Capturing information on curves and surfaces from their projected images
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Abstract

Obtaining complete information about the shape of an object by looking at it from a single direction is impossible in general. In this paper, we theoretically study obtaining differential geometric information of an object from orthogonal projections in a number of directions. We discuss relations between (1) a space curve and the projected curves from several distinct directions, and (2) a surface and the apparent contours of projections from several distinct directions, in terms of differential geometry and singularity theory. In particular, formulae for recovering certain information on the original curves or surfaces from their projected images are given.

1 Introduction

As is well known via triangulation, when we look at a point from two known viewpoints, we can then calculate where the point is. Let us turn our attention to the case of a surface. When we look at a surface, then we observe an apparent contour (a contour), and there are numerous studies on how to obtain information about surfaces from apparent contours (see [1, 2, 6], for example).

One cannot obtain complete information from a finite number of apparent contours, in general. However, to obtain the Gaussian curvature of a surface, information about the second order derivatives of the surface is required, and in [10, 11], Koenderink showed that one can obtain the Gaussian curvature of a surface as the product of the curvature of the contour and the normal curvature along a single direction. While Koenderink’s result needs more information than just the apparent contour, that is, it needs the normal curvature, this fact still suggests that we might be able to obtain some information about a surface from curvatures of small numbers of contours of the surface. It then is natural to ask how much information about the contour is enough to get the desired information about a surface.

\*2010 Mathematics Subject Classification. Primary 53A05; Secondary 53A04.
\*Keywords and Phrases: contour, differential geometry, singularity theory, curve, surface, Koenderink’s formulae, projection.
\*Partly supported by the JSPS KAKENHI Grants numbered 16J02200 and 18K03301.
In this paper, we consider an orthogonal projection of $\mathbb{R}^3$ to a plane:

$$\pi_\xi(x) = x - \langle x, \xi \rangle \xi : \mathbb{R}^3 \to \xi^\perp$$

for a unit vector $\xi \in \mathbb{R}^3$. The map $\pi_\xi$ is called the orthogonal projection in the direction $\xi$. Our interest is in getting local information on surfaces (or curves) from the curvatures of the contours (or the projected curves) with respect to orthogonal projections. In particular, we show how much information about contours (or image curves) is enough to recover the lower degree terms of the Taylor expansions of surfaces (or curves) at observed points. In addition, we construct some examples of sets of different surfaces whose information with respect to contours for certain orthogonal projections is exactly the same (Figure 3.3, 3.4, 3.5). See [1, 2, 3, 4, 5, 6, 12] for other approaches to these kinds of considerations.

Throughout the paper, we use the following notation for the Taylor coefficients of a given function. For a $C^\infty$ function $\psi : I \to \mathbb{R}$, we set

$$\text{coef}_0(\psi, t, k) = \text{coef}(\psi, t, k) = \left(\psi(0), \frac{\psi'(0)}{2}, \ldots, \frac{\psi^{(k)}(0)}{k!}\right)$$

($'=\frac{d}{dt}$ and $\psi^{(i)} = (\psi^{(i-1)}')'$ for $i = 1, 2, \ldots$), namely, if $h = a_0 + \sum_{i=1}^{k} (a_i / i!) t^i$, then $\text{coef}(\psi, t, k) = (a_0, a_1, \ldots, a_k)$. The data $\text{coef}(\psi, t, k)$ is called the $k$-th order information of $\psi$ at 0. We remark that the $k$-th order information of the given function $\psi$ at 0 represents the $k$-jet of $\psi$ at 0 in the terminology of singularity theory (cf. [9]).

### 1.1 Projections of curves

Let $I$ be an open interval containing 0, and let $\gamma : I \to \mathbb{R}^3$ ($\gamma(0) = (0, 0, 0)$) be a given unknown regular $C^\infty$ curve whose curvature does not vanish at 0. We remark that $\gamma$ has the orientation induced from that of $I$. Rotating the coordinate system of $\mathbb{R}^3$ if necessary, for any $k \in \mathbb{N}$, we may assume that $\gamma$ is locally written around 0 as

$$\gamma(t) = \left( t, \sum_{i=2}^{k} \frac{a_i}{i!} t^i, \sum_{i=3}^{k} \frac{b_i}{i!} t^i \right) + (O(k+1), O(k+1), O(k+1)), \quad (1.1)$$

where $a_i, b_i \in \mathbb{R}$ ($i = 2, \ldots, k$), and $O(k+1)$ stands for the terms whose degrees are greater than $k$. Specifically, $a_2$ and $b_3$ are important values of the space curve: the curvature and the torsion at 0. Set $\gamma_\xi = \pi_\xi \circ \gamma$ for a unit vector $\xi \in \mathbb{R}^3$. Our aim is to investigate how many conditions are enough to
recover the above coefficients in terms of the curvatures of $\gamma_\xi$ using a number of distinct directions $\xi$.

Since the setting is complicated for general choices of projection directions, we focus on the two singular cases where the kernel direction $\xi$ of an orthogonal projection is geometrically restricted. The following are our settings, and also abstracts of the results which will be given in Section 2:

1. We take two linearly independent vectors $\xi_1, \xi_2$, where each projected curve $\gamma_{\xi_i}$ ($i = 1, 2$) has an inflection point at 0. This implies that $\xi_1, \xi_2$ lie in the osculating plane, with the exception of the tangent line of $\gamma$ (Figure 2.1). Then the coefficients $a_i, b_i$ can be uniquely determined from the certain order of the information of the curvature functions of $\gamma_{\xi_i}$ ($i = 1, 2$) at 0. Namely, the coefficients $a_i, b_i$ can be uniquely determined by the information of the derivatives of the curvatures of the two projected curves.

2. We take $\xi_1$ as a tangent vector of $\gamma$ at 0. Then, $\gamma_{\xi_1} = \pi_{\xi_1} \circ \gamma$ has a singular point at 0 (Figure 2.2). We also take another vector $\xi_2$. Then the coefficients $a_i, b_i$ ($i \leq 5$) can be uniquely determined by the information of the curvature functions of $\gamma_{\xi_i}$ ($i = 1, 2$) at 0. Namely, the coefficients $a_i, b_i$ ($i \leq 5$) can be uniquely determined from the curvature functions of two projected curves from the tangential direction and another direction. The notion of the cuspidal curvature of a singular plane curve (introduced in [17]), especially, plays an important role.

1.2 Projections of surfaces

Let $U$ be an open subset of $\mathbb{R}^2$ containing $0 = (0, 0)$, and let $f : U \to \mathbb{R}^3$ ($f(0) = (0,0,0)$) be a given unknown regular $C^\infty$ surface. Without loss of generality, we may assume that $f$ is given by

$$f(u, v) = \left(u, v, h(u,v)\right), \quad h(u,v) = \frac{a_{20}}{2} u^2 + \frac{a_{02}}{2} v^2 + \sum_{i+j=3}^{k} \frac{a_{ij}}{i!j!} u^i v^j + O(k+1),$$

(1.2)

where $a_{ij} \in \mathbb{R}$ ($i, j = 0, 1, 2, \ldots, k$). We call $a_{20}, a_{02}$ (respectively, $a_{30}, a_{21}, a_{12}, a_{03}$) the second order (respectively, the third order) information of $f$ at 0. Taking vectors which are tangent to the image of $f$ at 0, we consider apparent contours of $f$ projected from the directions of these vectors. For tangent vector $\xi$ of the image of $f$ at 0, we set $f_\xi = \pi_\xi \circ f$. We call the set $S$ of singular points the contour generator (with respect to $\xi$), and $f_\xi(S)$ the contour (of $f$ with respect to $\xi$).
The following are abstracts of the results on surfaces which will be given in Section 3:

- We take three “general” (respectively, four “general”) distinct directions. Then the second order (respectively, the third order) information of a surface is uniquely determined by the 0-th order (respectively, the first order) information of the curvatures of the contours with respect to the directions. We remark that knowing the second order information is the same as knowing the pair of values of the mean and Gaussian curvatures. Moreover, formulae on the relations between the information of the surfaces and the curves are explicitly given.

