The calculation of beam-wall made of material with shape memory effect by the finite element method

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Abstract. In present article application of a method of finite elements for calculation of beam-wall, executed from a material with effect of memory of the form (SME) is considered. The equations of structural-analytical mesomechanics were used as defining relations.

1. Introduction
The introduction of new materials with previously unknown properties, the emergence of new engineering problems that require precise prediction of mechanical properties, urgently requires the creation and development of more advanced, in comparison with the known, approaches in the theory of deformation and destruction. It seems that such new approaches should be based on a rational synthesis of constructive ideas of mechanics of a deformable solid body and physical materials science, naturally combining the main achievements and methods of modern physics of plasticity and destruction as well as mechanics of a deformable solid.

The development of modern concepts of deformation of solids is based on the understanding of the interrelated process of deformation and destruction as multilevel and hierarchical, and requires of researchers to new approaches due to the need to analyze plastic deformation and the evolution of structural damage at different scale and structural levels, taking into account the localization and heterogeneity of deformation development. Alloys with SME have unique properties that are absent in most traditional materials used in industry. However, the unconventional behavior of these materials requires new models of the material in which the use of traditional methods of mechanics of a deformable solid body is possible. The development of the theory and methods for calculating engineering structures made of materials with the SME is an important task.

In this paper, we use the defining relations of the deformation type, derived from the general equations of structural-analytical mesomechanics [1–5].

In the framework of this approach, solutions of some boundary problems that allow an analytical solution are known. This is the problem of the bending of the beam, as well as the long thick-walled pipe under internal pressure. The following boundary-value problems were solved with numerical and numerical-analytical methods: calculation of a two-rod statically indefinable system with the finite element method, calculation of the truss structure with the numerical-analytical method of boundary elements [6, 7], calculation of a thin rectangular plate, working on a bend with the method finite differences [8]. In this paper, we consider a cantilever beam-wall loaded with a uniformly distributed load of constant intensity, the material of which undergoes inelastic martensitic deformations. Only
isothermal martensitic transformations are considered. The problem is solved in a physically non-linear, but geometrically linear formulation. To solve the problem, the finite element method is used in combination with the method of variable parameters of elasticity.

2. Defining relations at the stage of loading and the calculation algorithm
The defining relations for the material with the SME taking into account the simplified equations of the structural-analytical mesomechanics [4, 5] are represented as follows:

\[ e_i^r = \frac{\sigma_i}{E} + \left( \frac{2}{3} B \sigma_i^k k \left( \sigma_i - \sigma_n^{A\to M} \right) \right) \left( \sigma_n^{A\to M} - M_n - M_k \right) \]

where \( \sigma_i \) is the stress intensity; \( E \) is the modulus of elasticity; \( B \sigma_i^k \) is the material constant; \( k = \frac{T_0}{q_0} D_i \); \( T_0 \) is the thermodynamic equilibrium temperature; \( q_0 \) is the heat effect of the reaction; \( D_i \) is the distortion of the phase transformation; \( \sigma_n^{A\to M} = \frac{T_D - M_n}{k} \) is the voltage of the onset of direct martensitic transformation; \( T_D \) is the temperature of the onset of deformation, \( M_n \) and \( M_k \) are the temperatures of the beginning and end of the direct martensitic transformation, respectively; \( \sigma_n^{A\to M} = \frac{T_D - M_k}{k} \) is the voltage of the end of martensitic transformation, \( \dot{\sigma}_i = \frac{d\sigma_i}{dt} \); \( H(\ldots) \) is a Heaviside function.

Take the following assumptions:
1) isothermal loading from the austenitic state \( T_D=A \);
2) the material undergoes elastic and inelastic (phase) deformations;
3) accept the bilinear model of material deformation.

