Closed-form meromorphic solutions of some third order boundary layer ordinary differential equations

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Abstract

We introduce a general third order non-linear autonomous ODE which covers many ODEs coming from boundary layer problems, like the Falkner-Skan equation and the Cheng-Minkowycz equation. Using Wiman-Valiron theory and complex analytic methods recently developed, for the generic cases, it is shown that all their meromorphic solutions must be rational, or rational in one exponential, and then we find all of them explicitly. For a few non-generic cases, some solutions, which are meromorphic or singlevalued, are also obtained. Our results also explain why it is so difficult to obtain new closed-form solutions of the Falkner-Skan equation.

1 Introduction

In this paper, we use complex analytic methods to find meromorphic solutions of an equation which covers many ODEs stemming from boundary layer problems, such

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as the Falkner-Skan equation [19] and the Cheng-Minkowycz equation [7], namely the following third order non-linear ODE

\[ u'''(z) - Bu(z)u''(z) - Au'(z)^2 + \alpha u''(z) + \beta u'(z) + \gamma u(z) + \delta = 0, \quad A, B, \alpha, \beta, \gamma, \delta \in \mathbb{R}. \quad (1) \]

Boundary layer equations [17, 39] play quite an important role in fluid mechanics and engineering. They arise as simplifications of the Navier-Stokes equations for fluids near solid boundaries with sufficiently high Reynolds numbers, and have many applications in industry especially the design of airships and airplanes [35]. Various numerical methods [28, 40] have been adopted to study boundary layer equations, such as the Crank-Nicolson scheme, the Box scheme and their variations (see [28, 36] and references therein). Despite that, it is generally very difficult to construct exact solutions of the boundary layer equations. Fortunately, when we focus on the so-called similarity solutions [21], which have the property that the velocity profiles at different positions around the solid boundaries are the same apart from a scaling, the boundary layer (partial differential) equations can be simplified significantly and in certain cases they may be transformed into ordinary differential equations. In particular, if we consider the steady two-dimensional flow past a wedge of angle \( \pi \lambda \), then through similarity transformations, the boundary layer equations reduce to an ordinary differential equation subject to some boundary conditions

\[ f'''(\eta) + f(\eta)f''(\eta) + \lambda(1 - f'(\eta)^2) = 0, \quad (2) \]

\[ f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \quad (3) \]

which is called the Falkner-Skan equation [19, 30]. Here, \( \eta \) is the similarity variable and \( f(\eta) \) is the similarity stream function. For \( \lambda = 0 \), the boundary layer is around a flat plate parallel to a unidirectional flow with constant velocity, and the equation (2) reduces to the Blasius equation [3, 4, 26]. The famous Falkner-Skan equation (2) has been studied intensively by many people (including mathematicians like Weyl [44] as well as Swinnerton-Dyer and his collaborators [43, 41]) since 1931. One now knows that there exists a critical value \( \lambda_0 = -0.1988 \pm 0.0005 \) defining three regimes of solutions of (2) obeying (3) [19, 22, 42]:

(i) no solution for \( \lambda < \lambda_0 \);
(ii) two solutions for \( \lambda_0 < \lambda < 0 \);
(iii) one solution for \( 0 < \lambda \).

The rigorous treatment of the solvability of the boundary value problem (2) and (3) was given by Weyl [44]. He proved that, for any \( \lambda \geq 0 \), there exists a solution \( f(\eta) \) of (2) and (3) with the property that \( f' \) and \( f'' \) are respectively increasing.
and decreasing on \((0, \infty)\). A direct proof of this result was given by Coppel \[12\] whose argument shows that the equation \(2\) is solvable as well for the boundary conditions

\[ f(0) = a, f'(0) = b, \quad f'(+\infty) = 1, \quad a < 0, b < 0. \tag{4} \]

Apart from that, intensive study has been taken on the properties of solutions of \(2\) with various initial or boundary conditions, ranging from the existence of periodic solutions \[23\] and multiple solutions \[37\], oscillation of solutions \[23, 24\] to the solution dynamics and bifurcations \[43, 41\]. Although the existence of solutions of \(2\) has been verified, very few exact solutions have been derived except for \(\lambda = -1\) whose solutions can be expressed in terms of the parabolic cylinder functions \[13\] or the confluent hypergeometric functions \[45\]. Theorem 1 will explain why it is so difficult to find exact meromorphic solutions for the Falkner-Skan equation \(2\) since 1931 and Remark 3 will give an explicit infinite set of rational numbers for the coefficient \(\lambda\) that one should consider in order to find new exact meromorphic solutions of \(2\).

The similarity transformations are also applicable to boundary layer problems in other physical contexts, and the resulting equations may still be ordinary differential equations. For example, Cheng and Minkowycz \[7\] showed that the governing partial differential equations of free convective boundary layer flow over a vertical flat plate immersed in a porous medium can be reduced to

\[
\begin{cases}
  f'''(\eta) + \frac{a+1}{2} f(\eta)f''(\eta) - af'(\eta)^2 = 0, \quad a(a+1) \neq 0, \\
f(0) = 0, \quad f'(0) = 1, \quad f'(+\infty) = 0,
\end{cases}
\tag{5}
\]

which is now called the Cheng-Minkowycz equation \[32\].