- We give an example of a pair of different surfaces having the same information from the curvatures of contours with respect to two distinct directions. Namely, a surface \( f \) with two distinct directions \((\xi_1, \xi_2)\) and another surface \( \tilde{f} \) with two distinct directions \((\tilde{\xi}_1, \tilde{\xi}_2)\) are constructed, such that the information on the two contours of \( f \) with respect to \( \xi_1 \) and \( \xi_2 \) is the same as the information on the two contours of \( \tilde{f} \) with respect to \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) (Figures 3.3, 3.4, 3.5).

- We show that if the Gaussian curvature is positive, then there exist two directions such that the product of the contours of these directions gives the Gaussian curvature.

**Remark 1.1.** From the above results, we see that in order to judge the sign of the Gaussian at an observed point of a given surface, looking at it along general three directions in the tangent plane is necessary and sufficient.

## 2 Projections of space curves

Let \( \gamma : I \to \mathbb{R}^3 \) be a \( C^\infty \) curve \( (\gamma(0) = (0, 0, 0)) \), and let \( \gamma_\xi = \pi_\xi \circ \gamma \) for \( \xi \) with \( \pi \) given as in the introduction. We assume that the curvature of \( \gamma \) does not vanish at 0. We consider the following two cases. The first case is that the projection curve \( \gamma_\xi \) has an inflection point at 0. We remark that \( \gamma_\xi \) has the osculating plane. The second case is that one of the projection curve \( \gamma_\xi \) has a singular point, namely, the vector \( \xi \) is tangent to \( \gamma \) at 0.

### 2.1 Projections in the osculating plane

In this section, we consider the case that the curvature of \( \gamma \) does not vanish and \( \gamma_\xi \) has an inflection point at 0. Then it holds that \( \xi \) lies in the osculating plane, except for the tangent line of \( \gamma \) at 0. We remark that \( \gamma \) has the
orientation induced from that of $I$. Then rotating the coordinate system of $R^3$ if necessary, we may assume that $\gamma$ is written as in (1.1) and $\xi(\theta_j) = (\cos \theta_j, \sin \theta_j, 0)$, where $0 < \theta_j < \pi$ ($j = 1, 2, \ldots$). We give the orientation of $\xi^\perp$ as follows: We take a basis $\{X,Y\}$ of $\xi^\perp$. We say that $\{X,Y\}$ is a positive basis if $\{X,Y,\xi\}$ is a positive basis of $R^3$. We set the orientation of $\pi_x \circ \gamma$ to agree with that of $\gamma$ (see Figure 2.1). We set $\pi_x \circ \gamma = \gamma_{\theta_j}$, and also we set $s_j$ to be the arc-length of $\gamma_{\theta_j}$, and set $\kappa_{\theta_j}$ to be the curvature of $\gamma_{\theta_j} \subset \xi^\perp$ as a curve in the oriented plane $\xi^\perp$.

\[ \text{Figure 2.1: Orientations of } \xi^\perp \text{ and } \pi_x \circ \gamma. \]

Suppose that we are given the information of the curvatures of the contours from two distinct directions $0 < \theta_1, \theta_2 < \pi$. Set $\varphi = \theta_1 - \theta_2$ and $\tilde{\kappa}_{\theta_j} = (d\kappa_{\theta_j}/ds_j(0))^{1/3}$. We can determine the local information of space curves from local information of projected curves in two distinct directions:

**Theorem 2.1.** If $(d\kappa_{\theta_1}/ds_1(0), d\kappa_{\theta_2}/ds_2(0)) \neq (0,0)$, then the following hold.

1. $\theta_1, \theta_2$ and $b_3$ are uniquely determined from the second order information of $\kappa_{\theta_1}, \kappa_{\theta_2}$ at 0 and $\varphi = \theta_1 - \theta_2$.

2. The coefficients $a_{n-2}$ and $b_n$ are uniquely determined from the $(n-1)$-st order information of $\kappa_{\theta_1}, \kappa_{\theta_2}$ at 0 and $\varphi$ for $n \geq 4$.

The explicit formulae for $\theta_1, \theta_2 = \theta_1 + \varphi$ and $b_3$ are given as in (2.2) and (2.5) in the proof.

**Proof.** Note that the curvature $\kappa_{\theta_j}$ for a general parameter $t$ can be calculated by

\[ \kappa_{\theta_j}(t) = \frac{\det \left( \gamma'(t), \gamma''(t), \xi_{\theta_j}(t) \right)}{|\gamma'_{\theta_j}(t)|^3} \left( \frac{d}{dt} = \frac{d}{dt} \right). \]
Since $0 < \theta_1 < \pi$, it holds that $\sin \theta_1 \neq 0$, and by a direct calculation we have

$$
\text{coef}(\kappa_{\theta_1}, s_1, 3) = \left(0, -\frac{b_3}{\sin^3 \theta_1}, \frac{b_3 \sin \theta_1 + 6a_2 b_3 \cos \theta_1}{2 \sin^5 \theta_1}, \frac{-45a_2^2 b_3 \cos^2 \theta_1 + b_5 \sin^2 \theta_1 + 10(a_3 b_3 + a_2 b_4) \sin \theta_1 \cos \theta_1}{6 \sin \theta_1}\right).
$$

(2.1)

Then we see that the condition $(d\kappa_{\theta_1}/ds_1(0), d\kappa_{\theta_2}/ds_2(0)) \neq (0, 0)$ implies $b_3 \neq 0$, namely, the torsion of $\gamma$ at 0 does not vanish. By this argument, the fact $d\kappa_{\theta_1}/ds_1(0) \neq 0$ and $d\kappa_{\theta_2}/ds_2(0) \neq 0$ follows from the assumption $(d\kappa_{\theta_1}/ds_1(0), d\kappa_{\theta_2}/ds_2(0)) \neq (0, 0)$. Taking another direction $\theta_2 = \theta_1 + \varphi$ ($0 < \theta_2 < \pi$), we may consider $\kappa_{\theta_1}, \kappa_{\theta_2}, \varphi$ to be known. Since the equation

$$
\frac{d\kappa_{\theta_1}}{ds_1(0)} \frac{d\kappa_{\theta_2}}{ds_2(0)} = \sin^3(\theta_1 + \varphi)
$$

can be solved as

$$
\theta_1 = \cot^{-1} \left(\left(\frac{d\kappa_{\theta_1}}{ds_1(0)} \frac{1}{\sin \varphi} \left(\frac{d\kappa_{\theta_2}}{ds_2(0)}\right)^{1/3} - \cos \varphi\right)\right) \in (0, \pi),
$$

(2.2)

we obtain $\theta_1$ and $\theta_2$.

