Neutrosophic ∞–Ideals (∞-Subalgebras) of Subtraction Algebra

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Abstract

The connection between neutrosophy and algebra has been of great interest with respect to many researchers. The objective of this paper is to provide a connection between neutrosophic ∞–structures and subtraction algebras. In this regard, we introduce the concept of neutrosophic ∞–ideals in subtraction algebra. Moreover, we study its properties and find a necessary and sufficient condition for a neutrosophic ∞–structure to be a neutrosophic ∞–ideal.

Keywords: Subtraction algebra, ∞–structure, Neutrosophic ∞–ideal, Level set.

1. Introduction

Neutrosophic sets were introduced by Florentin Smarandache [11] as a new mathematical tool for dealing with uncertainty. They can be viewed as a generalization of the fuzzy sets that were introduced in 1965 by Lotfi Zadeh [14]. Where Zadeh defined fuzzy sets as mathematical model of vagueness in which an element belongs to a given set to some degree that is a number between 0 and 1 (both inclusive). Neutrosophy is a base of neutrosophic logic which is an extension of fuzzy logic where indeterminacy is included [13]. In neutrosophic logic [10], each proposition is estimated to have the degree of truth in a subset 𝑇, the degree of indeterminacy in a subset 𝑖, and the degree of falsity in a subset 𝐹. The study of neutrosophic sets and their properties have a great importance in the sense of applications as well as in understanding the fundamentals of uncertainty. Some related work can be found in [1, 2, 3, 12].

A crisp set 𝐴 in a universe 𝑋 can be defined in the form of its membership function 𝜇𝐴: 𝑋 → (0,1) where 𝜇𝐴(𝑥) = 1 if 𝑥 ∈ 𝐴 and 𝜇𝐴(𝑥) = 0 if 𝑥 ∉ 𝐴. A single valued neutrosophic set is an example of neutrosophic set which has many applications [10]. A new function, which is called negative-valued function, was introduced by Jun et al. [5] and they used it to construct ∞-structures. Some work related to neutrosophic ∞-structures can be found in [6, 7]. Schein [9] considered systems of the form (Φ, ∩, ∪), where Φ is a set of functions closed under the composition “∗” and hence (Φ, ∩) is a subtraction algebra. Jun et al. [4] introduced the concept of ideals in subtraction algebras and discussed the properties of these ideals. Some researchers worked on combining the notions of neutrosophic sets and subtraction algebra. For example, Ibrahim et al. introduced neutrosophic subtraction algebra (semigroups) and presented some results about them. Moreover, Park [8] discussed neutrosophic ideals of subtraction algebras by using single valued neutrosophic sets.

In this paper, we apply the concept of neutrosophic ∞-structures in subtraction algebras. And it is organized as follows: after an Introduction, in Section 2 and Section 3, we present some basic results about neutrosophic ∞-structures as well as about subtraction algebras that are used throughout the paper. In Section 4, we introduce neutrosophic ∞-ideals (∞-subalgebras) of subtraction algebra and prove that the intersection, the product, the homomorphic preimage, and onto homomorphc image are neutrosophic ∞-ideals. Finally, in Section 5, we prove a necessary and sufficient condition for ∞-structures to be neutrosophic ∞-ideals by introducing the (∈, ∩, ∪)–level sets.

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2. Neutrosophic $\mathcal{N}$-structures

In this section, we present some basic results about neutrosophic $\mathcal{N}$-structures. For more details about neutrosophy, we refer to [5, 6, 7].

**Definition 2.1.** [5] Let $S$ be a non-empty set. A function from $S \rightarrow [-1,0]$ is called a negative-valued function ($\mathcal{N}$-function) from $S$ to $[-1,0]$.

**Definition 2.2.** [7] Let $S$ be a non-empty set. A neutrosophic $\mathcal{N}$-structure over $S$ is defined as follows:

$$S_N = \left\{ \frac{x}{(T_N,I_N,F_N)} : x \in S \right\},$$

where $T_N,I_N,F_N$ are $\mathcal{N}$-functions on $S$ which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on $S$.

It is clear that for any $\mathcal{N}$-structure $S_N$ over $S$, $-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0$ for all $x \in S$.

**Definition 2.3.** [7] Let $S_N = \left\{ \frac{x}{(T_N,I_N,F_N)} : x \in S \right\}$ and $S_M = \left\{ \frac{x}{(T_M,I_M,F_M)} : x \in S \right\}$ be $\mathcal{N}$-structures over $S$.

1. $S_N$ is called a neutrosophic $\mathcal{N}$-substructure of $S_M$, denoted as $S_N \subseteq S_M$, if for all $x \in S$, $T_N(x) \geq T_M(x), I_N(x) \leq I_M(x), F_N(x) \geq F_M(x)$.

