New forms of BRST symmetry in rigid rotor

Sumit Kumar Rai * and Bhabani Prasad Mandal †

Department of Physics,
Banaras Hindu University,
Varanasi-221005, INDIA.

We derive the different forms of BRST symmetry by using the Batalin-Fradkin-Vilkovisky formalism in a rigid rotor. The so called “dual-BRST” symmetry is obtained from usual BRST symmetry by making a canonical transformation in the ghost sector. On the other hand, a canonical transformation in the sector involving Lagrange multiplier and its corresponding momentum leads to a new form of BRST as well as dual-BRST symmetry.

I. INTRODUCTION

Becchi-Rouet-Stora and Tyutin (BRST) symmetry [1] provides a basis for the modern quantization of gauge theory and is very important tool in characterizing various renormalizable field theoretic models. Path integral quantization of gauge theories [2] can be done both in Lagrangian formulation and Hamiltonian formulation. In both the formulation, phase space is extended by incorporating the Grassmanian odd ghost variables where the gauge invariance is ensured by the BRST symmetry.

The Hamiltonian approach developed by Batalin, Fradkin and Vilkovisky (BFV) [3] is a powerful technique to study the BRST quantization of constrained systems. The main features of BFV approach are: it does not require closure off-shell of the gauge algebra and therefore does not need an auxiliary field, this formalism heavily relies on BRST transformations which are independent of the gauge condition, this method is even appli-
cable to Lagrangians which are not quadratic in velocities and hence is more general than the strict Lagrangian approach. Being based on the Hamiltonian, the approach is closer to Hilbert space techniques and to unitarity. This method uses an extended phase space where the Lagrange multiplier and the ghosts are treated as dynamical variables. The generator of BRST symmetry for systems with first class constraints can be constructed from the constraints in a gauge independent way whose cohomology produces the physical states.

A great deal of work has been done on various models using BFV approach for the systems with first class constraints such as QED, U(1) gauge theory etc. This approach has also been applied to the systems with second class constraints such as Proca model, chiral Schwinger model etc by converting them to first class constraints using various techniques. A great deal of work has also been done on the various form of the BRST symmetry such as the non-local and non-covariant symmetry for QED, the non-local but covariant symmetry for QED and the local but non-covariant symmetry in Abelian gauge theories. In all the above mentioned symmetries, the variation of gauge fixing part vanishes which is defined as “dual” to the vanishing of the variation of kinetic part in the usual BRST. The so called dual-BRST (also called Co-BRST symmetry) where the gauge fixing part is independently invariant and the variation of the kinetic part cancels out with that of the ghost part of the effective action, was thought to be an independent symmetry. It has been shown recently, using BFV formalism that the dual BRST symmetry is not an independent symmetry but can be obtained by canonical transformation in the ghost sector of BRST symmetry for U(1) gauge theory.

In this paper, our purpose is to render the content of BFV-BRST technique more easily accessible to the non-expert by establishing the connection between the constrained systems, BRST and dual-BRST symmetries. In particular, we obtain the different forms of BRST symmetry by using BFV-BRST formulation in a simple system like rigid rotor. The canonical transformation in the ghost sector of the gauge fixed action gives rise to dual-BRST symmetry. On the other hand, a canonical transformation in the sector involving the Lagrange multiplier and its corresponding momentum leads to a new form
of BRST/dual-BRST symmetries. Our goal is to put this powerful technique of BFV approach into the framework of elementary quantum mechanics.

The plan of our paper is as follows. In Sec. II, we give a brief introduction to BFV formalism. Using this BFV approach, we generate the BRST symmetries in Sec. III-A. We obtain the new form of BRST symmetry in Sec. III-B. In Sec. IV, we have shown that the dual-BRST symmetry for a rigid rotor is obtained by making a canonical transformation in the ghost sector. A new form of dual-BRST symmetry is obtained in Sec. IV-A. Sec. V is devoted to conclusion and discussion.

