Soliton Resolution for the Short-pluse Equation

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Abstract

In this paper, we apply the steepest descent method to study the Cauchy problem for the focusing nonlinear short-pluse equation

\[ u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \]

\[ u(x, 0) = u_0(x) \in H^{1,1}(\mathbb{R}), \]

where \( H^{1,1}(\mathbb{R}) \) is a weighted Sobolev space. We construct the solution of the short-pluse equation via the solution of Riemann-Hilbert problem in the new scale \((y, t)\). In any fixed space-time cone of the new scale \((y, t)\) which stratify that \( v_1 \leq v \leq v_2 \in \mathbb{R}^- \) and \( \xi = \frac{v}{t} < 0 \),

\[ C(y_1, y_2, v_1, v_2) = \{(y, t) \in \mathbb{R}^2 | y = y_0 + vt, y_0 \in [y_1, y_2], v \in [v_1, v_2]\}, \]

we compute the long time asymptotic expansion of the solution \( u(x, t) \), which prove soliton resolution conjecture consisting of three terms: the leading order term can be characterized with an \( N(I) \)-soliton whose parameters are modulated by a sum of localized soliton-soliton interactions as one moves through the cone; the second \( t^{-1/2} \) order term coming from soliton-radiation interactions on continuous spectrum up to an residual error order \( O(|t|^{-1}) \) from a \( \mathcal{J} \) equation. Our results also show that soliton solutions of short-pluse equation are asymptotically stable.

Keywords: short pluse equation; Riemann-Hilbert problem, \( \mathcal{J} \) steepest descent method, soliton resolution, asymptotical stability.

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1 Introduction

In this paper, we study the long time asymptotic behavior for the initial value problem of the short pulse (SP) equation

\[ u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad (1.1) \]
\[ u(x, 0) = u_0(x), \quad (1.2) \]

where \( u(x, t) \) is a real-valued function, which represents the magnitude of the electric field, and the initial data \( u_0(x) \) belongs to the weighted Sobolev space

\[ H^{1,1}(R) = \{ f \in L^2(R); xf, f' \in L^2(R) \}. \]

The SP equation was proposed to describe the propagation of ultra-short optical pulses in silica optical fibers. Pulse propagation in optical fibers is usually modeled by the cubic
nonlinear Schrödinger (NLS) equation \[2\]. So NLS equation forms the basis for optimizing existing fiber links and suggesting new fiber communication systems in attempts to achieve high bit-rate data transmission. However, it is questionable that the validity of the NLS equation as a slowly varying amplitude approximation of Maxwell’s equations can describe the propagation of these very narrow pulseless. But the SP equation describes the evolution of a short pulse in nonlinear media if the pulse center is far from the nearest resonance frequency of the material’s susceptibility. In some sense it represents the opposite extreme from the NLS approximation since that results from expanding the susceptibility in the frequency while equation results from expanding the susceptibility in the wavelength, which means the SP equation provides an increasingly better approximation to the corresponding solution of the Maxwell equations as the pulse duration shortens \[3\]. However, we can find that the SP equation appeared first as one of Rabelo’s equations which describe photospherical surfaces, possessing a zero-curvature representation \[4\].

Some scholar found that the SP equation can be viewed as the short wave approximation to the modified Camassa-Holm equation \[5–8\]

\[
m_t + 
\left((u^2 - u_x^2)m\right)_x + u_x = 0, \quad m = u - u_{xx}.
\]

Actually, by introducing the new variables

\[
x' = \frac{x}{\epsilon}, \quad t' = t \epsilon, \quad u' = \frac{u}{\epsilon^2},
\]

passing to the limit \(\epsilon \to 0\) and retaining the main terms, we can reduce the modified Camassa-Holm equation \[1.3\] to the SP equation \[1.1\].

It has been shown that the SP equation \[1.1\] admits a Wadati-Konno-Ichikawa type Lax pair and is related to the sine-Gordon equation through a chain of transformations \[9\]. Then bi-Hamiltonian structure and the conservation laws were studied by Brunelli \[10, 11\]. Schäfer and Wayne proved the nonexistence theorem that the equation \[1.1\] doesn’t possess any solution representing a smooth localized pulse moving with constant shape and speed \[12\]. Sakovich and Sakovich also found the loop-soliton solutions of the SP equation \[1.1\] \[13\]. Matsuno found the connection between the SP equation and the sine-Gordon equation through the hodograph transformation, and further found many kinds of exact solutions including multi-soliton, multi-loop, multi-breather and Periodic solutions \[14, 15\]. And a lot of generalizations of the SP equation has been studied, for example, Pietrzyk, Kanattšikov and Bandelow introduced the vector SP equation \[16\] a two-component SP equation that
generalizes the scalar (1.1) and describes the propagation of polarized ultra-short light pulses in cubically nonlinear anisotropic optical fibers, which can be written as

\[ u_{n,xt} = c_n u_t + c_{nijk} (u_i u_j u_k)_{xx}, \tag{1.4} \]

where \( n, i, j, k = 1, 2 \), and the summation over the repeated indices is assumed. And there are others aspects of the SP equation have been addressed in the literature, including integrable semi-discrete and full-discrete analogues, well-posedness of the Cauchy problem and Riemann-Hilbert approach. Using the method of testing by wave packets, Okamoto prove the unique global existence of small solutions to the SP equation when the small initial data.

In 1974, Manakov first carried out the study on the long-time behavior of nonlinear wave equations solvable by the inverse scattering method. Then by using this method, Zakharov and Manakov give the first result for large-time asymptotic of solutions for the NLS equation with decaying initial value in 1976. The inverse scattering method also worked for long-time behavior of integrable systems such as KdV, Landau-Lifshitz and the reduced Maxwell-Bloch system. In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously obtain the long-time asymptotics behavior of the solution for the MKdV equation by deforming contours to reduce the original Riemann-Hilbert problem (RHP) to a model one whose solution is calculated in terms of parabolic cylinder functions. Since then this method has been widely applied to the focusing NLS equation, KdV equation, Fokas-Lenells equation, derivative NLS equation, short-pluse equation and Camassa-Holm equation etc.

In recent years, McLaughlin and Miller further presented a \( \tilde{\partial} \) steepest descent method which combine steepest descent with \( \tilde{\partial} \)-problem rather than the asymptotic analysis of singular integrals on contours to analyze asymptotic of orthogonal polynomials with non-analytical weights. When it is applied to integrable systems, the \( \tilde{\partial} \) steepest descent method also has displayed some advantages, such as avoiding delicate estimates involving \( L^p \) estimates of Cauchy projection operators, and leading the non-analyticity in the RHP reductions to a \( \tilde{\partial} \)-problem in some sectors of the complex plane which can be solved by being recast into an integral equation and by using Neumann series. Dieng and McLaughlin use it to study the defocusing NLS equation under essentially minimal regularity assumptions on finite mass initial data. Cussagna and Jenkins study the defocusing NLS equation with finite density initial data. This \( \tilde{\partial} \) steepest descent method also was successfully applied...
to prove asymptotic stability of N-soliton solutions to focusing NLS equation \[40\]; Jenkins et.al studied soliton resolution for the derivative nonlinear NLS equation for generic initial data in a weighted Sobolev space \[41\]. Their work provided the soliton resolution property for derivative NLS equation, which decomposes the solution into the sum of a finite number of separated solitons and a radiative parts when \(t \to \infty\). And the dispersive part contains two components, one coming from the continuous spectrum and another from the interaction of the discrete and continuous spectrum.

In our paper, we obtain the soliton resolution and long-time asymptotic behavior for the SP equation \(1.1\) with initial data \(u_0 \in H^{1,1}\) by using \(\bar{\partial}\) steepest descent method. This paper is arranged as follows. In section 2, we introduce two kinds of eigenfunctions to formulate the spectral singularity of the Lax pair for the short-pluse equation. The analytical and asymptotics of the eigenfunctions are further studied. In section 3, following the idea in \[35\], we construct a RH problem for \(M(z)\) to formulate the initial value problem of the short-pluse equation \(1.1\) in an alternative space variable \(y\) instead of the original space variable \(x\). In section 4, we introduce a function \(T(z)\) to define a new RH problem for \(M^{(1)}(z)\), which admits a regular discrete spectrum and two triangular decompositions of the jump matrix near critical point \(\pm z_0\). In section 5, by introducing a matrix-valued function \(R(z)\), we obtain a mixed \(\bar{\partial}\)-RH problem for \(M^{(2)}(z)\) by continuous extension to \(M^{(1)}(z)\). In section 6, we decompose \(M^{(2)}(z)\) into a model RHP problem for \(M^{RHP}(z)\) and a pure \(\bar{\partial}\) Problem for \(M^{(3)}(z)\). The \(M^{RHP}(z)\) can be obtained via an outer model \(M^{(out)}(z)\) for the soliton components to be solved in Section 7 and an inner model \(M^{(\pm z_0)}\) for the stationary phase point \(\pm z_0\) which are approximated by a solvable model for \(M^{sp}\) obtained in \[35\] in Section 8. In section 9, we compute the error function \(E(z)\) with a small-norm Riemann-Hilbert problem. In Section 10, we analyze the \(\bar{\partial}\)-problem for \(M^{(3)}(z)\). Finally, in Section 11, based on the result obtained above, a relation formula is found

\[
M(z) = M^{(3)}(z)E(z)M^{out}(z)T(z)^{-\sigma_3},
\]

from which we then obtain the soliton resolution and long-time asymptotic behavior for the short-pluse equation \(1.1\).
2 The spectral analysis

The SP equation (1.1) admits the Lax pair

$$\Phi_x = (\lambda \sigma_3 + L_0) \Phi, \quad \Phi_t = \left(\frac{1}{4\lambda} \sigma_3 + M_0\right) \Phi,$$

(2.1)

where

$$L_0 = \lambda u_x \sigma_1,$$

$$M_0 = \frac{\lambda}{4} u^2 \sigma_3 + \left(\frac{\lambda}{4} u^2 u_x + \frac{1}{2} u\right) \sigma_1,$$

and $\sigma_1$, $\sigma_2$ and $\sigma_3$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

In order to make the presentation close to the cases of the CH equation [32], we introduce the spectral parameter $z = i\lambda$.

First, we consider the symmetry of the eigenfunction $\Phi$. Note that

$$L_0(-z) = -L_0(z), \quad M_0(-z) = -M_0(z), \quad \sigma_2 \sigma_1 = -\sigma_1 \sigma_2, \quad \sigma_2 \sigma_3 = -\sigma_1 \sigma_3,$$

we find that all $\Phi(\bar{z})$, $\Phi(-z)$ and $\sigma_2 \Phi(z) \sigma_2$ satisfy the Lax pair (2.1) with the same asymptotics, which implies that

$$\Phi(\bar{z}) = \Phi(-z) = \sigma_2 \Phi(z) \sigma_2.$$

(2.2)

To study the long time asymptotic behaviors, usually we only use the $x$-part of Lax pair to analyze the initial value problem, and the $t$-part is used to determine the time evolution of the scattering data for the integrable equations by inverse scattering transform method. But unlike those of NLS and derivative NLS equations [38–40, 42], the Lax pair (2.1) for SP equation has singularities at $z = 0$ and $z = \infty$. In order to control the behavior of solutions of (2.1) and construct the solution $u(x,t)$ of the SP equation (1.1), we need use the $t$-part and the expansion of the eigenfunction as spectral parameter $z \to 0$. So we use two different transformations respectively to analyze these two singularities $z \to 0$. Because in the case $z = 0$ it has well property, we first consider this case.

**Case 1: $z=0$.**

Consider the Jost solutions of the Lax pair (2.1), which are restricted by the boundary conditions

$$\Phi_{\pm} \sim e^{i(xz - t/4z)\sigma_3}, \quad x \to \pm \infty.$$

(2.3)
By making transformation
\[ \mu_0^0 = \Phi_\pm e^{-i(zx-t/4z)\sigma_3}, \] (2.4)
we then have
\[ \mu_0^0 \sim I, \quad x \to \pm \infty. \]
Moreover, \( \mu_0^0 \) satisfy an equivalent Lax pair
\[
\begin{align*}
(\mu_0^0)_x &= iz[\sigma_3, \mu_0^0] + L_0 \mu_0^0, \tag{2.5} \\
(\mu_0^0)_t &= \frac{1}{4iz}[\sigma_3, \mu_0^0] + M_0 \mu_0^0, \tag{2.6}
\end{align*}
\]
from which we obtain its total differential form
\[
\begin{align*}
d\left(e^{-i(zx-t/4z)\sigma_3} \mu_0^0\right) &= e^{-i(zx-t/4z)\sigma_3} (L_0 dx + M_0 dt) \mu_0^0, \tag{2.7}
\end{align*}
\]
whose solutions can be expressed as Volterra type integrals
\[
\mu_0^0 = I + \int_{\pm \infty} dx e^{iz(x-y)\sigma_3} L_0(y) \mu_0^0(y) dy. \tag{2.8}
\]
Then we can show that

**Proposition 1.** As \( u(\cdot, t) \in H^1(\mathbb{R}) \) for all \( t \in \mathbb{R} \), the fundamental eigenfunctions \( \mu_\pm^0 \) exists and is unique.

Denote \( \mu_\pm^0 = ([\mu_\pm^0]_1, [\mu_\pm^0]_2) \), where \([\mu_\pm^0]_1\) and \([\mu_\pm^0]_2\) are the first and second columns of \( \mu_\pm^0 \) respectively. Then from (2.8), we can show that \([\mu_\pm^0]_1\) and \([\mu_\pm^0]_2\) are analysis in \( \mathbb{C}^- \), and \([\mu_\pm^0]_1\) and \([\mu_\pm^0]_2\) are analysis in \( \mathbb{C}^+ \).

It is necessary to discuss the asymptotic behaviors of the Jost solutions \( \mu_\pm^0 \) as \( z \to 0 \).

We consider the following asymptotic expansions
\[
\mu_\pm^0 = \mu_\pm^{0,(0)} + \mu_\pm^{0,(1)} z + \mu_\pm^{0,(2)} z^2 + \mathcal{O}(z^3), \quad \text{as} \; z \to 0, \; j = 1, 2, \tag{2.9}
\]
where \( \mu_\pm^{0,(k)} \) isn’t depend on \( z, k = 0, 1, 2, \ldots \).

