ANALYTIC \( m \)-ISOMETRIES AND WEIGHTED DIRICHLET-TYPE SPACES

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Abstract. Corresponding to any \((m-1)\)-tuple of semi-spectral measures on the unit circle, a weighted Dirichlet-type space is introduced and studied. We prove that every analytic \( m \)-isometry which satisfies a certain set of operator inequalities can be represented as the operator of multiplication by the coordinate function on such a weighted Dirichlet-type space. This extends a result of Richter as well as of Olofsson on analytic 2-isometries. We also prove that all left invertible \( m \)-concave operators satisfying the aforementioned operator inequalities admit a Wold-type decomposition. This result serves as a key ingredient in our model theorem and it also generalizes a result of Shimorin on a class of 3-concave operators.

1. Introduction

In what follows, \( \mathcal{H} \) and \( \mathcal{E} \) will denote complex separable Hilbert spaces. The notations \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{E}) \) will stand for the set of all bounded linear operators on \( \mathcal{H} \) and \( \mathcal{E} \) respectively. Let \( m \) be a positive integer. An operator \( T \) in \( \mathcal{B}(\mathcal{H}) \) is said to be an \( m \)-isometry (resp. \( m \)-concave) if \( \beta_m(T) = 0 \) (resp. \( \beta_m(T) \leq 0 \)), where the \( m \)-th defect operator \( \beta_m(T) \) is defined by

\[
\beta_m(T) := \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} T^{*j} T^j.
\]

Further, we set \( \beta_0(T) = I \). An operator \( T \) in \( \mathcal{B}(\mathcal{H}) \) is said to be expansive (norm-increasing) if \( \beta_1(T) \geq 0 \). The notion of \( m \)-isometric operators was introduced and studied by Agler in [1]. Later in a series of papers, Agler and Stankus studied the class of \( m \)-isometries extensively, see [2, 3, 4]. An operator \( T \) in \( \mathcal{B}(\mathcal{H}) \) is called analytic if \( \mathcal{H}_\infty(T) = \{0\} \), where \( \mathcal{H}_\infty(T) \), the hyper-range of \( T \), is defined to be \( \mathcal{H}_\infty(T) := \bigcap_{n=0}^{\infty} T^n(\mathcal{H}) \).

Note that a 1-isometry is simply an isometry on a Hilbert space. The classical Wold-Kolmogorov decomposition theorem says that an isometry can be uniquely written as the direct sum of a unitary operator and an analytic isometry, see [19, Theorem 1.1]. More generally, it follows from [31, Proposition 3.4] that every expansive \( m \)-isometry can be uniquely
written as the direct sum of a unitary operator and an analytic $m$-isometry. Thus the study of analytic $m$-isometries are natural starting point in order to investigate expansive $m$-isometries.

Let $H^2(\mathcal{E})$ denote the Hardy space of $\mathcal{E}$-valued holomorphic functions on the open unit disc $\mathbb{D}$, that is,

$$H^2(\mathcal{E}) = \left\{ \sum_{j=0}^{\infty} a_j z^j : a_j \in \mathcal{E}, \sum_{j=0}^{\infty} \|a_j\|^2_{\mathcal{E}} < \infty \right\}.$$  

The operator $M_z$ of multiplication by the coordinate function on $H^2(\mathcal{E})$ is an analytic isometry. From the Wold-Kolmogorov decomposition theorem, it follows that every analytic isometry $T$ in $B(H)$ is unitarily equivalent to the operator $M_z$ on the Hardy space $H^2(\mathcal{E})$ with $\mathcal{E} = \ker T^*$ (cf. [19, Theorem 1.1]). This is known as the model theorem for analytic isometries.

In 1991, Richter [24, Theorem 3.7, Theorem 5.1] proved that any cyclic analytic 2-isometry is unitarily equivalent to the operator of multiplication by the coordinate function on a Dirichlet-type space $D(\mu)$ for some finite positive Borel measure $\mu$ on the unit circle $\mathbb{T}$, where

$$D(\mu) := \left\{ f \in \mathcal{O}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) \, dA(z) < \infty \right\},$$

$P_\mu(z)$ denotes the Poisson integral of the measure $\mu$, $dA(z)$ denotes the normalized Lebesgue area measure on the open unit disc $\mathbb{D}$ and $\mathcal{O}(\mathbb{D})$ denotes the space of complex valued holomorphic functions on the open unit disc $\mathbb{D}$. Olofsson generalized this result of Richter and obtained a model for an arbitrary analytic 2-isometry by considering weighted Dirichlet-type spaces $D(\mu)$ associated to a positive $\mathcal{B}(\mathcal{E})$-valued operator measure $\mu$ on $\mathbb{T}$, see [20, Theorem 3.1 and 4.1].

Recently, Rydhe characterized the class of cyclic $m$-isometries in terms of shifts on abstract spaces of weighted Dirichlet-type. It was shown that every cyclic $m$-isometry is unitarily equivalent to the multiplication operator by the coordinate function on a Hilbert space $D^2_\mu$ (depending on the operator as well as on the cyclic vector chosen), induced by an allowable $m$-tuple $\vec{\mu} = (\mu_0, \ldots, \mu_{m-1})$ of distributions on the unit circle $\mathbb{T}$, see [28, Theorem 3.1]. Motivated by Rydhe’s model, in this article, we attempt to find Richter-type model as well as Olofsson-type model for a class of analytic $m$-isometries. In case of $m = 2$, this class coincides with the class of all analytic 2-isometries.

Let $\mathcal{E}$ be a Hilbert space and $\mathcal{O}(\mathbb{D}, \mathcal{E})$ denote the space of $\mathcal{E}$-valued holomorphic functions on $\mathbb{D}$. By a $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure on the unit circle $\mathbb{T}$, we mean a positive $\mathcal{B}(\mathcal{E})$-valued operator measure on $\mathbb{T}$, see [20, p. 719] for definition and basic properties. The notation $\mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$ stands for the set of all $\mathcal{B}(\mathcal{E})$-valued semi-spectral measures on $\mathbb{T}$. We will simply write semi-spectral measure in place of $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure whenever there is no possibility of ambiguity about the underlying space $\mathcal{E}$. The set $\mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathbb{C}))$ coincides with the set of all finite positive Borel measures on $\mathbb{T}$ and is abbreviated to $\mathcal{M}_+(\mathbb{T})$. For a $\mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$, $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$ and a positive integer $n$, we consider
$D_{\mu,n}(f)$, the *weighted Dirichlet integral of order* $n$, defined by

$$D_{\mu,n}(f) := \frac{1}{n!(n-1)!} \int_{\mathbb{D}} \langle P_{\mu}(z)f^{(n)}(z), f^{(n)}(z) \rangle (1 - |z|^2)^{n-1} dA(z),$$

where $f^{(n)}$ denotes the $n$-th order complex derivative of the function $f$, that is, $f^{(n)}(z) := \frac{d^n}{dz^n} f(z)$, and $P_{\mu}(z)$ denotes the *Poisson integral* of the semi-spectral measure $\mu$. In the case of $\mathcal{E} = \mathbb{C}$, the integral $D_{\mu,n}(\cdot)$ considered here coincides with the notion of weighted Dirichlet integral of order $n$ as described in [28]. The novelty here lies in considering the associated semi-inner product space $\mathcal{H}_{\mu,n}(\mathcal{E})$ and the Hilbert space $\mathcal{H}_{\mu}(\mathcal{E})$ defined below.

For any $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure $\mu$ on $\mathbb{T}$ and any positive integer $n$, we associate a natural linear space

$$\mathcal{H}_{\mu,n}(\mathcal{E}) := \{ f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : D_{\mu,n}(f) < \infty \}.$$

It is routine to verify that $\mathcal{H}_{\mu,n}(\mathcal{E})$ is a semi-inner product space with respect to the semi-inner product

$$\langle f, g \rangle = \frac{1}{n!(n-1)!} \int_{\mathbb{D}} \langle P_{\mu}(z)f^{(n)}(z), g^{(n)}(z) \rangle (1 - |z|^2)^{n-1} dA(z), \quad f, g \in \mathcal{H}_{\mu,n}(\mathcal{E}).$$

The space $\mathcal{H}_{\mu,1}(\mathcal{C})$, that is, the case of $n = 1$ and $\mathcal{E} = \mathbb{C}$, has been studied extensively in the literature, see for instance [24, 25, 5, 29], [13, Ch.7] and references therein. The properties of the space $\mathcal{H}_{\mu,1}(\mathcal{E})$ has been discussed in [20]. In Section 2 of this article, we explore various properties of the higher order weighted Dirichlet integrals $D_{\mu,n}(\cdot)$ and the associated spaces $\mathcal{H}_{\mu,n}(\mathcal{E})$.

One of the interesting properties of the functions in $\mathcal{H}_{\mu,n}(\mathcal{E})$ is that these can be approximated through $r$-dilations. For $0 < r < 1$, the $r$-dilation of a function $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$ is denoted by $f_r$ and is defined by $f_r(z) := f(rz)$, $z \in \mathbb{D}$. In Section 3 of this article, we show that $D_{\mu,n}(f_r - f) \to 0$ as $r \to 1$, for every $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$. In case $\mathcal{E} = \mathbb{C}$ and $\mu$ is a finite positive Borel measure in $\mathcal{M}_+(\mathbb{T})$, Richter and Sundberg established the approximation through $r$-dilations by showing that, for every $f \in \mathcal{H}_{\mu,1}(-1)$ and $0 < r < 1$, $D_{\mu,1}(f_r) \leq CD_{\mu,1}(f)$ with constant $C = 4$, see [25, Theorem 5.2]. Aleman improved this result by replacing the constant 4 by $5/2$ (cf. [3, Lemma 4.1]). This remained the best value known for the constant until Sarason, in [29, Proposition 3], proved that $D_{\mu,1}(f_r) \leq D_{\mu,1}(f)$ for every $f \in \mathcal{H}_{\mu,1}(\mathbb{C})$, and for every $0 < r < 1$, see also [13, Lemma 7.3.2]. In Section 3 using Theorem 1.1, we show that the result of Sarason remains true even for the functions in $\mathcal{H}_{\mu,n}(\mathcal{E})$, where $n$ is an arbitrary positive integer and $\mathcal{E}$ is an arbitrary complex separable Hilbert space.

**Theorem 1.1.** Let $\mathcal{E}$ be a complex separable Hilbert space and $\mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$. Then for every positive integer $n$ and $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$, we have

$$D_{\mu,n}(f_r) \leq D_{\mu,n}(f), \quad 0 < r < 1,$$

and consequently $D_{\mu,n}(f_r - f) \to 0$ as $r \to 1$. 
In Section 4, for \( m \geq 2 \), we consider \((m-1)\)-tuple \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \) and then we introduce the corresponding weighted Dirichlet-type space \( \mathcal{H}_\mu(\mathcal{E}) \) of \( \mathcal{E} \)-valued holomorphic functions in the following way:

\[
\mathcal{H}_\mu(\mathcal{E}) := \bigcap_{j=1}^{m-1} \mathcal{H}_{\mu_j,j}(\mathcal{E}) \cap H^2(\mathcal{E})
\]

with the norm \( \| \cdot \|_\mu \) given by

\[
\|f\|_\mu^2 = \|f\|_{\mu^2(\mathcal{E})}^2 + \sum_{j=1}^{m-1} D_{\mu,j}(f), \quad f \in \mathcal{H}_\mu(\mathcal{E}).
\]

The space \( \mathcal{H}_\mu(\mathcal{E}) \) with respect to the norm \( \| \cdot \|_\mu \) is shown to be a reproducing kernel Hilbert space. We also show that the multiplication operator \( M_z \) on the Hilbert space \( \mathcal{H}_\mu(\mathcal{E}) \) is a bounded analytic \( m \)-isometry, see Theorem 4.1. It is easily noted that the set of \( \mathcal{E} \)-valued polynomials is contained in \( \mathcal{H}_\mu(\mathcal{E}) \). It is therefore natural to ask: For \( m \geq 2 \) and an \((m-1)\)-tuple \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \), is the set of \( \mathcal{E} \)-valued polynomials dense in \( \mathcal{H}_\mu(\mathcal{E}) \)? In case \( m = 2 \), the Hilbert space \( \mathcal{H}_\mu(\mathcal{E}) \) coincides with the model space \( D(\mu_1) \), described by Olofsson in [20], and an affirmative answer to the question is known in this case, see [20, Corollary 3.1]. Using Theorem 1.1, we answer this question in affirmative in general, see Proposition 4.2. In Theorem 4.3, we show that the operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) is an analytic \( m \)-isometry satisfying the following set of operator inequalities:

\[
\beta_r(M_z) \geq \sum_{n=1}^{\infty} L_{M_z}^* L_{M_z}^n \beta_{r+1}(M_z) L_{M_z}^n, \quad r = 1, \ldots, m - 2,
\]

where for a left invertible operator \( T \) in \( \mathcal{B}(\mathcal{H}) \), we use the notation \( L_T \) to denote the left inverse \((T^*T)^{-1}T^*\) of \( T \). This set of operator inequalities turns out to be a characteristic property of an analytic \( m \)-isometry to represent it as the multiplication operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) for some \((m-1)\)-tuple \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \). In Section 6, we establish the following model theorem which is one of the key findings of this article.

**Theorem 1.2.** Let \( m \geq 2 \) and \( T \in \mathcal{B}(\mathcal{H}) \). Then \( T \) is an analytic \( m \)-isometry satisfying

\[
\beta_r(T) \geq \sum_{k=1}^{\infty} L_T^* L_T^k \beta_{r+1}(T) L_T^k, \quad r = 1, \ldots, m - 2
\]

if and only if \( T \) is unitarily equivalent to the multiplication operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) for some \((m-1)\)-tuple \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \) with \( \mathcal{E} = \ker T^* \).

