SENSOR DEPLOYMENT FOR PIPELINE LEAKAGE DETECTION VIA OPTIMAL BOUNDARY CONTROL STRATEGIES

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ABSTRACT. We consider a multi-agent control problem using PDE techniques for a novel sensing problem arising in the leakage detection and localization of offshore pipelines. A continuous protocol is proposed using parabolic PDEs and then a boundary control law is designed using the maximum principle. Both analytical and numerical solutions of the optimality conditions are studied.

1. Introduction.

1.1. Background and motivations. Pipeline transport is the transportation of mass flow (such as gas or crude oil) from one place to another through a pipeline network. Generally, pipelines are probably the most economical way of transporting any chemically stable substance over land, such as large quantities of oil, refined oil products or natural gas. Pipelines are also an important choice to transport oil from the platform to the tanker ships or directly to the land refinery factories in the offshore oil industry. Based on 2008 statistics, more than 6000 kilometers of pipelines have been constructed in China and more than 2000 kilometers of them are offshore pipelines [9].

The safety and security of pipeline networks is tightly controlled by government regulations and policies [29]. For example, it is a mandatory rule for pipeline operators in the State of Washington (US) to be able to detect and locate leaks of 8

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percent of maximum flow within no more than 15 minutes. Any failure in detecting pipeline leakages and delivering appropriate repairs in time may lead to serious environmental pollution and economic loss.

To ensure the operation safety of pipelines on land, field devices are installed at specific locations in the pipeline network to collect various physical data and transmit them to a central location (i.e., the control room) in real time using various communication systems, such as satellite channels, microwave links and even cellular phone connections. With these data provided by the Supervisory Control And Data Acquisition system (SCADA), various diagnostic approaches can be used to give extended information on leakage location and severity [35]. There has been a significant amount of research work on detecting and locating leakages in pipeline networks on land, including both the so-called hardware-based and software-based approaches, e.g., [5], [23], [35], [36].

However, this becomes much more challenging for the leakage detection or localization of offshore pipelines. First of all, since offshore pipelines on the seabed are covered by water, it is quite costly to add more sensors along the pipelines. Also, the GPS (global positioning system) signal is not available for real time communication.

Motivated by fluid mechanics, sensors can be categorized as Eulerian or Lagrangian sensors, where Eulerian sensors are fixed at specific locations while Lagrangian sensors float with the mass media being measured [30]. In contrast to ground pipelines, it is common in modern offshore pipeline transportation systems that data monitoring information is only collected at both terminals, while the interval between the terminals which is beneath the sea level, will not have Eulerian sensors due to extremely high cost. Once a leakage happens at any location within the interval, it would be quite challenging to detect and localize the leakage. Therefore, it is necessary to explore a new sensor deployment framework by combining both the Eulerian and Lagrangian sensors for health monitoring of offshore pipelines.

In this work, we consider the sensor deployment problem in a long pipeline in which each sensor can be regarded as an autonomous robot that contains sensors and a propulsion system. It moves along the pipeline together with the media flows. Meanwhile, it takes measurements and saves the data to the on-board memory through an embedded system. In this framework, initial diagnostic results should be achieved using only the terminal data to have a rough estimation (e.g., position and leakage rate) of leakages. Then, a sequence of Lagrangian sensors can be delivered to the leakage intervals to take further information, such as acoustic, pressure and flow rate signals. To make the Lagrangian sensors flexible to take measurements, we need to control the sensor distribution in a real-time way within the region of leakages. For example, more sensors are needed close to the leakage point than in other places within the pipeline. By utilizing the capability of short range communications between neighboring sensors, it is natural to consider the sensor deployment as a multi-agent system (MAS) problem (e.g., [31]). A schematic of the general idea proposed in this work is shown in Fig. 1.

Control and estimation of MAS has been a very popular topic in systems and control. One interesting topic of MAS is the deployment of smart sensors and much work has been done toward the consensus & formation of the agent group using only local information. The mathematical techniques used to demonstrate the convergence of either the consensus or formation of agents under given protocols are mainly the graph theory and finite dimensional dynamical systems on
graphs (e.g., [6, 21, 24, 27, 31]). When the agent population is large, the performance of the multi-agent system degrades, and thus continuation approaches can be introduced to derive unified protocols whose representations are usually governed by partial differential equations (PDEs) (e.g., [18, 19, 32]). In [13], the framework of partial difference equations (PDEs) over graphs is proposed to analyze the behavior of multi-agent systems equipped with decentralized control schemes. The resulting PDEs enjoy properties that are similar to those of well-known PDEs like the heat equation, which coincides with the graph Laplacian control approach proposed in [28]. Wave-like PDE models have been introduced in [4, 16, 17] to study the scaling laws of stability margin and robustness to external disturbances for large-scale vehicular formations. To achieve formation control of agents, linear diffusion-advection-reaction equations were investigated in [14] with dynamic boundary feedback control using backstepping design techniques [20]. To deal with the parameter uncertainties for MAS modeled by PDEs, adaptive control methods have been considered in [18, 19]. A systematic flatness-based motion planning using formal power series and suitable summability methods is considered in [25] for the finite-time deployment of multi-agent systems into planar formation profiles along predefined spatial-temporal paths governed by the Burgers equation. For sensor scheduling problems in terms of given physical dynamic models (both continuous and discrete cases), optimal discrete-valued control problems (e.g., [38], [40]) have been formulated to make optimal sensor deployments (e.g., [8], [11], [12], [37]).

1.2. Contribution. In this paper, we use the same idea in [14] to use the linear diffusion-advection-reaction PDE to model the protocol of the sensors. Instead of using the backstepping technique to design a boundary controller to stabilize the desired profile governed by the equilibrium of the PDE system, the optimal control of evolutionary PDE systems is considered in this work. For the optimal control of the linear diffusion-advection-reaction PDEs with boundary actuation, the calculus of variations method is used to derive the optimality condition, which consists of two coupled PDE boundary value problems. An iterative algorithm is then proposed to obtain the numerical solution. To the best of our knowledge, this is the first paper on sensor deployment arising in leakage detection of pipelines. For the optimal control of PDE systems, much work has been done (e.g., [1, 7, 10, 39, 26, 34, 22]).
1.3. Paper organization. We organize this paper as follows. In Section 2, we develop the mathematical model of the sensor deployment problem using the first order mass point kinetic assumption. Then, the continuation approach is used to obtain diffusion and diffusion-reaction protocols. In Section 3, the statement of the optimal control problem is given and the calculus of variations method is then used to derive the optimality condition. In Section 4, the analytical solutions are given for a particular special case using the method of separation of variables. For the general case, a numerical solution framework is needed. An iteration scheme is discussed in Section 5 that gives the numerical solution of the optimality condition, which consists of two coupled boundary value problems. We conclude the paper in Section 6 by stating remarks and future research work.

