STRUCTURE OF LARGE INCOMPLETE SETS IN ABELIAN GROUPS

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Abstract. Let \( G \) be a finite abelian group and \( A \) be a subset of \( G \). We say that \( A \) is complete if every element of \( G \) can be represented as a sum of different elements of \( A \). In this paper, we study the following question:

What is the structure of a large incomplete set?

The typical answer is that such a set is essentially contained in a maximal subgroup. As a by-product, we obtain a new proof for several earlier results.

1. Introduction

Let \( G \) be an abelian group and \( A \) be a subset of \( G \). We use \( S_A \) to denote the collection of all subset sums of \( A \)

\[ S_A := \{ \sum_{a \in B} a | B \subset A, |B| < \infty \} \]

We will keep this notation when \( A \) is sequence of (not necessarily different) elements of \( A \). In this case \( S_A \) is the collection of all subsequence sums of \( A \). \( Z_n \) denotes the cyclic group of order \( n \).

Example. Take \( G = \mathbb{Z}_{11} \). If \( A \) is the subset \( \{1, 2, 3\} \), then \( S_A = \{1, 2, 3, 4, 5, 6\} \). If \( A \) is the sequence \( \{1, 1, 3\} \), then \( S_A = \{1, 2, 3, 4, 5\} \).

Following Erdős [1], we say that \( A \) is complete if \( S_A = G \) and incomplete otherwise. If \( G \) is finite, the critical number of \( G \), \( c(G) \), is the smallest integer \( m \) such that any subset \( A \subset G \setminus \{0\} \) with size \( m \) is complete. This parameter has been studied for a long time and its exact value is known for most groups.

Theorem 1.1. Let \( G \) be a finite abelian group of order \( n = ph \), where \( p \) is the smallest prime divisor of \( G \). Then the following holds

- If \( p = 2 \) and \( h \geq 5 \) or \( G = Z_2 \oplus Z_2 \oplus Z_2 \), then \( c(G) = h \). If \( p = 2 \) and \( h \leq 4 \) and \( G \neq Z_2 \oplus Z_2 \oplus Z_2 \), then \( c(G) = h + 1 \).
- If \( h \) is a prime, then \( p + h - 2 \leq c(G) \leq p + h - 1 \). Furthermore, if \( h = p \geq 3 \) or \( h \geq 2p + 1 \), then \( c(G) = p + h - 2 \).
- If \( p \geq 3 \) and \( h \) is composite, then \( c(G) = p + h - 2 \).

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The first statement is due to Diderrich and Mann [7]. The second combines results of Mann and Wou [10] (who studied the case $h = p$) and Didderich [6]. The last statement has been known as Didderich conjecture, posed in [6] and was proved by Gao and Hamidoune [8], more than twenty years later.

In this paper, we would like to study the following question

\textit{What is the structure of a relatively large incomplete set?}

Technically speaking, we would like to have a characterization for incomplete sets of relatively large size. Such a characterization has been obtained recently in [9] for sets of size at least $n/(p + 2)$. In this paper, we will be able to treat much smaller sets. (In fact, our assumption on "relatively large" is almost sharp; see Theorem 2.9.) The method used in our proofs is different from those used in previous papers. As a by-product, one obtains a new proof for a good portion of Theorem 1.1, including a new proof for Didderich’s conjecture for large $n$ (see the remarks following Theorem 2.7).

\textit{Notation.} $\langle A \rangle$ denotes the subgroup generated by $A$. $E(X)$ denotes the expectation of a random variable $X$. All logarithms have natural base, if not specified otherwise.

2. The characterization of large incomplete sets

Let us start by a simple fact, whose proof is left as an exercise.

\textbf{Fact 2.1.} Let $p$ be a prime and $A$ be a sequence of $p - 1$ non-zero elements in $Z_p$. Then $S_A \cap \{0\} = Z_p$. On the other hand, there is a sequence of $p - 2$ non-zero elements of $Z_p$ such that $S_A \cap \{0\} \neq Z_p$.