Substituting (2.9) into the Lax pair (2.5) and (2.6), and comparing the coefficients, we obtain
\[
\begin{align*}
\mu_\pm^{0,(0)} &= I, \quad \mu_\pm^{0,(1)} = izu\sigma_1, \quad \mu_\pm^{0,(2)} = -\frac{u^2}{2}I + i(u^2u_x - 2u_t)\sigma_2. \tag{2.10}
\end{align*}
\]

**Case II:** \( z \to \infty \).
In order to control asymptotic behavior of the Lax pair (2.1) as \( z \to \infty \), we make a transformation

\[
\Phi_\pm = G\mu_\pm e^{iz\sigma_3},
\]  

(2.11)

where

\[
p(x, t, z) = x - \int_x^{+\infty} (\sqrt{m(y, t)} - 1)dy - \frac{t}{4z^2},
\]

(2.12)

\[
m(x, t) = 1 + u_x(x, t)^2,
\]

(2.13)

\[
G(x, t) = \sqrt{\frac{\sqrt{m} + 1}{2\sqrt{m}}} \left( \frac{1}{\sqrt{m} - 1} \frac{\sqrt{m} - 1}{u_x} \frac{1}{u_x} \right).
\]

(2.14)

Then the SP equation (1.1) is changed into an equivalent form

\[
(\sqrt{m})_t = \frac{1}{2}(u^2\sqrt{m})_x.
\]

And from (2.12), we have

\[
p_x = \sqrt{m}, \quad p_t = \frac{1}{2} u^2 \sqrt{m} - \frac{1}{4z^2}.
\]

(2.15)

The Lax pair (2.1) is changed into a new Lax pair

\[
(\mu_\pm)_x = ip_x[\sigma_3, \mu_\pm] + P\mu_\pm,
\]

(2.16)

\[
(\mu_\pm)_t = ip_t[\sigma_3, \mu_\pm] + Q\mu_\pm,
\]

(2.17)

where

\[
P = \frac{iux}{2m}\sigma_2, \quad Q = \frac{1}{4iz} \left( \frac{1}{\sqrt{m}} - 1 \right) \sigma_3 + \frac{iuxu^2}{4m} \sigma_2 - \frac{ux}{4iz\sqrt{m}\sigma_1},
\]

(2.18)

\[
\mu_\pm \sim I, \quad x \to \pm\infty.
\]

(2.19)

The Lax pair (2.16)-(2.17) can be written in to a total differential form

\[
d\left(e^{-izp\sigma_3}\mu_\pm\right) = e^{-izp\sigma_3}(Pdx + Qdt)\mu_\pm,
\]

(2.20)

which leads to two Volterra type integrals

\[
\mu_\pm = I + \int_x^{+\infty} e^{i(z(p(x) - p(y))\sigma_3} P(y)\mu_\pm(y)dy.
\]

(2.21)

Similarly, we denote \( \mu_\pm = ([\mu_\pm],[\mu_\pm]) \), then we can show that \( [\mu_-] \) and \( [\mu_+] \) are analysis in \( \mathbb{C}^- \), and \( [\mu_+] \) and \( [\mu_-] \) are analysis in \( \mathbb{C}^+ \). And the \( \mu_\pm \) admit the asymptotics

\[
\mu_\pm = I + \frac{D_1}{z} + O(z^{-2}), \quad z \to \infty,
\]

(2.22)
where the off-diagonal entries of the matrix \( D_1(x,t) \) are

\[
D_{12}(x,t) = D_{21}(x,t) = \frac{iu_{xx}}{4m\sqrt{m}}.
\]

(2.23)

Since \( \Phi_{\pm} \) are two fundamental matrix solutions of the Lax pair (2.1), there exists a linear relation between \( \Phi_+ \) and \( \Phi_- \), namely

\[
\Phi_+(z; x, t) = \Phi_-(z; x, t)S(z), \quad z \in \mathbb{C},
\]

(2.24)

where \( S(z) \) is called scattering matrix which only depend on \( z \). Form the symmetry relation (2.2) of \( \Phi_{\pm} \), the matrix \( S(z) \) also admits the symmetry

\[
S(z) = \begin{pmatrix}
\frac{a(z)}{b(z)} & b(z) \\
-b(z) & \frac{a(z)}{a(z)}
\end{pmatrix}.
\]

And combining (2.11) and (2.24) gives

\[
\mu_-(z) = \mu_+(z)e^{izp}\bar{\sigma}_3 S(z),
\]

(2.25)

which can be written as

\[
(\mu_-)_1, (\mu_-)_2 = (\mu_+)_1, (\mu_+)_2 \left( \begin{array}{cc}
\frac{a(z)}{b(z)} & \bar{e}^{2izp}b(z) \\
-e^{-2izp}b(z) & \frac{a(z)}{a(z)}
\end{array} \right),
\]

(2.26)

which implies that

\[
a(z) = \det (\mu_+_1, \mu_-)_2),
\]

(2.27)

and \( a(z) \) is analytical in \( C^+ \), and \( a(z) = -a(-\bar{z}) \). We introduce the reflection coefficient

\[
r(z) = \frac{b(z)}{a(z)},
\]

(2.28)

with symmetry \( r(-z) = \bar{r}(\bar{z}) \). The zeros of \( a(z) \) on \( \mathbb{R} \) are known to occur and they correspond to spectral singularities \( ? \). They are excluded from our analysis in the this paper. To deal with our following work, we assume our initial data satisfy this assumption.

**Assumption 1.** The initial data \( u_0(x) \in H^{1,1}(\mathbb{R}) \) and it generates generic scattering data which satisfy that

1. \( a(z) \) has no zeros on \( \mathbb{R} \).
2. \( a(z) \) only has finite number of simple zeros.
3. \( a(z) \) and \( r(z) \) belong \( H^{1,1}(\mathbb{R}) \).
We assume that $a(z)$ has $N$ simple zeros $z_n \in \mathbb{C}^+, n = 1, 2, ..., N$, $a(z_n) = 0$. Denote $\mathcal{Z} = \{z_n\}_{n=1}^{N}$ which is the set of the zeros in $\mathbb{C}^+$ of $a(z)$. From (2.22) and (2.27), we obtain the asymptotic of $a(z)$

$$a(z) = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty. \quad (2.29)$$

We can formulate a RH problem by defining the matrix function $M(x, t, z)$ with eigenfunctions $\mu_{\pm}$, while the reconstruction formula between the solution $u(x, t)$ and the RH problem can be found from the asymptotic of $\mu_{\pm}$ as $z \to 0$. So we need to calculate the relation between $\mu_{\pm}$ and $\mu_0^{\pm}$.

The relations (2.4) and (2.11) implies that there exist constant matrices $C_{\pm}(z)$ satisfying

$$\mu_{\pm}(x, t, z) = G^{-1}(x, t)\mu_0^{\pm}e^{i\int(x-t/4z)\sigma_3}C_{\pm}(z)e^{-iz\int(x,t,z)\sigma_3}, \quad (2.30)$$

which means $\mu_{\pm}(x, t, z)$ exists and is unique. Take $x \to \pm\infty$, we have

$$C_+ = I, \quad C_- = e^{icz\sigma_3}, \quad (2.31)$$

where

$$c = \int_{\mathbb{R}}(\sqrt{m} - 1)dy \quad (2.32)$$

is a conserved quantity under the dynamics governed by (1.1). Then we have

$$\mu_{\pm}(x, t, z) = G^{-1}(x, t)\mu_0^{\pm}e^{-iz\int_{\pm\infty}^{x}(\sqrt{m} - 1)dy}\sigma_3. \quad (2.33)$$

Since $\text{tr}(iz\sigma_3 + L_0) = \text{tr}(\frac{1}{4iz}\sigma_3 + M_0) = 0$, by the Abel formula, it holds that

$$\det(\Phi_{\pm})_x = \det(\Phi_{\pm})_t = 0, \quad (2.34)$$

which together with $\det(\mu_0^{\pm}) = \det(\Phi_{\pm})$ leads to

$$\det(\mu_0^{\pm})_x = \det(\mu_0^{\pm})_t = 0, \quad (2.35)$$

and

$$1 = \det(\mu_0^{\pm}) = \det(\Phi_{\pm}) = \det(S(z)). \quad (2.36)$$

Then we have $|a(z)|^2 + |b(z)|^2 = 1$, which is equivalent to $1 + |r(z)|^2 = \frac{1}{|a(z)|^2}$. In the absence of spectral singularities (real zeros of $a(z)$), there also exist $\nu \in (0, 1)$ such that $\nu < |a(z)| < 1/\nu$ for $z \in \mathbb{R}$, which implies $1 + |r(z)| > \nu^2 > 0$ for $z \in \mathbb{R}$. And from the asymptotic of the Jost solutions $\mu_0^{\pm}$ as $z \to 0$, we get the asymptotic of $a(z)$

$$a(z) = 1 + icz - \frac{c^2}{2}z^2 + \mathcal{O}(z^3), \quad z \to 0, \quad (2.37)$$

where $c$ is defined in (2.32).
3 The construction of a RH problem

Suppose that \( Z = \{z_n, \ n = 1, \cdots, N\} \) are simple zeros for \( a(z) \), we first calculate residue conditions. Since \( (\Phi_1^+, \Phi_2^+)(z) \) and \( (\Phi_1^-, \Phi_2^-)(z) \) are linearly dependent, there exists a constant \( b_k \) such that

\[
(\Phi_1^+, \Phi_2^+) = b_k (\Phi_1^-, \Phi_2^-),
\]

which implies that

\[
[\mu_+]_1 (z_k) = b_k e^{2iz_p(z_k)} [\mu_-]_2 (z_k). \tag{3.1}
\]

We denote norming constant \( c_k = b_k/a'(z_k) \), and the collection \( \sigma_d = \{z_k, c_k\}_{k=1}^N \) is called the scattering data.

We define a sectionally meromorphic matrix

\[
N(z;x,t) = \begin{cases} 
([\mu_+]_1, a(z)^{-1} [\mu_-]_2), & \text{as } z \in \mathbb{C}^+ , \\
(a(z)^{-1} [\mu_+]_1, [\mu_-]_2), & \text{as } z \in \mathbb{C}^- .
\end{cases}
\tag{3.2}
\]

which solves the following RHP.

**RHP1.** Find a matrix-valued function \( N(z;x,t) \) which satisfies:

- Analyticity: \( N(z;x,t) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \) and has single poles;
- Symmetry: \( \overline{N(z)} = N(-z) = \sigma_2 N(z) \sigma_2 \);
- Jump condition: \( N \) has continuous boundary values \( N_\pm \) on \( \mathbb{R} \) and

\[
N^+ (z;x,t) = N^- (z;x,t) V(z), \quad z \in \mathbb{R}, \tag{3.3}
\]

where

\[
V(z) = \begin{pmatrix} 1 & e^{2iz_p(z)} \\ e^{-2iz_p(z)} & 1 + |r(z)|^2 \end{pmatrix}; \tag{3.4}
\]

- Asymptotic behaviors:

\[
N(z;x,t) = I + O(z^{-1}), \quad z \to \infty; \tag{3.5}
\]

- Residue conditions: \( N \) has simple poles at each point in \( Z \cup \bar{Z} \) with:

\[
\text{Res } N(z) = \lim_{z \to z_n} N(z) \begin{pmatrix} 0 & c_n e^{2iz_p(z_k)} \\ 0 & 0 \end{pmatrix}, \tag{3.6}
\]

\[
\text{Res } N(z) = \lim_{z \to \bar{z}_n} N(z) \begin{pmatrix} 0 & 0 \\ -\bar{c}_n e^{-2i\bar{z}_n p(\bar{z}_n)} & 0 \end{pmatrix}. \tag{3.7}
\]

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We denote
\[ c_+(x, t) = \int_x^{+\infty} (\sqrt{m(k, t)} - 1) dk, \tag{3.8} \]
and consider the asymptotic of \( N(z; x, t) \)
\[ N(z; x, t) = G^{-1}(x, t) \left[ I + z(ic_+\sigma_3 + iu\sigma_1 + O(z^2)) \right], \quad z \to 0. \tag{3.9} \]
from which it is difficult to reconstruct the solution of the SP equation (1.1), since \( p(x, t, z) \) is still unknown. To overcome this, we introduce a new scale
\[ y(x, t) = x - \int_x^{+\infty} \left( \sqrt{m(k, t)} - 1 \right) dk = x - c_+(x, t). \tag{3.10} \]
The price to pay for this is that the solution of the initial problem can be given only implicitly, or parametrically: it will be given in terms of functions in the new scale, whereas the original scale will also be given in terms of functions in the new scale. By the definition of the new scale \( y(x, t) \), we define
\[ M(y, t, z) = N(x(y, t), t, z), \tag{3.11} \]
which satisfies the following RH problem.

**RHP2.** Find a matrix-valued function \( M(z) = M(y, t, z) \) which satisfies:
- **Analyticity:** \( M(z) \) is meromorphic in \( \mathbb{C} \setminus R \) and has single poles;
- **Symmetry:** \( M(\bar{z}) = M(-z) = \sigma_2 M(z) \sigma_2 \);
- **Jump condition:** \( M \) has continuous boundary values \( M^\pm \) on \( R \) and
  \[ M^+(z) = M^-(z) V(z), \quad z \in R, \tag{3.12} \]
where
  \[ V(z) = \begin{pmatrix} 1 & e^{2i(zy - \frac{t}{4z})} r(z) \\ e^{-2i(zy - \frac{t}{4z})} \bar{r}(z) & 1 + |r(z)|^2 \end{pmatrix}; \tag{3.13} \]
- **Asymptotic behaviors:**
  \[ M(z) = I + O(z^{-1}), \quad z \to \infty; \tag{3.14} \]
- **Residue conditions:** \( M \) has simple poles at each point in \( \mathcal{Z} \cup \bar{\mathcal{Z}} \) with:
  \[ \text{Res} \ M(z) = \lim_{z \to z_n} M(z) \begin{pmatrix} 0 & e^{-2i(z_n y - \frac{t}{4z_n})} \\ 0 & 0 \end{pmatrix}, \tag{3.15} \]
  \[ \text{Res} \ M(z) = \lim_{z \to \bar{z}_n} M(z) \begin{pmatrix} 0 & 0 \\ -e^{-2i\bar{z}_n (\bar{z} y - \frac{t}{4\bar{z}_n})} & 0 \end{pmatrix}. \tag{3.16} \]
From the asymptotic behavior of the functions $\mu_{\pm}$ and (3.9), we have following reconstruction formula of $u(x, t) = u(y(x, t), t)$:

$$
u(x, t) = \nu(y(x, t), t) = \lim_{z \to 0} \frac{(M(y, t, 0)^{-1}M(y, t, z))_1}{12iz}.$$  (3.17)

where

$$x(y, t) = y + c_+(x, t) = y + \lim_{z \to 0} \frac{(M(y, t, 0)^{-1}M(y, t, z))_1}{12iz} - 1.$$  (3.18)

### 4 Conjugation

In the jump matrix (3.13), we denote the oscillatory term

$$e^{2i(y - \frac{\xi}{4})} = e^{2it\phi(z)}, \quad \phi(z) = \frac{y}{t} - \frac{1}{4z}.$$  (4.1)

it will be found that the long-time asymptotic of RHP2 is affected by the growth and decay of the exponential function $e^{2it\phi}$ appearing in both the jump relation and the residue conditions. In this section, we introduce a new transform $M(z) \to M^{(1)}(z)$, from which we make that the $M^{(1)}(z)$ is well behaved as $|t| \to \infty$ along any characteristic line.

