NONUNIFORM SAMPLING AND RECOVERY OF BANDLIMITED FUNCTIONS IN HIGHER DIMENSIONS

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Abstract. We provide sufficient conditions on a family of functions \((\phi_\alpha)_{\alpha \in A} : \mathbb{R}^d \to \mathbb{R}^d\) for sampling of multivariate bandlimited functions at certain nonuniform sequences of points in \(\mathbb{R}^d\). We consider interpolation of functions whose Fourier transform is supported in some small ball in \(\mathbb{R}^d\) at scattered points \((\tau_j)_{j \in \mathbb{N}}\) such that the complex exponentials \(\left(e^{-i\langle x_j, \cdot \rangle}\right)_{j \in \mathbb{N}}\) form a Riesz basis for the \(L_2\) space of a convex body containing the ball. Recovery results as well as corresponding approximation orders in terms of the parameter \(\alpha\) are obtained.

1. Introduction

The theory of interpolation has long been of interest to approximation theorists, and has connections with many areas of mathematics including harmonic analysis, signal processing, and sampling theory. The theory of spline interpolation at the integer lattice was championed by I.J. Schoenberg, and typically falls under the heading of “cardinal spline interpolation.” More generally, cardinal interpolation is the study of schemes in which a given target function is interpolated at the multi-integer lattice in \(\mathbb{R}^d\). There has recently been an extensive interplay between the studies of cardinal interpolation, sampling theory of bandlimited functions, and radial basis function theory. For example, Schoenberg himself showed that bandlimited functions can be recovered by their cardinal spline interpolants in a limiting sense as the order of the spline tends to infinity. Similar analysis shows that such functions can also be recovered by cardinal Gaussian and Multiquadric interpolants.

Lately, the ideas of Schoenberg and many of his successors have been used to tackle problems in a broader setting, namely interpolation schemes at infinite point-sets that are nonuniform. The richness of the theory for lattices suggests a search for comparable results in the nonuniform setting. To that end, Lyubarskii and Madych \cite{Lyubarskii} considered univariate bandlimited function interpolation and recovery by splines, thus extending Schoenberg’s ideas to the nonuniform setting. Inspired by their work, Schlumprecht and Sivakumar \cite{Schlumprecht} showed analogous recovery results using translates of the Gaussian kernel. Then Ledford \cite{Ledford} gave sufficient conditions on a family of functions to yield similar convergence results for bandlimited functions in one dimension. One of the unifying themes in these works is the use of a special structure on the points, namely that they form Riesz-basis sequences (or complete interpolating sequences) for an associated Paley-Wiener space, or equivalently, the corresponding sequence of complex exponential functions forms a Riesz basis for a certain \(L_2\) space. We will discuss this in more detail later, but the main point here is that in one dimension, such sequences are characterized and relatively easy to come by. However, in higher dimensions, the problem becomes significantly more complicated as the existence of such Riesz-basis sequences is unknown even for nice domains such as the Euclidean ball.

Consequently, some intermediate steps have been done by Bailey, Schlumprecht, and Sivakumar \cite{Bailey} and Ledford \cite{Ledford}. The former consider Gaussian interpolation of bandlimited functions whose band lies in a ball of small radius \(\beta\), where the interpolation is done at a Riesz-basis sequence for some larger symmetric convex body. It is known that for certain types of these bodies, namely zonotopes, there are Riesz-basis sequences in any dimension. Moreover, one can approximate the Euclidean ball.

2000 Mathematics Subject Classification. Primary 41A05, 41A30, Secondary 42C30.

This work is part of the author’s doctoral dissertation. He thanks his advisors Th. Schlumprecht and N. Sivakumar for their guidance. Research partially supported by National Science Foundation grant DMS 1160633.
arbitrarily closely with such bodies. Precisely, given any \( \delta < 1 \), one can find a zonotope that lies between the ball of radius \( \delta \) and the unit ball, and there is an associated Riesz-basis sequence for the Paley-Wiener space over the zonotope. Ledford worked with squares in two dimensions using Poisson kernels for the interpolation scheme, and mentions an extension to cubes in higher dimensions. The use of cubes requires a careful analysis, and specific decay on the interpolator due to the geometry of the problem. It seems that with the techniques available, the geometry best suited to bandlimited function interpolation in higher dimensions is that of Paley-Wiener spaces over balls. In fact, the main theorem in [11] holds in higher dimensions for functions whose band lies in the unit ball, but as mentioned above, this may well be vacuous if there is no Riesz-basis for that space. It is at this point in the story that we take up our analysis. Inspired by Ledford’s conditions for univariate interpolation in [6] and the higher dimensional Gaussian interpolation results in [3], we give sufficient conditions on a family of functions to form interpolants for the Paley-Wiener space associated with some symmetric convex body in \( \mathbb{R}^d \), such as a zonotope, which also provides recovery of bandlimited functions whose Fourier transform is supported in a ball contained in the convex body.

The rest of the paper is laid out as follows. We begin with basic notations and facts in Section 2; we discuss our interpolation scheme and state the main theorem in Section 3. Then Section 4 provides several examples of families of functions that can be used for interpolation, while Section 5 contains the proof of the main result. Finally we conclude with a section of remarks on the problem.

2. Basic Notions

If \( \Omega \subset \mathbb{R}^d \) has positive Lebesgue measure, then let \( L_p(\Omega) \), \( 1 \leq p \leq \infty \), be the usual Lebesgue space over \( \Omega \) with its usual norm. If no set is specified, we mean \( L_p(\mathbb{R}^d) \). Similarly, let \( \ell_p := \ell_p(\mathbb{N}) \) be the usual sequence spaces indexed by the natural numbers.

Let \( C(\mathbb{R}^d) \) be the space of all continuous functions on \( \mathbb{R}^d \), and \( C_0(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : \lim_{\|x\| \to \infty} f(x) = 0 \} \), where \( \| \cdot \| \) is the Euclidean distance on \( \mathbb{R}^d \).

The Fourier transform of an integrable function \( f \) is given by

\[
\hat{f}(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) e^{-i\langle \xi, x \rangle} dx, \quad \xi \in \mathbb{R}^d,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \). The Fourier transform can be extended to an isometry of \( L_2 \) onto itself. We will denote by \( \mathcal{F}[f] \), the Fourier transform of a function \( f \in L_2 \).

Moreover, Parseval’s Identity states that

\[
\| \mathcal{F}[f] \|_{L_2} = \| f \|_{L_2}.
\]

If \( f \) is also continuous, and \( \mathcal{F}[f] \in L_1 \), then the following inversion formula holds:

\[
f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[f](\xi) e^{i\langle \xi, x \rangle} d\xi, \quad x \in \mathbb{R}^d.
\]

We consider interpolation of so-called bandlimited or Paley-Wiener functions. Precisely, for a bounded measurable set \( S \subset \mathbb{R}^d \) with positive Lebesgue measure, we define the associated Paley-Wiener space by

\[
PW_S := \{ f \in L_2 : \mathcal{F}[f] = 0 \text{ a.e. } \text{outside of } S \}.
\]

