Rigidity of Circle Packings with Crosscuts

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Abstract

Circle packings with specified patterns of tangencies form a discrete counterpart of analytic functions. In this paper we study univalent packings (with a combinatorial closed disk as tangent graph) which are embedded in (or fill) a bounded, simply connected domain. We introduce the concept of crosscuts and investigate the rigidity of circle packings with respect to maximal crosscuts. The main result is a discrete version of an identity theorem for analytic functions (in the spirit of Schwarz’ Lemma), which has implications to uniqueness statements for discrete conformal mappings.

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1 Introduction

The study of circle packings, as they are understood in this paper, was initiated by Paul Koebe as early as in 1936 in the context of conformal mapping, but the real success of the topic begun with William Thurston’s talk at the celebration of the proof of the Bieberbach conjecture in 1985. The publication of Ken Stephenson’s book [13] inspired further research and made the topic accessible to a wide audience. Since then many classical results in complex analysis found their discrete counterpart in circle packing.

In this paper we consider circle packings embedded in a bounded, simply connected domain. We introduce the concept of crosscuts for domain-filling circle packings, and study the rigidity of packings with respect to maximal crosscuts (for definitions see below). The main result is a discrete version of an identity theorem for analytic functions, which has implications to uniqueness results for boundary value problems for circle packings, and especially to discrete conformal mappings.

To be more specific, we recall that the tangency relations of a circle packing are encoded in a 2-dimensional simplicial complex $K$, referred to as the combinatorics of the packing. In this paper it is assumed that $K$ is a finite triangulation of a topological disk.

Circle packings are a mixture of flexibility and rigidity. Counting the degrees of freedom for the centers and the radii, and comparing this with the number of conditions caused by the tangency relations, we see that the first number exceeds the latter by $m + 3$, where $m$ is the

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number of boundary circles. In fact, the set of all circle packings for a fixed complex $K$ forms a smooth manifold of dimension $m + 3$ (Bauer et al. [11]). So the question arises which sort of conditions are appropriate to eliminate the flexibility of a packing and make it rigid. Motivated by our work on nonlinear Riemann-Hilbert problems, we are interested in boundary value problems for circle packings. These problems involve $m$ boundary conditions (one for each boundary circle) and three additional conditions, which can be imposed in different form on boundary circles and interior circles as well.

A standard boundary value problem of this kind consists in finding circle packings with (given combinatorics and) prescribed radii of its boundary circles. Somewhat surprisingly, this problem has always a locally univalent solution, and the solution is unique up to a rigid motion of the complete packing (see [13], Section 11.4, for details).

The existence of solutions is also known for a related more general problem, the discrete Beurling problem, where the radii of the boundary circles are prescribed as functions of their centers (see [16]), but the question of uniqueness has not yet been answered satisfactorily.

Last but not least there are several approaches to discrete conformal mapping via circle packing which fall into this category (see Stephenson [13], in particular Chap. 19 and 20, with many interesting comments on the history of this topic, also summarizing [4], [10], [14]).

In our favorite setting of discrete conformal mapping, the domain packing $P$ is a so-called maximal packing, which ‘fills’ the complex unit disk $\mathbb{D}$, while the range packing $P'$ is required to ‘fill’ a bounded, simply connected domain $G$. That a packing ‘fills a domain $G$’ basically means that all its circles lie in the closure $\overline{G}$ of that domain and all its boundary circles intersect (touch) the boundary $\partial G$ of $G$. For domains which are not Jordan this has to be complemented by a more subtle condition (see Definition 2).

In a series of papers, Oded Schramm proved several outstanding results about packings which fill a Jordan domain. His very general existence theorems do not only address packings of circles, but of much more general packable sets (for an explanation see [11]).

Surprisingly, much less is known about uniqueness. It is clear that uniqueness of a domain-filling (circle) packing can only be expected if one imposes additional conditions which eliminate the (three) remaining degrees of freedom. Whether this works depends on the type of normalization conditions and on the geometry of the domain. For example, in his uniqueness proofs, Schramm needs that the Jordan domain is (as he says) decent (see [12]).

This paper is devoted to the question which additional conditions are appropriate to make a domain filling circle packing unique. In analogy to the standard normalization of conformal mappings, it seems reasonable to fix the center of a distinguished circle (the so-called alpha-circle) at some point in $G$ and to require that the center of a neighboring circle lies on a given ray emerging from that point. Keeping the first condition, we have chosen another setting for the second one. This condition, involving crosscuts, is non-standard, more flexible and allows one to address other uniqueness problems too.

In order to give the reader a flavor of the result, we first state an analogous theorem for analytic functions. Recall that a crosscut of a domain $G$ in the complex plane $\mathbb{C}$ is an open Jordan arc $J$ in $G$ such that $\overline{J} = J \cup \{a, b\}$ with $a, b \in \partial G$ (see Pommerenke [8]). Slightly abusing terminology, we shall also denote $\overline{J}$ as a crosscut in $G$.2
**Theorem 1** (Identity Theorem for Analytic Functions). Let $J$ be a crosscut of a simply connected domain $G$, with $G^-$ and $G^+$ denoting the (simply connected) components of $G \setminus J$. If $f : G \to G$ is analytic, $f(z_0) = z_0$ for some $z_0 \in G^+$, and $f(G^-) \subset G^-$, then $f(z) = z$ for all $z \in G$.

*Proof.* Let $g : G \to \mathbb{D}$ be a conformal mapping of $G$ onto the unit disk $\mathbb{D}$ with $g(z_0) = 0$. Then $g$ maps the crosscut $J$ of $G$ to a crosscut of $\mathbb{D}$ (see [8], Prop.2.14) and the composition $g \circ f \circ g^{-1}$ satisfies the assumptions of the lemma with $G := \mathbb{D}$ and $z_0 := 0$. Hence it suffices to consider this special case.

Let $z_1$ be a point on $J$ with $|z_1| = \min_{z \in J} |z|$. Since $J$ is a crosscut in $\mathbb{D}$, and $0 = z_0 \in G^+$, we have

$$0 < |z_1| \leq \min \{|z| : z \in \overline{G^-}\} < 1.$$ 

By continuity, $f(G^-) \subset G^-$ and $z_1 \in \overline{G^-}$ imply that $f(z_1) \in \overline{G^-}$, and hence $|f(z_1)| \geq |z_1|$. Invoking Schwarz’ Lemma, we get $f(z) = cz$ in $\mathbb{D}$, where $c$ is a unimodular constant. Finally, the only rotation of $\mathbb{D}$ which maps $G^-$ into itself is the identity. \hfill \Box

Although Schwarz’ Lemma has already been investigated in the framework of circle packing (see [9], or [8] Chap. 13) the following interpretation of Theorem 1 is new. Though precise definitions will be deferred to the next section, we hope that Figure 1 helps to get an intuitive understanding of the setting.

![Figure 1: A domain-filling packing $\mathcal{P}$ with a crosscut and a maximal crosscut](image)

**Theorem 2** (Rigidity of Circle Packings with Crosscuts). Assume that a univalent circle packing $\mathcal{P} = \{D_v\}$ for a complex $K$ with vertex set $V$ fills a bounded, simply connected domain $G$. Let $J$ be a (maximal) crosscut of $\mathcal{P}$ in $G$, such that $G^-$ is a simply connected component of $G \setminus J$, and denote by $V^-$ and $V^+$ the sets of vertices of $K$ associated with circles in $G^-$ and $G^+ := G \setminus G^-$, respectively. Let $D_\alpha$ be an interior circle of $\mathcal{P}$ which is contained in $G^+$. 

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Assume further that a second univalent packing \( \mathcal{P}' = \{ D'_v \} \) for \( K \) is contained in \( G \), such that \( D_\alpha \) and \( D'_\alpha \) have the same center, and \( D'_v \subset G^- \) for all \( v \in V^- \). Then \( D'_v = D_v \) for all accessible vertices \( v \in V^- \).

We point out that everything hinges on the assumption about the common center of the two alpha-circles. Since we do not assume that \( \mathcal{P}' \) fills \( G \), it is solely this condition which prevents that \( \mathcal{P}' \) can be completely moved into \( G^- \).

The notion of accessible vertices will be explicated in Definition [1]. Here we only note that all vertices \( v \in V^- \) are accessible if and only if the complex \( K \) is strongly connected, which means \( K \) satisfies the following conditions (i) and (ii):

(i) Every boundary vertex has an interior neighbor.

(ii) The interior of \( K \) is connected.

Note that some authors of the circle packing community make the general assumption that the underlying complex \( K \) is strongly connected (see [13]). For circle packings with this simpler combinatoric structure the theorem yields complete rigidity with respect to crosscuts, i.e., \( D'_v = D_v \) for all \( v \in V \).

Figure 2 illustrates some effects which can be observed for packings with general combinatorics. The picture on the left shows an Apollonian packing \( \mathcal{P} \) with four generations. The highlighted line is a maximal crosscut, disks in the “lower domain” are the white ones, disks in the “upper domain” are the colored ones. The disk with the darkest color is the alpha-disk with fixed center. The accessible disks are those which can be connected with the alpha-disk by a chain of interior disks (see Definition 1).

![Figure 2: Some examples illustrating assumptions and assertions of Theorem 2](image)

The packing \( \mathcal{P}' \), depicted in the middle, satisfies the assumptions of the theorem. In this example, only the accessible disks of \( \mathcal{P}' \) (shown in darker colors) coincide with their partners in \( \mathcal{P} \). The non-accessible disks (shown in white and lighter colors) differ from the corresponding disks in \( \mathcal{P} \).
The example on the right illustrates that the result need not hold if the alpha-disk is a boundary disk. The depicted packing $P''$ satisfies all other assumptions (for the same crosscut), but, apart from the alpha disk, it is completely different from the packing $P$ shown on the left-hand side.

The result has an intuitive interpretation when we think of circle packings as dynamic structures: Suppose that $P$ fills $G$, and allow its circles to move (change position and size) in such a way, that they all remain in $G$, the center of the alpha-circle is fixed in $G^+$, and the circles in $G^-$ are not allowed to leave $G^-$. Then only those circles which are not accessible can be moved, while the core part of the packing is rigid.

In fact we shall even prove a stronger result, where the condition $D'_v \subset G^-$ need only be satisfied for $v$ in $U^-$, which stands for the set of those vertices $v \in V^-$ associated with circles $D_v$ touching the crosscut $J$.

In order to illustrate the analogies with Theorem 1, we interpret the result in the framework of discrete analytic functions: The circle packing $P$ filling $G$ is the domain packing, the packing $P'$ lies in $G$, so that $P \to P'$ defines a discrete analytic function from $G$ into itself. Fixing the centers of the alpha-circles of both packings at the same point $z_0$ corresponds to the normalization $f(z_0) = z_0$. Finally, the condition $D'_v \subset G^-$ for all $v \in V^-$ expresses the invariance of the subdomain $G^-$. Since the packing $P$ represents the identity function on $G$, it is natural to suppose that $P$ is univalent. Contrary to the continuous setting of Theorem 1 also $P'$ was assumed to be univalent in Theorem 2. It is challenging to investigate what happens when this condition is dropped.

Terminological remark. For our purposes it would be better to work with disk packings instead of circle packings. Though we stay with the traditional notion, we shall often speak of the disks in a circle packing. In order to avoid cumbersome formulations, we also say that a circle $\partial D$ lies in a domain $G$ when this holds for the open disk $D$ bounded by that circle. We already made use of this convention above.

## 2 Circle Packings

In order to make the paper self-contained we recall basic concepts and notions of topology and circle packing (for details we refer to Henle [5] and Stephenson [13]).

### Some Geometry.

If $A$ and $B$ are subsets of the (complex) plane, we say that $A$ intersects $B$ if $A \cap B \neq \emptyset$. If $A$ is a disk, then the phrase $A$ touches $B$ is in general used when $\overline{A} \cap \overline{B} \neq \emptyset$ and $A \cap B = \emptyset$. In this case we also say that the circle $\partial A$ touches $B$. As usual, the symbol $\partial$ denotes the boundary operator.

By a curve $\gamma$ we understand the image of a continuous mapping $\varphi : [a, b] \to \mathbb{C}$. The points $\varphi(a)$ and $\varphi(b)$ are said to be the initial point and the terminal point of $\gamma$, respectively; both are referred to as endpoints of $\gamma$. A Jordan arc and a Jordan curve are the homeomorphic images of a segment and a circle, respectively. By an open Jordan arc we mean a Jordan arc without its endpoints.
Let $J$ be an oriented Jordan curve. For $p, q \in J$ with $p \neq q$ we denote by $J(p,q)$ the (oriented) open subarc of $J$ with initial point $p$ and terminal point $q$. If $p, q, r$ are three pairwise different points on $J$, we say that $q$ lies between $p$ and $r$ on $J$ if $q \in J(p,r)$. Corresponding to whether $q$ lies between $p$ and $r$, or $q$ lies between $r$ and $p$, the orientation of the triplet $(p,q,r)$ with respect to $J$ is said to be positive or negative, respectively.

Let $G$ be a bounded, simply connected domain in $\mathbb{C}$. A conformal mapping $g : \mathbb{D} \to G$ of $\mathbb{D}$ onto $G$ has a continuous extension to $\mathbb{T}$ if and only if $\partial G$ is a closed curve, i.e., a continuous image of the unit circle $\mathbb{T}$ (see [8], Theorem 2.1). This extension (which we again denote by $g$) is a homeomorphism between $\mathbb{D}$ and $\overline{g} \mathbb{T}$ if (and only if) $G$ is a Jordan domain, i.e., $\partial G$ is a Jordan curve (see [8], Theorem 2.6).

In general, the conformal mapping $g$ induces a one-to-one correspondence between the points on $\mathbb{T}$ and certain equivalence classes of open Jordan arcs $\gamma$ in $G$ with terminal point $q$ on $\partial G$, so called prime ends. For the details we refer to Pommerenke [8], Chap 2, and Golusin [3], Section 2.3.

If $G$ contains a disk $D$ which touches the boundary $\partial G$ at some point $p \in \partial D \cap \partial G$, then every Jordan arc with starting point in $D$ and terminal point $p$ is contained in the same equivalence class. Hence there is a well defined prime end of $G$ associated with $p$ by $D$.

**Complexes.** The skeleton of a circle packing is a simplicial 2-complex $K$. Throughout this paper it is assumed that $K$ is a combinatorial closed disk, i.e., it is finite, simply connected and has a nonempty boundary. Simply speaking of a complex, we always mean a complex of this class. Properties of complexes which are relevant in circle packing are summarized in Lemma 3.2 of [13].

We denote the sets of vertices, edges and faces of $K$ by $V, E, F$, respectively. The edge adjacent to the vertices $u$ and $v$ is denoted by $e(u,v)$ or $\langle u,v \rangle$, where the first version stands for the non-oriented edge, while the second means the oriented edge from $u$ to $v$. Similarly, a face of $K$ with vertices $u, v, w$ is written as $f(u,v,w)$ (non-oriented) or $\langle u,v,w \rangle$ (oriented), respectively. Two vertices $u$ and $v$ are said to be neighbors if they are connected by an edge $e(u,v)$ in $E$.

