Deciding Whether a Regular Language is Generated by a Splicing System

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Abstract
Splicing as a binary word/language operation is inspired by the DNA recombination under the action of restriction enzymes and ligases, and was first introduced by Tom Head in 1987. Shortly thereafter, it was proven that the languages generated by (finite) splicing systems form a proper subclass of the class of regular languages. However, the question of whether or not one can decide if a given regular language is generated by a splicing system remained open. In this paper we give a positive answer to this question. Namely, we prove that, if a language is generated by a splicing system, then it is also generated by a splicing system whose size is a function of the size of the syntactic monoid of the input language, and which can be effectively constructed.

1 Introduction

In [10] Head described an language-theoretic operation, called splicing, which models DNA recombination, a cut-and-paste operation on DNA double-strands. Recall that a DNA single-strand is a polymer consisting of a series of the nucleobases Adenine (A), Cytosine (C), Guanine (G), and Thymine (T) attached to a linear, directed backbone. Due to the chemical structure of the backbone, the ends of a single-strand are called 3' end and 5' end. Abstractly, a DNA single-strand can be viewed as a string over the four letter alphabet \{A, C, G, T\}. The bases A and T, respectively C and G, are Watson-Crick-complementary, or simply complementary, which means they can attach to each other via hydrogen bonds. The complement of a DNA single-strand $\alpha = 3'\ldots a_1\ldots a_n-3'$ is the strand $\bar{\alpha} = 3'-\bar{a}_1\ldots\bar{a}_n-5'$ where $a_1,\ldots,a_n$ are bases and $\bar{a}_1,\ldots,\bar{a}_n$ denote their complementary bases, respectively; note that $\alpha$ and $\bar{\alpha}$ have opposite orientation. A strand $\alpha$ and its complement $\bar{\alpha}$ can bond to each other to form a DNA (double-)strand.

Splicing is meant to abstract the action of two compatible restriction enzymes and the ligase enzyme on two DNA double-strands. The first restriction enzyme recognizes a base-sequence $u_1v_1$, called its restriction site, in any DNA string, and cuts the string containing this factor between $u_1$ and $v_1$. The second restriction enzyme, with restriction site $u_2v_2$, acts similarly. Assuming that the sticky ends obtained after these cuts are complementary, the enzyme ligase aids then the recombination (catenation) of the first segment of one cut string with the second segment of another cut string. For example, the enzyme TaqI has restriction site TCGA, and the enzyme SciNI has restriction site GCCG. The enzymes cut double-strands

$$5'-\gamma-G\overline{C}\overline{G}C-3'$$
$$3'-\bar{\gamma}-G\overline{C}\overline{G}C-5'$$

and

$$5'-\gamma-C\overline{G}\overline{C}\overline{G}C-3'$$
$$3'-\bar{\gamma}-C\overline{G}\overline{C}\overline{G}C-5'$$

along the dotted lines, respectively, leaving the first segment of the left strand with a sticky end GC which is compatible to the sticky end CG of the second segment of the right strand. The
segments can be recombined to form either the original strands or the new strand
\[ 5' \rightarrow \alpha \rightarrow TC\,GC \rightarrow \delta \rightarrow 3' \]
\[ 3' \rightarrow \pi \rightarrow AG\,GC \rightarrow \beta \rightarrow 5' \]

A splicing system is a formal language model which consists of a set of initial words or axioms \( I \) and a set of splicing rules \( R \). The most commonly used definition for a splicing rule is a quadruple of words \( r = (u_1, v_1; u_2, v_2) \). This rule splices two words \( x_1u_1v_1y_1 \) and \( x_2u_2v_2y_2 \): the words are cut between the factors \( u_1, v_1 \), respectively \( u_2, v_2 \), and the prefix (the left segment) of the first word is recombined by catenation with the suffix (the right segment) of the second word, see Figure 1 and also [18]. A splicing system generates a language which contains every word that can be obtained by successively applying rules to axioms and the intermediated produced words.

\[
\begin{array}{c}
\begin{array}{c}
\text{x1} \\
\text{u1} \\
\text{v1} \\
\text{y1} \\
\end{array}
\begin{array}{c}
\text{x2} \\
\text{u2} \\
\text{v2} \\
\text{y2} \\
\end{array}
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{c}
\text{x1} \\
\text{u1} \\
\text{v2} \\
\text{y2} \\
\end{array}
\begin{array}{c}
\text{x2} \\
\text{u2} \\
\text{v1} \\
\text{y1} \\
\end{array}
\end{array}
\]

Figure 1: Splicing of the words \( x_1u_1v_1y_1 \) and \( x_2u_2v_2y_2 \) by the rule \( r = (u_1, v_1; u_2, v_2) \).

Example 1.1. Consider the splicing system \((I, R)\) with axiom \( I = \{ab\} \) and rules \( R = \{r, s\} \) where \( r = (a, b; \varepsilon, ab) \) and \( s = (ab, \varepsilon; ab) \); in this paper, \( \varepsilon \) denotes the empty word. Applying the rule \( r \) to two copies of the axiom \( ab \) creates the word \( aab \) and applying the rule \( s \) to two copies of the axiom \( ab \) creates the word \( abb \). More generally, the rule \( r \) or \( s \) can be applied to words \( a^ib^j \) and \( a^ib^j \) with \( i, j, k, \ell \geq 1 \) in order to create the word \( a^{i+1}b^{j+1} \) or \( a^{i+1}b^{j+1} \), respectively. The language generated by the splicing system \((I, R)\) is \( L(I, R) = a^+b^+ \).

The most natural variant of splicing systems, often referred to as finite splicing systems, is to consider a finite set of axioms and a finite set of rules. In this paper, by a splicing system we always mean a finite splicing system. Shortly after the introduction of splicing in formal language theory, Culik II and Harju [6] proved that splicing systems generate regular languages, only; see also [12, 17]. Gatterdam [7] gave \((aa)^*\) as an example of a regular language which cannot be generated by a splicing system; thus, the class of languages generated by splicing systems is strictly included in the class of regular languages. However, for any regular language \( L \) over an alphabet \( \Sigma \), adding a marker \( b \notin \Sigma \) to the left side of every word in \( L \) results in the language \( bL \) which can be generated by a splicing system [11]; e.g., the language \( b(aa)^* \) is generated by the axioms \( \{b, baa\} \) and the rule \( (baa, \varepsilon; b, \varepsilon) \).

This led to the question of whether or not one of the known subclasses of the regular languages corresponds to the class \( S \) of languages which can be generated by a splicing system. All investigations to date indicate that the class \( S \) does not coincide with another naturally defined language class. A characterization of reflexive splicing systems using Schützenberger constants has been given by Bonizzoni, de Felice, and Zizza [1–3]. A splicing system is reflexive if for all rules \((u_1, v_1; u_2, v_2)\) in the system we have that \((u_1, v_1; u_1, v_1)\) and \((u_2, v_2; u_2, v_2)\) are rules in the system, too. A word \( v \) is a Schützenberger constant of a language \( L \) if \( x_1vy_1 \in L \) and \( x_2vy_2 \in L \) imply \( x_1vy_2 \in L \) [19]. Recently, it was proven by Bonizzoni and Jonoska that every splicing language has a constant [5]. However, not all languages which have a constant are generated by splicing systems, e.g., in the language \( L = (aa)^+b^+ \) every word \( b^i \) is a constant, but \( L \) is not generated by a splicing system.

Another approach was to find an algorithm which decides whether a given regular language is generated by a splicing system. This problem has been investigated by Goode, Head, and Pixton [8, 9, 13] but it has only been partially solved: it is decidable whether a regular language is generated by a reflexive splicing system. It is worth mentioning that a splicing system by the original definition in [10] is always reflexive. A related problem has been investigated by Kim [16]: given a regular language \( L \) and a finite set of enzymes, represented by set of reflexive rules \( R \), Kim showed that it is decidable whether or not \( L \) can be generated from a finite set of axioms by using only rules from \( R \).
In this paper we settle the decidability problem, by proving that for a given regular language, it is indeed decidable whether the language is generated by a splicing system (which is not necessarily reflexive), Corollary 5.2. More precisely, for every regular language $L$ there exists a splicing system $(I_L, R_L)$ and if $L$ is a splicing language, then $L$ is generated by the splicing system $(I_L, R_L)$. The size of this splicing system depends on the size of the syntactic monoid of $L$. If $m$ is the size of the syntactic monoid of $L$, then all axioms in $I_L$ and the four components of every rule in $R_L$ have length in $O(m^2)$, Theorem 4.1. By results from [12, 13], we can construct a finite automaton which accepts the language generated by $(I_L, R_L)$, compare it with a finite automaton which accepts $L$, and thus, decide whether $L$ is generated by a splicing system. Furthermore, we prove a similar result for a more general variant of splicing that has been introduced by Pixton [17], Theorem 3.1.

The paper is organized as follows. In Section 2 we lay down the notation, recall some well-known results about syntactic monoids, and prove a pumping argument that is of importance for the proofs in the succeeding sections. Section 3 (Section 4) contains the proof that a regular language $L$ is generated by a Pixton splicing system (resp. classical splicing system) if and only if it is generated by one particular Pixton splicing system (resp. classical splicing system) whose size is bounded by the size of the syntactic monoid of $L$. Sections 3 and 4 can be read independently and overlap in some of their main ideas. The inclusion of both sections and the presentation order are chiefly for expository purposes: Due to the features of the Pixton splicing, Section 3 introduces the main ideas in a significantly more readable way. Finally, in Section 5 we deduce the decidability results for both splicing variants.

An extended abstract of this paper, including a shortened proof of Theorem 4.1 and Corollary 5.2 i.), has been published in the conference proceedings of DNA 18 in 2012 [15]. Theorem 3.1 and Corollary 5.2 ii.) have not been published elsewhere.

2 Notation and Preliminaries

We assume the reader to be familiar with the fundamental concepts of language theory, see [14].

Let $\Sigma$ be a finite set of letters, the alphabet; $\Sigma^*$ be the set of all words over $\Sigma$; and $\varepsilon$ denote the empty word. A subset $L$ of $\Sigma^*$ is a language over $\Sigma$. Throughout this paper, we consider languages over the fixed alphabet $\Sigma$, only. Let $w \in \Sigma^*$ be a word. The length of $w$ is denoted by $|w|$. (We use the same notation for the cardinality $|S|$ of a set $S$, as usual.) We consider the letters of $\Sigma$ to be ordered and for words $u, v \in \Sigma^*$ we denote the length-lexicographical order by $u \leq_L v$; i.e., $u \leq_L v$ if either $|u| \leq |v|$, or $|u| = |v|$ and $u$ is at most $v$ in lexicographical order. The strict length-lexicographic order is denoted by $<_L$; we have $u <_L v$ if $u \leq_L v$ and $u \neq v$.

For a length bound $m \in \mathbb{N}$ we let $\Sigma <^m$ denote the set of words whose length is at most $m$, i.e., $\Sigma <^m = \bigcup_{|w| \leq m} \Sigma^*$. Analogously, we define $\Sigma \leq^m = \bigcup_{|w| \leq m} \Sigma^*$.

If $w = xyz$ for some $x, y, z \in \Sigma^*$, then $x$, $y$, and $z$ are called prefix, factor, and suffix of $w$, respectively. If a prefix or suffix of $w$ is distinct from $w$, it is said to be proper.

Let $w = a_1 \ldots a_n$ where $a_1, \ldots, a_n$ are letters from $\Sigma$. By $w[i]$ for $0 \leq i \leq n$ we denote a position in the word $w$: if $i = 0$, it is the position before the first letter $a_1$, if $i = n$ it is the position after the last letter $a_n$, and otherwise, it is the position between the letters $a_i$ and $a_{i+1}$. We want to stress that $w[i]$ is not a letter in the word $w$. By $w[i; j]$ for $0 \leq i \leq j \leq n$ we denote the factor $a_i \ldots a_j$ which is enclosed by the positions $w[i]$ and $w[j]$. If $x = w[i; j]$ we say the factor $x$ starts at position $w[i]$ and ends at position $w[j]$. Whenever we talk about a factor $x$ of a word $w$ we mean a factor starting (and ending) at a certain position, even if the the word $x$ occurs as a factor at several positions in $w$. Let $x = w[i; j]$ and $y = w[i'; j']$ be factors of $w$. We say the factors $x$ and $y$ match (in $w$) if $i = i'$ and $j = j'$; the factor $x$ is covered by the factor $y$ (in $w$) if $i' \leq i \leq j \leq j'$; and the factors $x$ and $y$ overlap (in $w$) if $x \neq \varepsilon$, $y \neq \varepsilon$, and $i \leq i' < j$ or $i' \leq i < j$. In other words, if two factors $x$ and $y$ overlap in $w$, then they share a common letter of $w$. Let $x = w[i; j]$ be a factor of $w$ and let $p = w[k]$ be a position in $w$. We say the position $p$ lies at the left of $x$ if $k \leq i$; the position $p$ lies at the right of $x$ if $k \geq j$; and the position $p$ lies in $x$ if $i < k < j$.

Every language $L$ induces a syntactic congruence $\sim_L$ over words such that $u \sim_L v$ if and only
if for all words \( x, y \)
\[
xy ∈ L ⇔ xvy ∈ L.
\]
The syntactic class (with respect to \( L \)) of a word \( u \) is \( [u]_L = \{ v ∣ u ∼_L v \} \). The syntactic monoid of \( L \) is the quotient monoid
\[
M_L = \Sigma^*/\sim_L = \{ [u]_L ∣ u ∈ \Sigma^* \}.
\]

It is well known that a language \( L \) is regular if and only if its syntactic monoid \( M_L \) is finite. We will use two basic facts about syntactic monoids of regular languages.

**Lemma 2.1.** Let \( L \) be a regular language and let \( w \) be a word with \( |w| ≥ |M_L|^2 \). We can factorize \( w = αβγ \) with \( β ≠ ε \) such that \( α ∼_L αβ \) and \( γ ∼_L βγ \).

**Proof.** Consider a word \( w \) with \( n = |w| ≥ |M_L|^2 \). For \( i = 0, \ldots, n \), let \( X_i = w[0; i] \) be the syntactic classes of the prefixes of \( w \) and let \( Y_i = w[i; n] \) be the syntactic classes of the suffixes of \( w \). Note that \( X_i Y_i = [w]_L \). By the pigeonhole principle, there are \( i, j \) with \( 0 ≤ i < j ≤ n \) such that \( X_i = X_j \) and \( Y_i = Y_j \). Let \( α = w[0; i] \), \( β = w[i; j] \), and \( γ = w[j; n] \). As \( α ∈ X_i \) and \( αβ ∈ X_j \), we see that \( α ∼_L αβ \) and, symmetrically, \( γ ∼_L βγ \).

