Extended commutator algebra for the $q$-oscillator and a related Askey-Wilson algebra

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Abstract. Let $q$ be a nonzero complex number that is not a root of unity. In the $q$-oscillator with commutation relation $aa^+ - qa^+a = 1$, it is known that the smallest commutator algebra of operators containing the creation and annihilation operators $a^+$ and $a$ is the linear span of $a^+$ and $a$, together with all operators of the form $a^{+l}[a, a^+]^k$, and $[a, a^+]^k a^l$, where $l$ is a nonnegative integer and $k$ is a positive integer. That is, linear combinations of operators of the form $a^h$ or $(a^+)^h$ with $h \geq 2$ or $h = 0$ are outside the commutator algebra generated by $a$ and $a^+$. This is a solution to the Lie polynomial characterization problem for the associative algebra generated by $a^+$ and $a$. In this work, we extend the Lie polynomial characterization into the associative algebra $P = P(q)$ generated by $a$, $a^+$, and the operator $e^{\omega N}$ for some nonzero real parameter $\omega$, where $N$ is the number operator, and we relate this to a $q$-oscillator representation of the Askey-Wilson algebra $AW(3)$.

1 Introduction

Throughout, we consider operators on the Hilbert space $\ell_2$ over the complex field $\mathbb{C}$, which consists of all sequences $(z_n)_{n=0}^{\infty}$ of elements of $\mathbb{C}$ with the property $\sum_{n=0}^{\infty} |z_n|^2 < \infty$. We use the complete orthonormal basis of $\ell_2$ consisting of the ket vectors $|n\rangle$ for all $n \in \mathbb{N} := \{0, 1, \ldots\}$. Denote by $B(\ell_2)$ the Banach algebra of all bounded operators on $\ell_2$. In the traditional harmonic oscillator, the creation operator $a^+$ and the annihilation operator $a$ can be represented as elements of $B(\ell_2)$ acting on the basis ket vectors according
to the equations
\[
\begin{align*}
    a\ket{0} &= 0, \\
    a\ket{n} &= \sqrt{n}\ket{n-1}, \\
    a^+\ket{n} &= \sqrt{n+1}\ket{n+1},
\end{align*}
\]
\(n \in \mathbb{N}\setminus\{0\}\).

As a consequence, \(a^+\) and \(a\) satisfy the canonical commutation relation
\[
[a, a^+] = 1,
\]
where the operation \([-,-] : B(\ell_2) \times B(\ell_2) \to B(\ell_2)\) is the commutator given by the rule \([X,Y] := XY - YX\) for all \(X, Y \in B(\ell_2)\). The algebra \(B(\ell_2)\) is hence an abstract Lie algebra with respect to \([-,-]\) and the vector space operations.

**Problem 1.1.** What is the smallest space of operators containing \(a^+\) and \(a\) that is closed under the commutator and vector space operations? In other words, viewing \(B(\ell_2)\) as a Lie algebra under said operations, what is the Lie subalgebra of \(B(\ell_2)\) generated by \(a^+\) and \(a\)? Equivalently, what is the commutator algebra for the creation and annihilation operators of the classical harmonic oscillator?

The solution to Problem 1.1 is readily available from the theory of the classification of low-dimensional Lie algebras. As a consequence of the relation (1), the Lie subalgebra of \(B(\ell_2)\) generated by \(a^+\) and \(a\) is isomorphic to the Heisenberg algebra, which is the three-dimensional Lie algebra whose derived (Lie) algebra is contained in the center. The operators \(a^+, a\) and \(1 \in B(\ell_2)\) form a basis [14, Section 3.2.1].

### 1.1 The commutator algebra for \(a^+, a\) in the \(q\)-oscillator

Several modifications or generalizations to the relation (1) have been proposed. One is the replacement of the undeformed commutator by the \(q\)-deformed commutator \([a, a^+]_q := aa^+ - qa^+a\) for some scalar parameter \(q\). The result is the \(q\)-deformed harmonic oscillator or \(q\)-oscillator introduced in the classical works [4, 25]. Another is the \(\omega\)-commutator \([a, a^+]_\omega := e^{\omega}aa^+ - e^{-\omega}a^+a\) for some nonzero real parameter \(\omega\), which shall arise in later sections of this work. The latter deformed commutator also leads to a \(q\)-oscillator representation [29]. See, for instance, [2, 26] for physical interpretations or realizations of the \(q\)-oscillator. The significance of the \(q\)-deformed oscillator is that, together with the deformed quantum group \(SU_q(2)\), these two mathematical constructions were once considered as possible candidates for the role of oscillator and angular momentum, respectively, at very small scales, such as the Planck scale [29, p. 1155]. The natural continuation now is to extend Problem 1.1 for the \(q\)-oscillator.

**Problem 1.2.** Given a scalar \(q \neq 1\), what is the commutator algebra for the creation and annihilation operators of the \(q\)-oscillator?

A solution to Problem 1.2 has recently been obtained in [6, 11]. Some generalizations and alternative approaches were studied in [7, 8, 9, 10]. Before we describe the solution, we
first discuss some consequences of, and some algebraic structures related to, the $q$-deformed commutation relation

$$aa^+ - qa^+a = 1.$$ (2)

The $q$-deformed Heisenberg algebra is the unital associative algebra $\mathcal{H}(q)$ (with unit 1) abstractly defined by a presentation with generators $A, B$ and relation $AB - qBA = 1$. More succinctly, if the underlying field is $\mathbb{F}$, then $\mathcal{H}(q) := \mathbb{F} \langle A, B \rangle / (AB - qBA - 1)$.

The use of the symbol $\mathcal{H}(q)$ and of the term “$q$-deformed Heisenberg algebra” to both mean $\mathbb{F} \langle A, B \rangle / (AB - qBA - 1)$ is based on [16]. An excellent discussion may be found in [16, pp. 5–11] about which among the 410 references the authors cite, published not later than the year 2000, motivated the algebra $\mathbb{F} \langle A, B \rangle / (AB - qBA - 1)$, and how it has been consistently denoted by $\mathcal{H}(q)$ or referred to as the $q$-deformed Heisenberg algebra.

Some works published after [16] continue to refer to $\mathbb{F} \langle A, B \rangle / (AB - qBA - 1)$ as $\mathcal{H}(q)$ or as the $q$-deformed Heisenberg algebra. Said works come from varied fields of mathematics, such as Ring Theory [15, 17], Lie algebras [6, 8, 9, 10, 11], Mathematical Physics [7, 19], and algebraic curves [13]. All these, and most probably many more, refer to $\mathcal{H}(q) = \mathbb{F} \langle A, B \rangle / (AB - qBA - 1)$ as the $q$-deformed Heisenberg algebra. No confusion should arise with similar symbol and terminology from recent studies like [20, 21, 22, 23, 24], which are not the subject of this paper.

From this point onward, we assume that $\mathbb{F} = \mathbb{C}$. If $\mathcal{H}_0$ is the unital associative subalgebra of $\mathcal{B}(\ell_2)$ generated by $a^+$ and $a$, we immediately find that there exists a surjective homomorphism $\Psi : \mathcal{H}(q) \to \mathcal{H}_0$ of algebras such that $A \mapsto a$ and $B \mapsto a^+$. In the common realizations of $a^+$ and $a$ as operators on $\ell_2$, such as those in [1, 12, 27], it can be shown by routine arguments that the operator $[a, a^+]^n$ is not the zero operator for any $n \in \mathbb{N}$. This condition, together with an assumption that $q$ is nonzero and is not a root of unity, is enough to conclude, using [17, Theorem 6.7], that the homomorphism $\Psi$ is injective. i.e., The representation $\Psi$ of $\mathcal{H}(q)$ is faithful. Thus, we have:

**Proposition 1.3.** The subalgebra $\mathcal{H}_0$ of $\mathcal{B}(\ell_2)$ is isomorphic to $\mathcal{H}(q)$.

By this isomorphism, the algebraic properties of $\mathcal{H}(q)$ such as bases, gradations, and others, carry over to $\mathcal{H}_0$ with $A \mapsto a$ and $B \mapsto a^+$. We discuss these properties in the succeeding remarks.

**Remark 1.4.** By [17, Corollary 4.5], the elements

$$[a, a^+]^k,$$ (3)

$$[a, a^+]^k a^l,$$ (4)

$$(a^+)^l[a, a^+]^k,$$ (5)

where $k \in \mathbb{N}$, $l \in \mathbb{N} \setminus \{0\}$, form a basis for $\mathcal{H}_0$.

