Non-jumping Turán densities of hypergraphs

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Abstract

A real number α ∈ [0, 1) is a jump for an integer r ≥ 2 if there exists c > 0 such that no number in (α, α + c) can be the Turán density of a family of r-uniform graphs. A classical result of Erdős and Stone [7] implies that every number in [0, 1) is a jump for r = 2. Erdős [5] also showed that every number in [0, r!/r^r) is a jump for r ≥ 3 and asked whether every number in [0, 1) is a jump for r ≥ 3. Frankl and Rödl [8] gave a negative answer by showing a sequence of non-jumps for every r ≥ 3. After this, Erdős modified the question to be whether r!/r^r is a jump for r ≥ 3? What’s the smallest non-jump? Frankl, Peng, Rödl and Talbot [9] showed that 5√r/r^2 is a non-jump for r ≥ 3. Baber and Talbot [1] showed that every α ∈ [0, 2.299, 0.2316) ∪ [0, 0.2871, 8273) is a jump for r = 3. Pikhurko [14] showed that the set of all possible Turán densities of r-uniform graphs has cardinality of the continuum for r ≥ 3. However, whether r!/r^r remains the known smallest non-jump for r ≥ 3 remains open, and 5√r/r^2 has remained the known smallest non-jump for r ≥ 3. In this paper, we give a smaller non-jump by showing that 54√r/25√r^2 is a non-jump for r ≥ 3. Furthermore, we give infinitely many irrational non-jumps for every r ≥ 3.

Keywords: Jumping number, Turán density, hypergraph

1 Introduction

For a finite set V and a positive integer r we denote by \( \binom{V}{r} \) the family of all r-subsets of V. An r-uniform graph (r-graph) G is a set V(G) of vertices together with a set E(G) ⊆ \( \binom{V(G)}{r} \) of edges. The density of G is defined by d(G) = \( \frac{|E(G)|}{\binom{|V(G)|}{r}} \). For a family \( \mathcal{F} \) of r-graphs, an r-graph G is called \( \mathcal{F} \)-free if it does not contain an isomorphic copy of any r-graph of \( \mathcal{F} \). For a fixed positive integer n and a family of r-graphs \( \mathcal{F} \), the Turán number of \( \mathcal{F} \), denoted by \( ex(n, \mathcal{F}) \), is the maximum number of edges in an \( \mathcal{F} \)-free r-graph on n vertices. An averaging argument in [12] by Katona, Nemetz, and Simonovits shows that the sequence \( \frac{ex(n, \mathcal{F})}{\binom{n}{r}} \) is non-increasing. Hence \( \lim_{n \to \infty} \frac{ex(n, \mathcal{F})}{\binom{n}{r}} \) exists. The Turán density of \( \mathcal{F} \) is defined as

\[
\pi(\mathcal{F}) = \lim_{n \to \infty} \frac{ex(n, \mathcal{F})}{\binom{n}{r}}.
\]

If \( \mathcal{F} \) consists of a single r-graph \( F \), we simply write \( ex(n, \{F\}) \) and \( \pi(\{F\}) \) as \( ex(n, F) \) and \( \pi(F) \). Denote

\[
\Pi^* = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r-\text{uniform graphs}\},
\]

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\(\Pi_{fin} = \{ \pi(\mathcal{F}) : \mathcal{F} \text{ is a finite family of } r-\text{uniform graphs}, \} \)

and

\(\Pi_{t} = \{ \pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r-\text{uniform graphs and } |\mathcal{F}| \leq t \}.\)

Clearly,

\[\Pi_{1} \subseteq \Pi_{2} \subseteq \cdots \subseteq \Pi_{fin} \subseteq \Pi_{\infty}.\]

Finding good estimation for Turán densities in hypergraphs \((r \geq 3)\) is believed to be one of the most challenging problems in extremal combinatorics. The following concept concerns the accumulation points of the set \(\Pi_{\infty}\).

**Definition 1.1** A real number \(\alpha \in [0,1)\) is a jump for an integer \(r \geq 2\) if there exists a constant \(c > 0\) such that for any \(\epsilon > 0\) and any integer \(m \geq r\), there exists an integer \(n_{0}(\epsilon, m)\) such that any \(r\)-uniform graph with \(n > n_{0}(\epsilon, m)\) vertices and density \(\geq \alpha + \epsilon\) contains a subgraph with \(m\) vertices and density \(\geq \alpha + c\).

This concept describes where the set of ‘jumps’ is closely related to Turán densities. It was shown in [8] that \(\alpha\) is a jump for \(r\) if and only if there exists \(c > 0\) such that \(\Pi_{\infty} \cap (\alpha, \alpha + c) = \emptyset\). So every non-jump is an accumulation point of \(\Pi_{\infty}\). For 2-graphs, Erdős-Stone-Simonovits [6, 7] determined the Turán numbers of all non-bipartite graphs asymptotically. Their result implies that \(\Pi_{\infty}^{2} = \Pi_{fin}^{2} = \{0, \frac{1}{2}, \frac{2}{3}, \ldots, l - \frac{1}{l}, \ldots\}\).

