Classical and Quantum Aspects of Particle Propagation in External Gravitational Fields

Giorgio Papin

Department of Physics and Prairie Particle Physics Institute, University of Regina, Regina, Sask, S4S 0A2, Canada

In the study of covariant wave equations, linear gravity manifests itself through the metric deviation $\gamma_{\mu \nu}$ and a two-point vector potential $K_\lambda$ itself constructed from $\gamma_{\mu \nu}$ and its derivatives. The simultaneous presence of the two gravitational potentials is non contradictory. Particles also assume the character of quasiparticles and $K_\lambda$ carries information about the matter with which it interacts. We consider the influence of $K_\lambda$ on the dispersion relations of the particles involved, the particles’ motion, quantum tunneling through a horizon, radiation, energy-momentum dissipation and flux quantization.

INTRODUCTION

Fields for which quantum fluctuations are negligible relative to the expectation values of the fields themselves can be described by classical equations. This is the external field approximation. In electrodynamics radiation processes are greatly enhanced when incident photons are replaced by external fields. Processes like photon-electron collisions with pair production by a Coulomb field and bremsstrahlung owe their relevance in physics and astrophysics to the external field approximation.

In gravitation where field strengths are much lower than in electromagnetism and means to increase cross sections are very desirable, external field problems have been paid limited attention. This is presumably due to the large number of components of the gravitational potential and to the difficulties of finding manageable modifications for the basic fields of interest as required by the characteristics of the gravitational sources, like rotation and time-dependence. A variety of problems involving gravitational fields of weak to intermediate strength do not require the full-fledged use of general relativity and can be tackled using an external field approximation. Over the years solutions of covariant wave equations have been found [1–5] that are exact to first order in the metric deviation of general relativity and can be tackled using an external field approximation. Over the years solutions of covariant wave equations have been found [1–5] that are exact to first order in the metric deviation $\gamma_{\mu \nu}$ of the Minkowski metric, and do not depend on the choice of any field equations for $\gamma_{\mu \nu}$. They are a useful tool in the study of the interaction of gravity with quantum systems and have been applied to interferometry and gyroscope [2], the optics of particles [4], the observation of gravitational effects [2], spin currents [6], neutrino physics [7, 8] and radiative processes in astrophysics [9–12].

In all the solutions mentioned, gravity is contained in a phase factor whose main ingredient is the two-point potential $K_\lambda(z, x)$ that is a direct consequence of genuinely quantum equations and the expression, at the same time, of hidden symmetries of gravity. It is a novel feature of the approach. $K_\lambda$ describes gravity while remaining tied to a metric tensor. That is, $K_\lambda$ is known only if $\gamma_{\mu \nu}$ and its derivatives are known. $K_\lambda$ has other interesting aspects that are studied below. We show, in particular, that the potential $K_\lambda$ affects a particle’s dispersion relations, its motion, the WKB problem, energy-momentum dissipation, radiation and flux quantization.

An interesting development is that as gravity approaches the quantum domain, particles assume the character of quasiparticles and the gravitational potential carries information about the matter with which it interacts.

Spin terms that appear in the solutions of the covariant wave equations are not particularly relevant in what follows, thus we can simply consider the covariant Klein-Gordon (KG) equation. Similar conclusions can, of course, be reached starting from other known wave equations, whose solution is based, in any case, on the KG equation.

Neglecting curvature dependent terms and applying the Lanczos-De Donder condition

$$\gamma_{\alpha \nu}^{\nu} - \frac{1}{2} \gamma^\alpha_\alpha = 0,$$

we can write the covariant KG equation to $\mathcal{O}(\gamma_{\mu \nu})$ in the form

$$(\nabla_\mu \nabla^\mu + m^2) \phi(x) \simeq [\eta_{\mu \nu} \partial^\mu \partial^\nu + m^2 + \gamma_{\mu \nu} \partial^\mu \partial^\nu] \phi(x) = 0.$$  

(2)

Units $\hbar = c = k_B = 1$ are used unless specified otherwise. The notations are as in [12]. In particular, partial derivatives with respect to a variable $x_\mu$ are interchangeably indicated by $\partial_\mu$, or by a comma followed by $\mu$.

