Adapted gauge to a quasilocal measure of the black holes recoil

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(Dated: March 6, 2020)

We explore different gauge choices in the moving puncture formulation in order to improve the accuracy of a linear momentum measure evaluated on the horizon of the remnant black hole produced by the merger of a binary. In particular, motivated by constant values studies, we design a gauge via a variable shift parameter $m\eta(f(t))$ such that it takes a low asymptotic (and at the orbiting punctures) value, while about the standard value of 2 at the final hole horizon. This choice then follows the remnant black hole as it moves due to its net recoil velocity. We find that this choice keeps the accuracy of the binary evolution and, once the asymptotic value of the parameter $m\eta$ is chosen about or below 0.5, it produces more accurate results for the recoil velocity than the corresponding evaluation of the radiated linear momentum at infinity, for typical numerical resolutions. We also find that the choice of the $\delta_t$-gauge (at our working resolutions) is more accurate in this regard of computing recoil velocities than the $\delta_\eta$-gauge. Detailed studies of an unequal mass $q = m_1/m_2 = 1/3$ nonspinning binary are provided and then verified for other mass ratios ($q = 1/2, 1/3$) and spinning ($q = 1$) binary black hole mergers.

PACS numbers: 04.25.dg, 04.25.Nx, 04.30.Db, 04.70.Bw

I. INTRODUCTION

The discovery by numerical relativity computations [1, 2] that binary black hole mergers may impart thousand of kilometers per second speeds to the final black hole remnant had an immediate impact on the interest of observational astrophysics to search for signatures of such recoil for supermassive black holes in merged galaxies (See [3] for an early review). The interest in searching for observational effects extends to nowadays [4–6], including its incidence in statistical distributions [7–9] and binary formation channels as well as cosmological consequences [10].

More recently theoretical explorations evaluate the possibility to directly detect the effects of recoil on the gravitational waves observed by LIGO [11] and LISA [12, 13]. The use of numerical relativity waveforms to directly compare with the observation of gravitational waves require accurate modeling and good coverage of the parameter space [14]. There are already successful descriptions for the GW150914 [15, 16] and GW170104 [17] events and the analysis of the rest of the O1/O2 events [18] is well underway (Healy et al. 2020a, paper in preparation).

The accurate modeling of the final remnant of the merger of binary black holes is also of high interest for applications to gravitational waves modeling and tests of gravity as a consistency check [19, 20]. The computation of the final remnant mass and spin can be performed in three independent ways, a fit to the quasinormal modes of the final remnant Kerr black hole [21, 22], a computation of the energy and angular momentum carried away by the gravitational radiation to evaluate the deficit from the initial to final mass and spins, and a quasilocal computation of the horizon mass and spin using the isolated horizon formulas [23]. Comparison of the three methods has been carried out in [11, 25], concluding that at the typical resolutions used in production numerical relativity simulations the horizon quasilocal measures are an order of magnitude more accurate than the radiation or quasinormal modes fittings.

This lead to very accurate modeling of the final mass and spins from their initial binary parameters. In particular for nonprecessing binaries, the modeling [26] warrants errors typically 0.03% for the mass and 0.16% for the spin. Meanwhile the modeling of the final recoils leads to errors of the order of 5% since radiation of linear momentum is used to evaluate them. A similar accurate modeling of the recoil could be attempted by the use of a horizon quasilocal measure.

In reference [27] a quasi-local formula for the linear momentum of black-hole horizons was proposed, inspired by the formalism of quasi-local horizons. This formula was tested using two complementary configurations: (i) by calculating the large orbital linear momentum of the two black holes in an orbiting, unequal-mass, zero-spin, quasi-circular binary and (ii) by calculating the very small recoil momentum imparted to the remnant of the head-on collision of an equal-mass, anti-aligned-spin binary. The results obtained were consistent with the horizon trajectory in the orbiting case, and consistent with the radiated linear momentum for the much smaller head-on recoil velocity. A key observation we will explore in this paper is the dependence of the accuracy on a gauge parameter used in our simulations.

This paper is organized as follows, In Section II we study in detail the effects of choosing different (constant) values of $\eta$, the damping parameter in the shift evolution equation on the accuracy of the quasilocal measure of the
horizon linear momentum proposed in \cite{27}. We discuss in detail a prototype case of a nonspinning \( q = 1/3 \) binary. Other unequal mass cases and one spinning case are verified as well. In Section \( \text{III} \) we perform additional studies of the shift evolution equation for the alternative \( \partial_\eta \)-gauge, variable \( \eta \), and apply what we learned in the previous section to develop a variable shift parameter \( \eta \) and to more extreme unequal mass binary black hole mergers. In Section \( \text{IV} \) we discuss the benefits of the using of different values of \( \eta \) form the standard \( \eta = 2 \) for generic simulations, in particular for those that involve an accurate computation of the remnant recoil and we also conclude by noting the advantage of keeping the \( \partial_\eta \)-gauge for evolutions over the \( \partial_\theta \)-gauge.

II. NUMERICAL TECHNIQUES

Since the 2005 breakthrough work \cite{28} We obtain accurate, convergent waveforms and horizon parameters by evolving the BSSNOK \cite{29,31} system in conjuction with a modified 1+log lapse and a modified Gamma-driver shift condition \cite{28,32}.

\[
\partial_\alpha (\partial_k - \beta^i \partial_i) \alpha = -2\alpha K, \quad (1)
\]

\[
\partial_t \beta^a = \frac{3}{4} \hat{\Gamma}^a - \eta (x^k, t) \beta^a. \quad (2)
\]

with an initial vanishing shift and lapse \( \alpha(t = 0) = 2/(1 + \psi^3_{BL}) \).

