Observation of interaction-induced gauge fields in a Bose-Einstein condensate based on micromotion control in a shaken two-dimensional lattice

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We demonstrate an interaction-induced gauge field for Bose-Einstein condensates in a modulated two-dimensional optical lattice. The gauge field results from the synchronous coupling between the interactions and micromotion of the atoms. As a first step, we show that a coherent shaking of the lattice in two directions can couple the momentum and interactions of atoms and break the four-fold symmetry of the lattice. We then create a full interaction-induced gauge field by modulating the interaction strength in synchrony with the lattice shaking. When a condensate is loaded into this shaken lattice, the gauge field acts to preferentially prepare the system in different quasimomentum ground states depending on the modulation phase. We envision that these interaction-induced fields, created by fine control of micromotion, will provide a stepping stone to model new quantum phenomena within and beyond condensed matter physics.

Synthesizing gauge fields for cold atoms opens the door to investigate novel quantum phenomena associated with charged particles in an electromagnetic field [11]: examples include quantum Hall effects, topological matter and anyonic excitations [2]. Many experimental approaches have been developed in the past years to introduce gauge fields including Raman transitions [1], laser-assisted tunneling [3,4], lattice shaking [5,6] and magnetic field modulation [7].

As charged particles in motion also generate electromagnetic fields, a complete simulation of the particle-field system should include the feedback of the matter to the gauge field [5]. Such interactive coupling between matter and field is an important step to simulate gauge field models in condensed matter [9,11], and in high energy physics, as in Yang-Mills theories [12]. Theoretical mechanisms for introducing interactive gauge fields in quantum gases were suggested based on multi-component lattice gases [13–18], light-matter interactions [2,19,20], as well as lattice modulations [21,22]. Experimental realization, however, remains elusive.

Lattice shaking, in particular, has recently emerged as a promising experimental tool to generate gauge potentials in cold atom systems [23], leading to exciting observations, including the realization of Haldane model [24], topological bands [25] and ferromagnetically interacting superfluids [6]. In our recent work, lattice modulation induces quantum phase transition in the condensate, displaying interesting phenomena: domain formation [6], roton excitations [26], as well as critical dynamics that are both universal [27] and coherent [28]. In such systems, the superfluid remains long lived and the atomic interactions play an important role to establish the ordering of superfluid domains.

In this paper, we experimentally demonstrate an interaction-induced synthetic gauge potential in a Bose-Einstein condensate (BEC). The gauge potential $A(\rho)$ appears as the substitution,

$$q \rightarrow q - A(\rho)/\hbar$$

in the Hamiltonian, linking its dependence on the momentum, represented by the wavevector $q = (q_x, q_y)$, with $\rho$, the density coarse-grained over one unit cell. To create this interaction-dependent gauge potential we exploit fine control over the micromotion of atoms in a shaken 2D square optical lattice in the presence of periodically modulated interaction strength. For atoms condensed in a two-dimensional momentum state $q$, the resulting coupling yields a mean-field energy shift.
\[ \mathcal{E}_q = \eta_\theta \rho g_0, \] 

where \( g_0 = g(t) \) is the period-average of the interaction strength \( g(t) = 4\pi \hbar^2 a(t)/m \), \( a(t) \) is the scattering length, \( m \) is the atomic mass and \( 2\pi \hbar \) is Planck’s constant. The dimensionless interaction factor \( \eta_\theta \) accounts for the coupling between the micromotion and the inter-atomic interactions, as detailed below. A gauge potential in the form of Eq. (1) requires \( \eta_\theta \) to be linear in \( q \).

We perform the experiment in two stages. In the first stage we show the effect of micromotion on interactions by tuning the relative phase \( \theta_s \) between the lattice shaking in the \( x \)- and \( y \)-directions while keeping the scattering length stationary. The micromotion raises the kinetic energy along the direction of the polarization and can break the four-fold symmetry of the dispersion. In the second stage we generate a genuine interaction-induced gauge field by simultaneously modulating the scattering length with a phase \( \theta_y \) relative to the lattice shaking. This scheme creates a gauge field with a unit vector in the direction \( \Theta = \theta_y - \theta_s/2 \).

To demonstrate the impact of the interaction-momentum coupling on an interacting quantum gas, the condensate is prepared in a 2D shaken lattice with four degenerate minima in its dispersion. We show experimentally that the gauge potential can reduce the \( D4 \) symmetry of the system to \( D2 \) or \( D1 \) by controlling the modulation phases.