2. Modeling collective dynamics of agents. Given an agent \( i, i \in \{0, 1, 2, \ldots, n\} \), the mass point dynamics can be described as \( \frac{dx_i(t)}{dt} = u_i(t) \), where \( x_i(t) \) denotes the position of agent \( i \) at time \( t \) and \( u_i(t) \) is the control input for agent \( i \) at time \( t \). When the agent population is large, i.e., \( n \) is large, then we consider the following dynamical model using the continuation approach [26],

\[
\frac{\partial x(\theta, t)}{\partial t} = u(\theta, t), \quad \theta \in [0, 1],
\]

where \( x(\theta, t) \) represents the position of agent \( \theta \) at time \( t \), and \( u(\theta, t) \) is the control input for agent \( \theta \) at time \( t \). We consider the agent’s identification (ID) number \( \theta \) as the spatial variable of a PDE model for the collective dynamics. Note that \( \theta \) is only an auxiliary map (\( \theta : I \to [0, 1] \), where \( I = \{0, 1, 2, \ldots, n\} \)) to label the sensors and does not represent the spatial coordinate of the pipeline.

Usually, the following control protocol is used to achieve consensus,

\[
\frac{dx_i(t)}{dt} = \sum_{j \in N_i} [x_j(t) - x_i(t)],
\]

where \( N_i \) denotes the set of agents/neighbors that communicate with agent \( i \). By introducing a vector \( X(t) = [x_0(t), x_1(t), \ldots, x_n(t)]^T \), we can rewrite the multi-agent system into the following matrix representation

\[
\frac{dX(t)}{dt} = MX(t), \quad X(0) = [x_0(0), x_1(0), \ldots, x_n(0)]^T,
\]

where the matrix \( M \) is given by

\[
M = \begin{bmatrix}
-1 & 1 & 0 & 0 & \ldots \\
1 & -2 & 1 & 0 & \ldots \\
0 & 1 & -2 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \ldots & 1 & -1
\end{bmatrix}.
\]

Formally, equation (3) can be demonstrated to coincide with the heat equation [33]

\[
\frac{\partial x(\theta, t)}{\partial t} = \frac{\partial^2 x(\theta, t)}{\partial \theta^2},
\]

where each agent employs the diffusion-like feedback protocol, i.e., \( u(\theta, t) = \frac{\partial^2 x(\theta, t)}{\partial \theta^2} \).

The equilibrium \( \frac{\partial^2 x(\theta, t)}{\partial \theta^2} = 0 \) of this type of strategy only generates linear formations (the hollow circles and dot line shown in Fig. 2). In this work, we generalize...
the diffusion-based feedback protocol to a more complex case
\[
\begin{cases}
\frac{\partial x(\theta,t)}{\partial t} = \alpha \frac{\partial^2 x(\theta,t)}{\partial \theta^2} + \beta \frac{\partial x(\theta,t)}{\partial \theta} + \gamma x(\theta,t), \\
\left. \frac{\partial x(\theta,t)}{\partial t} \right|_{\theta=0} = f(t), \quad \left. \frac{\partial x(\theta,t)}{\partial t} \right|_{\theta=1} = g(t), \\
x(\theta,0) = x_0(\theta),
\end{cases}
\]
where the equations in the second line represents the boundary agents’ dynamics at \( \theta = 0 \) and \( \theta = 1 \), respectively. \( f \) and \( g \) represent the velocities of the end agents.

We discretize the PDE model spatially to obtain implementable control laws for the agents, e.g., the three-point central difference scheme. By using the continuous representation, PDE control techniques can be introduced to handle large agent systems. We do not need to make the number of sensors extremely large to cover the whole pipeline, but only to ensure the sensor group has a comparably dense formation. The advantage of using mobile sensors is to realize spatial coverage. In practice, less than ten sensors are sufficient to implement the PDE-based protocol but comparably dense formation is needed to guarantee the accuracy.

In order to make a standard Dirichlet boundary condition, we integrate the boundary condition of (6) over \([\sigma,t)\) \((0 \leq \sigma < t)\), then the boundary condition becomes
\[
\begin{align*}
x(0,t) &= x(0,\sigma) + \int_{\sigma}^{t} f(\tau) d\tau := F(t), \\
x(1,t) &= x(1,\sigma) + \int_{\sigma}^{t} g(\tau) d\tau := G(t),
\end{align*}
\]
where the initial time constant is usually set to be \( \sigma = 0 \). Thus, we consider a standard Dirichlet boundary control problem
\[
\begin{cases}
\frac{\partial x(\theta,t)}{\partial t} = \alpha \frac{\partial^2 x(\theta,t)}{\partial \theta^2} + \beta \frac{\partial x(\theta,t)}{\partial \theta} + \gamma x(\theta,t), \\
x(0,t) = F(t), \quad x(1,t) = G(t), \\
x(\theta,0) = x_0(\theta),
\end{cases}
\]
To simplify the problem, we suppose that the right end position is fixed at \( \bar{G} \), i.e., \( G(t) = \bar{G} \). The equilibria for (9) are much more general than linear in \( \theta \). These equilibria are governed by
\[
\begin{cases}
0 = \alpha \frac{d^2 \bar{x}(\theta)}{d\theta^2} + \beta \frac{d\bar{x}(\theta)}{d\theta} + \gamma \bar{x}(\theta), \\
\bar{x}(0) = \bar{F}, \quad \bar{x}(1) = \bar{G},
\end{cases}
\]
where \( \bar{F} \) is a designated point of the left end, for example, \( \bar{F} = 0 \). Although the linear reaction-advection-diffusion feedback protocol can generate very complex deployments (the solid circles and curve shown in Fig. 2), it does not guarantee stability for an arbitrary array of \((\alpha, \beta, \gamma)\) without the boundary control actuation. For example, when \( \alpha = 1, \beta = 0 \) and \( \gamma \) is sufficiently large, the solution of the uncontrolled system is unstable [33]. Now we assume the desired target is denoted by \( \bar{x}(\theta) \), then we can introduce a new variable \( \tilde{x}(\theta,t) := x(\theta,t) - \bar{x}(\theta) \) to denote the
difference, which satisfies
\[
\begin{cases}
\frac{\partial \tilde{x}(\theta, t)}{\partial t} = \alpha \frac{\partial^2 \tilde{x}(\theta, t)}{\partial \theta^2} + \beta \frac{\partial \tilde{x}(\theta, t)}{\partial \theta} + \gamma \tilde{x}(\theta, t), \\
\tilde{x}(0, t) = \tilde{F}(t) - \bar{\tilde{F}} := \tilde{F}(t), \quad \tilde{x}(1, t) = 0,
\end{cases}
\]
where $\tilde{F}(t)$ is the Dirichlet boundary control function. The stabilization problem using boundary actuation can be handled very well using the backstepping technique developed in [20]. In contrast, in this paper, we deploy an optimal control strategy.

