THE FINITISTIC DIMENSION CONJECTURES – A TALE OF 3.5 DECADES

BIRGE ZIMMERMANN HUISGEN

Abstract. We review the history of several homological conjectures, both from a chronological and a methodological point of view.

1. Introduction

Since the last five years have brought fresh initiatives in connection with the Finitistic Dimension Conjectures, providing not only new results but, at the same time, new homological methods, a synopsis of the recent developments appears helpful in keeping stock.

After some brief remarks on the origins of homological algebra, we will present the Finitistic Dimension Conjectures publicized by Bass in 1960, as well as their connections with other problems concerning the homology and structure theory of finite dimensional algebras and classical orders. We will give an overview over results to date, followed by a non-technical outline of the ideas involved. The methods applied in this context can be roughly categorized as follows; here \( \Lambda \) will stand for a finite dimensional algebra.

1. Detecting repetitions in the sequence \( \Omega^1(M), \Omega^2(M), \Omega^3(M), \ldots \) of syzygies of a \( \Lambda \)-module \( M \).
2. Passing numerical data along projective resolutions.
3. Finding the structure of second or higher syzygies.
4. Determining when the category \( \mathcal{P}_\infty \) of finitely generated \( \Lambda \)-modules of finite projective dimension is contravariantly finite in \( \Lambda \text{-mod} \).

2. A bit of history

I should really go back farther than I intend to, since the idea of using chain complexes to measure the distance of a given object from a situation which is considered ideal goes back to topology where, for instance, singular chain complexes were used to measure the discrepancy of a given space from simple connectedness. Around the middle of this century, algebraists started to adapt this idea, in the first place for use in commutative algebra, in particular, for a better understanding of the rings that arise as coordinate rings of algebraic varieties.

Officially, homological algebra came to life in the early 1950’s, with Cartan, Eilenberg, Nakayama, Auslander, Buchsbaum, Serre, and Nagata being among the principal instigators. However, there is one result going back to the last century which is, by all means, a homological one. In 1890, Hilbert’s celebrated work “Über die Theorie der algebraischen Formen” appeared [25], and in it a theorem which we now label

**Hilbert’s Syzygy Theorem (1890).** If \( k \) is a field, then \( \text{gl dim } k[X_1, \ldots, X_n] = n \).

*Key words and phrases.* Finite dimensional algebra, finitistic dimension, projective resolution, projective dimension, syzygy, contravariant finiteness.
Starting his argument with the wry comment “Der Beweis ist nicht mühelos”, Hilbert actually computed by brute force sets of relations for the iterated syzygies of a finitely generated module over $k[X_1, \ldots, X_n]$, and arrived at the conclusion that the $n$-th generation of relations is necessarily trivial. The ten pages of heavy-duty computations of Hilbert’s original proof leave us somewhat relieved that mathematicians like Emmy Noether provided a conceptual framework for the “modern” reasoning. On the side: it was also Hilbert who introduced the term “syzygy” into algebra in its present meaning (the term had previously occurred in a paper of Sylvester in 1853). Going back to the Greek roots “syn–zygon”, meaning “yoked together”, the term was first used by astronomers referring to configurations of three celestial bodies in a straight line.

The second milestone I want to mention is the

**Auslander-Buchsbaum-Serre Theorem (1955/56).** If $V$ is an algebraic variety over an algebraically closed field and $R$ its coordinate ring, then the global dimension of $R$ is finite if and only if $V$ is smooth. Moreover, in the smooth case, $\text{gl dim } R = \dim V$.

**Corollary (Auslander-Buchsbaum, 1959).** If $V$ is smooth, then all localizations of $R$ are unique factorization domains.

For details, see [42], [3], [4], [5]. The question of unique factorization had been pursued for a considerable period, and part of the credit should go to Nagata [40] and Zariski, who independently reduced the problem to varieties of dimension 3. It is this reduced situation which Auslander and Buchsbaum successfully tackled with homological methods [5].

As alluded to earlier, the general idea behind introducing homological dimensions, also into the representation theory of noncommutative rings, was to find a measure for the deviation of a given module category from the “ideal” categories arising in the Artin-Wedderburn situation which is characterized by the projectivity of all objects. This enterprise of measuring the complexity of a module category in terms of the projective dimensions of its objects and that of a ring by means of its global dimension was clearly very successful in that, in many cases, these dimensions do provide a very effective measure. Let me, however, remind you of an example where the global dimension does not at all live up to expectations.

**Example.** Let $R = k[X]/(X^2)$ where $k$ is a field. Then

- The projective dimension of the $R$-module $R/(X)$ is infinite; in fact, all syzygies of $R/(X)$ are isomorphic to $R/(X)$. Hence, $\text{gl dim } R$ is infinite. On the other hand, the category of $R$-modules is extremely simplistic, namely,
  - Each $R$-module is of the form $[\bigoplus$ copies of $R] \oplus [\bigoplus$ copies of $R/(X)]$.

In this and many other examples, a far more accurate measure of the complexity of the module category is provided by the finitistic dimensions of which I will remind you next:

$$(l)\text{fin dim } R = \sup\{p \text{ dim } M \mid M \text{ a fin. gen. left } R\text{-module with } p \text{ dim } M < \infty\}$$

$$(l)\text{Fin dim } R = \sup\{p \text{ dim } M \mid M \text{ an arbitrary left } R\text{-module with } p \text{ dim } M < \infty\}$$

Start by observing that in the previous example, $\text{Fin dim } R = 0$, which is more in line with the simplicity of the corresponding module category.

The little finitistic dimension, $\text{fin dim}$, occurred implicitly already in the proof of the Auslander-Buchsbaum-Serre Theorem, while Kaplansky suggested also studying the analogous invariant obtained by waiving the restriction to finitely generated modules. In exploring
these two new invariants, the first two natural questions appear to be as follows: Do they coincide? Are they always finite? Both questions were promptly answered in the negative for noetherian rings, even in the commutative noetherian situation.

