Abstract

This paper considers a nested stochastic distributed optimization problem. In it, approximate solutions to realizations of the inner-problem are leveraged to obtain a Distributed Stochastic Cubic Regularized Newton (DiSCRN) update to the decision variable of the outer problem. We provide an example involving electric vehicle users with various preferences which demonstrates that this model is appropriate and sufficiently complex for a variety of data-driven multi-agent settings, in contrast to non-nested models. The main two contributions of the paper are: (i) development of local stopping criterion for solving the inner optimization problem which guarantees sufficient accuracy for the outer-problem update, and (ii) development of the novel DiSCRN algorithm for solving the outer-problem and a theoretical justification of its efficacy. Simulations demonstrate that this approach is more stable and converges faster than standard gradient and Newton outer-problem updates in a highly nonconvex scenario, and we also demonstrate that the method extends to an EV charging scenario in which resistive battery losses and a time-of-use pricing model are considered over a time horizon.

1. Introduction

Motivation. As applications emerge which are high dimensional and described by large data sets, the need for powerful optimization tools has never been greater. In particular, agents in distributed settings are commonly given a global optimization task where they must sparingly exchange local information with a small set of neighboring agents for the sake of privacy and robust scalability. This architecture can, however, slow down convergence compared to centralized ones, which is concerning if obtaining the iterative update information is costly. Gradient-based methods are commonly used due to their simplicity, but they tend to be vulnerable to slow convergence around saddle points. Newton-based methods use second-derivative information to improve...
convergence, but they are still liable to be slow in areas where higher order terms dominate the objective function and even unstable when the Hessian is ill conditioned. A powerful tool for combating these Newton-based vulnerabilities is imposing a cubic regularization on the function’s second-order Taylor approximation, but the current work on this technique does not unify distributed, stochastic, and nonconvex elements. Motivated by this, we study the adaptation of the Stochastic Cubic Regularized Newton approach to solve a distributed nested optimization problem.

Literature Review. One of the most widely used stochastic optimization method is stochastic gradient-based (first-order) methods, see [16, 5, 6] as broad references. These methods are powerful because they necessitate only a small sampling of the data set to compute an update direction at each iterate. However, these first-order algorithms suffer from slow convergence around saddle-points [15], which are disproportionately more present in higher-dimensional nonconvex problems [14]. By contrast, higher-order Newton-based methods tend to perform more strongly across applications in terms of number of calls to an oracle or total iterations, see [23, 22] for examples in stochastic non-strongly convex and nonconvex settings, respectively, and [17, 3, 20] for various multi-agent examples.

An issue with many of the aforementioned algorithms is they are vulnerable to slow convergence or instability in the presence of saddle-points and/or an ill-conditioned Hessian matrix. A growing body of works thus focuses on using a cubic-regularization term in the second-order Taylor approximation of the objective function. Nesterov and Polyak laid significant groundwork for this method in [18], and substantial follow-ups are contained in [10, 11], which study adaptive batch sizes and the effect of inexactness in the cubic submodel on convergence. Excitement about this topic has grown substantially in the last few years, with [9] showing how the global optimizer of the nonconvex cubic submodel can be obtained under certain initializations of gradient descent, and [10] being one of the first thorough analyses of the algorithm in the traditional stochastic optimization setting. In [12], the authors consider the stochastic setting from an adaptive batch-size perspective and [21] is, to our knowledge, the only existing work in a distributed application, with an alternative approach that allows for a communication complexity analysis. Both [12] and [21] assume convexity, and [21] is nonstochastic. As far as we know, no current work has unified distributed, stochastic, and nonconvex elements, particularly in a nested optimization scenario.

Statement of Contributions. We begin the paper by formulating a nested distributed stochastic optimization problem, where approximate solutions to realizations of the inner-problem are needed to obtain iterative updates to the outer problem, and we motivate this model with an example based on electric vehicle charging preferences. The contributions of this paper are then twofold. First, we develop a stopping criterion for a Laplacian-gradient subsolver of the inner-problem. The stopping criterion can be validated locally by each agent in the network, and the relationship to solution accuracy aids the synthesis with the outer-problem update. Second, to that end, we formulate a distributed
optimization model of the stochastic outer problem and develop a cubic regularization of its second-order approximation. This formulation lends itself to obtaining a Distributed Stochastic Cubic-Regularized Newton (DiSCRN) algorithm, and we provide theoretical justification of its convergence.

2. Preliminaries

This section establishes notation and background concepts to be used throughout the paper.

First, we provide a brief background on the Cubic-Regularized Newton method. See [18] and [10, 11] for more information. Consider the problem of minimizing a (possibly nonconvex) function $f : \mathbb{R}^d \to \mathbb{R}$:

$$\min_{x \in \mathbb{R}^d} f(x).$$

One useful iterative model for minimizing $f(x^k)$ when the function is strictly convex at the current iterate $x^k$ (or, more accurately, if it is strictly convex on some neighborhood of $x^k$) is descent on a second-order Taylor expansion around $x^k$:

$$x^{k+1} = \arg\min_x \left\{ f(x^k) + (x - x^k) \top \nabla f(x^k) + \frac{1}{2} (x - x^k) \top \nabla^2 f(x^k)(x - x^k) \right\} = x^k - \nabla f(x^k) \nabla f(x^k).$$