Denote by $\mathfrak{L}(\mathcal{H}_0)$ the Lie subalgebra of $\mathcal{H}_0$ generated by $a^+$ and $a$, or equivalently, $\mathfrak{L}(\mathcal{H}_0)$ is the commutator algebra for the creation and annihilation operators of the $q$-oscillator.
The basis of $H_0$ consisting of the elements (3)–(5) has a very important role in the solution of Problem 1.2 as obtained in [6]. To describe this solution, we divide (3)–(5) into two groups. For the first group, we take $1 = [a,a^+]$ from (3), and from (4),(5), we consider those in which the exponent $k$ of $[a,a^+]$ is zero and at the same time, the exponent $l$ of either $a^+$ or $a$ is at least 2. That is,

$$1, \quad a^l, \quad (a^+)^l, \quad (l \in \mathbb{N}\{0,1\}).$$

(6)

Then the basis elements of $H_0$ in (3)–(5) that are not in (6) are

$$[a,a^+]^k, \quad (k \in \mathbb{N}\{0\}),$$

(7)

$$a, \quad [a,a^+]^k a^l, \quad (k,l \in \mathbb{N}\{0\}),$$

(8)

$$a^+, \quad (a^+)^l [a,a^+]^k, \quad (k,l \in \mathbb{N}\{0\}).$$

(9)

Remark 1.5. According to [6, Theorem 5.8], the elements (7)–(9) form a basis for $L(H_0)$. That is, if $\Gamma$ is the vector subspace of $H_0$ spanned by (6), then we have the direct sum decomposition

$$H_0 = \Gamma \oplus L(H_0).$$

(10)

In other words, any finite linear combination of (6) cannot be expressed as a finite linear combination of nested commutators in $a^+, a$, or equivalently not a Lie polynomial in $a^+, a$. Thus, the answer to Problem 1.2 is that the commutator algebra $L(H_0)$ is the set of all finite linear combinations of (7)–(9). The generators $a^+, a$ of $L(H_0)$ are clearly Lie polynomials in $a^+, a$, but then it is most natural to ask how the other basis elements of $L(H_0)$ from (7)–(9) can be expressed as Lie polynomials in $a^+, a$. To show this, we shall exhibit some relations from [6, 11] that express such basis elements as Lie polynomials or linear combinations of nested commutators of $a^+, a$, but before exhibiting these relations, we first discuss a convenient notation using the adjoint map.

Given a Lie algebra $L$ and $x \in L$, recall the adjoint map $ad : L \rightarrow L$ defined by $y \mapsto [x,y]$. By linearity of the Lie bracket in the second argument, $ad$ is a linear map for any $x \in L$, and by the skew-symmetry of the Lie bracket, $(-ad)(y) = -[x,y] = [y,x]$. For convenience in expressing nested commutators, we use the concept of the adjoint map in the following manner.

Remark 1.6. Let $x_1,x_2,\ldots,x_k,y_1,y_2,\ldots,y_l \in L$. If $k = 2$, the nested commutator $[x_1,[x_2,y]]$ can be expressed as the composition of two adjoint maps given by $(ad x_1) \circ (ad x_2)(y)$. In general, we have

$$((ad x_1) \circ (ad x_2) \circ \cdots \circ (ad x_k))(y) = [x_1,[x_2,[\cdots,[x_k,y]],[\cdots]]].$$

(11)

For the special case $x_1 = x_2 = \cdots = x_k$, we can write the left-hand side of (11) as $(ad x)^k(y)$. Similarly, if $l = 2$, the nested commutator of $[[x,y_1],y_2]$ can be expressed
as the composition of two adjoint maps given by \((-\text{ad } y_2) \circ (-\text{ad } y_1)(x)\). This can be generalized as

\[
((-\text{ad } y_1) \circ (-\text{ad } y_2) \circ \cdots \circ (-\text{ad } y_l))(x) = [[[\cdots [x, y_l], \cdots], y_2], y_1].
\] (12)

We also have the similar result that if \(y_1 = y_2 = \cdots = y_l\), then we can write the left-hand side of (12) as \((-\text{ad } y)^l(x)\).

**Remark 1.7.** From [6, Section 5], if \(q\) is nonzero and is not a root of unity, then the basis elements of \(\mathcal{L}(\mathcal{H}_0)\) from (7)–(9) can be expressed as Lie polynomials in \(a^+\) and \(a\) through the relations

\[
[a, a^+]^{k+2} = \frac{-q^k(1-q)}{1-q^{k+1}} \sum_{i=0}^{k} \left( (\text{ad } a^+) \circ (\text{ad } [a, a^+])^k \circ (\text{ad } a) \right) ([a, a^+])
\] (13)

\[
[a, a^+]^{k+1} a^l = \frac{-((\text{ad } [a, a^+])^k \circ (\text{ad } a)^{l+1}) (a^+)}{(1-q)^l(q^l-1)^k},
\] (14)

\[
(a^+)^l[a, a^+]^{k+1} = \frac{(\text{ad } a^+)^{l-1} \circ (\text{ad } [a, a^+])^k (a^+)}{(q-1)^{k+1}(1-q^{k+1})^{l-1}},
\] (15)

which hold for any \(k \in \mathbb{N}\), and any \(l \in \mathbb{N} \setminus \{0\}\). The operators \(a^+, a\), and \([a, a^+]\) are clearly Lie polynomials in \(a^+, a\), and so we need not consider them in the relations (13)–(15). The commutator table for these basis elements of \(\mathcal{L}(\mathcal{H}_0)\) can be gleaned from [6, Table 1], and a detailed discussion of the computational aspects of these relations can be found in [11, Section 2.1].

If \(q\) is a root of unity other than 1, there are special cases for the constructions in (13)–(15), and the basis (7)–(9) of \(\mathcal{L}(\mathcal{H}_0)\) is reduced according to some conditions in the exponents. These can be found in [11, Section 4.1]. If \(q = 0\), an entirely different basis and nested commutator constructions can be found in [6, Section 4]. These two cases, however, are not covered by the representation theorem [17, Theorem 6.7], and so the possibly non-faithful representation \(\mathcal{H}_0\) of \(\mathcal{H}(q)\) for these two cases may lead to further restrictions in the algebraic structure of \(\mathcal{L}(\mathcal{H}_0)\). We reserve these two cases as for a possibly interesting continuation of the results presented in this paper, and so we have the following.

**Assumption 1.8.** The scalar \(q\) is nonzero and is not a root of unity.

### 1.2 The objective of this study: extending the commutator algebra for \(a^+, a\)

As a natural continuation of Problem 1.2, we consider extending the generating set of the operator Lie algebra by another important operator from the \(q\)-oscillator. One possibility is the number operator \(N\) defined by the action \(N |n\rangle := n |n\rangle\) for any \(n \in \mathbb{N}\). The canonical commutation relations that relate \(N\) to \(a^+\) and \(a\) are

\[
[N, a^+] = a^+, \quad [a, N] = a,
\] (16)
which hold even for the $q$-oscillator. We note the simple nature of the relations (16), in which we can see that the result of performing the commutator operation on any of $a^+, a$ involving $N$ simply results to linear combinations of $a^+, a$. By an easy argument, an immediate consequence is that the extended commutator algebra for $a^+, a, N$ is simply the direct sum $CN \oplus L(H_0)$, where $CN$ is the set of all complex scalar multiples of $N$. As will be demonstrated in the results in this paper, it is the operator $q^{kN}$, defined by the action

$$q^{kN} |n\rangle := q^{kn} |n\rangle,$$

for some scalar $\kappa$, that gives a richer algebraic structure in extending the commutator algebra for $a^+, a$. By some choice of $\kappa$, our results for the extended commutator algebra for $a^+, a, q^{kN}$ turn out to be intimately related to a $q$-oscillator realization of an Askey-Wilson algebra. The three-parameter Askey-Wilson algebra $AW(3)$ was introduced in [29], and is said to be important by itself as a dynamical symmetry algebra in problems in which the Askey-Wilson polynomials arise as eigenfunctions [29, p. 1147]. The algebra $AW(3)$ and its extension to generalized families of similar algebras have been extensively studied in algebraic combinatorics, among many other fields. See, for instance, the comprehensive discussion in [28, Section 1] about how the Askey-Wilson algebra and its variants are studied in algebraic combinatorics, integrable systems, and quantum mechanics, or the discussion in [18, Section 1] about how $AW(3)$ is related to several significant quantum groups. The algebraic perspective in this work, however, differs from that in [18] or [28]. We do not study ring-theoretic generalizations or finite-dimensional representation theory, but the Lie structure of associative algebras, such as the perspective in the studies [5, 6, 11]. We show that our results about the extended commutator algebra for the $q$-oscillator shed light on some related Lie structure in a $q$-oscillator representation of the algebra $AW(3)$.

2 Preliminaries: Algebraic structure of the $q$-deformed Heisenberg algebra $\mathcal{H}(q)$

We review some important results about the algebra $\mathcal{H}(q)$, which has a presentation with generators $A, B$ and relation $AB - qBA = 1$. Let $C := [A, B]$. In terms of $A, B, C$, the basis (3)-(5) of $\mathcal{H}_0 = \mathcal{H}(q) \subseteq P$ can be rewritten as

$$C^k, \ C^k A^l, \ C^k B^l, \quad (k \in \mathbb{N}, \ l \in \mathbb{N}\backslash\{0\}).$$

For each $l \in \mathbb{N}$, define $\mathfrak{f}_{-l}$ as the span of all elements $C^k A^l$ ($k \in \mathbb{N}$), and $\mathfrak{f}_l$ as the span of $C^k B^l$ ($k \in \mathbb{N}$).

