Exact solutions of nonlinear delay reaction-diffusion equations with variable coefficients

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Abstract

A modified method of functional constraints is used to construct the exact solutions of nonlinear equations of reaction-diffusion type with delay and which are associated with variable coefficients. This study considers a most generalized form of nonlinear equations of reaction-diffusion type with delay and which are nonlinear and associated with variable coefficients. A novel technique is used in this study to obtain the exact solutions which are new and are of the form of traveling-wave solutions. Arbitrary functions are present in the solutions and they also contain free parameters, which make them suitable for usage in solving certain modeling problems, testing numerical and approximate analytical methods. The results of this study also find applications in obtaining the exact solutions of other forms of partial differential equations which are more complex. Specific examples of nonlinear equations of reaction-diffusion type with delay are given and their exact solutions are presented. Solutions of certain reaction-diffusion equations are also displayed graphically.

Keywords: Reaction-diffusion, Time delay, Exact solutions, Differential equations.

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1. Introduction

The roles which Nonlinear Partial Differential Equations (NPDEs) play are prominent in the description and analysis of real-life processes and phenomena (See e.g., [1,2]). Therefore, it is pivotal to seek for the ways of obtaining exact solutions of NPDEs for a proper and accurate analysis. Several processes and phenomena which occur in sciences and engineering lead to NPDEs as there are several conditions and parameters to be considered in the modeling of such systems. Reaction-Diffusion Equations (RDEs) are members of NPDEs. Reaction-diffusion systems can be described as mathematical models which find applications in diverse physical phenomena. In its simplest form in one spatial dimension, the RDE has the form

\begin{equation}
    u_t = D u_{xx} + G(u),
\end{equation}

where \( u(x,t) \) denotes the unknown function, \( G \) accounts for all local reactions, and \( D \) is a diffusion coefficient (which is a constant) (See e.g., [3]). RDEs are pervading in the mathematical modeling of the systems which occur in biology, chemistry, complex physics phenomena, engineering, and mechanics [4]. RDEs can also represent the chemical reactions and diffusion processes. Basically many real-life processes do not only depend on the present state but also past occurrences. Also, the dynamical systems are constituted by the time delay. The study of nonlinear delay RDEs provides a fundamental tool for quantitative and qualitative analyses of various dynamical systems such as those modeling infections. For RDEs with delay, the kinetic function \( G \) which denotes the chemical reactions rates is a function of both \( u = u(x,t) \) and \( w = u(x,t - \tau) \), which represent the sought after concentration function and delayed argument, respectively. Two special cases which can arise are \( G(u,w) = g(w) \) and \( G(u,w) = g(u) \). A system with local non-equilibrium media is described by \( G(u,w) = g(w) \). These are systems which possess inertial properties and reactions will always begin after a time \( \tau \). \( G(u,w) = g(u) \) represents the classical local equilibrium case (See e.g., [12]).

How to obtain the solutions of RDEs has recently attracted much attention. NPDEs are universal in nature and for finding their solutions, several methods have been employed which include spectral collocation and waveform relaxation [13,14,15], adomian decomposition [16,17], Tan-Cot [18], residual power series [19], perturbation [20,21]. However, there are disadvantages which are commonly associated with those methods. There are conditions which make the universal application of those listed methods and others to be impossible. The objects are different in their geometric shapes. The reaction kinetics and type of fluid flow are erratic. The worthlessness in the presence of singular points is indisputable. Also, efforts have been made on using certain recently proposed numerical and iterative methods, which are widely remarked for their accuracy and effectiveness in obtaining the solutions of high-
dimensional complex geometry nonlinear problems (See e.g., [6, 7, 8, 10, 11]). Obtaining the exact solutions is imperative for proper analysis of the processes which are under consideration (localization, nonuniqueness, blowup regimes, spatial, etc).

