Research Article

New Topological Approaches to Generalized Soft Rough Approximations with Medical Applications

Mostafa K. El-Bably1, Muhammad I. Ali2, and El-Sayed A. Abo-Tabl3,4

1Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt
2Mathematics Department, Islamabad Model College for Boys, G-11/1, Islamabad, Pakistan
3Mathematics Department, Faculty of Science, Assiut University, Assiut, Egypt
4Mathematics Department, College of Science and Arts, Methnab, Qassim University, Buridah, Saudi Arabia

Correspondence should be addressed to Mostafa K. El-Bably; mkamel_bably@yahoo.com

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1.Introduction

Mathematical modeling for the vagueness and uncertainty of data has many different methods, for instance, rough set theory [1], fuzzy set theory [2], soft set theory [3], and topology [4]. Pawlak [1] introduced the classical rough sets model in the early eighties to study vagueness of data, which originate from daily life situations. The key of this methodology is an equivalence relation which is constructed from the data of an information system. In general, it is very difficult to find an equivalence relation in such data. Therefore application of this technique is very limited. Therefore authors relaxed the condition of equivalence relation by some more general relations such as similarity relations (reflexive and symmetric) [5, 6], preorder relations (reflexive and transitive) [7], reflexive relations [8], general binary relations [8–14], topological approaches [15, 16], and coverings [17–19].

Soft set theory is another mathematical model to deal with uncertainty, when data is collected from real-life situations. This concept was introduced by Molodtsove [3]. This theory has applications in many fields, for instance, game theory, operations research, integration of Riemann, and measurement theory [3]. Recently, scientists and researchers have shown their inclination to the idea of soft sets to apply it in numerous areas. For more information about this theory and its applications, we refer the reader to the references (soft set theoretical concepts [20, 21], soft sets and soft topological spaces [22–24], soft rough sets and their applications [25–28], and medical applications of soft sets and their extensions [29–35]).

In rough set theory [1], basic requirement is to have an equivalence relation among the elements of the set under consideration. But in daily life situations it is not easy to find such an equivalence relation. Perhaps this limitation is
associated with rough set theory due to the lack of parameterization tools. The idea of soft rough sets was initiated and studied by Feng et al. in [24] which are very useful in intelligent systems. The concept of the lower (resp., upper) approximation of this theory is particularly useful to extract knowledge hidden in an information system. Decision-making has a crucial part in our daily life, and this method produces the best alternate among dissimilar selections. Chen et al. [34] proposed the choice values of objects in a soft set and considered how to use this notion to address decision-making problems. In [35], Roy and Maji generalized this method for new decision-making problems. There are several subsequent advances after Maji et al.’s work, such as the uni-int decision-making using soft set theory [36]; Jha et al.’s [37] neutrosophic soft set notion in decision-making problems for stock trending analysis, and medical applications [38].

Feng et al. replaced the classes of the equivalence relation by parameterizing subsets of a subset of the universe to define its approximations. In fact, Feng et al. have succeeded in proving that Pawlak’s rough set model is a specialization of the soft rough set as shown by Theorem 4.4 and Theorem 4.5 in [24]. It is worth noting that the concept of full soft sets deserves special attention for both theoretical and practical reasons. Theoretically, some typical properties of Pawlak’s rough sets hold for soft rough sets if and only if the underlying soft set in the soft approximation space is full. Pragmatically, it is justifiable to consider full soft sets in real-life applications. In fact, if a soft set is not full, it means that the available parameters are insufficient, and there exists at least one object which cannot be described by any of the parameters in the given soft set. With the help of soft rough approximations, some equivalent characterizations of full soft sets were given in [24]. In this paper, a new technique is given to define lower and upper approximations of a set with the help of topology generated by the given soft set; this is known mathematically as the notion of topological soft rough sets ($T_{SR}$ - sets).

The main contribution in the existing work is to present another model for soft rough sets without any restrictions and satisfy the characteristics of Pawlak’s rough sets. In other words, we propose a method for modifying soft rough sets from a topological point of view, so a new link between soft sets and general topology is proposed.

First, we discuss the concept of the topology of all definable sets in rough set theory [1] and in soft rough sets [24]. Accordingly, we are able to find the next very interesting questions:

What is the probability that a subset of the universe $U$ may be a definable set?
What is the probability that the lower approximation of a nonempty subset of $U$ may be an empty set?
What is the probability that the upper approximation of a proper subset of $U$ may be $U$?

Secondly, a general topology is generated from the soft set to modify and generalize soft rough sets proposed in [24]. The suggested techniques extend the way for more applications of the general topology in soft rough sets theory. In fact, we use the image of parameters as a subbasis for a unique topology generated by a soft set, denoted by $T_{SR}$. New generalized soft rough approximations, called “topological soft rough approximations” (briefly, $T_{SR}$-approximations), are defined. It is shown that accuracy of proposed technique is higher than soft rough sets, due to reduction of boundary region. The importance of proposed approximations is clear from the fact that these not only reduce the boundary region but also satisfy basic properties similar to rough sets. Several comparisons among the present method and the preceding one [34] are obtained. Numerous examples are suggested to exemplify the relations between the topological soft rough sets and soft rough sets.

Finally, some medical applications in the medical diagnosis of heart failure problems [39] are introduced. These applications illustrate the importance of the suggested methods in real-life problems. In fact, we apply a topological reduction for data set comprising the effect of five indicators for twenty patients with heart failure disease. Accordingly, we can identify the core factors of the heart failure diagnosis. A comparison has been made between proposed technique and some already existing in the literature which shows the usefulness of proposed technique. Two algorithms are given based on proposed technique. The proposed algorithms are tested on hypothetical data for the purpose of comparison with already existing methods.

2. Basic Concepts

The current section is devoted to present some elementary definitions and consequences that are applied through paper are mentioned.

2.1. Topological Space. A topology [4] of a set $U$ is defined by the collection $\tau$ of subsets of $U$ which fulfills the following three axioms:

$$(T1) \, \phi, \, U \in \tau.$$ $$(T2) A \text{ finite intersection of subsets of } \tau \text{ is a member in } \tau.$$ $$(T3) \text{ An arbitrary union of subsets of } \tau \text{ is a member in } \tau.$$ 

We call a pair $(U, \tau)$ “topological space” or “space” and the members of $U$ “points” of $\tau$, and the subsets of $U$ that belong to $\tau$ are said to be “open” sets and the complements of the open sets are called “closed” sets in the space. The collection of all closed sets denotes $\tau'$.

An interior $\text{int}(A)$ (resp., closure $\text{cl}(A)$) of a subset $A$ is given by a union of all open sets contained in $A$ (resp., intersection of all closed sets that contain $A$), formally:

$$\text{int}(A) = \cup \{V \in \tau: \, V \subseteq A\},$$
$$\text{cl}(A) = \cap \{Z \in \tau': \, A \subseteq Z\}.$$ (1)

A class $B \subseteq \tau$ is said to be a basis for $\tau$ if all nonempty open subset of $U$ can be represented as a union of subfamily of $B$. 
Evidently, any topology can have numerous bases, but the basis \( B \) generates a unique topology \( \tau \).

Each union of elements of \( B \) belongs to \( \tau \); therefore a basis of \( \tau \) entirely decides \( \tau \).

A family \( S \subseteq \tau \) is said to be a subbasis for a topological space \((U, \tau)\) if the collection of all finite intersections of \( S \) represents a basis for \((U, \tau)\).

For any class \( S \) of subsets of \( U \), \( S \) represents a subbasis for a unique basis \( B \) which generates a unique topology \( \tau \) on \( U \) such that for each \( i \in I \)

\[
B = \bigcap_{i} \{ S_i : S_i \in S \}, \\
\tau = \bigcup_{i} \{ B_i : B_i \in B \}.
\]

(2)

2.2. Pawlak Rough Set Theory. The current subsection presents some elementary notions pertaining to rough sets given by Pawlak [1].

**Definition 1.** [1] Consider \( U \) is a finite set called universe, and \( R \) is an equivalence relation on \( U \); we symbolize \( U/R \) to represent the collection of all equivalence classes of \( R \) and \( [s]_R \) to symbolize an equivalence class in \( R \) that contains an element \( s \in U \). Then, the pair \( A_R = (U, R) \) is said to be Pawlak’s approximation space and for any \( L \subseteq U \), we propose the lower and upper approximation of \( L \) by \( \overline{R}(L) = \{ s \in U : [s]_R \subseteq L \} \) and \( \overline{R}(L) = \{ s \in U : [s]_R \cap L \neq \phi \} \), respectively. Moreover, \( L \) is called a rough set if \( \overline{R}(L) \neq \overline{R}(L) \). Otherwise, it is an exact set.