3. **The PDE control problem.** We first give the following definitions for the $L_2$ inner product and linear operators that will be used later in this paper for the PDE control problem:

\[
\langle x_1(\theta), x_2(\theta) \rangle_{L_2} := \int_0^1 x_1(\theta)x_2(\theta)d\theta,
\]
\[
S x(\theta, t) := \int_0^1 S(\theta, \eta)x(\eta, t)d\eta,
\]
where $x_1(\theta)$ and $x_2(\theta)$ are square integrable functions in a Hilbert space and $S : L_2(0, 1) \to L_2(0, 1)$, $S(\theta, \eta) = S(\eta, \theta)$, $S(\theta, \eta) \geq 0$. Now we consider the following cost functional

\[
J_c(\tilde{F}) = \frac{1}{2} \int_0^T \left[ \langle \tilde{x}(\theta, t), Q\tilde{x}(\theta, t) \rangle + R\tilde{F}^2(t) \right] dt + \frac{1}{2} \langle \tilde{x}(\theta, T), S\tilde{x}(\theta, T) \rangle,
\]
subject to

\[
\begin{cases}
\frac{\partial \tilde{x}(\theta, t)}{\partial t} = \alpha \frac{\partial^2 \tilde{x}(\theta, t)}{\partial \theta^2} + \beta \frac{\partial \tilde{x}(\theta, t)}{\partial \theta} + \gamma \tilde{x}(\theta, t), \\
\tilde{x}(0, t) = \tilde{F}(t), \quad \tilde{x}(1, t) = 0,
\end{cases}
\]
where $Q : L_2(0, 1) \rightarrow L_2(0, 1)$, is symmetric and satisfies $\langle x, Qx \rangle \geq 0$ for all $x \in L_2(0, 1)$. The control weight factor $R$ is a strictly positive number. $S$ is symmetric operator, and $\langle x, Sx \rangle \geq 0$ for all $x \in L_2(0, 1)$. Note that the motivation for the cost function is to make $\tilde{x}(\theta, t)$ small, so that the system moves to an equilibrium, or steady-state.

**Theorem 3.1.** Let $\tilde{F}^*(t)$ be an optimal control for the problem (14)-(15) and $\tilde{x}^*(\theta, t)$ be the optimal trajectory corresponding to $\tilde{F}^*(t)$. Then the optimal control law is given by

$$\tilde{F}^*(t) = -\alpha R^{-1} \frac{\partial \lambda(\theta, t)}{\partial \theta} \bigg|_{\theta=0},$$

where the function $\lambda(\theta, t)$ satisfies the following costate equation,

$$\begin{cases} 
- \frac{\partial \lambda(\theta, t)}{\partial t} = \alpha \frac{\partial^2 \lambda(\theta, t)}{\partial \theta^2} - \beta \frac{\partial \lambda(\theta, t)}{\partial \theta} + \gamma \lambda(\theta, t) + Q \tilde{x}^*(\theta, t), \\
\lambda(0, t) = \lambda(1, t) = 0, \\
\lambda(\theta, T) = S \tilde{x}^*(\theta, T) = \int_0^1 S(\theta, \eta) \tilde{x}^*(\eta, T) d\eta.
\end{cases}$$

**Proof.** Let $\lambda(\theta, t)$ denote the costate variable corresponding to the optimal state and optimal control. Then we assume that a perturbation of the optimal control $\tilde{F}^*$ is $\tilde{F}$ such that

$$\tilde{F}(t) = \tilde{F}^*(t) + \epsilon \delta \tilde{F}(t),$$

where $\delta$ represents the perturbation operator and $\epsilon$ is an arbitrary constant. Let $\tilde{x}(\theta, t)$ is a solution of of (15) corresponding to the control $\tilde{F}(t)$. Because of the linearity of (15), it is easy to show that

$$\tilde{x}(\theta, t) = \tilde{x}^*(\theta, t) + \epsilon \delta \tilde{x}(\theta, t),$$

where $\delta \tilde{x}(\theta, t)$ is a solution of (15) corresponding to $\delta \tilde{F}(t)$ and $\delta \tilde{x}(\theta, 0) = 0$. Thus, the perturbed cost functional is

$$J_c(\tilde{F}^* + \epsilon \delta \tilde{F})$$

$$= \frac{1}{2} \int_0^T \langle \tilde{x}^*(\theta, t) + \epsilon \delta \tilde{x}(\theta, t), Q(\tilde{x}^*(\theta, t) + \epsilon \delta \tilde{x}(\theta, t)) \rangle \, dt$$

$$+ \frac{1}{2} \int_0^T R \left[ \tilde{F}^*(t) + \epsilon \delta \tilde{F}(t) \right]^2 \, dt$$

$$+ \frac{1}{2} \langle \tilde{x}^*(\theta, T) + \epsilon \delta \tilde{x}(\theta, T), S(\tilde{x}^*(\theta, T) + \epsilon \delta \tilde{x}(\theta, T)) \rangle.$$ 

We introduce a function of $\epsilon$ based on the perturbed cost functional by incorporating the PDE system using the Lagrangian multiplier $\lambda(\theta, t)$,

$$G_c(\epsilon) := J_c(\tilde{F}^* + \epsilon \delta \tilde{F})$$

$$+ \int_0^T \left\langle \lambda(\theta, t), \alpha \left[ \frac{\partial^2 \tilde{x}^*(\theta, t)}{\partial \theta^2} + \epsilon \frac{\partial^2 \delta \tilde{x}(\theta, t)}{\partial \theta^2} \right] \right\rangle \, dt.$$
Choosing the multiplier to satisfy