Let $\xi = \frac{y}{t} < 0$, then $z_0 = \sqrt{-\frac{1}{4\xi}} \in \mathbb{R}$, where $\pm z_0$ are the two critical points of the phase function $\phi(z)$. The case of $\xi > 0$ is discussed by Xu [35]. Then (4.1) can be written as

$$\phi(z) = -\frac{z}{4}\left(\frac{1}{z_0^2} + \frac{1}{z^2}\right), \quad \text{Re}(2it\phi) = -2t\text{Im}\phi = -2t\text{Im}z \left(\xi + \frac{1}{4|z|^2}\right).$$  (4.2)

The partition $\Delta_{z_0, \eta}^\pm$ of $\{1, \ldots, N\}$ for $z_0 \in \mathbb{R}$, $\eta = \text{sgn}(t)$ is defined as follow:

$$\Delta_{z_0, 1} = \Delta_{z_0, -1} = \{k \in \{1, \ldots, N\} ||z_k| < z_0\},$$

$$\Delta_{z_0, 1}^\pm = \Delta_{z_0, -1}^\pm = \{k \in \{1, \ldots, N\} ||z_k| > z_0\}.$$

This partition splits the residue coefficients $c_n$ in two sets which is shown in Figure. [1]
Figure 1: In the yellow region, $|e^{2it\theta}| \to \infty$ when $t \to \pm \infty$ respectively. And in white region, $|e^{2it\theta}| \to 0$ when $t \to \pm \infty$ respectively.

We define the following functions and notation which will be used later

\begin{equation}
  k(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2),
\end{equation}

\begin{equation}
  I_+ = (-\infty, -z_0] \cup [z_0, +\infty), \quad I_- = [-z_0, z_0]
\end{equation}

\begin{equation}
  \delta(z) = \delta(z, z_0, \eta) = \exp \left(i \int_{I_\eta} \frac{k(s)ds}{s - z} \right)
\end{equation}

\begin{equation}
  T(z) = T(z, z_0, \eta) = \prod_{k \in \Delta_{z_0, \eta}} \frac{z - \bar{z}_k}{z - z_k} \delta(z),
\end{equation}

\begin{equation}
  \beta^\pm(z, z_0, \eta) = -\eta k(\pm z_0) \log(\eta(z \mp z_0 + 1)) + \int_{I_\eta} \frac{k(s) - X_{\eta,\pm}(s)k(\pm z_0)}{s - z} ds,
\end{equation}

\begin{equation}
  T_0(\pm z_0) = T(\pm z_0, z_0, \eta) = \prod_{k \in \Delta_{z_0, \eta}} \frac{\pm z_0 - \bar{z}_k}{\pm z_0 - z_k} e^{i \beta^\pm(z_0, \pm z_0, \eta)}
\end{equation}

where $X_{\eta,\pm}(s)$ and $X_{\eta,-}(s)$ are the characteristic functions of the interval $\eta z_0 < \eta s < \eta z_0 + 1$ and $-\eta z_0 - 1 < \eta s < -\eta z_0$ respectively. In all of the above formulas, we choose the principal branch of power and logarithm functions.

**Proposition 2.** The function defined by (4.6) has following properties:

(a) $T$ is meromorphic in $\mathbb{C} \setminus I_\eta$, for each $n \in \Delta^-_{z_0, \eta}$, $T(z)$ has a simple pole at $z_n$ and a simple zero at $\bar{z}_n$;

(b) For $z \in \mathbb{C} \setminus I_\eta$, $\overline{T(z)}T(z) = 1$;
(c) For \( z \in I_\eta \), as \( z \) approaches the real axis from above and below, \( T \) has boundary values \( T_\pm \), which satisfy:
\[
T_+(z) = (1 + |r(z)|^2) T_-(z), \quad z \in I_\eta; \tag{4.9}
\]

(d) As \( |z| \to \infty \) with \( |\arg(z)| \leq c < \pi \),
\[
T(z) = 1 + \frac{i}{z} \left[ 2 \sum_{k \in \Delta_{\rho_0, \eta}} \text{Im}(z_k) - \int_{I_\eta} k(s) ds \right] + O(z^{-2}); \tag{4.10}
\]

(e) \( T(z) \) is continuous at \( z = 0 \), and as \( |z| \to 0 \),
\[
T(z) = T(0) (1 + zT_1) + O(z^2), \tag{4.11}
\]

where
\[
T_1 = 2 \sum_{k \in \Delta_{\rho_0, \eta}} \frac{\text{Im}(z_k)}{z_k} - \int_{I_\eta} \frac{k(s)}{s^2} ds; \tag{4.12}
\]

(f) As \( z \to \pm z_0 \), along \( z = \pm z_0 + e^{i\psi} l, \ l > 0, \ |\psi| \leq c < \pi \),
\[
|T(z, z_0, \eta) - T_0(\pm z_0, \eta)(\eta(\pm z_0))| \leq C|z_0|^{1/2}. \tag{4.13}
\]

Proof. The proof of above properties can be obtain by simple calculation, for details, see [42].

We now use \( T(z) \) to define a new matrix-valued function \( M^{(1)}(z) \)
\[
M^{(1)}(z) = M(z)T(z)^{-\sigma_3}, \tag{4.14}
\]

which then satisfies the following RH problem.  

**RHP3.** Find a matrix-valued function \( M^{(1)}(z) \) which satisfies:

- Analyticity: \( M^{(1)}(z) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \) and has single poles;
- Symmetry: \( M^{(1)}(z) = M^{(1)}(-z) = \sigma_2 M^{(1)}(z) \sigma_2 \);
- Jump condition: \( M^{(1)} \) has continuous boundary values \( M^{(1)}_\pm \) on \( \mathbb{R} \) and
\[
M^{(1)}_+(z) = M^{(1)}_-(z)V^{(1)}(z), \quad z \in \mathbb{R}, \tag{4.15}
\]

where
\[
as z \to R \setminus I_\eta, V^{(1)}(z) = \left( \begin{array}{cc}
\frac{1}{r(z)T(z)^2e^{-2it\theta}} & 0 \\
1 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & r(z)T(z)^{-2}e^{2it\theta} \\
0 & 1
\end{array} \right), \tag{4.16}
\]

\[
as z \to I_\eta \setminus \{ \pm z_0 \}, V^{(1)}(z) = \left( \begin{array}{cc}
\frac{r(z)T_-(z)^{-2}}{1 + |r(z)|^2}e^{2it\theta} & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\frac{1}{r(z)T_+(z)^2e^{-2it\theta}} & 0 \\
1 + |r(z)|^2 & 1
\end{array} \right); \tag{4.17}
\]

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Asymptotic behaviours:

\[ M^{(1)}(z) = I + O(z^{-1}), \quad z \to \infty; \quad (4.18) \]

Residue conditions: \( M^{(1)} \) has simple poles at each point in \( Z \cup \bar{Z} \) with:

For \( n \in \Delta^+_{z_0, \eta} \),

\[
\text{Res}_{z = z_n} M^{(1)}(z) = \lim_{z \to z_n} M^{(1)}(z) \begin{pmatrix}
0 & c_n T(z_n)^{-2} e^{-2i\theta_n t} \\
0 & 0
\end{pmatrix},
\quad (4.19)
\]

\[
\text{Res}_{z = \bar{z}_n} M^{(1)}(z) = \lim_{z \to \bar{z}_n} M^{(1)}(z) \begin{pmatrix}
0 & 0 \\
\bar{c}_n T(\bar{z}_n)^2 e^{2i\theta_n t} & 0
\end{pmatrix}.
\quad (4.20)
\]

For \( n \in \Delta^-_{z_0, \eta} \),

\[
\text{Res}_{z = z_n} M^{(1)}(z) = \lim_{z \to z_n} M^{(1)}(z) \begin{pmatrix}
c_n (1/T)'(z_n)^{-2} e^{2i\theta_n t} & 0 \\
0 & 0
\end{pmatrix},
\quad (4.21)
\]

\[
\text{Res}_{z = \bar{z}_n} M^{(1)}(z) = \lim_{z \to \bar{z}_n} M^{(1)}(z) \begin{pmatrix}
0 & \bar{c}_n T'(\bar{z}_k)^{-2} e^{-2i\theta_n t} \\
0 & 0
\end{pmatrix}.
\quad (4.22)
\]

where we denote \( \theta_n = \theta(z_n) \).

**Proof.** The analyticity, jump condition and asymptotic behaviours of \( M^{(1)}(z) \) is directly from its definition, the proposition and the properties of \( M \). As for residues, because \( T(z) \) is analytic at each \( z_n \) and \( \bar{z}_n \) for \( n \in \Delta^+_{z_0, \eta} \), from (3.15), (3.16) and (4.14) we obtain residue conditions at these point immediately.

For \( n \in \Delta^-_{z_0, \eta} \), we denote \( M(z) = (M_1(z), M_2(z)) \), then

\[ M^{(1)}(z) = \left( M_1^{(1)}(z), M_2^{(1)}(z) \right) = (M_1(z)T(z), M_2(z)T(z)^{-1}). \]

\( T(z) \) has a simple zero at \( \bar{z}_n \) and a pole at \( z_n \), so \( z_n \) is no longer the pole of \( M_1^{(1)}(z) \) with \( \bar{z}_n \) becoming the pole of it. And \( M_2^{(1)}(z) \) has opposite situation. It has pole at \( z_n \) and a removable singularity at \( \bar{z}_n \). The calculation of it is similar as it in [42]. \( \square \)
5 A mixed $\bar{\partial}$-RH problem

In this section, we make continuous extension to the jump matrix $V^{(1)}$, for this purpose, we introduce new contours defined as follow:

$$\Sigma_k = z_0 + e^{(2k-1)i\pi/4} R_+, \quad k = 1, 4,$$

$$\Sigma_k = z_0 + e^{(2k-1)i\pi/4} h, \quad h \in (0, (\sqrt{2})^{-1} z_0), \quad k = 2, 3,$$

$$\Sigma_k = z_0 + e^{(2k-1)i\pi/4} h, \quad h \in (0, (\sqrt{2})^{-1} z_0), \quad k = 5, 8,$$

$$\Sigma_k = -z_0 + e^{(2k-1)i\pi/4} R_+, \quad k = 6, 7,$$

$$\Sigma_k = e^{(2k-1)i\pi/4} h, \quad h \in (0, (\sqrt{2})^{-1} z_0), \quad k = 9, 10, 11, 12,$$

$$\Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_{12},$$

then the contour $\Sigma^{(2)}$ and real axis $\mathbb{R}$ separate complex plane $\mathbb{C}$ into ten open sectors denoted by $\Omega_k$, $k = 1, ..., 10$, starting with sector $\Omega_1$ between $I_\eta$ and $\Sigma_1$ and numbered consecutively continuing counterclockwise for $\eta = 1$ ($\eta = -1$ is similarly) as shown in Figure 2.

Additionally, let

$$\mu = \frac{1}{2} \min_{\lambda \neq \gamma \in \mathbb{Z} \cup \bar{\mathbb{Z}}} |\lambda - \gamma|.$$

Since there is no pole on the real axis, it holds that $\text{dist}(\mathbb{Z}, \mathbb{R}) > \mu$. Then we define

$$X_Z \in C_0^\infty(\mathbb{C}, [0, 1])$$

which only supported on the neighborhood of $\mathbb{Z} \cup \bar{\mathbb{Z}},$

$$X_Z(z) = \begin{cases} 1 & \text{dist}(z, \mathbb{Z} \cup \bar{\mathbb{Z}}) < \mu/3 \\ 0 & \text{dist}(z, \mathbb{Z} \cup \bar{\mathbb{Z}}) > 2\mu/3. \end{cases}$$

In order to deform the contour $\mathbb{R}$ to the contour $\Sigma^{(2)}$, we introduce a new unknown function $M^{(2)}$ as follow:

$$M^{(2)}(z) = M^{(1)}(z) R^{(2)}(z),$$

where $R^{(2)}(z)$ is chosen to satisfy the following conditions: First, $M^{(2)}$ has no jump on the real axis, so we choose the boundary values of $R^{(2)}(z)$ through the factorization of $V^{(1)}(z)$ in (4.15) where the new jumps on $\Sigma^{(2)}$ match a well known model RH problem; Second, we need to control the norm of $R^{(2)}(z)$, so that the $\bar{\partial}$-contribution to the long-time asymptotics of $u(y, t)$ can be ignored; Third the residues are unaffected by the transformation. So we
Figure 2: In the yellow region, $R^{(2)} \neq I$, in white region, $R^{(2)} = I$. 
choose $R^{(2)}(z)$ as

$$R^{(2)}(z) = \begin{cases} 
\begin{pmatrix} 1 & (-1)^m R_j(z) e^{2i\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_j, j = 1, 4, 7, 9; \\
\begin{pmatrix} 1 & 0 \\ (-1)^m R_j(z) e^{-2i\theta} & 1 \end{pmatrix}, & z \in \Omega_j, j = 3, 6, 8, 10; \\
I & z \in \Omega_2 \cup \Omega_5;
\end{cases}$$

(5.10)

where $m_1 = m_3 = m_7 = m_8 = 1$, $m_4 = m_6 = m_9 = m_{10} = 0$, and the function $R_j$, $j = 1, 3, 4, 6, 7, 8, 9, 10$, is defined in following proposition.

