Information form of the second law of thermodynamics

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An essential role of information in microscopic thermodynamics (e.g. Maxwell’s demon) opens a challenging question if there exists a formulation of the second law of thermodynamics based only on pure information ideas. Here, such a formulation is suggested for unitary processes by introducing information as a full-valuable physical quantity defining an (objective) microscopic information entropy as ‘information about microstate’. We show that various forms of entropy (Boltzmann, Shannon, Clausius) are in fact only a special cases of information entropy whose general form is found out. An observer plays here the role of a special (information) reference frame (IRF) towards which the entropy is defined. Some paradoxes or misunderstandings connected with the concept of entropy or the content of the second law arise by describing a situation without specifying a concrete IRF. Typically, the Boltzmann statistical approach cannot be symmetrically used towards the past as towards the future in one IRF. The information second law is full usable at meso- or microscopic scales: the information form of the generalized second law is found out too.

I. INTRODUCTION

As noticed by J.C. Maxwell in 1871, if a being (“demon”) has information about individual molecules of a system and is able to manipulate with them, the second law of thermodynamics can be violated [1]. The patch to this ”hole in law” has been found in the last decades: demon’s information about molecules has to be taken as a full-validated physical component of the description of the whole situation [2]–[7]. This idea has been demonstrated and verified in various experiments at mesoscopic and nanoscales [8]–[10], it has surprising applications in molecular biology [11]–[12] and nanoscale technologies [13]–[14]. A general physical theory of information addressing the basic problems of statistical physics (macroscopic time-asymmetry, objective microscopic definition of entropy) [4], however, does not exist yet.

In such a theory, information should be a basic concept expressing someone’s knowledge about a physical system [15]. To bring this idea closer, consider a system observed by several observers so that each of them has a different knowledge about it. The perfect observer [16] knows its microscopic state, x, others know less. The knowledge of individual observers can be quantified by using the ingenious idea by C.E. Shannon [17]. Imagine that the observers receive the message fully describing x. The value of information included in this message is different for each of them (for the perfect observer the message is worthless). Denote as \( I_O(x) \) this value for an observer \( O \). We can say that this value quantifies observers’ ignorance (lack of information) as to \( x \) before receiving the message.

If \( O = O(M) \) is a typical macroscopic observer then \( S = I_O(M) \) is the definition of entropy by E.T. Jaynes [18]. Entropy is then a function of microscopic state \( x \). Its evident dependence on a ”typical macroscopic observer” is, however, incompatible with objective physics.

But notice that the observer in the definition of \( I_O(x) \) plays only the role of some informational reference with respect to which information in the message is assessed. The fact that the value of a quantity is defined only with respect to another value of this quantity is not unfamiliar in physics.

To determine the position of a body, for example, we must do it in reference to positions of other bodies. The body has undoubtedly an objective location in space, its concrete meaning, however, is given in a concrete reference frame. The concept of microscopic information entropy, \( S = I_O(x) \), is analogical. It can be understood as an abstract objective quantity expressing information connected with a given microscopic state \( x \). Its concrete value and meaning, however, must be specified in a concrete information reference frame (IRF) represented, for example, by a macroscopic observer \( O(M) \).

We do not observe processes at the very microscopic level and information about them is usually given indirectly by observing other (usually some coarse-grained) quantities, which is the original idea by L. Botzmann [19]. Information gained by observing a system at an arbitrary spatial scale is what define a concrete IRF. It implies that the role of a macroscopic observer is not exclusive. We can have arbitrary information reference frames defined with respect to our experimental abilities or our intentions to model a concrete system. Information entropy - an objective quantity - has different values at these reference frames. A full microscopic description of a system (e.g. as a pure quantum state) is also a special (microscopic) information frame.

In this contribution, we use the idea of information reference frames to derive a novel formulation of the second law of thermodynamics which introduces information as a fundamental physical quantity without a need to define a particular observer and to introduce the concept of probabilities. The law is formulated for single deterministic and time-symmetric microscopic trajectories that play the role of adiabatic processes at arbitrary spatial scales.
The finding of an explicit relation between information and entropy allows us to derive a general form of the entropy dependence on the microscopic state of the system in a concrete IRF, \( S_T(x) = k_B (\ln |\Gamma|_\Psi + \Psi_T(x)) \), where \( \Gamma \) is a subset of the state space determined by a chosen IRF, \( |\Gamma|_\Psi \) is a special measure, and \( \Psi \) is a real function whose concrete choice (the so-called \( \Psi \)-representation) defines a concrete form of the entropy. A special choice of \( \Psi \) defines probabilities, \( p \) and the Shannon entropy, \( S = -k_B \ln p \).

The information form of the second law is in fact an information generalization of Boltzmann’s ingenious ideas [19,21]. The information approach, however, avoids the problematic point of Boltzmann’s statistical program: namely the fact that his statistical method gives peculiar results when applied towards the past. We argue that the way of thinking towards the past is not symmetric from information point of view: it must be done in another information reference frame. The paradox then disappears.

The use of our approach into thermodynamics defined at microscopic or mesoscopic scales is straightforward. We can connect the change of entropy with the dissipation work. The generalized second law is presented here in a pure information form.

The paper is organized as follows. The concept of value of information and the information reference frame are introduced and the information entropy is defined in Section II. The general definition of adiabatic processes is formulated and the information formula described the change of information entropy in these processes is found out in Section III. In Section IV, the concrete form of information entropy (i.e., its dependence on the microstate of the system and the used information reference frame) is derived. It generalizes the Boltzmann entropy. The information form of the second law of thermodynamics is discussed in Section V. Special forms of the information entropy (the so-called \( \Psi \)-representations) are found out in Section VI. The Shannon entropy and Clausius entropy of classical equilibrium thermodynamics. In Section VII, the concept of probability arising in a special \( \Psi \)-representation of the information entropy is studied, especially its referential meaning given by a special choice of the information reference frame. We show in Section VIII that some concepts of nonequilibrium statistical physics and the generalized second law of thermodynamics are special applications of the information second law.

II. INFORMATION REFERENCE FRAME AND ENTROPY

Information about a system \( S \) can be identified with a subset \( \Omega \) of its microscopic state space \( \Gamma_S \). Namely the set \( \Omega \) can be interpreted as including all microscopic states that are consistent with attainable information about the system observed (studied) by a concrete observer. A typical example is Boltzmann’s macrostate that includes all microstates that look the same for a macroscopic observer. A subset of the microscopic state space \( \Omega \subset \Gamma_S \) may represent, however, arbitrary information about the system gained at micro-, meso- or macroscopic scales.

