OPTIMAL EXPANSION TIMING DECISIONS IN MULTI-STAGE PPP PROJECTS INVOLVING DEDICATED ASSET AND GOVERNMENT SUBSIDIES

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Abstract. The topic of investment timing in multi-stage public-private partnership (PPP) projects has not been received much attention so far. This study investigates optimal expansion timing decisions in multi-stage PPP projects under an uncertain demand and where the first-stage greenfield project involving a dedicated asset is immediately implemented as the PPP contract is closed, whereas the timing of the later expansion is flexibly decided according to the demand. In this setting, the optimal multiple stopping timing theory is adopted to model the expansion framework. Furthermore, we integrate a government subsidy, including an investment subsidy and revenue subsidy, into the expansion timing decisions. Through a hypothetical three-stage investment plan for a sanitary sewerage project, the optimal expansion strategies and the value of the multi-stage project before and after the subsidy are provided using a least squares Monte Carlo simulation. Also, the influences of a dedicated asset on the expansion strategies and project value are illustrated. In addition, we compare the incremental value before and after the subsidy and earlier expansion derived from two types of subsidies. The comparisons show that there is more incremental value for the revenue subsidy, and that the investment subsidy brings an earlier expansion.

1. Introduction. Since the middle of the 1980s, there has been a significant growth in the number of large-scale infrastructure projects executed with a public-private partnership (PPP) concession agreement. With a PPP, a public entity (e.g., a government agency) delegates the construction and maintenance of an infrastructure
system to a private partner (called concessionaire), as an alternative means of financing a project [9]. PPP projects have been driven by the need to increase the efficiency of the delivery of public services and the desire to upgrade and extend public infrastructure even with fewer available public funds [8].

The investments made in PPP projects is typically large, and the environment is highly uncertain, making multi-stage investments an important method to hedge uncertainty [12]. It is worth noting that the previous projects of multi-phase projects usually require investment in dedicated assets. Here, dedicated assets are up-front or intermediate investments that can be used for future expansions in specific areas or markets [29]. Once deployed, these investments remain idle until future expansions are completed [16]. One of the things that needs to consider is that how the dedicated assets affect subsequent investment decisions and project value. The influence of a dedicated asset on the value of multi-stage build-operate-transfer (BOT, a type of PPPs) projects has been investigated in the literature [15].

Many infrastructures are characterized by demand uncertainty [10, 26, 28]. To respond to this uncertainty, it is important to find the right time to invest for infrastructures [27]. There are some publications that have explored investment timing decisions for infrastructure projects in the context of uncertainty [27, 21, 24, 6]. However, these studies on timing decisions focus on single-stage projects. To the best of our knowledge, timing flexibility in multi-stage PPP projects has not received much, if any, attention. To facilitate multi-stage investments, PPP concession contracts often grant the concessionaire abandonment rights or flexible expansion [13]. The value of multi-stage BOT projects that allow the concessionaire abandon investing has been evaluated by Huang and Pi [15].

The current paper addresses flexible expansion timing in multi-stage PPP projects, where it is assumed that the first-stage greenfield investment begins construction immediately when the concession contract has been closed and that the expansion timing for later projects will be decided in a flexible way, or abandoned according to the demand level, for up to the end of the $T$ year of the concession being granted. The rationality of the assumption is demonstrated in the following two factors: On the one hand, the necessity and feasibility of the first-stage project can be comprehensively assessed before the contract has been closed; therefore, the first-stage investment typically starts construction as soon as the contract has been finalized. On the other hand, the demand for later expansion projects is highly uncertain, so having flexible expansion timing or right to abandon these later expansion projects, which then allows for the decision maker to use future information, can reduce the uncertainty.

The optimal investment date for a private sector is likely going to be different from the date that would be chosen by a social planner [27]. As suggested by Pennings [25], a socially optimal investment timing would be earlier than what would be chosen by the private sector. From a government’s perspective, an early investment could satisfy public needs. However, this generally will not coincide with the private-sectors value-maximization perspective. If the valuation shows that a PPP project will not meet the concessionaires goal regarding return on investment, but the government nevertheless is keen to implement the project, then various types of subsidies are usually provided to the concessionaire to implement the project [17, 30, 31]. Some previous studies have investigated the impacts of a subsidy on the investment timing of PPP projects. Under demand uncertainty, Li and Cai [18] address the influences of government subsidies on private investment.
behaviors, including the choices of investment timing, capacity, and price, in which the results suggest that a lump-sum subsidy and unit subsidy can produce a timely investment. In a real options framework, Armada et al [1] analyze how certain subsidies could be optimally arranged to promote an immediate investment, and they further compare the performance of different subsidies about the project value and timing of the resulting cash flows. Similarly, this branch of the literature investigates the investment timing when there is a government subsidy for single-stage PPP projects. However, the current paper contributes by investigating the optimal investment timing with a government subsidy for multi-stage PPP projects.

It is known that optimizing the investment timing is an optimal stopping problem, as motivated by the analysis of American options [19]. In the present study, the optimal multiple stopping time theory driven by multi-exercise American options, is adopted to model the expansion decisions of multi-stage PPP projects. First, an optimal expansion timing framework that incorporates a dedicated asset investment is introduced to maximize the project value throughout its multiple stages. Subsequently, the governments subsidy mechanisms, including the investment subsidy and revenue subsidy, are incorporated into the expansion decision model to evaluate the influence of government subsidies on the expansion strategy. Finally, a hypothetical example using a sewage disposal project is presented to investigate the validity of the model. Compared with the general investment decision in the U.S. option framework, the difficulties in our model are seen in two aspects. One the one hand, the expected returns of each stage are path dependent and obtained by solving a differential equation system. On the other hand, optimizing the decisions for a multi-stage expansion is a multiple stopping time problem, which is more complicated than the classical single stopping time problem, and in our study, the optimized expansion strategies are given using least squares Monte Carlo simulations (LSM).

The current paper’s main results include: (i) The project value and the optimal expansion strategies throughout the projects phases are provided by the given example. The optimal expansion strategies are described by a sequence of critical demand thresholds at which it is optimal to invest. (ii) The impacts of dedicated assets and government subsidies on expansion strategies and project value are also examined. (iii) Furthermore, two types of subsidies’ comparisons that involve both the incremental project value before and after subsidies and the earlier expansion opportunities that can be derived from the subsidies. These results provide insights for the government and the concessionaire regarding when to enact the expansion projects, maximizing profits under uncertain demand, while also giving insights for the government regarding how to evaluate the impacts of different subsidies on private profit-maximizing investment decisions, and which subsidy type and size should be selected.

The contributions of the current study are as follows: First, the flexible expansion timing for multi-stage PPP projects is investigated. Up to now, this topic has rarely been examined; therefore, examining it can enrich the multi-stage investment theory when it comes to PPP projects. Second, we explore the impacts of the dedicated asset and government subsidy on the value of the project and optimal expansion strategies, and further compare the two types of subsidies in terms of the incremental value and earlier expansion opportunities that can be derived from the subsidies. Finally, for the complex calculations, the theoretical analysis and numerical algorithms suitable for the current problem are given in detail.
The paper is structured as follows: In Section 2, we illustrate the problem and assumptions. In Section 3, using the optimal multiple stopping theory, the model for multi-stage expansion timing decisions is first given for the case of no government subsidy, and following this, two subsidy arrangements are embedded in the optimal expansion timing model. In Section 4, the least squares Monte Carlo simulation is introduced to solve the optimal multiple stopping model. To verify the modeling results, an example of a three-stage sanitary sewerage system is given in Section 5. Finally, the Section 6 closes the paper by providing concluding remarks.

