Hall conductivity as a topological invariant

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Abstract

The object of the present work is to study the quantum Hall effect through its symmetries and topological aspects. We consider the model of an electron moving in a two-dimensional lattice in the presence of applied in-plane electric field and perpendicular magnetic field. We refer to this as the two dimensional electric-magnetic Bloch problem (EMB). The Hall conductivity quantizations beyond the linear response approximation is analyzed.

1 Introduction

The quantization of the Hall conductance (σ_H) in integer (IQHE) and fractional values [1] is an astonishing and unexpected result. For the IQHE, the exact quantization of the Hall conductivity was explained based on a topological argument by Thouless et al. [2]; σ_H in units of e²/h is a topological invariant, the so-called first Chern number [3], which can only take integer values if the Fermi level lies in a gap between Landau levels. The presence of a periodic potential broadens each Landau level in a series of minibands, separated by the corresponding minigaps. Contrary to the obvious suggestion that each subband carries a fraction of e²/h, Thouless et al. demonstrated that the contribution to σ_H of each filled miniband is also an integer multiple of e²/h. Clear experimental evidence for the internal structure of the Hofstadter butterfly spectrum has only been found very recently in the measurement of σ_H for lateral superlattices [4]. The work of Thouless et al. makes use of the Kubo linear response theory.
It is of our interest to consider the structure of the Hall current when a finite electric field \(E\) is applied, in particular we shall be able to demonstrate that even in this case the Hall conductivity remains an integer multiple of \(e^2/h\). From the study of electron wave packet propagation in similar physical conditions; it has been observed the existence of transitions from extended to localized states when the electric field switches from commensurate to incommensurate orientations relative to the lattice \[5\]. One interesting feature of our model is that it can be utilized in order to explore the conductivity behavior as a function of the electric field orientation \[7\].

2 Bloch-Floquet Wave Functions

We consider the EMB problem given by the time dependent Schrödinger equation

\[
S \left[ \alpha, \mathbf{k} \right] = \left[ \frac{1}{2m^*} \left( \Pi_x^2 + \Pi_y^2 \right) + V - \Pi_0 \right] \left[ \alpha, \mathbf{k} \right] = 0. \tag{1}
\]

Here \(m^*\) is the effective electron mass, \(\Pi_\mu = p_\mu + A_\mu\), \(p_\mu = (\hbar \partial/\partial t, -\hbar \nabla)\), the vector and scalar potentials, \(A_\mu = (\phi, A)\), are selected to yield: \(\nabla \times A = B\hat{k}\), and \(E = -\nabla \phi - \partial A/\partial t\). For simplicity we shall consider a periodic square lattice: \(V(x, y) = V_0 \left[ \cos \left(2\pi x/a\right) + \cos \left(2\pi y/a\right) \right]\). The energy and lengths are measured in units of \(\hbar \omega_c = \hbar eB/m^*\), and \(\ell_0 = \sqrt{\hbar/eB}\), respectively, where \(\omega_c\) is the cyclotron frequency and \(\ell_0\) the magnetic length, which is equivalent to setting \(\hbar = e = m^* = 1\).

A symmetry group of \(S\) is given by the electric-magnetic translation group \[6\], which consists of a combination of space translations and time evolution operations compensated by a gauge transformation. In general, the two magnetic translations and the electric-evolution operators do not commute amongst themselves, unless three commensurate conditions are imposed\[7\]: (1) The direction of the electric field, given by the angle \(\theta\) with respect to the lattice axis, is commensurate, i.e. \(\theta = \arctan m_1/m_2\) where \(m_1\) and \(m_2\) are relative prime numbers. Hence an extended lattice along the electric field preserves a spatial periodicity, the extended lattice parameter is \(b = a\sqrt{m_1^2 + m_2^2}\). (2) The magnetic flux through each unit cell of the extended lattice per flux quanta is a rational number \(p/q = 1/\sigma\) (3) Time in the evolution operator is restricted to multiple values of \(\tau = b/v_D\), where \(v_D = E/B\) is the drift velocity of the electron.

