Growth of aqueous foam on flexible membranes

Hiroyuki Shima

Department of Applied Physics, Graduate School of Engineering, Hokkaido University, Sapporo 060-8628, Japan
(Dated: May 30, 2009)

In this paper, I study the coarsening dynamics of two-dimensional dry foam sandwiched by deformable membranes. The time-varying deformation of the confining membranes gives rise to a significant alteration in the evolution of polygonal cells of bubbles when compared to the case of rigid membranes. This alteration is attributed to the correlation between the rate of inter-cell gas transfer and temporal fluctuation in surface curvature within a cell domain. The existing material constants are referred to understand the utility of the correlation effect toward the artificial control of the coarsening dynamics.

PACS numbers: 82.70.Rr, 81.10.Aj, 47.57.Bc, 02.40.-k

I. INTRODUCTION

Aqueous foam exhibits a good interplay between geometry and physics. With time, foam consisting of polyhedral bubbles evolves into the equilibrium structure, during which internal gas diffuses from a bubble to others through thin curved liquid interfaces [1, 2, 3, 4]. Diffusion is driven by the pressure difference between bubbles; assuming the constant diffusion coefficient, the pressure difference in two adjacent bubbles is proportional to the geometric curvature of their common boundary interface. Each boundary moves toward its concave sides due to the inter-bubble gas transfer, where the velocity of the boundary motion is again proportional to curvature [5]. As a result, some bubbles dilate while others shrink and eventually disappear, which results in a progressive increase in the average bubble size, i.e., the coarsening of foam.

Foam that we encounter in our daily life, such as shaving cream and beer head, consists of a three-dimensional agglomerate. Its coarsening dynamics [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] as well as equilibrium cellular structures [22, 24, 26] have required formidable efforts for clarification because of geometrical and topological complexity. This is partly the reason why a large degree of attention has been paid to two-dimensional counterparts, i.e., a foam monolayer confined between two membranes [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] as well as equilibrium liquid interfaces [1, 2, 3, 4]. Diffusion through thin curved liquid interfaces [1, 2, 3, 4] causes a topological change in the network of liquid interfaces.

The coarsening behavior on the flat plane alters drastically when the foam is constrained to a curved surface [22, 10]. In the latter case, the evolution of cells is characterized by the Gaussian curvature $K$ of the underlying surface. When the surface has a positive (negative) curvature, $n$-sided cells with $n < 6$ ($n > 6$) can be stationary, yielding $dS/dt = 0$, only if $S$ equals to a specific value that depends on $n$ and $K$ (see Eq. (10) below). Furthermore, the stability of those stationary cells is sensitive to the sign of $K$. For instance, no cell on a positively curved surface is stable; once a cell grows (shrinks) slightly under perturbation, then it keeps growing (shrinking). In contrast, all stationary cells on a negatively curved surface are stable; therefore, the equilibrium configuration consists of various $n$-sided cells each having a specific area determined by $n(> 6)$ and $K$. Such two-dimensional foam spreading over a curved surface could be realized on an elastic confining plate or on a phase boundary with another fluid medium that repels the foam.

The present article provides a further generalization of the coarsening on curved surfaces, i.e., the coarsening of foam constrained on a flexible surface exhibiting time-varying deformation. The successive deformation of the confining surface changes the pressure difference of adjacent cells that drives gas transfer across liquid interfaces. As a consequence of the correlation between surface deformation and inter-cell gas transfer, the growth rate of the cell shows an intrinsic difference from that of the rigid curved surface. Realistic material parameters are employed to prove that the correlation effect plays a dominant role in the actual coarsening dynamics on flexible membranes under appropriate physical conditions.

II. COARSENING ON A RIGID CURVED MEMBRANE

This section briefly reviews the coarsening theory of two-dimensional dry foam on a rigid curved surface. The term “dry” refers to the assumption that liquid films between adjacent cells are so thin that they can be treated as curves with no thickness and the vertices can be treated as points. This assumption has long succeeded...
in exploring the nature of coarsening dynamics of foam, while considering that the realistic shapes of films with finite thickness \([41][42][43][44]\) and their effects on permeability \([45]\) may encourage quantitative agreements with experiments; we shall revisit this point in Section V.