**Definition 2.** [1] Consider \( A_R = (U, R) \) is Pawlak approximation space and \( L \subseteq U \). Therefore, the boundary, positive, and negative regions and the accuracy of approximations of \( L \subseteq U \) are given, respectively, by

\[
\begin{align*}
BND_R(L) &= \overline{R}(L) - \overline{R}(L), \\
POS_R(L) &= \overline{R}(L), \\
NEG_R(L) &= U - \overline{R}(L), \\
\mu_R(L) &= \frac{|\overline{R}(L)|}{|\overline{R}(L)|},
\end{align*}
\]

where \( \overline{R}(L) \neq \phi \).

Thus, the probability that a subset of the universe \( U \) is a rough set is \( 1 - \mathcal{P} \).

Now, for the answer of the second question first the following result must be considered.

**Theorem 2.** If \( X \) is a subset of \( U \). Then \( \overline{R}(X) \) is an empty set if and only if \( X \) does not contain any nonempty element of \( \tau \).

Properties associated with rough sets can be seen in [1].

It is well known that the set of all definable subsets of the approximating space \((U, R)\) gives rise to a clopen topology \( \tau_c \) [8]. In this paper first, we will study how this topology is obtained and why in this topology each open set is closed as well.

As \( (U/R) = \{ [s]_R : s \in U \} \), now, for each \( A, B \in (U/R) \), \( A \cap B = \emptyset \) and \( U = A \cup B \). Thus \((U/R)\) may act as a basis for a \( \tau \) topology on \( U \).

**Theorem 1.** If the pair \((U, R)\) is Pawlak approximation space, then, \( \tau_c = \tau \).

**Proof.** Let \( A \in \tau_c \). Then \( A \) is a definable set, so \( A = A_1 \cup A_2 \cup \cdots \cup A_n \) where \( A_1, A_2, \ldots, A_n \in (U/R) \). Hence \( A \in \tau \). That is \( \tau_c \subseteq \tau \). Conversely, every \( B \in \tau \) is union of some elements of \((U/R)\), which are definable. Since union of definable sets is again definable, \( B \) is definable. This means \( B \in \tau_c \). So \( \tau \subseteq \tau_c \) as required. \( \square \)

Theorem 1 explains that topology of definable sets in any Pawlak’s approximation space is produced by the elements of the set \((U/R)\). In this topology every open set is closed because complement of any subset in the basis \((U/R)\) of this topology is the union of all remaining subsets.

Study of topology constructed by definable sets helps us to answer some very interesting questions such as the following:

What is the probability that a subset of \( U \) may be a definable set?

What is the probability that a nonempty subset of \( U \) has an empty lower approximation?

What is the probability that a proper nonempty subset of \( U \) has upper approximation equal to \( U \)?

Answer to the first question is a bit simple and the formula to find the probability that a subset of \( U \) may be a definable set is given as follows:

\[
\mathcal{P} = \frac{|\tau|}{2^n} \quad \text{where } |\tau| \text{ is the cardinality of } \tau \text{ and } n \text{ represents a number of elements in } U.
\]

**Proof.** Let \( \overline{R}(X) = \emptyset \). Then, by definition, there does not exist any \( x \in X \) such that \([x]_R \subseteq X\). This implies \([x]_R \notin X\), for each \( x \). Therefore, for each \( x \in X \), \([x]_R \notin X\). That is, no element of \( \tau \) is contained in \( X \). Conversely, let there exist some \( x \in X \) with \( x \in \overline{U}_x \in \tau \) such that \( \overline{U}_x \subseteq X \). This indicates \( x \in U \setminus \overline{U}_x \subseteq X \). Then by definition \( \overline{R}(X) = \cup_{x \in U} \overline{U}_x \neq \emptyset \), which is a contradiction, and therefore, the subset \( X \) does not contain any nonempty element of \( \tau \). \( \square \)
Now in any Pawlak approximation space, the probability $\mathcal{P}_\emptyset$ that lower approximation of a subset is an empty set can be obtained by the following formula:

$$\mathcal{P}_\emptyset = \frac{1 + N_X^r}{2^n},$$

where $N_X^r$ represents a number of subsets of $U$ which does not contain any nonempty element of $\tau$ and $n$ is the number of elements in $U$.

To find the answer to the last question, we may have to consider the following result.

**Theorem 3.** Let $X$ be a nonempty subset of $U$. Then $R(X) = U$ if and only if the subset $X$ intersects with every nonempty element of $\tau$.

**Proof.** Let $\overline{R}(X) = U$. Then, by definition, there does not exist any $x \in X$ such that $[x]_R \cap X = \emptyset$. Since nonempty elements of $\tau$ are union of some classes $[x]_R \in (U/R)$. As $x \in \bigcup [x]_R = U$, $X$ intersects with every $[x]_R$. Consequently, it intersects with every nonempty element of $\tau$. Conversely, let there exist some nonempty $\overline{U}_x \in \tau$ containing some $x \in U$ such that $\overline{U}_x \cap X = \emptyset$. As $\overline{U}_x$ is the union of some elements of $U/R$, there exists some class $[x]_R \in (U/R)$ such that $[x]_R \cap X = \emptyset$. So $[x]_R \not\in \overline{R}(X)$, which results in $\overline{R}(X) \neq U$, a contradiction; hence, $X$ intersects with every nonempty element of $\tau$.

Further, in any Pawlak approximation space, the probability $\mathcal{P}_U$ that the upper approximation of a subset is $U$ may be obtained by the following formula:

$$\mathcal{P}_U = \frac{N_{NX}^r}{2^n},$$

where $N_{NX}^r$ is the number of subsets of $U$ which intersect with every nonempty element of $\tau$, and $n$ represents the number of elements in $U$.

**Example 1.** Consider $U = \{1, 2, 3, 4\}$ is a set and $R$ represents an equivalence relation on $U$, such that $(U/R) = \{(1), (2, 3), (4)\}$; then $(U, R)$ represents Pawlak’s approximation space. Now $U/R$ can be a basis for a topology $\tau$ on $U$. Let us write the topology $\tau$ generated by $U/R$ as follows: $\tau = \{\emptyset, U, \{1\}, \{2, 3\}, \{4\}, \{1, 2, 3\}, \{1, 4\}, \{2, 3, 4\}\}$. The only definable subsets of $U$ are all elements of $\tau$. Now

$$|\tau| = \text{Number of elements in } \tau = \text{Number of definable subsets of } U.$$ (7)

Total number of subsets of $U = 2^4 = 16$.

Thus, the probability that a subset of $U$ is definable is given by

$$\frac{|\tau|}{2^n} = \frac{8}{16} = 0.5.$$ (8)

Next $\emptyset$, $\{2\}$, $\{3\}$ are the only subsets of $U$ which do not contain any nonempty element of $\tau$. Therefore, their lower approximation is empty.

$$\mathcal{P}_\emptyset = \frac{N_X^r}{2^n} = \frac{3}{2^4} = \frac{3}{16}.$$ (9)

Further the subsets $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$ are the only subsets of $U$, which intersect with every nonempty element of $\tau$. Therefore their upper approximation is $U$ and then

$$\mathcal{P}_U = \frac{N_{NX}^r}{2^n} = \frac{3}{2^4} = \frac{3}{16}.$$ (10)

**2.3. Soft Set Theory and Soft Rough Sets**

**Definition 3** (see [3]). Consider $U$ to be a set of items and $E$ to be a finite set of certain parameters in relative to the objects in $U$. Parameters represent attributes or characteristics of $U$ objects. A ‘soft set’ on $U$ is the pair $(F, A)$, where $A \subseteq E$, $P(U)$ symbolize the power set of $U$, and $F$ represents the map $F: A \rightarrow P(U)$. On the other hand, a soft set over $U$ is a parameterized collection of subsets of $U$. For $e \in A$, $F(e)$ represents the set of $e$-approximate elements of a soft set $(F, A)$. Note that, sometimes a soft set is indicated by $F_A$ and expressed as a set of ordered pairs $F_A = \{(e, F(e)) : e \in A\}$.

**Definition 4** (see [24]). Consider $F_A$ is a soft set on $U$. Thus, the pair $A = (U, F_A)$ is said to be a soft approximation space. Established on a soft approximation space $A$, we give the ”soft $A_\uparrow$ lower and soft $A_\downarrow$ upper” approximations of $X \subseteq U$, respectively, by

$$\overline{S}(X) = \{u \in U : \exists e \in A, [u \in F(e) \cap X]\},$$

$$\overline{S}(X) = \{u \in U : \exists e \in A, [u \in F(e), F(e) \cap X \neq \emptyset]\}.$$ (11)
Proposition 2 (see [24]). Suppose that $F_A$ is a soft set on $U$ and $A_s = (U, F_A)$ is a soft approximation space, then, for each $X \subseteq U$:

\[
\begin{align*}
S(X) &= \bigcup_{e \in A} \{ F(e) : F(e) \subseteq X \}, \\
\overline{S}(X) &= \bigcup_{e \in A} \{ F(e) : F(e) \cap X \neq \emptyset \}.
\end{align*}
\]

Properties associated with soft rough sets can be gotten in [34].

