On a weighted variable spaces $L_{p(x), \omega}$ for $0 < p(x) < 1$
and weighted Hardy inequality

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ABSTRACT. In this paper a weighted variable exponent Lebesgue spaces $L_{p(x), \omega}$ for $0 < p(x) < 1$ is investigated. We show that this spaces is a quasi-Banach spaces. Note that embedding theorem between weight variable Lebesgue spaces is proved. In particular, we show that $L_{p(x), \omega}(\Omega)$ for $0 < p(x) < 1$ isn’t locally convex. Also, in this paper a some two-weight estimates for Hardy operator are proved.

Keywords and phrases: Variable Lebesgue space, weights, quasi-Banach space, topology, embedding, Hardy operator.

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1. Introduction.

It is well known that the variable exponent Lebesgue space $L_{p(x)}$ for $p(x) \geq 1$ appeared in the literature for the first time already in [13]. Further development of this theory was connected with the theory of modular function spaces. Somewhat later, a more explicit version of these spaces, namely modular function spaces, were investigated by many mathematicians (see [12]). The next step in the investigation of variable exponent spaces was given in [16] and in [8]. But the variable exponent Lebesgue space for $0 < p(x) < 1$ very less studied. Note that the space $L_{p(x)}$ for $0 < p(x) < 1$ isn’t modular function spaces. The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [14], [17],[18]). For detailed information about variable exponent Lebesgue space $L_{p(x)}$ for $p(x) \geq 1$ we refer to [7].

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ and $\Omega$ be a Lebesgue measurable subset in $\mathbb{R}^n$ and $|x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$. Suppose that $p$ is a Lebesgue measurable function on $\Omega$ such that $0 < \underline{p} \leq p(x) \leq \bar{p} < 1$, $\underline{p} = ess \inf_{x \in \Omega} p(x)$, $\bar{p} = ess \sup_{x \in \Omega} p(x)$, and $\omega$ is a weight function on $\Omega$, i.e. $\omega$ is non-negative, almost everywhere (a.e.) positive function on $\Omega$. The Lebesgue measure of a set $\Omega$ will be denoted by $|\Omega|$. It is well known that $|B(0, 1)| = \frac{\pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right)}$, where $B(0, 1) = \{x : x \in \mathbb{R}^n; |x| < 1\}$. Further, in this paper all sets and functions are supposed Lebesgue measurable.
2. Preliminaries

**Definition 1.** By \( L_{p(x), \omega}(\Omega) \) we denote the set of measurable functions \( f \) on \( \Omega \) such that

\[
I_{p, \omega}(f) = \int_{\Omega} \left( |f(x)| \omega(x) \right)^{p(x)} \, dx < \infty.
\]

Note that the expression

\[
\|f\|_{L_{p(x), \omega}(\Omega)} = \|f\|_{p, \omega, \Omega} = \inf \left\{ \lambda > 0 : \frac{\int_{\Omega} \left( |f(x)| \omega(x) \right)^{p(x)} \, dx}{\lambda^p \int_{\Omega} \left( |f(x)| \omega(x) \right)^{p(x)} \, dx} \leq 1 \right\}
\]

(2.1)

defines a quasi-Banach spaces.

We note some main properties of this spaces.

1) For every \( 0 < \|f\|_{p, \omega, \Omega} < \infty \), \( I_{p, \omega} \left( \frac{f}{\|f\|_{p, \omega, \Omega}} \right) = 1 \).

If \( I_{p, \omega} \left( \frac{f}{\|f\|_{p, \omega, \Omega}} \right) < 1 \), we can find \( 0 < \lambda \leq \|f\|_{p, \omega, \Omega} \) such that \( I_{p, \omega} \left( \frac{f}{\lambda} \right) < 1 \). Indeed, let \( \lambda = \|f\|_{p, \omega, \Omega} I_{p, \omega}^{1/p} \left( \frac{f}{\|f\|_{p, \omega, \Omega}} \right) \). Then \( \lambda < \|f\|_{p, \omega, \Omega} \) and the inequality

\[
I_{p, \omega} \left( \frac{f}{\lambda} \right) = \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p, \omega, \Omega} I_{p, \omega}^{1/p} \left( \frac{f}{\|f\|_{p, \omega, \Omega}} \right)} \right)^{p(x)} \, dx 
\]

\[
\leq I_{p, \omega}^{1} \left( \frac{f}{\|f\|_{p, \omega, \Omega}} \right) \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p, \omega, \Omega}} \right)^{p(x)} \, dx = 1
\]

is valid. The obtained inequality contradicts to (2.1).

**Remark 1.** Note that property 1) for non-weighted case was proved in [15].

2) \( \min \left\{ \|f\|^{L}_{p, \omega, \Omega}, \|f\|^{P}_{p, \omega, \Omega} \right\} \leq I_{p, \omega}(f) \leq \max \left\{ \|f\|^{L}_{p, \omega, \Omega}, \|f\|^{P}_{p, \omega, \Omega} \right\} \).

Let \( \|f\|_{p, \omega, \Omega} \leq 1 \). Using the property 1) we have

\[
I_{p, \omega}(f) = \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p, \omega, \Omega}} \right)^{p(x)} \, dx \leq \|f\|^{L}_{p, \omega, \Omega} \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p, \omega, \Omega}} \right)^{p(x)} \, dx = \|f\|^{L}_{p, \omega, \Omega}.
\]

Conversely, \( I_{p, \omega}(f) \geq \|f\|^{P}_{p, \omega, \Omega} \). Analogously, is consider the case \( \|f\|_{p, \omega, \Omega} \geq 1 \).

3) The space \( L_{p(x), \omega}(\Omega) \) is real linear spaces.
By using of the property 1, we have

\[
\int_{\Omega} \left( \frac{|f(x) + g(x)| \omega(x)}{2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})} \right)^{p(x)} \, dx
\]

\[
\leq \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})} \right)^{p(x)} \, dx + \int_{\Omega} \left( \frac{|g(x)| \omega(x)}{2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})} \right)^{p(x)} \, dx
\]

\[
\leq \frac{1}{2} \left( \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p,\omega,\Omega}} \right)^{p(x)} \, dx + \int_{\Omega} \left( \frac{|g(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}} \right)^{p(x)} \, dx \right) = 1.
\]

Thus by Definition 1 $\|f + g\|_{p,\omega,\Omega} \leq 2^{1/p} (\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega})$. Therefore $f + g \in L_{p(x),\omega}(\Omega)$. Let $\alpha \in R \setminus \{0\}$ and $f \in L_{p(x),\omega}(\Omega)$. Now show that $\alpha f \in L_{p(x),\omega}(\Omega)$. We get

\[
\|\alpha f\|_{p,\omega,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|\alpha f(x)| \omega(x)}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\lambda |\alpha|} \right)^{p(x)} \, dx \leq 1 \right\}
\]

We substitute $\lambda = |\alpha| \mu$. Then

\[
\inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\lambda |\alpha|} \right)^{p(x)} \, dx \leq 1 \right\}
\]

\[
= \inf \left\{ |\alpha| \mu > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\mu} \right)^{p(x)} \, dx \leq 1 \right\}
\]

\[
= |\alpha| \inf \left\{ \mu > 0 : \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\mu} \right)^{p(x)} \, dx \leq 1 \right\} = |\alpha| \|f\|_{p,\omega,\Omega}.
\]

For $f = 0$ this fact is trivially. Hence implies that the variable Lebesgue space $L_{p(x),\omega}(\Omega)$ is real linear space.