The first order solution of (2) is

$$\phi(x) = (1 - i \Phi_G(x)) \phi_0(x),$$

(3)
where \( \phi_0(x) \) is a plane wave solution of the free KG equation

\[
(\partial_\mu \partial^\mu + m^2) \phi_0(x) = 0, \tag{4}
\]

and

\[
\Phi_G(x) = -\frac{1}{2} \int_p^x dz^\lambda \left( \gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z) \right) (x^\alpha - z^\alpha) k^\beta \tag{5}
\]

\[
+ \frac{1}{2} \int_p^x dz^\gamma \gamma_{\alpha\lambda}(z) k^\alpha = \int_p^x dz^\lambda K_\lambda(z, x),
\]

where \( P \) is an arbitrary point, henceforth dropped, and

\[
K_\lambda(z, x) = -\frac{1}{2} \left[ (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) (x^\alpha - z^\alpha) - \gamma_{\beta\lambda}(z) \right] k^\beta. \tag{6}
\]

The momentum of the plane wave solution \( \phi_0 \) of (4) is \( k^\alpha \) and satisfies the equation \( k_\alpha k^\alpha = m^2 \). There are no additional constraints on the solution of (2) represented by (3)–(6) except for the order of approximation in \( \gamma_{\mu\nu} \). Higher order solutions can be calculated following the procedure outlined in (12). Nonetheless our calculations are limited to \( O(\gamma_{\mu\nu}) \) as we are primarily concerned with gravitational fields of weak to moderate intensity.

It is easy to see that (3) is a solution of (2). By writing \( \phi \equiv \phi^{(1)} \) for the first order solution and differentiating (3) with respect to \( x_\mu \), we obtain

\[
\phi^{(1)}_{\mu\nu} = \phi_{0,\mu} - i \Phi_{G,\mu} \phi_0 - i \Phi_{G,\nu} \phi_0, \tag{7}
\]

and again

\[
\phi^{(1)}_{\mu\nu} = \phi_{0,\mu\nu} - i \Phi_{G,\mu\nu} \phi_0 - i \Phi_{G,\nu\mu} \phi_0 - i \Phi_{G,\mu\nu} \phi_0. \tag{8}
\]

The result then follows by substituting (8) in (2) and by noticing that \( \eta^{\mu\nu} \Phi_{G,\mu\nu} = \eta^{\mu\nu} k_\alpha \Gamma^{\alpha}_{\mu\nu} = 0 \) by virtue of (11) and \( k^\mu \Phi_{G,\mu} = \frac{1}{2} \gamma^{\mu\nu} k_\mu k_\nu \).

### QUASIPARTICLES

Equations (3), (5) and (6) are the byproduct of covariance (minimal coupling) and, ultimately, of Lorentz invariance and can therefore be applied to general relativity, in particular to theories in which acceleration has an upper limit [15–22] and that therefore allow the resolution of astrophysical [23–26] and cosmological singularities in quantum theories of gravity [27, 28]. They also are relevant to those theories of asymptotically safe gravity that can be expressed as Einstein gravity coupled to a scalar field [29].

Path-dependent field variables in electromagnetism and gravitation have been used in the works of Volkov [30], Bergmann [31], DeWitt [32] and Mandelstam [33]. From (8) we can derive expressions for the covariant derivatives of path-dependent quantities. The vector field \( K_\lambda \) plays a role similar to that of the vector potential in electromagnetism, but contains, at the same time, reference to matter through the momentum \( \phi_0 \). It is, in fact, this association that suggests the introduction of the notion of quasiparticle that in other areas of physics describes fields and particles whose properties are affected by the presence of other particles and media with which they interact.

\( K_\lambda(z, x) \) also satisfies Maxwell-type equations identically. By differentiating (3) with respect to \( z^\alpha \), we find [34]

\[
\tilde{F}_{\mu\lambda}(z, x) = K_{\lambda,\mu}(z, x) - K_{\mu,\lambda}(z, x) = R_{\mu\lambda,\beta}(z) J^{\alpha\beta}, \tag{9}
\]

where \( R_{\alpha\beta\lambda\mu}(z) = -\frac{1}{2} (\gamma_{\alpha\beta,\lambda\mu} + \gamma_{\mu\lambda,\beta\alpha} - \gamma_{\beta\lambda,\alpha\mu} - \gamma_{\alpha\lambda,\beta\mu}) \) is the linearized Riemann tensor satisfying the identity \( R_{\mu\nu\sigma\tau} + R_{\nu\mu\sigma\tau} + R_{\sigma\mu\nu\tau} = 0 \) and \( J^{\alpha\beta} = \frac{1}{2} \left[ (x^\alpha - z^\alpha) k^\beta - k^\alpha (x^\beta - z^\beta) \right] \) is the angular momentum about the base point \( x^\alpha \). Maxwell-type equations

