POLYFOLDS, COBORDISMS, AND THE STRONG WEINSTEIN CONJECTURE

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Abstract. We prove the strong Weinstein conjecture for closed contact manifolds that appear as the concave boundary of a symplectic cobordism admitting an essential local foliation by holomorphic spheres.

1. Introduction

Given a closed (co-orientable) contact manifold \((M, \xi)\) and a defining contact form \(\alpha\) (i.e. \(\xi = \ker \alpha\)) Weinstein conjectured in \cite{38} that the Reeb vector field \(R\), which is uniquely defined by \(i_R d\alpha = 0\) and \(\alpha(R) = 1\), admits a periodic solution. The Weinstein conjecture was proven by Taubes \cite{35} for all closed contact manifolds of dimension 3 and has been verified in higher dimensions in many situations most recently in the presence of contact connected sums in \cite{15, 16, 17}. We refer the reader to \cite{33} for the state of the art of the conjecture.

A stronger conjecture was given in \cite{1} that asks for a finite collection of periodic solutions of \(R\), a so-called null-homologous Reeb link, that oriented by \(R\) and eventually counted with a positive multiple of a period represents the trivial class in the homology of \(M\). We refer to the existence question of a null-homologous Reeb link as the strong Weinstein conjecture and remark that the stronger version of the conjecture is not covered by Taubes result. The aim of the present work is to confirm the strong Weinstein conjecture for closed contact manifolds \((M, \xi = \ker \alpha)\) that appear as the concave end of symplectic cobordisms with particular properties. This will generalize the results obtained in \cite{13}.

The notion of a symplectic cobordism was introduced in \cite{9, 22} in the context of symplectic field theory. A symplectic cobordism is a compact connected symplectic manifold \((W, \omega)\) with boundary that admits a Liouville vector field \(Y\) near \(\partial W\), which by definition satisfies \(L_Y \omega = \omega\). According to the boundary orientation induced by the orientation of the symplectic form the Liouville vector field \(Y\) points either in or out of \(W\). This decomposes the boundary of \(W\) into the concave boundary \(M^-\) (along which \(Y\) points in) and into the convex boundary \(M^+\) (along which \(Y\) points out). The Liouville vector field defines contact forms \(\alpha_-\) and \(\alpha_+\) by restricting \(i_Y \omega\) to the tangent bundles of \(M^-\) and \(M^+\), resp., so that \((M^-, \alpha_-)\) and \((M^+, \alpha_+)\) are particular examples of contact type hypersurfaces in \((W, \omega)\), cf. \cite{30}.

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A theorem of Hofer [19] relates the existence of closed Reeb orbits to the existence of (punctured finite energy) holomorphic curves in symplectic cobordisms, cf. [2, 13, 14]. In order to utilize this relation we will consider symplectic cobordisms that admit a so-called essential local foliation by holomorphic spheres. The definition is given in Subsection 2.3 below. The symplectic area of the holomorphic spheres of the foliation as it will turn out induces an upper bound on the total action of a null-homologous Reeb link, which is by definition the sum of the actions of the link components counted with the selected period multiplicities.

**Theorem 1.1.** If \((M, \alpha)\) is the concave boundary of a symplectic cobordism \((W, \omega)\) that admits an essential local foliation by holomorphic spheres of area \(\pi \varrho^2\), then there exists a null-homologous Reeb link in \((M, \alpha)\) of total action smaller than \(\pi \varrho^2\). If \((W, \omega)\) has no concave boundary it has no convex boundary either.

**Remark 1.2.** The only surface \((W, \omega)\), to which the theorem applies, is \(\mathbb{CP}^1\).

The qualitative content of the theorem is valid not only for one particular contact form on \(M\). In fact, the construction from [13] Section 3.3] allows to conclude for any contact form whose kernel is equal to \(\xi = \ker \alpha\). Each contact form that defines \((M, \xi)\) appears as a graph over the zero section in the symplectisation of \((M, \alpha)\). After a shift in the negative \(\mathbb{R}\)-direction the graph can be assumed to lie below the zero section. If \((M, \alpha)\) is the concave boundary of a symplectic cobordism \((W, \omega)\), then a positive constant multiple of any \(\xi\)-defining contact form can be realized as the concave boundary of a slightly modified symplectic cobordism. The cobordism is obtained by gluing the symplectic cobordisms that is cut out by the shifted graph and the zero section to the cobordism \((W, \omega)\) along \((M, \alpha)\). To express this circumstance we will say that \((M, \xi)\) is the concave boundary of \((W, \omega)\). Convex boundaries are handled similarly.

**Corollary 1.3.** The strong Weinstein conjecture holds for all contact manifolds that are the concave boundary of a symplectic cobordism that is provided with an essential local foliation by holomorphic spheres.

A contact manifold is called **semi-fillable** if it is a boundary component of a symplectic cobordism that has more then one convex but no concave boundary components. Examples of McDuff [29], Geiges [11, 12], and Massot-Niederkrüger-Wendl [28] show the existence of semi-fillable contact manifolds. Generalizing a result of McDuff [29] we obtain (cf. Remark 6.1):

**Corollary 1.4.** If \((M, \xi)\) is the concave boundary of a symplectic cobordism \((W, \omega)\) as in the theorem, then \((M, \xi)\) is not semi-fillable.

In [13] the following capacity for symplectic manifolds \((V, \omega)\) that are not of dimension 2 was introduced:

\[
c(V, \omega) = \sup_{(M, \alpha)} \inf \alpha.
\]

The supremum is taken over all closed contact type hypersurfaces \((M, \alpha)\) in \((V, \omega)\) and \(\inf \alpha\) is the minimal total action of a null-homologous Reeb link in \((M, \alpha)\).

**Corollary 1.5.** If \((W, \omega)\) as in the theorem has no concave (and hence no convex) boundary, then \(c(W, \omega) \leq \pi \varrho^2\).
In particular, the Gromov radius of \((W, \omega)\) as introduced in [18] is bounded by \(\pi \rho^2\) from above. This can also be derived from the following uniruledness result. This is because no embedding of an open ball into \(W\) can intersect \(\partial B \times \mathbb{C}P^1\), which is the boundary (in case if non-empty) of the local foliation domain \(B \times \mathbb{C}P^1\) that we require to exist, cf. Subsection 2.3.

