HOMOMORPHISMS INTO TOTALLY DISCONNECTED, LOCALLY
COMPACT GROUPS WITH DENSE IMAGE

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Abstract. Let $\phi : G \to H$ be a group homomorphism such that $H$ is a totally disconnected locally compact (t.d.l.c.) group and the image of $\phi$ is dense. We show that all such homomorphisms arise as completions of $G$ with respect to uniformities of a particular kind. Moreover, $H$ is determined up to a compact normal subgroup by the pair $(G, \phi^{-1}(L))$, where $L$ is a compact open subgroup of $H$. These results generalize the well-known properties of profinite completions to the locally compact setting. We go on to develop techniques to build Hecke pairs $(G, U)$ where $G$ is a finitely generated group. These techniques are applied to produce examples of compactly generated elementary t.d.l.c. groups of decomposition rank $\omega \cdot n + 2$ for each $n \geq 1$.

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1. Introduction

For $G$ a (topological) group, the **profinite completion** $\hat{G}$ of $G$ is the inverse limit of the finite (continuous) quotients of $G$. One can obtain other profinite groups by forming the inverse limit of a suitable set of finite quotients of $G$. Such a profinite group $H$ is always a quotient of $\hat{G}$, and obviously the composition map $G \to \hat{G} \to H$ has dense image.

On the other hand, one can ask what profinite groups $H$ are so that there is a (continuous) map $\psi : G \to H$ with dense image. Letting $\iota : G \to \hat{G}$ be the canonical inclusion, it turns out
there is always a continuous quotient map \( \tilde{\psi} : \hat{G} \to H \) so that \( \psi = \tilde{\psi} \circ \iota \); cf. \cite[Lemma 3.2.1]{10}. In this way one obtains a complete description of all profinite groups \( H \) and homomorphisms \( \psi : G \to H \) such that the image of \( G \) is dense in \( H \): all such groups and morphisms arise exactly by forming inverse limits of suitable sets of finite quotients of \( G \).

Our aim in the present paper is to extend the well-known and well-established description of homomorphisms to profinite groups with dense image, as described above, to homomorphisms into totally disconnected locally compact (t.d.l.c.) groups with dense image. That is to say, we will develop a theory of t.d.l.c. completions. Using the language of uniformities, we shall see that the answer generalizes the profinite case.

Our approach generalizes previous work by Schlichting (\cite{11}) and Belyaev (\cite[§7]{1}); Schlichting’s completion has also been studied in subsequent work, e.g. \cite{12}. The novel contributions of this work are to present a unified theory of t.d.l.c. completions, proving various properties are common to all completions, and to exhibit construction techniques giving interesting examples of t.d.l.c. groups.

1.1. Statement of results. We shall state our results in the setting of Hausdorff topological groups. If one prefers, the group \( G \) to be completed can be taken to be discrete. The topological group setting merely allows for finer control over the completions; for example, given a topological group \( G \), there is often an interesting difference between completions of \( G \) as a topological group and completions of \( G \) as a discrete group.

**Definition 1.1.** For a topological group \( G \), a completion map is a continuous homomorphism \( \psi : G \to H \) with dense image so that \( H \) is a t.d.l.c. group. We call \( H \) a t.d.l.c. completion of \( G \).

For a topological group \( G \), we first isolate the \( G \)-stable local filters. Each such filter \( S \) gives rise to a uniformity on \( G \), and we may then complete \( G \) with respect to this uniformity. Let \( \hat{G}_S \) be the completion and let \( \beta_{(G,S)} : G \to \hat{G}_S \) be the canonical homomorphism.

These \( G \)-stable local filters give t.d.l.c. completions.

**Proposition 1.2** (see Proposition 3.9). If \( G \) is a topological group and \( S \) is a \( G \)-stable local filter, then \( \hat{G}_S \) is a t.d.l.c. completion of \( G \) via \( \beta_{(G,S)} \).

We then show all t.d.l.c. completions indeed arise in this way.

**Theorem 1.3** (see Theorem 3.11). If \( G \) is a topological group and \( H \) is a t.d.l.c. completion of \( G \) via \( \phi : G \to H \), then there is a \( G \)-stable local filter \( S \) and a unique topological group isomorphism \( \psi : \hat{G}_S \to H \) such that \( \phi = \psi \circ \beta_{(G,S)} \).

We next consider completions of Hecke pairs. This is not a restriction; it merely allows us to organize our discussion. Recall that two subgroups \( U \) and \( V \) of a group \( G \) are commensurate if \( U \cap V \) has finite index in both \( U \) and \( V \).

**Definition 1.4.** A (topological) Hecke pair is a pair of groups \( (G,U) \) where \( G \) is a topological group and \( U \) is an open subgroup of \( G \) that is commensurated, that is, \( gUg^{-1} = U \) and \( U \) are commensurate for all \( g \in G \).

**Definition 1.5.** For a Hecke pair \( (G,U) \), a completion map for \( (G,U) \) is a continuous homomorphism \( \phi : G \to H \) with dense image so that \( H \) is a t.d.l.c. group and \( U \) is the preimage of a compact open subgroup of \( H \). We say that \( H \) is a t.d.l.c. completion of \( (G,U) \). When \( H \) is also second countable, we call \( H \) a t.d.l.c.s.c. completion.
A Hecke pair \((G, U)\) admits two canonical completions: The Belyaev completion, denoted \(\hat{G}_U\), and the Schlichting completions, denoted \(G/\!/U\). The canonical homomorphisms are denoted \(\beta_U\) and \(\beta_{G/U}\). These completions are the ‘largest’ and ‘smallest’ completions in the following sense:

**Theorem 1.6** (see Theorem 4.3). Suppose that \((G, U)\) is a Hecke pair and that \(H\) is a t.d.l.c. completion via \(\phi : G \rightarrow H\). Then there are unique continuous quotient maps \(\psi_1 : \hat{G}_U \rightarrow H\) and \(\psi_2 : H \rightarrow G/\!/U\) with compact kernels such that the following diagram commutes:

\[
\begin{array}{ccc}
G & \rightarrow & H \\
\downarrow \phi & \quad & \downarrow \beta_{G/U} \\
\hat{G}_U & \rightarrow & G/\!/U.
\end{array}
\]

The Hecke pair language can be eliminated as follows:

**Corollary 1.7.** Suppose that \(G\) is a topological group and that \(H\) is a t.d.l.c. completion via \(\phi : G \rightarrow H\). Letting \(U \leq G\) be the preimage of some compact open subgroup of \(H\), the pair \((G, U)\) is a Hecke pair, \(H\) is a t.d.l.c. completion of \((G, U)\), and there are unique continuous quotient maps \(\psi_1 : \hat{G}_U \rightarrow H\) and \(\psi_2 : H \rightarrow G/\!/U\) with compact kernels such that the following diagram commutes:

\[
\begin{array}{ccc}
G & \rightarrow & H \\
\downarrow \phi & \quad & \downarrow \beta_{G/U} \\
\hat{G}_U & \rightarrow & G/\!/U.
\end{array}
\]

The Belyaev completion enjoys a further property.

**Theorem 1.8** (see Theorem 4.7). Suppose that \((G, U)\) is a Hecke pair and \(H\) is a t.d.l.c. group with \(\phi : G \rightarrow H\) a continuous homomorphism such that \(\overline{\phi(U)}\) is profinite. There is then a unique continuous homomorphism \(\hat{\psi} : \hat{G}_U \rightarrow H\) such that \(\phi = \hat{\psi} \circ \beta_{(G,U)}\). If in addition \(\phi\) has dense image and \(\overline{\phi(U)}\) is open in \(H\), then \(\hat{\psi}\) is a quotient map.

In view of Theorem 1.6, the locally compact non-compact structure of a completion depends only on the Hecke pair. Supporting this claim, we isolate a variety of locally compact non-compact properties which are common to all completions of a given Hecke pair.

**Proposition 1.9** (see [13]). Let \((G, U)\) be a Hecke pair. For each of the following properties, either every completion of \((G, U)\) has the property, or every completion of \((G, U)\) fails to have the property.

1. Being \(\sigma\)-compact.
2. Being compactly generated.
3. Having a quotient isomorphic to \(N\) where \(N\) is any specified t.d.l.c. group that has no non-trivial compact normal subgroups.
4. Being amenable.
5. Being uniscalar.

Restricting to t.d.l.c.s.c. completions, being an elementary group, see [13], is also a property of Hecke pairs.
Proposition 1.10 (see Proposition 6.5). For \((G,U)\) a Hecke pair with \(G\) countable, either all t.d.l.c.s.c. completions of \((G,U)\) are elementary, or no t.d.l.c.s.c. completion of \((G,U)\) is elementary.

Our investigations conclude by developing techniques for building interesting Hecke pairs and therefore for building interesting t.d.l.c. groups. These techniques are then applied to produce compactly generated elementary groups with transfinite rank.

Theorem 1.11 (see Theorem 7.22). For each \(n \geq 1\), there is a compactly generated elementary group \(G_n\) with \(\xi(G_n) \geq \omega \cdot n + 2\).

Acknowledgments 1.12. The first named author would like to thank Aleksander Iwanow for pointing out the article [1] in response to an earlier preprint. The second named author would like to thank Pierre-Emmanuel Caprace for his helpful remarks and suggestions regarding the construction techniques of Section 7.

2. Preliminaries

All groups are taken to be Hausdorff topological groups and are written multiplicatively. We thus suppress the modifier “topological” when discussing topological groups. A quotient of a topological group must have closed kernel (so that the resulting quotient topology is Hausdorff). Topological group isomorphism is denoted \(\simeq\). We use “t.d.,” “l.c.,” and “s.c.” for “totally disconnected”, “locally compact”, and “second countable”, respectively. For a topological group \(G\), the set \(U(G)\) is the collection of compact open subgroups of \(G\).

We say two subgroups \(U\) and \(V\) of a group \(G\) are \textit{commensurate} if \(U \cap V\) has finite index in both \(U\) and \(V\). Given a closed subgroup \(U\) of \(G\), write \([U]\) for the \textit{commensurability class} of \(U\), meaning the set of closed subgroups that are commensurate to \(U\). We say \(g \in G\) \textit{commensurates} \(U\) if \(gUg^{-1}\) is commensurate with \(U\); equivalently, the action of \(g\) by conjugation preserves the commensurability class of \(U\). A \textit{commensurated subgroup} of \(G\) is a subgroup that is commensurated by every element of \(G\).

Definition 2.1. A \textit{Hecke pair} is a pair of groups \((G,U)\) where \(G\) is a topological group and \(U\) is an open subgroup of \(G\) that is commensurated. We say \((G,U)\) is \textit{countable} if \(G\) is a countable group. The pair \((G,U)\) is \textit{finitely generated} if there is a finite set \(F \subseteq G\) so that \(\langle F,U \rangle = G\). The pair \((G,U)\) is \textit{proper} if \(\bigcap_{g \in G} gUg^{-1} = \{1\}\).

2.1. Uniformities. Our approach to completions is via uniform spaces; our discussion of uniform spaces follows [2].

Definition 2.2. Let \(X\) be a set. A \textit{uniformity} \(\Phi\) is a set of binary relations \(\Phi\) on \(X\), called \textit{entourages}, with the following properties:

(a) Each \(A \in \Phi\) is reflexive, that is, \(\{(x,x) \mid x \in X\} \subseteq A\).
(b) For all \(A, B \in \Phi\), there exists \(C \in \Phi\) such that \(C \subseteq A \cap B\).
(c) For all \(A \in \Phi\), there exists \(B \in \Phi\) such that \(B \circ B := \{(x,z) \mid \exists y \in X : \{x,y,(y,z)\} \subseteq B\}\)

is a subset of \(A\).
(d) For all \(A \in \Phi\), there exists \(B \in \Phi\) such that the set \(\{(y,x) \mid (x,y) \in B\}\) is a subset of \(A\).

A set with a uniformity is called a \textit{uniform space}.
Definition 2.3. Let \((X, \Phi)\) be a uniform space. A filter \(f\) of subsets of \(X\) is called a minimal Cauchy filter if \(f\) is a \(\subseteq\)-least filter so that for all \(U \in \Phi\) there is \(A \in f\) so that \(A \times A \subseteq U\).

Definition 2.4 ([2, II.3.7]). The completion of a uniform space \((X, \Phi)\) is defined to be
\[
\hat{X} := \{f \mid f \text{ is a minimal Cauchy filter}\}
\]
along with the uniformity \(\hat{\Phi}\) given by entourages of the form
\[
\hat{U} := \{(f, g) \mid \exists A \in f \cap g \text{ so that } A \times A \subseteq U\}
\]
where \(U \in \Phi\).

There is a canonical, continuous \(i : X \rightarrow \hat{X}\) which has dense image defined by \(x \mapsto f_x\) where \(f_x\) is the minimal Cauchy filter containing the neighborhoods of \(x\).

A topological group has two canonical uniformities:

Definition 2.5. Suppose that \(G\) is a topological group and let \(B\) be a basis at 1. The left uniformity \(\Phi_l\) consists of entourages of the form
\[
U_l := \{(x, y) \mid x^{-1}y \in U\}
\]
where \(U \in B\). The right uniformity \(\Phi_r\) consists of entourages of the form
\[
U_r := \{(x, y) \mid xy^{-1} \in U\}
\]
where \(U \in B\).

The completion of \(G\) with respect to the right uniformity becomes a topological group exactly when the inverse function interacts nicely with the right uniformity.

Theorem 2.6 ([2, III.3.4 Theorem 1]). Suppose that \(G\) is a topological group and that \(\Phi_r\) is the right uniformity. The completion \(\hat{G}\) is a topological group if and only if the inverse map carries minimal \(\Phi_r\)-Cauchy filters to \(\Phi_r\)-Cauchy filters.