Subsequently, the term "exact solution" in relating to NPDEs will refer to where the solution can be expressed in:

(i) terms of elementary functions;
(ii) closed form with definite or indefinite integrals;
(iii) terms of solutions to Ordinary Differential Equations (ODEs) or systems of such equations.

Accepted form for exact solutions also includes the combinations of cases listed above (See e.g., [12, 22, 23]). A most general form of (1.1) is when nonlinear terms are present before the time derivative and the diffusion term. An awkward form occurs when the combination of the kinetic function $G$ and nonlinear terms are also associated with the variable derivative. This present study is motivated by how to construct the exact solutions for a most general form of nonlinear RDEs which are awkward in nature.

Let $G(u,w)$ be an arbitrary function which takes two arguments, $\tau > 0$ be the delay time, while $a(x), b(x), c(x), p(x)$ and $g(x)$ are some functions. This study considers nonlinear delay RDEs of the form

$$
\begin{align*}
[q(x) + c(x)g(u)]u_t &= [a(x)u_x]_t + [p(x) + b(x)G(u,w)]u_w, \\
&= w = u(x, t - \tau).
\end{align*}
$$

This is a most generalized form with the presence of variable coefficients and arbitrary constants. The exact solutions are obtained in the form of generalized traveling-wave solutions. A novel technique is presented for constructing the exact solutions of a most generalized form of nonlinear RDEs with variable coefficients and with delay. The exact solutions for coupled reaction-diffusion nonlinear system with delay are also obtained. The technique which is being presented in this paper can be applied for modeling a wide class of problems, such as diffusion of pollutants and population models due to the presence of arbitrary functions and free parameters. Besides it can be very useful to generate the testing problems for numerical modeling.

The whole paper comprises of five sections. Section 1 contains a brief introduction and literature survey. The rest of the manuscript is organized as follows: How to construct the exact solutions in the form of a generalized traveling-wave equation is presented in Section 2. Some examples are also given in Section 2 to validate the study. Some generalized and required transformations for the application of the proposed technique to more complex partial differential equations are presented in Section 3. Some special cases of and deductions from the main results of this study are discussed in Section 4. The study is substantiated in Section 5 with graphical analysis.

2. Solutions of generalized RDEs with delay

The aim is to find exact solutions of (1.2) which take the form

$$
\begin{align*}
u &= U(y), \quad y = t + \int h(x)dx,
\end{align*}
$$

which is the generalized traveling-wave equation. Substitution of (1.2) into (1.1) yields

$$
\begin{align*}
u = \int \frac{p(x)}{a(x)}dx + C_1, \quad v(x) = \exp \left(\int \frac{p(x)}{a(x)}dx\right)
\end{align*}
$$

where $C_1$ denotes an arbitrary constant.

Remark 2.1. Solving the Riccati equation (2.7) is not necessary for a given $h(x)$. Then the derived generalized traveling-wave equation (2.7), is said to be the exact solutions to certain RDEs with delay of the form (1.2) for which (2.3), (2.4), and (2.5) are true.

Consideration is given to the Riccati ODE (2.7) under two cases, which are degenerate and nondegenerate.

Degenerate case. For $k = 0$, the degeneration of Riccati equation (2.7) has the general solution which is given by

$$
\begin{align*}
u = \frac{\int v(x)q(x)dx + C_1}{a(x)v(x)}, \quad v(x) = \exp \left(\int \frac{p(x)}{a(x)}dx\right),
\end{align*}
$$

where $C_1$ denotes an arbitrary constant.

