Quantum nonlocality through entanglement plays a key role as a resource in quantum teleportation, cryptography and error-correcting codes. There exists, however, another nonlocal phenomenon: quantum nonlocality without entanglement [1]. It is connected with certain unentangled states (UOB) in n letter we analyze the set of orthonormal bases consisting of unextendible product basis [1–3]. In this work is involved with the criteria for recognizing such states are determined up to phase, to think about them unambiguously we must consider them to be elements of Hilbert spaces is an Unentangled Orthogonal Basis (UOB). Bennett et al, in their study of quantum nonlocality without entanglement, noted the lack of LOCC (local operations and classical communication) distinguishability for a specific 3 qubit UOB. In general, for n qubits, we prove that in its natural structure as a real variety, the space of UOB is a bouquet of products of Riemann spheres parametrized by a class of edge colorings of hypercubes. Its irreducible components of maximum dimension are products of $2^n - 1$ two-spheres. Using a theorem of Walgate and Hardy, we observe that the LOCC distinguishable UOB are exactly those in the maximum dimensional components.

Up to phase, the element $v_1 \otimes v_2 \cdots \otimes v_n$ is considered to be $[v_1] \otimes [v_2] \otimes \cdots \otimes [v_n]$. On $\mathbb{P}^1$ we define a real analytic fixed point free involution:

$$[v] \mapsto [\bar{v}],$$

which assigns to $[v]$ the line $[\bar{v}]$ perpendicular to it (i.e. $\langle v, \bar{v} \rangle = 0$). If $S$ is a set of elements of $\mathbb{P}^1$ then $\hat{S}$ denotes the set of $[s]$ for $[s]$ in $S$.

Our first goal is to turn the determination of all UOB into a combinatorial problem on the hypercube $Q_n$. We think of the vertices of the hypercube as the vectors in $\mathbb{R}^n$ with coordinates in the set $\{0, 1\}^n$, and consider this to be binary expansions of numbers $0, 1, \ldots, 2^n - 1$. We also view $Q_n$ as a graph with vertices 0, 1, ..., $2^n - 1$; its edges are the pairs of numbers whose binary expansions differ in exactly one digit (i.e. pairs with Hamming distance 1).

Let $u_0, u_1, \ldots, u_{2^n - 1}$ be a UOB, and write its states as

$$[u_j] = [u_{j1}] \otimes [u_{j2}] \otimes \cdots \otimes [u_{jn}].$$

As observed above, if $i \neq j$ then at least one pair $\{[u_{ki}], [u_{kj}]\}$ must be of the form $\{[v], [\bar{v}]\}$. We consider the subset of $\mathbb{P}^1$ that is the set $\mathcal{T} = \{[u_{kj}] | k = 1, \ldots, n, j = 0, \ldots, 2^n - 1\}$. We divide $\mathcal{T}$ into two disjoint pieces $\mathcal{T}_0$ and $\mathcal{T}_1$ such that $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$, $i = 0, 1$. This implies that if $[\bar{t}] \in \mathcal{T}_0$ and if $[\bar{t}] \in \mathcal{T}$ then $[\bar{t}] \in \mathcal{T}_1$ and vice-versa. To each $[u_j]$ we assign a vector $s_j = (s_{j1}, s_{j2}, \ldots, s_{jn})$ distinct such that its k–th coordinate is 0 if $[u_{kj}] \in \mathcal{T}_0$ or 1 if $[u_{kj}] \in \mathcal{T}_1$. We note that if we assign to $s_j$ the corresponding element

$$[s_j] = (s_{j1}s_{j2} \cdots s_{jn}),$$

then by its very definition $\{[s_j] | j = 0, \ldots, 2^n - 1\}$ is an orthonormal set. This implies that the two sets $\mathcal{T}_0$ and $\mathcal{T}_1$ each consist of exactly half of the elements of $\mathcal{T}$ and that $\mathcal{T}_i = \mathcal{T}_{\bar{i}}$. Reordering $\mathcal{T}$, let $\mathcal{T}_0 = \{t_1, \ldots, t_r\}$, such that $s_j$ is just the binary expansion of $j$. Assume a palette of colors $c_1, c_2, \ldots, c_r$ is available. From this palette, we assign to each vertex of $Q_n$ an n–tuple of colors taken
from $c_j$ with $1 \leq j \leq r$, such that if the $i$-factor of $u_j$ is $t_j$ or $t_j^*$, we assign to it the color $c_j$. This is equivalent to coloring the edges of $Q_n$. Indeed, let $a\rightarrow b$ be an edge, so $a$ and $b$ differ in exactly one component, which, by orthonormality, has the same color in both $a$ and $b$. We give the edge $a\rightarrow b$ that color. Conversely, given an edge-coloring of $Q_n$, we can assign an $n$-tuple of colors to each vertex as follows. For the vertex $a$ and component $i$, let $a^i$ be the unique vertex with all its components the same as those of $a$ except for the $i$-th which is opposite. We assign the $i$-th component of vertex $a$ the color of edge $a\rightarrow a^i$.