Furthermore, by (2.1), it holds that

$$
\sin \theta_i = -\frac{\tilde{b}}{\tilde{\kappa}_{\theta_i}}, \quad (i = 1, 2)
$$

(2.3)

where $\tilde{b} = b_3^{1/3}$ and $\tilde{\kappa}_{\theta_i} = (d\kappa_{\theta_i}/ds_1(0))^{1/3}$. Substituting (2.3) into the trigonometric identity

$$
\cos^2(\theta_1 - \theta_2) + \sin^2 \theta_1 + \sin^2 \theta_2 - 2 \sin \theta_1 \sin \theta_2 \cos(\theta_1 - \theta_2) - 1 = 0,
$$

we have

$$
(\tilde{\kappa}_{\theta_1}^2 - 2 \cos \varphi \tilde{\kappa}_{\theta_1} \tilde{\kappa}_{\theta_2} + \tilde{\kappa}_{\theta_2}^2)\tilde{b}^2 - \sin^2 \varphi \tilde{\kappa}_{\theta_1} \tilde{\kappa}_{\theta_2}^2 = 0.
$$

(2.4)

Note that $\tilde{\kappa}_{\theta_1}^2 - 2 \cos \varphi \tilde{\kappa}_{\theta_1} \tilde{\kappa}_{\theta_2} + \tilde{\kappa}_{\theta_2}^2 \geq 0$ and $\tilde{\kappa}_{\theta_1}^2 - 2 \cos \varphi \tilde{\kappa}_{\theta_1} \tilde{\kappa}_{\theta_2} + \tilde{\kappa}_{\theta_2}^2 = 0$ if and only if $\varphi = \tilde{\kappa}_{\theta_1} - \tilde{\kappa}_{\theta_2} = 0$ or $\tilde{\kappa}_{\theta_1} = \tilde{\kappa}_{\theta_2} = 0$. Thus by the assumption,
\[ b_3 = \left( \frac{\sin^2 \varphi \kappa_{\theta_1}^2 \kappa_{\theta_2}^2}{\kappa_{\theta_1}^2 - 2 \cos \varphi \kappa_{\theta_1} \kappa_{\theta_2} + \kappa_{\theta_2}^2} \right)^{3/2} \], \quad \left( \kappa_{\theta_i} = \left( \frac{d\kappa_{\theta_i}}{ds_i}(0) \right)^{1/3} \right) \quad (i = 1, 2).

Thus the claim (1) holds.

Since the pair of equations

\[ \frac{d^2 \kappa_{\theta_i}}{ds_i^2}(0) = \frac{b_4 \sin \theta_i + 6a_2 b_3 \cos \theta_i}{2 \sin^3 \theta_i} \quad (i = 1, 2) \]

is a linear system for \(a_2, b_4\), by \(\theta_1 \neq \theta_2\) we obtain \(a_2\) and \(b_4\) from (2.4) if \(b_3 \neq 0\). Thus the claim (2) when \(n = 4\) holds. Next, we show (2) when \(n \geq 5\). We assume that \(\varphi, a_1, \ldots, a_{n-3}\) and \(b_1, \ldots, b_{n-1}\) are known for \(n \geq 5\). Now we take the \((n - 2)\)-th derivative of \(\kappa_{\theta_i}\) with respect to \(s_i\). That value of the derivative at 0 is written in terms of \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\). In the formula of \(d^{n-2} \kappa_{\theta_i}/ds_i^{n-2}(0)\), as a polynomial of \(a_1, \ldots, a_n, b_1, \ldots, b_n\), we show that \(a_{n-1}, a_n\) do not appear, and \(a_{n-2}, b_n\) appear only to the first power linear terms. We set \(\kappa_{\theta_i} = \kappa\) and \(s_i = s\) for simplicity, for the moment. Let \(t(s)\) be the inverse function of

\[ s = \int_0^t |\gamma_{t_i}(t)|\, dt, \]

where \(\gamma' = d/dt\). By the formula for differentiation of a composition of functions (see [15, (3.4)] or [33, (3.6)] for example),

\[ \frac{d^{n-2}}{ds^{n-2}} \kappa(t(s)) = \sum_{k=0}^{n-2} \frac{d^k \kappa}{dt^k}(t(s)) \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (t(s))^{k-j} \frac{d^{n-2-j}}{ds^{n-2-j}}(t(s))^j, \]

and we look at the terms

\[ \frac{d^j t(s)}{ds^j}(0), \quad \frac{d^j \kappa}{dt^j}(0), \quad (j = n - 4, n - 3, n - 2). \]

Since \(dt/ds = |\gamma'_{t_i}|^{-1}\), it holds that

\[ \frac{d^j t}{ds^j} = \frac{d^{j-1} t}{ds^{j-1}} \left( \frac{1}{|\gamma_{t_i}'(t(s))|} \right) = \frac{d^{j-1} t}{ds^{j-1}} \left( \left\langle \gamma_{t_i}'(t(s)), \gamma_{t_i}'(t(s)) \right\rangle \right)^{-1/2} \]

\[ = -\frac{d^{j-2} t}{ds^{j-2}} \left( \left\langle \gamma_{t_i}'(t(s)), \gamma_{t_i}'(t(s)) \right\rangle \right)^{-3/2} \left\langle \gamma_{t_i}'(t(s)), \gamma_{t_i}''(t(s)) \right\rangle \frac{dt(s)}{ds} \]

\[ = -\frac{d^{j-2} t}{ds^{j-2}} \left( \left\langle \gamma_{t_i}'(t(s)), \gamma_{t_i}'(t(s)) \right\rangle \right)^{-2} \left\langle \gamma_{t_i}'(t(s)), \gamma_{t_i}''(t(s)) \right\rangle \frac{dt(s)}{ds}. \]
Continuing to calculate the derivative of the function by \( s \), we finally obtain a polynomial consisting of terms with some powers of \( \langle \gamma_{\theta_i}^j(t(s)), \gamma_{\theta_i}^k(t(s)) \rangle \) and

\[
\left\langle \gamma_{\theta_i}^{(\ell_1)}(t(s)), \gamma_{\theta_i}^{(\ell_2)}(t(s)) \right\rangle \quad (1 \leq \ell_1 \leq \ell_2 \leq j, \ 3 \leq \ell_1 + \ell_2 \leq j + 1).
\]

This implies that \( a_{n-1}, a_n, b_n \) do not appear in \( (d^2t(s)/ds^2)(0) \) \( (j = n-4, n-3, n-2) \). Moreover, although \( a_{n-2} \) may appear in \( (d^{n-2}t(s)/ds^{n-2})(0) \) from the term \( \langle \gamma_{\theta_i}^j, \gamma_{\theta_i}^{(n-2)} \rangle \), it does not actually appear, since \( \gamma_{\theta_i}^j(0) = (1, 0, 0) \) and the first component of \( \gamma_{\theta_i}^{(n-2)}(0) \) is 0. On the other hand, since \( \gamma \) is given by (1.1),

\[
\begin{align*}
\det(\gamma', \gamma^{(n-2)}, \xi_{\theta_i}) &= -b_{n-2} \sin \theta_i, \quad \det(\gamma', \gamma^{(n-1)}, \xi_{\theta_i}) = -b_{n-1} \sin \theta_i, \\
\det(\gamma', \gamma^{(n)}, \xi_{\theta_i}) &= a_2 b_{n-2} \cos \theta_i, \\
\det(\gamma'', \gamma^{(n-2)}, \xi_{\theta_i}) &= a_2 b_{n-1} \cos \theta_i, \\
\det(\gamma'', \gamma^{(n-1)}, \xi_{\theta_i}) &= a_2 b_{n-1} \cos \theta_i,
\end{align*}
\]

and \( a_{n-2}, a_{n-1}, a_n, b_n \) do not appear in \( \det(\gamma', \gamma^{(n)}, \xi_{\theta_i})(0) \) \( (j = 1, \ldots, n-3) \). This implies that they also do not appear in \( d^2\kappa/dt^2 \) \( (j = 1, \ldots, n-3) \).

