Elementary proofs of several results on false
discovery rate

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Abstract

We collect self-contained elementary proofs of four results in the litera-
ture on the false discovery rate of the Benjamini-Hochberg (BH) procedure
for independent or positive-regression dependent p-values, the Benjamini-
Yekutieli correction for arbitrarily dependent p-values, and the e-BH pro-
cedure for arbitrarily dependent e-values. As a corollary, the above proofs
also lead to some inequalities of Simes and Hommel.

1 False discovery rate

This short note contains proofs of four results on controlling the false discovery
rate (FDR) using the Benjamini-Hochberg (BH), Benjamini-Yekutieli (BY) and
e-BH procedures proposed by, respectively, Benjamini and Hochberg (1995),
Benjamini and Yekutieli (2001) and Wang and Ramdas (2022). All results are
known and various proofs exist in the literature. Our proofs are elementary,
concise, and self-contained. They may not be useful for experts in the field, but
they may become helpful for someone who is trained in probability theory but
not much in multiple hypothesis testing (e.g., a graduate student). This note
serves pedagogical purposes and does not contain any new results.

We first briefly define false discovery rate in multiple hypothesis testing. Let
$H_1, \ldots, H_K$ be $K$ hypotheses, and denote by $\mathcal{K} = \{1, \ldots, K\}$. The true data-
generating probability measure is denoted by $\mathbb{P}$. For each $k \in \mathcal{K}$, $H_k$ is called
a true null hypothesis if $\mathbb{P} \in H_k$, and the set $\mathcal{N} \subseteq \mathcal{K}$ be the set of indices of
true null hypotheses, which is unknown to the decision maker. Let $K_0 = |\mathcal{N}|$
be the number of true null hypotheses. A $p$-variable $P$ is a random variable
that satisfies $\mathbb{P}(P \leq \alpha) \leq \alpha$ for all $\alpha \in (0, 1)$. For each $k \in \mathcal{K}$, $H_k$ is associated
with p-value $p_k$, which is a realization of a random variable $P_k$, where $P_k$ is a $p$-
variable if $k \in \mathcal{N}$. For convenience, we will use the term “p-value” for both the
random variable $P_k$ and its realized value. We always write $P = (P_1, \ldots, P_K),$. 

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with the capital letters emphasizing that they are random variables. We also encounter e-values (Vovk and Wang (2021)) which will be defined in Section 4.

Let \( \mathcal{D} : [0, \infty)^K \to 2^K \) be a testing procedure, which reports the indices of rejected hypotheses based on observed p-values. The general aim is to reject non-null hypotheses among \( H_1, \ldots, H_K \). Write \( R_D := |\mathcal{D}| \), the number of discoveries (rejected hypotheses), and \( F_D := |\mathcal{D} \cap \mathcal{N}| \), the number of false discoveries and \( R_D \) is the false discovery proportion (FDP) with the convention \( 0/0 = 0 \). The quantities \( D, F_D \) and \( R_D \) are treated as random objects as they are functions of the p-values. Benjamini and Hochberg (1995) proposed to control FDR defined by \( \text{FDR}_D := \mathbb{E}[F_D/(R_D \lor 1)] \), where the expected value is taken under \( P \).

The assumption on the p-values or the e-values varies by sections. In Section 2, null p-values are iid uniform and independent of the non-null p-values. In Section 3, p-values satisfy a notion of positive dependence. In Sections 4 and 5, we deal with arbitrarily dependent e-values and p-values, respectively.

2 The BH procedure for iid null p-values

For \( k \in \mathcal{K} \), let \( p_{(k)} \) be the \( k \)-th order statistics of \( p_1, \ldots, p_K \), from the smallest to the largest. The BH procedure at level \( \alpha \in (0, 1) \) rejects all hypotheses with the smallest \( k^* \) p-values, where

\[
    k^* = \max \left\{ k \in \mathcal{K} : \frac{KP_{(k)}}{k} \leq \alpha \right\},
\]

with the convention \( \max(\emptyset) = 0 \), and accepts the rest.

