Generalized perturbations in neutrino mixing

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Abstract

We derive expressions for the neutrino mixing parameters that result from complex perturbations on (1) the Majorana neutrino mass matrix (in the basis of charged lepton mass eigenstates) and on (2) the charged lepton mass matrix, for arbitrary initial (unperturbed) mixing matrices. In the first case, we find that the phases of the elements of the perturbation matrix, and the initial values of the Dirac and Majorana phases, strongly impact the leading-order corrections to the neutrino mixing parameters and phases. For experimentally compatible scenarios wherein the initial neutrino mass matrix has $\mu - \tau$ symmetry, we find that the Dirac phase can take any value under small perturbations. Similarly, in the second case, perturbations to the charged lepton mass matrix can generate large corrections to the mixing angles and phases of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. As an illustration of our generalized procedure, we apply it to a situation in which nonstandard scalar and nonstandard vector interactions simultaneously affect neutrino oscillations.
1 Introduction

After decades of neutrino oscillation experiments, the mixing pattern in the lepton sector has been well established [1]. There are one small and two large mixing angles, and two mass-squared differences that differ by a factor of 30 in the neutrino sector. Numerous neutrino mixing scenarios have been proposed in the literature to explain such a nontrivial mixing pattern; for a recent review, see Ref. [2]. The most attractive scenarios are those with mixing patterns motivated by simple symmetries, such as tri-bimaximal mixing [3], bimaximal mixing [4], and golden ratio mixing [5]. All three mixing scenarios have $\theta_{13} = 0$, $\theta_{23} = 45^\circ$, and are a subset of the more general $\mu - \tau$ symmetry [6, 7]. However, recent measurements of $\theta_{13}$ by short-baseline reactor experiments Daya Bay [8], RENO [9], Double Chooz [10], and long-baseline accelerator experiments T2K [11], MINOS [12] strongly disfavor $\theta_{13} = 0$. Therefore, models with simple symmetries need additional features to explain the observed neutrino mixing pattern.

A modified approach to explain the data is to treat the simple mixing scenarios as the underlying model and add perturbations to accommodate the discrepancy between theoretical predictions and experimental data. In Ref. [7], we took $\mu - \tau$ symmetry (in the charged lepton basis), as the underlying model, and added real perturbations to the Majorana neutrino mass matrices to explain the data. We found that small perturbations can cause large corrections to $\theta_{12}$, and the experimental data can be explained by most $\mu - \tau$ symmetric mixing scenarios with perturbations of similar magnitude.

After the discovery that the mixing angle $\theta_{13}$ is relatively large, many scenarios without $\mu - \tau$ symmetry have been proposed [13]. Motivated by this, and by the fact that real perturbations have no effect on the Dirac and Majorana phases, in this paper we consider arbitrary initial mixing for the underlying model and generalize the perturbation results to the complex space. Under the assumption that the charged lepton mass matrix is diagonal, we derive analytic formulas for the leading-order (LO) corrections to the three mixing angles and the Dirac and Majorana phases. We find that the phases of the elements of the perturbation matrix, and the initial values of the Dirac and Majorana phases, strongly impact the LO corrections to the neutrino mixing parameters. We also perform a numerical study of
complex perturbations on initial neutrino mass matrices with $\mu - \tau$ symmetry. We find that the Dirac phase can take any value under small perturbations for experimentally compatible scenarios.

Since the mixings in both the charged lepton sector and neutrino sector contribute to the observed Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, we explore the case in which the charged lepton mass matrix is not diagonal, and consider small complex perturbations to the charged lepton mass matrix as well. Since the initial mixing matrix in the charged lepton sector is unconstrained, small perturbations in the charged lepton sector could have large effects on the 1-2 mixing in the charged lepton sector, which lead to large changes in the mixing angles and phases in the PMNS matrix.

In addition, as an application of our generalized perturbation procedure, we study neutrino oscillations with matter effects from both nonstandard scalar and nonstandard vector interactions. Nonstandard scalar interactions add small perturbations to the neutrino mass matrix, which yield corrections to the vacuum mixing angles and mass-squared differences. By using our formalism, we demonstrate how expressions for neutrino oscillation probabilities that simultaneously depend on nonstandard scalar and nonstandard vector interactions, can be obtained.

This paper is organized as follows. In Section 2, we work in the diagonal charged lepton basis and derive analytic formulas for the mixing parameters that result from real and complex perturbations. In Section 3, we perform a numerical analysis of complex perturbations to neutrino mass matrices with $\mu - \tau$ symmetry. In Section 4, we calculate corrections to the three mixing angles and phases in the PMNS matrix from perturbations in the charged lepton sector. In Section 5, we apply our perturbation results to study neutrino oscillations with both nonstandard scalar and nonstandard vector interactions. We summarize our results in Section 6.

2 Perturbations on the neutrino mass matrix

Our goal in this section is to obtain the LO corrections to all the physical parameters under small perturbations on the initial Majorana neutrino mass matrix assuming the charged
lepton mass matrix to be diagonal. The final (resultant) mass matrix can be written as the sum of an initial matrix $M_0$ and a perturbation matrix $E$, i.e.,

$$M = M_0 + E = U_0^* M_0 U_0^\dagger + \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix}, \quad (1)$$

where $M_0 = \text{diag}(m_1^0, m_2^0, m_3^0)$, and $U_0$ is the initial mixing matrix.

The final mass matrix can also be written as

$$M = U^* M U^\dagger, \quad (2)$$

where $U$ and $\overline{M}$ have the same form as $U_0$ and $M_0$. From neutrino oscillation experiments we know that $m_1$ and $m_2$ are nearly degenerate, so here we assume $|\epsilon_{ij}|, |\delta m^0_{21}| \ll |\delta m^0_{31}|$.

### 2.1 Real case

For simplicity, we consider the real case first. The mixing matrix $U_0$ can be written as

$$U_0 = R_{13}^0 R_{12}^0,$$ \quad (3)

where $R_{ij}^0$ is the rotation matrix in the $i - j$ plane with a rotation angle $\theta_{ij}^0$. Then Eq. (1) can be rewritten as

$$M = m_1^0 I + R_{23}^0 R_{13}^0 R_{12}^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m^0_{21} & 0 \\ 0 & 0 & \delta m^0_{31} \end{pmatrix} (R_{12}^0)^T (R_{13}^0)^T (R_{23}^0)^T + E$$

$$= m_1^0 I + R_{23}^0 R_{13}^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m^0_{21} & 0 \\ 0 & 0 & \delta m^0_{31} \end{pmatrix} (R_{12}^0)^T + E',$$

where $E' = (R_{13}^0)^T (R_{23}^0)^T E$. \quad (4)
where $E' = (R_{13}^0)^T (R_{23}^0)^T E R_{23}^0 R_{13}^0$, $\delta m_{ij}^0 = m_j^0 - m_i^0$, and $I$ is the $3 \times 3$ identity matrix. We employ the following notation:

$$
\epsilon_1 = \epsilon_{11}, \quad \epsilon_2 = \epsilon_{12} c_{23}^0 - \epsilon_{13} s_{23}^0, \quad \epsilon_3 = \epsilon_{12} s_{23}^0 + \epsilon_{13} c_{23}^0,
$$

$$
\epsilon_4 = \epsilon_{22} (c_{23}^0)^2 + \epsilon_{33} (s_{23}^0)^2 - 23 s_{2x23}^0,
$$

$$
\epsilon_5 = \epsilon_{23} c_{2x23}^0 + \frac{1}{2} (\epsilon_{22} - \epsilon_{33}) s_{2x23}^0,
$$

$$
\epsilon_6 = \epsilon_{22} (s_{23}^0)^2 + \epsilon_{33} (c_{23}^0)^2 + 23 s_{2x23}^0, \quad (5)
$$

where $c_{ij}^0$, $s_{ij}^0$ and $s_{2xij}^0$ denote $\cos \theta_{ij}^0$, $\sin \theta_{ij}^0$, $\cos(2\theta_{ij}^0)$ and $\sin(2\theta_{ij}^0)$, respectively. Then $\epsilon_{ij}' \equiv (E')_{ij}$ can be written explicitly as

$$
\epsilon_{11}' = \epsilon_1 (c_{13}^0)^2 + \epsilon_6 (s_{13}^0)^2 - 3 s_{2x13}^0, \quad \epsilon_{12}' = \epsilon_2 c_{13}^0 - \epsilon_5 s_{13}^0,
$$

$$
\epsilon_{13}' = \epsilon_3 c_{2x13}^0 + \frac{1}{2} (\epsilon_1 - \epsilon_6) s_{2x13}^0, \quad \epsilon_{22}' = \epsilon_4,
$$

$$
\epsilon_{23}' = \epsilon_2 s_{13}^0 + \epsilon_5 c_{13}^0, \quad \epsilon_{33}' = \epsilon_1 (s_{13}^0)^2 + \epsilon_6 (c_{13}^0)^2 + 3 s_{2x13}^0. \quad (6)
$$