Remark 2.1. In the basis (18) of $\mathcal{H}(q)$, we use basis elements of the form $C^k B^l$ for the product of powers of $C$ and $B$, instead of $B^l C^k$ which was used in [11, 17]. This may be justified by the reordering formula $B^l C^k = q^{-kl} C^k B^l$ (for any $k, l \in \mathbb{N}$) which is from [17, equation 18]. This change in normal form for this particular type of basis element is valid because of the assumption that $q$ is nonzero (Assumption 1.8).
Denote the set of all integers by \( \mathbb{Z} \). For any vector subspaces \( \mathfrak{A} \) and \( \mathfrak{B} \) of any algebra \( \mathfrak{A} \), we define \( \mathfrak{A}\mathfrak{B} \) as the span of all elements \( ab \) where \( a \in \mathfrak{A} \) and \( b \in \mathfrak{B} \). Following [17, Proposition 2.4, Corollary 4.5], the collection \( \{ \mathcal{H}_k : k \in \mathbb{Z} \} \) is a \( \mathbb{Z} \)-gradation of \( \mathcal{H}(q) \). That is, we have the direct sum decomposition

\[
\mathcal{H}(q) = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k,
\]

and that for any \( h, k \in \mathbb{Z} \),

\[
\mathcal{H}_h \mathcal{H}_k \subseteq \mathcal{H}_{h+k}.
\]

Thus, for instance, if \( m, k \in \mathbb{N} \) and \( n, l \in \mathbb{N}\{0\} \) with \( n > l \), then \( C^m B^n \cdot C^k A^l \in \mathcal{H}_{n-l} \). However, this does not give us any information about how to express \( C^m B^n \cdot C^k A^l \) as a linear combination of the basis elements \( C^h B^{n-l} \) \((h \in \mathbb{N})\) of \( \mathcal{H}_{n-l} \). In general, we are interested in the structure constants of \( \mathcal{H}(q) \) with respect to the basis (18). We discuss these in the following subsections.

### 2.1 Structure constants of \( \mathcal{H}(q) \) as an associative algebra

In this subsection, we recall the reordering formula and structure constants of \( \mathcal{H}(q) \) as an associative algebra, which is studied in [6, 11, 16, 17]. We present here a simplified discussion of [11, Section 2.1]. Define \( \{0\}_q := 0 \), and for each \( n \in \mathbb{N}\{0\} \), we recursively define \( \{n\}_q := 1 + q\{n-1\}_q \). That is, \( \{n\}_q = 1 + q + \cdots + q^{n-1} \). If \( q \neq 1 \), then \( \{n\}_q = \frac{1-q^n}{1-q} \). The Gaussian binomial coefficients or \( q \)-binomial coefficients are recursively defined by

\[
\begin{align*}
\binom{n}{0}_q &= 1, \\
\binom{0}{k+1}_q &= 0, \\
\binom{n+1}{k+1}_q &= \binom{n}{k}_q + q^{k+1}\binom{n}{k+1}_q,
\end{align*}
\]

for any \( n, k \in \mathbb{N} \). These \( q \)-binomial coefficients satisfy the symmetry property \( \binom{n}{k}_q = \binom{n}{n-k}_q \) for any \( k \in \{0, 1, \ldots, n\} \), and as a consequence also the properties

\[
\begin{align*}
\binom{n}{0}_q &= \binom{n}{n}_q = 1, \\
\binom{n}{1}_q &= \binom{n}{n-1}_q = \{n\}_q.
\end{align*}
\]

For each \( l \in \mathbb{N}\{0\} \) and each \( i \in \{0, 1, \ldots, l\} \) we define

\[
c_i(l) := (q-1)^{\bar{l}}(-1)^{l-i}(i+1)\binom{l}{i}_q.
\]
Let $k, m \in \mathbb{N}$, and let $l, n \in \mathbb{N}\{0\}$. As derived in [11, Section 2.1], the structure constants of the basis (18), given $l, n \in \mathbb{N}\{0\}$, may be determined from the relations

\begin{align*}
C^m A^n \cdot C^k A^l &= q^{kn} C^{m+k} A^{n+l}, \\
C^m B^n \cdot C^k B^l &= q^{-kn} C^{m+k} B^{n+l}, \quad (27) \\
C^m A^n \cdot C^k B^l &= \sum_{i=0}^{\min\{l,n\}} q^{(i+k)n-i} c_i(l) \cdot C^{m+i+k} A^{n-l}, \quad (n \geq l), \quad (29) \\
&= \sum_{i=0}^{\min\{l,n\}} q^{kn} c_i(n) \cdot C^{m+i+k} B^{l-n}, \quad (l > n), \quad (30) \\
C^m B^n \cdot C^k B^l &= \sum_{i=0}^{\min\{l,n\}} q^{-(i+k)n} c_i(l) \cdot B^{n-l} C^{m+k+i}, \quad (n \geq l), \quad (31) \\
&= \sum_{i=0}^{\min\{l,n\}} q^{-(i+k)n} c_i(n) \cdot C^{m+k+i} A^{l-n}, \quad (l > n). \quad (32)
\end{align*}

There is some simplification in the scalar coefficients regarding $c_i(l)$, since in [11] some other scalar coefficients were defined but are actually dependent on $c_i(l)$. We summarize in Table 1 below how the relations (27) to (32) give the structure constants of $\mathcal{H}(q)$ with respect to the basis (18).

| · | $C^k A^l$ | $C^k B^l$ |
|---|---|---|
| $C^m A^n$ | (27) | (29), (30) |
| $C^m B^n$ | (31), (32) | (28) |

Table 1: Relations that can be used to obtain the structure constants of the associative algebra $\mathcal{H}(q)$ with respect to the basis (18)

### 2.2 Structure constants of $\mathcal{H}(q)$ as a Lie algebra

We extend the information from Section 2.1 by determining the structure constants of $\mathcal{H}(q)$ as a Lie algebra, still with respect to the basis (18). Let $m, r \in \mathbb{N}$, and let $n, s \in \mathbb{N}\{0\}$. Using the computational techniques in the previous subsection, it is routine to show that the relations
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\[ [C^m A^n, C^r B^s] = \begin{cases} \min\{r,s\} \sum_{i=0}^{\min\{r,s\}} q^{rs} (1 - q^{E(i)}) c_i(n) C^{m+i+r}, & (n = s), \\ \min\{r,s\} \sum_{i=0}^{\min\{r,s\}} q^{nr+(s-n)(n-i)} (1 - q^{F(i)}) c_i(n) C^{m+i+r} B^{s-n}, & (n < s), \\ \min\{r,s\} \sum_{i=0}^{\min\{r,s\}} q^{nr+i(n-s)} (1 - q^{G(i)}) c_i(s) C^{m+i+r} A^{n-s}, & (n > s), \end{cases} \]

where

\[ E(i) := n(m + r - i), \]
\[ F(i) := n(2m + r + n - s) - in - ms, \]
\[ G(i) := s(m + 2r), \]

hold in $\mathcal{H}(q)$ where $i \in \{1, 2, \ldots, \min\{n, s\}\}$. Using (27), (28), we also have

\[ [C^m A^n, C^r A^s] = q^{nr} (1 - q^{ms-nr}) C^{m+r} A^{n+s}, \]
\[ [C^m B^n, C^r B^s] = q^{-nr} (1 - q^{nr-ms}) C^{m+r} B^{n+s}. \]

Similar to that done in Table 1, we summarize the structure constants of $\mathcal{H}(q)$ as a Lie algebra with respect to the basis (18) in Table 2 below. Because of the skew-symmetry of the Lie bracket, the relations (33) to (37) give us all the structure constants of $\mathcal{H}(q)$ as a Lie algebra.

| $[\cdot, \cdot]$ | $C^r A^s$ | $C^r B^s$ |
|------------------|----------|----------|
| $C^m A^n$       | (36)     | (33), (34), (35) |
| $C^m B^n$       | -        | (37)     |

Table 2: Relations that can be used to obtain the structure constants of $\mathcal{H}(q)$ as a Lie algebra with respect to the basis (18)

3 An extended operator algebra for the $q$-oscillator

Let $\omega$ be a positive real number. Earlier, the parameter $q$ was chosen to be any nonzero complex number that is not a root of unity. From this point onward, we further narrow down the choice of $q$ by setting $q = e^{-2\omega}$. Following [29, equations (6.5)–(6.6)], we have the relations

\[ e^{\omega} a a^+ - e^{-\omega} a^+ a = e^{\omega}, \]
\[ [a, a^+] = e^{-2\omega N} \]
for the $q$-oscillator. Define $K_0$ as the operator $e^{\omega N}$, i.e., $K_0 |n\rangle := e^{\omega n} |n\rangle$ for any $n \in \mathbb{N}$. With reference to the discussion in Section 1.2 about (17), we are here setting $\kappa = \frac{\omega}{\ln q}$. Denote by $\mathcal{P}_q$ the algebra (of operators on $\ell_2$) generated by $K_0, a, a^\dagger$. As a consequence of (38),(39), it can be shown by routine calculations that the relations

$$aa^\dagger = \frac{1 - e^{-2\omega} [a, a^\dagger]}{1 - e^{-2\omega}}, \quad (40)$$

$$a^\dagger a = \frac{1 - [a, a^\dagger]}{1 - e^{-2\omega}}, \quad (41)$$

$$a [a, a^\dagger] = e^{-2\omega} [a, a^\dagger] a, \quad (42)$$

$$a^\dagger [a, a^\dagger] = e^{2\omega} [a, a^\dagger] a^\dagger, \quad (43)$$

$$aK_0 = e^{\omega} K_0 a, \quad (44)$$

$$a^\dagger K_0 = e^{-\omega} K_0 a^\dagger, \quad (45)$$

$$[a, a^\dagger] K_0 = K_0 [a, a^\dagger], \quad (46)$$

$$K_0^2 [a, a^\dagger] = 1, \quad (47)$$

hold in $\mathcal{P}_q$. We are interested in the question of whether the relations (40)–(47) are enough to define a presentation for $\mathcal{P}_q$. In the interest of mathematical rigour, this is no trivial matter, and so in this section we prove that such a presentation can indeed be obtained for $\mathcal{P}_q$.

