DEGENERATION OF HODGE STRUCTURES OVER PICARD MODULAR SURFACES

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ABSTRACT. We study variations of Hodge structures over a Picard modular surface, and compute the weights and types of their degenerations through the cusps of the Baily-Borel compactification. The main tool is a theorem of Burgos and Wildeshaus.

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1. INTRODUCTION

The aim of the present work is to compute the weight of degeneration of a variation of Hodge structures through the cusps of a Picard modular surface. Such surfaces are PEL Shimura varieties; they are moduli spaces of abelian varieties of dimension 3 endowed with an action of an order of a quadratic imaginary field, and with some other additional structures.

Fix a Picard modular surface $S$, let $S^*$ be its Baily-Borel compactification,

$$j : S \hookrightarrow S^*$$

the canonical open immersion and

$$i : \text{pt} \hookrightarrow S^*$$

the inclusion of one fixed point of the boundary. Consider the "canonical construction" functor

$$\mu_S : \text{Rep}_{G,\mathbb{Q}} \longrightarrow \text{VHS}(S)_{\mathbb{Q}},$$

from the $\mathbb{Q}$-representations of the group $G$ of the Shimura datum underlying $S$ to (admissible) variations of $\mathbb{Q}$-Hodge structures over $S$ ([Pi90 1.18], see also [BW04 §2]).

The aim of the paper is to compute the weight of the $\mathbb{Q}$-Hodge structure

$$i^* R^k j_* \mu_S(F),$$

where
for all integers $k$ and all $G$-representations $F$. As an example, consider the universal abelian scheme $A$ over $S$ (which is of relative dimension 3), and take the $r$-fold fiber product:

$$f: A^r \to S.$$ 

Then the relative cohomology $R^p f_* \mathbb{Q}_A^r$ is a variation of Hodge structures over $S$ which belongs to the image of the functor $\mu_S$ (for all non-negative integers $p$ and $r$).

**Theorem 1.1.** If $p > 6r$ then the sheaf $R^p f_* \mathbb{Q}_A^r$ vanishes. For $p \leq 6r$ the following holds:

1. the $\mathbb{Q}$-Hodge structure $i^* R^0 j_* R^p f_* \mathbb{Q}_A^r$ has weight $\{p - j\}_{r \leq j \leq c_r}$,
2. the $\mathbb{Q}$-Hodge structure $i^* R^1 j_* R^p f_* \mathbb{Q}_A^r$ has weight $\{p + 1 - j\}_{0 \leq j \leq M_p}$,
3. the $\mathbb{Q}$-Hodge structure $i^* R^2 j_* R^p f_* \mathbb{Q}_A^r$ has weight $\{p + 3 + j\}_{0 \leq j \leq M_p}$,
4. the $\mathbb{Q}$-Hodge structure $i^* R^3 j_* R^p f_* \mathbb{Q}_A^r$ has weight $\{p + 4 + j\}_{c_p \leq j \leq C_r}$,
5. the $\mathbb{Q}$-Hodge structure $i^* R^k j_* R^p f_* \mathbb{Q}_A^r$ vanishes for $k \geq 4$.

where $c_p = 1$ if $p = 1, 6p - 1$ and 0 otherwise, $C_p = \min\{p, 2r, 6r - p\}$ and $M_p = p$ if $p \leq r$, $M_p = r + \left\lfloor \frac{p - 1}{2} \right\rfloor$ if $r < p \leq 3r$, $M_p = r + \left\lfloor \frac{5r - 2}{2} \right\rfloor$ if $3r < p \leq 5r$, and $M_p = 6r - p$ if $p > 5r$.

To compute the weight of a general $i^* R^k j_* \mu_S(F)$, one can extend the scalars from $\mathbb{Q}$ to $\mathbb{C}$. Then, as the functors $i^* R^k j_* \mu_S, C$ are linear, it is enough to consider irreducible representations of $G_C$.

In our case one has an isomorphism of $\mathbb{C}$-algebraic group

$$G_C \cong \text{GL}_3 \times \mathbb{G}_m,$$

so that (after having chosen a Borel and a maximal torus), the maximal weight of any irreducible representation corresponds to a list of four integers $(a, b, c, d)$ such that $a \geq b \geq c$. The following is our main result (see Theorem 4.11 in the text).

**Theorem 1.2.** Let $F_\lambda$ be the irreducible representation of $G_C$ of maximal weight $\lambda = (a, b, c, d)$. Then the following holds:

1. $i^* R^0 j_* \mu_{S,C}(F_\lambda)$ is of weight $-2a - b - 2d$,
2. $i^* R^1 j_* \mu_{S,C}(F_\lambda)$ is of weight $-a - 2b - 2d + 1$ and $-2a - c - 2d + 1$,
3. $i^* R^2 j_* \mu_{S,C}(F_\lambda)$ is of weight $-a - 2c - 2d + 3$ and $-2b - c - 2d + 3$,
4. $i^* R^3 j_* \mu_{S,C}(F_\lambda)$ is of weight $-b - 2c - 2d + 4$,
5. $i^* R^k j_* \mu_{S,C}(F_\lambda)$ vanishes for $k \geq 4$.

These computations of weights of degenerations are part of Wildeshaus’ program [Wil09], whose aim is the construction of motives associated to modular forms (generalizing the case of classical modular forms, which is due to Scholl [Sch94]). However, we do not explore this connection here; for explanations and references see Remark 1.13.

The main tool for our work is a theorem of Burgos and Wildeshaus [BW04], which works for a general Shimura variety and reduces the computation of weights of degenerations to a computation of cohomology of groups. In our case the groups are enough concrete to allow us to do all the computations explicitly.
1.3. **Organization of the paper.** In Section 2 we recall generalities about Picard surfaces and their Shimura datum. The standard reference is [Gor92].

In Section 3 we describe the boundary of Baily-Borel and toroidal compactifications, as well as the Shimura datum underlying the strata of these compactifications.

Section 4 contains the main results concerning the degeneration of Hodge structures.

1.4. **Notations and convention.** We will write \( \mathbb{R} \) for the field of real numbers, \( \mathbb{C} \) for the field of complex numbers and \( \mathbb{A}_f \) for the ring of finite adeles.

The morphism of complex conjugation will be written 
\[ z \mapsto \overline{z}. \]

We will write 
\[ \mathcal{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, \mathbb{C} \]
for the \( \mathbb{R} \)-algebraic group called Deligne’s torus. Here \( \text{Res}_{\mathbb{C}/\mathbb{R}} \) is the Weil restriction of scalars.

When \( L \supset K \) are fields and \( X \) is an object defined over \( K \) (for instance a variety or a group), we will write \( X_L \) for its base change to \( L \).

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2. **Picard datum**

In this section we recall the definition of a Picard datum \((G, X)\); the standard reference is [Gor92]. This is a Shimura datum of PEL type. We first construct the \( \mathbb{Q} \)-algebraic group \( G \), which after extending scalars becomes the direct product of a linear group by the multiplicative group (Remark 2.3).

2.1. **Notation.** Let \( E \) a quadratic imaginary field embedded in \( \mathbb{C} \), and \( n \) the only positive square-free integer such that
\[ E = \mathbb{Q}(i\sqrt{n}). \]

Let \( V = V_E \) be an \( E \)-vector space of dimension 3, and \( J \) be a hermitian form of signature \((2,1)\). In particular, one can find three orthogonal vectors \( v_1, v_2, v_3 \in V \) such that \( J(v_1, v_1) \) and \( J(v_2, v_2) \) are positive and \( J(v_3, v_3) \) is negative.

