On the Chow ring of Fano varieties of type S2

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Abstract
We show that certain Fano eightfolds (obtained as hyperplane sections of an orthogonal Grassmannian, and studied by Ito–Miura–Okawa–Ueda and by Fatighenti–Mongardi) have a multiplicative Chow–Künneth decomposition. As a corollary, the Chow ring of these eightfolds behaves like that of K3 surfaces.

Keywords Algebraic cycles · Chow groups · motives · Beauville’s splitting property · multiplicative Chow–Künneahl decomposition · Fano varieties of K3 type

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1 Introduction

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) = CH^i(X)_{\mathbb{Q}}$ denote the Chow group of codimension $i$ algebraic cycles modulo rational equivalence with $\mathbb{Q}$-coefficients, and let $A^i_{\text{hom}}(X)$ denote the subgroup of homologically trivial cycles. Intersection product defines a ring structure on $A^*(X) = \bigoplus A^i(X)$ [8]. In the case of K3 surfaces, this ring structure has a remarkable property:

Theorem 1.1 (Beauville–Voisin [2]) Let $S$ be a K3 surface. Let $D_i, D'_i \in A^1(S)$ be a finite number of divisors. Then

$$\sum_i D_i \cdot D'_i = 0 \text{ in } A^2(S) \iff \sum_i D_i \cdot D'_i = 0 \text{ in } H^4(S, \mathbb{Q}).$$

In the wake of this result (combined with results concerning the Chow ring of abelian varieties), Beauville has asked for which varieties the Chow ring behaves similarly to Theorem 1.1. (This is the problem of determining which varieties verify the “splitting property” of [1]; this circle of ideas has notably led to the famous “Beauville–Voisin
conjecture” concerning the Chow ring of hyperkähler varieties [1, 31].) We briefly state this problem here as follows:

**Problem 1.2** (Beauville [1]) Find a class $\mathcal{C}$ of varieties (containing $K3$ surfaces, abelian varieties and hyperkähler varieties, and stable under taking products), such that for any $X \in \mathcal{C}$, the Chow ring of $X$ admits a multiplicative bigrading $A^*_{(\ast)}(X)$, with

$$A^i(X) = \bigoplus_{j=0}^{i} A^i_{(j)}(X) \quad \text{for all } i.$$  

This bigrading should split the conjectural Bloch–Beilinson filtration, in particular

$$A^i_{\text{hom}}(X) = \bigoplus_{j \geq 1} A^i_{(j)}(X).$$

This question is hard to answer in practice, since we do not have the Bloch–Beilinson filtration at our disposal.

An interesting alternative approach to Problem 1.2 (as well as a reinterpretation of Theorem 1.1) is provided by the concept of multiplicative Chow–Künneth decomposition, giving rise to unconditional constructions of a bigraded ring structure on the Chow ring of many particular varieties [11, 12, 23, 28–30]. Thus one is led to the following problem, which is more concrete than Problem 1.2:

**Problem 1.3** (Shen–Vial [28]) Describe the class $\mathcal{C}'$ of varieties having a multiplicative Chow–Künneth decomposition.

To relate this to Problem 1.2, one might naively conjecture that $\mathcal{C} = \mathcal{C}'$ (and that for any $X \in \mathcal{C} = \mathcal{C}'$, the induced bigradings on $A^*(X)$ coincide). As a partial answer towards Problem 1.3, I have proposed the following:

**Conjecture 1.4** Let $X$ be a smooth projective Fano variety of $K3$ type (i.e. $\dim X = 2m$ and the Hodge numbers $h^{p,q}(X)$ are 0 for all $p \neq q$ except for $h^{m-1,m+1}(X) = h^{m+1,m-1}(X) = 1$). Then $X$ has a multiplicative Chow–Künneth decomposition.

