ORBITS OF $s$-REPRESENTATIONS WITH DEGENERATE GAUSS MAPPINGS

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ABSTRACT. In this paper we study tangentially degeneracy of the orbits of $s$-representations in the sphere. We show that an orbit of an $s$-representation is tangentially degenerate if and only if it is through a long root, or a short root of restricted root system of type $G_2$. Moreover these orbits provide many new examples of tangentially degenerate submanifolds which satisfy the Ferus equality.

1. Introduction

A submanifold is called tangentially degenerate if its Gauss mapping is degenerate. The investigation of tangentially degeneracy of submanifolds has long history. For example the classification of surfaces in $\mathbb{R}^3$ with degenerate Gauss mapping is equivalent to the classification of flat surfaces in $\mathbb{R}^3$. As a result, that is one of planes, cylinders, cones or tangent developable surfaces. In this paper we shall investigate the Gauss mapping of a submanifold in the sphere, that is defined as a mapping to a Grassmannian manifold. The definition of the Gauss mapping, which here we deal with, will be given in Section 2. Ferus [4] obtained a remarkable result for tangentially degeneracy of submanifolds in the sphere. He showed that there exists a number, so-called the Ferus number, such that if the rank of the Gauss mapping is less than the Ferus number, then a submanifold must be a totally geodesic sphere. However, in general it is still unknown whether there exist submanifolds which satisfy the Ferus equality, that is, the equality of the Ferus inequality. In their papers [9, 10, 11], Ishikawa, Kimura and Miyaoka studied submanifolds with degenerate Gauss mappings in the sphere via a method of isoparametric hypersurfaces. They showed that Cartan hypersurfaces and some focal submanifolds of homogeneous isoparametric hypersurfaces are tangentially degenerate. Moreover, some of them satisfy the Ferus equality.

A homogeneous isoparametric hypersurface in the sphere is obtained as an orbit of an $s$-representation of a compact symmetric pair of rank 2. Therefore we shall study submanifolds with degenerate Gauss mappings via a method of symmetric spaces. Our strategy is to investigate the space of relative nullity of the orbits. In fact, the index of relative nullity is equal to the rank of tangentially degeneracy. We will study the second fundamental form of the orbits of $s$-representations by restricted root systems, and determine their spaces of relative nullity. As a result, we will obtain that the space of relative nullity of the orbits through a long root, or a short root of restricted root system of type $G_2$, is coincide with the root space.

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of that root. Hence these orbits are tangentially degenerate. We note that these orbits are weakly reflective submanifolds as we showed in the previous paper [8]. Moreover, we will show that the orbits of \( s \)-representations with degenerate Gauss mapping are exhausted with above orbits. Finally we shall observe that these orbits provide many new examples of tangentially degenerate submanifolds in the sphere which satisfy the Ferus equality.

2. Submanifolds with degenerate Gauss mappings

Let \( f : M \to S^n \) be an immersion of an \( l \)-dimensional manifold \( M \) into an \( n \)-dimensional sphere \( S^n \). The Gauss mapping \( \gamma \) of \( f \) is defined as a mapping from \( M \) to a Grassmannian manifold \( G_{l+1}(\mathbb{R}^{n+1}) \) of all \((l+1)\)-dimensional subspaces in \( \mathbb{R}^{n+1} \) by:

\[
\gamma : M \to G_{l+1}(\mathbb{R}^{n+1}) \quad x \mapsto \mathbb{R} f(x) \oplus T f(x)(f(M)).
\]

We denote by \( r \) the maximal rank of the Gauss mapping \( \gamma \) of an immersion \( f \). If the Gauss mapping is degenerate, i.e. \( r < l \), then an immersed submanifold \( f(M) \subset S^n \) is said to be tangentially degenerate or developable. We note that \( \gamma \) is constant, i.e. \( r = 0 \), if and only if \( f(M) \) is a part of a totally geodesic sphere.

We denote by \( h \) and \( A \) the second fundamental form and the shape operator of \( f \), respectively. Chern and Kuiper [3] introduced the notion of the index of relative nullity at \( x \in M \), that is the dimension of the vector space

\[
\mathcal{N}_x = \{ X \in T_x(M) \mid h(X, Y) = 0, \forall Y \in T_x(M) \} = \bigcap_{\xi \in T^2_x(M)} \ker(A_\xi).
\]

It is easy to show \( \ker(d\gamma)_x = \mathcal{N}_x \), therefore the index of relative nullity is equal to the degeneracy of the Gauss mapping at each point.

Let \( f : M \to S^n \) be an immersion of a compact, connected manifold \( M \) of dimension \( l \). Ferus [4] showed that there exists a number \( F(l) \), which only depends on the dimension \( l \) of \( M \), such that the inequality \( r < F(l) \) implies \( r = 0 \). Then \( f(M) \) must be an \( l \)-dimensional great sphere in \( S^n \). Here the number \( F(l) \) is called the Ferus number and given by

\[
F(l) = \min\{ k \mid A(k) + k \geq l \},
\]

where \( A(k) \) is the Adams number, that is the maximal number of linearly independent vector fields at each point on the \((k-1)\)-dimensional sphere \( S^{k-1} \). Any positive integer \( k \) can be written as \((2s+1)2^t\) by some non-negative integers \( s \) and \( t \). We write \( t = c + 4d \) by some \( 0 \leq c \leq 3 \) and \( 0 \leq d \). In this situation the Adams number \( A(k) \) can be calculated by

\[
A(k) = 2^c + 8d - 1.
\]

Regarding the Ferus inequality, Ishikawa, Kimura and Miyaoka posed the following problem:

Problem 2.1 ([10]). (1) Is the inequality \( r < F(l) \) best possible for the implication \( r = 0 \)? Do there exist tangentially degenerate immersions \( M^l \to S^n \) with \( r = F(l) \)?
(2) If the above problem is true, classify tangentially degenerate immersions \( M^l \rightarrow S^n \) with \( r = F(l) \).

For these problems, they obtained the following results using isoparametric hypersurfaces in the sphere. It is well-known that the number of distinct principal curvatures of an isoparametric hypersurface in the sphere is 1, 2, 3, 4 or 6. A minimal isoparametric hypersurface with \( g = 3 \) is called a Cartan hypersurface.

**Theorem 2.2** ([9]). A homogeneous compact hypersurface in the real projective space \( \mathbb{R}P^n \) which is tangentially degenerate is projectively equivalent to a hyperplane or a Cartan hypersurface.

**Theorem 2.3** ([11]). When \( M \) is a homogeneous isoparametric hypersurface in the sphere with \( g = 6 \), then both focal submanifolds of \( M \) are tangentially degenerate. Moreover, these submanifolds satisfy the Ferus equality.

**Theorem 2.4** ([10]). When \( M \) is a homogeneous isoparametric hypersurface in the sphere with \( g = 4 \), then one of focal submanifolds of \( M \) is tangentially degenerate, and another one is not. Moreover, some of them satisfy the Ferus equality.

### 3. Orbits of \( s \)-representations

A linear isotropy representation of a Riemannian symmetric pair is called an \( s \)-representation. In the following section, we will study orbits of \( s \)-representations which are tangentially degenerate. For this purpose, we shall provide some fundamental notions of orbits of \( s \)-representations in this section.

Let \( G \) be a compact, connected Lie group and \( K \) a closed subgroup of \( G \). Assume that \( \theta \) is an involutive automorphism of \( G \) and \( G_\theta^0 = \{ g \in G \mid \theta(g) = g \} \) and \( G_\theta^0 \) is the identity component of \( G_\theta \). Then \( (G, K) \) is a compact symmetric pair with respect to \( \theta \). We denote the Lie algebras of \( G \) and \( K \) by \( g \) and \( k \), respectively. The involutive automorphism of \( g \) induced from \( \theta \) will be also denoted by \( \theta \). Then we have

\[
\mathfrak{k} = \{ X \in g \mid \theta(X) = X \}.
\]

Take an inner product \( \langle \ , \ \rangle \) on \( g \) which is invariant under \( \theta \) and the adjoint representation of \( G \). Set

\[
\mathfrak{m} = \{ X \in g \mid \theta(X) = -X \},
\]

then we have a canonical orthogonal direct sum decomposition

\[
g = \mathfrak{k} + \mathfrak{m}.
\]

Fix a maximal abelian subspace \( \mathfrak{a} \) in \( \mathfrak{m} \) and a maximal abelian subalgebra \( \mathfrak{t} \) in \( g \) containing \( \mathfrak{a} \). For \( \alpha \in \mathfrak{t} \) we set

\[
\tilde{\mathfrak{g}}_\alpha = \{ X \in g^C \mid [H, X] = \sqrt{-1}(\alpha, H)X \ (H \in \mathfrak{t}) \}
\]

and define the root system \( \tilde{\mathfrak{R}} \) of \( g \) by

\[
\tilde{\mathfrak{R}} = \{ \alpha \in \mathfrak{t} - \{0\} \mid \tilde{\mathfrak{g}}_\alpha \neq \{0\} \}.
\]

For \( \lambda \in \mathfrak{a} \) we set

\[
\mathfrak{g}_\lambda = \{ X \in g^C \mid [H, X] = \sqrt{-1}(\lambda, H)X \ (H \in \mathfrak{a}) \}.
\]
and define the restricted root system $R$ of $(\mathfrak{g}, \mathfrak{t})$ by
\[
R = \{ \lambda \in \mathfrak{a} - \{0\} \mid g_\lambda \neq \{0\}\}.
\]
Set
\[
\tilde{R}_0 = \tilde{R} \cap \mathfrak{t}
\]
and denote the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$ by $H \mapsto \bar{H}$. Then we have
\[
R = \{ \bar{\alpha} \mid \alpha \in \tilde{R} - \tilde{R}_0\}.
\]
We take a basis of $\mathfrak{t}$ extended from a basis of $\mathfrak{a}$ and define the lexicographic orderings $>_{\mathfrak{a}}$ on $\mathfrak{a}$ and $\mathfrak{t}$ with respect to these bases. Then for $H \in \mathfrak{t}$, $\bar{H} > 0$ implies $H > 0$. We denote by $\tilde{\mathfrak{F}}$ the set of simple roots of $\tilde{R}$ with respect to the ordering $>_{\mathfrak{a}}$. Set
\[
\tilde{\mathfrak{F}}_0 = \tilde{\mathfrak{F}} \cap \tilde{R}_0,
\]
then the set of simple roots $\mathfrak{F}$ of $R$ with respect to the ordering $>_{\mathfrak{a}}$ is given by
\[
\mathfrak{F} = \{ \bar{\alpha} \mid \alpha \in \tilde{\mathfrak{F}} - \tilde{\mathfrak{F}}_0\}.
\]
We set
\[
\tilde{\mathfrak{R}}^+ = \{ \alpha \in \tilde{R} \mid \alpha > 0\}, \quad R^+ = \{ \lambda \in R \mid \lambda > 0\}.
\]
We also set
\[
\mathfrak{k}_0 = \{ X \in \mathfrak{k} \mid [X, H] = 0 (H \in \mathfrak{a})\},
\]
and define
\[
\mathfrak{m}_\lambda = \mathfrak{m} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}), \quad m_\lambda = m \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda})
\]
for $\lambda \in R_+$. Under these notations, we have the following lemma.