4) Let $\|f\|_{p,\omega,\Omega} = 0$. Then we proved that $f = 0$ a.e. $x \in \Omega$. 

3
If \( \|f\|_{p,\omega,\Omega} = 0 \), then by (2.1) for all \( \lambda > 0 \), \( I_{p,\omega}\left(\frac{f}{\lambda}\right) \leq 1 \). For any \( \mu > 0 \) and \( \varepsilon \in (0, 1) \), we have

\[
I_{p,\omega}\left(\frac{f}{\mu}\right) = \int_{\Omega} \varepsilon^{p(x)} \left(\frac{|f(x)| \omega(x)}{\varepsilon \mu}\right)^{p(x)} \, dx \leq \varepsilon^{p} I_{p,\omega}\left(\frac{f}{\varepsilon \mu}\right) \leq \varepsilon^{p}.
\]

Since \( \varepsilon \) be any number from \((0,1)\), then \( I_{p,\omega}\left(\frac{f}{\mu}\right) = 0 \) for all \( \mu > 0 \). Therefore

\[
\int_{\Omega} \left(\frac{|f(x)| \omega(x)}{\mu}\right)^{p(x)} \, dx = 0 \quad \text{and thus} \quad f = 0 \text{ a.e. } x \in \Omega.
\]

5) Let \( |f(x)| \leq |g(x)| \) for a.e. \( x \in \Omega \). Then \( \|f\|_{p,\omega,\Omega} \leq \|g\|_{p,\omega,\Omega} \).

Indeed, by virtue of property 1) we have

\[
\int_{\Omega} \left(\frac{|f(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}}\right)^{p(x)} \, dx = \int_{\Omega} \left(\frac{|f(x)| \omega(x)}{|g(x)| \omega(x)}\right)^{p(x)} \, dx \leq \int_{\Omega} \left(\frac{|g(x)| \omega(x)}{\|g\|_{p,\omega,\Omega}}\right)^{p(x)} \, dx = 1.
\]

Thus by Definition 1 \( \|f\|_{p,\omega,\Omega} \leq \|g\|_{p,\omega,\Omega} \).

**Lemma 1.** Let \( 0 < p \leq p(x) \leq \overline{p} < 1 \) and \( f, g \in L_{p(x),\omega}(\Omega) \). Then

\[
\|f\| + \|g\|_{p,\omega,\Omega} \geq \|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}.
\]

**Proof.** First we show that the function \( h(t) = t^{r} \), for \( 0 < r < 1 \) and \( t > 0 \) is concave.

Let \( \alpha + \beta = 1 \), where \( \alpha, \beta \geq 0 \). We proved that \( (\alpha + \beta t)^{r} \geq \alpha + \beta t^{r} \). We consider the function \( F(t) = \frac{(\alpha + \beta t)^{r}}{\alpha + \beta t^{r}} \). Differentiating by \( t \) and after some calculation we have

\[
F'(t) = \frac{\alpha \beta p (\alpha + \beta t)^{r-1} (1 - t^{r-1})}{(\alpha + \beta t^{r})^{2}}.
\]

Since \( r - 1 < 0 \), then \( t = 1 \) is minimal value of the function \( F \) for all \( t > 0 \). Therefore \( F(t) \geq F(1) = 1 \). Thus \( (\alpha + \beta t)^{r} \geq \alpha + \beta t^{r} \). Taking \( t = t_{2} \) in last inequality we have \( (\alpha t_{1} + \beta t_{2})^{r} \geq \alpha t_{1}^{r} + \beta t_{2}^{r} \), i.e. the function \( h(t) = t^{r} \) is concave.

Now we show a requiring inequality. It is obvious that the case \( f = g = 0 \) a.e. \( x \in \Omega \) is trivial. Let \( \|f\|_{p,\omega,\Omega} > 0 \) and \( \|g\|_{p,\omega,\Omega} > 0 \). Using concavity property of power function and property 1), we get

\[
I_{p,\omega}\left(\frac{|f| + |g|}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}}\right) = \int_{\Omega} \left(\frac{|f(x)| + |g(x)|}{\|f\|_{p,\omega,\Omega} + \|g\|_{p,\omega,\Omega}}\right) \omega(x)^{p(x)} \, dx =
\]

4
Theorem 1. Let $s$ where $a, b > 0$ then we can see $s - s' = \frac{s}{s - 1}$. Differentiating by $t$ we have

$$G'(t) = t^{s-1} - \frac{1}{t^{s'+1}} = \frac{t^{ss'} - 1}{t^{s'+1}};$$

where $s + s' = ss' < 0$. Therefore the point $t = 1$ is maximal value of the function $G(t)$ for all $t > 0$. Thus $G(t) \leq G(1) = 1$, i.e., $\frac{t^{s}}{s} + \frac{t^{-s'}}{s'} \leq 1$. If we take $t = \frac{a^{1/s'}}{b^{1/s}}$, then

$$ab \geq \frac{a^{s}}{s} + \frac{b^{s'}}{s'},$$

where $a, b > 0$. 

Differentiating by $t$ we have

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where $s + s' = ss' < 0$. Therefore the point $t = 1$ is maximal value of the function $G(t)$ for all $t > 0$. Thus $G(t) \leq G(1) = 1$, i.e., $\frac{t^{s}}{s} + \frac{t^{-s'}}{s'} \leq 1$. If we take $t = \frac{a^{1/s'}}{b^{1/s}}$, then

$$ab \geq \frac{a^{s}}{s} + \frac{b^{s'}}{s'},$$

where $a, b > 0$. 

