 \in I, \\ pu_{i_0} + p & \text{if } i_0 \notin I, \end{cases}$$

hence in both cases we have

$$\nu_k a_{i_0 k} > pu_{i_0} - v_{i_0}.$$ But our hypothesis $\sum_{j=1}^N \nu_j a_j^+ = pu - v$ implies that

$$\nu_k a_{i_0 k} \leq pu_{i_0} - v_{i_0}.$$ This contradiction shows that $\nu_k \leq p - 1$. And since $k$ was arbitrary, the lemma is established. \qed

We recall the definition of the $A$-hypergeometric system of differential equations associated to the set $A = \{a_j^+\}_{j=1}^N$. Let $L \subseteq \mathbb{Z}^N$ be the lattice of relations on $A$,

$$L = \left\{ l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \mid \sum_{j=1}^N l_j a_j^+ = 0 \right\},$$

and let $\beta = (\beta_0, \ldots, \beta_{n+1}) \in \mathbb{C}^{n+2}$. The $A$-hypergeometric system with parameter $\beta$ is the system of partial differential operators in variables
Λ₁,...,Λₙ consisting of the box operators

\[ \Box_l = \prod_{l_j > 0} \left( \frac{\partial}{\partial \Lambda_j} \right)^{l_j} - \prod_{l_j < 0} \left( \frac{\partial}{\partial \Lambda_j} \right)^{-l_j} \quad \text{for } l \in L \]

and the Euler (or homogeneity) operators

\[ Z_i = \sum_{j=1}^{N} a_{ij} \Lambda_j \frac{\partial}{\partial \Lambda_j} - \beta_i \quad \text{for } i = 0, \ldots, n \]

and

\[ Z_{n+1} = \sum_{j=1}^{N} \Lambda_j \frac{\partial}{\partial \Lambda_j} - \beta_{n+1}. \]

Let \( A_I(\Lambda) = [A_{uv}(\Lambda)]_{u,v \in U_{\min}} \), where

\[ A_{uv}(\Lambda) = (-1)^{\mu_{i+1}} \sum_{\nu \in N^N} \Lambda_1^{\nu_1} \cdots \Lambda_N^{\nu_N} \prod_{j=1}^{N} \nu_j! \cdot \nu_N! \in \mathbb{Q}[\Lambda_1, \ldots, \Lambda_N]. \]

Note that in the notation of the Introduction we have \( A^S_{uv}(\Lambda) = A_{uv}(\Lambda) \). By Lemma 2.1, the polynomials \( A_{uv}(\Lambda) \) have \( p \)-integral coefficients. Lemma 2.1 also says that \( pu - v \) is very good in the sense of [1, Section 2]. We may therefore apply [1, Theorem 2.7] (or [1, Theorem 3.1] since this system is nonconfluent) to conclude that \( \bar{A}_{uv}(\Lambda) \) is a mod \( p \) solution of the \( A \)-hypergeometric system with parameter \( \beta = pu - v \) (or, equivalently, \( \beta = -v \) since we have reduced modulo \( p \)).

### 3. The zeta function

To make a connection between the matrix \( A(\Lambda) \) and the zeta function (1.2), we apply a consequence of the Dwork trace formula developed in [3] (see Equation (3.5) below). Let \( \gamma_0 \) be a zero of the series \( \sum_{i=0}^{\infty} t^i / p^i \) having \( \text{ord}_p \gamma_0 = 1 / (p - 1) \), where \( \text{ord}_p \) is the \( p \)-adic valuation normalized by \( \text{ord}_p p = 1 \). Let \( L_0 \) be the space of series

\[ L_0 = \left\{ \sum_{u \in [n+2]} c_u \gamma_0^{pu+1} x^u \left| \sum_{i=0}^{n} u_i - du_{n+1} = 0, c_u \in \mathbb{C}_p, \right. \{c_u\} \right. \text{is bounded} \right\}. \]

For \( I \subseteq \{0, \ldots, n\} \), let \( L_0^I \) be the subset of \( L_0 \) defined by

\[ L_0^I = \left\{ \sum_{u \in [n+2]} c_u \gamma_0^{pu+1} x^u \in L_0 \left| u_i > 0 \text{ for } i \in I \right. \right\}. \]
Let $AH(t) = \exp(\sum_{i=0}^{\infty} t^{p^i} / p^i)$ be the Artin-Hasse series, a power series in $t$ that has $p$-integral coefficients, and set

$$\theta(t) = AH(\gamma_0 t) = \sum_{i=0}^{\infty} \theta_i t^i.$$ 

We then have

$$\text{ord}_p \theta_i \geq \frac{i}{p - 1}. \tag{3.1}$$

We define the Frobenius operator on $L_0$. Put

$$\theta(\hat{\lambda}, x) = \prod_{j=1}^{N} \theta(\hat{\lambda}_j x^{a_j^+}), \tag{3.2}$$

where $\hat{\lambda}$ denotes the Teichmüller lifting of $\lambda$. We shall also need to consider the series $\theta_0(\hat{\lambda}, x)$ defined by

$$\theta_0(\hat{\lambda}, x) = \prod_{i=0}^{a-1} \prod_{j=1}^{N} \theta((\hat{\lambda}_j x^{a_j^+})^{p^i}) = \prod_{i=0}^{a-1} \theta(\hat{\lambda}^{p^i}, x^{p^i}). \tag{3.3}$$

Define an operator $\psi$ on formal power series by

$$\psi \left( \sum_{u \in \mathbb{N}^{n+2}} c_u x^u \right) = \sum_{u \in \mathbb{N}^{n+2}} c_{pu} x^u. \tag{3.4}$$

