Towards a covariant canonical formulation for closed topological defects without boundaries

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On the basis of the covariant description of the canonical formalism for quantization, we present the basic elements of the symplectic geometry for a restricted class of topological defects propagating on a curved background spacetime. We discuss the future extensions of the present results.

Running title: Towards a covariant....

I. INTRODUCTION

It is undeniable the current importance of the study of extended objects in physics, mainly in many high energy physics contexts. Such solitonic structures emerge, for example, in the context of cosmological phase transitions in the early universe, and playing a fundamental role at the Planck scale, where (super)string/M-theory drives a revolution without precedent in physics, such as to provide a quantum formulation for gravity. However, a great variety of aspects and questions are still not complete and properly understood, both at classical and quantum level, included the transition between the classical and quantum domains. This happens even for the simplest prototype for a extended object, the Dirac-Nambu-Goto model.

Within the general context of quantization, the canonical formalism is based in a Hamiltonian scheme that traditionally requires an explicit singling out of the time as evolution variable, which implies the spacetime to be topologically a direct product of space and time, as opposed to the general covariance imposed by the relativity theory. However, these difficulties have been overcome by means of a generalized Hamiltonian scheme in which space and time dimensions are treated on equal footing, leading up to a canonical formalism manifestly covariant ([1] and references therein). This covariant formalism has been applied for the analysis
of field theories such as Yang-Mills and General Relativity, originally in Ref. [1] by Witten et al, and more recently, incorporating the adjoint operators formalism, by ourselves in Ref. [2], open superstrings [3], the Wess-Zumino-Witten model [4], two-dimensional gravity models [5], among other ones; however, it has not been applied for the analysis of topological defects of arbitrary dimensions. That is due to several reasons: first, the procedures previously employed for field theories can not be extended directly for the natural geometrical quantities that describe the dynamics of a defect, and the translation turns out to be awkward and obscure. Additionally, within that covariant canonical formalism, one requires precisely of a covariant description of the geometry of deformations of the subject under study, which, in the case of topological defects (unlike field theories), is hardly in current development, and is yet not well understood. Attempts for overcoming such limitations include, for example, the formalism developed by Capovilla and Guven (CG) for the case of closed topological defects without boundaries [6], and subsequently for extended objects with edges [7]. Hence, the purpose of the present study is to employ the CG formalism for developing those elements which allow to establish a covariant quantization scheme described above, for topological defects, particularly those governed by a higher dimensional generalization of the Dirac-Nambu-Goto action (or branes), on a curved spacetime. We shall focus our attention, in this preliminary effort, on closed defects without physical boundaries.

In the covariant canonical formalism that we attempt to apply, the knowledge of the so called covariant space phase is crucial. Therefore, in Section IV, we outline the definition of such space, and the exterior calculus associated with. Although such definitions and concepts come entirely from Ref. [1], they will be suitably adjusted to the treatment of topological defects. In Section V, we incorporate the (self)-adjoint operators formalism, which allows us to obtain a covariantly conserved bilinear form on the phase space, which in turn, will be associated directly with the wanted symplectic form. In Section VI, a symplectic structure with certain invariance properties is explicitly constructed on the phase space. We have no conclusions in this letter, instead we have a lot of open questions for future works, and they are discussed in Section VII. In next section, we summarize the geometry of the embedding of the brane worldsheet in a background spacetime and of their deformations given in Reference [3].

II. GEOMETRY OF THE EMBEDDING AND THEIR DEFORMATIONS

2.1 The embedding
The $D$-dimensional brane dynamics is usually given by an oriented timelike worldsheet $m$ described by the embedding functions $x^\mu = X^\mu(\xi^a)$, $\mu = 0, \ldots, N-1$ and $a = 0, \ldots, D$, in a $N$-dimensional ambient spacetime $M$ endowed with the metric $g_{\mu \nu}$. Such functions specify the coordinates of the brane, and the $\xi^a$ correspond to internal coordinates on the worldsheet.

