ON THE PROOF OF UNIVERSALITY FOR ORTHOGONAL AND SYMPLECTIC ENSEMBLES IN RANDOM MATRIX THEORY

OVIDIU COSTIN, PERCY DEIFT AND DIMITRI GIOEV

Abstract. We give a streamlined proof of a quantitative version of a result from [DG1] which is crucial for the proof of universality in the bulk [DG1] and also at the edge [DG2] for orthogonal and symplectic ensembles of random matrices. As a byproduct, this result gives asymptotic information on a certain ratio of the $\beta = 1, 2, 4$ partition functions for log gases.

For $m \geq 2$, let

$$h(x) = \sum_{k=0}^{m-1} \beta_k x^{2k}$$

(1)

$$\beta_k = \frac{(2m)(2m-2)\cdots(2m-2k)}{(2m-1)(2m-3)\cdots(2m-2k-1)}, \quad 0 \leq k \leq m-1.$$  

For odd $q$ set

$$I(q) \equiv \frac{2}{\pi} \sin q \pi \frac{1}{2} \int_{-1}^{1} \frac{\cos(q \arcsin x)}{h(x)(1-x^2)} \, dx = \frac{4}{\pi} \int_0^{\pi/2} \frac{\sin q s \sin s h(\cos s)}{h(\cos s)} \, ds$$  

(2)

and

$$Q(q) \equiv I(q) + \frac{1}{2m}.$$  

(3)

For $n \equiv 2m - 1$, define the $(m-1) \times (m-1)$ matrix

$$T^{[m-1]} = I - (m!)^2 \frac{Q^{[m-1]} B^{[m-1]}}{m!(2m)!} \equiv I - K^{[m-1]}$$

(4)

where

$$Q^{[m-1]}_{ij} = Q(n-2i+2j), \quad B^{[m-1]}_{ij} = 2m \binom{n}{j-i}, \quad 1 \leq i, j \leq m-1.$$  

Here $\binom{n}{k} \equiv 0$ for $k < 0$.

In [DG1, Theorem 2.6], the authors prove the following result: for $m \geq 2$,

$$\det T^{[m-1]} \neq 0$$

(5)

(see Remark 2 after the proof of Theorem 1 below). Note that in the notation of [DG1], $T^{[m-1]} = T_{m-1}$.

In this paper we will give a streamlined proof of the following quantitative version of (5).

Theorem 1. For $m \geq 2$,

$$\det T^{[m-1]} \geq 0.0865.$$
Equation (6) plays a crucial role in proving universality in the bulk [DG1], and also at the edge [DG2], for orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) random matrix ensembles for a class of weights $w(x) = e^{-V(x)}$ where $V(x)$ is a polynomial $V(x) = \kappa_2 x^2 + \cdots$, $\kappa_2 > 0$. (Here $m$ is the same integer as in [4], [5].)

The situation is as follows. In [DG1, DG2], and also in [DGKV], the authors use the method of Widom [W], which is based in turn on [TW], together with the asymptotic analysis for orthogonal polynomials in [DKMVZ]. A new and challenging feature of the method in [W], which does not arise in the proof of universality in the case $\beta = 2$, is the appearance of the inverse of a certain matrix $C_{11}$ of fixed size $n = 2m - 1$ (see [DG1] (1.37) and Theorem 2.3 et seq.). In the scaling limit as $N \to \infty$, the matrix $C_{11}$ converges to a matrix $C_{11}^\infty$ and

$$
\det C_{11}^\infty = (\det T^{[m-1]})^2
$$

(see discussion from (2.13) up to Theorem 2.4 in [DG1]). Thus in order to control the scaling limit for $\beta = 1$ and 4, we need to show that $\det T^{[m-1]} \neq 0$.

It turns out that $\det T^{[m-1]}$ is related to partition functions for finite log gases in an external field $V$ at inverse temperatures $\beta = 1, 2, 4$

$$
Z_{V,\beta,k} = \frac{1}{k!} \int \cdots \int \prod_{1 \leq i < j \leq k} |x_i - x_j|^\beta e^{-\sum_{i=1}^k V(x_i)} \, dx_1 \cdots dx_k
$$

$$
= \frac{1}{k!} \int \cdots \int e^{-\beta \sum_{1 \leq i < j \leq k} \log |x_i - x_j| - \sum_{i=1}^k V(x_i)} \, dx_1 \cdots dx_k.
$$