Theorem 2.7 ([2, III.3.4 Theorem 1]). Suppose that \(G\) is a topological group, \(\Phi_r\) is the right uniformity, and the completion \(\hat{G}\) is a topological group. Then the following hold:

1. The map \(i : G \rightarrow \hat{G}\) is a continuous homomorphism with dense image.
2. Multiplication on \(\hat{G}\) is defined as follows: given \(f, f' \in \hat{G}\), then \(ff'\) is the minimal Cauchy filter of subsets of \(G\) generated by sets \(AB \subseteq G\) where \(A \in f\) and \(B \in f'\).

3. A General Construction for Completions

We first devise a procedure for producing completions. The goal of our construction is to apply Theorem 2.6. The subtlety in our construction is that we isolate left and right uniformities in a possibly coarser group topology and complete with respect to these.

Definition 3.1. Let \(G\) be a group and let \(S\) be a set of open subgroups of \(G\). We say that \(S\) is a \(G\)-stable local filter if the following conditions hold:

(a) \(S\) is non-empty;
(b) Any two elements of \(S\) are commensurate;
(c) \(S\) is a filter in its commensurability class, that is, given a finite subset \(\{V_1, \ldots, V_n\}\) of \(S\), then \(\bigcap_{i=1}^n V_i \in S\), and given \(V \leq W \leq G\) such that \(|W : V|\) is finite, then \(V \in S\) implies \(W \in S\);
(d) \(S\) is stable under conjugation in \(G\), that is, given \(V \in S\) and \(g \in G\), then \(gVg^{-1} \in S\).
We say $S$ is a $G$-stable local filter of $[U]$ if in addition $S \subseteq [U]$.

**Remark 3.2.** If $S$ is a $G$-stable local filter, then $S$ is a filter on $[V]$ for any $V \in S$. Furthermore, $[V]$ must be stable under the conjugation action of $G$, hence $V$ is a commensurated subgroup.

The $G$-stable local filters give rise to left and right uniformities on $G$.

**Definition 3.3.** Let $S$ be a $G$-stable local filter. The right $S$-uniformity is

$$\mathcal{R}_S := \{ R_U \mid U \in S \}$$

where $R_U := \{ (x, y) \in G^2 \mid Ux = Uy \}$.

The left $S$-uniformity is

$$\mathcal{L}_S := \{ L_U \mid U \in S \}$$

where $L_U := \{ (x, y) \in G^2 \mid xU = yU \}$.

One verifies that both $\mathcal{L}_S$ and $\mathcal{R}_S$ are uniformities. Additionally, they induce a group topology on $G$ potentially coarser than the original topology, and in this new group topology, $\mathcal{L}_S$ and $\mathcal{R}_S$ are precisely the left and right uniformities.

We remark that, in general, $G$ is not Hausdorff with respect to either uniformity, but the quotient group $G/K$ is Hausdorff with respect to the induced uniformities where $K$ is the normal subgroup $\bigcap_{U \in S} U$.

**Definition 3.4.** Let $S$ be a $G$-stable local filter. We define a right $S$-Cauchy filter $f$ in $G$ to be a minimal Cauchy filter with respect to the uniformity $\mathcal{R}_S$. In other words, $f$ is a filter of subsets of $G$ with the following properties:

(a) For every $V \in S$, there is exactly one right coset $Vg$ of $V$ in $G$ such that $Vg \in f$;
(b) Every element of $f$ contains a right coset of some element of $S$.

Left $S$-Cauchy filters are defined similarly with left cosets in place of right cosets. Notice that for each $g \in G$, there is a corresponding principal right $S$-Cauchy filter $f_g$ generated by $\{ Vg \mid V \in S \}$. Where the choice of $S$ is clear, we will write ‘Cauchy’ to mean ‘$S$-Cauchy’.

**Lemma 3.5.** Let $(G, U)$ be a Hecke pair, $N$ be a subgroup of $U$ of finite index, and $g \in G$. Then there are $h_1, \ldots, h_n \in G$ so that for all $h \in G$, the set $Ng \cap hN$ is either empty or a union of finitely many right cosets of $N \cap \bigcap_{i=1}^n h_iNh_i^{-1}$.

**Proof.** Suppose $Ng \cap hN \neq \emptyset$ and put $R := N \cap g^{-1}Ng$. For all $h \in G$ we have $(Ng \cap hN)R = Ng \cap hN$, so $Ng \cap hN$ is a union of left cosets of $R$ in $G$. The left cosets of $R$ in $G$ that are subsets of $Ng$ are exactly those of the form $gtR$ for $t \in g^{-1}Ng$; indeed,

$$xR \subseteq Ng \iff g^{-1}xR \subseteq g^{-1}Ng \iff g^{-1}x \in g^{-1}Ng.$$

Since $R$ has finite index in $g^{-1}Ng$, we deduce that only finitely many left cosets of the form $gtR$ exist. It now follows the set $\{ Ng \cap hN \mid Ng \cap hN \neq \emptyset \}$ is finite.

Say that $h_1, \ldots, h_n \in G$ are so that

$$\{ Ng \cap hN \mid Ng \cap hN \neq \emptyset \text{ and } h \in G \} = \{ Ng \cap h_1N, \ldots, Ng \cap h_nN \}.$$ Setting $M := N \cap \bigcap_{i=1}^n h_iNh_i^{-1}$, we see that $M(Ng \cap hN) = Ng \cap hN$ for all $h \in G$ with $Ng \cap hN \neq \emptyset$. Therefore, $Ng \cap hN$ is a union of right cosets of $M$. That this union is finite follows just as in the previous paragraph. \[ \square \]

**Lemma 3.6.** Let $G$ be a topological group, $S$ be a $G$-stable local filter, and $f$ be a set of subsets of $G$. Then $f$ is a right $S$-Cauchy filter in $G$ if and only if $f$ is a left $S$-Cauchy filter in $G$. 

Proof. By symmetry, it suffices to assume \( f \) is a right Cauchy filter and prove that \( f \) is a left Cauchy filter.

Fixing \( V \in S \), the filter \( f \) contains some right coset \( Vg \) of \( V \). Applying Lemma 3.5 to \( V \) and \( g \), we produce a finite intersection \( W \) of conjugates of \( V \) such that \( W \leq V \) and the set \( Vg \cap hV \) is either empty or a union of right cosets of \( W \) for any \( h \in G \). Since \( S \) is closed under conjugation and finite intersection, we additionally have \( W \in S \).

Since \( f \) is right Cauchy, there is a unique right coset \( Wk \) of \( W \) contained in \( f \), and since \( \emptyset \not\in f \), it must be the case that \( Wk \subseteq Vg \). Observing that \( Vg = \bigcup_{h \in G} Vg \cap hV \), there is some \( h \in G \) so that \( Wk \) intersects \( Vg \cap hV \). Lemma 3.5 then ensures that indeed \( Wk \subseteq Vg \cap hV \), hence \( hV \in f \). We conclude that \( f \) contains a left coset of \( V \) for every \( V \in S \). Since \( f \) is a filter and \( \emptyset \not\in f \), in fact \( f \) contains exactly one left coset of \( V \) proving (a) of the definition of a left \( S \)-Cauchy filter.

Given any element \( A \in f \), then \( A \) contains \( Vg \) for some \( V \in S \); in particular, \( A \) contains the left coset \( g(g^{-1}Vg) \) of \( g^{-1}Vg \in S \), proving (b). We thus deduce that \( f \) is also a left \( S \)-Cauchy filter. \( \square \)

Taking inverses in the group \( G \) sends right Cauchy filters to left Cauchy filters and vice versa, so it preserves the set of Cauchy filters which are both left and right Cauchy filters. Lemma 3.6 ensures all right Cauchy filters are also left Cauchy filters, whereby the following corollary holds:

**Corollary 3.7.** For \( G \) a topological group and \( S \) a \( G \)-stable local filter, the set of right \( S \)-Cauchy filters in \( G \) is preserved by the map on subsets induced by taking the inverse.

With Corollary 3.7 in hand, we may now apply Theorem 2.6 to produce a completion of \( G \) with respect to \( R_S \), denoted \( \hat{G}_S \). Specifically,

- The elements of \( \hat{G}_S \) are the (left) \( S \)-Cauchy filters in \( G \).
- The set \( \hat{G}_S \) is equipped with a uniformity with entourages of the form
  
  \[ E_U := \{(f, f') \mid \exists g \in G : Ug \in f \cap f' \} \quad (U \in S) \]

  and topology generated by this uniformity.
- The map \( \beta_{(G, S)} \) given by
  
  \[ G \to \hat{G}_S; \quad g \mapsto f_g \]

  is continuous, since the topology induced by \( R_S \) is coarser than the topology on \( G \). The image is dense, and the kernel is \( \bigcap S \). Where there is no ambiguity, we will simply write \( \beta \) for the map \( \beta_{(G, S)} \).
- There is a unique continuous group operation on \( \hat{G}_S \) such that \( \beta_{(G, S)} \) is a homomorphism. In fact, we can define multiplication on \( \hat{G}_S \) as follows: given \( f, f' \in \hat{G}_S \), then \( ff' \) is the minimal Cauchy filter of subsets of \( G \) generated by sets \( AB \subset G \) where \( A \in f \) and \( B \in f' \).

**Definition 3.8.** For \( G \) a topological group and \( S \) a \( G \)-stable local filter, we call \( \hat{G}_S \) the **completion** of \( G \) with respect to \( S \).

We now establish a correspondence between t.d.l.c. completions and completions with respect to a \( G \)-stable local filter.

**Proposition 3.9.** If \( G \) is a group and \( S \) is a \( G \)-stable local filter, then the following hold:
(1) The topological group $\hat{G}_S$ is a t.d.l.c. completion of $G$. 

(2) For $V \in S$, the subgroup $\beta(V)$ is compact and open and is naturally isomorphic as a topological group to the profinite completion of $V$ with respect to the quotients $V/N$ such that $N \in S$ and $N \leq V$.

Proof. For $V \in S$, set $B_V := \{f \in \hat{G}_S \mid V \in f\}$. From the definition of the topology of $\hat{G}_S$, the collection $\{B_V \mid V \in S\}$ is a base of identity neighborhoods in $\hat{G}_S$. Moreover, each $B_V$ is closed under multiplication and inverse, so $B_V$ is a subgroup of $\hat{G}_S$. Therefore, $\hat{G}_S$ has a base of identity neighborhoods consisting of open subgroups.

For $V \in S$, define $\mathcal{N}_V := \{\bigcap_{v \in V} vWv^{-1} \mid V \geq W \in S\}$. The set $\mathcal{N}_V$ is precisely the set of elements of $S$ that are normal in $V$; these necessarily have finite index in $V$. Form $\hat{V}_S$, the profinite completion of $V$ with respect to the finite quotients $\{V/N \mid N \in \mathcal{N}_V\}$. Representing $\hat{V}_S$ as a closed subgroup of $\prod_{N \in \mathcal{N}_V} V/N$ in the usual way, we define $\theta : B_V \to \hat{V}_S$ by setting $\theta(f) := (Ng)_{N \in \mathcal{N}_V}$ where $Ng$ is the unique coset of $N$ in $f$. One verifies that $\theta$ is an isomorphism of topological groups.

The set $\{B_V \mid V \in S\}$ is thus a basis at 1 of compact open subgroups, so $\hat{G}_S$ is a t.d.l.c. group. Since $\beta$ has a dense image, $\hat{G}_S$ is a t.d.l.c. completion of $G$, verifying (1). The density of $\text{img}(\beta)$ additionally implies that each $V \in S$ is so that $\beta(V) = B_V$, and (2) now follows from the previous paragraph.

For $V \in S$, we will often abuse notation slightly and say that the profinite completion $\hat{V}_S$ equals the compact open subgroup $\beta(G,S)(V)$ of $\hat{G}_S$. Notice that the proof of Proposition 3.9 ensures that $V$ is precisely the preimage of $\hat{V}_S$ under $\beta(G,S)$.

Remark 3.10. The completion we have produced generalizes the usual notion of the profinite completion of a group. Indeed, taking $S$ to be the collection of finite index subgroups of $G$, Proposition 3.9 implies $\hat{G}_S$ is exactly the profinite completion of $G$.

Proposition 3.9 gives one method for producing t.d.l.c. completions of a group $G$. In fact, just as in the profinite case, we see that all t.d.l.c. completions of $G$ arise in this way.

Theorem 3.11. If $G$ is a group and $H$ is a t.d.l.c. completion of $G$ via $\phi : G \to H$, then the set of all preimages of compact open subgroups of $H$ is a $G$-stable local filter $S$, and moreover, there is a unique topological group isomorphism $\psi : \hat{G}_S \to H$ such that $\phi = \psi \circ \beta(G,S)$.

Proof. Setting $S := \{\phi^{-1}(M) \mid M \in \mathcal{U}(H)\}$, one verifies that $S$ is a $G$-stable local filter.

For $f \in \hat{G}_S$, define $\tilde{f} := \bigcap\{\phi(Vg) \mid g \in G, V \in S, \text{ and } Vg \in f\}$. Fix $L$ a compact open subgroup of $H$. Every closed subgroup $P$ of $L$ of finite index contains $\phi(V)$ for some $V \in S$, namely $V := \phi^{-1}(P)$. Since $L$ admits a basis of finite index closed subgroups, it follows that $1 = \bigcap\{\phi(V) \mid V \in S\}$ is the trivial group. For general $f$, we conclude that $|\tilde{f}| \leq 1$ since $\tilde{f}$ is an intersection of compact cosets of a basis at 1. The set $\tilde{f}$ is the intersection of compact sets so that any finite intersection of which is non-empty since $f$ is a filter. Hence $|\tilde{f}| = 1$. 


We define $\psi : \hat{G}_S \to H$ by setting $\psi(f)$ to be the unique element of $\hat{f}$; one verifies that $\psi$ is an injective homomorphism and that $\phi = \psi \circ \beta$. To see that $\psi$ is continuous, it suffices to consider the basis at 1 of compact open subgroups in $H$: For a compact open subgroup $P$ of $H$, we see that $\psi^{-1}(P) = \hat{V}_S$ where $V = \phi^{-1}(P)$, hence $\psi$ is continuous. The equation $\phi = \psi \circ \beta_{(G,S)}$ additionally determines $\psi$ uniquely as a continuous map since $\beta_{(G,S)}(G)$ is dense in $\hat{G}_S$.