For any vertex $v \in V$ we denote by $E(v)$ the set of edges adjacent to $v$. This set is endowed with a natural cyclic (counterclockwise) ordering, so that for $e_1, e_2 \in E(v)$ definitions like $\{e \in E(v) : e_1 < e \leq e_2\}$ make sense.

Any edge $e$ of $K$ is adjacent to one or two faces. In the first case $e$ is a boundary edge, otherwise it is an interior edge of $K$. Boundary vertices are those vertices of $K$ which are adjacent to a boundary edge. The sets of boundary edges and boundary vertices are denoted $\partial E$ and $\partial V$, respectively, the vertices in $V \setminus \partial V$ are called interior vertices. We point out that a boundary vertex can be adjacent to many other boundary vertices, and that an edge which connects two boundary vertices need not be a boundary edge (cf. Figure 3 left).

We let $B(v)$ denote the smallest sub-complex of $K$ which contains a vertex $v$ and all its neighbors. If $v$ is an interior vertex $B(v)$ is said to be the flower of $v$, if $v$ is a boundary vertex we speak of an incomplete flower. Note that $B(v)$ need not contain all edges which connect neighbors of $v$ (see Figure 3).

Since $K$ is a triangulation with non-void boundary, it must have at least three boundary vertices. The natural cyclic ordering of boundary edges, corresponding to the orientation of
the boundary of the triangulated surface, induces a cyclic ordering of the boundary vertices. With respect to this ordering, any boundary vertex has a **precursor** and a **successor** which are well-defined.

Speaking of a **chain**, we mean a finite sequence \((c_1, \ldots, c_n)\) of vertices, edges or faces, such that neighboring elements \(c_j\) and \(c_{j+1}\) are adjacent to a common edge (if the \(c_j\) are vertices or faces) or a common vertex (if the \(c_j\) are edges), respectively.

![Figure 3: The sub-complex of a (incomplete) flower and a corresponding packing](image)

On page 4 we have illustrated some limitations of Theorem 2. The reason for the observed effects is the relative independence of some substructures from the rest of the packing. This is described more precisely in the following definition.

**Definition 1.** Let \(K\) be a complex with a distinguished interior vertex, the **alpha-vertex** \(v_\alpha\). Then a vertex \(v \in V\) is called **accessible** (from \(v_\alpha\)) if there is a chain of vertices \((v, v_1, \ldots, v_n, v_\alpha)\) such that \(v_1, \ldots, v_n\) are interior vertices. The set of all accessible vertices of \(K\) is denoted by \(V^*\), the set of all edges \(e(u, v) \in E\) with \(u, v \in V^*\) by \(E^*\), and the set of all faces \(f(u, v, w) \in F\) with \(u, v, w \in V^*\) by \(F^*\). The **kernel** \(K^*\) of \(K\) is defined as the simplicial-2-complex arising from \(V^*, E^*, F^*\), that is \(K^*(V^*, E^*, F^*) \subset K\).

Recall that a complex \(K\) is **strongly connected**, if the interior of \(K\) is connected, and every boundary vertex has an interior neighbor. The following lemma establishes a relation between this property and accessible vertices, as already stated on page 4.

**Lemma 1.** Let \(K\) be a complex with a distinguished interior alpha-vertex \(v_\alpha\). All vertices of \(K\) are accessible, i.e., \(K = K^*\), if and only if \(K\) is strongly connected.

**Proof.** Assume that all vertices of \(K\) are accessible. Let \(v \neq w\) be two interior vertices of \(K\). Since \(v\) and \(w\) are both accessible, there are two chains of vertices \((v, v_1, \ldots, v_i, v_\alpha)\) and \((w, w_1, \ldots, w_j, v_\alpha)\) such that \(v_1, \ldots, v_i, w_1, \ldots, w_j\) are interior vertices, hence the interior of \(K\) is connected. Let \(u\) be a boundary vertex of \(K\). Because \(u\) is accessible, there is a chain of vertices \((u, u_1, \ldots, u_n, v_\alpha)\) such that \(u_1, \ldots, u_n\) are interior vertices, hence every boundary vertex of \(K\) has an interior neighbor, and \(K\) is strongly connected.
Conversely, assume that $K$ is strongly connected. Let $v$ be an interior vertex of $K$. Since the interior of $K$ is connected, we find a chain of vertices $(v, v_1, \ldots, v_n, v_\alpha)$ such that $v_1, \ldots, v_n$ are interior vertices. If $w$ is a boundary vertex of $K$, it has some neighboring interior vertex $u$, and there exists a chain of interior vertices $(u, u_1, \ldots, u_n, v_\alpha)$ from $u$ to $v_\alpha$. Then the chain $(w, u, u_1, \ldots, u_n, v_\alpha)$ connects $w$ with $v_\alpha$. Hence any vertex of $K$ is accessible. □

Now, all vertices of the kernel $K^*$ are accessible per definition, so if we can show that $K^*$ is a complex, then it is also strongly connected. In order to prove this we provide the following two lemmas.

**Lemma 2.** Let $K^*$ be the kernel of a complex $K$, and let $V^*, V$ be their vertex sets, respectively. Then every vertex $v \in \partial V \cap V^* \neq \emptyset$ has exactly two other vertices of $\partial V \cap V^*$ as neighbors in $K^*$. Moreover all accessible neighbors of $v$ form an incomplete flower $B^*(v) \subset K^*$ around $v$.

**Proof.** Since all neighbors of accessible interior vertices of $K$ are accessible, we have $\partial V \cap V^* \neq \emptyset$. Let $v \in \partial V \cap V^*$ be such a (boundary) vertex. Because $v$ is accessible there must be a neighbor $u$ of $v$ in $K$, which is an accessible interior vertex in $K$, $u \in V^* \setminus \partial V$. Looking at the incomplete flower $B(v)$ of $v$ in $K$ it becomes clear that there must be an edge chain $C$ in $B(v)$, connecting two different boundary vertices $w_1, w_2 \in \partial V$, such that $C$ contains $u$ and no other boundary vertices of $K$ except $w_1, w_2$. Hence $w_1$ and $w_2$ are accessible, $w_1, w_2 \in V^*$, and $v$ has at least two other boundary vertices of $\partial V \cap V^*$ as neighbors.

Assume now that there is a third boundary vertex $w_3 \in \partial V$, different from $w_1$ and $w_2$, which is an accessible neighbor of $v$. Let $C_1, C_2 \subset C$ be the chains of vertices connecting $u$ with $w_1, w_2$, respectively. Since $v$ and $w_3$ are accessible, there are chains $(v, u, c_1, \ldots, c_i, v_\alpha)$ and $(w_3, c'_1, \ldots, c'_j, v_\alpha)$, such that $c_1, \ldots, c_i, c'_1, \ldots, c'_j$ are accessible interior vertices of $K$. The concatenation $C_3 := (v, u, c_1, \ldots, c_i, v_\alpha, c'_j, \ldots, c'_1, w_3, v)$ must encircle either $w_1$ or $w_2$ (see Figure 4 left), which is impossible because both are boundary vertices. Hence, every boundary vertex $v \in \partial V \cap V^*$ has exactly two other boundary vertices of $\partial V \cap V^*$ as neighbors, and the (incomplete) flower $B^*(v)$ around $v$ with respect to $K^*$ has the structure depicted in Figure 4 (middle), with $\{v_1, \ldots, v_n\} \in V^* \setminus \partial V$ and $n \geq 1$. □

**Lemma 3.** The kernel $K^*$ of a complex $K$ is a triangulation.

**Proof.** Let $K^*(V^*, E^*, F^*)$ be the kernel of a complex $K$. Lemma 3.2 of [13] tells us that $K^*$ is a triangulation if and only if it has the following properties (i)–(vi) — what we will check on the run.

(i) $K^*$ must be connected. As we already used above, every two vertices $v, w \in V^*$ can be connected by a chain of accessible vertices via the alpha-vertex.

(ii) Every edge of $E^*$ must belong to either one or two faces of $F^*$. Since $E^* \subset E$ and $F^* \subset F$, it is impossible that an edge of $E^*$ belongs to more than two faces of $F^*$. So it remains to show that there is no isolated edge, which does not belong to a face. Let $e = e(v, w)$, that is $v, w \in V^*$. If $v$ is an interior vertex of $K$, then all its neighbors are accessible, too, so $B(v) \subset K^*$. If $v$ is a boundary vertex of $K$, than all its accessible neighbors form an incomplete flower $B^*(v) \subset K^*$ around $v$ (Lemma 2). In both cases $e$ is contained in at least one face of $F^*$. 

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(iii) Every vertex \( v \) of \( K^* \) belongs to at most finitely many faces, and these form an ordered chain in which each face shares an edge from \( v \) with the next. The first assertion holds, because \( K^* \) is a subset of \( K \). The second part follows easily by considering the flower \( B(v) \) (if \( v \) is an interior vertex) or the incomplete flower \( B^*(v) \) (if \( v \) is a boundary vertex).

(iv) Every vertex \( v \) of \( K^* \) belongs either to no boundary edge, or to exactly two boundary edges. Using once more the flower around \( v \) immediately shows this property.

(v),(vi) Any two faces are either disjoint, share a single vertex, or share a single edge, and all of them are properly oriented. This follows directly from \( K^* \subset K \).

Figure 4: Constructions for the proof of Lemma 2 and 4

The crucial properties of the kernel \( K^* \) are summarized in the following lemma.

**Lemma 4.** The kernel \( K^* \) of a complex \( K \) is a strongly connected complex with \( \partial V^* = \partial V \cap V^* \).

**Proof.** In order to prove \( \partial V^* = \partial V \cap V^* \) let \( v \) be an accessible interior vertex of \( K \). By definition of accessible vertices the flower \( B(v) \subset K \) around \( v \) must also lie in \( K^* \), hence \( v \) is an interior vertex of \( K^* \), which implies \( \partial V^* = \partial V \cap V^* \).

Since \( K^* \) is a finite triangulation with nonempty boundary (Lemma 3), it is a complex (in our sense) if it is simply connected. Because every boundary vertex of \( K^* \) has exactly two other boundary vertices of \( K^* \) as neighbors (Lemma 2), \( K^* \) is simply connected if and only if the boundary of \( K^* \) is connected.

Assume that the boundary of \( K^* \) is not connected. This implies that there is a boundary vertex \( v \in \partial V^* \), which is enclosed by a closed chain of boundary vertices different from \( v \) (see Figure 4, right). Because \( K^* \) is a subset of \( K \), the vertex \( v \) must be enclosed by the boundary chain of \( K \). Hence \( v \) is an interior vertex of \( K \), a contradiction to \( \partial V^* = \partial V \cap V^* \).

Since \( K^* \) is a complex whose vertices are all accessible, Lemma 1 tells us that \( K^* \) is strongly connected.
Each vertex \( v \in V \) has an associated disk \( D_v \in \mathcal{P} \), such that \( \mathcal{P} = \{ D_v : v \in V \} \).

If \( \langle u, v \rangle \in E \) is an edge of \( K \), then the disks \( D_u \) and \( D_v \) touch each other.

If \( \langle u, v, w, \rangle \in F \) is a positively oriented face of \( K \), then the centers of the disks \( D_u, D_v, D_w \) form a positively oriented triangle in the plane.

A circle packing is called \emph{univalent}, if its disks are \emph{non-overlapping}, \( D_u \cap D_v = \emptyset \) for all \( u, v \in V \) with \( u \neq v \). In this paper all circle packings are assumed to be univalent.

Since the structure of the underlying complex \( K \) carries over to the associated packing \( \mathcal{P} \), all related attributes can be applied to the disks \( D_v \) as well – so we shall speak of boundary disks, interior disks, neighboring disks, etc.

The \emph{contact point} of two neighboring disks \( D_u, D_v \) is defined by \( c(u, v) := D_u \cap D_v \). The \emph{contact points of a packing} \( \mathcal{P} \) for the complex \( K = (V, E, F) \) are the points \( c(u, v) \) with \( e(u, v) \in E \).

We denote by \( D \) the union of all disks in \( \mathcal{P} \), \( D := \bigcup_{v \in V} D_v \). If \( \mathcal{P} \) is univalent and \( p \) and \( q \) are different points of \( \partial D \), there is at most one disk \( D_v \) whose boundary \( \partial D_v \) contains \( p \) and \( q \). If such a disk exists, we define \( \delta(p, q) \) as the positively oriented open subarc of \( \partial D_v \) from \( p \) to \( q \), and \( \delta[q, p] := \delta(p, q) \). In addition we set \( \delta(p, p) := \emptyset \) and \( \delta[p, p] := \{ p \} \). Note that \( \delta(p, q) \) and \( \delta[q, p] \) are complementary subarcs of \( \partial D_v \), provided that \( p \neq q \).

If \( \langle u, v, w, \rangle \) is a face of \( K \), the \emph{interstice} \( I(u, v, w) \) of \( \mathcal{P} \) is the Jordan domain bounded by the arcs \( \delta_u := \delta(c(u, v), c(u, w)) \), \( \delta_v := \delta(c(v, w), c(v, u)) \) and \( \delta_w := \delta(c(w, u), c(w, v)) \) (see Figure 5, left).

![Figure 5: Definition of the interstice \( I := I(u, v, w) \) and the carrier \( D^* \) of two packings.](image)

Besides the union \( D \) of all disks in a packing \( \mathcal{P} \) we need the \emph{carrier} of \( \mathcal{P} \), which is the compact set

\[
D^* := \overline{D} \cup \bigcup_{f(u, v, w) \in F} I(u, v, w)
\]

(see Figure 5, middle and right). Note that this definition is somewhat different from Stephenson’s (cp. [13] p.58). The carrier is essential in the next definition.
Definition 2. Let $G$ be a bounded, simply connected domain. We say that a (univalent) circle packing $\mathcal{P}$ is contained in $G$ (or lies in $G$) if the interior of $D^*$ is a subset of $G$. A packing $\mathcal{P}$ contained in $G$ is said to fill $G$ if every boundary disk of $\mathcal{P}$ touches $\partial G$.

If $G$ is a Jordan domain, $\mathcal{P}$ is contained in $G$ if and only if any disk of $\mathcal{P}$ is a subset of $G$. For general domains the latter condition alone would be too week, since then it could happen that “spikes” of $\partial G$ (think of $G$ as a slitted disk) penetrate into the packing, sneaking through between two boundary disks at their contact point. This is prevented by our definition; in particular it guarantees that $\partial G \cap I = \emptyset$ for every interstice $I$ of $\mathcal{P}$.

What happens when $\partial G$ meets a contact point of two boundary disks is explored in the following lemma (an explanation of associated prime ends is given on page 6).

