LOG-LEVEL COMPARISON PRINCIPLE FOR SMALL BALL PROBABILITIES

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We prove a new variant of comparison principle for logarithmic $L_2$-small ball probabilities of Gaussian processes. As an application, we obtain logarithmic small ball asymptotics for some well-known processes with smooth covariances.

1 Introduction

The theory of small deviations of random functions is currently in intensive development. In this paper we consider the most explored case of Gaussian processes in $L_2$-norm.

Suppose we have a Gaussian random function $X(x), x \in \Omega$, with zero mean and covariance function $G_X(x,y) = EX(x)X(y)$ for $x,y \in \Omega$. Let $\mu$ be a measure on $\Omega$. Set

$$||X||_\mu = \left( \int_{\Omega} X^2(x) \mu(dx) \right)^{1/2}. $$

(if $\mu$ is the Lebesgue measure, the index $\mu$ will be omitted). The problem is to define the behavior of $P\{|||X||_\mu \leq \varepsilon\}$ as $\varepsilon \to 0$.

The study of small deviation problem was initiated by Sytaya [S] and continued by many authors. The history of the problem in 20th century is described in reviews by Lifshits [L12] and Li and Shao [LS]. Latest results can be found in [L13].

According to the well-known Karhunen-Loève expansion, we have in distribution

$$||X||^2_\mu = \int_{\Omega} X^2(x) \mu(dx) \overset{d}{=} \sum_{n=1}^{\infty} \lambda_n \xi_n^2, \quad (1.1)$$

where $\xi_n, n \in \mathbb{N}$, are independent standard normal r.v.’s, and $\lambda_n > 0, n \in \mathbb{N}, \sum_n \lambda_n < \infty$, are the eigenvalues of the integral equation

$$\lambda f(x) = \int_{\Omega} G_X(x,y)f(y)\mu(dy), \quad x \in \Omega. \quad (1.2)$$

Thus we need to study the asymptotic behavior as $\varepsilon \to 0$ of $P\{\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2\}$. The answer heavily depends on the available information about the eigenvalues $\lambda_n$. We underline that the explicit formulas for these eigenvalues are known only for a limited number of Gaussian processes.

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A quite important contribution to $L_2$-small ball problem was made by Li [Li] who established the so-called comparison theorem for sums (1.1). In a slightly sharpened form, see [GHLT2], it reads as follows.

**Proposition 1.** Let $(\lambda_n)$ and $(\tilde{\lambda}_n)$, $n \in \mathbb{N}$, be two positive summable sequences. If the infinite product $P = \prod_{n=1}^{\infty} (\lambda_n/\tilde{\lambda}_n)$ converges, then, as $r \to 0$,

$$
P \left\{ \sum_{n=1}^{\infty} \tilde{\lambda}_n \xi_n^2 \leq r \right\} \sim P^{\frac{1}{2}} \cdot P \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\}.
$$

(1.3)

Thus, if we know sufficiently sharp asymptotics of $\lambda_n$, we can, in principle, calculate the asymptotics of $P\{||X||_\mu \leq \varepsilon\}$ up to a constant. Such asymptotics of eigenvalues can be obtained if the function $X$ is a one-parameter Gaussian process (i.e. $\Omega$ is an interval) and the covariance $G_X$ is the Green function of a boundary value problem for ordinary differential operator. This approach was developed in [NN1] for the case of ”separated” boundary conditions and in a recent work [Na3] in the general case. Note that if, in addition, the eigenfunctions of the covariance kernel can be expressed in terms of elementary or special functions, the sharp constants in the small ball asymptotics can be calculated explicitly by the complex variable methods [Na1] (see also [GHLT1], [Na3] and [NP]).

In general case, we cannot expect to derive the exact asymptotics. So, we need so-called logarithmic asymptotics, that is the asymptotics of $\ln P\{||X||_\mu \leq \varepsilon\}$ as $\varepsilon \to 0$. It was shown in [NN2] (see also [KNN] and [Na2]) that in some cases this asymptotics is completely determined by the one-term asymptotics of $\lambda_n$. This gives a logarithmic version of the comparison theorem.

