PROPERTIES OF BASINS OF ATTRACTION FOR PLANAR DISCRETE COOPERATIVE MAPS

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Abstract. It is shown that locally asymptotically stable equilibria of planar cooperative or competitive maps have basin of attraction $B$ with relatively simple geometry: the boundary of each component of $B$ consists of the union of two unordered curves, and the components of $B$ are not comparable as sets. The boundary curves are Lipschitz if the map is of class $C^1$. Further, if a periodic point is in $\partial B$, then $\partial B$ is tangential to the line through the point with direction given by the eigenvector associated with the smaller characteristic value of the map at the point. Examples are given.

1. Introduction. Fixed points and periodic points of planar maps often have basins of attraction that have very complex boundary. This is the case even if the map is smooth. An example is the planar map $F(x, y) = (x^2 - y^2 - 1, 2xy)$, $(x, y) \in \mathbb{R}^2$ (on the complex plane $\mathbb{C}$, $f(z) = z^2 - 1$) which has two repelling fixed points and a single minimal period-two point, namely $\{(-1, 0), (0, 0)\}$. The basin of attraction of the minimal period-two point has fractal boundary [18, 19]. See [17, 19, 20, 21, 22] for further properties of the basins of attraction for general maps in the plane or in higher dimension. Next we describe briefly results in recent articles that concern sets attracted to a point or to infinity, where the boundary of such sets is shown to have specific properties.

A. Berger and A. Duh studied in [1] difference equations

$$x_{n+1} = g(x_{n-1}, x_n), \quad n = 1, 2, \ldots \quad (1)$$

where $g$ is strictly increasing in each argument and convex in the first quadrant of the plane. Thus a map corresponding to (1) is given by $T(x, y) = (y, g(x, y))$. It is shown in [1] that the phase space is partitioned into two basins of attraction, one of which is bounded and convex (but the empty set is a possibility). In the case of non-empty bounded basin, its boundary is shown to consist of points that converge under iteration to a period-two solution.

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G. Karakostas in [11] investigated\textsuperscript{1} the \( m \) dimensional system

\begin{equation}
x_i(n + 1) = x_i(n) + x_i(n) \sum_{\ell=1}^{m} a_{ij} x_j(n), \quad n \in \mathbb{N}_0, \quad i = 1, 2, \ldots, m.
\end{equation}

where the matrix \( A = (a_{ij}) \) is such that \( C := -A \) is a nonsingular M-matrix. The map associated to (2) has the form \( T(x) = x + \text{diag}(x) (Ax) \), where \( \text{diag}(x) \) is a diagonal matrix with diagonal \( x \) in \( \mathbb{R}^m \). Theorem 1 in [11] states that there exists a set \( \mathcal{U} \subset \mathbb{R}^m \) that is non-empty, closed, invariant under the map of (2), such that \( \mathcal{U} \) is attracted by the origin and the complement of \( \mathcal{U} \) consists of initial points of unbounded orbits. Furthermore, the origin is on the boundary of \( \mathcal{U} \). Remarkably, a characterization of points on the boundary of \( \mathcal{U} \) is also given (the characterization is technical; for details see [11]).

Planar maps \( T(x, y) = (f(x, y), g(x, y)) \), where \( f \) and \( g \) are continuous functions defined on some subset of \( \mathbb{R}^2 \) with non-empty interior, such that \( f \) and \( g \) satisfy certain monotonicity conditions are called cooperative (\( f(x, y) \) and \( g(x, y) \) are non-decreasing in all arguments) or competitive (\( f(x, y) \) is non-decreasing in \( x \) and non-increasing in \( y \), and \( g(x, y) \) is nonincreasing in \( x \) and nondecreasing in \( y \)). In several papers [2, 6, 14, 15] results form [12, 13] are used to find the basins of attraction of fixed points of several competitive and cooperative maps. It is shown that for those maps, the boundaries of the basins of attractions of different attractors are curves given by the global stable manifolds of saddle fixed points or saddle period-two points or non-hyperbolic points of stable type [13, 15] or non-hyperbolic period-two points. Further, results from [12, 13] ensured that these boundary curves are ordered with respect to either the northeast or southeast partial orders. A common characteristic of the models studied in [2, 6, 14, 15] is that the basins of attraction are connected sets. In many cases, the results in [12, 13] suffice to give a description of basins of attraction consisting of one component. However, connectedness of the basin of attraction, even for cooperative or competitive planar maps, is not necessarily satisfied, as it is shown in Example 1 in this paper. See [3] for a class of difference equations with basins that are not connected.

In the present work we show that the basin of attraction of a locally asymptotically stable fixed point or a singular point on the boundary of the region of competitive or cooperative map is in general the disjoint union of a finite or countably infinite number of connected sets, only one of which contains the fixed point. We give a precise description of the boundaries of these connected sets. We illustrate results in this paper with an example of the basin of attraction consisting of three connected components. The example can be easily modified to produce basins with many finite number of components, or even a countably infinity number of them. Thus we show that the basin of attraction of a locally asymptotically stable fixed point of a strongly competitive or cooperative map is the union of connected components ordered in either south-east or south-west ordering whose boundaries are Lipschitz curves. In comparison to the complexity of the basin of attraction of general planar maps [18, 19, 20], the boundaries of the basins of attraction of strongly competitive and strongly cooperative maps cannot be fractals, as will be shown below.

It is shown in this paper that, in stark contrast to the general case of planar maps, basins of attraction \( \mathcal{B} \) of fixed points and periodic points of cooperative

\textsuperscript{1} The article [11] has the word cooperative in its title, but it is not the same concept as the one used in the present paper.
or competitive maps have a simple geometry. In particular, when $B$ contains a neighborhood of the periodic orbit, it is then bounded by unordered curves (in the sense of north-east order in the cooperative case, which is to say that they are the graphs of decreasing functions). Moreover, at any fixed or periodic points on $\partial B$, the boundary of the set $B$ is tangential to a line with direction of an eigenvector associated with a characteristic value of the map at the point in question. If $B$ has more than one connected component, then any two components are non-comparable as sets, and if the map is of class $C^1$, then the curves bounding $B$ are Lipschitz.

Kolmogorov type models that appear in population dynamics may feature the origin as a fixed point with a substantial basin of attraction. Livadiotis et al in [16] introduce the notion of planar competition model with strong Allee effect, which is a Kolmogorov type model where the origin has a basin of attraction bounded by a curve. The example studied in [16] is cooperative. A related class of triangular monotone systems is studied in [4]. However, there is no discussion of properties of the boundary curve of the basin of the origin other than the observation that it is obtained from stable manifolds of certain fixed points.

