Spatial Dynamics of a Nonlocal Dispersal Population Model in a Shifting Environment

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Abstract This paper is concerned with the spatial dynamics of a nonlocal dispersal population model in a shifting environment where the favorable region is shrinking. It is shown that the species becomes extinct in the habitat if the speed of the shifting habitat edge \( c > c^*(\infty) \), while the species persists and spreads along the shifting habitat at an asymptotic speed \( c^*(\infty) \) if \( c < c^*(\infty) \), where \( c^*(\infty) \) is determined by the nonlocal dispersal kernel, diffusion rate and the maximum linearized growth rate. Moreover, we demonstrate that for any given speed of the shifting habitat edge, the model system admits a nondecreasing traveling wave with the wave speed at which the habitat is shifting, which indicates that the extinction wave phenomenon does happen in such a shifting environment.

Keywords Spreading speed · Extinction wave · Nonlocal dispersal · Shifting environment

Mathematics Subject Classification 35K57 · 35R20 · 92D25

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1 Introduction

In this paper, we are interested in the following nonlocal dispersal population model in a shifting environment:

\[
\frac{\partial u(t, x)}{\partial t} = d \left( J * u - u \right)(t, x) + u(t, x)(r(x - ct) - u(t, x)), \quad t > 0, x \in \mathbb{R}, \tag{1.1}
\]

where \( u(t, x) \) stands for the population density of the species under consideration at time \( t \) and location \( x \), and \( J(x) \) is the spatial convolution kernel having the property that

\[ J \in C(\mathbb{R}, \mathbb{R}^+) \text{ is symmetric and compactly supported, and } \int_{\mathbb{R}} J(x)dx = 1. \]

Note that \( J(x - y) \) denotes the probability distribution of the population jumping from location \( y \) to location \( x \). Then \( \int_{\mathbb{R}} J(x - y)u(t, y)dy \) is the rate at which individuals are arriving to location \( x \) from all other places, while \( \int_{\mathbb{R}} J(y - x)u(t, x)dy = u(t, x) \) is the rate at which they are leaving location \( x \) to all other sites. It follows that

\[ Au(t, x) := (J * u - u)(t, x) = \int_{\mathbb{R}} J(x - y)u(t, y)dy - u(t, x) \]

can be viewed as a nonlocal dispersal operator modeling the free and large-range migration of the species (see Ignat and Rossi 2007; Murray 2003), and \( d > 0 \) is the dispersal rate. The reaction term describes the logistic-type growth of the species which depends on the density \( u \) and on the shifting habitat with a fixed speed \( c > 0 \). Throughout this paper, we always assume that

\[ \text{(H) The resource function } r(\xi) \text{ is continuous and nondecreasing, } r(\pm\infty) \text{ are finite, and } r(-\infty) < 0 < r(\infty). \]

Thus, the shifting environment may be divided into the favorable region \( \{x \in \mathbb{R} : r(x - ct) > 0\} \) and the unfavorable region \( \{x \in \mathbb{R} : r(x - ct) \leq 0\} \), both shifting with speed \( c > 0 \). Specifically, we see that when time increases, the unfavorable region is expanding and the favorable region is shrinking. This kind of problem comes from considering the threats associated with global climate change and the worsening of the environment resulting from industrialization which lead to the shifting or translating of the habitat ranges, and recently has attracted much attention, see, e.g., Berestycki et al. (2009), Gonzalez et al. (2010), Hu and Zou (2017), Lei and Du (2017), Potapov and Lewis (2004), Li et al. (2014), Hu and Li (2015), Li et al. (2016a), Zhou and Kot (2011), Parr et al. (2012), Scheffer et al. (2012), Vo (2015), Berestycki and Fang (2018) and Du et al. (2015), where certain classical reaction–diffusion models in a continuous/discrete media or with a free boundary and integro-difference models were addressed. Model (1.1) may also be derived from some specific epidemiological models by the arguments similar to those in Fang et al. (2016), where the authors deduced a classical reaction–diffusion Fisher–KPP equation in a wavelike environment from the consideration of pathogen spread.

It is well known that nonlocal dispersal problem (1.1) with space-time homogeneous growth rate \( r > 0, \) i.e., \( u_t = d(J * u - u) + u(r - u) \), has been fully investigated.
for the spatial spreading dynamics. Here we refer to Coville (2007), Schumacher (1980b), Yagisita (2009) and Carr and Chmaj (2004) for the existence and uniqueness of monotone traveling wave solutions, Schumacher (1980a) and Lutscher et al. (2005) for the spreading speed and Li et al. (2010) for the construction of new types of entire solutions. Roughly speaking, the slowest speed

\[ c^* = \min_{\lambda > 0} \frac{d \left( \int_{\mathbb{R}} J(y)e^{\lambda y} dy - 1 \right) + r}{\lambda} \]

for monotone wave fronts connecting \( r \) and 0 is also the spreading speed for a large class of solutions. More precisely, let \( u(t, x; u_0) \) be the nonnegative solution with compactly supported initial data \( u_0 \), then \( \lim_{t \to \infty, |x| \geq ct} u(t, x; u_0) = 0 \) for any \( c > c^* \) and \( \lim_{t \to \infty, |x| \leq ct} u(t, x; u_0) = r \) for any \( 0 < c < c^* \). Ecologically, the spreading speed can be understood as the asymptotic rate at which a species, initially introduced in a bounded domain, expands its spatial range as time evolves, while a traveling wave describes the propagation of a species as a wave with a fixed shape and a fixed speed. These two fundamental concepts along with some new types of entire solutions have been widely used for the description of species invasion and disease transmission. Regarding the nonlocal dispersal equations with time and/or space periodic dependence, we refer the readers to Shen and Zhang (2010, 2012b) and Rawal et al. (2015) for spreading speeds, Shen and Zhang (2012a), Coville et al. (2013), Rawal et al. (2015) and Bates and Chen (1999) for traveling wave solutions and Li et al. (2016b) for new types of entire solutions.

When Eq. (1.1) is used to model the population dynamics of a species, it is assumed that the underlying environment is continuous and the internal interaction of the organisms is nonlocal. Conversely, if we assume that the organisms move randomly between the adjacent spatial locations, then it is more effective to use the following classical reaction–diffusion equation

\[ u_t(t, x) = d \Delta u(t, x) + u(t, x)(r(x - ct) - u(t, x)), \quad t > 0, x \in \mathbb{R}, \quad (1.2) \]

and if the species live in patchy environments, the lattice differential equation of the form

\[ u_t(t, x) = d[u(t, x + 1) - 2u(t, x) + u(t, x - 1)] \\
+ u(t, x)(r(x - ct) - u(t, x)), \quad t > 0, x \in \mathbb{Z}, \quad (1.3) \]

is more meaningful. Note that Eqs. (1.1)–(1.3) are neither homogeneous nor periodic, but possess special heterogeneity with the form of “spatial shifting” at a constant speed.” Therefore, we cannot directly apply the abstract theory developed for monotone semiflows in Liang and Zhao (2007, 2010) and Fang et al. (2017) to address the issue of spreading speeds and traveling wave solutions. Certain ad hoc techniques that fit the equation itself are necessarily needed. Recently, Li et al. (2014) and Hu and Li (2015) studied the spatial dynamics of systems (1.2) and (1.3), respectively, and they showed that the long-term behavior of solutions depends on the speed of the shifting habitat edge \( c \) and a number \( c^*(\infty) \), where \( c^*(\infty) = 2\sqrt{dr(\infty)} \) for (1.2) and
More accurately, they demonstrated that if \( c > c^*(\infty) \), then the species will become extinct in the habitat, and if \( 0 < c < c^*(\infty) \), then the species will persist and spread along the shifting habitat at the asymptotic spreading speed \( c^*(\infty) \). Very recently, by the monotone iterative method, Hu and Zou (2017) proved that (1.2) admits a monotone traveling wave solution connecting 0 to \( r(\infty) \) with the speed being the habitat shifting speed, which indeed accounts for an extinction wave. Here we remark that by a change of variable 
\[
v(t, x) = u(t, -x),
\]
such a forced traveling wave for (1.2) can also be obtained from Fang et al. (2016, Theorem 2.1(i)), as applied to the resulting equation. In addition, Li et al. (2016a) considered this type of problem using an integro-diﬀerence equation model in an expanding or contracting habitat. In this paper, we propose to extend the above existing results on systems (1.2) and (1.3) to our nonlocal dispersal problem (1.1). To summarize, we first study the persistence and spreading speed properties by applying the comparison principle and constructing an appropriate subsolution and then establish the existence of traveling wave solutions by constructing super-/subsolution and using the method of monotone iteration. We should point out that the combination of nonlocal effects and shifting environment makes the analysis on model (1.1) more diﬃcult. In particular, the construction of some appropriate subsolutions to study the spreading speed and traveling waves is highly nontrivial.

The rest of the paper is organized as follows. In Sect. 2, we give some preliminaries including the uniqueness and existence of solutions and the comparison principle. Section 3 is devoted to the study of the persistence and spreading speed. In Sect. 4, we investigate traveling wave solutions.

### 2 Preliminaries

Let

\[
\mathbb{Y} = \{ \psi \in C(\mathbb{R}, \mathbb{R}) : \psi \text{ is bounded and uniformly continuous on } \mathbb{R} \}
\]

with norm \( \|\psi\| = \sup_{x \in \mathbb{R}} |\psi(x)| \), and \( \mathbb{Y}_+ = \{ \psi \in \mathbb{Y} : \psi(x) \geq 0, \forall x \in \mathbb{R} \} \). It is easily seen that \( \mathbb{Y}_+ \) is a closed cone of \( \mathbb{Y} \) and its induced partial ordering makes \( \mathbb{Y} \) into a Banach lattice. In view of the property (J), we see that \( J * u - u : \mathbb{Y} \to \mathbb{Y} \) is a bounded linear operator with respect to the norm \( \| \cdot \| \). It then follows that the system

\[
\begin{cases}
\frac{\partial u(t, x)}{\partial t} = d(J * u - u)(t, x), & t > 0, x \in \mathbb{R}, \\
u(0, x) = \psi(x), & x \in \mathbb{R}, \psi \in \mathbb{Y}
\end{cases}
\]

(2.1)

generates a strongly continuous semigroup \( P(t) \) on \( \mathbb{Y} \), which is also strongly positive in the sense of \( P(t)\mathbb{Y}_+ \subseteq \mathbb{Y}_+ \) and \( [P(t)\psi](x) \gg 0 \) if \( \psi(x) \geq 0 \) has a nonempty support and \( t > 0 \). According to Weng and Zhao (2006), the unique mild solution of system (2.1) is given by

\[
[P(t)\psi](x) = e^{-dt} \sum_{k=0}^{\infty} \frac{(dt)^k}{k!} a_k(\psi)(x),
\]

(2.2)
where \( a_0(\psi)(x) = \psi(x) \) and \( a_k(\psi)(x) = \int_{\mathbb{R}} J(x - y)a_{k-1}(\psi)(y)dy \) for any integer \( k \geq 1 \). On the other hand, Ignat and Rossi (2007, Section 2) showed that the fundamental solution of (2.1) can be decomposed as

\[
G(t, x) = e^{-dt} \delta_0(x) + R_t(x),
\]

(2.3)

where \( \delta_0(\cdot) \) is the delta measure at zero and \( R_t(x) = R(t, x) \) is smooth defined by

\[
R(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-dt}(e^{d\widehat{T}(\xi)t} - 1)e^{ix\xi}d\xi
\]

with \( i = \sqrt{-1} \) and \( \widehat{J} \) being the Fourier transform of \( J \). Moreover, the solution of (2.1) can also be written as

\[
u(t, x) = \int_{\mathbb{R}} G(t, y)\psi(x - y)dy = e^{-dt} \psi(x) + \int_{\mathbb{R}} R_t(y)\psi(x - y)dy, \quad t \geq 0, x \in \mathbb{R}.
\]

(2.4)

It then follows that \( u(t, \cdot) \) is as regular as \( \psi \) is, and hence, the nonlocal dispersal operator \( J * u - u \) does not have the regularizing effect for solutions of the Cauchy problem (2.1). Further, we have the following properties about \( G(t, x) \).

