STRONG CONVERGENCE RATES FOR COX-INGERSOLL-ROSS PROCESSES – FULL PARAMETER RANGE

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Abstract. We study strong (pathwise) approximation of Cox-Ingersoll-Ross processes. We propose a Milstein-type scheme that is suitably truncated close to zero, where the diffusion coefficient fails to be locally Lipschitz continuous. For this scheme we prove polynomial convergence rates for the full parameter range including the accessible boundary regime. The error criterion is given by the maximal $L_p$-distance of the solution and its approximation on a compact interval. In the particular case of a squared Bessel process of dimension $\delta > 0$ the polynomial convergence rate is given by $\min(1, \delta/2p)$.

1. Introduction

In recent years, strong approximation has been intensively studied for stochastic differential equations (SDEs) of the form

\[ dX_t^x = (a - bX_t^x) \, dt + \sigma \sqrt{X_t^x} \, dW_t, \quad X_0^x = x, \quad t \geq 0, \]

with a one-dimensional Brownian motion $W$, initial value $x \geq 0$, and parameters $a, \sigma > 0, b \in \mathbb{R}$. It is well known that these SDEs have a unique non-negative strong solution. Such SDEs often arise in mathematical finance, e.g., as the volatility process in the Heston model. Moreover, they were proposed as a model for interest rates in the Cox-Ingersoll-Ross (CIR) model. In the particular case of $b = 0$ and $\sigma = 2$ the solution of SDE (1) is a squared Bessel process, which plays an important role in the analysis of Brownian motion.

Strong approximation is of particular interest due to the multi-level Monte Carlo technique, see [12, 13, 17]. In this context, a sufficiently high convergence rate with respect to an $L_2$-norm is crucial. The multi-level Monte Carlo technique is used for the approximation of the expected value of a functional applied to the solution of an SDE. In mathematical finance, such a functional often represents a discounted payoff of some derivative and the price is then given by the corresponding expected value.

Strong convergence of numerical schemes for the SDE (1) has been widely studied in the past twenty years. Various schemes have been proposed and proven to be strongly convergent, see, e.g., [1, 10, 15, 18, 20, 23]. In recent years, the speed of convergence in terms of polynomial convergence rates has been intensively studied, see [2, 3, 6, 7, 11, 16, 21, 26]. In all these articles the approximation error is measured with respect to the $L_p$-norm either at a single time point, at the discretization points, or on a compact interval. Moreover, all these results impose conditions on $p$ (appearing in the $L_p$-norm) and the quantity

\[ \delta = \frac{4a}{\sigma^2} \in [0, \infty[ , \]

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which depends on the two parameters $a, \sigma > 0$ appearing in (1). None of these results yield a polynomial convergence rate for $\delta < 1$, cf. Figure 1. By the Feller test, the solution remains strictly positive, i.e., $P(\forall t > 0 : X_t^x > 0) = 1$, if and only if $\delta \geq 2$. Roughly speaking, the smaller the value of $\delta$, the closer the solution to the boundary point zero, where the diffusion coefficient is not even locally Lipschitz continuous.

![Figure 1](image.png)

**Figure 1.** The colored dashed lines show the convergence rates of the known best upper bounds for the error criterion in (3) for different values of $p$, see [11,21]. The corresponding solid lines show the convergence rates obtained in this paper. The cross shows the convergence rate at $\delta = 1$ for all $p \in [1, \infty]$, see [10].

The main aim of this paper is to construct a numerical scheme for the SDE (1) and to prove its convergence at a polynomial rate for all parameters. Let $T > 0$ and define the approximation scheme $\bar{Y}^{x,N} = (\bar{Y}^{x,N}_t)_{0 \leq t \leq T}$, which uses $N \in \mathbb{N}$ increments of Brownian motion, by

$$
\bar{Y}^{x,N}_t = x \quad \text{and} \quad \bar{Y}^{x,N}_{(n+1)T/N} = \Theta_{\text{Mil}} \left( \bar{Y}^{x,N}_{nT/N}, T/N, W_{(n+1)T/N} - W_{nT/N} \right),
$$

for $n = 0, \ldots, N - 1$, where the one-step function $\Theta_{\text{Mil}} : \mathbb{R}_0^+ \times [0,T] \times \mathbb{R} \to \mathbb{R}_0^+$ is given by

$$
\Theta_{\text{Mil}}(x,t,w) = \left( \max \left( \sqrt{\sigma^2/4 \cdot t}, \sqrt{\max(\sigma^2/4 \cdot t, x)} + \sigma/2 \cdot w \right) \right)^2 + \left( a - \sigma^2/4 - b \cdot x \right) \cdot t \right)^+.
$$

This yields a discrete-time approximation of the SDE (1) on $[0,T]$ based on a grid of mesh size $T/N$. Furthermore, between two grid points we use constant interpolation to get a continuous-time approximation, i.e.,

$$
\bar{Y}^{x,N}_t = \bar{Y}^{x,N}_{nT/N}, \quad t \in [nT/N, (n + 1)T/N],
$$
for \( n = 0, \ldots, N - 1 \). We refer to \( \bar{Y}_{x,N} \) as truncated Milstein scheme. Let us mention that this scheme coincides with the classical Milstein scheme as long as it is “away” from zero. Moreover, it is suitably truncated close to zero.

**Theorem 1** (Main Result). Let \( \delta > 0 \) be according to (2). For every \( 1 \leq p < \infty \) and every \( \varepsilon > 0 \) there exists a constant \( C > 0 \) such that

\[
\sup_{0 \leq t \leq T} \left( \mathbb{E} \left( \left| X_t^N - \bar{Y}_{x,N}^t \right|^p \right) \right)^{1/p} \leq C \cdot (1 + x) \cdot \frac{1}{N^{\min(1, \delta)/(2p) - \varepsilon}}
\]

for all \( N \in \mathbb{N} \) and for all \( x \geq 0 \).

![Figure 2](image-url). Numerical results for the truncated Milstein scheme with \( \sigma = 2, a = \delta = 1/2, b = T = 1, \) and \( x = 1/20 \) are shown for \( p = 1, 2 \). The error given by (3) is estimated based on \( 10^5 \) replications of consecutive differences. The dashed lines show the corresponding convergence rates from Theorem 1. For a numerical study of various Euler- and Milstein-type schemes (with \( 2/\delta < 3 \) and \( p = 1 \)) we refer to [1, Fig. 3].

The results of Theorem 1 are illustrated in Figure 1. Observe that the convergence rate in Theorem 1, given by \( \min(1, \delta)/(2p) \), is monotonically decreasing in \( 1/\delta \) and \( p \). This is the same monotonicity as in previous results. The numerical results with \( \delta = 1/2 \) for the truncated Milstein scheme presented in Figure 2 indicate that the convergence rate in Theorem 1 is sharp for \( p = 1 \). Furthermore, these results suggest that the \( L_p \)-norm affects the convergence rate, as expected from Theorem 1. We think that the convergence rate in Theorem 1 is sharp for \( p = 1 \) in the full parameter range but might be too pessimistic for \( p > 1 \). Note that a Milstein-type scheme, the drift-implicit Euler scheme, converges at rate \( 1/2 \) for all \( p \geq 1 \) in the particular situation of \( \delta = 1 \) and \( b = 0 \), see [16]. Let us stress that a convergence rate of \( 1/2 \) is optimal for various global error criteria in case of SDEs satisfying standard Lipschitz assumptions, see, e.g., [10, 25].
Let us mention that Theorem 1 actually holds for a whole class of approximation schemes. More precisely, a one-step approximation scheme satisfies the error bound (3) if it is $L_1$-Lipschitz continuous, has a Milstein-type local discretization error and is uniformly bounded, see (A1)-(A3) for the precise assumptions. Our analysis is based on the $L_1$-norm since CIR processes are in general not locally $L_p$-Lipschitz continuous in the initial value if $p > 1$, see Proposition 2. The results for $L_1$ are then extended to $p > 1$ by using classical interpolation techniques. Due to this interpolation the convergence rate in Theorem 1 is divided by $p$.