An alternative moving puncture evolution can be achieved \cite{33} by choosing \cite{34}

\[
\partial_\alpha (\partial_k - \beta^i \partial_i) \alpha = -2\alpha K, \quad (3)
\]

\[
\partial_0 \beta^a = (\partial_t - \beta^i \partial_i) \beta^a = \frac{3}{4} \hat{\Gamma}^a - \eta (x^k, t) \beta^a. \quad (4)
\]

In the subsequent, we will refer to this first order equations for the shift (2) as the \( \partial_\eta \)-gauge and to (4) as the \( \partial_\theta \)-gauge.

The parameter \( \eta \) (with dimension of one-over-mass: \( 1/m \)) in the shift equation regulates the damping of the gauge oscillations and traditionally has been given the value of \( \eta = 2/m \) as a compromise between the accuracy and stability of binary black hole evolutions. We have found in \cite{35} that coordinate dependent measurements, such as spin and linear momentum direction, become more accurate as \( \eta \) is reduced (and resolution \( h \to 0 \)). However, if \( \eta \) is too small (\( \eta \ll 1/m \)), the runs may become unstable. Similarly, if \( \eta \) is too large (\( \eta \gg 10/m \)), then grid stretching effects can cause the remnant horizon to continuously grow, eventually leading to an unacceptable loss in accuracy at late times.

We use the TWOPOUNCURES \cite{35} thorn to compute initial data. We evolve these black-hole-binary data-sets using the LAZEV \cite{37} implementation of the moving puncture formalism \cite{28}. We use the Carpet \cite{38,39} mesh refinement driver to provide a ‘moving boxes’ style mesh refinement and we use AHFINDERDIRECT \cite{40} to locate apparent horizons. We compute the magnitude of the horizon spin using the isolated horizon (IH) algorithm detailed in Ref. \cite{11} (as implemented in Ref. \cite{42}). Once we have the horizon spin, we can calculate the horizon mass via the Christodoulou formula \( m_\text{H} = \sqrt{m_\text{irr}^2 + S_H^2 / (4m_\text{irr}^2)} \), where \( m_\text{irr} = \sqrt{A / (16\pi)} \) and \( A \) is the surface area of the horizon. We measure radiated energy, linear momentum, and angular momentum, in terms of \( \psi_4 \), using the formulae provided in Refs. \cite{43,44} and extrapolation to to \( \mathcal{I}^+ \) is performed with the formulas given in Ref. \cite{45}.

In Reference \cite{27} we introduced an alternative quasilocal measurement of the linear momentum of the individual (and final) black holes in the binary that is based on the coordinate rotation and translation vectors

\[
P_{[i]} = \frac{1}{8\pi} \int_{AH} \xi_{[i]}^a R^b (K_{ab} - K_{\gamma ab}) d^2V, \quad (5)
\]

where \( K_{ab} \) is the extrinsic curvature of the 3D-slice, \( d^2V \) is the natural volume element intrinsic to the horizon, \( R^a \) is the outward pointing unit vector normal to the horizon on the 3D-slice, and \( \xi_{[i]}^a = \delta_{[i]}^a \).

We tested this formula using two complementary configurations: (i) by calculating the large orbital linear momentum of the two unequal-mass (\( q = 1/3 \), nonspinning, black holes in a quasi-circular orbit and (ii) by calculating the very small recoil momentum imparted to the remnant of the head-on collision of an equal-mass, anti-aligned-spin binary. We obtain results consistent with the horizon trajectory in the orbiting case, and consistent with the net radiated linear momentum for the much smaller head-on recoil velocity when we reduced the gauge parameter from \( m\eta = 2 \) to \( m\eta = 1 \).

Here we explore this initial results in much more detail, allowing for even smaller values of \( \eta \) and assessing convergence of the results with both, numerical resolution and values of \( \eta \to 0 \). This will allow us to assess when the quasilocal measure of linear momentum (5) can be considered more accurate than the measure of radiated linear momentum at \( \mathcal{I}^+ \).

A. Results for a \( q = 1/3 \) nonspinning binary

As a prototypical case of study we will consider a binary with mass ratio \( q = m_1/m_2 = 1/3 \) and spinless black holes starting at an initial coordinate separation \( D = 9m \), with \( m = m_1 + m_2 \) the total mass of the system. From this separation the binary performs about 6 orbits before the merger into a single final black hole at around \( t = 725m \).

The final mass and final spin are measured very accurately by the horizon quasilocal formulas \cite{24,42}; figs. 1 provide a visualization of their respective values after merger into the final settling black hole remnant. They display smaller variations versus time with increasing resolutions.
FIG. 1. The horizon measure of the mass and spin after merger of a $q = 1/3$ nonspinning binary versus time for $m\eta = 2$ at resolutions n100, n120, n140.

