ADDMC: Exact Weighted Model Counting with Algebraic Decision Diagrams

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Abstract. We compute exact literal-weighted model counts of CNF formulas. Our algorithm employs dynamic programming, with Algebraic Decision Diagrams as the primary data structure. This technique is implemented in ADDMC, a new model counter. We empirically evaluate various heuristics that can be used with ADDMC. We also compare ADDMC to state-of-the-art exact model counters (Cachet, c2d, d4, miniC2D, and sharpSAT) on the two largest CNF model counting benchmark families (BayesNet and Planning). ADDMC solves the most benchmarks in total within the given timeout.

Keywords: Weighted model counting · Algebraic decision diagram · Dynamic programming

1 Introduction

Model counting is a fundamental problem in artificial intelligence, with applications in machine learning, probabilistic reasoning, and verification [13,4,22]. Given an input set of constraints, with the focus in this paper on Boolean constraints, the model counting problem is to count the number of satisfying assignments. Although this problem is #P-Complete [36], a variety of tools exist that can handle industrial sets of constraints, cf. [28,23].

Dynamic programming is a powerful technique that has been applied both across computer science [16] and specifically to model counting [2,27]. The key idea is to solve a large problem by solving a sequence of smaller subproblems and then incrementally combining these solutions into the final result. Dynamic programming provides a natural framework to solve a variety of problems defined on sets of constraints: subproblems can be formed by partitioning the constraints into sets, called clusters. This framework has also been instantiated into algorithms for database-query optimization [21] and SAT-solving [35,1,25]. Techniques for local computation can also be seen as a variant of this framework, e.g. in theorem proving [37] or probabilistic inference [31].

In this work, we study two algorithms that follow this dynamic-programming framework and can be adapted for model counting: bucket elimination [11] and Bouquet’s Method [5]. Bucket elimination aims to minimize the amount of information needed to be carried between subproblems. When this information must be stored in an uncompressed table, bucket elimination will, with some
carefully chosen sequence of clusters, require the minimum possible amount of intermediate data (as governed by the treewidth of the input formula [2]). Intermediate data, however, need not necessarily be stored uncompressed. Several works have shown that using compact representations of intermediate data can dramatically improve bucket elimination for Bayesian inference [26,30,8]. Moreover, it has been observed that using compact representations – in particular, Binary Decision Diagrams (BDDs) – can allow Bouquet’s Method to outperform bucket elimination for SAT-solving [25]. Compact representations are therefore promising to improve dynamic-programming-based algorithms for model counting [2,27].

In particular, we consider the use of Algebraic Decision Diagrams (ADDs) [3] for model counting in a dynamic-programming framework. An ADD is a compact representation of a real-valued function as a directed acyclic graph. ADDs have been used in stochastic model checking [19] and stochastic planning [15]. ADDs have also been used for model counting [14], but not within a dynamic-programming framework. For functions with logical structure, an ADD representation can be exponentially smaller than the explicit representation. The construction of an ADD depends heavily on the choice of an order on the variables that form its domain, called a diagram variable order. Some variable orders may produce ADDs that are exponentially smaller than others for the same real-valued function. A variety of techniques exist in prior work to heuristically find diagram variable orders [33,18].

In addition to the diagram variable order, both bucket elimination and Bouquet’s Method require another order on the variables to build and arrange the clusters of input constraints; we call this a cluster variable order. We show in Section 5.1 that the choice of heuristics to find cluster variable orders has a significant impact on the runtime performance of both bucket elimination and Bouquet’s Method.

The primary contribution of this work is a dynamic-programming framework for weighted model counting that utilizes ADDs as a compact data structure. In particular:

– We lift the BDD-based approach of [25] for Boolean satisfiability to an ADD-based approach for weighted model counting.
– We implement this algorithm using ADDs and a variety of existing heuristics to produce ADDMC, a new weighted model counter.
– We perform an experimental comparison of ADDMC with a variety of other model counters and show that ADDMC solves the most benchmarks within the given timeout.

In Section 3, we outline the theoretical foundation for performing weighted model counting with ADDs. In Section 4, we present an algorithm for performing weighted model counting through dynamic-programming techniques, and discuss a variety of existing heuristics that can be used in the algorithm. In Section 5.1, we compare the performance of various heuristics in ADDMC and demonstrate that Bouquet’s Method is competitive with bucket elimination. In Section 5.2, we compare ADDMC against state-of-the-art model counters Cachet,
c2d, d4, miniC2D, and sharpSAT on two benchmark families: probabilistic inference (BayesNet) and planning (Planning). In particular, ADDMC is able to solve 87% of benchmarks while the next best solver d4 solves 84%. ADDMC is the fastest solver on 44% of the benchmarks. Through further analysis, we observe that ADDMC is less competitive on the Planning benchmarks but is a dramatic improvement over other counters on the BayesNet benchmarks. Finally, we conclude in Section 6.

