Numerical simulation of Oldroyd-B fluid with application to hemodynamics

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Abstract
Oldroyd-B viscoelastic fluid is numerically simulated using the stabilized Galerkin least squares finite element method. The instabilities due to the connective nature of the Oldroyd-B model is treated using the discrete viscous elastic split stress method. The model is used to study the behavior of the flow of blood through an abdominal aortic segment. The results show that the viscoelastic nature of the blood should be considered specially at low shear rates.

Keywords
Abdominal aortic aneurysm, viscoelastic fluids, Galerkin least squares, discrete viscous elastic split stress method, wall shear stress

Introduction
One of the most common problems which occur in the abdominal aortic artery is the abdominal aortic aneurysms (AAAs) where the artery has a balloon-shaped expansion, and hence, the increase in the lumen diameter reach to 50% of its normal diameter.¹ One of the causes of the AAAs is the loss of arterial wall integrity.² In men, it is found that AAA happens commonly in those above the fifth decade.³

In this work, the geometry is considered as a two-dimensional (2D) planar channel with non-deformable walls. Although this assumption is less realistic than other assumptions such as axisymmetric or three-dimensional, this assumption is acceptable for geometries with spatial symmetry. There are many works for blood flow simulations that consider the assumption of planar channel for arteries.⁴⁻⁷ Many studies⁸⁻¹⁴ related to AAAs considered the blood as a purely viscous fluid with constant viscosity, that is, Newtonian fluid. In this work, the blood is modeled as a viscoelastic fluid where the Oldroyd-B constitutive model is used to represent the viscoelastic property of the blood.

The stabilized Galerkin least squares (GLS) method is used¹⁵ since high Reynolds numbers are expected in blood flows. Due to lack of enough diffusion in the Oldroyd model, the discrete viscous elastic split stress (DEVSS) method¹⁶ is used to suppress the associated instabilities. An in-house built FORTRAN code is built to solve the governing equations, and the results are visualized using the Techplot software. This article is organized as follows: the problem’s geometry is described in “Problem geometry” section while the governing equations and stabilization techniques are described in “Mathematical modeling,” “The incompressibility constraint,” and “The DEVSS method” sections. Numerical parameters of the problem are stated in “Numerical simulation” section. Finally, results are

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Figure 1. Geometry of the artery.

presented and discussed in “Results and discussion” section.

Problem geometry

Figure 1 describes the upper half of geometry used in the current work in which a 2D channel with single aneurysm is considered. Equation (1) is the mathematical description of the upper wall geometry:

\[
y = \left\{ \begin{array}{ll}
\frac{D}{2} \left[ 1 + \sin\left( \frac{2\pi (x - L_1)}{L_2} \right) \right] & \quad 0 \leq x \leq L_1 \\
\frac{D}{2} & \quad L_1 \leq x \leq L_2 \\
\frac{D}{2} & \quad L_2 \leq x \leq L_T
\end{array} \right.
\]  

(1)

in which \( D \) is the artery diameter at the inlet section and \( L_T \) is the artery length with \( L_1 = 2.5D \), \( L_2 = 5D \), \( L_T = 7.5D \), and \( D_1 = 2D \). In this study, the diameter of the artery is taken to be 8 mm.

Mathematical modeling

Balance laws

Considering non-polar incompressible, viscous fluids, the balance of linear momentum is

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot (\mathbf{V} \otimes \mathbf{V}) \right) = -\nabla p + \mu_s \Delta \mathbf{V} + \nabla \cdot \tau_p
\]  

(2)

and the continuity equation reads

\[
\nabla \cdot \mathbf{V} = 0
\]  

(3)

where \( \rho \) is the fluid density, \( \mathbf{V} \) is the velocity vector, \( p \) is the pressure, \( \mu_s \) is the solvent zero-shear-rate viscosity, and \( \tau_p \) is the polymeric contribution to the extra stress tensor. For Newtonian fluids, \( \tau_p \) vanishes.

