LINEAR BOUND FOR MAJORITY COLOURINGS OF DIGRAPHS

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Abstract. Given $\eta \in [0,1]$, a colouring $C$ of $V(G)$ is an $\eta$-majority colouring if at most $\eta d^+(v)$ out-neighbours of $v$ have colour $C(v)$, for any $v \in V(G)$. We show that every digraph $G$ equipped with an assignment of lists $L$, each of size at least $k$, has a $2/k$-majority $L$-colouring. For even $k$ this is best possible, while for odd $k$ the constant $2/k$ cannot be replaced by any number less than $2/(k+1)$. This generalizes a result of Anholcer, Bosek and Grytczuk [1], who proved the cases $k = 3$ and $k = 4$ and gave a weaker result for general $k$.

1. Introduction

Given a digraph $G$, we write $V(G)$ and $E(G)$ for the vertex and edge set of a digraph $G$, respectively. For $v \in V(G)$, we denote by $d^+(v)$ the out-degree of $v$. Given $\eta \in [0,1]$, a (not necessarily proper) colouring $C$ of $V(G)$ is an $\eta$-majority colouring if at most $\eta d^+(v)$ out-neighbours of $v$ have colour $C(v)$, for any $v \in V(G)$. A $1/2$-majority colouring is referred to simply as a majority colouring. This concept was introduced by van der Zypen [5], who asked whether every digraph has a majority colouring with a bounded number of colours. This question was answered by Kreutzer, Oum, Seymour, van der Zypen and Wood [4], who showed that 4 colours always suffice.

We consider the list-colouring version of this problem. For a set $S$, we denote by $P(S)$ the power set of $S$. Given a digraph $G$ and an assignment $L : V(G) \to P(\mathbb{N})$ of lists to vertices of $G$, an $L$-colouring $C : V(G) \to \mathbb{N}$ of $G$ is a colouring of $V(G)$ such that $C(v) \in L(v)$ for every $v \in V(G)$. If $G$ has an $\eta$-majority $L$-colouring for any such assignment $L$ whose lists are all of size at least $k$, we say that $G$ is $\eta$-majority $k$-choosable. Anholcer, Bosek and Grytczuk [1] showed that every graph $G$ is $1/n$-majority $n^2$-choosable for every $n \geq 2$. Theorem 1 improves on this result.

Theorem 1. For any integer $k \geq 2$, every digraph $G$ is $2/k$-majority $k$-choosable.

Theorem 1 was proved independently by Girão, Kittipassorn and Popielarz [2]. The case $k = 2$ is trivial. Previously, Anholcer, Bosek and Grytczuk [1] showed that Theorem 1 holds in the cases $k = 3$ and $k = 4$ and conjectured

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that $2/k$ can be replaced by $1/2$ when $k = 3$. Theorem 1 is best possible when $k$ is even, as shown by the example of a $k/2$-regular tournament on $k + 1$ vertices (that is, all vertices have both in-degree and out-degree equal to $k/2$). If we make all lists equal, then some vertex must have an out-neighbour of the same colour, and this out-neighbour represents $2/k$ of its out-neighbourhood. When $k$ is odd, a similar example shows that we cannot replace $2/k$ by any number less than $2/(k + 1)$.

2. Proof of Theorem 1

We denote by $vw$ an edge from a vertex $v$ of a digraph to another vertex $w$. The proof of Theorem 1 relies on Lemma 2, which follows.

Lemma 2. Let $k \geq 2$ be an integer and let $G$ be a digraph on a vertex set $V = S \cup T$, such that $G[S]$ is strongly connected, $G[T]$ is edgeless and there are no edges from $T$ to $S$. Let $C_T$ be any colouring of $T$ and let $L : S \rightarrow \mathcal{P}(\mathbb{N})$ be an assignment of lists, each of size at least $k$, to vertices in $S$. Then there exists an extension $C$ of $C_T$ to $V$ with $C(v) \in L(v)$ for each $v \in S$, such that no vertex $v \in S$ has more than $2d^+(v)/k$ out-neighbours with the same colour as $v$.