Our experiments utilize Bose-Einstein condensates of \( N = 30,000 \) cesium atoms prepared in a harmonic trap with horizontal frequencies \( \omega_x \approx \omega_y = 2\pi \times 8 \text{Hz} \) and tight vertical confinement of \( \omega_z = 2\pi \times 200 \text{Hz} \). We load the atoms into a 2D, square optical lattice with lattice spacing \( d = \pi/q \) and 532 nm. The lattice depths \( V_0 \) along both directions are equal and small enough to maintain superfluidity of the gas. The lattice can then be shaken with identical peak-to-peak amplitudes \( s \) and angular frequencies \( \omega \) along both axes, see Fig. 1(a). The shaking frequency is chosen to be slightly higher than the excitation gap at zero momentum in the lattice [6].

When the shaking amplitude \( s \) exceeds a critical amplitude \( s_c \), the single particle dispersion \( E_{\text{kin}} \) develops four minima at momenta \( q = (\pm q^*, \pm q^*) \) and \( (\pm q^*, \mp q^*) \), where \( q^* \) can be controlled by \( s \), see Fig. 1(b). This four-fold degeneracy is the result of the \( D4 \) symmetry of the lattice, a 2D generalization of previous experiment in 1D [6, 24, 25]. Similar to the 1D system, the change in the dispersion induces a phase transition in which the condensate segregates into domains, each of which contains atoms occupying one of the four minima. Since the single particle Hamiltonian is separable along the two lattice axes, the kinetic energy is independent of the shaking polarization \( \theta_s \), defined as the relative phase between the two shaking lattices, see Fig. 1(c).

![FIG. 2. Interaction-momentum coupling due to micromotion. (a) Examples of micromotion for linear shaking (\( \theta_s = 0^\circ \)). Snapshots of the density \( |\psi_q(x,y,t)|^2 \) within a single 2D lattice site are shown for two states, \((+q^*, +q^*)\) (red) and \((-q^*, +q^*)\) (black), within a shaking period \( \tau \). (b) As a result of the micromotion, the mean microscopic density \( \langle n_q(t) \rangle \) oscillates and reaches a maximum when the wavefunction is most localized, and a minimum when it is most delocalized. Each curve is colored as in Fig. 1(c); note that the density oscillations of the white state are identical to the plot of the black curve. Dashed lines show the averaged densities. (c) Maps of the interaction factor \( \eta_\theta \) equal to the time-averaged microscopic density (see text), for different polarizations. The colored dots mark the ground states after accounting for the interaction factor. Note that circular polarization retains the \( D4 \) symmetry of the single particle dispersion.](image)
between the four kinetic energy minima, as shown in Fig. 2(b). Since the typical dynamics of the condensate (for instance, the formation of domains after the phase transition) occur on timescales spanning many shaking periods, they are predominantly sensitive to the interaction energy, \( E_\text{q} = \bar{\rho} g(t) \langle n_q(t) \rangle \), where the bar denotes time-averaging over one shaking period. Therefore, we define the interaction factor,

\[
\eta_\text{q} = \frac{1}{g_0} \bar{g}(t) \langle n_q(t) \rangle
\]

which accounts for the interplay between the interaction strength and the micromotion, see Eq. [2].

In the first stage of our experiments, with static interactions \( g(t) = g_0 \), we control the interaction-momentum coupling by tuning the macroscopic polarization, as shown in Fig. 2(c). To leading order in \( q/q_L \) the interaction factor is,

\[
\eta_\text{q} = \alpha + \beta s^2 \cos \theta_s q_x q_y,
\]

where \( \alpha \) and \( \beta \) are dimensionless constants that depend on the shaken lattice parameters \( V_0 \). The strength of this effect is greatest for linear shaking (\( \theta_s = 0^\circ \) or \( 180^\circ \)), with which the on-diagonal states (momentum along the axis of lattice motion) experience much stronger density modulation, leading to a higher interaction factor than the off-diagonal states (momentum perpendicular to the lattice motion), whose density is more consistent over time. This effect causes domains to form preferentially in the off-diagonal wells.