This conjecture is answered positively in a few scattered cases [10, 19–21]. The aim of the present note is to provide some more evidence for Conjecture 1.4, by considering certain Fano eightfolds studied by Ito–Miura–Okawa–Ueda [14] and Fatighenti–Mongardi [7]. Following [7], we say that a variety of type $S2$ is a smooth divisor in a certain very ample linear system $L$ on the orthogonal Grassmannian $\text{OGr}(3,8)$. Varieties of type $S2$ are Fano eightfolds of $K3$ type (cf. subsection 2.2 below). The main result of this note is a verification of Conjecture 1.4 for varieties of type $S2$:

**Theorem** (=Theorem 4.1) Let $X$ be a variety of type $S2$. Then $X$ has a multiplicative Chow–Künneth decomposition.

This is proven by first showing that for a general variety $X$ of type $S2$, certain genus 7 $K3$ surfaces that are naturally associated to $X$ are also related to $X$ on the level of Chow motives (Theorem 3.1).
As a nice bonus, the theorem implies that the Chow ring of these Fano eightfolds behaves like that of K3 surfaces:

**Corollary** (=Corollary 4.2) Let $X$ be a variety of type $S_2$. Let $R^5(X) \subset A^5(X)$ denote the subgroup

$$R^5(X) := \langle A^1(X) \cdot A^4(X), A^2(X) \cdot A^3(X), c_5(T_X), \text{Im}(A^*(OGr(3,8)) \to A^*(X)) \rangle.$$

Then the cycle class map induces an injection

$$R^5(X) \hookrightarrow H^{10}(X, \mathbb{Q}) \cong \mathbb{Q}^4.$$

Since $A^5_{hom}(X)$ is infinite-dimensional (for $X$ general, there is an isomorphism $A^5_{hom}(X) \cong A^2_{hom}(S)$, where $S$ is an associated K3 surface, cf. Theorem 3.1), this is just as remarkable as Theorem 1.1.

It would be interesting to test Conjecture 1.4 for the other Fano varieties of K3 type exhibited in [7] and [3]. I hope to return to this in the near future.

**Conventions** In this note, the word variety will refer to a reduced irreducible scheme of finite type over the field of complex numbers $\mathbb{C}$. All Chow groups will be with $\mathbb{Q}$-coefficients, unless indicated otherwise: For a variety $X$, we will write $A_j(X) := \text{CH}_j(X)_{\mathbb{Q}}$ for the Chow group of dimension $j$ cycles on $X$ with rational coefficients. For $X$ smooth of dimension $n$, the notations $A_j(X)$ and $A^{n-j}(X)$ will be used interchangeably. The notation $A^i_{hom}(X)$ will be used to indicate the subgroups of homologically trivial cycles.

We will write $\mathcal{M}_{\text{int}}$ for the contravariant category of Chow motives (i.e., pure motives as in [25, 27]).

## 2 Preliminaries

### 2.1 MCK decomposition

**Definition 2.1** (Murre [24]) Let $X$ be a smooth projective variety of dimension $n$. We say that $X$ has a CK decomposition if there exists a decomposition of the diagonal

$$\Delta_X = \Pi^n_X + \Pi^{n-1}_X + \cdots + \Pi^1_X$$

in $A^n(X \times X)$, such that the $\Pi^i_X$ are mutually orthogonal idempotents and $(\Pi^i_X)_*H^i(X) = H^i(X)$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

**Remark 2.2** The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [15, 24].

**Definition 2.3** (Shen–Vial [28]) Let $X$ be a smooth projective variety of dimension $n$. Let $\Delta_X \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{sm} := \{ (x,x,x) \mid x \in X \} \subset X \times X \times X.$$
An MCK decomposition is a CK decomposition \(\{\Pi_X^i\}\) of \(X\) that is multiplicative, i.e. it satisfies

\[
\Pi_X^k \circ \Delta_X^{sm} \circ (\Pi_X^i \times \Pi_X^j) = 0 \quad \text{in} \quad A^{2n}(X \times X \times X) \quad \text{for all} \quad i + j \neq k.
\]

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.)