Lemma 5. Let $G$ be a bounded, simply connected domain, and let $\mathcal{P}$ be a circle packing contained in $G$. Then every contact point $c(u, v) \in \partial G$ is associated with the same prime end by both $D_u$ and $D_v$.

Proof. Let $c = c(u, v)$ be a contact point of $\mathcal{P}$ which lies on the boundary of $G$. Then there exists a vertex $w \in V$ such that $f(u, v, w)$ is a face in the complex of $\mathcal{P}$, and we denote by $I = I(u, v, w)$ the corresponding interstice.

For $\varepsilon > 0$, let $B_\varepsilon$ be an open disk centered at $c$ with radius $\varepsilon$ and define

$$\tilde{B}_\varepsilon := B_\varepsilon \cap (D_u \cup D_v \cup T).$$

If $\varepsilon$ is sufficiently small, $\tilde{B}_\varepsilon \setminus \{c\}$ is a Jordan domain contained in $G$, and we have $D_u \cap B_\varepsilon \subset \tilde{B}_\varepsilon$, $D_v \cap B_\varepsilon \subset \tilde{B}_\varepsilon$ (see Figure 6, left). As a Jordan domain $\tilde{B}_\varepsilon \setminus \{c\}$ has a unique prime end $c^*$ corresponding to its boundary point $c$, so the prime ends of $G$ associated with $c$ by the disks $D_u$ and $D_v$, respectively, must coincide.

A packing which fills the unit disk $\mathbb{D}$ is called maximal. A celebrated result, the Koebe-Andreev-Thurston-Theorem (which can be traced back to Koebe’s paper [6]), tells us that any complex
boundary interstice
In order to define the $\delta$ boundary arc

Lemma 6. Let $\delta$ be the union of the arcs $D_\delta$ (arc degenerates to a point) of $G$. To provide some more notation, let $c$ be a circle packing which fills a bounded, simply connected domain $G$. By definition, every boundary disk $D_k$ touches $\partial G$ in a non-void (possibly uncountable) set $G_k$ of points, and $G_k$ must be contained in the closure $\delta[c^-_k, c^+_k]$ of the exterior boundary arc $\delta(c^-_k, c^+_k)$ of $D_k$. Let $\delta_k := \delta[g^-_k, g^+_k]$ be the smallest subarc (we admit that this ‘arc’ degenerates to a point) of $\delta[c^-_k, c^+_k]$ which contains $G_k$. Since $G_k$ is a closed set, we have $g^-_k, g^+_k \in G_k$.

In order to define the boundary interstice $I_k$ between two consecutive boundary disks $D_k$ and $D_{k+1}$ (see Figure 6, right) we distinguish two cases. If $g^+_k = c^+_k$, we set $I_k := \emptyset$. Otherwise we let $\delta$ be the union of the arcs $\delta(g^+_k, c^+_k)$ (a subarc of $\partial D_k$) and $\delta(c^-_k, g^-_{k+1})$ (a subarc of $\partial D_{k+1}$). The open Jordan arc $\delta$ is contained in $G$ with different endpoints on $\partial G$, hence it is a crosscut. The set $G \setminus \delta$ consists of two simply connected components $G_1$ and $G_2$. One of these components contains all disks of $\mathcal{P}$, the other one is (by definition) the boundary interstice $I_k$.

Lemma 7. $I_k \cap \mathcal{D} = \emptyset$ for all $k = 1, ..., m$.

Proof. Let $k \in \{1, ..., m\}$ be fixed. If $I_k = \emptyset$ the assertion is trivially fulfilled. Let $I_k \neq \emptyset$ and let $\delta$ be the crosscut defined above, so that $G \setminus \delta$ consists of exactly two simply connected domains $G_1 = I_k$ and $G_2$.

Clearly every disk of $\mathcal{P}$ is contained either in $G_1$ or $G_2$. We assume that there is a disk $D_u$ in $G_1$ (remember $D_u \subset G_2$). Because $K$ is connected there is a chain $C$ of vertices $\{u, ..., v\}$, where $v$ is the vertex associated with $D_k$. Because $D_u \subset G_1$ and $D_k \subset G_2$ there have to be two consecutive vertices $w_1, w_2$ in $C$, so that $D_{w_1}$ is contained in $G_1$ and $D_{w_2}$ in $G_2$. The contact point $c(w_1, w_2)$ must lie on $\partial G_1 \setminus \delta$, because there are no contact points of $\mathcal{P}$ on $\delta$ according to Lemma 6.
Let \( w_3 \) be a vertex, so that \( f(w_1, w_2, w_3) \) is a face of \( K \). The interstice \( I := I(w_1, w_2, w_3) \) is contained either in \( G_1 \) or \( G_2 \), because it is disjoint to \( \partial G \). Moreover both arcs \( \partial D_{w_1} \cap \partial I \) and \( \partial D_{w_2} \cap \partial I \) (up to their endpoints) lie in the same domain as \( I \), without being contained in the boundary of \( G \). This implies, that both disks \( D_{w_1} \) and \( D_{w_2} \) are contained either in \( G_1 \) or \( G_2 \), a contradiction. Hence, \( I_k \cap D = \emptyset \) for all \( k = 1, \ldots, m \). 

Last but not least we state a result about glueing simply connected domains along a common boundary arc. The proof is left as an exercise (see [8]).

**Lemma 8.** Let \( G_1 \) and \( G_2 \) be simply connected domains with locally connected boundaries. If \( G_1 \) and \( G_2 \) touch each other along a Jordan arc \( J \) with endpoints \( a, b \), i.e., \( G_1 \cap G_2 = \emptyset \) and \( \overline{G_1} \cap \overline{G_2} = J \), then \( (G_1 \cup J \cup G_2) \setminus \{a, b\} \) is a simply connected domain and its boundary is locally connected.

### 3 Crosscuts

Before we introduce crosscuts of a (univalent) circle packing which fills a domain \( G \), we define crosscuts of its complex.

**Definition 3.** A (combinatoric) crosscut of a complex \( K \) is a sequence \( L = (e_0, e_1, \ldots, e_l) \) of edges in \( K \) with the following properties (i)–(iv):

- (i) The edges are pairwise different, if \( 0 \leq j < k \leq l \) then \( e_j \neq e_k \).
- (ii) For \( 1 \leq j \leq l \) the edges \( e_{j-1} \) and \( e_j \) are adjacent to a common face of \( K \).
- (iii) Three consecutive edges are not adjacent to the same face of \( K \).
- (iv) The edges \( e_0 \) and \( e_l \) are boundary edges.

---

Figure 7: A crosscut \( L \) of \( K \), the vertex sets \( V_L^- \), \( V_L^+ \), \( U_L^+ \), and a corresponding packing.
It is easy to see that only the first and the last edge of a crosscut can be boundary edges of $K$. Because $e_0 \neq e_l$ we have $l \geq 1$. When one edge of a face $f$ belongs to $L$, then $L$ must contain exactly two edges of $f$, and these are subsequent members of $L$. So a crosscut can also be represented by a sequence $(f_1, \ldots, f_l)$ of faces, where $e_{j-1}$ and $e_j$ are adjacent to $f_j$. Since the three edges of a face are not allowed to be consecutive members of $L$, all faces $f_j$ must be pairwise different.

After removing the edges of a crosscut $L$ from $K$, the remaining graph consists of two edge-connected components $K^+_L$ and $K^-_L$. We assume that $K^-_L$ ‘lies to the right’ and $K^+_L$ ‘lies to the left’, respectively, when we move along the edges $e_0, e_1, \ldots, e_l$ in this order. The vertex sets of $K^-_L$ and $K^+_L$ are denoted by $V^-_L$ and $V^+_L$, respectively, and we call them the lower and the upper vertices of $K$ with respect to $L$. The set $U^+_L$ is constituted by all vertices $v$ in $V^+_L$ which are adjacent to an edge in $L$. These vertices and the corresponding disks are said to be the upper neighbors of $L$. A corresponding definition is made for the set $U^-_L$ of lower neighbors of $L$ (see Figure 7).

Given a (combinatorial) crosscut $L$ of a complex $K$ and a circle packing $\mathcal{P}$ for $K$ which fills a domain $G$, we define several related (geometric) crosscuts $J$ of $\mathcal{P}$ in $G$. To begin with, we associate with every edge $e_j = e(u,v)$ in $L$ the contact point $x_j := \overline{D_u \cap D_v}$ of the disks $D_u, D_v \in \mathcal{P}$. The common tangent to $\partial D_u$ and $\partial D_v$ at $x_j$ is denoted $\tau_j$. The set $X := \{x_0, \ldots, x_l\}$ of all contact points associated with edges of $L$ has a natural ordering, induced by the ordering of edges in the crosscut. Since the indexing of the elements fits with this ordering, we write $x_j < x_k$ if $j < k$.

The polygonal crosscut $J^+_L$ is build from the common tangents $\tau_i$ of circles at their contact points $x_i$ as follows. Let $i \in \{1, \ldots, l\}$ and assume that $x_{i-1}$ and $x_i$ are consecutive contact points of the pairs $D_u, D_v$ and $D_v, D_w$, respectively. Then the three circles $\partial D_u, \partial D_v, \partial D_w$ bound an interstice $I := I(u,v,w)$. The tangents $\tau_{i-1}$ and $\tau_i$ intersect each other at a point $s_i$ in $I$, and the union of the closed segments $[s_i, s_{i+1}]$ for $i = 1, \ldots, l-1$ is a Jordan arc in $G$ (see Figure 8).

![Figure 8: Local construction and global view of a polygonal crosscut](image)
In order to complete this arc to a crosscut in $G$ we look at the boundary disks $D_k$ and $D_{k+1}$ which touch each other at $x_0$. If $x_0$ is not a boundary point of $G$ we define $s_0$ as the endpoint of the largest segment $(x_0, s_0)$ on the tangent $\tau_0$ which is contained in $I_k$. Since there is no disk of $\mathcal{P}$ intersecting $I_k$ (Lemma 7) we see that $[x_0, s_0] \subset G$ is disjoint to $\mathcal{P}$ and $s_0 \in \partial G$. If $x_0$ is a boundary point of $G$ we set $s_0 := x_0$.

A similar construction is made for the point $s_{l+1}$ as (“the first”) intersection point of the tangent $\tau_1$ with $\partial G$. Here $x_0 \neq x_l$ ensures that $[s_0, s_1]$ and $[s_l, s_{l+1}]$ live in two different boundary interstices. Although this does not exclude $s_0 = s_{l+1}$, it guarantees that $s_0$ and $s_{l+1}$ are endpoints of the segments $[s_1, s_0]$ and $[s_l, s_{l+1}]$, belonging to different prime ends $s_0^*$ and $s_{l+1}^*$, respectively.

Finally, the union of the closed segments $[s_k, s_{k+1}]$ for $k = 0, \ldots, l$ forms the desired polygonal crosscut $J_L^0 := \bigcup_{k=0}^l [s_k, s_{k+1}]$ in $G$. It can easily be verified that $J_L^0$ is a (topologically closed) Jordan arc which meets $\partial D$ at the contact points $x_k$ – more precisely we have $X \subset J_L^0 \cap \partial D \subset X \cup \{s_0, s_{l+1}\}$. The open set $G \setminus J_L^0$ has two simply connected components $G_0^+$ and $G_0^-$, containing the disks associated with $V_L^+$ and $V_L^-$, respectively.

It is clear that, for a fixed combinatorial crosscut $L$ of $K$, the statement of Theorem 2 depends on the choice of the geometric crosscut $J$: the assertion becomes the stronger, the larger the domain $G_J^-$ is. Unfortunately, there exists (in general) no crosscut $J$ which maximizes $G_J^-$, since the boundary of the largest domain $G_J^-$ need not be a Jordan curve. We therefore extend the concept of crosscuts somewhat, defining the maximal crosscut $J_L^*$ in $\mathcal{P}$ as follows.

![Diagram of a maximal crosscut](image)

**Figure 9:** Construction of a maximal crosscut (which is not a Jordan arc)

Recall that $U_L^+$ is the vertex set of upper neighbors of $L$. If $x_k$ and $x_{k+1}$ are contact points of the disks $D_u, D_v$ and $D_w$ respectively, then either $v \in U_L^+$ or $u, w \in U_L^+$. The interstice $I(u, v, w)$ is bounded by three (topologically closed) circular arcs $\alpha_u, \alpha_v$ and $\alpha_w$, respectively. If $v \in U_L^+$ we connect $x_{k-1}$ with $x_k$ by the arc $a_k := \alpha_v$, in the second case we connect these points by the concatenation $a_k := \alpha_u \cup \alpha_w$ (see Figure 6). In addition we connect $x_0$ and $x_l$ with $\partial G$ by arcs $a_0 := \delta(g_j^+, x_0)$ and $a_{l+1} := \delta(x_l, g_k^*)$ of those circles $\partial D_j$ and $\partial D_k$ which are
upper neighbors of \( L \) and contain \( x_0 \) and \( x_l \), respectively. The union \( J^+_L := \bigcup_{k=0}^{l+1} a_k \) of these arcs is a curve which we call the \textit{maximal crosscut} in \( \mathcal{P} \) with respect to \( L \).

The maximal crosscut \( J^+_L \) is composed from a finite number of circular (topologically closed) arcs \( \omega_i \) which are linked at the \textit{turning points} \( t_i \) of \( J^+_L \), and every contact point \( x_k \) lies exactly on one arc \( \omega_i \) (see Figure 9). If \( J^+_L \) is not a Jordan arc, \( G \setminus J^+_L \) may consist of several connected components (see Figure 9, right), one of them containing all disks associated with vertices \( v \) in \( V^- \). We call this component \( G^-_L \) the \textit{maximal lower domain} for \( L \) with respect to \( \mathcal{P} \), and we set \( G^-_L := G \setminus G^+_L \). For the sake of brevity we define \( \omega := J^+_L \) and \( \Omega := G^-_L \).

Since the curve \( \omega \) can have multiple points (see Figure 9, right) there is no natural ordering of the \textit{points} on \( \omega \). However, considering \( \omega \) as part of the boundary of \( \Omega \), we can introduce an ordering of the \textit{terminal points} \( q \in \omega \) of open Jordan arcs \( \gamma(p, q) \) in \( \Omega \). In order to describe this procedure we need the following result.

**Lemma 9.** For any combinatorial crosscut \( L \) the maximal lower domain \( \Omega = G^-_L \) is simply connected and has a locally connected boundary.

**Proof.** Let \( G_i^- \) be the lower domain with respect to the polygonal crosscut \( J_i \) in \( \mathcal{P} \). Then \( G \setminus J_0^i \) consists of two simply connected domains \( G_i^- \) and \( G_i^+ \), respectively. The maximal lower domain \( G^-_L \) is constructed by gluing a finite number of simply connected domains along straight line segments to \( G_0^- \). Hence the assertion follows from Lemma 8.

