A CHARACTERIZATION OF LINEARIZABILITY FOR HOLOMORPHIC $\mathbb{C}^\ast$-ACTIONS

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Abstract. Let $G$ be a reductive complex Lie group acting holomorphically on $X = \mathbb{C}^n$. The (holomorphic) Linearization Problem asks if there is a holomorphic change of coordinates on $\mathbb{C}^n$ such that the $G$-action becomes linear. Equivalently, is there a $G$-equivariant biholomorphism $\Phi : X \to V$ where $V$ is a $G$-module? There is an intrinsic stratification of the categorical quotient $X/G$, called the Luna stratification, where the strata are labeled by isomorphism classes of representations of reductive subgroups of $G$. Suppose that there is a $\Phi$ as above. Then $\Phi$ induces a biholomorphism $\varphi : X/G \to V/G$ which is stratified, i.e., the stratum of $X/G$ with a given label is sent isomorphically to the stratum of $V/G$ with the same label.

The counterexamples to the Linearisation Problem construct an action of $G$ such that $X/G$ is not stratified biholomorphically to any $V/G$. Our main theorem shows that, for a reductive group $G$ with $G^0 = \mathbb{C}^\ast$ the existence of a stratified biholomorphism of $X/G$ to some $V/G$ is not only necessary but also sufficient for linearization. In fact, we do not have to assume that $X$ is biholomorphic to $\mathbb{C}^n$, only that $X$ is a Stein manifold.

1. Introduction

The problem of linearizing the action of a reductive group $G$ on $\mathbb{C}^n$ has attracted much attention both in the algebraic and holomorphic settings ([Huc90], [KR04], [Kra96],[Kut19]). The first results were obtained in the algebraic category. If $X$ is an affine $G$-variety, then the quotient is the affine variety $X/G$ with coordinate ring $O_{\text{alg}}(X)^G$. An early high point is a consequence of Luna’s slice theorem [Lun73]. Suppose that $X/G$ is a point and that $X$ is contractible. Then $X$ is algebraically $G$-isomorphic to a $G$-module. The structure theorem for the group of algebraic automorphisms of $\mathbb{C}^2$ shows that any action on $\mathbb{C}^2$ is linearizable [Kra96, Section 5]. As a consequence of a long series of results by many people, it was finally shown in [KR14] that an effective action of a positive dimensional $G$ on $\mathbb{C}^3$ is linearizable. The case of finite groups acting on $\mathbb{C}^3$ remains open.

The first counterexamples to the algebraic linearization problem were constructed by Schwarz [Sch89] for $n \geq 4$. His examples came from negative solutions to the equivariant Serre problem, i.e., there are algebraic $G$-vector bundles with base a $G$-module which are not isomorphic to the trivial ones (those of the form $\text{pr} : W \times W' \to W$ where $G$ acts diagonally on the $G$-modules $W$ and $W'$). It is interesting to note that in these counterexamples, the nonlinearizable actions may have the same stratified quotient as a $G$-module.

By the equivariant Oka principle of Heinzner and Kutzschebauch [HK95], all holomorphic $G$-vector bundles over a $G$-module are trivial. Thus the algebraic counterexamples to linearization are not counterexamples in the holomorphic category. But Derksen and Kutzschebauch [DK98] showed that for $G$ nontrivial, there is an $N_G \in \mathbb{N}$ such that there are nonlinearizable holomorphic actions of $G$ on $\mathbb{C}^n$, for every $n \geq N_G$.

Consider Stein $G$-manifolds $X$ and $Y$. There are the categorical quotients $X/G$ and $Y/G$. We have the Luna stratifications of $X/G$ and $Y/G$ labeled by isomorphism classes of representations.
of reductive subgroups of $G$ (see Section 2). We say that a biholomorphism $\varphi: X/G \to Y/G$ is \textit{stratified} if it sends the Luna stratum of $X/G$ with a given label to the Luna stratum of $Y/G$ with the same label. If $\Phi: X \to Y$ is a $G$-biholomorphism, then the induced mapping $\varphi: X/G \to Y/G$ is stratified. The counterexamples of Derksen and Kutzschebauch are actions on $\mathbb{C}^n$ whose quotients are not isomorphic, via a stratified biholomorphism, to the quotient of a linear action. Jointly with F. Lárusson we have shown in [KLS17] that, under a mild assumption (largeness), this is the only way to get a counterexample to linearization. To formulate this result let us recall some notions from [KLS17]. Suppose that we have a stratified biholomorphism $\varphi: X/G \to V/G$ where $V$ is a $G$-module. Then we may identify $X/G$ and $V/G$ and call the common quotient $Z$. We have quotient mappings $\pi_X: X \to Z$ and $\pi_Y: V \to Z$. Assume there is an open cover $\{U_i\}_{i \in I}$ of $Z$ and $G$-equivariant biholomorphisms $\Phi_i: \pi_X^{-1}(U_i) \to \pi_Y^{-1}(U_i)$ over $U_i$ (meaning that $\Phi_i$ descends to the identity map of $U_i$). We express the assumption by saying that $X$ and $V$ are \textit{locally $G$-biholomorphic over a common quotient}. Equivalently, our original $\varphi: X/G \to V/G$ locally lifts to $G$-biholomorphisms of $X$ to $V$. From [KLS17] we have:

\textbf{Theorem 1.1.} Suppose that $X$ is a Stein $G$-manifold, $V$ is a $G$-module and $X$ and $V$ are \textit{locally $G$-biholomorphic over a common quotient}. Then $X$ and $V$ are $G$-biholomorphic.