In order to obtain the final mixing matrix that diagonalizes $M$, we use a procedure that is similar to that in Ref. [14]. We first put zeros in the 2-3 and 1-3 entries of the matrix in the square bracket of Eq. (4) by using rotations $R_{23} (\delta_{23}^0)$ and $R_{13} (\delta_{13}^0)$, respectively. To LO in $O(|\epsilon_{ij}|/|\delta m_{31}^0|)$, we have

$$
\delta_{23}^0 \approx \frac{\epsilon_{23}^0}{\delta m_{31}^0}, \quad \delta_{13}^0 \approx \frac{\epsilon_{13}^0}{\delta m_{31}^0}, \quad (7)
$$

and the LO correction to $m_3^0$ is

$$
\delta m_3 = \epsilon_{33}^0. \quad (8)
$$

Note that since $|\epsilon_{ij}| \ll |\delta m_{31}^0|$, after the two rotations in the 2-3 and 1-3 planes, the matrix in the square bracket of Eq. (4) becomes block diagonal and the 1-2 submatrix remains unchanged to leading order. Hence we can rewrite Eq. (4) as

$$
M = m_1^0 I + V \begin{pmatrix} M' & 0 \\ 0 & \delta m_{31}^0 \end{pmatrix} V^T + O(|\epsilon_{ij}|^2/|\delta m_{31}^0|), \quad (9)
$$

where $V = R_{23}^0 R_{13}^0 R_{23} (\delta_{23}^0) R_{13} (\delta_{13}^0) R_{12}^0$, and

$$
M' = \begin{pmatrix}
\epsilon_{11}' (c_{12}^0)^2 + \epsilon_{12}' (s_{12}^0)^2 - \epsilon_{13}' s_{2x12}^0 & \epsilon_{12}' c_{2x12}^0 + \frac{1}{2} (\epsilon_{11}' - \epsilon_{12}') s_{2x12}^0 \\
\epsilon_{12}' c_{2x12}^0 + \frac{1}{2} (\epsilon_{11}' - \epsilon_{12}') s_{2x12}^0 & \epsilon_{11}' (s_{12}^0)^2 + \epsilon_{12}' (c_{12}^0)^2 + \epsilon_{12}' s_{2x12}^0 + \delta m_{21}^0
\end{pmatrix} \quad (10)
$$
which can be diagonalized by the rotation $R_{12}(\xi')$ with

$$\xi' = \frac{1}{2} \arctan \frac{2\epsilon'_{12}c_{21}^0 - (\epsilon'_{22} - \epsilon'_{11})s_{21}^0}{(\epsilon'_{22} - \epsilon'_{11})c_{21}^0 + 2\epsilon'_{12}s_{21}^0 + \delta m_{21}^0}.$$

(11)

The corrections to $m_1$ and $m_2$ can be written as

$$\delta m_i = \frac{\epsilon'_{11} + \epsilon'_{22}}{2} \pm \frac{1}{2} \left[ \delta m_{21}^0 - \sqrt{\Delta} \right],$$

(12)

where $\Delta = (\delta m_{21}^0)^2 + 4(\epsilon'_{12})^2 + (\epsilon'_{22} - \epsilon'_{11})^2 + 2\delta m_{21}^0 \left[ 2\epsilon'_{12}s_{21}^0 + (\epsilon'_{22} - \epsilon'_{11})c_{21}^0 \right]$, and the plus (minus) sign is for $i = 1$ (2). The final mass matrix is diagonalized by the following mixing matrix

$$U = R_{23}^0 R_{13}^0 R_{23}(\delta'_{23}) R_{13}(\delta'_{13}) R_{12}^0 R_{12}(\xi').$$

(13)

By comparing it to the standard parametrization, we find the LO corrections to the three mixing angles to be

$$\delta\theta_{13} = \delta'_{13} = \frac{\epsilon'_{13}}{\delta m_{31}^0},$$

$$\delta\theta_{23} = \delta'_{23} = \frac{\epsilon'_{23}}{\epsilon_{13}^0 \delta m_{31}^0},$$

$$\delta\theta_{12} = \xi' = \frac{1}{2} \arctan \frac{2\epsilon'_{12}c_{21}^0 - (\epsilon'_{22} - \epsilon'_{11})s_{21}^0}{(\epsilon'_{22} - \epsilon'_{11})c_{21}^0 + 2\epsilon'_{12}s_{21}^0 + \delta m_{21}^0},$$

(14)

where we have ignored the next-to-leading-order correction to $\theta_{12}$, which is $\mathcal{O}(|\epsilon_{ij}|/|\delta m_{31}^0|)$.

For $\theta_{13}^0 = 0$ and $\theta_{23}^0 = \pi/4$, it is easy to verify that the corrections in Eq. (14) yield the results of Ref. [7] for the LO corrections, which were obtained using degenerate perturbation theory. As noted in Ref. [7], the near degeneracy of $m_1$ and $m_2$ ($|\delta m_{21}^0| \ll |\delta m_{31}^0|$) implies that $\delta\theta_{12}$ can be large for small perturbations ($|\epsilon_{ij}| \ll |\delta m_{31}^0|$).

### 2.2 Complex case

For the complex case, the most general form for $U_0$ is

$$U_0 = R_{23}(\theta_{23}^0) U_{13}(\theta_{13}^0, \delta^0) R_{12}(\theta_{12}^0) P(\phi_2^0, \phi_3^0),$$

(15)

\footnotetext[1]{Also, for the next-to-leading-order correction to $\theta_{12}$, we obtain Eq. (14) of Ref. [7], except that $\delta m_{21}^0$ in the denominator should be replaced by $\delta m_{21}^{(1)} \equiv (m_2^0 + \delta m_2) - (m_1^0 + \delta m_1)$.
where

\[
R_{23}(\theta_{23}^0) = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23}^0 & s_{23}^0 \\
0 & -s_{23}^0 & c_{23}^0
\end{pmatrix}, \quad U_{13}(\theta_{13}^0, \delta^0) = \begin{pmatrix}
c_{13}^0 & 0 & e^{-i\delta^0}s_{13}^0 \\
0 & 1 & 0 \\
-e^{i\delta^0}s_{13}^0 & 0 & c_{13}^0
\end{pmatrix},
\]

\[
R_{12}(\theta_{12}^0) = \begin{pmatrix}
c_{12}^0 & s_{12}^0 & 0 \\
-s_{12}^0 & c_{12}^0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad P(\phi_2^0, \phi_3^0) = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\phi_2^0/2} & 0 \\
0 & 0 & e^{i\phi_3^0/2}
\end{pmatrix}.
\]  

(16)

Because of the nonzero Majorana phases, in general, the mixing matrix would not remain unchanged if we subtract the identity matrix multiplied by a constant from the mass matrix. Hence we use a slightly different procedure to obtain the LO corrections for the complex case. We rewrite the final mass matrix as

\[
M = U_0^*M_0U_0^\dagger + E = U_0^* \left[ M_0 + \tilde{E} \right] U_0^\dagger,
\]