**Corollary 1.6.** If \((W, \omega)\) as in the theorem has no contact type boundary components, then through each point of \(W\) there passes a (nodal) holomorphic sphere for any compatible almost complex structure that coincides with \(J_B \oplus i\) on \(U \times \mathbb{C}P^1\), where \(U\) is a collar neighborhood of \(\partial B\) in \(B\).

In order to prove the theorem we use holomorphic spheres corresponding to the given local foliation. The associated moduli space is non-empty and regular in a neighborhood of the holomorphic spheres that come from the local foliation. The assumption of being essential results in uniqueness properties of the moduli space. In order to ensure global regularity properties of the moduli space, which are obstructed by bubbling off of multiply covered holomorphic spheres of negative first Chern class, semi-positivity of \((W, \omega)\) could be used. In order to get an unrestricted statement we will employ the regularity theory developed by Hofer-Wysocki-Zehnder instead. The space of (not necessarily holomorphic) stable curves has a polyfold structure, see [27]. Using abstract perturbations (see [25]) the moduli space can be approximated by solution spaces of perturbed Fredholm problems that carry the structure of a smooth branched orbifold with weights. This enables us to conclude as in [13].

In Section 2 we formulate the definition of an essential local foliation by holomorphic spheres. Examples and applications are presented in Section 3. The content of Section 4-5 is a description of the moduli space of holomorphic curves relevant in our situation in view of the application of the theory of polyfolds and abstract perturbations. The proof of Theorem 1.1 is given in Section 6.

2. **Definition**

2.1. **Completion.** A collar neighborhood of a concave boundary \((M, \alpha)\) of a symplectic cobordism \((W, \omega)\) is symplectomorphic to \(\left([0, \varepsilon) \times M, d(e^{\alpha}a)\right)\) for some \(\varepsilon > 0\). Let \(\mathcal{T}\) denote the set of all smooth strictly increasing functions \((\mathbb{R}, (\varepsilon, 0])\) for smooth strictly increasing functions \((\mathbb{R}, (\alpha, 0])\) that allow a smooth extension to \(\mathbb{R}\) that restricted to \([0, \infty)\) coincides with \(a \mapsto e^\alpha\).

A **concave end** is a symplectic manifold of the form \(\left((\mathbb{R}, (\varepsilon, 0]) \times M, d(\tau\alpha)\right)\) with \(\tau \in \mathcal{T}\). Gluing the concave end to \((W, \omega)\) along \((M, \alpha)\) results in a symplectic manifold \((W', \omega')\) for \(\tau \in \mathcal{T}\). In order to reflect the conformal nature we will call \((W', \omega')\) the **completion** of \((W, \omega)\).

2.2. **Holomorphic spheres with nodes.** Consider a Riemann surface \((S, j)\) with finitely many connected components. A **nodal pair** is a subset of \(S\) that consists of two distinct points. Nodal pairs are required to be pairwise disjoint. Denote by \(D\) a finite set of nodal pairs such that that the quotient space \(S/D\) is connected. A **nodal Riemann surface** is a triple \((S, j, D)\). The nodal Riemann surface \((S, j, D)\) is said to be of genus zero if each connected component of \(S\) is diffeomorphic to the 2-sphere and if the points of a nodal pair do not both lie on the same component of \(S\); to phrase it differently, if \(S/D\) is simply connected.

Let \(J\) denote an almost complex structure on the completion of \(W\). We consider a nodal Riemann surface \((S, j, D)\) of genus zero. A **nodal \(J\)-holomorphic sphere**
is a smooth map \( u: S \to W' \) that solves the non-linear Cauchy-Riemann equation \( Tu \circ j = J(u) \circ Tu \) and that descends to a continuous map on the quotient \( S/D \). In case \( D \) is empty \( u \) is called un-noded.

2.3. Essential holomorphic foliations. Let \((B, \omega_B)\) be an open connected symplectic manifold. If \( B \) has non-empty boundary we require that \( \partial B \) is closed and connected. Denote by \( \omega_{\text{FS}} \) the Fubini-Study form on \( \mathbb{C}P^1 \), which integrates to total area \( \pi \). A local foliation by holomorphic spheres in \((W, \omega)\) is a symplectic embedding
\[
(B \times \mathbb{C}P^1, \omega_B + \varrho^2 \omega_{\text{FS}}) \to (W, \omega),
\]
where \( \varrho \) is a positive real number. If the boundary of \( B \) is non-empty we assume that \( \partial B \times \mathbb{C}P^1 \) is mapped diffeomorphically onto \( \partial W \setminus (M_- \cup M_+) \) assuming that besides the concave and the convex boundary a further boundary component (a posteriori equal to \( \partial B \times \mathbb{C}P^1 \)) exist.

The local foliation \( B \times \mathbb{C}P^1 \) is assumed to be equipped with a compatible almost complex structure of the form \( J_B \oplus i \). All almost complex structures on the completion \( W' \) are assumed to restrict to the split structure \( J_B \oplus i \) on \( B \times \mathbb{C}P^1 \) and are called admissible.

We remark that by a theorem of Moser [32] for any positive area for \( \sigma \) on \( S^2 \) with \( \sigma(S^2) = \pi \varrho^2 \) there exists a diffeomorphism of \( S^2 \) along which \( \sigma \) pulls back to \( \varrho^2 \omega_{\text{FS}} \). The area of a holomorphic sphere in the local foliation is \( \pi \varrho^2 \).

**Definition 2.1.** A local foliation \( B \times \mathbb{C}P^1 \subset W \) as readily defined is called essential if for all admissible compatible almost complex structures \( J \) on \( W' \) any nodal \( J \)-holomorphic sphere in \( W' \) that is
- homologous to \( * \times \mathbb{C}P^1 \) for \( * \in B \),
- non-constant restricted to any component of \( S \),
- and intersects \( B \times \mathbb{C}P^1 \) non-trivially
is un-noded and up to a pre-composition with a Möbius transformation equal to \( z \mapsto (b, z) \) for some \( b \in B \).