**Lemma 2.1** \( G(t, x) = G(t, -x) \) for all \( t \geq 0 \) and \( x \in \mathbb{R} \). Further, \( \int_{\mathbb{R}} G(t, y)dy = 1 \) and \( \|G(t, \cdot)\|_{L^p(\mathbb{R})} \leq 3 \) for any \( t \geq 0 \) and \( p \in [1, \infty] \).

**Proof** By the symmetry of \( J \), we have

\[
\widehat{J}(\xi) = \int_{\mathbb{R}} J(x)e^{-ix\xi}dx = \int_{\mathbb{R}} J(x)\cos(x\xi)dx,
\]

which implies that \( \widehat{J}(-\xi) = \widehat{J}(\xi) \) and \( -1 \leq \widehat{J}(\xi) \leq 1 \). Then a direct computation yields that

\[
R_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-dt}(e^{d\widehat{T}(\xi)t} - 1)e^{ix\xi}d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-dt}(e^{d\widehat{T}(\xi)t} - 1)[\cos(x\xi) + i \sin(x\xi)]d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-dt}(e^{d\widehat{T}(\xi)t} - 1) \cos(x\xi)d\xi.
\]

Therefore, \( R_t(-x) = R_t(x) \). By (2.3), we obtain

\[
G(t, x) = e^{-dt} \delta_0(x) + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-dt}(e^{d\widehat{T}(\xi)t} - 1) \cos(x\xi)d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} e^{d(\widehat{T}(\xi)-1)t} \cos(x\xi)d\xi.
\]
Clearly, \( G(t, x) = G(t, -x) \) for all \( t \geq 0 \) and \( x \in \mathbb{R} \). Note that \( u(t, x) \equiv C \) for \((t, x) \in [0, \infty) \times \mathbb{R} \) is a solution of (2.1), where \( C \) is some positive constant. On the other hand, by (2.4), the solution of (2.1) with initial value \( \psi(x) = C \) can be expressed by \( C = \int_{\mathbb{R}} G(t, y)Cdy \), which indicates that \( \int_{\mathbb{R}} G(t, y)dy = 1 \) for all \( t \geq 0 \). The conclusion of \( ||G(t, \cdot)||_{L_p(\mathbb{R})} \leq 3 \) with \( p \in [1, \infty) \) originates from Ignat and Rossi (2007, Remark 2.1).

Let \( f(x, u) = u(r(x) - u) \). For any \( 0 \leq u_1, u_2 \leq r(\infty) \) and \(-\infty < x < \infty \), since \( r(-\infty) \leq r(x) \leq r(\infty) \) and \( r(-\infty) < 0 < r(\infty) \) due to (H), we have

\[
|f(x, u_1) - f(x, u_2)| = |r(x) - (u_1 + u_2)||u_1 - u_2|
\leq (|r(x)| + |u_1| + |u_2|)|u_1 - u_2|
\leq (\max\{|-r(-\infty), r(\infty)\} + 2r(\infty))|u_1 - u_2|,
\]

which indicates that \( f(x, u) \) is Lipschitz continuous in \( u \in [0, r(\infty)] \). Choose \( \rho > 2r(\infty) - r(-\infty) \), then \( \rho u + f(x, u) \) is nondecreasing in \( u \in [0, r(\infty)] \). Consider the equivalent equation obtained by adding the linear term \( \rho u(t, x) \) to both sides of (1.1):

\[
\frac{\partial u(t, x)}{\partial t} + \rho u(t, x) = d(J \ast u - u)(t, x) + u(t, x)(\rho + r(x - ct) - u(t, x)). \tag{2.5}
\]

Set \( \mathbb{Y}_{r(\infty)} := \{ \psi \in \mathbb{Y} : 0 \leq \psi(x) \leq r(\infty), \forall x \in \mathbb{R} \} \). The mild solution of Eq. (2.5) or (1.1) with \( u(0, \cdot) = u_0(\cdot) \in \mathbb{Y}_{r(\infty)} \) can be expressed as a fixed point of the nonlinear integral equation in \( C(\mathbb{R}_+, \mathbb{Y}_{r(\infty)}) \):

\[
u(t, x) = [N_0u](t, x) \triangleq \left[ e^{-\rho t} P(t)u_0 \right](x)
+ \int_0^t e^{-\rho(t-s)} P(t-s)u(s, x)(\rho + r(x - cs) - u(s, x))ds. \tag{2.6}
\]

With the expression of \( P(t) \), a direct calculation shows that

\[
\frac{\partial[P(t)\psi](x)}{\partial t} = -d[P(t)\psi](x) + d\int_{\mathbb{R}} J(y)[P(t)\psi](x - y)dy,
\]

which indicates that the right side of (2.6) is differentiable with respect to \( t \). Thus, \( u(t, x) \) is a classical solution of Eq. (2.5) or (1.1).

**Definition 2.2** \( u \in C([0, T), \mathbb{Y}_+) \) with \( 0 < T \leq \infty \) is called a supersolution (subsolution) of (2.6) if \( u(t, x) \geq (\leq)[N_0u](t, x) \) for all \( t \in [0, T) \) and \( x \in \mathbb{R} \).

**Remark 2.1** If \( u \in C([0, T), \mathbb{Y}_+) \) being \( C^1 \) in \( t \in (0, T) \) satisfies (2.5) or (1.1) with “=” being replaced by “\( \geq \)” (“\( \leq \)”) and \( u(0, x) \geq (\leq)u_0(x) \), then it follows from the positivity of \( P(t) \) that \( u \) is a supersolution (subsolution) of (2.6). Moreover, we can easily verify that \( u \equiv r(\infty) \) and \( u \equiv 0 \) are a supersolution and a trivial subsolution of (2.6), respectively.

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Now we consider the sequence $\{u^n(t, x)\}_{n=0}^{\infty}$ generated by

$$u^{n+1}(t, x) = [\mathcal{N}_0u^n](t, x),$$

(2.7)

where $u^0(t, x) = 0$ or $u^0(t, x) = r(\infty)$.

**Theorem 2.3** Let $u_0 \in \mathcal{Y}_{r(\infty)}$. Then Eq. (2.6) admits a unique solution $u \in C(\mathbb{R}_+, \mathcal{Y}_{r(\infty)})$. Moreover, the comparison principle holds for (2.6), i.e., if $u_1(t, x)$ and $u_2(t, x)$ are two solutions of (2.6) associated with initial value $u_{10}, u_{20} \in \mathcal{Y}_{r(\infty)}$, respectively, with $u_{10}(x) \leq u_{20}(x)$ for all $x \in \mathbb{R}$, then $u_1(t, x) \leq u_2(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. If we further assume that $u_{10} \neq u_{20}$, then $u_1(t, x) < u_2(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}$.

**Proof** This proof is based on a classical super subsolution method and we only give a sketch here. Define $u^{n+1}(t, x) = [\mathcal{N}_0u^n](t, x)$ with $u^0(t, x) = 0$, and $\bar{u}^{n+1}(t, x) = [\mathcal{N}_0\bar{u}^n](t, x)$ with $\bar{u}^0(t, x) = r(\infty)$. Then we can show by induction that

$$0 \leq u^1(t, x) \leq \cdots \leq u^n(t, x) \leq \cdots \leq \bar{u}^1(t, x) \leq \bar{u}(\infty),$$

which implies that the pointwise limits
d

$$\underline{u}(t, x) := \lim_{n \to \infty} u^n(t, x) \text{ and } \bar{u}(t, x) := \lim_{n \to \infty} \bar{u}^n(t, x)$$

both exist and satisfy that $0 \leq \underline{u} \leq \bar{u} \leq r(\infty)$. Moreover, both $\underline{u}$ and $\bar{u}$ are solutions of (2.6) in $C(\mathbb{R}_+, \mathcal{Y}_{r(\infty)})$. We now prove $\underline{u}(t, x) = \bar{u}(t, x)$. Note that for any $\psi \in \mathcal{Y}$, $\|a_0(\psi)\| = \|\psi\|, \|a_1(\psi)\| = \|\int_\mathbb{R} J(y)a_0(\psi)(\cdot - y)dy\| \leq \|\psi\|$, by induction, we can claim that $\|a_k(\psi)\| \leq \|\psi\|$ for all $k = 0, 1, 2, \ldots$. By using (2.2), we obtain

$$\|P(t)\psi\| \leq e^{-\rho t} \sum_{k=0}^{\infty} \frac{(dt)^k}{k!} \|a_k(\psi)\| \leq \|\psi\|, \quad \forall t \geq 0.$$  

(2.8)

Therefore, by (2.6) and (2.8), a direct calculation yields that

$$0 \leq \bar{u}(t, x) - \underline{u}(t, x)$$

$$\leq (\rho + \max\{-r(-\infty), r(\infty)\}) + 2r(\infty) \int_0^t e^{-\rho(t-s)} P(t-s)[\bar{u}(s, x) - \underline{u}(s, x)]ds$$

$$\leq (\rho + \max\{-r(-\infty), r(\infty)\}) + 2r(\infty) \int_0^t e^{-\rho(t-s)} \|P(t-s)[\bar{u}(s, \cdot) - \underline{u}(s, \cdot)]\|ds$$

$$\leq (\rho + \max\{-r(-\infty), r(\infty)\}) + 2r(\infty) \int_0^t e^{-\rho(t-s)} \|\bar{u}(s, \cdot) - \underline{u}(s, \cdot)\|ds,$$

which shows that

$$0 \leq e^{\rho t} \|\bar{u}(t, \cdot) - \underline{u}(t, \cdot)\| \leq (\rho + \max\{-r(-\infty), r(\infty)\})$$

$$+ 2r(\infty) \int_0^t e^{\rho s} \|\bar{u}(s, \cdot) - \underline{u}(s, \cdot)\|ds.$$
Then Gronwall’s inequality implies that $0 \leq e^{\rho t} \|\tilde{u}(t, \cdot) - u(t, \cdot)\| \leq 0$. This implies that $u(t, x) = \tilde{u}(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. The comparison principle is a straightforward consequence of the construction for solutions. Using the strongly positivity of $P(t)$, we can easily prove the last conclusion of this theorem. \qed

The following result is a simple consequence of Theorem 2.3.

**Corollary 2.4** Let $u, v \in C(\mathbb{R}_+, Y_{r(\infty)})$ be the supersolution and subsolution of (2.6) for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, respectively. If $u(0, x) \geq v(0, x)$ for all $x \in \mathbb{R}$, then $u(t, x) \geq v(t, x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

**Proof** According to the positivity of $P(t)$ and the choice of $\rho$ (i.e., $\rho > 2r(\infty) - r(-\infty)$), which guarantees the monotonicity of $\rho u + f(x - ct, u)$ with respect to $u$, we can claim that the nonlinear operator $\mathcal{N}_0$ defined by (2.6) is order preserving in the sense that

$$u(t, x) \geq [\mathcal{N}_0 u](t, x) \geq [\mathcal{N}_0^k u](t, x) \geq \cdots \geq \lim_{k \to \infty} [\mathcal{N}_0^k u](t, x) =: \tilde{u}(t, x)$$

and

$$v(t, x) \leq [\mathcal{N}_0 v](t, x) \leq [\mathcal{N}_0^k v](t, x) \leq \cdots \leq \lim_{k \to \infty} [\mathcal{N}_0^k v](t, x) =: \tilde{v}(t, x).$$

Clearly, $\tilde{u}(0, x) = u(0, x)$ and $\tilde{v}(0, x) = v(0, x)$. Moreover, both $\tilde{u}(t, x)$ and $\tilde{v}(t, x)$ are the solutions of (2.6), and hence, Theorem 2.3 implies that $u(t, x) \geq \tilde{u}(t, x) \geq \tilde{v}(t, x) \geq v(t, x)$ because $\tilde{u}(0, x) \geq \tilde{v}(0, x)$, which implies the requested result. \qed

### 3 Persistence and Spreading Speeds

In this section, we first show that the species will become extinct in the long run if the edge of the habitat shifts relatively fast. For $r(x) > 0$ and $\lambda > 0$, we define

$$\phi(x; \lambda) = \frac{d}{\lambda} \left( \int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1 \right) + r(x).$$

Clearly, it follows from (J) that $\phi(x; \lambda) > 0$ and $\phi(x; \lambda) \to \infty$ as $\lambda \to 0$. Since $J$ is symmetric and compactly supported and $\int_{\mathbb{R}} J = 1$, we further have

$$\phi(x; \lambda) = \frac{d}{\lambda} \left( \int_{\mathbb{R}} J(y) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} dy - 1 \right) + r(x)$$

$$= \frac{d}{\lambda} \int_{\mathbb{R}} J(y) \sum_{m=1}^{\infty} \frac{(\lambda y)^{2m}}{(2m)!} dy + r(x)$$

$$= d \sum_{m=1}^{\infty} \frac{\lambda^{2m-1}}{(2m)!} \int_{\mathbb{R}} J(y) y^{2m} dy + r(x) \to \infty \text{ as } \lambda \to \infty$$

and

$$\frac{\partial^2 \phi(x; \lambda)}{\partial \lambda^2} = d \sum_{m=1}^{\infty} \frac{(2m-1)(2m-2)\lambda^{2m-3} \int_{\mathbb{R}} J(y) y^{2m} dy}{(2m)!} + \frac{2r(x)}{\lambda^3} > 0.$$
It then easily follows that for each fixed $x$, $\phi(x; \lambda)$ has only one minimum denoted by $c^*(x)$, i.e.,
\[ c^*(x) = \min_{\lambda > 0} \phi(x; \lambda) = \phi(x; \lambda^*(x)) > 0, \]
where $\lambda^*(x)$ denotes the unique point where the minimum occurs.