However, this statement is not always satisfactory. Indeed, if $\lambda_n$ decrease too fast (for example, exponentially) then it is difficult to calculate even the one-term asymptotics of eigenvalues in terms of covariance. In this paper we present a new variant of the log-level comparison principle based on the asymptotics of the counting function

$$
\mathcal{N}(\lambda) = \# \{n : \lambda_n < \lambda\}.
$$

**Theorem 1.** Let $(\lambda_n)$ and $(\tilde{\lambda}_n)$, $n \in \mathbb{N}$, be two positive summable sequences with counting functions $\mathcal{N}(\lambda)$ and $\mathcal{N}(\lambda)$, respectively. Suppose $\mathcal{N}$ satisfies

$$
\liminf_{x \to 0} \frac{\int_0^x \mathcal{N}(\lambda) d\lambda}{\int_0^x \mathcal{N}(\lambda) d\lambda} > 1 \quad \text{for any} \quad h > 1.
$$

(1.4)

If

$$
\mathcal{N}(\lambda) \sim \mathcal{N}(\lambda), \quad \lambda \to 0,
$$

(1.5)

then, as $r \to 0$,

$$
\ln P \left\{ \sum_{n=1}^{\infty} \tilde{\lambda}_n \xi_n^2 \leq r \right\} \sim \ln P \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\}.
$$

(1.6)

**Remark 1.** For the power-type decreasing of $\lambda_n$ the one-term asymptotics of $\mathcal{N}(\lambda)$ as $\lambda \to 0$ provides the one-term asymptotics of $\lambda_n$. So, Proposition 2.1 [NN2], Theorem 4.2 [KNN] and
Theorem 4.2 \[Na^2\] are particular cases of this statement. But for the super-power decreasing of \(\lambda_n\) the condition (1.5) is weaker than \(\lambda_n \sim \tilde{\lambda}_n\).

**Remark 2.** The assumption (1.4) is satisfied, for example, for \(\mathcal{N}(\lambda)\) regularly varying of order \(p \in (-1, 0)\) at zero, see \[KNN\]. However, if \(\mathcal{N}(\lambda) = \frac{1}{\lambda^\sigma} \), \(\sigma > 1\), then the relation (1.6) fails as it is shown in \[KNN\], Proposition 4.4. Thus, the assumption (1.4) cannot be removed.

The structure of the paper is as follows. In §2 we prove Theorem 1. In §3 we apply it to derive the logarithmic \(L_2\)-small ball asymptotics for the Gaussian processes with smooth covariances. Another example of the process with super-power decreasing of eigenvalues is considered in \[NSH\].

## 2 Proof of Theorem 1

We will use the following result which is a particular case of Theorem 2 from \[Lf1\].

**Proposition 2.** Let \((\lambda_n), n \in \mathbb{N}\), be a positive summable sequence. Define, for \(t, u \geq 0\),

\[
L(t) = \sum_{n=1}^{\infty} \ln f(u\lambda_n).
\]

Then, as \(r \to 0\),

\[
P\left\{ \sum_{n=1}^{\infty} \lambda_n f^2 \leq r \right\} \sim \frac{\exp(L(u) + ur)}{\sqrt{2\pi u^2 L''(u)}},
\]

where \(u = u(r)\) is any function satisfying

\[
\lim_{r \to 0} \frac{L'(u) + r}{\sqrt{L''(u)}} = 0.
\]

We begin by the asymptotic analysis of \(L'(u)\) as \(u \to \infty\). Clearly

\[
L'(u) = -\sum_{n=1}^{\infty} \frac{\lambda_n}{1 + 2u\lambda_n} = \int_{0}^{A} \frac{\lambda d\mathcal{N}(\lambda)}{1 + 2u\lambda}.
\]

(here \(A \geq \lambda_1\); without loss of generality, one can suppose \(A = 1\)). Note that this integral converges since \((\lambda_n)\) is summable.

Integrating by parts twice we obtain

\[
L'(u) = -\int_{0}^{1} \frac{\mathcal{N}(\lambda) d\lambda}{(1 + 2u\lambda)^2} = -\frac{\mathcal{M}(1)}{(1 + 2u)^2} - 4u \int_{0}^{1} \frac{\mathcal{M}(\lambda) d\lambda}{(1 + 2u\lambda)^3}
\]

where \(\mathcal{M}(\lambda) = \int_{0}^{\lambda} \mathcal{N}(t) dt\).
Note that
\[ u|L'(u)| \geq u \int_0^{1/u} \frac{N(\lambda)}{(1 + 2u\lambda)^2} \, d\lambda = \int_0^{1} \frac{N(\frac{1}{u})}{(1 + 2t)^2} \geq \frac{1}{3} N(\frac{1}{u}) \to \infty, \quad u \to \infty, \]
while, given \( a < 1 \),
\[ \frac{\mathcal{M}(1)}{(1 + 2u)^2} + 4u \int_a^1 \frac{\mathcal{M}(\lambda)}{(1 + 2u\lambda)^3} = O(u^{-2}), \quad u \to \infty, \]
and hence
\[ L'(u) = -4u \int_0^a \frac{\mathcal{M}(\lambda)}{(1 + 2u\lambda)^3} \cdot (1 + o(u^{-1})) = -4 \int_0^a \frac{\mathcal{M}(\lambda)}{(1 + 2t)^3} \cdot (1 + o(u^{-1})), \quad u \to \infty. \tag{2.3} \]