In fact, the $G$-biholomorphism induces the identity on the common quotient.

\textbf{Theorem 1.2.} Suppose that $X$ is a Stein $G$-manifold and $V$ is a $G$-module satisfying the following conditions.

1. There is a stratified biholomorphism $\varphi$ from $X/G$ to $V/G$.
2. $V$ (equivalently, $X$) is large.

Then, by perhaps changing $\varphi$, one can arrange that $X$ and $V$ are \textit{locally $G$-biholomorphic over $X/G \simeq V/G$, hence $\varphi$ lifts to a biholomorphism $\Phi: X \to G$}.

In [KLS17, Sec. 5] we establish the following:

\textbf{Theorem 1.3.} Let $X$ be a Stein $G$-manifold, $V$ a $G$-module and $\varphi: X/G \to V/G$ a stratified biholomorphism. Suppose that

1. $\dim V/G = 1$ ([Jia92]) or
2. $G = \text{SL}_2(\mathbb{C})$.

Then the conclusion of Theorem 1.2 holds.

The main point of this paper is to show that we can replace (1) or (2) above by the condition $\dim G = 1$.

\textbf{Theorem 1.4.} Let $G$ be reductive with $G^0 = \mathbb{C}^*$. Let $X$ be a Stein $G$-manifold, let $V$ be a $G$-module and let $\varphi: X/G \to V/G$ be a stratified biholomorphism. Then, after perhaps changing $\varphi$ by an automorphism of $V/G$, there is a biholomorphic $G$-equivariant lift $\Phi$ of $\varphi$ from $X$ to $V$.

2. Some general results

We start with some background. For more information, see [Lun73] and [Sno82, Section 6]. Let $X$ be a normal Stein space with a holomorphic action of a reductive complex Lie group $G$. The categorical quotient $Z = X/G$ of $X$ by the action of $G$ is the set of closed orbits in $X$ with a reduced Stein structure that makes the quotient map $\pi: X \to Z$ the universal $G$-invariant holomorphic map from $X$ to a Stein space. Since $X$ is normal, $Z$ is normal. If $U$ is an open subset of $Z$, then $\mathcal{O}_X(\pi^{-1}(U))^G \simeq \mathcal{O}_Z(U)$. We say that a subset of $X$ is $G$-saturated if it is a union of fibers of $\pi$. If $X$ is the Stein space associated to a normal affine variety $X'$, then $Z$ is just the complex space corresponding to the affine variety with coordinate ring $\mathcal{O}_{\text{alg}}(X')^G$. 
If $Gx$ is a closed orbit, then the stabilizer (or isotropy group) $G_x$ is reductive. We say that closed orbits $Gx$ and $Gy$ have the same isotropy type if $G_x$ is $G$-conjugate to $G_y$. Thus we get the isotropy type stratification of $Z$ with strata $Z_{(H)}$ indexed by conjugacy classes $(H)$ of reductive subgroups of $G$. The inverse image of $Z_{(H)}$ is denoted $X^{(H)}$. If $Z$ is irreducible, then there is an open dense stratum $Z_{pr}$, the principal stratum, and $X_{pr}$ denotes the corresponding set of closed orbits in $X$, the set of principal orbits. We say that the $G$-action is stable if the set of closed $G$-orbits is dense in $X$. Then $X_{pr} = \pi^{-1}(Z_{pr})$. We say that $X$ is $k$-principal if $X \setminus \pi^{-1}(Z_{pr})$ has codimension $k$ in $X$.

Assume that $X$ is smooth and let $Gx$ be a closed orbit. Then we can consider the slice representation which is the action of $G_x$ on $T_xX/T_x(Gx)$. We say that closed orbits $Gx$ and $Gy$ have the same slice type if they have the same isotropy type and, after arranging that $G_x = G_y$, the slice representations are isomorphic representations of $G_x$. The slice type (Luna) strata are locally closed smooth subvarieties of $Z$. The Luna stratification is finer than the isotropy type stratification, but the Luna strata are unions of connected components of the isotropy type strata [Sch80, Proposition 1.2]. Hence if the isotropy strata are connected, the Luna strata and isotropy type strata are the same. This occurs for the case of a $G$-module [Sch80, Lemma 5.5]. Alternatively, one can show directly that in a $G$-module, the isotropy group of a closed orbit determines the slice representation (see [Sch80, proof of Proposition 1.2]).