**Definition 2.2.** In the context of Notation 2.1 we define
\[ G = GU(V, J) \]
to be the \( \mathbb{Q} \)-algebraic group of the \( E \)-linear automorphisms of \( V \) which respect the hermitian form \( J \) up to scalar.

**Remark 2.3.** Let \( V_\mathbb{Q} \) be \( V \) viewed as a \( \mathbb{Q} \)-vector space. Then the algebra \( E \otimes E \) acts on the \( E \)-vector space \( V_\mathbb{Q} \otimes_{\mathbb{Q}} E \), and hence induces a decomposition
\[ V_\mathbb{Q} \otimes_{\mathbb{Q}} E = V_i \oplus V_{-i}, \]
where
\[ V_i = \{ w \in V \otimes Q E, \quad (a \otimes 1)w = (1 \otimes a)w \quad \forall a \in E \}, \]
and
\[ V_i = \{ w \in V \otimes Q E, \quad (a \otimes 1)w = (1 \otimes \pi)w \quad \forall a \in E \}. \]
Write
\[ \pi_i : V \otimes Q E \to V_i \]
for the projector whose kernel is \( V_{i-} \).
Let \( B = \{ v_1, v_2, v_3 \} \) be an \( E \)-basis of \( V \); then
\[ B_i = \{ \pi_i(v_1), \pi_i(v_2), \pi_i(v_3) \} \]
is a basis of \( V_i \). One has an isomorphism
\[ \phi_B : G_E \xrightarrow{\sim} GL_3, E \times \mathbb{G}_m, E \]
given by
\[ g \mapsto (g(v_1), \chi_g), \]
where the restriction \( g|_{V_i} \) of \( g \) to \( V_i \) is written in the basis \( B_i \) and \( \chi_g \) is the scalar such that \( J(g \cdot g) = \chi_g J(\cdot, \cdot) \).

2.4. Convention. Let \( B = \{ v_1, v_2, v_3 \} \) be an \( E \)-basis of \( V \). Then
\[ \tilde{B} = \{ v_1, i\sqrt{n}v_1, v_2, i\sqrt{n}v_2, v_3, i\sqrt{n}v_3 \} \]
is a \( \mathbb{Q} \)-basis of \( V \). For any \( \mathbb{Q} \)-algebra \( R \), and any \( R \)-point \( g \in G(R) \), we have will write
\[ g = \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & (a_{13}, b_{13}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & (a_{23}, b_{23}) \\ (a_{31}, b_{31}) & (a_{32}, b_{32}) & (a_{33}, b_{33}) \end{pmatrix}, \]
where \( a_{jk} \in R \) is the coordinate of \( g(v_k) \) with respect to \( v_j \) and \( b_{jk} \in R \) is the coordinate of \( g(v_k) \) with respect to \( i\sqrt{n}v_j \). Note that, by definition of \( G \), we have
\[ g(i\sqrt{n}v_k) = i\sqrt{n}g(v_k), \]
in particular the \( a_{jk} \in R \) and \( b_{jk} \in R \) determine \( g \).

Definition 2.5. Let \( B = \{ v_1, v_2, v_3 \} \) be an \( E \)-basis of \( V \) such that \( J(v_1, v_1) \) and \( J(v_2, v_2) \) are positive and \( J(v_3, v_3) \) is negative (see Notation 2.1). Consider the morphism of algebraic groups
\[ h_{v_1, v_2, v_3} : S \to G_R \]
given by
\[ (z_1, z_2) \mapsto \begin{pmatrix} \frac{z_1 + z_2}{2} & z_1 - z_2 \& \frac{z_1 - z_2}{2i\sqrt{n}} \\ 0 & 0 & \frac{z_1 + z_2}{2i\sqrt{n}} \\ 0 & 0 & \frac{z_1 - z_2}{2i\sqrt{n}} \end{pmatrix}, \]
where \( G \) is the group of Definition 2.2 and the morphism is written in the basis \( B \) using Convention 2.4.
We will write \( X \) for the topological space of the \( G(\mathbb{R}) \)-conjugacy class of the morphism \( h_{v_1, v_2, v_3} \).
Note that $X$ does not depend on the choice of $v_1, v_2, v_3$ and that the morphism $h_{v_1, v_2, v_3}$ only depends on the $E$-line passing through $v_3$. One can check that $h_{v_1, v_2, v_3}$ is a morphism of algebraic groups using the following computation.

**Lemma 2.6.** The formal identity
\[
\begin{pmatrix}
\frac{a+b}{2i} & -\frac{a-b}{2i} \\
\frac{a-b}{2i} & \frac{a+b}{2i}
\end{pmatrix}
\begin{pmatrix}
\frac{c+d}{2i} & -\frac{c-d}{2i} \\
\frac{c-d}{2i} & \frac{c+d}{2i}
\end{pmatrix}
= \begin{pmatrix}
\frac{ac+bd}{2i} & -\frac{ac-bd}{2i} \\
\frac{ac-bd}{2i} & \frac{ac+bd}{2i}
\end{pmatrix}
\]
holds.

**Definition 2.7.** The pair $(G, X)$ will be called a Picard datum. It is a pure Shimura datum in the sense of [Pin90, Definition 2.1].

When $K \subset G(\mathbb{A}_f)$ is a neat subgroup [Pin90, §0.5] we will write $S = Sh^K(G, X)$ for the induced Shimura variety. It is a complex¹ smooth and quasi-projective surface which we call a 'Picard modular surface'; see [Gor92] for details and proofs.

**Remark 2.8.** The Picard modular surface $S$ is the fine moduli space of polarised abelian varieties of dimension 3 endowed with an action of an order of $E$ and some additional structures (depending also on $K$). In particular, there is a universal abelian scheme $f: A \to S$; see [Gor92] for details and proofs.

### 3. Compactifications and boundary

The aim of this section is to describe the Shimura data underlying the strata of the boundary of the Baily-Borel and toroidal compactifications of a Picard modular surface $S$ (Definition 2.7).

These strata are associated to parabolic subgroups of the group $G$ introduced in Definition 2.2 (for generalities on strata of compactifications of Shimura varieties see [Pin90, Chapter 4]).

We start by describing these parabolics (3.1-3.3), then the Shimura datum associated to each stratum (3.4-3.10) and deduce the geometry of the boundary (3.12-3.14).

**Lemma 3.1.** Let $(V, J)$ be as in Notation 2.1. Then there exist infinitely many isotropic vectors in $V$.

Moreover if $\mathfrak{v}$ be any non-zero isotropic vector, then there exists a positive rational number $b$ and an isomorphism
\[
(V, J) \xrightarrow{\sim} (E^3, J_b)
\]
sending $\mathfrak{v}$ to the first vector of the canonical base of $E^3$, where $J_b$ is the hermitian form
\[
J_b = \begin{pmatrix}
0 & 0 & 1 \\
0 & b & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

¹In fact, it has a (canonical) model over the quadratic imaginary field $E$ [Gor92].
Proof. For the first part, let us diagonalize the hermitian form $J$. Then we have to look for rational solutions of an equation of the form
\[ \sum_{i=1}^{6} c_i t_i^2 = 0 \]
with $c_i$ integers, four positive and two negative. This indeed has a solution (and thus infinitely many) by [Ser77, corollaire 2, p. 77, chap. 4].

