Dynamic transition from insulating state to $\eta$-pairing state in a composite system

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Time dynamics of Hermitian many-body quantum systems has long been an elusive subject. In contrast, the exceptional point (EP) in a non-Hermitian system admits a peculiar dynamics: the final state being a particular eigenstate, coalescing state. In this work, we study the dynamic transition from a trivial insulating state to an $\eta$-pairing state in a composite Hubbard system. The system is consisted of two subsystems A and B, which is connected by unidirectional hoppings. We show that the dynamic transition from an insulating state to an $\eta$-pairing state occurs by the probability flow from A to B: the initial state is prepared as an insulating state of A, while B is left empty. The final state is $\eta$-pairing state in B but empty in A. Analytical analyses and numerical simulations show that the speed of relaxation of off-diagonal long-range order (ODLRO) pair state depends on the order of the EP, which is determined by the number of pairs and the fidelity of the scheme is immune to the irregularity of the lattice.

I. INTRODUCTION

Experimental advances in atomic physics, quantum optics, and nanoscience have made it possible to realize artificial systems. It is fascinating that some of them are described by Hubbard model to a high degree of accuracy [1, 2]. Then one can experimentally realize and simulate the physics of the model. The Hubbard model is a simple lattice model with particle interaction and has been intensely investigated in various contexts ranging from quantum phase transition to high temperature superconductivity. Direct simulations of such a simple model is not only helpful to solve important problems in condensed matter physics, but also to the engineering design of quantum devices. Importantly, the availability of experimental controllable Hubbard systems provides an unprecedented opportunity to explore the nonequilibrium dynamics in interacting many-body systems.

Very recently, it has been demonstrated that nonequilibrium many-body dynamics provides an alternative way to access a new exotic quantum state with energy far from the ground state [3, 4]. It makes it possible to design interacting many-body systems that can be used to perform practical tasks in principle, such as the preparation of certain desirable many-body quantum states. Unlike traditional protocols based on cooling down mechanism, quenching dynamics is a potentially vast field. It provides many ways to take a system out of equilibrium, such as applying a driving field or pumping energy or particles in the system through external reservoirs. In recent work Ref. [5], a scheme has been proposed to realize quantum mold casting, i.e., engineering a target quantum state on demand by the time evolution of a trivial initial state. The underlying mechanism is pumping fermions from a trivial subsystem to the one with topological quantum phase.

In this work, we extended this approach to interacting many-body systems. In general, time dynamics of Hermitian many-body quantum systems has long been an elusive subject, due to the complexity induced by the particle-particle interactions. Nevertheless, the exceptional point (EP) in a non-Hermitian system admits a peculiar dynamics: the final state being a particular eigenstate, coalescing state. The key point is the exceptional dynamics, which allows the particle pumping from the source to the center subsystem, realizing the dynamical preparation of many-body quantum states. In present work, we study the dynamic transition from a trivial insulating state to an $\eta$-pairing state in a composite Hubbard system. The system is consisted of two subsystems A and B, which is connected by unidirectional hoppings. Based on the performance of the system at EP, a scheme that produces a nonequilibrium steady superconducting-like state is proposed. Specifically, for an initial state with fully filled in A, but empty in B, unidirectional hoppings can drive it to the resonant coalescing state that favors superconductivity manifested by the ODLRO. Such a dynamical scheme can be realized no matter how the shapes of the two sublattices of the composite system. Therefore, our finding is distinct from the previous investigations [3, 4], and offer a quantum casting mechanism for generating superconductivity through nonequilibrium dynamics. On the other hand, the remarkable observation from our work can trigger further studies of both fundamental aspects and potential applications of composite non-Hermitian many-body systems.

The rest of this paper is organized as followed. In Section II, we present the model and its properties relating to $\eta$ operators, or $\eta$-symmetry. Section III is devoted to doublon effective Hamiltonian which captures the physics in a fixed energy shell. In Section IV, we present the Jordan form with high-order EP based on the effective Hamiltonian. In Section V numerical simulations are performed to estimate the efficiency of our scheme in various values of correlation strengths. Section
II. MODEL AND $\eta$ OPERATORS

