Abstract. The notion of polytopal map between two polytopal complexes is defined. Surprisingly, this definition is quite simple and extends naturally those of simplicial and cubical maps. It is then possible to define an induced chain map between the associated chain complexes. Finally, we use this new tool to give the first combinatorial proof of the splitting necklace theorem of Alon. The paper ends with open questions, such as the existence of Sperner’s lemma for a polytopal complex or the existence of a cubical approximation theorem.

Introduction

At the very beginning of algebraic topology, there is a simple construction that allows to derive from a simplicial map a chain map that carries a formal sum of oriented simplices onto a formal sum of their image. This chain map can then be treated as an algebraic object and the powerful machinery of algebra can be used to derive properties of the chain map, and then of the starting simplicial map itself. Ehrenborg and Hetyei have shown in 1995 [5] that a similar construction is possible for cubical maps, that is the analog of simplicial maps for cubical complexes. A natural question is then to ask whether it is possible to define in a similar way polytopal maps between two polytopal complexes and to define naturally an associated chain map that would keep the essential properties of the polytopal map. To our knowledge, this definition is something new.

The purpose of the present work is to show that it is possible. We give a very natural definition of a polytopal map and we will see that this definition contains both the notions of simplicial and cubical maps, although our definition of a cubical map is then more restrictive than which of Ehrenborg and Hetyei. This point will be discussed. Then we show how to derive a chain map from it. Finally, we use this new framework to give the first purely combinatorial proof of the celebrated splitting necklaces theorem, which states that an open necklace with \( t \) types of beads and a multiple of \( q \) of each type can always be fairly divided between \( q \) thieves using no more than \( t(q - 1) \) cuts. According to Ziegler [13], a combinatorial proof in a topological context is a proof using no simplicial approximation, no homology, no continuous map.

The plan of the paper is the following. In the first section (Section 1), we recall the definition of a polytopal complex (Subsection 1.1), we define the notion of polytopal map (Subsection 1.2), the notion of oriented polytope and the notion of boundary operator for a polytopal complex, inducing the notion of chain complex (Subsection 1.3). It is possible to define the cartesian product of two chains - a thing that is not possible for simplicial complexes (Subsection 1.4). Then we will see how a polytopal map induces a chain map. (Subsection 1.5). In Subsection 1.6, an homotopy equivalence of polytopal maps is proved. In the second section (Section 2), we prove combinatorially the splitting necklace theorem. Actually, we get also a direct proof of the generalization found by Alon, Moshkovitz and Safra [2] when there is not necessarily a multiple of \( q \) beads of each type. In the last section (Section 3), open questions are presented (Sperner’s lemma for a polytopal complex, cubical approximation theorem).

Acknowledgement: The author thanks Robin Chapman for his valuable comments, in particular his comment on a definition of a polytopal map, given in a first version of this paper, that was not a true generalization of the notion of simplicial map.

1. Polytopal and chain maps

1.1. Polytopal complexes. A polytopal complex \( \mathcal{P} \) is a collection of finite sets (called faces) on a vertex set \( V(\mathcal{P}) \) such that

\[ \text{Date: July 2, 2008.} \]
(1) For every $\sigma \in P$ the elements of $\sigma$ can be represented as the vertices of a finite dimensional polytope, where the faces contained in $\sigma$ are exactly the vertex sets of the faces of this polytope.

(2) If $\sigma, \tau \in P$ then $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau \in P$.

For every face $\sigma$ the dimension of the polytope associated to $\sigma$ is called the dimension of $\sigma$ and denote it by $\dim \sigma$. Any face $\tau$ of $\sigma$ such that $\dim \tau = \dim \sigma - 1$ is called a facet of $\sigma$. When all polytopes are simplices, the polytopal complex is called a simplicial complex, and when all polytopes are cubes, it is called a cubical complex. When $P$ and $Q$ are two polytopal complexes, $P \times Q$ is the $(\dim P + \dim Q)$-polytopal complex whose faces are all $\sigma \times \tau$ where $\sigma \in P$ and $\tau \in Q$. When the cartesian product of $P$ is taken $s$ times by itself, we use the notation $P^s$.

Finally, we emphasize the fact that we assume that there is no degeneracy in the sense that the faces of the polytopes are always true faces: two distinct faces of the same polytope have distinct supporting affine subspaces.

1.2. Polytopal maps. We give now the definition of a polytopal map.

Let $P$ and $Q$ be two polytopal complexes. A map $\lambda : V(P) \rightarrow V(Q)$ is a polytopal map if for every $d$-face $\sigma$ of $P$, $\lambda(\sigma)$ is a subset of a $d'$-face of $Q$, with $d' \leq d$.

In Figure 1 a polytopal map in the particular case of cubical complexes is illustrated. Note that the image of a cube is not necessarily a cube.

When $P$ and $Q$ are two simplicial complexes, the classical definition of a simplicial map is the following: a map $\lambda : V(P) \rightarrow V(Q)$ is a simplicial map if for every simplex $\sigma$ of $P$, $\lambda(\sigma)$ is a simplex of $Q$. It is straightforward to check that, in this case, this definition coincides with the one given above.