This implies that every \(\alpha \in [0,1)\) is a jump for \(r = 2\). For \(r \geq 3\), Erdős [5] proved that every \(\alpha \in [0, r! / r^{r}]\) is a jump. Furthermore, Erdős proposed the jumping constant conjecture: Every \(\alpha \in [0,1)\) is a jump for every integer \(r \geq 2\). In [8], Frankl and Rödl disproved the Conjecture by showing that \(1 - \frac{1}{r^{r-1}}\) is not a jump for \(r \geq 3\) and \(l > 2r\). However, there are still a lot of unknowns on whether a number is a jump for \(r \geq 3\). A well-known open question of Erdős is whether \(r! / r^{r}\) is a jump for \(r \geq 3\) and what is the smallest non-jump? Another question raised in [9] is whether there is an interval of non-jumps for some \(r \geq 3\)? Both questions seem to be very challenging. Frankl-Peng-Rödl-Talbot [9] showed that \(\frac{3r!}{27r}\) is a non-jump for \(r \geq 3\). Baber and Talbot [1] showed that for \(r = 3\) every \(\alpha \in [0.2299, 0.2316] \cup [0.2871, \frac{8}{27})\) is a jump. Pikhurko [14] showed that \(\Pi_{\infty}\) has cardinality of the continuum for \(r \geq 3\). However, whether \(\frac{r!}{2r^{r}}\) is a jump remains open. Regarding the first question, we determine a non-jump smaller than \(\frac{3r!}{27r}\) for \(r \geq 3\).

**Theorem 1.2** \(\frac{12}{25}\) is not a jump for \(r = 3\).

In [13], a way to generate non-jumps for every \(p \geq r\) based on a non-jump for \(r\) was given. The following result was shown there.

**Theorem 1.3** [13] Let \(p \geq r \geq 3\) be positive integers. If \(\alpha \cdot \frac{r!}{r^{r}}\) is a non-jump for \(r\), then \(\alpha \cdot \frac{p!}{p^{p}}\) is a non-jump for \(p\).

Combining Theorems 1.2 and 1.3 we will get

**Corollary 1.4** \(\frac{54r!}{25r^{r}}\) is a non-jump for \(r \geq 3\).
Chung and Graham [10] proposed the conjecture that every element in $\Pi_{\text{fin}}$ is a rational number. Baber and Talbot [2], and Pikhurko [14] disproved this conjecture independently by showing that there is an irrational number in $\Pi_{\text{fin}}$. Baber and Talbot asked if there is an irrational number in $\Pi_1$. Recently, Yan and Peng [16] showed that there is an irrational number in $\Pi_3$ and Wu-Peng [15] showed that there is an irrational number in $\Pi_4$. Pikhurko [14] showed that $\Pi_{\infty}$ is closed which implies that every non-jump is a Turán density (a Turán density may not be a non-jump). Brown and Simonovits [4] showed that the Lagrangian of an $r$-uniform hypergraph is in $\Pi_{\infty}$ also indicating the existence of irrational numbers in $\Pi_{\infty}$. No irrational non-jump has been previously given. In this paper, we will give an infinite sequence of irrational non-jumps for $r = 3$.

Theorem 1.5 Let $k \geq 2$ be an integer. Then $\alpha_k = \frac{2k - 6k^2 + 4k^4 - k\sqrt{4k^2 - 1} + 4k^2\sqrt{4k - 1}}{(2k^2 + 1)^2}$ is not a jump for $r = 3$.

Combining Theorem 1.5 and Theorem 1.3, we can also get corresponding non-jumps for $r \geq 3$.

The proof of Theorem 1.2 and 1.5 will be given in Section 3 and Section 4, respectively. Both proofs applied an approach developed by Frankl and Rödl in [8]. The crucial part in our proof is to give a ‘proper’ construction. In the following section, we will introduce some preliminary results and sketch the idea of the proof.

2 Preliminaries and Sketch of the proof

2.1 Karush-Kuhn-Tucker Conditions

Let us consider the optimisation problem:

maximise $f(x)$

subject to $g_i(x) \leq 0$, $i = 1, \ldots, m,$

(3.1)

where $x \in \mathbb{R}^n$ and $f$ and $g_i$ are differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i$. Let $\nabla f(x)$ be the gradient of $f$ at $x$ i.e., the vector in $\mathbb{R}^n$ whose $i$th coordinate is $\frac{\partial}{\partial x_i} f(x)$. We say that KKT conditions hold at $x^* \in \mathbb{R}^n$ if there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

1. $\nabla f(x^*) = \sum_{i=1}^m \lambda_i \nabla g_i(x^*)$,
2. $\lambda_i \geq 0$ for $i = 1, \ldots, m$,
3. $\lambda_i g_i(x^*) = 0$ for $i = 1, \ldots, m$.

We call the constraints linear if $g_1, \ldots, g_m$ are all affine functions.

Theorem 2.1 ([3], [11]) If the constraints of (3.1) are linear, then any optimal solution to (3.1) must satisfy the KKT conditions.

2.2 Properties of the Lagrangian function

In this section we will give the definition of the Lagrangian of an $r$-uniform graph, which is a helpful tool in our proof.
Definition 2.2 For an $r$-uniform graph $G$ with vertex set $\{1,2,\ldots,n\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, define the Lagrangian function

$$\lambda(G, \vec{x}) = \sum_{\{i_1,\ldots,i_r\} \in E(G)} x_{i_1}x_{i_2}\ldots x_{i_r}.$$ 

Let $S = \{\vec{x} = (x_1,x_2,\ldots,x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1,2,\ldots,n\}$. The Lagrangian of $G$, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$ 

A vector $\vec{x} \in S$ is called a feasible vector on $G$, and $x_i$ is called the weight of the vertex $i$. A feasible vector is called optimal if $\lambda(G, \vec{y}) = \lambda(G)$.

Fact 2.3 If $G_1 \subseteq G_2$, then

$$\lambda(G_1) \leq \lambda(G_2).$$

Fact 2.4 (i) Let $G$ be an $r$-graph on $[n]$. Let $\vec{x} = (x_1,x_2,\ldots,x_n)$ be an optimal vector on $G$. Then

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r\lambda(G)$$

for every $i \in [n]$ satisfying $x_i > 0$.

