Treatment of Constraints in Stochastic Quantization Method and Covariantized Langevin Equation

KENJIIKEGAMI

Graduate school of Science and Technology,
Chiba University,
1-33 Yayoi-cho, Inageku, Chiba 263, Japan

TADAHIKOKIMURA AND RIUII MOCHIZUKI

Department of Physics, Faculty of Science,
Chiba University,
1-33 Yayoi-cho, Inageku, Chiba 263, Japan

ABSTRACT

We study the treatment of the constraints in stochastic quantization method. We improve the treatment of the stochastic consistency condition proposed by Namiki et al. by suitably taking account of the Ito calculus. Then we obtain an improved Langevin equation and the Fokker-Planck equation which naturally leads to the correct path integral quantization of the constrained system as the stochastic equilibrium state. This treatment is applied to $O(N)$ non-linear $\sigma$ model and it is shown that singular terms appearing in the improved Langevin equation cancel out the $\delta^n(0)$ divergences in one loop order. We also ascertain that the above Langevin equation, rewritten in terms of independent variables, is actually equivalent to the one in the general-coordinate-transformation-covariant and vielbein-rotation-invariant formalism.

* To be published in Nucl. Phys. B
1. Introduction

Stochastic quantization method (SQM) was first proposed by Parisi and Wu. They showed that the method could be applied to gauge theory without the gauge fixing procedure. That is, in SQM it is not necessary to introduce the Faddeev-Popov ghost fields. Nevertheless the method produces the same contributions as those due to ghost fields, which was perturbatively confirmed first for Yang-Mills field and recently for non-Abelian anti-symmetric tensor field.

How to handle the constrained system in SQM was discussed by Namiki et al. in ref.5. They constructed Langevin equation for the system under the holonomic constraints by imposing the stochastic consistency condition. In the path integral quantization method the constraints introduce a determinant factor into path integral measure, which requires that in SQM Langevin equation for the constrained system is constructed so that the equilibrium Fokker-Planck distribution derived from the Langevin equation has the same determinant factor. They showed that the equilibrium distribution coincided with the path integral distribution. Nevertheless they had to use the 5-dimensional stochastic path integral representation of the transition probability distribution of stochastic process and could not derive Fokker-Planck equation directly from Langevin equation, because they did not take account of Ito calculus in their treatment of stochastic consistency condition. One of the main purposes of this paper is to improve Langevin equation of ref.5 by suitably taking account of Ito calculus, and to show that the improved Langevin equation leads to the Fokker-Planck equation which directly gives the correct path integral representation as the equilibrium distribution. Our improvement introduces some singular terms proportional to $\delta(0)$ in the Langevin equation. So, the Langevin equation of ref.5 is correct in the dimensional regularization scheme, but not in other regularization schemes and the general path integral distribution with constraints cannot be directly obtained as the equilibrium Fokker-Planck distribution.

On the other hand, if the constraints are solved explicitly and the system is described in terms of the independent variables only, the action of the system proves to have generally field-dependent metric. In this case we must apply the general-coordinate-transformation (GCT)-covariant and vielbein-rotation (VR)-invariant
Langevin equation to the system.\[^7\] It was not clear whether the latter Langevin equation is equivalent to the above Langevin equation for the constrained system. This point will be clarified in this paper.

Next we discuss about O(N) non-linear $\sigma$ model as an example of such constrained system. The model was studied by many authors.\[^9\] We apply the above two methods to the model and clarify the role of singular terms introduced into the Langevin equation with the stochastic perturbation theory. It will be shown that in both these methods these singular terms are necessary to cancel the $\delta^n(0)$ divergences appearing in one-loop expansion.

This paper is organized as follows. In section 2 we study the improved treatment of the constraints in SQM and show directly that the equilibrium Fokker-Planck distribution coincides with the path integral distribution. In section 3 it is shown that the improved Langevin equation for the constrained system is equivalent to the GCT-covariant and VR-invariant Langevin equation. In section 4 we apply both the improved Langevin equation for the constrained system and the GCT-covariant and VR-invariant Langevin equation to O(N) non-linear $\sigma$ model and examine the cancellation of $\delta^n(0)$ divergences. In section 5 we give conclusion and summary. In Appendix, we ascertain that an assumption, which is introduced in section 3, is satisfied at least in O(N) non-linear $\sigma$ model.

2. Constrained system in SQM

In this paper we consider the system with variables $q_i(x)(i = 1, 2, \cdots, N)$, regular Lagrangian $L(q_i, \partial_\mu q_i)$ ($\mu = 1, 2, \cdots, n$) and a set of constraints

$$F_a(q_i) = 0, \quad (a = 1, 2, \cdots, M; N > M). \quad (2.1)$$

In the path integral quantization method the transition amplitude is given by\[^5\]\[^10\]

$$\langle f \mid i \rangle = \int DqDf \sqrt{detD_{ab}} \exp[-\int d^n x (L(q, \partial q) - J_a F_a)]. \quad (2.2)$$

$$D_{ab} = \frac{\partial F_a}{\partial q_i} \frac{\partial F_b}{\partial q_i}. \quad (2.3)$$
Following the method “time by time constraint” proposed in ref.5, we quantize the above singular system in SQM. The treatment of the stochastic consistency condition is improved by taking account of Ito calculus. It will be shown that the improvement is essential for the acquisition of the Fokker-Planck equation which directly leads to the path integral representation as the equilibrium state.

