TITS GROUPS OF IWAHORI-WEYL GROUPS AND PRESENTATIONS OF
HECKE ALGEBRAS

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Abstract. Let $G$ be a connected reductive group over a non-archimedean local field $F$ and $I$ be an Iwahori subgroup of $G(F)$. Let $I_n$ be the $n$-th Moy-Prasad filtration subgroup of $I$. The purpose of this paper is to give some nice presentations of the Hecke algebra of connected, reductive groups with $I_n$-level structure; and to introduce the Tits group of the Iwahori-Weyl group of groups $G$ that split over an unramified extension of $F$.

The first main result of this paper is a presentation of the Hecke algebra $\mathcal{H}(G(F),I)$, generalizing the previous work of Iwahori-Matsumoto on the affine Hecke algebras. For split $GL_n$, Howe gave a refined presentation of the Hecke algebra $\mathcal{H}(G(F),I)$. To generalize such a refined presentation to other groups requires the existence of some nice lifting of the Iwahori-Weyl group $W$ to $G(F)$. The study of a certain nice lifting of $W$ is the second main motivation of this paper, which we discuss below.

In 1966, Tits introduced a certain subgroup of $G(k)$, which is an extension of $W$ by an elementary abelian 2-group. This group is called the Tits group and provides a nice lifting of the elements in the finite Weyl group. The “Tits group” $T$ for the Iwahori-Weyl group $W$ is a certain subgroup of $G(F)$, which is an extension of the Iwahori-Weyl group $W$ by an elementary abelian 2-group. The second main result of this paper is a construction of Tits group $T$ for $W$ when $G$ splits over an unramified extension of $F$. As a consequence, we generalize Howe’s presentation to such groups. We also show that when $G$ is ramified over $F$, such a group $T$ of $W$ may not exist.

1. Introduction

1.1. Presentations of Hecke algebras. Let $G$ be a connected reductive group over a non-archimedean local field $F$. Let $I$ be an Iwahori subgroup of $G(F)$ and $W$ be the Iwahori-Weyl group of $G(F)$. Then $G(F) = \cup_{w \in W} wI$. The group $W$ is a quasi-Coxeter group, namely, it is a semidirect product of an affine Weyl group $W_\infty$ with a group $\Omega$ of length-zero elements. The Iwahori-Hecke algebra $\mathcal{H}_0 = \mathcal{H}(G(F),I)$ is the $\mathbb{Z}$-algebra of the compactly supported, $I$-biinvariant functions on $G(F)$. The Iwahori-Matsumoto presentation of $\mathcal{H}_0$ reflects the quasi-Coxeter group structure of $W$: the generators of $\mathcal{H}_0$ are the characteristic functions $1_{IwI}$, where $w$ runs over elements in $W$, and the relations are given by multiplications of the characteristic functions via the condition on the length functions of $W$. See Theorem 4.1 for the precise statement.

The representations of $G(F)$ which are generated by the Iwahori-fixed vectors gives to the representations of the Iwahori-Hecke algebra $\mathcal{H}_0$. Let $n \in \mathbb{N}$ and $I_n$ be the $n$-th congruence subgroup of $I$. Let $\mathcal{H}_n = \mathcal{H}(G(F),I_n)$ be the $\mathbb{Z}$-algebra of the compactly supported, $I_n$-biinvariant functions on $G(F)$. It plays a role in the study of representations of $G(F)$ with deeper level structure.

One main purpose of this paper is to establish some nice presentations of $\mathcal{H}_n$. The first main result is the generalization of the Iwahori-Matsumoto presentation to $\mathcal{H}_n$: the generators are the characteristic functions on the $I_n$-double cosets on $G(F)$ and the multiplications of the characteristic functions are given via the conditions on the length function of $W$. We refer to Theorem 4.2 for the precise statement. As a consequence, we show that the algebra $\mathcal{H}_n$ is finitely generated.

In [13], Howe discovered a nice presentation of $\mathcal{H}_n$ when $G = GL_n(F)$. Here the generators are the characteristic functions $1_{gI_n}$ for $g \in I/I_n$ and $1_{I_n m(w) I_n}$, where $w$ runs over elements
of \( W \) of length 0 and 1, and \( m(w) \) is a nice representative of \( w \) in \( G(F) \). This presentation is a refinement of the Iwahori-Matsumoto presentation and has some nice applications to the representation theory of \( p \)-adic groups. Howe’s presentation was later generalized by the first-named author to split groups. We observed that such refined presentation requires the existence of the nice lifting of the Iwahori-Weyl group \( W \) to \( G(F) \). Such a lifting, which we introduce in §3, is motivated by Tits work on finite Weyl groups. We call such a lifting the Tits group of the Iwahori-Weyl group \( W \) and call the refined presentation of \( H_n \) the Howe-Tits presentation. In Theorem 4.7, we show that if the Tits group for \( W \) exists, then the algebra \( H_n \) admits the Howe-Tits presentation.

1.2. Tits groups of the finite Weyl groups and Iwahori-Weyl groups. Now we come to the second main purpose of this paper: the study of the Tits groups.

We first make a short digression and discuss Tits groups of finite Weyl groups. Let \( G \) be a connected reductive group split over a field \( \mathfrak{F} \) and \( W_0 \) be its absolute Weyl group. Tits in [23] introduced the Tits group \( \mathcal{T} \) of \( W_0 \). It is a subgroup of \( G(\mathfrak{F}) \), which is an extension of \( W_0 \) by \( T_2 \), where \( T_2 \) is the elementary abelian subgroup generated by \( \alpha^\vee(-1) \), where \( \alpha \) runs over all the roots in \( G \). Moreover, for any \( w \in W_0 \), there exists a nice lifting \( n_w \in \mathcal{T} \). These liftings have nice properties:

1. \( n_w^2 = \alpha^\vee(-1) \) for any simple root \( \alpha \).
2. The set \( \{n_s\} \) for simple reflections \( s \) satisfies the Coxeter relations, i.e., for any simple reflections \( s \) and \( s' \), we have

\[
n_s n_{s'} \cdots = n_{s'} n_s \cdots,
\]

where each side of the expression above has \( k(s, s') \) factors. Here \( k(s, s') \) is the order of \( ss' \).

We refer to the recent work of Reeder, Levy, Yu and Gross [17], Adams and the second-named author [3] and Rostami [19] for some further study of the elements \( n_w \) and its applications to supercuspidal representations of \( p \)-adic groups.

Now let us come back to the group \( G(F) \). Our second main result of this paper is the construction of a Tits group \( \mathcal{T} \) of the Iwahori-Weyl group of a connected, reductive group \( G \) that is \( \bar{F} \)-split. We establish in Theorem 6.4 that

Theorem 1.1. We have the short exact sequence

\[
1 \longrightarrow S_2 \longrightarrow \mathcal{T} \longrightarrow W \longrightarrow 1.
\]

Moreover, for any \( w \in W \), there exists a lifting \( n_w \in \mathcal{T} \) such that

- For any affine simple reflection \( s_a \), \( n_a^2 = b'(-1) \) where \( b \) is the gradient of \( a \).
- We have \( n_w = n_{s_{i_1}} n_{s_{i_2}} \cdots n_{s_{i_k}} n_\tau \) for any reduced expression \( w = s_{i_1} s_{i_2} \cdots s_{i_k} n_\tau \), where \( \tau \in \Omega \) and \( s_{i_1}, \ldots, s_{i_k} \) are simple reflections.

We refer to §3.3 for the definition of the elementary abelian 2-group \( S_2 \).

As a consequence, we have the Howe-Tits presentation of \( H_n \) for groups that are \( \bar{F} \)-split. It is also worth pointing out that for ramified groups, such a \( \mathcal{T} \) may not exist. We give an example in §5.2.

1.3. The difficulty and strategy. In this subsection, we describe the strategy that goes into the construction of the Tits group of the Iwahori-Weyl group \( W \) of \( G \) over \( F \).

The Tits group of the finite absolute Weyl group is constructed via a “pinning” of \( G(\mathfrak{F}) \). Roughly speaking, a pinning gives a collection of isomorphisms from additive group \( \mathbb{G}_a \) to the simple root subgroups of \( G \). Given a pinning, one may define the lifting of simple reflections \( n_a \) and check that the conditions (1) & (2) in §1.2 are satisfied. The Tits group of the finite Weyl group is generated by the \( n_a \) where \( s \) varies over the finite simple reflections.

When \( G_F \) is not quasi-split, the group need not admit a “pinning” analogous to the one discussed above, and hence there is no natural choice of representatives for the elements of the relative or affine Weyl group over \( F \).

We construct the Tits group of the Iwahori-Weyl group of \( G \) over \( F \) in two steps. We first construct the Tits group of Iwahori-Weyl group over \( \bar{F} \), where \( \bar{F} \) is the completion of the maximal unramified extension of \( F \) contained in a fixed separable closure of \( F \). Next, we “descend” this
construction down to $F$. The advantage of this approach is that the group $G_F$ is always quasi-split and admits a nice system of pinnings analogous to the one discussed in the preceding paragraph.

We now explain these two steps in more detail.

(1) Let $G$ be a connected, reductive group over $F$ such that $G_F$ is $\hat{F}$-split and let $T$ be a maximal $F$-torus in $G$ that is $\hat{F}$-split. Let $\mathfrak{a}$ be a $\sigma$-stable alcove in the apartment $\mathcal{A}(T, \hat{F})$ and let $\mathcal{S}$ be the set of affine simple reflections through the walls of $\mathfrak{a}$. To choose representatives of the elements of $\mathcal{S}$, we introduce an affine pinning: for each affine simple root $\check{a}$ with gradient $\check{b}$, this is a homomorphism $x_{\check{a}} : G_\check{a} \to U_\check{b}$ such that the image of $m(s_\check{a}) = x_{\check{a}}(1)x_{-\check{a}}(1)x_{\check{a}}(1)$ in the affine Weyl group is $s_\check{a}$. We then show that this set of representatives satisfy Coxeter relations and furthermore, $m(s_\check{a})^2 = \check{b}^\varphi(-1)$ for each affine simple reflection $\check{a}$. We show that the group generated by $\{\check{b}(\varphi) \mid \check{b} \in \Phi(G, T)\}$ yields a Tits group of the Iwahori-Weyl group over $\hat{F}$. We also include an example here of a wildly ramified unitary group over $\hat{F}$.

(2) We now explain the descent step. Let $\sigma$ denote the Frobenius morphism on $G_F$, such that the $F$-structure it yields is $G$. Let $\mathfrak{a} = \check{a}^\sigma$ and let $\mathcal{S}$ be the set of reflections through the walls of $\mathfrak{a}$. Then $\mathcal{S}$ generates the Coxeter group $W_{sf}$ and $W = W_{sf} \times \Omega_a$, where $\Omega_a$ is the stabilizer of the alcove $\mathfrak{a}$. By the work of Lusztig [16] it is known that the elements of $\mathcal{S}$ correspond to certain “nice” $\sigma$-orbits in $\mathcal{S}$. We construct an affine pinning over $\hat{F}$ such that the set of representatives $\{m(\check{s}) \mid \check{s} \in \mathcal{S}\}$ obtained using this pinning is $\sigma$-stable for each of these nice $\sigma$-orbits $\mathcal{S}$. This descent argument yields a set of representatives in $G(F)$ for the elements of $\mathcal{S}$ that satisfy Coxeter relations. This is done in §6.2.

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2. Preliminaries

2.1. Notation. Let $F$ be a non-archimedean local field with $\mathcal{O}_F$ its ring of integers, $p_F$ its maximal ideal, $\varpi_F$ a uniformizer, and $k = \mathbb{F}_q$ its residue field. Let $p$ be the characteristic of $k$. Let $\hat{F}$ be the completion of a separable closure of $F$. Let $\hat{F}$ be the completion of the maximal unramified subextension with valuation ring $\mathcal{O}_F$ and residue field $k$. Note that $\varpi_F$ is also a uniformizer of $\hat{F}$. Let $\Gamma = \text{Gal}(\hat{F}/F)$ and $\Gamma_0 = \text{Gal}(\hat{F}/\hat{F})$.

Let $G$ be a connected, reductive group over $F$. By Steinberg’s Theorem (see [22, Theorem 56]), $G_F$ is quasi-split. Let $\sigma$ denote the Frobenius action on $G(\hat{F})$ such that $G(F) = G(\hat{F})^\sigma$. Let $A$ be a maximal $F$-split torus of $G$ and $S$ be a maximal $\hat{F}$-split $F$-torus of $G$ containing $A$. Let $T = Z_G(S)$. Then $T$ is defined over $F$ and is a maximal $F$-torus of $G$ containing $S$. Let $\tilde{F}$ be the field of invariants of the kernel of the representation of $\Gamma_0$ on $X^*(T)$. This extension is Galois over $\hat{F}$. Hence $T$ and $G$ are split over $\tilde{F}$. By [21, Chapter V, §4, Proposition 7], there
exists a uniformizer \( \varpi_F \) of \( F \) with \( \mathrm{Nm}_{F/F}(\varpi_F) = \varpi_F \), where \( \mathrm{Nm}_{F/F} \) is the norm map. Fix one such.

Let \( \hat{\Phi}(G, T) \) be the set of roots of \( T_F \) in \( G_F \). Then the set of relative roots of \( S \) in \( G_F \), denoted by \( \hat{\Phi}(G, S) \), is the set of the restrictions of the elements in \( \hat{\Phi}(G, T) \) to \( S \). Let \( \hat{W}_0 \) denote the relative Weyl group of \( G \) with respect to \( S \) and let \( W(G, T) \) denote the absolute Weyl group of \( G \).

Let \( B(G, \hat{\Phi}) \) (resp. \( B(G, F) \)) denote the enlarged Bruhat-Tits building of \( G(\hat{\Phi}) \) (resp. \( G(F) \)). Then \( B(G, \hat{\Phi}) \) carries an action of \( \sigma \) and \( B(G, F) = B(G, \hat{\Phi})^\sigma \). Let \( \mathcal{A}(S, \hat{\Phi}) \) be the apartment in \( B(G, \hat{\Phi}) \) corresponding to \( S \). Let \( \hat{\alpha} \) be a \( \sigma \)-stable alcove in \( \mathcal{A}(S, \hat{\Phi}) \). Let \( \hat{v}_0 \) be a special vertex contained in the closure of \( \hat{\alpha} \). Set \( \alpha = \hat{\alpha}^\sigma \); this is an alcove in the apartment \( \mathcal{A}(A, F) \) (see [8, §5.1]).

Let \( \hat{\Phi}_{af}(G, S) \) denote the set of affine roots of \( G(\hat{\Phi}) \) relative to \( S \). Let \( V = X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \). The choice of \( \hat{v}_0 \) also allows us to identify \( \mathcal{A}(S, \hat{\Phi}) \) with \( V \) via \( \hat{v}_0 \mapsto 0 \in V \), which we now do. We then view \( \hat{\alpha} \subset V \). Let \( \hat{\Delta} \subset \hat{\Phi}_{af}(G, S) \) be the set of affine roots such that the corresponding vanishing hyperplanes form the walls of \( \hat{\alpha} \). The Weyl chamber in \( V \) that contains \( \hat{\alpha} \) then yields a set of simple roots for \( \hat{\Phi}(G, S) \) which we denote as \( \hat{\Delta}_0 \). Clearly \( \hat{\Delta}_0 \subset \hat{\Delta} \).

### 2.2. Iwahori-Weyl group over \( \hat{\Phi} \)

Let \( \hat{I} \) be the Iwahori subgroup associated to \( \hat{\alpha} \). Let \( \kappa_{\hat{I}, \hat{\Phi}} : T(\hat{\Phi}) \to X_*(T)_{\Gamma_0} \) denote the Kottwitz homomorphism. The map \( \kappa_{\hat{I}, \hat{\Phi}} \) is surjective and its kernel \( T(\hat{\Phi})_1 \) is the unique parahoric subgroup of \( T(\hat{\Phi}) \). By [14, §7.2], we have the following commutative diagram

\[
\begin{array}{ccc}
T(\hat{\Phi}) & \xrightarrow{\kappa_{\hat{I}, \hat{\Phi}}} & X_*(T) \\
\downarrow{\mathrm{Nm}_{F/F}} & & \downarrow{\mathrm{pr}} \\
T(\hat{\Phi}) & \xrightarrow{\kappa_{\hat{I}, \hat{\Phi}}} & X_*(T)_{\Gamma_0}.
\end{array}
\]

Let \( \hat{W} = N_G(S)(\hat{\Phi})/T(\hat{\Phi})_1 \) be the Iwahori-Weyl group of \( G(\hat{\Phi}) \) with length function \( \hat{l} \). This group fits into an exact sequence

\[ 1 \to X_*(T)_{\Gamma_0} \to \hat{W} \to \hat{W}_0 \to 1. \]

Recall that we have chosen a special vertex \( \hat{v}_0 \). With this, we have a semi-direct product decomposition

\[ \hat{W} \cong X_*(T)_{\Gamma_0} \rtimes \hat{W}_0. \]

Let \( \hat{\Delta} = \{ s_\hat{a} \mid \hat{a} \in \hat{\Delta} \} \) be the set of simple reflections with respect to the walls of \( \hat{\alpha} \). Let \( \hat{S}_0 = \{ s_\hat{a} \mid \hat{a} \in \hat{\Delta}_0 \} \). Let \( \hat{W}_{af} \subset \hat{W} \) be the Coxeter group generated by \( \hat{S} \). Let \( \hat{T}_{sc}, N_{sc} \) denote the inverse images of \( T \cap G_{sc} \), resp. \( N_G(S) \cap G_{sc} \) in \( G_{sc} \). Let \( S_{sc} \) denote the split component of \( T_{sc} \). Then \( \hat{W}_{af} \) may be identified with the Iwahori-Weyl group of \( G_{sc} \). It fits into the exact sequence

\[ 1 \to \hat{W}_{af} \to \hat{W} \to X^*(Z(\hat{G})^\Gamma_0) \to 1. \]

Let \( \Omega_{\hat{a}} \) be the stabilizer of \( \hat{a} \) in \( \hat{W} \). Then \( \hat{\Omega}_{\hat{a}} \) maps isomorphically to \( X^*(Z(\hat{G})^\Gamma_0) \) and we have a \( \sigma \)-equivariant semi-direct product decomposition

\[ \hat{W} \cong \hat{W}_{af} \rtimes \hat{\Omega}_{\hat{a}}. \]

Let \( \hat{l} \) be the length function on \( \hat{W} \). Then \( \hat{l}(s) = 1 \) for all \( s \in \hat{S} \) and \( \hat{\Omega}_{\hat{a}} \) is the set of elements of length 0 in \( \hat{W} \).

### 2.3. Iwahori-Weyl group over \( F \)

Let \( \hat{I} \) be the Iwahori subgroup of \( G(F) \) associated to \( \hat{\alpha} \). Then \( \hat{l} = \hat{I} \). Let \( M = Z_G(A) \) and \( M(F)_1 \) be the unique parahoric subgroup of \( M(F) \). We may identify \( M(F)_1 \) with the kernel of the Kottwitz homomorphism \( M(F) \to X^*(Z(M)^\Gamma_0) \). Let \( W = N_G(A)(F)/M(F)_1 \) denote the Iwahori-Weyl group of \( G(F) \) with length function \( \hat{l} \).

By [18, Lemma 1.6], we have a natural isomorphism \( W \cong \hat{W}_0^\sigma \). It is proved in [18, Proposition 1.11 & sublemma 1.12] that

\( \text{(a) for } w, w' \in W, \hat{l}(ww') = \hat{l}(w) + \hat{l}(w') \) if and only if \( \ell(ww') = \ell(w) + \ell(w') \).
The semi-direct product decomposition of $\tilde{W}$ in (2.3) is $\sigma$-equivariant and yields a decomposition
\[ W \cong \tilde{W}_a^X \rtimes \Omega_s. \]
Let $W_{af} = W_a^X$ and let $S$ be the set of reflections through the walls of $\mathfrak{a}$. Then $(W_{af}, S)$ is a Coxeter system. The group $\Omega_s$, which is the stabilizer of the alcove $\mathfrak{a}$, is isomorphic to $\Omega_s^X$ and is the set of length $0$ elements is $W$.

