Divergence of an integral of a process with small ball estimate

Yuliya Mishura\textsuperscript{a}, Nakahiro Yoshida\textsuperscript{b,c}

\textsuperscript{a}Taras Shevchenko National University of Kyiv
\textsuperscript{b}Graduate School of Mathematical Sciences, University of Tokyo
\textsuperscript{c}Japan Science and Technology Agency CREST

Abstract

The paper contains sufficient conditions on the function $f$ and the stochastic process $X$ that supply the rate of divergence of the integral functional $\int_0^T f(X_t)^2 dt$ at the rate $T^{1-\epsilon}$ as $T \to \infty$ for every $\epsilon > 0$. These conditions include so called small ball estimates which are discussed in detail. Statistical applications are provided.

Keywords: integral functional, rate of divergence, small ball estimate, statistical applications

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1. Introduction

The problem of convergence or divergence of perpetual integral functionals

$$\int_0^\infty g(X(t)) \, dt$$

for several classes of stochastic processes and several classes of functions $g$ appears when studying a variety of issues. Let $X = \{X(t), t \geq 0\}$ be a one-dimensional stochastic process with continuous trajectories, and let

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\textit{Email addresses:} myus@univ.kiev.ua (Yuliya Mishura), nakahiro@ms.u-tokyo.ac.jp (Nakahiro Yoshida)
$g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then for any $T > 0$ the integral functional

$$\int_0^T g(X(t)) \, dt$$

is defined. However, its properties and asymptotic behavior as $T \to \infty$ depend crucially on the properties of the process $X$ and of function $g$. The asymptotic behavior of the integral functional $\int_0^T g(X(t)) \, dt$ is very different even for one-dimensional Markov processes and depends on their transient or recurrent properties. On the one hand, conditions of existence of the perpetual integral functionals and in the case of non-existence, the rate of divergence, were studied, in the stochastic framework, for various classes of one- and multidimensional semimartingale and only partially for non-semimartingale stochastic processes, e.g., in [2, 4, 6, 7, 10, 11, 17]. In the papers where the rate of the divergence was studied, corresponding normalizing factors were suitable for the central or other functional limit theorems.

On the other hand, this question arises in the parametric statistical estimation because such integral functionals appear as the denominators of the residual terms, see, e.g., [8] for fractional models and [9] both for Wiener diffusions and fractional diffusions. In such case we need the divergence of this integral with some fixed rate and with probability 1, in order to get strongly consistent estimators. In this connection, the aim of the present paper is to investigate the rate of convergence of the integral $\int_0^T f(X_t)^2 \, dt$ to infinity as $T \to \infty$ depending on the properties of the measurable function $f$ and stochastic process $X$. It is well-known that in the case when $X$ is stationary, ergodic and $\mathbb{E} f^2(X_0)$ is finite, then $T^{-1} \int_0^T f(X_t)^2 \, dt \to \mathbb{E} f^2(X_0)$ as $T \to \infty$, and then the rate of divergence is evident, $T^{-1+\epsilon} \int_0^T f(X_t)^2 \, dt \to \infty$ for any $\epsilon > 0$. If process $X$ is not ergodic, the situation is more involved and the conditions on $f$ and $X$ should be much more complicated. In our approach, these conditions include so called small ball estimates. Since these conditions are interesting both themselves and from the point of view of various applications, see e.g. [13], we consider them even in those examples where the processes are ergodic, see examples 2.5 and 2.6.

The paper is organized as follows. Section 2 contains the basic conditions for the function $f$ and for the process $X$. Conditions for $X$ include small ball estimates which are presented in two versions, more mild and then more strong, with various examples. In Section 3 the main divergence theorem is proved, and then two modifications and several examples are provided. Sec-
tion presents two statistical applications. Section contains some auxiliary results.

2. Main conditions. Discussion of small ball estimates

Let \( X = \{X_t, t \geq 0\} \) be a real-valued stochastic process, and \( f = f(x) : \mathbb{R} \to \mathbb{R} \) be a measurable function. Let the following assumption hold:

(A) For any \( T > 0 \) the integral

\[
\mathbb{I}_T = \int_0^T f(X_t)^2 dt
\]

is correctly defined. Our goal is to establish sufficient conditions on function \( f \) and process \( X \) that supply the divergence to infinity:

\[
T^{-1+\epsilon} \mathbb{I}_T \to \infty \quad a.s. \quad (2.1)
\]
as \( T \to \infty \) for every \( \epsilon > 0 \).

Since we have two objects involving into the problem: function \( f \) and stochastic process \( X \), and respective assumptions will be non-trivial, let us consider them separately, with comments and examples.

2.1. Assumptions on function \( f \)

Concerning function \( f \), introduce the following notations: denote the sets \( H_+(x, \eta) = [x, x+\eta) \) and \( H_-(x, \eta) = (x-\eta, x] \). Basic assumptions on function \( f \) will be as follows.

(A1) (i) There exist positive constants \( K \) and \( \eta^* \) such that

\[
K(\eta) := \inf_{x \in \mathbb{R}} \min_{H \in \{H_+, H_-\}} \sup_{y \in H(x, \eta)} |f(y)| \geq \eta^K
\]

for every \( \eta \in (0, \eta^*) \).

(ii) Function \( f \) is from class \( C^1(\mathbb{R}) \) and for some constant \( C_0 \) we have that

\[
|f'(x)| \leq C_0(1 + |x|)^{C_0} \quad (x \in \mathbb{R}).
\]

Let us consider the equivalent form of assumption (A1), (i), sufficient condition for it and the simplest examples and a counterexample.
Lemma 2.1.  (1) Assumption (A1), (i) is equivalent to any of the following conditions:

(A3) There exist positive constants $K, C$ and $\eta_*$ such that

$$K_1(\eta) := \inf_{x \in \mathbb{R}} \min_{H \in \{H_+, H_-\}} \sup_{y \in H(x, \eta)} |f(y)| \geq C \eta^K$$

for every $\eta \in (0, \eta_*)$.

(A4) There exist positive constants $K$ and $\eta_*$ such that

$$K_2(\eta) := \inf_{x \in \mathbb{R}} \sup_{y \in (x, x+\eta)} |f(y)| \geq \eta^K$$

for every $\eta \in (0, \eta_*)$.

(A5) There exist positive constants $K, C$ and $\eta_*$ such that

$$K_3(\eta) := \inf_{x \in \mathbb{R}} \sup_{y \in (x, x+\eta)} |f(y)| \geq C \eta^K$$

for every $\eta \in (0, \eta_*)$.

(2) Let there exist such positive constants $\eta_0, \delta$ and $d \in \mathbb{N}$ such that $f \in C^{(d)}(\mathbb{R})$ and

$$\inf_{x \in \mathbb{R}} \max_{0 \leq i \leq d} \inf_{y \in [x, x+\eta_0]} \left| f^{(i)}(y) \right| \geq \delta. \tag{2.2}$$

Then assumption (A1), (i) holds.

Proof.  (1) Let us prove equivalence of conditions (A1), (i) and (A3). Indeed, if (A1), (i) holds then (A3) holds with $C = 1$. Inversely, if (A3) holds with $C \geq 1$ then (A1), (i) obviously holds, and if (A3) holds with $C < 1$, we can take $K' = K + 1$ and $\eta_*' = \eta_* \wedge C$. Equivalence of (A4) and (A5) can be established similarly. Now, let us prove equivalence of (A1), (i) and (A4). Indeed, let (A1), (i) hold. Since $K_2(\eta) \geq K(\eta/2)$, then $K_2(\eta) \geq 2^{-K} \eta^K$, whence (A5) consequently (A4) holds. Inversely, let (A4) hold. Then both values $\inf_{x \in \mathbb{R}} \sup_{y \in H_-(x, \eta)} |f(y)| \geq K_2(\eta) \geq \eta^K$ and $\inf_{x \in \mathbb{R}} \sup_{y \in H_+(x, \eta)} |f(y)| \geq K_2(\eta) \geq \eta^K$, whence $K(\eta) \geq \eta^K$.