Lemma 3.1. (1) We have orthogonal direct sum decompositions
\[
\mathfrak{t} = \mathfrak{t}_0 + \sum_{\lambda \in R^+} \mathfrak{t}_\lambda, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\lambda \in R^+} \mathfrak{m}_\lambda.
\]
(2) If $H \in \mathfrak{a}$ and $\langle \lambda, H \rangle \neq 0$, then $\text{ad}(H)$ gives a linear isomorphism between $\mathfrak{m}_\lambda$ and $\mathfrak{t}_\lambda$.

We define a subset $D$ of $\mathfrak{a}$ by
\[
D = \bigcup_{\lambda \in R} \{ H \in \mathfrak{a} \mid \langle \lambda, H \rangle = 0\}.
\]
A connected component of $\mathfrak{a} - D$ is a Weyl chamber. We set
\[
C = \{ H \in \mathfrak{a} \mid \langle \lambda, H \rangle > 0 (\lambda \in F)\}.
\]
Then $C$ is an open convex subset of $\mathfrak{a}$ and the closure of $C$ is given by
\[
\bar{C} = \{ H \in \mathfrak{a} \mid \langle \lambda, H \rangle \geq 0 (\lambda \in F)\}.
\]
For a subset $\Delta \subset F$, we define
\[
C^\Delta = \{ H \in \bar{C} \mid \langle \lambda, H \rangle > 0 (\lambda \in \Delta), \langle \mu, H \rangle = 0 (\mu \in F - \Delta)\}.
\]
Lemma 3.2. (1) For $\Delta_1 \subset F$, the decomposition
\[
\overline{C^{\Delta_1}} = \bigcup_{\Delta \subset F} C^\Delta
\]
is a disjoint union. In particular, $\bar{C} = \bigcup_{\Delta \subset F} C^\Delta$ is a disjoint union.
(2) For $\Delta_1, \Delta_2 \subset F$, $\Delta_1 \subset \Delta_2$ if and only if $C^{\Delta_1} \subset C^{\Delta_2}$.

For each $\lambda \in F$, we take $H_\lambda \in a$ such that
\[
\langle H_\lambda, \mu \rangle = \begin{cases} 
1 & (\mu = \lambda), \\
0 & (\mu \neq \lambda) 
\end{cases} \quad (\mu \in F).
\]

Then, for $\Delta \subset F$, we have
\[
C^\Delta = \left\{ \sum_{\lambda \in \Delta} t_\lambda H_\lambda \bigg| t_\lambda > 0 \right\}.
\]

We set
\[
R^\Delta = R \cap (F - \Delta)z, \quad R^+_{\Delta} = R^\Delta \cap R_+.
\]

Under these notations, we have the following lemma.

**Lemma 3.3** ([6]). Fix a subset $\Delta \subset F$. For $H \in C^\Delta$ we have the following:

1. $R^\Delta = \{ \lambda \in R \mid \langle \lambda, H \rangle = 0 \}$,
2. $R^+_{\Delta} = \{ \lambda \in R_+ \mid \langle \lambda, H \rangle = 0 \}$.

Now we shall study an orbit $\text{Ad}(K)H$ of the linear isotropy representation of $(G, K)$ through $H \in m$. We set
\[
Z^H_K = \{ k \in K \mid \text{Ad}(k)H = H \}.
\]

Then $Z^H_K$ is a closed subgroup of $K$ and the orbit $\text{Ad}(K)H$ is diffeomorphic to the coset manifold $K/Z^H_K$. The Lie algebra $\mathfrak{z}^H_K$ of $Z^H_K$ is given by
\[
\mathfrak{z}^H_K = \{ X \in \mathfrak{k} \mid [H, X] = 0 \}.
\]

An orbit $\text{Ad}(K)H$ is a submanifold of the hypersphere $S$ of radius $\| H \|$ in $m$. From [6], $\text{Ad}(K)H$ is connected. Since
\[
m = \bigcup_{k \in K} \text{Ad}(k)\bar{C},
\]
without loss of generality we may assume $H \in \bar{C}$. Moreover, from Lemma 3.2 there exists $\Delta \subset F$ such that $H \in C^\Delta$. From Lemma 3.1 we have the following lemma.

**Lemma 3.4** ([6]). For $\Delta \subset F$ and $H \in C^\Delta$, the tangent space $T_H(\text{Ad}(K)H)$ of the orbit $\text{Ad}(K)H$ at $H$ and the normal space $T^\perp_H(\text{Ad}(K)H)$ in the hypersphere can be expressed as

\[
T_H(\text{Ad}(K)H) = \sum_{\mu \in R_+ - R^+_\Delta} m_\mu,
\]

\[
T^\perp_H(\text{Ad}(K)H) = a \cap H^\perp + \sum_{\nu \in R^+_\Delta} m_\nu = \text{Ad}((Z^H_K)_0)(a \cap H^\perp),
\]

where $(Z^H_K)_0$ is the identity component of the stabilizer $Z^H_K$ of $H$ in $K$. 

4. Orbits of $s$-representations with degenerate Gauss mappings

4.1. Tangentially degenerate orbits. Let $(G, K)$ be a compact symmetric pair. We assume that $(G, K)$ is irreducible, namely $K$ acts irreducibly on $m$. We consider the orbit $\text{Ad}(K)H$ through $H \in a$. In this section, we study the orbits with degenerate Gauss mappings. Since the tangentially degeneracy of the orbit is invariant under scalar multiples on the vector space $m$, we do not discriminate the difference of the length of a vector $H$. When $(G, K)$ is of rank 1, $K$ acts on the sphere in $m$ transitively. Therefore we only consider a symmetric pair whose rank is greater than or equal to 2. The following theorem is the main result of this paper.

**Theorem 4.1.** An orbit of an $s$-representation is tangentially degenerate if and only if it is through a long root (any root when all roots have the same length), or a short root of restricted root system of type $G_2$. Let $\lambda \in R$ be such a root. Then the tangentially degeneracy of the orbit $\text{Ad}(K)\lambda$ is $\ker(d\gamma)_\lambda = m_\lambda$.

To prove this theorem, we show the following proposition first.

**Proposition 4.2.** If the orbit $\text{Ad}(K)H$ through $H \in a$ is tangentially degenerate, then $H$ is a constant multiple of a restricted root.

**Proof.** First we note that

$$A_\xi = \text{Ad}(k)^{-1} A_{\text{Ad}(k)\xi} \text{Ad}(k)$$

for any $\xi \in a \cap H^\perp$ and $k \in (Z^H_{K})_0$. From this we have

$$\bigcap_{\xi \in T^H_{(\text{Ad}(K)H)}} \ker A_\xi = \bigcap_{\xi \in \text{Ad}((Z^H_{K})_0(a \cap H^\perp))} \ker A_\xi$$

$$= \bigcap_{\xi \in a \cap H^\perp, k \in (Z^H_{K})_0} \ker A_{\text{Ad}(k)\xi}$$

$$= \bigcap_{\xi \in a \cap H^\perp, k \in (Z^H_{K})_0} \ker(\text{Ad}(k)A_\xi \text{Ad}(k)^{-1})$$

$$= \bigcap_{\xi \in a \cap H^\perp, k \in (Z^H_{K})_0} \ker(\text{Ad}(k)^{-1})A_\xi \ker \text{Ad}(k)$$

$$= \bigcap_{\xi \in a \cap H^\perp, k \in (Z^H_{K})_0} \text{Ad}(k) \ker A_\xi$$

$$= \bigcap_{k \in (Z^H_{K})_0} \text{Ad}(k) \bigcap_{\xi \in a \cap H^\perp} \ker A_\xi.$$

For $\xi \in a \cap H^\perp$ the set of eigenvalues of $A_\xi$ is given by

$$\left\{ \frac{\langle \lambda, \xi \rangle}{\langle \lambda, H \rangle} \bigg| \lambda \in R^\Delta - R^\Delta_+ \right\},$$

and the eigenspace associated with eigenvalue $-\langle \lambda, \xi \rangle/\langle \lambda, H \rangle$ is given by

$$\sum_{\langle \mu, \xi \rangle = -\langle \lambda, \xi \rangle/\langle \lambda, H \rangle} m_\mu.$$
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See [8] for details. The space $\ker A_\xi$ is nothing but the eigenspace associated with 0-eigenvalue. Thus

$$\ker A_\xi = \sum_{\langle \mu, \xi \rangle = 0} m_\mu.$$  

Therefore we have

$$\bigcap_{\xi \in a \cap H} \ker A_\xi = \bigcap_{\xi \in a \cap H} \sum_{\langle \mu, \xi \rangle = 0} m_\mu = \sum_{\mu / H} m_\mu,$$

hence

$$(4.1) \quad \bigcap_{\xi \in T^\perp H (\Ad(K) H)} \ker A_\xi = \bigcap_{k \in (Z_H^+)_0} \Ad(k) \sum_{\mu / H} m_\mu \subset \sum_{\mu / H} m_\mu.$$  

Consequently, if $\Ad(K)H$ is tangentially degenerate, then $H$ must be a constant multiple of a restricted root. □

From the above proposition, hereafter, we may consider the orbit through a restricted root, i.e., we may put $H = \lambda \in R_+$. We set

$$\Delta = \{ \mu \in F \mid \langle \mu, \lambda \rangle > 0 \}.$$  

Then we have $\lambda \in C^\Delta$. If $2\lambda \notin R_+$, then $t_0 + t_\lambda$ is a Lie subalgebra of $\mathfrak{g}$. We denote by $K(\lambda)$ the analytic subgroup of $K$ which corresponds to $t_0 + t_\lambda$.

**Proposition 4.3.** If $\lambda \in R_+$ satisfies

(a) $2\lambda \notin R_+$,
(b) $\lambda + \nu \notin R$ and $\lambda - \nu \notin R$ for all $\nu \in R^\Delta_+$,

then $\Ad(K)\lambda$ is tangentially degenerate.

**Proof.** Since the tangent space of the orbit $\Ad(K)\lambda$ at $\lambda$ is given as in [3.3], the image of $\lambda$ by the Gauss mapping $\gamma$ is

$$\gamma(\lambda) = R\lambda + \sum_{\mu \in R_+ - R^\Delta_+} m_\mu,$$

and its orthogonal complement in $\mathfrak{m}$ is

$$\gamma(\lambda)^\perp = a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu.$$  

From a rule of the bracket product of root spaces and the assumption (b), we have

$$\left[ t_0, a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu \right] \subset \sum_{\nu \in R^\Delta_+} m_\nu, \quad \left[ t_\lambda, a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu \right] = \{ 0 \}.$$  

Therefore

$$\left[ t_0 + t_\lambda, a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu \right] \subset a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu.$$  

This yields

$$\Ad(K(\lambda)) \left( a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu \right) = a \cap \lambda^\perp + \sum_{\nu \in R^\Delta_+} m_\nu.$$
Hence
\[ \text{Ad}(K(\lambda)) \cdot \gamma(\lambda) = \gamma(\lambda). \]
Since \( \gamma \) is \( K \)-equivariant, we have
\[ \gamma(\text{Ad}(k)\lambda) = \text{Ad}(k)\gamma(\lambda) = \gamma(\lambda) \]
for any \( k \in K(\lambda) \). This means that \( \gamma \) is constant on \( \text{Ad}(K(\lambda))\lambda \). It is clear that \( \text{Ad}(K(\lambda))\lambda \) is not a point, since \( T_{\lambda}(\text{Ad}(K(\lambda))\lambda) = m_{\lambda} \). Consequently \( \text{Ad}(K(\lambda)) \) is tangentially degenerate. \( \square \)

We denote by \( \delta \in R_+ \) the highest root of \( R \).