**Theorem 3.1** Assume that $c > c^*(\infty) \triangleq \min_{\lambda > 0} \frac{d(f_{\lambda} J(y)e^ydy - 1) + r(\infty)}{\lambda}$. Let $u(t, x, \psi)$ be the unique solution of (1.1) with $u(0, x, \psi) = \psi(x)$, $\forall x \in \mathbb{R}$. If $\psi \in \mathbb{Y}_{r(\infty)}$ has a compact support and $\sup_{x \in \mathbb{R}} \psi(x) < r(\infty)$, then $\lim_{t \to \infty} u(t, x, \psi) = 0$ uniformly for $x \in \mathbb{R}$.

**Proof** Let $c > c^*(\infty)$ be given, and let $U(x - ct)$ be the nondecreasing positive traveling wave solution of (1.1) with $U(-\infty) = 0$ and $U(+\infty) = r(\infty)$, the existence of which is guaranteed by Theorem 4.5. Since $\psi(x)$ has a compact support and $\psi(x) < r(\infty)$, $\forall x \in \mathbb{R}$, there exists a sufficiently large $x_0 > 0$ such that $U(x + x_0) \geq \psi(x)$, $\forall x \in \mathbb{R}$. Set $v(t, x) = U(x - ct + x_0)$ and $z = x - ct + x_0$, $\forall t \geq 0, x \in \mathbb{R}$. Since $r(\cdot)$ is nondecreasing due to (H), it follows from (4.1) that
\[
\frac{\partial v(t, x)}{\partial t} = -cU'(z) = d\left(\int_{\mathbb{R}} J(y)U(z - y)dy - U(z)\right) + U(z)[r(x - ct + x_0) - U(z)] \\
\geq d(J \ast v - v)(t, x) + v(t, x)[r(x - ct) - v(t, x)], \quad \forall t > 0, x \in \mathbb{R},
\]
which indicates that $v(t, x)$ is an upper solution of (1.1). By the standard comparison theorem, we then have $u(t, x, \psi) \leq U(x - ct + x_0)$, $\forall t \geq 0, x \in \mathbb{R}$. For any $\epsilon > 0$, since $U(-\infty) = 0$, we can choose a large number $M > 0$ such that $U(-M + x_0) < \epsilon$. Thus, by the monotonicity of $U$, we obtain
\[
u(t, x, \psi) \leq U(x - ct + x_0) \leq U(-M + x_0) < \epsilon, \quad \forall t \geq 0, x \leq -M + ct. \tag{3.1}
\]

Note that $c^*(\infty)$ is the spreading speed of the following homogeneous nonlocal dispersal equation (see, e.g., Lutscher et al. 2005, Theorem 3.2):
\[
w_t = d(J \ast w - w) + w(r(\infty) - w). \tag{3.2}
\]

Let $w(t, x, \psi)$ be the unique solution of (3.2) with $w(0, x, \psi) = \psi(x)$, $\forall x \in \mathbb{R}$. Fix $c_1 \in (c^*(\infty), c)$, we then have $\lim_{t \to \infty} \sup_{x \geq c_1 t} w(t, x, \psi) = 0$. It is easy to see that $w(t, x, \psi)$ is an upper solution of (1.1) due to (H), and hence, $u(t, x, \psi) \leq w(t, x, \psi)$, $\forall t \geq 0, x \in \mathbb{R}$. It follows that $\lim_{t \to \infty} \sup_{x \geq c_1 t} u(t, x, \psi) = 0$. Thus, there exists $T_1 > 0$ such that
\[
u(t, x, \psi) < \epsilon, \quad \forall t \geq T_1, x \geq c_1 t. \tag{3.3}
\]

Let $T = \max\left\{T_1, \frac{M}{c_1 - c}\right\}$. Then $-M + ct \geq c_1 t$, $\forall t \geq T$, and it follows from (3.1) and (3.3) that $u(t, x, \psi) < \epsilon$, $\forall t \geq T, x \in \mathbb{R}$. This shows that $\lim_{t \to \infty} u(t, x, \psi) = 0$ uniformly for $x \in \mathbb{R}$. □
In the rest of this section, we consider the case that the edge of habitat is moving at a speed less than \( c^*(\infty) \). The construction of a suitable subsolution plays a key role in our theoretical analysis. To proceed, we introduce an auxiliary function that can be found in Weinberger (1982), see also Li et al. (2014) and Hu and Li (2015). For \( \lambda > 0 \) and \( \gamma > 0 \), define

\[
v(x; \lambda, \gamma) = \begin{cases} 
    e^{-\lambda x} \sin(\gamma x), & \text{if } 0 \leq x \leq \pi/\gamma, \\
    0, & \text{elsewhere}. 
\end{cases}
\]

Note that \( v(x; \lambda, \gamma) \) is nonnegative, continuous in \( x \in \mathbb{R} \) and continuously differentiable when \( x \neq 0, \pi/\gamma \). Moreover, \( v(x; \lambda, \gamma) \) takes the maximum at the point \( x = \sigma(\lambda, \gamma) := (1/\gamma) \arctan(\gamma/\lambda) \in (0, \pi/\gamma) \). Note that \( \sigma(\lambda, \gamma) \) is strictly decreasing in \( \lambda > 0 \) and also that the maximum \( v(\sigma(\lambda, \gamma); \lambda, \gamma) \in (0, 1) \).

Define

\[
\varphi(\lambda, \gamma) = d \int_{\mathbb{R}} J(y) e^{\lambda y} \frac{\sin(\gamma y)}{\gamma} \, dy. \tag{3.4}
\]

Assume that the compact support of the kernel \( J \) is \([-L, L]\). Since \( J \) is symmetric, we have

\[
\varphi(\lambda, \gamma) = d \int_{0}^{L} J(y) \left( e^{\lambda y} - e^{-\lambda y} \right) \frac{\sin(\gamma y)}{\gamma} \, dy.
\]

From now on, \( \gamma > 0 \) will be assumed to be sufficiently small so that both \( \sin(\gamma y) > 0 \) and \( \cos(\gamma y) > 0 \) for \( y \in (0, L] \). Particularly, we can choose \( \gamma \in (0, \frac{\pi}{2L}) \). Obviously, \( \varphi(\lambda, \gamma) > 0 \). Further, a direct computation leads to

\[
\frac{\partial \varphi(\lambda, \gamma)}{\partial \lambda} = d \int_{0}^{L} y J(y) (e^{\lambda y} + e^{-\lambda y}) \frac{\sin(\gamma y)}{\gamma} \, dy > 0,
\]

which indicates that \( \varphi(\lambda, \gamma) \) is increasing in \( \lambda > 0 \). Let

\[
\phi_{\gamma}(l; \lambda) := \frac{d \left( \int_{\mathbb{R}} J(y) e^{\lambda y} \cos(\gamma y) \, dy - 1 \right) + r(l)}{\lambda} \tag{3.5}
\]

and

\[
c_{\gamma}^*(l) = \min_{\lambda > 0} \phi_{\gamma}(l; \lambda).
\]

Clearly, \( \phi_{\gamma}(l; \lambda) < \phi(l; \lambda) \) and \( \phi_{\gamma}(l; \lambda) \) converges to \( \phi(l; \lambda) \) uniformly for \( \lambda \) in any bounded interval as \( \gamma \to 0 \). Moreover, we have \( c_{\gamma}^*(l) < c^*(l) \) and the convergence \( c_{\gamma}^*(l) \to c^*(l) \) as \( \gamma \to 0 \).

**Lemma 3.2** Assume that \( c \in (0, c^*(\infty)) \). For any \( 0 < \delta < \frac{c^*(\infty) - c}{2} \), let \( l \) be the point such that \( c^*(l) = c^*(\infty) - \delta \), and small \( \gamma \in (0, \frac{\pi}{2L}) \) such that \( c^*(l) - c_{\gamma}^*(l) \leq \delta \). Choose \( 0 < \lambda_1 < \lambda_2 < c^*(l) \) satisfying \( \varphi(\lambda_1, \gamma) = c + \delta \) and \( \varphi(\lambda_2, \gamma) = c_{\gamma}^*(l) - 2\delta \). Then for any \( \lambda \in [\lambda_1, \lambda_2] \) and small \( a > 0 \), \( w(t, x) = a v(x - l - \varphi(\lambda, \gamma)t) \) is a continuous subsolution of (2.5). Furthermore, if \( u_0(x) \geq av(x - l; \lambda, \gamma) \), then
Clearly, \( w(t, x) \geq au(x - l - \varphi(\lambda, \gamma)t; \lambda, \gamma) \) for all \( t > 0 \) and \( x \in \mathbb{R} \), where \( u(t, x) \) is the solution of \((2.5)\) with \( u(0, x) = u_0(x) \).

Before we give the proof of this lemma, we first explain why there exist \( 0 < \lambda_1 < \lambda_2 < \lambda^*(l) \) such that \( \varphi(\lambda_1, \gamma) = c + \delta \) and \( \varphi(\lambda_2, \gamma) = c^*(l) - 2\delta \). Since \( c^*(l) = \min_{\lambda > 0} \phi(l; \lambda) = \phi(l; \lambda^*(l)) \), we have \( \frac{\partial \phi(l; \lambda)}{\partial \lambda} \big|_{\lambda = \lambda^*(l)} = 0 \). It then follows that
\[
d \int_{\mathbb{R}} J(y) y e^{\lambda^*(l)y} \, dy = \frac{d \left( \int_{\mathbb{R}} J(y) e^{\lambda^*(l)y} \, dy - 1 \right) + r(l)}{\lambda^*(l)} = c^*(l).
\]
Since \( J \) is symmetric and compactly supported, we can take \( \gamma > 0 \) small enough such that
\[
\varphi(\lambda^*(l), \gamma) = d \int_{0}^{L} J(y) y \left( e^{\lambda^*(l)y} - e^{-\lambda^*(l)y} \right) \frac{\sin(\gamma y)}{\gamma y} \, dy
\approx d \int_{0}^{L} J(y) y \left( e^{\lambda^*(l)y} - e^{-\lambda^*(l)y} \right) \, dy
= d \int_{-L}^{L} J(y) y e^{\lambda^*(l)y} \, dy = c^*(l),
\]
where \([{-L}, {L}] = \text{supp}(J)\). On the other hand, it is easy to verify that
\[
\varphi(0, \gamma) = 0 \text{ and } 0 < c + \delta < c^*(\infty) - 4\delta = c^*(l) - 3\delta \leq c^*_\gamma(l) - 2\delta < c^*(l).
\]

In view of these facts, we can further obtain the existence of such \( \lambda_1 \) and \( \lambda_2 \) by the continuity and monotonicity of \( \varphi(\cdot, \gamma) \).