3. Examples and Applications

In [13] the strong Weinstein conjecture was verified for contact manifolds that appear as the concave boundary of a semipositive symplectic cobordism that can be capped off along a convex boundary component in a particular way. The construction of the cap from [13, Section 5.1] generalizes to the present context as follows.

3.1. Holomorphic foliations via caps. We consider a non-empty closed connected contact manifold \((N, \alpha_N)\). For \( \varepsilon > 0 \) sufficiently small we denote by
\[
(V, \omega_V) = \left( (-\varepsilon, 0] \times N, d(e^\varepsilon \alpha_N) \right)
\]
a cylindrical subset of the symplectization of \((N, \alpha_N)\). We equip \((V, \omega_V)\) with an almost complex structure \( J_V \) that is compatible with the contact form \( \alpha_N \), i.e. \( J_V \) is invariant under translation, restricts to a compatible complex structure on the symplectic bundle \((\ker \alpha_N, d\alpha_N)\), and sends the Liouville vector field \( \partial_\varepsilon \) to the Reeb vector field of \( \alpha_N \).

Moreover, let \((Q, \omega_Q)\) be a closed connected symplectic manifold and denote by \( J_Q \) a compatible almost complex structure on \((Q, \omega_Q)\). The rational area spectrum of the symplectic form \( \omega_Q \) is given by the image of the map \( H_2(Q; \mathbb{Q}) \to \mathbb{R} \)
obtained by integration against $\omega_Q$. The spectrum is countable and hence constitutes a residual subset of $\mathbb{R}$.

**Proposition 3.1.** Let $(W, \omega)$ be a symplectic manifold with boundary. We assume that $(W, \omega)$ admits a local foliation

$$(V \times Q \times \mathbb{C}P^1, \omega_V + \omega_Q + q^2\omega_{FS})$$

with $\partial W = N \times Q \times \mathbb{C}P^1$ that is equipped with the almost complex structure

$$J_V \oplus J_Q \oplus i.$$

The local foliation is essential provided $\pi_0^2$ is not a rational spectral value of $\omega_Q$.

**Proof.** Let $u: S \to W$ be a nodal holomorphic sphere that intersects the local foliation $V \times Q \times \mathbb{C}P^1$ non-trivially. The restriction of $u$ to $u^{-1}(V \times Q \times \mathbb{C}P^1)$ can be projected to the $(-\varepsilon, 0]$-factor of $V$. The composition of the resulting map with the exponential map is subharmonic and has an interior maximum. By the maximum principle and an open-closed argument applied to each component of $S$ the image $u(S)$ is contained in $\{a\} \times N \times Q \times \mathbb{C}P^1$ for a suitable $a \in (-\varepsilon, 0]$. Because the projection of $u$ to the exact symplectic manifold $(-\varepsilon, 0] \times N$ must be constant (by compatibility) the image of $u$ is in fact contained in $\{v\} \times Q \times \mathbb{C}P^1 \equiv Q \times \mathbb{C}P^1$ for a suitable $v \in V$.

In view of the definition in Section 2.3 we assume in addition that $u$ is homologous to $\mathbb{C}P^1 \equiv \ast \times \mathbb{C}P^1$ in $W$ and that $u$ is non-constant on each component of $S$. Choose an ordering on the components of $S = S_1 \sqcup \ldots \sqcup S_k$ and write $u_j$ for the restriction of $u$ to the component $S_j$. Further, denote by $u_j^Q$ and $\varphi_j$ the projections of $u_j$ to $Q$ and $\mathbb{C}P^1$, resp. According to Künneth’s formula with respect to $Q \times \mathbb{C}P^1$ we get

$$[\mathbb{C}P^1] = \sum_{j=1}^k [u_j^Q] + \sum_{j=1}^k d_j [\mathbb{C}P^1]$$

in $H_2 W$, where $d_j$ is the degree of the holomorphic map $\varphi_j$. Because $\pi_0^2$ is not in the rational area spectrum of $\omega_Q$ an application of $\omega_W$ shows that

$$\sum_{j=1}^k \omega_Q([u_j^Q]) = 0 \quad \text{and} \quad \sum_{j=1}^k d_j = 1.$$

Because the symplectic energy is non-negative by compatibility we get $k = 1$, $d_1 = 1$, and $u_1^Q$ is constant. Therefore, there exists $q \in Q$ and an automorphism $\varphi$ of $\mathbb{C}P^1$ such that $u(z) = (v, q, \varphi(z))$ for all $z \in \mathbb{C}P^1$. $\square$

**Remark 3.2.** The construction generalizes to a symplectic manifold $(V, \omega_V)$ that has a weakly convex contact type boundary, see Remark 6.1 or is the negative half-symplectisation of the stable Hamiltonian structure $(\omega_N, d\theta)$ that is induced by a symplectic fibration $\theta: N \to S^1$ on the boundary $N$ of $V$ with respect to $\omega_N := \omega_V|_{TN}$, see [5, p. 877]. In the second case one requires that $\pi_0^2$ is not a rational area spectral value of $\omega_F + \omega_Q$, where $F$ is the typical fibre of $\theta$, which is symplectic with respect to $\omega_F := \omega_V|_{TF}$. The compatible almost complex structures $J_V$ we are now considering are translation invariant, turn the fibres of $\theta$ into a holomorphic submanifold in each level of $(\varepsilon, 0] \times N$, and send the Reeb vector field of the stable Hamiltonian structure $(\omega_N, d\theta)$ to $\partial_u$, see [5, Section 5].
3.2. Stabilized Weinstein conjecture. In the following we will construct a class of symplectic cobordisms with an essential local foliation as described. Let \((P, \omega_P)\) be a symplectic filling with connected boundary, i.e. a non-empty symplectic cobordism with one contact type boundary component that is convex. Consider a closed hypersurface
\[
\Sigma \subset P \times Q \times C
\]
that is of contact type with respect to \(\omega_P + \omega_Q + dx \wedge dy\). The induced contact form on \(\Sigma\) is denoted by \(\alpha_\Sigma\). Each component of \(\Sigma\) bounds a relatively compact open domain the so-called bounded domain. We require that the bounded domains are pairwise disjoint. We denote the union of the domains by \(D_\Sigma\).