This closed form expression for $x^{k+1}$ breaks down when $f$ is nonconvex due to some eigenvalues of $\nabla^2 f(x^k)$ having negative sign. Further, when $\nabla^2 f(x^k)$ is nearly-singular, the update becomes very large in magnitude and can lead to instability. For this reason, consider amending the second-order model with a cubic-regularization term, to obtain the cubic-regularized, third-order model of $f$ at $x^k$ as:

$$m_k(x) \triangleq \left\{ f(x^k) + (x - x^k) \top \nabla f(x^k) + \frac{1}{2} (x - x^k) \top \nabla^2 f(x^k)(x - x^k) + \frac{\rho}{6} \|x - x^k\|^3 \right\}.$$
Here, $\rho$ is commonly taken to be the Lipschitz constant of $\nabla^2 f(x)$, which we will formalize in Section 3. The update is naturally given by a minimizer to this model: $x^{k+1} = \arg\min_x m_k(x)$. Unfortunately, this model does not beget a closed-form minimizer as in (2), nor is it convex if $f$ is not convex. The model does, however, become convex for $x$ very far from $x^k$, which can be seen by computing the Hessian of $m_k$ as $\nabla^2 m_k(x) = \nabla^2 f(x^k) + \rho\|x - x^k\| I_n$. Additionally, $m_k$ is an over-estimator for $f$, i.e. $m_k(x) \geq f(x)$, $\forall x$. This is seen by considering the cubic term and recalling Lipschitz properties of $\nabla^2 f$; we describe this observation in more detail later in the paper. Therefore, $m_k$ possesses some advantages over other simpler submodels as it possesses properties of a more standard Newton-based, second-order model while being sufficiently conservative.

Finally, [9] recently showed that simply initializing $x = x^k - r \nabla f(x^k) / \|\nabla f(x^k)\|$ for $r \geq 0$ is sufficient to show that gradient descent on $m_k$ converges to the global minimizer of (3) (under light conditions on $r$ and the gradient step size).

3. Problem Formulation

This section details the two problem formulations which are of interest, where the first problem $P_1$ takes the form of a stochastic approximation whose cost is a parameterization of the cost of the second problem $P_2$. Problem $P_2$ is a separable resource allocation problem in which $n$ agents $i \in N$ must collectively obtain a solution that satisfies a linear equality constraint while minimizing the sum of their local costs. (This problem commonly appears in real-time optimal dispatch for electric grids with flexible loads and distributed generators, see e.g. [1].) Thus, $P_1$ can be treated as a nested optimization, with an objective $F$ that takes stochastic arguments, and is not necessarily available in closed form if $P_2$ cannot be solved directly and/or the distribution $\mathcal{D}$ being unknown. These problems are stated as

$$P_1 : \min_{x \in \mathbb{R}^d} F(x) = \mathbb{E}_{\chi \sim \mathcal{D}} [F_\chi(x)].$$

$$P_2 : \min_{p \in \mathbb{R}^n} f(x, p) = \sum_{i=1}^n f_i(x, p_i),$$

subject to $\sum_{i=1}^n p_i = P_{\text{ref}} + \hat{x} = P_{\text{ref}} + \sum_{i=1}^n \hat{x}_i$.

In $P_1$, each $f_i : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, and $F_\chi(x) \equiv f(x, p^*)$, where $p^*$ is the solution to $P_2$ for particular realizations $\hat{x}_i$, where $\chi_i \sim \mathcal{D}_i$, i.e. $\chi \sim \mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$. The elements $p_i \in \mathbb{R}$ of $p \in \mathbb{R}^n$ and terms $\hat{x}_i$ are each associated with and locally known by agents $i \in N$, and $P_{\text{ref}} \in \mathbb{R}$ is a given constant known by a subset of agents (we discuss its interpretation shortly with an example). First, for $F_\chi$
to be well defined, it helps if solutions \( p^* \) to \( \mathcal{P}_2 \) are unique for fixed \( x \) and \( \hat{\chi} \), which we now justify with convexity assumptions for \( f_i \).

**Assumption 1. (Function Properties: Inner-Problem Argument).** The local cost functions \( f_i \) are twice differentiable and \( \omega_i \)-strongly convex in \( p_i \) for any fixed \( x \). Further, the second derivatives are lower and upper bounded:

\[
0 < \omega_i \leq \nabla^2 f_i(x, p_i) \leq \theta_i, \quad \forall x \in \mathbb{R}^d, p_i \in \mathbb{R}, \text{ and } i \in \mathcal{N}.
\]

This implies \( \forall x \in \mathbb{R}^d, p_i, \hat{p}_i \in \mathbb{R} \) and \( i \in \mathcal{N} \):

\[
\omega_i \|p_i - \hat{p}_i\| \leq \|\nabla_p f_i(x, p_i) - \nabla_p f_i(x, \hat{p}_i)\| \leq \theta_i \|p_i - \hat{p}_i\|.
\]

We also use the shorthands \( \omega \triangleq \min_i \omega_i \) and \( \theta \triangleq \max_i \theta_i \).

This assumption will be required of our analysis in Section 4.1. We now state some additional assumptions.

**Assumption 2. (Function Properties: Lipschitz Outer-Problem Argument).** The functions \( f_i \) have \( l_i \)-Lipschitz gradients and \( \rho_i \)-Lipschitz Hessians:

\[
\|\nabla f_i(x, p_i) - \nabla f_i(y, p_i)\| \leq l_i \|x - y\|, \quad \forall x, y \in \mathbb{R}^d, \forall p_i \in \mathbb{R},
\]

\[
\|\nabla^2 f_i(x, p_i) - \nabla^2 f_i(y, p_i)\| \leq \rho_i \|x - y\|, \quad \forall x, y \in \mathbb{R}^d, \forall p_i \in \mathbb{R}.
\]

We also use the shorthands \( l \triangleq \max_i l_i \) and \( \rho \triangleq \max_i \rho_i \).