We provide three proofs of the main result of Benjamini and Hochberg (1995). The first one uses an argument based on the optional stopping theorem. This proof was provided by Storey (2002, Chapter 5). The second one is based on a replacement technique. The third one uses the argument from the proof of Theorem 2 below as explained in Remark 1.

**Theorem 1.** If the null p-values are iid uniform on \([0, 1]\) and independent of the non-null p-values, then BH procedure at level \( \alpha \in (0, 1) \) has FDR equal to \( \alpha K_0/K \).

**First proof of Theorem 1.** Let \( \mathcal{D} \) be the BH procedure at level \( \alpha \). For \( t \in [0, 1] \), let \( F(t) = |\{k \in \mathcal{N} : P_k \leq t\}| \), \( R(t) = |\{k \in \mathcal{K} : P_k \leq t\}| \lor 1 \), and

\[
    t_\alpha = \sup\{t \in [0, 1] : Kt \leq \alpha R(t)\}.
\]

Since \( R \) is upper semicontinuous, we know \( Kt_\alpha = \alpha R(t_\alpha) \). Take \( k \in \mathcal{K} \). If \( P_k \leq t_\alpha \), then there exists \( \ell \geq R(P_k) \) such that \( KP_{(\ell)}/\ell \leq \alpha \), which means \( \ell \leq k^* \), and hence \( H_k \) is rejected by the BH procedure. If \( P_k > t_\alpha \), then \( KP_k > \alpha R(P_k) \), and hence \( H_k \) is not rejected by the BH procedure. To summarize, each \( H_k \) is rejected by the BH procedure if and only if \( P_k \leq t_\alpha \).
For \( t \in [0, 1] \), let
\[
\mathcal{F}_t = \sigma(\{P_k \leq s\}, \ldots, \{P_K \leq s\} : s \in [t, 1]).
\]
Note that for \( k \in \mathcal{N} \) and \( s \leq t \),
\[
\mathbb{E}[\mathbbm{1}_{\{P_k \leq s\}} | \mathcal{F}_t] = \mathbb{E}[\mathbbm{1}_{\{P_k \leq s\}}] = \frac{s}{t} \mathbbm{1}_{\{P_k \leq t\}}.
\]
Hence, for \( s \leq t \), we have \( \mathbb{E}[F(s) | \mathcal{F}_t] = sF(t)/t \), which implies that \( t \mapsto F(t)/t \) is a backward martingale. Note that \( t_\alpha \) is a stopping time with respect to the filtration \( (\mathcal{F}_t)_{t \in [0, 1]} \). The optional stopping theorem gives \( \mathbb{E}[F(t_\alpha)/t_\alpha] = \mathbb{E}[F(1)] = F(1) = K_0 \). Hence, using \( F_D = F(t_\alpha) \) and \( R_D = R(t_\alpha) \),
\[
\mathbb{E} \left[ \frac{F_D}{R_D} \right] = \mathbb{E} \left[ \frac{F(t_\alpha)}{R(t_\alpha)} \right] = \frac{\alpha}{K} \mathbb{E} \left[ \frac{F(t_\alpha)}{t_\alpha} \right] = \frac{K_0}{K^\alpha},
\]
and this completes the proof. \( \square \)

