Relationships among three popular measures of differential risks: relative risk, risk difference, and odds ratio

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1. Introduction

The relative risk, risk difference, and odds ratio are three major measures used to assess differences in the risk of diseases between different groups. These measures – which play important roles in research and practice in the biomedical, behavioral, and social sciences – have been extensively discussed in statistics,[1,2] epidemiology,[3-7] and biomedical[8] literature. Although straightforward to interpret when used independently, there is considerable confusion and frequent misinterpretation of the measures when they are used together.[9,10] As popular as they are, relationships among the three measures have never been made clear and remain elusive. For example, here is an excerpt from Kraft and colleagues:[11]

"...genetic profiles based on sets of risk markers can potentially identify rare high-risk and low-risk subgroups with large relative differences in risk (that is, with odds ratios greater than 10). These profiles can also have a high population attributable risk (PAR; also known as population attributable fraction).” (p.264)

In this excerpt the authors incorrectly assumed that a larger odds ratio would imply a higher PAR, which, in turn, would give rise to a larger relative risk. Their line of thinking is apparently logical, since all three measures have traditionally been viewed as equivalent measures of differential risks such that a larger value in any one measure would naturally imply larger values in the other two measures. This paper discusses the properties of the three measures, systematically assesses relationships between the three pairs of measures (i.e., relative risk and odds ratio, relative risk and risk difference, and risk difference and odds ratio), and presents examples to clarify the misconception in the above statement as well as other pitfalls when interpreting the relationships between the different measures.
2. Relationship between relative risk and odds ratio

Let \( p_1 \) and \( p_2 \) denote disease prevalence in two groups of interest. For simplicity, we assume that \( 0 < p_1, p_2 < 1 \), that is, there exist two subgroups with potentially different prevalence rates for the disease of interest. The relative risk \( r \), risk difference \( d \), and odds ratio \( \theta \) between the groups are defined as:

\[
r = \frac{p_1}{p_2}, \quad d = p_1 - p_2, \quad \theta = \frac{p_1(1-p_2)}{(1-p_1)p_2}.
\]

(1)

Note that the relative risk is also called the risk ratio in the literature, \([6]\) but for convenience we use the term ‘relative risk’ hereafter.

From the definitions above, we immediately see that the three measures have quite different ranges; both the relative risk and odds ratio vary between 0 and \( \infty \), while the risk difference is limited to a much smaller interval between -1 and 1. Despite the fact that the relative risk and odds ratio have the same range, they represent totally different measures of differential risks and, therefore, have quite different interpretations. For example, if \( p_1 = 0.40 \) and \( p_2 = 0.25 \), then the relative risk is \( r = 1.60 \), but the odds ratio is \( \theta = 2.00 \).

Given two prevalence rates \( p_1 \) and \( p_2 \), we can calculate both the relative risk and odds ratio. However, there is generally more than one pair of prevalence rates \( (p_1, p_2) \) that yields any pre-specified relative risk (or odds ratio). For example, \( (p_1, c p_2) \) yields the same relative risk \( r = p_1 / p_2 \) for any value of \( c \) when \( 0 < c < 1 \), such as \( c = 0.2 \) or \( c = 0.8 \). Thus, before discussing the potential relationship between the relative risk and odds ratio, we must make sure that there is a unique pair of prevalence rates \( (p_1, p_2) \) that gives rise to the specified relative risk and odds ratio. The following theorem indicates that this is the case.

**Theorem 1.** The relative risk \( r \) and the odds ratio \( \theta \) satisfy one of the following conditions:

\[
r = \theta - 1, \quad \theta < r < 1, \quad \theta > r > 1.
\]

(2)

Given any specific pair of measures \( (r, \theta) \) satisfying either of the inequalities in (2), there exists a unique pair of prevalence rates \( (p_1, p_2) \) that gives rise to the \( r \) and \( \theta \).

[NOTE: The three theorems presented in this paper are new results. The proofs for these theorems will be published in an upcoming paper; they are available from the authors on request.]

The shaded areas in Figure 1 show the relationship between the relative risk and odds ratio. The only situation in which a one-to-one correspondence between the two measures occurs is the special case in which \( r = \theta - 1 \). For all other situations, the relationship between the two measures is not unique. For example, if \( r = 1.5 \), the odds ratio can be any number \( \theta > 1.5 \); and for \( \theta = 1.5 \), the relative risk can be anywhere in the interval \( 1 < r < 1.5 \). As shown in Figure 1, the odds ratio \( \theta \) can be arbitrarily large for any \( r > 1 \) and arbitrarily small for any \( r < 1 \).

The result of Theorem 1 has significant implications for case-control studies. Case-control designs are widely employed to study the risk of disease, especially rare diseases. The odds ratio is the most popular measure of differential risk in such studies, because, unlike relative risk and risk difference, it can be estimated from both prospective and retrospective studies. Several authors have discussed methods to approximate the relative risk using the odds ratio from case-control studies, \([2,4,6,8]\).