Given an $r$-graph $G$, and $i,j \in V(G)$, define

$$L_G(j \setminus i) = \{e : e \notin e \cup \{j\} \in E(G) \text{ and } e \cup \{i\} \notin E(G)\}.$$ 

Fact 2.5 (i) Let $G$ be an $r$-graph on $[n]$. Let $\vec{x} = (x_1,x_2,\ldots,x_n)$ be a feasible vector on $G$, and $i,j \in [n]$, $i \neq j$ satisfy $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$. Let $\vec{y} = (y_1,y_2,\ldots,y_n)$ be defined by letting $y_\ell = x_\ell$ for every $\ell \in [n] \setminus \{i,j\}$ and $y_i = y_j = \frac{1}{2}(x_i + x_j)$. Then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$. Furthermore, if the pair $\{i,j\}$ is contained in an edge of $G$, $x_i > 0$ for each $1 \leq i \leq n$, and $\lambda(G, \vec{y}) = \lambda(G, \vec{x})$, then $x_i = x_j$.

We also note that for an $r$-graph $G$ with $n$ vertices, if we take $\vec{u} = (u_1,\ldots,u_n)$, where each $u_i = \frac{1}{n}$, then

$$\lambda(G) \geq \lambda(G, \vec{u}) = \frac{|E(G)|}{n^r} \geq \frac{d(G)}{r^r} - \epsilon$$

for $n \geq n'(\epsilon)$, where $n'(\epsilon)$ is a sufficiently large integer. On the other hand, the blow-up of an $r$-uniform graph $G$ will allow us to construct $r$-uniform graphs with large number of vertices and density close to $r!\lambda(G)$.

Definition 2.6 Let $G$ be an $r$-uniform graph with $V(G) = \{1,2,\ldots,t\}$ and $(n_1,\ldots,n_t)$ be a positive integer vector. Define the $(n_1,\ldots,n_t)$ blow-up of $G$, $(n_1,\ldots,n_t) \otimes G$ as a t-partite $r$-uniform graph with vertex set $V_1 \cup \ldots \cup V_t$, $|V_i| = n_i, 1 \leq i \leq t$, and edge set $E((n_1,\ldots,n_t) \otimes G) = \{\{v_{i_1},v_{i_2},\ldots,v_{i_r}\} \in E(G) \text{ and } v_{i_k} \in V_{i_k} \text{ for } 1 \leq k \leq r\}$.

Remark 2.7 (i) Let $G$ be an $r$-uniform graph with $t$ vertices and $\vec{y} = (y_1,\ldots,y_t)$ be an optimal vector for $\lambda(G)$. Then for any $\epsilon > 0$, there exists an integer $n_1(\epsilon)$, such that for any integer $n \geq n_1(\epsilon)$,

$$d([ny_1],[ny_2],\ldots,[ny_t]) \otimes G \geq r!\lambda(G) - \epsilon.$$ (1)
Let us also state a fact which follows directly from the definition of the Lagrangian.

**Fact 2.8** ([8]) For every \( r \)-uniform graph \( G \) and every positive integer \( n \), \( \lambda((n,n,\ldots,n) \otimes G) = \lambda(G) \) holds.

Lemma 2.9 in [8] gives a necessary and sufficient condition for a number \( \alpha \) to be a jump.

**Lemma 2.9** ([8]) The following two properties are equivalent.

1. \( \alpha \) is a jump for \( r \).

2. There exists some finite family \( \mathcal{F} \) of \( r \)-uniform graphs satisfying \( \pi(\mathcal{F}) \leq \alpha \) and \( \lambda(F) > \frac{\alpha}{r!} \) for all \( F \in \mathcal{F} \).

### 2.3 Sketch of the proofs of Theorem 1.2 and 1.5

The general approach in proving Theorem 1.2 and Theorem 1.5 is sketched as follows: Let \( \alpha \) be a number to be proved to be a non-jump for \( r = 3 \). Assuming that \( \alpha \) is a jump for \( r = 3 \), we will derive a contradiction by the following steps.

Step 1. Construct a ‘proper’ 3-uniform graph \( G^*(t) \) with the Lagrangian at least \( \frac{\alpha}{6} + \epsilon \) for some \( \epsilon > 0 \). Then we ‘blow up’ it to a 3-uniform graph, say \( \bar{m} \otimes G^*(t) \) with large enough number of vertices and density \( \geq \alpha + \epsilon \) (see Remark 2.7). If \( \alpha \) is a jump, then by Lemma 2.9, there exists some finite family \( \mathcal{F} \) of 3-uniform graphs with Lagrangians \( \geq \frac{\alpha}{6} \) and \( \pi(\mathcal{F}) \leq \alpha \). So \( \bar{m} \otimes G^*(t) \) must contain some member of \( \mathcal{F} \) as a subgraph.

Step 2. We show that any subgraph of \( G^*(t) \) with the number of vertices not greater than \( \max \{|V(F)|, F \in \mathcal{F}\} \) has the Lagrangian \( \leq \frac{\alpha}{6} \) and derive a contradiction.

The crucial part is to construct an \( r \)-uniform graph satisfying the properties in both Steps 1 and 2. Generally, whenever we find such a construction, we can obtain a corresponding non-jump. This method was first developed by Frankl and Rödl in [8]. The technical part in the proof is to show that the construction satisfies the property in Step 2.