Note that for \( m = 2 \), the operator inequalities in (1) are vacuously satisfied. In case \( \ker T^* \) is of dimension one in above theorem, the associated \((m-1)\)-tuple of measures is uniquely determined. In the general case, the associated \((m-1)\)-tuple of semi-spectral measures is determined up to conjugation by a unitary operator, see Proposition 6.3 for the details.
For the operator $M_z$ on $H_\mu(E)$, we note that $\ker M_z^* = E$. Since the set of all $E$-valued polynomials is dense in $H_\mu(E)$, it follows that the operator $M_z$ on $H_\mu(E)$ has the wandering subspace property. Following [31, Definition 2.4], we say that an operator $T$ in $B(H)$ has the wandering subspace property if it satisfies
\[
\bigvee \{ T^n x : x \in \ker T^*, \, n \geq 0 \} = H.
\]

The term “wandering subspace” is attributed to Halmos [15]. In the literature, this property has been studied extensively, for instance see [23, 6, 31, 21, 16, 17, 18, 7] and references therein. In view of Theorem 1.2, it is necessary that an analytic $m$-isometry $T$ with $m \geq 2$, which satisfies the operator inequalities in (1), must have the wandering subspace property. In Section 5 of this article, we show that all analytic left invertible $m$-concave operators which satisfy the operator inequalities in (1) have the wandering subspace property. This plays a crucial role in proving Theorem 1.2.

**Theorem 1.3.** Let $T$ be an analytic left invertible $m$-concave operator in $B(H)$ for some $m \geq 2$. If $T$ satisfies
\[
\beta_r(T) \geq \sum_{k=1}^\infty L_T^{*k} \beta_{r+1}(T)L_T^k, \quad r = 1, \ldots, m-2,
\]
then $T$ has the wandering subspace property.

As a consequence of this theorem, it is obtained in Theorem 5.7 that any left invertible $m$-concave operator satisfying (1) admits a Wold-type decomposition (see Section 5 for definition). For $m = 3$, we find that Theorem 1.3 improves the result in [31, Corollary 3.10], see Remark 5.8 for details. Note that for $m = 2$, Theorem 1.3 gives us the wandering subspace property for every analytic $2$-concave operator, see [23, Theorem 1]. Since both, the analyticity and the concavity property, of an operator $T$ in $B(H)$ remain unchanged under the restriction of $T$ to its invariant subspaces, it follows that for any $B(E)$-valued semi-spectral measure $\mu$ on $T$, $M_z|_W$ has the wandering subspace property for every $M_z$ invariant subspace $W$ of $H_\mu(E)$. At this point, it is natural to ask if $\mu = (\mu_1, \ldots, \mu_{m-1})$ is an $(m-1)$ tuple of $B(E)$-valued semi-spectral measures on $T$ with $m \geq 3$, whether $M_z|_W$ has the wandering subspace property or not, for every $M_z$ invariant subspace $W$ of $H_\mu(E)$. Unfortunately, we are not able to answer this question right now as the conditions (1) in Theorem 1.3 may not be stable under the restriction of $T$ to its invariant subspaces. We hope to discover the answer to this question in our future work.

2. Properties of Weighted Dirichlet integral of order $n$

The symbols $\mathbb{Z}, \mathbb{N}$ and $\mathbb{Z}_+$ will denote the set of all integers, positive integers and non-negative integers respectively. By a $B(E)$-valued semi-spectral measure $\mu$ on $T$, we mean a finitely additive set function from the Borel $\sigma$-algebra of $T$ into the set of all positive operators in $B(E)$ such that $\mu_{x,y}(\cdot) := \langle \mu(\cdot)x, y \rangle$ defines a regular complex Borel measure on $T$ for every $x, y \in E$. As in the previous section, the notation $\mathcal{M}_+(\mathbb{T}, B(E))$ stands for the set
of all $B(\mathcal{E})$-valued semi-spectral measures on $\mathbb{T}$. For $\mu \in \mathcal{M}_+(\mathbb{T}, B(\mathcal{E}))$, consider the Poisson integral $P_\mu$ of $\mu$ given by

$$P_\mu(z) := \int_\mathbb{T} P(z, \zeta) d\mu(\zeta), \quad z \in \mathbb{D};$$

where $P(z, \zeta) = \frac{1-|z|^2}{|z-\zeta|^2}$, $z \in \mathbb{D}$, $\zeta \in \mathbb{T}$ is the Poisson kernel for the open unit disc $\mathbb{D}$. Note that for each $z \in \mathbb{D}$, $P_\mu(z)$ is a positive operator in $B(\mathcal{E})$ and for every $x, y \in \mathcal{E}$, the function $\langle P_\mu(z) x, y \rangle$ is the Poisson integral of the complex Borel measure $\mu_{x, y}$ on the unit circle $\mathbb{T}$. Thus the map $z \mapsto \langle P_\mu(z) x, y \rangle$ is a complex valued harmonic function on $\mathbb{D}$. For a positive integer $n$, consider the weighted Dirichlet integral $D_{\mu, n}(f)$ of order $n$, for $f \in O(\mathbb{D}, \mathcal{E})$ defined by

$$D_{\mu, n}(f) = \frac{1}{n!(n-1)!} \int_\mathbb{D} \langle P_\mu(z) f^{(n)}(z), f^{(n)}(z) \rangle (1 - |z|^2)^{n-1} dA(z).$$

It will be also useful to consider the weighted Dirichlet integral $D_{\mu, 0}(f)$ of order 0, given by

$$D_{\mu, 0}(f) = \lim_{R \to 1} \int_\mathbb{T} \langle P_\mu(R \zeta) f(R \zeta), f(R \zeta) \rangle d\sigma(\zeta),$$

provided the limit exists, where $d\sigma$ denotes the normalized arc length measure on the unit circle $\mathbb{T}$. We will see in Corollary 2.3 that $D_{\mu, 0}(f) < \infty$, that is, the corresponding limit exists and is finite, whenever $D_{\mu, n}(f) < \infty$ for some $n \in \mathbb{N}$.

The following lemma, a straightforward generalization of [28, Lemma 3.2 and Proposition 3.4] from the case of $\mathcal{E} = \mathbb{C}$ to an arbitrary complex separable Hilbert space $\mathcal{E}$, provides a formula for computing $D_{\mu, n}(f)$ for an arbitrary function $f$ in $O(\mathbb{D}, \mathcal{E})$, the space of all $\mathcal{E}$-valued functions holomorphic in some neighbourhood of the closed unit disc $\overline{\mathbb{D}}$. This can be obtained by computing $D_{\mu, n}(f)$ for any $\mathcal{E}$-valued polynomial $f$ and then using the observation that the power series of every function $f$ in $O(\mathbb{D}, \mathcal{E})$ converges uniformly on the closed unit disc $\overline{\mathbb{D}}$, one gets the desired formula for every $f$ in $O(\mathbb{D}, \mathcal{E})$, cf. [20, Equation (3.2)]. The second part of the lemma gives a relationship among the weighted Dirichlet integrals $D_{\mu, n}(f)$ for every $f$ in $O(\mathbb{D}, \mathcal{E})$, which can be easily verified from the formula obtained for $D_{\mu, n}(\cdot)$ in the first part.

**Lemma 2.1.** Let $f \in O(\mathbb{D}, \mathcal{E})$ and $\mu$ be a $B(\mathcal{E})$-valued semi-spectral measure on the unit circle $\mathbb{T}$.

(i) If $f(z) = \sum_{j=0}^\infty \hat{f}(j) z^j$ for $z \in \mathbb{D}$ then

$$D_{\mu, n}(f) = \sum_{k, l = n}^\infty \binom{k \wedge l}{n} \langle \hat{\mu}(l-k) \hat{f}(k), \hat{f}(l) \rangle, \quad n \in \mathbb{Z}_+,$$

where $k \wedge l = \min\{k, l\}$, and $\hat{\mu}(j) = \int_\mathbb{T} \zeta^{-j} d\mu(\zeta)$ for every $j \in \mathbb{Z}$. Moreover, the series on the right hand side is absolutely convergent.
(ii) The following identity holds:

\[ D_{\mu,n+1}(zf) - D_{\mu,n+1}(f) = D_{\mu,n}(f), \quad n \in \mathbb{Z}_+. \]

In Proposition 2.2, it is noted that the following refined version of the identity in Lemma 2.1 (ii) holds for functions in \( \mathcal{O}(\mathbb{D}, \mathcal{E}) \):

\[ D(\mu, n + 1, R, zf) - R^2 D(\mu, n + 1, R, f) = R^2 D(\mu, n, R, f) \]

where for \( 0 < R < 1 \) and for any \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \), the refined weighted Dirichlet integral of order \( n \) is defined by

\[
D(\mu, n, R, f) := \begin{cases} 
\frac{1}{n!(n-1)!} \int_{R\mathbb{D}} \langle P_\mu(z)f^{(n)}(z), f^{(n)}(z) \rangle (R^2 - |z|^2)^{n-1} dA(z), & n \geq 1, \\
\int_{\mathbb{T}} \langle P_\mu(R\zeta)f(R\zeta), f(R\zeta) \rangle d\sigma(\zeta), & n = 0.
\end{cases}
\]

Before we come to the refined version of the concerned identity, in the following proposition, we show that the refined weighted Dirichlet integral \( D(\mu, n, R, f) \) is equal to the weighted Dirichlet integral of order \( n \) of the function \( f_R \), the \( R \)-dilation of \( f \), given by \( f_R(z) = f(Rz), \ z \in \mathbb{D} \), with respect to an appropriate measure.

**Proposition 2.2.** Let \( \mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E})) \) and \( 0 < R < 1 \). Consider the \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measure \( \lambda_R \) on \( \mathbb{T} \) defined by \( d\lambda_R(\zeta) = P_\mu(R\zeta)d\sigma(\zeta) \). Then for every \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \) and \( n \in \mathbb{Z}_+ \), it follows that

\[ D(\mu, n, R, f) = D_{\lambda_R,n}(f_R). \]

Furthermore, for every \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \) and \( n \in \mathbb{N} \), it follows that \( D(\mu, n, R, f) \) increases to \( D_{\mu,n}(f) \) as \( R \) increases to 1.

**Proof.** For any \( x \in \mathcal{E} \), note that \( (\lambda_R)_{x,x} \) is a finite positive Borel measure on \( \mathbb{T} \) and is given by \( d(\lambda_R)_{x,x}(\zeta) = \langle P_\mu(R\zeta)x, x \rangle d\sigma(\zeta) \). Since the function \( z \mapsto \langle P_\mu(Rz)x, x \rangle \) is a harmonic function on a neighborhood of the closed unit disc \( \overline{\mathbb{D}} \), it follows that

\[ P_{(\lambda_R)_{x,x}}(z) = \langle P_\mu(Rz)x, x \rangle, \quad z \in \mathbb{D}. \]

Also, since \( P_{(\lambda_R)_{x,x}}(z) = \langle P_{\lambda_R}(z)x, x \rangle \) for every \( z \in \mathbb{D} \) and \( x \in \mathcal{E} \), it follows from the above equality that

\[ P_{\lambda_R}(z) = P_\mu(Rz), \quad z \in \mathbb{D}. \tag{2} \]

Note that \( f_R^{(n)}(w) = R^n f^{(n)}(Rw) \) for every \( w \in \mathbb{D} \) and for all \( n \in \mathbb{Z}_+ \). For \( n \geq 1 \), by a change of variables together with (2), we get that

\[
D_{\lambda_R,n}(f_R) = \int_{\mathbb{D}} \langle P_{\lambda_R}(w)f_R^{(n)}(w), f_R^{(n)}(w) \rangle (1 - |w|^2)^{n-1} dA(w) \\
= \int_{R\mathbb{D}} \langle P_\mu(z)f^{(n)}(z), f^{(n)}(z) \rangle (R^2 - |z|^2)^{n-1} dA(z) \\
= D(\mu, n, R, f).
\]
In case of \( n = 0 \), first note that \( f_n \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \) and the map \( w \mapsto \mu(Rw) \) is continuous on the closed unit disc \( \overline{\mathbb{D}} \). Thus it follows from (2) that
\[
\langle P_{\lambda R}(r\zeta)f_{R}(r\zeta), f_{n}(r\zeta) \rangle \to \langle P_{\mu}(R\zeta)f(\zeta), f(R\zeta) \rangle \quad \text{as} \quad r \to 1.
\]
Hence an application of dominated convergence theorem will give us
\[
D_{\lambda R, 0}(f_{n}) = \int_{\mathbb{T}} \langle P_{\mu}(R\zeta)f(R\zeta), f(R\zeta) \rangle \, d\sigma(\zeta) = D(\mu, 0, R, f).
\]
This completes the proof of the first part of the proposition. For the last part, let \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \) and \( n \in \mathbb{N} \). Note that for each \( z \in \mathbb{D} \), the function \( \chi_{\mathbb{R}^d}(z)\langle P_{\mu}(z)f^{(n)}(z), f^{(n)}(z)\rangle(R^2 - |z|^2)^{n-1} \) increases to \( \langle P_{\mu}(z)f^{(n)}(z), f^{(n)}(z)\rangle(1^2 - |z|^2)^{n-1} \) as \( R \) increases to 1, where \( \chi_{A}(z) \) denotes the characteristic function supported on \( A \). Hence by an application of monotone convergence theorem desired result follows. \( \square \)

The following proposition establishes a difference identity which is a refinement of that in Lemma 2.1(ii) for \( \mathcal{E} \)-valued holomorphic functions on the open unit disc \( \mathbb{D} \).