Now consider the case of a $G$-module $V$. Let $\pi: V \to Z$ be the categorical quotient and let $\text{Aut}(Z)$ denote the strata preserving holomorphic automorphisms of $Z$. Let $\text{Aut}(V)$ be the group of holomorphic automorphisms of $V$. Let $\mathfrak{A}(Z)$ denote the strata preserving vector fields on $Z$ and similarly define $\mathfrak{A}(V)$. Then we have a morphism $\pi_*: \mathfrak{A}(V)^G \to \mathfrak{A}(Z)$ by restricting $A \in \mathfrak{A}(V)^G$ to $O(Z) = O(V)^G$. Let $\Phi$ be a biholomorphism of Stein $G$-manifolds. We say that $\Phi$ is $\sigma$-equivariant if there is an automorphism $\sigma \in \text{Aut}(G)$ such that $\Phi \circ g = \sigma(g) \circ \Phi$ for all $g \in G$.

We say that $V$ has the infinitesimal lifting property (ILP) if every strata preserving holomorphic vector field $B \in \mathfrak{A}(Z)$ is $\pi_*(A)$ for some holomorphic vector field $A \in \mathfrak{A}(V)^G$. The scalar action of $\mathbb{C}^*$ on $V$ induces an action of $\mathbb{C}^*$ on $Z$ whose attractive fixed point is the image $\ast$ of $0 \in V$. We denote by $\text{Aut}_{ql}(Z)$ the quasi-linear automorphisms of $Z$, i.e., those which commute with the $\mathbb{C}^*$-action. If $\theta \in \text{Aut}(Z)$, then we get a family $\theta_t \in \text{Aut}(Z)$ where $\theta_t = t^{-1} \circ \theta \circ t$, $t \in \mathbb{C}^*$. We say that $V$ has the deformation property (DP) if for any $\theta \in \text{Aut}(Z)$, the limit $\theta_0$ of $\theta_t$ exists as $t \to 0$. Then $\theta_0 \in \text{Aut}_{ql}(Z)$. Finally, we say that $V$ has the lifting property (LP) if any $\theta \in \text{Aut}(Z)$ has a lift $\Theta$ to $V$. Here $\Theta$ need not be equivariant, but, of course, it has to send the fiber over any $z \in Z$ to the fiber over $\theta(z)$.

From Section 5 of [KLS17] we have the following result.

**Theorem 2.1.** Let $X$ be a Stein $G$-manifold and let $\varphi: X/\!/G \to Z$ be a strata preserving biholomorphism. Suppose that $V$ has the ILP and DP. Then, after perhaps changing $\varphi$ by an element of $\text{Aut}_{ql}(Z)$, $\varphi$ has a $G$-equivariant biholomorphic lift $\Phi: X \to V$.

Here are some results about the ILP.

**Proposition 2.2.** Let $V$ be a $G$-module. Let $V_0$ be a $G$-complement to $V^G$ in $V$.

1. $V$ has the ILP if and only if $V_0$ has the ILP.
2. If $(V, G^0)$ has the ILP then so does $(V, G)$.

**Proof.** See [Sch80, Lemma 7.1(1), Proposition 8.2].

Let $N(V) = \pi^{-1}(\pi(0))$ denote the nullcone of $V$. Let $H$ denote a principal isotropy group, i.e., $(H)$ is the conjugacy class of the stratum $Z_{pr}$. Then $Z$ is the quotient of $V^H$ by $N = N_G(H)/H$ where the principal isotropy groups of $V^H$ are trivial so that the action is stable [LR79]. The strata of $V/\!/G$ are the same as those of $V^H/\!/N$, with a change of label [Sch80, Thm. 11.3]. Thus
if $H$ is normal in $G$, then the ILP and/or the DP for $(V^H, N = G/H)$ implies that for $(V, G)$ (and vice versa).

**Proposition 2.3.** Let $V$ be a $G$-module where $G^0 = \mathbb{C}^*$ and $V^G = (0)$. Assume that $\dim Z \geq 3$ and that $N(V)$ has codimension at least 2 in $V$. Then $V$ has the ILP.

*Proof.* We may reduce to the case that the $G$-action is stable and faithful and that $G = \mathbb{C}^*$. Let $B \in \mathfrak{a}(Z)$ and let $\ast$ denote $\pi(0)$. Let $C$ be the invariant vector field on $V$ corresponding to a generator of $g$. Since the isotropy groups of closed nonzero orbits are finite, by Proposition 2.2(2) and the slice theorem, there is an open cover $U_i$ of $Z \setminus \{\ast\}$ and lifts $A_i$ of $B$ to $\pi^{-1}(U_i)$. The differences $A_i - A_j$ are multiples of $C$ with coefficients in $\mathcal{O}(U_i \cap U_j)$. Thus the obstruction to glueing the $A_i$ is in $H^1(Z \setminus \{\ast\}, \mathcal{O}_Z)$. Since $Z$ is Cohen-Macaulay of dimension at least 3, the obstruction vanishes and there is a lift $A'$ of $B$ to $\mathfrak{a}(V \setminus N(V))^G$. By Hartog’s theorem, $A'$ extends to $A \in \mathfrak{a}(V)^G$ which is a lift of $B$. □