It remains to show that $\psi$ is an isomorphism of topological groups. The group $\hat{G}_S$ has a base of identity neighborhoods of the form $\{\hat{V}_S \mid V \in S\}$. For each $V \in S$, say $\phi^{-1}(P) = V$ for $P \in \mathcal{U}(H)$, the image $\phi(V)$ is dense in $P$, and the image $\beta_{(G,S)}(V)$ is dense in $\hat{V}_S$. Since $\phi = \psi \circ \beta_{(G,S)}$, we infer that $\psi(\hat{V}_S)$ is also a dense subgroup of $P$. The map $\psi$ is continuous and $\hat{V}_S$ is compact, so in fact $\psi(\hat{V}_S) = P$.

The map $\psi$ is therefore an open, continuous, and injective homomorphism with open dense image. Since $H$ is a Baire space, it follows $\psi$ is also onto, whereby it is an isomorphism of topological groups.

**Remark 3.12.** Theorem 3.11 shows that, up to isomorphism, all t.d.l.c. completions of a group $G$ have the form $\hat{G}_S$ for $S$ some $G$-stable local filter.

We conclude this section by observing a general property of $G$-stable local filters with an inclusion relation.

**Proposition 3.13.** If $G$ is a group and $S$ and $S'$ are $G$-stable local filters, then the following are equivalent:

1. $S'$ is a subset of $S$;
2. There is a continuous homomorphism with compact kernel $\pi : \hat{G}_S \to \hat{G}_{S'}$ such that $\beta_{(G,S')} = \pi \circ \beta_{(G,S)}$.

In the case (1) and (2) hold, the map $\pi$ is a uniquely determined quotient map.

**Proof.** Suppose that $S'$ is a subset of $S$. Given an $S$-Cauchy filter $f$, we define

$$f' := \{K \in f \mid \exists g \in G \ \exists W \in S' \ gW \subseteq K\}.$$  

One verifies that $f'$ is an $S'$-Cauchy filter, so we obtain a map $\pi : \hat{G}_S \to \hat{G}_{S'}$ via $f \mapsto f'$. Moreover, the map $\pi$ is a homomorphism with dense image such that $\beta_{(G,S')} = \pi \circ \beta_{(G,S)}$.

Given $V \in S'$, we see that $\pi(\hat{V}_S) = \hat{V}_{S'}$ and that $\pi^{-1}(\hat{V}_{S'}) = \hat{V}_S$. Therefore, $\pi$ is a continuous open map, and since it has dense image and t.d.l.c. groups are Baire spaces, $\pi$ is onto. The group $\hat{V}_S$ contains the kernel of $\pi$, so $\pi$ also has compact kernel. The uniqueness of $\pi$ as a continuous homomorphism follows from the fact that $G$ has dense image in both $\hat{G}_S$ and $\hat{G}_{S'}$. We have now shown that (1) implies (2), and that in this situation $\pi$ has the additional specified properties.

Conversely, suppose that there exists a continuous map $\pi : \hat{G}_S \to \hat{G}_{S'}$ such that $\beta_{(G,S')} = \pi \circ \beta_{(G,S)}$. Taking $V \in S'$, the group $\pi^{-1}(\hat{V}_{S'})$ is a compact open subgroup $\hat{G}_S$ such that the preimage under $\beta_{(G,S)}$ is $V$. On the other hand, $\hat{G}_S$ has a basis at 1 of subgroups $\hat{Y}_S$ for $Y \in S$, so $\pi^{-1}(\hat{V}_{S'}) \geq \hat{Y}_S$ for some $Y \in S$. It now follows that $V \geq Y$ and $|V : Y| < \infty$, and thus, $V \in S$ since $S$ is a filter in its commensurability class. As $V \in S'$ was arbitrary, we conclude that $S' \subseteq S$, verifying that (2) implies (1).
Notice that if \( S \neq S' \), then \( \beta_{(G,S')} \) and \( \beta_{(G,S)} \) are not equivalent as completion maps for \( G \), in the sense that there does not exist a topological isomorphism \( \pi : \hat{G}_S \to \hat{G}_{S'} \) such that \( \beta_{(G,S')} = \pi \circ \beta_{(G,S)} \).

4. The Belyaev and Schlichting completions

We now consider the setting of Hecke pairs. We shall see this is not a restriction; Hecke pairs merely give a useful way to organize our discussion.

**Definition 4.1.** A completion map for a Hecke pair \((G,U)\) is a continuous homomorphism \( \phi : G \to H \) with dense image so that \( H \) is a t.d.l.c. group and \( U \) is the preimage of a compact open subgroup of \( H \). We say that \( H \) is a t.d.l.c. completion of \((G,U)\). When \( H \) is also second countable, we call \( H \) a t.d.l.c.s.c. completion.

For a Hecke pair \((G,U)\), one only considers \( G \)-stable local filters of \([U]\). There are two canonical \( G \)-stable local filters of \([U]\) defined as follows: The Belyaev filter for \([U]\) is \( S_B := [U] \). The Schlichting filter for \((G,U)\) is

\[
S_{G/U} := \{ V \in [U] \mid \exists g_1, \ldots, g_n \in G \text{ so that } \bigcap_{i=1}^n g_i U g_i^{-1} \subseteq V \}.
\]

We stress that the Schlichting filter depends on the specific Hecke pair \((G,U)\), while the Belyaev filter only depends on the group \( G \) and the commensurability class \([U]\).

**Definition 4.2.** The Belyaev completion for \((G,U)\), denoted \( \hat{G}_U \), is defined to be \( \hat{G}_{S_B} \). The canonical inclusion map \( \beta_{(G,S_B)} \) is denoted \( \beta_U \). The Schlichting completion for \((G,U)\), denoted \( G/U \), is defined to be \( \hat{G}_{S_{G/U}} \). The canonical inclusion map \( \beta_{(G,S_{G/U})} \) is denoted \( \beta_{G/U} \).

Given any \( G \)-stable local filter \( S \) of \([U]\) which contains \( U \), we have \( S_{G/U} \subseteq S \subseteq S_B \). The Belyaev filter is thus maximal while the Schlichting filter is minimal for \( G \)-stable local filters which contain \( U \). The Belyaev and Schlichting completions are thus maximal and minimal completions of a Hecke pair in the following strong sense:

**Theorem 4.3.** Suppose that \((G,U)\) is a Hecke pair and that \( H \) is a t.d.l.c. completion via \( \phi : G \to H \). Then there are unique continuous quotient maps \( \psi_1 : \hat{G}_U \to H \) and \( \psi_2 : H \to G/U \) with compact kernels such that the following diagram commutes:

\[
\begin{array}{ccc}
\hat{G}_U & \xrightarrow{\psi_1} & H \\
\beta_U \downarrow & \ & \beta_{G/U} \downarrow \\
G & \xrightarrow{\phi} & \hat{G}_{G/U} \\
\end{array}
\]

**Proof.** Set \( S := \{ \phi^{-1}(V) \mid V \in \mathcal{U}(H) \} \); observe that \( S \) contains \( U \) since \( H \) is a t.d.l.c. completion of the pair \((G,U)\). By Theorem 3.11 \( S \) is a \( G \)-stable local filter, and there is a unique topological group isomorphism \( \psi : \hat{G}_S \to H \) so that \( \phi = \psi \circ \beta_{(G,S)} \).
Since \( S \) contains \( U \), Proposition 3.13 ensures there are unique continuous quotient maps \( \pi_1 \) and \( \pi_2 \) with compact kernels so that the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\beta_U} & G_{U} \\
\downarrow{\beta_{(G,S)}} & & \downarrow{\beta_{G/U}}
\end{array}
\] \[
\begin{array}{ccc}
\hat{G}_U & \xrightarrow{\pi_1} & G_S \\
\downarrow{\pi_2} & & \downarrow{G/U}
\end{array}
\]

It follows that \( \psi_1 := \psi \circ \pi_1 \) and \( \psi_2 := \pi_2 \circ \psi^{-1} \) make the desired diagram commute and both are continuous quotient maps with compact kernels. Uniqueness follows since \( \psi, \pi_1 \), and \( \pi_2 \) are unique.

This theorem easily extends to the general case.

**Corollary 4.4.** Suppose that \( G \) is a group and that \( H \) is a t.d.l.c. completion via \( \phi : G \to H \). Letting \( U \leq G \) be the preimage of some compact open subgroup of \( H \), the pair \( (G,U) \) is a Hecke pair, \( H \) is a t.d.l.c. completion of \( (G,U) \), and there are unique continuous quotient maps \( \psi_1 : \hat{G}_U \to H \) and \( \psi_2 : H \to G/U \) with compact kernels such that the following diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & \hat{G}_U \\
\downarrow{\beta_{G/U}} & & \downarrow{\beta_{(G,S)}}
\end{array}
\] \[
\begin{array}{ccc}
\hat{G}_U & \xrightarrow{\psi_1} & H \\
\downarrow{\psi_2} & & \downarrow{G/U}
\end{array}
\]

**Proof.** Fix \( L \) a compact open subgroup of \( H \) and set \( U := \phi^{-1}(L) \). The subgroup \( U \) is an open commensurated subgroup of \( G \), so \( (G,U) \) is a Hecke pair. Since \( \phi \) has dense image, we conclude that \( H \) is a completion of \( (G,U) \). The final claim of the corollary now follows from Theorem 4.3.

Theorem 4.3 shows all possible completions of a Hecke pair \( (G,U) \) differ only by a compact normal subgroup. The locally compact, non-compact structure of a t.d.l.c. completion thus depends only on the Hecke pair - indeed only on the commensurability class \( [U] \); contrast this with the many different profinite completions a group can admit. We give precise statements illustrating this phenomenon in Section \( \text{\ref{sec:6}} \).

Corollary 4.4 shows the problem of classifying all continuous homomorphisms with dense image from a specified group \( G \) (possibly discrete) to an arbitrary t.d.l.c. group can, in principle, be broken into two steps:

1. Classify the commensurability classes \( \alpha \) of open subgroups of \( G \) that are invariant under conjugation; this typically amounts to classifying commensurated open subgroups.
2. For each such class \( \alpha \), form the Belyaev completion \( \hat{G}_U \) for some (any) \( U \in \alpha \) and classify the quotients of \( \hat{G}_U \) with compact kernel.

A number of researchers have already considered (1). Shalom and Willis classify commensurated subgroups of many arithmetic groups; see \[ \text{\cite{12}} \]. Other examples include classifications of commensurated subgroups of almost automorphism groups \[ \text{\cite{7}} \] and Burger-Mozes simple universal groups \[ \text{\cite{8}} \].
4.1. Further properties. We note several additional properties of completions of Hecke pairs.

The Schlichting completion has a natural description. Suppose \((G, U)\) is a Hecke pair and let \(\sigma_{(G, U)} : G \to \text{Sym}(G/U)\) be the permutation representation given by left multiplication. We consider \(\text{Sym}(G/U)\) to be equipped with the topology of pointwise convergence.

**Proposition 4.5.** For \((G, U)\) a Hecke pair, there is a unique topological group isomorphism \(\psi : G/U \to \sigma_{(G, U)}(G)\) so that \(\sigma_{(G, U)} = \psi \circ \beta_{G/U}\).

**Proof.** For \(Y \subseteq G\), set \(\hat{Y} := \sigma_{(G, U)}(Y)\). The orbits of \(\sigma(U)\) are finite on \(G/U\), so \(\hat{U}\) is a profinite group. On the other hand, \(\hat{U} = \text{Stab}_{\hat{G}}(U)\), hence it is open. It now follows that \(\hat{G}\) is a t.d.l.c. completion of \(G\).

The basis for the topology on \(\hat{G}\) is given by stabilizers of finite sets of cosets. Such stabilizers are exactly of the form \(\bigcap_{g \in F} \sigma(g)\hat{U}\sigma(g^{-1})\) with \(F \subseteq G\) finite. For every \(V \in \mathcal{U}(\hat{G})\), the subgroup \(\sigma^{-1}_{(G, U)}(V)\) therefore contains \(\bigcap_{g \in F} gUg^{-1}\) for some \(F \subseteq G\) finite. The \(G\)-stable local filter \(\mathcal{S} := \{\sigma^{-1}_{(G, U)}(V) \mid V \in \mathcal{U}(G)\}\) is thus exactly the Schlichting filter \(\mathcal{S}_{G/U}\). Theorem 4.11 now implies the proposition.

Via Theorem 4.3 and Proposition 4.5 we recover a result from the literature.

**Corollary 4.6** (Shalom–Willis [12], Lemma 3.5 and Corollary 3.7). Suppose that \(G\) is a t.d.l.c. group and that \(H\) is a t.d.l.c. completion via \(\phi : G \to H\). If \(U\) is the preimage of a compact open subgroup of \(H\), then there is a unique quotient map \(\psi : H \to G/U\) so that \(\sigma_{(G, U)} = \psi \circ \phi\).

We next show the Belyaev completion satisfies a stronger universality property.

**Theorem 4.7.** Suppose that \((G, U)\) is a Hecke pair and \(H\) is a t.d.l.c. group with \(\phi : G \to H\) a continuous homomorphism such that \(\phi(U)\) is profinite. There is then a unique continuous homomorphism \(\psi : \hat{G}_{U} \to H\) such that \(\phi = \psi \circ \beta_{(G, U)}\). If in addition \(\phi\) has dense image and \(\phi(U)\) is open in \(H\), then \(\psi\) is a quotient map.

**Proof.** Let \(\beta := \beta_{(G, U)}\), set \(L := \phi(U)\), and put \(\mathcal{F}_{U} := \{\phi^{-1}(V) \cap U \mid V \in \mathcal{U}(H)\}\). Given \(f \in \hat{G}_{U}\), we define \(\hat{f} := \bigcap\{\phi(gW) \mid g \in G, W \in \mathcal{F}_{U}, gW \in f\}\).

