1. Introduction

Throughout fix $n, l_1, ..., l_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^{d} l_i = n$.

1A. Webs, and Schur–Weyl and Howe duality. The so-called Schur–Weyl duality has played a key role ever since the early days of representation theory. It relates representations of the symmetric group $S_m$ and the general linear group $GL_m = GL_m(\mathbb{C})$, and has been generalized in many ways. The representation used to relate these two groups is $(\mathbb{C}^n)^{\otimes m}$.

Mathematics Subject Classification 2020. Primary: 17B37; Secondary: 18M05, 18M15, 20C08.

Keywords. Representations of quantum general linear groups and Levi subalgebras, annular webs, quantum skew Howe duality, diagrammatic presentation, generalized blob algebras.
Let $L = \text{GL}_{d_1} \times \ldots \times \text{GL}_{d_q} \subseteq \text{GL}_n$, and let us for simplicity stay over $\mathbb{C}$ for now. Two generalizations of Schur–Weyl duality are of crucial importance for this paper. Firstly, the Schur–Weyl duality of $(\mathbb{Z}/d\mathbb{Z}) \wr S_m$ (that is, type $G(d,1,m)$) and $L$ from [ATY95], [SS99] and [Hu01] (see also [MS16] for a nice and self-contained discussion of this duality). Here the underlying representation is again $(\mathbb{C}^n)^{\otimes m}$. Secondly, skew (type $A$) Howe duality, see [How89] and [How95], relating $\text{GL}_N$ and $\text{GL}_n$ via their action on the exterior algebra $\Lambda^\bullet (\mathbb{C}^N \otimes \mathbb{C}^n)$.

As explained in [CKM14], a diagrammatic interpretation of skew Howe duality is given by (exterior $\text{GL}_m$) webs. (The same diagrammatics goes under many names, including birdtracks [Cvi08] or spiders [Kup96].) In some sense, in [CKM14] skew Howe duality relating $\text{GL}_{N \in \mathbb{Z}_{\geq 0}}$ and $\text{GL}_n$ takes the form of an equivalence between the category of webs and the category of $\text{GL}_m$-representations generated by $\{ \Lambda^k \mathbb{C}^n \mid k \in \{1, \ldots, n\} \}$, with the web category being obtained by using all $\text{GL}_N$ for $N \in \mathbb{Z}_{\geq 0}$. After additive idempotent completion webs even give a diagrammatic interpretation of all finite dimensional $\text{GL}_m$-representations.

In this paper we show that an explicit quotient of the category of annular (exterior $\text{GL}_m$) webs is equivalent to the category of $L$-representations generated by the set $\{ \Lambda^k \mathbb{C}^n \mid k \in \{1, \ldots, n\} \}$. As before, additive idempotent completion gives a description of all finite dimensional $L$-representations. This, in some sense, is a form of what could be called annular skew Howe duality (we avoid the notion affine as its meaning is context depending) or skew type $G(d,1,m \in \mathbb{Z}_{\geq 0})$ Howe duality.

1B. The main result and relations to other works. We now give a few details and change to the universal enveloping algebras. We consider a Levi subalgebra of the form $\ell = \mathfrak{gl}_l \oplus \ldots \oplus \mathfrak{gl}_q \subseteq \mathfrak{gl}_n$ (in this paper we write $\ell$ instead of the usual notation $I$ for readability). Let $\mathbb{K}_q$ denote a field containing an element $q \in \mathbb{K}_q$ that is not a root of unity and additional variables $U = \{ u_1, \ldots, u_d \}$ and their inverses, and let further $\mathbb{K}_1$ denote a field of characteristic zero containing $U$ and their inverses. With these ground fields the category of finite dimensional $U_q(\ell)$-representations over $\mathbb{K}_q$ respectively of finite dimensional $U_1(\ell)$-representations over $\mathbb{K}_1$ are semisimple. (We should warn the reader: as explained in the main body of the text there are some nontrivial quantization issues and we carefully need to distinguish the two cases over $\mathbb{K}_q$ and $\mathbb{K}_1$.)

In Section 4 we define a $\mathbb{K}_q$-linear category of annular webs $\text{AWeb}_q \mathfrak{gl}_n$ as well as a quotient $\text{AWeb}_q \mathfrak{l}$ by evaluating essential circles using the variables $U$ and their inverses. Similarly over $\mathbb{K}_1$, where we write $\text{AWeb}_1 \mathfrak{gl}_n$ and $\text{AWeb}_1 \mathfrak{l}$. Let $\text{Fund}_q \mathfrak{l}$ respectively $\text{Fund}_1 \mathfrak{l}$ denote the categories of $U_q(\ell)$- and $U_1(\ell)$-representations generated by the exterior powers of the vector representation. Our main result is Theorem 6B.3 showing that $\text{AWeb}_q \mathfrak{l}$ is equivalent to $\text{Fund}_q \mathfrak{l}$ and that $\text{AWeb}_1 \mathfrak{l}$ is pivotally equivalent to $\text{Fund}_1 \mathfrak{l}$. The main ingredients in the proof of Theorem 6B.3 are the usual diagrammatic ideas, the Schur–Weyl type dualities from [SS99] as well as the explosion trick, which utilizes the semisimplicity.

An almost direct consequence of Theorem 6B.3 is that the endomorphism algebras of annular webs corresponding to tensor products of the vector representation can be described explicitly. As we will see in Section 6D these are given by certain row quotients of Ariki–Koike algebras (Ariki–Koike algebras are Hecke algebras of $(\mathbb{Z}/d\mathbb{Z}) \wr S_m$, see for example [AK94], [BM93], [Che87]) as studied in [LV21]. In the special case of two row quotients, which corresponds to $\ell = \mathfrak{gl}_1 \oplus \ldots \oplus \mathfrak{gl}_1$ being the Cartan subalgebra, these algebras are Martin–Woodcock’s generalized blob algebras [MW03]. We thus obtain a web description of generalized blob algebras, see Section 6E for details. This web description allows us to answer two conjectures of Cautis–Kamnitzer [CK18, Conjectures 10.2 and 10.3] affirmatively, up to technicalities as detailed in Remark 6F.5.

Let us also mention that our work is inspired by [Que15] (which gives another, very honest, version of annular skew Howe duality), the aforementioned paper [CK18] as well as [QW18] (which proves Theorem 6B.3 for $K_q = \mathbb{C}$ and the special case of the Cartan subalgebra). With respect to [Que15] and [QW18], which are partially motivated by skein theory, we should also warn the reader that the monoidal structure on $\text{AWeb}_q \mathfrak{l}$ coming from skein theory and the one on $\text{Fund}_q \mathfrak{l}$ coming from the Hopf algebra structure of $U_q(\ell)$ do not seem to be compatible, see Section 6C for a discussion.

Acknowledgments. We would like to thank Mikhail Khovanov and Alvaro Martinez Ruiz for helpful discussions, and the referee for their meticulous and careful reading (multiple times!) of the manuscript, and helpful suggestions.
2. Notations and conventions

We start by specifying our notations.

**Notation 2.1.** Recall that we fixed \( n, l_1, \ldots, l_d \in \mathbb{Z}_{\geq 0} \) with \( \sum_{i=1}^{d} l_i = n \). These will be used via the general linear Lie algebra \( \mathfrak{gl}_n \) and a Levi subalgebra \( \ell \) given by \( \ell = \mathfrak{gl}_{l_1} \oplus \cdots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n \). For \( l_1 = \cdots = l_d = 1 \), so that \( \ell = \mathfrak{gl}_1 \oplus \cdots \oplus \mathfrak{gl}_1 \) is the Cartan subalgebra, we will write \( \mathfrak{h} \) instead of \( \ell \).

**Notation 2.2.** We now specify our underlying ground rings.

(a) For essential circles, see Section 4C below, we need extra polynomial elements (these can be ignored otherwise). We denote by \( U = \{ u_1, \ldots, u_d \} \) which variables will play this role.

(b) Let \( \mathbb{Z}_v = \mathbb{Z}[v, v^{-1}, U, U^{-1}] \) for some indeterminate \( v \).

(c) We let \( \mathbb{K}_q \) denote a field containing \( U \) and \( U^{-1} \) and an element \( q \) which is not a root of unity. Let also \( \mathbb{K}_q \) be a field of characteristic zero containing \( U \) and \( U^{-1} \). (Note that \( \text{char}(\mathbb{K}_q) \) is allowed to be a prime, but we assume that \( \text{char}(\mathbb{K}_1) = 0 \).

(d) We also see \( \mathbb{K}_q \) as the specialization and scalar extension \( \otimes_{\mathbb{Z}_v} \mathbb{K}_q \) given by \( v \mapsto q \), and \( \mathbb{K}_1 \) as the specialization and scalar extension \( \otimes_{\mathbb{Z}_v} \mathbb{K}_1 \) given by \( v \mapsto 1 \). We will apply scalar extension to \( \mathbb{Z}_v \)-linear categories, and since this will play an important role we will indicate the specialization accordingly. The two important specializations for \( \mathbb{K}_q \) and \( \mathbb{K}_1 \) are distinguished by using \( q \) respectively 1 as a subscript.

(e) If not specified otherwise, then \( \otimes \) denotes the tensor product over the ground ring, which is either \( \mathbb{Z}_v \), \( \mathbb{K}_q \) or \( \mathbb{K}_1 \).

Not everything in this paper is defined over \( \mathbb{Z}_v \), and statements for \( \mathbb{K}_q \) are not strictly related to the ones for \( \mathbb{K}_1 \). If we use \( \mathbb{Z}_v \), then we can specialize without any problem. But if we do not work over \( \mathbb{Z}_v \), then, to not double parts of the text, we use the following crucial simplification:

**Notation 2.3.** Throughout (on the diagrammatic and the representation theoretical sides) we always specify our conventions for \( \mathbb{K}_q \) and leave the analog conventions and lemmas for \( \mathbb{K}_1 \) implicit; the ones for \( \mathbb{K}_1 \) are always the \( q = 1 \) versions of the ones for \( \mathbb{K}_q \). When lemmas etc. are the same for \( \mathbb{K}_q \) and \( \mathbb{K}_1 \), then we will use simply \( q \) (e.g. we write \( \mathbb{K}_q \) and not \( \mathbb{K}_q \) and \( \mathbb{K}_1 \)) to indicate this. We will stress whenever the statements for \( \mathbb{K}_q \) and \( \mathbb{K}_1 \) are significantly different.

We will use quantum numbers, factorials and binomials viewed as elements of \( \mathbb{Z}_v \). That is, for \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z}_{\geq 0} \) we let \( [0] = 0, [0]! = 1 = [0], [a] = [-a] \) for \( a < 0 \) and otherwise

\[
[a] = v^{a-1} + v^{a-3} + \cdots + v^{a-3} + v^{-a+1}, \quad [b]! = [b][b-1]\cdots[1], \quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a][a-1]\cdots[a-b+1]}{[b]!}.
\]

The following will be used silently throughout.

**Lemma 2.4.** All quantum binomials are invertible in \( \mathbb{K}_q \).

**Proof.** Easy and omitted. \( \square \)

**Notation 2.5.** We work with strict pivotal (thus, monoidal) categories, were we strictify categories if necessary (by the usual strictification theorems this restriction is for convenience only). We have two directions of composition, vertical \( \circ_v \) and horizontal \( \circ_h \), as well as a duality \( * \) operation. The monoidal unit is denoted by \( 1 \), and identity morphisms are denoted by \( \text{id} \). We will also distinguish objects and morphisms using different fonts, e.g. \( K \) and \( f \).

As we will see, e.g. in Section 6C below, it will turn out to be important to carefully distinguish the monoidal product on the various categories we consider. This will only play a role on the level of morphisms, and we use various symbols for monoidal products between morphisms if necessary.

**Notation 2.6.** We now summarize the diagrammatic conventions that we will use in this paper.
The following illustration of the interchange law summarizes our reading conventions:

\[(\text{id} \circ h \circ g) \circ v (f \circ h \circ \text{id}) = (f \circ h \circ \text{id}) \circ v (\text{id} \circ h \circ g).\]

That is, we read diagrams from bottom to top and left to right.

(b) As we will recall below, webs are certain types of labeled (with numbers \(a \in \mathbb{Z}\geq 0\)) and oriented graphs. Some labels and orientations are determined by others, and we will often omit orientations and labels that can be recovered from the given data to avoid clutter.

(c) If labels or orientations are omitted altogether, then the displayed webs are a shorthand for any web of the same shape and legit labels and orientations.

(d) We use webs with edges labeled by \(a \in \mathbb{Z}\), where we use the convention that edges of label 0 are omitted from the illustrations, and edges with label not in \(\mathbb{Z}_{\geq 0}\) set the web to zero. (We will use negative labels, but for objects and not for edges in webs.)

(e) We also use strands labeled by objects, i.e. for \(K = (k_1, \ldots, k_m)\)\(\in\mathbb{Z}^m\) for \(m \in \mathbb{Z}_{\geq 0}\) for the objects of \(\text{Dia}\), where we use the notations \(-k \leftrightarrow \downarrow k\) and \(-k \leftrightarrow \uparrow k\) to indicate an arbitrary (but finite) number of parallel strings.