2 Preliminaries

In this section, we introduce weighted model counting, the central problem of this work, and Algebraic Decision Diagrams, the primary data structure we use to solve weighted model counting.

2.1 Weighted Model Counting

The central problem of this work is to compute the weighted model count of a Boolean function, which we now define.

**Definition 1.** Let \( \varphi : 2^X \rightarrow \{0, 1\} \) be a Boolean function over a set of variables \( X \), and let \( W : 2^X \rightarrow \mathbb{R} \) be an arbitrary function. The **weighted model count** of \( \varphi \) w.r.t. \( W \) is

\[
W(\varphi) = \sum_{\tau \in 2^X} \varphi(\tau) \cdot W(\tau).
\]

The function \( W : 2^X \rightarrow \mathbb{R} \) is called a **weight function**. In this work, we focus on so-called **literal-weight functions**, where the weight of a model can be expressed as the product of weights associated with all satisfied literals. That is, where the weight function \( W \) can be expressed for all \( \tau \in 2^X \) as

\[
W(\tau) = \prod_{x \in \tau} W^+(x) \cdot \prod_{x \in X \setminus \tau} W^-(x)
\]

for some functions \( W^+(x), W^-(x) : X \rightarrow \mathbb{R} \). One can interpret these literal-weight functions \( W \) as assigning a real number weight to each literal: \( W^+(x) \) to \( x \) and \( W^-(x) \) to \( \neg x \). It is common to restrict attention to weight functions whose range lies in the positive reals or in \([0, 1]\).

When \( \varphi \) is given as a CNF formula, computing the literal-weighted model count is \#P-Complete [36]. Several algorithms and tools for weighted model counting directly reason about the CNF representation. For example, Cachet uses DPLL search combined with component caching and clause learning to perform weighted model counting [28].

If \( \varphi \) is given in a compact representation – e.g., as a Binary Decision Diagram (BDD) [6] or as a Sentential Decision Diagram (SDD) [10] – computing the literal-weighted model count can be done in time polynomial in the size of the representation. One recent tool for weighted model counting that exploits this is
miniC2D, which compiles the input CNF formula into an SDD and then performs a polynomial-time count on the SDD [23]. Nevertheless, these usually-compact representations may be exponential in the size of the corresponding CNF formula in the worst case.

2.2 Algebraic Decision Diagrams

The central data structure we use in this work is an Algebraic Decision Diagram (or ADD) [3], a compact representation of a function as a directed acyclic graph. Formally, an ADD is a tuple \((X, S, \pi, G)\), where \(X\) is a set of Boolean variables, \(S\) is an arbitrary set (called the carrier set), \(\pi : X \to \mathbb{N}\) is an injection (called the diagram variable order), and \(G\) is a rooted directed acyclic graph satisfying the following three properties. First, every terminal node of \(G\) is labeled with an element of \(S\). Second, every non-terminal node of \(G\) is labeled with an element of \(X\) and has two outgoing edges labeled 0 and 1. Finally, for every path in \(G\), the labels of the visited non-terminal nodes must occur in increasing order under \(\pi\). ADDs were originally designed for use in matrix multiplication and shortest path algorithms [3] and have since been used for stochastic model checking [19] and stochastic planning [15]. In this work, we do not need arbitrary carrier sets; it is sufficient to consider ADDs with \(S = \mathbb{R}\).

An ADD \((X, S, \pi, G)\) is a compact representation of a function \(f : 2^X \to S\). Although there are many ADDs representing each such function \(f\), for each injection \(\pi : X \to \mathbb{N}\) there is a unique minimal ADD that represents \(f\) with \(\pi\) as the diagram variable order, called the canonical ADD. ADDs can be minimized in polynomial time, so it is typical to only work with canonical ADDs. Given two ADDs representing functions \(f\) and \(g\), the ADDs representing \(f + g\) and \(f \cdot g\) can also be computed in polynomial time.

The choice of diagram variable order can have a dramatic impact on the size of the ADD. A variety of techniques exist to heuristically find diagram variable orders. Moreover, since Binary Decision Diagrams (or BDDs) [6] can be seen as ADDs with carrier set \(S = \{0, 1\}\), there is significant overlap with the techniques to find variable orders for BDDs. We discuss these heuristics in more detail in Section 4.