- If \( R \) is commutative noetherian local, then \( \text{fin dim } R = \text{depth } R \). The argument is implicit in Auslander’s and Buchsbaum’s paper [4]. So in particular, a commutative noetherian local ring has coinciding big and little finitistic dimensions if and only if it is Cohen-Macaulay.

- If \( R \) is commutative noetherian, then \( \text{Fin dim } R = K \text{dim } R \). The inequality \( \geq \) was established by Bass in 1962 [10]; the other was filled in by Gruson and Raynaud in 1971 [23]. In particular, \( \text{Fin dim } R \) may be infinite: indeed, commutative noetherian rings of infinite Krull dimension are well known to exist, Nagata having supplied the first examples.

However, no examples settling these questions were available in the noncommutative artinian situation. (Of course, commutative artinian rings are uninteresting in this context, since all the invariants in question are zero in that case.) This led to the following conjectures, which were publicized as “problems” by Bass in 1960 and later – restricted to finite dimensional algebras – promoted to the status of conjectures.

**Finitistic Dimension Conjectures.** Let \( \Lambda \) be a finite dimensional algebra over a field \( k \).

(I) \( \text{fin dim } \Lambda = \text{Fin dim } \Lambda \);

(II) \( \text{fin dim } \Lambda < \infty \).

I will start with a very compressed overview of results pertaining directly to the conjectures. Next, I will indicate how strongly the conjectures are interwoven with other representation theoretic problems, and finally, I will revisit existing results, again sketching them in fairly broad strokes, but this time from a methodological standpoint.

3. Synopsis of results

Throughout, \( \Lambda \) will be a finite dimensional algebra over a field \( k \), with Jacobson radical \( J \).

- If \( \text{fin dim } \Lambda = 0 \), then Conjecture I holds, i.e., \( \text{Fin dim } \Lambda = 0 \) as well. This easy observation is embedded in a far stronger result of Bass [9]. To verify it, suppose that \( \text{Fin dim } \Lambda > 0 \), which guarantees the existence of a left \( \Lambda \)-module \( M \) with \( p \text{dim } M = 1 \). Let \( f : P \to M \) be a projective cover. Then the kernel of \( f \) is nonzero and projective, and contained in the radical of \( P \). In particular, this forces (a copy of) a principal left ideal \( \Lambda e \), where \( e \) is a primitive idempotent, to be contained in \( JP \). Clearly, \( \Lambda e \) is then contained in \( JP_0 \), where \( P_0 \) is a finitely generated projective submodule of \( P \), and \( P_0/\Lambda e \) is a finitely generated module of projective dimension 1.

- If \( J^2 = 0 \), the second Conjecture is valid; more precisely, we then have \( \text{Fin dim } \Lambda \leq 1 + \sup \{ p \text{dim } S \mid S \in \Lambda \text{-mod simple of finite projective dimension} \} \). This is due to Mochizuki [39]. For a proof, it suffices to observe that the first syzygy of any \( \Lambda \)-module is semisimple.

- A strengthening of the previous result: If the projective dimension of \( J^2 \), viewed as a right \( \Lambda \)-module, is finite, Conjecture II is true. Small derived this from Mochizuki’s result, with the following change of rings trick [13]: applied to \( I = J^2 \): If \( R \) is any ring, \( I \) a nilpotent two-sided ideal, and \( M \) a left \( R \)-module with \( \text{Tor}_m^R(R/I, M) = 0 \) for all \( m > 0 \), then \( p \text{dim}_R M = p \text{dim}_{R/I} M/IM \).
The next series of events in this connection occurred only after a long time gap.

- Suppose that $\Lambda$ is a monomial relation algebra, i.e., $\Lambda = k\Gamma/I$, where $\Gamma$ is a quiver and $I$ an ideal generated by certain paths in $k\Gamma$. Then Conjecture II holds.

The first to prove this were Green, Kirkman, and Kuzmanovich in 1991 [21]. Simplified proofs and explicit bounds on $\text{Fin dim } \Lambda$ were given by Igusa-Zacharia [32], Cibils [13], and the author [26]. We will comment on the ideas involved later.

- Conjecture II holds whenever $J^3 = 0$. It is even sufficient to replace the hypothesis by the weaker condition $\text{p dim } J^3 < \infty$.

This was proved by Green and the author in 1991 [22]. Simplified arguments are due to Fuller-Saorín [18] and Igusa-Todorov [31]. Moreover, the following generalization appeared in two installments, the first of which is due to Dräxler-Happel [14], and the second to Wang [46]: If $J^{2i+1} = 0$ and $\Lambda/J^i$ has finite representation type, then $\text{fin dim } \Lambda < \infty$.

- Denote by $P^\infty$ the full subcategory of $\Lambda$-mod the objects of which are the modules of finite projective dimension. If $P^\infty$ is contravariantly finite in $\Lambda$-mod (the definition will follow in Section 8 below), Conjecture II is true. This was shown by Auslander and Reiten, again in 1991 [7].

- Conjecture I fails, even for monomial relation algebras. In fact, for each $n \geq 2$, there exists a monomial relation algebra $\Lambda$ such that $\text{fin dim } \Lambda = n$ and $\text{Fin dim } \Lambda = n + 1$. Examples were given by the author in 1992 [27]. A simplified proof for the underlying theory was developed by Butler [12].

In order to widen the horizon, I would like to point out a few of the numerous connections between the Finitistic Dimension Conjectures and other problems. Moreover, I would like to include some immediately contingent questions in the discussion.