The goal of this paper is to study meromorphic solutions of the third-order differential equation \(1\) which covers both the Falkner-Skan equation \(2\) and the Cheng-Minkowycz equation \(5\). It is known \[5\] that if the first order autonomous ODE

\[ P(w, w') = 0, \tag{6} \]

where \(P\) is a polynomial, has a general solution single-valued around all its singularities which depend on the initial conditions (the so-called Painlevé property \[8\]), then this solution must belong to the class \(W\), which consists of elliptic functions and their degeneracies (rational functions of one exponential \(\exp(kz), k \in \mathbb{C}\) and rational functions of \(z\)). Further, if the second order ODE

\[ w'' = F(w', w, x), \tag{7} \]
with $F$ rational in $w'$ and $w$, analytic in $x$, possesses the Painlevé property, and its general solution has no fixed singularities, then this solution is meromorphic (unless otherwise specified, meromorphic functions refer to functions defined on the whole complex plane $\mathbb{C}$ without singularities other than poles) [34]. Therefore, meromorphic functions are the natural building blocks to construct particular solutions of higher-order ODEs. It is for this reason that we will study meromorphic solutions of the equation (1), and unless otherwise specified, the independent variable $z$ is assumed to be complex.

Our goal is not to determine which equations (1) possess the Painlevé property, a question fully solved by Chazy [6] and Cosgrove [14], but only to determine all their solutions in a particular class, independently of their Painlevé property.

In terms of the construction of meromorphic solutions, Eremenko [18] has proved that there exists a class of autonomous algebraic ODEs such that all their meromorphic solutions (if they exist) belong to the class $W$, and then all of them could be explicitly obtained using either the sub-equation method [10] or the Hermite decomposition [16, 25].

This method has been successfully applied to many differential equations [33, 46], like the Swift-Hohenberg equation [11], whose homoclinic solutions have been intensively studied in [38]. However, it turns out that the arguments based on Nevanlinna theory used in [11, 18] cannot be applied to the equation (1) because it has two terms with top degree (see Theorem A for the definition), and hence we have to make use of generalized Wiman-Valiron theory [2] instead, which is the novelty of this paper.

The structure of this paper is as follows. In Section 2, the main results are presented. For generic cases (see Definition 2.1), all meromorphic solutions of (1) are shown to belong to the class $W$ and then explicitly derived. This conclusion does not hold for non-generic cases and several examples are provided. It is also shown that if one wants to obtain other new meromorphic solutions of the Falkner-Skan equation (2), then one has to consider some specific $\lambda$ such that $\lambda = 1 - 1/r, r \in \mathbb{Z}\{0\}$ or $\lambda = 2 + 6j/[(j - 1)(j - 6)], j \in \mathbb{N}\{0\}\{1, 2, 3, 6\}$ (see Remark 4). A few non-generic cases that pass the Painlevé test are discussed in Section 3. In Section 4, we recall the generalized Wiman-Valiron theory for meromorphic functions with direct tracts [2]. Section 5 is devoted to the proof of the main results, where Eremenko’s method is generalized. The main tools applied in the proof are the generalized Wiman-Valiron theory and Painlevé analysis. We summarize the main results of this paper in Section 6. Finally, a brief summary on Painlevé test is provided in Appendix A.
2 Main Results

We first recall Eremenko’s result on the classification of meromorphic solutions to a class of autonomous algebraic ODEs.

**Theorem A.** [18] If an autonomous algebraic ODE

\[ \sum_{\lambda \in I} w^{i_0}(w')^{i_1} \cdots (w^{(n)})^{i_n} = 0, \]  

(8)

where \( I \) consists of finite multi-indices of the form \( \lambda = (i_0, i_1, \ldots, i_n) \), \( i_k \in \mathbb{N} \), satisfies

1) there is only one top degree term (the degree of each term in (8) is defined as \( |\lambda| = i_0 + i_1 + \cdots + i_n \)),

2) there is no nonnegative integer Fuchs indices (see Appendix A for the definition),

then all its meromorphic solutions belong to the class \( W \).

**Remark 1.** Since equation (1) has two terms with top degree, Theorem A cannot be applied to it.

When applied to equation (1), this theorem yields the following results.

1. For \( AB \neq 0 \), there exist two top degree terms, Theorem A cannot be applied.
2. For \( A = B = 0 \), the equation (1) is linear with constant coefficients.
3. For \( A = 0, B \neq 0 \), the equation (1) obeys conditions i) and ii) of Theorem A, hence all its meromorphic solutions belong to the class \( W \).
4. For \( A \neq 0, B = 0 \), by Wiman-Valiron theory [29], we conclude that any entire solution of (1) must be a polynomial. On the other hand, we can find that equation (1) does not have any polynomial solution with degree greater than 2, while it is easy to construct its polynomial solutions whose degrees do not exceed 2. Therefore, it remains to consider meromorphic solutions \( u \) of (1) with at least one pole \( z_0 \) in \( \mathbb{C} \). By direct computation, we find that \( u \) has the following Laurent series expansion around \( z_0 \)

\[ u = \sum_{n=0}^{+\infty} u_n (z - z_0)^{n-1}, \quad u_0 = -\frac{6}{A}. \]  

(9)

The Fuchs indices of (9) are \(-1, 1, 6\) (corresponding to the arbitrary coefficients \( z_0, u_1, u_6 \)) and the conditions for (9) to exist are \( \alpha = \gamma = 0 \). In this case, equation (1) becomes

\[ f''' - Af'^2 + \beta f' + \delta = 0, \]  

(10)
whose general solution is \[ f(z) = \frac{\beta}{2A}z - \frac{6}{A} \zeta(z - z_0; g_2, g_3) + c_1, \] (11)

where \( \zeta(z; g_2, g_3) \) is the Weierstrass zeta function with

\[
\begin{cases}
  g_2 = -\frac{A\delta}{A^2c} - \frac{\beta^2}{12A}, \\
  g_3 = \frac{A^3}{108} - \frac{A\beta\delta}{36} - \frac{\beta^3}{216},
\end{cases}
\] (12)

and \( z_0, c, c_1 \in \mathbb{C} \) are arbitrary.