Using integration by parts, we can obtain the following simplification

Then, the necessary condition for optimality is

Note that,

Using integration by parts, we can obtain the following simplification

Using integration by parts, we can obtain the following simplification

Choosing the multiplier to satisfy $\lambda(0, t) = \lambda(1, t) = 0$ and noting that $\delta \tilde{x}(0, t) = \delta \tilde{F}(t)$ and $\delta \tilde{x}(1, t) = 0$, we have

Similarly, we have

$$
\left\langle \lambda(0, t), \beta \frac{\partial \delta \tilde{x}(\theta, t)}{\partial \theta} \right\rangle = -\beta \left\langle \frac{\partial \lambda(\theta, t)}{\partial \theta}, \delta \tilde{x}(\theta, t) \right\rangle.
$$
and
\[ \int_0^T \left\langle \lambda(\theta, t), \frac{\partial \delta \hat{x}(\theta, t)}{\partial t} \right\rangle dt = \langle \lambda(\theta, T), \delta \hat{x}(\theta, T) \rangle - \int_0^T \left\langle \frac{\partial \lambda(\theta, t)}{\partial t}, \delta \hat{x}(\theta, t) \right\rangle dt, \]
where we have noted that \( \delta \hat{x}(\theta, 0) = \delta \hat{x}_0(\theta) = 0. \)

Now we are ready to compute
\[ \frac{dG_\epsilon(c)}{dc} \bigg|_{c=0} = \frac{1}{2} \int_0^T [\langle \delta \hat{x}(\theta, t), Q \hat{x}^*(\theta, t) \rangle + \langle \hat{x}^*(\theta, t), Q \delta \hat{x}(\theta, t) \rangle] dt \]
\[ + \int_0^T [R \hat{F}^*(t) \delta \hat{F}(t)] dt \]
\[ + \frac{1}{2} \langle \delta \hat{x}(\theta, T), S \hat{x}^*(\theta, T) \rangle + \frac{1}{2} \langle \hat{x}^*(\theta, T), S(\delta \hat{x}(\theta, T)) \rangle + \int_0^T \left[ \alpha \left. \frac{\partial \lambda(\theta, t)}{\partial \theta} \right|_{\theta=0} \delta \hat{F}(t) \right] dt \]
\[ + \int_0^T \alpha \left( \frac{\partial^2 \lambda(\theta, t)}{\partial \theta^2}, \delta \hat{x}(\theta, t) \right) dt \]
\[ + \int_0^T [-\beta \left( \frac{\partial \lambda(\theta, t)}{\partial \theta}, \delta \hat{x}(\theta, t) \right) + \gamma \langle \lambda(\theta, t), \delta \hat{x}(\theta, t) \rangle] dt \]
\[ - \langle \lambda(\theta, T), \delta \hat{x}(\theta, T) \rangle + \int_0^T \left( \frac{\partial \lambda(\theta, t)}{\partial t}, \delta \hat{x}(\theta, t) \right) dt = 0, \]
where we have used the following property,
\[ \langle \delta \hat{x}(\theta, t), Q \hat{x}^*(\theta, t) \rangle = \langle \hat{x}^*(\theta, t), Q(\delta \hat{x}(\theta, t)) \rangle, \tag{25} \]
\[ \langle \delta \hat{x}(\theta, T), S \hat{x}^*(\theta, T) \rangle = \langle \hat{x}^*(\theta, T), S(\delta \hat{x}(\theta, T)) \rangle. \tag{26} \]

Thus, substituting the costate equation (17) into (24) completes the proof. \( \square \)

4. Analytical solutions for \( \beta = \gamma = 0. \)

4.1. Solution of the state equation. In this section, we set \( \beta = \gamma = 0 \) to simplify the computational procedure. This is a linear problem and the optimality conditions can be solved analytically. First we consider the following system
\[ \begin{cases} \frac{\partial \hat{x}(\theta, t)}{\partial t} = \alpha \frac{\partial^2 \hat{x}(\theta, t)}{\partial \theta^2}, \\ \hat{x}(0, t) = \hat{F}(t), \quad \hat{x}(1, t) = 0, \\ \hat{x}(\theta, 0) = \hat{x}_0(\theta). \end{cases} \] \( \tag{27} \)

Through the linear transformation \( \hat{x}(\theta, t) = \hat{\lambda}(\theta, t) + (1 - \theta) \hat{F}(t) \), the above equation (27) becomes
\[ \begin{cases} \frac{\partial \hat{\lambda}(\theta, t)}{\partial t} = \alpha \frac{\partial^2 \hat{\lambda}(\theta, t)}{\partial \theta^2} + P(\theta, t) \\ \hat{\lambda}(0, t) = 0, \hat{\lambda}(1, t) = 0, \\ \hat{\lambda}(\theta, 0) = \hat{x}_0(\theta), \end{cases} \] \( \tag{28} \)

where \( P(\theta, t) = (\theta - 1)F'(t) \) and \( \hat{x}_0(\theta) = x_0(\theta) + (\theta - 1)F'(0). \)
This system can be solved using the method of separation of variables, i.e., the solution can be decomposed as

$$\dot{\lambda}(\theta, t) = v(\theta)\dot{y}(t) = \sum_{n=1}^{\infty} v_n(\theta)y_n(t).$$  \hspace{1cm} (29)

Then, the uncontrolled PDE system (28) without the term $P(\theta, t)$ becomes

$$\begin{align*}
\begin{cases}
\dot{v}(\theta)\dot{y}(t) = \alpha v''(\theta)y(t), \theta \in (0, 1), \\
\ddot{\lambda}(0, t) = v(0)y(t) = 0, \dot{\lambda}(1, t) = v(1)y(t) = 0, \\
\ddot{\lambda}(\theta, 0) = v(\theta)y(0) = \ddot{\lambda}(\theta),
\end{cases}
\end{align*}$$  \hspace{1cm} (30)

where $\dot{y}(t)$ represents the time derivative and $v''(\theta)$ represents the spatial derivative, i.e., $\dot{y}(t) = dy(t)/dt$ and $v''(\theta) = d^2v(\theta)/d\theta^2$. We separate the time and space variables to obtain

$$\begin{align*}
\begin{cases}
\dot{y}(t) = \alpha v''(\theta), \\
v(0) = v(1) = 0,
\end{cases}
\end{align*}$$  \hspace{1cm} (31)