(17)

where \( \tilde{E} = U_0^T E U_0 \) can be explicitly written as

\[
\tilde{E} = \begin{pmatrix}
a & be^{i\phi_2^0/2} & de^{i\phi_3^0/2} \\
b & ce^{i\phi_2^0/2} & fe^{i(\phi_2^0+\phi_3^0)/2} \\
d & fe^{i(\phi_2^0+\phi_3^0)/2} & ge^{i\phi_3^0}
\end{pmatrix},
\]

(18)

with

\[
a = \epsilon_4(s_{12}^0)^2 + [\epsilon_1(c_{13}^0)^2 - \epsilon_3 s_{2\times 13}^0 e^{i\theta_1^0} + \epsilon_6(s_{13}^0)^2 e^{2i\delta^0}] (c_{12}^0)^2 + (\epsilon_5 s_{13}^0 s_{13}^0 - \epsilon_2 s_{13}^0) s_{2\times 12}^0,
\]

\[
b = \epsilon_2 c_{13}^0 c_{2\times 12}^0 + [\epsilon_1(c_{13}^0)^2 - \epsilon_4 + \epsilon_6(s_{13}^0)^2 e^{2i\delta^0}] c_{12}^0 s_{12}^0 - [\epsilon_3 s_{2\times 12}^0 s_{13}^0 c_{13}^0 + \epsilon_5 c_{2\times 12}^0 s_{13}^0] e^{i\theta_1^0},
\]

\[
c = \epsilon_4(c_{12}^0)^2 + [\epsilon_1(c_{13}^0)^2 - \epsilon_3 s_{2\times 13}^0 e^{i\theta_1^0} + \epsilon_6(s_{13}^0)^2 e^{2i\delta^0}] (s_{12}^0)^2 - (\epsilon_5 s_{13}^0 e^{i\theta_1^0} - \epsilon_2 s_{13}^0) s_{2\times 12}^0,
\]

\[
d = (\epsilon_1 c_{13}^0 e^{i\delta^0} - \epsilon_2 s_{13}^0) s_{13}^0 e^{-i\theta_1^0} - \epsilon_6 c_{13}^0 e^{i\delta^0} + \epsilon_3 c_{12}^0 s_{13}^0 - \epsilon_5 c_{13}^0 s_{12}^0,
\]

\[
f = (\epsilon_1 s_{12}^0 c_{13}^0 + \epsilon_2 c_{12}^0 s_{13}^0) e^{-i\theta_1^0} - \epsilon_6 s_{13}^0 c_{13}^0 e^{i\delta^0} + \epsilon_3 s_{12}^0 c_{13}^0 + \epsilon_5 c_{13}^0 c_{12}^0,
\]

\[
g = \epsilon_6(c_{13}^0)^2 + \epsilon_3 s_{2\times 13}^0 e^{-i\theta_1^0} + \epsilon_1(s_{13}^0)^2 e^{-2i\delta^0},
\]

(19)

Similar to the real case, we apply a unitary matrix \( U_\delta \) to \( N \equiv M_0 + \tilde{E} \) such that there are zeros in the 2-3 and 1-3 entries of the matrix \( U_\delta^T N U_\delta \). Since \( |\epsilon_{ij}| \ll |\delta m_{31}^0| \), to LO in
\[ \mathcal{O}(|\epsilon_{ij}|/|\delta m^0_{31}|), \]  

\[ U_\delta = \begin{pmatrix} 1 & 0 & \delta_{13} \\ 0 & 1 & \delta_{23} \\ -\delta^*_{13} & -\delta^*_{23} & 1 \end{pmatrix}, \]  

(20)

where

\[ \delta_{13} \approx \frac{|d|e^{-i\phi_{13}}}{|m^0_3 - m^0_1e^{-2i\phi_{13}}|}, \quad \delta_{23} \approx \frac{|f|e^{-i\phi_{23}}}{|m^0_3 - m^0_1e^{-2i\phi_{23}}|}, \]  

(21)

with \( \tan \phi_{13} = \frac{m^0_2 + m^0_0}{m^0_3 - m^0_1} \tan \left[ \arg(d) + \phi^0_0/2 \right] \) and \( \tan \phi_{23} = \frac{m^0_2 + m^0_0}{m^0_3 - m^0_1} \tan \left[ \arg(f) + \phi^0_0 + \phi^0_2/2 \right] \). After block-diagonalization, the LO correction to \( m_3 \) is

\[ \delta m_3 = |m^0_3 + g e^{i\phi^0_2} - m^0_3|. \]  

(22)

Note that the 1-2 submatrix of \( N \) remains unchanged to leading order after the block-diagonalization. Using the procedure described in Appendix A, we diagonalize this submatrix using the unitary matrix

\[ U_{12}(\xi, \phi) = \begin{pmatrix} c_\xi & s_\xi e^{-i\phi} & 0 \\ -s_\xi e^{i\phi} & c_\xi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]  

(23)

where

\[ \phi = \arctan \frac{|a + m^0_1| \sin(\phi_a - \phi_b) - |c e^{i\phi^0_2} + m^0_2| \sin(\phi_c - \phi_b)}{|a + m^0_1| \cos(\phi_a - \phi_b) + |c e^{i\phi^0_2} + m^0_2| \cos(\phi_a - \phi_b)}, \]  

(24)

\[ \xi = \frac{1}{2} \arctan \frac{2|b|}{|c e^{i\phi^0_2} + m^0_2| \cos(\phi_a - \phi_b) - |a + m^0_1| \cos(\phi_a - \phi_b)}, \]  

(25)

with \( \phi_a = \arg(a + m^0_1), \phi_b = \arg(b) + \phi^0_0/2 \) and \( \phi_c = \arg(c e^{i\phi^0_2} + m^0_2) \). In addition, we obtain the LO corrections to \( m_1 \) and \( m_2 \) as

\[ \delta m_1 = \left( a + m^0_1 \right) c^2_\xi + (c e^{i\phi^0_2} + m^0_2) s^2_\xi e^{2i\phi} - 2b s_\xi c_\xi e^{i\phi} - m^0_1, \]

\[ \delta m_2 = \left( a + m^0_1 \right) s^2_\xi e^{-2i\phi} + (c e^{i\phi^0_2} + m^0_2) c^2_\xi + 2b s_\xi c_\xi e^{-i\phi} - m^0_2. \]  

(26)
The final mixing matrix that diagonalizes $M$ and makes the diagonal elements real and non-negative can be written as

$$U = U_0 U_{12}(\xi, \phi) P,$$

where $P = \text{diag}(e^{i\omega_1/2}, e^{i\omega_2/2}, e^{i\omega_3/2})$, and

$$\omega_1 = -\arg \left[ (a + m_1^0) c_\xi^2 + (ce^{i\phi_0} + m_2^0) s_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi} \right],$$

$$\omega_2 = -\arg \left[ (a + m_1^0) s_\xi^2 e^{-2i\phi} + (ce^{i\phi_0} + m_2^0) c_\xi^2 + 2bs_\xi c_\xi e^{-i\phi} \right],$$

$$\omega_3 = -\arg \left( m_3^0 + ge^{i\phi_0} \right).$$

As shown in Appendix B, the right-multiplication of $U_{12}(\xi, \phi)$ does not change $\theta_{13}$ and $\theta_{23}$. Hence, the LO corrections to $\theta_{13}$ and $\theta_{23}$ come from the right-multiplication of $U_\delta$. Since $\delta_{13}$ and $\delta_{23}$ are suppressed by a factor of $|\epsilon_{ij}|/|\delta m_{31}^2|$, while $\xi$ and $\phi$ are not, the LO corrections to $\theta_{12}$ and the Dirac phases come from the right-multiplication of $U_{12}(\xi, \phi)$, and the LO corrections to the Majorana phases come from both $U_{12}(\xi, \phi)$ and $P$.