(2) Let assumption (2.2) hold. Let us fix $x \in \mathbb{R}$. Without loss of generality,
assume that
\[ \inf_{y \in (x,x+\eta_0)} |f^{(d)}(y)| \geq \delta. \]

If, additionally, \( \inf_{y \in (x,x+\eta_0)} |f^{(d-1)}(y)| \geq \frac{\delta\eta}{2d} \), then we proceed with \( f^{(d-2)} \). If \( \inf_{y \in (x,x+\eta_0)} |f^{(d-1)}(y)| < \frac{\delta\eta}{2d} \), then we check in which of four intervals \([x,x+\eta_0/4],[x+\eta_0/4,x+\eta_0/2],[x+\eta_0/2,x+3\eta_0/4],[x+3\eta_0/4,x+\eta_0]\) there exists a point \( y \) satisfying inequality \( |f^{(d-1)}(y)| < \frac{\delta\eta}{2d} \). Let, for example, \( y \in [x+\eta_0/4,x+\eta_0/2] \). Then for any \( z \in [x+3\eta_0/4,x+\eta_0] \) we have that \( |f^{(d-1)}(z) - f^{(d-1)}(y)| \geq \frac{\delta\eta}{4d} \), therefore for any \( z \in [x+3\eta_0/4,x+\eta_0] \) we have that \( |f^{(d-1)}(z)| \geq \frac{\delta\eta}{2d} \). Then we continue the same way with \( |f^{(d-1)}| \), and in the worst case, the smallest value that we can obtain, is: \( |f(z)| \geq \frac{\delta\eta^2}{2d} \) for \( z \) in some interval of the diameter \( \frac{\eta d}{2d} \). However, even this worst case means that we can put in assumption \( \eta_0 = \frac{\eta d}{2d} \), \( K = d \) and \( C = \frac{\delta}{2d} \) in assumption \((A5)\), which is equivalent to \((A1),(i)\), as it was already established. So, the proof follows.

\[ \square \]

**Example 2.2.** Consider the classes of functions satisfying assumption \((2.2)\). Obviously, any polynomial function \( P_m(x) \) of \( m \)th power satisfies \((2.2)\) because at least one of its derivatives is a non-zero constant. Also, any linear combination of the form \( \sum_{i=1}^{K} (a_i \sin(\alpha_i x) + b_i \cos(\beta_i x)) \) satisfies this assumption as well as the rational function \( \frac{P_m(x)}{Q_n(x)} \) with \( m > n \) and \( Q_n(x) \neq 0 \). An example of \( f \) that satisfies \((A1)\) is

\[ f(x) = 1_{\{x \neq 0\}} x^3 \sin\left(\frac{1}{x}\right). \]

A periodic version

\[ f(x) = 1_{\{x \in \pi\mathbb{Z}\}} (\sin(x))^3 \times \sin\left(\frac{1}{\sin x}\right) \]

is also an example that satisfies \((A1)\) and has infinitely many clusters of null points in every neighborhood of \( \infty \). Exponential function \( e^x \) does not satisfy this assumption around \(-\infty\).
2.2. Assumptions on process $X$, with examples

The first group of assumptions describes the processes bounded in $L^\infty$. It is formulated as follows.

(A2) (i) (Hölder continuity in $L^\infty$) $X$ is continuous a.s. and there exists a positive constant $\rho$ such that

$$\sup_{s,t \in \mathbb{R}^+ : s < t < s + 1} \frac{\|X_t - X_s\|_r}{|t - s|^{\rho}} < \infty$$

for every $r > 1$.

(ii) (boundedness in $L^\infty$) $\sup_{t \in \mathbb{R}^+} \|X_t\|_r < \infty$ for every $r > 1$.

(iii) (relaxed small ball estimate) There exist positive constants $\Delta_*$, $\gamma$, $\lambda$, $\mu$, $K_1$, $K_2$ and $K_3$, such that

$$\sup_{s \geq 0} \mathbb{P} \left[ \sup_{t \in [s, s + \Delta]} |X_t - X_s| \leq \eta \right] \leq K_1 \exp \left( -K_2 \eta^{-\lambda} \Delta^{\mu} \right)$$

for all $\Delta \in (0, \Delta_*)$ and $\eta \in (0, K_3 \Delta^\gamma)$.

Why are we considering so complicated “relaxed small ball estimate” (A2), (iii)? The reason is that a wide class of processes satisfies this, albeit a little more complicated, but milder condition, while a simpler but more rigid analogue, condition (A2), (iv) is satisfied by a narrower one. However, we will discuss both conditions.

Example 2.3. Consider the class of processes satisfying assumption (A2), (iii) (relaxed small ball estimate). In order to do this, let us combine Theorem 4.4 from [12] with assumptions from [13]. More precisely, let $X = \{X_t, t \geq 0\}$ be a centered Gaussian process. We assume now that its variance distance satisfies two-sided power bounds: there exist $H \in (0, 1]$, and $C_1, C_2, C_3 > 0$ such that for any $s, t \geq 0$, $|t - s| \leq C_3$ we have that

$$C_1 |t - s|^{2H} \leq \mathbb{E} (X_t - X_s)^2 \leq C_2 |t - s|^{2H}. \quad (2.3)$$

Let us work within this assumption. Note that Theorem 4.4 [12], in a little bit adapted form, states the following: let $\{Z_t, t \in [0, \Delta]\}$ be a centered Gaussian
process. Then for any \(0 < a \leq 1/2\) and \(\eta > 0\)

\[
\mathbb{P}\left\{ \sup_{0 \leq t \leq \Delta} |Z_t| \leq \eta \right\} \leq \exp\left\{ -\frac{\eta^4}{16a^2 \sum_{2 \leq i, j \leq 1/a} (E\xi_i \xi_j)^2} \right\},
\]

provided that \(a \sum_{2 \leq i \leq 1/a} E\xi_i^2 \geq 32\eta^2\), where \(\xi_i = Z_{ia}\Delta - Z_{(i-1)a}\Delta\). Now let us fix \(s \geq 0, \Delta > 0\), and put \(Z_t = X_{t+s} - X_s, 0 \leq t \leq \Delta\). Then

\[
\xi_i = Z_{ia}\Delta - Z_{(i-1)a}\Delta = X_{ia}\Delta - X_{(i-1)a}\Delta,
\]

and it follows from assumption (2.3) that

\[
C_1(a\Delta)^{2H} \leq E\xi_i^2 \leq C_2(a\Delta)^{2H},
\]

and so the inequality \(a \sum_{2 \leq i \leq 1/a} E\xi_i^2 \geq 32\eta^2\) is fulfilled if \(C_1(a\Delta)^{2H} \geq 32\eta^2\), or, that is the same,

\[
a \geq \left( \frac{4\sqrt{2}}{\sqrt{C_1}} \right)^{1/H} \eta^{1/H} \frac{1}{\Delta}.
\]  

(2.4)

Together with the inequality \(a \leq 1/2\) we get that

\[
\eta \leq C_4 \Delta^H,
\]

where \(C_4 = \frac{\sqrt{C_1}}{2\Delta^H \sqrt{2}}\). Additionally, we assume that the increments of \(X\) are positively correlated, more exactly, for any \(s_i, t_i \in \mathbb{R}^+, i = 1, 2, s_1 \leq t_1 \leq s_2 \leq t_2\)

\[
E(X_{t_1} - X_{s_1})(X_{t_2} - X_{s_2}) \geq 0.
\]  

(2.5)

Note that positive correlation immediately implies that

\[
\sum_{2 \leq i, j \leq 1} (E\xi_i \xi_j)^2 \leq \max_{2 \leq i, j \leq 1} E\xi_i \xi_j E(X_{\Delta+s} - X_s)^2 \leq \max_{2 \leq i \leq 1} E\xi_i^2 C_2 \Delta^{2H} \leq C_2^2 a^{2H} \Delta^{4H}.
\]  

(2.6)

Now put \(\Delta_* = C_3, \eta_* = 1, \Delta \leq \Delta_*, \eta \leq C_4 \Delta^H, a = C_5 \eta^{1/H} \frac{1}{\Delta}\), where \(C_5 =
\[
\left( \frac{\Delta \xi}{\sqrt{C_3}} \right)^{1/H}. \quad \text{Then}
\]
\[
\frac{\eta^4}{16a^2 \sum_{2 \leq i,j \leq H} (\mathbb{E} \xi_i \xi_j)^2} \geq \frac{\eta^4}{16C_2^2 a^{2+2H} \Delta^{4H}} \geq \frac{\eta^4}{16C_2^2 C_3^2 + 2H \eta^{2/H + 2} \Delta^{2H-2}} = C_6 \Delta^{2-2H} \eta^{2/H-2},
\]

where \( C_6 = \frac{1}{16C_2^2 C_3^2 + 2H} \). It means that assumption \((A2), (iii)\) holds with \( \Delta_* = C_3, \eta_* = 1, K_1 = 1, K_2 = C_6, K_3 = C_4, \gamma = H, \mu = 2 - 2H, \lambda = -1 \).

Evidently, assumptions \((2.3)\) and \((2.5)\) hold for fractional Brownian motion with \( H > \frac{1}{2} \). According to \([3]\) and \([13]\), a subfractional Brownian motion with \( H > \frac{1}{2} \) also satisfies \((2.3)\) and \((2.5)\). Note, however, that assumption \((A2), (ii)\) fails for these processes.

The next example supplies us with four classes of the processes satisfying all assumptions \((A2), (i) - (iii)\).