As a motivating example consider the difference equation from [2]
\begin{equation}
x_{n+1} = x_n^3 + x_{n-1}^3, \quad x_0, x_1, x_2, \ldots, n = 0, 1, \ldots,
\end{equation}
which has associated map
\begin{equation}
F(x, y) = (y, x^3 + y^3), \quad (x, y) \in \mathbb{R}^2.
\end{equation}
The fixed points of the map are $(0, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, where the origin is locally asymptotically stable and other two fixed points are saddle points. There are no periodic points. By using results from [13], it is shown in [2] that the basin of attraction $B$ of $(0, 0)$ is unbounded, and its boundary is the union of the stable manifolds of the two nonzero fixed points, see Fig. 1(a). Notice that $F$ is cooperative and $F^2$ is strongly cooperative.

A variation of (3) is the difference equation
\begin{equation}
x_{n+1} = x_n^3 + x_{n-1}^9, \quad x_0, x_1, x_2, \ldots, n = 1, 2, \ldots
\end{equation}
whose associated map
\begin{equation}
G(x, y) = (y^3, x^3 + y^3), \quad (x, y) \in \mathbb{R}^2,
\end{equation}
has three fixed points: the point $(0, 0)$ which is locally asymptotically stable (LAS), and the points $(-0.617, -0.851)$ and $(0.617, 0.851)$ which are saddle points. In addition, there are two repelling minimal period-two points $(-1.349, 1.105)$, $(1.349, -1.105)$. It can be shown with results from [13] that the basin $B$ of the origin is bounded, and that $\partial B$ consists of the union of stable manifolds of the two nonzero fixed points, and that the period two points are endpoints to both manifolds. See Figure 1 (b). The map $G$ is cooperative and $G^2$ is strongly cooperative.

The previous examples suggest the question of whether the geometry of the basin of locally asymptotically stable fixed or periodic points of planar monotone (cooperative or competitive) maps is particularly simple and amenable to a “nice” characterization.

We note that the maps in (4) and (6) are (locally) invertible, and that in each of both cases the boundary of the basin $B$ of the origin contains two saddle points. This allows, by using the results from [13] for example, the characterization of $\partial B$ as the union of stable manifolds of the saddle points. However, local invertibility of
(a) The basin of attraction $B$ of the zero fixed point $o$ of the map $T(x, y) = (y, x^3 + y^3)$. Note that $B$ is unbounded, and $\partial B$ contains two fixed points $p_1$ and $p_2$ which are saddle points. The union of the stable manifolds of $p_1$ and $p_2$ gives $\partial B$. (b) The basin of attraction $B$ of the zero fixed point of the map $T(x, y) = (y^3, x^3 + y^3)$. The set $B$ is bounded, and $\partial B$ contains two fixed points $p_1$ and $p_2$ (saddles) and a repelling minimal period-two point $q_1$ and $q_2$. The union of the stable manifolds of $p_1$ and $p_2$ gives $\partial B$.

a cooperative or competitive map, which was critical in the proofs using the results from [13] is not always true. Also, there is the question of the components of the basin of attraction in other cases, in addition to the possible presence of other fixed points (perhaps nonhyperbolic) on the boundary of the basin.

In general, the basin of attraction $B(E)$ of locally asymptotically stable fixed point $E$ of a map $T$ satisfies

$$B(E) = \bigcup_{k=0}^{\infty} T^{-k}B_0(E),$$

where $B_0(E)$ is a largest connected invariant set containing $E$, and $T^0$ is the identity map [7, 21]. The problem of characterization of $B(E)$ is finding the properties of $T^{-k}B_0(E)$ for an arbitrary map. In this paper we show that if $T$ is a monotone (cooperative or competitive) map, one can characterize those components of the basin of attraction. Our main results will show that the previous two examples are indicative of the structure of such basin of attraction. In addition, we will show that the components of the basin of attraction form an unordered chain of non-invariant sets which eventually map into $B_0(E)$.

This paper is organized as follows. In the rest of this section we give some basic notions about monotone maps in the plane. The second section presents our main results and some corollaries. The third section presents examples and the fourth section gives proofs of the main results.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $y \not= x$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$. If $x \preceq y$, the order interval $[x, y]$ is the set $\{z : x \preceq z \preceq y\}$. A map $T$ on a nonempty set $R \subset \mathbb{R}^2$
is a continuous function $T : \mathcal{R} \to \mathcal{R}$. A point $\bar{x}$ in $\mathcal{R}$ is a fixed point of $T$ if $T(\bar{x}) = \bar{x}$. The basin of attraction of a fixed point $\bar{x}$ of a map $T$, denoted as $B(\bar{x})$, is defined as the set of all initial points $x_0$ for which the sequence of iterates $T^n(x_0)$ converges to $\bar{x}$. The map $T$ is monotone if $x \leq y$ implies $T(x) \leq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x < y$ implies $T(x) < T(y)$ for all $x, y \in \mathcal{R}$. Throughout this paper we shall use the North-East ordering (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{NE} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \succeq_{SE} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. A map $T$ on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called cooperative, and a map monotone with respect to the South-East ordering is called competitive. If $T$ is continuously differentiable on an open set, a sufficient condition for $T$ to be strongly cooperative (respectively, strongly competitive) is that at every point of the set, the Jacobian matrix of $T$ has positive entries (respectively, positive diagonal entries and negative off-diagonal entries). For $x \in \mathbb{R}^2$, define $Q_i(x)$ for $i = 1, \ldots, 4$ to be the usual four quadrants based at $x$ and numbered in a counterclockwise direction, for example, $Q_1(x) = \{ y \in \mathbb{R}^2 : x \preceq_{NE} y \}$. A set $A$ is said to be order-convex if for every $x, y \in A$, the order interval $[x, y]$ is a subset of $A$. A general reference for difference equations and maps is [5]. For some basic notions about monotone discrete systems in the plane, see [2, 8, 9, 10, 12, 13, 14, 23].

2. Main results. The main result applies to cooperative maps whose $k$th power (for some $k \geq 1$) is strongly cooperative on the interior of an order interval. Smoothness of the map is not assumed, but it is considered later in Theorems 2.2 and 2.3. Unbounded domains are discussed in Remark 2, competitive maps in Remark 3, and periodic points in Remark 5.