**Lemma 2.2.** Let \( L \) be a regular language. Every element \( X ∈ M_L \) contains a word \( x ∈ X \) with \( |x| < |M_L| \).

**Proof.** We define a series of sets \( S_i ⊆ M_L \). We start with \( S_0 = \{ 1 \} \) (here, \( 1 = [ε]_L \)) and let \( S_{i+1} = S_i ∪ \{ X · [α]_L ∣ X ∈ S_i ∧ α ∈ Σ \} \) for \( i ≥ 0 \). It is not difficult to see that \( X ∈ S_i \) if and only if \( X \) contains a word \( x ∈ X \) with \( |x| ≤ i \). As \( S_i ⊆ S_{i+1} \) and \( M_L \) is finite, the series has a fixed point \( S_n \) such that \( S_i = S_n \) for all \( i ≥ n \). Let \( n \) be the least value with this property, i.e., \( S_{n-1} ⊆ S_n \) or \( n = 0 \). Observe that \( n < |M_L| \) as \( S_0 ⊆ S_1 ⊆ ⋯ ⊆ S_n \). Every element \( X ∈ M_L \) contains some word \( w ∈ X \), thus \( X ∈ [w]_L ⊆ S_n \). Concluding that \( X \) contains a word with a length of at most \( n < |M_L| \).

### 2.1 A Pumping Algorithm

Consider a regular language \( L \), a word \( αβγ \) where \( α ∼_L αβ \) and \( γ ∼_L βγ \), due to Lemma 2.1, and a large even number \( j \). In the proofs of Theorem 3.1 and Lemma 4.8, we need a pumping argument to replace all factors \( αβγ \) by \( αβ^2γ \) in a word \( z \) in order to obtain a word \( \tilde{z} \); thus, \( z ≡_L \tilde{z} \). As \( αβγ \) may be a factor of \( αβ^2γ \), we cannot ensure that \( αβγ \) is not a factor of \( \tilde{z} \). However, we can ensure that if \( αβγ = \tilde{z}[k; k'] \) is a factor of \( \tilde{z} \), then either \( a) \ αβ^{2/3} \) is a factor of \( \tilde{z} \) starting at position \( \tilde{z}[k] \) or \( b) \ β^{2/3}γ \) is a factor of \( \tilde{z} \) ending at position \( \tilde{z}[k'] \); i.e., either \( a \) is succeeded by a large number of \( β \)'s or \( γ \) is preceded by a large number of \( β \)'s. The next lemma is a technical result whose purpose is to show that for any word \( z \) there exists a word \( \tilde{z} \) such that the above-mentioned property holds and \( \tilde{z} \) is generated by applying several successive pumping steps \( αβγ → αβ^2γ \) to \( z \).

**Lemma 2.3.** Let \( z, α, β, γ \) be words with \( β ≠ ε \), let \( ℓ = |αβγ| \), and let \( j > |z| + ℓ \) be an even number. The following algorithm will terminate and output \( \tilde{z} \).

1. \( \tilde{z} := z \).
2. if \( \tilde{z}[k; k + ℓ] = αβγ \) for some \( k \) such that neither
   - \( a) \ αβ^{2/3} \) is a factor of \( \tilde{z} \) starting at position \( \tilde{z}[k] \) nor
   - \( b) \ β^{2/3}γ \) is a factor of \( \tilde{z} \) ending at position \( \tilde{z}[k + ℓ] \),
   then let \( \tilde{z} := \tilde{z}[0; k] · αβ^2γ · \tilde{z}[k + ℓ; |z|]; \) (replace the factor \( \tilde{z}[k; k + ℓ] = αβγ \) in \( z \) by \( αβ^2γ \))
3. repeat step 2 until there is no such factor \( αβγ \) in \( \tilde{z} \) left.
Before we prove Lemma 2.3, let us recall a basic fact about primitive words. A word $p$ is called primitive if there is no word $x$ and $i \geq 2$ such that $p = x^i$. The primitive root of a word $w \neq \varepsilon$ is the unique primitive word $p$ such that $w = p^i$ for some $i \geq 1$. For primitive $p$, it is well known that if $pp = xyy$, then either $x = p$ and $y = \varepsilon$, or $x = \varepsilon$ and $y = p$. In other words, whenever $p$ is a factor of $p^m$ starting at position $p^n[i]$, then $i \in |p| \cdot \mathbb{N}$.

For a word $w = xy$ we employ the notations $x^{-1}w = y$ and $wy^{-1} = x$. If $x$ is not a prefix of $w$, then the $x^{-1}w$ (resp. $wy^{-1}$) is undefined.

**Proof of Lemma 2.3.** Let $p$ be the primitive root of $\beta$ and let $m$ such that $\beta = p^m$.

First, observe that if, during the computation, a factor $\alpha\beta\gamma = z[k; k + \ell]$ is covered by a factor $\alpha\beta\gamma$ in $z$, then either (a) or (b) holds. Indeed, if $\alpha\beta\gamma = (\alpha\beta\gamma)[i; i + \ell]$ for some $i$, then $\beta$ is a factor of $\beta^i$ starting at position $\beta[i]$. As mentioned above, $i \in |p| \cdot \mathbb{N}$ and either position $\beta[i]$ is preceded or succeeded by $\beta^i/2$. Therefore, (a) or (b) is satisfied.

Let $z_0 = z$, let $z_n$ be the word $\tilde{z}$ after the $n$-th pumping step in the algorithm, and let $y = p^{m-2} = \beta p^{-2}$. For each $n$, we will define a unique factorization

$$z_n = x_0yx_{n,1} \cdots yx_{n,n}$$

where $p$ is a suffix of $x_{n,i}$ for $i = 0, \ldots, n - 1$ and $p$ is a prefix of $x_{n,i}$ for $i = 1, \ldots, n$. This factorization is defined inductively: naturally, we start with $x_0 = z_0 = z$. Assume $z_n$ is factorized in the above manner. Let $\alpha\beta\gamma = z_n[k; k + \ell]$ be the factor, such that neither (a) nor (b) holds, which we replace in the $(n + 1)$-st step (if there is no such factor, the algorithm terminates and we do not have to define $z_{n+1}$). By contradiction, assume that $\alpha$ starting at position $z_n[k]$ is covered by the $i$-th factor $y = p^{m-j-2}$ in the factorization of $z_n$ for some $1 \leq i \leq n$. By the first observation, the factor $\beta\gamma = z_n[k + |\alpha|; k + \ell]$ must overlap with $x_i$. However, as $p$ is a prefix of $x_i$, the factor $\beta = z_n[k + |\alpha|; k + |\alpha|\beta]$ has to cover the prefix $p$ of $x_i$ or it has to cover one of the $p$‘s in $y$. This implies that $\gamma$ is preceded by $p^{m-j/2} = \beta^{j/2}$ and (b) holds — contradiction.

Symmetrically, $\gamma$ is not covered by one of the factors $y$ in $z_n$ neither.

Thus, $\beta = z_n[k + |\alpha|; k + |\alpha|\beta]$ is covered by some $x_{n,i}$ in the factorization of $z_n$ and $x_{n,i}$ can be factorized $x_{n,i} = u\beta v$ where $u \neq \varepsilon$ and $v \neq \varepsilon$. Note that the length of $x_{n,i}$ has to be at least $|\beta| + 2$. Now, let $x_{n+1,h} = x_{n,h}$ for $h = 0, \ldots, i - 1$, let $x_{n+1,h+1} = x_{n,h}$ for $h = i + 2, \ldots, n$, let $x_{n+1,i+1} = up$, and let $x_{n+1,i+1} = pv$. Observe that this defines the desired factorization. Also note that

$$|x_{n+1,i}| = |u| + |p| = |x_{n,i}| - |\beta| - |v| + |p| \leq |x_{n,i}| - |v| < |x_{n,i}|$$

and, symmetrically, $|x_{n+1,i+1}| < |x_{n,i}|$. Thus, in each pumping step, we replace one of the factors $x_{n,i}$ by two strictly shorter factors $x_{n+1,i}$ and $x_{n+1,i+1}$. As we have noted above, in a factor $x_{n,i}$ cannot be pumped anymore, if it is shorter than $|\beta| + 2$. Eventually, all the the factors will be too short and the pumping algorithm will stop.

3. Pixton’s Variant of Splicing

In this section we use the definition of the splicing operation as it was introduced in [17]. A triplet of words $r = (u_1, u_2; v) \in (\Sigma^*)^3$ is called a (splicing) rule. The words $u_1$ and $u_2$ are called left and right site of $r$, respectively, and $v$ is the bridge of $r$. This splicing rule can be applied to two words $w_1 = x_1u_1y_1$ and $w_2 = x_2u_2y_2$, that each contain one of the sites, in order to create the new word $z = x_1vy_2$, see Figure 2. This operation is called splicing and it is denoted by $(w_1, w_2) \vdash_r z$.

![Figure 2: Splicing of the words $x_1u_1y_1$ and $x_2u_2y_2$ by the rule $r = (u_1, u_2; v)$.](attachment:image.png)
For a rule $r$ we define the splicing operator $\sigma_r$ such that for a language $L$
\[
\sigma_r(L) = \{ z \in \Sigma^* \mid \exists w_1, w_2 \in L : (w_1, w_2) \vdash_r z \}
\]
and for a set of splicing rules $R$, we let
\[
\sigma_R(L) = \bigcup_{r \in R} \sigma_r(L).
\]
The reflexive and transitive closure of the splicing operator $\sigma_R^*$ is given by
\[
\sigma_R^0(L) = L, \quad \sigma_R^{i+1}(L) = \sigma_R(L) \cup \sigma_R(\sigma_R^i(L)), \quad \sigma_R^*(L) = \bigcup_{i \geq 0} \sigma_R^i(L).
\]
A finite set of axioms $I \subseteq \Sigma^*$ and a finite set of splicing rules $R \subseteq (\Sigma^*)^3$ form a splicing system $(I, R)$. Every splicing system $(I, R)$ generates a language $L(I, R) = \sigma_R^*(I)$. Note that $L(I, R)$ is the smallest language which is closed under the splicing operator $\sigma_R$ and includes $I$. It is known that the language generated by a splicing system is regular, see [17]. A (regular) language $L$ is called a splicing language if a splicing system $(I, R)$ exists such that $L = L(I, R)$.

A rule $r$ is said to respect a language $L$ if $\sigma_r(L) \subseteq L$. It is easy to see that for any splicing system $(I, R)$, every rule $r \in R$ respects the generated language $L(I, R)$. Moreover, a rule $r \not\in R$ respects $L(I, R)$ if and only if $L(I, R \cup \{r\}) = L(I, R)$. We say a splicing $(w_1, w_2) \vdash_r z$ respects a language $L$ if $w_1, w_2 \in L$ and $r$ respects $L$; obviously, this implies $z \in L$, too.

Pixton introduced this variant of splicing in order to give a simple proof for the regularity of languages generated by splicing systems. As Pixton’s variant of splicing is more general than the classic splicing, defined in the introduction and in Section 4, his proof of regularity also applies to classic splicing systems. For a moment, let us call a classic splicing rule a quadruple and a Pixton splicing rule a triplet. Consider a quadruplet $r = (u_1, v_1; u_2, v_2)$. It is easy to observe that whenever we can use $r$ in order to splice $w_1 = x_1u_1v_1y_1$ with $w_2 = x_2u_2v_2y_2$ to obtain the word $z = x_1u_1v_2y_2$, we can use the triplet $s = (u_1v_1, u_2v_2; u_1v_2)$ in order to splice $(w_1, w_2) \vdash_s z$ as well. However, for a triplet $s = (u_1, u_2; v)$ where $v$ is not a concatenation of a prefix of $u_1$ and a suffix of $u_2$, there is no quadruplet $r$ that can be used for the same splicings. Moreover, the class of classical splicing languages is strictly included in the class of Pixton splicing languages; e.g., the language
\[
L = cx^a + dx^b f
\]
over the alphabet $\{a, b, c, d, e, f, x\}$ is a Pixton splicing language but not a classical splicing language, see [4]. For the rest of this section we focus on Pixton’s splicing variant and by a rule we always mean a triplet.

The main result of this section states that if a regular language $L$ is a splicing language, then it is created by a particular splicing system $(I, R)$ which only depends on the syntactic monoid of $L$.

**Theorem 3.1.** Let $L$ be a splicing language and $m = |M_L|$. The splicing system $(I, R)$ with $I = \Sigma^{<m^2+6m} \cap L$ and
\[
R = \left\{ r \in \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{<m^2+10m} \mid r \text{ respects } L \right\}
\]
generates the language $L = L(I, R)$.

As the language generated by the splicing system $(I, R)$ is constructible, Theorem 3.1 implies that the problem whether or not a given regular language is a splicing language is decidable. A detailed discussion of the decidability result is given in Section 5.

Let $L$ be a formal language. Clearly, every set of words $J \subseteq L$ and set of rules $S$ where every rule in $S$ respects $L$ generates a subset $L(J, S) \subseteq L$. Therefore, in Theorem 3.1 the inclusion $L(I, R) \subseteq L$ is obvious. The rest of this section is devoted to the proof of the converse inclusion.
3.1 Rule Modifications

Let us start with the simple observation that we can extend the sites and the bridge of a rule \( r \) such that the new rule respects all languages which are respected by \( r \).

Lemma 3.2. Let \( r = (u_1, u_2; v) \) be a rule which respects a language \( L \). For every word \( x \), the rules \((xu_1, u_2; xv), (u_1x, u_2; v), (u_1, xu_2; v), \) and \((u_1, u_2x; vx)\) respect \( L \) as well.

Proof. Let \( s \) be any of the four rules \((xu_1, u_2; xv), (u_1x, u_2; v), (u_1, xu_2; v), \) or \((u_1, u_2x; vx)\). In order to prove that \( s \) respects \( L \) we have to show that, for all \( w_1, w_2 \in L \) and \( z \in \Sigma^* \) such that \( (w_1, w_2) \vdash_s z \), we have \( z \in L \). Indeed, if \( (w_1, w_2) \vdash_s z \), then \( (w_1, w_2) \vdash_r z \) and as \( r \) respects \( L \), we conclude \( z \in L \). □

Henceforth, we will refer to the rules \((xu_1, u_2; xv)\) and \((u_1, u_2x; vx)\) as extensions of the bridge and to the rules \((u_1x, u_2; v)\) and \((u_1, xu_2; v)\) as extensions of the left and right site, respectively.