If we want to quantify information we need some referential information that plays the role of an information unit. Consider that an observer knows that the actual microstate belongs into a set \( \Gamma \). The set \( \Gamma \) plays the role of referential information. The value of information that the actual microstate, \( x \), belongs into a set \( \Omega \subset \Gamma \) can be defined with respect to referential information "\( x \in \Omega \)" as follows. Let the observer receive the message that "\( x \in \Omega \)". The information value of this message (for this observer) is a real number \( I_T(\Omega) \geq 0 \). If \( \Omega = \Gamma \) this value is obviously zero (the message is worthless), i.e., \( I_T(\Gamma) = 0 \). The larger the information value \( I_T(\Omega) \) the more valuable information is gained by the observer after receiving the message "\( x \in \Omega \)".

It is important to say that the referential set \( \Gamma \) is not an arbitrarily chosen subset of the state space. It plays the role of an 'information content' of a concrete observer, whereas 'observer' may be an experimental device, a robot with sensors, a typical macroscopic observer, etc. If an observer connected with referential information \( \Gamma \) gains additional information about the system, say "\( x \in \Omega_0 \)" the referential set changes into \( \Gamma' = \Gamma \cap \Omega_0 \).

A collection of referential sets \( \Gamma(t) \) for various \( t \) during a single microscopic process is called the information reference frame (IRF) in which this process is studied. Let an observer (or observers) study a process in between times \( t_i \) and \( t_f \). The IRF may be formed by all \( \Gamma(t), \ t \in (t_i,t_f) \), if the referential set can be defined at each time moment (e.g., when the process can be continuously observed). An extreme case of IRF describing a process from \( t_i \) to \( t_f \) is the IRF formed only by \( \Gamma(t_i) \) and \( \Gamma(t_f) \).

The important situation occurs if referential sets can be identified with values of some physical quantities \( \alpha \), we call them the observable indicators. The IRF is then defined by time evolution of these quantities (see Fig. 1). The set \( \Gamma = \Gamma(\alpha) \) then includes all microscopic states that are consistent with the value of \( \alpha \). Typical indicators are various coordinates defining the thermodynamic state [22,23] regardless at which length scale the observation is done (volume of a gas, quantities at small hydrodynamic cells, distance of the ends of a macromolecule [24]).

The observable indicators may be fully controlled by an observer (e.g., driven by an external agent [25,27]) or may be understood as special degrees of freedom of a fully autonomous system. In the latter case, the microstate \( x \) of the system equals \( x = (s,\alpha) \), where \( s \) are internal (not-observed) degrees of freedom (see Fig. 1), and

\[
\Gamma(\alpha) = \{(s,\alpha), \ s \in \Gamma(\alpha)\}. \tag{1}
\]

The set \( \Gamma \) may represent a "grain" in a coarse-grained modeling [28] and, as an extreme case, it can be even identical with the actual microstate of the system, \( \Gamma = \{x\} \), which means that an observer has a complete information about the system. Information about the sys-
tem can be also formed by all quantum states $|\psi_i\rangle$ in a quantum projector $\sum |\psi_i\rangle\langle\psi_i|$ describing an incomplete knowledge about a system in a mixed state $\{\rho\}$. A special case of IRF is the IRF$_{\text{mic}}$ defined by $\Gamma_{\text{mic}}(t) = \{x(t)\}$.

The important point of the presented approach is that referential sets (and IRF) may not be connected with any observed indicators. An important example of such a situation arises if the microscopic evolution of the system is deterministic. Let us denote $\mathcal{F}$ the evolution operator $x_i = \mathcal{F}(x_i)$, where $x_i$, $x_f$ is the initial and final microstate, respectively. Let $\Gamma_i$, $\Gamma_f$ be the referential sets at times $t_i$, $t_f$, respectively. Denote $G_i \subset \Gamma_i$ the set of all states $x \in \Gamma_i$ so that $\mathcal{F}(x) \in \Gamma_f$, and $G_f \equiv \mathcal{F}(G_i)$ (see Fig. 2).

We define the so-called *dynamic* information reference frame IRF’ as $\Gamma'(t_i) = G_i$, $\Gamma'(t_f) = G_f$. The referential sets here cannot be defined by current values of some observable indicators. If $\Gamma(t)$ are defined by current values of observable indicators $\alpha$ the sets $G(t)$ are given by knowledge of the process $\alpha_i \rightarrow \alpha_f$. The transformation of the reference frame into the dynamic one, IRF $\rightarrow$ IRF’, plays important role in what follows.

**Entropy.** From an information theoretic perspective, entropy is associated with observer’s ignorance (lack of information) as to the microstate of the system. In other words, the observer has some (incomplete) information about the microstate of the system and entropy measures the value of additional information that is necessary to ”add” to this knowledge to determine the actual microstate precisely. The knowledge of observer is identified here with referential information. It implies that

$$S_T(x) = k_B I_T(\{x\}),$$

where $k_B$ is Boltzmann’s constant and $I$ is a referential set from this IRF.

From thermodynamic point of view, entropy is a *state* quantity what means that $S$ can depend *only* on such information that concerns the system at a concrete time. It implies that each referential set from an IRF must be interpreted as information about the system at a concrete time moment only. If a set of the dynamic IRF defined above, i.e. $G(t)$, is used to define entropy it must be interpreted as special information about a concrete time moment that is not determined by observable indicators.

The information entropy defined by Eq. (2) may be understood as an objective, microscopic quantity (it depends on $x$) whose meaning is simply ”information about the actual microstate”. The well-known doubts concerning objectivity of entropy become irrelevant here because its observer-dependence is nothing else but a usual dependence of any physical quantity on some reference element (unit, frame, etc.). The concrete value of entropy is the value of information evaluated always in a concrete IRF. Notice that the entropy is, for example, always zero in the IRF$_{\text{mic}}$.

**Properties of information value.** Consider a physical phenomenon that can happen if and only if $x \in \Omega$. A very large value of $I_T(\Omega)$ means that the realization of this phenomenon is very rare for this observer (the value of message ”this number will win in lotto” is very high). Hence $I_T(\Omega) \rightarrow \infty$ means that the observer can observe this phenomenon with an overwhelmingly low probability, i.e. practically cannot observe this phenomenon at all.