**Proof** By Definition 2.2, we need to justify that \( w(t, x) \leq [N_0 w](t, x) \) for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\). Notice that for \( t > 0 \) and \( x < l + \varphi(\lambda, \gamma)t \) or \( x > l + \varphi(\lambda, \gamma)t + \pi/\gamma \), \( w(t, x) \equiv 0 \), then the proof is trivial. Now we consider the case where \( t > 0 \) and \( l + \varphi(\lambda, \gamma)t \leq x \leq l + \varphi(\lambda, \gamma)t + \pi/\gamma \). At present,
\[
w(t, x) = a u(x - l - \varphi(\lambda, \gamma)t; \lambda, \gamma) = a e^{-\lambda(x - l - \varphi(\lambda, \gamma)t)} \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)].
\]

Clearly, \( w(t, x) \) is continuously differentiable with respect to \( t \) in such case. According to Remark 2.1, it suffices to prove that for \( t > 0 \) and \( l + \varphi(\lambda, \gamma)t \leq x \leq l + \varphi(\lambda, \gamma)t + \pi/\gamma \), there holds that
\[
\frac{\partial w(t, x)}{\partial t} \leq d \left( \int_{\mathbb{R}} J(y) w(t, x - y) \, dy - w(t, x) \right) + w(t, x) \left( r(x - ct) - w(t, x) \right). \tag{3.6}
\]

By a direct calculation, we obtain
\[
\frac{\partial w(t, x)}{\partial t} = a \varphi(\lambda, \gamma) e^{-\lambda(x - l - \varphi(\lambda, \gamma)t)} \left( \lambda \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)] \right. \\
- \gamma \cos[\gamma(x - l - \varphi(\lambda, \gamma)t)]
\]
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Note that if \( y \in \text{supp}(J) = [-L, L] \), we have

\[
l + \varphi(\lambda, \gamma)t - L \leq x - y \leq l + \varphi(\lambda, \gamma)t + \pi/\gamma + L.
\]

It then follows that

\[
-\frac{\pi}{2} \leq -\gamma L \leq \gamma(x - y - l - \varphi(\lambda, \gamma)t) \leq \pi + \gamma L \leq \frac{3\pi}{2}.
\]

For such \( t \) and \( x \), there holds

\[
w(t, x - y) \geq ae^{-\lambda(x-y-l-\varphi(\lambda, \gamma)t)} \sin[\gamma(x - y - l - \varphi(\lambda, \gamma)t)].
\]

Therefore,

\[
d \left( \int_{\mathbb{R}} J(y) w(t, x - y) dy - w(t, x) \right) = d \left( \int_{-L}^{L} J(y) w(t, x - y) dy - w(t, x) \right)
\geq \int_{-L}^{L} J(y) e^{\lambda y} e^{-\lambda(x-y-l-\varphi(\lambda, \gamma)t)} \sin[\gamma(x - y - l - \varphi(\lambda, \gamma)t)] dy
\geq e^{-\lambda(x-y-l-\varphi(\lambda, \gamma)t)} \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)]
= ade^{-\lambda(x-y-l-\varphi(\lambda, \gamma)t)} \left( \int_{\mathbb{R}} J(y) e^{\lambda y} \cos(\gamma y) dy \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)] - \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)] \right).\]

To prove claim (3.6), one only need to show

\[
\lambda \varphi(\lambda, \gamma) \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)]
\leq d \left( \int_{\mathbb{R}} J(y) e^{\lambda y} \cos(\gamma y) dy - 1 \right) \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)]
+ \left( \gamma \varphi(\lambda, \gamma) - d \int_{\mathbb{R}} J(y) e^{\lambda y} \sin(\gamma y) dy \cos[\gamma(x - l - \varphi(\lambda, \gamma)t)] \right)
+ \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)] \left( r(x - ct) - a \nu(x - l - \varphi(\lambda, \gamma)t; \lambda, \gamma) \right),\]

which is equivalent to

\[
\lambda \varphi(\lambda, \gamma) \leq d \left( \int_{\mathbb{R}} J(y) e^{\lambda y} \cos(\gamma y) dy - 1 \right) + r(x - ct) - a \nu(x - l - \varphi(\lambda, \gamma)t; \lambda, \gamma)
\]

due to (3.4) and \( \sin[\gamma(x - l - \varphi(\lambda, \gamma)t)] > 0 \) for \( l + \varphi(\lambda, \gamma)t < x < l + \varphi(\lambda, \gamma)t + \pi/\gamma \).
Note that (3.7) holds naturally when \( x = l + \varphi(\lambda, \gamma)t \) or \( l + \varphi(\lambda, \gamma)t + \pi/\gamma \).
According to the hypothesis, \( \varphi(\lambda_1, \gamma) = c + \delta \) and \( \varphi(\lambda_2, \gamma) = c^*_\gamma(l) - 2\delta \) for some \( 0 < \lambda_1 < \lambda_2 < \lambda_*(l) \). Then for \( \lambda \in [\lambda_1, \lambda_2] \), \( x > l + \varphi(\lambda_1, \gamma) t \geq l + \varphi(\lambda, \gamma) t = l + (c + \delta) t > l + c t \) with \( t > 0 \), that is, \( x - c t > l \). Since \( r(\cdot) \) is nondecreasing, then \( r(x - c t) \geq r(l) \). Notice also that \( \nu(x - l - \varphi(\lambda, \gamma) t; \lambda, \gamma) \leq 1 \). Therefore, in order to prove (3.8), we only need to verify

\[
\lambda \varphi(\lambda, \gamma) \leq d \left( \int_{\mathbb{R}} J(y) e^{\lambda y} \cos(\gamma y) dy - 1 \right) + r(l) - a,
\]

which, together with (3.5), indicates that

\[
a \leq \lambda (\phi_\gamma(l; \lambda) - \varphi(\lambda, \gamma)).
\]

Based on the previous parameter setting, we have

\[
\phi_\gamma(l; \lambda) - \varphi(\lambda, \gamma) \geq c^*_\gamma(l) - \varphi(\lambda_2, \gamma) = c^*_\gamma(l) - (c^*_\gamma(l) - 2\delta) = 2\delta.
\]

Thus, (3.10) holds as long as we select \( 0 < a \leq 2\lambda_1 \delta \), which, in return, shows that for \( \lambda \in [\lambda_1, \lambda_2] \) and sufficiently small \( a > 0 \), \( w(t, x) = a \nu(x - l - \varphi(\lambda, \gamma) t; \lambda, \gamma) \) is a continuous subsolution of (2.5). Furthermore, if \( u_0(x) \geq \nu(x - l; \lambda, \gamma) \), then it follows from Corollary 2.4 that \( u(t, x) \geq \nu(x - l - \varphi(\lambda, \gamma) t; \lambda, \gamma) \) for all \( t > 0 \) and \( x \in \mathbb{R} \).

The subsequent result shows that if the edge of the habitat suitable for species growth is shifting at a speed \( c < c^*(\infty) \), then the species will persist in the space and spread to the right at the asymptotic speed \( c^*(\infty) \).

**Theorem 3.3** Assume that \( c^*(\infty) > c > 0 \). Let \( u(t, x) \) be the solution of (1.1) with \( u(0, \cdot) = u_0(\cdot) \in Y_{r(\infty)} \). Then the following statements are valid:

(i) if \( \sup_{x \in \mathbb{R}} u_0(x) < r(\infty) \) and there exists a constant \( K_1 \) such that \( u_0(x) = 0 \) for all \( x \leq K_1 \), then for any \( \xi > 0 \),

\[
\lim_{t \to \infty} \sup_{x \leq (c - \xi) t} u(t, x) = 0;
\]

(ii) if there exists a constant \( K_2 \) such that \( u_0(x) = 0 \) for all \( x \geq K_2 \), then for any \( \xi > 0 \),

\[
\lim_{t \to \infty} \sup_{x \geq (c^*(\infty) + \xi) t} u(t, x) = 0;
\]

(iii) if \( u_0(x) > 0 \) on a closed interval, then for each \( \xi \in \left( 0, \frac{c^*(\infty) - c}{2} \right) \), there holds

\[
\lim_{t \to \infty} u(t, x) = r(\infty).
\]
Proof  (i) By Theorem 2.3, we have $0 \leq u(t, x) \leq r(\infty)$ for all $t \geq 0$ and $x \in \mathbb{R}$. In view of the first part of the proof of Theorem 3.1, we see that for any $\epsilon > 0$, there exists a large number $M > 0$ such that

$$u(t, x) < \epsilon, \quad \forall t \geq 0, x \leq -M + ct.$$  

On the other hand, for any given $\zeta > 0$, there exists $T_2 > 0$ such that for $t \geq T_2$, $(c - \zeta)t \leq -M + ct$. In fact, this can be done if we let $T_2 \geq M/\zeta$. Thus, we have that

$$u(t, x) < \epsilon, \quad \forall t \geq T_2, x \leq (c - \zeta)t.$$  

This completes the proof of (i).

(ii) For any $\zeta > 0$, let $\lambda_{\zeta} > 0$ be the smaller positive solution of $\phi(\infty; \lambda) = c^*(\infty) + \zeta/2$. In other words, we have $d \left( \int_{\|y\|} J(y) e^{\lambda_{\zeta} y} dy - 1 \right) + r(\infty) = \lambda_{\zeta} (c^*(\infty) + \zeta/2)$. Note that $\hat{u}(t, x) = A e^{-\lambda_{\zeta} (x - (c^*(\infty) + \zeta/2)t)}$ with $A > 0$ being a constant satisfies

$$\hat{u}_t(t, x) = d(J \ast \hat{u} - \hat{u})(t, x) + r(\infty) \hat{u}(t, x). \quad (3.11)$$

According to (H), $u_t = d(J \ast u - u) + u(r(x - ct) - u) \leq d(J \ast u - u) + r(\infty) u$. This implies that $u(t, x)$ is a subsolution of the linear equation (3.11). Since $u_0 \in \mathcal{Y}_r(\infty)$ and $u_0(x) = 0$ for all $x \geq K_2$ for some constant $K_2$, we can choose $A$ large enough such that $u_0(x) \leq \hat{u}(0, x) = A e^{-\lambda_{\zeta} x}$, and hence, the comparison principle yields that

$$0 \leq u(t, x) \leq A e^{-\lambda_{\zeta} (x - (c^*(\infty) + \zeta/2)t)} = A e^{-\lambda_{\zeta} (x - (c^*(\infty) + \zeta)t)} e^{-\frac{\lambda_{\zeta} \zeta}{2} t}, \quad \forall t \geq 0, x \in \mathbb{R}.$$  

It then follows that $0 \leq u(t, x) \leq A e^{-\lambda_{\zeta} \frac{\zeta}{2} t}, \forall t \geq 0, x \geq (c^*(\infty) + \zeta)t$, and hence, $\lim_{t \to \infty} \sup_{x \geq (c^*(\infty) + \zeta)t} u(t, x) = 0$.

(iii) Choose $\delta$ small enough with $0 < \delta < \min \left\{ \frac{r(\infty)}{\lambda^2(\infty)}, \frac{c^*(\infty) - c}{5} \right\}$ and let $l$, $\lambda_1$, $\lambda_2$ and $\gamma$ be as in Lemma 3.2. Then Lemma 3.2 implies that for any $\lambda \in [\lambda_1, \lambda_2]$ and small $\alpha > 0$, $\frac{\alpha}{\nu(\sigma(\lambda, \gamma); \lambda, \gamma)} \nu(x - l - \varphi(\lambda, \gamma) t; \lambda, \gamma)$ is a continuous subsolution of (1.1).