**Theorem 2.1.** Let $\mathcal{R}$ be an order interval in $\mathbb{R}^2$ with nonempty interior, and let $T : \text{int}(\mathcal{R}) \to \text{int}(\mathcal{R})$ be a cooperative map such that for some power $k \geq 1$, $T^k$ is strongly cooperative. Suppose $\bar{x} \in \mathcal{R}$, and set $B := \{ x \in \text{int}(\mathcal{R}) : T^m(x) \to \bar{x} \text{ as } m \to \infty \}$. If there exists an open set $\mathcal{O}'$ in $\mathbb{R}^2$ containing $\bar{x}$ such that $\mathcal{O} := \mathcal{O}' \cap \text{int}(\mathcal{R}) \subset B$, then

(i) The boundary of each connected component $B'$ of $B$ is the union of two curves $C_-$ and $C_+$ (termed the lower and upper boundary curves of $B'$, respectively).

(ii) Points on a boundary curve that are interior to $\mathcal{R}$ have common endpoints, and these are their only common points.

(iii) Denote with $B_*$ the connected component of $B$ whose closure in $\mathcal{R}$ contains $\bar{x}$. The set $B_*$ is $T$-invariant. The intersection of each boundary curve of $B_*$ with the interior of $\mathcal{R}$ is $T$-invariant.

(iv) If $B'$ is a component of $B$ such that $B' \neq B_*$, then there exists a positive integer $n$ that depends on $B'$ such that $T^{n-1}(B') \cap B_* = \emptyset$ and $T^n(B') \subset B_*$. If $C_+ = \cap B'$ and $C_+ = \cap B_*$, respectively, set $\tilde{C}_+ := C_+ \cap \text{int}(\mathcal{R})$ and $\tilde{C}_+ := C_+ \cap \text{int}(\mathcal{R})$. Then $T^n(\tilde{C}_+) \subset \tilde{C}_-$ and $T^n(\tilde{C}_+) \subset \tilde{C}_+$.

The hypothesis on $T^k$ being strongly cooperative is satisfied by maps associated with second order scalar difference equations of the form

$$x_{n+1} = f(x_{n-1}, x_n), \quad n = 1, 2, \ldots, \quad x_0, x_1 \in I$$
where \( I \) is an interval in \( \mathbb{R} \) and \( f : I \times I \to I \) is increasing in both variables. Indeed, in this case the associated map \( T(x, y) = (y, f(x, y)) \) is not strongly monotonic, but \( T^2 \) is. Also note that cooperative or competitive triangular planar maps \( T \) such as those in [4] do not satisfy the hypothesis in Theorem 2.1 that some power of the map is strongly monotonic.

**Remark 1.** If in Theorem 2.1 the point \( \bar{x} \) is in \( \text{int}(\mathcal{R}) \), then \( \bar{x} \) is a fixed point and the set \( B \) is the *basin of attraction of \( \bar{x} \) in \( \text{int}(\mathcal{R}) \).* However, the map need not be defined at \( \bar{x} \) for Theorem 2.1 to apply, see Example 2 in Section 3.

See Figure 2 for visual illustration of Theorem 2.1.

**Remark 2.** The conclusions of Theorem 2.1 are valid for maps \( T \) on unbounded domains \( \mathcal{R} \) if either one is of the forms \( \{ x : x \leq p \} \), \( \{ x : p \leq x \} \), or \( \mathbb{R}^2 \). To prove this, consider the natural extension of the partial order to the extended plane \( \bar{\mathbb{R}}^2 = [\infty, \infty] \times [\infty, \infty] \). The set \( \mathcal{R} \) is a subset of \( \bar{\mathbb{R}}^2 \). Also modify the notion of boundary curve so that points common to a boundary curve and the boundary of the domain of the map may have one or both coordinates equal to \( -\infty \) or \( +\infty \). The proof is essentially the same as that for Theorem 2.1. See Example 1 in Section 3, where the domain of the map is \( \mathbb{R}^2 \).

![Figure 2](image-url)

**Figure 2.** (a) the basin \( B \) of the fixed point \( \bar{x} \) has three components \( B', B_*, B'' \) whose closure is in \( \text{int}(\mathcal{R}) \) and such that \( B' \ll_{se} B_* \ll_{se} B'' \). Each component has boundary curves \( C_+ \) and \( C_- \). (b) The set \( B \) has only one component, which has part of its boundary in \( \partial \mathcal{R} \). Also, \( \bar{x} \in \partial \mathcal{R} \).

The point \( p \) is an endpoint of both boundary curves \( C_- \) and \( C_+ \). The point \( p \) is a fixed point of \( T \).

By (i) and (iii) of Theorem 2.1, the set of endpoints of the boundary curves \( C_+ \) and \( C_- \) of \( B_* \) that belong to \( \text{int}(\mathcal{R}) \) is invariant. Such set has at most two points in \( \text{int}(\mathcal{R}) \), hence any such point is periodic with period two. Therefore we have the following result.

**Corollary 1.** Let \( p \) be an endpoint of a boundary curve of \( B_* \). If \( p \) is in \( \text{int}(\mathcal{R}) \), then \( p \) is a fixed point or a minimal period-two point of \( T \).

From (ii.) of Theorem 2.1 and Corollary 1 we have the following result.

**Corollary 2.** If there are no period-two points in \( Q_2(\bar{x}) \cup Q_4(\bar{x}) \) other than \( \bar{x} \), then there is only one component of \( B \), and the corresponding boundary curves have endpoints on \( \mathcal{R} \).
Smoothness of the map implies that the boundary of the basin $B$ in Theorem 2.1 is guaranteed to have additional properties.

**Theorem 2.2.** Let $R$, $T$ and $B$ be as in Theorem 2.1. Assume the hypotheses of Theorem 2.1. Suppose $z$ is a minimal period $k$ point of $T$ in $\text{int}(R) \cap \partial B$, and that $T$ is of class $C^1$ in a neighborhood of $z$. If the Jacobian matrix of $T^k$ at $z$ has positive entries, then $\partial B$ is tangential at $z$ to the line $\ell$ with direction given by the eigenspace associated to the characteristic value of $T$ at $z$ with the smallest modulus.

**Theorem 2.3.** Assume the hypotheses of Theorem 2.1. Suppose $T$ is a continuously differentiable map on $\text{int}(R)$ such that the Jacobian matrix at every point in $\text{int}(R)$ has positive entries. Let $B'$ be a component of the basin $B$ of $\bar{x}$, and let $C_-$ and $C_+$ be the corresponding boundary curves. Then,

i. Each of the curves $C_-$ and $C_+$ of $B'$ is the graph of a Lipschitz function of a real variable.

ii. If $C_-$ and $C_+$ intersect at a hyperbolic periodic point $p \in \text{int}(R)$, then $p$ is a source.

**Remark 3.** A version of Theorems 2.1, 2.2 and 2.3 and Corollaries 1 and 2 are valid for maps $T$ that are competitive (instead of cooperative). To obtain these results, replace the word cooperative by the word competitive, and replace the north-east partial order by the south-east partial order and vice-versa. With these modifications, the proofs carry over word for word, so those will be omitted. See [6, 15] for examples of the basins of attraction for competitive maps.