At each point of $m$, $e_a \equiv X^\mu_\nu \partial_\mu \equiv e^\mu_\nu \partial_\mu$, generate a basis of tangent vectors to $m$; thus, the induced $(D+1)$-dimensional worldsheet metric is given by $\gamma_{ab} = e^a_\mu e^b_\nu g_{\mu \nu} = g(e_a, e_b)$. Furthermore, the $(N-D)$ vector fields $n^i$ normal to $m$, are defined by $g(n^i, n^j) = \delta^{ij}$, $g(n^i, e^a) = 0$. (1)

Tangential indices are raised and lowered by $\gamma_{ab}$ and $\gamma^{ab}$, respectively, whereas normal vielbein indices by $\delta^{ij}$ and $\delta_{ij}$, respectively, and this fact will be used implicitly below. The collection of vectors $\{e_a, n^i\}$, which can be used as a basis for the spacetime vectors, satisfies the generalized Gauss-Weingarten equation:

$$D_a e_b = \gamma_{ab}^c e_c - K_{ab}^i n_i, \quad D_a n_i = K_{ab}^i e_b + \omega_{a ij} n_j,$$

where $D_a \equiv e^\mu_\nu D_\mu$ ($D_\mu$ is the torsionless covariant derivative associated with $g_{\mu \nu}$); thus, the connection coefficients $\gamma_{ab}^c$ compatible with $\gamma_{ab}$ is given by $\gamma_{ab}^c = g(D_a e_b, e^c) = \gamma^c_{ba}$, and the $i$th extrinsic curvature of the worldsheet by $K^i_{ab} = -g(D_a e_b, n^i) = K^i_{ba}$. Similarly the extrinsic twist potential of the worldsheet is defined by $\omega_{a ij} = g(D_a n^i, n^j) = -\omega_{a ji}$. Such a potential allows us to introduce a worldsheet covariant derivative ($\tilde{\nabla}_a$) defined on fields $(\Phi^i_j)$ transforming as tensors under normal frame rotations:

$$\tilde{\nabla}_a \Phi^i_j \equiv \nabla_\mu \Phi^i_j - \omega_{a ik} \Phi^i_k - \omega_{a jk} \Phi^i_k,$$

where $\nabla_\mu$ is the (torsionless) covariant derivative associated with $\gamma_{ab}$.

### 2.2 Deformations of the intrinsic geometry

The physically observable measure of the deformation of the embedding $m$ is given by the orthogonal projection of the infinitesimal spacetime variation $\xi^a \equiv \delta X^a = n^i_\mu \phi^i$, characterized by $N - D$ scalar fields $\phi^i$. Defining the vector field $\delta \equiv n_i \phi^i$, the displacement induced in the tangent basis $\{e_a\}$ along $\delta$ depends on $\phi^i$ and on their first derivatives:

$$D_\delta e_a = \beta_{ab} e^b + J_{ab} n^i,$$

where $D_\delta \equiv \delta^\mu D_\mu$, and

$$\beta_{ab} = g(D_\delta e_a, e_b) = K^i_{ab} \phi^i, \quad J_{ab} = g(D_\delta e_a, n_i) = \tilde{\nabla}_a \phi^i;$$

(4)
similarly, the deformation in the induced metric on \( m \) is given by

\[
D_\delta \gamma_{ab} = 2\beta_{ab} = 2K_{ab}^i \phi^i, \quad D_\delta \gamma^{ab} = -2\beta^{ab},
\]

(5)

For the case treated here, this is sufficient about the deformations of the intrinsic geometry.

### 2.3 Deformations of the extrinsic geometry

Introducing a covariant deformation derivative as \( \tilde{D}_\delta \Psi^i \equiv D_\delta \Psi^i - \gamma_{ij} \Psi^j \), where \( \gamma_{ij} = g(D_\delta n_i, n_j) = -\gamma_{ij} \), the covariant measure of the deformations of the quantities characterizing the extrinsic geometry are given by

\[
D_\delta n_i = -J_{\alpha i} e^a + \gamma_{ij} n^j, \quad \tilde{D}_\delta n_i = -J_{\alpha i} e^a = -\left(\nabla_a \phi^i\right) e^a,
\]

\[
\tilde{D}_\delta K_{ab}^i = -\nabla_a \nabla_b \phi^i + [K_{ac}^i K_{c}^j - g(R(e_a, n_j)e_b, n_i)]\phi^j,
\]

\[
\tilde{D}_\delta \omega_{ij} = D_\delta \omega_{ij} - \nabla_a \gamma_{ij} = -K_{ab}^i \nabla_b \phi^i + K_{ab}^i \nabla_b \phi^j + g(R(n_k, e_a)n_j, n_i)\phi^k,
\]

(6)

which depend on second derivatives of \( \phi^i \); the notation \( g(R(Y_1, Y_2)Y_3, Y_4) = R_{\mu\nu\alpha\beta} Y_\mu^1 Y_\nu^2 Y_\alpha^3 Y_\beta^4 \) is used, where \( R_{\mu\nu\alpha\beta} \) is the Riemann tensor of spacetime. Other useful formulae and more details can be found directly in Ref. [6].