\[
\tilde{F}_{\mu\lambda,\sigma} + \tilde{F}_{\lambda\sigma,\mu} + \tilde{F}_{\sigma\mu,\lambda} = 0 \tag{10}
\]

and

\[
\tilde{F}^{\mu\lambda}_{\alpha\beta} \equiv j^\mu = \left( R^{\mu\lambda}_{\alpha\beta} j^{\alpha\beta} \right) \lambda = R^{\mu\lambda}_{\alpha\beta,\lambda} (x^\alpha - z^\alpha) k^\beta + R^\mu_{\beta} k^\beta, \tag{11}
\]
can be obtained from \( (9) \) using the Bianchi identities \( R_{\mu
u\sigma\rho} + R_{\mu\rho\sigma\nu} + R_{\mu\nu\rho\sigma} = 0 \). The current \( j^\mu \) satisfies the conservation law \( j^\mu \partial_\mu = 0 \). Equations \( (10) \) and \( (11) \) are identities and do not represent additional constraints on \( \gamma_{\mu\nu} \).

One finds, in particular, that the "electric" and "magnetic" components of \( \tilde{F}_{\mu\nu} \) are

\[
\tilde{F}_{\mu\nu} = R_{\alpha\beta\gamma\delta} J^{\alpha\beta}, \quad \tilde{H}_{\mu} = \epsilon_{ijk} R^{kJ}_{\alpha\beta} J^{\alpha\beta},
\]

where \( \epsilon_{ijk} \) is the Levi-Civita symbol.

In this work we investigate some of the consequences that follow from \( (10) \) and \( (11) \) and from the close association of \( K_\lambda \) with matter.

The recombination of ten \( \gamma_{\mu\nu} \) into four \( K_\lambda \) is a remarkable phenomenon, even though knowledge of all \( \gamma_{\mu\nu} \) is still needed, in general, to calculate \( K_\lambda \). It follows from \( (6) \) that the gravitational field is described by \( K_\lambda \) along a particle world line and that this path. By substituting \( (3) \) into the first term of \( (14) \), we obtain

\[
\tilde{F}_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} + \partial^\alpha \gamma_{\alpha\beta} \partial_\beta \gamma_{\mu\nu} = 0,
\]

where \( \gamma_{\alpha\beta} \) is the Levi-Civita symbol.

The last term in \( (13) \) can be eliminated by a gauge transformation. We then obtain \( \partial^2 K_\lambda = 0 \), irrespective of the value of \( k_\alpha \).

**DISPERSION RELATIONS AND PARTICLE MOTION**

By using Schroedinger’s logarithmic transformation \( \phi = e^{-iS} \) we can pass from the KG equation \( (2) \) to the quantum Hamilton-Jacobi equation. We find to first order in \( \gamma_{\mu\nu} \)

\[
i(\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu \partial_\nu S - (\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu S \partial_\nu S + m^2 = 0,
\]

where

\[
S = k^3 \left\{ x_\beta + \frac{1}{2} \int dz \gamma_{\beta\lambda}(z) - \frac{1}{2} \int dz (\gamma_{\alpha\lambda}(z) - \gamma_{\beta\lambda}(z)) (x^\alpha - z^\alpha) \right\} + k_\alpha x^\alpha + A + B,
\]

\[
\equiv k_\alpha x^\alpha + A + B.
\]

It is well-known that the Hamilton-Jacobi equation is equivalent to Fresnel’s wave equation in the limit of large frequencies \( \omega \). However, at smaller, or moderate frequencies the complete equation \( (14) \) should be used. We follow this path. By substituting \( (3) \) into the first term of \( (14) \), we obtain

\[
i(\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu \partial_\nu S = i\eta^{\mu\nu} \partial_\mu (k_\nu + \Phi_{G,\nu}) - i\gamma^{\mu\nu} \partial_\mu k_\nu = i\eta^{\mu\nu} \Phi_{G,\mu} = i\eta^{\mu\nu} \gamma^{\mu\nu} \Gamma_{\mu\nu} = 0,
\]

on account of \( (1) \). This part of \( (14) \) is usually neglected in the limit \( \hbar \rightarrow 0 \). Here it vanishes as a consequence of solution \( (3) \). The remaining terms of \( (14) \) yield the classical Hamilton-Jacobi equation