By comparing $U$ to the standard parametrization, we obtain the LO corrections to the three mixing angles:

$$\delta\theta_{13} = \frac{|d|s_{12}^0 \cos(\delta_0 - \frac{\phi_0}{2} - \phi_{13}) + |f|s_{12}^0 \cos(\delta_0 + \frac{\phi_0}{2} - \phi_{23})}{|m_3^0 - m_1^0 e^{-2i\phi_{13}}|}$$

$$\delta\theta_{23} = -\frac{|d|s_{12}^0 \cos(\frac{\phi_0}{2} + \phi_{13}) + |f|s_{12}^0 \cos(\frac{\phi_0}{2} - \phi_{23})}{|m_3^0 - m_1^0 e^{-2i\phi_{13}}|}$$

$$\delta\theta_{12} = \arcsin \sqrt{\sin^2(\theta_{12}^0 + \xi) - \sin(2\theta_{12}^0) \sin(2\xi) \sin^2 \frac{\phi_0^2 + 2\phi}{4} - \theta_{12}^0},$$

where $t_{ij}^0$ denotes $\tan^{0}_{ij}$. The LO corrections to the three phases can be written as

$$\Delta\delta = \alpha - \beta,$$

$$\Delta\phi_2 = -2(\alpha + \beta) + \omega_2 - \omega_1,$$

$$\Delta\phi_3 = -2\beta + \omega_3 - \omega_1.$$
where
\[ \alpha = -\arctan \frac{\tan \theta_{12} \tan \xi \sin(\phi_2^0/2 + \phi)}{1 - \tan \theta_{12} \tan \xi \cos(\phi_2^0/2 + \phi)}, \]  
and
\[ \beta = \arctan \frac{\tan \xi \sin(\phi_2^0/2 + \phi)}{\tan \theta_{12}^0 + \tan \xi \cos(\phi_2^0/2 + \phi)}. \]  

From Eq. (31), we see that \( \delta \theta_{12} \) varies from \(-\xi\) to \(+\xi\) depending on the initial Majorana phase \( \phi_2^0 \) and the perturbation phase \( \phi \). Since \( \xi \) and \( \phi \) depend only on the ratios of linear combinations of \( \epsilon_{ij} \)'s and \( \delta m_{21}^0 \), large corrections to \( \theta_{12} \) and the Dirac and Majorana phases are possible even for small perturbations. However, corrections can be small in special cases, e.g., if \( \phi_2^0 \) is close to 180° for the inverted hierarchy, \( \phi \) approaches 90° and \( \xi \) is suppressed by a factor of \( |\epsilon_{ij}|/(m_2^0 + m_1^0) \), so that the corrections to \( \theta_{12} \) and the Dirac and Majorana phases are also small.

Note that the corrections in the complex case are strongly dependent on the phases of \( \epsilon_{ij} \), and the initial values of the Dirac and Majorana phases. If we take \( \epsilon_{ij} \)'s to be real, and set \( \delta^0 = \phi_2^0 = \phi_3^0 = 0 \) in Eqs. (29), (30) and (31), we recover Eq. (14).

### 3 Perturbations to \( \mu - \tau \) symmetry

As an illustration of our analytic results, we study perturbations on initial neutrino mass matrices with \( \mu - \tau \) symmetry. There are four classes of mixing with \( \mu - \tau \) symmetry [7]: (a) \( \theta_{23}^0 = 45^\circ, \theta_{13}^0 = 0 \); (b) \( \theta_{23}^0 = 45^\circ, \theta_{12}^0 = 0 \); (c) \( \theta_{23}^0 = 45^\circ, \theta_{12}^0 = 90^\circ \); (d) \( \theta_{23}^0 = 45^\circ, \delta^0 = \pm 90^\circ \).

In Ref. [15], it was shown that the initial class (a) can be perturbed to class (d) for a specific model. Here we reproduce the results of Ref. [15] by applying our general perturbation formulas. The complex neutrino mass matrix of Ref. [15] can be written (in our phase convention) as
\[
M = m_0 \begin{pmatrix}
1 + 2\delta'' & 0 & 0 \\
0 & \delta'' & -(1 + \delta'') \\
0 & -(1 + \delta'') & \delta''
\end{pmatrix} + m_0 \begin{pmatrix}
2\delta' & \delta'' & -\delta''^* \\
\delta'' & 0 & 0 \\
-\delta''^* & 0 & 0
\end{pmatrix},
\]  
(37)
where $m_0$ is a common mass parameter, $\delta''$, $\delta'$ are real and $|\delta'|, |\delta''| \ll |\delta'''|$. We treat the first term on the right-hand side of Eq. (37) as the initial mass matrix and the second term as the perturbation. The initial mass matrix has class (a) $\mu - \tau$ symmetry. In the standard parametrization, we have $\theta^0_{23} = \frac{\pi}{4}, \theta^0_{12} = \theta^0_{13} = \phi^0_2 = 0$ and $\phi^0_3 = \pi$. The three initial masses are $m^0_1 = m^0_2 = m_0(1 + 2\delta'''),$ and $m^0_3 = m_0$. In this case, Eq. (19) is greatly simplified:

$$a = 2m_0\delta', \quad b = \sqrt{2}m_0\text{Re}(\delta''), \quad c = f = g = 0, \quad d = i\sqrt{2}m_0\text{Im}(\delta'').$$

(38)

From Eqs. (21), (24) and (25), we find

$$\delta_{23} = 0, \quad \delta_{13} \approx \frac{\text{Im}(\delta'')}{\sqrt{2}\delta''}, \quad \phi = 0, \quad \xi = \frac{1}{2} \arctan \frac{\sqrt{2}\text{Re}(\delta'')}{-\delta'}. \quad (39)$$

Then the final mixing matrix can be written as

$$U = R_{23}(\frac{\pi}{4})P(0, \pi)R_{13}(\delta_{13})R_{12}(\xi) \quad (40)$$

$$= R_{23}(\frac{\pi}{4})U_{13}(\delta_{13}, \frac{\pi}{2})R_{12}(\xi)P(0, \pi).$$

Hence, the final mixing angles and the Dirac phase are

$$\theta_{23} = \frac{\pi}{4}, \quad \theta_{12} = \frac{1}{2} \arctan \frac{\sqrt{2}\text{Re}(\delta'')}{-\delta'}, \quad \theta_{13} = \frac{\text{Im}(\delta'')}{\sqrt{2}\delta''}, \quad \delta = \frac{\pi}{2}, \quad (41)$$

as in Ref. [13]. Note that the initial class (a) is perturbed to class (d), and that the large change of the Dirac phase $\delta$ coincides with the deviation of $\theta_{13}$ from 0.

The general form of the neutrino mass matrix with class (d) $\mu - \tau$ symmetry and its associated generalized CP symmetry has been recognized in Ref. [16], and deviations from it were discussed in Ref. [17]. It has been shown in Ref. [16] that the general forms of the neutrino mass matrices with class (a) and (d) $\mu - \tau$ symmetry are (in our phase convention)

$$M_a = \begin{pmatrix} x & y & -y \\ y & z & -w \\ -y & -w & z \end{pmatrix}, \quad \text{and} \quad M_d = \begin{pmatrix} u & r & -r^* \\ r & s & -v \\ -r^* & -v & s^* \end{pmatrix}, \quad (42)$$

respectively. Here $x, y, z, w, r, s$ are complex and $u, v$ are real. Hence, any perturbation matrix of the form

$$E = \begin{pmatrix} \text{Re}(\epsilon_{11}) - i\text{Im}(x) & \epsilon_{12} & -\epsilon_{12}^* + 2i\text{Im}(y) \\ \epsilon_{12} & \epsilon_{22} & \text{Re}(\epsilon_{23}) + i\text{Im}(w) \\ -\epsilon_{12}^* + 2i\text{Im}(y) & \text{Re}(\epsilon_{23}) + i\text{Im}(w) & \epsilon_{22}^* - 2i\text{Im}(z) \end{pmatrix}, \quad (43)$$
Table 1: Best-fit values and $2\sigma$ ranges of the oscillation parameters [18], with $\delta m^2 \equiv m^2_2 - m^2_1$ and $\Delta m^2 \equiv m^2_3 - (m^2_1 + m^2_2)/2$.

