On the Physics of a Cool Pion Gas

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At finite temperature, the Nambu-Goldstone bosons of a spontaneously broken chiral symmetry travel at a velocity \( v < 1 \). This effect first appears at order \( \sim T^4 \) in an expansion about low temperature, and can be related to the appearance of two distinct pion decay constants in a thermal bath. We discuss some consequences on the thermodynamics of a gas of massless pions.

1. Introduction

In these proceedings, we extend some previous work of ours \[1\]. The starting point is very simple. In the vacuum one invokes Lorentz invariance to define the pion decay constant, \( f_\pi \sim 93 \text{ MeV} \), by

\[
\langle 0 | A_\mu^a | \pi^b (P) \rangle = i f_\pi \delta^{ab} P_\mu ,
\]  

(1)

where \( A_\mu^a \) is the axial-vector current, and the pion has euclidean momentum \( P^\mu = (p^0, \vec{p}) \). At finite temperature, Lorentz invariance is lost and they are \emph{a priori} two distinct pion “decay constants”: the temporal component has one,

\[
\langle 0 | A_0^a | \pi^b (P) \rangle_T = i f_t^a \delta^{ab} p^0 ,
\]  

(2)

and, assuming O(3) invariance, the spatial part of the current has another,

\[
\langle 0 | A_i^a | \pi^b (P) \rangle_T = i f_s^a \delta^{ab} p^i .
\]  

(3)

This is familiar from nonrelativistic systems, such as discussed by Leutwyler \[2\]; in a similar context, this has been recognized by Kirchbach and Riska \[3\].

As in the vacuum, the pion mass shell is defined using current conservation,

\[
\partial_\mu \langle 0 | A^{\mu a} | \pi^b \rangle = 0 \longrightarrow f_t^a \omega^2 = f_s^a p^2 .
\]  

(4)

Then, quite trivially, \( f_t^a \neq f_s^a \) implies the velocity \( v \)

\[
v^2 = \text{Re}(f_s^a / f_t^a) < 1 .
\]  

(5)

\(^\dagger\)That \( v \leq 1 \) is required by causality. It is possible that in some background \( v > 1 \), but this is not what our work is about.
This, again, is familiar from other contexts, like the propagation of light in a medium – \( v < 1 \) corresponds to an index of refraction \( n > 1 \). But dealing with Nambu-Goldstone bosons has some non-trivial consequences. For instance, the \( f_\pi \)'s develop an imaginary part at finite temperature. From (4) one can conclude that the damping rate of massless pions vanishes at zero momentum \( \langle \pi \pi \rangle \), an expression of the Goldstone theorem.\(^3\)

It is easy to extend these considerations to include explicit symmetry breaking \( \langle \pi \pi \rangle \). For soft pions, \( p \ll f_\pi \), at low temperature, \( T \ll f_\pi \) – christened cool pions \( \langle \pi \pi \rangle \) – the dispersion relation is of the form

\[
\omega^2 = v^2 p^2 + m^2
\]

If \( v \neq 1 \), there is both a dynamic (position of the pole in the complex \( \omega \) plane, at \( p = 0 \)) and a static mass (position of the pole in the complex \( p \) plane, at \( \omega = 0 \)), the two being related by

\[
m_{dy} = v \cdot m_{st} \leq m_{st}
\]

Incidentally, one can define two Gell-Mann - Oakes - Renner relations:

\[
\text{Re} f_\pi^2 m_{dy}^2 = 2 m_q \langle \bar{q}q \rangle_T \tag{8}
\]

or

\[
\text{Re} f_\pi^4 m_{st}^2 = 2 m_q \langle \bar{q}q \rangle_T \tag{9}
\]

A pion dispersion relation like (6) has been particularly advocated by Shuryak \( \langle \pi \pi \rangle \) (see also Gale and Kapusta \( \langle \pi \pi \rangle \)), following a different line of thought. Note that \( v < 1 \) implies a flattening of the dispersion relation at finite temperature. Experimentally such an effect might produce an enhancement of dileptons from \( \pi \pi \) annihilations \( \langle \pi \pi \rangle \).