\[
(\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu S \partial_\nu S - m^2 = \gamma^{\mu\nu} k_\mu k_\nu - 2k^{\mu\nu} \Phi_{G,\mu} = 0,
\]

because \( k^{\mu\nu} \Phi_{G,\mu} = 1/2 \gamma^{\mu\nu} k_\mu k_\nu \). Equation \( (3) \) is therefore a solution of the more general quantum equation \( (14) \). It also follows that the particle acquires a generalized "momentum"

\[
P_\mu = k_\mu + \Phi_{G,\mu} = k_\mu + \frac{1}{2} \gamma_{\alpha\mu} k^\alpha - \frac{1}{2} \int dz (\gamma_{\mu\lambda}(z) - \gamma_{\beta\lambda}(z)) k^\beta,
\]

that satisfies the dispersion relation

\[
P_\mu P^\mu \equiv m_c^2 = m^2 \left( 1 + \gamma_{\alpha\mu}(x) u^\alpha u^\mu - \frac{1}{2} \int dz (\gamma_{\mu\lambda}(z) - \gamma_{\beta\lambda}(z)) u^\alpha u^\beta \right).
\]
The integral in (19) vanishes because $u^\mu u^\beta$ is contracted on the antisymmetric tensor in round brackets. The effective mass $m_e$ is not in general constant. In this connection too we can speak of quasiparticles. The medium in which the scalar particles propagate is here represented by space-time.

We have already used some of the properties of (18) and (19) elsewhere. $P_\mu$ of (18) describes the geometrical optics of particles correctly and gives the correct deflection predicted by general relativity. On using the relations

$$
\Phi_{G,\mu} = K_\mu(x, x) + \int z^\lambda \partial_\mu K_\lambda(z, x),
$$

and

$$
\Phi_{G,\mu\nu} = K_{\mu,\nu}(x, x) + \partial_\nu \int z^\lambda \partial_\mu K_\lambda(z, x) = k_\alpha \Gamma^\alpha_{\mu\nu},
$$

and by differentiating (18) we obtain the covariant derivative of $P_\mu$

$$
\frac{DP_\mu}{Ds} = m \left[ \frac{du_\mu}{ds} + \frac{1}{2} (\gamma_{\alpha\mu,\nu} - \gamma_{\mu\nu,\alpha}) u^\alpha u^\nu \right]
$$

This result is independent of any choice of field equations for $\gamma_{\mu\nu}$. We see from (22) that, if $k_\mu$ follows a geodesic, then $\frac{DP_\mu}{Ds} = 0$ and $\frac{Dm^2}{Ds} = 0$. The classical equations of motion are therefore contained in (22), but it would require the particle described by (2) to just choose a geodesic, among all the paths allowed to a quantum particle.

We also obtain, from (14),

$$
\sqrt{(\partial_\mu S)^2} = \pm \sqrt{-m^2 + (\partial_\mu S)^2 - \gamma_{\mu\nu} \partial_\mu \partial_\nu S} = \sqrt{-m^2 + k_\mu^2},
$$

which, in the absence of gravity, gives $k_\mu^2 = -m^2 + k_0^2$, as expected. Remarkably, (18) is an exact integral of (22) which can itself be integrated exactly to give the particle’s motion

$$
X_\mu = x_\mu + \frac{1}{2} \int z^\lambda \gamma_{\mu\lambda} \left\{ (\gamma_{\alpha\lambda,\mu} - \gamma_{\mu\lambda,\alpha}) (x^\alpha - z^\alpha) \right\}.
$$

Higher order approximations to the solution of (2) can be obtained by writing

$$
\phi(x) = \Sigma_n \phi(n)(x) = \Sigma_n e^{-i\hat{\Phi}_G \phi(n-1)},
$$

where the operator $\hat{\Phi}_G$ is

$$
\hat{\Phi}_G(x) = -\frac{1}{2} \int z^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) (x^\alpha - z^\alpha) \hat{k}^\beta
$$

$$
+ \frac{1}{2} \int z^\lambda \gamma_{\alpha\lambda} \hat{k}^\alpha,
$$

and $\hat{k}^\alpha = i\partial^\alpha$.

The solution (24) plays a dynamical role akin to Feynman’s path integral formula. In (24), however, it is the solution itself that is varied by successive approximations, rather than the particle’s path.

