Kernels of Polarizations of Abelian Varieties Over Finite Fields

Everett W. Howe

Dedicated to Professor Frans Oort on the occasion of his sixtieth birthday.

Abstract. Suppose $C$ is an isogeny class of abelian varieties over a finite field $k$. In this paper we give a partial answer to the question of which finite group schemes over $k$ occur as kernels of polarizations of varieties in $C$. We show that there is an element $I_C$ of a finite two-torsion group that determines which Jordan-Hölder isomorphism classes of finite commutative group schemes over $k$ contain kernels of polarizations. We indicate how the two-torsion group can be computed from the characteristic polynomial of the Frobenius endomorphism of the varieties in $C$, and we give some relatively weak sufficient conditions for the element $I_C$ to be zero. Using these conditions, we show that every isogeny class of simple odd-dimensional abelian varieties over a finite field contains a principally polarized variety. As a step in the proofs of these theorems, we prove that if $K$ is a CM-field and $A$ is a central simple $K$-algebra with an involution of the second kind, then every totally positive real element of $K$ is the reduced norm of a positive symmetric element of $A$.

This is a reproduction of a preprint, dated 27 August 1995, of a paper that appeared in the Journal of Algebraic Geometry in 1996. The only changes between the original 1995 preprint and this document are some adjustments to the font size and margins, the inclusion of this note, the addition of a current address, the updating of my email address, and the inclusion of a link to my homepage. The numbering of theorems and so forth are consistent with the published version. — EWH, 14 January 2020

1. Introduction

Every abelian variety over an algebraically closed field is isogenous to a principally polarized variety, but over non-algebraically-closed base fields the situation is more complicated. In this paper we study the case where the base field is finite, and we provide some relatively weak sufficient conditions for an isogeny class of abelian varieties over a finite field to contain a principally polarized variety.
finite field to contain a principally polarized variety. Our results are easiest to describe for simple varieties. Suppose $\mathcal{C}$ is an isogeny class of simple abelian varieties over a finite field $F_q$. The isogeny class $\mathcal{C}$ is determined by its Weil number, which is an algebraic integer $\pi \in \mathbb{C}$ such that $\pi \overline{\pi} = q$. The field $K = \mathbb{Q}(\pi)$ is either totally real or a so-called CM-field, that is, a totally imaginary quadratic extension of a totally real field $K^+$.

(1.1) Theorem. Let $\mathcal{C}$ and $K$ be as above. If $K$ is totally real then $\mathcal{C}$ contains a principally polarized variety. Suppose $K$ is a CM-field. If a finite prime of $K^+$ ramifies in $K/K^+$, or if there is a prime of $K^+$ that divides $\pi - \overline{\pi}$ and that is inert in $K/K^+$, then $\mathcal{C}$ contains a principally polarized variety.

From Theorem 1.1 we obtain an easily-stated result.

(1.2) Theorem. Every simple odd-dimensional abelian variety over a finite field is isogenous to a principally polarized variety.

The proof of Theorem 1.1 involves the idea of an attainable group scheme. If $k$ is a finite field and $\mathcal{C}$ is an isogeny class of abelian varieties over $k$, we call a group scheme $X$ over $k$ attainable in $\mathcal{C}$ if there is a polarization of a variety in $\mathcal{C}$ whose kernel is isomorphic to $X$. The proof of Theorem 1.1 depends on our being able to identify the image of the attainable group schemes in the Grothendieck group $G(Ker_{\mathcal{C}})$ of the category $Ker_{\mathcal{C}}$ whose objects are the kernels of isogenies between elements of $\mathcal{C}$. To every isogeny class $\mathcal{C}$ we associate a commutative ring $R$ of a type we will call a real/CM-order; if $A$ is any variety in $\mathcal{C}$ then $R$ is isomorphic to the subring of $End A$ generated over $\mathbb{Z}$ by the Frobenius and Verschiebung endomorphisms of $A$. In §3 we define an isomorphism between $G(Ker_{\mathcal{C}})$ and the Grothendieck group $G(Mod_R)$ of finite-length modules over $R$, so that our goal is to identify the attainable elements of $G(Mod_R)$, that is, the classes in $G(Mod_R)$ that contain the images of attainable group schemes. Some simple considerations show that the attainable elements lie in a certain subgroup $Z(R)$ of $G(Mod_R)$ that we define in §5. The attainable elements must also be effective; that is, an attainable element must be the class in $G(Mod_R)$ of an actual $R$-module. In §5 we define a finite two-torsion quotient $B(R)$ of $Z(R)$, and we prove the following theorem.

(1.3) Theorem. There is an element $I_C$ of the group $B(R)$ such that the elements of $G(Mod_R)$ that are attainable in $\mathcal{C}$ are precisely the effective elements of $Z(R)$ that map to $I_C$ in $B(R)$. In particular, the isogeny class $\mathcal{C}$ contains a principally polarized variety if and only if $I_C = 0$.

In order to apply Theorem 1.3 to actual isogeny classes, we must be able to calculate the obstruction group $B(R)$ and the obstruction element $I_C$. In §6 we indicate how $B(R)$ can be calculated from the Weil number of $\mathcal{C}$. In §7 we show that $I_C$ must lie in a certain calculable subgroup of $B(R)$. In many circumstances this subgroup is the zero group (see Proposition 7.2), and this fact allows us to prove Theorems 1.1 and 1.2.

Stronger results for isogeny classes of ordinary abelian varieties are obtained in [2]. The stronger results are made possible by a theorem of Deligne ([1]) that provides an equivalence between the category of varieties in an ordinary isogeny class $\mathcal{C}$ and a certain
category of $R$-modules. When dealing with arbitrary abelian varieties we no longer have this tool, so one of the main contributions of the present paper is Theorem 3.1, which says that in general we still at least have an isomorphism between the Grothendieck groups of $\text{Ker}_C$ and $\text{Mod}_R$ that behaves well with respect to endomorphisms. The other difficulty we face when working with arbitrary varieties is that their endomorphism algebras are more complicated than those of ordinary varieties; this is why we must prove the results on reduced norms found in §4.

**Convention.** If $A$ and $B$ are varieties over a field $k$, then when we speak of a morphism from $A$ to $B$ we always mean a $k$-morphism.

**Acknowledgment.** The author thanks Vladimir Platonov for helpful conversations and suggestions.

## 2. Preliminaries

Suppose $k$ is a finite field with $q = p^n$ elements, where $p$ is prime. If $A$ is a $g$-dimensional abelian variety over $k$, we let $h_A$ denote the characteristic polynomial of the Frobenius endomorphism of $A$. The polynomial $h_A$ is a monic polynomial of degree $2g$ with integer coefficients, and all of its roots in the complex numbers have magnitude $q^{1/2}$. It follows from the theorem of Honda and Tate (see [10]) that the polynomial $h_A$, the endomorphism algebra $E_A = (\text{End} A) \otimes \mathbb{Q}$ of $A$, the center $K_A$ of this algebra, and the subring $R_A$ of $K_A$ that is generated by the Frobenius and Verschiebung endomorphisms $F$ and $V$ of $A$, all depend only on the isogeny class of $A$. Thus, if $C$ is an isogeny class of abelian varieties over $k$ we can speak of $h_C$, $E_C$, $K_C$, and $R_C$. We note that the ring $K_C$ is a product of number fields that is generated over $\mathbb{Q}$ by the Frobenius endomorphism and that $E_C$ is a product of central simple algebras over the factors of $K_C$. By piecing together the reduced norms from the factors of $E_C$ to the factors of $K_C$, we get a homomorphism from $E_C^\ast$ to $K_C^\ast$ that we denote by $\text{Nrd}$ and continue to call the reduced norm. Detailed information about the structure of the endomorphism algebras of abelian varieties over finite fields can be found in [11, Ch. 1 and 2], [10], [3, §21, Application I, pp. 193–203], and [6].

An order is a commutative ring that has no nilpotent elements and that is free and finitely generated as a $\mathbb{Z}$-module. Equivalently, an order is a subring of finite index in the integral closure of $\mathbb{Z}$ in a finite product of number fields. An order $R$ is called real if $R \otimes \mathbb{Q}$ is a product of totally real number fields; it is called CM if $R \otimes \mathbb{Q}$ is a product of CM-fields; and it is called real/CM if $R \otimes \mathbb{Q}$ is a product of real and CM-fields. If $R$ is a real/CM-order then there is an involution $-$ on $R \otimes \mathbb{Q}$ that is complex conjugation on each factor of $R \otimes \mathbb{Q}$, and we say that $R$ is proper if $\overline{R} = R$. We let $R^+$ denote the ring of elements of $R$ that are fixed by the involution. If $C$ is an isogeny class of abelian varieties over a finite field, then the ring $R_C = \mathbb{Z}[F, V]$ is a proper real/CM-order and the involution $-$ interchanges $F$ and $V$.

Let $R$ be any commutative ring, let $M$ be an $R$-module, and let $G$ be an abelian group. A pairing $e: M \times M \to G$ is called $R$-balanced if for all $m$ and $n$ in $M$ and for all $a \in R$ we have $e(am, n) = e(m, an)$. If $R$ has an involution $-$, then the pairing $e$ is called $R$-semi-balanced if for all $m$ and $n$ in $M$ and for all $a \in R$ we have $e(am, n) = e(m, \overline{an})$. 


3. Grothendieck groups

Let $p$ be a prime number, let $k$ be a field with $q = p^a$ elements, and suppose $\mathcal{C}$ is an isogeny class of abelian varieties over $k$. Let $\text{Ker}_\mathcal{C}$ be the category whose objects are finite commutative group schemes that can be embedded (as closed sub-group-schemes) in some variety in the isogeny class $\mathcal{C}$, and whose morphisms are morphisms of group schemes. We see that the objects in $\text{Ker}_\mathcal{C}$ are those group schemes that can be written $\ker \varphi$ for some isogeny $\varphi: A \to B$ of elements of $\mathcal{C}$. The category $\text{Ker}_\mathcal{C}$ splits into a product of four subcategories whose objects are, respectively, reduced group schemes whose Cartier duals are reduced, reduced group schemes whose Cartier duals are local, local group schemes whose Cartier duals are reduced, and local group schemes whose Cartier duals are local. We will denote these subcategories by $\mathcal{K}_{rr}$, $\mathcal{K}_{rt}$, $\mathcal{K}_{tr}$, and $\mathcal{K}_{tt}$, respectively.

Suppose $X$ is an object of $\mathcal{K}_{rr}$ or of $\mathcal{K}_{rt}$. Then $X$ is completely determined by the $\text{Gal}(\overline{k}/k)$-module $X(\overline{k})$, where $\overline{k}$ is an algebraic closure of $k$. The Galois action on $X(\overline{k})$ is determined by the action of the $q$th-power automorphism of $\overline{k}$, which is identical to the action of the Frobenius endomorphism on $X(\overline{k})$. Since the action of Frobenius on $X$ agrees with the action of Frobenius on the varieties in $\mathcal{C}$, the group $X(\overline{k})$ can be viewed as a module over the ring $R = R_\mathcal{C} = \mathbb{Z}[F, V]$, where we let $F$ act as the Frobenius and $V$ as $q/F$. A morphism between two objects of $\mathcal{K}_{rr} \times \mathcal{K}_{rt}$ corresponds to a group homomorphism between their groups of $\overline{k}$-valued points that is equivariant under the Frobenius endomorphisms of the two groups. Thus, the functor from $\mathcal{K}_{rr} \times \mathcal{K}_{rt}$ to the category $\text{Mod}_R$ of finite $R$-modules that sends a group scheme $X$ to the $R$-module $X(\overline{k})$ is fully faithful and exact.

If $M$ is a finite $R$-module we define the dual module of $M$ to be the $R$-module $\hat{M} = \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$, where for every $r \in R$ and $\psi \in \hat{M}$ we let $(r\psi)(m) = \psi(rm)$ for all $m \in M$. Note that if $\mathfrak{p}$ is a maximal ideal of $R$ then $R/\mathfrak{p}$ and $R/\overline{\mathfrak{p}}$ are dual to one another. If $X$ is an object of $\mathcal{K}_{tt}$ then its Cartier dual $\hat{X}$ is an object of $\mathcal{K}_{rt}$, and the functor from $\mathcal{K}_{tt}$ to $\text{Mod}_R$ that sends $X$ to $\hat{X}(\overline{k})$ is fully faithful and exact.