2. Problem description and assumptions. An N-stage PPP project is planned to be invested in the \([0, T]\) period. The N-stage investment project is composed of a first-stage greenfield project and subsequent \(N - 1\) expansion projects. The concession period of the project is denoted by \(T_c\), which includes the construction period and the operation period. After the construction is completed for each stage project, it enters the operation period of the corresponding stage, during which the concessionaire obtains income to recover investment costs and gain profits. Until \(T_c\), multiple projects are transferred to the government together. The schematic diagram of a three-stage PPP project is shown in Figure 1.

![Figure 1. The schematic diagram of a three-stage PPP project.](image)

The current paper makes the following assumptions:

**Assumption 1.** The demand at time \(s\), \(X_s\), behaves according to the following geometric Brownian motion (GBM), \(\forall t \in [0, T_c], s \in [t, T_c]:\)

\[
\begin{align*}
    dX_s &= \alpha X_s ds + \sigma X_s dW_s \\
    X_t &= x
\end{align*}
\]

where \(\alpha\) is the demand drift, \(\sigma\) is the demand volatility, \(dW_s\) is an increment of a Brownian motion, and \(x\) is the demand at time \(t\). The GBM has been widely used to model demand evolution and fluctuation in infrastructure projects \([2, 11, 14]\) and hence works for the purpose of the current paper.

**Assumption 2.** The load of the post-project is allocated as the remaining demand after the full-load operation of the pre-projects.

**Assumption 3.** There exists a dedicated asset investment in the first project. That is, the investment cost of the first stage includes two parts: one is the cost that matches the project capacity in the first stage, and the other is the cost of the dedicated assets for later stages.

**Assumption 4.** The first-stage greenfield project is constructed at the initial time \(t = 0\) when the contract is closed.

**Assumption 5.** The investment timing for subsequent stages is flexibly decided according to the demand change, and the concessionaire can abandon the expansion under an insufficient level of demand.
From the perspective of maximizing the project value, the current study addresses the optimal expansion timing for the decision making of multi-stage PPP projects with and without a government subsidy. That is, the goal is to optimize $N - 1$ opportunities, maximizing the total income of $N$ stages (if the right to abandon is exercised, the optimized times of the expansion opportunities will be less than $N - 1$).

3. Model formulation.

3.1. The basic model for expansion timing decisions. It is assumed that the construction of a PPP project is divided into $N$ stages and that the procurement of $N$ stages is implemented together. The planned capacity of the $i$-th ($i = 1, 2, \cdots, N$) item is denoted by $m_i$. We assume that the $i$-th stage project is implemented at time $t$ and that the construction period is $\nu$; therefore, the operation period is $[t + \nu, T_c]$. Here, $T_c$ is the concession period, as mentioned above. Because the load of the $i$-th stage is allocated as the remaining demand after the full-load operation of the previous projects, the operation load of the $i$-th stage at time $s$ ($s \in [t + \nu, T_c]$) is the difference between the demand $X_s$ and the total loads of the previous projects under the limit of the capacity $m_i$, that is, $\min \left\{ (X_s - \sum_{k=1}^{i-1} m_k)^+, m_i \right\}$, here being $(\cdot)^+ = \max\{\cdot, 0\}$. Set the output price is $p$ and the operation cost per unit output is $c$, then the cash flow of the $i$-th stage at time $s$ denoted by $f_i(s, X_s)$ is the following:

$$f_i(s, X_s) = (p - c) \cdot \min \left\{ (X_s - \sum_{k=1}^{i-1} m_k)^+, m_i \right\}$$  \hspace{1cm} (2)

Therefore, the expected return of the $i$-th project invested at time $t$, $u_i(t, x)$, is the following:

$$u_i(t, x) = E_t^x \left[ \int_{t+\nu}^{T_c} f_i(s, X_s) e^{-\rho(s-t)} ds \right]$$  \hspace{1cm} (3)

Here, $\rho$ is the discount rate, and $E_t^x[\cdot]$ is the conditional expectation given the demand $x$ at time $t$, that is, $E_t^x[\cdot] = E[\cdot|X_t = x]$.

The construction cost is related to the capacity of the project, and following the result of [5], it is assumed that the construction cost being matched with the project capacity of the $i$-th stage, $I(m_i)(i = 1, 2, \cdots, N)$, is represented by the following:

$$I(m_i) = bm_i^\gamma$$  \hspace{1cm} (4)

Here, $b$ and $\gamma$ are constants, and $b > 0, 0 < \gamma \leq 1$, and $\gamma < 1$ represents the economy of scale.

Besides the cost being matched with the project capacity, the construction cost of the first stage also includes the dedicated asset that can be attributable to future expansions, which is represented by $I_d$. Thus, the construction cost of the first stage, $K_1$, equals the sum of $I(m_1)$ and $I_d$, that is, $K_1 = I(m_1) + I_d$. Accordingly, the construction cost of a later expansion is less than $I(m_i)(i = 2, 3, \cdots, N)$. Suppose the cost for the dedicated asset can be broken down, estimated, and allocated proportionally to the capacity for the subsequent stages of investment; then, the construction cost for a later expansion denoted by $K_i(i = 2, 3, \cdots, N)$ is calculated
by $K_i = I(m_i) - \frac{m_i}{\sum_{k=2}^{m_i} m_k} I_d$. To sum up, the construction cost of each stage, $K_i (i = 1, 2, \ldots, N)$, is the following:

$$K_i = \begin{cases} I(m_i) + I_d, & i = 1 \\ I(m_i) - \frac{m_i}{\sum_{k=2}^{m_i} m_k} I_d, & i = 2, 3, \ldots, N \end{cases} \quad (5)$$

From above, the expected net revenue of the $i$-th ($i = 1, 2, \ldots, N$) stage project invested at time $t$, $\pi_i(t, x)$, is yielded by the following:

$$\pi_i(t, x) = u_i(t, x) - K_i \quad (6)$$

According to the Assumption 4, the first-stage project is built at the initial time $t = 0$, the expected net revenue is $\pi_1(0, x)$.

The aim is to find the optimal expansion times that could maximize the project value, that is, the total revenues of multi-stage investments. It is an optimal multiple stopping time problem when attempting to optimize the investment opportunities for multi-state projects [4]. The project value and optimal expansion timing of the $N$ stage investments can be expressed as the following optimal multiple stopping time problem (7):

$$V = \pi_1(0, x) + \sup_{\vec{\tau} \in \mathcal{S}^{(N-1)}} E \left[ \sum_{i=2}^{N} e^{-\rho \tau_i} \pi_i(\tau_i, X_{\tau_i}) \right] \quad (7)$$

where $\vec{\tau} = (\tau_2, \tau_3, \ldots, \tau_N)$ represents a stopping time vector, $\tau_i (i = 2, 3, \ldots, N)$ is respectively the stopping time of the $i$-th stage project, which denotes the investment opportunity for the corresponding project, and $\mathcal{S}^{(N-1)}$ is the set of admissible stopping time vectors. The investments for two stages are at least separated by a certain time, also known as refracting period denoted by $\delta$. The first stage of the project is constructed at the initial time, so the investment opportunity of the second stage (the first expansion) is after $\delta$. Moreover, the concessionaire has the right to abandon the expansion, it might be desirable not to exercise all the rights of an expansion with maturity $T$, so the stopping times of a vector $\vec{\tau}$ might not all have their values in the interval $[\delta, T]$. Therefore, the set of admissible stopping time vectors with length $N - 1$ and refracting period $\delta > 0$, $\mathcal{S}^{(N-1)}$, is defined by the following:

$$\mathcal{S}^{(N-1)} = \{ \vec{\tau} \in (\tau_2, \tau_3, \ldots, \tau_N) \mid \tau_2 \geq \delta, \tau_{i+1} - \tau_i \geq \delta (i = 2, 3, \ldots, N - 1) \} \quad (8)$$

3.2. Expansion timing decisions with government subsidies. The purpose of a subsidy is to make the concessionaire execute investment at an earlier time; this can be achieved by reducing the investment cost or by increasing the value of the underlying cash flows. In the current study, we investigate the investment and revenue subsidy. An investment subsidy reduces the investment cost, and a revenue subsidy increases the value of the underlying cash flows.