For the calculation the program in the Mathsad package is compiled. The calculation was carried out by the finite element method in combination with the method of variable parameters of elasticity, which is widely used in solving problems of the theory of plasticity. The algorithm for solving the problem was as follows:

1. Solve the basic equation of the finite element method (FEM) \( [K]\{\Delta\}=[F] \) where \( [K] \) is the stiffness matrix of the structure, \( \{\Delta\} \) is the vector of displacements of nodes, \( \{F\} \) is the vector of nodal loads.
2. For elements experiencing phase transformation in which \( e_i>e_i^c \) to find the secant module \( E_c = \frac{\Phi(e_i)}{e_i} \); \( \Phi(e_i) \) is the equation of the strain diagram.

According to it, we determine the variable parameters of elasticity \( E_\phi = \frac{E_c}{1+\alpha} \), \( \alpha = \frac{1}{3} \frac{2\mu E}{E} \),

\[ \mu_\phi = \frac{1-2\alpha}{2+2\alpha}, \ G_\phi = \frac{E_c}{2(1+\mu_\phi)} \]

- recalculate the internal stiffness matrix taking into account these parameters

\[ G = \begin{bmatrix} 1 & \mu_\phi & 0 \\ \mu_\phi & 1 & 0 \\ 0 & 0 & \frac{1-\mu_\phi}{2} \end{bmatrix} \]

- recalculate the element stiffness matrix \( [K^e] \)
3. Recalculate the system stiffness matrix $[K']$

4. Solve the basic FEM equation again and find the displacements of the nodes $\{\Delta\}$, stresses and strains in the elements $[\sigma], [\varepsilon]$, stress intensity and strain intensity $\sigma_i, \varepsilon_i$.

5. To compare the results of the beginning and end of the calculation. In case of dissatisfaction with a given accuracy, go to the second approximation. Repeat steps 2, 3, 4, 5. The process is carried out until the convergence criterion is met.

Figure 1 shows the block diagram of the solution of the physically nonlinear FEM problem with the method of variable elastic parameters. First, the elastic problem is solved with a single load and the $F_T$ load is determined at which $\max \sigma_i = \sigma_T$. Then it is necessary to specify the load $F_i = F_T \cdot \alpha$, where $\alpha > 1$, in this problem we take $\alpha = 2$. Under such a load, the structure is guaranteed to undergo phase transformations. Next, an elastic calculation is carried out at a given load $F$. From the condition $\sigma_i \geq \Phi(\varepsilon_i)$, the elements in which the phase transformations begin are determined. For these elements, the new parameters of elasticity are calculated, the element stiffness matrix is recalculated. For elements that remain elastic, the element stiffness matrix remains the same. Then the system stiffness is formed, the basic FEM equation is solved and the nodal displacements $\{\Delta\}$ are determined.

According to the calculation results, the criterion for the end of the calculation of $Eps$ is determined. If $Eps < \eta$, the calculation is completed, if not, we go to the next iteration. Here $\eta$ is the specified calculation accuracy.

Figure 1. Block diagram of the method of variable parameters of elasticity.

3. Boundary problems for SMA

Formulation of the problem. Determine the stress-strain state of the cantilever beam-wall, loaded on the upper edge with a uniformly distributed load, at which stresses that are twice the phase yield point occur in the most loaded element. The length of the beam-wall is $a = 12$ mm, width $b = 10$ mm, thickness $\delta = 10$ mm. The plate is made of TiNi alloy, which has a shape memory effect.
Initial data are presented in table 1.

