RECIPEPROCAL TRANSFORMATIONS AND DEFORMATIONS OF INTEGRABLE HIERARCHIES

A. SERGYEYEV

Abstract. We present changes of variables that transform new integrable hierarchies found by Szablikowski and Błaszak using the R-matrix deformation technique [J. Math. Phys. 47 (2006), paper 043505] into known Harry-Dym-type and mKdV-type hierarchies.

Introduction

Recently, Szablikowski and Błaszak [6] came up with a new class of integrable hierarchies which is constructed as follows. Consider the Lax operator

\[ L = \sum_{i=0}^{N} u_i D_x^i + D_x^{-1} \circ u_{-1} \]  

and the related (formal) spectral problem

\[ L \psi = \lambda \psi. \]  

Here \( D_x = d/dx \) is the total \( x \)-derivative.

Following [6] consider the following hierarchy of equations compatible with (2):

\[ \psi_t^q = \left( P_{\geq 1}(L^q) + \epsilon [L^q]_0 D_x \right) \psi, \quad q = 1, 2, \ldots \]  

where for any formal series \( K = \sum_{j=-\infty}^{k} a_j D_x^j \) we have \( P_{\geq s}(K) \overset{\text{def}}{=} \sum_{j=s}^{k} a_j D_x^j \) and \( [K]_i = a_i \), and \( \epsilon \) is an arbitrary constant.

The compatibility conditions for (2) and (3) read

\[ L \psi_t^q = \left[ P_{\geq 1}(L^q) + \epsilon [L^q]_0 D_x, L \right], \quad q = 1, 2, \ldots \]  

Note that the hierarchy (4) with \( \epsilon = 0 \) was discovered by Kupershmidt [2].

The goal of the present work is to show that, under a suitable change of dependent and independent variables \( x, t_i, \) and \( u_k \), the “deformed” hierarchy (4) can be transformed into the linear extension of the undeformed \( (\epsilon = 0) \) hierarchy (4) in the senses of [4], i.e., into the said undeformed hierarchy plus a system of linear PDEs on the background of the latter. We also present a link among the deformed hierarchy (4) and the Harry-Dym-type hierarchy (5). Finally, we present the dispersionless limit of these results.

1. Main results

Let \( L \) be as above and suppose that we have

\[ [D_{ti} - P_{\geq 1}(L^i) - \epsilon [L^i]_0 D_x, D_{tj} - P_{\geq 1}(L^j) - \epsilon [L^j]_0 D_x] = 0, \quad i, j = 0, 1, 2, \ldots. \]  

Let \( \chi \) be an arbitrary (smooth) solution of the system

\[ \chi_{t_i} = \left( P_{\geq 1}(L^i) + \epsilon [L^i]_0 D_x \right) \chi, \quad i = 0, 1, 2, \ldots \]  

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such that \( \chi_x \neq 0 \).

Introduce new set of independent variables \((z, \{\tau_i\})\), where \( z = \chi \) and \( \tau_i = t_i \), instead of \((x, \{t_i\})\).

Define new dependent variables \( v_i \) by means of the formula

\[
\bar{L} = L|_{D_x = \chi_x D_z} = \sum_{i=0}^{N} u_i (\chi_x D_z)^i + D_z^{-1} \circ \chi_x^{-1} u_{i-1} \equiv v_N D_z^N + D_z^{-1} \circ v_{i-1} + \sum_{i=0}^{N-1} v_i D_z^i
\]  

We have the following generalization of Lemma 4, i) of [5]:

**Theorem 1.** The transformation \((x, \{t_i\}, \{u_j\}) \to (z, \{\tau_i\}, \{v_j\})\) sends the system

\[
L \psi = \lambda \psi, \quad \psi_t = (P_{\geq 1}(L^i) + \epsilon [L^i]_0 D_x) \psi, \quad i = 0, 1, 2, \ldots
\]

into

\[
\bar{L} \psi = \lambda \psi, \quad \psi_{\tau_i} = (P'_{\geq 2}(\bar{L}^i)) \psi, \quad i = 0, 1, 2, \ldots
\]

and hence sends the hierarchy

\[
L_{\tau_i} = [P'_{\geq 2}(\bar{L}^i), \bar{L}], \quad i = 0, 1, 2, \ldots
\]

into the Harry-Dym-type hierarchy

\[
\bar{L}_{\tau_i} = [P'_{\geq 2}(\bar{L}^i), \bar{L}], \quad i = 0, 1, 2, \ldots
\]

Here and below for any formal series \( M = \sum_{j=-\infty}^{k} b_j D_z^i \) we set \( P'_{\geq s}(M) = \sum_{j=s}^{k} b_j D_z^i \).