**Remark 4.** If the boundary of the set $B_*$ in Theorem 2.1 has a fixed or periodic saddle point, the local stable manifold can be extended to a global stable manifold by using topological arguments or results such as those in [13]. In these cases it is possible to obtain a description of $\partial B_*$. But often the sufficient conditions for global stable manifold are difficult to verify or are not applicable at all. In these cases, Theorems 2.1, 2.2, 2.3 and Corollaries 1 and 2 give the existence of invariant Lipschitz curves where other methods fail.

**Remark 5.** The results of this section are applicable to locally asymptotically stable minimal period $k$ points $p$ of a map $T$. To do this, consider the iterates $p$, $T(p), \ldots, T^{k-1}(p)$ as a fixed points of $T^k$. The basin of the orbit of $p$ is then the union of the basins of points of the orbit as fixed points of $T^k$.

3. **Examples.** In this section we provide two applications. Example 1 is a discussion of the global dynamics of a strongly cooperative map whose domain is $\mathbb{R}^2$. We show that the origin is locally asymptotically stable, with basin of attraction that has more than one component. This is the only example of cooperative map known to the authors with the property that the basin of attraction of a point consists of several components. A feature of the method used to produce the example is that it can be used to generate other examples with basins of attraction consisting of many components, even a countably infinite number of them. In Example 2 we consider a class of parametrized competitive maps defined on the nonnegative quadrant minus the origin. The maps have the origin as a singular point that has a substantial basin of attraction. Our results in this paper can be applied to characterize the boundary of the set attracted to the origin. This characterization is valid for all values of the parameters.
Example 1. We begin by defining a cooperative map $U$ on the plane for which the origin is a LAS fixed point with unbounded basin of attraction. Then a map $V$ is defined as a specific perturbation of $U$, so that the origin has bounded basin of attraction consisting of three components.

Consider the map

$$U(x, y) := (0.5 (x + y) + x^3 + y^3, 0.35 (x + y) + x^5 + y^5), \quad (x, y) \in \mathbb{R}^2. \tag{8}$$

This is a strongly cooperative map for which the origin is LAS, as can be easily determined from analysis of the jacobian matrix. The basin of attraction $B$ of the origin consists of a single unbounded component. This is a consequence of the relation $T(x, -x) = (0, 0)$ for $x \in \mathbb{R}$, hence the line $\{(x, -x) : x \in \mathbb{R}\}$ is a subset of $B$. That there cannot be any other components now follows from Theorem 2.1. See Figure 3.

![Figure 3. Basin of attraction of the origin o for the map U in (8). The points p and q are saddle fixed points.](image)

We now consider a perturbation of $U$ of the form

$$V(x, y) = U(x, y) + \Delta(x, y). \tag{9}$$

We shall choose $\Delta$ so that $V$ is a strongly cooperative map with the origin being a LAS fixed point with basin of attraction having more than one component. One way to accomplish this is by further specializing $\Delta$ to have the form

$$\Delta(x, y) := \left(\frac{\phi(x) - \phi(y)}{2}, \frac{-\phi(x) + \phi(y)}{2}\right) = \frac{1}{2} (\phi(x) - \phi(y)) (1, -1), \tag{10}$$

where $\phi$ is a smooth real valued odd function of a real variable to be chosen later. Since $\phi$ is an odd function we have,

$$\Delta(x, -x) = (\phi(x), -\phi(x)) = \phi(x) (1, -1). \tag{11}$$

Since $U(x, -x) = (0, 0)$, the dynamics of $V(x, y)$ on the line $x + y = 0$ are exactly the dynamics of $\phi$ on the real line.

We shall require that $\phi(0) = 0$, which is necessary for the origin to be a fixed point of $V(x, y)$. Also desirable is a small value of $|\phi'(0)|$ so the origin retains local stability after perturbing the original map. The function $\phi$ must give a cooperative $V$, which can be ensured by choosing $\phi$ with suitable growth restrictions. Consider the function (see Figure 4)

$$\phi(t) := \frac{2.5 t^3 + 0.00075 t^7}{(t^2 + 0.1)^2(t^2 + 1)} \tag{12}$$
Figure 4. Graphs of $\phi$ from (12) and the identity function on the nonnegative semi axis. $\phi$ has locally asymptotically stable fixed points $0$, $b = 2.06$, and a repelling fixed point $a = 0.95$. The real numbers $c = 6.03$ and $d = 12.80$ are pre-images of $a$. The basin of attraction of $0$ on the semi-axis consists of the intervals $0 \leq t < a$ and $c < t < d$. All decimal numbers have been rounded to two decimals.

With $\phi$ as in (12) the map $V(x, y)$ is strongly cooperative on its domain. See Figure 5 for a graphical illustration. The map $V$ has a locally asymptotically stable fixed point $o(0, 0)$ and saddle fixed points $r(-0.404, -0.297)$ and $s(0.404, 0.297)$ as well as the period-two points $q_1(-0.953, 0.953)$, $p_1(0.953, -0.953)$, $q_2(-2.067, 2.067)$, $p_2(2.067, -2.067)$ and eventually period-two points $p_3(6.034, -0.6034)$, $q_3(-6.034, 6.034)$, $q_4(-12.798, 12.798)$, $p_4(12.798, -12.798)$. See Figure 6. The invariant component of the basin of attraction of the origin is bounded by the global stable manifolds of two saddle fixed points which have endpoints at period-two points.

**Example 2.** Consider maps of the form

$$T(x, y) := \left( \frac{x^3}{\alpha x + (1 - \alpha) y}, \frac{y^3}{(1 - \delta) x + \delta y} \right), \quad (x, y) \in \mathbb{R}_+^2 \setminus \{0, 0\}, \quad \alpha, \delta \in (0, 1).$$

(13)

The origin $o$ is a singular point, and there are three fixed points, namely $a(\alpha, 0)$, $d(0, \delta)$ and $b(1, 1)$. A straightforward calculation gives that $a$ and $d$ are saddle points, each with an open semi-axis as unstable manifold. Also $b$ is a repeller, with characteristic values $2$, $4 - \alpha - \delta$, and corresponding eigenvectors $(1, 1)$ and...
Figure 6. (a) Three components of the basin of attraction of the zero fixed point $o$ of the map $V(x, y)$ in Example 1. Here $r, s$ are saddle fixed points, $p_1$ and $q_1$ is a saddle period-two point, $p_2$ and $q_2$ are repelling fixed points, and $p_3, p_4, q_3, q_4$ are eventual period-two points. The boundary of the invariant part of the basin of attraction consist of stable manifolds of saddle fixed points with a period-two endpoints. In addition, there are two eventually period-two points which are end points of another piece of the basin of attraction which is mapped into the invariant part. (b) The invariant component of the basin of attraction of the origin $o$.

$$(\alpha - 1, 1 - \delta).$$

The ray $\{(x, x) : x > 0\}$, is invariant, more specifically we have

$$T(x, x) = (x^2, x^2) \text{ for all } x > 0, \quad \alpha, \; \delta \in (0, 1). \quad (15)$$

The following is a complete characterization of the global dynamics of map (13) for all allowed values of the parameters. See Figure 7.