Let us assume that the monolayer foam is confined between two rigid membranes with spatially uniform Gaussian curvature \(K\). The gap between the membranes is smaller than the typical length of boundary curves. The growth-rate of the area of an \(n\)-sided cell with internal pressure \(p\) is described by

\[
\frac{dS}{dt} = -\gamma \sum_{j=1}^{n} \Delta p_j \ell_j, \tag{1}
\]

where \(\Delta p_j \equiv p - p_j\) is the pressure difference between the cell and its \(j\)th neighbor, \(\ell_j\) is the length of the \(j\)th boundary curve separating the two cells, and \(\gamma > 0\) is a diffusion constant. Equation (1) captures the simple experiments; we shall revisit this point in Section V.

Finite thickness \([41, 42, 43, 44]\) and their effects on permeability \([45]\) against perturbation. Therefore, we observe that the realistic shapes of films with curvature \(K\) change slightly larger than \(K\) due to perturbation, then the quantity in the square brackets in Eq. (1) becomes positive. Therefore, we obtain \(dS/dt > 0\) after the perturbation, which signifies a persistent growth in the cell. On the contrary, all stationary cells are stable for \(K < 0\), since \(S\) being larger (smaller) than \(S^*\) makes \(dS/dt\) negative (positive). In this context, the case of a flat plane is marginal, in which the stationary cell of \(n = 6\) is neither stable nor unstable against perturbation.

Let us assume that the Gaussian curvature \(K\) of the confining membrane is spatially uniform and varies continuously from \(K = K_0\) at \(t = 0\) to \(K_1 = K_0 + \delta K\) at \(t = \delta t\). The increment \(\delta S\) of the cell area obtained at \(t = \delta t\) is the sum of the two contributions:

\[
\delta S = \delta S_{\text{dif}} + \delta S_{\text{cur}}, \tag{6}
\]

where

\[
\delta S_{\text{dif}} = \frac{dS_{\text{dif}}}{dt} \bigg|_0 \delta t, \tag{7}
\]

and

\[
\delta S_{\text{cur}} = \frac{dS_{\text{cur}}}{dK} \bigg|_{K_0} \delta K, \tag{8}
\]

with \(\delta K = (dK/dt) \delta t\). The subscripts 0 and \(K_0\) imply differentiations at \(t = 0\) and \(K = K_0\), respectively. \(dS_{\text{dif}}/dt\) in Eq. (7) is identified with \(dS/dt\) appearing in Eq. (1), since both describe the growth rate before the deformation occurs. Equation (8) has an explicit form that is given as

\[
\delta S_{\text{cur}} = \frac{S_0^2}{2(2\pi - K_0 S_0)} \delta K, \tag{9}
\]

and it shall be proved in Appendix A. From Eqs. (6)–(9), the area \(S_1 = S_0 + \delta S\) at \(t = \delta t\) is readily evaluated.

Next, we consider the area growth within the time duration \([\delta t, 2\delta t]\). The increment \(\delta S_{\text{cur}}\) in this interval is obtained by replacing \(K_0\) and \(S_0\) in Eq. (9) by \(K_1\) and \(S_1\), respectively. On the other hand, some caution is required in deriving the form of \(\delta S_{\text{dif}}\) at this stage because of the correlation between internal pressure and membrane deformation. The pressure \(p\) of a given cell after deformation is expressed by

\[
p_1 = p_0 + \delta p_{\text{dif}} + \delta p_{\text{cur}}, \tag{10}
\]
where \( \delta p_{\text{diff}} \) is the pressure increment that would be obtained provided \( \delta K = 0 \), and \( \delta p_{\text{cur}} \) is the one provided no diffusion occurs during deformation. We denote by \( p' \) and \( S^j \) the counterparts of the \( j \)th neighbor, both of which obey the similar expressions of \( p \) and \( S \), i.e.,
\[
p'_j = p'_0 + \delta p_{\text{diff}}^{j} + \delta p_{\text{cur}}^{j}, \quad S^j_1 = S^0_0 + \delta S_{\text{diff}}^{j} + \delta S_{\text{cur}}^{j}.
\]
(11)
To analyze the deformation effect on the pressure difference \( \Delta p_j = p_1 - p'_1 \), we take notice of the fact that the number of gas molecules in the cell is preserved during deformation if no diffusion occurs. This conservation law is formally represented by \( p_0 + \delta p_{\text{cur}} = p_0 S_0 \), or equivalently
\[
p_0 - p'_0 = \sigma \kappa_0^j,
\]
(12)
which correlates \( \delta p_{\text{cur}} \) to \( \delta S_{\text{cur}} \). We also see from the Laplace-Young law that
\[
(p_0 + \delta p_{\text{cur}}) \cdot (S_0 + \delta S_{\text{cur}}) = p_0 S_0,
\]
(13)
where \( \kappa_0^j \) is the geodesic curvatures of the \( j \)th boundary observed at \( t = 0 \), and \( \kappa_{\text{st}}^j \) is the fictitious one that would be observed if \( \delta K = 0 \) during \([0, \delta t] \). From Eqs. (12)–(13), it follows that
\[
\Delta p_j = \sigma \kappa_0^j + \left[ \frac{\delta S_{\text{cur}}}{S_0} p_0 - \frac{\delta S_{\text{diff}}}{S_0^2} \left( p_0 - \sigma \kappa_0^j \right) \right].
\]
(14)
Substituting Eq. (14) in Eq. (1) and applying Gauss-Bonnet’s theorem, we obtain
\[
\frac{d S_{\text{diff}}}{d t} \bigg|_{\delta t} = \gamma \sigma \left[ \frac{\pi}{3} (n - 6) + K_0 S_0 + \delta K S_0 + K_0 \delta S_{\text{diff}} \right]
\]
\[
- \gamma \frac{\delta S_{\text{cur}}}{S_0} p_0 \sum_{j=1}^{n} \ell_j + \gamma \sum_{j=1}^{n} \frac{\delta S_{\text{cur}}^j}{S_0^2} \left( p_0 - \sigma \kappa_0^j \right) \ell_j,
\]
(15)
where the second-order terms with respect to increments were neglected. It is to be noted that all the increments in the right side are those obtained in the previous interval \([0, \delta t] \).

