TOWARDS A NON-ARCHIMEDEAN ANALYTIC ANALOG OF THE BASS–QUILLEN CONJECTURE

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Abstract We suggest an analog of the Bass–Quillen conjecture for smooth affinoid algebras over a complete non-archimedean field. We prove this in the rank-1 case, i.e. for the Picard group. For complete discretely valued fields and regular affinoid algebras that admit a regular model (automatic if the residue characteristic is zero) we prove a similar statement for the Grothendieck group of vector bundles $K_0$.

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Introduction

For a ring $A$ let us denote by $\text{Vec}_r(A)$ the set of isomorphism classes of finitely generated projective modules of rank $r$. The Bass–Quillen conjecture predicts that for a regular noetherian ring $A$ the inclusion into the polynomial ring $A[t_1, \ldots, t_n]$ induces a bijection

$$\text{Vec}_r(A) \xrightarrow{\sim} \text{Vec}_r(A[t_1, \ldots, t_n])$$

for all $n, r \geq 0$. Based on the work of Quillen and Suslin on Serre’s problem the conjecture has been shown in case $A$ is a smooth algebra over a field [14].

In this note we discuss a potential extension of this conjecture to affinoid algebras in the sense of Tate. Let $K$ be a field which is complete with respect to a non-trivial non-archimedean absolute value and let $A/K$ be a smooth affinoid algebra. In rigid

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geometry a building block is the ring of power series converging on the closed unit disc

\[ A(t_1, \ldots, t_n) = \left\{ f = \sum c_k t^k \in A[[t_1, \ldots, t_n]] \mid c_k \xrightarrow{|k| \to \infty} 0 \right\}, \]

which serves as a replacement for the polynomial ring in algebra.

Using these convergent power series the following positive result in analogy with Serre’s problem is obtained in [15].

Example 1 (Lütkebohmert). All finitely generated projective modules over \( K \langle t_1, \ldots, t_n \rangle \) are free.

Unfortunately, over more general smooth affinoid algebras one has the following negative example [8, 4.2].

Example 2 (Gerritzen). Assume the ring of integers \( K^\circ \) of \( K \) is a discrete valuation ring with prime element \( \pi \). For the smooth affinoid \( K \)-algebra \( A = K \langle t_1, t_2 \rangle / (t_1^2 - t_2^3 - \pi) \) the map

\[ \text{Pic}(A) \to \text{Pic}(A[t]) \]

is not bijective.

This example shows that for our purpose the ring of convergent power series \( A(t) \) is not entirely appropriate. Let \( \pi \in K \setminus \{0\} \) be an element with \( |\pi| < 1 \). As an improved non-archimedean analytic replacement for the polynomial ring over \( A \) we are going to use the pro-system of affinoid algebras \( \varprojlim_{t \mapsto \pi} A(t) \). It represents an affinoid approximation of the non-quasi-compact rigid analytic space \( (\mathbb{A}^1_A)^{an} \) since

\[ \varprojlim_{t \mapsto \pi} A(t) = H^0((\mathbb{A}^1_A)^{an}, \mathcal{O}). \]

Note that the latter non-affinoid \( K \)-algebra is harder to control, compare [9, Ch. 5] and [3].

As a non-archimedean analytic analog of the Bass–Quillen conjecture one might ask:

Question 3. Is the map

\[ \text{Vec}_r(A) \to \varprojlim_{t \mapsto \pi} \text{Vec}_r(A(t)) \]

a pro-isomorphism for \( A/K \) a smooth affinoid algebra?

We give a positive answer for \( r = 1 \).

Theorem 4. For \( A/K \) a smooth affinoid algebra the map

\[ \text{Pic}(A) \to \varprojlim_{t \mapsto \pi} \text{Pic}(A(t)) \]

is an isomorphism of pro-abelian groups.

This is stronger than the statement that \( \text{Pic}(A) \to \lim_{t \mapsto \pi} \text{Pic}(A(t)) \) is an isomorphism. The latter has the following consequence, which we will prove in § 3:
Corollary 5. For $A/K$ a smooth affinoid algebra the map

$$\text{Pic}(A) \to \text{Pic}((\mathbb{A}^1_A)^{an})$$

is an isomorphism.

The Picard group $\text{Pic}(A)$ of an affinoid algebra $A$ is isomorphic to the cohomology group $H^1(\text{Sp}(A), \mathcal{O}^*)$.

In case the residue field of $K$ has characteristic zero, one has the exponential isomorphism $\exp : \mathcal{O}(1) \to \mathcal{O}^*(1)$, where $\mathcal{O}(1) \subset \mathcal{O}$ is the subsheaf of rigid analytic functions $f$ with $|f|_{\text{sup}} < 1$ and $\mathcal{O}^*(1) \subset \mathcal{O}^*$ is the subsheaf of functions $f$ with $|1 - f|_{\text{sup}} < 1$. Based on this isomorphism [8, Satz 4] reduces Theorem 4 in case of characteristic zero to a vanishing result for the additive rigid cohomology group $H^1(\text{Sp}(A), \mathcal{O}(1))$ which is established in [1]. As the articles [1] and [2] are written in German and are not easy to read, we give a simplified proof of their main results in § 1 based on the cohomology theory of affinoid spaces [17].

However in case $\text{ch}(K) > 0$ this approach using the exponential isomorphism does not apply. Instead, in § 2 we explain how to pass from a vanishing result for the additive cohomology groups to a vanishing result for the multiplicative cohomology groups in the absence of an exponential isomorphism. Based on the latter vanishing the proof of Theorem 4 is given in § 3.

In § 4 we prove the following stable version of Question 3. Assume that $K$ is discretely valued, and hence its valuation ring is noetherian. Let $A^\circ$ denote the subring of power bounded elements in $A$. By a regular model for a regular affinoid $K$-algebra $A$ we mean a proper morphism of schemes $\mathcal{X} \to \text{Spec}(A^\circ)$ which is an isomorphism over $\text{Spec}(A)$ and such that $\mathcal{X}$ is regular.

Theorem 6. Let $K$ be discretely valued, and let $A/K$ be a regular affinoid algebra. Assume that $A$ admits a regular model; this is automatic if the residue field of $K$ has characteristic zero. Then

$$K_0(A) \to \lim_{\to \to} K_0(A(t))$$

is a pro-isomorphism.

The proof of Theorem 6 uses ‘pro-cdh-descent’ [12, 16] for the $K$-theory spectrum of schemes and resolution of singularities in the residue characteristic zero case; so it is rather non-elementary. Of course, in the cases where Theorem 6 applies it comprises Theorem 4, as there is a surjective determinant map $\det : K_0 \to \text{Pic}$.

Notations

We denote the supremum seminorm [5, § 3.1] of a rigid analytic function $f$ on an affinoid space $X$ by $|f|_{\text{sup}}$. For a real number $r > 0$ we denote by $\mathcal{O}_X(r) \subset \mathcal{O}_X$ the subsheaf of functions of supremum seminorm $< r$. We often omit the subscript $X$ if no confusion is possible. We write $\mathcal{O}^\circ \subset \mathcal{O}$ for the subsheaf of functions of supremum norm $\leq 1$.