This paper is organized as follows. In Section 3 we recall some basic properties of the solution of SDE (1). In Section 4 we provide a general framework for the analysis of one-step approximation schemes and prove strong convergence rates under suitable assumptions on such a one-step scheme. In Section 5 we study the truncated Milstein scheme and show that it satisfies the assumptions of Section 4.

2. Notation

We use $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. Moreover, $x^+ = \max(0, x)$ denotes the positive part of $x \in \mathbb{R}$. We use $\overset{d}{=} \text{to denote equality in distribution.}$ We do not explicitly indicate if statements are only valid almost surely, e.g., equality of two random variables. For functions $f$ and $g$ taking non-negative values we write $f \preceq g$ if there exists a constant $C > 0$ such that $f \leq C \cdot g$. Moreover, we write $f \simeq g$ if $f \preceq g$ and $g \preceq f$. From the context it will be clear which objects the constant $C$ is allowed to depend on.

3. Preliminaries

In this section we set up the framework and provide some basic facts about the SDE (1) that will be frequently used. To simplify the presentation we only consider the SDE (1) with $\sigma = 2$. Moreover, we only study approximation on the unit interval $[0, 1]$ instead of $[0, T]$. Both simplifications are justified by the following two remarks.

Remark 1 (Reduction to $\sigma = 2$). Consider the particular case of SDE (1) given by

$$d\hat{X}^x_t = (\delta - b\hat{X}^x_t) dt + 2\sqrt{\hat{X}^x_t} dW_t, \quad \hat{X}^x_0 = \hat{x}, \quad t \geq 0,$$

with $\hat{x} = x \cdot 4/\sigma^2$ and $\delta$ given by (2). Then we have

$$X^x_t = \frac{\sigma^2}{4} \cdot \hat{X}^{\hat{x}}_t$$

for all $t \geq 0$.

Remark 2 (Reduction to $T = 1$). Consider the instance of SDE (1) given by

$$d\tilde{X}^x_t = (\tilde{a} - \tilde{b}\tilde{X}^x_t) dt + \tilde{\sigma} \sqrt{\tilde{X}^x_t} d\tilde{W}_t, \quad \tilde{X}^x_0 = x, \quad t \geq 0,$$

with $\tilde{a} = T \cdot a$, $\tilde{b} = T \cdot b$, $\tilde{\sigma} = \sqrt{T} \cdot \sigma$, and Brownian motion $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$ given by

$$\tilde{W}_t = 1/\sqrt{T} \cdot W_{t/T}.$$

Then we have

$$X^x_t = \tilde{X}^{\tilde{x}}_{t/T}$$

for all $t \geq 0$. 

Throughout the rest of the paper \((\Omega, \mathcal{F}, P)\) denotes the underlying probability space that is assumed to be complete and \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) denotes a filtration that satisfies the usual conditions, i.e., \(\mathcal{F}_0\) contains all \(P\)-null-sets and \(\mathcal{F}\) is right-continuous. Moreover, \(W = (W_t)_{t \geq 0}\) denotes a scalar Brownian motion with respect to \(\mathcal{F}\). Finally, the process \(X^x = (X^x_t)_{t \geq 0}\) with initial value \(x \geq 0\) is given by the SDE
\[
dX^x_t = (\delta - bX^x_t) \, dt + 2\sqrt{X^x_t} \, dW_t, \quad X^x_0 = x, \quad t \geq 0,
\]
with fixed parameters \(\delta > 0\) and \(b \in \mathbb{R}\).

**Remark 3.** Let us mention that
\[
E(X^x_t) = x \cdot e^{-bt} + \delta \cdot \left\{ \begin{array}{ll}
(1 - e^{-bt})/b, & \text{if } b \neq 0, \\
t, & \text{if } b = 0,
\end{array} \right.
\]
for \(t \geq 0\), see, e.g., [8].

**Remark 4 (Marginal distribution of CIR processes).** Let \(Z\) be \(\chi^2\)-distributed with \(\delta\) degrees of freedom, i.e., \(Z\) admits a Lebesgue density that is proportional to
\[
z \mapsto \frac{z^{\delta/2 - 1} \cdot \exp(-z/2)}{1_{\mathbb{R}^+}(z)}.
\]
If \(b = 0\), we have
\[
X^0_t \overset{d}{=} t \cdot X^0_1 \overset{d}{=} t \cdot Z
\]
for \(t \geq 0\), according to [27, Prop. XI.1.6] and [27, Cor. XI.1.4]. In the general case \(b \in \mathbb{R}\) we obtain
\[
X^0_t \overset{d}{=} \psi(t) \cdot Z
\]
for \(t \geq 0\) with
\[
\psi(t) = \left\{ \begin{array}{ll}
(1 - e^{-bt})/b, & \text{if } b \neq 0, \\
t, & \text{if } b = 0,
\end{array} \right.
\]
due to [14, Eq. (4)].

**Remark 5.** For every \(1 \leq p < \infty\) there exists a constant \(C > 0\) such that
\[
\left( E \left( \sup_{0 \leq t \leq 1} |X^x_t|^p \right) \right)^{1/p} \leq C \cdot (1 + x)
\]
for \(x > 0\), since the coefficients of SDE (4) satisfy a linear growth condition, see, e.g., [22, Thm. 2.4.4].

**Remark 6 (Monotonicity in the initial value).** The comparison principle for one-dimensional SDEs yields
\[
X^x_t \leq X^y_t
\]
for all \(t \geq 0\) and \(0 \leq x \leq y\), see, e.g., [27, Thm. IX.3.7].

4. **Strong Approximation**

In this section we prove strong convergence rates for suitable one-step approximation schemes for the SDE (4). At first, we introduce the notion of a one-step scheme and identify sufficient conditions for such a scheme to be strongly convergent.

A one-step scheme for the SDE (4) with initial value \(x \geq 0\) is a sequence of approximating processes \(Y^{x,N} = (Y^{x,N}_t)_{t \geq 0}\) for \(N \in \mathbb{N}\) that is defined by a Borel-measurable one-step function
\[
\Theta: \mathbb{R}^+_0 \times [0,1] \times \mathbb{R} \rightarrow \mathbb{R}^+_0
\]
in the following way. The process $Y_{x,N}$ is defined at the grid of mesh size $1/N$ by
\[ Y_{x,N}^0 = x \quad \text{and} \quad Y_{x,N}^{(n+1)/N} = \Theta \left( Y_{x,N}^{n/N}, 1/N, \Delta W_N^n \right) \]
for $n \in \mathbb{N}_0$, where
\[ \Delta W_N^n = W_{(n+1)/N} - W_{n/N}. \]
Moreover, between two grid points we use constant interpolation, i.e.,
\[ Y_{x,N}^t = Y_{x,N}^{n/N}, \quad t \in [n/N, (n + 1)/N[, \]
for $n \in \mathbb{N}_0$. The value of $Y_{x,N}^n$ for $n \in \mathbb{N}$ is thus a function of the previous value $Y_{x,N}^{(n-1)/N}$, the time increment and the Brownian increment. Clearly, Euler and Milstein-type schemes are such one-step schemes. We refer to \cite[Sec. 2.1.4]{10} for one-step schemes in a more general setting.

