ON RANDOMLY PLACED ARCS ON THE CIRCLE

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ABSTRACT. We completely describe in terms of Hausdorff measures the size of the set of points of the circle that are covered infinitely often by a sequence of random arcs with given lengths. We also show that this set is a set with large intersection.

1. Introduction and statement of the results

Let us consider a nonincreasing sequence $\ell = (\ell_n)_{n \geq 1}$ of positive reals converging to zero. In 1956, A. Dvoretzky [7] raised the question to find a necessary and sufficient condition on the sequence $\ell$ to ensure that the whole circle $T = \mathbb{R}/\mathbb{Z}$ is covered almost surely by a sequence of random arcs with lengths $\ell_1$, $\ell_2$, etc. To be specific, let $A(x, l)$ denote the open arc with center $x \in T$ and length $l > 0$, that is, the set of all $y \in T$ such that $d(y, x) < l/2$, where $d$ denotes the usual quotient distance on $T$. Then, let $(X_n)_{n \geq 1}$ be a sequence of random points independently and uniformly distributed on $T$ and let

$$E_{\ell} = \limsup_{n \to \infty} A(X_n, \ell_n).$$

Dvoretzky’s problem amounts to finding a necessary and sufficient condition to ensure that

$$\text{a.s. } E_{\ell} = T. \quad (1)$$

This longstanding problem, along with several of its extensions, has raised the interest of, notably, P. Billard, P. Erdős, J.-P. Kahane, P. Lévy and B. Mandelbrot, and has found various applications, such as the study of multiplicative processes and that of some random series of functions, see [11, 12].

In 1972, L. Shepp completely solved Dvoretzky’s problem by showing that (1) holds if, and only if,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \ldots + \ell_n) = \infty. \quad (2)$$

Still, several related questions remained open. For example, when the series in (2) converges, it is natural to ask which proportion of the circle is actually covered by the random arcs. In other words, what is the size of the set $E_{\ell}$? A first answer may be given by computing the value of its Lebesgue measure $\mathcal{L}(E_{\ell})$. This is in fact trivial, since Fubini’s theorem and the Borel-Cantelli lemma directly imply that

$$\text{a.s. } \mathcal{L}(E_{\ell}) = \begin{cases} 0 & \text{if } \sum_n \ell_n < \infty \\ 1 & \text{if } \sum_n \ell_n = \infty. \end{cases} \quad (3)$$

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A typical way of refining the description of the size of the set $E_\ell$ is then to compute the value of its Hausdorff dimension. This has recently been done by A.-H. Fan and J. Wu \cite{FanWu} in the particular case where $\ell_n = a/n^\alpha$, for some $a > 0$ and $\alpha > 1$. Their result states that

$$\forall a > 0 \quad \forall \alpha > 1 \quad \text{a.s.} \quad \dim E_{(a/n^\alpha)} = \frac{1}{\alpha}.$$ 

Corollary 1 below ensures that actually, for any nonincreasing sequence $\ell = (\ell_n)_{n\geq1}$ converging to zero, the Hausdorff dimension of the set $E_\ell$ is almost surely equal to

$$s_\ell = \sup\{s \in (0,1) \mid \sum_n \ell_n^s = \infty\} = \inf\{s \in (0,1) \mid \sum_n \ell_n^s < \infty\},$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = 1$. Corollary 1 follows from Theorem 1 below, which in fact gives the value of the Hausdorff $g$-measure of the set $E_\ell$ for any gauge function $g$ and not only the monomial functions used to define the Hausdorff dimension, see Subsection 1.1. Therefore, this theorem provides a complete description of the size of the set $E_\ell$ in terms of Hausdorff measures, for any sequence $\ell$.

We also show in this note that the set $E_\ell$ enjoys a remarkable property originally introduced by K. Falconer \cite{Falconer}, namely, it is a set with large intersection. Roughly speaking, this means that $E_\ell$ is “large and omnipresent” in the circle in some strong measure theoretic sense, see Subsection 1.2. Sets with large intersection have been shown to arise on other occasions in probability theory – more precisely in multifractal analysis of stochastic processes, see \cite{Falconer} – as well as in other fields of mathematics, namely, number theory and dynamical systems, see \cite{Falconer, Falconer, Falconer} and the references therein.

