Strong commutativity preserving maps
on Lie ideals of semiprime rings

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Abstract

Let \( R \) be a 2-torsion free semiprime ring and \( U \) a nonzero square closed Lie ideal of \( R \). In this paper it is shown that if \( f \) is either an endomorphism or an antihomomorphism of \( R \) such that \( f(U) = U \), then \( f \) is strong commutativity preserving on \( U \) if and only if \( f \) is centralizing on \( U \).

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1 Introduction

Throughout the present paper \( R \) will denote a unitary associative ring. As usual, for \( x, y \) in \( R \), we write \([x, y] = xy - yx\), and we will use the identities

\[
[x, [x, y]] = [x, y] + [x, z]y, \quad [x, yz] = [x, y]z + y[x, z].
\]

For any \( a \in R \), \( d_a \) will denote the inner-derivation defined by \( d_a(x) = [a, x] \) for all \( x \in R \).

A ring \( R \) is said to be semiprime if \( aRa = 0 \) implies that \( a = 0 \). An ideal \( P \) of \( R \) is prime if \( aRb \subseteq P \) implies that \( a \in P \) or \( b \in P \). Recall that a ring \( R \) is semiprime if and only if its zero ideal is the intersection of its prime ideals. Moreover, if the zero ideal of \( R \) is prime, then \( R \) is said to be a prime ring. An additive subgroup \( U \) of a ring \( R \) is a Lie ideal if \([U, R] \subseteq U \). Moreover, if \( u^2 \in U \) for all \( u \in U \), then \( U \) is called a square closed Lie ideal. Since \((u + v)^2 \in U \) and \([u, v] \in U \), we see that \( 2uv \in U \) for all \( u, v \in U \). For a subset \( S \) of \( R \), denote by \( \text{ann}_R(S) \) the two-sided annihilator of \( S \) -i.e. \( \{x \in R/|xS = xS = \{0\}\} \).

For every ideal \( J \) of a semiprime ring \( R \), it is known that \( \text{ann}_R(J) \) is invariant under all derivations and \( J \cap \text{ann}_R(J) = 0 \).
A map $f : R \rightarrow R$ is centralizing on $S$ if $[f(x), x] \in Z(R)$ for all $x \in S$; in particular if $[f(x), x] = 0$ for all $x \in S$, then $f$ is called commuting on $S$. A map $f : R \rightarrow R$ is called commutativity preserving on $S$ if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$, for all $x, y \in S$. In particular, if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$, then $f$ is called strong commutativity preserving on $S$.

Recently, M. S. Samman [4] proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing on the entire ring. Moreover, he proved that if $R$ is a 2-torsion free semiprime ring, then a centralizing antihomomorphism of $R$ onto itself must be strong commutativity preserving. The purpose of this paper is to extend the results of [4] to square closed Lie ideals.

2 Preliminaries and results

In order to prove our main theorems, we shall need the following results.

**Lemma 1** Let $R$ be a 2-torsion free semiprime ring and $U$ a nonzero Lie ideal of $R$. If $[U, U] = 0$, then $U \subseteq Z(R)$.

**Proof.** Let $u \in U$; since $[u, rt] \in U$ for all $r, t \in R$, then $[u, [u, rt]] = 0$. Hence $u[u, rt] = [u, rt]u$. Therefore

$$ur[u, t] + u[u, r]t = r[u, t]u + [u, r]tu.$$  

As $u[u, r] = [u, r]u$ and $[u, t]u = u[u, t]$, then

$$ur[u, t] + [u, r]ut = ru[u, t] + [u, r]tu.$$  

It follows that $2[u, r][u, t] = 0$ for all $u \in U$ and $r, t \in R$. Since $R$ is 2-torsion-free, thus

$$[u, r][u, t] = 0, \text{ for all } u \in U \text{ and } r, t \in R. \quad (1)$$  

Replace $t$ by $sr$ in (1) to get $[u, r]R[u, r] = 0$ for all $u \in U, r, t \in R$. The fact $R$ is semiprime implies that $U \subseteq Z(R)$.  