We consider a composite system, described by the Hamiltonian

$$H = H_A + H_B + H_{AB},$$

with Hermitian terms

$$H_A = \frac{U}{2} \sum_{i=1}^{N_a} \left( a_{i,\downarrow}^{\dagger} a_{i,\uparrow} + a_{i,\uparrow}^{\dagger} a_{i,\downarrow} \right),$$

$$H_B = \sum_{\sigma=\uparrow, \downarrow} \sum_{i,j=1}^{N_b} \left( t_{ij} b_{i,\sigma}^{\dagger} b_{j,\sigma} + H.c. \right) + U \sum_{i=1}^{N_b} b_{i,\uparrow}^{\dagger} b_{i,\downarrow} b_{i,\downarrow}^{\dagger} b_{i,\uparrow},$$

and non-Hermitian term

$$H_{AB} = \sum_{\sigma=\uparrow, \downarrow} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \kappa_{ij} b_{j,\sigma}^{\dagger} a_{i,\sigma},$$

where $a_{i,\sigma}$ and $b_{j,\sigma}$ are fermion operators with spin $\frac{1}{2}$, polarization $\sigma = \uparrow, \downarrow$ in lattices $N_a$ and $N_b$, respectively. The parameters $t_{ij}$ ($i, j \in N_b, i \neq j$) and $\kappa_{ij}$ ($i \in N_a, j \in N_b$) are intra- and inter-cluster hopping strengths, and taken to be real in this paper. Here both $H_A$ and $H_B$ are Hermitian, describing the source system and the central system, respectively. $H_A$ is an interaction-free system with trivial flat band, while $H_B$ is a standard Hubbard model, which is restricted to be bipartite lattice. In particular, the key features of the setup are (i) $H_{AB}$ is non-Hermitian, representing a unidirectional tunneling between two systems $H_A$ and $H_B$. (ii) The on-site potential of a pair fermion in $H_A$ is identical to the on-site repulsion in $H_B$. The schematic of the system is presented in Fig. [I].

We define two $\eta$ operators for two subsystems

$$\eta_A = \sum_{i=1}^{N_a} \xi_i a_{i,\downarrow}^{\dagger} a_{i,\uparrow}^{\dagger},$$

$$\eta_B = \sum_{i=1}^{N_b} \xi_i b_{i,\uparrow} b_{i,\downarrow},$$

where $\xi_i = \pm 1$ can be taken arbitrarily, since there are no tunneling between any two sites in system $A$, while $\xi_i = 1$ and $-1$, for the different sublattice $i$ belongs to in the bipartite lattice $N_a$. It can be shown both operators satisfy

$$[H_A, \eta_A] = U \eta_A^{\dagger}, [H_B, \eta_B] = U \eta_B,$$

which can be utilized to construct the eigenstates of $H_A$, $H_B$, $H_A + H_B$,

$$|n\rangle_A = \frac{1}{\sqrt{\Omega_{A,n}}} \left( \eta_A^\dagger \right)^n |Vac\rangle,$$

$$|m\rangle_B = \frac{1}{\sqrt{\Omega_{B,m}}} \left( \eta_B^\dagger \right)^m |Vac\rangle,$$

where $\Omega_{A,n} = (n!)^2 C_{N_a}^n$ and $\Omega_{B,m} = (m!)^2 C_{N_b}^m$ are normalization factors.

$$H_A |n\rangle_A = n U |n\rangle_A, H_B |m\rangle_B = m U |m\rangle_B,$$

and

$$(H_A + H_B) |n\rangle_A |m\rangle_B = (n + m) U |n\rangle_A |m\rangle_B.$$

We can find that the set of eigenstates $|n\rangle_A |m\rangle_B$ are degenerate for fixed $m + n$.

In general, an $\eta$ pairing state can be regarded as a condensate of bound pair fermions as hardcore boson. However, state $|n\rangle_A$ is trivial since it is just one of multi-fold degenerate eigenstates. In addition, fully filled state $|N_a\rangle_A$ and $|N_b\rangle_B$ are insulating states and can be easily prepared. The desirable states are $|N_a\rangle_A |m\rangle_B$ and $|0\rangle_A |m\rangle_B$ with $1 < m < N_b$, since both two states possess ODLRO in the subsystem B.
III. DOUBLON EFFECTIVE HAMILTONIAN

Like the most interacting many-body systems, the exact solution of $H$ is rare although state $|n_A, m_B\rangle$ is eigenstate of $H_A + H_B$. In order to capture the physics of our scheme, we will consider the problem in an energy shell. In a Hermitian system, one can employ the perturbation method to get the effective Hamiltonian. However, the corresponding theory has not been well established for the non-Hermitian system, especially for unidirectional hopping perturbations. In the Appendix VII we have obtained the effective Hamiltonian of a two-site non-Hermitian system from the time evolution operator.

