HYLOMORPHIC SOLITONS ON LATTICES

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To our friend and colleague Louis Nirenberg, with affection and admiration.

Abstract. This paper is devoted to the study of solitons whose existence is related to the ratio energy/charge. These solitons are called hylomorphic. In the first part of the paper we prove an abstract theorem on the existence of hylomorphic solitons which can be applied to the main situations considered in literature. In the second part, we apply this theorem to the nonlinear Schrödinger and Klein Gordon equations defined on a lattice.

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1. **Introduction.** Roughly speaking a *solitary wave* is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time. A *soliton* is a solitary wave which exhibits some strong form of stability so that it has a particle-like behavior (see e.g. [1], [5], [8], [15], [19], [20]).

Today, we know (at least) three mechanisms which might produce solitary waves, vortices and solitons:

- Complete integrability, (e.g. Kortewg-de Vries equation);
- Topological constraints, (e.g. Sine-Gordon equation);
- Ratio energy/charge: (e.g. the nonlinear Schroedinger equation (NS) and the nonlinear Klein-Gordon equation (NKG)).

Following [2], [3], [5], the third type of solitary waves or solitons will be called *hylomorphic*. This class includes the $Q$-balls which are spherically symmetric solutions of NKG and vortices which might be defined as *spinning* $Q$-balls. Also it includes solitary waves and vortices which occur, by the same mechanism, in NS and in gauge theories; a bibliography on this subject can be found in the review papers [8], [5], and, for the vortices, in [9].

This paper is devoted to the proof of a general abstract theorem which can be applied to the main situations considered in the literature (see e.g. [5] and [8]). However this theorem can be also applied to the NS and to the NKG defined on a lattice (namely eq. (72) when $V$ satisfies (78) and eq. (90) when $W$ satisfies (93)). These results are new.

The paper is organized as follows. In section 2 we give the definition of hylomorphic solitons and describe their general features; in section 3 we prove some abstract results on the existence of hylomorphic solitons; in section 4 and in section 5 we apply the abstract results to NS and to NKG defined on a lattice.

2. **Hylomorphic solitary waves and solitons.**

2.1. **An abstract definition of solitary waves and solitons.** Solitary waves and solitons are particular *orbits* of a dynamical system described by one or more partial differential equations. The states of this system are described by one or more *fields* which mathematically are represented by functions

$$ u : \mathbb{R}^N \to V $$

where $V$ is a vector space with norm $| \cdot |_V$ which is called the internal parameters space. We denote our dynamical system by $(X, T)$ where $X$ is the set of the states and $T : \mathbb{R} \times X \to X$ is the time evolution map. If $u_0 \in X$, the evolution of the system will be described by the function

$$ U(t,x) = T_t u_0(x), \ t \in \mathbb{R}, \ x \in \mathbb{R}^N. $$

Now we can give a formal definition of solitary wave:

**Definition 1.** An orbit $U(t,x)$ is called solitary wave if it has the following form:

$$ U(t,x) = h(t,x)u_0(\gamma(t)x) $$

where

$$ \gamma(t) : \mathbb{R}^N \to \mathbb{R}^N $$

is a one parameter group of isometries which depends smoothly on $t$ and

$$ h(t,x) : V \to V $$
is a group of (linear) transformation of the internal parameter space which depends smoothly on \( t \) and \( x \). In particular, if \( \gamma (t) x = x \), \( U \) is called standing wave.

For example, consider a solution of a field equation which has the following form

\[
U (t, x) = u_0(x - x_0 - vt)e^{i(v \cdot x - \omega t)}; \quad u_0 \in L^2(\mathbb{R}^N, \mathbb{C}); \tag{3}
\]

In this case

\[
\gamma (t) x = x - x_0 - vt.
\]

\[
h(t, x) = e^{i(v \cdot x - \omega t)}.
\]

In this paper we are interested in standing waves, so (3) takes the form

\[
U (t, x) = u_0(x - x_0)e^{-i\omega t}; \quad u_0 \in L^2(\mathbb{R}^N, \mathbb{C}); \tag{4}
\]

The solitons are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well known notions from the theory of dynamical systems.

A set \( \Gamma \subset X \) is called invariant if \( u_0 \in \Gamma \Rightarrow \forall t, T_t u_0 \in \Gamma \); an invariant set \( \Gamma \subset X \) is called isolated if it has a neighborhood \( N \) such that:

\[
\text{if } \Delta \text{ is an invariant set and } \Gamma \subset \Delta \subset N, \text{ then } \Gamma = \Delta.
\]

**Definition 2.** Let \((X, d)\) be a metric space and let \((X, T)\) be a dynamical system. An isolated invariant set \( \Gamma \subset X \) is called stable, if \( \forall \varepsilon > 0, \exists \delta > 0, \forall u \in X, d(u, \Gamma) \leq \delta, \implies d(T_t u, \Gamma) \leq \varepsilon. \)

Now we are ready to give the definition of (standing) soliton:

**Definition 3.** A standing wave \( U(t, x) \) is called soliton if there is an isolated invariant set \( \Gamma \) such that

- (i) \( \forall t, U(t, \cdot) \in \Gamma \)
- (ii) \( \Gamma \) is stable
- (iii) \( \Gamma \) has the following structure

\[
\Gamma = \{u(x - x_0) : u \in \mathcal{K}, x_0 \in F\} \cong \mathcal{K} \times F \tag{5}
\]

where \( \mathcal{K} \subset X \) is a compact set and \( F \subseteq \mathbb{R}^N \) is not necessarily compact.

A generic field \( u \in \Gamma \) can be written as follows

\[
u_\theta(x - x_0),
\]

where \( \theta \) belongs to a set of indices \( \Xi \) which parametrize \( \mathcal{K} \) and \( x_0 \in F \).

In the generic case of many concrete situations, \( \Gamma \) is a manifold, then, (iii) implies that it is finite dimensional and \( (\theta, x_0) \) are a system of coordinates.

For example, in the case (3), we have

\[
\mathcal{K} = \{u_0(x)e^{i\theta} : \theta \in \mathbb{R}/(2\pi\mathbb{Z})\}
\]

\[
\Gamma = \{u_0(x - x_0)e^{i\theta} : \theta \in \mathbb{R}/(2\pi\mathbb{Z}), x_0 \in \mathbb{R}^N\} \cong \mathcal{K} \times \mathbb{R}^N
\]

\[
\gamma (t) x = x
\]

\[
\theta (t) = -\omega t;
\]

then \( U_0(t, x) = u_0(x - x_0)e^{-i\omega t} \in \Gamma \) is a soliton if \( \Gamma \) is stable.
The proof of (ii) of definition 3, namely that a solitary wave has enough stability to be a soliton, is a delicate question and in many cases of interest it is open. Moreover the notion of stability depends on the choice of the space $X$ and on the choice of its metric $d$ and hence different choices might lead to more or less satisfactory answers.

2.2. Integrals of motion and hylomorphic solitons. The existence and the properties of hylomorphic solitons are guaranteed by the interplay between energy $E$ and another integral of motion which, in the general case, is called *hylenic charge* and it will be denoted by $C$. Notice that $E$ and $C$ can be considered as functionals defined on the phase space $X$.

Thus, we make the following assumptions on the dynamical system $(X, T)$:

- **A-1.** It is variational, namely the equations of the motions are the Euler-Lagrange equations relative to a Lagrangian density $L[u]$.
- **A-2.** The equations are invariant for time translations.
- **A-3.** The equations are invariant for a $S^1$-action acting on the internal parameter space $V$ (cf. (1)).

By A-1, A-2 and A-3 and Noether’s theorem (see e.g. [14], [5], [8]) it follows that our dynamical system has 2 first integrals:

- the invariance with respect to time translations gives the conservation of the energy which we will shall denote by $E(u)$;
- The invariance A-3 gives another integral of motion called *hylenic charge* which we shall denote by $C(u)$.