**Assumption 3. (Function Properties: Bounded Variance Outer-Problem Argument).** The function \( F_\chi \) possesses the following bounded variance properties:

\[
\mathbb{E} \left[ \|\nabla F_\chi(x) - \nabla F(x)\|^2 \right] \leq \sigma_1^2,
\]

\[
\mathbb{E} \left[ \|\nabla^2 F_\chi(x) - \nabla^2 F(x)\|^2 \right] \leq \sigma_2^2,
\]

\[
\|\nabla F_\chi(x) - \nabla F(x)\|^2 \leq M_1 \text{ almost surely},
\]

\[
\|\nabla^2 F_\chi(x) - \nabla^2 F(x)\|^2 \leq M_2 \text{ almost surely}.
\]

**Assumption 4. (Function Properties: Lipschitz Interconnection of Variables).** The gradient and Hessian of the function \( f \) with respect to \( x \) are Lipschitz in \( p \); that is, there exists constants \( \psi_g, \psi_H > 0 \) such that

\[
\|\nabla_x f(x, p) - \nabla_x f(x, \hat{p})\| \leq \psi_g \|p - \hat{p}\|,
\]

\[
\|\nabla^2_{xx} f(x, p) - \nabla^2_{xx} f(x, \hat{p})\| \leq \psi_H \|p - \hat{p}\|,
\]

\( \forall x \in \mathbb{R}^d, p, \hat{p} \in \mathbb{R}^n \).

Assumption 1 is relatively common in the convex optimization literature, and it lends itself to obtaining approximate solutions to \( \mathcal{P}_2 \) very quickly with stopping criterion guarantees. Assumption 2 is unanimously leveraged in literature on Cubic-Regularized Newton methods, as the constant \( \rho \) pertains directly to the cubic submodel, while Assumption 3 is a common assumption in the

5
stochastic optimization literature [19]. We note that, although Assumptions 2 and 3 do not give a direct relationship with the local functions $f_i(x, p_i)$, they do imply an implicit relationship between $x, p, D$ in the sense that solutions $p^\star$ to $P_2$ (and therefore the distributions $D_i$) must be “well-behaved” in some sense. This relationship, along with a broader interpretation of the model $P_1$ and $P_2$, is illustrated more concretely in the following real-world power distribution example.

**Example 1. (EV Drivers with PV Generators).** Consider two EV drivers who each have an EV charging station and a PV generator. The goal of this small grid system is to consume net zero power from the perspective of the tie line to the bulk grid, thus $P_{ref} = 0$. The distributions $D_1, D_2$ represent the power output distributions of the PVs, and we consider two scenarios for these in this example: (1) a “sunny day” scenario, where the realizations $\chi_1, \chi_2 \sim D_1, D_2$ of PVs 1 and 2 are deterministic, and (2) a “cloudy day” scenario, where intermittent cloud cover induces some uncertainty in the moment-to-moment PV generation.

Let $A \in \{\text{sunny, cloudy}\}$ indicate the weather forecast. The model is then fully described as

$$D_i = \begin{cases} \delta_{1.5}, & A = \text{sunny}, \\ \mathcal{U}[0, 1.5], & A = \text{cloudy} \end{cases} \quad \text{for both } i = 1, 2,$$

$$f_1(x, p_1) = (2x + p_1 - 1)^2, \quad f_2(x, p_2) = (x + p_2 - 2)^2.$$

For $x = 0$, these quadratic functions have local minima at $p^\star = (p_1^\star, p_2^\star) = (1, 2)$, which is interpreted as drivers 1 and 2 preferring to charge at rates of 1 unit and 2 units, respectively, if there are no external incentives. On a sunny day, both PVs deterministically produce $\hat{\chi}_1, \hat{\chi}_2 = 1.5$, which effectively balances the unconstrained $p^\star$ and both drivers can charge at their preference to maintain $\sum_i p_i = P_{ref} + \sum_i \hat{\chi}_i$.

However, on cloudy days the generation of the PVs is no longer deterministic. Thus, the variable $x$ comes in to play, which can represent a government credit that the drivers value differently. The role of $x$ is to shift the cost functions such that the unconstrained minima are near lower charging values in consideration of the lower expected generation from PVs 1 and 2. The optimal $x^\star$ to $P_1$ is the value which gives the lowest expected cost of an instance of $P_2$ given $\hat{\chi}_1, \hat{\chi}_2$ realizations from the $A = \text{cloudy}$ distributions $D_1, D_2$. A more complete model of $P_2$ could include power flow constraints; in this work, we relax these for simplicity.

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2See [4] for an example where quadratic costs to EV users are induced by resistive energy losses in the battery model and [2] for a broad reference on modeling generator dispatch.
4. Distributed Formulation and Algorithm

In this section, we develop the inner-loop algorithm used to solve $P_2$. We then synthesize inexact solutions to $P_2$ with the DiSCRN algorithm for $P_1$.

4.1. Inner Loop Gradient Solver

For this section, consider $x$ to be fixed and known by all agents. Further, let $\hat{\chi}_i$ be fixed (presumably from a realization of $D_i$) and known only to agent $i$. We adopt the following assumption on the initial condition $p_0$.