### THE GRAVITATIONAL WKB PROBLEM

We now study the propagation of a scalar field in a gravitational background. We know, from standard quantum mechanics, that $S$ develops an imaginary part when the particle tunnels through a potential. This imaginary contributions is interpreted as the transition amplitude across the classically forbidden region, which is therefore given by

$$
\mathcal{T} = \exp \left\{ -2Im(S) \right\} = \exp \left\{ -2Im \left[ \ln \left( \Sigma_n \exp (-i\hat{\Phi}_G \phi(n-1)) \right) \right] \right\}.
$$
To \(O(\gamma_{\mu\nu})\), (26) becomes

\[
T = \exp \left\{ -2Im \left[ x_\beta + \frac{1}{2} \oint dz^\lambda \gamma_{\beta\lambda}(z) - \frac{1}{2} \oint dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))(x^\alpha - z^\alpha) \right] k^\beta \right\},
\]  

(27)

for a space-time path traversing the gravitational background from \(-\infty\) to \(+\infty\) and back as it must in order to make (27) invariant. Assuming a Boltzmann distribution for the particles \(T = e^{-k_0/T}\), where \(T\) is the temperature and the Boltzmann constant \(k_B = 1\), we find, in general coordinates,

\[
T = k_0/Im \left\{ 2k^\beta \left[ x_\beta + \frac{1}{2} \oint dz^\lambda \gamma_{\beta\lambda}(z) - \frac{1}{2} \oint dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))(x^\alpha - z^\alpha) \right] \right\}.
\]  

(28)

The intended application here is to the propagation problem in Rindler space given by

\[
ds^2 = (1 + ax)^2 \left( dx^0 \right)^2 - (dx)^2,
\]  

(29)

with a horizon at \(x = -1/a\), where \(a^2 = a_\alpha a^\alpha\) is the constant proper acceleration measured in the rest frame of the Rindler observer. We note that, a priori, our approach is ill-suited to treat this problem that frequently in the literature is tackled starting from exact, or highly symmetric solutions of the KG equation [39]. In fact the external field approximation \(|\gamma_{\mu\nu}| < |\eta_{\mu\nu}|\) may become inadequate close to the horizon, from where the imaginary part of \(\xi e^{\pm i\pi/2}\), for a round trip the horizon is crossed twice and each time \(a^0 \rightarrow a^0 - i\pi/2\) because of (30). The remaining term of (28) gives \(k_1 \Delta x^\prime = k_1 x^\prime - (-k_1)(-x^\prime) = 0\). The final result is therefore

\[
T = \frac{a}{2\pi},
\]  

(34)

which is independent of \(k^1\) and coincides with the usual Unruh temperature [41, 42]. This result, with the replacement \(a \rightarrow a/\sqrt{1 - a^2/A^2}\), where \(A = 2m\) is the maximal acceleration, also confirms a recent calculation [43] regarding particles whose acceleration has an upper limit. Equation (31) comes in fact from the term \(k_0 \Delta t'\) that does not
contain derivatives of $\gamma_{\mu\nu}$. The difference from [43], as well as from [40], is however represented by the form of (27) of the decay rate [38] which carries a factor 2 in the exponential as required by our invariant approach.

Despite its limitations, the external field approximation already reproduces (34) at $O(\gamma_{\mu\nu})$. Additional terms of [26] are expected to contain corrections to [33]. We note, however, that for a closed space-time path the last integral in (27) and (28) becomes $\int_{-\infty}^{\infty} d\sigma^{\alpha\beta} R_{\mu\nu\alpha\beta} J^{\alpha\beta}$, where $\Sigma$ is the surface bounded by the path [3], and has an imaginary part if $R_{\mu\nu\alpha\beta}$ has singularities. This eventuality may call for a complete quantum theory of gravity [44].