Definition 5 (see [24]). Suppose that $F_A$ is a soft set on $U$ and $A_s = (U, F_A)$ a soft approximation space. Then, $F_A$ is said to be a "full soft set" if $F_A = \bigcup_{e \in A} F(e)$. It is clear that if $F_A$ is a full soft set, then $\forall x \in U, \exists e \in A$ such that $x \in F(e)$.

Proposition 2 (see [24]). If $F_A$ is a full soft set on $U$ and $A_s = (U, F_A)$ is a soft approximation space, then, the subsequent conditions are true:

(i) $\overline{S}(U) = \overline{S}(U) = U$

(ii) $X \subseteq \overline{S}(X), \forall X \subseteq U$

(iii) $\overline{S}(\{x\}) \neq \emptyset, \forall x \in U$

Now, we present and study the idea of the topology of all definable sets generated by soft set $F_A$. Moreover, we illustrate the condition in which this topology is well defined. Now, again, we can find the answers to the questions which are related to the probability associated with subsets of a set in soft rough sets.

The following example illustrates that the condition "full soft set" in Proposition 2 is necessary to achieve the properties (i)-(iii).

Example 2. Consider $A_s = (U, F_A)$ to be a soft approximation space, such that $U = \{a, b, c, d\}$, and $A_s$ to be a soft set on $U$ where $A = \{e_1, e_2, e_3\}$ and $F_A = \{(e_1, \{a\}), (e_2, \{b, c\}), (e_3, \{a, b\})\}$. Then, it is clear that $F_A$ is not full soft set and thus we have $\overline{S}(U) = \overline{S}(U) = \{a, b, c\} \neq U$. Also, if $X = \{d\}$, then $\overline{S}(X) = \overline{S}(X) = \phi \neq X$.

Theorem 4. If $F_A$ is a full soft set on $U$ and $A_s = (U, F_A)$ is a soft approximation space, then, the collection $\tau_D = \{X \subseteq U : \overline{S}(X) = \overline{S}(X)\}$ is a quasi-discrete topology on $U$.

Proof. Since $F_A$ is a soft set on $U$, by using the properties of the soft approximations in [24], we get

(71) $\overline{S}(U) = \overline{S}(U)$ and $\overline{S}(\phi) = \overline{S}(\phi)$. Therefore, $U, \phi \in \tau_D$.

(72) Let $\{X_i : i \in I\}$ be a class of members in $\tau_D$, Then, $\cup_{i \in I} X_i \in \tau_D$.

(73) Let $\{X_i : i = 1, 2, 3, \ldots, n\}$ be a class of finite members in $\tau_D$. Then, $\cap_{i = 1}^n X_i \in \tau_D$. Now, we need to prove that $\tau_D$ is a quasi-discrete as follows: Let $X \in \tau_D$; then $\overline{S}(X) = \overline{S}(X)$. By taking the complement to both sides, we obtain $\overline{S}(\overline{S}(X)) = \overline{S}(\overline{S}(X))$ and this implies $\overline{S}(X^c) = \overline{S}(X^c)$. Thus, $X^c \in \tau_D$.

Remark 1. According to Proposition 2, the condition "full soft set" in the above theorem is necessary to construct the topology $\tau_D$. If the soft set is not full then $U$ may not be a definable set. Moreover there may be a subset which has same lower and upper approximations but is not definable.

For any soft approximation space $A_s = (U, F_A)$, such that $F_A$ be a full soft set, the probability of a subset of $U$ is proposed by

$\mathcal{P} = (|\tau_D|/2^n)$, where $|\tau_D|$ is the cardinality of $\tau_D$ and $n$ represents a number of elements in $U$.

Thus, the probability that a subset of $U$ is a rough set is $1 - \mathcal{P}$.

Now, for the answer of the second question, first the following results must be considered.

Lemma 1. Suppose that $F_A$ is a full soft set on $U$. Hence, we get

(1) $\forall x \in U, \exists B_x \in \tau_D$ such that $x \in B_x$ and $B_x$ is called an open set containing $x$.

(2) $\forall x \in U, \exists e \in A$ such that $x \in F(e)$.

(3) $\forall x \in A, \exists B_x \in \tau_D$ such that $F(e) \subseteq B_x$.

Proof. The proof of (1) is obvious and from the definition of $S$ and $\overline{S}$ (Definition 4), the proofs of (2) and (3) are straightforward.

Now, for any soft approximation space $A_s = (U, F_A)$, such that $F_A$ is a full soft set, the probability $\mathcal{P}_A$ that lower approximation of a subset is an empty set can be obtained by the following formula:
\[ \mathcal{P}_\varnothing = \frac{1 + \Lambda_{\varnothing}^{F_A}}{2^n}, \]

where \( \Lambda_{\varnothing}^{F_A} \) is the number of subsets of \( U \) which does not contain any nonempty element of \( F_A \) and \( n \) represents a number of elements in \( U \).

In order to find the answer of the last question we may have to consider the following.

**Theorem 5.** Consider \( A_s = (U, F_A) \) is a soft approximation space such that \( F_A \) is a full soft set and \( X \) is a subset of \( U \). Thus, \( \tilde{S}(X) = U \) if and only if \( X \) intersects with every nonempty element of \( \tau_D \).

**Proof.** Firstly, since \( F_A \) is a full soft set, then, \( \forall e \in A, \cup_{e \in A} F(e) = U \). Let \( \tilde{S}(X) = U \); then \( \tilde{S}(X) = \cup_{e \in A} F(e) \) such that \( F(e) \cap X \neq \phi \). Therefore, by Lemma 4, \( \exists B_e \in \tau_D \) such that \( F(e) \subseteq B_e \) and \( F(e) \cap X \neq \phi \). Accordingly, \( \forall B_e \in \tau_D, B_e \cap X \neq \phi \) which means that \( X \) intersects with every nonempty element of \( \tau_D \).

Consider \( \Lambda_{\varnothing}^{F_A} \) is the number of subsets of \( U \) which intersect with every nonempty element of \( \tau_D \). Conversely, let \( \exists B_e \in \tau_D \) such that \( B_e \cap X = \phi \) and \( \tilde{S}(X) \neq U \). Thus \( \exists e \in A \) such that \( F(e) \subseteq B_e \) and \( F(e) \cap X = \phi \). Accordingly, \( \tilde{S}(X) \neq U \) which is a contradiction.

Note that if \( \tilde{S}(X) = U \) then no need for \( X \) to intersect with \( F(e), \forall e \in A \) in general as Example 3 illustrates. Clearly, the subsets \( \{b, c\} \) and \( \{b, c, d\} \) do not intersect with \( F(e) \), although their soft upper approximation is \( U \).

Further in any soft approximation space, \( A_s = (U, F_A) \), such that \( F_A \) is a full soft set; the probability \( \mathcal{P}_U \) that upper approximation of a subset is \( U \) can be obtained by the following formula:

\[ \mathcal{P}_U = \frac{\Lambda_{\varnothing}^{F_A}}{2^n}, \]

where \( \Lambda_{\varnothing}^{F_A} \) is the number of subsets of \( U \) which intersect with every nonempty element of \( \tau \) and \( n \) is the number of elements in \( U \).

The next example explains the previous discussion.

**Example 3.** Consider \( A_s = (U, F_A) \) is a soft approximation space, such that \( U = \{a, b, c, d\} \) and \( F_A \) is a full soft set on \( U \), where \( A = \{e_1, e_2, e_3, e_4\} \) and \( F_A = \{[e_1, \{a\}], [e_2, \{a, b\}], [e_3, \{c\}], [e_4, \{b, d\}]\} \). Then, we obtain the topology of all soft definable subsets by \( \tau_D = \{U, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\} \). Therefore, \( |\tau_D| \) = number of elements in \( \tau_D \) = number of all definable subsets of \( U \). Thus, the probability that a subset of \( U \) is definable is given by \( \mathcal{P} = \frac{|\tau_D|/2^n} = \frac{\frac{1}{16}}{\frac{1}{16}} = (1/4)^{16} = (1/4) = 0.25 \).

Now, the subsets \( \{b\} \) and \( \{d\} \) are the only subsets of \( U \) that do not contain any nonempty element of \( F_A \). Therefore, their lower approximation is empty.

Accordingly, \( \mathcal{P}_\varnothing = \frac{(1 + \Lambda_{\varnothing}^{F_A})}{2^n} = \frac{(1 + 2)}{2^4} = (3/16) \). Further, the subsets \( \{b, c\}, \{a, b, c\}, \{a, c, d\}, \) and \( \{b, c, d\} \) are the only subsets of \( U \), which intersect with every nonempty element of \( \tau_D \). Therefore, their upper approximation is \( U \) and accordingly \( \mathcal{P}_U = \frac{\Lambda_{\varnothing}^{F_A}}{2^n} = \frac{4}{2^4} = (1/4) = 0.25 \).

