When is $Y_{\text{obs}}$ missing and $Y_{\text{mis}}$ observed?

JC Galati

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Abstract

In statistical modelling of incomplete data, missingness is encoded as a relation between datasets $Y$ and missingness patterns $R$. The partitioning of $Y$ into observed and missing components is often denoted $Y_{\text{obs}}$ and $Y_{\text{mis}}$. We point out a mathematical defect in this notation which results from two different mathematical relationships between $Y$ and $R$ not being distinguished, $(Y_{\text{obs}}, Y_{\text{mis}}, R)$ in which $Y_{\text{obs}}$ values are always observed, and $Y_{\text{mis}}$ values are always missing, and the overlaying of a missingness pattern onto the marginal distribution for $Y$, denoted $(Y_{\text{obs}}, Y_{\text{mis}})$. With the latter, $Y_{\text{obs}}$ and $Y_{\text{mis}}$ each denote mixtures of observable and unobservable data. This overlaying of the missingness pattern onto $Y$ creates a link between the mathematics and the meta-mathematics which violates the stochastic relationship encoded in $(Y, R)$. Additionally, in the theory there is also a need to compare partitions of $Y$ according to different missingness patterns simultaneously. A simple remedy for these problems is to use four symbols instead of two, and to make the dependence on the missingness pattern explicit. We explain these and related issues.

Key words and phrases: incomplete data, missing data, ignorable, ignorability, missing at random, multiple imputation.

1 Introduction

The modern framework for statistical modelling of incomplete data was introduced by Rubin (1976). Alongside the vector of data random variables, $Y$, a vector of response binary random variables $R$ was introduced, and conditions were given under which inferences could be based on the marginal density $f(y)$ alone. Note that $U$ and $M$ were used in Rubin (1976) to denote what we have called $Y$ and $R$, respectively.

Intrinsic to this approach is the partitioning of a realisation $y$ of $Y$ into values that are observed and values that are missing according to some missingness pattern $r$. In Rubin (1976) the subscripts ‘(1)’ and ‘(0)’ were introduced to denote this partition. These were replaced with the subscripts in $Y_{\text{obs}}$ and $Y_{\text{mis}}$ in Rubin (1987) and $Y_{\text{obs}}$ and $Y_{\text{mis}}$ in Little and Rubin (1987) and Schafer (1997). Over three decades the latter notation has become a de facto standard in the exposition of statistical methods for incomplete data typically aimed at practicing statisticians and other investigators, so it is important for there to be a clear understanding of what it means.

2 There are two different relationships between $Y$ and $R$

The following is an extract from Little and Rubin (1987 pp 89–90); also see Little and Rubin (2002, pp 118-119):

“Here to keep the notation simple we will be somewhat imprecise in our treatment of these complications. ...
The actual observed data consists of the values of the variables \((Y_{\text{obs}}, R)\). The distribution of the observed data is obtained by integrating \(Y_{\text{mis}}\) out of the joint density of \(Y = (Y_{\text{obs}}, Y_{\text{mis}})\) and \(R\). That is,

\[
f(Y_{\text{obs}}, R|\theta, \psi) = \int f(Y_{\text{obs}}, Y_{\text{mis}}|\theta) f(R|Y_{\text{obs}}, Y_{\text{mis}}, \psi) \, dY_{\text{mis}}. \tag{5.11}
\]

In the extract above, the authors stated that their intention was to keep the notation simple. But setting \(Y = (Y_{\text{obs}}, Y_{\text{mis}})\) encodes missingness into the notation as attributes of the data vector \(Y\) instead of as the vector \((Y_{\text{obs}}, Y_{\text{mis}}, R)\) in the mathematical relation \((Y, R)\). This, in fact, significantly complicates rather than simplifies the notation, particularly in regard to the domain of the marginal density, \(f(Y|\theta)\), for \(Y\). In the product of functions \(f(-|\theta)f(-|-, \psi)\) on the right hand side of (5.11), the factor \(f(-|\theta)\) is shorthand for the composition of functions \(f \circ \pi_Y\), where \(\pi_Y\) is the projection sending a realisation \((\mathbf{y}, \mathbf{r})\) of \((Y, R)\) to the realisation \(\mathbf{y}\) of \(Y\). When unpacked this way, the notation in (5.11) specifies that

\[
(Y_{\text{obs}}, Y_{\text{mis}}) = \pi_Y(Y_{\text{obs}}, Y_{\text{mis}}, R). \tag{1}
\]

This is not straightforward to interpret because the formal mathematical relation of missingness exists in the domain of \(\pi_Y\), and this missingness relation is not preserved by the projection \(\pi_Y\).

Note that the failure of \(\pi_Y\) to preserve the missingness relation is not simply because \(\pi_Y\) is a many-to-one function. Even if the domain of \(\pi_Y\) is restricted to include only pairs pertaining to a specific missingness pattern \(\mathbf{r}\), the two missingness relationships still differ. The formal definition of ‘observed’ and ‘missing’ encoded in \((Y, R)\) on the right hand side of (1) is an absolute concept: every data item in the range of \((Y, R)\) is stamped irrevocably either as ‘observed’ or ‘missing’. On the left hand side, however, ‘observed’ and ‘missing’ mean ‘observed this time’ and ‘missing this time’, respectively. This is a different concept which at the meta-mathematical level is inconsistent with \((Y, R)\) at the stochastic level (delete): in the density functions \(f(\mathbf{y}_{\text{obs}}, \mathbf{y}_{\text{mis}})\) and \(f(\mathbf{y}_{\text{mis}}|\mathbf{y}_{\text{obs}})\), the notations \(\mathbf{y}_{\text{obs}}\) and \(\mathbf{y}_{\text{mis}}\) denote arbitrary realisations, which entails holding fixed the missingness pattern determining the partition of \(Y\) while allowing \(\mathbf{y} = (\mathbf{y}_{\text{obs}}, \mathbf{y}_{\text{mis}})\) in the marginal distribution to vary (in contradiction of the stochastic relationship encoded in \((Y, R)\)).