For the above system, Langevin equation is

\[
 dq_i(t) \equiv q_i(t + dt) - q_i(t) = - \frac{\delta S}{\delta q_i} dt - \frac{\partial F_a}{\partial q_i} \lambda_a dt + dW_i, \tag{2.4}
\]

\[
 S \equiv \int d^n x L, \tag{2.5}
\]

where \( \lambda_a \) is Lagrange multiplier and \( dW_i \) is defined as

\[
 dW_i(t) \equiv W_i(t + dt) - W_i(t), \tag{2.6}
\]

\[
 W_i(t) \equiv \int_0^t dt' \eta_i(t'), \tag{2.7}
\]

\[
 \langle dW_i(x, t) dW_j(x', t) \rangle = 2 \delta_{ij} \delta(x - x') dt, \tag{2.8}
\]

\[
 \langle \eta_i(x, t) \eta_j(x', t') \rangle = 2 \delta_{ij} \delta^n(x - x') \delta(t - t'). \tag{2.9}
\]

and called Wiener process. From (2.8) we may regard \( dW \) as order \( \sqrt{dt} \). Lagrange multiplier \( \lambda_a \) is determined by the stochastic consistency condition \([5]\)

\[
 \dot{F}_a(q(t)) = 0, \tag{2.10}
\]

where dot denotes fictitious-time derivative. Besides, we demand the initial condition

\[
 F_a(q(t = t_0)) = 0, \tag{2.11}
\]

in order to have \( F = 0 \) at any \( t \). In SQM, for fictitious-time derivative of any function
unless we use Stratonovich calculus. In ref. 5 the above fact is not taken into account. (Their Langevin equation is correct only in the dimensional regularization scheme. In other regularization schemes, the Langevin equation is incorrect and the Fokker-Planck distribution does not coincide with the path integral distribution.) If \( \lambda dt \) in eq. (2.4) does not contain terms of order \( \sqrt{dt} \), \( dq = dW \) to order \( \sqrt{dt} \). From (2.4) we get

\[
dF_a = \frac{\partial F_a}{\partial q_i} \left\{ -(\frac{\delta S}{\delta q_i} + \lambda_a \frac{\partial ^2 F_a}{\partial q_i \partial q_j}) dt + dW_i \right\} + \frac{1}{2} (1-2b) \frac{\partial ^2 F_a}{\partial q_i \partial q_j} dW_i dW_j + O((\sqrt{dt})^3), \tag{2.13}
\]

to order \( dt \). Here we use the generalized Ito formula and the product of Wiener process \( dW \) and any function \( f(q) \) is defined as

\[
\{ f(q) \, dW \}(t) \equiv \{ bf(q(t+dt)) + (1-b)f(q(t)) \}dW(t),
\]

\[0 \leq b \leq 1,\tag{2.14}\]

where \( b = \frac{1}{2} \) corresponds to Stratonovich calculus and \( b = 0 \) to Ito calculus. Requiring \( dF_a = 0 \), we get

\[
\lambda_a dt = D_{ab}^{-1} \left\{ \frac{\partial F_b}{\partial q_i} \left( -\frac{\delta S}{\delta q_i} dt + dW_i \right) + \frac{1}{2} (1-2b) \frac{\partial ^2 F_a}{\partial q_i \partial q_j} dW_i dW_j \right\}, \tag{2.15}
\]

\[
D_{ab} \equiv \frac{\partial F_a}{\partial q_i} \frac{\partial F_b}{\partial q_i}. \tag{2.16}
\]

Here \( \lambda_a dt \) contains \( D_{ab}^{-1} \frac{\partial F_b}{\partial q_i} dW_i \) of order \( \sqrt{dt} \), which is inconsistent with the above assumption that \( \lambda_a dt \) does not contain terms of order \( \sqrt{dt} \). So we assume alternatively that the \( \lambda_a dt \) term contains terms of order \( \sqrt{dt} \) like eq. (2.15). Then, from
\begin{equation}
 dq_i = dW_i - \frac{\partial F_a}{\partial q_i} \lambda_a dt \nonumber \\
 = dW_i - \frac{\partial F_a}{\partial q_i} D^{-1}_{ab} \frac{\partial F_b}{\partial q_j} dW_j, \tag{2.17}
\end{equation}