In this work, our aim is the dynamics for a special initial state, with the subsystem A being fully occupied. It motivates us to consider pure doublon states in the subsystem B, which has the same energy shell with that of the initial state. For a subspace spanned by a set of basis of doublon, $\{\prod_{j}(b_{j,\uparrow}^\dagger b_{j,\downarrow}^\dagger |\text{Vac}\rangle\}$, where $\{j\}$ denotes all possible set of integer number $j \in [1, N_a]$ with dimension $\text{dim}\{j\} \in [1, N_a]$, the effective Hamiltonian can be written as

$$H_{\text{eff}}^B = -\frac{4\eta^2}{U} \sum_{i,j=1}^{N_a} (\eta_{B,i} \cdot \eta_{B,j} - \frac{1}{4}) + U \sum_{i=1}^{N_a} \left( \frac{1}{2} + \eta_{B,i}^z \right),$$

(11)

in the case of $U \gg |t_{ij}|$. Here pseudo-spin operator $\eta_{B,j} = (\eta_{B,j,\uparrow}/2 + \eta_{B,j,\downarrow}/2i, \eta_{B,j,\downarrow}/2i - \eta_{B,j,\uparrow}/2i, \eta_{B,j,\downarrow})$ with $\eta_{B,j,\uparrow} = \beta b_{j,\uparrow}^\dagger b_{j,\downarrow}^\dagger$ and $\eta_{B,j,\downarrow} = (n_{B,j,\uparrow} + n_{B,j,\downarrow} - 1)/2$. Similarly, for a subspace spanned by a set of basis of doublon, $\{\prod_{j}(\prod_{l} a_{j,\uparrow}^\dagger a_{j,\downarrow}^\dagger |\text{Vac}\rangle\}$, where $\{j\}$ denotes all possible set of integer number $j \in [1, N_a]$ with dimension $\text{dim}\{j\} \in [1, N_a]$, the effective Hamiltonian can be written as

$$H_{\text{eff}}^A = U \sum_{i=1}^{N_a} \left( \frac{1}{2} + \eta_{A,i}^z \right)$$

(12)

the corresponding obeying the Lie algebra, i.e., $[\eta_{\pm,i}, \eta_{A,j}^\pm] = 2\eta_{A,j}^\pm \delta_{ij}$ and $[\eta_{A,i}^\pm, \eta_{A,j}^\pm] = \pm 2\eta_{A,i}^\pm \delta_{ij}$.

Now, it turns to establish the effective Hamiltonian $H_{\text{eff}}^{AB}$ arising from the non-Hermitian term $H_{AB}$. Unlike the Hermitian term, there is no unquestioned perturbation theory for the non-Hermitian perturbation, especially near the EP. In this work, we present the effective Hamiltonian $H_{\text{eff}}^{AB}$ through a dynamic way. In the Appendix VII we show that, for the given initial state with full filling A lattice and empty B lattice, the dynamics obeys the effective Hamiltonian

$$H_{\text{eff}} = H_{\text{eff}}^A + H_{\text{eff}}^B + H_{\text{eff}}^{AB}$$

(13)

with

$$H_{\text{eff}}^{AB} = \sum_{l} \frac{4\kappa^2}{U} \eta_{A,l}^\dagger \eta_{B,l}^\dagger,$$

(14)

where

$$\eta_{A,l} = (-1)^l a_{l,\uparrow} a_{l,\uparrow}, \eta_{B,l}^\dagger = (-1)^l b_{l,\uparrow}^\dagger b_{l,\uparrow}^\dagger.$$ (15)

It is clear that $H_{\text{eff}}^{AB}$ describes a unidirectional hopping of a doublon or magnon from the point of view of spin wave.

Defining a total pseudo-spin operator

$$\eta^z = \sum_{i=1}^{N_a} \eta_{A,i}^z + \sum_{i=1}^{N_b} \eta_{B,i}^z,$$ (16)

we note that $\eta^z$ is conservative for the Hamiltonian $H_{\text{eff}}$ due to the commutation relation

$$[\eta^z, H_{\text{eff}}] = 0,$$ (17)

which ensures that the Hilbert space of $H_{\text{eff}}$ can be decomposed into many invariant subspaces labeled by the eigenvalues of $\eta^z$, i.e., $2\eta^z = -N_a - N_b, -N_a - N_b + 1, \ldots, N_a + N_b - 1, N_a + N_b$. In this work, we only focus on the subspace with $\eta^z = (N_a - N_b)/2$ ($N_a < N_b$), which contains the initial state with fully filling A sublattice and empty B sublattice, i.e., $\prod_{i=1}^{N_a} a_{i,\uparrow}^\dagger a_{i,\downarrow}^\dagger |\text{Vac}\rangle$.