TABLE I. Difference between final black hole and ADM masses. Values for $m\eta = 0.0$ are extrapolated.

| resolution | $m\eta = 2.0$ | $m\eta = 1.0$ | $m\eta = 0.5$ | $m\eta = 0.0$ |
|------------|----------------|----------------|----------------|----------------|
| 1/100      | 0.0204587      | 0.0204567      | 0.0204678      | 0.0204584      |
| 1/120      | 0.0204562      | 0.0204525      | 0.0204495      | 0.0204511      |
| 1/140      | 0.0204540      | 0.0204547      | 0.0204511      | 0.0204541      |

In Figs. 1 the convergence of the Hamiltonian constraint (Momentum constraints show a very similar convergent behavior) and the merger gravitational waveforms $(\ell, m) = (2, 2)$-mode are displayed. Both show highly convergent behavior, therefore they are well in the numerical convergence regime and resolving the binary system accurately.

As shown in the Tables I and II the computed final mass and spin of the remnant black hole are well in the convergence regime at typical computation resolutions [16, 46].

TABLE II. Difference between final black hole spin and initial ADM angular momentum. Values for $m\eta = 0.0$ are extrapolated.

| resolution | $m\eta = 2.0$ | $m\eta = 1.0$ | $m\eta = 0.5$ | $m\eta = 0.0$ |
|------------|----------------|----------------|----------------|----------------|
| 1/100      | -0.1918491     | -0.1918684     | -0.1918335     | -0.191856      |
| 1/120      | -0.1918424     | -0.1918509     | -0.1918414     | -0.1918464     |
| 1/140      | -0.1918336     | -0.1918402     | -0.1918365     | -0.1918378     |

As shown in the Tables I and II, the computed final mass and spin of the remnant black hole are well in the convergence regime at typical computation resolutions [16, 46].

Table I displays the differential between the initial total ADM mass minus final horizon mass and Table II displays the loss of ADM angular momentum from its initial value to the final horizon spin for different resolutions and different $(\eta$-values) gauges. The computations give consistent values to 5-decimal places for each resolution, showing we are deep in the convergence regime and also versus $\eta$, showing (as expected) that those computations are gauge-invariant.

The corresponding computation of radiated energy and angular momentum from the waveforms extrapolated to an observer at infinity and summed over all $(\ell, m)$-modes up to $\ell = 6$ are displayed in Tables III and IV showing consistent approximate 3rd order convergence for the three resolutions n100, n120, and n140. When using the horizon values as reference, the convergence order increases, and is over 4th order for the radiated angular momentum. In all cases, the computations are consistent.
in the first 3 digits. While taking as reference the horizon values the convergence is over 4th order. Consistent first 4-digits are computed in all cases. In particular, very weak dependence on $\eta$ is found, again as expected on the ground of gauge invariance of the gravitational waveform extrapolated to an observer at infinite location.

Tables III and IV also show that both radiative quantities, energy and angular momentum show very small variations with respect to the extrapolated $\eta \to 0$ values, as expected from gauge invariant quantities.

Since the formula (5) is not gauge invariant when applied to the horizon of the final black hole we expect to find stronger variation with $\eta$ when we use it to evaluate the linear momentum of the remnant. We will pursue this exploration in more detail next in order to assess what values of $\eta$ allow us to compute the recoil velocity of the final black hole with a good accuracy. We are interested in particular, for our typical numerical simulations resolutions, what values of $\eta$ can produce more accurate values of the recoil from the horizon by use of (6) than from the evaluation of the radiated linear momentum at infinity.

Our starting point is the gauge choices that we have been using regularly in our systematic studies of binary black hole mergers ($m\eta = 2$ and Eqs. (2)) and numerical resolutions labeled by the resolution at the extraction level of radiation as $n_{100}$, $n_{120}$, $n_{140}$, corresponding to wavezone resolutions of $h = 1/1.00 m, 1/1.20 m, 1/1.40 m$, respectively [16] [40]. We use 10 levels of refinement with an outer boundary at 400M. For each of these three resolutions we add a set of simulations by decreasing $\eta$ by a factor of two, i.e. $\eta = 1/2, 1/4, 1/8$. The results of these nine simulations are displayed in Fig. 3. For $\eta = 2/m$, the curves are very fiat versus time after the merger with the higher resolution run, $n_{140}$, notably stable. However their values for the evaluation of the recoil fall short compared to the estimate coming from the extrapolation of the radiative linear momentum to infinite resolution, represented by the solid black lines at about 177km/s.

The progression towards smaller $\eta$ shows closer agreement with that extrapolated value. The time dependence shows variations as we approach the smaller $\eta$ but still converging with resolution towards the expected 177km/s value and flatter for $n_{140}$, but clearly the limit $\eta \to 0$ requires much higher resolutions, as shown in Fig. 4. In our regime, reaching $\eta = 1/m$ or $\eta = 0.5/m$ seems a good compromise of accuracy versus cost of the simulation.

Fig. 4 shows the progression of $m\eta = 2.0, 1.5, 1.0, 0.5, 0.25$ for simulations with resolution $n_{140}$. Notably they lie in a roughly linear convergence towards the expected higher recoil velocity value 177km/s, but as we reach the smaller $m\eta = 0.25$ value it overshoots slightly, an effect of the required higher resolution needed to resolve accurately smaller values of $\eta$. In what follows we will restrict ourselves to values of $m\eta = 2.0, 1.0, 0.5$ to make sure we are in a convergence regime for our standard resolutions $n_{100}, n_{120}, n_{140}$.

Note that we have verified that the simulation with $n_{140}$ and $\eta = 0$ does not crash, but leads to inaccurate results.