**Proposition 1.** Let $T$ be as in (13). For all values of $\alpha$ and $\delta$ in $(0, 1)$, the set $B := \{(x, y) : T^n(x, y) \to (0, 0)\}$ is bounded by north-east ordered Lipschitz curves $C_+$ and $C_-$, which have endpoints $a, b$ and $d, b$ respectively. Also, $C_+$ and $C_-$ are tangential to the line $y = x$ at the point $b$. If $(x, y) \neq b$ is in $C_+$ (resp. $C_-$) then $T^n(x, y) \to a$ (resp. $T^n(x, y) \to d$), while if $(x, y)$ is in the complement of the closure of $B$, then $\|T^n(x, y)\| \to \infty$.

**Proof.** We begin by verifying that the origin has a relative neighborhood that is a subset of $B$. This can be seen as follows. The relations $T(x, 0) \succeq_{se} (x, 0)$ for $0 < x < \alpha$, and $T(0, y) \succeq_{se} (0, y)$ for $0 < y < \delta$ imply that for $(u, v)$ with $0 < u < x$ and $0 < v < y$, $T^n(0, y) \succeq_{se} T^n(u, v) \succeq_{se} T^n(x, 0)$. Since $T^n(x, 0) \to (0, 0)$ and $T^n(0, y) \to (0, 0)$, we have $T^n(u, v) \to (0, 0)$. Thus the set $O' = \{(x, y) : 0 < x < \alpha, \; 0 < y < \delta\}$ satisfies $O' \subset B$. Therefore the hypotheses of Theorems 2.1, 2.2 and 2.3 are satisfied.

We now show that $B$ has only one component. By Theorem 2.1, all components of $B$ are non-comparable in the south-east ordering, therefore they are comparable in the north-east ordering. By (15) the open line segment $L := \{(x, x) : 0 < x < 1\}$ consists of points $(x, x)$ such that $T^n(x, x) \to (0, 0)$. Also by (15) the ray $S :=$
\{(x,x) : 1 < x < \infty\} consists of points \((x,x)\) such that \(\|T^n(x,x)\| \to \infty\). For any point \((z,w)\) with \(z > 1\) or \(w > 1\) one may choose \(x\) so that \((x,x) \preceq (z,w)\) or \((z,w) \preceq (x,x)\). It follows that \(T^n(x,x) \preceq T^n(z,w)\) or \(T^n(z,w) \preceq T^n(x,x)\). Since \(T^n(x,x) = (x^{2^n}, x^{2^n})\), we have \(\|T^n(z,w)\| \to \infty\). In particular, it follows that \(B\) has only one component.

Note \(\{a, d, b\} \subset \partial B\). Let \(C^-\) and \(C^+\) be as in Theorem 2.1. Since no points outside of the unit square belong to \(B\), it follows that \(b\) is an endpoint of both \(C^+\) and \(C^-\). Also \(d\) is an endpoint of \(C^-\) and \(a\) is an endpoint of \(C^+\), due to the fact that the axes are unstable manifolds of \(a\) and \(d\). The rest of the proposition follows from Theorems 2.1, 2.2 and 2.3, and their corollaries.

4. Proofs. Proof of Theorem 2.1.

It is sufficient to consider the case where \(T\) is strongly monotonic. To see this, let \(T, \mathcal{B}, k\) and \(\mathcal{O}\) be as in Theorem 2.1, and let \(\mathcal{B}_k := \{x \in \text{int}(\mathbb{R}) : T^{mk}(x) \to \bar{x} \text{ as } m \to \infty\}\). If \(x \in \mathcal{B}_k\), then \(T^{mk}(x) \in \mathcal{O}\) for \(m\) large enough, which implies \(x \in \mathcal{B}\). Thus \(\mathcal{B}_k \subset \mathcal{B}\), and since \(\mathcal{B} \subset \mathcal{B}_k\) it follows \(\mathcal{B} = \mathcal{B}_k\). Without loss of generality we assume for the rest of this section that \(T\) is a strongly monotonic map \((k = 1)\).

We prove several claims first. The first two claims are about certain properties of \(\mathcal{B}\) and its boundary set.

Claim 1. The set \(\mathcal{B}\) is open and order convex, and it has either a finite or countably infinite number of connected components.

Proof. If \(x \in \mathcal{B}\), then for sufficiently large \(m \in \mathbb{N}\) we have \(T^m(x) \in \mathcal{O}\). Then \(x\) is an element of \((T^m)^{-1}(\mathcal{O})\), which is an open subset of \(\mathcal{B}\). Thus \(\mathcal{B}\) is open. If \(\{x,z\} \subset \mathcal{B}\), then by monotonicity of \(T\), for every \(y \in \text{int}(\mathbb{R})\) and all \(m \in \mathbb{N}\), \(x \preceq y \preceq z\) implies \(T^m(x) \preceq T^m(y) \preceq T^m(z)\). Hence \(T^m(y) \to \bar{x}\) and we conclude \(\mathcal{B}\) is order-convex. If the number of connected components of \(\mathcal{B}\) is not finite, choose a point in each of the components with rational entries. The collection of such points is countable, hence so it the collection of components of \(\mathcal{B}\). \(\square\)
Claim 2. The set $\partial B$ does not contain a linearly ordered line segment contained in $\text{int}(R)$.

Proof. Arguing by contradiction, suppose $\partial B$ contains a North-east linearly ordered line segment $L(x, z) \subset \text{int}(R)$. Choose $y$ a point in $L(x, z)$ with $y \neq x, z$. Then $T(x) \ll_{ne} T(y) \ll_{ne} T(z)$ by strong monotonicity of $T$. But then $V \ll_{ne} T(y) < \ll_{ne} W$ for some open neighborhoods $V$ of $T(x)$ and $W$ of $T(z)$. Now both $V$ and $W$ contain points in $B$, say $v$ and $w$. In particular, $v \ll_{ne} T(y) \ll_{ne} w$. Since $B$ is order-convex, it follows that $T(y) \in B$, which contradicts invariance of $\partial B$.

