On the critical points of the energy functional on vector fields of a Riemannian manifold

Giovanni Nunes Jaime Ripoll

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Abstract

Given a compact Lie subgroup $G$ of the isometry group of a compact Riemannian manifold $M$ with a Riemannian connection $\nabla$, it is introduced a $G$–symmetrization process of a vector field of $M$ and it is proved that the critical points of the energy functional

$$F(X) := \frac{\int_{M} \|\nabla X\|^2\,dM}{\int_{M} \|X\|^2\,dM}$$

on the space of $G$–invariant vector fields are critical points of $F$ on the space of all vector fields of $M$, and that this inclusion may be strict in general. One proves that the infimum of $F$ on $S^3$ is not assumed by a $S^3$–invariant vector field. It is proved that the infimum of $F$ on a sphere $S^n$, $n \geq 2$, of radius $1/k$, is $k^2$, and is assumed by a vector field invariant by the isotropy subgroup of the isometry group of $S^n$ at any given point of $S^n$. It is proved that if $G$ is a compact Lie subgroup of the isometry group of a compact rank 1 symmetric space $M$ which leaves pointwise fixed a totally geodesic submanifold of dimension bigger than or equal to 1 then all the critical points of $F$ are assumed by a $G$–invariant vector field.

Finally, it is obtained a characterization of the spheres by proving that on a certain class of Riemannian compact manifolds $M$ that contains rotationally symmetric manifolds and rank 1 symmetric spaces, with positive Ricci curvature $\text{Ric}_M$, $F$ has the lower bound $\text{Ric}_M / (n - 1)$ among the $G$–invariant vector fields, where $G$ is the isotropy subgroup of the isometry group of $M$ at a point of $M$, and that his lower bound is attained if and only if $M$ is a sphere of radius $1/\sqrt{\text{Ric}_M}$.

1 Introduction

Let $M$ be a compact, orientable, $n$–dimensional, $n \geq 2$, $C^\infty$ manifold with a Riemannian metric and let $\nabla$ be the Riemannian connection of $M$. 
In this paper we study the critical points of the energy of $\nabla$ acting on the space $C^\infty(TM)$ of $C^\infty$ vector fields of $M$ with unit $L^2$ norm. Precisely, we study the critical points of the functional

$$F(X) = \int_M \|\nabla X\|^2 \, dM$$

on the space of vector fields $X \in C^\infty(TM)$ such that

$$\int_M \|X\|^2 \, dM = 1.$$

It is well known that the critical values of $F$ are the eigenvalues of the so-called rough Laplacian $-\text{div} \nabla$ of $M$ [11] and it follows from the spectral theory for linear elliptic operators that they constitute an increasing sequence $0 \leq \delta_1 < \delta_2 < \cdots \to +\infty$ (counted with multiplicity) which are assumed by $C^\infty$ eigenvector fields. Moreover, if $M$ has no parallel vector fields then the infimum $\delta_1$ of $F$ is positive.

The search of geometric estimates of the spectrum of elliptic linear operators is an active topic of investigation in Geometric Analysis, the Laplacian being already a classical and well-studied one [10]. As to the study of the rough Laplacian operator, it seems to the authors that no attention has been paid so far. We have not found in the literature a description of its eigenvalues even in the simplest Riemannian spaces as the spheres.

In this paper we study the critical points of $F$ on Riemannian compact manifolds admitting a nontrivial isometry group. A simple but fundamental idea here is to introduce a process of symmetrization of a vector field of $M$ by a compact Lie subgroup $G$ of the isometry group $\text{Iso}(M)$ of $M$. We then use this symmetrization to prove that, apart from some exceptional cases where the symmetrization process leads to zero vector fields, the critical points of $F$ on the space of $G$-invariant vector fields are in the spectrum of $F$ (Proposition 1). What is somewhat surprising is that this inclusion may be strict, even for the infimum: We prove that the infimum of $F$ in the unit sphere $\mathbb{S}^3$ is not realized by a $\mathbb{S}^3$-invariant vector field, considering $\mathbb{S}^3$ as a Lie subgroup of $O(4)$ (Remark 3); in other words, left invariant vector fields of $\mathbb{S}^3$ (with a bi-invariant metric) are critical points of $F$ but are not ones with least energy. This raises the problem of whether the infimum or more generally the spectrum of $F$ is assumed by $G$-invariant vector fields for a given compact Lie subgroup $G$ of the isometry group of $M$.

We prove that the orthogonal projection $V$ of a nonzero vector $v$ of $\mathbb{R}^{n+1}$ on $T\mathbb{S}^n(1/k)$, where $\mathbb{S}^n(1/k)$ is a sphere of radius $1/k$, realizes the
infimum of $F$ on $C^\infty (T\mathbb{S}^n(1/k))$ and that this infimum is $k^2$ (Theorem 4) (we note that $V$ is invariant by the isotropy subgroup of $O(n+1)$ that leaves fixed the point $v/(k|v|) \in \mathbb{S}^n(1/k)$). Moreover, we prove that if $X$ is a Killing field of $\mathbb{S}^n(1/k)$ then $F(X) = (n-1)k^2$. It then follows that $X$ minimizes the energy if and only if $n=2$. As a consequence, if $\mathcal{S}_n \subset C^\infty (T\mathbb{S}^n(1/k))$ denotes the space of eigenvector fields associated to $k^2$ then $\dim \mathcal{S}_2 \geq \dim O(3) + \dim \mathbb{R}^3 = 6$. By using Fourier series it is proved in [2] that actually $\dim \mathcal{S}_2 = 6$.