The radiation of linear momentum in terms of the Weyl scalar $\psi_4$, as given by the formulas in [47], can be computed in a similar fashion as we compute the energy and angular momentum radiated. For this study, we do not remove the initial burst of spurious radiation from the linear momentum calculation since we are interested in

### Table III. Energy radiated away in gravitational waves up to $\ell = 6$ for the $q = 1/3$ nonspinning binary. Values for $m\eta = 0.0$ are extrapolated. Convergence order versus resolution and versus horizon values.

| resolution | $m\eta = 2.0$ | $m\eta = 1.0$ | $m\eta = 0.5$ | $m\eta = 0.0$ |
|------------|---------------|---------------|---------------|---------------|
| 1/100      | 0.0201710     | 0.0201723     | 0.0201713     | 0.0201718     |
| 1/120      | 0.0202959     | 0.0202978     | 0.0202945     | 0.0202966     |
| 1/140      | 0.0203590     | 0.0203591     | 0.0203563     | 0.0203590     |
| Infinite   | 0.0204650     | 0.0204542     | 0.0204577     | -             |
| order      | 3.031         | 3.228         | 3.084         | -             |
| AH-order   | 3.307         | 3.234         | 3.177         | -             |

### Table IV. Angular momentum radiated away in gravitational waves up to $\ell = 6$ for the $q = 1/3$ nonspinning binary. Values for $m\eta = 0.0$ are extrapolated. Convergence order versus resolution and versus horizon values.

| resolution | $m\eta = 2.0$ | $m\eta = 1.0$ | $m\eta = 0.5$ | $m\eta = 0.0$ |
|------------|---------------|---------------|---------------|---------------|
| 1/100      | -0.1907468    | -0.1907321    | -0.1907042    | -0.1907632    |
| 1/120      | -0.1912766    | -0.1912961    | -0.1912509    | -0.1912825    |
| 1/140      | -0.1915672    | -0.1915651    | -0.1915477    | -0.1915675    |
| Infinite   | -0.1921693    | -0.1919591    | -0.1921454    | -             |
| order      | 2.555         | 3.376         | 2.6161        | -             |
| AH-order   | 4.785         | 4.503         | 4.6200        | -             |
are $3\%$ away for $n_{140}$. The horizon evaluations lying
pute recoils, since the corresponding radiative quantities
nearly within $1\%$. This provides an effective way to com-
value by a few percent, but the values at $m_{\eta}$
their extrapolation to
values. We observe
variations with respect to the gauge choices. We observe
close to a linear dependence on $\eta$ of the recoil values.
their extrapolation to $\eta \to 0$ overshoots the expected
value by a few percent, but the values at $m_{\eta} = 0.5$ are
nearly within $1\%$. This provides an effective way to com-
compute recoils, since the corresponding radiative quantities
are $3\%$ away for $n_{140}$. The horizon evaluations lying

comparing to the final velocity of the merged BH. The
burst will impart a (usually) small kick to the center of
mass of the system.
Table\textbf{V} shows that the radiation of linear momentum
converges with resolution (at an approximate 2.6-2.7th
order) at similar rates than the radiated energy and mo-
moment (roughly 3rd order), and still varies little with
$\eta$. This is expected on gauge invariance grounds.

The values extrapolated to infinite resolution lie in the
176-177km/s range, consistently for all three values of
$m_{\eta} = 2, 1, 0.5$. Extrapolations of the recoil velocities to
$\eta \to 0$ are very close to their values at $m_{\eta} = 2, 1, 0.5$
for all three resolutions, again confirming the gauge in-
dependence of the results.

Table\textbf{VI} displays the crucial result of recoil velocities
very close to their desired values for low resolutions when
$m_{\eta} = 0.5$. They do not vary so much with resolution, as
expected for horizon quantities, when compared to the
variations with respect to the gauge choices. We observe
close to a linear dependence on $\eta$ of the recoil values.
Their extrapolation to $\eta \to 0$ overshoots the expected
value by a few percent, but the values at $m_{\eta} = 0.5$ are
nearly within $1\%$. This provides an effective way to com-
pute recoils, since the corresponding radiative quantities
are $3\%$ away for $n_{140}$. The horizon evaluations lying
closer to the expected values by a factor $3$ over the ra-
diative ones holds for all three resolutions.

The coordinate velocities do not benefit systematically
from the small $\eta$ gauges, but still provide a good bulk
value as shown in Table\textbf{VII}. This shows the benefits of
having a quasilocal measure of the momentum of the hole
over its horizon compared to the local coordinate velocity
of the puncture.

### B. Validation for other mass ratios ($q = 1/2, 1/5$)

In order to first validate our technique to extract
the recoil velocity of the remnant black hole from spin-
ing binaries, we have considered another unequal mass
$(q = 1/2)$ binary. The resulting recoil will be along the
orbit plane and due entirely to the asymmetry pro-
duced by the unequal masses. The results are presented
in Table\textbf{VIII} Assuming the extrapolation to infinite res-
olution of the radiative linear momentum computations is
the most accurate one leads to a recoil of $154.3 \pm 0.1 \text{km/s}$. Even for the lowest computed resolution, $n_{100}$, the hori-
zon evaluation for $m_{\eta} = 0.5$ at 159.6km/s is a better
approximation to that value than any radiative evolu-
tion at the same resolution (145.9km/s), with errors of
the order of $3\%$. This is also true for the other two resolu-
tions $n_{120}$, and $n_{140}$. Although, as we have seen before,
the improvement of those horizon values are obtained by
lowering the value of $m_{\eta} \to 0$, rather than by higher res-
olution, as the horizon quasilocal measure has essentially
already converged at those resolutions.