We now proceed to define functions $\phi_{\pm}$ of a real variable that are key to establishing further properties of the boundary of $B$. Denote with $\pi_1$ the projection operator on $\mathbb{R}^2$ given by $\pi_1(x, y) = x$. Let $I := \pi_1(B)$, that is, $I$ is the set consisting of all $t$ in $\mathbb{R}$ for which there exists $y$ in $\mathbb{R}$ such that $(t, y) \in B$. The set $I$ is open in $\mathbb{R}$, and it has a finite or countable number of connected components (intervals). For each connected component of $I$ choose a rational number $q$ in the component, and label the component as $I_q$. Let $Q$ be the set consisting of all such indices $q$. Then for each $q \in Q$, the sets $I_q$ are open in $\mathbb{R}$, pairwise disjoint, and satisfy $I = \bigcup_{q \in Q} I_q$.

Define for each $t \in I$,

$$\phi_-(t) := \inf \{y : (t, y) \in B\} \quad \text{and} \quad \phi_+(t) := \sup \{y : (t, y) \in B\}. \quad (16)$$

Note that the definition of $\phi_{\pm}$ implies $\text{graph}(\phi_{\pm}) \subset \partial B$ and

$$B = \{(t, y) \in \mathbb{R} : t \in I \text{ and } \phi_-(t) < y < \phi_+(t)\}. \quad (17)$$

Properties of $\phi_{\pm}$ are presented in Claims 3–8 below.

Claim 3. The functions $\phi_{\pm}$ are non-increasing on $I$.

Proof. Suppose this is not the case, so there exist $t_1, t_2$ in $I$, $t_1 < t_2$, such that $\phi_+(t_1) < \phi_+(t_2)$. Choose $y_1$ and $y_2$ so that $\phi_-(t_1) < y_1 < \phi_+(t_1)$ for $\ell = 1, 2$, and $y_2 > \phi_+(t_1)$. Then $(t_1, \phi_+(t_1))$ belongs to the order interval $[(t_1, y_1), (t_2, y_2)]$. Since $(t_1, y_1, y_2) \in B$ for $\ell = 1, 2$, it follows that $(t_1, \phi_+(t_1)) \in B$, which is a contradiction. Thus $\phi_+$ is non-increasing on $I$. The proof of the corresponding statement for $\phi_-$ is similar.

For each $q \in Q$ the restriction of the function $\phi_-$ (resp. $\phi_+$) to $I_q$ is nonincreasing, hence it has a natural extension to the closure of $I_q$ in the extended real line given by choosing the value at each endpoint of $I_q$ as the one-sided limit of $\phi_-$ (resp. $\phi_+$). We denote such extensions with $\phi^-_q$ and $\phi^+_q$. It is a consequence of Claim 3 that for $q \in Q$, the functions $\phi^-_q$ and $\phi^+_q$ are non-increasing, and their graphs are contained in $\partial B$.

Claim 4. For every $q \in Q$, (i) $\phi^+_q(t) < \phi^+_q(t)$ for $t \in I_q$, and (ii) $\phi^-_q(t) = \phi^+_q(t)$ for $t \in \partial I_q \setminus \partial \pi_1(R)$.

Proof. Statement (i) of Claim 4 follows from the definition of $\phi_{\pm}$. To prove (ii) of Claim 4, suppose that for some $q \in Q$ and some endpoint $t$ of $I_q$ with $t \notin \partial \pi_1(R)$, the inequality $\phi^-_q(t) < \phi^+_q(t)$ holds. In this case the line segment joining $(t, \phi^+_q(t))$ to $(t, \phi^-_q(t))$ is a $\ll_{ne}$-linearly ordered subset of $\partial B \setminus \text{Int}(R)$, which contradicts Claim 2. Therefore $\phi^-_q(t) = \phi^+_q(t)$.
Claim 5. Each of the sets $\bigcup_{q \in Q} (\text{graph}(\phi_q^\|-) \cap \text{int}(\mathcal{R}))$ and $\bigcup_{q \in Q} (\text{graph}(\phi_q^\|+) \cap \text{int}(\mathcal{R}))$ is invariant under $T$.

Proof. Let $t \in \text{clos}(I)$. If $(t', y') := T(t, \phi_+(t))$, then there is a curve in $B$ with endpoint at $(t, \phi_+(t))$, so this is true about $(t', y') := T(t, \phi_+(t))$. Thus $t' \in \text{clos}(I)$. If $t' \in \partial I$, then $\phi_-(t') = \phi_+(t')$ by claim 4, so in particular $T(t, \phi_+(t)) \in \text{graph}(\phi_+)$. Now suppose $t \in I$. For all $\delta > 0$ small enough, $(t, \phi_+(t) - \delta) \in B$ and consequently $(t'_\delta, y'_\delta) := T(t, \phi_+(t) - \delta) \in B$. By monotonicity of $T$, $(t'_\delta, y'_\delta) \ll_{ne} (t', y')$. But $\phi_-$ is non-increasing, so necessarily $(t', y') \in \text{graph}(\phi_+)$. \hfill \Box

The next result and the formula is needed for further reference.

Claim 6. $\partial B \cap \text{int}(\mathcal{R}) = \text{int}(\mathcal{R}) \cap \left( \bigcup_{q \in Q} \text{graph}(\phi_q^\|\) \cup \text{graph}(\phi_q^\|-) \right)$.

Claim 7. For $q \in Q$ let $\phi$ be either $\phi_q^\|-\) or $\phi_q^\|+)$. If $\text{graph}(\phi) \subset \text{int}(\mathcal{R})$, then $\phi$ is decreasing.

Proof. Arguing by contradiction, if $\phi(t_1) \leq \phi(t_2)$ for $t_1, t_2 \in \text{clos}(I)$ with $t_1 < t_2$, then $T(t_1, \phi(t_1)) <_{ne} T(t_2, \phi(t_2))$ by strong monotonicity. The latter relation together with the invariance of $\text{graph}(\phi)$ imply that $\phi$ is not non-decreasing, a contradiction. \hfill \Box

Claim 8. For every $q \in Q$, $\phi_q^\|-\) and $\phi_q^\|)$ are continuous on $\text{clos}(I_q)$.

Proof. Suppose $\phi_+$ is not continuous at some $t_0$ in $\text{clos}(I)$. By the monotonic character of $\phi$, the discontinuity is of the “jump” variety. More specifically, assume that $\phi_+$ is defined on an interval $t_0 < t < t_0 + \delta$ for some $\delta > 0$, and $y_0 > y_+$, where $y_0 := \phi_+(t_0)$ and $y_+ := \lim_{t \to t_0} \phi_+(t)$. In this case, $(t_0, y_+) \in B$, which is not possible. \hfill \Box

Now we prove statements (i)–(iv) of Theorem 2.1. Let $(\alpha, \beta) := \pi_1(\mathcal{R})$ be the projection of $B$ onto the first coordinate. If $B'$ is a component of $B$, then $\pi_1(B')$ is an interval such that $I^q = \pi_1(B')$ for some $q \in Q$. Define the curves $C_{\pm}$ by cases as follows (see Figure 8).

