TRANSLATION-LIKE ACTIONS YIELD REGULAR MAPS

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Abstract. For finitely generated groups \( H \) and \( G \), we observe that \( H \) admits a translation-like action on \( G \) implies there is a regular map, which was introduced in Benjamini, Schramm and Timár’s joint paper, from \( H \) to \( G \). Combining with several known obstructions to the existence of regular maps, we have various applications. For example, we show that the Baumslag-Solitar groups do not admit translation-like actions on the classical lamplighter group.

1. Introduction

The concept of translation-like actions was introduced by Whyte to solve a geometric version of the von Neumann conjecture [19]. It serves as a geometric analogy of subgroup containment for finitely generated groups: if \( H \) is a subgroup of a finitely generated group \( G \), then the natural right action of \( H \) on \( G \) is a translation-like action. Later on, answering a question of Whyte, Seward solved a geometric version of Burnside’s problem which is formulated using translation-like actions [17]. Recently, translation-like actions play an important role in studying the question which finitely generated groups admit a weakly aperiodic shift of finite type [13]. Similar approaches (using notions from geometric group theory) to this question also appeared in [4, 6].

Despite the above success of application of this concept in various problems, it seems to us that not too much is known on whether a group \( H \) admits a translation-like action on another finitely generated group \( G \) assuming \( H \) is not a subgroup of \( G \). For known examples, see [5, 13, 17, 19]. The purpose of this note is to observe that translation-like actions yield regular maps, which was introduced in [3].

Theorem 1.1 (Main theorem). Let \( G \) and \( H \) be finitely generated groups. If \( H \) admits a translation-like action on \( G \), then \( H \to_{\text{reg}} G \).

It is well-known that there are several ways to rule out existence of regular maps between spaces: asymptotic dimension, Dirichlet harmonic functions, growth and separation (see [3, last paragraph in section 1]). By the above theorem, we also get obstructions to admitting translation-like actions for a pair of groups. We discuss these in Section 3.

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2. Definitions and proof of the main theorem

First, we recall the general definition of translation-like actions following Seward [17, Definition 1.1], but here we use left actions. Note that the original definition is due to Whyte [19, Definition 6.1].

**Definition 2.1** (Translation-like actions). Let $H$ be a group and let $(X, d)$ be a metric space. A left action $\ast$ of $H$ on $X$ is translation-like if it satisfies the following two conditions:

(i) The action is free (i.e. $h \ast x = x$ implies $h = 1_H$).
(ii) For every $h \in H$ the set $\{d(x, h \ast x) \mid x \in X\}$ is bounded.

Next, we recall the definition of regular maps following [3].

**Definition 2.2** (Regular maps). Let $(Z, d_Z), (X, d_X)$ be metric spaces, and let $\kappa < \infty$. A (not necessarily continuous) map $f : Z \to X$ is $\kappa$-regular if the following two conditions are satisfied.

(i) $d_X(f(z_0), f(z_1)) \leq \kappa (1 + d_Z(z_0, z_1))$ holds for every $z_0, z_1 \in Z$;
(ii) For every open ball $B = B(x_0, 1)$ with radius 1 in $X$, the inverse image $f^{-1}(B)$ can be covered by $\kappa$ open balls of radius 1 in $Z$.

A regular map is a map which is $\kappa$-regular for some finite $\kappa$. Write $Z \to_{reg} X$ if there is a regular map from $Z$ to $X$.

Note that as mentioned in [3, p. 5], if there is a quasi-isometry between bounded degree graphs $Z$ and $X$, then there is a regular map from $Z$ to $X$ and also the other direction. Hence being quasi-isometric implies $Z \to_{reg} X$ and $X \to_{reg} Z$, but the converse does not hold.

From now on, we fix a finitely generated group $H$ and take $(X, d)$ to be a finitely generated group $G$ using a left-invariant word length metric $d$ associated to some finite symmetric generating set $T$, i.e. $d_G(x, y) := \ell_G(x^{-1}y)$ for all $x, y \in G$, where $\ell_G$ is the word length associated to $T$.

If $H \acts G$ is a translation-like action, then we can define a map $L : H \times G \to G$ by setting $L(h, x) = x^{-1}(h \ast x)$ for all $x \in G, h \in H$.

Note that $\acts$ is an action implies that $L(h_1, x)L(h_2, h_1 \ast x) = L(h_2h_1, x)$ for all $h_1, h_2 \in H$ and $x \in G$.

Write $c(g, x) := L(g, x)^{-1}$, then $c : H \times G \to G$ is a cocycle in the usual sense, i.e. $c(h_1h_2, x) = c(h_1, h_2 \ast x)c(h_2, x)$ for all $h_1, h_2 \in H, x \in G$. And the two conditions in the definition of translation-like action are just the following.

1. For all $x \in G$, the map $H \ni h \mapsto c(h, x) \in G$ is 1-1.
2. For all $h \in H$, the set $\{c(h, x) : x \in G\}$ is bounded, i.e. $\sup_{x \in G} \ell_G(c(h, x)) =: \lambda_h < \infty$.

We are ready to prove our main theorem.

**Proof of Theorem 1.1.** Fix any $x \in G$, define $f : H \to G$ by $f(h) := c(h^{-1}, x)^{-1}$. We claim this map is a regular map.
Fix a symmetric generating set $S$ for $H$ and define $\kappa := \max_{s \in S} \lambda_s + \#T$, where $\lambda_h := \sup_{x \in G} \ell_G(c(h,x))$ for all $h \in H$.