Thus $\|f\|_{p,x,\Omega} \leq \|f\|_{p,x,\Omega} + \|g\|_{p,x,\Omega}$ holds for every $f \in L_{p(x),\omega}(\Omega)$.
Putting $a = \frac{|f(x)|\omega(x)}{\|f\|_{p, \omega, \Omega}}$, $b = \frac{|g(x)|\omega^{-1}(x)}{\|g\|_{p', \omega^{-1}, \Omega}}$, $s = s(x) = p(x)$, $s' = s'(x) = p'(x)$ in inequality (2.3) and using the property 1) we have

$$\int_{\Omega} \frac{|f(x)g(x)|}{\|f\|_{p, \omega, \Omega} \|g\|_{p', \omega^{-1}, \Omega}} \, dx \geq \int_{\Omega} \frac{1}{p(x)} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p, \omega, \Omega}} \right)^{p(x)} \, dx + \int_{\Omega} \frac{1}{p'(x)} \left( \frac{|g(x)| \omega^{-1}(x)}{\|g\|_{p', \omega^{-1}, \Omega}} \right)^{p'(x)} \, dx$$

$$\geq \frac{1}{p} \int_{\Omega} \left( \frac{|f(x)| \omega(x)}{\|f\|_{p, \omega, \Omega}} \right)^{p(x)} \, dx + \frac{1}{p'} \int_{\Omega} \left( \frac{|g(x)| \omega^{-1}(x)}{\|g\|_{p', \omega^{-1}, \Omega}} \right)^{p'(x)} \, dx = \frac{1}{p} + \frac{1}{p'}.$$

Thus the inequality (2.2) is proved.

**Remark 2.** Note that in the proof of Lemma 1, the expression $\|g\|_{p', \omega^{-1}, \Omega}$ was used for negative values of the conjugate function. It should be understood as follows

$$\|g\|_{p', \omega^{-1}, \Omega} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|g(x)| \omega^{-1}(x)}{\lambda^{-1}} \right)^{-p'(x)} \, dx \leq 1 \right\}$$

$$= \inf \left\{ \frac{1}{\mu} > 0 : \int_{\Omega} \left( \frac{|g(x)| \omega^{-1}(x)}{\mu} \right)^{-p'(x)} \, dx \leq 1 \right\} =$$

$$= \sup \left\{ \mu > 0 : \int_{\Omega} \left( \frac{|g(x)| \omega^{-1}(x)}{\mu} \right)^{p'(x)} \, dx \leq 1 \right\}.$$

**Theorem 2.** Let $0 < p \leq p(x) \leq q(x) \leq \bar{q} < 1$ and $r(x) = \frac{p(x)q(x)}{q(x) - p(x)}$. Suppose that $\omega_1$ and $\omega_2$ are weights functions defined in $\Omega$ and satisfying the condition

$$\left\| \frac{\omega_1}{\omega_2} \right\|_{r, \Omega} < \infty.$$

Then the inequality

$$\|f\|_{p, \omega_1, \Omega} \leq \left( A + B + \|\lambda \omega_2\|_{L_{\infty}(\Omega)} \right)^{1/2} \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r}(\Omega)} \|f\|_{q, \omega_2, \Omega},$$

holds for every $f \in L_{q(x)}(\Omega)$, where $\Omega_1 = \{x \in \Omega : p(x) < q(x)\}$, $\Omega_2 = \{x \in \Omega : p(x) = q(x)\}$ and $A = \sup_{x \in \Omega_1} \frac{p(x)}{q(x)}$, $B = \sup_{x \in \Omega_1} \frac{q(x) - p(x)}{q(x)}$ and

$$\left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r}(\Omega_1)} = \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{r}(\Omega)} + \left\| \frac{\omega_1}{\omega_2} \right\|_{L_{\infty}(\Omega_2)}.$$
Proof. We have

$$\|f\|_{p, \omega_1, \Omega_2} = \|f \, \omega_1 \omega_2 \|_{p, \omega_1, \Omega_2} \leq \|\omega_1 \|_{L_\infty(\Omega_2)} \|f \omega_2\|_{p, \Omega_2}$$

Thus

$$\frac{\omega_1}{\omega_2} \|f \chi_{\Omega_2} \|_{p, \omega_2, \Omega} \leq \frac{\omega_1}{\omega_2} \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} \|f\|_{p, \omega_2, \Omega}.$$ 

Therefore

$$\left\| \frac{f}{\|f\|_{p, \omega_2, \Omega}} \right\|_{L_\infty(\Omega_2)} \leq \|\chi_{\Omega_2}\|_{L_\infty(\Omega)} \leq 1.$$

By virtue of property 1)

$$\int_{\Omega_2} \left( \frac{|f(x)| \omega_1(x)}{\|f\|_{L_\infty(\Omega_2)} \|f\|_{p, \omega_2, \Omega}} \right)^p(x) dx \leq \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}^p = \|\chi_{\Omega_2}\|_{L_\infty(\Omega)}.$$ 

It is well known that the inequality (2.3) for $s > 1$ is Young’s inequality, i.e.

$$ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'},$$

where $s' = s - 1$. We take $s = s(x) = \frac{q(x)}{p(x)}$, $a = \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{p(x)}$ and $b = \left[ \frac{\omega_1(x)}{\omega_2(x)} \|f\|_{q, \omega_2, \Omega_1} \right]^{p(x)}$.

Thus $s'(x) = \frac{q(x)}{q(x) - p(x)}$ and from inequality (2.5), we have

$$(\frac{|f(x)| \omega_1(x)}{\|f\|_{q, \omega_2, \Omega_1}})^{p(x)} \leq \frac{p(x)}{q(x)} \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{q(x)} + \frac{q(x) - p(x)}{q(x)} \left[ \frac{\omega_1(x)}{\omega_2(x)} \|f\|_{q, \omega_2, \Omega_1} \right]^{r(x)}.$$ 

Obviously, $1 \leq A + B \leq 2$. Integrating by $\Omega_1$, using the property 1), we get

$$\int_{\Omega_1} \left( \frac{|f(x)| \omega_1(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{p(x)} dx$$

$$\leq A \int_{\Omega_1} \left( \frac{|f(x)| \omega_2(x)}{\|f\|_{q, \omega_2, \Omega_1}} \right)^{q(x)} dx + B \int_{\Omega_1} \left[ \frac{\omega_1(x)}{\omega_2(x)} \|f\|_{q, \omega_2, \Omega_1} \right]^{r(x)} dx \leq A + B. \quad (2.6)$$
From (2.4) and (2.6) implies that

\[
\int_{\Omega} \left( \frac{|f(x)| \omega_1(x)}{L_{r(\omega_2)}^{1/p}(\Omega) \| f \|_{q, \omega_2, \Omega}} \right)^{p(x)} \, dx = \int_{\Omega_1} \left( \frac{|f(x)| \omega_1(x)}{L_{r(\omega_2)}^{1/p}(\Omega) \| f \|_{q, \omega_2, \Omega}} \right)^{p(x)} \, dx
\]

\[
+ \int_{\Omega_2} \left( \frac{|f(x)| \omega_1(x)}{L_{r(\omega_2)}^{1/p}(\Omega) \| f \|_{q, \omega_2, \Omega}} \right)^{p(x)} \, dx \leq \int_{\Omega_1} \left( \frac{|f(x)| \omega_1(x)}{L_{r(\omega_2)}^{1/p}(\Omega_1) \| f \|_{q, \omega_2, \Omega_1}} \right)^{p(x)} \, dx
\]