Several tools exist for efficiently manipulating ADDs. Here we use the tool CUDD [32], which supports carrier sets \(S = \{0, 1\}\) and (using floating-point arithmetic) \(S = \mathbb{R}\). In particular, CUDD offers APIs for carrying out multiplication and addition of ADDs.

3 Using ADDs for Weighted Model Counting with Early Projection

An ADD with carrier set \(\mathbb{R}\) can be used to represent both a Boolean function \(\varphi : 2^X \to \{0, 1\}\) and a weight function \(W : 2^X \to \mathbb{R}\). ADDs are thus a natural candidate as a data structure for weighted model counting algorithms.
In this section, we outline theoretical foundations for performing weighted model counting with ADDs. We consider first the general case of weighted model counting. We then specialize to literal-weighted model counting of CNF formulas and show how the technique of early projection can take advantage of such factored representations for $\varphi$ and $W$.

3.1 General Weighted Model Counting

We assume that the Boolean formula $\varphi$ and weight function $W$ are represented as ADDs. The goal is to compute $W(\varphi)$, the weighted model count of $\varphi$ w.r.t. $W$. To do this, we define two operations on functions $2^X \to \mathbb{R}$ that can be efficiently computed using the ADD representation: product and projection. These operations are combined in Theorem 1 to perform weighted model counting.

First, we define the product of two functions.

**Definition 2.** Let $X$ and $Y$ be sets of variables. The product of $A : 2^X \to \mathbb{R}$ and $B : 2^Y \to \mathbb{R}$ is the function $A \cdot B : 2^{X \cup Y} \to \mathbb{R}$ defined for all $\tau \in 2^{X \cup Y}$ by

$$(A \cdot B)(\tau) = A(\tau \cap X) \cdot B(\tau \cap Y).$$

Note that the operator $\cdot$ is commutative, associative, and has an identity element $1 : 2^\emptyset \to \mathbb{R}$ (that maps $\emptyset$ to 1). If $\varphi : 2^X \to \{0,1\}$ and $\psi : 2^Y \to \{0,1\}$ are Boolean formulas, their product $\varphi \cdot \psi$ is the Boolean function corresponding to their conjunction $\varphi \land \psi$. As mentioned earlier, CUDD enables implementation of the operator $\cdot$ for ADDs.

Second, we define the projection of a real-valued function, which reduces the number of Boolean variables by “summing out” a single variable. Variable projection in real-valued functions is similar to variable elimination in Bayesian networks [38].

**Definition 3.** Let $X$ be a set of variables and let $x \in X$. The projection of $A : 2^X \to \mathbb{R}$ w.r.t. $x$ is the function $\exists_x A : 2^{X \setminus \{x\}} \to \mathbb{R}$ defined for all $\tau \in 2^{X \setminus \{x\}}$ by

$$(\exists_x A)(\tau) = A(\tau) + A(\tau \cup \{x\}).$$

One can check that projection is commutative, i.e., $\exists_x \exists_y A = \exists_y \exists_x A$ for all variables $x, y \in X$ and functions $A : 2^X \to \mathbb{R}$. CUDD enables implementation of the projection operator for ADDs via ADD addition.

We are now ready to use product and projection to perform weighted model counting, through the following theorem.

**Theorem 1.** Let $\varphi : 2^X \to \{0,1\}$ be a Boolean function over a set of variables $X = \{x_1, \ldots, x_n\}$, and let $W : 2^X \to \mathbb{R}$ be an arbitrary weight function. Then

$$W(\varphi) = (\exists_{x_1} \cdots \exists_{x_n} (\varphi \cdot W))(\emptyset).$$
Theorem 1 suggests that \( W(\varphi) \) can be computed by constructing an ADD for \( \varphi \) and another for \( W \), computing the ADD for their product \( \varphi \cdot W \), and performing a sequence of projections to obtain the final weighted model count. Unfortunately, we see in Section 5 that this “monolithic” approach is infeasible on many benchmarks: the ADD representation for \( \varphi \cdot W \) is often too large, even with the best possible diagram variable order.

Instead, we show in the following section a technique for avoiding the construction of an ADD for \( \varphi \cdot W \) by rearranging the products and projections.

### 3.2 Early Projection

A key technique in symbolic computation is that of early projection: when performing a product followed by a projection (as in Theorem 1), it is sometimes possible and advantageous to perform the projection first. Early projection is possible when one component of the product does not depend on the projected variable. Early projection has been used in a variety of settings, including database-query optimization [17], symbolic model checking [7], and satisfiability solving [24]. The formal statement is as follows.