to order $\sqrt{dt}$. From (2.17) the modified expression of $dF$ is

\begin{equation}
 dF_a = \frac{\partial F_a}{\partial q_i} \{(- \frac{\delta S}{\delta q_i} - \lambda_b \frac{\partial F_b}{\partial q_i}) dt + dW_i\} + \frac{1}{2} (1 - 2b) \frac{\partial^2 F_a}{\partial q_i \partial q_j} K_{ik} dW_k K_{jl} dW_l, \tag{2.13'}
\end{equation}

\begin{equation}
 K_{ij} \equiv \delta_{ij} - R_{ij}, \quad R_{ij} \equiv \frac{\partial F_a}{\partial q_i} D^{-1}_{ab} \frac{\partial F_b}{\partial q_j}, \tag{2.18}
\end{equation}

to order $dt$. Here $R_{ij}, K_{ij}$ are projection operators, vertical each other. The final term in RHS of (2.13') does not exist in ref.5. From the consistency condition (2.10) the correct expression of $\lambda dt$ becomes

\begin{equation}
 \lambda_a dt = D^{-1}_{ab} \left\{ \frac{\partial F_b}{\partial q_i} (- \frac{\delta S}{\delta q_i}) dt + dW_i\right\} + \frac{1}{2} (1 - 2b) \frac{\partial^2 F_b}{\partial q_i \partial q_j} K_{ik} dW_k K_{jl} dW_l \right\}. \tag{2.15'}
\end{equation}

The above expression is surely correct, because, with the help of (2.15'), $dq$ has the same terms as (2.17) to order $\sqrt{dt}$. From (2.8) and (2.15'), Langevin equation (2.4) becomes

\begin{equation}
 \dot{q}_i = K_{ij}(\frac{\delta S}{\delta q_j} + \eta_j) - (1 - 2b) \frac{\partial^2 F_b}{\partial q_i \partial q_j} K_{kl} \delta^n(0), \tag{2.19}
\end{equation}

If we multiply (2.19) by $\frac{\delta F_a}{\delta q_i}$, we obtain the expression (2.13') or (2.10). Therefore, the consistency condition (2.10) is embedded in eq.(2.19) and eq.(2.19) means $N-M$ independent differential equations. In ref.5 Langevin equation did not have the singular term in (2.19). The same equilibrium Fokker-Planck distribution as (2.2) could not be derived directly from the Langevin equation of ref.5, while, due to the singular term, we can derive the correct Fokker-Planck equation directly from Langevin equation (2.19) as shown below.
In order to construct the Fokker-Planck equation, we introduce the expectation value of fictitious-time derivative of arbitrary function \( g(q) \)

\[
\langle \dot{g}(q(t)) \rangle \equiv \int Dq \, g(q) \dot{P}(q,t),
\]

(2.20)

where \( P(q,t) \) is the transition probability distribution. Using integral by parts and generalized Ito formula, we obtain the Fokker-Planck equation

\[
\dot{P}(q,t) = \frac{\delta}{\delta q_i} K_{ij} \{ \frac{\delta S}{\delta q_j} - \frac{\partial^2 F_a}{\partial q_j \partial q_l} D_{ab}^{-1} \frac{\partial F_b}{\partial q_l} \delta^n(0) + \frac{\delta}{\delta q_j} \} P(q,t).
\]

(2.21)

due to the singular term in (2.19). Eq.(2.21) cannot be derived from the Langevin equation of ref.5. The probability distribution can include any function \( f(F_a) \) because \( \frac{\delta F_a}{\delta q_i} \) is vertical to the projection operator \( K_{ij} \). In the equilibrium limit \( t \to \infty \), the probability distribution must satisfy \( \dot{P} = 0 \). In the limit the equation has a solution

\[
P(q) = \int Dv_a \sqrt{\text{det} D_{ab}} \exp(-S - \int d^n x \, v_a F_a),
\]

(2.22)

where we chose \( \int Dv \exp(-v_a F_a) \) as \( f(F_a) \) in accordance with the initial condition (2.11) in Minkowski space. The above equilibrium probability distribution coincides with eq.(2.2). However, it is strange that the Langevin equation has divergent term of \( \delta^n(0) \). In section 4 we examine O(N) non-linear \( \sigma \) model as an example of the system under constraint (2.1). In the same example we also show perturbatively that the term proportional to \( \delta^n(0) \) in eq.(2.19) is needed.