| Parameter       | $\theta_{12}(^\circ)$ | $\theta_{13}(^\circ)$ | $\theta_{23}(^\circ)$ | $\delta m^2 (10^{-5}\text{eV}^2)$ | $|\Delta m^2| (10^{-3}\text{eV}^2)$ |
|-----------------|------------------------|------------------------|------------------------|-----------------------------------|-----------------------------------|
| Normal hierarchy| $33.7^{+2.4}_{-2.1}$  | $8.86^{+0.73}_{-0.77}$ | $41.4^{+6.6}_{-2.6}$  | $7.54^{+0.46}_{-0.39}$           | $2.43^{+0.12}_{-0.13}$           |
| Inverted hierarchy| $33.7^{+2.4}_{-2.1}$  | $8.91^{+0.70}_{-0.82}$ | $42.4^{+9.5}_{-3.2}$  | $7.54^{+0.46}_{-0.39}$           | $2.38^{+0.12}_{-0.13}$           |

perturbs the initial mass matrix with class (a) $\mu - \tau$ symmetry to class (d) $\mu - \tau$ symmetry.

We now perform a numerical search to find perturbations that fit the experimental data for initial neutrino mass matrices with $\mu - \tau$ symmetry. We select class (d) and scan $\theta_{12}^0$ and $\theta_{13}^0$ over the range $[0, 90^\circ]$. Since the initial mass matrices of classes (a), (b) and (c) do not depend on $\delta^0$, the perturbation results of class (d) will cover the other classes, e.g., the perturbation results for bimaximal mixing would be the same as that of class (d) with $\theta_{13}^0 = 0$ and $\theta_{12}^0 = 45^\circ$. Since we work in the basis in which the charged lepton mass matrix is diagonal, the mixing matrix in the neutrino sector is the same as the observed PMNS matrix. We also choose $m_1 = 0$ for the normal hierarchy (or $m_3 = 0$ for the inverted hierarchy), so the best-fit values from the global fit in Table 1 define the other two final masses and the three final mixing angles.

We characterize the size of the perturbation as the root-mean-square (RMS) value of the perturbations,

$$
\epsilon_{\text{RMS}} = \sqrt{\frac{\text{Tr}[E^\dagger E]}{9}} = \sqrt{\frac{\sum_{i,j=1}^{3} |\epsilon_{ij}|^2}{9}}, \quad (44)
$$

where $i$ and $j$ sum over neutrino flavors. $\epsilon_{\text{RMS}}$ is determined by the three initial masses, two initial Majorana phases, two final Majorana phases and one final Dirac phase.

The initial Dirac phase in class (d) is fixed to be $\pm 90^\circ$. To evaluate the change in the Dirac phase due to the perturbations, we fix $\theta_{12}^0 = 45^\circ$ and scan over $\delta$ and $\theta_{13}^0$ to find the minimum RMS value of the perturbation, $\epsilon_{\text{RMS}}^{\text{min}}$, that results in the best-fit parameters. The results for $\delta^0 = 90^\circ$ are shown in Fig. 1. The results for $\delta^0 = -90^\circ$ (or $270^\circ$) are symmetric to those of $\delta^0 = 90^\circ$ with $\delta \rightarrow 360^\circ - \delta$. From Fig. 1 we see that for $\theta_{13}^0 \leq 20^\circ$ it is possible for the final Dirac phase to have any value under small perturbations ($\epsilon_{\text{RMS}}^{\text{min}} \lesssim 10$ meV), i.e.,
the correction to the Dirac phase can be large for small perturbations.

4 Perturbations in the charged lepton sector

In the basis in which the charged lepton mass matrix is not diagonal, the observed PMNS mixing matrix is

\[
U_{PMNS} = U_l^\dagger U_\nu,
\]

where \(U_l\) and \(U_\nu\) are the mixing matrices in the charged lepton and neutrino sectors, respectively. For an arbitrary charged lepton mass matrix \(M_l\), we have

\[
(M_l)^\dagger M_l = U_l \overline{M_l}^2 (U_l)^\dagger,
\]

where \(\overline{M_l} = \text{diag} (m_e, m_\mu, m_\tau)\).

Suppose the charged lepton mass matrix is also the result of small perturbations to an initial mass matrix, i.e., \(M_l = M_l^0 + E_l\), where \((E_l)_{ij} \equiv \epsilon_{ij}^l\) and \(|\epsilon_{ij}^l| \ll m_\tau\). If the initial
mixing matrix in the charged lepton sector is $U_\ell^0$, i.e.,

$$ (M_\ell^0)^\dagger M_\ell^0 = U_\ell^0 (M_\ell^0)^2 (U_\ell^0)^\dagger, \quad (47) $$

then to LO, we get

$$ (M_\ell)^\dagger M_\ell \approx U_\ell^0 (M_\ell^0)^2 (U_\ell^0)^\dagger + (M_\ell^0)^\dagger E_\ell + E^\dagger M_\ell^0 = U_\ell^0 \left[ M_\ell^0 + N^l \right] (U_\ell^0)^\dagger, \quad (48) $$

where $N^l = (U_\ell^0)^\dagger \left[ (M_\ell^0)^\dagger E_\ell + E^\dagger M_\ell^0 \right] U_\ell^0$. Note that since $U_\ell^0$ is unconstrained, the size of each element of the $N^l$ matrix could be of order $m_\tau |\epsilon_{ij}^l|$.

If $(M_\ell^0)^2 + N^l$ is diagonalized by a unitary matrix $U_\delta^l$, i.e.,

$$ (M_\ell^0)^2 + N^l = \left( \begin{array}{ccc} (m_\ell^0)^2 + N_{11}^l & N_{12}^l & N_{13}^l \\ (N_{11}^l)^* & (m_\ell^0)^2 + N_{22}^l & N_{23}^l \\ (N_{13}^l)^* & (N_{23}^l)^* & (m_\ell^0)^2 + N_{33}^l \end{array} \right) = U_\delta^l \left( \begin{array}{ccc} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{array} \right) (U_\delta^l)^\dagger, \quad (49) $$

then the PMNS matrix can be written as

$$ U_{PMNS} = (U_\ell^0 U_\delta^l)^\dagger U_\nu^0 = (U_\delta^l)^\dagger U_0, \quad (50) $$

where $U_0 = (U_\ell^0)^\dagger U_\nu^0$ has the most general form of Eq. (13). Since $N_{ij}^l \sim m_\tau |\epsilon_{ij}^l|$, the 2-3 and 1-3 mixing angles in $U_\delta^l$ are suppressed by a factor of $N_{ij}^l/m_\tau^2 \sim |\epsilon_{ij}^l|/m_\tau$. To LO in $O(|\epsilon_{ij}^l|/m_\tau)$, $U_\delta^l$ can be parametrized as

$$ U_\delta^l = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & \delta_{23}^l e^{-i\phi_{23}^l} \\ 0 & -\delta_{23}^l e^{i\phi_{23}^l} & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & \delta_{13}^l e^{-i\phi_{13}^l} \\ 0 & 1 & 0 \\ -\delta_{13}^l e^{i\phi_{13}^l} & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} \cos \theta_{12}^l & \sin \theta_{12}^l e^{-i\phi_{12}^l} & 0 \\ -\sin \theta_{12}^l e^{i\phi_{12}^l} & \cos \theta_{12}^l & 0 \\ 0 & 0 & 1 \end{array} \right), \quad (51) $$

where

$$ \delta_{13}^l \approx \frac{|N_{13}^l|}{m_\tau^2}, \quad \delta_{23}^l \approx \frac{|N_{23}^l|}{m_\tau^2}, \quad \theta_{12}^l \approx \frac{1}{2} \arctan \frac{2 |N_{12}^l|}{m_\mu^2 + N_{22}^l - N_{11}^l}, \quad (52) $$
\[
\text{and } \phi_{ij} \approx - \arg N^l_{ij}.
\]