**Theorem 2 (Early Projection).** Let \( X \) and \( Y \) be sets of variables, \( A : 2^X \to \mathbb{R} \), and \( B : 2^Y \to \mathbb{R} \). For all \( x \in X \setminus Y \),

\[
\exists_x (A \cdot B) = (\exists_x A) \cdot B.
\]

The use of early projection in Theorem 1 is quite limited when \( \varphi \) and \( W \) have already been represented as ADDs; since on many benchmarks, both \( \varphi \) and \( W \) depend on most of the variables. If \( \varphi \) is a CNF formula and \( W \) is a literal-weight function, however, both \( \varphi \) and \( W \) can be rewritten as the product of smaller functions. This can significantly increase the applicability of early projection.

Assume that \( \varphi \) is a CNF formula, i.e., given as a set of clauses. For every clause \( C \in \varphi \), observe that \( C \) is a Boolean function \( C : 2^{X_C} \to \{0, 1\} \) (where \( X_C \subseteq X \) is the set of variables that appear in \( C \)) that maps satisfying assignments to 1 and unsatisfying assignments to 0. One can check using Definition 2 that \( \varphi = \prod_{C \in \varphi} C \).

Similarly, assume that \( W : 2^X \to \mathbb{R} \) is a literal-weight function. For every \( x \in X \), define \( W_x : 2^{\{x\}} \to \mathbb{R} \) to be the function that maps \( \emptyset \) to \( W^-(x) \) and \( \{x\} \) to \( W^+(X) \). One can check using Definition 2 that \( W = \prod_{x \in X} W_x \).

When \( \varphi \) is a CNF formula and \( W \) is a literal-weight function, then, we can rewrite Theorem 1 as

\[
W(\varphi) = \left( \exists_{x_1} \cdots \exists_{x_n} \left( \prod_{C \in \varphi} C \cdot \prod_{x \in X} W_x \right) \right)(\emptyset). \tag{1}
\]

By taking advantage of the commutative and associative properties of the product operator and the commutative property of projection, it is possible to rearrange Equation (1) in order to apply early projection. We present an algorithm to perform this rearrangement in the following section.
4 Dynamic Programming for Model Counting

We now discuss an algorithm for performing literal-weighted model counting of CNF formulas using ADDs through dynamic-programming techniques.

Algorithm 1: ADD Weighted Model Counting

| Line | Description |
|------|-------------|
| 1    | $\pi \leftarrow \text{get-diagram-var-order}(\varphi)$ |
| 2    | $\rho \leftarrow \text{get-cluster-var-order}(\varphi)$ |
| 3    | $m \leftarrow \max_{x \in X} \rho(x)$ |
| 4    | for $i = m, m-1, \ldots, 2, 1$ |
| 5    | $\kappa_i \leftarrow \{C \in \varphi : \text{get-cluster-rank}(C, \rho) = i\}$ /* constructing cluster $\kappa_i$ of rank $i$ */ |
| 6    | $X_i \leftarrow \text{Vars}(\kappa_i) \setminus (\text{Vars}(\kappa_{i+1}) \cup \ldots \cup \text{Vars}(\kappa_m))$ /* the variables in $X_i$ will be projected at step $i$ in the next loop */ |
| 7    | $X_m \leftarrow X_m \cup (X \setminus (\text{Vars}(\kappa_1) \cup \ldots \cup \text{Vars}(\kappa_m)))$ /* collecting variables declared in $X$ which do not appear in $\varphi$ */ |
| 8    | for $i = 1, 2, \ldots, m$ |
| 9    | if $\kappa_i \neq \emptyset$ |
| 10   | $A \leftarrow \text{get-ADD}(\kappa_i, \pi)$ |
| 11   | for $x \in X_i$ |
| 12   | $A \leftarrow \exists x(A \cdot W_x)$ |
| 13   | if $i < m$ |
| 14   | $j \leftarrow \text{choose-cluster}(A, i)$ |
| 15   | $\kappa_j \leftarrow \kappa_j \cup \{A\}$ |
| 16   | return $A(\emptyset)$ |

Our algorithm is presented as Algorithm 1. Broadly, our algorithm partitions the clauses of a formula $\varphi$ into a sequence of clusters $\kappa_1, \kappa_2, \ldots, \kappa_m$. For each cluster, we construct the ADD corresponding to the conjunction of its clauses. These ADDs are then incrementally combined via the product operator. Throughout, each variable of the ADD is projected as early as Theorem 2 allows (where $X_i$ indicates the variables that can be projected after each iteration $i$). At the end of the algorithm, all variables have been projected and the resulting ADD has a single node representing the weighted model count. This algorithm can be seen as rearranging the terms of Equation 1 (according to the clusters) in order to perform the projections indicated by $X_i$ at each step $i$.