Now we set

$$M_\sigma = \{ u \in X : |C(u)| = \sigma \} .$$

Using the definition 3, we get the definition of hylomorphic soliton as follows:

**Definition 4.** Let $U$ be a soliton according to definition 3. $U$ is called a (standing) *hylomorphic soliton* if $\Gamma$ (as defined in (5)) coincides with the set of minima of $E$ on $M_\sigma$, namely

$$\Gamma_\sigma = \{ u \in M_\sigma : E(u) = c_\sigma \}$$

with

$$c_\sigma = \min_{u \in M_\sigma} E(u).$$

**Remark 5.** Suppose that the Lagrangian $L[u]$ is invariant for (a representation of) the Lorentz or the Galileo group. Then given a standing hylomorphic soliton, we can get a hylomorphic travelling soliton just by Galileo or a Lorentz transformation respectively.

We recall that in physics literature the solitons of definition 4 are called Q-balls [13] and were first studied in the pioneering paper [16]. The existence of stable solitary waves in particular cases has been established in [12] and [17]. The existence of hylomorphic solitons in more general equations has been proved in [3].

If the energy $E$ is unbounded from below on $M_\sigma$ it is still possible ([18], [11]) to have standing wave (see def. 1). Moreover there are also cases [17] in which it is possible to have solitons (see def. 3) which are only local minimizers [10]. These
solitons are not hylomorphic (def. 4) and they can be destroyed by a perturbation which send them out of the basin of attraction.

In the next section we analyze some abstract situations which imply $\Gamma_\sigma \neq \emptyset$ and the existence of hylomorphic solitons (definition 4).

3. Abstract results.

3.1. The general framework. We assume that $E$ and $C$ are two functionals on $\mathcal{D}(\mathbb{R}^N, V) (\equiv C_0^\infty(\mathbb{R}^N))$ defined by densities. This means that, given $u \in \mathcal{D}(\mathbb{R}^N, V)$, there exist density functions $\rho_{E,u}(x)$ and $\rho_{C,u}(x) \in L^1(\mathbb{R}^N)$ i.e. functions such that

$$E(u) = \int \rho_{E,u}(x) \, dx$$

$$C(u) = \int \rho_{C,u}(x) \, dx.$$ 

Also we assume that the energy can be written as follows

$$E(u) = \frac{1}{2} \int \rho_{E,u}^{(2)}(x) \, dx + \int \rho_{E,u}^{(3)}(x) \, dx$$

where $\rho_{E,u}^{(2)}$ is quadratic in $u$ and $\rho_{E,u}^{(3)}$ contains the higher order terms.

If we assume $\rho_{E,u}^{(2)} > 0$ for $u \neq 0$, then we can define the following norm:

$$\|u\|^2 = \int \rho_{E,u}^{(2)}(x) \, dx$$

and the Hilbert space

$$X = \{ \text{closure of } \mathcal{D}(\mathbb{R}^N, V) \text{ with respect to } \|u\| \}.$$

We assume that the energy $E$ and the charge $C$ can be extended as functional of class $C^2$ in $X$; in particular we will write $E$ as follows:

$$E(u) = \frac{1}{2} \langle Lu, u \rangle + K(u)$$

(7)

where $L : X \to X'$ is the duality operator, namely $\langle Lu, u \rangle = \|u\|^2$ and $K$ is superquadratic. Also, we assume that

$$C(0) = 0; \quad C'(0) = 0.$$

so that we can write

$$C(u) = \langle L_0 u, u \rangle + K_0(u)$$

(8)

where $L_0$ is a linear operator and $K_0$ is superquadratic.

For any $\Omega \subset \mathbb{R}^N$ we will write

$$E_\Omega (u) = \int_{\Omega} \rho_{E,u}(x) \, dx$$

$$C_\Omega (u) = \int_{\Omega} \rho_{C,u}(x) \, dx$$

$$\|u\|^2_\Omega = \int_{\Omega} \rho_{E,u}^{(2)}(x) \, dx$$

$$K_\Omega (u) = \int_{\Omega} \rho_{E,u}^{(3)}(x) \, dx.$$

In the general scheme described above, we make the following assumptions:
• (E-0) **(the main protagonists)** $E$ and $C$ are two functionals on $X$ of the form (7) and (8).

• (E-1) **(lattice translation invariance)** the charge and the energy are lattice translation invariant.

Namely we have that $\forall z \in \mathbb{Z}^N$

$$E(T_z u) = E(u)$$
$$C(T_z u) = C(u)$$

where $T_z : X \to X$ is a linear representation of the additive group $\mathbb{Z}^N$ defined as follows:

$$u(x) = u(x + Az)$$

$A$ is an invertible matrix which characterizes the representation $T_z$. Such a $T_z$ will be called **lattice transformation**.

• (E-2) **(coercivity)** if $E(u_n)$ and $C(u_n)$ are bounded, then $\|u_n\|$ is bounded

• (E-3) **(local compactness)** namely, if $u_n \rightharpoonup \bar{u}$, weakly in $X$, then for bounded $\Omega$

$$K_{\Omega} (u_n) \to K_{\Omega} (\bar{u})$$
$$C_{\Omega} (u_n) \to C_{\Omega} (\bar{u})$$

• (E-4) **(boundedness)** if $\|u\| \leq M$, then $K'_{\Omega} (u)$ and $C'_{\Omega} (u)$ are bounded in $X' (\Omega)$ for any $\Omega \subset \mathbb{R}^N$.

3.2. **The main theorems.** In the framework of the previous section, we want to investigate sufficient conditions which guarantee that the energy has a minimum on the set

$$\mathfrak{M}_\sigma = \{ u \in X : |C(u)| = \sigma \},$$

namely that $\Gamma_\sigma \neq \emptyset$ where

$$\Gamma_\sigma = \left\{ u \in \mathfrak{M}_\sigma : E(u) = \min_{v \in \mathfrak{M}_\sigma} E(v) \right\}.$$

In this section and in the next one we will study this minimization problem, namely we may think of $E$ and $C$ as two abstract functionals. In section 3.4 we will apply the minimization result to the case in which $E$ and $C$ are just the energy and the hylenic charge of a dynamical system.

We set

$$Q_0 = \{ x = (x_1, ..., x_N) \in \mathbb{R}^N : 0 \leq x_i < 1, \ i = 1, ..., N \} \ and \ (10)$$

$$Q = AQ_0 \ (11)$$

where $A$ is the matrix in (9). Also we set

$$e_0 = \lim_{\varepsilon \to 0} \inf \left\{ \frac{E_Q(u)}{|C_Q(u)|} : u \in X, \|u\|_Q \leq \varepsilon, \ |C_Q(u)| > 0 \right\}, \ (12)$$

The value $e_0 = +\infty$ is allowed.

We now set

$$\Lambda (u) = \frac{E(u)}{|C(u)|} \ (13)$$
\[ \Lambda_\sigma = \lim_{\varepsilon \to 0} \inf \{ \Lambda(u) : \sigma - \varepsilon \leq |C(u)| \leq \sigma + \varepsilon \}. \quad (14) \]

\[ \Lambda^* = \inf_{u \in \Lambda} \Lambda(u). \quad (15) \]

**Theorem 6.** Assume \((E-0,..,E-4)\). Moreover assume that

\[ 0 < \Lambda^* < e_0. \quad (16) \]

Then, there exists \(\bar{\sigma}\) such that

\[ \Gamma_{\bar{\sigma}} \neq \emptyset \quad (17) \]

where \(\Gamma_{\bar{\sigma}}\) is as in definition 4. Moreover

- if \(u_n\) is a sequence such that \(\Lambda(u_n) \to \Lambda_{\bar{\sigma}}\) and \(|C(u_n)| \to \bar{\sigma}\) then
  \[ \text{dist}(u_n, \Gamma_{\bar{\sigma}}) \to 0 \quad (18) \]

- \(\Gamma_{\bar{\sigma}}\) has the structure in (5), namely
  \[ \Gamma_{\bar{\sigma}} = \{ u(x-x_0) : u(x) \in \mathcal{K}, x_0 \in F \} \quad (19) \]

  with \(\mathcal{K}\) compact and \(F \subset \mathbb{R}^N\) is a closed set such that \(F = T_j F, \forall j \in \mathbb{Z}^N\).