The eigenstate base of this group is given by the Bloch-Floquet wave functions(BF) \[7\]

\[
\phi_{\alpha,k}(t, \mathbf{x}) = e^{\mathbf{k} \cdot \mathbf{x} - iE_\tau} u_{\alpha,k}(t, \mathbf{x}) \tag{2}
\]

\[
u_{\alpha,k}(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} e^{-ix\left(-y/2+b\right)} e^{iEtx/2} \sum_{\mu \in m} b_{\mu}^m e^{-i\eta E\mathbf{t} \cdot \mathbf{k}} \phi_{\mu} \tag{3}
\]

here \(\phi_{\mu} \equiv \phi_{\mu} \left( x - k_2 + 2\pi \frac{m+pl}{b} \right)\) is the harmonic oscillator wave function and \(b_{\mu}^m\) is a coefficient that satisfies a finite difference equation. For the particular
limit in which $E = 0$ and the Landau inter-level couplings are neglected, the equation coincides with the Harper equations; results for the generalized Harper equation are reported in reference [7].

Besides the finite difference equation, a useful alternative form of the generalized Harper’s equation in the presence of electric field can be derived within the present formalism, it reads

\[(E - iE \cdot \nabla_k)|\alpha, k\rangle = H|\alpha, k\rangle,\]

where $H$ is the Harper’s or generalized Harper’s hamiltonian [7].

### 3 Hall conductivity

Using the above described states, the current density operator takes the form

\[\hat{J} = \sum_{\alpha, \alpha'} \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2k'}{(2\pi)^2} b_\alpha^\dagger (k') b_\alpha (k) \langle \alpha', k'|H|\alpha, k\rangle,\]

where $b_\alpha^\dagger (k)$ and $b_\alpha (k)$ are the usual Fermi creation and annihilation operators. At low temperatures, it can be proved that the electric current becomes

\[J = \langle \Psi | \hat{J} | \Psi \rangle = \sum_{\alpha \leq \nu_F} \int \frac{d^2k}{(2\pi)^2} \left[ \nabla_k E_\alpha (k) - E \cdot \nabla_k A_\alpha (k) \right],\]

where $A_\alpha (k) = (2\pi)^2/(qa^2) \langle u_{\alpha, k}(t, x) | i\nabla_k | u_{\alpha, k}(t, x) \rangle$. Although it is clear that $A_\alpha (k)$ can be identified with the Berry connection, we notice that there is not any adiabatic approximation involved, the connection $A_\alpha (k)$ is evaluated using the states that exactly solve eq. (4). The first term on the right hand side of the previous equation represents the usual group velocity contribution. Additionally there is a novel contribution arising from the gradient of the Berry connection along the longitudinal electric field direction. Considering the diagonal matrix elements obtained from eq. (4), it yields $E_\alpha (k) = \Delta_\alpha (k) + E \cdot A_\alpha (k) - E k_y$ with $\Delta_\alpha (k) = \langle u_{\alpha, k} | H | u_{\alpha, k} \rangle$. Substituting this results into eq. (6), and considering completely filled subbands, it can be readily demonstrated that the current takes the form

\[\bar{J} = \frac{\nu_F}{2\pi p} E e_T - \frac{1}{2\pi p} E \times \sum_{\alpha \leq \nu_F} \int \frac{d^2k}{2\pi} \Omega_\alpha (k),\]

where $e_T$ is a unit vector transverse to $E$ and the Berry curvature is given by $\Omega_\alpha (k) = \nabla_k \times A_\alpha (k)$. Equation (7) shows that the longitudinal magnetoresistance exactly cancels, regardless of the value of $E$. On the other hand, the Hall conductance can be read from eq. (7) and written (restoring units) in a compact form as

\[\sigma_H = \frac{e^2}{h} \sum_{\alpha \leq \nu_F} \sigma_\alpha, \quad \sigma_\alpha = \frac{1}{p} \left( 1 - q \eta_\alpha \right),\]
where $\eta_\alpha$ is the $\alpha$ subband contribution to the conductance given as

$$\eta_\alpha = \frac{1}{q} \int \frac{d^2 k}{2\pi} \left[ \Omega_\alpha (k) \right]_3 = \frac{1}{q} \oint_{\text{CMBZ}} \frac{dk}{2\pi} \cdot A_\alpha (k).$$  \hspace{1cm} (9)

Here CMBZ denotes the contour of the MBZ and the Stoke’s theorem was used to write the second equality.