Let $G$ be an abelian group of size $n$ and $q$ be a prime divisor of $n$. Let $H$ be a subgroup of size $n/q$. A direct corollary of Fact 2.1 is the following

\textbf{Fact 2.2.} If $A$ is a nice incomplete subset of $G$ and $S_{A \cap H} = H$, then $H \setminus A$ has at most $q - 2$ elements. Consequently $A$ has at most $n/q + q - 2$ elements.

\textbf{Definition 2.3.} Let $G$ be an abelian group of size $n$. A subset $A$ of $G$ is \textit{nice} if there is a subgroup $H$ of $G$ such that $|G/H|$ is a prime and $S_{A \setminus H} = H$.

Given a subgroup $H$ in $G$ and an element $a \in G$, we use $a/H$ to represent the coset of $H$ which contains $a$. $a/H$ can be viewed as an element of the quotient group $G/H$. If $B$ is a subset of $G$, then $B/H := \{b/H | b \in B\}$ is a sequence in $G/H$.

\textbf{Fact 2.4.} If $A$ is a nice incomplete set in a finite abelian group $G$ of size $n$, then $|A| \leq \frac{n}{p} + p - 2$, where $p$ is the smallest prime divisor of $n$. Furthermore, $(A \setminus H)/H$ is an incomplete sequence in $Z_q = G/H$. 
Proof (Proof of Fact 2.4) If $A$ is a nice incomplete set then $|A| \leq \frac{n}{q} + q - 2$, for some prime divisor $q$ of $n$. On the other hand, it is easy to see that $\frac{n}{q} + q \leq \frac{n}{p} + p$, where $p$ is the smallest prime divisor of $n$. \hfill \blacksquare

Our leading idea is that relatively large incomplete sets are nice. A special case has been verified by Gao, Hamidoune, Lladó and Serra [9]. Their theorem can be reformulated in the current setting as follows

**Theorem 2.5.** Let $G$ be an abelian group of order $n = ph$, where $p \geq 5$ is the smallest prime divisor of $n$, $h \geq 15p$ is composite. Let $A$ be an incomplete subset of at least $n/p + 2 + p$ elements. Then $A$ is nice. Furthermore, there is a subgroup $H$ of size $n/p$ such that $S_A \cap H = H$.

For any positive $\varepsilon \leq 1$, define

$$C(\varepsilon) := \sqrt{\frac{40/\varepsilon^2}{\log(2/\varepsilon)}} \quad (1)$$

and let $n(\varepsilon)$ be the smallest integer $m$ such that for any $n \geq m$

$$n \geq C(\varepsilon) \sqrt{n \log n} > \frac{4}{\varepsilon^2} \quad (2)$$

Remark 2.6. $n(\varepsilon)$ is relatively small. One can take, say, $n(\varepsilon) = 500/\varepsilon^4$.

Now we are ready to state our first theorem.

**Theorem 2.7.** Let $\delta$ be a positive constant at most $1/6$ and $p_1 \leq p_2 \cdots \leq p_t$, $t \geq 2$, be primes satisfying three conditions

1. $p_2 \geq 3$;
2. $n := \prod_{i=1}^t p_i \geq n(\delta)$;
3. $p_1 \leq \frac{1}{3C(\delta)} \sqrt{n/\log n}$.

Let $G$ be an abelian group of order $n$ and $A$ be an incomplete subset of $G$ of size at least $(5/6 + \delta) \frac{n}{p_1}$. Then $A$ is nice and there is a subgroup $H$ such that $n/|H|$ is one of the $p_i$, $|A \setminus H| < 3p_1$ and $S_{A \cap H} = H$.

Remark 2.8. Let us have a few comments on this theorem.