Note that the second condition in Definition 2.2 is clear since $h \mapsto c(h,x)$ is 1-1. We are left to check the first condition holds.

$$d_G(f(h_1), f(h_2)) = d_G(c(h_1^{-1}, x)^{-1}, c(h_2^{-1}, x)^{-1}) = \ell_G(c(h_1^{-1}, x) c(h_2^{-1}, x)^{-1}) = \ell_G(c(h_1^{-1} h_2, h_2^{-1} x))$$

$$= \ell_G(c(s_1 \cdots s_k, h_2^{-1} x)), \text{ write } h_1^{-1} h_2 = s_1 \cdots s_k, \text{ where } k = \ell_H(h_1^{-1} h_2), s_i \in S.$$  

$$= \ell_G(c(s_1, s_2 \cdots s_k h_2^{-1} x) \cdots c(s_k, h_2^{-1} x))$$

$$\leq \sum_{i=1}^{k} \ell_G(c(s_i, s_{i+1} \cdots s_k h_2^{-1} x))$$

$$\leq \kappa \ell_H(h_1^{-1} h_2) = \kappa d_H(h_1, h_2).$$

Remark 2.3. Similar calculation was already used in [15, 16]. In fact, there is a connection between translation-like actions and continuous orbit equivalence theory, which may be worth exploring further.

3. Applications

As mentioned in the introduction, we use the obstruction to existence of regular maps, i.e. asymptotic dimension, which was first defined by Gromov [10] (see [1] for a nice survey), Dirichlet harmonic functions [2], growth and separation [3], to deduce results on non-existence of translation-like actions. Conversely, we may deduce results on the existence of regular maps between two groups. The above process would generate many (counter)examples, we just list a few of them here and we refer the readers to the above papers for definitions of the above invariants.

Corollary 3.1. If $G$ is a finitely generated non-amenable group, then $F_2 \rightarrow_{\text{reg}} G$, where $F_2$ is the non-abelian free group on two generators.

Proof. This is clear by Whyte’s solution to the geometric von Neumann conjecture, see [19].

Corollary 3.2. If $G$ is a finitely generated infinite group, then $\mathbb{Z} \rightarrow_{\text{reg}} G$.

Proof. This is clear by Seward’s solution to the geometric Burnside’s problem, see [17].

Corollary 3.3. $\mathbb{Z}^d$ does not admit translate-like actions on $F_2$ for all $d \geq 2$.
Proof. By theorem 1.1, we just need to show $\mathbb{Z}^d \not\rightarrow_{\text{reg}} F_2$ if $d \geq 2$. By [3, Corollary 3.3], $\lim_{n \to \infty} \text{sep}_{\mathbb{Z}^d}(n) = \infty$ if $d \geq 2$, while $\text{sep}_{F_2}(\cdot)$ is bounded by [3, Section 2]. Since the separation function is monotone non-decreasing with respect to regular maps by [3, Lemma 1.3], we deduce $\mathbb{Z}^d \not\rightarrow_{\text{reg}} F_2$. □

Corollary 3.3 answers a question of Cohen, see [5, p.4]. This question is one initial motivation for us to look at translation-like actions.

Corollary 3.4. The Baumslag-Solitar group $BS(m, n) := \langle s, t | st^ms^{-1} = t^n \rangle$ does not admit translation-like actions on the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ for any integers $m, n$.

Proof. Case 1: Assume $mn \neq 0$.

It suffices to show $BS(m, n) \not\rightarrow_{\text{reg}} (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ if $mn \neq 0$.

First, $\text{asdim}((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}) = 1$ by [9, Proposition on p.5]. Then note that the asymptotic dimension is monotone non-decreasing under regular maps, see [3, Section 6] or just apply [1, Theorem 29] in our setting. Therefore, we are left to show $\text{asdim}(BS(m, n)) \geq 2$ if $mn \neq 0$.

First, $BS(m, n)$ is always an infinite group, hence $\text{asdim}(BS(m, n)) > 0$ by [9, Lemma 1] or Corollary 3.2. We are left to show $\text{asdim}(BS(m, n)) \neq 1$. By [8, Corollary 1.2] (see also [7, 9, 12]), we just need to check $BS(m, n)$ does not contain free group as a subgroup of finite index. This is clear since $BS(m, n)$ is torsion-free when $mn \neq 0$ by [14] and any torsion-free virtually free groups are free groups by Stallings’ work [18].

Case 2: Assume $mn = 0$.

$BS(m, n) \cong \mathbb{Z} * (\mathbb{Z}/n\mathbb{Z})$ or $\mathbb{Z} * (\mathbb{Z}/m\mathbb{Z})$. This group contains a free group, then it does not admit a translation-like action on $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ by [13, Lemma 1.3] and [19, Theorem 6.2]. □

Remark 3.5. Note that to prove the above corollary, we can directly focus on $BS(1, n)$, which is amenable. Also note that if $m = n = 1$, $BS(1, 1) = \mathbb{Z}^2$, then we can also deduce $\mathbb{Z}^2 \not\rightarrow_{\text{reg}} (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ by [3, Proposition 6.1]. If $m = 0$ or $n = 0$, then $BS(m, n) \cong \mathbb{Z} * (\mathbb{Z}/n\mathbb{Z})$ or $\mathbb{Z} * (\mathbb{Z}/m\mathbb{Z})$, this group has asymptotic dimension one by [1, Theorem 84].

Corollary 3.4 answers [13, Conjecture 3] negatively. It also answers the geometric Gersten problem stated in [17] in the negative. The original geometric Gersten problem (stated under a further assumption compared with the one stated in [17]) was asked by Whyte in [19, p.107]. See [5] for more discussion on this conjecture. Note that the lamplighter group is a potential counterexample to the above conjectures was already suggested in [13].

We end the paper with the following question:

Question 3.6. Does [11, Theorem 2.3] still hold for regular maps (maybe up to some modification of the definition of fat bigons there)?
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