1.1. Size properties of the set $E_\ell$. A typical way of completely describing the size of a subset of the circle is to determine the value of its Hausdorff $g$-measure for any gauge function $g$, see \cite{Falconer, Falconer}. We call a gauge function any function $g$ defined on $[0, \infty)$ that is nondecreasing near zero, enjoys $\lim_{r \to 0^+} g(r) = g(0) = 0$ and is such that $r \mapsto g(r)/r$ is nonincreasing and positive near zero. For any gauge function $g$, the Hausdorff $g$-measure of a set $F \subseteq \mathbb{T}$ is then defined by

$$\mathcal{H}^g(F) = \lim_{\delta \to 0} \frac{\mathcal{H}^g_{\delta}(F)}{\mathcal{H}^g_{\delta}(\mathbb{T})} \quad \text{with} \quad \mathcal{H}^g_{\delta}(F) = \inf_{F \subseteq \bigcup_{p \geq 1} U_p, \sum_{p \geq 1} g(|U_p|) < \infty} \sum_{p \geq 1} g(|U_p|).$$

The infimum is taken over all sequences $(U_p)_{p \geq 1}$ of sets with $F \subseteq \bigcup_{p \geq 1} U_p$ and $|U_p| < \delta$ for all $p \geq 1$, where $|\cdot|$ denotes diameter. Note that if $g(r)/r$ goes to infinity at zero, every nonempty open subset of $\mathbb{T}$ has infinite Hausdorff $g$-measure and that, otherwise, $\mathcal{H}^g$ coincides up to a multiplicative constant with the Lebesgue measure on the Borel subsets of $\mathbb{T}$.

The size properties of $E_\ell$ are then completely described by the following result.

**Theorem 1.** Let $\ell = (\ell_n)_{n \geq 1}$ be a nonincreasing sequence of positive reals converging to zero and let $g$ be a gauge function. Then, with probability one, for any open subset $V$ of $\mathbb{T}$,

$$\mathcal{H}^g(E_\ell \cap V) = \begin{cases} 
\mathcal{H}^g(V) & \text{if } \sum_n g(\ell_n) = \infty \\
0 & \text{if } \sum_n g(\ell_n) < \infty.
\end{cases}$$
Recall that the Hausdorff dimension of a nonempty set $F \subseteq \mathbb{T}$ is defined with the help of the monomial functions $\text{Id}^s$ by
\[
\dim F = \sup\{s \in (0, 1) \mid \mathcal{H}^{\text{Id}^s}(F) = \infty\} = \inf\{s \in (0, 1) \mid \mathcal{H}^{\text{Id}^s}(F) = 0\},
\]
with the same convention regarding the infimum and the supremum of the empty set as in \cite{4}. Also, it is customary to let $\dim \emptyset = -\infty$. Using Theorem \ref{1} it is then possible to determine the value of the Hausdorff dimension of the set $E_\ell$, thereby generalizing the result of \cite{10}.

**Corollary 1.** For every nonincreasing sequence $\ell = (\ell_n)_{n \geq 1}$ of positive reals converging to zero, with probability one, $\dim E_\ell = s_\ell$, where $s_\ell$ is defined by \cite{3}.

Theorem \ref{1} and Corollary \ref{1} are proven in Sections 2 and 3, respectively.

**1.2. Large intersection properties of the set $E_\ell$.** Rigorously, the fact that $E_\ell$ is a set with large intersection means that it belongs to some classes $G^g(V)$ of subsets of the circle that we defined in \cite{2} Section 5. We refer to that paper, and also to \cite{2}, for a precise definition of those classes and we content ourselves with stressing on the fact that, for any gauge function $g$ and any nonempty open subset $V$ of the circle, one may define a class $G^g(V)$ of sets with large intersection in $V$ with respect to $g$ which, among other properties, enjoys the following.