We also prove that if $M$ is a compact rank 1 symmetric space and if $G$ is a Lie subgroup of $\text{Iso}(M)$ which leaves pointwise fixed a totally geodesic submanifold of dimension bigger than or equal to 1 then all the critical points of $F$ are realized by a $G$–invariant vector fields (Theorem 9). We note that there are many subgroups $G$ of $\text{Iso}(M)$ satisfying this condition. For example, any totally geodesic $m$–dimensional submanifold of $\mathbb{S}^n(1/k)$, $1 \leq m \leq n-2$, is the fixed point of a compact subgroup of $O(n+1)$ isomorphic to $O(n-m)$. More generally, any rank 1 symmetric space is plenty of totally geodesic submanifolds all of them being the fixed points of some compact Lie subgroup of the isometry group of the space (see [4]). We also mention the groups of reflections on totally geodesic submanifolds, which are finite groups isomorphic to $\mathbb{Z}_2$.

In the last part of the paper we obtain a characterization of the sphere as a space where $F$ attains the infimum of the energy on a certain class of Riemannian manifolds as explained next. We shall say that $M$ is two point symmetric with center $p \in M$ if the isotropy subgroup $\text{Iso}_p(M)$ of $\text{Iso}(M)$ at $p$ is isomorphic to the isotropy subgroup of the isometry group of a rank 1 symmetric space, that is, $\text{Iso}_p(M)$ is isomorphic to some of the following Lie groups: $O(n)$, $U(1) \times U(n-1)$, $Sp(1) \times Sp(n-1)$ or $\text{Spin}(9)$ (see [4] and Definition 11 ahead). Note that when $\text{Iso}_p(M) = O(n)$ then $M$ is a rotationally symmetric space (see [3]). Symmetric spaces of rank 1 satisfy the so called two point homogeneous property and hence are also known as two point homogeneous spaces.

We prove that if $M$ is a two point symmetric space with center $p$ and with Ricci curvature $\text{Ric}_M$ satisfying $\text{Ric}_M \geq (n-1)k^2$, then the infimum of $F$ on the space of $\text{Iso}_p(M)$–invariant vector fields is bigger than or equal to $k^2$ and the equality holds if and only if $M$ is a sphere of radius $1/k$ (Theorem 12).
2 A general result

Let $M$ be a compact Riemannian manifold of dimension $n \geq 2$. We choose on the full isometry group $\text{Iso}(M)$ of $M$ a fixed left invariant Riemannian metric. We consider on any Lie subgroup of $\text{Iso}(M)$ the left invariant Riemannian metric induced by the one of $\text{Iso}(M)$.

Let $G$ be a compact Lie subgroup of $\text{Iso}(M)$. Given a vector field $V \in C^\infty(TM)$, the $G$–symmetrization of $V$ is the vector field $V_G$ defined by setting, at a given $p \in M$,

$$
(V_G(p), u) = \frac{1}{\text{Vol}(G)} \int_G \langle (dg_p)^{-1}V(g(p)), u \rangle \, dg
$$

where $u \in T_pM$. We note that $G$ may be finite case in which $V_G$ is given by

$$
V_G(p) = \frac{1}{|G|} \sum_{g \in G} dg_p^{-1}(V(g(p)),
$$

where $|G|$ denotes the number of elements of $G$. Note that $V_G \in C^\infty(TM)$ is $G$–invariant, that is, $V_G(g(p)) = dg_p(V_G(p))$ for all $p \in M$ and $g \in G$. Moreover, by the linearity of $\text{div} \nabla$ and of the integration process we have

$$(\text{div} \nabla V)_G = \text{div} \nabla V_G.$$

In particular, if $V$ satisfies $\text{div} \nabla V = -\lambda V$ then $\text{div} \nabla V_G = -\lambda V_G$. We call $V_G$ the $G$–mean of $V$.

We observe that depending on $M$ and $G$ it may happen that $V_G \equiv 0$. This happens with any vector field $V$ on a rank 1 compact symmetric space $M$ if $G$ is the full isometry group of isometries of $M$ (this is consequence of Lemma 7). We prove:

**Proposition 1** Let $M$ be a compact $n$–dimensional Riemannian manifold and $G$ a compact Lie subgroup of $\text{Iso}(M)$. Then the eigenvalues and eigenvectors of $F$ restrict to the subspace of $G$–invariant vector fields of $C^\infty(TM)$ are also eigenvalues and eigenvectors of $F$ on $C^\infty(TM)$.