Second proof of Theorem 1. Let \( \alpha_r = \alpha r/K \) for \( r \in \mathcal{K} \). We can write
\[
\mathbb{E} \left[ \frac{F_D}{R_D} \right] = \mathbb{E} \left[ \frac{\sum_{k \in \mathcal{N}} \mathbbm{1}_{\{P_k \leq \alpha r\}}}{R_D} \right] = \sum_{k \in \mathcal{N}} \sum_{r=1}^K \frac{1}{r} \mathbb{E} \left[ \mathbbm{1}_{\{P_k \leq \alpha r\}} \mathbbm{1}_{\{R_D = r\}} \right]. \tag{1}
\]
For \( k \in \mathcal{N} \), let \( R_k \) be the number of rejection from the BH procedure if it is applied to \( P \) with \( P_k \) replaced by 0. Note that \( \{P_k \leq \alpha r, R_D = r\} = \{P_k \leq \alpha r, R_k = r\} \) for each \( k, r \). Hence, we have
\[
\mathbb{E}[\mathbbm{1}_{\{P_k \leq \alpha r\}}] = \mathbb{E}[\mathbbm{1}_{\{P_k \leq \alpha r, R_k = r\}}].
\]
Independence between \( P_k \) and \( (P_j)_{j \in \mathcal{K} \setminus \{k\}} \) implies that \( P_k \) and \( R_k \) are independent. For \( k \in \mathcal{N} \),
\[
\mathbb{E}[\mathbbm{1}_{\{P_k \leq \alpha r\}}] = \mathbb{P}(P_k \leq \alpha r) \mathbb{P}(R_k = r) = \frac{\alpha r}{K} \mathbb{P}(R_k = r). \tag{2}
\]
Putting this into (1), we get
\[
\mathbb{E} \left[ \frac{F_D}{R_D} \right] = \frac{\alpha}{K} \sum_{k \in \mathcal{N}} \sum_{r=1}^K \mathbb{P}(R_k = r) = \frac{K_0 \alpha}{K}, \tag{3}
\]
and this completes the proof. \( \square \)

3 The BH procedure for PRDS p-values

In what follows, inequalities should be interpreted component-wise when applied to vectors, and terms like “increasing” or “decreasing” are in the non-strict sense.
Definition 1. A set $A \subseteq \mathbb{R}^K$ is said to be increasing if $x \in A$ implies $y \in A$ for all $y \geq x$. The p-values $P_1, \ldots, P_K$ satisfy positive regression dependence on the subset $N$ (PRDS) if for any null index $k \in N$ and increasing set $A \subseteq \mathbb{R}^K$, the function $x \mapsto \mathbb{P}(P \in A \mid P_k \leq x)$ is increasing.

This version of PRDS is used by Finner et al. (2009, Section 4) which is weaker than the original one used in Benjamini and Yekutieli (2001), where “$P_k \leq x$” in Definition 1 is replaced by “$P_k = x$”; see also Lemma 1 of Ramdas et al. (2019). Below, we present a proof of the FDR guarantee of the BH procedure for PRDS p-values. This proof is found in Finner et al. (2009).

Theorem 2. If the p-values satisfy PRDS, then the BH procedure at level $\alpha$ has FDR at most $\frac{\alpha K \beta_1}{K}$.

Proof. Let $D$ be the BH procedure at level $\alpha$. Write $\alpha_r = \alpha r / K$ and $\beta_{k,r} = \mathbb{P}(R_D \geq r \mid P_k \leq \alpha_r)$ for $r, k \in K$, and set $\beta_{k,K+1} = 0$. By noting that $R_D$ is a decreasing function of the p-values, the PRDS property gives, for $k \in N$ and $r \in K$,

$$\mathbb{P}(R_D \geq r + 1 \mid P_k \leq \alpha_r) \geq \mathbb{P}(R_D \geq r + 1 \mid P_k \leq \alpha_{r+1}) = \beta_{k,r+1},$$

which leads to

$$\beta_{k,r} - \beta_{k,r+1} \geq \mathbb{P}(R_D \geq r \mid P_k \leq \alpha_r) - \mathbb{P}(R_D \geq r + 1 \mid P_k \leq \alpha_r) = \mathbb{P}(R_D = r \mid P_k \leq \alpha_r).$$

Using this inequality and (1), we get

$$\mathbb{E} \left[ \frac{F_D}{R_D} \right] = \sum_{k \in N} \sum_{r=1}^K \frac{1}{r} \mathbb{P}(P_k \leq \alpha_r) \mathbb{P}(R_D = r \mid P_k \leq \alpha_r)$$

$$\leq \sum_{k \in N} \sum_{r=1}^K \frac{\alpha}{K} \mathbb{P}(R_D = r \mid P_k \leq \alpha_r)$$

$$\leq \sum_{k \in N} \sum_{r=1}^K \frac{\alpha}{K} (\beta_{k,r} - \beta_{k,r+1}) = \sum_{k \in N} \frac{\alpha}{K} \beta_{k,1} = \frac{\alpha K_0}{K},$$

where the inequality (5) follows from $\mathbb{P}(P_k \leq \alpha_r) \leq \alpha_r$, and the last equality follows from $\beta_{k,1} = 1$.