**Assumption 5. (Feasibility of Inner-Problem Initial Condition).** The agents are endowed with an initial condition which is feasible with respect to the constraint of $P_2$; that is, they each possess elements $p_0^i$ of a $p^0$ satisfying $1_n^\top p^0 = P_{\text{ref}} + \hat{\chi}$.

The assumption is easily satisfied in practice by communicating $P_{\text{ref}}$ to one agent $i$ and setting $p_0^i = P_{\text{ref}} + \hat{\chi}_i$, with all other agents $j$ using $p_0^j = \hat{\chi}_j$. An alternative to this assumption consists of reformulating $P_2$ with distributed constraints and using a dynamic consensus algorithm as in [13], which would still retain exponential convergence. We impose Assumption 5 for simplicity.

Finally, we assume connectedness of the communication graph:

**Assumption 6. (Graph Properties).** The communication graph $G$ is connected and undirected; that is, a path exists between any pair of nodes and, equivalently, its Laplacian matrix $L = L^\top \succeq 0$ has rank $n-1$ with eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$.

The discretized Laplacian-flow dynamics are given by:

$$p^+ = p - \eta L \nabla_p f(x, p).$$

(4)

Note that these dynamics are distributed, as the sparsity of $L$ implies each agent need only know $\nabla_{p_i} f_i(x, p_i)$ and $\nabla_{p_j} f(x, p_j)$ for $j \in N_i$ to compute $p^+_i$. We now justify convergence of (4) to the solution $p^*$ of $P_2$:

**Proposition 1. (Convergence of Discretized Laplacian Flow).** Let $p^* \in \mathbb{R}^n$ be the unique minimizer of $P_2$. Given Assumption 2 on the feasibility of the initial condition, Assumption 2 on connectivity of the communication graph, and Assumption 7 on the Lipschitz gradient condition of the function gradients, then, under the dynamics (4) with $0 < \eta < \frac{2}{\theta \lambda_n}$, $p$ converges asymptotically to $p^*$.

**Proof.** Using a standard quadratic expansion around the current iterate $p$ (see e.g. §9.3 of [1]) and Lipschitz bounds yields $f(p^+)^\top - f(p)^\top \leq \theta \eta^2/2\|L \nabla_p f(x, p)\|^2 - \eta \nabla_p f(x, p)^\top L \nabla_p f(x, p)$. Careful treatment of the eigenspace of $L$ and some algebraic manipulation shows that $f(p^+)^\top - f(p)^\top$ is strictly negative for $\eta$ as in the statement. ☐
We now provide an additional result on exponential convergence of the state error with a further-constrained step size as compared to the statement in Proposition 1.

**Proposition 2. (Exponential Convergence with Bounded Error).** Let Assumptions 5, 6, and 7 hold as before. For $0 < \eta < \frac{\omega \lambda_2}{\theta^2 \lambda_n^2}$, the quantity $\|p - p^*\|$ converges exponentially to zero under the dynamics (4). For $\eta = \omega \lambda_2 / \theta^2 \lambda_n^2$, the rate is $\|p^* - p^*\| \leq \sqrt{\frac{1}{1 - \omega^2 \lambda_2^2 / \lambda_n^2 \theta^2}} \|p - p^*\|$, and $\|p^K - p^*\| \leq \Delta$ for $K \geq \log(\Delta / \|p^0 - p^*\|)/\log(\sqrt{1 - \omega^2 \lambda_2^2 / \lambda_n^2 \theta^2})$.

**Proof.** Consider $V(p) = \|p - p^*\|^2$. Substituting (4) and applying bounds via eigenvalues of $L$, using Assumption 1 and $\nu$-strong function convexity, we get $V(p^t) \leq (\eta^2 \lambda_n^2 \theta^2 - 2\eta \lambda_2 \omega + 1)V(p)$, with $0 < \eta < 2\lambda_2 \omega / \lambda_n^2 \theta^2$. The choice of $\eta = \omega \lambda_2 / \lambda_n^2 \theta^2$ implies the exponential convergence as in the statement. \hfill \Box

We note that the results of Propositions 1 and 2 simply build on a Laplacian-projected version of vanilla gradient descent. However, it lays some basic groundwork and supplements our main results in the next subsection.

With this, we are ready to transition to the discussion on obtaining a DiSCRN update to $\mathcal{P} 1$.

### 4.2. Outer-Loop Cubic-Newton Update

We endow each agent with a local copy $x_i$ of the variable $x$, and we let $x \in \mathbb{R}^{nd}$ be the stacked vector of these local copies. Thus, a distributed reformulation of $\mathcal{P} 1$ is

\[
\overline{\mathcal{P} 1}: \min_{x \in \mathbb{R}^{nd}} \overline{F}(x) = \mathbb{E}_{\chi \sim D} \left[ \overline{F}_\chi(x) \right],
\]