(I) If $\pi_1(\mathcal{R})$ and $\pi_1(B')$ have no common endpoints, $C_{\pm}$ is the curve given by the graph of $\phi_q^\pm_{\chi}$.

(II) If $\pi_1(\mathcal{R})$ and $\pi_1(B')$ have $\beta$ and only $\beta$ as common endpoint, $C_-$ is the curve given by the graph of $\phi_q^\|-\) and $C_+$ is the curve given by the graph of $\phi_q^\|+$, together with the line segment joining $(\beta, \phi_q^\|+(\beta))$ to $(\beta, \phi_q^\|-\(\beta)$.

(III) If $\pi_1(\mathcal{R})$ and $\pi_1(B')$ have $\alpha$ and only $\alpha$ as common endpoint, $C_+$ is the curve given by the graph of $\phi_q^\|+$ and $C_-$ is the curve given by the graph of $\phi_q^\|-\$, together with the line segment joining $(\alpha, \phi_q^\|-\(\alpha)$ to $(\alpha, \phi_q^\|+(\alpha)$.

(IV) If $\pi_1(\mathcal{R})$ and $\pi_1(B')$ have common endpoints $\alpha$ and $\beta$, $C_+$ is the curve given by the graph of $\phi_q^\|+$, together with the line segment joining $(\beta, \phi_q^\|+(\beta)$ to $(\beta, \phi_q^\|-\(\beta)$ and $C_-$ is the curve given by the graph of $\phi_q^\|-\$, together with the line segment joining $(\alpha, \phi_q^\|-\(\alpha)$ to $(\alpha, \phi_q^\|+(\alpha)$.

The different cases are illustrated in Figure 8. Statement (i) of Theorem 2.1 now follows from relation (17) and Claims 4, 7 and 8. Statement (ii) of Theorem 2.1 is a consequence of the order-convex character of $B$. Since $B$ is connected and contains the fixed point $\bar{x}$, it follows $T(B_+) \subset B$. This fact and Claim 5 imply statement
Then there exists an integer \( C \) be a closed convex double cone in \( \mathbb{R}^m \). Choose an integer \( \{n\} \). Let \( m \) containing \( x \) such that \( T^n(x) \in B_r \) and \( T^{-n}(x) \not\in B_r \). Now statement (iv) of Theorem 2.1 follows from Claim 5 and the fact that \( T \) maps connected sets to connected sets.

\[
\text{Figure 8. The four cases in the definition of } C_\pm.\]

**Lemma 4.1.** Let \( J \) be a \( 2 \times 2 \) matrix with positive entries. Let \( v \) be an eigenvector of \( J \) that is associated with the eigenvalue of \( J \) that has the smallest modulus. Let \( C \) be a closed convex double cone in \( \mathbb{R}^2 \) with vertex at the origin \( o \) such that \( v \not\in C \). Then there exists an integer \( m \) such that \( J^m(C) \subset Q_1(o) \cup Q_3(o) \).

**Proof.** Let \( \lambda_1 \) and \( \lambda_2 \) be eigenvalues of \( J \), with associated eigenvectors \( v_1, v_2 \). Assume \( |\lambda_1| < \lambda_2 \). We prove first that \( J^m(\partial C) \subset Q_1(o) \cup Q_3(o) \). If \( z \in \partial C \setminus \{0\} \), then \( z = \alpha_1 v_1 + \alpha_2 v_2 \) for some scalars \( \alpha_1, \alpha_2 \in \mathbb{R} \) with \( \alpha_2 \neq 0 \).

\[
J^m z = \lambda_1^m \alpha_1 v_1 + \lambda_2^m \alpha_2 v_2 = \lambda_2^m \left( \frac{\lambda_1}{\lambda_2} \right)^m \alpha_1 v_1 + \alpha_2 v_2 \tag{18}
\]

Note that \( v_2 \) has both coordinates with the same sign, by Perron-Frobenious Theorem. Since \( |\lambda_1/\lambda_2| < 1 \), it follows from (18) that for \( m \) large enough, \( J^m z \) has both coordinates with sign equal to the sign of \( \alpha_2 \). Hence \( J^m z \subset Q_1(o) \cup Q_3(o) \) and \( J^m(\partial C) \subset Q_1(o) \cup Q_3(o) \). Let \( \tilde{C} \) be the double cone in \( \mathbb{R}^2 \) that is complementary to \( C \). Note \( \tilde{C} \subset J(\tilde{C}) \), similarly \( \tilde{C} \subset J^m(\tilde{C}) \). The set \( J^m(C) \) is a double cone with boundary in \( Q_1(o) \cup Q_3(o) \), and such that \( v \in J^m(\tilde{C}) \). Since \( \mathbb{R}^2 = J^m(C) \cup J^m(\tilde{C}) \), it follows that \( J^m(C) \subset Q_1(o) \cup Q_3(o) \).

**Proof of Theorem 2.2.** We present here a proof for the case where the point \( p \) is a fixed point of \( T \). The case where \( p \) is a minimal period-\( m \) point may be treated by considering the map \( T^m \), for which \( p \) is a fixed point, and it is not given here. By Theorem 2.1, \( B = C_- \cup C_+ \). Without loss of generality we assume \( p \in C_+ \). If the conclusion is not true, then there exists a double cone \( C \) containing \( \ell \setminus \{p\} \) in its interior and a sequence \( \{x_n\} \) on the curve \( C_+ \) such that \( x_n \to p \) and \( x_n \not\in C \). Choose an integer \( m \) as in Lemma 4.1, and set \( J^m := \left( \begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix} \right) \). Let \( \{(t, \phi(t)) : t \in I\} \) be a parametrization of \( C_+ \) near \( p \), such that for some \( t_+ \) in the real interval \( I \), \((t_+, \phi(t_+)) = p\). Set

\[
L_n := \left( \begin{array}{c} \tilde{a} (t_n - t_+) + \tilde{b} (\phi(t_n) - \phi(t_+)), \tilde{c} (t_n - t_+) + \tilde{d} (\phi(t_n) - \phi(t_+)) \end{array} \right) \tag{19}
\]