We test for the presence of this interaction-momentum coupling by driving condensates across the phase transition with different shaking phases \( \theta_s \) and measuring the resulting quasimomentum distribution. Here, we use a lattice depth of \( V_0 = 8.86 \ E_\text{th} \), where \( E_\text{th} = \hbar^2 q_L^2/2m = h \times 1.33 \text{ kHz} \) is the recoil energy, and shaking frequency \( \omega = 2 \pi \times 8 \text{ kHz} \). After loading the condensate into the lattice, we linearly increase the shaking amplitude to \( s = 20 \text{ nm} \) over 100 ms, exceeding the critical amplitude \( s_c = 13 \text{ nm} \) to drive the condensate across the phase transition. We subsequently increase the shaking amplitude to \( s = 32 \text{ nm} \) over 10 ms and hold the gas for another 120 ms to ensure that domains have clearly formed, after which we perform a short (5 ms) time-off-flight which enables us to reconstruct the original, in-situ domain distribution \( L \). In particular, we can extract the density distributions \( n_i(r) \) of atoms occupying the quasimomentum state in the \( i \)th quadrant; for example, \( n_1 \) is the density in the \((+q^+, +q^+\rangle \) state. From these we calculate the pseudo-spin density along each lattice axis, \( j_x = n_1 + n_4 - n_2 - n_3 \) and \( j_y = n_1 + n_2 - n_3 - n_4 \).

Typical reconstructed domain images for various shaking polarizations are shown in Fig. 3(a). To better quantify the biasing of the domains toward particular wells for ensembles of many images, we introduce an imbalance factor \( D = (N_1 + N_3 - N_2 - N_4)/N_\text{tot} \), where \( N_i \) is the population in the \( i \)th quadrant and \( N_\text{tot} \) is the total atom number. We observe a clear, polarization-dependent biasing of the domains toward forming in off-diagonal wells, indicative of interaction-momentum coupling; see Fig. 3(b). For linear shaking, which maximizes the interaction-momentum coupling, the diagonal imbalance approaches 1 (-1) with \( \theta_s = 0^\circ \) (\( 180^\circ \)), as expected. Under these conditions, the \( D4 \) symmetry of the ground states is clearly broken by interactions. As the shaking polarization approaches circular, the imbalance is progressively reduced. In principle, any deviation from circular shaking would lead to a complete biasing of the gas toward off-diagonal wells. However, because of the finite ramp speed, the system is not adiabatic throughout the quantum phase transition \( L \), resulting in the observed partial bias. For precisely circular shaking (\( \theta_s = 90^\circ \)) the interaction-momentum coupling disappears and the \( D4 \)
FIG. 4. Interaction-induced synthetic field from synchronized shaking and interaction strength modulation. (a) The upper panel plots the mean, microscopic density for circular shaking (θ = 90°). The lower panel shows the modulated interaction strength g(t) = g₀ + g₁ cos(ωt − θₐ). The modulated interactions raise (lower) the energy of quasimomentum states whose density oscillates in phase (out of phase) with the interaction modulation. (b) Modulated interaction factors for θₐ = 90° (left) and θₐ = 45° (right). (c) Measurement of the average quasimomentum of the condensate (in units of q* = 0.08 q₀) in the x− and y−directions in the presence of the interaction-induced field. The solid curves show simultaneous, sinusoidal fits, which yield a phase offset of only ±0.3°, consistent with theory (see text). Error bars represent one standard error.

symmetry is restored, resulting in a diagonal imbalance of D=0.04(5) consistent with zero.

In the second stage of our experiments, we generate an interaction-induced gauge field by applying synchronized shaking and interaction strength modulation. We tune the magnetic field near a Feshbach resonance [3] to modulate the interaction strength as g(t) = g₀ − g₁ cos(ωt − θₐ) at the same frequency as the lattice shaking and with phase θₐ, see Fig. 4(a). In this case, the interaction-momentum coupling can be understood intuitively by comparing the microscopic density and the interaction strength during each shaking period, see Fig. 4(a). When the interaction strength oscillates in phase (out of phase) with the density, the interaction energy is maximized (minimized).

To quantify the interaction-induced field, the interaction factor can be decomposed as, see Eq. [3],

\[ \eta_q = \eta_q^{(0)} + \frac{g_1}{g_0} \eta_q^{(1)}, \]

where \( \eta_q^{(0)} = \langle \eta_q(t) \rangle \) is the static interaction factor and \( \eta_q^{(1)} = -\langle \eta_q(t) \rangle \cos(\omega t - \theta_g) \) is the modulated interaction factor. We use circular shaking (θₐ = 90°) so that the static interaction factor maintains the D4 symmetry. For small momentum \(|q| \ll q_L\) the modulated interaction factor takes the form [29],

\[ \eta_q^{(1)} = -\sqrt{\frac{\alpha \beta}{2}} s e_\theta \cdot q, \]

which corresponds to the interaction-induced gauge potential,

\[ A(\rho) = \sqrt{\frac{\alpha \beta}{2}} m s g_1 \rho e_\theta \]

whose magnitude depends on the density and whose direction is given by \( e_\theta \) with \( \Theta \equiv \theta_g - \theta_s/2 \). Salient examples of the modulated interaction factors from a complete numerical calculation are shown in Fig. 4(b).