Consequently, for \( f \in PW_S \), the inversion formula and the Riemann-Lebesgue Lemma imply that

\[
f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[f](\xi) e^{i\langle \xi, x \rangle} d\xi = \frac{1}{(2\pi)^d} \int_{S} \mathcal{F}[f](\xi) e^{i\langle \xi, x \rangle} d\xi,
\]

for every \( x \in \mathbb{R}^d \), and moreover that \( f \in C_0(\mathbb{R}^d) \).
A sequence of functions, \((\phi_j)_{j \in \mathbb{N}}\), is said to be a Riesz basis for a Hilbert space \(\mathcal{H}\) if every element \(h \in \mathcal{H}\) has a unique representation
\[
h = \sum_{j \in \mathbb{N}} a_j \phi_j, \quad \sum_{j \in \mathbb{N}} |a_j|^2 < \infty,
\]
and consequently by the Uniform Boundedness Principle, there exists a constant \(R_b\), called the basis constant, such that
\[
\frac{1}{R_b} \left( \sum_{j \in \mathbb{N}} |c_j|^2 \right)^{1/2} \leq \left\| \sum_{j \in \mathbb{N}} c_j \phi_j \right\|_{\mathcal{H}} \leq R_b \left( \sum_{j \in \mathbb{N}} |c_j|^2 \right)^{1/2},
\]
for every sequence \((c_j) \in l_2\). Given a bounded, symmetric, convex set \(Z\) with positive Lebesgue measure, we will consider interpolation at a given sequence of points, \(h \in H\) and consequently by the Uniform Boundedness Principle, there exists a constant \(R_b\), called the basis constant, such that
\[
\frac{1}{R_b} \left( \sum_{j \in \mathbb{N}} |c_j|^2 \right)^{1/2} \leq \left\| \sum_{j \in \mathbb{N}} c_j \phi_j \right\|_{\mathcal{H}} \leq R_b \left( \sum_{j \in \mathbb{N}} |c_j|^2 \right)^{1/2},
\]
for every sequence \((c_j) \in l_2\). Given a bounded, symmetric, convex set \(Z\) with positive Lebesgue measure, we will consider interpolation at a given sequence of points, \(X := (x_j)_{j \in \mathbb{N}}\) in \(\mathbb{R}^d\), that is a Riesz-basis sequence for \(L_2(Z)\), namely \((e^{i(x_j,\cdot)})_{j \in \mathbb{N}}\) forms a Riesz basis for \(L_2(Z)\). We note that in the literature, one will also see such sequences called complete interpolating sequences for the associated Paley-Wiener space, \(PW_Z\).

It is noted in \([3]\) that a necessary condition for a sequence \(X\) to be a Riesz-basis sequence for \(L_2(Z)\) is that it be separated, i.e. there exists a \(q > 0\) such that
\[
\|x_k - x_j\|, \quad \text{for all } k \neq j.
\]

Now we define two operators that will play an important part in our analysis. First, let \((e_j^*)_{j \in \mathbb{N}} \subset L_2(Z)\) be the coordinate functionals for \((e^{i(x_j,\cdot)})_{j \in \mathbb{N}}\). One can show that \((e_j^*)\) is also a Riesz basis for \(L_2(Z)\) with the same basis constant. Thus for every \(g \in L_2(Z)\), we can write
\[
g = \sum_{j \in \mathbb{N}} \langle g, e_j^* \rangle_Z e^{-i(x_j,\cdot)} = \sum_{j \in \mathbb{N}} \langle g, e^{-i(x_j,\cdot)} \rangle_Z e_j^*,
\]
where \(\langle \cdot, \cdot \rangle_Z\) is the usual inner product on \(L_2(Z)\). The final expression in \((7)\) combined with \((5)\) implies that if \(f \in PW_Z\), then
\[
\|f(x_j)\|_{\ell_2} \leq R_b \|\mathcal{F}[f]\|_{L_2(Z)}.
\]

Notice that for any \(a \in \mathbb{R}^d\), we have
\[
\left\| \sum_{j \in \mathbb{N}} \langle g, e_j^* \rangle_Z e^{-i(x_j,\cdot)} e^{-i(a,\cdot)} \right\|_{L_2(a+Z)} = \left\| \sum_{j \in \mathbb{N}} \langle g, e_j^* \rangle_Z e^{-i(a,x_j)} e^{-i(x_j,\cdot)} \right\|_{L_2(Z)} \leq R_b^2 \|g\|_{L_2(Z)},
\]
where \(R_b\) is the Riesz basis constant satisfying \((5)\).

Consequently, the following extension of \(g\) is locally square integrable and thus defined almost everywhere on \(\mathbb{R}^d\).
\[
E(g)(x) := \sum_{j \in \mathbb{N}} \langle g, e_j^* \rangle_Z e^{-i(x_j,\cdot)}, \quad x \in \mathbb{R}^d.
\]

If \(m \in \mathbb{N}\), then we define the prolongation operator \(A_m : L_2(Z) \to L_2(Z)\) via
\[
A_m(g)(\xi) := E(g)(2^m \xi) \chi_{Z \setminus 2^{-m}Z}(\xi), \quad \xi \in Z,
\]
where \(\chi_S\) is the function taking value 1 on the set \(S\) and 0 elsewhere.

It follows from \((9)\) that for \(g \in L_2(Z)\),
\[
\|A_m(g)\|^2_{L_2(Z)} = \int_{Z \setminus 2^{-m}Z} |E(g)(2^m u)|^2 du = 2^{-dm} \int_{2^{-m}Z \setminus 2^{m-1}Z} |E(g)(v)|^2 dv 
\leq 2^{-dm} N^m R_b^4 \|g\|^2_{L_2(Z)},
\]
where $\mathcal{N} = \mathcal{N}(2\mathbb{Z}, \mathbb{Z})$ is the minimum number of translates of $\mathbb{Z}$ which are needed to cover $2\mathbb{Z}$. The constant $\mathcal{N}$ can be bounded by a constant that depends only on $d$, and an induction argument shows that at most $\mathcal{N}^m$ translates of $\mathbb{Z}$ are required to cover $2^m\mathbb{Z}$.

As is customary, we use $C$ to denote a constant which may change from line to line, and subscripts may be used to denote dependence on a given parameter.

### 3. Interpolation Scheme

Suppose that $\mathbb{Z} \subset \mathbb{R}^d$ is a fixed bounded, convex set that is symmetric about the origin and has positive Lebesgue measure. Also assume that $X := (x_j)_{j \in \mathbb{N}}$ is a fixed but arbitrary Riesz-basis sequence for $L_2(\mathbb{Z})$ with basis constant $R_0$. We explore conditions on interpolation operators formed from translates of a single function that allow for recovery of bandlimited functions through a certain limiting process. The criteria here are inspired by so-called regular interpolators developed by Ledford [2]. The results therein are univariate by nature, and our analysis extends to sufficient conditions for interpolation schemes in higher dimensions.

**Definition 3.1.** We call a function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ a $d$-dimensional interpolator for $PW_Z$ if the following conditions hold.

1. $\phi \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $\hat{\phi} \in L_1(\mathbb{R}^d)$.
2. $\phi \geq 0$ and $\hat{\phi} \geq \varepsilon > 0$ on $\mathbb{Z}$.
3. Let $M_j := \sup_{u \in \mathbb{Z} \setminus \frac{1}{2}\mathbb{Z}} |\hat{\phi}(2^j u)|$. Then $(2^{-jd} \mathcal{N} j M_j) \in \ell_1$, where $\mathcal{N}$ is the covering number from [12].

It is important to note that for (I2), it is allowable for $\hat{\phi}$ to be negative everywhere and bounded away from 0 on $\mathbb{Z}$, in which case $-\phi$ satisfies the condition. Condition (I1) allows the use of the Fourier inversion formula [13], while (I2) allows one to show existence of an interpolant for a bandlimited function. Finally, (I3) is a technical condition that comes from a periodization argument that is ubiquitous throughout the proofs in the sequel.