**Definition 3.1.** Given a nonzero real number $\omega$, if $q = e^{-2\omega}$, then we define $\mathcal{P} = \mathcal{P}(q)$ as the unital associative algebra over $\mathbb{C}$ with generators $K, A, B, C$ subject to the relations

$$AB = \frac{1 - qC}{1 - q}, \quad (48)$$

$$BA = \frac{1 - C}{1 - q}, \quad (49)$$

$$AC = qCA, \quad (50)$$

$$BC = q^{-1} CB, \quad (51)$$

$$AK = q^{-\frac{1}{2}} KA, \quad (52)$$

$$BK = q^{\frac{1}{2}} KB, \quad (53)$$

$$CK = KC, \quad (54)$$

$$K^2 C = 1. \quad (55)$$

By the defining relations (48) to (55) of the algebra $\mathcal{P}$, and using the Diamond Lemma for Ring Theory [3, Theorem 1.2], the elements

$$K^h C^k, \quad K^h C^kB^l, \quad K^h C^k A^l, \quad (h, k \in \mathbb{N}, \ l \in \mathbb{N}\{0\}), \quad (56)$$

with the condition

$$k \in \mathbb{N}\{0\} \Rightarrow h \in \{0, 1\}. \quad (57)$$
form a basis for \( \mathcal{P} \).

Comparing (40)-(47) with (48)-(55), if we consider the assignment

\[
K \mapsto K_0, \quad A \mapsto a, \quad B \mapsto a^+, \quad C \mapsto [a, a^+],
\]

we find that the generators of \( \mathcal{P}_q \) satisfy the defining relations of \( \mathcal{P} \). Thus, we say that \( \mathcal{P}_q \), a concrete algebra of operators on \( \ell_2 \), is a representation of the abstractly defined algebra \( \mathcal{P} \). More precisely, there exists a homorphism \( \Psi_0 : \mathcal{P} \to \mathcal{P}_q \) of algebras that performs (58). Our goal, at this point, is to prove the faithfulness of the representation \( \mathcal{P}_q \) of \( \mathcal{P} \), or equivalently, the injectiveness of \( \Psi_0 \).

### 3.1 Deformed commutator maps and the faithfulness of the representation \( \mathcal{P}_q \)

Given an algebra \( \mathcal{A} \) over \( \mathbb{C} \), we recall, from [17], the deformed commutator mappings

\[
\theta_{\alpha,t} : \gamma \mapsto \alpha \gamma - t \gamma \alpha, \\
\eta_{\beta,t} : \gamma \mapsto \gamma \beta - t \beta \gamma,
\]
given \( \alpha, \beta \in \mathcal{A} \) and \( t \in \mathbb{C} \). Consequently, any (two-sided) ideal of \( \mathcal{A} \) is invariant under a deformed commutator mapping. We shall be concerned with such deformed commutator mappings on \( \mathcal{P} \) of very specific types. Given \( n \in \mathbb{Q} \), succeeding proofs and computations shall involve the deformed commutator mappings

\[
\theta_n := \theta_{A,q^n} : \gamma \mapsto A \gamma - q^n \gamma A, \\
\eta_n := \eta_{B,q^n} : \gamma \mapsto \gamma B - q^n B \gamma.
\]

We collect all such maps in \( \Phi := \{ \theta_n, \eta_n : n \in \mathbb{Q} \} \). Some subsets of \( \Phi \) shall be of importance in later proofs, such as \( \Phi_Z := \{ \theta_n, \eta_n \in \Phi : n \in \mathbb{Z} \} \), and the set difference \( \Phi \setminus \Phi_Z = \{ \theta_n, \eta_n \in \Phi : n \notin \mathbb{Z} \} \). Using the computational techniques in Section 2.1, the identities

\[
\theta_n(C^k) = q^k(1 - q^{n-k})C^kA, \\
\theta_n(C^k A^l) = q^k(1 - q^{n-k})C^k A^{l+1}, \\
\theta_n(C^k B^l) = q^k \frac{1 - q^{n-k}}{1 - q} C^k B^{l-1} - q^{k+1} \frac{1 - q^{n-k-l}}{1 - q} C^{k+1} B^{l-1}, \\
\eta_n(C^k) = q^{-k}(1 - q^{n+k})C^kB, \\
\eta_n(C^k B^l) = q^{-k}(1 - q^{n+k})C^k B^{l+1}, \\
\eta_n(C^k A^l) = q^{-k} \frac{1 - q^{n+k}}{1 - q} C^k A^{l-1} - q^{-k} \frac{1 - q^{n+k+l}}{1 - q} C^{k+1} A^{l-1},
\]

hold in \( \mathcal{H}(q) \) for any \( k \in \mathbb{N} \), and \( n \in \mathbb{Q} \) and any \( l \in \mathbb{N} \setminus \{0\} \). Setting \( n = k \) in (59) and \( n = -k \) in (62), we have

\[
\theta_k(C^k) = 0 = \eta_{-k}(C^k), \quad (k \in \mathbb{N}).
\]
Setting \( n = k + 1, \ l = 1 \) in (61) and \( n = -k, \ l = 1 \) in (64), we have
\[
\theta_{k+1}(C^kB) = q^k C^k, \quad \eta_{-k}(C^kA) = -q^{-k}C^{k+1}.
\]

In order to exhibit some more interesting properties of the deformed commutator map \( \theta_n \) that shall be useful in later computations and proofs, we first recall the notion of a \( q \)-derivative. It is the mapping
\[
D_q : f(x) \mapsto f(x) - f(qx) \over x - qx,
\]
where, in this work, we take \( f \) to be any element of the polynomial algebra \( \mathbb{C}[x] \) on the indeterminate \( x \). Throughout, we use the notation
\[
\{i\}_q! := \prod_{j=1}^{i} \{j\}_q.
\]

Using [16, Lemma C.3] and induction, it is routine to show that
\[
(D_q^i)(x^n) = \{i\}_q! \left( \begin{array}{c} n \\ i \end{array} \right)_q x^{n-i},
\]
for any \( i, n \in \mathbb{N} \). Moreover, for each \( k \in \mathbb{N} \), we have
\[
\theta_k(C^kB^l) = q^{k-l+1}C^{k+1} \cdot D_q(B^l).
\]
We note here that (69) is an adjusted form of a related formula from [17, Lemma 5.7].

As explained in Remark 2.1, the adjustment is due to our use of basis elements of the form \( C^kB^l \) as normal form instead of using the elements of the form \( B^lC^k \) which were used in [17]. It is possible to apply the mapping \( \theta_k \) on both sides of (69) and use the identity (68) to simplify the resulting right-hand side. Continuing with this manner of computation in a way that the number \( n \in \mathbb{N} \) of times the mapping \( \theta_k \) is applied to \( C^kB^l \) does not exceed \( l \), and by some routine computations and induction, we have
\[
\theta^n_k(C^kB^l) = q^{n(k-l+n)}\{n\}_q! \left( \begin{array}{c} l \\ n \end{array} \right)_q C^{k+n}B^{l-n}, \quad (k, n \in \mathbb{N}, \ l \in \mathbb{N}\{0\}, \ l \leq n.)
\]

The identity (70) suggests some computational significance of deformed commutator mappings in the sense that applying a deformed commutator mapping repeatedly reduces a basis element of \( \mathcal{H}(q) \) into a power of \( C \). In an interesting flow of computations and proofs, we show that this can be generalized—any nonzero element of \( \mathcal{H}(q) \), after an application of a suitable number of particular types of deformed commutator mappings, reduces to a power of \( C \).

We start with a few properties in the following proposition and develop our main algorithms in the three lemmas that follow. These are actually our reformulation of [17, Lemma 5.7-6.4]. In fact, we give a reformulation, an exposition, and a generalization all at the same time. Our techniques as presented here involve more explicit reference to computations and structure constants, as contrasted by the ring-theoretic approach in [17].
Lemma 3.2. Let $b \in \mathbb{N}\setminus\{0\}$. For each nonzero $U \in \bigoplus_{i=1}^{b} \mathcal{S}_{i}$, there exists a composition $\theta$ of elements of $\Phi_{Z}$, and some nonzero $V \in \mathcal{S}_{0} = \mathbb{C}[C]$ such that $\theta(U) = V$.

Proof. We use induction on $b$. Suppose $b = 1$. Let $U$ be a nonzero element of $\mathcal{S}_{1}$. Then there exists a nonzero $\beta \in \mathbb{C}[C]$ such that $U = \beta B$. Let $k$ be the polynomial degree of $\beta$. Choose any positive integer $n > k + 1$. If $c_{k}$ is the leading coefficient of the polynomial $\beta$, then by (61), $\theta_{n}(\beta B) = \beta'$ for some $\beta' \in \mathbb{C}[C]$ with polynomial degree $k + 1$, and with leading coefficient $-q^{k+1}\{n - k - 1\}q^{c_{k}} = -q^{k+1-\frac{q^{n-(k+1)}}{1-q}}c_{k}$, which is nonzero by Assumption 1.8 and because $n > k + 1$.

