Adversarial Sign-Corrupted Isotonic Regression

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Abstract

Classical univariate isotonic regression involves nonparametric estimation under a monotonicity constraint of the true signal. We consider a variation of this generating process, which we term adversarial sign-corrupted isotonic (ASCI) regression. Under this ASCI setting, the adversary has full access to the true isotonic responses, and is free to sign-corrupt them. Estimating the true monotonic signal given these sign-corrupted responses is a highly challenging task. Notably, the sign-corruptions are designed to violate monotonicity, and possibly induce heavy dependence between the corrupted response terms. In this sense, ASCI regression may be viewed as an adversarial stress test for isotonic regression. Our motivation is driven by understanding whether efficient robust estimation of the monotone signal is feasible under this adversarial setting. We develop ASCIFIT, a three-step estimation procedure under the ASCI setting. The ASCIFIT procedure is conceptually simple, easy to implement with existing software, and consists of applying the PAVA with crucial pre- and post-processing corrections. We formalize this procedure, and demonstrate its theoretical guarantees in the form of sharp high probability upper bounds and minimax lower bounds. We illustrate our findings with detailed simulations.

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1 Introduction

Isotonic regression is a classically studied nonparametric regression problem in which the underlying signal satisfies a monotonicity constraint. In the univariate case, this regression setup provides a flexible nonparametric generalization to simple linear regression. That is, the underlying signal may be non-linear, but still satisfies monotonicity as in the simple linear model. The classically studied isotonic regression generating process is formally described in Definition 1:

**Definition 1** (Classical isotonic regression). We consider \( n \) observations, \( \{Y_i \mid i \in [n]\} \), where each observation \( Y_i \) is generated from the following model:

\[
Y_i = \mu_i + \varepsilon_i \tag{1}
\]

s.t. \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \tag{2} \)

and \( \varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \tag{3} \)

The statistical goal under this classical setup is to estimate the underlying signal vector \( \mu := (\mu_1, \ldots, \mu_n)^T \), subject the monotonicity constraint in Equation (2), while \( \sigma \) is an unknown (nuisance) parameter. Throughout this paper we will adopt the convention, without loss of generality, that the signal vector is monotonically increasing (as per Equation (2)). Additionally we will assume that all estimation errors are computed under square loss (in Euclidean metric), in high probability.

1.1 Adversarial sign-corrupted isotonic (ASCI) regression

Our work in this paper is motivated by a variation of the classical isotonic regression estimation problem, per Definition 1. We refer to this newly proposed model as adversarial sign-corrupted isotonic (ASCI) regression. The generating process for this ASCI estimation problem is formalized in Definition 2.

**Definition 2** (Adversarial sign-corrupted isotonic (ASCI) regression). We consider \( n \) observations, \( \{R_i \mid i \in [n]\} \), where each observation \( R_i \) is generated from the following model:

\[
R_i = \xi_i(\mu_i + \varepsilon_i) \tag{4}
\]

s.t. \( 0 < \eta \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \tag{5} \)

and \( \varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \tag{6} \)

and \( \xi_i \in \{-1, 1\} \tag{7} \)

**Remark 1.** We note that the constant \( \eta > 0 \) is known to the observer and adversary as part of the generating process. This means that the true signal is positive, since it is uniformly bounded away from \( \eta \). It is an artefact of our method but as we will see later in Examples 3 and 4, highly non-trivial estimation tasks are still contained under this constraint.

**Remark 2.** Throughout this paper we will interchangeably use the terms ASCI regression, setting, setup, model, and generating process to refer to Definition 2.

By comparing Definitions 1 and 2, this ASCI regression generating process is a partial generalization of the classical isotonic regression. It can be briefly described as follows. Here the classical isotonic regression responses, \( \mu_i + \varepsilon_i \) in Equation (1), are sign-corrupted in a
manner chosen by an adversary, as captured by the multiplicative $\xi_i$ terms. Here the $\xi_i \in \{-1, 1\}$ are sign-corruptions for the true data generating process, i.e., $Y_i := \mu_i + \varepsilon_i$. It is important to note that the $\xi_i \in \{-1, 1\}$ for each $i \in [n]$, are chosen given that the adversary has full access to the true responses, i.e., $(\mu_i + \varepsilon_i \mid i \in [n])$. As such, Equation (1) in the classical isotonic regression setup represents a special case of Equations (4) and (7) by taking $\xi_i \overset{a.s.}{=} 1$ for each $i \in [n]$. However, we note that in this ASCI setting, in Equation (5) the monotonically increasing signal vector $\mu := (\mu_1, \ldots, \mu_n)^\top$ is bounded below by $\eta$, which is some fixed and known positive constant. As such this represents a restriction of the classical isotonic regression condition described in Equation (2). In summary, ASCI regression represents both a restriction and relaxation of the classical isotonic regression generating process. We will see why the restriction is necessary in this work, but we will later suggest possible ways in which it can be relaxed in future work.

1.2 Interesting special cases of ASCI regression

Interestingly, we note that even some special cases of this ASCI regression setup can result in highly non-trivial estimation tasks. Two particular ASCI regression special cases are formalized in Examples 3 to 4.

Example 3 (Two-component Gaussian mixture ASCI regression special case). We consider $n$ observations, $\{R_i \mid i \in [n]\}$, where each observation $R_i$ is generated from the following model:

$$R_i = \xi_i \mu_i + \varepsilon_i$$
(8)

s.t. $0 < \eta \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$
(9)

and $\varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$
(10)

and $\xi_i \overset{i.i.d.}{\sim} \text{Rademacher}(p)$, $p \in (0, 1)$, and $\xi_i \perp \varepsilon_i$
(11)

Remark 3. Note that $\xi_i \overset{i.i.d.}{\sim} \text{Rademacher}(p)$ for each $i \in [n]$, means that $\xi_i = +1$ with probability $p$, and $\xi_i = -1$ with probability $1 - p$. We note that the model defined in Example 3 is a special case of Definition 2. This is formally proved in Appendix B.3.

We note that in the univariate setting, Example 3 represents a generalization of the two-component Gaussian mixture model studied in detail in Balakrishnan et al. [2017, Section 3.2.1]. Our model generalizes their setting in the sense that we allow a different mean, i.e., $\mu_i$, for each of the $n$ univariate observations. Interestingly, in this more general univariate mixture setting, our proposed ASCIFIT estimator (see Section 2) provides an efficient alternative to the EM algorithm [Dempster et al., 1977]. Such models have extensive applications, e.g., community detection [Giraud and Verzelen, 2018, Royer, 2017].

Example 4 (Non-convex ASCI regression special case). We consider $n$ observations, $\{R_i \mid i \in [n]\}$, where each observation $R_i$ is generated from the following model:

$$R_i = \gamma_i + \varepsilon_i$$
(12)

s.t. $0 < \eta \leq |\gamma_1| \leq |\gamma_2| \leq \ldots \leq |\gamma_n|$ 
(13)

and $\varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$
(14)

Remark 4. Verifying that Example 4 is a special case of Definition 2 is not a priori obvious, and is formally proved in Appendix B.4.
In Example 4, one can think of this model being generated from the ASCI model as per Definition 2. In this special case, the adversary randomly chooses sign-corruptions independently of the error terms, i.e., $\xi_i \perp \perp \varepsilon_i$ for each $i \in [n]$. Under this setup, the resulting response term in Equation (12) is the same as the classical isotonic regression response, as seen by comparing to Equation (1). However, interestingly the adversarial sign-corruption is now absorbed into the revised monotonicity constraint $0 < \eta \leq |\gamma_1| \leq |\gamma_2| \leq \ldots \leq |\gamma_n|$, per Equation (13). As a result, this generating process is a highly non-convex constrained estimation problem. In this case ASCIFIT will allow one to recover $|\gamma_i|$, where as PAVA will not provide any information on the $|\gamma_i|$ (or $\gamma_i$), given the non-convex constraint.

1.3 Motivation and focus of our work

With the ASCI regression setup clearly defined, we turn our attention to describing the focus of our analysis in this work. This is summarized by the following core question of interest:

**Core question:** Given the adversarial sign-corrupted isotonic (ASCI) regression setup in Definition 2, can we find a computationally efficient estimator for $\mu$, and demonstrate its precise (non-asymptotic) statistical optimality?

To the best of our knowledge the ASCI model, and our core question of interest, have not been explicitly studied before in the literature. We note that this ASCI estimation problem is inherently challenging, and thus interesting, for three main reasons, i.e., Challenge I – Challenge III:

**Challenge I (Dependent responses):** in this estimation problem the adversary is free to choose the sign-corruption terms $\xi_i$, after observing all samples, possibly resulting in a strong dependence between the original isotonic responses. As such, any ASCI estimator must be able to handle arbitrary dependence structure between the sign-corrupted responses.

**Challenge II (Violating signal monotonicity):** qualitatively speaking, the sign-corruptions are in a sense ‘extreme’ in that by selectively changing the sign of the observations the adversary fundamentally ‘attacks’ the isotonic monotonicity constraint directly. It is this convex monotone constraint which classical isotonic estimators, i.e., PAVA, are designed to exploit.

**Challenge III (Interesting special cases):** The ASCI setting contains interesting non-trivial special cases as described in Examples 3 and 4. Naively applying typical least squares estimation techniques will be unable to provide any relevant information on the estimated quantity of interest.

Given these three formidable challenges posed by ASCI regression, any computationally and statistically efficient estimator here needs to utilize new techniques to exploit the potential non-convex structure in our setting. Our motivations here are thus driven by understanding the robustness of existing isotonic regression estimators under such adversarial settings. Moreover, for the ASCI setting to be worth studying, we wanted ensure practical algorithms for estimation under this adversarial setting, with sharp minimax (worst-case) statistical guarantees, both of which we were able to provide. We thus view the ASCI setting as stimulating prototype for more such research into adversarial robustness in isotonic regression.
1.4 Prior and related work

As noted, to the best of knowledge our core question of interest, i.e., isotonic regression under the proposed ASCI setup, has not been previously studied. Our work however builds on and utilizes known estimators from the classical isotonic regression literature. As such we limit our prior and related work summary on known risk bounds (and rates) for such isotonic regression estimators, and the efficient algorithms (i.e., the PAVA) and practical implementations thereof.

Isotonic regression (classical):

A lively historical overview of isotonic regression estimation from a computational lens is given in de Leeuw et al. [2009, Section 1]. In brief, we note that the origins of isotonic regression can be traced back to a number of independently written papers in the 1950s. In particular it was studied by Ayer et al. [1955], Brunk [1955]. Such estimators for “ordered parameters” were also analyzed in the series of papers van Eeden [1956, 1957a,b,c] which culminated into a PhD thesis in by the same author [van Eeden, 1958]. Shortly thereafter the articles [Bartholomew, 1959a,b] also investigated the related idea of hypothesis testing under monotonicity constraints. We refer the interested reader to the classical comprehensive references Barlow et al. [1972], Robertson et al. [1988], for further reading.

The classical isotonic regression setup per Definition 1 under square loss is a convex optimization problem. As such, it has a unique solution, i.e., the Euclidean projection onto the closed convex monotone cone given by the constraint in Equation (2). In this case, the non-asymptotic risk bounds for the least squares estimator (LSE) of the monotone parameters \( \mu_i \) are of the order \( n^{-2/3} \) in sample complexity. This convergence rate has been established across a number of papers including Birgé and Massart [1993], Chatterjee et al. [2015], Donoho [1990], Meyer and Woodroofe [2000], van de Geer [1990, 1993], Wang [1996], Zhang [2002]. Broadly speaking, these results typically vary in the generality of their underlying assumptions on the normality or independence of the error terms in classical isotonic regression. As noted in the excellent recent survey Guntuboyina and Sen [2018], the same risk rate for this (and for more general) LSEs was established using an alternative approach in Chatterjee [2014]. Moreover, in the case of minimax lower bounds, the matching risk rate (up to constant terms) for isotonic regression was established in Chatterjee et al. [2015] and also in Bellec and Tsybakov [2015], in both high probability and expectation terms.

Pool Adjacent Violators Algorithm (PAVA):

Rather remarkably, despite the nonparametric setup of classical isotonic regression, the LSE in this case has an explicit ‘max-min’ formulation [Barlow et al., 1972, Equation (1.9)]. However, in practice it is efficiently computed using the pool adjacent violators algorithm (PAVA). As described in Tibshirani et al. [2011] the PAVA in effect estimates the ordered parameters by scanning through the (sorted) observations. For each adjacent pair of observations, the monotonicity constraint is checked. If the constraint is ‘violated’ by a given observation, the average of the observations is used as the estimate, with appropriate (minimal) backtracking to ensure that any restrospectively incurred violations are similarly corrected for. Efficient PAVA implementations, e.g., as described in Best and Chakravarti [1990], Grotzinger and Witzgall [1984], have a computational complexity of \( O(n) \), where \( n \) is the sample size. Since we will use the PAVA in just one step in our proposed three-step estimator for the ASCI regression parameter \( \mu \), we will not detail it further here. However, such open-source PAVA implementation details can be found in de Leeuw et al. [2009], Pedregosa et al. [2011].
1.5 Main contributions

Our contributions in this paper are twofold and are summarized as follows:

- **Computable estimators with non-asymptotic upper bounds:** We propose a computationally efficient three-step algorithm ASCIFIT, to estimate the required parameter $\mu$, under the ASCI setting. Our ASCIFIT estimator converges at a $n^{-2/3}$ rate, with high probability. We illustrate our findings with extensive numerical simulations.

- **Sharp minimax lower bounds:** we provide matching high probability lower bounds (up to constant and log factors) under square loss, and thus demonstrate that our estimators are minimax optimal in this sense.

In particular, our upper bound proofs involve rather subtle theoretical details about the PAVA, and our use of method of moment techniques is quite unique in this literature. We believe these proof techniques will be of independent interest to researchers in isotonic regression. In particular, for similar adversarial estimation tasks, where traditional convex M-estimation techniques are infeasible.

1.6 Organization of the paper

The rest of this paper is organized as follows. In Section 2 we introduce ASCIFIT, our three-step estimation procedure for $\mu$. In Section 3 we provide high probability upper bounds on estimation rates using ASCIFIT. In Section 4 we establish sharp minimax lower bounds for the parameter estimation in our ASCI setting. In Section 5 we provide extensive numerical ASCI simulations, to illustrate our findings. In Section 6 we summarize our results and describe exciting future research directions.

1.7 Notation

Throughout this paper, we typically use lowercase for scalars in $\mathbb{R}$, e.g., $(x, y, z, \ldots)$, bold lowercase for vectors, e.g., $(x, y, z, \ldots)$, and bold uppercase for matrices, e.g., $(X, Y, Z, \ldots)$. We use $\lesssim$ and $\gtrsim$ to mean $\leq$ and $\geq$, respectively, up to positive universal constants. We denote $a \vee b := \max\{a, b\}$ for each $a, b \in \mathbb{R}$. We say that a sequence $a_n := \mathcal{O}(1)$ if there exists $C > 0, N \in \mathbb{N}$ such that $|a_n| < C$ for each $n > N$. Similarly, $a_n = \mathcal{O}(b_n)$ iff $\frac{a_n}{b_n} = \mathcal{O}(1)$. We say that a sequence $a_n = o(1)$ if $a_n \rightarrow 0$ as $n \rightarrow \infty$. Similarly, $a_n = o(b_n)$ iff $\frac{a_n}{b_n} = o(1)$. We denote the finite set $\{1, \ldots, n\}$ by $[n]$. We define the indicator function $I_{\Omega}(x)$ to take the value 1 when $x \in \Omega \subseteq \mathbb{R}^d$, and 0 otherwise. We say that a function $f : \Omega \rightarrow \mathbb{R}$ is increasing, if for all $u, v \in \Omega \subseteq \mathbb{R}$ such that $u \leq v$, implies $f(u) \leq f(v)$. We use strictly increasing in the case where these inequalities are strict. Similarly we note that $f$ is decreasing (or strictly decreasing) when these respective inequalities are reversed. We provide a useful notation summary table in Appendix A.1.

2 ASCIFIT: A three-step estimation procedure for $\mu$

As per our core question of interest, we now turn our attention to ASCIFIT, i.e. our proposed estimation procedure for $\mu$, under the ASCI setup. The Folded Normal distribution, and in particular its mean and variance, will be fundamental to ASCIFIT. As such, we first formalize the key properties of the Folded Normal distribution in Definition 5.
**Definition 5** (Folded Normal distribution). Suppose $R \sim \mathcal{N}(\mu, \sigma^2)$, and let $T := |R|$. We then say that $T \sim \text{FoldNorm}(\mu, \sigma)$, is a Folded Normal distribution. We denote the mean and variance of $T$, by $f(\mu, \sigma)$ and $g(\mu, \sigma)$, respectively. They are given as follows:

\[
\begin{align*}
    f(\mu, \sigma) & := \mathbb{E}(T) = \sigma \sqrt{\frac{2}{\pi}} \exp(-\mu^2/(2\sigma^2)) - \mu(1 - 2\Phi(\mu/\sigma)). \\
    g(\mu, \sigma) & := \text{Var}(T) = \mu^2 + \sigma^2 - f(\mu, \sigma)^2.
\end{align*}
\]

(15) (16)

**Remark 5.** We refer the reader to Elandt [1961], Tsagris et al. [2014] for more details. We only consider $\mu > \eta > 0$ per Equation (5), and we use the shorthand notation $f(\mu, \sigma)^2 := (f(\mu, \sigma))^2$.

We now describe ASCIFIT, our three-step procedure to estimate $\mu$ under the ASCI setting, as follows:

**ASCIFIT**: Three-step procedure to estimate $\mu$ under the ASCI setting

**Step I (Pre-processing and PAVA):**

Obtain an initial naive estimate of $\mu := (\mu_1, \ldots, \mu_n)^\top$ by fitting isotonic regression (using the PAVA) on $T_i := |R_i|$. Denote these estimates by $\hat{\mu}_{\text{naive}} := (\hat{T}_1, \ldots, \hat{T}_n)^\top$.

**Step II (Second moment matching):**

Estimate $\sigma$ in the following way. Pick the $\sigma$ solving the following equation, and denote the corresponding solution as $\hat{\sigma}$:

\[
G(\sigma) := \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma))^2 = \frac{1}{n} \sum_{i=1}^{n} T_i^2.
\]

(17)

Here $f^{-1}(\cdot, \sigma)$, denotes the inverse function of $f(\mu, \sigma)$ with respect to $\mu$, when we hold $\sigma$ fixed to the value $\sigma$.

**Step III (Post-processing via plug-in):**

From $\hat{\mu}_{\text{naive}}$ in Step I, and $\hat{\sigma}$ in Step II, compute $\hat{\mu}_{\text{ascifit}} := (\hat{\mu}_1, \ldots, \hat{\mu}_n)^\top$ as follows:

\[
\hat{\mu}_i := f^{-1}(\hat{T}_i \lor f(\eta, \hat{\sigma}), \hat{\sigma}), \text{ for each } i \in [n].
\]

(18)

2.1 Intuition for the three ASCIFIT steps

We now provide more precise intuition for each of the three ASCIFIT steps, i.e., Step I – Step III.