Experimentally, we test for the interaction-induced gauge field by measuring the bias toward particular quasimomenta as a function of the interaction phase θₐ. First, we prepare the condensate in a stationary lattice of depth \( V_0 = 4 E_R \) with a static scattering length of \( a_0 = 16 a_B \), where \( a_B \) is the Bohr radius, before ramping up the scattering length modulation amplitude to a maximum value of \( a_1 \approx 25 a_B \) over 25 ms. We then begin to circularly shake the lattice with frequency \( \omega = 2\pi \times 6.3 \) kHz, increasing the shaking amplitude to \( s = 26 \) nm over 70 ms, which drives the system across the phase transition. After a setting time of 10 ms, we measure the momentum distribution based on time-of-flight expansion [28].

The average quasimomentum after the phase transition shows a clear bias depending on the interaction modulation phase θₐ, indicative of the interaction-induced gauge field; see Fig. 4(c). Based on the form of the gauge potential shown in Eq. (6), we expect the biasing along the \( x− \) and \( y− \) axes to take the approximate forms \( \langle q_x \rangle \propto \cos(\theta_g - 45°) \) and \( \langle q_y \rangle \propto \sin(\theta_g - 45°) \). Simultaneous, sinusoidal fits to the data in Fig. 4(c) yield a phase consistent with this prediction. The magnitude of the bias in momentum does not reach \( q^* \), since it depends sensitively on the dynamics of crossing the phase
transition \cite{16} as well as the magnitude of the gauge potential. Experiments performed for a wide range of lattice depths and shaking frequencies exhibit similar results, consistent with the gauge potential described by Eq. \cite{17}.

In summary, we have presented an experimental implementation of an interaction induced gauge field based on a synchronous coupling between atomic micromotion and interaction effects. Our experiments have shown how this momentum-interaction coupling impacts the evolution of Bose condensates across an effectively ferromagnetic quantum phase transition. Importantly, the coupling allows a particular symmetry of the ground state to be selected in a manner consistent with expectations from theory. Our work presents a paradigm to guide the simulation of gauge field theories using ultracold atom systems.

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SUPPLEMENTARY INFORMATION

In this Supplement we sketch the theoretical treatment which demonstrates how our synthetic gauge potentials arise.

We describe our system, consisting of a Bose condensate in a shaken two-dimensional optical lattice with the many-body Hamiltonian

\[ H(t) = \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}, t) (H_0(t) - \mu) \hat{\psi}(\mathbf{r}, t) \]

\[ + \frac{g(t)}{2} \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t) , \]

where \( \hat{\psi}^\dagger(\mathbf{r}, t) \) creates (destroys) a boson at position \( \mathbf{r} = (x, y) \) and time \( t \), \( H_0(t) = -\frac{\hbar^2}{2m} \nabla^2 + V_L(\mathbf{r} - \delta \mathbf{t}) \) is the time-dependent single-particle Hamiltonian in a shaken lattice, \( \mu \) is the chemical potential of the Bose gas, and the interaction constant \( g(t) \) is periodically modulated.

Central for this paper is the assumption that the interaction energy is sufficiently small so that the system

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samples only the lowest Floquet band. This assumption is well justified for the experimental parameters considered. While the question of optimal preparation of many-body states is still a generally open question, we assume we can use the Floquet adiabatic preparation scheme.

**Single particle Floquet.**— Our lattice \( V_L (r) = V_L (x) + V_L (y) \) is separable, with a shaking function \( \delta r (t) = s [\sin (\omega t - \phi_x), \sin (\omega t - \phi_y)] / 2 \) composed of a pure sine-wave oscillating at frequency \( \omega = 2 \pi / \tau \); the phases \( \phi_x = -\phi_y = -\theta / 2 \) define the shaking polarization \( \theta \), which tunes the system from circular \((\theta_x = (j + 1/2) \pi \) where \( j = 0, 1, \ldots \) is an integer\) to linear shaking \((\theta_x = j \pi)\). Separability of the lattice transfers to the single particle Floquet band structure, since \( H_0 (t) = H_s (t) + H_g (t) \), where \( H_s (t) \) and \( H_g (t') \) commute at distinct times. The Floquet Hamiltonian \( H^\text{floq} = -i \hbar \ln \mathcal{U} (\tau) \) is therefore the sum of the Floquet Hamiltonian as calculated along each direction individually, i.e., \( H^\text{floq} = H_s^\text{floq} + H_g^\text{floq} \), with Floquet eigenstates given by product states \( | \psi_{x,y} \rangle = | \psi_{x} \rangle | \psi_{y} \rangle \), with corresponding energy \( E_{x,y}^\text{floq} = E_{x,y}^s + E_{x,y}^g \).