**Lemma 4.4 (13).** For \( \lambda \in R_+ \),
\[ \langle \lambda, \delta \rangle / \| \delta \|^2 = \begin{cases} 0 & \text{(when } \lambda \perp \delta), \\ 1 & \text{(when } \lambda = \delta), \\ 1/2 & \text{(otherwise).} \end{cases} \]
When \( \langle \lambda, \delta \rangle / \| \delta \|^2 = 0 \), then \( \lambda - \delta \) is not a root. When \( \langle \lambda, \delta \rangle / \| \delta \|^2 = 1/2 \), then \( \lambda - \delta \) is a root.

**Proof.** Since \( \delta \) is the highest root, clearly \( \lambda + \delta \) is not a root. We express \( \delta \)-series containing \( \lambda \) as \( \lambda + n\delta \) \((p \leq n \leq 0)\). Then
\[ -2\frac{\langle \lambda, \delta \rangle}{\| \delta \|^2} = p. \]
Now we shall show \( p = 0, -1 \) or \( -2 \). If we assume that \( p \leq -3 \), then \( \mu = \lambda - 3\delta \) is a root. Then, from the square norm of \( 3\delta = \lambda - \mu \), we have
\[ 9\| \delta \|^2 = \| \lambda \|^2 + \| \mu \|^2 - 2\langle \lambda, \mu \rangle. \]
From \( \| \lambda \| \leq \| \delta \|, \| \mu \| \leq \| \delta \| \) and Cauchy’s inequality
\[ -\langle \lambda, \mu \rangle \leq \| \lambda \|\| \mu \| \leq \| \delta \|^2, \]
we have \( 9\| \delta \|^2 \leq 4\| \delta \|^2 \). This is a contradiction.

In the case of \( p = 0 \), \( \lambda \) is perpendicular to \( \delta \) and \( \lambda - \delta \) is not a root. In the case of \( p = -1 \), \( \langle \lambda, \delta \rangle / \| \delta \|^2 = 1/2 \) and \( \lambda - \delta \) is a root. When \( p = -2 \), then \( \lambda = \delta \) from Cauchy’s inequality. \( \square \)

From Proposition 4.3 and Lemma 4.4 we have the following corollary.

**Corollary 4.5.** The orbit \( \text{Ad}(K)\delta \) through the highest root \( \delta \) of \( R \) is tangentially degenerate.

Since a long root is conjugate to the highest root under the action of the Weyl group, it satisfies the conditions of Proposition 4.3. Especially in the case where the lengths of all roots are equal, all roots satisfy the conditions of Proposition 4.3. We determine short roots satisfying the conditions of Proposition 4.3 in the following proposition.

**Proposition 4.6.** Short roots satisfying the conditions (a) and (b) of Proposition 4.3 are only short roots of the restricted root system of type \( G_2 \).
Proof. We will follow the notations of root systems in [2].
In the case of type $B$, the restricted root system is given by
$$R = \{ \pm e_i \mid 1 \leq i \leq p \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq p \}.$$ If we add $\pm e_j$ to a short root $\pm e_i$ ($i \neq j$), then it becomes a root again. Thus any short root does not satisfy the condition (b).

In the case of type $C$, the restricted root system is given by
$$R = \{ \pm 2e_i \mid 1 \leq i \leq p \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq p \}.$$ Short roots are $\pm e_i \pm e_j$. By the action of the Weyl group, it suffices to consider a short root $e_1 + e_2$. The set of roots which are perpendicular to $e_1 + e_2$ is
$$\{ \pm (e_1 - e_2) \} \cup \{ \pm 2e_i \mid 3 \leq i \leq p \} \cup \{ \pm e_i \pm e_j \mid 3 \leq i < j \leq p \}.$$ Since
$$(e_1 + e_2) + (e_1 - e_2) = 2e_1 \in R, \quad (e_1 + e_2) - (e_1 - e_2) = 2e_2 \in R,$$ $e_1 + e_2$ does not satisfy the condition (b).

In the case of type $G_2$, we can easily see that all short roots satisfy the conditions (a) and (b).

In the case of type $BC$, the restricted root system is given by
$$R = \{ \pm 2e_i \mid 1 \leq i \leq p \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq p \}.$$ We can see that short roots $\pm e_i \pm e_j$ do not satisfy the condition (b) by a similar way in the case of types $B$ and $C$.

The root system of $F_4$ contains a root system of type $B_2$ as a sub-system. Then a short root of type $F_4$ can be regarded as a short root of type $B_2$. Thus in this case a short root does not satisfy the condition (b). □

By the above discussion we obtained that the orbits stated in Theorem 4.1 are tangentially degenerate. In order to determine the spaces of relative nullity of these orbits and to show other orbits are not tangentially degenerate, we give the following criterion for an orbit of an $s$-representation to be tangentially degenerate.

**Proposition 4.7.** The orbit $\text{Ad}(K)\lambda$ through a restricted root $\lambda \in R$ is tangentially degenerate if and only if there exists a non-zero subspace of $\sum_{\mu/\lambda} m_\mu$ which is invariant under $\text{ad}(z^K_\lambda)$. More precisely,

$$(4.2) \quad \ker(d\gamma)_\lambda = \bigcap_{k \in (Z^K_\lambda)_0} \text{Ad}(k) \sum_{\mu/\lambda} m_\mu$$

and $\ker(d\gamma)_\lambda$ is the maximal subspace of $\sum_{\mu/\lambda} m_\mu$ which is invariant under $\text{ad}(z^K_\lambda)$.

**Proof.** From (4.11) we have (4.12) immediately. Thus the orbit $\text{Ad}(K)\lambda$ is tangentially degenerate if and only if the right-hand side of (4.2) is a non-zero vector space.

If there exists a non-zero subspace $V$ of $\sum_{\mu/\lambda} m_\mu$ which is invariant under $\text{Ad}((Z^K_\lambda)_0)$, then
$$\bigcap_{k \in (Z^K_\lambda)_0} \text{Ad}(k) \sum_{\mu/\lambda} m_\mu \supset \bigcap_{k \in (Z^K_\lambda)_0} \text{Ad}(k)V = V \neq \{0\}.$$
Hence $\text{Ad}(K)\lambda$ is tangentially degenerate. Conversely, we assume that $\text{Ad}(K)\lambda$ is tangentially degenerate. Then

$$\bigcap_{k \in (Z_K^\lambda)_0} \text{Ad}(k) \sum_{\mu / \lambda} m_\mu \subset \sum_{\mu / \lambda} m_\mu$$

is a non-zero subspace, and we denote it by $V$. Then for any $g \in (Z_K^\lambda)_0$ we have

$$\text{Ad}(g) V = \bigcap_{k \in (Z_K^\lambda)_0} \text{Ad}(g) \text{Ad}(k) \sum_{\mu / \lambda} m_\mu = \bigcap_{k \in (Z_K^\lambda)_0} \text{Ad}(gk) \sum_{\mu / \lambda} m_\mu = V.$$

Thus $V$ is invariant under $\text{Ad}((Z_K^\lambda)_0)$. Consequently, the orbit $\text{Ad}(K)\lambda$ is tangentially degenerate if and only if there exists a non-zero subspace of $\sum_{\mu / \lambda} m_\mu$ invariant under $\text{Ad}((Z_K^\lambda)_0)$. Since $Z_K^\lambda$ is the Lie algebra of a connected Lie group $(Z_K^\lambda)_0$, we obtain the assertion. \(\square\)

In particular, for an orbit of the adjoint representation of a compact Lie group we have the following corollary.

**Corollary 4.8.** An adjoint orbit of a compact, connected semisimple Lie group through a root $\alpha$ is tangentially degenerate if and only if there exists a non-zero subspace of $g \cap (g_\alpha \oplus g_{-\alpha})$ which is invariant under $\text{ad}(Z_K^\alpha)$.

**Lemma 4.9.** Let $\lambda$ be a root and $V$ a non-zero subspace of $m_\lambda$. Then $V$ is invariant under $\text{ad}(Z_K^\lambda)$ if and only if $V$ is invariant under $\text{ad}(t_0)$ and satisfies

$$\left[ \sum_{\nu \in R^+_\Delta} t_\nu, V \right] = \{0\}.$$

In addition, if the action of $t_0$ on $m_\lambda$ is irreducible then $V = m_\lambda$.

**Proof.** Since

$$Z_K^\lambda = \{ X \in \mathfrak{z} \mid [X, \lambda] = 0 \} = t_0 \oplus \sum_{\nu \in R^+_\Delta} t_\nu,$$

$V$ is invariant under $\text{ad}(Z_K^\lambda)$ if and only if $V$ is invariant under $\text{ad}(t_0)$ and

$$\left[ \sum_{\nu \in R^+_\Delta} t_\nu, V \right] \subset V \subset m_\lambda.$$

On the other hand,

$$\left[ \sum_{\nu \in R^+_\Delta} t_\nu, V \right] \subset \left[ \sum_{\nu \in R^+_\Delta} t_\nu, m_\lambda \right] \subset \sum_{\nu \in R^+_\Delta} (m_{\lambda+\nu} \oplus m_{\lambda-\nu}).$$

Hence we have

$$\left[ \sum_{\nu \in R^+_\Delta} t_\nu, V \right] \subset \left( m_\lambda \cap \sum_{\nu \in R^+_\Delta} (m_{\lambda+\nu} \oplus m_{\lambda-\nu}) \right) = \{0\}.$$
Lemma 4.10. The root space $m_\delta$ corresponds to the highest root $\delta$ is a subspace of $\sum_{\mu/\delta} m_\mu$ invariant under $\text{ad}(\mathfrak{z}_K^\delta)$.

Proof. The Lie algebra $\mathfrak{z}_K^\delta$ of $Z^\delta_K$ is given by

$$\mathfrak{z}_K^\delta = \{ X \in \mathfrak{t} \mid [X, \delta] = 0 \} = t_0 \oplus \sum_{(\nu, \delta) = 0} t_\nu.$$ 

From Lemma 4.4, we have $\delta \pm \nu \not\in R$ for any $\nu \in R_+$ which is perpendicular to $\delta$. Hence from Lemma 4.9, $m_\delta$ is invariant under $\text{ad}(\mathfrak{z}_K^\delta)$. $\Box$

From this lemma, we have the following proposition immediately.