**Example 2.4.** (Periodic Brownian bridge) Consider a process that in some sense is a periodic Brownian bridge. Namely, let \( X^{(k)} = \{X_t^{(k)}, t \in [k, k+1)\} \) be a sequence of independent Brownian bridges, constructed between the points \( (k, 0) \) and \( (k + 1, 0) \), \( k \geq 0 \). They satisfy the relation of a form

\[
X_t^{(k)} = (k + 1 - t) \int_k^t \frac{dW_u^{(k)}}{k + 1 - u}, t \in [k, k+1),
\]

where \( W^{(k)}, k \geq 0 \) is a sequence of independent Wiener processes, and let \( X_t = X_t^{(k)}, t \in [k, k+1) \). Evidently, we constructed a Gaussian process, and simple calculations show that its characteristics equal

\[
\mathbb{E}X_t = 0, \quad \mathbb{E} \left( X_t^{(k)} \right)^2 = (t - k)(k + 1 - t), \quad \mathbb{E} \left( X_t^{(k)} - X_s^{(k)} \right)^2 = (t - s)(1 + s - t).
\]

The middle equality means that assumption \((A2), (ii)\) holds, while last equality means that for \( s, t \in [k, k+1) \) and \( 0 \leq t - s \leq 1/2 \) we have that

\[
\frac{t - s}{2} \leq \mathbb{E} (X_t - X_s)^2 \leq t - s.
\]
Consider now \( t \) and \( s \) from neighbor intervals, and let \( s \in [k-1,k) \), \( t \in [k,k+1) \) and \( t - s < 1/2 \). Then, on the one hand, we have the following relations:

\[
\mathbb{E}(X_t - X_s)^2 = \mathbb{E}(X_t^{(k)})^2 + \mathbb{E}(X_s^{(k-1)})^2 = t - s - (t - k)^2 - (s - k)^2 \leq t - s.
\]

On the other hand, for \( s \leq k \leq t \) we have that \( (t - k)^2 + (s - k)^2 \leq (t - s)^2 \), and for \( t - s < 1/2 \) we have the inequality

\[
\mathbb{E}(X_t - X_s)^2 \geq t - s - (t - s)^2 \geq \frac{t - s}{2}.
\]

In particular, it means that

\[
\mathbb{E} \xi_i^2 \geq \frac{a \Delta}{2} \geq 32 \eta^2
\]

consequently the inequality \( a \sum_{2 \leq i \leq 1/a} \mathbb{E} \xi_i^2 \geq 32 \eta^2 \) holds provided that \( a < 1/2 \) and \( \eta \leq \frac{\Delta^{1/2}}{8 \sqrt{2}} \). All the relations above supply assumption \((A2),(i)\) and relations \((2.3)\) with \( \rho = H = 1/2 \). Note that the increments of \( X \) are not positively, but negatively correlated. Consider only interval \([0,1]\), other cases can be treated similarly. On this interval, it is easy to see that for any \( 0 \leq s \leq t \leq u \leq v \leq 1 \)

\[
\mathbb{E}(X_t - X_s)(X_v - X_u) = -(v - u)(t - s) < 0.
\]

In view of negative correlation of increments, we can not apply upper bound \((2.6)\). However, we can calculate and evaluate the sum \( S := \sum_{2 \leq i,j \leq 1/a}(\mathbb{E} \xi_i \xi_j)^2 \) explicitly:

\[
S = \sum_{2 \leq i \leq \frac{1}{a}} (\mathbb{E}(\xi_i)^2)^2 + \sum_{2 \leq i,j \leq \frac{1}{a},i \neq j} (\mathbb{E} \xi_i \xi_j)^2 = \left( \frac{1}{a} - 1 \right) (a \Delta (1 - a \Delta))^2
\]

\[
+ \left( \frac{1}{a} - 1 \right)^2 - \left( \frac{1}{a} - 1 \right) \right) a^4 \Delta^4 \leq a \Delta^2 + a^2 \Delta^4. \tag{2.7}
\]

Furthermore, \( a < 1/2, \Delta < 1 \), therefore, \( S < 2a \Delta^2 \). Therefore, taking into account \((2.4)\) with \( H = 1/2 \) and considering \( a = C_5 \frac{\eta^2}{\Delta} \) with \( \eta \leq C_4 \Delta^{1/2} \), we
get

\[
\frac{\eta^4}{16a^2\sum_{2 \leq i,j \leq n} (E\xi_i \xi_j)^2} = \frac{\eta^4}{16a^2S} \geq \frac{\eta^4}{32a^3\Delta^2} \geq \frac{\Delta}{32C_5^3\eta^2}.
\]

(2.8)

It means that assumption (A2), (iii) holds with \(\Delta_s = 1, K_1 = 1, K_2 = \frac{1}{32C_5^3}, K_3 = C_4, \gamma = 1/2, \mu = 1, \lambda = 2\).

**Example 2.5. (Stationary Ornstein–Uhlenbeck process)** Consider even more simple and natural example. Having in mind calculations from Example 2.4, we can omit some technical details. So, introduce a stationary Ornstein–Uhlenbeck process of the form

\[
X_t = \int_{-\infty}^{t} e^{\theta(s-t)} dW_s,
\]

where \(W\) is a two-sided Wiener process, \(\theta > 0\). For the technical simplicity, we put \(\theta = 1\). Then \(E X_t^2 = \frac{1}{2}\), and this process is Gaussian, therefore condition (A2), (ii) holds,

\[
X_t - X_s = (e^{-t} - e^{-s}) \int_{-\infty}^{s} e^z dW_z + e^{-t} \int_{s}^{t} e^z dW_z,
\]

for any \(s < t\), whence

\[
E (X_t - X_s)^2 = 1 - e^{s-t}.
\]

Evidently, on the one hand, \(1 - e^{s-t} \leq t - s\). On the other hand, we can state that \(1 - e^{-\theta} = e^{-\theta}(t-s) \geq e^{-1}(t-s)\) if \(t-s < 1\) (here \(\theta \in (-\infty, 0)\)). Therefore two-sided inequality (2.3) holds with \(H = 1/2\), and assumption (A2), (i) holds. In addition, Moreover, for any \(s \leq t \leq u \leq v\) we have that

\[
E (X_t - X_s)(X_v - X_u) = \frac{1}{2} (e^t - e^s) (e^{-u} - e^{-v}) < 0.
\]

So, the increments are negatively correlated. Let us evaluate the sum \(S\) from
\( \sum_{2 \leq i \leq \frac{1}{a}} (\mathbb{E}(\xi_i)^2)^2 + \sum_{2 \leq i, j \leq \frac{1}{a}, i \neq j} (\mathbb{E}\xi_i \xi_j)^2 = \left( \frac{1}{a} - 1 \right) (1 - e^{-a\Delta})^2 + \frac{1}{2} (1 - e^{-a\Delta})^2 (1 - e^{a\Delta})^2 \sum_{2 \leq i < j \leq \frac{1}{a}} e^{-2(j-i)\Delta} \leq \frac{1}{a} a^2 \Delta^2 + \frac{1}{2} ea^4 \Delta^4 \frac{1}{a^2} \leq \left( 1 + \frac{e}{2} \right) a\Delta^2. \)

(2.9)

Since we got the same upper bound as in the Example 2.4, up to a constant multiplier, we can make the same conclusions.

**Example 2.6.** (stationary fractional Ornstein-Uhlenbeck process) Let \( H \in (1/2, 1) \), and let \( B^H = \{ B^H_t, t \in \mathbb{R} \} \) be a two-sided fractional Brownian motion with Hurst index \( H \), that is, a centered Gaussian process with covariance function

\[ \mathbb{E}B^H_t B^H_s = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}). \]

Let \( -\infty \leq a < b \leq +\infty \), and let a measurable function \( h : [a, b] \to \mathbb{R} \) satisfy assumption

\[ \int_{[a,b]^2} |h(u)||h(v)||u - v|^{2H-2}du dv < \infty. \]

Then the integral \( \int_{[a,b]} h(z)dB^H_z \) is correctly defined and is a Gaussian random variable with zero mean and variance

\[ C_H \int_{[a,b]^2} h(u)h(v)|u - v|^{2H-2}du dv, \quad C_H = H(2H - 1). \]

(2.10)

In this connection, we can introduce a fractional Ornstein-Uhlenbeck process \( X_t = \int_{-\infty}^t e^{\theta(s-t)}dB^H_s, \theta > 0 \), that is a Gaussian stationary process with zero mean. For the technical simplicity, we put \( \theta = 1 \). Let’s calculate some of its quadratic moment characteristics, in order to evaluate the left-hand side of (2.6). This evaluation includes several constants whose value is not important, therefore we denote by \( C \) various constants whose value can change.
from line to line and even inside the same line. First, according to (2.10),

$$X^2_t = C_H e^{-2t} \text{E} \left[ e^{u+v} |u-v|^{2H-2} dudv \right] = (u' = -u, v' = -v)$$

$$= C_H e^{-2t} \int_{(-\infty,-t)^2} e^{-u'-v'} |u' - v'|^{2H-2} du'dv'$$

$$= (u' + t = z, v' + t = w) = C_H \int_{\mathbb{R}_+^{2}} e^{-z-w} |z-w|^{2H-2} dzdw = C_H I_0.$$  \hspace{1cm} (2.11)