**Proposition 3.** Take $\eta = -1$ as example, $R_j$; $\Omega_j \to C$, $j = 1, 3, 4, 6, 7, 8, 9, 10$ have boundary values as follow:

$$R_1(z) = \begin{cases} 
\frac{r(z)T(z)}{r(z_0)T_0(z_0)^{-2}(\eta(z - z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}, & z > z_0, \quad \text{z} \in \Sigma_1, \\
\frac{\bar{r}(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & 0 < z < z_0,
\end{cases}$$

(5.11)

$$R_3(z) = \begin{cases} 
\frac{r(z_0)T_0(z_0)^{-2}(\eta(z - z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}{r(z_0)T_0(z_0)^{-2}}, & z \in \Sigma_2, \\
\frac{\bar{r}(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & 0 < z < z_0,
\end{cases}$$

(5.12)

$$R_4(z) = \begin{cases} 
\frac{r(z)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & 0 < z < z_0, \\
\frac{r(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & z \in \Sigma_3,
\end{cases}$$

(5.13)

$$R_6(z) = \begin{cases} 
\frac{r(z_0)T_0(z_0)^{-2}(\eta(z - z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}{r(z)T(z)^2}, & z \in \Sigma_4, \\
\frac{\bar{r}(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & z > z_0,
\end{cases}$$

(5.14)

$$R_7(z) = \begin{cases} 
\frac{r(z)T(z)}{r(z_0)T_0(z_0)^{-2}(\eta(z + z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}, & z < -z_0, \\
\frac{\bar{r}(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & z \in \Sigma_6,
\end{cases}$$

(5.15)

$$R_8(z) = \begin{cases} 
\frac{r(z_0)T_0(z_0)^{-2}(\eta(z + z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}{\bar{r}(z)T(z)^2}, & 0 > z > -z_0, \\
\frac{\bar{r}(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & z \in \Sigma_5,
\end{cases}$$

(5.16)

$$R_9(z) = \begin{cases} 
\frac{r(z)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & 0 > z > -z_0, \\
\frac{r(z_0)T_0(z_0)^{-2}(\eta(z + z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}{1 + |r(z)|^2}, & z \in \Sigma_8,
\end{cases}$$

(5.17)

$$R_{10}(z) = \begin{cases} 
\frac{r(z_0)T_0(z_0)^{-2}(\eta(z + z_0))^{-2i\eta k(\zeta_0)^{(1 - X_2(\zeta))}}}{\bar{r}(z)T(z)^2}, & z \in \Sigma_7, \\
\frac{\bar{r}(z_0)T_0(z_0)^{-2}}{1 + |r(z)|^2}, & z < -z_0.
\end{cases}$$

(5.18)
And in the case of $\eta = -1$, $R_j$ is defined follow the in reverse order. $R_j$ have following property: for $j = 1, 3, 4, 6,$

$$|R_j(z)| \lesssim \sin^2(\arg(z - z_0)) + \langle \text{Re}(z) \rangle^{-1/2},$$

$$|\bar{\partial}R_j(z)| \lesssim |\bar{\partial}X_2(z)| + |p'_j(\text{Re}z)| + |z - z_0|^{-1/2},$$

and for $j = 7, 8, 9, 10,$

$$|R_j(z)| \lesssim \sin^2(\arg(z + z_0)) + \langle \text{Re}(z) \rangle^{-1/2},$$

$$|\bar{\partial}R_j(z)| \lesssim |\bar{\partial}X_2(z)| + |p'_j(\text{Re}z)| + |z + z_0|^{-1/2},$$

where

$$p_1(z) = p_7(z) = r(z), \quad p_3(z) = p_8(z) = \frac{r(z)}{1 + |r(z)|^2},$$

$$p_4(z) = p_9(z) = \frac{r(z)}{1 + |r(z)|^2}, \quad p_6(z) = p_{10}(z) = \bar{r}(z).$$

And

$$\bar{\partial}R_j(z) = 0, \quad \text{if } z \in \Omega_2 \cup \Omega_5 \text{ or dist}(z, \mathcal{Z} \cup \bar{\mathcal{Z}}) < \mu/3. \quad (5.25)$$

The proof of above proposition is similar to that in [40, 42]. In addition, from the definition of $k(z)$ in (4.3) and the symmetry of $r(z)$, we have that $k(z_0) = k(-z_0)$.

We now use $R^{(2)}$ to define the transformation (5.9), which satisfies the following mixed $\bar{\partial}$-RH problem.

**RHP4.** Find a matrix valued function $M^{(2)}(z; y, t)$ with following properties:

- **Analyticity:** $M^{(2)}(z; y, t)$ is continuous in $\mathbb{C}$, sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \bar{\mathcal{Z}})$ and meromorphic in $\Omega_2 \cup \Omega_5$;

- **Symmetry:** $M^{(2)}(z) = M^{(2)}(-z) = \sigma_2 M^{(2)}(z) \sigma_2$;

- **Asymptotic behaviours:**

$$M^{(2)}(z; y, t) = I + O(z^{-1}), \quad z \to \infty; \quad (5.26)$$

- **Jump condition:** $M^{(2)}$ has continuous boundary values $M^{(2)}_\pm$ on $\Sigma^{(2)}$ and

$$M^{(2)}_+(z; y, t) = M^{(2)}_-(z; y, t) V^{(2)}(z), \quad z \in \Sigma^{(2)}, \quad (5.27)$$

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where take $\eta = -1$ as an example, we have

$$V^{(2)}(z) = \begin{cases} 
\begin{pmatrix} 1 & R_1(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_9, \\
\begin{pmatrix} 1 & R_3(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_2, \\
\begin{pmatrix} 1 & R_4(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_3, \\
\begin{pmatrix} 1 & R_6(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_4 \cup \Sigma_{12}, \\
\begin{pmatrix} 1 & R_7(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_6 \cup \Sigma_{10}, \\
\begin{pmatrix} 1 & R_8(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_5, \\
\begin{pmatrix} 1 & R_9(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_8, \\
\begin{pmatrix} 1 & R_{10}(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_7 \cup \Sigma_{11}; 
\end{cases}$$

- $\bar{\partial}$-Derivative: For $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \bar{\mathcal{Z}})$ we have

$$\bar{\partial}M^{(2)} = M^{(1)}\bar{\partial}R^{(2)},$$

where

$$\bar{\partial}R^{(2)} = \begin{cases} 
\begin{pmatrix} 0 & (-1)^{m_i}\bar{\partial}R_j(z)e^{2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_j, j = 1, 4, 7, 9, \\
\begin{pmatrix} 0 & (-1)^{m_i}\bar{\partial}R_j(z)e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_j, j = 3, 6, 8, 10, \\
0 & 0, & z \in \Omega_2 \cup \Omega_5; 
\end{cases}$$

- Residue conditions: $M^{(2)}$ has simple poles at each point in $\mathcal{Z} \cup \bar{\mathcal{Z}}$ with:

1. When $n \in \Delta_{\mathcal{Z}_n}^{(2)}$,

$$\text{Res}_{z = z_n} M^{(2)}(z) = \lim_{z \to z_n} M^{(2)}(z) \begin{pmatrix} 0 & c_nT(z_n)^{-2}e^{-2it\theta_n} \\ 0 & 0 \end{pmatrix},$$

$$\text{Res}_{z = \bar{z}_n} M^{(2)}(z) = \lim_{z \to \bar{z}_n} M^{(2)}(z) \begin{pmatrix} 0 & c_nT(z_n)^{2}e^{2it\theta_n} \\ 0 & 0 \end{pmatrix}.$$
2. When \( n \in \Delta^+_{z_0, \eta} \),

\[
\text{Res}_{z = z_n} M^{(2)}(z) = \lim_{z \to z_n} M^{(2)}(z) \begin{pmatrix}
0 & c_n(1/T)'(z_n)^{-2}e^{2\theta_n t} \\
c_nT'(z_n)^{-2}e^{-2\theta_n t} & 0
\end{pmatrix},
\]

(5.33)

\[
\text{Res}_{z = \bar{z}_n} M^{(2)}(z) = \lim_{z \to \bar{z}_n} M^{(2)}(z) \begin{pmatrix}
0 & \bar{c}_nT'(ar{z}_k)^{-2}e^{2\bar{\theta}_n t} \\
\bar{c}_nT'(ar{z}_k)^{-2}e^{-2\bar{\theta}_n t} & 0
\end{pmatrix}.
\]

(5.34)

6 Decomposition of the mixed \( \partial - \text{RH} \) problem

To solve RHP4, we decompose it into a model RH Problem for \( M^{RHP}(z; y, t) \) with \( \partial R^{(2)} = 0 \) and a pure \( \partial \)-Problem with \( \partial R^{(2)} \neq 0 \). For the first step, we establish a RH problem for the \( M^{RHP}(z; y, t) \) as follows.

\textbf{RHP5.} Find a matrix-valued function \( M^{RHP}(z; y, t) \) with following properties:

- Analyticity: \( M^{RHP}(z; y, t) \) is analytical in \( \mathbb{C} \setminus (\Sigma^{(2)} \cup Z \cup \bar{Z}) \);
- Jump condition: \( M^{RHP} \) has continuous boundary values \( M^{RHP}_\pm \) on \( R \) and

\[
M^{RHP}_+(z) = M^{RHP}_-(z)V^{(2)}(z), \quad z \in R;
\]

(6.1)

- Symmetry: \( M^{RHP}(\bar{z}) = M^{RHP}(-z) = \sigma_2 M^{RHP}(z) \sigma_2 \);
- \( \partial \)-Derivative: \( \partial R^{(2)} = 0 \), for \( z \in \mathbb{C} \);
- Asymptotic behaviours:

\[
M^{RHP}(z; y, t) = I + \mathcal{O}(z^{-1}), \quad z \to \infty;
\]

(6.2)

- Residue conditions: \( M^{RHP} \) has simple poles at each point in \( Z \cup \bar{Z} \) with:

1. When \( n \in \Delta^+_{z_0, \eta} \),

\[
\text{Res}_{z = z_n} M^{RHP}(z) = \lim_{z \to z_n} M^{RHP}(z) \begin{pmatrix}
0 & c_nT(z_n)^{-2}e^{-2\theta_n t} \\
c_nT(z_n)^{-2}e^{2\theta_n t} & 0
\end{pmatrix},
\]

(6.3)

\[
\text{Res}_{z = \bar{z}_n} M^{RHP}(z) = \lim_{z \to \bar{z}_n} M^{RHP}(z) \begin{pmatrix}
0 & \bar{c}_nT'(ar{z}_k)^{-2}e^{2\bar{\theta}_n t} \\
\bar{c}_nT'(ar{z}_k)^{-2}e^{-2\bar{\theta}_n t} & 0
\end{pmatrix}.
\]

(6.4)

2. When \( n \in \Delta^-_{z_0, \eta} \),

\[
\text{Res}_{z = z_n} M^{RHP}(z) = \lim_{z \to z_n} M^{RHP}(z) \begin{pmatrix}
0 & c_n(1/T)'(z_n)^{-2}e^{2\theta_n t} \\
c_n(1/T)'(z_n)^{-2}e^{-2\theta_n t} & 0
\end{pmatrix},
\]

(6.5)

\[
\text{Res}_{z = \bar{z}_n} M^{RHP}(z) = \lim_{z \to \bar{z}_n} M^{RHP}(z) \begin{pmatrix}
0 & \bar{c}_nT'(ar{z}_k)^{-2}e^{-2\bar{\theta}_n t} \\
\bar{c}_nT'(ar{z}_k)^{-2}e^{2\bar{\theta}_n t} & 0
\end{pmatrix}.
\]

(6.6)

The existence and asymptotic of \( M^{RHP}(z) \) will shown in section 8.
We now use $M^{RHP}(z)$ to construct a new matrix function

$$M^{(3)}(z) = M^{(2)}(z)M^{RHP}(z)^{-1}. \quad (6.7)$$

which removes analytical component $M^{RHP}$ to get a pure $\bar{\partial}$-problem.

**RHP6.** Find a matrix-valued function $M^{(3)}(z; y, t)$ with following properties:

- **Analyticity:** $M^{(3)}(z; y, t)$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \overline{\mathcal{Z}})$ and meromorphic in $\Omega_2 \cup \Omega_5$.
- **Symmetry:** $M^{(3)}(\bar{z}) = M^{(3)}(-z) = \sigma_2 M^{(3)}(z) \sigma_2$;
- **Asymptotic behavior:**
  $$M^{(3)}(z; y, t) \sim I + \mathcal{O}(z^{-1}), \quad z \to \infty; \quad (6.8)$$
- **$\bar{\partial}$-Derivative:** For $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \overline{\mathcal{Z}})$ we have $\bar{\partial}M^{(3)} = M^{(3)}W^{(3)}$,
  $$W^{(3)} = M^{RHP}(z)\bar{\partial}R^{(2)}M^{RHP}(z)^{-1}. \quad (6.9)$$

**Proof.** By using properties of the solutions $M^{(2)}$ and $M^{RHP}$ for RHP4 and RHP5, the analyticity and asymptotics are obtained immediately. Since $M^{(2)}$ and $M^{RHP}$ have same jump matrix, we have

$$M^{(3)}(z; y, t) = M^{(2)}(z; y, t)M^{RHP}(z)^{-1} = I,$$

which means $M^{(3)}$ has no jumps and is everywhere continuous. We also can show that $M^{(3)}$ has no pole. For $\lambda \in \mathcal{Z} \cup \overline{\mathcal{Z}}$, let $\mathcal{N}$ denote the nilpotent matrix which appears in the left side of the corresponding residue condition of RHP4 and RHP5, we have the Laurent expansions in $z - \lambda$

$$M^{(2)}(z) = a(\lambda) \left[ \frac{\mathcal{N}}{z - \lambda} + I \right] + \mathcal{O}(z - \lambda),$$

$$M^{RHP}(z) = A(\lambda) \left[ \frac{\mathcal{N}}{z - \lambda} + I \right] + \mathcal{O}(z - \lambda),$$

where $a(\lambda)$ and $A(\lambda)$ are the constant row vector and matrix in their respective expansions. Then from $M^{RHP}(z)^{-1} = \sigma_2 M^{RHP}(z)^T \sigma_2$, we have

$$M^{(3)}(z) = \left\{ a(\lambda) \left[ \frac{\mathcal{N}}{z - \lambda} + I \right] \right\} \left\{ \left[ \frac{-\mathcal{N}}{z - \lambda} + I \right] \sigma_2 A(\lambda)^T \sigma_2 \right\} + \mathcal{O}(z - \lambda)$$

$$= \mathcal{O}(1), \quad (6.10)$$
which implies that $M^{(3)}$ has removable singularities at $\lambda$. And the $\bar{\partial}$-derivative of $M^{(3)}$ come from $M^{(3)}$ due to analyticity of $M^{RHP}$.