Example 2.1. Consider the case where $a = 1$ and $p(x) = q(x) = x$. Using (2.3) with $C_1 = 0$, gives $h(x) = x$. Substitution of the value of $h$ into (2.7) yields $y = t + x, b = 1$ and $c = -1$. Consequently, for functions $g(u)$ and $G(u,w)$ which are arbitrary, the nonlinear RDE with delay

$$
\begin{align*}
x - g(u)u_t &= u_x + [x + G(u,w)]u_w, \quad w = u(x, t - \tau),
\end{align*}
$$

is solved by

$$
\begin{align*}
u = U(y), \quad y = t + x,
\end{align*}
$$

where $U(y)$ is determined by the delay ODE

$$
\begin{align*}
u'' + [G(U,W) + g(U)]U_y = 0, \quad W = U(y - \tau),
\end{align*}
$$

which is obtained from (2.6) by setting $k = 0$.
Example 2.2. Consider the case where \( a = q = 1 \) and \( p(x) = \frac{1}{x} \). Using (2.2) with \( C_1 = 0 \), gives \( h(x) = \frac{1}{x^2} \). Substitution of the value of \( h \) into (2.1), (2.3) and (2.4) yields \( y = x + \frac{1}{x^2}, b = \frac{1}{x} \) and \( c = \frac{1}{x^4} \). Consequently, for functions \( g(u) \) and \( G(u, w) \) which are arbitrary, the nonlinear RDE with delay

\[
\left[ 4 - x^2 g(u) \right] u_t = 4 u_{xx} + \frac{4}{x} + 2 x G(u, w) u_x, \quad w = u(x, t - \tau),
\]

is solved by

\[
u = U(y), \quad y = t + \frac{1}{4} x^2,
\]

where \( U(y) \) is determined by (2.9).

Example 2.3. Consider the case where \( a = 1, p = 0 \), and \( q(x) = x \). Using (2.2) with \( C_1 = 0 \), gives \( h(x) = \frac{1}{x^2} \). Substitution of the value of \( h \) into (2.1), (2.3) and (2.4) yields \( y = x + \frac{1}{x^2}, b = \frac{1}{x} \) and \( c = \frac{1}{x^4} \). Consequently, for functions \( g(u) \) and \( G(u, w) \) which are arbitrary, the nonlinear RDE with delay

\[
\left[ 4x - x^4 g(u) \right] u_t = 4 u_{xx} + 2 x^2 G(u, w) u_x, \quad w = u(x, t - \tau),
\]

is solved by

\[
u = U(y), \quad y = t + \frac{1}{6} x^3,
\]

where \( U(y) \) is determined by (2.9).

Example 2.4. Consider the case where \( p = 0 \), and \( a(x) = q(x) = x \). Using (2.2) with \( C_1 = 0 \), gives \( h(x) = \frac{1}{x^2} \). Substitution of the value of \( h \) into (2.1), (2.3) and (2.4) yields \( y = x + \frac{1}{x^2}, b = \frac{1}{x} \) and \( c = \frac{1}{x^4} \). Consequently, for functions \( g(u) \) and \( G(u, w) \) which are arbitrary, the nonlinear RDE with delay

\[
\left[ x - \frac{1}{4} x^2 g(u) \right] u_t = \left[ u_{xx} \right] + \frac{1}{2} x G(u, w) u_x, \quad w = u(x, t - \tau),
\]

is solved by

\[
u = U(y), \quad y = t + \frac{1}{4} x^2,
\]

where \( U(y) \) is determined by (2.9).

Example 2.5. Consider the case where two constant coefficients are given as \( p(x) = 1 \) and \( q(x) = 0 \), while the third coefficient is given arbitrarily as \( a = a(x) \). Using (2.2) with \( C_1 = 0 \), gives \( h(x) = \frac{1}{x^2} \). Substitution of the value of \( h \) into (2.1), (2.3) and (2.4) yields \( y = t + \int \frac{1}{x} dx, \quad b = x \) and \( c = \frac{1}{x^3} \). Consequently, for functions \( g(u) \) and \( G(u, w) \) which are arbitrary, the nonlinear RDE with delay

\[
\left\{ \begin{array}{l}
\frac{\chi^2}{a(x)} g(u) u_t + \left[ a(x) u_{xx} \right] + \left[ 1 + x G(u, w) \right] u_x = 0, \\
w = u(x, t - \tau),
\end{array} \right.
\]

is solved by

\[
u = U(y), \quad y = t + \int \frac{x}{a(x)} dx,
\]

where \( U(y) \) is determined by (2.9). Substitution of the function \( a(x) = x^a \) into (2.10) produces a nonlinear RDE with delay

\[
x^{2-a} g(u) u_t + \left[ x^a u_{xx} \right] + [1 + xG(u, w)] u_x = 0, \quad w = u(x, t - \tau),
\]

where \( n \) represents any number.