In the undeformed case \((\epsilon = 0)\) we just recover Lemma 4, i) of [5]. The proof of Theorem 1 amounts to performing the change of variables \((x, \{t_i\}, \{u_j\}) \to (z, \{\tau_i\}, \{v_j\})\) in (6) and using the identity (see Lemma 3, i) of [5])

\[
P'_{\geq 2}(\bar{A}) = P_{\geq 1}(A) - (P_{\geq 1}(A) \chi) D_z
\]

where \( z = \chi \) and \( \bar{A} = A|_{D_x = \chi_x D_z} \), valid for any pseudodifferential operator \( A \). With this in mind, we readily see that the transformation in question indeed sends (6) into (7). Finally, as (8) (resp. (9)) are compatibility conditions for (6) (resp. (7)), (6) goes into (7), as desired.

Note that for \( N = 1 \) a transformation relating the hierarchies (8) and (9) was found in [1].

**Theorem 2.** Consider the reciprocal transformation from \( x \) and \( t_i \), \( i = 1, 2, \ldots \) to new independent variables \( z \) and \( \tau_i \), \( i = 1, 2, \ldots \), where \( \tau_i = t_i \), \( i = 1, 2, \ldots \), and \( z \) is defined by the formula

\[
dz = (u_N)^{-1/N} dx + \epsilon \sum_{q=1}^{\infty} (u_N)^{-1/N} [L^q]_0 dt_q,
\]

and introduce new dependent variables \( v_i \) related to \( u_i \) by means of the formulas

\[
\bar{L} = L|_{D_x = (u_N)^{-1/N} D_z} = \sum_{i=0}^{N} u_i ((u_N)^{-1/N} D_z)^i + D_z^{-1} \circ (u_N)^{-1/N} u_{i-1} \equiv D_z^N + D_z \circ v_{i-1} + \sum_{i=0}^{N-1} v_i D_z^i
\]

and

\[
v_N = (u_N)^{1/N}.
\]

Then the hierarchy (4) goes into the mKdV-type hierarchy of nonlinear PDEs for \( v_i \),

\[
\bar{L}_{\tau_q} = [P'_{\geq 1}(\bar{L}^q), \bar{L}], \quad q = 1, 2, \ldots
\]

along with a separate hierarchy for \( v_N \):

\[
(v_N)_{\tau_q} = -\epsilon [L^q]_0 z, \quad q = 1, 2, \ldots
\]
Moreover, if we now introduce new dependent variables 

\[ \text{Broer–Kaup system plus a linear equation for } \]

\[ \text{Upon setting } \]

\[ \text{where } \tilde{w} \text{ is defined by the first part of (11), that is, } \]

\[ \tilde{L} = L|_{D_x = (u_N)^{-1/N} D_x} = \sum_{i=0}^{N} u_i((u_N)^{-1/N} D_z)^i + D_z^{-1} \circ (u_N)^{-1/N} u_{-1} \]

while (3) becomes

\[ \psi_{\tau_q} = P_{\geq 1}(\tilde{L}^q)(\psi), \quad q = 1, 2, \ldots \]

Moreover, if we now introduce new dependent variables \( v_i \) using (11) then we have (see e.g. (4)) \( P_{\geq 1}(L^q) = P'_{\geq 1}(\tilde{L}^q) \), so the compatibility conditions for (16) and (17) are precisely (13).

On the other hand, if we pass from \((x, \{ t_q \})\) to \((z, \{ \tau_q \})\) in (15) and set \( v_N = u_N^{1/N} \), we obtain

\[ (v_N)_{\tau_q} = -\epsilon D_z([L^q]_0), \quad q = 1, 2, \ldots \]

and as \( P_{\geq 1}(L^q) = P'_{\geq 1}(\tilde{L}^q) \) implies [5] \([L^q]_0 = [\tilde{L}^q]_0\), we see that (18) becomes (13), as desired.

Therefore, the hierarchy (4) goes into (13) plus the equations (35), q.e.d. 

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**Proof.** First of all notice that [6] by virtue of (4) we have \((u_N)_{t_q} = \epsilon D_x(u_N)[L^q]_0 - \epsilon u_N D_x([L^q]_0)\), which is equivalent to

\[ \left( u_N^{-1/N} \right)_{t_q} = \epsilon D_x(u_N[L^q]_0). \]

From this equality it is immediate that the variable \( z \) above is well defined.