Denote by $\alpha_{\hat{\lambda}}$ the composition

$$\alpha_{\hat{\lambda}} := \psi^a \circ \text{"multiplication by $\theta_0(\hat{\lambda}, x)."}$$

The map $\alpha_{\hat{\lambda}}$ operates on $L_0$ and is stable on each $L_0^I$. The proof of Theorem 1.4 will be based on the following formula for the rational function $P_\lambda(t)$ defined in (1.2). By [3, Equation 7.12] we have

$$P_\lambda(qt) = \prod_{I \subseteq \{0,1,\ldots,n\}} \det(I - q^{n+1-|I|} I^{(\hat{\lambda})} \mid L_0^I)^{(n+1+|I|)}. \tag{3.5}$$

To exploit (3.5) we shall need $p$-adic estimates for the action of $\alpha_{\hat{\lambda}}$ on $L_0^I$. Expand (3.3) as a series in $x$, say,

$$\theta_0(\hat{\lambda}, x) = \sum_{w \in \mathbb{N}A} \theta_{0,w}(\hat{\lambda}) x^w. \tag{3.6}$$

Note that from the definitions we have $\theta_{0,w}(\hat{\lambda}) \in \mathbb{Q}_p(\zeta_q-1, \gamma_0)$. A direct calculation shows that for $v \in U^I$,

$$\alpha_{\hat{\lambda}}(x^v) = \sum_{u \in U^I} \theta_{0,qu-v}(\hat{\lambda}) x^u, \tag{3.7}$$
thus we need $p$-adic estimates for the $\theta_{0,qu-v}(\hat{\lambda})$ with $u, v \in U^I$.

Expand (3.2) as a series in $x$:

\[
\theta(\hat{\lambda}, x) = \sum_{w \in NA} \theta_w(\hat{\lambda})x^w,
\]

where

\[
\theta_w(\hat{\lambda}) = \sum_{\nu \in \mathbb{N}^N} \theta^{(w)}_\nu \hat{\lambda}^\nu
\]

with

\[
\theta^{(w)}_\nu = \begin{cases} 
\prod_{j=1}^N \theta_{\nu_j} & \text{if } \sum_{j=1}^N \nu_j a_j^+ = w, \\
0 & \text{if } \sum_{j=1}^N \nu_j a_j^+ \neq w.
\end{cases}
\]

From (3.1) we have the estimate

\[
\text{ord}_p \theta^{(w)}_\nu \geq \frac{\sum_{j=1}^N \nu_j}{p-1} = \frac{w_{n+1}}{p-1}.
\]

In particular, this implies the estimate

\[
\text{ord}_p \theta_w(\hat{\lambda}) \geq \frac{w_{n+1}}{p-1}.
\]

By (3.3) and (3.8) we have

\[
\theta_{0,u}(\hat{\lambda}) = \sum_{u^{(0)}, \ldots, u^{(a-1)} \in NA} \prod_{i=0}^{a-1} \theta_{u^{(i)}}(\hat{\lambda}^{p^i}).
\]

In particular, we get the formula

\[
\theta_{0,qu-v}(\hat{\lambda}) = \sum_{u^{(0)}, \ldots, u^{(a-1)} \in NA} \prod_{i=0}^{a-1} \theta_{w^{(i)}}(\hat{\lambda}^{p^i}).
\]

Applying (3.12) to the products on the right-hand side of (3.14) gives

\[
\text{ord}_p \left( \prod_{i=1}^{a-1} \theta_{w^{(i)}}(\hat{\lambda}^{p^i}) \right) \geq \sum_{i=0}^{a-1} \frac{w^{(i)}_{n+1}}{p-1}.
\]

This estimate is not directly helpful for estimating $\theta_{0,qu-v}(\hat{\lambda})$ because we lack information about the $w^{(i)}$. Instead we proceed as follows.

Fix $w^{(0)}, \ldots, w^{(a-1)} \in NA$ with

\[
\sum_{i=0}^{a-1} p^i w^{(i)} = qu - v.
\]
We construct inductively from \( \{ w^{(i)} \}_{i=0}^{a-1} \) a related sequence \( \{ \tilde{w}^{(i)} \}_{i=0}^{a} \subseteq U^I \) such that

\[
(3.17) \quad w^{(i)} = p \tilde{w}^{(i+1)} - \tilde{w}^{(i)} \quad \text{for } i = 0, \ldots, a - 1.
\]

First of all, take \( \tilde{w}^{(0)} = v \). Eq. (3.16) shows that \( w^{(0)} + \tilde{w}^{(0)} = p \tilde{w}^{(1)} \) for some \( \tilde{w}^{(1)} \in \mathbb{Z}^{n+2} \); since \( w^{(0)} \in \mathbb{N}A \) and \( \tilde{w}^{(0)} \in U^I \) we conclude that \( \tilde{w}^{(1)} \in U^I \). Suppose that for some \( 0 < k \leq a - 1 \) we have defined \( \tilde{w}^{(0)}, \ldots, \tilde{w}^{(k)} \in U^I \) satisfying (3.17) for \( i = 0, \ldots, k - 1 \). Substituting \( pw^{(i+1)} - \tilde{w}^{(i)} \) for \( w^{(i)} \) for \( i = 0, \ldots, k - 1 \) in (3.16) gives

\[
(3.18) \quad -\tilde{w}^{(0)} + p^k \tilde{w}^{(k)} + \sum_{i=k}^{a-1} p^i w^{(i)} = p^a u - v.
\]

Since \( \tilde{w}^{(0)} = v \), we can divide this equation by \( p^k \) to get \( \tilde{w}^{(k)} + w^{(k)} = p \tilde{w}^{(k+1)} \) for some \( \tilde{w}^{(k+1)} \in \mathbb{Z}^{n+2} \). Since \( w^{(k)} \in \mathbb{N}A \) and (by induction) \( \tilde{w}^{(k)} \in U^I \), we conclude that \( \tilde{w}^{(k+1)} \in U^I \). This completes the inductive construction. Note that in the special case \( k = a - 1 \), this computation gives \( \tilde{w}^{(a)} = u \).