- any \(u \in \Gamma_{\bar{\sigma}}\) solves the eigenvalue problem
  \[ E'(u) = \lambda C'(u). \quad (20) \]

In many concrete situations \(C(u)\) and \(\Lambda(u)\) behave monotonically with respect to the action of the dilatation group \(R_\theta\) defined by

\[ R_\theta u(x) = u(\theta x), \theta \in \mathbb{R}^+. \]

In this case we obtain a stronger result:

**Theorem 7.** Let the assumptions of Th. 6 be satisfied. Moreover suppose that there is the action of a group \(R_\theta, \theta \in \mathbb{R}^+\) such that \(\Lambda(R_\theta u)\) is decreasing in \(\theta\) while \(E(R_\theta u)\) and \(C(R_\theta u)\) are increasing. Then there exists \(\sigma_0\) such that, for any \(\bar{\sigma} \geq \sigma_0\), the same conclusions of Th. 6 hold.

The following proposition gives an expression of \(e_0\) (see (12)) which will be useful in the applications of Theorems 6 and 7.

**Proposition 8.** Let \(E\) and \(C\) be as in (7) and (8); then

\[ e_0 = \inf_{u \in X} \frac{1}{2} \frac{\langle L_0 u, u \rangle_Q}{\langle L_0 u, u \rangle_Q} \quad (21) \]

Proof. We have

\[ e_0 = \lim_{\varepsilon \to 0} \inf \left\{ \frac{1}{2} \frac{\langle L u, u \rangle_Q + K_Q(u)}{\langle L_0 u, u \rangle_Q + K_0 Q(u)} : u \in X, \| u \|_Q \leq \varepsilon, C_Q(u) > 0 \right\}; \]

set \(u = \varepsilon v\) with \(0 < \| v \|_Q \leq 1\); then

\[ e_0 = \lim_{\varepsilon \to 0} \inf_{\| v \|_Q \leq 1} \frac{1}{2} \frac{\langle L v, v \rangle_Q + K_Q(\varepsilon v)}{\langle L_0 v, v \rangle_Q + K_0 Q(\varepsilon v)} = \inf_{\| v \|_Q \leq 1} \frac{1}{2} \frac{\langle L u, u \rangle_Q}{\langle L_0 u, u \rangle_Q} \]
3.3. **Proof of the main theorems.** We shall first prove some technical lemmas.

**Lemma 9.** Let $Q$ be defined by (11) and $T_j q = q + A j$ ($q \in Q$). Then
\[ \mathbb{R}^N = \bigcup_{j \in \mathbb{Z}^N} T_j Q. \] (22)

*Proof.* Take a generic $x \in \mathbb{R}^N$ and set $y = A^{-1} x$. We can decompose $y$ as follows:
\[ y = q_0 + j \text{ where } j \in \mathbb{Z}^N \text{ and } q_0 \in Q_0 \text{ defined by (10)}. \] Then
\[ x = A y = A q_0 + A j = q + A j = T_j q \]
where $q := A q_0 \in Q$. Since $x$ is generic the lemma is proved. \[ \square \]

**Lemma 10.** The map
\[ \sigma \mapsto \Lambda_{\sigma}, \]
where $\Lambda_{\sigma}$ is defined in (14), is lower semicontinuous. Moreover
\[ \forall \sigma \in \mathbb{R}^+, \quad |C(u)| = \sigma \Longrightarrow E(u) \geq \sigma \Lambda_{\sigma} \] (23)

and
\[ \lim_{\sigma \to 0} \Lambda_{\sigma} \geq e_0 \] (24)

*Proof.* The semicontinuity of $\Lambda_{\sigma}$ is an immediate consequence of the definition. Moreover, by its definition, we have that
\[ \Lambda_{\sigma} \leq \inf_{u \in \mathbb{R}_\sigma} \Lambda_{\sigma}(u) = \inf_{u \in \mathbb{R}_\sigma} \frac{E(u)}{\sigma} \]
from which (23) follows. Let us prove (24). We set
\[ e_* = \lim_{\varepsilon \to 0} \inf \left\{ \frac{E(u)}{|C(u)|} : u \in X, \|u\| \leq \varepsilon, |C(u)| > 0 \right\}. \] (25)

Let $u_n \in X$ be a sequence such that
\[ \frac{E(u_n)}{|C(u_n)|} = e_* + o(1) \]
\[ \|u_n\| = o(1). \]

We can assume, passing eventually to a subsequence, that
\[ C(u_n) \geq 0. \]
If such a subsequence does not exist, we have $C(u_n) \leq 0$ and we argue in a similar way.

Now take $\varepsilon > 0$, then, for $n$ large enough
\[ \frac{E(u_n)}{C(u_n)} \leq e_* + \varepsilon, \]

So, if we set
\[ T_j Q = \Omega_j (j \in \mathbb{Z}^N), \]
by (22) (see also section 3.1) we have
\[ e_* + \varepsilon \geq \frac{E(u_n)}{C(u_n)} = \sum_{j \in \mathbb{Z}^N} \frac{E_{\Omega_j}(u_n)}{\sum_{j \in \mathbb{Z}^N} C_{\Omega_j}(u_n)} \geq \frac{\sum_{j \in I} E_{\Omega_j}(u_n)}{\sum_{j \in I} C_{\Omega_j}(u_n)} \] (26)
where
\[ \mathbb{I} = \{ j \in \mathbb{Z}^N : C_{\Omega_j} (u_n) > 0 \} . \]

Now, for every \( n \) large, it is possible to take \( j_n \in I \) such that
\[ \frac{E_{\Omega_{j_n}} (u_n)}{C_{\Omega_{j_n}} (u_n)} \leq e_* + \varepsilon. \tag{27} \]

To show this, we argue indirectly and assume that
\[ \forall j \in \mathbb{I}, \quad \frac{E_{\Omega_j} (u_n)}{C_{\Omega_j} (u_n)} > e_* + \varepsilon, \tag{28} \]
then you get
\[ \frac{\sum_{j \in \mathbb{I}} E_{\Omega_j} (u_n)}{\sum_{j \in \mathbb{I}} C_{\Omega_j} (u_n)} > \frac{\sum_{j \in \mathbb{I}} (e_* + \varepsilon) C_{\Omega_j} (u_n)}{\sum_{j \in \mathbb{I}} C_{\Omega_j} (u_n)} = e_* + \varepsilon. \tag{29} \]
This contradicts (26). Now set
\[ v_n(x) = u_n(x + Aj_n). \]
Then (27) gives
\[ \frac{E_Q (v_n)}{C_Q (v_n)} \leq e_* + \varepsilon. \tag{30} \]
Since \( \|u_n\| \) and consequently also \( \|u_n\|_Q \) are infinitesimal, from (30) and the definition of \( e_0 \) we obtain that
\[ e_0 \leq \frac{E_Q (v_n)}{C_Q (v_n)} \leq e_* + \varepsilon. \]
and so
\[ e_0 \leq e_* \tag{31} \]
Now set \( L = \lim_{\sigma \to 0} \Lambda_\sigma \); then there exists a sequence \( u_n \) such that \( \|C (u_n)\| \to 0 \) and
\[ \frac{E (u_n)}{\|C (u_n)\|} \to L. \] If \( E (u_n) \geq b > 0 \), then \( L = +\infty \) and (24) holds. Since
\[ E (u_n) = \frac{1}{2} \|u_n\|^2 + K (u_n) \]
if \( E (u_n) \to 0 \), we have that \( \|u_n\| \to 0 \). So, by the definition of \( e_* \) and (31), we have that \( L \geq e_* \geq e_0 \). \( \square \)

**Lemma 11. (Splitting property)** Let \( E \) and \( C \) be as in (7) and (8) and assume that E-3, E-4 are satisfied. Let \( w_n \rightharpoonup 0 \) weakly and let \( u \in X \); then
\[ E (u + w_n) = E (u) + E (w_n) + o(1) \]
\[ C (u + w_n) = C (u) + C (w_n) + o(1) \]