   If $S_N \subseteq S_M$ and $S_M \subseteq S_N$, we say that $S_N = S_M$.

2. The union of $S_N$ and $S_M$ is defined to be the $\mathcal{N}$-structure over $S$:

$$S_{N\cup M} = \left\{ \frac{x}{(T_N,T_M,I_N,I_M,F_N,F_M)} : x \in S \right\},$$

where $T_{N\cup M}(x) = T_N(x) \vee T_M(x), I_{N\cup M}(x) = I_N(x) \vee I_M(x), F_{N\cup M}(x) = F_N(x) \vee F_M(x)$ for all $x \in S$.

3. The intersection of $S_N$ and $S_M$ is defined to be the $\mathcal{N}$-structure over $S$:

$$S_{N\cap M} = \left\{ \frac{x}{(T_N,T_M,I_N,I_M,F_N,F_M)} : x \in S \right\},$$

where $T_{N\cap M}(x) = T_N(x) \wedge T_M(x), I_{N\cap M}(x) = I_N(x) \wedge I_M(x), F_{N\cap M}(x) = F_N(x) \wedge F_M(x)$ for all $x \in S$.

4. The complement of $S_N$ is defined to be the $\mathcal{N}$-structure over $S$:

$$S_{N^c} = \left\{ \frac{x}{(T_N^c,I_N^c,F_N^c)} : x \in S \right\},$$

where $T_{N^c} = 1 - T_N(x), I_{N^c} = 1 - I_N(x), F_{N^c} = 1 - F_N(x)$ for all $x \in S$.

**Definition 2.4.** [7] Let $X, Y$ be non-empty sets, $f:X \rightarrow Y$ be any function, and $X_N = \left\{ \frac{x}{(T_{f^{-1}(x)},I_{f^{-1}(x)},F_{f^{-1}(x)})} : x \in X \right\}$.

$Y_M = \left\{ \frac{y}{(T_{f^{-1}(x)},I_{f^{-1}(x)},F_{f^{-1}(x)})} : y \in Y \right\}$ be $\mathcal{N}$-structures over $X, Y$ respectively. Then

1. the $\mathcal{N}$-structure $X_{f^{-1}(M)} = \left\{ \frac{x}{(T_{f^{-1}(x)},I_{f^{-1}(x)},F_{f^{-1}(x)})} : x \in X \right\}$ over $X$ is defined as follows:

$$T_{f^{-1}(M)}(x) = T_M(f(x)), I_{f^{-1}(M)}(x) = I_M(f(x)), F_{f^{-1}(M)}(x) = F_M(f(x));$$

2. the $\mathcal{N}$-structure $Y_{(N)} = \left\{ \frac{y}{(T_{f^{-1}(x)},I_{f^{-1}(x)},F_{f^{-1}(x)})} : y \in Y \right\}$ over $Y$ is defined as follows:

$$T_{(N)}(y) = \begin{cases} T_M(x) & \text{if } f^{-1}(y) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}, \quad I_{(N)}(y) = \begin{cases} I_M(x) & \text{if } f^{-1}(y) \neq \emptyset; \\ -1 & \text{otherwise.} \end{cases},$$

and

$$F_{(N)}(y) = \begin{cases} F_M(x) & \text{if } f^{-1}(y) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}. $$

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Remark 2.1. Let $X, Y$ be non-empty sets, $f: X \to Y$ be any onto function, and $X_N = \{x_{\frac{-y}{(T_{\lambda}I_{\lambda})F_{\lambda}}}}: x \in X\}.$

$Y_M = \{y_{\frac{-y}{(T_{\lambda}I_{\lambda})F_{\lambda}}} = \{y \in Y\}$ be $N$-structures over $X, Y$ respectively. Then for all $y \in Y$

\[
T_{(N)}(y) = \bigwedge_{f(x) = y} T_M(x), \quad I_{(N)}(y) = \bigvee_{f(x) = y} I_M(x), \quad F_{(N)}(y) = \bigwedge_{f(x) = y} F_M(x).
\]

3. Subtraction algebra

In this section, we present some results related to subtraction algebra that are used throughout the paper. For more details, we refer to [4, 9, 15].

Definition 3.1. [15] An algebra $(X, -)$ is called a subtraction algebra if the single binary operation “−” satisfies the following identities: for any $x, y, z \in X$,

1. $x - (y - x) = x$;
2. $x - (x - y) = y - (y - x)$;
3. $(x - y) - z = (x - z) - y$.

We introduce an order relation “$\leq$” on subtraction algebras: $a \leq b$ if and only if $a - b = 0$; where $0 = a - a$ is an element that does not depend on the choice of $a \in X$.

It is clear that $a - 0 = a$ and $0 - a = 0$ for all $a \in X$.