Suppose that for all positive integers $c < b$, any nonzero element of $\bigoplus_{i=1}^{c} \mathcal{S}_{i}$ satisfies the statement. Let $U$ be a nonzero element of $\bigoplus_{i=1}^{b} \mathcal{S}_{i}$. Then there exists $U' \in \bigoplus_{i=1}^{b-1} \mathcal{S}_{i}$ and some $\beta \in \mathbb{C}[C]$ such that $U = U' + \beta B^{c}$. If $\beta = 0$, then we are done, and so we further assume $\beta \neq 0$. Let $k$ be the polynomial degree of $\beta$. By computations similar to those described for the case $b = 1$, we have $\theta_{n}(U) = \theta_{n}(U') + \beta' B^{c-1}$ for some $\beta' \in \mathbb{C}[C]$ with polynomial degree $k + 1$, and with leading coefficient $-q^{k+1}\{n - k - b\}q^{c_{k}} \neq 0$. Whenever $b \in \mathbb{N}\setminus\{0, 1\}$, the element $\theta_{n}(U')$ is in $\bigoplus_{i=1}^{b-2} \mathcal{S}_{i}$ by (61), and so no term in $\theta_{n}(U')$ can cancel the term $-q^{k+1}\{n - k - b\}q^{c_{k}}C^{k+1}B^{b-1}$ in $\beta' B^{b-1}$. Thus, $0 \neq \theta_{n}(U) \in \bigoplus_{i=1}^{b-1} \mathcal{S}_{i}$, and the element $\theta_{n}(U)$ satisfies the inductive hypothesis. That is, there exists a composition $\phi$ of elements of $\Phi_{Z}$, and some nonzero $V \in \mathcal{S}_{0} = \mathbb{C}[C]$ such that $\phi(\theta_{n}(U)) = V$. Take $\theta = \phi \circ \theta_{n}$. This completes the induction. □

Lemma 3.3. For any nonzero $U \in \mathbb{C}[C] = \mathcal{S}_{0}$, there exists a nonzero scalar $a$, some positive integer $N$ and a composition $\phi$ of elements of $\Phi_{Z}$ such that $\phi(U) = aC^{N}$.

Proof. For each nonzero $U \in \mathbb{C}[C]$, there exist numbers $k, n \in \mathbb{N}$ such that for some scalars $c_{k}, c_{k+1}, \ldots, c_{k+n}$, with $c_{k}$ and $c_{k+n}$ nonzero, we have

$$U = c_{k}C^{k} + c_{k+1}C^{k+1} + \cdots + c_{k+n}C^{k+n}. \quad (71)$$

That is, $n$ is the difference between the highest exponent of $C$ and the lowest exponent of $C$ that appear with nonzero scalar coefficient in $U$. We use induction on $n$. First, we consider the case $n = 0$. If $k = 0$, then take $\phi = \eta_{0} \circ \eta_{1}$, $a = q(q - 1)c_{0}$ and $N = 1$, while if $k \neq 0$, then we take $\phi$ as the empty composition or the identity map, $a$ as $c_{k}$ and $N$ as $k$, and we are done. Suppose that for some $n$, the statement holds for all nonzero elements of $\mathbb{C}[C]$ in the linear combination of which the difference $m$ between the highest exponent of $C$ and the lowest exponent is such that $m < n$. By routine computations that involve (61), (65), (66), there exist nonzero scalars $c_{i}$ with $i \in \{k + 1, \ldots, k + n\}$ such that

$$\left(\theta_{k+n+1} \circ \eta_{-k}\right)(U) = c_{k+1}c_{k+1}C^{k+1} + \cdots + c_{k+n-1}c_{k+n-1}C^{k+n-1} + (c_{k+n-1}c_{k+n} + c_{k+n}(1 - q^{n}))C^{k+n}. \quad (72)$$

The number of nonzero terms in the right-hand side of (72) is at most $k + n - (k + 1) = n - 1$. Thus, the inductive hypothesis applies to $V := \left(\theta_{k+n+1} \circ \eta_{-k}\right)(U)$, and so there exists
a nonzero scalar $a$, some positive integer $N$ and some composition $\phi_0$ of elements of $\Phi_Z$ such that $\phi_0(V) = aC^N$, and so

$$ (\phi_0 \circ \theta_{k+n+1} \circ \eta_{-k})(U) = \phi_0(V) = aC^N. $$

Take $\phi = \phi_0 \circ \theta_{k+n+1} \circ \eta_{-k}$, and this completes the induction. 

Our Lemmas 3.2 and 3.3 now culminate into the following result, which is our generalization of the rule (70), and also our reformulation and generalization of [17, Theorem 6.4].

**Corollary 3.4.** For any nonzero $U \in \mathcal{H}(q)$, there exists a composition $\psi$ of elements of $\Phi_Z$, some nonzero scalar $a$ and some positive integer $N$ such that $\psi(U) = aC^N$. 

**Proof.** Any nonzero $U \in \mathcal{H}(q)$ is a finite linear combination of the elements described in (18), and so there exist $a, b \in \mathbb{N}$ such that $U \in \bigoplus_{i=a}^{b} \mathfrak{H}_i$. By (62)–(64), there exists a composition $\eta$ of elements of $\Phi_Z$ such that $\eta(U)$ satisfies the hypotheses of Lemma 3.2, by which there exists a composition $\theta$ of elements of $\Phi_Z$ such that $(\theta \circ \eta)(U)$ satisfies the hypotheses of Lemma 3.3. Consequently, by Lemma 3.3, there exists a composition $\phi$ of elements of $\Phi_Z$ such that if $\psi := \phi \circ \theta \circ \eta$, and thus there exists a nonzero scalar $a$ and some positive integer $N$ such that $\psi(U) = aC^N$. 

As the reader may notice, our proof of Corollary 3.4 and our proof of the supporting lemmas constitute a logical flow that differs from the formulation in [17, Lemma 5.7-6.4]. First, we deal with the commutator mappings and basis elements, and eventually arbitrary nonzero elements, of $\mathcal{H}(q)$ without reference to any two-sided ideal. The proofs do not rely heavily on ring-theoretic notions and arguments such as divisibility in a ring. Our proofs are based on the structure constants and computational techniques as described in Section 2.1, which is based on the deeper discussion in [11]. Our only dilemma so far is that, since we are extending this reduction process into the bigger algebra $P$ in which $\mathcal{H}(q)$ is embedded, we have to take into account the effect of $K$ in the computations involving deformed commutator mappings. Thus, we have the following.

**Proposition 3.5.** For any $n \in \mathbb{Q}$ and any $U \in \mathcal{H}(q)$, we have

$$ \theta_n(KU) = q^{-\frac{1}{2}}K \theta_{n+\frac{1}{2}}(U), \quad (73) $$

$$ \eta_n(KU) = q^{\frac{1}{2}}K \eta_{n-\frac{1}{2}}(U). \quad (74) $$

**Proof.** By the linearity of deformed commutator mappings, letting $U$ be a basis element of $\mathcal{H}(q)$ among those listed in (18) will suffice. If $U = C^k$ for some $k \in \mathbb{N}$, we have that $\theta_n(K^k) = AKC^k - q^n KC^k A$, which by (52), becomes

$$ \theta_n(K^k) = q^{-\frac{1}{2}}K(AC^k - q^n C^k A) = q^{-\frac{1}{2}}K \theta_{n+\frac{1}{2}}(C^k). $$

The other cases for (73), which are $U = C^k A^l$ and the case $U = C^k B^l$ for some $k, l \in \mathbb{N}$ with $l \in \mathbb{N}\{0\}$, are proven similarly. The proof for the identity (74) involves similar routine computations. 

$\square$
Proposition 3.6. If \( \psi \) is (finite) composition of elements of \( \Phi \backslash \Phi _{\mathbb{Z}} \), then for any nonzero \( U \in \mathcal{H}(q) \),

\[ 0 \neq \psi (U) \in \mathcal{H}(q). \]

**Proof.** We use induction on the number \( t \) of deformed commutator mappings in the composition \( \psi \). Suppose \( t = 1 \), that is, for some non-integer \( m \in \mathbb{Q} \), either \( \psi = \eta _m \) or \( \psi = \theta _m \). Consider the case \( \psi = \eta _m \). Let \( U \) be any nonzero element of \( \mathcal{H}(q) \), and let \( a, b \in \mathbb{Z} \) such that \( U = V + W \) where \( V \in \bigoplus _{i=a}^{b-1} \mathcal{F}_i \) and \( W \in \mathcal{F}_b \). Without loss of generality, we assume \( W \neq 0 \). That is, \( b \) is the maximum index of a \( \mathbb{Z} \)-gradation subspace to which some terms in \( U \) belong. Consequently, there is a nonzero polynomial \( \gamma \in \mathbb{C}[C] \) such that \( W = \gamma B^b \) if \( b \in \mathbb{N} \), or \( W = \gamma A^{-b} \) if \( -b \in \mathbb{N} \setminus \{0\} \). In either case, according to (63), (64), \( \eta _m(W) \in \mathcal{F}_{b+1} \).