Proposition 4.11. Let $(G, K)$ be a compact symmetric pair. Then the orbit $\text{Ad}(K)\delta$ through the highest root $\delta$ is tangentially degenerate. Moreover, $\ker(d\gamma)_\delta = m_\delta$ except the case of type $BC$.

Similarly we also have the following proposition immediately.

Proposition 4.12. Let $(G, K)$ be a compact symmetric pair with restricted root system of type $G_2$. Then the orbit through any root $\lambda$ is tangentially degenerate. Moreover, $\ker(d\gamma)_\lambda = m_\lambda$.

Proposition 4.13. Let $G$ be a compact connected simple Lie group. An adjoint orbit of $G$ is tangentially degenerate if and only if it is through a long root, or a short root in the case of compact simple Lie group $G_2$.

Proof. We have already shown that the orbit through a long root, or a short root of the simple Lie group $G_2$ is tangentially degenerate. Therefore it suffices to show that, in the case of $G \neq G_2$, the orbit $\text{Ad}(G)\alpha$ through a short root $\alpha \in R_+$ is not tangentially degenerate.

Assume that $V$ is a subspace of $\mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ invariant under $\text{ad}(\mathfrak{z}_G^\alpha)$. Then the complexification $V^C \subset \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ of $V$ is a complex vector space which is invariant under $\text{ad}(\mathfrak{z}_G^\alpha)$. We take $v \in V^C$ and express as $v = X_\alpha + X_{-\alpha}$ ($X_{\pm \alpha} \in \mathfrak{g}_{\pm \alpha}$). In this case, from Proposition 4.6, there exists $\beta \in R_+$ which satisfies $\langle \beta, \alpha \rangle = 0$ and $\alpha \pm \beta \in R$. We take a non-zero vector $X_\beta \in \mathfrak{g}_\beta$. Then

$$[X_\beta, v] = [X_\beta, X_\alpha] + [X_\beta, X_{-\alpha}] \in (\mathfrak{g}_{\beta+\alpha} \oplus \mathfrak{g}_{\beta-\alpha}) \cap V^C = \{0\}.$$ 

This shows $X_{\pm \alpha} = 0$, since $[\mathfrak{g}_\beta, \mathfrak{g}_{\pm \alpha}] = \mathfrak{g}_{\beta \pm \alpha}$. Thus we obtain $V = \{0\}$. Hence from Corollary 4.8, $\text{Ad}(G)\alpha$ is not tangentially degenerate. $\Box$

In Proposition 4.11 we obtained the spaces of the relative nullity of the orbit through a highest root except the case of type $BC$. In the rest of this subsection, we shall study the space of relative nullity of the orbit through a highest root in the case of the restricted root system of type $BC_p$. In this case we can put

$$R = \{ \pm e_i \mid 1 \leq i \leq p \} \cup \{ \pm e_i \mid 1 \leq i \leq p \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq p \},$$

$\lambda = 2e_1$.

We already know that the space of relative nullity $N_\lambda$ of $\text{Ad}(K)\lambda$ satisfies

$$m_{2e_1} \subset N_\lambda \subset m_{2e_1} + m_{e_1}.$$
and invariant under $\text{ad}(\lambda_K)$. Since
\[
R^2_+ = \{ \mu \in R_+ \mid (\lambda, \mu) = 0 \} = \{ 2e_i \mid 2 \leq i \leq p \} \cup \{ e_i \mid 2 \leq i \leq p \} \cup \{ e_i \pm e_j \mid 2 \leq i < j \leq p \},
\]
we have
\[
\lambda_K = t_0 + \sum_{\mu \in R^2_+} t_\mu = t_0 + \sum_{2 \leq i \leq p} t_{2e_i} + \sum_{2 \leq i \leq p} t_{e_i} + \sum_{2 \leq i < j \leq p} t_{e_i \pm e_j}.
\]

First we determine the space of relative nullity of the orbit through a long root when $(G, K)$ is a Hermitian symmetric pair with restricted root system of type $BC$. For this purpose, we recall the following two lemmas.

**Lemma 4.14** ([12] Lemma 2.3). For a Hermitian symmetric space, the complex structure $J$ translates restricted root spaces as following:

\[
Jm_{e_i \pm e_j} = m_{e_i \mp e_j}, \quad Jm_{e_i} = m_{e_i}, \quad Ja = \sum_{i=1}^p m_{2e_i}.
\]

We denote the Hopf fibration by $\pi: S^{2n+1} \to CP^n$.

**Lemma 4.15** ([10] Lemma 2.2). Let $M \subset CP^n$ be a complex submanifold of complex dimension $k$. Then $\pi^{-1}(M)$ is a submanifold of dimension $2k+1$ with degenerate Gauss mapping of $S^{2n+1}$. Moreover, if $M$ is compact and not a complex projective subspace, then the rank of Gauss mapping is equal to $2k$.

Now we shall prove the following proposition.

**Proposition 4.16.** Assume that $p \geq 2$. Let $(G, K)$ be a Hermitian symmetric pair with restricted root system of type $BC_p$. Then the space of relative nullity $N_\lambda$ of the orbit through a long root $\lambda \in R$ is given by $N_\lambda = m_\lambda$.

**Proof.** Without loss of generality we can put $\lambda = 2e_1$, and we consider the orbit $\text{Ad}(K)\lambda$ through $\lambda$. The tangent space of $\text{Ad}(K)\lambda$ at $\lambda$ is given by

\[
T_\lambda(\text{Ad}(K)\lambda) = \sum_{\mu \in R_+ - R^2_+} m_\mu = m_{2e_1} + m_{e_1} + \sum_{2 \leq i \leq p} m_{e_i \pm e_i}.
\]

We denote by $\pi: S \to CP^n$ the Hopf fibration from the hypersphere $S$ in $m$ to the complex projective space. Then the image $\pi(\text{Ad}(K)\lambda)$ of the orbit $\text{Ad}(K)\lambda$ is a submanifold of $CP^n$, and its tangent space at $\pi(\lambda)$ is given by

\[
T_{\pi(\lambda)}(\pi(\text{Ad}(K)\lambda)) = m_{e_1} + \sum_{2 \leq i \leq p} m_{e_i \pm e_i}.
\]

Therefore from Lemma 4.14, $\pi(\text{Ad}(K)\lambda)$ is a complex submanifold of $CP^n$. Obviously $\pi(\text{Ad}(K)\lambda)$ is not a complex projective subspace when $p \geq 2$. Thus from Lemma 4.15, the index of the relative nullity of $\text{Ad}(K)\lambda \subset S$ is equal to 1. Hence $N_\lambda = m_{e_1}$. \qed

**Proposition 4.17.** In the case of $(G, K) = (Sp(2p+n), Sp(p) \times Sp(p+n))$ ($p \geq 2, n \geq 1$), the space of relative nullity of the orbit through a long root $\lambda \in R$ is given by $N_\lambda = m_\lambda$. 

Proof. We shall give the restricted root space decomposition of \((G, K) = (Sp(2p + n), Sp(p) \times Sp(p + n))\). We express \(g\) as
\[ g = sp(2p + n) = \{ X \in M_{2p+n}(H) \mid ^t \tilde{X} + X = 0 \}. \]

We define an involutive automorphism \(\theta\) on \(g\) by
\[ \theta : g \mapsto g; X \mapsto \begin{bmatrix} I_p & -I_{p+n} \\ -I_{p+n} & I_p \end{bmatrix} X \begin{bmatrix} I_p & -I_{p+n} \\ -I_{p+n} & I_p \end{bmatrix}, \]
where \(I_r\) denotes the \(r \times r\) identity matrix. Then the eigenspaces \(\mathfrak{t}\) and \(\mathfrak{m}\) of \(\theta\) associated to eigenvalues \(\pm 1\) are given by
\[ \mathfrak{t} = \left\{ \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \mid X \in sp(p), \; Y \in sp(p + n) \right\}, \]
\[ \mathfrak{m} = \left\{ \begin{bmatrix} -t \tilde{X} & X \\ X & t \tilde{X} \end{bmatrix} \mid X \in M_{p,p+n}(H) \right\}. \]

We take a maximal abelian subspace \(a\) of \(\mathfrak{m}\) by
\[ a = \left\{ \begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix} \mid T = t_1 E_{11} + \cdots + t_p E_{pp}, \; t_i \in \mathbb{R} \right\}, \]
where \(E_{ij}\) denotes a matrix whose \((i, j)\) element is 1 and all other elements are 0.

We define \(e_i \in a\) by
\[ e_i = \begin{bmatrix} E_{ii} \\ -E_{ii} \end{bmatrix}. \]

Then the restricted root system of \((g, \mathfrak{t})\) is of type \(BC_p\). We note that, when \(n = 0\), the restricted root system is of type \(C_p\).

In the case of type \(BC\), the restricted root spaces \(\mathfrak{t}_{e_i}\) and \(\mathfrak{m}_{e_i}\) which correspond to \(e_i\) are given by
\[ \mathfrak{m}_{e_i} = \left\{ \sum_{j=1}^n (x_j E_{i,2p+j} - \bar{x}_j E_{2p+j,i}) \mid x_j \in H \right\}, \]
\[ \mathfrak{t}_{e_i} = \left\{ \sum_{j=1}^n (y_j E_{i,2p+j} - \bar{y}_j E_{2p+j,i}) \mid y_j \in H \right\}. \]

In order to prove the proposition, we will show that \(N_\lambda\) does not contain \(m_{e_i}\)-component. We take \(X \in \mathfrak{m}_{e_i}\) arbitrarily. Then \([\mathfrak{t}_{e_2}, X] \subset \mathfrak{m}_{e_1} + \mathfrak{m}_{e_2} + \mathfrak{m}_{e_1 - e_2}\). Since \(N_\lambda\) is invariant under \(ad(\lambda)\), we have that if \(X \in N_\lambda\) then \([\mathfrak{t}_{e_2}, X] \subset N_\lambda \subset m_{2e_1} + m_{e_1}\). Therefore, if \(X \in N_\lambda\) then \([\mathfrak{t}_{e_2}, X] = \{0\}\). We can express \(X = \sum_{j=1}^n (x_j E_{1,2p+j} - \bar{x}_j E_{2p+j,1}) \in \mathfrak{m}_{e_1}\). Then
\[ [\mathfrak{t}_{e_2}, X] = \left\{ \left( \sum_{j=1}^n x_j \bar{y}_j \right) E_{1,p+2} - \left( \sum_{j=1}^n y_j \bar{x}_j \right) E_{p+2,1} \mid y_j \in H \right\}. \]

This yields \(X = 0\). Thus \(N_\lambda\) does not contain \(m_{e_1}\)-component. Hence \(N_\lambda = m_\lambda\).
4.2. Tangentially non-degenerate orbits. In the above subsection we have
proved that all orbits stated in Theorem [11] are tangentially degenerate. In this
subsection, we shall show that other orbits are not tangentially degenerate.

Proposition 4.18. Let \((G, K)\) be a Hermitian symmetric pair. (Then the restricted
root system of \((G, K)\) is of type \(C\) or \(BC\).) The orbit \(\text{Ad}(K)\lambda\) through \(\lambda = e_1 + e_2\)
is not tangentially degenerate.