3. Equivalence to GCT-covariant and VR-invariant Langevin equation

In this section we shall show that the improved Langevin equation (2.19) is actually equivalent to the Langevin equation in the GCT-covariant and VR-invariant formalism where the constraint (2.1) is explicitly solved and the equation is expressed in terms of independent variables only.
In general the system may have field-dependent metric (or kernel) \( G_{AB}(q) \) after the constraint is solved. According to ref.7, a system with field-dependent metric is described by the GCT-covariant and VR-invariant Langevin equation

\[
dq^A = X^A dt + E^A_m dW^m, \tag{3.1}
\]

where

\[
X^A \equiv -G^{AB} \frac{\delta S}{\delta q^B} + \frac{1}{\sqrt{G}} \frac{\delta}{\delta q^B} (\sqrt{G} G^{AB}) - 2b \frac{\delta E^A_m}{\delta q^B} E^B_n \delta^{mn},
\]

\[
G^{AB} = E^A_m E^B_n \delta^{mn}, \quad A, B, m, n = (1, 2, \cdots, N - M; x). \tag{3.2}
\]

\( G^{AB} \) is the inverse of metric \( G_{AB} \), the summation with respect to \( B \) includes space-time integration and \( dW^m \) is Wiener process defined in section 2. GCT-covariance and VR-invariance mean that Langevin equation is transformed covariantly under general coordinate transformation \( q \rightarrow q' = f(q) \) and is invariant under vielbein rotation \( E^A_m \rightarrow E^A_n \Lambda^m_n \). \( dq^A \) and \( E^A_m dW^m \) are not transformed covariantly and two extra terms in (3.2) are required to be GCT-covariant and VR-invariant.

In order to decompose variables into constraint variables and independent ones, we introduce a new set of variables \( \{Q^\mu\} (\mu = 1, 2, \cdots, N) \).\(^5\) \( Q^\mu \)'s are expressed in terms of \( q_i \)'s \( (i = 1, 2, \cdots, N) \) as

\[
\delta Q^\mu = e^\mu_i \delta q_i, \quad \text{or} \quad \frac{\partial Q^\mu}{\partial q_i} = e^\mu_i, \tag{3.3}
\]

where \( e^\mu_i \) is vielbein field defined as follows. First \( e^a_i \) and \( e_{a,i} \) \( (a = N - M + 1, \cdots, N) \) are defined as

\[
e^a_i = \frac{\partial F_a}{\partial q_i}, \quad e_{a,i} = D^{-1}_{ab} \frac{\partial F_b}{\partial q_i}, \tag{3.4}
\]

i.e. \( Q^a = F_a \). Then, \( e^A_i \) \( (A = 1, 2, \cdots, N - M) \) is chosen so as to satisfy

\[
e^A_i e_{a,i} = 0, \tag{3.5}
\]
and its inverse $e_{A,i}$ is defined as

$$e_{A,i} = (g^{-1})_{AB}e^B_i, \quad g^{AB} \equiv e^A_i e^B_i, \quad (3.6)$$

where we assume that $g^{AB}$ is non-singular. From the above definition it turns out that $e^\mu_i$ and $e_{\mu,i}$ satisfy the following relations

$$e^\mu_i e^\nu_i = \delta^{\mu}_{\nu}, \quad e^\mu_i e_{\mu,j} = \delta^{ij}, \quad (3.7)$$

$$e^A_i e^a_i = e^A_i e_{a,i} = e^A_i e_{a,i} = 0, \quad (3.8)$$

$$e^A_i e_{A,j} = K_{ij}, \quad e^a_i e_{a,j} = R_{ij}, \quad (3.9)$$

$$K_{ij} e^a_j = K_{ij} e_{a,j} = 0, \quad R_{ij} e^A_j = R_{ij} e_{A,j} = 0. \quad (3.10)$$

$$\text{det}(e^\mu_i) \neq 0. \quad (3.11)$$

From (3.11) the manifold spanned by $q_i$'s is identical with the one by $Q^\mu$'s. With the help of the same discussion as made about eq.(2.13'), $dQ^\mu$ is written as follows

$$dQ^\mu \equiv Q^\mu(t + dt) - Q^\mu(t) = \frac{\partial Q^\mu}{\partial q_i} dq_i + (1 - 2b) \frac{\partial^2 Q^\mu}{\partial q_i \partial q_j} K_{ij} dt \delta^n(0),$$

$$= e^\mu_i dq_i + (1 - 2b) \frac{\partial e^\mu_i}{\partial q_j} e^A_i e_{A,j} dt \delta^n(0). \quad (3.12)$$

Then, from (2.13') constraint variables $Q^a$'s satisfy Langevin equation

$$dQ^a = e^a_i \{K_{ij}(-\frac{\delta S}{\delta q_j} dt + dW_j) - (1 - 2b) \frac{\partial F^a}{\partial q_i} D^{-1}_{ab} \frac{\partial^2 F^b}{\partial q_j \partial q_l} K_{jl} dt \delta^n(0) \}$$

$$+ (1 - 2b) \frac{\partial e^a_i}{\partial q_j} e^A_i e_{A,j} dt \delta^n(0), \quad (3.13)$$

$$= 0,$$

to order $dt$. From (3.13) and the initial condition (2.11), constraint variables $Q^a$'s
are zero for all \( t \). As for the independent variables \( Q^A \)'s we get

\[
\begin{align*}
\dot{Q}^A &= e^A_i \{ K_{ij} \left( -\frac{\delta S}{\delta q_j} dt + dW_j \right) - (1 - 2b) \frac{\partial F_a}{\partial q_i} D^{-1}_{ab} \frac{\partial^2 F_b}{\partial q_j \partial q_i} K_{ij} dt \delta^n(0) \} \\
&\quad + (1 - 2b) \frac{\partial e^A_i}{\partial q_j} e^B_j e^B_i dt \delta^n(0), \\
&= -g^{AB} \frac{\delta S}{\delta Q^B} dt + (1 - 2b) \frac{\partial e^A_i}{\partial Q^B} e^B_i dt \delta^n(0) + e^A_i dW^i.
\end{align*}
\]