Since $u_0 \in \mathcal{Y}_r(\infty)$ and $u_0(x) > 0$ on a closed interval, it follows from Theorem 2.3 that $u(t, x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$. Let $c \in (0, c^*(\infty))$ be given and choose $0 < t_0 \leq \frac{\sigma(\lambda_1, \gamma)}{c}, \alpha > 0$ and $\gamma > 0$ sufficiently small such that $u(t_0, x) \geq \alpha$ for $x \in [l, l + 4\pi/\gamma]$. Define

$$W(0, x) = \begin{cases} \frac{\alpha}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)} \nu(x - l; \lambda_1, \gamma), & \text{if } l \leq x \leq l + \sigma(\lambda_1, \gamma), \\ \alpha, & \text{if } l + \sigma(\lambda_1, \gamma) \leq x \leq l + \frac{3\pi}{\gamma} + \sigma(\lambda_2, \gamma), \\ \frac{\alpha}{\nu(\sigma(\lambda_2, \gamma); \lambda_2, \gamma)} \nu(x - l - \frac{3\pi}{\gamma}; \lambda_2, \gamma), & \text{if } l + \frac{3\pi}{\gamma} + \sigma(\lambda_2, \gamma) \leq x \leq l + \frac{4\pi}{\gamma}, \\ 0, & \text{elsewhere}. \end{cases}$$

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Now we claim that for any $\varrho \in [0, 2\pi / \gamma]$, there holds

\[
W(0, x) \geq \frac{\alpha}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)} \nu(x - l - \varrho; \lambda_1, \gamma), \quad \forall x \in \mathbb{R} \quad \text{and} \quad (3.12)
\]

\[
W(0, x) \geq \frac{\alpha}{\nu(\sigma(\lambda_2, \gamma); \lambda_2, \gamma)} \nu(x - l - 3\pi / \gamma + \varrho; \lambda_2, \gamma), \quad \forall x \in \mathbb{R}. \quad (3.13)
\]

Here we only prove (3.12) since (3.13) can be obtained in a similar way. In the case where $l \leq x \leq l + \sigma(\lambda_1, \gamma)$, $W(0, x) = \frac{\alpha \nu(x - l - \varrho; \lambda_1, \gamma)}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)}$ and $-\frac{2\pi}{\gamma} \leq x - l - \varrho \leq x - l \leq \sigma(\lambda_1, \gamma)$. Note that $\nu(y; \lambda_1, \gamma)$ is nondecreasing in $y \in (-\infty, \sigma(\lambda_1, \gamma))$. Then $\nu(x - l - \lambda_1, \gamma) \geq \nu(x - l - \varrho; \lambda_1, \gamma)$, and hence, (3.12) holds. In the case where $l + \sigma(\lambda_1, \gamma) \leq x \leq l + \frac{3\pi}{\gamma} + \sigma(\lambda_2, \gamma)$, $W(0, x) = \alpha$. Since $\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)$ is the maximum of $\nu(\cdot; \lambda_1, \gamma)$ on $\mathbb{R}$, we then have $\frac{\nu(x - l - \varrho; \lambda_1, \gamma)}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)} \leq 1$, which indicates that (3.12) is true. In the case where $l + \frac{3\pi}{\gamma} + \sigma(\lambda_2, \gamma) \leq x \leq l + \frac{4\pi}{\gamma}$, $\sigma(\lambda_2, \gamma) \leq x - l - \varrho \leq \frac{4\pi}{\gamma}$. Thus, we have (3.12). In the case where $x < l$ or $x > l + \frac{4\pi}{\gamma}$, $W(0, x) = 0$. At the moment, either $x - l - \varrho < 0$ or $x - l - \varrho > \frac{2\pi}{\gamma}$, we then have $\nu(x - l - \varrho; \lambda_1, \gamma) = 0$, and hence, (3.12) holds naturally. Clearly, $u(t_0, x) \geq \alpha \geq W(0, x)$ for $x \in [l, l + 4\pi / \gamma]$. Further, by (3.12), (3.13) and Lemma 3.2, we have

\[
u(t, x) \geq \frac{\alpha}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)} \nu(x - l - \varphi(\lambda_1, \gamma)(t - t_0) - \varrho; \lambda_1, \gamma), \quad (3.14)
\]

\[
u(t, x) \geq \frac{\alpha}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)} \nu(x - l - \frac{3\pi}{\gamma} - \varphi(\lambda_1, \gamma)(t - t_0) + \varrho; \lambda_1, \gamma), \quad (3.15)
\]

for all $x \in \mathbb{R}, t \geq t_0$ and $0 \leq \varrho \leq 2\pi / \gamma$. Moreover, by the inductive argument similar to that in Li et al. (2014, pp. 1407 and 1408), we can show that for all $t \geq t_0$,

\[
u(t, x) \geq W(t - t_0, x), \quad (3.16)
\]

where

\[
W(t - t_0, x) = \begin{cases} \frac{\alpha}{\nu(\sigma(\lambda_1, \gamma); \lambda_1, \gamma)} \nu(x - l - \varphi(\lambda_1, \gamma)(t - t_0); \lambda_1, \gamma), & \text{if } l + \varphi(\lambda_1, \gamma)(t - t_0) \leq x \leq l + \varphi(\lambda_1, \gamma)(t - t_0) + \sigma(\lambda_1, \gamma), \\ \alpha, & \text{if } l + \varphi(\lambda_1, \gamma)(t - t_0) + \sigma(\lambda_1, \gamma) \leq x \leq l + \varphi(\lambda_2, \gamma)(t - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma}, \\ \frac{\alpha}{\nu(\sigma(\lambda_2, \gamma); \lambda_2, \gamma)} \nu(x - l - \frac{3\pi}{\gamma} - \varphi(\lambda_2, \gamma)(t - t_0); \lambda_2, \gamma), & \text{if } l + \varphi(\lambda_2, \gamma)(t - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma} \leq x \leq l + \frac{4\pi}{\gamma} + \varphi(\lambda_2, \gamma)(t - t_0), \\ 0, & \text{elsewhere.} \end{cases} \quad (3.17)
\]
Choose sufficiently large $t_1 > t_0$ as the initial time. It then follows from (2.6) that for any $t > t_1, u(t, x)$ satisfies

$$\begin{align*}
u(t, x) &= [e^{-\rho(t-t_1)} P(t-t_1)u(t_1, \cdot)](x) \\
&+ \int_{t_1}^{t} [e^{-\rho(t-s)} P(t-s)u(s, \cdot)(\rho + r(\cdot - cs) - u(s, \cdot))] (x) ds.
\end{align*}$$

(3.18)

According to (3.16), the nondecreasing monotonicity of $u(\rho + r(x - ct) - u)$ with respect to $u$ and the positivity of $P(t)$, we further get that

$$\begin{align*}
u(t, x) &\geq [e^{-\rho(t-t_1)} P(t-t_1)W(t_1-t_0, \cdot)](x) \\
&+ \int_{t_1}^{t} [e^{-\rho(t-s)} P(t-s)W(s-t_0, \cdot)(\rho + r(\cdot - cs) - W(s-t_0, \cdot))] (x) ds,
\end{align*}$$

(3.19)

where $t > t_1$. By the definition of $P(t)$ (see (2.2)), for the linear part, we obtain

$$\begin{align*}
[e^{-\rho(t-t_1)} P(t-t_1)W(t_1-t_0, \cdot)](x) &
\leq e^{-\rho(t-t_1)} e^{-d(t-t_1)} \sum_{k=0}^{\infty} \frac{[d(t-t_1)]^k}{k!} J^{(k)} \ast W(t_1-t_0, x) \\
&\leq e^{-\rho(t-t_1)} e^{-d(t-t_1)} \sum_{k=0}^{N} \frac{[d(t-t_1)]^k}{k!} J^{(k)} \ast W(t_1-t_0, x) \\
&= e^{-\rho(t-t_1)} e^{-d(t-t_1)} \left( W(t_1-t_0, x) + \frac{d(t-t_1)}{1!} \int_{-L}^{L} J(x_1) W(t_1-t_0, x-x_1) dx_1 \\
&\quad + \frac{[d(t-t_1)]^2}{2!} \int_{-L}^{L} \int_{-L}^{L} J(x_1)J(x_2) W(t_1-t_0, x-x_1-x_2) dx_1 dx_2 + \cdots \\
&\quad + \frac{[d(t-t_1)]^N}{N!} \int_{-L}^{L} \cdots \int_{-L}^{L} \prod_{i=1}^{N} J(x_i) W \left( t_1-t_0, x-\sum_{i=1}^{N} x_i \right) dx_1 dx_2 \cdots dx_N \right),
\end{align*}$$

(3.20)

where $J^{(0)} = \delta_0$ and $J^{(k)} \ast = J \ast J^{(k-1)} \ast$ for any $k \geq 1$, $N$ is some positive integer, and $[-L, L] := supp(J)$. In the case where $t \geq t_1$ and $x$ satisfies

$$l + \varphi(\lambda_1, \gamma)(t-t_0) + \sigma(\lambda_1, \gamma) + NL \leq x \leq l + \varphi(\lambda_2, \gamma)(t-t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma} - NL,$$

(3.20)

which does make sense by choosing $t_1 > t_0 + \frac{\sigma(\lambda_1, \gamma) - \sigma(\lambda_2, \gamma) + 2NL}{\varphi(\lambda_2, \gamma) - \varphi(\lambda_1, \gamma)}$, we have

\[ \text{@} Springer \]
$l + \varphi(\lambda_1, \gamma)(t_1 - t_0) + \sigma(\lambda_1, \gamma) \leq x \leq l + \varphi(\lambda_2, \gamma)(t_1 - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma}$ and

$$
l + \varphi(\lambda_1, \gamma)(t_1 - t_0) + \sigma(\lambda_1, \gamma) \leq x - \sum_{i=1}^{\tilde{N}} x_i \leq l + \varphi(\lambda_2, \gamma)(t_1 - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma},$$

where each $x_i \in [-L, L]$ and $\tilde{N} = 1, 2, \ldots, N$. Thus, we see from (3.17) that

$$W(t_1 - t_0, x) = \alpha, \quad W(t_1 - t_0, x - \sum_{i=1}^{\tilde{N}} x_i) = \alpha \text{ for } x_i \in [-L, L] \text{ and } \tilde{N} = 1, 2, \ldots, N.$$  

This, together with the fact that $\int_{-L}^{L} J(\gamma) \, dy = 1$, implies that for any $\epsilon > 0$, there exists sufficiently large $N_1 > 0$ such that for $N \geq N_1$,

$$[e^{-\rho(t-t_1)} P(t-t_1) W(t_1 - t_0, \cdot)](x)$$

$$\geq \alpha e^{-\rho(t-t_1)} e^{-d(t-t_1)} \sum_{k=0}^{N} \frac{[d(t-t_1)]^k}{k!}$$

$$= \alpha e^{-\rho(t-t_1)} e^{-d(t-t_1)} \left( e^{d(t-t_1)} - \sum_{k=N+1}^{\infty} \frac{[d(t-t_1)]^k}{k!} \right)$$

$$= \alpha e^{-\rho(t-t_1)} \left( 1 - e^{-d(t-t_1)} \sum_{k=N+1}^{\infty} \frac{[d(t-t_1)]^k}{k!} \right)$$

$$\geq \alpha (1 - \epsilon) e^{-\rho(t-t_1)}. \quad (3.21)$$

Regarding the nonlinear part, for any $s \in (t_1, t)$, we have

$$\left[ e^{-\rho(t-s)} P(t-s) W(s - t_0, \cdot)(\rho + r(-cs) - W(s - t_0, \cdot)) \right](x)$$

$$\geq e^{-\rho(t-s)} e^{-d(t-s)} \sum_{k=0}^{N} \frac{[d(t-s)]^k}{k!} J^{(k)} * W(s - t_0, x)(\rho + r(x - cs) - W(s - t_0, x))$$

$$= e^{-\rho(t-s)} e^{-d(t-s)} \left( W(s - t_0, x)(\rho + r(x - cs) - W(s - t_0, x)) \right.$$

$$+ \frac{[d(t-s)]}{1!} \int_{-L}^{L} J(x_1) W(s - t_0, x - x_1)(\rho + r(x - x_1 - cs) - W(s - t_0, x - x_1)) \, dx_1$$

$$+ \frac{[d(t-s)]^2}{2!} \int_{-L}^{L} J(x_2) \int_{-L}^{L} J(x_1) W(s - t_0, x - x_1 - x_2)(\rho + r(x - x_1 - x_2 - cs) - W(s - t_0, x - x_1 - x_2)) \, dx_1 \, dx_2 \right.$$
For any \( t > t_1 \) and \( x \) satisfying (3.20), since \( r(\cdot) \) is nondecreasing and \( \varphi(\lambda_1, \gamma) = c + \delta \), we then obtain that for all \( \tilde{N} = 1, 2, \ldots, N \),

\[
x - \sum_{i=1}^{\tilde{N}} x_i - ct \geq l + \varphi(\lambda_1, \gamma)(t - t_0) + \sigma(\lambda_1, \gamma) + NL - \tilde{N}L - ct
\]

which implies that \( r(x - \sum_{i=1}^{\tilde{N}} x_i - cs) \geq r(x - \sum_{i=1}^{\tilde{N}} x_i - ct) \geq r(l) \) for \( t_1 \leq s \leq t \) and \( \tilde{N} = 1, 2, \ldots, N \). By the assumption that \( c^*(l) = c^*(\infty) - \delta \), we have

\[
\frac{d\left(\int_{\mathbb{R}} J(y)e^{\lambda^*(\infty)y}dy - 1\right) + r(\infty)}{\lambda^*(\infty)} - \delta = \inf_{\lambda > 0} \frac{d\left(\int_{\mathbb{R}} J(y)e^{\lambda y}dy - 1\right) + r(l)}{\lambda} \leq \frac{d\left(\int_{\mathbb{R}} J(y)e^{\lambda^*(\infty)y}dy - 1\right) + r(l)}{\lambda^*(\infty)}.
\]