Consider first \(\hat{I} = \bigcap\{\phi(W) \mid W \in \mathcal{F}_{U}\}\). Every closed subgroup \(P\) of \(L\) of finite index contains \(V \cap P\) for some \(V \in \mathcal{U}(H)\). Therefore, \(\phi(W) \leq P\) for some \(W \in \mathcal{F}_{U}\), namely \(W := \phi^{-1}(V) \cap U\). Since \(L\) is profinite, this ensures that \(\hat{I}\) is the trivial group. For general \(f\), we conclude that \(|\hat{f}| \leq 1\) by the construction of \(\hat{f}\) as an intersection of cosets; since \(\hat{f}\) is the intersection of compact sets any finite intersection of which is non-empty, in fact \(|\hat{f}| = 1\). We thus define \(\psi : \hat{G}_{U} \to H\) by setting \(\psi(f)\) to be the unique element of \(\hat{f}\). One verifies that \(\psi\) is a homomorphism and that \(\phi = \psi \circ \beta\).

To see that \(\psi\) is continuous, fix \((V_{\alpha})_{\alpha \in I}\) a basis at \(1\) of compact open subgroups for \(H\), put \(W_{\alpha} := \phi^{-1}(V_{\alpha}) \cap U\), and observe, as in the previous paragraph, that \(\bigcap_{\alpha \in I} \phi(W_{\alpha}) = \{1\}\). Consider a convergent net \(f_{\beta} \to f\) in \(\hat{G}_{U}\). For each subgroup \(W_{\alpha}\), there is \(\eta_{\alpha} \in I\) so that \(f_{\gamma}f_{\gamma}^{-1}\) contains \(W_{\alpha}\) for all \(\gamma, \gamma' \geq \eta_{\alpha}\). We conclude that \(\psi(f_{\gamma}f_{\gamma}^{-1}) \in \phi(W_{\alpha})\) for all such \(\gamma, \gamma'\).
Since the intersection of the \( \phi(W_\alpha) \) is trivial and each \( \phi(W_\alpha) \) is compact, we see that \( \psi(f_b) \) is a convergent net in \( H \), hence \( \psi \) is continuous. Additionally, since \( \beta(G) \) is dense in \( \hat{G}_U \), the equation \( \phi = \psi \circ \phi \) determines the restriction of \( \psi \) to \( \beta(G,U)(G) \) and hence determines \( \psi \) uniquely as a continuous map.

Finally, suppose that \( \phi(U) \) is open in \( H \) and that \( \phi \) has dense image. For \( \hat{U} := \bar{\beta(U)} \), the group \( \hat{U} \) is a compact open subgroup of \( \hat{G}_U \), so \( \psi(\hat{U}) \) is compact in \( H \); in particular, \( \psi(\hat{U}) \) is closed in \( H \). We see also that \( \psi(\hat{U}) \geq \phi(U) \), so \( \psi(\hat{U}) \) is indeed open in \( H \). The image of \( \psi \) is therefore an open and dense subgroup of \( H \). As \( H \) is a Baire space, we conclude that \( \psi \) is onto, and it follows that \( \psi \) is a quotient map. \( \square \)

A standard universal property argument now shows that Theorem 4.7 characterizes the Belyaev completion up to topological isomorphism, so one can take Theorem 4.7 as the definition of the Belyaev completion.

We conclude this subsection by characterizing topological properties of completions of Hecke pairs.

**Proposition 4.8.** Suppose \((G,U)\) is a Hecke pair and \( S \) is a \( G \)-stable local filter of \([U]\). Then \( \hat{G}_S \) is first countable if and only if the set \( \{ V \in S \mid V \subseteq U \} \) is countable.

**Proof.** Let \( \beta : G \to \hat{G}_S \) be the completion map and set \( S_U := \{ V \in S \mid V \subseteq U \} \). If \( \hat{G}_S \) is first countable, then \( \hat{U}_S := \beta(U) \) has only countably many open subgroups. Since every element of \( S_U \) is the \( \beta \)-preimage of an open subgroup of \( \hat{U}_S \), it follows that \( S_U \) is countable. Conversely, if \( S_U \) is countable, then \( \hat{U}_S \) is the profinite completion of \( U \) with respect to a countable set of quotients via Proposition 3.9. The compact open subgroup \( \hat{U}_S \) is therefore first countable, and hence \( \hat{G}_S \) is first countable. \( \square \)

**Proposition 4.9.** Let \((G,U)\) be a Hecke pair with \( H \) a t.d.l.c. completion. Then,

1. \( H \) is \( \sigma \)-compact if and only if \( |G : U| \) is countable.
2. \( H \) is compactly generated if and only if \( G \) is generated by finitely many left cosets of \( U \).

**Proof.** Let \( \beta : G \to H \) be the completion map.

For (1), if \( H \) is \( \sigma \)-compact, then \( V := \beta(U) \) has only countably many left cosets in \( H \). Since \( U = \beta^{-1}(V) \), it follows that \( |G : U| \) is countable. Conversely, if \( |G : U| \) is countable, then since \( \beta(G) \) is dense in \( H \), there are only countably many left cosets of \( \beta(U) \) in \( H \), so \( H \) is \( \sigma \)-compact.

For (2), if \( H \) is compactly generated, then there exists a finite symmetric \( A \subseteq H \) such that \( G = \langle A \rangle V \) for \( V := \beta(U) \); see for instance [13 Proposition 2.4]. Since \( \beta(G) \) is dense, it follows we may assume that \( A = \beta(B) \) for a finite symmetric \( B \subseteq G \); see loc. cit. For every \( g \in G \), there thus exists \( v \in V \) and \( g' \in \langle B \rangle \) such that \( \beta(g) = \beta(g')v \). Since \( \beta^{-1}(V) = U \), it follows further that \( v = \beta(u) \) for some \( u \in U \). Thus, \( \beta((B,U)) = \beta(G) \), and as \( \ker \beta \leq U \), we infer that \( G = \langle B,U \rangle \). In particular, \( G \) is generated by finitely many left cosets of \( U \). Conversely, if \( G \) is generated by finitely many left cosets \( \{ b_1 U, b_2 U, \ldots, b_n U \} \) of \( U \), then \( \beta(\bigcup_{i=1}^{n} b_i U) \) generates a dense subgroup of \( H \), and hence the compact subset \( X := \bigcup_{i=1}^{n} \beta(b_i) \beta(U) \) generates a dense open subgroup of \( H \) and therefore generates \( H \). \( \square \)

Bringing together the previous two results gives a characterization of second countable completions.

**Proposition 4.10.** Let \((G,U)\) be a Hecke pair and \( S \) be a \( G \)-stable local filter of \([U]\). Then \( \hat{G}_S \) is a t.d.l.c.s.c. group if and only if \( \{ V \in S \mid V \subseteq U \} \) and \( |G : U| \) are countable.
Proof. Via \[\text{(5.3)}\], a locally compact group is second countable if and only if it is $\sigma$-compact and first countable. The proposition now follows from Propositions 4.9 and 4.8. \qed

When $G$ is a countable group, the requirement that $|G:U|$ is countable is trivially satisfied. This ensures there is always a t.d.l.c.s.e. completion of a countable Hecke pair $(G,U)$; the proof is straightforward and so omitted.

Corollary 4.11. If $(G,U)$ is a countable Hecke pair, then the Schlichting completion $G\sslash U$ is a t.d.l.c.s.e. group.

5. Factorization of completion maps

Given two completion maps $\beta_1 : G \to H_1$ and $\beta_2 : G \to H_2$ from $G$ to t.d.l.c. groups $H_1$ and $H_2$, it is natural to ask whether there is continuous homomorphism $\alpha : H_1 \to H_2$ such that $\beta_2 = \alpha \circ \beta_1$. (Note that $\alpha$ is necessarily unique in this situation and has dense image.)

By Theorem 3.11 we may assume that $H_i = \hat{G}_{S_i}$ for some $G$-stable local filter $S_i$ and that $\beta_i$ is the usual completion map. In view of this, Proposition 3.8 already characterizes when such an $\alpha$ exists which has a compact kernel. We now show there are natural necessary and sufficient criteria for $\beta_2$ to factor through $\beta_1$ in terms of $S_1$ and $S_2$. One can even identify when $\alpha$ is injective or has a discrete kernel.

Proposition 5.1. Let $G$ be a group with $G$-stable local filters $S_1$ and $S_2$. Then the following are equivalent:

(1) There is a continuous homomorphism $\alpha : \hat{G}_{S_1} \to \hat{G}_{S_2}$ such that $\beta_{(G,S_2)} = \alpha \circ \beta_{(G,S_1)}$.

(2) For all $V_2 \in S_2$, there exists $V_1 \in S_1$ such that $V_1 \leq V_2$.

Proof. Set $\beta_1 := \beta_{(G,S_1)}$ and $\beta_2 := \beta_{(G,S_2)}$.

Suppose that (1) holds and let $V_2 \in S_2$. There exists $L \in \mathcal{U}(\hat{G}_{S_2})$ such that $V_2 = \beta_2^{-1}(L)$. Since $\alpha$ is continuous, $\alpha^{-1}(L)$ is open in $\hat{G}_{S_1}$, so there exists $M \in \mathcal{U}(\hat{G}_{S_1})$ such that $M \leq \alpha^{-1}(L)$. We now infer that $\beta_1^{-1}(M) = V_1 \in S_1$, whereby $V_1 \leq \beta_1^{-1}\alpha^{-1}(L) = V_2$. Thus (1) implies (2).

Conversely, suppose that (2) holds. For $f \in \hat{G}_{S_1}$, we define $\hat{f}$ to be the element of $\hat{G}_{S_2}$ generated by the cosets

\[ \{V_2g \mid \exists V_1 \in S_1 \text{ so that } V_1g \in f \text{ and } V_1 \leq V_2 \}. \]

Condition (2) ensures that this definition specifies a unique element of $\hat{G}_{S_2}$. We now consider the map $\alpha : \hat{G}_{S_1} \to \hat{G}_{S_2}$ given by $\alpha(f) := \hat{f}$. Given $V_2 \in S_2$, we see that $\alpha((V_1)_{S_1}) \leq (V_2)_{S_2}$ whenever $V_1 \in S_1$ is so that $V_1 \leq V_2$. Since the subgroups $(V_i)_{S_i}$ form a basis of neighborhoods of the identity in $\hat{G}_{S_i}$, the map $\alpha$ is continuous in a neighborhood of the identity. It is straightforward to verify that $\alpha$ is an abstract group homomorphism, so in fact $\alpha$ is a continuous homomorphism. It is also immediate from the definition of $\alpha$ that $\beta_2 = \alpha \circ \beta_1$. \qed

Proposition 5.2. Let $G$ be a group with $G$-stable local filters $S_1$ and $S_2$. Then the following are equivalent:

(1) There is an injective continuous homomorphism $\alpha : \hat{G}_{S_1} \to \hat{G}_{S_2}$ such that $\beta_{(G,S_2)} = \alpha \circ \beta_{(G,S_1)}$.
(2) For all $V_2 \in \mathcal{S}_2$, there exists $V_1 \in \mathcal{S}_1$ such that $V_1 \leq V_2$, and for all $V_1 \in \mathcal{S}_1$, it is the case that $V_1 = \bigcap \{V_2 \in \mathcal{S}_2 \mid V_1 \leq V_2\}$.

**Proof.** Set $\beta_1 := \beta_{(G, \mathcal{S}_1)}$ and $\beta_2 := \beta_{(G, \mathcal{S}_2)}$.

Suppose that (1) holds. Let $V_1 \in \mathcal{S}_1$ and set $\mathcal{R} := \{V_2 \in \mathcal{S}_2 \mid V_1 \leq V_2\}$. The subgroup $(\hat{V}_1)_{\mathcal{S}_1}$ is a compact open subgroup of $\hat{G}_{\mathcal{S}_1}$, so $K := \alpha((\hat{V}_1)_{\mathcal{S}_1})$ is a compact, hence profinite, subgroup of $\hat{G}_{\mathcal{S}_2}$. It follows that $K$ is the intersection of all compact open subgroups of $\hat{G}_{\mathcal{S}_2}$ that contain $K$.

All compact open subgroups of $\hat{G}_{\mathcal{S}_2}$ are of the form $\hat{W}_{\mathcal{S}_2}$ for some $W \in \mathcal{S}_2$. For $\hat{W}_{\mathcal{S}_2} \geq K$ compact and open, we thus have $\beta_2^{-1}(K) \leq W$, and since $\beta_1 = \alpha \circ \beta_2$, we deduce further that $V_1 = \beta_1^{-1}(\hat{V}_1)_{\mathcal{S}_1} \leq W$. Hence, $K = \bigcap \{\hat{W}_{\mathcal{S}_2} \mid W \in \mathcal{R}\}$.

For $g \in \bigcap \mathcal{R}$, it is the case that $g \in \beta_2(W)$ for all $W \in \mathcal{R}$, so $\beta_2(g) \in K$. In other words,
\[ \alpha \circ \beta_1(g) \in \alpha((\hat{V}_1)_{\mathcal{S}_1}). \]

As $\alpha$ is injective, $\beta_1(g) \in (\hat{V}_1)_{\mathcal{S}_1}$, whereby $g \in \beta_1^{-1}((\hat{V}_1)_{\mathcal{S}_1}) = V_1$. The subgroup $V_1$ is thus the intersection of $\mathcal{R}$. The remainder of (2) follows from Proposition 5.1.