Equation (15) determines the diffusion-induced increment \( \delta S_{\text{diff}} = d S_{\text{diff}} / d t |_{\delta t} \) obtained at \( t = 2 \delta t \). The area growth for larger \( t \) can be evaluated by successively applying the procedure shown above. After deducing \( S_2 \) at \( t = 2 \delta t \), for instance, we rewrite the set \( \{ S_2, K_2, S_1, K_1, p_1 \} \) by \( \{ S_1, K_1, S_0, K_0, p_0 \} \) to calculate \( S_2 \) again, which provides the subsequent value of the area at \( t = 3 \delta t \). An explicit algorithm of pursuing the time-varying \( p \) is given in Appendix B.

It should be emphasized that in Eq. (15), the deformation effect manifests in the product \( \delta K S_0 \) in the square brackets and the two summations with respect to \( j \). In particular, the presence of the last two summations indicates that the diffusion-induced growth rate of a cell becomes dependent on the local environment around the cell. This situation is in contrast to the case of a rigid membrane described by Eq. (4), where the growth rate is determined only by the properties of the cell itself.

IV. DEFORMATION EFFECT ESTIMATION

To estimate the deformation effect in Eq. (15), we replace the fractions \( \delta S_{\text{cur}}^j / S_0^j \) by its mean value over \( n \) adjacent cells and the sum of edge lengths \( \sum \ell_j \) by the perimeter \( \mathcal{L} \) of an effective circular domain whose area equals to the original polygonal cell area \( S \). It follows that \( \mathcal{L} = t = \delta t \) is represented by
\[
\mathcal{L} = \sqrt{4 \pi S_1 - K_1 (S_1)^2},
\]
(16)
which will be derived in Appendix A (see Eq. (A4)). Straightforward calculation yields
\[
\frac{d S_{\text{diff}}}{d t} \bigg|_{\delta t} = \gamma \sigma \left[ \frac{\pi}{3} (n - 6) + K_0 S_0 \right] (1 - \beta) \]
\[
+ \gamma \sigma (\delta K S_0 + K_0 \delta S_{\text{diff}}) + \gamma p_0 (\beta - \alpha) \mathcal{L},
\]
(17)
where
\[
\alpha = \frac{\delta S_{\text{cur}}}{S_0}, \quad \beta = \frac{1}{n} \sum_{j=1}^{n} \frac{\delta S_{\text{cur}}^j}{S_0^j}.
\]
(18)
To proceed with the arguments, we consider sub-millimeter-scale bubbles of \( S \sim 0.1 \text{ mm}^2 \) under slightly time-varying curvature of \( \delta K \sim 10^{-3} \text{ mm}^{-2} \) per second with \( K = 1 \text{ mm}^{-2} \) at \( t = 0 \); these conditions are in the realm of laboratory experiments. Then, we have \( \beta \sim \delta S_{\text{cur}} / S_0 \ll 1 \), as a result of which Eq. (17) is simplified as
\[
\frac{d S_{\text{diff}}}{d t} \bigg|_{\delta t} = \gamma \sigma \left[ \frac{\pi}{3} (n - 6) + K(t) (S_0 + \delta S_{\text{diff}}) \right]
\]
\[
+ \gamma p_0 (\beta - \alpha) \mathcal{L}.
\]
(19)
The most important deviation of Eq. (19) from Eq. (4), i.e., from the growth rate equation for rigid membrane cases, is the presence of the term \( \gamma p_0 (\beta - \alpha) \mathcal{L} \). In fact, this term relates \( \delta S_{\text{cur}} \) obtained in the previous time interval, say, \([0, \delta t] \), to \( \delta S_{\text{diff}} \) obtained in the subsequent interval, \([\delta t, 2 \delta t] \). When the confining membranes are rigid, then this term vanishes since \( \alpha \propto \delta S_{\text{cur}} \propto \delta K = 0 \), and so does \( \beta \).

