Bounding twin-width for bounded-treewidth graphs, planar graphs, and bipartite graphs

Hugo Jacob† Marcin Pilipczuk‡

Twin-width is a newly introduced graph width parameter that aims at generalizing a wide range of “nicely structured” graph classes. In this work, we focus on obtaining good bounds on twin-width $\text{tww}(G)$ for graphs $G$ from a number of classic graph classes. We prove the following:

1. $\text{tww}(G) \leq 3 \cdot 2^{\text{tw}(G) - 1}$, where $\text{tw}(G)$ is the treewidth of $G$,
2. $\text{tww}(G) \leq \max(4 \cdot \text{bw}(G), \frac{9}{2} \cdot \text{bw}(G) - 3)$ for a planar graph $G$ with $\text{bw}(G) \geq 2$, where $\text{bw}(G)$ is the branchwidth of $G$,
3. $\text{tww}(G) \leq 183$ for a planar graph $G$,
4. the twin-width of a universal bipartite graph $(X, 2^X, E)$ with $|X| = n$ is $n - \log_2(n) + O(1)$.

An important idea behind the bounds for planar graphs is to use an embedding of the graph and sphere-cut decompositions to obtain good bounds on neighbourhood complexity.

1 Introduction

Twin-width is a graph parameter recently introduced by Bonnet et al [BKTW20], which has already proven to be very versatile and useful. It is defined via iterated contraction of vertices that are almost twins, while limiting the amount of errors that are carried on. Twin-width is known, for instance, to be bounded on classes of graphs of bounded treewidth, bounded rank-width, or excluding a fixed minor [BKTW20]. It is also possible to design algorithms on the contraction sequences, thus providing a common framework for efficient algorithms on several graph classes [BKTW20, BGK+21]. Twin-width is also linked to First Order logic, FO model checking is FPT for graphs of bounded twin-width, and FO transductions preserve twin-width boundedness [BKTW20] (see also [GPT21]).
However, finding good contraction sequences is hard [BBD21], and the arguments used to show the boundedness of twin-width do not necessarily provide a constructive way of obtaining a contraction sequence. This motivates more detailed comparisons of twin-width to other parameters (the case of poset width has already been considered [BH21] for instance).

Many currently known bounds on the twin-width, in particular for minor-closed graph classes such as planar graphs, rely on very general arguments and result in unreasonably large constants. Finding a better bound was explicitly mentioned as an open problem. In this paper, we present a few results we obtained while looking for an improved bound.

We first give some results on graphs of bounded treewidth: an exponential bound on the twin-width of a graph of bounded treewidth, and a linear bound on the twin-width of planar graphs of bounded treewidth. We then obtain a bound of 183 on the twin-width of planar graphs, which is, to the best of our knowledge, currently the best known bound. We were not able to prove a matching exponential lower bound for the twin-width of graphs of bounded treewidth. As a partial result in this direction, we determine the twin-width of universal bipartite graphs up to a constant additive term.

Independently of this work, Bonnet, Kwon, and Wood [EBjKW22] obtained a bound of 583 on the twin-width of planar graphs, among other results on more general classes such as bounded genus graphs.

2 Preliminaries

In the following \([n]\) denotes \(\{1, \ldots, n\}\). Given a set \(X\), \(|X|\) denotes its cardinality and \(2^X\) denotes the set of subsets of \(X\).

The subgraph induced by vertex subset \(A\) in graph \(G\) is denoted by \(G[A]\), \(G - A\) denotes \(G[V \setminus A]\). The neighbourhood of vertex \(v\) in \(G = (V, E)\) is \(N(v) = \{w \in V | \{v, w\} \in E\}\), and we extend this notation with \(N(X) = \bigcup_{x \in X} N(x)\). To emphasize that the neighbourhood is taken in graph \(G\), we use \(N_G\) instead of \(N\).

We call neighbourhood classes with respect to \(Y\) in \(X\) the set \(\Omega(X, Y) = \{N(x) \cap Y : x \in X\}\). Note that if \(|Y| = k\), then \(|\Omega(X, Y)| \leq 2^k\).

We call universal bipartite graph the bipartite graph \(B(n) = ([n], 2^{[n]}, \{(k, A \cup \{k\}) : k \in [n], A \in 2^{[n]\setminus\{k\}}\})\).