Example 3.1. Let $A_1 = \{0, 1\}$. Then $(A_1, -_1)$ is a subtraction algebra defined in Table 1.

| $-1$ | 0   | 1   |
|------|-----|-----|
| 0    | 0   | 0   |
| 1    | 1   | 0   |

Example 3.2. Let $A_2 = \{0, a, b, c\}$. Then $(A_2, -_2)$ is a subtraction algebra defined in Table 2.

| $-2$ | 0   | a   | b   | c   |
|------|-----|-----|-----|-----|
| 0    | 0   | 0   | 0   | 0   |
| a    | a   | 0   | a   | 0   |
| b    | b   | b   | 0   | 0   |
| c    | c   | b   | a   | 0   |

Definition 3.3. [4] A non-empty subset $A$ of a subtraction algebra $X$ is called a subalgebra of $X$ if for all $a, b \in A$, $a - b \in A$. 

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**Definition 3.4.** [4] A non-empty subset $A$ of a subtraction algebra $X$ is called an ideal of $X$ if it satisfies the following conditions.
(1) $a - x \in A$ for all $a \in A$ and $x \in X$;
(2) for all $a, b \in A$, whenever $a \lor b$ exists in $X$ then $a \lor b \in A$.

**Remark 3.1.** Every ideal of a subtraction algebra is a subalgebra. But the converse may not hold.

We illustrate Remark 3.1 by Example 3.3.

**Example 3.3.** Let $(A_2, -_2)$ be the subtraction algebra in Example 3.2. Then $\{0, c\}$ is a subalgebra of $A_2$ that is not an ideal of $A_2$. This is clear as $c - a = b \notin \{0, c\}$.

**Example 3.4.** Let $A_3 = \{0, d, e\}$. Then $(A_3, -_3)$ is a subtraction algebra defined in Table 3.

| $-_3$ | 0   | d   | e   |
|------|-----|-----|-----|
| 0    | 0   | 0   | 0   |
| d    | d   | 0   | d   |
| e    | e   | e   | 0   |

**Example 3.5.** Let $(A_3, -_3)$ be the subtraction algebra in Example 3.3. Then $\{0\}, \{0, d\}, \{0, e\}, A_3$ are the only subalgebras of $A_3$. Moreover, every subalgebra of $A_3$ is an ideal of $A_3$.

**Definition 3.5.** Let $(X, -_1), (Y, -_2)$ be subtraction algebras and $f: X \rightarrow Y$ be a function. Then

1. $f$ is a homomorphism if $f(x -_1 y) = f(x) -_2 f(y)$ and $f(x \lor y) = f(x) \lor f(y)$.
2. $f$ is an isomorphism if $f$ is a bijective homomorphism. In this case, we say that $X$ and $Y$ are isomorphic subtraction algebras and we write $X \cong Y$.

**Example 3.6.** Let $(X, -)$ be a subtraction algebra and Let $S$ be any subalgebra of Let $X$. Then $f: S \rightarrow X$ defined as $f(x) = x$ is a homomorphism.

**Example 3.7.** Let $(X, -_1), (Y, -_2)$ be subtraction algebras and $f: X \rightarrow Y$ be defined as $f(x) = 0$. Then $f$ is a homomorphism.

**4. Operations on neutrosopic $\mathbb{N}$-ideals ($\mathbb{N}$-subalgebra) of subtraction algebra**

In this section, we introduce neutrosophic $\mathbb{N}$-ideals ($\mathbb{N}$-subalgebras) of subtraction algebra, present some examples, and study different operations on them.

**Definition 4.1.** Let $(X, -)$ be a subtraction algebra. An $\mathbb{N}$-structure $X_\mathbb{N}$ over $X$ is called a neutrosophic $\mathbb{N}$-subalgebra of $X$ if the following conditions hold for all $x, y \in X$.

\[
T_\mathbb{N}(x - y) \leq T_\mathbb{N}(x) \lor T_\mathbb{N}(y), I_\mathbb{N}(x - y) \geq I_\mathbb{N}(x) \land I_\mathbb{N}(y), \text{ and } F_\mathbb{N}(x - y) \leq F_\mathbb{N}(x) \lor F_\mathbb{N}(y) .
\]

**Definition 4.2.** Let $(X, -)$ be a subtraction algebra. An $\mathbb{N}$-structure $X_\mathbb{N}$ over $X$ is called a neutrosophic $\mathbb{N}$-ideal of $X$ if the following conditions hold.

1. $T_\mathbb{N}(x - y) \leq T_\mathbb{N}(x), I_\mathbb{N}(x - y) \geq I_\mathbb{N}(x), \text{ and } F_\mathbb{N}(x - y) \leq F_\mathbb{N}(x) \text{ for all } x, y \in X$.