**Proposition 2.4.** If $V$ has the LP, then it has the DP.

*Proof.* This follows from [Sch14, Proposition 2.9]. (The proposition assumes that $V$ is 2-principal, but that is not needed for the conclusion about the DP.) □

**Proposition 2.5.** Let $H$ be a finite group and $W$ an $H$-module with quotient morphism $p: W \to W/H$. Let $U \subset W/H$ be open and let $\theta \in \text{Aut}(U)$. Then there is a lift $\Theta \in \text{Aut}(p^{-1}(U))$. Thus $W$ has the LP, hence also the DP.

*Proof.* See [KLM03, Theorem 5.4] (see also [Lya83]) or [Sch09, Theorem 3.1]. □

**Remark 2.6.** Let $\theta \in \text{Aut}(U)$ where $U$ is connected and let $\Theta \in \text{Aut}(\pi^{-1}(U))$ be a lift of $\theta$. Then $\Theta$ sends $H$-orbits to $H$-orbits and for $h \in H$, $\Theta \circ h \circ \Theta^{-1}$ induces the identity on $U$, hence it must be an element $\sigma(h)$ of $H$ where $\sigma$ is an automorphism of $H$. Thus $\Theta$ is $\sigma$-equivariant.

**Corollary 2.7.** Let $Y = V//G^0$ and let $\rho: Y \to Z = V//G$ be the canonical map. Suppose that $Y$ has no codimension one strata. If $(V, G^0)$ has the LP and $Y_{pr}$ is simply connected, then $(V, G)$ has the LP (and DP).

*Proof.* Let $\theta \in \text{Aut}(Z)$. Since $\rho$ is the quotient by a finite group, there is an open cover $U_i$ of $Y_{pr}$ by $G/G^0$-invariant open sets and $\theta_i \in \text{Aut}(U_i)$ which cover $\theta$. The compositions $\theta_i \circ \theta_j^{-1}$ take values in $G/G^0$, so we obtain a cocycle in $H^1(Y_{pr}, G/G^0)$. This corresponds to a covering space of $Y_{pr}$, which is trivial since $Y_{pr}$ is simply connected. Hence changing the $\theta_i$ by elements of $G/G^0$, we construct a lift $\tilde{\theta}$ of $\theta$ to $Y_{pr}$. Since $Y$ is normal, $\tilde{\theta}$ extends to $Y$ and preserves the strata of $Y$. Since $(V, G^0)$ has the LP, $\tilde{\theta}$ lifts to $\Theta \in \text{Aut}(V)$ which is a lift of $\theta$. □

**Theorem 2.8.** Suppose that $G$ is finite and $\varphi: X/G \to V/G$ is a strata preserving biholomorphism. Then, perhaps changing $\varphi$ by an element of $\text{Aut}_{\text{pr}}(Z)$, $\varphi$ has a $G$-equivariant biholomorphic lift $\Phi: X \to V$. The original $\varphi$ has a $\sigma$-equivariant lift for some $\sigma \in \text{Aut}(G)$.

*Proof.* This follows from Propositions 2.2 and 2.5, Theorem 2.1 and Remark 2.6. □

3. **The easy cases**

We return to the case where $G^0 = \mathbb{C}^*$.

**Theorem 3.1.** If the weights of $G^0$ on $V$ are all positive or all negative, then Theorem 1.4 holds.

*Proof.* All the closed $G$-orbits of $X$ lie in $X^{G^0}$ and similarly for $V$. Thus $\varphi$ is a strata preserving biholomorphism of $X^{G^0}/G$ with $V^{G^0}/G$. By Theorem 2.8, after modifying $\varphi$ by an element of
there is an equivariant biholomorphic lift $\Phi : X^{G^0} \to V^{G^0}$. Let $V_0$ be a $G$-complement to $V^{G^0}$ in $V$. By [HK95] the normal bundle of $X^{G^0}$ in $X$ is isomorphic to the normal bundle $V_0 \times V^{G^0} \to V^{G^0}$ in $V$. By [KLS17, Proposition 7] there is a $G$-biholomorphism $\Phi$ of a $G$-saturated neighborhood of $X^{G^0}$ to a $G$-saturated neighborhood of $V^{G^0}$ which induces $\varphi$ on $X^{G^0}$. But a $G$-saturated neighborhood of $X^{G^0}$ is all of $X$ and similarly for $V^{G^0}$. Hence $\Phi : X \to V$ is a $G$-equivariant biholomorphic lift of $\varphi$. □

**Theorem 3.2.** If the action of $G^0$ on $V$ has at least two strictly positive weights and at least two strictly negative weights, then Theorem 1.4 holds.

