Explicit polynomial solutions of the planar, unsteady Navier-Stokes equations

Tiemo Pedergnana, David Öttinger and George Haller

Department of Mechanical and Process Engineering,
ETH Zürich, Leonhardstrasse 21, 8092 Zürich, Switzerland

(Dated: August 14, 2019)
Abstract

We construct a class of spatially polynomial velocity fields that are exact solutions of the planar unsteady Navier–Stokes equation. These solutions can be used as simple benchmarks for testing numerical methods or verifying the feasibility of flow-feature identification principles. We use simple examples from the constructed family to illustrate deficiencies of streamlines-based feature detection and of the Okubo–Weiss criterion in unsteady flows.

I. INTRODUCTION

In this paper, we address the following question: For what time-dependent vectors \( a_{kj}(t) \in \mathbb{R}^2 \) does the planar velocity field

\[
    u(x, t) = \sum_{j=0}^{n} \sum_{k=0}^{m} a_{kj}(t)x^k y^j
\]

solve the incompressible Navier–Stokes equation with the spatial variable \( x = (x, y) \in \mathbb{R}^2 \) and the time variable \( t \in \mathbb{R} \)? Answering this question would enable one to produce a large class of exact Navier-Stokes solutions for numerical benchmarking and for verifying theoretical results on simple, dynamically consistent, unsteady flow models. For linear velocity fields \((a_{kj}(t) \equiv 0 \text{ for } k, j > 1)\), general existence conditions for such solutions are detailed by Majda [1] and Majda and Bertozzi [2]. A more specific form of these spatially linear solutions is given by Craik and Criminale [3], who obtain that any differentiable function \( a_{00}(t) \) and any differentiable, zero-trace matrix \( A(t) \) generates a linear Navier–Stokes solution \( u(x, t) \) in the form

\[
    u(x, t) = a_{00}(t) + A(t)x,
\]

provided that \( \dot{A}(t) + A^2(t) \) is symmetric. Note that all such solutions are universal (i.e. independent of the Reynolds number) because the viscous forces vanish identically on them. Reviews of exact Navier–Stokes solutions tend to omit a discussion of the Craik–Criminale solutions, although they list several specific spatially linear steady solutions for concrete physical settings (see Berker [4], Wang [5–7], and Drazin and Riley [8]). For solutions with at least quadratic spatial dependence, no general results of the specific form (1) have apparently been derived.

In related work, Perry and Chong [9] outline a recursive procedure for determining local
Taylor expansions of solutions of the Navier–Stokes equations up to any order when specific boundary conditions are available. Bewley and Protas [10] show that near a straight boundary, the resulting Taylor coefficients can all be expressed as functions and derivatives of the skin friction and the wall pressure. The objective in these studies is, however, a recursive construction of Taylor coefficients for given boundary conditions, rather than a derivation of the general form of exact polynomial Navier–Stokes velocity fields of finite order. We also mention the work of Bajer and Moffatt [11], wherein the authors construct exact, steady three-dimensional Navier–Stokes flows with quadratic spatial dependence for which the flux of the velocity field through the unit sphere vanishes pointwise.

The linear part of the velocity field (1) can already have arbitrary temporal complexity, but remains spatially homogeneous by construction. The linear solution family (2) identified by Craik and Criminale [3] cannot, therefore, produce bounded flow structures. As a consequence, these linear solutions cannot yield Navier–Stokes flows with finite coherent vortices or bounded chaotic mixing zones.

Higher-order polynomial vector fields of the form (1), however, are free from these limitations, providing an endless source of unsteady and dynamically consistent examples of flow structures away from boundaries. We will illustrate this on specific examples whose instantaneous streamlines look deceptively simple, yet fail to signal the correct nature of fluid particle behavior.

II. SOLUTION PROCEDURE

We rewrite the incompressible Navier-Stokes equation in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} = \nabla \left[ -\frac{1}{\rho} p(\mathbf{x}, t) \right],$$

which shows that the left-hand side of (3) is a gradient (i.e., conservative) vector field. By classic results in potential theory, a vector field is conservative on a simply connected domain if and only if its curl is zero. For a two-dimensional vector field, this zero-curl condition is equivalent to the requirement that the Jacobian of the vector field is zero, as already noted in the construction of linear Navier–Stokes solutions by Craik and Criminale [3].