The above Langevin equation is not invariant under vielbein rotation \( e^A_i \rightarrow e^A_j \Lambda^j_i (Q) \) because Wiener process \( dW^i \) is defined in a manifold spanned by original variables \( q_i \)'s and we must not consider the rotation in the manifold. If we perform field-dependent rotation in the manifold, \( \Lambda^i_j dW^j \) is not Wiener process, i.e. \( \langle \Lambda^i_j dW^j \rangle \neq 0 \). In order to reduce (3.14) to the form of (3.1) we decompose the vielbein as follows:

\[
e^A_i = E^A_I (Q) e^I_i (Q), \quad I = (1, 2, \cdots, N - M),
\]

\[
e^I_i e^J_j = \delta^{IJ}, \quad e^A_i e^B_i = E^A_I E^B_J \delta^{IJ} = g^{AB},
\]

and define \( dW^I \) by

\[
dW^I \equiv e^I_i dW^i - 2b \frac{\partial e^I_i}{\partial Q^B} e^B_J e^J_i dt \delta^n(0).
\]

Then, we obtain

\[
\langle dW^I \rangle = 0, \quad \langle dW^I (x) dW^J (y) \rangle = 2 \delta^{IJ} dt \delta^n (x - y).
\]

\( dW^I \) is desirable Wiener process and with \( dW^I \) Langevin equation is written as

\[
\begin{align*}
\dot{Q}^A &= -g^{AB} \frac{\delta S}{\delta Q^B} dt + \frac{\partial}{\partial Q^B} (E^A_I e^I_i) E^B_J e^J_i dt \delta^n(0) - 2b \frac{\partial E^A_I}{\partial Q^B} E^B_J \delta^{IJ} dt \delta^n(0) + E^A_I dW^I.
\end{align*}
\]

From now on we regard \( E^A_I \) as vielbein corresponding to \( E^A_m \) in (3.1). In fact the
above Langevin equation is invariant under vielbein rotation

\[ E^A_I \rightarrow E^A_J A^J_I. \quad (3.20) \]

Langevin equation (3.19) is a little different from GCT-covariant and VR-invariant Langevin equation (3.1). Here we assume

\[ e^A_i \nabla_B e^B_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial Q^B} (\sqrt{g} e^B_i) e^A_i = 0, \quad (3.21) \]

where \( \nabla_B \) is covariant derivative in Riemannian manifold spanned by \( Q^A \). Eq.(3.21) is usually presumed, because \( g^{AB} \) satisfies the metric condition \( \nabla_A g^{BC} = 0 \) and the latter condition makes eq.(3.21) naturally understandable. We ascertain in Appendix that the above assumption is satisfied in \( O(N) \) non-linear \( \sigma \) model. With (3.21), Langevin equation (3.19) is reduced to

\[ dQ^A = -g^{AB} \frac{\delta S}{\delta Q^B} dt + \frac{1}{\sqrt{g}} \frac{\partial}{\partial Q^B} (\sqrt{g} g^{AB}) \delta^n(0) dt - 2b \frac{\partial E^A_I}{\partial Q^B} E^B_J \delta^{IJ} \delta^n(0) dt + E^A_I dW^I, \quad (3.22) \]

\[ g \equiv det(g^{AB}). \quad (3.23) \]

Covariant derivative in (3.21) does not include spin connection because, as mentioned above, the rotation must not be considered in the manifold spanned by \( q_i \)’s. Eq.(3.22) is equivalent to GCT-covariant and VR-invariant Langevin equation (3.1). Thus it is ascertained that Langevin equation (2.19) for the constrained system is equivalent to GCT-covariant and VR-invariant Langevin equation (3.1). That Langevin equation also has divergent term including \( \delta^n(0) \) and, in the next section, we apply the Langevin equation to \( O(N) \) non-linear \( \sigma \) model and see the cancellation of \( \delta^n(0) \) divergences.
4. **O(N) non-linear σ model**

