SEQUENTIALLY COHEN-MACAULAY MODULES AND LOCAL COHOMOLOGY

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INTRODUCTION

Let $I \subset R$ be a graded ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$ where $K$ is a field, and fix a term order $<$. It has been shown in [17] that the Hilbert functions of the local cohomology modules of $R/I$ are bounded by those of $R/\text{in}(I)$, where $\text{in}(I)$ denotes the initial ideal of $I$ with respect to $<$. In this note we study the question when the local cohomology modules of $R/I$ and $R/\text{in}(I)$ have the same Hilbert function. A complete answer to this question can be given for the generic initial ideal $\text{Gin}(I)$ of $I$, where $\text{Gin}(I)$ is taken with respect to the reverse lexicographical order and where we assume that $\text{char}(K) = 0$. In this case our main result (Theorem 3.1) says that the local cohomology modules of $R/I$ and $R/\text{Gin}(I)$ have the same Hilbert functions if and only if $R/I$ is sequentially Cohen-Macaulay.

In Section 1 we give the definition of sequentially CM-modules which is due to Stanley [18], and in Theorem 1.4 we present Peskine’s characterization of sequentially CM-modules in terms of Ext-groups. This characterization is used to derive a few basic properties of sequentially CM-modules which are needed for the proof of the main result.

In the following Section 2 we recall some well-known facts about generic initial modules, and also prove that $R/\text{Gin}(I)$ is sequentially CM, see Theorem 2.2. Section 3 is devoted to the proof of the main theorem, and in the final Section 4 we state and prove a squarefree version (Theorem 4.1) of the main theorem. Its proof is completely different from that of the main theorem in the graded case. It is based upon a result on componentwise linear ideals shown in [2] and the fact (see [11]) that the Alexander dual of a squarefree componentwise linear ideal defines a sequentially CM simplicial complex.

1. Sequentially Cohen-Macaulay modules

We introduce sequentially Cohen-Macaulay modules and derive some of their basic properties. Throughout this section we assume that $R$ is a standard graded Cohen-Macaulay $K$-algebra of dimension $n$ with canonical module $\omega_R$.

The following definition is due to Stanley [18, Section II, 3.9].

**Definition 1.1.** Let $M$ be a finitely generated graded $R$-module. The module $M$ is sequentially Cohen-Macaulay if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$$
of $M$ by graded submodules of $M$ such that each quotient $M_i/M_{i-1}$ is CM, and \( \dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1}) \).

The following observation follows immediately from the definition:

**Lemma 1.2.** (a) Suppose that $M$ is sequentially CM with filtration $0 = M_0 \subset M_1 \subset \ldots M_r = M$. Then for any $i = 0, \ldots, r$, the module $M/M_i$ is sequentially CM with filtration $0 = M_i/M_i \subset M_{i+1}/M_i \subset \ldots \subset M_r/M_i$.

(b) Suppose that $M_1 \subset M$ and $M_1$ is CM, and $M/M_1$ is sequentially CM with \( \dim M_1 < \dim M/M_1 \). Then $M$ is sequentially CM.

In order to simplify notation we will write $E^i(M)$ for $\text{Ext}^i_R(M, \omega_R)$.

**Proposition 1.3.** Suppose that $M$ is sequentially CM with a filtration as in Lemma 1.2, and assume that $d_i = \dim M_i/M_{i-1}$. Then

(a) $E^{n-d_i}(M) \cong E^{n-d_i}(M/M_{i-1})$, and is CM of dimension $d_i$ for $i = 1, \ldots, r$, and $E^0(M) = 0$ if $j \notin \{n - d_1, \ldots, n - d_r\}$.

(b) $E^{n-d_i}(E^{n-d_i}(M)) \cong M_i/M_{i-1}$ for $i = 1, \ldots, r$.