On simple connected domains, therefore, a sufficient and necessary condition for \( \mathbf{u}(\mathbf{x}, t) \)
to be a Navier–Stokes solution is given by

\[
\nabla \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} \right] = \left( \nabla \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} \right] \right)^T,
\]

which no longer depends on the pressure. Substituting (1) into (4) and equating equal powers of \(x_1\) and \(x_2\) in the off-diagonal elements of the matrices on the opposite sides of the resulting equation, we obtain conditions on the unknown coefficients of the spatially polynomial velocity field \(\mathbf{u}(\mathbf{x}, t)\).

III. UNIVERSAL SOLUTIONS

By a universal solution of equation (3) we mean a solution \(\mathbf{u}(\mathbf{x}, t)\) on which viscous forces identically vanish, rendering the pressure \(p(\mathbf{x}, t)\) independent of the Reynolds number. Note that all spatially linear solutions of (3) are universal. More generally, a solution \(\mathbf{u}(\mathbf{x}, t)\) is a universal solution of the planar Navier–Stokes equation if it satisfies

\[
\Delta \mathbf{u} \equiv 0,
\]

i.e. if it is a harmonic solution. Hence when looking for universal solutions of the form (1) which satisfy (4), we look for solutions \(\mathbf{u}(\mathbf{x}, t)\) whose components are harmonic polynomials in \(\mathbf{x}\) with time-dependent coefficients. We will allow these solutions to be piecewise smooth with piecewise constant vorticity. Since the viscous terms in the Navier–Stokes vanish for harmonic flows, the universal solutions we seek will also be piecewise smooth solutions of Euler’s equation. General existence conditions for such solutions are given by Majda and Bertozzi [2]. Our main result is as follows:

**Theorem 1.** A \(n\)th-order polynomial velocity field \(\mathbf{u}(\mathbf{x}, t) = (u, v)\) of the variable \(\mathbf{x} = (x, y)\) is a universal solution of the planar Navier–Stokes equation with piecewise constant vorticity \(\omega = \omega(\mathbf{x})e_3, \omega \in \mathbb{R}\), if and only if it is of the form

\[
\mathbf{u}(\mathbf{x}, t) = \mathbf{h}(t) + \sum_{k=1}^n \begin{pmatrix} a_k(t) & b_k(t) \\ b_k(t) & -a_k(t) \end{pmatrix} \begin{pmatrix} \text{Re}[(x + iy)^k] \\ \text{Im}[(x + iy)^k] \end{pmatrix}.
\]

**Proof.** Since \(\mathbf{u}(\mathbf{x}, t)\) is assumed to satisfy (5), it is undetermined up to a spatially constant and a linear part. The requirement of piecewise constant vorticity \(\omega = \omega(\mathbf{x})e_3, \omega \in \mathbb{R}\), determines the form of the linear part of the sum in (6). Furthermore, for a planar harmonic,
divergence-free velocity field with piecewise constant vorticity \( \omega = \omega(x)e_3, \omega \in \mathbb{R} \), equation (4) is identically satisfied at all points where the vorticity is differentiable. To see this, note that at all such points, we have

\[
\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial y} = 0. \tag{7}
\]

Since \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \), eq. (7) implies that

\[
\begin{align*}
\frac{\partial^2 v}{\partial t \partial x} &= \frac{\partial^2 u}{\partial t \partial y}, \\
\frac{\partial^2 v}{\partial x \partial y} &= \frac{\partial^2 u}{\partial y^2}, \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 v}{\partial x^2}.
\end{align*} \tag{8-10}
\]

Since the velocity field \( u(x,t) \) is divergence-free, we also have

\[
\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}. \tag{11}
\]

Since the velocity field \( u(x,t) \) is harmonic, eq. (4) is satisfied if and only if the \( 2 \times 2 \) matrix

\[
\nabla \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] \tag{12}
\]

is symmetric. To verify the symmetry of this matrix, first note that the off-diagonal elements of \( \nabla \frac{\partial u}{\partial t} \) are equal by formula (8). Using formulas (9)-(11), we obtain by direct calculation that the off-diagonal elements of \( \nabla \left[ (u \cdot \nabla) u \right] \) are also equal. To prove that \( u(x,t) \) is necessarily of the form given in (6), we recall from Andrews, Askey and Roy \[12\] that a basis of the space of \( k \)-th-order, homogeneous harmonic polynomials of two variables is given by

\[
\{ \text{Re}[(x + iy)^k], \text{Im}[(x + iy)^k] \}. \tag{13}
\]