The assertion of Lemma 9 guarantees that any (fixed) conformal mapping \( g : \mathbb{D} \to \Omega \) has a continuous extension to \( \mathbb{D} \), which we again denote by \( g \) (see 8 Theorem 2.1). With respect to this mapping, we let \( \sigma_i \subset \mathbb{T} \) denote the preimage of the circular arcs \( \omega_i \) with \( i = 1, \ldots, n \). Then \( \sigma := \bigcup_{i=1}^n \sigma_i \) is the preimage of the maximal crosscut \( \omega \).

By the Prime End Theorem, the mapping \( g \) induces a bijection \( g^* \) between \( \mathbb{T} \) the set of prime ends of \( \Omega \). We denote by \( \omega^* := g^*(\sigma) \) the set of prime ends associated with \( \Omega \), and, for \( i = 1, \ldots, n \), we let \( \omega^*_i := g^*(\sigma_i) \) be the subsets of \( \omega^* \) corresponding to the arcs \( \sigma_i \).

Note that the preimages \( \sigma_i \) of the circular arcs \( \omega_i \) are topologically closed subarcs of \( \mathbb{T} \), and that the preimage \( \mathbb{T} \setminus \sigma \) of \( \partial \Omega \setminus \omega \) is not empty. Therefore \( \sigma_i \) and \( \sigma_j \), and thus \( \omega^*_i \) and \( \omega^*_j \), are disjoint if \( |i - j| > 1 \), while their intersection contains exactly one element if \( |i - j| = 1 \).

Further we see that the arcs \( \sigma_1, \sigma_2, \ldots, \sigma_n \) (in this order) are arranged in clockwise direction on \( \mathbb{T} \). It is therefore just natural to order the \textit{points} on the arc \( \sigma \) (and hence on each subarc \( \sigma_i \)) also in \textit{clockwise} direction. The mapping \( g^* \) transplants this ordering from \( \sigma \) to the set \( \omega^* \) of prime ends. If \( \gamma^*_1 = g^*(s_1) \) and \( \gamma^*_2 = g^*(s_2) \) are two prime ends of \( \omega^* \), the notion \( \gamma^*_1 \leq \gamma^*_2 \) refers to the ordering \( s_1 \leq s_2 \) of the associated points on \( \sigma \).

**Remark.** Every \( \omega_i \) without its endpoints is an open Jordan arc, so the interior points of \( \omega_i \) and \( \sigma_i \) corresponds one-to-one. Let \( \gamma \in \Omega \) be an open Jordan arc with terminal point \( q \) on \( \omega \), then the associated unique prime end \( \gamma^* \) in \( \omega^* \) must lie in \( \omega^*_i \), if \( q \) is an interior point of \( \omega_i \). Only if \( q \) is an endpoint of \( \omega \) there is a chance that the prime end \( \gamma^* \) is not contained in \( \omega^*_i \), because now \( \gamma^* \) depends on how \( \gamma \) approaches \( q \).
4 Loners

So far we have studied properties of a single circle packing $P$ which fills $G$. In the next step we consider pairs $(P, P')$ of packings which are subject to the assumptions of Theorem 2.

**Definition 4.** A pair $(P, P')$ of univalent circle packings for the complex $K$ is said to be *admissible* (for the crosscut $L$ of $K$ in $G$ with alpha-vertex $v_a$) if it satisfies the following conditions:

(i) The packing $P$ fills the bounded, simply connected domain $G$, and the packing $P'$ is contained in $G$ (see Definition 2).

(ii) For all vertices $v \in U_L^-$ (the lower neighbors of $L$) the disks $D_v'$ are contained in $G_L^-$ (the maximal lower domain of $G$ for $L$ with respect to $P$).

(iii) The centers of the alpha-disks of $P$ and $P'$ coincide and lie in $G_L^+ := G \setminus G_L^-$. Though it would be more precise to speak of an admissible sixtuple $(K, L, G, P, P', v_a)$, we shall use the term “admissible” generously, for instance saying that “$L$ is an admissible crosscut for $(P, P')$”.

Recall that $U_L^+$ denotes the vertex set of those disks in $P$ which lie in $G_L^+$ and touch the crosscut (“upper neighbors of $L$”). In the next step we are going to explore the interplay of the disks $D_v$ and $D_v'$ for $v, w \in U_L^+$.

**Definition 5.** Let $(P, P')$ be an admissible pair of circle packings for the complex $K$ with crosscut $L$. A vertex $v$ in $U_L^+$ is called a *loner*, if $D_v' \cap D_v = \emptyset$ for all $w \in U_L^+$ with $w \neq v$.

The concept of loners was introduced by Schramm [12] in a similar but somewhat different context. The main characteristic of a loner is the following.

**Lemma 10.** Let $v$ in $U_L^+$ be a loner of the admissible pair $(P, P')$ with complex $K$ and crosscut $L$. Then $D_v' \cap (G_L^+ \setminus D_v) = \emptyset$.

**Proof.** Let $u \in U_L^-$ and $w \in U_L^+$ be neighbors of $v$, and let $p$ and $q$ be the contact points of the disks $D_v'$ with $D_u'$ and $D_v$ with $D_w$, respectively. Clearly $p \neq q$, otherwise $D_u'$ had to intersect $D_v$ or $D_w$, a contradiction to condition (ii) of the admissible pair $(P, P')$.

Assume that $p$ is a boundary point of $D_v$. Then $\partial D_v$ and $\partial D_u'$ have a common tangent at $p$, otherwise $D_u'$ had to intersect $D_v$, a contradiction to condition (ii) of the admissible pair $(P, P')$. It follows that either $\overline{D_v'} \setminus \{p\} \subset D_v$ or $D_v' = D_v$ or $\overline{D_u} \setminus \{p\} \subset D_v'$. The latter implies that $q \in D_v'$, hence $D_v' \cap D_w \neq \emptyset$, which is impossible since $v$ is a loner. The other two cases imply the statement we want to prove.

Assume that $p$ is not a boundary point of $D_v$. Suppose that the assertion of Lemma 10 were false, i.e., there is some point $r$ in $D_v'$ which is also contained in $G_L^+ \setminus D_v$. Because $p$ lies in the maximal lower domain $G_v^-$, and $r$ lies in the upper domain $G_v^+$, both subarcs $\delta(p, r)$ and $\delta(r, p)$ of $D_v'$ must intersect the maximal crosscut $J_v^+$ at points $r_1$ and $r_2$, respectively. Since the vertex $v$ is a loner, we have $r_1, r_2 \in \partial D_v$. If $r_1 = r_2$, the boundary of $D_v'$ is the union of $\delta[p, r_1]$ and $\delta[r_2, p]$, hence $D_v' \cap G_v^+ = \emptyset$, a contradiction to $r \in D_v'$. If $r_1 \neq r_2$, we have $\partial D_v' \cap D_v = \delta(r_2, r_1)$, hence $r$ must be contained in $D_v$, a contradiction to $r \in G_L^+ \setminus D_v$. \hfill $\Box$
In Section 6 the property of loners described in Lemma 10 will allow us to move the crosscut $L$ through the packing, reducing in every step the number of circles in $G_L^+$. The next result is crucial for the applicability of this procedure.

**Lemma 11 (Existence of Loners).** Every admissible pair $(\mathcal{P}, \mathcal{P}')$ of circle packings with crosscut $L$ has a loner.

The proof is divided into several steps; the first part uses the geometry of disks, then we employ some topology, and finally everything is reduced to pure combinatorics. We start with some preparations.

Recall the definition of the contact points $x_k$: If $L = (e_0, \ldots, e_l)$ and $e_k = \langle u, v \rangle$, for some $k \in \{0, \ldots, l\}$, then $x_k := D_u \cap D_v$. Using the same notation, the corresponding contact points of disks in $\mathcal{P}'$ are given by $y_k := D'_u \cap D'_v$, where $Y := \{y_0, \ldots, y_l\}$ is the set of all such contact points.

The contact points $x_k$ form an ordered set on the maximal crosscut $\omega := J_L^+$, which is the upper boundary of the maximal lower domain $\Omega := G_L^-$. Since every $x_k$ lies on exactly one arc $\omega_i$, the set $X$ of contact points splits into classes $X_i := \{x_k \in X : x_k \in \omega_i\}$, $i = 1, \ldots, n$. The set $Y$ of the contact points of $\mathcal{P}'$ is divided accordingly, $Y_i := \{y_k \in Y : x_k \in \omega_i\}$ (the $x_k$ is no typo here). Like $X$, the set $Y$ is endowed with a natural ordering, we write $y_j < y_k$ if $j < k$.

Our next aim is to construct a Jordan arc $\alpha$ which is contained in $\overline{\Omega}$ and carries the contact points $y_k$ in their natural order.

**Lemma 12.** If $(\mathcal{P}, \mathcal{P}')$ is an admissible pair, then there exist oriented Jordan arcs $\alpha_k$ from $y_{k-1}$ to $y_k$ such $\alpha := \bigcup_{k=1,\ldots,l} \alpha_k$ is a Jordan arc in $\overline{\Omega}$ and $\alpha \cap \omega \subset Y$.

**Proof.** Let $k \in \{1, \ldots, l\}$. In order to determine the arc $\alpha_k$ of $\alpha$ which connects $y_{k-1}$ with $y_k$ we remark that both points lie on the boundary of one and the same disk $D'_v \in \mathcal{P}'$. We distinguish two cases:

**Case 1.** If $v \in V_L^-$, then the disk $D'_v$ is contained in $\Omega$, and we choose the segment $\alpha_k := [y_{k-1}, y_k]$ (see Figure 10, left).

![Figure 10: Construction of the Jordan arc $\alpha$ in Case 1 (left) and Case 2 (middle, right)]
Case 2. If \( v \in V_L^+ \), then \( e_{k-1}, e_k \) and a third edge \( \langle u, w \rangle \) of \( K \) form a face of \( K \), and the (neighboring) disks \( D'_u \) and \( D'_w \) are both contained in \( \Omega \). So we let \( z_k := D'_u \cap D'_w \) and connect \( y_{k-1} \) with \( y_k \) by \( [y_{k-1}, z_k] \cup [z_k, y_k] \subset \overline{\Omega} \) (see Figure 10, middle).

It is clear that all open segments \( (y_{k-1}, y_k) \), \( (y_{k-1}, z_k) \), \( (z_k, y_k) \) for \( k = 1, \ldots, l \) are pairwise disjoint, and that \( y_k \neq z_j \). However, it is possible that two endpoints \( z_k \) and \( z_j \) coincide for \( j \neq k \), in which case the concatenation of the arcs \( \alpha_k \) is not a Jordan arc.

If this happens, the point \( z := z_j = z_k \) is the contact point of two disks \( D'_u \) and \( D'_w \) with \( u, w \in V_L^- \). A little thought shows that then \( z \) can neither lie on the boundary of \( G \) nor on \( \omega \), and hence it must be an interior point of \( \Omega \). This allows one to resolve the double point of \( \alpha \) at \( z \) without destroying its other properties (see Figure 10, right.)

In the next step we transform the existence of loners to a topological problem. Technically this is much simpler when \( \alpha \) and \( \omega \) are disjoint. We consider this ‘regular case’ in Section 4.1. The ‘critical case’, where intersections of \( \alpha \) and \( \omega \) are admitted, will be treated in Section 4.2.

4.1 The Regular Case

Here we assume that \( \alpha \cap \omega = \emptyset \), which implies that all contact points \( y_k \ (k = 0, \ldots, l) \) lie in the lower domain \( \Omega \).

We fix \( i \in \{1, \ldots, n\} \) and denote by \( y'_i \) and \( y'^+_i \) the smallest and the largest member of \( Y_i \) with respect to the natural ordering of \( Y \), respectively. Both points (which may coincide), as well as all elements of \( Y_i \), lie on the same circle \( \partial D'_v \) associated with a vertex \( v = v(i) \in V \).

Let \( \delta'_i \) be the negatively oriented topologically closed subarc of \( \partial D'_v \) from \( y'_i \) to \( y'^+_i \). We consider the largest subarcs \( \nu_i \) and \( \pi_i \) of \( \delta'_i \) which are contained in \( \overline{\Omega} \setminus \omega \) and have initial points \( y'_i \) (for \( \eta_i \)) and \( y'^+_i \) (for \( \pi_i \)), respectively (see Figure 11).

![Figure 11: The arcs \( \nu_i \) and \( \pi_i \) and their intersection with the boundary of \( G_L^+ \)](image)

Lemma 13. If there exists no loner, then the terminal points \( \nu^+_i \) and \( \pi^+_i \) of \( \nu_i \) and \( \pi_i \), respectively, lie on \( \omega \) for \( i = 1, \ldots, n \).
Proof. If one of the arcs \( \nu_i \) or \( \pi_i \) does not intersect \( \omega \), then both coincide with \( \delta'_i \). In this case, the disk \( D'_{v(i)} \) is separated from \( G^+_L \) by the union of the arcs \( \alpha \) and \( \delta'_i \), which implies that \( D'_{v(i)} \) cannot intersect any disk \( D_w \) with \( w \in U^+_L \), so that \( v(i) \) is a loner.

Since (with the exception of their endpoints) the circular arcs \( \nu_i \) \( (i = 2, \ldots, n) \) and \( \pi_i \) \( (i = 1, \ldots, n - 1) \) lie in \( \Omega \) and have terminal points \( \nu_i^+ \) and \( \pi_i^+ \) on \( \omega \), they define prime ends \( \nu_i^* \) and \( \pi_i^* \) in \( \omega^* \). Because the arcs \( \nu_1 \) and \( \pi_n \) need not lie in \( \Omega \), a modified definition is needed for the prime ends \( \nu_1^* \) and \( \pi_n^* \). To do so we replace \( \nu_1 \) and \( \pi_n \) by slightly perturbed circular arcs \( \nu_i^0 \) and \( \pi_n^0 \), respectively, which have the same endpoints as \( \nu_1 \) and \( \pi_n \), respectively, and lie in \( \Omega \) (with the exception of their endpoints). Then \( \nu_i^* \) and \( \pi_n^* \) are defined as the prime ends associated with the terminal points of \( \nu_i^0 \) and \( \pi_n^0 \), respectively. Clearly such arcs \( \nu_i^0 \) and \( \pi_n^0 \) exist, and for all sufficiently small \( \varepsilon \) they define the same prime ends \( \nu_1^*, \pi_n^* \in \omega^*, \) respectively.

Since the set of prime ends \( \omega^* \) is endowed with a natural ordering, we can compare the prime ends \( \nu_i^* \) and \( \pi_i^* \).