From last inequality we have

\[
1 \geq \int_{\Omega} \left( \frac{|f(x)| \omega_1(x)}{\left( A + B + \| \chi_{\Omega_2} \|_{L_{\infty}(\Omega)} \right)^{1/p(x)} \| \omega_1 \|_{L_{r(\omega_2)}(\Omega_1) \| f \|_{q, \omega_2, \Omega}}} \right)^{p(x)} \, dx
\]

\[
\geq \int_{\Omega} \left( \frac{|f(x)| \omega_1(x)}{\left( A + B + \| \chi_{\Omega_2} \|_{L_{\infty}(\Omega)} \right)^{1/p(x)} \| \omega_1 \|_{L_{r(\omega_2)}(\Omega_1) \| f \|_{q, \omega_2, \Omega}}} \right)^{p(x)} \, dx.
\]

Thus

\[
\| f \|_{p, \omega_1, \Omega} \leq \left( A + B + \| \chi_{\Omega_2} \|_{L_{\infty}(\Omega)} \right)^{1/p} \| \omega_1 \|_{L_{r(\omega_2)}(\Omega_1) \| f \|_{q, \omega_2, \Omega}}.
\]

The theorem is proved.

**Remark 3.** Note that Theorem 2 in the case \( \omega_1 = \omega_2 = 1 \) and \( |\Omega| < \infty \) was proved in [15]. In the case \( 1 \leq p \leq p(x) \leq q(x) \leq \bar{q} < \infty \) for general measures Theorem 2 was proved in [4].

The following theorems are known.

**Theorem 3.** [1] Let \( 1 \leq p \leq p(x) \leq q(y) \leq \bar{q} < \infty \) for all \( x \in \Omega_1 \subset \mathbb{R}^n \) and \( y \in \Omega_2 \subset \mathbb{R}^m \). If \( p(x) \in C(\Omega_1) \), then the inequality

\[
\left\| \| f \|_{L_p(\Omega_1)} \right\|_{L_q(\Omega_2)} \leq C_{p,q} \left\| \| f \|_{L_p(\Omega_2)} \right\|_{L_p(\Omega_1)}
\]
is valid, where \( C_{p,q} = \left( \| \chi_{\Delta_1} \|_\infty + \| \chi_{\Delta_2} \|_\infty + \frac{p}{q} - \frac{p}{q} \right) \left( \| \chi_{\Delta_1} \|_\infty + \| \chi_{\Delta_2} \|_\infty \right), q = \text{ess inf}_{\Omega_2} q(x), \quad \mathcal{F} = \text{ess sup}_{\Omega_2} q(x), \quad \Delta_1 = \{ (x, y) \in \Omega_1 \times \Omega_2 : p(x) = q(y) \}, \quad \Delta_2 = \Omega_1 \times \Omega_2 \setminus \Delta_1 \) and \( C(\Omega_1) \) is the space of continuous functions in \( \Omega_1 \) and \( f : \Omega_1 \times \Omega_2 \to \mathbb{R} \) is any measurable function such that

\[
\| f \|_{q, \Omega_2} = \inf \left\{ \mu > 0 : \int_{\Omega_1} \left( \frac{\| f(x, \cdot) \|_{q(\cdot), \Omega_2}}{\mu} \right)^{p(x)} \, dx \leq 1 \right\} < \infty.
\]

The following lemmas are known.

**Lemma 2.** [6] Let \( 0 < s < 1, -\infty < a < b \leq \infty \) and \( f \) is non-negative and decreasing function defined on \((a, b)\). Then

\[
\left( \int_a^b f(x) \, dx \right)^s \leq s \int_a^b f^s(x) (x - a)^{s-1} \, dx.
\]

**Lemma 3.** [6] Let \( 0 < s < 1, -\infty \leq a < b < \infty \) and \( f \) is non-negative and increasing function defined on \((a, b)\). Then

\[
\left( \int_a^b f(x) \, dx \right)^s \leq s \int_a^b f^s(x) (b - x)^{s-1} \, dx.
\]

3. **On a topology of the spaces** \( L_{p(x), \omega} \) **for** \( 0 < p(x) < 1 \)

Now we formulate some definitions which be characterized of the topology in general vector spaces.

**Definition 2.** A subset \( G \) of a vector space \( X \) is called convex if, for any \( x_1, x_2, \ldots, x_m \in G, \sum_{i=1}^m \alpha_i x_i \in G \), where \( \sum_{i=1}^m \alpha_i = 1 \) and \( \alpha_i \geq 0, i = 1, 2, \ldots, m \). In particular, the subset contains the average \( \frac{1}{m} \sum_{i=1}^m x_i \).

**Definition 3.** A topological vector space \( X \) is called locally convex if the convex open sets are a base for the topology, i.e., any open set \( U \subset X \) around a point, there is a convex open set \( C \) containing that point such \( C \subset X \).
We show that the weighted variable Lebesgue spaces $L_{p(x),\omega}(\Omega)$ isn’t locally convex.

**Lemma 4.** Let $0 < p \leq p(x) \leq q(x) \leq q < 1$ and $\omega$ be a weight function defined on $\Omega$ and $0 < \omega(x) < \infty$ a.e. $x \in \Omega$. Then weighted variable Lebesgue spaces $L_{p(x),\omega}(\Omega)$ isn’t locally convex.

**Proof.** It is obvious that $\rho(f, g) = \int_\Omega [[f(x) - g(x)]^\omega(x)]^{p(x)} \, dx$ is defined a metric on $L_{p(x),\omega}(\Omega)$. We consider any open ball neighborhoods $0$:

$$U_R(0) = \{ f \in L_{p(x),\omega}(\Omega) : \rho(f, 0) = I_{p(x),\omega}(f) < R \}.$$

We will show that, for any $\varepsilon > 0$, the $\varepsilon$-ball neighborhoods zero contains functions whose average lies outside the ball of radius $R$.