**Remark 2.4** The small diagonal (seen as a correspondence from \(X \times X\) to \(X\)) induces the multiplication morphism

\[
\Delta_X^{sm} : h(X) \otimes h(X) \to h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

Suppose \(X\) has a CK decomposition

\[
h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

By definition, this decomposition is multiplicative if for any \(i, j\) the composition

\[
h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}
\]

factors through \(h^{i+j}(X)\). It follows that if \(X\) has an MCK decomposition, then setting

\[
A^i_{(i,j)}(X) := (\Pi_X^{2i-j} )_* A^i(X),
\]

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends \(A^i_{(i,j)}(X) \otimes A^j_{(i,j)}(X)\) to \(A^{i+j}_{(i+j,j)}(X)\).

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “weak splitting property” [1]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [28, Section 8] and also [10, 11, 23, 29, 30].

### 2.2 Varieties of type S2

**Notation 2.5** Let \(\text{OGr}(3,8)\) denote the orthogonal Grassmannian of 3-dimensional isotropic subspaces of an 8-dimensional vector space equipped with a bilinear form. As explained in [14, Section 2] and [7, Section 3.7], there are morphisms
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where $F_1 \cong F_2$ are the two connected components of $\text{OGr}(4,8)$, and the $p_i$ are $\mathbb{P}^3$-fibrations. The Picard group of $\text{OGr}(3,8)$ has rank 2 and is generated by the pullbacks $(p_i)\ast \text{Pic}(F_i), i = 1, 2$. The line bundle

$$L := (p_1)\ast \mathcal{O}_{F_1}(1) \otimes (p_2)\ast \mathcal{O}_{F_2}(1) \in \text{Pic}(\text{OGr}(3,8))$$

is very ample, and there are natural isomorphisms

$$H^0(\text{OGr}(3,8), L) \cong H^0(F_1, (p_1)\ast L) \cong H^0(F_2, (p_2)\ast L).$$

**Definition 2.6** A variety of type $S_2$ is by definition a smooth hypersurface in the linear system $|L|$ on $\text{OGr}(3,8)$ (cf. Notation 2.5).

**Proposition 2.7** Let $X$ be a a variety of type $S_2$. Then $X$ is a Fano variety of dimension 8. The Hodge diamond of $X$ is

$$
\begin{array}{cccccccc}
1 & & & & & & & \\
& 2 & & & & & & \\
& & 3 & & & & & \\
& & & 4 & & & & \\
0 & \ldots & \ldots & 1 & 24 & 1 & \ldots & \ldots & 0 \\
& & & 4 & & & & \\
& & & 3 & & & & \\
& & & 2 & & & & \\
& & & 1 & & & & \\
\end{array}
$$

(where all empty entries are 0).

**Proof** This is [7, Section 3.7]. The Hodge numbers are computed in [6, Proposition A.1.1].

**Remark 2.8** Varieties of type $S_2$ are part of a long list of Fano varieties of K3 type given in [7].

**Proposition 2.9** Given $s \in H^0(\text{OGr}(3,8), L)$ a general section, let $s_1, s_2$ be the induced sections of $(p_1)\ast L$ resp. $(p_2)\ast L$ under the isomorphism (1). Let $S_i \subset F_i$ denote the zero loci of $s_i (i = 1, 2)$. Then $S_1$ and $S_2$ are K3 surfaces of genus 7. Moreover,
(i) \( S_1 \) and \( S_2 \) are \( L \)-equivalent: one has equality in the Grothendieck ring of varieties
\[
(S_1 - S_2) \cdot L^3 = 0 \quad \text{in} \ K_0(\text{Var})
\]
(ii) \( S_1 \) and \( S_2 \) are derived equivalent (i.e. their derived categories of coherent sheaves are isomorphic);
(iii) for \( s \) very general, \( S_1 \) and \( S_2 \) are not isomorphic.

**Proof** Except for (ii), this is contained in [14]. The hypothesis that \( s \) be very general in (iii) is made to ensure that the \( S_i \) have Picard number 1 (cf. [14, Section 4]).

Statement (ii) (which we merely state for illustration, and will not use below) is announced in [7] and proven in [3]. \( \square \)

**Remark 2.10** Proposition 2.9 (ii) implies in particular, via [13], that \( S_1 \) and \( S_2 \) have isomorphic Chow motives. This will be proven directly below (Corollary 3.2), without appealing to the derived equivalence.