Let \(\varrho\) be a positive real number such that \(\pi \varrho^2\) is not in the rational area spectrum of \((Q, \omega_Q)\) and greater than the minimal area of a closed disc in \(C\) about the origin that contains the image of the projection map \(\Sigma \subset P \times Q \times C \to C\). We define the symplectic cap to be
\[
(C, \omega_C) = \left( P \times Q \times CP^1 \setminus D_\Sigma, \omega_P + \omega_Q + \varrho^2 \omega_{FS} \right).
\]
Applying Theorem 1.1 to \((C, \omega_C)\) with the components of \(D_\Sigma\) glued back that correspond to the concave boundary components we see that \((C, \omega_C)\) can not have a convex boundary. In other words \((\Sigma, \alpha_\Sigma)\) is the concave boundary of \((C, \omega_C)\).

Remark 3.3. Let the dimension of \(P \times Q \times C\) be 2n. If the \((n-1)\)-st power of the symplectic form \(\omega = \omega_P + \omega_Q + dx \wedge dy\) has a primitive \(\mu\), as it is the case if \(\omega_P\) is exact, the helicity (see [31]) can be used as in [39] to show that \(\Sigma\) is the concave boundary of \((C, \omega_C)\). Indeed, by Stokes theorem the symplectic volume of \((D_\Sigma, \omega)\) equals \(\int_{\Sigma} \mu \wedge \omega\), where \(\Sigma\) is equipped with the boundary orientation induced by the symplectic orientation of \((D_\Sigma, \omega)\). On the other hand, the restriction of \(\omega\) to \(T\Sigma\) equals \(d\alpha_\Sigma\), where \(\alpha_\Sigma\) is the contact form induced by the local Liouville vector field \(Y\), so that
\[
\left(\mu|_{T\Sigma} - \alpha_\Sigma \wedge (d\alpha_\Sigma)^{n-2}\right) \wedge d\alpha_\Sigma
\]
is an exact form on \(\Sigma\). Consequently, \(\int_{\Sigma} \mu \wedge \omega\) equals the contact volume of \((\Sigma, \alpha_\Sigma)\), so that
\[
i_Y \omega^n|_{T\Sigma} = n\alpha_\Sigma \wedge (d\alpha_\Sigma)^{n-1}
\]
is a positive volume form on \((\Sigma, \alpha_\Sigma)\). Hence, the boundary orientation on \(\Sigma\) equals the orientation induced by \(\alpha_\Sigma\), i.e. the local Liouville vector field \(Y\) points out.

Consequently, if \((A, \omega_A)\) is a symplectic cobordism such that \((\Sigma, \alpha_\Sigma)\) appears as convex boundary gluing along \((\Sigma, \alpha_\Sigma)\) yields a symplectic manifold
\[
(W, \omega_W) = (A, \omega_A) \cup_{(\Sigma, \alpha_\Sigma)} (C, \omega_C)
\]
to which Theorem 1.1 applies. As an example we phrase.

Corollary 3.4. The Gromov radius of
\[
\left( P \times Q \times D^2, \omega_P + \omega_Q + dx \wedge dy \right)
\]
is lower or equal than \(\pi\). The strong Weinstein conjecture holds true for any closed contact type hypersurface in
\[
\left( P \times Q \times C, \omega_P + \omega_Q + dx \wedge dy \right).
\]
In particular, we get the (very) stabilized strong Weinstein conjecture for hypersurfaces of contact type in $Q \times \mathbb{C}^\ell$ for $\ell \geq 2$, cf. Floer-Hofer-Viterbo [10]. If $H$ is a Donaldson hypersurface in $(Q, \omega_Q)$ (see [5]) so that the complement of $H$ has the structure of a Stein manifold $(P, \omega_P)$ then the strong Weinstein conjecture follows for contact type hypersurfaces in $(Q \setminus H) \times \mathbb{C}$. By [3] it is possible to construct symplectic hypersurfaces $H$ that lie in the complement of a given compact isotropic submanifold in $(Q, \omega_Q)$.

3.3. Cotangent bundles. We consider the unit sphere $S^{2m+1}$ in $\mathbb{C}^{m+1}$. The Weinstein conjecture holds true for any closed hypersurface $\Sigma$ that is of contact type in $T^* S^{2m+1}$, cf. [36]. The contact structure on $\Sigma$ is taken to be the one induced from $T^* S^{2m+1}$. Notice, that no assumption is made on the bounded component of the complement $T^* S^{2m+1} \setminus \Sigma$ in view of the zero section, cf. [20] [34]. We claim that the strong Weinstein conjecture holds for $\Sigma$ as well.

Indeed, $S^{2m+1}$ embeds into $\mathbb{C}^{m+1} \times \mathbb{C}P^m$ as a Lagrangian submanifold $L$ via the map that sends $z \in S^{2m+1} \subset \mathbb{C}^{m+1}$ to $(z, [\bar{z}])$. The map $S^{2m+1} \ni w \mapsto [w]$, where $[w]$ denotes the complex line through $w$ and the origin, is the so-called Hopf fibration, along which the Fubini-Study form $\omega_{FS}$ pulls back to $dx \wedge dy$. Because the complex conjugation $z \mapsto \bar{z}$ is anti-symplectic the symplectic form $dx \wedge dy + \omega_{FS}$ on $\mathbb{C}^{m+1} \times \mathbb{C}P^m$ vanishes pulled back to $S^{2m+1}$. Hence, the $(2m+1)$-dimensional submanifold $L \subset \mathbb{C}^{m+1} \times \mathbb{C}P^m$ is Lagrangian. Using the fibrewise radial Liouville flow of $T^* S^{2m+1}$ we can bring $\Sigma$ into a small neighborhood of the zero-section. Hence, with Weinstein’s tubular neighborhood theorem ([37]) $\Sigma$ can be realized as a contact type hypersurface in $\mathbb{C}^{m+1} \times \mathbb{C}P^m$ with the characteristic foliation to be conjugate to the one induced by the inclusion $\Sigma \subset T^* S^{2m+1}$. The claim follows with Corollary [34].