**Proof.** We have to show that \( \lim_{n \to \infty} |E (u + w_n) - E (u) - E (w_n)| = 0 \). By (7), we have that
\[ \lim_{n \to \infty} |E (u + w_n) - E (u) - E (w_n)| \]
\[ \leq \lim_{n \to \infty} \frac{1}{2} \left( \|u + w_n\|^2 - \|u\|^2 - \|w_n\|^2 \right) + \lim_{n \to \infty} |K (u + w_n) - K (u) - K (w_n)|. \]
Let us consider each piece independently:
\[ \lim_{n \to \infty} \|u + w_n\|^2 - \|u\|^2 - \|w_n\|^2 = \lim_{n \to \infty} |2 (u, w_n)| = 0. \]
Choose $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ such that
\[ E_{B_R}(u) < \varepsilon, \ K_{B_R}(u) < \varepsilon \text{ and } \|u\|_{B_R} < \varepsilon \]
where
\[ B_R = \mathbb{R}^N - B_R(0) \text{ and } B_R(0) = \{ x \in \mathbb{R}^N : |x| < R \} . \]
Then, by the local compactness assumption E-3 (see section 3.1), we have that
\[
\lim_{n \to \infty} |K(u + w_n) - K(u) - K(w_n)| = \lim_{n \to \infty} |K_{B_R}(u + w_n) + K_{B_R}(u + w_n) - K_{B_R}(u) - K_{B_R}(w_n)|
\]
\[ = \lim_{n \to \infty} |K_{B_R}(u + w_n) - K_{B_R}(w_n)| \leq \lim_{n \to \infty} |K_{B_R}(u + w_n) - K_{B_R}(w_n)| + \varepsilon . \]
By (E-4) and the intermediate value theorem we have that for a suitable $\theta \in (0, 1)$
\[ |K_{B_R}(u + w_n) - K_{B_R}(w_n)| \leq \left\| K'_{B_R}(\theta u + (1 - \theta) w_n) \right\|_{X'_{B_R}} \cdot \|u\|_{B_R} \leq M \cdot \varepsilon \]
Then
\[ \lim_{n \to \infty} |K(u + w_n) - K(u) - K(w_n)| \leq \varepsilon + M \cdot \varepsilon \]
and since $\varepsilon$ is arbitrary, this limit is 0. Then we have proved the splitting property for $E$. The splitting property for $C$ is obtained arguing in the same way we did with $K$. \hfill \Box

**Lemma 12.** Assume (E-0,…,E-4) and let $\sigma^+$ satisfy the following assumptions:
\[ \exists \sigma \leq \sigma^+ : \Lambda_\sigma < e_0 \quad (32) \]
\[ \forall \sigma \geq \sigma^+ : \sigma^+ \Lambda_{\sigma^+} \leq \sigma \Lambda_\sigma . \quad (33) \]
Then, there exists $\bar{\sigma} \in (0, \sigma^+]$ such that
\[ \Gamma_\sigma \neq \emptyset \quad (34) \]
where $\Gamma_\sigma$ is as in definition 4. Moreover
- if $u_n$ is a sequence such that $\Lambda(u_n) \to \Lambda_\sigma$ and $|C(u_n)| \to \bar{\sigma}$ then
  \[ \text{dist}(u_n, \Gamma_\sigma) \to 0 \quad (35) \]
- $\Gamma_\sigma$ has the structure in (5), namely
  \[ \Gamma_\sigma = \{ u(x - x_0) : u(x) \in \mathcal{K}, \ x_0 \in F \} \quad (36) \]
  with $\mathcal{K}$ compact and $F \subset \mathbb{R}^N$ is a closed set such that $F = T_j F, \ \forall j \in \mathbb{Z}^N$.
- Any $u \in \Gamma_\sigma$ solves the eigenvalue problem
  \[ E'(u) = \lambda C'(u) . \quad (37) \]

**Proof.** By (32) and Lemma 10, we have that $\min_{\sigma \in (0, \sigma^+]} \Lambda_\sigma$ exists. Let $\bar{\sigma} \in (0, \sigma^+]$ be such that
\[ \Lambda_\sigma = \min_{\sigma \in (0, \sigma^+]} \Lambda_\sigma ; \quad (38) \]
by (32), we have that
\[ \Lambda_\sigma < e_0 . \quad (39) \]
Let $u_n \in X$ be a sequence such that
\begin{align}
\Lambda(u_n) &= \Lambda_\sigma + o(1) \\
|C(u_n)| &= \bar{\sigma} + o(1).
\end{align}
(40) (41)

In order to fix the ideas, we may assume that
\begin{equation}
C(u_n) = \bar{\sigma} + o(1).
\end{equation}
(42)

If, on the contrary no subsequence of $C(u_n)$ converges to $\bar{\sigma}$, then $C(u_n) \to -\bar{\sigma}$ and we argue in a similar way. The proof consists of two steps.

Step 1. We prove that for a suitable sequence $\{z_n\} \subset \mathbb{Z}^N$ we have
\begin{equation}
u_n(x) = \bar{\nu}(x - A z_n) + w_n(x - A z_n)
\end{equation}
where $\bar{\nu} \neq 0$ and $w_n(x) \to 0$ weakly in $X$.

We decompose $\mathbb{R}^N$ as in (22). Take $\varepsilon > 0$, then, for $n$ large enough
\begin{equation}
\frac{E(u_n)}{C(u_n)} \leq \Lambda_\sigma + \varepsilon.
\end{equation}
(43)

Arguing as in (26) and (27) (replacing $e_\star$ with $\Lambda_\sigma$), for $n$ large, it is possible to take $j_n \in I$ such that
\begin{equation}
\frac{E(u_n)}{C(u_n)} \leq \Lambda_\sigma + \varepsilon.
\end{equation}
(44)

We set
\begin{equation}
v_n(x) = u_n(x + A j_n).
\end{equation}
(45)

By (40) and (41), $E(u_n)$ and $C(u_n)$ are bounded. So, also $E(v_n)$ and $C(v_n)$ are bounded and, by (E-2), $\|v_n\|$ is bounded. Let $\bar{u}$ be the weak limit of $v_n$. We want to show that $\bar{u} \neq 0$.

Clearly $C_Q(v_n) = C_{\Omega_{j_n}}(u_n)$. Then, since $j_n \in I$, we have that $C_Q(v_n) > 0$ and, for $n$ large, by (44) we have
\begin{equation}
\frac{E_Q(v_n)}{C_Q(v_n)} \leq \Lambda_\sigma + \varepsilon.
\end{equation}
(46)

We claim that the sequence $C_Q(v_n)$ does not converge to 0; in fact if $C_Q(v_n) \to 0$, then, by (46), we have that $E_Q(v_n) \to 0$. Since
\begin{equation}
E_Q(v_n) = \frac{1}{2} \|v_n\|_Q^2 + K_Q(v_n),
\end{equation}
we have that $\|v_n\|_Q \to 0$; so, by definition of $e_0$, and by (46), we have
\begin{equation}
\Lambda_\sigma + \varepsilon \geq \lim_{n \to \infty} \frac{E_Q(v_n)}{C_Q(v_n)} \geq e_0
\end{equation}
and this fact contradicts (39) if $\varepsilon > 0$ is small enough.

Since $C_Q(v_n)$ does not converge to 0, by (E-3) with $\Omega = Q$, we have that $C_Q(\bar{u}) > 0$ and we can conclude that $\bar{u} \neq 0$. Now set
\begin{equation}
w_n = v_n - \bar{u}
\end{equation}
and so $w_n(x) \to 0$ weakly in $X$.

Step 2. Next we will prove that
\begin{equation}
v_n \to \bar{u} \text{ strongly in } X.
\end{equation}
namely that \( w_n \to 0 \) strongly in \( X \). So, by (7), it will be enough to show that
\[
E(w_n) \to 0.
\] (47)

By (40), (41) and lemma 11
\[
\Lambda_\sigma = \frac{E(\bar{u} + w_n)}{C(\bar{u} + w_n)} + o(1) = \frac{E(\bar{u}) + E(w_n)}{\sigma} + o(1)
\] (48)
and so
\[
E(\bar{u}) + E(w_n) = \sigma \Lambda_\sigma + o(1) .
\] (49)

Now we set
\[
\sigma_1 = |C(\bar{u})|,
\]
\[
\sigma_2 = \lim |C(w_n)| .
\]

We consider three cases.

Case 1: \( |C(\bar{u})| = \sigma_1 \geq \sigma^+ \). Then
\[
E(\bar{u}) \geq \sigma_1 \Lambda_{\sigma_1} \text{ (by (23))}
\]
\[
\geq \sigma^+ \Lambda_{\sigma^+} \text{ (by (33))}
\]
\[
\geq \sigma^+ \Lambda_{\sigma} \text{ (by (38))}
\]
\[
\geq \tilde{\sigma} \Lambda_{\sigma}.
\]

and by (49)
\[
E(w_n) = \tilde{\sigma} \Lambda_{\sigma} + o(1) - E(\bar{u}) \leq o(1) ,
\]
and so \( E(w_n) \to 0 \).