Let us postulate the two important properties of $I_T$:

(i) If $\Omega \subset \Gamma_2 \subset \Gamma_1$ then the observer who knows $\Gamma_1$ needs
additional information $x \in \Gamma_2$ to become equivalent to that one who knows $\Gamma_2$, i.e., we postulate

$$I_{\Gamma_1}(\Omega) = I_{\Gamma_2}(\Omega) + I_{\Gamma_1}(\Gamma_2).$$  \hspace{1cm} (3)

(ii) The smaller is the set $\Omega$ with respect to $\Gamma$ the higher must be the value of information detecting $\Omega$. That is why we postulate

$$I_{\Gamma}(\Omega) = \Phi_{\Gamma}(M_{\Gamma}(\Omega)) \geq 0,$$  \hspace{1cm} (4)

where $\Phi_{\Gamma}$ is a decreasing real function and $M_{\Gamma}$ is a real measure of $\Omega$ on $\Gamma$ so that $I_{\Gamma}(\Gamma) = 0$. Concerning the measure $M_{\Gamma}$ we suppose that $M_{\Gamma}(\Omega) \to 0$ if the size of $\Omega$ goes to zero. It implies $^{32}$ that $M_{\Gamma}$ is expressed via a real, non-negative function $f_{\Gamma}$ so that $M_{\Gamma}(\Omega) = \int_{\Omega} f_{\Gamma}(z) dz$ in continuous, and $M_{\Gamma}(\Omega) = \sum_{z_i \in \Omega} f_{\Gamma}(z_i)$ in discrete cases.

The special case of $I_{\Gamma}$ fulfilling the postulated properties is $I_{\Gamma}^{Sh}(\Omega) = -\ln p(\Omega | \Gamma)$ (Shannon’s formula), where $p(\Omega | \Gamma)$ is the probability measure on $\Gamma$, i.e., $M_{\Gamma}(\Omega) = p(\Omega | \Gamma)$, $\Phi_{\Gamma} = -\ln$. The information value $I_{\Gamma}$, however, can be defined without any connotation to probabilities.

III. ADIABATIC PROCESSES

Any process governed only by inner dynamics of an isolated system is adiabatic. The concept of adiabatic processes is, however, broader. Namely, it is any process during which the exchange of energy with its surrounding exists only in the form of work. Work is defined in classical macroscopic thermodynamics, its generalization to meso- or microscopic processes is far from being easy and straightforward.

The well-accepted, scale-independent definition of processes during which the system-surrounding interaction is realized only with the exchange of work concerns situations when the evolution of observable indicators is firmly given by a prescribed protocol $^{26,27}$ and no other external intervention exists. Though there exists an exchange of energy with the system environment, such a process is fully deterministic and reversible. It motivates us to define the adiabatic process as any deterministic and reversible process $x_i \to x_f$, where $x$ is the full (microscopic) state of the system.

A concrete realization of the "prescribed-protocol" interaction means the existence of macroscopic massive bodies or some device that are insensitive to microscopic interactions. The work then can be defined for the system alone (supplied work) while its transfer into the surrounding cannot be quantify anyhow (see Fig. 3b). The above definition of the adiabatic process as any deterministic and reversible (unitary) process, however, does not exclude possibility of defining the work performed in the surrounding.

Let $S$ be a system and we study processes beginning at various microscopic states of this system, $x_i \in \Gamma_S$.

During any process, the system interacts with its environment, $E$, so that $S + E$ is a fully isolated system (universe). Suppose that the environment and the system-environment interaction is designed in such a way that the initial (microscopic) state of the environment is always $e_i$ and its final state (say, at a sufficiently large time) is always $e_f$. The transferred work into this environment, $W_{tr}$, is a function of $(e_f, e_i)$ (see Fig. 3a).

The universe $S + E$ is an isolated system, hence the process $(x_i, e_i) \to (x_f, e_f)$ is deterministic and reversible. Since $e_i$ and $e_f$ are fixed, the process $z_i \to z_f$ is deterministic and reversible too (though the trajectory of the environment and its interaction with the system is different for various initial conditions of the system). This idea is a generalization of the concept of adiabatic accessibility $^{34}$ into microscopic scales.

Information conservation. — During adiabatic evolution, information about possible microstates of the system has to be conserved. The reason is that the evolution is deterministic and reversible which means that the configuration at any time moment carries information about that at any other time.

Let $\mathcal{F}_\tau$, $\tau \in \mathbb{R}$ be a set of one-to-one mappings expressing the time shift of the system microstate during time evolution, i.e. $x(t+\tau) = \mathcal{F}_\tau(x(t))$ (hence the inverse mapping of $\mathcal{F}_\tau$ is $\mathcal{F}_{-\tau}$). Since the value of information about microstates is always related to referential information the law of conservation of information must be expressed as follows:

$$\forall \tau \forall \Omega \subset \Gamma : I_{\mathcal{F}_{\tau}(\Gamma)}(\mathcal{F}_\tau(\Omega)) = I_{\Gamma}(\Omega).$$  \hspace{1cm} (5)

The referential set, however, usually does not vary according the rule $\Gamma \to \mathcal{F}_\tau(\Gamma)$. If it is defined by values of some observable indicators, i.e. $\Gamma(\alpha)$, the set $\Gamma(\alpha(t+\tau))$ is usually different from $\mathcal{F}_\tau(\Gamma(\alpha))$ (see Figs. 1b).
Adiabatic entropy change. — The transformation into a dynamic reference frame IRF', IRF → IRF', can be written by the use of Eq. (3). If \( \Omega \subseteq G_i \) we get
\[
I_{\Gamma_i}(\Omega) = I_{G_i}(\Omega) + I_{\Gamma_i}(G_i),
\]
and
\[
I_{\Gamma_i}(F_r(\Omega)) = I_{G_i}(F_r(\Omega)) + I_{\Gamma_i}(G_f).
\]
Since \( G_f = F_r(G_i) \) the law of conservation of information cancels the first terms on the right-hand sides of the equations and we get that \( I_{\Gamma_i}(\Omega) - I_{\Gamma_i}(F_r(\Omega)) \) does not depend on \( \Omega \), i.e.
\[
I_{\Gamma_i}(F_r(\Omega)) - I_{\Gamma_i}(\Omega) = I_{\Gamma_i}(G_f) - I_{\Gamma_i}(G_i) \equiv \Delta J, \tag{6}
\]
where \( \Delta J = J' - J \), \( J \equiv I_{\Gamma_i}(G_i), J' \equiv I_{\Gamma_i}(G_f) \).