In fact, this shows the strong Weinstein conjecture for all cotangent bundles over closed manifolds of the form $X \times S^{2m+1}$ with $m \geq 1$, which admit a Lagrangian embedding into $T^* X \times \mathbb{C}^{m+1} \times \mathbb{C}P^m$. To get products with $S^2$ observe that $S^2$ embeds as a Lagrangian surface into the unit ball in $\mathbb{C}^2$ blown up in two points, [5] Section 6 Example (3). To the blown up ball the complement of the unit ball in $\mathbb{C} \times (\mathbb{C} \cup \infty)$ is glued on. With the construction of a symplectic cap (the cap being $(\mathbb{C} \setminus B_1) \times \mathbb{C}P^1$) and Theorem [4] it follows that any closed hypersurface of contact type in the cotangent bundle of $X \times S^2$ satisfies the strong Weinstein conjecture.

Similarly, because for any closed orientable 3-manifold $Y$ the connected sum $L = Y \#(S^1 \times S^2)$ admits a Lagrangian embedding into $\mathbb{C}^3$ the strong Weinstein conjecture holds true for $T^* L$, see [7]. This is of particular interest for contact type hypersurfaces in $T^* Y$ that miss one fibre. Examples are given by energy surfaces of classical mechanical systems on $Y$ with a sign changing potential function, cf. [34].

4. Stable curves and the moduli space

We consider a symplectic cobordism $(W, \omega)$ that admits an essential local foliation by holomorphic spheres $B \times \mathbb{C}P^1 \subset W$, see Section [23]. We denote by $(M, \alpha)$ the concave boundary of $(W, \omega)$ and associate the completion $(W', \omega_r)$ attaching the negative half-symplectisation of $(M, \alpha)$ as described in Section [24]. Moreover, we assume the contact form $\alpha$ to be non-degenerate, that is along periodic solutions of the Reeb vector field of $\alpha$ the linearised Poincaré return map has no eigenvalue equal to 1. Let $J$ be an admissible compatible almost complex structure on $(W', \omega_r)$ that is compatible with the contact forms $\alpha$ and $\alpha_+$ on the concave
end and in a neighborhood of $M_+$, resp., as described at the beginning of Section 3.

The aim of this section is to study stable holomorphic one-marked curves of genus zero in $(W', \omega_\tau, J)$.

4.1. Stable maps. Let $(S, j, D)$ be a nodal Riemann surface of genus zero. The set of points, the so-called nodal points, that belong to a nodal pair is denoted by $|D|$. We provide $(S, j, D)$ with a finite set $M$ of pairwise distinct points in $S \setminus |D|$, the so-called marked points. The points in $|D| \cup M$ are called special. A connected component $C$ of $S$ is called stable if the number of special points in $C$ is greater or equal than $3 - 2\text{genus}(C)$.

A stable map $(S, j, D, M, u)$ in $W'$ is a continuous map $u: S \rightarrow W'$ defined on a marked nodal Riemann surface $(S, j, D, M)$ that descends to a continuous map on the quotient $S/D$ such that:

- The map $u$ is of Sobolev-class $H^3_{\text{loc}}$ on $S \setminus |D|$ and of weighted Sobolev class $H^{3, \delta}$ near the nodal points $|D|$ for some $\delta \in (0, 2\pi)$, see [27] Definition 1.1.
- The cohomological integral $\int_C u^* \omega_\tau$ is non-negative for any connected component $C$ of $S$ and for one $\tau \in T$ (and hence for all by Stokes theorem).
- If a connected component $C$ of $S$ is not stable, then $\int_C u^* \omega_\tau > 0$.

4.2. The moduli space. Two stable maps $(S, j, D, M, u)$ and $(S', j', D', M', u')$ are said to be equivalent if there exists a diffeomorphism $\varphi: S \rightarrow S'$ such that $\varphi^* j' = j$, $\varphi^* D = D'$, $\varphi^* M = M'$, and $u' \circ \varphi = u$. Often we will write $u$ for $(S, j, D, M, u)$. The equivalence class $[S, j, D, M, u]$ of a stable map is called a stable curve and is denoted by $u$. We denote by $Z$ the set of all one-marked genus zero stable curves $u$ in $W'$ such that the map induced by $u$ on the quotient $S/D$ is homologous to $* \times \mathbb{C}P^1$ for some $* \in B$.

A stable curve $u$ is called holomorphic if it can be represented by a stable map $u$ that is holomorphic. The definition does not depend on the choice of $u$. We denote by $\mathcal{M}$ the moduli space of all holomorphic stable curves $u \in Z$.

Observe that for all $b \in B$ the class of $(\mathbb{C}P^1, i, \emptyset, \infty, z \mapsto (b, z))$ represents an element of $\mathcal{M}$ that we will call a standard sphere. Due to the stability condition and the local foliation $B \times \mathbb{C}P^1$ being essential all non-standard holomorphic curves in $\mathcal{M}$ do not intersect $B \times \mathbb{C}P^1$. We will identify the subset of standard spheres in $\mathcal{M}$ with $B \times \mathbb{C}P^1$, where the $\mathbb{C}P^1$-factor corresponds to the marked point. The complement is denoted by $\mathcal{M}_{\text{cut}} = \mathcal{M} \setminus B \times \mathbb{C}P^1$.

4.3. A priori uniform bounds. The Dirichlet-energy of all nodal holomorphic spheres $u$ induced by stable holomorphic curves $u \in \mathcal{M}$ equals

$$\int_S u^* \omega_\tau = [\omega_\tau]\left([\mathbb{C}P^1]\right) = \pi g^2$$

for all $\tau \in T$.

The moduli space $\mathcal{M}$ admits upper bounds in the following sense. Because the almost complex structure $J$ is compatible with the contact form $\alpha_+$ the maximum principle implies that no stable holomorphic curve can intersect a neighborhood
(ε, 0] × M, of M+, cf. the first part of the proof of Proposition 3.1. Similarly, because only standard curves u ∈ M intersect the foliation domain B × CP¹ we can bound M cut away from ∂B if the boundary of B is not empty.

As it will turn out (see Section 5.5) there are no lower bounds for M along the concave end. This will prove Theorem 1.1 as the following lemma shows:

Lemma 4.1. If there exists a sequence u_k in M such that u_k(S_k) intersects (−∞, −k] × M non-trivially, then (M, α) admits a null-homologous Reeb link of total action less than πϱ².

