Superconductivity in the repulsive Hubbard model: an asymptotically exact weak-coupling solution

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We study the phase diagram of the Hubbard model in the limit where the onsite repulsive interaction, is much smaller than the bandwidth. We present an asymptotically exact expression for $T_c$, the superconducting transition temperature, in terms of the correlation functions of the non-interacting system which is valid for arbitrary densities so long as the interactions are sufficiently small. Our strategy for computing $T_c$ involves first integrating out all degrees of freedom having energy higher than an unphysical initial cutoff $\Omega_0$. Then, the renormalization group (RG) flows of the resulting effective action are computed and $T_c$ is obtained by determining the scale below which the RG flows in the Cooper channel diverge. We prove that $T_c$ is independent of $\Omega_0$. Using this method, we find a variety of unconventional superconducting ground states in two and three dimensional lattice systems and present explicit results for $T_c$ and pairing symmetries as a function of the electron concentration.

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\section{I. INTRODUCTION}

The Hubbard model is widely studied as the paradigmatic model of strongly correlated electrons\textsuperscript{1,2}. However, in more than one dimension (1D) there is controversy concerning even the basics of the phase diagram of the model. Most theoretical work on the model has focused on intermediate to strong interactions, $U \sim W$, since this is the physically relevant range of parameters for any of the intended applications of the model to real solid state systems. (Here, $U$ is the repulsion between two electrons on the same site, and $W$ is the bandwidth in the limit $U = 0$.) However, for such strong interactions the only well controlled solutions are numerical and the application of determinental quantum Monte Carlo methods\textsuperscript{3,4} and the Density-Matrix-Renormalization-Group\textsuperscript{5} have been limited by the fermion sign\textsuperscript{6} and two-dimensional entanglement problems\textsuperscript{5} respectively.

Here, we study the limit of weak interactions, $U/t \to 0$, where we compute the phase diagram and obtain expressions for the critical temperatures which, assuming the validity of certain assumptions discussed below, are asymptotically exact. To be explicit, we consider the Hubbard model

\begin{equation}
H = H_0 + U \sum_i c_{i\uparrow}^\dagger c_{i\downarrow} c_{i\downarrow} c_{i\uparrow}
\end{equation}

\begin{equation}
H_0 = -t \sum_{<i,j>,\sigma} [c_{i\sigma}^\dagger c_{j\sigma} + h.c.] - t' \sum_{(i,j),\sigma} [c_{i\sigma}^\dagger c_{j\sigma} + h.c.]
\end{equation}

for a variety of lattice systems in two and three dimensions. Here, $c_{i\sigma}^\dagger$ creates an electron with spin polarization $\sigma$ on lattice site $i$, and $<i,j>$ and $(i,j)$ signify, respectively, pairs of nearest-neighbor and next-nearest-neighbor sites.

Since the Cooper instability is the only generic instability of a Fermi liquid, except for certain fine tuned values of $t'/t$ and the electron density $n$, the only ordered states that can be stabilized by weak interactions are superconducting states. For repulsive interactions, $W \gg U > 0$, the superconducting transition temperature has an asymptotic expansion

\begin{equation}
T_c \sim W \exp \left\{-\alpha_2(t/U)^2 - \alpha_1(t/U) - \alpha_0 \right\} \times \left[1 + \mathcal{O}(t/U) \right] \\
\sim W \exp \left\{-1/[\rho \exp(W/T_c)] \right\} [1 + \mathcal{O}(t/U)]
\end{equation}

where $\alpha_n$ are dimensionless functions of $t'/t$, $n$ and $\rho$ is the density of states at the Fermi energy. The principal result we report here is to give an explicit prescription for computing $\alpha_2$ and $\alpha_1$ as a function of the electron density, $n$, and the “band structure”. On the basis of the present analysis, we conclude that the resulting phase diagram is asymptotically exact in the sense that

\begin{equation}
\lim_{U \to 0} \{ (t/U)^2 \ln[W/T_c] \} = \alpha_2.
\end{equation}

We will also explain why we are unable to give a prescription for computing $\alpha_0$. In the process of computing $\alpha_2$, one determines the symmetry of the superconducting ground state (e.g. s-wave, p-wave, d-wave, etc.) and the form of the pair wavefunction.

There are, of course, special situations in which a variety of different non-superconducting ordered phases occur. While these situations are potentially significant in what they imply about the behavior of the system at intermediate $U$, in the small $U$ limit they always involve a large degree of fine tuning of parameters. The canonical example is the case of a square lattice, in which the model with $t'=0$ has a non-generic particle-hole symmetry which leads to perfect nesting of the Fermi surface when the mean electron density per site is $n=1$, where

\begin{equation}
n \equiv N^{-1} \sum_{j\sigma} < c_{j\sigma}^\dagger c_{j\sigma} >,
\end{equation}
These special situations are thus, in some sense, not really a part of the weak coupling problem, but rather a piece of the strong correlation problem that persists to weak coupling. If one’s principal interest is in extrapolating well controlled weak-coupling calculations to the range of strong interactions, this degree of fine tuning of the bandstructure is a small price to pay to gain access to phases which have broad ranges of stability for intermediate to large $U$. However, if we focus on the small $U$ limit in its own right, then for any fixed, non-zero value of $t'/t$ to antiferromagnetic insulating phase occurs.

An interesting interplay between various possible ordered phases can also occur when the Fermi energy is coincident with a van-Hove singularity. While these singularities occur for generic values of $t'/t$, the density must be fine tuned in order for the singularity to lie sufficiently close to the Fermi energy to matter. The study of the behavior of the system in weak coupling tuned near a van-Hove singularity has been explored by several authors, again as a route to understanding strong-coupling phases and the interplay between phases in a regime of parameters in which perturbative renormalization group (RG) methods can be applied. However, again, as we are focussing on the physics of a system with small $U/t$, we do not treat the interplay with non-superconducting orders.

The original idea of obtaining superconductivity from repulsive interactions dates back to the pioneering work of Kohn and Luttinger, who derived an effective attractive interaction from the Friedel oscillations of a 3D electron gas. RPA calculations for a repulsive Hubbard model found that near a spin-density-wave instability there was an effective interaction which favored d-wave superconductivity. The treatment of this problem from the standpoint of the renormalization group was presented by Zanchi and Schulz, and others. Furthermore, the problem of competing instabilities of electronic systems has extensively been studied via the numerical functional renormalization group (FRG) method. While these works have made significant progress in our understanding of superconductivity from repulsive interactions, we present here an asymptotic analysis of the problem and show explicitly the way the final expressions for the superconducting transition temperature are independent of the initial choice of cutoff. Our analysis is based on the renormalization group framework established by Shankar and Polchinski. In section VII we present a more complete discussion of the relationship between the work presented here and previous analyses of this problem.

This paper is organized as follows. In section II, we present the results of our perturbative RG treatment. Expressions for the quantity $\alpha_2$ are presented in the spin singlet and triplet pairing channels. These results are then applied to a variety of systems in section III. Both lattice and continuum systems in $d = 2$ and $d = 3$ are considered. In section IV, we present the overall strategy of our RG calculations. In section V, we describe the technical aspects of the perturbative renormalization of the effective interactions in the Cooper channel, and discuss the one-loop RG flows of the effective pairing vertex in section VI. In section VII we discuss the relation of the present work to previous closely related approaches, and we conclude, in section VIII with a few remarks about future directions. There is an appendix with technical details.

To simplify our notation, we henceforth adopt units in which $\hbar = 1$, and the volume of the unit cell $\nu = 1$.

## II. RESULTS

Before discussing the derivation, we articulate the final results of our analysis, i.e. we give an explicit method to compute $\alpha_2$ exactly. (It is more complicated to compute $\alpha_1$, and since it is subdominant; we defer that part of the discussion to the more technical portion of the paper.) The results are expressed in terms of $\chi(\vec{k})$, the static susceptibility of the non-interacting system,

$$
\chi(\vec{k}) = -\int \frac{d^d q}{(2\pi)^d} \frac{[f(\epsilon_{\vec{k}+\vec{q}}) - f(\epsilon_{\vec{q}})]}{[\epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{q}}]} \tag{5}
$$

evaluated in the limit $T \to 0$, and

$$
\rho = \int \frac{d^d q}{(2\pi)^d} \delta(\epsilon_{\vec{q}}) = \lim_{|\vec{k}| \to 0} \chi(\vec{k}) \tag{6}
$$

the density of states at the Fermi energy $E_F$. Here $\epsilon_{\vec{q}}$ is the band-dispersion measured relative to the Fermi energy, $E_F$, and $d = 2$ or 3 denotes the number of spatial dimensions. The integrals run over the appropriate first Brillouin. In general, $\chi$ must be computed numerically, but this is a straightforward computation.