Taking into account Gaussian property of $X$, we conclude that assumption (A2), (ii) holds. Further, for any $0 \leq s \leq t$ applying suitable change of variables

$$X_t - X_s = (e^{-t} - e^{-s}) \int_{-\infty}^{s} e^{z} dB_{z}^{H} + e^{-t} \int_{s}^{t} e^{z} dB_{z}^{H},$$

whence

$$\text{E}(X_t - X_s)^2 = C_H (e^{-t} - e^{-s})^2 \int_{(-\infty, s)^2} e^{u+v} |u-v|^{2H-2} dudv$$

$$+ 2C_H e^{-t} (e^{-t} - e^{-s}) \int_{-\infty}^{s} \int_{s}^{t} e^{u+v} |u-v|^{2H-2} dudv + C_H e^{-t} \int_{[s,t]^2} e^{u+v} |u-v|^{2H-2} dudv$$

$$= C_H \left( (e^{-t} - 1)^2 I_0 + 2e^{s-t} (e^{s-t} - 1) \int_{0}^{\infty} \int_{0}^{t-s} e^{-u+v} |u-v|^{2H-2} dudv \right.$$

$$+ e^{2s-2t} \int_{0}^{t-s} \int_{0}^{t-s} e^{u+v} |u-v|^{2H-2} dudv \bigg).$$ \hspace{1cm} (2.12)

Now, on the one hand, taking into account the above relations, the fact that $0 \leq 1 - e^{-x} \leq x$ for $x > 0$ and evident relation $e^{s-t} - 1 < 0$, we can get that for any $0 \leq s \leq t$

$$0 \leq \text{E} (X_t - X_s)^2 \leq C_H \left( (t-s)^2 I_0 + \int_{[0,t-s]^2} |z-w|^{2H-2} dzdw \right) \hspace{1cm} (2.13)$$

On the other hand, $2H < 2$, therefore \( \frac{(e^{s-t} - 1)^2}{(t-s)^2} \) → 0 as $t \to s$. Due to Lemma
5.3

\[
e^{s-t} (e^{s-t} - 1) \int_0^\infty \int_0^{t-s} e^{-u+v} |u-v|^{2H-2} du dv \to 0
\]
as \( t \to s \). Finally,

\[
e^{2s-2t} \int_0^{t-s} \int_0^{t-s} e^{u+v} |u-v|^{2H-2} du dv \to \frac{1}{H(2H-1)}
\]
as \( t \to s \). Then it follows from the limit relations above and (2.12) that

\[
\frac{\mathbb{E}(X_t - X_s)^2}{(t-s)^{2H}} \to 1
\]
as \( t \to s \), therefore, there exists \( d > 0 \) such that

\[
\mathbb{E}(X_t - X_s)^2 \geq \frac{1}{2}(t-s)^{2H}
\]
for \( t-s < d \). It follows from (2.13) and (2.14) that on some interval we have two sided inequality (2.13). In particular, it means that assumption (A2), (i) holds with \( \rho = H \). Further, as always, we are interested in values \( s = ia\Delta, t = (i+1)a\Delta, 2 \leq i \leq \frac{1}{a} - 1 \). In this case we get the following relations

\[
0 \leq \mathbb{E} (X_{(i+1)a\Delta} - X_{ia\Delta})^2 \leq C \left( a^2 \Delta^2 I_0 + (a\Delta)^{2H} \right).
\]

Since \( 2H < 2 \), for \( a\Delta < 1 \) we conclude that \( (\mathbb{E}(\xi_i^2))^2 \leq C(I_0 + 1)^2 a^{4H} \Delta^{4H} \), and the 1st term of the sum \( S \) (see (2.7)) can be bounded as

\[
\sum_{2 \leq i \leq \frac{1}{a}} (\mathbb{E}(\xi_i^2))^2 \leq C(I_0 + 1)^2 a^{4H-1} \Delta^{4H}.
\]

On the other hand, the inequality

\[
a \sum_{2 \leq i \leq 1/a} \mathbb{E}\xi_i^2 \geq 32\eta^2
\]
is supplied by the inequality \( a \geq \frac{(\eta^2)^{1/H}}{\Delta} \). Taking into account that we need to consider \( a < 1/2 \), we can put \( \Delta_\ast = 2d \) and \( \eta \leq \frac{\Delta^{H/2}}{2^{1+H}} \).
Now, for any $0 \leq u \leq v \leq s \leq t$

$$\mathbb{E}(X_t - X_s)(X_v - X_u) = C_H \left( (e^{-t} - e^{-s})(e^{-v} - e^{-u}) \int_s^t \int_u^v \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{z+w}|z-w|^{2H-2} \, dz \, dw + (e^{-t} - e^{-s})e^{-v} \int_s^t \int_u^v \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{z+w}|z-w|^{2H-2} \, dz \, dw + e^{-t}(e^{-v} - e^{-u}) \int_s^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{z+w}|z-w|^{2H-2} \, dz \, dw \right)$$

$$+ e^{-t-v} \int_s^t \int_u^v \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{z+w}|z-w|^{2H-2} \, dz \, dw) = C_H \left( (e^{-t} - 1)(e^{u-v} - 1) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-z-w}|z-w+s-u|^{2H-2} \, dz \, dw + (e^{-t} - 1) \int_0^{\infty} \int_0^{v-u} e^{-z-w}|z-w+s-v|^{2H-2} \, dz \, dw + (e^{u-v} - 1) \int_0^{t-s} \int_0^{\infty} e^{-z-w}|z-w+t-u|^{2H-2} \, dz \, dw \right)$$

Taking into account that we are interested in the values

$$u = ia\Delta, v = (i+1)a\Delta, s = ja\Delta, t = (j+1)a\Delta$$

for some $2 \leq i < j \leq \frac{1}{a}$, we get for $\xi_k = X_{(k+1)a\Delta} - X_{ka\Delta}, k = i, j$ that

$$\mathbb{E}\xi_i\xi_j = I_{ij}^{(1)} + 2I_{ij}^{(2)} + I_{ij}^{(3)},$$

where

$$I_{ij}^{(1)} = C_H(e^{-a\Delta} - 1)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-z-w}|z-w+(j-i)a\Delta|^{2H-2} \, dz \, dw,$$

$$I_{ij}^{(2)} = C_H(e^{-a\Delta} - 1) \int_0^{\infty} \int_0^{a\Delta} e^{-z-w}|z-w+(j-i)a\Delta|^{2H-2} \, dz \, dw,$$

$$I_{ij}^{(3)} = C_H \int_0^{a\Delta} \int_0^{a\Delta} e^{-z-w}|z-w+(j-i+1)a\Delta|^{2H-2} \, dz \, dw.$$
According to Lemma 5.2 from Appendix 5, integral
\[
\int_{\mathbb{R}^2} e^{-z-w}|z-w+(j-i)a\Delta|^{2H-2}dzdw
\]
is bounded by some constant. Therefore, \( I^{(1)}_{ij} \leq C(a\Delta)^2 \). Furthermore, according to the L'Hôpital’s rule
\[
\lim_{x \to 0} x^{-1} \int_0^x \int_0^x e^{-z-w}|z-w+(j-i)x|^{2H-2}dzdw
\]
\[
= \lim_{x \to 0} \int_0^x e^{-z-w}|x-w+(j-i)x|^{2H-2}dzdw = \int_0^\infty e^{-w}w^{2H-2}dw.
\]
Therefore,
\[
I^{(2)}_{ij} \leq C(a\Delta)^2.
\]
Finally, and under assumption that \( a\Delta \to 0 \)
\[
I^{(3)}_{ij} = C_H \int_0^{a\Delta} \int_0^{a\Delta} e^{-z-w}|z-w+(j-i)a\Delta|^{2H-2}dzdw
\]
\[
\sim C_H \int_0^{a\Delta} \int_0^{a\Delta} |z-w+(j-i)a\Delta|^{2H-2}dzdw
\]
\[
= C_H (a\Delta)^{2H} \int_0^1 \int_0^1 |z-w+(j-i)|^{2H-2}dzdw \sim C(a\Delta)^{2H},
\]
and, according to Lemma 5.1, \( \int_0^1 \int_0^1 |z-w+(j-i)|^{2H-2}dzdw \) is bounded by some constant not depending on \( j-1 \). Due to the fact that \( 2H < 2 \), we can state the following: there exists \( \Delta_* > 0 \) such that for \( \Delta < \Delta_* \), due to the fact that \( a < 1/2 \), \( \mathbb{E}\xi_i \xi_j > 0 \). It means that we are exactly in conditions of Example 2.3 and can produce the same conclusions.

