SUBSTITUTES, BOUSFIELD LOCALIZATION, HIGHER BRAIDED OPERADS, AND BAEZ-DOLAN STABILIZATION

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Abstract. This short note reports on joint work with Michael Batanin towards a general machine for proving Baez-Dolan Stabilization Theorems for various models of higher categories, based on substitutes, Bousfield localization, and homotopical Beck-Chevalley squares. I provide a road map to our recent papers, and include new results proving Baez-Dolan Stabilization Theorems for Tamsamani weak \( n \)-categories, higher Segal categories, Ara’s \( n \)-quasi-categories, and cartesian models of Segal and complete Segal objects due to Bergner and Rezk. I also attempt to clarify the connection to higher braided operads, and our more general stabilization machinery.

1. Introduction

In 1995, Baez and Dolan introduced the stabilization hypothesis, which loosely states that \( k \)-tuply monoidal weak \( n \)-categories are the same as \((k + 1)\)-tuply monoidal weak \( n \)-categories as long as \( k \geq n + 2 \) [BD95]. Here \( k \)-tuply monoidal signifies the additional structure you get on a weak \( n \)-category from reindexing from an \((n + k)\)-category with one cell in each dimension \(< k\). For example, if \( \mathcal{C} \) is a 2-category with one object and one morphism, and we reindex two levels, then we obtain a 0-category (i.e., a set) with two commuting operations, corresponding to horizontal and vertical composition in the 2-cells of \( \mathcal{C} \). By the Eckmann-Hilton argument, this yields the structure of a commutative monoid. Reindexing three levels, from a 3-category with only one cell in dimensions 0, 1, and 2, does not yield any additional structure on the resulting 0-category.

In [BW15], we sketched a proof of the stabilization hypothesis depending on the homotopy theory of \( k \)-operads (which encode \( k \)-tuply monoidal structure). We made good on this promise by proving the [BW20b, Theorem 14.2.1]:

**Theorem A** (Baez-Dolan Stabilization). Let \( 0 \leq n \) and \( \mathcal{M} \) an \( n \)-truncated monoidal combinatorial model category with cofibrant unit. Then \( i_l : B_k(\mathcal{M}) \to B_{k+1}(\mathcal{M}) \) and \( (j_k) : B_k(\mathcal{M}) \to E_\infty(\mathcal{M}) \) are left Quillen equivalences for \( k \geq n + 2 \).

Here \( \mathcal{M} \) should be thought of as a model category of weak \( n \)-categories (e.g., Rezk’s model via \( \Theta_n \)-spaces), \( B_k(\mathcal{M}) \) is the category of algebras over a \( k \)-operad \( G_k \) (the cofibrant replacement of the terminal \( k \)-operad) encoding \( k \)-tuply monoidal weak \( n \)-categories, and \( i \) and \( j \) are comparison functors (based on suspension and symmetrization) between \( k \)-operads, \((k+1)\)-operads, and symmetric operads, previously constructed by Batanin [Bat10]. To say \( \mathcal{M} \) is \( n \)-truncated means its simplicial mapping spaces are \( W_n \)-local (defined below).

Such a result (but requiring a standard system of simplices on \( \mathcal{M} \)) had previously been proven by Batanin [Bat17], but we deduce Theorem A from a much stronger result [BW20b, Theorem 14.1.2], where \( SO \) is the category of symmetric operads:
Theorem B. Let $\mathcal{M}$ a combinatorial monoidal model category with cofibrant unit. For $k \geq 3$ and $2 \leq n + 1 \leq k$, the symmetrization functor $\text{sym}_k : Op_k^{W_n}(\mathcal{M}) \to SO_i(\mathcal{M})$ and the suspension functor $\Sigma : Op_k^{W_n}(\mathcal{M}) \to Op_m^{W_n}(\mathcal{M})$ (for $k < m \leq \infty$) are left Quillen equivalences. Moreover, for $1 \leq n \leq \infty$, the braided symmetrization functor $\text{bsym}_n^{W} : Op_2^{W_n}(\mathcal{M}) \to BO(\mathcal{M})$ is a left Quillen equivalence with the category of braided operads.

Here $Op_k^{W_n}(\mathcal{M})$ denotes the category of locally constant $k$-operads, relative to the localizer $W_n$ that encodes $n$-types. As developed by Cisinski, a fundamental localizer [BW20b, Definition 9.1.1] is a class of functors between small categories that contains all identity functors, satisfies the two out of three property, is closed under retracts, contains functors $A \to 1$ where 1 is the terminal category and $A$ is a category with terminal objects, and such that, if $u/c : A/c \to B/c$ is in $W$ for each object $c \in C$ (where $u$ is a morphism in $\text{Cat}/C$) then $u$ is in $W$.