Next, for a language \( L \), let us investigate the syntactic class of a rule \( r = (u_1, u_2; v) \). The syntactic class (with respect to \( L \)) of \( r \) is the set of rules \( [r]_L = [u_1]_L \times [u_2]_L \times [v]_L \) and two rules \( r \) and \( s \) are syntactically congruent (with respect to \( L \)), denoted by \( r \sim_L s \), if \( s \in [r]_L \).

Lemma 3.3. Let \( r = (u_1, u_2; v) \) be a rule which respects a language \( L \). Every rule \( s \in [r]_L \) respects \( L \).

Proof. Let \( r = (u_1, u_2; v) \) and \( s = (\tilde{u}_1, \tilde{u}_2; \tilde{v}) \). Thus, \( u_1 \sim_L \tilde{u}_1 \) for \( i = 1, 2 \) and \( v \sim_L \tilde{v} \). For \( \tilde{w}_1 = x_1 \tilde{u}_1 y_1 \in \tilde{L} \) and \( \tilde{w}_2 = x_2 \tilde{u}_2 y_2 \in L \) we have to show that \( z \in x_1 \tilde{u}_1 y_1 \). For \( i = 1, 2 \), let \( w_i = x_i y_i y_i \) and note that \( w_i \sim_L \tilde{w}_i \); hence, \( w_i \in L \). Furthermore, \( (w_1, w_2) \vdash_r x_1 y_2 = z \in L \) as \( r \) respects \( L \) and \( \tilde{z} \in L \). □

Consider a splicing \((x_1 u_1 y_1, x_2 u_2 y_2) \vdash_r x_1 y_2 \) which respects a regular language \( L \) as shown in Figure 3 left side. The factors \( u_1 y_1 \) and \( x_2 u_2 \) may be relatively long but they do not occur as factors in the resulting word \( x_1 y_2 \). In particular, it is possible that two long words are spliced and the outcome is a relatively short word. Using Lemmas 3.2 and 3.3, we can find shorter words in \( L \) and a modified splicing rule which can be used to obtain \( x_1 y_2 \).

![Figure 3: The factors \( u_1 y_1 \) and \( x_2 u_2 \) can be replaced by short words.](image)

Lemma 3.4. Let \( r = (u_1, u_2; v) \) be a rule which respects a regular language \( L \) and \( w_1 = x_1 u_1 y_1 \in \tilde{L} \), \( w_2 = x_2 u_2 y_2 \in L \). There is a rule \( s = (\tilde{u}_1, \tilde{u}_2; v) \) which respects \( L \) and words \( \tilde{w}_1 = x_1 \tilde{u}_1 \in \tilde{L} \), \( \tilde{w}_2 = \tilde{u}_2 y_2 \in L \) such that \( |\tilde{u}_1|, |\tilde{u}_2| < |L| \). More precisely, \( \tilde{u}_1 \in [u_1]_L \) and \( \tilde{u}_2 \in [x_2 u_2]_L \).

In particular, whenever \( (w_1, w_2) \vdash_r x_1 y_2 = z \), there is a splicing \((\tilde{w}_1, \tilde{w}_2) \vdash_s z \) which respects \( L \) where \( \tilde{w}_1, \tilde{w}_2, \) and \( s \) have the properties described above.
Proof. By Lemma 3.2, the rule \((u_1 y_1, x_2 u_2; v)\) respects \(L\). Choose \(\tilde{u}_1 \in [u_1 y_1]_L\) and \(\tilde{u}_2 \in [x_2 u_2]_L\) as shortest words from the syntactic classes, respectively; as such, \(|\tilde{u}_1|, |\tilde{u}_2| < |M_L|\) (Lemma 2.2) and \(\tilde{w}_1 = x_1 \tilde{u}_1 \in L\), \(\tilde{w}_2 = \tilde{u}_2 y_2 \in L\). Furthermore, by Lemma 3.3, \(s = (\tilde{u}_1, \tilde{u}_2; v)\) respects \(L\).

Another way of modifying a splicing \((w_1, w_2) \rightarrow_r z\) is to extend the bridge of \(r\) to the left until it covers a prefix of \(w_1\). Afterwards, we can use the same method we used in Lemma 3.4 and replace \(w_1\) by a short word, see Figure 4. As the splicing operation is symmetric, we can also extend the bridge of \(r\) rightwards and replace \(w_2\) by a short word, even though Lemma 3.5 does not explicitly state this.

![Figure 4: The word \(x_1 u_1 y_1\) can be replaced by a short word as long as we extend the bridge of the splicing rule accordingly.](image)

**Lemma 3.5.** Let \(r = (u_1, u_2; v)\) be a rule which respects a regular language \(L\) and let \(w_1 = x_1 u_1 y_1 \in L\). Every rule \(s = (\tilde{v}, u_2; x_1 v)\), where \(\tilde{w} \in [w_1]_L \subseteq L\), respects \(L\). In particular, there is a rule \(s\), as above, where \(|\tilde{w}| < |M_L|\).

**Proof.** By Lemma 3.2, we see that \((x_1 u_1 y_1, u_2; x_1 v)\) respects \(L\) and, by Lemma 3.3, \(s = (\tilde{w}, u_2; x_1 v)\) respects \(L\). If \(\tilde{w} \in [w_1]_L\) is a shortest word from the set, then \(|\tilde{w}| < |M_L|\) by Lemma 2.2.

### 3.2 Series of Splicings

We are now investigating words which are created by a series of successive splicings which all respect a regular language \(L\). Observe, that if a word is created by two (or more) successive splicings, but the bridges of the rules do not overlap in the generated word, then the order of these splicings is irrelevant. The notation in Remark 3.6 is the same as in the Figure 5.

![Figure 5: The word \(x_1 v_1 w' v_2 y_3\) can be created either by using the right splicing first or by using the left splicing first.](image)

**Remark 3.6.** Consider rules \(r = (u_1, u_2; v_1)\) and \(s = (u'_2, u_3; v_2)\) and words \(w_1 = x_1 u_1 y_1\), \(w_2 = x_2 u_2 w' y_2\), and \(w_3 = x_3 u_3 y_3\). The word \(z = x_1 v_1 w' v_2 y_3\) can be obtained by the splicings

\[
(w_1, w_2) \rightarrow_r x_1 v_1 w' u'_2 y_2 = z',
\]

\[
(w_2, w_3) \rightarrow_s x_2 u_2 w' v_2 y_3 = z'',
\]

which makes the order of the splicing steps irrelevant.

Now, consider a word \(z\) which is created by two successive splicings from words \(w_i = x_i u_i y_i\) for \(i = 1, 2, 3\) as in Figure 6. If no factor of \(w_1\) or of the bridge in the first splicing is a part of \(z\), then we can find another splicing rule \(s\) such that \((w_3, w_2) \rightarrow_s z\) and the bridge of \(s\) is the bridge used in the second splicing.

**Lemma 3.7.** Let \(L\) be a language, \(w_i = x_i u_i y_i \in L\) for \(i = 1, 2, 3\), and \(r_1 = (u_1, u_2; v_1), r_2 = (u_3, u_4; v_2)\) be rules respecting \(L\). If there are splicings

\[
(w_1, w_2) \rightarrow_r x_1 v_1 y_2 = w_4 = x_4 u_4 y_4,
\]

\[
(w_3, w_2) \rightarrow_r x_3 v_2 y_4 = z,
\]

where \(y_4\) is a suffix of \(y_2\), then there is a rule \(s = (u_3, \tilde{u}_2; v_2)\) which respects \(L\) and \((w_3, w_2) \rightarrow_s z\).
Therefore, we may assume that a splicing language \( L \) of the form of \((I, R)\) and its generated language \( L(J, S) \). Let \( n \) be the length of the longest word in \( J \) and let \( \mu \) be the length-lexicographical largest word that is a component of a rule in \( S \). Define \( W_\mu = \{ w \in \Sigma^* \mid w \leq _{\ell} \mu \} \) as the set of all words that are at most as large as \( \mu \), in length-lexicographical order. Furthermore, let \( I = \Sigma^T \cap L \) be a set of axioms and let

\[
R = \{ r \in W_\mu \mid r \text{ respects } L \}
\]

be a set of rules. It is not difficult to see that \( J \subseteq I, S \subseteq R, \) and \( L = L(I, R) \). Whenever convenient, we may assume that a splicing language \( L \) is generated by a splicing system which is of the form of \((I, R)\).

Now, consider the creation of a word \( xyz \in L \) by splicing in \((I, R)\). The creation of \( xyz \) can be traced back to a word \( z_1 = x_1 y_1 \) where either \( z_1 \in I \) or where \( z_1 \) is created by a splicing that affects \( z \), i.e., the bridge in this splicing overlaps with the factor \( z \) in \( x_1 y_1 \). The next lemma describes this creation of \( xyz = z_{k+1} \) by \( k \) splicings in \((I, R)\), and shows that we can choose the rules and words which are used to create \( z_{k+1} \) from \( z_1 \) such that the words and bridges of rules are not significantly longer than \( \ell = \max \{ |x|, |y| \} \).

**Lemma 3.8.** Let \( L \) be a splicing language, let \( \ell, n \in \mathbb{N} \), let \( m = |M_L| \), and let \( \mu \) be a word with \( |\mu| \geq \ell + 2m \) such that for \( I = \Sigma^T \cap L \) and \( R = \{ r \in W^3_\mu \mid r \text{ respects } L \} \) we have \( L = L(I, R) \).

Let \( z_{k+1} = x_{k+1} y_{k+1} \), with \( |x_{k+1}|, |y_{k+1}| \leq \ell \), be a word that is created by \( k \) splicings from a word \( z_1 = x_1 y_1 \) where either \( z_1 \in I \) or where \( z_1 \) is created by a splicing \((w_0, w_0)' \) with \( w_0, w_0' \in L, s \in R, \) and the bridge of \( s \) overlaps with \( z \) in \( z_1 \). Furthermore, for \( i = 1, \ldots, k \) the intermediate splicings are either

(i) \( (w_i, z_i) \vdash r_i, x_{i+1} z_{i+1} = z_i+1, w_i \in L, r_i \in R, y_{i+1} = y_i, \text{ and the bridge of } r_i \text{ is covered by the prefix } x_{i+1} \)

(ii) \( (z_i, w_i) \vdash r_i, x_{i+1} z_{i+1} = z_i+1, w_i \in L, r_i \in R, x_{i+1} = x_i, \text{ and the bridge of } r_i \text{ is covered by the suffix } y_{i+1}. \)
There are rules and words creating $z_{k+1}$, as above, satisfying in addition:

1. There is $k' < k$ such that for $i = 1, \ldots, k'$ all splicings are of the form (i) and for $i = k' + 1, \ldots, k$ all splicings are of the form (ii).

2. For $i = 1, \ldots, k$ the following bounds apply: $|x_i|, |y_i| < \ell + 2m$, $|w_i| < m$, $r_i \in \Sigma^{\leq 2m} \times \Sigma^{\leq \ell + m}$.

In particular, if $n \geq m$, then $w_1, \ldots, w_k \in I$.

Proof. Statement 1 follows by Remark 3.6 Note that if $k = 0$, then statement 2 is trivially true. By the first statement, $x_{k+1} = x_{k+2} = \cdots = x_{k+1}$ and $y_1 = y_2 = \cdots = y_{k+1}$. Let us consider the splicings of the form (i) which are the steps $i = 1, \ldots, k'$. The notations we employ in order to prove the second statement for $i = 1, \ldots, k'$ are chosen to match the notations in Figure 7.

Figure 7: The $i$-th splicing step in the proof of Lemma 3.8 where $v_i = u_{i+1}\delta_i u_{i+1}$ and $x_{i+1} = u_{i+1}\delta_i u_{i+1}$.

Let $r_i = (w_i, u_i; v_i)$ where $w_i \in \Sigma^{< m} \cap L$ (Lemma 3.5) and $x_i = u_i x_i'$; (by Lemma 3.2, we extended the site $u_i$ to cover a prefix of $x_i$) such that $u_{i+1} x_i' = u_{i+1} x_i'$ with $u_{i+1} = \varepsilon$ and $x_{i+1} = x_{i+1} = x_{i+1}$. Lemma 3.7 justifies the assumption that every splicing occurs at the left of the preceding splicing, i.e., $x_i'$ is a proper suffix of $x_i$. Note that, as $|x_i'| \leq \ell$, the length of $x_i'$ is bounded by $\ell$. Now, choose $\delta_i u_{i+1}$ such that $x_{i+1} = \delta_i u_{i+1}$; thus, $u_{i+1} = v_i$.

For $i = 2, \ldots, k'$ we replace $u_i$ by a shortest word from $[u_i]_L$. Note that this does not change the fact that all rules respect $L$ (Lemma 3.3). We also replace the prefix of $x_i$ and $v_{i-1}$ by this factor. (There is no need to change $v_{k'}$ as $|v_{k'}| = |\delta_{k'}| \leq |x_{k'})$. Therefore, $|x_i| < |x_i'| + m < \ell + m$ and $r_i \in \Sigma^{< m} \times \Sigma^{< m} \times \Sigma^{< \ell + m}$ if $i \neq 1$ (Lemma 2.2). We do not change $u_1$ yet as this may affect the splicing $(w_0, u_0') \vdash_1 z_1$ if it exists. Note that, for $i = 2, \ldots, k'$, we have actually proven a stronger bound than claimed in statement 2 of Lemma 3.8. Even though we have not proven the bound for $r_1$ yet, we have already established $r_1 \in \Sigma^{< m} \times \Sigma^{< m} \times \Sigma^{< \ell + m}$. Symmetrically, we can consider statement 2 to be proven for $i = k'+ 2, \ldots, k$, i.e., only the prefix $x_i$ and the suffix $y_1 = y_{k'+1}$ have not been modified yet.