### III. THE DIRAC-NAMBU-GOTO ACTION AND ITS DEFORMATIONS

The simplest phenomenological theory of a topological defect is given by the most simple generally covariant action, proportional the area swept out by the worldsheet:

\[
S = -\sigma \int d^D\xi \sqrt{-\gamma},
\]

(9)

where \( \sigma \) is the brane tension. If we restrict us to closed defects without physical boundaries, the corresponding equations of motion are given by

\[
\Delta X^\mu + \Gamma^\mu_{\alpha\beta}(X^\nu) \gamma^{\alpha\beta} e_\alpha^a e_\beta^b = 0,
\]

(10)

where \( \Delta = \frac{1}{\sqrt{-\gamma}} \partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b) \), and \( \Gamma^\mu_{\alpha\beta}(X^\nu) \) are the spacetime Christoffel symbols evaluated on \( m \). Because all but \( N - D \) linear combinations of these equations are identically satisfied, they are entirely equivalent to
the $N - D$ equations

$$K^i = \gamma^{ab} K_{ab}^i = 0,$$

(11)

which describe extremal surfaces. Furthermore, using Eqs. (5), (7), and (11), one can find the deformation equations of motion in terms of the scalar fields $\phi^i$:

$$-\tilde{\Delta} K^i = -\gamma^{ab} \tilde{\Delta} K_{ab}^i - K_{ab}^i \mathcal{D} \gamma^{ab} = [\tilde{\Delta} \delta_{ij} - (M^2)^{ij}] \phi_j = 0,$$

(12)

where the d’Alembertian $\tilde{\Delta} = \tilde{\nabla}^a \tilde{\nabla}_a$, and the effective mass matrix $(M^2)^{ij} = -K_{ab}^i K_{ab}^j - g(R(e_a, n_i)e^a, n_j)$. Note that the linear operator $E_{ij} \equiv [\Delta \delta_{ij} - (M^2)^{ij}]$ takes vector fields into themselves, and the mass matrix is symmetric, $(M^2)^{ij} = (M^2)^{ji}$. By direct substitution of the relation $\phi^i = n^i \delta X^\mu$ into Eq. (12), it is a straightforward matter to write down the corresponding equations for spacetime variations $\xi^\mu$:

$$n^\mu D^a D_a \xi^\nu + (\tilde{\nabla}^a n^\mu_a) D_a \xi^\mu + [h^{\alpha\beta} \perp^\lambda \mu R^\sigma_{\alpha\lambda\beta} n^\sigma_a - 2\omega^a_{ij} \perp^\mu \nu n^i_a \perp^\mu_j \xi^\nu] = 0;$$

(13)

where we have exploited the projection tensor $h^{\mu\nu} \equiv \gamma^{ab} e^\mu_a e^\nu_b = g^{\mu\nu} - \perp^{\mu\nu}$, with $\perp^{\mu\nu} = n^\mu n^\nu$. The normal projection of previous equation on $n_{\nu i}$, gives finally

$$(E \xi^\mu)_\nu \equiv [\perp_{\mu\nu} D^a D_a + (n_{\nu i} \tilde{\nabla}^a n^i_a) D_a + h^{\alpha\beta} \perp^\lambda \nu R^\sigma_{\sigma\alpha\lambda\beta} - 2\omega^a_{ij} n^i_a \tilde{\nabla}^j a n^j_\mu] \xi^\mu = 0;$$

(similarly the linear operator $E$ is taking vector fields into themselves. As we shall see in Section 5.2, one can to find from both equations (12) and (13), a local continuity equation on the phase space.

IV. COVARIANT PHASE SPACE AND THE EXTERIOR CALCULUS

In according to Witten [1, 3], in a given physical theory, the classical phase space is the space of solutions of the classical equations of motion, which corresponds to a manifestly covariant definition. The basic idea of the covariant description of the canonical formalism is to construct a symplectic structure on such a phase space, instead of choosing $p$’s and $q$’s.