If \( \theta^l_{12} \) is also very small, the LO corrections to the three mixing angles in the PMNS matrix are

\[
\begin{align*}
\delta \theta_{13} &= -\theta^l_{12} s^0_{23} \cos(\delta^0 - \phi^l_{12}) - \delta^l_{13} c^0_{23} \cos(\delta^0 - \phi^l_{13}), \\
\delta \theta_{23} &= -\delta^l_{23} \cos \phi^l_{23} - \delta^l_{13} s^0_{23} t^0_{13} \cos(\delta^0 - \phi^l_{13}) + \theta^l_{12} c^0_{23} t^0_{13} \cos(\delta^0 - \phi^l_{12}), \\
\delta \theta_{12} &= \frac{1}{c^0_{13}} (\delta^l_{13} s^0_{23} \cos \phi^l_{13} - \theta^l_{12} c^0_{23} \cos \phi^l_{12}).
\end{align*}
\]

(53)

However, in general, since \( N^l_{ij} \sim m_\tau |\epsilon^l_{ij}| \), and if \( |\epsilon^l_{ij}| \sim m^2_\mu/m_\tau = 6 \text{ MeV} \), \( \theta^l_{12} \) could be very large, which will give large corrections to the mixing angles in the PMNS matrix. Thus, the situation in the charged sector is similar to that in the neutrino sector: the near degeneracy of \( m_e \) and \( m_\mu \) (on the scale of \( m_\tau \)) can lead to large corrections in 1-2 space.

For large \( \theta^l_{12} \), the analytical expressions for the corrections to the mixing angles in the PMNS matrix are cumbersome. Here, as an illustration, we consider the very simple scenario in which

\[
U^l_\delta = \begin{pmatrix}
\cos \theta^l_{12} & \sin \theta^l_{12} & 0 \\
-\sin \theta^l_{12} & \cos \theta^l_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(54)

Then from Eq. (50), the final mixing angles in the PMNS matrix are given by

\[
c^0_{13} c^0_{23} = c^0_{13} c^0_{23},
\]

(55)

\[
s^2_{13} = (s^0_{13})^2 (c^l_{12})^2 + (s^0_{23})^2 (c^0_{13})^2 (s^l_{12})^2 - 2 s^0_{13} c^0_{13} s^0_{23} s^l_{12} c^l_{12} \cos \delta^0, \]

(56)

\[
c^2_{13} s^2_{12} = [(c^l_{12} c^0_{13} s^0_{12} - s^l_{12} c^0_{13} c^0_{23})^2 + (s^l_{12})^2 (s^0_{12})^2 (s^0_{23})^2 - 2 s^l_{12} c^0_{13} s^0_{12} (c^l_{12} c^0_{13} s^0_{12} - s^l_{12} c^0_{13} c^0_{23}) \cos \delta^0],
\]

(57)

where \( c^l_{12} \) denotes \( \cos \theta^l_{12} \), and \( s^l_{12} \) denotes \( \sin \theta^l_{12} \). As an example, if \( \theta^l_{12} \) is the Cabibbo angle and the initial PMNS matrix has bimaximal symmetry (\( \theta^0_{12} = 45^\circ, \theta^0_{13} = 0 \)), then the resulting \( \theta_{12} \) and \( \theta_{13} \) are consistent with the observed values to within 2\( \sigma \).

There are eight parameters in Eqs. (55), (56) and (57). We use the best-fit values in Table I for the normal hierarchy to fix \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \). Then for given values of \( \theta^0_{13} \) and \( \delta^0, \)
Figure 2: Dependence of $\theta_{23}^0$ on $\theta_{13}^0$ for small perturbations in the charged lepton sector when $U^l_3$ is given by Eq. (54), and the three mixing angles in the PMNS matrix are fixed by the best-fit values of the global fit in Table 1 for the normal hierarchy.

the other three unknown parameters $\theta_{23}^0$, $\theta_{12}^l$ and $\theta_{12}^0$ are determined by the three equations. First, we obtain $\theta_{23}^0$ from Eq. (55) for a given $\theta_{13}^0$. Note that the constraints on $\theta_{23}^0$ and $\theta_{13}^0$ are symmetric for fixed $\theta_{13}^0$ and $\theta_{23}^0$, which can be seen from Fig. 2. Then we scan $\theta_{12}^l$ from $[-90^\circ, 90^\circ]$ to find solutions to Eq. (56) for a given $\delta_0$. For each solution of $\theta_{12}^l$, we obtain $\theta_{12}^0$ from Eq. (57) by scanning $\theta_{12}^0$ from $[-90^\circ, 90^\circ]$. Note that we only scan the first and fourth quadrants of $\theta_{12}^l$ [$\theta_{12}^0$], because Eq. (56) [Eq. (57)] is only sensitive to the relative sign between the cosine and sine of $\theta_{12}^l$ [$\theta_{12}^0$]. Once we obtain $\theta_{23}^0$, $\theta_{12}^l$ and $\theta_{12}^0$ for given values of $\theta_{13}^0$ and $\delta_0$, the resulting PMNS matrix is completely determined (except for the diagonal Majorana phase matrix) from Eqs. (50), (54) and (15). By comparing the PMNS matrix with the standard parametrization, the resulting Dirac phase $\delta$ is also obtained for given values of $\theta_{13}^0$ and $\delta_0$. The dependence of $\theta_{12}^l$, $\theta_{12}^0$ and $\delta$ on $\delta_0$ for different values of $\theta_{13}^0$ is shown in Fig. 3. From Figs. 2 and 3 we see that the initial mixing angles and the initial Dirac phase can be very different from their observed values in the PMNS matrix due to small perturbations in the charged lepton sector.

Generally, perturbations in both the charged lepton and neutrino sectors will be present.
In this case, one must first use the procedure described in this section to find the corrections to the initial PMNS matrix from perturbations in the charged lepton sector alone, then use the new PMNS matrix to rotate to the basis in which the final charged lepton mass matrix is diagonal, and ultimately use the procedure described in Section 2 to find the final corrections to the parameters in the PMNS matrix from perturbations in the neutrino sector.

5 Neutrino oscillations with nonstandard interactions

We now apply our generalized perturbation procedure to a phenomenological study of neutrino oscillations that are affected by nonstandard scalar and nonstandard vector interactions simultaneously.

As $\nu_e$ propagate in matter, they scatter on electrons via the V-A interaction. This is described by the MSW potential \[19\], which is added to the vacuum oscillation Hamiltonian:

$$H = \frac{1}{2E_\nu} U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} \sqrt{2}G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(58)

where $G_F$ is the Fermi constant, $N_e$ is the electron number density in the medium, and $U$ and $m_i$ are the mixing matrix and eigenmasses in vacuum, respectively.

New physics beyond the Standard Model can be probed by studying model-independent nonstandard interactions in neutrino oscillation experiments; for a recent review see Ref. [20]. Most studies of nonstandard interactions are focused on the vector interaction, which can be described by effective four-fermion operators of the form

$$\mathcal{L}_V = \frac{G_F}{\sqrt{2}} \epsilon_{\alpha\beta}^V \left[ \bar{\nu}_\alpha \gamma^\mu (1 - \gamma^5) \nu_\beta \right] \left[ \bar{f} \gamma_\rho (1 \pm \gamma^5) f \right] + \text{h.c.},$$

(59)

where $f = u, d, e$ is a charged fermion field, and $\epsilon_{\alpha\beta}^V$ are dimensionless parameters that denote the strength of the deviation from the standard interactions. Similar to the MSW term, the matter effect due to the nonstandard vector interaction modifies the oscillation Hamiltonian by additional potential terms, $\sqrt{2}G_F N_f \epsilon_{\alpha\beta}^V$.