The simple reflections $S$ of $W_{af}$ are certain elements in $\tilde{W}_{af}$. The explicit description is as follows. For any $\sigma$-orbit $X$ of $\tilde{S}$, we denote by $W_X$ the parabolic subgroup of $W_{af}$ generated by the simple reflections in $X$. It is proved by Lusztig [16, Theorem A.8] that there exists a natural bijection $s \mapsto X$ from $S$ to the set of $\sigma$-orbits of $\tilde{S}$ with $W_X$ finite such that the element $s \in W_{af} \subset W_{af}$ equals to $\tilde{w}_X$.

2.4. Moy-Prasad filtration subgroups. Let $\tilde{I}$ be the Iwahori subgroup of $G(\tilde{F})$ associated to the alcove $\tilde{a}$. Recall that we have chosen a special point $\tilde{v}_0$ in $A(S, \tilde{F})$, using which we have identified $A(S, \tilde{F})$ with $V$. Let $(\phi_0)_{a \in \Phi(G, S)}$ be the corresponding valuation of root datum of $(T, (U_a)_{a \in \Phi(G, S)})$ (see [7, §6.2]). For $\tilde{v} \in \tilde{\mathfrak{a}}$, $\tilde{a} \in \Phi(G, S)$ and $r \in \mathbb{R}$, let $U_{\tilde{a}}(\tilde{F})_{\tilde{v}, r}$ denote the filtration of the root subgroup $U_{\tilde{a}}$ (see [8, §4.3 - §4.6]). More precisely,
\[ U_{\tilde{a}}(\tilde{F})_{\tilde{v}, r} = \{ u \in U_{\tilde{a}}(\tilde{F}) | (\tilde{a}, \tilde{v}) + \phi_0(u) \geq r \}. \]

The subgroup $U_{\tilde{a}}(\tilde{F})_{\tilde{v}, 0}$ does not depend on the choice of $\tilde{v} \in \tilde{\mathfrak{a}}$ and we may denote it as $U_{\tilde{a}}(\tilde{F})_{\tilde{a}, 0}$. Note that $\tilde{I}$ is generated by $T(\tilde{F})_{1}$ and $U_{\tilde{a}}(\tilde{F})_{\tilde{a}, 0, \tilde{a} \in \Phi(G, S)}$.

Let $I_n$ be the $n$-th Moy-Prasad filtration subgroup of $\tilde{I}$. In particular, for $n \geq 1$, $I_n$ is a normal subgroup of $\tilde{I}$. $\mathcal{T}^{NR}$ denote the Neron-Raynaud model of $T$, a group scheme of finite type over $\mathcal{O}_F$ with connected geometric fibers such that $\mathcal{T}^{NR}(\mathcal{O}_F) = T(\tilde{F})_1$. Let $\mathcal{T}_n = \text{Ker}(\mathcal{T}^{NR}(\mathcal{O}_F) \to \mathcal{T}^{NR}(\mathcal{O}_F/\mathcal{p}^n))$. Then $I_n$ is generated by $\mathcal{T}_n$ and $U_{\tilde{a}}(\tilde{F})_{\tilde{a}, n, \tilde{a} \in \Phi(G, S)}$, where $\tilde{v}_n$ is the barycenter of $\tilde{a}$.

Let $I_n = I_n^n$. Then for $n \geq 1$, $I_n$ is a normal subgroup of $I$.

2.5. The subgroup $\tilde{P}_s$. Let $s \in S$. Let $X$ be the $\sigma$-stable orbit in $\tilde{S}$ corresponding to $s$ (see §2.3). Let $P_s = \bigcup_{\tilde{a} \in W_s} \tilde{I} \tilde{a} \tilde{I} \supset I$ be the parahoric subgroup of $G(\tilde{F})$ associated to $X$. This is the parahoric subgroup attached to $\tilde{a}_s = \overline{\tilde{a}}^W_X$, where $\overline{\tilde{a}}$ is the closure of the alcove $\tilde{a}$. Then $P_s$ is generated by $T(\tilde{F})_1$ and $U_{\tilde{a}}(\tilde{F})_{\tilde{a}, 0, \tilde{a} \in \Phi(G, S)}$. Let $P_{s, n}$ be the $n$-th Moy-Prasad filtration subgroup of $P_s$. It is generated by $\mathcal{T}_n$ and $U_{\tilde{a}}(\tilde{F})_{\tilde{a}, n, \tilde{a} \in \Phi(G, S)}$ where $\tilde{v}_n$ is the barycenter of $\tilde{a}_s$.

2.6. The Hecke algebra $H(G(F), I_n)$. Let $H_n = H(G(F), I_n)$ be the Hecke algebra of compactly supported, $I_n$-biinvariant Z-valued functions on $G(F)$. Note that $I_0 = I$. The algebra $H_0$ is the Iwahori-Hecke algebra.

3. A Tits group associated to an Iwahori-Weyl group

3.1. Tits group associated to an absolute Weyl groups. In this subsection, we assume that $\tilde{S}$ is any field and $G$ is a reductive group split over $\tilde{S}$. Let $T$ denote a maximal $F$-split torus in $G$.

We follow [23]. For any root $a$, we denote by $a'$ the corresponding coroot. Let $S_2$ be the elementary abelian two-group generated by $\{ a'(-1) \}$ for all roots $a$. Associated to any pinning of $G$, we have the Tits group $\mathcal{T}_{\mathfrak{n}}$. This is a subgroup of $N_G(T)$, generated by $\{ n_a \}$, where $s$ runs over the simple reflections in the absolute Weyl group $W(G, T)$ and $n_a$ is a certain lift of $s$ to $N_G(T)$.

Below are some properties on the Tits group $\mathcal{T}_{\mathfrak{n}}$:

1. $n_a n_{a'} = a'(-1)$ for any simple root $a$.
2. The set $\{ n_a \}$ for simple reflections $s$ satisfies the Coxeter relations, i.e., for any simple reflections $s$ and $s'$, we have
\[ n_{s} n_{s'} \cdots = n_{s'} n_{s} \cdots. \]
where each side of the expression above has $k(s, s')$ factors. Here $k(s, s')$ is the order of $ss'$ in $W(G, T)$.

(3) The map $n_s \mapsto s$ induces a short exact sequence

$$1 \rightarrow T_2 \rightarrow \mathcal{T}_{\text{fin}} \rightarrow W(G, T) \rightarrow 1.$$ 

For any $w \in W(G, T)$, we may define $n_w = n_{s_1} \cdots n_{s_k} \in \mathcal{T}_{\text{fin}}$, where $s_1, \ldots, s_k$ is a reduced expression of $w$. As a consequence of (2), the definition of $n_w$ is independent of the choice of the reduced expression of $w$. We call the liftings $\{n_w\}_{w \in W(G, T)}$ a Tits cross-section of $W(G, T)$ in $\mathcal{T}_{\text{fin}}$.

### 3.2. A Tits group of Iwahori-Weyl group over $\dot{F}$

Motivated by the construction of the Tits group of the absolute Weyl group, we introduce the Tits groups of Iwahori-Weyl groups.

For each $\dot{b}$ in the relative root system $\Phi(G, S)$, we set

$$\dot{b}_a = \begin{cases} \dot{b}, & \text{if } \dot{b} \text{ is reduced;} \\ \dot{b}/2, & \text{otherwise.} \end{cases}$$

Note that any element $\dot{w} \in \dot{W}$ can be written as $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_n} \dot{\tau}$, where $\dot{s}_{i_1}, \ldots, \dot{s}_{i_n} \in \dot{S}$ and $\dot{\tau} \in \Omega_\dot{a}$. If $n = \ell(\dot{w})$, then we say that $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_n} \dot{\tau}$ is a reduced expression of $\dot{w}$ in $\dot{W}$.

**Definition 3.1.** Let $\dot{S}_2$ be the elementary abelian two-group generated by $\dot{b}^\prime (-1)$ for $\dot{b} \in \Phi(G, S)$. A Tits group of $\dot{W}$ is a subgroup $\dot{T}$ of $N_{\dot{G}}(S)(\dot{F})$ such that

1. The natural projection $\dot{\phi} : N_{\dot{G}}(S)(\dot{F}) \rightarrow \dot{W}$ induces a short exact sequence

$$1 \rightarrow \dot{S}_2 \rightarrow \dot{T} \rightarrow \dot{W} \rightarrow 1.$$ 

2. There exists a Tits cross-section $\{m(\dot{w})\}_{\dot{w} \in \dot{W}}$ of $\dot{W}$ in $\dot{T}$ such that

- (a) for $\dot{a} \in \dot{A}$, $m(\dot{s}_a) = \dot{b}_a(\dot{a})^{-1}$, where $\dot{b}_a$ is the gradient of $\dot{a}$.

- (b) for any reduced expression $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_n} \dot{\tau}$ in $\dot{W}$, we have $m(\dot{w}) = m(\dot{s}_{i_1}) \cdots m(\dot{s}_{i_n}) m(\dot{\tau})$.

It is easy to see that the condition (2) (b) in Definition 3.1 is equivalent to

Condition (2)(b)$^\prime$: $m(\dot{w}\dot{w}') = m(\dot{w})m(\dot{w}')$ for any $\dot{w} \in \dot{W}_{af}$ and $\dot{w}' \in \dot{W}$ with $\ell(\dot{w}\dot{w}') = \ell(\dot{w}) + \ell(\dot{w}')$.

Suppose that a Tits group $\dot{T}$ of $\dot{W}$ exists and $\dot{\phi} : \dot{T} \rightarrow \dot{W}$ is the projection map. Let $\dot{T}_{af} = \dot{\phi}^{-1}(\dot{W}_{af})$. This is the subgroup of $\dot{T}$ generated by $\dot{S}_2$ and $m(\dot{w})$ for $w \in \dot{W}_{af}$. We have the following commutative diagram

$$
\begin{array}{ccc}
1 & \rightarrow & \dot{S}_2 \\
\downarrow & & \downarrow \\
\dot{T}_{af} & \rightarrow & \dot{W}_{af} & \rightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \rightarrow & \dot{S}_2 & \rightarrow & \dot{T} & \rightarrow & \dot{W} & \rightarrow & 1.
\end{array}
$$

For $\dot{\tau} \in \Omega_\dot{a}$, any lifting $m'(\dot{\tau})$ of $\dot{\tau}$ in $G(\dot{F})$ lies in the normalizer of $\dot{I}$, where $\dot{I}$ is the Iwahori subgroup attached to the alcove $\dot{a}$. This is a special case of the fact that for $g \in G(\dot{F})$ and $x \in A(S, \dot{F}^*)$, $g \dot{P}_x g^{-1} = \dot{P}_{tx}$, where $\dot{P}_x$ is the parahoric subgroup attached to $x$.

Let us add some comments on $\dot{b}_a$. In this paper, we will construct the Tits group for connected reductive groups split over $\dot{F}$. For these groups, there is no difference between $\dot{b}_a$ and $\dot{b}$. We expect that Tits groups (in the Definition 3.1) exist for tamely ramified groups. Then one needs to use $\dot{b}_a$ instead of $\dot{b}$ in condition (2) (a), e.g., for the tamely ramified unitary groups.

### 3.3. Tits groups over $F$

The Tits group of $W$ is defined as follows.

Note that any element $w \in W$ can be written as $w = s_{i_1} \cdots s_{i_n} \tau$, where $s_{i_1}, \ldots, s_{i_n} \in S$ and $\tau \in \Omega_a$. If $n = \ell(w)$, then we say that $w = s_{i_1} \cdots s_{i_n} \tau$ is a reduced expression of $w$ in $W$.

**Definition 3.2.** Let $S_2 = \dot{S}_2^2$. A Tits group of $W$ is a subgroup $T$ of $N_G(A)(F)$ such that

1. The natural projection $\phi : N_G(A)(F) \rightarrow W$ induces a short exact sequence

$$1 \rightarrow S_2 \rightarrow T \rightarrow W \rightarrow 1.$$ 

2. There exists a Tits cross-section $\{m(w)\}_{w \in W}$ of $W$ in $T$ such that
(a) for $a \in \Delta$, $m(s_a)^2 = b^*(a^* - 1)$, where $b$ is the gradient of $a$.

(b) for any reduced expression $w = s_{i_1} \cdots s_{i_n}$ in $W$, we have $m(w) = m(s_{i_1}) \cdots m(s_{i_n}) m(\tau)$.

Note that in general, $S_2$ is larger than than the subgroup generated by $b^*(a^* - 1)$ for $b \in \Phi(G,A)$.

Suppose that a Tits group $T$ of $W$ exists and $\phi : T \to W$ is the projection map. Let $T_{af} = \phi^{-1}(W_{af})$. This is the subgroup of $T$ generated by $S_2$ and $m(w)$ for $w \in W$. We have the following commutative diagram

$$
\begin{array}{c}
1 \\ \\
\downarrow \\
1
\end{array}
\begin{array}{c}
\xrightarrow{\phi} \\
\xleftarrow{\phi} S_2
\end{array}
\xrightarrow{\phi} \xrightarrow{\phi} T_{af} \xrightarrow{\phi} W_{af} \xrightarrow{\phi} 1
$$

We would like to point out that unlike Tits groups of absolute Weyl groups for split reductive groups, the Tits groups of $\tilde{W}$ and $W$ may not exist in general. See §5.2.

4. Two presentations of the Hecke algebra $H(G(F), I_n)$

We first recall the Iwahori-Matsumoto presentation of the Iwahori-Hecke algebras:

**Theorem 4.1.** For $w \in W$, let $\dot{w}$ be any representative of $w$ in $G(\tilde{F})$. The Hecke algebra $H_0$ is a free module with basis $\{1 \in \tilde{\omega} \in W\}$ and the multiplication is given by the following formulas:

1. $\tilde{1}_{\tilde{w}} \tilde{1}_{\tilde{w}'}, = 1_{\tilde{w} = \tilde{w}'}$ if $\ell(ww') = \ell(w) + \ell(w')$.

2. $\tilde{1}_{\tilde{w}} \tilde{1}_{\tilde{w}'}, = (q^{\ell(s)} - 1) \tilde{1}_{\tilde{w}} + q^{\ell(s)} \tilde{1}_{\tilde{w}'},$ for $s \in S$ and $w, w' \in W$ with $sw < w$.

The first main result of this section is the following similar presentation for $H$ for $n \geq 1$. We call it the Iwahori-Matsumoto presentation for $H$.

**Theorem 4.2.** Let $n \geq 1$. The algebra $H_n$ is generated by $\mathbb{1}_{I_n g I_n}$ for $g \in G(F)$ subject to the following relations:

1. If $g$ and $g'$ are in the same $I_n \times I_n$-coset of $G(F)$, then $\mathbb{1}_{I_n g I_n} = \mathbb{1}_{I_n g' I_n}$.

2. If $\ell(\pi(gg')) = \ell(\pi(g)) + \ell(\pi(g'))$, then

$$
\mathbb{1}_{I_n g I_n} \mathbb{1}_{I_n g' I_n} = \mathbb{1}_{I_n g g' I_n}.
$$

3. If $\pi(g) = s \in S$, then

$$
\mathbb{1}_{I_n g I_n} \mathbb{1}_{I_n g' I_n} = \begin{cases} q^{\ell(s)} \mathbb{1}_{I_n g g' I_n}, & \text{if } \pi(gg') = \pi(g'), \\
q^{\ell(s)} \sum_{\sigma \in P_{s,n} / I_n} \mathbb{1}_{I_n gg' I_n}, & \text{if } \pi(gg') < \pi(g').
\end{cases}
$$

In the above, $P_{s,n} = P_{s,n}^r$ with $P_{s,n}$ as in §2.5. Note that if $sw < w$, then $(I_{sI})(I_{wI}) = I_{wI} \cup I_{swI}$. For any $g \in I_{sI}$ and $g' \in I_{wI}$, there are two possibilities: either $(I_n g I_n)(I_n g' I_n) \subset I_{wI}$ or $(I_n g I_n)(I_n g' I_n) \subset I_{swI}$. Thus there are two cases for the multiplication $\mathbb{1}_{I_n g I_n} \mathbb{1}_{I_n g' I_n}$.

4.1. Collection of some results from [12]. We define the map

$$
\pi : G(F) \to W, \quad g \mapsto w \quad \text{for } g \in I_{\dot{w}I}.
$$

It is proved in [12, Lemma 4.5 & Lemma 4.6] that for any $g \in G(F)$, $(I_n g I_n / I_n)^\pi = I_n g I_n / I_n$ and $\mathbb{1}_{I_n g I_n / I_n} = \mathbb{1}_{I_n g I_n / I_n}$.

Moreover, it is proved in [12, §4.4] that for $g, g' \in G(F)$ with $\ell(\pi(gg')) = \ell(\pi(g)) + \ell(\pi(g'))$, the multiplication map in $G(F)$ induces a bijection

$$
I_{n g I_n} \times I_{n g' I_n} \cong I_{n gg' I_n}.
$$

Here $I_{n g I_n} \times I_{n g' I_n}$ be the quotient of $I_{n g I_n} \times I_{n g' I_n}$ by the action of $I_n$ defined by $a \cdot (z, z') = (za^{-1}, az')$. In this case,

$$
\mathbb{1}_{I_n g I_n} \mathbb{1}_{I_n g' I_n} = \mathbb{1}_{I_n g g' I_n}.
$$

**Corollary 4.3.** Let $n \geq 1$. The algebra $H_n$ is generated by $\mathbb{1}_{I_n g I_n}$ for $g \in I$ and $\mathbb{1}_{I_n \dot{w} I_n}$ for $w \in W$ with $\ell(w) \leq 1$. In particular, $H_n$ is finitely generated.
Proof. Note that $\mathcal{H}_n$ is spanned by $\mathbb{I}_I g_I t_n$ for $g \in G(F)$.

Suppose that $g \in \mathbb{I}_I F$ and $w = s_i \cdots s_i \tau$ for $s_i, \ldots, s_i \in S$ and $\tau \in W$ with $\ell(\tau) = 0$. Then $g = g_1 \cdots g_n g'$, with $g_i \in I(s_i) t_I$ for $1 \leq i \leq n$ and $g' \in I(\tau) t_I$. Then $\mathbb{I}_I g_I t_n = \mathbb{I}_I g_I t_n^* \ast \cdots \ast \mathbb{I}_I g_n g_I t_n$. For any $s \in S$ and $g \in I(s)$, we have $g = i_1 \tilde{s}_I 2$ for some $i_1, i_2 \in I$. Then $\mathbb{I}_I g_I t_n = \mathbb{I}_I i_1 I_n^* \ast \mathbb{I}_I i_2 I_n^* \ast \mathbb{I}_I i_3 I_n^*$.

For any $\tau \in W$ with $\ell(\tau) = 0$ and $g \in I(\tau) t_I$, we have $g = i \tau r$ for some $i \in I$. Then $\mathbb{I}_I g_I t_{I_n^*} = \mathbb{I}_I i \tau r I_n^* \ast \mathbb{I}_I i \tau r I_n^*.

Note that $\Omega_{\infty}$ is finitely generated. Let $\{\tau_1, \ldots, \tau_l\}$ be a generating set of $\Omega_{\infty}$. Then $\mathcal{H}_{n}$ is generated by $\mathbb{I}_I g_I t_n$ for $g \in I$, $\mathbb{I}_I i \tau r I_n^*$, for $s \in S$ and $\mathbb{I}_I i \tau r I_n^*$ for $1 \leq i \leq l$. Thus $\mathcal{H}_{n}$ is finitely generated.  

4.2. The subgroup $\hat{\mathcal{P}}_{s,n}$. Let $\hat{\mathcal{P}}_{s,n}$ be as in §2.5.

Lemma 4.4. For $n \geq 1$, $\hat{\mathcal{P}}_{s,n}$ is a normal subgroup of $\hat{I}$.