We chose our shaking frequency \( \hbar \omega = E_{x,y} + \delta \) to be near resonant with the zero-momentum band gap \( E_{x,y} \) of the two lowest bands, and choose \( \delta > 0 \). This allows us to approximate \( H^\text{floq} \) along direction \( x, y \) by a two-band model \([S31]\):

\[
H_{x,y}^\text{floq} (q_{x,y}) = \left( \begin{array}{cc} E^s (q_{x,y}) & e^{i q_{x,y} \Omega_{x,y}} \Omega_{x,y} \psi_{x,y} \\ e^{-i q_{x,y} \Omega_{x,y}} \Omega_{x,y} & E^g (q_{x,y}) - \omega \end{array} \right).
\]

The coupling \( \Omega_{x,y} \sim s \) can be used to drive the single particle dispersion along each axis from a single well at \( q_{x,y} = 0 \) to a double-well structure at \( q_{x,y} = \pm q^* \) at the critical shaking amplitude \( s = s_c \).

Diagonalizing the \( 2 \times 2 \) Hamiltonian in Eq. \((S2)\) gives two Floquet bands. We consider only the band adiabatically connected to the \( s \) band in the limit of zero shaking. Since the rotating-wave approximation is valid, \( \psi_{x,y}^\text{floq} \) is composed of only an \( s \)-band term coupled to a \( p \)-band term rotating as \( e^{i \omega t} \). We denote the Floquet wavefunction for this state as \( \psi_{x,y}^\text{floq} (x,t) = c_{q}^{(s)} u_{s}^{(s)} (x) + e^{i (\omega t - \phi_{x,y})} c_{q}^{(p)} u_{p}^{(p)} (x) \), where \( u_{s}^{(s,p)} \) and \( c_{q}^{(s,p)} \) are respectively the Bloch eigenfunction and Floquet coefficient for the \( s,p \) bands in question. For notational convenience, we will henceforth drop the superscript \( \text{floq} \) on Floquet states. We will also sometimes drop the subscript on \( q_{x,y} \) and \( \phi_{x,y} \); this is meant to imply that there are two equations, one for \( \{x, q_x, \phi_x\} \), and one for \( \{y, q_y, \phi_y\} \).

We conclude with a key observation: because of the separability of \( H_0 (t) \), the relative shaking phase between the \( x \) and \( y \) components will not enter in the Floquet energy.

**Effects of time dependent interactions.**— We now consider a Bose condensate, and turn to the effects of many body interactions through the mean field interaction energy, \( \mathcal{E}_I (t) = \frac{g(t)}{2} \int n^2 (r,t) dr \). Ultracold bosons will tend to occupy the combination of the four kinetic energy minima which also minimizes the total, time-averaged interaction energy \( \tilde{\mathcal{E}}_I = \frac{1}{\tau} \int_0^\tau \mathcal{E}_I (t) dt \) over one period \( \tau = 2 \pi / \omega \).

Since the interaction energy is minimized when all of the atoms occupy the same quasi-momentum state, we consider the marginal interaction energy for a particular quasi-momentum state in the ground Floquet band,

\[
\mathcal{E}_q (t) = \frac{\partial \mathcal{E}_I}{\partial N_q} = \rho g (t) \langle n_q (t) \rangle .
\]

The factor,

\[
\langle n_q (t) \rangle = d^2 \int_0^d \int_0^d | \psi_q (x,y,t) |^4 dx dy ,
\]

is the density enhancement factor that characterizes the increase in interaction energy due to the microscopic density modulation induced by the lattice structure. Here, the Floquet wavefunctions are normalized such that \( \int_0^d \int_0^d | \psi_q (x,y,t) |^2 dx dy = 1 \). Without any lattice, the density enhancement factor would take its minimum value \( \langle n_q (t) \rangle = 1 \). In a shaking lattice, the density enhancement factor becomes greater than one and oscillates at the shaking frequency \( \omega \).