In the rest of this paper, we will focus on the “generic” case of equation (1) that is defined as follows.

**Definition 2.1.** The “generic” case of equation (1) is defined to be any case for which \( A, B \in \mathbb{C} \) obey satisfy all of the following three conditions

i) \( AB(A + B) \neq 0 \); (13)

ii) \( \frac{B}{A + B} \notin \mathbb{Z} \setminus \{0\} \); (14)

iii) \( \frac{7A + 8B \pm \sqrt{25A^2 + 16AB - 32B^2}}{2(A + 2B)} \notin \mathbb{N} \cup \{0\} \). (15)

The main results are as follows.

**Theorem 1.** For any generic case, any meromorphic solution of the equation (1)

\[ Au'(z)^2 + Bu(z)u''(z) = u'''(z) + \alpha u''(z) + \beta u'(z) + \gamma u(z) + \delta, \]

where \( AB \neq 0 \), is either a rational function of the form

\[ u(z) = \frac{u_0}{z - z_0} + P(z), \] (16)

where \( P(z) \) is a polynomial of degree at most two, or a simply periodic function of the form

\[ u(z) = \frac{h_0}{e^{kz - \zeta_0}} + c_0, \quad h_0, k, \zeta_0 \in \mathbb{C}^*, c_0 \in \mathbb{C}. \] (17)

This theorem will be proven in Section 5.

**Remark 2.** We note that the solution (16) is a polynomial when \( u_0 \) is zero.

**Corollary 1.** For generic cases (see Theorem 1), all nonconstant meromorphic solutions of the ODE (1) can be explicitly constructed and they are listed in Table 1 (without loss of generality, \( \alpha \) can be set to 0 by a translation and the arbitrary constant \( z_0 \) is absorbed in \( z \) by representing \( z - z_0 \) as \( z \)).
Table 1: All meromorphic solutions of the ODE (1) in the generic case, in which \( \zeta_0 \in \mathbb{C}^\ast, b \in \mathbb{C} \) are arbitrary.

| Nonconstant meromorphic solutions of the ODE (1) | Constraints on the parameters |
|--------------------------------------------------|--------------------------------|
| \( u(z) = \frac{h_0}{e^{kz - \zeta_0}} + c_0 \)   | \( \begin{cases} \gamma = \delta = 0 \\ c_0 = -\frac{Bk}{2\zeta_0 (\beta + k^2)} \\ h_0 = -\frac{Bk}{2\zeta_0 (\beta + k^2)} \\ \beta(A + 2B) + k^2(A - B) = 0 \end{cases} \) |
| \( u(z) = -\frac{\gamma z^2}{4(A - B)} + \frac{\beta z}{2(A - B)} - \frac{6}{z(A + 2B)} \) | \( \begin{cases} \gamma = 0 \\ \beta^2(A - 2B) + 4\delta(A - B)^2 = 0 \end{cases} \) or \( \begin{cases} \gamma \neq 0 \\ A - B = 0 \end{cases} \) |
| \( u(z) = az^2 + bz + c \) | \( \begin{cases} 2A + B \neq 0 \\ a = \frac{\gamma}{2(2A + B)} \neq 0 \\ \beta = 0 \\ c = \frac{(2A + B)(A\beta^2 - \delta)}{2A\gamma} \end{cases} \) or \( \begin{cases} 2A + B = 0 \\ \beta = \gamma = 0 \\ A(b^2 - 4ac) = \delta = 0 \end{cases} \) |
| \( u(z) = az + b \) | \( \begin{cases} \gamma = 0 \\ Aa^2 - \beta a - \delta = 0 \end{cases} \) |

**Remark 3.** Chazy [6] introduced 13 classes of third order ODEs in his classification of certain third-order differential equations with the Painlevé property. For comprehensive discussions on these equations, we also refer the readers to the reference [14] in which the solutions of Chazy equations IX and X are constructed in terms of hyperelliptic functions of genus 2. Nevertheless, equation (1) is not covered by Chazy’s classes except two particular cases: \( A = B, \beta = \gamma = 0 \)’ and \( A = 3, B = 2, \alpha = \beta = \gamma = \delta = 0 \), which correspond to Chazy equations II and III respectively and will be treated in Section 3. This also implies that equation (1) does not possess the Painlevé property except for specific choices of the coefficients.