**Intuition for Step I:**

Here, we begin with the pre-processing operation $T_i := |R_i|$. This serves the critical dual purpose of removing the effect of the sign-corruptions $\xi_i$, and also induces independence of the resulting observations $(\hat{T}_1, \ldots, \hat{T}_n)^\top$. This helps directly address Challenge I and Challenge II under the ASCI setup. To better understand this dual effect, note that in the ASCI setup, the $\xi_i \in \{-1, 1\}$ may be arbitrarily chosen by the adversary (without a precise distributional assumption). However, the critical information in our model is given by pre-processing each observation, $R_i$, as $T_i := |R_i|$. More specifically we have that $T_i = |\xi_i (\mu_i + \varepsilon_i)| = |\mu_i + \varepsilon_i|$. Since $\mu_i + \varepsilon_i \overset{\text{i.n.i.d.}}{\sim} \mathcal{N}(\mu_i, \sigma^2)$, per Definition 5 we have that $T_i \overset{\text{i.n.i.d.}}{\sim} \text{FoldNorm}(\mu_i, \sigma)$, per
Definition 5. We note that our pre-processed observations \( \{T_1, \ldots, T_n\} \) are all i.n.i.d.\(^1\), since they have a common variance \( \sigma^2 \) but varying means \( \mu_i \) for each observation \( i \in [n] \). Moreover, fitting an isotonic regression to \( T_i \) intuitively estimates \( f(\mu_i, \sigma) \) which are the mean of each \( T_i \). This step is formally justified by the results of Zhang [2002].

**Intuition for Step II:**

This is motivated by second moment matching to estimate \( \sigma \). Specifically, using the fact that the expected value of \( \frac{1}{n} \sum_{i=1}^{n} T_i^2 \), is \( \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \). The left hand side of Equation (17) directly estimates the term \( \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \). In **Step II** it is not clear a priori whether such an inverse function \( f^{-1}(\cdot, \sigma) \) exists, or whether there exists a unique positive solution for \( \sigma \) in Equation (17). We will demonstrate that both assertions are true, and that the unique solution \( \hat{\sigma} \), to estimate \( \sigma \), can be computed efficiently with a binary search approach. We would like to note here that estimating \( \sigma \) is not an easy problem (it is not by coincidence that in classical isotonic regression that \( \sigma \) is viewed as a nuisance parameter). This difficulty explains why we need to impose some additional assumptions on the vector \( \mu \) and on \( \sigma \) later on. Next, we provide the intuition on why we use the factor \( \hat{T}_i \lor f(\eta, \sigma) \) in **Step II**, for each \( i \in [n] \). This is summarized in Proposition 6.

**Proposition 6** (Reason for the “\( \lor f(\eta, \sigma) \)”-correction in **Step II**). The need for defining the \( \lor f(\eta, \sigma) \) in Equation (17) in **Step II** in ASCIFIT, is that the solution to the problem

\[
\arg\min_{\hat{T}_1, \ldots, \hat{T}_n} \sum_{i=1}^{n} (T_i - \hat{T}_i)^2 \quad \text{s.t.} \quad f(\eta, \sigma) \leq \hat{T}_1 \leq \ldots \leq \hat{T}_n, \tag{19}
\]

is related to the solution to

\[
\arg\min_{\hat{T}_1, \ldots, \hat{T}_n} \sum_{i=1}^{n} (T_i - \hat{T}_i)^2 \quad \text{s.t.} \quad \hat{T}_1 \leq \ldots \leq \hat{T}_n, \tag{20}
\]

as \( \hat{T}_i := \hat{T}_i \lor f(\eta, \sigma) \).

To understand the significance of Proposition 6, first note that we apply the PAVA to the \( T_i \) values in **Step I**. As such, the corresponding least squares PAVA estimates, \( \hat{T}_i \), actually project onto the unconstrained monotone cone, as per Equation (20). However, in our setup we actually want to solve the constrained non-negative monotone means, as per Equation (19). Fortunately, this is not an issue since we can simply post hoc correct each of the fitted unconstrained PAVA solutions as \( \hat{T}_i := \hat{T}_i \lor f(\eta, \sigma) \), for each \( i \in [n] \). This follows by adapting Németh and Németh [2012, Corollary 1] to our ASCIFIT setup. From all of the above discussion, intuitively it follows that the term \( \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma)^2 \) in (17) also estimates \( \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \), which explains why in **Step II** we equate that term to \( \frac{1}{n} \sum_{i=1}^{n} T_i^2 \).

**Intuition for Step III:**

To understand the need for this step, one needs to realize that the PAVA will estimate the means of \( T_i \) which are \( f(\mu_i, \sigma) \). Hence in order for us to go back at the original \( \mu_i \) scale, we need to invert the value of the PAVA estimates \( \hat{T}_i \). Ideally we would use the true value of \( \sigma \) in the inversion, but since it is unavailable to us, we use the plug-in estimate \( \hat{\sigma} \) as computed in **Step II**. In addition the term “\( \lor f(\eta, \hat{\sigma}) \)” in Equation (18) is present, since by assumption the value of each \( \mu_i \) (or sufficiently just \( \mu_1 \)) must be at least \( \eta_i \) after inverting.

\(^1\)i.e., independent but not identically distributed.
3 Analysis of ASCIFIT: Upper bounds

We have now described the details and key intuition behind our three-step ASCIFIT estimator \( \hat{\mu}_{\text{ascifit}} \) for \( \mu \). We now turn our attention to formalizing this intuition into least squares estimation risk bounds. More specifically, our end goal in this section is to describe our high probability non-asymptotic upper risk bound for \( \hat{\mu}_{\text{ascifit}} \), and understand its dependence on the sample complexity, and other ASCI parameters. We also provide summary sketch behind the main proof techniques used and what insight they offer for estimation purposes. Before we state the results we will define the rate of convergence \( r_{n,2}(\mu_n, \mu_1, \sigma) \), which plays an important role in all of the Theorems to follow. For an absolute constant \( C_2 > 0 \) define

\[
 r_{n,2}(\mu_n, \mu_1, \sigma) := \min \left[ 2\sigma^2 C_2^2, \frac{27}{4} \left( \frac{\mu_n - \mu_1}{n} \right)^2 (\sigma C_2)^2 + \frac{2\sigma^2 C_2^2}{n} (1 + \log n) \right]. \tag{21}
\]

Importantly, assuming that \( \mu_n - \mu_1, \sigma \) are constants not scaling with the sample size \( n \), we have that \( r_{n,2}(\mu_n, \mu_1, \sigma) \lesssim \max \{ (\frac{\sigma^2 V}{n})^2, \frac{\sigma^2 \log n}{n} \} \), where \( V := \mu_n - \mu_1 \) is the total variation of the underlying monotone signal. With this essential background, we are ready to state our first result in Theorem 7.

**Theorem 7** (Equation (17) has a unique root). Assume that there exist constants \( \psi, \Psi, C > 0 \) such that \( \psi \leq \sigma \leq \Psi \) and \( \frac{1}{n} \sum_{i=1}^n \mu_i^2 \leq C \), for each \( n \in \mathbb{N} \). In addition let \( r_{n,2}(\mu_n, \mu_1, \sigma) = o(1) \), where the quantity \( r_{n,2}(\mu_n, \mu_1, \sigma) \) is defined in (21). Then for sufficiently large \( n \), \( \delta = o \left( \left( r_{n,2}(\mu_n, \mu_1, \sigma) \right)^{-1} \right) \), and \( \gamma = o \left( n^{1/2} \right) \), under the ASCI setup, Equation (17) in ASCIFIT has a unique root \( \sigma^* \in \left[ 0, \sqrt{\frac{1}{n} \sum_{i=1}^n T_i^2} \right] \) for \( \sigma \) with probability at least \( 1 - \delta^{-1} - 2\gamma^{-2} \).

The key insight of Theorem 7 from a statistical perspective, is that our second moment matching approach in **Step II** will ensure that our proposed estimator \( \hat{\sigma} \), for \( \sigma \), will be unique with high probability. The core idea behind the proof of Theorem 7 is that the map \( \sigma \mapsto G(\sigma) := \sigma^2 + \frac{1}{n} \sum_{i=1}^n (f^{-1}(\hat{T}_i \lor f(\eta_i, \sigma), \sigma))^2 \) is monotone increasing over \( \sigma \geq 0 \). This enables the use of the intermediate value theorem to check that two endpoints of \( G(\sigma) - \frac{1}{n} \sum_{i=1}^n T_i^2 \), evaluated at \( \sigma \in \{ 0, \sqrt{\frac{1}{n} \sum_{i=1}^n T_i^2} \} \) have opposite sign with high probability. This has important practical implications for estimation purposes. In effect, it means that we can efficiently compute \( \hat{\sigma} \), by solving \( G(\sigma) = \frac{1}{n} \sum_{i=1}^n T_i^2 \) (per Equation (17)) using a binary search approach between the two identified endpoints. We would like to mention that while using the intermediate value theorem sounds like an easy task, it turns out that it is extremely hard to verify that \( G(0) \leq \frac{1}{n} \sum_{i=1}^n T_i^2 \), for which the bulk of the proof of Theorem 7 is dedicated to.

Although Theorem 7 gives us a high probability bound on estimating \( \hat{\sigma} \) uniquely, it is important to next understand how efficiently \( \hat{\sigma} \) estimates \( \sigma \). This is summarized in Theorem 8.

**Theorem 8** (\( \hat{\sigma} \) is close to \( \sigma \)). Under the assumptions of Theorem 7, we have that \( |\sigma - \hat{\sigma}| \lesssim (\delta r_{n,2}(\mu_n, \mu_1, \sigma))^{1/2} + \gamma n^{-1/2} \) with probability at least \( 1 - \delta^{-1} - 2\gamma^{-2} \), where \( \delta^{-1}, \gamma^{-2} \in (0,1) \).

From Theorem 8 we see that \( \hat{\sigma} \) converges to \( \sigma \) roughly at a \( n^{-1/3} \) rate. In both Theorems 7 and 8, we require that there exist constants \( \psi, \Psi, C > 0 \) such that \( \psi \leq \sigma \leq \Psi \) and \( \frac{1}{n} \sum_{i=1}^n \mu_i^2 \leq C \), for each \( n \in \mathbb{N} \). For transparency, we note that such assumptions are an artefact of our methodology and ensure that our risk bounds can be tightly controlled using the second moment matching approach. Given the highly adversarial corruptions and non-convex
constraints that can arise under ASCI estimation, e.g., in Example 4, these are slightly stronger assumptions required for classical convex isotonic regression setup. They effectively represent a trade-off for the flexibility, and simplicity of using ASCIFIT under these adversarial settings, whilst still ensuring precise control in the parameter estimation risk bounds.

\( \hat{\sigma} \) in our post-processing correction for \( \mathbf{\hat{\mu}}_{\text{ascifit}} := (\hat{\mu}_1, \ldots, \hat{\mu}_n)^\top \). That is, our final estimate for each \( \mu_i \), is given by \( \hat{\mu}_i := f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}, \hat{\sigma})) \). With this explicit form and our tightly controlled bounds in Theorem 7 and Equation (17) we are finally able derive the required least squares risk rate of \( \mathbf{\hat{\mu}}_{\text{ascifit}} \). This is summarized in Theorem 9. We will shortly discuss this result further in Section 4 when we derive high probability minimax lower bounds.

**Theorem 9** (\( \mathbf{\hat{\mu}}_{\text{ascifit}} \) is close to \( \mathbf{\mu} \)). Under the assumptions of Theorem 7 and Theorem 8, we have that

\[
\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}, \hat{\sigma})) - \mu_i)^2 \lesssim \delta \gamma n, \quad \gamma \geq 1,
\]

with probability at least \( 1 - \delta^{-1} - 2\gamma^{-2} \).

**Remark 6.** We note that \( \eta \) is currently absorbed in our constants in Theorems 7 to 9. The exact form is complicated (but the smaller the \( \eta \) the bigger the constants). For more details, please refer to Appendix D.

### 4 Lower bounds

We now derive high probability minimax lower bounds under the ASCI setting. We accordingly first introduce the relevant related notation and definitions here for classes of monotonic sequences. We denote \( \mathcal{S} \) := \{ \( \mathbf{\mu} := (\mu_1, \ldots, \mu_n)^\top \mid \mu_1 \leq \ldots \leq \mu_n \} \) to be the set of all non-decreasing sequences. We define \( k(\mathbf{\mu}) \geq 1 \), for \( \mathbf{\mu} \in \mathcal{S} \), to be the integer such that \( k(\mathbf{\mu}) \) is the number of inequalities \( \mu_i \leq \mu_{i+1} \) that are strict for \( i \in [n-1] \) (i.e., the number of ‘jumps’ of \( \mathbf{\mu} \)). The class of bounded monotone functions are \( \mathcal{S}^\uparrow(V^*) := \{ \mathbf{\mu} \in \mathcal{S} \mid V(\mathbf{\mu}) \leq V^* \} \), for some fixed \( V^* \geq 0 \), and \( V(\mathbf{\mu}) := \mu_n - \mu_1 \), is the total variation of any \( \mathbf{\mu} \in \mathcal{S} \). We focus on the ASCI-restricted class of monotone sequences, i.e., \( \mathcal{S}^\uparrow(V^*, \eta, C) := \{ \mathbf{\mu} \in \mathcal{S}^\uparrow(V^*) \mid \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C, \mu_1 > \eta > 0 \} \).

We closely follow the approach of Bellec and Tsybakov [2015, Proposition 4] but non-trivially adapt it to our ASCI setting by ensuring the monotonicity constraint in Equation (5) is satisfied in the lower bound construction. The proof uses well established techniques including the Varshamov-Gilbert bound Tsybakov [2009, Lemma 2.9], and Fano’s Lemma arguments using Tsybakov [2009, Theorem 2.7]. This leads to our minimax lower bound result in Proposition 10.

**Proposition 10** (Minimax lower bounds). Let \( n \geq 2, V^* > 0 \) and \( \sigma > 0 \), and define

\[
\overline{r}_{n,2}(V^*, \sigma) := \max \left\{ \left( \frac{\sigma^2 V^*}{n} \right)^{\frac{3}{2}}, \frac{\sigma^2}{n} \right\}.
\]

Then, there exist absolute constants \( c, c' > 0 \) such that:

\[
\inf_{\mathbf{\mu}} \sup_{\mathcal{S}^\uparrow(V^*, \eta, C)} \Pr_{\mu} \left( \frac{1}{n} \| \mathbf{\hat{\mu}} - \mathbf{\mu} \|^2 \geq c \overline{r}_{n,2}(V^*, \sigma) \right) > c'.
\]

Crucially, Proposition 10 demonstrates that our high probability upper bounds for ASCIFIT in Theorem 9 are sharp in the minimax sense, up to constants and log factors. This is evident by directly comparing \( \overline{r}_{n,2}(V^*, \sigma) \) to \( r_{n,2}(\mu_n, \mu_1, \sigma) \) per Equation (21).
5 Simulations

We now demonstrate our ASCIFIT estimation algorithm in action through a variety of simulations. Specifically, for simulation purposes we consider \( n \) observations, \( \{R_i \mid i \in [n]\} \), where each observation \( R_i \) is generated from Example 3. For sufficiently large \( n \), the ASCI model in Example 3 roughly translates to \((1 - p)\)-proportion of observations being independently sign-corrupted by the adversary. Moreover we assume per Equation (11) that the adversarial sign-corruptions, \( \xi_i \), are chosen independently of all true errors, \( \varepsilon_i \), for each \( i \in [n] \). The true monotone signal is defined to be \( \mu_i := \eta + (1 - \eta) \frac{i - 1}{n} \), for each \( i \in [n] \). We run this generating process over the following parameter grid: \( \eta := 0.2 \), \( p := 0.5 \), \( \sigma \in \{0.5, 1, 1.5, 2\} \), \( n \in \{100, 250, 500, 1000\} \). We perform 50 replications for each combination of simulation grid parameters. In each replication of this generating process we fit the ASCIFIT estimator \( \hat{\mu}_{\text{ascifit}} \), for \( \mu \). The main summary result from running our simulation, is shown in Figure 1.

To clarify, given \( \eta = 0.2 \), \( p = 0.5 \), Figure 1 plots the sample mean-MSE, \( \frac{1}{n} \| \hat{\mu}_{\text{ascifit}} - \mu \|_2^2 \), over 50 ASCIFIT replications for each value of \( n \in \{100, 250, 500, 1000\} \). Here the sample mean-MSE is a useful simulation proxy for the least squares error, our core theoretical risk measure of interest. This sample mean-MSE is plotted separately for each of the four sigma values, \( \sigma \in \{0.5, 1, 1.5, 2\} \). The mean-MSE value of each replication (± 2 standard errors) are shown using error bars in an effort to quantify replication uncertainty. The plot in Figure 1 is as expected in that all of the sample mean-MSE values show a steady decreasing trend in \( n \). Importantly the relative uncertainty in sample mean-MSE reduces in \( n \), as seen by the smaller error bars to the right of Figure 1. For smaller \( \sigma \) values, i.e., smaller variance in the underlying generating model, we see a much lower sample mean-MSE on average compared to higher \( \sigma \)-valued simulations. That is, our ASCIFIT estimator achieves better accuracy, with smaller underlying variability in the model, on average when other factors are held constant.

Finally, in order to precisely gauge how well the ASCIFIT estimator \( \hat{\mu}_{\text{ascifit}} \), actually fits the true signal \( \mu \), it is instructive to plot both directly on the original generating sample data. This is seen for one instance over our parameter grid of simulations in Figure 2.

Specifically, for \( \eta = 0.2 \), \( p = 0.5 \), \( n = 1000 \), \( \sigma = 1.5 \), Figure 2 plots the simulated true generating process, \( \mu \), against the ASCIFIT estimator, \( \hat{\mu}_{\text{ascifit}} \). Additionally both the original and sign-corrupted individual observations are plotted to emphasize the difficulty of this esti-

---

\footnote{Reproducible code for all figures in this paper is found at: \url{https://github.com/shamindras/ascifit}. All of the simulation results in this section were run on a personal Macbook laptop with macOS, Intel Core i9 CPU, and 64GB RAM. The total runtime for a single run of all simulations is approximately 90 minutes.}
Figure 2: Mean (sample) MSE of ASCIFIT as a function of $n, \sigma$.