For simplicity, we shall restrict our attention in this paper to systems which possess inversion symmetry and no spin-orbit coupling. This enables us to classify the possible superconducting states as having either even or odd parity, the former class consisting of spin singlet and the latter of spin triplet states. Among the even parity spin-singlet superconducting states, the s-wave states are those which transform trivially under a point group operation of the crystal, whereas the “d-wave” and higher angular momentum channel gap functions transform according to a non-trivial representation of the point group. The spin triplet states include p-wave and f-wave gap functions, all of which transform according to a non-trivial representation of the point group. Depending on the crystalline point group, the superconducting gap functions may transform as either a non-degenerate or an n-fold degenerate irreducible representation of the point group operations. We will present expressions for $\alpha_2$ for each of these states; for any given bandstructure and electron concentration, the physical low temperature phase is that one which produces the smallest value of $\alpha_2$.

In order to compute $\alpha_2$, we must solve, for each symmetry class, the eigenvalue problem displayed below. In
the spin-singlet channel, the eigenvalue problem corresponds to the integral equation
\[
\int \frac{d\tilde{q}}{S_F} \tilde{g}_{\tilde{k},\tilde{q}}^{n} \psi_{s,\tilde{q}}^{(n)} = \lambda_{n} \psi_{s,\tilde{k}}^{(n)}
\]
\[
\tilde{g}_{\tilde{k},\tilde{q}}^{n} = \rho U^2 \frac{\tilde{v}_{F}(\tilde{k})}{v_{F}(\tilde{q})} \left[ \chi(\tilde{k} + \tilde{q}) + c_1 \right] \sqrt{\frac{v_{F}(\tilde{q})}{v_{F}(\tilde{q})}}
\]
where \(c_1\) is of order \(t/U\), \(\tilde{k}\) designates a vector on the unperturbed Fermi surface, \(S_F \equiv \int d\tilde{p}\) is the “area” of the Fermi surface, \(v_{F}(\tilde{p})\) is the magnitude of the Fermi velocity at position \(\tilde{p}\), and the norm of the Fermi velocity is defined according to
\[
\frac{1}{v_{F}} = \int d\tilde{p} \frac{1}{v_{F}(\tilde{p})}.
\]
The only effect of \(c_1\) is to penalize the trivial s-wave state due to the bare onsite repulsive interaction. Eigenfunctions with higher angular momentum, such as the d-wave states, and appropriate “extended s-wave” states are unaffected by it. In the spin-triplet channel, the eigenstates obey
\[
\int \frac{d\tilde{q}}{S_F} \tilde{g}_{\tilde{k},\tilde{q}}^{t} \psi_{t,\tilde{q}}^{(n)} = \lambda_{n} \psi_{t,\tilde{k}}^{(n)}
\]
\[
\tilde{g}_{\tilde{k},\tilde{q}}^{t} = -\rho U^2 \left[ \frac{\tilde{v}_{F}(\tilde{k})}{v_{F}(\tilde{q})} \chi(\tilde{k} - \tilde{q}) \right] \sqrt{\frac{v_{F}(\tilde{q})}{v_{F}(\tilde{q})}}
\]
Assuming that there is at least one negative eigenvalue present in either Eq. 7 or 9, the quantity \(\alpha_2\) is obtained by the relation
\[
\alpha_2 = |\lambda_0|^{-1} (U/t)^2,
\]
where \(\lambda_0\) is the most negative eigenvalue. The zero-temperature gap function is proportional to \(\psi\):
\[
\Delta_{s(t)}(\tilde{k}) \sim T_c \frac{v_{F}(\tilde{k})}{\tilde{v}_{F}} \psi_{s(t)}(\tilde{k}).
\]
The computation of \(\alpha_1\) involves an analysis of less singular scattering processes in perturbation theory and will be addressed in section VI.

III. APPLICATION TO VARIOUS SYSTEMS

The expressions in Eqs. 7 and 9 hold for the Hubbard model and are independent of the microscopic details of the electronic structure. In this section we determine the superconducting ground states for a variety of inversion and spin-rotationally symmetric systems in \(d = 2\) and \(d = 3\).

There are two distinct effects of the band-structure which affect the asymptotic behavior of \(T_c\) : Firstly, the existence of a superconducting instability from repulsive interactions at all derives from the \(\tilde{k}\)-space structure of the effective interactions, and these depend strongly on \(n\) and the details of the band structure. Secondly, the dimensionless density of states at the Fermi energy, \(\rho_f\), varies (in 2d especially) with distance from a van-Hove point. To distinguish these two effects, in the figures we re-express the leading-order asymptotics as
\[
T_c \sim \exp \{- |\rho V_{\text{eff}}|^{-1}\}
\]
where
\[
V_{\text{eff}}(n) = |\lambda|/\rho = (U/t)^2 [\alpha_2 \rho]^{-1}.
\]

A. Rotationally invariant systems in \(d = 2\) and \(3\)

We begin by considering electrons in the continuum limit with quadratic dispersion (which is achieved in a lattice system in the limit \(n \ll 1\):
\[
\epsilon_{\tilde{k}} = \frac{k^2}{2m}.
\]
In 3 dimensions,
\[
\chi(q) = \rho \left[ k_f + \frac{(2k_f)^2 - q^2}{4q} \log \frac{q + 2k_f}{q - 2k_f} \right]
\]
\[
\simeq \rho \left( 1 - \frac{1}{2} \left( \frac{q}{2k_f} \right)^2 + \cdots \right)
\]
where \(\rho = mk_f/2\pi^2\). For such a rotationally-invariant system, each eigenfunction can be classified according to its angular moment \(\ell\), and the solutions are \((2\ell + 1)\)-fold degenerate. The expansion of \(\chi(q)\) in powers of \(q/2k_f\) is justified when seeking the lowest angular moment pairing solutions and it shall suffice to compare the s-wave and p-wave solutions which require only the leading order expansion about \(q = 0\). The non-degenerate s-wave eigenfunction has a uniform gap everywhere on the Fermi surface. From Eq. 8 it can easily be seen that since the susceptibility is a positive-definite quantity, a negative eigenvalue for an s-wave gap function does not exist. However, the 3-fold degenerate p-wave channel with \(\psi_{t}(\theta) \propto \cos \theta\), does have a negative eigenvalue, as can be seen by applying Eq. 9
\[
-\rho \int \frac{dQ'}{4\pi} \left( 1 - \frac{1}{2} \left( \frac{|\tilde{k} - \tilde{k}'|}{2k_f} \right)^2 \right) \psi_{t}(Q') = \lambda \psi_{t}(Q)
\]
The solution is \(\lambda = -\rho/16 < 0\). Thus, for a system with a quadratic dispersion and a spherical Fermi surface, the ground state is a p-wave superconductor to order \(U^2\). Physically, this can be understood from the fact that the dominant spin fluctuations are ferromagnetic. Indeed, such considerations have been applied extensively to superfuid Helium-336,37.
For a two-dimensional rotationally invariant system with quadratic dispersion,
\[ \chi(q) = \frac{m}{2\pi} \left[ 1 - \frac{\text{Re} \sqrt{q^2 - (2k_f)^2}}{q} \right]. \] (16)

Note that the susceptibility is a constant for \( q < 2k_f \). However, for a system with a circular Fermi surface, the furthest apart any two points on the Fermi surface can be is \( 2k_f \). Therefore, the pairing strengths correspond to the eigenvalues of a constant matrix. One sees immediately from this that the s-wave solution has a positive eigenvalue \( \lambda = m/(2\pi+c_1) \), and all higher angular momentum gap functions have zero eigenvalues. Therefore, for such a rotationally invariant system in two dimensions, superconductivity does not occur in any channel to order \( U^2 \). When higher order terms in perturbation theory are taken into account, it has been found that the leading instability occurs in the p-wave channel\(^\text{38}\).

**B. Lattice systems: d=2**

Various authors\(^\text{39–42}\) have studied the Hubbard model on the 2D square lattice in the weak coupling limit. Near half-filling, nesting effects lead to \( d_{x^2-y^2} \) pairing. In weak coupling at low doping, higher order non-quadratic terms in the quasiparticle dispersion\(^\text{39}\) lead to \( d_{xy} \) pairing for \( t' = 0 \) and for \( t' < - t/4 \), the p-wave state is favored\(^\text{40}\). In higher order perturbation theory\(^\text{41,42}\), \( U^3 \) vertex corrections have been shown to enhance the p-wave over the \( d_{xy} \) state. Here, we are interested in the behavior of the model as \( U/t \) approaches zero while keeping the density of electrons fixed, and we have studied the eigenvalue problems given in Eqs. \( \text{7} \) and \( \text{9} \) must be studied numerically. A finite number of points on the Fermi surface form the basis for the matrices \( g^*t \). These in turn are diagonalized and the pairing eigenvalues are determined for each electron concentration. For each of the systems considered below, we have found that a discretization of the Fermi surface with 500 – 800 points is more than sufficient to produce accurate results. Both spin and the lattice point group symmetries are used to classify the resulting eigenstates. For a tetragonal crystal (i.e. the point group \( D_{4h} \)) without spin-orbit coupling, there are 4 non-degenerate spin singlet states, and a two-fold degenerate triplet state:

\[
\begin{align*}
A_{1g} & : \psi \sim 1, \text{ or } (x^2 + y^2) \\
A_{2g} & : \psi \sim (x^2 - y^2)xy \\
B_{1g} & : \psi \sim (x^2 - y^2) \\
B_{2g} & : \psi \sim xy \\
E_u & : \psi \sim \{x, y\}
\end{align*}
\] (17)

where the left hand side labels the irreducible representation and the right hand side lists the basis functions (with the association \( x \rightarrow \sin k_x, x^2 \rightarrow \cos k_x, \) etc).