O(N) non-linear σ model is defined by action

\[ S = \frac{1}{2} \int d^nx \partial_\mu \Phi_i \partial_\mu \Phi_i, \quad (\mu = 1, 2, \cdots, n; i = 1, 2, \cdots, N), \quad (4.1) \]

and constraint

\[ F = \Phi_i \Phi_i - \frac{1}{\alpha} = 0, \quad (4.2) \]

where \( \alpha \) is constant. The constraint is an example of (2.1). Applying Langevin equation (2.19) to the model, we obtain

\[ \dot{\Phi}_i = (\delta_{ij} - \alpha \Phi_i \Phi_j)(\partial^2 \Phi_j + \eta_j) - (1 - 2b)\alpha(N - 1)\Phi_i \delta^n(0). \quad (4.3) \]

The above equation includes (2.19)-type constraint (4.2) and means \( N-1 \) independent equations. From eq.(4.2) \( \Phi_i \) has non-zero vacuum expectation value. We shift the field

\[ \phi_i \equiv \Phi_i - < \Phi_i >, \quad (4.4) \]

\[ v_i \equiv \frac{1}{\sqrt{\alpha}} < \Phi_i >, \quad v_i v_i = 1, \quad (4.5) \]

and with the shifted field \( \phi \) Langevin equation is written

\[ \dot{\phi}_i = \{ K_{ij} - \sqrt{\alpha}(v_i \phi_j + v_j \phi_i) - \alpha \phi_i \phi_j \} \{ \partial^2 \phi_j + \eta_j \} \]

\[ - (1 - 2b)\alpha(N - 1)(\frac{1}{\sqrt{\alpha}}v_i + \phi_i)\delta^n(0), \quad (4.6) \]

\[ K_{ij} \equiv \delta_{ij} - v_i v_j. \quad (4.7) \]

Going to momentum space and integrating eq.(4.6) with respect to \( t \), we get
\[ \phi_i(k, t) = \int d\tau G_{ij}(k, t - \tau) \left[ K_{ji} \eta_i(k, \tau) + I_j(k, \tau) + J_j(k, \tau) \right] \]

\[ - (1 - 2b)\alpha(N - 1) \left\{ \frac{v_j}{\sqrt{\alpha}} + \phi_j(k, \tau) \right\} \delta^n(0), \]  
(4.8)

\[ G_{ij}(k, t) \equiv [\exp\{-K k^2 t\}]_{ij} = \exp(-k^2 t)K_{ij} + (\delta_{ij} - K_{ij}), \]  
(4.9)

\[ I_i(k, t) \equiv -\sqrt{\alpha} \int \frac{d^n p d^n q}{(2\pi)^n} \delta^n(k - p - q)(v_i \delta_{lm} + v_m \delta_{il}) \phi_l(p, t) \left\{ -q^2 \phi_m(q, t) + \eta_m(q, t) \right\}, \]  
(4.10)

\[ J_i(k, t) \equiv -\alpha \int \frac{d^n p d^n q d^n r}{(2\pi)^{2n}} \delta^n(k - p - q - r) \phi_i(p, t) \phi_j(q, t) \left\{ -r^2 \phi_j(r, t) + \eta_j(r, t) \right\}, \]  
(4.11)

After solving the above equation by iteration, we express the result graphically

\textit{Fig. 1}

where we denote \( \eta \) by a cross or an encircled cross, \( G \) a line and \( \alpha \delta(0) \) a bullet, respectively. We calculated the one-loop corrections of the two-point function and obtained six \( \delta(0) \)-divergent diagrams.

\textit{Fig. 2}

Each of the six diagrams contributes respectively

\( (2a) \)

\[ 2\alpha N \delta^n(k + k')K_{ij} \frac{1}{(k^2 + k'^2)^2} \int d^n p, \]

\( (2b) \)

\[ -4\alpha N \delta^n(k + k')\theta(0)K_{ij} \frac{1}{(k^2 + k'^2)^2} \int d^n p, \]
\[(2c)\]
\[2\alpha v^2 \delta^n(k + k') K_{ij} \frac{1}{(k^2 + k'^2)^2} \int d^n p,\]

\[(2d)\]
\[-4\alpha v^2 \delta^n(k + k') K_{ij} \frac{1}{(k^2 + k'^2)^2} \int d^n p,\]

\[(2e)\]
\[4\alpha v^2 \delta^n(k + k') \theta(0) K_{ij} \frac{1}{(k^2 + k'^2)^2} \int d^n p,\]

\[(2f)\]
\[-2\alpha (1 - 2b)(N - 1)(2\pi)^n \delta^n(0) \delta^n(k + k') K_{ij} \frac{1}{(k^2 + k'^2)^2}.\]

The contributions from all the $\delta(0)$-divergent diagrams cancel out if we put $\theta(0) = b$.\[^{13}\] It turns out that $\delta^n(0)$-divergent term of eq.(2.19) or eq.(4.3) is necessary to the cancellation of the $\delta(0)$ divergences in one-loop order.

Now we apply Langevin equation (3.1) to the model. As discussed in section 3, GCT-covariant and VR-invariant Langevin equation is equivalent to eq.(2.19). We show that the $\delta^n(0)$ divergences really cancel out.