Conversely, suppose that (2) holds, let $\alpha$ be as given by Proposition 5.1 and fix $f \in \ker \alpha$. For each $V_1 \in \mathcal{S}_1$, we may write $f = \beta_1(g)u$ for $g \in G$ and $u \in (\hat{V}_1)_{\mathcal{S}_1}$, since $\beta_1(G)$ is dense in $\hat{G}_{\mathcal{S}_1}$, and it follows that $\beta_2(g) \in \alpha((\hat{V}_1)_{\mathcal{S}_1})$. For any $V_2 \in \mathcal{S}_2$ such that $V_2 \geq V_1$, we now infer that $g \in V_2$, so $g \in V_1$ via (2). Recalling that $f = \beta_1(g)u$, it follows that $f \in (\hat{V}_1)_{\mathcal{S}_1}$. The subgroups $(\hat{V}_1)_{\mathcal{S}_1}$ form a basis at 1, so indeed $f = 1$. We conclude that $\alpha$ is injective. \(\Box\)

**Proposition 5.3.** Let $G$ be a group with $G$-stable local filters $\mathcal{S}_1$ and $\mathcal{S}_2$. Then the following are equivalent:

(1) There is a continuous homomorphism $\alpha : \hat{G}_{\mathcal{S}_1} \to \hat{G}_{\mathcal{S}_2}$ with discrete kernel such that $\beta_{(G, \mathcal{S}_2)} = \alpha \circ \beta_{(G, \mathcal{S}_1)}$.

(2) For all $V_2 \in \mathcal{S}_2$, there exists $V_1 \in \mathcal{S}_1$ such that $V_1 \leq V_2$, and there is $V_1 \in \mathcal{S}_1$ so that for all $W_1 \in \mathcal{S}_1$ there is $W_2 \in \mathcal{S}_2$ so that $V_1 \cap W_2 \leq W_1$.

**Proof.** Set $\beta_1 := \beta_{(G, \mathcal{S}_1)}$ and $\beta_2 := \beta_{(G, \mathcal{S}_2)}$.

Suppose $\ker \alpha$ is discrete, so there is $V_1 \in \mathcal{S}_1$ such that $\ker \alpha \cap (\hat{V}_1)_{\mathcal{S}_1} = \{1\}$. The map $\alpha$ restricts to a topological group isomorphism from $(\hat{V}_1)_{\mathcal{S}_1}$ to its image. Take $W_1 \in \mathcal{S}_1$ and set $Y := V_1 \cap W_1$. The subgroup $(\hat{Y})_{\mathcal{S}_1}$ is open in $(\hat{V}_1)_{\mathcal{S}_1}$, so $\alpha((\hat{Y})_{\mathcal{S}_1})$ is open in $\alpha((\hat{V}_1)_{\mathcal{S}_1})$. There thus exists $W_2 \in \mathcal{S}_2$ such that $\alpha((\hat{V}_1)_{\mathcal{S}_1}) \cap (\hat{W}_2)_{\mathcal{S}_2} \leq \alpha((\hat{Y})_{\mathcal{S}_1})$. Since $\alpha$ is injective on $(\hat{V}_1)_{\mathcal{S}_1}$, it follows that
\[ (\hat{V}_1)_{\mathcal{S}_1} \cap \alpha^{-1}((\hat{W}_2)_{\mathcal{S}_2}) \leq (\hat{Y})_{\mathcal{S}_1}. \]

We deduce that $V_1 \cap W_2 \leq Y \leq W_1$ since $\alpha \circ \beta_1 = \beta_2$. This verifies the second condition of (2), and the first condition is given by Proposition 5.1.

Conversely, suppose that (2) holds and let $\alpha$ be as given by Proposition 5.1. Taking $V_1 \in \mathcal{S}_1$ that witnesses the second claim of (2), we claim that $(\hat{V}_1)_{\mathcal{S}_1}$ intersects $\ker \alpha$ trivially, which demonstrates that $\ker \alpha$ is discrete. Suppose that $f \in (\hat{V}_1)_{\mathcal{S}_1} \cap \ker \alpha$ and suppose for contradiction that $f$ is non-trivial. There is then some $W_1 \in \mathcal{S}_1$ and $g \in G \setminus W_1$ so that $gW_1 \leq f$; we may assume that $W_1 \leq V_1$. Since $f \in (\hat{V}_1)_{\mathcal{S}_1}$, we additionally have $g \in V_1$.

Appealing to (2), there is $W_2 \in \mathcal{S}_2$ such that $V_1 \cap W_2 \leq W_1$, and there is $Y \in \mathcal{S}_1$ such that $Y \leq V_1 \cap W_2$. Letting $g' \in G$ be so that $g'Y \in f$, it must be the case that $g'Y_1 \leq W_2$ since $\alpha(f)$
is trivial. On the other hand, $g'Y \subseteq gW_1$, so $g' \in V_1$. We now see that $g' \in V_1 \cap W_2 \leq W_1$ and that $g'W_1 = gW_1$. The coset $gW_1$ thus contains the identity element of $G$, whereby $g \in W_1$, a contradiction. We conclude $(V_1)_1 \cap \ker \alpha = \{1\}$ as claimed.

6. Invariant properties of completions

By Theorem 4.3, the locally compact, non-compact structure of a t.d.l.c. completion of a Hecke pair $(G,U)$ is independent of the choice of t.d.l.c. completion. We can now make some precise statements of the consequences of this fact.

Here are some basic consequences of Theorem 4.3 that do not require any further preparatory results to prove.

Proposition 6.1. Let $(G,U)$ be a Hecke pair. For each of the following properties, either every completion of $(G,U)$ has the property, or every completion of $(G,U)$ does not have the property.

1. Being $\sigma$-compact.
2. Being compactly generated.
3. Having a quotient isomorphic to $N$ where $N$ is any specified t.d.l.c. group that has no non-trivial compact normal subgroups.
4. Being amenable.

Proof. By Theorem 4.3, it suffices to show for any t.d.l.c. group $H$ and compact normal subgroup $K$ of $H$, the group $H$ has the property if and only if $H/K$ does.

Claims (1) and (2) are immediate.

For (3), if $\pi : H \rightarrow N$ is a quotient map, then $\pi(K)$ is a compact normal subgroup of $N$. Since $N$ has no non-trivial compact normal subgroup, we deduce that $N$ is a quotient of $H$ if and only if $N$ is a quotient of $H/K$.

For (4), recall that every compact subgroup is amenable and that if $L$ is a closed normal subgroup of the locally compact group $H$, then $H$ is amenable if and only if both $H/L$ and $L$ are amenable. Since $K$ is compact, we deduce that $H$ is amenable if and only if $H/K$ is amenable. □

6.1. Elementary groups. We next consider the property of being an elementary group. Recall the class $\mathcal{E}$ of elementary groups is the smallest class of t.d.l.c.s.c. groups with the following properties:

(a) $1 \in \mathcal{E}$.
(b) If $G$ is a t.d.l.c.s.c. group and $K$ is a closed normal subgroup of $G$ such that $K \in \mathcal{E}$ and $G/K$ is profinite or discrete, then $G \in \mathcal{E}$.
(c) If $G$ is an increasing union of a sequence $(O_i)_{i \in \mathbb{N}}$ of open subgroups of $G$ such that $O_i \in \mathcal{E}$ for all $i$, then $G \in \mathcal{E}$.

Note in particular that if $G$ is a t.d.l.c.s.c. group that is non-discrete, compactly generated, and topologically simple, then $G$ is not in $\mathcal{E}$. The class $\mathcal{E}$ is thus strictly smaller than the class of all t.d.l.c.s.c. groups.

The class $\mathcal{E}$ can be defined in several other ways, as it exhibits a number of closure properties. Intuitively, it is the class of t.d.l.c.s.c. groups that are constructed out of profinite and discrete groups by ‘elementary’ operations.

Theorem 6.2 (See [13, Theorem 1.3]). The class $\mathcal{E}$ is the smallest class of t.d.l.c.s.c. groups that has all of the following properties:
(1) $\mathcal{E}$ contains all countable discrete groups.

(2) If $G$ is a t.d.l.c.s.c. group and $H \trianglelefteq G$ is a closed normal subgroup, then $G \in \mathcal{E}$ if and only if both $H \in \mathcal{E}$ and $G/H \in \mathcal{E}$.

(3) If $G \in \mathcal{E}$, $H$ is a t.d.l.c.s.c. group, and $\psi : H \to G$ is a continuous, injective homomorphism, then $H \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under taking compact subgroups.

(4) If $G$ is a t.d.l.c.s.c. group that is residually in $\mathcal{E}$, then $G \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under direct limits that result in a t.d.l.c.s.c. group.

(5) $\mathcal{E}$ is closed under local direct products.

(6) If $G$ is a t.d.l.c.s.c. group and $(C_i)_{i \in \mathbb{N}}$ is an $\subseteq$-increasing sequence of elementary closed subgroups of $G$ such that $N_G(C_i)$ is open for each $i$ and $\bigcup_{i \in \mathbb{N}} C_i = G$, then $G \in \mathcal{E}$.

The class of elementary groups comes with a canonical successor ordinal valued rank called the decomposition rank and denoted $\xi(G)$; see [13, Section 4]. The key property of the decomposition rank that we will exploit herein is that it is well-behaved under natural constructions. For a t.d.l.c. group $G$, recall that the discrete residual of $G$ is defined to be $\text{Res}(G) := \bigcap \{ O \leq G \mid O \text{ open} \}$.

**Proposition 6.3.** Let $G$ be a non-trivial elementary t.d.l.c.s.c. group. Then the following holds.

1. If $H$ is a t.d.l.c.s.c. group, and $\psi : H \to G$ is a continuous, injective homomorphism, then $\xi(H) \leq \xi(G)$. ([13, Corollary 4.10])

2. If $L \trianglelefteq G$ is closed, then $\xi(G/L) \leq \xi(G)$. ([13, Theorem 4.19])

3. If $G = \bigcup_{i \in \mathbb{N}} O_i$ with $(O_i)_{i \in \mathbb{N}}$ an $\subseteq$-increasing sequence of compactly generated open subgroups of $G$, then $\xi(G) = \sup_{i \in \mathbb{N}} \xi(\text{Res}(O_i)) + 1$. If $G$ is compactly generated, then $\xi(G) = \xi(\text{Res}(G)) + 1$. ([13, Lemma 4.12])

4. If $G$ is residually discrete, then $\xi(G) \leq 2$. ([13, Observation 4.11])

We now consider elementary completions of Hecke pairs. To this end, we require a general lemma about the decomposition rank.

**Lemma 6.4.** Suppose $G \in \mathcal{E}$ and $G$ lies in a short exact sequence of topological groups

$$\{1\} \to N \to G \to Q \to \{1\}.$$ 

Then $\xi(G) \leq \xi(N) + \xi(Q)$.

**Proof.** We argue by induction on $\xi(Q)$ for all such $G$. The base case, $\xi(Q) = 1$, is trivial as in this case $Q = \{1\}$. Suppose that the lemma holds up to $\beta$ and that $\xi(Q) = \beta + 1$. Let $\pi : G \to Q$ be the usual projection and fix $(O_i)_{i \in \mathbb{N}}$ an increasing exhaustion of $G$ by compactly generated open subgroups. Put $W_i := \pi(O_i)$ and note the $W_i$ form an increasing exhaustion of $Q$ by compactly generated open subgroups.

In view of Proposition 6.3, it suffices to consider $\xi(\text{Res}(O_i))$ for an arbitrary $i$. Fix $i \in \mathbb{N}$, form $R := \text{Res}(O_i)$, and put $M := \overline{RN}$. It follows that $M/N \leq \text{Res}(W_i)$, hence $\xi(M/N) \leq \beta$. The induction hypothesis implies $\xi(M) \leq \xi(N) + \beta$, and since $R \to M$, we conclude $\xi(R) \leq \xi(N) + \beta$. Appealing to Proposition 6.3, $\xi(G) \leq \xi(N) + \beta + 1$, verifying the lemma.

We now argue that the property of being elementary is a property of the Hecke pair.

**Proposition 6.5.** For $(G, U)$ a countable Hecke pair, the following hold:

1. Either all t.d.l.c.s.c. completions of $(G, U)$ are elementary, or all t.d.l.c.s.c. completions of $(G, U)$ are non-elementary.
(2) If every t.d.l.c.s.c. completion is elementary, then $\xi(G//U)$ has the least rank of any completion, and for any t.d.l.c.s.c. completion $H$ of $(G,U)$, 
\[ \xi(H) \leq 2 + \xi(G//U). \]

In particular, if any completion has transfinite rank, then every completion has the same rank.

Proof. Let $S$ be the Schlichting completion of $(G,U)$; by Corollary 4.11, $S$ is a t.d.l.c.s.c. group. For any other t.d.l.c.s.c. completion $H$, Theorem 4.3 supplies a quotient map $H \to S$ with compact kernel. Since the class of elementary groups is closed under extensions, we see that if $S$ is elementary, then so is $H$. Conversely, if $S$ is not elementary, then $H$ is also not elementary since the class of elementary groups is closed under quotients. This proves (1).

For (2), Proposition 6.3 implies $\xi(H) \geq \xi(S)$ for any t.d.l.c.s.c. completion $H$, so $\xi(S)$ has the least decomposition rank of any t.d.l.c.s.c. completion. On the other hand, since the kernel of the quotient map $H \to S$ is profinite, hence residually discrete, Lemma 6.4 ensures that $\xi(H) \leq 2 + \xi(S)$. If $\xi(S)$ is a transfinite ordinal, then $2 + \xi(S) = \xi(S)$, so in this case $\xi(H) = \xi(S)$. This proves (2).

The following definitions are now sensible; in view of Corollary 4.11 it is natural to restrict to countable Hecke pairs $(G,U)$.

**Definition 6.6.** A Hecke pair $(G,U)$ is called elementary if $G$ is countable and some (all) t.d.l.c.s.c. completions are elementary. For an elementary Hecke pair $(G,U)$, the decomposition rank of $(G,U)$ is $\xi(G,U) := \xi(G//U)$.

6.2. The scale function and flat subgroups. We conclude this section by considering the scale function and flat subgroups in relation to completions; these concepts were introduced in [14] and [15] respectively (although the term ‘flat subgroup’ is more recent).