Example 2.7. (Tempered fractional Brownian motion) There are several approaches how to introduce a tempered fractional Brownian motion. For the detail see [1, 15, 16]. We shall introduce it as follows. Let \( \theta > 0 \), \( \alpha > 0 \). Consider a process
\[ Y_t = \int_{-\infty}^t e^{-\theta(t-s)}(t-s)^\alpha dW_s, t \geq 0. \]
Process $Y$ is stationary and Gaussian, with the following characteristics:

$$
\mathbb{E} Y_t = 0, \quad \mathbb{E} Y_t^2 = \int_0^\infty e^{-2\theta z} z^{2\alpha} \, dz.
$$

As usual, without loss of generality, put $\theta = 1$. Let us calculate

$$
\mathbb{E} Y_0 Y_t = \mathbb{E} \int_{-\infty}^0 e^z (1-z)^\alpha \, dW_z \int_t^\infty e^{-t-z} (t-z)^\alpha \, dW_z
$$

$$
= \int_{-\infty}^0 e^z e^{-t-z} (1-z)^\alpha \, dz = e^{-t} \int_0^\infty e^{-z} z^\alpha (1+z)^\alpha \, dz
$$

$$
= t^{2\alpha+1} e^{-t} \int_0^\infty e^{-2zt} z^\alpha (1+z)^\alpha \, dz \to 0
$$

as $t \to \infty$. So, $Y$ is an ergodic process. We shall not provide the small ball calculations since they are very tedious.

**Remark 2.8.** All examples are about the case where $t_0 = 0$. However, since we consider asymptotics of the integral, process $X$ can be arbitrary till some fixed $t_0 > 0$ and satisfy assumption (A2) after this moment.

Now let us introduce more simple and stronger small ball estimate.

(A2), (iv) (stronger small ball estimate) There exist positive constants $\eta_*, \Delta_*$, $\lambda$, $\mu$, $K_1$ and $K_2$, such that

$$
\sup_{s \in \mathbb{R}^+} \mathbb{P} \left[ \sup_{t \in [s, s+\Delta]} \left| X_t - X_s \right| \leq \eta \right] \leq K_1 \exp \left( -K_2 \eta^{-\lambda} \Delta^\mu \right)
$$

for all $\eta \in (0, \eta_*)$ and $\Delta \in (0, \Delta_*)$.

As one can see, the difference is that in (A2), (iii) $\eta$ is adapted to $\Delta$, and it means that we consider $\eta$ under the curve $\eta = K_3 \Delta^\gamma$, while in (A2), (iv) we consider a whole rectangle $\eta \in (0, \eta_*)$, $\Delta \in (0, \Delta_*)$. All previous examples do not work, however, let us consider two other examples.

**Example 2.9.** One of the simplest examples of the processes $X$ satisfying assumptions (A2), is $X_t = \xi \varphi(t)$, where $\xi$ is a random variable satisfying the following conditions:

(i) All moments of $\xi$ are uniformly bounded;
There exist positive constants $\lambda_0$, $K_3 \geq 1$ and $K_4$, such that for any $x > 0$

$$P[|\xi| \leq x] \leq K_3 \exp\left(-K_4 x^{-\lambda_0}\right),$$

and function $\varphi$ is periodic with period 2, and equals $\varphi(t) = t_1_{\{0 \leq t \leq 1\}} + (2 - t)_1_{\{1 \leq t \leq 2\}}$. In this case process $X$ is continuous since $\varphi$ is continuous, and condition (i) is fulfilled with $\rho = 1$ because $|\varphi(t) - \varphi(s)| \leq |t - s|$, condition (ii) is supplied by (j) because $\varphi$ is a bounded function. Furthermore,

$$\sup_{t \in [s, s+\Delta]} |X_t - X_s| \geq |\xi| \frac{\Delta}{2},$$

since $\sup_{t \in [s, s+\Delta]} |\varphi(t) - \varphi(s)| \geq \frac{\Delta}{2}$, for any $0 < \Delta \leq 2$. Hence

$$\sup_{s \in \mathbb{R}^+} \mathbb{P}\left[ \sup_{t \in [s, s+\Delta]} |X_t - X_s| \leq \eta \right] = \sup_{s \in \mathbb{R}^+} \mathbb{P}\left[ |\xi| \frac{\Delta}{2} \leq \eta \right]$$

$$= \sup_{s \in \mathbb{R}^+} \mathbb{P}\left[ |\xi| \leq \frac{2\eta}{\Delta} \right] \leq K_3 \exp\left(-K_4 \left(\frac{2\eta}{\Delta}\right)^{-\lambda_0}\right),$$

and so condition (iii) follows from (jj) with any $\eta_* > 0$, $0 < \Delta_* < 2$, $\lambda = \mu = \lambda_0$, $K_1 = K_3$, $K_2 = \frac{K_2}{2\lambda_0}$. In this case we have a small ball estimate in time, but in some sense, uniformly ball estimate in space. We can modify this example in the following way: let $\Omega = [0, 2]$, and consider the same $\xi$ but shift the functions $\varphi$ in a random way, namely, let

$$\tilde{\varphi}(t, \omega) = \varphi(t + \omega).$$

Then $\sup_{t \in [s, s+\Delta]} |\tilde{\varphi}(t, \omega) - \tilde{\varphi}(s, \omega)| \geq \frac{\Delta}{2}$, for any $0 < \Delta \leq 2$, and we have the same estimates as before.

3. Divergence theorems

The first result describes the conditions of divergence to infinity for the processes bounded in $L^\infty$. We prove it under relaxed small ball estimate $(A2), (iii)$, however, this theorem is certainly true if to replace relaxed small ball estimate with $(A2), (iv)$. 

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Theorem 3.1. Let the function $f$ satisfy assumption (A1), (i) - (ii), and the process $X$ satisfy assumptions (A2), (i) - (iii).

Then (2.1) holds, i.e.,

$$T^{-1+\epsilon} \| \alpha \|_T \to \infty \text{ a.s.}$$

as $T \to \infty$ for every $\epsilon > 0$.

Proof. Let $\epsilon > 0$. Fix positive numbers $\epsilon(i)$ ($i = 0, 1, 2, 3, 4$) such that

$$\lambda \epsilon(0) > \mu \epsilon(1), \quad \epsilon(0) > \gamma \epsilon(1), \quad \epsilon(1) < \epsilon(2), \quad \epsilon(3) < \epsilon(2) \cdot \frac{\epsilon(2) \rho}{2}, \quad (3.1)$$

and that

$$2K \epsilon(0) - \epsilon(1) + \epsilon(2) < \frac{1}{2} \epsilon.$$ 

Such numbers $\epsilon(i)$ ($i = 0, 1, 2, 3, 4$) exist; for example, let $\delta \downarrow 0$ for

$$\epsilon(0) = \delta^4, \quad \epsilon(1) = \delta^5, \quad \epsilon(2) = \delta, \quad \epsilon(3) = \delta^2, \quad \epsilon(4) = \delta^3.$$ 

Let $\eta_n = n^{-\epsilon(0)} \wedge (K_n^{3n^{-\gamma \epsilon(1)}})$. The 2nd value, $K_n^{3n^{-\gamma \epsilon(1)}}$ is necessary in order to apply assumption (iii), the relaxed small ball estimate. However, we shall deal with the 1st value, $n^{-\epsilon(0)}$, therefore, assume that $n$ is sufficiently large, such that

$$\log n > \frac{\log \left( \frac{1}{K_n} \right)}{\epsilon(0) - \gamma \epsilon(1)}.$$ 

In this case $n^{-\epsilon(0)} < K_n^{3n^{-\gamma \epsilon(1)}}$, and $\eta_n = n^{-\epsilon(0)}$. Then

$$c_n := \frac{1}{4} \min \left\{ \inf_{x \in \mathbb{R}} \sup_{y \in H_{+}(x, \eta_n)} |f(y)|, \inf_{x \in \mathbb{R}} \sup_{y \in H_{-}(x, \eta_n)} |f(y)| \right\} \geq \frac{1}{4} \eta_n^K$$ 

for large $n$. Let $\Delta_n = n^{-\epsilon(1)}$, $s_n^j = (j - 1) \Delta_n$ and $t_n^j = j \Delta_n$ for $j, n \in \mathbb{N}$. Let $I_j^n = [s_j^n, s_j^n + \Delta_n/2]$. Let

$$A_j^n = \left\{ \sup_{s,t \in I_j^n} |X_t - X_s| > \eta_n \right\}.$$
For \( \omega \in A^n_j \), there exist \( \tau(\omega), \sigma(\omega) \in I^n_j \) such that \( \sigma(\omega) < \tau(\omega) \) and that \( \left| X_{\tau(\omega)}(\omega) - X_{\sigma(\omega)}(\omega) \right| > \eta_n \). Therefore, by the mean-value theorem, if \( X_{\sigma(\omega)}(\omega) < X_{\tau(\omega)}(\omega) \), then

\[
\max_{t \in I^n_j} |f(X_t(\omega))| \geq \sup_{t \in [\sigma(\omega), \tau(\omega)]} |f(X_t(\omega))| \geq \sup_{x \in H(X_{\sigma(\omega)}, \eta_n)} |f(x)| \geq 4c_n.
\]

Similarly, if \( X_{\tau(\omega)}(\omega) < X_{\sigma(\omega)}(\omega) \), then we consider \( H(X_{\sigma(\omega)}, \eta_n) \) and conclude that

\[
\max_{t \in I^n_j} |f(X_t(\omega))| \geq 4c_n. \tag{3.2}
\]

Thus, inequality (3.2) is always valid for \( \omega \in A^n_j \).