If $0 < r < 1$, functions of the from $1 + f$ with $|f|_{\text{sup}} < r$ are invertible, and we denote by $\mathcal{O}^*(r) \subset \mathcal{O}^*$ the subsheaf of invertible functions of this form.
We use similar notations $K(r), K^o, K^o(r)$ for corresponding elements of the field $K$ or complete valued extensions of $K$.

If $a$ is an analytic point of an affinoid space [11, § 2.1], we denote the completion of its residue field by $F_a$.

For the closed polydisk $\text{Sp}(K(t_1, \ldots, t_d))$ of radius 1 and dimension $d$ over $K$ we use the notation $\mathbb{B}^d_K$ or simply $\mathbb{B}^d$.

An affinoid algebra $A/K$ is called smooth if $A \otimes_K K'$ is regular for all finite field extensions $K \subset K'$. As a general reference concerning the terminology of rigid spaces we refer to [5].

1. Vanishing of additive cohomology (after Bartenwerfer)

The aim of this section is to give new, more conceptual proofs of the main results of [1] and [2]. Our techniques are based on the cohomology theory for affinoid spaces as developed by van der Put, see [17] and [11]. Let $K$ be a field which is complete with respect to the non-archimedean absolute value $|\cdot|: K \to \mathbb{R}$. We assume that the absolute value $|\cdot|$ is trivial. All affinoid spaces we consider in this section are assumed to be integral.

Let $\mathcal{M}, N$ be sheaves of $O^o$-modules on the affinoid space $X = \text{Sp}(A)$. We say that $\mathcal{M}$ is weakly trivial if there exists $r \in (0, 1)$ with $O(r)\mathcal{M} = 0$. Note that this just means that there exists $f \in K^o \setminus \{0\}$ with $f\mathcal{M} = 0$. The weakly trivial $O^o$-modules form a Serre subcategory of the abelian category of all sheaves of $O^o$-modules. We say that an $O^o$-morphism $u: \mathcal{M} \to N$ is a weak isomorphism if $\ker(u)$ and $\coker(u)$ are weakly trivial. Note that the weak isomorphisms are exactly those morphisms which are invertible up to multiplication by elements of $K^o \setminus \{0\}$. We say that $\mathcal{M}$ is weakly locally free (wlf) if there is a finite affinoid covering $X = \bigcup_{i \in I} U_i$ and weak isomorphisms $(O^o_{U_i})^{n_i} \cong \mathcal{M}|_{U_i}$ for each $i \in I$.

Note that for $\mathcal{M}$ wlf the $O_X$-module sheaf $\mathcal{M} \otimes_{O^o_X} O_X$ is coherent and locally free, i.e. locally free of finite type.

**Lemma 7.** Let $\psi: \mathcal{M} \to N$ be an $O^o$-morphism of wlf sheaves on $X = \text{Sp}(A)$, and let $f \in A^o$. If

$$f \coker(\psi \otimes 1: \mathcal{M} \otimes_{O^o} O \to N \otimes_{O^o} O) = 0,$$

then there exists $r \in (0, 1)$ such that $f K(r) \coker(\psi) = 0$.

**Proof.** By the definition of weak local freeness, we may assume without loss of generality that $\mathcal{M} = (O^o)^m$ and $N = (O^o)^n$. Let $C$ be the cokernel of $\psi$. By Tate’s acyclicity theorem [5, Corollary 4.3.11] we get an exact sequence

$$H^0(X, \mathcal{M} \otimes_{O^o} O) \to H^0(X, N \otimes_{O^o} O) \to H^0(X, C \otimes_{O^o} O),$$

where the right hand $A$-module is $f$-torsion by assumption. Let $e_1, \ldots, e_n \in N(X)$ be the canonical basis elements. So we deduce that $f e_1, \ldots, f e_n$ have preimages $l_1, \ldots, l_n \in H^0(X, \mathcal{M} \otimes_{O^o} O) = A^m$. Choose $r \in (0, 1)$ such that $K(r)l_1, \ldots, K(r)l_n \subset (A^o)^m$. □

**Proposition 8.** Let $\mathcal{M}$ be an $O^o$-module sheaf on $X = \text{Sp}(A)$ such that $\mathcal{M} \otimes_{O^o_X} O_X$ is coherent and locally free as $O_X$-module sheaf. Then the following are equivalent:
(i) $\mathcal{M}$ is wlf.

(ii) For each finite set of points $R \subset X$ there is an injective $\mathcal{O}^o$-linear morphism $\Psi : (\mathcal{O}^o)^n \to \mathcal{M}$ and $f \in \mathcal{O}^o(X)$ with $f(x) \neq 0$ for all $x \in R$ such that $f \coker(\Psi) = 0$.

(iii) For each point $x \in X$ there is an injective $\mathcal{O}^o(X)$-linear morphism $\Psi_x : (\mathcal{O}^o)^n \to \mathcal{M}$ and $f_x \in \mathcal{O}^o(X)$ with $f_x(x) \neq 0$ such that $f_x \coker(\Psi) = 0$.

**Proof.** Clearly, (ii) implies (iii). We first prove (iii) implies (i). Choose for each point $x \in X$ a map $\Psi_x$ and $f_x$ as in (iii). There is a finite set of points $x_1, \ldots, x_k \in X$ such that we get a Zariski covering

$$X = \bigcup_{i \in \{1, \ldots, k\}} \{x \in X \mid f_{x_i}(x) \neq 0\}.$$  

By [5, Lemma 5.1.8] there exists $\epsilon \in \sqrt{|K^*|}$ such that the $U_i = \{x \in X \mid |f_{x_i}(x)| \geq \epsilon\}$ cover $X$. Then the morphisms $\Psi_{x_i}|_{U_i}$ are weak isomorphisms, so $\mathcal{M}$ is wlf.

We now prove that (i) implies (ii). As $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X$ is locally free, there exists a finitely generated projective $A$-module $M$ with $M^c = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X$, [5, § 6.1]. By $A_R$ we denote the semilocal ring which is the localization of $A$ at the finitely many maximal ideals $R$. Choose a basis $b_1, \ldots, b_n$ of the free $A_R$-module $M \otimes_A A_R$. Without loss of generality we can assume $b_1, \ldots, b_n$ are induced by elements of $\mathcal{M}(X)$. We claim that the latter elements give rise to a morphism $\Psi$ as in (ii). Indeed, by elementary algebra we find $f' \in A^o$ such that $f'(x) \neq 0$ for all $x \in R$ and such that

$$f' \coker(A^n \to M) = 0.$$  

We conclude by Lemma 7. 

**Proposition 9.** Let $\phi : X \to Y$ be a finite étale morphism of affinoid spaces over $K$ and let $\mathcal{M}$ be a wlf $\mathcal{O}_X^o$-module. Then $\phi^* \mathcal{M}$ is a wlf $\mathcal{O}_Y^o$-module.