Constitutive laws

For viscoelastic fluids, the extra stress tensor is related to the velocity gradient via a differential equation. In the Oldroyd-B model, polymeric contribution is

\[
\lambda \left( \frac{\partial \tau_p}{\partial t} + (\mathbf{V} \cdot \nabla) \tau_p - (\nabla \mathbf{V}) \tau_p - \tau_p (\nabla \mathbf{V})^T \right) + \tau_p = \mu_p \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T \right)
\]  

(4)

where \( \mu_p \) is the polymer zero-shear-rate viscosity and \( \lambda \) is the relaxation time.

Nondimensionalization scheme

The following nondimensionalization scheme is used

\[
x^* = \frac{x}{D}, \quad y^* = \frac{y}{D}, \quad V^* = \frac{V}{V_m}, \quad t^* = \frac{t V_m}{D},
\]

\[
p^* = \frac{p - p_{exit}}{\rho V_m^2}, \quad \text{and} \quad \tau^* = \frac{\tau}{\rho V_m^2}
\]

in which \( V_m \) is the inlet average velocity and \( p_{exit} \) is the exit pressure. The velocity vector is \( \mathbf{V} = (u, v) \) and the polymeric stress tensor is

\[
\tau_p = \begin{bmatrix}
S & Q \\
Q & T
\end{bmatrix}
\]  

(5)

in which \( S, Q, \) and \( T \) are the axial, shear, and normal stresses, respectively. The non-dimensional form of equations (2) and (4) are (the asterisk is dropped for clarity)

\[
\frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot (\mathbf{V} \otimes \mathbf{V}) = -\nabla p + \frac{\beta}{\text{Re}} \Delta \mathbf{V} + \nabla \cdot \tau_p
\]  

(6)

\[
\frac{\partial \tau_p}{\partial t} + (\mathbf{V} \cdot \nabla) \tau_p - (\nabla \mathbf{V}) \tau_p - \tau_p (\nabla \mathbf{V})^T + \frac{\tau_p}{\text{We}} = (1 - \beta) \frac{\text{Re}}{\text{We}} \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T \right)
\]  

(7)

where \( \text{Re} = (\rho V_m D)/\mu \) is the Reynolds number, \( \mu = \mu_s + \mu_p \) is the total viscosity, \( \text{We} = \lambda (V_m/D) \) is the Weissenberg number, and \( \beta = \mu_s/\mu \) is the viscosity ratio.

The incompressibility constraint

To recover the link between the continuity and momentum equations, a modified form of the continuity equation is used using the pressure stabilized technique:

\[
\nabla \cdot \mathbf{V} = \epsilon \nabla^2 p
\]  

(8)

in which, \( \epsilon \) is a controlling parameter used to stabilize the pressure. In this study, the value of stabilization parameter is chosen to be 0.0001, where the time step values range from 0.0001 to 0.00005 according to the value of Reynolds number. The main advantage with this technique is that equal order basis functions could be used for all primitive variables.
The DEVSS method

Since the additional polymeric stress induces numerical instabilities, the DEVSS method is used in which the discrete form of the momentum equation is modified by adding extra diffusion. Introduce the Newtonian-like stress tensor \( D \) defined as

\[
D = \mu_p (\nabla V + (\nabla V)^T) - D
\]

where \( \nabla V \) is the momentum equation which leads to a symmetric influence matrix. The momentum equation, equation (2), is reformulated as follows

\[
\rho \frac{\partial V}{\partial t} + \rho (V \cdot \nabla V) = -\nabla p + \nabla \cdot (\mu_e (\nabla V + (\nabla V)^T) + \nabla \cdot (\mu_p (\nabla V + (\nabla V)^T) - D)
\]

(9)

The extra variable \( D \) can be calculated from the following non-dimensional equation

\[
\frac{(1 - \beta)}{Re} \left( \nabla V + (\nabla V)^T \right) - D = 0
\]

(10)

We can rewrite equation (9) in the non-dimensional form as follows

\[
\frac{\partial V}{\partial t} + \nabla \cdot (V \otimes V) = -\nabla p + \nabla \left( \frac{\nabla V + (\nabla V)^T}{Re} + \tau_p - D \right)
\]

(11)

This is the stabilized form of the balance of momentum equation.

Numerical simulation

The set of equations (7), (8), (10), and (11) are solved numerically using the finite elements method. The GLS technique is used in which the basis function is modified which leads to stable solutions. The unsteady terms are discretized using the first-order explicit Euler scheme. The time step is chosen small enough to obtain accurate and stable solutions. Zero values for the flow variables are used as an initial guess.