To distinguish these different concepts, the mathematical relationship \((Y_{\text{obs}}, Y_{\text{mis}}, R)\) will be called formally missing and the relationship \((Y_{\text{obs}}, Y_{\text{mis}})\) temporally missing. We note that there is an additional need in the theory to extend the temporal missingness relationship to all of \(Y \times R\) by partitioning the components of \(Y\) in all of \(Y \times R\) according to some fixed missingness pattern \(\mathbf{r}\) (that is, over all the possible missingness patterns). The use of this will be illustrated in the derivations in Appendix C and Appendix E.

Informally, the distinction between formal and temporal missingness is that the former is what is defined formally by the relation \((Y, R)\), whereas with the latter the data variables \(Y\) have been partitioned according to some missingness pattern \(\mathbf{r}\) simply for the purpose of considering \(Y\) from a particular point of view, and there is no requirement or expectation that the formal relationship \((Y, R)\) is, or can be, preserved.

Note that when formal missingness is intended, \(Y_{\text{obs}}\) makes sense only as one part of a pair \((Y_{\text{obs}}, R)\), and this pair denotes a stochastic function more general than a random vector. On the other hand, with temporal missingness, both \(Y_{\text{obs}}\) and \(Y_{\text{mis}}\) denote marginal distributions of \(Y\) that are each mixtures of formally observable and formally unobservable values.
There are also two different functions \( f(\mathbf{y}_{mis} | \mathbf{y}_{obs}) \)

(For this and subsequent sections see Appendix A for definitions of notation.)

A statement that is equivalent to a missing at random (MAR) assumption is often written in the following (or a similar) way:

\[
p(\mathbf{y}_{mis} | \mathbf{y}_{obs}, \mathbf{r}) = f(\mathbf{y}_{mis} | \mathbf{y}_{obs}).
\] (2)

Sometimes (2) is assumed to hold for just the realised observed values \((\overline{\mathbf{y}}_{obs}, \overline{\mathbf{r}})\) and at other times it is assumed to hold for all possible observable values \((\mathbf{y}_{obs}, \mathbf{r})\) under repeated sampling from \((Y, R)\) (see Seaman et. al. (2013) for details).

Despite the function on the right hand side of (2) being denoted ‘\(f(\mathbf{y}_{mis} | \mathbf{y}_{obs})\)’, technically the functions being compared are \(p(\mathbf{y}_{mis} | \mathbf{y}_{obs}, \mathbf{r})\) on the left hand side and \(f(-|-) \circ \pi_Y|\Omega_r(\mathbf{y}_{mis}, \mathbf{y}_{obs}, \mathbf{r})\) on the right hand side, where \(\pi_Y|\Omega_r\) is the restriction of the projection \(\pi_Y\) to the domain of \(p(\mathbf{y}_{mis} | \mathbf{y}_{obs}, \mathbf{r})\) and \(f(-|-)\) denotes the function \(f(\mathbf{y}_{mis} | \mathbf{y}_{obs})\) derived from the marginal density \(f(\mathbf{y}_{mis}, \mathbf{y}_{obs})\) for \(Y\). Note that \(f(-|-) \circ \pi_Y|\Omega_r(\mathbf{y}_{mis}, \mathbf{y}_{obs}, \mathbf{r}) \neq f(\mathbf{y}_{mis} | \mathbf{y}_{obs})\) because these functions have different domains. This mathematical distinction is a minor technicality, but the distinction is important stochastically. We will illustrate this shortly, but first we distinguish between these two functions by giving them different notation:

\[
f^{(T)}(\mathbf{y}_{mis} | \mathbf{y}_{obs}) := f(\mathbf{y}_{mis} | \mathbf{y}_{obs})
\] (3)

\[
f^{(F)}(\mathbf{y}_{mis} | \mathbf{y}_{obs}) := f(-|-) \left( \pi_Y(\mathbf{y}_{mis}, \mathbf{y}_{obs}, \mathbf{r}) \right)
\] (4)

The stochastic difference between \(f^{(T)}\) and \(f^{(F)}\) is that realisations drawn according to the former come from the range of the projection \(\pi_Y\), but realisations of the latter come from the domain of \(\pi_Y\). That is, the realisations come from different sides of equation (1). In particular, an update to a realisation \((\mathbf{y}_{obs}, \mathbf{y}_{mis}, \mathbf{r})\) according to \(f^{(F)}\) has the form of a three tuple \((\mathbf{y}_{obs}, \mathbf{y}_{mis}^*, \mathbf{r})\) with the missingness pattern \(\mathbf{r}\) remaining unchanged. However, an update to the same realisation \((\mathbf{y}_{obs}, \mathbf{y}_{mis}, \mathbf{r})\) according to \(f^{(T)}\) has the form of a two tuple \((\mathbf{y}_{obs}, \mathbf{y}_{mis}^*)\), and to maintain consistency with \((Y, R)\), a subsequent updating of the missingness pattern \(\mathbf{r}\) to \(\mathbf{r}^*\) according to the missingness mechanism \(g(\mathbf{r} | \mathbf{y})\) is required to complete the triple \((\mathbf{y}_{obs}, \mathbf{y}_{mis}^*, \mathbf{r}^*)\).