If we chose \( \Omega = \{ x \} \), \( x \in G_i \), we get the change of the entropy distribution \( \Delta S(x) = S_{\Gamma_i}(F_r(x)) - S_{\Gamma_i}(x) \) so that
\[
\Delta S(x) = k_B \Delta J. \tag{7}
\]

The right-hand side of Eq. (7) depends only on \( \Gamma_i, \Gamma_f \) and the mapping \( F_r \). It implies that the change of entropy is the same for each \( x \in G_i \),
\[
\forall x, x' \in G_i : \Delta S(x) = \Delta S(x') \equiv \Delta S. \tag{8}
\]
The change of entropy during an adiabatic process that is observed as \( \alpha_i \rightarrow \alpha_f \) must begin at \( G_i \) hence \( \Delta S \) is determined by the change of \( \alpha \), i.e. \( \Delta S(\alpha_i \rightarrow \alpha_f) \). It is a very important result: though the information entropy is a microscopic quantity its change is determined only by the change of quantities defined at the scale where the system is studied.

IV. CONCRETE FORM OF THE INFORMATION ENTROPY

Let \( \Gamma \) be a subset of a system state space and \( x \in \Gamma \). The value of information entropy \( S_{\Gamma}(x) \) is uniquely determined by \( x \) and \( \Gamma \). To find out this dependence we suppose that there exists a sufficiently small \( \tau_0 \) so that for each \( 0 < \tau < \tau_0 \) the element \( F_r(x) \in \Gamma \), where \( F_r \) represent an adiabatic process. Hence we can choose the same referential sets, \( \Gamma(t) = \Gamma \), for each \( t \in (0, \tau_0) \). It allows us to form the IRF for each \( \tau (\tau < \tau_0) \) so that
\[
\Gamma(0) \equiv \Gamma_i = \Gamma \text{ and } \Gamma(\tau) \equiv \Gamma_f = \Gamma.
\]

Since \( \Gamma \) is fixed in this consideration we omit the index \( \Gamma \) in all quantities, i.e. \( S \equiv S_{\Gamma} \) and the quantities in Eq. (4). \( M \equiv M_{\Gamma}, f \equiv f_{\Gamma}, \) and \( \Phi \equiv \Phi_{\Gamma} \). Since \( S(x) = k_B I_{\Gamma}(f(x)) = k_B \Phi(f(x)) \) we get
\[
f(x) = \Phi^{-1}(\psi(x)), \tag{9}
\]
where \( \Phi^{-1} \) is the inverse function of \( \Phi \) and \( \psi(x) \equiv k_B^{-1} S(x) \).

Suppose that the state space is discrete. Then the measure \( M(\Omega) = \sum_{x \in \Omega} f(x) \) and Eq. (6) becomes
\[
\Phi(\sum_{x \in \Omega} \Phi^{-1}(\psi(x) + \Delta J)) = \Phi(\sum_{x \in \Omega} \Phi^{-1}(\psi(x))) + \Delta J
\]
(we use Eq. (7) to express that \( \psi(F(x)) = \psi(x) + \Delta J \)).

The validity of Eq. (10) for all \( \tau \in (0, \tau_0) \) determines a possible form of \( I_{\Gamma i} \) (see Appendix), namely
\[
I_{\Gamma}(\Omega) = -\gamma \ln \sum_{x \in \Omega} e^{-\gamma^{-1} k_B S_i(x)}, \tag{11}
\]
where \( \gamma \) is a positive constant. It is easy to check that this form of \( I_{\Gamma} \) fulfills Eq. (9) for any \( \Gamma_i, \Gamma_f \) and \( \Delta J \) if the entropy distribution fulfills Eq. (7).

To fulfill the condition \( I_{\Gamma}(\Gamma) = 0 \) the entropy has to have a form
\[
S_{\Gamma}(x) = k_B \gamma (\ln |\Gamma|_{\psi} + \Psi_{\Gamma}(x)), \tag{12}
\]
where \( \Psi_{\Gamma} \) is a real function on the state space of the system that may depend also on \( \Gamma \), \( \gamma \) is a constant, and \( |\Gamma|_{\psi} \) is the special measure of the set \( \Gamma \),
\[
|\Gamma|_{\psi} = \sum_{x \in \Gamma} e^{-\Psi_{\Gamma}(x)} \tag{13}.
\]
The sums \( \sum_{x \in \Omega} \) in Eqs. (12) must be replaced by integrals, \( \int_{\Omega} dx \), in the case of the continuous state space. The constant \( \gamma \) only rescales the information value and information entropy. In what follows we fix \( \gamma = 1 \).

The distribution of \( \Psi_{\Gamma} \) on \( \Gamma \) depends on the mappings \( F_r \). Namely putting Eq. (12) into Eq. (9) we get
\[
\Delta \Psi(x) = \Psi_{\Gamma_f}(x(t + \tau)) - \Psi_{\Gamma_i}(x(t)) = \Delta J(\tau) + \ln |\Gamma|_{\psi} / |\Gamma|_{\psi}. \tag{14}
\]
It implies that
\[
\forall x \in G_i(F) : \Delta \Psi(x) = C(\Gamma_i, \Gamma_f, F). \tag{15}
\]
This condition is analogical to Eq. (8) which means that the change of function \( \Psi \) is similarly defined on the observation scale.

The only entropy distribution that fulfills Eq. (7) for an arbitrary \( F \) arises if we choose \( \Psi_{\Gamma}(x) = C(\Gamma) \), i.e. it does not depend on \( x \). We get
\[
S_{\Gamma}(x) = k_B \ln |\Gamma| = S_B, \ (x \in \Gamma) \tag{16}
\]
where \( |\Gamma| \) is the volume of the set \( \Gamma \). That is we get the Boltzmann entropy \( S_B \) as a special case of the information entropy [12].

Probabilistic interpretation. — The formula (11) can be interpreted otherwise. If we define
\[
p_{\Gamma}(\Omega) \equiv e^{-\gamma^{-1} I_{\Gamma}(\Omega)} > 0. \tag{17}
\]
and write simply $p_T(x)$ instead of $p_T(\{x\})$, Eq. (11) can be written as

$$p_T(\Omega) = \sum_{x \in \Gamma} p_T(x),$$

and $\sum_{x \in \Gamma} p_T(x) = 1$ because of the condition $I_T(\Gamma) = 0$.