**Definition 2.11** Let \( X \) be a variety of type \( S_2 \), and assume that the sections \( s_i \) as in Proposition 2.9 define smooth surfaces \( S_i \). We say that \( S_1 \) and \( S_2 \) are associated to \( X \).

2.3 The Franchetta property

**Definition 2.12** [9] Let \( X \to B \) be a smooth projective morphism. We say that \( X \to B \) has the Franchetta property if the following holds: any \( \Gamma \in A^\ast(X) \) which is fibrewise homologically trivial is fibrewise rationally trivial.

**Theorem 2.13** Let \( B \) be the Zariski open in \( \mathbb{P}H^0(\text{OGr}(3,8), L) \) parametrizing smooth dimensionally transverse hypersurfaces, and let \( B_0 \subset B \) such that for each fibre \( X_b = V(s_b) \) with \( b \in B_0 \) the zero locus \( S_b := V((pr_1)_*(s)) \subset F_1 \) is a smooth K3 surface. Let
\[
S \to B_0
\]
denote the universal family of sections of type \( S_b \).

The families
\[
S \to B_0, \quad S \times_{B_0} S \to B_0
\]
have the Franchetta property.

**Proof** This is not a surprising result: indeed, \( S \to B_0 \) contains the general K3 surface of genus 7 (i.e. there is a dominant morphism \( B_0 \to \mathcal{F}_7 \) to the moduli stack of genus 7 K3 surfaces), and the Franchetta property for \( S \to \mathcal{F}_7 \) and for \( S \times_{\mathcal{F}_7} S \) were already proven in [26] resp. in [9]. However, in [9, 26] the standard Mukai parametrization of genus 7 K3 surfaces was used, which is different from the parametrization \( B_0 \) used here, and so we need to do some extra work to prove Theorem 2.13.

The argument for \( S \to B_0 \) is similar to that of [26]. We use the following:
Lemma 2.14 [14] Let $\pi : X \to F_1$ denote the restriction of $p_1 : \text{OGr}(3, 8) \to F_1$ to $X$. The morphism $\pi : X \to F_1$ is a (Zariski locally trivial) $\mathbb{P}^3$-bundle over $S_1 \subset F_1$, and a (Zariski locally trivial) $\mathbb{P}^2$-bundle over $U := F_1 \setminus S_1$.

**Proof** This is [14, Lemma 2.1].

Let

$$\tilde{B} := \mathbb{P}^0(\text{OGr}(3, 8), \mathcal{L}),$$

and let us consider the universal family

$$S \to \tilde{B}$$

of possibly singular sections. There is an inclusion as a Zariski open $B_0 \subset B$. It follows from Lemma 2.14 that $\tilde{S}$ is a $\mathbb{P}^r$-fibration over $F_1$ (indeed, given a point $y \in F_1$ let $O_y := (p_1)^{-1} \cong \mathbb{P}^3 \subset \text{OGr}(3, 8)$ denote the fibre over $y$. Since $\mathcal{L}$ is base-point free, there exists a section of $\mathcal{L}$ not containing the whole fibre $O_y$, hence there is a surface $S_b \subset F_1$ avoiding the point $y$: every point $y \in Y$ imposes one condition on $\tilde{B}$).

Reasoning as in [26, Lemma 2.1], this implies that

$$\text{Im} \left( A^*(\tilde{S}) \to A^*(S_b) \right) = \text{Im} \left( A^*(F_1) \to A^*(S_b) \right) \quad \forall b \in B_0.$$

Since $A^2(F_1) \cong \mathbb{Q}$ is generated by intersections of divisors, this settles the Franchetta property for $S \to B_0$.

Next, we claim that the family $S \to B_0$ verifies property $(*_2)$ of [9, Definition 5.6]. This claim, combined with [9, Proposition 5.7] and the Franchetta property for $S \to B_0$ implies that

$$\text{Im} \left( A^*(\tilde{S} \times_B \tilde{S}) \to A^*(S_b \times S_b) \right) = \langle A^1(S_b), \Delta_{S_b} \rangle \quad \forall b \in B_0.$$

The right-hand side is known to inject into cohomology [31, Proposition 2.2], and so we are done.