Due to this stochastic difference between \(f^{(T)}\) and \(f^{(F)}\), it is important to emphasise that the correct statement of equation (2) is that:

\[
p(\mathbf{y}_{mis} | \mathbf{y}_{obs}, \mathbf{r}) = f^{(F)}(\mathbf{y}_{mis} | \mathbf{y}_{obs}).
\] (5)

4 Conceptual difficulties for the reader

The difference between \(f^{(T)}\) and \(f^{(F)}\) and the failure in the literature to distinguish between these densities and between variables which are formally and temporally missing creates unnecessary potential conceptual difficulties for a reader, and this can make it difficult for a reader to obtain a coherent conceptual picture of how the related statistical methods work. We outline some of these difficulties below.

**Difficulty 1.** The construction of the distribution \(f(\mathbf{y}_{mis} | \mathbf{y}_{obs})\) requires identification of variables \(\mathbf{y}_{obs}\) and \(\mathbf{y}_{mis}\) in the domain of the marginal density \(f(\mathbf{y})\) for \(Y\), and this requires the reader to deal with two inconsistent definitions of missingness simultaneously that are not distinguished in the notation: temporal, \((\mathbf{y}_{obs}, \mathbf{y}_{mis})\), pertaining to the marginal distribution for \(Y\) and formal, \((\mathbf{y}_{obs}, \mathbf{y}_{mis}, \mathbf{r})\), as defined by \((Y, R)\). 

Note that difficulty 1 is not due to the encoding of ‘observed’ and ‘missing’ into the labels ‘obs’ and ‘mis’, as opposed to ‘(1)’ and ‘(0)’ used in Rubin (1976), but rather
because the same labels are used both in the domain and in the range of the projection \( \pi_Y \) in equation (1).

**Difficulty 2.** If \((\tilde{Y}_{\text{obs}}, \tilde{Y}_{\text{mis}}, \tilde{r})\) denotes the particular realised values of \((Y, R)\), then the distribution \( f^{(T)}(\tilde{y}_{\text{mis}|\tilde{y}_{\text{obs}}}) \) is the wrong distribution conceptually for ignorable multiple imputation.

As we noted in Section 2 an update to \( y_{\text{mis}} \) according to \( f^{(T)}(y_{\text{mis}|\tilde{y}_{\text{obs}}}) \) arises as a two tuple \((\tilde{y}_{\text{obs}}, y_{\text{mis}}^*)\) and requires an update to the missingness pattern to form a completed three tuple \((\tilde{y}_{\text{obs}}, y_{\text{mis}}^*, r^*)\) to maintain consistency with \((Y, R)\). Therefore, a sequence of imputations drawn according to \( f^{(T)}(y_{\text{mis}|\tilde{y}_{\text{obs}}}) \) that is consistent with \((Y, R)\) has the form:

\[
(\tilde{y}_?, y^{(1)}?, r^{(1)}), (\tilde{y}_?, y^{(2)}?, r^{(2)}), \ldots, (\tilde{y}_?, y^{(m)}?, r^{(m)})).
\]  (6)

This contrasts with a sequence of imputations drawn according to \( f^{(F)}(y_{\text{mis}|y_{\text{obs}}}) \) which conceptually has the correct form:

\[
(\tilde{y}_{\text{obs}}, y^{(1)}_{\text{mis}}, \tilde{r}), (\tilde{y}_{\text{obs}}, y^{(2)}_{\text{mis}}, \tilde{r}), \ldots, (\tilde{y}_{\text{obs}}, y^{(m)}_{\text{mis}}, \tilde{r}).
\]  (7)

**Difficulty 3.** Standard conventions for interpreting mathematical notation leads to ‘\( f \)' in the notation ‘\( f(y_{\text{mis}|y_{\text{obs}}}) \)' being interpreted as the density \( f^{(T)} \) and not the density \( f^{(F)} \) as is required by equation (2) (see equation (5)).

Note that difficulty 3 does not apply to equation (2) because the context allows the reader to interpret the function on the right hand side correctly as \( f^{(F)} \) (if the reader examines the notation carefully). However, this is definitely not the case with the standalone notation ‘\( f(y_{\text{mis}|y_{\text{obs}}}) \)', and it is this latter notation which permeates much of the published literature on ignorable multiple imputation methodology.

**Difficulty 4.** Failure to distinguish between formal and temporal missingness clashes with the standard statistical convention of inferring the identity of a density function through the denotation of the variables in its domain.

It is common to infer from the notation ‘\( f(x_1, x_2) \)' for a joint density that ‘\( f(x_2) \)' denotes a marginal density. However, the notation ‘\( p(y_{\text{obs}}, r) \)' is ambiguous because the interpretation of \( y_{\text{obs}} \) as formally observed leads to one function \( p \), but the interpretation of \( y_{\text{obs}} \) as temporally observed leads to a different function \( p \) with a different domain.

5 Additional limitations and notational inconsistencies

Omitting from the notation the dependence of ‘\( \text{obs} \)' and ‘\( \text{mis} \)' on a specific missingness pattern \( r \) implicitly assumes that \( r \) is the only missingness pattern of interest to the reader. This prevents the expression of the mathematical relationships between missingness patterns that exist within equation (2). Understanding these relationships at a conceptual level is useful for a reader to comprehend the primary implications of a MAR assumption in practice where one missingness pattern per unit is observed, and several different missingness patterns are realised overall.