We now define formally the notion of twin-width of a graph. A trigraph is a triple \(G = (V, E, R)\) where \(E\) and \(R\) are disjoint sets of edges on \(V\), the (usual) edges and the red edges respectively. The notion of induced subgraph is extended to trigraphs in the obvious way. We denote by \(R(v)\) the red neighbourhood of \(v\). A trigraph \((V, E, R)\) such that \((V, R)\) has maximum degree at most \(d\) is a \(d\)-trigraph. Any graph \((V, E)\) can be seen as the trigraph \((V, E, \emptyset)\). Given a trigraph \(G = (V, E, R)\) and two vertices \(u, v\) of \(V\), the trigraph \(G' = (V', E', R')\) obtained by the contraction\(^1\) of \(u, v\) into a new vertex \(w\) is

\(^1\)The vertices are not required to be adjacent.
defined as the trigraph on vertex set \( V' = V \setminus \{u, v\} \cup \{w\} \), such that \( G - \{u, v\} = G' - \{w\} \), and such that \( N_G'(w) = N_G(u) \cap N_G(v) \) and \( R_G'(w) = R_G(u) \cup R_G(v) \cup (N_G(u) \Delta N_G(v)) \), where \( \Delta \) denotes the symmetric difference. A \( d \)-\emph{contraction sequence} of \( G \) is a sequence of trigraph contractions starting with \( G \) ending with the single-vertex trigraph, such that all intermediate trigraphs have maximum red degree \( d \). The \emph{twin-width} of graph \( G \) is the minimum \( d \) such that there exists a \( d \)-contraction sequence, it is denoted \( \text{tww}(G) \).

We use the notation of [Cou18] for tree decompositions. Given a rooted tree \( T \), \( N_T \) denotes its nodes, \( \leq_T \) denotes its \emph{ancestor relation} which is a partial order on \( N_T \) where the root is the maximal element, and the leaves are the minimal elements. For a fixed node \( u \) of \( T \), we denote by \( p(u) \) its parent (minimal strict ancestor), by \( T_\leq(u) \) the set \( \{w \in N_T | w \leq_T u\} \) and similarly for \( T_\geq(u), T_>(u) \). A tree \( T \) is \emph{normal} for graph \( G \) if \( V(G) = N_T \), and for each edge of \( G \), its endpoints are comparable under \( <_T \). We denote by \( (T, f) \) a \emph{tree decomposition} of \( G \) where \( T \) is a rooted tree, \( f \) maps \( N_T \) to \( 2^{V(G)} \) and satisfies the following conditions: every vertex of \( G \) is contained in at least one \emph{bag} \( f(u) \), for every edge of \( G \) there is a bag containing its two endpoints, and for every vertex of \( G \), the nodes \( u \) such that \( f(u) \) contains it induced a connected subgraph of \( T \). \( (T, f) \) is \emph{normal} if \( T \) is normal for \( G \), \( f(u) \subseteq T_\geq(u) \) and \( u \in f(u) \), for every \( u \in N_T \). \( f^*(u) \) denotes \( f(u) \setminus \{u\} \). \( (T, f) \) is \emph{clean} if it is normal, \( f^*(u) = N_G(T_\leq(u)) \cap T_>(u) \) for every node \( u \) of \( T \), and \( p(u) \in f(u) \) for every node \( u \) of \( T \) except its root. The \emph{width} of \( (T, f) \) is \( \max_{u \in N_T} |f(u)| - 1 \), and the treewidth of a graph is the minimum width over its tree decompositions. It is denoted by \( \text{tw}(G) \).

Let \( \Sigma \) be a sphere \( \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\} \). A \emph{\( \Sigma \)-plane} graph \( G \) is a planar graph embedded in \( \Sigma \) without crossing edges. To simplify notations, we do not distinguish vertices and edges from the points of \( \Sigma \) representing them. An \( O \)-arc is a subset of \( \Sigma \) homeomorphic to a circle. An \( O \)-arc in \( \Sigma \) is a \emph{noose} if it meets \( G \) only in vertices and intersects every face at most once. The length of a noose is the number of vertices it meets. Every noose \( O \) bounds two open discs \( \Delta_1, \Delta_2 \) in \( \Sigma \), i.e., \( \Delta_1 \cap \Delta_2 = \emptyset \) and \( \Delta_1 \cup \Delta_2 \cup O = \Sigma \).