The helper function $\text{get-ADD}(\kappa_i, \pi)$ constructs the ADD representing the product of all clauses in cluster $\kappa_i$ (including, possibly, ADDs that were added to $\kappa_i$ on line 15 of prior iterations) using $\pi$ as the diagram variable order. The remaining functions that appear within Algorithm 1, namely $\text{get-cluster-rank}$, $\text{choose-cluster}$, $\text{get-diagram-var-order}$, and $\text{get-cluster-var-order}$, represent heuristics that can be used to tune the specifics of Algorithm 1.
Before discussing the various heuristics, we assert the correctness of Algorithm 1 in the following theorem.

**Theorem 3.** Let $\varphi$ be a nonempty CNF formula over variables $X$ and let $W : 2^X \rightarrow \mathbb{R}$ be a literal-weight function. Assume that get-diagram-var-order returns an injection $X \rightarrow \mathbb{N}$, and moreover that for every call to get-cluster-rank and choose-cluster the following properties hold:

1. $1 \leq \text{get-cluster-rank}(C, \rho) \leq m$,
2. $i < \text{choose-cluster}(A, i) \leq m$, and
3. for all $k \in \mathbb{N}$ s.t. $i < k < \text{choose-cluster}(A, i)$, the sets of variables $X_k$ and $\text{Vars}(A)$ are disjoint.

Then Algorithm 1 returns $W(\varphi)$.

Condition 1 ensures that $\{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ is a partition of the clauses of $\varphi$. Condition 2 ensures that line 15 always places $A$ in a cluster that has not yet been processed. Condition 3 ensures that every variable $x$ projected in every step $i$ does not appear in later clusters (i.e., $x \notin \text{Vars}(\kappa_j)$ for all $i < j \leq m$). These three invariants are sufficient to prove that Algorithm 1 indeed computes the weighted model count of $\varphi$ w.r.t. $W$. All the heuristics we describe in this section satisfy the conditions of Theorem 3.

In the remainder of this section, we discuss various existing heuristics that can be used to implement get-diagram-var-order, get-cluster-var-order, get-cluster-rank, and choose-cluster. We compare the performance of these heuristics for Algorithm 1 in Section 5.

### 4.1 Heuristics for get-diagram-var-order and get-cluster-var-order

The heuristic chosen for get-diagram-var-order indicates the diagram variable order to use in every ADD constructed by Algorithm 1. The heuristic chosen for get-cluster-var-order indicates the variable order which, when combined with the heuristic for get-cluster-rank, is used to order the clauses of $\varphi$ (BE orders clauses by the smallest variable that appears, while BM orders them by the largest variable that appears). In this work, we consider seven possible heuristics for each variable order: Random, MCS, LexP, LexM, InvMCS, InvLexP, and InvLexM.

A simple heuristic for these variable orders is to randomly order the variables, i.e., for a formula over variables $X$, sample an injection $X \rightarrow \{1, 2, \ldots, |X|\}$ uniformly at random. We call this the Random heuristic. Random is a baseline for comparison of the other variable order heuristics.

For the remaining heuristics, we must define the Gaifman graph $G_{\varphi}$ of a formula $\varphi$. The Gaifman graph of $\varphi$ has a vertex for every variable in $\varphi$. Two vertices are connected by an edge if and only if the corresponding variables appear in the same clause of $\varphi$.

One such heuristic is called Maximum-Cardinality Search [33]. At each step of the heuristic, the next variable chosen is the variable adjacent in $G_{\varphi}$ to the
greatest number of previously chosen variables (breaking ties arbitrarily). We call this the 
MCS heuristic for variable order.

Another such heuristic is called \textit{Lexicographic search for perfect orders} [18]. Each vertex of \( G_\varphi \) is assigned a set of vertices (called the \textit{label}), which is initially empty. At each step of the heuristic, the next variable chosen is the variable \( x \) whose label is lexicographically smallest among the unchosen variables (breaking ties arbitrarily). Then \( x \) is added to the label of its neighbors in \( G_\varphi \). We call this the \texttt{LexP} heuristic for variable order.

A similar heuristic is called \textit{Lexicographic search for minimal orders} [18]. As before, each vertex of \( G_\varphi \) is assigned an initially-empty label. At each step of the heuristic, the next variable chosen is again the variable \( x \) whose label is lexicographically smallest (breaking ties arbitrarily). In this case, \( x \) is added to the label of every variable \( y \) where there is a path \( x, z_1, z_2, \ldots, z_k, y \) in \( G_\varphi \) s.t. every \( z_i \) is unchosen and the label of \( z_i \) is lexicographically smaller than the label of \( y \). We call this the \texttt{LexM} heuristic for variable order.