Proof. It is sufficient to prove that if \(X \in \mathfrak{m}_{e_1+e_2}\) satisfies \([\mathfrak{t}_{e_1-e_2}, X] = \{0\}\), then
\(X = 0\). From the assumption,
\[
0 = J[\mathfrak{t}_{e_1-e_2}, X] = [\mathfrak{t}_{e_1-e_2}, JX].
\]
Therefore we have
\[
0 = \langle a, [\mathfrak{t}_{e_1-e_2}, JX] \rangle = \langle [a, \mathfrak{t}_{e_1-e_2}], JX \rangle = \langle \mathfrak{m}_{e_1-e_2}, JX \rangle.
\]
From Lemma [4.14] we have \(JX \in \mathfrak{m}_{e_1-e_2}\). This implies \(JX = 0\), hence \(X = 0\). \(\Box\)

In the case of \((G, K) = (F_4, SU(2) \cdot Sp(3))\), \((G, K)\) is a compact symmetric pair
which corresponds to a normal real form. In this case, we shall show that the orbit
through a short root is not tangentially degenerate (Proposition 4.19).

For this purpose, we shall recall some definitions. A real form \(\mathfrak{g}\) of a semisimple
Lie algebra \(\mathfrak{g}\) over \(C\) is called normal if in each Cartan decomposition \(\mathfrak{g} = \mathfrak{t} + \mathfrak{m}\) the
space \(\mathfrak{m}\) contains a maximal abelian subalgebra of \(\mathfrak{g}\). It is known that there exists
a normal real form for each semisimple Lie algebra over \(C\), moreover that is unique
up to isomorphism ([5, Ch. IX, Theorem 5.10]).

A compact symmetric pair \((G, K)\) is called compact symmetric pair corresponds
to a normal real form if the dual \((\mathfrak{g}^*, \mathfrak{t})\) of the orthogonal symmetric Lie algebra
\((\mathfrak{g}, \mathfrak{t})\) of \((G, K)\) is a normal real form of the complexification \(\mathfrak{g}^C\) of \(\mathfrak{g}\).

Proposition 4.19. Let \((G, K)\) be a compact symmetric pair which corresponds to
a normal real form with a restricted root system of type \(B\), \(C\), or \(F_4\). Then the orbit
through a short root is not tangentially degenerate.

Proof. Since \((G, K)\) is a compact symmetric pair which corresponds to a normal
real form, \(\mathfrak{t}\) and \(\mathfrak{m}\) can be expressed as
\[
\mathfrak{t} = \sum_{\alpha \in R_+} \mathbb{R} F_\alpha, \quad \mathfrak{m} = \mathfrak{t} \oplus \sum_{\alpha \in R_+} \mathbb{R} G_\alpha, \quad \mathfrak{t}_\alpha = \mathbb{R} F_\alpha, \quad \mathfrak{m}_\alpha = \mathbb{R} G_\alpha,
\]
where \(F_\alpha = (E_\alpha - E_{-\alpha})/\sqrt{2}\) and \(G_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})/\sqrt{2}\). Here \(E_\alpha \in \mathfrak{g}_\alpha\) satisfies
that, for \(\alpha, \beta \in R\), if \(\alpha + \beta \in R\) then \([E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}\) and \(N_{\alpha,\beta}\) is non-zero
real number which satisfies \(N_{\alpha,\beta} = -N_{-\alpha,-\beta}\).

When \(\alpha\) is a short root, as we showed in the proof of Proposition 4.10 there exists
\(\beta \in R_+\) such that \(\alpha \perp \beta\) and \(\alpha \pm \beta \in R\). Then we have
\[
[\mathfrak{t}_\beta, \mathfrak{m}_\alpha] = \mathbb{R}(N_{\alpha,\beta} G_{\alpha+\beta} - N_{-\alpha,\beta} G_{\alpha-\beta}) \neq \{0\}.
\]
Thus, from Lemma 4.9, the orbit \(\text{Ad}(K)\alpha\) through \(\alpha\) is not tangentially degenerate. \(\Box\)

From Proposition 4.19 in the case of \((G, K) = (F_4, SU(2) \cdot Sp(3))\), the orbit
through a short root is not tangentially degenerate.

Proposition 4.20. In the case of \((G, K) = (SO(2p+n), S(O(p) \times O(p+n)))\) \((p \geq 2, n \geq 1)\), the orbit \(\text{Ad}(K)\lambda\) through a short root \(\lambda\) is not tangentially degenerate.
Proof. In this case the restricted root system $R$ of $(G, K)$ is of type $B_n$, that is

$$R = \{ \pm e_i \mid 1 \leq i \leq p \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq p \}.$$  

Without loss of generality we can put $\lambda = e_1$. The action of $\mathfrak{t}_0 = \mathfrak{o}(n)$ on $\mathfrak{m}_\lambda = \mathbb{R}^n$ is irreducible, thus $\mathfrak{m}_\lambda$ is the only non-zero subspace of $\mathfrak{m}_\lambda$ invariant under $\mathfrak{t}_0$. Restricted root spaces $\mathfrak{m}_{e_i}, \mathfrak{t}_{e_i} (1 \leq i \leq p)$ are given by

$$\mathfrak{m}_{e_i} = \left\{ \begin{array}{c|c}
X & \begin{array}{c}
\hline
& \hline
\end{array} \\
1 & \hline
\end{array} \right\} \quad X = x_1 E_{i1} + \cdots + x_n E_{in}, \ x_j \in \mathbb{R} \right\}. $$

$$\mathfrak{t}_{e_i} = \left\{ \begin{array}{c|c}
X & \begin{array}{c}
\hline
& \hline
\end{array} \\
1 & \hline
\end{array} \right\} \quad X = x_1 E_{i1} + \cdots + x_n E_{in}, \ x_j \in \mathbb{R} \right\}. $$

Therefore, when $i \geq 2$, we have that $e_i$ is perpendicular to $e_1$ and

$$[\mathfrak{t}_{e_i}, \mathfrak{m}_{e_1}] = \mathfrak{R} \begin{array}{c|c}
-E_{i1} & \begin{array}{c}
\hline
& \hline
\end{array} \\
1 & \hline
\end{array} \right\} \subset \mathfrak{m}_{e_1-e_1} \oplus \mathfrak{m}_{e_1+e_1}. $$

Hence, from Lemma 4.9, the orbit $\text{Ad}(K) \lambda$ is not tangentially degenerate.  

\[\square\]

**Proposition 4.21.** In the case of $(G, K) = (\text{Sp}(2p+n), \text{Sp}(p) \times \text{Sp}(p+n))$ $(p \geq 2, n \geq 0)$, the orbit $\text{Ad}(K) \lambda$ through a restricted root $\lambda = e_1 + e_2$ is not tangentially degenerate.

**Proof.** When $n \geq 1$, the restricted root system of $(G, K)$ is of type $BC_p$. And when $n = 0$, the restricted root system is of type $C_p$. However, we shall consider both cases uniformly. In order to prove the proposition, it suffices to show that $\{0\}$ is the only subspace of $\mathfrak{m}_{e_1+e_2}$ invariant under $\text{ad}(\mathfrak{g}_{\lambda}^\mathbb{R})$.

Let $V$ be a subspace of $\mathfrak{m}_{e_1+e_2}$ invariant under $\text{ad}(\mathfrak{g}_{\lambda}^\mathbb{R})$. We take $X \in V$ arbitrarily. Then $[\mathfrak{t}_{e_1-e_2}, X] \subset V \subset \mathfrak{m}_{e_1+e_2}$. On the other hand, $[\mathfrak{t}_{e_1-e_2}, X] \subset \mathfrak{m}_{2e_1} \oplus \mathfrak{m}_{2e_2}$. Therefore we have $[\mathfrak{t}_{e_1-e_2}, X] = \{0\}$.

Under the notation of the proof of Proposition 4.17 restricted root spaces $\mathfrak{m}_{e_1+e_j}$ and $\mathfrak{t}_{e_i-e_j}$ are given by

$$\mathfrak{m}_{e_1+e_j} = \{ x(E_{i,p+j} + E_{p+i,j}) - \bar{x}(E_{p+j,i} + E_{j,p+i}) \mid x \in \mathbb{H} \},$$

$$\mathfrak{t}_{e_i-e_j} = \{ y(E_{i,j} + E_{p+i,p+j}) - \bar{y}(E_{j,i} + E_{p+j,p+i}) \mid y \in \mathbb{H} \}. $$

We put $X = x(E_{1,p+2} + E_{p+1,2}) - \bar{x}(E_{p+2,1} + E_{2,p+1}) \in V$. Then

$$[\mathfrak{t}_{e_1-e_2}, X] = \{ (x\bar{y} - y\bar{x})(E_{1,p+1} + E_{p+1,1}) + (\bar{x}\bar{y} - \bar{x}y)(E_{2,p+2} + E_{p+2,2}) \mid y \in \mathbb{H} \}. $$

Therefore $x$ must be zero for the right-hand side to be $\{0\}$. Hence $V = \{0\}$. Consequently we have that $\{0\}$ is the only subspace of $\mathfrak{m}_{e_1+e_2}$ invariant under $\text{ad}(\mathfrak{g}_{\lambda}^\mathbb{R})$.

\[\square\]

Next we shall show when

$$(G, K) = (E_6, SU(2) \cdot SU(6)), \quad (E_7, SU(2) \cdot SO(12)), \quad (E_8, SU(2) \cdot E_7),$$

the orbit through a short root $\lambda$ is not tangentially degenerate. In these cases, $G/K$ is a compact quaternionic symmetric space whose restricted root system is of type $F_4$. See Appendix in detail.

Let $\nu$ be in $R_+$ such that $\nu \perp \lambda$. Note that $[\nu, \mathfrak{m}_\lambda] = \{0\}$. We take $X \in \mathfrak{m}_\lambda$ arbitrarily. From Lemma 4.9 it is sufficient to prove that $[\mathfrak{t}_\nu, X] = 0$ implies
$X = 0$. Now we assume that $[\mathfrak{t}_\nu, X] = 0$. Then, from the Jacobi identity and (2) of Lemma 3.1 we have

$$0 = [\nu, [\mathfrak{t}_\nu, X]] = [[\nu, \mathfrak{t}_\nu], X] + [\mathfrak{t}_\nu, [\nu, X]] = [m_\nu, X].$$

Hence $[\mathfrak{t}_\nu + m_\nu, X] = 0$. Applying the inverse $\Phi^{-1}$ of the Cayley transform to the equality above, we have

$$\left[ \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \nu} (RF_\alpha + RG_\alpha), \Phi^{-1}(X) \right] = 0.$$

Here we used Lemma 5.6. Since

$$\left[ \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \nu} (RF_\alpha + RG_\alpha) \right] = \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \nu} (g_\alpha + g_{-\alpha}),$$

we have

$$\left[ \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \nu} (g_\alpha + g_{-\alpha}), \Phi^{-1}(X) \right] = 0.$$

Using Lemma 5.6 again, we have

$$\Phi^{-1}(X) \in \Phi^{-1}(m_\lambda) \subset \Phi^{-1}(\mathfrak{m}_\lambda + \mathfrak{m}_\lambda) = \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (RF_\alpha + RG_\alpha) \subset \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (g_\alpha + g_{-\alpha}).$$

Hence it is sufficient to prove that if

$$Y \in \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (g_\alpha + g_{-\alpha})$$

satisfies

$$(4.3) \left[ \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \nu} (g_\alpha + g_{-\alpha}), Y \right] = 0,$$

then $Y$ must be 0.