Nondegenerate case. When \( k \neq 0 \), let

\[
h = \frac{1}{k} \psi',
\]

where \( \psi \) is referred to [24, 25]. Substituting (2.11) into (2.7) produces

\[
a(x) \psi'' + [p(x) + a'(x)] \psi' - q(x) \psi = 0,
\]

which is simplified to obtain linear ODE of second-order

\[
a(x) \psi'' + [p(x) + a'(x)] \psi' - k(x) \psi = 0.
\]

Interested readers in the exact solutions of (2.12) for varieties of the functions \( a(x), p(x) \) and \( q(x) \), are referred to [24, 25].

Example 2.6. Consider the case where \( a = q = 1 \) and \( p = 0 \). The equation (2.72) has the general solution which is given by

\[
\psi = \begin{cases} 
\Lambda_1 \cos(\lambda x) + \Lambda_2 \sin(\lambda x), & \text{if } k = -\lambda^2 < 0, \\
\Lambda_1 \cosh(\lambda x) + \Lambda_2 \sinh(\lambda x), & \text{if } k = \lambda^2 > 0,
\end{cases}
\]

where \( \Lambda_1 \) and \( \Lambda_2 \) are arbitrary constants. Setting \( \Lambda_1 = 1, \Lambda_2 = 0, \) and \( k = -1 \) \((< 0)\) in (2.13), and using formula (2.11) gives

\[
h(x) = \tan x.
\]

Substituting the function \( h \) into (2.2) and (2.4) gives

\[
b(x) = \tan x, \quad c(x) = -\tan^2 x.
\]

Hence, for arbitrary functions \( g(u) \) and \( G(u, w) \), the nonlinear RDE with delay

\[
\left[ 1 - \tan^2 x g(u) \right] u_t = u_{xx} + \tan x G(u, w) u_x,
\]

is solved by

\[
u = U(y), \quad y = t - \ln \cos x,
\]

where \( U(y) \) is determined by the delay ODE

\[
U'' + [G(U, W) + g(U) + 1] U' = 0, \quad W = U(z - \tau).
\]

Constructing exact solutions when the function \( h(x) \) is given

Table 1 gives the list of other possible ways which do not involve integrating the Riccati equation (2.7), for constructing solutions which are exact to equations of the form (1.2). Among the functions \( h(x), a(x), p(x) \), and \( q(x) \), three functions which include \( h(x) \) are assumed to be given. The unknown function which remains is then derived from (2.7). The whole nonlinear RDE of the form (1.2) is determined by using (2.3) and (2.4).
which are arbitrary, the nonlinear RDE

driving the unknown function which is described in Table 1. Consequently, an elucidation is given for the third way of deriving the unknown function when out of the functions $h(x), a(x), p(x),$ and $q(x),$ three functions which include $h(x)$ are given. Note that $k$ and $C_2$ are arbitrary constants.

| No. | Given functions | Derived function |
|-----|----------------|-----------------|
| 1   | $h = h(x), a = a(x), \ p = p(x).$ | $q(x) = [ah]' + ph + kah^2.$ |
| 2   | $h = h(x), a = a(x), \ q = q(x).$ | $p(x) = \left(q - ah' - kah^2\right)/h - a'.$ |
| 3   | $h = h(x), p = p(x), \ q = q(x).$ | $a(x) = \frac{\mu q h p dx c_2}{p}.$ where $\mu = h \exp(k \int h dx).$ |

Example 2.7. Illustration is given by using the third way of deriving the unknown function which is described in Table 1. The $h = h(x)$ is arbitrary, while $C_2 = 0, q = 0$, and $p = -1.$ Two possible cases are being considered.