In addition, consider nonstandard scalar interactions, which may arise from a Lagrangian
Figure 3: Dependence of $\theta_{12}$ (top), $\theta_{12}^0$ (middle) and $\delta$ (bottom) on $\delta^0$ for different values of $\theta_{13}^0$ for small perturbations in the charged lepton sector when $U^l_\delta$ is given by Eq. (5.1), and the three mixing angles in the PMNS matrix are fixed by the best-fit values of the global fit in Table 1 for the normal hierarchy.
of the form

\[ \mathcal{L}_S = \lambda^\alpha_\nu \bar{\nu}_\alpha \nu^\beta \phi + \lambda_f \bar{f} f \phi, \]  

(60)

where \( \phi \) is a new scalar field, and \( \lambda^\alpha_\nu \) and \( \lambda_f \) are dimensionless coupling constants for neutrinos and charged fermions, respectively. In a mean field approximation, the nonstandard scalar interaction will shift elements of the neutrino mass matrix by \( \epsilon_{\alpha\beta} \approx \frac{\lambda^\alpha_\nu}{m_\phi} \lambda_f N_f \),

(61)

where \( m_\phi \) is the mass of the scalar field, \( N_f \) is the number density of the charged fermion \( f \), which is assumed to be nonrelativistic.

Tests of the inverse square law of the gravitational force put stringent \( m_\phi \)-dependent limits on the coupling of a new scalar field to the nucleon field \[22\]. For \( m_\phi \) in the range, \( 10^{-6} \) eV to \( 10^{-10} \) eV, the current experimental upper limit of \( \lambda_N \) varies from \( 10^{-21} \) to \( 10^{-22} \) \[23\]. Since

\[ \epsilon_{\alpha\beta} \approx 0.46 \text{ meV} \left( \frac{\lambda_\nu}{10^{-4}} \right) \left( \frac{\lambda_N}{10^{-21}} \right) \left( \frac{N_f}{N_A / \text{cm}^3} \right) \left( \frac{10^{-6} \text{ eV}}{m_\phi} \right)^2, \]  

(62)

and \( N_f \sim 1 N_A / \text{cm}^3 \sim 10^{10} \) eV\(^3\) on Earth and \( N_f \sim 100 N_A / \text{cm}^3 \sim 10^{12} \) eV\(^3\) in the solar core where most solar neutrinos are produced, in these environments, a \( \lambda_\nu \) of order \( 10^{-3} \) gives a mass shift of order 1 meV for \( m_\phi = 10^{-6} \) eV. Such \( \epsilon_{\alpha\beta} \) values are possible for much smaller values of \( \lambda_\nu \) when \( m_\phi < 10^{-6} \) eV.

In the presence of both nonstandard scalar and nonstandard vector interactions, the effective Hamiltonian for neutrino oscillations can be written as

\[ H_{\text{eff}} = \frac{1}{2E_\nu} M_{\text{eff}}^\dagger M_{\text{eff}} + \sqrt{2} G_F N_e \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sqrt{2} G_F N_f \begin{pmatrix} \epsilon_{ee} V_{ee} & \epsilon_{e\mu} V_{e\mu} & \epsilon_{e\tau} V_{e\tau} \\ \epsilon_{\mu e} V_{\mu e} & \epsilon_{\mu\mu} V_{\mu\mu} & \epsilon_{\mu\tau} V_{\mu\tau} \\ \epsilon_{\tau e} V_{\tau e} & \epsilon_{\tau\mu} V_{\tau\mu} & \epsilon_{\tau\tau} V_{\tau\tau} \end{pmatrix}, \]  

(63)

where the effective mass matrix has the form

\[ M_{\text{eff}} = U^* \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U^\dagger + \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix}. \]  

(64)

We apply our generalized perturbation procedure to the study of both nonstandard scalar and nonstandard vector interactions. By incorporating the the corrections to the vacuum
oscillation parameters (arising from the scalar interaction) into the nonstandard vector interaction formulas, we immediately obtain new formulas for oscillation probabilities with both nonstandard scalar and nonstandard vector interactions. Taking the oscillations of $\nu_\mu$ in long baseline experiments as an example, the result for the $\nu_\mu$ survival probability is [21]

$$P_{\mu\mu} \simeq 1 - s_{23}^2 \left[ \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right]$$

$$- |\epsilon_{\mu\tau}^V| \cos \phi_{\mu\tau}^V s_{23}^2 \left[ s_{23}^2 (\sqrt{2} G_F N_e L) \sin \frac{\Delta m_{31}^2 L}{2E} + 4 c_{23}^2 \frac{2 \sqrt{2} G_F N_e E}{\Delta m_{31}^2} \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right]$$

$$+ (|\epsilon_{\mu\mu}^V| - |\epsilon_{\tau\tau}^V|) s_{23}^2 c_{23} \left[ \frac{\sqrt{2} G_F N_e L}{2} \sin \frac{\Delta m_{31}^2 L}{2E} - 2 \frac{2 \sqrt{2} G_F N_e E}{\Delta m_{31}^2} \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right], \quad (65)$$

where $\Delta m_{31}^2 = m_3^2 - m_1^2$, $s_{23} = \sin 2\theta_{12}$, $c_{23} = \cos 2\theta_{12}$, and $\phi_{\mu\tau}^V = \arg \epsilon_{\mu\tau}^V$. After replacing $\Delta m_{31}^2 \to \Delta m_{31}^2 + 2 (m_3^2 - m_1^2)$ and $\theta_{23} \to \theta_{23} + \delta \theta_{23}$, where the shifts in $m_\nu$ and $\theta_{23}$ can be easily obtained from our perturbation results in Section 2, the new formula for both nonstandard scalar and nonstandard vector interactions is as follows:

$$P_{\mu\mu} \simeq 1 - s_{23}^2 \left[ \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right]$$

$$- 2 \delta \theta_{23} \sin 4 \theta_{23} \sin^2 \frac{\Delta m_{31}^2 L}{4E} - \frac{(m_3^2 - m_1^2) L}{2E} s_{23}^2 \sin \frac{\Delta m_{31}^2 L}{2E}$$

$$- |\epsilon_{\mu\tau}^V| \cos \phi_{\mu\tau}^V s_{23}^2 \left[ s_{23}^2 (\sqrt{2} G_F N_e L) \sin \frac{\Delta m_{31}^2 L}{2E} + 4 c_{23}^2 \frac{2 \sqrt{2} G_F N_e E}{\Delta m_{31}^2} \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right]$$

$$+ (|\epsilon_{\mu\mu}^V| - |\epsilon_{\tau\tau}^V|) s_{23}^2 c_{23} \left[ \frac{\sqrt{2} G_F N_e L}{2} \sin \frac{\Delta m_{31}^2 L}{2E} - 2 \frac{2 \sqrt{2} G_F N_e E}{\Delta m_{31}^2} \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right]. \quad (66)$$

We see that cancellations between the nonstandard scalar and vector terms are possible, a study of which is beyond the scope of this paper.

### 6 Summary

We introduced a generalized procedure to study complex perturbations on Majorana neutrino mass matrices. In the charged lepton basis, we derived analytic formulas for the corrections to the three mixing angles, and the Dirac and Majorana phases for arbitrary initial mixing. Since $m_1$ and $m_2$ are nearly degenerate, the corrections to $\theta_{12}$ and the Dirac and Majorana phases could be very large. We performed a numerical analysis on the mass matrices with
$\mu-\tau$ symmetry to illustrate our analytical results, and found that the final Dirac phase can take any value under small perturbations.

We also studied the scenario in which the charged lepton mass matrix is not diagonal, and considered perturbations on the charged lepton mass matrix. We found that small perturbations in the charged lepton sector give small mixing in the 1-3 and 2-3 sectors, but the mixing in the 1-2 sector could be potentially large due to the near degeneracy of $m_e$ and $m_\mu$ (on the scale of $m_\tau$), which could lead to large corrections to all three mixing angles in the PMNS matrix.

In addition, we showed that using our generalized perturbation procedure, it is straightforward to study neutrino oscillations with both nonstandard scalar and nonstandard vector interactions.

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A Diagonalization of a $2 \times 2$ complex symmetric matrix

We show how to diagonalize a $2 \times 2$ complex symmetric matrix

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

so that

$$U^T M U = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix},$$

where $m_1$, $m_2$ are non-negative real numbers, and $U$ is an unitary matrix.