**Remark 4.** According to Theorem 1 and Corollary 1 if one wants to construct other nongeneric meromorphic solutions of the Falkner-Skan equation (2), then one has to consider some specific \( \lambda \) such that \( \lambda = 1 - 1/r, r \in \mathbb{Z} \setminus \{0\} \) or at least one of \( (8 - 7\lambda \pm \sqrt{\lambda(25\lambda - 16) - 32})/(4 - 2\lambda) \) is a nonnegative integer except 1, 2, 3 and 6, i.e., \( \lambda = 2 + 6j/[\{(j - 1)(j - 6)\}, j \in \mathbb{N} \cup \{0\} \setminus \{1, 2, 3, 6\} \), such as \( \lambda = -2 \) \((j = 4), -11/2 (j = 5), 9 (j = 7) \) or \( 38/7 (j = 8) \), because we will show in Section 3 that (2) admits general solution for \( \lambda = -1 \), which corresponds to \( j = 2, 3 \). In particular, one has to focus on \( \lambda = 1 - 1/r, r \in \mathbb{Z} \setminus \{0\} \) for physically meaningful meromorphic
solutions as $\lambda \pi$ represents the angle of the wedge which requires $-1 < \lambda < 2$. Otherwise, one has to concentrate on the solutions of [2] with more complicated singularities other than poles.

**Remark 5.** For non-generic cases, the conclusion of Theorem [1] does not hold and this is shown through the examples below. However, we do not know what happens in general if one of the conditions [13]–[15] is dropped.

**Example 1.** Let $\alpha = \beta = \gamma = \delta = 0$, $A = -B = 1$ (condition i) fails), then the equation
\[ u''' = (u')^2 - uu'', \]
has an entire solution
\[ u(z) = e^{z-z_0} - 1, \quad z_0 \in \mathbb{C}. \]

**Example 2.** Let $\alpha = \beta = \gamma = \delta = 0$, $A = 3$, $B = nA/(1 - n)$, where $n = -2$ (condition ii) fails), then the equation [11] reduces to the well-known Chazy equation III
\[ u''' = -3(u')^2 + 2uu'', \quad (18) \]
and its only globally meromorphic solution is the rational solution [6, p. 335]
\[ u(z) = c/(z - z_0)^2 - 6/(z - z_0), \quad c, z_0 \in \mathbb{C}, \]
with a double pole at $z_0$.

**Example 3.** Let $\alpha = \beta = \gamma = 0$, $A = B = -2$, then the general solution of
\[ u'''(z) = -2u'(z)^2 - 2u(z)u''(z) - \delta, \]
which has Fuchs indices $j = -1, 2, 3$ (condition iii) fails), is given by $u = w'/w$, where $w$ satisfies
\[ w'' - (\delta z^2/2 + c_1 z + c_2)w = 0, \quad c_1, c_2 \in \mathbb{C}, \]
and hence $u$ can be expressed in terms of the parabolic cylinder functions ($\delta \neq 0$), the Airy functions ($\delta = 0, c_1 \neq 0$), exponentials ($\delta = c_1 = 0, c_2 \neq 0$) or polynomials ($\delta = c_1 = c_2 = 0$).
3 Special Cases with General Solutions

Let us first require equation (11) to pass the Painlevé test (see Appendix A) because it provides necessary conditions for the Painlevé property [8]. It is noted that the indicial equation (see Appendix A) of (1) is

\[(i + 1) \left( i^2 + \frac{-7A - 8B}{A + 2B} i + 6 \right) = 0 \quad (19)\]

and the diophantine equation \(i_1 i_2 = 6\) has four solutions

\[(i_1, i_2) = (-1, -6), (1, 6), (-2, -3), (2, 3)\]

for the respective relations \(B/A = -7/11, 0, -2/3, 1.\) The case \((i_1, i_2) = (-1, -6)\) displays movable multivaluedness (double Fuchs index) while the case \((i_1, i_2) = (1, 6),\) which corresponds to \(B = 0,\) has been solved in Section 2. Hence, only two cases remain to be dealt with.

For \((i_1, i_2) = (2, 3),\) i.e., \(B = A,\) further conditions for equation (11) to pass the Painlevé test are \(\beta = \gamma = 0,\) then equation (11) reduces to a special case of Chazy equation II and admits the first integral

\[u' - \frac{A}{2} u^2 + \frac{\delta}{2} z^2 + k_1 z + k_0 = 0,\]

where \(k_1, k_0\) are the integration constants. Therefore, equation (11) has a singlevalued general solution given by

\[
\begin{align*}
 u &= \begin{cases} 
 - \frac{2}{A} w'_1(z - z_0), & \text{if } \delta \neq 0, \\
 - \frac{2}{A} w'_2(z - z_0), & \text{if } \delta = 0, k_1 \neq 0, \\
 - \frac{2}{A} w_2(z - z_0), & \text{if } k_1 = k_0 = 0, \delta = k_1 = 0, \\
 - \frac{1}{A} z - z_0, & \text{if } \delta = k_1 = k_0 = 0, \\
 \frac{2k \tan[k(z - z_0)]}{A}, & \text{if } k^2 = \frac{A k_0}{2}, \delta = k_1 = 0, k_0 \neq 0,
\end{cases}
\end{align*}
\]

where \(z_0\) is arbitrary and \(w_i, i = 1, 2,\) is the general solution of

\[w'' + \frac{1}{2} A \left( -\frac{\delta z^2}{2} + k_1 z + k_0 \right) w = 0, \quad (21)\]

which can be expressed in terms of the parabolic cylinder functions (\(\delta \neq 0\)) or the Airy functions (\(\delta = 0, k_1 \neq 0\)). Hence, the solution given in [12] of the Falkner-Skan equation (2) with \(\lambda = -1\) is rediscovered.