The salient finding of Eq. (19) is the fact that by referring realistic material constants such as \( p_0 = 10^5 \text{ Pa}, \sigma = 10^3 \text{ N/m}, \) and \( \gamma = 10^{-9} \text{ m/(Pa-sec)} \) \[44, 49\], we obtain
\[
\gamma p_0 (\beta - \alpha) \mathcal{L} \sim 10^{-5} \text{ mm}^2/\text{sec},
\]
(20)
and
\[
\gamma \sigma = 10^{-6} \text{ mm}^2/\text{sec}.
\]
(21)
Therefore, the correlation-related term given in (20) may be larger than (or comparable to, at least) the coefficient \( \gamma \sigma \) under the present physical conditions. This means that the term should be dominant in rate equation (19),
V. SUMMARY AND PERSPECTIVES

Our results are based on the assumptions that all cell boundaries have invariant material constants $\gamma$ and $\sigma$, and the shape of each $j$th boundary curve is described by a constant $\kappa_j$. In realistic foam, these parameters are determined by the nature of liquid films that retain three-dimensional geometry across the gap containing the foam. In other words, each cell boundary is not a truly one-dimensional curve, but a three-dimensional film with finite thickness whose value mainly depends spatially within the film. Therefore, membrane deformation will induce spatial fluctuations of $\gamma$ and $\sigma$ over the foam and that of $\kappa_j$ in each $j$th film. The consideration of these fluctuations may enhance the quantitative precision of the coarsening theory we have developed.

It is interesting to point out that membrane deformation may cause a flow of liquid through the films, as analogous to foam drainage in response to gravity and capillarity. In fact, the deformation of the confining membranes leads to the rearrangement of the liquid film network as well as cell configuration, and thus, inducing pressure gradient in the liquid. Recently, it was shown that fluid flow on a surface with time-varying surface curvature exhibits three kinds of dynamical responses depending on geometric and material constants. This article can provide a starting point for the analysis of the coupling between the membrane deformation and coarsening dynamics in two-dimensional cellular structures.

Acknowledgments

Illuminating discussions with Kousuke Yakubo, Akira Shudo, and Satoshi Tanda are greatly acknowledged. This study was supported by a Grant-in-Aid for Scientific Research from MEXT, Japan, and by Executive Office of Research Strategy in Hokkaido University.

APPENDIX A: DEFORMATION-INDUCED INCREMENT OF CELL AREA

In this Appendix, we derive Eq. (9), which is the expression of the deformation-induced increment $\delta S_{\text{cur}}$ of a cell area. For a general curved surface, the Gaussian curvature of a point on the surface is defined by

$$K = \lim_{r \to 0} \frac{3 [2\pi r - \mathcal{L}(r)]}{\pi r^3},$$

where $r$ is the radius of a geodesic circle around the point and $\mathcal{L}(r)$ is the lenght of its perimeter. For a spherical surface with radius $R$, for instance, it follows from Fig. 1 that $\mathcal{L}(r) = 2\pi a$, $a = R \sin \theta$ and $r = R \theta$. Therefore, we obtain $K = 1/R^2 > 0$ and

$$\mathcal{L}(r) = \frac{2\pi}{\sqrt{K}} \sin \left( \sqrt{K} r \right).$$

Equation (A2) holds not only when $K > 0$ but $K \leq 0$, as far as $K$ is constant with the circular region.
Now we consider the area \( S \) of the circular region. It is given by \( S = \int_0^\infty L(r^2, K) \, dr^2 \), and thus
\[
S = \frac{2\pi}{K} \left[ 1 - \cos \left( \sqrt{K} r \right) \right].
\]
(A3)