It then follows that \( r(l) \geq r(\infty) - \delta \lambda^*(\infty) \), and hence,

\[
r\left(x - \sum_{i=1}^{\tilde{N}} x_i - cs\right) \geq r(\infty) - \delta \lambda^*(\infty).
\]

Similar to (3.21), we can obtain that for the above \( \varepsilon > 0 \), when \( t \geq s \geq t_1 \) and \( x \) satisfies (3.20), there exists large \( N_2 > 0 \) such that \( N \geq N_2 \),

\[
\left[e^{-\rho(t-s)}P(t-s)W(s - t_0, \cdot)(\rho + r(\cdot - cs) - W(s - t_0, \cdot))\right](x)
\]

\[
\geq e^{-\rho(t-s)}e^{-d(t-s)}\alpha(\rho + r(\infty) - \delta \lambda^*(\infty) - \alpha) \sum_{k=0}^{N} \frac{[d(t-s)]^k}{k!}
\]

\[
= e^{-\rho(t-s)}\alpha(\rho + r(\infty) - \delta \lambda^*(\infty) - \alpha) \left(1 - e^{-d(t-s)} \sum_{k=N+1}^{\infty} \frac{[d(t-s)]^k}{k!}\right)
\]

\[
\geq e^{-\rho(t-s)}\alpha(\rho + r(\infty) - \delta \lambda^*(\infty) - \alpha)(1 - \varepsilon).
\]
Let $N \geq \max\{N_1, N_2\}$. In view of (3.19), (3.21) and (3.22), we then conclude that for $t \geq t_1$ and $x$ satisfying (3.20), there holds

$$u(t, x) \geq v^1(t),$$

where

$$v^1(t) = (1 - \varepsilon)ae^{-\rho(t-t_1)} + (1 - \varepsilon) \int_{t_1}^t e^{-\rho(t-s)} \alpha(\rho + r(\infty) - \delta\lambda^*(\infty) - \alpha)ds.$$  

(3.23)

Using (3.19), by induction, we can further derive that for sufficiently large $t \geq t_1$ and $x$ satisfying

$$l + \varphi(\lambda_1, \gamma)(t - t_0) + \sigma(\lambda_1, \gamma) + mNL \leq x \leq l + \varphi(\lambda_2, \gamma)(t - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma} - mNL,$$  

(3.24)

there holds

$$u(t, x) \geq v^m(t)$$

with $v^m$ being defined recursively by

$$v^m(t) = (1 - \varepsilon)ae^{-\rho(t-t_1)} + (1 - \varepsilon) \int_{t_1}^t e^{-\rho(t-s)} v^{m-1}(s)(\rho + r(\infty) - \delta\lambda^*(\infty) - v^{m-1}(t))ds, \quad m \geq 2.$$  

(3.25)

Clearly, $0 \leq v^m(t) \leq r(\infty)$ for all $m \geq 1$. Next, we study the asymptotic behavior of the sequence \{v^m(t)\} as $t \to \infty$. Let $h_{m-1}(t) = (1 - \varepsilon)v^{m-1}(t)(\rho + r(\infty) - \delta\lambda^*(\infty) - v^{m-1}(t))$, where $m = 1, 2, \ldots$ and $v^0(t) \equiv \alpha$. First, we rewrite (3.25) in its differential form:

$$\begin{cases}
\frac{dv^m(t)}{dt} = -\rho v^m(t) + h_{m-1}(t), & t > t_1, \\
v^m(t_1) = (1 - \varepsilon)\alpha, & m \geq 1.
\end{cases}$$  

(3.26)

In view of (3.23), we see that $v^1(\infty) = \lim_{t \to \infty} v^1(t) = \frac{(1 - \varepsilon)\alpha(\rho + r(\infty) - \delta\lambda^*(\infty) - \alpha)}{\rho}$. Moreover, according to the iteration relation (3.25), by induction, we can conclude that $v^m(\infty) = \lim_{t \to \infty} v^m(t)$ exists for all $m \geq 1$, and so does $h_{m-1}(\infty) = \lim_{t \to \infty} h_{m-1}(t)$, $\forall m \geq 1$. Note that the idea for proving the existence of $v^m(\infty)$ is the same as what we will do next to show $v^m(\infty) = \frac{h_{m-1}(\infty)}{\rho}$, where $v^m(t)$ satisfies (3.26). Specifically, we can choose $t_1$ large enough such that $h_{m-1}(t) \sim h_{m-1}(\infty)$ for all $t > t_1$, i.e., for any $\varepsilon > 0$, there exists $t_1 \gg 1$ such that if $t > t_1$, then $h_{m-1}(\infty) - \varepsilon < h_{m-1}(t) < h_{m-1}(\infty) + \varepsilon$. Using (3.25), we then have

$$(1 - \varepsilon)ae^{-\rho(t-t_1)} + (h_{m-1}(\infty) - \varepsilon) \frac{1 - e^{-\rho(t-t_1)}}{\rho}$$
\[ < v^m(t) = (1 - \epsilon)\alpha e^{-\rho(t-t_1)} + \int_{t_1}^t e^{-\rho(t-s)} h_{m-1}(s) ds \]

\[ < (1 - \epsilon)\alpha e^{-\rho(t-t_1)} + (h_{m-1}(\infty) + \epsilon) \frac{1 - e^{-\rho(t-t_1)}}{\rho}. \]

Letting \( t \to \infty \), we see that \( \frac{h_{m-1}(\infty) - \epsilon}{\rho} \leq v^m(\infty) \leq \frac{h_{m-1}(\infty) + \epsilon}{\rho} \), which implies that \( v^m(\infty) = \frac{h_{m-1}(\infty)}{\rho} \) since \( \epsilon > 0 \) is arbitrary, and hence,

\[ v^m(\infty) = \frac{(1 - \epsilon)v^{m-1}(\infty)(\rho + r(\infty) - \delta\lambda^*(\infty) - v^{m-1}(\infty))}{\rho}. \tag{3.27} \]

In the following, we further show that \( \{v^m(\infty)\}_{m \in \mathbb{N}} \) is a increasing sequence. In fact, by (3.27), we have \( v^m(\infty) \geq \frac{(1 - \epsilon)v^{m-1}(\infty)(\rho - \delta\lambda^*(\infty))}{\rho} \). Since the parameters \( \epsilon, \delta, \alpha > 0 \) can be sufficiently small, we see that \( v^m(\infty) \geq v^{m-1}(\infty), \forall m > 1 \) and \( v^1(\infty) > 0 \). Therefore, \( \{v^m(\infty)\}_{m \in \mathbb{N}} \) is a monotone and bounded sequence, and hence, \( \lim_{m \to \infty} v^m(\infty) \) exists and it is positive. Now, letting \( m \to \infty \) in both sides of (3.27), we have

\[ \lim_{m \to \infty} v^m(\infty) = r(\infty) - \delta\lambda^*(\infty) - \frac{\epsilon\rho}{1 - \epsilon}. \]

For an arbitrarily small \( t > 0 \), we choose \( M \) such that

\[ v^M(\infty) \geq r(\infty) - \delta\lambda^*(\infty) - \frac{\epsilon\rho}{1 - \epsilon} - t. \tag{3.28} \]

We now choose \( t_1 \) large enough such that for \( t \geq t_1 \),

\[ l + \varphi(\lambda_1, \gamma)(t - t_0) + \sigma(\lambda_1, \gamma) + MNL \]

\[ \leq x \leq l + \varphi(\lambda_2, \gamma)(t - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma} - MNL. \tag{3.29} \]

Furthermore,

\[ \lim_{t \to \infty} \inf_{t \geq t_1, x \text{ satisfies (3.29)}} u(t, x) \geq v^M(\infty). \tag{3.30} \]

For any given \( \zeta \in \left(0, \frac{c^*(\infty) - c}{2}\right)\), pick \( \delta \) small enough with \( \delta < \zeta/4 \), by Lemma 3.2, \( \varphi(\lambda_1, \gamma) = c + \delta \) and \( c^*(\infty) = c^*(l) + \delta \leq c^*(l) + 2\delta = \varphi(\lambda_2, \gamma) + 4\delta \). Thus, we can choose sufficiently large \( t \geq t_1 \) such that

\[ l + \varphi(\lambda_1, \gamma)(t - t_0) + \sigma(\lambda_1, \gamma) + MNL \]

\[ \leq (c + \zeta) t < (c^*(\infty) - \zeta) t \]

\[ \leq l + \varphi(\lambda_2, \gamma)(t - t_0) + \sigma(\lambda_2, \gamma) + \frac{3\pi}{\gamma} - MNL. \]
It follows that
\[
\lim_{t \to \infty} \inf_{(c+\varsigma)t \leq x \leq (c^*(\infty)-\varsigma)t} u(t, x) \geq v^M(\infty) \geq r(\infty) - \delta \lambda^*(\infty) - \frac{\epsilon \rho}{1 - \epsilon} - \iota.
\]

Since all the parameters $\delta, \epsilon, \iota$ can be chosen arbitrarily small, we obtain that
\[
\lim_{t \to \infty} \inf_{(c+\varsigma)t \leq x \leq (c^*(\infty)-\varsigma)t} u(t, x) \geq r(\infty).
\]

On the other hand, Theorem 2.3 implies that $0 \leq u(t, x) \leq r(\infty)$ for all $t > 0$ and $x \in \mathbb{R}$. In particular, we have
\[
\lim_{t \to \infty} \sup_{(c+\varsigma)t \leq x \leq (c^*(\infty)-\varsigma)t} u(t, x) \leq r(\infty),
\]
which implies that the statement (iii) is valid. □

We remark that the persistence obtained in Theorem 3.3 should be understood from the viewpoint of “by moving”, that is, the species will move toward the better resource with speed $c^*(\infty)$ which is larger than the shifting speed $c$. Indeed, for any given location $x$, since the resource function $r(x-ct)$ will become negative eventually as time goes, we see that the population at this location will vanish.

### 4 Forced Traveling Waves

In this section, we consider the positive traveling wave solutions of (1.1) with the wave speed at which the habitat is shifting.

Letting $u(t, x) = U(\xi)$ with $\xi = x - ct$, we see from (1.1) that $U(\xi)$ satisfies

\[
-cU'(\xi) = d \left( \int_{\mathbb{R}} J(y)U(\xi - y)dy - U(\xi) \right) + U(\xi)(r(\xi) - U(\xi)), \quad \xi \in \mathbb{R},
\]

where the symbol prime stands for the derivative. Recall that $c > 0$ is the habitat shifting speed. As explained in Hu and Zou (2017), we impose the following boundary conditions

\[
\lim_{\xi \to -\infty} U(\xi) = 0, \quad \lim_{\xi \to \infty} U(\xi) = r(\infty).
\]

Then the nondecreasing positive solution of (4.1) satisfying (4.2) is called a forced traveling wave of (1.1). This type of traveling wave solutions illustrates how the species under consideration would die out at every point. By Remark 2.1, we have $0 \leq U(\xi) \leq r(\infty), \forall \xi \in \mathbb{R}$. Moreover, by the strong maximum principles for nonlocal equations (Coville 2007, Theorem 2.12), we get $0 < U < r(\infty)$ in $\mathbb{R}$.