Let \( \beta = \frac{2\epsilon(3)}{\epsilon(2)} \) and let \( r > (\rho - \beta)^{-1} \), equivalently, \( \rho - \frac{1}{r} > \beta \). By (A2) (i),

\[
\mathbb{E}[|X_t - X_s|^r] \leq B(r)|t - s|^\rho \quad (t \in [s, s + 1])
\]

where

\[
B(r) = \left( \sup_{s, t \in \mathbb{R}^+ : s < t < s + 1} \frac{||X_t - X_s||_r}{|t - s|^\rho} \right)^r.
\]

Then by the Garsia-Rodemich-Ramsey inequality, there exists a constant \( C(r) \) (independent of \( s \)) such that

\[
\mathbb{P}\left[ \sup_{t_1, t_2 \in [s, s + 1] : t_1 \neq t_2} \frac{|X_{t_2} - X_{t_1}|}{|t_2 - t_1|^\beta} \geq h \right] \leq \frac{C(r)B(r)M}{h^r} \tag{3.3}
\]

for all \( h > 0 \), where

\[
M = \int_{[0,1]} \int_{[0,1]} |t_2 - t_1|^{r \rho - r \beta - 2} dt_1 dt_2; \tag{3.4}
\]

\( M \) is finite since \( r \rho - r \beta - 2 > -1 \). Since

\[
\sup_{t \in [s, s + \epsilon(2)]} \frac{|X_t - X_s|}{|t - s|^\beta} \geq \epsilon(2) \sup_{t \in [s, s + \epsilon(2)]} |X_t - X_s|,
\]

by setting \( h = \epsilon(3) \) in (3.3), and taking into account the definition of \( \beta \), we
obtain

\[
\sup_{s \in \mathbb{R}_+} \mathbb{P} \left[ \sup_{t \in [s, s+n^{-\epsilon}(2)]} |X_t - X_s| \geq n^{-\epsilon(3)} \right] 
\leq \sup_{s \in \mathbb{R}_+} \mathbb{P} \left[ \sup_{t \in [s, s+n^{-\epsilon}(2)]} \frac{|X_t - X_s|}{|t - s|^{\beta}} \geq n^{-\epsilon(3)+\epsilon(2)\beta} \right] 
= \sup_{s \in \mathbb{R}_+} \mathbb{P} \left[ \sup_{t \in [s, s+n^{-\epsilon}(2)]} \frac{|X_t - X_s|}{|t - s|^{\beta}} \geq n^{-\epsilon(3)} \right] \leq \frac{C(r)B(r)M}{n^{\epsilon(3)\beta}}
\]

for all \( n \in \mathbb{N} \).

Obviously,

\[
\mathbb{P} \left[ \sup_{t \in [s, s+n^{-\epsilon}(2)]} \left| f(X_t) - f(X_s) \right| \geq c_n \right] \leq \mathbb{P} \left[ |X_s| \geq n^{\epsilon(4)} \right] 
+ \mathbb{P} \left[ |X_s| \leq n^{\epsilon(4)}, \sup_{t \in [s, s+n^{-\epsilon}(2)]} |X_t - X_s| \leq n^{-\epsilon(3)}, \sup_{t \in [s, s+n^{-\epsilon}(2)]} \left| f(X_t) - f(X_s) \right| \geq c_n \right] 
+ \mathbb{P} \left[ \sup_{t \in [s, s+n^{-\epsilon}(2)]} |X_t - X_s| \geq n^{-\epsilon(3)} \right].
\]

By choosing a sufficiently large \( r \) in (3.5), we know

\[
\sup_{s \in \mathbb{R}_+} \mathbb{P} \left[ \sup_{t \in [s, s+n^{-\epsilon}(2)]} |X_t - X_s| \geq n^{-\epsilon(3)} \right] = O(n^{-L}) \tag{3.7}
\]
as \( n \to \infty \) for every \( L > 0 \). Moreover, from (A2), (ii) we have

\[
\sup_{s \in \mathbb{R}_+} \mathbb{P}[|X_s| \geq n^{\epsilon(4)}] = O(n^{-L}) \tag{3.8}
\]
as \( n \to \infty \) for every \( L > 0 \). By Taylor’s formula applied to \( f \), we obtain

\[
\sup_{t \in [s, s+n^{-\epsilon}(2)]} \left| f(X_t) - f(X_s) \right| \leq \sup_{t \in [s, s+n^{-\epsilon}(2)]} C_0 \left( 1 + |X_s| + |X_t - X_s| \right) C_0 |X_t - X_s| 
\leq C_0 (2 + n^{\epsilon(4)}) C_0 n^{-\epsilon(3)}
\]

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whenever \( |X_s| \leq n^{-\epsilon(4)} \) and \( \sup_{t \in [s, s + n^{-\epsilon(2)}]} |X_t - X_s| \leq n^{-\epsilon(3)} \). We also have
\[
C_0(2 + n^{\epsilon(4)})C_0 n^{-\epsilon(3)} < n^{-K\epsilon(0)/4} = \eta_n^K / 4 \leq c_n
\]
for large \( n \). Therefore, under \((A1), (ii)\) and \((A2), (i)-(ii)\),
\[
\sup_{s \in \mathbb{R}_+} \mathbb{P} \left[ \sup_{t \in [s, s + n^{-\epsilon(2)}]} |f(X_t) - f(X_s)| \geq c_n \right] = O(n^{-L}) \tag{3.9}
\]
as \( n \to \infty \) for every \( L > 0 \).

Let
\[
\tau^n_j = \inf \{ t \geq s^n_j; |f(X_t)| \geq 4c_n \} \land (s^n_j + \Delta_n/2),
\]
and let
\[
B^n_j = \left\{ \sup_{t \in [\tau^n_j, \tau^n_j + n^{-\epsilon(2)}]} |f(X_t) - f(X_{\tau^n_j})| < 3c_n \right\}. \tag{3.10}
\]
Let \( J_n = [n^{1+\epsilon(1)}] + 1 \). Now (3.9) gives the estimate
\[
\mathbb{P} \left[ \left( \bigcap_{j=1}^{J_n} B^n_j \right)^c \right]
\leq 3 \sum_{j=1}^{J_n} \sum_{i=1}^{[n^{-\epsilon(1)+\epsilon(2)}]+1} \mathbb{P} \left[ \sup_{t \in [s^n_j + (i-1)n^{-\epsilon(2)}, s^n_j + in^{-\epsilon(2)}]} |f(X_t) - f(X_{s^n_j + (i-1)n^{-\epsilon(2)}})| \geq c_n \right] = O(n^{-L}) \tag{3.11}
\]
as \( n \to \infty \) for every \( L > 0 \).

By \((A2), (iii)\) (recall that \( \eta_n < K_3 \Delta_n^2 \)), we have
\[
\sup_{j \leq J_n} \mathbb{P} \left[ \sup_{s, t \in I^n_j} |X_t - X_s| \leq \eta_n \right] = O(n^{-L})
\]
as \( n \to \infty \) for every \( L > 0 \). That is,

\[
P \left[ \left( \bigcap_{j \leq J_n} A_j^\circ \right)^c \right] = O(n^{-L}) \tag{3.12}
\]
as \( n \to \infty \) for every \( L > 0 \).