(I) Degenerate case $k = 0.$ It is obtained that $a(x) = \int h dx/h.$ Consequently, for functions $g(u)$ and $G(u, w)$ which are arbitrary, the nonlinear RDE with delay

$$c(x)g(u)u_t = [a(x)u_{1t} + [b(x)G(u, w) - 1]u_x], \quad (1.17)$$

where $a(x) = \int h dx/h, b(x) = \int h dx,$ and $c(x) = -h \int h dx,$ is solved by

$$u = U(y) = y = t + \int h dx. \quad (1.18)$$

Here, $U(y)$ in (1.18) is determined by (2.9).

(II) Nondegenerate case $k \neq 0.$ A new function $f = f(x)$ is introduced by setting $f(x) = \exp(k \int h dx).$ Therefore, it is obtained that $a(x) = \int h f dx/h f.$ Consequently, for arbitrary functions $g(u)$ and $G(u, w),$ the equation (1.17), where $a(x) = \int h f dx/h f, b(x) = \int h f dx,$ and $c(x) = -\int h f dx f,$ is solved by (2.18). Here, $U(y)$ is determined by (2.6).

Example 2.8. An elucidation is given for the third way of deriving the unknown function which is described in Table 1. Consider Example 2.7 with $h(x) = 1/x.$

Degenerate case ($k = 0$): For functions $g(u)$ and $G(u, w)$ which are arbitrary, the nonlinear RDE

$$\ln \left(\frac{1}{\sqrt{x}}\right) g(u)u_t = [x \ln x u_{1t}]_x + [\ln x G(u, w) - 1]u_x,$$

is solved by

$$u = U(y), \quad y = t + \ln x,$$

where $U(y)$ is determined by (2.9).

Nondegenerate case ($k \neq 0$): For arbitrary functions $g(u)$ and $G(u, w),$ the equation

$$\frac{1}{kx} g(u)u_t + \left[\frac{x}{k} u_{1x}\right]_x + \left[\frac{1}{k} G(u, w) - 1\right] u_x = 0,$$

is solved by (2.18). Here, $U(y)$ is determined by (2.6).

3. Some transformations and generalization

The following parameters are assumed to be chosen such that the below relations are satisfied:

$$b(x, t) = a(x, t)h(x),$$
$$c(x, t) = -a(x, t)h^2(x),$$
$$q(x, t) = [a(x, t)h(x)]' + p(x, t)h(x) + ka(x, t)h^2(x),$$

where the constant is $k$.

(I) One spatial dimension nonlinear RDEs type with delay

Consider the nonlinear RDEs with delay which is given as

$$[q(x, t) + c(x, t)g(u)] u_t = [a(x, t)u_{1t}]_t + p(x, t)u_x + b(x, t)h(x)G(h, u, w, u_t, h), \quad (3.2)$$

where the arbitrary function $G(h, u, w, \theta)$ takes four arguments, $w = ut(x, t - \tau)$ and the delay in time is $\tau > 0.$ The equation (3.2) has

$$u = U(y), \quad y = t + \int h(x) dx, \quad (3.3)$$

as its general solution. Here, $U(y)$ is determined by the delay ODE

$$U'' + (g(U) - k)U' + G(h, U, W, U'_y) = 0, \quad W = U(z - \tau).$$

This can be easily verified by substituting the functions in (3.3) into (3.2), while taking into account the relations (3.1).

Example 3.1. An illustration is given by substituting the functions in (3.3) into (3.2), while taking into account the relations (3.1).