Piecing together the functors defined above, we get an exact fully faithful functor $\mathcal{P}$ from the category $\mathcal{K}_{rr} \times \mathcal{K}_{rt} \times \mathcal{K}_{tr}$ to $\text{Mod}_R$. The functor $\mathcal{P}$ transforms Cartier duality into the duality of $\text{Mod}_R$ that we defined above, and for every object $X$ of $\mathcal{K}_{rr} \times \mathcal{K}_{rt} \times \mathcal{K}_{tr}$, the cardinality of the $R$-module $\mathcal{P}(X)$ is equal to the rank of the group scheme $X$.

We will need only two facts concerning the category $\mathcal{K}_{\ell\ell}$. The first is that if it contains any non-zero object then it contains exactly one simple object, namely the unique simple local-local finite commutative group scheme $\mathfrak{a}_p$. The second is that if $\mathcal{K}_{\ell\ell}$ contains $\mathfrak{a}_p$ then the varieties in $\mathcal{C}$ are not ordinary. In this case $F$ and $V$ are not coprime in $R$, and there is a unique prime of $R$ that contains them both: the kernel of the map from $R$ to $\mathbb{Z}/p\mathbb{Z}$ that sends both $F$ and $V$ to zero.

Recall that the Grothendieck group $G(\text{Mod}_R)$ of $\text{Mod}_R$ is defined to be the quotient of the free abelian group on the isomorphism classes of objects in $\text{Mod}_R$ by the subgroup generated by the expressions $M - M' - M''$ for all exact sequences $0 \to M' \to M \to M'' \to 0$ in $\text{Mod}_R$. We define the Grothendieck group $G(\text{Ker}_\mathcal{C})$ similarly. If $M$ is a
proof. Let \( (3.2) \) Lemma. Let \( \mathcal{C} \) denote its class in \( G(\text{Mod}_R) \); likewise, we let \( [X]_\mathcal{C} \) denote the class in \( G(\text{Ker}_\mathcal{C}) \) of an object \( X \) of \( \text{Ker}_\mathcal{C} \). Both \( G(\text{Mod}_R) \) and \( G(\text{Ker}_\mathcal{C}) \) are free abelian groups on the simple objects in their respective categories. An element of one of these Grothendieck groups is said to be effective if it is a sum of positive multiples of simple objects.

Suppose \( X \) is a simple object of \( \text{Ker}_\mathcal{C} \). If \( X \) is not local-local, define \( \epsilon([X]_\mathcal{C}) \) to be the element \( [\mathcal{P}(X)]_R \) of \( G(\text{Mod}_R) \); if \( X \) is local-local, then by our previous remarks the group \( \mathbb{Z}/p\mathbb{Z} \) can be made into an \( R \)-module \( M \) by letting \( F \) and \( V \) act as zero, and we define \( \epsilon([X]_\mathcal{C}) \) to be \( [M]_R \). Extending \( \epsilon \) by linearity, we get a homomorphism from all of \( G(\text{Ker}_\mathcal{C}) \) to \( G(\text{Mod}_R) \). The exactness of \( \mathcal{P} \) shows that for all objects \( X \) of \( \text{Ker}_\mathcal{C} \) that have no local-local part, we have \( \epsilon([X]_\mathcal{C}) = [\mathcal{P}(X)]_R \).

We know that the product of fields \( K = R \otimes \mathbb{Q} \) is the localization of \( R \) with respect to the multiplicative set of non-zero-divisors. Suppose \( \alpha \) is an invertible element of \( K \), say \( \alpha = a/b \) where \( a \) and \( b \) are elements of \( R \) that are not zero-divisors. The principal element generated by \( \alpha \) is the element \( \text{Pr}_R(\alpha) = [R/aR]_R - [R/bR]_R \) of \( G(\text{Mod}_R) \). It is easy to see that \( \text{Pr}_R(\alpha) \) is well-defined and that \( \text{Pr}_R \) is a homomorphism from \( K^* \) to \( G(\text{Mod}_R) \).

(3.1) Theorem. Suppose \( A \) is a variety in \( \mathcal{C} \), let \( E = (\text{End} \, A) \otimes \mathbb{Q} \), and let \( \text{Nrd}: E^* \to K^* \) be the reduced norm. If \( \alpha \) is an isogeny in \( \text{End} \, A \), then

\[ \epsilon([\ker \alpha]_\mathcal{C}) = \text{Pr}_R(\text{Nrd}(\alpha)). \]

Proof. The abelian variety \( A \) is isogenous to a variety \( A' \) that may be written \( A' = B_1^{n_1} \cdots B_r^{n_r} \), where the \( B_i \) are simple abelian varieties that are not isogenous to one another. It is not too difficult to show that the theorem will be true for \( A \) if and only if it is true for \( A' \), and that the theorem will be true for \( A' \) if and only if it is true for each of the varieties \( B_i^{n_i} \). Thus we need only consider the case where \( A \) may be written as the \( n \)-th power of a simple variety \( B \). Since both sides of the theorem’s equality are multiplicative, it will be enough to prove the theorem for a set of endomorphisms that generate the multiplicative group \( E^* \). In particular, we can restrict our attention to isogenies \( \alpha \) with \( \text{Nrd}(\alpha) \in R \). We must show that for every such \( \alpha \) we have

\[ \epsilon([\ker \alpha]_\mathcal{C}) = [R/\text{Nrd}(\alpha)R]_R. \]  \hspace{1cm} (1)

To show that equality (1) holds we will express both sides of the equality in terms of the \( \mathbb{Z} \)-basis \( \{[M]_R : M \text{ is a simple } R \text{-module} \} \) of \( G(\text{Mod}_R) \). If a particular simple \( R \)-module \( M \) occurs the same number of times on both sides, we will say that equality (1) holds at \( M \). For a fixed prime number \( \ell \), we will say that equality (1) holds at \( \ell \) if equality (1) holds at every simple \( R \)-module with cardinality a power of \( \ell \).

(3.2) Lemma. Let \( \ell \neq p \) be a prime number. Then equality (1) holds at \( \ell \).

Proof. Let \( h \) be the characteristic polynomial of the Frobenius endomorphism of \( B \). Since \( B \) is simple, \( h \) is a power of an irreducible polynomial \( P \) of degree \( e \), say \( h = P^d \). The endomorphism algebra \( E \) of \( A \) is a \( n \times n \) matrix algebra over the division algebra \( D = \)
the restriction of the endomorphism \( \alpha \) these objects are left modules over \( B \) of \( E \).

(3.3) Lemma. Equality (1) holds at \( \ell \).

Proof. Let \( D_p = D \otimes K_p \) be a \( d \times d \) matrix algebra over \( K_p \). The algebra \( E_p = E \otimes K_p \) is an \( n \times n \) matrix algebra over \( D_p \), and is thus also an \( nd \times nd \) matrix algebra over \( K_p \).

Let \( T' \) be the \( \ell \)-adic Tate module of \( B \) and let \( T \) be the \( \ell \)-adic Tate module of \( A \), so that \( T \) is the sum of \( n \) copies of \( T' \). Let \( U' = T' \otimes Q \) and let \( U = T \otimes Q \). The endomorphism algebra \( D \) acts on \( U' \), so the \( Q_\ell \)-module \( U' \) is also a module over \( K_\ell = K \otimes Q_\ell = \bigoplus_{p|\ell} K_p \) and \( D_\ell = D \otimes Q_\ell = \bigoplus_{p|\ell} D_p \). As a \( K_\ell \)-module, \( U' \) is free of rank \( d \), so that \( U' = \bigoplus_{p|\ell} U'_p \) where \( U'_p \) is a \( d \)-dimensional \( K_p \) vector space. The action of the endomorphism algebra \( D_\ell \) on \( U' \) is given by the action of each \( d \times d \) matrix algebra \( D_p \) on its corresponding vector space \( U'_p \). Similarly, \( E_\ell = E \otimes Q_\ell = \bigoplus_{p|\ell} E_p \) acts on \( U \), and the action is given by the action of each \( E_p \) on its corresponding \( nd \)-dimensional vector space.

Let \( X \) be the \( \ell \)-part of the group scheme \( \ker \alpha \) and let \( \alpha_\ell \) be the endomorphism of \( U \) obtained from the endomorphism \( \alpha \) of \( A \). Then the group \( X(\overline{k}) \) is isomorphic to \( T/\alpha_\ell(T) \), and the actions of Frobenius and Verschiebung on \( X(\overline{k}) \) correspond to their actions on \( T/\alpha_\ell(T) \). In other words, the \( R \)-module \( T/\alpha_\ell(T) \) is isomorphic to \( \mathcal{P}(X) \), where \( \mathcal{P} \) is the functor we defined above. Therefore, to prove that equality (1) holds at the prime \( \ell \) we need only show that

\[
[T/\alpha_\ell(T)]_{R_\ell} = [R_\ell / \text{Nrd}_\ell(\alpha)R_\ell]_{R_\ell},
\]

where \( R_\ell = R \otimes \mathbb{Z}_\ell \), where \( \text{Nrd}_\ell \) denotes the reduced norm from \( E_\ell^* \) to \( K_\ell^* \), and where we view \( \alpha \) as an element of \( E_\ell^* \).

Let \( \mathcal{O} \) be the ring of integers of \( K \), let \( \mathcal{O}_\ell = \mathcal{O} \otimes \mathbb{Z}_\ell \), and let \( S \) be any \( \mathcal{O}_\ell \)-lattice in \( U \) that contains \( T \). It is easy to show that \([S/\alpha_\ell(S)]_{R_\ell} = [T/\alpha_\ell(T)]_{R_\ell}\) and that \([\mathcal{O}_\ell / \text{Nrd}_\ell(\alpha)\mathcal{O}_\ell]_{R_\ell} = [R_\ell / \text{Nrd}_\ell(\alpha)R_\ell]_{R_\ell}\). Therefore we need only prove equality (2) with \( R \) replaced with \( \mathcal{O} \) and with \( T \) replaced with \( S \). Since \( \mathcal{O}_\ell = \bigoplus_{p|\ell} \mathcal{O}_p \) and since \( S \) splits accordingly into a sum of \( \mathcal{O}_p \)-modules \( S_p \), we need only prove equality (2) with \( R_\ell \) replaced by \( \mathcal{O}_p \), with \( T \) replaced by \( S_p \), and with \( \alpha_\ell \) replaced by \( \alpha_p \), where \( \alpha_p \) is the restriction of the endomorphism \( \alpha_\ell \) to \( U_p \). It follows from [9, §I.5, Lem. 3, p. 17] that \([S_p/\alpha_p(S_p)]_{\mathcal{O}_p} = [\mathcal{O}_p / \det_{\text{End}_{K_p} U_p}(\alpha_p)\mathcal{O}_p]_{\mathcal{O}_p} \), so it will suffice for us to show that \( \det_{\text{End}_{K_p} U_p}(\alpha_p) = \text{Nrd}_{K_p}(\alpha) \). But this last equality follows from the facts that the action of \( E_p \) on \( U_p \) gives us an isomorphism \( E_p \cong \text{End}_{K_p} U_p \) and that the reduced norm on a matrix algebra is the determinant. Thus, equality (1) holds at \( \ell \). \( \square \)

(3.3) Lemma. Equality (1) holds at \( p \).