The values of $p_T$ thus can be understood as a probability distribution over the referential set $\Gamma$ so that the probability

$$P(x \in \Omega) = e^{-I_T(\Omega)}$$

when putting $\gamma = 1$. It implies the Shannon-like expression of entropy,

$$S_T(x) = -k_B \ln p_T(x).$$

Notice that Eq. (19) implies that $I_T(\Omega) \to \infty$ if and only if $P(x \in \Omega) \to 0$.

V. THE SECOND LAW OF THERMODYNAMICS

Consider an observer, say Alice, who knows data about a system $S$ (possibly meso- or microscopic) only at a given time $t_i$; she performs a measurement at $t_i$ and gains some data $\alpha(t_i) = \alpha_i$ about the system. Alice knows that the next evolution of $S$ is adiabatic (its microscopic evolution is deterministic and reversible). Alice asks which value of $\alpha$ can be detected at a time $t_f > t_i$.

Each possible value of $\alpha(t_f) \equiv \alpha_*$ forms a subset $G(\alpha_*) \subset \Gamma_i$ of all possible initial microstates that realize the evolution $\alpha_i \to \alpha_*$. (this subset may be empty). Alice does not know, however, in which subset the actual microstate occurs. Nevertheless, information that $x_i \in G(\alpha_*)$ has different values for various $\alpha_*$, namely $I(\alpha_*) = I_T(G(\alpha_*))$. The largest this value the smaller probability $P$ that $x_i$ belongs into this subset as implied by Eq. (19).

$$P(x_i \in G(\alpha_*)) = e^{-I(\alpha_*)}.$$  

Hence if $I(\alpha_*) \to \infty$ then the result $\alpha(t_i) = \alpha_i$ cannot occur since its probability is zero.

In macroscopic thermodynamics, the second law can be expressed as impossibility of adiabatic processes during which the entropy decreases, i.e. during which $\Delta S < 0$. Let us study the situation when the entropy decreases by the use of Eq. (7) in the macroscopic limit $k_B \to 0$ so that $\Delta S \neq 0$.

If $\Delta S/k_B \to -\infty$, Eq. (7) implies $\mathcal{J} - \mathcal{J}' \to \infty$, i.e. $\mathcal{J} \to \infty$ since $\mathcal{J}' \geq 0$. $\mathcal{J} = I_T(G_i) = I(\alpha_f)$ is the value of information that $x_i \in G_i$, i.e. that the process $\alpha_i \to \alpha_f$ will happen. Eq. (21) implies that if $\Delta S < 0$ then the probability of this process is zero. Hence only macroscopic adiabatic processes with non-decreasing entropy are possible. Eq. (7) thus expresses the second law of thermodynamics.

This formulation of the second law is formally very close to Boltzmann’s statistical derivation of this law. The main argument is probabilistic too: if $\Delta S < 0$ and $|\Delta S| \gg k_B$ the "target" set $G_f$ is extremely smaller than the set of all initial possibilities ($\Gamma_i$) and there is an overwhelmingly small probability of "hitting" it (to realize the process $\alpha_i \to \alpha_f$).

The information formulation, however, can avoid the principle problem of Boltzmann, i.e. the fact that the statistical argumentation can be used in the opposite direction to make a paradox. To show it, imagine another observer, say Bob, who has information about the same system $S$ at time $t_f$ without communicating with Alice. Bob detects the value $\alpha(t_f) = \alpha_f$ and asks which value of $\alpha$ was at time $t_i < t_f$. If $\Delta S = S(t_f) - S(t_i) > 0$ we get from Eq. (7) in the macroscopic limit that $\mathcal{J}' \to \infty$. The probability that $x_i$ was in $\Gamma_i$ is thus overwhelmingly small (see Fig. 3). Bob must conclude that the past of the system could not be so that $\alpha(t_i)$ was $\alpha_i$. This conclusion is, however, false.

Nevertheless, the situation of Alice and Bob is not symmetric from information point of view. Bob can receive a message from Alice about the situation at $t_i$ while Alice cannot have a message from Bob about the situation at $t_f$ (information cannot be send into the past). The conclusion of Alice is thus based on all possibly attainable information (at her observation scale) about the system at $t_i$. The situation of Bob is different. There are two possibility concerning his information state (see Fig. 4): (i) Bob has a record about the situation at $t_i$ (e.g. Alice’s message). Then his information about the system is not only $\alpha(t_f)$ (his measurement) but also $\alpha(t_i)$ (the record). Having this information Bob cannot use $\Gamma_f$ as the set of all possibilities corresponding to his information about the system. If he knows that $\alpha(t_i) = \alpha_i$ and...
\(\alpha(t_f) = \alpha_f\) then all possibilities are given by sets \(G_i\) and \(G_f\). Bob thus uses the dynamic referential frame, IRF', in which \(\Gamma'(t_i) = G_i, \Gamma'(t_f) = G_f\) (see Fig. II). In this referential frame \(\Delta S' = 0\) and Eqs. (19) in IRF gives that \(P(x_i \in G_i) = P(x_f \in G_f)\). No paradox arises.

(ii) There is no record about the situation at \(t_i\). Bob concludes from Eq. (7) that \(\alpha\) at \(t_i\) could not be \(\alpha_f\). His conclusion, however, cannot be verified anyhow: Bob cannot send a message into the past and no record about the situation at \(t_i\) exists. Whenever a record about the situation at \(t_i\) appears (e.g. in an indirect physical proof) Bob must shift his consideration into the dynamic referential frame defined by sets \(\Gamma' = G\) and we get again the situation (i).

VI. \(\Psi\)-REPRESENTATIONS OF ENTROPY

The function \(\Psi_T\) has a unique meaning at the observable scale since its change is determined only by the process \(\alpha_i \rightarrow \alpha_f\) as implied by Eq. (15). A concrete choice of the function \(\Psi_T\) defines the information entropy and, consequently, the information value \(Z_T\). We call it the \(\Psi\)-representation. The simplest \(\Psi\)-representations are related to several kinds of entropy:

**Boltzmann entropy.** It arises in the \(C\)-representation in which the function \(\Psi_T\) is constant (Eq. (16)).

**Shannon entropy.** The probabilistic interpretation of information entropy as given by Eqs. (19) can be interpreted as a special \(\Psi\)-representation, called the \(\Psi\)-representation. We get it if the first term in Eq. (12) is identically zero, i.e. \(|\Gamma|_\Psi = 1\) for any arbitrary \(\Gamma\). If denote \(\pi_T(x) \equiv \exp(-\Psi_T(x)) > 0\), we get that the information entropy in \(p\)-representation, \(S_p\), now has the form of Shannon entropy, Eq. (20), \(\sum_{x \in \Gamma} \pi_T(x) = 1\) and \(\pi_T = p_T\) are probabilities.