**Lemma 14.** If \( (\mathcal{P}, \mathcal{P}') \) has no loner, the prime ends \( \nu_i^* \) and \( \pi_i^* \) form an interlacing sequence with respect to the prime end ordering of \( \omega^* \),

\[
\nu_1^* \leq \pi_1^* \leq \nu_2^* \leq \ldots \leq \nu_n^* \leq \pi_n^*.
\]

**Proof.** Let \( y_- := y_0 \) and \( z_- \) be the initial and terminal points of \( \nu_1 \), while \( y_+ := y_r \) and \( z_+ \) are the initial and terminal points of \( \pi_n \), respectively. We have \( z_-, z_+ \in \omega \) due to Lemma \[13\]. Further, let \( \omega_0^* \) be the set of all prime ends \( \gamma^* \) of \( \omega^* \) with \( \nu_1^* \leq \gamma^* \leq \pi_n^* \), and denote the set of all corresponding points on \( \omega \) by \( \omega_0 \). The set \( \omega_0 \) is a curve or a single point. Together with the Jordan arcs \( \nu_1, \alpha \) and \( \pi_n \) it forms the boundary of a simply connected domain \( \Omega_0 \subseteq \Omega \) with locally connected boundary. Let \( \Omega_0^* \) be the set of all prime ends associated with points on \( \partial \Omega_0 \). Because \( \Omega_0 \setminus \omega_0 \) is an open Jordan arc, the points \( y_-, y_+ \) are associated with uniquely determined prime ends \( y_1^*, y_2^* \) of \( \Omega_0 \).

Contrary to this, the points \( z_-, z_+ \) may be associated with several prime ends of \( \Omega_0 \). In order to explain which one we choose, let again \( \nu_1^*, \pi_n^* \) be small perturbations (as explained above) of \( \nu_1, \pi_n \), respectively, so that both arcs are crosscuts in \( \Omega_0 \). We define \( z_-^*, z_+^* \) as the prime ends in \( \omega^* \) associated with the terminal points \( z_- \) and \( z_+ \) of \( \nu_1^*, \pi_n^* \), respectively.

We have \( n > 1 \), because otherwise a loner would exist. It follows that \( y_- \neq y_+ \), so \( y_1^* \neq y_2^* \). From \( \alpha \cap \omega = \emptyset \) we get \( z_- \neq y_-, z_+ \notin \{y_-, y_+\} \), hence \( z_-^*, z_+^* \notin \{y_1^*, y_2^*\} \). If \( z_-^* = z_1^* =: z^* \), we directly get \( \omega^* \subset \Omega_0^* =: z^* \). This implies \( \nu_1^* = \pi_1^* = \nu_2^* = \ldots = \pi_n^* = z^* \), so the lemma holds. (We consider this case here, though Lemma \[15\] shows, that it cannot occur.) If \( z_-^* \neq z_+^* \), the prime ends \( y_-^*, y_+^*, z_-^* \) and \( z_+^* \) are pairwise distinct and with respect to the (cyclic) ordering of \( \Omega_0 \) we have \( y_-^* < y_+^* < z_-^* < z_+^* < y_2^* \). Therefore \( \Omega_0 \) can be mapped conformally onto a rectangle \( Q \) (with appropriately chosen aspect ratio) such that \( y_-^*, y_+^*, z_-^* \) and \( z_+^* \) correspond to the four corners of \( Q \) (see \[8\]), what is depicted in Figure \[12\]. Any of the arcs \( \nu_i \) \( (i = 2, \ldots, n) \) and \( \pi_i \) \( (i = 1, \ldots, n - 1) \) is mapped onto a crosscut of \( Q \) which connects two opposite sides of this rectangle. Since these Jordan arcs cannot cross each other in the interior of \( Q \), the ordering of their initial points on one side of \( Q \) is transplanted to the ordering of their terminal points on the opposite side of \( Q \). Translated back to \( \Omega_0 \), this implies
that the ordering of the prime ends $\nu_{i}^{*}$ and $\pi_{i}^{*}$ is the same as the ordering of the initial points $y_{i}^{-}$ and $y_{i}^{+}$ of $\nu_{i}$ and $\pi_{i}$, respectively, along the Jordan curve $\alpha$. By construction, the latter points form an interlacing sequence.

\[ \nu_{i}^{*}, \pi_{i}^{*} \]

\[ y_{i}^{-}, y_{i}^{+} \]

By construction, the latter points form an interlacing sequence.

Figure 12: Construction of $\Omega_{0}$ and $Q$ from $\omega, \alpha$ and $\nu_{1}, \pi_{n}$

Lemma 15. If both prime ends $\nu_{i}^{*}$ and $\pi_{i}^{*}$ belong to $\omega_{i}^{*}$, then the corresponding vertex $v(i)$ is a loner.

Proof. Let $v := v(i)$. It follows from $\nu_{i}^{*}, \pi_{i}^{*} \in \omega_{i}^{*}$ that $\nu_{i}^{+}, \pi_{i}^{+} \in \omega_{i} \subset \partial D_{v}$. If $\pi_{i}^{+} \neq \nu_{i}^{+}$, the positively oriented open subarc $\delta_{i}^{+}$ of $P_{i}^{+}$ from $\pi_{i}^{+}$ to $\nu_{i}^{+}$ lies in $D_{v}$. If $\pi_{i}^{+} = \nu_{i}^{+}$, we set $\delta_{i}^{+} := \emptyset$. In both cases the union of $\alpha_{i}, \pi_{i}, \delta_{i}^{+}$ and $\nu_{i}$ is a Jordan curve which does not intersect the disks $D_{u}$ with $u \in U_{L}^{+}$ and $u \neq v$. So either $D_{v}'$ is disjoint to all such disks $D_{u}$, or one of the disks $D_{u}$ is contained in $D_{v}'$. In the latter case the prime ends $\nu_{i}^{*}$ and $\pi_{i}^{*}$ cannot both belong to the same set $\omega_{i}^{*}$. 

Proof of Lemma 17. After these preparations we are ready to harvest the fruits: Assume that $(\mathcal{P}, \mathcal{P}')$ has no loner. Then, by Lemma 13, the endpoint $\nu_{i}^{+}$ of the arc $\nu_{i}$ must lie on $\omega$ and hence $\nu_{i}$ is associated with a prime end $\nu_{i}^{*} \in \omega^{*}$. If $\nu_{i}^{*} \in \omega_{k}^{*}$, we choose the smallest such $k$ and set $l(i) := k$. Similarly, we denote by $r(i)$ the smallest number $k$ for which $\pi_{i}^{*} \in \omega_{k}^{*}$.

Lemma 14 tells us that $r(i) \geq l(i)$ and $l(i+1) \geq r(i)$. In conjunction with Lemma 15 we conclude that the first condition implies $r(i) \geq l(i) + 1$. Starting with $l(1) \geq 1$, we get inductively that $r(i) \geq i + 1$ for $i = 1, \ldots, n$, ending up with the contradiction $r(n) \geq n + 1$. This proves Lemma 11 in the regular case.

4.2 The Critical Case

The second case, where we admit that $\alpha \cap \omega \neq \emptyset$, will be reduced to the regular case by an appropriate deformation of the Jordan arc $\alpha$. 

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Definition 6. A contact point \( y \in Y \) is called regular if \( y \notin \omega \), otherwise it is said to be critical.

If \( y \in Y \) is a critical contact point, then \( y \in \alpha \cap \omega \neq \emptyset \), and hence \( y \in \omega_j \) for some \( j \). Since \( y = \partial D'_u \cap \partial D'_v \) with some \( u \in U_L^+ \) and \( v = v(i) \in U_L^+ \), we see that \( y \) cannot be an endpoint of \( \omega_j \) (turning point of \( \omega \)) – otherwise \( D'_u \) would not be contained in \( \Omega \). Moreover, the circles \( \partial D'_u, \partial D'_v \), and \( \omega_j \) must be mutually tangent at \( y \). The arc \( \omega_j \) is a subset of the circle \( \partial D_w \) (with \( w = v(j) \in U_L^+ \)). Hence either \( D'_v \subset D_w \) (with \( D'_v = D_w \) admitted) or \( D_w \) is a proper subset of \( D'_v \).

In the next step we modify the Jordan arc \( \alpha \) in a neighborhood of \( y \) and redefine the arcs \( \nu_i \) and \( \pi_i \) (connecting \( y \) with \( \omega \)) introduced in the regular case.

Let \( \varepsilon \) be a sufficiently small positive number. Denote by \( z \) the \( \varepsilon \)-shift of \( y \) in the direction of the center of \( D_u' \). Append to \( D'_v \) an equilateral open triangular domain \( T \) with one vertex at \( z \), two vertices on \( \partial D'_v \), and symmetry axis through \( y \) and \( z \) (see Figure 13).

For \( y \notin \{y_0, y_l\} \) let \( \nu_i \) (and \( \pi_i \)) be the largest positively (negatively) oriented subarc of \( \partial (D'_v \cup T) \) which has initial point \( z \) and is contained in \( \Omega \). For \( y \in \{y_0, y_l\} \) (and only then) it can happen that \( y \) is a boundary point of \( G \). Therefore we define \( \nu_i := [z, y] \) in the case \( y = y_0 \), and \( \pi_i := [y, z] \) in the case \( y = y_l \). The case \( y_0 = y_l \) can never occur, because \( l \geq 1 \).

Denote by \( \nu_i^+ \) and \( \pi_i^+ \) the terminal points of \( \nu_i \) and \( \pi_i \). Clearly, \( \nu_i^+, \pi_i^+ \in \omega \), so let \( \nu_i^*, \pi_i^* \in \omega^* \) be their associated prime ends.

We see, that the statement of Lemma 13 holds in the critical case, too. Moreover, for the critical case, Lemma 14 can be proved in exactly the same way as for the regular case, we just have to apply the adapted definitions of \( \nu_i^* \) and \( \pi_i^* \). All what is missing is the following “critical” version of Lemma 15.

Lemma 16. Assume that \( \partial D'_u \) with \( v = v(i) \in U_L^+ \) contains a critical contact point \( y \in Y \cap \omega \). Then \( v \) is a loner if and only if \( \nu_i^* \) and \( \pi_i^* \) belong to \( \omega_i^* \).
Proof. We use the notations introduced above, with \( \varepsilon > 0 \) fixed and sufficiently small. We distinguish two cases.

Case 1. Let \( D'_v \subset D_w \) (see Figure 13, left). Then \( v \) is a loner if and only if \( w = v \), and this holds, if and only if \( j = i \) and \( \nu^*_i, \pi^*_i \in \omega^*_i \).

Case 2. Let \( D_w \subset D'_v \) and \( D_w \neq D'_v \) (see Figure 13, right). Then \( D'_v \) intersects at least two “upper” disks (namely \( D_w \) and one of its neighbors), so that \( v \) is not a loner. According to our construction, we have \( \nu^*_i \leq y^* \leq \pi^*_i \) (where \( y^* \in \omega^*_j \) is the prime end corresponding to \( y \) and \( w = v(j) \)), but both equalities are never fulfilled at the same time, and \( \nu^*_i, \pi^*_i \notin \omega^*_j \) for \( w = v(j) \). Therefore \( \nu^*_i \in \omega^*_m \) and \( \pi^*_i \in \omega^*_n \) with \( m \leq j \leq n \), but \( m < n \), so the prime ends \( \nu^*_i \) and \( \pi^*_i \) cannot both belong to the same class \( \omega^*_i \).

\( \square \)

Remark. If \( D'_v \) has several critical contact points \( y_k \in Y \cap \omega_j \) with the same arc \( \omega_j \), then \( D'_v \) must be tangent to \( D_w \) with \( w = v(j) \) at two different points. This implies that \( D'_v = D_w \), which explains why the criterion is independent of the choice of \( y \).

After replacing all critical contact points \( y_k \) by the shifted points \( z_k \), and modifying the construction of the curve \( \alpha \) accordingly, Lemma 11 can be proved completely the same way as in the regular case.

In Section 5 we need the following generalization of Lemma 11. We point out that \( v(i) = v(j) \) is allowed in assertion (i).

**Lemma 17.** Let \( D_{v(i)} = D'_{v(i)} \) and \( D_{v(j)} = D'_{v(j)} \) with \( 1 \leq i < j \leq n \). Then, in each of the following cases (i)-(iii), there exists a loner \( v(k) \) which is different from \( v(i) \) and \( v(j) \), such that \( k \) satisfies the corresponding conditions:

(i) If \( 1 \leq i < j - 1 \leq n - 1 \), then \( i < k < j \),

(ii) If \( i > 1 \), then \( 1 \leq k < i \),

(iii) If \( j < n \), then \( j < k \leq n \).

**Proof.** The proof differs only slightly from the proof of Lemma 11. For example, in order to prove (i) we need only replace the first inequality \( l(1) \geq 1 \) by \( l(i+1) \geq i + 1 \) (which follows from \( D_{v(i)} = D'_{v(i)} \)) and, assuming that no loner \( v(k) \) with \( i < k < j \) exists, proceed inductively for \( k = i+1, \ldots, j \) until we arrive at \( r(j) \geq j + 1 \). The last condition contradicts \( D_{v(j)} = D'_{v(j)} \).

If \( v(k) = v(i) \) or \( v(k) = v(j) \), we repeat the procedure, replacing \( i \) (in the first case) or \( j \) (in the second case) by \( k \), respectively. Iterating this a number of times, if necessary, we eventually find a loner \( v(k) \) which is different from \( v(i) \) and \( v(j) \), because for all \( m = 2, 3, \ldots, n - 1 \) we have \( v(m - 1) \neq v(m) \) and \( v(m) \neq v(m + 1) \).

\( \square \)

### 5 Structure of Upper Neighbors

In this section we analyze the structure of the set of upper neighbors \( U_L^+ \) and its subset of loners in more detail.
Two consecutive (non-oriented) edges \( e_{j-1} \) and \( e_j \) of \( L = (e_0, \ldots, e_l) \) can be represented as \( e_{j-1} = e(u,v) \) and \( e_j = e(v,w) \). The third edge of the face \( f(u,v,w) \) is considered as oriented from \( u \) to \( w \), and we set \( e_0^0 := \langle u, w \rangle \). The set of edges \( e_0^0 \) splits into two classes. We define \( E^-_L \) as the set of those \( e_0^j \) where the face \( \langle u, v, w \rangle \) is oriented counter-clockwise, whereas \( E^+_L \) consists of those edges with clockwise orientation of \( \langle u, v, w \rangle \), respectively. After renumbering the elements of \( E^-_L \) and \( E^+_L \), without changing their order, we get two sequences of oriented edges \( E^-_L = \{e^1_{-1}, \ldots, e^1_{-p}\} \) and \( E^+_L = \{e^1_{+1}, \ldots, e^1_{+q}\} \) (with \( p + q = l \)), which are called the sequences of lower and upper accompanying edges of the crosscut \( L \), respectively.

Here are some basic properties of \( E^-_L, E^+_L \), which follow quite easy from the definition of \( L \) (proofs are left as exercises). The oriented edges in \( E^-_L \cup E^+_L \) are pairwise disjoint; the corresponding non-oriented edges can appear at most twice, and either both in \( E^-_L \) or both in \( E^+_L \).