We shall prove the above claim for each of the three cases.

**Proposition 4.22.** In the case of $(G, K) = (E_6, SU(2) \cdot SU(6))$, the orbit $Ad(K)\lambda$ through a short root $\lambda$ is not tangentially degenerate.

**Proof.** We may put $\lambda = \pi(\Phi(\alpha_1))$. Then $\lambda$ is a short root, and

$$(\pi \Phi)^{-1}(\lambda) = \{ \alpha \in \tilde{R} \mid \pi(\Phi(\alpha)) = \lambda \} = \{ \alpha_1, \alpha_6 \}.$$

We set $\nu = \pi(\Phi(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6))$. Then $\nu$ is a short root perpendicular to $\lambda$, and

$$(\pi \Phi)^{-1}(\nu) = \{ \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 \}.$$

Now we assume that

$$Y = x_1E_{\alpha_1} + y_1E_{-\alpha_1} + x_2E_{\alpha_6} + y_2E_{-\alpha_6} \in \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (g_\alpha + g_{-\alpha}).$$
satisfies the condition \([4.3]\). We note that the set of roots of the form \(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = \alpha\) where \(\alpha \in (\pi \Phi)^{-1}(\lambda)\) is
\[
\{(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) + \alpha_1, (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) - \alpha_6\}.
\]
Therefore we have
\[
[E_{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}, Y]
= x_1 N_{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1} E_{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6} + y_2 N_{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, -\alpha_6} E_{\alpha_3 + \alpha_4 + \alpha_5}.
\]
This shows that the condition \([E_{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}, Y] = 0\) yields \(x_1 = y_2 = 0\). Similarly the condition \([E_{-(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)}, Y] = 0\) yields \(y_1 = y_2 = 0\). Hence we obtain \(Y = 0\).

The following two propositions can be proved in a similar way to the proof of Proposition \([4.22]\) So we write only the essentials of their proofs.

**Proposition 4.23.** In the case of \((G, K) = (E_7, SU(2) \cdot SO(12))\), the orbit \(\text{Ad}(K)\lambda\) through a short root \(\lambda\) is not tangentially degenerate.

**Proof.** We may put \(\lambda = \pi(\Phi(\alpha_4))\). Then \(\lambda\) is a short root, and
\[
(\pi \Phi)^{-1}(\lambda) = \{\alpha_4, \alpha_4 + \alpha_5, \alpha_2 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5\}.
\]
We set \(\nu = \pi(\Phi(\alpha_3 + \alpha_4))\). Then \(\nu\) is a short root perpendicular to \(\lambda\), and
\[
(\pi \Phi)^{-1}(\nu) = \{\alpha_3 + \alpha_4, \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}.
\]
We get the assertion from the following: The set of roots of the form \(\alpha_3 + \alpha_4 \pm \alpha\) where \(\alpha \in (\pi \Phi)^{-1}(\lambda)\) is
\[
\{(\alpha_3 + \alpha_4) - \alpha_4, (\alpha_3 + \alpha_4) + (\alpha_2 + \alpha_4 + \alpha_5)\}.
\]
The set of roots of the form \(\alpha_3 + \alpha_4 + \alpha_5 \pm \alpha\) where \(\alpha \in (\pi \Phi)^{-1}(\lambda)\) - \{\alpha_4, \alpha_2 + \alpha_4 + \alpha_5\} is
\[
\{(\alpha_3 + \alpha_4 + \alpha_5) - (\alpha_4 + \alpha_5), (\alpha_3 + \alpha_4 + \alpha_5) + (\alpha_2 + \alpha_4)\}.
\]

**Proposition 4.24.** In the case of \((G, K) = (E_8, SU(2) \cdot E_7)\), the orbit \(\text{Ad}(K)\lambda\) through a short root \(\lambda\) is not tangentially degenerate.

**Proof.** We may put \(\lambda = \pi(\Phi(\alpha_1))\). Then \(\lambda\) is a short root, and
\[
(\pi \Phi)^{-1}(\lambda) = \left\{ \alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_3 + \alpha_4 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \right\}.
\]
We set \(\nu = \pi(\Phi(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7))\). Then \(\nu\) is a short root perpendicular to \(\lambda\), and
\[
(\pi \Phi)^{-1}(\nu) = \left\{ \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_1 + \alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \right\}.
\]
We get the assertion from the following: The set of roots of the form \( \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \pm \alpha \) where \( \alpha \in (\pi\Phi)^{-1}(\lambda) \) is
\[
\begin{cases}
(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) - (\alpha_1 + \alpha_3), \\
(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) + (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)
\end{cases}
\].

The set of roots of the form \( \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \pm \alpha \) where \( \alpha \in (\pi\Phi)^{-1}(\lambda) \) is
\[
\begin{cases}
(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) - (\alpha_1 + \alpha_3 + \alpha_4), \\
(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)
\end{cases}
\].

The set of roots of the form \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \pm \alpha \) where \( \alpha \in (\pi\Phi)^{-1}(\lambda) \) is
\[
\begin{cases}
(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5), \\
(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) + (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)
\end{cases}
\].

\[\square\]

### 4.3. List of tangentially degeneracy

At the end of this section, we give the list of symmetric pairs whose ranks are equal or greater than 2 such that the orbits of their \( s \)-representations have degenerate Gauss mappings. All of them are orbits through long roots except the case of type \( G_2 \). In the case of type \( G_2 \) both of orbits through a long root and a short root have degenerate Gauss mappings, and both of them have the same dimension and the same rank of Gauss mapping. In Table I we denote the dimension of the orbit by \( l \) and the rank of Gauss mapping by \( r \). Then tangentially degeneracy is equal to \( l - r \).

In the list, we can find many orbits which satisfy the equality \( r = F(l) \). In order to observe this we state some properties of the Ferus number. The definition of the Ferus number immediately implies \( F(l) \leq l \).

**Lemma 4.25.** \( F(l) \leq F(l + 1) \).

**Proof.** The relation \( \{ k \mid A(k) + k \geq l + 1 \} \subseteq \{ k \mid A(k) + k \geq l \} \) implies
\[
F(l + 1) = \min \{ k \mid A(k) + k \geq l + 1 \} \geq \min \{ k \mid A(k) + k \geq l \} = F(l).
\]

\[\square\]

**Lemma 4.26.** \( F(2^r) = 2^r \).
Proof. It is sufficient to show \( A(k) + k < 2^q \) for \( k < 2^t \). We write \( k = 2^q - (2s + 1)2^t \) by some non-negative integers \( s \) and \( t \), and \( t = c + 4d \) by some \( 0 \leq c \leq 3 \) and \( d \geq 0 \). Then \( t < q \) and we get

\[
A(k) = A(2^q - 2^t(2s + 1)) = A(2^q - 2^t - (2s + 1)) = 2^c + 8d - 1.
\]

Thus

\[
A(k) + k = 2^q - \{2^{c+4d}(2s + 1) - 2^c - 8d\}.
\]

Here

\[
2^{c+4d}(2s + 1) - 2^c - 8d + 1 \quad \geq \quad 2^{c+4d} - 2^c - 8d + 1
\]

\[
= 2^c(2^d - 1) - 8d + 1
\]

\[
\geq 2^d - 8d \geq 1
\]

Therefore we obtain \( A(k) + k < 2^q \) \( \square \)
Proposition 4.27. Assume \( q \geq 1 \) and write \( q = c + 4d \) \((0 \leq c \leq 3, d \geq 0)\). Then
\[
F(2^a + a) = 2^a
\]
holds for any \( 0 \leq a \leq 2^c + 8d - 1 \).

Proof. Since \( q \geq 1 \), we have \( c \geq 1 \) or \( d \geq 1 \). Thus \( A(2^q) = 2^c + 8d - 1 \geq 1 \). This shows \( A(2^q) + 2^q = 2^q + 2^c + 8d - 1 \). From Lemmas 4.25 and 4.26 we get
\[
2^q \geq F(2^q + 2^c + 8d - 1) \geq F(2^q) = 2^q.
\]
\[ \square \]

The above proposition shows the following equalities:
\[
\begin{align*}
F(2^q + 1) &= 2^q \quad (q \geq 1), \\
F(2^q + 2) &= 2^q \quad (q \geq 2), \\
F(2^q + 3) &= 2^q \quad (q \geq 2), \\
F(2^q + 4) &= 2^q \quad (q \geq 3).
\end{align*}
\]

By the use of the above equalities, we can see many orbits of the \( s \)-representations which satisfy the Ferus equality \( F(l) = r \) in Table 1. For example, the orbits of the \( s \)-representations of the following symmetric pairs through a long root satisfy \( F(l) = r \):

\[
\begin{align*}
\{ &\mathfrak{su}(2^{q-1} + 2), \mathfrak{so}(2^{q-1} + 2)\} &\quad (q \geq 1), \\
&\mathfrak{su}(2^{q-2} + 2)^2, \mathfrak{su}(2^{q-2} + 2) &\quad (q \geq 2), \\
&\mathfrak{su}(2^{2q-3} + 2), \mathfrak{sp}(2^{2q-3} + 2) &\quad (q \geq 3), \\
&\mathfrak{e}_6, \mathfrak{f}_4 &\quad (q \geq 1), \\
&\mathfrak{su}(2^{q-1} + 1), \mathfrak{u}(2^{q-1} + 1) &\quad (q \geq 1), \\
&\mathfrak{sp}(2^{q-2} + 1)^2, \mathfrak{sp}(2^{q-2} + 1) &\quad (q \geq 2), \\
&\mathfrak{sp}(2^{2q-3} + 1), \mathfrak{sp}(2^{q-3} + 1) \oplus \mathfrak{sp}(2^{q-3} + 1) &\quad (q \geq 3), \\
&\mathfrak{su}(2^{q-2} + 1), \mathfrak{su}(2^{q-2} + 1) \oplus \mathfrak{su}(2^{q-2} + 1) \oplus \mathfrak{R} &\quad (q \geq 2), \\
&\mathfrak{so}(4(2^{q-3} + 1)), \mathfrak{u}(2(2^{q-3} + 1)) &\quad (q \geq 3), \\
&\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{R}, \\
&\mathfrak{so}(2(2^{q-2} + 2)), \mathfrak{so}(2^{q-2} + 2) \oplus \mathfrak{so}(2^{q-2} + 2) &\quad (q \geq 2), \\
&\mathfrak{so}(2^{2q-3} + 2)^2, \mathfrak{so}(2^{2q-3} + 2)) &\quad (q \geq 3), \\
&\mathfrak{e}_6 \oplus \mathfrak{e}_6, \mathfrak{e}_6 &\quad (q \geq 1), \\
&\mathfrak{e}_7, \mathfrak{su}(8) &\quad (q \geq 3), \\
&\mathfrak{e}_7 \oplus \mathfrak{e}_7, \mathfrak{e}_7 &\quad (q \geq 3), \\
&\mathfrak{e}_8, \mathfrak{so}(16) &\quad (q \geq 3), \\
&\mathfrak{e}_8 \oplus \mathfrak{e}_8, \mathfrak{e}_8 &\quad (q \geq 3), \\
&\mathfrak{e}_7, \mathfrak{su}(2) \oplus \mathfrak{so}(12)) &\quad (q \geq 3), \\
&\mathfrak{e}_8, \mathfrak{su}(2) \oplus \mathfrak{e}_7 &\quad (q \geq 3), \\
&\mathfrak{su}(2p + n), \mathfrak{su}(p) \oplus \mathfrak{su}(p + n) \oplus \mathfrak{R} &\quad (4p + 2n - 3 = 2^q + 1, p \geq 2, n \geq 1, q \geq 1), \\
&\mathfrak{sp}(2p + n), \mathfrak{sp}(p) \oplus \mathfrak{sp}(p + n)) &\quad (8p + 4n - 5 = 2^q + 3, p \geq 2, n \geq 1, q \geq 2).
\end{align*}
\]
Furthermore the orbits of s-representations of symmetric pairs

\((g_2, so(4))\) and \((g_2 \oplus g_2, g(2))\)

through a long root or a short root satisfy the Ferus equality \(F(5) = 4\) or \(F(10) = 8\).