A \emph{branch decomposition} \( (T, \mu) \) of a graph \( G \) consists of a ternary tree \( T \) (internal vertices of degree 3) and a bijection \( \mu : L \rightarrow E(G) \) from the set \( L \) of leaves of \( T \) to the edge set of \( G \). For every edge \( e \) of \( T \), the \emph{middle set} of \( e \) is a subset of \( V(G) \) corresponding to the common vertices of the two graphs induced by the edges associated to the leaves of the two connected components of \( T - e \). The width of the decomposition is the maximum cardinality of the middle sets over all edges of \( T \). An optimal decomposition is one with minimum width, which is called \emph{branchwidth} and denoted by \( \text{bw}(G) \).

For a \( \Sigma \)-plane graph \( G \), a \emph{sphere-cut decomposition} \( (T, \mu) \) is a branch decomposition such that for every edge \( e \) of \( T \), there exists a noose \( O_e \) meeting \( G \) only on the vertices of the middle set of \( e \) and such that the two graphs induced by the edges associated to the leaves of the two connected components of \( T - e \) are each on one side of \( O_e \). The following result is stated in [DPBF10] as a consequence of the results of Seymour and Thomas [ST94], and Gu and Tamaki [GT08].

**Lemma 2.1.** Let \( G \) be a connected \( \Sigma \)-plane graph of branchwidth at most \( \ell \) without
vertices of degree one. There exists a sphere-cut decomposition of $G$ of width at most $\ell$, and it can be computed in time $O(|V(G)|^3)$.

A sphere-cut decomposition $(T, \mu)$ can be rooted by subdividing an edge $e$ of $T$ into two edges $e', e''$ with middle vertex $s$, and adding a root $r$ connected to $s$. The middle set of $e'$ and $e''$ is the middle set of $e$, and $\{r, s\}$ has an empty middle set. For every edge $e$ of $T$, the subtree that does not contain the root is called the lower part, we denote by $G_e$ the subgraph induced by the edges associated to the leaves of the lower part. For an internal node $v$ of $T$, the edge on the path to $r$, is called the parent edge, and the other two are called children edges. There can be at most 2 vertices common to the middle sets of these three edges [DPBF10].

We slightly extend sphere-cut decompositions to cover the case of connected graphs with minimal degree one and branchwidth at least 2. Consider a connected graph $G$, let $G'$ be its maximal induced subgraph with no vertex of degree one. Note that $G'$ must be connected and that the graph $H$ induced by the edges $E(G) \setminus E(G')$ is a forest where each tree has only a vertex in common with $G'$, which we will consider as its root. We can first compute a sphere-cut decomposition $(T', \mu')$ of $G'$ and then for each root $r$ of a tree $H_i$ in $H$, we can find an edge $e$ of $T'$ such that $r$ is in its middle set (it exists because $r$ has degree at least 2 in $G'$), and attach an optimal branch decomposition of $H_i$ on $e$. This does not increase the branchwidth because $r$ was already in the middle set of $e$. Once this is done for all trees $H_i$ in $H$, we obtain a branch decomposition $(T, \mu)$ of $G$, such that there exists a noose meeting exactly the middle set of each edge of $T$. However, the nooses do not correspond to cycles in the radial graph anymore since we have to embed the $H_i$ in faces of $G'$.

**Lemma 2.2.** Let $G$ be a connected $\Sigma$-plane graph of branchwidth $\ell \geq 2$. There exists a sphere-cut decomposition of $G$ of width $\ell$, and it can be computed in time $O(|V(G)|^3)$.

**Proof.** Computing $G'$ and $H$ can be done in time $O(|E(G)|)$ and the optimal decompositions of the trees in $H$ can be produced in total time $O(|V(G)|^3)$.

\[ \square \]

# 3 Twin-width of graphs of bounded treewidth

The following result reuses a method to bound clique-width described in [Cou18, Proposition 13].

**Lemma 3.1.** For an undirected graph $G$, $\text{tww}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$.

**Proof.** We consider a connected graph $G$ as the twin-width of a disconnected graph is simply the maximum twin-width of one of its connected components.

We consider a clean tree decomposition $(T, f)$ of $G$ of width $\text{tw}(G)$ (this is always possible [Cou18, Lemma 3, Lemma 5]).
We proceed by induction on the tree $T$. Consider node $v$ with children $u_1, \ldots, u_k$.