### 3. Topological Soft Rough Approximations of Soft Rough Sets

The current section is devoted to introduction of topological soft rough approximations in view of topological structure. Firstly, it will be seen that soft sets and topological spaces have a very close relationship. The concept of topological soft rough approximations will be presented and their properties will be studied. On the other hand, it will be shown that accuracy of the proposed approach is better than existing techniques. Besides, we will give answers to some important questions about the probability in topological soft rough sets.

**Definition 6.** Consider \( F_A \) is a soft set on \( U \) and \( K = \cup F(e) \). Thus, we propose the following:

(i) If \( \mathcal{S}_{F_A} = \{F(e): e \in A\} \) may denote a subbasis;

(ii) If \( \mathcal{B}_{F_A} = \{A \cap B: (A, B) \in \mathcal{S}_{F_A} \times \mathcal{S}_{F_A}\} \) may denote a basis for the topology \( T_{SR} \) defined as the following.

If \( F_A \) is a soft set on \( U \) and \( K = \cup F(e) \), then the topology \( T_{SR} \) can be defined on \( K \) with a basis \( \mathcal{B}_{F_A} \). That is, \( T_{SR} = \cup \{B: B \in \mathcal{B}_{F_A}\} \). This topology may be called topology generated by \( F_A \) and we call it “soft rough topology” (in brief, SR topology).

**Remark 2.** There are three cases of a subbasis \( \mathcal{S}_{F_A} \):

(i) If \( \mathcal{S}_{F_A} \) is a partition of \( U \), then \( \mathcal{S}_{F_A} = \mathcal{B}_{F_A} \) and will be a basis for a quasi-discrete (clopen) topology (in which all open sets are closed).

(ii) If \( \mathcal{S}_{F_A} \) is a covering (not partition) of \( U \), then \( \mathcal{S}_{F_A} \neq \mathcal{B}_{F_A} \) and \( \mathcal{B}_{F_A} \) will be a basis for a general topology.

(iii) If \( \mathcal{S}_{F_A} \) is not a covering (not partition) of \( U \), then \( \mathcal{S}_{F_A} \neq \mathcal{B}_{F_A} \) and \( \mathcal{B}_{F_A} \) will be a basis for a general topology.

The following examples explain Remark 2.

**Example 4.** If \( A_s = (U, F_A) \) is a soft approximation space, such that \( U = \{a, b, c, d\} \) and \( F_A \) is a soft set on \( U \) where
\[ A = \{e_1, e_2, e_3\} \text{ and } F_A = \{(e_1, \{a, b\}), (e_2, \{c\}), (e_3, \{d\})\}, \]
then we get that the subbasis of \( T_{SR} \) is \( S_{F_A} = \{(e_1, \{d\}), (e_2, \{a, b\})\} \) and the basis is \( B_{F_A} = \{(e_1, \{d\}), (e_2, \{a, b\})\} \). Clearly, \( S_{F_A} \neq B_{F_A} \).

Accordingly, the topology generated by \( B_{F_A} \) is

\[ T_{SR} = \{U, \phi, [c], \{d\}, \{a, b, c\}, \{a, b, d\}\}. \quad (17) \]

Obviously, \( T_{SR} \) is a quasi-discrete topology.

**Example 5.** Suppose that \( A_1 = (U, F_A) \) is a soft approximation space, such that \( U = \{a, b, c, d\} \) and \( F_A \) is a soft set on \( U \) where \( A = \{e_1, e_2, e_3\} \) and \( F_A = \{(e_1, \{a, b\}), (e_2, \{c\}), (e_3, \{d\})\} \). Then, the subbasis of \( T_{SR} \) is \( S_{F_A} = \{(d), \{a, b\}, \{a, c\}\} \) and the basis is \( B_{F_A} = \{\{a\}, \{d\}, \{a, b, c\}\} \). Evidently, \( S_{F_A} \neq B_{F_A} \).

Accordingly, the topology \( T_{SR} \) generated by \( B_{F_A} \) is

\[ T_{SR} = \{U, \phi, [a], \{d\}, \{a, b, c\}, \{a, b, d\}\}. \quad (18) \]

Obviously, \( T_{SR} \) is not a quasi-discrete topology.

**Example 6.** Consider \( A_2 = (U, F_A) \) is a soft approximation space, such that \( U = \{a, b, c, d\} \) and \( F_A \) is a soft set on \( U \) where \( A = \{e_1, e_2, e_3\} \) and \( F_A = \{(e_1, \{a\}), (e_2, \{b, c\}), (e_3, \{c\})\} \). Then, the subbasis of \( T_{SR} \) is \( S_{F_A} = \{\{a\}, \{a, c\}\} \) and the basis is \( B_{F_A} = \{\{a\}, \{a, c\}, \{b, c\}\} \). Undoubtedly, \( S_{F_A} \neq B_{F_A} \).

Accordingly, the topology \( T_{SR} \) generated by \( B_{F_A} \) is

\[ T_{SR} = \{U, \phi, [a], [c], \{a, c\}, [b, c]\}. \quad (19) \]

Obviously, \( T_{SR} \) is not a quasi-discrete topology.

**Definition 7.** Consider \( A_2 = (U, F_A) \) is a soft approximation space and \( T_{SR} \) is the SR topology on \( U \). The triple \( A_{T_{SR}} = (U, F_A, T_{SR}) \) is said to be a "topological soft rough approximation space" (briefly, \( T_{SR} \) approximation space).

**Definition 8.** Consider \( A_{T_{SR}} = (U, F_A, T_{SR}) \) to be a \( T_{SR} \) approximation space. Therefore, for each \( X \subseteq U \) we suggest the topological soft rough approximations, "\( T_{SR} \) lower" and "\( T_{SR} \) upper," respectively, by

\[ S_{T_{SR}}(X) = \bigcup \{G \subseteq T_{SR} : G \subseteq X\}, \]
\[ \overline{S}_{T_{SR}}(X) = \bigcap \{H \subseteq T_{SR} : X \subseteq H\}. \quad (20) \]

**Remark 3**

(i) In general, \( S_{T_{SR}}(X) \) and \( \overline{S}_{T_{SR}}(X) \) represent the interior and closure of \( X \) associated with the topology \( T_{SR} \) respectively.

(ii) If \( S_{F_A} \) is a partition on \( U \), then \( S_{F_A} = B_{F_A} = (U/R) \) and hence \( S_{T_{SR}}(X) \) and \( \overline{S}_{T_{SR}}(X) \) are identical with Pawlak’s rough set approximations. Therefore, it can be said that the proposed approach is equivalent to Pawlak’s approach only in case \( S_{F_A} \) is a partition of \( U \). Accordingly, we can say that Pawlak’s rough set model is a specialization of proposed model. Example 4 illustrated this fact.

(iii) If \( S_{F_A} \) is not a partition on \( U \), then \( S_{F_A} \neq B_{F_A} \neq (U/R) \) and hence \( \overline{S}_{T_{SR}}(X) \) and \( S_{T_{SR}}(X) \) will be different from Pawlak’s approximations as illustrated in Example 7.

**Example 7.** Suppose that \( U = \{a, b, c, d\} \) is a set of students reading some languages. Let \( A = \{e_1, e_2, e_3\} \) and \( A_2 = (U, F_A) \) be a soft approximation space, where \( F_A \) is a soft set on \( U \). Consider the next information system in Table 1.

| German | French | English |
|--------|--------|---------|
| 1      | 2      | 3       |

Hence, for every \( X \subseteq U \), we express the "\( T_{SR} \) positive, \( T_{SR} \) negative, and \( T_{SR} \) boundary" regions and the "\( T_{SR} \) accuracy" of the \( T_{SR} \) approximations, respectively, by

\[ POS_{T_{SR}}(X) = \overline{S}_{T_{SR}}(X), \]
\[ NEG_{T_{SR}}(X) = U - S_{T_{SR}}(X), \]
\[ BND_{T_{SR}}(X) = \overline{S}_{T_{SR}}(X) - S_{T_{SR}}(X), \]
\[ \mu_{T_{SR}}(X) = \begin{cases} \frac{|S_{T_{SR}}(X)|}{|\overline{S}_{T_{SR}}(X)|} & \text{if } X \neq \phi, \\ 1, & \text{if } X \text{ is a } T_{SR} \text{ - definable}. \end{cases} \quad (23) \]

**Remark 4**

(i) It is clear that \( 0 \leq \mu_{T_{SR}}(X) \leq 1 \), for any \( X \subseteq U \).

(ii) If \( \overline{S}_{T_{SR}}(X) = S_{T_{SR}}(X) \), then \( BND_{T_{SR}}(X) = \phi \) and \( \mu_{T_{SR}}(X) = 1 \). Thus \( X \subseteq U \) is said to be "\( T_{SR} \) definable" or "\( T_{SR} \) exact" set; otherwise \( X \) is called a "\( T_{SR} \) rough" set.

The core objective of the following propositions is to discuss the basic properties of \( T_{SR} \) rough approximations \( S_{T_{SR}} \) and \( \overline{S}_{T_{SR}} ".
According to the characteristics of the interior and closure, we can demonstrate the subsequent results, so we omit the proof.