In case of SDE (3) the classical Euler-Maruyama scheme would be given by
\[ \Theta(x, t, w) = x + (\delta - bx) \cdot t + 2\sqrt{x} \cdot w. \]
However, this is not well-defined since $\Theta(x, t, w)$ might be negative. Hence classical schemes need to be appropriately modified to ensure positivity if a square root term is used.

We consider the following conditions on one-step functions for $p \in [1, \infty[$.

(A1) There exists a constant $K > 0$ such that
\[ (\mathbb{E}(|\Theta(x_1, t, W_t) - \Theta(x_2, t, W_t)|^p))^{1/p} \leq (1 + Kt) \cdot |x_1 - x_2| \]
for all $x_1, x_2 \geq 0$ and $t \in [0, 1]$.

We say that a one-step function $\Theta$ satisfying (A1) is $L_p$-Lipschitz continuous.

We define $\Delta_{\text{loc}} : \mathbb{R}_0^+ \times [0, 1] \to \mathbb{R}_0^+$ by
\[ \Delta_{\text{loc}}(x, t) = \begin{cases} t, & \text{if } x \leq t, \\ t^{3/2} \cdot x^{-1/2}, & \text{if } t \leq x \leq 1, \\ t^{3/2} \cdot x, & \text{if } x \geq 1. \end{cases} \]

(A2) There exists a constant $C > 0$ such that
\[ (\mathbb{E}(|\Theta(x, t, W_t) - X_t^x|^p))^{1/p} \leq C \cdot \Delta_{\text{loc}}(x, t) \]
for all $x \geq 0$ and $t \in [0, 1]$.

In Proposition \cite[3]{10} we will show that the local discretization error of a single Milstein-step is bounded by $\Delta_{\text{loc}}$. Hence we say that a one-step function $\Theta$ satisfying (A2) has an $L_p$-Milstein-type local discretization error for the SDE (1).

Remark 7. Under standard Lipschitz assumptions on the coefficients of an SDE, the Euler-Maruyama method and the Milstein method are $L_p$-Lipschitz continuous for all $p \in [1, \infty[$. Moreover, for all $p \in [1, \infty[$ the Euler-Maruyama method and the Milstein method have a local error of order $t$ and $t^{3/2}$, respectively, see, e.g., \cite[Chap. 1.1]{10}.

4.1. $L_1$-convergence. In this section we prove strong convergence rates with respect to the $L_1$-norm for suitable one-step schemes. The reason why we restrict ourselves to $L_1$ in this section is indicated in Section \cite[123]{10}.

Proposition 1 (Average local error). There exists a constant $C > 0$ such that
\[ \mathbb{E}(\Delta_{\text{loc}}(X_s^x, t)) \leq C \cdot (1 + x) \cdot t \cdot \begin{cases} 1, & \text{if } s \leq t, \\ (t/s)^{\min(1, \delta)/2} \cdot (1 + \ln(s/t) \cdot 1_{\{1\}}(\delta)), & \text{if } t \leq s, \end{cases} \]
for all $t \in [0, 1]$, $s \in [0, 1]$, and $x \geq 0$. 
Proof. Consider the situation of Remark 4 and let $c > 0$. Observe that for $\varepsilon \in [0, c]$ we have
\begin{equation}
\mathbb{P}(Z \leq \varepsilon) \leq \varepsilon^{\delta/2}
\end{equation}
and
\begin{equation}
\mathbb{E}\left(Z^{-1/2} \cdot 1_{\{Z \geq \varepsilon\}}\right) \leq \begin{cases} 1, & \text{if } \delta > 1, \\ 1 + \ln(c/\varepsilon), & \text{if } \delta = 1, \\ \varepsilon^{(\delta-1)/2}, & \text{if } \delta < 1. \end{cases}
\end{equation}
Furthermore, we clearly have
\begin{equation}
\psi(s) \asymp s
\end{equation}
for $s \in [0, 1]$.

For $s \leq t$ the claim follows from (7) with $p = 1$. In the following we consider the case $t \leq s$. Let $U_1, U_2 : \mathbb{R}_+^n \times [0, 1] \to \mathbb{R}_+$ be given by
$$U_1(x, t) = t^{3/2} \cdot x$$
and
$$U_2(x, t) = \begin{cases} t, & \text{if } x \leq t, \\ t^{3/2} \cdot x^{-1/2}, & \text{if } t \leq x, \end{cases}$$
such that
\begin{equation}
\Delta_{\text{loc}}(x, t) \leq U_1(x, t) + U_2(x, t).
\end{equation}
Using (7) with $p = 1$ we obtain
\begin{equation}
\mathbb{E}(U_1(X_s^0, t)) \leq t^{3/2} \cdot (1 + x)
\end{equation}
for $t \in [0, 1]$, $s \in [0, 1]$, and $x \geq 0$. Combining (6), (11), (12), and (13) yields
$$t \cdot \mathbb{P}(X_s^0 \leq t) = t \cdot \mathbb{P}(Z \leq t/\psi(s)) \leq t \cdot (t/s)^{\delta/2}$$
for $0 < t \leq s \leq 1$ and
$$t^{3/2} \cdot \mathbb{E}\left((X_s^0)^{-1/2} \cdot 1_{\{X_s^0 \geq t\}}\right) = t^{3/2} \cdot \mathbb{E}\left(\psi(s)^{-1/2} \cdot 1_{\{Z \geq t/\psi(s)\}}\right) \leq \frac{t^{3/2}}{\sqrt{s}} \cdot \begin{cases} 1, & \text{if } \delta > 1, \\ 1 + \ln(s/t), & \text{if } \delta = 1, \\ (t/s)^{(\delta-1)/2}, & \text{if } \delta < 1, \end{cases}$$
for $0 < t \leq s \leq 1$. Moreover, due to Remark 6 and monotonicity of $U_2(\cdot, t)$ we have
\begin{equation}
\mathbb{E}(U_2(X_s^x, t)) \leq \mathbb{E}(U_2(X_s^0, t))
\end{equation}
\begin{equation}
\leq t \cdot (t/s)^{\delta/2} + t \cdot \begin{cases} (t/s)^{1/2}, & \text{if } \delta > 1, \\ (t/s)^{1/2} \cdot (1 + \ln(s/t)), & \text{if } \delta = 1, \\ (t/s)^{\delta/2}, & \text{if } \delta < 1, \end{cases}
\end{equation}
for $0 < t \leq s \leq 1$ and $x \geq 0$. Combining (14), (15), and (16) completes the proof. \hfill \Box