The mean interaction energy per particle is,

\[
\mathcal{E}_q = \overline{\mathcal{E}_q (t)} ,
\]

where

\[
\langle n_q (t) \rangle = \frac{1}{g_0} \overline{g (t) \langle n_q (t) \rangle} ,
\]

is the interaction factor discussed in the main text. With \( g (t) = g_0 - g_1 \cos (\omega t - \theta) \), we see that \( \eta_q = \rho_q (0) + \frac{g_1}{g_0} \rho_q (1) \) naturally decomposes into a static term \( \eta_q (0) = \langle n_q (t) \rangle \) and a dynamic term \( \eta_q (1) = -\cos (\omega t - \theta) \langle n_q (t) \rangle \).

We now calculate the interaction factor, starting with the factorizable Floquet wavefunction \( \psi_q (x,y,t) = \psi_q (x,t) \psi_q (y,t) \), where \( \psi_q (x,t) = c_{q}^{(s)} u_{s}^{(s)} (x) + e^{i (\omega t - \phi_{x,y})} c_{q}^{(p)} u_{p}^{(p)} (x) \); factorizability implies

\[
\langle n_q (t) \rangle = \langle n_{qs} (t) \rangle \langle n_{qs} (t) \rangle .
\]

The density can then be expressed as

\[
n_q (x,t) = | \psi_q (x,t) |^2 = n_q (x) + \delta n_q (x,t) ,
\]

where

\[
n_q (x) = \left| c_{q}^{(s)} u_{s}^{(s)} (x) \right|^2 + \left| c_{q}^{(p)} u_{p}^{(p)} (x) \right|^2 .
\]
and

\[ \delta n_q(x, t) = \delta n_q(x) \cos(\omega t - \phi) \]  
\[ = 2Re \left[ e^{(s) q_u(x)} \left( e^{(p) q} (x) \right)^* e^{-i(\omega t - \phi)} \right], \quad (S11) \]

so

\[ \langle n_q(t) \rangle = \langle n_q \rangle + \langle \delta n_q \rangle \sin(\omega t - \phi) + \mathcal{O}(\delta n_q^2). \]

When calculating the static interaction factor,

\[ \eta_q^{(0)} = \langle n_q(t) \rangle = \langle n_{qs}(t) \rangle \langle n_{qs}(t) \rangle \]  
\[ (S12) \]

Finally, the dynamic interaction factor \( \eta_q^{(1)} \), is given by

\[ \eta_q^{(1)} = -\langle n_{qs} \rangle \langle \delta n_{qs} \rangle \cos(\omega t - \theta_g) \sin(\omega t - \phi_g) - \langle n_{qs} \rangle \langle \delta n_{qs} \rangle \cos(\omega t - \theta_g) \sin(\omega t - \phi_g) \]  
\[ = -\langle n_{qs} \rangle \langle \delta n_{qs} \rangle \sin(\Theta)/2 - \langle n_{qs} \rangle \langle \delta n_{qs} \rangle \sin(\Theta + \theta_s)/2 + \mathcal{O}(q^2) \]  
\[ (S15) \]

Expanding in small momentum, and taking \( \theta_s = 90^\circ \), we have

\[ \eta_q^{(1)} = -\sqrt{3}/2s \langle q_x \cos \Theta + q_y \sin \Theta \rangle \]  
\[ = -\sqrt{3}/2q \cdot \mathbf{e}_\Theta \]  
\[ (S16) \]

This analysis has yielded Eq. (6) in the main text. Again, a complete numerical calculation of the interaction factor can be calculated, and is shown in in Figs. 4(c) and 4(d) of the main text.

The energy shift due to the dynamic interaction factor can be understood as an interaction-induced synthetic gauge field by noting that a charged particle of mass \( m \) in a gauge potential \( \mathbf{A} \) experiences a momentum-dependent energy shift,

\[ E = -\frac{\mathbf{q} \cdot \mathbf{A}}{m}, \]  
\[ (S17) \]

where we have incorporated the hypothetical charge of the particle in the gauge potential itself. Equating this form with the mean interaction energy per particle from the dynamic interaction factor, \( E_a = -\sqrt{\frac{a^3}{2}} s \rho g \mathbf{e}_\Theta \), yields the interaction-induced synthetic gauge field,

\[ \mathbf{A}(\rho) = \sqrt{\frac{a^3}{2}} m s g \mathbf{e}_\Theta \]  
\[ (S18) \]

presented as Eq. (7) of the main text.