**Proof.** (a) We proceed by induction on $r$. From the short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ we obtain the long exact sequence

\[
\cdots \to E^j(M/M_1) \to E^j(M) \to E^j(M_1) \to E^{j+1}(M/M_1) \to \cdots
\]

From [3, Theorem 3.3.10] it follows that $E^j(M_1) = 0$ if $j \neq n - d_1$, and that $E^{n-d_1}(M_1)$ is CM of dimension $d_1$. Thus we get an exact sequence

\[
(1) \quad 0 \to E^{n-d_1}(M/M_1) \to E^{n-d_1}(M) \to E^{n-d_1}(M_1)
\]

\[
\to E^{n-d_1+1}(M/M_1) \to E^{n-d_1+1}(M) \to 0,
\]

and isomorphisms $E^j(M/M_1) \cong E^j(M)$ for all $j \neq n - d_1, n - d_1 + 1$.

By Lemma 1.2, the module $M/M_1$ is sequentially CM and has a CM filtration of length $r - 1$. Hence by induction hypothesis we have $E^{n-j}(M/M_1) = 0$ for $j \notin \{d_2, \ldots, d_r\}$. This implies that $E^{n-d_1}(M/M_1) = E^{n-d_1+1}(M/M_1) = 0$, and hence by (1) we have $E^{n-d_1}(M) \cong E^{n-d_1}(M_1)$, and $E^{n-d_1+1}(M) = 0$. Summing up we conclude that $E^{n-d_1}(M) \cong E^{n-d_1}(M_1)$ and $E^j(M) \cong E^j(M/M_1)$ for $j \neq n - d_1$. Thus the assertion follows from the induction hypothesis and the fact that $E^{n-d_1}(M_1)$ is CM of dimension $d_1$.

(b) follows from (a) and [3, Theorem 3.3.10] since for any CM-module $N$ of dimension $d$ one has $N \cong E^{d-E^d}(E^{-d}(N))$. \(\square\)

It is quite surprising that 1.3 has a strong converse. The following theorem is due to Peskine. Since there is no published proof available we present here a proof for the convenience of the reader.

**Theorem 1.4.** The following two conditions are equivalent:

(a) $M$ is sequentially CM;

(b) for all $0 \leq i \leq \dim M$, the modules $E^{n-i}(M)$ are either 0 or CM of dimension $i$. 

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The implication (a) ⇒ (b) follows from [3]. For the other direction we first need to show

**Lemma 1.5.** Let \( t = \text{depth} M \), and suppose that \( E^{n-t}(M) \) is CM of dimension \( t \). Then there exists a natural monomorphism \( \alpha : E^{n-t}(E^{n-t}(M)) \to M \), and the induced map \( E^{n-t}(\alpha) : E^{n-t}(M) \to E^{n-t}(E^{n-t}(E^{n-t}(M))) = E^{n-t}(M) \) is an isomorphism.

**Proof.** We write \( R = S/I \), where \( S \) is a polynomial ring. Let \( \mathfrak{m} \) be the graded maximal ideal of \( R \), and \( \mathfrak{n} \) the graded maximal ideal of \( S \). By the Local Duality Theorem (see [3, Theorem 3.6.10]) we have

\[
\text{Ext}_R^i(M, \omega_R) \cong \text{Hom}_R(H_{\mathfrak{m}}^{n-i}(M), E_R(K)),
\]

and

\[
\text{Ext}_S^i(M, \omega_S) \cong \text{Hom}_S(H_{\mathfrak{n}}^{m-i}(M), E_S(K)),
\]

where \( m = \dim S \). Since \( H_{\mathfrak{n}}^i(M) \cong H_{\mathfrak{m}}^i(M) \), and since \( \text{Hom}_S(R, E_S(K)) \cong E_R(K) \), we see that

\[
\text{Hom}_S(H_{\mathfrak{n}}^i(M), E_S(K)) \cong \text{Hom}_S(H_{\mathfrak{m}}^i(M), E_S(K)) \\
\cong \text{Hom}_R(H_{\mathfrak{m}}^i(M), \text{Hom}_S(R, E_S(K))) \\
\cong \text{Hom}_R(H_{\mathfrak{m}}^i(M), E_R(K)).
\]

Therefore (2) and (3) imply that

\[
\text{Ext}_R^{n-t}(M, \omega_R) \cong \text{Ext}_S^{n-t}(M, \omega_S),
\]

and we hence may as well assume that \( R \) is a polynomial ring.