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**FIG. 1:** Density of states as a function of electron concentration. Figure 1 shows \( \rho \) as a function of electron concentration \( n \) for a tight-binding model on a square lattice. Figure 2 shows the pairing strengths for the 2D Hubbard model on a square lattice with \( t' = 0 \) as a function of \( n \). (Particle-hole symmetry assures that the phase diagram is invariant under \( n \rightarrow 2 - n \).) We see clearly that near half-filling (\( n = 1 \)), the dominant form of superconductivity has \( d_{x^2-y^2} \) symmetry. At about \( n = 0.6 \), we see that the favored configuration changes to \( d_{xy} \) pairing. Thus, there are two distinct superconducting ground states that occur on a square lattice with a near-neighbor hopping as a function of concentration for the particle-hole invariant system at \( t' = 0 \): there is a \( d_{x^2-y^2} \) ground state for \( 1 > n > 0.6 \), and a \( d_{xy} \) ground state for \( n < 0.6 \). The results shown in Fig. 2 are in qualitative agreement with the spin fluctuation exchange studies of the Hubbard model of Scalapino \textit{et al.}\(^\text{24}\). Figure 3 shows the gap function for the d-wave state at \( n = 1.1 \). Notice that

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**FIG. 2:** Pairing strengths for the 2D Hubbard model at \( t' = 0 \) as a function of electron concentration.
it has a shape that differs substantially from the simple $(\cos k_x - \cos k_y)$ form.

For $t' \neq 0$, on the square lattice, the particle-hole symmetry is destroyed and the van Hove singularities occur away from half-filling. Figure 4 shows the pairing strengths on the square lattice at $t' = -0.3$ as a function of doping. Again, the dominant configuration which occurs near the half-filled system is $d_{x^2-y^2}$ pairing. The van Hove singularity occurs in this system at $n = n_{vH} \approx 0.72$. In a narrow window of densities near $n_{vH}$, the d-wave order is suppressed and the p-wave order dominates. Upon further decreasing the electron concentration away from the van Hove singularity, the $d_{x^2-y^2}$ ground state again gives way to a p-wave superconducting state at $n \approx 0.5$. Qualitatively similar results have been found for the square lattice $t,t'$ Hubbard model using functional renormalization group analysis for fillings close to the van Hove singularities.

It is worth examining the singular behavior near $n_{vH}$ (i.e. near the point $\mu = 4t'$) in more detail. For the square lattice with $t' \neq 0$, the susceptibility at $\vec{q} = (0,0)$ diverges logarithmically as $\mu \to 4t'$:

$$\chi(0) \sim \frac{1}{2\pi^2 t} \ln \left| \frac{t}{\mu - 4t'} \right|$$  \hspace{1cm} (18)

For a large momentum transfer, $\vec{Q} = (\pi, \pi)$, the susceptibility varies as

$$\chi(\vec{Q}) \sim \frac{1}{2\pi^2 t} \ln \left| \frac{t}{\mu - 4t'} \right| \ln \left| \frac{t}{2|t'|} \right|$$  \hspace{1cm} (19)

for $|\mu - 4t'| << t'$. Thus, a finite $t'$ acts to suppress the nesting of the Fermi surface (for the perfectly nested Fermi surface at $t' = 0$, $\chi(\vec{Q})$ diverges much more strongly as $\ln^2 (t'/\mu)$). Taking the dominant scattering processes at momenta $\vec{q} = (0,0)$ and $\vec{Q} = (\pi, \pi)$ into account, one finds that the effective d-wave pairing strength near the van Hove singularity is roughly

$$V_{eff} \approx \frac{U^2}{2\pi^2 t} \ln \left| \frac{t}{\mu - 4t'} \right| \ln \left| \frac{t}{2t'} \right| - V_0$$  \hspace{1cm} (20)

where $V_0$ is a subdominant contribution which arises from the intermediate momentum transfers on the Fermi sur-
FIG. 6: (upper left): Energy contours of the triangular lattice nearest-neighbor tight-binding model. The dashed black line marks the zone boundary. Blue contours correspond to the band bottom and red contours occur near the top of the band. (upper right) Density of states on the triangular lattice. When the Fermi level is at the van Hove point, the volume of the Fermi surface is 3/4 of the zone. (lower left) Energy contours of the honeycomb lattice nearest-neighbor tight-binding model. Due to the particle-hole symmetry, only the $e > 0$ contours are shown. (lower right) Density of states of the honeycomb model. The van Hove singularities occur at a filling of 3/8 and 5/8.

FIG. 7: The two distinct f-wave irreducible representations on the triangular and honeycomb lattices. (left) B_{1u}: $\psi \sim k_y (k_x^2 - 3k_y^2)$. (right) B_{2u}: $\psi \sim k_x (k_x^2 - 3k_y^2)$. The B_{1u} gap function is the dominant gap when there are two disconnected Fermi pockets in the system, each centered around the zone corners. The B_{1u} gap function has the opposite sign on each pocket.

face. For $t' << t$, the d-wave pairing strength is enhanced as $|\mu - 4t'| \to 0$. However, for $t/2t' > e$ the d-wave pairing strength decreases as the van Hove singularity is approached. This behavior is illustrated in Fig. 6. It reflects the fact that the d-wave pairing is driven by an interaction which is stronger at large momentum transfer. An interaction which is greater at small momentum transfer suppresses the d-wave pairing.

Next we consider the nearest-neighbor tight-binding model on the 2D triangular and honeycomb lattices. Figure 6 shows the basic electronic structure on these lattices. Both lattice systems have the hexagonal point group (D$_{6h}$) symmetry and therefore the same irreducible representations characterize their gap functions. In the singlet (even parity) channel, the following are the irreducible representations:

Singlet channel
\[
A_{1g} : \quad \psi \sim k_x^2 + k_y^2 \\
E_{2g} : \quad \psi \sim \{(k_x^2 - k_y^2), (2k_x k_y)\} \\
A_{2g} : \quad \psi \sim k_x k_y (k_x^2 - 3k_y^2)(k_y^2 - 3k_x^2)
\]

(21)

Note that the d-wave function is a two-dimensional representation (this is in fact the only two dimensional representation in the singlet channel for this system, so long as we restrict our superconductivity to be only in the basal plane). For the triplet channel, we have

Triplet channel
\[
E_{1u} : \quad \psi \sim \{k_x, k_y\} \\
B_{1u} : \quad \psi \sim k_y (k_y^2 - 3k_x^2) \\
B_{2u} : \quad \psi \sim k_x (k_x^2 - 3k_y^2)
\]

(22)

The p-wave gaps form a two-dimensional irreducible representation. There are two distinct one-dimensional f-wave gap functions that belong to the B_{1u}, B_{2u} representations and are shown in Fig. 7.

Figs 8 and 9 show the pairing strengths for the triangular and honeycomb lattice respectively. For both systems it is seen that the dominant pairing instabilities are either the two-component d-wave representation, or the non-degenerate f-wave representation.
For the triangular lattice at modest electron concentration, or the honeycomb lattice far away from half-filling, the Fermi surface is hexagonal in shape and is simply connected. In this regime, the dominant pairing configuration is d-wave pairing that consists of a $d_{x^2-y^2}$ component degenerate with a $d_{xy}$ component. In our analysis, the magnitude of each of the two d-wave gap functions and their relative phase cannot be determined, since they are determined by non-linear effects captured, for instance in the Landau Ginzburg theory. However, it is reasonable to expect that in order to gain condensation energy, the system will spontaneously break time-reversal symmetry and form a $d + id$ superconductor.

For the triangular lattice near the top of the band, and for the honeycomb lattice close to half-filling, the Fermi surfaces form disjoint pockets. In this concentration regime, we have found for both systems that the triplet f-wave gap function is the ground state. The f-wave state which is favored has its lines of nodes along the lines connecting the zone center to the midpoints of the zone edges (see Fig. 7). However, since these lines of nodes never cross the Fermi surfaces which are centered on the corners of the Brillouin zone, the f-wave pairing produces a fully gapped state on the Fermi surface. The gap changes sign between the two distinct Fermi surfaces. It has been argued in the past based on the spin-fluctuation exchange mechanism that the strong magnetic excitations associated with such disjoint, reasonably well-nested Fermi pockets results in an effective pairing interaction which is repulsive between the two pockets. Consequently, a solution which encodes a sign change of the gap among the two Fermi surfaces will naturally be favored. The transition between the f-wave and d-wave pairing states occurs as the electron density is varied across the van Hove filling. We note, finally, that the f-wave solution has a substantially higher transition temperature than the other superconducting gap functions found in these lattice systems.