- Using Theorem 2.7 and Facts 2.2, we can recover a large portion of Theorem 1.1. To see this, consider an incomplete set $A$ which does not contain zero. If $|A| \leq n/p_1$, there is nothing to prove. If $|A| \geq n/p_1$, and $n = |G|$ satisfies the assumptions of Theorem 2.7, apply this theorem to obtain the subgroup $H$. As $A$ does not contain zero, then $|A \cap H| \leq |H| - 1$. By Facts 2.2, $|A| \leq n/q + q - 3$, where $q = n/|H|$. But $n/q + q \leq n/p_1 + p_1$, so $|A| \leq n/p_1 + p_1 - 3$.
- The third assumption $p_1 \leq \frac{1}{3C(\delta)} \sqrt{n/\log n}$ in Theorem 2.7 can be voided if we assume $t \geq 3$ (i.e., $n/p_1$ is composite) and $n$ sufficiently large. In that case $p_1 \leq n^{1/3} \ll \sqrt{n/\log n}$. It follows that the assumptions of Theorem
2.7 are satisfied whenever \( p_1 \geq 3 \), \( h \) is composite and \( n \) is sufficiently large. Thus, we have a new proof of Didderich conjecture for sufficiently large \( n \).

- Unlike Theorem 2.5, one cannot conclude that \( H \) has size \( n/p_1 \). It is easy to give examples where \( |G|/|H| \) can be any of the \( p_i \).

The next question is to find the best lower bound on \( |A| \) that guarantees niceness. Our second theorem gives an almost complete answer for this question.

**Theorem 2.9.** For any positive constant \( \delta \) there is a positive constant \( D(\delta) \) such that the following holds. Let \( p_1 \leq p_2 \leq \cdots \leq p_t \), \( t \geq 3 \), be primes such that \( p_1 p_2 \leq \frac{1}{100} \sqrt{n \log n} \), where \( n := \prod_{i=1}^t p_i \). Let \( G \) be an abelian group of order \( n \) and \( A \) be an incomplete subset of \( G \) with cardinality at least \( (1 + \delta) \frac{n}{p_1 p_2} \). Then \( A \) is nice.

Furthermore, the lower bound \( (1 + \delta) \frac{n}{p_1 p_2} \) cannot be replaced by \( \frac{n}{p_1 p_2} + n^{1/4 - \alpha} \), for any constant \( \alpha \).

Finally, let us discuss the case when \( G = \mathbb{Z}_n \), where \( n \) is a prime. This case has not been covered by the results presented so far. Olson [13], improving upon a result of Erdős and Heilbronn [2], shows that \( c(\mathbb{Z}_n) \leq \sqrt{4n - 3} + 1 \). His bound was improved by da Silva and Hamidoune [3] to \( \sqrt{4n - 7} \). As far as characterization results are concerned, we know of the following two results.

**Theorem 2.10.** Let \( n \) be a prime and \( A \) be an incomplete subset of \( \mathbb{Z}_n \) of size at least \( (2n)^{1/2} \). Then there is some non-zero element \( b \in \mathbb{Z}_n \) such that

\[
\sum_{a \in bA} ||a|| \leq n + O(n^{3/4} \log n).
\]

**Theorem 2.11.** Let \( n \) be a prime and \( A \) be an incomplete subset of \( \mathbb{Z}_n \) of size at least \( 1.99 n^{1/2} \). Then there is some non-zero element \( b \in \mathbb{Z}_n \) such that

\[
\sum_{a \in bA} ||a|| \leq n + O(n^{1/2}).
\]

Theorem 2.10 is due to Deshouillers and Freiman [5]. Theorem 2.11 is due to Nguyen, Szemerédi and Vu [11]. The error term in this \( O(n^{1/2}) \) is best possible, as shown by a construction in [4].

The rest of the paper is organized as follows. Section 3 contains the main lemma to the proofs, which states that if \( A \) is sufficiently large, then \( S_A \) contains a subgroup of size comparable to \( |A| \). The proofs of the theorems come in Sections 4 and 5. Section 6 is devoted to concluding remarks.

### 3. The Existence of a Large Subgroup in \( S_A \)

Our key tool is the following statement, which asserts that if \( A \) is a sufficiently large subset of \( G \), then \( S_A \) contains a large subgroup of \( G \). Recall the definition of \( C(\epsilon) \) and \( n(\epsilon) \) from (1) and (2).
Theorem 3.1. Let $0 < \epsilon < 1$ be a constant and $G$ be an abelian group of size $n$, where $n \geq \max\{\frac{1}{\epsilon^2}, C(\epsilon)\sqrt{n \log n}\}$. Let $A$ be a subset of $G$ with at least $\max\{\frac{1}{\epsilon^2}, C(\epsilon)\sqrt{n \log n}\}$ elements. Then $S_A$ contains a subgroup of size at least $(1 - \epsilon)|A|$.