Let

\[
F_0 : 0 \to F_{n-t} \to \cdots \to F_1 \to F_0 \to 0
\]

be the minimal graded free resolution of \( M \). Note that \( \omega_R = R(-n) \) since \( R \) is a polynomial ring. Then the \( \omega_R \)-dual \( F^* \) of \( F_0 \), the free complex

\[
0 \to F^*_{n-t} \to \cdots \to F^*_1 \to F^*_0 \to 0
\]

with \( F^*_i = \text{Hom}_R(F_{n-i}, \omega_R) \), and \( H_0(F^*) = E^{n-t}(M) \).

Let \( G_* \) be the minimal graded free resolution of \( E^{n-t}(M) \). Then there exists a comparison map \( \varphi_* : F^* \to G_* \), which extends the identity on \( H_0(F^*) = E^{n-t}(M) = H_0(G_0) \).

Since by assumption \( E^{n-t}(M) \) is CM of dimension \( t \), the complex \( G_* \) has the same length as \( F^*_* \), namely \( n - t \). Thus the \( \omega_R \)-dual \( \varphi_*^* : G^*_i \to F_0 \) of \( \varphi_* \) induces a natural homomorphism \( \alpha = H_0(\varphi^*_*) : E^{n-t}(E^{n-t}(M)) = H_0(G^*_i) \to H_0(F_0) = M \).

Here \( G^*_i = \text{Hom}_R(G_{n-i}, \omega_R) \) and \( \varphi^*_i = \text{Hom}_R(\varphi_{n-i}, \omega_R) \) for all \( i \).

Since \( E^{n-t}(M) \) is CM by assumption, the complex \( G^*_* \) is exact, and hence a free resolution of \( E^{n-t}(E^{n-t}(M)) \), and so the induced map \( E^{n-t}(E^{n-t}(E^{n-t}(M))) \to E^{n-t}(M) \) is given by \( H_0(\varphi^{**}) = H_0(\varphi) = \text{id} \).

It remains to be shown that \( \alpha \) is a monomorphism. Let \( C_* \) be the mapping cone of \( \varphi^*_* : G^*_* \to F^*_0 \). Since \( F^*_* \) and \( G^*_* \) are acyclic, it follows that \( H_1(C_0) \cong \text{Ker}(\alpha) \) and \( H_i(C_0) = 0 \) for \( i > 1 \). Notice that \( \varphi^*_{n-t} \) is an isomorphism, since this is the case for
Let $\varphi_0$. Hence the chain map $C_{n-t+1} \rightarrow C_{n-t}$ is split injective, and so by cancellation we get a new complex of free $R$-modules

$$\tilde{C}_i : 0 \rightarrow D_{n-t} \rightarrow C_{n-t-1} \rightarrow \cdots \rightarrow C_0 \rightarrow 0,$$

where $D_{n-t} = \text{Coker}(C_{n-t+1} \rightarrow C_{n-t})$. Again we have $H_1(\tilde{C}_i) \cong \text{Ker}(\alpha)$ and $H_i(\tilde{C}_i) = 0$ for $i > 1$.

Now suppose that $\text{Ker}(\alpha) \neq 0$, and let $P$ be a minimal prime ideal of the support of $\text{Ker}(\alpha)$. Since $\text{Ker}(\alpha) \subset E^{n-t}(E^{n-t}(M))$, and since $E^{n-t}(E^{n-t}(M))$ is a CM-module of dimension $t$, it follows that $P$ is a minimal prime ideal of $E^{n-t}(E^{n-t}(M))$ with height $P = n - t$. Therefore $L_* = \tilde{C}_* \otimes R_P$, is a complex of length $n - t$ with depth($L_i$) = $n - t$ for all $i$, depth($H_1(L_*)) = 0$ and $H_i(L_*)) = 0$ for $i > 0$. By the Peskine-Szpiro lemme d’acyclicité [10] this implies that $\tilde{C}$ is acyclic, a contradiction.