**Proof.** It is enough to establish the ILP and DP. For the ILP we may reduce to the case of $(V,G^0)$ and assume that $V^{G^0} = (0)$. Then the null cone $\mathcal{N}(V)$ has codimension at least two in $V$ and by Propositions 2.2 and 2.3, $(V,G^0)$ and $(V,G)$ have the ILP. To establish the LP (hence DP) it suffices to prove that $(V,G^0)$ has the LP and that $Y_{pr}$ is simply connected, where $Y = V/\!/G^0$ (the latter may be false, but we get around this). We may assume that $V^{G^0} = (0)$.

Let $p_1, \ldots, p_k$ be the positive weights of $V$ and let $q_1, \ldots, q_l$ be the negative weights with $k + l = n$. We may assume that the GCD of the $p_i$ and $q_j$ is 1. We first deal with the codimension one strata of $Y$. A codimension one stratum occurs whenever there are $n - 1$ weights with GCD $r$ such that the remaining weight is prime to $r$. Thus the variable $x$ corresponding to the remaining weight, call it $d$, always occurs in any invariant to a power a multiple of $r$. Now let $F$ be $\mathbb{Z}/r\mathbb{Z} \subset G^0$. Then the quotient of $V$ by $F$, call it $V'$, has the same weights as $V$, except that $d$ has been replaced by $rd$. This eliminates the corresponding codimension one stratum. We can do the same process for any other codimension one stratum of $Z$. Let $F$ be the resulting cyclic subgroup of $G^0$.

**Example 3.3.** Suppose that the weights are $\{-2, -6, 3, 6\}$ all with multiplicity one. Then $F \approx \mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with the first (resp. second) factor acting on the first (resp. third) coordinate.

Now $V/F = V'$ is a $G^0$-module which is 2-principal. Let $\theta \in \text{Aut}(Y)$. Then $\theta$ preserves the strata of $V'/\!/G/F$). By [Sch14, Theorem 1.12] there is a lift $\Theta' \in \text{Aut}(V')$ of $\theta$. Since $\theta$ preserved the strata of $Y$, $\Theta'$ preserves the strata of the action of $F$. Since $F$ is finite, we have a lift of $\Theta'$ to $\Theta \in \text{Aut}(V)$. Hence $(V,G^0)$ has the LP. Dividing by a finite subgroup we can make the action of $G^0$ on $V'$ faithful. Then we have a fibration $\mathbb{C}^* \to V'_{pr} \to Y_{pr}$. Since $V'_{pr}$ is simply connected the exact homotopy sequence gives $0 \to \pi_1(Y_{pr}) \to \pi_0(\mathbb{C}^*)$, hence $\pi_1(Y_{pr})$ is trivial as required. Now we can apply Corollary 2.7 to establish the DP. □

4. The hard case

**Theorem 4.1.** If $G^0$ acts with weights of both signs, but only one strictly positive weight or only one strictly negative weight, then Theorem 1.4 holds.

This covers the remaining cases of Theorem 1.4. Without loss of generality we assume that there is one strictly positive weight and let $n$ be the number of strictly negative weights. It is easy to see that the ILP and DP hold in the case that $n = 1$, so we assume that $n \geq 2$. Then $G$ has to centralize $G^0$.

We need two local versions of the LP, as follows.

**Proposition 4.2.** Let $\theta \in \text{Aut}(U)$ where $U$ is a connected neighborhood of $V^G$ in $Z$ and $\theta$ is the identity on $V^G$. Then, modulo $\text{Aut}(Z)$, $\theta$ has a $G$-equivariant lift $\Theta$ to $\pi^{-1}(U)$.

**Proposition 4.3.** Let $\theta \in \text{Aut}(U)$ where $U$ is a connected neighborhood of $V^{G^0}/G$ in $Z$ and $\theta$ is the identity on $V^{G^0}/G$. Then, modulo $\text{Aut}(Z)$, $\theta$ has a $G$-equivariant lift $\Theta$ to $\pi^{-1}(U)$. 
Proof of Theorem 4.1. Now \( \varphi \) induces a biholomorphic map of \( X^G \) and \( V^G \). Let \( V_0 \) be a \( G \)-complement to \( V^G \) in \( V \). By [HK95] the normal bundle to \( X^G \) in \( X \) is equivariantly trivial, hence \( G \)-isomorphic to the normal bundle \( V_0 \times V^G \to V^G \) of \( V^G \) in \( V \). By [KLS17, Proposition 7], there is a \( G \)-biholomorphism \( \Psi : \Omega' \to \Omega \) where \( \Omega' \) is a \( G \)-saturated neighborhood of \( X^G \) in \( X \) and \( \Omega \) is a \( G \)-saturated neighborhood of \( V^G \) in \( V \). Moreover, \( \Psi \) agrees with \( \varphi \) on \( X^G \). We may assume that \( U' = \Omega' \cap G \) and \( U = \Omega \cap G \) are connected. Let \( \theta = \varphi \circ \psi^{-1} \) where \( \psi : U' \to U \) is induced by \( \Psi \). By Proposition 4.2 there is a \( \tau \in \text{Aut}(Z) \) such that \( \tau \circ \theta \) lifts to \( \Theta \in \text{Aut}(\Omega)^G \). Then \( \Phi_1 := \Theta \circ \Psi \) is an equivariant lift of \( \tau \circ \varphi \). Replace \( \varphi \) by \( \tau \circ \varphi \).