$K_\lambda$ IN INTERACTION

i. Poynting vector. The question we ask in this section is whether the vector $K_\lambda$ is redundant, or plays a role in radiation problems. Using $F_{\mu\nu}$, we can construct, for instance, a "Poynting" vector. Assuming, for simplicity, that $j_\mu = 0$ in (11), using known vector identities, integrating over a finite volume and reverting to normal units, we obtain from (10) and (11) the conservation equation

$$\int \frac{1}{c} \frac{\partial}{\partial t} \left( E^2 + H^2 \right) dV = -2 \oint \tilde{S} \cdot d\Sigma,$$

where $\Sigma$ is the surface bounding $V$ and $\tilde{S} = \tilde{E} \times \tilde{H}$ is the gravitational Poynting vector. Both sides of (35) acquire, in fact, the dimensions of an energy flux after multiplication by $G/c^3$. We can now calculate the flux of $\tilde{S}$ at the particle assuming that the momentum of the free particle is $k = k^3$ and that the source in $V$ emits a plane gravitational wave in the $x$-direction. In this case the wave is determined by the components $\gamma_{22} = -\gamma_{33}$ and $\gamma_{23}$, and we find $\tilde{E}_1 = 0$, $\tilde{E}_2 = 2R_{2033} J^{03} + 2R_{0231} J^{31}$, $\tilde{E}_3 = 2R_{0303} J^{03} + 2R_{0331} J^{31}$, $\tilde{H}_1 = 0$, $\tilde{H}_2 = -4R_{2103} J^{03} - 4R_{3113} J^{13}$, $\tilde{H}_3 = 4R_{2103} J^{03} + 4R_{3113} J^{13}$. It also follows that $R_{0203} = R_{0231} = R_{2103} = R_{2113} = -\gamma_{23}/2$ and $R_{0303} = R_{0331} = R_{3103} = R_{3113} = \gamma_{22}/2$. The action of $\tilde{S}$ on the quantum particle is directed along the axis of propagation of the wave and results in a combination of oscillations and rotations about the point $x^\alpha$ with angular momentum given by $2J^{03} = (x^0 - z^0)k - k^0(x^3 - z^3)$, $2J^{13} = (x^1 - z^1)k$ and $2J^{23} = (x^2 - z^2)k$. A similar motion also occurs in the case of Zitterbewegung [45]. Reverting to normal units, the energy flux associated with this process is

$$\Phi = (\omega^4 G/c^3) \{ (\gamma_{23}^2)(J^{03})^2 + J^{31} J^{03} + (\gamma_{22}^2)(J^{03})^2 - J^{31} J^{03} - (J^{31})^2 \}$$

and increases rapidly with the wave frequency $\omega$ and the particle’s angular momentum.

ii. Electromagnetic radiation. Let us assume that a spinless particle has a charge $q$. Acceleration, whatever its cause, makes the particle radiate electromagnetic waves. The four-momentum radiated away by the particle, while passing through the driving gravitational field $F_{\mu\nu}$, is given by the formula

$$\Delta p^\alpha = -\frac{2q^2}{3c} \int \frac{du_{\beta}}{ds} \frac{du_{\alpha}}{d\sigma} dx^\alpha = -\frac{2q^2}{3c} \int \left( \tilde{F}_{\mu\nu} u^\nu \right) \left( \tilde{F}^{\nu\delta} u_\delta \right) dx^\alpha,$$

that can be easily expressed in terms of the external fields on account of the equation of motion of the charge in the accelerating field [34]. At this level of approximation the particle can distinguish uniform acceleration which gives $\Delta p^\alpha \sim g^2 dx^\alpha$, where $g$ is a constant, from a non-local gravitational field and radiates accordingly. This is explained by the presence of $R_{\mu\nu\alpha\beta}$ in [12] and is a direct consequence of our use of the equation of geodesic deviation in (37).

When the accelerating field is the wave discussed above, the incoming gravitational wave and the emitted electromagnetic wave have the same frequency $\omega$ and the efficiency of the gravity induced production of photons increases as $\omega^4 k^2$.

iii. Flux quantization. Flux quantization is the typical manifestation of processes in which the wave function is non-integrable. Of interest is here the presence of the free particle momentum $k^\alpha$ in $K_\lambda$.

Let us consider for simplicity the case of a rotating superfluid. Then $\gamma_{01} = -\Omega_{22}/c$, $\gamma_{02} = \Omega_{21}/c$ and the remaining metric components vanish. The angular velocity $\Omega$ is assumed to be constant in time and $k_3 = 0$. Without loss of generality, we can also choose the reference point $x^\mu = 0$. We find $K_0 = K_3 = 0$ and

$$K_1 = -\frac{1}{2} \left[ \gamma_{01,2} z^2 - \gamma_{01} \right] k^0 - \frac{1}{2} \left[ -\gamma_{01,2} z^0 \right] k^2$$