II. BFV FORMALISM

This method provides a general procedure to quantize systems with first class constraints. We recapitulate the essence of this approach in terms of finite number of phase space variables. The action under such considerations can be written as

\[ S = \int dt \left( p^\mu \dot{q}_\mu - H_0 - \lambda^a \Omega_a \right), \]

(2.1)

where \((q^\mu, p_\mu)\) are the canonical variables describing the theory. \(H_0\) is the Hamiltonian and \(\lambda^a\) are the Lagrange multiplier associated with first class constraints, \(\Omega_a\). In this approach, Lagrange multipliers \(\lambda^a\) are dynamical variables and therefore, treated as the canonical variables. We introduce conjugate canonical momenta \(p^a_\lambda\) to \(\lambda^a\) where \(p^a_\lambda\) must be imposed as new constraints such that the dynamics of the theory does not change. BFV method introduces a pair of canonically conjugate ghosts \((C^a, P_a)\) for each constraints of the theory. These ghosts follow the anticommutation relation as follows

\[ \{C^a(x, t), P^b(y, t)\} = -i \delta^{ab} \delta(x - y), \]

(2.2)

where \(C\) and \(P\) have ghost number 1 and -1 respectively. The nilpotent generator, \(Q\) of BRST symmetry in an extended phase space of the system with first class constraints has the general form

\[ Q = C_a \Omega^a + \frac{1}{2} P^a j^b_c C_b C_c, \]

(2.3)
where the $f^{bc}_a$ is a structure constant, $\Omega^a$ is the first class constraint. According to the Fradkin-Vilkovisky theorem \cite{3} which states that the generating functional in the extended phase space is given as

$$Z_\Psi = \int \mathcal{D}\varphi \exp(iS_{eff}),$$

(2.4)

where the effective action, $S_{eff}$ is

$$S_{eff} = \int dt \left(p^\mu \dot{q}_\mu + \dot{C}^a P_a + p^a \dot{\lambda}_a - H_\Psi\right).$$

(2.5)

$\mathcal{D}\varphi$ is the Liouville measure on the phase space. $H_\Psi$ is the extended Hamiltonian given as

$$H_\Psi = H_0 + \{Q, \Psi\}.$$  

(2.6)

$\Psi$ is the gauge fixed fermion and $Z_\Psi$ does not depend upon the choice of $\Psi$.

III. BFV-BRST APPROACH IN RIGID ROTOR

A. BRST symmetry

We consider a rigid rotor in 2+1 dimension. The constraint equation is $(r - a) = 0$. The canonical Hamiltonian for such a system can be written as \cite{12}

$$H_c = H_0 + \lambda (r - a),$$

(3.1)

where $H_0 = \frac{p_\theta^2}{2mr^2}$, and $p_\theta = (xp_y - yp_x)$. The action in a finite phase space can be written as

$$S = \int dt \left[p_r \dot{r} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2mr^2} - \lambda (r - a)\right].$$

(3.2)

Using Dirac’s prescriptions \cite{13} for constraint analysis, it is trivial to see that the system has only two first class constraints, namely the primary constraint $p_\lambda = 0$ and the secondary constraint $(r - a) = 0$.

Using BFV approach, we introduce a pair of canonically conjugate ghosts $(C, P)$ with ghost number 1 and -1 respectively, for the first class constraint, $p_\lambda = 0$ and another pair
of canonically conjugate anticommuting ghosts \((\bar{C}, \bar{P})\) with ghost number -1 and 1 respectively for other constraint, \((r - a) = 0\). The effective action in the extended phase space using Eq. (2.5) becomes

\[
S_{\text{eff}} = \int dt \left[ p_r \dot{r} + p_\theta \dot{\theta} + p_\lambda \dot{\lambda} + \dot{C} P + \dot{\bar{C}} \bar{P} - \frac{p_\theta^2}{2m r^2} - \{Q, \Psi\} \right]. \tag{3.3}
\]

The symmetry generator for the rigid rotor from Eq. (2.3) is

\[
Q_b = i \left[ C(r - a) + \bar{P} p_\lambda \right]. \tag{3.4}
\]

Using the relation \(\delta_b \phi = [\phi, Q_b]_\pm \) (+ sign for bosonic and - for fermionic nature of \(\phi\)), the BRST charge given in Eq. (3.4) will generate the following BRST transformations

\[
\begin{align*}
\delta_b p_r &= C, & \delta_b \lambda &= -\bar{P}, & \delta_b \bar{C} &= p_\lambda, \\
\delta_b \theta &= 0, & \delta_b p_\lambda &= 0, & \delta_b p_\theta &= 0, \\
\delta_b r &= 0, & \delta_b C &= 0, & \delta_b \bar{P} &= 0, \\
\delta_b P &= (r - a).
\end{align*}
\tag{3.5}
\]