**Proposition 2.3.** Let \( \mu \in \mathcal{M}_{+}(\mathbb{T}, \mathcal{B}(\mathcal{E})) \), \( n \in \mathbb{Z}_{+} \) and \( 0 < R < 1 \). Then for any \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \),
\[
D(\mu, n + 1, R, zf) - R^2D(\mu, n + 1, R, f) = R^2D(\mu, n, R, f).
\]

**Proof.** Consider the \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measure \( \lambda_{R} \) on the unit circle \( \mathbb{T} \) defined by \( d\lambda_{R}(\zeta) = P_{\mu}(\zeta)d\sigma(\zeta) \). By Proposition 2.2, we have \( D(\mu, n, R, f) = D_{\lambda R, n}(f_{R}) \). Note that \( R^{n}(zf)^{(n+1)}(Rw) = (zf_{R})^{(n+1)}(w) \) for every \( w \in \mathbb{D} \). Again, by a change of variable, it is easy to verify that \( D(\mu, n + 1, R, zf) = R^2D_{\lambda R, n+1}(zf_{R}) \). Hence, it follows that for every \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \),
\[
D(\mu, n + 1, R, zf) - R^2D(\mu, n + 1, R, f) = R^2(D_{\lambda R, n+1}(zf_{R}) - D_{\lambda R, n+1}(f_{R})).
\]
Since \( f_{R} \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \), an application of Lemma 2.1(ii) yields that
\[
D(\mu, n + 1, R, zf) - R^2D(\mu, n + 1, R, f) = R^2D_{\lambda R, n}(f_{R}).
\]
This completes the proof. \( \square \)

Let \( \mu \) be a \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \) and \( n \) be an arbitrary but fixed positive integer. For the weighted Dirichlet integral \( D_{\mu,n}(\cdot) \), we consider the associated weighted Dirichlet-type space \( \mathcal{H}_{\mu,n}(\mathcal{E}) \) defined by
\[
\mathcal{H}_{\mu,n}(\mathcal{E}) := \{ f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : D_{\mu,n}(f) < \infty \}.
\]
It is straightforward to see that \( \mathcal{H}_{\mu,n}(\mathcal{E}) \) is a linear subspace of \( \mathcal{O}(\mathbb{D}, \mathcal{E}) \), containing the set of all \( \mathcal{E} \)-valued polynomials. Further, \( \mathcal{H}_{\mu,n}(\mathcal{E}) \) is a semi-inner product space induced by the semi-norm \( \sqrt{D_{\mu,n}(\cdot)} \). If \( \mathcal{E} = \mathbb{C} \) and \( \mu \) is the Lebesgue measure \( \sigma \) on \( \mathbb{T} \) then the integral \( D_{\sigma,1}(\cdot) \) becomes the usual Dirichlet integral and the space \( \mathcal{H}_{\sigma,1}(\mathbb{C}) \) coincides with
the classical Dirichlet space on $\mathbb{D}$. More generally, the set of all monomials $\{z^j : j \geq 0\}$ forms an orthogonal set in $H_{\sigma,n}(\mathbb{C})$ and

$$D_{\sigma,n}(z^j) = \begin{cases} 0, & 0 \leq j \leq n - 1, \\ \binom{j}{n}, & j \geq n. \end{cases}$$

The space $H_{\mu,n}(\mathcal{E})$ is a central topic of study in this section. In Proposition 2.6 we will show that the coordinate function $z$ is a multiplier of $H_{\mu,n}(\mathcal{E})$, that is, $f \in H_{\mu,n}(\mathcal{E})$ implies $zf \in H_{\mu,n}(\mathcal{E})$. Converse is also proved in Lemma 2.9(ii). To achieve this goal, we introduce another family of semi-norms which is a generalization of the class of norms introduced in [32] to the case of vector valued functions.

Suppose $Q$ is a positive operator in $B(\mathcal{E})$ and $\alpha \in \mathbb{R}$. Let $D_{\alpha,Q}(\mathcal{E}) := \{ f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : \| f \|_{D_{\alpha,Q}(\mathcal{E})} < \infty \}$, where for any $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathcal{O}(\mathbb{D}, \mathcal{E})$, the semi-norm $\| f \|_{D_{\alpha,Q}(\mathcal{E})}$ is defined by

$$\| f \|^2_{D_{\alpha,Q}(\mathcal{E})} = \sum_{k=0}^{\infty} (k + 1)^\alpha \langle Qa_k, a_k \rangle.$$  \hspace{1cm} (3)

It is easily verified that the space $D_{\alpha,Q}(\mathcal{E})$ equipped with the semi-norm $\| \cdot \|_{D_{\alpha,Q}(\mathcal{E})}$ is a semi-inner product space. Furthermore, if $Q$ is invertible then $\| \cdot \|_{D_{\alpha,Q}(\mathcal{E})}$ turns out to be a norm and the space $D_{\alpha,Q}(\mathcal{E})$ becomes a Hilbert space. Note that, if $Q$ is invertible then the two norms $\| \cdot \|_{D_{\alpha,Q}(\mathcal{E})}$ and $\| \cdot \|_{D_{\alpha,I}(\mathcal{E})}$ are equivalent and the associated spaces $D_{\alpha,Q}(\mathcal{E})$ and $D_{\alpha,I}(\mathcal{E})$ coincide. For any real number $\beta$ with $\alpha \leq \beta$, it is straightforward to verify that $\| f \|_{D_{\alpha,Q}(\mathcal{E})} \leq \| f \|_{D_{\beta,Q}(\mathcal{E})}$. Consequently, in this case, we have $D_{\beta,Q}(\mathcal{E}) \subseteq D_{\alpha,Q}(\mathcal{E})$. When $\mathcal{E} = \mathbb{C}$ and $Q = 1$, the Hilbert space $D_{\alpha,Q}(\mathcal{E})$ is simply denoted by $D_{\alpha}$. Many classical functional Hilbert spaces are given by $D_{\alpha}$; for example, $D_{-1}$, $D_0$ and $D_1$ are the Bergman space, the Hardy space and the Dirichlet space on the open unit disc $\mathbb{D}$ respectively. From [3], it can be easily seen that

$$f \in D_{\alpha,Q}(\mathcal{E}) \text{ if and only if } f' \in D_{\alpha-2,Q}(\mathcal{E}).$$ \hspace{1cm} (4)

For $\alpha < 0$, the semi-norms $\| \cdot \|_{D_{\alpha,Q}(\mathcal{E})}$ and $\| \cdot \|_{\alpha,Q}$ on $D_{\alpha,Q}(\mathcal{E})$ are equivalent (a straightforward generalization of [32] Lemma 2), where $\| f \|_{\alpha,Q}$ is given by

$$\| f \|^2_{\alpha,Q} := \int_{\mathbb{D}} \langle Qf(z), f(z) \rangle (1 - |z|^2)^{-\alpha-1} dA(z).$$ \hspace{1cm} (5)

For any $\mu \in \mathcal{M}_+(\mathbb{T}, B(\mathcal{E}))$, the following lemma establishes a relationship between the spaces $H_{\mu,n}(\mathcal{E})$ and $D_{n,\mu(\mathbb{T})}(\mathcal{E})$.

**Lemma 2.4.** Let $\mu$ be a $B(\mathcal{E})$-valued semi-spectral measure on the unit circle $\mathbb{T}$. Then,

$$H_{\mu,n}(\mathcal{E}) \subseteq D_{n-1,\mu(\mathbb{T})}(\mathcal{E}), \quad n \geq 1,$$

$$D_{n+1,\mu(\mathbb{T})}(\mathcal{E}) \subseteq H_{\mu,n}(\mathcal{E}), \quad n \geq 2.$$
Proof. From [26, p. 236], it is easy to see that the Poisson kernel satisfies the following estimates:

\[
\frac{1 - |z|^2}{4} \leq P(z, \zeta) \leq \frac{4}{(1 - |z|^2)}, \quad z \in \mathbb{D}, \ \zeta \in \mathbb{T}.
\]

This in turn implies that

\[
\mu(\mathbb{T}) \frac{1}{4} (1 - |z|^2)^n \leq P_\mu(z)(1 - |z|^2)^{n-1} \leq 4\mu(\mathbb{T})(1 - |z|^2)^{n-2}, \quad n \in \mathbb{N}, \ z \in \mathbb{D}.
\]

(6)

Using (5) along with the above estimates in (6), it follows that for \( f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \),

\[
D_{\mu,n}(f) \geq \frac{1}{4n!(n-1)!} \|f^{(n)}\|_{-\mu,\mu(\mathbb{T})}^2, \quad n \geq 1,
\]

(7)

\[
D_{\mu,n}(f) \leq \frac{4}{n!(n-1)!} \|f^{(n)}\|_{-\mu,\mu(\mathbb{T})}^2, \quad n \geq 2.
\]

(8)

Note that for \( \alpha < 0 \), the semi-norm \( \|f\|_{\alpha,\mu(\mathbb{T})} \) is equivalent to the semi-norm \( \|f\|_{\mathcal{D}_{\alpha,\mu(\mathbb{T})}(\mathcal{E})} \). Using this fact together with (1), (7), and (8), we obtain the desired conclusion. \( \square \)

Corollary 2.5. Let \( n \in \mathbb{N} \) and \( \mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E})) \). If \( \mu(\mathbb{T}) \) is invertible then \( \mathcal{H}_{\mu,n}(\mathcal{E}) \subseteq H^2(\mathcal{E}) \).

Proof. Note that \( \mathcal{D}_{-1,\mu(\mathbb{T})}(\mathcal{E}) \subseteq \mathcal{D}_{0,\mu(\mathbb{T})}(\mathcal{E}) \) for every \( n \in \mathbb{N} \). Since \( \mu(\mathbb{T}) \) is invertible, it follows that the two norms \( \|\cdot\|_{\mathcal{D}_{0,\mu(\mathbb{T})}(\mathcal{E})} \) and \( \|\cdot\|_{\mathcal{D}_{0,1}(\mathcal{E})} \) are equivalent and the associated spaces \( \mathcal{D}_{0,\mu(\mathbb{T})}(\mathcal{E}) \) and \( \mathcal{D}_{0,1}(\mathcal{E}) \) are equal. By definition, the space \( \mathcal{D}_{0,1}(\mathcal{E}) \) is equal to the Hardy space \( H^2(\mathcal{E}) \) of \( \mathcal{E} \)-valued holomorphic functions on the open unit disc \( \mathbb{D} \). The corollary is now immediate in the view of Lemma 2.4. \( \square \)

Now we are ready to show that the coordinate function \( z \) is a multiplier for \( \mathcal{H}_{\mu,n}(\mathcal{E}) \). The proof is divided into two different cases. Although the proof for the case of \( n = 1 \) follows from an argument in [20, Theorem 3.1], nevertheless we provide an alternative proof for the sake of completeness.

Proposition 2.6. Let \( \mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E})) \) and \( n \in \mathbb{N} \). The coordinate function \( z \) is a multiplier for the semi-inner product space \( \mathcal{H}_{\mu,n}(\mathcal{E}) \).

Proof. The proof is divided into two cases.

Case \( n = 1 \): Let \( f \in \mathcal{H}_{\mu,1}(\mathcal{E}) \). Note that \( f = f(0) + zg \) for some \( g \in \mathcal{O}(\mathbb{D}, \mathcal{E}) \). As \( P_\mu(z) \) is a positive operator, by an application of triangle inequality, it is straightforward to verify that

\[
\langle P_\mu(z)f(z), f(z) \rangle \leq 2\langle P_\mu(z)f(0), f(0) \rangle + 2\langle P_\mu(z)zg(z), zg(z) \rangle, \quad z \in \mathbb{D}.
\]

Let \( R \in (0,1) \). Since the map \( z \mapsto \langle P_\mu(Rz)f(0), f(0) \rangle \) is a positive harmonic function on a neighborhood of the closed unit disc \( \overline{\mathbb{D}} \), by an application of the mean value property for harmonic functions, it follows that

\[
\int_{\mathbb{T}} \langle P_\mu(R\zeta)f(0), f(0) \rangle d\sigma(\zeta) = \langle P_\mu(0)f(0), f(0) \rangle = \langle \mu(\mathbb{T})f(0), f(0) \rangle.
\]
Thus we get that
\[ D(\mu, 0, R, f) = \int_{\mathbb{T}} \langle P_\mu(R\zeta)f(R\zeta), f(R\zeta) \rangle d\sigma(\zeta) \]
\[ \leq 2\langle \mu(\mathbb{T})f(0), f(0) \rangle + 2R^2 \int_{\mathbb{T}} \langle P_\mu(R\zeta)g(R\zeta), g(R\zeta) \rangle d\sigma(\zeta) \]
\[ = 2\langle \mu(\mathbb{T})f(0), f(0) \rangle + 2R^2D(\mu, 0, R, g). \]
Applying Proposition 2.3 to the function \( g \), we find that
\[ R^2D(\mu, 0, R, g) \leq D(\mu, 1, R, zg). \]
Thus we have
\[ D(\mu, 0, R, f) \leq 2\langle \mu(\mathbb{T})f(0), f(0) \rangle + 2D(\mu, 1, R, zg). \]
Note that \( D(\mu, 1, R, zg) = D(\mu, 1, R, f) \). This together with Proposition 2.3 will give us
\[ D(\mu, 1, R, zf) \leq 2\langle \mu(\mathbb{T})f(0), f(0) \rangle + 3D(\mu, 1, R, f). \]
(9)

Now taking limit \( R \to 1 \) on both the sides of (9), we get \( D_{\mu,1}(zf) \leq 2\langle \mu(\mathbb{T})f(0), f(0) \rangle + 3D_{\mu,1}(f) \). This shows that \( zf \in \mathcal{H}_{\mu,1}(E) \) whenever \( f \in \mathcal{H}_{\mu,1}(E) \).

**Case** \( n \geq 2 \) : Consider \( d\nu(z) := (1 - |z|^2)^{n-1}P_{\mu}(z)dA(z) \), a \( \mathcal{B}(E) \)-valued weighted area measure on the unit disc \( \mathbb{D} \). Note that an \( \mathcal{E} \)-valued holomorphic function \( f \) on the unit disc \( \mathbb{D} \) is in \( \mathcal{H}_{\mu,n}(E) \) if and only if \( f^{(n)} \) is in \( L^2_{\mu}(\mathbb{D}, \mathcal{E}, d\nu) \), where
\[ L^2_{\mu}(\mathbb{D}, \mathcal{E}, d\nu) := \left\{ g \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : \int_{\mathbb{D}} \langle P_\mu(z)g(z), g(z) \rangle (1 - |z|^2)^{n-1}dA(z) < \infty \right\}. \]

Let \( f \in \mathcal{H}_{\mu,n}(E) \). We easily see that \( zf^{(n)} \in L^2_{\mu}(\mathbb{D}, \mathcal{E}, d\nu) \). In order to show that \( zf \in \mathcal{H}_{\mu,n}(E) \), using the relation \( (zf)^{(n)} = zf^{(n)} + nf^{(n-1)} \), it suffices to prove that \( f^{(n-1)} \in L^2_{\mu}(\mathbb{D}, \mathcal{E}, d\nu) \). By Lemma 2.4, \( \mathcal{H}_{\mu,n}(E) \subseteq \mathcal{D}_{n-1,\mu(\mathbb{T})}(E) \) and hence \( f \in \mathcal{D}_{n-1,\mu(\mathbb{T})}(E) \). By repeated use of (4), one obtains \( f^{(n-1)} \in \mathcal{D}_{-(n-1),\mu(\mathbb{T})}(E) \) and therefore \( \|f^{(n-1)}\|_{-(n-1),\mu(\mathbb{T})} < \infty \). Using (5) and (3), we get
\[ 4\|f^{(n-1)}\|_{-(n-1),\mu(\mathbb{T})} = 4\int_{\mathbb{D}} \langle \mu(\mathbb{T})f^{(n-1)}(z), f^{(n-1)}(z) \rangle (1 - |z|^2)^{n-2}dA(z) \]
\[ \geq \int_{\mathbb{D}} \langle P_\mu(z)f^{(n-1)}(z), f^{(n-1)}(z) \rangle (1 - |z|^2)^{n-1}dA(z). \]

Hence \( f^{(n-1)} \in L^2_{\mu}(\mathbb{D}, \mathcal{E}, d\nu) \), completing the proof of the proposition.