subject to $(L \otimes I_d) x = 0_{nd}$,

where $\overline{F}_\chi : \mathbb{R}^{nd} \to \mathbb{R}$ is analogous to $F_\chi : \mathbb{R}^d \to \mathbb{R}$ in the sense that each agent evaluates $f_i(x_i, p_i^t)$ with its local copy of $x_i$. Note that the constraint $(L \otimes I_d) x = 0_{nd}$ imposes $x_i = x_j, \forall i, j$ (Assumption 0), so $\overline{F}_\chi$ and $F_\chi$ are equivalent in the agreement subspace (and $\overline{\mathcal{P} 1}$ is equivalent to $\mathcal{P} 1$). Since our problem is nested and stochastic, there is a lack of access to a closed form expression for $\overline{F}$ and $\overline{F}_\chi$. Thus, we introduce an *empirical-risk, approximate* objective function. To this end, let $F_S^\Delta(x) = 1/S \sum_{s=1}^{S} F_{\chi^s}^\Delta(x)$ be the approximation of $\overline{F}$ for $S$ samples of $\chi^s \sim D$, where $\tilde{F}_{\chi^s}^\Delta \equiv \sum_{s=1}^{S} f_s(x_i, p_i^t)$ and $\|\tilde{p}^s - p^*\| \leq \Delta$ for realization $\chi^s$. In this sense, $F_{\chi^s}^\Delta$ implicitly depends on $\tilde{p}^s$, and the $\Delta$ superscript is a slight abuse of notation. For now, the reader can consider $\Delta$ to be a sufficiently small design parameter describing the inexactness of the obtained solutions to $\mathcal{P} 2$; we build on this later. Ultimately, we intend to use batches of $F_S^\Delta$ rather than the exact $\overline{F}$ to implement DiSCRN. Consider then the cubic regularized submodel of $F_S^\Delta$ at some $x^k$:

\[
m_S^k(x) = F_S^\Delta(x) + (x - x^k)^\top g^k + \frac{1}{2}(x - x^k)^\top H^k(x - x^k) + \sum_{i=1}^{n} \frac{p_i}{6} \|x_i - x_i^k\|^3, \quad (5)
\]
where \( g^k = \nabla F_S(x^k), H^k = \nabla^2 F_S(x^k) \). Note that there is a slight difference between (5) and the more standard cubic submodel (3) in that the regularization terms are \textit{directly separable}; this is crucial for a distributed implementation, and our forthcoming analysis justifies that convergence can still be established. We are interested in finding \( x^+ \) which minimizes (5) in the agreement subspace:

\[
P_3 : \min_{x \in \mathbb{R}^{nd}} m^k_S(x), \quad \text{subject to } (L \otimes I_d)x = 0_{nd}.
\]

Therefore, we prescribe the Decentralized Gradient Descent dynamics from [24]:

\[
x^{+,t+1} = W x^{+,t} - \alpha_t \nabla_x m^k_S(x^{+,t}),
\]

where \( W = I_{nd} - (1/\lambda_n L \otimes I_d) \) and \( \alpha_t \sim 1/t \). Per Proposition 3 and Theorem 2 of [24], \( x^{+,t} \) under the dynamics (6) converges asymptotically to a stationary point of \( P_3 \) with \( O(1/k) \) convergence in the agreement subspace, i.e. \( \|x_i - \bar{x}\| \) approaches zero at a rate \( O(1/k) \), where \( \bar{x} = \text{mean} (x_i) \).

We remark that one could formulate the Lagrangian of \( P_3 \) and use a saddle-point method to obtain a useful update \( x^+ \). This is more parallel to the work of [9], which achieves the global solution via gradient descent in the centralized setting. However, even the existence of a Lagrangian saddle-point is in question when the duality gap is nonzero, so further study is required on that approach.

The above discussion serves to set up the following condition on \( x^{k+1} \):

\textbf{Condition 1. (Subsolver Output).} Let \( x^{k+1} \) be the output of a subsolver for \( P_3 \). Then,

(i) \( x^{k+1} \) satisfies \( (L \otimes I_d)x^{k+1} = 0_{nd} \).

(ii) For an arbitrarily small constant \( c > 0 \) and some \( \varepsilon > 0 \), \( x^{k+1} \) satisfies

\[
m^k_S(x^{k+1}) - m^k_S(x^k) < -c\varepsilon \|x^{k+1} - x^k\| - c\sqrt{\rho}\|x^{k+1} - x^k\|^2.
\]

Part (i) is implied in a linear convergence sense by the result of [24] for the subsolver (6). The (ii) condition is straightforwardly implied by any subsolver that is guaranteed to strictly decrease \( m^k_S \), e.g. (7), because \( c \) can be taken arbitrarily small. However, it can be seen in the statement of Theorem 1 that small \( c \) implies a direct tradeoff with \( \Delta \) (becomes small) and/or \( S \) (becomes large).

We now give a brief outline of the entire algorithm.

\textbf{DiSCRN Algorithm}

1. Initialize \( x^0 \) s.t. \( (L \otimes I_d)x^0 = 0_{nd} \)
2. Realize \( \chi_s \) and initialize \( p^0 \) per Assumption 5
3. Implement (6) until \( |p_i^+ - p_i| \leq \Delta_n \lambda_2 \omega / \sqrt{n}, \forall i \)
4. Repeat from step 2 \( S \) times, storing \( p^s \leftarrow p^+ \) at each \( s \)
5. Compute locally required elements of \( g^k, H^k \)
6. Compute an $x^{k+1}$ satisfying Condition 2 e.g. via (7); repeat from step 2

The DiSCRN Algorithm describes a fully distributed algorithm, as each step can be performed with only local information. Ostensibly, $x^0$ could be initialized arbitrarily, but the first outer-loop would be a “garbage” update until agreement is obtained in step 8. Note that Step 8 relates to a distributed stopping criterion for the subsolver of $\mathcal{P} 2$; this condition produces a solution $p^*$ in finite iterations which is sufficiently close to $p^*$ for the sake of our analysis. This is detailed more in Theorem 1 and its proof.