Two consecutive edges \( e_{j-1}^\pm \) and \( e_j^\pm \) are linked at a common vertex. The vertex set of all edges in \( E^+_L \) is precisely the set \( U^+_L \) of upper neighbors of \( L \).

Figure 14 shows two examples. The involved crosscut on the right models the fourth generation of the Hilbert curve. With the exception of boundary edges, all edges in \( E^-_L \) (lighter color) and in \( E^+_L \) (darker color) appear with both orientations (not shown in the picture).

![Figure 14: The upper and the lower accompanying edges of a crosscut](image)

When we arrange the elements of \( U^+_L \) in the order they are met along the edge path \( E^+_L \) we get the sequence \( S^+_L \) of upper accompanying vertices. A similar definition is made for the sequence \( S^-_L \) of lower accompanying vertices. The geometry of circle packings causes some combinatorial obstructions for these sequences.

**Lemma 18.** The sequence \( S^+_L \) of upper accompanying vertices cannot contain the pattern \((\ldots, u, \ldots, v, \ldots, u, \ldots, v, \ldots)\) with \( u \neq v \).

**Proof.** If the sequence \( S^+_L \) contains the pattern \((\ldots, u, \ldots, v, \ldots, u, \ldots)\), the oriented curve \( \omega \) has three subarcs \( \omega_i, \omega_j, \omega_k \) with \( i < j < k \) such that \( \omega_i, \omega_k \subset \partial D_u \) and \( \omega_j \subset \partial D_v \). But then \( \omega \)
cannot contain a subarc of $\partial D_v \setminus \omega_j$ (see Figure 15 left), which would be necessary to append another $v$ to the sequence.

**Figure 15**: Illustrations to Lemma 18 and Lemma 20

**Definition 7.** A vertex $v \in U^+_L$ which appears only once in the sequence $S^+_L$ is called *simple*, the other elements in $U^+_L$ are said to be *multiple* vertices.

If $v$ is a multiple vertex in $U^+_L$, there are sequences $M := \{e^+_i, e^+_i, \ldots, e^+_j\} \subset E^+_L$ of accompanying edges such that $v$ is the initial vertex of $e^+_i$, as well as the terminal vertex of $e^+_j$ with $i < j$. Any such sequence is called a *loop* for $v$. We say that a loop $M$ *meets a vertex* $u$, if $u$ is adjacent to an edge in $M$ and $u \neq v$. The *set of vertices* met by $M$ is denoted by $V_M$. A loop $M$ also generates a *sequence of vertices* $U_M = \{v, v_1, \ldots, v_m, v\}$ when we arrange the elements of $V_M$ in the order they are met along the edge path $M$.

**Lemma 19.** Every loop $M$ of a multiple vertex $v$ meets a simple vertex $u$.

**Proof.** We consider the sequence $U_M = \{v, v_1, \ldots, v_m, v\}$ of vertices in $V_M$, arranged in the order as they are met by the edge path $M$. Let $w$ denote the element of this sequence with the earliest second appearance (this does *not* mean the first element which appears twice). Since $w$ cannot appear twice in direct succession, there exists a vertex $u$ in between the first two symbols $w$.

In order to show that $u$ is a simple vertex, we remark that $U_M$ is a subsequence of the sequence $S^+_L$ of upper accompanying vertices. By definition of $w$, there cannot be a second $u$ in $S^+_L$ between the two symbols $w$ next to $u$, and by Lemma 18, the sequence $S^+_L$ cannot contain a second $u$ outside these two $w$'s.

Since loners are vertices in $U^+_L$, it makes sense to speak of simple and multiple loners.
**Lemma 20.** Let \( v \) be a multiple loner with \( D'_v \neq D_v \). If \( u \neq v \) is a vertex which is met by a loop of \( v \), then \( u \) is a loner and \( D'_u \cap D_u = \emptyset \).

**Proof.** Let \( M \) be a loop of \( v \) with \( U_M = \{v, v_1, ..., v_m, v\} \). Let \( i \) be the smallest index, so that \( y_i \) is a contact point of \( v_1 \), and let \( j \) be the largest index, so that \( y_j \) is a contact point of \( v_m \). According to the ordering of \( Y \) and \( U_M \) (as subsequences of \( S_L^+ \)), \( y_{i-1} \) and \( y_{j+1} \) are contact points of \( D'_v \). Let \( u \in \{v_1, ..., v_m\} \) with \( u \neq v \).

The disk \( D'_u \) is enclosed by the union of the subarc \( \delta' := \delta[y_{i-1}, y_{j+1}] \) of \( D'_v \) and the subarc \( \alpha' \subset \alpha \) which connects the points \( y_{i-1} \) and \( y_{j+1} \) on \( \alpha \) (see Figure 15). Since \( v \) is a loner with \( D'_v \neq D_v \), it is clear that \( y_{i-1}, y_{j+1} \notin D_v \), and hence either \( D'_v \cap D_v = \emptyset \) or \( \partial D'_v \cap \partial D_v \) consists of one or two points. In both cases \( \delta' \) does not intersect \( D_v \). Therefore the union \( \alpha' \cup \delta' \) is contained in \( \overline{D} \), hence \( u \) is a loner. In particular \( D'_u \cap D_u = \emptyset \), which proves the last assertion. \( \square \)

Combining Lemma 11, Lemma 17 (applied recursively), Lemma 19 and Lemma 20 (applied recursively), the essence of this section can be summarized in the following lemma.

**Lemma 21.** Let \((P, P')\) be an admissible pair of circle packings with crosscut \( L \).

(i) The pair \((P, P')\) contains a simple loner \( v \in U^+_L \).

(ii) Every loop of a multiple loner \( v \) meets a simple loner \( u \), and if \( D'_v \neq D_v \) then \( D'_u \neq D_u \).

## 6 Proof of the Main Theorem

After all these preparations we are eventually in a position to prove Theorem 2. To begin with, we use the concept of loners and combinatorial surgery to modify the crosscut \( L \). In every step of this procedure the number of vertices in \( V^+_L \) is reduced. At the end we get a special combinatorial structure which is called a slit. Roughly speaking, this is a chain of vertices connecting the alpha-vertex with a boundary vertex. We shall prove that the disks of both packings coincide along a slit.

Then a subdivision procedure generates a sequence of slits, such that any accessible boundary vertex appears among their ends. So we get \( D'_v = D_v \) for all accessible \( v \in \partial V \), and finally a well-known theorem tells us that \( D'_v = D_v \) for all accessible \( v \in V \).

### 6.1 Combinatorial Reduction

Let \( L \) be a combinatoric crosscut of the complex \( K \). In this section we describe how a simple vertex \( v \in U^+_L \) can be “shifted” from \( V^+_L \) to \( V^-_L \) such that we get a new crosscut \( L' \) with \( |V^+_L| < |V^+_L| \). Depending on the properties of \( v \) we distinguish three cases.

**Case 1.** Let \( v \in U^+_L \) be a simple interior vertex.

**Case 2.** Let \( v \in U^+_L \) be a simple boundary vertex, and assume that neither the initial nor the terminal edge of \( L \) are adjacent to \( v \).

**Case 3.** Let \( v \in U^+_L \) be a simple boundary vertex, and assume that either the initial or the terminal edge of \( L \) are adjacent to \( v \).
Remark. The case where the initial and the terminal edge of $L$ are adjacent to $v$ cannot appear. Indeed, otherwise either $v$ is a multiple vertex (which is not considered) or all edges adjacent to $v$ must belong to $L$. The latter implies that $v$ is the only vertex in $V^+_L$, which is not allowed.

**Reduction of Type 1.** In order to modify the crosscut $L = (e_0, e_1, \ldots, e_l)$ in Case 1, we consider the flower $B = B(v)$ of $v$. Since $v$ is simple, the set of edges adjacent to $v$ consists of a subsequence $S = (e_i, \ldots, e_j)$ (with $0 \leq i \leq j \leq l$) of $L$ and a complementary sequence, which we denote by $S' = (e'_1, \ldots, e'_k)$ (with $k \geq 1$). Replacing in $L$ the sequence $S$ by $S'$, we get a new edge sequence

$$L' = (e_0, \ldots, e_{i-1}, e'_1, \ldots, e'_k, e_{j+1}, \ldots, e_l).$$

The reader can easily convince herself (see Figure 16, left), that the sequence $L'$ is a crosscut for $K$ with $|V^+_L| < |V^+_L'|$.

![Figure 16: Modification of the crosscut $L$ in Case 1 (left), Case 2 (middle) and Case 3(right)](image)

**Reduction of Type 2.** In Case 2 the flower of $v$ is incomplete. Nevertheless, the edges in $L$ which are adjacent to $v$ form again a sequence of consecutive edges in this incomplete flower, because $v$ is simple. However, the local modification of $L$ in a neighborhood of $v$ described above does not result in a crosscut $L'$, since the complementary sequence $S' = S'_1 \cup S'_2$ consists of exactly two connected components $S'_1 = (e'_1, \ldots, e'_k)$ and $S'_2 = (e''_1, \ldots, e''_m)$ (see Figure 16, middle). Replacing in $L$ the sequence $S$ by $S'_1$ or $S'_2$, we get a new edge sequence $L'$ or $L''$, respectively, with

$$L' = (e_0, \ldots, e_{i-1}, e'_1, \ldots, e'_k), \quad L'' = (e''_1, \ldots, e''_m, e_{j+1}, \ldots, e_l).$$

Both $L'$ and $L''$ are new crosscuts of $K$, but only one ($L'$, say) contains $v_\alpha$ among its upper vertices, so we choose this one as the new crosscut. Clearly $|V^+_L'| < |V^+_L|$. 

**Reduction of Type 3.** If either the initial or the terminal edge of $L$ are adjacent to $v$, then the Type 1 reduction applied to the incomplete flower of $v$ results in an admissible crosscut $L'$, which has one vertex (namely $v$) less in $V^+_L$ than in $V^+_L'$ (see Figure 16, right).

Remark. No matter which type of reduction we used, the sets $U^-_L$ and $U^-_L'$ of lower neighbors before and after the reduction, respectively, always fulfill $U^-_L \setminus U^-_L' = \{v\}$. 

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In order to not lose the normalization, we will only reduce vertices different from \( v_\alpha \). This leads to a situation where none of the above reductions can be applied, namely when \( v_\alpha \) is the only simple vertex in \( U_L^+ \). This special case will be explored in Section 6.2.

6.2 Slits

The next definition and the following lemma describe the situation when all but exactly one vertex of \( V \) are multiple.

**Definition 8.** A combinatoric slit of the complex \( K = (V,E,F) \) is a sequence \( S = (v_1,v_2,\ldots,v_s) \) of vertices in \( V \) which satisfies the following conditions (i)–(iv):

(i) The vertices of \( S \) are pairwise different, \( v_j \neq v_k \) if \( 1 \leq j < k \leq s \).

(ii) For \( j = 1,\ldots,s-1 \), the edges \( e_j := e(v_j,v_{j+1}) \) belong to \( E \).

(iii) For \( j = 1,\ldots,s \), the vertices \( v_{j-1} \) and \( v_{j+1} \) are the only neighbors of \( v_j \) in \( K \) which belong to \( S \) (where \( v_0 := \emptyset \) and \( v_{s+1} := \emptyset \)).

(iv) The vertex \( v_1 \) is a boundary vertex, and \( v_j \) are interior vertices for \( j = 2,\ldots,s \).

The vertices \( v_1 \) and \( v_s \) are referred to as the initial vertex and the terminal vertex of \( S \), respectively. The sequence \( E_S := (e_1,\ldots,e_{s-1}) \) (see (ii)) is said to be the edge sequence of \( S \).

The vertices \( v_1 \) and \( v_s \) are referred to as the initial vertex and the terminal vertex of \( S \), respectively. The sequence \( E_S := (e_1,\ldots,e_{s-1}) \) (see (ii)) is said to be the edge sequence of \( S \).

**Lemma 22.** Assume that the interior vertex \( v \) is the only simple vertex in \( U_L^+ \). Then the sequence of upper accompanying vertices \( S_L^+ \) has the symmetric form \((v_1,\ldots,v_{s-1},v,v_{s-1},\ldots,v_1)\) and \( S = (v_1,\ldots,v_{s-1},v) \) is a slit.

**Proof.** By definition of a multiple vertex, any vertex in \( U_L^+ \) except \( v \) must appear at least twice in the sequence \( S_L^+ \). If there are vertices which show up twice at a position left of \( v \), we choose one, say \( u \), whose appearances have minimal distance in the sequence \( S_L^+ = (\ldots,u,\ldots,u,\ldots,v,\ldots) \). Since neighboring vertices of \( S_L^+ \) must be different, there exists \( w \neq u \) such that \( S_L^+ = (\ldots,u,\ldots,w,\ldots,u,\ldots,v,\ldots) \). Because \( v \) is assumed to be simple and \( w \) is a multiple vertex, we have \( w \neq v \) and \( w \) must appear again at another place in \( S_L^+ \). By Lemma 18 this can only happen in between the two occurrences of \( u \), which is in conflict with the minimal distance property of \( u \).

Similarly, the assumption that there exists a vertex which appears in \( S_L^+ \) twice at a position right of \( v \) leads to a contradiction. Hence, with the only exception of \( v \), any vertex of \( U_L \) appears in \( S_L^+ \) exactly once on either side of \( v \). Applying Lemma 18 again, we see that the ordering of the vertices left of \( v \) must be reverse to the ordering on the right of \( v \), so that \( S_L^+ \) has the symmetric form claimed in the lemma.

Moreover we have shown that \( v_1,\ldots,v_{s-1},v \) are pairwise different, which is condition (i) of Definition 8. The second condition (ii) is trivial.

In order to verify condition (iv), it remains to show that \( v_j \) is an interior vertex for \( j = 2,\ldots,s-1 \), because \( v_1 \) is obviously a boundary vertex, while \( v_s := v \) is an interior vertex, by assumption. Assume \( v_j \) is a boundary vertex. The flower of \( v_j \) is incomplete and it is clear
that \(v_{j-1}\) and \(v_{j+1}\) are neighbors of \(v_j\). On the one hand, the subsequence \((v_{j-1}, v_j, v_{j+1})\) of \(S_L^+\) forces the crosscut \(L\) to look locally like shown in Figure 17 left. On the other hand, the subsequence \((v_{j+1}, v_j, v_{j-1})\) of \(S_L^+\) forces \(L\) to look locally like shown in the middle of Figure 17, a contradiction. Hence \(v_j\) must be an interior vertex and its flower must look qualitatively like shown in Figure 17 right.