We need the following result:

**Lemma 2** On the hypothesis of the Proposition 1 assume that $W \in C^\infty(TM)$ satisfies $W_G \equiv 0$. Let $V$ be a $G$–invariant vector field. Then

$$
\int_M \langle W(x), V(x) \rangle \, dx = 0.
$$
Proof. Since the elements of $G$ are isometries and $V$ is $G$–invariant we have, for all $g \in G$

$$
\int_M \langle W(x), V(x) \rangle \, dx = \int_M \langle W(g(x)), V(g(x)) \rangle \, dx \\
= \int_M \langle dg_x^{-1}(W(g(x))), dg_x^{-1}(V(g(x))) \rangle \, dx \\
= \int_M \langle dg_x^{-1}(W(g(x))), V(x) \rangle \, dx.
$$

It follows from Fubini’s theorem that

$$
\int_M \langle W(x), V(x) \rangle \, dx = \frac{1}{\text{Vol}(G)} \int_G \int_M \langle dg_x^{-1}(W(g(x))), V(x) \rangle \, dx \, dg \\
= \int_M \left( \frac{1}{\text{Vol}(G)} \int_G dg_x^{-1}(W(g(x))) \, dg, V(x) \right) \, dx \\
= \int_M \langle W_G(x), V(x) \rangle \, dx = 0.
$$

proving the lemma. ■

Proof of the Proposition[1] Let $X$ an eingenvector of $\text{div} \, \nabla$ on the space of the $G$-invariant vector fields associated to the eigenvalue $\lambda$. Then

$$
\int_M \langle -\text{div} \, \nabla X, W \rangle \, dx = \lambda \int_M \langle X, W \rangle \, dx,
$$

for all $G$-invariant vector field $W$. For proving that

$$
\int_M \langle -\text{div} \, \nabla X, V \rangle \, dx = \lambda \int_M \langle X, V \rangle \, dx,
$$

holds for any given vector field $V \in C^\infty(TM)$ we write $V = Z + V_G$ with $Z = V - V_G$ so that

$$
\int_M \langle -\text{div} \, \nabla X, V \rangle \, dx = \int_M \langle -\text{div} \, \nabla X, Z \rangle \, dx + \int_M \langle -\text{div} \, \nabla X, V_G \rangle \, dx.
$$

Noting that $Z$ has zero $G$–mean and that $\text{div} \, \nabla X$ is a $G$–invariant because $X$ is we have, by Lemma[2] that the first term of the hand side of the equality above is zero and then

$$
\int_M \langle -\text{div} \, \nabla X, V \rangle \, dx = \int_M \langle -\text{div} \, \nabla X, V_G \rangle \, dx = \lambda \int_M \langle X, V_G \rangle \, dx \\
= \lambda \int_M \langle X, V - Z \rangle \, dx = \lambda \int_M \langle X, V \rangle \, dx
$$

proving the proposition. ■
Remark 3 Considering a bi-invariant metric on the unit sphere \( S^3 \) with the Lie group structure, \( S^3 \) is a Lie subgroup of the isometry group \( O(4) \) of \( S^3 \). It follows from Proposition 1 that the \( S^3 \)-invariant vector fields are eigenvectors of the rough Laplacian of \( S^3 \). Clearly the \( S^3 \)-invariant vector fields are the left (and right) invariant vector fields of \( S^3 \). The orbits of a left invariant vector field constitute a Hopf fibration of \( S^3 \).

We have that the energy of a left invariant vector field is 2. Indeed, the Bochner-Yano formula for a vector field \( X \in C^\infty(TS^3) \), namely,

\[
\int_{S^3} |\nabla X|^2 \, dx = \int_{S^3} \left( \text{Ric}(X,X) + 2|\text{Kill}(X)|^2 - (\text{div} \, X)^2 \right) \, dx
\]

\[
= \int_{S^3} \left( 2|X|^2 + 2|\text{Kill}(X)|^2 - (\text{div} \, X)^2 \right) \, dx,
\]

where \( \text{Kill}(X) \) is the \((0,2)\)-tensor

\[
\text{Kill}(X)(U,V) = \frac{\langle \nabla_U X, V \rangle + \langle \nabla_V X, U \rangle}{2}, \quad U, V \in C^\infty(TS^3)
\]

(see [12]), when \( X \) is a Killing field, since \( \text{Kill}(X) \equiv 0 \) and \( \text{div} \, X \equiv 0 \), gives

\[
F(X) = \frac{\int_{S^3} |\nabla X|^2 \, dx}{\int_{S^3} |X|^2 \, dx} = 2.
\]

As we will see in the next section, the infimum of \( F \) on \( S^n \) is 1, for all \( n \). Thus, the \( S^3 \)-invariant vector fields do not realize the infimum of energy on \( S^3 \).

3 The infimum of the energy on a sphere

In this section we determine the infimum of \( F \) in the case that \( M \) is a sphere. Denoting by \( S^n(1/k) \) the sphere of radius \( 1/k \) in \( \mathbb{R}^{n+1} \) we prove:

Theorem 4 The infimum of \( F \) on \( C^\infty(TS^n(1/k)) \), \( n \geq 2 \), is \( k^2 \), and is assumed by the orthogonal projection on \( TS^n(1/k) \) of a constant nonzero vector field of \( \mathbb{R}^{n+1} \). Moreover, a Killing field of \( S^n(1/k) \) realizes the infimum \( k^2 \) of energy if and only if \( n = 2 \).