Numerical parameters

The physiological condition described in Table 1 is used in the current work.

Boundary conditions

The no-slip boundary condition is imposed over the walls. A fully developed flow is assumed at the artery inlet. At the artery exit, the normal component of the Cauchy stress tensor is set to zero, namely, the pressure, \( p \), is set to zero as well as the normal stress \( \tau \).

According to the theory of characteristics, two components of the stress have to be specified at the inlet. Using the fully developed flow assumption at the inlet, the axial and shear stresses are

\[
S = \left( \frac{2We(1 - \beta)}{Re} \right) \left( \frac{\partial V}{\partial y} \right)^2
\]

(12)

\[
Q = \left( \frac{(1 - \beta)}{Re} \right) \frac{\partial V}{\partial y}
\]

(13)

Results and discussion

Results are obtained for both cases, namely, the Newtonian case (\( \tau_p = 0 \)) and the viscoelastic case. In both cases, different values for the blood volumetric flow rate are considered as shown in Table 2. The corresponding Reynolds and Weissenberg numbers are described in Table 2 as well.

Grid-independent solution

Different mesh sizes are used to reach the mesh-independent solution. The computational domain shown in Figure 2 is discretized using 4000 bilinear quadrilateral iso-parametric elements. Table 3 includes the number of elements for each grid. The wall shear stress (WSS) is calculated using all meshes for the volumetric flow rate \( Q = 3 \text{ cm}^3/\text{s} \). Figure 3 shows that the last two solutions are very close to each other assuring that the finer the mesh, the better the solution.

| Table 1. Blood physiological conditions. |
|---|---|---|---|
| \( \rho \) | 1050 kg/m³ | \( \mu_p \) | \( 4 \times 10^{-4} \) Pa s |
| \( \mu_s \) | 0.00319 Pa s | \( \lambda \) | 0.06 s |

| Table 2. Blood volumetric flow rate and corresponding Reynolds and Weissenberg numbers. |
| Flow rate in cm³/s | Re | We |
|---|---|---|
| 0.1 | 4.65 | 0.01492 |
| 0.5 | 23.27 | 0.0746 |
| 1 | 46.54 | 0.1492 |
| 2 | 93.09 | 0.2984 |
| 3 | 139.7 | 0.447 |

| Table 3. Number of elements for each grid. |
| Grid | Number of elements |
|---|---|
| 1 | 1000 |
| 2 | 1960 |
| 3 | 3240 |
| 4 | 4000 |
| 5 | 4840 |
Viscoelastic flow patterns

Parametric study is performed to investigate the effect of the blood volumetric flow rate and, consequently, the $Re$ and $We$ on the velocity contours and the formulation of recirculation zones. In this study, the steady state solutions are considered. It is found that the time required for obtaining the steady solution is dependent on the Reynolds number. For example, the number of time steps for steady state for $Q = 0.1 \text{ cm}^3/\text{s}$ is 10,000 and the time step in this case is 0.0001 while the number of time steps is 50,000 for $Q = 1 \text{ cm}^3/\text{s}$ at the same time step. Figure 4 shows low $Re$ flows in which there are no vortices as indicated in cases (a) and (b), and the main stream fills the aneurysm. The vortices begin to appear for higher $Re$, and it is noted that the vortex size increases with the increase of flow rate till it fills the entire aneurysm. The formulation of vortices proves to have a great effect on the integrity of the arterial wall.\(^2\)

WSS distribution

The WSS, defined as the sum of the Newtonian contribution and the polymeric contribution, is one of the most important factors that affects the integrity of the arterial wall is the WSS. Boyd et al.\(^2\) found that the minimum values of the WSS have more substantial effect than large values. Figure 5 shows the spatial variation of the WSS for different flow rates. The minimum values for WSS occur in the region of the

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**Figure 2.** Computational domain.