The rest is basic linear algebra. \hfill \Box

Definition 3.2. Let $D$ be an isotropic $E$-line of $V$. An $E$-basis $w_1, w_2, w_3$ of $V$ is called a parabolic basis adapted to $D$ if $w_1$ generates $D$, and the matrix representing $J$ in this basis is of the form $J_b$ for some $b$ (following notations of Lemma 3.1).

Proposition 3.3. Let $D$ be an isotropic $E$-line of $V$ (see Lemma 3.1), and define $Q_D$ to be the subgroup of $G$ stabilizing $D$. Then
\[ Q_D = G \cap \left\{ \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & (a_{13}, b_{13}) \\ (0, 0) & (a_{22}, b_{22}) & (a_{23}, b_{23}) \\ (0, 0) & (0, 0) & (a_{33}, b_{33}) \end{pmatrix} \right\}, \]
where the coordinates are written using Convention 2.4 and we are using a parabolic basis adapted to $D$ (Definition 3.2).

The unipotent radical of $Q_D$ is
\[ R_u(Q_D) = \left\{ \begin{pmatrix} (1, 0) & (-ba_{23}, bb_{23}) & (-\frac{1}{2}(a_{23}^2 + nbb_{23}), b_{13}) \\ (0, 0) & (1, 0) & (a_{23}, b_{23}) \\ (0, 0) & (0, 0) & (1, 0) \end{pmatrix} \right\}, \]
with Lie algebra
\[ \text{Lie } R_u(Q_D) = \left\{ \begin{pmatrix} (0, 0) & (-ba_{23}, bb_{23}) & (0, b_{13}) \\ (0, 0) & (0, 0) & (a_{23}, b_{23}) \\ (0, 0) & (0, 0) & (0, 0) \end{pmatrix} \right\}. \]

The torus
\[ T_{m,D} = \left\{ \begin{pmatrix} \frac{\lambda_1 + \lambda_2}{2}, \frac{\lambda_1 - \lambda_2}{2 \sqrt{n}} \\ 0, 0 \\ 0, 0 \end{pmatrix}, \begin{pmatrix} \frac{\lambda_3 + \lambda_4}{2}, \frac{\lambda_3 - \lambda_4}{2 \sqrt{n}} \\ 0, 0 \\ 0, 0 \end{pmatrix}, \begin{pmatrix} \lambda_5, \lambda_6, \lambda_7 \\ 0, 0 \\ 0, 0 \end{pmatrix} \right\} \]
is a maximal torus of $G$ defined over $\mathbb{C}$, and the torus
\[ T_D = \left\{ \mu \cdot \begin{pmatrix} (\lambda, 0) & (0, 0) & (0, 0) \\ (0, 1) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (\lambda^{-1}, 0) \end{pmatrix} \right\} \]
is a maximal split torus defined over $\mathbb{Q}$. There is only one Borel $B_D$ of $G$ such that $Q_D \supseteq B_D \supseteq T_D$ (and it is $Q_D$ itself).

Moreover $Q_D$ is a admissible parabolic of $G$ in the sense of [Pin90] Definition 4.5] and the admissible parabolics are exactly those subgroups of the form $Q_D'$ for some isotropic $E$-line $D'$ of $V$. 
Proof. Let \( w_1, w_2, w_3 \) be a parabolic basis adapted to \( D \). The group \( Q_D \) stabilizes the line \( D \) and so it has to stabilize also \( D^\perp \) the plan orthogonal to \( D \). As \( D \) is generated by \( w_1 \) and \( D^\perp \) is generated by \( w_1 \) and \( w_2 \) we deduce the description of \( Q_D \) in the statement.

The unipotent radical of \( Q_D \) is

\[
R_uQ_D = G \cap \left\{ \begin{pmatrix} 1, 0 & (a_{12}, b_{12}) & (a_{13}, b_{13}) \\ 0, 0 & (1, 0) & (a_{23}, b_{23}) \\ (0, 0) & (0, 0) & (1, 0) \end{pmatrix} \right\}.
\]

By imposing the condition of being elements of the group \( G \), we find the equations in the statement.

A computation shows that the elements of \( T_{m,D} \) belong to \( G \) and hence to \( Q_D \). This torus is of dimension 4 and so, by Remark 2.3, it is maximal.

All maximal torus over \( \mathbb{C} \) are conjugated, in particular on the diagonal of any such torus \( T \) we will have the same coordinates appearing in \( T_{m,D} \). In particular, a subtorus of \( T \) that is defined and splits over \( \mathbb{Q} \) must verify \( \lambda_1 = \lambda_2 \) and \( \lambda_3 = \lambda_4 \), hence it is of dimension at most 2. As \( T_D \) has dimension 2, it is a maximal split torus defined over \( \mathbb{Q} \).

Note that by the description of \( Q_D, R_uQ_D \) and \( T_{m,D} \) we gave above, we have \( Q_D/R_uQ_D \cong T_{m,D} \). In particular \( Q_D \) has dimension 7 (and it is connected). By Remark 2.3 \( B_D \) must have dimension 7, hence \( B_D = Q_D \).

By Remark 2.3 the adjoint group of \( G \) is simple up to isogeny (namely it is a \( \mathbb{Q} \)-form of \( \text{SL}_3 \)), in particular the admissible parabolics of \( G \) are the maximal \( \mathbb{Q} \)-parabolics of \( G \). It is clear that a subgroup of the form \( Q_{D'} \) is a parabolic. Let us show now that any \( \mathbb{Q} \)-parabolic of \( G \) is contained in one subgroup of the form \( Q_{D'} \). Following notations from Remark 2.3 a parabolic \( P \) defined over \( E \) has to stabilize a line of \( V \); moreover it is of dimension at least 7. If \( P \) is moreover defined over \( \mathbb{Q} \), then it has to stabilize a line \( l \) of \( V \), and hence also the orthogonal plan \( l^\perp \). This line has to be isotropic, otherwise these two conditions force \( P \) to be of dimension at most 6.

\[ \square \]

**Lemma 3.4.** Let \( (Q_D, B_D, T_D) \) be as in Proposition 5.3. Consider the cocharacter

\[
\lambda_D : \mathbb{G}_{m, \mathbb{Q}} \rightarrow T_D
\]

that in a parabolic basis adapted to \( D \) (Definition 3.2) is given by

\[
t \mapsto \begin{pmatrix} (t, 0) & (0, 0) & (0, 0) \\ (0, 0) & (1, 0) & (0, 0) \\ (0, 0) & (0, 0) & (t^{-1}, 0) \end{pmatrix}
\]

(we write coordinates using Convention 2.4). Then \( \lambda_D \) is the cocharacter associated to the data \( (Q_D, B_D, T_D) \) in the general formalism of [Pin90, §4.1].