Proof. Note that $\hat{\mathcal{P}}_{s,n}$ is a normal subgroup of the parahoric subgroup $\hat{\mathcal{P}}_s$. Since $\hat{I} \subset \hat{\mathcal{P}}_s$, we have that $\hat{\mathcal{P}}_{s,n}$ is stable under the conjugation action of $\hat{I}$.

It remains to show that $\hat{\mathcal{P}}_{s,n} \subset \hat{\mathcal{P}}_s$. Recall that $\tilde{v} \in I_n$ is the barycenter of the facet $\mathfrak{a}_s$ in the closure of the base alcove $\mathfrak{a}$. Let $\Phi^+(G, S)$ be the set of positive roots in $\Phi(G, S)$. Then, using [8, §4.2.22], it follows that $0 \leq (\tilde{a}, \tilde{v}_i) \leq 1$ for any $a \in \Phi^+(G, S)$.

By definition, $\hat{\mathcal{P}}_{s,n}$ is generated by $\mathcal{T}_n$ and $U_0(\hat{F})\tilde{v}_n$. Let $u \in U_0(\hat{F})\tilde{v}_n$. If $\tilde{a} \in \Phi^+(G, S)$, then the condition $\langle \tilde{a}, \tilde{v}_i \rangle + \phi(u) \geq 0$ implies that $\phi(u) \geq n - 1 \geq 0$. If $-a \in \Phi^+(G, S)$, then the condition $\langle \tilde{a}, \tilde{v}_i \rangle + \phi(u) \geq 0$ implies that $\phi(u) \geq n - 1$. In both cases, $U_0(\hat{F})\tilde{v}_n \subset \hat{I}$. Therefore $\hat{\mathcal{P}}_{s,n} \subset \hat{I}$.

Lemma 4.5. Let $g \in G$ with $\pi(g) = s \in S$. Set $\hat{\mathcal{P}}_{s,n} = \hat{\mathcal{P}}_{s,n}^*$. Then for $n \geq 1$, $g I_n g^{-1} = I_n g I_n g^{-1} = P_{s,n} g I_n g^{-1} = P_{s,n}$ and it is a normal subgroup of $\hat{I}$.

Proof. For $n \geq 1$, $\tilde{I}_n$ is stable under the conjugation action of $\tilde{s}_I \tilde{I}_n = I_n \tilde{s}_I \tilde{I}_n \subset \hat{I}$. Therefore $\tilde{I}_n (\tilde{s}_I \tilde{I}_n)^{-1} = (\tilde{s}_I \tilde{I}_n)^{-1} I_n = I_n \tilde{s}_I \tilde{I}_n^{-1}$.

On the other hand, $\mathcal{T}_n \subset \tilde{I}_n$. Let $\tilde{a} \in \hat{\Phi}(G, S)$ and let $u \in U_0(\hat{F})\tilde{v}_n$. So $\langle \tilde{a}, \tilde{v}_i \rangle + \phi(u) \geq 0$. We claim that $u \in \tilde{I}_n \subset \tilde{s}_I \tilde{I}_n$. Suppose $u \notin \tilde{I}_n$. Then $\langle \tilde{a}, \tilde{v}_i \rangle + \phi(u) \geq n - 1 \geq 0$, so $\langle \tilde{a}, \tilde{v}_i \rangle + \phi(u) \geq 0$. If $\phi(u) \geq n - 1$, then $\tilde{a} + \phi(u) - n$ belongs to $\Phi_{\infty}$, from the set of affine roots that vanish at $\hat{v}_n$. Write $\Phi_{\infty} = (\Phi_{\infty} \cap \Phi^+_{\infty}(G, S)) \cup (\Phi_{\infty} \cap \Phi^-_{\infty}(G, S))$. Then, since $\mathcal{W}$ is the Weyl group of $\Phi_{\infty}$ and $s$ is the longest element in $\mathcal{W}$, we follow that $\tilde{a} + \phi(u) - n = s(\tilde{a}) + \phi(u) - n$ is a positive affine root. Then $u \in \tilde{s}_I \tilde{I}_n$. Therefore $\tilde{s}_I \tilde{I}_n = \tilde{s}_I \tilde{I}_n \tilde{s}_I \tilde{I}_n^{-1} = (\tilde{s}_I \tilde{I}_n \tilde{s}_I \tilde{I}_n^{-1}) I_n$. We have $g I_n g^{-1} = (i \tilde{s}_I \tilde{I}_n i \tilde{s}_I \tilde{I}_n^{-1})^{-1} = (i \tilde{s}_I \tilde{I}_n i \tilde{s}_I \tilde{I}_n^{-1})^{-1}$ for some $i \in \hat{I}$. Thus $g I_n g^{-1} I_n = i \tilde{s}_I \tilde{I}_n i \tilde{s}_I \tilde{I}_n^{(-1)} I_n = i \tilde{s}_I \tilde{I}_n i \tilde{s}_I \tilde{I}_n^{(-1)} I_n^{-1} = i \tilde{s}_I \tilde{I}_n i \tilde{s}_I \tilde{I}_n^{(-1)}$. Thus $\hat{\mathcal{P}}_{s,n} = \tilde{s}_I \tilde{I}_n \tilde{s}_I \tilde{I}_n I_n^\sigma = g I_n g^{-1} I_n / I_n \sigma = g I_n g^{-1} I_n / I_n$. Here the last equality follows from §4.1.

Thus $g I_n g^{-1} I_n = \hat{\mathcal{P}}_{s,n} = \hat{\mathcal{P}}_{s,n}$. This is a normal subgroup of $I$ since $\hat{\mathcal{P}}_{s,n}$ is a normal subgroup of $I$. As $P_{s,n} \subset \hat{I}$ and $I_n$ is a normal subgroup of $I$, we also have $(g I_n g^{-1}) I_n = I_n (g I_n g^{-1})$.  

Proposition 4.6. Let $n \geq 1$. Let $g, g' \in G$ with $\pi(g) = \pi(g') = s \in S$. The multiplication map on $G$ induces a surjective map $I_n g I_n \times I_n g' I_n \rightarrow P_{s,n} g g' I_n$.

Moreover, each fiber contains exactly $q^d(s)$ elements.

Proof. By Lemma 4.5, $I_n g I_n g' I_n = (I_n g I_n g^{-1}) g' I_n = P_{s,n} g g' I_n$. If $\pi(g') = s$, then $P_{s,n} = I_n (g') I_n (g')^{-1}$ and $P_{s,n} g g' I_n = I_n (g') I_n (g')^{-1} (g') I_n = I_n g g' I_n$. 

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By §4.1, \( I_n g I_n / I_n = q^{\ell(s)} \). Since the map \( I_n g I_n \times I_n I_n g' I_n / I_n \rightarrow P_{s,n} g g' I_n / I_n \) is equivariant under the left action of \( I_n \) and \( I_n \) acts transitively on \( P_{s,n} g g' I_n / I_n \), all the fibers have the same cardinality and the cardinality equals to

\[
\| (I_n g I_n \times I_n I_n g' I_n / I_n) / (P_{s,n} g g' I_n / I_n) \| = q^{2\ell(s)} / q^{2\ell(s)} = q^{\ell(s)}.
\]

If \( \pi(g') = 1 \), then \( g' = g^{-1} i \) for some \( i \in I \) and we have the following commutative diagram

\[
\begin{array}{ccc}
I_n g I_n \times I_n I_n g' I_n & \rightarrow & P_{s,n} g g' I_n \\
\downarrow & & \downarrow \\
I_n g I_n \times I_n I_n g^{-1} i & \rightarrow & P_{s,n} i.
\end{array}
\]

Thus it suffices to consider the case where \( g' = g^{-1} \). Since \( n \geq 1 \), by Lemma 4.4 and Lemma 4.5, \( g I_n g^{-1} \subset P_{s,n} \subset I \). Since \( I_n \) is a normal subgroup of \( I \), \( I_n \) is stable under the conjugation action of \( g I_n g^{-1} \). Thus \( g^{-1} I_n g \) is stable under the conjugation action of \( I_n \). Conjugation by \( g^{-1} \), we have that \( g^{-1} I_n g \cap I_n \) is a normal subgroup of \( I_n \). Thus for any \( p \in P_{s,n} \), the inverse image of \( I_n p I_n \) in \( I_n g I_n \times I_n I_n g' I_n \) equals to

\[
\{ (I_n g a, b g^{-1} I_n) \mid a, b \in I_n / (g^{-1} I_n g \cap I_n), g a b g^{-1} \in I_n p I_n \} / I_n.
\]

Let \( p' \in P_{s,n} \). Then since \( P_{s,n} = (g I_n g^{-1}) I_n \), we have \( p' I_n = (g I_n g^{-1}) I_n \) for some \( i \in I_n \). Note that \( I_n \) is stable under the conjugation action of \( g I_n g^{-1} \). Thus \( (g I_n g^{-1}) I_n p I_n = I_n (g I_n g^{-1}) I_n p I_n \) and the inverse image of \( I_n p I_n \) in \( I_n g I_n \times I_n I_n g' I_n \) equals to

\[
\{ (I_n g a, b g^{-1} I_n) \mid a, b \in I_n / (g^{-1} I_n g \cap I_n), g a b g^{-1} \in I_n p I_n \} / I_n.
\]

In particular all the fibers have the same cardinality. So the cardinality of each fiber equals to

\[
\| (I_n g I_n \times I_n I_n g' I_n / I_n) / (P_{s,n} / I_n) \| = q^{2\ell(s)} / q^{2\ell(s)} = q^{\ell(s)}.
\]

The statement is proved. \( \square \)

4.3. Proof of Theorem 4.2. We choose a representative \( g \) for each \( I_n \times I_n \)-orbit on \( G(F) \). We denote the set of representatives by \( \mathcal{Y} \). Then the set \( \{ 1_{I_n g I_n} : g \in \mathcal{Y} \} \) is a basis of \( \mathcal{H}_n \) as a free \( \mathbb{Z} \)-module. In particular, the set \( \{ 1_{I_n g' I_n} : g \in \mathcal{Y} \} \) generates \( \mathcal{H}_n \) as an algebra.

For any \( g_1, g_2 \in \mathcal{Y} \), we have

\[
1_{I_n g_1 I_n} * 1_{I_n g_2 I_n} = \sum_{g_{1,2} \in \mathcal{Y}} c_{g_1, g_2, g_{1,2}} 1_{I_n g_{1,2} I_n} \in \mathcal{H}_n
\]

for some \( c_{g_1, g_2, g_{1,2}} \in \mathbb{Z} \). We denote this relation by \( \ast_{g_1, g_2} \). It is tautological that the equalities \( \ast_{g_1, g_2} \) for \( g_1, g_2 \in \mathcal{Y} \) form a set of relations for the algebra \( \mathcal{H}_n \).

By Corollary 4.3, the algebra \( \mathcal{H}_n \) is generated by \( 1_{I_n g I_n} \) for \( g \in \mathcal{Y} \) with \( \ell(\pi(g)) \leq 1 \). Thus the equalities \( \ast_{g_1, g_2} \) for \( g_1, g_2 \in \mathcal{Y} \) with \( \ell(\pi(g_1)) \leq 1 \) form a set of relations for the algebra \( \mathcal{H}_n \).

If \( \ell(\pi(g_1)) = 0 \), then for any \( g_2 \in \mathcal{Y} \), \( \ell(\pi(g_1 g_2)) = \ell(\pi(g_2)) \). Thus the equality \( \ast_{g_1, g_2} \) is obtained from the relations \( (0) \) and \( (1) \) in Theorem 4.2.

If \( \ell(\pi(g_1)) = 1 \), then \( \pi(g_1) = s \) for some \( s \in \mathcal{S} \). Let \( w = \pi(g_2) \). If \( sw > w \), then the equality \( \ast_{g_1, g_2} \) is obtained from the relations \( (0) \) and \( (1) \) in Theorem 4.2. If \( sw < w \), then \( \pi(g_1 g_2) = \pi(g_1) \pi(g_2) < \pi(g_2) \). In either case, the equality \( \ast_{g_1, g_2} \) is obtained from the relations \( (0) \) and \( (2) \) in Theorem 4.2.

Theorem 4.2 is proved.

4.4. The Howe-Tits presentation of \( \mathcal{H}_n \). In [13], Howe discovered a nice presentation for the Hecke algebra \( \mathcal{H}_n \). This presentation was later generalized to split groups by the first-named author in [10]. This nice presentation of \( \mathcal{H}_n \) has found applications in the representation theory of \( p \)-adic groups. For instance, this presentation was used to establish a variant of a Hecke algebra isomorphism of Kazhdan for sufficiently close local fields, which in turn was used to study the local Langlands correspondence for connected reductive groups in characteristic \( p \) with an understanding of the local Langlands correspondence of such groups in characteristic 0 (see [5, 15, 10, 2, 11]).

Before stating the theorem, we first introduce some structure constants.
For $\tau, \tau' \in \Omega_\varphi$ and $s \in \mathbb{S}$, let
\[
c_{\tau, \tau'} = m(\tau)m(\tau')m(\tau\tau')^{-1}
\]
\[
c_{\tau, s} = m(\tau)m(s)m(\tau s\tau^{-1})^{-1}.
\] (4.1)

Recall that the Tits’ axiom (T3) (see [7, §1.2.6]) says that $m(s)I m(s)^{-1} \subset I \cup I m(s)I$. In particular, if $g \in I$ but $g \not\in I \cap m(s)I m(s)^{-1}$, this axiom implies that $m(s)gm(s)^{-1} \in I m(s)I$. Hence there exist $g_1, g_2$ in $I$ such that $m(s)gm(s)^{-1} = g_1m(s)g_2$. We have the following theorem.

**Theorem 4.7.** Let $T$ be a Tits group of $W$ and $\{m(w)\}_{w \in W}$ is a Tits cross-section of $W$ in $T$.

The Hecke algebra $\mathcal{H}_n$ has generators

1. $\mathbb{I}_{m(s)I_n}, s \in \mathbb{S}$,
2. $\mathbb{I}_{m(\tau)I_n}, \tau \in \Omega_\varphi$,
3. $\mathbb{I}_{I_n, g_1I_n}, g \in I$,

subject to the following relations:

(A) (i) For $s, s'$ distinct elements of $\mathbb{S}$ with $s \cdot s'$ of order $k(s, s')$,
\[
\mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(s')I_n} \cdots \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(s')I_n} \cdots = \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(s')I_n} \cdots
\]

(ii) For $s \in \mathbb{S}$, $\mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(s^2)I_n} = q(s) \sum_{x \in \mathbb{P}_n, I_n} \mathbb{I}_{I_n, xI_n}$.

(B) (i) For $\tau, \tau' \in \Omega_\varphi$, $\mathbb{I}_{I_n, m(\tau)I_n} \star \mathbb{I}_{I_n, m(\tau')I_n} = \mathbb{I}_{I_n, c_{\tau, \tau'}I_n} \star \mathbb{I}_{I_n, m(\tau')I_n}$.

(ii) For $\tau \in \Omega_\varphi$ and $s \in \mathbb{S}$,
\[
\mathbb{I}_{I_n, m(\tau)I_n} \star \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(\tau^{-1})I_n} \star \mathbb{I}_{I_n, c_{\tau, \tau^{-1}}I_n} = \mathbb{I}_{I_n, c_{\tau, \tau^{-1}}I_n} \star \mathbb{I}_{I_n, m(\tau s\tau^{-1})I_n}.
\]

(iii) For $\tau \in \Omega_\varphi$ and $g \in I$, $\mathbb{I}_{I_n, m(\tau)I_n} \star \mathbb{I}_{I_n, gI_n} \star \mathbb{I}_{I_n, m(\tau^{-1})I_n} \star \mathbb{I}_{I_n, c_{\tau, \tau^{-1}}I_n} \mathbb{I}_{I_n, m(\tau^{-1})I_n} = \mathbb{I}_{I_n, m(\tau)gm(\tau^{-1})I_n}$.

(C) (i) $\mathbb{I}_{I_n, m(s)I_n}$ is the identity element of $\mathcal{H}(G(F), I_n)$.

(ii) For $g, g' \in I$, $\mathbb{I}_{I_n, gI_n} \star \mathbb{I}_{I_n, g'I_n} = \mathbb{I}_{I_n, gg'I_n}$.

(iii) For $s \in \mathbb{S}$ and for $g \in I \cap m(s)I m(s)^{-1}$,
\[
\mathbb{I}_{I_n, m(s)I_n} \star g \mathbb{I}_{I_n, I_n} = \mathbb{I}_{I_n, m(s)gm(s)^{-1}I_n} \star \mathbb{I}_{I_n, m(s)I_n}
\]

(iv) For $s \in \mathbb{S}$ and for $g \in I \setminus (I \cap m(s)I m(s)^{-1})$, let $g_1, g_2$ are elements of $I$ such that $m(s)gm(s)^{-1} = g_1m(s)g_2$. Then
\[
\mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, gI_n} \star \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, g_1I_n} = q(s)\left(\mathbb{I}_{I_n, g_1I_n} \star \mathbb{I}_{I_n, m(s)I_n} \star \mathbb{I}_{I_n, g_2I_n}\right).
\]

Proof. Let $\mathcal{H}_n$ be the quotient of the free $\mathbb{Z}$-algebra generated by the elements (1) - (3), by the subalgebra generated by relations (A) - (C) stated in the theorem. For clarity, to distinguish the elements of $\mathcal{H}_n$ from the elements of $\mathcal{H}_m$, we denote the generators of $\mathcal{H}_n$ as $\mathbb{I}_{I_n, m(s)I_n}, s \in \mathbb{S}$, $\mathbb{I}_{I_n, m(\tau)I_n}, \tau \in \Omega_\varphi$, $\mathbb{I}_{I_n, gI_n}, g \in I$. By Theorem 4.2, the relations (A)-(C) in Theorem 4.7 are satisfied for $\mathcal{H}_n$. Thus we have an algebra homomorphism $\mathcal{H}_n \rightarrow \mathcal{H}_m$. This map is surjective by Corollary 4.3.

For $w \in W$, write $w = w_1 \tau$ for $w_1 \in W_{\text{aff}}$ and $\tau \in \Omega_\varphi$. Let $m(w_1) = m(s_{i_1}) \cdots m(s_{i_\ell})$ for a chosen reduced expression of $w_1$ and let $m(w) = m(w_1)m(\tau) \in T$. Define $\mathbb{I}_{I_n, m(w)I_n} = \mathbb{I}_{I_n, m(s_{i_1})I_n} \star \cdots \star \mathbb{I}_{I_n, m(s_{i_\ell})I_n} \star \mathbb{I}_{I_n, m(\tau)I_n}$. Then relation (A)(i) implies that this expression is independent of the choice of reduced expression for $w_1$. For $g \in G(F)$, write $g = x m(w) y$ for $w \in W$ and $x, y \in I$. Define $\mathbb{I}_{I_n, gI_n} = \mathbb{I}_{I_n, xI_n} \star \mathbb{I}_{I_n, m(w)I_n} \star \mathbb{I}_{I_n, yI_n}$.

We show that

(a) Let $w \in W$ and $x, y, x_1, y_1 \in I$ such that $x m(w) y = x_1 m(w) y_1$. Then
\[
\mathbb{I}_{I_n, xI_n} \star \mathbb{I}_{I_n, m(w)I_n} \star \mathbb{I}_{I_n, yI_n} = \mathbb{I}_{I_n, x_1I_n} \star \mathbb{I}_{I_n, m(w)I_n} \star \mathbb{I}_{I_n, y_1I_n}.
\]
Since $yy_1^{-1} = m(w)^{-1}(x^{-1}x_1)m(w)$, $yy_1^{-1} \in I \cap m(w)^{-1} Im(w)$. Therefore,

\[
\hat{I}_{m(w)I} = \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \cdots = \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I}.
\]

using relations $B(iii)$ and $C(i)$. It is easy to check that $a \in Ad(m(s_i)) \cdots m(s_{i-1}))^{-1}(I) \cap I$ implies that $a \in Ad(m(s_i)) \cdots m(s_{i-1}))^{-1}(I) \cap I$ and $Ad(m(s_i))(a) \in I \cap Ad(m(s_i)) \cdots m(s_{i-1}))^{-1}(I)$. Using the above and relation $C(iii)$ repeatedly, we have

\[
\hat{I}_{m(w)I} = \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I}.
\]

Now (a) follows from $C(i)$ and $C(ii)$.