**Proof.** Let $X = \text{Sp}(A)$ and $Y = \text{Sp}(B)$. The $\mathcal{O}_Y$-module sheaf $\phi^*(\mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y = \phi^*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X)$ is coherent and locally free. For $y \in Y$ let $R$ be the finite set $\phi^{-1}(y)$ and let $M \subset B$ be the maximal ideal corresponding to $y$. From Proposition 8 we deduce that there is an injective $\mathcal{O}_Y^o$-linear morphism

$$\Psi : (\mathcal{O}_X^o)^n \to \mathcal{M}$$  

whose cokernel is killed by some $f \in A^o$ which does not vanish on $R$. Then as the induced homomorphism $\phi^* : B \to A$ is finite the prime ideals of $B$ containing the ideal $I = (\phi^*)^{-1}(Af)$ are exactly the preimages of the prime ideals in $A$ which contain $f$, see [7, § V.2.1]. So we can find $g \in I \cap B^o$ which is not contained in $M$. Then the cokernel of the injective morphism

$$\phi^*(\Psi) : \phi^*(\mathcal{O}_X^o)^n \to \phi^*(\mathcal{M}).$$  

is $g$-torsion. By Proposition 8 we see that it suffices to show that $\phi^*(\mathcal{O}_X^o)$ is wlf.

Note that for $V \subset Y$ an affinoid subdomain $\mathcal{O}_X^o(\phi^{-1}(V))$ is the integral closure of $\mathcal{O}_Y^o(V)$ in $A \otimes_B \mathcal{O}_Y(V) = \mathcal{O}_X(\phi^{-1}(V))$ [5, Theorem 3.1.17]. As the field extension $Q(B) \to Q(A)$ is separable, it is not hard to bound this integral closure as follows. Let $b_1, \ldots, b_d \in \mathcal{O}^o(X)$
induce a basis of the free $B_M$-module $A \otimes B B_M$. This basis induces an injective $O_Y^\circ$-linear morphism
\[
\Psi : (O_Y^\circ)^d \to \phi_* (O_X^\circ).
\]
Let $\delta$ be the discriminant of $b_1, \ldots, b_d$. Then by [7, Lemma V.1.6.3] the cokernel of $\Psi$ is $\delta$-torsion.

As the point $y \in Y$ was arbitrary we conclude from Proposition 8 that $\phi_* (O_X^\circ)$ is wlf. 

Remark 10. Let $X = \text{Sp}(A)$ be an affinoid rigid space over $K$, and let $X^\text{an}$ be the Berkovich spectrum of $A$. The analytic points of $X$ are in canonical bijection with the points of the topological space $X^\text{an}$, and there is a morphism of topoi $(\sigma_*, \sigma^*) : X^\sim \to X^\text{an,~}$. The left adjoint $\sigma^*$ identifies $X^\text{an,~}$ with the full subcategory of $X^\sim$ consisting of overconvergent sheaves, and for any sheaf $\mathcal{M}$ on $X$ the counit $\sigma^* \sigma_* \mathcal{M} \to \mathcal{M}$ is identified with the canonical map $\mathcal{M}^\text{oc} \to \mathcal{M}$. The stalk of $\sigma_* \mathcal{M}$ in a point of $X^\text{an}$ is precisely the stalk of $\mathcal{M}$ in the corresponding analytic point. Finally, for an overconvergent abelian sheaf $\mathcal{M}$ on $X$ one has a natural isomorphism $H^*(X, \mathcal{M}) \simeq H^*(X^\text{an}, \sigma_* \mathcal{M})$ and similarly for higher direct images. Using this, van der Put’s base change theorem for overconvergent sheaves can be deduced from the ordinary proper base change theorem in topology. See [18, 19] for all this.

The following proposition is a simple consequence of Tate’s acyclicity theorem [5, Corollary 4.3.11].

Proposition 11. Let $X = \text{Sp}(A)$ be an affinoid space.

(i) For any finite affinoid covering $U$ of $X$ the Čech cohomology groups $H^i(U, O^\circ)$ are weakly trivial (as $K^\circ$-modules) for all $i > 0$.

(ii) The canonical map
\[
H^i(V, O_X(r)^\text{oc}|_V) \to H^i(V, O_V(r))
\]
is surjective for every affinoid subdomain $V \subset X$, every $r > 0$ and integer $i > 0$.

Proof. (i): Note that for each affinoid open subdomain $U$ of $X$ the Čech complex $(C(U, O), d)$ consists of complete normed $K$-vector spaces and the differential is continuous. To be concrete, we work with the supremum norm. The continuous morphism $d^{i-1} : C^{i-1}(U, O) \to Z^i(U, O)$
is surjective by [5, Corollary 4.3.11], so it is open according to [6, Theorem I.3.3.1]. In other words there exists \( r \in (0, 1) \) such that \( Z^i(\mathcal{U}, \mathcal{O}(r)) \) is contained in \( d^{i-1}(\mathcal{C}^{i-1}(\mathcal{U}, \mathcal{O}^v)) \). This means that \( H^j(\mathcal{U}, \mathcal{O}) \) is \( K(r) \)-torsion.

(ii): In order to show part (ii) of the proposition it suffices to show that for each finite covering \( \mathcal{U} = (U_l)_{l \in L} \) of \( V \) by rational subdomains of \( X \) the map

\[
H^i(\mathcal{U}, \mathcal{O}_X(r)^{oc}) \to H^i(\mathcal{U}, \mathcal{O}(r))
\]

is surjective. This is a consequence of

**Claim 12.**

(i) For \( i > 0 \) the image of \( d^{i-1} : C^{i-1}(\mathcal{U}, \mathcal{O}(r)) \to Z^i(\mathcal{U}, \mathcal{O}(r)) \) is open.

(ii) The image of \( Z^i(\mathcal{U}, \mathcal{O}_X(r)^{oc}) \to Z^i(\mathcal{U}, \mathcal{O}(r)) \) is dense.

Part (i) of the claim is a consequence of Proposition 11(i). For part (ii) of the claim note that for each rational subdomain

\[ U = \{ |g_1| \leq |g_0|, \ldots, |g_r| \leq |g_0| \} \]

of \( X \) the image of \( \mathcal{O}_X^{oc}(U) \to \mathcal{O}(U) \) is dense. To see this observe that for \( \epsilon > 1 \) and \( \epsilon \in |K^*|^g \) the set \( U \) is a Weierstrass domain inside \( \{ |g_1| \leq \epsilon|g_0|, \ldots, |g_r| \leq \epsilon|g_0| \} \).