For \((i_1, i_2) = (-2, -3),\) i.e., \(B = -2A/3,\) by using the perturbative Painlevé method, it has been proved in [9] that a necessary condition for equation (11) to possess the Painlevé property is \(\beta = \gamma = \delta = 0.\) In this case, by a scaling in the independent variable \(z,\) equation (11) reduces to the famous Chazy equation...
III (18) whose general solution is singlevalued and can be expressed in terms of two solutions to the hypergeometric equation. In addition, the general solution of Chazy equation III (18) has a movable natural boundary that is a circle with center and radius depending on the initial conditions.

Remark 6. We have rediscovered a particular solution \[ f(z) = \sqrt{6} \tanh \left( \frac{t}{\sqrt{6}} \right) \] of the Cheng-Minkowycz equation (5) for \( a = -1/3 \), i.e., \( A = B \). This case \((a = -1/3)\) has certain special interest as it is related to a horizontal line source embedded in a porous medium.

Remark 7. For generic values of the three constants of integration, none of the general solutions of (1) is elliptic or degenerate for the cases \( B = 0 \), \( B = -2A/3 \) and \( B = A \).

4 Generalized Wiman-Valiron Theory

Wiman-Valiron theory [29] describes the asymptotic behaviours of transcendental entire functions in certain discs around points of maximum modulus and has found many applications in complex differential equations [29]. It has been generalized to certain class of meromorphic functions by Bergweiler, Rippon and Stallard [2]. In this section, we shall recall some of the main results in [2] as they play an essential role in the proof of Theorem [1].

Definition 4.1 (2). Suppose \( \Omega \) is an unbounded domain in the complex plane \( \mathbb{C} \) such that its boundary consists of piecewise smooth curves and \( \mathbb{C} \setminus \Omega \) is unbounded. Let \( y \) be a complex-valued function whose domain of definition contains the closure \( \overline{\Omega} \) of \( \Omega \). Then \( \Omega \) is called a direct tract of \( y \) if the following two conditions hold

- \( y \) is analytic in \( \Omega \) and continuous in \( \overline{\Omega} \);
- there exists \( R > 0 \) such that \( |y(z)| = R \) for \( z \in \partial \Omega \) while \( |f(z)| > R \) for \( z \in \Omega \).

It will be shown in Section 5 that every transcendental meromorphic function with finitely many poles on the complex plane has a direct tract.

Let \( K > 0 \) and \( y \) be a transcendental meromorphic function on \( \mathbb{C} \) which has a direct tract \( \Omega \). Let

\[ M(r) = M(r, \Omega, y) = \max \{|y(z)| : |z| = r, z \in \Omega\}, \]
then the derivative
\[ a(r) = d \log M(r) / d \log r = \frac{rM'(r)}{M(r)} \]
exists for all \( r > 0 \) outside a countable set \( \Lambda \subseteq (0, \infty) \). By a theorem due to Fuchs [20], if \( \infty \) is not a pole of \( y \), then we have
\[
\frac{\log M(r)}{\log r} \to \infty \text{ and } a(r) \to \infty, \text{ as } r \to \infty, r \notin \Lambda. \quad (22)
\]
Assume \( r_0 = \inf \{|z| : z \in \Omega\} \), and for any \( r > r_0 \), \( z_r \) is chosen such that \( |z_r| = r \) and \( |y(z_r)| = M(r) \).

**Theorem B** ([2]). Let \( y \) be a transcendental meromorphic function on \( \mathbb{C} \) with a direct tract \( \Omega \). Then for every \( \sigma > 1/2 \), there exists a set \( F \subset [1, \infty) \) of finite logarithmic measure such that for \( r \in [r_0, \infty) \setminus F \), the disk
\[
D_r = \{ z : |z - z_r| < ra^{-\sigma}(r) \}
\]
is contained in \( \Omega \). Moreover, we have
\[
y^{(k)}(z) = \left( \frac{a(r)}{z} \right)^k \left( z \frac{z_r}{z_r} \right)^{a(r)} y(z)(1 + o(1)), \quad z \in D_r,
\]
as \( r \to \infty, r \notin F \).

**Lemma 1** ([2]). For any \( \beta > 0 \), we have
\[
(a(r))^\beta = o(M(r)),
\]
as \( r \to \infty \) outside a set of finite logarithmic measure.

## 5 Proof of Theorem [1]

**Lemma 2.** If \( A + B \neq 0 \), then any transcendental meromorphic solution of equation [1] has infinitely many poles.

**Proof.** We prove by contradiction. Suppose \( u \) has finitely many poles on \( \mathbb{C} \), then there exists \( L > 0 \) such that \( u \) is analytic in \( D = \{ z : |z| \geq L \} \). Choose \( R > \max_{|z|=L} |u(z)|. \) Let \( \Omega \) be a component of the set \( \{ z : |u(z)| > R, |z| > L \} \), then \( \Omega \) is a direct tract of \( u \). Apply Theorem [3] to the equation [1] with \( z = z_r \), then, as \( r \to \infty, r \notin F \), we obtain
\[
A\left( \frac{a(r)}{z_r} \right)^2 u(z_r)^2(1 + o(1)) + Bu(z_r)\left( \frac{a(r)}{z_r} \right)^2 u(z_r)(1 + o(1))
\]
\[
= \left( \frac{a(r)}{z_r} \right)^3 u(z_r)(1 + o(1)) + \alpha \left( \frac{a(r)}{z_r} \right)^2 u(z_r)(1 + o(1)) + \beta \left( \frac{a(r)}{z_r} \right) u(z_r)(1 + o(1))
\]
\[
+ \gamma u(z_r) + \delta.
\]
Take the modulus on both sides of equation (23), we then have