4. Implications and related problems

- If $\text{fin dim } \Lambda < \infty$, the Nakayama Conjecture holds for $\Lambda$. This latter conjecture asserts that a finite dimensional algebra $\Lambda$ of finite dominant dimension (i.e., possessing a minimal injective resolution $0 \to \Lambda \to Q_0 \to Q_1 \to \cdots$ in which all the terms $Q_i$ are projective) is quasi-Frobenius. This implication was pointed out by Tachikawa in [44]. I include the easy argument: Denote the homomorphisms in the above resolution by $f_i : Q_i \to Q_{i+1}$, and suppose that $\text{fin dim } \Lambda = d < \infty$. Assuming that all of the $Q_i$ are projective, the kernel of $f_i$ has projective resolution

$$0 \to \Lambda \to Q_0 \to \cdots \to Q_{i-1} \to \ker f_i \to 0.$$  

Combining this with the fact that $\text{p dim } \ker f_{i+1} \leq d$, we see that $\Lambda$ is a direct summand of $Q_0$ and thus injective.

- In case $\text{fin dim } \Lambda < \infty$, finiteness of the left injective dimension of $\Lambda$ implies finiteness of the right injective dimension. (It is well known that these dimensions coincide when both are finite.) This was pointed out by Auslander.

- Problem: Find explicit bounds on the finitistic dimensions of $\Lambda$ in terms of $\text{dim}_k \Lambda$.

Recall the following non-constructive result due to Schofield [41]: Given a field $k$, there exists a map $f : \mathbb{N} \to \mathbb{N}$ such that, for each finite dimensional $k$-algebra $\Lambda$, either $\text{gl dim } \Lambda = \infty$ or else $\text{gl dim } \Lambda \leq f(\text{dim}_k \Lambda)$. It was subsequently observed by Jensen and Lenzing [37] that an analogous statement holds if the global dimension of $\Lambda$ is replaced by $\sup\{\text{p dim } M |$
\( M \in \Lambda\text{-mod}, \text{length } M \leq b \}, \) where \( b \) is a fixed positive integer. Clearly this gives rise to the hope for a corresponding result for finitistic dimensions, at least for those known to be finite. Closely related is the problem of finding test classes of modules on which the finitistic dimension is known to be attained. In contrast to the global dimension, the finitistic dimensions are in general not attained on the cyclic modules, not even on the \( b \)-generated modules for any prefixed integer \( b \) (see Section 7 below).

To underline the usefulness of such bounds, we briefly describe an old problem on classical orders which can be translated into a question concerning a rather specialized, but still poorly understood, class of finite dimensional algebras.

- Problem: Establish bounds on the global dimension of classical orders over discrete valuation rings. Let \( \mathcal{O} \subseteq M_n(K) \) be a classical order over \( D \), where \( D \) is a discrete valuation ring with quotient field \( K \); moreover, denote by \( \pi \) a uniformizing parameter of \( D \). In 1970, Tarsy conjectured that the finiteness of the global dimension of \( \mathcal{O} \) implies \( \text{gl dim } \mathcal{O} \leq n - 1 \), where \( n \) is the matrix size of the order [45]. This conjecture may have been based merely on a paucity of examples; nonetheless, it appears to have had a stimulating effect on the area. In [36], V. A. Jategaonkar observed that, up to isomorphism, the number of orders in \( M_n(K) \) of finite global dimension is finite for any \( n \), whence a bounding function of \( n \) on the finite global dimensions arising inside \( M_n(K) \) does exist. Moreover, positive results relating to the conjecture were obtained for specialized classes of orders by her [35, 36], Kirkman-Kuzmanovich [38], and Fujita [16]. In particular, the conjecture received some credence in the situation where \( \mathcal{O} \) is tiled, meaning that \( \mathcal{O} \) contains a full set of \( n \) orthogonal idempotents. However, in [16], Fujita exhibited a class of tiled classical orders in \( M_n(K), n \geq 6 \), having global dimension \( n \), thus refuting the conjecture. Recently, Jansen and Odenthal constructed examples showing that a replacement of \( n - 1 \) by \( n \) will not save the conjecture; in fact, they obtained tiled classical orders \( \mathcal{O}_n \subseteq M_n(K), n \geq 8 \) and even, such that \( \text{gl dim } \mathcal{O}_n = 2n - 8 \) [34]. So this leaves the problem of suggesting another plausible bound in terms of the matrix size. A suggestion for tackling this question was given by Green-Kirkman-Kuzmanovich in [21], where they show the following homological connection between a classical order \( \mathcal{O} \) and the finite dimensional algebra \( \Lambda = \mathcal{O}/\pi \mathcal{O} \) over the field \( D/(\pi) \); namely, \( \text{fin dim } \mathcal{O} = 1 + \text{fin dim } \Lambda \). Moreover, Kirkman and Kuzmanovich observed that \( \text{gl dim } \Lambda = \infty \) whenever \( n \geq 2 \) and \( \mathcal{O} \) is tiled. The reason why this approach has a chance of being more viable lies in the fact that deleting \( \pi \mathcal{O} \) gives rise to a considerable simplification of the situation; in other words, the homology of \( \Lambda \) should be more transparent than that of \( \mathcal{O} \).

- Question: What is the structure of modules in \( \Lambda\text{-mod} \) having finite projective dimension, in contrast to the structure of those of infinite projective dimension? One of the major problems encountered in connection with the Finitistic Dimension Conjectures lies in the fact that, short of computing projective resolutions, one has no means of recognizing modules of finite projective dimension. A promising key to such structure theorems was provided by Auslander-Reiten in [7] for the case where \( \mathcal{P}^{\infty} \) is contravariantly finite in \( \Lambda\text{-mod} \) (see Section 8 below).

- Problem: Develop algorithms for computing or estimating homological dimensions. In [2], Anick-Green laid the foundations for an algorithm to construct projective resolutions; this work was later combined with an adapted theory of Gröbner bases by Farkas-Feustel-Green [15] and implemented on the computer by Feustel and Green. A low-effort algorithm for obtaining the finitistic dimensions of monomial relation algebras, up to an error of 1, was
presented by the author in [26]. Furthermore, a computer program for the computation of global dimensions of classical orders was developed at the University of Stuttgart under the direction of K. W. Roggenkamp.