**Proof.** Any cocharacter \( \lambda : \mathbb{G}_{m, \mathbb{Q}} \rightarrow T_D \) is of the form

\[
t \mapsto \begin{pmatrix} t^a & (0, 0) & (0, 0) \\ (0, 0) & (1, 0) & (0, 0) \\ (0, 0) & (0, 0) & (t^{-a}, 0) \end{pmatrix}
\]

By [Pin90, §4.1], the image of \( \lambda_D \) has to be contained in the derived group of \( G \), so \( b = 0 \).
Consider the action of $G_{m,R}$ over Lie $G$ induced by $\lambda_D$. By [Pin90, §4.1], the sub-Lie algebra $\text{Lie} Q_D \subset \text{Lie} G$ coincide with the sum of the eigenspaces associated to eigenvalues of non-negative weights, hence $a \geq 0$. Note that in the decomposition $\text{Lie} Q_D = (\text{Lie} G)_0 + (\text{Lie} G)_a + (\text{Lie} G)_{2a}$ each of the three eigenspaces is non-trivial. Also note that we have $\text{Lie} R_a(Q_D) = (\text{Lie} G)_a + (\text{Lie} G)_{2a}$.

On the other hand, as $G$ is reductive, $R_a(Q_D)$ is the unipotent radical of a group belonging to a Shimura datum (it will be the group $P_D$ of Lemma 3.3) and the decomposition $\text{Lie} R_a(Q_D) = (\text{Lie} G)_a + (\text{Lie} G)_{2a}$ is the one induced by the Shimura datum (see [Pin90, §4.8, 4.9, 4.10]). In particular, as the weights allowed in a mixed Shimura datum are 0, −1 et −2 (see [Pin90, Definition 2.1]), we must have $a = 1$.

\section*{3.5. Notation.}
Following [Pin90, §4.2, 4.3], we write the following morphisms of algebraic groups

$$h_0 : S_C \to S_C \times GL_{2,C},$$

$$(z_1, z_2) \mapsto \left(\frac{z_1 + z_2}{z_1 - z_2}, \frac{z_2 - z_1}{z_1 - z_2}\right),$$

and

$$h_\infty : S_C \to S_C \times GL_{2,C},$$

$$(z_1, z_2) \mapsto \left(1, i(z_1 z_2 - 1)\right).$$

We will consider also the "weight morphism"

$$p : G_{m,R} \to S$$

given by

$$z \mapsto (z, z).$$

\begin{lemma}
Let $D$ be an isotropic $E$-line of $V$, and $B = \{w_1, w_2, w_3\}$ a parabolic basis adapted to $D$ (see Definition 3.2). Consider the map

$$\omega_{w_1,w_2,w_3} : S_C \times GL_{2,C} \to G_C$$

given (in the basis $B$ and using Convention 2.4) by

$$\left(\begin{array}{c} z_1, z_2 \end{array} \right), \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \left(\begin{array}{ccc} (d, 0) & (0, 0) & (0, \frac{-b}{\sqrt{\gamma}}) \\ (0, 0) & (\frac{z_1 + z_2}{2}, \frac{z_2 - z_1}{2\sqrt{\gamma}}) & (0, 0) \\ (0, \frac{-b}{\sqrt{\gamma}}) & (0, 0) & (a, 0) \end{array} \right).$$

Then $\omega_{w_1,w_2,w_3}$ is the only map verifying the following properties:

- it is a morphism of algebraic groups defined over $R$;
- the equality $\omega_{w_1,w_2,w_3} \circ h_0 = h_{w_2,w_1+w_3,w_1-w_3}$ holds;
- the cocharacters $\lambda_D : (h_{w_2,w_1+w_3,w_1-w_3} \circ p)$ and $\omega_{w_1,w_2,w_3} \circ h_\infty \circ p$ are conjugated on to each other by an element of $Q_D(C)$.

Here the morphism $h_{w_2,w_1+w_3,w_1-w_3}$ is defined in Definition 2.5, the group $Q_D$ is defined in Proposition 3.3, the cocharacter $\lambda_D$ is defined in Lemma 3.4, and the morphisms $h_0, h_\infty, p$ are defined in Notation 3.5.

Proof. Existence and uniqueness of such a morphism come from [Pin90, Proposition 4.6]. The properties are easy to check. Note that the cocharacters
\[ t \mapsto \begin{pmatrix} (t^2, 0) & (0, 0) & (0, 0) \\ (0, 0) & (t, 0) & (0, 0) \\ (0, 0) & (0, 0) & (1, 0) \end{pmatrix} \text{ and } t \mapsto \begin{pmatrix} (t^2, 0) & (0, 0) & (0, \frac{i(1-t^2)}{\sqrt{n}}) \\ (0, 0) & (t, 0) & (0, 0) \\ (0, 0) & (0, 0) & (1, 0) \end{pmatrix}. \]
are conjugated one to the other by
\[ \begin{pmatrix} (1, 0) & (0, 0) & (0, -\frac{i}{\sqrt{n}}) \\ (0, 0) & (1, 0) & (0, 0) \\ (0, 0) & (0, 0) & (1, 0) \end{pmatrix}. \]
which belongs to \( Q_D(\mathbb{C}) \).

**Definition 3.7.** Let us keep the notation from Lemma 3.6, we will write \( h_{B, \infty} : \mathcal{S}_C \rightarrow Q_{D, C} \) for the morphism \( \omega_{w_1, w_2, w_3} \circ h_{\infty} \); explicitly
\[ (z_1, z_2) \mapsto \begin{pmatrix} (z_1 z_2, 0) & (0, 0) & \frac{i(1-z_1 z_2)}{\sqrt{n}} \\ \frac{z_1 + z_2}{2} & \frac{z_1 - z_2}{2i \sqrt{n}} & (0, 0) \\ (1, 0) \end{pmatrix}. \]

**Lemma 3.8.** The smallest normal \( \mathbb{Q} \)-subgroup of the group \( Q_D \) (Proposition 3.3) containing the image of the morphism \( h_{B, \infty} \) (Definition 3.7) is
\[ P_D = G \cap \left\{ \begin{pmatrix} (z_1 z_2, 0) & (0, 0) & \frac{i(1-z_1 z_2)}{\sqrt{n}} \\ \frac{z_1 + z_2}{2} & \frac{z_1 - z_2}{2i \sqrt{n}} & (0, 0) \\ (1, 0) \end{pmatrix} \right\} = G \cap \left\{ \begin{pmatrix} (a^2 + b^2, 0) & (a, \frac{b}{\sqrt{n}}) & (1, 0) \end{pmatrix} \right\}. \]
Moreover, \( P_D(\mathbb{R}) \) is path connected and \( W_D \), the unipotent radical of \( P_D \), coincides with \( R_u Q_D \), the unipotent radical of \( Q_D \).

**Proof.** First of all, note that the equality \( W_D = R_u Q_D \) holds a priori by [Pin90 proof of Lemma 4.8]. Now, the image of the morphism \( h_{B, \infty} \) is the group
\[ \text{Im} = \left\{ \begin{pmatrix} (a^2 + b^2, 0) & (0, 0) & \frac{i(1-a^2-b^2)}{\sqrt{n}} \\ (a, \frac{b}{\sqrt{n}}) & (0, 0) & (1, 0) \end{pmatrix} \right\}. \]
Note that \( P_D \) described in the statement is a normal subgroup \( Q_D \) and it contains \( \text{Im} \). On the other hand, the group \( P_D \) has to contain \( W_D \). We deduce that \( P_D \) cannot be smaller.