Using standard formulae for such partition functions (see e.g. [AvM], (4.4), (4.17), (4.20)), together with [DG1] (2.18), one finds (see [St, Remark 2.4], [DG1, Remark 1.5]) that for ensembles of (even) size $N$

$$
\det C_{11} = \left( \frac{1}{2^N (N/2)!} \frac{Z_{2V,4,N/2} Z_{V,1,N}}{Z_{2V,2,N}} \right)^2.
$$

Thus

$$
\lim_{N \to \infty} \frac{Z_{2V,4,N/2} Z_{V,1,N}}{2^N (N/2)! Z_{2V,2,N}} = \det T^{[m-1]} \neq 0.
$$

Formula (4), together with (7), raises the possibility of using the methods of statistical mechanics to prove (6), (5). The estimates in [3] show that the partition functions $Z_{V,\beta,k}$ have, for certain constants $\alpha_{V,\beta}$, leading order asymptotics of the form $e^{\alpha_{V,\beta} k^2 (1+o(1))}$ as $k \to \infty$, and moreover, their combined contributions to $\det C_{11}$ cancel to this order. In order to achieve cancellation at subsequent orders, and so prove (4), (5), one needs higher order asymptotics for the $Z_{V,\beta,k}$’s, but, unfortunately such asymptotics are known only for $\beta = 2$ (see [CM]). Regarding (4), we take the contrary point of view, i.e., (10) and (11) provide new information on the asymptotics of partition functions for log gases at inverse temperatures $\beta = 1$ and 4.

Much of the analysis in [DG1] involves estimating $Q(q)$ in two regions: $3 \leq q \lesssim \sqrt{m}$ and $\sqrt{m} \lesssim q \leq 4m - 5$. In this note, using bounds on

$$
W(x) = \frac{2}{\pi} \int_0^x \frac{\sin q s}{\sin s} \, ds
$$

which are uniform in $q = 3, 5, \cdots$ and in $0 \leq x \leq \pi/2$ (see Lemma 4 below), we are able to estimate $Q(q)$ uniformly for $q = 3, 5, \cdots, 4m - 5$ and so avoid many of the
We use the following result. For a matrix $X$ let $r(X) = \sup\{|\lambda| : \lambda \in \text{spec} X\}$ denote the spectral radius of $X$. As is well known, for any operator norm $\| \cdot \|$ on \{X\},
\begin{equation}
  r(X) = \lim_{j \to \infty} \|X^j\|^{1/j} = \inf_{j \geq 1} \|X^j\|^{1/j}.
\end{equation}

**Lemma 2.** Assume $K$ and $K'$ are $J$-dimensional matrices with real entries such that $|K_{ij}| \leq K'_{ij}$, $1 \leq i, j \leq J$, and $r(K') < 1$. Then $r(K) < 1$ and
\begin{equation}
  \det(I - K) \geq \det(I - K') > 0.
\end{equation}

**Proof.** The following is true: if $r(X) < 1$, then
\begin{equation}
  \det(I - X) = e^{-\sum_{i=1}^{\infty} \frac{1}{i} \text{tr}(X^i)}.
\end{equation}
This result is usually stated in the form that (14) holds if $\|X\| < 1$ (see e.g. [ReSi]). To obtain (14) for $r(X) < 1$ from the case $\|X\| < 1$ simply apply (14) to $\mu X$ for $\mu$ small and observe that for any fixed $\mu$ satisfying $r(X) < \rho < 1$, $\|X^l\| \leq \rho^l$ for $l$ sufficiently large: then (14) follows for $r(X) < 1$ by analytic continuation $\mu \to 1$.

Equip $\mathbb{R}^J$ with the $l_\infty$-norm $\| \cdot \|_\infty$ (any $l_p$-norm, $1 \leq p \leq \infty$ would do) and for a matrix $X$ mapping $\mathbb{R}^J \to \mathbb{R}^J$ denote the associated operator norm by $\|X\|$. For $\phi = \{\phi_j\} \in \mathbb{R}^J$ we denote the vector with coordinates $\{|\phi_j|\}$ by $|\phi|$. We claim that $r(K) \leq r(K')$. Indeed, for $\phi \in \mathbb{R}^J$, $\|(K')^l|\phi\| \leq \|(K')^l\| \|\phi\|$ and so $r(K) \leq r(K')$.