**Theorem 2** ($L_1$-convergence of one-step schemes). Let $Y_{x,N}^x$ be a one-step scheme given by (8) and (9), and assume that (A7) and (A3) are fulfilled for $p = 1$. Then there exists a constant $C > 0$ such that
\begin{equation}
\sup_{0 \leq t \leq 1} \mathbb{E}\left(|X_t^x - Y_t^{x,N}|\right) \leq C \cdot (1 + x) \cdot \frac{1 + \ln N}{N^{\min(1, \delta)/2}}
\end{equation}
for all $N \in \mathbb{N}$ and for all $x \geq 0$. 
Proof. For notational convenience, we set
\[ X_{n}^{x,N} = X_{n/N}^{x} \quad \text{and} \quad \hat{Y}_{n}^{x,N} = Y_{n/N}^{x,N} \]
for \( n = 0, \ldots, N \). Furthermore, we define
\[ e_{n}^{x,N} = \mathbb{E} \left( \left| X_{n}^{x,N} - \hat{Y}_{n}^{x,N} \right| \right) \]
for \( n = 0, \ldots, N \). Then we have \( e_{0}^{x,N} = 0 \) and
\[
e_{n+1}^{x,N} \leq \mathbb{E} \left( \left| X_{n+1}^{x,N} - \Theta \left( X_{n}^{x,N}, 1/N, \Delta W_{n}^{N} \right) \right| \right)
+ \mathbb{E} \left( \left| \Theta(X_{n}^{x,N}, 1/N, \Delta W_{n}^{N}) - \hat{Y}_{n+1}^{x,N} \right| \right)
= \mathbb{E} \left( \left| X_{1/N}^{x} - \Theta \left( \bar{x}, 1/N, W_{1/N} \right) \right| \right)_{\bar{x} = X_{n}^{x,N}}
+ \mathbb{E} \left( \left| \Theta(\bar{x}, 1/N, W_{1/N}) - \Theta(\tilde{y}, 1/N, W_{1/N}) \right| \right)_{(\bar{x}, \tilde{y}) = (X_{n}^{x,N}, \hat{Y}_{n}^{x,N})}
\]
for \( n = 0, \ldots, N - 1 \). Using \((A2)\) and \((A1)\) with \( p = 1 \) we obtain
\[
e_{n+1}^{x,N} \leq C_{1} \cdot \mathbb{E} \left( \Delta \text{loc}(X_{n}^{x,N}, 1/N) \right) + (1 + K/N) \cdot \mathbb{E} \left( \left| X_{n+1}^{x,N} - \hat{Y}_{n}^{x,N} \right| \right).
\]
Moreover, applying Proposition \([\text{II}]\) yields
\[
e_{n+1}^{x,N} \leq (1 + K/N) \cdot e_{n}^{x,N} + C_{1}C_{2} \cdot (1 + x) \cdot \frac{1}{N} \cdot \left\{ \begin{array}{ll}
1, & \text{if } n = 0, \\
\frac{1 + \ln(N) \cdot 1_{\{1\}}(\delta)}{(n + 1) \min(1, \delta)/2}, & \text{if } n \geq 1,
\end{array} \right.
\]
and hence
\[
e_{n+1}^{x,N} \leq (1 + K/N) \cdot e_{n}^{x,N} + 2C_{1}C_{2} \cdot (1 + x) \cdot \frac{1}{N} \cdot \frac{1 + \ln(N) \cdot 1_{\{1\}}(\delta)}{(n + 1) \min(1, \delta)/2}
\]
for \( n = 0, \ldots, N - 1 \). Recursively, we get
\[
e_{n}^{x,N} \leq 2 C_{1}C_{2} \cdot (1 + x) \cdot \frac{1}{N} \cdot \left( 1 + \ln(N) \cdot 1_{\{1\}}(\delta) \right) \cdot \sum_{k=1}^{n} \frac{(1 + K/N)^{n-k}}{k \min(1, \delta)/2}
\]
and hence
\[
e_{n}^{x,N} \leq 2 C_{1}C_{2} e^{K} \cdot (1 + x) \cdot \left( 1 + \ln(N) \cdot 1_{\{1\}}(\delta) \right) \cdot \frac{1}{N} \sum_{k=1}^{N} k^{- \min(1, \delta)/2}
\]
\[
\leq 4 C_{1}C_{2} e^{K} \cdot (1 + x) \cdot \frac{1 + \ln(N) \cdot 1_{\{1\}}(\delta)}{N \min(1, \delta)/2}
\]
for \( n = 0, \ldots, N \). This yields
\[
\max_{n=0,\ldots,N} \mathbb{E} \left( \left| X_{n}^{x,N} - \hat{Y}_{n}^{x,N} \right| \right) \leq 4 C_{1}C_{2} e^{K} \cdot (1 + x) \cdot \frac{1 + 1_{\{1\}}(\delta) \cdot \ln N}{N \min(1, \delta)/2}
\]
for all \( N \in \mathbb{N} \) and for all \( x \geq 0 \). Since the coefficients of SDE \([\text{II}]\) satisfy a linear growth condition we have
\[
\mathbb{E} \left( |X_{n}^{x} - X_{n}^{\tilde{t}}| \right) \leq (1 + x) \cdot \sqrt{|s - \tilde{t}|}
\]
for all \( x \geq 0 \) and \( s, \tilde{t} \in [0, 1] \), see, e.g., \([22] \text{ Thm. 2.4.3}\). Combining \((17)\) and \((18)\) completes the proof.
Remark 8. Consider the proof of Theorem 2. An analysis of the global error by adding local errors is a classical technique for ordinary differential equations. For such a technique in the context of SDEs under standard Lipschitz assumptions we refer to [24, Chap. 1.1]. In case of SDE (4), it is crucial to control the average local error, see Proposition 1.

4.2. $L_p$-convergence. In this section we extend the result from Theorem 2 to arbitrary $p > 1$ by using interpolation of $L_p$-spaces. For this we need the following additional assumption on a one-step scheme.

(A3) For every $1 \leq q < \infty$ there exists a constant $C > 0$ such that

$$\sup_{0 \leq t \leq 1} \left( E \left( \left| Y_{t}^{x,N} \right|^{q} \right) \right)^{1/q} \leq C \cdot (1 + x)$$

for all $x \geq 0$ and $N \in \mathbb{N}$.

We say that a one-step scheme $Y_{t}^{x,N}$ satisfying (A3) is uniformly bounded.

Remark 9. Under standard linear growth conditions on the coefficients of the SDE, the Euler-Maruyama method and the Milstein method are uniformly bounded, see, e.g., [22, Lem. 2.7.1] for the Euler-Maruyama method and $q = 2$.

Remark 10 (Interpolation of $L_p$-spaces). Let $1 \leq p < \infty$ and $0 < \varepsilon < 1/p$. Set

$$q = 1 + \frac{1 - 1/p}{\varepsilon}.$$  \hfill (19)

Note that $p \leq q < \infty$. An application of Hölder’s inequality yields

$$(E(\left| Z \right|^p))^{1/p} = \left( E \left( |Z|^{(1/p-\varepsilon)p} \right) \right)^{1/p} \leq \left( E \left( |Z|^{q} \right) \right)^{1/p-\varepsilon}$$

for all random variables $Z$.

Corollary 1 ($L_p$-convergence of one-step schemes). Consider the situation of Theorem 2 and assume in addition that (A3) is fulfilled. Furthermore, let $1 \leq p < \infty$ and $\varepsilon > 0$. Then there exists a constant $C > 0$ such that

$$\sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} - Y_{t}^{x,N} \right|^{p} \right) \right)^{1/p} \leq C \cdot (1 + x) \cdot \frac{1}{N^{\min(1,\beta)/(2p-\varepsilon)}}$$

for all $N \in \mathbb{N}$ and for all $x \geq 0$.