Let $V(\xi) = U(-\xi), \forall \xi \in \mathbb{R}$. It then follows from (4.1) and the symmetry of $J$ that

\[
cV'(\xi) = d \left( \int_{\mathbb{R}} J(y)V(\xi - y)dy - V(\xi) \right) + V(\xi)(r(-\xi) - V(\xi)), \quad \xi \in \mathbb{R}.
\]
Correspondingly, we have
\[ \lim_{\xi \to -\infty} V(\xi) = r(\infty), \quad \lim_{\xi \to \infty} V(\xi) = 0. \] (4.4)

In the following, by the combination of super-/sub-solutions and monotone iterations, we will prove that (4.3) admits a nonincreasing solution satisfying (4.4) for any given \( c > 0 \), which gives rise to the nondecreasing traveling wave solution of (1.1) connecting 0 to \( r(\infty) \).

In order to construct a subsolution for (4.3), we introduce an auxiliary nonlocal dispersal equation of ignition type (see, e.g., Coville 2003, page 2 or Coville 2012, page 1 for the definition). For any small \( \varepsilon \in (0, r(\infty)/5) \), define an ignition nonlinearity
\[
f_\varepsilon(u) = \begin{cases} u(r(\infty) - \varepsilon - u), & \text{if } u \geq 0, \\ 0, & \text{if } -\varepsilon \leq u < 0. \end{cases}
\]

We consider the following problem:
\[
\frac{\partial u(t, x)}{\partial t} = d(J^* u - u)(t, x) + f_\varepsilon(u(t, x)). \tag{4.5}
\]

According to Coville (2003, 2012), Eq. (4.5) admits a decreasing traveling wave solution \( V_\varepsilon(\xi) (\xi = x - c_\varepsilon t) \) connecting \( r(\infty) - \varepsilon \) to \( -\varepsilon \) with speed \( c_\varepsilon \), that is, \((V_\varepsilon, c_\varepsilon)\) satisfies
\[
\begin{align*}
-c_\varepsilon V_\varepsilon'(\xi) &= d \left( \int \! J(y) V_\varepsilon(\xi - y) \, dy - V_\varepsilon(\xi) \right) + f_\varepsilon(V_\varepsilon(\xi)), \\
V_\varepsilon(-\infty) &= r(\infty) - \varepsilon, \quad V_\varepsilon(+\infty) = -\varepsilon, \quad V_\varepsilon'(\xi) < 0.
\end{align*}
\] (4.6)

Then we have the following observation.

**Lemma 4.1** Let \( c^*(\infty) \) be the minimal wave speed of the monotone traveling wave connecting \( r(\infty) \) to 0 for the homogeneous nonlocal Fisher–KPP equation (see, e.g., Carr and Chmaj 2004; Schumacher 1980b):
\[

u_t = d(J^* u - u) + u(r(\infty) - u). \tag{4.7}
\]

Then \( \lim_{\varepsilon \to 0^+} c_\varepsilon = c^*(\infty) \).

**Proof** For any \( \eta_1, \eta_2 \in \mathbb{R} \), we have
\[
\int_{\eta_1}^{\eta_2} \int \! J(y) [V_\varepsilon(\xi - y) - V_\varepsilon(\xi)] \, dy \, d\xi
\]
\[
= - \int_{\eta_1}^{\eta_2} \int \! J(y) y \int_0^1 V_\varepsilon'(\xi - \tau y) \, d\tau \, dy \\
= - \int \! J(y) y \int_0^1 \int_{\eta_1}^{\eta_2} V_\varepsilon'(\xi - \tau y) \, d\xi \, d\tau \\
= \int_{-L}^L \! J(y) y \int_0^1 [V_\varepsilon(\eta_1 - \tau y) - V_\varepsilon(\eta_2 - \tau y)] \, d\tau \, dy.
\]
where $[-L, L]$ is the compact support of $J$. Letting $\eta_1 \to -\infty$ and $\eta_2 \to \infty$ in sequence, since $J$ is symmetric, we then obtain

$$
\int_{-\infty}^{\infty} \int_{\mathbb{R}} J(y)[V_\varepsilon(\xi - y) - V_\varepsilon(\xi)]dyd\xi = r(\infty)\int_{-L}^{L} J(y)dy = 0.
$$

Further, integrating the first equation of (4.6) from $-\infty$ to $\infty$, for any $\varepsilon \in (0, r(\infty)/5)$, we can conclude that

$$
c_\varepsilon = \frac{1}{r(\infty)} \int_{-\infty}^{\infty} f_\varepsilon(V_\varepsilon(\xi))d\xi \geq \frac{1}{r(\infty)} \int_{-\infty}^{\infty} f_{r(\infty)/5}(V_{r(\infty)/5}(\xi))d\xi > 0. \quad (4.8)
$$

We first prove that $c_\varepsilon$ is nonincreasing in $\varepsilon > 0$. Let, $\varepsilon_1 > \varepsilon_2 > 0$, and $u_1(t, x) = V_{\varepsilon_1}(x - c_{\varepsilon_1}t)$ and $u_2(t, x) = V_{\varepsilon_2}(x - c_{\varepsilon_2}t)$ be the decreasing traveling wave solution of (4.5) with $\varepsilon$ being replaced by $\varepsilon_1$ and $\varepsilon_2$, respectively. Noting that $V_{\varepsilon_1}(-\infty) < V_{\varepsilon_2}(-\infty), V_{\varepsilon_1}(+\infty) < V_{\varepsilon_2}(+\infty)$ and $f_{\varepsilon_1}(u) \leq f_{\varepsilon_2}(u)$. Since any translation of a wave profile is also a wave profile, we can always assume that $V_{\varepsilon_1}(x) \leq V_{\varepsilon_2}(x)$, $\forall x \in \mathbb{R}$. Then the comparison principle implies that $V_{\varepsilon_1}(x - c_{\varepsilon_1}t) \leq V_{\varepsilon_2}(x - c_{\varepsilon_2}t)$, $\forall x \in \mathbb{R}, t > 0$, and hence, we have $c_{\varepsilon_1} \leq c_{\varepsilon_2}$. Similarly, we can further show that for any small $\varepsilon > 0$, $c_\varepsilon \leq c^*(\infty)$. Thus, $\lim_{\varepsilon \to 0^+} c_\varepsilon$ exists. Set $\tilde{c} := \lim_{\varepsilon \to 0^+} c_\varepsilon$, we then have $c \leq c^*(\infty)$.

Recalling that for any $\varepsilon \in (0, r(\infty)/5), -r(\infty)/5 \leq -\varepsilon \leq V_\varepsilon \leq r(\infty) - \varepsilon \leq r(\infty)$. By (4.8) and the first equation of (4.6), a direct computation yields that there exists a constant $M_1 > 0$ such that $|V_\varepsilon'| \leq M_1$. Moreover, differentiating (4.6) with respect to $\xi$, we can get that $|V_\varepsilon''| \leq M_2$ for some $M_2 > 0$. Since $c_\varepsilon \to \tilde{c}$ as $\varepsilon \to 0^+$, by the uniform boundedness of $|V_\varepsilon'|$ and $|V_\varepsilon''|$, there exists a sequence $\varepsilon_n \to 0$ such that $V_{\varepsilon_n} \to \tilde{V}$ in $C^1_{loc}(\mathbb{R})$, where $\tilde{V}$ satisfies

$$
-\tilde{c}\tilde{V}'(\xi) = d \left( \int_{\mathbb{R}} J(y)\tilde{V}(\xi - y)dy - \tilde{V}(\xi) \right) + \tilde{V}(\xi)(r(\infty) - \tilde{V}(\xi)).
$$

Further, $\tilde{V}$ is nonincreasing on $\mathbb{R}$ and $0 \leq \tilde{V} \leq r(\infty)$. Without loss of generality, we can normalize $V_{\varepsilon_n}$ by $V_{\varepsilon_n}(0) = \frac{r(\infty)}{2}$, it then follows that $\tilde{V}(0) = \frac{r(\infty)}{2}$. Thus, we have $\tilde{V}(-\infty) = r(\infty)$ and $\tilde{V}(+\infty) = 0$. This implies that $\tilde{V}(x - \tilde{c}t)$ is a traveling wave of (4.7) connecting $r(\infty)$ to 0 with speed $\tilde{c}$, and hence, $\tilde{c} \geq c^*(\infty)$. Consequently, $\tilde{c} = c^*(\infty)$.

**Lemma 4.2** Fix a sufficiently small $\varepsilon \in (0, r(\infty)/5)$. Then for any $c > -c^*(\infty)$, $V(\xi) := \max\{V_\varepsilon(\xi), 0\}$ is a subsolution of (4.3), i.e., $V$ satisfies the following inequality:

$$
d \left( \int_{\mathbb{R}} J(y)V(\xi - y)dy - V(\xi) \right) - cV'(\xi) + V(\xi)(r(-\xi) - V(\xi)) \geq 0 \quad (4.9)
$$

for any $\xi \neq \xi_0$, where $\xi_0$ is the point satisfying $V_\varepsilon(\xi_0) = 0$ and $V_\varepsilon$ fulfills (4.6).
Proof According to Lemma 4.1, \( \lim_{\varepsilon \to 0^+} c_\varepsilon = c^*(\infty) \). It then follows that for sufficiently small \( \varepsilon \in (0, r(\infty)/5) \), we have \( c_\varepsilon > -c \) due to \( c > -c^*(\infty) \). Let us fix such an \( \varepsilon \). Without loss of generality, we can assume that \( V_\varepsilon(\xi_0) = 0 \) and \( r(-\xi_0) \geq r(\infty) - \varepsilon \). This can be realized by some appropriate translation of \( V_\varepsilon \) if necessary.

If \( \xi < \xi_0 \), \( V(\xi) = V_\varepsilon(\xi) > 0 \), since \( r \) is nondecreasing and \( V_\varepsilon'(\xi) < 0 \), we have

\[
d \left( \int_{\mathbb{R}} J(y)V(\xi - y)dy - V(\xi) \right) - cV'(\xi) + V(\xi)(r(-\xi) - V(\xi)) \geq 0 \tag{4.10}
\]

which shows that (4.9) holds for \( \xi < \xi_0 \).

If \( \xi > \xi_0 \), \( V(\xi) = 0 \), then

\[
d \left( \int_{\mathbb{R}} J(y)V(\xi - y)dy - V(\xi) \right) - cV'(\xi) + V(\xi)(r(-\xi) - V(\xi)) = 0.
\]

Hence, (4.9) also holds for \( \xi > \xi_0 \). \( \square \)

Next, we construct a supersolution for Eq. (4.3).

**Lemma 4.3** Choose \( \xi_1 > \xi_0 \) large enough such that \( r(-\xi_1) < 0 \). There exists \( \mu_1 > 0 \) such that

\[
d \left( \int_{\mathbb{R}} J(y)e^{\mu_1 y}dy - 1 \right) + c\mu_1 + r(-\xi_1) = 0.
\]

Then for any \( c > 0 \), \( \tilde{V}(\xi) := \min\{r(\infty), r(\infty)e^{-\mu_1(\xi - \xi_1)}\} \) satisfies

\[
d \left( \int_{\mathbb{R}} J(y)\tilde{V}(\xi - y)dy - \tilde{V}(\xi) \right) - c\tilde{V}'(\xi) + \tilde{V}(\xi)(r(-\xi) - \tilde{V}(\xi)) \leq 0 \tag{4.10}
\]

for any \( \xi \neq \xi_1 \).

**Proof** Let

\[
h(\mu) = d \left( \int_{\mathbb{R}} J(y)e^{\mu y}dy - 1 \right) + c\mu + r(-\xi_1).
\]
By a direct calculation, we have

\[
    h(0) = r(-\xi_1) < 0, \quad h(\mu) \to +\infty \text{ as } \mu \to \infty,
    
    h'(0) = c > 0, \quad h''(\mu) > 0, \forall \mu \in \mathbb{R},
\]

which implies that there exists \( \mu_1 > 0 \) such that \( h(\mu_1) = 0 \).