**Proposition 3.** If \( A_{T} = (U, F_{A}, T_{SR}) \) is a \( T_{SR} \) approximation space and \( X, Y \subseteq \emptyset \), then, the \( T_{SR} \) lower and \( T_{SR} \) upper approximations operators satisfy the next properties:

(i) \( S_{T_{SR}}(\emptyset) = \bar{S}_{T_{SR}}(\emptyset) = \emptyset \).

(ii) \( S_{T_{SR}}(U) = \bar{S}_{T_{SR}}(U) = U \).

(iii) \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(Y) \) if \( X \subseteq Y \).

(iv) \( S_{T_{SR}}(X \cap Y) \subseteq S_{T_{SR}}(X) \cap S_{T_{SR}}(Y) \).

(v) \( S_{T_{SR}}(X \cup Y) \supseteq S_{T_{SR}}(X) \cup S_{T_{SR}}(Y) \).

(vi) \( S_{T_{SR}}(X^c) = S_{T_{SR}}(X)^c \).

(vii) \( S_{T_{SR}}(X \times Y) = S_{T_{SR}}(X) \times S_{T_{SR}}(Y) \).

(viii) \( S_{T_{SR}}(X \times Y) = S_{T_{SR}}(X) \times \bar{S}_{T_{SR}}(Y) \).

Proposition 4. If \( A_{T_{SR}} = (U, F_{A}, T_{SR}) \) is a \( T_{SR} \) approximation space and \( X \subseteq U \), then:

(i) \( S_{T_{SR}}(S_{T_{SR}}(X)) = S_{T_{SR}}(X) \).

(ii) \( \bar{S}_{T_{SR}}(S_{T_{SR}}(X)) = \bar{S}_{T_{SR}}(X) \).

(iii) \( S_{T_{SR}}(X) \subseteq \bar{S}_{T_{SR}}(S_{T_{SR}}(X)) \).

(iv) \( \bar{S}_{T_{SR}}(S_{T_{SR}}(X)) \subseteq S_{T_{SR}}(X) \).

Remark 5. The inclusion relations in Proposition 4 may be strict, as shown in Example 8.

**Example 8.** Suppose that \( F_{A} \) is a soft set on \( U \) and \( A_{i} = (U, F_{A}) \) is a soft approximation space, where \( U = \{a, b, c, d\} \) and \( A = \{e_{1}, e_{2}, e_{3}, e_{4}, e_{5}\} \) such that \( F_{A} = \{(e_{1}, \{b\}), (e_{2}, \{a, b\}), (e_{3}, \{c, d\}), (e_{4}, \{a, c, d\}), (e_{5}, \{a, b, c\})\} \). Therefore, we get the following.

The subsbasis of \( T_{SR} \) is \( S_{F_{A}} = \{\{b\}, \{t(a, b)n, q\} \}

\( \{c, d\} \} \times \{7, a, b, c\} \} \).

Accordingly, we get

\[ T_{SR} = \{\{a, b, c, d\} \subseteq \{a, b, c\} \subseteq \{a, b\} \subseteq \{a\} \subseteq \emptyset \} \cdot \]

The complement of \( T_{SR} \) is

\[ T_{SR}^c = \{\{a, b, c, d\} \subseteq \{a, b, c\} \subseteq \{a, b\} \subseteq \{a\} \subseteq \emptyset \} \cdot \]

Thus, \( A_{T_{SR}} = (U, S, T_{SR}) \) is a \( T_{SR} \) approximation space. Now, let \( X = \{a, c\} \) and \( Y = \{a, b, d\} \). Then, \( S_{T_{SR}}(X) = \{a, c\} \) and \( S_{T_{SR}}(Y) = \{a, c, d\} \), which means that \( S_{T_{SR}}(X) \neq S_{T_{SR}}(Y) \). Also, \( S_{T_{SR}}(X) = \{a, b, d\} \) and \( S_{T_{SR}}(Y) = \{a, b, c\} \), which means that \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(Y) \).

The following theorem establishes a relationship between approximations of a set in soft rough sets [24] and topological soft rough sets.

**Theorem 6.** If \( A_{T_{SR}} = (U, F_{A}, T_{SR}) \) is a \( T_{SR} \) approximation space and \( X \subseteq U \), then:

(i) \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(X) \).

(ii) \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(X) \).

(iii) \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(X) \).

Proof: We shall verify only the first item and the other likewise. Let \( x \in S_{T_{SR}}(X) \); then \( \exists e \in A \), such that \( x \in F(e) \subseteq X \) and \( F(e) \in S_{F_{A}} \). Accordingly, \( F(e) \in T_{SR} \) such that \( x \in F(e) \) and \( F(e) \subseteq X \); this implies \( F(e) \subseteq S_{T_{SR}}(X) \), and therefore, \( x \in S_{T_{SR}}(X) \). Hence, \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(X) \).

**Corollary 1.** If \( A_{T_{SR}} = (U, F_{A}, T_{SR}) \) is a \( T_{SR} \) approximation space and \( X \subseteq U \), then:

(i) \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(X) \).

(ii) \( S_{T_{SR}}(X) \subseteq S_{T_{SR}}(X) \).

**Corollary 2.** If \( A_{T_{SR}} = (U, F_{A}, T_{SR}) \) is a \( T_{SR} \) approximation space and \( X \subseteq U \) and if \( X \) is a soft exact set, then it is a \( T_{SR} \) exact set.

**Remark 6**

(i) According to the above results, it is easy to see that boundary region in case of topological soft rough sets is smaller than boundary for soft rough sets.
Theorem 7. Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ to be a $T_{SR}$ approximation space and $X \subseteq U$. Therefore, we have the following:

(i) If the subset $X$ is a roughly $T_{SR}$ definable set, then $X$ is roughly soft $A_{SR}$ definable.

(ii) If the subset $X$ is an internally $T_{SR}$ definable set, then $X$ is internally soft $A_{SR}$ definable.

(iii) If the subset $X$ is an externally $T_{SR}$ definable set, then $X$ is externally soft $A_{SR}$ definable.

(iv) If the subset $X$ is a totally $T_{SR}$ definable set, then $X$ is totally soft $A_{SR}$ definable.

Proof. Only the first statement will be proved and the other statements can be made by a similar way. (i) Suppose that the subset $X$ is a roughly soft $A_{SR}$ definable set; then $S_{T_{SR}}(X) \neq \emptyset$ and $S_{T_{SR}}(X) = X$ and thus $BND_{T_{SR}}(X) = \emptyset$ and $\mu_{T_{SR}}(X) = 1$. Obviously, $X$ is a $T_{SR}$ exact (definable) set (according to our approach) although it is a soft rough set.

Next we define some very important notions.

Definition 10. Consider $A_{T_{SR}} = (U, F_A, T_{SR})$ is a $T_{SR}$ approximation space and $X \subseteq U$. Thus, we describe the subsequent four elementary sorts of $T_{SR}$ soft rough sets as follows:

The subset $X$ represents

(i) a roughly $T_{SR}$ definable set if $S_{T_{SR}}(X) \neq \emptyset$ and $S_{T_{SR}}(X) \neq U$,

(ii) an internally $T_{SR}$ indefinable set if $S_{T_{SR}}(X) = \emptyset$ and $S_{T_{SR}}(X) \neq U$,

(iii) an externally $T_{SR}$ indefinable set if $S_{T_{SR}}(X) \neq \emptyset$ and $S_{T_{SR}}(X) = U$,

(iv) a totally $T_{SR}$ indefinable set if $S_{T_{SR}}(X) = \emptyset$ and $S_{T_{SR}}(X) = U$.

The axiomatic significance of this classification is given as follows:

(i) If the subset $X$ is a roughly $T_{SR}$ definable set, then we can identify for some members of $U$ that they belong to $X$, and for other members of $U$ that they belong to $X^c$, by using existing knowledge from the $T_{SR}$ approximations proposed in this paper. Moreover, it clarifies the significance of the proposed approach in defining approximations of sets; for example, let $X$ be a totally soft $A_{SR}$ definable set. Then, we get $S(X) = \emptyset$ and $S(X) = U$. Thus, we are incapable of identifying what are the elements of $U$ that belong to $X$ or $X^c$.

(ii) The inverse of Theorem 7 does not hold, generally, as demonstrated in Example 9 and Subsection 4.1.

Now, once again, we can find the probability for different types of subsets in topological soft rough sets. Firstly, to answer the first question, we consider the following results.

Theorem 8. If $F_A$ is a soft set on $U$, $A_{T_{SR}} = (U, F_A, T_{SR})$ is a $T_{SR}$ approximation space, and $X \subseteq U$, then, the class $\tau_D = \{X \subseteq U: \exists \phi \in \tau_{TSR}: S_{T_{SR}}(X) = S_{T_{SR}}(\phi)\}$ gives rise a topology on $U$.