Hence the most general form of \( u(x,t) \) whose components are harmonic polynomials in \( x \) is of the form

\[
u(x,t) = \sum_{k=1}^{n} \begin{pmatrix} a_k(t) & b_k(t) \\ c_k(t) & d_k(t) \end{pmatrix} \begin{pmatrix} \text{Re}[(x + iy)^k] \\ \text{Im}[(x + iy)^k] \end{pmatrix}. \tag{14}
\]

By the Cauchy-Riemann equations for the real and imaginary part of a holomorphic complex function \( f(x,y) \), we have

\[
\begin{align*}
\frac{\partial \text{Re}[f]}{\partial x} &= \frac{\partial \text{Im}[f]}{\partial y}, \\
\frac{\partial \text{Re}[f]}{\partial y} &= -\frac{\partial \text{Im}[f]}{\partial x}.
\end{align*} \tag{15-16}
\]
Since \( f(x, y) = (x + iy)^k \) is holomorphic, requiring the divergence of \( \mathbf{u}(x, y) \) in (6) to vanish is equivalent to the requirement

\[
\sum_{k=0}^{n} \left( a_k \frac{\partial \text{Re}[f]}{\partial x} + b_k \frac{\partial \text{Im}[f]}{\partial x} + c_k \frac{\partial \text{Re}[f]}{\partial y} + d_k \frac{\partial \text{Im}[f]}{\partial y} \right) = \sum_{k=0}^{n} \left[ (a_k + d_k) \frac{\partial \text{Re}[f]}{\partial x} + (c_k - b_k) \frac{\partial \text{Re}[f]}{\partial y} \right] \equiv 0.
\]

For formula (18) to hold at order \( k = 0 \), any constant term \( a_{00} = h(t) \) can be selected. The same formulas requires \( a_1(t) = -d_1(t) \) at order \( k = 1 \). If the vorticity \( \omega \) is constant, we must additionally have \( c_1(t) \equiv b_1(t) + \omega(x) \). Finally, for \( k \geq 2 \), formula (18) requires \( a_k(t) \equiv -d_k(t) \) and \( b_k(t) \equiv c_k(t) \). These terms then all generate zero vorticity because by equations (15)-(16), we have

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \sum_{k=2}^{n} \left( b_k \frac{\partial \text{Re}[f]}{\partial x} - a_k \frac{\partial \text{Im}[f]}{\partial x} \right) - \left( a_k \frac{\partial \text{Re}[f]}{\partial y} + b_k \frac{\partial \text{Im}[f]}{\partial y} \right) = \sum_{k=2}^{n} b_k \left( \frac{\partial \text{Re}[f]}{\partial x} - \frac{\partial \text{Im}[f]}{\partial y} \right) - a_k \left( \frac{\partial \text{Im}[f]}{\partial x} + \frac{\partial \text{Re}[f]}{\partial y} \right) = 0.
\]

This concludes the proof of formula (6).

III. EXAMPLES

We now give some examples of dynamically consistent flow fields covered by the general formula (6). On these exact solutions, we also illustrate how a commonly used, frame-dependent vortex criterion, the Okubo-Weiss criterion \cite{13, 14}, fails to correctly identify the true nature of unsteady fluid particle motion. The Okubo-Weiss criterion, in its original two-dimensional form (Okubo \cite{13}, Weiss \cite{14}) defines the quantity

\[
Q(x, y, t) = \left[ \frac{\partial u(x, y, t)}{\partial x} \right]^2 + \frac{\partial v(x, y, t)}{\partial x} \frac{\partial u(x, y, t)}{\partial y},
\]

for a two-dimensional velocity field \( \mathbf{u}(x, t) = (u(x, y, t), v(x, y, t)) \). According to the criterion, an elliptic (or vortical) region is defined by the requirement \( Q(x, y, t) < 0 \). Conversely, a hyperbolic (or stretching) region is defined by the requirement \( Q(x, y, t) > 0 \).