Proof of 1.4. We proceed by induction on $n - t$. Let $t = \text{depth }M$, and let $M_1$ be the image of $E^{n-t}(E^{n-t}(M)) \rightarrow M$. By [1.3], the module $M_1$ is a CM-module of dimension $t$.

Consider the short exact sequence $0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$. As in the proof of [1.3] we get an exact sequence

$$0 \rightarrow E^{n-t}(M/M_1) \rightarrow E^{n-t}(M) \rightarrow E^{n-t}(M_1) \rightarrow E^{n-t+1}(M/M_1) \rightarrow E^{n-t+1}(M) \rightarrow 0,$$

and isomorphisms $E^j(M/M_1) \cong E^j(M)$ for all $j \neq n - t, n - t + 1$.

Since $E^{n-t}(M) \rightarrow E^{n-t}(M_1)$ is an isomorphism (see [1.3]), we deduce from the above exact sequence that $E^{n-t}(M/M_1) = 0$, and that $E^{n-t+1}(M/M_1) \cong E^{n-t+1}(M)$. Thus we have $E^j(M/M_1) \cong E^j(M)$ for $j < n - t$, and $E^j(M/M_1) = 0$ for $j \geq n - t$. Hence, by induction hypothesis, $M/M_1$ is sequentially CM, and so is $M$ by [1.2].

An immediate application of 1.4 is

Corollary 1.6. Let $M$ be a finite direct sum of sequentially CM-modules. Then $M$ is sequentially CM.

As a consequence of [1.3] and [1.3] we get

Corollary 1.7. A filtration of a sequentially CM module satisfying the conditions of [1.3] is uniquely determined.

Proof. Let $t = \text{depth }M$. The first module $M_1$ in the filtration must be the image of $E^{n-t}(E^{n-t}(M)) \rightarrow M$. Then one makes use of an induction argument to $M/M_1$ to obtain the desired result.

Notice that $M_1 = H^0_m(M)$ if depth $M = 0$. Thus, [1.3] together with [1.2] imply

Corollary 1.8. An $R$-module $M$ is sequentially CM if and only if $M/H^0_m(M)$ is sequentially CM.

In what follows we denote by $E^*(M) = \bigoplus E^j(M)$. Then we get
Corollary 1.9. Suppose that $x \in R$ is a homogeneous $M$- and $E^*(M)$-regular element. Then $M$ is sequentially CM if and only if $M/xM$ is sequentially CM.

Proof. Since $x$ is $E^*(M)$-regular, the long exact Ext-sequence derived from

$$0 \longrightarrow M(-1) \xrightarrow{x} M \longrightarrow M/xM \rightarrow 0$$

splits into short exact sequences

$$0 \longrightarrow E^{n-i}(M) \xrightarrow{x} E^{n-i}(M)(1) \longrightarrow E^{n-i+1}(M/xM) \rightarrow 0.$$

It follows that $E^{n-i}(M)$ is CM of dimension $i$ if and only if $E^{n-i+1}(M/xM)$ is CM of dimension $i - 1$. Thus [4.4] implies the assertion.

In conclusion we would like to remark that the same theory is valid in the category of finitely generated $R$-modules, where $R$ is a local CM ring and a factor ring of a regular local ring.

2. Generic initial modules

In this section we recall a few facts on generic initial modules, which are mostly due to Bayer and Stillman, and can be found in [7].

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$ of characteristic 0, and let $M$ be a graded module with graded free presentation $F/U$. Throughout this section let $<$ be a term order that refines the partial order by degree and that satisfies $x_1 > x_2 > \cdots > x_n$. We fix a graded basis $e_1, \ldots, e_m$ of $F$, and extend the order $<$ to $F$ as follows: Let $ue_i$ and $ve_j$ be monomials (i.e. $u$ and $v$ are monomials in $R$). We set $ue_i > ve_j$ if either $\deg(ue_i) > \deg(ve_j)$, or the degrees are the same and $i < j$, or $i = j$ and $u > v$.