With $\Phi_N$ substituted by means of constraint (4.2), action (4.1) becomes

\[
S_{N-1} = \int d^n x g^{\alpha\beta}(\Phi) \partial_\mu \Phi_\alpha \partial^\mu \Phi_\beta, \quad (4.12)
\]

\[
g^{\alpha\beta}(\Phi) = \delta_{\alpha\beta} + \frac{\alpha \Phi_\alpha \Phi_\beta}{1 - \alpha \Phi_\gamma \Phi_\gamma}, \quad \alpha, \beta, \gamma = 1, 2, \ldots, N - 1. \quad (4.13)
\]

We can use Langevin equation (3.1) for the system, because the assumption (3.21) is satisfied here as shown in Appendix. As we calculate the one-loop contributions to two-point functions, we need Langevin equation to order $\alpha$, which is

\[
\dot{\Phi}_\alpha = \partial^2 \Phi_\alpha + \alpha \Phi_\alpha \partial_\mu \Phi_\beta \partial_\mu \Phi_\beta - \alpha (N - 1) \delta^n(0) \Phi_\alpha + \alpha b N \delta^n(0) \Phi_\alpha + \xi_\alpha, \quad (4.14)
\]

where $\eta_\alpha$ is white noise. We calculate two-point functions in one-loop order and show that there remains no $\delta^n(0)$ divergence. Integral equation corresponding to
eq. (4.14) is

\[
\Phi_\alpha(x, t) = \int d\tau G(x, t - \tau) \left[ \eta_\alpha(x, \tau) - \frac{\alpha}{2} \Phi_\alpha(x, \tau) \Phi_\beta(x, \tau) \eta_\beta(x, \tau) \right] 
\]

\[
+ \alpha \Phi_\alpha(x, \tau) \partial_\mu \Phi_\beta(x, \tau) \partial_\mu \Phi_\beta(x, \tau) + \alpha \left\{ bN - (N - 1) \right\} \delta^4(0) \Phi_\alpha(x, \tau) \right], 
\]

where

\[
G(x, t) \equiv \int \frac{d^4k}{(2\pi)^4} \exp(-ikx)\theta(t)\exp(-k^2t), 
\]

which can be solved by iteration as eq.(4.8). Eq.(4.16) leads to the following vertices

**Fig. 3**

In Fig.3 we denote G by a line, \(\eta\) a cross or an encircled cross and \(\alpha\delta^n(0)\) a bullet, respectively. The above vertices contribute to the propagator shown diagramatically in Fig.4.

**Fig. 4**

In one-loop order the \(\delta^n(0)\)-divergent contributions from Figs.(4c), (4d) and (4e) are

\[(4c) \quad \frac{-2\alpha N}{(k^2 + k'^2)^2} \theta(0) \delta_\alpha \delta_\beta \delta^n(k + k') \int d^n p , \]

\[(4d) \quad \frac{2\alpha(N - 1)}{(k^2 + k'^2)^2} \delta_\alpha \delta_\beta \delta^n(k + k') \int d^n p + 2nd div. , \]

\[(4e) \quad \frac{2\alpha(1 - N + bN)}{(k^2 + k'^2)^2} (2\pi)^n \delta_\alpha \delta_\beta \delta^n(k + k') \delta^n(0) . \]

With \(\theta(0) = b\), the sum of the above contributions is zero, which coincides with the result of the constrained system. It seems that in O(N) non-linear \(\sigma\) model both Langevin equations (2.19) and (3.1) lead to the correct results.
5. Conclusion

We constructed Langevin equation (2.19) for constrained system. In the derivation of (2.19), the improved treatment of the stochastic consistency condition for constraints was essential. The Langevin equation can be applied to any system obeying (2.1)-type constraint. From the equation we directly derived Fokker-Planck equation and obtained eq.(2.22) as the equilibrium distribution, which coincides with the one obtained in path integral method. Owing to Ito calculus, the Langevin equation contains $\delta^n(0)$-type singular terms and we showed explicitly, in $O(N)$ non-linear $\sigma$ model, that the singular terms are necessary to the cancellation of $\delta^n(0)$ divergences in one loop order.

Furthermore, we ascertained that eq.(2.19) is equivalent to GCT-covariant and VR-invariant Langevin equation (3.1). We applied eq.(3.1) to O(N) non-linear $\sigma$ model and showed that the singular terms in GCT-covariant and VR-invariant Langevin equation are also necessary to the cancellation of $\delta^n(0)$ divergences in one-loop order.

6. Acknowledgments

We thank Prof.S.Kawasaki for careful reading of this manuscript and also thank Dr.A.Nakamura for valuable discussion.
7. APPENDIX

Here we show that assumption (3.21) is satisfied in O(N) non-linear $\sigma$ model. We choose polar coordinates as new variables $Q^\mu$'s in section 3 and immediately recognize that $e_i^A \nabla_B e_i^B$ vanishes. If the assumption (3.21) is ascertained in the above special case, it is satisfied for any $\{Q^\mu\}$ because of the covariance of the assumption.