Proof. We may assume $\varepsilon < 1/p$. According to Remark 10 we have

$$\sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} - Y_{t}^{x,N} \right|^{p} \right) \right)^{1/p} \leq \sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} - Y_{t}^{x,N} \right| \right)^{1/p-\varepsilon} \right) \cdot \left( \sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} \right|^{q} \right) \right)^{1/q} + \sup_{0 \leq t \leq 1} \left( E \left( \left| Y_{t}^{x,N} \right|^{q} \right) \right)^{1/q} \right)^{q \varepsilon}$$

with $q$ given by (19). Using (7) and (A3) we get

$$\sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} - Y_{t}^{x,N} \right|^{p} \right) \right)^{1/p} \leq \sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} - Y_{t}^{x,N} \right| \right)^{1/p-\varepsilon} \cdot (1 + x)^{q \varepsilon} \right) \cdot \left( \sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} \right|^{q} \right) \right)^{1/q} + \sup_{0 \leq t \leq 1} \left( E \left( \left| Y_{t}^{x,N} \right|^{q} \right) \right)^{1/q} \right)^{q \varepsilon} \cdot \left( \sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} \right|^{q} \right) \right)^{1/q} + \sup_{0 \leq t \leq 1} \left( E \left( \left| Y_{t}^{x,N} \right|^{q} \right) \right)^{1/q} \right)^{q \varepsilon} \cdot \left( \sup_{0 \leq t \leq 1} \left( E \left( \left| X_{t}^{x} \right|^{q} \right) \right)^{1/q} + \sup_{0 \leq t \leq 1} \left( E \left( \left| Y_{t}^{x,N} \right|^{q} \right) \right)^{1/q} \right)^{q \varepsilon}$$

It remains to apply Theorem 2 and to observe that $1/p - \varepsilon + q \varepsilon = 1$. \hfill $\Box$
4.3. Regularity in the initial value. In this section we study continuity properties of the solution of SDE (11) in the initial value. It turns out that the solution is in general not locally Lipschitz continuous in the initial value with respect to the $L_p$-norm if $p > 1$. This is the reason why we have restricted ourselves in Section 4.1 to the case $p = 1$. For results on local Lipschitz continuity in the initial value for more general SDEs we refer to [9]. However, in case of SDE (11) these results are restricted to $\delta \geq 2$.

The following lemma implies that the solution of SDE (11) is Lipschitz continuous in the initial value with respect to the $L_1$-norm on any compact time interval.

**Lemma 1.** We have
\[ E(|X^x_t - X^y_t|) = e^{-bt} \cdot |x - y| \]
for all $x, y \geq 0$ and for all $t \geq 0$.

**Proof.** Let $x \geq y \geq 0$. According to Remark 6 we have
\[ E(|X^x_t - X^y_t|) = E(X^x_t - X^y_t) = E(X^x_t) - E(X^y_t) \]
for all $t \geq 0$. Using (5) completes the proof. □

**Remark 11.** The proof of Lemma 1 is a general technique to obtain $L_1$-Lipschitz continuity in the initial value for a large class of one-dimensional SDEs. The comparison principle reduces this problem to the computation of expected values.

**Example 1** (One-dimensional squared Bessel process). In [16], it was shown that for $\delta = 1$ and $b = 0$ we have
\[ X^x_t = \left(W_t + \sqrt{x} - \min_{0 \leq s \leq t} W_s + \sqrt{x}\right)^2 \]
for $t \geq 0$ and $x \geq 0$. This yields
\[ |X^x_t - X^0_t| = \begin{cases} 0, & \text{if } \inf_{0 \leq s \leq t} W_s + \sqrt{x} \leq 0, \\ (W_t + \sqrt{x})^2 - (W_t - \inf_{0 \leq s \leq t} W_s)^2, & \text{if } \inf_{0 \leq s \leq t} W_s + \sqrt{x} \geq 0. \end{cases} \]
Using this one can show
\[ \left(E\left(|X^x_t - X^0_t|^p\right)\right)^{1/p} \approx x^{(1+1/p)/2} \]
for $x \in [0, 1]$, where $1 \leq p < \infty$ and $t > 0$.

In the rest of this section we assume
\[ 0 < \delta < 2 \quad \text{and} \quad b = 0, \]
i.e., the solution of SDE (11) is a squared Bessel process of dimension $\delta$. For $1 \leq p < \infty$ define the maximal Hölder exponent by
\[ \alpha_{ex}(\delta, p) = \sup \left\{ \alpha \geq 0 : \exists C > 0 \ \forall x \in [0, 1] : \left(E\left(|X^x_t - X^0_t|^p\right)\right)^{1/p} \leq C \cdot x^\alpha \right\}. \]
Note that replacing the time point $t = 1$ in the definition of $\alpha_{ex}$ to an arbitrary time point $t > 0$ does not affect the value of $\alpha_{ex}$, see Remark 1 and Remark 2. From Lemma 1 and Example 1 we already have
\[ \alpha_{ex}(\delta, 1) = 1 \]
and
\[ \alpha_{ex}(1, p) = (1 + 1/p)/2. \]
Proposition 2. We have

\[
\frac{1}{p} \leq \alpha_{\text{ex}}(\delta, p) \leq \frac{1}{p} + \frac{\delta}{2} - \frac{\delta}{(2p)}.
\]

In particular, we have \(\alpha_{\text{ex}}(\delta, p) < 1\) if and only if \(p > 1\).

Proof. If \(p = 1\), then (22) follows from (20). Note that for \(x \in [0, 1]\) we have

\[
P(\forall t \in [0, 1] : X^x_t > 0) \asymp x^{1-\delta/2}. \tag{23}
\]

This follows from [5, p. 75], where the density of the first hitting time of zero is given for Bessel processes. Let \(1 < p < \infty\) and let \(1 < q < \infty\) be the dual of \(p\), i.e., \(1/p + 1/q = 1\). Using Hölder’s inequality, Remark 6, Lemma 1, and (23) we get

\[
\left( E \left( \left| \varphi_{\text{Mil}}(X^x_t, W_t) - X^0_t \right|^p \right) \right)^{1/p} \geq E \left( \left| X^x_t - X^0_t \right| \cdot 1(\forall t \in [0, 1] : X^x_t > 0) \right) \cdot \left( P (\forall t \in [0, 1] : X^x_t > 0) \right)^{-1/q}
\]

\[
\asymp x \cdot x^{-1/q(1-\delta/2)}
\]

for \(x \in [0, 1]\), which shows the upper bound in (22). The lower bound follows by interpolation using Remark 10, Lemma 1 and (7), cf. the proof of Corollary 1. \(\square\)

Remark 12. Let us mention that the upper bound in (22) is sharp for \(p = 1\) and \(\delta = 1\), see (20) and (21).

5. Tamed Milstein Scheme

In this section we introduce a truncated Milstein scheme and prove that this scheme satisfies the assumptions of Theorem 2 and Corollary 1.

Consider \(\varphi_{\text{Mil}} : \mathbb{R}_+ \times [0, 1] \times \mathbb{R} \to \mathbb{R}\) given by

\[
\varphi_{\text{Mil}}(x, t, w) = x + (\delta - bx) \cdot t + 2\sqrt{\delta} \cdot w + (w^2 - t)
\]

\[
= (\sqrt{\delta} + w)^2 + (\delta - 1 - bx) \cdot t.
\]

This function models a single Milstein-step. Moreover, it only preserves positivity if \(b \leq 0\) and \(\delta \geq 1\). Hence it is not a valid one-step scheme in general. However, the analysis of \(\varphi_{\text{Mil}}\) is an important step since we will construct valid one-step schemes that are close to \(\varphi_{\text{Mil}}\).