**Table 1. Initial data.**

| Name                             | Formula                                      |
|---------------------------------|----------------------------------------------|
| Characteristic temperatures     | $M_n = 330K$, $M_s = 320K$                  |
|                                 | $A_n = 370K$, $A_s = 380K$                  |
| Material constant $k$           | $k = 0.29 \text{ K MPa}^{-1}$               |
| Constant $B_\theta$             | $B_\theta = 0.06 \cdot 10^{-2} \text{ MPa}^{-1}$ |
| Elastic modulus $E$             | $E = 7.42 \cdot 10^4 \text{ MPa}$          |
| Deformation temperature $T_D$   | $T_D = A_e = 380K$                          |
| Tension of the beginning of straight martensitic transformations (phase yield strength) $\sigma_{H11}^{A \rightarrow M} = \sigma_{T1}$ | $\sigma_{H11}^{A \rightarrow M} = \sigma_{T1} = 172 \text{ MPa}$ |
| Ferroelasticity module (tangent module) | $E_{\Phi \gamma} = E_K = 414 \text{ MPa}$ |
| Deformation at yield stress $\varepsilon_T$ | $\varepsilon_T = \frac{\sigma_{H11}^{A \rightarrow M}}{E} = \frac{\sigma_{T1}}{E} = 2.324 \cdot 10^{-3}$ |
| Calculation accuracy for displacements $\eta$ | $\eta = 10^{-4}$ |

4. **Decision**

To solve the problem, we use a linear triangular element. The number of nodes of the finite element model is 90, the number of elements is 139. The node numbers of the calculated beam-wall are presented in Table 2, where zero numbers mean fixings, and loaded nodes are presented in line 1.

**Table 2. The node numbers of the finite element mesh.**

|      | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| 1    | 0 | 6 | 12| 18| 24| 30| 36| 42| 48| 54 | 60 | 66 | 72 | 78 | 84 | 90 |
| 2    | 0 | 5 | 11| 17| 23| 29| 35| 41| 47| 53 | 59 | 65 | 71 | 77 | 83 | 89 |
| 3    | 0 | 4 | 10| 16| 22| 28| 34| 40| 46| 52 | 58 | 64 | 70 | 76 | 82 | 88 |
| 4    | 0 | 3 | 9 | 15| 21| 27| 33| 39| 45| 51 | 57 | 63 | 69 | 75 | 81 | 87 |
| 5    | 0 | 2 | 8 | 14| 20| 26| 32| 38| 44| 50 | 56 | 62 | 68 | 74 | 80 | 86 |
| 6    | 0 | 1 | 7 | 13| 19| 25| 31| 37| 43| 49 | 55 | 61 | 67 | 73 | 79 | 85 |

We approximate the material with a linear hardening diagram shown in figure 2.
It took 53 iterations to obtain the required displacement accuracy $\eta = 10^{-4}$.

As a result of the calculation, we obtain the values of stress intensity $\sigma_i$, strain intensity $\varepsilon_i$ and displacement $\Delta$, presented in table 3, in figure 3, respectively.

Table 3. Stress intensity $\sigma_i$, MPa at the grid nodes.

| № node | magnitude, $\Delta$, mm |
|--------|-------------------------|
| 171    | 115                     |
| 196    | 100                     |
| 176    | 95                      |
| 177    | 85                      |
| 176    | 80                      |
| 190    | 75                      |
| 183    | 70                      |
| 157    | 55                      |
| 141    | 50                      |
| 137    | 45                      |
| 131    | 40                      |
| 124    | 35                      |
| 101    | 30                      |
| 83     | 25                      |
| 67     | 20                      |
| 53     | 15                      |
| 41     | 10                      |
| 26     | 5                       |
| 16     | 1                       |

Figure 3. The intensity of deformation and displacement: a) strain intensity $\varepsilon_i$ and the region of origin of martensitic crystals as a result of the transformation (indicated by a circle); b) the positions of the nodes before the deformation (crosses) and after the deformation (circles) when moving nodes $\Delta$, mm.

Table 4. The results of the calculation of the maximum values.

| № node | magnitude, $\Delta$, mm | № node | magnitude, $\varepsilon_i$, % | № node | magnitude, $\sigma_i$, MPa |
|--------|-------------------------|--------|-----------------------------|--------|---------------------------|
| 90     | 1.15                    | 1      | 1.05                        | 1      | 239                       |

5. Conclusion
The problem of calculating the stress-strain state of a beam-wall made of a material with inelastic martensitic transformations is solved with the finite element method in combination with the method of variable elastic parameters.

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