...mission task. Moreover, for comparison purposes we also plot the naive estimator, i.e., $\hat{\mu}_{\text{naive}}$. Here $\hat{\mu}_{\text{naive}}$ represents the estimator by stopping at Step I in ASCIFIT. That is, estimating $\mu$, by simply fitting isotonic regression (using the PAVA) on $T_i := |R_i|$. Furthermore, since $p = 0.5$, as expected, on average roughly half of the true responses are adversarially sign-corrupted. Despite this, one can see that ASCIFIT is relatively stable and reasonably recovers the true signal. This shows more directly (in such an instance), the robustness of ASCIFIT under such randomized adversarial sign-corruptions. Moreover since $n = 1000$, we can see that ASCIFIT indeed fits well with increasing sample complexity. In addition it highlights the importance of Step II and Step III in ASCIFIT.

6 Conclusion

In this paper we have considered a variation of the original isotonic regression problem in which the observations can be adversarially corrupted in their sign value. In this ASCI setting, adversarially refers to the fact that the sign-corruptions can be chosen to have strong dependence with the error terms in the original model. Our simple three-step estimation procedure, ASCIFIT, is easy to implement with existing software and has sharp non-asymptotic minimax guarantees on the estimation error, under square loss. For future directions we note that that true signal is required to be strictly positive for our guarantees to hold. We believe this restriction can be lifted if one uses unimodal regression instead of isotonic regression in Step I. However, sharp risk guarantees need to first be proven similar to Zhang [2002] under this unimodal setting. It would also be interesting to see if the moment matching technique could be extended subgaussian error terms. We leave these exciting directions for future work.

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Appendix

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A Mathematical Preliminaries

In this appendix we provide detailed proofs of all key statements from the main paper. Since our work relies a variety of core ideas from isotonic regression we first introduce some common definitions which will be referred to in subsequent proofs.
A.1 Notation Summary

To ensure that the Appendix is can be read in a standalone manner, we consolidate key notation used in the paper in Table 1. Unless stated otherwise $K \subseteq \mathbb{R}^d$ is a closed, non-empty convex set, and $\Omega \subseteq \mathbb{R}^d$.

Table 1: Notation and conventions used in our paper

| Variables and inequalities |
|----------------------------|
| $a \wedge b$ | $\min\{a, b\}$ for each $a, b \in \mathbb{R}$ |
| $a \vee b$ | $\max\{a, b\}$ for each $a, b \in \mathbb{R}$ |
| scalars $x, y, z \in \mathbb{R}$ |
| vectors $x, y, z \in \mathbb{R}^d$ |
| matrices $X, Y, Z \in \mathbb{R}^{d \times m}$ |
| $\ll$ | $\leq$ up to positive universal constants |
| $\gg$ | $\geq$ up to positive universal constants |
| $a_n = O(1)$ | $(\exists C > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|a_n| < C)$ |
| $a_n = O(b_n)$ | $\frac{a_n}{b_n} = O(1)$ |
| $a_n = o(1)$ | $(\forall C > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|a_n| < C)$ |
| $a_n = o(b_n)$ | $\frac{a_n}{b_n} = o(1)$ |
| $X_n = o_P(1)$ | $(\forall \varepsilon > 0)(\exists C > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(\Pr(|X_n| \geq C) \leq \varepsilon)$ |

Functions and sets

- Indicator function $I_{\Omega}(x)$: Takes value 1 when $x \in \Omega$, and 0 otherwise
- $\Pi_K : \mathbb{R}^d \to K$ ($\ell^2$)-projection of any $x \in \mathbb{R}^d$ onto $K$
- $f : \Omega \to \mathbb{R}$ is increasing: If $\forall u, v \in \Omega$ such that $u \leq v \implies f(u) \leq f(v)$
- $f : \Omega \to \mathbb{R}$ is decreasing: If $\forall u, v \in \Omega$ such that $u \leq v \implies f(u) \geq f(v)$
- $\Phi : \mathbb{R} \to [0, 1]$ Cumulative density function of $\mathcal{N}(0, 1)$
- $\phi : \mathbb{R} \to \mathbb{R}$ Probability density function of $\mathcal{N}(0, 1)$
- $S^\dagger \{ \mu := (\mu_1, \ldots, \mu_n)^T \mid \mu_1 \leq \ldots \leq \mu_n \}$
- $S^\dagger_+ \{ \mu \in S^\dagger \mid \mu_1 \geq 0 \}$
- $S^\dagger(V^*) \{ \mu \in S^\dagger \mid V(\mu) \leq V^* \}$
- $V(\mu) \mu_n - \mu_1$ for $\mu \in S^\dagger$
- $S^\dagger_{k^*} \{ \mu \in S^\dagger \mid k(\mu) \leq k^* \}$
- $S^\dagger(V^*, \eta, C) \{ \mu \in S^\dagger(V^*) \mid \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C, \mu_1 > \eta > 0 \}$

A.2 Useful miscellaneous results

Here we prove some useful standard results that are used in several of the remaining proofs. For reader convenience, we provide short proofs to ensure that our work is self-contained.

We start with some elementary inequalities, which will be used repeatedly. First, in Lemma 11 we introduce a differencing inequality we use repeatedly to construct lower bounds.
Lemma 11 (Difference of squares lower bound). For each \(a, b, l \in \mathbb{R}\), such that \(b, l \geq 0\) and \(a - b \geq l\), the following holds:
\[
a^2 - b^2 \geq la \geq l^2
\] (24)

Proof of Lemma 11. We proceed as follows. First note that since \(b, l \geq 0\) by assumption, we have that \(a - b \geq l \iff a \geq b + l \geq 0\). Now observe:
\[
a^2 - b^2 = (a + b)(a - b) \geq l(a + b) \quad \text{(since} \ a - b \geq l \geq 0 \ \text{by assumption.)}
\]
\[
\geq la \quad \text{(since} \ b \geq 0 \ \text{by assumption.)}
\]
\[
\geq l^2 \quad \text{(since} \ a \geq l \).
\]
as required.

Lemma 12 (Lower bound via difference of squares). For each \(a, b, C, K \in \mathbb{R}\), such that \(b \geq 0, a^2 - b^2 \geq C > 0, a \in [0, K]\), the following holds:
\[
a - b \geq \frac{C}{2K}
\] (25)

Proof of Lemma 12. We proceed as follows. First note that since \(a^2 - b^2 \geq C > 0\) by assumption, we have that \(a > 0\), and hence \(a > b, K > 0\) since both \(a, b\) are non-negative. Now observe:
\[
a^2 - b^2 \geq C \quad \text{(by assumption.)}
\]
\[
\implies a - b \geq \frac{C}{a + b} \quad \text{(since} \ a > 0, b \geq 0 \implies a + b > 0.)
\]
\[
\geq \frac{C}{2a} \quad \text{(since} \ a \geq b.)
\]
\[
\geq \frac{C}{2K} \quad \text{(since} \ a \leq K.)
\]
as required.

Lemma 13 (Maximum difference square inequality). For each \(a, b, c \in \mathbb{R}\) such that \(b \leq c\) the following inequality holds:
\[
((a \lor b) - c)^2 \leq (a - c)^2
\] (26)

Proof of Lemma 13. Under the assumption that \(a, b, c \in \mathbb{R}\) such that \(b \leq c\), let \(d := a \lor b\). We then observe:
\[
(d - c)^2 \leq (a - c)^2
\]
\[
\iff d^2 - a^2 \leq 2dc - 2ac \quad \text{ (expanding and simplifying.)}
\]
\[
\iff (d + a)(d - a) \leq 2c(d - a)
\] (27)

So we need to equivalently prove that Equation (27). To that end we only need to consider 2 cases. Namely \(a \geq b\), and \(a < b\). Note that in the first case \(a \geq b \implies d := a \lor b = a\). In this
case, both LHS/RHS of Equation (27) are 0, and the statement holds. Next consider the case \( a < b \). Here we have \( a < b \iff d := a \lor b = b > a \). We then observe the following:

\[
\begin{align*}
    a + d &= a + b \quad \text{(since } d = b) \\
    &\leq 2b \quad \text{(since } a < b \text{ by assumption)} \\
    &\leq 2c \quad \text{(since } b \leq c \text{ by assumption)}
\end{align*}
\]

That is, we have that \( a + d \leq 2c \). Substituting back to Equation (27) we have that

\[
\left(d + a\right)\left(d - a\right) \leq 2c\left(d - a\right),
\]

which is what we wanted to show. Which completes the proof of Equation (26), as required. \( \square \)

**Lemma 14** (Square sum inequality). For each \( a, b \in \mathbb{R} \) the following holds:

\[
(a + b)^2 \leq 2(a^2 + b^2) \tag{28}
\]

**Proof of Lemma 14.** We proceed as follows:

\[
\begin{align*}
    (a + b)^2 &= a^2 + 2ab + b^2 \tag{29} \\
    &\leq a^2 + b^2 + 2|ab| \quad \text{(since } x \leq |x| \text{ for each } x \in \mathbb{R}) \\
    &\leq a^2 + b^2 + 2(|a|^2 + |b|^2) \quad \text{(by AM-GM we have } 2|ab| \leq |a|^2 + |b|^2) \\
    &= 2(a^2 + b^2) \tag{30}
\end{align*}
\]

as required. \( \square \)

As a result of Lemma 14 we obtain Corollary 15.

**Corollary 15.** For random variables \( X_1, X_2 \) the following holds:

\[
\text{Var}(X_1 - X_2) \leq 2(\text{Var}(X_1) + \text{Var}(X_1)) \tag{31}
\]

**Proof of Corollary 15.** First let the centered versions of the random variables be denoted as

\[
\tilde{X}_i := X_i - \mathbf{E}(X_i), \text{ for each } i \in [2]. \tag{32}
\]

It then follows that:

\[
\begin{align*}
    \text{Var}(X_1 - X_2) &= \text{Var}(X_1 - X_2 + \mathbf{E}(X_2) - \mathbf{E}(X_1)) \tag{33} \\
    &= \text{Var}(\tilde{X}_1 - \tilde{X}_2) \tag{34} \\
    &= \mathbf{E}\left((\tilde{X}_1 - \tilde{X}_2)^2\right) \quad \text{(since } \tilde{X}_1, \tilde{X}_2 \text{ are both centered)} \tag{35} \\
    &= \mathbf{E}\left(2(\tilde{X}_1^2 + \tilde{X}_2^2)\right) \quad \text{(using Lemma 14)} \tag{36} \\
    &= 2\left(\mathbf{E}\left(\tilde{X}_1^2\right) + \mathbf{E}\left(\tilde{X}_2^2\right)\right) \quad \text{(linearity of expectation)} \tag{37} \\
    &= 2(\text{Var}(X_1) + \text{Var}(X_2)) \quad \text{(since } \tilde{X}_1, \tilde{X}_2 \text{ are both centered)} \tag{38}
\end{align*}
\]

as required. \( \square \)

The following is a standard result from real analysis, which we use repeatedly.
**Lemma 16** (B-Lipschitz characterization via bounded derivative). Let \( f : I \to \mathbb{R} \) be continuous and once differentiable, where \( I \subseteq \mathbb{R} \) is an interval (possibly unbounded).

\[
f \text{ is B-Lipschitz, with } B > 0 \iff \left( \exists B > 0 \right) \left( \forall x \in \mathbb{R} \right) : \left( |f'(x)| \leq B \right)
\]  

(35)

**Proof of Lemma 16.** We prove both directions. In both parts we assume that \( f : I \to \mathbb{R} \) be continuous and once differentiable, where \( I \subseteq \mathbb{R} \) is an interval (possibly unbounded).

\((\implies)\). Suppose that \( f \) is B-Lipschitz, with \( B > 0 \). We then have that, for some fixed (but arbitrary) \( c \in I \):

\[
|f(x) - f(c)| \leq B |x - c| \quad \text{(by definition of B-Lipschitz property.)}
\]

\[
\implies \left| \frac{f(x) - f(c)}{x - c} \right| \leq B
\]

\[
\implies |f'(c)| \leq B
\]

Since \( c \in I \) is arbitrary, indeed \( |f'(x)| \leq B \), for each \( x \in I \), as required.

\((\iff)\). Suppose that \( |f'(x)| \leq B \), with \( B > 0 \). Further let \( x, y \in I \), such that \( x < y \). Since \( f \) is differentiable on \( I \), we have:

\[
|f(x) - f(y)| \leq |f'(c)| |x - y| \quad \text{(by the mean value theorem, for some } c \in (x, y).)
\]

\[
\leq B |x - y| \quad \text{(by assumption.)}
\]

Which implies that \( f \) is B-Lipschitz, as required. \(\square\)

**Lemma 17** (Standard normal upper bound). Let \( \phi(x), \Phi(x) \) respectively denote the probability density function, and cumulative density function of a standard normal variable. Then the following inequality holds:

\[
\frac{x \phi(x)}{2 \Phi(x) - 1} \leq \frac{1}{2}, \text{ for each } x \geq 0
\]

(36)

With equality if and only if \( x = 0 \).

**Proof of Lemma 17.** We first note that at \( x = 0 \), that \( \frac{x \phi(x)}{2 \Phi(x) - 1} \) is an indeterminate form of type \( \frac{0}{0} \). As such we have:

\[
\lim_{x \to 0} \frac{x \phi(x)}{2 \Phi(x) - 1} = \lim_{x \to 0} \frac{\frac{\partial}{\partial x} x \phi(x)}{\frac{\partial}{\partial x} 2 \Phi(x) - 1}
\]

\[
= \lim_{x \to 0} \frac{\phi(x) + x \phi'(x)}{2 \phi(x)}
\]

\[
= \lim_{x \to 0} \frac{\phi(x)}{2 \phi(x)} \quad \text{(by continuity of } \phi(x) \text{ at } x = 0.)
\]

\[
= \frac{1}{2}
\]

(37)
With our given function now defined to be \( \frac{1}{2} \) at \( x = 0 \), we now proceed to prove our given inequality. Observe that we can equivalently reformulate it as:

\[
\Phi(x) - x\phi(x) - \frac{1}{2} \geq 0
\]  

(38)

Setting \( h(x) := \Phi(x) - x\phi(x) - \frac{1}{2} \), we observe that \( h(0) = \Phi(0) - \frac{1}{2} = 0 \). We need to show that \( h(x) \geq 0 \), for each \( x \geq 0 \), which will imply the result. We will show that \( h(x) \) is increasing, i.e., or equivalently that \( h'(x) \geq 0 \), for each \( x \geq 0 \). We then have that:

\[
h'(x) = \phi(x) - (\phi(x) + x\phi'(x)) \\
= -x\phi'(x) \\
= -x \left( -x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \right) \\
= x^2 \phi(x) \\
\geq 0
\]

(39)

Note that the inequality in Equation (39) is strict when \( x > 0 \) and equality holds if and only if \( x = 0 \). This means the function is strictly increasing and bounded away from 0 when for each \( x \geq 0 \), and equal to 0 only when \( x = 0 \), as required. \( \square \)

### A.3 The Folded Normal Distribution

For convenience, we begin by quickly recalling the definition of the Folded Normal distribution.

**Definition 5** (Folded Normal distribution). Suppose \( R \sim \mathcal{N}(\mu, \sigma^2) \), and let \( T := |R| \). We then say that \( T \sim \text{FoldNorm}(\mu, \sigma) \), is a **Folded Normal** distribution. We denote the mean and variance of \( T \), by \( f(\mu, \sigma) \) and \( g(\mu, \sigma) \), respectively. They are given as follows:

\[
f(\mu, \sigma) := \mathbb{E}(T) = \sigma \sqrt{2/\pi} \exp(-\mu^2/(2\sigma^2)) - \mu (1 - 2\Phi(\mu/\sigma)).
\]

(15)

\[
g(\mu, \sigma) := \text{Var}(T) = \mu^2 + \sigma^2 - f(\mu, \sigma)^2.
\]

(16)

**Remark 7.** We note that Equation (15) can be equivalently written as follows:

\[
\sigma \sqrt{2/\pi} \exp(-\mu^2/(2\sigma^2)) + \mu (1 - 2\Phi(-\mu/\sigma))
\]

(40)

Note that this equivalence follows from the symmetry of the standard normal CDF, i.e., \( \Phi(x) = 1 - \Phi(-x) \) for each \( x \in \mathbb{R} \). For our purposes we typically use the form of Equation (15).

### A.4 Properties of the folded normal mean: \( f(\mu, \sigma) \)

Let’s start setting up some notation. First we note as previously \( T_i := |R_i| = |\mu_i + \varepsilon_i| \). Where we then have \( T_i \sim \text{FoldNorm}(\mu_i, \sigma^2) \). Now denote \( f(\mu_i, \sigma) := \mathbb{E}(T_i) \), for each \( i \in [n] \). Moreover the \( T_i \) random variables are all mutually independent, but not identically distributed (since their mean’s, i.e., \( f(\mu_i, \sigma) \) differ for each \( i \in [n] \)). Since we run PAVA on \((T_1, \ldots, T_n)\) we have the resulting estimators \((\hat{T}_1, \ldots, \hat{T}_n)\). We will also denote the population level error terms for this transformed (mean centered) response as \( \delta_i := T_i - f(\mu_i, \sigma) \). We note that the \((\delta_1, \ldots, \delta_n)\) are all mutually independent, but not identically distributed.
Lemma 18 (Properties of the Folded Normal mean). Suppose \( R \sim \mathcal{N}(\mu, \sigma^2) \). Let \( T \overset{a.s.}{=} |R| \), then \( T \sim \text{FoldNorm}(\mu, \sigma^2) \) per Definition 5. We denote the mean of the Folded Normal distribution by \( f(\mu, \sigma) := \mathbb{E}(T) \). Given this setup, and fixing \( \sigma > 0 \), we note the following important properties of \( f(\mu, \sigma) \):

\[
\begin{align*}
    f(\mu, \sigma) &\geq 0 \text{ for each } \mu \in \mathbb{R} \\
    f(\mu, \sigma) &\geq \mu \text{ for each } \mu \in \mathbb{R} \\
    f(\mu, \sigma) &\text{ is strictly increasing in } \mu \in \mathbb{R}_{>0} \\
    \frac{\partial f(\mu, \sigma)}{\partial \mu} &\in (0, 1) \text{ is for each } \mu \in \mathbb{R}_{>0} \\
    f(\mu, \sigma) &\text{ is } 1\text{-Lipschitz for each } \mu \in \mathbb{R}_{>0} \\
    f(\mu, \sigma)^2 &\leq \mu^2 + \sigma^2 \text{ for each } \mu \in \mathbb{R}_{\geq0}
\end{align*}
\]

Additionally for \( \mu_1 \leq \ldots \leq \mu_n \) we have that the relationship holds for \( V(f, \mu, \sigma) \), i.e., the total variation of the mean of the Folded Normal distribution:

\[
V(f, \mu, \sigma) := \sum_{i=1}^{n-1} |f(\mu_{i+1}, \sigma) - f(\mu_i, \sigma)| \leq \mu_n - \mu_1
\]

Proof of Lemma 18. We prove each property (Equations (41) to (47)) in turn. As per the assumption \( \sigma > 0 \) is fixed and that \( R \sim \mathcal{N}(\mu, \sigma^2) \) for \( \mu \in \mathbb{R} \).