Furthermore, since

\[
\frac{d^{n-2}\kappa}{dt^{n-2}} = \sum_{j+k \leq n} c_{jk} \left( \frac{1}{|\gamma_{\theta_i}|^3} \right)^{(n-2-j-k)} \det(\gamma^{(j)}, \gamma^{(k)}, \xi_{\theta_i})
+ \frac{1}{|\gamma_{\theta_i}|^3} \left( \det(\gamma', \gamma^{(n)}, \xi_{\theta_i}) + (n-3) \det(\gamma'', \gamma^{(n-1)}, \xi_{\theta_i}) \right)
+ m \det(\gamma'', \gamma^{(n-2)}, \xi_{\theta_i})
\]

\( (n \geq 5) \) holds, where \( m = (n-5)(n-2)/2 \) and \( c_{jk} \) are natural numbers, this implies that \( a_{n-1}, a_n \) do not appear in (2.6). Moreover, since the coefficient of \( d^{n-2}\kappa/dt^{n-2} \) in (2.6) is \( (dt/ds)^{n-2} \), setting

\[
A_i = \frac{1}{|\gamma_{\theta_i}'(0)|^3} \left( \frac{dt}{ds}(0) \right)^{n-2},
\]

the equations (2.6) at \( s = 0 \) for \( i = 1, 2 \) are

\[
\begin{align*}
\frac{d^{n-2}\kappa_{\theta_1}}{ds_1^{n-2}}(0) &= -A_1(b_n \sin \theta_1 + mb_3 a_{n-2} \cos \theta_1) + B_1, \\
\frac{d^{n-2}\kappa_{\theta_2}}{ds_2^{n-2}}(0) &= -A_2(b_n \sin \theta_2 + mb_3 a_{n-2} \cos \theta_2) + B_2,
\end{align*}
\]
where \( B_i \) \((i = 1, 2)\) are terms consisting of \( a_1, \ldots, a_{n-3} \) and \( b_1, \ldots, b_{n-1} \). The equation (2.9) can be solved when \( A_1 A_2 b_3 \sin(\theta_1 - \theta_2) \neq 0 \). Since \( dt/ds(0) = |\gamma_0'(0)|^{-1} \), we have the assertion.

Since obtaining \( a_2, a_3, b_3 \) is equivalent to obtaining the curvature, the first derivative of the curvature and the torsion, results of this type for the perspective projection can be found in [2, 14] and [6, Theorem 8]. Since we can detect the coefficients of the Taylor expansion of \( \gamma \), using our result, one can easily construct the desired curve whose projections have the prescribed curvatures.

### 2.2 Projections in the tangential direction and another direction

In this section, we consider the case that \( \xi_1 \) is tangent to \( \gamma \) at 0. In this case, \( \gamma_{\xi_1} = \pi_{\xi_1} \circ \gamma \) has a singular point at 0. To consider differential geometric invariants of the singular curve, we describe the cuspidal curvature of singular plane curves introduced in [17] (see also [18]). Let \( c : I \rightarrow \mathbb{R}^2 \) be a plane curve, and \( c'(0) = 0 \). The curve \( c \) is said to be \( A \)-type at 0 if \( c''(0) \neq 0 \). Let \( c \) be an \( A \)-type curve at 0. Then

\[
\mu = \left| \frac{\det(c''(0), c'''(0))}{|c''(0)|^{5/2}} \right|,
\]

does not depend on the choice of parameter, and is called the cuspidal curvature.

Let \( \gamma : I \rightarrow \mathbb{R}^3 \) be a \( C^\infty \) curve with non-zero curvature at 0. We assume that \( \gamma_{\xi_1} \) has a singular point at 0. Since the curvature of \( \gamma \) does not vanish, by the Frenet formula, \( \gamma_{\xi_1}(0) \) is an \( A \)-type curve at 0. We also assume that there exists an integer \( N \) such that \( \det((\gamma_{\xi_1})'', (\gamma_{\xi_1})^{(i)})(0) = 0 \) \((i = 3, 5, \ldots, 2N - 1)\) and \( \det((\gamma_{\xi_1})'', (\gamma_{\xi_1})^{(2N+1)})(0) \neq 0 \). We give the positively oriented \( xyz \)-coordinate system for \( \mathbb{R}^3 \), and rotating this coordinate system, we give a \( yz \)-coordinate system for \( \xi_1^+ \) as follows. We set the \( y \)-axis as the direction of \( (\gamma_{\xi_1})''(0) \), and set the \( x \)-axis as the direction of \( \xi_1 \). We give an orientation of \( \gamma_{\xi_1} \) so that \( \det((\gamma_{\xi_1})'', (\gamma_{\xi_1})^{(2N+1)})(0) > 0 \), and also so that of \( \gamma \) agrees with that of \( \gamma_{\xi_1} \) (see Figure 2.2). Then we may assume that \( \gamma \) is given by (1.1) with \( a_2 > 0, b_3 \geq 0 \). Then \( \mu = b_3/a_2^{3/2} \). On the other hand, we consider a unit vector \( \xi_2 = (\sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2, \cos \theta_1) \) \((\cos \theta_1 \neq 0)\) which is not tangent to \( \gamma \) at 0. Since we take the above \( xyz \)-coordinate, \( \theta_1, \theta_2 \) are known.

**Theorem 2.2.** Suppose that \( \cos \theta_1 \neq 0 \), then the followings hold.
Figure 2.2: Orientations of $\xi_1$ and $\pi_{\xi_1} \circ \gamma$.

(1) The coefficients $a_2$ and $b_3$ are uniquely determined by the cuspidal curvature $\mu$ and the 0-th order information of $\kappa_{\xi_2}$ at 0.

(2) In addition to (1), $a_3$ is uniquely determined by the cuspidal curvature $\mu$ and the first order information of $\kappa_{\xi_2}$ at 0.

Proof. We remark that $\cos \theta_1 \neq 0$, and then $|\sin \theta_1 \cos \theta_2| < 1$, equivalently $\cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_2 \neq 0$. The curvature $\kappa_{\xi_2}$ of the plane curve $\gamma_{\xi_2}$ satisfies

$$
\text{coef}(\kappa_{\xi_2}, s, 1) = \left( \frac{a_2 \cos \theta_1}{(\cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_2)^{3/2}}, \frac{Q(\theta)}{(\cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_2)^{3}} \right),
$$

(2.10)

where

$$
Q(\theta) = -b_3 \cos^2 \theta_1 \cos^2 \theta_2 \sin \theta_1 \sin \theta_2 - b_3 \sin \theta_1 \sin^3 \theta_2 \\
+ \cos^3 \theta_1 \cos \theta_2 (a_3 \cos \theta_2 - 3a_2^2 \sin \theta_2) + \cos \theta_1 \sin \theta_2 (3a_2^2 \cos \theta_2 + a_3 \sin \theta_2).
$$

Since we know $\mu = b_3/a_2^{3/2}$ and $\theta_1, \theta_2$, we obtain $a_2$ and $b_3$ from the first component of (2.10) if $\cos \theta_1 \neq 0$. Furthermore, we also obtain $a_3$ from the second component of (2.10), under the assumption $\cos \theta_1 (\cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_2) \neq 0$.