We choose the gauge fixed fermion as

\[
\Psi = \left[ \bar{P} \lambda + \bar{C} \left( p_r + \frac{\xi}{2} p_\lambda \right) \right], \tag{3.6}
\]

and calculate

\[
\{Q_b, \Psi\} = \lambda(r - a) + \bar{P} P - C \bar{C} + p_\lambda \left( p_r + \frac{\xi}{2} p_\lambda \right). \tag{3.7}
\]

Putting it into Eq. (3.3), the effective action becomes

\[
S_{\text{eff}} = \int dt \left[ p_r \dot{r} + p_\theta \dot{\theta} + p_\lambda \dot{\lambda} + \dot{C} P + \dot{\bar{C}} \bar{P} - \frac{p_\theta^2}{2m r^2} - \lambda(r - a) \\
- \bar{P} P + C \bar{C} - p_\lambda \left( p_r + \frac{\xi}{2} p_\lambda \right) \right]. \tag{3.8}
\]

The generating functional \(Z_\Psi\) corresponding to the above effective theory can be written as

\[
Z_\Psi = \int \mathcal{D}\varphi \exp \left[ i \int dt \left\{ p_r \dot{r} + p_\theta \dot{\theta} + p_\lambda \dot{\lambda} + \dot{C} P + \dot{\bar{C}} \bar{P} - \frac{p_\theta^2}{2m r^2} - \lambda(r - a) \\
- \bar{P} P + C \bar{C} - p_\lambda \left( p_r + \frac{\xi}{2} p_\lambda \right) \right\} \right]. \tag{3.9}
\]
The effective action in Eq. (3.8) is invariant under the BRST transformations given in Eq. (3.5). We integrate \( Z_\Psi \) in Eq. (3.9) over \( \mathcal{P}, \bar{\mathcal{P}} \) to obtain
\[
Z_\Psi = \int \mathcal{D}\varphi' \exp \left[ i \int dt \left\{ p_r \dot{r} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2mr^2} - \lambda (r - a) + p_\lambda (\dot{\lambda} - p_r) - \frac{\xi}{2} p_\lambda^2 + C \bar{C} + \bar{C} \dot{C} \right\} \right]
\] (3.10)
and integrate Eq. (3.10) over \( p_\lambda \) to obtain
\[
Z_\Psi = \int \mathcal{D}\varphi'' \exp \left[ i \int dt \left\{ p_r \dot{r} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2mr^2} - \lambda (r - a) - \frac{1}{2\xi} (\dot{\lambda} - p_r)^2 + C \bar{C} + \bar{C} \dot{C} \right\} \right].
\] (3.11)
which is same as the action mentioned in Ref. [12]. The difference is that this approach does not require any auxiliary field and is done in an gauge independent way. The BRST symmetry after integrating over \( \mathcal{P} \) and \( \bar{\mathcal{P}} \) becomes
\[
\delta_b p_r = C, \quad \delta_b \lambda = -\bar{C}, \quad \delta_b \bar{C} = p_\lambda,
\]
\[
\delta_b \theta = 0, \quad \delta_b p_\lambda = 0, \quad \delta_b p_\theta = 0,
\]
\[
\delta_b r = 0, \quad \delta_b C = 0,
\] (3.12)
which is similar to the BRST symmetries mentioned in Ref. [12]. In the effective action given by Eq. (3.11), we observe that the term \( \frac{1}{2\xi} (\dot{\lambda} - p_r)^2 \) is like a gauge fixing term.

B. New form of BRST symmetry

In this section, we make a canonical transformation in \((p_\lambda, \lambda)\) sector as follows
\[
p'_\lambda = p_\lambda + \frac{2}{\xi} p_r,
\]
\[
\lambda' = \lambda,
\] (3.13)
to find a new form of BRST transformations. The path integral measure does not change as the Jacobian equals to 1 in this sort of transformation. The BRST transformation for the new variable \( p'_\lambda \) now becomes
\[
\delta p'_\lambda = \frac{2}{\xi} \delta p_r = \frac{2}{\xi} C.
\] (3.14)
The BRST transformation for $\bar{C}$ is expressed in terms of new defined variable $p'_\lambda$. This gives rise to new form of BRST symmetry \[14\]

\[
\begin{align*}
\delta_b p_r &= C, & \delta_b \lambda &= -\dot{C}, & \delta_b \theta &= 0, \\
\delta_b p_\theta &= 0, & \delta_b r &= 0, & \delta_b C &= 0, \\
\delta_b \bar{C} &= p_\lambda + \frac{2}{\xi} p_r, & \delta_b p_\lambda &= -\frac{2}{\xi} \delta_b p_r &= -\frac{2}{\xi} C,
\end{align*}
\]

(3.15)

which leaves the action given in Eq. (3.10) invariant. It can be easily seen that they are nilpotent and can reduce to the original form of BRST symmetry.