Proposition 3 (Error of a Milstein-step). For every \(1 \leq p < \infty\) there exists a constant \(C > 0\) such that

\[
\left( E \left( \left| \varphi_{\text{Mil}}(x, t, W_t) - X^x_t \right|^p \right) \right)^{1/p} \leq C \cdot \Delta_{\text{loc}}(x, t)
\]

for all \(x \geq 0\) and \(t \in [0, 1]\), where \(\Delta_{\text{loc}}\) is given by (10).

The proof of Proposition 3 exploits the following simple lemma, which is a refinement of (7).

Lemma 2. For every \(1 \leq p < \infty\) there exists a constant \(C > 0\) such that

\[
\left( E \left( \sup_{0 \leq s \leq t} \left| X^x_s \right|^p \right) \right)^{1/p} \leq C \cdot (x + t)
\]

for all \(x \geq 0\) and \(t \in [0, 1]\).

Proof. According to (5) we have

\[
E(X^x_t) \leq x + t
\]
for \( x \geq 0 \) and \( t \in [0,1] \). Combining this with (7) and a Burkholder-Davis-Gundy-type inequality [22, Thm. 1.7.2] we obtain
\[
(E \left( \sup_{0 \leq s \leq t} |X_s^x|^2 \right))^{1/2} \leq x + (1 + x) t + \sqrt{t} \cdot \sqrt{t + t}
\]
for \( x \geq 0 \) and \( t \in [0,1] \), i.e., (24) holds for \( p = 2 \). In the same way we get (24) for \( p = 4,8,16, \ldots \), which suffices. \( \square \)

**Proof of Proposition 3.** We may assume \( 2 \leq p < \infty \). From (7), a Burkholder-Davis-Gundy-type inequality [22, Thm. 1.7.2], and (24) we get
\[
\left( E \left( \sup_{0 \leq s \leq t} |X_s^x - x|^p \right) \right)^{1/p} \leq (1 + x) t + \sqrt{t} \cdot \sqrt{t + t} \leq (1 + x) t + \sqrt{t}
\]
for \( x \geq 0 \) and \( t \in [0,1] \). Moreover, we obtain
\[
\left( E \left( |\varphi_{t,x} - x|^p \right) \right)^{1/p} \leq (1 + x) t + \sqrt{t} \cdot \sqrt{t + t} \leq (1 + x) t + \sqrt{t}
\]
for \( x \geq 0 \) and \( t \in [0,1] \). Combining (25) and (26) yields the claim for \( x \leq t \). Furthermore, according to (25) the error of the drift term satisfies
\[
\left( E \left( \left| \int_0^t (\delta - bx) \, ds - \int_0^t (\delta - bx) \, ds \right|^p \right) \right)^{1/p} \leq (1 + x) t^2 + \sqrt{x} \cdot t^{3/2} \leq \Delta_{\text{loc}}(x,t)
\]
for \( x \geq 0 \) and \( t \in [0,1] \). Define the stopping time
\[
\tau^x = \inf \{ s \geq 0 : |X_s^x - x| = x/2 \}
\]
for \( x \geq 0 \). Using Markov’s inequality we get
\[
P(\tau^x \leq t) = P \left( \sup_{0 \leq s \leq t} |X_s^x - x| \geq x/2 \right) \leq \frac{E \left( \sup_{0 \leq s \leq t} |X_s^x - x|^p \right)}{(x/2)^p}
\]
for \( t \geq 0 \) and \( x > 0 \), and hence (25) implies
\[
P(\tau^x \leq t) \leq \frac{(xt)^p + (tx)^{p/2}}{x^p} = t^p + (t/x)^{p/2} \leq t^{p/2} + (t/x)^{p/2} \leq 2 \cdot \frac{t^{p/2}}{\min(1,x^{p/2})}
\]
for \( x \geq t \) and \( t \in [0,1] \). By quadrupling \( p \) we obtain
\[
P(\tau^x \leq t) \leq \frac{t^{2p}}{\min(1,x^{2p})}
\]
for \( x \geq t \) and \( t \in [0,1] \). Define the stopped process \( \tilde{X}^x = (\tilde{X}_t^x)_{t \geq 0} \) by
\[
\tilde{X}_t^x = X_{t \wedge \tau^x}^x
\]
for \( x \geq 0 \). Clearly, \( \tilde{X}_t^x \in [x/2,3x/2] \) for \( t \geq 0 \). Itô’s lemma shows
\[
\sqrt{\tilde{X}_t^x} = \sqrt{x} + \int_0^{t \wedge \tau^x} \frac{(\delta - 1) - bx}{2\sqrt{X_s^x}} \, ds + W_{t \wedge \tau^x}
\]
and hence
\[
\left( E \left( \left| \sqrt{\tilde{X}_t^x} - (\sqrt{x} + W_{t \wedge \tau^x}) \right|^p \right) \right)^{1/p} \leq (1 + x) t/\sqrt{x}
\]
for \( t \geq 0 \) and \( x > 0 \). Combining the Cauchy-Schwarz inequality with (24) and (27) yields
\[
\left( \mathbb{E} \left( \left| \sqrt{X^x_t} - (\sqrt{x} + W_t) \right|^p \cdot 1_{\{\tau^x \leq t\}} \right) \right)^{1/p} \\
\leq \left( \mathbb{E} \left( \left| \sqrt{X^x_t} - (\sqrt{x} + W_t) \right|^{2p} \right) \right)^{1/(2p)} \cdot \left( \mathbb{P}(\tau^x \leq t) \right)^{1/(2p)} \\
\leq \left( \sqrt{x + t + \sqrt{x} + \sqrt{t}} \cdot \frac{t}{\min(1, x)} \right)
\]
and hence
\[
\left( \mathbb{E} \left( \left| \sqrt{X^x_t} - (\sqrt{x} + W_t) \right|^p \cdot 1_{\{\tau^x \leq t\}} \right) \right)^{1/p} \leq \left( \sqrt{x + t + \sqrt{x} + \sqrt{t}} \cdot \frac{t}{\min(1, x)} \right) \cdot (\sqrt{x + 1/\sqrt{x}})
\]
(29)

for \( x \geq t \) and \( t \in [0, 1] \). Moreover, we have
\[
\left( \mathbb{E} \left( \left| \sqrt{X^x_t} - (\sqrt{x} + W_t) \right|^p \right) \right)^{1/p} \leq \left( \mathbb{E} \left( \left| \sqrt{X^x_t} - (\sqrt{x} + W_t) \right|^p \cdot 1_{\{\tau^x > t\}} \right) \right)^{1/p} \\
+ \left( \mathbb{E} \left( \left| \sqrt{X^x_t} - (\sqrt{x} + W_t) \right|^p \cdot 1_{\{\tau^x \leq t\}} \right) \right)^{1/p}
\]

such that (28) and (29) yield
(30)

for \( x \geq t \) and \( t \in [0, 1] \). Using a Burkholder-Davis-Gundy-type inequality [22, Thm. 1.7.2] and (30) we obtain
\[
\left( \mathbb{E} \left( \left| \int_0^t \sqrt{X^x_s} \, dW_s - \int_0^t (\sqrt{x} + W_s) \, dW_s \right|^p \right) \right)^{1/p} \\
\leq \sqrt{t} \cdot \sup_{0 \leq s \leq t} \left( \mathbb{E} \left( \left| \sqrt{X^x_s} - (\sqrt{x} + W_s) \right|^p \right) \right)^{1/p} \\
\leq \Delta_{\text{loc}}(x, t)
\]
for \( x \geq t \) and \( t \in [0, 1] \). □