Figure 17: A sequence \(S_L^+\) with only one simple interior vertex generates a slit

To verify condition (iii) let \(j \in \{2, \ldots, s-1\}\) be fixed. Looking at the behavior of the crosscut \(L\) in the flower of \(v_j\), it becomes clear that any edge \(e(v_{j-1}, v_{j+1})\) (with the convention \(v_s := v\)) belonging to \(E\) must be contained in \(L\) twice, a contradiction. Furthermore, all other neighbors of \(v_j\) belong to \(V_L^-\) and hence not to \(V_L^+ \supset S_L^+\). A similar result can be derived by looking at the local behavior of \(L\) in the flower of \(v\) and the incomplete flower of \(v_1\), now using the subsequences \((v_{s-1}, v_s, v_{s-1})\) and \((v_1, v_2, \ldots, v_2, v_1)\) of \(S_L^+\), respectively. \(\square\)

The following lemma explains why we are interested in slits.

**Lemma 23.** Let \((\mathcal{P}, \mathcal{P}')\) be an admissible pair of circle packings for the complex \(K\) with crosscut \(L\) and alpha-vertex \(v_\alpha\). Then there exists a slit \(S = (v_1, \ldots, v_s, v_\alpha) \subset V_L^+\) with terminal vertex \(v_\alpha\) such that \(D'_v = D_v\) for all \(v \in S\).

**Proof.** To begin with, we invoke Lemma 21 which tells us that the pair \((\mathcal{P}, \mathcal{P}')\) has a simple loner \(v_\lambda\). The idea is to use the reduction procedures of the last section to shift \(v_\lambda\) from \(V_L^-\) to \(V_L^+\) which results in a new crosscut \(L'\).

As we remarked earlier (on page 27), the one and only lower neighbor of \(L\) which has not already been a lower neighbor of \(L\) is the simple loner \(v_\lambda\). Therefore Lemma 10 guarantees that \(L'\) is admissible for \((\mathcal{P}, \mathcal{P}')\). In order to find the appropriate type of reduction we distinguish the following cases:

**Case 1.** There exists a simple interior loner \(v_\lambda\) which is different from the alpha-vertex \(v_\alpha\).

**Case 2.** There exists a simple boundary loner \(v_\lambda\).

**Case 3.** The only simple loner \(v_\lambda\) is the alpha-vertex \(v_\alpha\).

In Case 1 we apply the reduction of Type 1, while in Case 2 either the reduction of Type 2 or Type 3 can be applied, respectively, depending on whether \(v_\lambda\) is adjacent to the initial or the terminal edge of \(L\), or not. In any case we get a new combinatoric crosscut \(L'\) of \(K\). Applying
the reduction in Case 1 and Case 2 recursively as long as possible, the number of vertices in 
$V_L^+$ decays in every step at least by one, so that we eventually arrive at Case 3.

The alpha-vertex $v_α$ is a loner if and only if $D_α’ = D_α$. This implies, by Lemma 17, that there 
exists another loner $v_µ$. Since $v_α$ is the only simple loner, $v_µ$ must be a multiple loner. If 
$D_µ’ ≠ D_µ$, then according to Lemma 21(i), the vertex set $V_M$ of any loop $M$ of $v_µ$ contains 
a simple loner, i.e., $M$ meets $v_α$. Because $D_α’ = D_α$, assertion (ii) of this lemma tells us that 
$D_µ’ = D_µ$.

Applying Lemma 17 and Lemma 21 repeatedly in this manner, we see that all vertices in 
$U_L^+ \{v_α\}$ must be multiple loners and hence that $D_μ = D_v$ for all $v ∈ U_L^+$. Furthermore $v_α$ 
is the only simple vertex in $U_L^+$, so, by Lemma 22 we just constructed a slit $S ⊂ V_L^+$ with 
terminal vertex $v_α$.

In the next step we are going to construct crosscuts from slits. To begin with, we introduce 
some more notations.

![Figure 18: The left and right neighboring edges of $v$ in a slit $S$](image)

Let $S = (v_1, \ldots, v_s)$ be a slit. For any vertex $v$ in $S$ we define the subsets $E_S^−(v)$ and $E_S^+(v)$ of 
$E(v)$ as follows. For $v = v_1$, the (boundary) vertex $v_1$ has two adjacent boundary edges $e^-_1$ and 
$e^+_1$ in $E(v_1)$, such that $e^-_1$ is the predecessor of $e^+_1$ in the chain of boundary edges. We set (the 
meaning of the inequalities is explained on page 6)

$$E_S^−(v_1) := \{ e ∈ E(v_1) : e(v_1, v_2) < e ≤ e^-_1 \},$$

$$E_S^+(v_1) := \{ e ∈ E(v_1) : e^-_1 ≤ e < e(v_1, v_2) \}.$$

If $v = v_j$, with $j = 2, \ldots, s - 1$, we define

$$E_S^−(v_j) := \{ e ∈ E(v_j) : e(v_j, v_{j+1}) < e < e(v_{j-1}, v_j) \},$$

$$E_S^+(v_j) := \{ e ∈ E(v_j) : e(v_{j-1}, v_j) < e < e(v_j, v_{j+1}) \},$$

and for the terminal vertex $v_s$ of $S$ we let

$$E_S^−(v_s) = E_S^+(v_s) := \{ e ∈ E(v_s) : e(v_{s-1}, v_s) < e < e(v_{s-1}, v_s) \}.$$

The edges in

$$E_S^− := \bigcup_{j=1}^{s-1} E_S^−(v_j) \text{ and } E_S^+ := \bigcup_{j=1}^{s-1} E_S^+(v_j)$$
are called the left and the right neighbors of $S$, respectively. Note that condition (iii) in Definition 8 guarantees that every edge $e$ which is a neighbor of a slit $S$ has exactly one adjacent vertex in $S$.

**Lemma 24.** If $S = (v_1, \ldots, v_s, v)$ is a slit in $K$, then there exists a combinatoric crosscut $L$ such that $v \in S^-_L$, and $S^-_L = (v_1, \ldots, v_{s-1}, v_s, v_{s-1}, \ldots, v_1)$ is the sequence of lower accompanying vertices of $L$.

**Proof.** Walking along the slit $S$ from $v_1$ to $v_s$ and back to $v_1$, we build the crosscut $L$ from the concatenation of the edge sequences

$$E_S^-(v_1), \ldots, E_S^-(v_s), e(v_s, v), E_S^+(v_s), \ldots, E_S^+(v_1).$$

It is easy to see that all edges in $L$ are pairwise different, so that $L$ satisfies condition (i) of Definition 3. Condition (ii) can easily be verified and (iv) is obvious. In order to prove (iii) we assume that three edges of $L$ would form a face of $K$. Since these edges are neighbors of $S$, exactly one vertex of every edge must belong to $S$, which is impossible. The construction also guarantees that the sequence $S^-_L$ of lower accompanying edges of $L$ has the desired form and that $v$ belongs to $S^+_L$ (see, for example, Figure 19, left).

Figure 19: Constructing crosscuts from one slit (left) and two slits (middle, right)

A crosscut $L$ can also be constructed from glueing two slits $S_1$ and $S_2$ with a common terminal vertex $v$. This procedure is somewhat more complicated, in particular when the “right side” of $S_1$ is close to the “left side” of $S_2$. In those cases we cannot glue the cuts at their common terminal vertex $v$, since then the resulting edge sequence $L$ would contain some edges more than once. Instead we modify the procedure by glueing $S_1$ and $S_2$ at some appropriately chosen vertex $u$ in $S_2$ or $S_1$ which has a neighbor in $S_1$ or $S_2$, respectively. Figure 19 (middle, right) illustrates the result, showing an associated circle packing and the related maximal crosscuts.

**Lemma 25.** Let $S_1 = (v_1, \ldots, v_t, v)$ and $S_2 = (w_1, \ldots, w_s, v)$ be slits in $K$ with $S_1 \cap S_2 = \{v\}$. Assume further that $E_{S_1}^-(v_1) \cap E_{S_2}^-(w_1) = \emptyset$. Then there exists a combinatoric crosscut $L$ and a vertex $u \in (S_1 \cup S_2) \cap U^+_L$ such that

$$S^-_L = (w_1, w_2, \ldots, w_{\sigma}, u_1, \ldots, u_k, v_\tau, v_{\tau-1}, \ldots, v_1), \quad 1 \leq \tau \leq t, \quad 1 \leq \sigma \leq s,$$

(1)
where \((w_\sigma, u_1, \ldots, u_k, v_\tau)\) is a (positively oriented) chain of neighbors of \(u\).

**Proof.** We set \(v_{t+1} := v\) and \(w_{s+1} := v\). Let \(i\) be the smallest number in \(\{1, \ldots, t + 1\}\) for which \(E^+_{S_1}(v_i)\) contains an edge \(e(v_i, w)\) with \(w \in S_2\). Then let \(j\) be the smallest number in \(\{1, \ldots, s + 1\}\) for which \(E^-_{S_2}(w_j)\) contains an edge \(e(w_j, v_i)\). If \(i \neq 1\) and \(j \neq s + 1\) we set \(\tau := i - 1, \sigma := j\) and \(u := v_i\). If \(i \neq 1\) but \(j = s + 1\), then \(i = t\) must hold (otherwise \(v\) would have more than one neighbor in \(S_1\)), and we set \(\tau := t, \sigma := s\) and \(u := v\). If \(i = 1\) we set \(\tau := 1, \sigma := j - 1\) and \(u := w_j\). In the last case we have \(j > 1\), since otherwise \(i = j = 1\) would contradict the assumption \(E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset\).

In every case \(1 \leq \tau \leq t\) and \(1 \leq \sigma \leq s\) hold, and \(u\) is well defined. We now build \(L\) as the concatenation of the edge sequences

\[
E^-_{S_2}(w_1), \ldots, E^-_{S_2}(w_\sigma), E^*(u), E^+_{S_1}(v_\tau), \ldots, E^+_{S_1}(v_1),
\]

where \(E^*(u) = (e(u, w_\sigma), e(u, u_1), \ldots, e(u, u_k), e(u, v_\tau))\) is the negatively oriented chain of edges in the set \(\{e' \in E(v) : e(u, w_\sigma) \leq e' \leq e(u, v_\tau)\}\).

Because \(S_1, S_2\) are slits, all edges in the “\(E^+_{S_1}\) -part” and in the “\(E^-_{S_2}\) -part” of \(L\) are pairwise different. Furthermore, it cannot happen that such an edge is contained in both parts (according to the definition of \(u\)), or that it belongs to \(E^*(u)\) (by definition of \(E^*(u)\)). Hence, \(L\) satisfies condition (i) of the crosscut definition (page 13).

Condition (ii) can easily be verified and (iv) is trivial. In order to prove (iii) we assume that three edges of \(L\) form a face of \(K\). By definition of \(u\), the sequence \((w_1, w_2, \ldots, w_\sigma, u, v_\tau, \ldots, v_1)\) divides \(K\) into two parts \(K_1, K_2\). All edges of the “\(E^+_{S_1}\) -part” and of the “\(E^-_{S_2}\) -part” have exactly one vertex lying in \(S_0^1 \cup S_0^2\) and one in \(K_1\), so three of them can never form a face of \(K\). All edges of \(E^*(u)\) \(\backslash\) \(\{e(u, v_\tau), e(u, w_\sigma)\}\) have exactly one vertex lying in \(S_0^1 \cup S_0^2\) and one in \(K_2\), so again three of them can never form a face of \(K\). The only remaining edges are \(e(u, v_\tau), e(u, w_\sigma)\), but two edges cannot form a face, and a combination of edges from more than one of the three distinguished edge types can clearly never form a face. Hence, \(L\) is a crosscut with \(u \in (S_1 \cup S_2) \cap U^+_L\), and \(S^-_L\) has the form \([1]\). \(\square\)

The operation described in the proof is well defined by the slits \(S_1\) and \(S_2\), and will be referred to as reflected concatenation \(S_1 \ominus S_2\) of \(S_1\) with \(S_2\). It delivers a crosscut \(L\), a vertex \(u\), and the reduced slits \(S_0^1, S_0^2\). Note that the reflected concatenation is not commutative.

### 6.3 Subdivision by Disk Chains

Let \(v_\beta\) be an arbitrary accessible boundary vertex. In this section we describe an approach which allows us to apply Lemma 23 recursively, until we find a slit \(S\) with initial vertex \(v_\beta\) such that \(D^+_v = D_v\) for all \(v \in S\), so especially \(D^+_v = D_v\). During this procedure we construct a sequence of crosscuts \(L_j\) such that \(V^+_L\) contains \(v_\beta\) and the number of elements in \(V^+_L\) is strictly decreasing for increasing \(j\). This procedure will be crucial for proving the following lemma, and finally Theorem 2.

**Lemma 26.** Let \((P, P')\) be an admissible pair with complex \(K\), interior alpha vertex \(v_\alpha\) and crosscut \(L\). Then \(D^+_v = D_v\) for all accessible boundary vertices \(v \in \partial P^*\).
Proof. To begin with, let $S_0 = (v_1, \ldots, v_s, v_\alpha)$ be a slit according to Lemma 23. Let $v_\beta$ be an accessible boundary vertex. If $v_1 = v_\beta$ then $D'_\beta = D_\beta$ and we are done. So let us assume that $v_\beta \notin S_0$.

By Lemma 24 there exists a crosscut $L_1$ such that $S^-_{L_1} = (v_1, \ldots, v_s, v_s, v_s, \ldots, v_1)$ and $v_\alpha \in S^+_{L_1}$. Applying Lemma 23 again, but now with respect to the crosscut $L_1$, we get another slit $S_1 = (w_1, \ldots, w_s, v_\alpha) \subset V_{L_1}$, such that $D'_v = D_v$ for all $v \in S_1$. If $v_1 = v_\beta$ then $D'_\beta = D_\beta$ and we are done. So suppose that $v_\beta \notin S_1$.

The three boundary vertices $v_1$, $w_1$ and $v_\beta$ are pairwise different, and we assume, without loss of generality, that they are oriented such that $w_1 < v_\beta < v_1$. This ensures the condition $E_{S_1}^+(v_1) \cap E_{S_1}^-(w_1) = \emptyset$ of Lemma 25, because otherwise $v_\beta$ could be either accessible or a boundary vertex, but not both. Since, except $v_\alpha$, all vertices of $S_0$ belong to $V_{L_0}$, we have $S_0 \cap S_1 = \{v_\alpha\}$. Consequently, the reflected concatenation $S_0 \odot S_1$ of $S_0$ with $S_1$ is well defined.

It delivers a crosscut $L_2$, a vertex $v_{\alpha_2}$, and reduced slits $S^-_2 \subset S_0, S^+_2 \subset S_1$ with common terminal vertex $v_{\alpha_2}$. By Lemma 25 the vertex $v_{\alpha_2}$ belongs to $S_1$ or $S_2$ and the set $U_{L_2}$ of lower neighbors of $L_2$ consists solely of elements of $S_0 \cup S_1$ and of (lower) neighbors of $v_{\alpha_2}$. Since $D'_v = D_v$ for all $v \in S_0 \cup S_1$, this implies that $L_2$ is an admissible crosscut for $(\mathcal{P}, \mathcal{P}')$. Moreover, the order of $S_0$ and $S_1$ in the reflected concatenation has been chosen such that $v_\beta$ belongs to $V^+_{L_2}$.