Thus $\|K^l\| \leq \|(K')^l\|$ and so $r(K) \leq r(K') < 1$ by (12). It follows that (14) is valid for $K$ and $K'$. But clearly $|\text{tr}(K^l)| \leq \text{tr}((K')^l)$ and (13) is now immediate. $\square$

The function $h(x)$ in (11) has the following properties (see [DG1 Proposition 6.2]): for $0 < x < 1$

(i) $h$ solves the differential equation
\[ x(x^2 - 1)h' + (2m - 1 - 2(m - 1)x^2)h = 4m \]
\begin{equation}
  \frac{4m}{2m - 1} = h(0) \leq h(x) \leq h(1) = 4m
\end{equation}

(ii) $h(x) = \frac{4mx^{2m-1}}{\sqrt{1 - x^2}} \int_x^1 \frac{t^{2m-2}}{\sqrt{1 - t^2}} dt$.

Property (i) reflects the fact that $h$ is a hypergeometric function,
\[ h(x) = \frac{4m}{2m - 1} {}_2F_1(1, -m + 1, -m + 3/2; x^2) \]
(see [DG1 (6.11)]) and (iii) follows by integrating (i). Property (ii) follows from (i) and (11).

Set
\begin{equation}
  u(x) \equiv u(x; m) = \frac{1}{h(x)} - \frac{1 - x^2}{2} + \frac{1}{4m}.
\end{equation}
Note that the function $u(x)$ is closely related to the function $y_m$ which plays a prominent role in [DG1]: we have
$$u(x) = \sqrt{1 - x^2} y_m(\arcsin x) + \frac{1}{2m}, \quad 0 \leq x \leq 1.$$  

Also note that using the elementary identities for $q = 3, 5, \cdots$,
$$\frac{2}{\pi} \int_0^{\pi/2} \sin qs \sin s \, ds = 0, \quad W\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \int_0^{\pi/2} \frac{\sin qs}{\sin s} \, ds = 1$$
we have from (2), (16)

(17) \quad I(q) = \frac{4}{\pi} \int_0^{\pi/2} \frac{\sin qs}{\sin s} u(\cos s) \, ds - \frac{1}{2m}.

The main technical result in our proof of Theorem 1 is the following.

**Lemma 3.** The function $u(x) = u(x; m)$, $m \geq 2$, has the following properties.

(i) $u(x)$ is unimodal for $x \in [0, 1]$. More precisely, there exists $x_0 \in (0, 1)$ such that $u'(x) < 0$ for $0 < x < x_0$ and $u'(x) > 0$ for $x_0 < x < 1$.

(ii) $u(0) = 0$, $u(1) = \frac{1}{2m}$.

(iii) For $0 \leq x \leq 1$,
$$-\frac{1}{4m} < u(x) \leq \frac{1}{2m}.$$  

The proof of Lemma 3 is given after the proof of Theorem 1 below. We also need the following elementary result from Fourier analysis.

**Lemma 4.** For $q \geq 3$, $0 \leq x \leq \pi/2$,
$$0 \leq W(x) \leq \frac{\sqrt{3}}{\pi} + \frac{2}{3} < 1.218.$$

**Proof.** As the factor $\sin s$ in $W(x) = \frac{2}{\pi} \int_0^x \frac{\sin qs}{\sin s} \, ds$ is increasing, a standard argument in the analysis of the Gibbs phenomenon shows that for $0 \leq x \leq \pi/2$,
$$0 \leq W(x) \leq \frac{2}{\pi} \int_0^{\pi/q} \frac{\sin qs}{\sin s} \, ds = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{q \sin(t/q)} \, dt.$$  
But for $0 \leq t \leq \pi/2$, $q \mapsto q \sin(t/q)$ is increasing, and so for $q \geq 3$ and $0 \leq x \leq \pi/2$, $0 \leq W(x) \leq \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{3 \sin(t/3)} \, dt = \frac{\sqrt{3}}{\pi} + \frac{2}{3}$. \hfill $\Box$

Assuming Lemma 3, we now prove Theorem 1. By (9), (17), integrating by parts and using Lemma 3 (ii),

$$Q(q) = \frac{4}{\pi} \int_0^{\pi/2} \frac{\sin qs}{\sin s} u(\cos s) \, ds$$
$$= 2(W(s) u(\cos s)) \bigg|_0^{\pi/2} + 2 \int_0^{\pi/2} W(s) \, u'(\cos s) \sin s \, ds$$
$$= 2 \left( \int_{\arccos x_0}^{\arccos x_0} + \int_{\arccos x_0}^{\pi/2} \right) W(s) \, u'(\cos s) \sin s \, ds.$$