In all that follows $U$ will be a square closed Lie ideal of $R$ and $M$ will denote the ideal of $R$ generated by $[U, U]$, that is $M = R[U, U]R$.

**Lemma 2** Let $R$ be a 2-torsion free semiprime ring and $d$ a derivation of $R$. If $a$ in $R$ satisfies $ad(U) = 0$, then $ad(M) = 0$.  

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Proof. Let \( P \) be an arbitrary prime ideal of \( R \), and note that \( \overline{R} = \frac{R}{P} \) is prime. If \([U,U] \subseteq P\) or \( \operatorname{char}(\overline{R}) = 2 \), then \( 2ad(R)M \subseteq P \) and \( 2\operatorname{Mad}(R) \subseteq P \). Assume now that \([U,U] \not\subseteq P\) and \( \operatorname{char}(\overline{R}) \neq 2 \). The fact that \( R \) is 2-torsion-free and \( ad(U) = \{0\} \) implies that \( aUd(v) = \{0\} \) for all \( v \in U \) and thus \( \overline{aUd(U)} = 0 \). As \([U,U] \not\subseteq P\), then \( \overline{U} \not\subseteq \overline{Z(R)} \). Since \([\overline{U},U] \neq 0 \) from \([4, \text{Lemma 4}]\) either \( \overline{d(U)} = 0 \) or \( \overline{a} = 0 \), that is \( d(U) \subseteq P \) or \( a \in P \). If \( d(U) \subseteq P \), then \( d[r,u] \in P \) for all \( r \in R \) and \( u \in U \). Replace \( r \) by \( rv \), where \( v \in U \), to get \( d(R)[U,U] \subseteq P \). Thus \( d(R)R[U,U] \subseteq P \) which yields \( d(R) \subseteq P \) because \([U,U] \not\subseteq P\). In conclusion \( ad(R) \subseteq P \). Consequently, \( ad(R)M \subseteq P \) and \( \operatorname{Mad}(R) \subseteq P \). We now know that \( 2ad(R)M \subseteq P \) and \( 2\operatorname{Mad}(R) \subseteq P \) for all prime ideals \( P \) of \( R \), hence \( 2ad(R)M = 2\operatorname{Mad}(R) = \{0\} \). By 2-torsion-freeness we conclude that \( ad(R)M = \operatorname{Mad}(R) = \{0\} \). If we set \( J = \operatorname{ann}_R(\operatorname{ann}_R(M)) \), then obviously \( ad(R)J = 0 \). Since \( R \) is semiprime, then \( d(J) \subseteq J \) so that \( ad(J) \subseteq J \cap \operatorname{ann}_R(J) \). Once again using the semiprimeness of \( R \), we conclude that \( J \cap \operatorname{ann}_R(J) = 0 \) so that \( ad(J) = 0 \). Since \( M \subseteq J \), this leads us to \( ad(M) = 0 \).

Lemma 3 Let \( R \) be a 2-torsion free semiprime ring. If \( z \in U \) is such that \( z[U,U] = 0 \), then \( [z,U] = 0 \).

Proof. If \([U,U] = 0\), then \( U \subseteq Z(R) \) by Lemma 1 and therefore \([z,U] = 0\).

Now suppose that \([U,U] \neq 0\); from \( z[U,U] = 0 \) we get \( zd_u(u) = 0 \) for all \( u,v \in U \). Using Lemma 2, we find that \( zd_u(x) = 0 \) for all \( u \in U \), \( x \in M = R[U,U]R \). But \( zd_u(u) = 0 \) assures that \( zd_x(u) = 0 \) for all \( u \in U \), \( x \in M \) and once again using Lemma 2, we get \( zd_x(M) = 0 \), for all \( x \in M \). Hence \( zd_x(y) = 0 \) for all \( x,y \in M \) and thus \( z[x,y] = 0 \) for all \( x,y \in M \).