As the coordinate function \( z \) is a multiplier for \( \mathcal{H}_{\mu,n}(E) \), we obtain that the difference identity as described in Lemma 2.1(ii) remains valid for a larger class of functions, namely for functions in \( \mathcal{H}_{\mu,n}(E) \).

**Proposition 2.7.** Let \( \mu \) be a \( \mathcal{B}(E) \)-valued semi-spectral measure on the unit circle \( \mathbb{T} \) and \( n \) be a positive integer. Then for every \( f \) in \( \mathcal{H}_{\mu,n}(E) \), \( D_\mu(zf) = D_\mu(\mu, n) - D_\mu(n-1) = D_\mu(n-1)(f) \).

**Proof.** Let \( 0 < R < 1 \) and \( f \in \mathcal{H}_{\mu,n}(E) \). By Proposition 2.3, we have that
\[ D(\mu, n, R, zf) - R^2D(\mu, n, R, f) = R^2D(\mu, n - 1, R, f). \]
(10)
Note that, by Proposition 2.6, $zf \in \mathcal{H}_{\mu,n}(\mathcal{E})$. Thus by taking limit as $R \to 1$ on the both sides of (10), we obtain the desired identity. \hfill \Box

As an immediate corollary of the above proposition, we get the following interesting inclusion.

**Corollary 2.8.** Let $\mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$ and $n \in \mathbb{N}$ and $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$. If $D_{\mu,n}(f) < \infty$, then $D_{\mu,n-1}(f) < \infty$. Consequently, we have $\mathcal{H}_{\mu,j+1}(\mathcal{E}) \subseteq \mathcal{H}_{\mu,j}(\mathcal{E})$ for every $j \in \mathbb{N}$.

In the following lemma, we present the converse of Proposition 2.6. For any $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$, let $Lf$ be the $\mathcal{E}$-valued function on the unit disc $\mathbb{D}$, defined by

$$Lf(z) = \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{D}.$$ 

Note that $Lf \in \mathcal{O}(\mathbb{D}, \mathcal{E})$ and $D_{\mu,n}(zLf) = D_{\mu,n}(f)$ for every $n \in \mathbb{N}$ and $\mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$.

**Lemma 2.9.** Let $\mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$ and $n \in \mathbb{N}$. Then for every $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$, we have

(i) $D_{\mu,n}(Lf) \leq D_{\mu,n}(f)$.

(ii) $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ if and only if $zf \in \mathcal{H}_{\mu,n}(\mathcal{E})$.

**Proof.** Suppose $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$. As an application of Proposition 2.3 for any $0 < R < 1$ and $n \in \mathbb{N}$, we obtain

$$R^2 D(\mu, n, R, f) \leq D(\mu, n, R, zf).$$

Now taking limit as $R \to 1$ on both sides, we get that $D_{\mu,n}(f) \leq D_{\mu,n}(zf)$. Since $D_{\mu,n}(zLf) = D_{\mu,n}(f)$, it follows that $D_{\mu,n}(Lf) \leq D_{\mu,n}(f)$ for every $f \in \mathcal{O}(\mathbb{D}, \mathcal{E})$. This shows that $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ whenever $zf \in \mathcal{H}_{\mu,n}(\mathcal{E})$. The remaining part follows from Proposition 2.6. \hfill \Box

The following lemma can be thought of as a generalization of [24, Lemma 3.3]. This will be an essential ingredient in proving Theorem 1.1. We will see in Corollary 3.3 of Section 3 that this lemma is also valid for $n \geq 1$ and $j = 0$.

**Lemma 2.10.** Let $n \geq 2$ and $\mu$ be a $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$. Then for any function $f$ in $\mathcal{H}_{\mu,n}(\mathcal{E})$, we have

$$\sum_{k=1}^{\infty} D_{\mu,j}(L^k f) = D_{\mu,j+1}(f), \quad 1 \leq j \leq n - 1. \quad (11)$$

**Proof.** Let $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ and $k \in \mathbb{N}$. It follows from Lemma 2.9 that $L^k f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ and hence by Corollary 2.8, $D_{\mu,j}(L^k f)$ is also finite for all $j = 1, \ldots, n - 1$. Let $0 < R < 1$ and $j \in \{1, \ldots, n - 1\}$. Since $D(\mu, j + 1, R, f) = D(\mu, j + 1, R, zf)$, by Proposition 2.3, we obtain that

$$D(\mu, j + 1, R, f) = R^2 D(\mu, j + 1, R, Lf) + R^2 D(\mu, j, R, Lf).$$

Applying this equality repeatedly, one obtains

$$D(\mu, j + 1, R, f) = R^{2k} D(\mu, j + 1, R, L^k f) + \sum_{i=1}^{k} R^{2i} D(\mu, j, R, L^i f). \quad (12)$$
By Corollary 2.8, we have \( f \in \mathcal{H}_{\mu,j+1}(E) \) and consequently from Lemma 2.9 we have \( L^k f \in \mathcal{H}_{\mu,j+1}(E) \). Note that \( D(\mu, j + 1, R, L^k f) \) increases to \( D_{\mu,j+1}(L^k f) \) as \( R \to 1 \). Now by repeated applications of Lemma 2.9 we get \( D(\mu, j + 1, R, L^k f) \leq D_{\mu,j+1}(L^k f) \leq D_{\mu,j+1}(f) \). Thus, it follows that \( \lim_{k \to \infty} R^{2k} D(\mu, j + 1, R, L^k f) = 0 \). Hence, by (12), the series \( \sum_{i=1}^{\infty} R^{2i} D(\mu, j, R, L^i f) \) is convergent and

\[
\sum_{i=1}^{\infty} R^{2i} D(\mu, j, R, L^i f) = D(\mu, j + 1, R, f). \tag{13}
\]

Since for all \( i \in \mathbb{N} \), \( D(\mu, j, R, L^i f) \) increases to \( D_{\mu,j}(L^i f) \) as \( R \to 1 \), an application of the monotone convergence theorem completes the proof. \( \square \)

3. Approximations through dilations

Let \( \mu \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(E)) \) and \( n \) be a positive integer. For \( 0 < r < 1 \) and \( f \in \mathcal{O}(\mathbb{D}, E) \), let \( f_r \) denote the \( r \)-dilation of \( f \), that is, \( f_r(z) := f(rz) \), \( z \in \mathbb{D} \). In this section, we will show that for every \( f \in \mathcal{H}_{\mu,n}(E) \), we have \( f_r \to f \) in \( \mathcal{H}_{\mu,n}(E) \) as \( r \to 1 \). A standard approach to obtain this is to find a positive constant \( C \) such that \( D_{\mu,n}(f_r) \leq CD_{\mu,n}(f) \) holds for every \( 0 < r < 1 \). In case of \( n = 1 \) and \( E = \mathbb{C} \), in [29, Proposition 3], Sarason proved that \( D_{\mu,1}(f_r) \leq D_{\mu,1}(f) \) for every \( f \in \mathcal{H}_{\mu,1}(\mathbb{C}) \) and \( 0 < r < 1 \), (see also [25, Theorem 5.2], [5, Lemma 4.1] and [13, Lemma 7.3.2]). In what follows, we show that \( D_{\mu,n}(f_r) \leq D_{\mu,n}(f) \) for any positive integer \( n \) and any complex separable Hilbert space \( E \). Before we provide a proof of this, we choose to draw a proof for the base case, that is, the case of \( n = 1 \) and arbitrary \( E \) in the following lemma.

**Lemma 3.1.** Let \( \mu \) be a \( \mathcal{B}(E) \)-valued semi-spectral measure on \( \mathbb{T} \). Then for any \( f \in \mathcal{H}_{\mu,1}(E) \) and \( 0 < r < 1 \), the inequality \( D_{\mu,1}(f_r) \leq D_{\mu,1}(f) \) holds.

**Proof.** Let \( p \) be a \( E \)-valued polynomial given by \( p(z) = \sum_{j=0}^{d} c_j z^j \). By Lemma 2.1 we have

\[
D_{\mu,1}(p) = \sum_{k,l=1}^{d} (k \wedge l) \langle \hat{\mu}(l - k)c_k, c_l \rangle. \tag{14}
\]

Consider the matrix \( A = ((A_{k,l}))_{k,l=0}^{\infty} \), where \( A_{k,l} = (k \wedge l) \hat{\mu}(l - k) \) for \( k, l \geq 0 \). In view of (14), it follows that the matrix \( A \) is formally positive semi-definite. Let \( \sigma^* A \) be the infinite matrix whose \((k, l)\)-th element is given by \( (\sigma^* A)_{k,l} = A_{k+1,l+1} \) for \( k, l \geq 0 \). By Proposition 2.7 we also have

\[
D_{\mu,1}(z^2 p) - 2D_{\mu,1}(zp) + D_{\mu,1}(p) = D_{\mu,0}(zp) - D_{\mu,0}(p) = 0.
\]

This is equivalent to saying that \( (\sigma^* - I)^2 A = 0 \). Thus from [31, Theorem 3.11], it follows that the matrix \( ((1 - r^{k+l})A_{k,l}))_{k,l=0}^{\infty} \) is formally positive semi-definite for every \( 0 < r < 1 \). This gives us \( D_{\mu,1}(p_r) \leq D_{\mu,1}(p) \) for every \( E \)-valued polynomial \( p \) and \( 0 < r < 1 \). By a simple uniform limit argument, it follows that for each \( r \in (0, 1) \), we have

\[
D_{\mu,1}(f_r) \leq D_{\mu,1}(f),
\]
whenever $f$ is a $\mathcal{E}$-valued holomorphic function defined on a neighbourhood of the closed unit disc, that is, $f \in \mathcal{O}(\overline{D}, \mathcal{E})$. Now let $f \in \mathcal{H}_{\mu,1}(\mathcal{E})$ and $r$ be an arbitrary but fixed number in $(0, 1)$. Let $0 < R < 1$ and $\lambda_R$ be the $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$ given by $d\lambda_R(\zeta) = P_{\mu}(R\zeta)d\sigma(\zeta)$. Since $f_R$ is in $\mathcal{O}(\overline{D}, \mathcal{E})$, we obtain $D_{\lambda_R,1}((f_R)_r) \leq D_{\lambda_R,1}(f_R)$. Since $(f_R)_r = (f_r)_R$, it follows that $D_{\lambda_R,1}((f_r)_R) \leq D_{\lambda_R,1}(f_R)$.

Thus we obtain that $D(\mu, 1, R, f_r) \leq D(\mu, 1, R, f)$. This holds for every $R \in (0, 1)$.

Now we provide the proof of the approximation result, Theorem 1.1

**Proof of Theorem 1.1** Using induction, we shall first prove that $D_{\mu,n}(f_r) \leq D_{\mu,n}(f)$ for each $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ and $0 < r < 1$. Lemma 3.1 precisely deals with the case $n = 1$. Fix a positive integer $n \geq 2$ and let the claim holds for every $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ and $0 < r < 1$. Let $f \in \mathcal{H}_{\mu,n+1}(\mathcal{E})$ and $0 < r < 1$. Note that $(L_{f_r})(z) = r(L_f)_r(z)$ for every $z \in \overline{D}$. A simple induction argument will give us $(L_{f_r})_k(z) = r^k(L_{f_r})_k(z)$ for every $k \in \mathbb{N}$, $z \in \overline{D}$. Thus we have $L_{f_r} = r^k(L_{f_r})_k$ for every $k \in \mathbb{N}$. It follows that $D_{\mu,n}(L_{f_r})_k = r^{2k}D_{\mu,n}(L_{f_r})_k$ for every $k \in \mathbb{N}$. As $L$ acts contractively on $H_{\mu,n+1}(0, 1)$, we have $L_f \in \mathcal{H}_{\mu,n+1}(\mathcal{E})$ for every $k \in \mathbb{N}$, see Lemma 2.9. Using Corollary 2.8, we have $L_k f \in \mathcal{H}_{\mu,n}(\mathcal{E})$ for every $k \in \mathbb{N}$. Now applying induction hypothesis, we obtain that

$$D_{\mu,n}(L_{f_r})_k = r^{2k}D_{\mu,n}(L_{f_r})_k \leq r^{2k}D_{\mu,n}(L_{f_r})_k < D_{\mu,n}(L_{f_r})_k, \text{ } k \in \mathbb{N}.$$  

An application of Lemma 2.10 will give us

$$D_{\mu,n+1}(f_r) = \sum_{k=1}^{\infty} D_{\mu,n}(L_{f_r})_k \leq \sum_{k=1}^{\infty} D_{\mu,n}(L_{f_r})_k = D_{\mu,n+1}(f).$$

This completes the proof for the first part of Theorem 1.1.

The technique to prove the remaining part of Theorem 1.1 is standard, see for instance [13 Theorem 7.3.1]. Nevertheless we include the details for the sake of completeness. For every positive integer $n$, using parallelogram identity, we have

$$D_{\mu,n}(f_r - f) + D_{\mu,n}(f_r + f) = 2D_{\mu,n}(f_r) + 2D_{\mu,n}(f), \text{ } f \in \mathcal{H}_{\mu,n}(\mathcal{E}).$$

Using the first part of Theorem 1.1, we have $D_{\mu,n}(f_r) \leq D_{\mu,n}(f)$ for every $0 < r < 1$. Note that for every $z \in \overline{D}$, we have $f_r^{(n)}(z) \to f^{(n)}(z)$ as $r \to 1$. Applying Fatou’s lemma, we obtain $D_{\mu,n}(2f) \leq \liminf_{r \to 1} D_{\mu,n}(f_r + f)$. Hence it follows that

$$\limsup_{r \to 1} D_{\mu,n}(f_r - f) \leq 0.$$  

Since $D_{\mu,n}(f_r - f) \geq 0$ for every $0 < r < 1$, we conclude that $\lim_{r \to 1} D_{\mu,n}(f_r - f) = 0$. This completes the proof of the theorem.