At this point, we will supplement the sketch of the existing results given above, and discuss the principal ideas behind the arguments.

5. Method 1: Repetitions in the structure of syzygies

This method was used in proving the first generation of results for monomial relation algebras.

The idea is to take advantage of repetitions likely to occur in the sequence of syzygies of a module. Recall the trivial example given to motivate the introduction of the finitistic dimensions; in that example, all syzygies recurred ad infinitum. While this example is particularly simplistic, the phenomenon as such is not at all atypical. However, usually it does not occur in quite as clean-cut a form.

To measure repetitiveness in the structure of syzygies, we introduce the following repetition index of a module \( M \) in \( \Lambda\text{-Mod} \):

\[
\rho(M) = \inf \{ i \in \mathbb{N} | \text{each non-projective indecomposable summand of } \Omega^i(M) \text{ occurs as a summand of } \Omega^j(M) \text{ for infinitely many } j \}.
\]

In case the above set is empty, we set \( \rho(M) = \infty \). This concept is not altogether new; without introducing the repetition index, Jans [33] referred to modules \( M \) with the property that \( \text{add}(\bigoplus_{i \geq 1} \Omega^i(M)) \) has finite representation type as modules having an “ultimately closed” projective resolution; clearly such modules have finite repetition index. Another slightly different concept measuring repetition is due to Fuller and Wang [19]: they say that the projective resolution of a module \( M \) “has a strongly redundant image from an integer \( n \)” in case each indecomposable direct summand of \( \Omega^n(M) \) occurs as a summand of some \( \Omega^j(M) \) with \( j > n \); this number \( n \) may differ from the repetition index by 1.

- The following facts were implicitly proved by Igusa and Zacharia in [32]:
  
  (a) If \( \rho = \rho((\Lambda/J)_\Lambda) \), then (l) \( \text{Fin dim } \Lambda \leq \rho \).
  
  (b) If \( \Lambda \) is a monomial relation algebra, then \( \rho((\Lambda/J)_\Lambda) \leq \text{dim}_k J \).

Remarks: (i) The inequality under (a) remains true whenever \( \rho = \rho(E_\Lambda) \), where \( \text{Soc } E \) contains all simple right \( \Lambda \)-modules [20]. Interestingly, injective cogenerators usually yield better bounds on \( \text{Fin dim } \Lambda \) than \( \Lambda/J \). On the side, we mention that Fuller and Wang [19] extended a version of (a) to noetherian rings.

(ii) If \( \Lambda = \mathcal{O}/\pi \mathcal{O} \) for a tiled classical order \( \mathcal{O} \) as above, then finiteness of the global dimension of \( \mathcal{O} \) implies that all \( \Lambda \)-modules have finite repetition index; this was observed by Goodearl and the author [20]. We conjecture that the repetition indices of injective cogenerators for \( \text{Mod-} \Lambda \) are always finite when \( \Lambda = \mathcal{O}/\pi \mathcal{O} \). If confirmed, this would show that the little finitistic dimensions of tiled classical orders are necessarily finite.
**Example for Method 1.** Let $D = k[[\pi]]$, where $k$ is a field and $\pi$ an indeterminate, and let $K = k((\pi))$. If we define $\mathcal{O}$ to be the following subring of $M_5(K)$

$$
\mathcal{O} = \begin{pmatrix}
D & D & D & D & D \\
\pi & D & D & D & D \\
\pi^2 & \pi & D & D & D \\
\pi^2 & \pi^2 & \pi & D & \pi \\
\pi^3 & \pi^2 & \pi & \pi & D 
\end{pmatrix}
$$

then $\Lambda = \mathcal{O}/\pi\mathcal{O}$ has the quiver

```
1 ----> 4 \\
|      |
|      |
2 ----> 3 ----> 5
```

and indecomposable left projectives $\Lambda e_1, \ldots, \Lambda e_5$ with the following graphs:

```
1
|    |
| \  |
2 ----> 4 1 ----> 3
|    |
2     3
|    |
3 ----> 4 5
|    |
3 ----> 4 5 1
|    |
5
```

See [26, 11] for an introduction to these graphs; similar graphical representations of modules were developed by Alperin [1] and Fuller [17]. Here we mention only that the entries in the $l$-th row of vertices in the graph of a module $M$ give the indices of the simple direct summands of $J^{l-1}M/J^lM$. It is readily checked that the syzygies of $S_1 = \Lambda e_1/Je_1$ have the following graphs:

```
\Omega^1(S_1)    \Omega^2(S_1)    \Omega^3(S_1)
```

which shows that $\rho(S_1) = 1$. We leave it as an easy exercise for the reader to check that $\rho(S_2 \oplus \cdots \oplus S_5) = 2$. This implies that

$$
\text{fin dim } \mathcal{O} = 1 + \text{fin dim } \Lambda \leq 3.
$$

(Here, in fact, the global dimension of $\mathcal{O}$ is 3.)
6. **Method 2: Passing numerical data along projective resolutions**

This method applies to the case where $J^3 = 0$ as well as to monomial relation algebras (see \[22, 28, 26\]).

The underlying idea is roughly this: Let $M$ be a finitely generated left $\Lambda$-module which is contained in the radical of a projective module. (Note that all syzygies of finitely generated modules are of this form.) The goal is to find a package of numerical data for $M$ which determines whether $M$ is zero, and allows to compute the analogous package of data for $\Omega^1(M)$. The numerical invariants I have in mind are multiplicities of the simple modules in the various radical layers of $M$, which can be conveniently arranged in a suitable group $\text{Mat} \ Z$ of integer matrices. Suppose, for the moment, that for each module $M$ as above, we have found a suitable matrix $[M] \in \text{Mat} \ Z$, together with a $\mathbb{Z}$-linear map $L : \text{Mat} \ Z \rightarrow \text{Mat} \ Z$ such that

$$[M] = 0 \iff M = 0$$

$$L([M]) = [\Omega^1(M)].$$

Then, clearly, the projective dimension of $M$ is less than or equal to $m$ if and only if $L^{m+1}([M]) = 0$, and hence $\text{fin dim} \ \Lambda \leq \text{rank}_\mathbb{Z} (\text{Mat} \ Z)$.