Thus, by Lemma 3 and Lemma 4

$$Q(q) \leq 2 \int_{\arccos x_0}^{\arccos x_0} W(s) u'(\cos s) \sin s \, ds$$

$$\leq 2 \cdot 1.218 \cdot (u(1) - u(x_0))$$

$$\leq 2 \cdot 1.218 \cdot \left( \frac{1}{2m} + \frac{1}{4m} \right) = \frac{3}{2} \cdot \frac{1.218}{m}$$

On the other hand

$$Q(q) \geq 2 \int_{\arccos x_0}^{\pi/2} W(s) u'(\cos s) \sin s \, ds$$

$$\geq 2 \cdot 1.218 \cdot (u(x_0) - u(0)) \geq \frac{1}{2} \cdot \frac{1.218}{m}$$

and thus

$$|Q(q)| \leq \frac{3}{2} \cdot \frac{1.218}{m} = \frac{1.827}{m}.$$

Recalling the definitions of $Q^{[m-1]}$ and $B^{[m-1]}$, we have for $1 \leq i, j \leq m - 1$

$$|(Q^{[m-1]}B^{[m-1]})_{ij}| \leq \left| \sum_{l=1}^{j} Q(n - 2i + 2l) 2m \binom{n}{j-l} \right| \leq 2 \cdot 1.827 \cdot \sum_{l=0}^{j-1} \binom{n}{l}$$

and hence

$$|(Q^{[m-1]}B^{[m-1]})_{ij}| \leq 2 \cdot 1.827 \cdot L_{ij}, \quad 1 \leq i, j \leq m - 1,$$

where $L$ is the rank 1 matrix with entries $L_{ij} = \sum_{l=0}^{j-1} \binom{n}{l}$, independent of $i$. Hence $L$ has only 1 non-zero eigenvalue $\lambda_1(L)$ and we find

$$r(L) = \lambda_1(L) = \sum_{k=1}^{m-1} L_{1k} = \sum_{k=1}^{m-1} \sum_{l=1}^{k} \binom{n}{l-1}$$

$$= \sum_{l=0}^{m-1} (m - l - 1) \binom{2m-1}{l} = \frac{m}{2} \binom{2m-1}{m-1} - 2^{2m-3} \leq \frac{m}{2} \binom{2m-1}{m-1}. $$

In the second last step, we have used the elementary formula preceding (6.7) in [DG1].

Assembling the above results and recalling the definition of $K^{[m-1]}$, we obtain for $1 \leq i, j \leq m - 1$,

$$|K^{[m-1]}_{ij}| = \frac{(ml)^2}{m(2m)!} |(Q^{[m-1]}B^{[m-1]})_{ij}| \leq K'_{ij}$$

where

$$K'_{ij} = 2 \cdot 1.827 \cdot \frac{(ml)^2}{m(2m)!} L_{ij}, \quad 1 \leq i, j \leq m - 1,$$

and by (18), the only non-zero eigenvalue of $K'$ satisfies

$$\lambda_1(K') = r(K') = 2 \cdot 1.827 \cdot \frac{(ml)^2}{m(2m)!} r(L)$$

$$\leq 2 \cdot 1.827 \cdot \frac{(ml)^2}{m(2m)!} m \frac{2m-1}{2} \left( \frac{2m-1}{m-1} \right) = \frac{1.827}{2} = 0.9135 < 1.$$
Thus by Lemma 2
\[ \det(1 - K^{m-1}) \geq \det(1 - K') = 1 - \lambda_1(K') \geq 0.0865. \]
This completes the proof of Theorem 1.