**Corollary 3.2.** Let $\mu$ be a $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure on the unit circle $\mathbb{T}$, $n \in \mathbb{N}$ and $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$. Then there exists a sequence of polynomials $\{p_k\}$ such that $D_{\mu,n}(p_k - f) \to 0$ as $k \to \infty$. 
Corollary 3.3. Let $\mu$ be a $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure on the unit circle $\mathbb{T}$, $n \in \mathbb{N}$ and $f \in \mathcal{H}_{\mu,n}(\mathcal{E})$. Then the following statements hold:

(i) $D_{\mu,n}(L^k f) \to 0$ as $k \to \infty$.

(ii) $\sum_{k=1}^{\infty} D_{\mu,0}(L^k f) = D_{\mu,1}(f)$.

Proof. Let $\epsilon > 0$. By Corollary 3.2 there exists a polynomial $p$ such that $D_{\mu,n}(f - p) < \epsilon$. Using Lemma 2.9 we obtain that for all $k \geq \deg(p) + 1$,

$$D_{\mu,n}(L^k f) = D_{\mu,n}(L^k (f - p)) \leq D_{\mu,n}(f - p) < \epsilon.$$ 

This completes the proof of the first part. For the second part, note that by Lemma 2.9 and Corollary 2.8 $D_{\mu,0}(L^k f)$ is finite for each $k \in \mathbb{N}$. Using Proposition 2.7, we get that for each $j \in \mathbb{N}$,

$$\sum_{k=1}^{j} D_{\mu,0}(L^k f) = \sum_{k=1}^{j} (D_{\mu,1}(z L^k f) - D_{\mu,1}(L^k f))$$

$$= \sum_{k=1}^{j} (D_{\mu,1}(L^{k-1} f) - D_{\mu,1}(L^k f))$$

$$= D_{\mu,1}(f) - D_{\mu,1}(L^j f).$$

The proof is now completed by applying the first part. \qed

4. Weighted Dirichlet-type Spaces and analytic $m$-isometric operators

Let $\mu$ be a $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$, $j \in \mathbb{N}$ and $f$ be an arbitrary but fixed function in $\mathcal{H}_{\mu,j}(\mathcal{E})$. For any $n \in \mathbb{Z}_+$, we consider $\triangle^n D_{\mu,j}(f)$, the $n$-th order forward difference of $D_{\mu,j}(f)$, defined by

$$\triangle^n D_{\mu,j}(f) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D_{\mu,j}(z^k f), \quad f \in \mathcal{H}_{\mu,j}(\mathcal{E}).$$

Note that $\triangle^{k+1} D_{\mu,j}(f) = \triangle^k D_{\mu,j}(zf) - \triangle^k D_{\mu,j}(f)$ holds for every $k \in \mathbb{Z}_+$. Now an induction argument together with the application of Proposition 2.7 it follows that $\triangle^n D_{\mu,j}(f) = D_{\mu,j-n}(f)$ for every $0 \leq n \leq j$. Since $f \in \mathcal{H}_{\mu,j}(\mathcal{E})$, by Proposition 2.6 and Corollary 2.8 we
have $D_{\mu,0}(zf) < \infty$ and $D_{\mu,0}(f) < \infty$. It is also straightforward to verify that $D_{\mu,0}(zf) - D_{\mu,0}(f) = 0$. Hence we obtain the following

$$
\Delta^n D_{\mu,j}(f) = \begin{cases} 
D_{\mu,j-n}(f), & 0 \leq n \leq j, \\
0, & n \geq j + 1, 
\end{cases} \quad f \in \mathcal{H}_{\mu,j}(\mathcal{E}), \ j \in \mathbb{N}. \tag{15}
$$

Let $m \geq 2$ and $\mu = (\mu_1, \ldots, \mu_{m-1})$ be an $(m - 1)$-tuple of $\mathcal{B}(\mathcal{E})$-valued semi-spectral measures on $\mathbb{T}$. In this section we will introduce a Hilbert space $\mathcal{H}_\mu(\mathcal{E})$, called weighted Dirichlet-type space associated to $(m - 1)$-tuple of semi-spectral measures, on which the operator $M_z$ acts as an analytic $m$-isometry. Let $\mathcal{H}_\mu(\mathcal{E})$ denote the linear space given by

$$
\mathcal{H}_\mu(\mathcal{E}) := \bigcap_{j=1}^{m-1} \mathcal{H}_{\mu,j}(\mathcal{E}) \cap H^2(\mathcal{E})
$$

$$
= \{ f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : D_{\mu,j}(f) \leq \infty \text{ for } j = 1, \ldots, m-1 \} \cap H^2(\mathcal{E}).
$$

We associate a norm $\| \cdot \|_\mu$ to the linear space $\mathcal{H}_\mu(\mathcal{E})$ given by

$$
\| f \|^2_\mu := \| f \|^2_{H^2(\mathcal{E})} + \sum_{j=1}^{m-1} D_{\mu,j}(f),
$$

where $\| f \|^2_{H^2(\mathcal{E})}$ denotes the Hardy norm of $f$ for any $f \in H^2(\mathcal{E})$. Note that, if there exists a $j \in \{1, \ldots, m-1\}$ such that $\mu_j(\mathbb{T})$ is invertible then by Corollary 2.5, $\mathcal{H}_{\mu,j}(\mathcal{E}) \subseteq H^2(\mathcal{E})$, and therefore $\mathcal{H}_\mu(\mathcal{E})$ coincides with $\cap_{j=1}^{m-1} \mathcal{H}_{\mu,j}(\mathcal{E})$. It is straightforward to verify that the linear space $\mathcal{H}_\mu(\mathcal{E})$ is a Hilbert space with respect to the norm $\| \cdot \|_\mu$. Let $z \in \mathbb{D}$ and $x \in \mathcal{E}$. Consider the evaluation map $ev_{z,x} : \mathcal{H}_\mu(\mathcal{E}) \to \mathbb{C}$ defined by $ev_{z,x}(f) = \langle f(z), x \rangle$, $f \in \mathcal{H}_\mu(\mathcal{E})$. Since $\| f \|^2_{H^2(\mathcal{E})} \leq \| f \|_\mu$ for every $f \in \mathcal{H}_\mu(\mathcal{E})$, it follows that the evaluation map $ev_{z,x}$ is bounded for every $z \in \mathbb{D}$ and $x \in \mathcal{E}$. Thus the Hilbert space $\mathcal{H}_\mu(\mathcal{E})$ is a reproducing kernel Hilbert space (see [8, 22] for definition and other basic properties of reproducing kernel Hilbert spaces).

**Theorem 4.1.** Suppose $m \geq 2$ and $\mu = (\mu_1, \ldots, \mu_{m-1})$ is an $(m - 1)$-tuple of $\mathcal{B}(\mathcal{E})$-valued semi-spectral measures on $\mathbb{T}$. Then the multiplication operator $M_z$ on $\mathcal{H}_\mu(\mathcal{E})$ is a bounded, analytic $m$-isometry.

**Proof.** By Corollary 2.9, $zf \in \mathcal{H}_\mu(\mathcal{E})$ whenever $f \in \mathcal{H}_\mu(\mathcal{E})$. Since $\mathcal{H}_\mu(\mathcal{E})$ is a reproducing kernel Hilbert space, by closed graph theorem, it follows that $M_z$ on $\mathcal{H}_\mu(\mathcal{E})$ is bounded. As $\mathcal{H}_\mu(\mathcal{E})$ is contained in $\mathcal{O}(\mathbb{D}, \mathcal{E})$, the operator $M_z$ on $\mathcal{H}_\mu(\mathcal{E})$ is analytic. Note that

$$
\sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \| zf \|^2_\mu = \sum_{k=1}^{m-1} \Delta^m D_{\mu,k}(f), \quad f \in \mathcal{H}_\mu(\mathcal{E}).
$$

In view of (15), it follows that the operator $M_z$ on $\mathcal{H}_\mu(\mathcal{E})$ is an $m$-isometry.

Following Lemma 2.9(ii), we see that every function $f$ in $\mathcal{H}_\mu(\mathcal{E})$ has the following decomposition.

$$
f(z) = f(0) + zg(z), \quad g \in \mathcal{H}_\mu(\mathcal{E}).
$$
Since \( \langle f(0), zg \rangle_\mu = 0 \), for every \( g \in \mathcal{H}_\mu(\mathcal{E}) \), it follows that
\[
\ker M_z^* = \mathcal{E}.
\] (16)

It is straightforward to see that \( \mathcal{H}_\mu(\mathcal{E}) \) contains the set of all \( \mathcal{E} \)-valued polynomials. In the following proposition, we show that the set of all \( \mathcal{E} \)-valued polynomials is dense in \( \mathcal{H}_\mu(\mathcal{E}) \). In the case of \( m = 2 \), that is, when \( \mu = \mu_1 \), this result follows from [23, Theorem 1] together with (16), see also [20, Corollary 3.1]. Here we obtain that this result remains true even when \( \mu \) is an arbitrary \((m - 1)\)-tuple of semi-spectral measures with \( m \geq 2 \).

**Proposition 4.2.** Let \( m \geq 2 \) and \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) be an \((m - 1)\)-tuple of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \). Let \( f \in \mathcal{H}_\mu(\mathcal{E}) \). Then \( f_r \), the \( r \)-dilation of \( f \), converges to \( f \) in \( \mathcal{H}_\mu(\mathcal{E}) \) as \( r \to 1 \), that is, \( \|f_r - f\|_\mu \to 0 \) as \( r \to 1 \). Consequently, the set of all \( \mathcal{E} \)-valued polynomials is dense in \( \mathcal{H}_\mu(\mathcal{E}) \).

**Proof.** Applying Theorem 4.1, we obtain that \( D_{\mu,j}(f_r - f) \to 0 \) as \( r \to 1 \) for each \( j = 1, \ldots, m - 1 \). Since \( f \in H^2(\mathcal{E}) \), it is straightforward to verify that \( \|f_r - f\|_{H^2(\mathcal{E})} \to 0 \) as \( r \to 1 \). Hence it follows that \( \|f_r - f\|_\mu \to 0 \) as \( r \to 1 \). So, for any given \( \epsilon > 0 \), there exists a \( R \in (0,1) \) such that \( \|f_R - f\|_\mu < \frac{\epsilon}{2} \). Note that \( f_R \) is in \( \mathcal{O}(\mathbb{D}, \mathcal{E}) \). So the power series expansion of \( f_R \) about origin converges uniformly on a neighbourhood of the closed unit disc \( \mathbb{D} \). Let \( s_n(f_R) \) be the \( n \)-th partial sum of the associated power series of \( f_R \). Clearly \( \|f_R - s_n(f_R)\|_{H^2(\mathcal{E})} \to 0 \) as \( n \to \infty \). By Lemma 2.1(i), we obtain that \( D_{\mu,j}(f_R - s_n(f_R)) \to 0 \) as \( n \to \infty \) for every \( j = 1, \ldots, m - 1 \). This gives us that for any given \( \epsilon > 0 \) there exists \( k \in \mathbb{N} \) such that \( \|f_R - s_n(f_R)\|_\mu < \frac{\epsilon}{2} \), for every \( n \geq k \). Consequently, we obtain that \( \|f - s_k(f_R)\|_\mu < \epsilon \). □

**Remark 4.3.** Note that, if \( \mathcal{E} = \mathbb{C} \) and \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) is an \((m - 1)\)-tuple of finite positive Borel measures in \( \mathcal{M}_+(\mathbb{T}) \) and by Theorem 4.1 and Corollary 5.3, it follows that the operator \( M_z \) on \( \mathcal{H}_\mu(\mathbb{C}) \) is a cyclic, analytic \( m \)-isometry. In this case, the space \( \mathcal{H}_\mu(\mathbb{C}) \) coincides with \( D_{\mu,z}^2 \), where \( \overrightarrow{\mu} = (\sigma, \mu_1, \ldots, \mu_{m-1}) \) is an \( m \)-tuple of finite positive Borel measures in \( \mathcal{M}_+(\mathbb{T}) \), as described in Rydhe’s model for cyclic \( m \)-isometry, see [28, p. 735].

Using Lemma 2.9(i), we observe that \( L \) is a bounded operator on \( \mathcal{H}_\mu(\mathcal{E}) \). Thus the operator \( L \) is a left inverse of \( M_z \) with \( \ker L = \ker M_z^* \) (see (16)). Hence the operator \( L \) on \( \mathcal{H}_\mu(\mathcal{E}) \) coincides with the operator \( L_{M_z} = (M_z^* M_z)^{-1} M_z^* \). In what follows, we will use the notations \( L \) and \( L_{M_z} \) interchangeably, when the underlying Hilbert space is \( \mathcal{H}_\mu(\mathcal{E}) \). The following theorem shows that the operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) satisfies (11). In Section 6 we will show that this set of operator inequalities in (11) plays a key role in identifying the operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) among the class of all analytic \( m \)-isometries.