To prove the claim, we reason as above: given two different points $y_1, y_2 \in F_1$, let $O_{y_1}, O_{y_2} \subset \text{OGr}(3, 8)$ denote the fibres. Given the definition of $\mathcal{L}$, one readily finds that restriction induces a surjection

$$H^0(\text{OGr}(3, 8), \mathcal{L}) \twoheadrightarrow H^0(O_{y_1}, \mathcal{L}|_{O_{y_1}}) \oplus H^0(O_{y_2}, \mathcal{L}|_{O_{y_2}}),$$

i.e. two different points $y_1, y_2 \in F_1$ impose 2 independent conditions on $\tilde{B}$. This proves the claim. \hfill $\Box$

### 3 An isomorphism of motives

**Theorem 3.1** Let $X \subset \text{OGr}(3, 8)$ be a variety of type $S_2$, and assume that $X$ has an associated K3 surface $S$. There is an isomorphism of motives
\( h(X) \cong h(S)(3) \oplus \bigoplus 1(*) \) in \( \mathcal{M}_{\text{rat}} \).

(In particular, one has \( A^i_{\text{hom}}(X) = 0 \) for all \( i \neq 5 \).)

**Proof** Without loss of generality we may assume \( S = S_1 \) with respect to the notation introduced above. Let \( \pi : X \to F_1 \) and \( U := F_1 \setminus S_1 \) be as in Lemma 2.14, i.e. \( \pi \) is generically a \( \mathbb{P}^2 \)-fibration but degenerates to a \( \mathbb{P}^3 \)-fibration over \( S_1 \). Then one has an isomorphism of motives

\[
h(X) \cong \bigoplus_{j=0}^{2} h(F_1)(j) \oplus h(S)(3) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

(2)

Since \( F_1 \cong \text{Spin}(8) \) is a 6-dimensional quadric (cf. [14, Remark 2.4]), the motive \( h(F_1) \) is isomorphic to a sum of twisted Lefschetz motives \( \bigoplus 1(*) \), and so the theorem follows from (2).

The isomorphism (2) can be proven using Voevodsky motives (as I did in [22, Proof of Theorem 2.1, equation (4)], where the situation is completely similar), but also follows directly by applying [16, Corollary 3.2]. \( \square \)

**Corollary 3.2** Let \( X \) be a variety of type \( S_2 \), and assume \( X \) has associated K3 surfaces \( S_1, S_2 \). Then

\[
h(S_1) \cong h(S_2) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

**Proof** Theorem 3.1 implies that there are isomorphisms of motives

\[
\Gamma_i : h(X) \cong t(S_i)(3) \oplus \bigoplus 1(*) \quad \text{in} \quad \mathcal{M}_{\text{rat}} \quad (i = 1, 2),
\]

and so one gets an isomorphism

\[
t(S_1) \oplus \bigoplus 1(*) \cong t(S_2) \oplus \bigoplus 1(*) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

This means that one can write

\[
t(S_1) \cong M_{tr} \oplus M_{alg},
\]

(3)

with \( M_{tr} \) being a submotive of \( t(S_2) \) and \( M_{alg} \) a submotive of \( \bigoplus 1(*) \), hence \( M_{alg} \) is of the form \( \bigoplus 1(*) \). Taking cohomology on both sides of (3), one finds that \( M_{alg} = 0 \).

Applying the same argument to \( t(S_2) \), one finds that \( t(S_1) \cong t(S_2) \) and hence (as both \( S_i \) are K3 surfaces, so have the same Betti numbers) one concludes that \( h(S_1) \cong h(S_2) \). \( \square \)

### 4 MCK for varieties of type \( S_2 \)

**Theorem 4.1** Let \( X \) be a variety of type \( S_2 \). Then \( X \) has a multiplicative Chow–Künneth decomposition.
The Chern classes $c_j(T_X)$, as well as cycles in the image of the restriction $A^*(\text{OGr}(3, 8)) \to A^*(X)$, are in $A^*_0(X)$. 