Eliminating \( r \) from Eqs. (A2) and (A3) yields
\[
S(K, L) = \frac{2\pi - \sqrt{4\pi^2 - KL^2}}{K},
\]
(A4)

which converges to \( L^2/(4\pi) \) in the limit of \( K \to 0 \). Finally, we obtain
\[
\frac{\partial S}{\partial K} \bigg|_L = -\frac{2\pi}{K^2} + \frac{\sqrt{4\pi^2 - KL^2}}{K^2} - \frac{L^2}{2K\sqrt{4\pi^2 - KL^2}},
\]
and equivalently,
\[
\frac{\partial S}{\partial K} \bigg|_L = \frac{S^2}{2(2\pi - KS)},
\]
(A6)

which completes the proof.

APPENDIX B: SUCCESSIVE RELATION FOR INTERNAL PRESSURE

The internal pressure \( p_\mu \) at \( t = \mu \delta t \) (\( \mu \geq 1 \)) is given by the following procedure. First, \( \delta p_{\text{cur}} \) is deduced from Eq. (12) as
\[
\delta p_{\text{cur}} = -p_{\mu-1} \frac{\delta S_{\text{cur}}}{S_{\mu-1}}.
\]
(B1)

Next, \( \delta p_{\text{diff}} \) is derived by substituting Eq. (13) into the Gauss-Bonnet theorem, which leads to
\[
-\delta p_{\text{diff}} L_\mu + \sum_{j=1}^n \left( p_0^j \delta \ell^j + \delta p_{\text{diff}}^j \ell^j_1 \right) = \sigma K_{\mu-1} \delta D_{\mu-1},
\]
(B2)

where \( \delta \ell^j = \ell^j_1 - \ell^j_0 \). The summations in Eq. (B2) are canceled out if the sign of the summed terms is positive or negative depending on \( j \). As a result, we obtain
\[
\delta p_{\text{diff}} = -\sigma K_{\mu-1} \frac{\delta D_{\mu-1}}{L_\mu},
\]
(B3)

which gives \( p_\mu = p_{\mu-1} + \delta p_{\text{diff}} + \delta p_{\text{cur}} \) for all \( \mu \geq 1 \) through the successive relations.
[35] J. von Neumann, in *Metal Interfaces* (American Society for Metals, Cleveland, 1952).
[36] M. Durand and H. A. Stone, Phys. Rev. Lett. 97, 226101 (2006).
[37] S. J. Cox, G. Graner, and M. F. Vaz, Soft Matter 4, 1871 (2008).
[38] J. P. Raven and P. Marmottant, Phys. Rev. Lett. 102, 084501 (2009).
[39] J. E. Avron and D. Levine, Phys. Rev. Lett. 69, 208 (1992).
[40] P. Peczak, G. S. Grest, and D. Levine, Phys. Rev. E 48, 4470 (1993).
[41] E. Terriac, J. Etrillard and I. Cantat, Europhys. Lett. 74, 909 (2006).
[42] A. Eri and K. Okumura, Phys. Rev. E 76, 060601(R) (2007).
[43] P. Grassia, G. Montes-Atenas, L. Lue and T. G. Green, Eur. Phys. J. E 25, 39 (2008).
[44] J. Marchalot, J. Lambert, I. Cantat, P. Tabeling, and M. C. Juilien, EPL 83, 64006 (2008).
[45] E. Lorenceau, N. Louvet, F. Rouyer, and O. Pitois, Eur. Phys. J. E 28, 293 (2009).
[46] R. S. Millman and G. D. Parker, *Elements of Differential Geometry* (Prentice-Hall, Englewood Cliffs, NJ, 1977).
[47] R. D. Kamien Rev. Mod. Phys. 74, 953 (2002).
[48] J. Plateau, *Statique Expérimentale et Théorique des Liquides Soumis aux Seules Forces Moléculaires* (Gauthier-Villars, Paris, 1873).
[49] H. M. Princen, J. Th. G. Overbeek, and S. G. Mason, J. Coll. Int. Sci. 24, 125 (1967).
[50] S. Hutzler and D. Weaire, Phil. Mag. Lett. 80, 419 (2000).
[51] M. U. Vera and D. J. Durian, Phys. Rev. Lett. 88, 088304 (2002).
[52] A. Saint-Jalmes, Soft Matter 2, 836 (2006).
[53] K. Feitosa and D. J. Durian, Eur. Phys. J. E 26, 309 (2008).
[54] M. Arroyo and A. DeSimone, Phys. Rev. E 79, 031915 (2009).