Using the Cauchy theorem and (1.4), we conclude that for any given \( h > 1 \) and sufficiently large \( u \)
\[ \frac{L'(u)}{L'(hu)} = \frac{\int_0^u \frac{\mathcal{M}(\lambda)}{(1 + 2u\lambda)^3} \cdot (1 + o(u^{-1}))}{\int_0^u \frac{\mathcal{M}(\lambda)}{(1 + 2u\lambda)^3} \cdot (1 + o(u^{-1}))} = \frac{\mathcal{M}(\frac{\lambda}{u})}{\mathcal{M}(\frac{\lambda}{hu})} \cdot (1 + o(u^{-1})) \geq 1 + \delta(h - 1). \tag{2.4} \]

Let us define the function \( \tilde{L}(u) \) using the sequence \( (\tilde{\lambda}_n) \) instead of \( (\lambda_n) \). Due to (1.5), given \( \varepsilon > 0 \), we can find \( a > 0 \) such that \( |\tilde{\lambda}(t) - \lambda(t)| \leq \varepsilon \) for \( t < a \), and (2.3) gives for sufficiently large \( u \)
\[ \left| \frac{\tilde{L}'(u)}{L'(u)} - 1 \right| = \left| \int_0^a \frac{\mathcal{M}(\lambda)}{(1 + 2u\lambda)^3} \cdot (1 + o(u^{-1})) \right| \leq 2\varepsilon. \tag{2.5} \]

Now we observe that \( L''(u) = 4 \int_0^1 \frac{\lambda N(\lambda)}{(1 + 2u\lambda)^3} \, d\lambda \geq 0 \), and therefore, the equation \( L'(u) + r = 0 \) has for sufficiently small \( r \) the unique solution \( u(r) \) such that \( u(r) \to \infty \) as \( r \to 0 \). In a similar way we define \( \tilde{u}(r) \) as the solution of the equation \( \tilde{L}'(u) + r = 0 \).

Since \( \tilde{L}'(\tilde{u}(r)) \equiv L'(u(r)) \), relations (2.4) and (2.5) imply for arbitrary \( \varepsilon > 0 \) and for sufficiently small \( r \)
\[ \delta \left( \frac{u(r)}{\tilde{u}(r)} - 1 \right) \leq \left| \frac{\tilde{L}'(\tilde{u}(r))}{L'(u(r))} - 1 \right| \leq 2\varepsilon, \]
and, similarly, \( \delta \left( \frac{\tilde{u}(r)}{u(r)} - 1 \right) \leq 2\varepsilon. \) Thus, \( \tilde{u}(r) \sim u(r) \) as \( r \to 0 \).

Since \( u(r) \) trivially satisfies (2.2) we can apply formula (2.2) with \( u = u(r) \). Naturally, the replacement of \( u(r) \) by its one-term asymptotics \( \tilde{u}(r) \) breaks the relation (2.2). Therefore, this replacement does not work to extract the explicit form of exact small ball asymptotics from (2.1). Fortunately, it works for the logarithmic asymptotics. Namely, we have
\[ L(u) = \frac{1}{2} \int_0^1 \ln(1 + 2u\lambda) \, d\mathcal{N}(\lambda) = -u \int_0^1 \frac{\mathcal{N}(\lambda) \, d\lambda}{1 + 2u\lambda}, \]
and formula (2.1) gives, as $r \to 0$ and $u = u(r)$,

$$\ln P \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\} \sim L(u) + ur = L(u) - uL'(u) = -2u^2 \int_0^1 \frac{N(\lambda) \lambda d\lambda}{(1 + 2u\lambda)^2}.$$