We set $U^\text{sat} = \bigcup_r U : m^r = \{ f \in F : fm^r \in U \text{ for some } r \}$.

From now on let $<$ denote the reverse lexicographic order. In the next proposition we collect all the results which will be needed later.

Proposition 2.1. For generic choice of coordinates one has:

(a) $\dim F/\text{Gin}(U) = \dim F/U$ and $\text{depth } F/\text{Gin}(U) = \text{depth } F/U$;
(b) $x_n$ is $F/U$ regular if and only if $x_n$ is $F/\text{Gin}(U)$ regular;
(c) $\text{Gin}(U)^\text{sat} = \text{Gin}(U^\text{sat})$.

Proof. After a generic choice of coordinates we may assume that $\text{Gin}(U) = \text{in}(U)$. The first statement in (a) is true for any term order, while the second statement about the depth and assertion (b) follow from [6, Theorem 15.13] because we may assume that the sequence $x_n, x_{n-1}, \ldots, x_{n-t+1}$ is $M$-regular if depth $M = t$.

By the module version of [6, Proposition 15.24], and by [7, Proposition 15.12] one has that

$$\text{Gin}(U)^\text{sat} = \bigcup_r (\text{Gin}(U) : x_n^r) = \bigcup_r \text{Gin}(U : x_n^r).$$
On the other hand for a generic choice of coordinates we have $U^{sat} = \bigcup_i (U : x_i^n)$.
Therefore $\bigcup_r \operatorname{Gin}(U : x_i^n) = \operatorname{Gin}(\bigcup_r (U : x_i^n)) = \operatorname{Gin}(U^{sat})$, which yields the last assertion.

The following result will be crucial for the proof of the main theorem of this paper.

**Theorem 2.2.** The module $F/\operatorname{Gin}(U)$ is sequentially CM.

*Proof.* Observe that, since we assume $\operatorname{char}(K) = 0$, we have $\operatorname{Gin}(U) = \bigoplus_j I_j e_j$ where for each $j$, $I_j$ is a strongly stable ideal, cf. [7, Theorem 15.23]. Hence $F/\operatorname{Gin}(U) \cong \bigoplus_j R/I_j$, so that, by [7, Proposition 15.23], one only has to prove that $R/I$ is sequentially CM for any strongly stable ideal $I \subset R$.

Recall that a monomial ideal is strongly stable if for all $u \in G(I)$ and all $i$ such that $x_i$ divides $u$ one has $x_j(u/x_i) \in I$ for all $j < i$. Here $G(I)$ denotes the unique minimal set of monomial generators of $I$.

For a monomial $u$ we let $m(u) = \max\{i : x_i \text{ divides } u\}$, and $s = \max\{m(u) : u \in G(I)\}$. Let $R' = K[x_1, \ldots, x_s]$, and let $J \subset R'$ be the unique monomial ideal such that $I = JR$. It is clear that $J$ is a strongly stable ideal in $R'$. Thus it follows that $J^{sat} = \bigcup_r (J : x_i^n)$. Note that $J^{sat}$ contains $J$ properly and is strongly stable. Let $I_1 = \bigcup_r (I : x_i^n)$. Then $I_1 = J^{sat}R$, and since the extension $R' \to R$ is flat, we have $I_1/I \cong (J^{sat}/J) \otimes_{R'} R$. Now $J^{sat}/J$ is a non-trivial 0-dimensional CM module over $R'$, and therefore $M_1 = I_1/I \subset R/I$ is an $(n-s)$-dimensional CM-module over $R$. Next we observe that $(R/I)/M_1 = R/I_1$ and that $I_1$ is strongly stable. Since $\dim R/I_1 \geq n - \max\{m(u) : u \in G(I_1)\} > n - s$, the assertion of the theorem follows from [17].