Here we also provide the computation of the horizon
evaluations for the mass and spin of the remnant black
hole in Table\textbf{IX}. Those tables display the excellent agree-
ment between the horizon and radiative computation of
the energy and angular momentum (with convergence rates of the 3-4th order). It also displays the agreement
of those radiative computations for the $q = 2$ and the
$\eta = 0.5$ cases, as expected on the ground of gauge invar-
ance at the extrapolated infinite observer location. The
Table also shows the robustness of the horizon computa-

![FIG. 4. The horizon measure of the linear momentum after merger of a $q = 1/3$ nonspinning binary lowering values of $\eta = 2 \to 0$ at resolution n140. The reference value of $V_f$ is found by extrapolation to infinite resolution of the radiated linear momentum.](image-url)
as shown in Table X. The horizon evaluation for the traditional \( \eta = 2 \) and for the \( \eta = 0.5 \) case. Extrapolation to infinite resolution and order of convergence is also given for the horizon and radiative extraction.

| Run resolution | Radiation \( m\eta = 2.0 \) | Horizon \( m\eta = 2.0 \) | Radiation \( m\eta = 0.5 \) | Horizon \( m\eta = 0.5 \) |
|----------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1/100          | 145.45                      | 127.80                      | 145.91                      | 159.57                      |
| 1/120          | 149.45                      | 121.02                      | 149.64                      | 152.48                      |
| 1/140          | 151.38                      | 118.77                      | 151.48                      | 152.92                      |
| Infinite       | 154.28                      | 117.08                      | 154.39                      | -                           |
| Order          | 3.31                        | 5.49                        | 3.18                        | -                           |

TABLE IX. Comparison of the computation of the horizon mass and spin of the remnant of a \( q = 1/2 \), nonspinning binary with the radiation of the energy and angular momentum for the \( \eta = 2 \) and for the \( \eta = 0.5 \) cases. Extrapolation to infinite resolution and order of convergence is also given for the horizon computation and the radiative extraction.

| \( m\eta = 2.0 \) resolution | \( E_{\text{rad}} \) | \( J_{\text{rad}} \) | \( M_f \) | \( \alpha_f \) |
|-------------------------------|---------------------|---------------------|-----------|-----------|
| 1/100                         | 2.9994e-02          | -3.0497e-01         | 0.9612621 | 0.6234437 |
| 1/120                         | 3.0162e-02          | -3.0600e-01         | 0.9612540 | 0.6234529 |
| 1/140                         | 3.0238e-02          | -3.0629e-01         | 0.9612533 | 0.6234557 |
| Infinite                      | 3.0337e-02          | -3.0646e-01         | 0.9612532 | 0.6234575 |
| Order                         | 3.70                | 6.36                | 13.36     | 6.09      |

| \( m\eta = 0.5 \) resolution | \( E_{\text{rad}} \) | \( J_{\text{rad}} \) | \( M_f \) | \( \alpha_f \) |
|-------------------------------|---------------------|---------------------|-----------|-----------|
| 1/100                         | 2.9984e-02          | -3.0516e-01         | 0.9612422 | 0.6234394 |
| 1/120                         | 3.0144e-02          | -3.0583e-01         | 0.9612527 | 0.6234500 |
| 1/140                         | 3.0220e-02          | -3.0615e-01         | 0.9612553 | 0.6234520 |
| Infinite                      | 3.0329e-02          | -3.0662e-01         | 0.9612566 | 0.6234528 |
| Order                         | 3.42                | 7.39                | 7.20      | 8.62      |

Computations at any of the used resolutions (generally 5 digits) and an overconvergence due to those small differences.

We complete our nonspinning studies by simulating a smaller mass ratio \((q = 1/5)\) binary. The recoil from radiation of linear momentum extrapolates to about 139km/s as shown in Table X. The horizon evaluation for \( m\eta = 2 \) underevaluates this by about 30%, while for \( m\eta = 2 \) the horizon formula is about 5% this value for the medium and high resolution runs. Given the smaller mass ratio, the low resolution run is not as accurate.

We also provide as a reference the computation of the \( m\eta = 2 \) of the horizon evaluations for the mass and spin of the remnant black hole in Table X. Those tables display the excellent agreement between the horizon and radiative computation of the energy and angular momentum (with high convergence orders). We find excellent agreement of those radiative computations for the \( \eta = 2 \) and the \( \eta = 0.5 \) cases, as expected from the gauge invariance of the waveforms extrapolated to an infinite observer location. The table also shows the robustness of the horizon computations at medium and high resolutions (generally 3 digits) and an overconvergence due to those small differences.