Suppose $\varepsilon > 0$ and $m \geq 1$. We select $m$ disjoint intervals $A_1, A_2, \ldots, A_m$ in $\Omega$, which need not cover of all $\Omega$. We put $f_k = \left( \frac{\varepsilon}{\omega(A_k)} \right)^{1/p(x)} \chi_{A_k}$, where $\omega(A_k) = \int_{A_k} [\omega(x)]^{p(x)} \, dx$ and $k = 1, 2, \ldots, m$. Then $I_{p,\omega}(f_k) = \frac{\varepsilon}{\omega(A_k)} \int_{A_k} [\omega(x)]^{p(x)} \, dx = \varepsilon$, and so every $f_k$ is at distance $\varepsilon$ from $0$. But, since the functions $f_k$ are supported on disjoint sets, their average $g_m = \frac{1}{m} \sum_{i=1}^{m} f_i$ satisfies

$$I_{p,\omega}(g_m) = \int_\Omega g_m^{p(x)}(x) \, dx = \int_\Omega \frac{1}{mp(x)} \left( \sum_{i=1}^{m} f_i \right)^{p(x)} \, dx$$

$$\geq \frac{1}{mp(x)} \sum_{i=1}^{m} \int_\Omega (f_i(x) \omega(x))^{p(x)} \, dx = \frac{\varepsilon}{mp} \sum_{i=1}^{m} 1 = m^{1-p} \varepsilon.$$

Then $I_{p,\omega}(g_m) \to \infty$, for $m \to \infty$ (depending on $\varepsilon$). Therefore $\rho(g_m, 0) \to \infty$, for $m \to \infty$. Thus the distance between $g_n$ and $0$ can be made as large as desired.

The Lemma 4 is proved.

**Theorem 4.** Let $0 < p \leq p(x) \leq q(x) \leq q < 1$ and $\omega$ be a weight function defined on $\Omega$ and $0 < \omega(x) < \infty$ a.e. $x \in \Omega$. Then $[L_{p(x),\omega}(\Omega)]^* = \{0\}$, where $\ast$ - be denoted dual space of $L_{p(x),\omega}(\Omega)$, i.e., is the space of continuous linear functionals from $L_{p(x),\omega}(\Omega)$ to $R$.

**Proof.** We argue by contradiction. Let $\varphi \neq 0$ and $\varphi \in [L_{p(x),\omega}(\Omega)]^*$. Let $\tilde{B}(0, t) = \Omega \cap B(0, t)$, where $0 < t < \infty$. 
Suppose that \( \varphi \) is linear continuous functional defined in \( L_{p(x), \omega}(\Omega) \). Then we can find an \( f \in L_{p(x), \omega}(\Omega) \) such that \( \varphi(f) = 1 \). Now, the map \( t \mapsto f\chi_{\hat{B}(0,t)} \) is continuous, since \( |f|^{p(x)} \omega(x) \) is integrable:

\[
\int_{\hat{B}(0,t_2)} |f(x)|^{p(x)} \omega(x) \, dx - \int_{\hat{B}(0,t_1)} |f(x)|^{p(x)} \omega(x) \, dx = \int_{\Omega \cap B_{t_1,t_2}} |f(x)|^{p(x)} \omega(x) \, dx, \quad \text{for } t_1 < t_2,
\]

where \( B_{t_1,t_2} = \{ x : t_1 \leq |y| < t_2 \} \). Thus we may choose \( t \in (t_1, \infty) \) such that \( \varphi(f\chi_{\hat{B}(0,t)}) = \varphi(f\chi_{\Omega \setminus \hat{B}(0,t)}) = \frac{1}{2} \). Next, notice that \( g = f\chi_{\hat{B}(0,t)} \) and \( h = f\chi_{\Omega \setminus \hat{B}(0,t)} \) satisfy

\[
\int_{\Omega} |f(x)|^{p(x)} \omega(x) \, dx = \int_{\hat{B}(0,t)} |f(x)|^{p(x)} \omega(x) \, dx + \int_{\Omega \setminus \hat{B}(0,t)} |f(x)|^{p(x)} \omega(x) \, dx = I_{p, \omega}(g) + I_{p, \omega}(h).
\]

Thus, at least one of \( I_{p, \omega}(g) \) or \( I_{p, \omega}(h) \) is less than \( \frac{1}{2} I_{p, \omega}(f) \). Let’s say that \( I_{p, \omega}(g) \leq \frac{1}{2} I_{p, \omega}(f) \). Then, \( f_1 = 2g \) satisfies

\[
\varphi(f_1) = 1 \quad \text{and} \quad I_{p, \omega}(f_1) \leq 2^{p} I_{p, \omega}(g) \leq 2^{p-1} I_{p, \omega}(f).
\]

By induction, we can find a sequence \( \{f_n\}_{n \geq 1} \) in \( L_{p(x), \omega}(\Omega) \) with

\[
\varphi(f_n) = 1 \quad \text{and} \quad I_{p, \omega}(f_n) \leq 2^{p-1} I_{p, \omega}(f).
\]

It is obvious that \( \overline{p} - 1 < 0 \) and \( f_n \to 0 \) in \( L_{p(x), \omega}(\Omega) \) while \( T(f_n) = 1 \). Thus, \( T = 0 \) is the only continuous linear functional.

**Theorem 5.** Let \( 0 < p \leq p(x) \leq q(x) \leq \overline{p} < 1 \) and \( \omega \) be a weight function defined on \( \Omega \) and \( 0 < \omega(x) < \infty \) a.e. \( x \in \Omega \). Then the spaces \( L_{p(x), \omega}(\Omega) \) is complete.