Summing Eq. (3.17) over \( i = 0, \ldots, a - 1 \) and using \( \tilde{w}^{(0)} = v \), \( \tilde{w}^{(a)} = u \), gives

\[
(3.19) \quad \sum_{i=0}^{a-1} w^{(i)} = pu - v + (p - 1) \sum_{i=1}^{a-1} \tilde{w}^{(i)},
\]

hence

\[
(3.20) \quad \sum_{i=0}^{a-1} w^{(i)}_{n+1} - v_{n+1} = pu_{n+1} - v_{n+1} + \sum_{i=1}^{a-1} \tilde{w}^{(i)}_{n+1}.
\]

For \( w^{(0)}, \ldots, w^{(a-1)} \) as in (3.16), we thus get from (3.15)

\[
(3.21) \quad \ord_p \left( \prod_{i=0}^{a-1} \theta_{w^{(i)}}(\hat{x}^{p^i}) \right) \geq \frac{pu_{n+1} - v_{n+1}}{p - 1} + \sum_{i=1}^{a-1} \tilde{w}^{(i)}_{n+1}.
\]

Since \( \tilde{w}^{(i)} \in U^I \), we have

\[
(3.22) \quad \tilde{w}^{(i)}_{n+1} = \mu_I + 1 \quad \text{if} \quad \tilde{w}^{(i)} \in U^I_{\min} \quad \text{and} \quad \tilde{w}^{(i)}_{n+1} \geq \mu_I + 2 \quad \text{if} \quad \tilde{w}^{(i)} \not\in U^I_{\min}.
\]

From (3.21) and (3.22) we get the following result.

**Lemma 3.23.** For \( u, v \in U^I \) and \( w^{(0)}, \ldots, w^{(a-1)} \) as in (3.16), we have

\[
(3.24) \quad \ord_p \left( \prod_{i=0}^{a-1} \theta_{w^{(i)}}(\hat{x}^{p^i}) \right) \geq \frac{pu_{n+1} - v_{n+1}}{p - 1} + (a - 1)(\mu_I + 1).
\]
Furthermore, if any of the terms $\tilde{w}^{(1)}, \ldots, \tilde{w}^{(a-1)}$ of the associated sequence satisfying (3.17) is not contained in $U^I_{\text{min}}$, then

$$\text{ord}_p \left( \prod_{i=0}^{a-1} \theta_{w^{(i)}}(\hat{\lambda}^p) \right) \geq \frac{p u_{n+1} - v_{n+1}}{p - 1} + (a - 1)(\mu_I + 1) + 1.$$  

Our desired estimate for $\theta_{0, qu-v}(\hat{\lambda})$ now follows from (3.14).

**Corollary 3.26.** For $u, v \in U^I$ we have

$$\text{ord}_p \theta_{0, qu-v}(\hat{\lambda}) \geq \frac{p u_{n+1} - v_{n+1}}{p - 1} + (a - 1)(\mu_I + 1).$$

4. The Action of $\alpha_{\hat{\lambda}}$ on $L^I_0$

In this section, we use Corollary 3.26 to study the action of $\alpha_{\hat{\lambda}}$ on $L^I_0$. From (3.7) and the formula of Serre\[8, Proposition 7\] we have

$$\text{det}(I - t \alpha_{\hat{\lambda}} | L^I_0) = \sum_{m=0}^{\infty} a^I_m t^m,$$

where

$$a^I_m = (-1)^m \sum_{U_m \subseteq U^I} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{u \in U_m} \theta_{0, qu-\sigma(u)}(\hat{\lambda}) \in \mathbb{Q}_p(\zeta_{q-1}, \gamma_0),$$

the outer sum is over all subsets $U_m \subseteq U^I$ of cardinality $m$, and $S_m$ is the group of permutations on $m$ objects.

**Proposition 4.3.** The coefficient $a^I_m$ is divisible by $q^{m(\mu_I + 1)}$ and satisfies the congruence

$$a^I_m \equiv (-1)^m \sum_{U_m \subseteq U^I_{\text{min}}} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{u \in U_m} \theta_{0, qu-\sigma(u)}(\hat{\lambda}) \pmod{pq^{m(\mu_I + 1)}}.$$

In particular, $a^I_m \equiv 0 \pmod{pq^{m(\mu_I + 1)}}$ if $m > |U^I_{\text{min}}|$.

**Proof.** If $U_m \subseteq U^I$ is a subset of cardinality $m$ and $\sigma$ is a permutation of $U_m$, then by (3.27)

$$\text{ord}_p \left( \prod_{u \in U_m} \theta_{0, qu-\sigma(u)}(\hat{\lambda}) \right) \geq m(a - 1)(\mu_I + 1) + \sum_{u \in U_m} \frac{p u_{n+1} - \sigma(u)_{n+1}}{p - 1}$$

$$= m(a - 1)(\mu_I + 1) + \sum_{u \in U_m} u_{n+1}$$

$$\geq ma(\mu_I + 1)$$
since \( u \in U_m \) implies \( u_{n+1} \geq \mu_I + 1 \). It follows from (4.2) that \( a_{m,n}^{(\mu_I+1)} \) is divisible by \( q^{\mu_I+1} \). Furthermore, \( u_{n+1} \geq \mu_I + 2 \) if \( u \not\in U_{\min}^I \), so

\[
\text{ord}_p \left( \prod_{u \in U_m} \theta_{0,qu-\sigma(u)}(\hat{\lambda}) \right) \geq m(a_{I} + 1) + 1 \quad \text{if} \quad U_m \not\subseteq U_{\min}^I.
\]

The congruence (4.4) now follows from (4.2). \( \square \)

As an immediate corollary of Proposition 4.3, we have the following result.