**Condition 2. (Assumptions and Conditions for Theorem 1).** Let $F$ satisfy Assumption 2 on Lipschitz gradients and Hessians, and Assumption 3 on variance conditions, and let $f$ satisfy Assumption 4 on Lipschitz interconnection of $x$ and $p$, and Assumption 1 on the Lipschitz condition of the function gradients with respect to $p$. Further, let Assumption 5 on the feasibility of the initial condition for $\mathcal{P} 2$, and Assumption 4 on connectivity of the communication graph, each hold. Let $x^{k+1}$ be the output of a subsolver for $\mathcal{P} 3$ that satisfies Condition 1 with $c$, and let $\tilde{p}^* \leftarrow p^*$, where $p^*$ is the returned value under the dynamics (11) satisfying $|p^*_i - p_i| \leq \Delta n \omega^2 / \sqrt{n}$, $\forall i$.

**Theorem 1. (Convergence of DiSCRN).** Let the circumstances of Condition 2 apply here. For $S \geq \max \{ M_1 \zeta, M_2 \bar{\kappa} \bar{\omega} \sigma^2 \bar{\sigma}^2 \} O((\epsilon^{1.5} \bar{\epsilon}^{-1}))$ with $\bar{\epsilon} + \psi \Delta \leq \bar{\epsilon} \epsilon$ and $\bar{\epsilon} \sqrt{\bar{\sigma}} + \psi \bar{\lambda} \Delta \leq c \sqrt{\bar{\epsilon}}$, then, under the DiSCRN algorithm dynamics and for all $\zeta > 0$:

1. $\bar{F}(x^{k+1}) < \bar{F}(x^k)$ with probability $\geq 1 - \zeta$.

2. There is a unique accumulation point $\bar{F}^*$ of the sequence $\{\bar{F}(x^k)\}$ with probability $1 - \zeta$ and, for sufficiently large $k$, the value of $\bar{F}(x^k)$ is bounded in probability: $\mathbb{P}(|\bar{F}(x^k) - \bar{F}^*| \geq \kappa) \leq \zeta$ for any $\kappa > 0$.

3. If $\bar{F}$ is radially unbounded, then the sequence $\{x^k\}$ converges to a point $x^*$ such that $\bar{F}(x^*) = \bar{F}^*$ with probability $1 - \zeta$.

**Proof.** First, we aim to obtain the bound $\|\tilde{p}^* - p^*\| \leq \Delta$ for each instance $s$ of $\mathcal{P} 2$. The Lipschitz condition of Assumption 1 implies

$$\omega \|p - p^*\| \leq \|\nabla_p f(x,p) - \nabla_p f(x,p^*)\|,$$
$$\lambda \omega \|p - p^*\| \leq \|\nabla p f(x,p) - \nabla p f(x,p^*)\|$$

$$\quad = \|\nabla p f(x,p)\| = 1/\eta \|p^+-p\| \leq \Delta \omega.$$

Finally, $1/\sqrt{n}$ comes from breaking $p^+-p$ into components and since, for $v \in \mathbb{R}^n$, if $|v_i| \leq c/\sqrt{n}$ implies $\|v\| \leq c$.

Turning to $\mathcal{P} 1$, let $g_s^k = \frac{1}{S} \sum_{s=1}^S \nabla f_s(x^k, p^*_i)$ and $H^k = \frac{1}{S} \sum_{s=1}^S \nabla^2_{x,x} f_s(x^k, p_i^*)$. Lemma 4 of [19] justifies that for arbitrary $\bar{\epsilon} > 0$, choosing $S \geq \max \{ M_1 \zeta, M_2 \bar{\kappa} \bar{\omega} \sigma^2 \bar{\sigma}^2 \} O((\epsilon^{1.5} \bar{\epsilon}^{-1}))$ implies that
\(\|g^k - \nabla \bar{F}(x^k)\| \leq \bar{c} \varepsilon \) and \(\|(H^k_x - \nabla^2_{xx} \bar{F}(x^k)) v\| \leq \bar{c} \varepsilon \sqrt{\rho \varepsilon} \|v\|, \forall v \) with probability \(1 - \zeta\).

Let \(\phi^k_g = g^k - g^k_\varepsilon, \phi^k_H = H^k - H^k_\varepsilon\), where \(g^k\) and \(H^k\) use the inexact estimates \(\bar{p}^k\) satisfying \(\|\bar{p}^k - p^*\| \leq \Delta\). Substitutions and applying Assumption 11 gives:

\[
\|g^k - \nabla_x \bar{F}(x^k)\| \leq \|g^k - \nabla_x \bar{F}(x^k)\| + \|\phi^k_g\|
\leq \bar{c} \varepsilon + \psi_g \Delta,
\]

\[
\|(H^k - \nabla^2_{xx} \bar{F}(x^k)) v\| \leq \|(H^k - \nabla^2_{xx} \bar{F}(x^k)) v\| + \|\phi^k_H\|
\leq \bar{c} \varepsilon \sqrt{\rho \varepsilon} + \psi_H \Delta, \forall v.
\]