IV. JORDAN FORM WITH HIGH-ORDER EP

In the above, we know that there are many degenerate eigenstates for $H_A + H_B$, which may become coalescing states when proper non-Hermitian term is added [9]. For non-Hermitian operators, when EP appears, there are eigenstates coalesce into one state, leading to an incomplete Hilbert space [10][13]. Mathematically, it relates to the Jordan block form in the matrix [14][17]. Remarkably, the peculiar features around EP have sparked tremendous attention to the classical and quantum photonic systems. The corresponding intriguing dynamical phenomena include asymmetric mode switching [18], topological energy transfer [19], robust wireless power transfer [20], and enhanced sensitivity [21][24] depending on their EP degeneracies. Many works have been devoted to the formation of the EP and corresponding topological characterization in both theoretical and experimental aspects [25][28]. In this work, we employ the EP dynamics to prepare states with ODLRO. We start with the Jordan form with high-order EP.

Considering two degenerate eigenstates $|A\rangle$ and $|B\rangle$ of the Hermitian Hamiltonian $H_A + H_B$, where

$$|A\rangle = |N_a\rangle_A |0\rangle_B = \frac{1}{\sqrt{\Omega_{A,N_a}}} \left( \eta_{A}^\dagger \right)^{N_a} |\text{Vac}\rangle,$$ (18)

$$|B\rangle = |0\rangle_A |N_b\rangle_B = \frac{1}{\sqrt{\Omega_{B,N_b}}} \left( \eta_{B}^\dagger \right)^{N_b} |\text{Vac}\rangle,$$ (19)

we have

$$H |B\rangle = N_b U |B\rangle, H^\dagger |A\rangle = N_a U |A\rangle,$$ (20)
due to the facts
\[ H_{AB} |0\rangle_A |N_a\rangle_B = 0, (H_{AB})^\dagger |N_a\rangle_A |0\rangle_B = 0. \] (21)

It means that two states |A\rangle and |B\rangle are mutually biorthogonal conjugate and \( \langle A | B \rangle \) is the biorthogonal norm of them. On the other hand, we have
\[ \langle A | B \rangle = 0. \] (22)

The vanishing norm indicates that state \( |B\rangle (|A\rangle) \) is coalescing state of \( H(H^\dagger) \), or Hamiltonians \( H \) and \( H^\dagger \) gets an EP.

However, it is a little hard to determine the corresponding Jordan block form and the order of the EP. In the following, we estimate the order in large \( U \) limit. At first, the above analysis for two states |A\rangle and |B\rangle is applicable for the effective Hamiltonian \( H_{\text{eff}} \). This means that there is an EP in the invariant subspace with \( \eta^2 = (N_a - N_b)/2 \), and dimension \( C_{N_a}^{N_b} \). The order of such an EP is determined by the corresponding Jordan block. Second, when we consider a complete set of degenerate eigenstates of the Hermitian Hamiltonian \( H_A + H_B \) in this subspace, which are denoted as \{ |n\rangle_A |m\rangle_B \} \( n \in [0, N_a] \), \( n \leq N_b \) with fixed \( m + n = N_a \), the effective Hamiltonian can be expressed as an \( (N_a + 1) \times (N_a + 1) \) matrix \( M \) with nonzero matrix elements
\[ (M)_{N_a+1-n, N_a-n} \]
\[= (N_a - n)_{1B} |n\rangle_A \langle H_{\text{eff}} | n + 1\rangle_A |N_a - n + 1\rangle_B \]
\[= \frac{4\kappa^2 N_a - n}{U} \frac{N_b}{\sqrt{(n + 1)(N_b - N_a + n + 1)}} \] (23)
with \( n = [0, N_a - 1] \), and
\[ (M)_{N_a+1-n, N_a+1-n} \]
\[= (N_a - n)_{1B} |n\rangle_A \langle H_{\text{eff}} | n\rangle_A |N_a - n\rangle_B \]
\[= N_a U \] (24)
with \( n = [0, N_a] \). It is obviously an \( (N_a + 1) \)-order Jordan block, satisfying
\[ \left[(M - N_a U I)_{N_a}^{N_a + 1}\right]_{ij} = \prod_{n=0}^{N_a-1} \frac{4\kappa^2 N_a - n}{U} \frac{N_b}{N_b} \]
\[\times \sqrt{(n + 1)(N_b - N_a + n + 1)\delta_{N_a+1,1}}. \] (25)
where \( I \) is the unit matrix. In other words, matrix \( (M - N_a U I) \) is a nilpotent matrix, i.e.,
\[ (M - N_a U I)^{N_a+1} = 0. \] (26)