First, we diagonalize $M$ with a unitary matrix $V$ of the form

$$V = \begin{pmatrix} c_\xi & s_\xi e^{-i\phi} \\ -s_\xi e^{i\phi} & c_\xi \end{pmatrix},$$
where \(c_\xi\) and \(s_\xi\) denote \(\cos \xi\) and \(\sin \xi\), respectively:

\[
V^T M V = \begin{pmatrix}
ac_\xi^2 + cs_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi} & (ae^{-i\phi} - ce^{i\phi})s_\xi c_\xi + b(c_\xi^2 - s_\xi^2) \\
( ae^{-i\phi} - ce^{i\phi} )s_\xi c_\xi + b(c_\xi^2 - s_\xi^2) & as_\xi^2 e^{-2i\phi} + cc_\xi^2 + 2bs_\xi c_\xi e^{-i\phi}
\end{pmatrix}.
\]

(70)

The diagonalization condition is

\[
( ae^{-i\phi} - ce^{i\phi} )s_\xi c_\xi + b(c_\xi^2 - s_\xi^2) = 0,
\]

(71)

which implies the phase \(\phi\) is

\[
\phi = \arctan \frac{|a| \sin(\phi_a - \phi_b) - |c| \sin(\phi_c - \phi_b)}{|a| \cos(\phi_a - \phi_b) + |c| \cos(\phi_c - \phi_b)},
\]

(72)

and the mixing angle \(\xi\) is

\[
\xi = \frac{1}{2} \arctan \frac{2|b|}{|c| \cos(\phi_c + \phi - \phi_b) - |a| \cos(\phi_a - \phi - \phi_b)}.
\]

(73)

Also, the two eigenvalues can be written as

\[
m_1 = |ac_\xi^2 + cs_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi}|,
\]

\[
m_2 = |as_\xi^2 e^{-2i\phi} + cc_\xi^2 + 2bs_\xi c_\xi e^{-i\phi}|.
\]

(74)

The final unitary matrix that diagonalizes \(M\) is

\[
U = V \begin{pmatrix}
e^{i\omega_1/2} & 0 \\
0 & e^{i\omega_2/2}
\end{pmatrix},
\]

(75)

where

\[
\omega_1 = -\arg \left(ac_\xi^2 + cs_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi}\right),
\]

\[
\omega_2 = -\arg \left(as_\xi^2 e^{-2i\phi} + cc_\xi^2 + 2bs_\xi c_\xi e^{-i\phi}\right).
\]

(76)

B Right-multiplication

We calculate the change of mixing parameters when a general initial mixing matrix \(U_0\) (see Eq. 15) is multiplied by the following unitary matrix from the right:

\[
U_{12}(\xi, \phi) = \begin{pmatrix}
c_\xi & s_\xi e^{-i\phi} & 0 \\
-s_\xi e^{i\phi} & c_\xi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(77)
Multiplying $U_0$ on the right by $U_{12}$ yields

$$
U = R_{23}(\theta^0_{23})U_{13}(\theta^0_{13}, \delta^0)R_{12}(\theta^0_{12}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\delta^0/2} & 0 \\ 0 & 0 & e^{i\delta^0/2} \end{pmatrix} U_{12}(\xi, \phi)
$$

$$
= \begin{pmatrix}
    c_{13}^0 \tilde{C}_{12} & c_{13}^0 \tilde{S}_{12} e^{i\delta^0/2} & e^{-i(\delta^0 - \frac{\delta^0}{2})} s_{13}^0 \\
    -c_{23}^0 \tilde{S}_{12} - s_{23}^0 s_{13}^0 \tilde{C}_{12} e^{i\delta^0} & (c_{23}^0 \tilde{C}_{12} - s_{23}^0 s_{13}^0 \tilde{S}_{12} e^{i\delta^0}) e^{i\delta^0/2} & e^{i \frac{\theta^0}{4}} c_{13}^0 s_{23}^0 \\
    s_{23}^0 \tilde{S}_{12} - c_{23}^0 s_{13}^0 \tilde{C}_{12} e^{i\delta^0} & (s_{23}^0 \tilde{C}_{12} + s_{23}^0 s_{13}^0 \tilde{S}_{12} e^{i\delta^0}) e^{i\delta^0/2} & e^{i \frac{\theta^0}{4}} c_{13}^0 c_{23}^0
\end{pmatrix}, \quad (78)
$$

where

$$
\tilde{C}_{12} = c_{12}^0 c_\xi - s_{12}^0 s_\xi e^{i \frac{\phi^0 + 2\phi}{2}}
$$

and

$$
\tilde{S}_{12} = s_{12}^0 c_\xi + c_{12}^0 s_\xi e^{i \frac{\phi^0 + 2\phi}{2}}
$$

are complex. Comparing $U$ to the standard parametrization, we find that

$$
\theta_{23} = \theta^0_{23}, \quad \theta_{13} = \theta^0_{13},
$$

and

$$
\theta_{12} = \arcsin(|\tilde{S}_{12}|) = \arcsin \sqrt{\sin^2(\theta^0_{12} + \xi) - \sin(2\theta^0_{12}) \sin(2\xi) \sin^2 \frac{\phi^0 + 2\phi}{4}}. \quad (82)
$$

Note that after the right-multiplication the phases of the resulting mixing matrix are not in the standard form. Defining

$$
\alpha = \text{arg}(\tilde{C}_{12}) = -\arctan \frac{\tan \theta^0_{12} \tan \xi \sin(\phi^0_{2}/2 + \phi)}{1 - \tan \theta^0_{12} \tan \xi \cos(\phi^0_{2}/2 + \phi)},
$$

$$
\beta = \text{arg}(\tilde{S}_{12}) = \arctan \frac{\tan \xi \sin(\phi^0_{2}/2 + \phi)}{\tan \theta^0_{12} + \tan \xi \cos(\phi^0_{2}/2 + \phi)},
$$

we can write $U$ as

$$
U = \begin{pmatrix}
    c_{13}^0 |\tilde{C}_{12}| e^{i\alpha} & c_{13}^0 |\tilde{S}_{12}| e^{i(\frac{\delta}{4} - \beta)} & e^{-i(\delta^0 - \frac{\delta^0}{2})} s_{13}^0 \\
    -c_{23}^0 |\tilde{S}_{12}| e^{i\beta} - s_{23}^0 s_{13}^0 |\tilde{C}_{12}| e^{i(\delta^0 + \alpha)} & (c_{23}^0 |\tilde{C}_{12}| e^{-i\alpha} - s_{23}^0 s_{13}^0 |\tilde{S}_{12}| e^{i(\delta^0 - \beta)}) e^{i\frac{\delta}{2}} & e^{i \frac{\theta^0}{4}} c_{13}^0 s_{23}^0 \\
    s_{23}^0 |\tilde{S}_{12}| e^{i\beta} - c_{23}^0 s_{13}^0 |\tilde{C}_{12}| e^{i(\delta^0 + \alpha)} & (-s_{23}^0 |\tilde{C}_{12}| e^{-i\alpha} - c_{23}^0 s_{13}^0 |\tilde{S}_{12}| e^{i(\delta^0 - \beta)}) e^{i\frac{\delta}{2}} & e^{i \frac{\theta^0}{4}} c_{13}^0 c_{23}^0
\end{pmatrix}. \quad (85)
$$
On removing the unphysical phases $\phi_e = \alpha$ and $\phi_\mu = \phi_\tau = \beta$ from the rows, the phases in the second and third columns match the form of the standard parametrization, with the Majorana phases shifted by

$$\Delta \phi_2 = -2(\alpha + \beta),$$

$$\Delta \phi_3 = -2\beta,$$

and the Dirac phase shifted by

$$\Delta \delta = \alpha - \beta.$$  \hspace{1cm} (86)

$$\hspace{1cm} (87)$$

$$\hspace{1cm} (88)$$

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