In particular, the element $\hat{I}_{m(w)I}$ is well-defined.

We prove that Relation (0) of Theorem 4.2 holds for $\hat{H}_n$. Let $g = xm(w)y$ with $x, y \in I$ and $g' \in L_gI$. Then $g' = x'y'm(w)y'$ for some $x', y' \in I$ with $x' \in L_nx$ and $y' \in L_y$. By (a) and relation (C(i), (ii)), we have

\[
\hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I} = \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I}.
\]

We prove that Relation (1) of Theorem 4.2 holds for $\hat{H}_n$. We need to show that if $l(\pi(g')) = l(\pi(g)) + l(\pi(y'))$, then

\[
\hat{I}_{m(w)I} * \hat{I}_{m(w)I} = \hat{I}_{m(w)I}.
\]

To prove this claim, we may easily reduce ourselves to the case when $l(\pi(g)) \leq 1$. If $l(\pi(g)) = 0$, then (4.2) follows from relations $B(i)$, $B(ii)$, $B(iii)$, $C(i)$ and $C(ii)$. If $l(\pi(g)) = 1$, we may assume $g = m(s)$ for a suitable $s \in S$. Let $\pi(g') = w$. Write $g' = x'm(w)y'$ for $x', y' \in I$. Then $gg' = m(s)x'm(w)y'$. Since $l(sw) = l(s) + 1$, we see that $m(s)Im(w) \subset Im(s)m(w)I$. In particular, $I \subset m(s)^{-1} Im(s)m(w)Im(w)^{-1}$. Write $x' = x_1x_2'$ for $x_1 \in m(s)^{-1} Im(s)$ and $x_2' \in m(w)Im(w)^{-1}$. Then

\[
sgg' = m(s)x_1x_2'm(s)^{-1}m(w)m(w)^{-1}x_2'm(w)y'
\]

and

\[
\hat{I}_{m(s)x_1x_2'} = \hat{I}_{m(s)x_1x_2'} * \hat{I}_{m(s)x_1x_2'}.
\]

In this case, (4.2) holds using relations $C(i)$, $C(ii)$ and $C(iii)$.

We prove that relation (2) of Theorem 4.2 holds for $\hat{H}_n$. We may assume $g = m(s)$. Let $\pi(g') = w$. In this case, $l(sw) < l(w)$. Write $g' = x'm(w)y'$. Then $m(s)x'm(w) \in Im(w) \cup Im(s)m(w)I$. Now $m(s)x'm(w) \in Im(s)m(w)I$ if and only if $\pi(gg') < \pi(g')$. Further, writing $x' = x_1x_2'$ for $x_1 \in m(s)^{-1} Im(s)$ and $x_2' \in m(w)Im(w)^{-1}$, using $C(iii)$ and $C(ii)$, we have

\[
\hat{I}_{m(s)x_1x_2'} = \hat{I}_{m(s)x_1x_2'} * \hat{I}_{m(s)x_1x_2'}.
\]

Since $l(sw) < l(w)$, we have $w = sw'$ for a suitable $w' \in W$. Then $\hat{I}_{m(w)I} = \hat{I}_{m(w)I} * \hat{I}_{m(w)I}$. Using this in (4.3) and using Relation A(ii) finishes the proof of Relation (2) when $\pi(gg') < \pi(g')$. Next, $m(s)x'm(w) \in Im(w)I$ if and only if $\pi(gg') = \pi(g')$. In this case, we may write $x' = x_1m(s)^{-1}x_2$ where $x_1 \in m(s)^{-1} Im(s)$ and $x_2' \in m(w)Im(w)^{-1}$. Then

\[
m(s)g' = (m(s)x_1m(s)^{-1}m(w)) (m(w)^{-1}x_2'm(w)y')
\]

Now

\[
\hat{I}_{m(w)I} * \hat{I}_{m(w)I} = q_{l(s)}(\hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I}).
\]

In the above, the first equality uses $C(iv)$ and $C(i)$, the second one uses $C(iii)$ and $C(ii)$. 

\[
\hat{I}_{m(w)I} * \hat{I}_{m(w)I} = q_{l(s)}(\hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I} * \hat{I}_{m(w)I}).
\]
We have verified that relations (0) - (2) of Theorem 4.2 hold for $\hat{H}_n$. This concludes the proof of the theorem. □

5. Tits Groups over $\bar{F}$

5.1. The Tits group of the relative Weyl group over $\bar{F}$. We begin with a discussion on the Tits group of the relative Weyl group of $G$ over $\bar{F}$, which is probably well-known, but a reference discussing this does not seem to be available in literature.

Let $G$ be a connected, reductive group over $\mathbb{F}$. In this section, we prove the existence of the Tits group of a finite relative Weyl group of $G_\mathbb{F}$.

We consider a Steinberg pinning $(x_0)_\mathbb{F}$ of $G_\mathbb{F}$ relative to $S$ (see [8, §4.1.3]). It has the following properties:

1. $x_0: G_a \to U_{\mathbb{F}}$ is a $\bar{F}$-isomorphism.
2. For each $\bar{a} \in \Delta_0$ and $\gamma \in \text{Gal}(\bar{F}/\bar{F})$, $x_0(\bar{a}) = \gamma \circ x_0 \circ \gamma^{-1}$.

This extends to a Chevalley-Steinberg system of pinnings $x_0: G_\mathbb{F} \to U_{\mathbb{F}}$ for all $\bar{a} \in \Phi(G, T)$, which is compatible with the action of $\text{Gal}(\bar{F}/\bar{F})$. From this, we get a set of pinnings for $\bar{a} \in \Phi(G, S)$, which we briefly recall. Let $U_\mathbb{F}$ be the root subgroup of the root $\bar{a}$. When $2\bar{a}$ is not a root, we have $x_{\bar{a}}: \text{Res}_{\bar{F}a/\bar{F}} G_{\mathbb{F}} \xrightarrow{\cong} U_a$. If $2\bar{a}$ is a root, let $H_0(\bar{F}_a, \bar{F}_{2\bar{a}}) = \{(u, v) \in \bar{F}_a \times \bar{F}_{2\bar{a}} \mid u \cdot \gamma(u) = v + \gamma_0(v)\}$, where $\gamma_0$ is the non-trivial $\bar{F}_{2a}$-automorphism of $\bar{F}_a$. We have

$$x_{\bar{a}}: \text{Res}_{\bar{F}_{2\bar{a}}/\bar{F}} H_0(\bar{F}_a, \bar{F}_{2\bar{a}}) \xrightarrow{\cong} U_a.$$ 

For any $\bar{a} \in \Delta_0$, let

$$n_{x_{\bar{a}}} := x_{\bar{a}}(1)x_{-\bar{a}}(1)x_{\bar{a}}(1).$$

(5.1)

We note here that we have used the convention of [8, §4.1.5], and $n_{x_{\bar{a}}} \in N_G(S)(\bar{F})$ (This is different from the convention used in [20] where $n_{x_{\bar{a}}} := x_{\bar{a}}(1)x_{-\bar{a}}(-1)x_{\bar{a}}(1)$).

Now, let $\bar{a} \in \Delta_0$. If $2\bar{a}$ is not a root, we set

$$n_{x_{\bar{a}}} := x_{\bar{a}}(1)x_{-\bar{a}}(1)x_{\bar{a}}(1).$$

(5.2)

Next, suppose $2\bar{a}$ is a root. By [21, Chapter V, §4, Proposition 7], there exists $c \in \bar{F}_a$ such that $c\gamma_0(c) = 2$. Set

$$n_{x_{\bar{a}}} = x_{\bar{a}}(c, 1)x_{-\bar{a}}(c, 1)x_{\bar{a}}(c, 1).$$

(5.3)

Note that if the residue characteristic of $F$ is not 2, such a $c$ in fact lies in $\mathbb{Q}_c$, and when the characteristic of $F$ is 2, $c = 0$. By [8, §4.1.11], we have

$$n_{x_{\bar{a}}} = \prod n_{x_{\bar{a}}}^{-1} n_{x_{\bar{a}}} n_{x_{\bar{a}}}^{-1}.$$ 

(5.4)

where the product is indexed by the family of sets $\{a, a^\prime\}$ with $a, a^\prime \in \Phi(G, T)$ such that $a + a^\prime$ is a root and $a|_S = a^\prime|_S = \bar{a}$.

Let $\mathcal{T}_\text{fin}$ be the group generated by the elements $\{n_{x_{\bar{a}}} \mid \bar{a} \in \Delta_0\}$. Note that $S_2 = \langle a^\vee(-1) \mid \bar{a} \in \Delta_0 \rangle$ is contained in $\mathcal{T}_\text{fin}$. Then the elements $\{n_{x_{\bar{a}}} \mid \bar{a} \in \Delta_0\}$ satisfy Coxeter relations is a consequence of [7, §6.1.3] (see [4, Proposition IV.6]). Furthermore, we have that

$$n_{x_{\bar{a}}}^2 = \begin{cases} a^\vee(-1), & \text{if } 2\bar{a} \text{ is not a root;} \\ 1, & \text{if } 2\bar{a} \text{ is a root.} \end{cases}$$

Let $\hat{S}_2 = \langle a^\vee(-1) \mid \bar{a} \in \Phi(G, S) \rangle$. Then $\mathcal{T}_\text{fin}$ fits into a short exact sequence

$$1 \to \hat{S}_2 \to \mathcal{T}_\text{fin} \to W(G, S) \to 1.$$
5.2. An example of $G_{\bar{F}}$ for which $\bar{\mathcal{T}}$ does not exist. In this subsection, we will give an example of a wildly ramified unitary group over $\bar{F}$ for which the Tits group $\bar{\mathcal{T}}$ does not exist.

Let $\bar{F} = \mathbb{Q}_2$, $\bar{F} = \mathbb{Q}^\text{un}_2$, $\bar{F} = \bar{F}(\sqrt{-1})$. Then $\bar{F}$ is a wildly ramified quadratic extension of $\bar{F}$. Let $G$ be a connected reductive group over $\bar{F}$ with $G_{\bar{F}} = U_6 \subset \text{Res}_{\bar{F}/F} \text{GL}_6$. Let $\gamma$ denote the generator of $\text{Gal}(\bar{F}/F)$. Then

$$\bar{\Phi}(G, S) = \{ \pm e_i \pm e_j | 1 \leq i < j \leq 3 \} \cup \{ \pm 2e_i | 1 \leq i \leq 3 \},$$

$$\bar{\Phi}_{ad}(G, S) = \{ \pm e_i \pm e_j + \frac{1}{2} Z | 1 \leq i < j \leq 3 \} \cup \{ \pm 2e_i + Z | 1 \leq i \leq 3 \}.$$

The hyperplanes with respect to the roots $\tilde{a}_1 := e_1 - e_2, \tilde{a}_2 := e_2 - e_3, \tilde{a}_3 := 2e_3, \tilde{a}_0 := -e_1 - e_2 + \frac{1}{2}$ form a alcove in $\mathcal{A}(S, \bar{F})$ which we denote as $\tilde{\mathcal{A}}$. Let $\tilde{\Delta}_0 = \{ \tilde{a}_i | 1 \leq i \leq 3 \}$ and $\tilde{\Delta} = \tilde{\Delta}_0 \cup \{ \tilde{a}_0 \}$. Let $\tilde{s}_i = \tilde{s}_{a_i}, 0 \leq i \leq 3$.

Suppose $\bar{\mathcal{T}}$ can be defined. Then $\bar{\mathcal{T}}$ contains representatives $m(\tilde{s}_i), 0 \leq i \leq 3$ that satisfy Coxeter relations and such that $m(\tilde{s}_i)^2 = (\tilde{b}_i)^\gamma(-1) \in \tilde{S}_2$.

The Coxeter relations involving the reflection $\tilde{s}_0$ are $\tilde{s}_0 \tilde{s}_1 \tilde{s}_0 = \tilde{s}_2 \tilde{s}_0 \tilde{s}_2$ and $\tilde{s}_0 \tilde{s}_3 = \tilde{s}_3 \tilde{s}_0$. Additionally, we have $\tilde{s}_0^2 = 1$.

Let $t \in T(\bar{F})$. We may write $t = \text{diag}(d_1, d_2, d_3, \gamma(d_3)^{-1}, \gamma(d_2)^{-1}, \gamma(d_1)^{-1})$. Then

$$\tilde{s}_1(t) = \text{diag}(d_2, d_1, d_3, \gamma(d_3)^{-1}, \gamma(d_1)^{-1}, \gamma(d_2)^{-1}),$$

$$\tilde{s}_2(t) = \text{diag}(d_1, d_3, d_2, \gamma(d_2)^{-1}, \gamma(d_1)^{-1}, \gamma(d_3)^{-1}),$$

$$\tilde{s}_3(t) = \text{diag}(d_1, d_2, \gamma(d_3)^{-1}, d_3, \gamma(d_2)^{-1}, \gamma(d_1)^{-1}),$$

$$\tilde{s}_0(t) = \text{diag}(\gamma(d_2)^{-1}, \gamma(d_1)^{-1}, d_3, \gamma(d_3)^{-1}, d_1, d_2).$$

To see the last equality, we note that a reduced expression for the image of $\tilde{s}_0$ in $W(G, S)$ is $\tilde{s}_2 \tilde{s}_3 \tilde{s}_1 \tilde{s}_2 \tilde{s}_3 \tilde{s}_2$. This equality can also be seen by noting that the image of $\tilde{s}_0$ in $W(G, S) \subset W(G, T)$ represents the permutation $(1, 5)(2, 6)$ in the symmetric group $S_6$.

For $\sigma$ a permutation in the symmetric group $S_6$, let $g_{\sigma}$ denote the corresponding permutation matrix in $\text{GL}_6(\bar{F})$ whose entries are all 0 or 1. The element $m(\tilde{s}_0) \in U_6(\bar{F}) \subset \text{GL}_6(\bar{F})$ can be written as a product $t_0 \cdot g_{(1,5)(2,6)}$ where

$$t_0 = \text{diag}(d_1, d_2, d_3, \gamma(d_3)^{-1}, \gamma(d_2)^{-1}, \gamma(d_1)^{-1}).$$

Now $m(\tilde{s}_0)^2 \in \tilde{S}_2$ implies that $d_2 = \pm \gamma(d_1)$. Next, the relation $m(\tilde{s}_0)m(\tilde{s}_1) = m(\tilde{s}_1)m(\tilde{s}_0)$ implies that $t_0 g_{(1,5)(2,6)} = \tilde{s}_1(t_0)m(\tilde{s}_1)g_{(1,5)(2,6)}m(\tilde{s}_1)^{-1}$. So

$$\tilde{s}_1(t_0)g_{(1,5)(2,6)} = g_{(1,5)(2,6)}g_{(1,5)(2,6)}m(\tilde{s}_1)^{-1}.$$
5.3. Affine pinning. Recall that we have chosen a special vertex $\tilde{e}_0$ and we view $\tilde{a} \in V$. For each reflection $\tilde{s}$ in $\tilde{S}$ there is a unique affine root such that the reflection in the hyperplane with respect to this affine root is $\tilde{s}$. Let $\tilde{\Delta}$ denote this collection of affine roots. The Weyl chamber in $V$ that contains $\tilde{a}$ determines a set of simple roots of $\tilde{\Phi}(G,T)$ which we denote as $\Delta_0$. Note that $\Delta_0 \subset \tilde{\Delta}$.

We consider a pinning $\{x_\alpha\}_{\alpha \in \Delta_0} \subset G_F$ relative to $T$ (see [8, §4.1.3]), that is, for each $\tilde{a} \in \Delta_0$, we fix a $\tilde{F}$-isomorphism $x_\alpha : G_a \to U_\alpha$, where $U_\alpha$ is the root subgroup of the root $\tilde{a}$. This extends to a Chevalley system of pinnings $x_\alpha : G_a \tilde{\to} U_\alpha$ for all $\tilde{a} \in \tilde{\Phi}(G,T)$.

Let $\tilde{a} \in \Delta \setminus \Delta_0$ and let $\tilde{b} \in \tilde{\Phi}(G,T)$ be the gradient of $\tilde{a}$. Let $m_{\tilde{\omega}} : G_a \to G_a$ denote $\tilde{F}$-morphism given by multiplication by $\tilde{\omega}_F$, where $\tilde{\omega}_F$ is a uniformizer of $\tilde{F}$. Define $x_\tilde{a} := x_\tilde{b} \circ m_{\tilde{\omega}}$. Note that $x_\tilde{a}$ is a $\tilde{F}$-isomorphism from $G_a$ to $U_\tilde{b}$.

The set $\{x_\alpha | \tilde{a} \in \Delta\}$ is called an affine pinning of $G_F$. For $\tilde{a} \in \tilde{\Delta}$ define

$$n_{\tilde{s}_a} := x_\tilde{a}(1)x_{-\tilde{a}}(1)x_\tilde{a}(1).$$

We note here again that we use the convention of [8, §4.1.5] and $n_{\tilde{s}_a} \in N_G(T)(\tilde{F})$.

Lemma 5.1. The set $\{n_{\tilde{s}} | \tilde{s} \in \tilde{S}\}$ satisfies the Coxeter relations.

Remark 5.2. For a different proof of the Coxeter relations for the affine Weyl group of a split reductive group, see [10, Proposition 3.1].

Proof. Let $\tilde{s} = s_a$ and $\tilde{s}' = s_{a'}$ for $\tilde{a}, \tilde{a}' \in \tilde{\Delta}$ with gradients $\tilde{b}, \tilde{b}'$ respectively. Let $\tilde{\Phi}_{\tilde{b}, \tilde{b}'} \subset \tilde{\Phi}(G,T)$ denote the rank 2 root system spanned by $\tilde{b}, \tilde{b}'$. If $\tilde{\Phi}_{\tilde{b}, \tilde{b}'}$ is a product of rank 1 root systems, then the Coxeter relation is obvious, so we may and do assume that $\tilde{\Phi}_{\tilde{b}, \tilde{b}'}$ is irreducible.

Set $k = k(\tilde{s}, \tilde{s}')$. We put the elements of $\tilde{\Phi}_{\tilde{b}, \tilde{b}'}$ in “circular order” as required in [7, Proposition 6.1.8], that is, we can enumerate the elements of $\tilde{\Phi}_{\tilde{b}, \tilde{b}'}$ as $b_1, b_2, \ldots, b_{2k}$ so that $b_1 = \tilde{b}, b_k = \tilde{b}'$, and for $1 < i < 2k$,

$$\tilde{\Phi}_{\tilde{b}, \tilde{b}'} \cap (Q_i b_{i-1} + Q_i b_{i+1}) = \{b_{i-1}, b_i, b_{i+1}\}.$$  

By [7, §6.1.3], for any $u \in U_{\tilde{b}}(\tilde{F}) \setminus \{1\}$ and $u' \in U_{\tilde{b}'}(\tilde{F}) \setminus \{1\}$, there exists unique triples $(u_1, m(u), u_2) \in U_{\tilde{b}}(\tilde{F}) \times N_G(S)(\tilde{F}) \times U_{\tilde{b}'}(\tilde{F})$ and $(u'_1, m(u'), u'_2) \in U_{\tilde{b}'}(\tilde{F}) \times N_G(S)(\tilde{F}) \times U_{\tilde{b}}(\tilde{F})$ such that $u = u_1 m(u) u_2$ and $u' = u'_1 m(u')u'_2$. By [7, Proposition 6.1.8, Part (9)],

$$m(u) \cdot m(u')^{-1} \cdots = m(u')^{-1} \cdot m(u) \cdots,$$

where each side has $k$ factors.