For \( \xi \in Z^i(\mathcal{U}, \mathcal{O}(r)) \) we find \( \xi' \in C^{i-1}(\mathcal{U}, \mathcal{O}) \) with \( d(\xi') = \xi \), using again [5, Corollary 4.3.11]. Find a sequence \( \xi_j' \in C^{i-1}(\mathcal{U}, \mathcal{O}_X^{oc}) \) such that its image in \( C^{i-1}(\mathcal{U}, \mathcal{O}) \) converges to \( \xi' \). Then \( d(\xi_j') \in Z^i(\mathcal{U}, \mathcal{O}^{oc}) \) is a sequence approximating \( \xi \). By [11, Lemma 2.3.1] for large \( j \) we have \( d(\xi_j') \in Z^i(\mathcal{U}, \mathcal{O}_X(r)^{oc}) \).

**Theorem 13 (Bartenwerfer/van der Put).** We have

\[
H^i(\mathbb{B}^d, \mathcal{O}(r)) = 0
\]

for all \( r > 0 \) and integers \( i > 0 \).

This is proven by Bartenwerfer [2, Theorem] and using different methods by van der Put [17, Theorem 3.15]. For the convenience of the reader, we sketch van der Put’s proof.

**Idea of proof (van der Put).** Using Tate’s acyclicity theorem the theorem is equivalent to the following two statements:

- for all \( r > 0 \) and integers \( i > 0 \) the cohomology group
  \[
  H^i(\mathbb{B}^d, \mathcal{O}/\mathcal{O}(r)) = 0,
  \]
- \( H^0(\mathbb{B}^d, \mathcal{O}) \to H^0(\mathbb{B}^d, \mathcal{O}/\mathcal{O}(r)) \) is surjective.

The sheaf \( \mathcal{O}/\mathcal{O}(r) \) is overconvergent by [17, Lemma 1.5.2]. So we can apply base change [11, Theorem 2.7.4] for the linear fibrations \( \phi : \mathbb{B}^d \to \mathbb{B}^{d-1} \). Using the fact that for any fiber \( \phi^{-1}(a) \cong \mathbb{B}_{r_a}^1 \) over an analytic point \( a \) of \( \mathbb{B}^{d-1} \) we have

\[
(\mathcal{O}_{\mathbb{B}^d}/\mathcal{O}_{\mathbb{B}^d}(r))|_{\phi^{-1}(a)} \cong \mathcal{O}_{\mathbb{B}_{r_a}^1}/\mathcal{O}_{\mathbb{B}_{r_a}^1}(r),
\]
compare Lemma 25, we reduce the theorem to the case \( d = 1 \). In fact, by what is said and using the one-dimensional case of the theorem we get that
\[
\phi_*(\mathcal{O}_{\mathbb{B}^d}/\mathcal{O}_{\mathbb{B}^d}(r)) = \bigoplus_{N} \mathcal{O}_{\mathbb{B}^{d-1}}/\mathcal{O}_{\mathbb{B}^{d-1}}(r),
\]
\[
R^j\phi_*(\mathcal{O}_{\mathbb{B}^d}/\mathcal{O}_{\mathbb{B}^d}(r)) = 0 \quad (j > 0)
\]
and we conclude by the Leray spectral sequence and by induction on \( d \).

In the one-dimensional case the theorem follows from an explicit computation based on the Mittag–Leffler decomposition.

**Corollary 14.** The cohomology group
\[
H^i(\mathbb{B}^d, \mathcal{O}^o)
\]
is \( K(1) \)-torsion for all integers \( i > 0 \).

Indeed, for any \( \alpha \in K(1) \) the multiplication by \( \alpha \) on \( H^i(\mathbb{B}^d, \mathcal{O}^o) \) factors through \( H^i(\mathbb{B}^d, \mathcal{O}(1)) \) which vanishes by Theorem 13.

**Remark 15.** In fact, in [4, Theorem] Bartenwerfer shows that \( H^i(\mathbb{B}^d, \mathcal{O}^o) = 0 \) for every \( i > 0 \).

**Lemma 16.** Let \( X = \text{Sp}(A) \) be an affinoid space such that the cohomology group \( H^i(X, \mathcal{O}^o) \) is weakly trivial for some \( i > 0 \). Then for any wlf \( \mathcal{O}^o \)-module \( \mathcal{M} \) the cohomology group \( H^i(X, \mathcal{M}) \) is weakly trivial.

**Proof.** Below we are going to construct for every point \( x \in X \) a function \( f_x \in A^o \) with \( f_x(x) \neq 0 \) and with \( f_x H^i(X, \mathcal{M}) = 0 \). As the \( f_x \) generate the unit ideal in \( A \), there exist finitely many points \( x_1, \ldots, x_r \in X \) and \( c_1, \ldots, c_r \in A^o \) with
\[
c_1 f_{x_1} + \cdots + c_r f_{x_r} =: c \in K^o \setminus \{0\}.
\]
Then \( c \, H^i(X, \mathcal{M}) = 0 \).

In order to construct such \( f_x \) for given \( x \in X \) we use Proposition 8 in order to find an injective \( O_X^o \)-linear morphism \( \Psi : (O^o)^n \to \mathcal{M} \) and \( f' \in O^o(X) \) with \( f'(x) \neq 0 \) and such that \( f' \, \text{coker}(\Psi) = 0 \). From the long exact cohomology sequence corresponding to the short exact sequence
\[
0 \to (O^o)^n \xrightarrow{\Psi} \mathcal{M} \to \text{coker}(\Psi) \to 0
\]
it follows that we can take any nonzero \( f_x \in K(r) f' \), where \( r \in (0, 1) \) is chosen such that \( K(r) \, H^i(X, \mathcal{O}^o) = 0 \).

**Theorem 17.** For \( X/K \) a smooth affinoid space and for \( \mathcal{M} \) a wlf \( O_X^o \)-module the cohomology groups \( H^i(X, \mathcal{M}) \) are weakly trivial (as \( K^o \)-modules) for all \( i > 0 \).

**Proof.** By Lemma 16 we can assume without loss of generality that \( \mathcal{M} = \mathcal{O}^o \). We use induction on \( i > 0 \). The base case \( i = 1 \) is handled in the same way as the induction step,
so let us assume \( i > 1 \) and that we already know weak triviality of \( H^j(U, \mathcal{O}^\phi) \) for all \( 0 < j < i \) and smooth affinoid spaces \( U/K \).

Since \( X/K \) is smooth, \([13, \text{Satz} 1.12]\) implies that there exists a finite affinoid covering \( U = (U_l)_{l \in L} \) and finite étale morphisms \( \phi_l : U_l \to \mathbb{B}^d \). From the Čech spectral sequence

\[
E_2^{pq} = H^p(U, H^q(\mathcal{O}^\phi)) \Rightarrow H^{p+q}(X, \mathcal{O}^\phi)
\]

we see that \( H^i(X, \mathcal{O}^\phi) \) has a filtration whose associated graded piece \( \text{gr}^0 \) is a subquotient of \( H^p(U, H^{i-p}(\mathcal{O}^\phi)) \). By Proposition 11(i), \( \text{gr}^i \) is weakly trivial. By our induction assumption, \( H^{i-p}(\mathcal{O}^\phi)(U) \) is weakly trivial for \( 0 < p < i \) and for \( U \) an intersection of opens in \( U \), hence \( \text{gr}^{i-p} \) is weakly trivial for these \( p \). It thus suffices to show that \( \text{gr}^0 \) is weakly trivial or that \( H^i(U_l, \mathcal{O}^\phi_{U_l}) \) is weakly trivial for all \( l \in L \).