The easiest instance where this method works to advantage is the case where $J^3 = 0$ and all the simple left modules, $S_1, \ldots, S_n$, have infinite projective dimension. In that case, we choose $\text{Mat} \ Z$ to be the group of all $2 \times n$ matrices over $\mathbb{Z}$ and $[M]$ to be the matrix listing the multiplicities of the $S_i$ in $M/JM$ and in $JM$ as entries of the first and second row, respectively. Then we can easily find a linear map $L$ as above, which tells us that $\text{fin dim} \ \Lambda \leq 2n$ in this case. Examples readily show that this bound is sharp. If some of the simples $S_i$ have finite projective dimension, a similar choice of matrices still yields a bound on $\text{fin dim} \ \Lambda$, but not in terms of the vector space dimension of $\Lambda$; in fact, the left finitistic dimension of $\Lambda$ is bounded above by

$$1 + \sup \{p \dim S_i \mid p \dim S_i < \infty \} + 2 \cdot \text{card} \{S_i \mid p \dim S_i = \infty \}.$$

(See \[22\].)

Considering several matrix groups simultaneously, however, does lead to a general bound in terms of the vector space dimension for algebras $\Lambda$ with vanishing radical cube \[28\]. Namely:

- If $J^3 = 0$ and $n$ is again the number of isomorphism types of simple $\Lambda$-modules, then

$$\text{fin dim} \ \Lambda \leq n^2 + 1 \leq (\dim_k \Lambda)^2 + 1.$$

**Example for Method 2.** Let $\Gamma$ be the following quiver:
Whenever \( I \subseteq k\Gamma \) is an ideal containing all paths of length 3,
\[
\text{fin dim} \ (k\Gamma/I) \leq 42.
\]

7. Method 3: Finding the structure of second or higher syzygies

This method provides the theory in the background of the above-mentioned examples refuting the first Finitistic Dimension Conjecture.

“Usually”, the submodules of projectives which arise as \( m \)-th syzygies get progressively simpler as \( m \) grows. The key to results derived from such simplifications lies in the observation that, if one completely understands \( m \)-th syzygies, one can determine the finitistic dimensions up to an error of \( m - 1 \). Monomial relation algebras provide an instance where this method succeeds with \( m = 2 \). The following was proved in [26]:

- If \( \Lambda = k\Gamma/I \) is a monomial relation algebra and \( M \) an arbitrary left \( \Lambda \)-module, then \( \Omega^2(M) \) is isomorphic to a direct sum of copies of principal left ideals \( \Lambda p \) for suitable paths \( p \) in \( k\Gamma \setminus I \) of positive length.

The original proof of this result is somewhat rough going; those who prefer it smoother we refer to [12]. This structure of second syzygies permits to compute both the big and the little finitistic dimension of \( \Lambda \) up to an error of 1. Namely, if
\[
s = \sup \{ p \dim \Lambda q \mid q \text{ a path of length } \geq 1 \text{ with } p \dim \Lambda q < \infty \},
\]
and \( s = -1 \) if this set is empty, then
\[
\text{fin dim} \ \Lambda, \ \text{Fin dim} \ \Lambda \in [s + 1, s + 2].
\]

There is a straightforward graphical method for computing \( s \), which can easily be carried out by hand for algebras of moderate \( k \)-dimensions, say of dimensions \( \leq 50 \) (see [26]).

Due to the fact that the appearance of a principal left ideal \( \Lambda p \) in the second syzygy of a module \( M \) can be related to the structure of the first syzygy [27], the above theorem, moreover, allows us to build monomial relation algebras with prescribed finitistic dimensions differing by at most 1.

Further examples resulting from this approach are as follows: There exist monomial relation algebras for which the little finitistic dimension is not attained on a cyclic module; more strongly, for each positive integer \( b \), there exists a monomial relation algebra whose little finitistic dimension is not attained on a module of length at most \( b \). On top of this, the mentioned phenomena may depend on the choice of base field. See [29].

**Example for Method 3.** Let \( \Gamma \) be the quiver

![Quiver Diagram](quiver.png)

and let \( \Lambda = k\Gamma/I \) be the monomial relation algebra whose indecomposable projectives have the following graphs:
(Again we refer to [26] for an interpretation of these graphs.) In other words, the ideal \( I \) is generated by all those paths which do not make an appearance in any of the above graphs. Using the method of [26], one readily checks that the only paths \( q \) with \( \text{pdim} \Lambda q < \infty \) are \( q = \epsilon \) and \( q = \mu \), and that \( \text{pdim} \Lambda \epsilon = 1 \), \( \text{pdim} \Lambda \mu = 0 \). Thus we obtain \( s = 1 \) and conclude that

\[ 2 \leq \text{finit dim} \Lambda \leq \text{Fin dim} \Lambda \leq 3. \]

In fact, the little finitistic dimension of \( \Lambda \) is 3, since \( \text{pdim} \Lambda e_1/\Lambda(\alpha + \beta) = 3 \).