Proof. Let \( D, E, \) and \( h = P^d \) be as in the proof of Lemma 3.2. Let \( L \) be an unramified complete extension field of \( Q_p \) with residue field \( k \), let \( \sigma \) be the automorphism of \( \ell \) induced by the automorphism \( x \mapsto x^p \) of \( k \), let \( W \) be the ring of integers of \( L \), and let \( \mathfrak{A} = W[f, v] \) where \( f \) and \( v \) are two indeterminates subject to the relations \( fv = vf = p, f\lambda = \lambda^p f, \) and \( \lambda v = v\lambda^p \) for every \( \lambda \in W \). Let \( T' \) be the Dieudonné module associated to \( B \) and \( T \) the Dieudonné module associated to \( A \) (see [11, Ch. 1 and 2] or [4]); \( T' \) and \( T \) are left \( \mathfrak{A} \)-modules, and \( T \) is the direct sum of \( n \) copies of \( T' \). Let \( U' = T' \otimes Q \) and \( U = T \otimes Q \); these objects are left modules over \( Q = \mathfrak{A} \otimes Q = L[f, v] \).
Every endomorphism $\beta$ of $B$ gives us an endomorphism $\beta_p$ of $T'$ and of $U'$, so $U'$ is a module over $K_p = K \otimes Q_p$. The splitting $K_p = \bigoplus_{p \mid p} K_p$ of $K_p$ into a sum of local fields gives us a splitting $U' = \bigoplus_{p \mid p} U'_p$ of $U'$ into $K_p$-modules $U'_p$. The Frobenius endomorphism $F \in K$ of $B$ and the element $f^a$ of $\mathfrak{B}$ (where $a = [k : F_p]$) have identical actions on $U'$. If we let $P_p$ be the irreducible factor of $P$ in $Q_p[X]$ that is the minimal polynomial over $Q_p$ of $F \in K_p$, then $\mathfrak{B}$ acts on $U'_p$ via reduction to $\mathfrak{B}_p = \mathfrak{B}/P_p(f^a)\mathfrak{B}$, which is a central simple algebra of dimension $a^2$ over $K_p$. Similarly, the $\mathfrak{B}$-module $U$ splits into a direct sum of submodules $U'_p$, each of which is a sum of $b$ copies of $U'_p$.

Let $X$ be the $p$-part of the group scheme $\text{ker} \alpha$. Then the Dieudonné module attached to $X$ is isomorphic to $T/\alpha_p(T)$. The simple group schemes occurring in $X$ correspond to the Jordan-Hölder factors of the left $\mathfrak{B}$-module $T/\alpha_p(T)$. Let $T_p$ be the image of $T$ in $U_p$, and let $S = \bigoplus T_p \subset \bigoplus U_p = U$. It is not hard to show that the left $\mathfrak{A}$-modules $S/\alpha_p(S)$ and $T/\alpha_p(T)$ are Jordan-Hölder isomorphic to one another, and $S/\alpha_p(S) \cong \bigoplus T_p/\alpha_p(T_p)$, where $\alpha_p$ is the restriction of $\alpha$ to $T_p$. For each $p$, let $\mathcal{O}_p$ denote the ring of integers of $K_p$, so that $R_p = R \otimes \mathbb{Z}_p$ is a subring of finite index in $\bigoplus \mathcal{O}_p$; it is again not hard to show that $[R_p]/\text{Nrd}(\alpha)R_pR = \sum_{p \mid p}[\mathcal{O}_p/\text{Nrd}_p(\alpha)\mathcal{O}_p]_R$, where for every term in the sum we view $\alpha$ as an element of $E_p = E \otimes K_p$ and where $\text{Nrd}_p$ is the reduced norm from $E_p^*$ to $K_p^*$. Therefore, to show that equality (1) holds at $p$ we need only prove the following lemma.

**Lemma.** Let notation be as above and let $p$ be any prime of $K$ lying over $p$. Let $Y$ be the finite group scheme corresponding to the Dieudonné module $T_p/\alpha_p(T_p)$. Then

$$\epsilon([Y]_c) = [\mathcal{O}_p/\text{Nrd}_p(\alpha)\mathcal{O}_p]_R. \quad (3)$$

**Proof.** Let $q$ be the prime of $R$ that lies under the prime $p$. Then the only simple $R$-module that occurs in $\mathcal{O}_p/\text{Nrd}_p(\alpha)\mathcal{O}_p$ is $R/q$. We will show that $R/q$ is also the only simple $R$-module that occurs in $\epsilon([Y]_c)$.

First suppose that $p$ does not contain $F$. Recall that the prime $p$ corresponds to an irreducible factor $P_p \in Q_p[X]$ of the polynomial $P$; similarly, the prime $q$ corresponds to an irreducible factor $Q_q \in F_p[X]$ of $P$ modulo $p$, and in fact $(P_p \mod p)$ is a power of $Q_q$. Since $F$ is not in $p$ the constant coefficient of $P_p$ is not a multiple of $p$, and the constant coefficient of $Q_q$ is non-zero. Let $N$ be a simple $\mathfrak{A}$-module on which $f^a$ satisfies $P_p$. Since $N$ is simple it is killed by $p$, and $v$ must act as zero on $N$ because $fv = p$ and $f$ acts invertibly on $N$. Also, since a power of $Q_q(f^a)$ kills $N$, $f^a$ must in fact satisfy $Q_q$ on $N$. This shows that $\mathfrak{A}$ must act on $N$ via reduction to the ring

$$W[f,v]/(p,v,Q_q(f^a)) = k[f]/(Q_q(f^a)),$$

where for every $\lambda \in k$ we have $f\lambda = \lambda^pf$. This ring is a simple algebra over its center, which is $F_p[f^a]/(Q_q(f^a)) \cong F_p[X]/(Q_q(X)) \cong R/q$. There is exactly one simple representation of this algebra, so $N$ is the unique simple left $\mathfrak{A}$-module on which $f^a$ satisfies $P_p$. Let $Z$ be the simple reduced-local group scheme corresponding to $N$. One can show that
the action of the Frobenius on $Z(\overline{k})$ satisfies the polynomial $Q_\alpha$, so $P(Z) \cong R/\alpha$. Thus, $R/\alpha$ is the only simple $R$-module occurring in $e([Y]_C)$.

Next, suppose $p$ does not contain $V$, so that the conjugate ideal $\overline{p}$ of $p$ does not contain $F$. If we take duals everywhere and apply the preceding argument, we find that $R/\overline{p}$ is the only simple $R$-module that occurs in $e([\overline{Y}]_C)$. Taking duals again, we find that $R/\alpha$ is the only simple $R$-module occurring in $e([Y]_C)$.

Finally, suppose $p$ contains both $F$ and $V$, so that $q$ is the unique prime of $R$ that contains $F$ and $V$ and $R/q \cong \mathbb{Z}/p\mathbb{Z}$. Let $N$ be any simple quotient of the $A$-module $T_p$. Then $f$ and $v$ must both act as zero on $N$, so $N$ must be the simple $A$-module $k$. This $A$-module corresponds to the local-local group scheme $\alpha_p$, and we know $e([\alpha_p]_C) = [R/\alpha]_R$; thus, $R/\alpha$ is the only simple $R$-module occurring in $e([Y]_C)$.

In every case $R/\alpha$ is the only $R$-module occurring on either side of (3). Let $N$ be the unique simple left $A$-module that is a quotient of $T_p$. We must show that the number of occurrences of $N$ in $T_p/\alpha_p(T_p)$ is equal to that of $R/\alpha$ in $O_p/Nrd_p(\alpha)O_p$. We can find these numbers simply by looking at the cardinalities of the objects involved.

Let $m$ denote the $W$-length of $N$. Since the unique simple $W$-module is $k$ and $#k = p^a$, the cardinality of $N$ is $p^{am}$. On the other hand, the cardinality of $R/\alpha$ is equal to the rank of the group scheme corresponding to $N$, and the general Dieudonné theory tells us that this rank is $p^m$; thus we have $#R/\alpha = p^m$. Therefore, to finish the proof of the lemma we must show that the cardinality of $T_p/\alpha_p(T_p)$ is the $a$th power of that of $O_p/Nrd_p(\alpha)O_p$.

To aid us in computing $T_p/\alpha_p(T_p)$ we choose an $O_p$-lattice $S$ in $U_p$. It is not hard to show that $\#T/\alpha_p(T) = \#S/\alpha_p(S)$, and a standard lemma shows that the $O_p$-modules $S/\alpha_p(S)$ and $O_p/\det_{\text{End}_{K_p}} U_p(\alpha_p)O_p$ are Jordan-Hölder isomorphic; here we view $U_p$ as a $K_p$-vector space and we view $\alpha_p$ as a $K_p$-linear automorphism of $U_p$. Thus to prove the lemma we must show that the cardinality of $O_p/\det_{\text{End}_{K_p}} U_p(\alpha_p)O_p$ is the $a$th power of that of $O_p/Nrd_p(\alpha)O_p$, and to prove this we need only show that $\det_{\text{End}_{K_p}} U_p(\alpha_p) = \text{Nrd}_p(\alpha)^a$.

We know from [11, p. 527] that the dimension of $U'_p$ as a $K_p$-vector space is $ad$. Since $U_p$ is the sum of $n$ copies of $U'_p$, we have $\dim_{K_p} U_p = an$. On the other hand, $D_p$ is a central simple algebra of dimension $d^2$ over $K_p$, so $E_p$ is a central simple algebra of dimension $(nd)^2$ over $K_p$. Let $\overline{K}_p$ be an algebraic closure of $K_p$. The $\overline{K}_p$-algebra $E_p \otimes_{K_p} \overline{K}_p$ is an $nd \times nd$ matrix algebra, the map $E_p \otimes_{K_p} \overline{K}_p \to \text{End}_{\overline{K}_p}(U_p \otimes_{K_p} \overline{K}_p)$ is an embedding of a $nd \times nd$ matrix algebra into an $and \times and$ matrix algebra, and the reduced norm on $E_p \otimes_{K_p} \overline{K}_p$ is the determinant map. The embedding of the first algebra into the second differs from the block diagonal embedding by an inner automorphism. It is clear that if we map an element $\beta$ of $E_p \otimes_{K_p} \overline{K}_p$ to the second algebra and take its determinant, we get the $a$th power of the determinant of $\beta$ as an element of $E_p \otimes_{K_p} \overline{K}_p$. Thus $\det_{\text{End}_{K_p}} U_p(\alpha_p) = \text{Nrd}_p(\alpha)^a$, and the proofs of Lemmas 3.4 and 3.3 are complete.

Together, Lemmas 3.2 and 3.3 complete the proof of Theorem 3.1.

\begin{itemize}
\item[(3.5)] Theorem. The homomorphism $e$ is an isomorphism.
\end{itemize}
Proof. We know that \( \epsilon \) takes the class of a simple group scheme to the class of a simple \( R \)-module, so if \( \epsilon \) were not an isomorphism there would be a simple \( R \)-module \( M \) such that \( M \) does not occur in \( \epsilon(P) \) for any \( P \in G(\ker \epsilon) \). But if \( M \) is a simple \( R \)-module killed by a prime \( \ell \) then \( M \) occurs in \( \epsilon(\ker \ell) \) by Theorem 3.1. Thus \( \epsilon \) is an isomorphism. \( \square \)

4. Reduced norms

Suppose \( K \) is a CM-field whose maximal real subfield \( K^+ \) has degree \( e_0 \) over \( \mathbb{Q} \). Let \( \sigma \) be the non-trivial automorphism of \( K/K^+ \). Let \( A \) be a central simple algebra over \( K \) and let \( \tau \) be an involution of \( A \) of the second kind — that is, \( \tau \) should extend the involution \( \sigma \) of \( K \). An element \( x \) of \( A \) is symmetric if \( \tau(x) = x \), and we denote by \( A^+ \) the set of symmetric elements of \( A \). One can show that all of the roots of the characteristic polynomial of a symmetric \( x \in A \) are totally real algebraic numbers. We call a symmetric \( x \in A \) totally positive if all of the roots of its characteristic polynomial are positive under every embedding of \( \overline{\mathbb{Q}} \) into \( \mathbb{C} \). An \( x \) in \( A^+ \) is totally positive if and only if \( x \) is sent to a positive real number under every ring homomorphism from \( K/K^+ \) into \( K^+ \) to \( \mathbb{C} \). We denote the sets of totally positive elements of \( A^+ \) and of \( K^+ \) by \( A_0 \) and \( K_0 \), respectively. The Hasse-Schilling-Maass theorem ([7, Thm. 33.15, p. 289]) shows that \( \text{Nrd}(A^*) = K^* \), where \( \text{Nrd} \) is the reduced norm from \( A^* \) to \( K^* \). Our next result, which is related to [5, §6.7, Ex. 2, p. 368], describes the set of reduced norms of totally positive elements.