Replace \( y \) by \( yz \) to get \( zy[x,z] = 0 \), so that \( zM[x,z] = 0 \). In view of \( zM[x,z] = 0 \), we then obtain \( [x,z]M[x,z] = 0 \). Since an ideal of a semiprime ring is semiprime, \([x,z] = 0 \) for all \( x \in M \). As \( R[U,U] \subseteq M \), then \( [z,r[u,v]] = 0 \) for all \( r \in R \), \( u,v \in U \). Using \([u,v] \in M \), it then follows that \([z,r][u,v] = 0 \). Replace \( r \) by \( rs \) in the least equality, we find that \([z,r]s[u,v] = 0 \) so that \([z,r]R[u,v] = 0 \), for all \( u,v \in U \), \( r \in R \). In particular \([z,v]R[z,v] = 0 \), proving \([z,v] = 0 \) for all \( v \in U \) and thus \([z,U] = 0 \).

Now we are ready for our first theorem.

Theorem 1 Let \( R \) be a 2-torsion free semiprime ring and \( U \) a nonzero square closed Lie ideal of \( R \). Suppose that \( f \) is an endomorphism of \( R \) such that \( f(U) = U \). Then \( f \) is strong commutativity preserving on \( U \) if and only if \( f \) is centralizing on \( U \).
**Proof.** From \([x, 2xy] = [f(x), f(2xy)]\) for all \(x, y \in U\), it follows that \((x - f(x))[x, y] = 0\) for all \(x, y \in U\). Replacing \(y\) by \(2uy\) where \(u, y \in U\), we get

\[(x - f(x))U[x, y] = 0 \text{ for all } x, u \in U.\] \hspace{1cm} (2)

As \(2[U, U]R \subseteq U\) (because \(2[u, v]r = 2[u, vr] - 2v[u, r]\)), then (2) implies that

\[(x - f(x))[U, U]R[x, y] = 0 \text{ for all } x, y \in U.\] \hspace{1cm} (3)

Let \(P\) be an arbitrary prime ideal of \(R\). It follows from (3) that for each \(x \in U\), either \((x - f(x))[U, U] \subseteq P\) or \([x, U] \subseteq P\). The two sets of elements of \(U\) for which these conditions hold are additive subgroups of \(U\) whose union is \(U\), hence one must be equal to \(U\). Therefore \((x - f(x))[U, U] \subseteq P\) for all \(x \in U\) and all prime ideals \(P\)-i.e., \((x - f(x))[U, U] = \{0\}\) for all \(x \in U\). Since \(f(U) \subseteq U\), then \(u - f(u) \in U\) for all \(u \in U\) and Lemma 3 yields

\([u - f(u), v] = 0 \text{ for all } u, v \in U.\)

Consequently, \([f(u), u] = 0\) for all \(u \in U\) so that \(f\) is commuting on \(U\). Accordingly, \(f\) is centralizing on \(U\).

Conversely, suppose that \([f(x), x] \in Z(R)\) for all \(x \in U\). By linearization \([x, f(y)] + [y, f(x)] \in Z(R)\) for all \(x, y \in U\). Using \([x, f(x^2)] + [x^2, f(x)] \in Z(R)\) together with 2-torsion-freeness, we find that \((x + f(x))[x, f(x)] \in Z(R)\), for all \(x \in U\). Hence \([(x + f(x))[x, f(x)], x] = 0\) and therefore \([x, f(x)]^2 = 0\). Since \([x, f(x)] \in Z(R)\), this yields \([x, f(x)]R[x, f(x)] = 0\) and the semiprimeness of \(R\) forces

\([x, f(x)] = 0 \text{ for all } x \in U.\)