The use of uppercase letters to denote both variable realisations of random vectors as well as the random vectors themselves is common in the literature on incomplete data methods. This is contrary to the recommendations in Halperin et. al. (1965). It is also another potential source of conceptual confusion for readers because the notation ‘\( f(Y) \)' ordinarily would be understood to mean the composition \( f \circ Y \) of the density function \( f \) with the random variable \( Y \), whereas a density function is something that is integrated to calculate probabilities for \( Y \).
The use of a capital ‘P’ to denote a probability density function also seems fairly common in the literature on methods for incomplete data. This too is contrary to widely understood usage of the notation where a capital P denotes the probability measure and is a function of events (subsets of outcomes), whereas the density is a corresponding function of outcomes which is integrated over subsets to calculate values for P. This is a further potential source of confusion for readers.

6 A remedy

Missing data is a common problem across a broad range of medical and public health research, and in other fields of empirical research as well. Consequently, there is a broad range of stakeholders with an interest in being able to read and understand the literature on the relevant statistical methods. The defects and ambiguities in the notation which we have identified potentially undermines its purpose to disseminate the requisite information in a clear and logically coherent manner to a broad range of stakeholders, which we have identified potentially undermines its purpose to disseminate the requisite information in a clear and logically coherent manner to a broad range of stakeholders, because these limitations likely make this literature difficult, if not impossible, for certain subsets of these stakeholders to read.

Fortunately the remedy is straightforward. What is needed are four symbols rather than two to denote the partition of Y into observable and unobservable components; one pair each for the two different relationships between Y and R. Additionally, the dependence of the partition on a definite missingness pattern needs to be made explicit. Notation for this purpose is defined formally in Appendix A. In practice, all one needs to understand is that four symbols are needed

\[ Y^{ob(r)} \text{ and } Y^{mi(r)} \] (to denote formal missingness),

\[ Y^{ot(r)} \text{ and } Y^{mi(r)} \] (to denote temporal missingness).

Demonstrations of how this allows the difficulties discussed in Section 5 to be overcome are given in Appendices B to E.

7 Some further remarks

As in Seaman et. al. (2013), we have retained Rubin’s (1976) original notation ‘f’ and ‘g’ for the marginal density and missingness mechanism in a selection model factorization \( f(y)g(r|y) \) of the full density. We have also retained the lowercase ‘p’ from Molenberghs et. al. (2015 p 95) for the factors of the pattern-mixture factorization \( p(r) p(y|r) \), but we have introduced ‘h’ for the full densities involving both Y and R, because we feel that this is clearer than denoting every density with the generic symbol ‘p’.

Our notation resolves the ambiguity present in ‘\( f(y_{mis}|y_{obs}) \)’ because by the definitions of \( y^{ob(r)} \) and \( y^{mi(r)} \), the notation ‘\( f(y^{mi(r)}|y^{ob(r)}) \)’ can denote only the function \( f^{(F)} \) in [4] and not the function \( f^{(T)} \) in [3]. Alternatively, through use of the notation ‘\( f(y^{mi(r)}|y^{ot(r)}) \)’ the marginal distribution for Y is freed from \((Y,R)\) in a way that does not conflict with the stochastic relationship imposed by \((Y,R)\). Note that we would not expect a casual reader of the literature to understand the distinction between these notations, and we recommend that authors always explain that imputations drawn according to \( f(y^{mi(r)}|y^{ob(r)}) \) represent triples \((\tilde{y}^{ob(r)},y^{mi(r)},\tilde{r})\) from \( \Omega_{r} \) and not pairs \((\tilde{y}^{ot(r)},y^{mi(r)})\) from \( \Omega_{y} \).

The additional ambiguities noted earlier are resolved in our notation as well. For example, the two functions corresponding to the notation ‘\( p(y_{obs},r) \)’ are denoted by \( h(y^{ob(r)},r) \) and \( h(y^{ot(r)},r) \), respectively, in our notation (see equations (18) and (19) in Appendix B). We have not addressed explicitly the notation ‘\( p(r|y_{obs}) \)’ because this is
the subject of a separate investigation currently in preparation (delete) treated in detail in Galati (2019).

We have maintained consistency with the recommendation in Halperin et al. (1965) to distinguish between random vectors and their realisations with uppercase and lowercase letters, respectively. However, we saw no need to consider vectors specifically to be column matrices in the present circumstances. Also, one non-standard feature of our notation is that we use $\Omega$ to denote the range of the random vector $(Y, R)$ instead of its domain. This is because we have a need explicitly to refer to subsets of $\Omega$, and in likelihood theory the probability spaces of interest typically are defined entirely on the range of $(Y, R)$ via density functions (and often there is no need to refer explicitly to some underlying sample space).

We hope that identification and elaboration of the notational issues raised in this paper will assist readers to navigate more easily the existing literature on statistical methods for incomplete data, and to assist future authors to improve the clarity of their expositions.

8 Appendix A (Definition of Notation)

8.1 Random Vectors

Throughout, $Y$ denotes a random vector modelling the observed and unobserved data comprising all units in the study jointly, and $R$ denotes a random vector of binary response random variables of the same dimension as $Y$, where ‘1’ means observed. Joint distributions for the pair of random vectors $(Y, R)$ will be referred to as full distributions.

Note. We have no need to distinguish between vectors interpreted as column matrices versus row matrices, and so for our purposes we do not give vectors column matrix interpretation and dispense with the common $'$ and $T$ notations.

Note. Typically a data analyst thinks of a given $y$ as comprising a rectangular matrix with each column pertaining to a specific ‘variable’ (for example, blood pressure) and each row pertaining to a specific unit (for example, an individual in the study). In our notation, the data matrix is shaped so that there is a single row with the data for the various units placed side by side in sets of columns.