An expression similar to eq. (7) was originally discussed by Thouless and Kohmoto [2]. However, their work is based on the Kubo linear response theory, consequently only terms linear in $E$ appear in that case. Our result reduces to that of Thouless and Kohmoto if we drop the electric field, i.e. if we evaluate the conductivity in eq. (9) using the zero order solution of eq. (4). As far as the gap condition is satisfied the arguments of Thouless and Kohmoto can be repeated in order to show that the expression in eq. (9) is related to the first Chern number, and consequently $\eta_\alpha$ is quantized. In the case of intermediate electric field intensities, although modified, the band structure is preserved [7] and it is then expected that the quantization of the band conductivities should be in principle observable. However as the electric field is increased the high density of levels makes it practically impossible to resolve the conductance contributions for each miniband, as far as the original Hofstadter miniband is concerned this fact can be evaluated as a nonlinear contribution to $\sigma_\alpha$ that will lead to a break down of its quantization. Higher order corrections to the Hall conductivity can be calculated by a perturbative evaluation of eq. (4).

We observe that the term $-iE \cdot \nabla k$ in eq. (4) can be considered as a perturbative potential; hence a series expansion in $E$ can be worked out. The Hall conductance for the $\alpha$–band is expanded as

$$\sigma_\alpha = \sigma_\alpha^{(0)} + \sigma_\alpha^{(1)} E + \sigma_\alpha^{(2)} E^2 + \ldots, \quad \eta_\alpha = \eta_\alpha^{(0)} + \eta_\alpha^{(1)} E + \eta_\alpha^{(2)} E^2 + \ldots.$$  \hspace{1cm} (10)

Table 1 shows the values of $\eta_\alpha^{(0)}$ for various selections of the inverse magnetic flux $\sigma$. It is verified that $\eta_\alpha^{(0)}$ always yields an integer value; here $\alpha = (\mu, n)$ labels the Landau level $\mu$ and the $n$ internal miniband. For $\sigma = 1/3$ we find: $\eta_{\mu,1}^{(0)} = 1$, $\eta_{\mu,2}^{(0)} = -2$, and $\eta_{\mu,3}^{(0)} = 1$, regardless of $\mu$. The numbers $q$ and $p$ are relatively prime, so there must be integer numbers $u$ and $v$ such that $up + vq = 1$. We can identify $v \equiv \eta_\alpha^{(0)}$, hence $(1 - q\eta_\alpha^{(0)})/p$ is also an integer, but according to eq. (8) this number is the Hall conductance of the subband $\alpha$; thus $\sigma_\alpha^{(0)}$ is quantized in units of $e^2/h$. If the coupling between Landau levels is small, the sum rules $\sum_n \eta_{\mu,n}^{(0)} = 0$, and $\sum_n \sigma_{\mu,n}^{(0)} = 1$ are satisfied guaranteeing that the Hall conductivity of a completely filled Landau level takes the value $e^2/h$.