See [13, Section 3] for relationships between invariants of space curves and projected plane curves.
3 Projections of surfaces

In this section, we consider the local information of surfaces and contours. Let $U \subset \mathbb{R}^2$ be a neighborhood of the origin $0 = (0, 0)$. Let $f : U \to \mathbb{R}^3$ ($f(0) = (0, 0, 0)$) be a $C^\infty$ surface with non-vanishing Gaussian curvature at $0$. Without loss of generality, we may assume that $f$ is written in the form (1.2) with $a_{20}a_{02} \neq 0$, $a_{20} > a_{02}$, $a_{20} > 0$. We set the unit normal vector $\nu$ of $f$ so that it satisfies $\nu(0, 0) = (0, 0, 1)$.

3.1 Obtaining information about surfaces from contours

Let $\xi$ be a unit vector which is tangent to $f$ at $0$. Then we may assume $\xi = \xi(\theta_1) = (\cos \theta_1, \sin \theta_1, 0)$, where $0 < \theta_1 < \pi$. The set $S$ of singular points of the map $\pi_\xi(\theta_1) \circ f$ is

$$S = \{(u, v) \mid \cos \theta_1 h_u + \sin \theta_1 h_v = 0\}.$$  

We assume

$$p(\theta_1) = a_{20} \cos^2 \theta_1 + a_{02} \sin^2 \theta_1 \neq 0,$$

which implies that the direction $\xi(\theta_1)$ is not an asymptotic direction of $f$. By this assumption,

$$\left((\cos \theta_1 h_u + \sin \theta_1 h_v), (\cos \theta_1 h_u + \sin \theta_1 h_v)\right)(0, 0)$$

$$= (a_{20} \cos \theta_1, a_{02} \sin \theta_1) \neq (0, 0),$$

and there exists a regular parametrization of $S$. For the sake of taking this parametrization, we set an orientation of $S$ as follows. First, we give an orientation of the normal plane $\xi(\theta_1) \perp \xi(\theta_1)$ such that $X = (-\sin \theta_1, \cos \theta_1, 0)$, $Y = (0, 0, 1)$ is a positive basis. Next, we put an orientation on $(\pi_\xi(\theta_1) \circ f)(S)$ so that it agrees with the direction of $X$, and also put that of $S$ agreeing with $(\pi_\xi(\theta_1) \circ f)(S)$ (see Figure 3.1). Let $k_\theta$ be the curvature of the contour from a direction $\theta$.

**Lemma 3.1.** The first order information $k_\theta$ with respect to the arc-length parameter $s$ (suitably oriented) are

$$\text{coef}(k_{\theta_1}, 1, s) = \left(\frac{a_{20}a_{02}}{p(\theta_1)} \cdot \frac{q(\theta_1)}{p(\theta_1)^3}\right),$$  

(3.2)
where

\[
q(\theta_1) = a_{03}a_{20}^3 \cos^3 \theta_1 - 3a_{02}a_{12}a_{20}^2 \cos^2 \theta_1 \sin \theta_1 \\
+ 3a_{02}^2a_{20}a_{21} \cos^2 \theta_1 \sin^2 \theta_1 - a_{02}^3a_{30} \sin^3 \theta_1.
\]  (3.3)

Proof. Since \(a_{02} \sin \theta_1 \neq 0\), we can take a parametrization \(C(t) = (t, c(t))\) of \(S\). Then since \(\pi_{\xi(\theta_1)} \circ f \circ C\) lies on the plane \(\xi(\theta_1)^\perp\),

\[
k_{\theta_1}(0) = \frac{\det (\hat{C}'', \hat{C}'''', \xi(\theta_1))}{|\hat{C}'|^3}(0),
\]

\[
\frac{dk_{\theta_1}}{ds}(0) = \frac{\det (\hat{C}'', \hat{C}'''', \xi(\theta_1))}{|\hat{C}'|^4}(0) + 3 \left\langle \hat{C}'', \hat{C}'''', \xi(\theta_1) \right\rangle \frac{\det (\hat{C}'', \hat{C}'''', \xi(\theta_1))}{|\hat{C}'|^6}(0)
\]

where \(\hat{C}' = f \circ C = f(t, c(t))\) and \(t' = d/dt\). Since \(h'(t, c(t)) = 0\) at \(t = 0\), it holds that

\[
det (\hat{C}'', \hat{C}^{(i)}', \xi(\theta_1)) = (c' \cos \theta_1 + \sin \theta_1)(h \circ C)^{(i)} \quad (i = 2, 3),
\]

where \(\hat{C}^{(2)} = \hat{C}''\) and \(\hat{C}^{(3)} = \hat{C}'''\). By (3.1), it holds that

\[
\text{coef}(c(t), 2, t) = \left(0, -\frac{a_{20} \cos \theta_1}{a_{02} \sin \theta_1}, \frac{1}{a_{02} \sin^3 \theta_1} \tilde{q}(\theta_1)\right),
\]

Figure 3.1: Orientations of \(\xi^\perp\) and the contour.
where
\[ \tilde{q}(\theta_1) = -a_{12}a_{20}^2 \cos^3 \theta_1 - a_{03}a_{20}^2 \cos^2 \theta_1 \sin \theta_1 + 2a_{02}a_{20}a_{21} \cos^2 \theta_1 \sin \theta_1 + 2a_{02}a_{12}a_{20} \cos \theta_1 \sin^2 \theta_1 - a_{02}a_{20}a_{21} \cos^2 \theta_1 - a_{02}a_{21} \sin^3 \theta_1. \]

Summarizing up the above calculation, we have the assertion.

The inner product of \((\pi_{\xi(\theta_1)} \circ f \circ C)'(0)\) and \(X\) is
\[ \langle (\pi_{\xi(\theta_1)} \circ f \circ C)'(0), X \rangle = \frac{-1}{a_{02} \sin \theta_1} p(\theta_1). \] (3.4)

Let \(s\) be the arc-length parameter of \(S\) where the orientation is given in the above manner. Thus we remark that by (3.4), if \(a_{02} \sin \theta_1 p(\theta_1)\) is negative, \(s\) is the opposite direction to the above parameter \(t\).

**Remark 3.2.** If \(a_{20}a_{02} \neq 0\) and \(p(\theta_1) \neq 0\), then \(q(\theta_1) = 0\) if and only if the contour has a vertex at \((\pi_{\xi(\theta_1)} \circ f \circ C)(0)\), and \(\xi(\theta_1) = (\cos \theta_1, \sin \theta_1, 0)\) is called the cylindrical direction of \(f\) at the origin (see [7] for details).