(Proof of Equation (41).) We have that \( T := |R| \geq 0 \) a.s. \( \implies f(\mu, \sigma) := \mathbb{E}(T) = \mathbb{E}(|R|) \geq 0 \) by the monotonicity of expectation, as required. \( \blacksquare \)

(Proof of Equation (42).) We have that \( R \leq |R| \) a.s. \( \implies \mu := \mathbb{E}(R) \leq \mathbb{E}(|R|) = \mathbb{E}(T) =: f(\mu, \sigma) \) again by the monotonicity of expectation, as required. \( \blacksquare \)

(Proof of Equation (43).) For any \( \mu > 0 \) we have that:

\[
\begin{align*}
    f(\mu, \sigma) &:= \sigma \sqrt{\frac{2}{\pi}} \exp \left( -\frac{\mu^2}{2\sigma^2} \right) - \mu \left( 1 - 2\Phi \left( \frac{\mu}{\sigma} \right) \right) \quad \text{(per Equation (15))} \\
    \implies \frac{\partial f(\mu, \sigma)}{\partial \mu} &= -\frac{\mu}{\sigma} \sqrt{\frac{2}{\pi}} \exp \left( -\frac{\mu^2}{2\sigma^2} \right) - 1 + 2\Phi \left( \frac{\mu}{\sigma} \right) + \frac{2\mu}{\sigma} \phi \left( \frac{\mu}{\sigma} \right) \\
    &= 2\Phi \left( \frac{\mu}{\sigma} \right) - 1 \\
    &> 0 \quad \text{(since } \mu, \sigma > 0 \text{ and } \Phi \left( \frac{\mu}{\sigma} \right) > \frac{1}{2})
\end{align*}
\]

as required. \( \blacksquare \)

(Proof of Equation (44).) By the previous proof, we note that \( \frac{\partial f(\mu, \sigma)}{\partial \mu} > 0 \). Also using the previous proof and noting that \( \Phi(x) > \frac{1}{2} \) for each \( x > 0 \), it follows that \( \frac{\partial f(\mu, \sigma)}{\partial \mu} = -1 + 2\Phi \left( \frac{\mu}{\sigma} \right) < 1 \). Combining both parts we have that \( \frac{\partial f(\mu, \sigma)}{\partial \mu} \in (0, 1) \), as required. \( \blacksquare \)

(Proof of Equation (45).) By the previous proof, we note that \( \frac{\partial f(\mu, \sigma)}{\partial \mu} \in (0, 1) \implies \left| \frac{\partial f(\mu, \sigma)}{\partial \mu} \right| \leq
for each $\mu > 0$. It follows by the mean value theorem, that $f(\mu, \sigma)$ is 1-Lipschitz as required.

(Proof of Equation (46).) Observe that from Equation (16) we have that $g(\mu, \sigma) := \text{Var}(T) = \mu^2 + \sigma^2 - f(\mu, \sigma)^2$. Since $\text{Var}(T) \geq 0$, it follows that $f(\mu, \sigma)^2 \leq \mu^2 + \sigma^2$ for each $\mu \in \mathbb{R}$, as required.

(Proof of Equation (47).) Let $i \in [n]$ be arbitrary. Now note that by the Equation (45) property it follows that, we then have that:

\[
V(f, \mu, \sigma) = \sum_{i=1}^{n-1} |f(\mu_{i+1}, \sigma) - f(\mu_i, \sigma)|
\]

(by definition)

\[= \sum_{i=1}^{n-1} f(\mu_{i+1}, \sigma) - f(\mu_i, \sigma)
\]

(using Equation (43) and monotonicity of $\mu_1 \leq \ldots \leq \mu_n$)

\[= f(\mu_n, \sigma) - f(\mu_1, \sigma)
\]

(by telescoping sum)

\[\leq |\mu_n - \mu_1|
\]

(using Equation (45))

\[= \mu_n - \mu_1
\]

(by monotonicity of $\mu_1 \leq \ldots \leq \mu_n$)

as required.

Thus all properties specified in Equations (41) to (47) are now proved.

A.5 Properties of the folded normal variance: $g(\mu, \sigma)$

Lemma 19 (Properties of the Folded Normal variance). Let $T \sim \text{FoldNorm}(\mu, \sigma^2)$ per Definition 5, and let $g(\mu, \sigma) := \text{Var}(T)$. Given this setup, and fixing $\sigma > 0$, we note the following properties of $g(\mu, \sigma)$:

\[g(\mu, \sigma) \leq \sigma^2, \text{ for each } \mu \in \mathbb{R}\]  \hspace{1cm} (48)

\[g(\mu, \sigma) \geq g(0, \sigma), \text{ for each } \mu \in \mathbb{R}_{>0}\]  \hspace{1cm} (49)

\[\text{Var}(T^2) = 4\mu^2\sigma^2 + 2\sigma^4, \text{ for each } \mu \in \mathbb{R}\]  \hspace{1cm} (50)

Proof of Lemma 19. We prove each properties specified in Equations (48) to (50) in turn.

(Proof of Equation (48).) We have for each $\mu \geq 0$

\[g(\mu, \sigma) := \mu^2 + \sigma^2 - f(\mu, \sigma)^2 \hspace{1cm} \text{(per Equation (16))}
\]

\[\leq \sigma^2 \hspace{1cm} \text{(since } f(\mu, \sigma)^2 \geq \mu^2 \text{ using Equation (42))}
\]

as required.

(Proof of Equation (49).) First note that $g(0, \sigma) = \sigma^2 - f(0, \sigma)^2 = \sigma^2 - \left(\frac{2}{\pi}\right)\sigma^2$. It then
follows that:

\[ g(\mu, \sigma) \geq g(0, \sigma) \]

\[ \iff \mu^2 + \sigma^2 - f(\mu, \sigma)^2 \geq \sigma^2 - \left(\frac{2}{\pi}\right)\sigma^2 \]

\[ \iff \mu^2 + \left(\frac{2}{\pi}\right)\sigma^2 \geq f(\mu, \sigma)^2 \]  \ (51)

We will then prove the equivalent statement Equation (51). Since \( \mu, \sigma > 0 \) in our case, let \( \nu := \frac{\mu}{\sigma} > 0 \) in what follows. Then dividing both sides of Equation (51) by \( \nu \) we obtain:

\[ \sqrt{\nu^2 + \frac{2}{\pi}} \geq \nu(2\Phi(\nu) - 1) + \sqrt{\frac{2}{\pi} e^{-\nu^2/2}} \]

Let us then define

\[ g(\nu) := \sqrt{\nu^2 + \frac{2}{\pi}} - \nu(2\Phi(\nu) - 1) - \sqrt{\frac{2}{\pi} e^{-\nu^2/2}} \]  \ (52)

Taking the derivative of \( g(\nu) \) with respect to \( \nu \) we obtain:

\[ g'(\nu) = \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} - 2\Phi(\nu) + 1 - \frac{2\nu}{\sqrt{2\pi}} e^{-\nu^2/2} + \nu \sqrt{\frac{2}{\pi}} e^{-\nu^2/2} \]

\[ = \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} - 2\Phi(\nu) + 1 \]  \ (53)

\[ = \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} - 2\Phi(\nu) + 1 \]  \ (54)

As such all global and local extrema are obtained by setting \( g'(\nu) = 0 \), that is:

\[ g'(\nu) = 0 \]

\[ \iff \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} = 2\Phi(\nu) - 1 \]  \ (55)

Now, \( \nu = 0 \) is a clear solution, at which our function is exactly equal to 0. Also, we need to look at \( \nu = \infty \), where we also have an identity. So we need to take care of other possible roots to the equation \( \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} = 2\Phi(\nu) - 1 \). Now observe that since when \( \nu \geq 0 \) the function \( \Phi(\nu) \) is concave and therefore \( 2\Phi(\nu) - 1 = 2(\Phi(\nu) - \Phi(0)) \leq 2\nu \Phi(0) = \sqrt{\frac{2}{\pi}} \nu \). Thus for any non-zero solution \( \bar{\nu} \) to the equation \( \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} = 2\Phi(\nu) - 1 \), we must have \( \frac{\nu}{\sqrt{\nu^2 + \frac{2}{\pi}}} \leq \bar{\nu} \sqrt{\frac{2}{\pi}} \).

This implies that \( \bar{\nu}^2 \geq \frac{\pi}{2} - \frac{2}{\pi} \). Now, going back to the original function we need to show

\[ \sqrt{\bar{\nu}^2 + \frac{2}{\pi}} \geq \bar{\nu}(2\Phi(\bar{\nu}) - 1) + \sqrt{\frac{2}{\pi}} e^{-\bar{\nu}^2/2} = \frac{\bar{\nu}^2}{\sqrt{\bar{\nu}^2 + \frac{2}{\pi}}} + \sqrt{\frac{2}{\pi}} e^{-\bar{\nu}^2/2}. \]

The latter is equivalent to \( \bar{\nu}^2 + \frac{2}{\pi} \geq \bar{\nu}^2 + \sqrt{\frac{2}{\pi}} e^{-\bar{\nu}^2/2} \sqrt{\bar{\nu}^2 + \frac{2}{\pi}} \), which is equivalent to \( \frac{2}{\pi} e^{\bar{\nu}^2} \geq \bar{\nu}^2 + \frac{2}{\pi} \), since the function \( \nu \mapsto \frac{2}{\pi} e^{\nu^2} - \nu^2 \) is increasing for positive \( \nu \) it suffices to check that
$\frac{2}{\pi} e^{\bar{\nu}^2} \geq \bar{\nu}^2 + \frac{2}{\pi}$ for $\bar{\nu} = \frac{\pi}{2} - \frac{2}{\pi}$ (since as we know from before $\bar{\nu}$ is at least that value). This is true, and completes the proof, as required. ■

(Proof of Equation (50).) By direct calculation we have:

$$\text{Var} (T^2) = \mathbb{E}(T^4) - (\mathbb{E}(T^2))^2$$

$$= \mathbb{E}(R^4) - (\mathbb{E}(R^2))^2$$

$$= (\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4) - (\mu^2 + \sigma^2)^2$$

$$= 4\mu^2\sigma^2 + 2\sigma^4$$

as required. ■

Thus all properties specified in Equations (48) to (50) are now proved. □

A.6 Properties of the inverse folded normal mean: $f^{-1}(\mu, \sigma)$

Lemma 20 (Properties of the Folded Normal mean inverse). Suppose $R \sim \mathcal{N}(\mu, \sigma^2)$. Let $T \stackrel{a.s.}{=} |R|$, then $T \sim \text{FoldNorm}(\mu, \sigma^2)$ per Definition 5. We denote the mean of the Folded Normal distribution by $f(\mu, \sigma) := \mathbb{E}(T)$. Given this setup, and fixing $\sigma > 0$, we note the following important properties of $f^{-1}(u, \sigma)$ (which denotes the inverse with respect to $\mu$ function of $f(\mu, \sigma)$ when $\sigma$ is held fixed):

$$f^{-1}(u, \sigma) \text{ exists,} \quad (57)$$

$$\frac{\partial}{\partial \sigma} f^{-1}(u, \sigma) = -\frac{\sqrt{2/\pi} \exp(-\mu^2/(2\sigma^2))}{2\Phi(\mu/\sigma) - 1}, \quad (58)$$

$$\frac{\partial}{\partial u} f^{-1}(u, \sigma) = 1/(2\Phi(\mu/\sigma) - 1), \quad (59)$$

$$f^{-1}(u, \sigma) \text{ is a Lipschitz function for each } u > f(\eta, \sigma) > 0 \text{ for a fixed } \sigma, \quad (60)$$

where in the above $u = f(\mu, \sigma)$ (or in other words $\mu = f^{-1}(u, \sigma)$).

Proof of Lemma 20. We prove each properties specified in Equations (57) to (60) in turn.

(Proof of Equation (57).) Note that for a fixed $\sigma > 0$ the function $f(\mu, \sigma)$ is invertible (as it is increasing, per Lemma 18), as required. ■

(Proof of Equation (58).) In order to find the derivative of $\frac{\partial}{\partial \sigma} f^{-1}(\cdot, \sigma)$, we can parametrize as follows:

$$u = f(\mu, \sigma) \quad (61)$$

$$v = \sigma \quad (62)$$

We will use the inverse function theorem which says that under certain conditions $\mu = F(u, v) = F(u, \sigma)$ and $\sigma = G(u, v) = v$, for some functions $F$ and $G$. Note that for a fixed $\sigma > 0$ the function $f(\mu, \sigma)$ is invertible (per Equation (57)). Thus

$$\frac{\partial}{\partial \sigma} f^{-1}(u, \sigma) = \frac{\partial \mu}{\partial \sigma} = \frac{\partial F(u, v)}{\partial v} = \frac{-\partial v}{\partial \sigma} \frac{\partial F(u, v)}{\partial v} = -\frac{\sqrt{2/\pi} \exp(-\mu^2/(2\sigma^2))}{J} \quad (63)$$
Where $J$ is the Jacobian of the transformation

$$J = \begin{vmatrix} \frac{\partial u}{\partial \mu} & \frac{\partial u}{\partial \sigma} \\ \frac{\partial v}{\partial \mu} & \frac{\partial v}{\partial \sigma} \end{vmatrix}$$

$$= 2\Phi(\mu/\sigma) - 1$$

It follows that

$$\frac{\partial}{\partial \sigma} f^{-1}(u, \sigma) = -\sqrt{2/\pi} \exp(-\mu^2/(2\sigma^2)) \frac{1}{2\Phi(\mu/\sigma) - 1}$$

(64)

As required.

(Proof of Equation (59).) We similarly evaluate the derivative $\frac{\partial}{\partial u} f^{-1}(u, \sigma)$ as follows:

$$\frac{\partial}{\partial u} f^{-1}(u, \sigma) = \frac{\partial}{\partial u} \mu = \frac{\partial}{\partial \sigma} \frac{1}{J}$$

(65)

(66)

(67)

As required.

(Proof of Equation (60).) We note that Equation (59) implies that

$$\frac{\partial}{\partial u} f^{-1}(u, \sigma) \leq \frac{1}{2\Phi(\eta/\sigma) - 1}$$

(68)

since $\mu \geq \eta > 0$ under our setting. In this case, this holds for each $u \geq f(\eta, \sigma) > 0$, for a fixed $\sigma$. Since this derivative is bounded by this constant, it follows that $f^{-1}(u, \sigma)$ is Lipschitz by applying Lemma 16.

Thus all properties specified in Equations (57) to (60) are now proved.

\[\square\]

A.7 Properties of: $J(\sigma)$

Definition 21 ($J(\sigma)$). Let $\eta > 0$ be fixed, and $\sigma \geq 0$ per Equations (4) and (5), respectively. We define the function, $J : \mathbb{R}_{\geq 0} \to \mathbb{R}$, as:

$$J(\sigma) := \begin{cases} 0 & \text{if } \sigma = 0 \\ \sigma \left( \frac{1}{2} - \frac{\eta/\sigma \phi(\eta/\sigma)}{2\Phi(\eta/\sigma) - 1} \right) & \text{otherwise} \end{cases}$$

(69)

In order to prove the key properties of $J(\sigma)$, we will first need to prove a useful result in Lemma 22.

Lemma 22. We define the function, $M : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, as:

$$M(x) := \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{x \phi(x)}{2\Phi(x) - 1} & \text{otherwise} \end{cases}$$

(70)

Note that $M(0) = \frac{1}{2}$, by Equation (37). Then $M(x)$ is strictly decreasing for each $x > 0$. 

27
Proof of Lemma 22. In order to show that \( M(x) \) is decreasing for each \( x > 0 \), we will show that \( M'(x) < 0 \) for each \( x > 0 \). To see this, first observe that:

\[
M'(x) = \frac{(2\Phi(x) - 1)(\phi(x) + x\phi'(x)) - 2x\phi^2(x)}{(2\Phi(x) - 1)^2} = \frac{(2\Phi(x) - 1)(\phi(x) - x^2\phi(x)) - 2x\phi^2(x)}{(2\Phi(x) - 1)^2} \quad \text{(since } \phi'(x) + x\phi(x) = 0) \\
= \frac{\phi(x)}{(2\Phi(x) - 1)^2} \left( (2\Phi(x) - 1)(1 - x^2) - 2x\phi(x) \right) \\
= \frac{\phi(x)}{(2\Phi(x) - 1)^2} \left( 2\left(\Phi(x) - \frac{1}{2}\right)(1 - x^2) - 2x\phi(x) \right). \tag{71}
\]

Now we see that:

\[
2\left(\Phi(x) - \frac{1}{2}\right) = 2(\Phi(x) - \Phi(0)) = 2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{2}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt \leq \frac{2x}{\sqrt{2\pi}}, \tag{72}
\]

where the last inequality in Equation (72) followed from the fact that \( e^{-\frac{t^2}{2}} \leq 1 \) for each \( t \geq 0 \). It then follows that:

\[
\left(2\left(\Phi(x) - \frac{1}{2}\right)(1 - x^2) - 2x\phi(x)\right) = \left(\frac{2}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt\right)(1 - x^2) - \frac{2x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \leq \frac{2x}{\sqrt{2\pi}} (1 - x^2) - \frac{2x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{(using Equation (72))} \\
< 0, \tag{73}
\]

Where Equation (73) followed by observing that since \( 1 - x^2 < 1 - \frac{x^2}{2} < e^{-\frac{x^2}{2}} \) for each \( x > 0 \). Now since \( \frac{\phi(x)}{(2\Phi(x) - 1)^2} > 0 \) for each \( x > 0 \), we have by applying Equation (73) to Equation (72) that \( M'(x) < 0 \), for each \( x > 0 \), as required. \( \square \)

Lemma 23 (Properties of \( J(\sigma) \)). Let \( J(\sigma) \) be defined as per Equation (69). Then \( J(\sigma) \) satisfies the following properties:

\[
J(\sigma) > 0 \text{ for each } \sigma \in \mathbb{R}_{>0} \text{ and } 0 \text{ if and only if } \sigma = 0 \tag{74}
\]

\[
J(\sigma) \text{ is continuous for each } \sigma \in \mathbb{R}_{>0} \tag{75}
\]

\[
\text{For any } 0 < \sigma_1 < \sigma_2, \quad \min_{\sigma \in [\sigma_1, \sigma_2]} J(\sigma) \geq \sigma_1 \left(1 - \frac{\eta/\sigma_2 \phi(\eta/\sigma_2)}{2\Phi(\eta/\sigma_2) - 1}\right) > 0 \tag{76}
\]