Thus \(f\) is commuting on \(U\) and therefore \([f(x), y] = [x, f(y)]\) for all \(x, y \in U\). As \(R\) is 2-torsion-free, then \([f(x), xy] = [x, f(xy)]\) and thereby \((f(x) - x)[f(x), y] = 0\) for all \(x, y \in U\). Replacing \(y\) by \(2uy\) where \(u \in U\), we get \((f(x) - x)u[f(x), y] = 0\), so that \((f(x) - x)U[x, f(y)] = 0\). Since \(f(U) = U\), then \((f(x) - x)U[x, y] = 0\) for all \(x, y \in U\). From \(2[U, U]R \subseteq U\), it then follows that

\[(f(x) - x)[U, U]R[x, y] = 0 \text{ for all } x, y \in U.\]

Reasoning as in the first part of the proof, we find that \([f(z) - z, u] = 0\) for all \(z, u \in U\), and therefore \([f(z), u] = [z, u]\) for all \(z, u \in U\). Consequently, for \(y, z \in U\), this leads us to \([f(z), f(y)] = [z, f(y)] = [z, y]\), proving that \(f\) is strong commutativity preserving on \(U\). \(\blacksquare\)

**Remark.** From the proof of Theorem 1, one can easily see that the condition
$f(U) \subseteq U$ is sufficient to prove that $f$ is strong commutativity preserving implies that $f$ is commuting on $U$ and therefore centralizing on $U$.

We easily derive the Proposition 2.1 of [4], for 2-torsion free semiprime rings, as a corollary to Theorem 1.

**Corollary 1** Let $f$ be an epimorphism of a 2-torsion free semiprime ring $R$. Then $f$ is strong commutativity preserving if and only if $f$ is centralizing.

In [3] it is proved that if $R$ is a 2-torsion free prime ring and $T$ an automorphism of $R$ which is centralizing on a Lie ideal $U$ of $R$ and nontrivial on $U$, then $U$ is contained in the center of $R$. Accordingly, in the special case when $U = R$, Theorem 2 gives a commutativity criterion as follows.

**Corollary 2** Let $f$ be a nontrivial automorphism of a 2-torsion free prime ring $R$. If $f$ is strong commutativity preserving, then $R$ is commutative.

To end this paper, the following theorem gives a condition under which an antihomomorphism becomes strong commutativity preserving.

**Theorem 2** Let $R$ be a 2-torsion free semiprime ring and $U$ a square closed Lie ideal of $R$. If $f$ is an antihomomorphism of $R$ such that $f(U) = U$, then $f$ is centralizing on $U$ if and only if $f$ is strong commutativity preserving on $U$.

**Proof.** Suppose $[U, U] \neq 0$ and then $M = R[U, U]R$ is a nonzero ideal of $R$. If $f$ is centralizing on $U$, then reasoning as in the proof of Theorem 1 we find that $f$ is commuting on $U$, so that $[f(x), y] = [x, f(y)]$ for all $x, y \in U$. Since $R$ is 2-torsion-free, using $[f(x), 2xy] = [x, f(2xy)]$ together with $f(U) = U$ we get

$$x[x, y] = [x, y]f(x) \text{ for all } x, y \in U. \quad (4)$$

Replace $y$ by $2uy$ in (4), where $u \in U$, and once again using 2-torsion-freeness, we get $[x, u][x, y + f(y)] = 0$. Write $2uv$ instead of $u$ in this equality, with $v \in U$, to find that $[x, u]v[x, y + f(y)] = 0$. Hence

$$[x, u]U[x, y + f(y)] = 0 \text{ for all } x, u, y \in U. \quad (5)$$

Since $f(U) \subseteq U$, replacing $u$ by $y + f(y)$ in (5), we conclude that

$$[x, y + f(y)]U[x, y + f(y)] = 0 \text{ for all } x, y \in U. \quad (6)$$

If we set $T(U) = \{x \in R/ [x, R] \subseteq U\}$, then $[T(U), R] \subseteq U \subseteq T(U)$ and from [[2], Lemma 1.4, p. 5] it follows that $T(U)$ is a subring of $R$. Moreover,
Replacing $y$ and thus $x$ and therefore