\[
\left| (A + B) \left( \frac{a(r)}{z_r} \right)^2 \right| M^2(r)(1 + o(1)) = \left| \left( \frac{a(r)}{z_r} \right)^3 u(z_r)(1 + o(1)) + \alpha \left( \frac{a(r)}{z_r} \right)^2 u(z_r)(1 + o(1)) \\
+ \beta \left( \frac{a(r)}{z_r} \right) u(z_r)(1 + o(1)) + \gamma a(r) u(z_r) + \delta \right| \\
\leq a^3(r) |u(z_r)| + |\alpha| a^2(r) |u(z_r)| + |\beta| a(r) |u(z_r)| \\
+ |\gamma| |u(z_r)| + |\delta| \\
\leq K'a^3(r) M(r)
\]

for some $K' > 0$ and sufficiently large $r \notin F$. After dividing by $M(r)$, we get

\[
\left| (A + B) \left( \frac{a(r)}{z_r} \right)^2 \right| M(r)(1 + o(1)) \leq K'a^3(r). 
\]

According to Lemma 1 for any $\varepsilon \in (0,1)$, when $r \notin F$ is sufficiently large, the inequality

\[
|A + B| \left( \frac{a(r)}{r} \right)^2 M(r)(1 + o(1)) \leq M^\varepsilon(r)
\]

holds, and further by (22), we have $M(r) > r^{1-\varepsilon}$ for all large $r$ and it follows from $a(r) \to \infty$ as $r \notin \Lambda$ that

\[
A + B = 0.
\]

Thus, we get a contradictation.

Corollary 2. If $A + B \neq 0$, then the ODE (1) has no transcendental entire solutions.

Proof of Theorem 1 Let $u$ be a meromorphic solution of (1). If $u$ has no poles in $\mathbb{C}$, then according to Corollary 2 $u$ must be a polynomial. Because of condition ii), by substituting $u$ into the equation (1) and comparing the coefficients of the top degree terms, we conclude that the degree of $P$ is at most two.

From now on, we consider the solution $u$ with at least one pole in $\mathbb{C}$. Assume $z_0$ is a pole of $u$ with the Laurent expansion

\[
u = \sum_{n=0}^{+\infty} u_n(z - z_0)^{n+p}, \quad u_0 \neq 0, -p \in \mathbb{N}.
\]

Because of condition ii), by substituting the above Laurent expansion into equation (1), we find that
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1) \( p = -1, \ u_0 = -\frac{6}{A+2B} \);

2) the Fuchs indices (see Appendix A) of the equation \( (1) \) are \( j = -1, j_2, j_3 \), where
\[
 j_{2,3} = \frac{7A + 8B \pm \sqrt{25A^2 + 16AB - 32B^2}}{2(A + 2B)}.
\]

Therefore, because of condition iii), all the \( u_n, n \geq 1 \), are uniquely determined by \( u_0 \). This implies the existence of only one formal Laurent expansion with a pole at \( z_0 \) that satisfies the equation \( (1) \) (see Appendix A).

Now we distinguish two cases.

**Case 1.** \( u \) is a nonconstant rational function.

If \( u \) has at least two poles, say at \( z_1 \) and \( z_2 (z_1 \neq z_2) \), then both of \( u_1 = u(z - z_0 + z_1) \) and \( u_2 = u(z - z_0 + z_2) \) are solutions of \( (1) \) with a pole at \( z_0 \).

As there is a unique Laurent expansion around \( z_0 \), we must have \( u_1 \equiv u_2 \) on \( \mathbb{C} \), which implies \( \forall z \in \mathbb{C}, u(z) = u(z + z_2 - z_1) \), and \( u \) is periodic. This is a contradiction because a nonconstant rational function cannot be periodic.

If \( u \) has only one pole and condition ii) is enforced, then
\[
u(z) = \frac{u_0}{z - z_0} + P(z), \tag{24}
\]
where \( P(z) \) is a polynomial. To further prove \( \deg(P) \leq 2 \), one may substitute \( (24) \) into the equation \( (1) \) and consider the coefficient of the term with top degree in \( z \).

**Case 2.** \( u \) is a transcendental meromorphic function.

According to Lemma \( (2) \) \( u \) has infinitely many poles on \( \mathbb{C} \). With the same argument as that in Case 1, we conclude that \( u \) is periodic. As the set of all poles of \( u \) is \( \{ z_0 + w | w \in \Gamma \} \), where \( \Gamma \) is a non-trivial discrete subgroup of \( (\mathbb{C}, +) \) \( [27, \text{p. 57}] \), we conclude that \( u \) is either a doubly periodic (elliptic) function or a simply periodic function.

If \( u \) is elliptic, then using the same argument as in Case 1 it must have only one pole in the fundamental parallelogram and this pole is known to be simple. This is impossible because the sum of residues of all poles inside the fundamental parallelogram of an elliptic function is zero.