Next, pass in (2) and (3) from the independent variables \((x, \{ t_q \})\) to \((z, \{ \tau_q \})\). Then (2) takes the form

\[ \tilde{L}\psi = \lambda \psi, \]

where \( \tilde{L} \) is defined by the first part of (11), that is,

\[ \tilde{L} = L|_{D_x = (u_N)^{-1/N} D_x} = \sum_{i=0}^{N} u_i((u_N)^{-1/N} D_z)^i + D_z^{-1} \circ (u_N)^{-1/N} u_{-1} \]

Thus, the hierarchy (4) does not produce any substantially new integrable systems: its contents is formed by the linear extensions (35) of the undeformed mKdV-type hierarchy (13).

To show how the above transformation works, set \( N = 1 \) and consider the so-called extended Broer–Kaup system (see Example 4 in [6]) with \( L = u D_x + v + D_x^{-1} \circ w \). From \( L_{t_i} = [P_{\geq 1}(L^i) + \epsilon [L^i] D_x, L] \) with \( i = 1, 2 \) we obtain

\[ \left( \begin{array}{c} u \\ v \\ w \end{array} \right)_{t_1} = \left( \begin{array}{c} \epsilon u_x v - \epsilon w_x \\ u w_x + \epsilon v_x \\ u_x w + u w_x + v w_x + \epsilon v_x \end{array} \right) \]

\[ u_{t_2} = \epsilon u_x v^2 - 2 \epsilon uv v_x - 2 \epsilon u_x^2 w_x - \epsilon u_x v_{xx} \]

\[ v_{t_2} = 2uv_x w + 2uv_x w_x + 2u_x^2 w_x + uu_x v_x + u_x^2 v_x + \epsilon u_x^2 v_x + 2w x v + \epsilon u_x v_x \]

\[ w_{t_2} = 2uu_x w + 2uu_x w_x + 2uu_x w - uu_x w_x - uu_x w_x - u_x^2 w_{xx} + 2u_x w_x + \epsilon u_x w_x + 2u_x w_x + 4uv w_x + \epsilon v w_x + \epsilon v w_x \]

Upon setting \( \epsilon = 0 \) and \( u = 1 \) we recover [6] from the second \((t_2)\) flow the standard Kaup–Broer system.

However, using the transformation from Theorem 2 we can reduce the system (19) to the standard Broer–Kaup system plus a linear equation for \( u \). Indeed, let us pass from the independent variables \( x \) and \( t_1 \) to \( z \) and \( \tau_1 \) defined by the formulas \( \tau_1 = t_1, \tau_2 = t_2 \), and

\[ dz = (1/u) dx + \epsilon(v/u) dt_1 + (\epsilon/u)(2wu + uw_x + v^2) dt_2 \]

(we ignored here the times \( t_i \) with \( i \neq 1, 2 \)).

We have \( D_x = (1/u) D_z \), so \( L \) becomes

\[ L = u D_x + v + D_x^{-1} \circ w = D_z + v + D_z^{-1} \circ r, \]

where \( r = wu \).
Upon passing from the variables \((x, t_1, t_2, u, v, w)\) to \((z, \tau_1, \tau_2, u, v, r)\) the system (19) takes the form

\[
\begin{pmatrix}
u \\
v_z \\
r \tau_1 \\
r_z \\
u_{r_2} \\
v_{r_2} \\
r_{r_2}
\end{pmatrix} = \begin{pmatrix}
-\epsilon v_z \\
v_z \\
r_z \\
r_z \\
-\epsilon(2r + v_z + v^2) \\
2u u x w + 2u v v x + 2u^2 w x + uu x v x + u^2 v_{xx} + \epsilon v^2 v_x + 2\epsilon u w_x w + \epsilon v_x^2 \\
2u v w + 2u v w + 2u w w - 3uu x w_x - uu x w - u^2 w_{xx} + 2\epsilon u x w^2 + 2\epsilon u v_x w + \epsilon u v_x w + \epsilon u w_x w
\end{pmatrix}
\]  

(20)

It is immediate that the last two equations of (20), namely those for \(v_{r_2}\) and \(r_{r_2}\), form the standard (non-extended) Broer–Kaup system for \(v\) and \(r\), while the equation determining \(u_{r_2}\) is a linear equation that can be readily solved for given \(v\) and \(r\).