Now, let $x_1 = u_1 x_1'$ (as above) and, symmetrically, let $y_1 = y_{k'+1} u_{k'+1}$ where $u_{k'+1}$ is the left site of $r_{k'+1}$. If $k' = 0$ (or $k' = k$), then $u_1$ (resp. $u_{k'+1}$) can be considered empty and $x_1' = x_{k+1}$ (resp. $y_{k'+1} = y_{k+1}$). If $z \in I$ we replace $u_1$ and $u_{k'+1}$ by shortest words from their syntactic classes, respectively, and the claim holds. Otherwise, $(w_0, u_0') \vdash_s z_1$ where $s = (u_0, u_0', v)$, $w_0 = u_0 u_0$, and $u_0' = u_0'y$, by Lemma 3.4. Thus,

$$z_1 = u_1 x_1' y_{k'+1} u_{k'+1} = x v y.$$

In the case when $v$ does not overlap with the prefix $x_1$ of $z_1$, replace $u_1$ by a shortest word from its syntactic class. If $v$ and the prefix $u_1$ overlap, let $u_1 = \delta_1 \delta_2$ such that $\delta_2$ is the overlap and replace $\delta_1$ by a shortest word from their syntactic classes, respectively. In both cases, $|u_1| < 2m$ (Lemma 2.2) and if $v$ was modified, it got shorter; hence, we still have $v \in W_m$. Observe that $|x_1| < \ell + 2m$ and $r_1 \in \Sigma^{< m} \times \Sigma^{< 2m} \times \Sigma^{< \ell + m}$. Analogously, $u_{k'+1}$ and $r_{k'+1}$ can be treated in order to conclude the prove of statement 2.

\section{Proof of Theorem 3.1}

Let $L$ be a splicing language and $m = |M_L|$. Throughout this section, by $\sim_L$ we denote the equivalence relation $\sim_L$ and by $[\cdot]_L$ we denote the corresponding equivalence classes $[\cdot]_L$. 

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Recall that Theorem 3.1 claims that the splicing system \((I, R)\) with \(I = \Sigma^{<m^2+6m} \cap L\) and 
\[
R = \left\{ r \in \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{<m^2+10m} \mid r \text{ respects } L \right\}
\]
generates \(L\). The proof is divided in two parts. In the first part, Lemma 3.9, we prove that \(L\) is generated by a splicing system \((I, R')\) where all sites of rules in \(R'\) are shorter than \(2m\), but we do not care about the lengths of the bridges. The second part will then conclude the proof by showing that there are no rules in \(R'\) with bridges of length greater than or equal to \(m^2 + 10m\) which are essential for the creation of the language \(L\) by splicing.

**Lemma 3.9.** Let \(L, m,\) and \(I\) as above. There is \(n \in \mathbb{N}\) such that the splicing system \((I, R')\) with 
\[
R' = \left\{ r \in \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{\leq n} \mid r \text{ respects } L \right\}
\]
generates the language \(L = L(I, R')\).

**Proof.** As \(I \subseteq L\) and every rule in \(R'\) respects \(L\), it is clear that \(L(I, R') \subseteq L\) for any \(n\); we only need to prove the converse inclusion.

As \(L\) is a splicing language, \(L = L(J, S)\) for some splicing system \((J, S)\). Let \(n\) be larger than the length of every bridge of every rule in \(S\) and \(n \geq 4m^2\).

In order to prove \(L \subseteq L(I, R')\) we use induction on the length of words in \(L\). For all \(w \in L\) with \(|w| < m^2 + 6m\), by definition, \(w \in I \subseteq L(I, R')\).

Now, consider \(w \in L\) with \(|w| \geq m^2 + 6m\). The induction hypothesis states that every word \(w' \in L\) with \(|w'| < |w|\) belongs to \(L(I, R')\). Factorize \(w = x\alpha\beta\gamma\delta y\) such that \(|x|, |y| = 3m, |\alpha\beta\gamma| = m^2, |\beta| \geq 1, \alpha \sim \alpha\beta\), and \(\gamma \sim \beta\gamma\).

The proof idea is to use a pumping argument on \(\alpha\beta\gamma\) in order to obtain a very long word. This word has to be created by a series of splicings in \((J, S)\). We show that these splicings can be modified in order to create \(w\) by splicing from a set of strictly shorter words and with rules from \(R'\). Then, the induction hypothesis implies \(w \in L(I, R')\).

Choose \(j\) sufficiently large (\(j > n\) and \(J\) does not contain words of length \(j\) or more). We let \(z = \alpha\beta\gamma\delta\) and investigate the creation of \(xzy \in L\). As \(z\) is not a factor of a words in \(J\), every word in \(L\) which contains \(z\) is created by some splicing in \((J, S)\). Thus, we can trace back the creation of \(xzy\) by splicing to the point where the factor \(z\) is affected for the last time. Let \(z_{k+1} = x_{k+1}z_{k}y_{k+1}\), where \(x_{k+1} = x\) and \(y_{k+1} = y\), be created by \(k\) splicings from a word \(z_1 = x_1y_1\) where \(x_1y_1\) is created by a splicing \((u_0, w_0) \vdash_s z_1\) with \(u_0, w_0 \in L, s \in S\), and the bridge of \(s\) overlaps with \(z\) in \(z_1\). Furthermore, for \(i = 1, \ldots, k\) the intermediate splicings are either

i) \((w_i, z_i) \vdash_r x_{i+1}z_{i+1}y_{i+1} = z_{i+1}, w_i \in L, r_i \in S, y_{i+1} = y_i\), and the bridge of \(r_i\) is covered by the prefix \(x_{i+1}\) or

ii) \((z_i, w_i) \vdash_r x_{i+1}z_{i+1}y_{i+1} = z_{i+1}, w_i \in L, r_i \in S, x_{i+1} = x_i\), and the bridge of \(r_i\) is covered by the suffix \(y_{i+1}\).

Following Lemma 3.8 (with \(\ell = 3m\)), we may assume that \(w_1, \ldots, w_k \in I, r_1, \ldots, r_k \in \Sigma^{<2m} \times \Sigma^{<4m}\), thus \(r_1, \ldots, r_k \in R'\), and \(|x_1|, |y_1| < 5m\). Furthermore, we may use the same words and rules in order to create \(w = x_{k+1}\alpha\beta\gamma\delta y_{k+1}\) from \(x_1\alpha\beta\gamma\delta y_1\) by splicing, i.e., if \(x_1\alpha\beta\gamma\delta y_1\) belongs to \(L(I, R')\), so does \(w\).

Now, consider the first splicing \((w_0, w_0') \vdash_s z_1 = x_1y_1\). By Lemma 3.4, we assume \(s = (u_1, u_2; v)\) such that \(w_0 = xu_1, w_0' = uy_2y\) and \(|u_1|, |u_2| < m\) (\(x\) and \(y\) are newly chosen words).

Hence,
\[
z_0 = xvy = x_1\alpha\beta\gamma\delta y_1.
\]
where \(x\) is a proper prefix of \(x_1\alpha\beta\gamma\delta\) and \(y\) is a proper suffix of \(\alpha\beta\gamma\delta y_1\).

We will now pump down the factor \(\beta^j\) to \(\beta\) in order to obtain the words \(\bar{x}, \bar{v}, \bar{y}\) from \(x, v, y\), respectively, as follows:

1. If \(v\) overlaps with \(\beta^j\) but does neither cover \(\alpha\) nor \(\gamma\), extend \(v\) (Lemma 3.2) such that \(v = \alpha\beta^j\gamma\). Observe that, now, the factor \(\alpha\beta^j\gamma\) is covered by either \(xv\) or \(vy\).
2. If $\alpha \beta \gamma$ is covered by one of $x$, $v$, or $y$, then replace this factor by $\alpha \beta$ or $\beta \gamma$, respectively. Otherwise, by symmetry, assume that $\alpha \beta \gamma$ is covered by $x v$ and, therefore, we can factorize
\[
x = x_1 \alpha \beta_1 \beta_4 \quad \quad v = \beta_2 \beta_2 \gamma v'
\]
where $\beta_2 \beta_2 = \beta$ and $j_1 + j_2 + 1 = j$. The results of pumping are the words $\bar{x} = x_1 \alpha \beta_1$, $\bar{v} = \beta_2 \gamma v'$, and $\bar{y} = y$.

Let $u_1$ and $u_2$ be the sites of $s$ that may have been altered due to the extension of $v$ and, by Lemma 3.4, assume $|u_1|, |u_2| < m$. If we used an extension for $v$, then $|\bar{v}| = m^2$. No matter whether we used an extension, $t = (\bar{u}_1, \bar{u}_2, t') \in R'$ and $(\bar{u}_1, \bar{u}_2 \bar{y}) \vdash t \hat{x} \alpha \beta_2 \gamma v_1$ as desired. Observe that $\hat{x}$ is a prefix of $x_1 \alpha \beta_2 \gamma$ and $\bar{y}$ is a suffix of $\alpha \beta_2 \gamma v_1$ and recall that $|x_1|, |y_1| < 5m$. Therefore, $|\bar{x} \bar{u}_1|, |\bar{u}_2 \bar{y}| < |\alpha \beta_2 \gamma| + 6m = |w|$ and, by induction hypothesis, $\bar{x} \bar{u}_1$ and $\bar{u}_2 \bar{y}$ belong to $L(I, R')$.

We conclude that $x_1 \alpha \beta_2 \gamma v_1$ as well as $w$ belong to $L(I, R')$.

We are now prepared to prove the main result.

Proof of Theorem 3.1. Recall that for a splicing language $L$ with $m = |M_L|$, we intend to prove that the splicing system $(I, R)$ with $I = \Sigma^{<m^2+6m} \cap L$ and
\[
R = \left\{ r \in \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{<m^2+10m} \mid r \text{ respects } L \right\}
\]
generates the language $L = L(I, R)$. Obviously, $L(I, R) \subseteq L$. By Lemma 3.9, there is a finite set of rules $R' \subseteq \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^*$ such that $L(I, R') = L$.

For a word $\mu$ we let $W_\mu = \{ w \in \Sigma^* \mid |w| \leq \ell \mu \}$, as we did before. Define the set of rules where every component is length-lexicographically bounded by $\mu$
\[
R_\mu = \{ r \in \Sigma^{<2m} \times \Sigma^{<2m} \times W_\mu \mid r \text{ respects } L \}
\]
and the language $L_\mu = L(I, R_\mu)$; clearly, $L_\mu \subseteq L$. For two words $\mu \leq \ell \nu$ we see that $R_\mu \subseteq R_\nu$, and hence, $L_\mu \subseteq L_\nu$. Thus, if $L_\mu = L$ for some word $\mu$, then for all words $v$ with $\mu \leq \ell v$, we have $L_v = L$. As $L = L(I, R')$, there exists a word $\mu$ such that $L_\mu = L$. Let $\mu$ be the smallest word, in the length-lexicographic order, such that $L_\mu = L$. Note that if $|\mu| < m^2 + 10$, then $R_\mu \subseteq R$ and $L = L_\mu \subseteq L(I, R)$.

For the sake of contradiction assume $|\mu| \geq m^2 + 10m$. Let $\nu$ be the next-smaller word than $\mu$, in the length-lexicographic order, and let $S = R_\nu$. Note that $L(I, S) \subseteq L$ and $R_\mu \cap S$ contains only rules whose bridges are $\mu$.

Choose $w$ from $L \setminus L(I, S)$ as a shortest word, i.e., for all $w' \in L$ with $|w'| < |w|$, we have $w' \in L(I, S)$. Factorize $w = x z y$ with $|x| = |y| = 3m$; note that $|w| \geq m^2 + 6m$ since, otherwise, $w \in L$. Factorize $\mu = \delta_1 \alpha \beta_1 \gamma \delta_2$ with $|\delta_1|, |\delta_2| \geq 5m$, $|\alpha \beta_1 \gamma| = m^2$, $\beta \neq \varepsilon$, $\alpha \sim \alpha \beta$, and $\gamma \sim \beta \gamma$, by Lemma 2.1.

Next, we will use a pumping argument on all factors $\alpha \beta \gamma$ in $z$. As in the proof of Lemma 3.9, this new word has to be created by a series of splicings in $(I, R_\mu)$ and we will show that these splicings can be modified in order to create $w$ from strictly shorter words and with rules from $S$. This will contradict the assumption that $w$ is a shortest word from $L \setminus L(I, S)$.

Let $j$ be a sufficiently large even number ($j > 4 |\mu| + |z|$ will do). We define a word $\bar{z}$ which is the result of applying the pumping algorithm from Lemma 2.3 on $z$, as discussed in Section 2.1. The pumping algorithm replaces the occurrences of $\alpha \beta \gamma$ in $z$ by $\gamma \beta \gamma$ such that for every factor $\bar{z}[k, k + m^2] = \alpha \beta \gamma$, either

(a) $\alpha \beta^{j/2}$ is a factor of $\bar{z}$ starting at position $\bar{z}[k]$

(b) $\beta^{j/2} \gamma$ is a factor of $\bar{z}$ ending at position $\bar{z}[k + m^2]$

holds. In particular, if $\delta_1 \alpha \beta_1 \gamma \delta_2$ is a factor of $z$ either (a) $\gamma \delta_2$ is a prefix of a word in $\beta^*$ or (b) $\delta_1 \alpha \beta_1 \gamma \delta_2$ is a suffix of a word in $\beta^*$. By induction and as $\alpha \beta \gamma \sim \alpha \beta \gamma$, it is easy to see that $z \sim \bar{z}$ and $x \bar{y} \in L$.

Let us trace back the creation of $x \bar{y} \in L$ by splicing in $(I, R_\mu)$ to a word $x_1 \bar{y} v_1$ where either $x_1 \bar{y} v_1 \in I$ or where $x_1 \bar{y} v_1$ is created by a splicing that affects $\bar{z}$. Let $z_{k+1} = x_{k+1} \bar{y} v_{k+1}$, where
$x_{k+1} = x$ and $y_{k+1} = y$, be created by $k$ splicings from a word $z_1 = x_1 \tilde{z} y_1$ where either $x_1 \tilde{z} y_1 \in I$ or $x_1 \tilde{z} y_1$ is created by a splicing $\langle w_0, w_0' \rangle \vdash_{s} z_1$ with $w_0, w_0' \in L$, $s \in R_\mu$, and the bridge of $s$ overlaps with $\tilde{z}$. Furthermore, for $i = 1, \ldots, k$ the intermediate splicings are either

(i) $(w_i, z_i) \vdash_{r_i} x_{i+1} \tilde{z} y_{i+1} = z_{i+1}$, $w_i \in L$, $r_i \in R_\mu$, $y_{i+1} = y_i$, and the bridge of $r_i$ is covered by the prefix $x_{i+1}$ or

(ii) $(z_i, w_i) \vdash_{r_i} x_{i+1} \tilde{z} y_{i+1} = z_{i+1}$, $w_i \in L$, $r_i \in R_\mu$, $x_{i+1} = x_i$, and the bridge of $r_i$ is covered by the suffix $y_{i+1}$.