We need some preliminary lemmas.

Lemma 5 Let \( X \) be a vector field of \( \mathbb{R}^{n}(1/k) \) with zero \( G \)-mean \( (X_G \equiv 0) \), where \( G \) is the isotropy subgroup of \( \text{Iso}(\mathbb{R}^{n}(1/k)) \) that leaves fixed a point
$v \in \mathbb{S}^n(1/k)$. Then the function $f(p) := \langle X(p), v \rangle$, $p \in \mathbb{S}^n(1/k)$, has zero mean in $\mathbb{S}^n(1/k)$ that is,

$$\int_{\mathbb{S}^n(1/k)} f(x) dx = 0.$$ 

**Proof.** Using the formula of coarea to integrate $f$ on $\mathbb{S}^n(1/k)$ along the level sets of $h : \mathbb{S}^n(1/k) \to \mathbb{R}$, $h(p) = d(p, v)$, where $d$ is the distance in $\mathbb{S}^n(1/k)$, since $\|\text{grad } h\| = 1$, we have

$$\int_{\mathbb{S}^n(1/k)} f(x) dx = \int_{0}^{\pi} \left( \int_{h^{-1}(t)} f(x) \right) dt. \quad (1)$$

We note that $h^{-1}(t)$ is a geodesic sphere $\mathbb{S}^n_{t-1}$ of $\mathbb{S}^n(1/k)$ centered at $v$ and with radius $t$ and that these spheres are the orbits of $G$. Given $t \in (0, \pi/k)$, choose $p \in h^{-1}(t)$, and denote by $H$ the subgroup of isotropy of $G$ at $p$. Let

$$\psi : \frac{G}{H} \to \mathbb{S}^n_{t-1}$$

be given by $\psi(gH) = g(p)$. We note that, up to a factor multiplying the metric of $G$, $\psi$ is an isometry. Setting

$$\tilde{f} := f \circ \psi : \frac{G}{H} \to \mathbb{R}$$

we then obtain

$$\int_{h^{-1}(t)} f(x) dx = \int_{\frac{G}{H}} \tilde{f}(gH) d(gH). \quad (2)$$

Let

$$\phi : G \to \frac{G}{H},$$

be the projection of $G$ over $G/H$, and set $\overline{f} := \tilde{f} \circ \phi$. Using the coarea formula to integrate $\overline{f}$ on $G$ along the fibers of $\phi$, we obtain

$$\int_{g \in G} \overline{f}(g) dg = \int_{\frac{G}{H}} \left( \int_{gH \frac{\text{Jac } \phi}} \frac{1}{\|\text{Jac } \phi\|} \overline{f}(gh) d(gh) \right) d(gH)$$

$$= \int_{\frac{G}{H}} \left( \int_{gH} \overline{f}(gh) d(gh) \right) d(gH).$$

Since $\overline{f}$ is constant on $gH$, $g \in G$, we get

$$\int_{G} \overline{f}(g) dg = \text{Vol}(H) \int_{\frac{G}{H}} \tilde{f}(gH) d(gH).$$

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or by (2)
\[ \int_{h^{-1}(t)} f(x)dx = \frac{1}{\text{Vol}(H)} \int_G f(g)dg. \] (3)

But
\[ \int_G f(g)dg = \int_G (f \circ \psi)(gH) dg = \int_G f(g(p)) dg \]
\[ = \int_G \langle X(g(p)), v \rangle dg = \int_G \langle dg_p^{-1}X(g(p)), dg_p^{-1}v \rangle dg \]
\[ = \int_G \langle dg_p^{-1}X(g(p)), v \rangle dg = \langle X_G(p), v \rangle = 0. \]

Therefore from (3)
\[ \int_{h^{-1}(t)} f = 0, \]
for any \( t \in (0, \pi/k) \), and this, with (1), proves the lemma.

**Lemma 6** Let \( V \in C^\infty(TS^n(1/k)) \), \( n \geq 2 \), with
\[ \int_{S^n(1/k)} |V|^2 = 1 \]
be given. Let \( G \) be the isotropy subgroup of \( \text{Iso}(S^n(1/k)) \) at some point of \( S^n(1/k) \). If \( V \) is nonzero and has zero \( G \)-mean, then \( F(V) \geq (n-1)k^2 \).