Let us now show that the group \( P_D(\mathbb{R}) \) is path connected. First note that, as subgroup of \( \text{GL}_3, \mathbb{C} \), it coincides to the set of elements of the form
\[ \left\{ \begin{pmatrix} z_1^2 & w_1 & w_2 \\ z & w_3 \end{pmatrix} \right\}, \]
respecting the hermitian form $J_b$ up to a scalar (see Definition 3.2). In particular it is generated by the two subgroups
\[
\begin{pmatrix}
|z|^2 & z \\
-1 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & -bt + i bu \\
bt^2 + bu^2 & t + i u
\end{pmatrix}
\]
which are both path-connected.

**Definition 3.9.** Let $p : \mathbb{G}_m, R \to \mathbb{S}$ be as in Definition 3.5, $h_{B,\infty}$ as in Definition 3.7 and $P_D$ as in Lemma 3.8. Following [Pin90, Lemma 4.8], we define the unipotent algebraic group $U_D$ and the topological space $X_D$ as follows. Define
\[ U_D = \exp(W_{-2} \text{Lie } P_D), \]
where $W_{-2} \text{Lie } P_D$ is the subspace of $\text{Lie } P_D$ where, for $t \in \mathbb{G}_m$, the action of $h_{B,\infty} \circ p(t)$ is given by the multiplication by $t^2$. Define $X_D$ as the orbit of $h_{B,\infty}$ under the action by conjugation of $P_D(\mathbb{R}) U_D(\mathbb{C})$.

**Proposition 3.10.** The topological space $X_D$ (see Definition 3.9) is connected and the pair $(P_D, X_D)$ is a (mixed) Shimura datum (see [Pin90, Definition 2.1]). Moreover $(P_D, X_D)$ is a proper rational boundary component of $(G, X)$, and any proper rational boundary component of $(G, X)$ is of the form $(P_{D'}, X_{D'})$ for some isotropic $E$-line $D'$ of $V$ (see [Pin90, §4.11] for generalities on proper rational boundary components).

**Proof.** The space $X_D$ is connected as $P_D(\mathbb{R})$ is (Lemma 3.8). The general theory [Pin90, §4.1-4.11] and our previous results in this section imply that $(P_D, X_D)$ is a proper rational boundary component of $(G, X)$. As any admissible parabolic is of the form $Q_{D'}$ (Proposition 3.3) then any proper rational boundary component of $(G, X)$ is of the form $(P_{D'}, X_{D'})$.

**Lemma 3.11.** The unipotent groups $W_D$ (Lemma 3.8) and $U_D$ (Definition 3.10) are of dimension respectively 3 and 1.

**Proof.** By Lemma 3.8 $W_D$ coincides with the unipotent group $R_u Q_D$. Using the description of $R_u Q_D$ in Proposition 3.3 we have that $W_D$ has dimension 3.

Following the proof of Lemma 3.4 we have $\text{Lie } R_u (Q_D) = (\text{Lie } G)_a + (\text{Lie } G)_{2a}$ and $\text{Lie } U_D = (\text{Lie } G)_{2a}$. Then we can conclude combining this with the description of $\text{Lie } R_u Q_D$ in Proposition 3.3.

**Proposition 3.12.** Let $(P_D, X_D)$ be the Shimura datum of Proposition 3.10. $W_D$ the unipotent radical of $P_D$ and $U_D$ the unipotent group as in Definition 3.9. Consider the quotients of Shimura data (in the sense of [Pin90, Proposition 2.9])
\[(P_D, X_D)/W_D = (P_D/W_D, X_D')\]
and
\[(P_D, X_D)/U_D = (P_D/U_D, X_D^2).\]
Then, as complex analytic variety, $X_D'$ is a point, $X_D^2$ is an affine space of dimension 1 and $X_D$ is an affine space of dimension 2.
Proof. Consider the unipotent quotients $P_D \to P_D / U_D \to P_D / W_D$. As $X_D$ is connected (Proposition 3.10), by [Pin90] Remark 2.9 the topological spaces $X_D$ and $X_D^2$ are also connected. On the other hand $P_D / W_D$ is commutative, so $X_D^2$ is a finite number of points, hence a point.

Then, by the general results on unipotent extensions [Pin90 §2.18, 2.19], $X_D$ is a $\mathbb{C}$-vector space whose real dimension coincides with the one of $(W_D / U_D)$. We conclude using the Lemma 3.11.

Again by [Pin90 §2.18, 2.19], $X_D$ is a vector bundle over $X_D^2$ whose fiber have complex dimension coinciding with the one of $U_D$. By Lemma 3.11 this is a line bundle, and as $X_D^2$ is contractible, the line bundle is trivial. □

Corollary 3.13. Let $S$ be a Picard modular surface as defined in Definition 2.7, $\partial S$ be the boundary of the Baily-Borel compactification of $S$ and $\partial S^T$ be the boundary of the toroidal compactification of $S$. Then the Shimura data underlying the strata of $\partial S$ are of the form $(P_D / W_D, X_D^1)$ and the Shimura data underlying the strata $\partial S^T$ are of the form $(P_D / U_D, X_D^2)$. In particular, as complex varieties, $\partial S$ is a finite number $N$ of points and $\partial S^T$ is a disjoint union of $N$ smooth and proper curves of genus 1.

Remark 3.14. The number $N$ in the previous proposition is computed in several cases in [Sto12]. Note also that the union of the $N$ points (or the union of the $N$ curves) is actually defined over the imaginary quadratic field $E$, but a priori each point is not.

The geometry of $\partial S^T$ appears already in [Lar92] and [Bel02 Chapter 1].

Proof. The general theory ([Pin90 Chapter 6] and [Wil00 Lemma 1.7]), Proposition 3.10 and Proposition 3.12 imply the Baily-Borel case, as well as the fact that the Shimura data underlying the strata of $\partial S^T$ are of the form $(P_D, X_D) / U_D^\sigma$, with $U_D^\sigma$ a subgroup of the unipotent group $U_D$ (Definition 3.9).

If $U_D^\sigma$ were trivial, then the boundary would be of dimension 2 by Proposition 3.12 which is impossible as $S$ is a surface. On the other hand $U_D$ has dimension 1 by Lemma 3.11 hence $U_D^\sigma = U_D$. □

4. Degeneration of Hodge structures

This section contains the main result, Theorem 4.11. We study how variations of Hodge structures over a Picard modular surface degenerate through the cusps of its Baily-Borel compactification. More precisely, we describe the types of the Hodge structures $R^{k+i} j_* \mu_S(F)$ for all $G$-representations $F$ (see Notation 4.1). Remark 4.2 shows that these structures have a geometric interest.

We start by reducing the problem to a combinatorial question (4.3-4.9). The main ingredient is a theorem of Burgos and Wildeshaus, which in this case has a simplified version via Lemma 4.3. We then deal with this combinatorial question (4.7-4.9) and finally describe the Hodge structures we are interested in (4.10-4.12). The last part (4.13-4.15) explains how to deduce Theorem 1.1 from Theorem 4.11.

The following notation will be used throughout the section.
4.1. Notation. Let $S$ be a Picard modular surface (Definition 2.7), $S^*$ be its Baily Borel compactification and

\[ j : S \hookrightarrow S^* \]

be the canonical open immersion. By Corollary 4.13 the boundary of $S^*$ is a finite set of points. Let us fix one of these points, and let

\[ i : \text{pt} \hookrightarrow S^* \]

be the inclusion.