Taking \( N_a = N_b = 4 \), for example, the matrix has the form
\[ M = \frac{2\kappa^2}{U} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + 4 U I, \] (27)
which possesses a single eigenvector \( (0 \ 0 \ 0 \ 0 \ 1)^T \).

The dynamics for any states in this subspace is governed by the time evolution operator
\[ U(t) = e^{-i M t} = e^{-i N_a U t} \sum_{i=0}^{N_a} \frac{1}{i!} [-i (M - N_a U I)]^i. \] (28)
It indicates that for the initial state \( |\Psi(0)\rangle = |A\rangle \), we have
\[ |\Psi(t)\rangle = e^{-i M t} |A\rangle \]
\[= e^{-i N_a U t} (1 \ f_1 \ ... \ f_{N_a})^T, \] (29)
where the elements
\[ f_q = \sqrt{A_N^q A_{N_b}^q} \left(-\frac{4 i k^2 t}{U N_b}\right)^q, \]
\[ q \in [1, N_a] \]
\[ A_N^q = \frac{N_a!}{(N_a - q)!}, A_{N_b}^q = \frac{N_b!}{(N_b - q)!}, \] (30)
and
\[ ||\Psi(t)|| \approx 1 + \sum_{q=1}^{N_a} |f_q|^2 \approx |f_{N_a}| \] (32)
at large \( t \gg \frac{U N_a}{\kappa^2} \). Setting the target state as
\[ |\Psi_{\text{target}}\rangle = |B\rangle, \] (33)
we have the fidelity
\[ F(t) = \frac{|\langle \Psi_{\text{target}} | \Psi(t) \rangle|}{||\Psi(t)||} \approx |f_{N_a}| \approx 1, \] (34)
which indicates that state |\Psi_{\text{target}}\rangle becomes dominant in the evolved state |\Psi(t)\rangle at large \( t \gg \frac{U N_a}{\kappa^2} \). The increasing behavior of \( ||\Psi(t)|| \) obeys \( ||\Psi(t)||^2 \propto t^{2 N_a} \) within large \( t \) region is also a dynamic demonstration for the order of the Jordan block. This analytical analysis shows that the speed of relaxation of ODLRO pair state depends on the order of the EP, which is determined by the number of pairs. Furthermore, we would like to address two points.
(i) The non-Hermitian effective Hamiltonian \( H_{\text{eff}}^{AB} \) is obtained from the simplest case with \( N_a = N_b = 1 \) in the Appendix VII. Its validity still needs to be proved for large systems. (ii) The power behavior of \( ||\Psi(t)|| \) requires large \( t \). However, in practice, \( F(t) \) may approach unit before this time domain. This is discussed in the Appendix VIII for small \( U \).

V. DYNAMIC TRANSITION

The above analysis provides a prediction about the dynamic transition from an insulating state to an \( \eta \)-pairing state in a composite system. The composite system is consisted of two parts (or two layers), one is a trivial system (source system) constructed by a set of isolated
sites, while the other is a Hubbard model (central system), which supports $\eta$-pairing eigenstates. Initially, two subsystems are separated and the source system is fully filled by electrons, being in an insulating state, while the central system is empty. The decoupling between two subsystems can be achieved in two ways, i.e., the hopping terms between two subsystems directly under the resonant energy of the central system; (b) switching off the hopping terms between two subsystems directly under the resonant condition. The post-quench Hamiltonian is then $H_A + H_B + H_{AB}$. According to our analysis, both quench dynamics should result in steady superconducting state, realizing the dynamic transition from an insulating state to an $\eta$-pairing state.