We assume that for each $u_i$, we have contracted $V(T_{\leq}(u_i))$ into $A_i$ consisting of at most $|\Omega(T_{\leq}(u_i), f^*(u_i))|$ vertices such that their incident red edges have both endpoints within $A_i$.

We will contract these sets of vertices into a set $C$ consisting of at most $|\Omega(T_{\leq}(v), f^*(v))|$ vertices.

Let $B_0 = \emptyset$. We will inductively obtain for each $i \in [k]$ a vertex set $B_i$ of size at most
\[
|\Omega\left(\bigcup_{j=1}^i T_{\leq}(u_j), f^*(v)\right)|,
\]
by contracting vertices of $\bigcup_{j=1}^i A_i$.

For each $i \in [k]$, we first contract vertices of $A_i$ that have the same neighbourhood in $f^*(v)$, this produces $\tilde{A}_i$ consisting of at most $|\Omega(T_{\leq}(u_i), f^*(u_i))|$ vertices. Doing so will produce at most $|\tilde{A}_i|$ red edges incident to $v$, which now has at most $|B_{i-1}| + |\tilde{A}_i|$ incident red edges. We then contract vertices of $\tilde{A}_i \cup B_{i-1}$ that have the same neighbourhood in $f^*(v)$, producing $B_i$ consisting of at most
\[
|\Omega\left(\bigcup_{j=1}^i T_{\leq}(u_j), f^*(v)\right)|
\]
vertices. Note that the red degree of a vertex resulting from one of these contractions is at most $|\tilde{A}_i| - 1 + |B_{i-1}| - 1 + |\{v\}| \leq |B_{i-1}| + |\tilde{A}_i|$. Vertex $v$ now has $|B_i|$ incident red edges.

After this we can contract $v$ with the vertex of $B_k$ having the same neighbourhood in $f^*(v)$ if it exists. This produces $C$ consisting of at most $|\Omega(T_{\leq}(v), f^*(v))|$ vertices and such that their incident red edges remain within $C$.

In all of the described steps, the red degree of a vertex is at most $3 \cdot 2^{tw(G) - 1}$:

- Vertices in $A_i$ have red degree at most $|A_i| \leq |\Omega(T_{\leq}(u_i), f^*(u_i))| \leq 2^{tw(G)}$.
- Vertices in $\tilde{A}_i$ have red degree at most $|\tilde{A}_i| \leq |\Omega(T_{\leq}(u_i), f^*(u_i))| \leq 2^{tw(G) - 1}$.
- $v$ has red degree at most $|B_{i-1}| + |\tilde{A}_i| \leq |\Omega\left(\bigcup_{j=1}^{i-1} T_{\leq}(u_j), f^*(v)\right)| + |\Omega(T_{\leq}(u_i), f^*(u_i)) - \{v\})| \leq 3 \cdot 2^{tw(G) - 1}$.
- When contracting $B_{i-1} \cup \tilde{A}_i$, vertices have red degree at most $|B_{i-1}| + |\tilde{A}_i| \leq 3 \cdot 2^{tw(G) - 1}$.

Since the property is trivial on leaves of the tree, we conclude that $tw(G) \leq 3 \cdot 2^{tw(G) - 1}$.

Using sphere-cut decompositions, we establish the following lemma.

**Lemma 3.2.** For an undirected connected planar graph $G$ with $bw(G) \geq 2$, $tw(G) \leq \max(4 \cdot bw(G), \frac{9}{2} bw(G) - 3) \leq \max(4tw(G) + 4, \frac{9}{2} tw(G) + \frac{3}{2})$.

For an undirected connected planar graph $G$ with $bw(G) \leq 1$, $tw(G) = 0$.

This mainly relies on the following result.
Claim 3.1. If $N$ is a noose with $|V(N)| = k$ that separates a plane graph $G$ in $G_1$ and $G_2$, then $\Omega(V(G_1) \setminus V(G_2), V(G_2)) = \Omega(V(G_1) \setminus V(N), V(N))$ and $|\Omega(V(G_1) \setminus V(N), V(N))| \leq 4k - 4 =: h(k)$.