3.2.1. Investment subsidy. We assume that the investment subsidy is given as a lump sum when the investment is implemented, and the subsidy amount of the $i$-th stage, $S_i^c (i = 1, 2, \ldots, N)$, is a proportion of the corresponding construction cost, as follows:

$$S_i^c = \theta K_i \quad (9)$$

where $\theta$ is the subsidy proportion that is agreed upon in advance of the contract, and $K_i$ is the construction cost of the $i$-th stage project as expressed by (5). Because the concessionaire can give up the expansion under insufficient demand, the expansion
subsidy, \( S_i^c(i = 2, 3, \ldots, N) \), is essentially depends on whether the expansion is exercised or not; therefore, the total investment subsidy amount of the \( N \)-stage project, \( S^c \), also depends on the expansion decisions.

With the subsidy, the construction cost of the \( i \)-th stage of the concessionaire is reduced from \( K_i \) to \( K_i - S_i^c \), so the expected net revenue with the investment subsidy of the \( i \)-th stage project invested at time \( t \), \( \pi^c_i(t,x)(i = 1, 2, \ldots, N) \), is the following:

\[
\pi^c_i(t,x) = u_i(t,x) - K_i + S_i^c \quad (10)
\]

Similar to the optimal multiple stopping time problem (7) without the subsidy, the project value \( V^c \) with the investment subsidy is represented by the following expression (11):

\[
V^c = \pi^c_1(0,x) + \sup_{\tau \in \mathcal{S}^{(N-1)}} E \left[ \sum_{i=2}^{N} e^{-\rho\tau_i} \pi^c_i(\tau_i,X_{\tau_i}) \right] \quad (11)
\]

Accordingly, the subsidy makes the investment opportunity more valuable by an incremental value equal to \( \Delta V^c \), as follows:

\[
\Delta V^c = V^c - V \quad (12)
\]

The difference between the subsidy amount \( S^c \) and the incremental value \( \Delta V^c \) is the value of the option to defer, denoted by \( A^c \):

\[
A^c = S^c - \Delta V^c \quad (13)
\]

### 3.2.2. Revenue subsidy

The revenue subsidy is that the concessionaire is given a price subsidy per unit product. It reduces the uncertainty of earnings by increasing the cash flow during the operating period. A main differences between a revenue subsidy and investment subsidy is when they occur. Here, an investment subsidy is assumed to be paid immediately, and the revenue subsidy is assumed to be paid over time in the future.

Suppose the subsidy per unit product is \( p^* \), increasing the net revenue per unit from \( p - c \) to \( p + p^* - c \), then the cash flow of the \( i \)-th stage project at time \( s(s \in [t + \nu, T_c]) \) denoted by \( f^R_i(s,X_s) \), is represented by the following:

\[
f^R_i(s,X_s) = (p + p^* - c) \cdot \min \left\{ (X_s - \sum_{k=1}^{i-1} m_k) + , m_i \right\}
\]

Consequently, the expected revenue with the subsidy, \( u^R_i(t,x) \), and the revenue subsidy amount, \( S^R_i(t,x) \), of the \( i \)-th stage project invested at time \( t \), are respectively, yielded by the following:

\[
u^R_i(t,x) = E^{t,x} \left[ \int_{t+\nu}^{T_c} (p + p^* - c) \cdot \min\{ (X_s - \sum_{k=1}^{i-1} m_k) + , m_i \} e^{-\rho(s-t)} ds \right]
\]

\[
S^R_i(t,x) = E^{t,x} \left[ \int_{t+\nu}^{T_c} p^* \cdot \min\{ (X_s - \sum_{k=1}^{i-1} m_k) + , m_i \} e^{-\rho(s-t)} ds \right]
\]

It is obvious that the relation between \( u^R_i(t,x) \) and \( S^R_i(t,x) \) is the following:

\[
S^R_i(t,x) = \frac{p^*}{p + p^* - c} u^R_i(t,x)
\]
Therefore, the expected net revenue with the revenue subsidy of the $i$-th stage project invested at time $t$, $\pi_{Ri}(t, x)$, is the following:

$$\pi_{Ri}(t, x) = u_{Ri}(t, x) - K_i$$

(14)

Like the investment subsidy, the revenue subsidies for the expansions $S_{Ri}(t, x)(i = 2, 3, \ldots, N)$ are also liability contingent on whether or not to exercise the expansions will be exercised, and the total revenue subsidy amount of the $N$-stage projects, $S_{R}$, depends on the expansion decisions. The project value $V^R$ with the revenue subsidy is as shown in the following expression (15):

$$V^R = \pi_{R1}(0, x) + \sup_{\tau \in S^{(N-1)}} E \left[ \sum_{i=2}^{N} e^{-\rho \tau_i} \pi_{Ri}(\tau_i, X_{\tau_i}) \right]$$

(15)

Also, the subsidy makes the investment opportunity more valuable by an amount equal to $\Delta V^R$, as follows:

$$\Delta V^R = V^R - V$$

(16)

The value of the option to defer, $A^R$, is the following:

$$A^R = S^R - \Delta V^R$$

(17)

4. Theoretical analysis and numerical computation. We illustrate the computation of the optimal multiple stopping time problem without a government subsidy (7), and a similar procedure can be applied to solve the problems with an investment or revenue subsidy. First, the theoretical analysis of investment revenue $u_i(t, x)(i = 1, 2, \ldots, N)$ represented by (3) is given, obtaining that $u_i(t, x)$ satisfies a differential equation. Subsequently, the finite difference method is used to solve the differential equation of $u_i(t, x)(i = 1, 2, \ldots, N)$, and then the numerical solution of $\pi_i(t, x)(i = 1, 2, \ldots, N)$ in equation (6) can be obtained. Following this, the LSM is introduced to solve the optimal multiple stopping time problem (7).

4.1. Theoretical analysis and numerical computation of investment revenue.

4.1.1. Theoretical analysis. From equation (3), we can find that $u_i(t, x)$, the expected return of the $i$-th project invested at time $t$, is path dependent because its value is related to the demand after time $t$. From the principle of dynamic programming and Itô formula, it can be concluded that $u_i(t, x)$ satisfies the following differential equation demonstrated in Theorem 4.1.

**Theorem 4.1.** The expected return of the $i$-th project invested at time $t$, $u_i(t, x)(i = 1, 2, \ldots, N)$, satisfies the following differential equation, $\forall (t, x) \in (0, T_c - \nu) \times (0, +\infty)$, as follows:

$$\frac{\partial u_i}{\partial t}(t, x) + \alpha x \frac{\partial u_i}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u_i}{\partial x^2}(t, x) - \rho u_i(t, x) + F_i(t, x) = 0$$

(18)

The boundary conditions are the following:

$$u_i(t, 0) = 0, \quad u_i(t, X_{\text{max}}) = \frac{(p - c) m_i}{\rho} (e^{-\rho \nu} - e^{-\rho(T_c - t)})$$

(19)

Here, $X_{\text{max}}$ is set to be large enough to make the operator $\min \{ \cdot \}$ of (2) equal to $m_i$.