The localizer $W_n$ is the smallest localizer containing the unique functor from the $(n+1)$-sphere (viewed as a category) to the terminal category. That minimal fundamental localizers such as $W_n$ exist is a theorem of Cisinski. We recall that a category $A$ is said to be $W$-aspherical if the unique functor from $A$ to 1 is in $W$.

2. Substitudes and Left Bousfield Localization

To study the homotopy theory of $Op_k^{W_n}(\mathcal{M})$, we encode categories of $k$-operads as algebras over substitudes. A substitute [BW20b, Definition 5.1.1] is equivalent to the data of a colored operad $P$ with a category $A$ of unary operations. This means one can encode structures with substitudes that cannot be encoded with colored operads. We use techniques from [BB17] and [WY18a] to transfer model structures from presheaf categories $[A, \mathcal{M}]$ to categories algebras over what we call $\Sigma$-free tame unary substitudes with faithful unit, a class that includes categories of $k$-operads. Notably, we prove a transfer theorem more general in two ways than those that have appeared previously. First, it works for substitudes rather than only for colored operads [BW20b, Theorem 8.1.7]. Secondly, it works when the base, $V$, is only a semi-model category [BW20b, Theorem 2.2.1].

We generalize work of Cisinski [Cis09] to prove the existence of left Bousfield localizations $[A, \mathcal{M}]^{W}$ for any proper fundamental localizer $W$, with respect to the projective, injective, or Reedy model structure on presheaves [BW20b, Theorem 9.3.5]. In these local model structures, local objects $F : A \to \mathcal{M}$ are $W$-locally constant presheaves, i.e., for any $W$-aspherical category $A'$, and any functor $u : A' \to A$, the induced functor $u^*(F) : A' \to \mathcal{M}$ is isomorphic to a constant presheaf in $Ho[A', \mathcal{M}]$. The local equivalences are morphisms $u : A \to B$ inducing right Quillen equivalences on categories of locally constant presheaves.

$W_{\infty}$ is the minimal fundamental localizer making categories with terminal objects $W$-aspherical. Equivalently, $W_{\infty}$ is the class of functors whose nerve is a weak equivalence. $W_{\infty}$-locally constant functors $F$ are those taking all morphisms $f$ in $A$ to weak equivalences. This is analogous to [CW18] where the local objects are the homotopy functors (i.e., those preserving weak equivalences). If $\mathcal{M}$ is $n$-truncated then $[A, \mathcal{M}]^{W_r} \to [A, \mathcal{M}]^{W_{\infty}}$ is a Quillen equivalence for all $r \geq n + 1$.

To get from $Op_k(\mathcal{M})$ to $Op_k^{W_n}(\mathcal{M})$, we must left Bousfield localize. Unfortunately, categories of algebras over substitudes are often not left proper. To remedy this, we develop a theory of left Bousfield localization that does not require left properness, and results in a semi-model structure. A semi-model category [BW20b,
Definition 2.1.1] has three classes of morphisms that satisfy all of the model category axioms except that we only know that trivial cofibrations with cofibrant domain lift against fibrations, and that morphisms with cofibrant domain admit factorizations into trivial cofibrations followed by fibrations. Because semi-model categories admit cofibrant replacement, and because the subcategory of cofibrant objects behaves exactly like a model category, every result about model categories has a semi-model categorical analogue, and semi-model categories are equally useful in practice. We state our localization theorem \[BW20a, \text{Theorem A}\]:

Theorem C (Bousfield localization without left properness). Suppose that \(M\) is a combinatorial semi-model category whose generating cofibrations have cofibrant domain, and \(C\) is a set of morphisms of \(M\). Then there is a semi-model structure \(L_C(M)\) on \(M\), whose weak equivalences are the \(C\)-local equivalences, whose cofibrations are the same as \(M\), and whose fibrant objects are the \(C\)-local objects. Furthermore, \(L_C(M)\) satisfies the universal property that, for any left Quillen functor of semi-model categories \(F: M \to N\) taking \(C\) into the weak equivalences of \(N\), then \(F\) is a left Quillen functor when viewed as \(F: L_C(M) \to N\).

This theorem is of independent interest for a host of applications, detailed in \[BW20a\], as lack of left properness has bedeviled researchers seeking to left Bousfield localize for years. Examples in \[BW20a\], show that sometimes the classes of morphisms above do not satisfy the model category axioms, so only a semi-model structure is possible. Examples of semi-model structures abound \[Bat17, BW15, BW21, GW18, HW20, Whi14, Whi15, Whi21, WY16, WY17, WY18a, WY18b, WY19a, WY19b, WY20\].