When $P$ and $Q$ are two cubical complexes, the classical definition of a cubical map is the following (see Fan [6] or Ehrenborg and Hetyei [5] for instance): a map $\lambda : V(P) \rightarrow V(Q)$ is a cubical map if the following conditions are both fulfilled:

(1) for every cube $\sigma$ of $P$, $\lambda(\sigma)$ is a subset of the vertices of a cube of $Q$

(2) $\lambda$ takes adjacent vertices to adjacent vertices or the same vertex.

Our polytopal map when the polytopal complexes are cubical complexes is a cubical map in the sense of Fan or Ehrenborg and Hetyei, but the converse is not true as the following example shows.

In a first version of this work, presented at the TGGT 08 conference in Paris, the author was not conscious of this fact and claimed erroneously that his notion of a polytopal map contains the notion of a cubical map by Fan, Ehrenborg and Hetyei.
Whether it is possible to define a more general notion of a polytopal map which would contain a cubical map in this more general sense is an open question (see Section 3 for a complementary discussion).

1.3. Oriented polytopes and chain complex of polytopal complexes. Let $P$ be a $d$-dimensional polytope. $\epsilon : (\triangle_d^+) \rightarrow \{-1, 0, +1\}$ is an orientation of $P$ if the following subsequent conditions are fulfilled:

1. $\epsilon(v_0, \ldots, v_d) = 0$ if and only if $v_0, \ldots, v_d$ are affinely dependent.
2. $\epsilon(v_0, \ldots, v_d, v_0') = \epsilon(v_0, \ldots, v_d, v_d')$ if $v_i'$ and $v_i''$ are in the same open half-space (of $\mathbb{R}^d$) delimited by the supporting hyperplane of $v_0, \ldots, v_i, \ldots, v_d$, and $\epsilon(v_0, \ldots, v_d, v_0') = -\epsilon(v_0, \ldots, v_d, v_d')$ if not (the hat means a missing element).
3. $\epsilon(v_0, v_1, \ldots, v_d) = \text{sign}(\pi)\epsilon(v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(d)})$ for every permutation $\pi$.

Lemma 1. Every $d$-dimensional polytope admits exactly two distinct orientations. Moreover, if $\epsilon$ is one of them, $-\epsilon$ is the other one.

Proof. One can define such an $\epsilon$ by the sign of a determinant, it is enough to prove that an $\epsilon$ is completely defined by its value on a $d$-simplex. But this is obvious since it is possible to go from a $d$-simplex to any other simplex of the polytope by replacing one vertex after the other, while keeping the affinely independence. $\square$

For every $d$-face $\sigma$ of a polytopal complex $K$, an orientation $\epsilon$ of $\sigma$ is an orientation of the associated polytope. We say then that $(\sigma, \epsilon)$ is an oriented face of $P$. When $\tau$ is a facet of $\sigma$, one defines the induced orientation of an orientation $\epsilon$ of $\sigma$ by

$$\epsilon|_{\tau}(v_0, \ldots, v_{d-1}) := \epsilon(v_0, \ldots, v_{d-1}, v_d)$$

for any $v_1, \ldots, v_{d-1} \in \tau$ and $v_d \in \sigma \setminus \tau$.

One can now define the chain complex of a polytopal complex $P$. The group of formal sums of oriented $d$-faces with coefficients in $\mathbb{Z}$ is denoted by $C_d(P)$, where $-(\sigma, \epsilon) = (\sigma, -\epsilon)$. The boundary operator is defined on an oriented $d$-face $(\sigma, \epsilon)$:

$$\partial(\sigma, \epsilon) := \sum_{\tau \text{ facets of } \sigma} (\tau, \epsilon|_{\tau}).$$

Lemma 2. $\partial \circ \partial = 0$.

Proof. Let $\eta$ be a $(d-2)$-face of $\sigma$, and let $\tau_1$ and $\tau_2$ the two facets of $\sigma$ containing $\eta$. The lemma will be a consequence of the equality $(\epsilon|_{\tau_1})|_{\eta} = -(\epsilon|_{\tau_2})|_{\eta}$.

This equality is obvious: take $v_0, v_1, \ldots, v_{d-2}$ be $d-2$ affinely independent vertices of $\eta$, take moreover $w_1$ (resp. $w_2$) be an independent vertex of $\tau_1$ (resp. $\tau_2$) and $v$ another independent vertex of $\sigma$; then $\epsilon(v_0, \ldots, v_{d-2}, w_1, v) = -\epsilon(v_0, \ldots, v_{d-2}, w_2, v)$. $\square$

Hence, the $C_d(P)$'s provide a chain complex $C(P)$.

1.4. Product of chains. An interesting property of the polytopal complexes is that the cartesian product of two polytopal complexes is still a polytopal complex. It is possible to exploit this property on the level of chains identifying $C_d(K) \otimes C_d(L)$ and $C_{d+d'}(K \times L)$. It will be used in Subsections 1.6 and 2.3.

Remark: This is a combinatorial interpretation of the classical tensor product of chain complexes (see p. 338 of the book by Munkres [12]).