It is clear from equation (5.5) that there exist $u \in U_{-\tilde{b}_i}$ and $u' \in U_{-\tilde{b}_i}$ such that $n_{\tilde{s}_a} = m(u)$ and $n_{\tilde{s}_a'} = m(u')^{-1}$. Now the statement follows from (5.6). $\Box$

Now we prove the following existence result of Tits group over $\tilde{F}$.

Proposition 5.3. Suppose that $G$ is split over $\tilde{F}$. Let $\{x_\alpha | \tilde{a} \in \tilde{\Delta}\}$ be an affine pinning of $G_F$ and $\{n_{\tilde{s}_a} | \tilde{a} \in \tilde{\Delta}\}$ as in (5.5). Let $\tilde{T}$ be the group generated by $\tilde{S}_2$, $\{n_{\tilde{s}_a} | \tilde{a} \in \tilde{\Delta}\}$ and $\tilde{\lambda}(\tilde{\omega}_F)$ for $\tilde{\lambda} \in X_S(T)$. Then $\tilde{T}$ is a Tits group of $\tilde{W}$.

Proof. By direct calculation, $n_{\tilde{x}_a}^2 = \tilde{b}'(-1)$, where $\tilde{b}$ is the gradient of $\tilde{a}$.

Now we define the lifting $n_{\tilde{x}_a}$.

If $\tilde{w} \in \tilde{W}_{\tilde{d}}$, then we set $n_{\tilde{x}_a} = n_{\tilde{s}_{i_1}} \cdots n_{\tilde{s}_{i_k}}$, where $\tilde{s}_{i_1}, \ldots, \tilde{s}_{i_k}$ is a reduced expression of $\tilde{w}$ in $\tilde{W}$. By Lemma 5.1, the element $n_{\tilde{x}_a}$ is independent of the choice of reduced expression. If $\tilde{r} \in \Omega_{\tilde{d}}$, we may write $\tilde{r}$ as $t_{\tilde{\lambda}}\tilde{y}$ with $\tilde{\lambda} \in X_S(T)$ and $\tilde{y} \in \tilde{W}_0$. We then set $n_{\tilde{x}_a} = \tilde{\lambda}(\tilde{\omega}_F)n_{\tilde{y}}$. Note that any element $\tilde{w} \in \tilde{W}$ is of the form $\tilde{w} = \tilde{w}_{\tilde{d}}\tilde{y}$ for some $\tilde{w}_{\tilde{d}} \in \tilde{W}_{\tilde{d}}$ and $\tilde{y} \in \tilde{W}_0$. We set $n_{\tilde{x}_a} = n_{\tilde{x}_{\tilde{d}}}$. The collection $\{n_{\tilde{x}_a} | \tilde{w} \in \tilde{W}\}$ satisfies condition (2) in Definition 3.1.

Now we check condition (1). Note that $\tilde{S}_2$ is a normal subgroup of $\tilde{T}$. For any $\tilde{w} \in \tilde{W}$ and $\tilde{\lambda} \in X_S(T)$, we have $n_{\tilde{x}_a}(\tilde{\omega}_F)n_{\tilde{x}_a}^{-1} = \tilde{\lambda}(\tilde{\omega}_F)$, where $\tilde{\lambda} = \tilde{w}(\tilde{\lambda}) \in X_S(T)$. Let $\tilde{T}'$ be the subgroup of $\tilde{T}$ generated by $n_{\tilde{x}_a}$ for $\tilde{a} \in \tilde{\Delta}$. Then any element in $\tilde{T}'$ is of the form $t_{\tilde{d}} n(\tilde{\omega}_F)$ for some $t_{\tilde{d}} \in \tilde{S}_2$, $n \in \tilde{T}'$ and $\tilde{\lambda} \in X_S(T)$. If $\tilde{\Phi}(t_{\tilde{d}} n(\tilde{\omega}_F)) = 1$, then $\tilde{\lambda} \in \tilde{Z}(\tilde{\Phi}(G,T))$. 

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Let $G\sigma F\sigma$ be a Frobenius morphism on $X$. Assuming that $G\sigma F\sigma$ is a Frobenius morphism, for any $\sigma$, let $G = G\sigma F\sigma$. Therefore, $\ker\tilde{\phi}$ is contained in the subgroup generated by $S_2$ and $T'$.

Any element of $\mathcal{T}'$ is of the form $n^1_{s_{i_1}} \cdots n^1_{s_{i_k}}$. Since $n^2_{s_{i_1}} \in S_2$, we have $n^1_{s_{i_1}} \cdots n^1_{s_{i_k}} = n^2_{s_{i_1}} \cdots n^2_{s_{i_k}} S_2$.

It remains to show that

(a) If $\hat{s}_{i_1} \cdots \hat{s}_{i_k} = 1$, then $n_{s_{i_1}} \cdots n_{s_{i_k}} \notin S_2$.

We argue by induction on $k$. By the deletion condition of Coxeter groups (see [6, Chapter IV, §1.3 - §1.5]), there exists $\hat{s}_{i_{l+1}}, \ldots, \hat{s}_{i_k}$ such that $n_{s_{i_1}} \cdots n_{s_{i_k}} = n_{s_{i_1}} \cdots n_{s_{i_{l}}}$ and $\hat{s}_{i_{l}} = \hat{s}_{i_{l+1}}$ for some $l$. We have $n_{s_{i_{l+1}}} \cdots n_{s_{i_k}} n_{s_{i_{l+1}}} \cdots n_{s_{i_k}} = n_{s_{i_{l+1}}} \cdots n_{s_{i_{l}}} n_{s_{i_{l+1}}} \cdots n_{s_{i_k}} S_2$.

Note that $\hat{s}_{i_{l+1}} \cdots \hat{s}_{i_k} \hat{s}_{i_{l+1}} \cdots \hat{s}_{i_k} = 1$. Since there are only $k - 2$ simple reflections involved, by inductive hypothesis, $n_{s_{i_{l+1}}} \cdots n_{s_{i_{l}}} n_{s_{i_{l+1}}} \cdots n_{s_{i_k}} \in S_2$. Hence $n_{s_{i_1}} \cdots n_{s_{i_k}} \notin S_2$.

Condition (1) of Definition 3.1 is verified. \hfill $\square$

6. Tits group over $F$

6.1. The strategy. The main result of this section is the existence of Tits groups over $F$ for any connected reductive group defined over $F$ and splits over $\bar{F}$. The strategy is as follows.

1. Let $G$ connected, reductive group over $F$ and let $\sigma$ be the Frobenius morphism on $G$ with $G(\bar{F})^\sigma = G(F)$. We first construct an affine pinning such that the set $\{n_{s_{\hat{a}}} | \hat{a} \in \hat{\Delta}\}$ is $\sigma$-stable for any $\sigma$-orbit $\hat{\Delta}$ with $W_\Delta$ finite. This result has two consequences. First, when $G$ is semisimple and simply connected, it yields a definition of the Tits group of its Iwahori-Weyl group, which is the affine Weyl group, over $F$. Second, we may construct a Tits group $\tilde{T}$ over $\bar{F}$ that is stable under the action of a given quasi-split Frobenius morphism $\sigma$;

2. We then choose a suitable Frobenius morphism $\sigma^*$ for each inner form and show that there exists a Tits cross-section in $\tilde{T}$ that is “compatible” with the Frobenius morphism $\sigma^*$;

3. Finally, we use the descent argument to show that $\tilde{T}^{\sigma^*} \subset G(\bar{F})^{\sigma^*}$ is a Tits group of the Iwahori-Weyl group over $F$ of the group $G(\bar{F})^{\sigma^*}$.

6.2. Affine pinnings and Frobenius morphisms. We have proved in Lemma 5.1 that given any affine pinning $\{x_\hat{a} | \hat{a} \in \hat{\Delta}\}$, the set $\{n_{s_{\hat{a}}} | \hat{a} \in \hat{S}\}$ satisfies the Coxeter relations, where $n_{s_{\hat{a}}} = x_\hat{a}(1)x_{-\hat{a}}(1)x_{\hat{a}}(1)$. By §2.3, for any $s \in S$, there exists a $\sigma$-orbit $\hat{\Delta}$ of $S$ with $W_\Delta$ finite such that $s = \bar{x}_{\hat{a}} \in \bar{W}$. In this section, we show the following.

Proposition 6.1. Let $G$ be a connected reductive group defined over $F$ that splits over $\bar{F}$. Let $\sigma$ be a Frobenius morphism on $G_F$. There exists an affine pinning $\{x_\hat{a} | \hat{a} \in \hat{\Delta}\}$ such that the set $\{n_{s_{\hat{a}}} | \hat{a} \in \hat{\Delta}\}$ is $\sigma$-stable for any $\sigma$-orbit $\hat{\Delta}$ of $\hat{\Delta}$ with $W_\Delta$ finite.

Remark 6.2. (1) If $\sigma$ is a quasi-split Frobenius, that is, if $G(\bar{F})^\sigma$ is quasi-split, then $W_\Delta$ is finite for any $\sigma$-orbit $\hat{\Delta}$ in $\hat{\Delta}$.

(2) Assuming that $G$ is absolutely simple, the finiteness assumption on $W_\Delta$ fails only for inner forms of type $A$, that is, if $G_F$ is split of type $A$, and if $\sigma$ is such that $G(\bar{F})^\sigma$ is a group over $F$ whose adjoint group is $\text{PGL}_1(D)$ for a suitable division algebra $D$. Such a group is anisotropic over $F$ with trivial affine Weyl group and its Iwahori-Weyl group has only length zero elements.

(3) Recall from §2.3 that the elements of $S$ are in bijection with $\sigma$-orbits $\hat{\Delta}$ such that $W_\Delta$ is finite. So, it suffices to consider such orbits to construct the Tits group over $F$. However, while the proof below uses the assumption that the $\sigma$-orbit $\hat{\Delta}$ is such that $W_\Delta$ is finite, we will show in Proposition 6.7 through a different argument that the finiteness assumption on $W_\Delta$ can be dropped.

Proof. Let $\{x_\hat{a} | \hat{a} \in \hat{\Delta}\}$ be an affine pinning and $\hat{\Delta}$ be a $\sigma$-orbit in $\hat{\Delta}$ such that $W_\Delta$ is finite. Let $k = \# \hat{\Delta}$.

Fix $\hat{a} \in \hat{\Delta}$ and let $\hat{b}$ be the gradient of $\hat{a}$. Since $W_\Delta$ is finite, we have $\bar{x}_{\hat{a}} \in \bar{W}_\Delta^\sigma$. In particular, $W_\Delta^\sigma \neq 1$. Thus $\bar{b}_{\hat{a}} \neq 0$ and hence is a root $\hat{b}$ in $\Phi(G, \hat{A})$. We show that
(a) There exists $u \in \mathcal{D}_p^\sigma$ such that $\sigma^k(x_a(u)) = x_a(u)$.

Let $v \in \mathcal{A}(A, F) \subset \mathcal{A}(T, \hat{F})$ and $r \in \mathbb{R}$. For $\tilde{b} \in \Phi(G, T)$, let $U_b(\hat{F})_{v,r} \subset U_b(\hat{F})$ denote the filtration of root subgroup $U_b(\hat{F})$ as in [8, $\S$4.3]. We recall the definition of the filtration of the root subgroup $U_b(\hat{F})$ (cf. [8, $\S$5.1.16 - 5.1.18]). Let $\Phi^b := \{\tilde{c} \in \Phi(G, T) \mid \tilde{c}|_A = b \text{ or } 2b\}$. This is a $\sigma$-stable positively closed subset of $\Phi(G, T)$; that is if $\tilde{c}_1, \tilde{c}_2 \in \Phi^b$ such that $\tilde{c} + \tilde{c}'$ is a root, then $\tilde{c} + \tilde{c}' \in \Phi^b$. For any fixed ordering, the subset

$$U_b(\hat{F})_{v,r} := \prod_{\tilde{c} \in \Phi^b, \tilde{c}|_A = a} U_{\tilde{c}}(\hat{F})_{v,r} \prod_{\tilde{c} \in \Phi^b, \tilde{c}|_A = 2b} U_{\tilde{c}}(\hat{F})_{v,2r}$$

(6.1)

is a subgroup of $U_b(\hat{F})$. Let $U_b(\hat{F})_{v,r} := U_b(\hat{F})_{v,r} \cap U_b(\hat{F})$.

We have

$$U_b(\hat{F})_{v,r} = U_b(\hat{F})_{\tilde{v}_0, r + (v - \tilde{v}_0)},$$

(6.2)

where $\tilde{v}_0$ is the special vertex in $\mathcal{A}(S, \hat{F})$ fixed in $\S2.1$.

The pinning $x_a : G_a \to U_b(\hat{F})$ satisfies $x_a(\mathcal{D}_F^\sigma) = U_b(\hat{F})_{\tilde{v}_0, a_0}$ and $x_a(p_F) = U_b(\hat{F})_{\tilde{v}_0, a_0}$ for a suitable $\tilde{v}_0 < a_0$.

Using (6.2) and then adjusting $r$ if necessary, we ensure

$$x_a(\mathcal{D}_F^\sigma) = U_b(\hat{F})_{v,r}, \quad x_a(p_F) = U_b(\hat{F})_{v,r}$$

(6.3)

for a suitable $r \in \mathbb{R}$. In particular, $U_b(\hat{F})_{v,r} \neq U_b(\hat{F})_{v,r}$. Now [8, Proposition 5.1.19] implies that $U_b(\hat{F})_{v,r} \neq U_b(\hat{F})_{v,r}$. In other words, there exists $u' \in U_b(\hat{F})_{v,r}$ such that $u' \notin U_b(\hat{F})_{v,r}$ and $U_b(\hat{F})_{v,r} \neq U_b(\hat{F})_{v,r}$.

Now that $\sigma^k$ fixes every element of $\mathcal{X}$ (and hence also every element of $\Phi^b$) and $\sigma^k(u') = u'$. By (6.1), we have

$$u' \notin \prod_{\tilde{c} \in \Phi^b, \tilde{c}|_A = a} U_{\tilde{c}}(\hat{F})_{v,r} \prod_{\tilde{c} \in \Phi^b, \tilde{c}|_A = 2b} U_{\tilde{c}}(\hat{F})_{v,2r},$$

and

$$u' \notin \prod_{\tilde{c} \in \Phi^b, \tilde{c}|_A = a} U_{\tilde{c}}(\hat{F})_{v,r} \prod_{\tilde{c} \in \Phi^b, \tilde{c}|_A = 2b} U_{\tilde{c}}(\hat{F})_{v,2r}.$$

Thus there exists $\tilde{c} \in \Phi^b, \tilde{c}|_A = b$ such that $U_{\tilde{c}}(\hat{F})_{v,r} \neq U_{\tilde{c}}(\hat{F})_{v,r}$. Since $\tilde{c} = \sigma^k(\tilde{b})$ for a suitable $\tilde{i} < k$, we also have

$$U_b(\hat{F})_{v,r} \neq U_b(\hat{F})_{v,r}.$$

Let $u'' \in U_b(\hat{F})_{v,r} \nsubseteq U_b(\hat{F})_{v,r}$ and $u = x_a(u'')$. Then $\sigma^k(x_a(u)) = x_a(u)$. By (6.3), $u \notin \mathcal{D}_p^\sigma$.

(a) is proved.

Since $x_a(u)x_{-a}(u^{-1})x_a(u) = \sigma^k(x_a(u)) \sigma^k(x_{-a}(u^{-1})) \sigma^k(x_a(u)) \in N_G(T)(\hat{F})$, we have

$$x_a(u) \sigma^k(x_{-a}(u^{-1})) x_a(u) = \sigma^k(x_a(u)) \sigma^k(x_{-a}(u^{-1})) \sigma^k(x_a(u)) \in N_G(T)(\hat{F}).$$

The uniqueness assertion in [8, $\S$6.1.2, (2)] implies that $\sigma^k(x_{-a}(u^{-1})) = x_{-a}(u^{-1})$.

Let $x'_a = x_a \circ m_a$, where $m_a$ is the multiplication by $u$. We consider the pinning $\{x'_a, \sigma \circ x'_a, \ldots \sigma^{k-1} \circ x'_a\}$. Then $x'_a(1) = x_a(u)$ and $x'_{a}(1) = x_{-a}(u^{-1})$. For $\tilde{c} \in \mathcal{X}$, let $n'_{s\tilde{c}} = x'_a(1)x'_{-\tilde{c}}(1)x'_{\tilde{c}}(1)$ be the representative in $N_G(S)(\hat{F})$ of $s\tilde{c}$ obtained using this pinning. Then $\sigma^k(n'_{s\tilde{c}}) = n'_{s\tilde{c}} \in S_{\tilde{c}}$ and the set $\{n'_{s\tilde{c}} \mid \tilde{c} \in \mathcal{X} = \{n'_{s\tilde{c}}, \sigma(n'_{s\tilde{c}}), \ldots, \sigma^{k-1}(n'_{s\tilde{c}})\}$ is $\sigma$-stable. \qed

6.3. The Frobenius morphism for each inner form. In this rest of this section, let $G$ denote a connected, reductive group over $F$ that is quasi-split over $F$ and split over $\hat{F}$. Let $\sigma$ denote the Frobenius morphism on $G_F$ so that the $\sigma$-structure it yields is $G$. We will later construct for each $F$-isomorphism class of inner twists of $G$ a suitable Frobenius morphism $\sigma^*$ and let $G^* = G_F^*$ be the $F$-group in the given isomorphism class of inner twists.

By Proposition 6.1, there exists an affine pinning $\{x_{\tilde{a}} \mid \tilde{a} \in \tilde{A}\}$ such that the set $\{n_{\tilde{a}} \mid \tilde{a} \in \tilde{S}\}$ is $\sigma$-stable. For $\tilde{\lambda} \in X_*(T)$, let $n_{\tilde{\lambda}} = \tilde{\lambda}(\varphi_F)$. Then $\sigma(n_{\tilde{\lambda}}) = n_{\sigma(\tilde{\lambda})}$. Let $\mathcal{T}$ be the Tits group of $W$ generated by $\tilde{S}_2$, $\{n_{\tilde{a}} \mid \tilde{a} \in \tilde{S}\}$ and $\{n_{\tilde{\lambda}} \mid \tilde{\lambda} \in X_*(T)\}$. Then $\mathcal{T}$ is stable under the action of $\sigma$. 
We will choose a suitable Frobenius morphism $\sigma^*$ for each $F$-isomorphism class of inner twists of $G$ such that $\hat{T}$ is stable under the action of $\sigma^*$. Finally, we will show that $\hat{T}^{\sigma^*}$ is a Tits group over $F$ for $G^*$.

6.3.1. The group $(\Omega_{\mathfrak{a}, \text{ad}})_\sigma$. The $F$-isomorphism classes of inner twists of $G$ is parametrized by $H^1((\sigma), G_{\text{ad}}(\hat{F}))$. By [9, Lemma 2.1.2] and §2.3-2.4, we have

$$H^1((\sigma), G_{\text{ad}}(\hat{F})) \cong H^1((\sigma), \Omega_{\mathfrak{a}, \text{ad}}) = (\Omega_{\mathfrak{a}, \text{ad}})_\sigma.$$ 

Now, we describe the group $\Omega_{\mathfrak{a}, \text{ad}}$ in more detail. We may assume that $G_{\mathfrak{F}_{\mathfrak{a}, \text{ad}}}$ is $\hat{F}$-simple. We will use the same labeling of the roots in $\Phi(G, T)$ as in [6] and we denote the indexing set of simple reflections by $I$. With this, the set $\{ s_\alpha \mid \alpha \in \Delta_0 \}$ is identified with $\{ \check{s}_i \mid i \in I \}$. Note that we have used the letter $I$ for the indexing set for the simple reflections; this should not cause any confusion, since the Iwahori subgroup will not be mentioned in the rest of this paper. For $J \subset I$, let $\hat{y}_J$ be the maximal element in the subgroup generated by $\check{s}_i$, $i \in J$.