3. Web categories in the plane

This section serves as a reminder on web categories and their basic properties. Details can be found in many books and papers, e.g. [TV17] for general diagrammatics, and [CKM14] or, closer to our conventions, [LT21] and the references therein for web categories. The proofs of the statements below are easy or can be found in loc. cit.

3A. Preliminaries. Let \(R\) be a commutative ring with unit.

**Definition 3A.1.** For the purpose of this section, a diagram category \(\text{Dia}\) is a pivotal \(R\)-linear category with objects \(\uparrow k\)-generated by \(\downarrow k = (\uparrow k)^*\) for \(k \in \mathbb{Z}_{\geq 0}\) with \(\mathbb{1}\) being the empty word, and a braid group action on upwards objects, meaning morphisms \(\hat{R}_{k,l}: \uparrow k \circ h \uparrow l \rightarrow \uparrow l \circ h \uparrow k\) for each simple braid group generator that satisfy the braid relations.

We also write \(K = (k_1, \ldots, k_m)\) for \(m \in \mathbb{Z}_{\geq 0}\) for the objects of \(\text{Dia}\), where we use the notations \(-k \leftrightarrow \downarrow k\) and \(-k \leftrightarrow \uparrow k\) for \(k > 0\). We illustrate the (co)evaluation morphisms as

\[
\begin{align*}
    &\kern-3.9em\hat{k} & &\hat{k} & &\kern-3.9em\hat{k} \\
\begin{array}{ccc}
    & \uparrow k & \downarrow k \\
    \kern-1.2em\hat{k} & \kern-1.2em\hat{k} & \kern-1.2em\hat{k}
\end{array}
\end{align*}
\]

and the braid group action as \((k,l)\)-crossings (overcrossings and undercrossings):

\[
\begin{array}{ccc}
    & \kern-3.9em\hat{k} & \kern-3.9em\hat{\downarrow} & \kern-3.9em\hat{\uparrow} & \kern-3.9em\hat{k} \\
\begin{array}{ccc}
    & \uparrow k & \downarrow k & \uparrow k & \downarrow k \\
    \kern-1.2em\hat{k} & \kern-1.2em\hat{k} & \kern-1.2em\hat{k} & \kern-1.2em\hat{k}
\end{array}
\end{array}
\]

**Definition 3A.2.** The diagrammatic antiinvolution \((-)^\downarrow\) on a diagram category is defined on objects by \((K \circ h \circ L \circ h \ldots)^\downarrow = K^* \circ h\circ L^* \circ h \ldots\) and on morphisms as in (3A.3) below. The diagrammatic involution \((-)^*\) on a monoidal diagram category is defined on objects by \((K \circ h \circ L \circ h \ldots)^* = \ldots \circ h \circ L \circ h \circ K\) and on morphisms as in (3A.3) below.

\[(3A.3)\]

\[
\begin{align*}
    (\hat{k} \hat{l})^\downarrow &= \hat{l} \hat{k}, & (\hat{k} \hat{l})^* &= \hat{l} \hat{k}.
\end{align*}
\]
Lemma 3A.4. The diagrammatic antiinvolution is an antiinvolution, and the diagrammatic involution is an involution.

Proof. Easy and omitted. □

Definition 3A.5. Mating in Dia is the process of applying (co)evaluation morphisms. For example,

\[
\begin{align*}
\text{mating} & \quad \rightarrow \\
\text{left mate} & \quad =
\end{align*}
\]

To be precise, a mate of a morphism \( f \) is any morphism obtained from \( f \) by applying (co)evaluation morphisms. A left and right mate of \( f \) is a mate that only uses left or right (co)evaluation morphisms. Finally, the left and right mate of a crossing is as indicated above.

Note that mating produces many morphisms from a given set of morphisms.

Lemma 3A.6. Suppose that \( \text{End}_{\text{Dia}}(↑k) \cong \text{End}_{\text{Dia}}(↑l) \cong R \). If the left mate of the \((k, l)\)-overcrossing is invertible, then all mates of the \((k, l)\)-overcrossing and its inverse span a pivotal subcategory equivalent to \( R \)-linear (labeled) tangles. In general, if the left mate of the \((k, l)\)-overcrossing is invertible, then all mates of the \((k, l)\)-overcrossing and its inverse span a pivotal subcategory equivalent to \( R \)-linear (labeled) framed tangles.

Proof. Note that the assumption \( \text{End}_{\text{Dia}}(↑k) \cong R \) implies the Reidemeister I relation up to scalars, which is enough to copy the argument in [Kas95, Theorem X11.2.2]. The second claim follows similarly. □

We say \( \text{Hom}_{\text{Dia}}(K, L) \) is determined by \( \text{Hom}_{\text{Dia}}(K', L') \) if there exists an isomorphism of \( R \)-modules between them.

Lemma 3A.7. If the left mates of the \((k, l)\)-overcrossings are invertible, then all hom-spaces in \( \text{Dia} \) are determined by upwards hom-spaces.

Proof. By Lemma 3A.6, the assumptions imply that

\[
\begin{align*}
f & \rightarrow \\
\end{align*}
\]

is an isomorphism. This easily generalizes. □

Definition 3A.8. A left invertible morphism \((↑k)\circ h ↑l \rightarrow (↑k)\circ h ↑l\) is called explosion.

Up to rescaling, explosion morphisms and their left inverses can be illustrated by

\[
\begin{align*}
\text{respectively} & \quad \\
\end{align*}
\]

Lemma 3A.9. If the left mates of the \((k, l)\)-overcrossings are invertible and all explosion morphisms exist, then all hom-spaces in \( \text{Dia} \) are determined by end-spaces between objects of the form \( 1^\circ h \ldots \circ h 1 \) (k factors).

Proof. Turn all strands upwards using Lemma 3A.7, and then explode the strands inductively. □

3B. Exterior \( \mathfrak{gl}_n \)-webs. We now recall the category of exterior \( \mathfrak{gl}_n \) webs.

Definition 3B.1. The (exterior \( \mathfrak{gl}_n \)) web category \( \text{Web}_{\mathfrak{gl}_n} \) is the diagram category for \( R = \mathbb{Z}_v \) with \( \circ h \)-generating objects of categorical dimension \( \left[ \begin{array}{c} n \\ k \end{array} \right] \) and \( \circ v \)-\( \circ h \)-generating morphisms
such that the left mates of the \((k, l)\)-overcrossings are invertible. The relations imposed on \(\text{Web}_{n, \gl_n}\) are isotopies (not displayed zigzag and trivalent-slide relations, see e.g. [LT21, Section 2] for details), the exterior relation, associativity, coassociativity, digon removal, and dumbbell-crossing relation. That is, we take the quotient by the \(\circ_{n, \gl_n}\)-ideal generated by isotopies and

\[
\begin{align*}
> n & \quad k + l + m = k + l + m, \\
> n & \quad k + l + m = k + l + m, \\
> n & \quad k + l + m = k + l + m,
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{tikz_diagram1} \\
\end{array} & = 0, \\
\begin{array}{c}
\includegraphics{tikz_diagram2} \\
\end{array} & = \left[ \frac{k + l}{k} \right], \\
\begin{array}{c}
\includegraphics{tikz_diagram3} \\
\end{array} & = \left[ \frac{k + l}{k} \right].
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{tikz_diagram4} \\
\end{array} & = (-1)^{kl} \sum_{k = a = b} (\nu)^{k - a} (l - b), \\
\begin{array}{c}
\includegraphics{tikz_diagram5} \\
\end{array} & = (-1)^{kl} \sum_{k = a = b} (\nu)^{-(k - a)} (l - b), \\
\begin{array}{c}
\includegraphics{tikz_diagram6} \\
\end{array} & = (-1)^{kl} \sum_{k = a = b} (\nu)^{k + b}.
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{tikz_diagram7} \\
\end{array} & = (v, v^{-1}) \\
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{tikz_diagram8} \\
\end{array} & = v^{k(-k+n+1)}.
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{tikz_diagram9} \\
\end{array} & = 0, \\
\begin{array}{c}
\includegraphics{tikz_diagram10} \\
\end{array} & = 0
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{tikz_diagram11} \\
\end{array} & = v^{k(k-n-1)}, \\
\begin{array}{c}
\includegraphics{tikz_diagram12} \\
\end{array} & = 0.
\end{align*}
\]

We call morphisms in \(\text{Web}_{n, \gl_n}\) (exterior \(\gl_n\)) webs.

Remark 3B.2. The dumbbell-crossing relation is not new and can be deduced from Green’s book on the Schur algebra [Gr80] via an interpretation of webs as elements in the Schur algebra. Consequently, this relation is also called the Schur relation.

It follows for example from [LT21, Sections 2 and 5] that there is a well-defined functor from the pivotal category of topological webs (where webs are defined as plane labeled oriented trivalent graphs up to planar isotopy) to \(\text{Web}_{n, \gl_n}\) which we will use to draw webs in a topological fashion.

Lemma 3B.3. In \(\text{Web}_{n, \gl_n}\) we have

\[
\begin{align*}
(3B.4) & \quad \begin{array}{cc}
\begin{array}{c}
\includegraphics{tikz_diagram13} \\
\end{array} & = (-1)^{kl} \sum_{b = a = k = l} (\nu)^{k - b}, \\
\end{array} \\
\begin{array}{cc}
\begin{array}{c}
\includegraphics{tikz_diagram14} \\
\end{array} & = (-1)^{kl} \sum_{b = a = k = l} (\nu)^{-(k - a)} (l - b), \\
\end{array} \\
\begin{array}{cc}
\begin{array}{c}
\includegraphics{tikz_diagram15} \\
\end{array} & = (-1)^{kl} \sum_{b = a = k = l} (\nu)^{k + b}.
\end{array}
\end{align*}
\]

Proof. This is explained in [LT21, Section 5]. \qed

For completeness, the \(k = l = 1\) case of (3B.4) is

\[
\begin{align*}
(3B.5) & \quad \begin{array}{ccc}
\begin{array}{c}
\includegraphics{tikz_diagram16} \\
\end{array} & = v, \\
\end{array} \\
\begin{array}{ccc}
\begin{array}{c}
\includegraphics{tikz_diagram17} \\
\end{array} & = v^{-1}, \\
\end{array}
\end{align*}
\]

Lemma 3B.6. The following hold in \(\text{Web}_{n, \gl_n}\).

(a) The crossings satisfy the Reidemeister II and III relations, and the Reidemeister I relation holds up to scalars, that is,

\[
\begin{align*}
\begin{array}{cc}
\begin{array}{c}
\includegraphics{tikz_diagram18} \\
\end{array} & = v^{k(-k+n+1)}.
\end{array}
\end{align*}
\]

(b) Square switches, see e.g. [LT21, Lemma 5.6]. Various other relations that we do not explicitly use, see e.g. [LT21, Section 2], also hold.
(c) Explosion holds for $\text{Web}_{q\mathfrak{gl}_n}$ (but not in $\text{Web}_{e\mathfrak{gl}_n}$), that is

$$
\begin{array}{c}
\begin{array}{c}
k+l \\
k+l
\end{array}
= \left[\frac{k+l}{k}\right]^{-1} \cdot
\begin{array}{c}
k+l \\
k+l
\end{array},
\end{array}
$$

(together with its $(-)^2$-dual. Moreover, the $(k, l)$-overcrossings satisfy explosion as well, i.e.

$$
\begin{array}{c}
l \kern-30pt \strut
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\end{array}
= [k]!^{-1}[l]!^{-1} \cdot
\begin{array}{c}
k \kern-30pt \strut
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\end{array},
\end{array}
$$

as well as a similar formula for the $(k, l)$-undercrossings. $\square$

Note that thus Lemma 3A.6 applies in $\text{Web}_{q\mathfrak{gl}_n}$. Let us also note the following, partially explaining why explosion works well in practice:

**Lemma 3B.9.** The additive idempotent completion of $\text{Web}_{q\mathfrak{gl}_n}$ is semisimple.

**Proof.** By Lemma 3A.6 and the existence of certain projectors, see e.g. [RT16] or [TVW17, Section 2.3]. To be precise (and to fix notation for the rest of the paper), the projectors are

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}
\end{array}
= [k]!^{-1}[l]!^{-1} \cdot
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}
\end{array},
\end{array}
$$

where the dots indicate an inductive construction as illustrated on the right (the order of how these are constructed is irrelevant due to associativity and coassociativity). $\square$

## 4. Annular webs

This section discusses our main diagram categories of this paper. Similar constructions have appeared in many texts, e.g. [CK18] or [QW18].

### 4A. The annular web category.

The following definition does not use any $\circ_h$ structure.

**Definition 4A.1.** The (exterior $\mathfrak{gl}_n$) annular web category $\text{AWeb}_{e\mathfrak{gl}_n}$ is the category obtained from $\text{Web}_{e\mathfrak{gl}_n}$ by adding extra $\circ_v$-generators

$$
\begin{array}{c}
\begin{array}{c}
k_2 \kern-30pt \strut
\begin{array}{c}
\begin{array}{c}
k_1 \\
k_1
\end{array}
\end{array}
\end{array}
\end{array}
= [k]!^{-1}[l]!^{-1} \cdot
\begin{array}{c}
\begin{array}{c}
k_2 \kern-30pt \strut
\begin{array}{c}
\begin{array}{c}
k_1 \\
k_1
\end{array}
\end{array}
\end{array}
\end{array},
\end{array}
$$

for each $K = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ to $\text{Web}_{e\mathfrak{gl}_n}$ modulo the $\circ_v$-ideal generated by the relations

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}\end{array}
= [k]!^{-1}[l]!^{-1} \cdot
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array},
\end{array}
$$

(4A.2)
together with the \((-\uparrow)\) and \((-\leftrightarrow)\)-duals of the bottom two relations.