**Proof.** Denote by \( \nabla \) the Riemannian connection of \( S^n(1/k) \). Write
\[ V = \sum_{l=1}^{n+1} q_l e_l, \]
where \( \{e_l\} \) is an orthonormal basis of the \( \mathbb{R}^{n+1} \). Fix \( p \in S^n(1/k) \) and let \( \{E_j\} \) be an orthonormal frame of \( S^n(1/k) \) on a neighborhood of \( p \). Then, for each \( i \) we have
\[ \nabla_{E_i}V = \sum_{l=1}^{n+1} E_i(q_l)e_l - \langle V, E_i \rangle kp \]
and thus
\[
\langle \nabla V, \nabla V \rangle = \sum_{i=1}^{n} \langle \nabla E_i V, \nabla E_i V \rangle \\
= \sum_{i=1}^{n} \left( - \langle V, E_i \rangle^2 k^2 + \sum_{l=1}^{n+1} \langle E_i(a_l) e_l, E_i(a_l) e_l \rangle \right) \\
= -k^2 |V|^2 + \sum_{l=1}^{n+1} |\text{grad}(a_l)|^2.
\]

We then have
\[
F(V) = -k^2 + \sum_{l=1}^{n+1} \int_{S^n(1/k)} |\text{grad}(a_l)|^2.
\]

By Lemma 5 the functions $a_l$ have zero mean on $S^n(1/k)$. Since the first positive eigenvalue of $S^n(1/k)$ is equal to $nk^2$, from Poincaré inequality we obtain
\[
F(V) \geq -k^2 + nk^2 \sum_{l=1}^{n+1} \int_{S^n(1/k)} |a_l|^2 \\
= -k^2 + nk^2 \int_{S^n(1/k)} |V|^2 = (n-1)k^2.
\]

Lemma 7 Assume that a Lie subgroup $H$ of $O(n) = \text{Isom}(S^n(1/k))$ acts transitively on $S^n(1/k)$. Then
\[
\int_H \langle hu, v \rangle \, dh = 0
\]
for any fixed vectors $u, v \in S^n(1/k)$.

Proof. Let $u, v \in S^n(1/k)$ be given. There is $g \in H$ such that $g(v) = -v$. Since the left translation $L_{g^{-1}} : H \to H$, $L_{g^{-1}}(h) = g^{-1}h$ is an orientation preserving isometry of $H$ we have
\[
\int_H \langle hu, v \rangle \, dh = \int_H \langle g^{-1}hu, v \rangle \, (L_{g^{-1}})^* \, dh = \int_H \langle hu, gv \rangle \, dh = -\int_H \langle hu, v \rangle \, dh
\]
which proves (4). \qed
The next lemma will also be used to prove Theorem 12. We make use of the following terminology: An orbit of highest dimension of a compact Lie group $G$ acting on a compact manifold $M$ is called a principal orbit (except to the exceptional orbits, see [5]). We say that $G$ acts with cohomogeneity one if the principal orbits of $G$ have codimension 1. The principal orbits of $G$ foliate a open dense subset of $M$ whose complementary has zero dimension ($M$).

**Lemma 8** Let $M^n$ be a compact, orientable Riemannian manifold. Let $G$ be a compact Lie subgroup of $\text{Iso}(M)$ acting with cohomogeneity one on $M$. Assume moreover that the subgroup of isotropy of $G$ at any point of a principal orbit of $G$ acts transitively (by the derivative) on the spheres centered at origin of the tangent space of the orbit at the point. Then any $G$–invariant vector field is orthogonal to the principal orbits of $G$.

**Proof.** Let $p \in M$ be such that $G(p)$ is a principal orbit of $G$ and let $v \in T_pG(p)$ be any fixed vector. Since $X$ is $G$–invariant we have

$$\langle X(p), v \rangle = \langle dg_p^{-1}(X(g(p))), v \rangle = \langle X(g(p)), dg_p v \rangle$$

for all $g \in G$, so that

$$\langle X(p), v \rangle = \frac{1}{\text{Vol}(G)} \int_G \langle X(g(p)), dg_p v \rangle dg.$$

Denoting by $H$ be the isotropy subgroup of $G$ at $p$ we have by coarea formula

$$\int_G \langle X(g(p)), dg_p v \rangle dg = \int_{G/H} \left( \int_{gH} \langle X((gh)(p)), d(gh)_p v \rangle d(gh) \right) d(gH).$$

Moreover

$$\int_{gH} \langle X((gh)(p)), d(gh)_p v \rangle d(gh) = \int_H \langle X(g(p)), d (gh)_p v \rangle dh$$

$$= \int_H \langle dg_p^{-1}X(g(p)), dh_p v \rangle dh = 0$$

by the previous lemma, since the action $h \mapsto dh_p$ of $H$ on $T_pG(p)$ is transitive on the spheres of $T_pG(p)$. Then

$$\langle X(p), v \rangle = \frac{1}{\text{Vol}(G)} \int_G \langle X(g(p)), dg_p v \rangle dg = 0.$$
This implies that \( X(p) \in (T_pG(p))^\perp \) since \( v \) is arbitrary, concluding with the proof of the lemma. ■

**Proof of Theorem 4.** A calculation shows that if \( X \in C^\infty(TS^n(1/k)) \) is the orthogonal projection on \( TS^n(1/k) \) of a vector \( v \in \mathbb{R}^{n+1} \) then \( \text{div} \nabla X = k^2 X \). We will prove that \( k^2 \) is the infimum of \( F \) and hence proving the theorem. We may assume that \( v \in S^n(1/k) \).

Let \( G \) be the isotropy subgroup of \( \text{Iso}(S^n(1/k)) \) at \( v \) and \( W \in C^\infty(TS^n(1/k)) \) assuming the infimum of \( F \). It follows from Lemma 6 that if \( W \) has zero \( G \)-mean, then \( F(W) \geq (n - 1)k^2 \) and the theorem is proved in this case. We may then assume that \( W_G \) is non zero. Since \( \text{div} W_G = (\text{div} W)_G \), \( F \) assumes its infimum at \( W_G \) too.