We construct the solution $M^{RHP}$ of the RHP5 in the following form

$$M^{RHP} = \begin{cases} 
E(z)M^{(\text{out})}(z) & z \notin U_{\pm z_0}, \\
E(z)M^{(\pm z_0)}(z) & z \in U_{\pm z_0}, \\
E(z)M^{(-\pm z_0)}(z) & z \in U_{-\pm z_0},
\end{cases} \tag{6.11}$$

where $U_{\pm z_0}$ are the neighborhoods of $\pm z_0$, respectively

$$U_{\pm z_0} = \left\{ z : |z \mp z_0| \leq \min \left\{ \frac{z_0}{2}, \frac{\mu}{3} \right\} \right\} . \tag{6.12}$$

This implies that $M^{RHP}$ and $M^{(\pm z_0)}$ have no poles in $U_{\pm z_0}$, since $\text{dist}(Z, \mathbb{R}) > \mu$. This decomposition splits $M^{RHP}$ into two parts: $M^{(\text{out})}$ solves a model RHP obtained by ignoring the jump conditions of RHP5, which will be solved in next Section 7. While $M^{(\pm z_0)}$, whose solution can be approximated with parabolic cylinder functions if we let $M^{(\pm z_0)}$ exactly match to the $M^{(2)}$ and a parabolic cylinder model in $U_{\pm z_0}$, these results will given in Section 8. And $E(z)$ is a error function, which is a solution of a small-norm RH problem and we discuss it in Section 9.

And from the RHP5, whose jump matrix admits the following estimates.

**Proposition 4.** For the jump matrix $V^{(2)}(z)$, we have the following estimate

$$\| V^{(2)} - I \|_{L^\infty(\Sigma_0^{(2)})} = O(e^{-\frac{2\mu}{3}|z|}), \tag{6.13}$$

$$\| V^{(2)} - I \|_{L^\infty(\Sigma_0^{(2)})} = O(e^{-\frac{\mu}{3}|z|}), \tag{6.14}$$

where the contours are defined by

$$\Sigma_1^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4, \quad \Sigma_2^{(2)} = \Sigma_5 \cup \Sigma_6 \cup \Sigma_7 \cup \Sigma_8,$$

$$\Sigma_0^{(2)} = \Sigma_10 \cup \Sigma_{11} \cup \Sigma_{12} \cup \Sigma_9.$$

**Proof.** We prove (6.13) for the case when $\eta = +1$ and $z \in \Sigma_9$, other cases can be shown in a similar way. By using definition of $V^{(2)}$ and (5.19), we have

$$\| V^{(2)} - I \|_{L^\infty(\Sigma_0)} \approx \| R_1 e^{2it\theta} \|_{L^\infty(\Sigma_0)} . \tag{6.15}$$

Note that $|z| < \sqrt{2}z_0/2$ for $z \in \Sigma_9$, together with (4.11), we find that

$$|e^{2it\theta}| = e^{-2t\Im(z)(\xi + \frac{1}{4z^2})} \leq e^{-2t|z|} \to 0, \quad \text{as } t \to +\infty,$$

which together with (6.15) gives (6.14). And the calculation of $\Sigma_0^{(2)}$ is similar. \qed
This proposition means that the jump matrix $V^{(2)}$ uniformly goes to $I$ on both $\Sigma_{\pm z_0}$ and $\Sigma_{0}$, so outside the $U_{z_0} \cup U_{-z_0}$ there is only exponentially small error (in $t$) by completely ignoring the jump condition of $M^{RHP}$. And note that unlike the neighborhood of $\pm z_0$, $V^{(2)} \to I$ as $z \to 0$, it has uniformly property. So we doesn’t need to consider the neighborhood of $z = 0$ alone.

7 Outer model RH problem

In this section, we build a outer model RH problem and show that its solution can approximated with a finite sum of soliton solutions. Note that from the reconstruct formula (3.17), we only need the property of $M^{(out)}$ as $z \to 0$. We can introduce following outer model problem.

**RHP7.** Find a matrix-valued function $M^{(out)}(z; y, t)$ with following properties:

- **Analyticity:** $M^{(out)}(z; y, t)$ is analytical in $\mathbb{C} \setminus (\Sigma^{(2)} \cup Z \cup \bar{Z})$;
- **Symmetry:** $\overline{M^{(out)}(\bar{z})} = M^{(out)}(-z) = \sigma_2 M^{(out)}(z) \sigma_2$;
- **Asymptotic behaviours:**
  \[
  M^{(out)}(z; y, t) \sim I + O(z^{-1}), \quad z \to \infty;
  \]
  (7.1)
- **Residue conditions:** $M^{(out)}$ has simple poles at each point in $Z \cup \bar{Z}$ satisfying the same residue relations (6.3)-(6.6) with $M^{RHP}(z)$.

Before showing the existence and uniqueness of solution of the above RHP7, we first consider the reflectionless case of the RHP1. In this case, $M$ has no contour, the RHP1 reduces to the following RH problem.

**RHP8.** Given discrete data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N$, and $Z = \{z_k\}_{k=1}^N$. Find a matrix-valued function $m(z; y, t|\sigma_d)$ with following properties:

- **Analyticity:** $m(z; y, t|\sigma_d)$ is analytical in $\mathbb{C} \setminus (\Sigma^{(2)} \cup Z \cup \bar{Z})$;
- **Symmetry:** $\overline{m(z; y, t|\sigma_d)} = m(-z; y, t|\sigma_d) = \sigma_2 m(z; y, t|\sigma_d) \sigma_2$;
- **Asymptotic behaviours:**
  \[
  m(z; y, t|\sigma_d) \sim I + O(z^{-1}), \quad z \to \infty;
  \]
  (7.2)
- **Residue conditions:** $m(z; y, t|\sigma_d)$ has simple poles at each point in $Z \cup \bar{Z}$ satisfying
  \[
  \text{Res}_{z=z_0} m(z; y, t|\sigma_d) = \lim_{z \to z_0} m(z; y, t|\sigma_d) \tau_k, \quad (7.3)
  \]
  \[
  \text{Res}_{z=\bar{z}_0} m(z; y, t|\sigma_d) = \lim_{z \to \bar{z}_0} m(z; y, t|\sigma_d) \bar{\tau}_k, \quad (7.4)
  \]
where $\tau_k$ is a nilpotent matrix satisfies
\[
\tau_k = \begin{pmatrix} 0 & \gamma_k \\ 0 & 0 \end{pmatrix}, \quad \hat{\tau}_k = \sigma_2 \tau_k \sigma_2, \quad \gamma_k = c_k e^{-2 t \theta_k}.
\] (7.5)

Moreover, the solution satisfies
\[
\| m(z; y, t|\sigma_d)^{-1} \|_{L^\infty(C \setminus (\mathbb{Z} \cup \bar{\mathbb{Z}}))} \lesssim 1.
\] (7.6)

**Proposition 5.** The RHP8 exists an unique solution.

**Proof.** The uniqueness of solution follows from the Liouville’s theorem. The symmetries of $m(z; y, t|\sigma_d)$ means that it admits a partial fraction expansion of following form
\[
m(z; y, t|\sigma_d) = I + \sum_{k=1}^{N} \left[ \frac{1}{z - z_k} \begin{pmatrix} \nu_k(y, t) & 0 \\ 0 & \varsigma_k(y, t) \end{pmatrix} + \frac{1}{z - \bar{z}_k} \begin{pmatrix} \varsigma_k(y, t) & 0 \\ -\nu_k(y, t) & 0 \end{pmatrix} \right].
\] (7.7)

By using a similar way to Appendix B in [?], we can show the existence of the solution for the RHP8. Since $\det(m(z; y, t|\sigma_d))=1$, $\| m(z; y, t|\sigma_d) \|_{L^\infty(C \setminus (\mathbb{Z} \cup \bar{\mathbb{Z}}))}$ is bounded. And from (7.7), we simply obtain (7.6).

In reflectionless case, the transmission coefficient admits following trace formula
\[
a(z) = \prod_{k=1}^{N} \frac{z - z_k}{z - \bar{z}_k},
\] (7.8)
whose poles can be split into two parts. Let $\triangle \subseteq \{1, 2, ..., N\}$, and define
\[
a_\triangle(z) = \prod_{k \in \triangle} \frac{z - z_k}{z - \bar{z}_k},
\]
we make a renormalization transformation
\[
m_\triangle(z|D) = m(z|\sigma_d) a_\triangle(z)^{-\sigma_3},
\] (7.9)
where the scattering data are given by
\[
D = \{(z_k, c'_k)\}_{k=1}^{N}, \quad c'_k = c_k a_\triangle(z)^2.
\] (7.10)

It is easy to see that the transformation (7.9) splits the poles between the columns of $m_\triangle(z|D)$ according to the choice of $\triangle$, and it satisfies the following modified discrete RH problem.
RHP9. Given discrete data (7.10), find a matrix-valued function $m^\Delta(z; y, t|D)$ with following properties:

- Analyticity: $m^\Delta(z; y, t|D)$ is analytical in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{\bar{Z}})$;
- Symmetry: $m^\Delta(z; y, t|D) = \sigma_2 m^\Delta(-\bar{z}; y, t|D) \sigma_2 = m^\Delta(-\bar{z}; y, t|D)$;
- Asymptotic behaviours:
  $$m^\Delta(z; y, t|D) \sim I + O(z^{-1}), \quad z \to \infty;$$
  \hspace{1cm} (7.11)
- Residue conditions: $m(z; y, t|\sigma_d)$ has simple poles at each point in $\mathcal{Z} \cup \mathcal{\bar{Z}}$ satisfying
  $$\text{Res}_{z = z_k} m^\Delta(z; y, t|D) = \lim_{z \to z_k} \frac{m^\Delta(z; y, t|D)}{z - z_k} = \gamma_k e^{-2it\theta_k}, \quad z \in \Delta,$$
  \hspace{1cm} (7.14)

Since (7.9) is a explicit transformation of $m(z; y, t|\sigma_d)$, by Proposition 3, we obtain the existence and uniqueness of the solution of the RHP9.

In the RHP9, take $\Delta = \Delta_{zn, \eta}$ and replace the scattering data $D$ with scattering data

$$\hat{D} = \{(z_k, \tilde{c}_k)\}_{k=1}^N, \quad \tilde{c}_k = c_k \delta(z_k)^2,$$

\hspace{1cm} (7.15)

then we have

**Corollary 1.** There exists and unique solution for the RHP7, moreover,

$$M^{(\text{out})}(z; y, t) = m^{\Delta_{zn, \eta}}(z; y, t|\hat{D}),$$

where scattering data $\hat{D}$ is given by (7.15).

If $u_{\text{sol}}(y, t) = u_{\text{sol}}(y, t; D)$ denotes the $N$-soliton solution of the SP equation encoded by the RHP8, by using (7.9), we also have the reconstruction formula

$$u_{\text{sol}}(x, t|D) = u_{\text{sol}}(y(x, t), t|D) = \lim_{z \to 0} \frac{\left(m^\Delta(0; y, t|D)^{-1} m^\Delta(z; y, t|D)\right)}{iz}|_{z}, \quad \text{as } t \to 0^+,$$

\hspace{1cm} (7.16)
which show that each normalization encodes $u_{sol}(y,t)$ in the same way. If we choosing $\triangle$ appropriately, the asymptotic limits $|t| \to \infty$ with $\xi = y/t$ bounded are under better asymptotic control. Then we consider the long-time behavior of soliton solutions.

Give pairs points $y_1 \leq y_2 \in \mathbb{R}$ and velocities $v_1 \leq v_2 \in \mathbb{R}^-$, we define a cone

$$C(y_1, y_2, v_1, v_2) = \{(y, t) \in \mathbb{R}^2 | y = y_0 + vt, y_0 \in [y_1, y_2], v \in [v_1, v_2]\}.$$ \hspace{9.5cm} (7.17)

and denote

$$I = \left\{ z : -\frac{1}{4v_1} < |z|^2 < -\frac{1}{4v_2} \right\},$$

$$Z(I) = \{ z_k \in Z : z_k \in I \}, \quad N(I) = |Z(I)|,$$

$$Z^-(I) = \left\{ z_k \in Z : |z|^2 > -\frac{1}{4v_2} \right\}, \quad Z^+(I) = \left\{ z_k \in Z : |z|^2 < -\frac{1}{4v_1} \right\},$$

$$c_k(I) = c_k \prod_{\text{Re}z_n \in I_n \setminus I} \left( \frac{z_k - z_n}{z_k - \bar{z}_n} \right)^2 \exp \left[ -\frac{1}{\pi i} \int_{I_n} \log[1 + |\tau(z)|^2] \frac{d\tau}{z - \bar{z}} \right]. \hspace{9.5cm} (7.18)$$

We can show the following lemma.

**Lemma 1.** Fix reflectionless data $D = \{(z_k, c_k')\}_{k=1}^N, D(I) = \{(z_k, c_k'(I)) | z_k \in Z(I)\}$. Then for $|t| \to \infty$ with $(y,t) \in C(y_1, y_2, v_1, v_2)$, we have

$$m^{\Delta \eta_0, \eta}(z; y, t|D) = \left( I + \mathcal{O}(e^{-2\mu(I)|t|}) \right) m^{\Delta \eta_0, \eta}(z; y, t|D(I)), \hspace{9.5cm} (7.19)$$

Figure 3: (a) In the example here, the original data has four pairs zero points of discrete spectrum, but inside the cone $C$ only three pairs points with $Z(I) = \{z_1\}$; (b) The cone $C(y_1, y_2, v_1, v_2)$

**Lemma 1.** Fix reflectionless data $D = \{(z_k, c_k')\}_{k=1}^N, D(I) = \{(z_k, c_k'(I)) | z_k \in Z(I)\}$. Then for $|t| \to \infty$ with $(y,t) \in C(y_1, y_2, v_1, v_2)$, we have

$$m^{\Delta \eta_0, \eta}(z; y, t|D) = \left( I + \mathcal{O}(e^{-2\mu(I)|t|}) \right) m^{\Delta \eta_0, \eta}(z; y, t|D(I)), \hspace{9.5cm} (7.19)$$

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where \( \mu(I) = \min_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(I)} \left\{ \text{Re}(z_k) \frac{\overline{\Delta}}{\sqrt{2\nu_2}} (|z| + \frac{1}{2\sqrt{\nu_2}}) \right\} \).