5.1. \( L_1 \)-convergence. Recall that \( \varphi_{\text{Mil}} \) is given by
\[
\varphi_{\text{Mil}}(x, t, w) = h(x, t, w) + (\delta - 1 - bx) \cdot t
\]
with
\[
h(x, t, w) = (\sqrt{x} + w)^2.
\]
Consider the one-step function \( \Theta_{\text{Mil}} : \mathbb{R}^+_0 \times [0, 1] \times \mathbb{R} \to \mathbb{R}^+ \) given by
(31)
\[
\Theta_{\text{Mil}}(x, t, w) = \left( \tilde{h}(x, t, w) + (\delta - 1 - bx) \cdot t \right)^+
\]
with
\[
\tilde{h}(x, t, w) = \left( \max \left( \sqrt{t}, \sqrt{\max(t, x) + w} \right) \right)^2.
\]
The corresponding one-step scheme is denoted by \( \tilde{Y}^{x, N} \). We refer to this scheme as truncated Milstein scheme. Let us mention that we have separated the nonlinear parts of \( \varphi_{\text{Mil}} \) and \( \Theta_{\text{Mil}} \), namely \( h \) and \( \tilde{h} \), from the linear drift term. The following lemma shows that \( \tilde{h} \) is \( L_1 \)-Lipschitz continuous with constant 1 and that \( \tilde{h} \) is close to \( h \) in a suitable way, cf. (A1) and (A2). The proof of Lemma 3 is postponed to the appendix.
Lemma 3. We have
\[ \text{E} \left( \left| \tilde{h}(x_1, t, W_t) - \tilde{h}(x_2, t, W_t) \right| \right) \leq |x_1 - x_2| \]
for all \( x_1, x_2 \geq 0 \) and \( t \in [0, 1] \). Furthermore, for every \( 1 \leq p < \infty \) there exists a constant \( C > 0 \) such that
\[ \left( \text{E} \left( \left| \tilde{h}(x, t, W_t) - \tilde{h}(x, t, W_t) \right|^p \right) \right)^{1/p} \leq C \cdot \Delta_{\text{loc}}(x, t) \]
for all \( x \geq 0 \) and \( t \in [0, 1] \).

The following theorem shows that the truncated Milstein scheme satisfies the assumptions of Theorem 2.

Theorem 3 (\( L_1 \)-convergence of truncated Milstein scheme). There exists a constant \( K > 0 \) such that
\[ \text{E} \left( |\Theta_{\text{Mil}}(x_1, t, W_t) - \Theta_{\text{Mil}}(x_2, t, W_t)| \right) \leq (1 + Kt) \cdot |x_1 - x_2| \]
for all \( x_1, x_2 \geq 0 \) and \( t \in [0, 1] \), i.e., (A1) is fulfilled for \( p = 1 \). Furthermore, for every \( 1 \leq p < \infty \) there exists a constant \( C > 0 \) such that
\[ \left( \text{E} \left( |\Theta_{\text{Mil}}(x, t, W_t) - X^p_t| \right) \right)^{1/p} \leq C \cdot \Delta_{\text{loc}}(x, t) \]
for all \( x \geq 0 \) and \( t \in [0, 1] \), i.e., (A2) is fulfilled for every \( 1 \leq p < \infty \).

Proof. Using \(|y^+ - z^+| \leq |y - z|\) for \( y, z \in \mathbb{R} \) we obtain
\[ \text{E} \left( |\Theta_{\text{Mil}}(x_1, t, W_t) - \Theta_{\text{Mil}}(x_2, t, W_t)| \right) \leq \text{E} \left( \left| \tilde{h}(x_1, t, W_t) - \tilde{h}(x_2, t, W_t) + (x_2 - x_1) \cdot bt \right| \right) \leq (1 + bt) \cdot |x_1 - x_2| \]
due to the first part of Lemma 3. Moreover, using \(|z^+ - y| \leq |z - y|\) for \( y \geq 0 \) and \( z \in \mathbb{R} \) we have
\[ \left( \text{E} \left( |\Theta_{\text{Mil}}(x, t, W_t) - X^p_t| \right) \right)^{1/p} \leq \left( \text{E} \left( |\tilde{h}(x, t, W_t) + (\delta - 1 - bx) \cdot t - X^p_t| \right) \right)^{1/p} \leq \left( \text{E} \left( |\tilde{h}(x, t, W_t) - X^p_t| \right) \right)^{1/p} + \left( \text{E} \left( \left| \tilde{h}(x, t, W_t) - \tilde{h}(x, t, W_t) \right|^p \right) \right)^{1/p}. \]
Applying Proposition 3 and the second part of Lemma 3 yields the second statement.

Remark 13. Let us stress that there is some freedom regarding the particular form of the truncated Milstein scheme. For instance, the proof of Theorem 3 shows that the positive part in (31) may be replaced by the absolute value.

5.2. \( L_q \)-convergence. In this section we show that the truncated Milstein scheme \( Y^x, N \) defined by the one-step function \( \Theta_{\text{Mil}} \) given in (31) is uniformly bounded and hence satisfies the assumptions of Corollary 1.