**Definition 6.7.** For $G$ a t.d.l.c. group, the scale function $s : G \to \mathbb{Z}$ is defined by 
\[ s(g) := \min\{|gUg^{-1} : gUg^{-1} \cap U| \mid U \in \mathcal{U}(G)\}. \]

A compact open subgroup $U$ of $G$ is tidy for $g \in G$ if it achieves $s(g)$. We say $g$ is uniscalar if $s(g) = s(g^{-1}) = 1$.

A subset $X$ of $G$ is flat if there exists a compact open subgroup $U$ of $G$ such that for all $x \in X$, the subgroup $U$ is tidy for $x$; in this case, we say $U$ is tidy for $X$. If $X$ is a finitely generated flat subgroup, the rank of $X$ is the least number of generators for the quotient group $X/\{x \in X \mid s(x) = 1\}$.

The scale function and flatness are clearly locally compact non-compact phenomena. In relation to t.d.l.c. completions, they indeed only depend on the Hecke pair $(G,U)$; in fact, they only depend on the commensurability class $[U]$.

**Proposition 6.8.** Suppose that $(G,U)$ is a Hecke pair and that $H$ is a t.d.l.c. completion via $\phi$. Then the following hold:

(1) For $\hat{s}$ and $s$ the scale functions for $\hat{G}_U$ and $H$, $s \circ \phi = \hat{s} \circ \beta_U$.

(2) For $X \subseteq G$, the subset $\phi(X)$ is flat if and only if $\beta_U(X)$ is flat.

(3) If $K \leq G$ is a finitely generated subgroup, then $\phi(K)$ is flat with rank $k$ if and only if $\beta_U(K)$ is flat with rank $k$. 
We conclude that $\beta$ is a flat subgroup of $\hat{G}$, where $\pi : \hat{G} \rightarrow H$ is a quotient map with compact kernel. The result [9] Lemma 4.9] ensures that $s \circ \pi = \hat{s}$, hence $s \circ \phi = \hat{s} \circ \beta_U$, proving (1).

Appealing again to [9] Lemma 4.9], if $U$ is tidy for $g$ in $\hat{G}$, then $\pi(U)$ is tidy for $\pi(g)$ in $H$, and conversely if $V$ is tidy for $\pi(g)$ in $H$, then $\pi^{-1}(V)$ is tidy for $g$ in $\hat{G}$. Therefore, if $\beta_U(X)$ has a common tidy subgroup, then so does $\phi(X)$. Conversely, if $\phi(X)$ has a common tidy subgroup $V$ in $H$, then $\pi^{-1}(V)$ is a common tidy subgroup for $\beta_U(X)$. We conclude that $\phi(X)$ has a common tidy subgroup if and only if $\beta_U(X)$ does, verifying (2).

Finally, if $K$ is a subgroup of $G$, then $\phi(K)$ is a flat subgroup of $H$ if and only if $\beta_U(K)$ is a flat subgroup of $\hat{G}$ by (2). The rank of $\phi(K)$ is the number of generators of the factor $\phi(K)/L_H$ where $L_H := \{ x \in \phi(K) \mid s(x) = 1 \}$. Letting $L_{\hat{G}}$ be the analogous subgroup of $\hat{G}$, it follows from (1) that the map $\pi$ induces an isomorphism $\tilde{\pi} : \beta_U(K)/L_{\hat{G}} \rightarrow \phi(K)/L_H$. We conclude that $\beta_U(K)$ has rank $k$ if and only if $\phi(K)$ has rank $k$, proving (3).

The next corollary is immediate from Proposition 6.8 and the fact the scale function is continuous (see [14]).

Corollary 6.9. For $(G,U)$ a Hecke pair, either all t.d.l.c. completions are uniscalar or no t.d.l.c. completion is uniscalar.

7. A construction method and an application

In this section, we demonstrate general techniques for building interesting countable Hecke pairs and hence t.d.l.c. groups via completions of these Hecke pairs. The advantage of building Hecke pairs as opposed to directly constructing t.d.l.c. groups is that the potentially intractable or tedious topological considerations in a construction become tractable combinatorial considerations. As an application of these techniques, we build compactly generated elementary groups with decomposition rank up to $\omega^2 + 1$.

7.1. Preliminaries. For a group $H$ and a set $X$, we write $H^{<X}$ for the group of finitely supported functions from $X$ to $H$ with pointwise multiplication. The restricted wreath product of $H$ with a permutation group $(G,X)$ is then

$$H \wr (G,X) := H^{<X} \rtimes G$$

where $G$ acts on $H^{<X}$ via its action on $X$. If $H$, $X$ and $G$ are all countable, then $H \wr (G,X)$ is countable.

We observe a useful property of wreath products.

Lemma 7.1 [14, Lemma 3.6]. Suppose $(G,X)$ is a transitive permutation group and $H$ is a group. If $N \trianglelefteq H \wr (G,X)$ intersects $G$ non-trivially, then $[H,H]^{<X} \leq N$.

The (restricted) wreath product (in the imprimitive action) of two permutation groups $(H,Y)$ with $(G,X)$ is defined to be the permutation group $(H \wr (G,X), X \times Y)$ where

$$(\alpha,g).(x,y) := (g.x, \alpha(g.x).y).$$

One easily verifies the definition gives a group action. Wreath products of permutation groups in imprimitive action are also associative; this allows us to omit parentheses when iterating wreath products.

For a sequence of permutation groups $((H_i, X_i))_{i \in \mathbb{N}}$ and $k \in \mathbb{N}$, we write

$$(K_k, Y_k) := (H_k, X_k) \wr \cdots \wr (H_0, X_0).$$
For each \( k \geq 0 \), the group \( K_k \) embeds into \( K_{k+1} \), so we may take a direct limit \( \lim_{\rightarrow} K_k \). We denote this direct limit by \( \bigoplus_{i=0}^{\infty} (H_i, X_i)_{i \geq 0} \) and call it the \textbf{left wreath product} of the sequence \( ((H_i, X_i))_{i \in \mathbb{N}} \); we will often suppress the modifier “left”. When the sequence \( ((H_i, X_i))_{i \in \mathbb{N}} \) consists of copies of the same permutation group \((H, X)\), we write \( \bigoplus_{i=0}^{\infty} (H, X) \) for \( \bigoplus_{i=0}^{\infty} (H_i, X_i)_{i \in \mathbb{N}} \).

The associativity of the wreath product gives the following identity for each \( k \geq 0 \):

\[
(\bigoplus_{i=0}^{\infty} (H_i, X_i))_{i \geq k} \triangleleft (K_k, Y_k) = \bigoplus_{i=0}^{\infty} (H_i, X_i)_{i \geq 0}.
\]

7.2. \textbf{Wreath products of Hecke pairs.} We wish to build Hecke pairs via wreath products. As wreath products require permutation groups, a natural definition arises.

\textbf{Definition 7.2.} A triple \((G, U, X)\) is called a \textbf{permutation Hecke pair} if \((G, U)\) is a countable proper Hecke pair, \((G, X)\) is a permutation group with \(G\) and \(X\) countably infinite, and the induced action of \(U\) on \(X\) has finite orbits. The permutation Hecke pair is called \textbf{transitive} if \((G, X)\) is a transitive permutation group. It is called \textbf{finitely generated} if \((G, U)\) is a finitely generated Hecke pair.

\textbf{Remark 7.3.} Our definition of a permutation Hecke pair is somewhat restrictive. We make these restrictions as we are interested in building examples of t.d.l.c.s.c. groups from Hecke pairs, with an emphasis on the non-compact properties of these completions. There is therefore no advantage in allowing \(U\) to have a non-trivial normal core. Restricting to countable proper Hecke pairs ensures our examples will be second countable.

Given a countable proper Hecke pair \((G, U)\), there is always a canonical permutation Hecke pair \((G, U, G/U)\) where \(G \curvearrowright G/U\) by left multiplication.

Permutation Hecke pairs are well-behaved under wreath products.

\textbf{Lemma 7.4.} If \((H_0, A_0, X_0)\) and \((H_1, A_1, X_1)\) are permutation Hecke pairs, then

\[ (H_0 \trianglelefteq (H_1, X_1), A_0 \trianglelefteq (A_1, X_1), X_1 \times X_0) \]

is a permutation Hecke pair where \(H_0 \trianglelefteq (H_1, X_1) \cap X_1 \times X_0\) via the imprimitive action.

\textbf{Proof.} Let \(G := H_0 \trianglelefteq (H_1, X_1)\) and let \(L := A_0 \trianglelefteq (A_1, X_1)\). Taking \((p, q) \in X_1 \times X_0\), the \(L\)-orbit of \((p, q)\) is contained in \(A_1 p \times A_0 q\), by the definition of the imprimitive action. Since both \(A_0\) and \(A_1\) have finite orbits, we deduce that \(L\) has finite orbits of \(X_1 \times X_0\).

To show that \(L\) is commensurated in \(G\), it suffices to show that \(L\) is commensurated by both \(H_1\) and by \(H_0 < X_1\). First consider \(h \in H_1\). The element \(h\) normalizes \(A_0 < X_1\), and \(h\) commensurates \(A_1\). We thus have

\[ L \cap hLh^{-1} = A_0 < X_1 \times (A_1 \cap hA_1 h^{-1}), \]

which has finite index in both \(L\) and \(hLh^{-1}\). Hence, \(h \in \text{Comm}_G(L)\).

Consider next \(f \in H_0 < X_1\) and let \(Z\) be the support of \(f\). Since the orbits of \(A_1\) on \(X_1\) are finite, the group \(A' := \text{Stab}_{A_1}(Z)\) has finite index in \(A_1\). The element \(f\) commensurates \(A_0^Z\); indeed, \(A_0^Z\) is commensurated by \(H_0 < X_1\). Furthermore, \(f\) centralizes both \(A_0 < X_1 < Z\) and \(A'\). The element \(f\) thus commensurates the group

\[ L' = A_0 \trianglelefteq (A', X_1) = (A_0^Z \times A_0 < X_1 \setminus Z) \times A', \]

which is a finite index subgroup of \(L\). We conclude that \(f \in \text{Comm}_G(L') = \text{Comm}_G(L)\), hence \(L\) is a commensurated subgroup of \(G\).
Since the normal cores of $A_1$ and $A_0$ in $H_1$ and $H_0$ are trivial, it follows that the normal core of $L$ in $G$ is trivial. Hence, $(G, L, X_1 \times X_0)$ is a permutation Hecke pair, completing the proof of the lemma.

**Definition 7.5.** For permutation Hecke pairs $(H_0, A_0, X_0)$ and $(H_1, A_1, X_1)$, the **wreath product** of $(H_0, A_0, X_0)$ with $(H_1, A_1, X_1)$ is the permutation Hecke pair

$$(H_0, A_0, X_0) \rtimes (H_1, A_1, X_1) := (H_0 \rtimes (H_1, X_1), A_0 \rtimes (A_1, X_1), X_1 \times X_0).$$

The $A_i$ or $X_i$ parameter is sometimes suppressed when clear from context.

Wreath products of permutation Hecke pairs are clearly associative. Properties of Hecke pairs also pass to the wreath product.

**Observation 7.6.** Suppose that $(H_0, A_0, X_0)$ and $(H_1, A_1, X_1)$ are permutation Hecke pairs. If $(H_0, A_0, X_0)$ and $(H_1, A_1, X_1)$ are finitely generated and transitive, then $(H_0, A_0, X_0) \rtimes (H_1, A_1, X_1)$ is finitely generated and transitive.

We call a permutation Hecke pair $(G, U, Y)$ **elementary** if $(G, U)$ is an elementary Hecke pair.

**Lemma 7.7.** If $(H_0, A_0, X_0)$ and $(H_1, A_1, X_1)$ are permutation Hecke pairs which are elementary, then $(H_0, A_0, X_0) \rtimes (H_1, A_1, X_1)$ is elementary.

**Proof.** Set $G := H_0 \rtimes (H_1, X_1)$, put $U := A_0 \rtimes (A_1, X_1)$, and let $K$ be a t.d.l.c. completion of $(H_0, A_0, X_0) \rtimes (H_1, A_1, X_1)$ via $\sigma : G \to K$. For $Y \subseteq G$, write $\hat{Y} := \overline{\sigma(Y)}$.

Let $\hat{H}_0^y$ be the copy of $H_0$ in $G$ given by functions supported on $x \in X_1$ and let $A_0^x$ be the copy of $A_0$ in $H_0^y$. Observing that $\hat{H}_0^y \cap \hat{U} = \hat{A}_0^x$, the group $\hat{H}_0^y$ is a completion of $(H_0, A_0)$, so it is elementary. Additionally,

$$\hat{H}_0^y \leq \hat{H}_0^{<X_1} =: L,$$

and $L = \bigcup_{Y \in I} \hat{H}_0^y$ where $I$ is the set of finite subsets of $X_1$.

We argue $K_Y := \hat{H}_0^y$ is elementary for each $Y \in I$ by induction on $|Y|$. The previous paragraph verifies the base case, so suppose $|Y| = n + 1$ and fix $y \in Y$. Setting $P := H_0^{\setminus \{y\}}$ and $N := \hat{H}_0^y$, we see that $K_Y = P N$, so $K_Y = U P N$ for $U$ a compact open subgroup of $K_Y$. Therefore, $U P U P \cap N \simeq K_Y / N$.

The group $P$ is elementary by the induction hypothesis. Theorem 5.2 thus ensures that $U P$ is elementary, hence $K_Y / N$ is also elementary. It now follows that $K_Y$ is elementary, finishing the induction.

Since $X_1$ is countable, Theorem 5.2 implies that $L$ is elementary. On the other hand, $\hat{H}_1$ is a completion of $(H_1, A_1)$, so it is also elementary. Letting $\pi : \hat{G} \to \hat{G} / L$ be the usual projection, it follows that $\pi(\hat{H}_1) = \hat{G} / L$, hence $\hat{G} / L$ is elementary. Appealing again to Theorem 5.2 the group $\hat{G}$ is elementary, proving the lemma.