**Proof.** Let \( \{f_n\}, n \in N \) be a sequence in \( L_{p(x), \omega}(\Omega) \) such that

\[
\|f_n - f_m\|_{p, \omega, \Omega} \to 0, \quad \text{for } n, m \to \infty.
\]

From properties 1) implies that

\[
\int_{\Omega} (|f_n - f_m| \omega(x))^{p(x)} \, dx \to 0, \quad \text{for } n, m \to \infty.
\]

We choose the subsequence \( \{n_k\} \) such that

\[
A = \sum_{k=1}^{\infty} \int_{\Omega} (|f_{n_{k+1}} - f_{n_k}| \omega(x))^{p(x)} \, dx < \infty.
\]
Then for any $\ell \in \mathbb{N}$

$$\int_{\Omega} \left[ \sum_{k=1}^{\ell} \left( |f_{n_{k+1}} - f_{n_k}| \omega(x) \right) \right]^{p(x)} dx \leq \sum_{k=1}^{\ell} \int_{\Omega} \left( |f_{n_{k+1}} - f_{n_k}| \omega(x) \right)^{p(x)} dx \leq A.$$  

If $\ell \to \infty$, then by monotone convergence theorem

$$\int_{\Omega} \left[ \sum_{k=1}^{\infty} \left( |f_{n_{k+1}} - f_{n_k}| \omega(x) \right) \right]^{p(x)} dx \leq A.$$  

Therefore,

$$\sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \omega(x) < \infty, \text{ a.e. } x \in \Omega.$$  

Hence, by completeness of $\mathbb{R}$, $f_{n_k}$ converges a.e. $x \in \Omega$. We define a measurable function $f$ by

$$f(x) = \begin{cases} \lim_{k \to \infty} f_{n_k}, & \text{for a.e. } x \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

Since $\int_{\Omega} (|f_n - f_m| \omega(x))^{p(x)} dx \to 0$, for $n, m \to \infty$, then $|f_n - f_m|^{p(x)} \to 0$, $n, m \to \infty$. Given $\varepsilon > 0$ we can find $N_\varepsilon$ so that $n \geq N_\varepsilon$ implies

$$\left| \left\{ x : |f_n(x) - f_m(x)|^{p(x)} \right\} \right| = \int_{\left\{ x : |f_n(x) - f_m(x)|^{p(x)} \right\}} dx \leq \varepsilon, \text{ for } m \geq n.$$  

In particular, $\left| \left\{ x : |f_n(x) - f_{n_k}(x)|^{p(x)} \right\} \right| \leq \varepsilon$, for $k \to \infty$. Hence, by Fatou’s lemma for $n \geq N_\varepsilon$, we have

$$\left| \left\{ x : |f_n(x) - f(x)|^{p(x)} \right\} \right| \leq \lim_{k \to \infty} \inf \left\{ x : |f_n(x) - f_{n_k}(x)|^{p(x)} \right\} \leq \varepsilon.$$  

Hence $f \in L_{p(x), \omega}(\Omega)$ and $\int_{\Omega} |(f_n - f) \omega(x)|^{p(x)} dx \to 0$, for $n \to \infty$.

This completes the proof of Theorem 5.

**Remark 4.** Note that from property 5) and Theorem 5 implies that the spaces $L_{p(x), \omega}(\Omega)$ is ideal.
4. Main results.

We consider the classical Hardy operator and its dual operator defined as

\[ H f(x) = \frac{1}{x} \int_{0}^{x} f(t) \, dt, \quad H^* f(x) = \frac{1}{x} \int_{x}^{\infty} f(t) \, dt \]

where \( f \) is nonnegative function on \((0, \infty)\).

**Lemma 5.** Let \( 0 < p \leq p_n \leq \bar{p} \leq 1, \ p_n \geq p_{n+1} \) and \( \{x_n\}_{n \geq 1} \) be any non-negative sequence of real numbers such that \( x_n^{p_n} \geq x_{n+1}^{p_{n+1}} \) for any \( n \in \mathbb{N} \).

Then

\[
\left( \sum_{n=1}^{\infty} \frac{x_n}{x_n^{p_n}} \right)^{\frac{p}{p_n}} \leq \sum_{n=1}^{\infty} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] \leq \sum_{n=1}^{\infty} x_n^{p_n}. \quad (4.1)
\]

**Proof.** First we proved that

\[
\left( \sum_{n=1}^{m} \frac{x_n}{x_n^{p_n}} \right)^{\frac{p_n}{p_n}} \leq \sum_{n=1}^{m} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] \quad (4.2)
\]

We consider the function \( h(t) = \frac{(1 + t)^q - 1}{t^q} \), where \( t \geq 0 \) and \( 0 < q < 1 \). It is obvious that

\[
h'(t) = \frac{q [1 - (1 + t)^{q-1}]}{t^{q+1}} \geq 0 \text{ for all } t \geq 0.
\]

In particular, the function \( h(t) \) is monotone increasing in the segment \([0, B]\). Therefore \( h(t) \leq h(B) \), i.e.,

\[
(1 + t)^q \leq 1 + t^q \left( B^{-1} + 1 \right)^q - B^{-q} \text{ for any } 0 \leq t \leq B. \quad (4.3)
\]

Since \( x_1^{p_1} \geq x_2^{p_2} \), then \( x_2 \leq x_1^{p_1} \). Therefore taking \( t = \frac{x_2^{p_2}}{x_1^{p_1}}, \ B = 1 \) and \( q = p_2 \) in (4.3), we have

\[
\left( \frac{x_1^{p_1}}{x_1^{p_1}} + x_2 \right)^{p_2} \leq x_1^{p_1} + x_2^{p_2} (2^{p_2} - 1). \quad (4.4)
\]

It is obvious that the inequality (4.4) be inequality (4.2) for \( m = 2 \). By the condition of Lemma 2 \( p_2 \geq p_3 \) and so \( 2^{p_3} \leq 2^{p_2} \). Since \( x_3 \leq \frac{x_1^{p_1} + x_2^{p_2}}{2} \) from (4.3) and (4.4) for \( t = \frac{x_3}{x_1^{p_1} + x_2^{p_2}}, \ B = \frac{1}{2} \) and \( q = p_3 \), we get

\[
\left( \frac{x_1^{p_1} + x_2^{p_2}}{x_1^{p_1} + x_2^{p_2} + x_3} \right)^{p_3} \leq \left( \frac{x_1^{p_1} + x_2^{p_2}}{x_1^{p_1} + x_2^{p_2}} \right)^{p_3} + x_3^{p_3} (3^{p_3} - 2^{p_3})
\]

\[
\leq x_1^{p_1} + x_2^{p_2} (2^{p_2} - 1) + x_3^{p_3} (3^{p_3} - 2^{p_3}) \leq x_1^{p_1} + x_2^{p_2} (2^{p_2} - 1) + x_3^{p_3} (3^{p_3} - 2^{p_3}).
\]
The last inequality is (4.1) for \( m = 3 \). Clearly \( x_1^{m+1} + x_2^{m+1} + \ldots + x_m^{m+1} + x_{m+1} \geq (m + 1)x_{m+1} \). Hence \( x_{m+1} \leq \frac{x_1^{m+1} + x_2^{m+1} + \ldots + x_m^{m+1}}{m} \). Therefore taking

\[
t = \frac{x_{m+1}}{\frac{a_1}{x_{m+1}^{m+1}} + \frac{a_2}{x_{m+1}^{m+1}} + \ldots + \frac{a_m}{x_{m+1}^{m+1}}}, \quad B = \frac{1}{m}
\]

in (4.3), we have

\[
\left( \sum_{n=1}^{m+1} \frac{p_n}{x_n^{m+1}} \right)^{p_{m+1}} = \left( \sum_{n=1}^{m} \frac{p_n}{x_n^{m+1}} + x_{m+1} \right)^{p_{m+1}} \leq \left( \sum_{n=1}^{m} \frac{p_n}{x_n^{m+1}} \right)^{p_{m+1}} + x_{m+1}^p (m+1)^{p_{m+1} - m_{m+1}} \leq \sum_{n=1}^{m} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] + x_{m+1}^p [(m+1)^{p_{m+1}} - m_{m+1}]
\]

By the induction principle the inequality (4.2) is proved for any \( m \in \mathbb{N} \).