**Proposition 1.** For any gauge function $g$ and any nonempty open $V \subseteq \mathbb{T}$,
\begin{enumerate}
\item[(a)] the class $G^g(V)$ is closed under countable intersections;
\item[(b)] every set $F \in G^g(V)$ enjoys $\mathcal{H}^g(F) = \infty$ for any gauge $\overline{g}$ with $\overline{g} \prec g$;
\item[(c)] $G^g(V) = \bigcap_{\mathbb{R}} G^\overline{g}(V)$ where $\overline{g}$ is a gauge function enjoying $\overline{g} \prec g$;
\item[(d)] $G^g(V) = \bigcap_{U} G^g(U)$ where $U$ is a nonempty open subset of $V$;
\item[(e)] every $G_\delta$-set with full Lebesgue measure in $V$ belongs to the class $G^g(V)$.
\end{enumerate}

The notation $\overline{g} \prec g$ means that $\overline{g}/g$ monotonically tends to infinity at zero. In view of Proposition \ref{1} every set in the class $G^g(V)$ has infinite Hausdorff $g$-measure in every nonempty open subset of $V$ for any gauge function $\overline{g} \prec g$, and any countable intersection of such sets enjoys the same property. Therefore, the classes $G^g(V)$ provide a rigorous way of stating that a set is large and omnipresent in $V$ in a strong measure theoretic sense.

In order to describe the large intersection properties of the set $E_\ell$, we shall make use of the following result, which gives a simple sufficient condition for a limsup of arcs to be a set with large intersection in the circle. It may be seen as the analog for the periodic setting of the ubiquity result established in \cite{4}.

**Proposition 2.** Let $(y_n)_{n \geq 1}$ be a sequence in $\mathbb{T}$ and let $(r_n)_{n \geq 1}$ be a sequence of positive reals converging to zero. Then, for any gauge function $g$,
\[
\mathcal{L} \left( \limsup_{n \to \infty} A(y_n, 2g(r_n)) \right) = 1 \quad \Rightarrow \quad \limsup_{n \to \infty} A(y_n, 2r_n) \in G^g(\mathbb{T}).
\]

Proposition \ref{2} may be interpreted as follows. Given that any gauge function $g$ is bounded below by the identity function (up to a multiplicative constant), the limsup of the arcs $A(y_n, 2r_n)$ may be seen as a “reduced” version of the limsup of the arcs $A(y_n, 2g(r_n))$. If the latter limsup is large and omnipresent enough to contain Lebesgue-almost every point of the circle, then its reduced version is also large and omnipresent, in the weaker sense that it belongs to the class $G^g(\mathbb{T})$. We refer to Section \ref{4} for a proof of Proposition \ref{2}.
The large intersection properties of the set $E_\ell$ are then completely described by the following result.

**Theorem 2.** Let $\ell = (\ell_n)_{n \geq 1}$ be a nonincreasing sequence of positive reals converging to zero and let $g$ be a gauge function. Then, almost surely, for any nonempty open subset $V$ of $\mathbb{T}$,

$$E_\ell \in G^g(V) \iff \sum_n g(\ell_n) = \infty.$$ 

The remainder of this note is organized as follows: Theorems 1 and 2 are established in Section 2, Corollary 1 is proven in Section 3, and the proof of Proposition 2 is given in Section 4. Before detailing the proofs, let us mention that we shall basically only make use of the main properties of the classes $G^g(V)$ given by Proposition 1, the ubiquity result given by Proposition 2 and the value (3) of the Lebesgue measure of the set $E_\ell$. In particular, unlike the authors of [10], we do not need to call upon any specific result on the spacings between the random centers $X_n$ of the arcs. This also means that our method can straightforwardly be extended to the case of random balls on the $d$-dimensional torus for any $d \geq 2$.

2. Proofs of Theorems 1 and 2

Theorems 1 and 2 follow from four lemmas which we now state and prove. Throughout the section, $\ell = (\ell_n)_{n \geq 1}$ is a nonincreasing sequence of positive reals converging to zero.

**Lemma 1.** For any gauge function $g$,

$$\sum_n g(\ell_n) < \infty \implies \forall V \text{ open } \mathcal{H}^g(E_\ell \cap V) = 0.$$ 

**Proof.** For any $\delta > 0$, there is an integer $n_0 \geq 1$ such that $0 < \ell_n < \delta$ for any $n \geq n_0$. Moreover, the set $E_\ell$ is covered by the arcs $A(X_n, \ell_n)$ for $n \geq n_0$, so that $\mathcal{H}^g(E_\ell) \leq \sum_{n=n_0}^{\infty} g(\ell_n)$. If the series $\sum_n g(\ell_n)$ converges, then letting $n_0$ tend to infinity and $\delta$ go to zero yields $\mathcal{H}^g(E_\ell) = 0$. \hfill $\square$

**Lemma 2.** For any gauge function $g$,

$$\sum_n g(\ell_n) < \infty \implies \forall V \neq \emptyset \text{ open } E_\ell \notin G^g(V).$$ 