### 3 Proof of Theorem 1.2

**Proof.** Suppose that \( \frac{12}{25} \) is a jump for \( r = 3 \). By Lemma 2.9 there exists a finite collection \( \mathcal{F} \) of 3-uniform graphs satisfying the following:

i) \( \lambda(F) > \frac{2}{25} \) for all \( F \in \mathcal{F} \), and

ii) \( \pi(\mathcal{F}) \leq \frac{12}{27} \).

Let \( G(t) = (V,E) \) be the 3-uniform defined as follows. The vertex set \( V = V_1 \cup V_2 \cup V_3 \), where \( |V_1| = |V_2| = \frac{t}{4} \) and \( |V_3| = \frac{t}{5} \) and the value of \( t \) will be determined later. The edge set of \( G(t) \) is

\[
\left( V_1 \times V_2 \times V_3 \right) \bigcup \left( \left( \frac{V_1}{2} \right) \times V_2 \right) \bigcup \left( \left( \frac{V_2}{2} \right) \times V_3 \right),
\]
i.e., the edges consisting of one vertex from each $V_1, V_2$ and $V_3$, or two vertices from $V_1$ and one vertex from $V_2$, or two vertices from $V_2$ and one vertex from $V_3$. Then

$$|E(G(t))| = \frac{2t^3}{25} - \frac{3t^2}{25}. \quad (2)$$

We will apply the following lemma from [8].

**Lemma 3.1** [8] For any $c \geq 0$ and any integer $s \geq r$, there exists $t_0(s,c)$ such that for every $t \geq t_0(s,c)$, there exists an $r$-uniform graph $A = A(s,c,t)$ satisfying:

1. $|V(A)| = t$,
2. $|E(A)| \geq ct^{r-1}$,
3. For all $V_0 \subset V(A)$, $r \leq |V_0| \leq s$, we have $|E(A) \cap (V_0) | \leq |V_0| - r + 1$.

Set $s = \max_{F \in F} |V(F)|$ and $c = 1$. Let $r = 3$ in Lemma 3.1, $t_0(s,1)$ be given as in Lemma 3.1 and $\frac{2t^3}{5} \geq t_0(s,1)$. The 3-uniform graph $G^*(t)$ is obtained by adding $A(s,1, \frac{2t^3}{5})$ to the 3-uniform hypergraph $G(t)$ in $V_1$. Then

$$\lambda(G^*(t)) \geq \lambda(G^*(t), \left(\frac{1}{t}, \frac{1}{t}, \ldots, \frac{1}{t}\right)) = \frac{|E(G^*(t))|}{t^3}.$$ 

In view of the construction of $G^*(t)$ and equation (2), we have

$$\frac{|E(G^*(t))|}{t^3} \geq \frac{|E(G(t))|}{t^3} + \left(\frac{2t^3}{5}\right)^2 / t^3 \geq \frac{2}{25} + \frac{1}{25t}. \quad (3)$$

Now suppose $\vec{y} = (y_1, y_2, \ldots, y_t)$ is an optimal vector of $\lambda(G^*(t))$. Let $n$ be large enough. By Remark 2.7, 3-uniform graph $S_n = (\lfloor ny_1 \rfloor, \ldots, \lfloor ny_n \rfloor) \otimes G^*(t)$ has density at least $\frac{12}{25} + \frac{1}{25t}$. Since $\pi(F) \leq \frac{12}{25}$, some member $F$ of $\mathcal{F}$ is a subgraph of $S_n$ for $n$ sufficiently large. For such $F \in \mathcal{F}$, there exists a subgraph $M$ of $G^*(t)$ with $|V(M)| \leq |V(F)| \leq s$ so that $F \subset (s,s,\ldots, s) \otimes M$. By Fact 2.3 and Fact 2.8, we have

$$\lambda(F) \leq \lambda((s,s,\ldots, s) \otimes M) \leq \lambda(M). \quad (4)$$

The following lemma will be proved in Section 3.1.

**Lemma 3.2** Let $M$ be any subgraph of $G^*(t)$ with $|V(M)| \leq s$. Then

$$\lambda(M) \leq \frac{2}{25} \quad (5)$$

holds.

Assuming that Lemma 3.2 is true and applying Lemma 3.2 to (4), we have

$$\lambda(F) \leq \frac{2}{25}.$$
which contradicts our choice of $F$, i.e., contradicts that $\lambda(F) > \frac{2}{25}$ for all $F \in \mathcal{F}$.

To complete the proof of Theorem 1.5, what remains is to show Lemma 3.2.

### 3.1 Proof of Lemma 3.2

By Fact 2.3, we may assume that $M$ is an induced subgraph of $G^*(t)$. Let

$$U_i = V(M) \cap V_i = \{v^i_1, v^i_2, \ldots, v^i_{s_i}\}.$$ 

So $s = s_1 + s_2 + s_3$. Let $\bar{z} = (z_1, z_2, \ldots, z_s)$ be an optimal vector for $M$. Without loss of generality, assume that $v^i_1, v^i_2, \ldots, v^i_{s+2}$ have the $s + 2$ largest weights. Then replacing the $s$ edges in $M[U_1]$ by $v^i_1v^i_2v^i_3, v^i_1v^i_2v^i_4, \ldots, v^i_1v^i_2v^i_{s+2}$ doesn’t decrease the Lagrangian. So we have the following claim similar to Claim 4.4 in [8].

**Claim 3.3** If $N$ is the 3-uniform graph formed from $M$ by removing the edges contained in $U_1$ and inserting the edges $v^i_1v^i_2v^i_3$, where $3 \leq j \leq s_1$, then $\lambda(M) \leq \lambda(N)$.