Let $\check{\nu}$ be the half-sum of the positive coroots in any positive system. Let $i \in I$. If $\nu_i$ is minuscule, we denote by $\check{\nu}_{\text{ad},(i)} = t_{\check{\nu}} \check{y}_J(\check{i}) \in \Omega_{\mathfrak{a}, \text{ad}}$ the corresponding element. Here $\check{y}_J(\check{i}) = \check{y}_{J(i)} \check{y}_J$. Note that if $\check{\nu}_{\text{ad},(i)} = \check{\nu}_{\text{ad},(j)}$ for some $k \in \mathbb{N}$, then we also have that $\check{y}_J(\check{i}) = \check{y}_J(\check{j})$.

The description of $\Omega_{\mathfrak{a}, \text{ad}}$ is given in the following table. We list according to the type of the local Dynkin diagram of $G_{\mathfrak{F}_{\mathfrak{a}, \text{ad}}}$. We only list the types for which $\Omega_{\mathfrak{a}, \text{ad}}$ is non-trivial. In the last column, we make a choice of generator $\check{\nu}_{\text{ad},0}$ in the case where $\Omega_{\mathfrak{a}, \text{ad}}$ is cyclic. Such element $\check{\nu}_{\text{ad},0}$ will be used later.

| Type | $\Phi(G, T)$ | $\Omega_{\mathfrak{a}, \text{ad}}$ | Elements | Generator |
|------|-------------|----------------------------------|----------|-----------|
| $A_n$ | $A_n$ | $\mathbb{Z}/(n+1)\mathbb{Z}$ | $\{ 1, \check{\nu}_{\text{ad},(i)} \mid 1 \leq i \leq n \}$ | $\check{\nu}_{\text{ad},(1)}$ |
| $B_n$ | $B_n$ | $\mathbb{Z}/2\mathbb{Z}$ | $\{1, \check{\nu}_{\text{ad},(1)}\}$ | $\check{\nu}_{\text{ad},(1)}$ |
| $C_n$ | $C_n$ | $\mathbb{Z}/2\mathbb{Z}$ | $\{1, \check{\nu}_{\text{ad},(n)}\}$ | $\check{\nu}_{\text{ad},(n)}$ |
| $D_n$ | $D_n$, $2 \nmid n$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\{1, \check{\nu}_{\text{ad},(1), \check{\nu}_{\text{ad},(n-1), \check{\nu}_{\text{ad},(n)}}\}$ | N/A |
| $D_n$ | $D_n$, $2 \mid n$ | $\mathbb{Z}/4\mathbb{Z}$ | $\{1, \check{\nu}_{\text{ad},(1), \check{\nu}_{\text{ad},(n-1), \check{\nu}_{\text{ad},(n)}}\}$ | $\check{\nu}_{\text{ad},(n)}$ |
| $E_6$ | $E_6$ | $\mathbb{Z}/3\mathbb{Z}$ | $\{1, \check{\nu}_{\text{ad},(1), \check{\nu}_{\text{ad},(2)}}\}$ | $\check{\nu}_{\text{ad},(1)}$ |
| $E_7$ | $E_7$ | $\mathbb{Z}/2\mathbb{Z}$ | $\{1, \check{\nu}_{\text{ad},(7)}\}$ | $\check{\nu}_{\text{ad},(7)}$ |

Table 1: The group $\Omega_{\mathfrak{a}, \text{ad}}$

If $\sigma$ acts trivially on $\Omega_{\mathfrak{a}, \text{ad}}$, then $(\Omega_{\mathfrak{a}, \text{ad}})_\sigma \cong \Omega_{\mathfrak{a}, \text{ad}}$. If the action of $\sigma$ on $\Omega_{\mathfrak{a}, \text{ad}}$ is nontrivial, then

$$(\Omega_{\mathfrak{a}, \text{ad}})_\sigma = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } G_{\mathfrak{F}_{\mathfrak{a}, \text{ad}}} \text{ is of type } A_{2n+1} \text{ or } D_n; \\
1, & \text{otherwise.} \end{cases}$$

6.3.2. The construction of suitable Frobenius morphism $\sigma^*$. Let $j : G_{\mathfrak{F}_{\mathfrak{a}, \text{ad}}} \to G_{\mathfrak{F}_{\mathfrak{a}, \text{ad}}}$ denote the adjoint quotient. This induces maps $T \to T_{\text{ad}}$, $\hat{W} \to \hat{W}_{\text{ad}}$ and $\Omega_{\mathfrak{a}} \to \Omega_{\mathfrak{a}, \text{ad}}$, and we will denote all these maps by $j$ as well. The exact sequence $1 \to Z \to T \to T_{\text{ad}} \to 1$ induces exact sequences

$$1 \to X_*(Z^{0}) \to X_*(T) \xrightarrow{j} X_*(T_{\text{ad}})$$

and

$$1 \to X_*(Z^{0}) \to \Omega_{\mathfrak{a}} \xrightarrow{j} \Omega_{\mathfrak{a}, \text{ad}},$$

where $Z$ is the center of $G$ and $Z^{0}$ is the maximal torus in the center of $G$.

We will construct a suitable Frobenius morphism $\sigma^*$ associated to each $F$-isomorphism class of inner twists of $G$. It suffices to consider the case where $G_{F, \text{ad}}$ is $F$-simple.

We first discuss the case where $G_{F, \text{ad}}$ is $\hat{F}$-simple.
We choose as follows the element $\tilde{v}_{\text{ad}}$ in $\Omega_{\tilde{\text{ad}}}$ whose image in $(\Omega_{\tilde{\text{ad}}})_{\sigma}$ the parametrizes the inner twist $G^*$ of $G$. If $G_{\text{ad}}$ is of type $A_{2n+1}$ or $D_{2n+1}$ for some $n \in \mathbb{N}$ and the $\sigma$-action on $\Omega_{\tilde{\text{ad}}}$ is nontrivial, then $\Omega_{\tilde{\text{ad}}}$ is nontrivial, then $(\Omega_{\tilde{\text{ad}}})_{\sigma} = \mathbb{Z}/2\mathbb{Z}$. In this case, we take $\tilde{v}_{\text{ad}} = 1$ if $G^*$ is quasi-split over $F$ and $\tilde{v}_{\text{ad}} = \tilde{v}_{\text{ad},0}$ if $G^*$ is not quasi-split over $F$. Here $\tilde{v}_{\text{ad},0}$ is the generator of $\Omega_{\tilde{\text{ad}}}$ listed in Table 1. In other cases, we may take $\tilde{v}_{\text{ad}}$ to be any element in $\Omega_{\tilde{\text{ad}}}$ that corresponds to the $F$-isomorphism class of $G^*$. The choice of $\tilde{v}_{\text{ad}}$ is not essential, but will simplify some calculations in the rest of this section.

Let $\tilde{v}_{\text{ad}} = t_{\tilde{\eta}_{\text{ad}}} \tilde{z}$. We construct suitable liftings of $t_{\tilde{\eta}_{\text{ad}}}$ and $\tilde{z}$.

The lifting of $t_{\tilde{\eta}_{\text{ad}}}$ is constructed as follows. Note that the quotient $X_+(T_{\text{ad}})/j(X_+(T))$ is finite. Consider the element $t_{\tilde{\eta}_{\text{ad}}} \in X_+(T_{\text{ad}})$. Let $k \geq 1$ be the smallest integer such that $k \tilde{\eta}_{\text{ad}} \in j(X_+(T))$.

Write

$$k \tilde{\eta}_{\text{ad}} = j(\tilde{\eta})$$

for some $\tilde{\eta} \in X_+(T)$. Let $\tilde{v} \in T$. Note that $\tilde{v} \in W$, but need not lie in $\Omega_{\tilde{\text{ad}}}$. We know that $n_{\tilde{\eta}_{\text{ad}}} = \tilde{\eta}_{\text{ad}}(w_F)$. Set

$$g_0 = \tilde{\eta}(w_F^{1/k}).$$

Note that $g_0 \in T(\tilde{F}) \subset G(\tilde{F})$ and $j(g_0 \tilde{\eta}) = n_{\tilde{\eta}_{\text{ad}}}$. Note that for each root $\tilde{\alpha} = \tilde{\Phi}(G, T)$, we have $\tilde{\alpha}(\tilde{\eta}) \in \tilde{\mathbb{Z}}$ because $\tilde{j}(\tilde{\alpha}) = \tilde{\alpha} \tilde{\eta} = \tilde{j}(\tilde{\eta})$, so $k(\tilde{\alpha}, \tilde{\eta}_{\text{ad}}) \in \mathbb{Z}$. In particular, by conjugation by $g_0$, preserves $G(\tilde{F})$.

Now we construct a lifting $g_z$ of $\tilde{z}$ in $\tilde{T}$. If $G_{\text{ad}}$ is of type $D_n$ with $n$ even, then we set $g_z = n_z$. Otherwise, $\Omega_{\tilde{\text{ad}}}$ is a cyclic group. From our construction, $\tilde{v}_{\text{ad}} = \tilde{v}_{\text{ad},0}$ for $0 \leq i < |(\Omega_{\tilde{\text{ad}}})_{\sigma}|$. We then have $\tilde{z} = \tilde{z}_i$. Let $g_0 = g_0 \tilde{g}_z \in G(\tilde{F})$. We set

$$\sigma^* = \text{Ad}(g_0) \circ \sigma.$$

Next, suppose $G_{\text{ad}}$ is not simple.

By our assumption $G_{\text{ad}}$ is simple. We may write $G_{\text{ad}} = \text{Res}_{L_0/F} G'_{\text{ad}}$, where $L_0$ is a finite unramified extension of $F$ of degree $k$ contained in $\tilde{F}$ and $G'_{\text{ad}}$ is $\tilde{F}$-simple. Then

$$G_{\text{ad}} = G^{(1)}_{\text{ad}} \times \cdots \times G^{(k)}_{\text{ad}},$$

where $G^{(1)}_{\text{ad}} \cong \cdots \cong G^{(k)}_{\text{ad}} \cong G_{\text{ad}}'$. We may also write $\tilde{\Omega}_{\tilde{\text{ad}}} = \tilde{\Omega}^{(1)}_{\tilde{\text{ad}}} \times \cdots \times \tilde{\Omega}^{(k)}_{\tilde{\text{ad}}}$, where $\Omega^{(1)}_{\text{ad}} \cong \cdots \cong \Omega^{(k)}_{\text{ad}}$ are as in Table 1. Then the projection map $\tilde{\Omega}^{(1)}_{\tilde{\text{ad}}} \to (\tilde{\Omega}_{\tilde{\text{ad}}})_{\sigma}$ is surjective. In fact, $(\tilde{\Omega}^{(1)}_{\tilde{\text{ad}}})_{\sigma} \cong (\tilde{\Omega}_{\tilde{\text{ad}}})_{\sigma}$. Let $\tilde{v}_{\text{ad}} = t_{\tilde{\eta}_{\text{ad}}} \tilde{z} \in \tilde{\Omega}^{(1)}_{\tilde{\text{ad}}}$ such that its image in $(\tilde{\Omega}_{\tilde{\text{ad}}})_{\sigma}$ parametrizes the isomorphism class of $G^*$. We construct $g_0 \in G(\tilde{F})$ as above. More precisely, write $\tilde{v}_{\text{ad}} = t_{\tilde{\eta}_{\text{ad}}} \tilde{z}$. Then $\tilde{\eta}$ and $g_0$ have been constructed in (6.4). If $G^{(1)}_{\text{ad}}$ is of type $D_n$ with $n$ even, then we set $g_z = n_z$.

Otherwise we have $\tilde{v}_{\text{ad}} = \tilde{v}_{\text{ad},0}$ for $0 \leq i < |(\tilde{\Omega}^{(1)}_{\tilde{\text{ad}}})_{\sigma}|$. We also have $\tilde{z} = \tilde{z}_i$. Set $g_z = n_z$. Let $g_0 = g_0 g_z$ and let $\sigma^* = \text{Ad}(g_0) \circ \sigma$.

6.3.3. The action of $\sigma^*$ on $\tilde{T}$. It is easy to see that $\tilde{S}_2$ is stable under the action of $\sigma^*$. For each $\tilde{\lambda} \in X_+(T)$, we have

$$\sigma^*(\tilde{\lambda}) = \text{Ad}(g_0) (\sigma(\tilde{\lambda})) = n_z (\sigma(\tilde{\lambda})) = n_{\sigma^*(\lambda)}.$$

Note that $\sigma^*$ acts as $\text{Ad}(\tilde{\eta}_{\text{ad}}) \circ \sigma$ on $\tilde{W}_{\text{ad}}$. So for any $\tilde{y} \in \tilde{W}_0$, $\sigma^* (\tilde{y}) = \tilde{y} \tilde{g} (\tilde{y}) \tilde{g}^{-1}$, where $\tilde{g} = \text{Ad}(\tilde{z})(\sigma(\tilde{y}))$. Note that $\tilde{y}_{\text{ad}} = -\tilde{y} (\tilde{y}_{\text{ad}}) \tilde{z}$. We are generated by $\tilde{S}_2$, $\tilde{m}(\tilde{y})$ for $\tilde{y} \in \tilde{W}_0$ and $\tilde{n}_{\lambda}$ for $\lambda \in X_+(T)$, by the following lemma, we have

$$\sigma^*(\tilde{T}) = \tilde{T}.$$

Lemma 6.3. Let $\tilde{y} \in \tilde{W}_0$. Then for any lifting $m(\tilde{y}) \in \tilde{T}$, we have

$$\sigma^*(m(\tilde{y})) = n_{\tilde{\eta}_{\text{ad}}} (\tilde{y} \tilde{g}(\tilde{y})) \tilde{g}^{-1}.$$
Proof. We have $\Ad(\tilde{\eta})(\sigma(\tilde{y})) = t_{\eta-y'(\tilde{\eta})} \tilde{\eta} \sigma(\tilde{y}) \tilde{\eta}^{-1}$. Now

$$j(\tilde{\eta} - y'(\tilde{\eta})) = j(\tilde{\eta}) - j(y'(\tilde{\eta})) = k(\tilde{\eta}_{ad}) - k(y'(\tilde{\eta}) = k(\tilde{\eta}_{ad}) - k(y'(\tilde{\eta}))$$

Here the last equality follows from the fact that $\tilde{\eta}_{ad} - y'(\tilde{\eta}_{ad}) \in \tilde{Z} \Phi^\vee(G,T)$ and the restriction of the map $j$ to $\tilde{Z} \Phi^\vee(G,T)$ (which is just $X_*(T_w)$) is the identity map. Hence there exists $\tilde{\mu} \in X_*(\mathbb{Z}^0)$ such that

$$\tilde{\eta} - y'(\tilde{\eta}) = k(\tilde{\eta}_{ad} - y'(\tilde{\eta}_{ad})) + \tilde{\mu}.$$ 

Since $\tilde{\mu} \in X_*(\mathbb{Z}^0)$, $y'(\tilde{\mu}) = \tilde{\mu}$ and thus

$$(y')^i(\tilde{\eta}) - (y')^{i+1}(\tilde{\eta}) = k((y')^i(\tilde{\eta}_{ad}) - (y')^{i+1}(\tilde{\eta}_{ad})) + \tilde{\mu}$$

for any $i$. Let $l$ be the order of $y'$. Then

$$l \tilde{\mu} = \sum_{i=0}^{l-1} ((y')^i(\tilde{\eta}_{ad}) - (y')^{i+1}(\tilde{\eta}_{ad})) + \tilde{\mu} = \sum_{i=0}^{l-1} ((y')^i(\tilde{\eta}) - (y')^{i+1}(\tilde{\eta})) = 0.$$

So $\tilde{\mu} = 0$ and $\tilde{\eta} - y'(\tilde{\eta}) = k(\tilde{\eta}_{ad} - y'(\tilde{\eta}_{ad}))$. Then

$$\sigma'((m(\tilde{y}))(m(\tilde{y}))^{-1}(m(\tilde{y}))(m(\tilde{y}))^{-1}.$$ 

It remains to show that $g_{\tilde{\eta}} \Ad(y')(g_{\tilde{\eta}}^{-1}) = n_{\tilde{\eta}_{ad} - y'(\tilde{\eta}_{ad})}$. 

With the definition of $g_{\tilde{\eta}}$ in (6.4), we have

$$g_{\tilde{\eta}} \Ad(y')(g_{\tilde{\eta}}^{-1}) = (\tilde{\eta} - y'(\tilde{\eta}))(e_F^{1/k}) = (\tilde{\eta}_{ad} - y'(\tilde{\eta}_{ad}))(e_F) = n_{\tilde{\eta}_{ad} - y'(\tilde{\eta}_{ad})}.$$ 

This finishes the proof of the lemma. \hfill \qedsymbol

Now we state the main result of this section.

**Theorem 6.4.** Let $G$ be a connected reductive group, quasi-split over $F$ and split over $\tilde{F}$. Let $\sigma^*$ be the Frobenius morphism associated to a given $F$-isomorphism class of inner twists of $G$. Then $\tilde{T}^{\sigma^*}$ is a Tits group of the Iwahori-Weyl group of $G(\tilde{F})^{\sigma^*}$.

In the rest of this section, we will prove this theorem. The proof involves, among other things, some identities on the finite Tits groups, which we now summarize.

### 6.4. Some identities in finite Tits groups.

In this subsection, let $\tilde{F}$ be any field and let $G$ be a split, connected, reductive group over $\tilde{F}$. Let $T_{\text{fin}}$ be the Tits group of the absolute Weyl group $W_0$ of $G(\tilde{F})$ and $\{n_w\}_{w \in W_0}$ is a Tits cross-section of $W$ on $T_{\text{fin}}$. In the application to the proof of Theorem 6.4, $\tilde{F}$ is the field $\tilde{F}$ and $T_{\text{fin}}$ is the subgroup of $\tilde{T}$ generated by $n_s$ for $s \in W_0$. However, the identities on the finite Tits group hold in the general setting.

Let $\{s_i \mid i \in I\}$ be the set of simple reflections of the absolute Weyl group and $\{a^\vee_i \mid i \in I\}$ be the set of simple coroots. Then $n_{2s_i} = a^\vee_i(-1)$. For any subset $J \subset I$, let $\rho_J^\vee$ be the half sum of positive coroots spanned by $\{a^\vee_j \mid j \in J\}$ and $y_J$ be the maximal element in the subgroup generated by $s_i$ for $i \in J$. We will simply write $\rho^\vee$ for $\rho_J^\vee$. For any $i \in I$, we set

$$y(i) = y_I(i) y_J.$$

We are interested in the power of $n_{y(i)}$ when $\omega^\vee_i$ is a minuscule coweight. This is calculated using the following result.

**Proposition 6.5** (Proposition 3.2.1 of [19]). Let $u, v \in W_0$. Then

$$n_u n_v = n_{uv} \Pi_{a>0,n(a)<0,n(u)<0} a^\vee a(-1).$$

The following corollary is an easy consequence of the proposition above and some results in [1, §3].