The final condition is the following:

$$u_i(T_c - \nu, x) = 0$$

(20)
where
\[ F_i(t, x) = (p - c) \cdot \left\{ x e^{-(\rho - \alpha)u} \left[ \Phi \left( g(S_k^{-1}m) \right) - \Phi \left( g(S_k^{-1}m) \right) \right] - e^{-\rho u} \left[ (S_k^{-1}m) \cdot \Phi \left( h(S_k^{-1}m) \right) - (S_k^{-1}m) \cdot \Phi \left( h(S_k^{-1}m) \right) \right] \right\} \]

\( \Phi \) is the standard normal distribution function, and
\[ g(z) = \frac{\ln(x/z)}{\sigma \sqrt{\nu}} + \frac{\alpha \sqrt{\nu}}{\sigma} + \frac{1}{2} \sigma \sqrt{\nu}, \quad h(z) = \frac{\ln(x/z)}{\sigma \sqrt{\nu}} - \frac{1}{2} \sigma \sqrt{\nu}. \]

**Proof.** See the Appendix A.

4.1.2. Finite difference method for investment revenue. We first discretize this PDE (18) in the spatial direction. Let the interval \([0, X_{\text{max}}]\) be divided uniformly into \(N + 1\) sub-intervals with \(0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = X_{\text{max}}\) \((X_{\text{max}}\) is large enough) and interval length \(\Delta x\). The finite difference scheme is proposed to solve the PDE (18), for \(\forall n = 1, 2, \ldots, N\), the discrete scheme is:
\[ -\frac{\partial u^h_i}{\partial t}(t, x_n) + d_{n,n-1}u^h_i(t, x_{n-1}) + d_{n,n}u^h_i(t, x_n) + d_{n,n+1}u^h_i(t, x_{n+1}) = F_i(t, x_n) \]  

(21)

where
\[ d_{n,n-1} = -1 \frac{\sigma^2 x^2}{2 (\Delta x)^2} \]  

(22a)
\[ d_{n,n} = \rho + \frac{\alpha x}{\Delta x} + \frac{\sigma^2 x^2}{(\Delta x)^2} \]  

(22b)
\[ d_{n,n+1} = -\frac{\alpha x}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x^2}{(\Delta x)^2} \]  

(22c)

These form an \(N \times N\) linear system for \(u_i(t) = (u^h_i(t, x_1), u^h_i(t, x_2), \ldots, u^h_i(t, x_N))^T\) and \(F_i(t) = (F_i(t, x_1) - d_{1,0}u^h_i(t, x_0), F_i(t, x_2), \ldots, F_i(t, x_{N-1}), F_i(t, x_N) - d_{1,N+1}, u^h_i(t, x_{N+1}))^T\) with \(u^h_i(t, x_0)\) and \(u^h_i(t, x_{N+1})\) in both (21) and \(F_i(t)\) being equal to the given boundary conditions in (19). Let \(D_n (n = 1, 2, \ldots, N)\) be \(1 \times N\) row vectors defined by:
\[ D_1 = (d_{1,1}, d_{1,2}, 0, \ldots, 0), \]
\[ D_n = (0, \ldots, 0, d_{n,n-1}, d_{n,n}, d_{n,n+1}, 0, \ldots, 0), n = 2, 3, \ldots, N - 1, \]
\[ D_N = (0, \ldots, 0, d_{N,N-1}, d_{N,N}), \]

where \(d_{n,n-1}, d_{n,n}, d_{n,n+1} (n = 1, 2, \ldots, N)\) are defined by (22a)-(22c) and those entries which are not defined are zero. Using \(D_n(n = 1, 2, \ldots, N)\), the discrete scheme (21) can be rewritten as:
\[ -\frac{\partial u^h_i}{\partial t}(t, x_n) + D_n u_i(t) = F_i(t) \]  

(23)

This is a first-order linear ordinary differential equation (ODE).

Furthermore, we discretize the ODE (23) in time direction. To this end, \([0, T_{c-\nu}]\) is divided into a set of partition points satisfying \(T_{c-\nu} = t_0 < t_1 > \cdots > t_K = 0\). Applying the two-level implicit time-stepping method with a splitting parameter \(\theta \in [1/2, 1]\) to obtain
\[ \frac{u^h_i(t_{k+1}, x_n) - u^h_i(t_k, x_n)}{\Delta t} + \theta D_n u^{k+1}_i + (1 - \theta) D_n u^k_i = \theta F^{k+1}_i + (1 - \theta) F^k_i \]  

(24)
for \( k = 0, 1, \ldots, K-1 \), where \( \Delta t = -t_{k+1} - t_k > 0 \), \( u_k^i = (u_k^i(t_k, x_1), u_k^i(t_k, x_2), \cdots, u_k^i(t_k, x_N))^T \) is the approximation vector of \( u_i(t) \) at \( t = t_k \) and \( F_k^i = F_i(t_k) \). Set
\[
D = (D_1, D_2, \ldots, D_N)^T
\]
and \( E \) is a \( N \times N \) identity matrix. Then, (24) can be expressed as
\[
(E + \theta \cdot \Delta t \cdot D)u_{i+1}^k = (E - (1 - \theta) \cdot \Delta t \cdot D)u_k^i + \theta F_{i+1}^k + (1 - \theta)F_i^k
\]
(25)
Through analyzing the elements of matrix \( D \) and it is found that the matrix \( E + \theta \cdot \Delta t \cdot D \) is a \( M \) matrix. That means that equation system (24) satisfy the principle of discrete maximum value, and the discrete scheme (24) is monotonous.

4.2. Least squares Monte Carlo simulation. Since the least squares Monte Carlo method (LSM) is used to evaluate the stopping time problem of U.S. options [20], it has been extended to optimal multiple stopping time problems [7, 22, 23]. A difference between these references and our problem is that there is a refraction period of two consecutive investments is set to \( \gamma \). It has been extended to optimal multiple stopping time problems [7, 22, 23].

4.2.1. Symbolic descriptions. Partition investment interval \( [0, T] \): \( 0 = s_0 < \delta = s_1 < s_2 < \cdots < s_J = T \), and \( [\delta, T] \) is partitioned uniformly with the decision step \( \Delta = \frac{T-\delta}{J} \). Here, \( \{s_1, s_2, \ldots, s_J\} \) is the set of the discretized expansion opportunities. Because the refraction period of two consecutive investments is \( \delta \), if an expansion time is \( s_j \), the next expansion opportunity will be after \( s_j + \delta \). Let \( \gamma = \lceil \frac{s}{\Delta} \rceil \), and \( \lceil \cdot \rceil \) is the ceiling function, then \( \gamma \Delta \geq \delta \). Therefore, the refraction period of two investments is set to \( \gamma \Delta \) for the numerical calculations.

The variables used in the numerical calculations are described as follows, where \( \{\omega_1, \omega_2, \cdots, \omega_L\} \) represents \( L \) sample paths generated by the Monte Carlo simulation:
\[
\bar{X}_j = (X_j(\omega_1), X_j(\omega_2), \cdots, X_j(\omega_L))^T \): is the demand vector of all sample paths at time \( s_j \), and the component is the demand for each sample path respectively;
\[
\bar{V}^r_j = (V^r_j(\omega_1), V^r_j(\omega_2), \cdots, V^r_j(\omega_L))^T \): is the project value vector of all sample paths for \( r \) remaining expansion rights at time \( s_j \), and the component is the project value for each sample path respectively;
\[
\bar{C}^r_j(\omega_l) \): is the continuation value of the sample path \( \omega_l \) for \( r \) remaining expansion rights when the investment is not performed at time \( s_j \);
\[
\bar{D}^r_j(\omega_l) \): is the continuation value of the sample path \( \omega_l \) for \( r \) remaining expansion rights after executing investment at time \( s_j \);
\[
\bar{E}^r_j(\omega_l) \): is the net revenue of the sample path \( \omega_l \) for \( r \) remaining expansion rights at time \( s_j \).

4.2.2. Calculation steps. 1. Through the Monte Carlo simulation, generate \( L \) sample paths for the stochastic demand given by the stochastic differential equation (1), and store those samples of \( L \) sample paths at each time \( s_j(j = 1, 2, \cdots, J) \) as a \( L \times J \) dimension matrix \( Q \).