By Claim 3.3, the proof of Lemma 3.2 will be completed if we show that $\lambda(N) \leq \frac{2}{25}$. By Lemma 2.5, we can obtain an optimal vector $\bar{z}$ of $\lambda(N)$ such that $w(v^i_1) = w(v^i_2) \overset{\text{def}}{=} \frac{a}{2}, w(v^i_3) = w(v^i_4) = \cdots = w(v^i_{s_1}) \overset{\text{def}}{=} \frac{b}{s_1 - 2},$ (6)

where $w(v)$ denotes the component of $\bar{z}$ corresponding to vertex $v$.

Let $c, d$ be the sum of the components of $\bar{z}$ corresponding to all vertices in $U_2$ and $U_3$, respectively. Note that

$$a + b + c + d = 1.$$ 

Then

$$\lambda(N) \leq \left(\frac{a^2}{4} + ab + \frac{b^2}{2}\right)c + (a + b)cd + \frac{c^2d}{2} + \frac{a^2}{4}b = \lambda(a, b, c, d).$$

From now on, we assume that $(a, b, c, d)$ is an optimal vector for $\lambda(a, b, c, d)$.

If $c = 0$, then

$$\lambda(a, b, c, d) = \frac{a^2b}{4} \leq \frac{1}{8} \left(\frac{a + a + 2b}{3}\right)^3 \leq \frac{1}{27}.$$ 

So we may assume that $c > 0$. If $a = 0$, then

$$\lambda(a, b, c, d) = \frac{b^2c}{2} + bcd + \frac{c^2d}{2} \leq \lambda.$$ 

If $b = 0$, then $\lambda = \frac{c^2d}{2} \leq \frac{2}{27}$. Similarly we have $d > 0$. So we may assume that $b, c, d > 0$ in this case. By Theorem 2.1 we have

$$\frac{\partial \lambda}{\partial b} = \frac{\partial \lambda}{\partial c} = \frac{\partial \lambda}{\partial d} = \cdots$$
so

\[ bc + cd = \frac{b^2}{2} + bd + cd = \frac{c^2}{2} + bc. \]

Combining with \( b + c + d = 1 \), we have \( b = c = 2d = 0.4 \), and \( \lambda = \frac{2}{25} \). So we may assume that \( a > 0 \).

If \( b = 0 \), then

\[ \lambda(a, b, c, d) = \frac{a^2c}{4} + acd + \frac{c^2d}{2} < \frac{a^2c}{2} + acd + \frac{c^2d}{2} \leq \frac{2}{25} \]

as we have shown that \( \frac{b^2}{2} + bcd + \frac{c^2}{2} \leq \frac{2}{25} \). So we may assume that \( b > 0 \).

We will prove that \( d > 0 \) next. If \( d = 0 \), then

\[ \lambda(a, b, c, d) = \left( \frac{a^2}{4} + ab + \frac{b^2}{2} \right)c + \frac{a^2}{4}b = \lambda. \]

By Theorem 2.1, we have

\[ \frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial b}. \]

So

\[ \left( \frac{a}{2} + b \right)c + \frac{ab}{2} = (a + b)c + \frac{a^2}{4}, \]

i.e., \( a = 2b - 2c \). Since \( a + b + c = 1 \), then \( c = 3b - 1 \) and \( a = 2 - 4b \). So

\[ \lambda \leq \left( \frac{a^2}{4} + ab + \frac{b^2}{2} \right)c + \frac{a^2}{4}b \]

\[ = \frac{11b^3}{2} - \frac{21b^2}{2} + 6b - 1 = f(b). \]

\[ f'(b) = \frac{33b^2}{2} - 21b + 6. \]

Since \( a, c > 0 \), then \( \frac{1}{3} \leq b \leq \frac{1}{2} \). Therefore \( f(b) \) is increasing in \( \left[ \frac{1}{3}, \frac{7 - \sqrt{5}}{11} \right] \) and decreasing in \( \left[ \frac{7 - \sqrt{5}}{11}, \frac{1}{2} \right] \). Then \( \lambda < f \left( \frac{7 - \sqrt{5}}{11} \right) < 0.076 \).

So we assume that \( a, b, c, d > 0 \), then we have

\[ \frac{\partial \lambda(a, b, c, d)}{\partial a} = \left( \frac{a}{2} + b \right)c + cd + \frac{ab}{2}, \]

\[ \frac{\partial \lambda(a, b, c, d)}{\partial b} = (a + b)c + cd + \frac{a^2}{4}, \]

\[ \frac{\partial \lambda(a, b, c, d)}{\partial c} = \frac{a^2}{4} + ab + \frac{b^2}{2} + ad + bd + cd, \]

\[ \frac{\partial \lambda(a, b, c, d)}{\partial d} = ac + bc + \frac{c^2}{2}, \]

\[ d = 1 - a - b - c. \]

By Theorem 2.1, we have

\[ \frac{\partial \lambda(a, b, c, d)}{\partial a} = \frac{\partial \lambda(a, b, c, d)}{\partial b}, \]