(4.1) Theorem. Let \( K \) be a CM-field and let \( A \) be a central simple \( K \)-algebra with an involution \( \tau \) of the second kind that fixes \( K^+ \). Then \( \text{Nrd}(A_0) = K_0 \).

Proof. We note for future reference that the fact that \( A \) has an involution of the second kind gives us information about its local invariants. Landherr’s theorem ([8, Thm. 10.2.4, p. 355]) tells us that if \( q \) is a prime of \( K \) such that \( q = \overline{q} \) then \( \text{inv}_q(A) = 0 \), while if \( q \neq \overline{q} \) then \( \text{inv}_q(A) + \text{inv}_{\overline{q}}(A) = 0 \).

It is clear that \( \text{Nrd}(A_0) \subset K_0 \), so we are left to show that for every \( y \in K_0 \) there is an \( x \in A_0 \) with \( \text{Nrd}(x) = y \). The first step of our proof will be to find a monic polynomial of degree \( d = \sqrt{[A:K]} \) that has constant term \((-1)^d y \) and that satisfies local conditions at a set \( S \) of primes of \( K^+ \) that we now define. The polynomial we construct will depend on \( A \) only insofar as \( A \) determines \( S \). Let \( S_{\infty} \) be the set of infinite primes of \( K^+ \). Let \( S_1 \) be the set of finite primes \( p \) of \( K^+ \) that lie under \( q \) of \( K \) with \( \text{inv}(A) \neq 0 \); Landherr’s theorem tells us that every such \( p \) splits in \( K/K^+ \). If the set \( S_1 \) is empty, replace it with a set consisting of one finite prime of \( K^+ \) that splits in \( K/K^+ \). Let \( S_2 \) be the set of finite primes \( p \) of \( K^+ \) such that \( p \) does not split in \( K/K^+ \) and such that the algebraic group \( \text{SU}(A, \tau) \) over \( K^+ \) is not \( K^+ \)-quasisplit (that is, no geometric Borel subgroup of \( \text{SU}(A, \tau) \otimes_{K^+} K^+ \) should be definable over \( K^+ \)). The set \( S_2 \) is finite by [5, Thm. 6.7, p. 291]. For each \( p \in S = S_{\infty} \cup S_1 \cup S_2 \) we will define a monic polynomial \( f_p \) in \( K_p^+[X] \) of degree \( d \).

Suppose \( p \in S_{\infty} \). Let \( \varphi : K^+ \to K_p^+ = \mathbb{R} \) be the embedding of \( K^+ \) into \( \mathbb{R} \) corresponding to \( p \). By assumption \( \varphi(y) > 0 \). Let \( \alpha_1, \ldots, \alpha_d \) be \( d \) distinct positive real numbers whose
product is \( y \), and let \( f_p = (X - \alpha_1) \cdots (X - \alpha_d) \). Clearly \( f_p \) has constant coefficient \((-1)^d y\).

Suppose \( p \in S_1 \). Let \( f_p \) be any monic irreducible polynomial in \( K_p^+[X] \) with degree \( d \) and constant coefficient \((-1)^d y\); such polynomials exist by \([7, \text{Cor. 33.13, p. 288}]\).

Suppose \( p \in S_2 \). Let \( \alpha_1, \ldots, \alpha_d \) be distinct elements of \( K_p^+ \) whose product is \( y \) and let \( f_p = (X - \alpha_1) \cdots (X - \alpha_d) \). Clearly the constant term of \( f_p \) is \((-1)^d y\).

We want to find a polynomial \( f \in K^+[X] \) that approximates each of the \( f_p \) that we have just defined. The next lemma, which is a slight generalization of Krasner’s lemma (see \([7, \text{Lem. 33.8, p. 284}]\)), will give us a neighborhood of the coefficients of \( f_p \) for all \( p \in S \).

**(4.2) Lemma.** Let \( K \) be a field complete with respect to a valuation and let \( f = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \) be a separable monic polynomial in \( K[X] \). Let \( L \) be the \( n \)-dimensional \( K \)-algebra \( K[X]/(f) \), let \( \alpha \) be the image of \( X \) in \( L \), and use the valuation on \( K \) to define a topology on \( L \). Let \( U \) be a neighborhood of \( \alpha \) in \( L \). Then there is a neighborhood \( V \) of \((a_0, a_1, \ldots, a_{n-1}) \in K^n \) such that for any \((b_0, b_1, \ldots, b_{n-1}) \) in \( V \) the following statement is true: Let \( g = X^n + b_{n-1}X^{n-1} + \cdots + b_0 \), let \( M = K[X]/(g) \), and let \( \beta \) be the image of \( X \) in \( M \). Then there is a \( K \)-algebra isomorphism \( i: M \to L \) such that \( i(\beta) \in U \).

**Proof.** Let \( c: L \to K^n \) be the continuous map that sends an element of \( L \) to the \( n \)-tuple of the coefficients of its characteristic polynomial over \( K \), and let \( \overline{c}: L \otimes_K K^{\text{sep}} \to (K^{\text{sep}})^n \) be the map obtained from \( c \), where \( K^{\text{sep}} \) is a separable closure of \( K \). Applying \([5, \text{Lem. 6.25, p. 363}]\), we see that \( \overline{c} \) is a local isomorphism near \( \alpha \), and we can find a \( \text{Gal}(K^{\text{sep}}/K) \)-invariant neighborhood \( W \subset L \otimes_K K^{\text{sep}} \) of \( \alpha \) on which \( \overline{c} \) is an isomorphism. Thus, we can take \( V \) to be \( c(U \cap W) \).

Suppose \( p \) is a prime in \( S \). Apply Lemma 4.2 to the field \( K_p^+ \) and the polynomial \( f_p \). If \( p \) is a finite prime, take \( U \) to be the open set \( L \) (in the notation of the lemma), while if \( p \) is infinite take \( U \) to be the open subset \((0, \infty)^d \) of \( L \cong \mathbb{R}^d \). Let \( V_p \) be the neighborhood of the coefficients of \( f_p \) that is given by the lemma. Let \( f \) be any monic degree \( d \) polynomial in \( K^+[X] \) that has constant term \((-1)^d y \) and whose coefficients lie in the neighborhood \( V_p \) for every \( p \in S \). Let \( L^+ = K^+[X]/(f) \), and let \( x \) be the image of \( X \) in \( L^+ \).

**(4.3) Lemma.** The \( K^+ \)-algebra \( L^+ \) is a totally real number field, \( x \) is totally positive, and \( N_{L^+/K^+}(x) = y \).

**Proof.** The polynomial \( f \) is irreducible, because for every \( p \) in the non-empty set \( S_1 \) we have \( K_p^+[X]/(f) \cong K_p^+[X]/(f_p) \), and \( f_p \) is irreducible. Therefore \( L^+ \) is a field. The field \( L^+ \) is totally real because for every infinite prime \( p \) of \( K^+ \) we have an isomorphism \( i_p: K_p^+[X]/(f) \cong K_p^+[X]/(f_p) \cong \mathbb{R}^d \). The element \( x \) of \( L^+ \) is totally positive because for every infinite \( p \) Lemma 4.2 says that \( i_p(x) \) lies in the subset \((0, \infty)^d \) of \( \mathbb{R}^d \). And finally, the constant term of \( f \) tells us that \( N_{L^+/K^+}(x) = y \).

The compositum of the fields \( L^+ \) and \( K \) is a CM-field \( L \) with maximal real subfield \( L^+ \), and the non-trivial automorphism \( \rho \) of \( L \) over \( L^+ \) extends the automorphism \( \sigma \) of \( K \). Clearly \( x \) is fixed by \( \rho \). We will show in several steps that we can embed \((L, \rho)\) in \((A, \tau)\) as a \( K \)-algebra with involution.
(4.4) Lemma. There is an embedding \( L \rightarrow A \) of \( K \)-algebras without involution.

Proof. Since \( L \) has degree \( d \) over \( K \) and \( \dim_K(A) = d^2 \), we see by [7, Cor. 28.10, p. 240] that we can embed \( L \) in \( A \) if \( L \) is a splitting field for \( A \). By [7, Thm. 32.15, p. 278], the field \( L \) splits \( A \) if and only if for every prime \( \mathfrak{p} \) of \( L \) we have \( (L_{\mathfrak{p}} : K_{\mathfrak{p}}) \text{inv}_q(A) = 0 \) in the Brauer group of \( K_{\mathfrak{q}} \), where \( \mathfrak{q} \) is the prime of \( K \) lying under \( \mathfrak{p} \). Since \( \text{inv}_q(A) \) is killed by \( d \), it will be enough for us to show that if \( \mathfrak{q} \) is a prime of \( K \) with \( \text{inv}_q(A) \neq 0 \) then \( \mathfrak{q} \) is inert in \( L \), for then there will be only one prime \( \mathfrak{p} \) of \( L \) over \( \mathfrak{q} \) and \((L_{\mathfrak{p}} : K_{\mathfrak{p}}) = d \).

So suppose \( \mathfrak{q} \) is a prime of \( K \) with \( \text{inv}_q(A) \neq 0 \). Let \( \mathfrak{p} \) be the prime of \( K^+ \) lying under \( \mathfrak{q} \); the prime \( \mathfrak{p} \) is an element of the set \( S_1 \). By construction we have \( K_{\mathfrak{p}}^+[X]/(f) \cong K_{\mathfrak{q}}^+[X]/(f_{\mathfrak{p}}) \), and since \( f_{\mathfrak{p}} \) is irreducible for \( \mathfrak{p} \in S_1 \) this last ring is a field; in other words, the prime \( \mathfrak{p} \) of \( K^+ \) is inert in \( L^+ = K^+[X]/(f) \). But \( \mathfrak{p} \) splits in \( K/K^+ \), so \( \mathfrak{q} \) is inert in \( L/K \). \( \square \)

(4.5) Lemma. For every prime \( \mathfrak{p} \) of \( K^+ \), there is an embedding

\[
(L \otimes_{K^+} K_{\mathfrak{p}}^+, \rho) \rightarrow (A \otimes_{K^+} K_{\mathfrak{p}}^+, \tau)
\]

of \( K_{\mathfrak{p}}^+ \)-algebras with involution.

Proof. First suppose \( \mathfrak{p} \) is an infinite prime, and let \( \mathfrak{q} \) be the prime of \( K \) lying over \( \mathfrak{p} \). Then there is an isomorphism \( L \otimes_{K^+} K_{\mathfrak{p}}^+ \rightarrow \mathbb{C}^d \) that takes \( \rho \) to the involution that is complex conjugation on each copy of \( \mathbb{C} \). There is also an isomorphism \( A \otimes_{K^+} K_{\mathfrak{p}}^+ \rightarrow M_d(\mathbb{C}) \) that takes \( \tau \) to the conjugate transpose involution. Then the first of these algebras with involution is isomorphic to the diagonal of the second.

Next suppose \( \mathfrak{p} \) is a finite prime that splits into distinct primes \( \mathfrak{q} \) and \( \overline{\mathfrak{q}} \) of \( K \). Then there is an isomorphism \( i: A \otimes_{K^+} K_{\mathfrak{p}}^+ \rightarrow (A \otimes_K K_{\mathfrak{q}}) \times (A \otimes_K K_{\overline{\mathfrak{q}}}) \). Let \( B \) be the \( K_{\mathfrak{p}}^- \)-algebra \( A \otimes_K K_{\mathfrak{q}} \). Since \( \text{inv}_q(A) + \text{inv}_{\overline{\mathfrak{q}}}(A) = 0 \), the \( K_{\mathfrak{p}}^+ \)-algebra \( A \otimes_K K_{\overline{\mathfrak{q}}} \) is isomorphic to the opposite algebra of \( B \), and the isomorphism \( i \) takes \( \tau \) to the involution of \( B \times B^{opp} \) that switches factors. Similarly, there is an isomorphism \( L \otimes_{K^+} K_{\mathfrak{p}}^+ \rightarrow (L \otimes_K K_{\mathfrak{q}}) \times (L \otimes_K K_{\overline{\mathfrak{q}}}) \) that takes \( \rho \) to the involution that switches factors. Therefore, to show that \( (L \otimes_{K^+} K_{\mathfrak{p}}^+, \rho) \) embeds in \( (A \otimes_{K^+} K_{\mathfrak{p}}^+, \tau) \) we need only show that \( L \otimes_K K_{\mathfrak{p}} \) embeds in \( A \otimes_K K_{\mathfrak{q}} \). But we already know that \( L \) embeds in \( A \), so we are done.