**Proof.** Let us assume that the series $\sum_n g(\ell_n)$ converges. Then, one may build a gauge function $\overline{g}$ such that $\overline{g} < g$ and the series $\sum_n \overline{g}(\ell_n)$ converges too, for example by adapting a construction given in the proof of [1, Theorem 3.5]. By Lemma 1, the set $E_\ell$ has Hausdorff measure zero in $V$ and thus cannot belong to the class $G^\overline{g}(V)$, due to Proposition 1(b). \hfill $\square$

**Lemma 3.** For any gauge function $g$,

$$\sum_n g(\ell_n) = \infty \implies \text{ a.s. } \forall V \neq \emptyset \text{ open } E_\ell \in G^g(V).$$ 

**Proof.** If the series $\sum_n g(\ell_n)$ diverges, then $\sum_n g(\ell_n/2)$ diverges as well (because $r \mapsto g(r)/r$ is nonincreasing near zero). Hence, thanks to (3), the limsup of the arcs $A(X_n, 2g(\ell_n/2))$ has Lebesgue measure one with probability one. We conclude using Proposition 2 and Proposition 1(b). \hfill $\square$

**Lemma 4.** For any gauge function $g$,

$$\sum_n g(\ell_n) = \infty \implies \text{ a.s. } \forall V \text{ open } \mathcal{H}^g(E_\ell \cap V) = \mathcal{H}^g(V).$$ 

Proof. If the series $\sum_n g(\ell_n)$ diverges, then $\sum_n g(\ell_n/2)$ diverges as well (because $r \mapsto g(r)/r$ is nonincreasing near zero). Hence, thanks to (3), the limsup of the arcs $A(X_n, 2g(\ell_n/2))$ has Lebesgue measure one with probability one. We conclude using Proposition 2 and Proposition 1(b). \hfill $\square$
Proof. We may obviously assume that $V$ is nonempty. Let us suppose that the series \( \sum_n g(\ell_n) \) diverges. Then, again by following a construction given in the proof of [1] Theorem 3.5, it is possible to build a gauge function $g$ such that $g \prec g$ and the series \( \sum_n g(\ell_n) \) diverges too, provided that $g \prec \text{Id}$. Therefore, thanks to Lemma 3, the set $E_\ell$ belongs to the class $\mathcal{G}(V)$. Hence, $\mathcal{H}^g(E_\ell \cap V) = \infty = \mathcal{H}^g(V)$, owing to Proposition 1. In the case where $g \neq \text{Id}$, the Hausdorff $g$-measure coincides, up to a multiplicative constant, with the Lebesgue measure on the Borel subsets of the circle and the result follows from 3. \( \square \)

3. Proof of Corollary 1

In order to prove Corollary 1 let us consider a nonincreasing sequence $\ell = (\ell_n)_{n \geq 1}$ of positive reals converging to zero. Theorem 1 along with the definition 4 of the real $s_\ell$, implies that for any real $s \in (0, 1)$, with probability one,
\[
\mathcal{H}^{s_\ell}(E_\ell) = \begin{cases} 
\infty & \text{if } s < s_\ell \\
0 & \text{if } s > s_\ell.
\end{cases}
\]

Let us assume that $s_\ell \in (0, 1]$. Then, for all $m$ large enough, with probability one, the set $E_\ell$ has infinite Hausdorff $Id^{s_\ell - 1/m}$-measure, so that its Hausdorff dimension is at least $s_\ell - 1/m$. Therefore, the dimension of $E_\ell$ is almost surely at least $s_\ell$. Likewise, if $s_\ell \in [0, 1)$, then $E_\ell$ has $Id^{s_\ell + 1/m}$-measure zero with probability one for all $m$ large enough, so that its Hausdorff dimension is almost surely at most $s_\ell$. As a result, with probability one, $\dim E_\ell = s_\ell$ if $s_\ell \in (0, 1]$ and $\dim E_\ell \leq 0$ if $s_\ell = 0$.