By the Lemma 6, \( W_G \) is orthogonal to the orbits of \( G \), which are geodesic spheres centered at \( v \). We then have \( W_G = \langle W_G, \text{grad} s \rangle \text{grad} s \), where \( s \) is the distance in \( S^n(1/k) \) to \( v \). Define \( h \in C^2([0, \pi/k]) \) by \( h(t) = \langle W_G, \text{grad} s \rangle(x) \), where \( x \in S^n(1/k) \) is such that \( t = s(x) \). Since \( W_G \) is \( G \)-invariant \( h \) is well defined and we have \( W_G = h(s) \text{grad} s \).

If \( \phi \in C^2([0, \pi/k]) \) is a primitive of \( h \) and \( f : S^n(1/k) \to \mathbb{R} \) is defined by \( f(x) = \phi(s(x)) \) then we have

\[
\text{grad} f = \phi' \text{grad} s = h \text{grad} s = W_G.
\]

Applying Reilly’s formula to \( f \) (see [9]) we obtain

\[
\int_{S^n(1/k)} (\Delta f)^2 \, dx = \int_{S^n(1/k)} \text{Ric}_{S^n(1/k)}(\text{grad} f, \text{grad} f) \, dx + \int_{S^n(1/k)} |\text{Hess} (f)|^2 \, dx = \int_{S^n(1/k)} \text{Ric}_{S^n(1/k)}(W_G, W_G) \, dx + \int_{S^n(1/k)} |\text{Hess} (f)|^2 \, dx,
\]

where \( \Delta f \) and \( |\text{Hess} (f)| \) are the Laplacian and the norm of the Hessian of \( f \). Note that \( |\text{Hess} (f)| = |\nabla W_G| \) and then, since \( (\Delta f)^2 \leq n |\text{Hess} (f)|^2 \), assuming that \( |W_G|_{L^2} = 1 \), we obtain

\[
(n - 1) \int_{S^n(1/k)} |\nabla W_G|^2 \, dx \geq \int_{S^n(1/k)} W_G^2 \text{Ric}_{S^n(1/k)}(W_G, W_G) \, dx \geq (n - 1)k^2.
\]

It follows that \( F(W_G) \geq k^2 \), proving the first part of the theorem. For the last part, it follows from Bochner-Yano formula of Remark 3 that if \( X \) is a
Killing field in $S^n(1/k)$ then $F(X) = (n-1)k^2$ so that $F(X) = k^2$ if and only if $n = 1$. ■

4 The critical points of the energy on a rank 1 compact symmetric space

In this section we prove:

**Theorem 9** Let $M$ be a rank 1 compact symmetric space. Let $G$ be a compact Lie subgroup of $\text{Iso}(M)$ that leaves pointwise fixed a totally geodesic submanifold of $M$ with dimension bigger than or equal to 1. Then all the critical points of the energy in $M$ are assumed by a $G$–invariant vector field.

If $V$ is a vector field on $M$ and if $h : M \to M$ is a diffeomorphism, we denote by $V^h$ the $h$–related vector field to $V$, that is, $V^h(p) = (dh_p)^{-1}(V(h(p)))$, $p \in M$. We use the following lemma.

**Lemma 10** Let $p \in M$, $v \in T_pM$ and $V \in C^\infty(TM)$ be given. Assume that $g(p) = p$ and $dg_p(v) = v$ for all $g \in G$. Given $q \in M$ assume that $h \in \text{Iso}(M)$ is such that $h(p) = q$ and $dh_q^{-1}(V(q)) = v$. Let $x_n \in M$ be a sequence converging to $p$. Then

$$V^h(p) = \lim_{n \to \infty} \frac{\int_G dg_z^{-1}(V^h(g(x_n))))}{\text{Vol}(G(x_n))},$$

where $\text{Vol}(G(x_n))$ is the $k$–dimensional Hausdorff measure of $G(x_n)$, $k = \dim G(x_n) \geq 0$.

**Proof.** We will prove that

$$\left\langle V^h(p), Z(p) \right\rangle = \left\langle \lim_{n \to \infty} \frac{\int_G dg_z^{-1}(V^h(g(x_n))))}{\text{Vol}(G(x_n))}, Z(p) \right\rangle, Z \in C^\infty(TM).$$

Given $Z \in C^\infty(TM)$ we have, for a given $n$,

$$\left\langle \frac{\int_G dg_z^{-1}(V^h(g(x_n))))}{\text{Vol}(G(x_n))}, Z(x_n) \right\rangle = \int_G \frac{dg_z^{-1}(V^h(g(x_n))))}{\text{Vol}(G(x_n))}, Z(x_n) \right\rangle.$$

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so that
\[ \inf_{g \in G} \left( \langle dg^{-1}(V^h(g(x_n))), Z(x_n) \rangle \right) \leq \frac{\int_{G} \langle dg^{-1}(V^h(g(x_n))), Z(x_n) \rangle \omega}{\text{Vol}(G(x_n))} \leq \sup_{g \in G} \left( \langle dg^{-1}(V^h(g(x_n))), Z(x_n) \rangle \right). \]