where $\lambda$ is a constant. We obtain the eigenvalue problem

$$\begin{align*}
\begin{cases}
v''(\theta) + \lambda v(\theta) = 0, \\
v(0) = v(1) = 0,
\end{cases}
\end{align*}$$  \hspace{1cm} (32)

with the solution given by $v(\theta) = C_1 \cos \sqrt{\lambda} \theta + C_2 \sin \sqrt{\lambda} \theta$, where $C_1$ and $C_2$ are constants. Using the boundary conditions, we can obtain $C_1 = 0, v(1) = C_2 \sin \sqrt{\lambda} = 0$. Let $\mu := \sqrt{\lambda}$, then we have $\sin \mu = 0$. The solution of $\mu$ can be arranged as an increasing sequence $\{\mu_n\}_{n=1}^{\infty}$ and the eigenfunctions are $v_n(\theta) = \sin \mu_n \theta$, where $\mu_n = n\pi, n = 1, 2, \ldots$. For the initial condition, we have $\dot{\lambda}(0) = \lim_{t \to 0} \ddot{\lambda}(\theta, t) = \sum_{n=1}^{\infty} v_n(\theta)\dot{y}_n(0)$, where

$$y_n(0) = \int_{0}^{1} v_n(\theta)\dot{\lambda}_n(0)d\theta.$$  \hspace{1cm} (33)

We introduce a trial function $v(\theta, t) = \phi(t)v_n(\theta)$, where $\phi(T) = 0$. Multiplying $v(\theta, t)$ by both sides of the PDE, we obtain

$$\frac{\partial \ddot{\lambda}}{\partial t}(\theta, t)\phi(t)v_n(\theta) - \alpha \frac{\partial^2 \dot{\lambda}}{\partial \theta^2}(\theta, t)\phi(t)v_n(\theta) = P(\theta, t)\phi(t)v_n(\theta).$$  \hspace{1cm} (34)

Then, using integration by parts and noting that $v_n(1) = v_n(0) = 0, \dot{\lambda}(1, t) = 0,$ $\dot{\lambda}(0, t) = 0, v'_n(0) = \mu_n$ and $v''_n(\theta) = -\mu_n^2 v_n(\theta)$, we obtain the weak form

$$\int_{0}^{1} \int_{0}^{T} \frac{\partial \ddot{\lambda}}{\partial t}(\theta, t)\phi(t)v_n(\theta)dtd\theta$$

$$= \int_{0}^{1} [\phi(T)v_n(\theta)\dot{\lambda}(\theta, T) - \dot{\phi}(0)v_n(\theta)\dot{\lambda}(\theta, 0)]d\theta - \int_{0}^{1} \int_{0}^{T} \ddot{\lambda}(\theta, t)\phi'(t)v_n(\theta)dtd\theta$$

$$= \int_{0}^{1} [-\phi(0)v_n(\theta)\dot{\lambda}_n(0)]d\theta - \int_{0}^{1} \int_{0}^{T} \dot{\lambda}(\theta, t)\phi'(t)v_n(\theta)dtd\theta$$

$$= -y_n(0)\phi(0) - \int_{0}^{T} y_n(t)\phi'(t)dt,$$  \hspace{1cm} (35)
By noting that $\phi$ is an ordinary differential equation for $v_n(\theta)$. The solution of this ODE is
\[
\int_0^T \int_0^1 \frac{\partial^2 \hat{\lambda}}{\partial \theta^2} (\theta, t) \phi(t) v_n(\theta)d\theta dt + \int_0^T \int_0^1 \frac{\partial \hat{\lambda}}{\partial \theta} (\theta, t) \phi(t) v_n(\theta)d\theta dt
= -\int_0^T \int_0^1 \frac{\partial \hat{\lambda}}{\partial t} (\theta, t) \phi(t) v_n(\theta)d\theta dt
= -\int_0^T \int_0^1 \hat{\lambda}(1, t) \phi(t) v_n(1) d\theta dt - \int_0^T \int_0^1 \hat{\lambda}(0, t) \phi(t) v_n(0) d\theta dt
= -\int_0^T \int_0^1 \hat{\lambda}(1, t) \phi(t) v_n(1) d\theta dt + \int_0^T \int_0^1 \hat{\lambda}(0, t) \phi(t) v_n(0) d\theta dt
= -\int_0^T \int_0^1 \hat{\lambda}(\theta, t) \phi(t) v_n(\theta) d\theta dt
= -\int_0^T \int_0^1 \phi(t) \mu_n^2 v_n(\theta) d\theta dt.
\]

We further simplify the weak form to obtain
\[
\int_0^1 \int_0^T \frac{\partial \hat{\lambda}}{\partial t} (\theta, t) \phi(t) v_n(\theta)d\theta dt - \alpha \int_0^T \int_0^1 \frac{\partial^2 \hat{\lambda}}{\partial \theta^2} (\theta, t) \phi(t) v_n(\theta)d\theta dt
= -y_n(0) \phi(0) - \int_0^T y_n(t) \phi(t) dt + \alpha \int_0^T \phi(t) \mu_n^2 v_n(t) dt
= -y_n(0) \phi(0) + y_n(0) \phi(0) + \int_0^T \dot{y}_n(t) \phi(t) dt + \alpha \int_0^T \phi(t) \mu_n^2 v_n(t) dt
= \int_0^T \dot{y}_n(t) \phi(t) dt + \alpha \int_0^T \phi(t) \mu_n^2 v_n(t) dt
= \int_0^1 \int_0^T P(\theta, t) \phi(t) v_n(\theta) d\theta dt.
\]

By noting that $\phi(t)$ is not trivially equal to zero for any $t$, we have the following ordinary differential equation for $y_n(t)$
\[
\begin{cases}
\dot{y}_n(t) + \alpha \mu_n^2 y_n(t) = \int_0^1 P(\theta, t) v_n(\theta)d\theta := \hat{F}_n(t), \\
y_n(0) = \int_0^1 v_n(\theta) \hat{\lambda}_0(\theta)d\theta.
\end{cases}
\]