C. Spinning black holes

In order to further verify our technique to extract the recoil velocity of the remnant black hole from spinning binaries, we have considered an equal mass \((q = 1)\), antialigned (opposite) spin \((\alpha_{1,2} = \pm 0.8)\) binary. The resulting recoil will be along the orbital plane and due entirely to the asymmetry produced by the opposing spins. The results are presented in Table XI showing that assuming the extrapolation to infinite resolution of the radiative linear momentum computations is the most accurate one, leading to a recoil of 403 ± 1km/s, even for the lowest computed resolution. The horizon evaluation for \( m\eta = 0.5 \) at 420km/s is as good to that value than any radiative evolution at the same resolution (388km/s),

TABLE X. Comparison of the computation of the recoil velocity of the remnant of a \( q = 1/2 \), nonspinning binary by traditional radiation of linear momentum and the horizon formula \( E_{\text{rad}} \) averaged between \( t = 1550M \) and \( t = 1850M \) for the traditional \( \eta = 2 \) and for the \( \eta = 0.5 \) case. Extrapolation to infinite resolution and order of convergence is also given for the horizon and radiative extraction.

| Run resolution | Radiation \( m\eta = 2.0 \) | Horizon \( m\eta = 2.0 \) | Radiation \( m\eta = 0.5 \) | Horizon \( m\eta = 0.5 \) |
|----------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1/100          | 129.40                      | 98.67                        | 130.21                      | 209.00                      |
| 1/120          | 133.91                      | 102.53                       | 133.19                      | 131.93                      |
| 1/140          | 135.94                      | 101.62                       | 136.70                      | 143.28                      |
| Infinite       | 138.57                      | 101.20                       | -                           | 146.78                      |
| Order          | 3.72                        | 7.46                         | -                           | 10.16                       |

TABLE XI. Comparison of the computation of the recoil mass and spin of the remnant of a \( q = 1/5 \), nonspinning binary with the radiation of the energy and angular momentum for the \( \eta = 2 \) and for the \( \eta = 0.5 \) cases. Extrapolation to infinite resolution and order of convergence is also given for the horizon computation and the radiative extraction.

| \( m\eta = 2.0 \) resolution | \( E_{\text{rad}} \) | \( J_{\text{rad}} \) | \( M_f \) | \( \alpha_f \) |
|-------------------------------|---------------------|---------------------|-----------|-----------|
| 1/100                         | 1.2367e-02          | -1.5454e-01         | 0.982172  | 0.416669  |
| 1/120                         | 1.2251e-02          | -1.4872e-01         | 0.982349  | 0.416663  |
| 1/140                         | 1.2264e-02          | -1.4803e-01         | 0.982372  | 0.416953  |
| Infinite                      | 1.2267e-02          | -1.4788e-01         | 0.982378  | 0.416670  |
| Order                         | 11.59               | 11.41               | 10.78     | -15.49    |

| \( m\eta = 0.5 \) resolution | \( E_{\text{rad}} \) | \( J_{\text{rad}} \) | \( M_f \) | \( \alpha_f \) |
|-------------------------------|---------------------|---------------------|-----------|-----------|
| 1/100                         | 1.3103e-02          | -1.6762e-01         | 0.981579  | 0.415720  |
| 1/120                         | 1.2413e-02          | -1.5479e-01         | 0.982210  | 0.416697  |
| 1/140                         | 1.2268e-02          | -1.4859e-01         | 0.982356  | 0.416629  |
| Infinite                      | 1.2209e-02          | -1.3922e-01         | 0.982421  | 0.416620  |
| Order                         | 8.12                | 7.30                | 7.59      | 14.39     |
TABLE XII. Comparison of the computation of the recoil velocity of the remnant of a $q = 1$, $\alpha = \pm 0.8$ binary by traditional radiation of linear momentum and the horizon formula averaged between $t = 1050M$ and $t = 1350M$ for the traditional $\eta = 2$ and for the $\eta = 0.5$ case. Extrapolation to infinite resolution and order of convergence is also given for the radiative extraction.

| Run resolution | Radiation $m\eta = 2.0$ | Horizon $m\eta = 2.0$ | Radiation $m\eta = 0.5$ | Horizon $m\eta = 0.5$ |
|----------------|-------------------------|------------------------|-------------------------|------------------------|
| $1/100$        | 387.92                  | 335.17                 | 388.13                  | 419.89                 |
| $1/120$        | 394.16                  | 339.21                 | 394.16                  | 421.02                 |
| $1/140$        | 397.43                  | 331.75                 | 397.16                  | 419.67                 |
| Infinite order | 403.42                  | -                      | 402.00                  | -                      |

TABLE XIII. Comparison of the computation of the horizon mass and spin of the remnant of a $q = 1$, $\alpha = \pm 0.8$ binary with the radiation of the energy and angular momentum for the $\eta = 2$ and for the $\eta = 0.5$ cases. Extrapolation to infinite resolution and order of convergence is also given for the radiative extraction.

| $m\eta = 2.0$ | $E_{rad}$ | $J_{rad}$ | $M_f$ | $\alpha_f$ |
|---------------|-----------|-----------|-------|------------|
| $1/100$       | 3.8716e-02 | -3.4232e-01 | 0.9507072 | 0.6841343 |
| $1/120$       | 3.8980e-02 | -3.4320e-01 | 0.9507092 | 0.6841341 |
| $1/140$       | 3.9108e-02 | -3.4362e-01 | 0.9507097 | 0.6841324 |
| Infinite Order | 3.9303e-02 | -3.4421e-01 | 0.9507100 | 0.6841343 |

| $m\eta = 0.5$ | $E_{rad}$ | $J_{rad}$ | $M_f$ | $\alpha_f$ |
|---------------|-----------|-----------|-------|------------|
| $1/100$       | 3.8716e-02 | -3.4232e-01 | 0.9507084 | 0.6841309 |
| $1/120$       | 3.8978e-02 | -3.4318e-01 | 0.9507107 | 0.6841340 |
| $1/140$       | 3.9105e-02 | -3.4358e-01 | 0.9507104 | 0.6841337 |
| Infinite Order | 3.9297e-02 | -3.4411e-01 | 0.9507103 | 0.6841337 |

with errors of the order of 4%. As we have seen before, the improvement of those horizon values are obtained by lowering the value of $m\eta \to 0$, rather than by higher resolution, as the horizon quasilocal measure has essentially already converged at those resolutions.