**Theorem 4.4.** Let \( m \geq 2 \) and \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) be an \((m - 1)\)-tuple of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( \mathbb{T} \). Then the operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) satisfies
\[
\langle \beta_r(M_z) f, f \rangle = D_{\mu,0}(f) + \sum_{n=1}^{\infty} \langle \beta_{r+1}(M_z) L_{M_z}^n f, L_{M_z}^n f \rangle, \quad f \in \mathcal{H}_\mu(\mathcal{E}), \quad r = 1, \ldots, m - 1.
\] (17)
In particular,
\[ \beta_r(M_z) \geq \sum_{n=1}^{\infty} L^n_{M_z} \beta_{r+1}(M_z) L^n_{M_z}, \quad r = 1, \ldots, m - 1. \]

Proof. Since \( \|zf\|_{H^2(E)} = \|f\|_{H^2(E)} \) for every \( f \in H_{\mu}(E) \), it follows that for \( r = 1, \ldots, m - 1 \),
\[ \langle \beta_r(M_z)f, f \rangle = \sum_{j=1}^{m-1} \Delta^r D_{\mu,j}(f), \quad f \in H_{\mu}(E). \]

Note that if \( \mu \in M_+(T, \mathcal{B}(E)) \) and \( f \in H_{\mu,j}(E) \) for some \( j \in \mathbb{N} \), then in view of (15), we get that
\[ \Delta^{s+1} D_{\mu,j}(L^n f) = \begin{cases} D_{\mu,j-s-1}(L^n f), & \text{if } j - s - 1 \geq 0, \quad n \in \mathbb{N}, \quad s \in \mathbb{Z}_+. \\ 0, & \text{otherwise} \end{cases} \]

Consecutively using Lemma 2.10 and Corollary 3.3 we get the following identity:
\[ \Delta^s D_{\mu,j}(f) = \sum_{n=1}^{\infty} \Delta^{s+1} D_{\mu,j}(L^n f) = \begin{cases} D_{\mu,0}(f), & s = j, \quad f \in H_{\mu,j}(E), \quad j \in \mathbb{N}, \quad s \in \mathbb{Z}_+. \\ 0, & s \neq j \end{cases} \] (18)

Using (18), we obtain that for any \( f \in H_{\mu}(E) \),
\[ \langle \beta_r(M_z)f, f \rangle - \sum_{n=1}^{\infty} \langle \beta_{r+1}(M_z)L^n f, L^n f \rangle = \sum_{j=1}^{m-1} \Delta^r D_{\mu,j}(f) - \sum_{j=1}^{m-1} \sum_{n=1}^{\infty} \Delta^{r+1} D_{\mu,j}(L^n f) = D_{\mu,0}(f). \]

This establishes (17) and completes the proof of the theorem. \( \square \)

5. The Wandering Subspace Property for a Class of \( m \)-Concave Operators

The main aim of this section is to establish Theorem 1.3 that is, the wandering subspace property for left invertible analytic \( m \)-concave operators satisfying the operator inequalities (1). This not only becomes a key tool to prove the main theorem of Section 6 but it also provides an alternative way to obtain the density of polynomials in the Hilbert space \( H_{\mu}(E) \). Before proceeding to the proof, we note down the following lemma which will be crucial in the proof of Theorem 1.3. This lemma can essentially be found in [14, Corollary 2.4 and Theorem 2.5].

Lemma 5.1. Let \( T \) be an \( m \)-concave operator in \( \mathcal{B}(\mathcal{H}) \) for some \( m \in \mathbb{N} \). Then,
\[ T^* T^n \leq \sum_{j=0}^{m-1} \binom{n}{j} \beta_j(T), \quad n \geq m. \]

Moreover, \( \beta_{m-1}(T) \geq 0 \).
An operator $T$ in $\mathcal{B}(\mathcal{H})$ is left invertible if and only if $T^*T$ is invertible. Note that $L_T$ is a left inverse of $T$ satisfying $\ker L_T = \ker T^*$. Furthermore, if $L$ is a left inverse of $T$ satisfying $\ker L = \ker T^*$ then it follows that $L = L_T$. The following lemma for a class of left invertible operators will be essential in proving Theorem 1.3.

**Lemma 5.2.** Let $m \geq 2$ and let $T$ be a left invertible operator in $\mathcal{B}(\mathcal{H})$ satisfying

(i) $\beta_{m-1}(T) \geq 0$, 

(ii) $\beta_r(T) \geq \sum_{n=1}^{\infty} L_T^n \beta_{r+1}(T) L_T^n$, $r = 1, \ldots, m - 2$.

Then the following inequalities hold

$$\sum_{n=r+1}^{\infty} \binom{n-1}{r} L_T^n \beta_{r+1}(T) L_T^n \leq I, \quad r = 0, \ldots, m - 2.$$ 

**Proof.** It is evident from the hypothesis that $\beta_r(T) \geq 0$ for $r = 1, \ldots, m - 1$. Also, for $r = 0, \ldots, m - 2$, set $\Psi(r) = \sum_{n=r+1}^{\infty} \binom{n-1}{r} L_T^n \beta_{r+1}(T) L_T^n$. We claim that $\Psi(r) \leq I$ for $r = 0, \ldots, m - 2$. To this end, note that, for any $r \in \{1, \ldots, m - 2\}$, we have

$$\Psi(r-1) = \sum_{i=r}^{\infty} \binom{i-1}{r-1} L_T^i \beta_r(T) L_T^i \geq \sum_{i=r}^{\infty} \binom{i-1}{r-1} \sum_{n=1}^{\infty} L_T^{n+i} \beta_{r+1}(T) L_T^{n+i}$$

$$= \sum_{p=r+1}^{\infty} \left( \sum_{i=r}^{p-1} \binom{i-1}{r-1} \right) L_T^p \beta_{r+1}(T) L_T^p$$

$$= \sum_{p=r+1}^{\infty} \binom{p-1}{r} L_T^p \beta_{r+1}(T) L_T^p = \Psi(r),$$

where the second last equality follows from a combinatorial identity, known as the Hockey-stick identity (see [12, p. 46]). Thus the above inequality shows that

$$\Psi(m-2) \leq \Psi(m-3) \leq \cdots \leq \Psi(0).$$

Hence, in order to prove $\Psi(r) \leq I$ for $r = 0, \ldots, m - 2$, it suffices to show that

$$\Psi(0) = \sum_{i=1}^{\infty} L_T^i \beta_1(T) L_T^i \leq I.$$

To this end, it follows from [23, p. 209] that,

$$\|x\|^2 = \sum_{i=0}^{n-1} \|PL_T^i x\|^2 + \|L_T^n x\|^2 + \sum_{i=1}^{n} \|DL_T^i x\|^2,$$

where $D$ is the positive square root of $\beta_1(T)$ and $P = I - TL_T$. Thus we have,

$$\left\langle \sum_{i=1}^{\infty} L_T^i \beta_1(T) L_T^i x, x \right\rangle = \sum_{i=1}^{\infty} \|DL_T^i x\|^2 \leq \|x\|^2.$$
Consequently, \( \sum_{i=1}^{\infty} L_T^i \beta_1(T) L_T^i \leq I \). This completes the proof. \( \square \)

We are now ready to present the proof of the main result (Theorem 1.3) of this section. The techniques involved in the proof are motivated from those in [24, Theorem 1].

**Proof of Theorem 1.3.** Since \( T \) is \( m \)-concave, by Lemma 5.1, \( \beta_m(T) \geq 0 \). This together with (1) implies that \( \beta_1(T) \geq 0 \), i.e. \( T \) is expansive. We claim that
\[
\left\{ T^n(\ker T^*) : n \in \mathbb{Z}_+ \right\} = \mathcal{H}.
\]
Using Lemmas 5.1 and 5.2, note that for \( k, l \in \mathbb{N} \) with \( l \geq k \geq m \), we have
\[
\inf_{k \leq n \leq l} \left( \| T^n L_T^n x \|^2 - \| L_T^n x \|^2 \right) \sum_{n=k}^{l} \frac{1}{n} \leq \sum_{n=k}^{l} \frac{\| T^n L_T^n x \|^2 - \| L_T^n x \|^2}{n} \leq \sum_{n=k}^{l} \sum_{j=1}^{m-1} \frac{1}{j} \left( n - 1 \right) \langle L_T^n \beta_j(T) L_T^n x, x \rangle = \sum_{j=1}^{m-1} \frac{1}{j} \sum_{n=k}^{l} \left( n - 1 \right) \langle L_T^n \beta_j(T) L_T^n x, x \rangle \leq \left( \sum_{j=1}^{m-1} \frac{1}{j} \right) \| x \|^2.
\]
Since \( \{ \| L_T^n x \| \} \) is a decreasing sequence of non-negative numbers and \( \sum_{n=1}^{\infty} \frac{1}{n} \) is a divergent series, we have \( \lim \inf \| T^n L_T^n x \| = \| L_T^n x \| \). Thus the sequence \( \{ T^n L_T^n x \} \) is bounded in \( \mathcal{H} \) and therefore there exists a subsequence \( \{ T^n L_T^n x_k \} \), converging to \( y \) (say) in weak topology of \( \mathcal{H} \). As \( T^n \) is expansive, the range ran\( (T^n) \) is a closed subspace of \( \mathcal{H} \) and \( y \in \text{ran}(T^n) \) for each \( n \in \mathbb{N} \). Since \( T \) is analytic, \( y = 0 \). Thus \( (I - T^n L_T^n) x \to x \) weakly. Note that
\[
(I - T^j L_T^j) = \sum_{p=0}^{j-1} T^p (I - TL_T) L_T^p, \quad j \geq 1,
\]
and \( (I - TL_T) \) is the orthogonal projection onto \( \ker T^* \). Thus it follows that \( (I - T^n L_T^n) x \in \bigvee \{ T^n(\ker T^*) : n \in \mathbb{Z}_+ \} \). Hence \( x \in \bigvee \{ T^n(\ker T^*) : n \in \mathbb{Z}_+ \} \). Thus we conclude that \( T \) has the wandering subspace property. \( \square \)

**Corollary 5.3.** Let \( m \geq 2 \) and \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) be an \( (m-1) \)-tuple of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures on \( T \). The multiplication operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) has the wandering subspace property. In particular, the set of \( \mathcal{E} \)-valued polynomials is dense in \( \mathcal{H}_\mu(\mathcal{E}) \).

**Proof.** By Theorem 4.1, \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) is a bounded, analytic \( m \)-isometry. Using Theorem 4.4 and Theorem 1.3 we get that \( M_z \) has wandering subspace property. Since \( \ker M_z^* = \mathcal{E} \) (see (10)), the corollary is proved. \( \square \)
Remark 5.4. We note here that the operator inequalities in (1) are not necessary for an expansive analytic $m$-isometry to have the wandering subspace property. For a counterexample (this example is due to Shailesh Trivedi) consider $m = 3$. Let $0 < \varepsilon < 1$ and $p(z) = 1 + z + \frac{\varepsilon}{2}z^2$ for $z \in \mathbb{C}$. Define $\lambda_0 := \sqrt{1 - \varepsilon}$, $\lambda_1 := \sqrt{2\varepsilon}$ and $\lambda_n := \sqrt{\frac{p(n-1)}{p(n-2)}}$ for $n \geq 2$. Let $\{e_n : n \in \mathbb{Z}_+\}$ be an orthonormal basis for $\ell^2(\mathbb{Z}_+)$. Define a linear operator

$$S_\lambda : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$$

by the rule

$$S_\lambda(e_n) = \begin{cases} \lambda_0 e_0 + \lambda_1 e_1 & \text{if } n = 0, \\ \lambda_{n+1} e_{n+1} & \text{if } n \geq 1. \end{cases}$$

From [7, Example 3.1] (with $a = 1$ and $b = \varepsilon/2$), we note that $S_\lambda$ is an analytic expansive 3-isometry having wandering subspace property. We claim that if $\varepsilon$ is in a small neighbourhood of 0 then $S_\lambda$ does not satisfy (1). From [7, Example 3.1], we also note that $L^k_{S_\lambda} = S_\mu$, where $S_\mu : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ is a linear operator such that

$$S_\mu(e_n) = \begin{cases} \mu_0 e_0 + \mu_1 e_1 & \text{if } n = 0, \\ \mu_{n+1} e_{n+1} & \text{if } n \geq 1. \end{cases}$$

with $\mu_0 = \frac{\lambda_0}{\lambda_5 + \lambda_7}$, $\mu_1 = \frac{\lambda_3 + \lambda_2}{\lambda_5 + \lambda_7}$ and $\mu_n = \frac{1}{\lambda_n}$ for $n \geq 2$. It follows that

$$L^k_{S_\lambda} e_0 = \mu_0^k e_0, \quad k \in \mathbb{N}. \quad (19)$$

We have

$$\langle \beta_1(S_\lambda)e_0, e_0 \rangle = \|S_\lambda e_0\|^2 - \|e_0\|^2 = \lambda_0^2 + \lambda_1^2 - 1 = \varepsilon \quad (20)$$

and from (19), one gets

$$\sum_{k=1}^{\infty} \langle L^k_{S_\lambda} \beta_2(S_\lambda) L^k_{S_\lambda} e_0, e_0 \rangle = \sum_{k=1}^{\infty} \mu_0^{2k} \langle \beta_2(S_\lambda)e_0, e_0 \rangle$$

$$= \frac{\mu_0^2}{1 - \mu_0^2} \left(1 - 2(\lambda_0^2 + \lambda_1^2) + \lambda_0^2 + \lambda_1^2(\lambda_0^2 + \lambda_1^2)\right)$$

$$= \frac{2(1 - \varepsilon)}{3 + \varepsilon} \quad (21)$$

If $\varepsilon$ is in a small neighbourhood of 0 then we note that the quantity in (20) is lesser than that in (21) and hence the claim stands verified.

We find that any left invertible $m$-concave operator in $B(\mathcal{H})$ satisfying (1) admits a *Wold-type decomposition*, see Theorem 5.7. Shimorin introduced the notion of *Wold-type decomposition* in order to study operators close to isometries, see [31]. An operator $T$ in $B(\mathcal{H})$ is said to admit a *Wold-type decomposition* if the following statements hold:

(i) $\mathcal{H}_\infty(T)$, the hyper range of $T$, is a reducing subspace for $T$ and $T|_{\mathcal{H}_\infty(T)}$ is a unitary operator.

(ii) The operator $T|_{\mathcal{H}_\infty(T)^\perp}$ has the wandering subspace property.
It follows from [23, Theorem 1] that every 2-concave operator admits a Wold-type decomposition, see [31, Theorem 3.6]. In the same paper [31], Shimorin asked the following question:

**Question 5.5.** [31, p. 185] If an operator $T$ in $\mathcal{B}(\mathcal{H})$ is expansive and $m$-concave for some $m \geq 3$, then does $T$ admit a Wold-type decomposition?

The answer is not yet known even for the class of expansive $m$-isometries. Recently, it has been shown that there are plenty of non-expansive cyclic analytic 3-isometries which fail to have the wandering subspace property (see [7]). In view of the above question, in case of $m$-concave operators, the best known result till now to our knowledge is the following theorem due to Shimorin (see [31, Theorem 3.8]).

**Theorem 5.6** (Shimorin). Let $T \in \mathcal{B}(\mathcal{H})$ be expansive and satisfy the operator inequality

$$T^*T^2 - 3T^*T + 3I - T^*T - P_{\ker T^*} \leq 0,$$

where $P_{\ker T^*}$ is the orthogonal projection of $\mathcal{H}$ onto $\ker T^*$ and $T' = (T^*T)^{-1}$ is the Cauchy dual of $T$. Then $T$ is a 3-concave operator and admits a Wold-type decomposition.