8.2 Sample Spaces

Let $\mathcal{R} = \{r_1, r_2, \ldots, r_k\}$ be the set of distinct missingness patterns with $r_1 = 1$ denoting the ‘all ones’ vector corresponding to the complete cases. For convenience, we let $r_0 = 0$ denote the ‘all zeros’ vector corresponding to non-participants, where it may or may not be the case that $r_j = r_0$ for some $j \in \{1, 2, \ldots, k\}$. (We exclude $j = 0$ so as to avoid ever having $P(r_0) = 0$.) Note that the dot product $r_j \cdot r_j$ gives the number of values observed when the $j^{th}$ missingness pattern is realised and, in particular, $r_1 \cdot r_1$ gives the number of variables in $R$ (and also in $Y$). Let $\mathcal{Y} = \text{range}(Y)$ be the set of realisable datasets, where a realizable dataset contains complete data including all values that may or may not be observable.

Let $\Omega = \mathcal{Y} \times \mathcal{R} = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ be the full sample space of realisable pairs of datasets and missingness patterns, where $\Omega_j = \mathcal{Y} \times \{r_j\}$ for $r_j \in \mathcal{R}$. When the subscript $j$ of $r$ is omitted, we denote $\Omega_j$ by $\Omega_r$. Let $\pi_Y$ and $\pi_R$ denote the projections $(y, r) \mapsto y$ and $(y, r) \mapsto r$, respectively.

Realisations which represent a specific realisable dataset or missingness pattern only are denoted $\bar{y}$ and $\bar{r}$, respectively.
8.3 Projections on \( \mathcal{Y} \) and \( \Omega_j \)

For \( j = 1, 2, \ldots, k \), let \( \pi(r_j) : \mathcal{Y} \to \mathcal{Y}^{\pi(r_j)} \) and \( \pi(-r_j) : \mathcal{Y} \to \mathcal{Y}^{\pi(-r_j)} \) denote the projections extracting from each \( y \) vector the vectors of its observed and unobserved values, respectively, according to the missingness pattern \( r_j \). (In logic, ‘\(-\)’ is commonly used for negation.) By convention we set \( \pi(r_0) = \pi(-r_1) = \emptyset \). To apply these projections correctly over \( \Omega \), we define the following mappings

\[
o: \mathcal{R} \to \{ \pi(r) \circ \pi_Y : \Omega_r \to \mathcal{Y}^{\pi(r)} \}
\]

and use an abbreviated notation to refer to the images of \((y, r) \in \Omega \) under these mappings:

\[
y^{ob(r)} := (y, r)^{\pi(\pi_R(y, r))} \quad \text{over } \mathcal{Y}
\]

\[
y^{mi(r)} := (y, r)^{\pi(\pi_R(y, r))} \quad \text{over } \Omega
\]

Additionally, for \( r \in \mathcal{R} \) and \( y \in \mathcal{Y} \) set

\[
y^{ot(r)} := \begin{cases} y^{\pi(r)} & \text{over } \mathcal{Y} \\ (y, r)^{\pi(\pi_R(y, r))} & \text{over } \Omega \end{cases}
\]

\[
y^{mt(r)} := \begin{cases} y^{\pi(-r)} & \text{over } \mathcal{Y} \\ (y, r)^{\pi(\pi_R(y, r))} & \text{over } \Omega. \end{cases}
\]

Note. The notations in (8) and (9) and on the right hand sides of (10)−(13) may seem unwieldy. Note that these notations are needed solely for the purpose of carefully defining the four symbols \( y^{ob(r)} \), \( y^{mi(r)} \), \( y^{ot(r)} \) and \( y^{mt(r)} \). It is only these latter four symbols that are needed for working with densities for the distributions for \((Y, R)\) themselves.

Note. The vectors \( y^{ob(r)} \) and \( y^{ot(r)} \) have length \( r \cdot r \) while the vectors \( y^{mi(r)} \) and \( y^{mt(r)} \) have length \( r_1 \cdot r_1 - r \cdot r \). Note that these lengths vary from missingness pattern to missingness pattern.

Note. The projections \( ob(r) \) and \( mi(r) \) apply solely on the range of \((Y, R)\) and are always consistent with the missingness relation \((Y, R)\). Each missingness pattern \( r_j \) gives projections \( ob(r_j) \) and \( mi(r_j) \) on \( \Omega_j \), and each(delete) these are pieced together over all missingness patterns to give a single pair of functions on all of \( \Omega \).

Note. The projections \( ot(r) \) and \( mt(r) \) apply on either \( Y \) or \( (Y, R) \) as the context dictates. Each \( r_j \) gives a distinct pair of projections \( ot(r_j) \) and \( mt(r_j) \) on all of \( \mathcal{Y} \) or all of \( \Omega \), as the case may be. In the latter case, these(delete) \( ot(r_j) \) and \( mt(r_j) \) are consistent with \((Y, R)\) on \( \Omega_j \) and inconsistent with \((Y, R)\) elsewhere on \( \Omega \). The ‘t’ in ‘ot’ and ‘mt’ can be taken to mean ‘temporally’ or ‘this time’.

Note. The notation ‘\( f(y^{mt(r)}(y^{ot(r)}) \)’ is ambiguous because as defined by (12) and (13) this can denote either the function \( f(T) \) (see (3)) defined on \( \mathcal{Y} \) or a function (not \( f(T) \)) defined on all of \( \Omega \), but which one is intended should be clear from the context. However, the notation ‘\( f(y^{mt(r)}(y^{ob(r)}) \)’ is unambiguous because by (10) and (11) this must denote \( f(T) \).