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and we get $a = 2b - 2c$. Therefore $d = 1 - a - b - c = 1 - 3b + c$. By
\[
\frac{\partial \lambda(a, b, c, d)}{\partial b} = \frac{\partial \lambda(a, b, c, d)}{\partial d},
\]
we get $\frac{c^2}{2} = cd + \frac{a^2}{4} = c - 3bc + c^2 + b^2 - 2bc + c^2$, so
\[
c = \frac{5b - 1 \pm \sqrt{19b^2 - 10b + 1}}{3}.
\]
By
\[
\frac{\partial \lambda(a, b, c, d)}{\partial b} = \frac{\partial \lambda(a, b, c, d)}{\partial c},
\]
we get
\[
c = \frac{13b^2 - 6b}{8b - 4}.
\]
Therefore,
\[
\frac{13b^2 - 6b}{8b - 4} = \frac{5b - 1 \pm \sqrt{19b^2 - 10b + 1}}{3}.
\]
By direct calculation, we have $(-b^2 + 10b - 4)^2 = \left( \pm (8b - 4) \sqrt{19b^2 - 10b + 1} \right)^2$. Simplifying, we get
\[
9b(5b - 2)(9b - 4)(3b - 2) = 0.
\]
If $b = \frac{2}{5}$, then $c = \frac{13b^2 - 6b}{8b - 4} = \frac{2}{5}$ and $a = 2b - 2c = 0$, a contradiction.
If $b = \frac{4}{5}$, then $c = \frac{13b^2 - 6b}{8b - 4} = \frac{2}{5}$, $a = 2b - 2c = \frac{4}{5}$ and $d = -\frac{1}{5}$, a contradiction.
If $b = \frac{3}{5}$, then $c < \frac{1}{3}$ and $a = 2b - 2c > \frac{2}{3}$ and $d < 0$, a contradiction. \(\square\)

4 Proof of Theorem 1.5

Let $B(2k, n)$ be the $3$-graph with vertex set $[n]$ and edge set $E(B(2k, n)) = \{ e \in \binom{[n]}{3} : e \cap [2k] \neq \emptyset \}$. Let $\alpha_k = \frac{2k^2 + 4k^3 + \sqrt{4k^2 - 1}}{(2k^2 + 1)^2}$. We first show that $\alpha_k = 6 \lim_{n \to \infty} \lambda(B(2k, n))$.

Let $\bar{x} = \{x_1, x_2, \ldots, x_n\}$ be an optimal vector of $\lambda(B(2k, n))$. Let $x_1 + x_2 + \cdots + x_{2k} = a$ and
$b = 1 - a$. Then
\[
\lim_{n \to \infty} \lambda(B(2k, n)) = \left( \frac{a}{2k} \right)^3 \binom{2k}{3} + \left( \frac{a}{2k} \right)^2 \binom{2k}{2} (1 - a) + a \left( \frac{1 - a}{2} \right)^2 = f(a)
\]
\[
f'(a) = \left( \frac{1}{4k^2} + \frac{1}{2} \right) a^2 - \left( \frac{1}{2k} + 1 \right) a + \frac{1}{2}.
\]
Note that $f(a)$ is increasing in $[0, \frac{2k^2 + k - \sqrt{4k - 1}}{2k^2 + 1}]$ and decreasing in $[\frac{2k^2 + k - \sqrt{4k - 1}}{2k^2 + 1}, 1]$. Therefore
\[
f\left( \frac{2k^2 + k - \sqrt{4k - 1}}{2k^2 + 1} \right) = \frac{2k - 6k^3 + 4k^4 - k \sqrt{4k - 1} + 4k^2 \sqrt{4k - 1}}{6(2k^2 + 1)^2} = \frac{\alpha_k}{6}.
\]
Since $k \geq 1$ and $4k - 1$ is not a square number (a square number is 0 or 1 mod(4)), then $\alpha_k$ is an irrational number.

**Proof of Theorem 1.5.** Suppose that $\alpha_k$ is a jump. By Lemma 2.9 there exists a finite collection $\mathcal{F}$ of 3-uniform graphs satisfying the following:

i) $\lambda(F) > \frac{\alpha_k}{6}$ for all $F \in \mathcal{F}$, and

ii) $\pi(F) \leq \alpha_k$.

Let $G(t) = (V, E)$ be the 3-uniform defined as follows. The vertex set $V = V_1 \cup V_2 \cdots \cup V_{2k} \cup V_{2k+1}$, where $|V_1| = |V_2| = \cdots = |V_{2k}| = \frac{2k+1-\sqrt{4k-1}}{4k^2+2}t$ and $|V_{2k+1}| = \frac{k\sqrt{4k-1}-1-k}{2k^2+1}t$. The edge set of $G(t)$ is

$$
\bigcup_{1 \leq i_1 < i_2 < i_3 \leq 2k} (V_{i_1} \times V_{i_2} \times V_{i_3}) \bigcup_{1 \leq i_1 < i_2 \leq 2k} (V_{i_1} \times V_{i_2} \times V_{2k+1}) \bigcup_{1 \leq i_1 \leq 2k} (V_{i_1} \times \left(\frac{V_{2k+1}}{2}\right)).
$$

Then

$$
|E(G(t))| = \frac{2k - 6k^3 + 4k^4 - k\sqrt{4k-1} + 4k^2\sqrt{4k-1}}{6(2k^2 + 1)^2}t^3 + \frac{-3k - 6k^2 + 18k^3 + 3k\sqrt{4k-1} - 6k^2\sqrt{4k-1} - 6k^3\sqrt{4k-1}}{6(2k^2 + 1)^2}t^2.
$$

Let $\vec{u} = (u_1, \ldots, u_t)$, where $u_i = \frac{1}{t}$ for $1 \leq i \leq t$, then

$$
\lambda(G(t)) \geq \lambda(G(t), \vec{u}) = \frac{|E(G)|}{t^3} = \frac{2k - 6k^3 + 4k^4 - k\sqrt{4k-1} + 4k^2\sqrt{4k-1}}{6(2k^2 + 1)^2} + \frac{-3k - 6k^2 + 18k^3 + 3k\sqrt{4k-1} - 6k^2\sqrt{4k-1} - 6k^3\sqrt{4k-1}}{6(2k^2 + 1)^2t}
$$

(7)

$$
= \frac{\alpha_k - c_0}{6} - \frac{c_0}{t},
$$

where $c_0 = \frac{3k + 6k^2 - 18k^3 - 3k\sqrt{4k-1} + 6k^2\sqrt{4k-1} + 6k^3\sqrt{4k-1}}{6(2k^2 + 1)^2} > 0$.