Next suppose \( \mathfrak{p} \) is a finite prime at which \( \text{SU}(A, \tau) \) is quasisplit. Then the proof of [5, Lem. 6.26, p. 365] shows that \( (L \otimes_{K^+} K_{\mathfrak{q}}^+, \rho) \) can be embedded in \( (A \otimes_{K^+} K_{\mathfrak{p}}^+, \tau) \).

We are left to consider the finite primes of \( K^+ \) that do not split in \( K \) and at which \( \text{SU}(A, \tau) \) is not quasisplit. These are exactly the primes in \( S_2 \). Let \( \mathfrak{p} \) be one such prime and let \( \mathfrak{q} \) be the prime of \( K \) lying over \( \mathfrak{p} \). Now, \( (L \otimes_{K^+} K_{\mathfrak{q}}^+, \rho) \) is isomorphic to \( (K_{\mathfrak{q}}[X]/(f), \sigma) \), where \( \sigma \) acts trivially on \( X \), and this last algebra with involution is isomorphic to \( (K_{\mathfrak{q}}[X]/(f_{\mathfrak{p}}), \sigma) \). Thus, to embed \( (L \otimes_{K^+} K_{\mathfrak{q}}^+, \rho) \) into \( (A \otimes_{K^+} K_{\mathfrak{q}}^+, \tau) \) we must find an element \( \alpha \) of \( A \otimes_{K^+} K_{\mathfrak{q}}^+ \) that is fixed by \( \tau \) and that has minimal polynomial \( f_{\mathfrak{p}} \).

By Landherr’s theorem \( \text{inv}_q(A) = 0 \), so \( A \otimes_{K^+} K_{\mathfrak{q}}^+ = A \otimes_K K_{\mathfrak{q}} \) is a matrix algebra over \( K_{\mathfrak{q}} \). From [8, Thm. 8.7.4, pp. 301–302] and [8, Thm. 7.6.3, p. 259] we see that there is an isomorphism \( i: A \otimes_K K_{\mathfrak{q}} \rightarrow M_d(K_{\mathfrak{q}}) \) and a diagonal matrix \( a \in M_d(K_{\mathfrak{q}}) \) with
entries in $K_p^+$ such that $i$ takes the involution $\tau$ to the involution $\eta$ of $M_d(K_q)$ defined by $\eta(x) = ax^*a^{-1}$, where $x^*$ is the conjugate transpose of $x$. Recall that $f_p$ was defined to be $(X - \alpha_1) \cdots (X - \alpha_d)$ where the $\alpha_i$ were distinct elements of $K_p^+$. Let $\alpha$ be the diagonal matrix $\langle \alpha_1, \ldots, \alpha_d \rangle \in M_d(K_q)$. Then $\alpha$ is fixed by $\eta$ and has minimal polynomial $f_p$.

Using this $\alpha$, we can embed $(L \otimes_{K_p} K_p^+, \rho)$ into $(A \otimes_{K_p} K_p^+, \tau)$, and we are done. $\square$

The notation of the next lemma is independent of that of the rest of this section. The notation is chosen to agree with that of the lemma’s proof, which is to be found in [5].

**Lemma.** Suppose $L/K$ is a quadratic extension of number fields, suppose $F$ is an extension of $K$ that is linearly disjoint from $L$ over $K$, suppose $A$ is a central simple $L$-algebra with an involution $\tau$ of the second kind that fixes $K$, and suppose $(F : K)^2 = (A : L)$. Let $P$ be the compositum of $F$ and $L$, so that the non-trivial automorphism of $L$ over $K$ extends to an involution $\sigma$ of $P$ that fixes $F$. Then there is an embedding $\theta: (P, \sigma) \rightarrow (A, \tau)$ of algebras with involution if and only if there is an embedding $\epsilon: P \rightarrow A$ of algebras without involution and for each prime $p$ of $K$ there exists an embedding $\theta_p: (P \otimes_K K_p, \sigma) \rightarrow (A \otimes_K K_p, \tau)$ of algebras with involution.

**Proof.** The ‘only if’ statement is clear. The ‘if’ statement is proven in [5]. The argument is found at the end of §6.7, from the last paragraph of page 366 onward. $\square$

Together, Lemmas 4.4, 4.5, and 4.6 show that there is an embedding $(L, \rho) \rightarrow (A, \tau)$ of algebras with involution. View $x \in L$ as an element of $A$ via this embedding. Then $x$ is fixed by $\tau$, and in fact $x$ is a totally positive element of $A$. Furthermore, $\text{Nrd}(x) = N_{L/K}(x) = N_{L^+/K}(x) = y$ by Lemma 4.3, so Theorem 4.1 is proved. $\square$

The second theorem we will prove in this section involves central simple algebras over a totally real number field. Let $K$ be a totally real number field of degree $e$ over $\mathbb{Q}$ and let $A$ be a central simple $K$-algebra with an involution $\tau$ of the first kind — that is, $\tau$ should be the identity on $K$. Let $A^+$ denote the set of elements of $A$ that are fixed by $\tau$. Suppose that the form $x \mapsto \text{Tr}_{A/\mathbb{Q}}(x\tau(x))$ is positive definite, and suppose that $A$ is ramified at all of the infinite primes of $K$. Then $A$ is isomorphic to a matrix algebra over a quaternion algebra $D$ over $K$, say $A \cong M_d(D)$, and one can use this fact to show that all of the roots of the characteristic polynomial of a symmetric element of $A$ are totally real algebraic numbers. We call an element $x$ of $A^+$ **totally positive** if all of the roots of its characteristic polynomial are positive under every embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$. An $x$ in $A^+$ is totally positive if and only if $x$ is sent to a positive real number under every ring homomorphism from $K(x)$ to $\mathbb{C}$. We denote the sets of totally positive elements of $A^+$ and of $K$ by $A_0$ and $K_0$, respectively. The Hasse-Schilling-Maass theorem shows that $\text{Nrd}(A^*) = K_0$.

**Theorem.** Let $K$ be a totally real number field and let $A$ be a central simple $K$-algebra with an involution $\tau$ of the first kind such that $x \mapsto \text{Tr}_{A/\mathbb{Q}}(x\tau(x))$ is positive definite. Suppose $A$ is ramified at all of the infinite primes of $K$. Then $\text{Nrd}(A_0) = (K_0)^2$.

**Proof.** Let $x \mapsto x^*$ be the conjugate transpose involution on $M_d(D)$. By [8, Thm. 8.7.4, pp. 301–302] and [8, Thm. 7.6.3, p. 259], there is an isomorphism $i: A \rightarrow M_d(D)$ and a
diagonal matrix $a \in M_d(D)$ with $a^* = \pm a$ such that the isomorphism $i$ takes the involution \( \tau \) to the involution \( \eta \) of \( M_d(D) \) defined by \( \eta(x) = axa^{-1} \). The argument at the bottom of page 195 of \([3]\) show that \( a^* = -a \) is impossible when \( A \) ramifies at an infinite prime, so we must have \( a^* = a \). This shows that \( a \) is a diagonal matrix with entries in \( K \).

It is now easy to show that \( Nrd(A_0) \supset (K_0)^2 \). Suppose \( y \in K_0 \). Let \( x \in M_d(D) \) be the diagonal matrix with \( y \) in the upper left corner and with ones elsewhere on the diagonal. Clearly \( i^{-1}(x) \) is fixed by \( \tau \) and is totally positive, and \( Nrd(i^{-1}(x)) = Nrd_D/K(y) = y^2 \). Therefore \( Nrd(A_0) \supset (K_0)^2 \).

Now we show the reverse inclusion holds. Suppose \( x \in A \) is fixed by \( \tau \) and is totally positive. By \([8, \text{Thm. 7.6.3, p. 259}]\), there is an invertible element \( b \) of \( M_d(D) \) such that \( c = bi(x)\eta(b) \) is diagonal, and since \( i(x) \) and \( c \) represent isomorphic Hermitian forms on a \( d \)-dimensional \( D \)-vector space, \( c \) must be totally positive. Write the diagonal matrix \( c \) as \( \langle c_1, \ldots, c_d \rangle \) with \( c_i \in D \). Since \( c \) is fixed by \( \eta \), each of the elements \( c_i \) must be fixed by the standard involution of \( D \), so each \( c_i \) is in \( K \). Furthermore, since \( c \) is totally positive, each \( c_i \) must lie in \( K_0 \). Thus, the reduced norm of \( c \), which is \( (c_1 \cdots c_d)^2 \), is in \( (K_0)^2 \).

Since

\[
Nrd_{M_d(D)/K}(c) = Nrd(x)Nrd_{M_d(D)/K}(b)Nrd_{M_d(D)/K}(\eta(b)) = Nrd(x)(Nrd_{M_d(D)/K}(b))^2
\]

and since \( Nrd_{M_d(D)/K}(b) \in K_0 \) by Hasse-Schilling-Maass, we see that \( Nrd(x) \in (K_0)^2 \). \( \square \)

5. Kernels of polarizations

We begin this section by defining the obstruction group \( B(R) \). Suppose \( R \) is a proper real/CM-order. In §3 we defined a duality \( M \mapsto \hat{M} = \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \) on \( \text{Mod}_R \). We use this duality to define an involution \( \hat{-} \) on \( G(\text{Mod}_R) \) by setting \([M]_R = [\hat{M}]_R \) for \( R \)-modules \( M \). An element \( P \) of \( G(\text{Mod}_R) \) is symmetric if \( P = \overline{P} \). By considering finite \( R \)-modules to be finite \( R^+ \)-modules, we obtain the norm homomorphism \( N_{R/R^+} \) from \( G(\text{Mod}_R) \) to \( G(\text{Mod}_{R^+}) \), which satisfies \( N_{R/R^+}([M]_R) = [M]_{R^+} \) for every finite \( R \)-module \( M \). We define \( Z(R) \) to be the set of symmetric elements of the kernel of the homomorphism \( G(\text{Mod}_R) \to G(\text{Mod}_{R^+}) \otimes (\mathbb{Z}/2\mathbb{Z}) \) obtained from the norm.

Let \( K = R \otimes \mathbb{Q} \) and write \( \hat{K} = K_1 \times \cdots \times K_r \) as a product of number fields. If \( K_1 \) is a CM-field define \( K_1^\dagger \) to be the multiplicative group of totally positive elements of \( K_1^+ \), and if \( K_1 \) is real let \( K_1^\dagger \) be the group of squares of totally positive elements of \( K_1 \). Let \( K^\dagger \) be the group \( K_1^\dagger \times \cdots \times K_r^\dagger \). Consider the subgroup \( \text{Pr}_R(K^\dagger) \) of \( G(\text{Mod}_R) \), where \( \text{Pr}_R \) is the homomorphism that was defined in §3. Clearly the elements of \( \text{Pr}_R(K^\dagger) \) are symmetric, and in fact \( \text{Pr}_R(K^\dagger) \subseteq Z(R) \). It suffices to verify this in the case where \( R \) is a domain. In this case we must show that for every \( a \in R^+ \cap K^\dagger \), the image of \( R/aR \) in \( G(\text{Mod}_{R^+}) \) lies in \( 2G(\text{Mod}_{R^+}) \). If \( R \) is real then this follows from the fact that every \( a \in K^\dagger \) is a square. If \( R \) is a CM-order, then this follows from the fact that \( R = R^+[F] \) is a free \( R^+ \)-module of rank two, so that \( R/aR \cong (R^+/aR^+)^2 \) as \( R^+ \)-modules.