Note that

$$|L(u)| \geq 2u^2 \int_0^{1/u} \frac{N(\lambda) \lambda d\lambda}{(1 + 2u\lambda)^2} = 2 \int_0^{1} \frac{N(\lambda \lambda t) dt}{(1 + 2t)^2} \to \infty, \quad u \to \infty,$$

while, given $a < 1$,

$$u^2 \int_a^1 \frac{N(\lambda) \lambda d\lambda}{(1 + 2u\lambda)^2} = O(1), \quad u \to \infty,$$

and hence, using $\bar{u}(r) \sim u(r)$, we obtain

$$\frac{\ln P \left\{ \sum_{n=1}^{\infty} \bar{\lambda}_n \xi_n^2 \leq r \right\}}{\ln P \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\}} \sim \bar{u}^2(r) \int_0^{1} \frac{N(\lambda \lambda t) dt}{(1 + 2u(\lambda \lambda t))^2} \sim \int_0^a \frac{N(\lambda) \lambda d\lambda}{(1 + 2u\lambda)^2}.$$
Remark 3. According to [KNN, Theorem 2.2], under assumptions of Theorem 2 \( \psi \) is also a slowly varying function. Moreover, since \( \varphi(z) \to \infty \) as \( z \to 0 \), we have for arbitrary large \( A \) and \( x > 1/A \)

\[
\psi(x) \geq \int_{x}^{Ax} \frac{\varphi(z) \, dz}{z} = \int_{1}^{A} \varphi(xt) \frac{dt}{t} \sim \varphi(x) \int_{1}^{A} \frac{dt}{t} = \ln(A) \cdot \varphi(x),
\]

which implies

\[
\varphi(x) = o(\psi(x)), \quad x \to 0,
\]

(3.5)

Proof. The relation (3.1) clearly implies

\[
\lim_{x \to 0} \frac{c x}{\int_{0}^{x} \mathcal{N}(t) \, dt} = c \quad \text{for any} \quad c > 0,
\]

and (1.4) follows. By Theorem 1, when deriving the small ball asymptotics the relation (3.2) allows us to put \( \varphi \) instead of \( \mathcal{N} \) in all formulas. So, we obtain

\[
L'(u) \sim -\frac{\varphi(\lambda)}{(1 + 2u \lambda)^2} = -\frac{\varphi(\frac{t}{u})}{u} \cdot \int_{0}^{\infty} \chi_{[0,u]}(t) \cdot \frac{\varphi(\frac{t}{u})}{\varphi'(\frac{1}{u})} \cdot \frac{dt}{(1 + 2t)^2}.
\]

The integrand tends pointwise to \( \frac{1}{(1+2t)^2} \) as \( u \to \infty \). Moreover, for any \( p > 0 \) the function \( t \to t^p \varphi(\frac{t}{u}) \) is increasing in a neighborhood of the origin, and hence

\[
\chi_{[0,u]}(t) \cdot \frac{\varphi(\frac{t}{u})}{\varphi'(\frac{1}{u})} \leq Ct^{-p}.
\]

This provides a summable majorant, and the Lebesgue Theorem gives

\[
L'(u) \sim -\frac{\varphi(\frac{1}{u})}{u} \cdot \int_{0}^{\infty} \frac{dt}{(1 + 2t)^2} = -\frac{\varphi(\frac{1}{u})}{2u}, \quad u \to \infty,
\]

and (3.4) follows.

Further,

\[
L(u) \sim -\int_{0}^{u} \frac{\varphi(\frac{t}{u}) \, dt}{1 + 2t}.
\]

Note that, given \( A > 0 \),

\[
\int_{0}^{A} \frac{\varphi(\frac{t}{u}) \, dt}{1 + 2t} = O(\varphi(1/u)), \quad u \to \infty,
\]
while for \( u > A \)

\[
\int_A^u \frac{\varphi(\frac{1}{u})}{1 + 2t} dt = \int_1^{u/A} \frac{\varphi(\frac{Ax}{u})}{\frac{Ax}{u} + 2x} dx = \frac{1}{2} \int_1^{u/A} \varphi(Ax/u) \frac{dx}{x} \cdot (1 + o_A(1)) = \\
= \frac{\psi(\frac{1}{u})}{2} (1 + o_A(1)) \sim \frac{\psi(\frac{1}{u})}{2} (1 + o_A(1)).
\]

Finally, since \( ur \sim \frac{\varphi(\frac{1}{u})}{2} \) as \( r \to 0 \), the relation (3.5) gives \( L(u) + ur \sim -\frac{\psi(\frac{1}{u})}{2} \), and (3.3) follows.