The solution of this ODE is
\[
y_n(t) = \int_0^1 v_n(\xi) \hat{\lambda}_0(\xi) e^{-\alpha \mu_n^2 t} d\xi + \int_0^t e^{-\alpha \mu_n^2 (t-s)} \alpha \mu_n \hat{F}_n(s) ds,
\]
and then the solution of the PDE is
\[
\hat{\lambda}(\theta, t) = \sum_{n=1}^\infty v_n(\theta) y_n(t)
= \int_0^1 \sum_{n=1}^\infty v_n(\theta) v_n(\xi) \hat{\lambda}_0(\xi) e^{-\alpha \mu_n^2 t} d\xi.
\]
We can use the method of separation of variables shown in Section 4.1, i.e., the Remark 1. Here, $|v_n|$, $|\hat{\lambda}_0(\xi)|$, $|\tilde{F}_n(s)|$ and $|\alpha|$ are all bounded, so the series
\[ \sum_{n=1}^{\infty} v_n(\theta) v_n(\xi) \hat{\lambda}_0(\xi) e^{-\alpha n^2 s(t-s)} \] and its solution is
\[ \tilde{\lambda}(\theta, t) = \hat{\lambda}(\theta, T - t). \]

4.2. Solution of the costate equation. To solve the costate equation in the simplified form ($\beta = \gamma = 0$) of (17), we need to introduce new variables
\[ \tau = T - t, \quad \tilde{\lambda}(\theta, \tau) = \lambda(\theta, T - \tau). \]

Then the costate equation becomes
\[
\begin{align*}
\frac{\partial \tilde{\lambda}(\theta, \tau)}{\partial \tau} & = \alpha \frac{\partial^2 \tilde{\lambda}(\theta, \tau)}{\partial \theta^2} + Q \tilde{x}^*(\theta, T - \tau), \\
\tilde{\lambda}(0, \tau) & = \tilde{\lambda}(1, \tau) = 0, \\
\tilde{\lambda}(\theta, 0) & = S \tilde{x}^*(\theta, T) = \int_0^1 S(\theta, \eta) \tilde{x}^*(\eta, T) d\eta.
\end{align*}
\]

We use $\hat{Q}(\theta, \tau)$ and $\hat{\lambda}_0(\theta)$ to denote $Q \tilde{x}^*(\theta, T - \tau)$ and $S \tilde{x}^*(\theta, T)$, respectively. Then the above equation becomes
\[
\begin{align*}
\frac{\partial \tilde{\lambda}(\theta, \tau)}{\partial \tau} & = \alpha \frac{\partial^2 \tilde{\lambda}(\theta, \tau)}{\partial \theta^2} + \hat{Q}(\theta, t), \\
\tilde{\lambda}(0, \tau) & = \tilde{\lambda}(1, \tau) = 0, \\
\tilde{\lambda}(\theta, 0) & = \hat{\lambda}_0(\theta).
\end{align*}
\]

We can use the method of separation of variables shown in Section 4.1, i.e., the spatial-temporal separation, $\dot{\lambda}(\theta, \tau) = w(\theta) z(\tau) = \sum_{n=1}^{\infty} w_n(\theta) z_n(\tau)$, where $w_n(\theta) = \sin \sigma_n \theta$ and $\sigma_n = n \pi$, $(n = 1, 2, \ldots)$. Here $z_n(t)$ satisfies the equation
\[
\begin{align*}
\dot{z}_n(\tau) + \alpha \sigma_n^2 z_n(\tau) & = \int_0^1 \hat{Q}(\theta, \tau) w_n(\theta) d\theta := \hat{q}_n(\tau), \\
z_n(0) & = \int_0^1 \hat{\lambda}_0(\theta) w_n(\theta) d\theta,
\end{align*}
\]
and its solution is
\[ z_n(\tau) = \int_0^1 w_n(\xi) \hat{\lambda}_0(\xi) e^{-\alpha \sigma_n^2 \tau} d\xi + \int_0^\tau e^{-\alpha \sigma_n^2 (\tau-s)} \hat{q}_n(s) ds. \]

Finally, we have
\[ \tilde{\lambda}(\theta, \tau) = \int_0^1 \sum_{n=1}^{\infty} w_n(\theta) w_n(\xi) \hat{\lambda}_0(\xi) e^{-\alpha \sigma_n^2 \tau} d\xi + \int_0^\tau \sum_{n=1}^{\infty} w_n(\theta) \hat{q}_n(s) e^{-\alpha \sigma_n^2 (\tau-s)} ds. \]

By inverting the time coordinate $\lambda(\theta, t) = \tilde{\lambda}(\theta, T - t)$, we obtain the costate $\lambda(\theta, t)$. 

5. **Numerical experiments.** To obtain the optimal control law and the solution of the system state profile, we need to solve the state and the costate equations simultaneously. These two equations are coupled through the boundary control. For general system coefficients, an analytical solution is not available and thus a numerical method must be used to obtain an approximate solution. The algorithm, which is based on the iterations between the state and costate equations, can be summarized as follows:

1. Give the initial guess of $\tilde{F}_{[0]}^*(t)$ in the admissible set; i.e., set $j = 0$ to start the iteration;
2. Solve the state equation to obtain $\tilde{x}_{[j]}(\theta, t)$ with the control $\tilde{F}_{[j]}^*(t)$:
   \[
   \begin{aligned}
   \frac{\partial \tilde{x}_{[j]}^*(\theta, t)}{\partial t} &= \alpha \frac{\partial^2 \tilde{x}_{[j]}^*(\theta, t)}{\partial \theta^2} + \beta \frac{\partial \tilde{x}_{[j]}^*(\theta, t)}{\partial \theta} + \gamma \tilde{x}_{[j]}^*(\theta, t), \\
   \tilde{x}_{[j]}^*(0, t) &= \tilde{F}_{[j]}^*(t), \quad \tilde{x}_{[j]}^*(1, t) = 0, \quad \tilde{x}_{[j]}^*(\theta, 0) = \tilde{x}_0(\theta);
   \end{aligned}
   \]  

3. Reverse the time scale from $t$ to $T - \tau$ and solve the costate equation to obtain $\lambda_{[j]}(\theta, \tau)$ and $\lambda_{[j]}(\theta, t)$:
   \[
   \begin{aligned}
   \frac{\partial \lambda_{[j]}(\theta, \tau)}{\partial \tau} &= \alpha \frac{\partial^2 \lambda_{[j]}(\theta, \tau)}{\partial \theta^2} - \beta \frac{\partial \lambda_{[j]}(\theta, \tau)}{\partial \theta} + \gamma \lambda_{[j]}(\theta, \tau) + Q \tilde{x}_{[j]}^*(\theta, T - \tau), \\
   \lambda_{[j]}(0, \tau) &= \tilde{\lambda}_{[j]}(1, \tau) = 0, \\
   \tilde{\lambda}_{[j]}(\theta, 0) &= S \tilde{x}_{[j]}^*(\theta, T) = \int_0^1 S(\theta, \eta)\tilde{x}_{[j]}^*(\eta, T)d\eta;
   \end{aligned}
   \]  