2. For each sample path, go back in time until the first timestep has been reached. At each timestep \( s_j(j = J, J-1, \cdots, 1) \), calculate the execution value and the continuation value for \( r(r = 1, 2, \cdots, N-1) \) remaining expansion rights and then obtain the optimal exercise policy of the expansion and the project value. The detailed processes are as follows:
(i) The exercise strategy and project value at time $s_j$.

Because $s_j$ is the last investment opportunity, the continuation values at $s_j$ are zero, that is, $C_j^r(\omega_l) = D_j^r(\omega_l) = 0$ ($r = 1, 2, \cdots, N-1; t = 1, 2, \cdots, L$). Because there are $N-r+1$ expansion rights, the investment for $r$ remaining expansion rights is the $N-r+1$-th investment (the first investment is for the first-stage greenfield project at initial time $t = 0$). The investment net revenue $E_j^r(\omega_l)$ can be obtained from the numerical solution derived from the finite differential method of $\pi_{N-r+1}(t, x)$ in equation (6): $E_j^r(\omega_l) = \pi_{N-r+1}(s_j, X_j(\omega_l))$. If $E_j^r(\omega_l) > 0$, the investment for the expansion is performed at time $s_j$. The project value at time $s_j$ is that of $V_j^r(\omega_l) = \max\{0, E_j^r(\omega_l)\}$.

(ii) Move back through time, calculate the continuation value and execution value at time $s_j$ ($j = J-1, J-2, \cdots, 1$), and then obtain the exercise strategy and project value.

Because of the refraction time of $\gamma \Delta$ between two consecutive investments, there are two situations for the next investment opportunity that depend on whether or not to implement the investment at time $s_j$. If the investment is not executed at time $s_j$, the next investment opportunity is $s_{j+1}$; otherwise, it is time $s_{j+\gamma}$. Accordingly, the continuation value is divided into two types. In one case that the investment is not performed at time $s_j$, the continuation value refers to the conditional expectation on the information $F_j^{(l)}$ available at time $s_j$ of the project value at time $s_{j+1}$, discounted to the time $s_j$, that is, $C_j^r(\omega_l) = E[e^{-\rho \Delta} V_{j+1}^r | F_j^{(l)}]$. In the other case that investment is exerted at time $s_j$, the continuation value is the conditional expectation on the information $F_j^{(l)}$ of the project value at time $s_{j+\gamma}$, discounted to the time $s_j$, that is, $D_j^r(\omega_l) = E[e^{-\rho \gamma \Delta} V_{j+\gamma}^r | F_j^{(l)}]$.

The difficulty in calculating the continuation values, $C_j^r(\omega_l)$ and $D_j^r(\omega_l)$, lies in estimating the conditional expectations. In the least squares Monte Carlo method, a polynomial of the current demand $X_j(\omega_l)$ is obtained through a least squares regression and is then used to approximate the conditional expectation. The two continuation values are expressed mathematically as the following:

$$C_j^r(\omega_l) = E[e^{-\rho \Delta} V_{j+1}^r | F_j] \approx \sum_k a_{j,k}^r L_k(X_j(\omega_l))$$

$$D_j^r(\omega_l) = E[e^{-\rho \gamma \Delta} V_{j+\gamma}^r | F_j] \approx \sum_k \bar{a}_{j,k}^r L_k(X_j(\omega_l))$$

where the basis functions $L_k(\cdot)$ are represented by polynomials of the underlying risks, which are standard in the LSM literature [20, 3]. The coefficients $a_{j,k}^r, \bar{a}_{j,k}^r$ are obtained by regressing the project value $e^{-\rho \Delta} \bar{V}_{j+1}^r (e^{-\rho \gamma \Delta} \bar{V}_{j+\gamma}^r)$, onto basis \{${L_k(\bar{X}_j)}$\}. Literature [20] proves that a limited number of base functions can approximate conditional expectations very well.

The execution value in multi-exercise options is the sum of the net revenue of the investment and the continuation value after executing the investment with one expansion right less than the original one, that is, $E_j^r(\omega_l) + D_j^{r-1}(\omega_l)$. Now, the exercise strategy is made by comparing $E_j^r(\omega_l) + D_j^{r-1}(\omega_l)$ with $C_j^r(\omega_l)$. If the execution value exceeds the continuation value, that is, $E_j^r(\omega_l) + D_j^{r-1}(\omega_l) > C_j^r(\omega_l)$, the investment for expansion is performed; otherwise, the investment opportunity will be continued in the future. The project value is the maximum value of the

OPTIMAL EXPANSION TIMING DECISIONS IN MULTI-STAGE PPP PROJECTS 2075
continuation and execution values, as follows:

\[ V_j^\pi(\omega_l) = \max\{E_j^\pi(\omega_l) + D_j^{r-1}(\omega_l), C_j^r(\omega_l)\} \]  

The optimal stopping time of the \( i \)-th stage project associated to the simulated path, \( \tau^\pi_{i} (i = 2, 3, \ldots, N), \) is determined by

\[ \{\tau^\pi_{i}\}_{i=2,3,\ldots,N} = \inf \{\{s_j\}_{j=1,2,\ldots,J} : E_j^{N-i-1}(\omega_l) + D_j^{N-i}(\omega_l) > C_j^{N-i+1}(\omega_l)\} \]

3. Obtain the multi-stage project value and optimal expansion strategies.

Once the optimal stopping time set is determined, the estimate of the \( N - 1 \)-stage expansion project, denoted by \( V_E \), is given by averaging the sum of the net revenue relative to the optimal stopping time along each sample path, discounted to time 0:

\[ V_E = \frac{1}{L} \sum_{l=1}^{L} \sum_{i=2}^{N} e^{-\rho \tau^\pi_{i}} \pi_i(\tau^\pi_{i}, X_{\tau_i}(\omega_l)) \]

The project value of the \( N \)-stage project, \( V \), is equal to the sum of the project value of the first stage implemented at the initial time, \( \pi_1(0, x) \), and the value of the \( N - 1 \)-stage expansion project \( V_E \), that is,

\[ V = \pi_1(0, x) + \frac{1}{L} \sum_{l=1}^{L} \sum_{i=2}^{N} e^{-\rho \tau^\pi_{i}} \pi_i(\tau^\pi_{i}, X_{\tau_i}(\omega_l)) \]  

Because there are more than one expansion rights, the optimal strategy cannot be described by a single threshold, such as in the classic U.S. option. To illustrate the optimal exercise strategy, a strategy matrix \( \Lambda \) is introduced. For \( J \) expansion opportunities and \( N - 1 \) expansion rights, the matrix \( \Lambda \) has \( N - 1 \) rows and \( J \) columns, and the matrix element \( \lambda_{ij} \) \( (i = 1, 2, \ldots, N-1; j = 1, 2, \ldots, J) \) denotes the threshold for the \( i \)-th expansion at the \( j \)-th opportunity. At \( s_j (j = 1, 2, \ldots, J) \), the concessionaire exercises the \( i \)-th \( (i = 1, 2, \ldots, N - 1) \) expansion when the demand is higher than \( \lambda_{ij} \). The threshold values \( \lambda_{ij} \) can be extracted by just searching for the smallest samples corresponding to the optimal stopping time of \( i \)-th expansion at time \( s_j \). In this way, it can be verified that the empirical errors of the optimal thresholds are relatively small if the number of paths is sufficient [7].

5. Application of the proposed model: A numerical example. This section applies the proposed model to a hypothetical three-stage investment plan for a sanitary sewerage project. This application serves the following purposes: to see if a flexible expansion time can create project value, to present optimal expansion strategies, and to assess the impact of dedicated assets and government subsidies on the project value and optimal expansion strategies.