Following Lemma 3.8 (with $\ell = 3m$), we may assume that $w_1, \ldots, w_k \in I$, $r_1, \ldots, r_k \in \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{<4m}$, thus $r_1, \ldots, r_k \in S$, and $|x_1|, |y_1| < 5m$. Furthermore, we may use the same words and rules in order to create $w = x_{k+1} \tilde{z} y_{k+1}$ from $x_1 \tilde{z} y_1$ by splicing. As $w$ does not belong to $L(I, S)$, the word $x_1 \tilde{z} y_1$ must not belong to $L(I, S)$ either. If $z_1$ was in $I$, then $x_1 \tilde{z} y_1 \in I$ as well, as $z$ is at most as long as $\tilde{z}$.

Therefore, $z_1$ is created by a splicing $\langle w_0, w_0' \rangle \vdash_{s} z_1$ where $s = (u_1, u_2; v)$, $w_0 = x_1 u_1$, and $w_0' = u_2 y$ where $|u_1|, |u_2| < m$, by Lemma 3.4 (here, $x$ and $y$ are newly chosen words). We have

$$z_1 = x_1 \tilde{z} y_1 = xvy$$

where $x$ is a proper prefix of $x_1 \tilde{z}$ and $y$ is a proper suffix of $\tilde{z} y_1$. Recall that either $s \in S$ or $v = \mu$.

However, we will see next that if $v \neq \mu$, there is also a rule $\tilde{s} \in S$ and slightly modified words which can be used in order to create $x_1 \tilde{z} y_1$ by splicing. In this case $\mu = \delta_1 x_1 \beta \gamma_2 \delta_2$ is a factor of $z_1$. As $|\delta_1|, |\delta_2| \geq 5m > |x_1|, |y_1|$, the factor $\beta \gamma$ is covered by $\tilde{z}$ and, as such, the pumping algorithm ensured that either (a) $\alpha$ is succeeded by $\beta \gamma_2$ or (b) $\gamma$ is preceded by $\beta \gamma_2$. Due to symmetry, we only consider the former case, in which $\gamma \delta_2$ is a prefix of a word in $\beta^+$. Let us shorten the bridge $v$ such that $\tilde{s} = (u_1, u_2; \delta_1 \alpha \gamma \delta_2)$. Note that $\tilde{s} \in S$ (as $\alpha \sim \alpha \beta$ and by Lemma 3.3). Furthermore, as $j$ is large enough, $y = \beta_2 \beta \gamma y'$ where $\beta_2$ is the suffix of $\beta$ such that $\gamma \delta_2 \delta_2 \beta_2 \in \beta^+$ and $\ell \geq |\gamma|$. Note that this implies $\beta_2 \gamma$ is a prefix of $y$, which allows us to add an additional $\beta$. Therefore, $\langle w_0, u_2 \beta \beta_2 \beta \gamma y' \rangle \vdash_{\tilde{z}} z_1$ where $u_2 \beta \beta_2 \beta \gamma y' \in L$. This observation justifies the assumption that $v \neq \mu$ and $s \in S$ which we will make for the remainder of the proof.

Next, we will pump down the factors $\beta \gamma$ to $\beta \gamma$ in $\tilde{z}$ again. At every position where we pumped up before, we are now pumping down (in reverse order) in order to obtain the words $\tilde{x}$, $\tilde{v}$, $\tilde{y}$ from the words $x$, $v$, $y$, respectively. The pumping in each step is done as in the proof of Lemma 3.9:

1. If $v$ overlaps with $\beta^j$ (in the factor that we are pumping down) but it neither covers $\alpha$ nor $\gamma$, extend $v$ (Lemma 3.2) such that $v = \alpha \beta \gamma$. Observe that, now, the factor $\alpha \beta \gamma$ is covered by either $xv$ or $vy$.

2. If $\alpha \beta \gamma$ is covered by one of $x$, $v$, or $y$, then replace this factor by $\alpha \beta$ or $\beta \gamma$, respectively. Otherwise, by symmetry, assume that $\alpha \beta \gamma$ is covered by $xv$ and, therefore, we can factorize

$$x = x' \alpha \beta \gamma = y'$$

where $\beta_1 \beta_2 = \beta$ and $j_1 + j_2 + 1 = j$. The results of pumping are the words $x' \alpha \beta_1$, $\beta_2 \gamma y'$.

Let $\tilde{u}_1$ and $\tilde{u}_2$ be the sites of $s$ that may have been altered due to extensions and, by Lemma 3.4, assume $|\tilde{u}_1|, |\tilde{u}_2| < m$. If we used an extension for $v$ in at least one of the steps, then $|\tilde{v}| \leq m^2$. No matter whether or not we used an extension, $t = (\tilde{u}_1, \tilde{u}_2; \tilde{v}) \in S$ and $(\tilde{x} \tilde{u}_1, \tilde{u}_2 \tilde{y} \tilde{u}_2) \vdash_{t} x_1 \tilde{z} y_1$. As $|\tilde{x} \tilde{u}_1|, |\tilde{u}_2 \tilde{y}| < |z| + 6m = |w|$, $\tilde{x} \tilde{u}_1$ and $\tilde{u}_2 \tilde{y}$ belong to $L(I, S)$. We conclude that $x_1 \tilde{z} y_1$ as well as $w$ belong to $L(I, S)$ — the desired contradiction.

4 The Case of Classical Splicing

In this section, we consider the splicing operation as defined in [18]. This is the most commonly used definition for splicing in formal language theory. The notation we use has been employed in previous papers, see e.g., [2, 9]. Throughout this section, a quadruplet of words
$r = (u_1, v_1; u_2, v_2) \in (\Sigma^*)^4$ is called a (splicing) rule. The words $u_1v_1$ and $u_2v_2$ are called left and right site of $r$, respectively. This splicing rule can be applied to two words $w_1 = x_1u_1v_1y_1$ and $w_2 = x_2u_2v_2y_2$, that each contain one of the sites, in order to create the new word $z = x_1u_1v_2y_2$, see Figure 8. This operation is called splicing and it is denoted by $(w_1, w_2) \vdash_r z$. The splicing position of this splicing is $z[i|x_1u_1|]$; that is the position between the factors $x_1u_1$ and $v_2y_2$ in $z$.

![Figure 8: Splicing of the words $x_1u_1v_1y_1$ and $x_2u_2v_2y_2$ by the rule $r = (u_1, v_1; u_2, v_2)$.](image)

Just as in Section 3, for a rule $r$ we define the splicing operator $\sigma_r$ such that for a language $L$

$$\sigma_r(L) = \{ z \in \Sigma^* \mid \exists w_1, w_2 \in L: (w_1, w_2) \vdash_r z \}$$

and for a set of splicing rules $R$, we let

$$\sigma_R(L) = \bigcup_{r \in R} \sigma_r(L).$$

The reflexive and transitive closure of the splicing operator $\sigma_R^*$ is given by

$$\sigma_R^0(L) = L, \quad \sigma_R^{i+1}(L) = \sigma_R^i(L) \cup \sigma_R(\sigma_R^i(L)), \quad \sigma_R^*(L) = \bigcup_{i \geq 0} \sigma_R^i(L).$$

A finite set of axioms $I \subseteq \Sigma^*$ and a finite set of splicing rules $R \subseteq (\Sigma^*)^4$ form a splicing system $(I, R)$. Every splicing system $(I, R)$ generates a language $L(I, R) = \sigma_R^*(I)$. Note that $L(I, R)$ is the smallest language which is closed under the splicing operator $\sigma_R$ and includes $I$. It is known that the language generated by a splicing system is regular, see [6, 17]. A (regular) language $L$ is called a splicing language if a splicing system $(I, R)$ exists such that $L = L(I, R)$.

A rule $r$ is said to respect a language $L$ if $\sigma_r(L) \subseteq L$. It is easy to see that for any splicing system $(I, R)$, every rule $r \in R$ respects the generated language $L(I, R)$. Moreover, a rule $r \notin R$ respects $L(I, R)$ if and only if $L(I, R \cup \{r\}) = L(I, R)$. We say a splicing $(w_1, w_2) \vdash_r z$ respects a language $L$ if $w_1, w_2 \in L$ and $r$ respects $L$; obviously, this implies $z \in L$, too.

The main result of this section states that, if a regular language $L$ is a splicing language, then it is generated by a particular splicing system $(I, R)$ which only depends on the syntactic monoid of $L$.

**Theorem 4.1.** Let $L$ be a splicing language and $m = |M_L|$. The splicing system $(I, R)$ with

$$I = \Sigma^{<m^2 + 6m} \cap L$$

and

$$R = \left\{ r \in \Sigma^{<m^2 + 10m} \times \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{<m^2 + 10m} \middle| r \text{ respects } L \right\}$$

generates the language $L = L(I, R)$.

As the language generated by the splicing system $(I, R)$ is constructible, Theorem 4.1 implies that the problem whether or not a given regular language is a splicing language is decidable. A detailed discussion of the decidability result is given in Section 5.

Let $L$ be a formal language. Clearly, every set of words $J \subseteq L$ and set of rules $S$ where every rule in $S$ respects $L$ generates a subset $L(J, S) \subseteq L$. Therefore, in Theorem 4.1 the inclusion $L(I, R) \subseteq L$ is obvious. The rest of this section is devoted to the proof of the converse inclusion $L \subseteq L(I, R)$. The proof uses many ideas that have been employed in the Section 3. However,
there are some challenges we encounter solely while considering the classic splicing variant. The additional complexity comes from having to handle the first and fourth components of rules, which in the case of classical splicing occur both in the words used for splicing and the splicing result. In contrast, in Pixton splicing the sites of a rule do not occur in the splicing result, whereas the bridge is not a factor of the words used for splicing. The structure of this section is the same as Section 3.

In Section 4.1 we will present techniques to obtain rules that respect a regular language \( L \) from other rules that respect \( L \), and we show how we can modify a splicing step, such that the words used for splicing are not significantly longer than the splicing result; similar results can be found in [8,9]. In Section 4.2 we use these techniques to show that a long word \( z \in L \) can be obtained by a series of splicings from a set shorter words from \( L \) and by using rules which satisfy certain length restrictions. Finally, in Section 4.3 we prove Theorem 4.1.

4.1 Rule Modifications

The first lemma states us that we can extend the sites of a rule \( r \) such that the extended rule respects all languages that are respected by \( r \).

**Lemma 4.2.** Let \( r = (u_1; v_1; u_2, v_2) \) be a rule which respects a language \( L \). For every word \( x \), the rules \( (xu_1, v_1; u_2, v_2) \), \( (u_1, v_1; xu_2, v_2) \), \( (u_1, v_1; u_2, xv_2) \), and \( (u_1, v_1; u_2, xv_2) \) respect \( L \) as well.

**Proof.** Let \( s \) be any of the rules \( (xu_1, v_1; u_2, v_2) \), \( (u_1, v_1; xu_2, v_2) \), \( (u_1, v_1; u_2, xv_2) \). In order to prove that \( s \) respects \( L \), we have to show that, for all \( w_1, w_2 \in L \) and \( z \in \Sigma^* \) such that \( (w_1, w_2) \vdash z \), we have \( z \in L \), too. Indeed, if \( (w_1, w_2) \vdash z \), then \( (w_1, w_2) \vdash z \), and as \( r \) respects \( L \), we conclude \( z \in L \) .

Henceforth, for a rule \( r = (u_1, v_1; u_2, v_2) \), we will refer to the rules \( (xu_1, v_1; u_2, v_2) \) and \( (u_1, v_1; u_2, xv_2) \) as extensions of the left site of \( r \) and to \( (u_1, v_1; xu_2, v_2) \) and \( (u_1, v_1; u_2, v_2) \) as extensions of the right site of \( r \).

Next, for a language \( L \), let us investigate the syntactic class of a rule \( r = (u_1, v_1; u_2, v_2) \). The syntactic class (with respect to \( L \)) of \( r \) is the set of rules \( [r]_L = [u_1]_L \times [v_1]_L \times [u_2]_L \times [v_2]_L \) and two rules \( r \) and \( s \) are syntactically congruent (with respect to \( L \)), denoted by \( r \sim_L s \), if \( s \in [r]_L \).

**Lemma 4.3.** Let \( r \) be a rule which respects a language \( L \). Every rule \( s \in [r]_L \) respects \( L \).

**Proof.** Let \( r = (u_1, v_1; u_2, v_2) \) and \( s = (\bar{u}_1, \bar{v}_1; \bar{u}_2, \bar{v}_2) \). Thus, \( u_i \sim_L \bar{u}_i \) and \( v_i \sim \bar{v}_i \) for \( i = 1, 2 \). For \( \tilde{w}_1 = x_1u_1v_1y_1 \in L \) and \( \tilde{w}_2 = x_2u_2v_2y_2 \in L \) we have to show that \( \tilde{z} = x_1\tilde{u}_1\tilde{v}_2y_2 \in L \). For \( i = 1, 2 \), let \( \tilde{w}_i = x_iu_1v_1y_1 \) and note that \( w_i \sim_L \tilde{w}_i \); hence, \( w_i \in L \). Furthermore, \( (w_1, w_2) \vdash r, x_1u_1v_2y_2 = z \in L \) as \( r \) respects \( L \), and \( \tilde{z} \in L \) as \( z \sim_L \tilde{z} \).

Consider a splicing \( (x_1u_1v_1y_1, x_2u_2v_2y_2) \vdash r, x_1u_1v_2y_2 \) which respects a regular language \( L \), as shown in Figure 9 on the left site. The factors \( v_1y_1 \) and \( x_2u_2 \) may be relatively long but they do not occur as factors in the resulting word \( x_1u_1v_2y_2 \). In particular, it is possible that two long words are spliced and the outcome is a relatively short word. Using the Lemmas 4.2 and 4.3, we can find shorter words in \( L \) and a modified splicing rule which can be used to obtain \( x_1u_1v_2y_2 \).