**Proof** Let

$$\mathcal{X} \to B$$

denote the universal family of smooth hyperplane sections of $\text{OGr}(3, 8)$ (i.e. $B$ is a Zariski open in the space of sections $\mathbb{P}H^0(\text{OGr}(3, 8), \mathcal{L})$, where notation is as above). Let $B_0 \subset B$ and $S \to B_0$ be as in Sect. 2.3. In view of a standard spread argument (cf. [33, Lemma 3.2]), it suffices to prove the theorem for $X_b$ with $b \in B_0$.

Theorem 3.1 gives us an isomorphism

$$h(X_b) \cong h(S_b)(3) \oplus \bigoplus_1^7 \mathbb{I}(\ast) \quad \text{in } \mathcal{M}_{\text{rat}},$$

(4)

for each $b \in B_0$. Even better, this isomorphism exists universally. Indeed, the maps

$$\Psi_b : h(S_b)(3) \to h(X_b)$$

that enter into the isomorphism (4) clearly exist universally (they are defined in terms of pullback and pushforward). A Hilbert schemes argument as in [32, Proposition 3.7] (cf. also [18, Proposition 2.11] for the precise form used here) then implies that the $\Gamma_b$ also exist universally, i.e. there exists $\Gamma \in A^5(\mathcal{X} \times_{B_0} \mathcal{S}) \oplus A^*(\mathcal{X})$ inducing fibrewise isomorphisms

$$\Gamma|_b : h(X_b) \cong h(S_b)(3) \oplus \bigoplus_1^7 \mathbb{I}(\ast) \quad \text{in } \mathcal{M}_{\text{rat}},$$

(5)

for each $b \in B_0$.

One can readily construct a universal CK decomposition for $X$, i.e. there exist cycles $\pi^1_X \in A^8(\mathcal{X} \times_{B_0} \mathcal{X})$ such that the restriction

$$\pi^1_{X_b} := \pi^1_X|_b \in A^8(X_b \times X_b)$$

defines a CK decomposition for $X_b$ for each $b \in B$. (This is a standard construction, cf. for instance [32, Lemma 3.6]. In brief, one observes that for any $i < 8$, $H^i(X_b) \cong H^i(\text{OGr}(3, 8))$ is algebraic, and so $\pi^i_{X_b}$ is of the form $\sum_k a^i_{jk} \otimes a^k$, where the $a^i_{jk}$ are a basis for $H^i(\text{OGr}(3, 8))$ and the $a^k$ form a dual basis for $H^{18-i}(X_b) \cong H^{18-i}(\text{OGr}(3, 8))$. One then defines $\pi^i_X, i < 8$ by restricting appropriate cycles in $OGr(3, 8) \times OGr(3, 8) \times B$. The cycles $\pi^i_X, i \geq 8$ are defined as the transpose of $\pi^{18-i}$, and the remaining cycle $\pi^8_X$ is defined as the difference $\Delta_X = \sum_{i \neq 8} \pi^i_X$.)

It remains to check that this is an MCK decomposition, i.e. one needs to check that the relative correspondence

$$\Phi_{ijk} := \pi^k_X \circ \Delta^m_X \circ ((pr_{13})^*(\pi^i_X) \cdot (pr_{24})^*(\pi^j_X)) \in A^{16}(\mathcal{X} \times_{B} \mathcal{X} \times_{B} \mathcal{X}), \quad i + j \neq k$$

is fibrewise equal to 0. (Here $\Delta^m_X \in A^{16}(\mathcal{X} \times_{B} \mathcal{X} \times_{B} \mathcal{X})$ denotes the relative small diagonal.) By a standard spread argument [33, Lemma 3.2], it suffices to prove that the fibrewise restriction $\Phi_{ijk}|_b$ is 0 for all $b \in B_0$. The assumption $i + j \neq k$ implies that $\Phi_{ijk}$ is fibrewise homologically trivial. Thus, the image

$$\langle \Gamma, \Gamma, \Gamma \rangle_s(\Phi_{ijk}) \in A^7(S \times_{B_0} S \times_{B_0} S) \oplus A^*(S \times_{B_0} S) \oplus A^*(S) \oplus A^*(B_0)$$

is 0 for all $b \in B_0$. The assumption $i + j \neq k$ implies that $\Phi_{ijk}$ is fibrewise homologically trivial.