It remains to establish that $E_\ell$ is almost surely nonempty when $s_\ell = 0$. Note that the set $E_{1/n}$, obtained by picking $\ell_n = 1/n$, has Lebesgue measure one with probability one, by virtue of 3. Let us assume that this property holds. Furthermore, note that $\ell_n = \mathcal{O}(1/n)$ as $n$ goes to infinity, thanks to Olivier’s theorem [13]. In particular, $\ell_n \leq 1/n$ for any integer $n$ greater than or equal to some $n_1 \geq 1$. Let $I_1 = A(X_{n_1}, \ell_{n_1}/2)$. The union over $n > \max\{n_1, 8/\ell_{n_1}\}$ of the arcs $A(X_n, 1/n)$ has full Lebesgue measure in the circle, so its intersection with the arc $A(X_{n_1}, \ell_{n_1}/4)$ is nonempty. Therefore, there is an integer $n_2 > \max\{n_1, 8/\ell_{n_1}\}$ such that $A(X_{n_2}, 1/n_2) \subseteq I_1$. Then, let $I_2 = A(X_{n_2}, \ell_{n_2}/2)$. Repeating this procedure, one may obtain a nested sequence of open arcs $I_n$ and the intersection of their closures yields a point that belongs to the set $E_\ell$.

4. Proof of Proposition 2

Let $g$ be a gauge function such that the limsup of the arcs $A(y_n, 2g(r_n))$ has Lebesgue measure one. Thus, following the terminology of 4, the family $(k + y_n, g(r_n))_{(k,n) \in \mathbb{Z} \times \mathbb{N}}$ is a homogeneous ubiquitous system in $\mathbb{R}$. Here, each $y_n$ is the only real in $[0, 1)$ such that $\phi(y_n) = y_n$, where $\phi$ denotes the canonical surjection from $\mathbb{R}$ onto $\mathbb{T}$. Thanks to 4 Theorem 2, the set of all reals $x$ such that $|x - k - y_n| < r_n$ for infinitely many $(k,n) \in \mathbb{Z} \times \mathbb{N}$ belongs to the class $G^{s}(\mathbb{R})$ of sets with large intersection in $\mathbb{R}$ with respect to the gauge function $g$, which is defined in 4. Equivalently, the inverse image under $\phi$ of the limsup of the arcs $A(y_n, 2r_n)$ belongs to $G^{s}(\mathbb{R})$, which ensures that this limsup belongs to the class $G^{s}(\mathbb{T})$, see [5 Section 5].
References

[1] A. Durand, Propriétés d’ubiquité en analyse multifractale et séries aléatoires d’ondelettes à coefficients corrélés, PhD thesis, Université Paris 12, 240 pages, 2007, available at http://tel.archives-ouvertes.fr/tel-00185375/en/.
[2] A. Durand, Large intersection properties in Diophantine approximation and dynamical systems, submitted, 24 pages, 2008, arXiv:0803.3852.
[3] A. Durand, Random wavelet series based on a tree-indexed Markov chain, Comm. Math. Phys., 27 pages, 2008, doi:10.1007/s00220-008-0504-7.
[4] A. Durand, Sets with large intersection and ubiquity, Math. Proc. Cambridge Philos. Soc. 144(1):119–144, 2008.
[5] A. Durand, Singularity sets of Lévy processes, Probab. Theory Relat. Fields, 28 pages, 2008, doi:10.1007/s00440-007-0134-6.
[6] A. Durand, Ubiquitous systems and metric number theory, Adv. Math. 218(2):368–394, 2008.
[7] A. Dvoretzky, On covering a circle by randomly placed arcs, Proc. Nat. Acad. Sci. USA 42:199–203, 1956.
[8] K.J. Falconer, Sets with large intersection properties, J. London Math. Soc. (2) 49(2):267–280, 1994.
[9] K.J. Falconer, Fractal geometry: Mathematical foundations and applications, 2nd ed., John Wiley & Sons Inc., New York, 2003.
[10] A.-H. Fan and J. Wu, On the covering by small random intervals, Ann. Inst. H. Poincaré Probab. Statist. 40(1):125–131, 2004.
[11] J.-P. Kahane, Some random series of functions, 2nd ed., Cambridge University Press, Cambridge, 1985.
[12] J.-P. Kahane, Random coverings and multiplicative processes, in Fractal geometry and stochastics (Greifswald/Koserow, 1998), Progr. Probab. 46, Birkhäuser, Basel, pp. 125–146, 2000.
[13] L. Olivier, Remarques sur les séries infinis et leur convergence, J. Reine Angew. Math. 2:31–44, 1827.
[14] C.A. Rogers, Hausdorff Measures, Cambridge University Press, Cambridge, 1970.

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