Remark 3.2. The bound $(1 - \epsilon)|A|$ is asymptotically sharp, as $A$ itself can be a subgroup. The lower bound $C\sqrt{n \log n}$ for $|A|$ is sharp up to the logarithmic term. To see this, consider $G = \mathbb{Z}_{p^2}$ and $A = \{0, 1, \ldots, p\}$. It is clear that $|A| > p = \sqrt{|G|}$. On the other hand, $S_A$ does not contain any proper subgroup of $G$. It is interesting to see whether the log term can be removed.

Remark 3.3. The theorem also holds for non-abelian group, see Theorem 3.9 at the end of this section.

By definition of $n(\epsilon)$, if $n \geq n(\epsilon)$ then

$$n > C(\epsilon)\sqrt{n \log n} > \frac{4}{\epsilon^2}.$$ 

In this case we have the following corollary, which is easier to use.

Corollary 3.4. Let $0 < \epsilon < 1$ be a constant and $G$ be an abelian group of size $n \geq n(\epsilon)$. Let $A$ be a subset of $G$ with at least $C(\epsilon)\sqrt{n \log n}$ elements. Then $S_A$ contains a subgroup of size at least $(1 - \epsilon)|A|$.

To prove Theorem 3.1, we use the following result of Olson [12] (see also [15, Chapter 12]). Let $A$ be a set and $l$ be a positive integer, we define

$$lA := \{a_1 + \ldots a_l | a_i \in A\}.$$ 

Also recall that $<A>$ denotes the subgroup generated by $A$.

Theorem 3.5. Let $G$ be a finite abelian group, $l$ be a positive integer and $0 \in A$ be a finite subset of $G$. Then either $lA = <A>$ or $|lA| \geq |A| + (l - 1)(\frac{|A|}{2} + 1)$.

Since $|A| + (l - 1)(\frac{|A|}{2} + 1) \geq (l + 1)|A|/2$, the following corollary is immediate.

Corollary 3.6. Let $G$ be a finite abelian group, $l$ be a positive integer and $0 \in A$ be a finite subset of $G$ such that $(l + 1)|A| \geq 2|G|$, then

$$lA = <A>.$$ 

We also need the following result of Olson [12], which refines an earlier result of Szemerédi [14] (Szemerédi proved the theorem for an unspecified constant instead of 3).

Theorem 3.7. Let $G$ be an abelian group of order $n$ and $A$ be subset of at least $3\sqrt{n}$ elements. Then $0 \in S_A$.

We also need the following simple lemma:

Lemma 3.8. Let $G$ be an abelian group and $A$ be a subset of $G$. Let $l$ be a positive integer and $S$ a subset of $G$ such that every element of $S$ can be represented as the sum of two different elements of $A$ in at least $2l - 1$ ways (not counting permutations). Then $lS \subset S_A$. 
Proof (Proof of Lemma 3.8) Let \( x_1, \ldots, x_l \) be (not necessarily different) elements of \( S \). We represent \( x_1 + \cdots + x_l \) as a sum of different elements of \( A \) using the greedy algorithm. To start, represent \( x_1 = a_1 + a_1' \) where \( a_1 \neq a_1' \) are different elements of \( A \). Assume that we have represented \( x_1 = a_1 + a_1', \ldots, x_i = a_i + a_i' \) where \( 1 \leq i < l \) and \( a_1, a_1', \ldots, a_i, a_i' \) are all different. Now look at \( x_{i+1} \). Each of the \( 2i \) elements \( a_1, a_1', \ldots, a_i, a_i' \) appear in at most one representation of \( x_{i+1} \). Since \( x_{i+1} \) has at least \( 2l - 1 > 2i \) representations, we can find a representation \( x_{i+1} = a_{i+1} + a_{i+1}' \) where both \( a_{i+1} \) and \( a_{i+1}' \) are different from \( a_1, a_1', \ldots, a_i, a_i' \). This concludes the proof.