We perform numerical simulations on finite system with the following considerations. (i) The analysis in last section only predicts the results for large $U$ within large time domain. The efficiency of the scheme should be estimated from numerical simulations. (ii) The existence of $\eta$-pairing eigenstates are independent of the distribution of the hoppings for sublattice B. The evolved states $|\Psi(t)\rangle$ for initial states $|2\rangle_A |0\rangle_B$ and $|3\rangle_A |0\rangle_B$ in two finite systems are computed by exact diagonalization. We focus on the Dirac probability

$$P(t) = ||\Psi(t)||^2,$$

and the fidelity

$$F(t) = \frac{1}{\sqrt{P}} |\langle \Psi_{\text{target}} | \Psi(t) \rangle|,$$

with the target states being $|0\rangle_A |2\rangle_B$ and $|0\rangle_A |3\rangle_B$, respectively. The lattice geometry and numerical results are plotted in Fig. 2. We plot the fidelity as function of $U$ for three typical instants. We find that there exits an optimal $U \approx 5$, at which the fidelity gets the maximal value. We also plot the probability $\ln P(t)$ as function of $\ln t$ to demonstrate the EP dynamic behavior. From the results of linear fitting, it can be seen that the slope of the line deviates from the predicted value for the cases with small $U$, especially for larger system. This indicates the speed of relaxation of pair state depends on the order of the EP and the fidelity of the scheme is immune to the irregularity of the lattice.
VI. SUMMARY AND DISCUSSION

In summary, we have extended the scheme of quantum casting to interacting many-body systems. Unlike the previous work Ref. [8] on non-interacting system, the present scheme does not require the scan on the chemical potential of the source system. Our findings offer a method for the efficient preparation of correlated states and are expected to be necessary and insightful for quantum engineering. The key point is the exceptional dynamics, which allows the particle pumping from the source to the center subsystem, realizing the dynamical preparation of many-body quantum states. It is due to the resonance between the initial state and the target state. Accordingly, there is a class of initial states (see Fig. 3) evolving to the same final state after long time. Numerical simulations for finite $U$ and small-size system support this conclusion. In this sense, such a scheme can be applied to other interacting many-body systems. On the other hand, considering a quench process with the pre-quench Hamiltonian being $H_A + H_B$, and the post-quench Hamiltonian being $H_A + H_B + H_{AB}$, the Loschmidt echo $|L|^2 = |\langle \Psi (0) | \Psi (t) \rangle|^2$ should turn to zero after a long time for a finite system. It may predict an asymptotic dynamic quantum phase transition (DQPT) \[29\] in thermodynamic limit, i.e., $|L|^2$ decays rapidly, rather than vanishes at a finite instant in a standard DQPT. The final answer depends on the scaling behavior of $|L|^2$, that is an open question at the present stage.

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VII. APPENDIX

In this Appendix, we present a derivation of the effective Hamiltonian in the doublon subspace for the tunneling term between two subsystems A and B. We will obtain the effective Hamiltonian from the time evolution operator rather than the perturbation method due to the concern with the availability of it for a non-Hermitian system at exceptional point.

Consider a two-site Hamiltonian

$$H_{\text{conn}} = \sum_{\sigma = \uparrow, \downarrow} \kappa b_\sigma^{\dagger} a_{\sigma} + U b_\uparrow^{\dagger} b_\downarrow^{\dagger} b_{\uparrow} b_{\downarrow} + \frac{U}{2} (a_{\uparrow}^{\dagger} a_{\uparrow} + a_{\downarrow}^{\dagger} a_{\downarrow}),$$

which describes the connection between any two sites among A and B subsystems. We neglect the subscripts of the operators for the sake of simplicity. We start from the matrix representation of the Hamiltonian in the invariant subspace spanned by the basis set

$$|1\rangle = |\uparrow \downarrow \rangle_A \otimes |0 \rangle_B = a_{\uparrow}^{\dagger} a_{\downarrow}^{\dagger} |\text{Vac}\rangle,$$

$$|2\rangle = |0 \uparrow \downarrow \rangle_A \otimes |b_{\uparrow}^{\dagger} b_{\downarrow}^{\dagger} |\text{Vac}\rangle,$$

$$|3\rangle = |\uparrow \uparrow \rangle_A \otimes |b_{\downarrow}^{\dagger} |\text{Vac}\rangle,$$

$$|4\rangle = |\downarrow \downarrow \rangle_A \otimes |b_{\uparrow}^{\dagger} |\text{Vac}\rangle,$$