4. Update the control input $\tilde{F}_{[j+1]}^*(t)$:
   \[
   \begin{cases}
   \text{time re-scaling: } & \lambda_{[j]}(\theta, t) = \tilde{\lambda}_{[j]}(\theta, T - t), \\
   \text{control update: } & \tilde{F}_{[j+1]}^*(t) = -R^{-1} \alpha \frac{\partial \lambda_{[j]}(\theta, t)}{\partial \theta} \bigg|_{\theta = 0};
   \end{cases}
   \]  

5. If the following conditions are satisfied:
   \[
   \begin{aligned}
   &\sup_{\theta, t} |\tilde{x}_{[j+1]}(\theta, t) - \tilde{x}_{[j]}(\theta, t)| \leq \epsilon_1; \\
   &\sup_{\theta, t} |\lambda_{[j+1]}(\theta, t) - \lambda_{[j]}(\theta, t)| \leq \epsilon_2; \\
   &\sup_{\theta, t} |\tilde{F}_{[j+1]}^*(t) - \tilde{F}_{[j]}^*(t)| \leq \epsilon_3;
   \end{aligned}
   \]  

   let $\tilde{x}(\theta, t) = \tilde{x}_{[j+1]}(\theta, t)$, $\lambda(\theta, t) = \lambda_{[j+1]}(\theta, t)$ and $\tilde{F}^*(t) = \tilde{F}_{[j+1]}^*(t)$ and stop.
6. Let $j = j + 1$. If $j > N_{\text{max}}$ where $N_{\text{max}}$ is a given integer, then stop. Otherwise, goto Step 2.

**Remark 2.** The numerical solution is based on the iteration procedures in the above algorithm which can be considered as a Roson-type method and widely used to solve optimal control of diffusive models. For further details on the convergence of the Roson-type numerical algorithms, one may read the recent volumes [3, 15] on computational optimal control of systems governed by PDEs.

**Remark 3.** The result in Theorem 3.1 can be extended to the case of convex constrained control sets where the Pontryagin Maximum Principle of the optimal control problem can be established without making too many changes to Theorem 3.1. More details on general optimal control of PDEs can be found in [34].
In addition, the numerical algorithm can be modified to apply the Rosen-type projected gradient algorithm to approximate the optimal control over a certain convex set [1, 2].

In the numerical illustration, we let $\alpha = 0.1$, $\beta = 0$, $\gamma = 0.4$, $T = 3$ and the initial profile is $\tilde{x}(\theta, 0) = \sin(\pi \theta)$, respectively. The weighting functions are set to be $Q(\theta, \eta) = \theta^2 + \eta^2$ and $S(\theta, \eta) = \theta^2 + \eta^2$. To start the numerical iterations, we first set the control function to be zero, then the behavior of the freely driven system is shown in Fig. 3a while the corresponding costate behavior is shown in Fig. 3b, where the uncontrolled trajectory $\tilde{x}(\theta, t)$ is used for the term in the costate equation (48). The Matlab function `pdepe` is used to generate the numerical evolutions of both the state trajectory $\tilde{x}(\theta, t)$ (forward-in-time) and the costate trajectory $\lambda(\theta, t)$ (backward-in-time), where $t \in [0, T]$. Now we can continue this iteration process by updating the control input (which is set to be zero) following the optimal control law (16). Then, one can immediately update the state and costate equations using the updated control input sequence. Some of the iterations for the control input sequences are shown in Fig. 4 where convergence can be observed after 13 iterations. We compare the final profile $\tilde{x}(\theta, T)$ in Fig. 5 associated with the control sequences of each iteration. One can observe that the control input obtained using the optimality condition can improve the stability and convergence. The evolutions of both the state and the costate dynamics are shown in Fig. 6a and Fig. 6b, respectively.

One can observe that the compatibility condition of the Dirichlet control and the initial profile is not satisfied at the left boundary end. In practice, this is impossible and a modification function $m(t) = 1 - \exp(-t/\tau)$ (by changing the control function locally at $\theta = 0$, $\tilde{F}(t) \leftarrow \tilde{F}(t)m(t)$) can be introduced to satisfy the compatibility condition, i.e., $\tilde{F}(0) = \tilde{x}(0, 0)$, where $\tau$ is a positive constant.

6. Conclusions and future work. We proposed in this paper a methodology to solve the sensor deployment problem along the offshore pipelines to detect and localize the leakage points. By introducing the PDE protocol technique, we can handle the MAS when the population of autonomous sensors becomes quite large. To increase the flexibility of sensor distributions, the PDE protocol is more complex than the heat equation and the linear reaction-advection-diffusion PDE is used instead with a boundary control designed using the optimal control of PDEs.
Figure 4. The iteration process for the control input $\tilde{F}(t)$

Figure 5. The iteration process for $\tilde{x}_{\text{end}}(\theta) := \tilde{x}(\theta, T)$
Notice from Fig. 3a, that the simulation case is stable without boundary stabilizing actuation, but the dissipation is comparably slow. As the next step, the closed-loop control synthesis will be explored to give a feedback control formulation, where a nonlinear Riccati PDE will be solved to obtain the feedback gain kernel. In addition, this paper only considers a simplified case by converting the unique dynamic boundary conditions to normal Dirichlet boundary conditions (7)-(8). A new theoretical work should be done toward the new boundary condition for the future work, including the solution, and feedback control synthesis of the PDE with the boundary conditions with time derivatives.