We show how this identification works for two oriented polytopes. The extension for all chains is done through bilinearity. Let $(\sigma, \epsilon)$ and $(\sigma', \epsilon')$ be two oriented polytopes, the first one in $K$ and the second one in $L$. Let $v_0, v_1, \ldots, v_d$ (resp. $v'_0, v'_1, \ldots, v'_d$) be $d + 1$ independent vertices of $\sigma$ (resp. $d' + 1$ independent vertices of $\sigma'$). Note that all pairs $(v_i, v'_i)$ are vertices of the polytope $\sigma \times \sigma'$. Now let $\epsilon \times \epsilon'$ be the orientation of $\sigma \times \sigma'$ such that

$$(\epsilon \times \epsilon')(v_0, v'_0), (v_1, v'_0), \ldots, (v_d, v'_0), (v_0, v'_d), (v_1, v'_d), \ldots, (v_d, v'_d)) := \epsilon(v_0, \ldots, v_d)\epsilon'(v'_0, v'_1, \ldots, v'_d).$$

Lemma 3. $\epsilon \times \epsilon'$ is a well defined orientation of $\sigma \times \sigma'$, in the sense that it is independent of the choice of the independent vertices of $\sigma$ and $\sigma'$.

Sketch of proof. A simple way to see it is through the determinant representation. A simple computation leads to the conclusion. $\square$
Define
\[(\sigma, \epsilon) \otimes (\sigma', \epsilon') := (\sigma \times \sigma', \epsilon \times \epsilon').\]

A useful relation is given by the following lemma.

**Lemma 4.** Let \( c \in C_d(K) \) and \( c' \in C_d(L) \). Then
\[\partial c \otimes c' = (-1)^d \partial c \otimes c' + c \otimes \partial c'.\]

**Sketch of proof.** Let \((\sigma, \epsilon)\) (resp. \((\sigma', \epsilon')\)) be an oriented \(d\)-dimensional polytope of \(K\) (resp. an oriented \(d'\)-dimensional polytope \(L\)). Take a facet \(\tau\) of \(\sigma\) and a facet \(\tau'\) of \(\sigma'\). Compute the orientation induced on \(\tau \times \tau'\) and on \(\sigma \times \tau'\) by \(\epsilon \times \epsilon'\) to conclude. \(\square\)

1.5. **Chain maps.** A polytopal map induces naturally a chain map of the corresponding chain complexes. Recall that a chain map is such that \(\lambda_\# \circ \partial = \partial \circ \lambda_\#\).

**Theorem 1.** Let \(\lambda : V(P) \to V(Q)\) be a polytopal map. Then it is possible to build a chain map \(\lambda_\# : C(P) \to C(Q)\) such that
\[
\begin{align*}
\bullet & \quad \lambda_\#(v, \epsilon) = (\lambda(v), \epsilon') \text{ for every } v \in V(P), \text{ with } \epsilon'(\lambda(v)) = \epsilon(v) \text{ and} \\
\bullet & \quad \text{for every oriented } d\text{-face } (\sigma, \epsilon) \text{ of } P \text{ and any oriented } d'\text{-face } (\sigma', \epsilon') \text{ of } Q \text{ such that } \lambda(\sigma) \subseteq \sigma' \text{ and } d' \leq d, \text{ either } d' < d \text{ and } \lambda_\#(\sigma, \epsilon) = 0, \text{ or } d' = d \text{ and there exists an } \alpha_\sigma \in \mathbb{N} \text{ such that} \\
& \quad \lambda_\#(\sigma, \epsilon) = \alpha_\sigma (\sigma', \epsilon').
\end{align*}
\]

Moreover, this construction is functorial: id induces identity on the level of chain complexes, and \((\lambda \circ \mu)_\# = \lambda_\# \circ \mu_\#\).

**Proof.** Let us postpone the discussion of the functoriality. The proof works by induction on \(d\).

For \(d = 0\), there is nothing to prove.

Suppose \(d \geq 1\) and let \((\sigma, \epsilon)\) be an oriented \(d\)-face of \(P\). Let \(\sigma'\) be a \(d'\)-face of \(Q\) such that \(\lambda(\sigma) \subseteq \sigma'\) and \(d' \leq d\).

A facet \(\tau\) of \(\sigma\) is vanishing if \(\lambda(\tau)\) is included in a face of \(\sigma'\) of dimension at most \(d - 2\). By induction, if \(\tau\) is vanishing, then \(\alpha_\tau = 0\). Note that if \(\tau\) is non-vanishing, then there is a unique facet of \(\sigma'\) containing \(\lambda(\tau)\). Denote by \(f(\tau)\) this facet.

By induction, \(\lambda_\#\) is defined for \(\partial(\sigma, \epsilon)\): one has
\[\lambda_\#(\partial(\sigma, \epsilon)) = \sum_{\tau \text{ non-vanishing facet of } \sigma} \alpha_\tau(f(\tau), \epsilon(\tau)),\]
where \(\epsilon(\tau)\) is a certain orientation of \(f(\tau)\). Choose the \(\epsilon(\tau)\) such that if \(f(\tau_1) = f(\tau_2)\) then \(\epsilon(\tau_1) = \epsilon(\tau_2)\).

Then, regrouping the \(\tau\) having same image, one defines the \(\beta_{\tau'} \in \mathbb{N}\) associated to the facets \(\tau'\) of \(\sigma'\), such that
\[\lambda_\#(\partial(\sigma, \epsilon)) = \sum_{\tau' \text{ facet of } \sigma'} \beta_{\tau'}(\tau', \epsilon(\tau')),\]
where \(\epsilon(\tau')\) is a certain orientation of \(\tau'\).