Additionally, the variable orders produced by each of the heuristics MCS, \texttt{LexP}, and \texttt{LexM} can be inverted. We call these new heuristics \texttt{InvMCS}, \texttt{InvLexP}, and \texttt{InvLexM}.

4.2 Heuristics for \texttt{get-cluster-rank}

The heuristic chosen for \texttt{get-cluster-rank} indicates the strategy for clustering the clauses of \( \varphi \). In this work, we consider three possible heuristics to be chosen for \texttt{get-cluster-rank} that satisfy the conditions of Theorem 3: \texttt{Mono}, \texttt{BE}, and \texttt{BM}.

One simple case is when \texttt{get-cluster-rank} is constant on all clauses, e.g., when \( \text{get-cluster-rank}(C, \rho) = m \) for all \( C \in \varphi \). In this case, all clauses of \( \varphi \) are placed in \( \kappa_m \), so line 10 of Algorithm 1 combines every clause of \( \varphi \) into a single ADD before performing projections. This corresponds to the monolithic approach we discussed in Section 3.1. We thus call this the \texttt{Mono} heuristic for \texttt{get-cluster-rank}. Notice that the performance of Algorithm 1 with \texttt{Mono} does not depend on the heuristic for \texttt{get-cluster-var-order} or \texttt{choose-cluster}. This heuristic has previously been applied to ADDs in the context of knowledge compilation [14].

A more complex heuristic assigns the rank of each clause to be the smallest \( \rho \)-rank of the variables that appear in the clause. That is, \( \text{get-cluster-rank}(C, \rho) = \min_{x \in \text{Vars}(C)} \rho(x) \). This heuristic corresponds to \textit{bucket elimination} [11], so we call this the \texttt{BE} heuristic. In this case, notice that every clause containing \( x \in X \) can only appear in \( \kappa_j \) with \( j \leq \rho(x) \). It follows that \( x \) has always been projected by the end of iteration \( \rho(x) \) of Algorithm 1 using \texttt{BE}.

A closely related heuristic assigns the rank of each clause to be the largest \( \rho \)-rank of the variables that appear in the clause. That is, \( \text{get-cluster-rank}(C, \rho) = \max_{x \in \text{Vars}(C)} \rho(x) \). This heuristic corresponds to \textit{Bouquet’s Method} [5], so we call this the \texttt{BM} heuristic. We can make no similar guarantees about when each variable \( x \) is projected in Algorithm 1 using \texttt{BM}.
4.3 Heuristics for choose-cluster

The heuristic chosen for choose-cluster indicates the strategy for combining the ADDs produced from each cluster. In this work, we consider two possible heuristics to use for choose-cluster that satisfy the conditions of Theorem 3: List and Tree.

One heuristic is when choose-cluster selects to place $A$ in the closest cluster that satisfies the conditions of Theorem 3, namely the next cluster to be processed. That is, $\text{choose-cluster}(A, i) = i + 1$. Under this heuristic, Algorithm 1 equivalently builds an ADD for each cluster and then combines the list of ADDs one-by-one, in-order, projecting variables as early as possible. At every iteration, there is a single intermediate ADD representing the combination of previous clusters. We call this the List heuristic.

Another heuristic is when choose-cluster selects to place $A$ in the furthest cluster that satisfies the conditions of Theorem 3. That is, choose-cluster$(A, i)$ returns the smallest $j > i$ such that $X_j$ and $\text{Vars}(A)$ are not disjoint (or returns $m$, if $\text{Vars}(A) = \emptyset$). At every iteration, there may be many intermediate ADDs representing the combination of previous clusters. We call this the Tree heuristic.

Notice that the computational structure of Algorithm 1 can be represented by a tree of clusters, where the cluster $\kappa_i$ is the child of $\kappa_j$ whenever the ADD produced from cluster $\kappa_i$ is placed in $\kappa_j$ after line 14 (i.e., $j = \text{choose-cluster}(A, i)$ at iteration $i$). These trees are always left-deep under the List heuristic, but can be more complex using the Tree heuristic.

A clustering heuristic is a valid combination of a get-cluster-rank heuristic and a choose-cluster heuristic. The five clustering heuristics are: Mono, BE—List, BE—Tree, BM—List, and BM—Tree.