Proof of Lemma 23. We prove each property (Equations (74) to (76)) in turn. Throughout these proofs, we write:

\[
J(\sigma) := J_1(\sigma)J_2(\sigma), \text{ where } J_1(\sigma) := \sigma, \text{ and } J_2(\sigma) := \frac{1}{2} - \frac{\eta/\sigma \phi(\eta/\sigma)}{2\Phi(\eta/\sigma) - 1}. \tag{77}
\]

(Proof of Equation (74).) Observe that both \( J_1(\sigma), J_2(\sigma) \) are zero if and only if \( \sigma = 0 \). In the case of \( J_2(\sigma) \) this follows from Lemma 17. Now for \( \sigma > 0, J_1(\sigma) := \sigma > 0 \), by assumption. And the fact that \( J_2(\sigma) > 0 \), for \( \sigma > 0 \) again follows directly from Lemma 17. As such, \( J(\sigma) > 0 \) for each \( \sigma > 0 \), since it is the product of two strictly positive functions over this
support, as required. ■

(Proof of Equation (75).) \( J_1(\sigma) \) is continuous for \( \sigma > 0 \). Moreover since \( \phi(x), \Phi(x) \) for a standard normal are continuous over their support, \( \mathbb{R} \), it follows that \( J_2(\sigma) \) is also continuous for \( \sigma > 0 \). As such, \( J(\sigma) \) is continuous for each \( \sigma > 0 \), since it is the product of two continuous functions, as required. ■

(Proof of Equation (76).) Note that for any two fixed \( \sigma_1, \sigma_2 \), such that \( 0 < \sigma_1 < \sigma_2 \), the interval \([\sigma_1, \sigma_2]\) is compact. From Equation (75), we know that \( J(\sigma) \) is continuous for \( \sigma > 0 \), and so it attains its minimum (and maximum) on this interval. Moreover, from Equation (74), it follows that \( \min_{\sigma \in [\sigma_1, \sigma_2]} J(\sigma) > 0 \). Now we note that \( \sigma \mapsto J_1(\sigma) := \sigma \), is increasing in \( \sigma \). Moreover for each \( \sigma > 0 \), we have that \( J_2(\sigma) := \frac{1}{2} - M(\frac{\eta}{\sigma^2}) \), where the function \( M \) is as defined in Equation (70). Moreover it follows from Lemma 22 that \( J_2(\sigma) \) is strictly decreasing for each \( \sigma > 0 \). By the non-negativity of \( J(\sigma) \) over its domain, we have that \( \min_{\sigma \in [\sigma_1, \sigma_2]} J(\sigma) \geq J_1(\sigma_1)J_2(\sigma_2) = \sigma_1 \left( \frac{1}{2} - \frac{\eta/\sigma_2 \phi(\eta/\sigma_2)}{2 \Phi(\eta/\sigma_2)} \right) > 0 \), as required. ■

Thus all properties specified in Equations (74) to (76) are now proved. □

A.8 Properties of: \( G(\sigma) \)

Definition 24 (\( G(\sigma) \)). Under the setup of ASCI generating process per Definition 2, and per the ASCIFIT model we define the function, \( G : \mathbb{R}_{\geq 0} \to \mathbb{R} \), as:

\[
G(\sigma) := \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} \left( f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma) \right)^2 \tag{78}
\]

Lemma 25 (Properties of \( G(\sigma) \)). Under the setup of ASCI generating process per Definition 2, and with \( G(\sigma) \) defined as per Definition 24, we note the following important properties of \( G(\sigma) \):

\[
\frac{\partial}{\partial \sigma} G(\sigma) = \frac{4}{n} \sum_{i=1}^{n} \sigma \left( \frac{1}{2} - \frac{f^{-1}(\hat{T}_i, \sigma)/\sigma \phi(f^{-1}(\hat{T}_i, \sigma)/\sigma) \mathbb{I}(\hat{T}_i \geq f(\eta, \sigma))}{2 \Phi(f^{-1}(\hat{T}_i, \sigma)/\sigma) - 1} \right), \tag{79}
\]

\( G(\sigma) \) is increasing for \( \sigma \geq 0 \), and strictly increasing for \( \sigma > 0 \). \( \quad \tag{80} \)

Proof of Lemma 25. We prove each property (Equations (79) and (80)) in turn. Throughout these proofs, \( J(\sigma) \) is as defined in Definition 21, and \( G(\sigma) \) is as defined in Definition 24.
(Proof of Equation (79).) Using the definition, we have:

\[
\frac{\partial}{\partial \sigma} G(\sigma) = 2\sigma - \frac{2}{n} \sum_{i=1}^{n} \sqrt{\frac{2}{\pi} f^{-1}(\hat{T}_i, \sigma)} \exp\left(-f^{-1}(\hat{T}_i, \sigma)^2/(2\sigma^2)\right) \mathbb{1}(\hat{T}_i \geq f(\eta, \sigma)) \frac{1}{2\Phi(f^{-1}(\hat{T}_i, \sigma)/\sigma - 1)} \\
\text{(using Equation (58))}
\]

\[
= 2\sigma - \frac{4\sigma}{n} \sum_{i=1}^{n} \frac{f^{-1}(\hat{T}_i, \sigma)/\sigma \phi(f^{-1}(\hat{T}_i, \sigma)/\sigma) \mathbb{1}(\hat{T}_i \geq f(\eta, \sigma))}{2\Phi(f^{-1}(\hat{T}_i, \sigma)/\sigma - 1)}
\]

\[
= \frac{4}{n} \sum_{i=1}^{n} \sigma \left( \frac{1}{2} - \frac{f^{-1}(\hat{T}_i, \sigma)/\sigma \phi(f^{-1}(\hat{T}_i, \sigma)/\sigma) \mathbb{1}(\hat{T}_i \geq f(\eta, \sigma))}{2\Phi(f^{-1}(\hat{T}_i, \sigma)/\sigma - 1)} \right)
\]

as required.

(Proof of Equation (80).) Now per Lemma 17, we have that \( x \mapsto x \phi(x)/(2\Phi(x) - 1) \leq 1/2 \) for all \( x \geq 0 \), and moreover it is decreasing for \( x > 0 \). Therefore the derivative, \( \frac{\partial}{\partial \sigma} G(\sigma) \), is bounded from below by 0. As such \( G(\sigma) \) is increasing in \( \sigma \), for \( \sigma \geq 0 \). In fact, since \( \eta, \sigma > 0 \), it follows that \( \frac{\partial}{\partial \sigma} G(\sigma) \) is bounded from below by \( J(\sigma) := \sigma \left( \frac{1}{2} - \frac{\eta/\phi(\eta/\sigma)}{2\Phi(\eta/\sigma) - 1} \right) > 0 \), for each \( \sigma > 0 \), using Lemma 23. It follows that \( G(\sigma) \) is strictly increasing in \( \sigma \), for \( \sigma > 0 \), as required.

Thus all properties specified in Equations (79) and (80) are now proved. □
B Proofs of Section 1

B.1 Mathematical Preliminaries

Lemma 26 (Symmetrization with Rademacher random variables). Suppose that $\varepsilon$ is a symmetric distribution i.e. $\varepsilon \overset{d}{=} -\varepsilon$, $\xi \sim \text{Rademacher}(\alpha)$, with $\alpha \in [0, 1]$. If $\xi \perp \varepsilon$ then $\xi \varepsilon \overset{d}{=} \varepsilon$.

Proof of Lemma 26. Let us define $Q := \xi \varepsilon$. We then have the following:

\[
\begin{align*}
\Pr (Q \geq q) := & \Pr (\xi \varepsilon \geq q) \\
= & \Pr (\xi \varepsilon \geq q \mid \xi = -1) \Pr (\xi = -1) + \Pr (\xi \varepsilon \geq q \mid \xi = 1) \Pr (\xi = 1) \\
= & \Pr (-\varepsilon \geq q) (1 - \alpha) + \Pr (\varepsilon \geq q) (\alpha) \\
= & \Pr (\varepsilon \geq q) (1 - \alpha) + \Pr (\varepsilon \geq q) (\alpha) \\
= & \Pr (\varepsilon \geq q) (1 - \alpha + \alpha) \\
= & \Pr (\varepsilon \geq q)
\end{align*}
\]

So we have that $Q := \xi \varepsilon \overset{d}{=} \varepsilon$, as required. \qed

The setting can be simplified if the adversary chooses the sign-corruptions independent of the error terms. To see this, first note that $\varepsilon_i$ are centered (i.e. symmetric) Gaussian random variables. Now, if the $(\xi_1, \ldots, \xi_n)$ are picked independently from $(\varepsilon_1, \ldots, \varepsilon_n)$, the ASCI generating process response reduces to $R_i = \xi_i \mu_i + \varepsilon_i$. That is our setting encompasses this more simplified setting, and is shown formally in Corollary 27. Further, we note that in the case where $\xi_i \overset{a.s.}{=} 1$ then and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$, then this is equivalent to the standard univariate isotonic regression setup.

Corollary 27. In the case where $\xi_i \perp \varepsilon_i$, for each $i \in [n]$ we have that the ASCI generating process simplifies to $R_i = \xi_i \mu_i + \varepsilon_i$.

Proof of Corollary 27. We note that the underlying adversarial generating process is given by $R_i = \xi_i (\mu_i + \varepsilon_i) = \xi_i \mu_i + \xi_i \varepsilon_i$, for each $i \in [n]$. Now since $\xi_i \perp \varepsilon_i$ we have by applying Lemma 26 for each $i \in [n]$ that $\xi_i \varepsilon_i \overset{i.i.d.}{\sim} \varepsilon_i$. And so the required adversarial model can be written as $R_i = \xi_i \mu_i + \varepsilon_i$, as required. \qed

B.2 Important Model Definitions

First, we formally (redefine) the generating model described in Example 3.

Definition 28 (Two-component Gaussian mixture ASCI special case from Example 3). We consider $n$ observations, $\{R_i \mid i \in [n]\}$, where each observation $R_i$ is generated from the following model:

\[
\begin{align*}
R_i &= \xi_i \mu_i + \varepsilon_i \\
\text{s.t. } 0 < \eta &\leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \\
\text{and } &\varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \\
\text{and } &\xi_i \overset{i.i.d.}{\sim} \text{Rademacher}(p), p \in (0, 1), \text{ and } \xi_i \perp \varepsilon_i
\end{align*}
\]
Second, we formally define the generating model described in Example 4.

**Definition 29** (Non-convex generating model from Example 4). We consider \( n \) observations, \( \{ R_i | i \in [n] \} \), where each observation \( R_i \) is generated from the following model:

\[
R_i = \gamma_i + \epsilon_i \tag{86}
\]

s.t. \( 0 < \eta \leq |\gamma_1| \leq |\gamma_2| \leq \ldots \leq |\gamma_n| \) \tag{87}

and \( \epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2) \) \tag{88}

**Remark 8.** From a simulation perspective, each \( \gamma_i \) is generated first subject to Equation (87), then \( \xi_i \) is sampled independently, and both are added to give each response \( R_i \).

Third, we introduce an alternative model as per Definition 30.

**Definition 30** (Alternative non-convex model).

\[
R_i = \xi_ia_i + \epsilon_i \tag{89}
\]

s.t. \( 0 < \eta \leq a_1 \leq a_2 \leq \ldots \leq a_n \) \tag{90}

and \( \epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2) \) \tag{91}

and \( \xi_i = \text{sgn}(\gamma_i) \) \tag{92}

and \( \xi_i \perp \epsilon_i \) \tag{93}

and \( a_i = |\gamma_i| \) \tag{94}

Finally, for convenience we recall Definition 2 as follows.

**Definition 2** (Adversarial sign-corrupted isotonic (ASCI) regression). We consider \( n \) observations, \( \{ R_i | i \in [n] \} \), where each observation \( R_i \) is generated from the following model:

\[
R_i = \xi_i(\mu_i + \epsilon_i) \tag{4}
\]

s.t. \( 0 < \eta \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \) \tag{5}

and \( \epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2) \) \tag{6}

and \( \xi_i \in \{-1, 1\} \) \tag{7}

**B.3 Proof justification for Example 3**

**Proposition 31** (Justification for Example 3). Under the model generating process described in Example 3 (i.e., per Definition 28), the following model definition inclusion holds.

\[
\text{Definition 28} \subseteq \text{Definition 2} \tag{95}
\]

**Remark 9.** Here, each definitional inclusion is to be read as the former generating model definition being a special case of the latter generating model definition.

**Proof of Example 3.** Our basic strategy is to show each model inclusion in turn.

(Definition 28 \( \subseteq \) Definition 2). Observe that Equations (83) and (84) are definitionally equivalent to Equations (5) and (6), respectively. Moreover we have that Equation (85) is a special case of Equation (7). Finally, from Equation (85) we have that \( \xi_i \overset{i.i.d.}{\sim} \text{Rademacher}(p), p \in \)
(0,1), and \( \xi_i \perp \epsilon_i \). Thus from Corollary 27, it follows that Equation (82) is a special case of Equation (82).

In summary we have shown that Equation (95) holds, from which it follows that Example 3 (or equivalently Definition 28) is a special case of Definition 2, as required.

\[ \Box \]

\[ \text{B.4 Proof justification for Example 4} \]

We now provide a formal proof justification that Example 4 is a special case of the generating process described in Definition 2.

**Proposition 32** (Justification for Example 4). Under the model generating process described in Example 4, the following model definition inclusion holds.

\[
\text{Definition 29} = \text{Definition 30} \subseteq \text{Definition 2} \quad (96)
\]

**Remark 10.** As with Proposition 31, each definitional inclusion is to be read as the former generating model definition being a special case of the latter generating model definition. In the case of equivalence, we note that both inclusions hold between the model definitions.

**Proof of Example 4.** Our basic strategy is to show each model inclusion in turn.

(Definition 29 = Definition 30) This follows by construction. Observe that Equations (92) and (94) imply that \( \xi_i a_i = \text{sgn} (\gamma_i) |\gamma_i| = \gamma_i \), so that Equations (86) and (89) are equivalent. In addition from Equation (94), we have that \( a_i = |\gamma_i| \) and thus Equations (87) and (90) are equivalent, as are Equations (88) and (91). As such the equality is established between the two generating model definitions.

(Definition 30 \( \subseteq \) Definition 2). Observe that by Equations (6) and (91) are definitionally equivalent. Observe from Equation (92) that \( \xi_i = \text{sgn} (\gamma_i) \in \{-1,1\} \) which is a special case of Equation (7). For each observation \( i \in [n] \) using Equation (94) that by setting \( a_i := \mu_i \) that Equations (5) and (90) are equivalent. Finally since \( \xi_i \perp \epsilon_i \) from Equation (93), we note that Equation (89) is a special case of Equation (4) by applying Corollary 27 to the observation \( R_i \), for each \( i \in [n] \).

In summary we have shown that Equation (96) holds, from which it follows that Definition 29 is a special case of Definition 2, as required.

\[ \Box \]
C Proofs of Section 2

C.1 Mathematical Preliminaries

**Theorem 33** (Projection onto the nonnegative monotone cone). Suppose that $S^\uparrow \subseteq \mathbb{R}^n$ is the monotone cone, that is,

$$S^\uparrow := \{ \mu := (\mu_1, \ldots, \mu_n)^\top \in \mathbb{R}^n \mid \mu_1 \leq \cdots \leq \mu_n \}.$$  

and $S^\uparrow_+ \subseteq \mathbb{R}^n$ is the nonnegative monotone cone, that is,

$$S^\uparrow_+ := \{ \mu := (\mu_1, \ldots, \mu_n)^\top \in S^\uparrow \mid \mu_1 \geq 0 \}.$$  

Then for an arbitrary $v \in \mathbb{R}^n$ it holds that

$$\Pi_{S^\uparrow_+}(v) = (\Pi_{S^\uparrow}(v))^+,$$

where for any $z \in \mathbb{R}^n$, $z^+ \in \mathbb{R}^n$ stands for the lattice operation defined by the order induced by the nonnegative orthant in $\mathbb{R}^n$. That is, we define the operation componentwise as $(z^+_i) := (z)_i \lor 0$ for each component index $i \in [n]$.

**Proof of Theorem 33.** See Németh and Németh [2012, Corollary 1] for details.

**Remark 11.** In effect, Theorem 33 basically states that in order to project onto the nonegative monotone cone, $K$, one can instead first project onto the monotone cone, $W$, first, and then take the non-negative part along each component. This is useful, since one can leverage algorithms like the PAVA which already efficiently handle projection onto the unrestricted monotone cone, $W$.

C.2 Proof of Proposition 6

**Proposition 6** (Reason for the “$\lor f(\eta, \sigma)$”-correction in Step II). The need for defining the $\lor f(\eta, \sigma)$ in Equation (17) in Step II in ASCIFIT, is that the solution to the problem

$$\arg \min_{\tilde{T}_1, \ldots, \tilde{T}_n} \sum_{i=1}^n (T_i - \tilde{T}_i)^2 \text{ s.t. } f(\eta, \sigma) \leq \tilde{T}_1 \leq \cdots \leq \tilde{T}_n,$$  

is related to the solution to

$$\arg \min_{\hat{T}_1, \ldots, \hat{T}_n} \sum_{i=1}^n (T_i - \hat{T}_i)^2 \text{ s.t. } \hat{T}_1 \leq \cdots \leq \hat{T}_n,$$

as $\tilde{T}_i := \hat{T}_i \lor f(\eta, \sigma)$.

**Proof of Proposition 6.** This follows along the following lines. First subtract $f(\eta, \sigma)$ from all $\tilde{T}_i$ to bring the first problem to

$$\arg \min_{\tilde{T}_i} \sum_{i=1}^n ((T_i - f(\eta, \sigma)) - \tilde{T}_i)^2 \text{ s.t. } 0 \leq \tilde{T}_1 \leq \cdots \leq \tilde{T}_n,$$  

...
where $\bar{T}_i = \tilde{T}_i - f(\eta, \sigma)$. Now the solution to the unrestricted problem

$$
\arg \min_{T_i^*} \sum_{i=1}^{n} ((T_i - f(\eta, \sigma)) - T_i^*)^2 \text{ s.t. } T_1^* \leq \ldots \leq T_n^*,$$

is $T_i^* = \bar{T}_i - f(\eta, \sigma)$. Next we apply Theorem 33, we see that $\bar{T}_i = T_i^* \vee 0$, so that $\bar{T}_i = \tilde{T}_i + f(\eta, \sigma) = T_i^* \vee 0 + f(\eta, \sigma) = (T_i^* + f(\eta, \sigma)) \vee f(\eta, \sigma) = \tilde{T}_i \vee f(\eta, \sigma)$ which is what we wanted to show. \qed
D Proofs of Section 3

D.1 Mathematical Preliminaries

The key idea to prove this theorem here is to apply [Zhang, 2002, Theorem 2.2(ii)] to our specific setting. To ensure our work is self-contained, we translate this result into the notation of our paper:

**Theorem 34** (Theorem 2.2 (ii) [Zhang, 2002]). Let \( R_{n,p}(f, \mu, \sigma, \sigma_p) := \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( |\hat{T}_i - f(\mu_i, \sigma)|^p \right) \right)^{\frac{1}{p}} \).