By induction, one has also
\[(\partial \circ \lambda_\#)(\partial(\sigma, \epsilon)) = 0\]
since \(\partial\) and \(\lambda_\#\) commute when applied to a \((d - 1)\)-chain.

Hence,
\[\partial \sum_{\tau' \text{ facet of } \sigma'} \beta_{\tau'}(\tau', \epsilon(\tau')) = 0,\]
which implies that, for any two facets \(\tau'_1\) and \(\tau'_2\) sharing a common \((d - 2)\)-face, one has \(\beta_{\tau'_1} = \beta_{\tau'_2}\) and the induced orientations of \(\epsilon(\tau'_1)\) and \(\epsilon(\tau'_2)\) on the common \((d - 2)\)-face are opposed. Therefore, all the \(\beta_{\tau'}\) are equal\(^3\) — call this common value \(\alpha_\sigma\) — and the \(\epsilon(\tau')\) are induced by a common orientation \(\epsilon'\) of \(\sigma'\). Then define \(\lambda_\#(\sigma, \epsilon)\) to be \(\alpha_\sigma(\sigma', \epsilon')\).

\(^2\)By convention, a \(-1\)-face is the empty set, in the case when \(d = 1\).

\(^3\)As noted by Robin Chapman, the equality of the \(\beta_{\tau'}\) is also a direct consequence of acyclicity, but we want to keep the combinatorial track.
It remains to check that one has $(\lambda_\# \circ \partial)(\sigma, \epsilon) = (\partial \circ \lambda_\#)(\sigma, \epsilon)$. By definition of $\epsilon'$ and $\alpha_\sigma$ one rewrites Equation (1):

$$(\lambda_\# \circ \partial)(\sigma, \epsilon) = \sum_{\tau' \text{ facet of } \sigma'} \alpha_\sigma(\tau', \epsilon'|_{\tau'})$$

Since $\sum_{\tau' \text{ facet of } \sigma'}(\tau', \epsilon'|_{\tau'})$ is precisely $\partial(\sigma', \epsilon')$, one has

$$(\lambda_\# \circ \partial)(\sigma, \epsilon) = \partial(\lambda_\#(\sigma, \epsilon)).$$

Now, it remains to show that the identity as a polytopal map induces a identity as chain map, but it is straightforward by following the induction scheme above. The same holds for the composition. □

In Figure 2 an example of a polytopal map with $\alpha_\sigma = 2$ is given.

Remark: Note the similarity with the notion of cellular map between CW-complexes: a polytopal map maps a polytope to a polytope of smaller dimension, and the $\alpha_\sigma$ is reminiscent of the degree of a cellular map. Nevertheless, a polytopal map is something purely combinatorial, and the image of a polytope is not necessarily a polytope.

When one works with simplicial maps, the $\lambda_\#$ is necessarily the classical one. Indeed, one can easily prove by induction that the $\alpha_\sigma$ above is equal to $-1$, $0$ or $+1$. The same holds for cubical maps. Moreover, given a polytopal map $\lambda$, the definition of a chain map and Theorem 4 above are enough to define inductively $\lambda_\#(\sigma, \epsilon)$. Start by the vertices, then define $\lambda_\#$ on the edges, then on the 2-faces, and so on...

Finally, one can show that if $\lambda(\sigma) \neq \sigma'$ (with the notation used in the theorem above), then $\lambda_\#(\sigma, \epsilon) = 0$ (but the converse is not necessarily true).

1.6. Homotopy equivalence. An important notion when dealing with induced homology maps is the notion of homotopic maps. Two polytopal maps $\lambda, \mu : K \to L$ are homotopic when there is a path $P_n = v_0 \ldots v_n$ (seen as a 1-dimensional cubical map) and a polytopal map $\phi : K \times P_n \to L$ such that for every $v \in V(K)$ we have $\lambda(v) = \phi(v, v_0)$ and $\mu(v) = \phi(v, v_n)$. If we can take $P_n$ to be a path of length one, i.e. a standard 1-cube $\square_1$, then we call $\lambda$ and $\mu$ elementary homotopic maps.

Take Definition 30, page 285, in the paper by Ehrenborg and Heytei [5] to see that this notion contains the notion of homotopic cubical maps. It contains also the notion of homotopic simplicial maps. In this context, two “elementary homotopic maps” $\lambda$ and $\mu$ are so-called contiguous simplicial maps: for each simplex $v_0, \ldots, v_d$ of $K$, the points

$$(\lambda(v_0), \ldots, \lambda(v_d), \mu(v_0), \ldots, \mu(v_d))$$
span a simplex of \( L \). Starting with \( \lambda \) and substituting progressively the image by \( \lambda \) of the vertices of \( K \) by their image by \( \mu \), we see that two contiguous simplicial maps are homotopic polytopal maps. Actually, the length of this path is at most the chromatic number of the 1-skeleton of \( K \); since, at each step, we can substitute the image of a stable of this graph.

We prove now that homotopic polytopal maps induce homotopic chain maps, that is, there is a morphism \( D: C_i(K) \to C_{i+1}(L) \) such that
\[ D \circ \partial + \partial \circ D = \lambda \# - \mu \# . \]

**Lemma 5.** If the polytopal maps \( \lambda, \mu: K \to L \) are homotopic, then the induced chain maps \( \lambda \#, \mu \# \) are chain homotopic.