Case 2: \( \sigma_2 = |C(w_n)| + o(1) \geq \sigma^+ \). Then
\[
E(w_n) \geq |C(w_n)| \Lambda_{|C(w_n)|} \text{ (by (23))}
\]
\[
\geq \sigma_2 \Lambda_{\sigma_2} + o(1) \text{ (by lemma 10)}
\]
\[
\geq \sigma^+ \Lambda_{\sigma^+} + o(1) \text{ (by (33))}
\]
\[
\geq \sigma^+ \Lambda_{\sigma} + o(1) \text{ (by (38))}
\]
\[
\geq \tilde{\sigma} \Lambda_{\sigma} + o(1) .
\]

Then by (49) you get
\[
\tilde{\sigma} \Lambda_{\sigma} = E(\bar{u}) + E(w_n) + o(1) \geq E(\bar{u}) + \tilde{\sigma} \Lambda_{\sigma} + o(1)
\]
and this is a contradiction since \( E(\bar{u}) > 0 \); thus case 2 cannot occur.

Case 3: \( \sigma_1, \sigma_2 \leq \sigma^+ \). In this case, we have by (49) and (23)
\[
\tilde{\sigma} \Lambda_{\sigma} = E(\bar{u}) + E(w_n) + o(1) \geq \sigma_1 \Lambda_{\sigma_1} + \sigma_2 \Lambda_{\sigma_2} + o(1)
\]
\[
\geq (\sigma_1 + \sigma_2) \Lambda_{\sigma}.
\]

Then
\[
\sigma_1 + \sigma_2 \leq \tilde{\sigma} .
\] (50)

Now the opposite inequality can be obtained by splitting the charge as in lemma 11:
\[
\tilde{\sigma} = |C(\bar{u} + w_n)| + o(1) \leq |C(\bar{u})| + |C(w_n)| + o(1) = \sigma_1 + \sigma_2 + o(1) .
\] (51)

From (50) and (51) we get
\[
\sigma_1 + \sigma_2 = \tilde{\sigma} .
\] (52)

Now we claim that
\[
\sigma_1 > 0 .
\] (53)
Arguing by contradiction assume $\sigma_1 = 0$, then by (52) we have $\sigma_2 = \bar{\sigma}$ and by (49) and (23)

$$\bar{\sigma} \Lambda_{\bar{\sigma}} = E(\bar{u}) + E(w_n) + o(1) \geq E(\bar{u}) + \sigma_2 \Lambda_{\sigma_2} + o(1) = E(\bar{u}) + \bar{\sigma} \Lambda_{\bar{\sigma}} + o(1)$$

and this contradicts $E(\bar{u}) > 0$.

Now it is not restrictive to suppose that

$$\bar{\sigma} = \min \left\{ \sigma : \Lambda_{\sigma} = \min_{\tau \in (0, \sigma^+]} \Lambda_{\tau} \right\}.$$  \hspace{1cm} (54)

We claim that $\sigma_2 = 0$. In fact, arguing by contradiction assume that $\sigma_2 > 0$, then, by (52), $\sigma_1 < \bar{\sigma}$ and, by (54), $\Lambda_{\sigma_2} - \Lambda_{\bar{\sigma}} = \delta > 0$. So we have

$$\sigma \Lambda_{\sigma} = E(\bar{u}) + E(w_n) + o(1) \geq \sigma_1 \Lambda_{\sigma_1} + \sigma_2 \Lambda_{\sigma_2} + o(1)$$

$$\geq \sigma_1 (\Lambda_{\bar{\sigma}} + \delta) + \sigma_2 \Lambda_{\bar{\sigma}} + o(1) = \bar{\sigma} \Lambda_{\bar{\sigma}} + \sigma_1 \delta + o(1)$$

and this is a contradiction since $\sigma_1 \delta > 0$, so we have $\sigma_2 = 0$.

Since $\sigma_2 = 0$, then $\sigma_1 = \bar{\sigma}$, and by (49) and (23)

$$E(w_n) = \bar{\sigma} \Lambda_{\bar{\sigma}} - E(\bar{u}) + o(1) \leq \bar{\sigma} \Lambda_{\bar{\sigma}} - \sigma_1 \Lambda_{\sigma_1} + o(1) = o(1)$$

from which we get (47).

By the preceding results we easily get the conclusions (34,...,37). In fact:

- Consider the sequence $v_n$ defined in (45). We have seen in steps 1, 2 that $v_n \to \bar{u}$ strongly in $X$. Then, since $E$ and $C$ are continuous, we have

$$\frac{E(v_n)}{C(v_n)} = \frac{E(\bar{u})}{C(\bar{u})} + o(1) = \frac{E(\bar{u})}{\bar{\sigma}} + o(1).$$  \hspace{1cm} (55)

Moreover by (40)

$$\frac{E(v_n)}{C(v_n)} = \Lambda_{\bar{\sigma}} + o(1) \leq \inf \left\{ \frac{E(u)}{\bar{\sigma}} : C(u) = \bar{\sigma} \right\} + o(1).$$  \hspace{1cm} (56)

From (55) and (56) we deduce that

$$\bar{u} \in \Gamma_{\bar{\sigma}}.$$  \hspace{1cm} (57)

- By steps 1, 2 and (57), we clearly get (35). Moreover, if we take a sequence $\{u_n\} \subset \Gamma_{\bar{\sigma}}$, by using again steps 1, 2, we get that there exists a subsequence, which we continue to call $u_n$, and $\{j_n\} \subset F$ such that

$$v_n \to \bar{u} \in \Gamma_{\bar{\sigma}} \text{ strongly in } X, \ v_n(x) = u_n(Aj_n + x).$$

Then also (36) holds.

Finally (37) clearly follows by the definition of $\Gamma_{\bar{\sigma}}$. \hspace{1cm} \(\square\)

**Proof of Th. 6.** We prove that the assumptions (32) and (33) of Lemma 12 are satisfied.

First, we observe that, by (16), $\Lambda_{\sigma} \geq \Lambda^* > 0$, then

$$\sigma \Lambda_{\sigma} \to \infty \text{ for } \sigma \to \infty.$$  \hspace{1cm} (58)

Now set

$$\tau_n = \sup \{ \sigma : \sigma \Lambda_{\sigma} \geq n \}.$$  \hspace{1cm} (59)

Then, by definition

$$\tau_n \Lambda_{\tau_n} \leq n.$$  \hspace{1cm} (60)

By (58)

$$\tau_n \in \mathbb{R} \text{ and } \tau_n \to \infty \text{ for } n \to \infty.$$  \hspace{1cm} (61)
Now by (16) there exists \( u_0 \in X \) such that

\[
\Lambda(u_0) < \epsilon_0. \tag{62}
\]

By (61) there exists \( \bar{u} \) such that

\[
\tau_{\bar{u}} \geq |C(u_0)|.
\]

and by (59) and (60)

\[
\tau_{\bar{n}} \Lambda_{\tau_{\bar{n}}} \leq \bar{n} \leq \sigma \Lambda_{\sigma} \text{ for } \sigma \geq \tau_{\bar{n}}. \tag{63}
\]

Set \( \sigma_+ = \tau_{\bar{n}} \), then by (62) and (63) we get

\[
\sigma = |C(u_0)| \leq \sigma^+, \quad \Lambda_{\sigma} < \epsilon_0
\]

\[
\sigma^+ \Lambda_{\sigma^+} \leq \sigma \Lambda_{\sigma} \text{ for } \sigma \geq \sigma^+.
\]

Then the assumptions (32) and (33) of Lemma (12) are satisfied. \( \square \)

**Remark 13.** By the proof of this theorem, we can see that the assumption \( \Lambda^* > 0 \) is used only to get (58). This assumption can be replaced by the following one

\[
\|u_n\| \to \infty \Rightarrow E(u_n) \to \infty \tag{64}
\]

In fact (64) implies (58). To show this, we argue indirectly and assume that there exists a sequence \( \sigma_n \to \infty \) such that \( \sigma_n \Lambda_{\sigma_n} \) is bounded; so there exists a sequence \( u_n \) such that

\[
|C(u_n)| \to \infty \tag{65}
\]

and

\[
|C(u_n)| \Lambda(u_n) = E(u_n) \text{ is bounded.} \tag{66}
\]

By (65) and (E-4), we have that (for a subsequence) \( \|u_n\| \to \infty \); then, by (64), \( E(u_n) \to \infty \). This contradicts (66), then we conclude that (58) holds.