We call morphisms in $AWeb_{d, gl_n}$ annular (exterior $gl_n$) webs, and $\rho_K$ and $\rho^K$ are called coils.

**Remark 4A.5.** We think of the coils as crossings in front of the annulus, e.g.

$$\rho_{(1, 1)} \leftrightarrow 1 1 .$$

This convention comes because we follow [SS99] later on for computations. Using the inverse braiding compared to the definitions in [SS99, Theorem 3.2] translate to coils passing behind the annulus.

**Remark 4A.6.** The name annular webs comes from the interpretation of the pictures in Definition 4A.1 as embedded in an annulus. For example,

The following elements defined via Lagrange interpolation play a crucial role later on:

**Definition 4A.7.** For $i \in \{1, \ldots, d\}$ define

$$pr^w_i = \prod_{j \neq i} \frac{\rho_1 - u_j}{u_i - u_j} = \prod_{j \neq i} \frac{\sqrt{y_i - y_j}}{u_i - u_j} \in \text{End}_{AWeb_{d, gl_n}}(1),$$

which we call web block projectors.

**4B. Properties of annular webs.** We can endow $AWeb_{d, gl_n}$ with a monoidal structure $\circ_h^A$ as follows. On objects $K o_h K' = K o^K_h K'$ is just the concatenation, i.e. if $K = (k_1, \ldots, k_r)$ and $K' = (k'_1, \ldots, k'_s)$, then $K o_h K' = (k_1, \ldots, k_r, k'_1, \ldots, k'_s)$. On morphisms $o^K_h$ we use

$$f \circ^K_h g = \begin{array}{c}
\ldots \\
f \\
\ldots \\
\end{array} \begin{array}{c}
\ldots \\
g \\
\ldots \\
\end{array} \begin{array}{c}
\ldots \\
f \\
\ldots \\
\end{array},$$

using the $(k, l)$-crossings from (3B.4) and their mates so that $f$ is in the front and $g$ is in the back.

**Remark 4B.2.** (4B.1) is a standard construction in skein theory, see e.g. [PS98].

**Lemma 4B.3.** The monoidal structure $\circ_h^A$ and $\mathbf{1} = \emptyset$ endows $AWeb_{d, gl_n}$ with the structure of a pivotal category with duality given by cups and caps.
Proof. Easy and omitted.

Lemma 4B.4. The following holds in $\text{AWeb}_{v\mathfrak{gl}}_n$.

(a) We have an annular digon removal, that is

\[
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(together with its $(\_)^3$-dual. Various other annular versions of the relations in Lemma 3B.6 hold as well (but are not stated since we do not use them).

(b) All half-slides of merges, splits, cups and caps, e.g.

\[
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(c) Annular explosion holds for $\text{AWeb}_{q\mathfrak{gl}}_n$ (but not in $\text{AWeb}_{v\mathfrak{gl}}_n$), that is

\[
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k + l \\
k + l
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
l \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{K} \\
\text{K}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

(together with its $(\_)^{\pm}$-dual.

Proof. We get

(4B.5)

by (plain) explosion and (4A.3). The claimed relations can be proven using this. □

The thin coils suffice as $\circ_{v}$-generators:

Lemma 4B.6. The morphisms $\rho_{(k,K)}$ and $\rho^{(k,K)}$ in $\text{AWeb}_{q\mathfrak{gl}}_n$ (but not in $\text{AWeb}_{v\mathfrak{gl}}_n$) can be defined inductively from $\rho_{(\pm 1,K)}$ and $\rho^{(\pm 1,K)}$. Moreover, the morphisms $\rho_{(-1,K)}$ and $\rho^{(-1,K)}$ in $\text{AWeb}_{v\mathfrak{gl}}_n$ can be defined from $\rho^{(1,-1,K,-1)}$ and $\rho_{(1,-1,K,-1)}$. 

Proof. The pictures

\[
\begin{align*}
K & \quad k = [k]! - 1 \cdot k^1, \\
K & \quad K
\end{align*}
\]

define the morphisms as claimed. Their inverses are the \((\_\_\_\_\_\_\_\_\_)^{++}\)-duals of these pictures. □

The following lemma compares \(\text{AWeb}_{\mathfrak{g}l}_n\) to the construction in [CK18]:

**Lemma 4B.7.** The category \(\text{AWeb}_{\mathfrak{g}l}_n\) is equivalent as a diagram category to the affinization \(\text{Aff}(\text{Web}_{\mathfrak{g}l}_n)\) (in the sense of e.g. [MS21, Definition 2.1]) of \(\text{Web}_{\mathfrak{g}l}_n\).

**Proof.** We only sketch the proof: as often in diagrammatic algebra matching a generator-relation presentation with a “all diagrams” definition is lengthy and we omit some details.

First, there is an essentially surjective functor \(\Gamma\) from \(\text{AWeb}_{\mathfrak{g}l}_n\) to \(\text{Aff}(\text{Web}_{\mathfrak{g}l}_n)\) that puts a plane web into the annulus. Next, \(\text{Aff}(\text{Web}_{\mathfrak{g}l}_n)\) is defined by adjoining more morphisms and relations to \(\text{AWeb}_{\mathfrak{g}l}_n\), namely one coil and its inverse for each \(K\) and relations [MS21, equation (2.5)]. But using the coils in Definition 4A.1 one can define these more general coils following [MS21, equation (2.5), left] which satisfy [MS21, equation (2.5), right], showing that \(\Gamma\) is full. Faithfulness of \(\Gamma\) can then be deduced from Theorem 6B.3 below, by showing that the functor therein factors through \(\text{Aff}(\text{Web}_{\mathfrak{g}l}_n)\) via \(\Gamma\).

Alternatively, one can match the generator-relation presentation of \(\text{AWeb}_{\mathfrak{g}l}_n\) with the generator-relation presentation from [HäOl01], and then the topological presentation of \(\text{Aff}(\text{Web}_{\mathfrak{g}l}_n)\) with the topological presentation of [HäOl01] and the result follows via [HäOl01, Proposition 10]. (With the caveat that [HäOl01] only discusses tangles and the description therein needs to be extended to webs. That is straightforward but lengthy.) □

4C. Quotient by essential circles. We now define quotients of \(\text{AWeb}_{\mathfrak{g}l}_n\).

**Definition 4C.1.** The left and right essential \(k\)-circles are defined to be

\[
c_k^\leftarrow = \kern-2.5em \begin{array}{c}
\phantom{1}
\end{array} \kern-2.5em \begin{array}{c}
\phantom{1}
\end{array} \kern-2.5em \begin{array}{c}
\phantom{1}
\end{array} \\
= \begin{array}{c}
\phantom{k! - 1}
\end{array}, \quad c_k^\rightarrow = \begin{array}{c}
\phantom{k! - 1}
\end{array} \kern-2.5em \begin{array}{c}
\phantom{1}
\end{array} \kern-2.5em \begin{array}{c}
\phantom{1}
\end{array} \kern-2.5em \begin{array}{c}
\phantom{1}
\end{array}
\]

We also say essential circles for short.

Note that essential circles are nontrivial endomorphism of \(\mathbb{1}\). We want to evaluate them. To this end, let \(e_k\) denote the \(k\)th elementary symmetric polynomial in \(n\) variables, i.e. \(e_k = e_k(Z_1, \ldots, Z_n)\).

**Definition 4C.2.** The Levi evaluation for \(\ell\) of the essential circles is defined to be

\[
(4C.3)
\]

which are elements in \(\text{End}_{\text{AWeb}_{\mathfrak{g}l}_n}(\mathbb{1})\).
Example 4C.4. As an extreme case take \( \ell = \mathfrak{gl}_n \). The formulas in (4C.3) then become

\[
\begin{array}{c}
\frac{1}{k} \quad \leftarrow v^{-kn} \binom{n}{k} u_1^k, \\
\frac{1}{k} \quad \leftarrow v^{kn} \binom{n}{k} u_1^{-k}.
\end{array}
\]

The appearing scalars are multiples of the categorical dimension of \( \uparrow_k \), which is the value of the usual circle in the web calculus. Since we want to eventually evaluate essential circles to these scalars, this might be a hint of a connection to annular webs obtained from evaluation representations for the circle in the web calculus. Since we want to eventually evaluate essential circles to these scalars, this

The appearing scalars are multiples of the categorical dimension of \( \uparrow_k \), which is the value of the usual circle in the web calculus. Since we want to eventually evaluate essential circles to these scalars, this might be a hint of a connection to annular webs obtained from evaluation representations for the affine Lie algebra, see e.g. [Que15, Section 3].

The quotient \( \text{AW}_{q,\ell} \) of \( \text{AW}_{q,\mathfrak{gl}_n} \) by an ideal \( \mathcal{I}_\ell \) defined later in Section 6B gives a diagrammatic description of quantum \( \ell \)-representations.

For \( K_1 \) it will turn out that \( \mathcal{I}_\ell \) is the two-sided \( \mathcal{O}_e' - \mathcal{O}_e' \)-ideal generated by the Levi evaluations (4C.3). Being careful with the underlying monoidal structure (details are given in Theorem 6B.3), the same holds for \( K_q \), hence the name.

5. Representation theory of Levi subalgebras

This section discusses the representation categories of this paper. The below is (partially) well-known and we will be brief whenever appropriate. A lot of details and also background can be found in texts such as [Jan96].

5A. The general linear representation category. We start with notations regarding the general linear quantum algebra. Let \( U_q(\mathfrak{gl}_n) \) be the divided power quantum enveloping algebra for \( \mathfrak{gl}_n \), where we use the conventions, excluding the Hopf algebra structure, from [APW91] in the special case of \( \mathfrak{gl}_n \) (using \( K_q^\pm 1 = L_q^\pm 1 F_q^\pm 1 \)). The algebra \( U_q(\mathfrak{gl}_n) \) specializes to either the \( K_q \)-algebra \( U_q(\mathfrak{gl}_n) \), for which we now recall the relevant formulas, or the \( K_1 \)-algebra \( U_1(\mathfrak{gl}_n) \).

The algebra \( U_q(\mathfrak{gl}_n) \) is generated by \( L_i^\pm 1 \) for \( i \in \{1, \ldots, n\} \) (these are inverses) and \( E_i, F_i \) for \( i \in \{1, \ldots, n-1\} \) and the Hopf algebra structure used in this paper is

\[
\Delta(E_i) = E_i \otimes L_i L_{i+1}^{-1} + 1 \otimes E_i, \quad \epsilon(E_i) = 0, \quad S(E_i) = -E_i L_i^{-1} L_{i+1},
\]

\[
\Delta(F_i) = F_i \otimes 1 + L_i^{-1} L_{i+1} \otimes F_i, \quad \epsilon(F_i) = 0, \quad S(F_i) = -L_i^{-1} F_i L_{i+1},
\]

with \( L_i^\pm 1 \) being group like.

The vector representation \( V_q = V_q(\mathfrak{gl}_n) = K_q \{ v_1, \ldots, v_n \} \) of \( U_q(\mathfrak{gl}_n) \) is the given \( K_q \)-vector space with action

\[
L_i^\pm 1 \cdot v_j = q^{\pm \delta_{i,j}} v_j, \quad E_i \cdot v_j = \delta_{i,j-1} v_{j-1}, \quad F_i \cdot v_j = \delta_{i,j+1} v_{j+1}.
\]

Let \( TV_q \) be the tensor algebra. The \( k \)th quantum exterior power \( \bigwedge^k_q V_q \) is defined as the degree \( k \) part (in the usual sense) of the quantum exterior algebra given by

\[
\bigwedge^k_q V_q = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigwedge^k_q V_q = TV_q/\langle v_h \otimes v_i, v_j \otimes v_i + q^{-1} v_i \otimes v_j | i < j \rangle \text{two-sided } \otimes \text{-ideal}.
\]

The exterior powers are \( U_q(\mathfrak{gl}_n) \)-representations by using the Hopf algebra structure, and so is \( 1 = K_q \) itself and all the duals of the above, denoted by using negative powers: \( \bigwedge^{-k} q V_q = (\bigwedge^k q V_q)^* \) for \( k \in \mathbb{Z}_{< 0} \).

Lemma 5A.2. For \( k \in \mathbb{Z}_{\geq 0} \), the \( K_q \)-vector space \( \bigwedge^k_q V_q \) has a basis given by \( \{ v_S = v_{i_1} \otimes \ldots \otimes v_{i_k} | S = (i_1 < \ldots < i_k) \text{ for } i_j \in \{1, \ldots, n\} \} \). If \( -k \in \mathbb{Z}_{\geq 0} \), then the \( K_q \)-vector space \( \bigwedge^k_q V_q \) has a basis given by \( \{ v_S^* = v_{i_k} \otimes \ldots \otimes v_{i_1} | S = (i_1 < \ldots < i_k) \text{ for } i_j \in \{1, \ldots, n\} \} \).