**Lemma 7.8.** Suppose $(H_0, A_0, X_0)$ and $(H_1, A_1, X_1)$ are finitely generated transitive permutation Hecke pairs so that $H_0$ is perfect. If $(H_0, A_0, X_0)$ and $(H_1, A_1, X_1)$ are elementary and $A_1$ is infinite, then $(G, U, Z) := (H_0, A_0, X_0) \rtimes (H_1, A_1, X_1)$ is elementary with $\xi(G, U) \geq \xi(H_0, A_0) + 1$. 

Under mild additional assumptions, we obtain a lower bound on the rank.
Proof. Suppose that $E$ is the Schlichting completion of $(G,U)$ via $\sigma : G \to E$. For $Y \subseteq G$, write $\hat{Y} := \sigma(Y)$.

The t.d.l.c.s.c. group $E$ is compactly generated, so

$$\xi(E) = \xi(\text{Res}(E)) + 1$$

by Proposition 6.3. Letting $O \trianglelefteq E$ be an open normal subgroup, $O \cap \hat{A}_1$ has finite index in $\hat{A}_1$ so $O \cap \sigma(H_0 \langle H_1, X_1 \rangle)$ must meet $\sigma(A_1)$ non-trivially. In view of Lemma 7.11, we deduce that $\sigma([H_0, H_0]^{<X}) = \sigma(H_0^{<X}) \leq O$. It now follows that $\sigma(H_0^{<X}) \leq \text{Res}(E)$, and a fortiori, $\sigma(H_0^0) \leq \text{Res}(K)$, where $H_0^0$ is the collection of functions supported on $x$. Since $\hat{H}_0^0$ is a t.d.l.c.s.c. completion of $(H_0, A_0)$, we conclude that $\xi(\text{Res}(E)) \geq \xi(H_0, A_0)$, and the lemma follows.

We now upgrade these results to wreath products of infinite sequences. We first introduce some notation that will be used for the rest of this subsection.

**Notation 7.9.** Let $((H_i, A_i, X_i))_{i \in \mathbb{N}}$ be a sequence of permutation Hecke pairs. For each $k \geq i \geq 0$, we have a permutation Hecke pair

$$(J_{[k,i]}, I_{[k,i]}, Y_{[k,i]}) := (H_k, A_k, X_k) \wr \cdots \wr (H_i, A_i, X_i).$$

When $i = 0$, we simply write $(J_k, I_k, Y_k)$.

We put $J_{(\infty, k)} := \infty((H_i, X_i)^{> i})$ and $I_{(\infty, k)} := \infty((A_i, X_i)^{> i})$; the groups $J_{(\infty, k)}$ and $I_{(\infty, k)}$ are defined in the obvious manner. We write $J$ and $I$ for $J_{(\infty, 0)}$ and $I_{(\infty, 0)}$. Observe that $J = J_{(\infty, k)} \wr (J_k, Y_k)$ and $I = I_{(\infty, k)} \wr (I_k, Y_k)$.

**Lemma 7.10.** If $((H_i, A_i, X_i))_{i \in \mathbb{N}}$ is a sequence of permutation Hecke pairs, then

$$(\infty((H_i, X_i)_{i \in \mathbb{N}}, \infty((A_i, X_i)_{i \in \mathbb{N}}))$$

is a countable proper Hecke pair.

**Proof.** Lemma 7.3 ensures that $(J_k, I_k)$ is a Hecke pair for each $k \geq 0$. For each $x \in J$, there is $k$ for which $x \in J_k$, so $x$ commensurates $I_k$. Furthermore, $J_k$ normalizes $I_{(\infty,k)}^{Y_k}$ in $J$, hence $I_{(\infty,k)} \wr (I_k \cap xI_kx^{-1}, Y_k)$ is a finite index subgroup of both $I$ and $xI_kx^{-1}$. We conclude that $x$ commensurates $I$. Thus, $I$ is commensurated in $J$. It follows that $I$ also has a trivial normal core in $J$, since $I_k$ has trivial normal core in $J_k$ for each $k$. The pair $(J,I)$ is therefore a countable proper Hecke pair.

**Definition 7.11.** For $((H_i, A_i, X_i))_{i \in \mathbb{N}}$, a sequence of permutation Hecke pairs, the left wreath product of the sequence $((H_i, A_i, X_i))_{i \in \mathbb{N}}$ is the Hecke pair

$$(\infty((H_i, X_i)_{i \in \mathbb{N}}, \infty((A_i, X_i)_{i \in \mathbb{N}})).$$

**Lemma 7.12.** If $((H_i, A_i, X_i))_{i \in \mathbb{N}}$ is a sequence of permutation Hecke pairs each of which is elementary, then $\infty((H_i, A_i, X_i))_{i \in \mathbb{N}}$ is elementary.

**Proof.** By repeated application of Lemma 7.11, we see that $(J_k, I_k)$ is elementary for all $k \in \mathbb{N}$. Let $E$ be the Schlichting completion of the pair $(J, I)$ via $\sigma : J \to E$. For $Y \subseteq J$, write $\hat{Y} := \sigma(Y)$.

For each $k \geq 0$, we have that $L_k := \hat{J}_{(\infty, k)}^{Y_k} \subseteq E$. We claim $E/L_k$ is a t.d.l.c.s.c. completion of $(J_k, I_k)$ via the induced map $\bar{\sigma} : J_k \to E/L_k$. It is immediate that $\bar{\sigma}$ has dense image and that

$$V := \bar{\sigma}(I_k) = \hat{L}_k/L_k.$$
is compact and open in \( E/L_k \). It remains to show that \( \sigma^{-1}(V) = I_k \).

Suppose \( x \in J_k \) is so that \( \sigma(x) \in V \) and say \( \sigma(x) = uy \) for \( u \in \hat{I} \) and \( y \in L_k \). Since \( \hat{I} \) is open in \( E \), we may find \( z \in J_{(\infty,k)}^{< Y_k} \) for which \( \sigma(xz) \in \hat{I} \). The group \( E \) is a completion of \( (J,I) \), so \( xz \in I \). Therefore, since \( z \in J_{(\infty,k)}^{< Y_k} \) while \( x \in J_k \), the element \( x \) must be contained in \( I_k \). We conclude \( \sigma^{-1}(V) = I_k \), whereby \( E/L_k \) is a completion of \( (J_k, I_k) \).

The group \( E/L \) with

\[
L := \bigcap_{k \geq 0} J_{(\infty,k)}^{< Y_k}
\]

is then residually elementary, so via Theorem 6.2, \( E/L \) is elementary. It now follows as in the previous paragraph that the group \( E/L \) is a t.d.l.c.s.c. completion of \( (J,I) \). The pair \( (J,I) \) is therefore elementary.

We now compute a lower bound on the rank of certain left wreath products.

**Proposition 7.13.** Suppose that \( (G,A,X) \) is a finitely generated transitive permutation Hecke pair. If \( (G,A,X) \) is elementary and \( G \) is centerless and perfect, then

\[
\xi((G,A,X)) \geq \xi(G,A) + \omega + 1.
\]

**Proof.** Let \( E \) be the Schlichting completion of the pair \( (J,I) \) via \( \sigma : J \to E \), and set \( \alpha := \xi(G,A) \). For \( Y \subseteq J \), write \( \hat{Y} := \sigma(Y) \).

We first argue by induction on \( k \geq 0 \) that

\[
\xi(J_{k+1},I_{k+1}) \geq \alpha + k.
\]

The base case is immediate since any t.d.l.c.s.c. completion of \( (J_1,I_1) \) is also a completion of \( (G,U) \). For \( k > 0 \), the construction of \( J_{k+1} \) ensures

\[
J_{k+1} = J_{(k+1,1)} \lhd J_1 = (J_k,I_k) \lhd (J_1,I_1).
\]

Since \( G \) is perfect, \( J_k \) is perfect, and additionally, the subgroup \( I_1 \) is infinite. Lemma 7.8 and the induction hypothesis thus imply

\[
\xi(J_{k+1},I_{k+1}) \geq \xi(J_k,I_k) + 1 \geq \alpha + k - 1 + 1.
\]

This finishes the induction.

For each \( k \geq 0 \), the subgroup \( \hat{J}_{k+1} \leq E \) is a completion of \( (J_{k+1},I_{k+1}) \), so \( \xi(\hat{J}_{k+1}) \geq \alpha + k \). Proposition 6.3 now implies

\[
\xi(E) \geq \sup\{\xi(\hat{J}_{k+1}) \mid k \geq 0\} \geq \sup\{\alpha + k \mid k \geq 0\} = \alpha + \omega.
\]

As the rank is always a successor ordinal, we conclude that \( \xi(E) \geq \alpha + \omega + 1 \). \( \square \)

### 7.3. HNN-extensions of Hecke pairs.

Suppose \( G \) is a group, \( K,L \) are subgroups of \( G \), and \( \psi : K \to L \) is an isomorphism. The **HNN-extension** of \( G \) relative to \( \psi \) is the group

\[
G*_{\psi} := \langle H,t \mid tkt^{-1} = \psi(k) \forall k \in K \rangle.
\]

An important property of \( G*_{\psi} \) is that \( G \) embeds into \( G*_{\psi} \) in the obvious way; see [5]. The HNN-extension is called ascending if one of \( K \) or \( L \) is equal to \( G \). In the case that \( K = G \), the subgroup \( \bigcup_{n \geq 0} t^{-n}Kt^n \) is a normal subgroup of \( G*_{\psi} \), and \( G*_{\psi} = \left( \bigcup_{n \geq 0} t^{-n}Kt^n \right) \rtimes \langle t \rangle \) where \( \mathbb{Z} = \langle t \rangle \) acts on \( \bigcup_{n \geq 0} t^{-n}Kt^n \) in the obvious way.

We wish to adapt HNN-extensions to the setting of Hecke pairs; we concentrate on ascending HNN extensions since in this case we can control when the resulting group is elementary.
The primary problem with taking HNN-extensions of Hecke pairs is that the commensurated subgroup can fail to be commensurated in the resulting extension. We record a setting in which this problem can be remedied.

Given a set $K$ and $α ∈ K^{<N}$, write $α^δ$ for the image of $α$ under the left shift: $α^δ(n) := α(n + 1)$. The map $α ↦ α^δ$ is a surjective but not injective map from $K^{<N}$ to itself.

**Lemma 7.14.** Suppose that $(G, U)$ is a Hecke pair and that $ψ : G → H$ is an isomorphism such that $H < G$. If there is $K ≤ C_U(H)$ so that $K ∩ H = \{1\}$ and $ψ(U)K ∼_c U$, then $ψ : G × K^{<N} → HK × K^{<N}$ via $(g, α) ↦ (ψ(g)α(0), α^δ)$ is an isomorphism, and

$$(G × K^{<N})^*_ψ, U × K^{<N}$$

is a Hecke pair.

**Proof.** The map $ψ$ is a well-defined epimorphism since $K$ centralizes $H$ and $ψ$ is an isomorphism. The map is additionally injective since $H ∩ K = \{1\}$. We have thus verified our first claim.

For the second claim, set $D := (G × K^{<N})^*_ψ$. Certainly, $G × K^{<N} ≤ \text{Comm}_D(U × K^{<N})$. We thus have only to show $t ∈ \text{Comm}_D(U × K^{<N})$. Computing the image yields

$$t(U × K^{<N})t^{-1} = ψ(U × K^{<N}) = ψ(U)K × K^{<N}.$$

Since $ψ(U)K$ is commensurate with $U$, we deduce that $ψ(U)K × K^{<N}$ is commensurate with $U × K^{<N}$. Hence, $((G × K^{<N})^*_ψ, U × K^{<N})$ is a Hecke pair. □

We make a definition encapsulating the hypotheses of Lemma 7.14.

**Definition 7.15.** A Hecke pair $(G, U)$ and an injective endomorphism $ψ : G → G$ are called **HNN-compatible** for $K ≤ G$ if $K ≤ C_U(ψ(G))$, $K ∩ ψ(G) = \{1\}$ and $ψ(U)K ∼_c U$.

**Definition 7.16.** For $(G, U)$ and $ψ$ HNN-compatible for $K$, the Hecke pair given by Lemma 7.14 is denoted by $(G, U)^{*}_ψ^K$.

**Proposition 7.17.** Suppose that $(G, U)$ is a countable Hecke pair and suppose that $(G, U)$ and $ψ : G → G$ are HNN-compatible for $K ≤ G$. Then the following hold:

1. If $(G, U)$ is finitely generated, then $(G, U)^{*}_ψ^K$ is finitely generated.
2. If $(G, U)$ is elementary, then $(G, U)^{*}_ψ^K$ is elementary.

**Proof.** For (1), there is a finite $F ⊆ G$ so that $⟨F, U⟩ = G$. Therefore,

$$(G × K^{<N})^*_ψ = ⟨⟨F, U × K^{<N}⟩,$$

and we conclude that $(G, U)^{*}_ψ^K$ is finitely generated.

For (2), let $E$ be a t.d.l.c.s.c. completion of $(G, U)^{*}_ψ^K$ via $σ : (G × K^{<N})^*_ψ → E$. For $Y ⊆ (G × K^{<N})^*_ψ$, write $Y := \overline{σ(Y)}$.