**Definition 1** A coloring of $Q_n$ is said to be admissible if for every pair of vertices there is a component, $i$, so that one vertex has a 0 in the $i$-th position and the other has a 1 and both are assigned the same color in that position.

If we have a coloring of $Q_n$ with colors $c_1$, ..., $c_k$ and $[u_1], ..., [u_k]$ are elements of $\mathbb{P}^1$ then we assign to each vertex $s = s_1s_2...s_n$ a product state (up to phase): if the $i$-component has color $c_i$ and $s_i = 0$ then put $[u_i]$ in the $i$-th position; if $s_i = 1$ put $[\hat{u}_i]$ in the $i$-th position. For example, for $n = 3$ we have the admissible coloring:

```
\begin{tabular}{ccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11 \\
\end{tabular}
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Here $c_1 = \text{green}$, $c_2 = \text{blue}$, $c_3 = \text{red}$, $c_4 = \text{orange}$, $c_5 = \text{purple}$, $c_6 = \text{cyan}$ and $c_7 = \text{brown}$. The procedure assigns the UOB:

$$[u_3] \otimes [u_2] \otimes [u_1], [u_3] \otimes [u_2] \otimes [\hat{u}_1],$$

$$[u_3] \otimes [\hat{u}_2] \otimes [u_4], [u_3] \otimes [\hat{u}_2] \otimes [\hat{u}_4],$$

$$[\hat{u}_3] \otimes [u_5] \otimes [u_6], [\hat{u}_3] \otimes [u_5] \otimes [\hat{u}_6],$$

$$[\hat{u}_3] \otimes [\hat{u}_5] \otimes [u_1], [\hat{u}_3] \otimes [\hat{u}_5] \otimes [\hat{u}_7].$$

(1)

We give the set of UOB of $H_n$, $U_n$, its subspace topology in the set of $2^n$–tuples of elements of the projective space on $H_n$, $\mathbb{P}(H_n)$.

**Proposition 2** Fix a palette of colors $c_1$, ..., $c_k$, ... To each admissible coloring, $C$, of $Q_n$ with $k$ colors the procedure above yields an injective, continuous mapping

$$\Phi_C : (\mathbb{P}^1)^k \rightarrow U_n.$$  

The union of the images of $\Phi_C$ running through all admissible colorings is all of $U_n$.

For each coloring $C$ the map $\Phi_C$ is a homeomorphism onto its image. Thus $U_n$ is a finite union of smooth manifolds diffeomorphic with $(\mathbb{P}^1)^k$ for $k$ running through the cardinalities of admissible colorings of $Q_n$. We introduce a partial order on the set of colorings of $Q_n$.

**Definition 3** If $C_1$, $C_2$ are colorings of $Q_n$ then $C_1 \lessdot C_2$ if the colors used in $C_1$ form a subset, $S$, of those used in $C_2$ and the set of edges that were colored in $C_2$ by color $c \notin S$ all have their color replaced by a color in $S$.

**Lemma 4** Up to changing the names of the admissible colors $C_1 \lessdot C_2$ if and only if the image of $\Phi_{C_1}$ is contained in that of $\Phi_{C_2}$.

We make some observations about this ordering. If $C$ is a coloring of $Q_n$ let $C(i)$ denote the colors of the edges with vertices that differ in the $i$-th position. We change the colors of each $C(i)$ so that $C(i) \cap C(j) = \emptyset$ if $i \neq j$. Thus in a maximal coloring every vertex has $n$ distinct colors. There is a unique minimal coloring (up to changing the names of the colors): the coloring with one color. This coloring yields the tensor product of the standard orthogonal bases of $\mathbb{C}^2$.

Theorem 6 implies that the admissible coloring of $Q_3$ above is maximal and has the maximum number of colors, 7. This implies that $U_n$ can be thought of as a bouquet of some fourteen dimensional real manifolds and some lower dimensional ones corresponding to maximal colorings with less than 7 colors. Here is an example of a maximal coloring of $Q_3$ with 6 colors:

In preparation for our main theorem we give a recursive algorithm for admissibly coloring $Q_n$ with $2^n - 1$ colors, which the theorem asserts is the maximum number. Also Theorem 6 implies this is the only way, up to permuting indices, to color $Q_n$ admissibly with $2^n - 1$ colors.