Proof. We will count the different possible neighbourhoods by size:

- The only possibility for size 0 is the empty neighbourhood.
- The possibilities for size 1 are the singletons of $V(N)$ and there are $k$ of them.
- For the neighbourhoods of size 2, we pick one vertex for each of them, and call $A$ the set of picked vertices. We now consider $G_1[A \cup V(N)]$ and smooth the vertices of $A$ in it, i.e. for each vertex $a$ of $A$ with incident edges $ua, av$, we remove vertex $a$ and edges $ua, av$ and replace them by edge $uv$, this operation preserves planarity and the resulting graph $H$ is an outerplanar graph on vertices $V(N)$ because they were on the outerface of $G_1[A \cup V(N)]$. Since the number of edges of $H$ is at most $2k - 3$ because it is outerplanar and is equal to $|A|$, the number of different neighbourhoods is bounded by $2k - 3$.

- For the neighbourhoods of size more than 3, we once again pick one vertex for each of them, and call $B$ the set of picked vertices. We now consider $G_1[B \cup V(N)]$ which is planar. We show $|B| \leq n_3(k) \leq k - 2$ by induction on $k = V(N)$, where $n_3(k)$ denotes the maximum number of vertices of $B$ of degree more than 3 we can have in $G_3[B \cup V(N)]$. First, if $k \leq 2$ then there are no such neighbourhoods, and if $k = 3$, there is exactly one. Then for $k > 3$,

$$n_3(k) = 1 + \max \left\{ \sum_{i=1}^{\ell} n_3(a_i + 1) : \ell \geq 3, \forall i \in [\ell], a_i \geq 1, \sum_{i=1}^{\ell} a_i = k \right\}$$

because after placing one vertex $v$ of degree $\ell \geq 3$, we must have subdivided our instance into $\ell$ smaller instances because edges incident to $v$ will not be crossed by other edges. Using the induction hypothesis, we have

$$n_3(k) \leq 1 + \sum_{i=1}^{\ell} (a_i - 1) \leq 1 + k - l \leq k - 2$$

By summing the previous bounds, we conclude that $|\Omega(V(G_1) \setminus V(N), V(N))| \leq 4k - 4$.

Note that this bound is tight: denote the vertices in their order on the noose by $[k]$, we can place vertices with neighbourhoods $\{\emptyset\} \cup \{\{i\} : i \in [k]\} \cup \{\{i, i + 1\} : i \in [k - 1]\} \cup \{\{1, i, i + 1\}, \{1, i + 1\} : i \in [2, k - 1]\}$.

Proof of Lemma 3.2. Consider a connected planar graph $G$. If $G$ has branchwidth at most 1, it cannot contain a path on 4 vertices as a subgraph, hence it is a star and has twin-width 0 (first contract twins and finish with the root). We now consider the case
when \(\text{bw}(G) \geq 2\). \(G\) admits a sphere-cut decomposition \((T, \mu)\) of width \(k := \text{bw}(G)\). We root \(T\) arbitrarily. We proceed by induction on \(T\). Consider a parent edge \(e\) with children edges \(e_1, e_2\). We assume that, for \(i \in \{1, 2\}\), \(V(G_{e_i} - V(N_{e_i}))\), has been contracted to a set \(A_i\) according to the neighbourhood in \(V(N_{e_i})\). Consequently, \(|A_i|\) is at most \(|\Omega(V(G_{e_i} - V(N_{e_i})), V(N_{e_i}))|\), and red edges incident to \(A_i\) have both endpoints in \(A_i\).

Let \(x := |V(N_{e_i}) \cap V(N_{e_1})|\) and \(y := |V(N_{e_i}) \cap V(N_{e_2})|\). Note that \(x + y - 2 \leq |V(N_{e_i})| \leq k\).

Let \(I := V(N_{e_1}) \cap V(N_{e_2}) \setminus V(N_{e_i})\), and \(z := |I|\).

For \(i \in \{1, 2\}\), we contract vertices of \(A_i\) that have the same neighbourhood in \(V(N_{e_i}) \setminus I\), and call the resulting set of vertices \(\tilde{A}_i\). The vertices of \(I\) now have red degree at most \(|\tilde{A}_1| + |\tilde{A}_2|\), while the vertices of \(\tilde{A}_i\) have red degree at most \(|I| + |\tilde{A}_i| - 1\).