By Theorem 2.8 and Remark 2.6, the restriction of \( \varphi \) to \( X^{G_0}/G \) has a \( \sigma \)-equivariant lift \( \varphi_0 : X^{G_0} \to V^{G_0} \) where \( \sigma \) is an automorphism of \( G/G^0 \). Now the restriction of \( \Phi_1 \) to \( \Omega' \cap X^{G_0} \) is an equivariant lift of \( \varphi_0 \). Thus \( \varphi_0 \) and \( \Phi_1 \) locally differ on \( \Omega' \cap X^{G_0} \) by multiplication by some \( g \in G/G^0 \). Then they differ on \( X^{G_0} \) by a fixed \( g \in G \). We can make \( \varphi_0 \) equivariant by replacing it by \( g \circ \varphi_0 \). Now by [HK95] and [KLS17, Proposition 7] again, we can find a \( G \)-equivariant biholomorphism \( \Psi : \Omega' \to \Omega \) which agrees with \( \varphi_0 \) on \( X^{G_0} \) where \( \Omega' \) is a \( G \)-saturated neighborhood of \( X^{G_0} \) in \( X \) and \( \Omega \) is a \( G \)-saturated neighborhood of \( V^{G_0} \) in \( V \). We may assume that \( U_0' = \Omega' \cap G \) and \( U = \Omega \cap G \) are connected. Let \( \psi : U_0' \to U \) be induced by \( \Psi \) and let \( \theta = \varphi \circ \psi^{-1} \).

By Proposition 4.3 there is a \( \tau \in \text{Aut}(Z) \) such that \( \tau \circ \theta \) has a \( G \)-equivariant lift \( \Theta \) on \( \Omega' \). Replace \( \varphi \) by \( \tau \circ \varphi \). Then \( \Phi := \Theta \circ \Psi \) is an equivariant lift of \( \varphi \) over \( U_0 \).

Let \( E \) be the Euler vector field on \( V \) and let \( \pi_*(E) \) denote its pushdown to \( \pi_*(Z) \). Let \( B \in \mathfrak{a}(X^G) \) denote the image of \( \pi_*(E) \) using \( \varphi^{-1} \). Using \( \Phi \) one gets a lift \( A_0 \) of \( B \) over the neighborhood \( U_0' \) of \( X^{G_0}/G \). Since the isotropy groups of closed orbits outside of \( X^{G_0} \) are finite, \( B \) has local lifts \( A_i \) over an open cover \( \{U_i'\} \) of \( X^G \). Since \( X^G \) is affine, we see that there is no obstruction to finding a global lift \( A \in \mathfrak{a}(X^G) \) of \( B \). Now one can use the argument for the proof of Theorem 1 of [KLS17] (see Remark 4 of loc. cit) to prove Theorem 4.1.

It remains to prove the two propositions. The rest of this section (and paper) is the proof of Proposition 4.3. The proof of Proposition 4.2 is similar. We may assume that the single strictly positive weight of \( V \) is \( d > 0 \) and that the other weights are negative. We write

\[
V = \mathbb{C} \oplus V^G \oplus W, \quad W = W_0 \oplus W', \quad W' = \bigoplus_{i=1}^k W_i
\]

where \( \mathbb{C} \) is the weight space of the weight \( d \) and the weights in \( W_i \) are \( -e_i, 0 = e_0 < e_1 < \cdots < e_k \). Let \( x \) be a coordinate function on \( \mathbb{C} \), let \( x_1, \ldots, x_p \) be coordinate functions on \( V^G \), let \( y_1, \ldots, y_d \) be coordinate functions on \( W_0 \) and let \( z_1, \ldots, z_n \) be coordinate functions on \( W' \) where each \( z_i \) is a coordinate function on some \( W_j \). Then \( x \) has weight \( -d \), the \( x_i \) and \( y_j \) have weight \( 0 \) and the \( z_i \) have weights in \( \{e_1, \ldots, e_k\} \). Note that \( G \) stabilizes \( \mathbb{C}, V^G \) and the \( W_i, i = 0, \ldots, k \).

Now let \( H \) denote the isotropy group of \( x_0 := (1, 0, 0) \in \mathbb{C} \ominus V^G \ominus W \). Then \( H \) contains the isotropy group \( \mathbb{Z}/d\mathbb{Z} \) of \( G^0 \) at \( x_0 \). It is easy to see that \( H/(\mathbb{Z}/d\mathbb{Z}) \simeq G/G^0 \). Since \( G \) centralizes \( G^0 \), \( H \) centralizes \( \mathbb{Z}/d\mathbb{Z} \). Let \( V' \) denote \( V^G \oplus W \).