**Lemma 5** Let $C_0$ and $C_1$ be admissible colorings of $Q_{n-1}$. Writing $Q_n$ as $0 \times Q_{n-1} \cup 1 \times Q_{n-1}$ and choosing a new color $c$ then we color $Q_n$ as follows: all first coordinates are colored with color $c$ if the first index is 0 (respectively 1) then the rest of the indices are colored as in $C_0$ (resp. $C_1$). This recipe yields an admissible coloring. In particular, if $C_0$ and $C_1$ both use $2^{n-1} - 1$ colors without any repetitions between the colors, then the number of colors is $2^n - 1$ for the coloring of $Q_n$.  




Below is an example of this method for $Q_5$ (it uses the algorithm starting with the $Q_3$ example above with 7 colors to get a $Q_4$ coloring with 15 colors and then another application to get 31 colors).

![Diagram of a 5-dimensional hypercube]

In the proof of the following result we will only use the admissibility of every 2-face of an admissible coloring.

**Theorem 6** (i) Let $Q_n$ be admissibly colored. Then there exists a subforest $F$ (i.e. a subgraph with no circuits) of $Q_n$ that has edges of every possible color in $Q_n$.

(ii) The maximum number of colors in an admissible coloring of $Q_n$ is $2^n - 1$.

(iii) $Q_n$ is admissibly colored with $2^n - 1$ colors if and only if some forest in $Q_n$ containing all of its colors each exactly once is a tree that contains all the vertices of $Q_n$.

(iv) If $Q_n$ is admissibly colored with $2^n - 1$ colors then every subcube $Q_m$ where $m < n$ is also admissibly colored with $2^m - 1$ colors.

**Proof.** We first show how one can derive (ii) and (iii) from (i). To prove (ii) we note that if a forest consists of $k$ disjoint trees and $m$ vertices then the number of edges is at most $m - k$. Thus if $F$ is the forest asserted in (i) then $m \leq 2^n$. As the number of colors is at most the number of its edges, we have that the number of colors is at most $2^n - k$, with $k$ the number of connected components (disjoint trees). This proves (ii).

To prove (iii) consider $F$, a subforest of $Q_n$ containing $2^n - 1$ edges. Then it must contain at least $2^n$ vertices and the number of connected components is 1. If $Q_n$ is admissibly colored and if $F$ is a tree containing all of its colors each exactly once and all of the vertices of $Q_n$ then since the number of edges is $2^n - 1$, that must be the number of colors.

We now prove (i) by induction on $n$. If $n = 1, 2$, the result is obvious. So we assume (i) for $n - 1 \geq 2$ and prove the result for $n$. Let $Q_n'$ be the set of elements of $Q_n$ with first coordinate $j$ with $j = 0$ or 1. We take each to be an $n - 1$ subcube and give each the coloring that it inherits from $Q_n$. The inductive hypothesis implies that for each of these cubes there is respectively a sub-forest $F \subset Q_n'$ and $G \subset Q_n'$ as in (i). From $G$ we delete all the edges with colors that are in $F$. We now take $H$ to be $F \cup G$ with a subset of edges not in the $Q_n'$ (we call such edges vertical) adjoined that contain all of the colors of $Q_n$ not contained in $F \cup G$ each exactly once. If we show that $H$ has no cycles then (i) is proved. Suppose on the contrary there is a cycle in $H$. Then it cannot stay in $F$ and vertical edges or in $G$ and vertical edges. Thus we may assume that it starts in $F$ at $p_1$ immediately goes vertical along $v_1$ then passes through $q_1, q_2, \ldots, q_k$ in $G$ and then goes vertical along the edge $v_2$ which connects to $q \in F$. The circuit may not be as yet closed but we now show that this is enough for a contradiction. In fact, we show that $v_1$ and $v_2$ must have the same color. Indeed, consider the following diagram:

\[
\begin{align*}
q_1 &\to q_2 \to q_3 \cdots \to q_{k-1} \to q_k \\
v_1 &\uparrow w_1 \uparrow w_2 \uparrow \cdots \uparrow w_{k-2} \uparrow v_2 \uparrow \\
p_1 &\to p_2 \to p_3 \cdots \cdots \to p_{k-1} \to q
\end{align*}
\]

In this diagram only the $q_i, p_1, q_1$ are guaranteed to be vertices in $H$ and only $v_1$ and $v_2$ are vertical edges in $H$. However, each of the

\[
q_i \to q_{i+1} \\
v_i \uparrow w_i \uparrow w_{i+1} \\
p_i \to p_{i+1}