8.4 Observable Data Events

Given \((y, r) \in \Omega \), we call

\[
\Omega_{(y, r)} = \{ (y^{ob(r)}, y^{mi(r)}, r) : y_r \in \mathcal{Y} \}
\]
the observed data event corresponding to \((y, r)\). The set \(\Omega_{(y, r)}\) consists of all datasets \(y_*\) which have the same observed values as \(y\) (as defined by the missingness pattern \(r\)). For a fixed \(r \in R\), the events in \((14)\) partition \(\Omega_r\), and over all \(r\) they give a partition of \(\Omega\). These observable data events are the classes of the equivalence relation defined by setting for all \((y_1, r_1), (y_2, r_2) \in \Omega\), \((y_1, r_1) \sim_{ob} (y_2, r_2)\) if, and only if, \(r_1 = r_2\) and \(y_{1,ob(r_1)} = y_{2,ob(r_2)}\).

8.5 Density Functions

We specify full distributions for \((Y, R)\) through density functions \(h : \Omega \to \mathbb{R}\), with probabilities being determined by integration: \(P(A) = \int_A h\) for any \(A \subseteq \Omega\) for which a probability can be defined (see Ash and Doléans-Dade (2000) or Shorack (2000) for details). Note that we suppress the dominating measure in the notation. Two different ways of factorizing \(h\) are useful:

\[
h(y, r) = f(y) g(r | y) = p(r) p(y | r). \tag{15}
\]

The first factorization in \((15)\) is called a selection model factorization of \(h\), and the factor \(g(r | y)\) is called the missingness mechanism. The second factorization in \((15)\) is called a pattern-mixture factorization, and for each \(r \in R\), we call the conditional density \(p(y | r)\) the pattern mixture component pertaining to \(r\).

Note. Technically, the symbols \(h, f, g\) and \(p\) denote density functions and \(h(y, r), f(y), g(r | y), p(r)\) and \(p(y | r)\) denote real numbers. Because it is common in statistics to use the same symbol to denote different densities, for example a joint density \(f(x_1, x_2)\) and a marginal density \(f(x_1)\), we adopt the usual convention and often refer to density functions by their values.

9 Appendix B (The observable data distribution)

To apply likelihood theory to incomplete data, from the model for the full data one must construct a model for just the observable data. This involves specifying a set of outcomes and a set of events for the observable data, and to each full density \(h\), a corresponding density on the set of outcomes for the observable data. Here we give an explicit construction for this probability space together with a step-by-step derivation of the density given in (5.11) in the extract quoted in Section [1].

The outcomes can be taken to be either the set of observable data events or the range of the map \((y, r) \mapsto (y_{ob(r)}, r)\) because there is a one-to-one correspondence between \(\{\Omega_{(y, r)}\}\) and \(\{(y_{ob(r)}, r)\}\). The latter seems to be preferred (Little and Rubin (1987), Little and Rubin (2002), Tsiatis (2006)):

\[
\Omega_{ob} := \bigcup_{j=1}^k (y_{ob(r_j)} \times \{ r_j \}). \tag{16}
\]

This is an irregularly-shaped set because as noted in Appendix A the vectors \(y_{ob(r_j)}\) typically have different lengths for different missingness patterns.

Under the one-to-one correspondence between \(\{y_{ob(r)}, r\}\) and \(\{\Omega_{(y, r)}\}\), events in \(\Omega_{ob}\) correspond to unions of observable data events in \(\Omega\). Restricting to observable data events gives the density for the probability distribution on \(\Omega_{ob}\):

\[
h(y_{ob(r)}, r) = \int f(y) g(r | y) dy_{mi(r)} = \int_{\Omega_{(y, r)}} h(y, r). \tag{17}
\]
This can be seen to be the required density simply by pulling events in $\Omega_{\text{ob}}$ back to unions of observable data events in $\Omega$ and integrating $h$ over these corresponding events for $(Y, R)$ by applying iterated integrals as per Fubini’s Theorem (Ash and Doléans-Dade (2000 p 101)). Note that we use $mt(r)$ and not $mt(r)$ in ‘$d^m_{\text{mi}}(r)$’ because the integrand $h$ is defined on all of $\Omega$ and the variables integrated out of $h$ are different for each subset $\Omega_j$.

The missingness-pattern-dependant processing being performed in the construction of the density in (17) does not correlate well with the selection-model factorization for $h$, and this can make the construction seem a little opaque. An alternative derivation is possible starting with a pattern-mixture factorization for $h$ and this can make the construction seem a little opaque. An alternative derivation is possible starting with a pattern-mixture factorization for $h$.

One way to do this is to start from $h(y, r) = p(r) p(y_{\text{mi}}(r), y_{\text{ob}}(r)| r)$, restrict $\Omega$ to $\Omega_j$: $h(y_j, r_j) = p(r_j) p(y_{\text{mi}}(r_j), y_{\text{ob}}(r_j)| r_j)$, marginalize to $y_{\text{ob}}(r_j)$:

$$h(y_{\text{ob}}(r_j), r_j) = \int p(r_j) p(y_{\text{mi}}(r_j), y_{\text{ob}}(r_j)| r_j) \, d y_{\text{mi}}(r_j)$$

$$= p(r_j) \int p(y_{\text{mi}}(r_j), y_{\text{ob}}(r_j)| r_j) \, d y_{\text{mi}}(r_j)$$

$$= p(r_j) p(y_{\text{ob}}(r_j)| r_j)$$

(18)