**Remark 3.2.** We note that (I1) could also be stated slightly differently. It is mainly needed to show that an interpolant exists. We may also replace it as follows (which we will do in Section 3.3).

(I1') $\phi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi(\xi) e^{i(x, \xi)} d\xi = \mathcal{F}^{-1} [\psi](x)$ for some $\psi \in L_1 \cap L_2$.

**Theorem 3.3.** Let $\mathbb{Z} \subset \mathbb{R}^d$ be bounded, convex, and symmetric about the origin with positive Lebesgue measure. Suppose that $X$ is a Riesz-basis sequence for $L_2(\mathbb{Z})$, and that $\phi$ is a $d$-dimensional interpolator for $PW_Z$.

(i) For every $f \in PW_Z$, there exists a unique sequence $a \in \ell_2$ such that

$$\sum_{j \in \mathbb{N}} a_j \phi(x_j - x_k) = f(x_k), \quad k \in \mathbb{N}.$$ 

(ii) Let $f$ and $a$ be as in (i). The interpolant of $f$,

$$\mathcal{I}_\phi f(x) = \sum_{j \in \mathbb{N}} a_j \phi(x - x_j), \quad x \in \mathbb{R}^d,$$

belongs to $C_0(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$.

(iii) The Fourier transform of $\mathcal{I}_\phi f$ is given by

$$\mathcal{F}[\mathcal{I}_\phi f](\xi) = \hat{\phi}(\xi) \sum_{j \in \mathbb{N}} a_j e^{-i(x_j, \xi)}, \quad \xi \neq 0.$$ 

Moreover, $\mathcal{F}[\mathcal{I}_\phi f] \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$.

(iv) The Interpolation Operator $\mathcal{I}_\phi : PW_Z \to L_2(\mathbb{R}^d)$ defined by

$$\mathcal{I}_\phi f(\cdot) = \sum_{j \in \mathbb{N}} a_j \phi(\cdot - x_j),$$
where $a$ is as in (i), is a well-defined, bounded linear operator from $PW_Z$ to $L_2(\mathbb{R}^d)$.

Now we turn to sufficient regularity conditions on a family of $d$-dimensional interpolators to provide convergence to bandlimited functions both in the $L_2$ and uniform norms. Assume that $\delta B_2 \subset Z \subset B_2$, where $B_2$ is the Euclidean ball in $\mathbb{R}^d$.

**Definition 3.4.** Let $\beta > 0$. Suppose $A \subset (0, \infty)$ is unbounded, and consider a family of $d$-dimensional interpolators for $PW_Z$, $(\phi_\alpha)_{\alpha \in A}$. We call this family regular for $PW_{\beta B_2}$ if the following hold.

(R1) If $S_\alpha := \sum_{j \in \mathbb{N}} M_j(\alpha)$ where $N$ is the covering number discussed above and $M_j(\alpha)$ is as in (I3), then there is a constant $C$, independent of $\alpha$, such that $S_\alpha \leq CM_\alpha$, where $M_\alpha$ := sup$_{u \in B_2 \setminus \delta B_2} \|\hat{\phi}_\alpha(u)\|$. 

(R2) Let $m_\alpha(\beta) := \inf_{u \in \beta B_2} |\hat{\phi}_\alpha(u)|$, and $\gamma_\alpha := \inf_{u \in B_2} |\hat{\phi}_\alpha(u)|$. Then

$$\frac{M_\alpha^3}{m_\alpha(\beta)\gamma_\alpha^2} \to 0, \text{ as } \alpha \to \infty.$$ 

**Remark 3.5.** All of the examples considered in Section 4 are radial basis functions that are decreasing with the radius. For such functions, it is evident that we can restate the condition (R2) as follows:

(R2') \(\frac{\hat{\phi}_\alpha(\delta)^3}{\hat{\phi}_\alpha(\beta)\hat{\phi}_\alpha(1)^2} \to 0, \text{ as } \alpha \to \infty.\)

We consider interpolation of bandlimited functions $f \in PW_{\beta B_2}$ for some $\beta < \delta$. Hence, $\mathcal{F}[f]$ has support in a subset of the convex body $Z$. The condition (R2) comes from exploiting the geometry of the problem, namely that $\beta B_2 \subset \delta B_2 \subset Z \subset B_2$. We now state our main result.

**Theorem 3.6.** Let $d \in \mathbb{N}$, $\delta \in (0, 1)$, and $\beta < \delta$. Suppose that $Z \subset \mathbb{R}^d$ is convex and symmetric about the origin such that $\delta B_2 \subset Z \subset B_2$, and let $X$ be a Riesz-basis sequence for $L_2(Z)$. Suppose that $(\phi_\alpha)_{\alpha \in A}$ is a family of $d$-dimensional interpolators for $PW_Z$ that is regular for $PW_{\beta B_2}$, and $\mathcal{I}_\alpha$ are the associated interpolation operators. Then for every $f \in PW_{\beta B_2}$,

$$\lim_{\alpha \to \infty} \|\mathcal{I}_\alpha f - f\|_{L_2(\mathbb{R}^d)} = 0,$$

and

$$\lim_{\alpha \to \infty} |\mathcal{I}_\alpha f(x) - f(x)| = 0, \text{ uniformly on } \mathbb{R}^d.$$ 

We note that the condition (R2) will generally give additional restrictions on $\delta$ and $\beta$. This will become clear from the examples.

4. Examples

Since the conditions given above are somewhat abstract, it is prudent to pause and discuss some examples that motivate the general result. Throughout this section, suppose that $Z$ and $X$ satisfy the hypothesis of Theorem 3.6. We begin with an example that is already known due to [3], the Gaussian kernel, and show that we recover the main result therein.

**4.1. Gaussians.** To fit the imposed condition of $\alpha$ tending to infinity, we use a different convention for the Gaussian kernel than [3]:

$$g_\alpha(x) := e^{-\frac{\|x\|^2}{(2 \alpha)^2}}, \quad \alpha \geq 1, \quad x \in \mathbb{R}^d.$$ 

Thus $\hat{g}_\alpha(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\alpha \|\xi\|^2}$. Conditions (I1)-(I3) are readily verified, and will be calculated in a subsequent example. Evidently, $\hat{g}_\alpha$ is radially decreasing, so $M_\alpha = (2\alpha)^{-\frac{d}{2}} e^{-\alpha \delta^2}$ and $M_j(\alpha) = \ldots$
Proof. Positivity is evident from (13) and the fact that $\parallel \cdot \parallel$ is some constant depending only on the dimension $d$. Finally, we check (R2'), noting that $m_\alpha(\beta) = \tilde{g}_\alpha(\beta)$, and $\gamma_\alpha = \hat{g}_\alpha(1)$.

\[ \frac{M_\alpha^3}{m_\alpha(\beta)\gamma_\alpha^2} \leq e^{\alpha(\beta^2 + 3\delta^2)}, \]

which requires $\beta < \sqrt{3\delta^2 - 2}$. This, in turn, requires $\delta > \sqrt{2/3}$ since $\beta$ must be positive. Consequently, the result of Theorem 3.6 coincides with the main theorem in [3], which we reproduce here in our terminology.

**Theorem 4.1** (cf. [3], Theorem 3.6). Let $\delta \in (\sqrt{2/3}, 1)$ and $\beta \in (0, \sqrt{3\delta^2 - 2})$. Then the set of Gaussians $\left( e^{-\frac{\parallel \cdot \parallel^2}{\alpha}} \right)_{\alpha \in [1, \infty)}$ is a family of $d$-dimensional interpolators for $PW_Z$ that is regular for $PW_{\beta B_2}$. In particular, for every $f \in PW_{\beta B_2}$, we have $\lim_{\alpha \rightarrow \infty} \mathcal{I}_\alpha f = f$ in $L_2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}^d$.