5.1. Project profile. Assume that a PPP sanitary sewerage project is planned and will be divided into three stages over 10 years. The investor will have exclusive rights to build and operate the sanitary sewerage systems in the designated service areas for 30 years. Suppose the capacity for each stage is 40,000 m³ per day. The parameters of the cost matching the capacity in (4) is set to be \( b = 2917.8 \) and \( \gamma = 0.9427 \), which are estimated from the construction data of a sanitary sewerage project in China, and using these two parameters, the unit of the cost \( I(m_i) \) is 10,000 CNY. The construction of the project involves three types of work: sewer systems, house or service connections, and water treatment and disposal plants. To allow for future expansion, the sewer systems, as well as treatment and disposal plants, require an upfront investment in the first stage for reserve capacity and
sewer transitions. The dedicated asset investment amount \( I_d \) for expansion in the first stage is assumed to be the following:

\[
I_d = \eta (I(m_2) + I(m_3)) \quad (\eta \in (0, 1))
\]

(28)

where \( \eta \) is the dedicated asset ratio, and \( I(m_2) \) and \( I(m_3) \) are, respectively, the cost matching the corresponding capacity of the second and third stage.

In summary, all parameters involved in the model are listed by the mathematical symbols in Table 1. Unless noted otherwise, in our analysis, we assume the parameter values in Table 1.

Table 1. Default parameters used in the calculations

| Constant              | Symbol | Value | Unit    |
|-----------------------|--------|-------|---------|
| Concession Period     | \( T_c \) | 30    | Year    |
| Investment period     | \( T \) | 10    | Year    |
| Planned investment times | \( N \) | 3     | time    |
| Construction period   | \( \nu \) | 1     | Year    |
| Refraction time       | \( \delta \) | 2     | Year    |
| Capacity of \( i \)-th stage | \( m_i \) | 40,000 | \( m^3/day \) |
| Unit price            | \( p \) | 1.8   | CNY/\( m^3 \) |
| Unit operational cost | \( c \) | 0.8   | CNY/\( m^3 \) |
| Construction cost parameter | \( b \) | 2917.8 |
| Construction cost parameter | \( \gamma \) | 0.9427 |
| Drift                 | \( \alpha \) | 6%    |
| Volatility rate       | \( \sigma \) | 15%   |
| Discount rate         | \( \rho \) | 8%    |
| Dedicated asset ratio | \( \eta \) | 10%   |

5.2. The project value and optimal expansion strategies. Figure 2 depicts the multi-stage project value for different demand levels. As one would expect, the project value increases as the demand increases. The comparison of the multi-stage project value between with and without flexible expansion timing is also displayed in this figure. Here, the multi-stage investment with flexible timing is more valuable than without flexible timing, and the difference between the two values is the value of flexibility.

The optimal exercise boundaries, which are the thresholds between the exercise and the continuation of the expansion rights, are shown in Figure 3. Here, \( i \) refers to the \( i \)-th of two expansion rights. For example, the exercise boundary of \( i=1 \) is the threshold of the first expansion right, and the concessionaire should exercise the expansion when this value is exceeded by the actual demand. Because the first-stage greenfield project is built at the initial time and the refraction period between the two consecutive investments is two years, the second investment (i.e., the first expansion) opportunity is after the second year. Similarly, the second expansion implementation is only possible after the fourth year. Then, the expansion investment opportunity is within \([2, 10]\).

The results show that the exercise boundary increases as \( i \) grows. Indeed, a larger \( i \) means a later investment; because the pre-projects have already loaded a certain amount of demand, the later project will only be invested in at a higher level of demand. It is also found that the boundary of the first expansion fluctuates at time
8, with lower demand on the left side and a higher demand on the right side. The reason behind this is that time 8 is a critical point for investment, meaning that if the investment is not carried out at this time, there will be no chance for the second expansion at a later period because of the refraction time being two years. Consequently, around the eighth year, the investor will reduce his or her earning expectations and implement the investment at a lower level of demand. However, if the expansion is not implemented at time 8, afterwards, the second expansion will not be carried out at a later time. For this scenario, the optimal strategy is to maximize the benefit of the first expansion to make sure the exercise boundary will rise again. Unlike the first expansion boundary, the second boundary changes smoothly except near the expiration date. This is because the second expansion is the last one, and the concessionaire may be less concerned about the worthlessness of the expansion right except near the maturity.

5.3. **Influences of dedicated asset.** Figure 4 shows that the influences of a dedicated asset on the project value and expansion strategy. For the sake of simplicity, only the first expansion is discussed, but it should be noted that there are similar results for the second one. The amount of the dedicated asset is reflected by the dedicated asset ratio $\eta$ in (28). Here, $\eta$ is changed to 0\%, 10\%, and 20\%. At 0\%, there is no dedicated asset investment in the first stage. The results reveal a higher
ratio for the dedicated asset results in lower project value, as well as lower exercise boundaries. From expression (5), we know that the dedicated asset increases the cost of the first stage and decreases the cost of later expansions, and in the notional amount, the increased cost in the first stage equals the total decreased cost in later stages. Because of the time value of money, the discounted cost for the same notional amount is greater in the first stage than in later stages. The higher the ratio of the dedicated asset, the higher the investment cost in the first stage and the lower cost for later expansions will be; this leads to a higher total cost, consequently reducing the project value. The implication is clear: reducing the upfront investment amount with a dedicated asset can increase the project value. Regarding the exercise boundary, because the cost of a later expansion decreases as the dedicated asset ratio increases, the investment for the expansion will be triggered at a lower demand level.

5.4. Influences of government subsidy.

5.4.1. Influences of the investment subsidy. Since investment subsidy amount depends on the expansion decisions which are affected by the demand, the influences of the demand on the subsidy amount and the project value are investigated. Moreover, with the investment subsidy, the later expansion projects will be built earlier under lower demand levels. It is worth exploring how the subsidy impacts the expansion strategies and project value.

Figure 5 demonstrates how the demand impacts both the subsidy amount $S^c$ and the project value $V^c$ of (11), hence influencing the incremental value, denoted by $\Delta V^c$ of (12). In this case, the subsidy proportion of (9) is set to be $\theta = 20\%$. The project values with and without the subsidy with respect to the demand are shown in Figure 5(a). As expected, the subsidy improves the project value, which is reflected by a higher value with the subsidy than that without; the difference between them is the incremental value before and after subsidy, which is illustrated in Figure 5(b), which also depicts the subsidy amount. Here, a higher demand improves the subsidy and the incremental value. Moreover, the incremental value is less than the subsidy amount. The difference between the subsidy amount and the incremental value refers to the value of the option to defer $A^c$ of (13). If the PPP is granted in the competitive procurement, the incremental value will correspond to the value that the concessionaire is willing to additionally pay for the PPP.
Accordingly, the net benefit for the concessionaire is the value of the option to defer.

The influences of the subsidy proportion $\theta$ on the subsidy amount, as well as the project value and the incremental value, are shown in Figure 6(a): here, the demand is assumed to be 30,000 $m^3$ per day. The proportion changes from the 0% to 40%, where 0% corresponds to no subsidy. As expected, both the subsidy amount and the project value, as well as the incremental value, increase with the subsidy proportion. Furthermore, from the figure, the value of the option to defer, which is the difference between the subsidy amount and the incremental value, increases with the proportion of the subsidy given. Figure 6(b) shows the optimal exercise boundaries of the first expansion for different subsidy proportions. From the gap of the exercise boundaries with and without the subsidy, the expansion with the subsidy is triggered at a lower demand level than that without. Moreover, the higher the proportion is, the lower the demand level triggering the expansion will be.
5.4.2. Influences of the revenue subsidy. Similar with the investment subsidy, we will discuss how the demand impacts the revenue subsidy and consequently how the revenue subsidy influence the expansion strategies and project value.