**Proof.** We denote

\[
\triangle^+(I) = \{ k | \text{Re}(z_k) < -v_2/4 \}, \quad \triangle^-(I) = \{ k | \text{Re}(z_k) > -v_1/4 \}.
\]

and take \( \triangle = \triangle_{z_0, \eta} \) with \( \eta = \text{sgn}(t) \) in the RHP\( \phi \), then for \( z \in \mathcal{Z} \setminus \mathcal{Z}(I) \) and \( (y, t) \in C(y_1, y_2, v_1, v_2) \), by using the residue coefficients (7.14), direct calculation shows that

\[
|\gamma_k| = |c_k e^{2\text{Re}(z) y_0} |e^{2t\text{Re}(z)(\frac{\overline{\Delta}}{\sqrt{2\nu_2}} + v_0)}| = |c_k e^{2\text{Re}(z) y_0} |e^{-2t\text{Re}(z)(\frac{\overline{\Delta}}{\sqrt{2\nu_2}} + v_0)}| |e^{(\frac{1}{2\sqrt{\nu_2}})(|z| - \frac{1}{2\sqrt{\nu_2}})|} .
\]

which leads to

\[
\| \tau_k^{\pm}(I) \| = \mathcal{O}(e^{-2\mu(I)|t|}), \quad t \to \pm \infty. \quad (7.20)
\]

Suppose that \( D_k \) is a small disks centred in each \( z_k \in \mathcal{Z} \setminus \mathcal{Z}(I) \) with radius smaller than \( \mu \). Denote \( \partial D_k \) is the boundary of \( D_k \). Then we can introduce a new transformation which can remove the poles \( z_k \in \mathcal{Z} \setminus \mathcal{Z}(I) \) and these residues change to near-identity jumps.

\[
\overline{m}^{\triangle_{z_0, \eta}}(z_t; y, t | D) = \begin{cases} \ m^{\triangle_{z_0, \eta}}(z; y, t | D) \left( I - \frac{\text{sgn}(t)}{z - z_k} \right) & z \in D_k, \\ \ m^{\triangle_{z_0, \eta}}(z; y, t | D) \left( I - \frac{\text{sgn}(t)}{z - z_k} \right) & z \in \bar{D}_k, \\ \ m^{\triangle_{z_0, \eta}}(z; y, t | D) & \text{elsewhere}. \end{cases} \quad (7.21)
\]

Comparing with \( m^{\triangle_{z_0, \eta}} \), the new matrix function \( \overline{m}^{\triangle_{z_0, \eta}}(z; y, t | D) \) has new jump in each \( \partial D_k \) which denote by \( \overline{V}(z) \). Then using (7.20), we have

\[
\| \overline{V}(z) - I \|_{L_\infty(\mathcal{Z})} = \mathcal{O}(e^{-2\mu(I)|t|}), \quad \mathcal{Z} = \bigcup_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(I)} (\partial D_k \cup \partial \bar{D}_k) . \quad (7.22)
\]

Since \( \overline{m}^{\triangle_{z_0, \eta}}(z; y, t | D) \) has same poles and residue conditions with \( m^{\triangle_{z_0, \eta}}(z; y, t | D(I)) \), then

\[
m_0(z) = \overline{m}^{\triangle_{z_0, \eta}}(z; y, t | D) m^{\triangle_{z_0, \eta}}(z; y, t | D(I))^{-1}
\]

has no poles, but it has jump matrix for \( z \in \overline{\mathcal{Z}} \),

\[
m_0^{\pm}(z) = m_0(z) V_{m_0}(z), \quad (7.23)
\]

where the jump matrix \( V_{m_0}(z) \) given by

\[
V_{m_0}(z) = m(z | D(I)) \overline{V}(z) m(z | D(I))^{-1}, \quad (7.24)
\]

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which, by using (7.22), also admits the same decaying estimate
\[
\| V m_0(z) - I \|_{L^\infty(S)} = \| \tilde{V}(z) - I \|_{L^\infty(S)} = \mathcal{O}(e^{-2\mu(I)|t|}), \quad t \to \pm \infty.
\]

Then by using the theory of small norm RH problem [? ? ], we find that \( m_0(z) \) exists and
\[
m_0(z) = I + \mathcal{O}(e^{-2\mu(I)|t|}), \quad t \to \pm \infty,
\]
which together with (7.21) gives the formula (7.19). \( \square \)

Using reconstruction formula to \( m^{\Delta z_0,n}(z; y, t|D) \), we immediately obtain the following result.

**Corollary 2.** Let \( u_{\text{sol}}(y, t; D) \) and \( u_{\text{sol}}(y, t; D(I)) \) denote the \( N \)-soliton solution of (3.1) corresponding to discrete scattering data \( D \) and \( D(I) \) respectively. As \( |t| \to \infty \) with \( (y, t) \in C(y_1, y_2, v_1, v_2) \), we have
\[
\lim_{z \to 0} \left( m^{\Delta z_0,n}(0; y, t|D) - m^{\Delta z_0,n}(z; y, t|D) \right)_{11} = u_{\text{sol}}(y(x, t), t; D)
\]
\[
= u_{\text{sol}}(y(x, t), t; D(I)) + \mathcal{O}(e^{-\mu(I)|t|}).
\] (7.25)

By using (3.8), we have
\[
c_{+}(x, t; D) = \lim_{z \to 0} \left( m^{\Delta z_0,n}(0; y, t|D) - m^{\Delta z_0,n}(z; y, t|D) \right)_{11} - 1
\]
\[
= c_{+}(x, t; D(I)) + \mathcal{O}(e^{-\mu(I)|t|}).
\] (7.26)

Now we come back to the outer model and obtain the following result.

**Corollary 3.** The RHP7 exists an unique solution \( M^{(\text{out})} \) with
\[
M^{(\text{out})}(z) = m^{\Delta z_0,n}(z|D^{(\text{out})})
\]
\[
= m^{\Delta z_0,n}(z; y, t|D(I)) \prod_{\text{Res} z_n \in \mathbb{C}_0 \setminus I} \left( \frac{z - z_n}{z - \tilde{z}_n} \right)^{-\sigma_3} \delta^{-\sigma_3} + \mathcal{O}(e^{-\mu(I)|t|}),
\] (7.27)
where \( D^{(\text{out})} = \{ z_k, c_k(z_0) \}_{k=1}^N \) with
\[
c_k(\xi) = c_k \exp \left[ -\frac{1}{\pi i} \int_{\gamma} \frac{\log(1 + |r(\zeta)|^2)}{\zeta - z} d\zeta \right].
\]
Then substitute (7.27) into (7.6) we immediately have
\[
\| M^{(\text{out})}(z)^{-1} \|_{L^\infty(C\setminus(Z\cup\bar{Z}))} \lesssim 1. \tag{7.28}
\]
Moreover, we have reconstruction formula
\[
\lim_{z \to 0} \left( \frac{M^{(\text{out})}(0)^{-1} M^{(\text{out})}(z)}{iz} \right)_{12} = u_{\text{sol}}(y, t; D^{(\text{out})}), \tag{7.29}
\]
where the \( u_{\text{sol}}(y, t; D^{\text{out}}) \) is the \( N \)-soliton solution of (1.1) corresponding to discrete scattering data \( \tilde{D} \). And
\[
u_{\text{sol}}(y, t; D^{(\text{out})}) = u_{\text{sol}}(y, t; D(I)) + \mathcal{O}(e^{-\mu |I| t}), \quad \text{for } t \to \pm \infty. \tag{7.30}
\]

8 A local solvable RH model near phase points

From the Proposition 4 in the neighborhood \( U_{\pm z_0} \) of \( \pm z_0 \), we find that \( V^{(2)} - I \) doesn’t have a uniformly small jump for large time, so we establish a local model for function \( E(z) \) with a uniformly small jump.

For soliton-free case when there are no discrete spectrum, the formula (4.6) and (4.8) reduce to \( T_0(\pm z_0) = \delta(\pm z_0) \). The RHP5 exactly reduces to a solvable model for the SP equation [35].

RHP10. Let \( \Sigma^{(2)} \) be the same contour in the RHP5. Find a matrix-valued function \( M^{sp}(z; \eta) \) such that
- Analyticity: \( M^{sp}(z; \eta) \) is analytical in \( \mathbb{C} \setminus \Sigma^{(2)} \);
- Symmetry: \( M^{sp}(z; \eta = 1) = \sigma_2 M^{sp}(-z; \eta = -1) \sigma_2 \);
- Asymptotic behaviors:
  \[
  M^{sp}(z; \eta) \sim I + \mathcal{O}(z^{-1}), \quad z \to \infty. \tag{8.1}
  \]
- Jump condition: \( M^{sp}(z; \eta) \) has continuous boundary values \( M^{sp}_\pm(z; \eta) \) on \( \Sigma^{(2)} \) and
  \[
  M^{sp}_+(z; \eta) = M^{sp}_-(z; \eta) V^{(sp)}(z; \eta), \quad z \in \Sigma^{(2)}, \tag{8.2}
  \]
where the jump matrix $V^{(sp)}(z, \eta)$, taking $\eta = -1$ as an example, is given by

$$
V^{(sp)}(z, \eta = -1) = \begin{cases} 
\begin{pmatrix} 1 & r(z_0) \delta^{-2}(z_0)(z - z_0)^{2i \kappa} e^{2i \theta(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_9, \\
\frac{\bar{r}(z_0) \delta^2(z_0)}{1 + |r(z_0)|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_2, \\
\begin{pmatrix} 1 & r(-z_0) \delta^{-2}(-z_0)(z + z_0)^{2i \kappa} e^{2i \theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_3, \\
\frac{\bar{r}(-z_0) \delta^2(-z_0)}{1 + |r(-z_0)|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_4 \cup \Sigma_{12}, \\
\begin{pmatrix} 1 & r(z_0) \delta^{-2}(z_0)(z - z_0)^{-2i \kappa} e^{-2i \theta(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_6 \cup \Sigma_{10}, \\
\frac{\bar{r}(z_0) \delta^2(z_0)}{1 + |r(z_0)|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_5, \\
\begin{pmatrix} 1 & r(-z_0) \delta^{-2}(-z_0)(z + z_0)^{-2i \kappa} e^{-2i \theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_8, \\
\frac{\bar{r}(-z_0) \delta^2(-z_0)}{1 + |r(-z_0)|^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_7 \cup \Sigma_{11}.
\end{cases}
$$

The proposition shows that the jump matrix $V^{(sp)}$ uniformly goes to $I$ outside the neighborhood of $\pm z_0$, then following the result in [32], the above RHP10 is solvable. The main contribution to the $M^{sp}(z; \eta)$ comes from a local RH problem near $\pm z_0$, see Figure [32]. We simply describe the process of construction for the solution of the RHP10, see [32] for the detail. We decompose the jump matrix $V^{(sp)} = (b_-)^{-1}b_+$, and set

$$w_\pm = \pm(b_\pm - I), \quad w = w_+ + w_-,$$

and let $\mu(z)$ is the solution of the operator equation $\mu(z) = I + C_w \mu(z)$, here $C_w$ is defined by

$$C_w f = C_+(fw_-) + C_-(fw_+),$$

with $C_\pm$ denoting the Cauchy projection operators. In the same way, we can define matrix functions $\mu_{\pm z_0}(z)$ and $w_{\pm z_0}(z)$ by using the jumps near phase points $\pm z_0$, then the solution
of the RHP10 is given by

\[ M^{sp}(z; \eta) = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}_{-}} \frac{\mu(s)w(s)}{s - z} ds \]

\[ = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}_{+}} \frac{\mu_{z_0}(s)w(s)}{s - z} ds + \frac{1}{2\pi i} \int_{\Sigma^{(2)}_{-}} \frac{\mu_{-z_0}(s)w(s)}{s - z} ds + O(|t|^{-1} \log |t|) \]

\[ = I + \frac{|t|^{-1/2}}{z - z_0} A(z_0, \eta) - \frac{|t|^{-1/2}}{z + z_0} A(-z_0, \eta) + O(|t|^{-1} \ln |t|), \quad (8.4) \]

Figure 4: The jump contour for the local RHP near phase points \( \pm z_0 \).

where

\[ A(z_0, \eta) = \left( \begin{array}{cc} 0 & -A_{12}^\eta \\ A_{21}^\eta & 0 \end{array} \right), \quad A(-z_0, \eta) = \left( \begin{array}{cc} 0 & -A_{21}^\eta \\ A_{12}^\eta & 0 \end{array} \right), \quad (8.5) \]

with

\[ A_{12}^+ = iz_0^{3/2} \beta_{12}, \quad A_{21}^+ = iz_0^{3/2} \beta_{21}, \quad A_{12}^- = -iz_0^{3/2} \beta_{21}, \quad A_{21}^- = -iz_0^{3/2} \beta_{12}, \quad (8.6) \]

\[ \beta_{12} = \frac{\sqrt{2\pi e^{-\kappa}}}{\kappa} i^{\kappa/4} e^{i\kappa/4}, \quad \beta_{21} = \frac{\beta_{12}}{\kappa}, \quad \arg(\beta_{12}) = \frac{\pi}{4} + \arg(\Gamma(-i\kappa)) + \arg(r_0). \quad (8.7) \]

In addition, it is shown that \( \| M^{sp}(z; \eta) \|_{\infty} \lesssim 1 \).

It is easy to check that the RHP5 and RHP10 have the same contour and jump matrices, we use \( M^{sp}(z; \eta) \) to define a local model in two circles \( z \in U_{\pm z_0} \)

\[ M^{(\pm z_0)}(z) = M^{(out)}(z) M^{sp}(z; \eta), \quad (8.8) \]

which is a bounded function in \( U_{\pm z_0} \) and has the same jump matrix as \( M^{RHP}(z, \eta) \).
9 The small norm RH problem for error function

In this section, we consider the error matrix-function $E(z)$. From the definition and , we can obtain a RH problem for the matrix function $E(z)$.