Let \( k \) be the polynomial degree of \( \gamma \) and let \( c_k \) be the leading coefficient. Using the identities (63), (64), we find that there exists a polynomial \( \gamma' \in \mathbb{C}[C] \) of degree either \( k \) or \( k + 1 \), with leading coefficient \( d = \frac{q^k}{(1-q)^k} (1 - q^{m+T})c_k \) (where \( k \) and \( T \) are integers, and \( i \) is either 0 or 1) such that \( \eta _m(W) = \gamma' B^{b+1} \) if \( b \in \mathbb{N} \), or \( \eta _m(W) = \gamma' A^{-b+1} \) if \( -b \in \mathbb{N} \setminus \{0\} \).

Since \( m + T \) is not an integer, \( m + T \) is not zero, and by Assumption 1.8, \( 1 - q^{m+T} \) is nonzero, and so is the leading coefficient \( d \) of \( \gamma' \). Then \( \eta _m(W) \neq 0 \). In the right-hand side of the equation \( \eta _m(U) = \eta _m(V) + \eta _m(W) \), none of the terms in \( \eta _m(V) \) can cancel the nonzero \( \eta _m(W) \) because, by (62) and (63), \( \eta _m(V) \in \bigoplus _{i=a+1}^{b} \mathcal{F}_i \), while \( \eta _m(W) \in \mathcal{F}_{b+1} \). Therefore, \( \eta _m(U) \neq 0 \). The case \( \psi = \theta _m \) is proven similarly. This completes the proof for \( t = 1 \).

Suppose the statement holds for any composition of \( n < t \) elements of \( \Phi \backslash \Phi _{\mathbb{Z}} \). We write \( \psi = \zeta \circ \phi \) for some \( \zeta \in \Phi \backslash \Phi _{\mathbb{Z}} \) and some composition \( \phi \) of \( t - 1 \) elements from \( \Phi \backslash \Phi _{\mathbb{Z}} \). By the inductive hypothesis, for any nonzero \( U \in \mathcal{H}(q) \), the element \( V := \phi (U) \) is nonzero. Then by the inductive hypothesis, \( 0 \neq \zeta (V) = \psi (U) \). Therefore, the statement holds for any value of \( t \).

**Theorem 3.7.** The algebra \( \mathcal{P}_q \) faithfully represents \( \mathcal{P} \).

**Proof.** Showing that \( \mathcal{P} \) is simple will suffice. To do this, we prove that, given a (two-sided) ideal \( \mathcal{T} \) of \( \mathcal{P} \), the following are equivalent.

(i) \( \mathcal{T} \neq 0 \).

(ii) \( C^M \in \mathcal{T} \) for some positive integer \( M \).

(iii) \( K \in \mathcal{T} \).

(iv) \( \mathcal{T} \) is a nonzero ideal.

The equivalence between (iii) and (iv) follows from the invertibility of \( K \). To see why the implication (ii) \( \Rightarrow \) (iii) is true, simply consider the fact that \( K = K^{1+2h}C^h \) since \( K^2C = 1 \) and \( K \) commutes with \( C \). Since \( K \) is a basis element of \( \mathcal{P} \), it is nonzero, and so we immediately have (iii) \( \Rightarrow \) (i). We now show (i) \( \Rightarrow \) (ii). Suppose \( \mathcal{T} \) is a nonzero
ideal of $\mathcal{P}$, and choose a nonzero $x \in \mathcal{T}$. If we write $x$ as a linear combination of the basis elements (56) of $\mathcal{P}$, then each basis element is of the form $K^{2h+i}U$ for some $h \in \mathbb{N}$, some $i \in \{0, 1\}$ and some basis element $U$ of $\mathcal{H}(q)$ from (18). Let $N$ be the maximum possible $h$ among the basis elements $K^{2h+i}U$ that appear in the linear combination. Since $K^2C = 1$ and $CK = KC$, by multiplying $C^N$ to the left of $x$, every term on the right-hand side becomes $C^NK^{2h+i}U = C^kK^iU = K^iC^kU$, for some $k \in \mathbb{N}$. Observe that $C^kU$ is also another basis element of $\mathcal{H}(q)$ from (18). Since $i$ is either 0 or 1, then

$$C^Nx = f + Kg, \quad (75)$$

for some $f, g \in \mathcal{H}(q)$. If $f = 0$, then we multiply both sides of (75) by $CK$, and we obtain $KC^{N+1}x = g$. Here, $g$ cannot be zero for this will contradict $x \neq 0$. Thus, we use Corollary 3.4 on $g$, and we are done. Suppose $f \neq 0$. If $g = 0$, then we can again make use of Corollary 3.4, and we are done for this case. Thus, we now assume that both $f$ and $g$ are nonzero. Using Corollary 3.4 on $f$, let $\psi$ be the composition of elements of $\Phi_Z$ such that, for some nonzero scalar $c_1$ and positive integer $H$, we have $\psi(f) = c_1C^H$. Thus, $\psi(C^Nx) = c_1C^H + \psi(Kg)$, and by (65),

$$\left(\theta_H \circ \psi\right)(C^Nx) = \left(\theta_H \circ \psi\right)(Kg). \quad (76)$$

Note here that $H \in \mathbb{Z}$ and so we can write $\left(\theta_H \circ \psi\right)$ as

$$\left(\theta_H \circ \psi\right) = \zeta_1 \circ \zeta_2 \circ \cdots \circ \zeta_r,$$

for some $r \in \mathbb{N}\setminus\{0\}$ and some $\zeta_1, \zeta_2, \ldots, \zeta_r \in \Phi_Z$. Consequently,

$$\left(\theta_H \circ \psi\right)(Kg) = \left(\zeta_1 \circ \zeta_2 \circ \cdots \circ \zeta_r\right)(Kg). \quad (77)$$

Using the identities in Proposition 3.5 repeatedly on the right-hand side of (77), there exists $N \in \mathbb{Q}$ and some $\xi_1, \xi_2, \ldots, \xi_r \in \Phi \setminus \Phi_Z$ such that

$$\left(\theta_H \circ \psi\right)(Kg) = q^NK \left(\xi_1 \circ \xi_2 \circ \cdots \circ \xi_r\right)(g). \quad (78)$$

Since $g \neq 0$, by Proposition 3.6, $g' := (\xi_1 \circ \xi_2 \circ \cdots \circ \xi_r)(g)$ is a nonzero element of $\mathcal{H}(q)$. Multiplying both sides of (78) by $q^{-N}KC$, we have $q^{-N}KC(\theta_H \circ \psi)(Kg) = g'$, and so (76) becomes

$$q^{-N}KC(\theta_H \circ \psi)(C^Nx) = g'. \quad (79)$$

We apply Corollary 3.4 to the nonzero element $g'$ of $\mathcal{H}(q)$, and so there exists a composition $\varphi$ of elements of $\Phi_Z$, some nonzero scalar $c_2$, and some positive integer $M$ such that (79) becomes

$$c_2^{-1}\varphi(q^{-N}KC(\theta_H \circ \psi)(C^Nx)) = C^M. \quad (80)$$

Since $x$ is an element of the ideal $\mathcal{T}$, which is invariant under deformed commutator mappings, the left-hand side of (80) is an element of $\mathcal{T}$. Therefore, $C^M \in \mathcal{T}$. \qed
By Proposition 1.3, we immediately have:

**Corollary 3.8.** The subalgebra of $\mathcal{P}$ generated by $A, B$ is isomorphic to $\mathcal{H}(q)$.

Hence, we identify $a$ as $A$ and $a^+$ as $B$. All the properties of $\mathcal{H}(q)$ are then satisfied in the subalgebra of $\mathcal{P}$ generated by $A, B$.

**Remark 3.9.** We rewrite in here the relations (13) to (15) in terms of $A, B, C$:

\[
[A, B]^{k+2} = \frac{-q^k(1 - q)}{1 - q^{k+1}} \sum_{i=0}^{k} \left( (\text{ad } B) \circ (-\text{ad } [A, B])^k \circ (\text{ad } A) \right) (C),
\]

(81)

\[
[A, B]^{k+1} A^l = -\frac{(-\text{ad } [A, B])^k \circ (-\text{ad } A)^{l+1}}{(1 - q)^l (q^l - 1)^k} (B),
\]

(82)

\[
[A, B]^{k+1} B^l = q^{l(k+1)} \frac{(\text{ad } B)^{l-1} \circ (\text{ad } [A, B])^{k+1}}{(q - 1)^{k+1} (1 - q^{k+1})^{l-1} l},
\]

(83)

where $k \in \mathbb{N}$ and $l \in \mathbb{N}\{0\}$. The relation (83) is (15) with the adjustment from $B^l C^k$ to $C^k B^l$ as discussed in Section 2. All such relations (81) to (83) hold in $\mathcal{P}$. Additionally, because of the embedding of $\mathcal{H}(q)$ in $\mathcal{P}$ as given in Corollary 3.8, by Remark 1.5, the elements

\[
C^k, \quad (k \in \mathbb{N}\{0\}),
\]

(84)

\[
A, \quad C^k A^l, \quad (k, l \in \mathbb{N}\{0\}),
\]

(85)

\[
B, \quad C^k B^l, \quad (k, l \in \mathbb{N}\{0\}),
\]

(86)

form a basis for the Lie subalgebra of $\mathcal{P}$ generated by $A, B$ (or by $A, B, C$), while any (finite) linear combination or

\[
1, \quad A^l, \quad B^l, \quad (l \in \mathbb{N}\{0, 1\}),
\]

(87)

is not a Lie polynomial in in $A, B$ (nor in $A, B, C$).