Figure 7 demonstrates the project value $V_R$, the subsidy amount $S_R$ and the increment value $\Delta V_R$ under different demands. Here, the subsidy price is set to $p^* = 0.2$ CNY/m$^3$. From Figure 7(a), it is found that the project value with the revenue subsidy is more valuable than that without the subsidy. In addition, from the gap between the values with and without the subsidy, it can be concluded that the incremental value increases as the demand rises, and this is further demonstrated in Figure 7(b). Besides this, Figure 7(b) also shows that the subsidy amount is more than the incremental value, which is also correct for the investment subsidy, and the difference between the subsidy amount and the incremental value refers to the value of the option to defer, i.e., $A_R$ of (17). Similarly, if the PPP is granted in the competitive procurement, the incremental value corresponds to the value that the concessionaire is willing to additionally pay for the PPP. Accordingly, the net benefit for the concessionaire is the value of the option to defer.

The influences of the subsidy price on the subsidy amount, the project value and incremental value are shown in Figure 8(a); here, the demand is also assumed to be 30,000 m$^3$ per day. The subsidy price per unit $p^*$ changes from 0 to 0.5 CNY/m$^3$. As expected, the subsidy amount and project and incremental values increase with the price. The difference between the subsidy amount and incremental value, that is, the value of the option to defer, increases with the subsidy price. Moreover, the optimal exercise boundaries of the first expansion for the different subsidy prices are displayed in Figure 8(b). From the gap of the exercise boundaries between with and without the subsidy, the expansion with the subsidy is shown to be triggered at a lower demand level. Moreover, the higher the subsidy price is, the lower the demand level for triggering the expansion is.

5.4.3. Comparison of the subsidy arrangements. A subsidy incentive induces two types of cash flows: the subsidy given to the concessionaire and the additional value of the opportunity to invest. We compare the investment subsidy and revenue subsidy in terms of subsidy amount, incremental value, and expansion timing.
Figure 8. The influences of the revenue subsidy price.

Figure 9. The comparison of the subsidy amount.

Figure 9 depicts the comparison of the subsidy amount that depends on demand. From the figure, the revenue subsidy amount is more sensitive to the level of demand. This disparity could be attributed to the different subsidy mechanisms. The revenue subsidy is given as a subsidy price-per-unit output; consequently, the revenue subsidy amount is significantly affected by the demand across the entire operation period. The investment subsidy is presented as a proportion of the corresponding construction cost. Since the investment of the first stage must be performed no matter what the future demand is, the subsidy amount for the first stage is not related to the demand. The only correlation between the investment subsidy and demand level is that the demand affects later expansion strategies, that is, whether and when the expansions will be implemented. As a result, it impacts the later subsidy amount. However, because the construction cost of the first stage is the largest in the three stages because of the dedicated asset in the first stage, the subsidy amount of the first stage is larger than that of later expansions. Therefore, the construction subsidy amount is less sensitive to the level of demand.

To verify which subsidy pattern is more preferable, the comparisons are investigated from two angles. One is to compare the value performance, which is represented by the incremental value before and after the subsidy. The other is to...
compare the expansion timing performance of the early investment, as reflected by the exercise boundary.

Let proportion \( \theta \) or price \( p^* \) change, we obtain a serial of computational results in the incremental values, the subsidy amounts, and the expansion boundaries of two type subsidies, whereby the initial demand \( X_0 = 30,000 \, m^3/\text{day} \), model parameters \( \alpha = 6\% \), \( \sigma = 15\% \), \( \rho = 8\% \). From these, the value and timing performances are shown respectively in Figure 10 and Figure 11. From Figure 10, we can conclude that the revenue subsidy outperforms the investment subsidy in value performance because it offers more incremental value under the same subsidy amount. In addition, the gap is widened as the subsidy amount increases. As for the timing performance, it is found from Figure 11 that there is a lower boundary for the investment subsidy, which means that there is an earlier expansion for the investment subsidy. In fact, this result of timing performance is consistent with the result of value performance. A more earlier expansion under the investment subsidy means that there is a higher value of the option to defer when there is no subsidy; consequently, from the expression of (13) or (17), this leads to a lower incremental value for the investment subsidy, as illustrated in the value performance. The two comparisons show that if the government is keener on expanding within an earlier timeframe, the construction subsidy, or a lump-sum subsidy pattern, should be given. If the government is concerned about incremental value, the revenue subsidy, or a unit subsidy pattern, is more suitable.

6. Concluding remarks. In the current paper, multi-stage PPP projects that include a first-stage greenfield project that involves a dedicated asset for future expansions and later expansion projects were addressed. We argued that because of the uncertainty regarding the level of demand and irreversibility associated with infrastructure projects, flexibility for the timing of expansion projects should be incorporated. Having flexible expansion timing was modeled as an optimal multiple stopping model. Also, we considered how government subsidies, including an investment subsidy and revenue subsidy, affect the expansion strategies and project value. An investment subsidy is given as a percentage of the construction cost, while a revenue subsidy is given as a price subsidy of the per-unit product. Finally, a hypothetical three-stage PPP sewerage project was presented to validate the practical application of the proposed model. The current paper provided a better framework.
for determining when to exercise the expansions optimally and the extent of the government subsidy provided to take a project forward as soon as possible.

Our main findings are as follows: First, the optimal expansion strategies reflected by the exercise boundaries were provided, according to which it can be decided when to exert the expansions maximizing the project value throughout its multiple stages. Meanwhile, the project value when there is timing flexibility was also presented; having flexible expansion timing makes the project more valuable. Second, a dedicated asset in the first stage reduces the project value while making an expansion come earlier. Third, the incremental value and the earlier expansion derived from the two kinds of subsidies were illustrated and compared. The revenue subsidy is better than the investment subsidy in incremental value while it is inferior in the performance of the earlier expansion.

However, there are some limitations that deserve further research efforts. First, it is assumed that the capacity of each stage of the project is predetermined in the concession contract. In the future, a simultaneous analysis of timing and capacity decisions for multi-stage projects would be an interesting topic. Second, besides the demand, other uncertainty sources should be considered. For instance, the uncertainty of the construction costs is also important. Third, in the current study, the optimized object of the expansion decisions is to maximize the economic benefit, but further research can consider the trade-off between economic and social benefits.

A. Proof of Theorem 4.1.

Proof of Theorem 4.1. The differential equations of $u_i(t, x)$ ($i = 1, 2, \ldots, N$) can be derived by the principle of dynamic programming and Itô formula:

$$\frac{\partial u_i}{\partial t}(t, x) + \alpha x \frac{\partial u_i}{\partial x}(t, x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u_i}{\partial x^2}(t, x) - \rho u_i(t, x) + E_t^t [f_i(t + \nu, X_{t+\nu})e^{-\rho \nu}] = 0$$

Denote $F_i(t, x) = E_t^t [f_i(t + \nu, X_{t+\nu})e^{-\rho \nu}]$, then

$$F_i(t, x) = (p - c) \cdot E_t^t [\min(X_{t+\nu} - \sum_{k=1}^{i-1} m_k)^+, m_i]$$

(29)

Next, we will compute $E_t^t [\min(X_{t+\nu} - \sum_{k=1}^{i-1} m_k)^+, m_i]$. 
Denote \( \Phi(\cdot) \), \( \varphi(\cdot) \) respectively represent the standard norm distribution function and density function, and
\[
g(z) = \frac{\ln(x/z)}{\sigma \sqrt{\nu}} + \frac{\alpha \sqrt{\nu}}{\sigma} + \frac{1}{2} \sqrt{\nu},
\]
\[
h(z) = \frac{\ln(x/z)}{\sigma \sqrt{\nu}} + \frac{\alpha \sqrt{\nu}}{\sigma} - \frac{1}{2} \sqrt{\nu}.
\]

The distribution function \( F(z) \) of the random variable \( X_{t+\nu} = xe^{(\alpha - \frac{x^2}{2})\nu + \sigma B_\nu} \) is:
\[
F(z) = P[X_{t+\nu} \leq z] = P[\ln X_{t+\nu} \leq \ln z] = \Phi(-\nu z).
\]

and its density function \( f(z) \) is \( f(z) = \varphi(h(z)) \frac{1}{z \sigma \sqrt{\nu}} \).