Proposition 4. For every \( 1 \leq q < \infty \) there exists a constant \( C > 0 \) such that
\[ \sup_{0 \leq t \leq 1} \left( \text{E} \left( \left| Y^x_N \right|^q \right) \right)^{1/q} \leq C \cdot (1 + x) \]
for all \( x \geq 0 \) and \( N \in \mathbb{N} \), i.e., (A3) is fulfilled.
Proof. Let \( x \geq 0, t \in [0, 1], \) and \( w \in \mathbb{R} \). At first, note that
\[
\hat{h}(x, t, w) \leq \left( \max \left( \sqrt{t}, \sqrt{t} + w \right) \right)^2 + \left( \max \left( \sqrt{t}, \sqrt{t} + w \right) \right)^2
\]
and
\[
(\delta - 1 - bx) \cdot t \leq (\delta + |b| x) \cdot t.
\]
Moreover, note that \( \max \left( \sqrt{t}, \sqrt{t} + w \right) \) is monotonically increasing in \( x \). Hence the auxiliary one-step function \( g: \mathbb{R}_0^+ \times [0, 1] \times \mathbb{R} \to \mathbb{R}_0^+ \) defined by
\[
g(x, t, w) = x + (\delta + |b| x + 3) \cdot t + 3w^2 + 2\sqrt{t} \cdot w
\]
satisfies
\[
\Theta_{\text{Mil}}(x_1, t, w) \leq g(x_2, t, w),
\]
for \( 0 \leq x_1 \leq x_2 \). This yields
(32)
\[
0 \leq \tilde{Y}_t^{x,N} \leq Z_t^{x,N}
\]
for all \( t \geq 0 \), where \( Z_t^{x,N} \) denotes the corresponding auxiliary scheme with piecewise constant interpolation. Moreover, the auxiliary scheme satisfies the integral equation
\[
Z_t^{x,N} = x + \int_0^t (\delta + |b| Z_s^{x,N} + 6) \, ds + \int_0^t \left( 2\sqrt{Z_s^{x,N}} + 6 (W_s - \bar{W}_s^{N}) \right) \, dW_s
\]
for \( t = 0, 1/N, 2/N, \ldots \), where \( (\bar{W}_s^{N})_{s \geq 0} \) denotes the piecewise constant interpolation of \( W \) at the grid of mesh size \( 1/N \), i.e., \( \bar{W}_s^{N} = W_{sN}/N \). Due to this integral equation we can now apply standard techniques exploiting the linear growth condition we obtain uniform boundedness, cf. [22, Lem. 2.6.1]. Let \( 2 \leq q < \infty \). Straightforward calculations show
\[
\left| Z_s^{x,N} \right|^q \leq 1 + x^q + \int_0^s \left| Z_u^{x,N} \right|^q \, du
\]
\[
+ \left| \int_0^s \left( 2\sqrt{Z_u^{x,N}} + 6 (W_u - \bar{W}_u^{N}) \right) \, dW_u \right|^q
\]
for \( x \geq 0 \) and \( s = 0, 1/N, \ldots, 1 \). This yields
\[
\sup_{0 \leq s \leq t} \left| Z_s^{x,N} \right|^q \leq 1 + x^q + \int_0^t \left| Z_u^{x,N} \right|^q \, du
\]
\[
+ \sup_{0 \leq s \leq t} \left| \int_0^s \left( 2\sqrt{Z_u^{x,N}} + 6 (W_u - \bar{W}_u^{N}) \right) \, dW_u \right|^q
\]
for \( x \geq 0 \) and \( t \in [0, 1] \). Using a Burkholder-Davis-Gundy-type inequality [22 Thm. 1.7.2] and the linear growth condition we obtain
\[
E \left( \sup_{0 \leq s \leq t} \left| Z_s^{x,N} \right|^q \right) \leq 1 + x^q + \int_0^t E \left( \sup_{0 \leq s \leq u} \left| Z_s^{x,N} \right|^q \right) \, du \]
\[
+ \int_0^t E \left( \left| \int_0^s \left( 2\sqrt{Z_u^{x,N}} + 6 (W_u - \bar{W}_u^{N}) \right) \, dW_u \right|^q \right) \, du
\]
\[
\leq 1 + x^q + \int_0^t E \left( \sup_{0 \leq s \leq u} \left| Z_s^{x,N} \right|^q \right) \, du
\]
for \( x \geq 0 \) and \( t \in [0, 1] \). Applying Gronwall’s lemma yields the desired inequality for the auxiliary scheme. Due to [22] this inequality also holds for the truncated Milstein scheme. \( \square \)
Lemma 4. Let $Z$ be standard normally distributed. We have  
\[ E \left( \left| \left( \max(1, \sqrt{x_1} + Z) \right)^2 - \left( \max(1, \sqrt{x_2} + Z) \right)^2 \right| \right) \leq |x_1 - x_2| \]  
for all $x_1, x_2 \geq 1$. Furthermore, for every $1 \leq p < \infty$ there exists a constant $C > 0$ such that  
\[ \left( E \left( \left| \left( \max(1, \sqrt{x} + Z) \right)^2 - (\sqrt{x} + Z)^2 \right|^p \right) \right)^{1/p} \leq C \cdot \frac{1}{\sqrt{x}} \]  
for all $x \geq 1$.

Proof. Let $\phi$ and $\Phi$ denote the Lebesgue density and the distribution function of the standard normal distribution, respectively. Moreover, let $f: \mathbb{R} \to \mathbb{R}$ be given by
\[
 f(x) = E \left( \left( \max(1, x + Z) \right)^2 \right) = \Phi(1-x) + x^2 \Phi(x-1) + 1 + (1-x) \phi(1-x) - \Phi(1-x).
\]
Hence the derivative of $f$ reads
\[
 f'(x) = \phi(1-x) - (1+x) \phi'(1-x) + 2x \Phi(x-1) + x^2 \phi(x-1)
 = 2 \phi(1-x) + 2x \Phi(x-1).
\]
For $x > 0$ we define $g(x) = f(\sqrt{x})$ such that
\[
 g'(x) = \frac{1}{\sqrt{x}} \cdot \phi(\sqrt{x} - 1) + \phi(\sqrt{x} - 1).
\]
Using
\[
 \frac{1}{x+1} \leq \frac{\sqrt{4+x^2} - x}{2}
\]
for $x \geq 0$ and $[4]$, we have
\[
 \frac{1}{x+1} \cdot \phi(x) + \Phi(x) \leq \frac{\sqrt{4+x^2} - x}{2} \cdot \phi(x) + \Phi(x) \leq 1
\]
for $x \geq 0$. This yields
\[
 g'(x) \leq 1
\]
for $x \geq 1$. For $x_1 \geq x_2 \geq 1$ we hence get
\[
 E \left( \left| \left( \max(1, \sqrt{x_1} + Z) \right)^2 - \left( \max(1, \sqrt{x_2} + Z) \right)^2 \right| \right) = g(x_1) - g(x_2) \leq x_1 - x_2,
\]
which shows the first claim. The Cauchy-Schwarz inequality implies
\[
 E \left( \left| \left( \max(1, \sqrt{x} + Z) \right)^2 - (\sqrt{x} + Z)^2 \right|^p \right) = E \left( \left| 1 - (\sqrt{x} + Z)^2 \right|^p \cdot 1_{\{\sqrt{x}+Z<1\}} \right) \leq \sqrt{E \left( \left| 1 - (\sqrt{x} + Z)^2 \right|^{2p} \right) \cdot \mathbb{P}(\sqrt{x}+Z<1)}.\]
Hence we get
\[
\left( E \left( \left( \max(1, \sqrt{x} + Z) \right)^2 - (\sqrt{x} + Z)^2 \right)^p \right)^{1/p} \\
\leq \left( 1 + 2x + 2 \cdot E \left( (Z^2)^p \right) \right)^{1/(2p)} \cdot (P(Z > \sqrt{x} - 1))^{1/(2p)} \\
\leq x \cdot (P(Z > \sqrt{x} - 1))^{1/(2p)}
\]
for \( x \geq 1 \). By using a standard tail estimate for the standard normal distribution, see, e.g., \[27, \text{p. 31}\], we get the second claim.

Remark 14. The upper bound in the second statement in Lemma 4 can be considerably improved due to the exponential decay of the density of the standard normal distribution. However, \( 1/\sqrt{x} \) suffices for our purposes.

Proof of Lemma 3. At first, note that
\[
h(x, t, w) = t \cdot \left( \frac{\sqrt{1/t}}{\sqrt{x}} + \frac{w}{\sqrt{t}} \right)^2
\]
and
\[
\tilde{h}(x, t, w) = t \cdot \left( \max \left( 1, \sqrt{\max \left( 1, \frac{x}{t} \right)} + \frac{w}{\sqrt{t}} \right) \right)^2
\]
Using (34) and the first part of Lemma 4 we obtain
\[
E \left( \left| h(x_1, t, W_t) - \tilde{h}(x_1, t, W_t) \right| \right) \leq t \cdot | \max \left( 1, \frac{x_1}{t} \right) - \max \left( 1, \frac{x_2}{t} \right) | \leq |x_1 - x_2|.
\]
Moreover, using (33), (34), and the second part of Lemma 4 we obtain
\[
\left( E \left( \left| h(x, t, W_t) - \tilde{h}(x, t, W_t) \right|^{p} \right) \right)^{1/p} \leq t \cdot C \cdot \frac{1}{\sqrt{x/t}} \leq \Delta_{\text{loc}}(x, t)
\]
for \( x \geq t \). Finally, we have
\[
\left( E \left( \left| h(x, t, W_t) - \tilde{h}(x, t, W_t) \right|^{p} \right) \right)^{1/p} \leq (E (\left| h(x, t, W_t) \right|^{p}))^{1/p} + (E (\left| \tilde{h}(x, t, W_t) \right|^{p}))^{1/p} \leq t = \Delta_{\text{loc}}(x, t)
\]
for \( x \leq t \). 

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