4.2. **Inverse Multiquadrics.** Our next example is a family of inverse multiquadrics. For an exponent $\nu > d/2$, we define the generalized inverse multiquadric with shape parameter $c > 0$ by

\[ \phi_{\nu,c}(x) := \frac{1}{(\parallel x \parallel^2 + c^2)^{\nu}}, \quad x \in \mathbb{R}^d. \]

We will consider regularity in the parameter $c$. This example requires a bit more work up front since the Fourier transform of the inverse multiquadrics is somewhat complicated. For now, suppose $\nu$ is fixed, and we suppress the dependence on $\nu$ and write $\phi_c$ for notational ease.

Since $\nu > d/2$, $\phi_c$ is integrable for all $c > 0$, and from [13, Theorem 8.15], we find that its Fourier transform is given by the equation

\[ \hat{\phi}_c(\xi) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \frac{\parallel \xi \parallel}{c} \right)^{\nu - \frac{d}{2}} K_{\nu - \frac{d}{2}}(c \parallel \xi \parallel), \quad \xi \neq 0, \]

where $K_\gamma$ is the univariate modified Bessel function of the second kind defined as follows.

\[ K_\gamma(r) := \int_0^\infty e^{-r \cosh t} \cosh(\gamma t) dt, \quad \gamma \in \mathbb{R}, r > 0. \]

It is important to note from the definition that $K_\gamma$ is symmetric with respect to its order. That is, $K_{-\gamma} = K_\gamma$. Additionally, $K_\gamma(r) > 0$ for all $r > 0$ and all $\gamma \in \mathbb{R}$.

We begin with some necessary properties of both the inverse multiquadrics and the modified Bessel functions of the second kind. First, we have the following differentiation formula:

\[ \frac{d}{dr} [r^\gamma K_\gamma(r)] = -r^\gamma K_{\gamma - 1}(r). \]

This leads to the following observation.

**Proposition 4.2.** The function $\hat{\phi}_c$ is always positive, and is radially decreasing. That is, if $\parallel x \parallel \leq \parallel y \parallel$, then $\hat{\phi}_c(x) \geq \hat{\phi}_c(y)$.

**Proof.** Positivity is evident from [13] and the fact that $K_\gamma(r) > 0$. To see that $\hat{\phi}_c$ is decreasing, set $r = \parallel x \parallel$, and note that (13) and (15) imply

\[ \frac{d}{dr} \left[ \hat{\phi}_c(r) \right] = \frac{2^{1-\nu}}{\Gamma(\nu)c^{\nu - \frac{d}{2}}} \frac{d}{dr} \left[ r^{\nu - \frac{d}{2}} K_{\nu - \frac{d}{2}}(cr) \right] = -\frac{2^{1-\nu}}{\Gamma(\nu)c^{\nu - \frac{d}{2} - 1}r^{\nu - \frac{d}{2} - 1}} K_{\nu - \frac{d}{2} - 1}(cr) < 0. \]

The following inequalities are summarized from [13, Section 5.1].
Proposition 4.3. (i) If \( \nu - d/2 \geq 1/2 \), then
\[
K_{\nu - \frac{d}{2}}(r) \geq \sqrt{\frac{\pi}{2}} r^{-1/2} e^{-r}, \quad r > 0.
\]
(ii) If \( \nu - d/2 < 1/2 \) and \( r > 1 \), then
\[
K_{\nu - \frac{d}{2}}(r) \geq C_{\nu, d} r^{-1/2} e^{-r}, \quad \text{where } C_{\nu, d} := \frac{\sqrt{\pi 3^{\nu - \frac{d}{2} - \frac{1}{2}}}}{2^{\nu - \frac{d}{2} + 1} \Gamma \left( \nu - \frac{d}{2} + \frac{1}{2} \right)}.
\]
(iii)
\[
K_{\nu - \frac{d}{2}}(r) \leq \sqrt{2\pi} r^{-1/2} e^{-r} e^{-\frac{\nu - \frac{d}{2}}{2}}, \quad r > 0.
\]
(iv)
\[
K_{\nu - \frac{d}{2}}(r) \leq 2^{\nu - \frac{4}{2} - 1} \Gamma \left( \nu - \frac{d}{2} \right) r^{\nu - \frac{d}{2}}, \quad r > 0.
\]

To show (I1), it is evident from the definition that \( \hat{\phi}_c \) is integrable, and the following proposition shows that \( \hat{\phi}_c \) is as well.

Proposition 4.4. For \( \nu > d/2 \) and \( c > 0 \), \( \hat{\phi}_c \in L_1(\mathbb{R}^d) \).

Proof. According to (13), we need only show that \( \int_{\mathbb{R}^d} \|\xi\|^{\nu - \frac{d}{2}} \hat{K}_{\nu - \frac{d}{2}}(c\|\xi\|) d\xi \) converges. We split this into two pieces, the integral over the Euclidean ball and the integral outside. By Proposition 4.3 (iv), we see that
\[
I_1 := \int_{B_2} \|\xi\|^{\nu - \frac{d}{2}} \hat{K}_{\nu - \frac{d}{2}}(c\|\xi\|) d\xi \leq C \int_{B_2} \|\xi\|^{\nu - \frac{d}{2}} \|\xi\|^{\frac{d}{2} - \nu} d\xi = C m(B_2),
\]
where \( C \) is a finite constant depending on \( \nu, d, \) and \( c, \) and \( m(B_2) \) is the Lebesgue measure of the Euclidean ball. Furthermore, by Proposition 4.3 (iii),
\[
I_2 := \int_{\mathbb{R}^d \setminus B_2} \|\xi\|^{\nu - \frac{d}{2}} \hat{K}_{\nu - \frac{d}{2}}(c\|\xi\|) d\xi \leq C \int_{\mathbb{R}^d \setminus B_2} \|\xi\|^{\nu - \frac{d}{2} - \frac{1}{2}} e^{-c\|\xi\|} e^{\frac{|\nu - \frac{d}{2}|^2}{2\nu}} d\xi,
\]
and the right hand side is a convergent integral. Again, \( C \) is a finite constant depending on \( \nu, d, \) and \( c. \) In the final inequality, we have used the fact that \( e^{\frac{|\nu - \frac{d}{2}|^2}{2\nu}} \leq e^{-\frac{|\nu - \frac{d}{2}|^2}{2\nu}}. \]

Next notice that (I2) follows from Proposition 4.2 and the fact that \( \hat{\phi}_c(1) > 0. \) Thus it remains to check (I3) and the regularity conditions. By Propositions 4.2 and 4.3 (iii),
\[
M_j(c) \leq |\hat{\phi}_c(2^{j-1}\delta)| \leq C_{\nu} \left( \frac{2^{j-1}\delta}{c} \right)^{\nu - \frac{d}{2}} e^{-c^2\delta} e^{\frac{|\nu - \frac{d}{2}|^2}{2\nu}}.
\]
The right hand side is summable for any fixed \( c, \) which yields (I3).