5 Empirical Evaluation

We aim to: (1) find good heuristic configurations for our tool ADDMC, and (2) compare ADDMC against state-of-the-art exact model counters Cachet [28], c2d [9], d4 [20], miniC2D [23], and sharpSAT [34]. We use the largest CNF model counting benchmark family BayesNet, which contains 1091 weighted benchmarks [29]. Although our focus is on weighted model counting, we also consider 557 unweighted Planning benchmarks, which constitute the second-largest CNF model counting benchmark family.\(^1\)

Our ADDMC source code and experimental data can be obtained from a repository.\(^2\)

5.1 Experiment 1: Comparing ADDMC Heuristics

ADDMC heuristic configurations are constructed from five options for clustering heuristics (Mono, BE-List, BE-Tree, BM-List, and BM-Tree) and seven

\(^1\) From http://www.cril.univ-artois.fr/KC/benchmarks.html
\(^2\) See https://github.com/vardigroup/ADDMC
options for variable order heuristics (\texttt{Random}, \texttt{MCS}, \texttt{InvMCS}, \texttt{LexP}, \texttt{InvLexP}, \texttt{LexM}, and \texttt{InvLexM}). Using one variable order heuristic for the cluster variable order and another for the diagram variable order gives us 245 configurations in total. We compare these configurations to find the best combination of heuristics.

We run each job on an exclusive node in a homogeneous Linux cluster. Each node has a Xeon E5-2650v2 CPU (2.60-GHz) with 32 GB of memory. The time cap for each job is 100-second.

**Result** Table 1 reports the best, median, and worst across all 245 \texttt{ADDMC} heuristic configurations. Evidently, some configurations perform quite well while others perform quite poorly. We choose \texttt{BM-Tree} with \texttt{LexP} cluster variable order and \texttt{InvMCS} diagram variable order as the representative configuration for Experiment 2.

**Table 1.** A comparison of the number of benchmarks solved by the best, median, and worst heuristic configurations after 100 seconds. The best configuration for each clustering heuristic is also shown. \texttt{ADDMC} has 245 possible heuristic configurations, and their performance varies widely on both benchmark families.

| Clustering | Cluster var | Diagram var | Benchmarks solved | Ranking   |
|------------|-------------|-------------|-------------------|-----------|
|            |             |             | \texttt{BayesNet} | \texttt{Planning} | \texttt{Total} |         |
| \texttt{BM-Tree} | \texttt{LexP} | \texttt{InvMCS} | 1060 | 326 | 1386 | Best (overall) |
| \texttt{BE-Tree} | \texttt{InvMCS} | \texttt{InvLexP} | 1022 | 348 | 1370 | Best \texttt{BE-Tree} |
| \texttt{BM-List} | \texttt{LexP} | \texttt{InvMCS} | 1036 | 331 | 1367 | Best \texttt{BM-List} |
| \texttt{BE-List} | \texttt{InvLexP} | \texttt{MCS} | 1000 | 265 | 1265 | Best \texttt{BE-List} |
| \texttt{BE-List} | \texttt{LexP} | \texttt{LexP} | 394 | 214 | 608 | Median |
| \texttt{Mono} | \texttt{LexM} | \texttt{LexP} | 85 | 154 | 239 | Best \texttt{Mono} |
| \texttt{BE-List} | \texttt{Random} | \texttt{Random} | 15 | 60 | 75 | Worst |

Figure 1 shows the best, median, and worst \texttt{ADDMC} heuristic configurations. The wide range of performance indicates that the choice of heuristics is essential to the competitiveness of \texttt{ADDMC}.

### 5.2 Experiment 2: Comparing Exact Model Counters

In the previous experiment, the best \texttt{ADDMC} heuristic configuration is \texttt{BM-Tree} clustering with \texttt{LexP} cluster variable order and \texttt{InvMCS} diagram variable order. We now compare \texttt{ADDMC} using this configuration against state-of-the-art ex-
act model counters: Cachet, c2d\(^3\), d4, miniC2D, and sharpSAT\(^4\). Again, we use two standard CNF model counting benchmark families: BayesNet (weighted) and Planning (unweighted).

We run each job on an exclusive node in a homogeneous Linux cluster. Each node has a Xeon E5-2650v2 CPU (2.60-GHz) with 32 GB of memory. The time cap for each job is 1000-second.

Note that Cachet computes long double floating-point counts for weighted model counting, whereas other model counters use double precision. For unweighted model counting, ADDMC uses 64-bit floats, while other model counters use arbitrary-precision integers. We count benchmarks on which the floating-point numbers used by ADDMC overflow as failures.

**Result** Table 2 summarizes the performance of six exact model counters. ADDMC solves the most benchmarks overall. Split by benchmark family, ADDMC solves fewer benchmarks than the other methods on the Planning benchmarks. One partial explanation is that the floating-point numbers used by ADDMC overflow on

\(^3\) c2d does not natively support weighted model counting. In order to compare c2d on the weighted BayesNet benchmarks, we use c2d to compile CNF into d-DNNF then use d-DNNF-reasoner to compute the weighted model counts.