In the present case, the phase space is the space of solutions of Eqs. (10) (or equivalently Eqs. (11)), and we shall call it $Z$. Any background quantity, such as those defined in Section 2.1, will be associated with zero-forms on $Z$. The deformation operator $\delta$ acts as an exterior derivative on $Z$, taking $k$-forms into $(k + 1)$-forms, and it should satisfy

$$\delta^2 = 0,$$

(14)
and the Leibniz rule
\[ \delta(AB) = \delta AB + (-1)^A A\delta B. \]  \hspace{1cm} (15)

In particular, \( \delta X^\mu \) is the exterior derivative of the zero-form \( X^\mu \), and it will be closed:
\[ \delta^2 X^\mu = 0. \]  \hspace{1cm} (16)

Furthermore, since \( \phi^i = n^i \phi^\mu X^\mu \), and \( n^i \) corresponds to a zero-forms on \( Z \), the scalar fields \( \phi^i \) are one-forms on \( Z \), and thus are anticommutating objects: \( \phi^i \phi^j = -\phi^j \phi^i \). This property allows us to verify that, being the vector field \( \delta = n^i \phi^i \), thus \( \delta^2 = n^i n^j \phi^i \phi^j \), which vanishes because of the commutativity of the zero-forms \( n^i \) and the anticommutativity of the \( \phi^i \) on \( Z \), in fully agreement with Eq. (14). It is important to mention, at this point, that the covariant deformation operator \( D_\delta \) (and subsequently \( \tilde{D}_\delta \)) also works as an exterior derivative on \( Z \), in the sense that maps \( k \)-forms into \( (k+1) \)-forms; however \( D_\delta^2 \) does not vanish necessarily.

In this manner, from Eq. (4) we can identify \( \beta_{ab} \) and \( J_{aj} \) as one-forms on \( Z \), and similarly for \( \gamma_{ij} \) in Section 2.3.

Which these preliminary, we can determine certain two-forms on \( Z \) that will be useful for our present proposes. Considering that \( \delta \equiv \delta X^\mu \partial_\mu \), and \( D_\delta \equiv \delta X^\mu D_\mu \), we can show that \( D_\delta (\delta X^\mu) \) vanishes:
\[ D_\delta (\delta X^\mu) = \delta X^\alpha [\partial_\alpha \delta X^\mu + \Gamma^\mu_{\alpha\lambda} \delta X^\lambda] = \delta^2 X^\mu + \Gamma^\mu_{\alpha\lambda} \delta X^\alpha \delta X^\lambda = 0, \]  \hspace{1cm} (17)

where the first term vanishes in according to Eq. (16), and the second one because of the symmetry of \( \Gamma^\mu_{\alpha\lambda} \) in the indices \( \alpha \) and \( \lambda \), and the anticommutativity of \( \delta X^\alpha \) and \( \delta X^\lambda \). Hence, Eq. (17) suggests that \( D_\delta \) is, as well as \( \delta \), a measure of the closeness of \( \delta X^\mu \) on \( Z \). Furthermore, using Eqs. (6), (17), and the Leibniz rule for \( D_\delta \), we find that:
\[ D_\delta \phi^i = D_\delta (n^i \delta X^\mu) = D_\delta n^i \delta X^\mu + n^i \delta D_\delta X^\mu = D_\delta n^i (n^\mu \phi^i) = \gamma^i_j \phi^j, \]
which implies that
\[ \tilde{D}_\delta \phi^i = 0. \]  \hspace{1cm} (18)

Although effectively \( \tilde{D}_\delta^2 \) does not vanish, Eq. (18) suggests that \( \tilde{D}_\delta \) is not only a covariant measure of the deformations, but also the measure of the closeness of \( \phi^i \) on \( Z \). In the CG deformation scheme, the \( \phi^i \) are considered, in a conventional sense, as scalar fields “living on the worldsheet”; in the present scheme such fields can be considered as closed (in the sense of Eq. (18)) one-forms, “living on the corresponding phase space \( Z \)”. The property (18) of the \( \phi^i \) will be essential for our present purposes.
V. SELF-ADJOINT OPERATORS AND A COVARIANTLY CONSERVED CURRENT ON $Z$

In this Section we shall construct, using the concept of self-adjoint operators, a worldsheet covariantly conserved two-form on $Z$.