So in order to show Theorem 17 we can assume without loss of generality that \( \mathcal{M} = \mathcal{O}^\phi_X \) and that there exists a finite étale morphism \( \phi : X \to \mathbb{B}^d \).

For all \( j > 0 \) we get morphisms

\[
R^j \phi_* (\mathcal{O}^\phi_X) \simeq R^j \phi_* (\mathcal{O}_X(1)) \leftarrow R^j \phi_* (\mathcal{O}_X(1)_{\text{oc}}).
\]

with a weak isomorphism on the left and a surjective morphism on the right. The surjectivity follows from Proposition 11(ii). By base change \([11, \text{Theorem} 2.7.4]\) the stalk \( R^j \phi_* (\mathcal{O}_X(1)_{\text{oc}})_a \simeq H^j(X_a, \mathcal{O}_X(1)_{\text{oc}}|_{X_a}) \) vanishes for every analytic point \( a \) of \( \mathbb{B}^d \). Since \( R^j \phi_* (\mathcal{O}_X(1)_{\text{oc}}) \) is overconvergent \([11, \text{Lemma} 2.3.2]\), it follows that \( R^j \phi_* (\mathcal{O}_X(1)_{\text{oc}}) = 0 \) and hence that \( R^j \phi_* (\mathcal{O}^\phi_X) \) is weakly trivial.

Combining this observation with the Leray spectral sequence we see that it suffices to show that \( H^i(\mathbb{B}^d, \phi_* (\mathcal{O}^\phi_X)) \) is weakly trivial for \( i > 0 \). From Proposition 9 we deduce that \( \phi_* (\mathcal{O}^\phi_X) \) is wlf as an \( \mathcal{O}^\phi_{\mathbb{B}^d} \)-module, so we conclude by using Theorem 13 and Lemma 16. \( \square \)

The following corollary, which we will apply in the next sections, was first shown in \([1]\) and \([2, \text{Folgerung} 3]\).

**Corollary 18** (Bartenwerfer). For \( X/K \) smooth affinoid there exists \( s \in (0, 1) \) such that the map

\[
H^i(X, \mathcal{O}(sr)) \to H^i(X, \mathcal{O}(r))
\]

vanishes for all \( r > 0 \) and integers \( i > 0 \).

**Proof.** Choose \( \pi \in K(1) \setminus \{0\} \) and write \( s' = |\pi| \). By Theorem 17 we can assume without loss of generality that \( \pi H^i(X, \mathcal{O}(1)) = 0 \) for \( i > 0 \). Now we claim \( s = s'^2 \) satisfies the requested property of the corollary. Indeed, for \( r > 0 \) set \( r' = \max\{ |\pi|^n \mid n \in \mathbb{Z}, |\pi|^n \leq r \} \). Then we get a commutative square

\[
\begin{array}{ccc}
H^i(X, \mathcal{O}(sr')) & \rightarrow & H^i(X, \mathcal{O}(r')) \\
\downarrow^j & & \downarrow^j \\
H^i(X, \mathcal{O}(1)) & \rightarrow & H^i(X, \mathcal{O}(1))
\end{array}
\]

where the lower horizontal map is multiplication by \( \pi \) and the vertical maps are induced by the isomorphisms \( \mathcal{O}(sr') \cong \mathcal{O}(1) \) and \( \mathcal{O}(r') \cong \mathcal{O}(1) \) given by multiplying with the
appropriate powers of \( \pi \). The morphism (3) is the composition of

\[
H^i(X, \mathcal{O}(sr)) \to H^i(X, \mathcal{O}(s' r')) \xrightarrow{=} 0 \to H^i(X, \mathcal{O}(r')) \to H^i(X, \mathcal{O}(r)). \]

\[\square\]

2. Vanishing of multiplicative cohomology

Given \( r' < r \) we write \( \mathcal{O}(r, r') := \mathcal{O}(r)/\mathcal{O}(r') \) and, if \( r' < r \leq 1 \), \( \mathcal{O}^*(r, r') := \mathcal{O}^*(r)/\mathcal{O}^*(r') \).

Lemma 19. For \( r' < r \leq 1 \) we have isomorphisms of sheaves of sets \( \mathcal{O}(r) \xrightarrow{\sim} \mathcal{O}^*(r) \) and \( \mathcal{O}(r, r') \xrightarrow{\sim} \mathcal{O}^*(r, r') \) given by \( f \mapsto 1 + f \). If \( r' \geq r^2 \), the latter isomorphism is an isomorphism of abelian sheaves.

Proof. Most of the claims are easy. To see that \( f \mapsto 1 + f \) induces a map on the quotient sheaves \( \mathcal{O}(r, r') \to \mathcal{O}^*(r, r') \) note that if \( f, g \) are functions of supremum seminorm \( < 1 \), then \( |f - g|_{\text{sup}} < r' \) if and only if \( |(1 + f)(1 + g)^{-1} - 1|_{\text{sup}} < r' \). Indeed, this follows from the computation \( |f - g|_{\text{sup}} = |(1 + f) - (1 + g)|_{\text{sup}} = |(1 + f)(1 + g)^{-1} - 1(1 + g)|_{\text{sup}} = |(1 + f)(1 + g)^{-1} - 1|_{\text{sup}} \), where we used that \( |1 + g|_{\text{sup}} = |(1 + g)^{-1}|_{\text{sup}} = 1 \). \[\square\]

Given an affinoid space \( X \), we consider the following condition on the real number \( 0 < s \leq 1 \):

The map \( H^i(X, \mathcal{O}(sr)) \to H^i(X, \mathcal{O}(r)) \) vanishes for all \( r > 0 \) and integers \( i > 0 \).

(4)

Proposition 20. Let \( X/K \) be smooth affinoid. Assume that \( s \) satisfies (4). Then the map

\[
H^1(X, \mathcal{O}^*(sr)) \to H^1(X, \mathcal{O}^*(r))
\]

vanishes for every \( r \in (0, s) \).

Proof. We first prove:

Lemma 21. Assume that \( s \) satisfies (4) for the affinoid space \( X \). For any integer \( i > 0 \), \( r \in (0, s) \), and \( \xi \in H^i(X, \mathcal{O}^*(sr)) \) there exists a decreasing zero sequence \( (r_n) \) in \( (0, s) \) with \( r_0 = r \) and a compatible system

\[
(\xi'_n) \in \lim_n H^i(X, \mathcal{O}^*(r_n))
\]

such that \( \xi'_0 \in H^i(X, \mathcal{O}^*(r)) \) is equal to the image of \( \xi \) under \( H^1(X, \mathcal{O}^*(sr)) \to H^1(X, \mathcal{O}^*(r)) \).