Finally, let \( B(R) \) be the subgroup \( \{ P + \overline{F} : P \in G(\text{Mod}_R) \} \) of \( Z(R) \). We define \( B(R) \) to be the group \( Z(R)/(B(R) \cdot \text{Pr}_R(K^\dagger)) \). Note that if \( i: R \to S \) is a homomorphism of
proper real/CM-orders then we get a group homomorphism $i^*: \mathcal{B}(S) \to \mathcal{B}(R)$ from the
map $G(\text{Mod}_S) \to G(\text{Mod}_R)$ obtained by viewing every $S$-module as an $R$-module. In
fact, $\mathcal{B}$ is a contravariant functor from the category of real/CM-orders to the category of
abelian groups, and we will show in §6 that $\mathcal{B}(R)$ is a finite two-torsion group.

**1.3 Theorem.** Let $\mathcal{C}$ be an isogeny class of abelian varieties over a finite field and let
$R = R_\mathcal{C}$. There is an element $I_\mathcal{C}$ of $\mathcal{B}(R)$ such that the elements of $G(\text{Mod}_R)$ that are
attainable in $\mathcal{C}$ are precisely the effective elements of $Z(R)$ that map to $I_\mathcal{C}$ in $\mathcal{B}(R)$. In
particular, the isogeny class $\mathcal{C}$ contains a principally polarized variety if and only if $I_\mathcal{C} = 0$.

**Proof.** It is clear that attainable elements must be effective, so the proof of Theorem 1.3
splits up into three steps. In the first step we will show that every attainable element of
$G(\text{Mod}_R)$ is contained in $Z(R)$. In the second step we will show that if $\lambda$ and $\mu$ are
both polarizations of varieties in $\mathcal{C}$ then $\epsilon([\ker \lambda]_\mathcal{C})$ and $\epsilon([\ker \mu]_\mathcal{C})$ differ by an element of
$B(R) \cdot \text{Pr}_R(K^\dagger)$. In the third step we will show that if $P$ is an effective element of $Z(R)$
that differs from an attainable element by an element of $B(R) \cdot \text{Pr}_R(K^\dagger)$, then $P$ is itself
attainable. The theorem will follow if we then take $I_\mathcal{C}$ to be the image in $\mathcal{B}(R)$ of any
attainable element of $Z(R)$.

**Step one.** Suppose $\lambda$ is a polarization of a variety in $\mathcal{C}$ and let $X$ be its kernel. There is a
non-degenerate alternating pairing $X \times X \to \mathbb{G}_m$ (see [3, §23]), so there is an isomorphism
between $X$ and its Cartier dual $\hat{X}$. Let $X_{\ell\ell}$ denote the local-local part of $X$ and let $Y$ denote
the non-local-local part of $X$. Then $X_{\ell\ell}$ and $Y$ are both self-dual. From §3 we know that
$\overline{\mathcal{P}(Y)} = \mathcal{P}((Y)$ (where $\mathcal{P}$ is the functor defined in §3) and it follows easily that
$\epsilon([X]_\mathcal{C}) = \epsilon([\hat{X}]_\mathcal{C}) = \epsilon([\hat{X}]_\mathcal{C})$. Therefore, attainable elements are symmetric.

Let $M$ be the $R$-module $\mathcal{P}(Y)$. The non-degenerate alternating pairing from $Y$ to $\mathbb{G}_m$
gives us a non-degenerate alternating $R$-semi-balanced pairing from $M$ to $\mathbb{Q}/\mathbb{Z}$, and we can view this pairing as an $R^+$-balanced pairing on the $R^+$-module $M$. Lemma 5.1 below shows that every simple $R^+$-module occurs in $M$ an even number of times, so $\epsilon([Y]_\mathcal{C})$ is in the kernel of the map $G(\text{Mod}_R) \to G(\text{Mod}_{R^+}) \otimes (\mathbb{Z}/2\mathbb{Z})$. We know from Riemann-Roch
(see [3, §16]) that the rank of $X$ is a square, and from what we have just seen the rank of $Y$ is a square.
Therefore the rank of $X_{\ell\ell}$ is a square, which shows that the simple group scheme $\alpha_p$ occurs an even number of times in $X_{\ell\ell}$. It follows that $\epsilon([X_{\ell\ell}]_\mathcal{C})$ is in the kernel of $G(\text{Mod}_R) \to G(\text{Mod}_{R^+}) \otimes (\mathbb{Z}/2\mathbb{Z})$, so $\epsilon([X]_\mathcal{C})$ is in this kernel as well. Thus step one will be completed if we prove the following lemma.

**5.1 Lemma.** Let $A$ be a commutative ring and $M$ an $A$-module with $\#M < \infty$. Suppose
there is a non-degenerate balanced alternating pairing $\epsilon: M \times M \to \mathbb{Q}/\mathbb{Z}$. Then every
simple $A$-module that occurs in the Jordan-Hölder decomposition of $M$ occurs an even
number of times.

**Proof.** We will prove the lemma by induction on the cardinality of $M$. The lemma is
certainly true if $M$ is the zero module. So suppose the lemma is true for all modules with
cardinality less than that of $M$. We will show that it holds for $M$ also.

Let $S$ be a simple $A$-module that occurs in $M$, let $\mathfrak{p}$ be the maximal ideal of $A$ that
annihilates $S$, and let $L$ be the finite field $A/\mathfrak{p}$. Identify $S$ with any one-dimensional
Then \( e \) gives us a balanced alternating pairing \( S \times S \to \mathbb{Q}/\mathbb{Z} \) on the \( L \)-vector space \( S \). An easy lemma shows that there are no non-degenerate balanced alternating pairings on a one-dimensional vector space over a finite field (see the proof of [2, Lem. 7.3, p. 2378]), so \( e \) restricted to \( S \) must be degenerate and hence identically zero. In other words, \( S \) is an isotropic submodule of the \( A \)-module \( M \).

Let \( S' \subset M \) be the annihilator of \( S \) under \( e \), so that \( S \subset S' \) by what we have just shown. The non-degenerate balanced pairing \( S \times M/S' \to \mathbb{Q}/\mathbb{Z} \) induced from \( e \) shows that \( M/S' \) and \( S \) are both simple modules killed by \( p \), so they are isomorphic. Thus in the Grothendieck group of \( A \) we have \([M]_A = [S'/S]_A + 2[S]_A\). The pairing \( e \) restricts to a non-degenerate balanced alternating pairing \( S'/S \times S'/S \to \mathbb{Q}/\mathbb{Z} \). The induction hypothesis shows that every simple module that occurs in \( S'/S \) occurs an even number of times. Therefore, the lemma holds for \( M \), and we are done.

The center \( K_C \) of the endomorphism algebra \( E_C \) is a product of number fields, say \( K_C = K_1 \times \cdots \times K_r \), and we may write \( E_C = E_1 \times \cdots \times E_r \), where each \( E_i \) is a central simple \( K_i \)-algebra. If \( \lambda \) is a polarization of a variety in \( C \), then the Rosati involution \( ' \) on \( E_C \) associated to \( \lambda \) splits into an involution on each of the \( E_i \). If \( K_i \) is a CM-field then \( ' \) is an involution of the second kind on \( E_i \). If \( K_i \) is totally real, then \( E_i \) ramifies at all real primes of \( K_i \) and \( ' \) is a positive involution on \( E_i \). We will use these facts in steps two and three.

**Step two.** Suppose \( \lambda: A \to \hat{A} \) and \( \mu: B \to \hat{B} \) are polarizations of varieties in \( C \) and let \( \varphi: A \to B \) be an isogeny from \( A \) to \( B \). Let \( \nu \) be the polarization \( \hat{\varphi} \mu \varphi \) of \( A \), where \( \varphi: \hat{B} \to \hat{A} \) is the dual isogeny of \( \varphi \). Let \( n \) be any positive integer such that \( \ker \lambda \subset \ker(n\nu) \) as group schemes. Then there is an isogeny \( \alpha: \hat{A} \to \hat{A} \) such that \( n\nu = \alpha\lambda \). A polarization is equal to its own dual isogeny, so we can equate the right-hand side of the last equality with its dual to get \( n\nu = \lambda\hat{\alpha} \). Using [3, §21, Application III, pp. 208–210] (see especially the final paragraph) and the fact that \( n\nu \) and \( \lambda \) are polarizations, we find that \( \hat{\alpha} \in \text{End}(A) \) is fixed by the Rosati involution and is totally positive.

The equality \( n\hat{\varphi} \mu \varphi = \lambda \hat{\alpha} \) translates into the equality

\[
[ker n]_C + [ker \hat{\varphi}]_C + [ker \mu]_C + [ker \varphi]_C = [ker \lambda]_C + [ker \hat{\alpha}]_C
\]

in \( G(Ker_C) \). Let \( Q = \epsilon([ker \varphi]_C) \), so that \( \overline{Q} = \epsilon([ker \hat{\varphi}]_C) \). Applying the isomorphism \( \epsilon \) to the equality above and using Theorem 3.1, we find that

\[
\Pr_R(\text{Nrd}(n)) + \overline{Q} + \epsilon([ker \mu]_C) + Q = \epsilon([ker \lambda]_C) + \Pr_R(\text{Nrd}(\hat{\alpha})).
\]

Now \( n \) and \( \hat{\alpha} \) are both fixed by the Rosati involution and are totally positive, so by Theorems 4.1 and 4.7 their reduced norms lie in the subgroup \( K^\dagger \) of \( K^\ast \). Since \( Q + \overline{Q} \) is an element of \( B(R) \), it is clear that \( \epsilon([ker \lambda]_C) \) and \( \epsilon([ker \mu]_C) \) differ by an element of \( B(R) \cdot \Pr_R(K^\dagger) \).

**Step three.** Now suppose \( P \) is an effective element of \( Z(R) \) such that

\[
P + Q + \overline{Q} = \epsilon([ker \lambda]_C) + \Pr_R(a)
\]
for some $Q \in G(\text{Mod}_R)$, some $a \in K^\dagger$, and some polarization $\lambda: A \to \hat{A}$ of a variety in $\mathcal{C}$. By Theorems 4.1 and 4.7, there is an $\alpha \in (\text{End} A) \otimes Q$ that is fixed by the Rosati involution associated to $\lambda$, that is totally positive, and such that $\text{Nrd}(\alpha) = a$. If we replace $\alpha$ by an integer multiple of itself and change $Q$ accordingly, we can assume that we have

$$P + Q + \overline{Q} = \epsilon([\ker \lambda]_C) + \text{Pr}_R(\text{Nrd}(\alpha))$$

where $Q$ is effective and where $\alpha$ is a totally positive isogeny of $A$ that is fixed by the Rosati involution. Then [3, §21, Application III, pp. 208–210] tells us that $\nu = \lambda \alpha$ is also a polarization of $A$, and we have

$$P + Q + \overline{Q} = \epsilon([\ker \nu]_C). \quad (4)$$

Let $X = \ker \nu$ and let $e: X \times X \to \mathbb{G}_m$ be the non-degenerate alternating pairing on $X$ whose existence is shown in [3, §23]. Let $Y$ be an element of $\text{Ker}_C$ such that $S = \epsilon([Y]_C)$ is a simple $R$-module that occurs in the effective element $Q$ of $G(\text{Mod}_R)$. Suppose we can find an embedding of $Y$ into $X$ such that the pairing $e$ restricted to $X \times Y$ is the trivial pairing, and let $\varphi$ be the natural isogeny from $A$ to $B = A/Y$. Then [3, §23, Cor. to Thm. 2, p. 231] shows that there is a polarization $\nu'$ of $B$ such that $\nu = \hat{\varphi} \nu' \varphi$. In $G(\text{Mod}_R)$ this gives us the equality

$$\epsilon([\ker \nu]_C) = S + \epsilon([\ker \nu']_C) + \overline{S}.$$ 

If we replace $Q$ by $Q - S$ and $\nu$ by $\nu'$, we will again have equality (4), but we will have decreased the number of simple $R$-modules that occur in $Q$. By applying this argument repeatedly, we can finally reduce equality (4) to the desired equality $P = \epsilon([\ker \nu]_C)$ for a polarization $\nu$ of a variety in $\mathcal{C}$. Thus step three and the proof of Theorem 1.3 are completed by the following proposition.