5.1. Adjoint operators and local continuity laws

The general relationship between adjoint operators and covariantly conserved currents has been already given in previous works (see for example [2] and references cited therein), however we shall discuss it in this section for completeness.

If $P$ is a linear partial differential operator which takes matrix-valued tensor fields into themselves, then, the adjoint operator of $P$, is that operator $P^\dagger$, such that

$$\text{Tr}\{f^{\rho\sigma\ldots}[P(g_{\mu\nu\ldots})]_{\rho\sigma\ldots} - [P^\dagger(f^{\rho\sigma\ldots})]^{\mu\nu\ldots}g_{\mu\nu\ldots}\} = \nabla^\mu J^\mu,$$

(19)

where $\text{Tr}$ denotes the trace and $J^\mu$ is some vector field. From this definition, if $Q$ and $R$ are any two linear operators, one easily finds the following properties:

$$(QR)^\dagger = R^\dagger Q^\dagger,$$

$$(Q + R)^\dagger = Q^\dagger + R^\dagger,$$

and in the case of a function $F$,

$$F^\dagger = F,$$

which will be used implicitly below.

From Eq. (19) we can see that this definition automatically guarantees that, if $P$ is a self-adjoint operator ($P^\dagger = P$), and the fields $f$ and $g$ correspond to a pair of solutions admitted by the linear system $P(f) = 0 = P(g)$, then we obtain the continuity law $\nabla_{\mu} J^\mu = 0$, which establishes that $J^\mu$ is a covariantly conserved current, bilinear on the fields $f$ and $g$. This fact means that for any self-adjoint homogeneous equation system, one can always construct a conserved current. Although this result has been established assuming only tensor fields and the presence of a single equation, such a result can be extended in a direct way to equations involving spinor fields, matrix fields, and the presence of more than one field.

In the present work, the indices appearing in Eq. (19), can correspond to spacetime, internal, and/or normal indices. Furthermore, since the fields $f$ and $g$ will be identified with deformations (which correspond
5.2. Self-adjointness of the operators governing the deformations

In this section we shall show that the operators $E^{ij}$ and $E$ in Eqs. (12) and (13) are indeed self-adjoint.

The case of $E^{ij}$ is relatively simple: let $\phi_i^1$ and $\phi_i^2$ be two scalar fields (which will be identified as a pair of solutions of Eqs. (12), we mean a pair of one-forms on $Z$), then it is very easy to prove the following identity,

$$
\phi_{1i} \Delta \phi_{2i} = \phi_{1i} \nabla^a \phi_{2i} - (\nabla^a \phi_{1i}) \phi_{2i},
$$

which implies that

$$
\phi_{1i} (\Delta \delta^{ij} - (M^2)^{ij}) \phi_{2j} = [\Delta \delta^{ij} - (M^2)^{ij}] \phi_{2j} + \nabla_a j^a,
$$

where we have considered the symmetry of the mass matrix; such a mass term does not contribute explicitly to $j^a$ (although it does implicitly). Eq. (20) shows that, in according to Eq. (19), $E^{ij}$ is self-adjoint. In this manner, if $\phi_{1i}$ and $\phi_{2i}$ corresponds to a pair of solutions of Eqs. (12), $\nabla_a j^a = 0$, which implies that, at each brane worldsheet point, $j^a$ is a covariantly conserved two-form on $Z$. This bilinear product is evidently antisymmetric in $\phi_{1i}$ and $\phi_{2i}$; thus, we can set $\phi_{1i} = \phi_{2i} = \phi$, without loosing generality, and write simply

$$
j^a = \phi \nabla^a \phi^i.
$$

On the other hand, for demonstrating the self-adjointness of the operator $\xi$ governing the deformations in Eq. (13), we need the following identities which are very easy of verifying. Let $\xi_i^\nu$ and $\xi_i^\mu$ be two spacetime vector fields, then:

$$
\xi_i^\nu \perp_{\mu \nu} D^a D_a \xi_2^\mu \equiv \nabla_a (\perp_{\mu \nu} \xi_i^\nu) D^a \xi_2^\mu - D^a (\perp_{\mu \nu} \xi_i^\nu) \xi_2^\mu + [D^a D_a (\perp_{\mu \nu} \xi_i^\nu)] \xi_2^\mu;
$$