Next, let \(\xi^k := x^{k+1} - x^k\) for notational convenience. The separable cubic regularized terms of \(m^k_S\) can be used to bound the true function value:

\[
\bar{F}(x^{k+1}) \leq \bar{F}(x^k) + \nabla \bar{F}(x^k)^\top \xi^k + \xi^k \bar{F}_k \nabla^2 \bar{F}(x^k) \xi^k
+ \sum_i \rho_i (x_i^{k+1} - x_i^k)^3 \Rightarrow
\]

\[
\bar{F}(x^{k+1}) - \bar{F}(x^k) \leq m^k_S(x^{k+1}) - m^k_S(x^k)
+ (\nabla \bar{F}(x^k) - g^k)^\top \xi^k + 1/2 \xi^k \bar{F}_k (\nabla^2 \bar{F}(x^k) - H^k) \xi^k
\leq m^k_S(x^{k+1}) - m^k_S(x^k)
+ (\bar{c} \varepsilon + \psi_g \Delta) \|\xi^k\| + (\bar{c} \varepsilon \sqrt{\rho \varepsilon} + \psi_H \Delta) \|\xi^k\|^2
\leq m^k_S(x^{k+1}) - m^k_S(x^k) + c \varepsilon \|\xi^k\| + c \varepsilon \sqrt{\rho \varepsilon} \|\xi^k\|^2 < 0,
\]

where the first inequality is implied by breaking up \(\bar{F}_x(x)\) in to its separable local functions, applying Assumption 2 and noting that the inequality carries through the expectation operator. Subsequent inequalities are directly obtained via substitutions. The lefthand inequality of the final line stems from the Theorem statement, and the righthand inequality of the final line from 11 of Condition 1.

As for statement 3, the existence of the unique accumulation point \(\bar{F}^*\) is a consequence of the monotonicity of a sequence of real numbers. Bounding the value of this point follows directly from the definition of an accumulation point.

Regarding statement 3 note that, if the sequence \(\{x^k\}_k\) is bounded, then the set of its accumulation points have a finite norm. This follows from radial unboundedness of \(\bar{F}\) due to \(\bar{F}(x^k) \in \{\bar{F}(x) | \bar{F}(x) \leq \bar{F}(x^0)\}\) being compact (where \(\bar{F}(x^0)\) is some fixed constant). Then, consider any accumulation point \(x^*_k\) of the sequence \(\{x^k\}_k\). Because of the monotonicity of \(\{\bar{F}(x^k)\}_k\), it follows directly that \(\bar{F}(x^*_k) = \bar{F}^*\).

\(\square\)

5. Simulations

Our simulation study considers two cases: (i) a synthetic nonconvex case, which demonstrates the superiority of DiSCRN over analagous gradient-based
and Newton-based approaches, and (ii) a quadratic convex case, which extends the setting to an electric vehicle charging scenario over a time-horizon in which the price of electricity follows a time-of-use model.

5.1. Nonconvex Scenario

The cost functions \( f_i \) for this scenario are represented as:

\[
 f_i(x, p_i) = \frac{1}{2} \alpha_i(x)p_i^2 + \beta_i(x)p_i + \gamma_i.
\]

Each \( \alpha_i : \mathbb{R} \rightarrow \mathbb{R} \) is quartic in \( x \) and generated according to (5), where each \( a_i^2 \) is determined such that \( \min_x \alpha_i(x) = \omega_i > 0 \) with \( \omega_i \in \mathcal{U}[1, 5] \) per Assumption [1]. The \( \beta_i : \mathbb{R} \rightarrow \mathbb{R} \) are (possibly nonconvex) quadratic (9), and \( \gamma_i = 0 \).

\[
\begin{align*}
  \alpha_i(x) &= a_i^1(x - z_i^1)(x - z_i^2)(x - z_i^3)(x - z_i^4) + a_i^2, \\
  &\quad a_i^1 \in \mathcal{U}[0.5, 1.5], z_i^1 \in \mathcal{U}[-2, -1], z_i^2 \in \mathcal{U}[-1, 0], \\
  &\quad z_i^3 \in \mathcal{U}[0, 1], z_i^4 \in \mathcal{U}[1, 2], \\
  \beta_i(x) &= b_i^1(x - z_i^5)(x - z_i^6), \\
  &\quad b_i^1 \in \mathcal{U}[-1, 1], z_i^5 \in \mathcal{U}[-2, 0], z_i^6 \in \mathcal{U}[0, 2],
\end{align*}
\]

We compare our DiSCRN method with gradient-based and Newton-based updates of the same batch sizes, where the gradient-like and Newton-like updates are computed via:

\[
\begin{align*}
  m_g^k(x) &= F^S(x^k) + (x - x^k)^\top g^k + \sum_i \frac{\eta_g}{2} \| x_i - x_i^k \|^2, \\
  m_H^k(x) &= F^S(x^k) + (x - x^k)^\top g^k + \frac{1}{2} (x - x^k)^\top H^k (x - x^k) + \sum_i \frac{\eta_H}{2} \| x_i - x_i^k \|^2.
\end{align*}
\]

We obtain \( x^{k+1} \) empirically for all three methods by implementing (7) until the updates become very small. We found that both \( \eta_g \) and \( \eta_H \) must be sufficiently large to ensure stability, and \( \nabla_x^2 F(x) \succ -\eta_H I_d \) to ensure \( m_H^k(x) \) bounded. We take \( \Delta = 0.1, S = 20, n = 40, |\mathcal{E}| = 120, P_{ref} = 40, D_i = \mathcal{U}[0, 1.5] \forall i, \rho = 50, \eta_g = 100, \eta_H = 50 \).