**Proof of Th. 7.** Arguing as in the proof of Th. 6, there exists \( \sigma_0^+ > 0 \) such that

\[
\Lambda(\bar{u}) = \min_{|C(u)| \leq \sigma_0^+} \Lambda(u). \tag{67}
\]

for a suitable \( \bar{u} \).

We shall show that

\[
|C(\bar{u})| = \sigma_0^+. \tag{68}
\]

Arguing by contradiction, assume that \( |C(\bar{u})| < \sigma_0^+ \). Then, since \( C(R\theta u) \) and \( \Lambda(R\theta u) \) are respectively increasing and decreasing in \( \theta \), for \( \varepsilon > 0 \) small enough and \( 1 < \vartheta < 1 + \varepsilon \), we have

\[
\sigma_0^+ \geq |C(R\theta \bar{u})| > |C(\bar{u})| \tag{69}
\]

\[
\Lambda(R\theta \bar{u}) < \Lambda(\bar{u}). \tag{70}
\]

Clearly (69) and (70) contradict (67). So (68) holds.

Now set \( \sigma_0 = \sigma_0^+ \) and take any other \( \sigma^+ \geq \sigma_0 \). Clearly (32) and (33) hold and we can argue as before. \( \square \)
3.4. Dynamical consequences of the main theorem. The above theorems can be applied to the case in which \((X, \| \cdot \|)\) is the state space of a dynamical system \((X, T)\) and it proves the existence of hylomorphic solitons; more exactly we have:

**Theorem 14.** Let \((X, T)\) be a dynamical system and let \(E\) and \(C\) be the energy and the charge. If \(X, E\) and \(C\) are as in section 3.1 and satisfy the assumptions of theorem 6, then \((X, T)\) has hylomorphic solitons. Moreover, if also the assumptions of Th. 7 are satisfied, there exists \(\sigma_0\) such that there are solitons for any charge \(\bar{\sigma} \geq \sigma_0\).

*Proof.* We consider Def. 4. We set

\[
\Gamma_\sigma = \{ u \in \mathcal{M}_\sigma : E(u) = c_\sigma \}
\]

with

\[
c_\sigma = \min_{u \in \mathcal{M}_\sigma} E(u).
\]

By theorem 6 \(\Gamma_\sigma \neq \emptyset\). In order to prove the existence of solitons we need to prove (ii) and (iii) of definition 3. (ii) follows by (36).

In order to prove stability, we use the Lyapunov criterium; we define the Lyapunov function

\[
V(u) := (E(u) - c_\sigma)^2 + (|C(u)| - \sigma)^2;
\]

then by (35)

\[
V(u_n) \to 0 \implies d(u_n, \Gamma) \to 0.
\]

(71)

Then, by the Lyapunov stability theorem \(\Gamma\) is stable.

The second statement follows directly from Th. 7. \(\Box\)

4. The nonlinear Schrödinger equation. We are interested to the nonlinear Schrödinger equation:

\[
i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x) \psi + \frac{1}{2} W'(\psi)
\]

(72)

where \(\psi : \mathbb{R}^N \to \mathbb{C} (N \geq 3)\), \(V : \mathbb{R}^N \to \mathbb{R}\), \(W : \mathbb{C} \to \mathbb{R}\) such that \(W(s) = F(|s|)\) for some smooth function \(F : [0, \infty) \to \mathbb{R}\) and

\[
W'(s) = \frac{\partial W}{\partial s_1} + i \frac{\partial W}{\partial s_2}, \quad s = s_1 + is_2
\]

(73)

namely

\[
W'(s) = F'(|s|) \frac{s}{|s|}.
\]

Equation (72) is the Euler-Lagrange equation relative to the Lagrangian density

\[
\mathcal{L} = \text{Re} \left( i\partial_t \psi \overline{\psi} \right) - \frac{1}{2} |\nabla \psi|^2 - V(x)|\psi|^2 - W(\psi)
\]

(74)

4.1. Existence results. We assume that \(W\) has the following form

\[
W(s) = \frac{1}{2} h^2 s^2 + N(s)
\]

(75)

where \(h^2 = W''(0)\) and \(N(s) = o(s^2)\). We make the following assumptions on \(W\):

- (W-i) **(Positivity)** \(W(s) \geq 0\)
- (W-ii) **(Nondegeneracy)** \(W\) has the form (75) and \(h > 0\)
- (W-iii) **(Hylomorphism)** \(0 < \inf_{s} \frac{W(s)}{s^2} < W''(0)\)
(W-iii) **(Growth condition)** there are constants \(c_1, c_2 > 0\), \(2 < p < 2N/(N-2)\) such that for any \(s > 0\):

\[
|N'(s)| \leq c_1 s^{p-1} + c_2 s^{2-\frac{2}{p}}.
\]

If we set

\[
\alpha^2 = \inf_s \frac{W(s)}{s^2},
\]

then the hylomorphy assumption (W-iii) reads

\[
0 < \alpha < h.
\]

This assumption implies that

\[
\exists \bar{s} : N(\bar{s}) < 0.
\]

We make the following assumptions on \(V\):

- (V-i) **(Positivity)** \(V \geq 0\) and \(V \in L^\infty\).
- (V-ii) **(Lattice invariance)** There exists an \(N \times N\) invertible matrix \(A\) such that

\[
V(x) = V(x - Az)
\]

for all \(x \in \mathbb{R}^N\) and \(z \in \mathbb{Z}^N\).

Here we want to use the results of the previous sections to study \((72)\). In this case the state \(u\) coincides with \(\psi\) and the general framework of the previous sections takes the following form:

\[
X = H^1(\mathbb{R}^N)
\]

where \(H^1(\mathbb{R}^N)\) is the usual Sobolev space and

\[
E(u) = \int \left(\frac{1}{2} |\nabla u|^2 + V(x) |u|^2 + W(u)\right) dx
\]

\[
= \int \left(\frac{1}{2} |\nabla u|^2 + \frac{h^2 u^2}{2} + V(x) |u|^2\right) dx + \int N(u) dx;
\]

\[
C(u) = \int u^2 dx
\]

\[
\|u\|^2 = \int \left(|\nabla u|^2 + au^2\right) dx.
\]

Then the energy \(E\) and the hylenic charge \(C\) have the form \((7)\) and \((8)\) respectively. We shall prove the following theorem

**Theorem 15.** Assume that \(W\) satisfies W-i),...W-iii) and that \(V\) satisfies V-i), V-ii). Moreover assume that

\[
\frac{\alpha^2}{2} + \|V\|_{L^\infty} < \frac{h^2}{2}
\]

where \(\alpha\) and \(h\) have been introduced in \((76)\) and \((75)\). Then equation \((72)\) admits hylomorphic solitons (see definition 4).

**Remark 16.** Observe that, when \(V = 0\), assumption \((82)\) reduces to the request \(\alpha < h\), which is the “usual” hylomorphy condition (see [3], [2], [7], [5]). Moreover, in this case it is possible to apply Th. 7 and to get the existence of solitons for any sufficiently large charge.
Remark 17. Actually, the assumptions (W-i,...,W-iii) are not the most general. For example the positivity assumption is not necessary. In the case $V = 0$, we refer to [4]. If $V \neq 0$, we do not know whether the assumptions used in [4] are sufficient.

We first obtain some estimates on $e_0$ and $\Lambda_*$ defined by (21) and (15).

Lemma 18. Assume that $W$ satisfies (W-i,...W-iii) and that $V$ satisfies (V-i, V-ii). Then

$$\frac{h^2}{2} \leq e_0 \leq \frac{h^2}{2} + \|V\|_{L^\infty}$$ \hfill (83)

$$\frac{\alpha^2}{2} \leq \Lambda_* \leq \frac{\alpha^2}{2} + \|V\|_{L^\infty}$$ \hfill (84)

Proof. By using (21), we clearly deduce that (83) holds.