**Clausius (equilibrium) entropy.** Another possible \(\Psi\)-representation we get if \(\Psi_T\) is a constant of motion, i.e. \(\Delta \Psi = 0\). If the system energy is conserved during the studied process we can identify \(\Psi_T(x) = \beta H(x)\), where \(H(x)\) is the system hamiltonian and \(\beta \equiv (k_B T)^{-1}\) with \(T\) being an artificially chosen constant. In this \(H\)-representation we have

\[ Z_T(\Omega) = -\ln \sum_{x \in \Omega} e^{-\beta H(x)} \], \( (H - \text{rep.}) \)  

and Eq. (12) becomes

\[ S_T(x) = T^{-1}(H(x) - F_T) = S_C, \quad (H - \text{rep.}) \]  

where \(F_T = -k_B T \ln \sum_x \exp(-\beta H(x))\) is the free energy. If \(x \in \Gamma_{eq}\), where \(\Gamma_{eq}\) is the equilibrium macrostate of the system, the hamiltonian can be identified with the system internal energy, \(U = H(x)\), the parameter \(T = T_{eq}\) is the thermodynamic temperature, and the information entropy becomes the classical (Clausius) entropy \(S_C\).

VII. PROBABILITIES

When assuming that the entropy in \(\Psi\)-representation equals the entropy in \(p\)-representation, i.e. \(S_T = S_p\), we find the probabilities in individual representations. In \(C\)-representation we get

\[ p_T(x) = |\Gamma|^{-1}. \quad (C - \text{rep.}) \]

Similarly in \(H\)-representation we have

\[ p_T(x) = \frac{e^{-\beta H(x)}}{Z_T} = p_{eq}(H(x)), \quad (H - \text{rep.}) \]

where \(Z_T = \sum_x e^{-\beta H(x)}\). It means that the probability of a microstate corresponds in the \(H\)-representation with the Boltzmann-Gibbs equilibrium distribution, \(p_{eq}(H)\). Its interpretation is, however, different. The microscopic energy of a system in thermal contact with a reservoir fluctuates and reaches the energy \(E\) with the probability \(p_{eq}(E)\). We, however, study an isolated system, i.e. \(H(x(t))\) is constant along its trajectory. The probability \(p_T(x)\) expresses in \(H\)-representation nothing but observer’s knowledge as to \(x\) in dependence on \(\Gamma(t)\) (that is expressed through \(Z_T\)).

During the unitary evolution the entropy change in \(p\)-representation, \(\Delta S = -(\ln p_T(F(x)) - \ln p_T(x))\), must be the same for all \(x \in G_i\), i.e.

\[ P_{out} = P_{in} e^{-k_B^{-1} \Delta S}, \]

where

\[ P_{in} = \sum_{x \in G_i} p_{in}(x), \quad P_{out} = \sum_{x' \in G_f} p_{T_{f'}}(x'). \]

This result looks strange. Namely we expect that the probability of occurrence of the system at a (micro)state \(x\) so that the process \(\alpha_i \rightarrow \alpha_f\) is realized, i.e. \(P_m\), must be the same as the probability of finding the final (micro)state that corresponds to the fact that the process \(\alpha_i \rightarrow \alpha_f\) has been realized, i.e. \(P_{out}\). However, if \(\Delta S \neq 0\) then \(P_{in} \neq P_{out}\).

The explanation of this (and other) seeming inconsistencies connected with Eq. (26) consists in understanding the entropy in information context. Namely information entropy is given only by information about the system concerning only one time moment. A correct interpretation of Eq. (26) should be done via the idea of two independent observers so that the first has information about the system at \(t_i\) only, the second does the same at \(t_f\) (Alice and Bob in Section VII). The first one observes \(\alpha_i\) and (in principle) deduce that the process \(\alpha_i \rightarrow \alpha_f\) happens with the probability \(P_{in}\). The second one observes \(\alpha_f\) and (in principle) deduce that the process \(\alpha_i \rightarrow \alpha_f\) has happened with the probability \(P_{out}\) (she/he does not know the past of the system).

If an observer knows that the process \(\alpha_i \rightarrow \alpha_f\) has happened this observer knows more than the previous
ones, i.e. she/he knows that \( x_i \in G_i \) and \( x_f \in G_f \). We can
describe this situation in the dynamic information reference frame, \( \text{IRF}' = \{ \Gamma'(t) \} \), in which the referential sets \( \Gamma'(t_i) = G_i \) and \( \Gamma'(t_f) = G_f \). In the IRF', \( \Delta S' = 0 \)
and the corresponding probabilities \( P'_\text{in} \) and \( P'_\text{out} \) are now identical.

A direct connection of information entropy and probabilities shows the necessity of defining also probabilities with respect to an information reference frame, IRF. The interpretation of ‘probability of an event’ thus depends on a chosen IRF too. It is the context in which Eq. (26) must be interpreted.

VIII. INFORMATION FORM OF THE GENERALIZED SECOND LAW

The expression of entropy in \( \mathcal{H} \)-representation, Eq. (23), can be interpreted in standard thermodynamics concepts in the case of a macroscopic system in thermal equilibrium. In this Section, we use the \( \mathcal{H} \)-representation to find the thermodynamic interpretation of the information entropy for an arbitrary (micro-, meso-, macroscopic) system in a general nonequilibrium state.

We study a more complex structure called here the supersystem that is perfectly isolated from surroundings so that its evolution is deterministic and reversible and its energy is constant. The observable indicators of the supersystem are its special degrees of freedom, \( \alpha \), so that the complete state of the supersystem \( x = (s, \alpha) \). The studied thermodynamic system is then defined by the (internal) degrees of freedom \( s \). We can write

\[
\mathcal{H}(x) = \epsilon(\alpha) + \mathcal{H}_{\text{int}}(s, \alpha),
\]

where \( \epsilon(\alpha) \) is the energy connected with the observable degrees of freedom and \( \mathcal{H}_{\text{int}} \) represents the rest of energy connected with internal degrees of freedom and their interactions with the observed ones. The energy of the studied thermodynamic system is identified as \( \mathcal{H}_{\text{int}} \).