Proof (Proof of Theorem 3.1) For each element \( x \in G \), let \( m_x \) be the number of ways to represent \( x \) as the sum of two different elements of \( A \) (not counting permutations). A double counting argument gives

\[
\sum_{x \in G} m_x = \binom{|A|}{2}. \tag{3}
\]

Notice that \( m_x \) is at most \( M := |A|/2 \). Set \( K := 2/\epsilon \). Let \( S_j \) be the collection of those \( x \) where \( K^{-j}M < m_x \leq K^{-j+1}M \) for \( j = 1, \ldots, j_0 \), where \( j_0 \) is the largest integer such that \( K^{-j_0}M \geq 1 \). Let \( S_{j_0+1} \) be the collection of those \( x \) where \( 1 \leq m_x \leq K^{-j_0}M \). By the definition of \( S_j \)

\[
\sum_{j=1}^{j_0+1} K^{-j+1}M|S_j| \geq \sum_{x \in G} m_x, \tag{4}
\]

which, together with (3) imply

\[
\sum_{j=1}^{j_0+1} K^{-j+1}M|S_j| \geq \binom{|A|}{2}. \tag{5}
\]

Call a set \( S_j \) small \((j = 1, \ldots, j_0 + 1)\) if it has at most \((1 - \epsilon)|A|\) elements and large otherwise. The contribution from the small \( S_j \) on the left hand side is at most

\[
\sum_{j=1}^{j_0+1} K^{-j+1}M(1 - \epsilon)|A| \leq \frac{K}{K - 1}M(1 - \epsilon)|A| = (1 - \frac{\epsilon}{2 - \epsilon})\frac{|A|^2}{2}
\]

taken into account the facts that \( K = 2/\epsilon \) and \( M = |A|/2 \). Since \( |A| \geq \frac{4}{\epsilon} \), we have

\[
(1 - \frac{\epsilon}{2 - \epsilon})\frac{|A|^2}{2} \leq (1 - \epsilon/2)\frac{|A|^2}{2} - \frac{|A|}{2}.
\]

From this and (5), we have

\[
\sum_{S_j \text{ large}} K^{-j+1}M|S_j| \geq \binom{|A|}{2} - (1 - \epsilon/2)\frac{|A|^2}{2} + \frac{|A|}{2} = \epsilon \frac{|A|^2}{4}. \tag{6}
\]

The bound \( |A| \geq C(\epsilon)\sqrt{n \log n} \) guarantees that

\[
\frac{\epsilon}{4} \frac{|A|^2}{4} \geq 5Kn \log_{2/\epsilon} n. \tag{7}
\]
(In fact, \( C(\epsilon) \) is defined so that this inequality holds.) Set \( l_j := K^{-j}M \). Since the number of large \( S_j \) is at most \( j_0 + 1 \leq \lceil \log_{2/\epsilon} |A|/2 \rceil + 1 \), (6), (7) and the pigeon hole principle imply that there is a large \( S_j \) such that
\[
l_j|S_j| \geq 4n.
\]

Notice that \( |S_j| \leq |G| = n \). It follows that \((|l_j/2| + 1)|S_j| \geq 2n\). Apply Corollary 3.4 to \( l := \lfloor l_j/2 \rfloor \) and \( S := S_j \cup \{0\} \), we can conclude that \( lS = \langle S \rangle \). On the other hand, by the definition of

\[
iS_j \subset S_A,\]

for all \( 1 \leq i \leq l \). Finally, \( 0 \in S_A \) by Olson’s theorem. Thus \( S_A \) contains \( \langle A \rangle \), which has at least \( (1 - \epsilon)\sqrt{n \log n} \) elements since \( S_j \) is large and \( |\langle S \rangle| \geq |S| \geq |S_j| \).

This concludes the proof.

All the tools used in the proof (Theorems 3.5 and 3.7, Lemma 3.8) hold for non-abelian groups. Thus, Theorem 3.1 also holds for this case. The proof requires only two simple modifications. First, in Lemma 3.8, \( 2l - 1 \) is replaced by \( 4l - 3 \). The reason is that in the proof, each of the elements \( a_1, a'_1, \ldots, a_i, a'_i \) can now appear in at most 2 representations of \( x_i+1 \). The second is that in the proof of Theorem 3.1, we need to fix an ordering on the elements of \( G \) and when we consider a sum \( x + y \), we always assume that \( x \) precedes \( y \) in this ordering. The rest of the proof remains the same.