In the following theorem we provide an improvement of Theorem 5.6. Also note that the inequalities stated in this theorem are nothing but the inequalities in (1).

**Theorem 5.7.** Let $T$ be a left invertible $m$-concave operator in $\mathcal{B}(\mathcal{H})$ for some $m \geq 2$. If $T$ satisfies the following inequalities:

$$\beta_r(T) \geq \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T)L_T^k, \quad r = 1, \ldots, m-2$$

then $T$ admits a Wold-type decomposition.

**Proof.** Since $T$ is $m$-concave, by Lemma 5.1, $\beta_{m-1}(T) \geq 0$. This together with (1) implies that $\beta_1(T) \geq 0$, i.e. $T$ is expansive. Hence, by [31, Proposition 3.4], $\mathcal{H}_\infty(T)$ is a reducing subspace for $T$ and $T|_{\mathcal{H}_\infty(T)}$ is a unitary operator. Define $S := T|_{\mathcal{H}_\infty(T)^\perp}$ and note that $L_S = L_T|_{\mathcal{H}_\infty(T)^\perp}$. It is straightforward to see that $S$ is analytic, $m$-concave and satisfies (1). The wandering subspace property of $S$ now follows from Theorem 1.3. This completes the proof. \qed

We also find that the result of Shimorin [31, Theorem 3.8] follows as a special case of Theorem 5.7. We provide the details below.

**A proof of Theorem 5.6.** Let $T$ be an expansive operator in $\mathcal{B}(\mathcal{H})$ satisfying the inequality

$$T^*T^2 - 3T^*T + 3I - L_T^*L_T - P \leq 0,$$

where $P$ is the orthogonal projection onto $\ker T^*$. Since $TL_T$ is an orthogonal projection onto the range of $T$, we have $I - P = TL_T = L_T^*T^*TL_T$. Note that (22) is equivalent to

$$\beta_2(T) \leq \beta_1(T) - L_T^*\beta_1(T)L_T.$$
From (23), we see that $T^* \beta_2(T)T \leq T^* \beta_1(T)T - \beta_1(T) = \beta_2(T)$, which in turn implies that $\beta_3(T) \leq 0$. Hence $T$ is 3-concave. Again, from (23), we get

$$\sum_{k=0}^{n} L_T^* \beta_2(T)L_T^k \leq \beta_1(T) - L_T^* \beta_1(T)L_T^{n+1} \leq \beta_1(T), \quad n \in \mathbb{N}. \quad (24)$$

Since $T$ is 3-concave, $\beta_2(T) \geq 0$. Hence, (24) in particular implies that

$$\sum_{k=1}^{\infty} L_T^* \beta_2(T)L_T^k \leq \beta_1(T).$$

Now applying Theorem [5.7] for $m = 3$, we see that $T$ admits a Wold-type decomposition. \qed

**Remark 5.8.** Note that by Theorem 4.1 and Theorem 4.4, for any $\mu = (\mu_1, \ldots, \mu_{m-1})$, an $(m-1)$-tuple of finite positive Borel measures on $\mathbb{T}$, the multiplication operator $M_\mu$ on $H_\mu(\mathbb{C})$ is an $m$-isometry and satisfies (11). Now let $m = 3$ and suppose that $M_z$ on $H_\mu(\mathbb{C})$ satisfies (22). Then using the equivalence of (22) and (23) we have

$$\langle \beta_2(M_z)1,1 \rangle \leq \langle (\beta_1(M_z) - L_{M_z}\beta_1(M_z)L_{M_z})1,1 \rangle = \beta_1(M_z)1,1 \rangle.$$

Thus, by (17), we obtain $D_{\mu_{2,0}}(1) \leq D_{\mu_{1,0}}(1)$, that is, $\mu_2(\mathbb{T}) \leq \mu_1(\mathbb{T})$. Hence we infer that the class of left invertible 3-concave operators which satisfy (11) is strictly larger than that of expansive operators which satisfy (22).

**Remark 5.9.** Let $T$ be a left invertible $m$-concave operator in $\mathcal{B}(\mathcal{H})$ with $m \geq 2$. Note that for $e \in \ker T^*$, we have

$$\left\langle \left(\beta_r(T) - \sum_{k=1}^{\infty} L_T^* \beta_{r+1}(T)L_T^k\right)T^n e, T^n e \right\rangle = \left\langle \left(T^n \beta_r(T)T^n - \sum_{k=1}^{n} T^n \beta_{r+1}(T)T^{n-k}e, e \right) \right.$$

$$= \left\langle \left(T^n \beta_r(T)T^n - \sum_{k=0}^{n-1} T^n \beta_{r+1}(T)T^k e, e \right) \right.$$

$$= \left\langle \beta_r(T)e, e \right\rangle, \quad r \in \mathbb{N}, n \in \mathbb{Z}_+. \quad (25)$$

Here the last equality follows from the relation $T^n \beta_r(T)T^n - \sum_{k=0}^{n-1} T^n \beta_{r+1}(T)T^k = \beta_r(T)$, which can easily be verified by induction on $n$. Further, suppose that $T$ is a unilateral weighted shift operator with non-zero weights (see [30] for definition and other basic properties of unilateral weighted shift). Then for any fixed non-zero vector $e \in \ker T^*$, the set $\{T^n e : n \in \mathbb{Z}_+\}$ forms an orthogonal basis of $\mathcal{H}$. Note that with respect to this basis, the operator $L_T^* T^{n-j} T^j L_T^k$ is diagonal for each $j, k \in \mathbb{Z}_+$ and consequently the operator $L_T^* \beta_n(T)L_T^k$ is also diagonal for each $n, k \in \mathbb{Z}_+$. Hence in view of (25), it follows that if $\beta_r(T) \geq 0$ for $r = 1, \ldots, m-2$, then $\beta_r(T) \geq \sum_{k=1}^{\infty} L_T^* \beta_{r+1}(T)L_T^k$ holds for $r = 1, \ldots, m-2$. Thus by an application of Lemma 5.1, we obtain that for any left invertible $m$-concave unilateral weighted shift $T$ with $m \geq 2$, the following two conditions are equivalent:

(i) $\beta_r(T) \geq \sum_{k=1}^{\infty} L_T^* \beta_{r+1}(T)L_T^k$ for every $r = 1, \ldots, m-2$,

(ii) $\beta_r(T) \geq 0$ for every $r = 1, \ldots, m-2$. 

6. Model for a Class of $m$-Isometries

In this section, we obtain a model for a class of analytic $m$-isometries satisfying (1). We start with a couple of lemmas which will be crucial for the proof of the main theorem of this section.

**Lemma 6.1.** Let $A$ and $T$ be two operators in $\mathcal{B}(\mathcal{H})$ and $\mathcal{E} = \ker T^*$. Suppose $A$ is positive and $T^* AT = A$. Then there exists a $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure $\mu$ on $\mathbb{T}$ such that

$$\langle AT^i x, T^j y \rangle = \int_{\mathbb{T}} \zeta^{l-j} d\mu_{x,y}(\zeta), \ l, j \in \mathbb{Z}_+, \ x, y \in \mathcal{E}.$$ 

**Proof.** Let $\mathcal{A}$ be the subspace of $\mathcal{H}$ given by $\overline{\text{ran}(A^{1/2})}$, the closure of the range of $A^{1/2}$. Consider the operator $S$ on $\text{ran}(A^{1/2})$ defined by $S(A^{1/2} x) := A^{1/2} T x$, for $x \in \mathcal{H}$. Since $T^* AT = A$, it follows that $S$ extends to an isometry on $\mathcal{A}$. By abuse of language, let it be denoted by $S$ itself. Suppose $V$ is a unitary extension of $S$ on some Hilbert space $\mathcal{K}$ containing $\mathcal{A}$ (see [19, Proposition I.2.3]) and $E$ is the spectral measure associated to $V$. Since $V$ is unitary, the support of $E$ is contained in $\mathbb{T}$. Let $P_\mathcal{E}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{E}$ and $P_\mathcal{A}$ denote the orthogonal projection of $\mathcal{K}$ onto $\mathcal{A}$. Now consider the $\mathcal{B}(\mathcal{E})$-valued semi-spectral measure $\mu$ on $\mathbb{T}$ defined by

$$\mu(\Delta) := P_\mathcal{E} A^{1/2} P_\mathcal{E} E(\Delta) A^{1/2} |_{\mathcal{E}},$$

for all $\Delta$ in the Borel sigma algebra of $\mathbb{T}$. Note that for every $x, y \in \mathcal{E}$, we have

$$\mu_{x,y}(\Delta) = \langle \mu(\Delta) x, y \rangle = \langle E(\Delta) A^{1/2} x, A^{1/2} y \rangle = E_{A^{1/2} x, A^{1/2} y}(\Delta).$$

Thus it follows that

$$\langle V^{*j} V^i A^{1/2} x, A^{1/2} y \rangle = \int_{\mathbb{T}} \zeta^{l-j} d\mu_{x,y}(\zeta), \ l, j \in \mathbb{Z}_+, \ x, y \in \mathcal{E}.$$ 

Since $V^n A^{1/2} x = S^n A^{1/2} x = A^{1/2} T^n x$, for every $x \in \mathcal{E}$ and $n \in \mathbb{Z}_+$, we obtain that

$$\langle AT^i x, T^j y \rangle = \langle V^i A^{1/2} x, V^{*j} A^{1/2} y \rangle = \int_{\mathbb{T}} \zeta^{l-j} d\mu_{x,y}(\zeta), \ l, j \in \mathbb{Z}_+, \ x, y \in \mathcal{E}.$$ 

This completes the proof. \qed

The following lemma shows that the weighted Dirichlet integral $D_{\mu,n}(\cdot)$ determines the semi-spectral measure $\mu$. For a scalar valued polynomial $p$ and $e \in \mathcal{E}$, we use the notation $pe$ to denote the $\mathcal{E}$-valued polynomial given by $pe(z) = p(z)e$ for every $z \in \mathbb{C}$.

**Lemma 6.2.** Let $\mu$ and $\nu$ be two $\mathcal{B}(\mathcal{E})$-valued semi-spectral measures on the unit circle $\mathbb{T}$ and $n \in \mathbb{Z}_+$. Suppose $D_{\mu,n}(pe) = D_{\nu,n}(pe)$ for every scalar valued polynomial $p$ and for every $e \in \mathcal{E}$, then $\mu = \nu$.

**Proof.** Let $D_{\mu,n}(pe) = D_{\nu,n}(pe)$ for every scalar valued polynomial $p$ and for each $e \in \mathcal{E}$. In view of Proposition 2.7, we find that $D_{\mu,0}(pe) = D_{\nu,0}(pe)$ for every scalar valued polynomial
$p$ and $e \in \mathcal{E}$. Now observe that

$$D_{\mu,0}(pe) = \lim_{R \to 1} \int_{\mathbb{T}} |p(R\zeta)|^2 P_{\mu,e,e}(R\zeta) d\sigma(\zeta).$$

It is well-known that for a positive measure $m \in \mathcal{M}_+(\mathbb{T})$, the measure $P_m(R\zeta)d\sigma(\zeta)$ converges to $m$ in weak*-topology as $R \to 1$ (see [27, Theorem 3.3.4]). Consequently, by continuity of $p$ on $\overline{\mathbb{D}}$, we obtain that

$$D_{\mu,0}(pe) = \int_{\mathbb{T}} |p(\zeta)|^2 d\mu_{e,e}(\zeta).$$

By our assumption along with the polarization identity, we find that for any scalar valued polynomials $p$ and $q$,

$$\int_{\mathbb{T}} p(\zeta)\overline{q(\zeta)}d\mu_{e,e}(\zeta) = \int_{\mathbb{T}} p(\zeta)\overline{q(\zeta)}d\nu_{e,e}(\zeta).$$

Thus it follows that $\mu_{e,e} = \nu_{e,e}$, that is $\langle \mu(\Delta)e, e \rangle = \langle \nu(\Delta)e, e \rangle$ for every Borel subset $\Delta \subseteq \mathbb{T}$ and $e \in \mathcal{E}$. Hence we get that $\mu(\Delta) = \nu(\Delta)$ for every Borel subset $\Delta \subseteq \mathbb{T}$. This completes the proof.

Before we state the main result of this section, in the following proposition, we prove that any $(m-1)$-tuple of $\mathcal{B}(\mathcal{E})$-valued semi-spectral measures $\mu$ on $\mathbb{T}$ determines the associated unitary equivalence class of multiplication operator $M_z$ on $\mathcal{H}_\mu(\mathcal{E})$ spaces. This generalizes the result in [20, Theorem 4.2] by Olofsson for the case of $m = 2$.

**Proposition 6.3.** Let $m \geq 2$, and $\mathcal{E}$ and $\mathcal{F}$ be complex separable Hilbert spaces. Let $\mu = (\mu_1, \ldots, \mu_{m-1})$ and $\nu = (\nu_1, \ldots, \nu_{m-1})$ be two $(m-1)$-tuples of measures in $\mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$ and $\mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{F}))$ respectively. Then the operators $M_z$ on $\mathcal{H}_\mu(\mathcal{E})$ and $M_z$ on $\mathcal{H}_\nu(\mathcal{F})$ are unitarily equivalent if and only if there exists a unitary $V$ in $\mathcal{B}(\mathcal{E}, \mathcal{F})$ such that $\mu_r(\Delta) = V^*\nu_r(\Delta)V$ for every Borel subset $\Delta \subseteq \mathbb{T}$ and for all $r = 1, \ldots, m-1$.