### 4 Commutator algebra for the generators $K, A, B, C$ of $\mathcal{P}$

In this section we find a description for the commutator algebra or Lie subalgebra of $\mathcal{P}$ generated by $K, A, B, C$. In the next two lemmas, we exhibit some important properties of the basis elements (56).

**Lemma 4.1.** Every basis element in (56), except the multiplicative identity $1$, is a Lie polynomial in $K, A, B, C$. 
Proof. For convenience in later computations, we use the notation
\[ p := q^{\frac{1}{2}}. \]  
(88)

Thus, the defining relations of \( \mathcal{P} \) become
\[
AB = \frac{1 - p^2C}{1 - p^2}, \tag{89}
\]
\[
BA = \frac{1 - C}{1 - p^2}, \tag{90}
\]
\[
AC = p^2CA, \tag{91}
\]
\[
BC = p^{-2}CB, \tag{92}
\]
\[
AK = p^{-1}KA, \tag{93}
\]
\[
BK = pKB, \tag{94}
\]
\[
CK = KC, \tag{95}
\]
\[
K^2C = 1. \tag{96}
\]

By some routine computations and induction, the relations (91) to (94) can be generalized into:
\[
A^lK^h = p^{-hl}K^hA^l, \tag{97}
\]
\[
B^lK^h = p^{hl}K^hB^l, \tag{98}
\]
\[
A^lC^k = p^{2kl}C^kA^l, \tag{99}
\]
\[
B^lC^k = p^{-2kl}C^kB^l, \tag{100}
\]

which hold for any \( h, k, l \in \mathbb{N} \). Our strategy is to segregate the basis elements \( U \neq 1 \) among (56) into the following:
\[
C^kA^l, \ C^kB^l, \ C^k, \quad (k, l \in \mathbb{N}\{0\}), \tag{101}
\]
\[
KC^kA^l, \ KC^kB^l, \quad (k, l \in \mathbb{N}\{0\}), \tag{102}
\]
\[
K^hA^l, \ K^hB^l, \ K^l, \ KC^l, \quad (h \in \mathbb{N}, \ l \in \mathbb{N}\{0\}). \tag{103}
\]

By (13) to (15), the basis elements (101) are all elements of \( \mathcal{L}(\mathcal{P}) \). Using the reordering formulas (97) and (98), and the commutativity of \( K \) with \( C \) in some routine computations, we have the relations
\[
KC^kA^l = \frac{p^l}{1 - p^l} \left[ C^kA^l, K \right], \tag{104}
\]
\[
KC^kB^l = \frac{1}{1 - p^l} \left[ K, C^kB^l \right], \tag{105}
\]

which hold for any \( k, l \in \mathbb{N}\{0\} \). But since \( C^kA^l, K, C^kB^l \in \mathcal{L}(\mathcal{P}) \) for any \( k, l \), the relations (104) and (105) imply that the basis elements (102) are all in \( \mathcal{L}(\mathcal{P}) \). We now
sketch the computations necessary to prove that basis elements (103) are in \( \mathcal{L}(\mathcal{P}) \). First, we consider those of the form \( K^h A^l, K^h B^l \) for the case \( h \in \mathbb{N} \setminus \{0\} \). Using induction and the reordering formulas (97) and (98), the relations
\[
K^h A^l = \frac{p^{hl}}{(1 - p)(1 - p^h)^{h-1}} \left( (\text{ad} K)^{h-1} \circ (\text{ad} A)^l \right) (K), \quad (h, l \in \mathbb{N} \setminus \{0\}), \tag{106}
\]
\[
K^h B^l = \frac{1}{(1 - p)(1 - p^h)^{h-1}} \left( (\text{ad} K)^{h-1} \circ (-\text{ad} A)^l \right) (K), \quad (h, l \in \mathbb{N} \setminus \{0\}), \tag{107}
\]
hold in \( \mathcal{P} \). In view of Remark 1.6, the relations (106), (107) imply that \( K^h A^l, K^h B^l \in \mathcal{L}(\mathcal{P}) \) whenever \( h, l \in \mathbb{N} \setminus \{0\} \).

We are missing the case \( h = 0 \) in the relations (106) and (107), and so we turn our attention to some other types of Lie polynomials in \( K, A, B, C \) on which we can use the reordering formulas (97) to (100) and also the defining relations of \( \mathcal{P} \) in order to show that \( A^l, B^l \in \mathcal{L}(\mathcal{P}) \). Using the reordering formulas (97) and (98), we have
\[
p^{l+1}[KB, KA^{l+1}] = p^{l+2}K^2(BA)A^l - K^2A^l(AB). \tag{108}
\]
Using (89), (90), we replace \( BA \) and \( AB \) in the right-hand side of (108). This gives us
\[
p^{l+1}(1 - p^2)[KB, KA^{l+1}] = K^2(p^{l+2}(1 - C)A^l - A^l(1 - p^2C)). \tag{109}
\]
By further manipulations that also make use of the relation (96), we turn (109) into
\[
A^l = \frac{1 - p^2}{p(1 - p)}[KA^{l+1}, KB] = \frac{1 - p^{l+2}}{p^{l+2}(1 - p)} K^2 A^l, \quad (l \in \mathbb{N} \setminus \{0\}). \tag{110}
\]
We have established previously that \( KA^{l+1}, KB, K^2A^l \in \mathcal{L}(\mathcal{P}) \), and so (110) asserts that \( A^l \in \mathcal{L}(\mathcal{P}) \). By a computational pattern similar to that done in (108) to (110), the relations
\[
B^l = \frac{1 - p^2}{1 - p} p^{l-1}[KA, KB^{l+1}] - \frac{1 - p^{l+2}}{1 - p} p^{l-2} K^2 B^l, \quad (l \in \mathbb{N} \setminus \{0\}), \tag{111}
\]
\[
K^{h+2} = \frac{1 - p^2}{1 - p^{h+2}}[K^{h+2}A, B] + \frac{1 - p^h}{1 - p^{h+2}} p^2 K^h, \quad (h \in \mathbb{N}), \tag{112}
\]
\[
KC^l = \frac{1 - p^2}{1 - p^2 l_p^{2l-2}}[KC^{l-1}A, B] + \frac{1 - p^{2l-2}}{1 - p^2l} K C^{l-1}, \quad (l \in \mathbb{N} \setminus \{0\}), \tag{113}
\]
can be shown to hold in \( \mathcal{P} \) by routine calculations. As established earlier, \( KA, KB^{l+1}, K^2B^l \in \mathcal{L}(\mathcal{P}) \). Then by (111), \( B^l \in \mathcal{L}(\mathcal{P}) \).

So far, the relations (106), (107), (110), (111) assert that all basis elements in (103) of the form \( K^h A^l, K^h B^l \) (with \( h \in \mathbb{N}, l \in \mathbb{N} \setminus \{0\} \)) are in \( \mathcal{L}(\mathcal{P}) \). For those of the form \( K^l \), we use (112). If \( h = 0 \) in (112), then we find a relation asserting that \( K^2 \) is a scalar multiple of \([K^2A, B]\), where \( K^2A \in \mathcal{L}(\mathcal{P}) \) as previously established. Thus, \( K^2 \in \mathcal{L}(\mathcal{P}) \). We can use the condition \( K, K^2 \in \mathcal{L}(\mathcal{P}) \) and (112) in an inductive argument which proves that \( K^l \in \mathcal{L}(\mathcal{P}) \) for all \( l \in \mathbb{N} \setminus \{0\} \). By a similar argument, (113) can be used to show that \( KC^l \in \mathcal{L}(\mathcal{P}) \) for any \( l \in \mathbb{N} \setminus \{0\} \). This completes all the cases needed in the proof.
Lemma 4.2. If \( U, V \) are any two basis elements from (56) neither of which is the multiplicative identity 1, and if \([U, V] = \sum_{i=1}^{t} c_i W_i\) for some nonzero scalars \( c_i \) and some basis elements \( W_i \) from (56), then \( W_i \neq 1 \) for any \( t \).

Proof. Since \([U, V] = 0\) if one of \( U \), \( V \) is 1, we further assume that \( U \) and \( V \) are of the form \( K^h C^k B^l \) or \( K^h C^k A^l \) for some \( h, k \in \mathbb{N} \) and some \( l \in \mathbb{N} \setminus \{0\} \). If \( U, V \in \mathcal{H}(q) \) (i.e., if \( h = 0 \)), then by Remark 3.9, we find that it is not possible for \( W_i \) to be 1 for any \( t \). Thus, we further assume the exponent of \( K \) in at least one of \( U \), \( V \) is positive.

It is routine to show, using (97), (98), that under all such remaining cases, either \([U, V] = K^N [U', V']\) or \([U, V] = K^N W'\) for some positive integer \( N \), where \( U', V', W' \) are among (84) to (86). If \([U, V] = K^N [U', V']\), then by Remark 3.9, \([U', V']\) is also one of (84) to (86), but \( K^N \) multiplied by any one of (84) to (86) is not 1. Neither is \( K^N W' \) for the last remaining case.