According the definition of conditional expectation, it yields
\[
E^{t,x}\left[ \min\{X_{t+\nu} - \sum_{k=1}^{i-1} m_k, m_i\} \right] = \int_{-\infty}^{+\infty} \min\{X_{t+\nu} - \sum_{k=1}^{i-1} m_k, m_i\} f(z) dz
\]
\[
= \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} (z - \sum_{k=1}^{i-1} m_k) f(z) dz + \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} m_i f(z) dz
\]
\[
= \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz - \sum_{k=1}^{i-1} m_k \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz + m_i \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz
\]
\[
= \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz - \sum_{k=1}^{i-1} m_k \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz + m_i \int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz
\]

By using the relationship \( f(z) = \varphi(h(z)) \frac{1}{z \sigma \sqrt{\nu}} \), we can get the following:
\[
\int_{\sum_{k=1}^{i} m_k}^{+\infty} f(z) dz = xe^{\alpha \nu} \left[ \Phi(g(\sum_{k=1}^{i} m_k)) - \Phi(g(\sum_{k=1}^{i-1} m_k)) \right],
\]
\[
\int_{\sum_{k=1}^{i-1} m_k}^{+\infty} f(z) dz = \Phi(h(\sum_{k=1}^{i} m_k)) - \Phi(h(\sum_{k=1}^{i-1} m_k)),
\]
\[
\int_{\sum_{k=1}^{i} m_k}^{+\infty} f(z) dz = \Phi(h(\sum_{k=1}^{i} m_k)),
\]

Substituting (30) and (31) into (29) yields
\[
F_i(t, x) = (p - c) \left( xe^{-(p - \alpha)\nu} \left[ \Phi(g(\sum_{k=1}^{i} m_k)) - \Phi(g(\sum_{k=1}^{i-1} m_k)) \right] \right.
\]
\[
- e^{-\rho \nu} \left[ (\sum_{k=1}^{i-1} m_k) \cdot \Phi(h(\sum_{k=1}^{i-1} m_k)) - (\sum_{k=1}^{i} m_k) \cdot \Phi(h(\sum_{k=1}^{i} m_k)) \right] \right)
\]
then equation (18) is proved. Moreover, from the expression of \( u_i(t, x) \) in (3), the boundary conditions (19) and final condition(20) can be easily obtained.

REFERENCES
[1] M. J. R. Armada, P. J. Pereira and A. Rodrigues, Optimal subsidies and guarantees in public-private partnerships, *Eur. J. Financ.*, 18 (2012), 469–495.
[2] L. E. T. Brandao and E. Saravia, The option value of government guarantees in infrastructure projects, *Constr. Manag. Econ.*, 26 (2008), 1171–1180.
[3] G. Cortazar, M. Gravet and J. Urzua, The valuation of multidimensional American real options using the LSM simulation method, *Comput. Oper. Res.*, 35 (2008), 113–129.
[4] E. Dahlgren and T. Leung, An optimal multiple stopping approach to infrastructure investment decisions, *J. Econ. Dynam. Contr.*, 53 (2015), 251–267.
[5] T. Dangl, Investment and capacity choice under uncertain demand, *Eur. J. Oper. Res.*, 117 (1999), 415–428.
[6] P. Doan, K. Menyah, Impact of irreversibility and uncertainty on the timing of infrastructure projects, J. Constr. Eng. Manage., 139 (2013), 331–338.

[7] U. Dör, Valuation of Swing Options and Examination of Exercise Strategies by Monte Carlo Techniques, Master thesis, University of Oxford, 2003.

[8] E. Engel, R. Fischer and A. Galetovic, The basic public finance of public-private partnerships, J. Euro. Econ. Assoc., 11 (2013), 83–111.

[9] B. C. Esty, Modern Project Finance: A Casebook, Princeton, Wiley, 2003.

[10] B. Flyvbjerg, M. Holm and S. Buhl, How common and how large are cost overruns in transport infrastructure projects? Transp. Rev., 23 (2003), 71–88.

[11] C. C. Gkochari, Optimal investment timing in the dry bulk shipping sector, Transp. Res. Part E, 79 (2015), 102–109.

[12] H. B. Herath, C. Park, Multi-stage capital investment opportunities as compound real options, Eng. Econ., 47 (2002), 1–27.

[13] Y. Huang, Project and Policy Analysis of Build-Operate-Transfer Infrastructure Developments, Ph.D thesis, University of California at Berkeley, 1995.

[14] Y. L. Huang and S. P. Chou, Valuation of the minimum revenue guarantee and the option to abandon in BOT infrastructure projects, Constr. Manag. Econ., 24 (2006), 379–389.

[15] Y. Huang and C. Pi, Valuation of multi-stage BOT projects involving dedicated asset investments: a sequential compound option approach, Constr. Manag. Econ., 27 (2009), 653–666.

[16] B. Klein and K. B. Leffler, The role of market forces in assuring contractual performance, J. Polit. Econ., 89 (1981), 615–641.

[17] Y. Kwak, Y. Chih and C. I. William, Towards a comprehensive understanding of public-private partnerships for infrastructure development, California Manag. Rev., 51 (2009), 51–78.

[18] S. Li and H. B. Cai, Government incentive impacts on private investment behaviors under demand uncertainty, Transp. Res. Part E, 101 (2017), 115–129.

[19] W. Li and S. Wang, Pricing American options under proportional transaction costs using a penalty approach and a finite difference scheme, J. Ind. Manag. Optim., 9 (2013), 365–389.

[20] F. A. Longstaff and E. S. Schwartz, Valuing American options by simulation: A simple least squares approach, Rev. Financ. Stud., 14 (2001), 113–147.

[21] L. J. Maseda, Real Options Analysis of Flexibility in a Hospital Emergency Department Expansion Project: A Systems Approach, Master thesis, MIT, 2008.

[22] N. Meinshausen and B. M. Hambly, Monte Carlo methods for the valuation of multiple exercise options, Math. Financ., 14 (2004), 557–583.

[23] S. Nadarajah, F. Margot and N. Secomandi, Comparison of least squares Monte Carlo methods with applications to energy real options, Eur. J. Oper. Res., 256 (2017), 196–204.

[24] R. Neufville, Y. S. Lee and S. Scholtes, Flexibility in hospital infrastructure design, Working Paper, MIT, 2008.

[25] E. Pennings, Optimal pricing and quality choice when investment in quality is irreverible, J. Ind. Econ., 52 (2004), 569–589.

[26] M. Skamris and B. Flyvbjerg, Inaccuracy of traffic forecasts and cost estimates on large transport projects, Transp. Policy, 4 (1997), 141–146.

[27] S. Szymanski, The optimal timing of infrastructure investment, J. Transp. Econ. Policy, 25 (1991), 247–258.

[28] Z. Tan and H. Yang, Flexible build-operate-transfer contracts for road franchising under demand uncertainty, Transp. Res. Part B, 46 (2012), 1419–1439.

[29] O. E. Williamson, The Economic Institutions of Capitalism, The Free Press, New York, 1985.

[30] Y. Xenidis and D. Angelides, The financial risks in build-operate-transfer projects, Constr. Manag. Econ., 23 (2005), 431–441.

[31] X. Zhang, Financial viability analysis and capital structure optimization in privatised public infrastructure projects, J. Constr. Eng. Manag., 131 (2005), 656–668.

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