$$
\xi_i^\nu n_{\mu i} \nabla^a n^\mu_{\nu j} D_a \xi_2^\mu \equiv \nabla_a [(n_{\nu i} \nabla^a n^\mu_{\nu j}) \xi_i^\nu] - D_a [(n_{\nu i} \nabla^a n^\mu_{\nu j}) \xi_i^\nu] \xi_2^\mu;
$$

furthermore, using the definition $\perp_{\mu \nu} \equiv n_{\mu i} n^i_{\nu}$, it is straightforward to show the following background identities:

$$
D^a \perp_{\mu \nu} = n_{\nu i} \nabla^a n^i_{\mu} + n_{\mu i} \nabla^a n^i_{\nu},
$$

$$
D^a D_a \perp_{\mu \nu} - 2D_a (n_{\nu i} \nabla^a n^i_{\mu}) = 2\omega^{i\mu} (n_{\nu i} \nabla_a n^i_{\nu j} - n_{\mu i} \nabla_a n^i_{\nu j}).
$$
In this manner, using Eqs. (22) and (23), and the explicit form of the operator \( E \) in Eq. (13), one finds that

\[
\xi_1^\nu (E \xi_2^\mu) = \left\{ \perp_{\mu\nu} D^a D_a \xi_1^\nu + 2(n_{\mu i} \nabla^a n_{\nu i}) \nabla_a \xi_1^\nu + [-2\omega^{aij} n_{\mu i} \nabla_a n_{\nu j} + h^{\alpha\beta} \perp_{\alpha\beta} \perp_{\mu\nu} \nabla^a n_{\mu i} + (n_{\mu i} \nabla^a n_{\nu i}) \nabla_a \xi_1^\nu + \nabla_a j^a] \right\} \xi_2^\mu
\]

which corresponds to Eq. (19) with \( E^\dagger = E \), and

\[
\tilde{j}^a \equiv \perp_{\mu\nu} \nabla^a \xi_1^\nu + (n_{\mu i} \nabla^a n_{\nu i}) \xi_1^\nu - [\perp_{\mu\nu} (\nabla^a \xi_1^\nu) \xi_2^\mu + (n_{\mu i} \nabla^a n_{\nu i}) \xi_1^\nu \xi_2^\nu].
\]

Similarly, if \( \xi_1^\nu \) and \( \xi_2^\mu \) corresponds to a pair of solutions admitted by Eqs. (13), \( (E \xi_1^\nu)_\mu = 0 = (E \xi_2^\mu)_\nu \), then from Eq. (24), we obtain the local continuity equation: \( \nabla_a j^a = 0 \). \( j^a \) in Eq. (25) is antisymmetric in \( \xi_1^\nu \) and \( \xi_2^\mu \); thus, setting \( \xi_1^\nu = \xi_2^\nu \), \( j^a \equiv \perp_{\mu\nu} \nabla^a \xi_1^\nu + (n_{\mu i} \nabla^a n_{\nu i}) \xi_1^\nu \), which has a remarkable simplification in terms of the fields \( \phi^i = n_{\mu i} \xi_1^\nu \):

\[
\tilde{j}^a \equiv \phi^i [n_{\mu i} \nabla^a \xi_1^\nu + (n_{\mu i} \nabla^a n_{\nu i}) \xi_1^\nu] = \phi^i [n_{\mu i} \nabla^a \xi_1^\nu - \omega^{aij} \phi^j] = \phi^i [\nabla^a (n_{\mu i} \xi_1^\nu) - \omega^{aij} \phi^j] = \phi^i \nabla^a \phi^i,
\]

in fully agreement with Eq. (21). Note that \( j^a \), with support confined to the brane worldsheet, transforms as a scalar under normal frame rotations. In next Section we shall discuss about the closedness of this bilinear form on \( Z \).

VI. THE SYMPLECTIC STRUCTURE ON \( Z \)

Strictly speaking, a (non-degenerate) closed two-form on \( Z \) is called a symplectic structure; the closedness holds if the exterior derivative of such a two-form vanishes. This property is equivalent to that Poisson brackets satisfy, in the usual Hamiltonian scheme, the Jacobi identity.