![Figure 9: Replacing \( v_1y_1 \) and \( x_2u_2 \) by short words.](image-url)
Lemma 4.4. Let \( r = (u_1, v_1; u_2, v_2) \) be a rule which respects a regular language \( L \) and \( w_1 = x_1v_1y_1 \in L, w_2 = x_2v_2y_2 \in L \). There is a rule \( s = (u_1, \tilde{v}_1; u_2, \tilde{v}_2) \) which respects \( L \) and words \( \tilde{v}_1 = x_1u_1y_1 \in L, \tilde{v}_2 = x_2u_2y_2 \in L \) such that \(|\tilde{v}_1|, |\tilde{v}_2| < |M_L| \). More precisely, \( \tilde{v}_1 \in [v_1y_1]_L \) and \( \tilde{v}_2 \in [x_2u_2]_L \).

In particular, whenever \( (w_1, w_2) \vdash_r x_1u_1v_2y_2 = z \), then there is a splicing \( (\tilde{w}_1, \tilde{w}_2) \vdash_s z \) which respects \( L \) where \( \tilde{w}_1, \tilde{w}_2 \), and \( s \) have the properties described above.

Proof. By Lemma 4.2, the rule \( (u_1, v_1; x_2, u_2, v_2) \) respects \( L \). Choose \( \tilde{v}_1 \in [v_1y_1]_L \) and \( \tilde{w}_2 \in [x_2u_2]_L \) as shortest words from the sets, respectively. By Lemma 2.2, \(|\tilde{u}_1|, |\tilde{w}_2| < |M_L| \) and \( w_1 = x_1u_1v_1 \in L, \tilde{w}_2 = \tilde{u}_2v_2y_2 \in L \). Furthermore, by Lemma 4.3, \( s = (u_1, \tilde{v}_1; \tilde{w}_2, v_2) \) respects \( L \).

4.2 Series of Splicings

Consider the creation of words by a series of splicings. Let us begin with a simple observation. In the case when a word is created by two (or more) successive splicings, but none of the splicing sites overlaps the position of the other splicing, the order of these splicings is irrelevant. Recall that the case when a word is created by two successive splicings, but none of the splicings sites have the properties described above.

Figure 10: The word \( x_1u_1v_2y_2 \) can be created either by using the right splicing first or by using the left splicing first.

Remark 4.5. Let \( w_1 = x_1u_1v_1y_1, w_2 = x_2u_2v_2y_2 \), where \( v_2 \) is a prefix of \( z_2 \), \( w_3 = x_3u_3v_3y_3 \) be words and \( r_1 = (u_1, v_1; u_2, v_2), r_2 = (u'_2, v'_2; u_3, v_3) \) be rules. In order to create the word \( z = x_1u_2z_2v_3y_3 \) by splicing, we may use splicings

\[
(w_1, w_2) \vdash_{r_1} x_1u_1z_2v_2y_2 = z', \quad (z', w_3) \vdash_{r_2} z \quad \text{or} \quad (w_2, w_3) \vdash_{r_2} x_2u_2z_2v_3y_3 = z'', \quad (w_1, z'') \vdash_{r_1} z.
\]

Now, consider a word \( z \) which is created by two successive splicings from words \( w_i = x_iu_iv_iy_i \), for \( i = 1, 2, 3 \) as in Figure 11. If no factor of \( w_1 \) is a part of \( z \), then we can find another splicing rule \( s \) such that \( (w_3, w_2) \vdash_s z \). This replacement will become crucial in the proof of Lemma 4.7.

Figure 11: If no part of \( x_1u_1v_1y_1 \) is a factor of the splicing result, then the two splicings can be reduced to one splicing.

Lemma 4.6. Let \( L \) be a language, \( w_i = x_iu_iv_iy_i \in L \) for \( i = 1, 2, 3 \), and \( r_1 = (u_1, v_1; u_2, v_2), r_2 = (u_3, v_3; u_4, v_4) \) be rules respecting \( L \). If there are splicings

\[
(w_1, w_2) \vdash_{r_1} x_1u_1v_2y_2 = w_4 = x_4u_4v_4y_4, \quad (w_3, w_4) \vdash_{r_2} x_3u_3v_4y_4 = z
\]
where \( v_3y_1 \) is a suffix of \( v_2y_2 \), then there is a rule \( s = (u_3, v_3; u_2\delta, \tilde{\delta}_1) \) which respects \( L \) and \( (u_3, w_2) \vdash s z \). Furthermore, \( v_4 = v_4 \) or \( v_4 \leq_{\ell} v_2 \).

**Proof.** Extension (Lemma 4.2) justifies the assumption that the factors \( u_1v_2 \) and \( u_1v_4 \) match in \( w_4 \); let \( w_4[i, j] = u_1v_2 \) and \( w_4[i', j'] = u_4v_4 \),

- if \( i < i' \) we extend \( u_4 \) in \( r_2 \) to the left by \( i' - i \) letters,
- if \( i > i' \) we extend \( u_1 \) in \( r_1 \) to the left by \( i - i' \) letters,
- if \( j < j' \) we extend \( v_2 \) in \( r_1 \) to the right by \( j' - j \) letters, and
- if \( j > j' \) we extend \( v_4 \) in \( r_2 \) to the right by \( j - j' \) letters.

Clearly, the extended factors \( u_1v_2 \) and \( u_1v_4 \) match in \( w_4 \). As \( v_3y_4 \) was a suffix of \( v_2y_2 \) before extension, now, \( v_4 \) is a suffix of \( v_2y_2 \) and \( y_4 = y_1 \). Additionally, either \( v_4 \) was not extended or \( v_4 \leq_{\ell} v_2 \) and \( v_2y_2 \) was not extended. Let \( \delta \) such that \( \delta v_4 = v_2 \), let \( s = (u_3, v_3; u_2\delta, v_4) \), and observe that \( (w_4, u_4) \vdash s z \).

Next, let us prove that \( s \) respects \( L \). Let \( w'_4 = \delta x_1u_3v_3y'_3 \in L \) and \( w'_2 = \delta x_2u_2v_4y'_2 = \delta x_2u_2v_2y'_2 \in L \). There are splicings

\[
(w_1, w'_2) \vdash r_1, x_1u_3v_3y'_3 = w'_4 = x_1u_4v_4y'_2, \quad (w'_3, w'_4) \vdash r_2, x'_3u_3v_3y'_3 = z', 
\]

and \( z' \in L \), concluding that \( s \) respects \( L \).

Consider a splicing system \((J, S)\) and its generated language \( L = L(J, S) \). Let \( n \) be the length of the longest word in \( J \) and let \( \mu \) be the length-lexicographically largest word that is a component of a rule in \( S \). Define \( W_{\mu} = \{ w \in \Sigma^* \mid w \leq_{\ell} \mu \} \) as the set of words which are at most as large as \( \mu \), in length-lexicographic order. Furthermore, let \( I = \Sigma^{\leq n} \cap L \) be a set of axioms and let

\[
R = \{ r \in W_{\mu} \mid r \text{ respects } L \}
\]

be a set of rules. It is not difficult to see that \( J \subseteq I, S \subseteq R \), and \( L = L(I, R) \). Whenever convenient, we will assume that a splicing language \( L \) is generated by a splicing system which is of the form of \((I, R)\).

Now, let us consider a word \( xyz \in L \) where the length of the middle factor \( z \) is at least \( |\mu| \).

The creation of \( xyz \) by splicing in \((I, R)\) can be traced back to a word \( x_1zy_1 = z_1 \) where either \( z_1 \in I \) or where \( z_1 \) is created by a splicing that affects the factor \( z \), i.e., the splicing position lies in the factor \( z \). The next lemma describes this creation of \( xyz \) as \( z_{k+1} \) by \( k \) splicings in \((I, R)\), and shows that we can choose the rules and words which are used to create \( z_{k+1} \) from \( z_1 \) such that the words are not significantly longer than \( \ell = \max \{|x|, |y|\} \) and such that the rules satisfy certain length restrictions.

**Lemma 4.7.** Let \( L \) be a splicing language, let \( \ell, n \in \mathbb{N} \), let \( m = |M_L| \), and let \( \mu \) be a word with \( |\mu| \geq \ell + 2m \) such that for \( I = \Sigma^{\leq n} \cap L \) and \( R = \{ r \in W_{\mu} \mid r \text{ respects } L \} \) we have \( L = L(I, R) \). Let \( z_{k+1} = x_{k+1}zy_{k+1} \) with \( |z| \geq |\mu| \) and \( |x_{k+1}|, |y_{k+1}| \leq \ell \) be a word that is created by \( k \) splicings from a word \( z_1 = x_1zy_1 \) where either \( z_1 \in I \) or \( z_1 \) is created by a splicing \((w_0, w'_0) \vdash s z_1 \) where \( w_0, w'_0 \in L, s \text{ respects } L, \) and the splicing position lies in the factor \( z \). Furthermore, for \( i = 1, \ldots, k \), the intermediate splicings are either

(i) \( (w_i, z_i) \vdash r_i, x_{i+1}zy_{i+1} = z_{i+1}, w_i \in L, r_i \in R, y_{i+1} = y_i, \) and the splicing position lies at the left of the factor \( z \) or

(ii) \( (z_i, w_i) \vdash r_i, x_{i+1}zy_{i+1} = z_{i+1}, w_i \in L, r_i \in R, x_{i+1} = x_i, \) and the splicing position lies at the right of the factor \( z \).

There are rules and words creating \( z_{k+1} \), as above, satisfying in addition:

1. There is \( k' \leq k \) such that for \( i = 1, \ldots, k' \) all splicings are of the form (i) and for \( i = k' + 1, \ldots, k \) all splicings are of the form (ii).
2. For $i = 1, \ldots, k'$ the following bounds apply: $|x_i| < \ell + 2m$, $|w_i| < \ell + 2m$, $r_i \in \Sigma^{<\ell + m} \times \Sigma^{<2m} \times \Sigma^{<2m} \times W_\mu$, and $x_{k'+1} = x_{k'+2} = \cdots = x_{k+1}$.

3. For $i = k'+1, \ldots, k$ the following bounds apply: $|y_i| < \ell + 2m$, $|w_i| < \ell + 2m$, $r_i \in W_\mu \times \Sigma^{<2m} \times \Sigma^{<2m} \times \Sigma^{<\ell + m}$, and $y_1 = y_2 = \cdots = y_{k+1}$.

In particular, if $n \geq \ell + 2m$, then $w_1, \ldots, w_k \in I$.

**Proof.** The first statement follows immediately by Remark 4.5 and the fact that $|z| \geq |\mu|$. The first statement also implies implies $x_{k'+1} = x_{k'+2} = \cdots = x_{k+1}$ and $y_1 = y_2 = \cdots = y_{k+1}$. Note that if $k' = 0$ (or $k' = k$), then statement 2 (resp. statement 3) is trivially true.

![Figure 12: The $i$-th splicing for $i \leq k'$ in the proof of Lemma 4.7 where $x_{i+1} = u_i x'_i$ and $v'_i$ is a prefix of $x'_i z$.](image)

The notation we employ in order to prove statement 2 is chosen such that it matches with Figure 12. For $i = 1, \ldots, k'$, let $r_i = (u_i, v_i; u'_i, v'_i)$. By extension (Lemma 4.2), we may assume that $w_i = u_i v_i$ and $x_i = u'_i x'_i$ such that $x_{i+1} = u_i x'_i$ and $v'_i$ is a prefix of $x'_i z$. Let $x'_{k'+1} = x_{k'+1} = x_{k+1}$ and $u'_{k'+1} = e$. By Lemma 4.6, we may assume that every splicing position lies at the left of the previous splicing position, i.e., $x'_i$ is a proper suffix of $x'_{i+1}$ and $|x'_i| \leq \ell$ as $|x'_{i+1}| \leq \ell$. Due to the modifications we made, we may have lost control of the lengths of $u_i$, $v_i$, and $u'_i$ but $v'_i$ still belongs to $W_\mu$ and $r_i$ respects $L$. Let $\delta_{i+1}$ such that $x'_i = \delta_{i+1} x'_i$; hence, $u_i = u'_{i+1} \delta_{i+1}$. The factor $\delta_{i+1}$ is the the part of $x_{i+1}$ which is added by the $i$-th splicing and is not modified afterwards; $x_{k'+1} = \delta_{k'+1} \cdots \delta_2 x'_1$. Now, for $i = 2, \ldots, k'$, we replace $u'_i$ by a shortest word from $[u'_i]_L$. (We also replace this prefix of $x_i$ and $u_{i-1}$.) Furthermore, we replace $v_i$ by a shortest word from $[v_i]_L$ for $i = 1, \ldots, k'$. By Lemma 2.2, we have $|u'_i|, |v_i| < m$. We do not replace $u'_1$ yet, as this might affect the word $w_0$ and the rule $s$ in the splicing $(w_0, w'_0) \vdash_s x_i z y_1$.

Observe that the words $z_i$, $w_i$, and the rules $r_i$ can still be used to create $z_{k+1}$ by splicing, in the way described in the claim. For $i = 2, \ldots, k'$, we have $|x_i| = |u'_i x'_i| < \ell + m$, $|w_i| \leq |x_{i+1} + v_i| < \ell + 2m$, and $r_i \in \Sigma^{<\ell + m} \times \Sigma^{<m} \times \Sigma^{<m} \times W_\mu$. We also have $|w_i| < \ell + 2m$ and $r_1 \in \Sigma^{<\ell + m} \times \Sigma^{<m} \times \Sigma^{<m} \times W_\mu$. Note that, except for the length of $x_1$, and the third component of $r_1$, we have proven statement 2 (of the lemma) and we actually have proven a stronger bound than claimed. Symmetrically, we can consider statement 3 to be proven except for $y_1 = y_{k'+1}$ and the second component of $r_{k'+1}$.

Let $x_1 = u'_1 x'_1$ as above and, symmetrically, let $y_1 = y_{k'+1} v'_{k'+1}$ where $v'_{k'+1}$ is the second component of $r_{k'+1}$. If $k' = 0$ (or $k' = k$), then $u'_1$ (resp. $v'_{k'+1}$) can be considered empty and $x'_1 = x_{k+1}$ (resp. $y'_{k'+1} = y_{k+1}$). If $z_1 \in I$, we replace $u'_1$ and $v'_{k'+1}$ by shortest words from their syntactic classes, respectively, and the claim holds by Lemma 2.2. Otherwise, $(w_0, w'_0) \vdash_s z_1$ where $u'_1$ is a prefix of $w_0$ and $v'_{k'+1}$ is a suffix of $w'_0$.