Set $s = \max_{F \in \mathcal{F}} |V(F)|$ and $c = k$. Let $r = 3$ in Lemma 3.1 and $t_0(s, k)$ be given as in Lemma 3.1. Take an integer $t > \frac{k\sqrt{4k-1} - 1 - k}{2k^2+1}t_0$. The 3-uniform graph $G^*(t)$ is obtained by adding $A(s, k)$ to the 3-uniform hypergraph $G(t)$ in $V_{2k+1}$. Then

$$
\lambda(G^*(t)) \geq \frac{|E(G^*(t))|}{t^3}.
$$
In view of the construction of $G^*(t)$ and equation (7), we have

$$\frac{|E(G^*(t))|}{t^3} \geq \frac{|E(G(t))|}{t^3} + \frac{k(\frac{\sqrt{4k-1}+1-k}{2k^2+1})^2}{6} \frac{2k-6k^3+4k^4-k\sqrt{4k-1}+4k^2\sqrt{4k-1}}{(2k^2+1)^2} + \frac{3k-18k^2+18k^3+24k^4+3k\sqrt{4k-1}+6k^2\sqrt{4k-1}-18k^3\sqrt{4k-1}}{6(2k^2+1)t} \geq \frac{1}{6} \left(\frac{2k-6k^3+4k^4-k\sqrt{4k-1}+4k^2\sqrt{4k-1}}{(2k^2+1)^2}\right) + \frac{c_1}{t} = \frac{\alpha_k + c_1}{t} \tag{8}$$

where $t$ is a sufficiently large integer and $c_1 = \frac{3k-18k^2+18k^3+24k^4+3k\sqrt{4k-1}+6k^2\sqrt{4k-1}-18k^3\sqrt{4k-1}}{6(2k^2+1)t} > 0$.

Now suppose $\vec{y} = (y_1, y_2, ..., y_t)$ is an optimal vector of $\lambda(G^*(t))$. Let $n$ be large enough. By Remark 2.7, 3-uniform graph $S_n = (\{ny_1, \ldots, |ny_t|\}) \otimes G^*(t)$ has density at least $\alpha_k + c_2$. Since $\pi(F) \leq \alpha_k$, some member $F$ of $\mathcal{F}$ is a subgraph of $S_n$ for $n$ sufficiently large. For such $F \in \mathcal{F}$, there exists a subgraph $M$ of $G^*(t)$ with $|V(M)| \leq |V(F)| \leq s$ so that $F \subset (n, n, \ldots, n) \otimes M$. By Fact 2.3 and Fact 2.8 we have

$$\lambda(F) = \lambda((n, n, \ldots, n) \otimes M) \leq \lambda(M). \tag{9}$$

Theorem 1.5 will follow from the following lemma to be proved in Section 4.1.

**Lemma 4.1** Let $M$ be any subgraph of $G^*(t)$ with $|V(M)| \leq s$. Then

$$\lambda(M) \leq \frac{1}{6} \alpha_k \tag{10}$$

holds.

Assuming that Lemma 4.1 is true and applying Lemma 4.1 to (9), we have

$$\lambda(F) \leq \frac{1}{6} \alpha_k$$

which contradicts our choice of $F$, i.e., contradicts that $\lambda(F) > \frac{1}{6} \alpha_k$ for all $F \in \mathcal{F}$. \hfill $\Box$

To complete the proof of Theorem 1.5 what remains is to show Lemma 4.1

### 4.1 Proof of Lemma 4.1

By Fact 2.3 we may assume that $M$ is an induced subgraph of $G^*(t)$. Let

$$U_i = V(M) \cap V_i = \{v_{i1}, v_{i2}, \ldots, v_{is_i}\}.$$  

So $s = s_1 + \cdots + s_{2k+1}$.

Similar to Claim 3.3 we have

**Claim 4.2** If $N$ is the 3-uniform graph formed from $M$ by removing the edges contained in $U_{2k+1}$ and inserting the edges $v_j^{2k+1}v_j^{2k+1}v_j^{2k+1}$, where $3 \leq j \leq s_{2k+1}$, then $\lambda(M) \leq \lambda(N)$.  

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By Claim 4.2, the proof of Lemma 4.1 will be completed if we show that \( \lambda(N) \leq \alpha_k \). By Lemma 2.3, there exists an optimal vector \( \vec{z} \) of \( \lambda(N) \) such that

\[
\begin{align*}
    w(v_1^{2k+1}) &= w(v_2^{2k+1}) = \frac{a}{2}, \quad w(v_3^{2k+1}) = w(v_4^{2k+1}) = \cdots = w(v_{2k+1}^{2k+1}) = \frac{b}{s_{2k+1} - 2},
\end{align*}
\]

where \( w(v) \) denotes the component of \( \vec{z} \) corresponding to vertex \( v \). Let \( w_1 \) be the sum of the components of \( \vec{z} \) corresponding to all vertices in \( U_i \). Then

\[
\lambda(N) \leq \left( \frac{w_1}{2k} \right)^3 \left( \frac{2k}{3} \right) + \left( \frac{w_1}{2k} \right)^2 \left( \frac{2k}{2} \right) (1 - w_1) + w_1 \left( \frac{a^2}{4} + ab + \frac{b^2}{2} \right) + \frac{a^2}{4} b,
\]

where \( w_1 + a + b = 1 \).