Proof. Easy and omitted. \( \square \)

Notation 5A.3. We also use the notation \( v_S \) from Lemma 5A.2 more generally for any \( S = (i_1, \ldots, i_k) \) for \( i_j \in \{1, \ldots, n\} \), and use the usual set operations on them. Recall that such expressions need to potentially be reordered using (5A.1) to match the basis of Lemma 5A.2.

We now consider so-called \( U_q(\mathfrak{gl}_n) \)-representations of type 1 which, as usual, is not a serious restriction, see e.g. [Jan96, Section 5.2] for details.
Definition 5A.4. Let $\text{Rep}_q\mathfrak{gl}_n$ denote the category of finite dimensional $U_q(\mathfrak{gl}_n)$-representations of type 1. We view $\text{Rep}_q\mathfrak{gl}_n$ as pivotal using the above Hopf algebra structure on $U_q(\mathfrak{gl}_n)$. Let further $\text{Fund}_q\mathfrak{gl}_n$ denote the full pivotal subcategory with objects of the form $\bigwedge^k_q V_q = \bigwedge^k_q V_q \otimes \cdots \otimes \bigwedge^m_q V_q$ for $k = (k_1, ..., k_m) \in \mathbb{Z}^m$ and $m \in \mathbb{Z}_{\geq 0}$.

We call $\text{Rep}_q\mathfrak{gl}_n$ the representation category of $U_q(\mathfrak{gl}_n)$ and $\text{Fund}_q\mathfrak{gl}_n$ its fundamental category. (We use the same terminology for $\ell$ defined below.) The following is crucial, but well-known, and will be used throughout. To state it let $X^+_{\mathfrak{gl}_n} \subset \mathbb{Z}^n$ denote the set of dominant integral $\mathfrak{gl}_n$-weight, i.e. tuples $\lambda = (\lambda_1, ..., \lambda_n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$.

Lemma 5A.5. We have the following.

(a) The category $\text{Rep}_q\mathfrak{gl}_n$ is semisimple, its simple objects can be indexed by $\lambda \in X^+_{\mathfrak{gl}_n}$ and their characters are given by Weyl’s character formula.

(b) $\text{Fund}_q\mathfrak{gl}_n$ is pivotally equivalent to $\text{Rep}_q\mathfrak{gl}_n$ upon additive idempotent completion.

We will denote the simple objects in Lemma 5A.5.a by $L(\lambda)$ for $\lambda \in X^+_{\mathfrak{gl}_n}$. We do not need them explicitly, but their construction is well-known (the $L(\lambda)$ are often called Weyl modules).

Proof. Claim (a). See e.g. [Jan96, Theorems 5.15 and 5.17] or [APW91, Section 6].

Claim (b). By (a) classical theory applies, see e.g. [Jan96, Theorems 5.15 and 5.17] or [APW91, Section 6] for details. \qed

Let us now list generating $U_q(\mathfrak{gl}_n)$-equivariant morphisms that will be the images of the elements of $\text{Web}_{\mathfrak{gl}_n}$. The notation is hopefully suggestive.

For tuples $S, T$ as in Notation 5A.3 let $|S < T| = \left| \{(s, t) \in S \times T \mid s < t \} \right|$ and $|S, N| = |S < N| - |N < S|$ for $N = (1, 2, ..., n)$. Using such tuples and this notation we define (here $k, l \in \mathbb{Z}_{\geq 0}$):

\begin{align*}
&\Lambda^{k+l}_{k,l} : \wedge^k_q V_q \otimes \wedge^l_q V_q \to \wedge^{k+l}_q V_q, \quad v_S \otimes v_T \mapsto \delta_{S,T}\delta_{S,T}(-q)^{-|T < S|} v_S \otimes v_T, \\
&\gamma^k_{k,l} : \wedge^k_q V_q \otimes \wedge^l_q V_q \to \wedge^k_q V_q \otimes \wedge^l_q V_q, \quad v_U \mapsto (-1)^{k+l} \sum_{S \cup T = S \cup t \mid S = k} (-q)^{|S < T|} v_S \otimes v_T, \\
&\Lambda^k_{k} : \wedge^k_q V_q \otimes \wedge^k_q V_q \to \mathbb{K} q, \quad v_S \otimes v_T \mapsto \delta_{S,T}, \\
&\gamma^k_{k} : \wedge^k_q V_q \otimes \wedge^k_q V_q \to \mathbb{K} q, \quad v_S \otimes v_T \mapsto q^{S,N} \delta_{S,T}, \\
&\cup^k_k : \mathbb{K} q \to \wedge^k_q V_q \otimes \wedge^k_q V_q, \quad 1 \mapsto \sum_{|S| = k} v_S \otimes v_S, \\
&\cup^{k+\ast}_{k} : \mathbb{K} q \to \bigwedge^k_q V_q \otimes \bigwedge^k_q V_q, \quad 1 \mapsto \sum_{|S| = k} q^{-|S,N|} v^{\ast}_S \otimes v_S.
\end{align*}

(5A.6)

Lemma 5A.7. The morphisms in (5A.6) are $\circ_{- \otimes}$-generators of $\text{Fund}_q\mathfrak{gl}_n$.

Proof. A careful check of the relations shows that these maps are $U_q(\mathfrak{gl}_n)$-equivariant. That they generate follows from Lemma 5A.5 and classical theory. \qed

We have an algebraic version of explosion:

Lemma 5A.8. Explosion holds for $\text{Fund}_q\mathfrak{gl}_n$, that is

$$\text{id}_{k+l} = \Lambda^{k+l}_{k,l} \Lambda^{k+l-1}_{k,l} \cdots \Lambda^{1}_{k} \Lambda^{1}_{k}.$$ 

Proof. A direct computation. \qed

We denote by $Y^1_{k,1}$ and $\Lambda^k_{k,1}$ the successive explosion of $k$ strands.

We also have the following $(1, 1)$-overcrossings and $(1, 1)$-undercrossings:

\begin{align*}
\tilde{R}_{1,1} = q \text{id}_{1,1} - Y^1_{1,1} \Lambda^2_{1,1} : V_q \otimes V_q \to V_q \otimes V_q, \quad v_i \otimes v_j & \mapsto \begin{cases} v_i \otimes v_i & \text{if } i = j, \\
v_i \otimes v_i & \text{if } i < j, \\
v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i > j,
\end{cases}
\end{align*}
The case

Lemma 5B.3. As nevertheless we state a few lemmas that we will use. For starters, note that all

Lemma 5B.2. We have

We endow

Lemma 5B.1. Let

Definition 5B.5. Let \( \text{Rep}_q \ell \) denote the category of finite dimensional \( U_q(\ell) \)-representations of type 1. We view \( \text{Rep}_q \ell \) as pivotal using the above Hopf algebra structure on \( U_q(\mathfrak{gl}_n) \). Let further \( \text{Fund}_q \ell \) denote the full pivotal subcategory with objects of the form \( \bigwedge^k_q V_q \) for \( k = (k_1, \ldots, k_m) \in \mathbb{Z}^m \) and \( m \in \mathbb{Z}_{\geq 0} \).

Lemma 5B.4. As \( U_q(\ell) \)-representations we have \( V_q \otimes^k \cong \bigwedge^k_q V_q \oplus W \) and no simple constituent of \( \bigwedge^k_q V_q \) appears in \( W \).

Proof. By looking at highest weight vectors and classical theory, this follows directly from the first and the second decomposition in Lemma 5B.3. □

Definition 5B.5. Let \( \text{Rep}_q \ell \) denote the category of finite dimensional \( U_q(\ell) \)-representations of type 1. We view \( \text{Rep}_q \ell \) as pivotal using the above Hopf algebra structure on \( U_q(\mathfrak{gl}_n) \). Let further \( \text{Fund}_q \ell \) denote the full pivotal subcategory with objects of the form \( \bigwedge^k_q V_q \) for \( k = (k_1, \ldots, k_m) \in \mathbb{Z}^m \) and \( m \in \mathbb{Z}_{\geq 0} \).
We write $\circ f^\ell$ for the monoidal structure on $\text{Rep}_q \ell$ and $\text{Fund}_q \ell$ on the morphism level, and $\circ_h$ on the object level. With contrast to (4B.1), a picture for the monoidal structure on $\text{Fund}_q \ell$ is

\begin{equation}
\begin{array}{ccc}
& f & \circ^\ell \circ_h & g & \\
\vdots & & & \vdots & \\
& f & \circ_h & g & \\
\end{array}
\end{equation}

Remark 5B.7. (5B.6) is a standard construction in the theory of Hopf algebras, see e.g. [TV17].

Lemma 5B.8. We have the following.

(a) The category $\text{Rep}_q \ell$ is semisimple, and its simple objects are of the form $L(\lambda_1) \otimes \ldots \otimes L(\lambda_d)$ with the factors being simple objects of $\text{Rep}_q \mathfrak{g}_{l_1}^i$.

(b) The additive idempotent completion of $\text{Fund}_q \ell$ is pivotally equivalent to $\text{Rep}_q \ell$.

Proof. Lemma 5A.5 applies componentwise. $\square$

We define the Levi $(1,1)$-overcrossings and Levi $(1,1)$-undercrossing to be

\begin{align*}
\hat{R}^\ell_{1,1} : & \mathbb{V}_q \otimes \mathbb{V}_q \rightarrow \mathbb{V}_q \otimes \mathbb{V}_q, & & v_{i,j} \otimes v_{k,l} \mapsto \begin{cases} v_{k,l} \otimes v_{i,j} & \text{if } i = k, \\ v_{k,l} \otimes v_{i,j} & \text{else}, \end{cases} \\
(\hat{R}^\ell_{1,1})^{-1} : & \mathbb{V}_q \otimes \mathbb{V}_q \rightarrow \mathbb{V}_q \otimes \mathbb{V}_q, & & v_{i,j} \otimes v_{k,l} \mapsto \begin{cases} (\hat{R}^\ell_{1,1})^{-1}(v_{i,j} \otimes v_{k,l}) & \text{if } i = k, \\ v_{k,l} \otimes v_{i,j} & \text{else}. \end{cases}
\end{align*}

In other words, $\hat{R}^\ell_{1,1}$ is the respective $\mathfrak{g}_{l_1}^i$ braiding within one block, and the swap map otherwise, and similarly for its inverse. Note that these maps are in general not $U_q(\mathfrak{g}_{l_1}^i)$-equivariant.

Example 5B.9. In the extreme case that $\ell = \mathfrak{h}$ the Levi $(1,1)$-overcrossings equals the Levi $(1,1)$-undercrossings equals the swap map. $\square$

Definition 5B.10. For $k,l \in \mathbb{Z}_{\geq 0}$ define $\hat{R}^\ell_{k,l}$ as the composition

\[ \bigwedge_q^k \mathbb{V}_q \otimes \bigwedge_q^l \mathbb{V}_q \xrightarrow{\iota} \mathbb{V}_q^{(k+l)} \xrightarrow{\pi} \mathbb{V}_q^{(k+1)} \xrightarrow{p} \bigwedge_q^k \mathbb{V}_q \otimes \bigwedge_q^l \mathbb{V}_q, \]

where are i and p are inclusion and projection, respectively, and $x$ is defined as in (3B.8) but with Levi crossings. Define $(\hat{R}^\ell_{k,l})^{-1}$ similarly.

We also get various mates of which we think as rotated versions of the ones in Definition 5B.10.

Lemma 5B.11. The Levi crossings are $U_q(\ell)$-equivariant, satisfy the Reidemeister II and III relations and various naturality relations.

Proof. This follows from a calculation, Lemma 5B.2 and Lemma 5A.9. $\square$

Define now the following coil maps:

\begin{align}
& r_{1,k} : \mathbb{V}_q \otimes \bigwedge_q^k \mathbb{V}_q \rightarrow \bigwedge_q^k \mathbb{V}_q \otimes \mathbb{V}_q, & & v_{i,j} \otimes w \mapsto u_i \hat{R}^\ell_{k,1,k}(v_{i,j} \otimes w), \\
& r^{1,k} : \bigwedge_q^k \mathbb{V}_q \otimes \mathbb{V}_q \rightarrow \mathbb{V}_q \otimes \bigwedge_q^k \mathbb{V}_q, & & v \otimes w \mapsto u^{-1}_i(\hat{R}^\ell_{1,k})^{-1}(w \otimes v_{i,j}).
\end{align}

Note that $r_{1,k}$ and $r^{1,k}$ are inverses. The coil maps are not $U_q(\mathfrak{g}_{l_1}^i)$-equivariant in general.

Lemma 5B.13. The morphisms in (5A.6) and (5B.12) are $\circ_h$-generators of $\text{Fund}_q \ell$.

Proof. The morphisms in (5A.6) are $U_q(\mathfrak{g}_{l_1}^i)$-equivariant, so they are also $U_q(\ell)$-equivariant, and one easily checks that the morphisms in (5B.12) are $U_q(\ell)$-equivariant. That these generate follows from Lemma 5A.7 and Lemma 5B.3. $\square$

For $K = \emptyset$ we will write $r_1 = r_{1,\emptyset}$.