Lemma 4.4. Evaluation at \( x_0 \) gives an isomorphism \( \lambda : \mathbb{C}[V]^G \to \mathbb{C}[V']^H \).

Proof. We may assume that \( V^G = (0) \). Clearly \( \lambda \) is injective. Let \( f \in \mathbb{C}[W]^H \) and let \( m \) be a monomial occurring in \( f \). Then \( m \) has degree \( a_0, \ldots, a_k \) in the variables of \( W_0, \ldots, W_k \). Thus \( H \cdot m \) is a sum of monomials of the same degrees. Now \( \sum a_i e_i \equiv 0 \mod d \) so there is a unique \( r \) such that \( -rd + \sum a_i e_i = 0 \). Then \( x^r (H \cdot m) \in \mathbb{C}[V]^G \) is \( \lambda^{-1}(H \cdot m) \). Hence \( \lambda \) is an isomorphism.

As a consequence of the lemma, \( Z = V/G \simeq Z' := V'/H \). If \( S = Z_{(L)} \) is a stratum of \( Z \), then \( S \) is the image of \( V'^{<L>} := V' \cap \pi^{-1}(S) \). We use similar notation for strata \( Z'_{(L')} \) of \( Z' \) and subsets \( V'^{<L'>} \) of \( V' \).
Let $p: V' \to Z'$ be the quotient mapping. Let $\theta \in \text{Aut}(U)$ where $U$ is a connected neighborhood of $V^G/G$ in $Z$. Let $U'$ denote the subset of $Z'$ corresponding to $U$ and let $\theta'$ denote the element of $\text{Aut}(U')$ corresponding to $\theta$. By Proposition 2.5 and Remark 2.6 there is a lift $\Theta' \in \text{Aut}(p^{-1}(U'))$ which is $\sigma$-equivariant for some automorphism $\sigma$ of $H$. Let $S = Z(L)$ be a stratum of $Z$. If $L$ is finite, then we may assume that $L \subset H$. Now

$$S = \pi(C \setminus \{0\} \times V^G \times W^{<L>}) = \pi(\{1\} \times V^G \times W^{<L>}) = p(V^G \times W^{<L>})$$

is a stratum of $Z'$. Moreover, $p^{-1}(S)$ is $C^*$-stable (scalar action) and $\Theta'$ preserves $p^{-1}(S \cap U')$. If $L$ is infinite, then $S$ is the image of

$$V^{<L>} = V^G \times W^{<L>}_0 = V^G \times W^{<L>_0}_0$$

where $L_0$ is a subgroup of $H$ containing $\mathbb{Z}/d\mathbb{Z}$ such that $L/\mathbb{C}^* \simeq L_0/(\mathbb{Z}/d\mathbb{Z})$. Thus $S$ is a subset of a stratum $S'$ of $Z'$. Now $p^{-1}(S)$ is $C^*$-stable (scalar action) and $p^{-1}(S \cap U')$ is $\Theta'$-stable since $\Theta'$ induces $\theta$. Then $d\Theta'(0)$ is $\sigma$-equivariant and induces elements $\theta_0 \in \text{Aut}(Z)$ and $\theta'_0 \in \text{Aut}(Z')$. Replace $\theta$ by $\theta_0^{-1} \circ \theta$, replace $\theta'$ by $(\theta'_0)^{-1} \circ \theta'$ and replace $\Theta'$ by $d\Theta'(0)^{-1} \circ \Theta'$. Then $\Theta' \in \text{Aut}(p^{-1}(U'))^H$ is a lift of $\theta'$.

Now $\Theta' = (x_1', \ldots, x_p', y_1', \ldots, y_q'; z_1', \ldots, z_n')$ where $x_1', \ldots, x_p'$ are $H$-invariant, $(y_1', \ldots, y_q') : V' \to W_0$ is $H$-equivariant where the $y_i'$ are $\mathbb{Z}/d\mathbb{Z}$-invariant and $z_i'$ has weight $e_j \mod d$ relative to $\mathbb{Z}/d\mathbb{Z}$ where $z_i$ is a coordinate for $W_j$. The $x_i'$ are the restrictions of $a_1, \ldots, a_p \in \mathcal{O}(\pi^{-1}(U))^G$. The $y_i'$ lift to elements $b_1, \ldots, b_q \in \mathcal{O}(\pi^{-1}(U))^G$ such that $(b_1, \ldots, b_q) : \pi^{-1}(U) \to W_0$ is $G$-equivariant. The main problem is that the $z_i'$ have lifts which may involve negative powers of $x$. It is clear that everything has holomorphic parameters coming from $V^G$, so we may reduce to the case that $V^G = (0)$. So $U'$ is now a neighborhood of $W_0/H$ in $Z'$ and $U$ is the corresponding open subset of $Z$. The morphism $\Theta' : p^{-1}(U') \to p^{-1}(U')$ is $H$-equivariant and preserves $W_0 \times \{0\}$. Let $\Theta$ denote the (rational) lift of $\Theta'$ to $\pi^{-1}(U)$. We denote the entries of $\Theta$ by $(x, b_1, \ldots, b_q, z_1', \ldots, z_n')$.