Proof. If u_k maps a component C of S_k into the concave end u_k|C is constant by Stokes theorem. Therefore, we find sequences z_k and w_k of points on S_k such that u_k(z_k) ∈ W and u_k(w_k) is contained in the concave end so that the projection of u_k(w_k) to the R-factor tends to −∞. With respect to a metric on W' that equals a product metric on the concave end (product with the Euclidean metric on the R-axis) the gradient of u_k blows up. This follows with a mean value argument as in [14, 22]. The bubbling off analysis from [1] shows the existence of a holomorphic building of height k−1 for some k−1 ≥ 1. The lowest level of the building is represented by a punctured finite energy surface in the symplectisation of (M, α) that has Hofer-energy less than πϱ² and positive punctures exclusively. Near the punctures the finite energy surface converges to cylinders over closed Reeb orbits of α exponentially fast. Hence, the projection of a component to M along the R-axis defines a 2-chain whose boundary is a Reeb link in (M, α) of total action less than πϱ², cf. [13, 19].

In other words, in order to prove Theorem 1.1 we have to exclude the case where there exists K > 0 such that for all u in M the image u(S) is contained in (−K, 0] × M ∪ W.

We remark that the proof of Lemma 4.1 uses the assumption α being non-degenerate. This is not a restriction as the arguments in Section 5 will show.

5. Polyfold structure

5.1. Topology of the space of stable curves. The space Z of stable curves has a natural topology as described in [27, Section 2.1/3.4] that is second countable, paracompact, and Hausdorff, see [27, Theorem 1.6]. The topology is induced by the $H^3$-topology of maps on nodal Riemann surfaces that have exponential decay in holomorphic polar coordinates near the nodes. Part of the construction is a choice of auxiliary marked points that stabilize all connected components of the domain if necessary. The additional points are fixed using local codimension-2 submanifolds in W' that are transverse to the image of the curve intersecting in a single point. It is required that the additional marked points are mapped to the intersection points. If a non-trivial automorphism group is acting on a stable curve the auxiliary marked points are supposed to be chosen equivariantly. The stabilization of the domain makes it possible to use the topology of the corresponding Deligne-Mumford space in terms of uniformizing families, see [27, Definition 2.12]. In order to describe the desingularization of the nodes (i.e. the gluing) uniformizing families are used to obtain uniformizers for the space of stable maps Z, see [27, Section 3.1/3.2].

We remark that the evaluation map ev : Z → W' that maps u to the value u(z) at the marked point z is continuous, see [25, p. 2290] or [27, p. 7].
5.2. The target space. Let \( u \) be a stable map that represents a class \( u \) in \( Z \). We denote by \( \xi \) a continuous section of \( \text{Hom}(\Lambda T^*S, u^*TW') \) such that for each \( z \in S \) the map \( \xi(z): T_zS \to T_{u(z)}W' \) is complex anti-linear with respect to \( J(u(z)) \). The section \( \xi \) is required to be of Sobolev class \( H^2_{\text{loc}} \) on \( S \setminus |D| \) and of weighted Sobolev class \( H^{2,\delta} \) near \( |D| \) for some \( \delta \in (0,2\pi) \), see [27, Section 1.2]. An equivalence \( \varphi \) of stable maps \( (S,j,D,M,u) \) and \( (S',j',D',M',u') \) is an equivalence of \( (S,j,D,M,u,\xi) \) and \( (S',j',D',M',u',\xi') \) if \( \xi' \circ T\varphi = \xi \). The equivalence class is denoted by \( \xi \) and the space of all equivalence classes \( \xi \) by \( \mathcal{W} \).

By [27, Theorem 1.9] \( \mathcal{W} \) has a natural topology that is second countable, paracompact, and Hausdorff so that the projection \( p: \mathcal{W} \to Z \) that maps the class \( \xi \) to the class \( u \), if \( \xi \) is a section along \( u \), is continuous. The Cauchy-Riemann operator \( \partial_j \) is a section of \( p \) whose value \( \partial_j u \) at a point \( u \in Z \) is the class represented by

\[
\left( S, j, D, M, u, \frac{1}{2} (Tu + J(u) \circ Tu \circ j) \right).
\]

Notice that the moduli space \( \mathcal{M} \) equals the zero set \( \{ \partial_j u = 0 \} \).

5.3. Polyfold Fredholm section. The space of stable curves \( Z \) is a polyfold (see [25, Section 3]) so that the evaluation map \( e: Z \to W' \) is sc-smooth, see [27, Theorem 1.7/1.8], [25, Theorem 1.10]. The projection map \( p: \mathcal{W} \to Z \) is a strong polyfold bundle, see [27, Theorem 1.10]. By [27, Theorem 1.11] the Cauchy-Riemann operator \( \partial_j: \mathcal{Z} \to \mathcal{W} \) is a sc-smooth, component proper Fredholm section which is naturally oriented. The Fredholm index of \( \partial_j \) equals the dimension of \( W \) because the first Chern class of \( (TW', J) \) evaluates to 2 on \([ \ast \times \mathbb{C}P^1] \).

On the level of objects the strong polyfold bundle structure of \( p: \mathcal{W} \to Z \) is obtained by gluing strong polyfold bundles in the sense of ep-groupoids. To understand the Cauchy-Riemann section \( \partial_j: \mathcal{Z} \to \mathcal{W} \) it is enough to consider a local \( M \)-polyfold bundle chart. We would like to apply this principle for standard holomorphic spheres \( u \in \mathcal{M} \). Represent \( u \) by the map \( u: z \mapsto (b, z) \) for some \( b \in B \). Denote by \( X \) the sc-Hilbert manifold of pairs \((v, z)\), where \( z \in \mathbb{C}P^1 \) is a marked point near \( \infty \) and \( v \) a map \( \mathbb{C}P^1 \to W' \) of class \( H^3 \) that is close to \( u \) mapping 0 and 1 into \( B \times 0 \) and \( B \times 1 \), resp. Moreover, let \( E \) be the sc-Hilbert space bundle over \( X \) with fibre consisting of all \( H^2 \)-maps from \( \mathbb{C}P^1 \) into the space of \( J(v) \)-anti-linear maps \( \Lambda T^*CP^1 \to v^*TW' \) for \( v \in X \). A \( M \)-polyfold bundle chart of \( p: \mathcal{W} \to Z \) about \( u \) is then given by \( E \to X \) because \( u \) is un-noded, so that the surrounding splicing core is the full ambient space and the sc-retration map equals the identity.