**Corollary 4.5.** The reciprocal roots of \( \det(I-t\alpha_\hat{\lambda} | L_0^I) \) are all divisible by \( q^{\mu_I+1} \).

Corollary 4.5 allows us to analyze the terms on the right-hand side of (3.5).

**Proposition 4.6.** The reciprocal roots of \( \det(I-q^{n+1-|I|}t\alpha_\hat{\lambda} | L_0^I) \) are divisible by \( q^{\mu_I+2} \) unless either \( |I| = n + 1 \) or \( |I| = n \) and \( n \) is divisible by \( d \), in which case they are divisible by \( q^{\mu_I+1} \).

**Proof.** Corollary 4.5 and the definition of \( \mu_I \) imply that the reciprocal roots of \( \det(I-q^{n+1-|I|}t\alpha_\hat{\lambda} | L_0^I) \) are divisible by \( q \) to the power

\[
(4.7) \quad n + 1 - |I| + \left\lceil \frac{|I|}{d} \right\rceil.
\]

If \( |I| = n + 1 \), this reduces to \( \mu + 1 \). Suppose \( |I| = n \). From the definition of \( \mu \) we have \( n = \mu d + r \) with \( 0 \leq r \leq d - 1 \). The expression (4.7) then reduces to \( 1 + \left\lceil (\mu d + r)/d \right\rceil \), which equals \( \mu + 2 \) if \( r > 0 \) and equals \( \mu + 1 \) if \( r = 0 \).

If \( |I| = n - 1 \), expression (4.7) reduces to \( 2 + \left\lceil (\mu d + r - 1)/d \right\rceil \), which equals \( \mu + 3 \) if \( r > 1 \) and equals \( \mu + 2 \) if \( r = 0, 1 \). Finally, note that expression (4.7) cannot decrease when \( |I| \) decreases, so expression (4.7) will be \( \geq \mu + 2 \) for \( |I| < n - 1 \). \( \square \)

From (3.5) and Proposition 4.6 we get the following result.

**Proposition 4.8.** If \( n \) is not divisible by \( d \), then

\[
(4.9) \quad P_\lambda(q^{-\mu}t) \equiv \det(I-q^{-\mu-1}t\alpha_\hat{\lambda} | L_0^S) \pmod{q}.
\]

If \( n \) is divisible by \( d \), then

\[
(4.10) \quad P_\lambda(q^{-\mu}t) \equiv \frac{\det(I-q^{-\mu-1}t\alpha_\hat{\lambda} | L_0^S)}{\prod_{i=0}^{n} \det(I-q^{-\mu}t\alpha_\hat{\lambda} | L_0^{S\{i}\}} \pmod{q}.
\]
5. Proof of Theorem 1.4

It follows from (4.9) that \( R_\lambda(q^{-\mu}t) \equiv 1 \pmod{q} \) if \( n \) is not divisible by \( d \). To establish Theorem 1.4, it remains to prove the congruence for \( Q_\lambda(q^{-\mu}t) \).

Consider the matrix

\[
B^I(\hat{\lambda}) = [B^I_{uv}(\hat{\lambda})]_{u,v \in U^I_{\min}}
\]

defined by

\[
B^I_{uv}(\hat{\lambda}) = \theta_{0,qu-v}(\hat{\lambda}).
\]

We have

\[
\det \left( I - tB^I(\hat{\lambda}) \right) = \sum_{m=0}^{[U^I_{\min}]} b^I_m t^m,
\]

where

\[
b^I_m = (-1)^m \sum_{u_m \subseteq U^I_{\min}} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{u \in U_m} \theta_{0,qu-\sigma(u)}(\hat{\lambda}).
\]

Proposition 4.3 implies \( b^I_m \equiv 0 \pmod{q^{m(\mu_I+1)}} \) and

\[
b^I_m \equiv a^I_m \pmod{pq^{m(\mu_I+1)}},
\]

which establishes the following result.

**Proposition 5.3.** We have

\[
\det (I - q^{-\mu}tB^I(\hat{\lambda})) \equiv \det (I - q^{-\mu}tB^S(\hat{\lambda})) \pmod{p}.
\]

Combined with Proposition 4.8, this gives the following congruences.

**Corollary 5.4.** If \( n \) is not divisible by \( d \), then

\[
P_\lambda(q^{-\mu}t) \equiv \det (I - q^{-\mu}tB^S(\hat{\lambda})) \pmod{p}.
\]

If \( n \) is divisible by \( d \), then

\[
P_\lambda(q^{-\mu}t) \equiv \frac{\det (I - q^{-\mu}tB^S(\hat{\lambda}))}{\prod_{i=0}^{d-1} \det (I - q^{-\mu}tB^S(\hat{\lambda}))} \pmod{p}.
\]

To simplify (5.5) and (5.6) further, we restate Lemma 3.23 in the special case where \( u, v \in U^I_{\min} \), i.e., \( u_{n+1} = v_{n+1} = \mu_I + 1 \).