**Proof.** Let $M_z^{(\mu)}$ and $M_z^{(\nu)}$ denote the multiplication operators on the Hilbert spaces $\mathcal{H}_\mu(\mathcal{E})$ and $\mathcal{H}_\nu(\mathcal{F})$ respectively. Suppose that $M_z^{(\mu)}$ and $M_z^{(\nu)}$ are unitarily equivalent. Then there exists a unitary operator $U : \mathcal{H}_\mu(\mathcal{E}) \to \mathcal{H}_\nu(\mathcal{F})$ such that $UM_z^{(\mu)} = M_z^{(\nu)}U$. Since ker $M_z^{(\mu)} = \mathcal{E}$ and ker $M_z^{(\nu)} = \mathcal{F}$, it follows that $U^*(\mathcal{F}) = \mathcal{E}$ and consequently we have $U(\mathcal{E}) = \mathcal{F}$. Let $V$ be the unitary operator in $\mathcal{B}(\mathcal{E}, \mathcal{F})$ given by $V = U|_{\mathcal{E}}$. Since $UM_z^{(\mu)} = M_z^{(\nu)}U$, for any scalar valued polynomial $p$ and $e$ in $\mathcal{E}$, $U(pe) = Up(M_z^{(\mu)})(e) = p(M_z^{(\nu)})(Ue) = p(Ve)$. Also, for $r = 1, \ldots, m-1$, by a routine verification, we see that

$$U\left(\beta_r(M_z^{(\mu)}) - \sum_{k=1}^{\infty} L_{M_z^{(\mu)}}^{k}\beta_{r+1}(M_z^{(\mu)})L_{M_z^{(\mu)}}^{k}\right) = \left(\beta_r(M_z^{(\nu)}) - \sum_{k=1}^{\infty} L_{M_z^{(\nu)}}^{k}\beta_{r+1}(M_z^{(\nu)})L_{M_z^{(\nu)}}^{k}\right)U.$$

Hence by Theorem 4.3 it follows that $D_{\mu_r,0}(f) = D_{\nu_r,0}(Uf)$ for all $f$ in $\mathcal{H}_\mu(\mathcal{E})$, and consequently, we obtain that $D_{\mu_r,0}(pe) = D_{\nu_r,0}(p(Ve))$ for any polynomial $p$ and $e \in \mathcal{E}$. Note that $V^*\nu_rV \in \mathcal{M}_+(\mathbb{T}, \mathcal{B}(\mathcal{E}))$ and

$$D_{\nu_r,0}(p(Ve)) = D_{V^*\nu_rV,0}(pe),$$
for every polynomial \( p \) and \( r = 1, \ldots, m - 1 \). Thus in view of Lemma 6.2, we conclude that 
\[ \mu_r(\Delta) = V^* \nu_r(\Delta)V \text{ for every Borel subset } \Delta \subseteq \mathbb{T} \text{ and for all } r = 1, \ldots, m - 1. \]

For the reverse implication, suppose \( V \) is a unitary map in \( \mathcal{B}(\mathcal{E}, \mathcal{F}) \) such that \( \mu_r(\cdot) = V^* \nu_r(\cdot)V \) for \( r = 1, \ldots, m - 1 \). Note that \( P_{\mu_r}(z) = V^* P_{\nu_r}(z)V \) for every \( z \in \mathbb{D} \) and for each \( r = 1, \ldots, m - 1 \). The map \( V \) induces a linear map \( U \) from \( \mathcal{H}_\mu(\mathcal{E}) \) into \( \mathcal{H}_\nu(\mathcal{F}) \) given by \( (Uf)(z) = V(f(z)) \) for \( f \in \mathcal{H}_\mu(\mathcal{E}) \) and \( z \in \mathbb{D} \). It is straightforward to verify that \( U \) is unitary from \( \mathcal{H}_\mu(\mathcal{E}) \) onto \( \mathcal{H}_\nu(\mathcal{F}) \) satisfying \( UM_z(\mu) = M_z(\nu)U \).

Now we are ready to prove the main theorem of this section, namely the Theorem 1.2. This provides a canonical model for the class of analytic \( m \)-isometries satisfying (11) and generalizes [24, Theorem 5.1] and [20, Theorem 4.1]. Note that, if \( T \) is an \( m \)-isometry then the approximate point spectrum \( \sigma_{ap}(T) \) is a subset of \( \mathbb{T} \) (see [24 Lemma 1.21]). As \( 0 \notin \sigma_{ap}(T) \), it follows that an \( m \)-isometry \( T \) is always left invertible.

**Proof of Theorem 1.2** The backward implication follows directly from Theorem 4.4 and Theorem 4.1.

For the forward implication assume that \( T \) is an analytic \( m \)-isometry satisfying (11). From Theorem 1.3 we obtain that \( T \) has the wandering subspace property. Thus the linear span of \( \{ T^n \mathcal{E} : n \in \mathbb{Z}_+ \} \) is dense in \( \mathcal{H} \), where \( \mathcal{E} = \ker T^* \). Also note that \( T^* \beta_m(T)T = \beta_m(T) \). From Lemma 5.1 it follows that \( \beta_m(T) \geq 0 \). Moreover, using the relation \( L_T T = I \)

together with \( T^* \beta_r(T)T - \beta_{r+1}(T) = \beta_r(T) \), it follows that for each \( r = 1, \ldots, m - 1 \),

\[
T^* \left( \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \right) T = \beta_r(T) + \beta_{r+1}(T) - \sum_{k=1}^{\infty} T^* L_T^k \beta_r(T) L_T^k \]

\[
= \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \cdot
\]

For \( r = 1, \ldots, m - 1 \), since the operator \( \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \) is positive by hypothesis, as an application of Lemma 6.1 there exists an \((m - 1)\)-tuple of \( \mathcal{B}(\mathcal{E}) \)-valued semi-spectral measures \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) on \( \mathbb{T} \) such that

\[
\langle \left( \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \right) T^l(x), T^j(y) \rangle = \int_{\mathbb{T}} \zeta^{l-j} d(\mu_r)_{x,y}(\zeta), \quad j, l \in \mathbb{Z}_+ , \quad x, y \in \mathcal{E}.
\]

We claim that \( T \) is unitarily equivalent to the multiplication operator \( M_z \) by the coordinate function on \( \mathcal{H}_\mu(\mathcal{E}) \). In view of Theorem 4.4 together with the polarization identity and (26), we have

\[
\langle \beta_{m-1}(M_z) z^l x, z^j y \rangle = \int_{\mathbb{T}} \zeta^{l-j} d(\mu_{m-1})_{x,y}(\zeta) = \langle \beta_{m-1}(T) T^l x, T^j y \rangle, \quad j, l \in \mathbb{Z}_+ , \quad x, y \in \mathcal{E}.
\]

(27)
Let \( e \in \mathcal{E} \). Since \( L_T T = I \) and \( L_T(e) = 0 \), we get that
\[
L_T^n T^j(e) = \begin{cases} T^{j-n}e & \text{if } n \leq j, \\ 0 & \text{if } n > j. \end{cases}
\]
In a similar manner, we also have
\[
L_M^n z^j e = \begin{cases} z^{j-n}e & \text{if } n \leq j, \\ 0 & \text{if } n > j. \end{cases}
\]
This combined with (27) gives
\[
\langle \beta_{m-1}(M_z) L_M^k z^j x, L_M^k z^j y \rangle = \langle \beta_{m-1}(T) L_T^k T^j x, L_T^k T^j y \rangle, \quad k, l, j \in \mathbb{Z}_+, x, y \in \mathcal{E}.
\]
Again using Theorem 4.4 along with the polarization identity and (26), one obtains
\[
\left\langle \left( \beta_r(M_z) - \sum_{k=1}^{\infty} L_M^k \beta_{r+1}(M_z) L_M^k \right) z^j x, z^j y \right\rangle = \left\langle \left( \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \right) T^j x, T^j y \right\rangle,
\]
for every \( j, l \in \mathbb{Z}_+, \) and \( r = 1, \ldots, m-2 \). Using this together with (28), inductively, we have
\[
\langle \beta_r(M_z) z^j x, z^j y \rangle = \langle \beta_r(T) T^j x, T^j y \rangle, \quad j, l \in \mathbb{Z}_+, x, y \in \mathcal{E},
\]
for every \( r = m-1, \ldots, 1 \). From the case of \( r = 1 \), it follows that
\[
\langle z^{l+1} x, z^{l+1} y \rangle - \langle z^l x, z^j y \rangle = \langle T^{l+1} x, T^{j+1} y \rangle - \langle T^l x, T^j y \rangle, \quad j, l \in \mathbb{Z}_+, x, y \in \mathcal{E}.
\]
We also have \( \langle x, y \rangle_M = \langle x, y \rangle \) and \( \langle z^l x, y \rangle = \langle T^l x, y \rangle = 0 \) for every \( l \geq 1 \) and \( x, y \in \mathcal{E} \). Hence inductively we obtain that
\[
\langle z^l x, y \rangle = \langle T^l x, y \rangle, \quad j, l \in \mathbb{Z}_+, x, y \in \mathcal{E}.
\]
Now consider the map \( U \) defined on the linear span of \( \{ T^n \mathcal{E} : n \in \mathbb{Z}_+ \} \) given by
\[
U \left( \sum_{j=0}^{k} T^j x_j \right) := \sum_{j=0}^{k} z^j x_j, \quad x_0, \ldots, x_k \in \mathcal{E}.
\]
From (29), it follows that \( U \) is an isometry from the linear span of \( \{ T^n \mathcal{E} : n \in \mathbb{Z}_+ \} \) onto the set of all \( \mathcal{E} \)-valued polynomials in \( \mathcal{H}_M(\mathcal{E}) \) and the equality \( UT = M_z U \) holds on the linear span of \( \{ T^n \mathcal{E} : n \in \mathbb{Z}_+ \} \). Since \( \mathcal{E} \)-valued polynomials are dense in \( \mathcal{H}_M(\mathcal{E}) \) (by Corollary 5.3), the map \( U \) extends as a unitary map from \( \mathcal{H} \) onto \( \mathcal{H}_M(\mathcal{E}) \) satisfying \( UT = M_z U \). This completes the proof. \( \Box \)

We conclude this section by showing that the adjoint of every analytic \( m \)-isometry with \( n \) dimensional kernel, \( n \in \mathbb{N} \), lies in the class \( B_n(\mathcal{D}) \) of Cowen-Douglas class operators associated to the open unit disc \( \mathcal{D} \), see [11] for the definition and other properties. Similar kinds of spectral behaviour for expansive analytic \( m \)-isometries have been studied in [9] Lemma 2.6 and Corollary 3.4.

**Proposition 6.4.** Let \( T \in B(\mathcal{H}) \) be an analytic \( m \)-isometry with \( \dim(\ker T^*) = n \), for some positive integers \( m \) and \( n \). Then it follows that
(i) \( \sigma(T) = \mathbb{D} \) and \( \sigma_{ap}(T) = \mathbb{T} \),
(ii) \( T^* \in B_n(\mathbb{D}) \),
where \( \sigma(T) \) and \( \sigma_{ap}(T) \) denote the spectrum and the approximate point spectrum of the operator \( T \) respectively.

Proof. Since \( T \) is an \( m \)-isometry, it follows that \( \sigma_{ap}(T) \subseteq \mathbb{T} \) (see [2, Lemma 1.21]). Thus \( T - \lambda I \) is bounded below for every \( \lambda \in \mathbb{D} \). This gives us that \( T - \lambda I \) is semi-Fredholm for every \( \lambda \in \mathbb{D} \). Since \( \ker T^* \) is \( n \) dimensional, using continuity of the semi-Fredholm index, we conclude that \( \dim(\ker(T^* - \lambda I)) = n \), for every \( \lambda \in \mathbb{D} \). Since the boundary of spectrum of an operator is contained in the approximate point spectrum (see [10, Ch.7,6.7]), it follows that \( \sigma(T) = \mathbb{D} \) and \( \sigma_{ap}(T) = \mathbb{T} \).

In order to show that \( T^* \in B_n(\mathbb{D}) \), it is sufficient to show that
\[
\bigvee \{ \ker(T^* - \lambda I) : \lambda \in \mathbb{D} \} = \mathcal{H}.
\]

Note that \( (T^* - \lambda I) \) is Fredholm for every \( \lambda \in \mathbb{D} \) with Fredholm index equal to \( n \). Now following [11, Proposition 1.11], we obtain that there exist holomorphic \( \mathcal{H} \)-valued functions \( \{ e_i(z) : i = 1, \ldots, n \} \) defined on some neighborhood \( \Omega \) of 0 such that \( \{ e_1(z), \ldots, e_n(z) \} \) forms a basis for \( \ker(T^* - zI) \) for every \( z \in \Omega \). As \( T \) is bounded below, it follows that \( T^k \) is also bounded below for every \( k \in \mathbb{N} \). Now from general properties of Fredholm index, see [10, Ch.11, 3.7], we obtain that \( \dim(\ker T^k) = kn \), for every \( k \in \mathbb{N} \). Furthermore, from the proof of [11, Lemma 1.22], one may infer that
\[
\text{span}\{e_1(0), \ldots, e_n(0), \ldots, e_1^{(k-1)}(0), \ldots, e_n^{(k-1)}(0)\} = \ker T^k, \quad k \in \mathbb{N}.
\]
Thus, it follows that \( \ker T^k \) is contained in \( \bigvee \{ \ker(T^* - \lambda I) : \lambda \in \mathbb{D} \} \) for every \( k \in \mathbb{N} \). Note that the hyper-range \( \mathcal{H}_\infty(T) \) of \( T \) is given by
\[
\left( \bigvee \{ \ker T^k : k \in \mathbb{N} \} \right)^\perp = \mathcal{H}_\infty(T).
\]
As \( T \) is analytic by assumption, we obtain that
\[
\bigvee \{ \ker(T^* - \lambda I) : \lambda \in \mathbb{D} \} = \mathcal{H}.
\]

In view of Theorem [11] and Proposition [6,4] together with [16], the following corollary is now immediate. The result in [24, Corollary 3.8(b,c)] can be seen as a special case of the following corollary.

**Corollary 6.5.** Let \( \mathcal{E} \) be a complex Hilbert space of dimension \( n \) for some \( n \in \mathbb{N} \) and \( \mu = (\mu_1, \ldots, \mu_{m-1}) \) be an \( (m - 1) \)-tuple of semi-spectral measures on \( \mathbb{T} \). The operator \( M_z \) on \( \mathcal{H}_\mu(\mathcal{E}) \) has the following properties:
(i) \( \sigma(M_z) = \mathbb{D} \) and \( \sigma_{ap}(M_z) = \mathbb{T} \).
(ii) \( M_z^* \in B_n(\mathbb{D}) \).

**Acknowledgement.** The authors are grateful to Sameer Chavan for his constant support and many valuable suggestions in the preparation of this article. The authors also wish to express their sincere thanks to Shailesh Trivedi for several comments and fruitful discussions.
We sincerely thank the anonymous referee for several constructive comments which improved the presentation of this article.

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