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(2) if \( x \lor y \) exists in \( X \) then
\[
T_N(x \lor y) \leq T_N(x) \lor T_N(y), \quad I_N(x \lor y) \geq I_N(x) \land I_N(y), \quad \text{and} \quad F_N(x \lor y) \leq F_N(x) \lor F_N(y).
\]

**Example 4.1.** Let \( (X, -) \) be any subtraction algebra and \( t, i, f \in [-1, 0] \). Then \( X_N = \left\{ \frac{x}{(t,i,f)} : x \in S \right\} \) is a neutrosophic \( \mathbb{N} \)-ideal of \( X \). We call this neutrosophic \( \mathbb{N} \)-ideal as **constant neutrosophic \( \mathbb{N} \)-ideal**.

**Example 4.2.** Let \( (X, -) \) be any non-trivial subtraction algebra and \( t, i, f \in [-1, 0] \) with \((t, i, f) \neq (0, -1, 0)\). Then
\[
X_N = \left\{ \frac{0}{(t,i,f)} - \frac{x}{(a,b,c)} : x \in X - \{0\} \right\}
\]
is a neutrosophic \( \mathbb{N} \)-ideal of \( X \).

**Corollary 4.1.** Let \( (X, -) \) be any non-trivial subtraction algebra (i.e., \( X \neq \emptyset \)). Then \( X \) has at least two neutrosophic \( \mathbb{N} \)-ideals.

Proof. The proof follows from Example 4.1 and Example 4.2.

**Remark 4.1.** Let \( (X, -) \) be a subtraction algebra. Then every neutrosophic \( \mathbb{N} \)-ideal of \( X \) is a neutrosophic \( \mathbb{N} \)-subalgebra of \( X \). But the converse may not hold.

We illustrate Remark 4.1 by Example 4.3.

**Example 4.3.** Let \( (A_2, -_2) \) be the subtraction algebra defined in Example 3.2 and define \( A_{2N} \) as follows:
\[
A_{2N} = \langle 0, a, b, c \rangle_{\mathbb{N}}^{\mathbb{N}}_{\mathbb{N}}^{\mathbb{N}}_{\mathbb{N}} \quad \text{where} \quad \mathbb{N} = \langle -0.8, -0.1, -0.7 \rangle \quad \text{and} \quad \mathbb{N} = \langle -0.8, -0.1, -0.7 \rangle.
\]
Then \( A_{2N} \) is a neutrosophic \( \mathbb{N} \)-subalgebra of \( A_2 \) that is not neutrosophic \( \mathbb{N} \)-ideal of \( A_2 \).

**Remark 4.2.** The results in this section are also valid for neutrosophic \( \mathbb{N} \)-subalgebras. But we restrict our proof to neutrosophic \( \mathbb{N} \)-ideals.

**Proposition 4.1.** Let \( (X, -) \) be a subtraction algebra and \( X_N \) be a neutrosophic \( \mathbb{N} \)-ideal (\( \mathbb{N} \)-subalgebra) of \( X \). Then for all \( x \in X \),
\[
T_N(0) \leq T_N(x), \quad I_N(0) \geq I_N(x), \quad \text{and} \quad F_N(0) \leq F_N(x).
\]

Proof. Since \( 0 = x - x \) for all \( x \in X \), it follows that \( T_N(0) = T_N(x - x) \leq T_N(x) \), \( I_N(0) = I_N(x - x) \geq I_N(x) \), and \( F_N(0) = F_N(x - x) \leq F_N(x) \).

**Theorem 4.1.** Let \( (X, -) \) be a subtraction algebra. Then \( X_N \) and \( X_{NC} \) are neutrosophic \( \mathbb{N} \)-ideals (\( \mathbb{N} \)-subalgebras) of \( X \) if and only if \( X_N \) is the constant neutrosophic \( \mathbb{N} \)-ideal of \( X \).

Proof. It is clear that if \( X_N \) is the constant neutrosophic \( \mathbb{N} \)-ideal of \( X \) then \( X_N \) and \( X_{NC} \) are neutrosophic \( \mathbb{N} \)-ideals (\( \mathbb{N} \)-subalgebras) of \( X \).

Let \( X_N \) and \( X_{NC} \) be neutrosophic \( \mathbb{N} \)-ideals (\( \mathbb{N} \)-subalgebras) of \( X \). Proposition 4.1 asserts that \( T_N(0) \leq T_N(x) \) and \( T_{NC}(0) \leq T_{NC}(x) \), \( I_N(0) \geq I_N(x) \) and \( I_{NC}(0) \geq I_{NC}(x) \), and \( F_N(0) \leq F_N(x) \) and \( F_{NC}(0) \leq F_{NC}(x) \). The latter implies that
\[
T_N(0) \leq T_N(x) \quad \text{and} \quad -1 - T_N(0) \leq -1 - T_N(x),
\]
\[
I_N(0) \geq I_N(x) \quad \text{and} \quad -1 - I_N(0) \geq -1 - I_N(x),
\]
\[
F_N(0) \leq F_N(x) \quad \text{and} \quad -1 - F_N(0) \leq -1 - F_N(x).
\]
We get now that \( T_N(x) = T_N(0) \), \( I_N(x) = I_N(0) \), and \( F_N(x) = F_N(0) \). Thus, \( X_N \) is the constant neutrosophic \( \mathbb{N} \)-ideal of \( X \).