Definition 5B.14. For $i \in \{1, \ldots, d\}$ define

\[ \text{pr}^i_\ell = \prod_{j \neq i} \frac{\ell_1 - u_j}{u_i - u_j} \in \text{End}_{\text{Fund}_q \ell}(\mathbb{V}_q), \]

which we call Levi block projectors.
**Lemma 5B.15.** We have \(\text{pr}_i^l \text{pr}_j^l = \delta_{i,j} \text{pr}_i^l\) and \(\text{id}_{V_q} = \sum_{i=1}^d \text{pr}_i^l\). These projectors realize the decomposition \(V_q \cong \bigoplus_{i=1}^d V_q(sl_i)\).

**Proof.** Note that \(r_1\) is given by multiplication by \(u_i\) on \(v_{i,j}\). Thus, the formula for \(\text{pr}_i^l\) is the usual Lagrange-type interpolation and the claims follow. \(\square\)

5C. Levi crossings. The Levi crossings are not \(U_q(gl_n)\)-equivariant in general, and there is no planar web picture for it. However, it will be helpful to have the following diagrammatic notation. For the Levi \((k,l)\)-overcrossings and the Levi \((k,l)\)-undercrossings we use:

\[
\text{over} : \hat{R}^{l}_{k,l} \leftrightarrow \begin{array}{c} l \\ k \\ l \\ k \end{array}, \quad \text{under} : \left(\hat{R}^{l}_{k,l}\right)^{-1} \leftrightarrow \begin{array}{c} k \\ k \\ l \\ l \end{array}.
\]

We also use rotated pictures for their mates.

By Lemma 5B.11, we have the Reidemeister II and III relations, e.g.

\[
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}, \quad \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.
\]

We can use this to define *Levi braids* associated to any braid word. Of particular importance will be the (positive) *Levi full twist* on \(k\) strands (denoted by a box notation). By definition, this map is the square of the positive lift, using Levi overcrossings, of the longest word in the symmetric group on \(\{1, ..., k\}\). For example, for \(k = 4\) this full twist is

\[
\text{ft} = \left(\begin{array}{c}
\end{array}\right)^2.
\]

We also have the usual naturality relations such as

\[
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} \quad \text{or} \quad \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array},
\]

including the various \((-)^4\) and \((-)^\ast\ast\)-duals.

However, we need to be careful with the *Reidemeister I relation* as \(\bigwedge^k V_q\) needs not to be simple as a \(U_q(\ell)\)-representation, see Lemma 5B.3. Nevertheless, we still have the following lemma.

**Lemma 5C.1.** The Levi crossings are diagonal matrices in the basis given by the decomposition of \(\bigwedge^k V_q\) from Lemma 5B.3.

**Proof.** Note that the decomposition of \(\bigwedge^k V_q\) from Lemma 5B.3 is multiplicity free, and Schur’s lemma applies. (We stress that Schur’s lemma in this setting does not need the underlying field to be algebraically closed, see e.g. [APW91, Corollary 7.4] or [AT17, Remark 2.29].) \(\square\)

One can check that the diagonal entries mentioned in Lemma 5C.1 are given by products of the Reidemeister I scalars in (3B.7).

6. THE EQUIVALENCE

We now state and prove our main result.
6A. A reminder on the \( \mathfrak{gl}_n \) story. We first recall the relationship between \( \mathfrak{gl}_n \) webs and the representation theory of \( U_q(\mathfrak{gl}_n) \). Define a functor
\[
\Gamma_q = \Gamma_q^{\text{ext}}(\mathfrak{gl}_n) : \text{Web}_q \mathfrak{gl}_n \to \text{Fund}_q \mathfrak{gl}_n
\]
sending the object \( K \) to \( V^K_q \) and the generating morphisms of \( \text{Web}_q \mathfrak{gl}_n \) to the following maps:
\[
\begin{align*}
& k \mapsto Y_{k,l}^{k+l}, & & l \mapsto Y_{k,l}^{k+l}, & & k + l \mapsto \cap_k \rightarrow \cap_k^-, & & k - l \mapsto \cup_k \rightarrow \cup_k^-, & & -k \mapsto \cap_k^-, & & -k \mapsto \cup_k^-.
\end{align*}
\]
and the downwards merges and splits to the respective mates.

**Theorem 6A.1.** The functor \( \Gamma_q \) is an equivalence of pivotal categories, and it induces an equivalence of pivotal categories between the additive idempotent completion of \( \text{Web}_q \mathfrak{gl}_n \) and \( \text{Rep}_q \mathfrak{gl}_n \).

**Proof.** That \( \Gamma_q \) is fully faithful is [CKM14, Theorem 3.3.1] and the fact that the relevant hom-spaces stay of the same dimension when restricting from \( \mathfrak{gl}_n \) to \( \mathfrak{sl}_n \). The second claim follows from Lemma 5A.5.(b), or [CKM14, Theorem 3.3.1] and flatness of restriction from \( \mathfrak{gl}_n \) to \( \mathfrak{sl}_n \). \( \square \)

6B. The statement. We now extend the functor \( \Gamma_q \) into a functor
\[
\mathbf{A} \Gamma_q = \mathbf{A} \Gamma_q^{\text{ext}}(\ell) : \text{AWeb}_q \mathfrak{gl}_n \to \text{Fund}_q \ell.
\]
On objects and the generators of \( \text{Web}_q \mathfrak{gl}_n \) the functor \( \mathbf{A} \Gamma_q \) is defined to be \( \Gamma_q \). We define \( \mathbf{A} \Gamma_q \) on the two coils \( \rho^K \) and \( \rho^K_\ell \) for \( k_1 = 1 \) by
\[
\begin{align*}
& k_2 \cdots 1 \mapsto \gamma_{1,K}, & & 1 \cdots k_m \mapsto \gamma_{1,K}^\ell.
\end{align*}
\]
We call the kernel of the functor \( \mathbf{A} \Gamma_q \) the **Levi ideal** and denote it by \( \mathcal{I}_\ell \).

**Lemma 6B.1.** The Levi ideal \( \mathcal{I}_\ell \) is a two-sided \( \circ_v \)-ideal in \( \text{AWeb}_q \mathfrak{gl}_n \).

**Proof.** The kernel is a two-sided \( \circ_v \)-ideal. \( \square \)

Thus, we get a well-defined category by the following definition.

**Definition 6B.2.** Let \( \text{AWeb}_q \ell \) denote the quotient of \( \text{AWeb}_q \mathfrak{gl}_n \) by the Levi ideal \( \mathcal{I}_\ell \).

**Theorem 6B.3.** Let \( \text{AdId}(\_ \_ \_) \) denote additive idempotent completion.

(a) We have the commuting diagram of categories
\[
\begin{align*}
\text{AWeb}_q \mathfrak{gl}_n \xrightarrow{(4C.3)} & \text{AWeb}_q \ell \xrightarrow{\text{AdId}(\_ \_ \_)} \text{AdId}(\text{AWeb}_q \ell) \\
\text{A} \Gamma_q & \cong \text{A} \Gamma_q^{\ell} \xrightarrow{\cong} \text{AdId}(\text{A} \Gamma_q^{\ell}) \xrightarrow{\cong} \text{AdId}(\text{A} \Gamma_q^{\ell}) \\
\text{Fund}_q \ell & \xrightarrow{\text{AdId}(\_ \_ \_)} \text{Rep}_q \ell.
\end{align*}
\]
The Levi ideal \( \mathcal{I}_\ell \) is the two-sided \( \circ_v \circ_h^{\ell} \)-ideal generated by the Levi evaluations. Here \( \circ_h^{\ell} \) is the pullback of the monoidal structure from \( \text{Fund}_q \ell \) to \( \text{AWeb}_q \mathfrak{gl}_n \).

(b) We have the commuting diagram of pivotal categories
\[
\begin{align*}
\text{AWeb}_1 \mathfrak{gl}_n \xrightarrow{(4C.3)} & \text{AWeb}_1 \ell \xrightarrow{\text{AdId}(\_ \_ \_)} \text{AdId}(\text{AWeb}_1 \ell) \\
\text{A} \Gamma_1 & \cong \text{A} \Gamma_1^{\ell} \xrightarrow{\cong} \text{AdId}(\text{A} \Gamma_1^{\ell}) \xrightarrow{\cong} \text{AdId}(\text{A} \Gamma_1^{\ell}) \\
\text{Fund}_1 \ell & \xrightarrow{\text{AdId}(\_ \_ \_)} \text{Rep}_1 \ell.
\end{align*}
\]
The Levi ideal \( \mathcal{I}_\ell \) is the two-sided \( \circ_v \circ_h^A \)-ideal generated by the Levi evaluations.
The proof of Theorem 6B.3 is postponed to Section 7, since we want to focus on applications of this theorem first. For the rest of the section we assume that Theorem 6B.3 holds with the exception of the next subsection where we only assume that $A\Gamma_q$ is well-defined.

6C. Monoidal behavior of the main functor. The categories $\text{AWeb}_q\mathfrak{gl}_n$ and $\text{Rep}_q\ell$ are endowed with monoidal structures that are natural from two different perspectives, as explained in Remark 4B.2. However, as we will elaborate now, these need not to be the same under the equivalence Theorem 6B.3.

Lemma 6C.1. Assume that the functor $A\Gamma_q$ is well-defined.

(a) The functor $A\Gamma_1$ is pivotal.

(b) The functor $A\Gamma_q$ is not monoidal (and thus not pivotal).

Proof. (a). Let us consider $q = 1$. In this case the braiding on $\text{Rep}_1\ell$ is given by permutation. Comparing (4B.1) and (5B.6), and observing that coils in $\text{Rep}_1\ell$ are permutations, up to diagonal entries of the form $u_i$ on blocks, shows that the functor $A\Gamma_1$ is monoidal. Pivotality is then clear.

(b). For $q \neq 1$, one can check that the images under $A\Gamma_q$ of (4B.1) is not the same as (5B.6). Explicitly, taking $k = L = (1), f = \rho_l$ and $g = \text{id}_L$ verifies that $A\Gamma_q$ is not monoidal. □

Remark 6C.2. We note that Lemma 6C.1 shows that the choice which side goes over or under in (4B.1) matters and gives different results on the representation theoretical side. This indicates that one might need to use the notion of module categories rather than monoidal categories to describe the representation theory associated to $A\text{Web}_q\mathfrak{gl}_n$. This is similar as in, for example, [ST19] or [ES18] (via [MS16, Remark 12]), so coideal subalgebras might play a role.

The quantization issue that we are facing in Lemma 6C.1 is also potentially related to the classification of K-matrices as, for example, in [Mud02] where all nondegenerate solutions to the reflection equation in the quantum case have satisfy a minimal polynomial of order $\leq 2$. Diagrammatically K-matrices correspond to coils, but for $\ell = \mathfrak{gl}_{l_1} \oplus \ldots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n$ these coils need to have a minimal polynomial of order $d$.

Remark 6C.3. One could use the equivalence in Theorem 6B.3 to pullback the monoidal structure of $\text{Rep}_q\ell$, resulting in a monoidal structure on $A\text{Web}_q\ell$ that is distinct from the one we give in Lemma 4B.3 above. This pullback monoidal structure would not satisfy the conditions in [MS21, (2.11)], which are necessary for it to be unique, see [MS21, Proposition 2.5]. Conversely, one could push the monoidal structure from $A\text{Web}_q\ell$ to $\text{Rep}_q\ell$, resulting in a monoidal structure on $\text{Rep}_q\ell$ satisfying [MS21, (2.11)].

6D. Ariki–Koike algebras and annular webs. The Ariki–Koike algebra $H_q^{m,d}$ from [AK94], [BM93], [Che87], using different conventions, is defined as follows.

Definition 6D.1. Fix $m \in \mathbb{Z}_{\geq 0}$, the number of strands, and let $H_q^{m,d}$ denote the $\mathbb{Z}_q$-algebra with algebra generators $T_0, T_1, \ldots, T_{m-1}$ modulo the two-sided ideal generated by

$$\prod_{k=1}^{d} (T_0 - u_k), \ T_0 T_1 T_0 T_1 - T_1 T_0 T_1 T_0, \ (T_i - q)(T_i + q^{-1}) \text{ if } i > 0, \ T_i T_j T_i - T_j T_i T_i \text{ if } |i - j| = 1, \ T_i T_j - T_j T_i \text{ if } |i - j| > 1,$$

where $i, j \in \{1, \ldots, m-1\}$.

The Ariki–Koike algebra acts on $1^{\otimes m}$.

Proposition 6D.2. We have a surjective $\mathbb{K}_q$-algebra homomorphism

$$\pi_{m,d}: H_q^{m,d} \to \text{End}_{A\text{Web}_q\ell}(1^{\otimes m}), \ T_0 \mapsto \begin{array}{c} 1 \\
\hline \hline \hline
1^{\otimes (m-1)} \\
\end{array}, \ T_i \mapsto \begin{array}{c} 1 \\
\hline \hline \hline
1^{\otimes (m-1)}
\end{array}.$$

Here the bottom left strand of the image of $T_i$ is the $i$th strand from the left.

Proof. By Theorem 6B.3, this follows from [SS99, Theorem 4.2] after adjustment of conventions. □
We define the usual Jucys–Murphy elements as follows.

**Definition 6D.3.** Define elements of $H_{q}^{m,d}$ by $X_1 = T_0$ and for $i \in \mathbb{Z}_{\geq 1}$ recursively $X_i = T_i X_{i-1} T_i$.