**Example 1.** Haller \cite{15} proposed the velocity field

\[
\mathbf{u}(x, t) = \begin{pmatrix} \sin 4t & \cos 4t + 2 \\ \cos 4t - 2 & -\sin 4t \end{pmatrix} x,
\]

\( \square \)
as a purely kinematic benchmark example for testing vortex criteria. By inspection of (6),
we immediately find that (22) is actually dynamically consistent, solving the Navier–Stokes
equation with \( h(t) \equiv 0 \), \( a_1(t) = \sin 4t \), \( b_1(t) = \cos 4t + 2 \), and \( \omega = -4 \) and \( a_k = b_k \equiv 0 \) for \( k \geq 2 \). More generally, formula (6) shows that the linear unsteady velocity field
\[
\mathbf{u}(\mathbf{x}, t) = h(t) + \begin{pmatrix}
-\sin(Ct) & \cos(Ct) - \frac{\omega}{2} \\
\cos(Ct) + \frac{\omega}{2} & \sin(Ct)
\end{pmatrix} \mathbf{x}
\]  
(23)
solves the Navier–Stokes equations for any constants \( \omega \) and \( C \), and any smooth function \( h(t) \). We set \( h(t) \equiv 0 \) for simplicity and pass to a rotating \( y \) coordinate frame via the transformation
\[
\mathbf{x} = \mathbf{M}(t)\mathbf{y}, \quad \mathbf{M}(t) = \begin{pmatrix}
\cos(\frac{C}{2}t) & \sin(\frac{C}{2}t) \\
-\sin(\frac{C}{2}t) & \cos(\frac{C}{2}t)
\end{pmatrix}
\]  
(24)
Differentiating the coordinate change (24) with respect to time and using (23) gives the form of (23) in the \( y \)-frame as
\[
\dot{\mathbf{y}} = \mathbf{u}(\mathbf{y}) = \begin{pmatrix}
0 & 1 + \frac{1}{2}(C - \omega) \\
1 - \frac{1}{2}(C - \omega) & 0
\end{pmatrix} \mathbf{y}
\]  
(25)
for fluid particles in the rotating frame. This transformed velocity field is steady, defining
an exactly solvable autonomous linear system of differential equations for particle motions.
The nature of its solutions depends on the eigenvalues \( \lambda_{1,2} = \pm \sqrt{1 - \frac{1}{4}(\omega - C)^2} \) of the coefficient matrix in (25). Specifically, for \( |\omega - C| < 2 \), we have a saddle-type flow with
typical solutions growing exponentially, while for \( |\omega - C| > 2 \), we have a center-type flow in
which all trajectories perform periodic motion. Mapped back into the original frame via the
time-periodic transformation, the center-type trajectories become quasiperiodic, as shown in
Fig. 7, which shows particle trajectories and instantaneous streamlines of the velocity field
(23) with \( h(t) \equiv 0 \), \( a_1(t) = \sin 4t \), \( b_1(t) = \cos 4t + \frac{1}{2} \), and \( \omega = -1 \) and \( a_k = b_k \equiv 0 \) for \( k \geq 2 \),
which gives the velocity field
\[
\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix}
\sin 4t & \cos 4t + \frac{1}{2} \\
\cos 4t - \frac{1}{2} & -\sin 4t
\end{pmatrix} \mathbf{x},
\]  
(26)
giving \( |\omega - C| > 2 \). This flow would, therefore, appear as an unbounded vortex in any flow-
visualization experiment involving dye or particles, yet its instantaneous streamlines suggest
FIG. 1: (a) A typical fluid trajectory generated by the linear unsteady velocity field \( \mathbf{h}(t) \equiv 0 \), \( a_1(t) = \sin 4t \), \( b_1(t) = \cos 4t + \frac{1}{2} \), and \( \omega = -1 \) and \( a_k = b_k \equiv 0 \) for \( k \geq 2 \), \( t_0 = 0 \), \( t_1 = 200 \) for the initial condition \( \mathbf{x}_0 = (2, 0) \). This flow would appear as an unbounded vortex in any flow-visualization experiment involving dye or particles. (b) The instantaneous streamlines of the same velocity field, shown here for \( t = 0 \), suggest a saddle point at the origin for all times (streamlines at other times look similar).