Now we consider obtaining the second order information of the surface by the contours of the projections of three distinct directions. Let \(f\) be the surface given by (1.2). Since the mean and the Gaussian curvatures of \(f\) are \((a_{20} + a_{02})/2\) and \(a_{20}a_{02}\) respectively, if the mean and the Gaussian curvatures are obtained, then one can regard that the second order information are obtained. Thus we consider obtaining the mean and the Gaussian curvatures. We set \(M\) and \(G\) to be the mean and the Gaussian curvatures at 0:
\[ M = \frac{a_{20} + a_{02}}{2}, \quad G = a_{20}a_{02}. \] (3.5)

We also take another direction \(\theta_2(\neq \theta_1, 0 < \theta_2 < \pi)\) which satisfies \(p(\theta_2) \neq 0\). By (3.2) we get
\[ \cos 2\theta_i = \frac{-2a_{20}a_{02} + (a_{20} + a_{02})k_{\theta_i}}{(a_{02} - a_{20})k_{\theta_i}} \quad (i = 1, 2). \] (3.6)

Substituting these formulae into the trigonometric identity
\[ \cos^2 2(\theta_i - \theta_j) + \cos^2 2\theta_i + \cos^2 2\theta_j - 2 \cos 2(\theta_i - \theta_j) \cos 2\theta_i \cos 2\theta_j - 1 = 0, \]
\((i, j = 1, 2)\) we get an equation \(P_{ij}(M, G) = 0\) where
\[ P_{ij}(M, G) = \left( M_{ij}^2 - G_{ij} \cos^2(\theta_i - \theta_j) \right) G^2 - 2G_{ij}M_{ij} \sin^2(\theta_i - \theta_j)GM + G_{ij}^2 \sin^4(\theta_i - \theta_j)M^2 + G_{ij}^2 \cos^2(\theta_i - \theta_j) \sin^2(\theta_i - \theta_j)G. \]
\[ (M, G) Q_{ij} \left( \frac{G}{M} \right) + G_{ij}^2 \cos^2(\theta_i - \theta_j) \sin^2(\theta_i - \theta_j) G \]  

(3.7) 

and 

\[ M_{ij} = \frac{k_{\theta_i} + k_{\theta_j}}{2}, \quad G_{ij} = k_{\theta_i} k_{\theta_j}, \]  

(3.8) 

\[ Q_{ij} = \begin{pmatrix} M_{ij}^2 - G_{ij} \cos^2(\theta_i - \theta_j) & -G_{ij} M_{ij} \sin^2(\theta_i - \theta_j) \\ -G_{ij} M_{ij} \sin^2(\theta_i - \theta_j) & G_{ij}^2 \sin^4(\theta_i - \theta_j) \end{pmatrix}. \] 

Since \( P_{ij}(M, G) = 0 \) is a quadratic curve, generally the values of \( G \) and \( M \) should be determined by the curvatures of the apparent contours from three distinct directions.

**Theorem 3.3.** Take three distinct directions \( 0 < \theta_1, \theta_2, \theta_3 < \pi \) satisfying \( p(\theta_i) \neq 0 \) and \( \det V \neq 0 \), where 

\[ V = \begin{pmatrix} \frac{M_{12}^2 - G_{12} \cos^2(\theta_1 - \theta_2)}{G_{12}^2 \sin^2(\theta_1 - \theta_2)} & \frac{M_{12}}{G_{12}} \sin^2(\theta_1 - \theta_2) \\ \frac{M_{23}^2 - G_{23} \cos^2(\theta_2 - \theta_3)}{G_{23}^2 \sin^2(\theta_2 - \theta_3)} & \frac{M_{23}}{G_{23}} \sin^2(\theta_2 - \theta_3) \\ \frac{M_{31}^2 - G_{31} \cos^2(\theta_3 - \theta_1)}{G_{31}^2 \sin^2(\theta_3 - \theta_1)} & \frac{M_{31}}{G_{31}} \sin^2(\theta_3 - \theta_1) \end{pmatrix}. \] 

Then \( G \) and \( M \) are given as follows:

\[ G = \frac{\det L}{\det V}, \quad M = -\frac{\det P}{\det V}, \]  

(3.9) 

where 

\[ L = \begin{pmatrix} \cos^2(\theta_1 - \theta_2) & \frac{M_{12}}{G_{12}} \sin^2(\theta_1 - \theta_2) \\ \cos^2(\theta_2 - \theta_3) & \frac{M_{23}}{G_{23}} \sin^2(\theta_2 - \theta_3) \\ \cos^2(\theta_3 - \theta_1) & \frac{M_{31}}{G_{31}} \sin^2(\theta_3 - \theta_1) \end{pmatrix}. \] 

and 

\[ P = \begin{pmatrix} \frac{M_{12}^2 - G_{12} \cos^2(\theta_1 - \theta_2)}{G_{12}^2 \sin^4(\theta_1 - \theta_2)} \cos^2(\theta_1 - \theta_2) & \frac{M_{12}}{G_{12}^2} \sin^2(\theta_1 - \theta_2) \\ \frac{M_{23}^2 - G_{23} \cos^2(\theta_2 - \theta_3)}{G_{23}^2 \sin^4(\theta_2 - \theta_3)} \cos^2(\theta_2 - \theta_3) & \frac{M_{23}}{G_{23}^2} \sin^2(\theta_2 - \theta_3) \\ \frac{M_{31}^2 - G_{31} \cos^2(\theta_3 - \theta_1)}{G_{31}^2 \sin^4(\theta_3 - \theta_1)} \cos^2(\theta_3 - \theta_1) & \frac{M_{31}}{G_{31}^2} \sin^2(\theta_3 - \theta_1) \end{pmatrix}. \]
Proof. A triplet of the equations \( P_{12}(M, G) = P_{23}(M, G) = P_{31}(M, G) = 0 \) is a system of equations

\[
W \begin{pmatrix} G^2 \\ GM \\ M^2 \end{pmatrix} = Gb,
\]

where \( W = (w_1, w_2, w_3) \) with

\[
w_1 = \begin{pmatrix} M_{12}^2 - G_{12} \cos^2(\theta_1 - \theta_2) \\ M_{23}^2 - G_{23} \cos^2(\theta_2 - \theta_3) \\ M_{31}^2 - G_{31} \cos^2(\theta_3 - \theta_1) \end{pmatrix}, \tag{3.10}
\]

\[
w_2 = -\begin{pmatrix} 2G_{12}M_{12} \sin^2(\theta_1 - \theta_2) \\ 2G_{23}M_{23} \sin^2(\theta_2 - \theta_3) \\ 2G_{31}M_{31} \sin^2(\theta_3 - \theta_1) \end{pmatrix}, \quad w_3 = \begin{pmatrix} G_{12}^2 \sin^4(\theta_1 - \theta_2) \\ G_{23}^2 \sin^4(\theta_2 - \theta_3) \\ G_{31}^2 \sin^4(\theta_3 - \theta_1) \end{pmatrix}, \tag{3.11}
\]

and

\[
b = \begin{pmatrix} G_{12}^2 \cos^2(\theta_1 - \theta_2) \sin^2(\theta_1 - \theta_2) \\ G_{23}^2 \cos^2(\theta_2 - \theta_3) \sin^2(\theta_2 - \theta_3) \\ G_{31}^2 \cos^2(\theta_3 - \theta_1) \sin^2(\theta_3 - \theta_1) \end{pmatrix}. \tag{3.12}
\]

Since

\[
\det W = -2G_{12}^2G_{23}^2G_{31}^2 \sin^2(\theta_1 - \theta_2) \sin^2(\theta_2 - \theta_3) \sin^2(\theta_3 - \theta_1) \det V,
\]

if \( k_{\theta_1}k_{\theta_2}k_{\theta_3} \neq 0 \), by Cramer’s rule, we get

\[
G = \frac{\det W_1}{\det W}, \quad M = \frac{\det W_2}{\det W}, \tag{3.13}
\]

where \( W_1 = (b, w_2, w_3) \), \( W_2 = (w_1, b, w_3) \). By a direct calculation, we have \( \blacksquare \).