8. Method 4: Contravariant finiteness

The following concept is due to Auslander and Smalø [8] and was further developed by Auslander-Buchweitz [6] in the commutative case, by Auslander-Reiten [7] in the context of finite dimensional algebras. A full subcategory \( \mathcal{A} \) of \( \Lambda\text{-mod} \) is called contravariantly finite if each module \( M \) in \( \Lambda\text{-mod} \) has an \( \mathcal{A} \)-approximation as follows: there exists a homomorphism \( f: A \to M \) with \( A \in \mathcal{A} \) such that each \( g \in \text{Hom}_\Lambda(B, M) \) with \( B \in \mathcal{A} \) factors through \( f \), i.e.,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & M \\
\exists & \Downarrow & \Downarrow g \\
& B & \\
\end{array}
\]

In the literature, the maps which we call \( \mathcal{A} \)-approximations are labeled right \( \mathcal{A} \)-approximations, to account for the dual concept. It is well known that, provided \( M \) has an \( \mathcal{A} \)-approximation, the \( \mathcal{A} \)-approximations of minimal length are isomorphic; it is thus unambiguous to speak of “the” minimal \( \mathcal{A} \)-approximation of \( M \) in that case. Recall that \( \mathcal{P}^\infty \) denotes the full subcategory of \( \Lambda\text{-mod} \) the objects of which are the modules of finite projective dimension. As we mentioned in the synopsis of results above, contravariant finiteness of \( \mathcal{P}^\infty \) in \( \Lambda\text{-mod} \) implies finiteness of \( \text{fin dim} \Lambda \) [7]. The major advantage of this particular approach to classes of algebras satisfying the second Finitistic Dimension Conjecture lies in the fact that, potentially, it yields a great deal of additional information as a byproduct.

- If \( \mathcal{P}^\infty \) is contravariantly finite in \( \Lambda\text{-mod} \), and \( A_1, \ldots, A_n \) are the minimal \( \mathcal{P}^\infty \)-approximations of the simple left \( \Lambda \)-modules, then \( \text{fin dim} \Lambda = \max\{\text{pdim} A_i \mid 1 \leq i \leq n\} \). Moreover, an object in \( \Lambda\text{-mod} \) has finite projective dimension if and only if it is a direct summand of a module \( X \) with a filtration \( X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_d = 0 \) with \( X_i/X_{i+1} \in \{A_1, \ldots, A_n\} \), up to isomorphism [7].

In other words, the minimal \( \mathcal{P}^\infty \)-approximations of the simples are the basic building blocks of the modules of finite projective dimension in case \( \mathcal{P}^\infty \) is contravariantly finite. The drawback of this approach lies in the following catch: For few classes of algebras is it known whether \( \mathcal{P}^\infty \) is contravariantly finite and, in general, the question is not at all easy to decide.
Furthermore, even when $\mathcal{P}^\infty$ is known to be contravariantly finite, describing or computing the minimal approximations of the simple modules is rather tricky. To date, the following results have been obtained along this line:

- If $\Lambda$ is stably equivalent to a hereditary algebra, then $\mathcal{P}^\infty$ is contravariantly finite [7].
- Igusa-Smalø-Todorov demonstrated with an example that $\mathcal{P}^\infty$ need not be contravariantly finite, not even for a monomial relation algebra with $J^3 = 0$ [30]. Their example will be presented below.
- If $\Lambda$ is left serial, meaning that the indecomposable projective left $\Lambda$-modules are uniserial, then $\mathcal{P}^\infty$ is contravariantly finite, and the minimal approximations of the simple left $\Lambda$-modules are completely understood [11]. In fact, given quiver and relations for $\Lambda$, they can be explicitly constructed. It turns out that, contrary to the widespread belief that the left module theory over left serial algebras is fairly simple, these minimal approximations attest to a rather sophisticated structure in general. We illustrate the shape of the resulting approximations with an example.

**Example for Method 4.** Let $\Gamma$ be the quiver

```
1
\downarrow \alpha
\downarrow \gamma
\downarrow \beta
\downarrow \delta
5
```

and let $I \subseteq k\Gamma$ be such that the graphs of the indecomposable projective left $\Lambda$-modules are as follows:

```
1 \quad 2 \quad 3 \quad 4 \quad 5
\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon
5 \quad 3 \quad 5 \quad 3 \quad 5
\epsilon \quad \gamma \quad \epsilon \quad \gamma \quad \epsilon
5 \quad 5 \quad 5 \quad 5
5 \quad 5 \quad 5 \quad 5
```

Then the minimal $\mathcal{P}^\infty$-approximation $A_1$ of the simple $S_1$ going with the vertex 1 has graph

```
4 \quad 2 \quad 1 \quad 2 \quad 4
\downarrow \delta \quad \beta \quad \beta \quad \delta
3 \quad 3 \quad 3 \quad 3
\gamma \quad \gamma \quad \gamma \quad \gamma
5 \quad 5 \quad 5 \quad 5
\epsilon \quad \epsilon \quad \epsilon \quad \epsilon
5 \quad 5 \quad 5 \quad 5
```
Happel and the author developed criteria for contravariant finiteness and infinite dimensional substitutes for minimal $\mathcal{P}^\infty$-approximations in case $\mathcal{P}^\infty$ fails to be contravariantly finite. To define these substitutes, consider the following slight variant of approximations as defined by Auslander-Smalø: Suppose that $\mathcal{C}$ is a subcategory of $\mathcal{A}$. A $\mathcal{C}$-approximation of $M$ inside $\mathcal{A}$ is a map $f \in \text{Hom}_\mathcal{A}(A, M)$ with $A \in \mathcal{A}$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{f} & M \\
\Downarrow & & \Downarrow \\
C & \xrightarrow{g} & \exists
\end{array}
$$

for all $C \in \mathcal{C}$ and $g \in \text{Hom}_\mathcal{A}(C, M)$. Note that, whenever $\mathcal{C}$ has finite representation type, $\mathcal{C}$-approximations of any module $M$ inside $\mathcal{A}$ exist. The following “phantoms” are to take over the role of minimal $\mathcal{P}^\infty$-approximations when the latter fail to exist.