REFERENCES

[1] N. Ahmed and K. Teo, Optimal Control of Distributed Parameter Systems, North Holland, 1981.
[2] S. Anita, V. Arnautu and V. Capasso, An Introduction to Optimal Control Problems in Life Sciences and Economics, Modeling and Simulation in Science, Engineering and Technology. Birkhäuser/Springer, New York, 2011.
[3] V. Arnautu and P. Neittaanmaki, Optimal Control from Theory to Computer Programs, Kluwer Academic, Dordrecht, 2003.
[4] P. Barooah, P. Mehta and J. Hespanha, Mistuning-based control design to improve closed-loop stability margin of vehicular platoons, IEEE Transactions on Automatic Control, 54 (2009), 2100–2113.
[5] S. Blazic, D. Matko and G. Geiger, Simple model of a multi-batch driven pipeline, Mathematics and Computers in Simulation, 64 (2004), 617–630.
[6] F. Bullo, J. Cortes and S. Martinez, Distributed Control of Robotic Networks (In Applied Mathematics Series), Princeton University Press, New York, 2009.
[7] M. Chen and D. Georges, Nonlinear optimal control of an open-channel hydraulic system based on an infinite-dimensional model, in Proceeding of the Conference on Decision and Control, vol. 5, 1999.
[8] H. Cho and G. Hwang, Optimal design for dynamic spectrum access in cognitive radio networks under rayleigh fading, Journal of Industrial and Management Optimization, 8 (2012), 821–840.
[9] E. Chow, L. Hendrix, M. Herberg, S. Itoh, B. Kong, M. Lall and P. Stevans, Pipeline Politics in Asia: The Intersection of Demand, Energy Markets, and Supply Routes, National Bureau of Asian Research, 2010.
[10] Y. Ding and S. Wang, Optimal control of open-channel flow using adjoint sensitivity analysis, Journal of Hydraulic Engineering-ASCE, 132 (2006), 1215–1228.
[11] Z. Feng, K. Teo and V. Rehbock, Branch and bound method for sensor scheduling in discrete time, *Journal of Industrial and Management Optimization*, 1 (2005), 499–512.

[12] Z. Feng, K. Teo and V. Rehbock, Hybrid method for a general optimal sensor scheduling problem in discrete time, *Automatica*, 44 (2008), 1295–1303.

[13] G. Ferrari-Trecate, A. Buffa and M. Gati, Analysis of coordination in multi-agent systems through partial difference equations, *IEEE Transactions on Automatic Control*, 51 (2006), 1058–1063.

[14] P. Frihauf and M. Krstic, Leader-enabled deployment onto planar curves: A pde-based approach, *IEEE Transactions on Automatic Control*, 56 (2011), 1791–1806.

[15] R. Glowinski, J. Lions and J. He, *Exact and Approximate Controllability for Distributed Parameter Systems: A Numerical Approach*, (Encyclopedia of Mathematics and its Applications) Cambridge University Press, Cambridge, 2008.

[16] H. Hao and P. Barooah, On achieving size-independent stability margin of vehicular lattice formations with distributed control, *IEEE Transactions on Automatic Control*, 57 (2012), 2688–2694.

[17] H. Hao, P. Barooah and P. Mehta, Stability margin scaling laws for distributed formation control as a function of network structure, *IEEE Transactions on Automatic Control*, 56 (2011), 923–929.

[18] J. Kim, K. Kim, V. Natarajan, S. Kelly and J. Bentsman, PdE-based model reference adaptive control of uncertain heterogeneous multiagent networks, *Nonlinear Analysis: Hybrid Systems*, 2 (2008), 1152–1167.

[19] J. Kim, V. Natarajan, S. Kelly and J. Bentsman, Disturbance rejection in robust PdE-based MRAC laws for uncertain heterogeneous multiagent networks under boundary reference, *Nonlinear Analysis: Hybrid Systems*, 4 (2010), 484–495.

[20] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, SIAM, Philadelphia, 2008.

[21] Z. Lin, *Distributed Control and Analysis of Coupled Cell Systems*, VDM Verlag, Germany, 2008.

[22] W. Litvinov, Optimal control of electroreheological clutch described by nonlinear parabolic equation with nonlocal boundary conditions, *Journal of Industrial and Management Optimization*, 7 (2011), 291–315.

[23] M. Liu, S. Zang and D. Zhou, Fast leak detection and location of gas pipelines based on an adaptive particle filter, *International Journal of Applied Mathematics and Computer Science*, 15.

[24] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks (In Applied Mathematics Series)*, Princeton University Press, New York, 2010.

[25] T. Meurer and M. Krstic, Finite-time multi-agent deployment: A nonlinear pde motion planning approach, *Automatica*, 47 (2011), 2534–2542.

[26] S. Moura and H. Fathy, Optimal boundary control & estimation of diffusion-reaction PDEs, in *Proceeding of the Conference on Decision and Control*, 2011, 921–928.

[27] R. Murray, Recent research in cooperative control of multi-vehicle systems, *Journal of Dynamical Systems, Measurement and Control*, 571–583.

[28] R. Olfati-Saber and R. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Transactions on Automatic Control*, 49 (2004), 1520–1533.

[29] P. Parfomak, *Pipeline Safety and Security: Federal Programs*, CRS Report for Congress, Washington, DC, 2008.

[30] M. Rafiee, Q. Wu and A. Bayen, Kalman filter based estimation of flow states in open channels using Lagrangian sensing, *Proceedings of the Conference on Decision and Control*, (2009), 8266–8271.

[31] W. Ren and Y. Cao, *Distributed Coordination of Multi-agent Networks*, (Communications and Control Engineering Series) Springer-Verlag, London, 2011.

[32] A. Sarlette and R. Sepulchre, A PDE viewpoint on basic properties of coordination algorithms with symmetries, in *Proceedings of the Conference on Decision and Control*, 2009, 5139–5144.

[33] J. Strikwerda, *Finite Difference Schemes and Partial Differential Equations, 2nd Edition*, SIAM, Philadelphia, 2004.

[34] F. Tröltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods and Applications* (Graduate Studies in Mathematics), American Mathematical Society, New York, 2010.
[35] G. Wang and H. Ye, "Leakage Detection and Localization of Long Distance Fluid Pipelines," Tsinghua University Press, Beijing, 2010. (In Chinese).

[36] Z. Wang, H. Zhang, J. Feng and S. Lun, "Present situation and prospect on leak detection and localization techniques for long distance fluid transport pipeline," Control and Instruments in Chemical Industry, 30 (2003), 5–10.

[37] S. Woon, V. Rehbock and R. Loxton, "Global optimization method for continuous-time sensor scheduling," Nonlinear Dynamics and Systems Theory, 10 (2010), 175–188.

[38] S. Woon, V. Rehbock and R. Loxton, "Towards global solutions of optimal discrete-valued control problems," Optimal Control Applications and Methods, 33 (2012), 576–594.

[39] K. Yu, K. Mak and K. Teo, "Airfoil design via optimal control theory," Journal of Industrial and Management Optimization, 1 (2005), 133–148.

[40] C. Yu, B. Li, R. Loxton and K. Teo, "Optimal discrete-valued control computation," Journal of Global Optimization, 56 (2013), 503–518.

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