Proof. According to Proposition 3, the proof is clear.

Note that $\tau_D$ represents a topology of all definable sets in $U$.

Lemma 2. If $A_{T_{SR}} = (U, F_A, T_{SR})$ is a $T_{SR}$ approximation space, then $\tau_D \subseteq \tau_{TSR}$.

Proof: Let $X \in \tau_D$; then $S_{T_{SR}}(X) = S_{T_{SR}}(\phi)$ and this implies $S_{T_{SR}}(X) = X$. Accordingly, $X \in \tau_{TSR}$.

Lemma 3. Let $A_{T_{SR}} = (U, F_A, T_{SR})$ be a $T_{SR}$ approximation space and $S_{F_A}$ be a partition of $U$; then $\tau_D = \tau_{TSR}$ and $S_{F_A}$ is a basis for $\tau_D$.

Proof. Firstly, using Lemma 2, $\tau_D \subseteq \tau_{TSR}$. Now, let $X \in \tau_{TSR}$ which is a quasi-discrete topology; then $X$ is an open and
closed subset. Therefore, \( S_{ \tau_{SR}} (X) = \overline{S}_{ \tau_{SR}} (X) = X \) and this implies \( X \in \tau_D \). Hence, \( T_{SR} \subseteq \tau_D \).

The subsequent example explains that the condition "\( S_{ \tau_{SR}} \) is a partition of \( U \)" is necessary condition.

**Example 10.** Consider Example 7; we get \( \tau_D = \{ U, \phi, \{a, c\}, \{b, d\}\} \) but \( T_{SR} = \{ U, \phi, \{d\}, t(a, c), n, q \{b, d\}h_{\{a, c, d\}} \} \). It is clear that \( T_{SR} \not\subseteq \tau_D \).

**Definition 11.** Suppose that \( A_{T_{SR}} = (U, F_A, T_{SR}) \) is a \( T_{SR} \) approximation space and \( \tau_D \) is a topology of all definable sets in \( U \). The probability \( P_D \) that a subset of \( U \) is definable is defined by

\[
P_D = \frac{|\tau_D|}{2^n},
\]

where \( |\tau_D| \) is the cardinality of \( \tau_D \) and \( n \) represents a number of elements of \( U \).

Therefore, the probability that a subset is a rough set \( X \subseteq U \) is \( 1 - P_D \).

**Example 11**

1. Consider Example 4; we have \( T_{SR} = \tau_D \). Thus, the probability \( P_D \) that a subset of the universe set is definable is given by

\[
P_D = \frac{|\tau_D|}{2^n} = \frac{8}{2^3} = \frac{1}{2} = 0.5.
\]

2. Consider Example 5; we have \( \tau_D = \{ U, \phi, \{d\}, t(a, b, c) \} \). Therefore, the probability \( P_D \) that a subset of the universe set is definable is given by

\[
P_D = \frac{|\tau_D|}{2^n} = \frac{4}{2^3} = \frac{1}{4} = 0.25.
\]

Secondly, to identify the probability that lower approximation of a nonempty subset of \( U \) may be an empty set, we propose the next results.

**Lemma 4.** If \( A_{T_{SR}} = (U, F_A, T_{SR}) \) is a \( T_{SR} \) approximation space, then, \( B_{\tau_{SR}} \subseteq \overline{S}_{\tau_{SR}} \).

**Proof.** The proof is straightforward.

**Theorem 9.** Consider \( A_{T_{SR}} = (U, F_A, T_{SR}) \) to be a \( A_{T_{SR}} = (U, F_A, T_{SR}) \) approximation space and \( X \subseteq U \). \( \overline{S}_{\tau_{SR}} (X) = \phi \) if and only if the proper subsets of \( X \) do not contain any nonempty element of \( T_{SR} \).

**Proof.** Using Definition 8, the proof is clear.

**Theorem 10.** Let \( A_{T_{SR}} = (U, F_A, T_{SR}) \) be a \( A_{T_{SR}} = (U, F_A, T_{SR}) \) approximation space and \( X \subseteq U \). \( \overline{S}_{\tau_{SR}} (X) = \phi \) if and only if the proper subsets of \( X \) are not element in \( B_{\tau_{SR}} \).

**Proof.** Firstly, if \( \overline{S}_{\tau_{SR}} (X) = \phi \) then \( \forall G \subseteq X, G \notin T_{SR} \), and this implies \( \forall G \subseteq X, G \notin B_{\tau_{SR}} \). Conversely, let \( \exists G \subseteq X, G \in B_{\tau_{SR}} \) such that \( \overline{S}_{\tau_{SR}} (X) = \phi \). Then, \( G \in T_{SR} \) such that \( G \subseteq X \) and this implies \( G \subseteq \overline{S}_{\tau_{SR}} (X) \) which contradicts assumption \( \overline{S}_{\tau_{SR}} (X) = \phi \). Accordingly, \( X \) does not contain any nonempty element of \( T_{SR} \).

**Definition 12.** Suppose that \( A_{T_{SR}} = (U, F_A, T_{SR}) \) is a \( T_{SR} \) approximation space. The probability \( \mathcal{P}_\emptyset \) that the \( T_{SR} \) lower approximation of a subset of the universe set is an empty set is defined by

\[
\mathcal{P}_\emptyset = \frac{1 + \Lambda_{T_{SR}}}{2^n}, \quad \text{where} \quad \Lambda_{T_{SR}} \text{ represents a number of subsets of } U \text{ which does not contain any nonempty element of } T_{SR} \text{ and } n \text{ is the number of elements in } U.
\]

Therefore, the probability that the \( T_{SR} \) lower approximation of \( X \subseteq U \) is not an empty set is \( 1 - \mathcal{P}_\emptyset \).

**Example 12**

1. Consider Example 4; obviously, the subsets \( \{a\} \) and \( \{b\} \) are the only subsets which do not contain any nonempty element of \( T_{SR} \). Thus, the probability \( \mathcal{P}_\emptyset \) that the \( T_{SR} \) lower approximation of a subset of the universe set is an empty set is defined by

\[
\mathcal{P}_\emptyset = \frac{1 + \Lambda_{T_{SR}}}{2^n} = \frac{1 + 2}{2^4} = \frac{3}{16} = 0.19.
\]

2. Consider Example 5; obviously, the subsets \( \{b\}, \{c\} \), and \( \{b, c\} \) are the only subsets which do not contain any nonempty element of \( T_{SR} \). Therefore, the probability \( P_\emptyset \) that the \( T_{SR} \) lower approximation of a subset of the universe set is an empty set is defined by

\[
P_\emptyset = \frac{1 + \Lambda_{T_{SR}}}{2^n} = \frac{1 + 3}{2^4} = \frac{4}{16} = 0.25.
\]

In order to find the probability that \( T_{SR} \) upper approximation of any subset is \( U \), we may have to suggest the subsequent result.
Theorem 11. Consider $A_{T_{SR}} = (U, F, A, T_{SR})$ is a $T_{SR}$ approximation space and $X \subseteq U$. $S(X) = U$ if and only if the subset $X$ intersects with every element in $T_{SR}$.

Proof. Obvious.

Now, for any $T_{SR}$ approximation space, the probability $\mathcal{P}_U$ that $T_{SR}$ upper approximation of a subset is $U$ can be obtained by the following formula:

$$\mathcal{P}_U = \frac{X^{T_{SR}}}{2^n},$$

where $X^{T_{SR}}$ is the number of subsets of $U$ which intersect with every nonempty element of $T_{SR}$ and $n$ is the number of elements in $U$.

The next example explains the above discussion. □

Example 13

(1) Consider Example 4; obviously, the subsets $\{a, c, d\}$ and $\{b, c, d\}$ are the only subsets which intersect with every nonempty element of $T_{SR}$. Thus, the probability $\mathcal{P}_U$ that $T_{SR}$ upper approximation of a subset is $U$ is defined by

$$\mathcal{P}_U = \frac{X^{T_{SR}}}{2^2} = \frac{2}{2} = \frac{2}{2} = 0.125. \quad (33)$$

(2) Consider Example 5; obviously, the subsets $\{a, d\}, \{a, b, d\}$, and $\{a, c, d\}$ are the only subsets which intersect with every nonempty element of $T_{SR}$. Thus, the probability $\mathcal{P}_U$ that $T_{SR}$ upper approximation of a subset is $U$ is defined by

$$\mathcal{P}_U = \frac{X^{T_{SR}}}{2^3} = \frac{3}{2} = \frac{3}{16} = 0.19. \quad (34)$$

4. Medical Application in Heart Failure

In the current article, we illustrate the significance of the suggested approach in decision-making problems for medical applications. Consequently, we apply it to the issue of heart failure. We have a data set with the results of five symptoms for twenty patients divided into twelve males ($P_3, P_6, P_9, P_{11}, \ldots, P_{17}, P_{19}$) and 8 females ($P_1, P_2, P_3, P_5, P_7, P_{10}, P_{18}, P_{20}$). The study was conducted at Om El-Kora Cardiac Center, Hospital of Heart Diseases, Tanta, Egypt. This research involved twenty patients who came to the hospital with various symptoms and underwent a thorough history, physical examination, lab tests, resting ECG, and conventional echo assessment. Finally, the diagnosis of heart failure was verified.