Let $P(m,d)$ denote the set of $d$-partitions of $m$ (which we identify with $d$-tuples of Young diagrams in the English convention), and for $\lambda \in P(m,d)$ let $Std(\lambda)$ denote the set of all standard $d$-tableaux of shape $\lambda$.

**Lemma 6D.4.** For all $\lambda \in P(m,d)$ there exists a simple $H_{q}^{m,d}$-representation $V_\lambda$, and these form a complete and nonredundant set of simple $H_{q}^{m,d}$-representations. Moreover, we have an $\mathbb{K}_q$-algebra isomorphism $H_{q}^{m,d} \cong \bigoplus_{\lambda \in P(m,d)} \text{End}_{\mathbb{K}_q}(V_\lambda)$. Finally, $V_\lambda$ can be given a $\mathbb{K}_q$-basis $\{v_T | T \in Std(\lambda)\}$ such that $X_i$ acts by

$$(6D.5) \quad X_i \cdot v_T = u_a q^{2b-2c} v_T,$$

where $a$ is the component of the entry $i$ in $T$, $b$ is the column number where $i$ appears and $c$ is the row number of $i$.

**Proof.** Our assumptions on the involved parameters imply that $H_{q}^{m,d}$ is semisimple and the lemma follows from [AK94, Theorem 3.7] and results in the same section, e.g. [AK94, Proposition 3.16].

Let $I_{>l_1,\ldots,l_d} \subset H_{q}^{m,d}$ denote the two-sided ideal generated by the idempotents, realizing the Artin–Wedderburn decomposition in Lemma 6D.4, for $d$-partitions of $m$ with strictly more than $l_i$ rows in the $i$th entry.

**Example 6D.6.** An important special case is $I_{>1,\ldots,1} \subset H_{q}^{m,d}$. In this case the $d$-partitions indexing the idempotents not in $I_{>1,\ldots,1}$ are of the form

$$\begin{bmatrix}
\square & \cdots & \square \\
\kappa_1 & \cdots & \kappa_d
\end{bmatrix}$$

for $(k_1,\ldots,k_d) \in \mathbb{Z}_{\geq 0}^d$. These are so-called one row $d$-partitions. \diamondsuit

**Remark 6D.7.** The image of $I_{>l_1,\ldots,l_d} \subset H_{q}^{m,d}$ under $\pi_{m,d}$ from Proposition 6D.2 is forcing a condition on the minimal polynomial of coils.

The following definition appears in [LV21, Section 2C]:

**Definition 6D.8.** Let $B_q^\ell$ be the algebra quotient of $H_q^{m,d}$ by $I_{>l_1,\ldots,l_d}$.

**Proposition 6D.9.** The map $\pi_{m,d}$ from Proposition 6D.2 induces a $\mathbb{K}_q$-algebra isomorphism

$$\pi_\ell : B_q^\ell \cong \text{End}_{\text{AWeb}_q}(1_{0,\ell,m}).$$

Thus, the kernel of $\pi_{m,d}$ is $\ker(\pi_{m,d}) = I_{>l_1,\ldots,l_d}$.

**Proof.** As in the proof of Proposition 6D.2. \qed

6E. Cartan subalgebras and generalized blob algebras. We now consider the case of the Cartan subalgebra in detail.

**Proposition 6E.1.** The functor $\Gamma_q$ descends to an equivalence of categories $\Gamma_q^h : \text{AWeb}_q h \to \text{Fund}_h$, and it induces an equivalence of categories between the additive idempotent completion of $\text{AWeb}_q h$ and $\text{Rep}_q h$.

**Proof.** Directly from Theorem 6B.3 and Lemma 6C.1. \qed

**Remark 6E.2.** Proposition 6E.1 should be compared with [QW18, Corollary 43], and can be seen as a quantum version of that corollary.

Now recall the so-called generalized blob algebra $B_q^{m,d}$ in the sense of [MW03] (which is a special case of Definition 6D.8):

**Definition 6E.3.** Let $B_q^{m,d}$ be the algebra quotient of $H_q^{m,d}$ by $I_{>1,\ldots,1}$. 

Proposition 6E.4. The map \( \pi_{m,d} \) from Proposition 6D.2 induces a \( \mathbb{K}_q \)-algebra isomorphism

\[
\pi_{m,d} : \mathbb{K}_q^{m,d} \xrightarrow{\cong} \text{End}_{\text{AWeb}_b}(1^{0,h,m}).
\]

Thus, the kernel of \( \pi_{m,d} \) is ker(\( \pi_{m,d} \)) = \( J_{1,\ldots,1} \).

Proof. By Proposition 6E.1, this follows from [ATY95, Theorem 3.1].

6F. On two conjectures about end-spaces in annular webs. Recall from Section 4C that \( e_k \) denotes the \( k \)th elementary symmetric polynomial in \( d \) variables. In the special case of the Cartan subalgebra we have \( d = n \), and we let \( e_k = e_k(u_1, \ldots, u_d) \).

Definition 6F.1. Define \( e_k^{(i)} = e_k \) and for \( i \in \mathbb{Z}_{\geq 1} \) recursively \( e_k^{(i)} = e_k^{(i-1)} + (q^2 - 1)(X_{i-1} e_k^{(i-1)} - X_{i-1} e_{k-1}^{(i-1)} + \ldots \pm X_k) \), and let \( J_{>1} \subset H_q^{m,d} \) denote the two-sided ideal generated by \( R_i = X_i - e_1^{(i)} X_i^{d-1} + \ldots \pm e_d^{(i)} \). Let \( J_2 \subset H_q^{m,d} \) denote the two-sided ideal generated by \( R_2 \).

Proposition 6F.2. The kernel of \( \pi_{m,d} \) is alternatively given by ker(\( \pi_{m,d} \)) = \( J_2 = J_{>1} \). The same holds for \( q = 1 \).

Proof. We start by making two claims.

Claim 1. \( R_2 \) acts on \( V_{\lambda} \) as zero if and only if the \( d \)-partition \( \lambda \) has at most one row per component.

Proof of the Claim 1. The case where \( \lambda \) has one node is easy, so assume that \( \lambda \) has at least two nodes. We use that \( V_{\lambda} \) has a \( \mathbb{K}_q \)-basis given by \( v_T \) for \( T \) a standard \( d \)-tableau of shape \( \lambda \) on which \( X_i \) acts by \( (6D.5) \). Using \( (6D.5) \), we can calculate the action of \( R_2 \) on the \( \mathbb{K}_q \)-basis given by the \( v_T \). There are three cases depending on the positions of 1 and 2 in \( T \) one needs to check:

\[
\begin{align*}
(i): (\ldots [1], \ldots, 2 \ldots) & \text{ or vice versa,} \\
(ii): (\ldots 1 2 \ldots), & \text{ (iii): (\ldots 1 2 \ldots).}
\end{align*}
\]

All of these are annoying but straightforward calculations, and details are omitted.

Claim 2. \( J_2 = J_{>1} \).

Proof of the Claim 2. To see this, we note that on \( v_T \in V_{\lambda} \) for \( \lambda \) a one row \( d \)-partition and \( T \) a standard \( d \)-tableau of shape \( \lambda \) we have

\[
e_k^{(i)} \cdot v_T = e_k(q^2 u_1, \ldots, q^2 u_d) v_T,
\]

where \( \alpha_r = |\{s < i \mid s \text{ is in the } r \text{th component of } T\}| \). Using this formula and \( (6D.5) \) one can recursively check that the above claim holds for \( R_i \) for \( i \geq 2 \), which implies that \( J_{>1} \subset J_2 \), and the proof of the claim is complete.

The first claim implies that \( J_2 = J_{>1} \), and this together with the second claim and Proposition 6E.4 proves the lemma.

Recall \( \text{Aff}(\text{Web}_q \mathfrak{g}_n) \) from Lemma 4B.7. Mimicking the construction of \( \text{AWeb}_b \mathfrak{g}^{\ell} \) from \( \text{AWeb}_b \mathfrak{g}_n \) as in Definition 6B.2, we denote by \( \text{Aff}(\text{Web}_q \mathfrak{g}_n) / \text{LI} \) (LI stand for Levi ideal) the quotient of \( \text{Aff}(\text{Web}_q \mathfrak{g}_n) \) by essential circles. The following will be compared with [CK18, Conjectures 10.2 and 10.3] in Proposition 6F.4 below.

Proposition 6F.3. We have \( \mathbb{K}_q \)-algebra isomorphisms

\[
H_q^{m,d} / J_2 \cong H_q^{m,d} / J_{>1} \xrightarrow{\cong} \text{End}_{\text{AWeb}_b}(1^{0,h,m}) \cong \text{End}_{\text{Aff}(\text{Web}_q \mathfrak{g}_n) / \text{LI}}(1^{0,h,m}),
\]

with the first map being induced by Proposition 6D.2 and the second being induced by Lemma 4B.7.

Proof. Combine Lemma 4B.7 and Proposition 6F.2.

Proposition 6F.4. Proposition 6F.3 answers [CK18, Conjectures 10.2 and 10.3] affirmatively (up to different ground rings; we address this in Remark 6F.5 below).

Proof. The common object of interest when comparing [CK18] to this paper is the annular web category and its various flavors:

(i) First, we have \( \text{AWeb}_q \mathfrak{g}_n \) and \( \text{AWeb}_q \mathfrak{g}^{\ell} \) and these are the main objects of study in this paper.

(ii) We also have \( \text{Aff}(\text{Web}_q \mathfrak{g}_n) \) and \( \text{Aff}(\text{Web}_q \mathfrak{g}_n) / \text{LI} \), the versions defined by affinization. These are studied in [CK18] for \( \mathfrak{s}_n \) instead of \( \mathfrak{g}_n \).
We first note that the difference between $\mathfrak{sl}_n$ instead of $\mathfrak{gl}_n$ plays no key role in the sense that the relevant hom-spaces are of the same dimension and all maps are defined verbatim. On the representation theoretical side this is well-known, for the webs see e.g. [TVW17, Remark 1.1].

In [CK18, Section 10] Cautis–Kamnitzer define a map from the affine Hecke algebra $H^q$ to the category $\text{Aff}(\text{Web}^q_{\mathfrak{gl}_n})$. In [CK18, Conjecture 10.3] they conjecture what the kernel of this map is. In Proposition 6F.3 we identify the kernel of the respective map from the Ariki–Koike algebra $H^q_{m,d}$ to $\text{AWeb}^q_{\ell}$. Thus, we get the following comparison diagram:

\[ H^q_{m,d} \xrightarrow{\text{map in Proposition 6D.2}} \text{Aff}(\text{Web}^q_{\mathfrak{gl}_n}) \xrightarrow{\text{map in [CK18, Section 10]}} \text{Rep}_q^\ell \xrightarrow{\text{Proposition 6F.3}} \text{AWeb}^q_{\ell}. \]

Comparison of definitions shows that this diagram commutes. Similarly for [CK18, Conjecture 10.2] which follows from the $q = 1$ version of the above.

\begin{proof}
\end{proof}

Remark 6F.5. Recall from Notation 2.2 that we work over a field containing variables $\mathbb{U} = \{u_1, \ldots, u_d\}$ as well as $\mathbb{U}^{-1}$. We use this crucially in (4C.3) where we evaluate essential circles to elementary symmetric polynomials $e_k$ in these variables and their inverses.

On the other hand, the ground used in [CK18, Section 10] is

\[ E = \mathbb{C}(q)[\tilde{e}_1, \ldots, \tilde{e}_n] \]

where $\tilde{e}_k$ is the $k$th elementary symmetric function (not the polynomial), and these elementary symmetric function compare to our variables $\mathbb{U}$.

Note that [CK18, Section 10] does not have $\mathbb{U}^{-1}$ and in this sense Proposition 6F.3 and [CK18, Conjectures 10.2 and 10.3] are strictly speaking not comparable.

6G. Working integrally. Note that our main result Theorem 6B.3 is not stated or proven over $\mathbb{Z}_v$, and we work over $\mathbb{K}_q$ in which case $\text{AWeb}^q_{\ell}$ and $\text{Rep}_q^\ell$ are semisimple. Working integrally, that is, over $\mathbb{Z}_v$ or even $\mathbb{Z}[v, v^{-1}, \mathbb{U}]$ needs some nontrivial extra steps:

(i) Theorem 6A.1 works over $\mathbb{Z}_v$, see [Eli15, Theorem 2.58], which uses the light ladder strategy from that paper and [AST18, Theorem 3.1]. Passing to an appropriate field (e.g. $\mathbb{F}_p$ for a prime $p$) one gets an equivalence of pivotal categories between the additive idempotent completion of $\text{Web}^q_{\mathfrak{gl}_n}$, and the category of tilting modules $\text{Tilt}_{\ell}^\mathfrak{gl}_n$.

(ii) The relation from diagram categories to tilting modules is a folk observation in the field, see e.g. [Eli15, Theorem 2.58], [AST18, Section 5A], [TW21, Proposition 2.28] or [Bod20, Theorem 1.1] for some examples.

(iii) Thus, it is tempting to conjecture that integral versions of Theorem 6B.3 and its consequences involve $\text{Tilt}_{\ell}^\mathfrak{gl}_n$, e.g. under appropriate assumptions on the underlying field the additive idempotent completion of $\text{AWeb}^q_{\ell}$ should be equivalent to $\text{Tilt}_{\ell}^\mathfrak{gl}_n$. However, there is a nontrivial catch: the quantization does not behave very well, see e.g. Section 6C or [CK18, Section 10]. As sketched in Remark 6C.2, this might indicate that quantum groups are not the correct objects to use in this setting.

