Comment on “Critical point scaling of Ising spin glasses in a magnetic field” by J. Yeo and M.A. Moore

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In a section of a recent publication, [J. Yeo and M.A. Moore, Phys. Rev. B 91, 104432 (2015)], the authors discuss some of the arguments in the paper by Parisi and Temesvári [Nuclear Physics B 858, 293 (2012)]. In this comment, it is shown how these arguments are misinterpreted, and the existence of the Almeida-Thouless transition in the upper critical dimension 6 reasserted.

In a recent paper 1 by Yeo and Moore about the long debated existence of the Almeida-Thouless instability in the short ranged Ising spin glass below the upper critical dimension six, the authors criticize in Sec. III. some of our statements and arguments in Ref. 3. In that paper we have demonstrated: firstly the incorrect reasoning of Ref. 4 about the disappearance of the Almeida-Thouless (AT) transition line when approaching the upper critical dimension from above; secondly we have computed the AT line staying exactly in six dimensions (and not by a limiting process); and thirdly the $\epsilon$-expansion was used to compute the AT line below six dimensions, and the relatively smooth behavior of it while crossing $d = 6$ (with fixed bare parameters) was exhibited. In what follows, we want to comment the discussion in Sec. III. of 1.

I. AT AND ABOVE SIX DIMENSIONS

The first order renormalization group (RG) equations for the six-dimensional model are worked out and solved in Sec. 3 of Ref. 3, the AT line follows from that calculation [see Eq. (37) in 3]:

$$h^2_{\text{AT}} = \frac{4}{(1 - w^2 \ln |r| + \frac{10}{3} w^2 \ln w)^4} \frac{w|r|^2 \approx 4}{(1 - w^2 \ln |r|)^4} \frac{w|r|^2}{4 w |r|^2}, \quad d = 6$$

(1)

where $w^2 \ll 1$ was used. [Note that a minus sign in the denominator of Eq. (13) has been left out in 1.] As it turns out from the discussion in Sec. 3 of Ref. 3, this approximation is valid if the scaling variable with zero scaling dimension (which is invariant under RG in $d = 6$) is small, i.e.

$$w^2 \frac{1}{1 + \frac{5}{3} w^2 \ln w^2 - w^2 \ln |r|} \ll 1,$$

(2)

and this condition is always satisfied whenever $|r| \ll 1$ and $w^2 \ll 1$; see also the middle part of Eq. (59) of that reference. Yeo and Moore forget all about this derivation of the six-dimensional AT line; they deduce it from Eq. (11) of 1 by the limit $\epsilon \to 0$, and finally they argue that “Eq. (11) is not valid for this limit”. We can absolutely agree with this last statement: the system at the upper critical dimension needs special care, physical quantities, like the critical magnetic field where replica symmetry breaking sets in, cannot be obtained by a limiting process of $\epsilon \to 0$.

The point is that $\epsilon$ in Eq. (11) may be small, but fixed, while $|r| \ll 1$, and the $|r|^{\epsilon/2}$ term in the denominator must be ignored. Taking account of this, the AT line above dimension six, Eq. (11) of 1, must be written (consistently with the approximations used to derive it) as:

$$h^2_{\text{AT}} \sim \frac{w|r|^{\frac{d-1}{2}}}{(2 w^2 + 1)^{\frac{d-1}{2}}}, \quad d > 6.$$

(3)

This is just Eq. (28) of Ref. 3. This equation for the AT line above six dimensions must be supplemented by the range of its applicability, otherwise false conclusions like Eq. (12) in 1 (which is obviously incompatible with (1)) could be deduced. For this reason, we briefly repeat the two steps needed for the derivation of (3):

1 We use here the notations of Ref. 1. In fact $|r|$ was called $\tau$ in 1, whereas $r$ had the role of the nonlinear scaling field associated with $\tau$. We also adapt here to the somewhat unconventional use of the symbol $\epsilon$ as $\epsilon = d - 6$. 
The RG equations for the three bare parameters, namely
\[ |r| = \left( 2 - \frac{10}{3} w^2 \right) |r|, \]
\[ w^2 = -\epsilon w^2 - 2 w^4, \]
\[ h^2 = \left( 4 + \frac{\epsilon}{2} + \frac{1}{3} w^2 \right) h^2 \]
are valid for \(|r| \ll 1\) and \(w^2 \ll 1\). One can introduce the nonlinear scaling fields \(\hat{w}\) satisfying exactly, by definition, the linearized (around the fixed point) and diagonalized RG equations. For the system in (4), and for its Gaussian fixed point, one readily finds
\[ g_{|r|} = 2 g_{|r|}, \quad g_{w^2} = -\epsilon g_{w^2}, \quad \text{and} \quad g_{h^2} = \left( 4 + \frac{\epsilon}{2} \right) g_{h^2}. \]
The relations between bare parameters and nonlinear scaling fields were published in (3), for completeness we repeat them here:
\[ |r| = g_{|r|} \left( 1 - \frac{2}{\epsilon} g_{w^2} \right)^{-\frac{1}{\epsilon}}, \quad w^2 = g_{w^2} \left( 1 - \frac{2}{\epsilon} g_{w^2} \right)^{-1}, \quad \text{and} \quad h^2 = g_{h^2} \left( 1 - \frac{2}{\epsilon} g_{w^2} \right)^{\frac{1}{\epsilon}}. \]