To check (R1), we may assume, without loss of generality, that \( c \) is large enough so that the final exponential term on the right hand side is at most 2, in which case we have (by Proposition 4.3 (ii))
\[
\sum_{j \in \mathbb{N}} N_j M_j(c) \leq C_{\nu} \sum_{j \in \mathbb{N}} N_j (2^{j-1}\delta)^{\nu - \frac{d}{2}} e^{-2^{2j-1}\delta} \leq D c^{\frac{d+1}{2}} e^{-2^{2j-1}\delta} \leq D \hat{\phi}_c(\delta),
\]
where \( D \) may depend on \( \nu \) and \( d. \)

Finally, we check (R2'). By Proposition 4.3 we find that
\[
\frac{\hat{\phi}_c(\delta)^3}{\hat{\phi}_c(\beta) \hat{\phi}_c(1)^2} \leq C_{\nu, d} \left( \frac{\delta}{\beta} \right)^{\nu - \frac{d+1}{2}} e^{-\nu} e^{c(\beta + 2 - 3\delta)}.
\]
Consequently, as long as $0 < \beta < 3\delta - 2$ and $\delta > 2/3$, (R2') is satisfied. We summarize this in the following theorem.

**Theorem 4.5.** Let $\nu > d/2$. Assume $\delta \in (2/3, 1)$ and $\beta \in (0, 3\delta - 2)$. Then the set of Inverse Multiquadrics $(\|x\|^2 + c^2)^{-\nu})_{c \in [1, \infty)}$ is a family of d-dimensional interpolators for $PW_{\beta B}$ that is regular for $PW_{\beta B}$. In particular, for every $f \in PW_{\beta B}$, we have $\lim_{c \to \infty} \mathcal{J}_c f = f$ in $L_2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}^d$.

4.3. A Broad Class of Examples. We end with a large class of examples which includes both the Gaussian and the Poisson kernel as specific cases. These classes provide natural extensions to the ideas in [3]. For any $p > 0$, we define the following function with parameter $\alpha$:

$$
\gamma_\alpha(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\alpha \|x\|^p} e^{i(x, \xi)} d\xi, \quad x \in \mathbb{R}^d,
$$

or in other words, $g_\alpha = \mathcal{F}^{-1} [e^{-\alpha \|\cdot\|^p}]$. We note that in the case $d = 1$ and $p \leq 2$, these classes correspond to the so-called $p$-stable random variables. It may be the case for some values of $p$ that $g_\alpha$ is not integrable; however, the point in the proof of Lemma 5.1 where we use the condition (I1) is still valid because we have defined $g_\alpha$ as an inverse Fourier integral as in (I1').

Condition (I2) is evident, and to check (I3), note that since $\hat{g}_\alpha$ is radially decreasing, $M_j(\alpha) = e^{-\alpha 2^{j-1}\|\cdot\|^p}$, and thus $(2^{-jd}N^j M_j(\alpha))_j \in \mathbb{N}$ is summable.

We now check the regularity conditions, which will give us bounds on $\beta$ and $\delta$ as in the previous examples. Note that $M_\alpha = e^{-\alpha \delta^p}$. Then as before,

$$
S_\alpha = \sum_{j \in \mathbb{N}} N^j e^{-\alpha 2^{j-1}\|\cdot\|^p} \leq C e^{-\alpha \delta^p} = C M_\alpha,
$$

where $C$ is some constant independent of $\alpha$.

Per Remark 3.5, we consider (R2') as follows:

$$
\frac{M_\alpha}{m_\alpha(\beta) \gamma_\alpha^2} = \frac{g_\alpha(\delta)^3}{g_\alpha(\beta) g_\alpha(1)^2} = e^{\alpha (\beta^p + 2 - 3\delta^p)}.
$$

Evidently, the right hand side tends to 0 as $\alpha \to \infty$ whenever $\beta < (3\delta^p - 2)^{\frac{1}{p}}$. We conclude the following.

**Theorem 4.6.** Let $p > 0$. Suppose $\delta \in \left(\left(\frac{2}{3}\right)^{\frac{1}{p}}, 1\right)$, and $0 < \beta < (3\delta^p - 2)^{\frac{1}{p}}$. Then $(g_\alpha)_{\alpha \in (0, \infty)}$ defined by (10) is a family of d-dimensional interpolators for $PW_{\beta B}$ that is regular for $PW_{\beta B}$. In particular, for every $f \in PW_{\beta B}$, we have $\lim_{\alpha \to \infty} \mathcal{J}_\alpha f = f$ in $L_2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}^d$.

Note that in the case $p = 2$, $g_\alpha$ is the Gaussian discussed in the first example, and the condition reads $0 < \beta < \sqrt{3\delta^2 - 2}$; so in this case, Theorems 4.1 and 4.6 coincide.

5. Proofs

5.1. Proof of Theorem 3.3. Throughout this section, assume that $Z$ is as in the statement of the theorem and that $X$ is a Riesz-basis sequence for $L_2(Z)$. Our first step in the proof is the following key lemma.

**Lemma 5.1.** Suppose that $\phi$ is a d-dimensional interpolator for $PW_B$, and let $A := (x_m - x_n)_{m, n \in \mathbb{N}}$. Then $A : \ell_2 \to \ell_2$ is a bounded, invertible, linear operator.

**Proof.** Linearity is plain, so we will take up boundedness first by looking at $\langle Aa, a \rangle_{\ell_2}$ for arbitrary $a \in \ell_2$. 


\[
\sum_{m,n \in \mathbb{N}} a_m \overline{a_n} \phi(x_m - x_n) = \sum_{m,n \in \mathbb{N}} a_m \overline{a_n} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) e^{i(x_m - x_n, \xi)} \, d\xi \\
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) \left| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right|^2 \, d\xi \\
= \frac{1}{(2\pi)^d} \left[ \int_{\mathcal{Z}} \hat{\phi}(\xi) \left| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right|^2 \, d\xi + \sum_{j \in \mathbb{N}} \int_{2^{j-1} \mathcal{Z} \setminus 2^j \mathcal{Z}} \hat{\phi}(2^j \xi) \left| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right|^2 \, d\xi \right] \\
\leq \frac{1}{(2\pi)^d} \left[ \sup_{u \in \mathcal{Z}} |\hat{\phi}(u)| R_0^2 \|a\|^2_{\ell_2} + \frac{1}{(2\pi)^d} \sum_{j \in \mathbb{N}} 2^{-jd} M_j 2^{-jd} N^j R_0^6 \|a\|^2_{\ell_2} \right].
\]

The last term on the right hand side is \(R_0^6 \|(2^{-2d} N^j M_j)\|_{\ell_1}\), which by (I3) is finite since \(2^{-2d} < 2^{-jd}\). Here we have used the Dominated Convergence Theorem in the second line and (I1) to write \(\phi(x_m - x_n)\) via the Fourier inversion formula.

To show invertibility, we will find a lower bound for the inner product. Indeed, using the Dominated Convergence Theorem again along with (I2), we find that

\[
\sum_{m,n \in \mathbb{N}} a_m \overline{a_n} \phi(x_m - x_n) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) \left| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right|^2 \, d\xi \\
\geq \frac{1}{(2\pi)^d} \int_{\mathcal{Z}} \hat{\phi}(\xi) \left| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right|^2 \, d\xi \\
\geq \frac{\varepsilon}{R_0^2 (2\pi)^d} \|a\|^2_{\ell_2}.
\]

We are now ready to supply the proof.