There are two ways to get from \([A, M]\) to \(Op_k^W(M)\). One can either localize first, then lift the resulting model structure (as in \[Whi17, Whi21\]), or one can lift first (using the transfer theorem) and then attempt to localize. As proven in \[BW21, \text{Theorem 5.6}\], these two approaches are equivalent (when both work).

In addition to these localization results, to prove Theorem B we develop a theory of homotopical Beck-Chevalley squares \[BW20b, \text{Theorem 4.2.2}\] to lift Quillen equivalences of presheaf categories to Quillen equivalences of algebras over substitutes. This vastly generalizing previous work on such problems (e.g., \[WY19a\]).

A square of right adjoints:

\[
\begin{array}{ccc}
A & \xleftarrow{\psi^*} & B \\
\beta^* & \downarrow & \alpha^* \\
C & \xleftarrow{\phi^*} & D
\end{array}
\]

is called \textit{Beck-Chevalley} if the natural transformation \(bc : \phi_!\beta^* \to \alpha^*\psi_!\) is an isomorphism. This implies that if \((\phi_!, \phi^*)\) is an adjoint equivalence and \(\beta^*, \alpha^*\) reflect isomorphisms then \((\phi_!, \phi^*)\) is adjoint equivalence.

The homotopical version of this machinery says that the square above is \textit{homotopy Beck-Chevalley} if \(L\phi_!R\beta^*(\_ \to R\alpha^*L\psi_!(\_))\) for \(\Delta\) is an isomorphism in \(\text{Ho}(\Delta)\). This occurs if \(\alpha^*\) preserves weak equivalences and \(\beta^*\) preserves cofibrant objects. This implies that if \((\phi_!, \phi^*)\) is a Quillen equivalence and \(\beta^*, \alpha^*\) reflect weak equivalences between fibrant objects, then \((\phi_!, \phi^*)\) is a Quillen equivalence.
We use this result to lift Quillen equivalences from presheaf categories $[A, \mathcal{V}]_{p_{\text{proj}}}^W \rightleftarrows [B, \mathcal{V}]_{p_{\text{proj}}}^W$ to algebras over substitudes $\text{Alg}_P^W(\mathcal{V}) \rightleftarrows \text{Alg}_Q^W(\mathcal{V})$. Specifically, if a given morphism of substitudes $(f, g) : (P, A) \to (Q, B)$ induces a homotopical Beck-Chevalley square, we see that if $g !, g ^*$ is a Quillen equivalence, then so is $(f, f ^*)$.

A crucial ingredient in the proof of Theorem B is that the morphisms comparing categories of $k$-operads, $(k + 1)$-operads, and symmetric operads, do induce homotopical Beck-Chevalley squares, both before and after localization [BW20b, Proposition 11.3.2, Proposition 14.1.1].

3. Higher Braided Operads

Locally constant $k$-operads are a model for higher braided operads [Bat07, Bat10], and the category of unary operations $Q_k$ has $Q_k \equiv \bigsqcup Q_k(m)$ such that the nerve of $Q_k(m)$ is homotopy equivalent to the unordered configuration space of points in $\mathbb{R}^k$. An analysis of this homotopy type [BW20b, Theorem 11.1.7] is the last ingredient in the proof of Theorems A and B, and the reason for the inequalities featuring $k$ and $n$. We also lift various equivalences of homotopy categories in this setting (known since [Bat10]) to Quillen equivalences [BW20b, Proposition 12.2.1].

Another consequence of Theorem B is:

Theorem D (Stabilization for Higher Braided Operads). If $\mathcal{M}$ is a $n$-truncated, combinatorial, monoidal model category with cofibrant unit, and $n + 2 \leq k \leq \infty$, then the symmetrization functor $\text{sym}_k : \text{Op}^W_n(\mathcal{M}) \to \text{SO}(\mathcal{M})$ and the suspension functor $\Sigma : \text{OP}^W_n(\mathcal{M}) \to \text{OP}^W_m(\mathcal{M})$ (for $k < m \leq \infty$) are left Quillen equivalences. Moreover, for $1 \leq n \leq \infty$, $b\text{sym}_2 : \text{OP}^W_n(\mathcal{M}) \to \text{BO}(\mathcal{M})$ is a left Quillen equivalence.

This is proven in [BW20b, Corollary 14.1.3].

4. Baez-Dolan Stabilization Theorems

Finally, we obtain a stabilization result for $(n + m, n)$-categories, rather than just weak $n$-categories, as a consequence of the stronger results listed above. We state this first for Rezk’s model of $(n + m, n)$-categories (where $Sp_m$ models $m$-types).