**Theorem 3.9.** For any constant \( 0 < \epsilon < 1 \) there are constant \( n_1(\epsilon) \) and \( C_1(\epsilon) \) such that the following holds. Let \( G \) be a group of size \( n \), where \( n \geq n_1(\epsilon) \). Let \( A \) be a subset of \( G \) with at least \( C_1(\epsilon)\sqrt{n \log n} \) elements. Then \( S_A \) contains a subgroup of size at least \( (1 - \epsilon)|A| \).

The values of \( n_1(\epsilon) \) and \( C_1(\epsilon) \) might be slightly different from that of \( n(\epsilon) \) and \( C(\epsilon) \), due to the modifications.

### 4. Proof of Theorem 2.7

**Lemma 4.1.** Let \( G \) be a finite additive group and \( A \) be a subset of \( G \) with cardinality at least \( \lceil |G|/2 \rceil + 2 \). Then \( S_A = G \).

**Proof** (Proof of Lemma 4.1) Let \( x \) be an arbitrary element of \( G \). There are exactly \( \lceil |G|/2 \rceil \) (unordered) pairs \((a, b)\) of different elements of \( G \) such that \( a + b = x \). The claim follows by the pigeon hole principle. One can improve the bound slightly but from our point of view it is not important.

**Proof** (Proof of Theorem 2.7) Let \( A_1 \) be an arbitrary subset of \( A \) with cardinality \( (1 + 2\delta)\frac{n}{3p_1} \). By the upper bound on \( p_1 \), we can assume that \( |A_1| \geq C(\delta)\sqrt{n \log n} \).
which enables us to apply Corollary 3.1 to $A_1$ and obtain a subgroup $H \subset S_{A_1}$

$$|H| \geq (1 - \delta)|A_1| = (1 - \delta)(1 + 2\delta)\frac{n}{3p_1} > \frac{n}{3p_1}.$$  

The assumption $p_3 \geq 3$ shows that $H > \frac{n}{p_1p_2}$. It follows that $|H| = n/q$ where $q$ is one of the $p_i$ ($1 \leq i \leq t$). Furthermore, 

$$q < 3p_1.$$  

Consider the sequence $B := \{a/H|a \in A\setminus A_1\}$ in the quotient group $G/H = Z_q$. If $B$ has at least $q - 1$ non-zero elements, then by Fact 2.1 $S_B$ contains $Z_q \setminus \{0\}$, which implies that

$$G \subset S_{A_1} + S_{A\setminus A_1} \subset S_A$$

a contradiction as $A$ is incomplete. Thus, $B$ has at most $q - 2$ non-zero elements. So we can conclude that all but at most $q - 2$ elements of $A\setminus A_1$ lie in $H$. Let $A_2$ denote the set of these elements. We have

$$|A_2| > |A\setminus A_1| - (q - 2) \geq |A| - |A_1| - 3p_1 + 2 \geq \left(\frac{5}{6} + \delta - (1 + 2\delta)\frac{1}{3}\right)\frac{n}{p_1} - 3p_1 + 2.$$  

The right most formula is

$$\left(\frac{1}{2} + \frac{1}{3}\delta\right)\frac{n}{p_1} - 3p_1 + 2 \geq \frac{n}{2p_1} + 2.$$  

since $\frac{1}{3}\delta \frac{n}{p_1} \geq 3p_1$ by the assumption $p_1 \leq \frac{1}{3C(\delta)}\sqrt{n/\log n}$ and the definition of $C(\delta)$.

On the other hand, $|H|$ is at most $\frac{n}{p_1}$. Thus, $|A_2| \geq |H|/2 + 2$ and so by Lemma 4.1, $S_{A_2} = H$. Notice that $A_2 \subset H \cap A$. Thus $S_{A \cap H} = H$ which means that $A$ is nice, completing the proof.  