4.1. The Experimental Results. The experimental findings are discussed in this subsection by adding a preparatory analysis performed on five heart disease symptoms for twenty patients, according to Thivagar and Richard [40]. Table 2 shows the data from the information system for twenty patients, addressing the heart failure issue. The columns reflect the heart disease diagnosis (where “Yes” indicates that the patient has symptoms and “No” indicates that the patient has none) [40], with condition attributes, such that $e_1$ indicates “the paroxysmal nocturnal dyspnea,” $e_2$ indicates “the breathlessness,” $e_3$ indicates “the orthopnea,” $e_4$ indicates “the ankle swelling,” $e_5$ indicates “the paroxysmal nocturnal dyspnea,” $e_4$ indicates “the orthopnea,” and $e_5$ indicates “the paroxysmal nocturnal dyspnea.” Attribute $D$ indicates “decision of heart failure.” For rows in Table 2, $P = \{P_1, P_2, P_3, \ldots, P_{20}\}$ represents the set of twenty patients. Therefore, the set of all attributes is

$$A = \{e_1, e_2, e_3, e_4, e_5\} \cup D$$

which is represented by columns.

Here 1 and 0 denote “yes” and “no,” respectively.

We apply the suggested method in the set of female’s patients only and the others similarly.

Accordingly, Table 3 represents the soft set of female’s patients, where the set of female’s patients is $U = \{P_1, P_2, P_3, P_5, P_7, P_{10}, P_{18}, P_{20}\}$ and the set of attributes is $A = \{e_1, e_2, e_3, e_4, e_5\}$.

Let $(F, A)$ be a soft set over given by Table 3; the basis, generated by $(F, A)$, is given by

$$B_{F_A} = \{(P_1), (P_{18}), (P_2, P_{20}), (P_4, P_{18}), (P_4, P_{20}), (P_{18}, P_{20}), (P_{18}, P_{18}), (P_{18}, P_{20}), (P_2, P_5, P_7, P_{18}, P_{20}), (P_4, P_{18}, P_{20}), (P_1, P_4, P_{18}, P_{20})\}. \quad (35)$$

Therefore, the topology generated by this base is
Table 2: Original medical information system.

| U/A | e_1 | e_2 | e_3 | e_4 | e_5 | D  |
|-----|-----|-----|-----|-----|-----|----|
| p_1 | 1   | 1   | 0   | 0   | 0   | Yes|
| p_2 | 0   | 0   | 0   | 1   | 1   | No |
| p_3 | 0   | 1   | 1   | 0   | 1   | Yes|
| p_4 | 1   | 1   | 1   | 1   | 0   | Yes|
| p_5 | 0   | 0   | 0   | 0   | 1   | No |
| p_6 | 0   | 0   | 0   | 0   | 1   | No |
| p_7 | 0   | 0   | 1   | 0   | 0   | No |
| p_8 | 1   | 0   | 0   | 1   | 0   | No |
| p_9 | 0   | 1   | 1   | 0   | 1   | Yes|
| p_{10} | 0   | 0   | 0   | 0   | 0   | No |
| p_{11} | 1   | 0   | 0   | 1   | 1   | No |
| p_{12} | 0   | 1   | 1   | 0   | 1   | No |
| p_{13} | 1   | 1   | 1   | 1   | 1   | Yes|
| p_{14} | 1   | 0   | 0   | 1   | 0   | Yes|
| p_{15} | 1   | 0   | 0   | 0   | 1   | No |
| p_{16} | 0   | 0   | 0   | 1   | 1   | No |
| p_{17} | 0   | 1   | 1   | 0   | 0   | Yes|
| p_{18} | 1   | 1   | 1   | 0   | 1   | Yes|
| p_{19} | 0   | 1   | 1   | 0   | 1   | Yes|
| p_{20} | 1   | 1   | 0   | 0   | 1   | Yes|

Table 3: The tabular form for female’s soft set (F, A).

| U/A | e_1 | e_2 | e_3 | e_4 | e_5 | D  |
|-----|-----|-----|-----|-----|-----|----|
| p_1 | 1   | 1   | 0   | 0   | 0   | Yes|
| p_2 | 0   | 0   | 0   | 1   | 1   | No |
| p_4 | 1   | 1   | 1   | 1   | 0   | Yes|
| p_5 | 0   | 0   | 0   | 1   | 0   | No |
| p_7 | 1   | 0   | 0   | 1   | 0   | No |
| p_{10} | 0   | 0   | 0   | 0   | 0   | No |
| p_{18} | 1   | 1   | 1   | 0   | 1   | Yes|
| p_{20} | 0   | 1   | 1   | 0   | 1   | Yes|

\[ T_{SR} = \{ U, \emptyset, \{ p_2 \}, \{ p_4 \}, \{ p_{18} \}, \{ p_2 \cup p_4 \}, \{ p_2 \cup p_{18} \}, \{ p_4 \cup p_{18} \}, \{ p_2 \cup p_4 \cup p_{18} \}, \{ p_1 \cup p_3 \}, \{ p_1 \cup p_4 \}, \{ p_1 \cup p_{18} \}, \{ p_1 \cup p_4 \cup p_{18} \}, \{ p_2 \cup p_3 \}, \{ p_2 \cup p_4 \}, \{ p_2 \cup p_{18} \}, \{ p_2 \cup p_4 \cup p_{18} \}, \{ p_3 \cup p_4 \}, \{ p_3 \cup p_{18} \}, \{ p_3 \cup p_4 \cup p_{18} \}, \{ p_1 \cup p_2 \}, \{ p_1 \cup p_3 \}, \{ p_1 \cup p_4 \}, \{ p_1 \cup p_{18} \}, \{ p_1 \cup p_2 \cup p_{18} \}, \{ p_1 \cup p_3 \cup p_{18} \}, \{ p_1 \cup p_4 \cup p_{18} \}, \{ p_1 \cup p_2 \cup p_3 \}, \{ p_1 \cup p_4 \cup p_{18} \}, \{ p_1 \cup p_2 \cup p_4 \}, \{ p_1 \cup p_3 \cup p_{18} \}, \{ p_1 \cup p_4 \cup p_{18} \}, \{ p_1 \cup p_2 \cup p_3 \cup p_{18} \} \} \]

(36)

The complement of \( T_{SR} \) is

\[ T_{SR}^c = \{ U, \emptyset, \{ p_{10} \}, \{ p_1 \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \}, \{ p_2 \}, \{ p_{15} \}, \{ p_{10} \}, \{ p_5 \}, \{ p_{10} \} \} \]

(37)

Now, we introduce a comparison between the boundary approximations “\( T_{SR} \) approximations” and the previous ones in Table 4.
We shall apply the nanotopology of $T_{SR}$ approximations to identify the key factors of “heart failure” using topological reduction of attributes in information system of Table 3.

First, let us extend the definition of “nanotopology” into “$T_{SR}$ nanotopology” using $T_{SR}$ approximations.

Definition 13. Consider $A_{T_{SR}} = (U, F_{A}, T_{SR})$ is a $T_{SR}$ approximation space, and $X \subseteq U$. Therefore, the class $N_{T_{SR}} = \{U, \phi, \Sigma_{T_{SR}}(X), \Sigma_{T_{SR}}(X), BND_{T_{SR}}(X)\}$ is called “$T_{SR}$ nanotopology” which represents a general topology generated by the soft rough set $X \subseteq U$. The basis of this topology is given the class $\beta_{T_{SR}} = \{U, \Sigma_{T_{SR}}(X), BND_{T_{SR}}(X)\}$.

Definition 14. Consider $A_{T_{SR}} = (U, F_{A}, T_{SR})$ is a $T_{SR}$ approximation space, and $N_{T_{SR}}$ is a $T_{SR}$ nanotopology with a basis $\beta_{T_{SR}}$.

(i) if $\beta_{T_{SR}} \subseteq \beta_{T_{SR}}$, then the attribute $e_k$ is called “dispensable”;

(ii) if $\beta_{T_{SR}} \neq \beta_{T_{SR}}$, then the attribute $e_k$ is not “dispensable.” Therefore, the core of attributes is $\text{CORE} = \{e_k\}$ which represents the common part of reduction.

Now, we apply the topological reduction for Table 3 to identify the key factors of “heart failure” as follows: We compute the $T_{SR}$ nanotopology to decision-making for two sets of patients:

$X = \{p_1, p_4, p_18, p_20\}$ which represents a set of patients that have the disease of heart failure and $Y = \{p_2, p_5, p_7, p_{10}\}$ which represents a set of patients that do not have the disease of heart failure.

We will make a topological reduction for first set $X$ and the second set $Y$ similarly.