Since the sequence \( \{p_n\}_{n \geq 1} \) is decreasing, then \( \lim_{n \to \infty} p_n = \overline{p} \). Therefore passing to the limit at \( m \to \infty \) in (4.2) we have the left part of inequality (4.1). By using the inequality \( n^{p_n} \leq (n-1)^{p_n} + 1 \), we have the right part of inequality (4.1).

The Lemma 2 is proved.

**Example 4.1.** Let \( x_n = \begin{cases} n^{-\frac{p}{2}}, & \text{for } n = k^2 \\ 0, & \text{for } n \neq k^2 \end{cases} \), and \( \overline{p} < \frac{p + 1}{2} \).

It is obvious that the sequence \( \{x_n^{p_n}\}_{n \geq 1} \) isn’t monotone and \( \sum_{n=1}^{\infty} x_n^{p_n} = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty \).

On the other hand \( n^{p_n} - (n-1)^{p_n} \sim \overline{p}_n n^{p_n-1} \sim n^{p_n-1} \) for \( n \to \infty \). Therefore

\[
\sum_{n=1}^{\infty} x_n^{p_n} [n^{p_n} - (n-1)^{p_n}] \sim \sum_{n=1}^{\infty} x_n^{p_n} n^{p_n-1} = \sum_{k=1}^{\infty} k^{-\frac{p}{p+2}} k_{p-2} \leq \sum_{k=1}^{\infty} k^{2\overline{p}-\overline{p}-2}.
\]

It is well known that the series \( \sum_{k=1}^{\infty} k^{2\overline{p}-\overline{p}-2} \) is converges if and only if \( \overline{p} < \frac{p + 1}{2} \). Thus for \( \overline{p} < \frac{p + 1}{2} \) the inequality (3.1) isn’t holds.

The example show that the condition of monotonicity of sequence \( \{x_n^{p_n}\}_{n \geq 1} \) is essential.
Remark 5. Note that Lemma 5 in the case \( p_1 = p_2 = \ldots = p_n = \ldots = p = \text{const} \) was proved in [5].

Theorem 6. Let \( x \in (0, \infty), \ 0 < p \leq p(x) \leq q(x) \leq \bar{q} < 1, \ r(x) = \frac{pp(x)}{p(x) - p} \) and \( f(x) \) are non-negative and decreasing function defined on \( (0, \infty) \). Suppose \( \omega_1 \) and \( \omega_2 \) are weight functions defined on \( (0, \infty) \).

Then for any \( f \in L_{p(x), \omega_1}(0, \infty) \) the inequality
\[
\| Hf \|_{L_{p(\cdot), \omega_2}(0, \infty)} \leq p^\frac{1}{p} c_{p,q} d_p \left\| \frac{t^{1/p'}}{\omega_2(t^{1/p})} \left\| \frac{\omega_1}{L_{q(t^{1/p})}} \right\|_{L_{\nu}(0, \infty)} \right\| \| f \|_{L_{p(\cdot), \omega_1}(0, \infty)},
\]
where \( c_{p,q} = \left( \| \chi_{\Delta_1} \|_{L_{\nu}(0, \infty)} + \| \chi_{\Delta_2} \|_{L_{\nu}(0, \infty)} + p \left( \frac{1}{q} - \frac{1}{\bar{q}} \right) \right) \left( \| \chi s_1 \|_{L_{\nu}(0, \infty)} + \| \chi s_2 \|_{L_{\nu}(0, \infty)} \right), S_1 = \{ x \in (0, \infty) : p(x) = p \}, S_2 = (0, \infty) \setminus S_1, \) and \( d_p = \left( 1 + \frac{p - p}{p} + \| \chi s_1 \|_{L_{\nu}(0, \infty)} \right)^{1/p} \)

Proof. Taking \( a = 0, b = x \) and \( s = p \) and apply Lemma 2 and property 5), we have
\[
\| Hf \|_{L_{p(\cdot), \omega_2}(0, \infty)} = \| \omega_2 Hf \|_{L_{q(\cdot)}(0, \infty)} = \left\| \frac{\omega_2}{x} \int_0^x f(t) dt \right\|_{L_{q(\cdot)}(0, \infty)}
\]
\[
\leq p^\frac{1}{p} \left\| \frac{\omega_2(x)}{x} \left( \int_0^x ft^{p-1} dt \right) \right\|_{L_{q(\cdot)}(0, \infty)}^{1/p}
\]
Now applied Theorem 3, we get
\[
\left\| \frac{\omega_2(x)}{x} \left( \int_0^x ft^{p-1} dt \right) \right\|_{L_{q(\cdot)}(0, \infty)}^{1/p}
\]
\[
= \left\| \int_0^\infty f^{\nu}(t) \chi_{(0, x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^\frac{p}{2} t^{p-1} dt \right\|_{L_{q(\cdot)}(0, \infty)}^{1/p}
\]
\[
= \left\| \int_0^\infty f^{\nu}(t) \chi_{(0, x)}(t) \left[ \frac{\omega_2(x)}{x} \right]^\frac{p}{2} t^{p-1} dt \right\|_{L_{q\nu(\cdot)}(0, \infty)}^{1/p}
\]
Finally, apply Theorem 2, we get

\[
\leq c_{p,q} \left( \int_0^\infty \left\| f(t) \, \chi_{(0,x)}(t) \, \left[ \frac{\omega_2(x)}{x} \right]^{p} \right\|_{L_{q}\,(0,\infty)} \, dt \right)^{1/p}
\]

\[
= c_{p,q} \left( \int_0^\infty f(t) \, t^{p-1} \left\| \chi_{(0,x)}(t) \, \left[ \frac{\omega_2(x)}{x} \right]^{p} \right\|_{L_{q}\,(0,\infty)} \, dt \right)^{1/p}
\]

\[
= c_{p,q} \left( \int_0^\infty f(t) \, t^{p-1} \left\| t \, \frac{\omega_2(x)}{x} \right\|_{L_{q}\,(t,\infty)} \, dt \right)^{1/p} = c_{p,q} \left\| f \, t^{1/p'} \left\| t \, \frac{\omega_2(x)}{x} \right\|_{L_{q}\,(t,\infty)} \right\|_{L_{q}\,(0,\infty)}.
\]

Thus

\[
\| Hf \|_{L_{q}\,(q(x),\infty)} \leq \frac{1}{p} c_{p,q} \left\| f \, t^{1/p'} \left\| t \, \frac{\omega_2(x)}{x} \right\|_{L_{q}\,(t,\infty)} \right\|_{L_{q}\,(0,\infty)} \| f \|_{L_{p}(\omega_1(0,\infty))}.
\]

Theorem 6 is proved.