3. THE MAIN THEOREM

As in the previous section we let $K$ be a field of characteristic 0, $R = K[x_1, \ldots, x_n]$ be the polynomial ring over $K$ and $M$ be a finitely generated graded $R$-module with graded free presentation $M = F/U$. We want to compare the Hilbert functions of the local cohomology modules of $F/U$ and $F/\operatorname{Gin}(U)$, where $\operatorname{Gin}(U)$ is taken with respect to the reverse lexicographical order. In general one has (see [L7]) a coefficientwise inequality $\operatorname{Hilb}(H^*_m(F/U)) \leq \operatorname{Hilb}(H^*_m(F/\operatorname{Gin}(U)))$. The main purpose of this section is to prove

**Theorem 3.1.** The following conditions are equivalent:

(a) $F/U$ is sequentially CM;
(b) for all $i \geq 0$ one has $\operatorname{Hilb}(H^*_m(F/U)) = \operatorname{Hilb}(H^*_m(F/\operatorname{Gin}(U)))$.

*Proof.* (a) $\Rightarrow$ (b): Set $M = F/U$ and $N = F/\operatorname{Gin}(U)$. We proceed by induction on $\dim M$. Suppose $\dim M = 0$, then $\dim N = 0$ and $\operatorname{Hilb}(M) = \operatorname{Hilb}(N)$. Since $H^*_m(M) = M$, $H^*_m(N) = N$ and $H^*_m(M) = H^*_m(N) = 0$ for $i > 0$, the assertion follows in this case.

Now suppose that $\dim M > 0$. Assume first that depth $M = 0$. We have $M/H^*_m(M) \cong F/U^{sat}$ and, by [L7], $N/H^*_m(N) = F/\operatorname{Gin}(U)^{sat} = F/\operatorname{Gin}(U^{sat})$. By [L8] we also know that $M/H^*_m(M)$ is sequentially CM. Thus, if the implication
(a) ⇒ (b) were known for modules of positive depth, it would follow that

\[ \text{Hilb}(H^i_m(M)) = \text{Hilb}(H^i_m(M/H^0_m(M))) = \text{Hilb}(H^i_m(N/H^0_m(N))) = \text{Hilb}(H^i_m(N)) \]

for all \( i > 0 \). Notice that \( H^0_m(M) = U^{sat}/U \) and \( H^0_m(N) = \text{Gin}(U^{sat})/\text{Gin}(U) \). However, since \( M = F/U \) and \( N = F/\text{Gin}(U) \), and since \( F/U^{sat} \) and \( F/\text{Gin}(U^{sat}) \) have the same Hilbert function, we conclude that also \( H^0_m(M) \) and \( H^0_m(N) \) have the same Hilbert function.

These considerations show that we may assume that depth \( M > 0 \). Accordingly, depth \( N > 0 \) by \([2.1]\), and \( N \) is sequentially CM by \([2.2]\). Since \( M \) and \( N \) are sequentially CM we have depth \( E^*(M) > 0 \) and depth \( E^*(N) > 0 \). We may assume that the coordinates are chosen generically so that \( \text{Gin}(U) = \text{in}(U) \) and that \( x_n \) is regular on \( E^*(M) \) and regular on \( E^*(N) \). According to \([1.3]\), \( M/x_nM = F/(U + x_nF) \) is sequentially CM. Therefore our induction hypothesis, together with \([7\), Proposition 15.12\], implies that the Hilbert functions of the local cohomology modules of \( M/x_nM \) and of \( F/\text{Gin}(U + x_nF) = F/(\text{Gin}(U) + x_nF) = N/x_nN \) are the same.

We have short exact sequences of graded \( R \)-modules

\[ 0 \rightarrow H^{i-1}_m(M/x_nM) \rightarrow H^i_m(M)(-1) \xrightarrow{x_n} H^i_m(M) \rightarrow 0, \]

and

\[ 0 \rightarrow H^{i-1}_m(N/x_nN) \rightarrow H^i_m(N)(-1) \xrightarrow{x_n} H^i_m(N) \rightarrow 0, \]

because \( x_n \) is regular on \( E^*(M) \) and \( E^*(N) \). Therefore applying the induction hypothesis to \( M/x_nM \) we get

\[ \text{Hilb}(H^i_m(M))(t-1) = \text{Hilb}(H^{i-1}_m(M/x_nM)) = \text{Hilb}(H^{i-1}_m(N/x_nN)) = \text{Hilb}(H^i_m(N))(t-1), \]

from which we deduce that \( \text{Hilb}(H^i_m(M)) = \text{Hilb}(H^i_m(N)) \).