A simple example is presented at Fig. 1. The supersystem is the whole structure including the gas and the piston. The position and velocity of the piston represent the observable indicators so that \( \epsilon(\alpha) = MgX + 1/2MV^2 \) with \( M \) being the mass of the piston. The gas is the own thermodynamic system. The change of \( \epsilon \) during a process measures the change of energy of the gas.

We study processes during which \( \Delta \mathcal{H} = 0 \) (the supersystem is isolated). The change of information entropy in \( \mathcal{H} \)-representation during the single (microscopic) process \( x_i \to x_f \) is given by Eq. (23),

\[
T \Delta S = \Delta \mathcal{H}_{\text{int}} - \Delta F(\alpha, T), \quad (\mathcal{H} - \text{rep.}) \tag{28}
\]

where \( F \) corresponds to the free energy of the internal degrees of freedom,

\[
F(\alpha, T) = -k_B T \ln \sum_{s' \in \Gamma(\alpha)} \exp(-\beta \mathcal{H}_{\text{int}}(s', \alpha)). \tag{29}
\]

During the process the energy \( W = \Delta \mathcal{H}_{\text{int}} = -\Delta \epsilon \) (since \( \Delta \mathcal{H} = 0 \)) is transferred from observable degrees of freedom into the internal ones, i.e. into the thermodynamic system. This energy transfer can be called the supplied work regardless if the system is macroscopic or microscopic \[41\]. The quantity \( T \Delta S \) thus can be identified with the dissipated work, \( W_{\text{dis}} = W - \Delta F \), defined for a single trajectory \[42\],

\[
T \Delta S = W_{\text{dis}}. \quad (\mathcal{H} - \text{rep.}) \tag{30}
\]

The parameter \( T \) is an arbitrarily chosen positive real number. Its value can be fixed and identified with a thermodynamic temperature with using the idea of contact temperature \[43\].

Namely if an arbitrary system (micro-, meso-, macroscopic) is at the state \( x = (s, \alpha) \) we can put it into a thermal contact with a sufficiently large thermal reservoir in equilibrium with the temperature \( T_{\text{eq}} \), fix the value of \( \alpha \) and keep relaxing the internal degrees of freedom into thermal equilibrium, i.e. \( (s, \alpha) \to (s_{\text{eq}}, \alpha) \). During the relaxation the energy \( Q_{\text{rel}} = I(s, \alpha, T_{\text{eq}}) \) is absorbed by the internal degrees of freedom. In dependence on \( T_{\text{eq}} \) this energy may be positive or negative. We can choose \( T_{\text{eq}} \) so that \( Q_{\text{rel}} = 0 \) and identify the parameter \( T \) with this temperature, \( T = T_{\text{eq}} \). The temperature \( T \) thus depends on a chosen state of the system, i.e. \( T = T(s, \alpha_0) \).

Generalized second law. — The change of energy \( \epsilon \) connected with observable degrees of freedom \( \alpha \) is the maximal energy that can be used by the observer who detects (and principally can control) these variables, i.e. the usable work, \( W_u = -W = \Delta \epsilon \). If \( \Delta S \geq 0 \) we get from Eq. (28) the familiar inequality of classical thermodynamics,

\[
W_u \leq -\Delta F. \tag{31}
\]

We can express the relation between \( W_u \) and \( \Delta F \) more precisely in a full general situation by using the transformation of the used IRF (that is defined by observable indicators \( \alpha \)) into the dynamic IRF' with \( \Gamma'(t) = G(t) \). In the IRF' we can define the free energy, \( F' \), and the entropy \( S' \), and we get \( \Delta S' = 0 \) and Eq. (28) gives \( \epsilon = -\Delta F' \). Using Eq. (22) we get \( \Delta F' = \Delta F + \Delta \mathcal{J} \), i.e.

\[
W_u = -\Delta F - k_B T \Delta \mathcal{J}. \tag{32}
\]

Since \( \Delta \mathcal{J} = J' - J \) and \( J' \geq 0 \) we get the general inequality

\[
W_u \leq -\Delta F + k_B T J, \tag{33}
\]

that generalizes the standard inequality Eq. (31) since \( J \geq 0 \). The additional term increases a possible value of the gained energy. This increase has a pure informational character since \( J = \mathcal{J}(G) \) is the value of information about such microscopic initial conditions leading to the demanded gain of energy.

It is instructive to use the inequality (33) in a situation when an observer repeats the experiment in which the
initial observed indicator is always $\alpha_i$ and the probability of occurrence of a concrete initial state of internal degrees of freedom (microstate), $s_i$, corresponds to the state of thermal equilibrium, $p_{eq}(H_{int}(s_i, \alpha_i))$. If $G_i \neq \Gamma$, the resulting value of $\alpha$ is not always $\alpha_f$ and various results of $\Delta \epsilon = W_u(s_i)$ can be expected. The observed averaged quantity,

$$\langle W_u \rangle = \sum_{s_i \in \Gamma} p_{eq}(s_i, \alpha_i)W_u(s_i),$$

where $W_u(s_i) = \epsilon(\alpha(s_i)) - \epsilon(\alpha_i)$ with $\alpha(s_i)$ being the final value of $\alpha$ if the initial $s$ is $s_i$ (if $s_i \in G_i$ then $\alpha(s_i) = \alpha_f$, see Fig. (2)). Other averaged values are defined similarly.

The use of Eq. (22) for determining $\mathcal{J} = \mathcal{T}_\Gamma(G_i)$ and averaging Eq. (33) gives the averaged form of this inequality, namely

$$\langle W_u \rangle \leq \langle \Delta F \rangle + k_B T I,$$

where $I = \langle \mathcal{J} \rangle$ is the mutual information. The inequality Eq. (34) is called the generalized second law of thermodynamics [14-16]. It is worth stressing that this concrete form is valid in the case when the averaging is done over the initial distribution corresponding to the thermal equilibrium. Its information form, Eq. (33), is valid for a single adiabatic process on an arbitrary length scale with an arbitrary initial conditions (e.g. that representing a highly nonequilibrium macroscopic state).