Fix $M \in \Lambda$-mod. A $\Lambda$-module $H$ is called a $\mathcal{P}^\infty$-phantom of $M$ if there exists a nonempty finite subclass $\mathcal{A}$ of $\mathcal{P}^\infty$ such that each $\text{add}(\mathcal{A})$-approximation of $M$ inside $\mathcal{P}^\infty$ has $H$ as a subfactor; direct limits of such modules $H$ are again labeled phantoms. In particular, phantoms of the latter type need not be finitely generated. The crucial point is that non-trivial phantoms always exist. Of course, the “best” phantom is the minimal $\mathcal{P}^\infty$-approximation of $M$ in the case of existence. The remaining case is covered by the following theorem:

- $M$ does not have a $\mathcal{P}^\infty$-approximation if and only if $M$ has $\mathcal{P}^\infty$-phantoms of countably infinite $k$-dimension [24].

This equivalence comes with instructions for finding phantoms over various classes of algebras. We illustrate these concepts with several examples. The first is a slight variant of the above-mentioned Igusa-Smalø-Todorov example.

**Further Examples for Method 4.** (a) Let $\Gamma$ be the quiver

$$
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
\xrightarrow{\beta} & & \\
2 & \xrightarrow{\gamma} & 
\end{array}
$$

and suppose that the indecomposable projectives have the following graphs:

$$
\begin{array}{ccc}
\Lambda e_1 & & \Lambda e_2 \\
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
\xrightarrow{\beta} & & \\
2 & \xrightarrow{\gamma} & 
\end{array} & & \begin{array}{ccc}
2 & \xrightarrow{\gamma} & \exists \\
2 & \xrightarrow{\beta} & \exists
\end{array}
\end{array}
$$

Then the simple left $\Lambda$-module $S_1$ fails to have a $\mathcal{P}^\infty$-approximation. The following graph represents an infinite dimensional $\mathcal{P}^\infty$-phantom $H_1$ of $S_1$:

$$
\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 1 \\
\xrightarrow{\beta} & & \xrightarrow{\beta} \\
2 & \xrightarrow{\beta} & 2 \\
\xrightarrow{\beta} & & \xrightarrow{\beta} \\
2 & & 
\end{array}
$$
This example might leave the impression that the existence of a module $H_1$ of finite projective dimension, with graph as above, already prevents $S_1$ from having a $\mathcal{P}^\infty$-approximation. The next example counters this impression.

(b) This time, the quiver $\Gamma$ is

$$
3 \overset{\delta}{\longrightarrow} 1 \overset{\alpha}{\underset{\beta}{\longrightarrow}} 2 \gamma
$$

and the indecomposable projectives in $\Lambda$-mod have the graphs

\[
\begin{array}{ccc}
\Lambda e_1 & \Lambda e_2 & \Lambda e_3 \\
\alpha & 1 \gamma & 2 \\
\beta & 2 & 3 \\
\gamma & 2 & 1 \\
\end{array}
\]

Note that the graphs of $\Lambda e_1$ and $\Lambda e_2$ are as under (a). In particular, there is again a unique infinite dimensional module $H_1$ of finite projective dimension having the graph displayed above. However, this time, $H_1$ is not a $\mathcal{P}^\infty$-phantom of $S_1$. In fact, $\mathcal{P}^\infty$ is contravariantly finite in this example, the minimal $\mathcal{P}^\infty$-approximation of $S_1$ being of the form

$$
1 \overset{\beta}{\underset{\alpha}{\longrightarrow}} 2 \overset{\delta}{\longrightarrow} 3
$$

Our final example is to indicate how instable both of the conditions, contravariant finiteness of $\mathcal{P}^\infty$ and failure thereof, are in general.

(c) Take over the quiver of example (b), as well as the graphs of the indecomposable projectives $\Lambda e_1$ and $\Lambda e_2$. Only the graph of $\Lambda e_3$ is modified, through the deletion of one relation.

\[
\begin{array}{ccc}
\Lambda e_1 & \Lambda e_2 & \Lambda e_3 \\
\alpha & 1 \gamma & 2 \\
\beta & 2 & 3 \\
\gamma & 2 & 1 \\
\end{array}
\]
We display two $\mathcal{P}^\infty$-phantoms of $S_1$ having infinite $k$-dimension:

\[
\begin{array}{c|c|c|c}
1 & 1 & 1 \\
\beta & \beta & \beta \\
2 & 2 & 2 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c|c|c}
3 & 3 & 3 \\
\delta & \delta & \delta \\
1 & 1 & 1 \\
\beta & \beta & \beta \\
2 & 2 & 2 \\
\end{array}
\]

In particular, these phantoms show that $S_1$ does not have a $\mathcal{P}^\infty$-approximation in this example.