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Proposition 4.2. Let \((X, -)\) be a subtraction algebra and \(X_N, X_M\) be neutrosophic \(N\)-ideals (\(N\)-subalgebras) of \(X\). Then \(X_{N|M}\) is a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X\).

Proof. Let \(x, y \in X\). Then
\[
T_{N|M}(x - y) = T_N(x - y) \lor T_M(x - y) \leq T_N(x) \lor T_M(x) = T_{N|M}(x);
\]
\[
I_{N|M}(x - y) = I_N(x - y) \land I_M(x - y) \geq I_N(x) \land I_M(x) = I_{N|M}(x);
\]
\[
F_{N|M}(x - y) = F_N(x - y) \lor F_M(x - y) \leq F_N(x) \lor F_M(x) = F_{N|M}(x).
\]
Suppose that \(x \lor y\) exists in \(X\). Then
\[
T_{N|M}(x \lor y) = T_N(x \lor y) \lor T_M(x \lor y) \leq T_N(x) \lor T_N(y) \lor T_M(x) \lor T_M(y) = T_{N|M}(x) \lor T_{N|M}(y);
\]
\[
I_{N|M}(x \lor y) = I_N(x \lor y) \land I_M(x \lor y) \geq I_N(x) \land I_N(y) \land I_M(x) \land I_M(y) = I_{N|M}(x) \land I_{N|M}(y);
\]
\[
F_{N|M}(x \lor y) = F_N(x \lor y) \lor F_M(x \lor y) \leq F_N(x) \lor F_N(y) \lor F_M(x) \lor F_M(y) = F_{N|M}(x) \lor F_{N|M}(y).
\]
Therefore, \(X_{N|M}\) is a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X\).

Corollary 4.2. Let \((X, -)\) be a subtraction algebra and \(X_{N_i}\) be a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X\) for \(i = 1, 2, ..., n\). Then \(X_{\bigcap_{i=1}^{n} N_i}\) is a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X\).

Remark 4.3. Let \((X, -)\) be a subtraction algebra and \(X_N, X_M\) be neutrosophic \(N\)-ideals (\(N\)-subalgebras) of \(X\). Then \(X_{NUM}\) may not be a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X\).

We illustrate Remark 4.3 by Example 4.4.

Example 4.4. Let \((A_2, -)\) be the subtraction algebra defined in Example 3.2 and define the neutrosophic \(N\)-ideals of \(X\) \(A_{2N}, A_{2M}\) as follows:
\[
A_{2N} = \langle 0, a, b, c \rangle
\]
\[
A_{2M} = \langle 0, a, b, c \rangle
\]
Then
\[
A_{2NUM} = \langle 0, a, b, c \rangle
\]
is not a neutrosophic \(N\)-ideal of \(X\) as \(-0.3 = I_{NUM}(c) = I_{NUM}(a \lor b) \geq I_{NUM}(a) \land I_{NUM}(b) = -0.2\).

Proposition 4.3. Let \((X, -), (Y, -)\) be subtraction algebras and \(X_N, Y_M\) be neutrosophic \(N\)-ideals (\(N\)-subalgebras) of \(X, Y\) respectively. Then \((X \times Y)_{N \times M}\) is a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X \times Y\). Here, for all \((x, y) \in X \times Y\),
\[
T_{N \times M}((x, y)) = T_N(x) \lor T_M(y), I_{N \times M}((x, y)) = I_N(x) \land I_M(y), \text{ and } F_{N \times M}((x, y)) = F_N(x) \lor F_M(y).
\]

Proof. The proof is straightforward.

Example 4.5. Let \((A_1, -)\) be the subtraction algebra defined in Example 3.1 and define \(A_{1N}\) as follows:
\[
A_{1N} = \langle 0, 1 \rangle
\]

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Then \((A_1 \times A_1)_{N \times N} = \langle (0,0), (0,1), (1,0), (1,1) \rangle\) is a neutrosophic \(N\)-ideal for the subtraction algebra \((A_1 \times A_1, -)\) presented in Table 4.

| \(-\)   | (0,0) | (0,1) | (1,0) | (1,1) |
|---------|-------|-------|-------|-------|
| (0,0)   | (0,0) | (0,0) | (0,0) | (0,0) |
| (0,1)   | (0,1) | (0,0) | (0,1) | (0,0) |
| (1,0)   | (1,0) | (1,0) | (0,0) | (0,0) |
| (1,1)   | (1,1) | (1,0) | (0,1) | (0,0) |

**Theorem 4.2.** Let \((X, -_1), (Y, -_2)\) be subtraction algebras, \(X_N, Y_M\) be neutrosophic \(N\)-ideals (\(N\)-subalgebras) of \(X, Y\) respectively, and \(f: X \rightarrow Y\) be a homomorphism. Then \(X_{f^{-1}(M)}\) is a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(X\).