2. Quantitative Results

To leading order in a low temperature expansion \( \sim T^2/f_\pi^2 \), and in the chiral limit \( m_\pi = 0 \),

\[
f_\pi(T) = f_\pi (1 - T^2/12 f_\pi^2) \tag{10}
\]

This result was first obtained, in a different context, by Binetruy and Gaillard \( \langle \pi \pi \rangle \) and subsequently derived by Gasser and Leutwyler \( \langle \pi \pi \rangle \) using chiral perturbation theory (\( \chiPT \)). Here, (10) implies that \( f_\pi^4 = f_\pi^4 \) to leading order. This is actually a consequence of chiral symmetry,\(^4\) as made particularly clear by the derivation of Dey, Eletsky and Ioffe \( \langle \pi \pi \rangle \). Using Current Algebra and PCAC, they showed that

\[
\langle A_\mu^a A_\nu^b \rangle_T \sim \langle V_\mu^a V_\nu^b \rangle_T, \quad \langle A_\mu^a A_\nu^b \rangle_T + T^2/6 f_\pi^2 \langle V_\mu^a V_\nu^b \rangle_T.
\]

From pion pole dominance, one then extracts (10) from (11). Besides the mixing with the vector-vector correlator, what is remarkable is that Lorentz invariance is manifest – hence that \( v = 1 \) – to order \( T^2/f_\pi^2 \).

\(^{3}\)In a nonlinear \( \sigma \) model, \( \gamma \sim \rho T^4/f_\pi^4 \).

\(^{4}\)This has been checked recently by Bochkarev and Kapusta \( \langle \pi \pi \rangle \) by comparing the predictions of the \( O(N) \) linear and non-linear (using different parametrizations) \( \sigma \) models.
Thus, the effect discussed here can only appear at next-to-leading order $\sim T^4$. In $\chi$PT this implies computing to two-loop order. For the sake of the argument, in [1] we instead made use of a weakly coupled linear $\sigma$ model, i.e. with a light $\sigma$ particle: $m_\sigma^2 = 2\lambda\sigma^2$ and $f_\sigma \equiv \sigma$, so that $O(T^4/f_\sigma^2 m_\sigma^2)$ corrections dominate over the $O(T^4/f_\sigma^2)$ ones. Expanding in powers of $T/m_\sigma$, a one-loop calculation then suffices to verify (5). The result for the $f_\pi$’s is

\begin{align}
 f^t_\pi & \sim (1 - t_1 + 3t_2 + it_3)f_\pi \\
 f^s_\pi & \sim (1 - t_1 - 5t_2 - it_3)f_\pi
\end{align}

with

\begin{align}
 t_1 &= T^2/12f_\pi^2, \quad t_2 = \frac{\pi^2 T^4}{45 f_\pi^2 m_\sigma^2}, \quad t_3 = \frac{m_\sigma^4}{32\pi f_\pi^2 \omega^2} \exp(-m_\sigma^2/4\omega T)
\end{align}

so that

\begin{align}
 v^2 & \sim 1 - 8t_2
\end{align}

which agrees with the direct calculation of the dispersion relation, as first carried out by Itoyama and Mueller [8].

There is no calculation of $f_\pi$ to next-to-leading order in $\chi$PT, but that $v < 1$ in the chiral limit is implicit in [1] where Gerber and Leutwyler computed, among other things, the three-loop corrections to the energy density $u$ of a massless pion gas:

\begin{align}
 u = \frac{1}{10} \pi^2 T^4 \left[ 1 + \frac{T^4}{108 f_\pi^4} \left( 7 \ln \frac{\Lambda_p}{T} - 1 \right) + O(T^6) \right]
\end{align}

Apart from the pion decay constant, the energy density depends on another scale in the chiral limit, $\Lambda_p \sim 275$ MeV [1]. The first term in (16) is the energy density of a non-interacting massless pion gas,

\begin{align}
 u_0 = 3 \int \frac{d^3 p}{(2\pi)^3} \omega(p)n_B(p) = \frac{1}{10} \pi^2 T^4
\end{align}

with $\omega(p)^2 = p^2$. Substituting in (17) a modified pion dispersion relation,

\begin{align}
 \omega^2 = v^2 p^2,
\end{align}

can mimic the effect of the pion interactions. The energy density of a gas of free quasi-pions,

\begin{align}
 u = \frac{u_0}{v^\omega},
\end{align}

reproduces (16), provided the following estimate of the quasi-pion mean velocity holds:

\begin{align}
 v \sim 1 - \frac{1}{3} \frac{T^4}{108 f_\pi^4} \left( 7 \ln \frac{\Lambda_p}{T} - 1 \right)
\end{align}