Proof. Put \( r_0 = r \) and inductively \( r_{n+1} = r_n^2/s \). Explicitly, \( r_n = (r/s)^{2^n} \). Since \( r < s \), the \( r_n \) form a decreasing zero sequence.

Put \( \xi_0 = \xi \). We will inductively construct elements \( \xi_n \in H^i(X, \mathcal{O}^*(sr_n)) \) such that the images of \( \xi_n \) and \( \xi_{n+1} \) in \( H^1(X, \mathcal{O}^*(r_n)) \) coincide. Denote this common image by \( \xi'_n \). Then \( (\xi'_n)_{n \geq 0} \) is the desired compatible system.
Assume that we have already constructed $\xi_n$. From the commutative diagram with exact rows

$$
\begin{align*}
H^i(X, O(sr_n)) &\longrightarrow H^i(X, O(s^2r_{n+1})) &\longrightarrow H^{i+1}(X, O(s^2r_{n+1})) \\
| &\quad | &\quad \\
H^i(X, O(sr_n)) &\longrightarrow H^i(X, O(sr_{n+1})) &\longrightarrow H^{i+1}(X, O(sr_{n+1})) \\
| &\quad | &\quad \\
H^i(X, O(r_n)) &\longrightarrow H^i(X, O(r_{n+1})) &\longrightarrow H^{i+1}(X, O(r_{n+1}))
\end{align*}
$$

we see that $H^i(X, O(sr_n, s^2r_{n+1})) \longrightarrow H^i(X, O(r_{n+1}))$ vanishes for $i > 0$. Since $sr_{n+1} \geq r^2_n$ and $s^2r_{n+1} = s^2r^2_n = (sr_n)^2$, we may apply Lemma 19 to deduce that also $H^i(X, O^*(sr_n, s^2r_{n+1})) \longrightarrow H^i(X, O^*(r_{n+1}))$ vanishes. From the commutative diagram with exact rows

$$
\begin{align*}
H^i(X, O^*(sr_n)) &\longrightarrow H^i(X, O^*(sr_{n+1})) \\
| &\quad \\
H^i(X, O^*(sr_{n+1})) &\longrightarrow H^i(X, O^*(r_{n+1})) &\longrightarrow H^{i+1}(X, O^*(r_{n+1}))
\end{align*}
$$

we deduce the existence of the desired element $\xi_{n+1} \in H^i(X, O^*(sr_{n+1}))$ such that the images of $\xi_n$ and $\xi_{n+1}$ in $H^i(X, O^*(r_{n+1}))$ coincide. 

**Lemma 22.** Let $X/K$ be smooth affinoid, and let $(\xi_n) \in \lim_n H^1(X, O^*(r_n))$ be a compatible system where the $r_n$ form a decreasing zero sequence in $(0, 1)$. Then there exists a finite affinoid covering $\mathcal{U}$ of $X$ such that $(\xi_n)$ lies in the image of $\lim_n H^1(\mathcal{U}, O^*(r_n))$.

**Proof.** Let $\mathcal{U}$ be a finite affinoid covering of $X$ such that $\xi_0$ lies in the image of $H^1(\mathcal{U}, O^*(r_0))$. We claim that then $\xi_n$ lies in the image of $H^1(\mathcal{U}, O^*(r_n))$ for all $n$. Recall that for any abelian sheaf $\mathcal{F}$ the map $H^1(\mathcal{U}, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F})$ is injective, and an element $\xi \in H^1(X, \mathcal{F})$ belongs to the image of this map if and only if $\xi|_U = 0$ in $H^1(U, \mathcal{F}|_U)$ for every $U \in \mathcal{U}$.

Fix $U \in \mathcal{U}$. We want to show that $\xi_n|_U = 0$ in $H^1(U, O^*(r_n))$. By Corollary 18 there exists $m \geq n$ such that $H^1(U, O(r_m)) \rightarrow H^1(U, O(r_n))$ vanishes. Under the sequence of maps

$$
H^1(U, O^*(r_m)) \rightarrow H^1(U, O^*(r_n)) \rightarrow H^1(U, O^*(r_0))
$$

we have $\xi_m|_U \mapsto \xi_n|_U \mapsto 0$. Hence the element $\xi_n|_U$ lifts to an element $\eta_m$ in $H^0(U, O^*(r_0, r_m))$. We claim that the image of $\eta_m$ in $H^0(U, O^*(r_0, r_n))$ has a preimage in $H^0(U, O^*(r_0))$. In view of the commutative diagram with exact rows

$$
\begin{align*}
H^0(U, O^*(r_0)) &\longrightarrow H^0(U, O^*(r_0, r_n)) &\longrightarrow H^1(U, O^*(r_n)) \\
| &\quad | &\quad \\
H^0(U, O^*(r_0)) &\longrightarrow H^0(U, O^*(r_0, r_m)) &\longrightarrow H^1(U, O^*(r_m))
\end{align*}
$$

this will imply that $\xi_n|_U = 0$. 

\[\square\]
To prove the claim, note that Lemma 19 gives bijections $H^0(U, \mathcal{O}^*(r_0)) \cong H^0(U, \mathcal{O}(r_0))$ and $H^0(U, \mathcal{O}^*(r_n)) \cong H^0(U, \mathcal{O}(r_n))$ and similarly for $r_n$ replaced by $r_m$. On the other hand, by the choice of $m$, the map $H^1(U, \mathcal{O}(r_m)) \to H^1(U, \mathcal{O}(r_n))$ vanishes. This implies the existence of the desired lift in view of the commutative diagram with exact rows

$$
\begin{array}{c}
H^0(U, \mathcal{O}(r_0)) \\
\downarrow \\
H^0(U, \mathcal{O}(r_0, r_n)) \\
\downarrow \\
H^0(U, \mathcal{O}(r_0, r_n)) \\
\downarrow \\
H^1(U, \mathcal{O}(r_n)) \\
\end{array}
\begin{array}{c}
H^1(U, \mathcal{O}(r_n)) \\
\downarrow \\
H^1(U, \mathcal{O}(r_m)) \\
\end{array}
\begin{array}{c}
\text{= 0} \\
\text{= 0} \\
\end{array}
$$

We can now finish the proof of Proposition 20. Using the two preceding lemmas, it suffices to show that $\lim_n H^1(U, \mathcal{O}^*(r_n))$ vanishes for every decreasing zero sequence $(r_n)$. Consider an element $(\xi_n)_n$ in this inverse limit, and choose representing Čech 1-cocycles $\zeta_n \in Z^1(U, \mathcal{O}^*(r_n))$. Then there exist 0-cochains $\eta_n \in C^0(U, \mathcal{O}^*(r_n))$ such that $\zeta_n = \xi_{n+1} \cdot \partial \eta_n$. Since $(r_n)$ is a zero sequence, the product $\prod_{k=0}^{\infty} \eta_{n+k}$ converges in $C^0(U, \mathcal{O}^*(r_n))$, and we get $\zeta_n = \partial(\prod_{k=0}^{\infty} \eta_{n+k})$, i.e. $\xi_n = 0$.