Since in our setting, \( \theta_1 - \theta_2, \theta_1 - \theta_3, k_{\theta_1}, k_{\theta_2}, k_{\theta_3} \) are known, all matrix elements of \( W \) and \( b \) are known. Thus we obtain \( G \) and \( M \) by (3.13). Moreover, we obtain \( \theta_1, \theta_2 \) and \( \theta_3 \) by (3.6). Since \( G = a_{20}a_{02} \) and \( M = (a_{20} + a_{02})/2 \), we obtain \( a_{20} \) and \( a_{02} \).

Next let us consider the third order terms of the surface. Let us take four distinct directions

\[
\theta_1, \theta_2, \theta_3, \theta_4.
\]

Then we obtain \( a_{30}, a_{21}, a_{12}, a_{03} \) by \( k_{\theta_i} \) \( (i = 1, 2, 3, 4) \) as follows. By (3.12) and (3.13), we see that

\[
A \begin{pmatrix} a_{30} \\ a_{21} \\ a_{12} \\ a_{03} \end{pmatrix} = d,
\]

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where \( A = (a_1, a_2, a_3, a_4) \) and

\[
\begin{align*}
 a_1 &= -a_{02}^2 \left( \sin^3 \theta_1, \sin^3 \theta_2, \sin^3 \theta_3, \sin^3 \theta_4 \right), \\
 a_2 &= 3a_{20}a_{02}^2 \left( \sin^2 \theta_1 \cos \theta_1, \sin^2 \theta_2 \cos \theta_2, \sin^2 \theta_3 \cos \theta_3, \sin^2 \theta_4 \cos \theta_4 \right), \\
 a_3 &= -3a_{20}^2a_{02} \left( \sin \theta_1 \cos^2 \theta_1, \sin \theta_2 \cos^2 \theta_2, \sin \theta_3 \cos^2 \theta_3, \sin \theta_4 \cos^2 \theta_4 \right), \\
 a_4 &= a_{20}^3 \left( \cos^3 \theta_1, \cos^3 \theta_2, \cos^3 \theta_3, \cos^3 \theta_4 \right), \\
 d &= \left[ p(\theta_1) \frac{dk_{\theta_1}}{ds}(0), p(\theta_2) \frac{dk_{\theta_2}}{ds}(0), p(\theta_3) \frac{dk_{\theta_3}}{ds}(0), p(\theta_4) \frac{dk_{\theta_4}}{ds}(0) \right].
\end{align*}
\]

Since \( \det A = 9a_{20}^6a_{02}^8 \prod_{i<j} \sin(\theta_i - \theta_j) \), and \( \theta_1, \ldots, \theta_4 \) are distinct, \( a_{20}a_{02} \neq 0 \), and it holds that \( \det A \neq 0 \). By Cramer’s rule, we get

\[
\begin{align*}
 a_{30} &= \frac{\det A_1}{\det A}, \quad a_{21} = \frac{\det A_2}{\det A}, \quad a_{12} = \frac{\det A_3}{\det A}, \quad a_{03} = \frac{\det A_4}{\det A},
\end{align*}
\]

where \( A_1 = (d, a_2, a_3, a_4), \ A_2 = (a_1, d, a_3, a_4), \ A_3 = (a_1, a_2, d, a_4), \ A_4 = (a_1, a_2, a_3, d) \). Thus we obtain \( a_{30}, a_{21}, a_{12}, a_{03} \) by \( k_{\theta_i} \) (\( i = 1, 2, 3, 4 \)).

### 3.2 Quadratic curves defined by two directions

In this section, under the above setting, we consider the quadratic curve

\[ C = C_{12}(M, G) = \{(M, G) \in \mathbb{R}^2 \mid P_{12}(M, G) = 0\} \]

in the \( MG \)-plane, where \( P_{12}(M, G) \) is defined in (3.7). This curve satisfies that for two points \( (M, G) \) and \( (\tilde{M}, \tilde{G}) \in C \), there exist surfaces \( f = (u, v, (a_{20}u^2 + a_{02}v^2)/2 + O(3) \) and \( \tilde{f} = (u, v, (\tilde{a}_{20}u^2 + \tilde{a}_{02}v^2)/2 + O(3) \)

\[
\begin{align*}
 (a_{20} + a_{02} = 2M, a_{20}a_{02} = G, \tilde{a}_{20} + \tilde{a}_{02} = 2\tilde{M}, \tilde{a}_{20}\tilde{a}_{02} = \tilde{G}),
\end{align*}
\]

there exist \( \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_1, \tilde{\theta}_2 \) such that \( \tilde{\theta}_1 - \tilde{\theta}_2 = \bar{\theta}_1 - \bar{\theta}_2 \) and \( k_{\tilde{\theta}_i} = k_{\tilde{\theta}_i} \) (\( i = 1, 2 \)), where \( k_{\tilde{\theta}_i} \) is the curvature of the contour in the direction \( \tilde{\theta}_i \) of the surface \( \tilde{f} \). Since we assume that \( a_{20} > a_{02} \) and \( a_{20} > 0 \), a point on \( C \) expresses the unique surface up to second order information. Hence we have a family of surfaces where curvatures of their contours with respect to two (moving) directions do not change.

**Proposition 3.4.** We have the following:

1. The curve \( C \) is a hyperbola (respectively ellipse) if \( G_{ij} > 0 \) (respectively \( G_{ij} < 0 \)).
(2) The curve $C$ is tangent to the $M$-axis.

(3) If $C$ is a hyperbola, then the two branches of $C$ lie on both sides of $\{G > 0\}$ and $\{G < 0\}$.

Proof. We have $\det Q_{ij} = -G^3_{ij} \cos^2(\theta_i - \theta_j) \sin^4(\theta_i - \theta_j)$, and (1) is proved. We have $\partial P_{ij}/\partial M(0,0) = 0$, which gives (2). By (1) and (2), it is clear that the assertion (3) holds.

The assertion (3) of Proposition 3.4 means that the above family can be divided into two continuous families whose Gaussian curvatures are always positive and always negative respectively.

Example 3.5. Let us set

$$a_{20} = 3 - \sqrt{3} + \sqrt{11 - 6\sqrt{3}}, \quad a_{02} = 3 - \sqrt{3} - \sqrt{11 - 6\sqrt{3}};$$

$$\tilde{a}_{20} = -6 + \sqrt{37}, \quad \tilde{a}_{02} = -6 - \sqrt{37},$$

and let us set

$$\theta_1 = (-1/2) \arccos \left( \frac{1}{13} \left( -4\sqrt{11 - 6\sqrt{3}} - \sqrt{3(11 - 6\sqrt{3})} \right) \right),$$

$$\tilde{\theta}_1 = -\frac{1}{2} \arccos \left( \frac{5}{\sqrt{37}} \right).$$

Then the curvatures of the contours of $f$ with respect to $\xi(\theta_1)$ and of $\tilde{f}$ with respect to $\xi(\tilde{\theta}_1)$ are 1, and the curvatures of the contours of $f$ with respect to $\xi(\theta_2)$ and of $\tilde{f}$ with respect to $\xi(\tilde{\theta}_2)$ are 2. The $C$ of $f$, $\tilde{f}$ can be drawn as in Figure 3.2. See Figures 3.3, 3.4 and 3.5 for these surfaces and their contours.