The Cauchy-Riemann operator \( \partial_j: \mathcal{Z} \to \mathcal{W} \) induces a local section \( f: X \to E \) near \( x = (u, \infty) \). The linearization \( f'(x): T_xX \to E_x \) is the vertical differential, which is the composition of the sc-differential \( T_xf: T_xX \to T_0E \) and the projection \( T_0E = T_xX \oplus E_x \to E_x \), see [23, Section 4.4]. By [27, p. 55 and Definition 5.5] and [24, Section 3] the linearization \( f'(x) \) is a sc-Fredholm operator and coincides with the linearized Cauchy-Riemann operator at \((u, \infty)\) in the \( C^\infty \)-sense (cf. [31]) on a dense subset, see [23, Proposition 2.14/2.15]. Because a sc-Fredholm operator is regularizing \( f'(x) \) is surjective with kernel being of dimension equal to \( \dim W \) that consists of smooth elements exclusively.

5.4. Abstract Perturbations. Let \( \lambda: \mathcal{W} \to \mathbb{Q}\cap(0, \infty) \) denote a \( \text{sc}^+\)-multisection. In a local representation \( P: E \to X \) of \( p: \mathcal{W} \to Z \) we write \( \Lambda: E \to \mathbb{Q}\cap(0, \infty) \) for
the sc+\textsuperscript{-}-multisection that corresponds to \( \lambda \). The local sc+\textsuperscript{-}-sections of \( P \) that are attached to \( \Lambda \) are denoted by \( s_\iota \). A pair \((\p_\iota, \lambda)\) is called transverse if in local representations \( f: X \to E \) of the Cauchy-Riemann operator the linearizations
\[
f'(u) - s_\iota'(u): T_u X \to E_u
\]
are surjective for all \( i \) and for all \( u \in Z \) that are contained in \( \{ \lambda(\p_\iota) > 0 \} \), see [24 Definition 4.7(1)]. Observe that in case \( f'(u) \) is onto for \( u \in M \) that has a simple representation by an un-noded holomorphic sphere map \( u \) we can choose one local section that is identically 0 in a neighborhood of \( u \) in \( X \), i.e. \( \lambda(0_u) = 1 \).

5.5. Gromov-Witten integration. In this section we give a proof of Theorem 1.1 in the case the contact form \( \alpha \) is non-degenerate. For \( K > 0 \) denote by \( W_K \) the open domain in \( W' \) that is obtained from \( W' \) by removing \( (-\infty, -K) \times M, [-1/K, 0] \times M_+ \), and \( B_K \times CP^1 \), where \( B_K \) is a subset of \( B \) that is diffeomorphic to either a ball of radius \( 1/K \) or, if \( \p B \) is not empty, a collar neighborhood \([ -1/K, 0 ] \times \p B \) of \( \p B \). We have to exclude uniform lower bounds for \( M \), see Lemma 4.1.

Arguing by contradiction we assume that there exists \( K > 0 \) such that for all non-standard curves \( u \in M_{\text{cut}} \) the image \( u(S) \) is contained in \( W_K \). There exists a neighborhood \( U \) of \( M_{\text{cut}} \) in \( Z \) such that for all \( u \in U \) the image \( u(S) \) is contained in \( W_K \), cf. Section 5.1. For example we can choose \( U = ev^{-1}(W_K) \) because of \( ev(M_{\text{cut}}) \subseteq W_K \). Moreover, by [4] and [24 Proposition 4.9 and Remark 4.10] \( M \) is a compact subset of \( Z \).

By [25 Theorem 4.17] there exists a small sc+\textsuperscript{-}-multisection \( \lambda \) such that \((\p_\iota, \lambda)\) is transverse. Notice, that for all standard spheres \( u \in M \) the isotropy group of \( u \) is trivial and \( f'(u) \) is onto, where \( f \) is a local representation of \( \p_\iota \). The proof of [25 Theorem 4.17] implies that \( \lambda \) can be chosen to be trivial for all standard spheres \( u \in M \), i.e. in a local representation \( \Lambda \) is identically 1 on the zero-section over the set of standard holomorphic spheres \( u \), cf. [25 Definition 3.35]. The support of \( \lambda \) can be assumed to be contained in \( U \). Therefore, the solution set
\[
S = \{ u \in Z \mid \lambda(\p_\iota u) > 0 \}
\]
of \((\p_\iota, \lambda)\) is an oriented compact branched suborbifold of dimension \( \dim W \) with boundary \( \p S \), see [26 Theorem 4.17] and [27 Section 1.4]. Moreover, \( S \setminus (B \times CP^1) \) is contained in \( U \) and \( S \) is equipped with the weight function
\[
\theta: Z \to \mathbb{Q} \cap [0, \infty): u \mapsto \lambda(\p_\iota u).
\]
A neighborhood of \( \p S \) in \( S \), which is equal to a neighborhood of \( \p M \) in \( M \), can be identified with a collar neighborhood of \( \p B \times CP^1 \) in \( B \times CP^1 \subset W \).

In order to reach the desired contradiction consider a top-dimensional form \( \Omega \) on \( W' \) with total volume \( \int_W \Omega = 1 \) that has support in the interior of \( B_K \times CP^1 \). Notice, that the evaluation map \( ev: S \to W' \) induces an embedding of \( B_K \times CP^1 \) into the solution space \( S \), which coincides with the moduli space \( M \) along the image \( B_K \times CP^1 \). Therefore,
\[
1 = \int_W \Omega = \int_{(S, \theta)} ev^* \Omega.
\]
Using a compactly supported diffeotopy in the interior of \( W' \) we can bring the support of \( \Omega \) into the complement of \( W_K \cup (B_K \times CP^1) \). Denoting by \( \Omega_1 \) the image of \( \Omega \) under pull back along the diffeotopy the difference \( \Omega - \Omega_1 \) has a primitive \( \mu \) with compact support in the interior of \( W' \). Because the support of \( \Omega_1 \) does not
lie in the image of the evaluation map \( \text{ev} \colon \mathcal{S} \to W' \) the restriction of \( \text{ev}^* \Omega_1 \) to \( \mathcal{S} \) vanishes. By Stokes theorem \cite{StokesTheorem}, Theorem 1.27,

\[
\int_{(\mathcal{S}, \partial)} \text{ev}^* \Omega = \int_{(\mathcal{S}, \partial)} \text{ev}^* \mu.
\]

But \( \text{ev}^* \mu \) restricted to the boundary \( \partial \mathcal{S} \) must vanish because \( \mu \) has compact support in the complement of \( \partial W' \). This is a contradiction. This finishes the proof of Theorem \ref{thm:main} for a non-degenerate contact form \( \alpha_1 \).