(b) ⇒ (a): We proceed again by induction on \( \dim M \). With the same arguments as in the proof of the first implication, we may assume that depth \( M > 0 \). Therefore depth \( N > 0 \), too, and since we are working with generic coordinates and \( N \) is sequentially CM by \([2.2]\), \( x_n \) is \( E^*(N) \)-regular, \( M \)- and \( N \)-regular. We shall show that \( x_n \) is also \( E^*(M) \)-regular. Since \( x_n \) is \( E^*(N) \)-regular, the long exact cohomology sequence derived from \( 0 \rightarrow N(-1) \xrightarrow{x_n} N \rightarrow N/x_nN \rightarrow 0 \) splits into short exact sequences

\[ 0 \rightarrow H^{i-1}_m(N/x_nN) \rightarrow H^i_m(N)(-1) \xrightarrow{x_n} H^i_m(N) \rightarrow 0, \]

We show by induction on \( i \) that the corresponding sequences for \( M \) are also exact. For \( i = 0 \) the assertion is trivial, since \( H^0_m(M) = 0 \). Now let \( i > 0 \), and assume that the assertion is true for all \( j < i \). Then \( H^{i-1}_m(M)(-1) \xrightarrow{x_n} H^{i-1}_m(M) \) is surjective, and we obtain the exact sequence

\[ 0 \rightarrow H^{i-1}_m(M/x_nM) \rightarrow H^i_m(M)(-1) \xrightarrow{x_n} H^i_m(M). \]

Suppose multiplication with \( x_n \) is not surjective, then there exists a degree \( a \) such that \( H^i_m(M)_{a-1} \xrightarrow{x_n} H^i_m(M)_{a} \) is not surjective. Using \([4\), and the hypothesis that
the local cohomology modules of $M$ and $N$ have the same Hilbert function, one has
\[
\dim_K H^{i-1}_m(M/x_n M)_a > \dim_K H^i_m(M)_{a-1} - \dim H^i_m(M)_a = \dim K H^i_m(N)_{a-1} - \dim H^i_m(N)_a = \dim_K H^{i-1}_m(N/x_n N)_a.
\]
This is a contradiction, since $M/x_n M = F/(U + x_n F)$ and $N/x_n N = F/\text{Gin}(U + x_n F)$, and consequently $\dim_K H^{i-1}_m(M/x_n M) > \dim_K H^{i-1}_m(N/x_n N)$.

Now it follows that $x_n$ is $E^*(M)$-regular, and also that the cohomology modules of $M/x_n M$ and $N/x_n N$ have the same Hilbert functions, whence our induction hypothesis implies that $M/x_n M$ is sequentially CM. Since $x_n$ is $E^*(M)$-regular, we finally deduce that $M$ is sequentially CM.

4. The squarefree case

In this section we will state and prove the squarefree analogue of Theorem 3.1. Let $\Delta$ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$, and let $I_\Delta \subset R$ be the Stanley-Reisner ideal of $\Delta$, where $R = K[x_1, \ldots, x_n]$ and $K$ is field of characteristic 0. The $K$-algebra $K[\Delta] = R/I_\Delta$ is the Stanley-Reisner ring of $\Delta$.

We recall the concept of symmetric algebraic shifting which was introduced by Kalai in [15]: Let $u \in R$ be a monomial, $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$. We define
\[
u^\sigma = x_{i_1}x_{i_2 + 1}\cdots x_{i_d + d - 1}.
\]
Note that $u^\sigma$ is a squarefree monomial (in a possibly bigger polynomial ring).

As usual the unique minimal monomial set of generators of a monomial ideal $I$ is denoted by $G(I)$.

The symmetric algebraic shifted complex of $\Delta$ is the simplicial complex $\Delta^\sigma$ whose Stanley-Reisner ideal $I_{\Delta^\sigma}$ is generated by the squarefree monomials $u^\sigma$ with $u \in G(I_\Delta)$.