C. Lattice systems: d=3

Next we consider 3 dimensional lattice systems. In particular, we shall consider the superconducting instabilities of the Hubbard model on the simple cubic (SC), body-centered cubic (BCC), face-centered cubic (FCC) and diamond lattices. We shall restrict our analysis to nearest neighbor tight binding dispersion in each of these cases. All of these lattices have the octahedral ($O_h$) point group symmetry which permits the following classification for the gap functions:

\begin{align}
A_{1g} : \psi \sim 1, \text{ or } (x^2 + y^2 + z^2) \\
A_{2g} : \psi \sim (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) \\
E_g : \psi \sim \{(2z^2 - x^2 - y^2), \sqrt{3}(x^2 - y^2)\} \\
T_{1g} : \psi \sim \{xy(x^2 - y^2), yz(y^2 - z^2), zx(z^2 - x^2)\} \\
T_{2g} : \psi \sim \{xy, yz, zx\}
\end{align}

(23)

for the singlet gap functions and

\begin{align}
A_{1u} : \psi \sim xyz \\
T_{1u} : \psi \sim \{x, y, z\} \\
T_{2u} : \psi \sim \{x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)\}
\end{align}

(24)

for the triplet states. Figure 10 shows the density of
FIG. 11: Pairing strengths of the dominant pairing configurations on the simple cubic lattice. For electron concentrations that are far from half-filling, the 3-fold degenerate p-wave \((T_{1u})\) state occurs whereas closer to half-filling the doubly-degenerate d-wave state \((E_g)\) is found.

FIG. 12: Pairing strengths for the dominant pairing configurations on the BCC lattice. For electron concentrations that are far from half-filling, the 3-fold degenerate p-wave \((T_{1u})\) state has the greatest strength. For intermediate concentrations, the \(E_g\) and \(T_{2g}\) d-wave solutions are found to have the strongest strengths.

states as a function of electron concentration on each of these lattice systems. With the exception of the BCC and FCC lattices, the density of states remains finite across van Hove singularities in 3 dimensional systems.

Figures 11 and 12 display the phase diagram on the SC and BCC lattice respectively as a function of electron concentration. At low concentrations, the p-wave solution has the highest \(T_c\) in both lattices whereas near half-filling, a d-wave solution has the higher transition temperature. On the SC lattice the \(E_g\) d-wave configuration is found near half filling. This state is the 3 dimensional analog of the \(d_{x^2-y^2}\) gap found on the square lattice near half-filling. Due to the underlying symmetry of the cubic lattice, the \(d_{x^2-y^2}\) gap must be degenerate with the \(d_{z^2-r^2}\) gap function. This degeneracy in turn implies that below \(T_c\), the system on the cubic lattice will spontaneously break time-reversal symmetry and form a \(d+i\)\(d\) gap function. On the BCC lattice (Fig. 12), the dominant pairing configuration near half-filling is the triply degenerate d-wave gap function \((T_{2g})\). At intermediate concentrations, the \(E_g\) gap functions have the largest \(T_c\) and again at low concentrations, the p-wave solution is favored.

IV. RG STRATEGY

At any given temperature, the thermodynamic properties of the model can be computed perturbatively in powers of \(U/t\) so long as \(U/t\) is small compared to a characteristic \(T\) dependent magnitude. (Indeed, we generally expect that perturbation theory is convergent at
finite temperature, with a finite radius of convergence. Dynamical properties of the system often depend nonanalytically on the strength of the interactions, even at elevated temperatures, but as long as we stick to thermodynamic quantities, this issue should not arise. Alternatively, even at elevated temperatures, but as long as we stick to thermodynamic quantities, this issue should not arise.) Alternatively, even at $T = 0$, if we introduce an artificial low energy cutoff, $\Omega_0$, in the spectrum, low order perturbation theory is reliable so long as $\Omega > \Omega_{PT}$, where $\Omega_{PT}$ can be obtained by looking at the most divergent terms in each order of perturbation theory, the familiar particle-particle ladders:

$$\rho[U] \log[W/\Omega_{PT}] = 1;$$

$$\Omega_{PT} = W \exp[-1/(\rho(E_F)|U|]].$$  \hspace{1cm} (25)$$

$\Omega_{PT}$ is a physical energy scale in the problem - it is the highest energy at which the bare interactions begin to be significantly renormalized by many-body effects.

With this in mind, we formulate the problem in terms of a Grassman path integral, which we express in terms of the normal modes of the quadratic piece of the action defined by $H_0$. As a first step, we integrate out all the modes with energies greater than a cutoff, $\Omega_0$, chosen so that

$$U^2/t \gg \Omega_0 \gg \Omega_{PT}. $$  \hspace{1cm} (26)$$

Because $\Omega_0 \gg \Omega_{PT}$, the interactions in the resulting effective action can be computed using straightforward perturbation theory. Moreover, we are guaranteed that in dimensionless units, all the effective interactions are still weak. Because $W \gg U^2/t$, the resulting effective action involves only modes within a parametrically narrow window, of width $\Omega_0$, about the Fermi surface. In particular, this effective action is of precisely the form assumed as the starting point for the perturbative RG analysis of the Fermi liquid. Specifically, the dispersion can be linearized about the Fermi surface, the effects of small irrelevant terms can be neglected, and the beta function for the marginaly relevant interactions can be computed to one loop order.

The second step is to compute the RG flows starting from the initial data obtained in the first perturbative step. These flows describe how the effective couplings change as we continue the process of integrating out high energy modes by reducing the cutoff below $\Omega_0$. These equations cease to be accurately governed by the perturbative beta function when one or more dimensionless interaction grows to be of order 1. However, the value of the cutoff, $\Omega^*$, at this point defines (up to a multiplicative constant of order 1), a characteristic energy scale in the problem. Assuming that a one parameter scaling theory describes the low energy physics, then all emergent energies in the problem, including $T_\phi$, the root-mean-squared gap magnitude, $\Delta_0$, etc., are all simply proportional to $\Omega^*$. (Without knowing more explicitly the crossover behavior from the Fermi liquid to the superconducting fixed point, it is not possible to obtain a precise value for the constants of order 1, and hence $\alpha_0$ in Eq. \[2\] cannot be computed by the present methods.)

Note that $\Omega_0$ in this treatment is not a physical energy scale, but rather a calculational convenience. It is important that the results should be independent of the value of $\Omega_0$. We will see that by choosing $\Omega_0 \ll U^2/t$, we make simple the manipulations that insure that our results are independent of $\Omega_0$, at least to the desired order in powers of $U/t$. This highlights an important difference between the present analysis and a conceptually similar treatment of the small $U$ problem considered by Zanchi and Schulz, in which a two step RG analysis of the Hubbard model was undertaken, but in which the intermediate scale (which they call $\ell_x$) is a physical $U$ independent measure of the proximity to a van-Hove point. (We return to this comparison in Sec. VII below.)

V. FIRST STAGE RENORMALIZATION: PERTURBATIVE RESULTS

The first step is to integrate out the states with energies down to $\Omega_0$, and to compute the resulting effective interactions perturbatively in powers of $U$. The important terms, which will serve as inputs for the second stage renormalization, are the electron self energy, $\Sigma(\tilde{k})$, and the two particle vertex, $\Gamma_{\sigma,\sigma'}(\tilde{k}, \tilde{q})$, in the particle-particle channel i.e. the scattering amplitude of a pair of particles with spin polarization $\sigma$ and $\sigma'$ and momenta $\tilde{k}$ and $-\tilde{k}$ into a pair of particles with the same spin polarization and momenta $\tilde{q}$ and $-\tilde{q}$. The two particle vertex can be decomposed into the singlet and triplet channels, $\Gamma_s$ and
FIG. 15: Two particle irreducible diagrams to third order in $U$ which contribute to the effective interaction in the particle-particle channel. The incoming set of electrons at the left of each diagram have momentum $\vec{k}, -\vec{k}$ and are scattered by the interaction to states with momenta $\vec{q}, -\vec{q}$. The dashed line denotes the bare Hubbard interaction, and the solid lines correspond to the bare electron propagators. The diagrams are constrained by the requirement that the spin must be flipped across the dashed line.

FIG. 16: Diagrams to $\mathcal{O}(U^4)$ which are used in determining the effective interaction in the particle-particle channel. The upper set of diagrams (L_a–L_h) show the contributions in the spin singlet channel, whereas the lower set (L_m, L_n) show the processes that contribute in the triplet channel.