Proof. Let \(x, a \in X\). Then
\[
\begin{align*}
T_{f^{-1}(M)}(x - a) &= T_M(f(x) - f(a)) = T_M(f(x) - f(a)) \\
I_{f^{-1}(M)}(x - a) &= I_M(f(x) - f(a)) = I_M(f(x) - f(a)) \\
F_{f^{-1}(M)}(x - a) &= F_M(f(x) - f(a)) = F_M(f(x) - f(a))
\end{align*}
\]

Therefore, \(X_{f^{-1}(M)}\) is a neutrosophic \(N\)-ideal of \(X\).

**Example 4.6.** Let \((X, -)\) be a subtraction algebra, \(S\) a subalgebra of \(X\), and \(X_N = \left\{ \frac{x}{[T_{N \cap f^{-1}(N)}]} : x \in X \right\}\) a neutrosophic \(N\)-ideal of \(X\). Then by Theorem 4.1 and by taking \(f: S \rightarrow X\) as \(f(x) = x\) for all \(x \in S\) we get that \(S_N\) is a neutrosophic \(N\)-ideal of \(S\). Where \(S_N = \left\{ \frac{x}{[T_{N \cap f^{-1}(N)}]} : x \in S \right\}\)

**Theorem 4.3.** Let \((X, -_1), (Y, -_2)\) be subtraction algebras, \(X_N, Y_M\) be neutrosophic \(N\)-ideals (\(N\)-subalgebras) of \(X, Y\) respectively, and \(f: X \rightarrow Y\) be a surjective homomorphism. Then \(Y_{f(N)}\) is a neutrosophic \(N\)-ideal (\(N\)-subalgebra) of \(Y\).

Proof. Let \(y, b \in Y\). Since \(f\) is surjective, it follows that \(T_{f(N)}(y - b) = \Lambda_{y-b=f(x)} T_N(x)\). Moreover, there exist \(a \in X\) such that \(b = f(a)\). We have that \(T_{f(N)}(y) = \Lambda_{y=f(x)} T_N(x) = T_N(x')\) for some \(x' \in X\) with \(f(x') = y\). We get now that \(y - b = f(x') - f(a) = f(x' - a)\). The latter implies that \(T_{f(N)}(y - b) \leq T_{N}(x' - a) \leq T_{N}(x') = T_{f(N)}(y')\). Similarly, we get that \(F_{f(N)}(y - b) \leq F_{f(N)}(y)\).

\[
I_{f(N)}(y - b) = \bigvee_{f(x') = y} I_N(x')
\]

Moreover, there exists \(a \in X\) such that \(b = f(a)\). We have that \(I_{f(N)}(y) = \bigvee_{f(x') = y} I_N(x')\) for some \(x' \in X\) with \(f(x') = y\). We get now that \(y - b = f(x') - f(a) = f(x' - a)\). The latter implies that \(I_{f(N)}(y - b) \geq I_N(x' - a) \geq I_N(x') = I_{f(N)}(y)\).

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Suppose that \( y \lor b \in Y \). We prove that \( T_{i(N)}(y \lor b) \leq T_{i(N)}(y) \lor T_{i(N)}(b) \) and \( I_{i(N)}(y \lor b) \geq I_{i(N)}(y) \land I_{i(N)}(b) \), \( F_{i(N)}(y \lor b) \leq F_{i(N)}(y) \lor F_{i(N)}(b) \) are done similarly.

We have \( T_{i(N)}(y \lor b) = \Lambda_{y \lor b \in f(x)} T_N(x) \). Moreover, there exists \( a \in X \) such that \( b = f(a) \). We have that \( T_{i(N)}(y) = \Lambda_{y \in f(x)} T_N(x) = T_N(x') \) and \( T_{i(N)}(b) = \Lambda_{b \in f(x)} T_N(x) = T_N(a) \) for some \( x' \), \( a \in X \) with \( f(x') = y \), \( f(a) = b \). We get now that \( y \lor b = f(x') \lor f(a) = f(x' \lor a) \). The latter implies that \( T_{i(N)}(y \lor b) \leq T_N(x' \lor a) \leq T_N(x') \lor T_N(a) = T_{i(N)}(y) \lor T_{i(N)}(b) \).