We note substantially improved performance of DiSCRN over the more traditional gradient-based and Newton-based approaches. In particular, the trajectory finds a minimizer in roughly half and one-third the number of outer-loop iterations required by Newton and gradient, respectively. It is clear that, for \( x^{k+1} \approx x^k \), the cubic regularization is less dominant than the squared regularizations, allowing the DiSCRN trajectory to be influenced more by the problem data \( g^k, H^k \). As for the parameters \( (\rho, \eta_g, \eta_H) \), \( \eta_H = 50 \) and \( \eta_g = 100 \) were roughly the lowest possible values without inducing instability. By contrast, reducing \( \rho \) to values \( \sim 10^{-1} \) was still stable for DiSCRN. We noticed a clear tradeoff between \( S \) and \( \Delta \), with small \( S \sim 10^0 \) requiring \( \Delta \sim 10^{-1} \) to converge and large \( S \sim 10^3 \) converging even for large \( \Delta \sim 10^2 \), which is implied.
Figure 1: Comparison of CRN method with gradient-based and Newton-based approaches. Top: empirical approximation of $F(x^k)$, obtained by averaging $f(x^k, p^\star)$ over 500 realizations of $\mathcal{P} 2$ at each $k$. Bottom: agents’ disagreement on the value of $x$, quantified by $\|(I - 11^T/n)x^k\|_2$.

by Theorem 1. Finally, DiSCRN achieves reduced disagreement compared to gradient and Newton; this could be in part due to (7) finding a stationary point of $\mathcal{P} 3$ faster, allotting more iterations where the consensus terms dominate the update.

5.2. Electric Vehicle Charging Scenario

Consider a system of $n = 25$ electric vehicle users attempting to satisfy a global load constraint while simultaneously minimizing the cost of (i) the actual economic cost of charging (or discharging) at each time $l \in \{1, \ldots, 60\}$ according to a time-of-use pricing model $P_l$ and (ii) a user-specified preferred charging rate, as in Example 1. The time-of-use pricing model is characterized by:

$$P_l = \begin{cases} 2, & l \in \{1, \ldots, 20\}, \\ 4, & l \in \{21, \ldots, 40\}, \\ 1, & l \in \{41, \ldots, 60\}. \end{cases}$$

The value $P_{\text{ref}} = 40$ is fixed for all $l$, as are the distributions $\mathcal{D}_i = \mathcal{U}[0.5, 1.5]$ for all $i$. The realizations $\chi_{i,l}$ at each time step $l$ represent some net generation/consumption quantity from each user, e.g. stochastic PV generation and residential load use.

The following quadratic cost model applies to each EV user at time $l$:

$$f_l^i(x, p_i) = a_i P_l p_i^2 + b_i P_l p_i + c_i (p_i - p_i^\star - d_i x)^2.$$
The total cost to be minimized for $\mathcal{P} 1$ is summed over the entire horizon:

$$F_\chi(x) \equiv 1/60 \sum_{l=1}^{60} \sum_i f_i^l(x, p^*_i).$$

The first two terms of $f_i^l$ are associated with the real economic cost of charging/discharging at time $l$, where $a_i, b_i > 0$ are physical constants associated with the battery model of each user $i$ (e.g., the internal resistance, charge capacity, and open circuit voltage, see e.g. [4]). The last term incentivizes charging close to the user’s preferred rate, $p_i$, with “tuning” coefficients $c_i, d_i > 0$, which weighs this cost against the economic cost, and $d_i > 0$, which allows for a shift in the preferred charging rate via an external incentive $x$.

Users may not only have batteries with different physical constants $a_i, b_i$, but also different preferences on $p_i, c_i, d_i$. For this study, we simply generate $a_i, b_i, c_i, d_i \in U[1, 3]$ and $p_i \in U[0, 2]$. We choose the simulation constants $\Delta = 0.1, S = 20, |\mathcal{E}| = 75, p = 0.1, \eta_g = 500, \eta_H = 1000$.

The results are plotted in Figure 2. We again note superior performance by the DiSCRN algorithm over the gradient-based and Newton-based methods, where the convergence is much more clearly linear in this convex case. However, a stronger takeaway of this study to note that the model was applicable for this extended time-horizon scenario. A subject of future work is to incorporate battery charging dynamics and constraints in this scenario which can be gradually learned by the algorithm, akin to a reinforcement learning setting.

Figure 2: Comparison of CRN method with gradient-based and Newton-based approaches for time-of-use pricing EV charging (convex) scenario. **Top:** price signal versus time $l$. **Bottom:** empirical approximation of $F(x^k)$, obtained by averaging $1/60 \sum_{l=1}^{60} f_i^l(x^k, p^*_i)$ over 500 realizations of $\mathcal{P} 2$ at each $k$. 

The price signal is shown in the top panel, with real-time pricing information. The bottom panel depicts the empirical function value $F_\chi(x^k)$, averaged over 500 realizations, for different methods: DiSCRN, gradient (grad), and Newton. The convergence is more clearly linear for the DiSCRN method compared to the gradient-based and Newton-based approaches.
6. Conclusion

Here, we studied a nested, distributed stochastic optimization problem and applied a Distributed Stochastic Cubic-Regularized Newton (DiSCRN) algorithm to solve it. In order to compute the DiSCRN update, a batch of approximate solutions to realizations of the inner-problem are obtained, and we developed a locally-checkable stopping criterion to certify sufficient accuracy of these solutions. The accuracy parameter is directly leveraged in the analysis of the outer-problem, and simulations justify both faster and more robust convergence properties than that of comparable gradient-like and Newton-like approaches. Future work involves developing and analyzing a saddle-point dynamics approach for solving $\mathcal{P}3$ (extending the work of [9]), extending the analysis to accommodate small disagreements in the agent states $x_i$, and exploring adaptive batch size techniques.

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