As an example we consider a set of stationary Gaussian processes \( R_{C,\alpha} (C, \alpha > 0) \), with zero mean-value and the spectral density

\[
K_{R_{C,\alpha}}(\xi) = \exp(-C|\xi|^\alpha), \quad \xi \in \mathbb{R}.
\]

The corresponding covariances \( G_{R_{C,\alpha}} \) are smooth functions. For example, it is well-known that

\[
G_{R_{C,1}}(s,t) = \frac{C}{\pi(C^2 + (s-t)^2)}, \quad G_{R_{C,2}}(s,t) = \frac{1}{2\sqrt{\pi C}} \exp\left(-\frac{(s-t)^2}{4C}\right).
\]

The eigenvalue asymptotics of the integral operators at the finite interval with kernels of this type was treated in remarkable paper [W]. We underline that for \( \alpha < 1 \) Theorem 1 [W] provides the asymptotics of \( \lambda_n \), while for \( \alpha \geq 1 \) Theorems 2 and 3 [W] give only the asymptotics of \( \ln(\lambda_n) \) that enables only to obtain the asymptotics of the counting function. In order to provide a unified approach, we find the asymptotics of \( N(\lambda) \) for all \( \alpha \). Namely, as \( \lambda \to 0 \),

\[
N(\lambda) \sim \varphi(\lambda) \equiv \begin{cases} \\
\frac{1}{\pi C^\frac{\alpha}{2}} \cdot \ln \frac{1}{\lambda}, & \alpha < 1; \\
\frac{1}{\pi C} \cdot \ln \left(\frac{1}{\lambda}\right), & \alpha = 1; \\
\frac{1}{2-2/\alpha} \cdot \frac{\ln \left(\frac{1}{\lambda}\right)}{\ln \left(\frac{1}{\varepsilon}\right)}, & \alpha > 1;
\end{cases}
\]

Here

\[
\mathcal{C} = \frac{K(\text{sech}(\pi/2C))}{K(\text{tanh}(\pi/2C))}
\]

while \( K \) is the complete elliptic integral of the first kind.

Let \( \alpha < 1 \). Then the relation (3.4) reads

\[
r \sim \frac{\ln \frac{1}{\lambda}(u)}{2\pi C^\frac{\alpha+1}{4} u} \iff u \sim \frac{\ln \frac{1}{\lambda}(\frac{1}{\varepsilon})}{2\pi C^\frac{\alpha}{4} r}.
\]

Therefore, (3.3) provides, after substitution \( r = \varepsilon^2 \),

\[
\ln P \{ \| R_{C,\alpha} \| \leq \varepsilon \} \sim -\frac{\alpha \ln \frac{\alpha+1}{\alpha}(u)}{2\pi(\alpha + 1)C^\frac{\alpha}{4}} \sim -\left( \frac{2}{C} \right)^\frac{1}{\alpha} \cdot \frac{\alpha \ln \frac{\alpha+1}{\alpha}(\frac{1}{\varepsilon})}{(\alpha + 1)\pi}, \quad \varepsilon \to 0.
\]

(3.6)
Similarly, for $\alpha = 1$

$$\ln P \{ \|R_{C,\alpha}\| \leq \varepsilon \} \sim -\frac{\ln^2(\frac{1}{\varepsilon})}{\pi \varepsilon}, \quad \varepsilon \to 0.\quad (3.7)$$

For $\alpha > 1$ (3.4) gives

$$r \sim \frac{1}{4 - \frac{2}{\alpha}} \cdot \frac{\ln(u)}{u \ln \ln(u)} \iff u \sim \frac{1}{4 - \frac{2}{\alpha}} \cdot \frac{\ln(\frac{1}{r})}{r \ln \ln(\frac{1}{r})},$$

and we obtain after simple calculation

$$\ln P \{ \|R_{C,\alpha}\| \leq \varepsilon \} \sim -\frac{1}{2 - \frac{2}{\alpha}} \cdot \frac{\ln^2(\frac{1}{\varepsilon})}{\ln \ln(\frac{1}{\varepsilon})}, \quad \varepsilon \to 0.\quad (3.8)$$

Thus, the logarithmic small ball asymptotics in this case does not depend on $C$.

**Remark 3.** The processes $R_{C,\alpha}$ were studied in [AILZ] where the order of decreasing in logarithmic scale was obtained for the small ball probabilities in sup-norm. We note that our approach gives an alternative proof of the key upper estimate in [AILZ] due to a trivial relation $\|X\| \leq \sup |X|$.

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