**Remark 4.28.** When \((G, K)\) is of rank 2, the results above were studied by Ishikawa, Kimura and Miyaoka [10].

5. **Appendix : Quaternionic symmetric spaces**

A \(4n\)-dimensional Riemannian manifold is called quaternion-Kähler if its holonomy group is contained in \(Sp(n) \cdot Sp(1)\). A quaternion-Kähler manifold is called quaternionic symmetric if it is a Riemannian symmetric space ([11], p. 396).

We will review a construction of a quaternionic symmetric space from a compact simple Lie algebra \(g\) whose rank is greater than or equal to 2 (see [13] in detail).

Set \(G = \text{Int}(g)\), which is a compact connected semisimple Lie group. We denote by \(\langle \cdot, \cdot \rangle\) a biinvariant Riemannian metric on \(G\). Take a maximal torus \(T\) in \(G\) and denote its Lie algebra by \(t\). For \(\alpha \in t\) we set \(\tilde{g}_\alpha\) as (3.1), and define root system \(\tilde{R}\) by (3.2). We have then \(g^C = t^C + \sum_{\alpha \in \tilde{R}} \tilde{g}_\alpha\).

For \(\alpha \in \tilde{R}\) we can take \(E_\alpha \in \tilde{g}_\alpha\) such that

\[E_\alpha - E_{-\alpha} \in g, \quad \sqrt{-1}(E_\alpha + E_{-\alpha}) \in g, \quad [E_\alpha, E_{-\alpha}] = -\sqrt{-1}\alpha,\]

\[\left\| \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \right\| = \left\| \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \right\| = 1,\]

and that if we define \(N_{\alpha, \beta}\) by \([E_\alpha, E_\beta] = N_{\alpha, \beta}E_{\alpha + \beta}\), then \(N_{\alpha, \beta} = -N_{-\alpha, -\beta}\) where we put \(N_{\alpha, \beta} = 0\) if \(\alpha + \beta \notin \tilde{R}\). Let \(\tilde{F}\) be a fundamental system of \(\tilde{R}\) and denote by \(\tilde{R}_+\) the set of positive roots with respect to \(\tilde{F}\). For \(\alpha \in \tilde{R}_+\) set

\[F_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \quad G_\alpha = \frac{\sqrt{-1}}{\sqrt{2}}(E_\alpha + E_{-\alpha}),\]

then we have

(5.1) \(g = t + \sum_{\alpha \in \tilde{R}_+} (RF_\alpha + RG_\alpha), \quad \|F_\alpha\| = \|G_\alpha\| = 1, \quad [F_\alpha, G_\alpha] = \alpha.\)

For each \(\alpha \in \tilde{R}_+\), we define a subalgebra \(g(\alpha)\) of \(g\) by

\[g(\alpha) = R\alpha + g \cap (\tilde{g}_\alpha + \tilde{g}_{-\alpha}) = R\alpha + RF_\alpha + RG_\alpha,\]

which is isomorphic to \(su(2)\). We denote the highest root by \(\delta \in \tilde{R}_+\). By Lemma 4.4

\[s = \exp \left( \frac{2\pi}{\|\delta\|^2} \delta \right)\]

is an involutive automorphism of \(g\). The fixed points set \(t\) of \(s\) in \(g\) is given by

\[t = t + RF_\delta + RG_\delta + \sum_{\alpha \perp \delta} (RF_\alpha + RG_\alpha)\]

\[= g(\delta) + t \cap \delta^+ + \sum_{\alpha \perp \delta} (RF_\alpha + RG_\alpha).\]
The subalgebras \( g(\delta) \) and \( t \cap \delta^\perp + \sum_{\alpha \perp \delta}(RF_\alpha + RG_\alpha) \) are ideals of \( \mathfrak{k} \). The \((-1)\)-eigenspace \( m \) of \( s \) is given by

\[
m = \sum_{\alpha \in \check{\mathbb{R}}_m^+} (RF_\alpha + RG_\alpha) \quad \text{where} \quad \check{\mathbb{R}}_m^+ = \left\{ \alpha \in \check{\mathbb{R}}_+ \mid \frac{\langle \alpha, \delta \rangle}{\|\delta\|^2} = \frac{1}{2} \right\}.
\]

Since there exists a subset \( \check{\mathbb{R}}_m^+ (\delta) \) in \( \check{\mathbb{R}}_m^+ \) such that (5.2)

\[
m = \sum_{\alpha \in \check{\mathbb{R}}_+(\delta)} (RF_\alpha + RG_\alpha + RF_{\delta - \alpha} + RG_{\delta - \alpha}),
\]

the dimension of \( m \) is a multiple of 4.

We also denote by \( s \) the involutive automorphism of \( G \) induced from \( s \). Since the fixed point set of \( s \) in \( G \) is closed and \( G \) is compact, the identity component \( K \) of the fixed points set is also compact. The Lie algebra of \( K \) coincides with \( \mathfrak{k} \) and \((G, K)\) is a compact symmetric pair. Hence the coset manifold \( G/K \) is a compact Riemannian symmetric space. Moreover \( G/K \) is a quaternionic symmetric space since (5.2) defines a quaternionic structure. Conversely it is known that every compact quaternionic symmetric space is obtained in this way. We omit its proof. See [13] in detail.

Quaternionic symmetric spaces have a similar property with Hermitian symmetric spaces as we shall mention below: Two roots \( \gamma_1, \gamma_2 \in \check{\mathbb{R}}_m^+ (\delta) \) are said to be strongly orthogonal if \( \gamma_1 \pm \gamma_2 \notin \check{\mathbb{R}} \).

**Proposition 5.1.** Let \( G/K \) be a compact quaternionic symmetric space of rank \( p \). Then there exist \( \check{\mathbb{R}}_+(\delta) \) which satisfies (5.2) and a subset \( \{\gamma_i\}_{1 \leq i \leq p} \) of \( \check{\mathbb{R}}_+(\delta) \) consisting of strongly orthogonal roots such that

\[
a = \sum_{i=1}^{p} RF_{\gamma_i}
\]

is a maximal abelian subspace of \( m \).

The proof requires some preparation.

**Lemma 5.2.** If \( \alpha, \beta \in \check{\mathbb{R}}_m^+ \) and \( \alpha + \beta \in \check{\mathbb{R}} \), then \( \alpha + \beta = \delta \).

**Proof.** Since \( \alpha, \beta \in \check{\mathbb{R}}_m^+ \), we have

\[
\frac{\langle \alpha + \beta, \delta \rangle}{\|\delta\|^2} = 1.
\]

Using Lemma 4.4, \( \alpha + \beta \in \check{\mathbb{R}} \) implies \( \alpha + \beta = \delta \). \( \square \)

**Corollary 5.3.** \( [\check{g}_\alpha, \check{g}_\beta] \subset \check{g}_\delta \) for \( \alpha, \beta \in \check{\mathbb{R}}_m^+ \).

**Proof.** If \( \alpha + \beta \in \check{\mathbb{R}} \), Lemma 5.2 implies \( [\check{g}_\alpha, \check{g}_\beta] = \check{g}_\delta \). If \( \alpha + \beta \notin \check{\mathbb{R}} \), then \( [\check{g}_\alpha, \check{g}_\beta] = \{0\} \).

If \( Q \) is any subset of \( \check{\mathbb{R}}_m^+ \), let

\[
m_Q = \sum_{\alpha \in Q} (\check{g}_\alpha + \check{g}_{-\alpha}).
\]

Remark that \( m_{\check{\mathbb{R}}_m^+} = m^C \). For the lowest root \( \gamma \) in \( Q \), put

\[
Q(\gamma) = \{\beta \in Q - \{\gamma\} \mid \beta \pm \gamma \notin \check{\mathbb{R}}\}.
\]
Then $\beta \pm \gamma \notin \tilde{R} \cup \{0\}$ for $\beta \in Q(\gamma)$.

**Lemma 5.4.** We denote by $\mathfrak{z}_{mQ}(E_\gamma + E_{-\gamma})$ the centralizer of $E_\gamma + E_{-\gamma}$ in $m_Q$. Then

$$\mathfrak{z}_{mQ}(E_\gamma + E_{-\gamma}) = m_{Q(\gamma)} + C(E_\gamma + E_{-\gamma}).$$

**Proof.** Since $\beta \pm \gamma \notin \tilde{R} \cup \{0\}$ for $\beta \in Q(\gamma)$, we have

$$[m_{Q(\gamma)}, \tilde{g}_\gamma + \tilde{g}_{-\gamma}] = \left[ \sum_{\beta \in Q(\gamma)} (\tilde{g}_\beta + \tilde{g}_{-\beta}), \tilde{g}_\gamma + \tilde{g}_{-\gamma} \right] = \{0\}.$$

Since $E_\gamma + E_{-\gamma} \in \tilde{g}_\gamma + \tilde{g}_{-\gamma}$, we get

$$[m_{Q(\gamma)}, E_\gamma + E_{-\gamma}] = \{0\}.$$

Hence we have

$$m_{Q(\gamma)} + C(E_\gamma + E_{-\gamma}) \subset \mathfrak{z}_{mQ}(E_\gamma + E_{-\gamma}).$$

Conversely let $X$ be in $\mathfrak{z}_{mQ}(E_\gamma + E_{-\gamma})$. Since $X \in m_Q$, we can express $X$ as

$$X = c_\gamma E_\gamma + c_{-\gamma} E_{-\gamma} + \sum_{\beta \in Q'} (c_\beta E_\beta + c_{-\beta} E_{-\beta}) \quad \text{where} \quad Q' = Q - \{\gamma\}.$$