The electron self-energy and two-particle vertices are

$$\Sigma(\vec{k}; \Omega_0) = U^2 \Sigma_2(\vec{k}; \Omega_0) + \mathcal{O}(U^3)$$

(28)

$$\Sigma^{(2)}(\vec{k}; \Omega_0) = \int_{\Omega_0} \frac{d\vec{q}\, d\omega}{(2\pi)^{d+1}} G(\vec{k} + \vec{q}, \omega) G(\vec{q}, \omega)$$

(29)

$$\equiv \chi(\vec{k}; \Omega_0)$$

The higher order vertex functions can likewise be evaluated by keeping a non-zero value of $\Omega_0$ where ever there is a logarithmic divergence in the $\Omega_0 \to 0$ limit, but setting $\Omega_0 = 0$ elsewhere. The expression in the singlet channel is

$$\Gamma_s(\vec{k}, \vec{q}; \Omega_0) = \frac{1}{2} \left[ \Gamma_{\uparrow, \downarrow}(\vec{k}, \vec{q}) + \Gamma_{\downarrow, \uparrow}(\vec{k}, \vec{q}) \right]$$

(27)

$$\Gamma_t(\vec{k}, \vec{q}) = \Gamma_{\uparrow, \uparrow}(\vec{k}, \vec{q})$$

The particle-hole bubble, $\chi$, is regular in the limit $\Omega_0 \to 0$, while the particle-particle bubble has the well known logarithmic divergence associated with the Cooper instability, but is otherwise regular:

$$\chi(\vec{k}; \Omega_0) = \chi(\vec{k}) + \mathcal{O}(\Omega_0)$$

$$P(\Omega_0) = \rho \log[A/\Omega_0] + \mathcal{O}(\Omega_0)$$

(35)

where $A$ is determined by the band-structure over the entire band. Since we will never need to keep terms higher order than $U^4$, and since, by assumption, $\Omega_0 \ll U^2$, the higher order terms in powers of $\Omega_0$ can be neglected henceforth.

The higher order vertex functions can likewise be evaluated by keeping a non-zero value of $\Omega_0$ where ever there is a logarithmic divergence in the $\Omega_0 \to 0$ limit, but setting $\Omega_0 = 0$ elsewhere. The expression in the singlet channel is

$$\Gamma_s^{(3)}(\vec{k}, \vec{q}; \Omega_0) = \rho^2 \log^2[A/\Omega_0] + [\gamma^{(3)}(\vec{k}) + \gamma^{(3)}(\vec{q})] \rho \log[A/\Omega_0] + \Gamma_s^{(3)}(\vec{k}, \vec{q}) + \mathcal{O}(\Omega_0)$$

(36)
where if we define \( \hat{p} \) to designate a vector on the (unperturbed) Fermi surface and \( S_F \equiv \int d\hat{p}_F \) equal to the “area” of the Fermi surface

\[
\gamma^{(3)}(\hat{k}) \equiv \int \frac{d\hat{p}_F}{S_F} \left( \frac{\bar{v}_F}{v_F(\hat{p})} \right) \chi(\hat{k} + \hat{p})
\]  
(37)

with \( v_F(\hat{p}) \) the magnitude of the Fermi velocity at position \( \hat{p} \) on the Fermi surface and the norm of the Fermi velocity defined according to

\[
\frac{1}{\bar{v}_F} \equiv \int \frac{d\hat{p}_F}{S_F} \left( \frac{1}{v_F(\hat{p})} \right).
\]  
(38)

The first term of Eq. 38 is obtained from diagram (L_d) in Fig. 16 whereas the terms proportional to \( \log[A/\Omega_0] \) are derived from diagrams (L_e, L_f) in Fig. 16 with the thick solid line treated to second order in \( U \). Finally, \( \tilde{\Gamma}^{(3)} \) contains all the non-singular contributions in the limit \( \Omega_0 \to 0 \), which are derived from the third-order two-particle-irreducible (2PI) diagrams (3a-3f in Fig. 15). They can be expressed as double momentum integrals over suitable products of quartets of \( G \)’s, as shown in the appendix, but in the interest of clarity, we do not display them here. In the triplet channel, the third-order correction to the vertex is non-singular and is obtained from diagram (3g) of Fig. 15.

\[
\Gamma^{(3)}(\hat{k}, \hat{q}; \Omega_0) = \tilde{\Gamma}^{(3)}(\hat{k}, \hat{q}) + \mathcal{O}(\Omega_0)
\]  
(39)

An explicit expression for this quantity is also derived in the appendix.

Similarly, we can obtain an expression for \( \Gamma^{(4)} \):

\[
\Gamma^{(4)}(\hat{k}, \hat{q}; \Omega_0) = \left\{ \rho^2 \log^2[A/\Omega_0] + \left[ \gamma_1^{(3)}(\hat{k}) + \gamma_1^{(3)}(\hat{q}) \right] \rho^2 \log^2[A/\Omega_0] \right. \\
+ \left[ \gamma_2^{(3)}(\hat{k}) + \gamma_2^{(3)}(\hat{q}) \right] \rho \log[A/\Omega_0] + \gamma_s^{(3)}(\hat{k}, \hat{q}) \rho \log[A/\Omega_0] \right\} + \tilde{\Gamma}^{(4)}(\hat{k}, \hat{q}) + \mathcal{O}(\Omega_0)
\]  
(40)

VI. SECOND STAGE RESULTS: RG ANALYSIS

The second stage of renormalization is carried out following the standard Fermi liquid renormalization group (RG) procedure of Shankar\(^9\) and Polchinski\(^22\). Notice that the effective action generated after the first stage of renormalization is precisely of the form assumed as the starting point of this renormalization procedure: the effective interactions are all small (so a perturbative RG approach is justified) and the remaining states lie in a narrow strip of width \( \Omega_0 \) about the Fermi surface so that the spectrum can be linearized without loss of accuracy. Various interactions (Fermi liquid parameters) are marginal at the non-interacting fixed point. These do not significantly affect our principle results. The only couplings that renormalize are those in the Cooper channel. These are governed by the one-loop RG equations which we write in matrix form as

\[
\frac{dq}{d\ell} = -g \ast g
\]  
(46)

where \( \ell \equiv \log[\Omega_0/\Omega] \),

\[
(g \ast h)_{\hat{k}, \hat{q}} \equiv \int \frac{d\hat{p}_F}{S_F} g_{\hat{k}, \hat{p}} h_{\hat{p}, \hat{q}},
\]  
(47)

where \( g \) designates a dimensionless matrix

\[
g_{\hat{k}, \hat{q}} \equiv \rho \sqrt{\frac{\bar{v}_F}{v_F(\hat{k})}} \Gamma(\hat{k}, \hat{q}) \sqrt{\frac{\bar{v}_F}{v_F(\hat{q})}}.
\]  
(48)

Here, we have left implicit the dependence of both \( g \) and \( \Gamma \) on the spin indices.
of the first stage of renormalization. Because the value of the interaction matrix, \( g \), is needed to remove the dependence on the arbitrarily chosen initial cutoff \( \Omega_0 \).

In integrating these equations, we start with an initial value of the interaction matrix, \( \phi^0 \), which is the output of the first stage of renormalization. Because \( g \) is a real symmetric matrix, it is also Hermitian, and so can be diagonalized:

\[
g_{\vec{k},\vec{q}}^0 = \sum_n \lambda_n^0 \psi_n^*(\vec{k}) \psi_n(\vec{q})
\]

where the eigenvalues, \( \lambda_n \), are real and the eigenfunctions form an orthonormal basis,

\[
\int \frac{d\vec{k}}{S_F} \psi_n^*(\vec{k}) \psi_m(\vec{k}) = \delta_{n,m}.
\]

As a result, each eigenvalue renormalizes independently:

\[
d\lambda_n = -\lambda_n^2; \quad \lambda_n(\Omega) = \frac{\lambda_n^0}{1 + \lambda_n^0 \log(\Omega_0/\Omega)}.
\]

To determine the physically important scale, \( \Omega^* \), at which the RG treatment breaks down, we must first identify the smallest eigenvalue of \( g^0 \), \( \lambda^0 \), for which \( \lambda^0_n \leq \lambda^0 \) for all \( n > 0 \). Assuming that \( \lambda^0_n \) is negative, then

\[
\Omega^* = \Omega_0 \exp[-1/|\lambda^0_0|]
\]

Asymptotically, both \( T_c \) and the zero temperature gap scale, \( \Delta_0 \), are then equal to a (unknown) number of order 1 times \( \Omega^* \). Under generic circumstances, we expect \( \lambda_0 \) to be non-degenerate, i.e. \( \lambda_0 < \lambda_1 \). The exception to this is the vicinity of a zero temperature phase transition between two superconducting states with different symmetries, where two eigenvalues cross. The properties of the infinitesimal neighborhood of such critical points will not be investigated further in this paper.

There is an apparent problem with Eq. (52), which is that it has an explicit dependence on \( \Omega_0 \). Since \( \Omega_0 \) was introduced as an unphysical calculational device, this dependence must be spurious. Fortunately, there is also an implicit dependence of \( \lambda^0 \) on \( \Omega_0 \), which just cancels this explicit dependence.