We then contract the vertices of \(I \cup \tilde{A}_1 \cup \tilde{A}_2\) that have the same neighbourhood in \(V(N_{e_i})\), and call \(A\) the resulting set of vertices. Contracted vertices have red degree at most \(|\tilde{A}_1| + |\tilde{A}_2| + |I| - 2\). Using Claim 3.1, we obtain the following inequalities:

\[
|\tilde{A}_1| + |\tilde{A}_2| + |I| - 2 \leq |\Omega(V(G_{e_1} - V(N_{e_1})), V(N_{e_1}) \setminus I)| + |\Omega(V(G_{e_2} - V(N_{e_2})), V(N_{e_2}) \setminus I)| \\
(4x - 4) + (4y - 4) \leq 4k
\]

\[
|\tilde{A}_1| + |\tilde{A}_2| + |I| - 2 \leq |\Omega(V(G_{e_1} - V(N_{e_1})), V(N_{e_1}) \setminus I)| + |\Omega(V(G_{e_2} - V(N_{e_2})), V(N_{e_2}) \setminus I)| \\
|z - 2| \leq 4x + 4y + z - 10 = \frac{3}{2}(x + y) + \frac{1}{2}(x + z) + \frac{1}{2}(y + z) - 10
\]

We have the following constraints on \(x, y, z\): \(x + y \leq k + 2\), \(|V(N_{e_1})| = x + z \leq k\), \(|V(N_{e_2})| = y + z \leq k\).

By summing inequalities, we obtain \(|\tilde{A}_1| + |\tilde{A}_2| + |I| - 2 \leq \frac{9}{2}k - 3\).

\(V(G_{e_i} - V(N_{e_i}))\) has been contracted to a set \(A\) of at most \(|\Omega(V(G_{e_i} - V(N_{e_i})), V(N_{e_i}))|\) vertices.

We conclude that \(\text{tww}(G) \leq \max(4k, \frac{9}{2}k - 3)\) \(\square\)

4 Twin-width of planar graphs

**Theorem 4.1.** The twin-width of planar graphs is at most 183.

**Proof.** We will make use of the argument used to decompose planar graphs in [UWY21, Lemma 5], and produce a \(d\)-contraction sequence of a planar graph \(G\) inductively on the decomposition, \(d \leq 183\). The embedding of the graph will be useful in our arguments to make use of Claim 3.1. Recall that \(h(k) = 4k - 4\).

We may suppose that \(G\) is connected since the twin-width of a graph is simply the maximum of the twin-width over its connected components. We denote by \(G^+\) a triangulation containing \(G\) as a spanning subgraph. Let \(T\) be a BFS spanning tree in \(G^+\) with root \(r\) on its outerface. Note that since \(G\) is a subgraph of \(G^+\), the plane embedding of \(G^+\) gives a plane embedding of \(G\) and its subgraphs.

For a cycle \(C\), we write \(C = [P_1, \ldots, P_k]\) if the \(P_i\) are pairwise disjoint, and the last
vertex of $P_i$ is adjacent to the first vertex of $P_{i+1}$ for $i \in [k]$, with $P_{k+1} = P_1$. For a path $P$, we write $P = [P_1, \ldots, P_k]$ if the $P_i$ are pairwise disjoint, and the last vertex of $P_i$ is adjacent to the first vertex of $P_{i+1}$ for $i \in [k-1]$.

The following version of Sperner’s Lemma is used to recursively decompose $G^+$.

**Lemma 4.1** (Sperner’s Lemma). Let $G$ be a near-triangulation whose vertices are coloured $1, 2, 3$, with the outerface $F = [P_1, P_2, P_3]$ where each vertex in $P_i$ is coloured $i$. Then $G$ contains an internal face whose vertices are coloured $1, 2, 3$.

We prove inductively the following:

**Lemma 4.2.** Let $P_1, \ldots, P_k$ for some $k \in [5]$ be pairwise disjoint vertical paths of $T$ such that $F = [P_1, \ldots, P_k]$ is a cycle in $G^+$, let $H$ be the subgraph of $G$ induced by the vertices of $F$ and the set $X$ of vertices in the (strict) interior of $F$, with $r \notin X$. Let $X^j$ denote the set of vertices of $X$ that are at a distance $j$ from $r$ in $T$. We can construct a partial $d$-contraction sequence of $H$ to $H'$ such that for each $j$, the vertices of $X^j$ are contracted to obtain a set of vertices $A^j$ in $H'$, $|A^j| \leq h(3k)$, the vertices of $A^j$ have red neighbours only in $A^{j-1}, A^j, A^{j+1}$, and $d \leq 183$.