Theorem 4.1. The suspension functor induces a left Quillen equivalence

$$i_! : B_k(\Theta_n Sp_m) \to B_{k+1}(\Theta_n Sp_m)$$

for $k \geq m + n + 2$ and, hence, an equivalence between homotopy categories of Rezk’s $k$-tuple monoidal $(n + m, n)$-categories and Rezk’s $(k + 1)$-tuple monoidal $(n + m, n)$-categories.

This is proven as [BW20b, Corollary 14.2.3], using that Rezk’s $\Theta_n Sp_m$ is a $(n + m)$-truncated, combinatorial, monoidal model category with cofibrant unit. The same hypotheses apply to other models of higher categories, including Tamassani weak $n$-categories, higher Segal categories, $n$-quasi-categories, and models of Bergner and Rezk for Segal and complete Segal objects in $\Theta_{n-1}$ spaces. These results are new, though we plan to add them to [BW20b].

Theorem 4.2. Let $\mathbb{M}$ be a combinatorial, monoidal model category with cofibrant unit. Then Simpson’s categories $PC^n(\mathbb{M})$ [Sim12, Theorem 19.3.2] are too, and hence satisfy our Stabilisation Theorem B. If $\mathbb{M}$ is furthermore left proper, then
Simpson’s localization $\text{Seg}^n(M)$, whose fibrant objects satisfy a Segal condition, satisfies our Stabilisation Theorem B, and its $m$-truncation $\tau_m\text{Seg}^n(M)$ satisfies our Theorem A. In particular, the Baez-Dolan stabilization hypothesis is true for Tam-samani weak $n$-categories (corresponding to $M = \text{Set}$, the trivial model structure) and higher Segal categories (corresponding to the Kan-Quillen model $M = s\text{Set}$).

Remark 4.3. Simpson also proved a Baez-Dolan stabilization result [Sim12, Theorem 23.0.3], but did not model $k$-tuply monoidal weak $n$-categories as algebras over a $k$-operad. Instead, he modeled them as $(k-1)$-connected weak $(n+k)$-categories. We conjecture that these two approaches are Quillen equivalent. Furthermore, we conjecture that ‘left proper’ could be dropped above, using Theorem C to produce semi-model structures for the localizations, and then proving versions of Theorems A and B that only require a semi-model structure to begin. Lastly, it should be mentioned that Simpson requires $M$ to be tractable, left proper, and cartesian (hence, to have cofibrant unit [Sim12, Definition 7.7.1]). Implicit in Theorem 4.2 is an extension of Simpson’s approach to the realm of combinatorial, monoidal model categories. When $M$ is monoidal but not cartesian, Simpson’s connection to enrichments is lost, but one still has model categories $\text{PC}^n(M)$ and $\text{Seg}^n(M)$, and our Baez-Dolan stabilization result.

We turn now to Ara’s model category of $n$-quasi-categories, as presheaves over $\Theta_n$ that are Quillen equivalent to Rezk’s model [Ara14, Theorem 8.4].

Theorem 4.4. Ara’s model category of $n$-quasi-categories [Ara14, Theorem 2.2, Corollary 8.5] satisfies the conditions of Theorem B and its truncation $\tau_m n\text{Qcat}$ satisfies the conditions of Theorems A and D. Hence, $n$-quasi-categories satisfy Baez-Dolan stabilization.

Lastly, we turn to two models introduced by Bergner and Rezk:

1. $\Theta_n Sp$-Segal categories, a combinatorial, cartesian model structure on functors $\Delta^{op} \to \Theta_n Sp$ whose fibrant objects satisfy a Segal condition [BR14, Theorem 5.2].

2. A combinatorial, cartesian model structure with all objects cofibrant, whose fibrant objects satisfy a subset of the conditions required of complete Segal spaces [BR14, Proposition 5.9].

Furthermore, they prove these two are equivalent to each other and to Rezk’s $\Theta_n Sp$ [BR14, Theorem 6.14, Corollary 7.1, Theorem 9.6]. Lastly, both can be left Bousfield localized to make them $m$-truncated.

Theorem 4.5. Both of the Bergner-Rezk model structures above satisfy the conditions of Theorem B, and their truncations satisfy the conditions of Theorems A and D. Hence, the Baez-Dolan stabilization hypothesis is true for these models.

These applications demonstrate the power of Theorems A, B, and D: the conditions are satisfied by all models of higher categories that we are aware of that possess a monoidal model structure. There are other model of higher categories, including $n$-relative categories, $n$-fold Segal spaces, and simplicial categories, which may or may not possess a good monoidal product. If any of them is endowed with a monoidal model structure in the future, we anticipate that our methods will prove the Baez-Dolan stabilisation hypothesis for that model. Furthermore, any model that is homotopically equivalent to a model where we have proven the Baez-Dolan
stabilization hypothesis will automatically satisfy stabilization on the homotopy category level.

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