Let $(G, X)$ be the Shimura datum underlying $S$ (Definition 2.7),

\[ (G_{\text{pt}}, X_{\text{pt}}) = (P_D/W_D, X^1_D) \]

be the Shimura datum underlying (the stratum containing) \( \text{pt} \) (Corollary 3.13) and $Q_D \supset P_D$ be the corresponding parabolic subgroup (see Proposition 3.8 and Lemma 4.3). Recall that $P_D$ and $Q_D$ have the same unipotent radical $W_D$ (Lemma 3.8).

By [Pin90, 1.18] (see also [BW04, §2]), there are linear tensor functors

\[ \mu_S : \text{Rep}_{G, Q} \longrightarrow \text{VHS}(S)_Q \]

and

\[ \mu_{\text{pt}} : \text{Rep}_{G_{\text{pt}}, Q} \longrightarrow \text{HS}(\text{pt})_Q, \]

(called canonical construction functors) from the $\mathbb{Q}$-representations of $G$ (resp. $G_{\text{pt}}$) to (admissible) variations of $\mathbb{Q}$-Hodge structures over $S$ (resp. over pt).

The functor $i^*j_* : \text{VHS}(S)_Q \longrightarrow \text{HS}(\text{pt})_Q$ from (admissible) variations of $\mathbb{Q}$-Hodge structures over $S$ to $\mathbb{Q}$-Hodge structures over pt is left exact and, for any integer $k \geq 0$, we write

\[ R^k i^*j_* : \text{VHS}(S)_Q \longrightarrow \text{HS}(\text{pt})_Q \]

for the derived functor. Note that $R^k i^*j_* = i^* R^k j_*$.

Remark 4.2. Let $V$ be as in Notation 2.1 and let $f : A \longrightarrow S$ be the universal abelian scheme (Remark 2.8). Then one has a canonical identification

\[ \mu_S(V^\vee) = R^1 f_* \mathbb{Q}_A. \]

In particular, as $\mu_S$ is a tensor functor, all the relative cohomology sheaves of any $r$-fold fiber product of $A$ over $S$ are in the image of the functor $\mu_S$ (as well as several interesting direct factors of these sheaves, e.g. primitive parts with respect to a Lefschetz decomposition).

Lemma 4.3. The $\mathbb{Q}$-algebraic group $Q_D/W_D$ is isogenous to the direct product of a compact torus and a $\mathbb{Q}$-split torus. In particular, any neat arithmetic subgroup of $Q_D/W_D(\mathbb{Q})$ is trivial.

Proof. Consider the compact torus defined over $\mathbb{Q}$

\[ T_n = \{ (a, b) : a^2 + nb^2 = 1 \}, \]

where $n$ is the integer as in Notation 2.1. The map

\[ \mathbb{G}_{m, \mathbb{Q}}^2 \times T_n^2 \longrightarrow Q/W_1 \]
given (using Convention [23]) by

\[
(\lambda, \lambda', (a, b), (a', b')) \mapsto \begin{pmatrix}
\lambda\lambda' (a, b) & \lambda' (a', b') \\
\lambda' (a, b) & \lambda^{-1} \lambda' (a, b)^{-1}
\end{pmatrix}
\]

is an isogeny. \( \square \)

**Theorem 4.4.** For any \( F \in \text{Rep}_G \), there is a canonical isomorphism of \( \mathbb{Q} \)-Hodge structures over \( \text{pt} \)

\[
R^{k, j}_* j_* (\mu_S(F)) = \mu_{\text{pt}}((H^k(W_D, F|_{Q_D}))|_{G_{\text{pt}}}),
\]

where \( F|_{Q_D} \) is \( F \) seen as representation of \( Q_D \) and \( H^k(W_D, \cdot) \) is the \( k \)-th derived of the functor that associates to a \( Q_D \)-representation its \( W_D \)-invariant part, and \( \cdot|_{G_{\text{pt}}} \) is again a restriction functor (a \( Q_D/W_D \)-representation is seen as a \( G_{\text{pt}} \)-representation).

**Proof.** This is [BW04, Theorem 2.9] in a simplified version, that holds because the arithmetic group appearing in loc. cit. has to be trivial by Lemma 4.3. \( \square \)

### 4.5. Notation

For any reductive group \( H \), we will write \( F_{\lambda, H} \) for the \( H \)-irreducible representation whose maximal weight is \( \lambda \). We will simply write \( F_{\lambda} \) if the group \( H \) can be deduced from the context.

**Theorem 4.6** (Kostant Theorem, see [War72] thm 2.5.2.1). Let \( H \) be a reductive group over \( \mathbb{C} \), \( B \) a Borel with unipotent radical \( W \), \( \Phi \) the associated root system, \( \Phi^+ \) the subset of the positive ones, \( \rho \) the half of the sum of positives roots, and \( \mathcal{R} \) be the Weyl group.

For any \( \sigma \in \mathcal{R} \), define the length of \( \sigma \) as

\[
l(\sigma) = \# \{ \alpha \in \Phi^+, \sigma^{-1} \alpha \notin \Phi^+ \}.
\]

Then one has an equality of \( B/W \)-representations

\[
H^k(W, F_{\lambda, H}|_B) = \bigoplus_{l(\sigma)=k} F_{\sigma(\lambda+\rho)-\rho, B/R}
\]

(following Notation [4.3]).

### 4.7. Lengths of roots

Let \( B \) be a parabolic basis adapted to \( D \) (Definition [3.2]). We have an isomorphism

\[
\phi_B : G_{\mathbb{C}} \cong GL_{3,\mathbb{C}} \times G_{m,\mathbb{C}}
\]

from Remark [26]. Write \( T_s \subset GL_{3,\mathbb{C}} \) for the subgroup of upper-triangular matrices and \( \Delta \subset GL_{3,\mathbb{C}} \) for the diagonal ones. To describe the root system let us choose \( \Delta \times G_{m,\mathbb{C}} \cong G_{m,\mathbb{C}}^4 \) as maximal torus, and \( T_s \times G_{m,\mathbb{C}} \) as Borel containing the torus. Note that \( \phi_B \) restricts to an isomorphism:

\[
\phi_B : G_{D,\mathbb{C}} \cong T_s \times G_{m,\mathbb{C}}.
\]

We write \( \lambda_1, \ldots, \lambda_4 \) for the four standard characters, which together form a basis for the lattice of characters. We also write \( e_{ij} = \lambda_i \lambda_j^{-1} \). The simple roots are \( e_{12} \) and \( e_{23} \), the other positive root is \( e_{13} \), and so

\[
\rho = e_{13}.
\]
The Weyl group $\mathcal{R}$ is the group of permutations of the first three coordinates of the characters. We will write elements of $\mathcal{R}$ with the standard notations for permutations; their lengths are given by

$$l(e) = 0, \ l(12) = l(23) = 1, \ l(123) = l(132) = 2, \ l(13) = 3.$$  

**4.8. Computation.** We keep notations from 4.7. Let $\lambda = (a, b, c, d)$ be any character written in the basis fixed in 4.7. Note that we have

$$\rho = (1, 0, -1, 0).$$

Let us compute $\sigma(\lambda + \rho) - \rho$ for all permutations $\sigma \in \mathcal{R}$.