(5.2) Proposition. Suppose $X$ and $Y$ are finite commutative group schemes over a finite field $k$, and suppose $Y$ is simple. Suppose further that there is a non-degenerate alternating pairing $e: X \times X \to \mathbb{G}_m$ and that $[X] - [Y] - [\hat{Y}]$ is an effective element of the Grothendieck group of finite commutative group schemes over $k$. Then there is an embedding of $Y$ into $X$ such that $e$ restricted to $X \times Y$ is the trivial pairing.

Proof. The pairing $e$ splits into three non-degenerate alternating pairings, one each on the reduced-reduced part of $X$, the product of the reduced-local and the local-reduced parts of $X$, and the local-local part of $X$. Therefore it is enough to prove the theorem when $X$ is either a reduced-reduced group, a product of a reduced-local and a local-reduced group, or a local-local group. Clearly we may assume that $Y$ is of the same type as $X$.

Suppose $X$ is reduced-reduced. Let $S$ be the group ring over $\mathbb{Z}$ of the Galois group $\text{Gal}(\overline{k}/k)$; the map $\sigma \mapsto \sigma^{-1}$ of the Galois group gives us an involution $-$ on $S$, and $S$ is commutative since $k$ is finite. The group schemes $X$ and $Y$ are completely determined by the $S$-modules $X(\overline{k})$ and $Y(\overline{k})$, and the pairing on $X$ gives us a pairing $X(\overline{k}) \times X(\overline{k}) \to$
Q/Z that is non-degenerate, alternating, and semi-balanced with respect to the involution on S. The proposition follows in this case from [2, Prop. 7.1, p. 2378].

Suppose X is a product of a reduced-local and a local-reduced group scheme. We may assume that Y is reduced-local. Let \( i: Y \to X \) be any embedding of Y into X; it is not hard to show that such embeddings exist. Then \( e \) restricted to \( Y \times Y \) cannot be non-degenerate, because otherwise we would have an isomorphism between the reduced-local group scheme Y and its local-reduced dual \( \hat{Y} \). Therefore \( e \) is trivial on \( Y \times Y \).

Suppose X is local-local, so that Y must be isomorphic to \( \alpha_p \), where \( p = \text{char } k \). If \( p > 2 \) then a straightforward Hopf algebra computation shows that there are no non-degenerate alternating pairings on \( \alpha_p \), so again we can take any embedding of \( \alpha_p \) into X. We are left with the case where \( p = 2 \) and \( Y = \alpha_2 \).

Let \( F = \text{End} \alpha_2 \cong k \), let \( U \) be the \( F \)-vector space \( \text{Hom}(\alpha_2, X) \) (where \( F \) acts by premultiplication), and let \( V \) be the \( F \)-vector space of alternating pairings on \( \alpha_2 \) (where \( F \) acts by premultiplication on the first factor). A computation shows that \( V \) is one-dimensional. Let \( i: \alpha_2 \to X \) be any embedding of \( \alpha_2 \) into X and let \( j \) be any embedding of \( \alpha_2 \) into the kernel of the composite map \( X \cong \hat{X} \to \hat{\alpha}_2 \), where the isomorphism \( X \cong \hat{X} \) is obtained from the pairing \( e \) and where \( \hat{X} \to \hat{\alpha}_2 \) is dual to \( i \). If \( j \) is a multiple of \( i \) in \( U \) then the pairing \( e \) restricted to \( i(Y) \times i(Y) \) is trivial and we are done. Otherwise, let \( p_1 \) and \( p_2 \) be the elements \( e \circ (i \times i) \) and \( e \circ (j \times j) \) of \( V \). If either \( p_1 \) or \( p_2 \) is trivial we are done, so assume they are not. Using the fact that \( i(\alpha_2) \) and \( j(\alpha_2) \) are orthogonal under \( e \), one can show that for every \( \beta \in F \) we have

\[
e \circ ((i + \beta j) \times (i + \beta j)) = p_1 + \beta^2 p_2.
\]

Since \( p_1 \) and \( p_2 \) are non-zero elements of the one-dimensional \( F \)-vector space \( V \) and since the squaring map \( F \to F \) is onto, we find that there is a choice of \( \beta \) so that the pairing \( e \) restricted to the image of the embedding \( i + \beta j \) is trivial. \( \square \)

6. The obstruction group

In this section we will determine how one can calculate the obstruction group \( \mathcal{B}(R_C) \) for an isogeny class \( \mathcal{C} \) of abelian varieties over a finite field \( k \). We begin by calculating \( \mathcal{B}(R) \) in the easiest cases: when \( R \) is totally real, and when \( R \) is the ring of integers of a CM-field.

(6.1) Proposition. Suppose \( R \) is an order in a product of totally real number fields. Then \( \mathcal{B}(R) \cong 0 \).

Proof. In this case the involution on \( R \) is trivial and \( R = R^+ \). We check that \( Z(R) = 2G(\text{Mod}_R) \) and \( B(R) = 2G(\text{Mod}_R) \). Then clearly \( \mathcal{B}(R) = 0 \). \( \square \)

(6.2) Proposition. Let \( O \) be the ring of integers of a CM-field \( K \). Then

\[
\mathcal{B}(O) \cong \text{Cl}^+(K^+)/N_{K/K^+}(\text{Cl}(K)),
\]
where \( \text{Cl}^+(K^+) \) denotes the narrow class group of \( K^+ \). If \( K/K^+ \) is ramified at a finite prime, then \( \text{Cl}^+(K^+)/N_{K/K^+}(\text{Cl}(K)) \) is the zero group. Otherwise, the Artin map provides an isomorphism

\[
\text{Cl}^+(K^+)/N_{K/K^+}(\text{Cl}(K)) \cong \text{Gal}(K/K^+).
\]

**Proof.** The map \( a \mapsto O/\alpha O \) gives an isomorphism between the ideal group of \( K^+ \) and \( \mathbb{Z}(O) \). Under this isomorphism, the subgroup of principal ideals of \( K^+ \) that are generated by totally positive elements is identified with \( \text{Pr}_O(K^+) \). Also, the subgroup of norms of ideals of \( K \) is identified with \( B(O) \). Therefore

\[
\text{Cl}^+(K^+)/N_{K/K^+}(\text{Cl}(K)) \cong B(O).
\]

The rest of the proposition is [2, Prop. 10.1, p. 2385]. \( \square \)

**Corollary.** Let \( K \) be a CM-field. If \( [K^+: \mathbb{Q}] \) is odd then \( B(O_K) \cong 0 \).

**Proof.** We see from [2, Lem. 10.2, p. 2385] that if \( [K^+: \mathbb{Q}] \) is odd then \( K/K^+ \) must be ramified at a finite prime. The corollary follows from Proposition 6.2. \( \square \)

The next proposition allows us to compute the obstruction group for one ring in terms of that of an overring.

**Proposition.** Suppose \( R \) and \( S \) are proper real/CM-orders and \( R \) is a subring of finite index in \( S \). Then

\[
\begin{array}{ccc}
Z(S)/B(S) & \rightarrow & B(S) \\
\downarrow N & & \downarrow i^* \\
Z(R)/B(R) & \rightarrow & B(R)
\end{array}
\]

is a push-out diagram, where \( i: R \rightarrow S \) is the inclusion map, where \( N \) is induced from the norm from \( G(\text{Mod}_S) \) to \( G(\text{Mod}_R) \), and where the horizontal maps are the natural reduction maps.

**Proof.** We start with the exact sequence

\[
Z(S) \xrightarrow{(N,-1)} Z(R) \oplus Z(S) \xrightarrow{1 \oplus N} Z(R) \longrightarrow 0
\]

where \( N \) is the norm map from \( Z(S) \) to \( Z(R) \). Let \( K = R \otimes \mathbb{Q} \); since \( R \) is of finite index in \( S \) we have \( K = S \otimes \mathbb{Q} \). From [2, Lem. 2.1, p. 2365] we have \( \text{Pr}_R = N \circ \text{Pr}_S \), and dividing the last two terms of the preceding sequence by the image of \( K^+ \) under \( \text{Pr}_S \) and \( \text{Pr}_R \) we get the exact sequence

\[
Z(S) \longrightarrow Z(R) \oplus (Z(S)/\text{Pr}_S(K^+)) \longrightarrow Z(R)/\text{Pr}_R(K^+) \longrightarrow 0. \quad (5)
\]

We also have an exact sequence

\[
B(S) \xrightarrow{(N,-1)} B(R) \oplus B(S) \xrightarrow{1 \oplus N} B(R) \longrightarrow 0. \quad (6)
\]
If we take the cokernel of the natural map from (6) to (5) we get the exact sequence

\[ Z(S)/B(S) \xrightarrow{(N,-\chi)} (Z(R)/B(R)) \oplus B(S) \xrightarrow{\psi \oplus i^*} B(R) \longrightarrow 0, \quad (7) \]

where \( \chi: Z(S)/B(S) \to B(S) \) and \( \psi: Z(R)/B(R) \to B(R) \) are the natural reduction maps. This sequence being exact is precisely what it means for the diagram of the proposition to be a push-out diagram.

The groups on the left-hand side of Proposition 6.4 can be replaced by finite groups. We need some notation to do this. For every proper real/CM-order \( R \) let \( H(R) = Z(R)/B(R) \). The group \( H(R) \) is a vector space over the field with two elements, and a basis for \( H(R) \) as an \( \mathbb{F}_2 \)-vector space is given by the images in \( H(R) \) of the classes of the simple \( R \)-modules of the form \( R/p \), where \( p \) is a maximal ideal of \( R \) that is fixed by the involution of \( R \) and whose residue field has even degree over the residue field of the prime of \( R^+ \) that lies under it; we call such maximal ideals _generating primes_ of \( H(R) \).

Now suppose \( R \) and \( S \) are as in Proposition 6.4. Let \( X \) denote the set of generating primes of \( H(R) \), and for each \( p \) in \( X \) let \( x_p \) be the image of the simple \( R \)-module \( R/p \) in \( H(R) \). Similarly, let \( Y \) denote the set of generating primes of \( H(S) \), and for each \( q \) in \( S \) let \( y_q \) be the image of \( S/q \) in \( H(S) \). Let \( X_{\text{good}} \) be the set of \( p \) in \( X \) such that \( S \otimes_R R/p \cong R/p \), and let \( Y_{\text{good}} \) be the set of primes \( q \) of \( Y \) such that \( S \otimes_R R/(R \cap q) \cong R/(R \cap q) \). It is not hard to see that every prime \( p \in X_{\text{good}} \) has exactly one prime \( q \) of \( S \) lying over it, that this prime of \( S \) lies in \( Y_{\text{good}} \), and that the residue fields of \( p \) and \( q \) are equal. Likewise, it is easy to see that every prime \( q \in Y_{\text{good}} \) lies over a prime \( p \in X_{\text{good}} \).

Let \( C_{\text{good}} \) and \( C_{\text{bad}} \) be the subspaces of \( H(R) \) spanned by the sets \( \{x_p \mid p \in X_{\text{good}}\} \) and \( \{x_p \mid p \in X \setminus X_{\text{good}}\} \), respectively, and let \( D_{\text{good}} \) and \( D_{\text{bad}} \) be the subspaces of \( H(S) \) spanned by the sets \( \{y_q \mid q \in Y_{\text{good}}\} \) and \( \{y_q \mid q \in Y \setminus Y_{\text{good}}\} \), respectively. It is easy to see from what we have noted above that the norm map from \( H(S) \) to \( H(R) \) induces an isomorphism between \( D_{\text{good}} \) and \( C_{\text{good}} \), and that the image of \( D_{\text{bad}} \) lies in \( C_{\text{bad}} \).

(6.5) **Proposition.** Let notation and assumptions be as in Proposition 6.4. Then

\[
\begin{array}{ccc}
D_{\text{bad}} & \longrightarrow & B(S) \\
\downarrow N & & \downarrow i^* \\
C_{\text{bad}} & \longrightarrow & B(R)
\end{array}
\]

is a push-out diagram.