Remark 2. Using Lemma 2 the calculations in [DG1] also yield a quantitative version of (15) but with a weaker bound. As above, we estimate $T^{m-1}$ elementwise with a rank one matrix so that we can estimate the determinant by estimating the only nonzero eigenvalue. We note that we cannot use [DG1, (6.22)] (the matrix in (6.22) is not rank one). For “small” $m$ we use [DG1] (6.55), (6.56), and for “large” $m$ we use [DG1] (6.55), (6.21). We claim that
\[ \det T^{m-1} \geq 0.02, \quad m \geq 2. \]
This estimate is not optimal, but we could not strengthen it compared to 60 by the methods in [DG1]. To prove (20) for $2 \leq m \leq 46$, we note that the RHS in [DG1] (6.16) is $< 0.98$ for $m$ in this range. (Note that our $Q(q)$ and $I(q)$ in [DG1] are related by $\hat{I}(q) = mQ(q) - 1$ and hence $|Q(q)| = \frac{1}{m}|1 + \hat{I}(q)| \leq \frac{1}{m}(1 + |\hat{I}(q)|)$.) To prove (20) for $m \geq 47$, we set $\delta \equiv 0.04$ and consider $q = 3, 5, \cdots, 4m - 5$ in the regions $\frac{\delta\sqrt{m+1/2}}{q} \leq 1 - \delta$ and $\frac{\delta\sqrt{m+1/2}}{q} > 1 - \delta$ separately. In the former $q$-region, by [DG1] (6.21), $|1 + \hat{I}(q)| \leq 1 + |\hat{I}(q)| \leq 1.96$. In the latter $q$-region, substituting $\frac{\delta}{\sqrt{m+1/2}} \leq \frac{1}{\pi} \epsilon$ in [DG1] (6.55), we note that the resulting estimate on $|1 + \hat{I}(q)|$ multiplied by $\left(1 - \frac{(m-1)^2}{m(2m-1)}\right)\frac{2}{2m-2}$, is $< 0.98$ in fact for $m \geq 44$. These facts together with Lemma 2 prove (20) (cf. [DG1] (6.56), (6.57)).

It remains to prove Lemma 3. A straightforward computation using [DG1] (i) and (ii) shows that $u$ is a solution of the equation
\[ x(1 - x^2)u'' - 4mu^2 + (2(m+1)x^2 + 1 - 2m)u - \frac{x^2}{2m} = 0. \]
Moreover as $h(x) > 0$, $u$ is smooth. By [DG1] (ii), and by differentiating (21), we find,
\[ u(0) = 0, \quad u'(0) = 0, \quad u''(0) = -\frac{1}{m(2m-3)} \]
\[ u(1) = \frac{1}{2m}, \quad u'(1) = \frac{2}{3} + \frac{1}{3m}. \]
Now observe that at a point $0 < x < 1$ where $u'(x) = 0$, we cannot have $4m(m+1)u(x) - 1 = 0$, i.e. $u(x) = \frac{1}{4m(m+1)}$. Indeed, substituting these values into (21), we find $-1 + (1 - 2m)(m + 1) = 0$, which is a contradiction. Next we show that
\[ (u'(x) = 0 \text{ for some } 0 < x < 1) \implies u''(x) = \frac{(4m(m+1)u(x) - 1)^2}{m(1 - 2mu(x))(1 - 4mu(x))}. \]
Indeed, differentiating (21), we find for such a point $x$
\[ u''(x) = \frac{1 - 4m(m+1)u(x)}{m(1 - x^2)}. \]
Setting $u'(x) = 0$ in (21) and solving for $(1 - x^2)$ in terms of $u(x)$, we obtain
\[ 1 - x^2 = -\frac{(1 - 2mu(x))(1 - 4mu(x))}{4m(m+1)u(x) - 1}. \]
Note that by the above argument, the denominator in \( \frac{1}{2m} \) is non-zero: also the numerator is non-zero as \( 1 - x^2 \neq 0 \). Substituting \( \frac{1}{2m} \) into \( \frac{1}{2m} \), we obtain \( \frac{1}{2m} \). Furthermore, the calculation shows that if \( u'(x) = 0 \) for some \( 0 < x < 1 \), then \( u''(x) \) is (finite and) non-zero.

From \( \frac{1}{2m} \), we see that for small \( x > 0 \), \( u(x) < 0 \). As \( u(1) = 0 \), there must be at least one point \( x \in (0, 1) \) where \( u(x) = 0 \). But it follows from \( \frac{1}{2m} \) that if \( u(x) = 0 \), \( x \in (0, 1) \), then \( u''(x) = \frac{1}{2m(1-x^2)} > 0 \). Hence \( u \) crosses the level zero at a unique point \( x_1 \in (0, 1) \). Next suppose that \( u'(x) = 0 \) for some \( x \in (0, x_1) \). Then by \( \frac{1}{2m} \), \( u''(x) > 0 \) as \( u'(x) < 0 \). Thus any critical point for \( u(x) \) in \((0, x_1)\), must be a local minimum. As \( u(x) \) clearly has a minimum on \((0, x_1)\), it follows that it has a unique minimum at \( x_0 \in (0, x_1) \), say, and no other critical points on \((0, x_1)\). Thus \( u'(x) < 0 \) for \( 0 < x < x_0 \), and \( u'(x) > 0 \) for \( x_0 < x \leq x_1 \).