Hence \( u \) is simply periodic, and again it must have only one pole in the period stripe and this pole is simple. Then \( u \) can be represented as \( h(e^{kz}) \), where \( k \in \mathbb{C}^* \) and \( h \) is a meromorphic function on \( \mathbb{C}^* \) which has only one simple pole on \( \mathbb{C}^* \).
Let us now prove by contradiction that \( h \) is rational. Suppose \( h \) has an essential singularity at infinity. If we let \( \zeta = e^{kz} \), the equation \([1]\) becomes
\[
A\zeta^2 h'^2 + Bh \left( \zeta^2 h'' + \zeta h' \right) = k\zeta^3 h(3) + 3k\zeta^2 h'' + \left( \frac{k^2 + \beta}{k} \right) \zeta h' + \frac{\gamma}{k^2} h + \frac{\delta}{k^2}, (25)
\]
where \( \alpha \) is set to 0 by a translation in \( u \) (or \( h \)). As \( h \) has only one pole in \( \mathbb{C}^* \), then there exists \( L > 0 \) such that \( h \) is analytic in \( D = \{ \zeta \in \mathbb{C} : |\zeta| > L \} \). Let \( R = \max_{|\zeta|=L} |h(\zeta)| \). Since \( h \) has an essential singularity at infinity, by the big Picard theorem, there exists some \( \xi_0 \) with \( |\xi_0| > L \) such that \( |h(\xi_0)| > R \), so the set \( S = \{ \xi \in \mathbb{C} : |h(\xi)| > R, |\xi| > L \} \) is non-empty. Let \( \Omega \) be a component of the set \( S \). Since \( h \) has an essential singularity at infinity, \( \mathbb{C}\setminus\Omega \) must be unbounded. We claim that \( \Omega \) is also unbounded and hence \( \Omega \) is a direct tract of \( h \). If \( \Omega \) is bounded, then \( |h(\xi)| = R \) on \( \partial\Omega \) while \( |h(\xi)| > R \) in \( \Omega \) which contradicts the maximum modulus principle. Apply Theorem [13] to the equation \([25]\) with \( \zeta = \zeta_r \), then, as \( r \to \infty, r \not\in F \), we obtain
\[
A\zeta_r^2 \left( \frac{a(r)}{\zeta_r} \right)^2 h(\zeta_r)^2 (1 + o(1)) + Bh(\zeta_r) \left[ \zeta_r^2 \left( \frac{a(r)}{\zeta_r} \right)^2 h(\zeta_r) + \zeta_r \left( \frac{a(r)}{\zeta_r} \right) h(\zeta_r) \right] (1 + o(1)) = k\zeta_r^3 \left( \frac{a(r)}{\zeta_r} \right)^3 h(\zeta_r)(1 + o(1)) + 3k\zeta_r^2 \left( \frac{a(r)}{\zeta_r} \right)^2 h(\zeta_r)(1 + o(1)) + \frac{(k^2 + \beta)}{k} \zeta_r \left( \frac{a(r)}{\zeta_r} \right) h(\zeta_r)(1 + o(1)) + \frac{\gamma}{k^2} h(\zeta_r) + \frac{\delta}{k^2}.
\]
With similar argument as in the proof of Lemma [2], for any \( \varepsilon \in (0,1) \), when \( r \not\in F \) is sufficiently large, the inequality
\[
|A + B| a^2(r) M_h(r)(1 + o(1)) \leq M_h^\varepsilon(r)
\]
holds, where \( M_h(r) = \max\{|h(\zeta)| : |\zeta| = r, \zeta \in \Omega \} \), and further by \([22]\), we have
\[
A + B = 0,
\]
and thus, we get a contradiction.

When \( h \) has an essential singularity at \( \zeta = 0 \), we may let \( \eta = e^{-kz} \) and consider the function \( g(\eta) = h(1/\eta) \). Then \( g \) is a meromorphic function on \( \mathbb{C}^* \) with an essential singularity at infinity and again similar arguments as above lead to a contradiction. As a consequence, neither 0 nor infinity is an essential singularity of \( h \), and hence \( h \) is a rational function of the form
\[
h(\zeta) = \frac{h_0}{\zeta - \zeta_0} + P_1(\zeta) + P_2(1/\zeta), \quad h_0, \zeta_0 \in \mathbb{C}^*,
\]
where \( P_1 \) and \( P_2 \) are polynomials, and \( u \) is expressed as

\[
u(z) = h(e^{kz}) = \frac{h_0}{e^{kz} - \zeta_0} + P_1(e^{kz}) + P_2(e^{-kz}), \quad k, h_0, \zeta_0 \in \mathbb{C}^*.
\]

Denote by

\[
P_1(w) = c_nw^n + \cdots + c_1w + c_0, \quad P_2(w) = d_mw^m + \cdots + d_1w + d_0,
\]

where \( c_n, d_m \in \mathbb{C}^* \) and \( n, m \in \mathbb{N} \cup \{0\} \) are the degrees of \( P_1 \) and \( P_2 \) respectively.

Assume \( m \geq 1 \), then as \( \zeta \to 0 \), from (25), we have

\[
A\zeta^2 \left( -m \frac{d_m}{\zeta^{m+1}} \right)^2 + B \frac{d_m}{\zeta^m} \left[ \zeta^2 m(m+1) \frac{d_m}{\zeta^{m+2}} + \zeta (-m) \frac{d_m}{\zeta^{m+1}} \right] + O(\zeta^{-2m+1}) = O(\zeta^{-m})
\]

and it can be simplified as

\[
m^2 d_m^2 (A + B) \zeta^{-2m} + O(\zeta^{-2m+1}) = O(\zeta^{-m})
\]

which contradicts to \( A + B \neq 0 \). Hence, \( m = 0 \) and \( P_2 \) is a constant.