Now we prove (84). First we show that:

$$\Lambda_* \geq \frac{\alpha^2}{2}.$$ \hfill (85)

In fact, by using (76), we get

$$\Lambda_* = \inf_u \frac{E(u)}{|C(u)|} = \inf_u \frac{\int \left( \frac{1}{2} |\nabla u|^2 + V(x) |u|^2 + W(u) \right) dx}{\int u^2 dx} \geq \inf_u \frac{\int |W(u)| dx}{\int u^2 dx} \geq \inf_u \frac{\int \frac{1}{2} \alpha^2 u^2 dx}{\int u^2 dx} = \frac{1}{2} \alpha^2.$$ \hfill (86)

Now we prove that

$$\Lambda_* \leq \frac{\alpha^2}{2} + \|V\|_{L^\infty}.$$ \hfill (87)

Take $\epsilon > 0$, then by (76), there exists $s_\epsilon > 0$ such that

$$W(s_\epsilon) < \frac{1}{2} s_\epsilon^2 (\alpha^2 + \epsilon).$$ \hfill (88)

Let $R > 0$ and set

$$u_{\epsilon,R} = \begin{cases} s_\epsilon & \text{if } |x| < R \\ 0 & \text{if } |x| > R + 1 \\ \frac{|x|}{R} s_\epsilon - (|x| - R) \frac{R + 1}{R} s_\epsilon & \text{if } R < |x| < R + 1 \end{cases}$$ \hfill (89)

Clearly

$$\frac{\int \frac{1}{2} |\nabla u_{\epsilon,R}|^2 dx}{\int |u_{\epsilon,R}|^2 dx} \leq O \left( \frac{1}{R} \right).$$ \hfill (90)

Then, by (87) and (89) we get
\[ \Lambda_* \leq \frac{\int \left( \frac{1}{2} |\nabla u_{e,R}|^2 + V(x) |u|^2 + W(u_{e,R}) \right) dx}{\int |u_{e,R}|^2 dx} \]

\[ \leq \frac{\int_{|x|<R} \left( W(u_{e,R}) + V(x) |u_{e,R}|^2 \right) dx}{\int_{|x|<R} |u_{e,R}|^2 dx} \]

\[ + \frac{\int_{R<|x|<R+1} \left( \frac{1}{2} |\nabla u_{e,R}|^2 + W(u_{e,R}) + V(x) |u_{e,R}|^2 \right) dx}{\int_{|x|<R} |u_{e,R}|^2 dx} \]

\[ \leq \frac{\int_{|x|<R} \left( W(u_{e,R}) + V(x) |u_{e,R}|^2 \right) dx}{\int_{|x|<R} |u_{e,R}|^2 dx} + \frac{c_1 R^{N-1}}{c_2 R^N} \]

\[ = \frac{\int_{|x|<R} \left( W(s_{\varepsilon}) + V(x) |s_{\varepsilon}|^2 \right) dx}{\int_{|x|<R} |s_{\varepsilon}|^2 dx} + O \left( \frac{1}{R} \right) \]

\[ \leq \frac{s_{\varepsilon}^2 (\alpha^2 + \varepsilon) R^N}{s_{\varepsilon}^2 R^N} + \frac{\|V\|_{L^\infty} s_{\varepsilon}^2 R^N}{s_{\varepsilon}^2 R^N} + O \left( \frac{1}{R} \right) \]

\[ = \frac{1}{2} (\alpha^2 + \varepsilon) + \|V\|_{L^\infty} + O \left( \frac{1}{R} \right) \]

Then, since \( \varepsilon > 0 \) is arbitrary, we easily get (86). Finally (84) follows from (85) and (86).

**Proof of Theorem 15:** By (83), (84) and (82) we deduce that \( 0 < \Lambda_* < \varepsilon_0 \). It can be shown, by standard calculations (see e.g. [7]), that under the assumptions W-i), W-ii) and W-iii) and V-i), V-ii), the functionals \( E \) and \( C \), defined by (79) and (81), satisfy (E-0,...,E-4) of section 3.1. Then, by using Theorem 6, we deduce that equation (72) admits hylomorphic solitons. Since these solitons \( u_0 \) are minimizers of the energy \( E \) on the manifold \( \{ u \in H^1(\mathbb{R}^N) : C(u) = \int u^2 dx = \sigma \} \), we get

\[ E'(u_0) = -\omega C'(u_0) \]

where \( \omega \) is a Lagrange multiplier. Then it can be easily seen that \( u_0 \) solves (72) and \( u_0 = \psi_0(x)e^{-ixt} \), where \( \omega \in \mathbb{R} \) and \( \psi_0(x) \) solve the equation

\[ -\frac{1}{2} \Delta \psi_0 + V(x) \psi_0 + \frac{1}{2} W'(\psi) = \omega \psi_0 \]

5. **The nonlinear Klein-Gordon equation.** In this section we will apply th. 6 to the existence of hylomorphic solitons of the nonlinear Klein-Gordon equation. We point out that the existence of such solitons for this equation has been recently stated in [3]. Here we consider the case in which \( W \) depends on \( x \) and it has a lattice symmetry.

More exactly, we consider the equation

\[ \Box \psi + W'(x, \psi) = 0 \]

(90)

where \( \Box = \partial_t^2 - \nabla^2 \), \( \psi : \mathbb{R}^N \rightarrow \mathbb{C} \) (\( N \geq 3 \)), \( W : \mathbb{R}^N \times \mathbb{C} \rightarrow \mathbb{R} \) and \( W' \) is the derivative with respect to the second variable as in (73). We assume \( W \) to be as
follows
\[ W(x, s) = \frac{1}{2} h(x)^2 s^2 + N(x, s), \] \[ x \in \mathbb{R}^N, \ s \in \mathbb{R}^+, \ h(x) \in L^\infty \] (91)

where
\[ h(x) \geq h_0 > 0 \] (92)
and \( N(x, s) = o(s^2) \). We make the following assumptions on \( W \):

• (NKG-i) **(Positivity)** \( W(x, s) \geq 0 \)

• (NKG-ii) **(Lattice invariance)** There exists an \( N \times N \) invertible matrix \( A \) such that
\[ W(x, s) = W(x - Az, s) \] (93)

for all \( x \in \mathbb{R}^N \) and \( z \in \mathbb{Z}^N \).

• (NKG-iii) **(Hylomorphy)** \( \exists \alpha, \bar{s} \in \mathbb{R}^+ \) such that
\[ W(x, \bar{s}) \leq \frac{1}{2} \alpha^2 \bar{s}^2 \] (94)

• (NKG-iii) **(Growth condition)** there are constants \( c_1, c_2 > 0, 2 < p < \frac{2N}{N-2} \) such that for any \( s > 0 \):
\[ |N'(x, s)| \leq c_1 s^{p-1} + c_2 s^{2-\frac{2}{p}}. \]

We shall assume that the initial value problem is well posed for (NKG). Eq. (90) is the Euler-Lagrange equation of the action functional
\[ S(\psi) = \int \left( \frac{1}{2} |\psi_t|^2 - \frac{1}{2} |\nabla \psi|^2 - W(x, \psi) \right) dxdt. \] (94)

The energy and the charge take the following form:
\[ E(\psi) = \int \left[ \frac{1}{2} |\psi_t|^2 + \frac{1}{2} |\nabla \psi|^2 + W(x, \psi) \right] dx \] (95)
\[ C(\psi) = -\text{Re} \int i \partial_t \psi \overline{\psi} dx. \] (96)

(\text{the sign “minus” in front of the integral is a useful convention).}

5.1. **The NKG as a dynamical system.** We set
\[ X = H^1(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C}) \]
and we will denote the generic element of \( X \) by \( u = (\psi(x), \hat{\psi}(x)) \); then, by the well posedness assumption, for every \( u \in X \), there is a unique solution \( \psi(t, x) \) of (90) such that
\[ \psi(0, x) = \psi(x) \]
\[ \partial_t \psi(0, x) = \hat{\psi}(x). \]

Thus, using our notation, we can write
\[ T_t u = U(t, x) = (\psi(t, x), \hat{\psi}(t, x)) \in C^1(\mathbb{R}, X). \]

Using this notation, we can write equation (90) in Hamiltonian form:
\[ \partial_t \psi = \hat{\psi} \] (97)
\[ \partial_t \hat{\psi} = \Delta \psi - W'(x, \psi). \] (98)

The energy and the charge, as functionals defined in \( X \), become
\[ E(u) = \int \left[ \frac{1}{2} |\psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(x, \psi) \right] dx \] (99)
We shall tacitly assume that \( W \) is such that \( E, C \) are \( C^1 \) in \( X \).