\]

is a 2 dimensional subcube of $Q_n$. Since the edge $q_i \to q_{i+1}$ is in $G$ and $p_i \to p_{i+1}$ is an edge of $Q_n^0$, the two edges have different colors. This implies that $w_i$ and $w_{i+1}$ have the same color (by admissibility). The argument applies to the first and last square also so we see that $v_1$ and $v_2$ have the same color contrary to the choice of edges to include.

Before we prove (iv) we recall a property of the forest $T$ that was found in the proof of (iii). There is no path in $T$ that starts in $F$ continues in $G$ and returns to $F$. We now prove (iv). We note that it is enough to prove this for codimension one subcubes with the inherited coloring. If we choose one such subcube we rotate it so that it is $Q_n^0$. We now consider the forests $T$ and $F$. Since $Q_n$ has $2^n - 1$ colors $T$ must be connected. According to (iii) we will be done if we show that $F$ is connected. To prove this we consider $x, y$ vertices in $F$. Since $T$ is connected there must be a path from $x$ to $y$ in $T$. This path cannot leave $F$ and return to $F$. Thus it stays in $F$.

**Theorem 7** Let $Q_n$ be admissibly colored with $2^n - 1$ distinct colors. Then there exists a direction for which all $2^{n-1}$ edges in that direction have the same color.

**Proof.** We first note that the theorem can be proved directly for $n = 2, 3$. We also observe that if $Q_3$ is colored admissibly with 7 colors then if 3 out of 4 of the edges in the same direction have the same color then so does the fourth. The proof is by induction. Suppose $n \geq 4$ and the lemma is true for $Q_{n-1}$. We suppose that we have a
maximal coloring of $Q_n$ with $2^n - 1$ distinct colors. As before, let us split the $Q_n$ into two $n - 1$ dimensional subcubes, the top and the bottom. Let us call them $Q^{(0)}$ for the bottom and $Q^{(1)}$ for the top. The edges between them we call vertical. If all the vertical edges are of the same color, we are done. So suppose that there are at least two distinct colors on the vertical edges. Let us call the vertical direction the $x_n$-direction, taking the naming convention as if the cube was embedded in $\mathbb{R}^n$ with vertices $\{0,1\}^n$.

The inductive hypothesis implies that there exists some direction, let us call it the $x_1$-direction, in which all the edges in $Q^{(0)}$ are of the same color, let us say the color red. We wish to show that all the edges in the $x_1$-direction in $Q^{(1)}$ are also red. Since not all vertical edges are of the same color, there must exist some 3 dimensional subcube $Q'$ of $Q_n$, which has edges in the $x_1$-direction, the vertical $x_n$-direction, and some other third direction $x_j$, such that not all vertical edges in $Q'$ are of the same color. The cube $Q'$ has the maximum, 7, colors, therefore one of its directions has all edges of the same color. It cannot be the $x_j$-direction because the $x_1$-direction bottom edges are red, so we cannot have the two bottom $x_j$-direction edges also of the same color by Theorem 6(iv) (we would have a face with only 2 colors on a maximally colored 3-cube). Our choice of $Q'$ implies that it is not the vertical $x_n$-direction that has all the same color. Hence all the $x_1$-direction edges in $Q'$ are of the same color, and so they are all red.

Next pick an “adjacent” cube $Q''$ with edges in the $x_1$-direction, $x_n$-direction and $x_k$-direction for some $k$, such that $Q''$ and $Q'$ share an $(x_1, x_n)$-face. The two bottom edges in the $x_1$-direction in $Q''$ are red, and also the two edges in the $x_1$-direction on the face it shares with $Q'$ are red. So $Q''$ has at least 3 red edges in the $x_1$-direction, and as it is colored with the maximum, 7, colors, all edges in the $x_1$-direction in $Q''$ are red. We repeat this procedure until we have shown that all edges in the $x_1$-direction in the top cube $Q^{(1)}$ are red completing the proof.