**Figure 3.** Grid-independent solution for WSS.

**Figure 4.** Viscoelastic flow streamlines for different flow rates: (a) flow rate = 0.1 $\text{cm}^3/\text{s}$, (b) flow rate = 0.5 $\text{cm}^3/\text{s}$, (c) flow rate = 1 $\text{cm}^3/\text{s}$, (d) flow rate = 2 $\text{cm}^3/\text{s}$, and (e) flow rate = 3 $\text{cm}^3/\text{s}$.
aneurysm due to the convective deceleration of the flow which results in small velocity gradients at the wall. The WSS changes its sign from negative values within the aneurysm to positive sign at the distal end of the aneurysm at which it reaches its maximum values.

**Wall pressure distribution**

Another important factor in hemodynamics is the blood pressure distribution at the arterial wall shown in Figure 6 for different flow conditions. The pressure is decreasing downstream till it reaches the aneurysm region in which it starts to build up until it reaches its maximum, which is proportional to the flow rate; at the distal end, it starts to decrease again toward the artery exit.

**Axial velocity**

To have more insights into the blood flow through the abdominal aneurysms, it is convenient to calculate the axial velocity on a vertical line at the mid-section of the aneurysm for different flow rate conditions as shown in Figure 7.

It is noted that the axial velocity increases with the increase of the Reynolds number due to the increase of the flow rate. The axial velocity takes negative values near the arterial wall, and these values appear clearly for high flow rates. This is due to the formation of the vortices which assert again that the aneurysm zone is a region which is worth to be studied.

**Viscoelastic versus Newtonian flows**

To show the importance of the inclusion of the viscoelasticity effects, the WSS is compared for both cases for different flow conditions. Figure 8 shows a difference of about 13% for the low flow rate case, while this difference reaches 40% in the distal end region of the aneurysm for higher flow rates as indicated in Figure 9.

This asserts that the effect of red blood cell in the blood flow simulation is important and should not be ignored, although there are many studies that treated the blood as a Newtonian fluid. This conclusion is consistent with Berger and Jou.
Summary and conclusion

In this contribution, mathematical modeling and finite element analysis are presented for viscoelastic Oldroyd-B fluid. The model is used to simulate blood flow in a rigid artery. In order to simulate low shear rate conditions as well as high shear rates, different flow conditions are considered ranging from volumetric flow rate of 0.1–3 cm³/s. Results show that there are no vortices observed at high shear rates (low volumetric flow rates), and hence, the effect of red blood cells sounds at the macroscale. However, the Oldroyd-B model is limited for low Reynolds number flow and still needs some modifications to account for shear dependent viscosity or shear-thinning property.

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References

1. Scotti CM, Jimenez J, Muluk SC, et al. Wall stress and flow dynamics in and abdominal aortic aneurysms: finite element analysis vs. fluid-structure interaction. *J Comput Methods Biomech Biomed Eng* 2008; 11: 301–322.
2. Budwig R, Elger D, Hooper H, et al. Steady flow in abdominal aortic aneurysms. *J Biomech Eng* 1993; 115: 418–423.
3. Kent KC. Clinical practice. Abdominal aortic aneurysms. *New Engl J Med* 2014; 371: 2101–2019.
4. Tian F, Zhu L, Fok P, et al. Simulation of a pulsatile non-Newtonian flow past a stenosed 2D artery with atherosclerosis. *J Comput Biol Med* 2013; 43: 1098–1113.
5. Razavi A, Shirani E and Sadeghi MR. Numerical simulation of blood pulsatile flow in a stenosed carotid artery using different rheological models. *J Biomech* 2011; 44: 2021–2030.
6. Ashrafizadeh M and Bakhshaei H. A comparison of non-Newtonian models for lattice Boltzmann blood flow simulations. *J Comput Math Appl* 2009; 58: 1045–1054.
7. Valencia A and Baeza F. Numerical simulation of fluid–structure interaction in stenotic arteries considering two layer nonlinear anisotropic structural model. *J Int Commun Heat Mass Transf* 2009; 36: 137–142.
8. Scherer P. Flow in axisymmetrical glass model aneurysms. *J Biomech* 1973; 6: 695–700.
9. Stehbens W. Flow disturbances in glass models of aneurysms at low Reynolds numbers. *Q J Exp Physiol* 1974; 59: 167–174.
10. Taylor T and Yamaguchi T. Three-dimensional simulation of blood flow in an abdominal aortic aneurysm–steady and unsteady flow cases. *ASME J Biomech Eng* 1994; 116: 89–97.
11. Bluestein D, Niu L, Schoephoerster RT, et al. Steady flow in an aneurysm model: correlation between fluid
dynamics and blood platelet deposition. *J Biomech Eng* 1996; 118: 280–286.