(iv) Note that the blob algebra is not defined integrally, and it is also not clear from the definition how to work integrally. Let us however point out that the description of $\mathcal{B}_q^{m,d}$ from [LV21, Theorem 2.15] works integrally and might play a role in the integral story.

We decided not to pursue these points further in this work.

7. Proof of the main theorem

7A. Well-definedness. Recall that the images of the coils are defined by explosion, that is, we mimic (4B.5) on the side of the representation theory. We define $u: V^\otimes_{q,k} \to V^\otimes_{q,k}$ as the map sending $v_{i_1,j_1} \otimes \cdots \otimes v_{i_k,j_k}$ to $u_{i_1} \cdots u_{i_k} v_{i_1,j_1} \otimes \cdots \otimes v_{i_k,j_k}$. We use a box to denote this map in illustrations.
Lemma 7A.1. We have

\[ A_G(\rho_{(k,K)}) = [k]^{-1}. \]

\[ \begin{array}{c}
\text{ft} \\
K \\
k \\
\end{array} \]

Proof. Using the Levi crossings introduced in Section 5C, we easily check that \( A_G \) sends the coil \( \rho_{(k,K)} \) to the claimed picture. \( \square \)

Lemma 7A.2. In \( \text{Fund}_q \) we have

\[ \begin{array}{c}
[k]^{-1}[l]^{-1} \cdot \text{ft} = [k+l]^{-1} \cdot \text{ft}.
\end{array} \]

Proof. We first suppose that the strands are oriented upward. The image of \( Y_{k+l}^{1,\ldots,1} \) is an \( U_q(\ell) \)-subrepresentation of \( \Lambda_{k+l}^{k+l} V_q \) and since the weight spaces of \( \Lambda_{k+l}^{k+l} V_q \) are of dimension one, we deduce that the image of a vector \( v_S \) by \( \text{ft} \circ Y_{k+l}^{1,\ldots,1} \) is a multiple of \( Y_{k+l}^{1,\ldots,1}(v_S) \), say \( s_S Y_{k+l}^{1,\ldots,1}(v_S) \).

The same arguments shows that the equality also holds with strands oriented downward. \( \square \)

Lemma 7A.3. The relation (4A.2) is satisfied after applying the functor \( A_G \).

Proof. Clear. \( \square \)

Lemma 7A.4. The relation (4A.3) is satisfied after applying the functor \( A_G \).

Proof. Using the associativity of splits and merges and the fact that Levi crossings satisfy the Reidemeister III relation, it remains to prove

\[ \begin{array}{c}
[k+l]^{-1} \cdot \text{ft} = [k]^{-1}[l]^{-1} \cdot \text{ft}.
\end{array} \]

which follows immediately from Lemma 7A.2. \( \square \)

We can show similarly that merges slide through coils.

Lemma 7A.5. The relation (4A.4) is satisfied after applying the functor \( A_G \).
Proof. The image of the right-hand side of (4A.4) is given, up to the multiplication by elements of $U$ that cancel out, by

\[
\left[ k \right]! - 2 \cdot \left[ k \right]^{-1}.
\]

Using Reidemeister II relations and the fact that mates of merges are splits and vice versa, it remains to prove the following equality:

\[
\left[ k \right]! - 2 \cdot \left[ k \right]^{-1} = \left[ k \right]^{-1}.
\]

In order to get rid of the explosions between the Levi full twists, we apply repeatedly Lemma 7A.2 with $l = 1$ and use the fact that the Levi full twist on $k$ strands can be obtain from the Levi full twist on $k - 1$ strands and some extra crossings. We can conclude using explosion as in Lemma 3B.6. The argument for the leftward oriented cap is similar and omitted. \(\square\)

We can show similarly that cups slide through coils.

**Lemma 7A.6.** The relations in (4C.3) are in the kernel of $A\Gamma_q$.

**Proof.** The image of the leftward oriented essential circle with a strand of thickness $k$ through $A\Gamma_q$ is given by

\[
\left[ k \right]!^{-1} \cdot \left[ k \right]!^{-2} \cdot \left[ k \right]!^{-1}.
\]

In this calculation the first equality is obtained from explosion of strands, and the last equality is obtained from Lemma 7A.2.
Now, since

\[
[k]!^{-1} \cdot \frac{n!}{n-1} \cdot \ldots \cdot 1 \cdot \ldots \cdot \frac{n!}{n-1} \cdot \ldots \cdot 1,
\]

it remains to compute the following scalar:

\[
[k]!^{-1} \cdot \sum_{k_1+\ldots+k_d=k} \sum_{1\leq j_1<\ldots<j_1,k_1 \leq l_1} u_1^{k_1} \ldots u_d^{k_d} \prod_{i=1}^d q^{-n+2(l_1+\ldots+l_d)k_1+\sum_{r=1}^{k_i} r}.
\]

We easily check that the twist sends \( v_{i,j} \) to \( q^{-n+2(l_1+\ldots+l_d)}v_{i,j} \) and therefore the previous scalar is equal to

\[
e_k(q^{-1}u_1, q^{-3}u_1, \ldots, q^{-2l_1+1}u_1, \ldots, q^{-1}u_d, q^{-3}u_d, \ldots, q^{-2l_d+1}u_d),
\]

which is what we needed to show. \( \square \)

**Lemma 7A.7.** The functor \( \mathbf{A} \Gamma_q \) is well-defined and descends to the functor \( \mathbf{A} \Gamma_\ell^q \). Moreover, for \( q \neq 1 \) the Levi ideal \( I_\ell \) contains the two-sided \( \circ_v \)-ideal and right \( \circ_h \)-ideal generated by the Levi evaluations, while for \( q = 1 \) the Levi ideal \( I_\ell \) contains the two-sided \( \circ_v \cdot \circ_h \)-ideal generated by the Levi evaluations.

**Proof.** We need to check that the relations in Definition 4A.1 are satisfied and that the Levi evaluations from (4C.3) are in the kernel of \( \mathbf{A} \Gamma_q \). This follows as a combination of Lemma 7A.5 and Lemma 7A.6. By Lemma 6C.1, the statement about the Levi ideal for \( q = 1 \) follows from that. To verify the claim for \( q \neq 1 \), we recall from Remark 4A.5 that the coils pass in front of the annulus. Now we observe that, for example, (the \( \rightsquigarrow \) refers to Remark 4A.5)

\[
\text{right } \circ_h \text{-ideal: } \rightsquigarrow \text{left } \circ_h \text{-ideal: } \rightsquigarrow
\]

In the top picture the essential circle is not pierced by the identity morphism, while in the bottom it is. More generally, multiplying essential circles from the right (but not from the left) by any morphism \( f \) produces a picture with \( f \) not interfering with the essential circles and the essential circles are on the outside. Thus, by our convention from Remark 4A.5, the essentially circle evaluations will not change by right multiplication. \( \square \)
Recall also the web block projectors \( \text{pr}_w^w \) from Definition 4A.7 respectively the Levi block projectors \( \text{pr}_s^s \) from Definition 5B.14. We use these to define a certain basis in the following definition, where \( S_k \) is the symmetric group on \( \{1, \ldots, k\} \):

**Definition 7B.1.** Fix \( k \in \mathbb{Z}_{\geq 0} \). Let \( s = (s_1, \ldots, s_k), t \in \{1, \ldots, d\}^k \) and let \( \sigma \) be a colored permutation from \( s \) to \( t \) such that the longest word of \( S_{l_i+1} \subset S_k \) does not appear in the permutation of color \( i \). (This condition is vacuous for \( l_i \geq k \) or if the color \( i \) does not appear strictly more than \( l_i \) times.) Define

\[
\nu_{s,t,\sigma}^{1} = \sigma \circ v (\text{pr}_{w_1}^w \circ \ldots \circ \text{pr}_{w_{s,t,\sigma}}^w), \quad n_{s,t,\sigma}^{1} = \sigma \circ v (\text{pr}_{s_1}^s \circ \ldots \circ \text{pr}_{s_{s,t,\sigma}}^s),
\]

where we view \( \sigma \) as an element of \( \text{AWeb}_1^\ell \) and \( \text{Fund}_1^\ell \), respectively. The respective sets (collecting these elements for all \( k \in \mathbb{Z}_{\geq 0} \)) of these are denoted by \( B(\text{AWeb}_1^\ell) \) and \( B(\text{Fund}_1^\ell) \).

**Lemma 7B.2.** We have \( A^{1}_1 (B(\text{AWeb}_1^\ell)) = B(\text{Fund}_1^\ell) \) and this set is a \( \mathbb{K}_1 \)-linear independent set in \( \prod_k \text{End}_{\text{Fund}_1^\ell}(V^\otimes_k) \).

**Proof.** We get \( A^{1}_1 (B(\text{AWeb}_1^\ell)) = B(\text{Fund}_1^\ell) \) directly from Lemma 6C.1 and the definition.

To prove faithfulness, let \( \text{pr}_s^s = \text{pr}_{w_1}^w \circ \ldots \circ \text{pr}_{w_{s,t,\sigma}}^w \). By Lemma 5B.15 and construction we have \( n_{s,t,\sigma}^{1} \circ \sigma \circ v \circ \text{pr}_s^s = \text{pr}_s^s \circ v \circ n_{s,t,\sigma}^{1} \circ \sigma \). By Lemma 5B.15 and construction we have \( n_{s,t,\sigma}^{1} \circ \sigma \circ v \circ \text{pr}_s^s = \text{pr}_s^s \circ v \circ n_{s,t,\sigma}^{1} \circ \sigma = 0 \) for \( q \neq s \) and \( r \neq t \). Thus, it suffices to show that \( (n_{s,t,\sigma}^{1})_{\sigma} \) is \( \mathbb{K}_1 \)-linear independent (that is, we can fix \( s \) and \( t \)). After sorting the colors of \( s \) and \( t \), it remains to verify faithfulness within a block, i.e. for \( gl_{l_i} \). Thus, classical theory applies: for \( gl_{l_i} \) and \( S_k \) it is known that Schur–Weyl duality gives the generalized Temperley–Lieb algebra as the endomorphism algebra. This algebra admits a description in terms of a quotient of \( S_k \) by the longest word of \( S_{l_i+1} \), see e.g. [Hä99, Section 3], and the elements of \( B(\text{Fund}_1^\ell) \) describe the associated standard-type basis of each block.

**Lemma 7B.3.** The set \( B(\text{AWeb}_1^\ell) \) \( \mathbb{K}_1 \)-linearly spans \( \prod_k \text{End}_{\text{AWeb}_1^\ell}(1^{\otimes_k}) \). Moreover, the set \( B(\text{Fund}_1^\ell) \) is a \( \mathbb{K}_1 \)-linear spanning set of \( \prod_k \text{End}_{\text{Fund}_1^\ell}(V^\otimes_k) \).

**Proof.** We first recall that the full antisymmetrizer is zero in \( \text{Fund}_1^\ell \), see e.g. [MS16, Lemma 11]. Thus, the same holds in \( \text{AWeb}_1^\ell \) by Lemma 7B.2. That is, in formulas we have

\[
\left( \sum_{\sigma \in S_{l_i+1}} (-1)^{l(\sigma)} \sigma (\text{pr}_s^s)_{\sigma}^{1}(l_i+1) = 0 \right) \Rightarrow \left( \sum_{\sigma \in S_{l_i+1}} (-1)^{l(\sigma)} \sigma (\text{pr}_s^s)_{\sigma}^{1}(l_i+1) = 0 \right),
\]

and both hold.

To address the first statement of the lemma, observe that \( \prod_k \text{End}_{\text{AWeb}_1^\ell}(1^{\otimes_k}) \) is generated as a \( \mathbb{K}_1 \)-algebra by crossings and coils since we can remove essential circles by Lemma 7A.7. Moreover, the sliding relations (4A.3) and (4A.4) imply that we have the usual \( \mathbb{K}_1 \)-linear spanning set given by first coils and then crossings. Observe next that the web block projectors \( \mathbb{K}_1 \)-linear span the subalgebra generated by coils, so it remains to see that the symmetric group part is \( \mathbb{K}_1 \)-linear spanned by \( \sigma \) such that the longest word of \( S_{l_i+1} \) does not appear in the permutation of color \( i \). This however is a consequence of the vanishing of the antisymmetrizer.

For the second statement of the lemma we use Lemma 5B.3, Lemma 5B.15 and Schur’s lemma. (As explained in the proof of Lemma 5C.1, Schur’s lemma still holds in this setting although \( \mathbb{K}_1 \) is not necessary algebraically closed.)

**Lemma 7B.4.** The set \( B(\text{AWeb}_1^\ell) \) is a \( \mathbb{K}_1 \)-basis of \( \prod_k \text{End}_{\text{AWeb}_1^\ell}(1^{\otimes_k}) \). Furthermore, the set \( B(\text{Fund}_1^\ell) \) is a \( \mathbb{K}_1 \)-basis of \( \prod_k \text{End}_{\text{Fund}_1^\ell}(V^\otimes_k) \).
Lemma 7C.4. The set $\mathcal{B}\text{ skein relations}$).