Example 3.2. An elucidation is given for the third way of deriving the unknown function which is described in Table 1. Consider Example 2.7 with $h(x) = 1/x.$

Degenerate case ($k = 0$): For functions $g(u)$ and $G(u, w)$ which are arbitrary, the nonlinear RDE

$$\ln \left(\frac{1}{\sqrt{x}}\right) g(u)u_t = [x \ln x u_{1t}]_x + [\ln x G(u, w) - 1]u_x,$$

is solved by

$$u = U(y), \quad y = t + \ln x,$$

where $U(y)$ is determined by (2.9).

Nondegenerate case ($k \neq 0$): For arbitrary functions $g(u)$ and $G(u, w),$ the equation

$$\frac{1}{kx} g(u)u_t + \left[\frac{x}{k} u_{1x}\right]_x + \left[\frac{1}{k} G(u, w) - 1\right] u_x = 0,$$

is solved by (2.18). Here, $U(y)$ is determined by (2.6).
Consider the system of nonlinear RDEs with delay which is given as
\[
\begin{align*}
[q(x,t) + c(x,t)g_1(u,v)]u_t &= [a(x,t)u_x]_x + p(x,t)u_t + h(x,t)G_1(h,u,v,w_1,u_x/h,v_{x}/h), \\
[q(x,t) + c(x,t)g_2(u,v)]v_t &= [a(x,t)v_x]_x + p(x,t)v_t + h(x,t)G_2(h,u,v,w_2,u_x/h,v_{x}/h),
\end{align*}
\tag{3.4}
\]
where \(G_1(h,u,v,w_1,\theta_1,\theta_2), G_2(h,u,v,w_2,\theta_1,\theta_2)\) are arbitrary functions which take six arguments, \(w_1 = u(x,t - \tau), w_2 = v(x,t - \tau)\), and the delay in time is \(\tau > 0\). The system of equations (3.4) is solved by
\[
u = U(y), v = V(y), \quad y = t + \int h(x)dx.
\tag{3.5}
\]
Here, \(U)\) and \(V)\) are determined by the coupled delay ODEs
\[
\begin{align*}
U_{xx} + (g_1(U,V) - k)U_x + G_1(h,U,V,W), V_x, V_y) &= 0, \\
V_{xx} + (g_2(U,V) - k)V_x + G_1(h,U,V,W), U_x, U_y) &= 0,
\end{align*}
\]
where \(W_i = U(y) - \tau)\) and \(W_2 = V(y) - \tau\). This can be easily verified by substituting the given functions in (3.5) into (3.4), while taking into account the relations (3.1).

**Example 3.2.** An example is given for the highlight of the degenerate case \(k = 0\). This is done by setting \(a(x,t) = 1, q(x,t) = 2\), and \(p(x,t) = 0\) in the third relation of equation (3.7). It is obtained that \(h(x) = 2x, b(x, t) = 2x\) and \(c(x,t) = -4x^2\). By setting \(G_1(h,u,v,w_1,\theta_1,\theta_2) = h f_1(u,v,w)\theta_1, G_2(h,u,v,w_1,\theta_1,\theta_2) = h f_2(u,v,w)\theta_2\) in (3.7), the coupled nonlinear reaction-diffusion system with delay
\[
\begin{align*}
2 - 4x^2 g(u,v) & u_t = u_{xx} + 2x f_1(u,v,w)u_x, \\
2 - 4x^2 g(u,v) & v_t = v_{xx} + 4x^2 f_2(u,v,w)v_x,
\end{align*}
\]
is solved by
\[
u = U(y), v = V(y), \quad y = t + x^2,
\]
where \(U)\) and \(V)\) are determined by the coupled ODEs with delay
\[
\begin{align*}
U_{xx} + g_1(U,V)U_x + G_1(h,U,V,W), V_x, V_y) &= 0, \\
V_{xx} + g_1(U,V)U_x + G_2(h,U,V,W), U_x, U_y) &= 0.
\end{align*}
\]

4. Some transformations and deductions

Special cases of (1.2) are being discussed in this section.

(I) Consider the case \(c(x) = p(x) = 0\).