Remark 1. In the proof of Theorem 2, the inequality (4) holds as an equality if $P_r$ is independent of $(P_k)_{k \in K \setminus \{r\}}$, and the inequality (5) holds as an equality if $P_k$ is uniformly distributed. Therefore, this argument also proves Theorem 1. See also Ramdas et al. (2019) for a superuniformity lemma which becomes useful when showing other FDR statements for PRDS p-values.

Next, we mention the Simes inequality as a corollary of Theorem 1-2. Define the function $S_K : [0, \infty)^K \to [0, \infty)$ of Simes (1986) as

$$S_K(p_1, \ldots, p_k) = \bigwedge_{k=1}^K \frac{K}{k} p_{(k)},$$
where $p_{(k)}$ is the $k$-th smallest order statistic of $p_1, \ldots, p_K$. The Simes function is closely linked to the BH procedure. In case $N = K$, that is, the global null, any rejection is a false discovery, and the FDP is 1 as soon as there is any rejection. Therefore, the FDR of the BH procedure $\mathcal{D}$ at level $\alpha$ is equal to $\mathbb{P}(\mathcal{D} \neq \emptyset)$, and $\mathcal{D} \neq \emptyset$ is equivalent to $S_K(\mathcal{P}) \leq \alpha$. The Simes inequality, shown by Simes (1986) for the iid case and Sarkar (1998) for a notion of positive dependence, follows directly from Theorems 1-2 and the above observation in the case of PRDS p-values.

**Corollary 1.** Suppose $N = K$. If the p-values are PRDS, then

$$\mathbb{P}(S_K(\mathcal{P}) \leq \alpha) \leq \alpha \quad \text{for all } \alpha \in [0, 1].$$

If the p-values are iid uniform on $[0, 1]$, then $S_K(\mathcal{P})$ is also uniform on $[0, 1]$.

The last statement on uniformity in Corollary 1 was originally shown by Simes (1986) using a concise proof by induction.

### 4 The e-BH procedure for arbitrary e-values

As introduced by Vovk and Wang (2021), an e-variable $E$ is a $[0, \infty]$-valued random variable satisfying $E[E] \leq 1$. E-variables are often obtained from stopping an e-process $(E_t)_{t \geq 0}$, which is a nonnegative stochastic process adapted to a pre-specified filtration such that $E[E_{\tau}] \leq 1$ for any stopping time $\tau$. In the setting of e-values, for each $k \in \mathcal{K}$, $H_k$ is associated with e-value $e_k$, which is a realization of a random variable $E_k$ being an e-variable if $k \in \mathcal{N}$, and an e-testing procedure $\mathcal{D} : [0, \infty]^K \to 2^K$ takes these e-values as input.

For $k \in \mathcal{K}$, let $e_{(k)}$ be the $k$-th order statistic of $e_1, \ldots, e_K$, sorted from the largest to the smallest so that $e_{(1)}$ is the largest e-value. The e-BH procedure at level $\alpha \in (0, 1)$, proposed by Wang and Ramdas (2022), rejects hypotheses with the largest $k^*$ e-values, where

$$k^* = \max \left\{ k \in \mathcal{K} : \frac{ke_{(k)}}{K} \geq \frac{1}{\alpha} \right\}.$$

In other words, the e-BH procedure is equivalent to the BH procedure applied to $(e_1^{-1}, \ldots, e_K^{-1})$.

Below we present a simple proof of the FDR guarantee of the e-BH procedure. Wang and Ramdas (2022) also considered the e-BH procedure on boosted e-values which we omit here.