**Proof of Theorem**\(^3\text{3}\)\textsuperscript{3} Note that (i) is a direct consequence of Lemma \(5.1\) and \(5.5\). The Fourier inversion formula \(3\) along with the Riemann-Lebesgue Lemma show that (iii) implies (ii), so we set out to prove (iii). To do so, we again use a standard periodization argument.

\[
\int_{\mathbb{R}^d} |\hat{\phi}(\xi)| \left| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right| \, d\xi \leq \sup_{u \in \mathcal{Z}} |\hat{\phi}(u)| \left\| \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \cdot)} \right\|_{L_1(\mathcal{Z})} + \sum_{j \in \mathbb{N}} 2^{-jd} M_j \left\| A_j \left( \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \cdot)} \right) \right\|_{L_1(\mathcal{Z})} \\
\leq \sup_{u \in \mathcal{Z}} |\hat{\phi}(u)| m(Z)^{\frac{j}{2}} R_0 \|a\|_{\ell_2} + \left\| \left(2^{-\frac{j}{2}} N^j M_j \right) \right\|_{\ell_1} m(Z)^{\frac{j}{2}} R_0 \|a\|_{\ell_2}.
\]

The final step used the Cauchy-Schwarz inequality, and the second term on the right hand side is finite because of (I3) and the fact that \(N^j \leq N^j\) since \(N > 1\), and \(2^{-\frac{j}{2}} \leq 2^{-jd}\). The argument
Proof. We note that since

\[
\int_{\mathbb{R}^d} \left| \hat{\phi}(\xi) \right|^2 \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \xi)} \right|^2 \, d\xi \leq \sup_{u \in \mathbb{Z}} \left| \hat{\phi}(u) \right|^2 R_u^2 \|a\|_{\ell_2}^2 + \sum_{j \in \mathbb{N}} 2^{-2j} \|N_j^2 M^2 \| \|a\|_{\ell_2}^2.
\]

The series on the right is \((2^{-jd} N_j^2 M_j)\|e\|_{\ell_2} \leq \|(2^{-jd} N_j^2 M_j)\|_{\ell_1}\), which is finite by (I3). Consequently, the function \(\hat{\phi} \sum_{n \in \mathbb{N}} a_n e^{i(x_n, \cdot)} \in L_1 \cap L_2\), and it follows from basic principles that it is the Fourier transform of \(\mathcal{F}_\phi f\), and \(\mathcal{F}_\phi f \in C_0 \cap L_2\).

Finally, to conclude (iv), simply notice from the final step above, Lemma 5.1, Plancherel’s Identity, and (8), that

\[
\|\mathcal{F}_\phi f\|_{L_2} = \|\mathcal{F}[\mathcal{F}_\phi f]\|_{L_2} \leq C \|a\|_{\ell_2} \leq C \|A^{-1}\|_{\ell_2} \|f(x_k)\|_{\ell_2} \leq C \|A^{-1}\|_{\ell_2} R_0 \|f\|_{L_2}.
\]

\[
\square
\]

5.2. Proof of Theorem 3.6 We now embark on a journey to prove our main result. Let \(Z, X, \delta\), and \(\beta\) be as in the statement of Theorem 3.6, and let \((\phi_\alpha)_{\alpha \in A}\) be a family of \(d\)-dimensional interpolators for \(PW_Z\) that is regular for \(PW_{\beta B_2}\). The first step is to show that there exists a constant \(C \leq \infty\) so that

\[
\|\mathcal{F}[\mathcal{F}_\alpha f]\|_{L_2(Z)} \leq C \left( \frac{M_{\alpha}}{\gamma_{\alpha}} \right) \|\mathcal{F}[f]\|_{L_2(Z)}, \quad \alpha \in A,
\]

for every \(f \in PW_Z\). We proceed in a series of steps following the techniques of [8].

To begin, define the function

\[
(17) \quad \Psi_\alpha(u) := \sum_{j \in \mathbb{N}} a_j e^{-i(x_j, u)} = \frac{1}{\hat{\phi}_\alpha(u)} \mathcal{F}[\mathcal{F}_\alpha f](u), \quad u \in \mathbb{R}^d,
\]

and let \(\psi_\alpha\) denote the restriction of \(\Psi_\alpha\) to \(Z\).

Remark 5.2. It is important to note that by uniqueness of the Riesz basis representation for a function on \(Z\), we have that \(\Psi_\alpha(u) = E(\psi_\alpha)(u)\) on \(\mathbb{R}^d\). That is, \(\Psi_\alpha\) is defined globally by its Riesz basis representation on the body \(Z\). This fact is crucial to the subsequent analysis.

Lemma 5.3. The following holds.

\[
\mathcal{F}[f] = \mathcal{F}[\mathcal{F}_\alpha f] + \sum_{m \in \mathbb{N}} 2^{dm} A_m^* \left( \hat{\phi}_\alpha(2^{m}\cdot) A_m(\psi_\alpha) \right) \quad \text{a.e. on } Z.
\]

Proof. We note that since \((e^{-i(x, \cdot)}\) is a Riesz basis for \(L_2(Z)\), it suffices to show that the inner product of both sides above with respect to the basis elements are all equal. First, by (3),

\[
\left\langle \mathcal{F}[f], e^{-i(x_j, \cdot)} \right\rangle_Z = (2\pi)^d f(x_j).
\]

On the other hand, the interpolation condition guarantees that

\[
(2\pi)^d f(x_j) = (2\pi)^d \mathcal{F}_\alpha f(x_j)
\]

\[
= \int_{\mathbb{R}^d} \mathcal{F}[\mathcal{F}_\alpha f](u)e^{i(x_j, u)} \, du
\]

\[
= \int_Z \mathcal{F}[\mathcal{F}_\alpha f](u)e^{i(x_j, u)} \, du + \sum_{m \in \mathbb{N}} \int_{2^m Z \setminus 2^{m-1} Z} \hat{\phi}_\alpha(u)\Psi_\alpha(u)e^{i(x_j, u)} \, du
\]

\[
=: I_1 + I_2.
\]
Evidently, \( I_1 = \langle \mathcal{F} \mathcal{I}_\alpha f, e^{-i(x,j_{\alpha})} \rangle_Z \). Now

\[
I_2 = \sum_{m \in \mathbb{N}} 2^{dm} \int_{Z \setminus \frac{Z}{Z}} \hat{\phi}_\alpha(2^m v) \hat{A}_m(\psi_\alpha)(v) A_m(e^{i(x,j_{\alpha})}) (v) dv
\]

\[
= \sum_{m \in \mathbb{N}} 2^{dm} \int_{Z \setminus \frac{Z}{Z}} \hat{\phi}_\alpha(2^m v) A_m(\psi_\alpha)(v) A_m(e^{i(x,j_{\alpha})}) (v) dv
\]

\[
= \sum_{m \in \mathbb{N}} 2^{dm} \left\langle A_m \left( \hat{\phi}_\alpha(2^m \cdot) A_m(\psi_\alpha) \right), e^{-i(x,j_{\alpha})} \right\rangle_Z,
\]

whence the identity.

We now define an operator that is implicit in the previous Lemma:

\[
\tau_\alpha : L_2(Z) \to L_2(Z), \quad \text{via} \quad \tau_\alpha(h) := \sum_{m \in \mathbb{N}} A_m^* \left( \hat{\phi}_\alpha(2^m \cdot) A_m(h) \right).
\]

**Proposition 5.4.** The operator \( \tau_\alpha \) defined by (18) is a bounded linear operator operator on \( L_2(Z) \) that is positive, (i.e. \( \langle \tau_\alpha(h), h \rangle_Z \geq 0 \) for all \( h \in L_2(Z) \)). Moreover, there exists a positive number \( C \), which is independent of \( \alpha \), so that

\[
\| \tau_\alpha \| \leq CM_\alpha.
\]

**Proof.** Linearity is plain, and positivity can be seen as follows.

\[
\langle \tau_\alpha(h), h \rangle_Z = \sum_{m \in \mathbb{N}} 2^{dm} \left\langle \hat{\phi}_\alpha(2^m \cdot) A_m(h), A_m(h) \right\rangle_Z = \sum_{m \in \mathbb{N}} 2^{dm} \int_{Z \setminus \frac{Z}{Z}} \hat{\phi}_\alpha(2^m u) |A_m(h)(u)|^2 du \geq 0,
\]

the final inequality stemming from the positivity of \( \hat{\phi}_\alpha \).