To see this most simply, first consider the case of the negative \( U \) Hubbard model. In this case, the lowest eigenvalue is clearly in the spin singlet channel, and it can be computed perturbatively in powers of \( U \) as

\[
\psi_0(\vec{k}) = \sqrt{\frac{v_F(\vec{k})}{v_F}} [1 + O(U)]
\]

\[
\lambda^0_i = \rho U + \rho U^2 [P(\Omega_0) + \rho'] + O(U^3)
\]

where

\[
\rho' = \int \frac{d\vec{k} d\vec{q}}{S_F^2} \chi(\vec{k} + \vec{q}).
\]

The logarithmic dependence of \( \rho \) on \( \Omega_0 \) insures that when the expression in Eq. (53) is inserted into Eq. (52) the result is independent of \( \Omega_0 \), at least to the stated order in powers of \( U \):

\[
T_c \sim \Omega^* = A \exp[-(1/|\rho U|) e^{-\rho'/\rho}] [1 + O(U)].
\]

The same analysis can be carried through for the more complicated case of the repulsive \( U \) Hubbard model. In this case, however, since the leading contribution to \( \lambda^0 \) is order \( U^2 \), the term responsible for canceling the \( \Omega_0 \) dependence of the prefactor is a logarithmically divergent fourth-order order contribution to the ‘dressed’ vertex. The dressed vertex is represented by the thick solid line in diagram (L\(_s\)) or (L\(_m\)) of Fig. 16. We shall consider the singlet and triplet channels separately in what follows.

In order to see that \( \Omega^* \) in the singlet channel does not depend on \( \Omega_0 \), it is helpful to discretize the points on the Fermi surface, so that the matrix, \( \Gamma_s \) in Eq. (52) and hence the matrix, \( \Gamma_s \) in Eq. (53) as well, is a \( N \times N \) matrix, where \( N \) is the number of k-points. (The continuum limit can easily be taken at the end of the calculation.) This \( N \times N \) matrix can then be partitioned into a \( 1 \times 1 \) block \( g_{s,0} \) which affects only the trivial s-wave states, and an \( (N - 1) \times (N - 1) \) block \( g_{s,1} \) containing the non-trivial superconducting states:

\[
g_s(\vec{k}_s, \vec{q}_s) = \left( \begin{array}{cc} g_{s,0} & T_s \\ T_s^* & g_{s,1} \end{array} \right)
\]

Here, \( T_s \) is a \( 1 \times (N - 1) \) matrix which connects the trivial s-wave subspace with the orthogonal space of sign-changing pair-fields.

The lowest order contributions to \( g_{s,0} \propto \Gamma_{s,0} \) comes from diagrams (L\(_a\),L\(_b\)) in Fig. 16 so \( \Gamma_{s,0} \sim O(U) \). On the otherhand, \( g_{s,1} \propto \Gamma_{s,1} \sim O(U^2) \). The diagrams which contribute to \( T_s(\vec{k}_s, \vec{q}_s) \) are shown in (L\(_c\)) and (L\(_d\)) in Fig. 16. The interaction vertex in these diagrams has the form of a product of two terms, one of which consists of a bare (and therefore momentum-independent) vertex operating on the incoming momenta (\( \vec{k}_s \)), and the other of which is a dressed vertex connected to the outgoing momenta (\( \vec{q}_s \)). Therefore, these terms, when viewed as a matrix, operate on a non-trivial singlet pairing configuration and produce the trivial s-wave solution. From Figs. 15 and 16 it is easy to see that \( T_s \sim O(U^3) \). In a similar
fashion, diagrams \((L_f)\) and \((L_s)\) contribute to \(T_s\) which is just the hermitian conjugate of \(T_s\). In diagonalizing \(g_t\), the off-diagonal matrix elements, \(T_s\), can be treated perturbatively. Indeed, it is clear that the leading effect of these terms on the eigenvalue problem is \(O(U^5)\), and so to the order we are working, they can be set to zero.

Within the non-trivial \((N-1)\)-dimensional subspace, the lowest-order term in perturbation theory which contribute to \(\Gamma_{s,1}\) is represented by the dressed vertex in diagram \((L_c)\) of Fig. 16. To lowest order in \(U/t\), this is the diagram in \((2a)\), which is \(U^2\chi(\mathbf{k}_i+\mathbf{k}_j;\Omega_0)\), and when all of diagrams \((2a)-(3g)\) are taken into account, the non-singular terms of \(O(U^3)\) which produce \(\Gamma_s(\mathbf{k}_i,\mathbf{k}_j)\) in Eq. 58 will also contribute to the vertex. We define the quantity \(\Gamma_{s,1}\) to be the piece of \(\Gamma_{s,1}\) which is non-singular in the limit \(\Omega_0 \rightarrow 0\)

\[
\Gamma_{s,1}(\mathbf{k}_i,\mathbf{k}_j) = U^2\chi(\mathbf{k}_i+\mathbf{k}_j) + U^3\Gamma_s(\mathbf{k}_i,\mathbf{k}_j) + O(U^4)
\]  

(57)

and correspondingly

\[
\hat{g}_{s,1}(\mathbf{k},\mathbf{q}) = \rho \exp \left[ \frac{\bar{v}_{F}}{v} \Gamma_{s,1}(\mathbf{k},\mathbf{q}) \right] \exp \left[ \frac{\bar{v}_{F}}{v} \hat{g}_{s,1}(\mathbf{k},\mathbf{q}) \right].
\]

(58)

The lowest order term in perturbation theory for \(\Gamma\) which has singular (logarithmic) \(\Omega_0\) dependence is the same diagram \((L_0)\) in Fig. 16 that gives the logarithm in Eq. 59; however, this term is purely a contribution to \(\Gamma_{s,0}\). The lowest order singular term in \(\Gamma_{s,1}\) which has the singular \(\Omega_0\) dependence derives from fourth-order diagram \((L_0)\) in Fig. 16. Thus, when all the diagrams are properly taken into account, we see that

\[
g_{s,1} = \hat{g}_{s,1} + \tilde{g}_{s,1} \log [A/\Omega_0] + O(U^5)
\]

(59)

Thus, if we express the results in terms of \(\lambda\), the eigenvalues of the non-singular part of the interaction, \(\hat{g}_{s,1}\), we find that

\[
T_c = A \exp \left\{ -1/|\lambda_{min}| \right\} \left[ 1 + O(U^2) \right]
\]

(60)

The dependence of \(T_c\) on the initial scale \(\Omega_0\) is eliminated by diagram \((L_0)\) in Fig. 16 which possesses the required logarithmic divergence \(\sim \log [A/\Omega_0]\). After including both diagrams \((L_m)\) and \((L_n)\) in Fig. 16 we find that the interaction vertex in the triplet channel is

\[
g_{t,1} = \hat{g}_{t,1} - \hat{g}_{t,1} \log [A/\Omega_0] + O(U^5)
\]

(62)

Thus, as was found in the singlet channel, the final expression for \(T_c\) is independent of \(\Omega_0\).

The general feature in perturbation theory which acts to eliminate the initial cutoff dependence of \(T_c\) is shown in Fig. 17. In Fig. 17(a), an arbitrary irreducible interaction vertex \(\Gamma\) in the particle-hole channel is shown. For the negative U Hubbard model, \(\Gamma\) would correspond simply to the bare vertex, and for the repulsive cases, \(\Gamma\) is the appropriate irreducible vertex in either the singlet or triplet channel as discussed above, and could be computed to an arbitrary order in perturbation theory. For each such \(\Gamma\), there corresponds a diagram of the form shown in 17(b) which acts to remove the dependence on \(\Omega_0\). In this diagram, the internal legs which separate each vertex \(\Gamma\) produces the desired logarithmic divergence factor \(P(\Omega_0)\). Since this is true for an arbitrary interaction vertex, the fact that the final expression for \(T_c\) is independent of the arbitrarily chosen initial scale \(\Omega_0\) true to all orders in perturbation theory for the interaction vertex (we note in passing that self energy corrections to the internal propagator lines produces \(O(U^2)\) corrections in the Hubbard model and do not affect the general structure discussed here).

VII. RELATION TO PREVIOUS WORK

The present work is, in part, a recasting of old work in a new framework, in terms of a more well controlled asymptotic analysis. Specifically, Kolm and Luttinger observed that even for a repulsive bare interaction, the momentum-dependent structure in the irreducible particle-particle vertex can give rise to a Cooper instability in a suitable channel. For a short range repulsive bare interaction, the resulting pairing interaction is mediated by an \(S = 1\) particle-hole channel which is the leading term of the Berk-Schrieffer spin fluctuation exchange interaction.

An important piece of the physics of correlated materials that is missing in the weak coupling limit is associated with “competing orders,” and the accompanying interplay between interactions in different “channels.” We stress that this is not a failure of the method of solution, but is something that is an intrinsic feature of a weakly perturbed Fermi liquid. In order to address this physics, various calculations have been undertaken using the Functional Renormalization Group (FRG) method which is in many ways similar in structure to the weak-coupling analysis undertaken here. In this approach, a single set of perturbative non-linear
flow equations are derived for the coupling constants defined both near to and far from the Fermi surface, and then the resulting flows are analyzed numerically starting from initial conditions corresponding, for instance, to the Hubbard model with intermediate couplings, \( U \sim 3 - 4t \).

As stressed in Ref.\(^\text{[30]}\), for these values, the effective interactions near the Fermi surface are typically large. On an intuitive level, the advantage of the FRG approach is that it does capture some physics of multiple intertwined scattering processes in a physically compelling manner.