Note that if \( b \leq w_1 \) or \( a = 0 \) or \( w_1 \geq \frac{1}{2} \), then

\[
\begin{align*}
    \lambda(N) &\leq \left( \frac{w_1}{2k} \right)^3 \left( \frac{2k}{3} \right) + \left( \frac{w_1}{2k} \right)^2 \left( \frac{2k}{2} \right) (1 - w_1) + w_1 \left( \frac{a^2}{4} + ab + \frac{b^2}{2} \right) \\
    &\quad + w_1 \left( \frac{a^2}{4} + a(1 - w_1) - (1 - w_1 - a)^2 \right) + \frac{a^2}{4} (1 - w_1 - a) \\&\leq \lim_{n \to \infty} \lambda(B(2k, n)) = \frac{\alpha_k}{6}.
\end{align*}
\]

So we may always assume that \( w_1 < \frac{1}{2} \). Since \( b = 1 - w_1 - a \), then

\[
\begin{align*}
    \lambda(N) &\leq \left( \frac{w_1}{2k} \right)^3 \left( \frac{2k}{3} \right) + \left( \frac{w_1}{2k} \right)^2 \left( \frac{2k}{2} \right) (1 - w_1) + w_1 \left( \frac{a^2}{4} + a(1 - w_1) - (1 - w_1 - a)^2 \right) + \frac{a^2}{4} (1 - w_1 - a) \\&\leq f(a),
\end{align*}
\]

where \( w_1 + a < 1 \).

\[
f'(a) = w_1 \left( \frac{a}{2} + (1 - w_1 - a) - a - (1 - w_1 - a)^2 \right) + \frac{a}{2} (1 - w_1 - a) - \frac{a^2}{4} = -3a^2 + \frac{a}{2} - aw_1.
\]

Note that \( w_1 < \frac{1}{2} \), then \( f(a) \) is increasing in \([0, \frac{2 - 4w_1}{3}]\) and decreasing in \([\frac{2 - 4w_1}{3}, 1]\). So

\[
f(a) \leq f\left( \frac{2 - 4w_1}{3} \right) = \left( \frac{w_1}{2k} \right)^3 \left( \frac{2k}{3} \right) + \left( \frac{w_1}{2k} \right)^2 \left( \frac{2k}{2} \right) (1 - w_1) + \frac{11w_1^3}{54} - \frac{5w_1^3}{9} + \frac{5w_1}{18} + \frac{1}{27} = g(w_1).
\]

Then \( g'(w_1) = (\frac{1}{18k^2} - \frac{7}{18})w_1^3 - (\frac{1}{2k} + \frac{1}{3})w_1 + \frac{5}{18} \). Solving \( g'(w_1) = 0 \), we obtain that \( w_1 = \frac{5 - \frac{7}{18} + \sqrt{\frac{49}{4} + \frac{1}{36}} - \frac{1}{3}}{2k} \). Note that \( -\sqrt{\frac{49}{4} + \frac{1}{36} - \frac{1}{3}} < 0 \) and \( \frac{1}{4k^2} - \frac{7}{18} < 0 \). We will show that \( \frac{\frac{49}{4} + \frac{1}{36} - \frac{1}{3}}{2k} > \frac{1}{2} \). It’s
sufficient to show that \(\sqrt{\frac{36}{81} + \frac{1}{9k} - \frac{1}{36k^2}} > \frac{1}{2} + \frac{1}{2k} - \frac{1}{4k^2}\). Note that

\[
\sqrt{\frac{36}{81} + \frac{1}{9k} - \frac{1}{36k^2}} > \frac{2}{3} > \frac{23}{36} > \frac{1}{2} + \frac{1}{2k} - \frac{1}{4k^2}
\]

holds for \(k \geq 3\). As for \(k = 2\), we have

\[
\sqrt{\frac{36}{81} + \frac{1}{9k} - \frac{1}{36k^2}} = \sqrt{\frac{165}{18}} > \frac{11}{16} = \frac{1}{2} + \frac{1}{2} - \frac{1}{36}.
\]

Since earlier discussion allows us to assume that \(w_1 < \frac{1}{2}\), therefore \(g(w_1)\) is increasing in \([0, \frac{1}{2}]\). Note that

\[
\left.\frac{11w_1^3}{54} - \frac{5w_1^2}{9} + \frac{5w_1}{18} + \frac{1}{27}\right|_{w_1 = \frac{1}{2}} = \frac{1}{16} = \frac{1}{2} (1 - w_1)^2 |_{w_1 = \frac{1}{2}}.
\]

Then

\[
\lambda(N) \leq g\left(\frac{1}{2}\right) = \left(\frac{w_1}{2k}\right)^3 \left(\frac{2k}{3}\right) + \left(\frac{w_1}{2k}\right)^2 \left(\frac{2k}{2}\right) (1 - w_1) + \left(\frac{w_1}{2k}\right) \left(\frac{2k}{2}\right) (1 - w_1) + \left(\frac{w_1}{2k}\right) \left(\frac{2k}{2}\right) (1 - w_1) + \frac{1}{2} |_{w_1 = \frac{1}{2}}
\]

\[
= \left(\frac{w_1}{2k}\right)^3 \left(\frac{2k}{3}\right) + \left(\frac{w_1}{2k}\right)^2 \left(\frac{2k}{2}\right) (1 - w_1) + \frac{1}{2} |_{w_1 = \frac{1}{2}}
\]

\[
\leq \lim_{n \to \infty} \lambda(B(2k, n)) = \frac{\alpha_k}{6}.
\]

\[
\square
\]

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