The nonlinear delay RDE
\[
q(x)u_t = [a(x)u_x]_x + b(x)G(u,w)u_x, \quad w = u(x,t - \tau),\tag{4.1}
\]
is obtained. Some existing results can be obtained as corollaries to (4.1) (See e.g., [23]).

(II) Consider the case \(c(x) = 0\) and \(G(u,w) = g(u)\).

Nonlinear RDE of the form
\[
q(x)u_t = [a(x)u_x]_x + p(x)u_x + b(x)g(u)\tag{4.2}
\]
is obtained. The equation (4.2) is a corollary to our results and it represents the classical local equilibrium case. This vindicates that our results are extension of some existing results where such cases were considered (See e.g., [23]).

5. Graphical illustration

Examples are given to show the graphical illustration of the solutions of certain RDEs.

**Example 5.1.** Consider the RDE
\[
x^2 u_t = u_{xx} - u_xu_x,\tag{5.1}
\]
The interval is \(0 \leq x \leq 1\), where the time \(t \geq 0\). The initial condition which is the solution at \(t = 0\) is given as
\[
u(x,0) = \sin(\pi x).
\]
Moreover, the Dirichlet boundary conditions are given as
\[
u(0,t) = 0 \quad \text{and} \quad \nu(1,t) = 0.
\]
Figure 1 shows the graphical illustration for (5.1). It displays how \(\nu\) changes with respect to \(x\) at \(t = 2\).

**Example 5.2.** The convection-diffusion equation is given as
\[
u_t = u_{xx} + b(x)u_x,\tag{5.2}
\]
which has the initial condition as $u(0, x) = \frac{1}{1 + (x - 5)^2}$ and the
Dirichlet boundary conditions $u(t, -\infty) = u(t, +\infty) = 0$, where

$$b(x) = 3\bar{u}(x)^2 - 2\bar{u}(x)$$

and $\bar{u}$ is implicitly defined by the relation

$$\frac{1}{\bar{u}} + \log \frac{1 - \bar{u}}{\bar{u}} = x.$$

The function $\bar{u}$ is known as an equilibrium solution of

$$u_t + (u^3 - u^2)_x = u_{xx},$$

where $\bar{u}(\infty) = 1$ and $\bar{u}(0) = 0$ (See e.g [27]). Figure 2 shows
the graphical illustration for (5.2). A peak is observed to occur at $x = 5$
as changes in $u$ with respect to $x$ is being displayed at $t = 2$.

**Example 5.3.** Consider a system of RDEs

$$u_t = u_{xx} + (1 - u - v)u_x,$$
$$v_t = v_{xx} + (1 - u - v)v_x,$$

(5.3)

with the initial conditions

$$u(0, x) = x^2,$$
$$v(0, x) = x(x - 2),$$

and Cauchy boundary conditions which are specified as

$$u_t(t, 0) = 0; u(t, 1) = 1,$$
$$v_t(t, 0) = 0; v(t, 1) = 0.$$

For the system of reaction-diffusion (5.3), the solution for $u$ is
displayed in Figure 3 while the solution for $v$ is displayed in
Figure 4. Observe that while $u$ has a positive slope, the slope
of $v$ is negative.
6. Conclusion

Recently, exact solutions of RDEs and reaction-diffusion systems have attracted great attention. In this paper, exact solutions are presented for a most generalized form of RDE with delay and which are associated with variable coefficients. The presence of arbitrary functions and free parameters in the solutions represents their feasible application in solving certain modeling problems such as diffusion of pollutants and population models, where the population is spatially distributed. It also makes the obtained solutions to be suitable for usage in testing the numerical and approximate analytical methods. The obtained results also find applications in finding the exact solutions of other form of partial differential equations which are more complex. The examples of delay RDEs with their exact solutions are displayed for elucidation. Graphical illustration of the solutions of some specific RDEs are given.

Abbreviations

NPDEs: Nonlinear Partial Differential Equations
RDEs: Reaction-Diffusion Equations

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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