and then put the pieces together over all of $\Omega_{\text{ob}}$: $h(y_{\text{ob}}(r), r) = p(r) p(y_{\text{ob}}(r)| r)$. Alternatively, for each $j$ one can marginalize over all of $\Omega$:

$$h(y_{\text{ob}}(r_j), r) = \int p(r) p(y_{\text{mi}}(r_j), y_{\text{ob}}(r_j)| r) \, d y_{\text{mi}}(r_j)$$

$$= p(r) \int p(y_{\text{mi}}(r_j), y_{\text{ob}}(r_j)| r) \, d y_{\text{mi}}(r_j)$$

$$= p(r) p(y_{\text{ob}}(r_j)| r)$$

(19)

restrict to $\Omega_j$: $h(y_{\text{ob}}(r_j), r_j) = h(y_{\text{ob}}(r_j), r_j) = p(r_j) p(y_{\text{ob}}(r_j)| r_j)$, and then put the pieces together over all of $\Omega_{\text{ob}}$:

$$h(y_{\text{ob}}(r), r) = p(r) p(y_{\text{ob}}(r)| r)$$

(20)

Note. In (19), for a given $j$ the density $h(y_{\text{ob}}(r_j), r)$ is a marginal density of $h$ with domain $y_{\text{ob}}(r_j) \times R$. There are $k$ of these distributions. On the other hand, there is only one density $h(y_{\text{ob}}(r), r)$ with domain $\Omega_{\text{ob}}$. For a given $j$, the function $h(y_{\text{ob}}(r_j), r)$ agrees with $h(y_{\text{ob}}(r), r)$ on the set $\Omega_j$, but comparison of these two functions on the rest of their domains is not well defined.

Note. Because (16) is irregularly shaped and not a Cartesian product, the stochastic function obtained by composing $(Y, R)$ with $(y, r) \mapsto (y_{\text{ob}}(r), r)$ is not a random vector. Tsiatis ([14] page 13) calls these ‘random quantities’. Stochastic functions more general than random vectors are called ‘random objects’ by Ash and Doléans-Dade (2000 p 178) and ‘random elements’ by Shorack (2000 p 90). To be applicable to incomplete data, the likelihood theory must be sufficiently general to cover these random quantities. See Shorack (2000 pp. 563–567) for a sufficiently general likelihood theory for the case of IID data.

Note. If $Y_{\text{obs}}$ is interpreted as formally observed and considered to vary over missingness patterns, then it denotes the composition of $(Y, R)$ with $(y, r) \mapsto y_{\text{ob}}(r)$. As was noted in Section 2 when interpreted this way $Y_{\text{obs}}$ alone is insufficient to model the observable data. This is because there is potential for clashes between the ranges from distinct missingness patterns. That is, we may have $r_j \neq r_{j'}$ with $y_{\text{ob}}(r_j) = y_{\text{ob}}(r_{j'})$ on the right hand side of (16).
Now in the sum on the right-hand side of (23), the terms for which all the entries of \( \mathbf{r} \) satisfy \( \pi_l(\mathbf{r}_i) \leq \pi_l(\mathbf{r}_j) \) for all \( l \in \{1, 2, \ldots, r_1 \cdot r_1\} \). Letting \( \mathbf{r}_i \leq_p \mathbf{r}_j \) if, and only if, all values that are defined to be observed according to pattern \( \mathbf{r}_i \) are defined to be observed according to pattern \( \mathbf{r}_j \). It is straightforward to check that this relation is reflexive, transitive and anti-symmetric.

Let \( \mathbf{r} \in \mathcal{R} \) and consider a full density \( h(\mathbf{y}, \mathbf{r}) = f(\mathbf{y}) g(\mathbf{r} | \mathbf{y}) = p(\mathbf{r}) p(\mathbf{y} | \mathbf{r}) \). Marginalising the latter factorisation over all missingness patterns gives the marginal density for \( \mathbf{Y} \) as a mixture of the pattern-mixture components:

\[
f(\mathbf{y}) = \sum_{j=1}^{k} p(\mathbf{r}_j) p(\mathbf{y} | \mathbf{r}_j). \tag{22}\]

Letting \( \mathbf{y} \in \mathcal{Y} \) and substituting \( \mathbf{y} = (\mathbf{y}^{\text{mit}(\mathbf{r})}, \mathbf{y}^{\text{ob}(\mathbf{r})}) \) into both sides of (22) gives:

\[
f(\mathbf{y}^{\text{mit}(\mathbf{r})}, \mathbf{y}^{\text{ob}(\mathbf{r})}) = \sum_{j=1}^{k} p(\mathbf{r}_j) p(\mathbf{y}^{\text{mit}(\mathbf{r})}, \mathbf{y}^{\text{ob}(\mathbf{r})} | \mathbf{r}_j). \tag{23}\]

Now in the sum on the right-hand side of (23), the terms for which all the entries of \( \mathbf{y}^{\text{mit}(\mathbf{r})} \) are labelled as formally missing according to \( (Y, R) \) are those with missingness patterns satisfying \( \mathbf{r}_j \leq_p \mathbf{r} \) (according to the partial order defined in (21)). Similarly, the terms for which all the entries of \( \mathbf{y}^{\text{ob}(\mathbf{r})} \) are labelled as formally observed according to \( (Y, R) \) are those with missingness patterns satisfying \( \mathbf{r} \leq_p \mathbf{r}_j \). By anti-symmetry, the only component on the right-hand side of (23) for which all labelling of the \( y \) values is formally correct is the single component with \( \mathbf{r}_i = \mathbf{r} \). Hence, provided \( \mathcal{R} \) contains at least two missingness patterns, one of \( \mathbf{y}^{\text{ob}(\mathbf{r})} \) and \( \mathbf{y}^{\text{mit}(\mathbf{r})} \) is a mixture of formally observable and formally unobservable data. (This shows that at least one of \( Y^{\text{ob}(\mathbf{r})} \) and \( Y^{\text{mit}(\mathbf{r})} \) is mixed. In most cases, this will be true of both.)