**Lemma 5.7.** For \( u, v \in U^I_{\min} \) and \( w^{(0)}, \ldots, w^{(a-1)} \) as in (3.16), we have

\[
\text{ord}_p \left( \prod_{i=0}^{a-1} \theta_{w^{(i)}}(\hat{\lambda}^{p^i}) \right) \geq a(\mu_I + 1).
\]
Furthermore, if any of the terms $\tilde{w}^{(1)}, \ldots, \tilde{w}^{(a-1)}$ of the associated sequence satisfying (3.17) is not contained in $U_{\text{min}}$, then

$$\text{ord}_p \left( \prod_{i=0}^{a-1} \theta_{w^{(i)}}(\hat{\lambda}^{p^i}) \right) \geq a(\mu_I + 1) + 1.$$  

Applying Lemma 5.7 to Equations (3.14) and (3.17) gives the following congruence: for $u, v \in U_{\text{min}}$,

$$\theta_{0,qu-v}(\hat{\lambda}) \equiv \sum_{\tilde{w}^{(1)}, \ldots, \tilde{w}^{(a-1)} \in U_{\text{min}}} \prod_{i=0}^{a-1} \theta_{p\tilde{w}^{(i+1)}-\tilde{w}^{(i)}}(\hat{\lambda}^{p^i}) \pmod{pq^{\mu_I+1}},$$

where $\tilde{w}^{(0)} = v$ and $\tilde{w}^{(a)} = u$.

Let $C_I^I(\hat{\lambda}) = [C_{uv}^I(\hat{\lambda})]_{u,v \in U_{\text{min}}}$ be the matrix

$$C_{uv}^I(\hat{\lambda}) = \theta_{pu-v}(\hat{\lambda}).$$

It is straightforward to check by induction on $a$ that the right-hand side of (5.10) is the $(u,v)$-entry in the matrix product

$$C^I(\hat{\lambda}^{p^a-1})C^I(\hat{\lambda}^{p^{a-2}}) \cdots C^I(\hat{\lambda}),$$

i.e., (5.10) implies the matrix congruence

$$B^I(\hat{\lambda}) \equiv C^I(\hat{\lambda}^{p^a-1})C^I(\hat{\lambda}^{p^{a-2}}) \cdots C^I(\hat{\lambda}) \pmod{pq^{\mu_I+1}}.$$

We make explicit the matrix $C^I(\hat{\lambda})$. Let $u, v \in U_{\text{min}}$. From (3.9) and (3.10) we have

$$\theta_{pu-v}(\hat{\lambda}) = \sum_{\nu \in [N]^N} \left( \prod_{j=1}^{N} \theta_{\nu_j} \right)^{\hat{\lambda}^{v_j}}.$$

Since $u, v \in U_{\text{min}}$, Lemma 2.1 implies that $\nu_j \leq p - 1$ for all $j$ in the sum on the right-hand side of (5.12). It then follows from the definition of $\theta(t)$ that $\theta_{\nu_j} = \gamma_{0}^{\nu_j}/\nu_j!$. Examining the last coordinate of the equation $\sum_{j=1}^{N} \nu_j a_j^+ = pu - v$ gives $\sum_{j=1}^{N} \nu_j = (p - 1)(\mu_I + 1)$, so (5.12) can be simplified to

$$\theta_{pu-v}(\hat{\lambda}) = \gamma_{0}^{(p-1)(\mu_I + 1)} \sum_{\nu \in [N]^N} \frac{\hat{\lambda}^{v}}{\nu_1! \cdots \nu_N!},$$

where $\nu_j = a_j^+ = pu - v$.
Using (2.3), we obtain the relation between the matrices $C^I(\hat{\lambda})$ and $A^I(\Lambda)$:

$$(5.14) \quad C^I(\hat{\lambda}) = (-\gamma_0^{p-1})^{\mu I + 1} A^I(\hat{\lambda}).$$

From (5.11), we then obtain a relation between $A^I(\hat{\lambda})$ and $B^I(\hat{\lambda})$:

$$(5.15) \quad B^I(\hat{\lambda}) \equiv (-\gamma_0^{p-1})^{a(\mu I + 1)} A^I(\hat{\lambda}^{p^{a-1}}) A^I(\hat{\lambda}^{p^{a-2}}) \cdots A^I(\hat{\lambda}) \pmod{pq^{\mu I + 1}}.$$

Recall that $\sum_{i=0}^{\infty} \gamma_0^{p^i} / p^i = 0$ and that $\text{ord}_p \gamma_0 = 1/p - 1$. In particular,

$$\text{ord}_p \frac{\gamma_0^{p^i}}{p^i} = \frac{p^i}{p - 1} - i,$$

and since the right-hand side of this expression is an increasing function of $i$ for $i \geq 1$ we have

$$\gamma_0 + \frac{\gamma_0^p}{p} \equiv 0 \pmod{\gamma_0 p^{p-1}}.$$

Multiplying by $p/\gamma_0$ then gives

$$-\gamma_0^{p-1} \equiv p \pmod{p^3}.$$

It follows from this that

$$(-\gamma_0^{p-1})^{a(\mu I + 1)} \equiv q^{\mu I + 1} \pmod{pq^{\mu I + 1}},$$

so (5.15) may be simplified to

$$(5.16) \quad B^I(\hat{\lambda}) \equiv q^{\mu I + 1} A^I(\hat{\lambda}^{p^{a-1}}) A^I(\hat{\lambda}^{p^{a-2}}) \cdots A^I(\hat{\lambda}) \pmod{pq^{\mu I + 1}}.$$

Corollary 5.4 now implies the following congruences.