Let \( \delta_i := T_i - f(\mu_i, \sigma) \) be independent, with \( \mathbb{E}(\delta_i) = 0 \) and \( \mathbb{E} \left( |\delta_i|^{p/2} \right) \leq \sigma_{p/2}^2 \), \( p \geq 1 \) then:

\[
R_{n,p}(f, \mu, \sigma, \sigma_p) \leq 2\frac{1}{p} \sigma_p C_p \min \left[ 1, \frac{3}{2} \left( \frac{3 \left( \frac{V(f, \mu, \sigma)}{n \sigma_p C_p} \right)}{3 - p} + \frac{1}{n} \int_0^n dx \left( x \vee \frac{1}{2} \right)^{\frac{p}{2}} \right) \right] \tag{98}
\]

where \( C_p \) are constants depending on \( p \) only in general.

**Proof of Theorem 34.** See Zhang [2002, Theorem 2.2(ii)] for details. Note that to translate between our notation and theirs respectively, we have \( \hat{T}_i \equiv y_i, \hat{T}_i \equiv \hat{f}_n(t_i), f(\mu_i, \sigma) \equiv f(t_i), \delta_i \equiv \varepsilon_i \) for each \( i \in [n] \). \( \square \)

**Corollary 35** (Upper bound for \( R_{n,2}^2(f, \mu, \sigma, \sigma_2) \)). In our setting, define \( X := \frac{1}{n} \sum_{i=1}^{n} \left( \hat{T}_i - f(\mu_i, \sigma) \right)^2 \). We then have:

\[
\mathbb{E} \left( X \right) \leq \min \left[ 2 \sigma^2 C_2^2, \frac{27}{4} \left( \frac{\mu_n - \mu_1}{n} \right)^{\frac{3}{2}} (\sigma C_2)^{\frac{3}{2}} + \frac{2 \sigma^2 C_2^2}{n} (1 + \log n) \right] \tag{99}
\]

where \( C_2 \) is a constant.

**Proof of Corollary 35.** Since \( X := \frac{1}{n} \sum_{i=1}^{n} \left( \hat{T}_i - f(\mu_i, \sigma) \right)^2 \), then \( X = R_{n,2}^2(f, \mu, \sigma, \sigma_2)^2 \), by definition in the setting of Theorem 34, assuming the relevant sufficient conditions are met. We now need to check the sufficient condition for Theorem 34. Here we have, for each \( i \in [n] \), that \( \delta_i := T_i - f(\mu_i, \sigma) \). Note that by definition \( \mathbb{E}(\delta_i) = \mathbb{E}(T_i) - f(\mu_i, \sigma) = 0 \). We observe that \( (\delta_1, \ldots, \delta_n) \) are independent since the original responses, i.e. \( (R_1, \ldots, R_n) \) are independent by assumption. And taking absolute values and centering are measurable transformations which preserve their independence. We note that as per Zhang [2002, Theorem 2.2(ii)], we are required to check the sufficient condition \( \mathbb{E} \left( |\delta_i|^{p/2} \right) \leq \sigma_{p/2}^2 \). In our case, with \( p = 2 \), this is equivalent to showing that \( \mathbb{E} \left( |\delta_i|^2 \right) \leq \sigma_2^2 \). Then for each \( i \in [n] \) we have:

\[
\mathbb{E} \left( |\delta_i|^2 \right) = \mathbb{E} \left( \delta_i^2 \right) = \mathbb{V} \mathbb{A} \mathbb{R} \left( T_i \right) \text{ (since } \delta_i \text{ are mean centered } T_i \text{ values.)}
\]

\[
= \var(g(\mu_i, \sigma)) \text{ (by definition.)}
\]

\[
\leq \sigma^2 \text{ (using Equation (48))}
\]

\[
= \sigma_2^2 \text{ (100)}
\]
As required, by defining $\sigma_2 := \sigma$. So we meet this sufficient condition. Additionally observe that
\[
\int_0^n \frac{dx}{(x \lor 1)} = \int_0^1 \frac{dx}{(x \lor 1)} + \int_1^n \frac{dx}{(x \lor 1)} \quad \text{(by truncation)}
\]
\[
= \int_0^1 dx + \int_1^n \frac{dx}{x}
\]
\[
= 1 + \log n \quad \text{(101)}
\]

Now, in our setting note that $V(f, \mu, \sigma) \leq \mu_n - \mu_1$ using Equation (47), it follows that:
\[
E(X) := R_{n,2}^2(f, \mu, \sigma, \sigma_2) \quad \text{(by definition.)}
\]
\[
\leq \min \left\{ 2\sigma_2^2 C_2^2, \frac{27}{4} \frac{(\mu_n - \mu_1)^2}{n} (\sigma_2 C_2)^{\frac{3}{2}} + \frac{2\sigma_2^2 C_2^2}{n} \int_0^n \frac{dx}{(x \lor 1)} \right\} \quad \text{(setting } p = 2 \text{ in Theorem 34.)}
\]
\[
= \min \left\{ 2\sigma_2^2 C_2^2, \frac{27}{4} \frac{(\mu_n - \mu_1)^2}{n} (\sigma_2 C_2)^{\frac{3}{2}} + \frac{2\sigma_2^2 C_2^2}{n} (1 + \log n) \right\} \quad \text{(using Equation (101))}
\]
\[
= \min \left\{ 2\sigma_2^2 C_2^2, \frac{27}{4} \frac{(\mu_n - \mu_1)^2}{n} (\sigma C_2)^{\frac{3}{2}} + \frac{2\sigma_2^2 C_2^2}{n} (1 + \log n) \right\} \quad \text{(since } \sigma_2 := \sigma \text{ per Equation (100))}
\]
\[
=: r_{n,2}(\mu_n, \mu_1, \sigma) \quad \text{(102)}
\]
as required.

\textbf{Lemma 36 (Concentration for mean Folded Normal).} \textbf{In our setting we assume that } \frac{1}{n} \sum_{i=1}^n \mu_i^2 \leq C, \text{ for each } n \in \mathbb{N}. \text{ Define } X := \frac{1}{n} \sum_{i=1}^n (T_i - f(\mu_i, \sigma))^2. \text{ We then have:}
\[
|X - E(X)| \leq 2\gamma \sigma \sqrt{\frac{5\sigma^2 + 4C}{n}} \quad \text{(103)}
\]
with probability at least $1 - \gamma^{-2}$, where $E(X) = \frac{1}{n} \sum_{i=1}^n g(\mu_i, \sigma) = \frac{1}{n} \sum_{i=1}^n (\mu_i^2 + \sigma^2 - f(\mu_i, \sigma)^2).$

\textbf{Proof of Lemma 36.} First we determine $E(X)$ as follows:
\[
E(X) := E\left( \frac{1}{n} \sum_{i=1}^n (T_i - f(\mu_i, \sigma))^2 \right) \quad \text{(by definition of } X) \]
\[
= \frac{1}{n} \sum_{i=1}^n E\left( (T_i - f(\mu_i, \sigma))^2 \right) \quad \text{(by since } f(\mu_i, \sigma) := E(T_i).) \]
\[
= \frac{1}{n} \sum_{i=1}^n \text{Var}(T_i) \quad \text{(by since } g(\mu_i, \sigma) := \text{Var}(T_i).) \]
\[
= \frac{1}{n} \sum_{i=1}^n g(\mu_i, \sigma) \quad \text{(using Equation (16))}
\]
\[
= \frac{1}{n} \sum_{i=1}^n (\mu_i^2 + \sigma^2 - f(\mu_i, \sigma)^2) \quad \text{(102)}
\]
as required. Next we determine \( \text{Var}(X) \) as follows:

\[
\text{Var}(X) := \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} (T_i - f(\mu_i, \sigma))^2 \right) \quad \text{(by definition of } X) \\
= \frac{1}{n^2} \sum_{i=1}^{n} \text{Var} \left( (T_i - f(\mu_i, \sigma))^2 \right) \quad \text{(by the independence of } T_i) 
\]

Now note that for each \( i \in [n] \) we have:

\[
\text{Var} \left( (T_i - f(\mu_i, \sigma))^2 \right) = \text{Var} \left( T_i^2 + f(\mu_i, \sigma)^2 - 2f(\mu_i, \sigma)T_i \right) \\
= \text{Var} \left( T_i^2 - 2f(\mu_i, \sigma)T_i \right) \quad \text{(by translation invariance.)} \\
\leq 2 \left( \text{Var} \left( T_i^2 \right) + \text{Var} \left( 2f(\mu_i, \sigma)T_i \right) \right) \quad \text{(using Corollary 15)} \\
= 2 \text{Var} \left( T_i^2 \right) + 8f(\mu_i, \sigma)^2 \text{Var} \left( T_i \right) \\
= 2 \text{Var} \left( T_i^2 \right) + 8f(\mu_i, \sigma)^2 g(\mu_i, \sigma) \quad \text{(since } g(\mu_i, \sigma) := \text{Var} \left( T_i \right)) \\
\leq 16f(\mu_i, \sigma)^2 \sigma^2 + 4\sigma^4 \quad \text{(using Equation (42))} \\
\leq 16(\mu_i^2 + \sigma^2)\sigma^2 + 4\sigma^4 \quad \text{(using Equation (46))} \\
= 16\mu_i^2 \sigma^2 + 20\sigma^4 \quad \text{(104)}
\]

Therefore we have that

\[
\text{Var}(X) = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var} \left( (T_i - f(\mu_i, \sigma))^2 \right) \\
\leq \frac{1}{n^2} \sum_{i=1}^{n} (16f(\mu_i, \sigma)^2 \sigma^2 + 4\sigma^4) \quad \text{(using Equation (42))} \\
\leq \frac{1}{n^2} \sum_{i=1}^{n} (16\mu_i^2 \sigma^2 + 20\sigma^4) \quad \text{(using Equation (104))} \\
\leq \frac{16C\sigma^2 + 20\sigma^4}{n} \quad \text{(assuming } \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C, \text{ for each } n \in \mathbb{N}.)
\]

From this it follows that:

\[
\Pr \left( |X - E(X)| \geq t \right) \leq \frac{\text{Var}(X)}{t^2}, \text{ for each } t > 0 \quad \text{(using Chebychev’s inequality)} \\
\leq \frac{16C\sigma^2 + 20\sigma^4}{nt^2}, \text{ for each } t > 0 \quad \text{(105)}
\]

It then follows that by setting the upper bound (RHS) to \( \gamma^{-2} \in (0, 1) \), that

\[
\frac{16C\sigma^2 + 20\sigma^4}{nt^2} = \frac{1}{\gamma^2} \quad \Rightarrow \quad t = \gamma\sigma \sqrt{\frac{5\sigma^2 + 4C}{n}}
\]

We then have that \( |X - E(X)| \leq 2\gamma\sigma \sqrt{\frac{5\sigma^2 + 4C}{n}} \), with probability at least \( 1 - \gamma^{-2} \), as required. \( \square \)
Our end goal is to show a the following high probability result described in Theorem 37.

**Theorem 37** (Concentration of fitted Folded Normal).

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \hat{T}_i - f(\mu_i, \sigma) \right)^2 \leq \delta r_{n,2}(\mu_n, \mu_1, \sigma)
\]

with probability at least \(1 - \delta^{-1}\).

**Proof of Theorem 37.** First to simplify notation we let \(X := \frac{1}{n} \sum_{i=1}^{n} \left( \hat{T}_i - f(\mu_i, \sigma) \right)^2\) represent the quantity of interest. Observe that \(X \geq 0\) a.s. by definition, so that \(|X| \overset{a.s.}{=} X\). Then for any \(t > 0\) we have:

\[
\Pr(X \geq t) \leq \frac{\mathbb{E}(X)}{t} \leq \frac{R_{n,2}(f(\mu, \sigma))}{t} \leq \frac{r_{n,2}(\mu_n, \mu_1, \sigma)}{t}
\]

It then follows that by setting the upper bound (RHS) to \(\delta^{-1} \in (0, 1)\), that

\[
\frac{r_{n,2}(\mu_n, \mu_1, \sigma)}{t} = \frac{1}{\delta} \implies t = \delta r_{n,2}(\mu_n, \mu_1, \sigma)
\]

We then have that \(|X| \overset{a.s.}{=} X \leq \delta r_{n,2}(\mu_n, \mu_1, \sigma)\), with probability at least \(1 - \delta^{-1}\), as required. \(\square\)

**Lemma 38** (Concentration of \(\frac{1}{n} \sum_{i=1}^{n} T_i^2\)). In our setting, define \(X := \frac{1}{n} \sum_{i=1}^{n} T_i^2\). We then have:

\[
|X - \mathbb{E}(X)| \leq 2\gamma \sigma \sqrt{\frac{2\sigma^2 + 4C^2}{n}}
\]

with probability at least \(1 - \gamma^{-2}\), where \(\mathbb{E}(X) = \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2)\).

**Proof of Lemma 38.** Let \(X := \frac{1}{n} \sum_{i=1}^{n} T_i^2\). First we determine \(\mathbb{E}(X)\) as follows:

\[
\mathbb{E}(X) := \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} T_i^2 \right) \quad \text{(by definition of } X) \]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(T_i^2) \quad \text{(by linearity of expectation.)}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \text{Var}(T_i) + (\mathbb{E}(T_i))^2 \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2 - f(\mu_i, \sigma)^2 + f(\mu_i, \sigma)^2) \quad \text{(using Equations (15) and (16))}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2)
\]
as required. Next we determine $\text{Var}(X)$ as follows:

$$\text{Var}(X) := \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} T_i^2 \right)$$  \hspace{1cm} \text{(by definition of $X$.)}

$$= \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(T_i^2)$$  \hspace{1cm} \text{(by the independence of $T_i$.)}

$$= \frac{1}{n^2} \sum_{i=1}^{n} (4\mu_i^2 \sigma^2 + 2\sigma^4)$$  \hspace{1cm} \text{(using Equation (50))}

$$\leq \frac{4\sigma^2 + 2\sigma^4}{n}$$  \hspace{1cm} \text{(assuming $\frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C$, for each $n \in \mathbb{N}$.)}

From this it follows that:

$$\text{Pr}(\|X - \mathbb{E}(X)\| \geq t) \leq \frac{\text{Var}(X)}{t^2}, \text{ for each } t > 0$$  \hspace{1cm} \text{(using Chebychev’s inequality)}

$$\leq \frac{4\sigma^2 + 2\sigma^4}{nt^2}, \text{ for each } t > 0$$ \hspace{1cm} \tag{108}

It then follows that by setting the upper bound (RHS) to $\gamma^{-2} \in (0, 1)$, that

$$\frac{4\sigma^2 + 2\sigma^4}{nt^2} = \frac{1}{\gamma^2} \implies t = \gamma \sigma \sqrt{\frac{2\sigma^2 + 4C}{n}}$$

We then have that $\|X - \mathbb{E}(X)\| \leq 2\gamma \sigma \sqrt{\frac{2\sigma^2 + 4C}{n}}$, with probability at least $1 - \gamma^{-2}$, as required.

\[\square\]

D.2 Proof of Theorem 7

Theorem 7 (Equation (17) has a unique root). Assume that there exist constants $\psi, \Psi, C > 0$ such that $\psi \leq \sigma \leq \Psi$ and $\frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C$, for each $n \in \mathbb{N}$. In addition let $r_{n,2}(\mu_n, \mu_1, \sigma) = o(1)$, where the quantity $r_{n,2}(\mu_n, \mu_1, \sigma)$ is defined in (21). Then for sufficiently large $n$, $\delta = o\left((r_{n,2}(\mu_n, \mu_1, \sigma))^{-1}\right)$, and $\gamma = o\left(n^{1/2}\right)$, under the ASCI setup, Equation (17) in ASCIFIT has a unique root $\sigma^* \in \left[0, \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2}\right]$ for $\sigma$ with probability at least $1 - \delta^{-1} - 2\gamma^{-2}$.

Proof of Theorem 7. First, under the ASCIFIT setup, we can rewrite Equation (17) as $H(\sigma) = 0$, where:

$$H(\sigma) := G(\sigma) - \frac{1}{n} \sum_{i=1}^{n} T_i^2.$$

$$G(\sigma) := \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\widehat{T}_i \lor f(\eta, \sigma)), \sigma)^2$$ \hspace{1cm} \tag{110}

Our goal in this proof is to show that $H(\sigma) = 0$ has a solution $\sigma^* \in \left[0, \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2}\right]$, which occurs with high probability. We note that per Lemma 25 that $G(\sigma)$ is increasing for $\sigma \geq 0$.
and strictly increasing for \( \sigma > 0 \) (per Equation (80)). Moreover to see that the equation \( H(\sigma) = 0 \) has a unique root we appeal to the Intermediate Value Theorem. Specifically we are required to find two values for \( \sigma \), i.e. \( \{\sigma_1, \sigma_2\} \), such that the following conditions hold:

\[
G(\sigma_2) \geq \frac{1}{n} \sum_{i=1}^{n} T_i^2 \tag{111}
\]

\[
G(\sigma_1) \leq \frac{1}{n} \sum_{i=1}^{n} T_i^2 \tag{112}
\]

By taking \( \sigma_2 := \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2} \), we observe that a.s.:

\[
G(\sigma_2) = \frac{1}{n} \sum_{i=1}^{n} T_i^2 + \frac{1}{n} \sum_{i=1}^{n} \left( f^{-1} \left( \hat{T}_i \vee f \left( \eta, \frac{1}{n} \sum_{i=1}^{n} T_i^2 \right) \right) - \frac{1}{n} \sum_{i=1}^{n} T_i^2 \right)^2 \geq 0 \text{ a.s.} \tag{113}
\]

\[
\geq \sum_{i=1}^{n} T_i^2 \tag{114}
\]

So indeed \( \sigma_2 := \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2} \) satisfies the required condition in Equation (111). We now claim that \( \sigma_1 := 0 \) will satisfy Equation (112). First observe that:

\[
G(0) = \frac{1}{n} \sum_{i=1}^{n} f^{-1} \left( \hat{T}_i \vee f(\eta, 0), 0 \right)^2 = \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2, \tag{115}
\]

we then want to show that

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2 \leq \frac{1}{n} \sum_{i=1}^{n} T_i^2, \tag{116}
\]

holds with high probability, to be specified later.