**Proof.** We would like to show that the sequence

\[ D_{\text{bad}} \xrightarrow{(N,-\chi)} C_{\text{bad}} \oplus B(S) \xrightarrow{\psi \oplus i^*} B(R) \longrightarrow 0 \quad (8) \]

is exact. Consider the natural map from this sequence to the sequence (7). We have already noted that the norm from \( H(S) \) to \( H(R) \) induces an isomorphism between \( D_{\text{good}} \) and \( C_{\text{good}} \). A simple diagram chase making use of this fact shows that (8) is exact. \( \square \)

One can show that every \( p \in X \setminus X_{\text{good}} \) is singular and that every \( q \in Y \setminus Y_{\text{good}} \) lies over a singular prime of \( R \), so \( D_{\text{bad}} \) and \( C_{\text{bad}} \) are finite. This leads to the following corollary:
(6.6) Corollary. Let $R$ be a proper real/CM-order. Then $B(R)$ is a finite two-torsion group.

Proof. Let $K = R \otimes \mathbb{Q}$ and write $K$ as a product of fields $K_1 \times \cdots \times K_r$. For each $i$ let $\mathcal{O}_i$ be the ring of integers of $K_i$, and let $S$ be the product of the $\mathcal{O}_i$. It is clear that the functor $\mathcal{B}$ applied to a product of rings gives the sum of $\mathcal{B}$ applied to each factor, so using Propositions 6.1 and 6.2 we can see that $\mathcal{B}(S) = \mathcal{B}(\mathcal{O}_1) \oplus \cdots \oplus \mathcal{B}(\mathcal{O}_r)$ is a finite two-torsion group. Now apply Proposition 6.5 to $R$ and $S$. Since the upper right and lower left groups in the diagram of Proposition 6.5 are finite two-torsion groups, so is $\mathcal{B}(R)$. \qed

We end this section with two comments. Suppose $R = R_C$ for an isogeny class $C$, and let $R_{CM}$ be the image of $R$ under the projection map from $R \otimes \mathbb{Q}$ to the product of the CM-fields occurring in $R \otimes \mathbb{Q}$. One can use Proposition 6.5 to show that the natural map $\mathcal{B}(R_{CM}) \rightarrow \mathcal{B}(R)$ is an isomorphism. Also, the ring $R_{CM}$ is itself of the form $R_{C'}$ for an isogeny class $C'$. In [2, §9] it is shown how the CM-order $R_{CM}$ is determined by the characteristic polynomial $h_{C'}$. (It is assumed in [2] that the isogeny classes contain only ordinary abelian varieties, but the only way this assumption is used in [2, §9] is in the fact that the rings associated with ordinary isogeny classes are automatically CM-orders.) Thus, one can actually calculate the obstruction group for an isogeny class $C$ if one knows $h_C$. Finally, we note that although the definition of $\mathcal{B}(R)$ found in [2] is different from the one given in this paper, the two definitions agree when $R$ is a proper CM-order that is locally free of rank two over $R^+$. In particular, one can show that the definitions agree when $R$ is a CM-order of the form $R_C$.

7. Restricting the obstruction element

Let $C$ be an isogeny class of abelian varieties over a finite field. In this section we show that $I_C$ must lie in a particular subgroup of $\mathcal{B}(R_C)$.

(7.1) Proposition. Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}$ in the product of fields $K_C$ and let $i: R_C \rightarrow \mathcal{O}$ be the inclusion map. Then the obstruction element $I_C \in \mathcal{B}(R_C)$ lies in the image of $\mathcal{B}(\mathcal{O})$ under the map $i^*: \mathcal{B}(\mathcal{O}) \rightarrow \mathcal{B}(R_C)$.

Proof. As usual we let $R = R_C$. By [11, Thm. 3.13, p. 534], there is a variety $A$ in $C$ such that $\mathcal{O} \subset \operatorname{End}(A)$. Let $\lambda$ be any polarization of $A$ and let $X = \ker \lambda$. We write $X_{\ell \ell}$ for the local-local part of $X$ and $Y$ for the non-local-local part of $X$. Theorem 1.3 shows that $I_C$ is the image of $X = X_{\ell \ell} \times Y$ in $\mathcal{B}(R)$.

We showed in step one of the proof of Theorem 1.3 that the image of $X_{\ell \ell}$ in $G(Ker_C)$ is an even multiple of the simple group scheme $\mathfrak{a}_p$. Therefore the image of $X_{\ell \ell}$ in $G(\operatorname{Mod}_R)$ is an even multiple of the simple $R$-module $\mathbb{Z}/p\mathbb{Z}$ on which $F$ and $V$ act as zero. Thus the image of $X_{\ell \ell}$ in $G(\operatorname{Mod}_R)$ is in $\mathcal{B}(R)$, so the image of $X_{\ell \ell}$ in $\mathcal{B}(R)$ is zero. We are left to consider $Y$.

Let $n$ be any positive integer such that $R \supset n\mathcal{O}$ and let $\mu$ be the polarization $n\lambda$ of $A$. Let $\mathcal{O}$ act on $\hat{A}$ by sending $a \in \mathcal{O} \subset \operatorname{End} A$ to the endomorphism $\hat{a}'$ of $\hat{A}$, where $'$ is the Rosati involution of $\operatorname{End} A$ associated to $\lambda$. Then the definition of the Rosati involution shows that $\lambda$ and $\mu$ are $\mathcal{O}$-equivariant, so $\mathcal{O}$ acts on $X$ and on the kernel $X'$ of $\mu$. Let $e$
and \( e' \) be the non-degenerate alternating pairings from \( X \times X \) and \( X' \times X' \) to \( G_m \) defined in \([3, \S23]\). The pairings \( e \) and \( e' \) are defined over \( k \), so for every \( \alpha \in R \), for every \( k \)-algebra \( S \), and for every pair of \( S \)-valued points \( x \) and \( y \) of \( X' \) we have \( e'(\alpha x, y) = e'(x, \alpha y) \); we express this fact by saying that the pairing \( e' \) is \( R \)-semi-balanced. We also know that for every \( k \)-algebra \( S \), if \( x \in X(S) \) and \( y \in X'(S) \) then \( e'(x, y) = e(x, ny) \); this is \([3, \text{item 4, p. 228}]\). Using this relation between the two pairings and the fact that \( e \) is \( R \)-semi-balanced, one can show that \( e(\alpha x, y) = e(x, \alpha y) \); this is \([3, \S23]\). The pairings \( e \) are inert in \( \mathcal{O} \) and for every \( \alpha \in \mathcal{O} \), for every \( k \)-algebra \( S \), and for every pair of points \( x, y \in X(S) \), we have \( e(ax, y) = e(x, \alpha y) \).

The ring \( \mathcal{O} \) acts on \( Y \), and since there are no non-zero morphisms between \( Y \) and \( X_{\ell \ell} \), the alternating \( \mathcal{O} \)-semi-balanced pairing \( e \) restricted to \( Y \times Y \) is still non-degenerate. Let \( M = \mathcal{P}(Y) \), where \( \mathcal{P} \) is the functor from \( \text{Ker}_C \) to \( \text{Mod}_R \) that was defined in \( \S3 \). Since \( \mathcal{O} \) acts on \( Y \), the \( R \)-module \( M \) is actually an \( \mathcal{O} \)-module, and our pairing from \( Y \times Y \) to \( G_m \) gives us a non-degenerate alternating \( \mathcal{O} \)-semi-balanced pairing from \( M \times M \) to \( Q/\mathbb{Z} \). As in the similar situation in step one of the proof of Theorem 1.3, this tells us that \( [M]_\mathcal{O} \) is a symmetric element of \( G(\text{Mod}_R) \), and Lemma 5.1 tells us that the image of \( M \) in \( G(\text{Mod}_{\mathcal{O}^+}) \otimes (\mathbb{Z}/2\mathbb{Z}) \) is zero. Thus \( [M]_\mathcal{O} \) is in \( Z(\mathcal{O}) \). The natural map from \( Z(\mathcal{O}) \) to \( B(R) \) factors through \( B(\mathcal{O}) \), so the image of \( [M]_R \) in \( B(R) \), which is the image of \( [M]_\mathcal{O} \) in \( B(\mathcal{O}) \). This proves the proposition.

The following proposition, which should be compared to \([2, \text{Prop. 11.3, p. 239}]\), gives sufficient conditions for \( i^* \) to be the zero map in the case where \( \mathcal{C} \) is simple.

(7.2) Proposition. Let \( \mathcal{C} \) be an isogeny class of simple \( g \)-dimensional abelian varieties over a finite field \( k \), let \( K = K_\mathcal{C} \), let \( \mathcal{O} \) be the ring of integers of \( K \), and let \( i: R_\mathcal{C} \rightarrow \mathcal{O} \) be the inclusion map. If \( K \) is totally real then \( i^* \) is the zero map. Suppose \( K \) is a CM-field. If \( K/K^+ \) is ramified at a finite prime, or if there is a prime of \( K^+ \) that divides \( (F - V) \) and that is inert in \( K/K^+ \), then \( i^* \) is the zero map. Otherwise, \( i^* \) is not the zero map and \( g \) is even.

Proof. If \( K \) is totally real then \( B(R) \cong 0 \) by Proposition 6.1, and there is nothing more to say. So suppose \( K \) is a CM-field. If \( K/K^+ \) is ramified at a finite prime then \( B(\mathcal{O}) \cong 0 \) by Proposition 6.2 and \( i^* \) must be the zero map. We are left with the case where \( K/K^+ \) is unramified at all finite primes; then Proposition 6.2 shows that \( B(\mathcal{O}) \) has two elements, so \( i^* \) is the zero map precisely when it kills the non-zero element of \( B(\mathcal{O}) \). We will use Proposition 6.5 to determine when this is the case.

Let us adopt the notation of Proposition 6.5 and apply the proposition with \( S = \mathcal{O} \). Let \( I \) be the non-zero element of \( B(\mathcal{O}) \). The generating primes of \( H(\mathcal{O}) \) are the primes of \( \mathcal{O} \) that are inert in \( K/K^+ \), and Proposition 6.2 tells us that for every prime \( q \) of \( \mathcal{O} \) that is inert in \( K/K^+ \), the image of \( y_q \) in \( B(\mathcal{O}) \) is \( I \). Now, Proposition 6.5 shows that \( i^*(I) = 0 \) if and only if there is a \( y \in D_{\text{bad}} \) that maps to \( I \) in \( B(\mathcal{O}) \) and to zero in \( C_{\text{bad}} \); for the moment let us call such a \( y \) an annihilating element. Since the basis elements \( y_q \) of \( D_{\text{bad}} \) all map to \( I \) in \( B(\mathcal{O}) \), and since the image of \( y_q \) in \( C_{\text{bad}} \) is either 0 or a basis element \( x_p \), we see that there is an annihilating \( y \in D_{\text{bad}} \) if and only if there is an annihilating \( y \) of the form \( y = y_q \). One can check that an element \( y_q \) maps to zero in \( C_{\text{bad}} \) if and only if the residue field of \( q \) has even degree over the residue field of the prime \( p = q \cap R \) of \( R \),
and this last condition is equivalent to the condition that complex conjugation on $\mathcal{O}/q$ act trivially on $R/p \subset \mathcal{O}/q$. Since $R = \mathbb{Z}[F,V]$, this will be the case if and only if $F$ and its complex conjugate $V$ are congruent to one another modulo $q$, if and only if $q$ divides $F - V$. Thus, $i^*$ is the zero map if and only if there is a prime of $K^+$ that is inert in $K/K^+$ and that divides $F - V$. Finally, if $i^*$ is not the zero map then Corollary 6.3 shows that $[K^+:\mathbb{Q}]$ is even, and since $g$ is a multiple of this degree, $g$ is even also. \hfill \Box

We close this section by noting that Theorems 1.1 and 1.2 follow immediately from Proposition 7.2, Proposition 7.1, and Theorem 1.3.

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