To prove the upper bound, notice that (12) implies that for \( h \in L_2(Z) \),

\[
\| \tau_\alpha(h) \|_{L_2(Z)} \leq \sum_{m \in \mathbb{N}} 2^{dm} \left\| A_m^* \left( \hat{\phi}_\alpha(2^m \cdot) A_m(h) \right) \right\|_{L_2(Z)}
\]

\[
\leq R^2 \sum_{m \in \mathbb{N}} 2^{dm} N^{\frac{\alpha}{2}} M_m(\alpha) \| A_m(h) \|_{L_2(Z)}
\]

\[
\leq R^2 \sum_{m \in \mathbb{N}} N^{\alpha} M_m(\alpha) \| h \|_{L_2(Z)}
\]

\[
= R^2 S_\alpha \| h \|_{L_2(Z)}
\]

\[
\leq CM_\alpha \| h \|_{L_2(Z)}.
\]

The final inequality comes from the condition (R1). \( \square \)

Next, note that positivity of \( \tau_\alpha \) and Lemma 5.3 imply that

\[
\| \mathcal{F}[f] \|_{L_2(Z)} \| \mathcal{I}_\alpha \psi_\alpha \|_{L_2(Z)} \geq \mathcal{F}[f], \psi_\alpha \|_{Z} \geq \langle \mathcal{F}[\mathcal{I}_\alpha f], \psi_\alpha \rangle_Z \geq \gamma_\alpha \| \psi_\alpha \|_{L_2(Z)}^2.
\]

Therefore,

\[
\| \mathcal{I}_\alpha \psi_\alpha \|_{L_2(Z)} \leq \frac{1}{\gamma_\alpha} \| \mathcal{F}[f] \|_{L_2(Z)}.
\]
From Lemma 5.3, Proposition 5.4, and (20), we see that

\[ \| \mathcal{F}[\mathcal{I}_\alpha f] \|_{L_2(Z)} \leq CM_\alpha \gamma_\alpha \| \mathcal{F}[f] \|_{L_2(Z)}. \]

Now we estimate \( \| \mathcal{F}[\mathcal{I}_\alpha f] \|_{L_2(\mathbb{R}^d \setminus Z)} \). We accomplish this by a familiar periodization argument.

\[ \| \mathcal{F}[\mathcal{I}_\alpha f] \|_{L_2(\mathbb{R}^d \setminus Z)}^2 = \sum_{m \in \mathbb{N}} \int_{2mZ \setminus 2m-1Z} |\hat{\phi}_\alpha(u)|^2 |\Psi_\alpha(u)|^2 du \]

\[ = \sum_{m \in \mathbb{N}} 2^{dm} M_m(\alpha)^2 \| A_m(\psi_\alpha) \|_{L_2(Z)}^2 \]

\[ \leq \sum_{m \in \mathbb{N}} C^m M_m(\alpha)^2 R_\alpha^4 \| \psi_\alpha \|_{L_2(Z)}^2 \]

\[ \leq R_\alpha^4 \frac{1}{\gamma_\alpha} \| \mathcal{F}[f] \|_{L_2(Z)}^2 \sum_{m \in \mathbb{N}} N^m M_m(\alpha)^2. \]

It remains to note that the covering number \( N \) must be larger than 1, so the series in the final expression above is majorized by

\[ \sum_{m \in \mathbb{N}} N^{2m} M_m(\alpha)^2 \leq \left( \sum_{m \in \mathbb{N}} N^m M_m(\alpha) \right)^2 = S^2 \leq CM^2. \]

The first inequality above comes from the fact that the \( \ell_2 \) norm is subordinate to the \( \ell_1 \) norm, and the final inequality comes from (R1). We conclude the following.

**Theorem 5.5.** There exists a constant \( C \), independent of \( \alpha \), such that

\[ \| \mathcal{F}[\mathcal{I}_\alpha f] \|_{L_2(\mathbb{R}^d)} \leq C M_\alpha \gamma_\alpha \| \mathcal{F}[f] \|_{L_2(Z)}, \quad f \in \text{PW}_Z. \]

Define a multiplication operator on \( L_2(Z) \) by

\[ T_\alpha : L_2(Z) \to L_2(Z), \quad h \mapsto \frac{\gamma_\alpha}{\hat{\phi}_\alpha} h. \]

Evidently \( \| T_\alpha \| \leq 1 \). Then we can rewrite Lemma 5.3 as

\[ \mathcal{F}[f] = \mathcal{F}[\mathcal{I}_\alpha f] + \sum_{m \in \mathbb{N}} 2^{dm} A^*_m \left( \frac{\hat{\phi}_\alpha(2^m \cdot)}{\gamma_\alpha} A_m(\gamma_\alpha \psi_\alpha) \right), \]

which by (17) is

\[ \mathcal{F}[\mathcal{I}_\alpha f] + \frac{1}{\gamma_\alpha} T_\alpha \circ \mathcal{F}[\mathcal{I}_\alpha f] = \left( I + \frac{1}{\gamma_\alpha} T_\alpha \right) \mathcal{F}[\mathcal{I}_\alpha f], \]

where \( I \) is the identity operator on \( L_2(Z) \).

**Proposition 5.6.** The map \( I + \frac{1}{\gamma_\alpha} T_\alpha \) is an invertible operator on \( L_2(Z) \), and

\[ \left( I + \frac{1}{\gamma_\alpha} T_\alpha \right)^{-1} \mathcal{F}[f] = \mathcal{F}[\mathcal{I}_\alpha f], \quad f \in \text{PW}_Z. \]

Moreover,

\[ \left\| \left( I + \frac{1}{\gamma_\alpha} T_\alpha \right)^{-1} \right\| \leq \frac{M_\alpha}{\gamma_\alpha}. \]
Proof. Surjectivity of the operator in question follows from the identity in Lemma 5.3. To see injectivity, suppose that \((I + \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha) h = 0\) for some nonzero \(h \in L_2(\mathbb{Z})\). Let \(f \in PW_\mathbb{Z}\) be the function so that \(\mathcal{F}[f] = h\). Then by Lemma 5.3 and positivity of \(\tau_\alpha\), we have
\[
0 = \langle \left(I + \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha\right) \mathcal{F}[f], T_\alpha \mathcal{F}[f] \rangle \geq \langle \mathcal{F}[f], T_\alpha \mathcal{F}[f] \rangle = 0.
\]
Consequently, \(T_\alpha \mathcal{F}[f] = 0\), which implies that \(\mathcal{F}[f] = 0\) since \(T_\alpha h = 0\) if, and only if, \(h = 0\).

Finally, the “moreover” statement follows from Theorem 5.5 and Proposition 5.4 imply given that Lemma 5.3 and the preceding argument imply that the inverse of \(I + \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha\) is the restriction of \(\mathcal{I}_\alpha[\cdot]\) to \(Z\).

□

At last we have the necessary ingredients to complete the proof.