Let $s = (u_0, v_0; u'_0, v'_0)$ and consider the overlap of the factor $u_0$ in the splicing $(w_0, w'_0) \vdash_s z_1$ with the prefix $u'_1$ of $w_0$. In case when $u_0$ does not overlap with $u'_1$, replace $u'_1$ by a shortest word from its syntactic class. If $u_0$ and $u'_1$ overlap, let $u'_1 = \delta_1 \delta_2$ such that $\delta_2$ is the overlap and replace $\delta_1$ and $\delta_2$ by shortest words from their syntactic classes, respectively. Note that if we modified $u_1$, it got shorter; hence, $s$ still belongs to $\bar{R}$. In any case, $|u'_1| < 2m$, $|x_1| < \ell + 2m$ (Lemma 2.2), and $r_1 \in \Sigma^{<\ell + m} \times \Sigma^{<m} \times \Sigma^{<2m} \times W_\mu$: thus, the second statement.

We may treat $v'_{k'+1}$ and $r_{k'+1}$ symmetrically in order to prove statement 3. □

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4.3 Proof of Theorem 4.1

Let $L$ be a splicing language and $m = |M_L|$. Throughout this section, by $\sim$ we denote the equivalence relation $\sim_L$ and by $\cdot$ we denote the corresponding equivalence classes $[\cdot]_L$.

Recall that Theorem 4.1 claims that the splicing system $(I, R)$ with $I = \Sigma^{\leq n} \cap L$ and

$$R = \left\{ r \in \Sigma^{\leq n+10m} \times \Sigma^{\leq n+10m} \times \Sigma^{\leq n+10m} \mid r \text{ respects } L \right\}$$

generates $L$. The proof is divided in two parts. In the first part, Lemma 4.8, we prove that the set of rules can be chosen as $\left\{ r \in (\Sigma^{\leq n+10m})^4 \mid r \text{ respects } L \right\}$ for some finite set of axioms. The second part concludes the proof of Theorem 4.1, by employing the length bound $2m$ for the second and third component of rules and by proving that the set of axioms can be chosen as $I = \Sigma^{\leq n+6m} \cap L$.

**Lemma 4.8.** Let $L$ and $m$ as above. There exists $n \in \mathbb{N}$ such that the splicing system $(I, R)$ with $I = \Sigma^{\leq n} \cap L$ and

$$R = \left\{ r \in (\Sigma^{\leq n+10m})^4 \mid r \text{ respects } L \right\}$$

generates the same language $L = L(I, R)$.

**Proof.** As every word in $I$ belongs to $L$ and every rule in $R$ respects $L$, the inclusion $L(I, R) \subseteq L$ holds (for any $n$).

Since $L$ is a splicing language, there exists a splicing system $(I', R')$ which generates $L$. Let $n'$ be a number larger than any word in $I'$ and larger than any component of a rule in $R'$ and let $n = n' + 6m$. Let $I = \Sigma^{\leq n} \cap L$ as in the claim and observe that $L(I, R') = L$.

For a word $\mu$ we let $W^i_{\mu} = \{ w \in \Sigma^* \mid w \leq_{\mu} \mu \}$, as we did before. Define the set of rules where every component is length-lexicographically bounded by $\mu$

$$R_{\mu} = \{ r \in W^4_{\mu} \mid r \text{ respects } L \}$$

and the language $L_{\mu} = L(I, R_{\mu})$; clearly, $L_{\mu} \subseteq L$. For two words $\mu \leq_{\ell} v$ we see that $R_{\mu} \subseteq R_{v}$, and hence, $L_{\mu} \subseteq L_{v}$. Thus, if $L_{\mu} = L$ for some word $\mu$, then for all words $v$ with $\mu \leq_{\ell} v$, we have $L_{v} = L$. As $L = L(I, R')$, there exists a word $\mu$ such that $L_{\mu} = L$ and $|\mu| + 6m \leq n$. Let $\mu$ be the smallest word, in the length-lexicographic order, such that $L_{\mu} = L$. Note that if $|\mu| < 2m + 10$, then $R_{\mu} \subseteq R$ and $L = L_{\mu} \subseteq L(I, R)$. For the sake of contradiction assume $|\mu| \geq m^2 + 10m$. Let $\nu$ be the next-smaller word than $\mu$, in the length-lexicographic order, and let $S = R_{\nu}$. Note that $L(I, S) \subseteq L$ and $L_{\mu} \setminus S$ contains only rules which have a component that is equal to $\mu$.

Choose $w$ from $L \setminus L(I, S)$ as a shortest word, i.e., for all $w' \in L$ with $|w'| < |w|$, we have $w' \in L(I, S)$. Factorize $w = xyz$ with $|x| = |y| = 3m$ and note that $|z| \geq |\mu|$, otherwise $w \in I$. Factorize $\mu = \delta_1 \alpha \beta \gamma \delta_2$ with $|\delta_1|, |\delta_2| \geq 5m, |\alpha \beta \gamma| = m^2$, $\beta \neq z$, $\alpha \sim \alpha \beta$, and $\gamma \sim \beta \gamma$ (Lemma 2.1).

We will show that there is a series of splicings which creates $w$ from a set of shorter words and by using splicing rules from $S$. This yields a contradiction to the choice of $w$. In order to find this series of splicings we investigate the creation of a word $x\bar{z}y$ where $\bar{z}$ is derived by using a pumping argument on all factors $\alpha \beta \gamma$ in $z$.

Let $j$ be a sufficiently large even number ($j > 4 |\mu| + |z|$ will suffice). We define a word $\bar{z}$ which is the result of applying the pumping algorithm from Lemma 2.3 on $z$, as discussed in Section 2.1. The pumping algorithm replaces the occurrences of $\alpha \beta \gamma$ in $z$ by $\alpha \beta^j \gamma$ such that for every factor $\bar{z}[k, k + m^2] = \alpha \beta \gamma$, either

(a) $\alpha \beta^j \gamma$ is a factor of $\bar{z}$ starting at position $\bar{z}[k]$ or
(b) $\beta^j \gamma$ is a factor of $\bar{z}$ ending at position $\bar{z}[k + m^2]$

holds. In particular, if $\delta_1 \alpha \beta \delta_2$ is a factor of $\bar{z}$ either (a) $\gamma \delta_2$ is a prefix of a word in $\beta^+$ or (b) $\delta_1 \alpha$ is a suffix of a word in $\beta^+$. By induction and as $\alpha \beta \gamma \sim \alpha \beta \gamma$, it is easy to see that $z \sim \bar{z}$ and $x\bar{z}y \in L$. 

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Let us trace back the creation of $xzy \in L$ by splicing in $(I, R_{\mu})$ to a word $x_1 zy_1$ where either $x_1 zy_1 \in I$ or where $x_1 zy_1$ is created by a splicing that affects $z$, i.e., the splicing position lies within the factor $z$. Let $z_{i+1} = x_{i+1} z y_{i+1}$, where $x_{i+1} = x$ and $y_{i+1} = y$, be created by $k$ splicings from a word $z_1 = x_1 zy_1$ where either $x_1 zy_1 \in I$ or $x_1 zy_1$ is created by a splicing $(w_0, w'_0) \mapsto s z_1$ with $w_0, w'_0 \in L$, $s \in R_{\mu}$, and the splicing position lies in the factor $z$. Furthermore, for $i = 1, \ldots, k$ the intermediate splicings are either

(i) $(w_i, z_i) \mapsto r_i, x_{i+1} z y_{i+1} = z_{i+1}, w_i \in L, r_i \in R_{\mu}, y_{i+1} = y$, and the splicing position lies at the left of the factor $z$ or

(ii) $(z_i, w_i) \mapsto r_i, x_{i+1} z y_{i+1} = z_{i+1}, w_i \in L, r_i \in R_{\mu}, x_{i+1} = x$, and the splicing position lies at the right of the factor $z$.

Note that $|z| \geq |z| \geq |\mu|$ and, therefore, we can apply Lemma 4.7 (with $\ell = 3m$). Thus, we may assume that $w_i \in I$ and $|x_i|, |y_i| < 5m$ for $i = 1, \ldots, k$.

Consider a rule $r_i$ in a splicing of the form (i). By Lemma 4.7, $r_i \in \Sigma^{4m} \times \Sigma^{2m} \times \Sigma^{2m} \times W_{\mu}$. Suppose the fourth component of $r_i$ covers a prefix of the left-most factor $\alpha \beta^{1/2}$ in $z$ which is longer than $\alpha$ (as $j$ is very large, it cannot fully cover $\alpha \beta^{1/2}$). By extension (Lemma 4.2), we may write $r_i = (u_i, v_i; u_2, v' \alpha)$. By Lemma 4.3 and as $\alpha \sim \alpha \beta$, we may replace this rule by $(u_1, v_1; u_2, v' \alpha)$. Let us summarize: if we replaced $\beta$ by $\beta'$ in $z$, which we pumped up before, may overlap with each other, the left-most (and right-most) position where we replaced $\beta$ by $\beta'$ is preceded by the factor $\alpha$ (resp. succeeded by the factor $\gamma$) in $z$.

Next, we show that all the rules $r_1, \ldots, r_k$ belong to $S$, now. By contradiction, suppose $r_i \notin S$ for some $i$ and, by symmetry, suppose this $i$-th splicing is of the form (i). Thus, the fourth component of $r_i$ has to be $\mu = \delta_1 \alpha \beta \gamma \delta_2$. As $|\delta_1| \geq 5m > |x_i|$, the factor $\alpha \beta \gamma \mu$ in $\beta$ covered by $z$. Let $k$ such that $\alpha \beta \gamma = z[k; k + m^2]$ is this factor in $z$. The pumping algorithm ensured that (a) $\alpha \beta^{1/2}$ is a factor of $z$ starting at position $\bar{k}$ or (b) $\beta^{1/2} \gamma$ is a factor of $z\tilde{z}$ ending at position $\bar{k} + m^2$. As $j/2$ is very large and the splicing position of $(w_i, z_i) \mapsto r_i, z_{i+1}$ is too close to the left end of $z_{i+1}$, case (b) is not possible. Thus, case (a) holds, the fourth component of $r_i$ overlaps in more than $|\alpha| \mu$ letters with the left-most factor $\beta^{1/2}$ in $z$, and we used the replacement above which ensured $r_i \in S$ — contradiction.

Let us summarize: if $x_1 zy_1$ was in $L(I, S)$, then $w \in L(I, S)$ as well, which would contradict the choice of $w$. If $z_1 = x_1 zy_1 \in I$, then $x_1 zy_1$, which is as long as most as $z_1$, would belong to $I$ and we are done. We only have to consider the case when $(w_0, w'_0) \mapsto x_1 zy_1$ and the splicing position lies within the factor $z$. We will show that, from this splicing, we derive another splicing $(\tilde{w}_0, \tilde{w}'_0) \mapsto x uv$ which respects $L(I, S)$ and, therefore, yields the contradiction.

Let $s = (u, v_1; u_2, v), w_0 = xuvu_1$ and $w'_0 = u_2vy$ where $|v_1|, |u_2| < m$, by Lemma 4.4 (here, $x$ and $y$ are newly chosen words). We have

$z_1 = x_1 zy_1 = xuvy$

where $xu$ is a proper prefix of $x_1 z$ and $vy$ is a proper suffix of $zy_1$.

We will see next that if $s \notin S$, then we can use a rule $\tilde{s} \in S$ and maybe slightly modified words in order to obtain $z_1$ by splicing. If $s \notin S$, then $u = \mu = v = \mu$. Suppose $u = \mu = \delta_1 \alpha \beta \gamma \delta_2$. Thus, the factor $\alpha \beta \gamma \mu$ of $\mu$ is covered by the factor $z$ in $z_1$ as $|\delta_1| \geq 5m > |x_1|$. Let $\alpha \beta \gamma = z[k; k + m^2]$ be this factor. (a) $\alpha \beta^{1/2}$ is a factor of $z$ starting at position $\bar{k}$ or (b) $\beta^{1/2} \gamma$ is a factor of $z\tilde{z}$ ending at position $\bar{k} + m^2$. As $j/2$ is very large and the splicing position of $(w_0, z_0) \mapsto r_0, z_1$ is too close to the left end of $z_1$, case (b) is not possible. Thus, case (a) holds, the first component is now shorter than $\mu$. Otherwise, (a) holds and $\gamma \delta_2 v$ is a prefix of a word in $\beta^+$. As $j$ is very large and $\gamma$ is a prefix of a word in $\beta^+$, we may extend $v$ (Lemma 4.2) such that we can write $\beta^+ \delta_2 = \beta^+ \delta_1 \delta_2$ and $v = \beta_2 \beta^+ \gamma$ where $\ell_1 \geq 1, \ell_2 \geq 0$ and $\beta_1 \beta_2 = \beta$. Now, we pump down one of the $\beta$ in the first component and $\beta^+ \delta_2$ in the fourth
component and we let \( \tilde{s} = (\delta_1, \alpha_2, \ldots, \beta_1, 1, 2; \beta_1, \beta_2) \sim s. \) As all components are shorter than \( \mu, \) we see that \( \tilde{s} \in S \)

\[
(\varepsilon t_1, \alpha_2, \tilde{\beta}_1, 1, 2; \beta_2) \sim z_1,
\]

i.e., we have shifted one of the occurrences of \( \beta \) from \( w_0 \) to \( w_0'. \) Note that \( \beta_2 \gamma \) is a prefix of \( \beta_2 \beta_2 \gamma \). Treating the fourth component analogously justifies the assumption that \( s \in S. \)

Next, we will pump down the factors \( \alpha \beta \gamma \) to \( \alpha \beta \gamma \) in \( \tilde{z} \) again. At every position where we pumped up before, we are now pumping down (in reverse order) in order to obtain the words \( \tilde{x}, \tilde{u}, \tilde{v}, \tilde{y} \) from the words \( x, u, v, y, \) respectively. For each pumping step do:

1. If \( \tilde{u} \) is covered by the factor \( \alpha \beta \gamma \) (which we pump down in this step), extend \( u \) to the left such that it becomes a prefix of \( \alpha \beta \gamma \). Symmetrically, if \( \tilde{v} \) is covered by the factor \( \alpha \beta \gamma \), extend \( v \) to the right such that it becomes a suffix of \( \alpha \beta \gamma \) (Lemma 4.2). Observe that extension ensures that the factor \( \alpha \beta \gamma \) is covered by either \( xu, uv, \) or \( v \).