As a final remark note that the result of Theorem 2 remains valid for the operators \(L\) of more general form than (1). Namely, Theorem 2 still holds if we replace \(L\) (1) by

\[
L = \sum_{i=-\infty}^{N} u_i D_x^i
\]

and define new dependent variables \(v_i\) using the following modification of the formula (11):

\[
\tilde{L} = L\big|_{D_x=(u_N)^{-1/N}D_x} = \sum_{i=-\infty}^{N} u_i ((u_N)^{-1/N}D_x)^i \equiv D_2^N + \sum_{i=-\infty}^{N-1} v_i D_x^i.
\]

2. Dispersionless case

Analog of Theorems 1 and 2 hold in the dispersionless case when \(D_x\) is replaced by a formal parameter \(p\) and the commutator \([,]\) is replaced by the Poisson bracket

\[
\{f, g\} = \partial f / \partial p D_x(g) - \partial g / \partial p D_x(f),
\]

see e.g. the discussion in [6] and references therein for further details.

The differential operator \(L\) (11) is now replaced by a function

\[
\mathcal{L} = \sum_{i=0}^{N} u_i p^i + p^{-1} u_{-1},
\]

and instead of the spectral problem (2) we have

\[
\mathcal{L}|_{p=\psi_x} = \lambda,
\]

while (3) is replaced by

\[
\psi_{l_q} = (P_{\geq 1}(\mathcal{L}'(p=\psi_x) + \epsilon \mathcal{L}_0)|_{p=\psi_x} \psi_x, \quad q = 1, 2, \ldots,
\]

where the projection \(P_{\geq s}\) is now now defined as follows:

\[
P_{\geq s}\left(\sum_{j=-\infty}^{N} a_j p^j\right) = \sum_{j=s}^{N} a_j p^j.
\]

Finally, the associated hierarchy, the counterpart of (4), has the form

\[
\mathcal{L}_{l_q} = \{P_{\geq 1}(\mathcal{L}'(q) + \epsilon(\mathcal{L}'(p=0D_x, \mathcal{L}), \quad q = 1, 2, \ldots
\]

We have the following results:
Consider the Theorem 4.

The transformation $v_i = (P'_{\geq 2}(\tilde{\mathcal{L}}^i))_{\tilde{\varphi} = \varphi_z}$, $i = 0, 1, 2, \ldots$ into

and hence sends the hierarchy

$\mathcal{L}_{\tilde{\varphi} = \varphi_z} = \lambda$, \hspace{1em} $\psi_{\tilde{\varphi} z} = (P'_{\geq 2}(\tilde{\mathcal{L}}^i))_{\tilde{\varphi} = \varphi_z}$, \hspace{1em} $i = 0, 1, 2, \ldots$

into the dispersionless Harry-Dym-type hierarchy

$\mathcal{L}_{\tilde{\varphi} z} = \{P'_{\geq 2}(\tilde{\mathcal{L}}^i), \tilde{\mathcal{L}}\}$, \hspace{1em} $i = 0, 1, 2, \ldots$

Theorem 4. Consider the reciprocal transformation from $x$ and $t_i$, $i = 1, 2, \ldots$ to new independent variables $z$ and $\tau_i$, $i = 1, 2, \ldots$, where $\tau_i = t_i$, $i = 1, 2, \ldots$, and $z$ is defined by the formula

and introduce new dependent variables $v_i$ related to $u_i$ by means of the formulas

$\tilde{\mathcal{L}} = \mathcal{L}_{\tilde{\varphi} = (u_N)^{-1/N}} = \sum_{i=0}^{N} u_i ((u_N)^{-1/N})^i + (u_N)^{-1/N} u_{-1} \equiv \tilde{p} + \tilde{p} v_{-1} + \sum_{i=0}^{N-1} v_i \tilde{p}^i$

and

$v_N = (u_N)^{1/N}$.

Then the hierarchy (26) goes into the hierarchy of nonlinear PDEs for $v_i$:

$\tilde{\mathcal{L}}_{v_i} = \{P'_{\geq 1}(\tilde{\mathcal{L}}^i), \tilde{\mathcal{L}}\}$, \hspace{1em} $q = 1, 2, \ldots$

and

$(-v_N)_{\tau q} = -\epsilon([\tilde{\mathcal{L}}^q]_0) z$, \hspace{1em} $q = 1, 2, \ldots$

Note that equations (34) are the compatibility conditions for the system

$\tilde{\mathcal{L}}_{\tilde{\varphi} = \varphi_z} = \lambda$

with

$\psi_{\tilde{\varphi} z} = (P_{\geq 1}(\tilde{\mathcal{L}}^q))_{\tilde{\varphi} = \varphi_z}$, \hspace{1em} $q = 1, 2, \ldots$.  

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