is

$$h = \begin{pmatrix}
U & 0 & 0 & 0 \\
0 & U & \kappa & -\kappa \\
\kappa & 0 & \frac{U}{2} & 0 \\
-\kappa & 0 & 0 & \frac{U}{2} \\
\end{pmatrix},$$

which contains a $2 \times 2$ Jordan block for nonzero $U$. The solution of matrix consists 3 eigenvectors $|\phi_{\phi}\rangle$, $|\phi_{\Delta}\rangle$ and $|\phi_{\Delta}\rangle$, with eigenvalues $U$, $U/2$, and $U/2$, respectively. The
explicit form of the vectors is

\[ |\phi_a\rangle = \begin{pmatrix} 1 \\ 0 \\ \frac{2\kappa}{\sqrt{b}} \\ -\frac{2\kappa}{\sqrt{b}} \end{pmatrix}, \quad |\phi_c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(40)} \]

\[ |\phi_3\rangle = \begin{pmatrix} -\frac{2\kappa}{1} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\phi_4\rangle = \begin{pmatrix} \frac{2\kappa}{0} \\ 0 \\ 1 \\ 1 \end{pmatrix}, \]

where $|\phi_c\rangle$ is coalescing vector and $|\phi_a\rangle$ is the corresponding auxiliary vector, satisfying

\[ (h - UI)|\phi_a\rangle = |\phi_c\rangle, \quad \text{(41)} \]

where $I$ is the unit matrix. We would like to point out that in the case of $U = 0$, $h$ contains a $3 \times 3$ Jordan block. The solution of matrix consists 2 eigenvectors $|\phi_c\rangle$ and $|\phi_4\rangle$, with the same eigenvalue 0. The explicit form of the vectors is

\[ |\phi_a\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\phi_c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\phi_4\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{(42)} \]

where $|\phi_c\rangle$ is coalescing vector and $|\phi_a\rangle$ is the corresponding auxiliary vector. In this work, we only focus on the case with nonzero $U$. However, one should consider the effect of 3-order EP when $U$ is very small. Then the time evolution operator in such an invariant subspace can be obtained as

\[ e^{-iht} = e^{-iU} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{4it\kappa^2}{U} & 1 & 0 & 0 \\ 0 & 0 & e^{\frac{i\mu t}{4}} & 0 \\ 0 & 0 & 0 & e^{\frac{i\mu t}{4}} \end{pmatrix}, \quad \text{(43)} \]

where $\Lambda = 1 - e^{\frac{i\mu t}{4}}$. The time evolution of the trivial initial state $|\phi_a\rangle$ can be expressed as

\[ |\Psi(t)\rangle = e^{-iht} |\phi_a\rangle = e^{-iU} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{4it\kappa^2}{U} & 1 & 0 & 0 \\ 0 & 0 & e^{\frac{i\mu t}{4}} & 0 \\ 0 & 0 & 0 & e^{\frac{i\mu t}{4}} \end{pmatrix}. \quad \text{(44)} \]

We find that

\[ \langle 2 |\Psi(t)\rangle = -e^{-iU} \frac{4\kappa^2}{U} \left( it + \frac{2}{U} \left( 1 - e^{\frac{i\mu t}{4}} \right) \right), \quad \text{(45)} \]

which can be valued within finite time scale as

\[ \langle 2 |\Psi(t)\rangle \approx -e^{-iU} \frac{4\kappa^2}{U} \left\{ \frac{it}{U^2}, \text{ large } U \right\}, \quad \text{(46)} \]

Note that the switching of powers of the variable $t$ is due to the cancellation of the linear $t$ term in the small $U$ limit. It accords with the above analysis about the order of Jordan block. In large $U$ limit, $U \gg \kappa$, $e^{-iht}$ reduces to a diagonal-block form

\[ e^{-iht} \approx e^{-iU} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{i\mu t}{4}} & 0 \\ 0 & 0 & 0 & e^{\frac{i\mu t}{4}} \end{pmatrix}. \quad \text{(47)} \]

Then in the doublon subspace spanned by $a^\dagger \downarrow a^\dagger \uparrow |\text{Vac}\rangle$ and $b^\dagger \downarrow b^\dagger \uparrow |\text{Vac}\rangle$, the effective Hamiltonian is

\[ H_{\text{eff}}^{\text{con}} = \frac{4\kappa^2}{U} b^\dagger \downarrow b^\dagger \uparrow a^\dagger \downarrow a^\dagger \uparrow + U I. \quad \text{(48)} \]

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