According to the definition of \( \overline{V}(\xi) \) and the assumption (J), we have

\[
    \int_{\mathbb{R}} J(y) \overline{V}(\xi - y) dy \leq \min \left\{ r(\infty), r(\infty) e^{-\mu_1 (\xi - \xi_1)} \int_{\mathbb{R}} J(y) e^{\mu_1 y} dy \right\}.
\]

When \( \xi < \xi_1, \overline{V}(\xi) = r(\infty) \), and hence

\[
    d \left( \int_{\mathbb{R}} J(y) \overline{V}(\xi - y) dy - \overline{V}(\xi) \right) - c \overline{V}'(\xi) + \overline{V}(\xi)(r(-\xi) - \overline{V}(\xi))
    
    \leq r(\infty)(r(-\xi) - r(\infty)) \leq 0.
\]

When \( \xi > \xi_1, \overline{V}(\xi) = r(\infty)e^{-\mu_1 (\xi - \xi_1)} \). It then follows from (H) that

\[
    d \left( \int_{\mathbb{R}} J(y) \overline{V}(\xi - y) dy - \overline{V}(\xi) \right) - c \overline{V}'(\xi) + \overline{V}(\xi)(r(-\xi) - \overline{V}(\xi))
    
    \leq r(\infty)e^{-\mu_1 (\xi - \xi_1)} \left[ d \left( \int_{\mathbb{R}} J(y)e^{\mu_1 y} dy - 1 \right) + c \mu_1 + r(-\xi) - r(\infty)e^{-\mu_1 (\xi - \xi_1)} \right]
    
    \leq r(\infty)e^{-\mu_1 (\xi - \xi_1)} \left[ d \left( \int_{\mathbb{R}} J(y)e^{\mu_1 y} dy - 1 \right) + c \mu_1 - r(-\xi_1) \right]
    
    \equiv 0.
\]

This completes the proof of (4.10).

Let \( BC(\mathbb{R}, \mathbb{R}) \) be the space of all bounded and continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \), and \( BC^+ = \{ v \in BC(\mathbb{R}, \mathbb{R}) : v(x) \geq 0 \text{ for all } x \in \mathbb{R} \} \). For any \( v, \tilde{v} \in BC(\mathbb{R}, \mathbb{R}) \), we denote \( v \geq \tilde{v} \) or \( \tilde{v} \leq v \) if \( v - \tilde{v} \in BC^+ \). According to Lemmas 4.2 and 4.3, we can easily verify that \( V \leq \overline{V} \) on \( \mathbb{R} \). Define the profile set

\[
    \Theta = \{ v \in BC(\mathbb{R}, \mathbb{R}) : \underline{V} \leq v \leq \overline{V} \}
\]

and the operator \( H : \Theta \to C(\mathbb{R}, \mathbb{R}) \) by

\[
    H(V)(\xi) = \beta V(\xi) + d \int_{\mathbb{R}} J(y)V(\xi - y) dy - dV(\xi) + V(\xi)(r(-\xi) - V(\xi)),
\]

where \( \beta = d + 2r(\infty) - r(-\infty) > 0 \). Then (4.3) can be rewritten as

\[
    cV'(\xi) = -\beta V(\xi) + H(V)(\xi). \quad (4.11)
\]
Clearly, if \( v \in \Theta \), then \( 0 \leq v(\xi) \leq r(\infty) \) for all \( \xi \in \mathbb{R} \). Let us introduce the following integral equation

\[
V(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V)(z)dz,
\]

(4.12)

which is well defined for \( V \in \Theta \) since \( H(V) \) is bounded and continuous with respect to \( V \in \Theta \). Moreover, it is easy to see that the solution of (4.12) is \( C^1 \) and satisfies (4.11). Thus, the existence of monotone solutions of (4.3)–(4.4) reduces to that of the fixed point of the operator \( F : \Theta \rightarrow C(\mathbb{R}, \mathbb{R}) \) defined as follows

\[
F(V)(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V)(z)dz.
\]

(4.13)

Now we summarize some properties of the operator \( F \).

**Lemma 4.4** \( F \) is a nondecreasing operator and maps \( \Theta \) to \( \Theta \). Moreover, if \( V \in \Theta \) is nonincreasing, then \( F(V)(\xi) \) is nonincreasing with respect to \( \xi \).

**Proof** For any \( V, \tilde{V} \in \Theta \) with \( V \geq \tilde{V} \), we have

\[
H(V)(\xi) - H(\tilde{V})(\xi) = [\beta - d + r(-\xi) - (V(\xi) + \tilde{V}(\xi))](V(\xi) - \tilde{V}(\xi))
\]

\[
+ d \int_{\mathbb{R}} J(y)[V(\xi - y) - \tilde{V}(\xi - y)]dy
\]

\[
\geq [r(-\xi) - r(-\infty) + 2r(\infty) - (V(\xi) + \tilde{V}(\xi))](V(\xi) - \tilde{V}(\xi))
\]

\[
\geq 0,
\]

which implies that \( F(V)(\xi) \geq F(\tilde{V})(\xi) \), \( \forall \xi \in \mathbb{R} \). It then follows that

\[
F(V)(\xi) \leq F(V)(\xi) \leq F(\tilde{V})(\xi)
\]

(4.14)

for all \( V \in \Theta \) and all \( \xi \in \mathbb{R} \). On the other hand, by Lemma 4.3, we have

\[
F(V)(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V)(z)dz
\]

\[
\leq \frac{1}{c} \left\{ \left( \int_{-\infty}^{\xi_1} + \int_{\xi_1}^{\xi} \right) e^{-\frac{\beta}{c}(\xi-z)} [c \bar{V}'(z) + \beta \bar{V}(z)]dz \right\}
\]

(4.15)

and by Lemma 4.2, it follows that

\[
F(V)(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V)(z)dz
\]

\[
\geq \frac{1}{c} \left\{ \left( \int_{-\infty}^{\xi_0} + \int_{\xi_0}^{\xi} \right) e^{-\frac{\beta}{c}(\xi-z)} [c \bar{V}'(z) + \beta \bar{V}(z)]dz \right\}
\]

(4.16)

\[
= \bar{V}(\xi).
\]

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Combining (4.14)–(4.16), we obtain $F(\Theta) \subseteq \Theta$.

If $V \in \Theta$ is nonincreasing, then for all $\xi \in \mathbb{R}$ and any $h > 0$, we have

$$H(V)(\xi + h) - H(V)(\xi) = [\beta - d - V(\xi + h) - V(\xi)](V(\xi + h) - V(\xi)) + r(-\xi - h)V(\xi + h) - r(-\xi)V(\xi) + d \int_{\mathbb{R}} J(y)[V(\xi + h - y) - V(\xi - y)]dy \leq [2r(\infty) - V(\xi + h) - V(\xi) + r(-\xi) - r(-\infty)](V(\xi + h) - V(\xi)) \leq 0,$$

which further leads to

$$F(V)(\xi + h) = \frac{1}{c} \int_{-\infty}^{\xi + h} e^{-\frac{\beta}{r}(\xi + h - z)} H(V)(z)dz = \frac{1}{c} \int_{0}^{\infty} e^{-\frac{\beta}{r}(\xi + h - z)} H(V)(\xi + h - z)dz \leq \frac{1}{c} \int_{0}^{\infty} e^{-\frac{\beta}{r}(\xi - z)} H(V)(\xi - z)dz = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{r}(\xi - z)} H(V)(z)dz = F(V)(\xi).$$

We then complete the proof. \(\Box\)

Now we are in a position to prove our main result in this section.

**Theorem 4.5** For any given $c > 0$, (4.1) admits a nondecreasing positive solution $U(\xi)$ satisfying (4.2). In other words, (1.1) has a nondecreasing forced traveling wave connecting $0$ and $r(\infty)$ with the wave speed at which the environment is shifting. If, in addition, $r$ is strictly increasing, then the wave profile is monotone increasing.

**Proof** Define the iterations:

$$V_1 = F(V), \quad V_{n+1} = F(V_n), \quad \forall n \geq 1.$$

Since $\overline{V} \in \Theta$ is nonincreasing on $\mathbb{R}$, by Lemma 4.4, we can conclude that $V_n \in \Theta$ and $V_n(\xi)$ is nonincreasing with respect to $\xi$ for each fixed $n = 1, 2, \ldots$, and

$$\underline{V}(\xi) \leq V_{n+1}(\xi) \leq V_n(\xi) \leq \overline{V}(\xi), \quad \forall \xi \in \mathbb{R}, \quad \forall n \geq 1.$$

Then the pointwise limit of the sequence \{V_n\} exists, denoted by $V$, i.e., for every $\xi \in \mathbb{R}$, $V(\xi) = \lim_{n \to \infty} V_n(\xi)$. Obviously, $V(\xi)$ is a nonincreasing and nonnegative function defined on $\mathbb{R}$ and
\[ V(\xi) \leq V(\xi) \leq \overline{V}(\xi). \] (4.17)

Moreover, \( H(V_n) \) converges pointwise to \( H(V) \).

We now show \( V \) is a fixed point of \( F \). Since

\[ |H(V_n)| \leq [\beta + 2d + r(\infty) + \max\{r(\infty), -r(-\infty)\}]r(\infty), \forall n \geq 1, \]

by (4.13) and the Lebesgue’s dominated convergence theorem, we have

\[
V(\xi) = \lim_{n \to \infty} V_{n+1}(\xi) = \lim_{n \to \infty} F(V_n)(\xi)
= \frac{1}{c} \lim_{n \to \infty} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V_n)(z)dz
= \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V)(z)dz = F(V)(\xi).
\]

It is easy to verify that \( V \in C^1(\mathbb{R}) \) satisfies (4.3). Next we prove that \( V \) meets (4.4).

Clearly, it follows from (4.17) as well as

\[
\lim_{\xi \to \infty} V(\xi) = \lim_{\xi \to \infty} \overline{V}(\xi) = 0,
\]

that

\[
\lim_{\xi \to \infty} V(\xi) = 0.
\]

Note that \( V(\xi) \) is nonincreasing in \( \mathbb{R} \) and \( 0 \leq V(\xi) \leq \overline{V}(\xi) \leq r(\infty) \).

Therefore, \( A := \lim_{\xi \to -\infty} V(\xi) \) exists. Then

\[
\lim_{\xi \to -\infty} H(V)(\xi)
= \lim_{\xi \to -\infty} \left[ \beta V(\xi) + d \int_{-L}^{L} J(y)V(\xi - y)dy - dV(\xi) + V(\xi)(r(-\xi) - V(\xi)) \right]
= \beta A + A(r(\infty) - A),
\]

where \([-L, L] = \text{supp}(J)\). Applying L'Hôpital’s rule, we obtain

\[
A = \lim_{\xi \to -\infty} V(\xi) = \lim_{\xi \to -\infty} F(V)(\xi)
= \frac{1}{c} \lim_{\xi \to -\infty} \int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-z)} H(V)(z)dz
= \frac{1}{c} \lim_{\xi \to -\infty} \frac{H(V)(\xi)}{\beta/c}
= A + \frac{A(r(\infty) - A)}{\beta},
\]
which implies that $A = 0$ or $A = r(\infty)$. However, if $A = 0$, by the monotonicity of $V$, we must have $V \equiv 0$, which contradicts the assumption $V \geq V$ and the definition of $V$. Thus, we have

$$\lim_{\xi \to -\infty} V(\xi) = r(\infty).$$

Now we can obtain the desired conclusion by using the relation $U(\xi) = V(\xi)$.

In the case where $r$ is strictly increasing, we assume, by contradiction, that there exist $\xi_2 < \xi_3$ such that $U(\xi_2) = U(\xi_3)$. Since $U(\xi)$ is nondecreasing, we have

$$U'(\xi) = 0 \text{ and } U(\xi) \equiv U(\xi_2), \forall \xi \in [\xi_2^+, \xi_3^-].$$

where $\xi_2 < \xi_2^+ < \xi_3^- < \xi_3$. In view of (4.1), it then follows that

$$d \int_{-L}^{L} J(y)[U(\xi_2^+ - y) - U(\xi_3^- - y)]dy + U(\xi_2^+)[r(\xi_2^+) - r(\xi_3^-)] = 0, \quad (4.18)$$

where $[-L, L] = supp(J)$. However, since $0 < U < r(\infty)$ is nondecreasing and $r$ is strictly increasing, we obtain

$$\int_{-L}^{L} J(y)[U(\xi_2^+ - y) - U(\xi_3^- - y)]dy \leq 0,$$

$$U(\xi_2^+)[r(\xi_2^+) - r(\xi_3^-)] < 0,$$

which contradicts (4.18). This shows that $U$ is monotone increasing. \qed

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