**Theorem 3.** For arbitrary e-values, the e-BH procedure at level $\alpha \in (0, 1)$ has FDR at most $\alpha K_0 / K$.

**Proof.** Let $\mathcal{D}$ be the e-BH procedure at level $\alpha$. By definition, the e-BH procedure $\mathcal{D}$ applied to arbitrary e-values $(E_1, \ldots, E_K)$ satisfies the following property

$$E_k \geq \frac{K}{\alpha K_0} \quad \text{if } k \in \mathcal{D}. \quad (6)$$
Using (6), the FDP of \( \mathcal{D} \) satisfies
\[
\frac{F_{\mathcal{D}}}{R_{\mathcal{D}}} = \frac{\left| \mathcal{D} \cap \mathcal{N} \right|}{R_{\mathcal{D}} \vee 1} = \sum_{k \in \mathcal{N}} \frac{1}{R_{\mathcal{D}} \vee 1} \leq \sum_{k \in \mathcal{N}} \frac{1_{(k \in \mathcal{D})} \alpha E_k}{K} \leq \sum_{k \in \mathcal{N}} \frac{\alpha E_k}{K},
\]
where the first inequality is due to (6). As \( E_k \leq 1 \) for \( k \in \mathcal{N} \), we have
\[
E \left[ \frac{F_{\mathcal{D}}}{R_{\mathcal{D}}} \right] \leq \sum_{k \in \mathcal{N}} E \left[ \frac{\alpha E_k}{K} \right] \leq \frac{\alpha K_0}{K},
\]
thus the desired FDR guarantee.

**Remark 2.** Any e-testing procedure \( \mathcal{D} \) satisfying (6) is said to be self-consistent by Wang and Ramdas (2022). The e-BH procedure dominates all other self-consistent e-testing procedures. From the proof of Theorem 3, any self-consistent e-testing procedure has FDR at most \( \frac{\alpha K_0}{K} \) for arbitrary e-values.

5 The BY correction for arbitrary p-values

The next theorem concerns the FDR guarantee of the BH procedure for arbitrary p-values. As shown by Benjamini and Yekutieli (2001), in the most adversarial scenario, the BH procedure needs to pay the price of a factor of \( \ell_K \), where
\[
\ell_K = \sum_{k=1}^{K} \frac{1}{k} \approx \log K.
\]
We provide two simple proofs. The first one is similar to the original proof of Benjamini and Yekutieli (2001). The second proof is based on the FDR of the e-BH procedure in Theorem 3; a similar argument, without explicitly using e-values, is given by Blanchard and Roquain (2008).

**Theorem 4.** For arbitrary p-values, the BH procedure at level \( \alpha \in (0, 1) \) has FDR at most \( \ell_K \alpha K_0 / K \).

**First proof of Theorem 4.** Note that (1) can be rearranged to
\[
E \left[ \frac{F_{\mathcal{D}}}{R_{\mathcal{D}}} \right] = \sum_{k \in \mathcal{N}} \left( \sum_{r=1}^{K} \frac{E \left[ 1_{\{p_k \leq \alpha R_D\}} 1_{\{R_D \leq r\}} \right]}{r(r+1)} + \frac{E \left[ 1_{\{p_k \leq \alpha R_D\}} \right]}{K+1} \right). \tag{7}
\]
For \( k \in \mathcal{N} \),
\[
E \left[ 1_{\{p_k \leq \alpha R_D\}} 1_{\{R_D \leq r\}} \right] \leq E \left[ 1_{\{p_k \leq \alpha r\}} 1_{\{R_D \leq r\}} \right] \leq E \left[ 1_{\{p_k \leq \alpha r\}} \right] \leq \frac{r}{K} \alpha,
\]
and hence (7) leads to
\[
E \left[ \frac{F_{\mathcal{D}}}{R_{\mathcal{D}}} \right] \leq \sum_{r=1}^{K} \frac{K_0 \alpha}{(r+1)K} + \frac{K_0 \alpha}{K+1} = \sum_{r=1}^{K} \frac{K_0 \alpha}{(r+1)K} = \sum_{r=1}^{K} \frac{1}{K+1} = \frac{\ell_K K_0}{K} \alpha.
\]
and this completes the proof. □