Letting \( n \to \infty \) we obtain
\[ \inf_{g \in G} \left( \langle dg^{-1}(V^h(g(p))), Z(p) \rangle \right) \leq \left( \lim_{n \to \infty} \frac{\int_{G} dg^{-1}(V^h(g(x_n))) dg}{\text{Vol}(G(x_n))} \right), \]
\[ \leq \sup_{g \in G} \left( \langle dg^{-1}(V^h(g(p))), Z(p) \rangle \right) \]
and, since \( g(p) = p \),
\[ \inf_{g \in G} \left( \langle dg^{-1}(V^h(p)), Z(p) \rangle \right) \leq \left( \lim_{n \to \infty} \frac{\int_{G} dg^{-1}(V^h(g(x_n))) dg}{\text{Vol}(G(x_n))} \right) \leq \sup_{g \in G} \left( \langle dg^{-1}(V^h(p)), Z(p) \rangle \right). \]

Since
\[ dg^{-1}(V^h(p)) = dg^{-1}(v) = v = V^h(p) \]
for all \( g \in G \) it follows that
\[ \langle V^h(p), Z(p) \rangle \leq \left( \lim_{n \to \infty} \frac{\int_{G} dg^{-1}(V^h(g(x_n))) dg}{\text{Vol}(G(x_n))} \right), \]
proving the lemma.

**Proof of Theorem 9.** Let \( V \in C^\infty(TM) \) be a critical point of \( F \) with unit \( L^2 \)-norm. Since \( V \) is non zero, there is \( q \in M \) such that \( V(q) \neq 0 \). Let \( G \) be a Lie subgroup of \( \text{Iso}(M) \) that leaves pointwise fixed a totally geodesic submanifold \( T \) of \( M \), \( \dim T \geq 1 \). Choose a \( p \in T \) and a nonzero vector \( v \in T_pT \). We have \( g(p) = p \) for all \( g \in G \) and we claim that \( dg_p(v) = v \) for all \( g \in G \) too. Up to a multiplication of \( v \) by a positive real number, we may assume that \( v \) is contained on a normal geodesic ball of \( T_pT \). Set \( w = dg_p(v) \). Then, since \( \exp_p v \in T \), where \( \exp_p \) it the Riemannian exponential, we have
\[ \exp_p v = g(\exp_p v) = \exp_{g(p)} dg_p(v) = \exp_p w \]
and then \( v = w \).

Since \( M \) is a compact rank 1 symmetric space, there is an isometry \( h \) of the \( M \) such that \( h(p) = q \) and \( dh_p^{-1}(V(q)) = v \). Let \( V^h \) be the vector field \( h \)-related to \( V \). We claim that \( V^h_G \) is not identically zero. Indeed, taking a sequence \( x_n \in M \) converging to \( p \), we have, by the Lemma 10

\[
V^h(p) = \lim_{n \to \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n))))}{\text{Vol}(G(x_n))} = \lim_{n \to \infty} \frac{V^h_G(x_n)}{\text{Vol}(G(x_n))}.
\]

Hence, if \( V^h_G \equiv 0 \) then \( V^h(p) = 0 \), a contradiction! This proves the theorem.

\[\Box\]

5 A characterization of spheres

Rotationally symmetric manifolds are well known and much used as models on comparison theorems on Geometric Analysis. We consider here a generalization of such manifolds which we call two point symmetric manifolds:

**Definition 11** We say that a Riemannian \( n \)-dimensional manifold \( M \), \( n \geq 2 \), is two point symmetric with center \( p \in M \) if the isotropy subgroup \( G := \text{Is}o_p(M) \) of the isometry group of \( M \) at \( p \) is isomorphic to the isotropy subgroup of the isometry group of a two point homogeneous space \( S \) at any point in \( S \).

We note that the above definition is a natural generalization of the way used in [3] to define rotationally symmetric spaces. We also observe that this definition is equivalent to the requirement that given points \( p_1, p_2, p_3, p_4 \in M \) that belong to a common geodesic sphere of \( M \) centered at \( p \), if \( d(p_1, p_2) = d(p_3, p_4) \), where \( d \) is the Riemannian distance in \( M \), then there is an isometry \( i \in G \) such that \( i(p_1) = p_3 \) and \( i(p_2) = p_4 \) (see [1], [4]). It is also known that this two point homogeneous characterization of \( M \) around \( p \) is equivalent to the isotropy of \( G \) at any point \( x \in M \) of a principal orbit \( G(x) \) of \( G \) acting transitively on the Euclidean spheres centered at the origin of \( T_xG(x) \) (4).