**Corollary 23.** For every $r \in (0, 1)$ we have $H^1(\mathbb{B}^d, \mathcal{O}^*(r)) = 0$.

**Proof.** By Theorem 13, $s = 1$ satisfies condition (4) for $X = \mathbb{B}^d$. Hence by Proposition 20, the identity map on $H^1(\mathbb{B}^d, \mathcal{O}^*(r))$ vanishes.

**Corollary 24.** Let $X/K$ be a smooth affinoid space. Then there exists $0 < r \leq 1$ such that $H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{O}^*/\mathcal{O}^*(r'))$ is injective for every $r' \in (0, r)$.

**Proof.** By Corollary 18 there exists $0 < s \leq 1$ satisfying (4). By Proposition 20 we can take $r = s^2$.

### 3. Homotopy invariance of Pic

In this section we prove Theorem 4. Given $0 < r \leq 1$, we set $\mathcal{O}^*(\infty, r) = \mathcal{O}^*/\mathcal{O}^*(r)$. Let $X = \text{Sp}(A)$ be an affinoid space, and let $p : X \times \mathbb{B}^1 \to X$ be the projection, $\sigma : X \to X \times \mathbb{B}^1$ the zero section.

**Lemma 25.** For any fiber $p^{-1}(a) \cong \mathbb{B}^1_{F_a}$ over an analytic point $a$ of $X$ we have $\mathcal{O}^*_{X \times \mathbb{B}^1}(\infty, r)|_{p^{-1}(a)} \cong \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r)$.

**Proof.** This follows easily from [11, Lemmas 2.7.1, 2.7.2].

**Lemma 26.** We have $R^1 p_* \mathcal{O}^*_{X \times \mathbb{B}^1}(\infty, r) = 0$.

**Proof.** The sheaf $\mathcal{O}^*_{X \times \mathbb{B}^1}(\infty, r)$ and hence its higher direct images are overconvergent (see [17, 1.5.3], [11, Lemma 2.3.2]). Hence it suffices to prove that for any analytic point
of $X$ the stalk $R^1 p_*O^\times_{X \times \Bbbk^1}(\infty, r)_a$ vanishes. By base change \cite[Theorem 2.7.4]{11} and Lemma 25, we have

$$R^1 p_*O^\times_{X \times \Bbbk^1}(\infty, r)_a \cong H^1(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r)).$$

In the exact sequence

$$H^1(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}) \rightarrow H^1(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r)) \rightarrow H^2(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(r))$$

the group on the left vanishes because the Tate algebra is a unique factorization domain, the group on the right vanishes by dimension reasons.

Fix $\pi \in K \setminus \{0\}$ with $|\pi| < 1$. Let $t$ denote the coordinate on $\Bbbk^1$. Then $t \mapsto \pi t$ induces a map $p_*O^\times_{X \times \Bbbk^1}(\infty, r) \rightarrow p_*O^\times_{X \times \Bbbk^1}(\infty, r)$.

**Lemma 27.** We have an isomorphism of pro-abelian sheaves

$$\varprojlim_{t \rightarrow \pi t} p_*O^\times_{X \times \Bbbk^1}(\infty, r) \cong O^\times_X(\infty, r)$$

**Proof.** Obviously, $O^\times_X(\infty, r) \xrightarrow{p^*} p_*O^\times_{X \times \Bbbk^1}(\infty, r) \xrightarrow{\sigma^*} O^\times_X(\infty, r)$ is the identity. Choose $n$ big enough such that $|\pi^n| \leq r$. We claim that the map

$$p_*O^\times_{X \times \Bbbk^1}(\infty, r) \rightarrow p_*O^\times_{X \times \Bbbk^1}(\infty, r)$$

induced by $t \mapsto \pi^n t$ factors through $O^\times_X(\infty, r) \xrightarrow{p^*} p_*O^\times_{X \times \Bbbk^1}(\infty, r)$. By overconvergence again it is enough to check this on the stalk at any analytic point $a$ of $X$ (consider the image of the composition of the first map with the projection to coker$(p^*)$). By base change and Lemma 25 we have $p_*O^\times_{X \times \Bbbk^1}(\infty, r)_a \cong H^0(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r))$. By Corollary 23 the natural map $H^0(B^1_{Fa}, O^*) \rightarrow H^0(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r))$ is surjective. Any element of $H^0(B^1_{Fa}, O^*)$ is of the form $u \cdot f(t)$ with $u \in F^*_a$, $f(0) = 1$, and $|f(t) - 1|_{\text{sup}} < 1$ (see \cite[Corollary 2.2.4]{5}). But then $|f(\pi^n t) - 1|_{\text{sup}} < |\pi^n| \leq r$. This implies that the map

$$H^0(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r)) \rightarrow H^0(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r))$$

induced by $t \mapsto \pi^n t$ factors through $F^*_a/F^*_a(r) \hookrightarrow H^0(B^1_{Fa}, O^\times_{\Bbbk^1_{Fa}}(\infty, r))$, concluding the proof.

**Proof of Theorem 4.** Note that Pic$(A) \cong H^1(X, O^*)$. Since $X = \text{Sp}(A)$ is assumed to be smooth, Corollary 24 implies that there exists $r \in (0, 1)$ such that the map $H^1(X \times \Bbbk^1, O^*) \rightarrow H^1(X \times \Bbbk^1, O^*(\infty, r))$ is injective. It thus suffices to show that

$$\sigma^* : \varprojlim_{t \rightarrow \pi t} H^1(X \times \Bbbk^1, O^\times_{X \times \Bbbk^1}(\infty, r)) \rightarrow H^1(X, O^\times_X(\infty, r))$$

is a pro-isomorphism.

Using the Leray spectral sequence, Lemma 26 yields an isomorphism

$$H^1(X \times \Bbbk^1, O^\times_{X \times \Bbbk^1}(\infty, r)) \cong H^1(X, p_*O^\times_{X \times \Bbbk^1}(\infty, r)).$$
We combine this with the pro-isomorphism
\[ \lim_{n \to \infty} \pi^1 H^1(X, p_* \mathcal{O}_X^* (\infty, r)) \cong H^1(X, \mathcal{O}_X^*(\infty, r)) \]

implied by Lemma 27 to finish the proof. \qed

**Proof of Corollary 5.** Write $X$ for $\text{Sp}(A)$, $U_n$ for the closed disk of radius $|\pi^{-n}|$, and $\mathbb{A}^{1,\text{an}}$ for the analytic affine line over $K$. Then $X \times U_n$, $n = 0, 1, \ldots$, is an admissible covering of $X \times \mathbb{A}^{1,\text{an}}$. Note that the pro-systems $\lim_{n \to \infty} \text{Pic}(X \times U_n)$ and $\lim_{n \to \infty} \text{Pic}(A(t))$ are naturally isomorphic. Taking the limit of the isomorphism of pro-abelian groups in Theorem 4 then gives the isomorphism

\[ \text{Pic}(X) \cong \lim_{n} \text{Pic}(X \times U_n). \]