**Proof.** If we have 3 vertices then there is no vertex in the interior of the triangle, the empty contraction sequence satisfies the properties.

Otherwise, we decompose $H$ using the argument of [UWY21], see Fig. 1. First, we colour the vertices of $H$ with $k$ colours as follows. For each vertex $v \in V(H)$, we assign colour $i \in [k]$ if the first vertex of $F$ on the path from $v$ to $r$ in $T$ is a vertex of $P_i$. This is well defined because $r$ is on the outerface of $G^+$.

We set up for Sperner’s Lemma with the following constructions:

- If $k = 1$ then, since $F$ is a cycle, $P_3$ has at least 3 vertices so we can write $P_3 = [u, R_2, v]$, and set $R_1 := u, R_3 := v$.
- If $k = 2$ then, since $F$ is a cycle, one of $P_1$ and $P_2$ has at least 2 vertices. W.l.o.g. assume it is $P_1$, then we write $P_1 = [u, R_2]$, and set $R_1 := u, R_3 := P_2$.
- If $k = 3$ then set $R_1 := P_1, R_2 := P_2, R_3 := P_3$.
- If $k = 4$ then set $R_1 := P_1, R_2 := P_2, R_3 := [P_3, P_4]$
- If $k = 5$ then set $R_1 := P_1, R_2 := [P_2, P_3], R_3 := [P_4, P_5]$

Note that $F = [R_1, R_2, R_3]$. We give colour $i$ to the vertices of $H$ whose first vertex of $F$ on their path to the root in $T$ is in $R_i$.

Applying Sperner’s Lemma, we obtain a triangular face of $G^+$, with vertices $v_1, v_2, v_3$ of the 3 colours. We denote $Q_i'$ the path in $T$ from $v_i$ to $r$ restricted to its vertices in $X$ (it might be empty). These paths delimit at most 3 faces $F_1, F_2, F_3$, each of which having at most 5 vertical paths around it. We can apply the induction hypothesis on each of the faces to obtain partial contraction sequences. We first apply them in an
arbitrary order (the contents of the faces are antiadjacent to each other). We denote by $A^j_\alpha$ the contracted sets of face $F_\alpha$.

For each $\alpha$ and increasing $j$, we contract all vertices of $A^j_\alpha$ that are in the same neighbourhood class with respect to $P_1, \ldots, P_k$ in $G$. Note that only vertices on layers $j - 1, j, j + 1$ of the $P_i$ may be adjacent and that there are at most 3 of the $P_i$s that are adjacent to $F_\alpha$. This gives us sets $\tilde{A}^j_\alpha$ of size at most $h(9)$ by Claim 3.1, since by removing the vertices of $Q'_i$ and keeping only vertices of layers $j - 1, j, j + 1$ we obtain a graph that is still planar and in which the cycle delimiting $F_\alpha$ gives a noose with at most 9 vertices (vertical paths have at most 1 vertex per layer).

Then for increasing $j$, we contract vertices of $\tilde{A}^j_1 \cup \tilde{A}^j_2 \cup \tilde{A}^j_3 \cup Q'_1 \cup Q'_2 \cup Q'_3$ that are in the same neighbourhood class with respect to $P_1, \ldots, P_k$ in $G$, see Fig. 2. This gives sets $A^j$ of size at most $h(15)$ by Claim 3.1, because we can deduce a noose from $F = [P_1, \ldots, P_k]$ and by keeping only the vertices of layers $j - 1, j, j + 1$ we have at most 15 vertices on the noose.

We now bound the red degree that may appear in our contraction sequence. When first contracting $A^j_\alpha$ the number of red edges of its vertices is at most $|\tilde{A}^{j-1}_\alpha| + |A^j_\alpha| + |A^{j+1}_\alpha| + 6 - 2$ where the 6 term bounds the number of vertices on the $Q'_i$ that are adjacent to vertices of $A^j_\alpha$, this amounts to at most $h(9) + 2h(15) + 4 = 148$.