Hence, let us determine the exterior derivative of the two-form \( j^a \) previously constructed in Eq. (21). We calculate first \( \tilde{D}_b \nabla_b \phi^i \); for which we can follow the calculation for obtaining Eq. (4.15) in Ref. [4], and to consider, of course, that in the present approach \( \tilde{D}_b \) acts as an exterior derivative on \( Z \), and, in particular, obeys the Leibniz rule (15):

\[
\tilde{D}_b \nabla_b \phi^i = D_b [D_b \phi^i - \omega_b^{ij} \phi^j] - \gamma^{ij} \nabla_b \phi^j
\]
\[ D_b D_b \phi^i - (D_b \omega_b^i) - \omega_b^i D_b \phi_j - \nabla_b (\gamma^{ij} \phi_j) + (\nabla_b \gamma^{ij}) \phi_j \]

\[ = \nabla_b D_b \phi^i - (D_b \omega_b^i - \nabla_b \gamma^{ij}) \phi_j, \]

the first term vanishes in according to Eq. (18), whereas the second term can be rewritten using Eq. (8),

\[ \tilde{D}_b \nabla_b \phi^i = [K_{bc} J^j - K_{bc} \gamma^{ci} - g(R(n_k, e_b)n_j, n_i) \phi^i] \phi_j \]

\[ = -K_{bc} J^c + \beta_{bc} J^c - g(R(n_k, e_b)n_j, n_i) \phi^i \phi_j, \]

\[ \text{Eq. (26)} \]

in the last line, we have considered that, in according to Eq. (21), \( j^a = \phi_i J^a_i = -J^{ai} \phi_i \) (\( J^{ai} \) and \( \phi_i \) are one-forms), and \( K_{bc} \phi_j = \beta_{bc} \) (see first of Eq. (4)). Now, it is very easy to find that:

\[ D_b j_b = \tilde{D}_b j_b = \tilde{D}_b (\phi^i \nabla_b \phi_i) = (\tilde{D}_b \phi^i) \nabla_b \phi_i - \phi^i \tilde{D}_b \nabla_b \phi_i, \]

where we have used again the Leibniz rule, and one more time the first term vanishes according to Eq. (18), and using Eq. (26) the second term reduces to

\[ D_b j_b = 2 \beta_{bc} J^c + g(R(n_k, e_b)n_j, n_i) \phi^i \phi^j \phi^k. \]

Thus, considering Eq. (27), and the second of Eq. (5), one finds that

\[ \delta j^a = D_b j^a = D_b (\gamma^{ab} j_b) = (D_b \gamma^{ab}) j_b + \gamma^{ab} D_b j_b \]

\[ = e^{\alpha \nu} R_{\mu \nu \alpha \beta} n^a_k n^a_j n^a_i \phi^k \phi^j \phi^i \]

\[ = e^{\alpha \nu} R_{\mu \nu \alpha \beta} \xi^\alpha \xi^\nu \xi^\beta. \]

\[ \text{Eq. (28)} \]

However, the projection of the spacetime Riemann tensor on the right-hand side of Eq. (28) vanishes, since considering the definitions \( 2D_{[\mu D_{\nu]} A_{\alpha} = R_{\mu \nu \alpha} A_{\rho} \) and \( D_b \equiv \xi^a D_a \), it can be rewritten as

\[ R_{\mu \nu \alpha \beta} \xi^\alpha \xi^\nu \xi^\beta = -2 \xi^\alpha D_b D_{\alpha} \xi_\nu = 2D_b (D_b \xi_\nu) = 0, \]

\[ \text{Eq. (29)} \]

where Eqs. (15) and (17) has been considered. Therefore, the symplectic current is closed on \( \Sigma \),

\[ \delta j^a = 0. \]

\[ \text{Eq. (30)} \]

Since we have explicitly assumed that the worldsheet is orientable, we can define integration in a direct way, and to construct the following two-form in terms of \( j^a \), which will correspond finally to the wanted symplectic structure:

\[ \omega \equiv \int_{\Sigma} \sqrt{-\gamma} j^a d\Sigma_a, \]

\[ \text{Eq. (31)} \]
where Σ is a spacelike section of \( m \), and corresponds to an initial value (hyper)surface for the configuration of the defect. Since \( j^a \) is covariantly conserved, \( \omega \) in Eq. (31) is independent of the choice of \( \Sigma \). On the other hand, since \( \delta \sqrt{-\gamma} = 0 \) (from this condition the equations of motion for the defect are obtained from Eq. (9)), the closeness of \( \omega \) holds if \( j^a \) itself is closed, and then, from Eqs. (30), and (31), \( \delta \omega = 0 \). Hence, \( \omega \) is the wanted symplectic structure on \( Z \).