On the other hand, the perturbative methods used are formally valid only in the asymptotic limit \( U/t \to 0 \). While, as far as we know, there has been no published work analyzing the FRG flows for the Hubbard model in this limit, were the FRG calculations carried out in this limit they would approximate the leading order behavior derived in the present paper using somewhat different methods.

Another feature of the weak coupling limit is that the dynamics is set by the bare bandwidth. Thus, the bare susceptibilities that enter the expression for the pairing interaction are those of the unperturbed Fermi gas, in which the only energy scale is the bandwidth. Again, a plausible extension of the present results involves replacing one or more of these susceptibilities with a dressed susceptibility, or even with an experimentally measured susceptibility. For example, near a magnetically ordered phase, such a susceptibility would reflect the enhancement of the magnetic fluctuations for \( \vec{k} \sim \vec{Q}_t \), the antiferromagnetic ordering vector, and still more importantly, the retardation effects implicit in the emergence of a new energy scale, “\( J \)”, associated with magnetic fluctuations. From this perspective, the various approaches to a “spin-fluctuation exchange” mechanism of superconductivity appear as natural extrapolations of the present results to a more strongly coupled regime. However, it is important to stress that such procedures represent uncontrolled approximations. There may well be competing channels or possibly a failure of the basic framework. One would like to have numerical calculations to test the utility of such phenomenological approximations. There are several specific studies we wish to discuss explicitly, as they have produced results which are particularly close to those obtained here:

The two step RG approach of Schulz\(^\text{[15]}\) has already been mentioned. In the case considered there, the role of \( \Omega_0 \) was played by a physical energy scale, defined as the energy scale below which the singular structures due to the proximate van Hove singularity could be ignored, and above which the system might as well be precisely tuned to the van Hove point. In particular, in that case, \( \Omega_0 \) (which they call \( \ell_z \)) takes on a \( U \) independent value which depends, instead, on \( t, t', \) and the chemical potential, \( \mu \). Strictly speaking, unless the parameters are fine-tuned to be parametrically close to the van-Hove point, this physics does not survive the small \( U \) limit, as we have shown; our method includes, already, all effects of arbitrary band-structure both at and away from the Fermi surface.

VIII. DISCUSSION AND FURTHER DIRECTIONS

The results we have obtained are asymptotically exact in the limit \( U/t \to 0 \), so long as the conventional RG treatment of the Fermi liquid is valid. In this limit, \( T_c \) tends rapidly to zero, so the present results cannot be directly associated with a mechanism of “high temperature superconductivity.” Moreover, it is clear that in most materials of interest, the interactions are moderate to strong, so the results cannot be said to have any direct relevance to these materials. However, some aspects of the results seem worth emphasizing which, when extrapolated to stronger coupling (where controlled calculations are not possible), may give insight into mechanisms of high temperature superconductivity in real materials.

Most importantly, we have shown that repulsive interactions combined with lattice induced band-structure effects generically do result in a superconducting ground-state with non-trivial transformation properties with respect to the point group symmetries of the crystal.

1) Where band-structure effects are weak, \( i.e. \) where the Fermi surface is nearly circular or spherical, the dominant superconducting instability is, generically, a two or three-fold degenerate spin-triplet \( p \)-wave. The driving force for this is more or less the same as originally envisaged by Kohn and Luttinger. However, here if we compare \( T_c \) with fixed small \( U/t \) at different band-fillings, the values of \( T_c \) found in this regime are small compared to the (still small) values obtained where the Fermi surface is more structured. Since in all the cases we studied, the \( p \)-wave state is a 2 or 3 fold degenerate representation of the point group, and the \( d \) vector is also arbitrary, there are many different possible forms of ordered state possible in principle. However, general considerations suggest that the generic ground-state will either break time-reversal symmetry, forming a “topological” \( p+ip \) superconductor (which corresponds to the A-phase of Helium-3 and presumably is the state that is observed in \( \text{Sr}_2\text{RuO}_4 \)), or the system will spontaneously break the relative “spin-orbit” symmetry as is the case in the B-phase of Helium-3. The perturbation which lifts the degeneracy between these two possibilities is the spin-orbit coupling of the normal state. Thus, an interesting extension of our work would be to incorporate the effects of spin-orbit coupling into the present analysis.

2) On the square or tetragonal lattice with \( n \) near 1, higher values of \( T_c \) are obtained in a spin singlet \( d \)-wave channel. This is encouragingly similar to what is found in the cuprate high temperature superconductors.

3) On the triangular, hexagonal, cubic, FCC, and BCC lattices, there is a range of electron concentrations for which the dominant superconducting state is a two or three-fold degenerate singlet \( d \)-wave state. General arguments suggest that, in this case, the superconducting
state will be a time-reversal symmetry breaking d+id superconducting state. While we do not know of a currently well characterized material in which such a superconducting state has been observed, the state is analogous to the “anyon superconducting state” originally proposed by Laughlin.

4) On both the triangular and hexagonal lattices, when there is more than one electron or hole pocket, the superconducting ground-state is a non-degenerate spin triplet state, in which there is a full gap everywhere on the Fermi surface, but the gap changes sign in going from one pocket to the other. This situation gives the highest transition temperatures (for fixed small $U/t$) that we have found. Moreover, the sign alternation on different pieces of the Fermi surface (although not the triplet character) is reminiscent of the proposed gap function symmetry in the Fe-pnictides.

Conversely, it is interesting to study what aspects of the physics of real materials are inconsistent with the behavior of the Hubbard model at small $U/t$. Taking the cuprate high temperature superconductors as an example, there are several features worth mentioning: 5) In the cuprates, $T_c$ is not very small in comparison to the microscopic energy scales. Indeed, it reaches values that are, within a factor of two or three, equal to the strong-microscopic energy scales. Indeed, it reaches values that are not very small in comparison to the microscopic energy scales. Moreover, there is a clear breakdown of Fermi liquid theory at temperatures above $T_c$, except possibly in the case of the most overdoped materials. Moreover, there is good evidence of substantial superconducting phase fluctuations persisting well above $T_c$. In weak coupling, by contrast, $T_c$ is an emergent energy which is exponentially smaller than any of the microscopic energies in the problem. (However, $T_c$ does extrapolate to a value of order $t$ in the limit $\rho U \sim 1$, assuming that the extrapolation remains valid even where the justification for the result fails.) 6) In the cuprates, there is a clear breakdown of Fermi liquid theory at temperatures above $T_c$, except possibly in the case of the most overdoped materials. Moreover, there is good evidence of substantial superconducting phase fluctuations persisting well above $T_c$. In weak coupling, by contrast, $T_c$ is an emergent energy which is exponentially smaller than any of the microscopic energies in the problem. (However, $T_c$ does extrapolate to a value of order $t$ in the limit $\rho U \sim 1$, assuming that the extrapolation remains valid even where the justification for the result fails.)

3) Direct measurements of the gap function in the cuprates suggest a gap function with the simple form, $\Delta_c = |\cos(k_x) - \cos(k_y)|$, at least in optimally doped materials. In real-space, this form implies the pair-field extends only to nearest-neighbor sites. In contrast, the weak-coupling gap function is more structured in k-space, as shown in Fig. 5. This reflects the non-local character of the induced pairing interactions and is, we believe, a generic feature of weak coupling.

4) In the cuprates, superconductivity emerges from a doped antiferromagnetic insulator with well defined spin-wave modes similar to those expected for a spin-1/2 Heisenberg antiferromagnet with exchange coupling $J < t$ (and possibly some higher order exchange coupling representing the fact that $U \sim 8t$ is not all that large). Moreover, there is direct evidence from neutron scattering that spin-wave-like collective excitations of the system with the same characteristic energy scale persist into the superconducting phase at energy scales larger than the gap, even when there is no corresponding broken symmetry. In weak coupling, an antiferromagnetic insulating state occurs only when two parameters are exponentially fine tuned, $|t'/t| < \delta$ and $|(n-1) - \delta|$, where $\delta \sim \exp \left[ -\alpha \sqrt{t/U} \right]$. A metallic antiferromagnetic state (as well as various other more exotic ordered phases) could occur with the fine tuning of only a single parameter, $|(n-n_{\text{ch}})| < \delta$, in the vicinity of a van-Hove singularity. However, outside of this exponentially narrow range of concentrations, there is no identifiable local antiferromagnetic order present at weak coupling. Indeed, without some form of exponential fine-tuning, "competing orders" is not a concept that occurs in the weak coupling limit.

There are many interesting directions in which the present work can be generalized.

1) Most straightforwardly, it can apply to the Hubbard model on lattice systems with other band-structures than those considered here.

2) It is clearly also possible to extend the same sort of analysis to situations in which there are more complicated interactions than the Hubbard $U$, such as multi-band models, where there are both intra-band and interband interactions, and even single band models with longer range interactions, such as a nearest-neighbor repulsion, $V$. Technically, these interactions complicate the analysis in the sense that a large number of other diagrams, involving the interaction between electrons of like spin, enter the pertubative analysis. Moreover, there are a variety of ways that the asymptotic analysis can be carried out, which will clearly lead to different physics. For instance, with interactions $U$ and $V$, the analysis can be carried out as $U/t \to 0$ with $U/V$ held fixed, or with $U/V \sim (U/t)^{15}$ where $y$ is an appropriate positive exponent.