Next we show that

\[
0 < u(x) < \frac{1}{2m} \quad \text{for } x_1 < x < 1.
\]

Indeed, if \( u(x) = \frac{1}{2m} \) for \( 0 < x < 1 \), then from \( \frac{1}{2m} \) we find \( u'(x) = \frac{2m+1}{2mx} > 0 \). But we know from \( \frac{1}{2m} \) that \( u(1) = \frac{1}{2m}, u''(1) > 0 \). Hence \( u(x) \) cannot cross the level \( \frac{1}{2m} \) for \( 0 < x < 1 \). This proves \( \frac{1}{2m} \).

To complete the proof that \( u \) is unimodal we show that \( u'(x) > 0 \) for \( x_1 < x < 1 \). Suppose \( u'(x_2) < 0 \) for some \( x_1 < x_2 < 1 \). Then as \( u(x_1) = 0 \) and \( u(x_2) < u(1) = \frac{1}{2m} \), there must exist \( x_1 < x^{-} < x_2 \) and \( x_2 < x^{+} < 1 \) such that \( u \) has a local maximum at \( x^{-} \) and a local minimum at \( x^{+} \). By \( \frac{1}{2m} \), we must have \( u(x^{-}) > \frac{1}{2m} \) and \( u(x^{+}) < \frac{1}{2m} \). This implies, in particular, that \( u(x) \) crosses the level \( \frac{1}{2m} \) at least one point \( x^{\#} \in (x^{-}, x^{+}) \) such that \( u'(x^{\#}) \leq 0 \). But by \( \frac{1}{2m} \), \( u(x) = \frac{1}{2m}, 0 < x < 1 \), implies \( u'(x) = \frac{1}{2x} > 0 \), which is a contradiction. Thus \( u'(x) \geq 0 \) on \((x_1, 1)\). On the other hand if \( u'(x_3) = 0 \) for some \( x_1 < x_3 < 1 \), then \( \frac{1}{2m} \), \( u''(x_3) \neq 0 \) and so \( u'(x) \) changes sign in a neighborhood of \( x_3 \), contradicting \( u'(x) \geq 0 \) on \((x_1, 1)\). Thus \( u'(x) > 0 \) for all \( x_1 \leq x < 1 \). This completes, in particular, the proof of part (i) of Lemma 3.

It remains to show that \( u(x) = u(x; m) > -\frac{1}{4m} \) for \( m \geq 2, x \in [0, 1] \). It turns out that \( x = x_m = \sqrt{\frac{m-1}{m+2}} \) plays a distinguished role. More precisely, as we now show,

\[
\frac{1}{4m} > u(x_m) \quad \Rightarrow \quad \left( u(x) > -\frac{1}{4m} \text{ for all } x \in [0, 1] \right).
\]

To see this, suppose \( u(x) = -\frac{1}{4m} \) for some \( x \in (0, 1) \): then from \( \frac{1}{2m} \) we obtain

\[
u'(x) = \frac{(m+2)x^2 - (m-1)}{2mx(1-x^2)}.
\]

Suppose \( u(x_m) > -\frac{1}{4m} \). If \( u(\hat{x}) \leq -\frac{1}{4m} \) for some \( 0 < \hat{x} < x_m \), then clearly

\[
u'(\hat{x}) = -\frac{1}{4m}, \quad u'(x^{\#}) \geq 0 \quad \text{for some } x^{\#} \in [\hat{x}, x_m].
\]