For small \( p_1 \) and \( p_2 \), we have:

\[
\theta = \frac{r}{1 - r} \approx \frac{p_1}{p_2} = \theta.
\]

Thus, for small prevalence rates, the relative risk will be close to the odds ratio. However, the reverse is usually not true; that is, the relative risk and odds ratio may still be close, even if \( p_1 \) and \( p_2 \) are not small. This relationship is presented in Figure 2, which shows the largest value that \( p_2 \) can take to ensure precision of the above formula for estimating \( r \) from \( \theta (\epsilon = |r - \theta|) \) as a function of the odds ratio \( \theta \) for several levels of the error bound \( \epsilon \) (assuming that \( 0 < p_2 < p_1 < 1 \)). For relatively large odds ratios (e.g., \( \theta \geq 3 \)), \( p_2 \) must remain very small in order to obtain a good approximation of the relative risk using the odds ratio. However, for relatively small \( \theta \), \( p_2 \) and \( p_1 \) can take on large values. For example, if \( \theta = 2 \) and \( p_2 = 0.2 \), then \( r = 1.154 \), which is quite close to \( \theta = 0.2 \). Theorem 1 ensures that such approximations of the relative risk using the odds ratio are meaningful, since each approximated relative risk, together with the odds ratio, corresponds to a unique set of prevalence rates.
3. Relationship between relative risk and risk difference

Given two prevalence rates $p_1$ and $p_2$, we can calculate both the relative risk and risk difference. The relationship between these measures is specified in Theorem 2.

**Theorem 2.** The relative risk and risk difference satisfy one of the following conditions:

$$r-1=d=0, \quad r-1<d<0, \quad 0<d<1-1/r. \quad (3)$$

Given any specific pair of measures $(r, d)$ satisfying either of the inequalities in (3), there exists a unique pair of prevalence rates $(p_1, p_2)$ that give rise to the $r$ and $d$.

The shaded areas in Figure 3 depict the relationship between the two measures. The only situation in which a one-to-one correspondence between the two measures occurs is the trivial case in which $r=0$ and $\theta=1$. For all other situations, the relationship between the two measures is not unique. For example, if $r=5$, the risk difference can range between $d=0$ and $d=0.8$. As shown in the Figure 3, for any $d>0$ the relative risk can be arbitrarily large.

Unlike the relationship between the relative risk and the odds ratio, it is not possible to estimate the relative risk using the risk difference (or vice versa), so assessing the degree of correspondence of the paired measures is of little practical value.

4. Relationship between risk difference and odds ratio

Unlike the relationship between the relative risk and the odds ratio and the relationship between the relative risk and the risk difference, a specific pair of odds ratio and risk difference measures $(\theta, d)$ does not give rise to a unique pair of prevalence rates $(p_1, p_2)$. This assertion follows from the following theorem.

**Theorem 3.** The odds ratio $\theta$ and risk difference $d$ satisfy one of the following conditions:

$$\theta - 1 = d = 0, \quad 0 < d < 1 - 1/r. \quad (4)$$

Any pair of measures $(\theta, d)$ satisfying either of the inequalities in (4) corresponds to two different pairs of prevalence rates $(p_1, p_2)$ and $(p_1', p_2')$, except for the special case of $\theta = [(1+d)/(1-d)]^2$ (when the two pairs coincide).

The shaded areas in Figure 4 show the relationship between the risk difference and the odds ratio. There is no unique set of prevalence rates corresponding to a given pair of risk difference and odds ratio measures, so the relationship of the risk difference and odds ratio is difficult to interpret and of little practical value.

Figure 2. The largest value that $p_2$ can take given the odds ratio and pre-specified precision

Figure 3. Relationship between risk difference and relative risk

Figure 4. Relationship between risk difference and odds ratio
5. Examples

In this section we use numerical examples to clarify misconceptions about the relationships among the risk ratio, risk difference, and odds ratio. For simplicity and without the loss of generality, we only consider the case in which 0 < \( p_1 < p_2 < 1 \), implying that \( d > 0, r > 1 \) and \( \theta > 1 \).

The discussion in the preceding sections makes it clear that one must be very careful when comparing the three different measures. In some cases, such as the relationship between the risk difference and the odds ratio, it is meaningless to even speak of such a relationship, because the same pair of measures may correspond to quite different sets of prevalence rates. Moreover, even if the measures arise from a unique set of prevalence rates – which occurs when comparing the relative risk to the odds ratio or comparing the relative risk to the risk difference – the relationship between the measures is not monotone (i.e., it varies for different ranges of the measures), so it is not possible to make qualitative comparisons of the magnitude of the measures. Unfortunately, these properties have not been made clear in the literature, leading to frequent misinterpretation of study findings.