**Theorem 7.** Let \(0 < p \leq p(x) \leq q(x) \leq q < 1\), \(r(x) = \frac{p \, p(x)}{p(x) - p}\) and \(f(x)\) are non-negative and increasing function defined on \((0, \infty)\). Suppose \(\omega_1\) and \(\omega_2\) are weight functions defined on \((0, \infty)\).

Then for any \(f \in L_{p}(\omega_1(0, \infty))\) the inequality

\[
\| Hf \|_{L_{q}\,(\omega_2(0,\infty))} \leq \frac{1}{p} \left\| f \, t^{1/p'} \left\| t \, \frac{\omega_2(x)}{x} \right\|_{L_{q}\,(t,\infty)} \right\|_{L_{q}\,(0,\infty)} \| f \|_{L_{p}(\omega_1(0,\infty))},
\]

where \(c_{p,q}\) and \(d_p\) the constants in Theorem 6.

**Proof.** Taking \(a = 0\), \(b = x\) and \(s = p\) and apply Lemma 3 and property 5), we have

\[
\| Hf \|_{L_{q}\,(\omega_2(0,\infty))} = \| \omega_2 Hf \|_{L_{q}\,(0,\infty)} = \left\| \frac{\omega_2}{x} \int_0^x f(t) \, dt \right\|_{L_{q}\,(0,\infty)}.
\]
Now applied Theorem 3, we get

\[ \leq \left( p \right)^{1/p} \left\| \frac{\omega_2(x)}{x} \left( \int_0^x f(t) \left( x-t \right)^{\frac{1}{p'}} dt \right) \right\|_{L_q(\cdot, \infty)}^{1/p}. \]

Finally, apply Theorem 2, we get

\[ \leq \left( p \right)^{1/p} \left\| \frac{\omega_2(x)}{x} \left( \int_0^x f(t) \left( x-t \right)^{\frac{1}{p'}} dt \right) \right\|_{L_q(\cdot, \infty)}^{1/p}. \]
Thus
\[ \|Hf\|_{L_q(\cdot, \omega_2(0, \infty))} \leq \frac{1}{p^2} c_{p,q} d_p \left\| \left\| \frac{(x-t)^{\frac{1}{p'}} \omega_2}{x} \right\|_{L_q(t, \infty)} \frac{1}{\omega_1} \right\|_{L_{r'}(0, \infty)} \|f\|_{L_p(\cdot, \omega_1(0, \infty))}. \]

The Theorem 7 is proved.

For the dual operator $H^*$ a theorem below is proved analogously.

**Theorem 8.** Let $x \in (0, \infty)$, $0 < p \leq p(x) \leq q(x) \leq \overline{q} < 1$, $r(x) = \frac{pp(x)}{p(x) - p}$ and $f(x)$ are non-negative and decreasing function defined on $(0, \infty)$. Suppose $\omega_1$ and $\omega_2$ are weight functions defined on $(0, \infty)$.

Then for any $f \in L_{p(x), \omega_1}(0, \infty)$ the inequality
\[ \|H^* f\|_{L_q(\cdot, \omega_2(0, \infty))} \leq \frac{1}{p^2} c_{p,q} d_p \left\| \left\| \frac{(t-x)^{\frac{1}{p'}} \omega_2}{x} \right\|_{L_q(x, \infty)} \frac{1}{\omega_1} \right\|_{L_{r'}(0, \infty)} \|f\|_{L_{p}(\cdot, \omega_1(0, \infty))}, \]

where $c_{p,q}$ and $d_p$ the constants in Theorem 6.

**Remark 6.** Note that Theorem 6, Theorem 7 and Theorem 8 in the case $p(x) = q(x) = p = \text{const}$ and $\omega_1(x) = \omega_2(x) = x^\alpha$ was proved in [6] (see also [5]). In the case $1 \leq p(x) \leq q(x) \leq \overline{q} < \infty$ Hardy inequality is very much studied (see [2], [3] and etc.). In the constant exponent case $1 \leq p \leq q \leq \overline{q} \leq \infty$ for detailed information we refer to [10]. Note that similar problem for Hardy maximal function was investigated in [9] and [11].

**Example 4.2.** Let $x \in (0, \infty)$, $0 < p(x) = p = \text{const} < 1$, $q(x) = \begin{cases} \frac{1}{4}, & \text{for } 0 < x < 1 \\ \frac{1}{2}, & \text{for } x \geq 1, \end{cases}$ $0 < p \leq q(x)$ and $p' = \frac{p}{p - 1}$. Suppose $\omega_1(x) = x^\alpha$, $\omega_2(x) = x^{\beta+1}$, $\beta < -2$, $\beta \neq -4$ and $\beta + 2 + \frac{1}{p'} < \alpha < \min \left\{ \frac{1}{p'}, \beta + 4 + \frac{1}{p'} \right\}$, where $r(x) = \infty$.

Then the pair $(\omega_1, \omega_2)$ satisfies the condition of Theorem 6.

**Example 4.3.** Let $x \in (0, \infty)$, $0 < p \leq p(x) \leq q(x) \leq \overline{q} < 1$ and $p' = \frac{p}{p - 1}$. Suppose $\omega_1(x) = x^{1/p'} \left\| \frac{\omega_2}{x} \right\|_{L_q(x, \infty)}$. Then condition $\|1\|_{L_{r'}(0, \infty)} < \infty$ is guaranteed the satisfy of condition of Theorem 6. Note that by Definition 1 the condition $\|1\|_{L_{r'}(0, \infty)} < \infty$ is equivalent to
\[ \int_0^\infty \delta^{\frac{p(p(x)}{p-1}} dx < \infty, \]
where \( \delta \in (0, 1) \). Then the pair \((\omega_1, \omega_2)\) satisfies the condition of Theorem 6.

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