We then observe that the number of contractions of pairs of vertices of $\tilde{A}^j_1 \cup \tilde{A}^j_2 \cup \tilde{A}^j_3$ that
may happen when obtaining \( A^j \) is at most 5 for the following reasons. We have at most two contractions to contract the potential vertices with empty neighbourhoods coming from each \( F_i \). Furthermore, at most 3 vertices of the \( P_3 \) can have adjacent vertices in two \( F_\alpha \) (the first vertices of \( F \) on the path from each \( v_i \) to \( r \) in \( T \)), so we may contract the two potential representatives of the neighbourhood classes consisting of a singleton of such a vertex in the two adjacent \( F_\alpha \). Since we know \( |A^j| \leq h(15) \) and each contraction may reduce the number of vertices by at most 1, we have \( |\tilde{A}_j^1| + |\tilde{A}_j^2| + |\tilde{A}_j^3| \leq h(15) + 5 \).

The red degree of a vertex of \( Q'_i \) is bounded by the sizes of the \( |\tilde{A}_j^\alpha| \) of its 3 adjacent layers on the two faces to which it is adjacent, this is because by always contracting to the same vertex in each neighbourhood class we can ensure that the number of red edges to this vertex is always increasing. If we add the size of the last face for each layer (positive terms), we can easily bound using the previous inequality, by \( 3(h(15) + 5) = 183 \).

When the outerface is reached, we can contract arbitrarily to a single vertex layer by layer, and then contract the path. Doing so we have red degree at most \( 3h(9) + 1 < 183 \) because there are only 3 vertices on the outerface.

We conclude that we have constructed a \( d \)-contraction sequence of \( G \) such that \( d \leq 183 \).
5 Bipartite graph

Theorem 5.1. The twin-width of the universal bipartite graph $B(n)$ is $n - \log(n) + O(1)$.

Proof. We first prove an upper bound. Let $k \in [n]$. We denote by $A$ a subset of $k$ vertices in $X = [n]$. First, contract vertices of $Y = 2^{[n]}$ that have the same neighbourhood in $A$. When this is done, vertices of $A$ have no incident red edges, while vertices of $X \setminus A$ have red edges going to all remaining vertices of $Y$ (there are $2^k$ such vertices).

At this point the red degree is at most $\max(2^k, n - k)$.

The vertices of $X \setminus A$ can then be contracted into a single vertex without creating new red edges. We can then contract all the remaining vertices of $Y$ into a new vertex of red degree $k + 1$. Finally, we contract $A$ onto the said vertex. This establishes that for any choice of $k$

$$\text{tww}(B(n)) \leq \max(2^k, n - k, k + 1).$$

By choosing $k = \lfloor \log(n) - 1 \rfloor$, we obtain $\text{tww}(B(n)) \leq n - \log(n) + O(1)$.

We now prove a lower bound. Consider a $(n - k)$-contraction sequence for $B(n)$. We focus on the moment before the first contraction with a vertex of $X$.

Note that the number of initial vertices contained in a current vertex of $Y$ with red degree $d$ is at most $2^d$, hence at most $2^{n-k}$.

Since a contracted vertex of $Y$ has red degree at least 1. From the bound on the red degree of vertices of $X$, we know that there are at most $n(n - k)$ red edges. More precisely, if we denote by $l_a$ for $a \in [n - f(n)]$ the number of vertices of $Y$ with red degree $a$, we have

$$\sum_{a=1}^{n-k} al_a \leq n(n - k).$$

The number of vertices that were contracted in $Y$ is therefore at most

$$\sum_{a=1}^{n-k} l_a 2^a = \sum_{a=1}^{n-k} al_a \cdot \frac{2^a}{a} \leq n(n - k) \cdot \frac{2^{n-k}}{n-k} = n2^{n-k}.$$

When contracting with a vertex of $X$ for the first time, the number of red edges that become incident to it is therefore at least

$$2^{n-1} - n2^{n-k} - 1.$$

This is bounded by $n - k$, which implies $k \leq \log_2(n) + O(1)$.

We can thus conclude that

$$\text{tww}(B(n)) = n - \log_2(n) + O(1).$$

$\square$
6 Conclusion

Although, we provide no lower bound matching our upper bound on the twin-width of graphs of bounded treewidth, we believe that the exponential dependency is necessary. One might want to consider k-trees with heavy branching in order to find such a lower bound.

As for the twin-width of planar graphs, it might be possible to improve the given bound with a more careful analysis. Another interesting prospect would be to adapt our arguments for planar graphs to graphs of bounded genus, for which properties of the embedding might also prove useful.

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