On \( A_j^\circ \cap B_j^\circ \) we have that \( |f(X_t)| \geq c_n \) on the interval of length at least \( n^{-\varepsilon(2)} \), therefore

\[
\int_{[s_n^j, t_n^j]} f(X_t)^2 dt \geq c_n^2 n^{-\varepsilon(2)}.
\]

Therefore,

\[
\mathbb{I}_n \geq \frac{1}{16} n^{-2K\varepsilon(0)} |n^{1+\varepsilon(1)}| n^{-\varepsilon(2)} \tag{3.13}
\]
on \( \bigcap_{j=1}^{J_n} (A_j^\circ \cap B_j^\circ) \). Thanks to (3.11), (3.12) and (3.13) with the inequality \( 2K\varepsilon(0)-\varepsilon(1)+\varepsilon(2) < \varepsilon/2 \), we obtain

\[
P \left[ \mathbb{I}_n < n^{1-\frac{3}{4} \varepsilon} \right] = O(n^{-L})
\]
as \( n \to \infty \) for every \( L > 0 \). In particular, by Borel-Cantelli’s lemma,

\[
P \left[ \limsup_{n \to \infty} \{ \mathbb{I}_n < n^{1-\frac{3}{4} \varepsilon} \} \right] = 0.
\]

Therefore,

\[
P \left[ \lim_{n \to \infty} (n^{-(1-\varepsilon)} \mathbb{I}_n) = \infty \right] = 1. \tag{3.14}
\]

This shows (2.1) for \( T = n \) but it is sufficient for proof of the theorem. \[\square\]

**Theorem 3.2.** The convergence (2.1) holds if we exclude condition (A2), (ii) of boundedness in \( L^\infty \), and instead add the condition

(A1)(iii) There exist constants \( Q > 0 \), \( p > 0 \) and \( C > 0 \) such that

\[
|f(x)| \geq Q|x|^p
\]
for any \( |x| \geq C \).
Proof. Analyzing proof of Theorem 3.1, we can see that condition (A2), (ii) is applied only when we construct the upper bound for the first term on the right-hand side of the inequality (3.6). In this connection, we can consider, instead of the first two terms on the left-hand side of (3.6), one term of the form

$$\sup_{s \in \mathbb{R}^+} P \left[ \sup_{t \in [t_n, t_n + n^{-\varepsilon(2)}]} |X_t - X_s| \geq n^{-\varepsilon(3)}, |X_s| \leq n^\varepsilon(4) \right] = O(n^{-L})(3.15)$$

It means that instead of (3.10) we should consider the events

$$\tilde{B}_n^j = \left\{ \sup_{t \in [\tau_n^j, \tau_n^j + n^{-\varepsilon(2)}]} |f(X_t) - f(X_{\tau_n^j})| < 3c_n \right\} \cup \left\{ |X_{\tau_n^j}| \geq n^\varepsilon(4) \right\}.$$  

Then the upper bound (3.11) still holds for $\tilde{B}_n^j$ in place of $B_n^j$, and moreover, on $A_n^j \cap \tilde{B}_n^j$ we have, as before that $|f(X_t)| \geq c_n$ on the interval of length at least $n^{-\varepsilon(2)}$, otherwise, for sufficiently large $n$

$$|X_t| \geq n^\varepsilon(4) - n^{-\varepsilon(3)} \geq 1/2n^\varepsilon(4),$$

whence

$$|f(X_t)| \geq \frac{Q}{2^p n^{p\varepsilon(4)},}$$

therefore, for sufficiently large $n$

$$\int_{[t_n^j, t_n^j]} f(X_t)^2 dt \geq c_n^2 n^{-\varepsilon(2)} \wedge \frac{Q}{2^p n^{p\varepsilon(4)} - \varepsilon(1)},$$

and we can conclude as in Theorem 3.1.

Note that without condition (A2), (ii), the diverging rate obtained here could be far from optimal. This is confirmed by the following statement.

**Theorem 3.3.** Let the function $f(x) = |x|^p, p > 1$, and let the process $X = \{X_t, t \geq 0\}$ be a real-valued stochastic process, satisfying the following conditions

(i) $X$ is self-similar with index $H \in (0, 1)$;

(ii) The random variable $\int_0^1 |X_t|^p dt$ satisfies assumption $\int_0^1 |X_t|^p dt \geq \xi,$
where \( \xi \) is a non-negative random variable with bounded density (particularly, \( \int_0^1 |X_t|^p dt \) itself has a bounded density). Then for any \( \epsilon \in (0, pH) \) we have that
\[
\liminf_{T \to \infty} T^{-1-\epsilon} \int_0^T |X_t|^p dt > 0 \quad \text{a.s.}
\]

**Proof.** Let constant \( C > 0 \) is an upper bound for the density of the random variable \( \xi \) from assumption \((ii)\). Then for any \( k \in \mathbb{N}, 0 < \epsilon < pH, \beta > 0 \) and \( x > 0 \) it follows from the self-similarity of the finite-dimensional distributions of \( X \) that
\[
P_{k,x} := P \left\{ \frac{\int_0^{k\beta} |X_t|^p dt}{k^{\beta(1+\epsilon)}} < x \right\} = P \left\{ \frac{\int_0^1 |X_{sk\beta}|^p ds}{k^{\beta\epsilon}} < x \right\}
\]
\[
= P \left\{ k^{\beta(pH-\epsilon)} \int_0^1 |X_s|^p ds < x \right\} = P \left\{ \int_0^1 |X_s|^p ds < \frac{x}{k^{\beta(pH-\epsilon)}} \right\}
\]
\[
\leq P \left\{ \xi < \frac{x^\frac{1}{p}}{k^{\beta(H-\frac{1}{p})}} \right\} \leq C \frac{x^\frac{1}{p}}{k^{\beta(H-\frac{1}{p})}}.
\]

If we choose \( \beta > \left( H - \frac{5}{p} \right)^{-1} \), then \( \Sigma_{k \geq 1} P_{k,x} \) converges, and it follows from Borel-Cantelli and the fact that \( x > 0 \) is arbitrary that
\[
\lim_{k \to \infty} \frac{\int_0^{k\beta} |X_t|^p dt}{k^{\beta(1+\epsilon)}} = \infty \quad \text{a.s.}
\]

Further, for any \( T \in [k^{\beta}, (k+1)^{\beta}] \)
\[
\frac{\int_0^T |X_t|^p dt}{T^{1+\epsilon}} \geq \frac{\int_0^{k\beta} |X_t|^p dt}{k^{\beta(1+\epsilon)}} \left( \frac{k}{k+1} \right)^{\beta(1+\epsilon)} \geq \frac{1}{2^{\beta(1+\epsilon)}} \frac{\int_0^{k\beta} |X_t|^p dt}{k^{\beta(1+\epsilon)}},
\]
therefore
\[
\liminf_{T \to \infty} \frac{\int_0^T |X_t|^p dt}{T^{1+\epsilon}} = +\infty \quad \text{a.s.}
\]
for any \( 0 < \epsilon < pH \). \( \square \)
Example 3.4. For example, for any $p > 1$

$$\liminf_{T \to \infty} T^{-1-\epsilon} \int_0^T |B_t^H|^p dt > 0 \quad \text{a.s.}$$

for any $0 < \epsilon < pH$, where $B_H^t$ is a fractional Brownian motion with Hurst index $H \in (0, 1)$. Indeed, $B_H^t$ is a self-similar process with the index $H$ of self-similarity, and in this case

$$\int_0^1 |B_t^H|^p dt \geq \xi = |\mathcal{N}(0, \sigma^2)|^p,$$

and

$$\sigma^2 = \frac{1}{2} \int_0^1 \int_0^1 (s^{2H} + u^{2H} - |s - u|^{2H}) duds = \frac{1}{2H + 2}.$$  

Obviously, $\xi$ has a bounded density. However, in this particular case we can say more and establish the exact rate of convergence. Indeed, consider $\epsilon = pH$. In this case, according to Theorem 3.3 [14].

$$\liminf_{T \to \infty} \frac{\sup_{0 \leq t \leq T} |B_s^H|^p (\log \log T)^{pH}}{T^{pH}} = c > 0 \quad \text{a.s.,}$$

Therefore,

$$P_{k,x} \leq 2 \frac{1}{\sigma \sqrt{2\pi}} \frac{x^{1-p}}{k^{p(H - \frac{1}{2})}}.$$  

where $c$ is a positive constant. Therefore,

$$\liminf_{T \to \infty} \frac{\int_0^T |B_s^H|^p ds}{T^{1+pH}} \leq \liminf_{T \to \infty} \frac{\sup_{0 \leq s \leq t} |B_s^H|^p}{T^{pH}} = 0 \quad \text{a.s.}$$  

A fortiori, for any $\epsilon > pH$

$$\liminf_{T \to \infty} \frac{\int_0^T |B_s^H|^p ds}{T^{1+\epsilon}} = 0 \quad \text{a.s.}$$

Concluding this section, we will consider a stationary $X$. Let $g_t = f(X_t)^2$. Suppose that $\{g_t\}_{t \in \mathbb{R}_+}$ is uniformly integrable, which is satisfied, for example,
under condition \((A2), (ii)\), and that process \(X\) is stationary. Then

\[
\frac{1}{T} \int_0^T g_t \, dt \to Z \ \text{a.s.}
\] (3.16)

as \(T \to \infty\) for some nonnegative random variable \(Z\) by Birkhoff’s individual ergodic theorem; \(Z\) is a random variable measurable to the invariant \(\sigma\)-field. Define the event \(A\) by

\[
A = \{Z = 0\}.
\] (3.17)

The family \(\{T^{-1} \int_0^T g_t \, dt\}_{T>0}\) of random variables is uniformly integrable, and hence

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T E[g_t 1_A] \, dt = \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T g_t \, dt 1_A \right]
\]

\[
= E \left[ \lim_{T \to \infty} \frac{1}{T} \int_0^T g_t \, dt 1_A \right] = E[Z 1_A] = 0.
\] (3.18)

Since \((g_s, 1_A) =^d (g_t, 1_A)\) for any \(s, t \in \mathbb{R}_+\) by stationarity of \(X\), (3.18) implies

\[
E[g_t 1_A] = 0
\] (3.19)

for any \(t \in \mathbb{R}_+\), and hence,

\[
E \left[ \int_0^T g_t \, dt 1_A \right] = 0
\] (3.20)

for any \(T \in \mathbb{R}_+\). On the other hand, if

\[
\sup_{T>0} \int_0^T g_t \, dt > 0 \ \text{a.s.},
\] (3.21)

then (3.20) is valid only when \(P(A) = 0\), i.e., \(Z > 0\) a.s. and then \(\int_0^T g_t \, dt\) diverges at the rate of \(T\) a.s. as \(T \to \infty\). In particular, under the conditions of Theorem 3.1, it holds that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T g_t \, dt > 0 \ \text{a.s.}
\]
This supplements Theorem 3.1’s result on the divergence of the integral.