With the same argument one shows that if \( (W, \omega) \) has no concave boundary \( (W, \omega) \) cannot have a convex boundary too.

6. Proof of Theorem \ref{thm:main}

We claim that there exist \( N \in \mathbb{N} \) and periodic solutions \( x_1, \ldots, x_N \) of the Reeb vector field \( R \) of \( \alpha \) that are of (not necessary prim) period \( T_1, \ldots, T_N \), resp., such that \( T_1 + \ldots + T_N < \pi \varrho^2 \) and \( [x_1] + \ldots + [x_N] = 0 \) in the homology of \( M \). With Section 5.5 and Lemma \ref{lem:9} the claim follows for a non-degenerate contact form \( \alpha \). If \( \alpha \) is degenerate we find a sequence of contact forms \( \alpha_k \) on \( M \) that are non-degenerate such that \( \alpha_k \) tends to \( \alpha \) in \( C^\infty \), see \cite[Proposition 6.1]{prop}. Observe, that the Reeb vector field \( R_k \) of \( \alpha_k \) tends to \( R \) in the \( C^\infty \)-topology too.

First of all we consider a sequence of periodic solutions \( x_k \) of \( R_k \) that are of period \( T_k < \pi \varrho^2 \). To the reparametrised sequence \( y_k(t) = x_k(T_k t) \) the Arzelà-Ascoli theorem applies so that a subsequence of \( y_k \) converges in \( C^\infty(\mathbb{R}/\mathbb{Z}) \) to a loop \( y \) in \( M \) that is tangent to \( \mathbb{R}R \) and has action

\[
T := \int_0^1 y^* \alpha = \lim_{k \to \infty} \int_0^1 y_k^* \alpha_k \leq \pi \varrho^2,
\]

because \( \int_0^1 y_k^* \alpha_k = T_k \). The loop \( x(t) := y(t/T) \) is a \( T \)-periodic solution of \( R \). To express this circumstance we will say that a subsequence of \( x_k \) converges to \( x \) after reparametrisation.

With this preliminaries we apply the established existence result to the non-degenerate contact form \( \alpha_k \) for each \( k \). This results in a sequence of periodic solutions \( x_1^k, \ldots, x_N^k \) of \( R_k \) that are of period \( T_1^k, \ldots, T_N^k \), resp., such that the periods sum up to total action less than \( \pi \varrho^2 \) and the loops represent the zero class in the homology of \( M \). By the flow-box theorem applied to \( R \) we get \( A_k > A/2 \) for \( k \) sufficiently large, where \( A \) and \( A_k \) denote the minimal action of a periodic solution of \( R \) and \( R_k \), resp. Hence, \( N_k \) is bounded from above by \( 2\pi \varrho^2 / A \) so that we can assume the number of link components to be constantly equal to \( K \in \mathbb{N} \). Therefore, we find a subsequence \( x_1^k, \ldots, x_K^k \) that converges after reparametrisation to periodic solutions \( x_1, \ldots, x_K \) of \( R \) with period \( T_1 + \ldots + T_K \leq \pi \varrho^2 \) such that \( [x_1^k] = [x_1], \ldots, [x_K^k] = [x_K] \) for \( k \) sufficiently large. In particular, the sum \( [x_1] + \ldots + [x_K] \) must vanish. The desired claim follows with the exception of a non-strict inequality for the total action of the null-homologous Reeb link. In order to obtain the strict inequality observe that we can realize \( (M, (1+\delta)\alpha) \) as the concave boundary of \( (W, \omega) \) for some small \( \delta > 0 \).

Q.E.D.

Remark 6.1. (Weak contact type boundary condition) In Theorem \ref{thm:main} the convex boundary can be replaced by a \( J \)-convex boundary in the sense of \cite{JConvex} or \cite{JConvex}, cf. \cite[Section 3.2 (C4)]{section} or \cite{section}. This means the boundary components \( M_+ \) oriented as the boundary of \( (W, \omega) \) admit a positive contact form \( \alpha_+ \) such that in
a neighborhood of \( M_+ \) there exists an almost complex structure \( J \) satisfying the following properties:

- \( J \) is tamed by \( \omega \),
- \( J \) leaves the kernel \( \xi_+ \) of \( \alpha_+ \) invariant, i.e. \( \xi_+ = TM \cap JTM \), and
- \( J \) restricted to the contact structure \( \xi_+ \) is tamed by \( \partial e^a \alpha_+ \).

The reason is that there exists a collar neighborhood \( U \) of \( M \) such that any \( J \)-holomorphic sphere that intersects \( U \) must be constant. Indeed, \( U \) can be chosen to be equal to \( (-\varepsilon, 0] \times M \) for \( \varepsilon > 0 \) sufficiently small such that the restriction of \( \partial \alpha \) to the boundary \( M \) equals \( -JR_+ \), where \( R_+ \) is the Reeb vector field of \( \alpha_+ \). Shrinking \( \varepsilon > 0 \) if necessary the symplectic form \( d(e^a \alpha_+) \) tames \( J \) on \( U \) and \(-d(de^a \circ J)\) is positive on \( J \)-complex lines.

**Proof of Corollary 1.6.** In the case \( (W, \omega) \) has no contact type boundary the proof of Theorem 1.1 shows that the evaluation map restricted to the solution space \( S \) is surjective. Sending the abstract perturbations to zero one obtains a family of corresponding nodal solutions through each point of \( W \) that admit converging subsequences in the topology of \( Z \), cf. [25, Theorem 4.17]. The limits are holomorphic nodal spheres. □

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