For the sake of completeness, and to verify the accuracy of the horizon evaluations for the mass and spin of the remnant black hole, we provide their computation in Tables XIII. Those tables display the excellent agreement between the horizon and radiative computation of the energy and angular momentum (with convergence rates of the 3-4th order). It also displays the agreement of those radiative computations for the $\eta = 2$ and the $\eta = 0.5$ cases, as expected on the ground of gauge invariance at the extrapolated infinite observer location. Further, the tables show the robustness of the horizon computations at any of the used resolutions (generally 5 digits).

A final note on the convergence studies carried out in this section is that we observe a good convergence rate of 3rd to 4th order for radiative quantities while for horizon quantities a more wider range of values, with sometimes overconvergence. This is due to the fact that the horizon evaluations (particularly for the mass and spin) lead to very accurate values and hence small differences between the three resolutions chosen for the simulations ($n_{100}$, $n_{120}$, and $n_{140}$). In order to seek very significant differences between resolutions, factors larger than 1.2 should be chosen (although requiring much larger computational resources). Note that nevertheless we have been able to prove that horizon quantities can be evaluated very accurately at any of the resolutions quoted above.

III. OTHER GAUGES STUDIES

Right after the breakthrough that allowed evolving binary black holes with the moving puncture formalism [25–34], several papers analyzed extensions of the basic gauges (2) - (4). In Refs. [48] and [34] several parametrizations of the shift conditions are studied, and displayed some (slight) preference for the $\partial_0$-gauge over the $\partial_1$-gauge (See Table I of Ref. [48] and Fig. 10 of Ref. [34]).

The sensitivity of the computed recoil on the gauge give us an opportunity to quantify the relative accuracy of the $\partial_0$-gauge versus the $\partial_1$-gauge.

A. $\partial_0$-gauge

In Fig. 5 we draw a comparative analysis of the final black hole horizon recoil as computed in the $\partial_0$ and $\partial_1$ gauges, at our highest resolution $n_{140}$. In all three cases, $\eta = 2, 1, 0.5$, the computation in the $\partial_1$-gauge is notably and systematically closer to the expected ($V_f \sim 177 \text{km/s}$) recoil velocity. This also provides a scale of the accuracy of the evaluation of the recoil for our new preferred value, $\eta = 0.5$.

Convergence with resolution does not resolve this discrepancies in favor of the $\partial_1$-gauge as displayed in Fig. 5 for $m\eta = 2$ in the $\partial_0$-gauge at $n_{100}$, $n_{120}$, and $n_{140}$ resolutions. With the limit $\eta \to 0$ being even harder to resolve than in the $\partial_1$-gauge case.

In conclusion we observe the clear advantages of working in the $\partial_1$-gauge over the $\partial_0$-gauge. This agrees with the generic experience for binary black holes simulations being more accurate in our standard $\partial_1$-gauge at the same resolutions, but now we have quantified it in the recoil computations example.

B. $\eta$-variable gauge

In addition to the changes in the (constant) values of $\eta$, different functional dependences for $\eta(x^k, t)$ have been proposed in [37, 49–54].

Here we use a modified form motivated by the results of [27] and this paper that the recoil velocities (of a merged binary) are more accurately computed when
where $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are the punctures location, and $s$ is a width of the Gaussian that can be conveniently chosen, for instance, $s = 2m$.

The choice to center the Gaussian correction to the $\eta_\infty$ at the center of mass is to provide a simple way of following the final black hole after merger, even if acquired a large recoil velocity. The motivation to set different values around the black hole is to provide enough accuracy and convergence in the strong field regime, while preserving the benefits of the coordinates adapted to recoil measurements away from the remnant hole.

In order to assess those statements we have considered two cases labeled as N10 and N12, respectively determined by the $\mp$ in

$$m\eta(x^k, t) = 1 \mp e^{-r^2/(2m)^2}$$  (8)

with the same reference asymptotic value of 1 at infinity and vanishing or taking the standard value of 2 at the (final) hole location.

The simulations (in the standard $\partial^0$-gauge) make use of this $\eta$ dependence during the whole run, not only during the post-merger phase, but the horizon of the final black hole is only found and evaluated for linear momentum after the merger occurs (about $t \sim 725m$).

The results of the two cases are displayed in Fig. 7. Each case has been studied at our standard n100, n120, n140 resolutions to convey an idea of the convergence. The upper panel displays a very good agreement with the expected recoil, particularly for the medium and high resolutions. The agreement is even better than the constant $m\eta = 1$ case, displayed in the central panel of Fig. 6, which lies in between the N12 and N10 cases. To confirm the improvements reached by this variable-$\eta$ gauge, the case N10, where the asymptotic value is the same, equal to 1, but where the $\eta$ is reduced near the black hole, reduces notably the accuracy necessary to compute the linear momentum of the horizon and convergence is still more challenging than in the previous N12 case.

In conclusion, a first exploration of an $\eta$-variable leads to immediate benefits and opens the possibilities for further refinement to have both, accuracy and precision improved in the numerical simulations of merging binary black holes.