$$\quad [x, y]r, s = [x, y]rs − s[x, y]r \in T(U) \quad \text{for all } r, s \in R;$$

and therefore $s[x, y]r \in T(U)$ so that $R[T(U), T(U)]R \subseteq T(U)$. In particular $R[U, U]R \subseteq T(U)$, which proves that $[M, R] \subseteq U$, where $M = R[U, U]R$.

In view of (6), if we set $[x, y + f(y)] = a$ then $aUa = 0$. Let $u \in U$, $m \in M$ and $r \in R$; from $[mau, r] \in [M, R] \subseteq U$ it follows that

$$\quad 0 = a[mau, r]a = a[ma, r]ua + ama[u, r]a = a[ma, r]ua = amaruA,$$

so that $amaRama = 0$. Using $2am \in 2[U, U]R \subseteq U$, we get Lemma 1.4, et $amaRama = 0$, hence $aMa = 0$. Since $a \in M$, we obviously get $a = 0$, which implies that

$$\quad [f(x), y] = [y, x], \quad \text{for all } x, y \in U. \quad \text{Accordingly,}$$

$$\quad [f(x), f(y)] = [f(y), x] = [x, y] \quad \text{for all } x, y \in U,$$

proving that $f$ is strong commutativity preserving on $U$.

Conversely, if $f$ is strong commutativity preserving on $U$, then

$$\quad [f(x), f(y)] = [x, y], \quad \text{for all } x, y \in U. \quad \text{(7)}$$

Replace $y$ by $2xy$ in (7) we obtain

$$\quad x[x, y] = [x, y]f(x). \quad \text{(8)}$$

Write $2uy$ instead of $y$ in (8), where $u \in U$, to find that

$$\quad xu[x, y] + x[x, u]y = u[x, y]f(x) + [x, u]yf(x).$$

Since $x[x, u]y = [x, u]f(x)y$ and $[x, y]f(x) = x[x, y]$ by (8), then

$$\quad xu[x, y] + [x, u]f(x)y = ux[x, y] + [x, u]yf(x)$$

and therefore

$$\quad [x, u][x + f(x), y] = 0 \quad \text{for all } x, y, u \in U. \quad \text{(9)}$$

Replacing $y$ by $x$ in (9), we obtain

$$\quad [x, u][x, f(x)] = 0 \quad \text{for all } x, u \in U. \quad \text{(10)}$$

As $f(U) \subseteq U$, write $2f(x)u$ instead of $u$ in (10) to get $[x, f(x)]u[x, f(x)] = 0$ and thus

$$\quad [x, f(x)]U[x, f(x)] = 0.$$
If we set \( a = [x, f(x)] \), then \( aUa = 0 \) and \( a \in M = R[U,U]R \). Reasoning as in the first part of our proof, we conclude that \( a = 0 \) so that \( [x, f(x)] = 0 \). Accordingly, \( f \) is commuting on \( U \) and therefore \( f \) is centralizing on \( U \).

**Remark.** In the particular case when \( U = R \), the implication that \( f \) is strong commutativity preserving implying that \( f \) is centralizing is still valid without conditions on characteristic of \( R \).

In [4], Proposition 2.4 M. S. Samman proved that if \( R \) is a 2-torsion free semiprime ring, then a centralizing antihomomorphism of \( R \) onto itself must be strong commutativity preserving. Applying Theorem 2, we obtain a more general result as follows

**Corollary 3**  Let \( R \) be a 2-torsion free semiprime ring. If \( f \) is an antihomomorphism of \( R \) onto itself, then \( f \) is centralizing if and only if \( f \) is strong commutativity preserving.

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