In agreement with our previous argument, $v = 1$ to $O(T^2)$. It is equivalent to recognize that there is no $T^6$ correction to the energy density (16). Also, $v < 1$ for $T < \sim 250$ MeV, so that $u > u_0$.

\[\text{This is the argument that led Shuryak to [1]. [9]}\]
What about massive pions? The most extensive work on the propagation of thermal pions is due to Schenk [12, 13]. Apparently, he found no evidence of (6). However, in fig. 5 of ref. [12] and fig. 7 of ref. [13], Schenk plots $R(p)$, the ratio of the quasiparticle energy, to the pion energy in free space, as a function of momentum. To two loop order, as $p$ increases from zero there is a dip in $R(p)$: it first decreases, and then increases, approaching one from below. This is only possible if the quasiparticle energy
\[ \omega(p)^2 = v^2(p) p^2 + m^2_\pi(T), \]
with $v(0) < 1$. A. Schenk, private communication, estimates that $v(0) \sim 0.87$ at $T \sim 150$ MeV.

3. Outlook

We conclude with two remarks. The first concerns spin waves in ferromagnets (magnons) and was brought to our attention by R. Brout. The other is on the behavior of $f_\pi$ near the critical temperature.

In two landmark papers, Dyson [14], improving on earlier ideas of Bloch [15], developed a formalism to describe the motion of magnons, and computed the low temperature corrections to the magnetization. Taking magnon interactions into account, he found
\[ M(T)/M(0) \sim 1 - a_0 T^{3/2} - a_1 T^{5/2} - a_2 T^{7/2} - a_3 T^4 + O(T^{9/2}) \quad (21) \]
Given the magnon dispersion relation
\[ \omega \sim c p^2 + O(p^4), \]
the $T^{3/2}$ term is the famous prediction of the simple Bloch theory, in which magnons are treated as non-interacting bosonic particles. The $a_1$ and $a_2$ terms are lattice effects (the $O(p^4)$ terms in (22)), and do not concern us. Only the $a_3$ term $\sim T^4$ is due to magnon interactions. What is striking is that there is no $T^3$ term in (21); this is reminiscent of the absence of $T^6$ term in (16). In the quasi-particle picture, this means that the parameter $c$ in (22), is not renormalized to leading order ($T^{3/2}$ in the present case) but only to next-to-leading order, or $\sim T^{5/2}$. Also, in both cases – magnons and massless pions – the correction to the “velocity” is proportional to the energy density.

These are precisely the kind of similarities that an effective Lagrangian approach, like $\chi$PT, to the dynamical properties of spin waves, both in ferromagnets and antiferromagnets, could shed light on [2, 16].

Now about $f_\pi$ near $T_c$? In a recent paper, Jeon and Kapusta [17], computed $f_\pi$ to $O(T^2)$, both at low $T$ and near the critical temperature $T_c$, in an $O(N)$ non-linear $\sigma$ model, to next-to-leading order in a large $N$ expansion. Here, we give a sketch of a linear $\sigma$ model derivation of their results.*

The relevant diagrams are shown below. At low $T \ll m_\sigma$, $\sigma$ mode propagation is Boltzmann suppressed in diagrams (a) and (b). The latter shrinks to a tadpole and both diagrams contribute to order $\sim T^2$. The result, taking into account wave function renormalization, is
\[ f_\pi(T) = \sigma(1 - T^2/12\sigma^2), \quad (23) \]
*The Lagrangian can be found in [1].
to be compared with the low $T$ dependance of the order parameter

$$\sigma(T) = \sigma(1 - T^2/8\sigma^2)$$  \hspace{1cm} (24)$$

Near $T_c$, the $\sigma$ particle is light, $m_\sigma \ll T$, and, in the high $T$ expansion, (b) does not contribute nor is there wave function renormalization to $O(T^2)$; hence, only the tadpole (a) contributes and, manifestly,

$$f_\pi(T) = \sigma(T) = \sigma(1 - T^2/4\sigma^2)$$  \hspace{1cm} (25)$$

for $T$ near and below $T_c$. This agrees with Jeon and Kapusta [17]. That $f_\pi$ and $\sigma$ vanish at the same temperature is hardly surprising. An interesting question is whether the critical exponents of $f_\pi$ and $\sigma$ are the same also in the critical regime, i.e. beyond the mean-field result (25).

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