Suppose \( n \geq 2 \), then when \( \zeta \) tends to infinity, from (25), we have

\[
A\zeta^2 (nc_n \zeta^{n-1})^2 + Bc_n \zeta^n \left[ \zeta^2 n(n-1)c_n \zeta^{n-2} + \zeta nc_n \zeta^{n-1} \right] + O(\zeta^{2n-1}) = O(\zeta^n)
\]

which implies that

\[
n^2 c_n^2 (A + B) \zeta^{-2n} + O(\zeta^{2n-1}) = O(\zeta^n).
\]

Again, it contradicts to \( A + B \neq 0 \). Similarly, one may also show that \( n \neq 1 \),

so we conclude that \( P_1 \) is also a constant and \( u \) is expressed as

\[
u(z) = \frac{h_0}{e^{kz} - \zeta_0} + c_0, \quad h_0, k, \zeta_0 \in \mathbb{C}^*, c_0 \in \mathbb{C}.
\]

This completes the proof of Theorem 1.

### 6 Conclusion

In this paper, we studied the third order ODE (1) which includes the Falkner-Skan equation and the Cheng-Minkowycz equation as special cases. For the generic case (see Definition 2.1), all meromorphic solutions of this equation were derived by using complex analytic methods. Certain non-generic cases of equation (1) with the Painlevé property were displayed as well. Our results show that, to find other closed-form solutions of equation (1) in the future, one can either study meromorphic solutions of the remaining non-generic cases or focus on non-meromorphic solutions which are expected to have more complicated singularities than poles.
DECLARATION OF COMPETING INTEREST

The authors declare that there is no competing interest.

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In this appendix, we give a brief summary on Painlevé test.

Let $I = (i_0, i_1, \ldots, i_n), i_k \in \mathbb{N} \cup \{0\}, 0 \leq k \leq n$ and

$$H(y, y', \ldots, y^{(n)}) = \sum_{I \in \Lambda} c_I y^{i_0}(y')^{i_1} \cdots (y^{(n)})^{i_n}, y = y(z), c_I \in \mathbb{C} \setminus \{0\}.$$ 

If $y = (z - z_0)^p, -p \in \mathbb{N}$, then

$$H(y, y', \ldots, y^{(n)}) = \sum_{I \in \Lambda} C_I (z - z_0)^{\alpha_I},$$

where $C_I \in \mathbb{C}, \alpha_I = i_0 p + i_1 (p - 1) + \cdots + i_n (p - n)$.

Next, let $A$ be the set of those negative integers $p$ such that $\min_{I \in \Lambda} \alpha_I$ is attained by at least two $I$’s. For each $p \in A$, denote by $\Lambda' = \{I' \in \Lambda | \alpha_{I'} = \min_{I \in \Lambda} \alpha_I \}$ and
then we define the **dominant terms** for each \( p \in A \) to be

\[
\hat{E} = \sum_{I \in \Lambda'} c_I y_0^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n}.
\]

Suppose \( u(z) = \sum_{n=0}^{+\infty} u_n (z - z_0)^{n+p} (u_0 \neq 0, -p \in \mathbb{N}) \) with a pole at \( z = z_0 \) is a meromorphic solution of

\[
H(y, y', \cdots, y^{(n)}) = 0.
\] (26)

Then if we plug \( y = u(z) \) into \( H \), we will get an expression of the form

\[
\hat{E} = \sum_{j=0}^{+\infty} E_j \chi^j + p = 0, \quad \chi = z - z_0, E_j \in \mathbb{C}.
\]

Since \( y = u(z) \) is a solution of \( H = 0 \), we must have \( E_j = 0 \), for all \( j \in \mathbb{N} \).

On the other hand, for \( j = 1, 2, \ldots \), we can express \( E_j \) as:

\[
E_j \equiv P(u_0; j) u_j + Q_j(\{u_l|l < j\}),
\] (27)

where \( P(u_0; j) \) is a polynomial in \( j \) determined by \( u_0 \) and \( Q_j \) is a polynomial in \( j \) with coefficients in \( u_l(l < j) \). In fact, it is known that [15] (see also [8, p. 15])

\[
P(u_0; j) = \lim_{\chi \to 0} \chi^{-j-q} \hat{E}'(u_0 \chi^p) \chi^{j+p},
\] (28)

where \( \hat{E}'(u) \) is defined by

\[
\hat{E}'(u)v := \lim_{\lambda \to 0} \frac{\hat{E}(u + \lambda v) - \hat{E}(u)}{\lambda}.
\] (29)

In order to have \( E_j = 0 \) for all \( j \in \mathbb{N} \), we must have for each \( j \), either

1) \( u_j \) is uniquely determined by \( P(u_0; j) \) and \( Q_j \), or

2) both \( P(u_0; j) \) and \( Q_j \) vanish,

otherwise there is no meromorphic function satisfying \( H(y, y', \cdots, y^{(n)}) = 0 \).

Therefore if the polynomial \( P(u_0; j) \) in \( j \) does not have any nonnegative integer root, then each \( u_j \) is uniquely determined by \( P(u_0; j) \) and \( Q_j \).

**Definition 6.1.** The zeros of \( P(u_0; j) \) are defined to be the **Fuchs indices** of the equation \( H(y, y', \cdots, y^{(n)}) = 0 \) and the **indicial equation** is defined as \( P(u_0; j) = 0 \).

**Definition 6.2.** The ODE \( (26) \) is said to pass the Painlevé test if all its Fuchs indices are distinct integers, and at every positive integer Fuchs index \( j \), the condition \( Q_j = 0 \) is obeyed.