A saddle point at the origin for all times. Similarly, the Okubo–Weiss criterion pronounces the entire plane hyperbolic for the flow \( \mathbf{h}(t) \equiv 0 \) for all times. Indeed, formula \( (21) \) gives

\[
Q \equiv 1 - \frac{\omega^2}{4} = \frac{7}{16} > 0.
\]  

(27)

**Example 2.** By the general formula \( (6) \), a simple quadratic extension of \( (22) \) is given by the universal Navier–Stokes solution

\[
\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} \sin 4t & \cos 4t + 2 \\ \cos 4t - 2 & -\sin 4t \end{pmatrix} \mathbf{x} + \alpha(t) \begin{pmatrix} x^2 - y^2 \\ -2xy \end{pmatrix},
\]  

(28)

where we have chosen \( a_2(t) \equiv \alpha(t) \) and \( b_2(t) \equiv 0 \) in the quadratic terms of \( (6) \), and selected \( h(t), \omega, a_k(t) \) and \( b_k \) for \( k > 2 \) as in Example 1. Selecting \( \alpha(t) \equiv -0.1 \) for simplicity, we find that the instantaneous streamlines now suggest a bounded spinning vortex enclosed by connections between two stagnation points. The Okubo–Weiss criterion also suggests a coherent vortex surrounding the origin at all times, as \( Q < 0 \) holds on the yellow domain shown in Fig. 2(a) for the initial time \( t = 0 \). In reality, however, the origin is again a
FIG. 2: (a) Instantaneous streamlines and Okubo–Weiss elliptic region (yellow) for the universal Navier–Stokes solution (28) with $\alpha(t) \equiv -0.1$ at time $t = 0$. Other time slices are similar. (b) Stable (blue) and unstable (red) manifolds of the fixed point of the Poincaré map (based at $t = 0$ with period $T = \pi/2$) for the Lagrangian particle motions under the same velocity field, superimposed on the structures shown in (a).

Saddle-type Lagrangian trajectory with transversely intersecting stable and unstable manifolds. Shown in Fig. 2(b) for the Poincaré map of the flow, the resulting homoclinic tangle creates intense chaotic mixing that depletes all but a measure zero set of initial conditions rapidly from the Okubo–Weiss vortical region. Therefore, the Navier–Stokes solution (28) with $\alpha(t) \equiv -0.1$ provides a clear false positive for coherent vortex detection based on streamlines or the Okubo–Weiss criterion.

Example 3. Building on the discussion of the stability of the $\mathbf{x} = \mathbf{0}$ fixed point of equation (23), we now select another specific velocity field of the form

$$
\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix}
\sin 4t & \cos 4t + \frac{1}{2} \\
\cos 4t - \frac{1}{2} & -\sin 4t
\end{pmatrix} \mathbf{x} + \alpha(t) \begin{pmatrix}
x^2 - y^2 \\
-2xy
\end{pmatrix},
$$

(29)

from the universal solution family (6). In the notation used for equation (23), we have $\omega = -1$ and $C = 4$, which gives $|C - \omega| > 2$. Therefore, as discussed in example 1, the origin of (29) is a centre-type fixed point under linearization for the Lagrangian particle motion. At the same time, both the instantaneous streamlines in Figure 29 (a) and the
FIG. 3: (a) Instantaneous streamlines and Okubo–Weiss elliptic region (yellow, lies outside the domain shown) for the universal Navier–Stokes solution (29) with $\alpha(t) \equiv -0.015$ at time $t = 0$. Other time slices are similar. (b) Stable manifold (blue) of the fixed point of the Poincaré map (based at $t = 0$ with period $T = \pi/2$) for the Lagrangian particle motions under the same velocity field.

Okubo–Weiss criterion suggest saddle-type (hyperbolic) behavior for the linearized flow, given that $Q > 0$ holds on the whole plane. By the KAM theorem [16], however, selection of the small parameter $\alpha(t) \equiv -0.015$ in (29) is expected to preserve the elliptic (vortical) nature of the the Lagrangian particle motion in the quadratic velocity field (29). Indeed, most quasiperiodic motions of the linearized system survive with the exception of resonance islands, as indicated by the Poincaré map with period $T = \pi/2$ for Lagrangian particle motions in Fig. 3(b). Therefore, the Navier–Stokes solution (29) with $\alpha(t) \equiv -0.015$ provides a false negative for coherent vortex detection based on streamlines or the Okubo–Weiss criterion. Again, we mean a coherent vortex here in a material sense, as in example 2.
IV. CONCLUSIONS