The zeros of the scaling function of the replicon mass, \(\hat{\Gamma}_R\), are the locations of the AT instability, \(\hat{\Gamma}_R\) depends on the bare parameters \(|r|, w^2\), and \(h^2\) through the RG invariants \(x \equiv g_{w^2} g_{|r|}^{-\frac{1}{\epsilon}}\) and \(y \equiv g_{h^2} g_{|r|}^{-\frac{1}{\epsilon}}\). The AT instability line can then be written as \(y = f(x)\) or
\[ g_{h^2} = g_{|r|}^{2+\frac{1}{\epsilon}} f \left( g_{w^2} g_{|r|}^{-\frac{1}{\epsilon}} \right) = \frac{g_{|r|}^2}{\sqrt{g_{w^2}}} g \left( g_{w^2} g_{|r|}^{-\frac{1}{\epsilon}} \right), \quad \text{with} \quad g(x) \equiv \sqrt{x} f(x). \]
The following remarks are now in order:

(i) This form of the AT line is generic for the system where the zero-external-magnetic-field symmetry is broken only by the linear replica symmetric invariant in the Lagrangian whose bare coupling constant is \(h^2\). (This model is used in Refs. 1, 2 too.) Eqs. (5) cannot be used, in this generic case, to replace nonlinear scaling fields by bare couplings, as they were derived from the one-loop RG equations in (4).

(ii) Eq. (14) of (5) formally agrees with (6), but the bare couplings are there instead of the \(g\)'s. In this form it is not correct.

(iii) The function \(g(x)\) of (6) can be calculated perturbatively, the 1-loop result was published in (2): \(g(x) = (-C' \epsilon)^x\) where \(-C' \epsilon > 0\) is analytic and positive around \(\epsilon = 0\). Putting this into (6), one gets
\[ g_{h^2} \sim g_{|r|}^{2+\frac{1}{\epsilon}} \sqrt{g_{w^2}}, \]
and inserting the inverse relations of those in Eq. (5) one immediately arrives at (6).

As it must be clear from the two-step process above, a mixture of renormalization and perturbation theory leads to Eq. (3). The leading, linear contribution to \(g(x)\) is free from a singularity at \(d = 6\), as it comes from an ultraviolet convergent one-loop graph. Triangular insertions in the next, two-loop graphs, however, certainly produce singular terms like \(g(x) \sim \frac{1}{x^2}\), their neglect is acceptable only if \(\frac{1}{x} = \frac{1}{g_{w^2} g_{|r|}} \ll 1\). Expressing this condition by the bare couplings, one can write the range of applicability of Eq. (3) as
\[ |r| \ll 1, \quad w^2 \ll 1, \quad \text{and most importantly} \quad \frac{1}{\epsilon} w^2 |r| \ll \left( 1 + \frac{2}{\epsilon} w^2 \right)^{-1-\frac{1}{\epsilon}} \ll 1. \]
The left-hand-side of the third condition becomes of order unity (1/2), and thus breaks down, when \(\epsilon \to 0\) while \(|r|\) and \(w^2 \ll 1\), but otherwise fixed. This is just the limit leading to Eq. (12) of (2) (and to the conclusion of the disappearance of the AT line for \(\epsilon \to 0\)), and is the source of the basic fault in the original arguments in (5). [See also Fig. 2(b) and the discussion around it in (5).] \(\epsilon\) in (5) may be small, but must be kept fixed. Simple first order perturbational result is obtained for \(w^2 \ll \epsilon\). The joint application of the perturbational method and RG (and not
RG alone as Yeo and Moore\textsuperscript{1} claim) provide (3) which is valid for $0 < \epsilon \ll w^2 \ll 1$ too. In this latter case the range of applicability of Eq. (3), according to (7), shrinks to zero as $-\ln |r| \gg \epsilon^{-1}$, together with the amplitude in (3). This phenomenon signals the appearance of the logarithmic correction in $d = 6$: $h^2_{AT} \sim (\ln |r|)^{-4} |r|^2$, and it is not an indication of the disappearance of the AT line.

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