$$\sigma(a + 1, b, c - 1, d) - (1, 0, -1, 0) = (a, b, c, d),$$

$$\sigma(12)(a + 1, b, c - 1, d) - (1, 0, -1, 0) = (b - 1, a + 1, c, d),$$

$$\sigma(23)(a + 1, b, c - 1, d) - (1, 0, -1, 0) = (a, c - 1, b + 1, d),$$

$$\sigma(13)(a + 1, b, c - 1, d) - (1, 0, -1, 0) = (c - 2, a + 1, b + 1, d),$$

$$\sigma(123)(a + 1, b, c - 1, d) - (1, 0, -1, 0) = (b - 1, c - 1, a + 2, d).$$

By Theorem 4.6 we deduce the following equalities of $Q_{D,\mathbb{C}}/W_{D,\mathbb{C}}$-representations:

$$H^0(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}}) = F_{a,b,c,d},$$

$$H^1(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}}) = F_{12}(a+1,c,d \oplus F_{a,c-1,b+1,d},$$

$$H^2(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}}) = F_{23}(c-a,b+1,d \oplus F_{b-1,c-1,a+2,d},$$

$$H^3(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}}) = F_{13}(c,b,a+2,d),$$

and $H^k(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}}) = 0$, for $k \geq 4$ (we are following Notation 4.5).

**Remark 4.9.** As $Q_{D,\mathbb{C}}/W_{D,\mathbb{C}}$ is isomorphic to the torus $G^4_{m,\mathbb{C}}$ (see 1.7), the irreducible representations of $Q_{D,\mathbb{C}}/W_{D,\mathbb{C}}$ are just characters. In particular the six representations on the right hand side above are 1-dimensional and explicit.

**4.10. Restriction to $S$, the types.** Consider $h_{B,\infty} : S_{\mathbb{C}} \to Q_{D,\mathbb{C}}$ of Definition 3.7 and the induced map

$$h_{B,\infty} : S_{\mathbb{C}} \to Q_{D,\mathbb{C}}/W_{D,\mathbb{C}}.$$  

Consider also $\phi_{B} : Q_{D,\mathbb{C}}/W_{D,\mathbb{C}} \xrightarrow{\sim} G^4_{m,\mathbb{C}}$ defined in 1.7. The composition

$$\phi_{B} \circ h_{B,\infty} : S_{\mathbb{C}} \to G^4_{m,\mathbb{C}},$$

is then given by

$$(z_1, z_2) \mapsto (z_1 z_2, z_1, 1, z_1 z_2).$$

Hence, for any character $\lambda = (a, b, c, d)$ (written in the basis fixed in 4.7), one deduces from 4.8 the following equalities of $S_{\mathbb{C}}$-representations

$$H^0(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}})_{S_{\mathbb{C}}} = F_{a+b+c+d,a+d},$$

$$H^1(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}})_{S_{\mathbb{C}}} = F_{a+b+d,b+d-1} \oplus F_{a+c+d-1,a+d},$$

$$H^2(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}})_{S_{\mathbb{C}}} = F_{a+c+d-1,c+d-2} \oplus F_{b+c+d-2,b+d-1},$$

$$H^3(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}})_{S_{\mathbb{C}}} = F_{b+c+d-2,c+d-2}$$

and $H^k(W_{D,\mathbb{C}}, F_{\lambda,\mathbb{C}|Q_{D,\mathbb{C}}})_{S_{\mathbb{C}}} = 0$, for $k \geq 4$ (we are following Notation 4.5).
Remark 4.12. This is Theorem 4.4 with the computations done in 4.10.

4.7 from $\lambda$ see Remark 2.3), but not over $\mu$

V 0 if it is negative) and

Lemma 4.13. Let $\lambda$ be the irreducible representation of $G_C$ of maximal weight $\lambda = (a, b, c, d)$ (written in the basis fixed in 4.7). Then the following holds:

1. $R^k \iota^* j_\ast \mu_{S,C}(F_\lambda)$ has type $(-a - b - d, -a - d)$,
2. $R^k \iota^* j_\ast \mu_{S,C}(F_\lambda)$ has types $(-a - b - d, -b - d + 1)$ and $(-a - c - d + 1, -a - d)$,
3. $R^k \iota^* j_\ast \mu_{S,C}(F_\lambda)$ has types $(-a - c - d + 1, -c - d + 2)$ and $(-b - c - d + 2, -b - d + 1)$,
4. $R^k \iota^* j_\ast \mu_{S,C}(F_\lambda)$ has type $(-b - c - d + 2, -c - d + 2)$,
5. $R^k \iota^* j_\ast \mu_{S,C}(F_\lambda)$ vanishes for $k \geq 4$.

Proof. This is Theorem 4.4 with the computations done in 4.10.

Remark 4.12. Note that the functor $R^k \iota^* j_\ast \mu_{S,C}$ is linear, hence the types of $R^k \iota^* j_\ast \mu_{S,C}(F)$ can be computed for any representation $F$ once we know its decomposition into irreducible factors.

Note also that all $C$-representations $F_\lambda$ are defined over $E$ (as $G$ splits over $E$, see Remark 2.3), but not over $\mathbb{Q}$. If we start with a $\mathbb{Q}$-irreducible representation $F$, then $F_\mathbb{E}$ will decompose in general in two factors, say $F_\mathbb{E} = F^1 \oplus F^2$ (e.g. $V_\mathbb{E} = F_1, 0, 0, 0 \oplus F_0, 0, -1, 1$), and the types of $R^k \iota^* j_\ast \mu_{S,C}(F^1 \oplus F^2_\mathbb{E})$ will respect the Hodge symmetry.

Lemma 4.13. Let $p$ and $r$ be two non-negative integers, and let us keep notations from 4.7. The $G_C$-irreducible representations contained in $\Lambda^p(F_{0,0,,-1,0}^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}} \oplus F^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}}_{1,0,0,,-1})$ are exactly the $F_{a,b,c,d}$ verifying

1. $r \geq a \geq b \geq c \geq -r$,
2. $3r + a_+ + b_- + c_- \geq -d \geq a_+ + b_+ + c_+$, and
3. $a + b + c + 2d = -p$.

where, for any integer $x$, we define $x_+$ (resp. $x_-$) as $x$ itself if it is positive (resp. if it is negative) and 0 otherwise.

Proof. In the representation $F_{0,0,,-1,0}^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}} \oplus F^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}}_{1,0,0,,-1}$ we have an explicit basis of 6$r$ elements that are eigenvectors for the action of the maximal torus. We deduce from them a basis $B_p$ of $\Lambda^p(F_{0,0,,-1,0}^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}} \oplus F^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}}_{1,0,0,,-1})$: each vector of $B_p$ corresponds to the choice of $p$ between the 6$r$ previous elements. Then each vector of $B_p$ is also an eigenvector for the action of the torus, whose weight the sum of the weights of the $p$ elements chosen. From this we deduce that $a, b, c, d$ is a weight for the action of the maximal torus on $\Lambda^p(F_{0,0,,-1,0}^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}} \oplus F^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}}_{1,0,0,,-1})$ if and only if the integers verify

1. $r \geq a, b, c \geq -r$,
2. $3r + a_- + b_+ + c_- \geq -d \geq a_+ + b_+ + c_+$, and
3. $a + b + c + 2d = -p$.