**Proof.** By transitivity, it is enough to prove it for elementary homotopic polytopal map. We denote by \( D \) substitute the image of a stable of this graph.

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Lemma 5. If the polytopal maps \( \lambda, \mu: K \to L \) are homotopic, then the induced chain maps \( \lambda \#, \mu \# \) are chain homotopic.

**Proof.** By transitivity, it is enough to prove it for elementary homotopic polytopal map. We denote by \( v_0 \) and \( v_1 \) the two vertices of \( P_1 \), which we identify with the 1-dimensional oriented simplex \([v_0,v_1]\) (standard notation for oriented simplices).

Let \( s : C(K) \to C(K \times P_1) \) defined by \( s(\sigma, \epsilon) = (\sigma, \epsilon) \otimes [v_0,v_1] \).

Using the definition of \( s \), we compute with the help of Lemma 3
\[ (s \circ \partial)(\sigma, \epsilon) = -(\partial(\sigma, \epsilon)) \otimes [v_0,v_1] . \]

On the other hand, one has
\[ (\partial \circ s)(\sigma, \epsilon) = (\partial(\sigma, \epsilon)) \otimes [v_0,v_1] + (\sigma, \epsilon) \otimes v_1 - (\sigma, \epsilon) \otimes v_0 . \]

Hence,
\[ (s \circ \partial + \partial \circ s)(\sigma, \epsilon) = (\sigma, \epsilon) \otimes v_1 - (\sigma, \epsilon) \otimes v_0 . \]

Defining \( D := \varphi \circ s \) satisfies the required property. \( \square \)

### 2. Splitting necklaces

#### 2.1. The necklace theorem

We turn now to the splitting necklace theorem. The first version of this theorem, for two thieves, was proved by Goldberg and West [7], and by Alon and West with a shorter proof [3]. The version with any number \( q \) of thieves (but still a multiple of \( q \) of beads of each type) was proved by Alon [1]. The version proved here is slightly more general and was proved by Alon, Moshkovitz and Safra [2] (they proved first a continuous version, and then proved that a ‘rounding-procedure’ is possible with flows).

Suppose that the necklace has \( n \) beads, each of a certain type \( i \), where \( 1 \leq i \leq t \). Suppose there are \( A_i \) beads of type \( i \), \( 1 \leq i \leq t \), \( \sum_{i=1}^t A_i = n \), where \( A_i \) is not necessarily a multiple of \( q \). A \( q \)-splitting of the necklace is a partition of it into \( q \) parts, each consisting of a finite number of non-overlapping sub-necklaces of beads whose union captures either \( \lfloor A_i/q \rfloor \) or \( \lceil A_i/q \rceil \) beads of type \( i \), for every \( 1 \leq i \leq t \).

**Theorem 2.** Every necklace with \( A_i \) beads of type \( i \), \( 1 \leq i \leq t \), has a \( q \)-splitting requiring at most \( t(q - 1) \) cuts.

By a well-known trick (see [1][3]), it is enough to prove it when \( q = p \) is prime. The necklace is identified with the interval \([0,n]\). The \( k \)th bead occupies uniformly the interval \([k-1,k]\).

#### 2.2. Encoding of the cuts – \( K \) – and of what get the thieves – \( L \)

We define a graph \( S := (V,E) \) with \( 1 + pn \) vertices. This graph is seen as a 1-dimensional polytopal complex. Define first the vertex set:
\[ V := \{(k,r) \in \{1,\ldots,n\}, r \in \{1,\ldots,p\}\} \cup \{o\} . \]

A vertex of \( S \) is either \( o \) – see it as the left origin of the necklace – or a position \( k \) of a cut and a thief \( r \). Second, define the edges:
\[ E := \{(k,r)(k',r') \mid |k-k'| = 1 \text{ and } r = r'\} \cup \{o(1,r) \mid r \in \{1,\ldots,p\}\} . \]

\( S \) is then a collection of \( p \) paths – \( P_1, P_2, \ldots, P_p \) – having each \( n \) edges, and a common vertex \( o \).

Consider the polytopal complex (actually, cubical complex) \( S^{(p-1)+1} \). Each vertex \( v \) of \( S \) is of the form \( (v_1,\ldots,v_{(p-1)+1}) \). Define \( K \) as the subcomplex of \( S^{(p-1)+1} \) such that always one of the \( v_j \)’s is of the form \( (n,r) \) with \( r \in \{1,\ldots,p\} \). A vertex \( v \) of \( K \) encodes a splitting of the necklace with at most \( t(p-1) \) cuts: each \( v_j \) gives a cut and there are at most \( t(p-1) \) such cuts that are different from \( n \); now, consider the
Figure 3. Encoding of a splitting as a vertex of $K$. Here the $v_j$ of the form $(n, r)$ is $v_3$ with $r = \text{Bob}$.

$j$th sub-necklace and take the $v_{j'} = (k', r')$, with the smallest $j'$, whose position $k'$ is on the right of this sub-necklace; the thief $r'$ is the one who gets this sub-necklace.

An example is given in Figure 3. Note that if the first two cuts were exchanged, then the first sub-necklace would get to Bob and the second one still to Alice, the other assignments remaining unchanged.

This encoding was proposed in the paper [11] for the case with two thieves.