We have derived an explicit form for spatially polynomial, universal, planar Navier–Stokes flows up to arbitrary order. Using these results, we have produced a strongly mixing Navier–Stokes flow whose analysis via instantaneous streamlines and the Okubo–Weiss criterion suggests a lack of mixing due to the presence of a coherent vortex. Likewise, we have constructed a Navier–Stokes flow that has a bounded coherent Lagrangian vortex despite the hyperbolic flow structure suggested by instantaneous streamlines and the Okubo–Weiss criterion. We expect such solutions to be useful as basic unsteady benchmarks for coherent structure detection criteria and numerical schemes. We also believe that the solutions obtained here provide a wealth of bounded, physically realistic flow patterns away from boundaries. For instance, the present solutions are expected to produce useful, dynamically consistent models of coherent structures in geophysical flows.

We finally note that the planar polynomial vector fields we have constructed immediately generate a large family of three-dimensional, incompressible Navier–Stokes solutions whose planar components are just these velocity fields. Their vertical component $w(x, y, t)$ satisfies a scalar advection-diffusion equation with diffusivity equal to the viscosity (cf. Majda and Bertozzi [2]). Any solution of this advection-diffusion equation complements our planar polynomial solutions $(u(x, y, t), v(x, y, t))$ to exact three-dimensional Navier–Stokes solutions.

V. ACKNOWLEDGEMENTS

We would like to acknowledge useful conversations with Mattia Serra on the subject of this paper. We would also like to thank Gabriel Provencher-Langlois for helpful discussions, and for generating Fig. [1] for an earlier version of this manuscript. This work was partially supported by the Turbulent Superstructures Program of the German National Science Foundation (DFG).
[1] A. Majda, Vorticity and the mathematical theory of incompressible fluid flow, Commun. Pure. Appl. Math. 39, S187 (1986).

[2] A. J. Majda and A. L. Bertozzi, Vorticity and incompressible flow (Cambridge University Press, 2002).

[3] A. D. D. Craik and W. O. Criminale, Evolution of wavelike disturbances in shear flows: a class of exact solutions of the Navier–Stokes equations, in Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 1830 (The Royal Society, 1986) pp. 13–26.

[4] R. Berker, Intégration des équations du mouvement d’un fluide visqueux incompressible, Handbuch der Physik VIII/2, 1 (1963).

[5] C. Y. Wang, Exact solutions of the unsteady Navier-Stokes equations, Appl. Mech. Rev. 42, S269 (1989).

[6] C. Y. Wang, Exact solutions of the Navier-Stokes equations-the generalized Beltrami flows, review and extension, Acta Mech. 81, 69 (1990).

[7] C. Y. Wang, Exact solutions of the steady-state Navier-Stokes equations, Ann. Rev. Fluid Mech. 23, 159 (1991).

[8] P. G. Drazin and N. Riley, The Navier-Stokes equations: a classification of flows and exact solutions, 334 (Cambridge University Press, 2006).

[9] A. E. Perry and M. Chong, A series-expansion study of the Navier–Stokes equations with applications to three-dimensional separation patterns, Journal of fluid mechanics 173, 207 (1986).

[10] T. R. Bewley and B. Protas, Skin friction and pressure: the footprints of turbulence, Physica D: Nonlinear Phenomena 196, 28 (2004).

[11] K. Bajer and H. K. Moffatt, On a class of steady confined stokes flows with chaotic streamlines, J. Fluid Mech. 212, 337 (1990).

[12] G. E. Andrews, R. Askey, and R. Roy, Special functions (Cambridge University Press, 2000).
[13] A. Okubo, Horizontal dispersion of floatable particles in the vicinity of velocity singularities such as convergences, in *Deep-Sea Res.* (Elsevier, 1970) pp. 445–454.

[14] J. Weiss, The dynamics of enstrophy transfer in two-dimensional hydrodynamics, *Physica D* **48**, 273 (1991).

[15] G. Haller, An objective definition of a vortex, *J. Fluid Mech.* **525**, 1 (2005).

[16] V. I. Arnold, *Mathematical methods of classical mechanics* (Springer, 1989).