$$K_2 = -\frac{1}{2} \left[ \gamma_{02,1} z^1 - \gamma_{02} \right] k^0 - \frac{1}{2} \left[ -\gamma_{02,1} z^0 \right] k^1.$$
Integrating over a loop of superfluid, the condition that the superfluid wave function be single-valued gives the quantization condition

$$\oint dz^λ K_λ = -\frac{\Omega z^0}{2c} \oint (k^2 dz^λ - k^1 dz^2) = \frac{\pi \Omega z^0}{c} k_φ = 2\pi n,$$

where $n$ is an integer, $k = \sqrt{k_1^2 + k_2^2}$ and $\theta = \sqrt{z_1^2 + z_2^2}$. The time integrating factor $z^0$, extended to $N$ loops, becomes $z^0 = 2\pi \varrho \varepsilon N/pc$, where $\varepsilon^2 = (pc)^2 + (mc^2)^2$ and $p = \hbar k$. The superfluid quantum of circulation satisfies the condition

$$\Omega(\pi \varrho^2)\varepsilon N/c^2 = n\hbar.$$

If the superfluid is charged, then the wave function is single-valued if the total phase satisfies the relation

$$\oint dz^λ K_λ + \frac{q}{c} \oint dz^λ A_λ = 2\pi n\hbar,$$

which, for $n = 0$ and zero external magnetic field, leads to $\oint \vec{H} \cdot d\Sigma = -2\pi^2 \Omega \varrho^2 \varepsilon N/qc$. In this case, therefore, rotation generates a magnetic flux through $\Sigma$ and, obviously, a current in the $N$ superconducting loops. No fundamental difference is noticed in this case from DeWitt’s original treatment of the problem \[46 - 49\].

CONCLUSIONS

The two-point potential $K_λ(z, x)$ plays a prominent role in the solution of covariant wave equations through the phase $Φ_λ$. It satisfies Maxwell-type equations identically, depends on the metric tensor and is complementary to it. The potential $K_λ$ suggests the introduction of the notion of quasiparticle because gravity affects in general the dispersion relations of the particles with which it interacts, as shown by \[19\], and because it carries with itself information about matter through the particle momentum $k_α$. Some particular aspects of the behaviour of $K_λ$ have been examined. We have found that when $j_\mu = 0$, scale invariance assures that a gas of gravitons satisfies Planck’s radiation law, but that this is no longer so, in principle, for non-pure gravitational fields.

$K_λ$ also determines the equations of motion of a particle through \[20, 21, 22\] and \[24\]. We have found that the motion follows a geodesic only if the quantum particle chooses, among all available paths, that for which $Dk_α/Ds = 0$. Along this particular path the principle of equivalence is obviously satisfied. We have then shown that the particle motion is contained in the solution \[3\] of the covariant KG equation.

We have also studied quantum mechanical tunneling through a horizon and derived a covariant and canonical invariant expression for the transition amplitude. Though the external field approximation looks ill-suited to deal with regions of space-time close to a gravitational horizon, the approximation reproduces the Unruh temperature exactly in the case of the Rindler metric. No corrections and no effects due to $k_α$ have been found to the standard result to $O(γ_{μν})$. Higher order approximations can be calculated by applying \[28\].

Because $F_{μν}$ satisfies Maxwell-like equations, it is also possible to define a Poynting vector and a flux of energy and angular momentum at the particle so that the particle’s motion can be understood as a sequence of oscillations and rotations similar to what found in the case of Zitterbewegung \[45\].

Use of $K_λ$ in problems where gravity accelerates a charged particle and electromagnetic radiation is produced offers a rather immediate relationship between the loss of energy-momentum by the quantum particle and the driving gravitational field. These processes could give sizeable contributions for extremely high values of $ω$. Astrophysical processes like photoproduction \[50\] and synchrotron radiation \[51\] have been discussed in the literature and are worthy of re-consideration in view of the present results. An advantage on the high frequency detection side, for which detection schemes are in general difficult to conceive, is represented by the efficiency of the graviton-photon conversion rate and by the high coupling afforded by a radio receiver over, for instance, a mechanical one. This would enable, in principle, a spectroscopic analysis of the signal.

In the last problem considered, we have calculated the flux of $K_λ$ in the typical quantum case of a non-integrable wave function. Here too, it is possible to isolate quantities of physical interest, like magnetic flux, or circulation, despite the non-intuitive character of $\oint dz^λ K_λ$. Unlike \[32\], our procedure and results are fully relativistic. They can be applied directly to boson condensates in boson stars \[52\].