References

[1] J. L. Alperin: ‘Diagrams for modules’, J. Pure Appl. Algebra 16 (1980) 111-119.
[2] D. J. Auzick and E. L. Green: ‘On the homology of path algebras’, Communic. in Algebra 15 (1987) 309-341.
[3] M. Auslander and D. Buchsbaum: ‘Homological dimension in noetherian rings’, Proc. Nat. Acad. Sci. USA 42 (1956) 36-38.
[4] ———: ‘Homological dimension in regular local rings’, Trans. Amer. Math. Soc. 85 (1957) 390-405.
[5] ———: ‘Unique factorization in regular local rings’, Proc. Nat. Acad. Sci. USA 45 (1959) 733-734.
[6] M. Auslander and R.-O. Buchweitz: ‘The homological theory of maximal Cohen-Macaulay approximations’, Memoire Soc. Math. France 38 (1989) 5-37.
[7] M. Auslander and I. Reiten: ‘Applications of contravariantly finite subcategories’, Advances in Math. 86 (1991) 111-152.
[8] M. Auslander and S. Smalo: ‘Preprojective modules over artin algebras’, J. Algebra 66 (1980) 61-122.
[9] H. Bass: ‘Finitistic dimension and a homological generalization of semiprimary rings’, Trans. Amer. Math. Soc. 95 (1960) 466-488.
[10] ———: ‘Injective dimension in noetherian rings’, Trans. Amer. Math. Soc. 102 (1962) 18-29.
[11] W. D. Burgess and B. Zimmermann Huisgen: ‘Approximating modules by modules of finite projective dimension’, preprint.
[12] M. Butler: Unpublished notes, 1992.
[13] C. Cibils: ‘The syzygy quiver and the finitistic dimension’, Communic. in Algebra 21 (1993) 4167-4171.
[14] P. Dräxler and D. Happel: ‘A proof of the generalized Nakayama conjecture for algebras with $J^{2l+1} = 0$ and $A/J$ representation finite’, J. Pure Appl. Algebra 78 (1992) 161-164.
[15] D. R. Farkas, C. D. Feustel, and E. L. Green: ‘Synergy in the theories of Gröbner bases and path algebras’, Canad. J. Math. 45 (1993) 727-739.
[16] H. Fujita: ‘Tiled orders of finite global dimension’, Trans. Amer. Math. Soc. 322 (1990) 329-341. Erratum: Trans. Amer. Math. Soc. 327 (1991) 919-920.
[17] K. R. Fuller: ‘Algebras from diagrams’, J. Pure Appl. Algebra 48 (1987) 23-37.
[18] K. R. Fuller and M. Saorín: ‘On the finitistic dimension conjecture for artinian rings’, manuscripta math. 74 (1991) 117-132.
[19] K. R. Fuller and Y. Wang: ‘Redundancy in resolutions and finitistic dimensions of Noetherian rings’, Communic. in Algebra 21 (1993) 2983-2994.
[20] K. R. Goodearl and B. Zimmermann Huisgen: ‘The syzygy type of finite dimensional algebras and classical orders’, in preparation.
[21] E. L. Green, E. E. Kirkman, and J. J. Kuzmanovich: ‘Finitistic dimension of finite dimensional monomial algebras’, J. Algebra 136 (1991) 37-51.
[22] E. L. Green and B. Zimmermann Huisgen: ‘Finitistic dimension of artinian rings with vanishing radical cube’, Math. Z. 206 (1991) 505-526.
[23] L. Gruson and L. Raynaud: ‘Critères de platitude et de projectivité. Techniques de “platification” d’un module’, Invent. Math. 13 (1971) 1-89.
[24] D. Happel and B. Zimmermann Huisgen: ‘Viewing finite dimensional representations through infinite dimensional ones’, in preparation.

[25] D. Hilbert: ‘Über die Theorie der algebraischen Formen’, Math. Ann. 36 (1890) 473-534.

[26] B. Zimmermann Huisgen: ‘Predicting syzygies over monomial relation algebras’, manuscripta math. 70 (1991) 157-182.

[27] ———: ‘Homological domino effects and the first finitistic dimension conjecture’, Invent. Math. 108 (1992) 369-383.

[28] ———: ‘Bounds on finitistic and global dimension for artinian rings with vanishing radical cube’, J. Algebra 161 (1993) 47-68.

[29] ———: ‘Field dependent homological behavior of finite dimensional algebras’, manuscripta math. 82 (1994) 15-29.

[30] K. Igusa, S. Smalø and G. Todorov: ‘Finite projectivity and contravariant finiteness’, Proc. Amer. Math. Soc. 109 (1990) 937-941.

[31] K. Igusa and G. Todorov: ‘On the finitistic global dimension conjecture for artin algebras’, preprint.

[32] K. Igusa and D. Zacharia: ‘Syzygy pairs in a monomial algebra’, Proc. Amer. Math. Soc. 108 (1990) 601-604.

[33] J. P. Jans: ‘Some generalizations of finite projective dimension’, Illinois J. Math. 5 (1961) 334-344.

[34] W. Jansen and C. Odenthal: work in progress.

[35] V. A. Jategaonkar: ‘Global dimension of triangular orders over a discrete valuation ring’, Proc. Amer. Math. Soc. 38 (1973) 8-14.

[36] ———: ‘Global dimension of tiled orders over a discrete valuation ring’, Trans. Amer. Math. Soc. 196 (1974) 313-330.

[37] C. U. Jensen and H. Lenzing: ‘Model Theoretic Algebra’, New York-London (1989) Gordon and Breach.

[38] E. E. Kirkman and J. J. Kuzmanovich: ‘Global dimensions of a class of tiled orders’, J. Algebra 127 (1989) 57-72.

[39] H. Mochizuki: ‘Finitistic global dimension for rings’, Pacific J. Math. 15 (1965) 249-258.

[40] M. Nagata: ‘A general theory of algebraic geometry over Dedekind domains II’, Amer. J. Math. 80 (1958) 382-420.

[41] A. Schofield: ‘Bounding the global dimension in terms of the dimension’, Bull. London Math. Soc. 17 (1985) 393-394.

[42] J.-P. Serre: ‘Sur la dimension homologique des anneaux et des modules Noethériens’, in Proc. Internat. Symposium on Algebraic Number Theory, Tokyo and Nikko 1955, pp. 175-189, Tokyo (1956) Science Council of Japan.

[43] L. W. Small: ‘A change of rings theorem’, Proc. Amer. Math. Soc. 19 (1968) 662-666.

[44] H. Tachikawa: ‘Quasi-Frobenius rings and generalizations’, Lecture Notes in Math. 351, Berlin-Heidelberg-New York (1973) Springer-Verlag.

[45] R. B. Tarsy: ‘Global dimension of orders’, Trans. Amer. Math. Soc. 151 (1970) 335-340.

[46] Y. Wang: ‘A note on the finitistic dimension conjecture’, Communic. in Algebra 22 (1994) 2525-2528.

Department of Mathematics, University of California, Santa Barbara, CA 93106, U.S.A.
E-mail address: birge@math.ucsb.edu