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is also fibrewise homologically trivial. But then the Franchetta property (Theorem 2.13), plus the fact that $A^7(S_b) = 0$ for dimension reasons, implies that
\[(\Gamma|_b,\Gamma|_b,\Gamma|_b)_*(\Phi|_b) = (\Gamma,\Gamma,\Gamma,\Phi|_b)|_b = 0 \text{ for all } b \in B_0.\]
Since $(\Gamma|_b,\Gamma|_b,\Gamma|_b)_*$ is injective (cf. isomorphism (5)), this proves that
\[\Phi|_b = 0 \text{ in } A^16(X \times B X \times B X), \text{ for all } b \in B_0 \text{ and } i + j \neq k.\]
This proves the first part of the theorem.

For the second part, one observes that the above argument (passing from the family $\mathcal{X}$ to the family $S$) shows that the family $\mathcal{X} \to B$ (and also the family $\mathcal{X} \times_B \mathcal{X} \to B$) has the Franchetta property (that is, any $\Gamma \in A^*(\mathcal{X})$ that is fibrewise homologically trivial is fibrewise zero). It follows that
\[\text{Im}(A^i(\mathcal{X}) \to A^i(X_b)) \subset A^i_{(0)}(X),\]
as one sees by applying the Franchetta property to
\[(\pi^i_{X_b})_*(a|X_b) = (\pi^i_{X_b})_*(a)|X_b, \text{ for } i \neq 2j,\]
where $a \in A^i(\mathcal{X})$. The inclusion (6) applies to the Chern classes $c_j(T_{X_b}) = c_j(T_{X/B}|_{X_b}$ and also to cycles in $\text{Im}(A^*(\text{OGr}(3,8)) \to A^*(X_b))$. This proves the second part of the theorem.

\[\square\]

**Corollary 4.2** Let $X$ be an eightfold as in Theorem 4.1. Let $R^5(X) \subset A^5(X)$ denote the subgroup
\[R^5(X) := \langle A^1(X) \cdot A^4(X), A^2(X) \cdot A^3(X), c_5(T_X), \text{Im}(A^*(\text{OGr}(3,8)) \to A^*(X)) \rangle.\]
Then the cycle class map induces an injection
\[R^5(X) \hookrightarrow H^{10}(X,\mathbb{Q}) \cong \mathbb{Q}^4.\]

**Proof** Since $A^i_{\text{hom}}(X) = 0$ for $i \neq 5$, we have $A^i(X) = A^i_{(0)}(X)$ for $i \neq 5$. Combined with Theorem 4.1, this implies that $R^5(X) \subset A^5_{(0)}(X)$. It only remains to check that the cycle class map induces an injection
\[A^5_{(0)}(X) \hookrightarrow H^{10}(X,\mathbb{Q}).\]
To this end, we observe that (by construction) the correspondence $\pi^5_{X}$ is supported on $V \times W \subset X \times X$, where $V$ resp. $W$ are subvarieties of dimension 5 resp. 3. As in [5], the action of $\pi^5_{X}$ on $A^5(X)$ factors over $A^0(W)$ (where $\widetilde{W} \to W$ is a resolution of singularities). In particular, the action of $\pi^5_{X}$ on $A^5_{\text{hom}}(X)$ factors over $A^0_{\text{hom}}(\widetilde{W}) = 0$ and so is zero. But the action of $\pi^5_{X}$ on $A^5_{(0)}(X)$ is the identity, and so
\[A^5_{(0)}(X) \cap A^5_{\text{hom}}(X) = 0.\]
\[\square\]
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