We quote the following properties of $I_{\Delta^\sigma}$ from [1] and [2]:

(i) $I_{\Delta^\sigma}$ is a strongly stable ideal in $R$;
(ii) one has the following inequality of graded Betti numbers:
\[
\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^\sigma});
\]
(iii) $I_\Delta$ and $I_{\Delta^\sigma}$ have the same graded Betti numbers if and only if $I_\Delta$ is componentwise linear.

Recall that an ideal $I \subset R$ is called componentwise linear if in each degree $i$, the ideal generated by the $i$-th graded component $I_i$ of $I$ has a linear resolution.

Let $\Delta^*$ denote the Alexander dual of $\Delta$, i.e., the simplicial complex
\[
\Delta^* = \{F \subset [n]: [n] \setminus F \not\in \Delta\}.
\]
It has been noted in [11, Theorem 9] that
\[
K[\Delta] \text{ is sequentially CM} \iff I_{\Delta^*} \text{ is componentwise linear}.
\]

**Theorem 4.1.** Let $\Delta$ be a simplicial complex. Then
(a) \( \text{Hilb}(H_m^i(K[\Delta])) \leq \text{Hilb}(H_m^i(K[\Delta^s])) \) for all \( i \).

(b) The local cohomology module of \( K[\Delta] \) and \( K[\Delta^s] \) have the same Hilbert function if and only if \( K[\Delta] \) is sequentially CM.

Proof. Part (a) is proved in [17]. For the proof of (b) we shall need the following result which also can be found in [17]: for all \( i \geq 0 \) and \( j \geq 0 \) one has

(5) \[ \dim_K H^i_m(K[\Delta])_{-j} = \sum_{h=0}^{n} \binom{n}{h} \binom{h+j-1}{j} \beta_{i-h+1,n-h}(K[\Delta^s]). \]

(Observe that \( H^i_m(K[\Delta])_{j} = 0 \) for \( j > 0 \) and all \( i \), as shown by Hochster, see [13] and [6, Theorem 5.3.8]).

Now suppose that \( K[\Delta] \) is sequentially CM. Then \( I_{\Delta^s} \) is componentwise linear, and hence \( \beta_{ij}(K[\Delta^s]) = \beta_{ij}(K[(\Delta^s)^s]) \) by Property (iii) of symmetric algebraic shifting. Since \( (\Delta^s)^s = (\Delta^s)^* \), Formula (5) shows that \( \dim_K H^i_m(K[\Delta])_{-j} = \dim_K H^i_m(K[\Delta^s])_{-j} \) for all \( i \) and \( j \), as desired.

For the vice versa, let \( H \) be the \((n+1) \times (n+1)\)-matrix with entries \( h_{ij} = \dim_K H^i_m(K[\Delta])_{-j}, \ i,j = 0, \ldots, n \), \( B \) the \((n+1) \times (n+1)\)-matrix with entries \( b_{hi} = \binom{n}{h} \beta_{i-h+1,n-h}(K[\Delta^s]) \) and \( A \) the \((n+1) \times (n+1)\)-matrix with entries \( a_{jh} = \binom{h+j-1}{j} \). Then (3) says that \( H' = AB \). Since \( A \) is invertible, we see that the numbers \( b_{hi} = \binom{n}{h} \beta_{i-h+1,n-h} \) are determined by the Hilbert functions of the local cohomology modules of \( K[\Delta] \). Thus, if \( \dim_K H^i_m(K[\Delta])_{j} = \dim_K H^i_m(K[\Delta^s])_{j} \) for all \( i \) and \( j \), then the numbers \( b_{hi} \) for \( \Delta^s \) and \( (\Delta^s)^s \) coincide, which in turn implies that their graded Betti numbers are the same (because \( \beta_{ij} = b_{n-j,i+n-j-1} \binom{n}{n-j} \)). By Property (iii) of symmetric algebraic shifting this implies that \( I^*_\Delta \) is componentwise linear, and hence \( K[\Delta] \) is sequentially CM. \( \square \)
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