**Proposition 19.** Let \( u_0(x) = (\psi_0(x), \hat{\psi}_0(x)) \in X \) be a critical point of \( E \) constrained on the manifold \( \mathcal{M}_\sigma = \{ u \in X : C(u) = \sigma \} \). Then there exists \( \omega \in \mathbb{R} \) such that \( \psi_0 \) satisfies the equation

\[
- \Delta \psi_0 + W'(x, \psi_0) = \omega^2 \psi_0
\]

(101) and

\[
U(t,x) = \left[ \begin{array}{c} \psi_0(x)e^{-i\omega t} \\ -i\omega \hat{\psi}_0(x)e^{-i\omega t} \end{array} \right]
\]

(102)
solves (97), (98).

**Proof.** Clearly

\[
E'(u_0) = -\omega C'(u_0)
\]

(103)
where \(-\omega\) is a Lagrange multiplier. We now compute the derivatives \( E'(u_0), C'(u_0) \).

For all \((v_0, v_1) \in X = H^1(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C})\), we have

\[
E'(u_0) \left[ \begin{array}{c} v_0 \\ v_1 \end{array} \right] = \text{Re} \int (\hat{\psi}_0 \overline{\nabla} + \nabla \psi_0 \overline{\nabla} + W'(x, \psi_0) \overline{\nabla}) dx
\]

\[
C'(u_0) \left[ \begin{array}{c} v_0 \\ v_1 \end{array} \right] = -\text{Re} \int (i\hat{\psi}_0 \overline{v_0} + iv_1 \overline{\psi_0}) dx
\]

\[
= -\text{Re} \int (i\hat{\psi}_0 \overline{v_0} + iv_1 \overline{\psi_0}) dx
\]

\[
= \text{Re} \int (i\hat{\psi}_0 \overline{v_0} - iv_1 \overline{\psi_0}) dx.
\]

Then (103) can be written as follows:

\[
\text{Re} \int \left( \nabla \overline{\psi_0} + W'(x, \psi_0) \overline{\nabla} \right) dx = \omega \text{Re} \int i\hat{\psi}_0 \overline{\nabla} dx
\]

\[
\text{Re} \int \hat{\psi}_0 \overline{\nabla} dx = -\omega \text{Re} \int i\hat{\psi}_0 \overline{\nabla} dx.
\]

Then

\[
- \Delta \psi_0 + W'(x, \psi_0) = i\omega \hat{\psi}_0
\]

\[
\psi_0 = -i\omega \hat{\psi}_0
\]

(104)

So we get (101). From (101) and (104) we easily verify that (102) solves (97), (98).

**5.2. Existence results for NKG.** The following Theorem holds:

**Theorem 20.** Assume that \( W \) satisfies NKG-i),...NKG-iii) and that

\[
\alpha < h_0
\]

(105)
where \( h_0 \) is defined by (91) and (92). Then equation (NKG) admits hylomorphic solitons having the following form

\[
U(t,x) = (\psi_0(x)e^{-i\omega t}, -i\omega \hat{\psi}_0(x)e^{-i\omega t}).
\]
In order to prove the existence of hylomorphic solitons, we will use Th. 6. Clearly the energy $E$ and the hylenic charge $C$ have the form (7) and (8) respectively, with

$$X = \left\{ u = \left( \psi, \hat{\psi} \right) \in H^1(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C}) \right\}$$

$$\langle Lu, u \rangle = \int \left( |\psi|^2 + |\nabla \psi|^2 + h^2 |\psi|^2 \right) dx; \quad K(u) = \int N(\psi) dx,$$

$$\langle L_0u, u \rangle = C(u) = - \text{Re} \int i\psi \hat{\psi} dx; \quad K_0(u) = 0.$$  \hfill (106)

Now let us compute $e_0$ and $\Lambda_\ast$ defined by (21) and (15).

**Lemma 21.** Assume that $W$ satisfies NKG-i,...NKG-iii), then

$$e_0 \geq h_0 \quad \text{and} \quad \Lambda_\ast \leq \alpha.$$  \hfill (108)

**Proof.** By (21) we have

$$e_0 = \inf \frac{\frac{1}{2} \langle Lu, u \rangle_Q}{\langle L_0 u, u \rangle_Q} = \inf \frac{\frac{1}{2} \int_Q \left( |\psi|^2 + |\nabla \psi|^2 + h(\bar{x}) |\psi|^2 \right) dx}{\text{Re} \int_Q i\psi \hat{\psi} dx} \hfill (110)$$

$$\geq \inf \frac{\frac{1}{2} \int_Q \left( |\psi|^2 + h_0^2 |\psi|^2 \right) dx}{\int_Q |\psi| \cdot |\psi| dx} \geq \inf \frac{h_0 \int_Q |\psi| \cdot |\psi| dx}{\int_Q |\psi| \cdot |\psi| dx} = h_0.$$  \hfill (110)

Then

$$e_0 \geq h_0$$

Let us now prove that

$$\Lambda_\ast \leq \alpha$$

Let $R > 0$; set

$$u_R = \begin{cases} \bar{s} & \text{if } |x| < R \\ 0 & \text{if } |x| > R + 1 \\ \frac{|x|}{R} \bar{s} - (|x| - R) \frac{R + 1}{R} \bar{s} & \text{if } R < |x| < R + 1 \end{cases} \hfill (111)$$

and set $\psi = u_R$, and $\hat{\psi} = \alpha u_R.$
Then

$$\Lambda_* = \inf_{\psi, \phi} \frac{\int \left( \frac{1}{2} |\hat{\psi}|^2 + \frac{1}{2} |\nabla \psi|^2 + W(x, \psi) \right) dx}{\text{Re} \int i\hat{\psi} \phi^{*} dx}$$

$$\leq \frac{\int \left( \frac{1}{2} \alpha^2 |u_R|^2 + \frac{1}{2} |\nabla u_R|^2 + W(x, u_R) \right) dx}{\alpha \int |u_R|^2 dx}$$

$$\leq \frac{\int_{|x| < R} \left( \frac{1}{2} \alpha^2 |u_R|^2 + \frac{1}{2} |\nabla u_R|^2 + W(x, u_R) \right) dx}{\alpha \int_{|x| < R} |u_R|^2 dx}$$

$$\quad + \frac{\int_{R < |x| < R + 1} \left( \frac{1}{2} \alpha^2 |u_R|^2 + \frac{1}{2} |\nabla u_R|^2 + W(x, u_R) \right) dx}{\alpha \int_{|x| < R} |u_R|^2 dx}$$

$$= \frac{1}{2} \alpha + \frac{\int_{|x| < R} W(x, \hat{s}) dx}{\alpha \int_{|x| < R} |\hat{s}|^2 dx} + O \left( \frac{1}{R} \right).$$

Then, by NKG-iii, we have

$$\Lambda_* \leq \frac{1}{2} \alpha + \frac{\int_{|x| < R} \frac{1}{2} \hat{s}^2 \alpha^2 R^N}{\alpha \int_{|x| < R} |\hat{s}|^2 dx} + O \left( \frac{1}{R} \right) = \alpha + O \left( \frac{1}{R} \right).$$

Then, we get

$$\Lambda_* \leq \alpha \quad \square$$

**Proof of Theorem 20:** By Lemma 21 and assumption (105) we deduce that $\Lambda_* < \epsilon_0$. By standard calculations it can be shown that under the assumptions NKG-i),...,NKG-iii) the functionals $E$ and $C$, defined by (79) and (81), satisfy (E1,2,3,4) of section 3.1. Then, by using Th. 6 and remark 13, we deduce that equation (90) admits hylomorphic solitons. Since these solitons are minimizers of the energy $E$ on the manifold $\{u \in X : C(u) = \sigma\}$, we easily get, by Proposition 19, that they are solutions of (90) of the type $U(t, x) = (\psi_0(x) e^{-i\omega t}, -i\omega \psi_0(x) e^{-i\omega t})$ with $\psi_0, \omega$ satisfying (101). \quad \square

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