Furthermore, since \( \hat{T}_i \vee \eta \) is the solution to an optimization problem we have that a.s.:

\[
\sum_{i=1}^{n} (\hat{T}_i \vee \eta - \eta)(T_i - \eta) = \sum_{i=1}^{n} (\hat{T}_i \vee \eta - \eta)^2. \tag{117}
\]

We see that Equation (117) holds since when you project any vector \( \mathbf{v} \in \mathbb{R}^n \) on a monotone cone \( K \subseteq \mathbb{R}^n \), then \( \Pi_K(\mathbf{v}) \mathbf{v} = \|\Pi_K(\mathbf{v})\|_2 \) per Bellec [2018, Equation 1.16]. Specifically, in our case we have that \( K = S^+_n := \{ \boldsymbol{\mu} := (\mu_1, \ldots, \mu_n)^\top \in \mathbb{R}^n : 0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \} \), and \( \mathbf{v} := (T_1 - \eta, \ldots, T_n - \eta)^\top \). We further observe that Equation (117) can be rewritten as
follows a.s.:
\[
\sum_{i=1}^{n} (\hat{T}_i \lor \eta - \eta)^2 = \sum_{i=1}^{n} (\hat{T}_i \lor \eta - \eta)(T_i - \eta) \quad \text{(Equation (117))}
\]
\[
\iff \sum_{i=1}^{n} (\hat{T}_i \lor \eta)^2 + 2\eta \sum_{i=1}^{n} (\hat{T}_i \lor \eta - \eta) + \eta^2 = \sum_{i=1}^{n} (\hat{T}_i \lor \eta)T_i - \eta \sum_{i=1}^{n} (\hat{T}_i \lor \eta - \eta) - \eta \sum_{i=1}^{n} \hat{T}_i + \eta^2 \quad \text{(expanding LHS/RHS.)}
\]
\[
\iff \sum_{i=1}^{n} (\hat{T}_i \lor \eta)^2 = \sum_{i=1}^{n} (\hat{T}_i \lor \eta) - \eta \left( \sum_{i=1}^{n} T_i - \sum_{i=1}^{n} \hat{T}_i \lor \eta \right) \quad \text{(118)}
\]
\[
\iff \sum_{i=1}^{n} (\hat{T}_i \lor \eta)^2 = \sum_{i=1}^{n} (\hat{T}_i \lor \eta)T_i - \eta \left( \sum_{i=1}^{n} \hat{T}_i - \sum_{i=1}^{n} \hat{T}_i \lor \eta \right) \quad \text{(119)}
\]
where to go from Equation (118) to Equation (119) we used the fact that \( \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} \hat{T}_i \). This holds since we know that \( \hat{T}_i \) are the PAVA solutions. Now we derive the following upper bound a.s.:
\[
\frac{1}{n} \left( \sum_{i=1}^{n} \hat{T}_i - \sum_{i=1}^{n} \hat{T}_i \lor \eta \right) = \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i - \frac{1}{n} \sum_{i=1}^{n} f(\mu_i, \sigma) + \frac{1}{n} \sum_{i=1}^{n} f(\mu_i, \sigma) - \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i \lor \eta
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i - f(\mu_i, \sigma)) + \frac{1}{n} \sum_{i=1}^{n} (f(\mu_i, \sigma) - \hat{T}_i \lor \eta) \quad \text{(120)}
\]
\[
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i - f(\mu_i, \sigma))^2} + \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \lor \eta - f(\mu_i, \sigma))^2}, \quad \text{(121)}
\]
where the transition between Equations (120) and (121) was by applying the Cauchy-Schwartz inequality to each summand. Note that for each \( i \in [n] \), we have that \( f(\mu_i, \sigma) \geq \mu_i \geq \eta \) per Equations (5) and (42). Then using Lemma 13 we have a.s.:
\[
\frac{1}{n} \sum_{i=1}^{n} \left( (\hat{T}_i \lor \eta) - f(\mu_i, \sigma) \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i - f(\mu_i, \sigma))^2. \quad \text{(122)}
\]
Hence by first using the inequality in Equation (122) to upper bound Equation (121), we can in turn upper bound the LHS of Equation (119) as follows a.s.:
\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \lor \eta)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \lor \eta)T_i + 2\eta \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i - f(\mu_i, \sigma))^2}. \quad \text{(123)}
\]
On the other hand we have by Theorem 37 that:
\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i - f(\mu_i, \sigma))^2 \leq \delta r_{n,2}(\mu_n, \mu_1, \sigma), \quad \text{(124)}
\]
with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$. Thus from Equation (123), we have:

$$\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)T_i + 2\eta (\delta r_{n,2}(\mu_n, \mu_1, \sigma))^\frac{1}{2}$$  \hspace{1cm} \text{(using Equation (124))}

$$= \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)\hat{T}_i + 2\eta (\delta r_{n,2}(\mu_n, \mu_1, \sigma))^\frac{1}{2}. \quad (125)$$

Note that the final equality in Equation (125) holds, since $\sum_{i=1}^{n} (\hat{T}_i \vee \eta)T_i = \sum_{i=1}^{n} (\hat{T}_i \vee \eta)\hat{T}_i$, by again since we know that $\hat{T}_i$ are the PAVA solutions. We then apply Cauchy-Schwartz to this summand of Equation (125) to obtain the following upper bound with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$.

$$\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2 \leq \frac{1}{n} \left( \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} \hat{T}_i^2 \right)^{\frac{1}{2}} + 2\eta (\delta r_{n,2}(\mu_n, \mu_1, \sigma))^\frac{1}{2}, \quad (126)$$

Now observe that since $\eta > 0$, the following holds a.s.:

$$\eta = |\eta| = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \eta^2} \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2} \quad (127)$$

Then using Equation (127) we have the following:

$$\eta \left( \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2} - \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2} \right)$$

$$\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2} \left( \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2} - \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2} \right) \quad \text{(using Equation (127))}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2 - \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2 \sqrt{\sum_{i=1}^{n} \hat{T}_i^2}. \quad (128)$$

By applying the upper bound derived in Equation (126) to Equation (128) we obtain the following:

$$\sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i \vee \eta)^2} - \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2} \leq 2 (\delta r_{n,2}(\mu_n, \mu_1, \sigma))^\frac{1}{2}, \quad (129)$$

with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$.

Now we will show that $\sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2}$ is a constant distance away from $\sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2}$, which will imply that for large $n$ the value at 0 is smaller than the target value, i.e., $G(\sigma_1) := G(0) \leq \frac{1}{n} \sum_{i=1}^{n} T_i^2$ as required per Equation (112).

On the other hand using Lemma 36, we have:

$$\left| \frac{1}{n} \sum_{i=1}^{n} (T_i - f(\mu_i, \sigma))^2 - \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2 - f(\mu_i, \sigma)^2) \right| \leq I(\gamma, C, \sigma), \quad (130)$$
with probability at least $1 - \gamma^{-2}$, where $l(\gamma, C, \sigma) := \gamma \sigma \sqrt{\frac{5\sigma^2 + 4C}{n}}$. Subtracting the inequalities in Equations (124) and (130) we then obtain:

$$
\frac{1}{n} \sum_{i=1}^{n} T_i^2 - \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2 + \frac{2}{n} \sum_{i=1}^{n} (\hat{T}_i - T_i) f(\mu_i, \sigma)
\geq \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2 - f(\mu_i, \sigma)^2) - l(\gamma, C, \sigma) - \delta r_{n,2}(\mu_n, \mu_1, \sigma)
$$

(131)

$$
\iff \frac{1}{n} \sum_{i=1}^{n} T_i^2 - \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2
\geq \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2 - f(\mu_i, \sigma)^2)
- \left( l(\gamma, C, \sigma) + \delta r_{n,2}(\mu_n, \mu_1, \sigma) + \frac{2}{n} \sum_{i=1}^{n} (\hat{T}_i - T_i) f(\mu_i, \sigma) \right).
$$

(132)

Now, in order sharpen the lower bound in Equation (132), we upper bound the term $\frac{2}{n} \sum_{i=1}^{n} (\hat{T}_i - T_i) f(\mu_i, \sigma)$ as follows:

$$
\frac{2}{n} \sum_{i=1}^{n} (\hat{T}_i - T_i) f(\mu_i, \sigma)
= \frac{2}{n} \sum_{i=1}^{n} (\hat{T}_i - T_i) (f(\mu_i, \sigma) - \hat{T}_i)
\quad \text{(since} \hat{T}_i \text{are the PAVA solutions.})
$$

$$
\leq \frac{2}{n} \sqrt{\sum_{i=1}^{n} (\hat{T}_i - T_i)^2} \sqrt{\sum_{i=1}^{n} (f(\mu_i, \sigma) - \hat{T}_i)^2}
\quad \text{(by Cauchy-Schwartz.)}
$$

$$
\leq \frac{2}{n} \sqrt{\sum_{i=1}^{n} (f(\mu_i, \sigma) - T_i)^2} \sqrt{\sum_{i=1}^{n} (f(\mu_i, \sigma) - \hat{T}_i)^2}
\quad \text{(since} \hat{T}_i \text{are PAVA, i.e., LSE solutions.)}
$$

$$
= 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} (f(\mu_i, \sigma) - T_i)^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (f(\mu_i, \sigma) - \hat{T}_i)^2}
\quad \text{(133)}
$$

$$
\leq 2 \left( l(\gamma, C, \sigma) \delta r_{n,2}(\mu_n, \mu_1, \sigma) \right)^{\frac{1}{2}},
$$

(134)

with probability at least with probability at least $1 - \delta^{-1} - \gamma^{-2}$, by the union bound. Note that to obtain Equation (134) we applied the bounds in Equations (146) and (130) to Equation (133). Now using the bound in Equation (134) in Equation (132) we conclude that:

$$
\frac{1}{n} \sum_{i=1}^{n} T_i^2 - \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2
\geq \frac{1}{n} \sum_{i=1}^{n} g(\mu_i, \sigma)
- \left( l(\gamma, C, \sigma) + \delta r_{n,2}(\mu_n, \mu_1, \sigma) + 2 \left( l(\gamma, C, \sigma) \delta r_{n,2}(\mu_n, \mu_1, \sigma) \right)^{\frac{1}{2}} \right)
$$

(135)
Now from Equation (49) we have that $g(\mu, \sigma) \geq g(0, \sigma) = \sigma^2 \left(1 - \frac{2}{\pi}\right)$, for each $\mu > 0$. Hence if $\sigma \geq \psi > 0$, then

\[ \frac{1}{n} \sum_{i=1}^{n} T_i^2 - \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2 \geq \psi^2 \left(1 - \frac{2}{\pi}\right) \left(l(\gamma, C, \sigma) + \delta r_{n,2}(\mu_n, \mu_1, \sigma) + 2(l(\gamma, C, \sigma)\delta r_{n,2}(\mu_n, \mu_1, \sigma))^{\frac{1}{2}}\right) > 0 \quad (136) \]

\[ \frac{1}{n} \sum_{i=1}^{n} T_i^2 - \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2 \geq \psi^2 \left(1 - \frac{2}{\pi}\right) > 0, \text{ with probability at least } 1 - \delta^{-1} - \gamma^{-2}. \]

Hence under the assumption that $r_{n,2}(f, \mu_n, \mu_1, \sigma) = o(1)$, for sufficiently large $n$ the above will be bigger than a constant. Now by Lemma 38 we have $\frac{1}{n} \sum_{i=1}^{n} T_i^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2) + 2\gamma\sigma\sqrt{\frac{2\sigma^2 + 4C}{n}}$ which is upper bounded by some constant for sufficiently large $n$ given our assumption that $\frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C$, for each $n \in \mathbb{N}$, and $\sigma \leq \Psi$ for some constants $C, \Psi > 0$. It follows that by applying Lemma 12 to Equation (136) we have:

\[ \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2} - \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2} \geq \kappa > 0, \quad (137) \]

for sufficiently large $n$, where $\kappa$ is some positive constant, with probability at least $1 - \delta^{-1} - 2\gamma^{-2}$.

Going back to equation (129), it follows that required equation will have a solution between $\left[0, \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_i^2}\right]$, with probability at least with probability at least $1 - \delta^{-1} - 2\gamma^{-2}$, as required.

\[ \square \]

D.3 Proof of Theorem 8

Lemma 39 (Upper and lower bounds for $\hat{\sigma}$). Assume that there exist constants $\psi, \Psi, C > 0$ such that $\psi \leq \sigma \leq \Psi$ and $\frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C$, for each $n \in \mathbb{N}$. Then under the \textsc{asci} setting per Definition 2, the following hold:

\[ \hat{\sigma} \geq K_1 \text{ with probability at least } 1 - \delta^{-1} - 2\gamma^{-2}, \text{ for sufficiently large } n. \quad (138) \]

\[ \hat{\sigma} \leq K_2 \text{ with probability at least } 1 - \delta^{-1} - 2\gamma^{-2}, \text{ for sufficiently large } n, \quad (139) \]

where $K_1, K_2 > 0$ are fixed constants and $\gamma^{-2}, \delta^{-1} \in (0, 1)$ are as in the proof of Theorem 7.

Proof of Lemma 39. We prove each property (Equations (138) and (139)) in turn.

(Proof of Equation (138).) We note that by assumption we have $0 < \psi \leq \sigma \leq \Psi$. We now want to show that $\hat{\sigma}$ is positively bounded away from 0, with high probability. First, observe that per Theorem 7 that $\hat{\sigma}$ uniquely solves $G(\hat{\sigma}) = \frac{1}{n} \sum_{i=1}^{n} T_i^2$, with high probability. Per Equation (115), we then have that:

\[ G(\hat{\sigma}) - G(0) = \frac{1}{n} \sum_{i=1}^{n} \hat{T}_i^2 - \frac{1}{n} \sum_{i=1}^{n} (\hat{T}_i + \eta)^2. \quad (140) \]

We then have by the Mean Value Theorem, and the fact that $G(\sigma)$ is increasing for each $\sigma \geq 0$ (per Lemma 25), that there exists a $\bar{\sigma} \in [0, \hat{\sigma}]$ such that

\[ G(\bar{\sigma}) - G(0) = G'(\bar{\sigma})\bar{\sigma}. \quad (141) \]
Now we have \( \tilde{\sigma} \leq \hat{\sigma} \), or equivalently that \( 2\tilde{\sigma} \leq 2\hat{\sigma} \). Since \( G'(\sigma) \leq 2\sigma \) using Equation (81), it follows that \( G'(\sigma) \leq 2\tilde{\sigma} \leq 2\hat{\sigma} \). Using this and Equation (141), we see that:

\[
G(\tilde{\sigma}) - G(0) = G'(\tilde{\sigma})\tilde{\sigma} \leq (2\tilde{\sigma})\hat{\sigma} \leq 2\hat{\sigma}^2,
\]

Now using Equation (142) and the proof of Theorem 7 we have that \( G(\tilde{\sigma}) - G(0) = \frac{1}{n} \sum_{i=1}^{n} T_i^2 - \frac{1}{n} \sum_{i=1}^{n} (\tilde{T}_i \vee \eta) \leq 0 \) is positively bounded away from 0 with high probability. So it follows that \( \hat{\sigma} \geq \sqrt{\frac{G(\tilde{\sigma}) - G(0)}{2}} > 0 \), with high probability, as required. \( \square \)

(Proof of Equation (139).) First, observe that per Theorem 7 that \( \tilde{\sigma} \) uniquely solves \( G(\tilde{\sigma}) = \frac{1}{n} \sum_{i=1}^{n} T_i^2 \), with high probability. By Definition 24 this implies that \( \tilde{\sigma} \leq \frac{1}{n} \sum_{i=1}^{n} T_i^2 \), with high probability. Moreover by Lemma 38 we have \( \frac{1}{n} \sum_{i=1}^{n} T_i^2 \leq \frac{1}{n} \sum_{i=1}^{n} (\mu_i^2 + \sigma^2) + 2\gamma\sigma\sqrt{\frac{2\sigma^2 + 4C}{n}} \) with probability at least \( \gamma^{-1} \), where \( 1 - \gamma^{-2} \) for \( \gamma \in (0, 1) \). This in turn is bounded, in high probability, by some constant, \( K_2 > 0 \) for sufficiently large \( n \) given our assumptions \( \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C \), for each \( n \in \mathbb{N} \), and \( \sigma \leq \Psi \), as required. \( \square \)

Thus all properties specified in Equations (138) and (139) are now proved. \( \square \)

**Theorem 8** (\( \tilde{\sigma} \) is close to \( \sigma \)). Under the assumptions of Theorem 7, we have that \( |\sigma - \tilde{\sigma}| \leq (\delta r_{n,2}(\mu_n, \mu_1, \sigma))^{1/2} + \gamma n^{-1/2} \) with probability at least \( 1 - \delta^{-1} - 2\gamma^{-2} \), where \( \delta^{-1}, \gamma^{-2} \in (0, 1) \).

**Proof of Theorem 8.** Recall our map \( G(\sigma) := \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\tilde{T}_i \vee f(\eta, \sigma), \sigma) - (f(\mu_i, \sigma), \sigma)) \) as originally defined in Equation (110). We will first try to show that \( G(\sigma) \) is close to \( \frac{1}{n} \sum_{i=1}^{n} T_i^2 \). First note that \( f^{-1}(\cdot \vee f(\eta, \sigma), \sigma) \) is a \( L := \frac{1}{2\Phi(\eta/\sigma)} \)-Lipschitz function per Lemma 20 and the fact that \( \sigma \) is a (both upper and lower) bounded quantity by assumption. Thus it follows that

\[
\left| f^{-1}(\tilde{T}_i \vee f(\eta, \sigma), \sigma) - f^{-1}(f(\mu_i, \sigma), \sigma) \right| \leq L \left| \tilde{T}_i \vee f(\eta, \sigma) - f(\mu_i, \sigma) \right|,
\]

and therefore

\[
\frac{1}{n} \sum_{i=1}^{n} \left( f^{-1}(\tilde{T}_i \vee f(\eta, \sigma), \sigma) - \mu_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} \left| f^{-1}(\tilde{T}_i \vee f(\eta, \sigma), \sigma) - f^{-1}(f(\mu_i, \sigma), \sigma) \right|^2 \leq \frac{L^2}{n} \sum_{i=1}^{n} (\tilde{T}_i - f(\mu_i, \sigma))^2 \quad \text{(using Equation (143))}
\]

In sum, we have established:

\[
\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\tilde{T}_i \vee f(\eta, \sigma), \sigma) - \mu_i)^2 \leq \frac{L^2}{n} \sum_{i=1}^{n} (\tilde{T}_i - f(\mu_i, \sigma))^2,
\]

We saw earlier by Theorem 37 we have that

\[
\frac{1}{n} \sum_{i=1}^{n} (\tilde{T}_i - f(\mu_i, \sigma))^2 \leq \delta r_{n,2}(\mu_n, \mu_1, \sigma),
\]

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with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$. Combining Equations (145) and (146) we have that

$$\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma) - \mu_i)^2 \leq L^2 \delta r_{n, 2}(\mu_n, \mu_1, \sigma)$$  \hspace{1cm} (147)

with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$. Thus by the triangle inequality, and reverse triangle inequality we have

$$\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma))^2 \in \left[ \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 - h_n, \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 + L^2 \delta r_{n, 2}(\mu_n, \mu_1, \sigma) + h_n \right]$$  \hspace{1cm} (148)

where $h_n := 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} \mu_i^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma) - \mu_i)^2}$. Given our assumption that $\frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C$, for each $n \in \mathbb{N}$, we have that:

$$h_n \leq 2L (C \delta r_{n, 2}(\mu_n, \mu_1, \sigma))^\frac{1}{2}$$  \hspace{1cm} (149)

with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$. Then combining Equations (148) and (149), we have that there exists some $l_1 \in [-2, 2]$ for sufficiently large $n$ such that

$$\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \lor f(\eta, \sigma), \sigma))^2 = \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 + l_1 L (2C \delta r_{n, 2}(\mu_n, \mu_1, \sigma))^\frac{1}{2}.$$  \hspace{1cm} (150)

with probability at least $1 - 2\delta^{-1}$, for $\delta^{-1} \in (0, 1)$, using the union bound.