4. Statistical Application

Let us consider two statistical applications of the divergence results. Namely, let us consider Ornstein–Uhlenbeck process \( Y = \{Y_t, t \geq 0\} \) with unknown drift parameter \( \theta > 0 \) that is the solution of the equation

\[
Y_t = Y_0 - \theta \int_0^t Y_s ds + \int_0^t g_s dW_s,
\]

where \( W = \{W_t, t \geq 0\} \) is a Wiener process, \( g : \mathbb{R}_+ \to \mathbb{R} \) is a measurable function such that \( 0 \leq c \leq |g_s| \leq C, s \geq 0 \). Since \( Y \) satisfies assumption (A2) and \( f(x) = x^2 \) satisfies assumption (A1), we can conclude that both \( T^{-1+\epsilon} \int_0^T Y_s^2 ds \to \infty \) and \( T^{-1+\epsilon} \int_0^T Y_s^2 g_s^2 ds \to \infty \) for any \( \epsilon > 0 \) as \( T \to \infty \).

Consider the equality

\[
\int_0^T Y_t dY_t = -\theta \int_0^T Y_s^2 ds + \int_0^T Y_s g_s dW_s,
\]

whence

\[
\frac{\int_0^T Y_t dY_t}{\int_0^T Y_s^2 ds} = -\theta + \frac{\int_0^T Y_s g_s dW_s}{\int_0^T Y_s^2 ds}.
\]

Furthermore, since \( \int_0^T Y_s^2 g_s^2 ds \to \infty \), we get from the strong law of large numbers for martingales that

\[
\frac{\int_0^T Y_s g_s dW_s}{\int_0^T Y_s^2 g_s^2 ds} \to 0 \text{ a.s. as } T \to \infty.
\]

However,

\[
c^2 \leq \frac{\int_0^T Y_s^2 g_s^2 ds}{\int_0^T Y_s^2 ds} \leq C^2,
\]

and it means that

\[
\frac{\int_0^T Y_s g_s dW_s}{\int_0^T Y_s^2 ds} \to 0 \text{ a.s. as } T \to \infty.
\]
We get that \(-\int_0^T Y_s dY_s\) is a strongly consistent estimator of \(\theta\).

Another example can be introduced as follows. Let the processes \(X\) and \(Y\) be observable, and satisfy the relation

\[ X_t = X_0 + \theta \int_0^t g(Y_s)ds + B_t^H, \]

without any restriction on \(\theta \in \mathbb{R}\) and \(H \in (0, 1)\), but assuming that \(g = f^2\), where \(f\) satisfies (A1) and \(Y\) satisfies (A2). Then

\[ \frac{X_T}{\int_0^T g(Y_s)ds} = \frac{X_0}{\int_0^T g(Y_s)ds} + \theta + \frac{B_T^H}{\int_0^T g(Y_s)ds}. \]

According to [8],

\[ \frac{B_T^H}{T^{H+\epsilon}} \to 0 \]

a.s. as \(T \to \infty\) for any \(\epsilon > 0\) while \(\frac{\int_0^T g(Y_s)ds}{T^{H+\epsilon}} \to \infty\) a.s. for \(H + \epsilon \leq 1\).

5. Appendix

Here we establish some auxiliary results.

Lemma 5.1. Let \(H \in (1/2, 1)\), \(x, y \geq 0\). Then there exists \(C > 0\) depending only on \(H\) such that

\[ \int_0^x |w - y|^{2H-2}dw \leq Cx^{2H-1}. \]

Proof. We consider only \(x > 0\). Let \(y = 0\). Then \(\int_0^x w^{2H-2}dw = (2H - 1)^{-1}x^{2H-1}\).

Let \(0 < y \leq x\). Then

\[ \int_0^x |w - y|^{2H-2}dw = \int_0^y (y - w)^{2H-2}dw + \int_y^x (w - y)^{2H-2}dw \]
\[ = (2H - 1)^{-1} (y^{2H-1} + (x - y)^{2H-1}) \leq 2(2H - 1)^{-1}x^{2H-1}. \]
Let \( y \geq x \). Then, since \( |a^\alpha - b^\alpha| \leq |a - b|^\alpha \) for \( \alpha \in (0, 1) \), we have that

\[
\int_0^x |w - y|^{2H-2} \, dw = \int_0^x (y - w)^{2H-2} \, dw
\]

\[
= 2H - 1 \left[ y^{2H-1} - (y - x)^{2H-1} \right] \leq (2H - 1)^{-1} x^{2H-1}.
\]

Lemma is proved.

**Lemma 5.2.** There exists such \( C > 0 \) that for any \( p > 0 \)

\[
I_5 := \int e^{-z-w} |z - w + p|^{2H-2} \, dz \, dw \leq C.
\]

Proof. Let us provide the following transformations:

\[
I_5 = \int_0^\infty e^{-z} \int_0^{z+p} e^{-w(z + p - w)^{2H-2}} \, dw \, dz + \int_0^\infty e^{-z} \int_{z+p}^\infty e^{-w(w - z - p)^{2H-2}} \, dw \, dz
\]

\[
= \int_0^\infty e^{-z} \int_0^{z+p} e^{z-w} x^{2H-2} \, dx \, dz + \int_0^\infty e^{-w} \int_0^{w-p} e^{-z(w - p - z)} x^{2H-2} \, dx \, dz
\]

\[
= e^{-p} \left( \int_0^p e^{x} x^{2H-2} \, dx \int_0^\infty e^{-2z} \, dz + \int_0^\infty e^{x} x^{2H-2} \int_{x-p}^\infty e^{-2z} \, dz \, dx \right)
\]

\[
+ \int_0^\infty e^{-w} \int_0^{w-p} e^{w-x} x^{2H-2} \, dx \, dw =: I_6 + I_7 + I_8.
\]

Since

\[
\lim_{p \to \infty} e^{-p} \int_0^p e^{x} x^{2H-2} \, dx = \lim_{p \to \infty} \frac{e^{p} p^{2H-2}}{e^p} = 0,
\]

the value \( I_6 = e^{-p} \int_0^p e^{x} x^{2H-2} \, dx \) is bounded. Further,

\[
I_7 = e^{-p} \int_0^\infty e^{x} x^{2H-2} \int_{x-p}^\infty e^{-2z} \, dz \, dx
\]

\[
= \frac{1}{2} e^{-p} \int_0^\infty e^{x} e^{-2x+2p} x^{2H-2} \, dx = \frac{1}{2} e^p \int_0^\infty e^{-x} x^{2H-2} \, dx,
\]

and

\[
\lim_{p \to \infty} \frac{\int_0^p e^{-x} x^{2H-2} \, dx}{e^{-p}} = \lim_{p \to \infty} \frac{e^{-p} p^{2H-2}}{e^{-p}} = 0,
\]

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therefore this value is bounded, too. Finally,

\[
I_8 = e^p \int_p^\infty e^{-2w} \int_0^{w-p} e^x x^{2H-2} dx dw \leq (2H - 1)^{-1} \int_p^\infty e^{-w} (w - p)^{2H-1} dw
\]

\[
= (w - p = x) =
\]

\[
= (2H - 1)^{-1} e^{-p} \int_0^\infty e^{-x} x^{2H-1} dx \to 0, p \to \infty.
\]

Therefore, this values is bounded, too. Lemma is proved.

**Lemma 5.3.** For any \( H \in (1/2, 1) \) we have the limit relation

\[
\lim_{x \to 0} x^{-1} \int_0^x \int_0^x e^{-u+v} |u - v|^{2H-2} du dv = \Gamma(2H - 1).
\]

**Proof.** Applying L’Hospital’s rule, we immediately get that

\[
\lim_{x \to 0} x^{-1} \int_0^x \int_0^x e^{-u+v} |u - v|^{2H-2} du dv = \lim_{x \to 0} \int_0^\infty e^{-u+x} |u - x|^{2H-2} du
\]

\[
= \lim_{x \to 0} \int_{-x}^\infty e^{-z} |z|^{2H-2} du = \Gamma(2H - 1),
\]

and lemma is proved.

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