**Second proof of Theorem 4.** Denote by $\alpha' = \alpha K$. Define the decreasing function $\phi : [0, \infty) \to [0, \alpha' K]$ by

$$\phi(p) = \frac{K}{\alpha' [K_p/\alpha]} \mathbb{1}_{\{p \leq \alpha\}} \text{ with } \phi(0) = K/\alpha'.$$

Let $E_k = \phi(P_k)$ for $k \in K$. It is straightforward to check that, for $\int_0^1 \phi(p) \, dp = 1$. Therefore, $E_k$ is an $e$-value for $k \in \mathcal{N}$. By Theorem 3, applying the e-BH procedure at level $\alpha'$ to $(E_1, \ldots, E_K)$ has an FDR guarantee of $\alpha' K_0/K$. Note that

$$\frac{k E_k}{K} \geq \frac{1}{\alpha'} \iff \frac{k}{[K P_k/\alpha]} \geq 1 \iff \frac{K P_k}{k} \leq \alpha.$$

Hence, the e-BH procedure at level $\alpha'$ applied to $(E_1, \ldots, E_K)$ is equivalent to the BH procedure applied to $(P_1, \ldots, P_K)$ at level $\alpha$. This yields the FDR guarantee $\alpha' K_0/K$ of the BH procedure. □

The second proof of Theorem 4 can be generalized to many other procedures, as long as they can be converted to an e-BH procedure via a transform similar to (8).

In a similar way to Corollary 1, the inequality of Hommel (1983) for arbitrary p-values follows from Theorem 4.

**Corollary 2.** Suppose $\mathcal{N} = K$. For arbitrary p-values, we have

$$\mathbb{P}(S_K(P) \leq \alpha) \leq \ell_K \alpha \quad \text{for all } \alpha \in [0, 1].$$

As observed by Hommel (1983) and Simes (1986), the inequalities in Corollaries 1 and 2 cannot be improved to obtain a smaller upper bound under their respective assumptions. Similarly, the FDR upper bound in Theorems 1-4 cannot be improved in general.

**References**

Benjamini, Y. and Hochberg, Y. (1995). Controlling the false discovery rate: A practical and powerful approach to multiple testing. *Journal of the Royal Statistical Society Series B*, 57(1), 289–300.

Benjamini, Y. and Yekutieli, D. (2001). The control of the false discovery rate in multiple testing under dependency. *Annals of Statistics*, 29(4), 1165–1188.

Blanchard, G. and Roquain, E. (2008). Two simple sufficient conditions for FDR control. *Electronic Journal of Statistics*, 2, 963–992.

Finner, H., Dickhaus, T. and Roters, M. (2009). On the false discovery rate and an asymptotically optimal rejection curve. *The Annals of Statistics*, 37(2), 596–618.

Hommel, G. (1983). Tests of the overall hypothesis for arbitrary dependence structures. *Biometrical Journal*, 25(5), 423–430.
Ramdas, A. K., Barber, R. F., Wainwright, M. J. and Jordan, M. I. (2019). A unified treatment of multiple testing with prior knowledge using the p-filter. *Annals of Statistics*, **47**(5), 2790–2821.

Sarkar, S. K. (1998). Some probability inequalities for ordered MTP2 random variables: A proof of the Simes conjecture. *Annals of Statistics*, **26**(2), 494–504.

Simes, R. J. (1986). An improved Bonferroni procedure for multiple tests of significance. *Biometrika*, **73**, 751–754.

Storey, J. (2002). *False Discovery Rates: Theory and Applications to DNA Microarrays*. PhD Thesis, Stanford University.

Vovk, V. and Wang, R. (2021). E-values: Calibration, combination, and applications. *Annals of Statistics*, **49**(3), 1736–1754.

Wang, R. and Ramdas, A. (2022). False discovery rate control with e-values. *Journal of the Royal Statistical Society Series B*, [https://doi.org/10.1111/rssb.12489](https://doi.org/10.1111/rssb.12489).