**RHP11.** Find a matrix-valued function $E(z)$ with following properties:

- **Analyticity:** $E(z)$ is analytical in $\mathbb{C} \setminus (\Sigma(E))$, where
  \[ \Sigma(E) = \partial U_{z_0} \cup \partial U_{-z_0} \cup (\Sigma(2) \setminus (U_{z_0} \cup U_{-z_0})); \]

- **Symmetry:** $E(\bar{z}) = E(-z) = \sigma_2 E(z) \sigma_2^{-1}$;

- **Asymptotic behaviours:**
  \[ E(z) \sim I + O(z^{-1}), \quad |z| \to \infty; \]  
  \[ (9.1) \]

- **Jump condition:** $E$ has continuous boundary values $E_{\pm}$ on $\Sigma(E)$ satisfying
  \[ E_+(z) = E_-(z)V^{(E)}; \]

where the jump matrix $V^{(E)}$ is given by

\[ V^{(E)}(z) = \begin{cases} 
    M^{(out)}(z)V^{(2)}(z)M^{(out)}(z)^{-1}, & z \in \Sigma(2) \setminus U_{\pm z_0}, \\
    M^{(out)}(z)M^{(sp)}(z)M^{(out)}(z)^{-1}, & z \in \partial U_{\pm z_0}, 
\end{cases} \]  
\[ (9.2) \]

which is shown in Figure 5.

We will show that for large times, the error function $E(z)$ solves following small norm RH problem.

![Diagram](image-url)

**Figure 5:** The jump contour $\Sigma(E)$ for the $E(z)$.
By using (7.28) and Proposition 4, we have the following estimates

$$|V^{(E)} - I| \lesssim \begin{cases} 
\exp \left\{ -t \frac{\pi}{2z_0} |z \mp z_0| \right\}, & z \in \Sigma^{(2)} \setminus U_{\pm z_0}, \\
\exp \left\{ \frac{|t|}{4\pi} \right\}, & z \in \Sigma^{(2)}_0.
\end{cases} \quad (9.3)$$

For $z \in \partial U_{\pm z_0}$, $M^{(out)}(z)$ is bounded, so by using (8.4), we find that

$$|V^{(E)} - I| = |M^{(out)}(z)^{-1}(M^{sp}(z) - I)M^{(out)}(z)| = O(|t|^{-1/2}). \quad (9.4)$$

Therefore, the existence and uniqueness of the RHP11 can shown by using a small-norm RH problem [29?], and we have

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^{(E)}} \frac{(I + \rho(s))(V^{(E)} - I)}{s - z} ds, \quad (9.5)$$

where the $\rho \in L^2(\Sigma^{(E)})$ is the unique solution of following equation:

$$(1 - C_E)\rho = C_E (I), \quad (9.6)$$

where $C_E$ is a integral operator defined by

$$C_E(f)(z) = C_-(f(V^{(E)} - I)), \quad (9.7)$$

where the $C_-$ is the usual Cauchy projection operator on $\Sigma^{(E)}$

$$C_-(f)(s) = \lim_{z \to \Sigma^{(E)}} \frac{1}{2\pi i} \int_{\Sigma^{(E)}} \frac{f(s)}{s - z} ds. \quad (9.8)$$

Then by (9.4) we have

$$\| C_E \| \leq \| C_- \| \| V^{(E)} - I \|_\infty \lesssim O(t^{-1/2}), \quad (9.9)$$

which means $\| C_E \| < 1$ for sufficiently large t, therefore $1 - C_E$ is invertible, and $\rho$ exists and is unique. Moreover,

$$\| \rho \|_{L^2(\Sigma^{(E)})} \lesssim \frac{\| C_E \|}{1 - \| C_E \|} \lesssim |t|^{-1/2}. \quad (9.10)$$

Then we have the existence and boundedness of $E(z)$. In order to reconstruct the solution $u(y, t)$ of (1.1), we need the asymptotic behavior of $E(z)$ as $z \to 0$ and the long time asymptotic behavior of $E(0)$. Note that when we estimate its asymptotic behavior, from (9.5) and (9.3) we only need to consider the calculation on $\partial U_{\pm z_0}$ because it approach zero exponentially on other boundary.
Proposition 6. As $z \to 0$, we have

$$E(z) = E(0) + E_1 z + O(z^2),$$

where

$$E(0) = I + \frac{1}{2\pi i} \int_{\Sigma(E)} \frac{(I + \rho(s)) (V^{(E)} - I)}{s} ds,$$

with long time asymptotic behavior

$$E(0) = I + |t|^{-1/2} H^{(0)} + O(|t|^{-1}),$$

and

$$H^{(0)} = \frac{1}{2\pi i} \int_{\partial U \pm z_0} \frac{M^{(out)}(s)^{-1} A(\pm z_0, \eta) M^{(out)}(s)}{s(\pm s - z_0)} ds$$

$$= \frac{1}{z_0} M^{(out)}(z_0)^{-1} A(z_0, \eta) M^{(out)}(z_0) + \frac{1}{z_0} M^{(out)}(-z_0)^{-1} A(-z_0, \eta) M^{(out)}(-z_0).$$

The last equality follows from a residue calculation. Moreover,

$$E(0)^{-1} = I + O(|t|^{-1/2}).$$

And

$$E_1 = -\frac{1}{2\pi i} \int_{\Sigma(E)} \frac{(I + \rho(s)) (V^{(E)} - I)}{s^2} ds,$$

satisfying long time asymptotic behavior condition

$$E_1 = |t|^{-1/2} H^{(1)} + O(|t|^{-1}),$$

where

$$H^{(1)} = \frac{1}{2\pi i} \int_{\partial U \pm z_0} \frac{M^{(out)}(s)^{-1} A(\pm z_0, \eta) M^{(out)}(s)}{s^2(\pm s - z_0)} ds$$

$$= \frac{1}{z_0^2} M^{(out)}(z_0)^{-1} A(z_0, \eta) M^{(out)}(z_0) - \frac{1}{z_0^2} M^{(out)}(-z_0)^{-1} A(-z_0, \eta) M^{(out)}(-z_0).$$

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10 Analysis of the pure $\bar{\partial}$-Problem

Now we consider the proposition and the long time asymptotics behavior of $M^{(3)}$. The RHP6 of $M^{(3)}$ is equivalent to the integral equation

$$M^{(3)}(z) = I + \frac{1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}M^{(3)}(s)}{z-s} dm(s) = I + \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{z-s} dm(s),$$

where $m(s)$ is the Lebesgue measure on the $\mathbb{C}$. If we denote $C_z$ is the left Cauchy-Green integral operator,

$$fC_z(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(s)W^{(3)}(s)}{z-s} dm(s),$$

then above equation can be rewritten as

$$M^{(3)}(z) = I (I - C_z)^{-1}.$$  \hspace{1cm} (10.2)

To proof the existence of operator $(I - C_z)^{-1}$, we have following Lemma.

**Lemma 2.** The norm of the integral operator $C_z$ decay to zero as $t \to \infty$:

$$\| C_z \|_{L^\infty \to L^\infty} \lesssim |t|^{-1/6},$$

which implies that $(I - C_z)^{-1}$ exists.

**Proof.** For any $f \in L^\infty$,

$$\| fC_z \|_{L^\infty} \leq \| f \|_{L^\infty} \frac{1}{\pi} \int_{\mathbb{C}} \frac{|W^{(3)}(s)|}{|z-s|} dm(s) \lesssim \| f \|_{L^\infty} \frac{1}{\pi} \int_{\mathbb{C}} \frac{|\bar{\partial}R^{(2)}(s)|}{|z-s|} dm(s).$$  \hspace{1cm} (10.4)

So we only need to estimate the integral

$$\frac{1}{\pi} \int_{\mathbb{C}} \frac{|\bar{\partial}R^{(2)}(s)|}{|z-s|} dm(s).$$

We only show the case $\eta = -1$. For $\bar{\partial}R^{(2)}(s)$ is a piece-wise function, we prove the case in the region $\Omega_1$, the other regions are similar. By using (5.20), we have

$$\int_{\Omega_1} \frac{|\bar{\partial}R^{(2)}(s)|}{|z-s|} dm(s) \leq F_1 + F_2 + F_3,$$  \hspace{1cm} (10.5)
where

\[ F_1 = \int_0^{+\infty} \int_{z_0 + \nu}^{+\infty} \frac{\partial X_Z(s) e^{-\frac{v t}{2(z_0^2 + s^2)}}}{\sqrt{(u - x)^2 + (v - y)^2}} du e^{-2tv\xi} dv; \]

(10.6)

\[ F_2 = \int_0^{+\infty} \int_{z_0 + \nu}^{+\infty} \frac{p'_1(u) e^{-\frac{v t}{2(z_0^2 + s^2)}}}{\sqrt{(u - x)^2 + (v - y)^2}} du e^{-2tv\xi} dv; \]

(10.7)

\[ F_3 = \int_0^{+\infty} \int_{z_0 + \nu}^{+\infty} \frac{((u - z_0)^2 + v^2)^{-1/4} e^{-\frac{v t}{2(z_0^2 + s^2)}}}{\sqrt{(u - x)^2 + (v - y)^2}} du e^{-2tv\xi} dv. \]

(10.8)

and we denote \( s = u + vi, \ z = x + yi. \)

In the following calculation, we will use the inequality

\[ \| s - z \|^{-1} \| L^2(z_0, +\infty) = \int_{z_0}^{+\infty} \frac{1}{|v - y|} \left( \frac{(u - x)^2}{|v - y|} + 1 \right)^{-1} d \left( \frac{u - x}{|v - y|} \right) \leq \frac{\pi}{|v - y|}. \]

(10.9)

To deal with the absolute value sign, we suppose \( y > 0. \) In fact \( y < 0 \) we can directly remove the absolute value sign and use the same way to estimates it.

For \( F_1, \) noting that \( -\frac{vt}{2(z_0^2 + s^2)} \) is a monotonic decreasing function of \( u, \) so

\[ F_1 \leq \int_0^{+\infty} \| s - z \|^{-1} \| L^2(z_0, +\infty) \| \partial X_Z(s) \| L^2(z_0, +\infty) e^{-\frac{vt}{2(z_0^2 + s^2)}} e^{-2tv\xi} dv \]

\[ \leq \int_0^{+\infty} (y - v)^{-1/2} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv \]

\[ = \int_0^{y} (y - v)^{-1/2} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv \]

\[ + \int_y^{+\infty} (v - y)^{-1/2} \exp \left( -\frac{v|t|y^2}{2z_0^2(y^2 + z_0^2)} \right) dv. \]

(10.10)

For the first item, note that \( e^{-z} \leq z^{1/6} \) for all \( z > 0, \) then

\[ \int_0^{y} (y - v)^{-1/2} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv \]

\[ \leq \int_0^{y} (y - v)^{-1/2} v^{-1/2} dv \| t \|^{-1/4} \leq |t|^{-1/6}. \]

(10.11)

For the last integral we make the substitution \( w = v - y \) then we get

\[ \int_y^{+\infty} (v - y)^{-1/2} \exp \left( -\frac{w|t|y^2}{2z_0^2(y^2 + z_0^2)} \right) dw \]

\[ \leq \int_0^{+\infty} w^{-1/2} \exp \left( -\frac{w|t|y^2}{2z_0^2(y^2 + z_0^2)} \right) dw \exp \left( -\frac{|t|y^2}{2z_0^2(y^2 + z_0^2)} \right) \lesssim |t|^{-1/2}. \]

(10.12)
Substituting (10.12) and (10.11) into (10.10) gives

\[ F_1 \lesssim |t|^{-1/4}. \]  

(10.13)

The \( F_2 \) has the same estimate with (10.13). And for \( F_3 \), we first have that

\[
\| (u - z_0)^2 + v^2 \|_{L^p(z_0, +\infty)}^{-1/4} = \left\{ \int_{z_0}^{+\infty} \left[ (u - z_0)^2 + v^2 \right]^{-p/4} dv \right\}^{1/p} \]

(10.14)

and

\[
\| s - z \|_{L^q(z_0, +\infty)}^{-1} = \left\{ \int_{z_0}^{+\infty} \left[ \left( \frac{u - x}{v - y} \right)^2 + 1 \right]^{-q/2} d \left( \frac{u - x}{v - y} \right) \right\}^{1/q} \lesssim |v - y|^{1/q-1}, \]

(10.15)

where \( p > 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then we have

\[
F_3 \leq \int_{0}^{+\infty} \| s - z \|_{L^q(z_0, +\infty)} \left\{ (u - z_0)^2 + v^2 \right\}^{-1/4} \| L^p \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv 
\lesssim \int_{0}^{+\infty} v^{1/p-1/2} |v - y|^{1/q-1} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv 
\lesssim \int_{0}^{y} v^{1/p-1/2} (y - v)^{1/q-1} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv 
+ \int_{y}^{+\infty} v^{1/p-1/2} (v - y)^{1/q-1} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv. \]  

(10.16)

For the first term, using the inequality \( e^{-z} \leq z^{-1/6} \) leads to

\[
\int_{0}^{y} v^{1/p-1/2} (y - v)^{1/q-1} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv 
\lesssim |t|^{-1/6} \int_{0}^{y} v^{1/p-1} (y - v)^{1/q-1} dv \lesssim |t|^{-1/6}. \]  

(10.17)

And for the second term, we estimate similarly as we estimate \( F_1 \). Let \( w = v - y \), then we
have
\[
\int_{y}^{+\infty} v^{1/p-1/2} (v - y)^{1/q-1} \exp \left( -\frac{v|t|}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv
\leq \int_{0}^{+\infty} w^{1/q-1} (w + y)^{1/p-1/2} \exp \left( -\frac{w|t|y^2}{2z_0^2(y^2 + z_0^2)} \right) dw \exp \left( -\frac{|t|y^3}{2z_0^2(y^2 + z_0^2)} \right)
\leq \int_{0}^{+\infty} w^{-1/2} \exp \left( -\frac{w|t|y^2}{2z_0^2(y^2 + z_0^2)} \right) dw \lesssim |t|^{-1/2}.
\]
(10.18)

Finally, we have
\[
F_3 \lesssim |t|^{-1/6}.
\]
(10.19)

Summary the results obtained above, we obtain the finally consequence.