5. PROOF OF THEOREM 2.9

Without loss of generality, we can assume that $\delta \leq 1/2$ and $A$ has exactly $(1 + \delta)\frac{n}{p_1p_2}$ elements. Let $A_1$ be a subset of $A$ of size $(1 + \delta/2)\frac{n}{p_1p_2}$. Setting $D(\delta)$ sufficiently large, one can assume that $n$ is sufficiently large and $|A_1| \geq C(\delta/4)\sqrt{n/\log n}$ (where $C$ is defined as in (1)), thanks to the assumption

$$p_1p_2 \leq \frac{1}{D(\delta)}\sqrt{n/\log n}.$$  

This enables us to apply Corollary 3.4 to $A_1$ and conclude that $S_{A_1}$ contains a subgroup $H$ of size at least

$$(1 - \delta/4)|A_1| = (1 - \delta/4)(1 + \delta/2)\frac{n}{p_1p_2} > \frac{n}{p_1p_2}.$$  

The critical point here is that $|H|$ is larger than $\frac{n}{p_1p_2}$. This forces $|H| = n/q$ where $q$ is one of the primes $p_i$. It would be easy to finish the proof now if $A$ had at least $(2 + \delta)\frac{n}{p_1p_2}$ (instead of only $(1 + \delta)\frac{n}{p_1p_2}$) elements. The reason is that in this case we still have $(1 + \delta/2)\frac{n}{p_1p_2}$ elements outside $A_1$ to play with. Arguing as in the
previous proof, we can show that most of these elements should be in \( H \) and span it. As we lack these extra elements, we need an additional trick that helps us to show that actually most elements of \( A_1 \) are already in \( H \). The heart of this trick is Lemma 5.1 below. Before presenting the lemma, let us make some observations.

Set \( A_2 := A \setminus A_1 \). As \( A_1 \) was chosen arbitrarily, \( A_2 \) is an arbitrary subset of \( A \) with \( \frac{\delta n}{2p_1p_2} \) elements. Since \( A \) is incomplete, \( |A_2 \setminus H| \leq q - 2 \), where \( q = |G|/|H| \leq p_1p_2 \).

By setting \( D(\delta) \) sufficiently large, we can assume

\[
p_1p_2 \leq \frac{\delta^2}{20} \frac{n}{p_1p_2} = \frac{\delta}{10} |A_2|
\]

which implies

\[
|A_2 \cap H| \geq |A_2| - p_1p_2 \geq (1 - \delta/10)|A_2|.
\]

To summarize, \( A \) has the property that for any subset \( A_2 \) of size \( \frac{\delta n}{2p_1p_2} = \frac{\delta}{2(1 + \delta)} |A| \), there is a maximal subgroup \( H \) of \( G \) such that \( |A_2 \cap H| \geq (1 - \delta/10)|A_2| \).

**Lemma 5.1.** Let \( S \) be a subset of \( G \) of size \( (1 + \delta)\frac{n}{p_1p_2} \) such that no maximal subgroup of \( G \) contains \( (1 - \delta/2) \) fraction of \( S \). Then there is a subset \( S' \subset S \) of size \( \frac{\delta}{2(1 + \delta)} |S| \) such that no maximal subgroup of \( G \) contains \( (1 - \delta/10) \) fraction of \( S' \).

Assuming the lemma for a moment, we can conclude the proof as follows. By the lemma and its preceding paragraph, there is a maximal subgroup \( H \) such that

\[
|H \cap A| \geq (1 - \delta/2)|A| = (1 - \delta/2)(1 + \delta)\frac{n}{p_1p_2} \geq (1 + \delta/4)\frac{n}{p_1p_2}
\]

as \( \delta \leq 1/2 \). Since \( |H| \leq n/p_1 \) and the smallest prime divisor \( p' \) of \( H \) is either \( p_1 \) or \( p_2 \), it is easy to verify that

\[
|H \cap A| \geq \frac{|H|}{p'} + p'.
\]

Thus we can apply Theorem 1.1 or Theorem 2.7 for \( H \) and \( A \cap H \) to deduce that \( A \cap H \) is complete in \( H \). Therefore, \( S_{A \cap H} = H \) and \( A \) is nice.