**Corollary 6.6.** (1) Suppose that $G_{\tilde{F},ad}$ is $\tilde{F}$-simple. Let $\omega^\vee_i$ be a minuscule coweight and $k$ be the order of $y(i)$ in $W_0$. Then $n^k_{y(i)}$ is the center of $G(\tilde{F})$. 
(2) Suppose that $G$ is of type $A_n$. For $0 \leq i \leq n$,
\[ n^{i+1}_{(1)} = \begin{cases} n^{i+1}_{(1)}, & \text{if } i \text{ is even}, \\ (a^+_1 + a^+_3 + \cdots + a^+_i)(-1)n^{i+1}_{(1)}, & \text{if } i \text{ is odd}. \end{cases} \]

(3) Suppose that $G$ is of type $D_n$ with $n$ odd. Then
\begin{align*}
(a) \quad n^{2}_{y(n)} &= \begin{cases} (a^+_2 + \cdots + a^+_{n-1})(-1)n^{y(n)}_{(1)}, & \text{if } n \equiv 1 \mod 4, \\ (a^+_2 + \cdots + a^+_{n-3} + a^+_n)(-1)n^{y(n)}_{(1)}, & \text{if } n \equiv 3 \mod 4. \end{cases} \\
(b) \quad n^3_{y(n)} &= \begin{cases} n^{y(n)}_{(n-1)}, & \text{if } n \equiv 1 \mod 4, \\ n^{y(n)}_{(n-1)}(a^+_{n-1} + a^+_n)(-1), & \text{if } n \equiv 3 \mod 4. \end{cases}
\end{align*}

(4) Suppose that $G_{\tilde{s}, \ad}$ is of type $D_{2n}$ and $(i,j,k) = \{1, 2n - 1, 2n\}$. Then there exists a central element $z$ of $G(\tilde{\mathfrak{g}})$ such that
\[ n_{y(i)}^{}n_{y(j)}^{} = z \cdot n_{y(k)}^{} = n_{y(j)}^{}n_{y(i)}^{}. \]

Proof. All the parts of the corollary are consequences of Proposition 6.5 and some explicit calculations. In the case where $G$ is almost simple over $\mathfrak{g}$, Adrian showed in [1, Proposition 3.3] that $n^{k}_{y(i)} = 1$, where $k$ is the order of $y(i)$ in $W_0$. This implies (1). Parts (2), (3) are direct consequences of Proposition 6.5. Part (4) is deduced from [1, Theorem 3.5]. □

6.5. The $\sigma^*$-stable liftings of $\tilde{\mathfrak{g}}$. In this subsection, we will prove the following result.

**Proposition 6.7.** Let $\tilde{s} \in \tilde{\mathfrak{g}}_0$ and $X$ be the $\sigma^*$-orbit of $\tilde{s}$. Then
\[ (\sigma^*)^{[X]}(n_{\tilde{s}}) = n_{\tilde{s}}. \]

As a consequence, we obtain the following stronger version of Proposition 5.1.

**Corollary 6.8.** There exists a set of representatives $\{m(\tilde{s}) \mid \tilde{s} \in \tilde{\mathfrak{g}}\}$ in $\tilde{T}$ that is $\sigma^*$-stable.

**Proof.** It suffices to consider the case where $G_{\tilde{F}, \ad}$ is $\tilde{F}$-simple. In this case, $\sigma^*$ acts transitively on the set of connected components of the affine Dynkin diagram of $G_F$. The case $\sigma^* = \sigma$ is already proved. Now we assume that $\sigma^* \neq \sigma$. Then each $\sigma^*$-orbit on $\tilde{\mathfrak{g}}$ contains a simple reflection in $\tilde{\mathfrak{g}}_0$. For each $\sigma^*$-orbit $X$, we fix a representative $\tilde{s}_X$ such that $\tilde{s}_X \in X \cap \tilde{\mathfrak{g}}_0$. Then any element $\tilde{s} \in \tilde{\mathfrak{g}}$ is of the form $\tilde{s} = (\sigma^*)^{[X]}(\tilde{s}_X)$ for a unique $\sigma^*$-orbit $X$ and a unique $l$ with $0 \leq l < |X|$. Then we set $m(\tilde{s}) = (\sigma^*)^{[X]}(n_{\tilde{s}_X}) \in \tilde{T}$. By Proposition 6.7, $\{m(\tilde{s}) \mid \tilde{s} \in \tilde{\mathfrak{g}}\}$ is $\sigma^*$-stable. □

6.5.1. Reduction step. We first explain how to reduce ourselves to the case when $G_{\tilde{F}, \ad}$ is $\tilde{F}$-simple. We keep notations as in §6.3.2. Note that
\[ \tilde{W}_{ad} = \tilde{W}_{ad}^{(1)} \times \tilde{W}_{ad}^{(2)} \times \cdots \times \tilde{W}_{ad}^{(k)}, \]
with $\tilde{W}_{ad}^{(1)} \equiv \tilde{W}_{ad}^{(2)} \equiv \cdots \equiv \tilde{W}_{ad}^{(k)}$ and $\sigma$ permutes these factors transitively. Write $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(1)} \times \cdots \times \tilde{\mathfrak{g}}^{(k)}$. The element $\tilde{s} = (\tilde{s}^{(1)}, \tilde{s}^{(2)}, \ldots, \tilde{s}^{(k)})$ and $n_{\tilde{s}} = n_{\tilde{s}^{(1)}}n_{\tilde{s}^{(2)}} \cdots n_{\tilde{s}^{(k)}}$. Note that $\sigma^{k}$ stabilizes the set $\tilde{\mathfrak{g}}^{(1)}$ and that $(\sigma^{k})^k$ acts as $\Ad(\tilde{\nu}_{ad}) \circ \sigma^k$ on $\tilde{\mathfrak{g}}^{(1)}$. Let $X^{(1)}$ be the $\Ad(\tilde{\nu}_{ad}) \circ \sigma^k$-orbit of $\tilde{s}^{(1)}$ in $\tilde{\mathfrak{g}}^{(1)}$. Note that $|X| = k|X^{(1)}|$. Thus $(\sigma^{k})^{[X]}(n_{\tilde{s}^{(1)}}) = n_{\tilde{s}^{(1)}}$ if and only if $(\Ad(\tilde{\nu}_{ad}) \circ \sigma^k)^{[X^{(1)}]}(n_{\tilde{s}^{(1)}}) = n_{\tilde{s}^{(1)}}$. In particular, we may reduce ourselves to the case when $k = 1$, i.e., the case when $G_{\tilde{F}, \ad}$ is $\tilde{F}$-simple.

In this case, $\Omega_{\tilde{F}, \ad}$ is as in Table 1 and we drop all the superscripts in the rest of the argument.

Next we show that it suffices to prove the equality (6.7) below. This equality only involves the elements from the finite Tits group.

If $\tilde{\nu}_{ad} = 1$, then $\sigma^* = \sigma$ and (a) follows from the fact that the set $\{n_{\tilde{s}} \mid \tilde{s} \in \tilde{\mathfrak{g}}\}$ is $\sigma$-stable.

Now we assume that $\tilde{\nu}_{ad} \neq 1$. Recall that $\tilde{\nu}_{ad} = \tilde{t}_{ad} \tilde{z} = \tilde{\nu}_{ad} \tilde{z} \in \Omega_{\tilde{F}, \ad}$. We have $(\tilde{\nu}_{ad} \circ \sigma)^{[X]} = \tilde{t}_{ad} \tilde{z} \sigma^{[X]} \in \tilde{W}_{ad} \times \sigma$ for some $\tilde{z} \in X_{r}(T_{ad})$. Since $(\sigma^{k})^{[X]}(\tilde{s}) = \tilde{s}$, we have $(\Ad(\tilde{z}) \circ \sigma)^{[X]}(\tilde{s}) = \tilde{s}$ and $\tilde{s}(\tilde{\xi}_{ad}) = \tilde{\xi}_{ad}$. Recall in §6.3.2, we have $k\tilde{\eta}_{ad} = j(\tilde{\eta})$ for some $\tilde{\eta} \in X_{r}(T)$. Since $\tilde{\xi}_{ad}$ is an
integral linear combination of the $\hat{W}_0$-orbit of $\hat{\eta}_{ad}$, we have $k\hat{\xi}_{ad} = j(\hat{\xi})$ for some $\hat{\xi} \in X_*(T)$. By (6.6) and Lemma 6.3, we have

$$(\sigma^*)^{[X]}(n_k) = \text{Ad}(\hat{\xi}(x_F^{1/k}))(\text{Ad}(g_z) \circ \sigma)^{[X]}(n_k).$$

By the proof of Lemma 6.3, we have $\text{Ad}(\hat{\xi}(x_F^{1/k}))(n_k) = n_k$. Thus it remains to prove that

$$(\text{Ad}(g_z) \circ \sigma)^{[X]}(n_k) = n_k. \quad (6.7)$$

6.5.2. The case where $\sigma^*(\hat{s}) = \hat{s}$. If $G_{F,ad}$ is of type $D_{2n}$, $g_z = n_{z_0}$. Otherwise $g_z = n_{z_0}^i$ for a suitable $0 \leq i < |(\Omega_{ad})_s|$. For type $D_{2n}$ and for the other types with $i = 1$, the statement follows from Coxeter relations. In more detail, since $\sigma^*(\hat{b}) = \hat{z}b$, we know that $\sigma^*(\hat{b}) = \pm b$. Since $\sigma^*$ preserves $\Delta_0$, we see that $\sigma^*(\hat{b}) = \hat{b}$. But $\sigma^*(\hat{b}) = \hat{z}(\sigma(\hat{b}))$. Now, since $\sigma(\hat{b}) \in \Delta_0$ and $\hat{z}(\sigma(\hat{b})) \in \Delta_0$, we see that $l(\hat{z}\sigma(\hat{b})) = l(\hat{z}) + 1 = l(s_k\hat{z})$. By Condition (2)(b)$^1$ of §3.2, $g_z n_{z_0}^i n_{s_k} = n_{z_0}^i n_{s_k} = n_{z_0}^i g_z$.

If $G_{F,ad}$ is of type $D_{2n+1}$ and $i > 1$, then the action of $\sigma$ on $\Omega_{ad}$ is trivial and $G_{F,ad}$ is of type $A_n$, $D_{2n+1}$ or $E_6$. We have $\text{Ad}(\hat{z}_0^i)(\hat{s}) = \hat{s}$ for $1 < i < |(\Omega_{ad})_s|$ and we need to prove that $n_{z_0}^i n_{z_0} n_{s_k} = n_k$.

If $G_{F,ad}$ is of type $A_n$, then since $\hat{z}_0^i$ does not have any fixed points on $\hat{S}$, there is no such $\hat{s}$ and the statement is trivial.

If $G_{F,ad}$ is of type $E_6$, then $|(\Omega_{ad})_s| = 3$. If $i > 1$, then $i = 2$ and $\hat{z}_0 = (\hat{z}_0^2)^2$. If $\hat{z}_0^2$ fixes $\hat{s}$, then necessarily $\hat{z}_0$ fixes $\hat{s}$. By Condition (2)(b)$^1$ of §3.2, $n_{z_0} n_{s_k} = n_{z_0} n_{s_k} = n_{z_0} n_{s_k} = n_{z_0} n_{s_k}$. Hence $g_z n_{z_0} n_{s_k} = n_{z_0} n_{s_k} = n_{z_0} n_{s_k}$.

If $G_{F,ad}$ is of type $D_{2n+1}$, then $|(\Omega_{ad})_s| = 4$. The element $\hat{z}_0^3$ has no fixed elements in $\hat{S}$. For $i = 2$, since $\hat{z}_0^2$ fixes $\hat{s}$, we have $\hat{s} = \hat{s}_k$ with $1 < k < 2n$. In this case, $\hat{z}_0(\hat{s}) = \hat{s}_{2n+1-k}$. By Condition (2)(b)$^1$ of §3.2, we have $n_{z_0} n_{s_k} n_{z_0} = n_{z_0} n_{s_k} n_{z_0} = n_{z_0} n_{z_0} n_{s_k} = n_{z_0} n_{z_0} n_{s_k} = n_{z_0} n_{z_0} n_{s_k}$. So $n_{z_0}^2 n_{z_0} n_{s_k} = n_s$.

6.5.3. The remaining cases. In this subsection, we assume that $\sigma^* \neq \sigma$. So in particular, we have $(\Omega_{ad})_s \neq 1$.

We first discuss the case where the $\sigma$-action on $\Omega_{ad}$ is trivial.

If $G_{F,ad}$ is of type $A_n$, then $g_z = n_{z_0}^i$ for some $i$ with $1 < i < |(\Omega_{ad})_s|$. We have $\text{Ad}(\hat{z}_0^i)(\hat{s}) = \hat{s}$. Since $\hat{z}_0$ acts transitively on the gradients of elements of $\Delta$, we see that $|(\Omega_{ad})_s|$ divides $i|X|$. By Corollary 6.6(2), we have $\text{Ad}(g_z) \circ \sigma)^{[X]}(n_k) = (\text{Ad}(g_z))^{[X]}(n_k) = (\text{Ad}(n_{z_0}^i)(n_k) = n_k$.

If $G_{F,ad}$ is of type $A_n$ and the $\sigma$-action on $\Omega_{ad}$ is trivial, then each $\sigma^*$-orbit on $\hat{S}$ is of size 1 or the order l of $\hat{z}$ in $\hat{W}_0$. The case where $\sigma^*(\hat{s}) = \hat{s}$ is handled in §6.5.2. If $\sigma^*(\hat{s}) \neq \hat{s}$, then by Corollary 6.6(1),

$$(\sigma^*)^i(n_k) = \text{Ad}(g_z)^{[X]}(n_k) = \text{Ad}(g_z)^{[X]}(n_k) = n_k.$$

Next we discuss the case where the $\sigma$-action on $\Omega_{ad}$ is non-trivial. Since $(\Omega_{ad})_s \neq 1$, $G_{F,ad}$ is of type $A_{2n+1}$ or type $D_n$.

If $G_{F,ad}$ is of type $A_{2n+1}$, then $g_z = n_{\theta(1)}$ and $(\sigma(g_z) = n_{\theta(2n+1)}$. We have $g_z \sigma(g_z) = (\hat{a}_1^\gamma + \hat{a}_2^\gamma + \cdots + \hat{a}_{2n+1}^\gamma)(-1) \in Z(\hat{F})$. Note that any $\sigma^*$-orbit on $\hat{S}$ is of size 1 or 2. The case where $\sigma^*(\hat{s}) = \hat{s}$ is handled in §6.5.2. If $\sigma^*(\hat{s}) \neq \hat{s}$, then

$$(\sigma^*)^2(n_k) = \text{Ad}(g_z \sigma(g_z))^{[X]}(n_k) = \text{Ad}((\hat{a}_1^\gamma + \hat{a}_2^\gamma + \cdots + \hat{a}_{2n+1}^\gamma)(-1))n_k = n_k.$$

If $G_{F,ad}$ is of type $D_n$ with $n$ odd, then $g_z = n_{\theta(n)}$ and $(\sigma(g_z) = n_{\theta(-n)}$. By Corollary 6.6(3), $g_z \sigma(g_z) = 1$ or $(\hat{a}_{n-1}^\gamma + \hat{a}_{n}^\gamma)(-1)$. In either case, $g_z \sigma(g_z) \in Z(\hat{F})$. We then follow the same argument as the type $A_{2n+1}$ case above.

If $G_{F,ad}$ is of type $D_n$ with $n$ even and $g_z = n_{\theta(n)}$, then $\sigma(g_z) = n_{\theta(-n)}$. By Corollary 6.6(4), $g_z \sigma(g_z) = (\hat{a}_{n-1}^\gamma + \hat{a}_{n}^\gamma)(-1) \in Z(\hat{F})$. We then follow the same argument as the type $A_{2n+1}$ case above.
If $G_{F,\text{ad}}$ is of type $D_n$ with $n$ even and $g_2 = n_{\tilde{y}(3)}$ or $n_{\tilde{y}(n)}$, then by Corollary 6.6(4), $g_2 \sigma(g_2) = z n_{\tilde{y}(1)}$ for some $z \in Z(F)$. Note that each $\sigma^\ast$-orbit on $\tilde{S}$ is of size 1 or 4. We have

\[
(\text{Ad}(g_2) \circ \sigma)^4 = \text{Ad}(g_2 \sigma(g_2)) \text{Ad}(\sigma^2(g_2 \sigma(g_2))) \circ \sigma^4 = \text{Ad}(zn_{\tilde{y}(1)}, \sigma^2(zn_{\tilde{y}(1)})) \circ \sigma^4 = \text{Ad}(zn_{\tilde{y}(1)}, n_{\tilde{y}(1)^2}) \circ \sigma^4 = \sigma^4.
\]

Thus

\[
(\sigma^*)^4(n_x) = (\text{Ad}(g_2) \circ \sigma)^4(n_x) = \sigma^4(n_x) = n_x.
\]

This finishes the verification of (6.7) in all the remaining cases and thus finishes the proof of Proposition 6.7.

6.6. The $\sigma^\ast$-fixed liftings of $\Omega^\ast_a$. Let $\tilde{\tau} = t_\lambda \tilde{y} \in \Omega^\ast_a$. We will set $m(\tilde{\tau}) = n_\chi m(\tilde{y})$ for a suitable $m(\tilde{y}) \in \tilde{\mathcal{F}}$.

Let us first choose $m(\tilde{y})$ when $G_{F,\text{ad}}$ is $\tilde{F}$-simple. If $G_{F,\text{ad}}$ is of type $D_n$ with $n$ even, then we set $m(\tilde{y}) = n_y$. Otherwise, $\Omega_\chi,\text{ad}$ is a cyclic group and $\tilde{y} = \tilde{z}_0^j$ for $0 \leq j < |\Omega_\chi,\text{ad}|$. Set $m(\tilde{y}) = n_{\tilde{y}}$ and $m(\tilde{\tau}) = n_\chi n_\tilde{y} m(\tilde{y})$.

If $G_{F,\text{ad}}$ is not $\tilde{F}$-simple, then

\[
W(G, T) = W(G, T)^{(1)} \times \cdots \times W(G, T)^{(k)},
\]

where $W(G, T)^{(1)} \equiv \cdots \equiv W(G, T)^{(k)}$ are irreducible finite Weyl groups and the action of $\sigma$ permutes transitively the irreducible factors $W(G, T)^{(1)}, \ldots, W(G, T)^{(k)}$. There exist $\tilde{y}^{(1)} \in W(G, T)^{(1)}$ such that $\tilde{y} = \tilde{y}^{(1)}(\sigma(\tilde{y}^{(1)})) \cdots \sigma^{k-1}(\tilde{y}^{(1)})$. Define $m(\tilde{y}^{(1)})$ as in the preceding paragraph.

More precisely, if $G_{F,\text{ad}}$ is of type $D_n$ with $n$ even, then we set $m(\tilde{y}^{(1)}) = n_\tilde{y}^{(1)}$. Otherwise, $\Omega_\chi^{(1)}$ is a cyclic group and $\tilde{y}^{(1)} = \tilde{z}_0^j$ for $0 \leq j < |\Omega_\chi^{(1)},\text{ad}|$. Set $m(\tilde{y}^{(1)}) = n_{\tilde{y}^{(1)}}$. Let

\[
m(\tilde{y}) = m(\tilde{y}^{(1)}) \sigma(m(\tilde{y}^{(1)})) \cdots \sigma^{k-1}(m(\tilde{y}^{(1)})).
\]

The main result of this subsection is the following.

Proposition 6.9. Let $\tilde{\tau} \in \Omega^\ast_a$. Then $\sigma^\ast(m(\tilde{\tau})) = m(\tilde{\tau})$.

6.6.1. Reduction step. We begin with a simple lemma.

Lemma 6.10. For each $\tilde{\tau} \in \Omega^\ast_a$, we have $\sigma^\ast(\tilde{\tau}) = \sigma(\tilde{\tau})$.

Proof. Let $\tilde{\tau} = t_\lambda \tilde{y}$. Then $\sigma^\ast(\tilde{\tau}) = t_{\sigma(\lambda)} t_{\eta_{\text{ad}} - \psi'(\eta_{\text{ad}})} \tilde{y}'$, where $\tilde{y}' = Ad(\tilde{z})(\sigma(\tilde{y}))$. Since $\Omega_\chi,\text{ad}$ is abelian, $\tilde{y}' = \sigma(\tilde{y})$ and $\sigma^\ast(\tilde{y}) = \tilde{z}(\sigma(\tilde{y})) - \tilde{y} - \tilde{z}(\sigma(\tilde{y})) + \tilde{y}'$ for a suitable $\tilde{y} \in X_\ast(Z^0)$. In particular, we have $\tilde{z}$ commutes with $\tilde{y}' = \sigma(\tilde{y})$.