### 11 Appendix D (Derivation of the MAR Identity)

Here we give a formal derivation of equation (2). Given \( h(\mathbf{y}, \mathbf{r}) = f(\mathbf{y}) g(\mathbf{r} | \mathbf{y}) \) factorised in selection model form together with observed data \( \Omega_{(\mathcal{Y}, \mathcal{R})} \), we say that the missingness mechanism \( g \) is missing at random (MAR) with respect to \( \Omega_{(\mathcal{Y}, \mathcal{R})} \). To MAR hold with respect to \( \Omega_{(\mathcal{Y}, \mathcal{R})} \) for all densities \( h(\theta, \psi) \in \mathcal{M} \). Everywhere MAR in Seaman et. al. (2013) is accommodated by requiring that MAR hold with respect to all observed data events (for all densities in \( \mathcal{M} \)).

Let \( h \) be as in (15) and let \( (\mathbf{y}, \mathbf{r}) \in \Omega \setminus \Omega_{\mathcal{R}_1} \) be a partially-observed realisation drawn according to \( h \). Partitioning \( \mathbf{y} \) into observable and unobservable components as defined by \( \mathbf{r} \) gives

\[
p(\mathbf{r}) p(\mathbf{y}^{\text{mit}(\mathbf{r})}, \mathbf{y}^{\text{ob}(\mathbf{r})} | \mathbf{r}) = f(\mathbf{y}^{\text{mit}(\mathbf{r})}, \mathbf{y}^{\text{ob}(\mathbf{r})}) g(\mathbf{r} | \mathbf{y}). \tag{24}\]

Note that the ‘\( f \)’ in \( f(\mathbf{y}^{\text{ob}(\mathbf{r})} | \mathbf{y}^{\text{ob}(\mathbf{r})}) \) denotes the function \( f^{(F)} \) (see (4)) and not the function \( f^{(T)} \). Factorizing the joint density for the \( y \) values on each side of (24) into
the product of a marginal and a conditional density, and then rearranging (provided all required denominators are non-zero) gives:

$$p(y_{mi(r)} \mid y_{ob(r)}, r) = \frac{f(y_{mi(r)} \mid y_{ob(r)}) f(y_{ob(r)} \mid r)}{p(r) p(y_{ob(r)} \mid r)}. \quad (25)$$

In (25) the function $f(y_{ob(r)})$ denotes the composition of the marginal density $f(y_{od(r)})$ with the projection $\pi_Y$ (suitably restricted).

If $g$ is MAR with respect to $\Omega_{(y, r)}$, then over $\Omega_{(y, r)}$ the only non-constant factor on the right hand side is $f(y_{mi(r)} \mid y_{ob(r)})$. Integrating both sides with respect to the $y_{mi(r)}$ variables and rearranging gives

$$p(y_{ob(r)} \mid r) = \frac{1}{p(r)} f(y_{ob(r)} \mid r) g(r \mid y) \quad (26)$$

because $\int p(y_{mi(r)} \mid y_{ob(r)}, r) \, dy_{mi(r)} = \int f(y_{mi(r)} \mid y_{ob(r)}) \, dy_{mi(r)} = 1$. Substituting (26) back into (25) then gives

$$p(y_{mi(r)} \mid y_{ob(r)}, r) = f(y_{mi(r)} \mid y_{ob(r)}). \quad (27)$$

12 Appendix E (Further analysis of the MAR Identity)

In this final Appendix we examine the MAR identity (27) more closely. For a fixed $(y, r) \in \Omega$, the domain of the densities in this equality is the observed data event $\Omega_{(y, r)}$. When restricted to this event, $\pi_Y$ gives a bijection onto a corresponding subset $\pi_Y (\Omega_{(y, r)})$ of $Y$. Combining the inverse of this bijection with (27) gives

$$f(y_{mi(r)} \mid y_{ob(r)}) = f(y_{mi(r)} \mid y_{ob(r)}) = p(y_{mi(r)} \mid y_{ob(r)}, r), \quad (28)$$

where the first equality is for values of functions with different domains. For notational simplicity, we relabel the missingness patterns, if necessary, so that $r = r_k$. Conditioning on the $y_{od(r)}$ variables in (23) yields

$$f(y_{mi(r)} \mid y_{ob(r)}) = p(r) p(y_{mi(r)} \mid y_{ob(r)} \mid r) + \sum_{j=1}^{k-1} p(r_j) p(y_{mi(r)} \mid y_{ob(r)} \mid r_j). \quad (29)$$

Substituting (28) into (29) and rearranging then gives:

$$p(y_{mi(r)} \mid y_{ob(r)}, r) = \frac{1}{1 - p(r)} \sum_{j=1}^{k-1} p(r_j) p(y_{mi(r)} \mid y_{ob(r)} \mid r_j). \quad (30)$$

When the data comprise $n$ IID draws with differing missingness patterns across units, holding $r$ fixed in (30) and letting $y$ vary shows that associations on the left hand side for which data are never observed are partially observed on the right hand side amongst units with missingness patterns different from $r$. This key feature of MAR is obscured in the notation on the right hand side of (2).

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