It remains to discuss gauge invariance of \( \omega \). Since all fields appearing in the definition of \( \omega \) transform homogeneously, like tensors, \( \omega \) is invariant under spacetime diffeomorphisms. Similarly, since \( \omega \) involves integration of a worldsheet density, \( \omega \) is also invariant under worldsheet reparametrizations. With respect to normal frame rotations, \( \phi_i \) and \( \hat{\nabla}^a \phi^i \) transform homogeneously, like vectors, and then \( j^a \) and \( \omega \), transforming as scalars, are invariant under such rotations.

**VII. DISCUSSIONS AND OPEN QUESTIONS**

In this manner, we have obtained a symplectic structure on \( Z \) for defects propagating on a curved background spacetime, and it emerges in a natural way and without additional assumptions. However, as pointed out in Refs. [6, 7, 8], the CG scheme employed in the present treatment fails to treat the topological defect as a source for gravity, which should be considered in a more complete description of deformations, and in the construction of the corresponding symplectic structure. Within this complete scheme, the deformations of the ambient spacetime itself will be taken into account, and it is possible that similar results can be obtained in such situation. We hope to address this subject in forthcoming works.

As a by-product, the worldsheet conserved current \( j^a \) obtained in Section 5.2, and used for generating a symplectic structure on \( Z \) in the present approach, can be considered, in a more ordinary sense, as a Noetherian current that allows us to obtain conserved currents associated with any continuous symmetries of the background.

It is opportune to comment on the role that the deformation gauge connection \( \gamma^{ij} \) plays on the phase space \( Z \). Such as in the CG scheme, in the present approach it never appears explicitly in any relevant physical quantity on \( Z \), such as \( j^a \) and \( \omega \), but it shows up in intermediate calculations, for example in Eqs. (18) and the first one of the previous section. Therefore, the connection neither needs to be calculated explicitly on \( Z \).

Finally, in spite of we have considered the simplest theory for an extended topological defect, the present results show that the symplectic geometry of the phase space possesses a rich underlying structure, deserving
a more wide investigation. In fact, using a different approach for the geometry of deformations of defects given by Carter, we have obtained results analogous to those presented here \cite{9}, and questions such as the existence of a symplectic potential, degenerate directions of the (pre-)symplectic structure, are explicitly discussed \cite{10}. These results will provide a very powerful geometrical approach for the quantization of such objects, and we hope to extend our achievements elsewhere.

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References

[1] C. Crncovic and E. Witten, in Three Hundred Years of Gravitation, edited by S. W. Hawking and W. Israel (Cambridge University Press. Cambridge, 1987); A. Ashtekar, L. Bombelli, O. Reula in Mechanics, Analysis and Geometry: 200 Years after Lagrange, Francaviglia Ed., Elsevier Science Publisher (1991); Lee J., R. M. Wald, J. Math. Phys. 31, 725 (1990).
[2] R. Cartas-Fuentevilla J. Math. Phys. 43, 644 (2002).
[3] E. Witten, Nucl. Phys. B276, 291 (1986).
[4] M. Chu, P. Goddard, I. Halliday, D. Olive, and Schwimmer, Phys. Lett. B 266, 71 (1991).
[5] K. S. Soh, Phys. Rev. D 49, 1906 (1994).
[6] R. Capovilla and J. Guven, Phys. Rev. D 51, 6736 (1995).
[7] R. Capovilla and J. Guven, Phys. Rev. D 57, 5158 (1998).
[8] J. Guven, Phys. Rev. D 48, 5562 (1993).
[9] R. Cartas-Fuentevilla, "Identically closed two-form for phase space quantization of Dirac-Nambu-Goto p-branes in a curved spacetime", to be published, Phys. Lett. B (2002).
[10] R. Cartas-Fuentevilla, "Global symplectic potentials on the Witten covariant phase space for bosonic extendons", submitted to Phys. Lett. B. (2002).