Let $l$ be the order of $\tilde{z}$. Then $z^{l+1}(\sigma(\lambda)) - z^l(\sigma(\lambda)) = \tilde{z}^l(\tilde{z}(\tilde{y})) = \tilde{z}^l(\tilde{z}(\tilde{y})) = \tilde{z}^l(\tilde{z}(\tilde{y})) = 0$ for any $i$. Since $\tilde{y}^{(l)} = 1$, we have $\tilde{z}(\tilde{z}(\tilde{y})) = \tilde{z}^2(\tilde{z}(\tilde{y})) = 0$. Thus

\[
l = \sum_{l=0}^{l-1} \tilde{z}^l(\tilde{y}) + \tilde{y}^{(l)}(\tilde{y}) + \tilde{y}^{(l-1)}(\tilde{y}) = 0
\]

and hence $\tilde{y} = 0$. So $\sigma^\ast(\tilde{\tau}) = t_{\sigma(\lambda)} \sigma(\tilde{y}) = \sigma(\tilde{\tau})$. 

Let $\tilde{\tau} = t_\lambda \tilde{y} \in \Omega^\ast_a \ast \Omega^\ast_a$. Then we have $\sigma(\tilde{y}) = \tilde{y}$ and $\lambda = \sigma^\ast(\tilde{y}) + \tilde{y} - \tilde{y}$ ad $- Ad(\tilde{z})(\sigma(\tilde{y}))(\tilde{y})$. By (6.6) and Lemma 6.3,

\[
\sigma^\ast(m(\tilde{\tau})) = \sigma^\ast(n_\chi m(\tilde{y})) = \sigma^\ast(n_\chi) \sigma^\ast(m(\tilde{y})) = n_{\sigma^\ast(\lambda) + \tilde{y} - \tilde{y}} - Ad(\tilde{z})(\sigma(\tilde{y}))(\tilde{y}) = n_\chi Ad(g_2) \sigma(m(\tilde{y})).
\]

To verify $\sigma^\ast(m(\tilde{\tau})) = m(\tilde{\tau})$, it remains to show

1. $\sigma(m(\tilde{y})) = m(\tilde{y})$;
2. $Ad(g_2)(m(\tilde{y})) = m(\tilde{y})$.

Now we show that it suffices to check the case where $G_{F,\text{ad}}$ is $\tilde{F}$-simple.
Lemma 6.11. We have

1. \( \sigma(m(\hat{y})) = m(\hat{y}) \) if and only if \( \sigma^k(m(\hat{y}^{(1)})) = m(\hat{y}^{(1)}) \).

2. \( \text{Ad}(g_z)(m(\hat{y})) = m(\hat{y}) \) if and only if \( \text{Ad}(g_z)(m(\hat{y}^{(1)})) = m(\hat{y}^{(1)}) \).

Proof. Note that \( \sigma(\hat{y}) = \hat{y} \) if and only if \( \sigma^k(\hat{y}^{(1)}) = \hat{y}^{(1)} \). By the definition of \( m(\hat{y}) \), we have \( \sigma(m(\hat{y})) = \sigma(m(\hat{y}^{(1)})) \sigma^2(m(\hat{y}^{(1)})) \cdots \sigma^k(m(\hat{y}^{(1)})) \). Further, (6.8) implies that

\[
\bar{T}_{\text{fin}} \cong \bar{T}_{\text{fin}}^{(1)} \times \bar{T}_{\text{fin}}^{(2)} \times \cdots \times \bar{T}_{\text{fin}}^{(k)},
\]

where \( \bar{T}_{\text{fin}}^{(i)} \) is the finite Tits group attached to \( W(G,T)^{(i)} \). Hence

\[
\sigma(m(\hat{y})) = \sigma^k(m(\hat{y}^{(1)})) \sigma(m(\hat{y}^{(1)})) \sigma^2(m(\hat{y}^{(1)})) \cdots \sigma^k(m(\hat{y}^{(1)})).
\]

Now it follows that \( \sigma(m(\hat{y})) = m(\hat{y}) \) if and only if \( \sigma^k(m(\hat{y}^{(1)})) = m(\hat{y}^{(1)}) \).

For (2), since \( g_z \in \bar{T}_{\text{fin}}^{(i)} \) and \( \sigma^i(m(\hat{y}^{(1)})) \in \bar{T}_{\text{fin}}^{(i)} \) for each \( 0 \leq i \leq k - 1 \), we see using (6.9) that \( g_z \) commutes with \( \sigma^i(m(\hat{y}^{(1)})) \) for all \( i \geq 1 \). Hence \( \text{Ad}(g_z)(m(\hat{y})) = m(\hat{y}) \) if and only if \( \text{Ad}(g_z)(m(\hat{y}^{(1)})) = m(\hat{y}^{(1)}) \). \( \square \)

6.6.2. Proof of Proposition 6.9 for \( \bar{F} \)-simple groups. We assume that \( G_{\bar{F},\text{ad}} \) is \( \bar{F} \)-simple and we drop the subscripts in the discussion below. In particular, we may assume \( \Omega_{\bar{F},\text{ad}}, \bar{v}_{\text{ad},0} \) are as in Table 1. So \( g_z = n_z \) if \( \Omega_{\bar{F},\text{ad}} \) is of type \( D_n \) with \( n \) even. Otherwise, \( g_z = n_z^1 \) for a suitable \( 0 \leq i < |\Omega_{\bar{F},\text{ad}}| \). Also \( m(\hat{y}) = m_0 \) if \( \Omega_{\bar{F},\text{ad}} \) is of type \( D_n \) with \( n \) even. Otherwise, \( m(\hat{y}) = n_z^i \) for a suitable \( 0 \leq i < |\Omega_{\bar{F},\text{ad}}| \).

We show that \( \text{Ad}(g_z)(m(\hat{y})) = m(\hat{y}) \).

When \( G_{\bar{F},\text{ad}} \) is of type \( D_n \), \( n \) even, this is a consequence of Corollary 6.6(4). Otherwise, \( g_z \) and \( m(\hat{y}) \) are both powers of \( n_z \) and the claim is obvious.

We show that \( \sigma(m(\hat{y})) = m(\hat{y}) \).

The proof involves a detailed case-by-case analysis. Recall that the representatives \( \{ n_\hat{a} \mid \hat{a} \in \hat{A} \} \) satisfies \( H(\hat{A}, \sigma) \). When \( G_{\bar{F},\text{ad}} \) is of type \( D_n \), \( n \) even, we have \( \sigma(m(\hat{y})) = \sigma(n_\hat{y}) = n_\hat{y} = m(\hat{y}) \). Next we consider the case where \( \Omega_{\bar{F},\text{ad}} \) is cyclic. If the action of \( \sigma \) on \( \Omega_{\bar{F},\text{ad}} \) is trivial, then \( \sigma(n_\hat{y}) = n_\hat{y} \) and \( \sigma(n_\hat{z}) = n_\hat{z} \). In this case, \( \sigma(m(\hat{y})) = \sigma(n_\hat{y}^i) = \sigma(n_\hat{z}^i) = m_0^i \). It remains to prove the claim when \( \Omega_{\bar{F},\text{ad}} \) is cyclic and the action of \( \sigma \) on \( \Omega_{\bar{F},\text{ad}} \) is non-trivial.

Recall that \( j : G_{\bar{F}} \to G_{\bar{F},\text{ad}} \) is the adjoint quotient. Let \( \bar{y} \in \Omega_{\bar{F},\text{ad}}^* \). If \( j(\bar{y}) = 1 \), then \( \bar{y} = 1 \) and \( m(\hat{y}) = 1 \). In this case, \( \sigma(m(\hat{y})) = 1 = m(\hat{y}) \).

Now we assume that \( j(\bar{y}) \neq 1 \). This happens when \( G_{\bar{F},\text{ad}} \) is of type \( A_{2n+1} \) or \( G_{\bar{F},\text{ad}} \) is of type \( D_{2n+1} \) and \( j = 2 \). In both these cases \( \sigma(\bar{v}_{\text{ad},0}) = \bar{v}_{\text{ad},0}^{-1} \) and \( \sigma(\bar{z}_0) = \bar{z}_0^{-1} \).

If \( G_{\bar{F},\text{ad}} \) is of type \( A_{2n+1} \), then \( m(\hat{y}) = n_{\bar{z}_0}^{n+1} \) and \( \bar{y}_{n+1} = \bar{y}(n+1) \). By Corollary 6.6(2), we have

\[
n_{\bar{z}_0}^{n+1} = \begin{cases} n_{\bar{z}_0}^{n+1}, & \text{if } n \text{ is even,} \\ (a_1^\nu + a_3^\nu + \cdots + a_n^\nu)(-1)n_{\bar{z}_0}^{n+1}, & \text{if } n \text{ is odd.} \end{cases}
\]

We have \( \sigma(\bar{y}(n+1)) = \bar{y}(n+1) \). Thus

\[
\sigma(n_{\bar{z}_0}^{n+1}) = \begin{cases} n_{\bar{z}_0}^{n+1}, & \text{if } n \text{ is even,} \\ (a_1^\nu + a_3^\nu + \cdots + a_n^\nu)(-1)n_{\bar{z}_0}^{n+1}, & \text{if } n \text{ is odd.} \end{cases}
\]

We identify \( \Omega_{\bar{F},\text{ad}} \) with \( X_*(T_{\text{ad}})/X_*(T_{\text{sc}}) \). Under this identification,

\[
j(\bar{y}) = (n+1)^\nu_0, \quad \bar{y}_{n+1} = a_1^\nu + 2a_2^\nu + 3a_3^\nu + \cdots + (2n+1)a_{n+1}^\nu
\]

\[
= \frac{a_1^\nu + 3a_3^\nu + 5a_5^\nu + \cdots + (2n+1)a_{2n+1}^\nu}{2} + \frac{a_3^\nu + 3a_5^\nu + \cdots + (2n+1)a_{2n+1}^\nu}{2} + \cdots + n\bar{a}_{2n}^\nu 
\]

mod \( X_*(T_{\text{sc}}) \).
Since \( \tilde{r} \in X_*(T)/X_*(T_{sc}) \) and \( j \) acts as identity on \( X_*(T_{sc}) \), \( \frac{a_1^j + a_2^j + \cdots + a_{2n+1}^j}{2} \in X_*(T) \). Hence\
\[
\left( \tilde{a}_1^j + \tilde{a}_2^j + \cdots + \tilde{a}_{2n+1}^j \right) (-1) = \left( \frac{\tilde{a}_1^j + \tilde{a}_2^j + \cdots + \tilde{a}_{2n+1}^j}{2} \right) (-1) = 1 \in G(\tilde{F}).
\]

Therefore we have \( \sigma(m(\tilde{y})) = \sigma(n^{\tilde{y}_{20}} + n^{\tilde{y}_{21}}) = m(\tilde{y}) \).

If \( G_{p, ad} \) is of type \( A_{2n+1} \), we have \( m(\tilde{y}) = n^{\tilde{a}_{2n+1}} \), and by Corollary 6.6(3), \( \sigma(m(\tilde{y})) = tm(\tilde{y}) \) where \( t = (\tilde{a}_2^j + \tilde{a}_{2n+1}^j)(-1) \). We claim that \( t = 1 \) in \( G(\tilde{F}) \). The argument is similar to type \( A_{2n+1} \). Consider the element \( j(\tilde{r}) \in \Omega_{ad} = X_*(T_{ad})/X_*(T_{sc}) \). Then\
\[
\tilde{v}_{ad, 0} = \tilde{a}_1^j + 2\tilde{a}_2^j + \cdots + (2n-1)\tilde{a}_{2n-1}^j + \frac{1}{2}(2n-1)\tilde{a}_{2n}^j + \frac{1}{2}(2n+1)\tilde{a}_{2n+1}^j \mod X_*(T_{sc}).
\]

Then \( j(\tilde{r}) \equiv 2\tilde{v}_{ad, 0} \equiv \frac{1}{2}a_1^j + \frac{1}{2}a_{2n+1}^j \mod X_*(T_{sc}) \). Since \( 2\tilde{r} \in X_*(T_{sc}) \) and \( j \) acts as identity on \( X_*(T_{sc}) \), we see that\n\[
\tilde{r} \equiv \frac{1}{2}\tilde{a}_2^j + \frac{1}{2}a_{2n+1}^j \mod X_*(T_{sc}).
\]

Since \( \tilde{r} \in X_*(T)/X_*(T_{sc}) \), we see that \( \frac{1}{2}a_2^j + \frac{1}{2}a_{2n+1}^j \in X_*(T) \). Hence\n\[
t = (\tilde{a}_2^j + \tilde{a}_{2n+1}^j)(-1) = \left( \frac{1}{2}\tilde{a}_2^j + \frac{1}{2}a_{2n+1}^j \right)(-1) = 1.
\]

Hence \( \sigma(m(\tilde{y})) = m(\tilde{y}) \).

This finishes the proof of Proposition 6.9.

6.7. Proof of Theorem 6.4. For \( s \in \tilde{S} \), let \( m(s) \in \tilde{T} \) be the lifting of \( \tilde{s} \) in Corollary 6.8. For any \( \tilde{r} \in \tilde{T} \), let \( m(\tilde{r}) \in \tilde{T} \) be the lifting of \( \tilde{r} \) constructed in §6.6. Then \( \sigma^*(m(\tilde{r})) = m(\tilde{r}) \) for all \( \tilde{r} \in \tilde{T} \) and \( \sigma^*(m(s)) = m(\sigma^*(s)) \) for all \( s \in \tilde{S} \).

We set \( T = \tilde{T}^\sigma^* \).

For any \( s \in \tilde{S} \), we have \( s = \tilde{w}_X \) for some \( \sigma \)-orbit \( X \) in \( \tilde{\Delta} \) with \( \tilde{W}_X \) finite (see §2.3). Let \( \tilde{w}_X = \tilde{s}_1 \cdots \tilde{s}_{n_s} \) be a reduced expression of \( \tilde{w}_X \) in \( \tilde{W}_{af} \). Then \( \tilde{w}_X = \sigma(\tilde{s}_1) \cdots \sigma(\tilde{s}_{n_s}) \) is again a reduced expression of \( \tilde{w}_X \) in \( \tilde{W}_{af} \).

We have\n\[
m(\tilde{w}_X) = m(\tilde{s}_1) \cdots m(\tilde{s}_{n_s}) = m(\sigma(\tilde{s}_1)) \cdots m(\sigma(\tilde{s}_{n_s})) = m(\sigma(\tilde{s}_1)) \cdots m(\sigma(\tilde{s}_{n_s}))
\]
\[
= \sigma(m(\tilde{w}_X)).
\]

In particular, \( m(s) = m(\tilde{w}_X) \in T = \tilde{T}^\sigma^* \).

Let \( w \in \tilde{W}_{af} \) and \( s_1 \cdots s_{n_s} \) be a reduced expression of \( w \) in \( W \). We set \( m(w) = m(s_1) \cdots m(s_{n_s}) \). Then \( m(w) \in T \). Suppose that \( s'_1 \cdots s'_{n_s} \) is another reduced expression of \( w \) in \( W \). By §2.3 (a), \( \ell(w) = \ell(s_1) + \cdots + \ell(s_{n_s}) = \ell(s'_1) + \cdots + \ell(s'_{n_s}) \). Since \( \{m(s) \mid s \in \tilde{S} \} \) satisfies the Coxeter relations, by condition (2)(b) \( m(s_1) \cdots m(s_{n_s}) = m(s'_1) \cdots m(s'_{n_s}) \). In other words, \( m(w) \) is independent of the choice of reduced expression in \( W \). Finally for \( w \in W \), we have \( w = u_1 \tau \) for a unique \( u_1 \in \tilde{W}_{af} \) and \( \tau \in \tilde{\Omega}_a = \Omega_a. \) We set \( m(w) = m(u_1)m(\tau) \). Then \( m(w) \in T \). In other words, the map \( \phi: T \to W \) is surjective. We have\n\[
ker(\phi) = \ker(\tilde{\phi}) \cap \tilde{T}_{af} = \tilde{S}_2 \cap \tilde{T}_{af} = S_2.
\]

It remains to show that for each \( a \in \Delta \), we have \( m(s_{a2}) = b^a(-1) \) where \( b \) is the gradient of \( a \). By [8, §5.1], we know that the elements of \( \Delta \) and in bijection with \( \sigma^* \)-orbits \( X \) of \( \tilde{\Delta} \) with \( \tilde{a}_{X(A,F)} \) non-constant for \( \tilde{a} \in X \). Let \( F \) denote the \( \mathbb{R} \)-vector space of affine linear functions on \( A(A,F) \). Then \( F \) may be identified with the \( \sigma^* \)-invariants of \( \tilde{F} \) where \( \tilde{F} \) is the \( \mathbb{R} \)-vector space of affine functions on \( A(T, \tilde{F}) \) (which we have identified with \( V = X_*(T) \otimes \mathbb{R} \) after choosing a special point). Under this identification, we have for \( a \in \Delta \),\n\[
a \mapsto \frac{1}{|X|} \sum_{\tilde{a} \in X} \tilde{a},
\]
where \( X \) is the \( \sigma^* \)-orbit on \( \Delta \) that corresponds to \( a \).
Fix a $\sigma^*$-invariant scalar product $(\cdot, \cdot)$ on $V = X_*(T) \otimes \mathbb{R}$ and we identify $V$ with $V^*$ via this inner product. We may extend this to a scalar product on $\mathcal{F}$ by setting $(\mathring{f}, \mathring{g}) = \langle D\mathring{f}, D\mathring{g}\rangle$, where $D\mathring{f} \in V$ is the gradient of $\mathring{f}$. For any $\mathring{f} \in \mathcal{F}$ with $D\mathring{f} \neq 0$, let $\mathring{f}^\vee = \frac{2\mathring{f}}{(\mathring{f}, \mathring{f})}$. 

Then for $a \in A$, we have $a^\vee = \frac{a}{(a, a)}$. Fix $\mathring{a} \in \mathring{A}$ whose restriction to $\mathcal{A}(A, F)$ is the affine root $a$. Then

$$\langle a^\vee, a^\vee \rangle = \frac{1}{|\mathcal{A}|} \sum_{\mathring{a}^\vee \mathcal{X}} (\mathring{a}, \mathring{a}^\vee).$$

This implies that

$$a^\vee = c_a \sum_{\mathring{a}^\vee \mathcal{X}} \mathring{a}^\vee, \quad b^\vee = c_a \sum_{\mathring{a}^\vee \mathcal{X}} \mathring{b}^\vee$$

(6.10)

where

$$c_a = \frac{\langle \mathring{a}, \mathring{a} \rangle}{\sum_{\mathring{a}^\vee \mathcal{X}} \langle \mathring{a}, \mathring{a}^\vee \rangle} = \frac{\langle \mathring{b}, \mathring{b} \rangle}{\sum_{\mathring{a}^\vee \mathcal{X}} \langle \mathring{b}, \mathring{b}^\vee \rangle}.$$

Now, let us prove that $m(s_a) = b^\vee(-1)$. We may easily reduce ourselves to the case where $G_{\mathcal{F}}$ is $\mathcal{F}$-simple and simply connected. Via a simple case-by-case analysis, each $\sigma^*$-orbit $\mathcal{X}$ consists of simple roots in $\mathring{A}$ whose corresponding Dynkin diagram is either a product of $A_1$’s or is a single copy of $A_2$.

In the former case, $c_a = 1$ and by (6.10),

$$m(s_a)^2 = m(s_a)^2 \cdot m(s_a)^2 \cdots m(s_a)^2 = b^\vee(1) b^\vee(1) \cdots b^\vee(1) = b^\vee(-1),$$

where $\mathcal{X} = \{\mathring{a}_1, \mathring{a}_2, \ldots, \mathring{a}_k\}$. In the latter case, $\mathcal{X} = \{\mathring{a}_1, \mathring{a}_2\}$ and $\mathring{a}_1 + \mathring{a}_2$ is an affine root. In this case, $c_a = 2$. By (6.10),

$$m(s_a)^2 = (m(s_a)^2 m(s_a)^2 m(s_a))^2 = 1 = b^\vee(-1).$$

Thus $\mathcal{T}$ is a Tits group of $W$ and $\{m(w) \mid w \in W\}$ is a Tits cross-section of $W$ in $\mathcal{T}$.

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