IV. DUAL-BRST SYMMETRY

We make the following canonical transformation in the ghost sector $(C^a, P_a)$ of the theory

\[
\begin{align*}
C & \to P \\
P & \to C \\
\bar{C} & \to \bar{P} \\
\bar{P} & \to \bar{C}
\end{align*}
\]

(4.1)

which does not change the effective action in Eq. (3.8). The generator of the BRST symmetry in Eq. (3.4) after the above canonical transformation becomes

\[
Q_d = i \left[ P(r - a) + \bar{C} \lambda \right].
\]

(4.2)

Now $Q_d$ generates the following new transformations

\[
\begin{align*}
\delta_d p_r &= P, & \delta_d \lambda &= -\bar{C}, & \delta_d \theta &= 0, \\
\delta_d p_\lambda &= 0, & \delta_d p_\theta &= 0, & \delta_d r &= 0, \\
\delta_d \bar{C} &= 0, & \delta_d P &= 0, & \delta_d \bar{P} &= p_\lambda, \\
\delta_d C &= (r - a).
\end{align*}
\]

(4.3)
The Jacobian of the canonical transformation in Eq. (4.3) is unity, so the Liouville measure in the generating functional does not change. The effective action given in Eq. (3.10) is symmetric under the transformations mentioned in Eq. (4.3).

We observe that the variation of gauge fixing part in Eq. (3.11) vanishes independently [i.e. \( \delta_d (\lambda - p_r) = 0 \)]. This is the dual-BRST symmetry as mentioned in the introduction part. We carry out the integration over \( \mathcal{P} \) and \( \bar{\mathcal{P}} \), in the generating functional given by Eq. (3.9) to get the following dual-BRST symmetries

\[
\begin{align*}
\delta_d p_r &= -\dot{\bar{C}}, \\
\delta_d \lambda &= -\dot{C}, \\
\delta_d \theta &= 0, \\
\delta_d \rho &= 0, \\
\delta_d \rho &= 0, \\
\delta_d \bar{C} &= 0,
\end{align*}
\]

under which the effective action given by Eq. (3.11) is invariant.

### A. New form of Dual-BRST symmetry

The above mentioned dual-BRST symmetry is obtained by the canonical transformation in the ghost sector. A new form of Dual-BRST symmetry can also be obtained by making a canonical transformation [Eq. (3.13)] in the sector of Lagrange multiplier and its momenta. Following the steps analogous to Sec III-B, we obtain the new form of dual-BRST symmetry in case of rigid rotor as

\[
\begin{align*}
\delta_d p_r &= -\dot{\bar{C}}, \\
\delta_d \lambda &= -\dot{C}, \\
\delta_d \theta &= 0, \\
\delta_d \rho &= 0, \\
\delta_d \rho &= 0, \\
\delta_d \bar{C} &= 0,
\end{align*}
\]

\[
\delta_d \rho = (r - a), \quad \delta_d \rho = \frac{2}{\xi} \delta_d p_r = -\frac{2}{\xi} \dot{\bar{C}}.
\]

Now \( p_\lambda \) is changing non-trivially in the above transformations. The transformations mentioned in Eq. (4.5) leave the action given in Eq. (3.10) invariant.
V. CONCLUSION

We study the BFV-BRST formulation in a very simple system, rigid rotor to demonstrate the techniques to obtain different forms of BRST symmetries. Dual-BRST symmetry is obtained by making a canonical transformation in the ghost sector of the effective action of rigid rotor. This is the local and covariant version of the kind of transformations considered by Lavelle and McMullan [9], Tang and Finkelstein [10] and Yang and Lee [11]. This implies BRST and dual-BRST are not independent symmetries in this case rather these are related through canonical transformation. On the other hand, a canonical transformation in the sector involving Lagrange multiplier and its momenta leads to a new form of BRST as well as dual-BRST symmetry. This simple technique can be applied to more complicated system to derive different form of BRST as well as dual-BRST symmetry which can simplify the renormalizable program.

Acknowledgment

We thankfully acknowledge the financial support from the Department of Science and Technology (DST), Government of India, under the SERC project sanction grant No. SR/S2/HEP-29/2007.

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