**Theorem 5.17.** If $n$ is not divisible by $d$, then

$$(5.18) \quad P_\lambda(q^{-\mu}t) \equiv \det (I - tA^S(\hat{\lambda}^{p^{a-1}}) A^S(\hat{\lambda}^{p^{a-2}}) \cdots A^S(\hat{\lambda})) \pmod{p}.$$

If $n$ is divisible by $d$, then

$$(5.19) \quad P_\lambda(q^{-\mu}t) \equiv \frac{\det (I - tA^S(\hat{\lambda}^{p^{a-1}}) A^S(\hat{\lambda}^{p^{a-2}}) \cdots A^S(\hat{\lambda}))}{\prod_{i=0}^{n} \det (I - tA^S(\hat{\lambda}^{p^{a-1}}) A^S(\hat{\lambda}^{p^{a-2}}) \cdots A^S(\hat{\lambda}))} \pmod{p}.$$

Since $A^S(\Lambda)$ is the matrix denoted $A(\Lambda)$ in the Introduction and since $\hat{\lambda}$ is the Teichmüller lifting of $\lambda$ we have $A^S(\hat{\lambda}) \equiv A^S(\lambda) \pmod{p}$. Theorem 1.4 now follows from (5.18), and (5.19) is equivalent to

$$(5.20) \quad P_\lambda(q^{-\mu}t) \equiv \frac{\det (I - tA^S(\lambda^{p^{a-1}}) A^S(\lambda^{p^{a-2}}) \cdots A^S(\lambda))}{\prod_{i=0}^{n} \det (I - tA^S(\lambda^{p^{a-1}}) A^S(\lambda^{p^{a-2}}) \cdots A^S(\lambda))} \pmod{p}.$$
6. The case \( d \mid n \)

We first give an example to show that Theorem 1.4 fails when \( d \mid n \). Consider the variety \( X \) in \( \mathbb{P}^n \) defined by the equation \( x_1 \cdots x_n = 0 \) (so we have \( d = n \) and \( \mu = 1 \)). A short calculation shows that its zeta function \( Z(X, t) \) has the form (1.2) with

\[
P(t)^{(1-n)} = \prod_{j=1}^{n-1} (1 - q^j t)^{(1-n)/(j-1)},
\]

In particular, we have \( R(q^{-1} t) \equiv (1 - t) \ (\text{mod } q) \), contradicting Theorem 1.4.

An elementary observation will allow us to simplify the denominator of (5.20). Assume that \( d \mid n \) and \( i \in \{0, 1, \ldots, n\} \). Then

\[
\mu_{S \setminus \{i\}} + 1 = \left\lceil \frac{n}{d} \right\rceil = \mu.
\]

This implies that \( U_{\text{min}}^{S \setminus \{i\}} \) is a singleton: \( U_{\text{min}}^{S \setminus \{i\}} = \{u^{(i)}\} \), where

\[
u^{(i)} = (1, \ldots, 1, 0, 1, \ldots, 1, \mu) \in \mathbb{N}^{n+2}
\]

with zero in the \( i \)-th entry. It follows that if we define a polynomial \( g_i(\Lambda) \in (\mathbb{Q} \cap \mathbb{Z}_p)[\Lambda] \) by the formula

\[
g_i(\Lambda) = (-1)^\mu \sum_{\nu \in \mathbb{N}^N \atop \sum_{j=1}^N \nu_j a_j^{(p-1)u^{(i)}}} \frac{\Lambda_1^{\nu_1} \cdots \Lambda_N^{\nu_N}}{\nu_1! \cdots \nu_N!}.
\]

then by (2.3), \( A^{S \setminus \{i\}}(\Lambda) \) is a one-by-one matrix with entry \( g_i(\Lambda) \). From (5.20) we then have the following result.

**Theorem 6.2.** If \( d \mid n \), then

\[
P_\lambda(q^{-\mu} t) \equiv \frac{\det (I - t \bar{A}^S(\lambda^{p^{a-1}}) \bar{A}^S(\lambda^{p^{a-2}}) \cdots \bar{A}^S(\lambda))}{\prod_{i=0}^{n} (1 - t \bar{g}_i(\lambda^{p^{a-1}}) \bar{g}_i(\lambda^{p^{a-2}}) \cdots \bar{g}_i(\lambda))} \ (\text{mod } p).
\]

7. Generic invertibility of \( \bar{A}^I(\Lambda) \)

The proof of generic invertibility follows the lines of our recent proof of generic invertibility for the Hasse-Witt matrix\cite{2}. We fix \( I \) and prove that the matrix \( \bar{A}^I(\Lambda) \) is generically invertible, in a sense made precise below.

We consider the following condition on the set \( \{a_j\}_{j=1}^N \). Suppose that \( N \geq \mu_I + |U_{\text{min}}^I| \) and that the elements \( \{a_j\}_{j=1}^{\mu_I} \) and \( \{a_j\}_{j=\mu_I+1}^{\mu_I+|U_{\text{min}}^I|} \) have the following property:
Hypothesis 7.1. For each \( u \in U^I_{\text{min}} \), there exists (a necessarily unique) \( k_u, 1 \leq k_u \leq |U^I_{\text{min}}| \), such that \( x^u = x^{a^I_{\mu_I+k_u}} \prod_{j=1}^{\mu_I} x^{a^I_j} \).

We show that such subsets always exists when, for example, \( \{ x^{a^I_j} \}_{j=1}^N \) consists of all monomials of degree \( d \). To fix ideas and simplify notation, suppose that \( |I| = h, 0 \leq h \leq n+1 \) and that \( I = \{0, 1, \ldots, h-1\} \). For \( j = 1, \ldots, \mu_I \) we take \( x^{a^I_j} = \prod_{d=1}^{j-1} x^a \) and we take \( \{ x^{a^I_j} \}_{j=\mu_I+1}^{\mu_I+|U^I_{\text{min}}|} \) to consist of all monomials of degree \( d \) that are divisible by the product \( x_{\mu_I d} \cdots x_{h-1} \). Then Hypothesis 7.1 is satisfied.