Consider now the polytopal complex $L := (\Delta_{p-1})^t$, where $\Delta_{p-1}$ denotes the $(p-1)$-dimensional simplex, whose vertices are $1, \ldots, p$. For each vertex $v$ of $K$, define $\lambda_i(v)$ as the thief who gets the largest amount of beads of type $i$ when one splits the necklace according to $v$ (using the positions of the beads on the necklace as a total order, one can avoid a tie: in case of equality, the thief with the lower position is considered as advantaged). This thief is called the $i$-winner.

2.3. Combinatorial polytopal Dold’s theorem and proof of the necklace theorem. Consider now $\nu$ the cyclic shift $r \mapsto r + 1$ modulo $p$. It induces free actions on $K$. The map

$$\lambda : K \rightarrow L$$

$$v \mapsto (\lambda_1(v), \ldots, \lambda_t(v))$$

is then an equivariant polytopal map, that induces an equivariant chain map $C(K) \rightarrow C(L)$ (one uses here Theorem 1 to derive the construction of this induced chain map).

Lemma 6. $\lambda$ is an equivariant polytopal map.

Proof. The equivariance is straightforward. Let us check that $\lambda$ is polytopal, that is, that the image of a $d$-cube $\sigma$ in $K$ is included in a face of $L$ of dimension at most $d$.

Take a $d$-cube $\sigma$. It is defined by $d$ cuts, each of them selected in one $S$ in the product $S^{t(p-1)+1}$. Among these $d$ cuts, consider those sliding on type 1 beads on $\sigma$. Denote by $d_1$ the number of such cuts. Recall that
among all the vertices of \( \sigma \). Call \( H \) a thief. Consider the hypergraph each cut is "signed", that is, indicates a thief \( r \) (a cut is of the form \((k,r)\), where \( k \) is a position and \( r \) is a thief). Let \( \sigma \) be this thief. Denote by \( m_1 \) the number of its hyperedges. We have then the following equality:

\[
\sum_{F \in H_1} |F| \leq d_1 + m_1. \tag{2}
\]

Call \( W \) the set of thieves that are 1-winner among the vertices of \( \sigma \).

Denote \( a_1 := \left\lfloor \frac{d_1}{p} \right\rfloor \). Take the vertex of \( \sigma \) that gives to the 1-winner the smallest number of beads of type 1 among all the vertices of \( \sigma \). Let \( w \) be this thief.

If \( w \) gets \( a_1 \) beads, all the thieves get this number of beads. Suppose then that \( w \) gets \( a_1 + b \) beads where \( b \geq 1 \). Select in each edge of \( H_1 \) an "head", which is precisely the thief who gets the corresponding bead in the present splitting. Let \( n_x \) be the number of thieves in \( W \) who get \( a_1 + x \) beads of type 1. Now, consider a thief \( r \neq w \) getting also exactly \( a_1 + b \) beads. There is at least one hyperedge in which he is not the head, otherwise one can easily show, using the assumption on \( w \), that this thief cannot be a 1-winner among the vertices of \( \sigma \).

Moreover, for a thief getting \( a_1 + x \) beads, there are at least \( b - x \) hyperedges in which he is not the head, since there exists a vertex of \( \sigma \) giving him at least \( a_1 + b \) beads (he can be a 1-winner).

Combining this two remarks, we get that \( n_0 - 1 + \sum_{x \leq b} (b - x) n_x \leq \sum_{F \in H_1} (|F| - 1) \), which implies according to equation \( 2 \) that \( n_b + \sum_{x < b} n_x \leq d_1 + 1 \). Since \( n_b + \sum_{x < b} n_x \) is the number of vertices in \( W \), i.e. the number of thieves that may be 1-winners, we get that the image of \( \sigma \) by \( \lambda \) in the first copy of \( \Delta_{p-1} \) is a face of \( \Delta_{p-1} \) having at most \( d_1 + 1 \) vertices, that is, being of dimension at most \( d_1 \).

The same holds for each type of beads. Finally, the dimension of the minimal face containing \( \lambda(\sigma) \) is at most \( d_1 + d_2 + \ldots + d_t = d \).

**Remark:** In the present case, \( \lambda \) is a very particular polytopal map, since the image of (the vertex set of) any cube is the vertex set of a polytope (and not simply included in the vertex set of polytope).

**Theorem 3.** Let \( \lambda \) be an equivariant polytopal map \( K \rightarrow L \). Then \( K \) has a \( t(p-1) \)-face \( \sigma \) whose image by \( \lambda \) is the \( t(p-1) \)-face of \( L \).

This theorem can be interpreted as a combinatorial polytopal Dold’s theorem. Dold’s theorem is a theorem that generalizes Borsuk-Ulam theorem for \( \mathbb{Z}_p \) action, the Borsuk-Ulam theorem being the case \( p = 2 \) (see \[4, 9\]).

Once this theorem is proved, we are done. Indeed, one has the following lemma.

**Lemma 7.** If \( K \) has a \( t(p-1) \)-face \( \sigma \) whose image by \( \lambda \) is the \( t(p-1) \)-face of \( L \), then \( \sigma \) has at least one vertex corresponding to a p-splitting.

**Proof.** Consider the hypergraphs \( H_i \), and the numbers \( d_i \) and \( m_i \) as in the proof of Lemma \[6\]. One of the \( d_i \) is smaller than \( p - 1 \) since \( \sum_i d_i = t(p - 1) \). Suppose w.l.o.g. that it is \( d_1 \). Note that according to the assumption on \( \lambda \) and \( \sigma \), each vertex of \( H_1 \) can be a 1-winner for a vertex of \( \sigma \).