Case 1 (patients having the heart failure disease).

According to Table 3, we get $\Sigma_{T_{SR}}(X) = \{p_1, p_4, p_{18}, p_{20}\}$, $\Sigma_{T_{SR}}(X) = U - \{p_2\}$, and $BND_{T_{SR}}(X) = \{p_5, p_7, p_{10}\}$. Thus, the basis of $T_{SR}$ nanotopology generated by the above $T_{SR}$ approximations is

\[ \beta_{T_{SR}} = \{U, \{p_1, p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\}. \]  

Step 1. When the attribute “the breathlessness ($e_1$)” is removed:

The topology generated by this base is

(1) Table 4 presents a comparison between the soft rough sets [24] and topological soft rough sets.

(2) For a comparison with the method given in [26], we have the following classes for each parameter:

For $e_1$ classes are $\{p_1, p_4, p_7, p_{18}\}, \{p_2, p_5, p_{10}, p_{20}\}$

For $e_2$ classes are $\{p_1, p_4, p_{18}, p_{20}\}, \{p_2, p_5, p_7, p_{10}\}$

For $e_3$ classes are $\{p_4, p_{18}, p_{20}\}, \{p_1, p_2, p_5, p_7, p_{10}\}$

For $e_4$ classes are $\{p_4, p_{18}, p_{20}\}, \{p_1, p_2, p_5, p_7, p_{10}\}$

Now, the intersection of all these classes is $\{p_1, p_4, p_7, p_{18}\}, \{p_2, p_5, p_{10}, p_{20}\}$, $\{p_4, p_{18}, p_{20}\}$. This means intersection of all equivalence relations gives rise to identity relation $R = \{(p_1, p_1), (p_2, p_2), (p_4, p_4), (p_7, p_7), (p_{18}, p_{18}), (p_{20}, p_{20})\}$. Therefore, every subset will be definable in approximation space $(U, R)$. Therefore, the method discussed in [26] fails to produce distinct lower and upper approximations in the given example.

(3) For a comparison with the method given in [25], we have the map $\varphi: U \rightarrow P(E)$ as the following $\varphi(p_1) = \{e_1, e_2\}, \varphi(p_2) = \{e_1, e_2\}, \varphi(p_3) = \{e_1, e_2\}$.

Clearly these sets are distinct. Therefore, we define the approximation of any subset $X$ of $U$ as per the following:

$$ X_{\varphi} = \{x \in X, \varphi(x) \neq \varphi(y) \text{ for all } y \in X\}, $$

$$ \bar{X}_{\varphi} = \{x \in X, \varphi(x) = \varphi(y) \text{ for some } y \in X\}. $$

So, all subsets of $U$ will be definable so this method also fails to produce distinct lower and upper approximations in the given example.

Decision-making is essential in the daily lives, and this process yields the best alternative from a variety of options. We give Algorithm 1 in table for a decision-making of an information system in terms of the $T_{SR}$ approximations.
Step 1: Input the soft set \((F, A)\).

Step 2: Take the class \(S_{F} = \{F(e) : \forall e \in A\}\) as a subbasis for a basis \(B_{F}\).

Step 3: Compute the basis \(B_{F} = \{A \cap B : (A, B) \in S_{F} \times S_{F}\}\) by Definition 6.

Step 4: Generate the topology \(T_{\sr} = \cup B : B \in B_{F}\) by Definition 6.

Step 5: Investigate the \(T_{\sr}\) upper approximations, say \(\overline{S}_{\sr}(X)\), and \(T_{\sr}\) lower approximations, say \(\underline{S}_{\sr}(X)\), for every \(X \subseteq U\), according to Definition 8.

Step 6: Determine the boundary region, say \(\text{BND}_{\sr}(X)\), from Step 2, according to Definition 9.

Step 7: Calculate the accuracy of the approximation, say \(\mu_{\sr}(X)\), from Step 2, according to Definition 9.

Step 8: Decide, exactly, rough sets and exact sets, using Definition 9.

Algorithm 1: A decision-making via \(T_{\sr}\) approximations.

\[
T_{\sr} = \{U, \emptyset, \{p_1\}, \{p_1, p_2\}, \{p_1, p_3, p_7, p_{10}\}, \{p_1, p_4, p_5, p_7, p_{10}\}, \{p_1, p_5, p_{7, 10}\}, \{p_1, p_5, p_{7, 10}\}, \{p_1, p_5, p_{7, 10}\}\}
\]

The complement of \(T_{\sr}\) is

\[
\overline{T}_{\sr} = \{U, \emptyset, \{p_1\}, \{p_1, p_2\}, \{p_1, p_3, p_7, p_{10}\}, \{p_1, p_4, p_5, p_7, p_{10}\}, \{p_1, p_5, p_{7, 10}\}, \{p_1, p_5, p_{7, 10}\}, \{p_1, p_5, p_{7, 10}\}\}
\]

Therefore, \(T_{\sr}\) approximations of \(X\) in this case are \(\overline{S}_{\sr}(X) = \{p_1, p_2, p_{10}, p_{20}\}\), \(\overline{S}_{\sr}(X) = U - \{p_2\}\), and \(\text{BND}_{\sr}(X) = \{p_2, p_5, p_{10}\}\). Thus, the basis of \(T_{\sr}\) nanotopology generated by the above \(T_{\sr}\) approximations is

\[
\beta_{\sr-e_1} = \{U, \emptyset, \{p_1, p_4, p_{10}, p_{20}\}, \{p_5, p_7, p_{10}\}\} \neq \beta_{\sr-e_2}
\]

Step 2. When the attribute “the orthopnea (\(e_2\))” is removed:

By the same way as in Step 1, we get

\[
\beta_{\sr-e_2} = \{U, \emptyset, \{p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\} \neq \beta_{\sr-e_3}
\]

Step 3. When the attribute “the paroxysmal nocturnal dyspnea (\(e_3\))” is removed: By the same way as in Step 1, we get

\[
\beta_{\sr-e_3} = \{U, \emptyset, \{p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\} \neq \beta_{\sr-e_4}
\]

Step 4. When the attribute “reduced exercise tolerance (\(e_4\))” is removed:

By the same way as in Step 1, we get

\[
\beta_{\sr-e_4} = \{U, \emptyset, \{p_4, p_{18}, p_{20}\}, \{p_5, p_7, p_{10}\}\} \neq \beta_{\sr-e_5}
\]

Step 5. When the attribute “the ankle swelling (\(e_5\))” is removed:

By the same way as in Step 1, we get

\[
\beta_{\sr-e_5} = \{U, \emptyset, \{p_4, p_{18}, p_{20}\}, \{p_2, p_5, p_{7, 10}\}\} \neq \beta_{\sr-e_6}
\]

Therefore, we get the attributes \([e_1, e_4]\) are dispensable and \([e_2, e_3, e_5]\) are not dispensable. Accordingly, the core of attributes is \(\text{CORE}(T_{\sr}) = \{e_2, e_3, e_5\}\), i.e., “the orthopnea, reduced exercise tolerance, and the ankle swelling” represent the main attributes that have close joining to the disease of the heart failure.

By the same manner, we can make a topological reduction to Table 2 (an information system of all “the heart failure” patients).

At the end of the paper, we give Algorithm 2 in table which can be used to make a topological reduction of attributes for information system in terms of the \(T_{\sr}\) approximations.

In literature many methods for reduction of parameters for soft sets have been given, for example, [32, 34, 41]. In all these methods only positive parameters are considered which result in a decision parameter as the sum of the values allotted to an alternative. Then on the basis of this decision parameter most suitable alternative is selected. In the present case of decision-making these methods fail. In given case
parameters are not all positive but a blend of positive and negative, decided by the experts. Moreover decision parameter is attached already.

5. Conclusion

In this article notion of topological soft rough sets is introduced, where topology generated from a soft set plays a vital role. Here notion of soft rough approximations is discussed and some of their properties are given. Their properties have been studied and their relationships with some other methods have been examined. In fact, the proposed approaches fulfill all axioms of Pawlak’s rough sets without adding extra restrictions as Propositions 3 and 4 illustrated. The proposed techniques depend basically on general topology and hence they open the way for applications of topology in soft rough sets. Further, we have answered some very important questions, such as how to determine the probability that a subset of the universe is definable in the classical rough sets and their extensions (like the soft sets and topological soft rough sets).

Finally, we have introduced medical applications, in the decision-making of medical diagnosis for heart failure problems [39], to illustrate the importance of current methods and also to compare proposed method and the previous ones. Moreover, we have succeeded in making a topological reduction for the data set covering the result of five symptoms for twenty patients with heart failure disease, and thus we identify the core factors of the heart failure diagnosis. Besides, two algorithms to our method have been obtained.

For future works, it is hoped that presented framework may be useful to study its application in COVID-19 and other diseases.

Data Availability
Not data were used to support this study.

Conflicts of Interest
The authors declare that they have no competing interests.

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