Theorem 7.2. If \( \{a^I_j\}_{j=1}^N \) satisfies Hypothesis 7.1, then the matrix \( \bar{A}^I(\Lambda) \) is invertible.

We begin with a reduction step. Define a related matrix \( D^I(\Lambda) = [D^I_{uv}(\Lambda)]_{u,v \in U^I_{\text{min}}} \) by setting
\[
D^I_{uv}(\Lambda) = \left( \sum_{j=1}^{\mu_I} \Lambda_j \right)^{-(p-1)} \Lambda_{\mu_I+k_u} A^I_{uv}(\Lambda) \in
\Lambda^{-1}_{\mu_I+k_u} A^I_{\mu_I+k_u} (\mathbb{Q} \cap \mathbb{Z}_p) [\Lambda^{-1}_1, \ldots, \Lambda^{-1}_{\mu_I}, \Lambda^{-1}_{\mu_I+k_u}, \Lambda_{\mu_I+2}, \ldots, \Lambda_{\mu_I+k_u}, \ldots, \Lambda_N],
\]
where \( A^I_{uv}(\Lambda) \) is given by (2.3). In other words, we obtain \( D^I(\Lambda) \) by multiplying row \( u \) of \( A^I(\Lambda) \) by \( \left( \Lambda_{\mu_I+k_u} \prod_{j=1}^{\mu_I} \Lambda_j \right)^{-(p-1)} \) and multiplying column \( v \) of \( A^I(\Lambda) \) by \( \Lambda_{\mu_I+k_u} \prod_{j=1}^{\mu_I} \Lambda_j \). It follows that
\[
\det D^I(\Lambda) = \left( \sum_{j=1}^{\mu_I} \Lambda_j \right)^{-(p-1)|U^I_{\text{min}}|} \left( \sum_{j=1}^{\mu_I} \Lambda_{\mu_I+j} \right)^{-(p-1)} \det A^I(\Lambda),
\]
hence to prove Theorem 7.2 it suffices to prove that
\[
(7.4) \quad \det D^I(\Lambda) \neq 0.
\]

For \( u \in U^I_{\text{min}} \), put
\[
L_u = \left\{ l = (l_1, \ldots, l_N) \in \mathbb{Z}^N \mid \sum_{k=1}^N l_k a^+_k = 0, \right. \\
l_j \leq 0 \text{ for } j = 1, \ldots, \mu_I \text{ and } j = \mu_I + k_u, l_j \geq 0 \text{ otherwise} \right\}.
\]

Lemma 7.5. If the monomial \( \Lambda^I \) appears in the Laurent polynomial \( D^I_{uv}(\Lambda) \), then \( l \in L_u \).
Proof. It follows from Hypothesis 7.1 that
\[ u = a_{\mu_I+k_u}^+ + \sum_{j=1}^{\mu_I} a_j^+ \quad \text{and} \quad v = a_{\mu_I+k_v}^+ + \sum_{j=1}^{\mu_I} a_j^+. \]
The formula for \( A_{uv}^I(\Lambda) \) is given in (2.3), where each exponent \( \nu_j \) satisfies \( 0 \leq \nu_j \leq p-1 \) by Lemma 2.1. The assertion of Lemma 7.5 now follows from the definition of \( D_{uv}^I(\Lambda) \).

Lemma 7.6. The Laurent polynomial \( D_{uv}^I(\Lambda) \) has no constant term if \( u \neq v \) and has constant term equal to \( (- (p-1)!)^{-(\mu_I+1)} \) if \( u = v \).

Proof. If \( u \neq v \), the definition of \( D_{uv}^I(\Lambda) \) shows that every monomial in \( D_{uv}^I(\Lambda) \) contains a negative power of \( \Lambda_{\mu_I+k_u} \) and a positive power of \( \Lambda_{\mu_I+k_v} \). If \( u = v \), then
\[ pu - v = (p-1)u = (p-1) \left( a_{\mu_I+k_u}^+ + \sum_{j=1}^{\mu_I} a_j^+ \right), \]
so (2.3) shows that the monomial
\[ (-1)^{\mu_I+1} \frac{A_{\mu_I+k_u}^I \prod_{j=1}^{\mu_I} \Lambda_j^{p-1}}{(p-1)!^{\mu_I+1}} \]
appears in \( A_{uv}^I(\Lambda) \). The assertion of the lemma then follows from the definition of \( D_{uv}^I(\Lambda) \).

We shall prove (7.4) by showing that \( \det \bar{D}^I(\Lambda) \) has a nonzero constant term. In fact, we shall show that the constant term of \( \det D^I(\Lambda) \) is a \( p \)-adic unit. By Lemma 7.5 and the following proposition, the constant term of \( \det D^I(\Lambda) \) is the determinant of the matrix whose \( (u, v) \)-entry is the constant term of \( D_{uv}^I(\Lambda) \). And by Lemma 7.6, this matrix of constant terms is a diagonal matrix whose diagonal entries are \( p \)-adic units. Thus the following proposition completes the proof of (7.4).