Defining similarly \( n_x \) and \( n_b \), as in the proof of Lemma \[6\]. One has \( n_b - 1 + \sum_{x < b} n_x (b - x) \leq p - 1 \). On the other hand, there are \( p \) vertices in \( H_1 \), thus \( n_b + \sum_{x < b} n_x = p \). Combining these two inequalities leads to \( n_b + \sum_{x < b} n_x (b - x) \leq n_b + \sum_{x < b} n_x \), that is \( \sum_{x < b} n_x (b - x) = \sum_{x < b} n_x \), which implies that \( n_x = 0 \) whenever \( x < b - 1 \). Thus, all thieves get \( a_1 \) or \( a_1 + 1 \) beads of type 1. The beads of type 1 are fairly divided between the thieves.

Moreover, we see that \( b = 1 \), and that \( d_1 \) is at least \( n_0 + n_1 - 1 = p - 1 \). It shows that \( d_1 = p - 1 \). Hence, since we can make the same reasoning for all \( d_i \) smaller than \( p - 1 \), we get that all \( d_i \) are equal to \( p - 1 \), and then that the division is fair for each type of beads.

Hence, one has only to prove Theorem \[3\]. If we were allowed to use homology, the proof would be a direct consequence of the Hopf-Lefschetz formula. Moreover, such a prove would work by contradiction. The following proof neither uses homology, nor works by contradiction (it is constructive in a logical sense).

**Proof of Theorem \[3\]** Its proof uses three ingredients:
(i) a sequence of $d$-chains $h_d$ in $C(K)$ such that

$$h_0 := (o, (n, 1), o, ..., o), \quad \partial h_{2l+1} = \sum_{r=1}^{p} \nu^r h_{2l} \quad \text{and} \quad \partial h_{2l+2} = (\nu - \nu^{-1}) h_{2l+1}. $$

(ii) an equivariant chain map $\eta_#: C(L) \to C \left( \mathbb{Z}_p^{*t(p-1))} \right)$, where $*t$ is the join operation.

(iii) a sequence of chain maps $\phi_{l#} : C_d \left( \mathbb{Z}_p^{*t(p-1))} \right) \to \mathbb{Z}_p$ such that $\phi_{0#}$ is equal to 1 for a vertex in the first copy of $\mathbb{Z}_p$ in $\mathbb{Z}_p^{*t(p-1))}$ and 0 elsewhere,

$$\phi_{(2l+1)#} \circ (\nu^{-1} - \nu) = \phi_{(2l+2)#} \circ \partial \quad \text{and} \quad \phi_{(2l+2)#} \circ \left( \sum_{r=1}^{p} \nu^r \right) = \phi_{(2l+1)#} \circ \partial. $$

(i) In each copy of $S$, we have $p$ paths $P_1, ..., P_p$ (defined above in Subsection 2.2). We orient each $P_r$ them from $o$ to its endpoint $(n, r)$. Define, with the notation introduced in Subsection 1.4

\begin{align*}
\tilde{h}_0 &:= o \quad (3) \\
\tilde{h}_{2l+1} &:= \sum_{r=1}^{p} \nu^r \tilde{h}_{2l} \otimes P_l \quad (4) \\
\tilde{h}_{2l+2} &:= (\nu - \nu^{-1}) \tilde{h}_{2l+1} \otimes P_l \quad (5)
\end{align*}

Note that $\tilde{h}_d \in C_d(S^d)$. Then define

$$h_d := \partial(\tilde{h}_d \otimes P_1 \otimes o \otimes ... \otimes o) + \tilde{h}_d \otimes o \otimes o \otimes ... \otimes o. $$

Using the fact that $(\nu - \nu^{-1}) \circ (\sum_{r=0}^{p-1} \nu^r) = 0$, the checking that $(h_d)$ satisfies the required relation is straightforward.

(ii) Define $sd(L)$ to be the barycentric subdivision of $L$. There is a natural chain map $sd_#$ that maps a face $(\sigma, \epsilon)$ of $L$ to the chain induced by $(\sigma, \epsilon)$ on $sd(L)$.

Then, consider the poset $\mathcal{P}$ of the faces of $L$, and let $\Delta(\mathcal{P})$ be the order complex of $\mathcal{P}$, that is the simplicial complex whose vertices are the elements of $\mathcal{P}$ and whose faces are the chains of $\mathcal{P}$. One has the classical isomorphism $sd(L) \simeq \Delta(\mathcal{P})$.

Define a chain map $g_# : C(\Delta(\mathcal{P})) \to C \left( \mathbb{Z}_p^{*t(p-1))} \right)$ by taking in each orbit of $\Delta(\mathcal{P})$ a face $\sigma$ of $L$ (recall that $\mathbb{Z}_p$ acts on $\mathcal{P}$) and by defining $g(\sigma)$ to be any vertex in the $(\dim \sigma + 1)$-copy of $\mathbb{Z}_p$ in $\mathbb{Z}_p^{*t(p-1))}$. The map $g$ is then an equivariant simplicial map.

Finally define $\eta_# := g_# \circ sd_#$.

(iii) This sequence can be constructed using methods presented in [10] [8].