**Lemma 4.5.** Let $\Theta$ be as above. Then there is a $\lambda \in \text{Aut}(Z)$ and a lift $\Lambda$ of $\lambda$ such that $\Lambda \circ \Theta \in \text{Aut}(\pi^{-1}(U))^G$.

The lemma implies Proposition 4.3 and completes our proof of Theorem 4.1, hence of Theorem 1.4.

Let $m$ be a monomial in $z_1, \ldots, z_n$ which is a generator of the covariants of $\mathbb{Z}/d\mathbb{Z}$ of weight $e_1 \mod d$ as a module over the $\mathbb{Z}/d\mathbb{Z}$-invariants. Then $m$ times a unique power of $x$ has weight $e_1$. The power of $x$ can be negative (see example below). Let $\overline{M}$ be the $\mathbb{C}[W]^G$-$\mathbb{Z}/d\mathbb{Z}$-module generated by the covariants of weights $e_1, \ldots, e_k \mod d$ and let $M$ be the $(\mathbb{C}[x, x^{-1}] \otimes \mathbb{C}[W]^G)$-module generated by the corresponding (rational) covariants of weights $e_1, \ldots, e_k$.

**Lemma 4.6.** Let $m \in M$ have weight $e_j$ and let $m_1, \ldots, m_k$ be elements of $M$ of weights $e_1, \ldots, e_k$. Then $m' = m(m_1, \ldots, m_k) \in M$.

**Proof.** Let $m \to \overline{m}$ denote the mapping from $M$ to $\overline{M}$ given by restriction to $x = 1$. Then the mapping from the covariants of $e_j$ in $M$ to those of weight $e_j \mod d$ in $\overline{M}$ is an isomorphism. Since $\overline{m} = \overline{m}(\overline{m_1}, \ldots, \overline{m_k}) \in \overline{M}$, it follows that $m' \in M$.

**Corollary 4.7.** Let $m \in M$. Then for any homogeneou element $f \in \mathbb{C}[\mathbb{C} \oplus W]^G$ of sufficiently high degree, $fm$ is polynomial and remains polynomial after any substitutions $fm \mapsto fm(m_1, \ldots, m_k)$.

**Example 4.8.** We have coordinate functions $(x, y, z, w)$ with weights $(-3, 1, 8, 12)$. Mod 3 the weights on $(y, z, w)$-space are $(1, 2, 0)$. Thus, mod 3, the invariants are generated by $y^3, z^3, yz$ and $w$. The $C^0$-invariants are then generated by $xy^3, x^2y^3, x^3yz$ and $x^4w$. The isotropy groups of closed orbits are $G^0$, $\mathbb{Z}/3\mathbb{Z}$ and $\{e\}$. 
Now we look at the generators for the covariants for the $\mathbb{Z}/3\mathbb{Z}$-action on $(y, z, w)$-space. Those corresponding to the representation where $\xi \in \mathbb{Z}/3\mathbb{Z}$ acts via multiplication are $y$ and $z^2$. Those corresponding to the dual representation are $y^2$ and $z$. Those corresponding to the trivial representation are $1$, $y^3$, $z^3$, $yz$ and $w$. Then taking into account the action of $G^n$ we see that $M$ is generated by the following covariants.

1. Weight 1: $y$ and $x^5z^2$.
2. Weight 8: $z$ and $x^{-2}y^2$.
3. Weight 12: $w$, $x^{-4}$, $x^{-3}y^3$, $x^4z^3$ and $x^{-1}yz$.

The elements of $M$ are closed under composition as per Lemma 4.6.

We call an element $m \in M$ good if $m$ is polynomial and remains polynomial after any substitutions as in Lemma 4.6, else we say that $m$ is bad. By Corollary 4.7, there are only finitely many homogeneous $m_i \in M$ which are bad. We are going to modify $\Theta$ by a sequence of meromorphic $G$-automorphisms $\Lambda$ of $V$ such that $\Lambda$ induces a strata preserving automorphism $\lambda \in \text{Aut}(Z)$.

Let us assume that $z_1, \ldots, z_\ell$ are the variables of $W_1$. Suppose that a monomial in the Taylor expansion of $z'_j$ (in the variables $z_1, \ldots, z_n$) contains one of the variables $z_1, \ldots, z_\ell$. Then it is an invariant times the variable and is obviously good since invariants are preserved under trivial representation are 1, $y$, $x$, $y^2$, $xy$, $x^2$, $yx$, $y^2x$, $xy^2$, $x^2y$, $yx^2$, $y^2x^2$, $xy^2x$, $x^2y^2$, $yx^2y$, $y^2x^2y$, etc. Hence there is a $\Lambda$ as claimed in Lemma 4.5.

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A CHARACTERIZATION OF LINEARIZABILITY FOR HOLOMORPHIC $\mathbb{C}^*$-ACTIONS

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