12. Yu SCM. Steady and pulsatile flow studies in abdominal aortic aneurysm models using particle image velocimetry. *Int J Heat Fluid Fl* 2000; 21: 74–83.

13. Finol EA and Amon CH. Flow—induced wall shear stress in abdominal aortic aneurysms: part I—steady flow hemodynamics. *J Comput Methods Biomech Biomed Eng* 2002; 5: 309–318.

14. Boutsianis E, Guala M, Olgac U, et al. CFDD and PTV steady flow investigation in an anatomically accurate abdominal aortic aneurysm. *J Biomech Eng* 2009; 131: 011008.

15. Tezduyar TE. Stabilized finite element formulations for incompressible flow computations. *J Adv Appl Mech* 1991; 28: 1–44.

16. Baaijens FPT. Mixed finite element methods for viscoelastic flow analysis: a review. *J Non-Newton Fluid Mech* 1998; 79: 361–385.

17. Guaily A and Epstein M. Boundary conditions for hyperbolic system of partial differential equation. *J Adv Res* 2013; 4: 321–329.

18. Gunzburger MD and Nicolaides RA. *Incompressible computational fluid dynamics trends and advances*. Cambridge: Cambridge University Press, 1993, pp.151–182.

19. Elhanafy A, Guaily A and Elsaid A. Pressure stabilized finite elements simulation for steady and unsteady Newtonian fluids. *J Appl Math Comput Mech* 2017; 16: 17–26.

20. Li X, Han X and Wang X. Numerical modeling of viscoelastic flows using equal low-order finite elements. *J Comput Meth Appl Mech Eng* 2010; 199: 570–581.

21. Bodnar T, Sequeira A and Pirkl L. Numerical simulations of blood flow in a stenosed vessel under different flow rates using a generalized Oldroyd-B model. In: 7th *International conference of numerical analysis and applied mathematics (ICNAAM 2009)*, Rethymno, Crete, Greece, 18–22 September 2009.

22. Leuprecht A and Perktold K. Computer simulation of non-Newtonian effects on blood flow in large arteries. *J Comput Methods Biomech Biomed Eng* 2001; 4: 149–163.

23. Gijsen FJH, Allanic E, van de Vosse FN, et al. The influence of the non-Newtonian properties of blood on the flow in large arteries: unsteady flow in a 90-degree curved tube. *J Biomech* 1999; 32: 705–713.

24. Yapici K, Karasozen B and Uludag Y. Finite volume simulation of viscoelastic laminar flow in a lid-driven cavity. *J Non-Newton Fluid Mech* 2009; 164: 51–65.

25. Tomé MF, Mangiavacchi N, Cuminato JA, et al. A finite difference technique for simulating unsteady viscoelastic free surface flows. *J Non-Newton Fluid Mech* 2002; 106: 61–106.

26. Boyd AJ, Kuhn CS, Lozowy RJ, et al. Low wall shear stress predominates at sites of abdominal aortic aneurysm rupture. *J Vasc Surg* 2016; 63: 1613–1619.

27. Berger SA and Jou LD. Flows in stenotic vessels. *J Ann Rev Fluid Mech* 2000; 32: 347–382.

### Appendix 1

#### Notation

- $D$: the artery diameter
- $D$: auxiliary variable
- $p$: pressure
- $p_{exit}$: the pressure at exit of the arterial segment
- $Q$: the shear stress
- $Re$: the Reynolds number
- $S$: the axial stress
- $T$: the normal stress
- $V$: velocity vector with components of $u, v$
- $V_n$: the mean velocity at the artery inlet
- $W_e$: the Weissenberg number
- $\beta$: the viscosity ratio
- $\epsilon$: the stabilization parameter
- $\lambda$: the relaxation time
- $\mu$: the total viscosity
- $\mu_p$: polymer viscosity
- $\rho$: blood density