3.3 Obtaining Gaussian curvature

According to Section 3.1, we can obtain all of the second order information of the surface by the contour of projections from three distinct directions. In particular, we can obtain the Gaussian curvature. In this section, we discuss obtaining just the Gaussian curvature $K$. 

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Figure 3.2: The hyperbola $C$

Figure 3.3: The surfaces $f$ (blue) and $\tilde{f}$ (green) viewed from $\xi(\theta_1)$ and $\xi(\tilde{\theta}_1)$, respectively. These contours have the same curvatures.

Figure 3.4: The surfaces $f$ and $\tilde{f}$ viewed from $\xi(\theta_2)$ and $\xi(\tilde{\theta}_2)$, respectively. These contours have the same curvatures.
By (3.2), we have
\[ k_{\theta_1}k_{\theta_2} = \frac{a_{20}a_{02}}{(a_{20}\cos^2\theta_1 + a_{02}\sin^2\theta_1)(a_{20}\cos^2\theta_2 + a_{02}\sin^2\theta_2)}. \]

Hence if
\[ \frac{a_{20}a_{02}}{(a_{20}\cos^2\theta_1 + a_{02}\sin^2\theta_1)(a_{20}\cos^2\theta_2 + a_{02}\sin^2\theta_2)} = 1, \tag{3.14} \]
then \( K = k_{\theta_1}k_{\theta_2} \). If \( \theta_1, \theta_2 \) satisfy (3.14), then we say that \( \xi_{\theta_1}, \xi_{\theta_2} \) are contour-conjugate each other. Now we consider the existence of the contour-conjugate. Since (3.14) is equivalent to
\[ a_{20} - a_{02} = 0 \quad \text{or} \quad a_{20}\frac{\cos^2\theta_2}{\sin^2\theta_2} = a_{02}\frac{\sin^2\theta_1}{\cos^2\theta_1}, \tag{3.15} \]
we have the following proposition.

**Proposition 3.6.** Let \( p \) be a point that is not flat umbilic on a regular surface. If \( p \) is an umbilic point, then any pair of two directions are contour-conjugates at \( p \). If \( K(p) > 0 \) and \( p \) is not an umbilic point, then any direction has two contour-conjugates at \( p \), and if \( K(p) < 0 \) there are no contour-conjugate for any direction at \( p \).

**Example 3.7.** Let us set
\[ f(u, v) = \left( u, v, \frac{u^2}{2} + v^2 \right) \]
and
\[ \theta_1 = \pi/4, \quad \theta_2 = \text{arccot}(\sqrt{2}). \]

Then, since \( \theta_1 \) and \( \theta_2 \) satisfy (3.15), they are contour-conjugate (see Figure 3.6), namely, the product of the curvatures with respect to these directions equals 2, the Gaussian curvature of \( f \) at 0.
A Normal curvature and Euler’s formula

In this appendix, we give a similar formula to Theorem 3.3 for the normal curvatures. To obtain \(a_{20}, a_{02}\), we have another expression by using Euler’s formula (see [16, p 214], for example). In the same setting as in Section 3.1, let \(\theta_1, \theta_2, \theta_3\) be the distinct angles \((0 < \theta_1, \theta_2, \theta_3 < \pi)\). Let \(k^n_i (i = 1, 2, 3)\) be the normal curvatures of \(f\) with respect to \(\xi_i\), and let \(M^n_{ij} = (k^n_i + k^n_j)/2\), \(G^n_{ij} = k^n_i k^n_j\). By Euler’s formula, we have

\[
\sin^4(\theta_i - \theta_j) M^2 - 2M^n_{ij} \sin^2(\theta_i - \theta_j) M + \cos^2(\theta_i - \theta_j) \sin^2(\theta_i - \theta_j) G + ((M^n_{ij})^2 - G^n_{ij} \cos^2(\theta_i - \theta_j)) = 0.
\]

For \(ij = 12, 23, 31\), these equations form a linear system

\[
\begin{pmatrix}
\sin^4(\theta_1 - \theta_2) & -2M^n_{12} \sin^2(\theta_1 - \theta_2) & \cos^2(\theta_1 - \theta_2) \sin^2(\theta_1 - \theta_2) \\
\sin^4(\theta_2 - \theta_3) & -2M^n_{23} \sin^2(\theta_2 - \theta_3) & 2 \cos^2(\theta_2 - \theta_3) \sin^2(\theta_2 - \theta_3) \\
\sin^4(\theta_3 - \theta_1) & -2M^n_{31} \sin^2(\theta_3 - \theta_1) & 2 \cos^2(\theta_3 - \theta_1) \sin^2(\theta_3 - \theta_1)
\end{pmatrix}
\begin{pmatrix}
M^2 \\
M \\
G
\end{pmatrix}
= \begin{pmatrix}
(M^n_{12})^2 - G^n_{12} \cos^2(\theta_1 - \theta_2) \\
(M^n_{23})^2 - G^n_{23} \cos^2(\theta_2 - \theta_3) \\
(M^n_{31})^2 - G^n_{31} \cos^2(\theta_3 - \theta_1)
\end{pmatrix}.
\]

By Cramer’s rule, we have an expression

\[
M = \frac{\det P^n}{\det V^n}, \quad G = \frac{2 \det L^n}{\det V^n},
\]

where

\[
V^n = \begin{pmatrix}
M^n_{12} & \cos^2(\theta_1 - \theta_2) & \sin^2(\theta_1 - \theta_2) \\
M^n_{23} & \cos^2(\theta_2 - \theta_3) & \sin^2(\theta_2 - \theta_3) \\
M^n_{31} & \cos^2(\theta_3 - \theta_1) & \sin^2(\theta_3 - \theta_1)
\end{pmatrix}.
\]
$$L^n = \begin{pmatrix}
\frac{(M^n_{12})^2 - G^n_{12} \cos^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 - \theta_2)} & M^n_{12} \sin^2(\theta_1 - \theta_2) \\
\frac{(M^n_{23})^2 - G^n_{23} \cos^2(\theta_2 - \theta_3)}{\sin^2(\theta_2 - \theta_3)} & M^n_{23} \sin^2(\theta_2 - \theta_3) \\
\frac{(M^n_{31})^2 - G^n_{31} \cos^2(\theta_3 - \theta_1)}{\sin^2(\theta_3 - \theta_1)} & M^n_{31} \sin^2(\theta_3 - \theta_1)
\end{pmatrix},$$

$$P^n = \begin{pmatrix}
\frac{(M^n_{12})^2 - G^n_{12} \cos^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 - \theta_2)} & \cos^2(\theta_1 - \theta_2) \sin^2(\theta_1 - \theta_2) \\
\frac{(M^n_{23})^2 - G^n_{23} \cos^2(\theta_2 - \theta_3)}{\sin^2(\theta_2 - \theta_3)} & \cos^2(\theta_2 - \theta_3) \sin^2(\theta_2 - \theta_3) \\
\frac{(M^n_{31})^2 - G^n_{31} \cos^2(\theta_3 - \theta_1)}{\sin^2(\theta_3 - \theta_1)} & \cos^2(\theta_3 - \theta_1) \sin^2(\theta_3 - \theta_1)
\end{pmatrix}. $$

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