Proposition 7. The solution \( M^{(3)}(z) \) of the RHP6 admits the following estimate
\[
||M^{(3)}(0) - I|| \lesssim |t|^{-1}.
\]
(10.20)

Proof. We only prove the case \( \eta = -1 \). Because of the boundedness of \( M^{RHP} \), we only need to estimate
\[
\frac{1}{\pi} \int_{C} \frac{\left| \partial R^{(2)}(s) \right|}{|z - s|} dm(s).
\]
For \( \partial R^{(2)}(s) \) is a piece-wise function, we show the case in the region \( \Omega_1 \), the other regions are similar. Denote \( s = u + vi \), then we have
\[
\int_{\Omega_1} \frac{\left| \partial R^{(2)}(s) \right|}{|s|} dm(s) = \int_{0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{\left| \partial R_1(s) \right| e^{-\frac{z_0^2 + 2v}{u^2 + v^2}} e^{-2tv\xi}}{(u^2 + v^2)\frac{1}{2}} dv du \leq I_1 + I_2 + I_3,
\]
(10.21)

where by using (5.20) we have
\[
I_1 = \int_{0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{\left| \partial X_2(s) \right| e^{-\frac{z_0^2 + 2v}{u^2 + v^2}} e^{-2tv\xi}}{(u^2 + v^2)\frac{1}{2}} dv du;
\]
(10.22)
\[
I_2 = \int_{0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{\left| p_1'(u)(s) \right| e^{-\frac{z_0^2 + 2v}{u^2 + v^2}} e^{-2tv\xi}}{(u^2 + v^2)\frac{1}{2}} dv du;
\]
(10.23)
\[
I_3 = \int_{0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{\left| (u - z_0)^2 + v^2 \right|^{-1/4} e^{-\frac{z_0^2 + 2v}{u^2 + v^2}}}{(u^2 + v^2)\frac{1}{2}} e^{-2tv\xi} dv du.
\]
(10.24)

For \( I_1 \), we divide it to two part:
\[
I_1 = \int_{z_0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{\left| \partial X_2(s) \right| e^{-\frac{z_0^2 + 2v}{u^2 + v^2}}}{(u^2 + v^2)\frac{1}{2}} e^{-2tv\xi} dv du + \int_{0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{\left| \partial X_2(s) \right| e^{-\frac{z_0^2 + 2v}{u^2 + v^2}}}{(u^2 + v^2)\frac{1}{2}} e^{-2tv\xi} dv du.
\]
(10.25)
For the first integral, we use (10.15) in the case of $y = 0$, then we have

\[
\int_{z_0}^{+\infty} \int_{z_0 + v}^{+\infty} |\bar{\partial}X(\xi)| e^{-\frac{u v}{2(u^2 + v^2)}} e^{-2tv\xi} dudv \\
\leq \int_{z_0}^{+\infty} \|s - z\|^{-1}_{L^2} \|\bar{\partial}X(\xi)\|_{L^2} e^{-\frac{u v}{2(u^2 + v^2)}} e^{-2tv\xi} dv \\
\lesssim \int_{z_0}^{+\infty} v^{-1/2} \exp \left\{ -\frac{|2t|v}{5z_0^2} \right\} dv \\
\lesssim \int_{z_0}^{+\infty} \exp \left\{ -\frac{|2t|v}{5z_0^2} \right\} dv \lesssim |t|^{-1}. 
\]

(10.26)

And for the second item,

\[
\int_{0}^{z_0} \int_{z_0 + v}^{+\infty} |\bar{\partial}X(\xi)| e^{-\frac{u v}{2(u^2 + v^2)}} e^{-2tv\xi} dudv \\
\leq \int_{0}^{z_0} \left( (v + z_0)^2 + v^2 \right)^{-1/2} \exp \left( \frac{|t|v}{(v + z_0)^2 + v^2} \right) dv \\
\lesssim \int_{0}^{z_0} \exp \left( \frac{|t|v}{z_0^2} \right) dv \lesssim |t|^{-1}. 
\]

(10.27)

And using the same way we can estimate $I_2$ and get same result. Finally we bound $I_3$, similarly we divide it to two parts:

\[
I_3 = \int_{z_0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{(u - z_0)^2 + v^2}{(u^2 + v^2)^{1/2}} e^{-\frac{u v}{2(u^2 + v^2)}} e^{-2tv\xi} dudv \\
+ \int_{0}^{z_0} \int_{z_0 + v}^{+\infty} \frac{(u - z_0)^2 + v^2}{(u^2 + v^2)^{1/2}} e^{-\frac{u v}{2(u^2 + v^2)}} e^{-2tv\xi} dudv. 
\]

(10.28)

For the first integral, we use (10.15) and (10.14) similarly, take $p > 2$ and $\frac{1}{p'} = \frac{1}{q} = 1:

\[
\int_{z_0}^{+\infty} \int_{z_0 + v}^{+\infty} \frac{(u - z_0)^2 + v^2}{(u^2 + v^2)^{1/2}} e^{-\frac{u v}{2(u^2 + v^2)}} e^{-2tv\xi} dudv \\
\leq \int_{z_0}^{+\infty} \|s\|^{-1}_{L^q} \|(u - z_0)^2 + v^2\|^{-1/4}_{L^p} \exp \left( -\frac{|v|t}{2} \left( \frac{1}{z_0^2} - \frac{1}{z_0^2 + v^2} \right) \right) dv \\
\lesssim \int_{z_0}^{+\infty} v^{-1/2} \exp \left\{ -\frac{|2t|v}{5z_0^2} \right\} dv \\
\lesssim \int_{z_0}^{+\infty} \exp \left\{ -\frac{|2t|v}{5z_0^2} \right\} dv \lesssim |t|^{-1}. 
\]

(10.29)
And for the second integral, note that
\[
\| (u^2 + v^2)^{-1/2} \exp \left\{ \frac{|tv|}{2(u^2 + v^2)} \right\} \|_{L^4(z_0 + v, +\infty)}
\]
\[
= \left\{ \int_{z_0}^{+\infty} (u^2 + v^2)^{-2} \exp \left\{ \frac{2|tv|}{u^2 + v^2} \right\} du \right\}^{1/4}
\]
\[
= \left\{ \int_{z_0}^{+\infty} |8tv|^{-1}u^{-1} \exp \left\{ \frac{2|tv|}{u^2 + v^2} \right\} du \right\}^{1/4}
\]
\[
\lesssim |t|^{-1/4}u^{-1/4} \left( \exp \left\{ \frac{|tv|}{2((v + z_0)^2 + v^2)} \right\} + 1 \right).
\] (10.30)

Then together with (10.14) we obtain
\[
\int_{z_0}^{z_0} 0 \int_{z_0 + v}^{+\infty} \frac{|(u - z_0)^2 + v^2|^{-1/4} e^{-\frac{2tv}{2(u^2 + v^2)}}}{(u^2 + v^2)^{1/2}} = e^{-2tv\xi} dudv
\]
\[
\lesssim \int_{z_0}^{z_0} 0 \| (u - z_0)^2 + v^2 \|_{L^{4/3}} \| (u^2 + v^2)^{-1/2} \exp \left\{ \frac{|tv|}{2(u^2 + v^2)} \right\} \|_{L^4} e^{-2tv\xi} dv
\]
\[
\lesssim |t|^{-1/4} \int_{0}^{z_0} v^{3/4 - 1/2}v^{-1/4} e^{-2tv\xi} \left( \exp \left\{ \frac{|tv|}{2((v + z_0)^2 + v^2)} \right\} + 1 \right) dv
\]
\[
\lesssim |t|^{-1/4} \int_{0}^{z_0} e^{-2tv\xi} dv \lesssim |t|^{-5/4}.
\] (10.31)

So we have
\[
I_3 \lesssim |t|^{-1}.
\] (10.32)

We come to the result by combining above equations.

To reconstruct the solution \( u(y, t) \) of the SP equation (1.1), we need the asymptotic behavior of \( M^{(3)}_1 \) given by
\[
M^{(3)}(z) = M^{(3)}(0) + M^{(3)}_1(x, t)z + O(z^2), \quad z \to 0
\] (10.33)
and
\[
M^{(3)}_1(y, t) = \frac{1}{\pi} \int_C \frac{M^{(3)}(s)W^{(3)}(s)}{s^2} dm(s).
\] (10.34)

The \( M^{(3)}_1 \) admits the following estimate.

**Lemma 3.** For all \( t \neq 0 \), we have
\[
|M^{(3)}_1(x, t)| \lesssim |t|^{-1}.
\] (10.35)
Proof. From Lemma 2 and (10.2), we have \( \| M^{(3)} \|_\infty \lesssim 1 \). And we only estimate the integral on \( \Omega_1 \) since the other estimates are similar. Like in the above Lemma, by (5.20) and (5.30) we obtain

\[
\frac{1}{\pi} \int_{\Omega_1} M^{(3)}(s) \tilde{\partial} R^{(2)}(s) |s|^{-2} dm(s) \lesssim \frac{1}{\pi} \int_{\Omega_1} |\tilde{\partial} R^{(2)}(s)||s|^{-2} dm(s) \lesssim I_4 + I_5 + I_6, \tag{10.36}
\]

where the last inequality is from (5.20) and we also have for \( s = u + vi \),

\[
I_4 = \int_0^{+\infty} \int_{z_0+v}^{+\infty} \frac{|\tilde{\partial} X(z) e^{-\frac{z^2}{2(u^2+v^2)}}|}{u^2+v^2} dvde^{-2tv\xi} du; \tag{10.37}
\]

\[
I_5 = \int_0^{+\infty} \int_{z_0+v}^{+\infty} \frac{\tilde{p}_1(u)(s)e^{-\frac{z^2}{2(u^2+v^2)}}}{u^2+v^2} dvde^{-2tv\xi} du; \tag{10.38}
\]

\[
I_6 = \int_0^{+\infty} \int_{z_0+v}^{+\infty} \frac{((u-z_0)^2 + v^2)^{-1/4} e^{-\frac{z^2}{2(u^2+v^2)}}}{u^2+v^2} dvde^{-2tv\xi} du. \tag{10.39}
\]

Note that for all \( s \) in \( \Omega_1 \)

\[
(u^2 + v^2)^{-1/2} \leq \frac{1}{z_0},
\]

so we have

\[
I_j \leq \frac{1}{z_0} I_{j-3}, \quad \text{for } j = 4, 5, 6. \tag{10.40}
\]

So from Proposition 7 we can easily get the result. \( \square \)

11 Soliton resolution for the SP equation

Now we begin to construct the long time asymptotics of the SP equation (1.1). Inverting the sequence of transformations (4.14), (5.9), (6.7) and (6.11), we have

\[
M(z) = M^{(3)}(z) E(z) M^{(\text{out})}(z) R^{(2)}(z)^{-1} T(z)^{-\sigma_3}, \quad z \in C \setminus U_{\pm z_0} \tag{11.1}
\]

To reconstruct the solution \( u(y, t) \) by using (3.17), we take \( z \to 0 \) along the imaginary axis. In this case, \( R^{(2)}(z) = I \), and we have

\[
u(x, t) = u(y(x, t), t) = -i \lim_{z \to 0} z \left( M(0)^{-1} M(z) \right)_{12}. \tag{11.2}
\]

Further using Propositions 2, 6 and 7 we can obtain the long time asymptotics behavior

\[
(M(0)^{-1} M(z) = M^{(\text{out})(0)^{-1} M^{(\text{out})(0) + M^{(\text{out})(0)^{-1} M^{(\text{out})(0) T_1^{-\sigma_3} z
\]

\[
+ M^{(\text{out})(0)^{-1} H^{(1)} M^{(\text{out})(0) z|t|^{-1/2} + O(|t|^{-1}), \tag{11.3}
\]
where $T(z)^{\otimes 3}$ is a diagonal matrix, then by corollary 2 and simply calculation we finally obtain following result.

**Theorem 1.** Let $q(x, t)$ be the solution for the initial-value problem (1.1)-(1.2) with generic data $u_0(x) \in H^{1,1}(\mathbb{R})$. For fixed $y_1, y_2, v_1, v_2 \in \mathbb{R}$ with $y_1 \leq y_2$ and $v_1 \leq v_2 \in \mathbb{R}^-$, we define two zones for spectral variable $z$

$$I = \{ z : -1/(4v_1) < |z|^2 < -1/(4v_2) \}, \quad N(I) = \{ z_k \in \mathbb{Z} : z_k \in I \}$$

(11.4)

and a cone for variables $y, t$

$$C(y_1, y_2, v_1, v_2) = \{ (y, t) \in \mathbb{R}^2 | y = y_0 + vt \text{, with } y_0 \in [y_1, y_2], \text{ v } \in [v_1, v_2] \},$$

which are shown in Figure 3. Denote $u_{sol}(y, t|D(I))$ be the $N(I)$ soliton solution corresponding to scattering data $\{ z_k, c_k(I) \}_{k=1}^{N(I)}$ which given in (7.18) and corresponding $c_+(x, t|D(I))$ defined by (7.20). Then as $|t| \to \infty$ with $(y, t) \in C(y_1, y_2, v_1, v_2)$, we have

$$u(x, t) = u(y(x, t), t) = u_{sol}(y(x, t), t|D(I)) - i|t|^{-1/2}f_{12}(y, t) + O(|t|^{-1}),$$

(11.5)

where

$$y(x, t) = x - c_+(x, t; D(I)) - iT_1^{-1} - i|t|^{-1/2}f_{11}(y, t) + O(|t|^{-1}),$$

(11.6)

$$f_{12}(y(x, t), t) = \begin{bmatrix} M^{(out)}(0)^{-1}H^{(1)}M^{(out)}(0) \end{bmatrix}_{12},$$

$$f_{11}(y(x, t), t) = \begin{bmatrix} M^{(out)}(0)^{-1}H^{(1)}M^{(out)}(0) \end{bmatrix}_{11}.$$

The long time asymptotic expansion (11.5) shows the soliton resolution of for the initial value problem of the short-pluse equation, which consisting of three terms: the leading order term can be characterized with an $N(I)$-soliton whose parameters are modulated by a sum of localized soliton-soliton interactions as one moves through the cone; the second $t^{-1/2}$ order term coming from soliton-radiation interactions on continuous spectrum up to an residual error order $O(|t|^{-1})$ from a $\partial_t$ equation. Our results also show that soliton solutions of short-pluse equation are asymptotically stable.

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