We consider the components of $[X, E_\gamma + E_{-\gamma}] = 0$ in the root space decomposition. Since the $tC$-component is

$$c_\gamma [E_\gamma, E_{-\gamma}] + c_{-\gamma} [E_{-\gamma}, E_\gamma] = (c_\gamma - c_{-\gamma})[E_\gamma, E_{-\gamma}],$$

we have $c_\gamma = c_{-\gamma}$, which implies that

$$X = c_\gamma (E_\gamma + E_{-\gamma}) + \sum_{\beta \in Q'} (c_\beta E_\beta + c_{-\beta} E_{-\beta}).$$

Put

$$Y = \sum_{\beta \in Q'} (c_\beta E_\beta + c_{-\beta} E_{-\beta}),$$

then $X = c_\gamma (E_\gamma + E_{-\gamma}) + Y$ and

$$0 = [X, E_\gamma + E_{-\gamma}] = [Y, E_\gamma + E_{-\gamma}]$$

$$= \sum_{\beta \in Q'} (c_\beta [E_\beta, E_\gamma] + c_\beta [E_\beta, E_{-\gamma}] + c_{-\beta} [E_{-\beta}, E_\gamma] + c_{-\beta} [E_{-\beta}, E_{-\gamma}]).$$

Here $[E_\beta, E_\gamma] \in \tilde{g}_\delta$ and $[E_{-\beta}, E_{-\gamma}] \in \tilde{g}_{-\delta}$ by Corollary 5.3. Since $\beta, \gamma \in \tilde{R}_m$, we have $\langle \beta - \gamma, \delta \rangle = 0$. Clearly we get $[E_\beta, E_\gamma] \in \tilde{g}_{\beta - \gamma}$ and $[E_{-\beta}, E_\gamma] \in \tilde{g}_{-\beta + \gamma}$. Since $\gamma$ is the lowest root in $Q$, we have $\beta - \gamma > 0$ for $\beta \in Q'$ and $- \beta + \gamma < 0$, which implies that $\beta - \gamma \neq \delta$, $- \beta + \gamma \neq \delta$. Hence, if $\beta - \gamma \in \tilde{R}$, then $c_\beta = 0$ and $c_{-\beta} = 0$. If $\beta + \gamma \in \tilde{R}$, then $\beta = \delta - \gamma$ by Lemma 10.2. In this case, $c_\beta = 0$ and $c_{-\beta} = 0$. Hence we get

$$Y = \sum_{\beta \in Q(\gamma)} (c_\beta E_\beta + c_{-\beta} E_{-\beta}) \in m_{Q(\gamma)}.$$

Therefore we get the assertion.  \qed
Proof of Proposition 5.1. We inductively define a sequence of subsets
\[ \tilde{R}_+^m = Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_s \supseteq Q_{s+1} = \emptyset \]
as follows: Let \( \gamma_i \) be the lowest root in \( Q_i \) and set \( Q_{i+1} = Q_i(\gamma_i) \). Since the cardinal numbers of \( \{Q_i\} \) are strictly monotone decreasing, the operation is finished at finitely many times. Hence we can define \( \gamma_1, \ldots, \gamma_s \in \tilde{R}_+^m \). Set
\[ \tilde{b} = \sum_{i=1}^s C(E_{\gamma_i} + E_{-\gamma_i}) \subset m^C. \]
We shall show that \( \tilde{b} \) is a maximal abelian subspace of \( m^C \). By the definition of \( \gamma_i \), two distinct roots \( \gamma_i \) and \( \gamma_j \) are strongly orthogonal. In particular \( \tilde{b} \) is an abelian subspace. In order to prove the maximality of \( \tilde{b} \), set \( m_i = m_{Q_i} \), and define a sequence of subspaces in \( m^C \) by
\[ m^C = m_1 = m_1 + \tilde{b} \supseteq m_2 + \tilde{b} \supseteq \cdots \supseteq m_s + \tilde{b} \supseteq m_{s+1} + \tilde{b} = \tilde{b}. \]
We shall show that if \( X \in m^C \) satisfies \( [X, \tilde{b}] = \{0\} \), then \( X \in \tilde{b} \). In order to prove this, it is sufficient to show that if \( X \in m_p + \tilde{b} \) then \( X \in m_{p+1} + \tilde{b} \). We can express \( X \in m_p + \tilde{b} \) as
\[ X = Y + Z \quad (Y \in m_p, \ Z \in \tilde{b}). \]
Since \( [X, \tilde{b}] = \{0\} \), we have
\[ 0 = [X, E_{\gamma_p} + E_{-\gamma_p}] = [Y, E_{\gamma_p} + E_{-\gamma_p}], \]
which implies that \( Y \in \tilde{m}_p(E_{\gamma_p} + E_{-\gamma_p}) = m_{p+1} + C(E_{\gamma_p} + E_{-\gamma_p}) \) by Lemma 5.4 Hence \( X = Y + Z \) is in \( m_{p+1} + \tilde{b} \).

Since \( \gamma_i + \gamma_j \neq \delta \), we can take a subset \( \tilde{R}_+(\delta) \) which satisfies (5.2) and contains \( \{\gamma_i\}_{1 \leq i \leq p} \).

Hence \( m \) is given by the following:
\[
m = a + \sum_{i=1}^p (RG_{\gamma_i} + RF_{\delta - \gamma_i} + RG_{\delta - \gamma_i})
+ \sum_{\alpha \in R_+(\delta) - \{\gamma_1, \ldots, \gamma_p\}} (RF_{\alpha} + RG_\alpha + RF_{\delta - \alpha} + RG_{\delta - \alpha})
\]
When the root system of \( G \) is not of type \( G_2 \), then \( \|\gamma_1\| = \cdots = \|\gamma_p\| \). Set
\[ b = l \cap \{\gamma_1, \ldots, \gamma_p\}^\perp, \quad b' = a + b, \]
then \( b' \) is a maximal abelian subalgebra of \( g \) containing \( a \). We define the Cayley transform \( \Phi \) by
\[ \Phi = \exp \left( \frac{\pi}{2} \text{ad} \left( \sum_{j=1}^p \frac{G_{\gamma_j}}{\|\gamma_j\|} \right) \right) \in \text{Aut}(g), \]
and set \( \lambda_i = \|\gamma_i\| F_{\gamma_i} \), then
\[ \Phi(\gamma_i) = \lambda_i, \quad \Phi(H) = H \quad (H \in b). \]
Hence the Cayley transform $\Phi$ maps $t$ onto $t'$. We denote by $R$ the restricted root system of $(G, K)$ with respect to $a$. Let $\pi : t' = a + b \to a$ be the orthogonal projection, then $R = \pi(\Phi(\tilde{R}))$. Since

$$\alpha \equiv \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \gamma_i \mod b \quad \text{for} \quad \alpha \in \tilde{R},$$

we have

$$\Phi(\alpha) \equiv \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i \mod b,$$

which implies that

$$\pi(\Phi(\alpha)) = \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i.$$

In particular

$$\{\lambda_1, \cdots, \lambda_p\} \subset R = \left\{ \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i \mid \alpha \in \tilde{R} \right\}.$$

The multiplicity $m(\lambda)$ of $\lambda = \pi(\Phi(\alpha)) \in \Sigma (\alpha \in \tilde{R})$ is given by

$$m(\lambda) = \# \{ \beta \in \tilde{R} \mid \langle \alpha, \gamma_i \rangle = \langle \beta, \gamma_i \rangle \}.$$

By (5.3), we have

$$\|\pi(\Phi(\alpha))\|^2 = \sum_{i=1}^{p} \frac{\langle \alpha, \gamma_i \rangle}{\|\gamma_i\|^2} \lambda_i \leq \|\alpha\|^2,$$

and the equality holds if and only if $\alpha \in \text{span}\{\gamma_1, \cdots, \gamma_p\}$. Hence $\|\pi(\Phi(\alpha))\|^2 = \|\alpha\|^2$ for any $\alpha \in \tilde{R}$ if and only if $p = \text{rank}(G)$.

**Lemma 5.5.** $R_{G, \gamma_i} \subset m_{\lambda_i}, \quad R_{\gamma_i} \subset t_{\lambda_i}$.

**Proof.** For $H = \sum x_j \lambda_j \in a$, we have

$$[H, G_{\gamma_i}] = \sum x_j [\|\gamma_i\|F_{\gamma_j}, G_{\gamma_i}] = x_i [\|\gamma_i\||F_{\gamma_i}, G_{\gamma_i}]$$

$$= x_i \|\gamma_i\| \frac{\gamma_i}{\|\gamma_i\|} = \langle H, \lambda_i \rangle \frac{\gamma_i}{\|\gamma_i\|},$$

$$[H, \frac{\gamma_i}{\|\gamma_i\|}] = \sum x_j [\|\gamma_j\|F_{\gamma_i}, \gamma_i] = -x_i [\|\gamma_j\| F_{\gamma_j}, \gamma_i]$$

$$= -\langle H, \lambda_i \rangle G_{\gamma_j},$$

where we used (5.1). \(\square\)

**Lemma 5.6.**

$$t_{\lambda} + m_{\lambda} = \Phi \left( \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (RF_{\alpha} + RG_{\alpha}) \right).$$

**Proof.** Since

$$t_{\lambda} + m_{\lambda} = \{ X \in g \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \quad (H \in a) \},$$

$$RF_{\alpha} + RG_{\alpha} = g \cap (\tilde{g}_\alpha + \tilde{g}_{-\alpha})$$

$$= \{ X \in g \mid [H, [H, X]] = -\langle \alpha, H \rangle^2 X \quad (H \in t) \},$$
we have
\[
\Phi \left( \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (RF_{\alpha} + RG_{\alpha}) \right)
\]
\[
= \Phi \left( \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} \{ X \in g \mid [H, [H, X]] = -\langle \alpha, H \rangle^2 X \ (H \in \mathfrak{t}) \} \right)
\]
\[
= \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} \{ Y \in g \mid [\Phi(H), [\Phi(H), Y]] = -\langle \Phi(\alpha), \Phi(H) \rangle^2 Y \ (H \in \mathfrak{t'}) \}
\]
\[
\subseteq \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} \{ Y \in g \mid [H, [H, Y]] = -\langle \pi(\Phi(\alpha)), H \rangle^2 Y \ (H \in \mathfrak{a}) \}
\]
\[
= \mathfrak{t}_{\lambda} + m_{\lambda}.
\]
Here \(\dim(\mathfrak{t}_{\lambda} + m_{\lambda}) = 2m(\lambda)\). Since \(\Phi\) is a linear isomorphism, we have
\[
\dim \Phi \left( \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (RF_{\alpha} + RG_{\alpha}) \right) = \dim \sum_{\alpha \in \tilde{R}, \pi(\Phi(\alpha)) = \lambda} (RF_{\alpha} + RG_{\alpha})
\]
\[
= 2\# \{ \alpha \in \tilde{R} \mid \pi(\Phi(\alpha)) = \lambda \}
\]
\[
= 2m(\lambda).
\]
Hence we get the assertion. \(\Box\)

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