We start with a set of variables $\{q_i\}(i=1,2,\cdots,N)$ obeying $(q_i q_i)^{1/2} = r_c (r_c = \text{constant})$ and introduce a new set of variables $\{Q^\mu\} (\mu=1,2,\cdots,N)$. In accordance with section 3

$$Q^a \equiv Q^N = (q_i q_i)^{1/2} - r_c.$$

If we choose $N-1$ angles of $N$ dimensional polar coordinates as $Q^A$'s $(A=1,2,\cdots,N-1)$

$$q_1 = r_c \cos Q^1,$$
$$q_2 = r_c \sin Q^1 \cos Q^2,$$
$$\vdots$$
$$q_{N-1} = r_c \sin Q^1 \sin Q^2 \cdots \sin Q^{N-2} \cos Q^{N-1},$$
$$q_N = r_c \sin Q^1 \sin Q^2 \cdots \sin Q^{N-2} \sin Q^{N-1},$$

we obtain the vielbein $e_i^A$

$$e_i^A \equiv \frac{\partial Q^A}{\partial q_i} = \frac{\cos Q^A \sin Q^{A+1} \cdots \sin Q^{i-1} \cos Q^i}{r_c \sin Q^1 \cdots \sin Q^{A-1}}, \quad (i > A, i \neq N),$$
$$e_i^A = 0, \quad (A > i),$$
$$e_i^A = \frac{-\sin Q^A}{r_c \sin Q^1 \cdots \sin Q^{A-1}}, \quad (i = A),$$
$$e^A_N = \frac{\cos Q^A \sin Q^{A+1} \cdots \sin Q^{N-1}}{r_c \sin Q^1 \cdots \sin Q^{A-1}},$$

and $e_i^a$

$$e_i^a \equiv \frac{\partial Q^N}{\partial q_i} = \sin Q^1 \cdots \sin Q^{i-1} \cos Q^i, \quad (i \neq N),$$
$$e^a_N = \sin Q^1 \cdots \sin Q^{N-1}.$$
The vielbein satisfies (3.5)

\[ e_i^A e_i^a = 0. \]

Vielbein \( e_i^A \) leads to metric

\[ g^{AB} \equiv e_i^A e_i^B = \frac{1}{r_c^2 \sin^2 Q_1 \cdots \sin^2 Q_{A-1}} \delta^{AB}. \]

By straightforward calculation we obtain

\[ e_i^A \nabla_B e_i^B = 0, \]

which holds also for other choices of \( \{Q^\mu\} \) due to the covariance of the equation. As we proved that assumption (3.21) is satisfied in O(N) non-linear \( \sigma \) model, we are allowed to use the equation (3.1) in section 4.
REFERENCES

1. G.Parisi and Y.Wu, Sci. Sin. 24 (1981)483

2. For a review, P.H.Damgaard and H.Hüffel, Phys. Rep. 152 (1987)227

3. M.Namiki, I.Ohba, K.Okana and Y.Yamanaka, Prog. Theor. Phys. 69 (1983) 1580

4. A.Nakamura, Prog. Theor. Phys. 86 (1991)925

5. M.Namiki, I.Ohba and K.Okano, Prog. Theor. Phys. 72 (1984) 350

6. K.Ito, Proc. Imp. Acad. 20 (1944)519

7. R.Graham, Phys. Lett. 109A (1985)209;
   R.Mochizuki, Mod. Phys. Lett. A5 (1990)2335;
   H.Rumpf, Phys. Rev. D33 (1986)942

8. R.Mochizuki, Prog. Theor. Phys. 85 (1991)407

9. J.Zinn-Justin, Nucl. Phys. B275[FS17] (1986) 135;
   N.Nakazawa and D.Ennyu, Nucl. Phys. B305[FS23] (1988) 516

10. L.D.Faddeev,Teoret. i Mat. Piz. 1 (1969)3;
    P.Senjanovich, Ann. of Phys. 100 (1976)227;
    T.Maskawa and H.Nakajima, Prog. Theor. Phys. 56 (1976)1295

11. L.Arnold, Stochastic Differential Equations (Wiley-Intersciense, New York, 1974)

12. T.L.Stratonovich, Conditional Markov processes and their application to the theory of optimal control (Elsevier, New York, 1968)

13. H.Kawara, M.Namiki, H.Okamoto and S.Tanaka, Prog. Theor. Phys. 84 (1990)749;
    N.Komoike, Prog. Theor. Phys. 86 (1991)575
Figure Captions

Fig.1 Vertices from eq.(4.8)

Fig.2 $\delta^n(0)$-divergent diagrams contributing to two-point function

Fig.3 Vertices from eq.(4.16)

Fig.4 Propagator including vertices shown in Fig.3