Similarly, using Lemma 38 we have that there exists some $l_2(\sigma, C, \gamma) \in \mathbb{R}$ such that

$$\frac{1}{n} \sum_{i=1}^{n} T_i^2 = \sigma^2 + \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 + l_2(\sigma, C, \gamma)n^{-1/2}.$$  \hspace{1cm} (151)

with probability at least $1 - \gamma^{-2}$, for $\gamma^{-2} \in (0, 1)$. Moreover per Theorem 7 we have that $\hat{\sigma}$ uniquely solves $G(\hat{\sigma}) = \frac{1}{n} \sum_{i=1}^{n} T_i^2$, with high probability. That is:

$$G(\hat{\sigma}) = \hat{\sigma}^2 + \frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \lor f(\eta, \hat{\sigma}), \hat{\sigma}))^2 = \frac{1}{n} \sum_{i=1}^{n} T_i^2.$$  \hspace{1cm} (152)

Then combining Equations (151) and (152), we conclude that

$$|G(\sigma) - G(\hat{\sigma})| \leq l_1 L (2C \delta r_{n, 2}(\mu_n, \mu_1, \sigma))^\frac{1}{2} + l_2(\sigma, C, \gamma)n^{-1/2}.$$  \hspace{1cm} (153)

We now consider two cases, namely $\hat{\sigma} > \sigma$ and $\sigma > \hat{\sigma}$. In the first case, with $\hat{\sigma} > \sigma$, we seek to show that $G'(\xi) \geq K_1 > 0$, in high probability, for each $\xi \in (\sigma, \hat{\sigma})$. Here $K_1$ represents a positive constant. By Lemma 39 both $\sigma$ and $\hat{\sigma}$ are upper and lower bounded by some constants which implies that $\xi$ is also upper and lower bounded by some constants call them $C_1$ and $C_2$, i.e., $C_1 \leq \xi \leq C_2$. Since $G'(\xi) \geq J(\xi) = \xi \left( \frac{1}{2} - \frac{n/\phi(n/\xi)}{2\Phi(n/\xi) - 1} \right)$. As we argued earlier $J(\xi)$ is positive and since it is a continuous function and the set $[C_1, C_2]$ is compact it achieves its minimum, which is strictly positive. Hence $G'(\xi) \geq K_1 > 0$.

Similarly, in the second case, with $\sigma > \hat{\sigma}$, we can also show that $G'(\xi) \geq K_2 > 0$, in high probability, for each $\xi \in (\sigma, \hat{\sigma})$. Where again, $K_2$ represents a positive constant.
Then by using the Mean Value Theorem we have that there exists some $\xi \in (\sigma, \hat{\sigma})$ such that $|G(\sigma) - G(\hat{\sigma})| = G'(\xi) |\sigma - \hat{\sigma}| > \min(K_1, K_2) |\sigma - \hat{\sigma}|$. Thus from equation (153) we have

$$l_1 L (2C\delta r_{n,2}(\mu_n, \mu_1, \sigma))^{1/2} + l_2(\sigma, C, \gamma)n^{-1/2} = |G(\sigma) - G(\hat{\sigma})| \geq \min(K_1, K_2) |\sigma - \hat{\sigma}|,$$  \hspace{1cm} (154)

and hence $|\sigma - \hat{\sigma}| \leq l_1 L (2C\delta r_{n,2}(\mu_n, \mu_1, \sigma))^{1/2} + l_2(\sigma, C, \gamma)n^{-1/2}$.

\section{Proof of Theorem 9}

\textbf{Theorem 9} ($\hat{\mu}_{\text{ascifit}}$ is close to $\mu$). Under the assumptions of Theorem 7 and Theorem 8, we have that

$$\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \sigma) - \mu_i)^2 \leq \delta r_{n,2}(\mu_n, \mu_1, \sigma) + \gamma^2 n^{-1},$$ \hspace{1cm} (22)

with probability at least $1 - \delta^{-1} - 2\gamma^{-2}$.

\textit{Proof of Theorem 9.} We will now consider $\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \sigma) - \mu_i)^2$. We observe that a.s.:

$$\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \sigma) - \mu_i)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \sigma) - f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma))^2$$

$$+ \frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma) - \mu_i)^2$$

$$+ 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma) - \mu_i)^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma) - f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma))^2},$$ \hspace{1cm} (156)

where the transition between Equations (155) and (156) was by applying adding and subtracting $f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma)$, then applying the triangle inequality, and finally applying the Cauchy-Schwartz inequality to the cross product summand.

We now set to upper bound the Equation (156) further. First, we saw in Equation (147) that $\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma) - \mu_i)^2 \leq L^2 \delta r_{n,2}(\mu_n, \mu_1, \sigma)$, with probability at least $1 - \delta^{-1}$, for $\delta^{-1} \in (0, 1)$. Next, we will tackle the term

$$\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \sigma) - f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma))^2.$$ \hspace{1cm} (157)

Note that map $\sigma \mapsto f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma)$ is a $L := \sqrt{2/\pi} \exp(-\mu^2/\sigma^2/2) \leq \sqrt{2/\pi} \frac{2\Phi(\mu/\sigma)}{2\Phi(\mu/\sigma)} - \text{Lipschitz per Lemma 16 and 20}$, and in addition both $\sigma, \hat{\sigma}$ are upper and lower bounded by constants. It
follows that
\[
\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \hat{\sigma}) - f^{-1}(\hat{T}_i \vee f(\eta, \sigma), \sigma))^2
\leq \frac{1}{n} \sum_{i=1}^{n} |L(\sigma - \hat{\sigma})|^2
= L^2(\sigma - \hat{\sigma})^2
\lesssim 2C\delta r_{n,2}(\mu_n, \mu_1, \sigma) + \gamma^2 n^{-1},
\]
(158)

where Equation (158) follows from Theorem 8 with probability at least \(1 - \delta^{-1} - 2\gamma^{-2}\).

Then applying the upper bounds in Equations (157) and (158) appropriately to each corresponding summand of Equation (156), we conclude that
\[
\frac{1}{n} \sum_{i=1}^{n} (f^{-1}(\hat{T}_i \vee f(\eta, \hat{\sigma}), \hat{\sigma}) - \mu_i)^2 \lesssim \delta r_{n,2}(\mu_n, \mu_1, \sigma) + \gamma^2 n^{-1},
\]
(159)

with probability at least \(1 - \delta^{-1} - 2\gamma^{-2}\). \(\square\)
E Proofs of Section 4

E.1 Mathematical Preliminaries

Since we adapt the lower bound construction from Bellec and Tsybakov [2015] for our ASCI setting, we first introduce the relevant related notation and definitions here first for classes of monotonic sequences. We denote $S^\uparrow := \{ \mu := (\mu_1, \ldots, \mu_n)^{\top} | \mu_1 \leq \ldots \leq \mu_n \}$ to be the set of all non-decreasing sequences. We define $k(\mu) \geq 1$, for $\mu \in S^\uparrow$, to be the integer such that $k(\mu) - 1$ is the number of inequalities $\mu_i \leq \mu_{i+1}$ that are strict for $i \in [n-1]$ (i.e., number of jumps of $\mu$). The class of monotone functions we will consider are $\mathcal{S}^\uparrow(\cdot) := \{ \mu \in S^\uparrow | V(\mu) \leq V^* \}$, for some fixed $V^* \in \mathbb{R}$, and $V(\mu) = \mu_n - \mu_1$, is the total variation of any $\mu \in S^\uparrow$. We also consider the restricted class of monotone sequences, $\mathcal{S}^\uparrow_{k^*} := \{ \mu \in S^\uparrow | k(\mu) \leq k^* \}$, and $\mathcal{S}^\uparrow_{(\cdot),\eta,C} := \{ \mu \in S^\uparrow(\cdot) | \frac{1}{n} \sum_{i=1}^{n} \mu_i^2 \leq C, \mu_1 > \eta > 0 \}$.

E.2 Proof of Proposition 40

We follow directly the proof technique and construction from Bellec and Tsybakov [2015, Proposition 4], but make suitable adaptations for our ASCI setup. We largely follow their notation to help readers align the commonalities and differences in the underlying constructions used. Our first lower bound result is stated in Proposition 40.

**Proposition 40** (Minimax lower bounds). Let $n \geq 2, V^* > 0$ and $\sigma > 0$. There exist absolute constants $c, c' > 0$ such that for any positive integer $k^* \leq n$ satisfying $(k^*)^3 \leq \frac{16\eta(V^*)^2}{\sigma^2}$ we have

$$\inf_{\mu} \sup_{\mathcal{S}^\uparrow_{k^*} \cap \mathcal{S}^\uparrow(\cdot,\eta,C)} \Pr_{\mu} \left( \frac{1}{n} \| \tilde{\mu} - \mu \|^2 \geq \frac{cc'\sigma^2 k^*}{n} \right) \geq c'$$

(160)

where $\eta > 0$ is a fixed positive constant per Equation (5), $C \geq (V^*)^2 + 4\gamma^2 + 2\eta^2$, $\gamma := \frac{1}{8} \sqrt{\frac{2k^*}{n}}$, $\Pr_{\mu}$ denotes the distribution of $(R_1, \ldots, R_n)^\top$ satisfying Equation (4), and $\inf_{\mu}$ is the infimum over all estimators.

**Proof of Proposition 40.** Let $n$ be a multiple of $k^* \in \mathbb{N}$. Then for any $\omega, \omega' \in \{0, 1\}^{k^*}$, using the Varshamov-Gilbert bound [Tsybakov, 2009, Lemma 2.9], there exists a set $\Omega \in \{0, 1\}^{k^*}$ such that:

$$0 = (0, \ldots, 0)^\top \in \Omega, \quad \log(|\Omega| - 1) \geq \frac{k^*}{8}, \quad \text{and} \quad D_{\text{HAMM}}(\omega, \omega') \geq \frac{k^*}{8}$$

(161)

for any two distinct $\omega, \omega' \in \Omega$. For each $\omega \in \Omega$, define a vector $u^\omega \in \mathbb{R}^n$ componentwise, for each component index $i \in [n]$ as follows:

$$u^\omega_i := \left\lfloor \frac{(i-1)k^*}{n} \right\rfloor V^* + \gamma \omega_{\left\lfloor \frac{(i-1)k^*}{n} \right\rfloor + 1},$$

(162)

$$\bar{u}^\omega_i := u^\omega_i + \eta,$$

(163)

where $\gamma := \frac{1}{8} \sqrt{\frac{\sigma^2 k^*}{n}}$ and $\eta > 0$ is a fixed positive constant per Equation (5). Importantly we note that $u^\omega_i$ per Equation (162) is precisely as constructed in Bellec and Tsybakov [2015,
Proposition 4). However, critically the construction in Equation (163) is adapted to our ASCI setting, by componentwise translation by $\eta > 0$. More compactly, it is also convenient to represent this construction as $\bar{u}^\omega := u^\omega + \eta$, where $\eta := (\eta, \ldots, \eta)^T \in \mathbb{R}^n$.

As per Bellec and Tsybakov [2015, Proposition 4] we first note the following properties for $u_i^\omega$, for each $i \in [n]$. For any $\omega \in \Omega$, $u^\omega$ is a piecewise constant sequence with $k(u^\omega) \leq k^*$, $u^\omega$ is a non-decreasing sequence because $\gamma \leq \frac{V^*}{2k^*}$, and by construction $V(u^\omega) \leq V^*$. Thus, $u^\omega \in S_{k^*} \cap S^I(V)$ for all $\omega \in \Omega$.

Now we observe the following corresponding properties of the $\eta$-translated sequence $\bar{u}^\omega$. First note that since for any $\omega \in \Omega$, $u^\omega$ is a piecewise constant non-decreasing sequence, so is $\bar{u}^\omega$, by translation invariance. Next, consider any arbitrary index $j \in [n]$ relating to a ‘jump’ in $u^\omega$, i.e., $u_j^\omega < u_{j+1}^\omega$ (note the strict inequality). We then have that:

$$u_j^\omega < u_{j+1}^\omega \quad \iff \quad u_j^\omega + \eta < u_{j+1}^\omega + \eta$$

$$\iff \bar{u}_j^\omega < \bar{u}_{j+1}^\omega \quad \text{(by using Equation (163))}$$

So any ‘jump’ in the original sequence $u^\omega$ corresponds to a jump in the $\eta$-translated sequence $\bar{u}^\omega$. That is, we have $k(\bar{u}^\omega) = k(u^\omega) \leq k^*$. In addition, we note that

$$V(\bar{u}^\omega) = u^\omega - \bar{u}^\omega = (u^\omega + \eta) - (\bar{u}^\omega + \eta) = u^\omega - \bar{u}^\omega = V(u^\omega) \leq V^* \quad \text{(by construction of } u^\omega.)$$

By construction we also have that $\bar{u}_1^\omega := u_1^\omega + \eta \geq \eta > 0$, since $u_1^\omega \geq 0$ by construction (in fact each component is non-negative). Finally, per our ASCI setting, we want to check if there exists a $C > 0$, such that $\frac{1}{n} \sum_{i=1}^{n} (\bar{u}_i^\omega)^2 \leq C$, for each $n \in \mathbb{N}$. Given $\bar{u}^\omega$ we observe the following for each component index $i \in [n]$:

$$(\bar{u}_i^\omega)^2 := (u_i^\omega + \eta)^2 \quad \text{(using Equation (162))}$$

$$\leq 2 \left( (u_i^\omega)^2 + \eta^2 \right) \quad \text{(using Lemma 14)}$$

$$= 2 \left( \left( \frac{|i-1|}{k} \gamma \omega \omega_{[(i-1)\frac{k}{n}]+1} \right)^2 \right) + \eta^2 \quad \text{(using Equation (162))}$$

$$\leq 2 \left( \left( \frac{|i-1|}{k} \gamma \omega \omega_{[(i-1)\frac{k}{n}]+1} \right)^2 + \eta^2 \right) \quad \text{(using Lemma 14)}$$

$$\leq 2 \left( \left( \frac{V^* k}{2k} \gamma \omega \omega_{[(i-1)\frac{k}{n}]+1} \right)^2 + \eta^2 \right) \quad \text{(since } \frac{i-1}{k} \leq 1 \text{ for each } i \in [n].)$$

$$\leq (V^*)^2 + 4\gamma^2 + 2\eta^2 \quad \text{(164)}$$

So indeed it follows from Equation (164) that:

$$\frac{1}{n} \sum_{i=1}^{n} (\bar{u}_i^\omega)^2 \leq (V^*)^2 + 4\gamma^2 + 2\eta^2 =: C$$

(165)
So that we have $\tilde{u}^\omega \in S^*_k \cap S^*(V^*, \eta, C)$. Moreover, for any $\omega, \omega' \in \Omega$, we observe that:

$$\left| \tilde{u}^\omega - \tilde{u}^\omega' \right|^2 := \left| (u^\omega + \eta) - (u^\omega' + \eta) \right|^2 \quad \text{(by construction } \tilde{u}^\omega := u^\omega + \eta \text{)}$$

$$= \left\| u^\omega - u^\omega' \right\|^2 \quad \geq \frac{\gamma^2}{8}$$

$$= \frac{\sigma^2 k^*}{512n}$$

Set for brevity $P_\omega = P_{\tilde{u}^\omega}$. The Kullback-Leibler divergence $D_{KL}(P_\omega \mid \mid P_{\omega'})$, between $P_\omega$ and $P_{\omega'}$, is equal to $\frac{n}{2\sigma^2} \left\| u^\omega - u^\omega' \right\|^2$ for all $\omega, \omega' \in \Omega$. Thus,

$$D_{KL}(P_\omega \mid \mid P_0) = \frac{\gamma^2 n D_{HAMM}(0, \omega)}{2k^*\sigma^2} \leq \frac{k^*}{128} \leq \frac{\log(|\Omega| - 1)}{16} \quad (23) \quad (166)$$

Applying Tsybakov [2009, Theorem 2.7] with $\alpha = 1/16$ completes the proof. \qed

### E.3 Proof of Proposition 10

From Proposition 40, in line with Bellec and Tsybakov [2015, Corollary 5], we immediately obtain the following result in Proposition 10. Once again, we utilize the technique of Bellec and Tsybakov [2015, Corollary 5] to obtain the following corollary. The important changes to ensure that we adapt to our ASCI setting are captured in Proposition 40 and our proof thereof.

**Proposition 10** (Minimax lower bounds). Let $n \geq 2, V^* > 0$ and $\sigma > 0$, and define $\tilde{r}_{n,2}(V^*, \sigma) := \max \left\{ \left( \frac{\sigma^2 V^*}{n} \right)^{\frac{3}{2}}, \frac{\sigma^2}{n} \right\}$. Then, there exist absolute constants $c, c' > 0$ such that:

$$\inf_{\hat{\mu}} \sup_{S^*(V^*, \eta, C)} \Pr_{\mu} \left( \frac{1}{n} \left\| \hat{\mu} - \mu \right\| \geq \sigma \tilde{r}_{n,2}(V^*, \sigma) \right) > c' \quad (23)$$

**Proof of Proposition 40.** As per Bellec and Tsybakov [2015, Corollary 5], to prove this corollary it is enough to note that if $\frac{16n(V^*)^2}{\sigma^2} \geq 1$, by choosing $k^*$ in Proposition 40 as the integer part of $\left( \frac{16n(V^*)^2}{\sigma^2} \right)^{\frac{3}{4}}$, we obtain the lower bound corresponding to $\left( \frac{\sigma^2 V^*}{n} \right)^{\frac{3}{2}}$ under the maximum in Equation (23). On the other hand, if $\frac{16n(V^*)^2}{\sigma^2} < 1$ the term $\frac{\sigma^2}{n}$ is dominant, so that we need to have the lower bound of the order $\frac{\sigma^2}{n}$, which is trivial (it follows from a reduction to the bound for the class composed of two constant functions). \qed