Proof of Theorem 5.6. By Proposition 5.6 and Lemma 5.3 on \(Z\), we have
\[
\mathcal{F}[f] - \mathcal{F}[\mathcal{I}_\alpha f] = \left[I - \left(I + \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha\right)^{-1}\right] (\mathcal{F}[f]) = \left(I + \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha\right)^{-1} \circ \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha (\mathcal{F}[f]).
\]
Therefore, if \(f \in PW_{\beta \mathbb{B}_2}\), Theorem 5.5 and Proposition 5.4 imply
\[
\|\mathcal{F}[f] - \mathcal{F}[\mathcal{I}_\alpha f]\|_{L_2(\mathbb{Z})} \leq \left\| \left(I + \frac{1}{\gamma_\alpha} \tau_\alpha \circ T_\alpha\right)^{-1} \left\| \frac{1}{\gamma_\alpha} \tau_\alpha \| T_\alpha \mathcal{F}[f] \|_{L_2(\mathbb{Z})}
\right.
\leq C \frac{M_\alpha}{\gamma_\alpha} M_\alpha \leq \frac{M_\alpha}{\gamma_\alpha} \| \mathcal{F}[f] \|_{L_2(\beta \mathbb{B}_2)}.
\]

Next, we estimate \(\|\mathcal{F}[\mathcal{I}_\alpha f]\|_{L_2(\mathbb{R}^d \setminus Z)}\) by familiar techniques.
\[
\|\mathcal{F}[\mathcal{I}_\alpha f]\|_{L_2(\mathbb{R}^d \setminus Z)} \leq \sum_{m \in \mathbb{N}} 2^{dm} M_m(\alpha)^2 \| A_m(\psi_\alpha) \|^2_{L_2(\mathbb{Z})}
\leq R_\delta^2 S_\delta \left\| \mathcal{F}[\mathcal{I}_\alpha f] \right\|^2_{L_2(\mathbb{Z})}
\leq C \frac{M_\alpha^2}{\gamma_\alpha^2} \| T_\alpha \mathcal{F}[\mathcal{I}_\alpha f] \|^2_{L_2(\mathbb{Z})}
\leq C \left[ \frac{M_\alpha}{\gamma_\alpha} \| T_\alpha (F[\mathcal{I}_\alpha f] - \mathcal{F}[f]) \|_{L_2(\mathbb{Z})} + \| T_\alpha \mathcal{F}[f] \|_{L_2(\beta \mathbb{B}_2)} \right]^2
\leq C \left[ \frac{M_\alpha^3}{\gamma_\alpha^2 m_\alpha(\beta)} + \frac{M_\alpha}{m_\alpha(\beta)} \right] \| \mathcal{F}[f] \|^2_{L_2(\beta \mathbb{B}_2)}.
\]

In order for \(\|\mathcal{F}[\mathcal{I}_\alpha f] - \mathcal{F}[f]\|_{L_2(\mathbb{R}^d)} \to 0\) as \(\alpha \to \infty\), it is necessary for all three of the ratios above to tend to 0. However, the largest of the ratios is \(\frac{M_\alpha^3}{\gamma_\alpha m_\alpha(\beta)}\). Indeed one obtains this by multiplying \(\frac{M_\alpha}{m_\alpha(\beta)}\) by \(\frac{M_\alpha^2}{\gamma_\alpha^2 m_\alpha(\beta)}\) which is at least 1 by definition. Similarly, \(\frac{M_\alpha^3}{\gamma_\alpha m_\alpha(\beta)} \leq \frac{M_\alpha^2}{\gamma_\alpha m_\alpha(\beta)} \frac{M_\alpha}{\gamma_\alpha} = \frac{M_\alpha^2}{\gamma_\alpha m_\alpha(\beta)}\). Therefore, condition (R2) implies convergence in the \(L_2\) norm.
To show uniform convergence on $\mathbb{R}^d$, use (3) to see that
\[
|\mathcal{I}_a f(x) - f(x)| = \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \int_{Z} (|\mathcal{F}[\mathcal{I}_a f](\xi) - \mathcal{F}[f](\xi)| e^{i(x,\xi)} d\xi + \int_{\mathbb{R}^d \setminus Z} |\mathcal{F}[\mathcal{I}_a f](\xi)| e^{i(x,\xi)} d\xi \right|
\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \||\mathcal{F}[\mathcal{I}_a f] - \mathcal{F}[f]|\|_{L_1(Z)} + \||\mathcal{F}[\mathcal{I}_a f]|\|_{L_1(\mathbb{R}^d \setminus Z)} \right).
\]

Due to the Cauchy-Schwarz inequality and convergence of the corresponding $L_2$ norm, $\||\mathcal{F}[\mathcal{I}_a f] - \mathcal{F}[f]|\|_{L_1(Z)} \to 0$ as $\alpha \to \infty$. A similar calculation to the one above and again using the Cauchy-Schwarz inequality gives that $\||\mathcal{F}[\mathcal{I}_a f]|\|_{L_1(\mathbb{R}^d \setminus Z)} \to 0$, which concludes the proof. $\square$

6. Remarks

Remark 6.1.

It is notable that there are many ways one could choose to periodize the integrals over $\mathbb{R}^d \setminus Z$, and consequently, the condition (I3) could well be formulated differently. For example, if one periodized using the annuli $mZ \setminus (m-1)Z$, then the condition would be that $(m^{-d}N(mZ \setminus (m-1)Z, Z)M_m) \in \ell_1$ once the definition of $M_m$ is modified suitably. However, this modification of $M_m$ essentially counteracts the change in annuli, and so does not give a substantially different condition.

Remark 6.2.

As a product of the proof of Theorem 5.6 one can deduce approximation rates in terms of the parameter $\alpha$ for functions $f \in PW_{\beta B_2}$ in terms of the ratio $\frac{M^2_{P,\alpha}(\beta B_2)}{m_{\alpha}(B_2)}$. In fact, all of the examples from Section 4 exhibit exponential approximation rates in this case.

Remark 6.3.

As mentioned before, the problem of finding Riesz-basis sequences for $L_2(Z)$ is generally quite complex and heavily depends on the geometry of the set $Z$. Zonotopes are Minkowski sums of line segments with one endpoint at the origin, and it is known (see, for example, [1] Theorem 4.1.10) that there are zonotopes satisfying the hypothesis of Theorem 5.6. It was also shown by Lyubarskii and Rashkovskii [9] that for such a zonotope $Z$, $L_2(Z)$ has a Riesz-basis sequence. Their proof is for $d = 2$, but the paper alludes to the general result in higher dimensions (see also the remarks on page 525 of [2]).

Remark 6.4.

It is worth discussing the limiting case briefly. All of Theorems 4.1, 4.5, and 4.6 hold in the case that we let $\delta = \beta = 1$, which is the case that $Z = B_2$. Indeed one needs only look at the end of the proof of Theorem 3.6 and see that the Dominated Convergence Theorem can be applied to show that $\lim_{\alpha \to \infty} \|T_\alpha g\|_{L_2(B_2)} = 0$ for $g \in L_2(B_2)$. However, as mentioned above, the result may be vacuous as it is unknown if there is any Riesz-basis sequence for $L_2(B_2)$. This is the primary reason for the analysis we have done here, to exploit the fact that we know there are Riesz-basis sequences for some convex bodies contained in the Euclidean ball.

Remark 6.5.

For further reading on the interesting problem of finding Riesz-basis sequences, the reader is referred to [3,10,14] for results in one dimension, and [1,2,9,12] for higher dimensions.

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