The condition $a \geq b \geq c$ corresponds to the choice of the Borel containing the maximal torus (as we did in 4.7).

We need to show now that any $a, b, c, d$ verifying the condition in the statement is the maximal weight of a subrepresentation of $\Lambda^p(F_{0,0,,-1,0}^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}} \oplus F^{\mathfrak{g}^{\mathfrak{g}^{\mathfrak{g}}}}_{1,0,0,,-1})$. Suppose $-d \leq p/2$ and $a, b, c$ negative (the other cases are analogous) and consider

$$W_{a,b,c,d} = (\bigwedge F_{0,0,,-1,0}) \otimes a \otimes (\bigwedge F_{0,0,,-1,0}) \otimes b \otimes (\bigwedge F_{0,0,,-1,0}) \otimes c \otimes (\bigwedge F_{0,0,,-1,0}) \otimes d.$$
The maximal weight of $W_{a,b,c,d}$ is $a, b, c, d$. To show that $W_{a,b,c,d}$ is a subrepresentation of $\bigwedge^p (F_{0,0,-1,0}^{\mathbb{Q}r} \oplus F_{1,0,0,-1}^{\mathbb{Q}r})$ it is enough to show that $F_{0,0,-1,0}$ is a subrepresentation of $F_{0,0,-1,0} \oplus F_{1,0,0,-1}$. This is indeed the case: the action canonical paring of the standard representation of $GL_3$ with its dual induces a non-zero morphism $F_{0,0,-1,0} \oplus F_{1,0,0,-1} \rightarrow F_{0,0,0,-1}$. \hfill $\square$

4.14. Proof of Theorem 1.1 First if all, we have the following identification

$$R^p f_* \mathbb{Q}_{A'} = \bigwedge^p (F_{0,0,-1,0}^{\mathbb{Q}r} \oplus F_{1,0,0,-1}^{\mathbb{Q}r})$$

see also Remark 4.12 and Remark 4.13

We deduce from Theorem 4.11 and Lemma 4.13 that

1. the $\mathbb{Q}$-Hodge structure $i^* R^0 j_* R^p f_* \mathbb{Q}_{A'}$ has weight $p - (a - c)$,
2. the $\mathbb{Q}$-Hodge structure $i^* R^1 j_* R^p f_* \mathbb{Q}_{A'}$ has weight $p + 1 + (b - c)$ and $p + 1 + (b - c)$,
3. the $\mathbb{Q}$-Hodge structure $i^* R^2 j_* R^p f_* \mathbb{Q}_{A'}$ has weight $p + 3 + (b - c)$ and $p + 3 + (b - c)$,
4. the $\mathbb{Q}$-Hodge structure $i^* R^3 j_* R^p f_* \mathbb{Q}_{A'}$ has weight $p + 4 + (a - c)$,
5. the $\mathbb{Q}$-Hodge structure $i^* R^k j_* R^p f_* \mathbb{Q}_{A'}$ vanishes for $k \geq 4$.

with $a, b, c$ varying between the numbers satisfying conditions in Lemma 4.13

First of all, note that the map

$$(a, b, c, d) \mapsto (-c, -b, -a, -3r - d)$$

gives a bijection between the values of $(a, b, c, d)$ appearing in the list of irreducible subrepresentations of $\bigwedge^p (F_{0,0,-1,0}^{\mathbb{Q}r} \oplus F_{1,0,0,-1}^{\mathbb{Q}r})$ and the values appearing in $\bigwedge^{6r-p} (F_{0,0,-1,0}^{\mathbb{Q}r} \oplus F_{1,0,0,-1}^{\mathbb{Q}r})$, so that we can suppose $p \leq 3r$.

Note now that $a - b, b - c$ and $a - c$ are non-negative integers bounded by $2r$ and $p$.

Let us start by studying the possible values $j \leq p$ of $a - b$ (the case $b - c$ is similar).

If $j \leq r$, then the triple $(a, b, c) = (j, 0, 0)$ (or $(a, b, c) = (j, 0, -1)$, depending on the parity of $j$) verifies the conditions in Lemma 4.13 and gives $a - b = j$.

If $r < j \leq M_p$, then $(a, b, c) = (r, r - j, r - j)$ (or $(a, b, c) = (r, r - j, r - j - 1)$, depending on the parity of $j$) verifies the conditions in Lemma 4.13 and gives $a - b = j$; it is also clear from this description that $a - b$ cannot take values bigger than $M_p$.

Let us now consider the possible values $j < p$ of $a - c$. If $0 < j \leq r$, then the triple $(a, b, c) = (j, 0, 0)$ (or $(a, b, c) = (j, 1, 0)$, depending on the parity of $j$) verifies the conditions in Lemma 4.13 and gives $a - c = j$.

If $j = 0$, then the triple $(a, b, c) = (0, 0, 0)$ if $p$ is even, or $(a, b, c) = (1, 1, 1)$ if $p$ is odd and at least 3 verifies the conditions in Lemma 4.13 and gives $a - c = 0$: this description shows also that if $p = 1$ then $a - c$ cannot be 0.

If $r < j \leq C_p$, then $(a, b, c) = (r, 0, r - j)$ (or $(a, b, c) = (r, 1, r - j)$, depending on the parity of $j$) verifies the conditions in Lemma 4.13 and gives $a - c = j$.

Remark 4.15. For any $G$-representation $F_{a,b,c,d}$, the sheaf $\mu_S(F_{a,b,c,d})$ is the realization of a relative Chow motive over $S$ of the form $M_S(A', p, n)$ for some integer $r$
and $s$ [Anc12, Theorem 4.7]. Here $A^r$ is the $r$-fold product of the universal abelian scheme $A$ over $S$ (Remark 2.8), and $p$ is a projector.

By [Wil12a, Corollary 2.13], the projector $p$ induces a direct factor $\partial M(A^r)^p$ of the boundary motive $\partial M(A^r)$ of the variety $A^r$ (seen over its field of definition). By [Wil09, Theorem 4.3] if $\partial M(A^r)^p$ avoids weights $-1$ and $0$ then one has a canonical direct factor of the interior motive of $A^r$. This is a Chow motive realizing to (a canonical direct factor of) the interior cohomology of $A^r$.

A necessary (and conjecturally sufficient) condition to have that weights $-1$ and $0$ are avoided is to check this on realization. More precisely, let $\overline{w}$ be the weight of $\mu_{S,C}(F_{a,b,c,d})$ and $w_{k,i}$ be a weight of $R^k i^* j_* \mu_{S,C}(F_{a,b,c,d})$, then one has to check that the list of integers

$$\{k - w_{k,i} + \overline{w}\}_{k,i}$$

does not contain $-1$ and $0$ (see [Wil12b, Theorem 1.2, Proposition 3.5]).

Using Theorem 4.11 one can compute this list and obtain

$$\{a - b, a - c, b - c, -1 - (a - b), -1 - (a - c), -1 - (b - c)\}.$$ 

This means that the representations of $G$ that satisfy the condition after realization are exactly those which do not belong to a wall of the Weyl chamber. Hence, one should be able to construct canonical interior motives associated to a 'generic' representation. Such a result would be the Picard analogue of Scholl Theorem [Sch94] for modular curves and Wildeshaus Theorem [Wil12b] for Hilbert modular varieties.

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