**Example 1. A larger relative risk does not imply a larger risk difference.**

Suppose \( (p_1, p_2) = (0.40, 0.25) \) and \( (\bar{p}_1, \bar{p}_2) = (0.54, 0.36) \). The risk differences for the two cases are 0.15 and 0.18, respectively, but the corresponding relative risks are 1.6 and 1.5. Thus, when changing from the first to the second pair of prevalence rates the risk difference increased, but the relative risk decreased.

**Example 2. A larger odds ratio does not imply a higher relative risk.**

Many publications, especially publications in epidemiology, assume that larger odds ratios correspond to higher relative risks. For example, the section of the paper by Kraft and colleagues\(^{[11]}\) quoted in the introduction indicated that a larger odds ratio implied a higher population attributable risk (PAR) and, thus, a higher relative risk (because PAR=1-1/\( r \)) in their study. This is not true. As shown in Table 1, when \( (p_1, p_2) = (3/6, 1/6) \), \( r = 3 \) and \( \theta = 5 \), but when \( (p_1, p_2) = (10/12, 5/12) \), \( r = 2 \) and \( \theta = 7 \); that is, a higher odds ratio was associated with a lower relative risk (and a lower risk difference).

**Table 1. Non-monotone relationships among odds ratio, relative risk, and risk difference**

| \( p_1 \) prevalence in target group | \( p_2 \) prevalence in comparison group | \( r \) relative risk | \( \theta \) odds ratio | \( d \) risk difference |
|--------------------------------------|----------------------------------------|----------------------|----------------------|-----------------------|
| 3/6                                  | 1/6                                    | 3                    | 5                    | 2/6                   |
| 10/12                                | 5/12                                   | 2                    | 7                    | 5/12                  |
| 4/10                                 | 1/10                                   | 4                    | 6                    | 3/10                  |

**Example 3. A larger odds ratio does not imply a larger risk difference.**

Table 1 also shows that when \( (p_1, p_2) = (3/6, 1/6) \), \( \theta = 5 \) and \( d = 1/3 \). However, when \( (p_1, p_2) = (4/10, 1/10) \), \( \theta = 7 \) and \( d = 3/10 \). Thus, an increase in the odds ratio does not imply an increase in the risk difference.

**Example 4. In this example, we consider a scenario where the odds ratio (represented by \( t \) in the following equations) becomes arbitrarily large while the relative risk approaches 1 (the theoretical minimum value of the relative risk) and the risk difference approaches 0 (the theoretical minimum value of the risk difference).**

For \( t > 0 \), let

\[
\begin{align*}
p_1(t) &= 1 - \exp(-t), \\
p_2(t) &= \frac{1 - \exp(-t)}{1 + \exp(-t)(t-1)}. \\
\end{align*}
\]

It is clear that \( 0 < p_1(t) < p_2(t) < 1 \) for any \( t > 0 \). The relative risk, odds ratio, and risk difference are

\[
\begin{align*}
r(t) &= 1 + \exp(-t)(t-1), \\
\theta(t) &= t, \\
d(t) &= \frac{1 - \exp(-t)\exp(-t)(t-1)}{1 + \exp(-t)(t-1)}. \\
\end{align*}
\]

Figure 5 shows the plots of \( r(t) \) and \( d(t) \) versus \( \theta(t) \). Both \( r(t) \) and \( d(t) \) first increase then decrease with \( t \), reaching their maximum values when \( t = 2.1496 \). For \( t > 2.1496 \), both \( r(t) \) and \( d(t) \) are decreasing functions of \( t \), with \( r(t) \) reaching the asymptote at 1 and \( d(t) \) reaching the asymptote of 0. Thus a larger odds ratio does not imply a larger relative risk or a larger risk difference. The plot shows the complexity of the changes in the relative risk and risk difference as the odds ratio increases.

**Figure 5. Relationship of the odds ratio \( \theta(t) \), relative risk \( r(t) \), and risk difference \( d(t) \) described in Example 4**

6. Conclusion

Although the risk difference, risk ratio, and odds ratio are widely popular in biomedical and psychosocial research, the relationships among the different
measures have not been made clear in the literature. Many researchers incorrectly assume that there is a monotone relationship among the different indices such that higher values for one index will correspond to higher values of the other indices. The examples presented in this paper demonstrate that this is not the case; the three measures of differential risks behave very differently and in general can only be interpreted within the unique confines of their definitions.

This misconception about the equivalence of the measures is particularly problematic when pooling results in a meta-analysis from individual studies on a given topic that have employed different measures of risk. The theorems and examples in this paper demonstrate that there is no logical relationship between any of these measures. With the exception of the odds ratio and relative risk, to consolidate findings from different studies, one must either combine studies using the same measure or recalculate each measure using the original prevalence rates.

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Authors’ contributions
CYF and HYW derived the theoretical results; BKW, LX, and HS, constructed the examples and graphs; and TXM drafted the manuscript. All authors read and approved the final manuscript.

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