Then, we show by induction that

$$(\phi_{(2l)\#} \circ \eta_\# \circ \lambda_\#) \left( \sum_{r=1}^{p} \nu^r \right) h_{2l} \equiv (-1)^l \mod p \quad \text{and} \quad (\phi_{(2l+1)\#} \circ \eta_\# \circ \lambda_\#) \left( \nu - \nu^{-1} \right) h_{2l+1} \equiv (-1)^{l+1} \mod p, $$

for $l = 0, ..., \lfloor (t(p-1)-1)/2 \rfloor$. Start with $l = 0$. One has $(\phi_{0\#} \circ \eta_\# \circ \lambda_\#)(\sum_{r=1}^{p} \nu^r h_{0}) = \phi_{0\#}(\sum_{r=1}^{p} \nu^r \eta_\#(1, 1, 1, ..., 1))$, where $(1, 1, 1, ..., 1) \in V((\Delta_{p-1})^t)$. Hence, $(\phi_{0\#} \circ \eta_\# \circ \lambda_\#)(\sum_{r=1}^{p} \nu^r h_{0}) = 1$. After that, the formulas above are proved by a straightforward induction.

Now, if $t(p-1) - 1$ is even, we have $(\phi_{(t(p-1)-1)\#} \circ \eta_\# \circ \lambda_\#) \left( \sum_{r=1}^{p} \nu^r \right) h_{t(p-1)-1} \neq 0$. It can be rewritten $(\phi_{(t(p-1)-1)\#} \circ \eta_\# \circ \lambda_\#)(\partial h_{t(p-1)-1}) \neq 0$, or equivalently $(\phi_{(t(p-1)-1)\#} \circ \partial \circ \eta_\#)(\lambda_\#(h_{t(p-1)-1})) \neq 0$, which shows that $\lambda_\#(h_{t(p-1)-1}) \neq 0$ and hence that there is an oriented $t(p-1)$-face $(\sigma, \epsilon)$ of $K$ whose image by $\lambda_\#$ is nonzero. The same holds if $t(p-1) - 1$ is odd.
3. Discussion

3.1. Cubical maps. We have outlined the fact that our definition of a polytopal map, when specialized for cubical complexes, does not lead to the cubical map defined in its full generality (see Subsection 1.2). Whether a more general version of a polytopal map is possible, which would generalize correctly the notion of a cubical map in its full generality is an open question.

An application might be a Sperner lemma for a polytopal complex that generalizes simultaneously the classical Sperner’s lemma and the cubical Sperner’s lemma found by Ky Fan [6].

Theorem 4 ((Classical) Sperner’s lemma). Let $\Delta$ be a $d$-dimensional simplex and let $\lambda : V(\Delta) \to \Delta$ be a labelling such that $\lambda(v)$ is a vertex of the minimal face containing $v$. Then there is a small $d$-simplex whose vertices are one-to-one mapped on the vertices of $\Delta$.

Theorem 5 (Cubical Sperner’s lemma). Let $\Lambda$ be a cubical subdivision of the $d$-dimensional cube $\square_d$ and let $\lambda : V(\Lambda) \to \square_d$ be a labelling such that

1. $\lambda(v)$ is a vertex of the minimal face containing $v$,
2. two adjacent vertices are mapped by $\lambda$ on adjacent vertices or the same vertex.

Then there is a small $d$-cube of $\Lambda$ whose vertices are one-to-one mapped on the vertices of $\square_d$.

3.2. Approximation theorem. A well-known fact about simplicial map is that if there is continuous map from a simplicial complex $K$ to another one $L$, then there is a subdivision $sd^N K$ and a simplicial map $\lambda : sd^N K \to L$ that is a simplicial approximation of $f$.

The author is not aware of a similar property for cubical maps.

The cubical subdivision $csd$ of a cube is the cartesian product of the barycentric subdivision of the 1-dimensional cubes whose product defines the cube.

We make the following conjecture, which seems to be a first step toward a cubical approximation theorem.

Conjecture 1. Let $P$ and $Q$ be two cubes, and let $f : V(P) \to V(Q)$. Then there exists $N \geq 0$ and a cubical map $g : csd^N P \to Q$ such that for each $v \in V(P)$ one has $f(v) = g(v)$ (where $Q$ is identified with its face complex).

As a starting point, let $P := [0, 1]$ and $Q := [0, 1] \times [0, 1]$. Define $f$ a continuous function mapping 0 to (0,0) and 1 to (1,1). One has to subdivide $P$ ($N := 1$).

In the case of simplicial complex, such a situation never arises.

3.3. Cup product. When one defines the cohomology ring of a simplicial complex, one needs to define the cup product of two cochains. Is it possible to get the cup product of two cochains of a polytopal complex without passing through a simplicial subdivision for instance?

3.4. Algorithms. Finally, the classical questions in this topic are:

- is there a constructive proof of the combinatorial polytopal Dold’s theorem (Theorem 3)? Note that from a purely logical point of view, our proof is constructive since we do neither use the choice axiom nor use contradiction. But our proof provides no algorithm.
- is there a constructive proof of the necklace theorem?
- is there a polynomial algorithm that solves the necklace problem?

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Université Paris Est, LVMT, École Nationale des Ponts et Chaussées, 6-8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-la-Vallée cedex 2, France.

E-mail address: frederic.meunier@enpc.fr