Now we prove Lemma 5.1, using a probabilistic argument. **Proof** (Proof of Lemma 5.1) Set \( s := |S| = (1 + \delta)\frac{n}{p_1p_2} \) and \( \epsilon := \delta/10 \). Consider a random subset \( S_1 \) of \( S \) obtained by selecting each element \( a \in S \) to be in \( S_1 \) with probability \( \rho := (1 + 2\epsilon)\frac{\delta}{2(1 + \delta)} \), independently. Let \( H \) be a subgroup of \( G \). By linearity of expectation and the assumption of the lemma, we have

\[
\mathbb{E}(|H \cap S_1|) = \rho |H \cap S| \leq \rho (1 - \delta/2)s = \rho (1 - 5\epsilon)s.
\]

On the other hand, \( \mathbb{E}(|S_1|) = \rho s \). Both \( H \cap S_1 \) and \( S_1 \) have binomial distribution. By property of the binomial distribution, there is a positive constant \( c_0 \) depending only on \( \epsilon \) such that with probability at least \( 1 - \exp(-c_0 \rho s) \)

\[
(1 - \epsilon)\rho s \leq |S_1| \leq (1 + \epsilon)\rho s.
\]

and

\[
|H \cap S_1| \leq (1 + \epsilon)\rho (1 - 5\epsilon)s.
\]
It is well known (and easy to prove) that the number of maximal subgroups of \( G \) is at most \(|G|=n\). If \( D(\delta) \) (and so \( n \)) is sufficiently large, then
\[
2n \leq \exp(c_0\rho s).
\]

Thus, we can use the union bound to conclude that there exists a set \( S_1 \) such that (8) holds and (9) holds simultaneously for every maximal subgroup \( H \). Let \( S' \) be any subset of \( S_1 \) of size \( \frac{2n}{2n_{1/2}} \leq 1+2\epsilon \rho s \). For any maximal subgroup \( H \)
\[
|S' \cap H|/|S'| \leq |S_1 \cap H|/|S'| \leq \frac{(1+\epsilon)(1-5\epsilon)\rho s}{1+2\epsilon \rho s} < (1-\epsilon) = (1-\delta/10).
\]

This concludes the proof of the lemma.

The following example shows that the lower bound \((1+\delta)n_{1/2}^{1/4-\alpha}\) cannot be reduced to \( n_{1/2}^{1/4-\alpha} \), for any fixed \( \alpha \).

Example. Take \( n := p^2q \) where \( 1 < p < q \) are large primes. Consider \( G = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_q \).
Given any \( \delta > 0 \) and any function \( D(\delta) \), by choosing \( q \) properly \( p \) we can guarantee that
\[
n^{1/2-\alpha} \leq p^2 \leq \frac{1}{D(\delta)} \sqrt{n/\log n}.
\]

We write an element \( a \in G \) as \( a = (x, y) \) where \( x \in \mathbb{Z}_{p^2} \) and \( y \in \mathbb{Z}_q \). Let \( m \) be the largest integer such that \( \sum_{i=0}^{m} i < p^2 - 1 \). Set
\[
A := \{(x, 0)|0 \leq x \leq m\} \cup \{(0, y)|0 \leq y \leq q-1\}.
\]

It is easy to show that \( A \) is incomplete and not nice, thanks to the fact that \( \sum_{i=0}^{m} i < p^2 - 1 \). On the other hand,
\[
|A| = m + q = m + \frac{n}{p^2} = m + \frac{n}{p_1 p_2} \geq n^{1/4-\alpha} + \frac{n}{p_1 p_2}.
\]

The proof of the theorem is complete.

6. Concluding remarks

One can use the additional trick in the proof of Theorem 2.9 to improve upon the constant \((5/6 + \delta)\) in Theorem 2.7. However, this requires some modification on the assumptions. We prefer to present Theorem 2.7 in the simplest way in order to illustrate the ideas.

One can also use the method presented here to study incomplete sets with size less than \( \frac{n}{p_1 p_2} \). However, the characterization obtained in this case is more technical and less appealing.
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