\(^4\) sharpSAT does not natively support weighted model counting. We are not aware of a straightforward method to have sharpSAT compute the weighted model counts of the BayesNet benchmarks.
Table 2. A comparison of ADDMC with a variety of existing model counters on 1648 weighted and unweighted benchmarks, sorted by the number of total benchmarks completed within 1000 seconds (since sharpSAT is an unweighted model counter, we did not evaluate it on the BayesNet weighted benchmarks). ADDMC solves the most benchmarks overall. ADDMC is less competitive on the Planning benchmarks but is a dramatic improvement on the BayesNet benchmarks.

| Model counter | BayesNet | Planning | Total |
|---------------|----------|----------|-------|
|               | Number   | Percentage | Number | Percentage | Number | Percentage |
| ADDMC         | 1077     | 98.72%    | 353    | 63.38%     | 1430   | 86.77%     |
| d4            | 938      | 85.98%    | 444    | 79.71%     | 1382   | 83.86%     |
| miniC2D       | 916      | 83.96%    | 426    | 76.48%     | 1342   | 81.43%     |
| c2d           | 839      | 76.90%    | 441    | 79.17%     | 1280   | 77.67%     |
| Cachet        | 776      | 71.13%    | 410    | 73.61%     | 1186   | 71.97%     |
| sharpSAT      | -        | -         | 428    | 76.84%     | -      | -          |

35 of these benchmarks, which we count as failures. On the other hand, ADDMC is a dramatic improvement over other tools on the BayesNet benchmarks. In particular, ADDMC is able to solve 114 benchmarks that no other tool we tested can. See Figure 2 and Figure 3 for a more detailed analysis of solver runtime on the Planning and BayesNet benchmarks respectively.

Regarding correctness, to compare model counts produced by different counters, we consider non-negative real numbers \(a \leq b\) equal when:

\[
\begin{cases}
    b - a < 10^{-3} & \text{if } a = 0 \text{ or } b < 10 \\
    b/a < 1 + 10^{-5} & \text{otherwise}
\end{cases}
\]  

(2)

The first case is mostly for BayesNet weighted model counts, which are at most 1. The second case is for Planning unweighted model counts, which can be large integers.

Within the equality tolerance in Equation (2), ADDMC produces the same model counts as other tools (Cachet, c2d, d4, miniC2D, and sharpSAT), with two exceptions. First, on the BayesNet benchmarks, d4 outputs some numbers that are different from other tools. A similar issue occurs with Cachet on the Planning benchmarks.

6 Discussion

We developed a dynamic-programming framework for weighted model counting that captures both bucket elimination and Bouquet’s Method. We implemented this algorithm in ADDMC, a new exact weighted model counter. We used ADDMC to compare bucket elimination and Bouquet’s Method across a variety of variable order heuristics on two standard model counting benchmark families and concluded that Bouquet’s Method is competitive with bucket elimination. Moreover, we demonstrated that ADDMC solved more benchmarks than state-of-the-art model counters (Cachet, c2d, d4, miniC2D, and sharpSAT) on the combined
Fig. 2. A cactus plot of the number of unweighted Planning benchmarks solved by various tools for a given timeout. ADDMC is outperformed by the state of the art on these benchmarks.

Fig. 3. A cactus plot of the number of weighted BayesNet benchmarks solved by various tools for a given timeout. ADDMC significantly outperforms the state of the art on these benchmarks.
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benchmark set of Planning and BayesNet. On further analysis, ADDMC is outperformed by the other tools on the Planning benchmarks but is the fastest tool on the BayesNet benchmarks. Thus ADDMC is valuable as part of a portfolio solver, and ADD-based approaches to model counting in general are promising and deserve further study. For example, it would be interesting to discover the fundamental reason why our approach works well on the BayesNet benchmarks but not on the Planning benchmarks.

Bucket elimination has been well-studied theoretically, with close connections to treewidth and tree-decompositions [11,8]. On the other hand, Bouquet’s Method is much less well-known. Previous work [25] has also observed that Bouquet’s Method can outperform bucket elimination using compact representations. An interesting direction for future work is to develop a theoretical framework to explain the relative performance between bucket elimination and Bouquet’s Method.

In this work, we focused on ADDs implemented in the library CUDD. There are other ADD libraries that may be fruitful to explore in the future. For example, the ADD library Sylvan [12] supports arbitrary-precision arithmetic and multicore operations on ADDs, which would allow us to compute large unweighted model counts (and thus improve the performance on the Planning benchmarks) and investigate the impact of parallelism on our techniques. It would also be interesting to explore dynamic-programming techniques for model counting using other compact representations, e.g., affine ADDs [30].

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