The subgroup $G$ is a completion of $(G, U)$ and hence is elementary. The group $G × K^{<N}$ contains $G$ as a cocompact normal subgroup, so $G × K^{<N}$ is also elementary via Theorem 6.2. The group $G × K^{<N}$ is also open in $E$. The union $∪_{k ≥ 0}t^{-k}(G × K^{<N}) t^k$ is then an open normal subgroup of $E$ which is elementary. Appealing to Theorem 6.2 second time, $E$ is elementary. □
Hecke pairs with HNN-compatible endomorphisms arise naturally from left wreath products. Seeing this requires a bit more terminology. Suppose \((G, U, X)\) is a transitive permutation Hecke pair and identify \(\omega^l(G, X) = (\omega^l(G, X)) \ltimes (G, X)\). For \(x \in X\), there is a canonical embedding \(\psi_x : \omega^l(G, X) \to (\omega^l(G, X)) \ltimes (G, X)\) defined by \(a \mapsto f_a\) where \(f_a \in \omega^l(G, X)^<X\) is the function that takes value \(a\) at \(x\) and \(1\) elsewhere. We call \(\psi_x\) the \(x\)-\textbf{contraction map} for \(\omega^l(G, X)\). The \(x\)-\textbf{contraction centralizer} is the subgroup \(K_x := \omega(l(U, X)^<X \setminus \{x\} \rtimes \text{Stab}_U(x)\).

\textbf{Lemma 7.18.} If \((G, U, X)\) is a permutation Hecke pair and \(x \in X\), then \(\omega^l(G, U, X)\) and the \(x\)-\textbf{contraction map} \(\psi_x\) are HNN-compatible for the \(x\)-\textbf{contraction centralizer} \(K_x\).

\textbf{Proof.} Put \(\omega^l(G, X)^x := \psi_x((\omega^l(G, X))\) and \(\omega^l(U, X)^x := \psi_x((\omega^l(U, X))\). The subgroup \(K_x\) is contained in \(\omega^l(U, X)\), centralizes \(\omega^l(G, X)^x\), and has trivial intersection with \(\omega^l(G, X)^x\). The subgroup \(\text{Stab}_U(x)\) has finite index in \(U\) since \(U\) has finite orbits on \(X\). Therefore, \(\omega^l(U, X)^x K_x\) is a finite index subgroup of \(\omega^l(U, X)\), and the lemma is verified.

\textbf{Proposition 7.19.} For \((G, U, X)\) a transitive permutation Hecke pair and \(x \in X\), the following hold:

1. If \((G, U)\) is generated by \(F\), then \(\omega^l(G, U, X)^{K_x}_{\psi_x}\) is generated by \(F \cup \{t\}\).
2. If \((G, U)\) is elementary, then \(\omega^l(G, U, X)^{K_x}_{\psi_x}\) is elementary.

\textbf{Proof.} For this proof, we use the language established in Notation \[7.19\]

For (1), setting \(L := \langle F, t \rangle\), it suffices to show that \(J_k \leq L\) for each \(k \geq 0\). We argue by induction. The base case, \(J_0 = G\), is immediate. For the successor step, we have \(J_{k+1} = G^{<Y_k} \rtimes J_k\). The induction hypothesis ensures \(J_k \leq L\), so we only need to show \(G^{<Y_k} \leq L\). For each \(g \in F\), the conjugate \(t^k gt^{-k}\) is in \(G^{<Y_k}\); indeed, there is \(y \in Y_k\) so that \(t^k gt^{-k}\) is a function which takes value \(g\) on \(y\) and \(1\) elsewhere. The group \(\langle t^k gt^{-k} \mid g \in F \rangle\) is thus the collection of functions supported on \(y\) in \(G^{<Y_k}\) and is contained in \(L\). Since \(J_k\) acts transitively on \(Y_k\), we conclude that \(G^{<Y_k} \leq L\), hence \(J_{k+1} \leq L\) finishing the induction.

For (2), the Hecke pair \(\omega^l(G, U, X)\) is elementary via Lemma \[7.12\]. Proposition \[7.17\] now implies that \(\omega^l(G, U, X)^{K_x}_{\psi_x}\) is also elementary.

We again obtain lower bounds on the rank.

\textbf{Proposition 7.20.} Let \((G, U, X)\) be a finitely generated transitive permutation Hecke pair and \(x \in X\). If \((G, U, X)\) is elementary and \(G\) is perfect, then

\[\xi(\omega^l(G, U, X)^{K_x}_{\psi_x}) \geq \xi(G) + \omega + 2.\]

\textbf{Proof.} Keep Notation \[7.19\] and let \(E\) be the Schlichting completion of \(\omega^l(G, U, X)^{K_x}_{\psi_x}\) via \(\sigma\). For \(Y \subseteq \omega^l(G, U, X)^{K_x}_{\psi_x}\), write \(\tilde{Y} := \sigma(Y)\).

Let \(J \leq \omega^l(G, U, X)^{K_x}_{\psi_x}\) be the canonical copy of \(\omega^l(G, X)\) in the HNN extension. As \(G\) is perfect, \(J\) is also perfect, and \(J = J \wr (J_1, Y_1)\). For \(O \leq E\) open, \(\tilde{J}_1 \cap O\) is a finite index subgroup of \(\tilde{I}_1\), hence \(O \cap \sigma(J)\) intersects \(\sigma(J_1)\) non-trivially. Lemma \[7.1\] now implies \(\sigma(J^{<Y_1}) \leq O\). In view of the action of \(t\) on \(J\), there is some \(y \in Y_1\) so that \(t^2 J t^{-2}\) is the collection of functions supported on \(y\). As the group \(O\) is invariant under conjugation by \(t\), we infer that \(\tilde{J} \leq O\), and thus, \(\tilde{J} \leq \text{Res}(E)\).
The group $E$ is compactly generated and elementary via Proposition 7.19 whereby $\xi(E) = \xi(\text{Res}(E)) + 1$ via Proposition 6.3. Proposition 7.13 now implies
\[
\xi\left(\bigoplus_i (G, U, X) * K_{x_i}\right) \geq \xi(G, U) + \omega + 2.
\]

7.4. Elementary groups with transfinite rank. We here build interesting examples of elementary groups via Hecke pairs; the inspiration for our approach is work of M. Brin on elementary amenable groups.

Perfect groups play an important part in our discussion. We thus require a technical lemma that produces perfect groups.

Lemma 7.21. Suppose that $(L \times \langle t \rangle, U)$ is a finitely generated proper elementary Hecke pair with $\langle t \rangle = \mathbb{Z}$ and $U \leq L$. Suppose further that $L$ is perfect and that there is a finite $F \subseteq L$ so that $\bigcup_{k \in \mathbb{Z}} t^k F t^{-k} \cup U$ generates $L$. Then there is a finitely generated proper elementary Hecke pair $(E, W)$ with $E$ perfect so that $\xi(L \times \langle t \rangle, U) \leq \xi(E, W)$.

Proof. Let the alternating group on five letters $A_5$ act in the usual way on $\{0, \ldots, 4\} =: [5]$ and form the wreath product
\[
M := (L \times \mathbb{Z}) \wr (A_5, [5]) = L^{[5]} \rtimes (\mathbb{Z} \wr (A_5, [5])).
\]
Take the subgroup $H := \langle (A_5)_{\mathbb{Z}^2(A_5, [5])} \rangle$ of $M$, where $\langle (A_5)_{\mathbb{Z}^2(A_5, [5])} \rangle$ is the normal closure of $A_5$ in $\mathbb{Z} \wr (A_5, [5])$, and for each $g \in L$, let $f_g \in L^{[5]}$ be the element of $M$ which takes value $g$ on 0 and is the identity elsewhere. Setting $\Omega := \{f_g \mid g \in F \cup U\}$, we define $E := \langle H \cup \Omega \rangle$.

To see $E$ is perfect, observe the function $r \in \mathbb{Z}^{[5]}$ defined by
\[
\begin{cases}
1 & \text{if } i = 0 \\
-1 & \text{if } i = 1 \\
0 & \text{otherwise.}
\end{cases}
\]
is an element of $H$. For each $g \in F$ and $n \in \mathbb{Z}$, the element $(nr)f_g(-nr) = f_{tn^gt^{-n}}$ is in $E$. Hence, $L^0$, the set of functions in $L^{[5]}$ supported on 0, is contained in $E$. As the groups $H$ and $L^0$ are perfect and $E = \langle H, L^0 \rangle$, the group $E$ is perfect.

Since $A_5 \leq E$ and $L^0 \leq E$, in fact $L^{[5]} \leq E$; in particular $W := U^{[5]}$ is a subgroup of $E$. Lemma 7.7 ensures that $(M, W)$ is an elementary Hecke pair, and it follows that $(E, W)$ is also an elementary Hecke pair. Observing that $H$ is finitely generated, the pair $(E, W)$ is additionally finitely generated. The group $U^{[5]}$ has trivial normal core in $L^{[5]}$, hence $W$ has trivial normal core in $E$. We thus conclude that $(E, W)$ is a finitely generated proper elementary Hecke pair.

Finally, define $\phi : L \times \mathbb{Z} \rightarrow E$ via $(g, n) \mapsto (f_g, nr)$ where $f_g \in L^{[5]}$ and $r$ are as defined above. This map is well-defined since $E$ contains $L^0$ and $r$. A calculation verifies that $\phi$ is an injective homomorphism. It is additionally the case that $\phi^{-1}(W) = U$.

Suppose that $S$ is the Schlichting completion of $(E, W)$ via $\sigma$ and let $V \leq S$ be a compact open subgroup so that $\sigma^{-1}(V) = W$. Putting $S' := \text{img}(\sigma \circ \phi)$, the map $\sigma \circ \phi : L \times \mathbb{Z} \rightarrow S'$ is injective with dense image. Furthermore,
\[
(\sigma \circ \phi)^{-1}(S' \cap V) = \phi^{-1}(\sigma^{-1}(S')) \cap \phi^{-1}(W) = U
\]
The group $S'$ is therefore a t.d.l.c.s.c. completion of $L \times \mathbb{Z}$, and it follows that $\xi(L \times \mathbb{Z}, U) \leq \xi(S') \leq \xi(S) = \xi(E, W)$. The lemma is now verified. \qed
Theorem 7.22. For each \( n \geq 1 \), there is a finitely generated proper elementary Hecke pair \((G_n, U_n)\) with \( \xi(G_n, U_n) \geq \omega \cdot n + 2 \).

Proof. We define inductively a sequence of finitely generated proper elementary Hecke pairs \((G_n, U_n)\) with a normal subgroup \( L_n \trianglelefteq G_n \) so that the following hold:

(i) \( \xi(G_n, U_n) \geq \omega \cdot n + 2 \);
(ii) \( L_n \) is perfect and \( L_n \rtimes Z = G_n \); and
(iii) letting \( t \) be a generator for \( Z \) in \( G_n \), there is a finite \( F \subseteq L_n \) so that
\[
\left( \bigcup_{k \in \mathbb{Z}} t^k F t^{-k} \cup U_n \right) = L_n.
\]

Fix \( H \) an infinite finitely generated perfect group with a non-trivial finite subgroup \( A \leq H \) and fix a transitive free permutation representation \((H, X)\). For concreteness, one can take Thompson’s group \( V, X := \text{Cay}(V) \), and \( A := A_5 \leq V \). Since \( A \) is finite, \((H, A, X)\) is a finitely generated transitive permutation Hecke pair which is elementary with rank at least 1. Applying Proposition 7.20 to \((H, A, X)\) and \( x \in X \), we obtain \( (G_1, U_1) := \omega l(H, A, X) \rtimes_{\psi_x} K_x \) which is a finitely generated elementary Hecke pair with \( \xi(G_1, U_1) \geq \omega + 2 \).

For the second requirement, \( \omega l(H, X) \) is perfect, so the normal subgroup
\[
L_1 := \bigcup_{k \geq 0} t^{-k} \left( \omega l(H, X) \right) t^k = \bigcup_{k \geq 0} t^{-k} \left( \omega l(H, X) \rtimes K_x^{\omega l(H, X)} \right) t^k
\]
of \( G_1 \) is perfect. It is clear that \( L_1 \rtimes Z = G_1 \). Condition (iii) is ensured by Proposition 7.19.

Suppose we have \((G_n, U_n)\) and \( L_n \) as hypothesized. Since \((G_n, U_n)\) satisfies (ii) and (iii), we may apply Lemma 7.21 to find a finitely generated proper elementary Hecke pair \((E, W)\) with \( E \) perfect so that \( \xi(E, W) \geq \xi(G_n, U_n) \geq \omega \cdot n + 2 \). We obtain a permutation Hecke pair \((E, W, X)\) by taking \( X := E/W \) and allowing \( E \rtimes X \) by left multiplication.

Applying Proposition 7.20 to \((E, W, X)\) and \( x \in X \) produces a finitely generated elementary Hecke pair
\[
(G_{n+1}, U_{n+1}) := \omega l(E, W, X) \rtimes_{\psi_x} K_x
\]
with
\[
\xi(G_{n+1}, U_{n+1}) \geq \xi(E, W) + \omega + 2 \geq \omega \cdot n + 2 + \omega + 2 = \omega \cdot (n + 1) + 2.
\]
Requirements (ii) and (iii) follow just as in the base case.

The construction is finished, and we conclude the theorem. \(\square\)

Corollary 7.23. There is an elementary group with rank at least \( \omega^2 + 1 \).

Proof. Let \((G_n, U_n)\) be the Hecke pairs given by Theorem 7.22 and let \( H_n \) be a t.d.l.c.s.c. completion of \((G_n, U_n)\). Fixing \( W_n \) a compact open subgroup of \( H_n \), form the local direct product \( E := \bigoplus_{i \geq 1} (H_n, W_n) \). Theorem 6.2 ensures the group \( E \) is elementary, and Proposition 6.3 implies
\[
\xi(E) \geq \sup \{ \xi(H_n) \mid n \geq 1 \} \geq \sup \{ \omega \cdot n + 2 \mid n \geq 1 \} \geq \sup \{ \omega \cdot n \mid n \geq 1 \} = \omega^2.
\]
Since the rank is always a successor ordinal, we conclude that \( \xi(E) \geq \omega^2 + 1 \). \(\square\)

Remark 7.24. The decomposition rank of an elementary group is always less than \( \omega_1 \), the first uncountable ordinal. A fundamental question concerning elementary groups is whether or not this bound can be improved. Theorem 7.22 shows the least upper bound must be greater than or equal to \( \omega^2 + 1 \).
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