We also recall that a two point homogeneous space is isometric to a rank 1 symmetric space. Accordingly to the symmetric space classification, it follows that a Riemannian manifold \( M \) is two point homogenous with center \( p \in M \) if \( \text{Iso}_p(M) \) is isomorphic to one of the following Lie groups: \( O(n) \), \( U(1) \times U(n) \), \( Sp(1) \times Sp(n) \) or \( \text{Spin}(9) \) (see [4]). When \( \text{Iso}_p(M) = O(n) \) the space \( M \) is rotationally symmetric.
Theorem 12 Let $M^n$ be a compact two point symmetric space with center $p$, $n \geq 2$, with positive Ricci curvature $\text{Ric}_M$ and assume that $\text{Ric}_M \geq (n - 1)k^2$, $k > 0$. Set $G = \text{Iso}_p(M)$. Then the infimum of $F$ on the $G$--invariant vector fields is bigger than or equal to $k^2$ and the equality holds if and only if $M$ is a sphere of radius $1/k$.

**Proof.** Denote by $s : M \to \mathbb{R}$ the distance in $M$ to $p$. Note that the level sets of $s$ are geodesic spheres centered at $p$ and that the mean curvature and the norm of the second fundamental form of these geodesic spheres depend only on $s$. Set $l = \max s$.

Let $V \in C^\infty(TM)$ a $G$--invariant vector field such that $F(V)$ is the positive infimum of $F$ on the space of $G$--invariant vector fields. Since the subgroup of isotropy of $G$ at a point $p$ of a principal orbit of $G$ acts transitively (by the derivative) on the spheres centered at origin of $T_pG(p)$ it follows from Lemma 8 that $V$ may be written on the form $V = \langle V, \text{grad} s \rangle \text{grad}$. Define $h$, $f$ and $\phi$ as in Theorem 4. The same reasoning used in this theorem allows us to conclude that $F(V) \geq k^2$, proving the first part of the theorem. By Theorem 4 we know that if $M$ is a sphere of radius $1/k$ then $F(V) = k^2$. We now prove the converse. Thus, assume that $F(V) = k^2$. From the inequality obtained from Reilly’s formula in Theorem 4 we obtain

$$ (\Delta f)^2 = n |\text{Hess}(f)|^2. \quad (5) $$

Given $s \in (0, l)$ denote by $|B|(s)$ and $H(s)$ the norm of the second fundamental form and the mean curvature of the geodesic sphere at a distance $s$ of $p$, with respect to the unit normal vector pointing to the center $p$. Since $f = \phi \circ s$, straightforward calculations give

$$ (\Delta f)^2 = (\phi'')^2 - 2(n - 1) \phi'' \phi' H + (n - 1)^2 (\phi')^2 H^2 $$

and

$$ |\text{Hess}(f)|^2 = (\phi'')^2 + (\phi')^2 |B|^2. $$

Then (5) is equivalent to

$$ (\phi'' + H \phi')^2 = \left( \phi' \right)^2 \left[ -\frac{n |B|^2}{n - 1} + n H^2 \right] \quad (6) $$

which implies that

$$ - \frac{|B|^2}{n - 1} + H^2 \geq 0 \quad (7) $$

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since $\phi'$ cannot be identically zero (otherwise $V$ would be, which is not the case). But, denoting by $\lambda_i$ the principal curvatures of the geodesic spheres, we have

$$-\frac{|B|^2}{n-1} + H^2 = \frac{-\left(\sum_{i=1}^{n-1} \lambda_i^2\right)}{n-1} + \frac{\left(\sum_{j=1}^{n-1} \lambda_j\right)^2}{(n-1)^2}$$

$$= -\sum_{i,j=1, i<j}^{n} (\lambda_i - \lambda_j)^2 \leq 0$$

so that

$$-\frac{|B|^2}{n-1} + H^2 = 0,$$

and, from (6),

$$\phi'' + H\phi' = 0. \quad (8)$$

Since

$$V = \text{grad } f = (h \circ s) \text{ grad } s$$

and $V$ is an infimum of $F$ we have

$$-\text{div } \nabla (h(s) \text{ grad } s) = k^2 h(s) \text{ grad } s.$$

A calculation gives

$$\text{div } \nabla (h \text{ grad } s) = \left(h''(s) - (n-1) H(s) h'(s) - |B|^2(s) h(s)\right) \text{ grad } s$$

so that

$$h'' - (n-1) H h' - |B|^2 h = -k^2 h. \quad (9)$$

But from (8) we have

$$h'' - (n-1) H h' - |B|^2 h = h'' + (n-1) \left[H^2 - |B|^2 / (n-1)\right] h = h''$$

and hence

$$h''(s) = -k^2 h(s).$$

It follows that $h$ is of the form

$$h(s) = A \cos(sk) + B \sin(k s)$$
and, since $\phi' = h$,

$$\phi(s) = A_k \cos(sk) + B_k \sin(ks).$$

From $f(x) = \phi(s(x))$ we then obtain using (8)

$$\Delta f = \phi'' + \phi' \Delta s = \phi'' - (n-1)H\phi' = \phi'' + (n-1)\phi'' = n\phi'' = -nk^2\phi = -nk^2 f.$$

Hence $nk^2$ is an eigenvalue of the usual Laplacian. Under the hypothesis $\text{Ric}_M \geq (n-1)k^2$ this implies that $M$ is a sphere of radius $1/k$ (8), finishing the proof of the theorem. ■

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