2. If \( \alpha \) or \( \beta \gamma \) is covered by one of \( x, u, v, \) or \( y, \) then replace this factor by \( \alpha \beta \) or \( \beta \gamma, \) respectively. Otherwise, let us show how to pump when \( \alpha \beta \gamma \) is covered by \( uv. \) The cases when \( \alpha \beta \gamma \) is covered by \( uv \) or \( v \) can be treated analogously. We can factorize \( \tilde{x} = x \alpha \beta \gamma \) and \( \tilde{u} = \beta_2 \beta_2 \gamma \) where \( \tilde{u}_2 = \beta \) and \( j_1 + j_2 + 1 = j. \) The pumping results are the words \( \tilde{x}' \alpha \beta \gamma \) and \( \beta_2 \gamma \)

Observe that, after reversing all pumping steps, \( \tilde{x} \sim xu, \tilde{u} \sim uy, \tilde{v} \sim v, \tilde{w} = x_i y z_1, \) and the rule \( t = (\tilde{u}, \tilde{v}_1, \tilde{u}_2, \tilde{v}) \) respects \( L. \) Furthermore, if we used extension for \( u \) (or \( v \)) in one of the steps, then \( |\tilde{u}| \leq m^2 \) (resp. \( |\tilde{v}| \leq m^2 \)); in any case \( t \in S. \) Recall that \( w \) was chosen as the shortest word from \( L(I, S). \) As \( |\tilde{x} w u|, |\tilde{w} u y| < |z| + 6m = |w|, \) the words \( \tilde{u}_w = \tilde{x} w u \) and \( \tilde{u}_y = \tilde{x} w u \) belong to \( L(I, S), \) and as \( (\tilde{u}_w, \tilde{u}_y) \vdash _1 t x_i z_1 y_1, \) we conclude that \( x_i y z_1 y_1 \) as well as \( w \) belong to \( L(I, S) \) — the desired contradiction.

Now, we can prove our main result.

**Proof of Theorem 4.1.** Recall that for a splicing language \( L \) with \( m = |M_L| \) we intend to prove that the splicing system \( (I, R) \) with \( I = \Sigma^{m^2 + 6m} \cap L \)

\[
R = \left\{ r \in \Sigma^{m^2 + 10m} \times \Sigma^{2m} \times \Sigma^{2m} \times \Sigma^{m^2 + 10m} \mid r \text{ respects } L \right\}
\]

generates the language \( L = L(I, R). \)

Obviously, \( L(I, R) \subseteq L. \) By Lemma 4.8, we may assume that \( L \) is generated by a splicing system \( (J, S) \) where

\[
S = \left\{ r \in (\Sigma^{m^2 + 10m})^\dagger \mid r \text{ respects } L \right\}.
\]

In order to prove \( L \subseteq L(I, R), \) we use induction on the length of words in \( L. \) For \( w \in L \) with \( |w| < m^2 + 6m, \) by definition, \( w \in I \subseteq L(I, R). \)

Now, consider \( w \in L \) with \( |w| \geq m^2 + 6m. \) The induction hypothesis states that every word \( w' \in L \) with \( |w'| < |w| \) belongs to \( L(I, R). \) Factorize \( w = x \alpha \beta \gamma \) such that \( |x| = |y| = 3m, \)

\[
|\alpha \beta \gamma| = m^2, \beta \neq \varepsilon, \alpha = 0 \alpha \beta \gamma, \text{ and } \gamma = 0 \beta \gamma \text{ (Lemma 2.1).}
\]

The proof idea is similar as in the proof of Lemma 4.8. We use a pumping argument on \( \beta \) in order to obtain a very long word. This word has to be created by a series of splicings in \( (J, S). \) We show that these splicings can be modified in order to create \( w \) by splicing from a set of strictly shorter words and with rules from \( R. \) Then, the induction hypothesis yields \( w \in L(I, R). \)

Choose \( j \) sufficiently large \( (j > |w| + m^2 + 10m \) and \( J \) does not contain words of length \( j \) or more. \) We let \( z = \alpha \beta \gamma \delta \) and investigate the creation of \( xzy \in L \) by splicing in \( (J, S). \) As \( z \) is not a factor of a word in \( J, \) we can trace back the creation of \( xzy \) by splicing to the point where the factor \( z \) is affected for the last time. Let \( x_{k+1} = x_{k+1} y_{k+1}, \) where \( x_{k+1} = x \) and \( y_{k+1} = y, \) be created by \( k \) splicings from a word \( z_1 = x_{z1} y_{z1} \) which is created by a splicing \( (w_0, w_0') \vdash _1 z_1 \) with \( w_0, w_0' \in L, s \in S, \) and the splicing position lies in the factor \( z. \) Furthermore, for \( i = 1, \ldots, k \) the intermediate splicings are either

(i) \( (w_i, z_i) \vdash _r x_{i+1} y_{i+1} = z_{i+1}, w_i \in L, r, \in S, y_{i+1} = y_i, \) and the splicing position lies at the left of the factor \( z \) or
(ii) \((z_i, w_i) \vdash_{r_i} x_{i+1}z_{y_{i+1}} = z_{i+1}, w_i \in L, r_i \in S, x_{i+1} = x_i\), and the splicing position lies at the right of the factor \(z\).

As \(|z| \geq m^2 + 10m\) we can apply Lemma 4.7. Thus, we may assume \(w_1, \ldots, w_k \in I, r_1, \ldots, r_k \in R\), and \(|x_1|, |y_1| < 5m\).

Consider a rule \(r_i\) in a splicing of the form (i). Suppose the fourth component of \(r_i\) covers a prefix of the factor \(\alpha \beta \) in \(z\) which is longer than \(\alpha \beta\) (as \(j\) is very large, it cannot fully cover \(\alpha \beta\)). By extension (Lemma 4.2), we may write \(r_i = (u_1, v_1; u_2, v' \alpha \beta\)) for some \(\ell \geq 1\). By Lemma 4.3 and as \(\alpha \sim \alpha \beta\), we may replace this rule by \((u_1, v_1; u_2, v' \alpha \beta) \in R\). Moreover, after we symmetrically treated rules of form (ii), these new rules \(r_1, \ldots, r_k\) and the words \(w_1, \ldots, w_k\) can be used in order to create \(w = x_{k+1} \alpha \beta \gamma \delta y_{k+1}\) from \(x_1 \alpha \beta \gamma \delta y_1\) by splicing. Thus, if \(x_1 \alpha \beta \gamma \delta y_1\) belongs to \(L(I, R)\), so does \(w\).

Now, consider the first splicing \((w_0, w_0') \vdash_s z_1 = x_1 z y_1\). By Lemma 4.4, let \(s = (u, v_1; u_2, v)\) such that \(w_0 = xuv_1, w_0' = u_2vy\) and \(|v_1|, |u_2| < m\) (here, \(x\) and \(y\) are newly chosen words). Hence,

\[ z_1 = xuvy = x_1 z y_1 = x_1 \alpha \beta \gamma \delta y_1 \]

where \(xu\) is a proper prefix of \(x_1z\) and \(vy\) is a proper suffix of \(zy_1\).

Next, we will pump down the factor \(\alpha \beta \gamma \) to \(\alpha \beta \gamma\) in \(z\) again in order to obtain the words \(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}\) from the word \(x, u, v, y\), respectively. The pumping is done as in the proof of Lemma 4.8:

1. If \(u\) is covered by the factor \(\alpha \beta \gamma\), extend \(u\) to the left such that it becomes a prefix of \(\alpha \beta \gamma\).

Symmetrically, if \(v\) is covered by the factor \(\alpha \beta \gamma\), extend \(v\) to the right such that it becomes a suffix of \(\alpha \beta \gamma\) (Lemma 4.2). Observe that extension ensures that the factor \(\alpha \beta \gamma\) is covered by either \(xu\), \(wv\), or \(vy\).

2. If \(\alpha \beta \) or \(\beta \gamma\) is covered by one of \(x, u, v, or y\), then replace this factor by \(\alpha \beta\) or \(\beta \gamma\), respectively. Otherwise, let us show how to pump when \(\alpha \beta \gamma\) is covered by \(xu\). The cases when \(\alpha \beta \gamma\) is covered by \(uv\) or \(vy\) can be treated analogously. We can factorize \(x = x' \alpha \beta \beta_1\) and \(u = \beta_2 \beta_3 \gamma u'\) where \(\beta_1 \beta_2 = \beta\) and \(j_1 + j_2 + 1 = j\). The pumping result are the words \(x' \alpha \beta_1\) and \(\beta_2 \gamma u', \) respectively.

Observe that, \(\tilde{x} \tilde{u} \sim xu, \tilde{v} \tilde{y} \sim vy, \tilde{x} \tilde{u} \tilde{v} \tilde{y} = x_1 \alpha \beta \gamma \delta y_1\), and the rule \(t = (\tilde{u}, v_1; u_2, \tilde{v})\) respects \(L\). Furthermore, if we used extension for \(u\) (or \(v\)), then \(|\tilde{u}| \leq m^2\) (resp. \(|\tilde{v}| \leq m^2\)). No matter whether we used extension, \(t \in R\). As \(|\tilde{x} \tilde{u} v_1|, |u_2 \tilde{v} \tilde{y}| < |z| + 6m = |w|\) and by induction hypothesis, the words \(\tilde{x} \tilde{u} v_1\) and \(\tilde{x} \tilde{u} \tilde{v} \tilde{y}\) belong to \(L(I, S)\). We conclude that \((\tilde{u}_0, \tilde{v}_0) \vdash_t x_{k+1} \alpha \beta \gamma \delta y_{k+1} \in L(I, R)\) and, therefore, \(w = x_{k+1} \alpha \beta \gamma \delta y_{k+1} \in L(I, R)\) as well.

5 Decidability

The main question we intended to answer when starting our investigation was, whether or not it is decidable if a given regular language \(L\) is a splicing language. If we can decide whether a splicing rule respects a regular language and if we can construct a (non-deterministic) finite automaton accepting the language generated by a given splicing system, then we can decide whether \(L\) is a classic splicing language (Pixon splicing language) as follows. We compute the splicing system \((I, R)\) as given in Theorem 4.1 (resp. Theorem 3.1) and we compute a finite automaton accepting the splicing language \(L(I, R)\). Theorem 4.1 (resp. Theorem 3.1) implies that \(L\) is a splicing language if and only if \(L = L(I, R)\). Recall that equivalence of regular languages is decidable, e.g., by constructing and comparing the minimal deterministic finite automata of both languages.

It is known from [8, 13] that it is decidable whether a classic splicing rule respects a regular language. Furthermore, there is an effective construction of a finite automaton which accepts the language generated by a Pixon splicing system [17]. As mentioned earlier, Pixon splicing systems are more general than classic splicing systems, which means the latter result applies to classic splicing systems, too. Such a construction for classic splicing systems is also given in [12].

Let us prove that it is decidable whether a Pixon splicing rule \(r\) respects a regular language \(L\). Actually, we will decide whether the set \([r]_L\) respects \(L\), which is equivalent by Lemma 4.3.
The proof can easily be adapted in order to prove that it is decidable whether a classic splicing rule respects $L$.

**Lemma 5.1.** Let $L$ be a regular language and let $r$ be a Pixton splicing rule. It is decidable whether $r$ respects $L$.

**Proof.** Let $\sim$ denote the equivalence relation $\sim_L$ and $[\cdot]$ denote the corresponding equivalence classes $[\cdot]_L$.

Let $r = (u_1, u_2; v)$. We define the two sets $S_1, S_2 \subseteq M_L$ as

$$S_1 = \{ X \in M_L \mid \exists Y: X[u_1]Y \subseteq L \}, \quad S_2 = \{ Y \in M_L \mid \exists X: X[u_2]Y \subseteq L \},$$

i.e., $[x_1]$ belongs to $S_1$ if and only if $x_1u_1y_1 \in L$ for some word $y_1$ and $[y_2]$ belongs to $S_2$ if and only if $x_2u_2y_2 \in L$ for some word $x_2$. We claim that $r$ respects $L$ if and only if $X[v]Y \subseteq L$ for all $X \in S_1$ and $Y \in S_2$, which is a property that can easily be decided.

Firstly, suppose $r$ respects $L$. For $X \in S_1$ and $Y \in S_2$ choose words $x_1 \in X$ and $y_2 \in Y$. By definition of $S_1$ and $S_2$, there is $y_1$ and $x_2$ such that $x_1u_iy_i \in L$ for $i = 1, 2$ and, as $r$ respects $L$, $x_1vy_2 \in L$. This implies $X[v]Y \subseteq L$.

Vice versa, suppose $X[v]Y \subseteq L$ for all $X \in S_1$ and $Y \in S_2$. For all $x_1u_iy_i \in L$ with $i = 1, 2$, we have $[x_1] \in S_1$ and $[y_2] \in S_2$. Therefore, $x_1vy_2 \in [x_1][v][y_2] \subseteq L$ and $r$ respects $L$.

These observations lead to the decidability results.

**Corollary 5.2.**

(i.) For a given regular language $L$, it is decidable whether or not $L$ is a classic splicing language. Moreover, if $L$ is a classic splicing language, a splicing system $(I, R)$ generating $L$ can be effectively constructed.

(ii.) For a given regular language $L$, it is decidable whether or not $L$ is a Pixton splicing language. Moreover, if $L$ is a Pixton splicing language, a splicing system $(I, R)$ generating $L$ can be effectively constructed.

**Final Remarks**

It has been known since 1991 that the class $S$ of languages that can be generated by a splicing system is a proper subclass of the class of regular languages. However, to date, no other natural characterization for the class $S$ exists. The problem of deciding whether a regular language is generated by a splicing system is a fundamental problem in this context and has remained unsolved. To the best of our knowledge, the problem was first stated in the literature in 1998 [11].

In this paper we solved this long standing open problem.

Regarding the complexity of the decision algorithm, let $L$ be a regular language given as syntactic monoid $M_L$ and $(I, R)$ be the splicing system described in Theorem 4.1 (resp. Theorem 3.1). An automaton which accepts $L(I, R)$ and is created as described in Section 5 has a state set of size in $2^{O(m^2)}$, where $m = |M_L|$. Deciding the equivalence of two regular languages, given as NFAs, is known to be PSPACE-complete [20]; hence, the naïve approach to decide whether or not $L = L(I, R)$ uses double exponential time $2^{2^{O(m^2)}}$. As there may be an exponential gap between an NFA accepting $L$ and the syntactic monoid $M_L$, the complexity, when considering an NFA as input, becomes triple exponential. Improving the complexity of the algorithm is subject of future research.

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