From Lemma 4.1, \( L_n \in Q_1(o) \cup Q_3(o) \). Also, \( L_n \neq (0, 0) \), since the null space of \( J \), if nontrivial, is contained in the cone \( C \). Then,

\[
\text{for each } n \geq 0, L_n \text{ has nonzero coordinates with the same sign.} \tag{20}
\]
Let $K$ be a compact interval containing $t_*$ in its interior and define, for $t \in K$ and $\epsilon_1, \epsilon_2 \in [-1, 1]$,
\[
\Phi_1(t, \epsilon_1) := (\tilde{a} + \epsilon_1)(t - t_*) + (\tilde{b} - \epsilon_1)(\phi(t) - \phi(t_*))
\]
\[
\Phi_2(t, \epsilon_2) := (\tilde{c} + \epsilon_2)(t - t_*) + (\tilde{d} - \epsilon_2)(\phi(t) - \phi(t_*)).
\]
(21)
The functions $\Phi_1$ and $\Phi_2$ are uniformly continuous on $K \times [-1, 1]$, and by (20), $\Phi_1(t, 0)$ and $\Phi_2(t, 0)$ are nonzero and have the same sign for all $n$. By uniform continuity, there exists $\delta > 0$ such that $\Phi_1(t, \epsilon_1)$ and $\Phi_2(t, \epsilon_2)$ are nonzero and have the same sign for all $t \in K$ and $|\epsilon_1| < \delta$, $|\epsilon_2| < \delta$. Without loss of generality we may assume that
\[
t_n > t_* \quad \text{and} \quad \phi(t_n) < \phi(t_*), \quad n \in \mathbb{N}.
\]
(22)
Let $J$ and $\tilde{J}$ be the jacobian matrices of $T$ and $T^m$ at $p$ respectively. Since $p$ is a fixed point, the chain rule gives $J = \tilde{J}^m$. Note the entries of $\tilde{J}$ are positive. Set $\tilde{J} := \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$. For each $n \in \mathbb{N}$ define $o_n^{(1)}$ and $o_n^{(2)}$ by
\[
(o_n^{(1)}, o_n^{(2)}) := T^m(t_n, \phi(t_n)) - T^m(t_*, \phi(t_*)) - L_n.
\]
(23)
Since $T$ is continuously differentiable,
\[
g_n^{(\ell)} := \frac{o^{(\ell)}}{|t_n - t_*| + |\phi(t_n) - \phi(t_*)|} \to 0 \quad \text{as} \quad n \to \infty, \quad \ell = 1, 2.
\]
(24)
Rearranging terms in (23) and using (19), (22) and (24) we have
\[
T^m(t_n, \phi(t_n)) - T^m(t_*, \phi(t_*)) = \left( (\tilde{a} + g_n^{(1)})(t_n - t_*) + (\tilde{b} - g_n^{(1)})(\phi(t_n) - \phi(t_*)), \tilde{c} + g_n^{(2)} + (\tilde{d} - g_n^{(2)})(\phi(t_n) - \phi(t_*)) \right)
\]
(25)
By (20), (22) and (24) and the assumption that $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are positive, both coordinates in the right-hand side of (25) have the same sign for large $n$, and therefore either $T(t_n, \phi(t_n)) \leq T(t_*, \phi(t_*))$ or $T(t_*, \phi(t_*)) \leq T(t_n, \phi(t_n))$. But this contradicts (i) of Theorem 2.1, which requires points on $C_+$ to be non-comparable.

**Proof of Theorem 2.3.** (i) Let $p \in C_+$, and let $\{(t, \phi(t)) : t \in I\}$ be a parametrization of $C_+$ near $p$, such that for some $t_* \in I$, $(t_*, \phi(t_*)) = p$. Here $I \subset \mathbb{R}$ is an interval. The function $\phi$ is decreasing. If $\phi$ is not Lipschitz at $t_*$, then there exists a sequence $\{t_n\}$ in $I$ such that $t_n \to t_*$ and
\[
\left| \frac{\phi(t_n) - \phi(t_*)}{t_n - t_*} \right| \to \infty \quad \text{as} \quad n \to \infty.
\]
(26)
Without loss of generality we may assume that $t_n > t_*$ and $\phi(t_n) < \phi(t_*)$ for all $n$, that is,
\[
t_n \downarrow t_* \quad \text{and} \quad \frac{\phi(t_n) - \phi(t_*)}{t_n - t_*} \to -\infty \quad \text{as} \quad n \to \infty.
\]
(27)
Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the jacobian matrix of $T$ at $p$. For each $n \in \mathbb{N}$ define $o_n^{(1)}$ and $o_n^{(2)}$ by
\[
(o_n^{(1)}, o_n^{(2)}) := \begin{pmatrix} a(t_n - t_*) + b(\phi(t_n) - \phi(t_*)), c(t_n - t_*) + d(\phi(t_n) - \phi(t_*)) \end{pmatrix}.
\]
(28)
Since $T$ is countinuously differentiable,

$$g_n^{(\ell)} := \frac{\phi^{(\ell)}}{|t_n - t_s| + |\phi(t_n) - \phi(t_s)|} \to 0 \text{ as } n \to \infty, \quad \ell = 1, 2. \quad (29)$$

Rearranging terms in (28) and using (27) and (29) we have

$$T(t_n, \phi(t_n)) - T(t_s, \phi(t_s)) = (t_n - t_s) \left( a + g_n^{(1)} (b - g_n^{(1)}) \frac{\phi(t_n) - \phi(t_s)}{t_n - t_s}, \quad c + g_n^{(2)} (d - g_n^{(2)}) \frac{\phi(t_n) - \phi(t_s)}{t_n - t_s} \right). \quad (30)$$

By (27) and (29) and the assumption that $a, b, c$ and $d$ are positive, both coordinates in the right-hand side of (30) are negative for large $n$, and therefore $T(t_n, \phi(t_n)) \leq T(t_s, \phi(t_s))$. But this contradicts (i) of Theorem 2.1, which requires points on $C_+$ to be non-comparable. Thus $\phi$ is Lipschitz.

(ii) We present the proof for the case when $p$ is a fixed point of $T$. Note $p$ is necessarily unstable since $p \in \partial B$. Since it is hyperbolic, it is either a saddle point or a source. If $p$ is a saddle point, then it has a local stable manifold $M^s$, which is tangential to $v$ with $v$ not comparable to the origin by the Krein-Rutman theorem [12]. But there exist points $x$ in $B_*$ that are arbitrarily close to $p$ and which belong to the union of quadrants $Q_2(p)$ and $Q_4(p)$. Furthermore, such points $x$ may be chosen to be comparable to points on $M^s$, which would prevent the iterates of such points from converging to $p$, thus contradicting the definition of stable manifold. $\square$

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