Proof. Combine Lemma 7B.2 and Lemma 7B.3. (Note that Lemma 7B.2 also proves that $B(\text{AWeb}_1)$ is a $\mathbb{K}_1$-linear independent set.)

Proposition 7B.5. The functor $\Gamma_1^\ell$ is fully faithful and the Levi ideal $\mathcal{I}_\ell$ is the two-sided $\circ_v - \circ_h^A$-ideal generated by the Levi evaluations.

Proof. Lemma 3A.9 implies that we need to show that $\bigsqcup_k\text{End}_{\text{AWeb}_1}(1^{\otimes_k})$ and $\bigsqcup_k\text{End}_{\text{Fund}_1}(V_q^\otimes_k)$ are isomorphic $\mathbb{K}_1$-vector spaces with $\Gamma_1^\ell$ inducing an isomorphism. This follows from Lemma 7B.2 and Lemma 7B.4. The proof is complete.

Proof of Theorem 6B.3. (b). Lemma 7A.7 shows that the functors $\Gamma_1$ and $\Gamma_1^\ell$ are well-defined and Lemma 6C.1 shows the statements involving the pivotal structure. Fully faithfulness follows from Proposition 7B.5, and Lemma 5B.8 ensures that we have that $\text{Ad Id}_{\Gamma_1}$ is equivalent to $\text{Rep}_{\ell}$. These statements taken together complete the proof using the usual properties of the additive idempotent completion.

7C. Proof for $q \neq 1$. We start with the following lemma.

Lemma 7C.1. The functor $\Gamma_q$ is full.

Proof. It is clear that the image of the crossings and the coils span. (Note that there is no issue with essential circles in $\text{Fund}_q$.)

For $q \neq 1$, we note that we can mimic Definition 7B.1 on the Levi side (the only difference is that we use a positive lift, in Levi crossings, of $\sigma$ instead of $\sigma$ itself) to define $n^q_{s,t,\sigma}$ as well as $B(\text{Fund}_q)$. Now we use that and Lemma 7C.1 to define $\nu^q_{s,t,\sigma}$ as well as $B(\text{AWeb}_q)$ by pulling back the elements from $n^q_{s,t,\sigma}$ by choosing a preimage.

Lemma 7C.2. The set $B(\text{AWeb}_q) \subset \bigsqcup_k\text{End}_{\text{AWeb}_q}(1^{\otimes_k})$ is $\mathbb{K}_q$-linearly independent. Moreover, the set $B(\text{Fund}_q)$ is $\mathbb{K}_q$-linearly independent in $\bigsqcup_k\text{End}_{\text{Fund}_q}(V_q^\otimes_k)$.

Proof. The claim on the Levi side can be proven verbatim as in Lemma 7B.2, so our focus is on the web side. However, by construction, the set $B(\text{AWeb}_q)$ is then sent to $B(\text{Fund}_q)$, so $B(\text{AWeb}_q)$ is $\mathbb{K}_q$-linearly independent because $B(\text{Fund}_q)$ is.

Lemma 7C.3. The set $B(\text{AWeb}_q) \mathbb{K}_q$-linearly spans $\bigsqcup_k\text{End}_{\text{AWeb}_q}(1^{\otimes_k})$. Moreover, the set $B(\text{Fund}_q)$ is a $\mathbb{K}_q$-linear spanning set of $\bigsqcup_k\text{End}_{\text{Fund}_q}(V_q^\otimes_k)$.

Proof. We can remove essential circles in front of the cylinder by definition of the monoidal product, see also (7A.8), and the Levi ideal. The essential circles in the back of the cylinder labeled by 1 can then be inductively removed by using

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

which is the classical skein relation (that holds in our setting by using (3B.5)). Using explosion, the rest of the argument is the same as in the proof of Lemma 7B.3 by using that the $q \neq 1$ basis agrees with $B(\text{AWeb}_1)$ on the associated graded by filtration by number of crossings (using the skein relations).

Lemma 7C.4. The set $B(\text{AWeb}_q)$ is a $\mathbb{K}_q$-basis of $\bigsqcup_k\text{End}_{\text{AWeb}_q}(1^{\otimes_k})$. Furthermore, the set $B(\text{Fund}_q)$ is a $\mathbb{K}_q$-basis of $\bigsqcup_k\text{End}_{\text{Fund}_q}(V_q^\otimes_k)$.

Proof. By Lemma 7C.2 and Lemma 7C.3.

Remark 7C.5. We do not have or need any explicit description of the elements of $B(\text{AWeb}_q)$ in terms of webs.

Proposition 7C.6. The functor $\Gamma_q^\ell$ is fully faithful and the Levi ideal $\mathcal{I}_\ell$ is the two-sided $\circ_v - \circ_h^A$-ideal generated by the Levi evaluations. Here $\circ_h^A$ is the pullback of the monoidal structure from $\text{Fund}_q$ to $\text{AWeb}_q$. 

ANNULAR WEBS AND LEVI SUBALGEBRAS 25
Proof. As in the proof of Proposition 7B.5. □

Proof of Theorem 6B.3.(a). Using the above statements, this can be proven verbatim as Theorem 6B.3.(b). □

References

[APW91] H.H. Andersen, P. Polo, and K.X. Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991. doi:10.1007/BF01245066.

[AST18] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Cellular structures using $U_q$-tilting modules. *Pacific J. Math.*, 292(1):21–59, 2018. URL: https://arxiv.org/abs/1503.00224, doi:10.2140/pjm.2018.292.21.

[AT17] H.H. Andersen and D. Tubbenhauer. Diagram categories for $U_q$-tilting modules at roots of unity. *Transform. Groups*, 22(1):29–89, 2017. URL: http://arxiv.org/abs/1409.2799, doi:10.1007/s00031-016-9363-z.

[AK94] S. Ariki and K. Koike. A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations. *Adv. Math.*, 106(2):216–243, 1994. doi:10.1006/aima.1994.1057.

[ATY95] S. Ariki, T. Terasoma, and H. Yamada. Schur–Weyl reciprocity for the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$. *J. Algebra*, 178(2):374–390, 1995. URL: http://arxiv.org/abs/hep-th/9310078, doi:10.1006/jabr.1995.1354.

[AK94] S. Ariki and K. Koike. A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and construction of its irreducible representations. *Adv. Math.*, 106(2):216–243, 1994. doi:10.1006/aima.1994.1057.

[ATY95] S. Ariki, T. Terasoma, and H. Yamada. Schur–Weyl reciprocity for the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$. *J. Algebra*, 178(2):374–390, 1995. URL: http://arxiv.org/abs/hep-th/9310078, doi:10.1006/jabr.1995.1354.

[BM93] M. Broué and G. Malle. Zyklotomische Heckealgebren. Number 212, pages 119–189. 1993. Représentations unipotentes génériques et blocs des groupes réductifs finis.

[CK18] S. Cautis and J. Kamnitzer. Quantum K-theoretic geometric Satake: the $SL_n$ case. *Compos. Math.*, 154(2):275–327, 2018. URL: https://arxiv.org/abs/1509.00112, doi:10.1112/S0010437X17007564.

[CKM14] S. Cautis, J. Kamnitzer, and S. Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014. URL: https://arxiv.org/abs/1210.6437, doi:10.1007/s00208-013-0984-4.

[Che87] I.V. Cherednik. A new interpretation of Gelfand–Tzetlin bases. *Duke Math. J.*, 54(2):563–577, 1987. doi:10.1215/S0012-7094-87-05423-8.

[Cvi08] P. Cvitanović. *Group theory*. Princeton University Press, Princeton, NJ, 2008. Birdtracks, Lie’s, and exceptional groups. doi:10.1515/9781400837670.

[ES18] M. Ehrig and C. Stroppel. Nazarov–Wenzl algebras, coideal subalgebras and categorified skew Howe duality. *Adv. Math.*, 331:58–142, 2018. URL: https://arxiv.org/abs/1310.6437, doi:10.1007/s00208-013-0984-4.

[Eli15] B. Elias. Light ladders and clasp conjectures. 2015. URL: https://arxiv.org/abs/1510.06840.

[Gr80] J.A. Green. *Polynomial representations of $GL_n$*. In Lecture Notes in Mathematics, 830. Springer-Verlag, Berlin-New York, 1980. doi:10.1007/3-540-4594-3.

[HäOl01] R. Häring-Oldenburg. Actions of tensor categories, cylinder braids and their Kauffman polynomial. *Topology Appl.*, 112 (2001), no. 3, 297–314. URL: https://arxiv.org/abs/q-alg/9712031, doi:10.1016/S0166-8641(00)00006-7.

[Härt99] M. Härterich. Murphy bases of generalized Temperley–Lieb algebras. *Arch. Math. (Basel)*, 72(5):337–345, 1999. doi:10.1007/s000130050341.

[How95] R. Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Conf. Proc.*, pages 1–182. Bar-Ilan Univ., Ramat Gan, 1995.

[How95] R. Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989. doi:10.2307/2001418.

[Hu01] J. Hu. Schur–Weyl reciprocity between quantum groups and Hecke algebras of type $G(r,1,n)$. *Math. Z.*, 238(3):505–521, 2001. doi:10.1007/s002090100264.

[Jan96] J.C. Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[Kass95] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. doi:10.1007/978-1-4612-0783-2.
[Kup96] G. Kuperberg. Spiders for rank 2 Lie algebras. Comm. Math. Phys., 180(1):109–151, 1996. URL: https://arxiv.org/abs/q-alg/9712003.

[LV21] A. Lacabanne and P. Vaz. Schur–Weyl duality, Verma modules, and row quotients of Ariki–Koike algebras. Pacific J. Math., 311(1):113–133, 2021. URL: https://arxiv.org/abs/2004.01065, doi:10.2140/pjm.2021.311.113.

[LT21] G. Latifi and D. Tubbenhauer. Minimal presentations of $\mathfrak{gl}_n$-web categories. 2021. URL: https://arxiv.org/abs/2112.12688.

[MW03] P.P. Martin and D. Woodcock. Generalized blob algebras and alcove geometry. LMS J. Comput. Math., 6:249–296, 2003. URL: https://arxiv.org/abs/math/0205263, doi:10.1112/S1461157000000450.

[MS16] V. Mazorchuk and C. Stroppel. $G(\ell, k, d)$-modules via groupoids. J. Algebraic Combin., 43(1):11–32, 2016. URL: https://arxiv.org/abs/1412.4494, doi:10.1007/s10801-015-0623-0.

[MS21] Y. Mousaaid and A. Savage. Affinization of monoidal categories. J. Éc. polytech. Math., 8:791–829, 2021. URL: https://arxiv.org/abs/2010.13598, doi:10.2140/pjm.2021.311.113.

[Mud02] A. Mudrov. Characters of $U_q(\mathfrak{gl}(n))$-reflection equation algebra. Lett. Math. Phys., 60(3):283–291, 2002. doi:10.1023/A:1016283527111.

[PS98] J.H. Przytycki and A.S. Sikora. Skein algebra of a group. In Knot theory (Warsaw, 1995), volume 42 of Banach Center Publ., pages 297–306. Polish Acad. Sci. Inst. Math., Warsaw, 1998.

[Que15] H. Queffelec. Skein modules from skew Howe duality and affine extensions. SIGMA Symmetry Integrability Geom. Methods Appl., 11:Paper 030, 36, 2015. URL: https://arxiv.org/abs/1309.4985, doi:10.3842/SIGMA.2015.030.

[QW18] H. Queffelec and P. Wedrich. Extremal weight projectors II. 2018. URL: https://arxiv.org/abs/1803.09883.

[RT16] D.E.V. Rose and D. Tubbenhauer. Symmetric webs, Jones–Wenzl recursions, and $q$-Howe duality. Int. Math. Res. Not. IMRN, (17):5249–5290, 2016. URL: https://arxiv.org/abs/1501.00915, doi:10.1093/imrn/rnv302.

[SS99] M. Sakamoto and T. Shoji. Schur–Weyl reciprocity for Ariki–Koike algebras. J. Algebra, 221(1):293–314, 1999. doi:10.1006/jabr.1999.7973.

[ST19] A. Sartori and D. Tubbenhauer. Webs and $q$-Howe dualities in types BCD. Trans. Amer. Math. Soc., 371(10):7387–7431, 2019. URL: https://arxiv.org/abs/1701.02932, doi:10.1090/tran/7583.

[TV17] V.G. Turaev and A. Virelizier. Monoidal categories and topological field theory, volume 322 of Progress in Mathematics. Birkhäuser/Springer, Cham, 2017. doi:10.1007/978-3-319-49834-8.

A.L.: Laboratoire de Mathématiques Blaise Pascal (UMR 6620), Université Clermont Auvergne, Campus Universitaire des Cézeaux, 3 place Vasarely, 63178 Aubière Cedex, France, www.normalesup.org/~lacabanne, ORCID 0000-0001-8691-3270
Email address: abel.lacabanne@uca.fr

D.T.: The University of Sydney, School of Mathematics and Statistics F07, Office Carslaw 827, NSW 2006, Australia, www.dttubbenhauer.com, ORCID 0000-0001-7265-5047
Email address: daniel.tubbenhauer@sydney.edu.au

P.V.: Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium, https://perso.uclouvain.be/pedro.vaz, ORCID 0000-0001-9422-4707
Email address: pedro.vaz@uclouvain.be