Hence it suffices to show that the natural map $\text{Pic}(X \times \mathbb{A}^{1,\text{an}}) \to \lim_{n} \text{Pic}(X \times U_n)$ is an isomorphism. The cohomological description of Picard groups yields a short exact sequence

\[ 0 \to \lim_{n} \mathcal{O}^*(X \times U_n) \to \text{Pic}(X \times \mathbb{A}^{1,\text{an}}) \to \lim_{n} \text{Pic}(X \times U_n) \to 0. \]

We have a natural decomposition $\mathcal{O}^*(X \times U_n) \cong \mathcal{O}^*(X) \oplus \mathcal{O}_0^*(X \times U_n)$ where $\mathcal{O}_0^*(X \times U_n)$ consists of those units that restrict to 1 on $X \subset X \times U_n$. Clearly, $\lim_{n} \mathcal{O}^*(X) = 0$ and it remains to prove that $\lim_{n} \mathcal{O}_0^*(X \times U_n)$ vanishes. Note that given $f \in \mathcal{O}_0^*(X \times U_{n+m})$, its restriction to $X \times U_n$ satisfies $|f|_{X \times U_n} - 1|_{\sup} < |\pi^m|$. Hence, given any sequence $(g_n)_{n=0}^{\infty}$ with $g_n \in \mathcal{O}_0^*(X \times U_n)$, the product

\[ f_n := \prod_{k=n}^{\infty} g_k|_{X \times U_n} \in \mathcal{O}_0^*(X \times U_n) \]

converges. By construction we have $g_n = f_n \cdot (f_{n+1}|_{X \times U_n})^{-1}$ for every $n \geq 0$. This shows the desired vanishing of the $\lim^1$-term. \qed

**4. $K_0$-invariance**

In this section we assume that $K$ is a complete discretely valued field. Then for an affinoid algebra $A/K$ the ring of power bounded elements $A^\circ$ is noetherian, excellent, and of finite Krull dimension, for excellence see [10, § I.9]. Let $\pi \in K^\circ$ be a prime element.

Let $\mathcal{X} \to \text{Spec}(A^\circ)$ be a proper morphism of schemes which is an isomorphism over $\text{Spec}(A)$. For an integer $n > 0$ set $\mathcal{X}_n = \mathcal{X} \otimes_{K^\circ} K^\circ/(\pi^n)$.

**Proposition 28.** There exists $n > 0$ such that

\[ K_0(\mathcal{X}) \to K_0(\mathcal{X}_n) \]

is injective.

**Proof.** Let $K(\mathcal{X}, \mathcal{X}_n)$ be the homotopy fiber of the map $K(\mathcal{X}) \to K(\mathcal{X}_n)$ between non-connective $K$-theory spectra [21, § IV.10] and let $K_1(\mathcal{X}, \mathcal{X}_n)$ be its homotopy groups. By ‘pro-cdh-descent’ [12, Theorem A] the natural map

\[ \lim_{n} K_0(A^\circ, A^\circ/(\pi^n)) \to \lim_{n} K_0(\mathcal{X}, \mathcal{X}_n) \]

is a pro-isomorphism. For each $n$ we have an exact sequence
\[
K_1(A^\circ) \to K_1(A^\circ/(\pi^n)) \to K_0(A^\circ, A^\circ/(\pi^n)) \to K_0(A^\circ) \xrightarrow{\sim} K_0(A^\circ/(\pi^n))
\]
where the left map is surjective [21, Remark III.1.2.3] and the right map is an isomorphism [21, Lemma II.2.2], since $A^\circ$ is $\pi$-adically complete. So $K_0(\mathcal{X}, \mathcal{X}_n)$ vanishes as a pro-system in $n$. By the exact sequence
\[
K_0(\mathcal{X}, \mathcal{X}_n) \to K_0(\mathcal{X}) \to K_0(\mathcal{X}_n)
\]
this finishes the proof of the proposition. \hfill \qed

**Lemma 29.** If $\mathcal{X}$ is a regular scheme we obtain a natural exact sequence
\[
G_0(\mathcal{X}_1) \to K_0(\mathcal{X}) \to K_0(A) \to 0,
\]
where $G_0$ is the Grothendieck group of coherent sheaves.

**Proof of Theorem 6.** In case the residue field of $K$ has characteristic zero, $A^\circ$ contains $\mathbb{Q}$ and is excellent. Hence there exists a blow-up $\mathcal{X} \to A^\circ$, whose center is (set theoretically) contained in the closed fiber $\text{Spec}(A^\circ/\pi)$, such that $\mathcal{X}$ is a regular scheme [20, Theorem 1.1]. So we can now assume in the general case that $\mathcal{X} \to \text{Spec}(A^\circ)$ is a regular model of $A$ in the sense of the introduction. Let $A^\circ(t) \subset A^\circ[[t]]$ be those formal power series for which the coefficients converge to zero. Note that $\mathcal{X}^\prime = \mathcal{X} \otimes_{A^\circ} A^\circ(t)$ is a regular scheme with generic fiber $\text{Spec}(A(t))$. Set $\mathcal{X}_n^\prime = \mathcal{X}^\prime \otimes_{K^\circ} K^\circ/(\pi^n)$.

Applying Lemma 29 to $\mathcal{X}$ and $\mathcal{X}^\prime$ we get a commutative diagram with exact rows
\[
\begin{array}{cccccc}
G_0(\mathcal{X}_1) & \to & K_0(\mathcal{X}) & \to & K_0(A) & \to 0 \\
\sigma^* \downarrow & & \sigma^* \downarrow & & \sigma^* \downarrow \\
G_0(\mathcal{X}_1^\prime) & \to & K_0(\mathcal{X}^\prime) & \to & K_0(A(t)) & \to 0
\end{array}
\]
where $\sigma$ is the zero section induced by $t \mapsto 0$. The left vertical arrow is an isomorphism by homotopy invariance of $G$-theory [21, Theorem II.6.5] as $\mathcal{X}_1' = \mathbb{A}^1_{\mathcal{X}_1}$. In order to prove Theorem 6 we have to show that
\[
\sigma^* \cdot \lim_{t \to \pi t} \cdot K_0(A(t)) \to K_0(A)
\]
is a pro-monomorphism. According to Proposition 28 we find $n > 0$ such that $K_0(\mathcal{X}^\prime) \to K_0(\mathcal{X}_n^\prime)$ is injective. So by a diagram chase it suffices to show that
\[
\sigma \cdot \lim_{t \to \pi t} \cdot K_0(\mathcal{X}_n^\prime) \to K_0(\mathcal{X}_n)
\]
is a pro-monomorphism, which is clear as the morphism $\mathcal{X}_n^\prime \xrightarrow{t \mapsto \pi^n t} \mathcal{X}_n^\prime$ factors through $\mathcal{X}_n$. \hfill \qed
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