As the Fermi energy sweeps through a Landau level, the partial contributions of the subband (although integer multiples of $e^2/h$) do not follow a monotonous behavior. If the Fermi energy lies in the $n$-minigap, the accumulated conductance $\zeta_\alpha$ is defined as $\zeta_{\mu,n} = \sum_{j=1}^n \sigma_{\mu,j}$. The $n$-minigap conductance satisfies the Diophantine equation $n = \lambda q + p \zeta_{\mu,n}$, ($|\lambda| \leq p/2$) in agreement with the results obtained by Thouless et al. [2].
Table 1: Zero and second order coefficients $\eta^{(0)}_\alpha$, and $\tilde{\eta}^{(2)}_\alpha$ ($\alpha = (\mu, n)$) for $\sigma = 1/3$, 1/5 and 1/7.

| $\sigma$ | $n$ | $\eta^{(0)}_\alpha$ | $\tilde{\eta}^{(2)}_\alpha$ | $n$ | $\eta^{(0)}_\alpha$ | $\tilde{\eta}^{(2)}_\alpha$ | $n$ | $\eta^{(0)}_\alpha$ | $\tilde{\eta}^{(2)}_\alpha$ |
|----------|-----|----------------------|-----------------------------|-----|----------------------|-----------------------------|-----|----------------------|-----------------------------|
| $1/3$    | 3   | 0.146                | 0.018                       | 6   | 1                     | 0.2760                      | 5   | 1                     | 0.0239                      |
| $1/5$    | 4   | 0.328                | -4.152                      | 4   | 1                     | -110.2766                   | 5   | 1                     | -111.7056                   |
| $1/7$    | 2   | -0.087               | -0.006                      | 3   | 1                     | -7.3303                     | 1   | 1                     | -0.0003                     |

We now turn our attention to the higher order correction to the Hall conductance. Restoring units, the second order contribution to the Hall conductance has the form

$$\sigma^{(2)}_H = \frac{e^2}{h} \tilde{\eta}^{(2)}_\alpha$$

where

$$\tilde{\eta}^{(2)}_\alpha = \sum_{\beta \neq \alpha} \oint_{\text{CMBZ}} dk \cdot \left( \eta^{(0)}_\alpha - \eta^{(0)}_\beta \right) h_1(k)$$

$$\times \left[ \frac{\langle \alpha^{(0)} | \frac{\partial H}{\partial k_x} | \beta^{(0)} \rangle \langle \beta^{(0)} | \frac{\partial H}{\partial k_i} | \alpha^{(0)} \rangle}{|E_\alpha - E_\beta|^4} + (x \leftrightarrow i) \right].$$

(11)

Here the function $h(k)$ can be selected as the gauge potential of a unit flux tube in $k$-space: $h(k) = \frac{1}{2\pi} (k_1, -k_2)/\left(k_1^2 + k_2^2\right)$. In Table 1 we quote some of the values obtained for $\tilde{\eta}^{(2)}_\alpha$. It is observed that as the internal structure of a band increases, the corrections to the conductance of the internal minibands increases abruptly. It can be easily proved that the sum rule $\sum_n \tilde{\eta}^{(2)}_{\mu,n} = 0$ applies. Consequently, even if each miniband presents $E^2$ corrections to the Hall conductance, the contributions from a completely filled Landau level cancel exactly, and the Hall resistance quantization remains valid to order $O(E^3)$. The nonlinear correction to $\sigma_H$ are then expected to be dominant for each miniband and as Table 1 suggests these become larger as the denominator in $\sigma$ increases.

4 Summary

We address the calculation of the Hall conductance beyond the linear response approximation. A closed expression for the Hall conductance, valid to all orders in $E$ is obtained. As far as the band structure is preserved the conductivity for each miniband takes integer values. However for intense electric fields the high density of levels makes it practically impossible to resolve the conductance contributions for each miniband, as far as each miniband is concerned this can be interpreted as nonlinear corrections to $\sigma_H$. The leading order contribution
is quantized in units of $e^2/h$. The first order correction exactly vanishes, while the second order correction shows a $\sigma^{(2)}_H \propto e^3/V_0^2 B$ dependence. $\sigma^{(2)}_H$ cancels for a completely filled Landau band. Hence the nonlinear correction to the conductance is expected to be of order $O(E^2)$ for a filled miniband, whereas for a complete Landau level the correction is expected to be of order $O(E^3)$.

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