5. Level sets and neutrosophic \( N \)-ideals (\( N \)-subalgebra) of subtraction algebra

In this section, we define \( (\alpha, \beta, \gamma) \) level sets of \( X_N \) and study their relation with \( N \)-ideals of \( X \).

Let \( X_N \) be a neutrosophic \( N \)-structure over \( X \) and let \( \alpha, \beta, \gamma \in [-1,0] \) be such that \(-3 \leq \alpha + \beta + \gamma \leq 0 \). Then \((\alpha, \beta, \gamma) \) level set of \( X_N \) is defined as follows:

\[
\{x \in X : T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}.
\]

**Remark 5.1.** The results in this section are also valid for neutrosophic \( N \)-subalgebras (instead of ideal we have subalgebra). But we restrict our proof to neutrosophic \( N \)-ideals.

**Proposition 5.1.** Let \((X, -)\) be a subtraction algebra, \( \alpha \in [-1,0] \), and \( X_N \) a neutrosophic \( N \)-ideal of \( X \). Then \( T_N^\alpha \) is either an empty set or an ideal of \( X \). Here, \( T_N^\alpha = \{x \in X : T_N(x) \leq \alpha\} \) is either an empty set or an ideal of \( X \).

**Proof.** Let \( x, y \in T_N^\alpha \neq \emptyset \). Since \( T_N(x - y) \leq T_N(x) \leq \alpha \), it follows that \( x - y \in T_N^\alpha \). Suppose that \( x \lor y \) exists in \( X \). Then \( T_N(x \lor y) \leq T_N(x) \lor T_N(y) \leq \alpha \). Thus, \( x \lor y \in T_N^\alpha \). Therefore, \( T_N^\alpha \) is an ideal of \( X \).

**Proposition 5.2.** Let \((X, -)\) be a subtraction algebra, \( \beta \in [-1,0] \), and \( X_N \) a neutrosophic \( N \)-ideal of \( X \). Then \( I_N^\beta \) is either an empty set or an ideal of \( X \). Here, \( I_N^\beta = \{x \in X : I_N(x) \geq \beta\} \).

**Proof.** The proof is similar to that of Proposition 5.1.

**Proposition 5.3.** Let \((X, -)\) be a subtraction algebra, \( \gamma \in [-1,0] \), and \( X_N \) a neutrosophic \( N \)-ideal of \( X \). Then \( F_N^\gamma \) is either an empty set or an ideal of \( X \). Here, \( F_N^\gamma = \{x \in X : F_N(x) \leq \gamma\} \).

**Proof.** The proof is similar to that of Proposition 5.1.

**Corollary 5.1.** Let \((X, -)\) be a subtraction algebra, \( \alpha, \beta, \gamma \in [-1,0] \), and \( X_N \) a neutrosophic \( N \)-ideal of \( X \). Then the \((\alpha, \beta, \gamma) \) level set of \( X_N \) is either an empty set or an ideal of \( X \).

**Proof.** We have the \((\alpha, \beta, \gamma) \) level set of \( X_N \) is \( T_N^\alpha \cap I_N^\beta \cap F_N^\gamma \). And by Propositions 5.1, 5.2, and 5.3, we have \( T_N^\alpha \), \( I_N^\beta \), and \( F_N^\gamma \) are either empty sets or ideals of \( X \). Thus, the \((\alpha, \beta, \gamma) \) level set of \( X_N \) is either empty or an intersection of ideals of \( X \) and hence, it is an ideal.

**Lemma 5.1.** Let \((X, -)\) be a subtraction algebra, \( \alpha, \beta, \gamma \in [-1,0] \), and \( X_N \) an \( N \)-structure over \( X \). If every non-empty \((\alpha, \beta, \gamma) \) level set of \( X_N \) is an ideal of \( X \) then \( X_N \) a neutrosophic \( N \)-ideal of \( X \).

**Proof.** Let \( x, a \in X \). Then there exist \( a', \beta', \gamma' \in [-1,0] \) such that \( T_N(a) = a', I_N(a) = \beta', F_N(a) = \gamma' \). Then \( a \) is in the \((a', \beta', \gamma') \) level set of \( X_N \) which is an ideal of \( X \). The latter implies that \( a - x \) is in the \((a', \beta', \gamma') \) level set of \( X_N \). We get now that \( T_N(a - x) \leq a' = T_N(a), I_N(a - x) \geq \beta' = I_N(a), F_N(a - x) \leq \gamma' = F_N(a) \).
Suppose that \( x \lor a \in X \). Then there exist \( \alpha', \beta', \gamma', \alpha'', \beta'', \gamma'' \in [-1,0] \) such that \( T_N(\alpha) = \alpha', I_N(\alpha) = \beta', F_N(\alpha) = \gamma' \), \( T_N(x) = \alpha'' \), \( I_N(x) = \beta'' \), \( F_N(x) = \gamma'' \). Let \( \alpha = \alpha' \lor \alpha'' \), \( \beta = \beta' \land \beta'' \), \( \gamma = \gamma' \lor \gamma'' \). Then \( a, x \) are in the \((\alpha, \beta, \gamma)\)-level set of \( X_N \) which is an ideal of \( X \). The latter implies that \( a \lor x \) is in the \((\alpha, \beta, \gamma)\)-level set of \( X_N \). Thus, \( T_N(a \lor x) \leq a = T_N(a) \lor T_N(x), \quad I_N(a \lor x) \geq \beta = I_N(a) \land I_N(x), \quad F_N(a \lor x) \leq \gamma = F_N(a) \lor F_N(x) \).