### C. Small mass ratio

Because of the technical similarities (although with complementary mass ratio applications), here we briefly discuss dealing with the small mass ratio binaries by the modeling of the damping parameter $\eta$. The proposal for $\eta$ in Eq. (6) is meant to be used for comparable masses $q > 1/10$ when we can still use a constant $\eta$ for evolutions. For smaller mass ratios this $\eta_\infty$ can be replaced by the $\eta(W)$ (the conformal factor $W = \sqrt{\chi} = \exp(-2\phi)$ suggested by [56]) used in Ref. [54] or a modification of it given below or yet other based on superposition of

\[
\eta(W) = \eta_\infty \left(1 - \frac{q}{10}\right)
\]
FIG. 7. The horizon measure of the linear momentum after merger of a \( q = 1/3 \) nonspinning binary for the three resolutions labeled as n100 (dotted), n120 (dashed), and n140 (solid) for \( \eta = \text{N12} \) (red) and N10 (blue), top to bottom respectively. The reference value of \( V_f \) is found by extrapolation to infinite resolution of the radiated linear momentum.

weighted Gaussians. Note that the recoil for \( q < 1/10 \) is small. This question has been already studied in [50, 52, 53, 57, 58]. Here we simply bring back some of those ideas, assuming we evaluate \( \eta(\vec{r}_1(t), \vec{r}_2(t)) \) parametrized by the black holes 1 and 2 punctures trajectories \((\vec{r}_1(t), \vec{r}_2(t))\).

The (initial) conformal factor evaluated at every time step is

\[
\psi_0 = 1 + \frac{m_1}{|\vec{r} - \vec{r}_1(t)|} + \frac{m_2}{|\vec{r} - \vec{r}_2(t)|}
\]  

(9)

And we can then define analogously to the \( \eta(W) \)

\[
m\psi = A + B \frac{\sqrt{\vec{\nabla} \psi_0}^b}{(1 - \psi_0)^a}.
\]  

(10)

We plot an example in Fig. 8. At the puncture \( m\eta = 2 \) and at the center of mass \( m\eta = 2.04 \), but it goes through a minimum \( m\eta = 0 \) and a this point, as well as the punctures, \( \eta \) is \( C^0 \). Since the gauge condition Eq. (2) involves an integration, this might still be fine for evolutions (as well as at the punctures).

A second alternative smoother behavior is the superposition of Gaussian (See also [58])

\[
\eta_G = \frac{A}{m} + \frac{B}{m_1} e^{-|\vec{r} - \vec{r}_1(t)|^2/s_1^2} + \frac{C}{m_2} e^{-|\vec{r} - \vec{r}_2(t)|^2/s_2^2},
\]  

(11)

which for the parameters of the previous example we display in Fig. 9, behaving like 1.25 at the first puncture, 1.75 at the second and is essentially 1 in between and far away from the binary.

The explicit application and evaluations in actual simulations of small mass ratio binary black hole mergers is left for an independent study (Healy et al. 2020b, paper in preparation).

IV. DISCUSSION

The purpose of this study was to assess what choices of \( \eta \) lead to accurate measures of the linear momentum of the horizon with the non-gauge-independent formula (5), and we found that for small values \( m\eta \leq 0.5 \) this is a reliable measure and can compete with the measurement at \( I^+ \) of the radiated momentum carried by the gravitational waves. As with the computations of the mass and angular momentum of the remnant via the horizon measure and at infinity, it is important to have two concurrent methods to assess errors of those measures. Further accuracy could be achieved by the use of a variable \( \eta \) (See Eq. (6)). Our results indicate that the choice of \( m\eta = 2 \) at the horizon with lower values asymptotically far produce the best results for evaluation of the recoil.

While these gauges were studied in detail for a non-spinning \( q = 1/3 \) binary and verified as control cases for the \( q = 1/2 \) and \( q = 1/5 \) binaries as well as for a \( q = 1 \)
spinning case, we expect these conclusions to be general and plan to apply the findings to simulations where the computation of recoil is important, including precessing binaries. The cross checking with radiated linear momentum will provide a control to its applicability and can be carried out concurrently in each simulation.

Finally, we have been able to assess the relative accuracy of the two original moving puncture choices for the shift, the $\delta_t$ and $\delta_\phi$ gauges regarding their accuracy to evaluate the recoil and found that the $\delta_t$-gauge seems to be superior, at the current typical numerical resolutions.

ACKNOWLEDGMENTS

The authors thank Y. Zlochower for discussions on the gauge choices. The authors gratefully acknowledge the National Science Foundation (NSF) for financial support from Grants No. PHY-1912632, No. PHY-1707946, No. ACI-1550436, No. AST-1516150, No. ACI-1516125, No. PHY-1726215. This work used the Extreme Science and Engineering Discovery Environment (XSEDE) [allocation TG-PHY060027N], which is supported by NSF grant No. ACI-1548562. Computational resources were also provided by the NewHorizons, BlueSky Clusters, and Green Prairies at the Rochester Institute of Technology, which were supported by NSF grants No. PHY-0722703, No. DMS-0820923, No. AST-1028087, No. PHY-1229173, and No. PHY-1726215. Computational resources were also provided by the Blue Waters sustained-petасcale computing NSF projects OAC-1811228, OAC-0832606, OAC-1238993, OAC-1516247 and OAC-1515969, OAC-0725070. Blue Waters is a joint effort of the University of Illinois at Urbana-Champaign and its National Center for Supercomputing Applications.

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