Theorem 4.3. The Lie subalgebra \( \mathcal{L}(\mathcal{P}) \) of \( \mathcal{P} \) generated by \( K, A, B, C \) is precisely the span of all the basis elements (56) except the multiplicative identity 1. i.e., The algebra \( \mathcal{P} \) can be decomposed as the direct sum

\[ \mathcal{P} = \mathbb{C}1 \oplus \mathcal{L}(\mathcal{P}). \] (114)

Proof. Let \( \mathcal{V} \) be the span of all basis elements (56) except 1. Thus, \( \mathcal{V} \) contains the generators \( K, A, B, C \) of \( \mathcal{L}(\mathcal{P}) \). Lemma 4.2 asserts that \( \mathcal{V} \) is a Lie subalgebra of \( \mathcal{P} \), while Lemma 4.1 implies \( \mathcal{V} \subseteq \mathcal{L}(\mathcal{P}) \). By the minimality of \( \mathcal{L}(\mathcal{P}) \) among all Lie subalgebras of \( \mathcal{P} \) that contain \( K, A, B, C \), we have \( \mathcal{V} = \mathcal{L}(\mathcal{P}) \).

5 On a \( q \)-oscillator representation of the Askey-Wilson algebra

We now discuss how our algebraic results are related to a Lie algebra structure in some representation of an Askey-Wilson algebra. In particular, we shall consider the following.

Definition 5.1. The Askey-Wilson algebra, denoted by \( AW(3) \), is the unital associative algebra over \( \mathbb{C} \) with generators \( K_0, K_1, K_2 \) and relations

\[ e^\omega K_0 K_1 - e^{-\omega} K_1 K_0 = K_2, \] (115)
\[ e^\omega K_2 K_0 - e^{-\omega} K_0 K_2 = b K_0 + c_1 K_1 + d_1, \] (116)
\[ e^\omega K_1 K_2 - e^{-\omega} K_2 K_1 = b K_1 + c_0 K_0 + d_0, \] (117)

where \( \omega \in \mathbb{R} \setminus \{0\} \). The scalars \( b, c_0, c_1, d_0, d_1 \in \mathbb{C} \) are called the structure constants of \( AW(3) \).

According to [29, pp. 1155–1156], if the conditions

\[ b = c_1 = d_0 = d_1 = 0, \quad c_0 = e^{2\omega} - 1, \] (118)
are imposed on the structure constants of $AW(3)$, then the algebra $AW(3)$ has a $q$-oscillator representation given by
\begin{align}
K_0 &= e^{\omega N}, \\
K_1 &= a + a^+.
\end{align}

In this representation, $K_1$ is said to be the $\omega$-analogue of the position operator.

**Definition 5.2.** Define $AW_q(3)$ as the subalgebra of $P_q$ generated by the representations of $K_0$ and $K_1$ given in (119),(120), with $q = e^{-2\omega}$.

Thus, the choice here of $q$ is nonzero and not a root of unity, and the results of the previous sections apply. Recall our notation $p = q^{1/2}$. By $q = e^{-2\omega}$ and (118), the relations (115)-(117) can be rewritten as
\begin{align}
 p^{-1}K_0K_1 - pK_1K_0 &= K_2, \\
 p^{-1}K_2K_0 - pK_0K_2 &= 0, \\
 p^{-1}K_1K_2 - pK_2K_1 &= \frac{1-p^2}{p^2}K_0.
\end{align}

By (121), the algebra $AW_q(3)$ contains the representation of $K_2 \in AW(3)$ as an operator on the $q$-oscillator. By Theorem 3.7, we can represent elements of $AW_q(3)$ in terms of the generators $K, A, B, C$ of $P$. Thus, (119), (120) become
\begin{align}
K_0 &= K, \\
K_1 &= A + B.
\end{align}

As for the generator $K_2$, we use (121) and the reordering formulas (52), (53). By some routine calculations, we have
\begin{align}
K_2 &= p^{-1}(1-p)KA + p^{-1}(1-p^3)KB.
\end{align}

Our next goal is to determine the relationship between the algebras $P$ and $AW_q(3)$, and as shall be seen in the computations and proofs that we will present, the Lie polynomials in the generators of these algebras play an important role.

**Lemma 5.3.** The Lie subalgebra $L(AW_q(3))$ of $AW_q(3)$ generated by $K_0, K_1, K_2$ is a Lie subalgebra of $L(P)$.

**Proof.** Using (93), (94), and by some routine calculations, we obtain
\begin{align}
KA &= \frac{p[A,K]}{1-p}, \\
KB &= \frac{[K,B]}{1-p}.
\end{align}
From (122)-(124), we have enough information to express \( K_2 \) as a Lie polynomial in \( K, A, B \) by routine computations. To summarize all relevant equations, we have

\[
\begin{align*}
K_0 &= K, \\
K_1 &= A + B, \\
K_2 &= [A - p^{-1}(1 + p + p^2)B, K].
\end{align*}
\]

An immediate consequence of (125)-(127) is that every Lie polynomial in \( K_0, K_1, K_2 \) is also a Lie polynomial in \( K, A, B \), which are enough to generate \( \mathcal{L}(\mathcal{P}) \) since \( C = [A, B] \). \( \square \)

The most natural continuation is to consider the converse: is it possible to write any Lie polynomial in the generators of \( \mathcal{P} \) as a Lie polynomial in the generators of \( AW_q(3) \)? We now show that we have an answer in the affirmative.

**Lemma 5.4.** The generators \( K, A, B \) of the Lie algebra \( \mathcal{L}(\mathcal{P}) \) are Lie polynomials in \( K_0, K_1, K_2 \).

**Proof.** Since \( K_0 = K \), we only need to show \( A, B \) are Lie polynomials in \( K_0, K_1, K_2 \). We start with evaluating \([K_0, K_1]\) using (125),(126) and the reordering formulas (52),(53). The result is a linear combination of \( KA \) and \( KB \), and so we can use (122) to solve for \( KA \) and \( KB \). From these routine computations, we have

\[
\begin{align*}
KA &= \frac{p^2K_2 + p(1+p+p^2) [K_1, K_0]}{(1-p)(1+p)^2}, \\
KB &= \frac{pK_2 + p [K_0, K_1]}{(1-p)(1+p)^2}.
\end{align*}
\]

The right-hand sides of (128),(129) are Lie polynomials in \( K_0, K_1, K_2 \), and so we have \( KA, KB \in \mathcal{L}(AW_q(3)) \). Our next step is to compute the commutator of these new Lie polynomials \( KA, KB \) with \( K_1 = A + B \). Using routine computations that make use of the reordering formulas (89),(90),(93),(94), we obtain

\[
\begin{align*}
KA^2 - \frac{p^2}{1-p^2}KC &= \frac{p}{1-p} [K_1, KA] + \frac{p}{1-p^2}K_0, \\
KB^2 - \frac{1}{1-p^2}KC &= \frac{[KB, K_1]}{1-p} + \frac{K_0}{p(1-p^2)}.
\end{align*}
\]

We also compute the commutator of \( K_1 = A + B \) with \( K_2 \) as given in (122). Moreover, using (89),(90),(93),(94), we obtain, by routine calculations, the relation

\[
KA^2 - p(1+p+p^2)KB^2 + p\frac{1+p^2}{1-p^2}KC = \frac{p^2}{(1-p)^2} [K_1, K_2] - \frac{1+p^2}{1-p^2}K_0.
\]

Because \( KA, KB \) are Lie polynomials in \( K_0, K_1, K_2 \), then so are the right-hand sides of (130)-(132). Denote these elements of \( \mathcal{L}(AW_q(3)) \) by \( X, Y, Z \), respectively. We can
eliminate $KA^2$ and $KB^2$ from (132) using (130),(131). By routine computations, we have the relation

$$KC = \frac{1+p}{p}X - (1+p)(1+p+p^2)Y - \frac{1+p}{p}Z \in \mathcal{L}(AW_q(3)).$$

We have thus shown that $KA$, $KB$, $KC$ are Lie polynomials in $K, K_0, K_1, K_2$. The significance of this is that by some routine computations that involve the reordering formulas (89)-(96), we have

$$A = \frac{[KC,KA]}{1-p},$$

$$B = \frac{p[KB,KC]}{1-p}. \quad (134)$$

Therefore, $A, B \in AW_q(3)$ as desired.

**Theorem 5.5.** The algebras $AW_q(3)$ and $\mathcal{P}$ are equal, and so are the corresponding Lie algebras $\mathcal{L}(AW_q(3))$ and $\mathcal{L}(\mathcal{P})$.

**Proof.** By Lemma 5.4, the generators $K, A, B$ and even $C = [A, B]$ of $\mathcal{L}(\mathcal{P})$ are in $\mathcal{L}(AW_q(3))$, and so we have the inclusion $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{L}(AW_q(3))$. Using Lemma 5.3, we further have $\mathcal{L}(\mathcal{P}) = \mathcal{L}(AW_q(3))$. By Lemma 4.1 each basis element $U \neq 1$ of $\mathcal{P}$ from (56) is a Lie polynomial in $K, A, B, C$, and so according to Lemma 5.4, $U \in \mathcal{L}(AW_q(3)) \subseteq AW_q(3)$. The remaining basis element $1 = K^2C$ is also in $AW_q(3)$ since $K, C \in \mathcal{L}(AW_q(3)) \subseteq AW_q(3)$. Therefore, $\mathcal{P}$ is contained in its subalgebra $AW_q(3)$, and is hence equal to $AW_q(3)$. 

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