Proposition 7.7. Let \( l^{(u)} = (l_1^{(u)}, \ldots, l_N^{(u)}) \in L_u \) for \( u \in U_{\min}^I \). One has
\[ \sum_{u \in U_{\min}^I} l^{(u)} = 0 \]
if and only if \( l^{(u)} = 0 \) for all \( u \in U_{\min}^I \).
Proof. The "if" part of the proposition is clear, so consider a set \( \{ \ell(u) \}_{u \in U_{\min}^I} \) with \( \ell(u) \in L_u \) satisfying (7.8). Since \( \ell(u) \in L_u \) we have

\[
\sum_{k=1}^{N} \ell_k^u \mathbf{a}_k^+ = \mathbf{0},
\]

which implies (since the last coordinate of each \( \mathbf{a}_k^+ \) equals 1)

\[
\sum_{k=1}^{N} \ell_k^u = 0.
\]

From the definition of \( L_u \), we have \( \ell_j^u \leq 0 \) for all \( u \) if \( j \leq \mu_I \) and \( \ell_j^u \geq 0 \) for all \( u \) if \( j \geq \mu_I + |U_{\min}^I| + 1 \), so (7.8) implies that

\[
\ell_j^u = 0 \text{ for all } u \text{ and all } j \leq \mu_I
\]

and

\[
\ell_j^u = 0 \text{ for all } u \text{ and all } j \geq \mu_I + |U_{\min}^I| + 1.
\]

To establish the proposition, it remains to show that \( \ell_j^u = 0 \) for all \( u \) if \( \mu_I + 1 \leq j \leq \mu_I + |U_{\min}^I| \).

Using (7.11) and (7.12), Equations (7.9) and (7.10) become

\[
\sum_{k=1}^{|U_{\min}^I|} \ell_{\mu_I+k}^u \mathbf{a}_k^+ = \mathbf{0}
\]

and

\[
\sum_{k=1}^{|U_{\min}^I|} \ell_{\mu_I+k}^u = 0
\]

for all \( u \). Since \( \ell(u) \in L_u \), we have \( \ell_{\mu_I+k_u}^u \leq 0 \) and \( \ell_{\mu_I+k_u}^u \geq 0 \) if \( k \neq k_u \).

If \( \ell_{\mu_I+k_u}^u = 0 \), then (7.14) implies that \( \ell_{\mu_I+k_u}^u = 0 \) for \( k \neq k_u \) as well, so \( \ell(u) = \mathbf{0} \).

If follows that if \( \ell(u) \neq \mathbf{0} \), then \( \ell_{\mu_I+k_u}^u < 0 \) so we can solve (7.13) for \( \mathbf{a}_k^+ \):

\[
\mathbf{a}_k^+ = \sum_{k=1}^{|U_{\min}^I|} \left( -\frac{\ell_{\mu_I+k_u}^u}{\ell_{\mu_I+k_u}^u} \right) \mathbf{a}_k^+.
\]

The coefficients on the right-hand side of (7.15) are nonnegative rational numbers that sum to 1, so (7.15) implies the following statement.
Lemma 7.16. If \( l^{(u)} \neq 0 \), then \( a^+_k \) is an interior point of the convex hull of the set \( \{a^+_k \mid l^{(u)}_{\mu_j+k} > 0 \} \).

Let \( Z = \{a^+_k \mid l^{(u)}_{\mu_j+k} \neq 0 \text{ for some } u \in U^I_{\text{min}} \} \). If \( a^+_k \notin Z \), then \( l^{(u)}_{\mu_j+k} = 0 \) for all \( u \in U^I_{\text{min}} \), so if \( Z = \emptyset \) then \( l^{(u)}_{\mu_j+k} = 0 \) for all \( k, 1 \leq k \leq |U^I_{\text{min}}| \), and all \( u \in U^I_{\text{min}} \), which establishes the proposition. So suppose \( Z \neq \emptyset \) and choose \( k_0, 1 \leq k_0 \leq |U^I_{\text{min}}| \), such that \( a^+_k \) is a vertex of the convex hull of \( Z \). If \( l^{(u)} \neq 0 \), then Lemma 7.16 implies that \( k_0 \neq k_u \). By the definition of \( L_u \), we therefore have \( l^{(u)}_{\mu_j+k_0} \geq 0 \) for all \( u \in U^I_{\text{min}} \).

Furthermore, since \( a^+_k \in Z \), we must have \( l^{(u)}_{\mu_j+k_0} > 0 \) for some \( u \in U^I_{\text{min}} \).

It follows that \( \sum_{u \in U^I_{\text{min}}} l^{(u)}_{\mu_j+k_0} > 0 \), contradicting (7.8). Thus we must have \( Z = \emptyset \). \( \square \)

References

[1] A. Adolphson and S. Sperber. Hasse invariants and mod \( p \) solutions of \( A \)-hypergeometric systems. J. Number Theory 142 (2014), 183–210.

[2] A. Adolphson and S. Sperber. \( A \)-hypergeometric series and the Hasse-Witt matrix of a hypersurface. Finite Fields and Their Applications 41 (2016), 55–63.

[3] A. Adolphson and S. Sperber. Distinguished-root formulas for generalized Calabi-Yau hypersurfaces. Preprint, available at arXiv:1602.03378.

[4] J. Ax. Zeroes of polynomials over finite fields. Amer. J. Math. 86 (1964), 255–261.

[5] N. Katz. On a theorem of Ax. Amer. J. Math. 93 (1971), 485–499.

[6] N. Katz. Algebraic solutions of differential equations (\( p \)-curvature and the Hodge filtration). Invent. Math. 18 (1972), 1–118.

[7] N. Katz. Une formule de congruence pour la fonction \( \zeta \). Exposée XXII in S. G. A. 7 II, Lecture Notes in Mathematics 340, Springer, Berlin-Heidelberg-New York, 1973.

[8] J.-P. Serre. Endomorphismes complètement continus des espaces de Banach \( p \)-adiques. Inst. Hautes Études Sci. Publ. Math. No. 12 (1962), 69–85.

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