But by \( \frac{1}{2m} \), \( u'(x^{\#}) < 0 \), which is a contradiction. Similarly if \( u(\hat{x}) \leq -\frac{1}{4m} \) for some \( x_m < \hat{x} < 1 \), there must exist a point \( x^{\#} \in [x_m, \hat{x}] \) such that \( u(x^{\#}) = -\frac{1}{4m}, \quad u'(x^{\#}) \leq 0 \). But this contradicts \( \frac{1}{2m} \) as above. This proves \( \frac{1}{2m} \).
To complete the proof of Lemma $8$, we must prove $u(x_m) \equiv u(x_m; m) > -\frac{1}{4m}$, $m \geq 2$. Set $s = 1 - x$. From \ref{8} (iii), we obtain
\[
h(x) = \frac{4m(1-s)^{2m-1}}{\sqrt{(2-s)}} \int_0^s \frac{(1-\tau)^{2m}}{\sqrt{\tau(2-\tau)}} d\tau \leq \frac{4m(1-s)^{2m}}{(2-s)(1-s)\sqrt{s}} \int_0^s \frac{(1-\tau)^{2m}}{\sqrt{\tau}} d\tau.
\]
Using the elementary inequality $\frac{1}{\sqrt{s}} \leq e^{\tau-s}$ for $0 \leq \tau \leq s$, we find
\[
h(x) \leq 4me^{-2ns} \int_0^s \frac{e^{2m\tau}}{\sqrt{\tau}} d\tau = \frac{4me^{-2m\mu}}{(1 - \frac{\mu^2}{4m})(1 - \frac{\mu^2}{4m})} \mu \int_0^\mu e^{\lambda^2} d\lambda.
\]
where
\[
\mu = \sqrt{2ms} = \sqrt{2m(1-x)}.
\]
In order to prove $u(x_m) > -\frac{1}{4m}$, $m \geq 2$, we see that it is sufficient to show that
\[
\frac{(1 - \frac{\mu^2}{4m})\mu\mu^2}{2} - \mu^2 + \frac{1}{1 - \frac{\mu^2}{4m}} > 0 \quad \text{for} \quad \mu = \mu_m = \sqrt{2m(1-x_m)}.
\]
By the inequality $\frac{1}{1 - \frac{\mu^2}{4m}} > 1 + \frac{\mu^2}{4m}$, and the elementary fact that $1 < \mu_m < \sqrt{3}$, $m \geq 2$, we see that it is sufficient to show
\[
F(\mu) \geq \frac{1}{m} G(\mu) \quad \text{for} \quad 1 \leq \mu \leq \sqrt{3}
\]
where
\[
F(\mu) \equiv \mu e^{\mu^2} + 2(1 - \mu^2) \int_0^\mu e^{\lambda^2} d\lambda, \quad G(\mu) \equiv \frac{\mu^2}{2} \left( \mu e^{\mu^2} - \int_0^\mu e^{\lambda^2} d\lambda \right).
\]
But $G(\mu)$ is clearly increasing and so it is enough to show
\[
F(\mu) \geq \frac{G(\sqrt{3})}{m} \quad \text{for} \quad 1 \leq \mu \leq \sqrt{3}.
\]
Differentiating $F(\mu)$ we find
\[
F(1) = e, \quad F'(1) = 3e - 4 \int_0^1 e^{\lambda^2} d\lambda > 2.304 > 0
\]
\[
F''(1) = 2e - 4 \int_0^1 e^{\lambda^2} d\lambda > -0.415
\]
\[
F'''(\mu) \geq 0 \quad \text{for} \quad \mu \geq 1.
\]
Thus for $1 \leq \mu \leq \sqrt{3}$
\[
F(\mu) \geq F(1) + F'(1)(\mu - 1) + \frac{F''(1)}{2}(\mu - 1)^2 \geq e - \frac{0.415}{2}(\sqrt{3} - 1)^2 > 2.607.
\]
On the other hand $G(\sqrt{3}) < 41.3$, and if we choose $m$ so that $2.607 > \frac{41.3}{m}$, then \ref{20} will hold. Clearly $m \geq 16$ satisfies this inequality. We conclude that $u(x_m) > -\frac{1}{4m}$ for $m \geq 16$. On the other hand, using Maple (only sums and products are involved), we find from \ref{8}, \ref{8}
\[
\min_{2 \leq m \leq 15} \left( u(x_m) + \frac{1}{4m} \right) > 0.0129 > 0.
\]
This completes the proof of Lemma $8$ and hence Theorem $1$.

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COSTIN: DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, 231 W. 18TH AVE., COLUMBUS, OH 43210
E-mail address: costin@math.ohio-state.edu

DEIFT: DEPARTMENT OF MATHEMATICS, COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER ST., NEW YORK, NY 10012
E-mail address: deift@cims.nyu.edu

GIOEV: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, HYLAN BLDG., ROCHESTER, NY 14627
E-mail address: gioev@math.rochester.edu