![Figure 20: Construction of the crosscut $L_{j+1}$ from $L_j$](image)

The general step of the procedure is as follows. Assume that we already have an admissible crosscut $L_j$, the alpha vertex $v_{\alpha_j}$, and the reduced slits $S^-_j$ and $S^+_j$, such that $v_\beta \in V^+_{L_j}$. Denoting by $v^-_j$ and $v^+_j$ the initial vertices of $S^-_j$ and $S^+_j$, respectively, we may assume that $v^-_j < v_\beta < v^+_j$, which will again be essential to ensure the special condition of Lemma 25.

Applying Lemma 23 we get a new slit $S_j \subset V^+_{L_j}$, such that $S^-_j, S_j$ and $S^+_j$ are pairwise disjoint, except at their common terminal vertex $v_{\alpha_j}$, and $D'_v = D_v$ for all $v \in S_j$. If $v_\beta \in S_j$ we are done. Otherwise we either have $v^-_j < v_\beta < v_j$ or $v_j < v_\beta < v^+_j$. In the first case we build the reflected concatenation $S^-_j \odot S_j$, in the second case we form $S_j \odot S^+_j$. The result is a new crosscut $L_{j+1}$, a corresponding alpha-vertex $v_{\alpha_{j+1}}$, and reduced slits $S^-_{j+1}, S^+_{j+1}$. If follows directly from the construction of the reflected concatenation that $v_{\alpha_{j+1}}, v_\beta \in V^+_{L_{j+1}}$.
Moreover, \( v_{\alpha, j+1} \in S_j^- \), and hence \( D'_{\alpha, j+1} \equiv D_{\alpha, j+1} \). To see that \( L_{j+1} \) is admissible for the pair \( (P, P') \) it remains to prove that \( D'_v \subset G^-_{L_{j+1}} \) for all \( v \in U^-_{L_{j+1}} \).

By Lemma \[25\] the set \( U^-_{L_{j+1}} \) of lower neighbors of \( L_{j+1} \) consists solely of elements of \( S_j^- \cup S_j^+ \) and of (lower) neighbors of \( v_{\alpha, j+1} \). Since \( D'_v = D_v \) for all \( v \in S_j^- \cup S_j^+ \cup \{v_{\alpha, j+1}\} \), and \( D_v \subset G^-_{L_{j+1}} \) for all \( v \in U^-_{L_{j+1}} \), the assertion follows.

The number of elements in \( V^+_L \) is strictly decreasing in every step, and hence the procedure must come to end. This can only happen if \( v_j \in S_j^* \) for some \( j^* \in \mathbb{N} \). Because \( D'_v = D_v \) for all \( v \in S_j \) with \( j \leq j^* \), we have shown \( D'_{v_{j^*}} = D_{v_{j^*}} \).

Now we are close to the end. By Lemma \[4\] the kernel \( K^* \) is a strongly connected complex with vertex set \( V^* \). Since we have shown that \( D'_v = D_v \) for all boundary vertices \( v \in \partial V^* \) of \( K^* \), and every boundary vertex of \( K^* \) is also a boundary vertex of \( K \) (that is \( \partial V^* = V^* \cap \partial V \)), Theorem 11.6 in Stephenson \[13\] (on the uniqueness of a locally univalent packing with prescribed combinatorics and given radii of boundary circles) tells us that \( D'_v = D_v \) for all \( v \in V^* \), which is the assertion of Theorem \[2\].

### 7 Concluding Remarks

All proofs in this paper work with (simple) geometric or combinatoric arguments, alone in the very last step we had recourse to a theorem established in the literature. For purists we mention that even this could have been avoided, at the expense of adding a few pages to this rather longish text.

Theorem \[2\] can be interpreted as uniqueness result for (the range packing of) discrete conformal mappings. Here is a simple version:

**Theorem 3.** Suppose that two univalent packings \( P \) and \( P' \) for \( K \) fill \( G \). If \( D'_\alpha \) and \( D_\alpha \) have the same center, and if \( D'_\beta \subset D_\beta \) for some boundary vertex \( v_\beta \), then \( D'_v = D_v \) for all vertices \( v \in V^* \).

The proof follows immediately from Theorem \[2\] applied to the maximal crosscut which separates the disk \( D_\beta \) from the rest of the packing \( P \) (see the leftmost image of Figure \[21\]). The condition \( D'_\beta \subset D_\beta \) can even be relaxed, it suffices to require that \( D'_\beta \) lies in the lower domain \( G_- \) with respect to this crosscut (see the second image of Figure \[21\]). Note that both figures show the packing \( P \) and a single disk \( D'_\beta \) of \( P' \) in \( G_- \).

We point out that the condition \( D'_\beta \subset G_- \) is always satisfied (possibly after exchanging the roles of \( P \) and \( P' \)), if the packings are normalized so that \( D'_\beta \) and \( D_\beta \) touch the boundary \( \partial G \) in a generalized sense at the same regular point (or, more generally, at the same regular prime end). Without explaining these concepts here (see \[7\]), we mention that a point which lies on a smooth subarc of \( \partial G \) is always regular, while a point at a re-entrant corner fails to be regular. The two pictures on the right of Figure \[21\] illustrate that uniqueness of domain-filling circle packings may be violated in that case. Both displayed packings \( P \) and \( P' \) fill a Jordan domain \( G \), \( D_\alpha \) and \( D'_\alpha \) have the same center, and \( D_\beta \) and \( D'_\beta \) touch \( \partial G \) at the same point. While this
type of normalization implies uniqueness of classical conformal mappings, the corresponding circle packings $\mathcal{P}$ and $\mathcal{P}'$ are completely different.

We further mention that for domain-filling circle packings $\mathcal{P}$ and $\mathcal{P}'$ the assertions of Theorem 2 and Theorem 3 can be strengthened to $D'_{v} = D_{v}$ for all $v \in V$, using the results of our forthcoming paper [7].

In the general setting of Theorem 2 a complete description which disks are uniquely determined by a crosscut seems not to be known. The figures below show some examples. The accessible disks are depicted in darker colors, the alpha-disk is the darkest one. By Theorem 2 these disks are uniquely determined (rigid) by the crosscut, but the rigid part also comprises the non-accessible disks shown in brighter color.

The example on the right is of special interest: a short crosscut separates only one non-accessible disk $D_{\beta}$ from the alpha-disk. Here the theorem yields rigidity for the dark (blue) disks, so that
$D_\beta$ seems to have some mysterious "remote action". However, a little thought shows that there is a chain of rigid disks (depicted in lighter color) which connects the cut with the alpha-disk and acts as "transmission line".

Isn’t it wonderful that simple circles can form such fascinating structures?

**Glossary**

$\ominus$, $S_1 \ominus S_2$ reflected concatenation of slit $S_1$ with slit $S_2$; p. 32

$\langle u, v, w \rangle$ oriented face of $K$ with vertices $u,v$ and $w$; p. 6

$\langle u, v \rangle$ oriented edge of $K$ from vertex $u$ to vertex $v$; p. 6

$\alpha_i, \alpha$ special Jordan arcs connecting $y_i^-$ and $y_i^+$, and their concatenation; p. 18

$B(v)$ the flower of the vertex $v$, a subcomplex of $K$; p. 6

$c(u,v)$ contact point of the disks $D_u$ and $D_v$, $c(u,v) = \overline{D_u} \cap \overline{D_v}$; p. 10

$c^-_k, c^+_k$ contact points of boundary disk $D_k$ with $D_{k-1}$ and $D_{k+1}$, respectively; p. 12

$D$ union of all disks in $\mathcal{P}$; p. 10

$D^*$ carrier of $\mathcal{P}$; p. 10

$D_k, D'_k$ boundary disks in $\mathcal{P}$ and $\mathcal{P}'$, respectively; p. 12

$D_v, D'_v$ disks in $\mathcal{P}$ and $\mathcal{P}'$, respectively; p. 6

$\partial$ boundary operator, applied to various objects

$\delta(p,q)$ positively oriented open circular arc from $p$ to $q$ on $\partial D$; p. 10

$\delta[p,q]$ positively oriented closed circular arc from $p$ to $q$ on $\partial D$; p. 10

$\delta(c^-_k, c^+_k)$ exterior boundary arc of $D_k$; p. 12

$\delta(c^+_k, c^-_k)$ interior boundary arc of $D_k$; p. 12

$\delta_k$ smallest subarc of $\delta[c^-_k, c^+_k]$ which contains $G_k$; p. 12

$E_S$ the edge sequence of the slit $S$; p. 28

$E$ the set of edges of the complex $K$; p. 6

$\partial E$ boundary edges of the complex $K$; p. 6

$E(v)$ the (cyclically ordered) sequence of edges adjacent to $v \in V$; p. 6

$E^+_-L(v)$ sequences of upper and lower accompanying edges of the crosscut $L$; p. 24

$E^+_S(v)$ sequences of edges adjacent to a vertex $v$ in a slit $S$; p. 30

$E^\pm_S$ sequences of left and right neighbor edges of slit $S$, respectively; p. 31

$e(u,v)$ non-oriented edge between vertices $u$ and $v$; p. 6

$e_j$ edges in a crosscut, $L = (e_0, e_1, \ldots, e_l)$; p. 13

$e^-_j, e^+_j$ lower and upper accompanying edges of the crosscut $L$, respectively; p. 24

$\eta_k, \eta$ segments connecting the centers of $D_k$ and $D_{k+1}$ and their concatenation; p. 12

$F$ set of faces of the complex $K$; p. 6

$f(u,v,w)$ non-oriented face with vertices $u,v$ and $w$; p. 6

$G$ Jordan domain to be filled with $\mathcal{P}$; p. 3

$G^-L, G^+_L$ lower and upper domains of $G$ with maximal crosscut $J^+_L$, $G^-L = \Omega$; p. 16

$G_k$ set of contact points of $D_k$ with $\partial G$; p. 12, $G_k := D_k \cap \partial G$

$g^-_k, g^+_k$ first and the last contact point of $D_k$ with $\partial G$; p. 12
I \_k \text{ boundary interstice between } D \_k \text{ and } D \_k +1; \text{ p. 12}
I (u, v, w) \text{ interstice between the disks } D \_u, D \_v \text{ and } D \_w; \text{ p. 10}
J \_L^0 \text{ polygonal (geometric) crosscut in } G \text{ for (combinatorial) crosscut } L \text{ in } K; \text{ p. 14}
J \_L^* \text{ maximal ‘crosscut’, the upper boundary of the lower domain } G \_L^-, J \_L^+ = \omega; \text{ p. 16}
K \text{ simplicial 2-complex, combinatorial disk, finite triangulation, } K = (V, E, F); \text{ p. 6}
K^* \text{ kernel of } K, \text{ largest sub-complex of } K \text{ with vertex set } V^*; \text{ p. 7}
L \text{ combinatorial crosscut, sequence of edges in } K; \text{ p. 13}
l (i) \text{ smallest label } k \text{ of prime end set } \omega^*_k \text{ associated with } \nu_i; \text{ p. 21}
M, M(\mu) \text{ loop of a multiple loner } v \_\mu, \text{ a sequence of edges; p. 25}
\Omega \text{ lower subdomain of } G \text{ with respect to a maximal crosscut, } \Omega = G \_L^-; \text{ p. 16}
\omega \text{ upper boundary of lower domain } \Omega, \text{ concatenation of the } \omega_i, \text{ maximal crosscut; p. 16}
\omega^* \text{ prime ends of } \Omega \text{ associated with } \omega; \text{ p. 16}
\omega_i^* \text{ classes of prime ends associated with the arcs } \omega; \text{ p. 16}
\nu_i, \pi_i \text{ negatively and positively oriented arcs on } \partial D \text{ from } y_i^-, y_i^+ \text{ to } \omega, \text{ respectively; p. 19}
\nu_i^+, \pi_i^+ \text{ terminal points of the arcs } \nu_i, \pi_i, \text{ respectively; p. 19}
\nu_i^*, \pi_i^* \text{ prime ends of } \Omega \text{ associated with } \nu_i, \pi_i, \text{ respectively; p. 20}
\mathcal{P} \text{ a univalent circle packing for } K \text{ filling } G; \text{ p. 9, p. 17}
\mathcal{P}' \text{ a univalent circle packing for } K \text{ in } G; \text{ p. 17}
r (i) \text{ largest label } k \text{ of prime end set } \omega^*_k \text{ associated with } \pi_i; \text{ p. 21}
S \text{ combinatorial slit, a sequence of vertices; p. 28}
S \_L^-, S \_L^+ \text{ sequences of lower and upper accompanying vertices of } L \text{, respectively; p. 24}
t_i \text{ turning points of the upper boundary } \omega, \text{ cusps of } \Omega; \text{ p. 16}
U \_L, U \_L^+ \text{ sets of lower and upper neighbors of } L, \text{ respectively, } U \_L^- \subset V \_L^-, U \_L^+ \subset V \_L^+; \text{ p. 14}
U_M \text{ sequence of the vertices in } V \_M \text{ for a loop } M; \text{ p. 25}
V \text{ vertex set of the complex } K; \text{ p. 6}
V^* \text{ the set of all accessible vertices of } K; \text{ p. 7}
\partial V \text{ boundary vertices of the complex } K; \text{ p. 16}
V \_L, V \_L^+ \text{ lower and upper vertices of } K \text{ with crosscut } L, \text{ respectively, subsets of } V; \text{ p. 14}
V \_M \text{ set of all vertices met by a loop } M; \text{ p. 25}
v_\alpha \text{ alpha vertex of } K, \text{ a distinguished interior vertex; p. 7}
v (i) \text{ vertex of the disk which contains the circular arc } \omega_i, \nu (i) \in U \_L^+; \text{ p. 19}
x_k, X \text{ contact points of upper with lower disks in } \mathcal{P}, \text{ the set of all } x_k; \text{ p. 14}
X_i \text{ sets of contact points } x_k \text{ on } \omega_i, X_i \subset X; \text{ p. 18}
y_-, y_+ \text{ initial point and terminal point of } \alpha, \text{ respectively; p. 20}
y_k, Y \text{ contact points of upper with lower disks in } \mathcal{P}', \text{ the set of all } y_k; \text{ p. 18}
y_i^-, y_i^+ \text{ minimal and maximal element of } Y_i, \text{ respectively; p. 19}
Y_i \text{ sets of contact points } y_k \text{ with } x_k \in \omega_i, Y_i \subset Y; \text{ p. 18}
z_-, z_+ \text{ terminal points of } \nu_i \text{ and } \pi_a, \text{ respectively; p. 20}
z_k \text{ shifted contact points when } y_k \text{ is critical; p. 19}
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