3) Perhaps the most interesting extension involves the fine-tuning discussed above, which may permit the study of various strong-coupling phenomena in the weak coupling limit. In particular, the physics of competing orders can be explored by performing an asymptotic analysis in the limit $U/t \to 0$ and $n \to n_{\text{ch}}$ in such a way that $|n-n_{\text{ch}}| \exp \left[ -\alpha \sqrt{t/U} \right] \to \text{constant}$. We expect that this will permit us to reproduce much of the interesting physics obtained in various FRG calculations, but in a more controlled fashion. In order to study the interplay between superconductivity and a Mott insulating (an-
tiferromagnetic) phase, an even more complex analysis must be performed, taking the limit $U/t \to 0$, $n \to 1$, and $t'/t \to 0$ in such a way that $|n - 1| \exp[\alpha \sqrt{1/U}] \to$ constant and $|t'/t| \exp[\alpha \sqrt{1/U}] \to$ another constant.

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Appendix: Perturbation theory in the Cooper channel

The perturbative corrections to the 2-particle interaction vertices in the particle-particle channel are shown in Figs. 14 and 16. In these diagrams two particles with momentum $\vec{k}$ and $-\vec{k}$ are scattered to states with momenta $\vec{q}$ and $-\vec{q}$. In the singlet channel, the incoming and outgoing pair of particles have the opposite spin whereas in the triplet channel, we require them to have the same spin. We first consider the diagrams in the singlet channel.

The first order contribution to the interaction give in diagram (La) of Fig. 16 is simply

$$\Gamma_s(L_a) = 1 \quad (A.1)$$

The first order term is independent of momentum and affects only the trivial s-wave pairing states. Next consider the higher-order terms. Before proceeding, we introduce as a notational convenience the following definition

$$\sum_p = \int_{\Omega_0} dp_0 d^dp \quad (A.2)$$

where $\rho_0$ is a Matsubara frequency. The second-order ladder diagram shown in (La) of Fig. 16 is

$$\Gamma_s(L_b) = \sum_p G_0(p) G_0(-p)$$

$$= \int_{\Omega_0} \frac{d^dp}{(2\pi)^d} \left[ \frac{1 - 2 f(\epsilon_\vec{p})}{2\epsilon_\vec{p}} \right]$$

$$= \rho \log [A/\Omega_0] + \mathcal{O}(\Omega_0) \quad (A.3)$$

where $G_0(p)$ is the non-interacting Matsubara Green function

$$G_0(p) = \frac{1}{ip_0 - \epsilon_\vec{p}} \quad (A.4)$$

Had we been interested in the negative U Hubbard model, $\Gamma_s(L_a)$ and $\Gamma_s(L_b)$ are the most important quantities and give rise to the BCS instability in the trivial s-wave channel. Note that $\Gamma(L_b)$ eliminates the dependence on the initial cutoff $\Omega_0$ as discussed in section VI.

For unconventional superconductivity that derives from repulsive interactions, we will need to consider higher order diagrams. The next set of diagrams which are most important correspond to the set of diagrams in (Le) of Fig. 16. Of these the second-order vertex shown in (2a) corresponds to

$$\Gamma_s(2a) = \sum_p G_0(p) G_0(k + q + p)$$

$$= \chi(\vec{k} + \vec{q}) + \mathcal{O}(\Omega_0) \quad (A.5)$$

Adding the two second-order terms, we find the expression in Eq. (42).

The most divergent third order term is given by diagram (Lc) in Fig. 16 which is simply

$$\Gamma_s(L_d) = \sum_p G_0(p) G_0(-p) \quad (A.6)$$

The next most important contribution comes from the diagrams in (Le) and (Lf) with the thick solid line treated to second order (i.e. using diagram (2a)). These produce

$$\Gamma_s^{(3)}(L_e) = \sum_{pp'} G_0(q + p + p') G_0(p') G_0(p) G_0(-p)$$

$$= \left[ \sum_p \chi(\vec{q} + \vec{p}) \left[ G_0(p) G_0(-p) \right] + \mathcal{O}(\Omega_0) \right]$$

$$+ \delta \Gamma_s(L_e) \quad (A.7)$$

with

$$\delta \Gamma_s(L_e) = \sum_p [\chi(\vec{q} + \vec{p}; i\rho_0) - \chi(\vec{q} + \vec{p}, 0)] G_0(p) G_0(-p)$$

and the integrals in $\delta \Gamma_s(L_e)$ are carried out with $\Omega_0 = 0$, since this quantity is non-singular, and $\chi(\vec{p}; i\rho_0)$ is the frequency-dependent susceptibility. In practice, this quantity will be evaluated numerically. Upon summing over the Matsubara frequencies and making use of the identity

$$\int_{\Omega_0} \frac{d^dp}{(2\pi)^d} \int_{|\epsilon| > \Omega_0} d\rho(\epsilon) \quad (A.9)$$

where $\epsilon$ is the energy relative to the Fermi surface, it follows that

$$\Gamma_s^{(3)}(L_e) = \gamma^{(3)}(\vec{q}) \rho \log [A/\Omega_0] + \delta \Gamma_s(L_e) \quad (A.10)$$

The remaining third order terms are non-singular and are obtained from diagram (Le) again, but this time, keeping only 3rd order contributions to the thick solid line, which
are obtained from diagrams (3a)-(3g). While diagram (3a) is simply
\[ \Gamma_s(3a) = \sum_{pp'} G_0(p)G_0(k + q + p)G_0(p')G_0(k + q') = \chi^2(\vec{k} + \vec{q}) + \mathcal{O}(\Omega_0), \quad (A.11) \]
diagrams (3b,3c) which are
\[ \Gamma_s(3b) = \Gamma_s(3c) = \sum_{pp'} G_0(p)G_0(q+p'-p)G_0(p')G_0(k + q + p') \]
are not expressible in simple closed form due to the internal frequency integration (but it is possible to obtain these quantities to arbitrary precision in a numerical computation). Diagram (3d) contributes
\[ \Gamma_s(3d) = \chi^2(\vec{k} - \vec{q}) \quad (A.13) \]
which is also non-singular, and lastly, the diagram of the form given in (3e) contributes
\[ \Gamma_s(3e) = -\sum_{pp'} G_0(p)G_0(p + k - q)G_0(p')G_0(p' - k - q) \quad (A.14) \]
Next, we consider the fourth-order terms. Again, the most singular contribution comes from the ladder shown in (L_9) of Fig. 16
\[ \Gamma(L_9) = \rho^3 \log^3 [A/\Omega_0] + \mathcal{O}(\Omega_0) \quad (A.15) \]
The next leading terms are obtained again from the diagrams (L_e, L_f), except now, we require that the thick black lines in these diagrams take the non-singular third-order vertex which is given by \( \tilde{\Gamma}_s^{(3)} \) described above:
\[ \Gamma_s^{(4)}(v) = \rho \log [A/\Omega_0] \gamma_s^{(4)}(\vec{q}) + \mathcal{O}(\Omega_0) \quad (A.18) \]
and a similar expression holds for \( \Gamma_s^{(4)}(L_f) \). The final term which is singular at fourth order is given by the diagram (L_k), where the thick solid line is approximated by the second-order contribution in (2a). This contributes
\[ \Gamma_s(L_k) = \sum_p \chi(\vec{k} + p)G_0(p)G_0(-p)\chi(p + \vec{q}) = U^4 \rho \log [A/\Omega_0] \gamma_s^{(4)}(\vec{k}, \vec{q}) + \mathcal{O}(\Omega_0)(A.19) \]
The remainder of terms that occur to fourth order do not make a singular contribution to the vertex. We shall not discuss them here.

Now, we consider the triplet channel which has considerably fewer diagrams. The lowest order contribution comes from the second-order diagram in (2b):
\[ \Gamma_t(2b) = -\chi(\vec{k} - \vec{q}) + \mathcal{O}(\Omega_0) \quad (A.20) \]
The third order correction to the vertex comes from diagram (3g) which is
\[ \Gamma_t(3g) = -\sum_{pp'} G_0(p)G_0(k - q + p)G_0(p')G_0(k + p + p') = -\sum_p G_0(p)G_0(k - q + p)\chi(k + p) + \mathcal{O}(\Omega_0)(A.21) \]
Lastly, the fourth-order term comes from diagram (L_n), treating the thick solid line to \( \mathcal{O}(U^2) \), ie. using diagram (2b). This produces
\[ \Gamma_t(L_n) = \rho \log [A/\Omega_0] \gamma_t^{(4)}(\vec{k}, \vec{q}) + \mathcal{O}(\Omega_0) \quad (A.22) \]
This contribution, although higher order, has an important conceptual importance: it has the correct logarithmic dependence on \( \Omega_0 \) to remove the dependence of the final expression for \( T_c \) on the initial cutoff.

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