At this point we see that up to permuting the components of $Q_n$ (and then putting them back in order of the algorithm), Lemma 4 yields all colorings with a maximum number of colors. Thus in our description of the set of all UOB as a bouquet of products of $\mathbb{P}^1$ given by the maps $\Phi_C$ for an admissible coloring of $Q_n$, the components of highest dimension $(2n+1) - 2$ are described up to permutation of factors and order as the images of $\Phi_C$ with $C$ given by the algorithm. Thus we have

**Theorem 8** The irreducible components of maximum dimension of the variety of UOB are up to permutation of factors the images of $\Phi_C$ with $C$ given by the algorithm in Lemma 4. In fact, after reordering factors we can write such a component as

$$B = \{[a] \otimes B_1, [\bar{a}] \otimes B_2\},$$

where $B_i, i = 1, 2$ are images of $\Phi_{C,i=1,2}$ respectively with $C_1, C_2$ colorings of $Q_{n-1}$ given by the algorithm in Lemma 4.

We now consider LOCC distinguishability of elements of an $n$-qubit UOB. We are given an unknown $n$-qubit state in a UOB, and allowed a protocol in which we can perform a sequence of local unitary transformations and local measurements on qubits (local operations), where the choice of which qubit to measure at each step can depend on the outcomes of the previous measurements (classical communication). We ask if this LOCC information can determine with certainty which basis element was presented. Let us consider the two families of UOB in three qubits corresponding to the first two displayed colorings above. The first is an example of a coloring, $C$, with the maximum, 7, colors. We consider the corresponding bases, of the form $\Phi_C([u_1], \ldots, [u_7])$, as in eq. $[1]$, and look at the basis state $[u_3] \otimes [u_2] \otimes [u_4]$. We note that if the first measurement is in the first qubit (after applying the local unitary transformation taking $[0]$ and $[1]$ to $[u_3]$ and $[\bar{u}_2]$ respectively), then the outcome is $[u_3]$ with certainty. From a second measurement in the second qubit (after applying the local unitary transformation taking $[0]$ and $[1]$ to $[u_2]$ and $[\bar{u}_2]$ respectively), the outcome is $[u_2]$ with certainty. Similarly the measurement in the third qubit must be $[u_4]$ with certainty. We therefore have the correct state with certainty. Notice that the order of measurement is critical. We now consider the second example which is a maximal coloring of $Q_3$ using 6 colors. This examples appears in $[1]$, where it is shown that there is no ordered set of local transformations and measurements for the UOB of the form $\Phi_C([u_1], \ldots, [u_6])$, with $[u_i] \neq [u_j]$, that will determine a basis element with certainty.

**Theorem 5** implies that the discussion above for $\Phi_C([u_1], \ldots, [u_{2n-1}])$ for an admissible coloring of $Q_n$ with $2^n - 1$ colors will work as long as the order is adapted to the algorithm, in Lemma 4 that is used to construct the coloring. Theorem 1 of Walgate and Hardy $[2]$ has the immediate implication that if $C$ is a maximal coloring of $Q_n$ with $k < 2^n - 1$ colors then there is no such ordered set of measurements that will identify with certainty a specific state in $\Phi_C([u_1], \ldots, [u_k])$, if all of the $u_i$ that appear in a given factor are distinct. This can also be seen as a direct consequence of Theorem 6 in $[3]$ and substantiates our claims.

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[1] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 59, 1070 (1999).
[2] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Communications in Mathematical Physics 238, 379 (2003).
[3] B. Somshubhro, A. Cosentino, N. Johnston, V. Russo, J. Watrous, and Y. Nengkun, arXiv:1408.6981 [quant-ph] (2014).
[4] J. Walgate and L. Hardy, Phys. Rev. Lett. 89, 147901 (2002).
[5] J. Lebl, A. Shakeel, and N. R. Wallach, to appear.
[6] N. R. Wallach, Contemp. Math. 305, 291 (2002).