Therefore, \( X_N \) is a neutrosophic \( N \)-ideal of \( X \).

**Theorem 5.1.** Let \((X, -)\) be a subtraction algebra, \( \alpha, \beta, \gamma \in [-1,0] \), and \( X_N \) an \( N \)-structure over \( X \). Then \( X_N \) is a neutrosophic \( N \)-ideal of \( X \) if and only if every non-empty \((\alpha, \beta, \gamma)\)-level set of \( X_N \) is an ideal of \( X \).

Proof. The proof follows from Corollary 5.1 and Lemma 5.1.

**Theorem 5.2.** Let \((X, -)\) be a subtraction algebra, \( \alpha, \beta, \gamma \in [-1,0] \), and \( X_N \) an \( N \)-structure over \( X \). Then the following statements are equivalent.

1. \( X_N \) is a neutrosophic \( N \)-ideal of \( X \);
2. \( T_N^\alpha, I_N^\beta, F_N^\gamma \) are either empty sets or ideals of \( X \);
3. Every non-empty \((\alpha, \beta, \gamma)\)-level set of \( X_N \) is an ideal of \( X \).

Proof. (1) \(\Rightarrow\) (2): If \( X_N \) is a neutrosophic \( N \)-ideal of \( X \) then by Propositions 5.1, 5.2, and 5.3, we have \( T_N^\alpha, I_N^\beta, F_N^\gamma \) are ideals of \( X \).

(2) \(\Rightarrow\) (3): If \( T_N^\alpha, I_N^\beta, F_N^\gamma \) are ideals of \( X \) then the \((\alpha, \beta, \gamma)\)-level set of \( X_N \) is intersection of ideals of \( X \) (intersection of \( T_N^\alpha, I_N^\beta, F_N^\gamma \)) and hence, it is an ideal of \( X \).

(3) \(\Rightarrow\) (1): By Lemma 5.1.

**Theorem 5.3.** Let \((X, -)\) be a subtraction algebra and \( \alpha, \beta, \gamma \in [-1,0] \) with \( (\alpha, \beta, \gamma) \neq (0, -1, 0) \). Then every ideal of \( X \) is an \((\alpha, \beta, \gamma)\)-level set of a neutrosophic \( N \)-ideal of \( X \).

Proof. Let \( I \) be an ideal of \( X \) and \( X_N = \left\{ x \in X \mid x \in \text{defined as follows.} \right\} \) be \( N \)-structure of \( X \) defined as follows. \( T_N(x), I_N(x), F_N(x) = \begin{cases} (\alpha, \beta, \gamma) \quad \text{if } x \in I; \\ (0, -1, 0) \quad \text{otherwise.} \end{cases} \) Let \( \alpha', \beta', \gamma' \in [-1,0] \). Then the \((\alpha', \beta', \gamma')\)-level set of \( X_N \) is given as follows:

\[
\begin{cases} I & \text{if } \alpha \leq \alpha' < 0, -1 < \beta' \leq \beta, \gamma \leq \gamma' < 0; \\ X & (\alpha', \beta', \gamma') = (0, -1, 0) \\ 0 & \text{otherwise.} \end{cases}
\]

And it is either an empty set or an ideal of \( X \). Therefore, by Lemma 5.1, \( X_N \) is a neutrosophic \( N \)-ideal of \( X \).

6. Conclusion

In this paper, we combined the notions of \( N \)-structures, neutrosophy, and subtraction algebra to introduce \( N \)-ideals (\( N \)-subalgebras) of subtraction algebras. Some operations on the defined notions were discussed. Moreover, the \((\alpha, \beta, \gamma)\)-level sets were introduced and used to find a necessary and sufficient condition for \( N \)-structures to be neutrosophic \( N \)-ideals (\( N \)-subalgebras).

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For future work, it would be interesting to check whether there is a relation between our results about $\aleph^{-}$-ideals ($\aleph$-subalgebras) of subtraction algebras and the results related to single valued neutrosophic subtraction algebras discussed by Chul Hwan Park in [8].

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