\section{IX. CONCLUDING DISCUSSION}

The statistical interpretation of the second law of thermodynamics is the most natural and logical explanation of why reversible microscopic behavior can manifest as irreversible at macroscopic scales. It is only a statistical reason why a drop of ink put into a bottle with water will always smear over the water, and why the dissolved ink does not return into the initial drop formation. Namely the macrostate with a higher Boltzmann entropy is overwhelmingly larger then that with a lower one (see Fig. 1). Hence the microstate wanders into this huge set in overwhelmingly many cases [37].

I argue that the statistical way of analyzing macroscopic processes at the microscopic level is nothing but an expression of information ideas. The set of possible microstates $\Gamma$ (e.g. Boltzmann’s macrostate) is information about the actual microstate $x$: it belongs into $\Gamma$. Entropy is information about the actual microstate, $\mathcal{S} \sim \mathcal{I}(x)$. The crucial idea is that the value of information (as well as value of other physical quantities) can be defined only with respect to referential information. Contrary to the majority of physical quantities that can be defined with respect to a firmly given referential element, the referential sets $\Gamma$ are usually defined by actual values of some physical quantities (observable indicators), $\alpha$, that vary in time. Moreover, the referential set changes (diminishes) whenever the observer (a human, an experimental device, robot, etc.) gains new information about the studied system. The information reference frame (IRF) formed by referential sets $\Gamma(t)$ thus resembles more referential frames of the general theory of relativity firmly connected with dynamics of physical fields [27].

The essential element of the information form of the second law of thermodynamics is the value of information gained at time $t$ that concerns the coarse-grained state of the system at another time $t'$. Let the observer know only the actual coarse-grained state at $t$, say the current values of some physical quantities $\alpha$. If she gets the message that this state will be/was $\alpha'$ at $t'$ the value of this message, $J$, depends on $\alpha$ and $\alpha'$ ($t, t'$ are given). This message reveals the process between $t$ and $t'$ if the situation at $t$ is known, i.e. we can write $J = J(\alpha, \alpha \rightarrow \alpha') \geq 0$. The second law of thermodynamics for adiabatic processes, Eq. (7), then can be written as

$$k_B^{-1}(S(t') - S(t)) = J(\alpha', \alpha' \rightarrow \alpha) - J(\alpha, \alpha \rightarrow \alpha').$$

In the macroscopic limit, $|\Delta \mathcal{S}|/k_B \rightarrow \infty$, the value $J(\alpha, \alpha \rightarrow \alpha') \rightarrow \infty$ if $S(t') < S(t)$, i.e. $P(\alpha, \alpha \rightarrow \alpha') \rightarrow 0$ due to Eq. (19). Hence the observer who knows that the observable indicator is $\alpha$ at $t$ must conclude that the observable indicator at $t'$ cannot be $\alpha'$. It looks paradoxically since we must get either $P(\alpha, \alpha \rightarrow \alpha') \rightarrow 0$ or $P(\alpha', \alpha' \rightarrow \alpha) \rightarrow 0$ if $\Delta \mathcal{S} \neq 0$ in the macroscopic limit, hence the occurrence of any couple of coarse-grained states $\alpha, \alpha'$ differing by a nonzero entropy appears as impossible. The explanation of this evident nonsensical result consists in the fact that the past and future are not symmetric: the observer can verify the situation at $t'$ if $t' > t$. It implies that if $P(\alpha, \alpha \rightarrow \alpha') \rightarrow 0$ and $t' > t$ the observer cannot detect $\alpha'$ at $t'$. Eq. (35) thus implies that the adiabatic evolution of a macroscopic system cannot be connected with a macroscopic decrease of entropy.

If $t' < t$ the fact that $P(\alpha, \alpha \rightarrow \alpha') \rightarrow 0$ does not make a controversy since the observer who detects $\alpha$ at $t$ then cannot verify the situation at $t'$ (we cannot move into the past). If there is, however, information that $\alpha$ was $\alpha'$ at $t'$ available at $t$, this information must occur in form of a record (e.g. a change in some brain cells) that exists at $t$. The observer at $t$ thus has information about the system that forms the dynamic referential frame IRF" in which $\Delta \mathcal{S}' = 0$ and Eq. (35) implies that $P'(\alpha', \alpha' \rightarrow \alpha) = P'(\alpha', \alpha' \rightarrow \alpha)$ may have an arbitrary value. The information form of the second law thus in-heres an important referential asymmetry between past and future though no such asymmetry occurs in microscopic physics [38].

The important question is why the change of the referential frame, $\text{IRF} \rightarrow \text{IRF}'$, ”automatically” happens whenever a record about the past appears. The explanation consists in the fact that the information reference frame is primarily defined by information. A collection of subsets $\Gamma(t) \subset \Gamma_S$ is only its mathematical expression. A record is a physical event that can be interpreted as information about a past moment. If so, any physical consideration must take into account this information.
An analogy with spatial reference frames is instructive here. Namely the reference frame in space is primarily done by a collection of referential bodies (an analogy of attainable information about a physical system). These bodies define the mathematical structure $R$ denoted by a collection of referential bodies (an analogy of here. Namely the reference frame in space is primarily described by other coordinates. Analogically, whenever the referential bodies changes the same physical point is presented approach. For example, the implementation of abatic processes only and the probabilities are defined its content is, however, somewhat different from the stan-
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The equality Eq. (10) holds for all sufficiently small $\tau > 0$. Since $\lim_{\tau \to 0} \Delta S = 0$ we denote $H(x_1, x_2, \ldots) = \Phi(\Phi^{-1}(x_1))$ and study the condition $H(x_1 + y, x_2 + y, \ldots) = H(x_1, x_2, \ldots) + y$ in the limit $y \to 0$. The derivation of this condition by $y$ gives $\phi(\sum_i \Phi^{-1}(x_i)) = \sum_i \phi(\Phi^{-1}(x_i))$ in the limit $y \to 0$, where $\phi(\nu) = (\Phi(\nu))^{-1}$. It implies that $\phi(\nu) = \kappa \nu$, where $\kappa$ is a constant. After integration we get $\Phi(\nu) = -\gamma \ln \nu + \kappa_0$, where $\gamma = -\kappa^{-1} > 0$ ($\Phi$ is a decreasing function) which implies $f(z) = e^{-\gamma^{-1} \Phi(z)} e^\gamma^{-1} \kappa_0$. Hence $I_I(\Omega) = -\gamma \ln \sum_{z \in \Omega} e^\gamma^{-1} \kappa_0 \nu \nu f(z)$, where $\gamma$ cannot depend on $\Omega$ to fulfill Eq. (0) for $\Gamma_i \neq \Gamma_f$.

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