Proof. For any colouring $C$ of $V$, we define the function $f_C : S \rightarrow \mathbb{R}$ by

$$f_C(v) = \frac{|\{w \in N^+(v) \mid C(w) = C(v)\}|}{d^+(v)}$$

for each vertex $v \in S$; i.e., $f_C(v)$ is the proportion of out-neighbours of $v$ which have the same colour as $v$ under $C$. Given $v \in S$, we write $d^+_S(v) = |N^+(v) \cap S|$.

Let $A$ be the non-negative real $S \times S$ matrix with entries $A_{vw} = 1/d^+_S(v)$ if $vw$ is an edge of $G$ and $A_{vw} = 0$ otherwise. We have $A\mathbf{j} = \mathbf{j}$ (where $\mathbf{j}$ is the vector of all 1’s). On the other hand if $A\mathbf{y} = \mathbf{y}$ for any vector $\mathbf{y}$, then choose $v \in S$ such that $|y_v|$ is maximal; now $|c\mathbf{y}_v| = |\sum_{w \in S} A_{vw} y_w| \leq \sum_{w \in S} A_{vw} |y_w| = |y_v|$ and so $|c| \leq 1$. Thus, the spectral radius of $A$ is 1.

By applying the Perron–Frobenius Theorem (see, e.g., [3, Theorem 8.8.1]) to $A^T$, noting that $G[S]$ is strongly connected, we obtain an eigenvector $\mathbf{x}$ of $A^T$ with positive entries and eigenvalue 1. We remark that by normalizing $\mathbf{x}$ we could obtain a stationary distribution of the uniform random walk on $G[S]$.

Consider an extension $C$ of $C_T$ with $C(v) \in L(v)$ for each $v \in S$ such that $\sum_{v \in S} x_v f_C(v)$ is minimized. We claim that $C$ satisfies the requirements of the lemma. For brevity we write $f$ for $f_C$. It suffices to show that $f(v) \leq 2/k$ for every $v \in S$. Observe that

\begin{equation}
\sum_{v \in S} x_v f(v) = \sum_{v \in E(G) : C(v) = C(w)} \frac{x_v}{d^+(v)}.
\end{equation}

Fix a vertex $v \in S$. We define $g : L(v) \rightarrow \mathbb{R}$ by

$$g(i) = \sum_{\substack{w \in N^+(v) \mid C(w) = i}} \frac{x_v}{d^+(v)} + \sum_{\substack{w \in N^-(v) \mid C(u) = i}} \frac{x_u}{d^+(u)}$$

for every $i$. Then $g(i)$ is a positive eigenvector of $A$ for the eigenvalue 1.
for $i \in L(v)$. Observe that if $v$ were recoloured with colour $i$, then (1) would change by $g(i) - g(C(V))$. By the minimality of $C$ and the definition of $g$ we have that $g(i) \geq g(C(v)) \geq x_v f(v)$. Since $A^\top x = x$,

$$x_v = \sum_{u \in N^-(v)} \frac{x_u}{d^-(u)} \geq \sum_{u \in N^-(v)} \frac{x_u}{d^+(u)}$$

and hence

$$2x_v \geq \sum_{i \in L(v)} g(i) \geq kx_v f(v).$$

Since $x_v > 0$, we have $f(v) \leq 2/k$. It follows immediately that $C$ satisfies the requirements of the lemma. \hfill \square

Proof of Theorem 1. We partition $V(G)$ into strongly connected components $S_1, S_2, \ldots, S_r$, where there are no edges from $S_i$ to $S_j$ for any $i < j$. Let $L : V(G) \rightarrow P(\mathbb{N})$ be an assignment of lists, each of size at least $k$, to vertices of $G$. We write $A_i$ for $\bigcup_{j \in [i]} S_j$ (taking $A_0 = \emptyset$); let $C_0$ be the unique colouring of $A_0$. For each $i = 0, 1, 2, \ldots, r - 1$ in turn, we apply Lemma 2 with $S = S_{i+1}$, $T = A_i$ and $G = G[A_{i+1}] \setminus G[A_i]$ to obtain an extension of $C_i$ to an $L$-colouring $C_{i+1}$ of $A_{i+1}$ such that no $v \in S_{i+1}$ (and hence no $v \in A_{i+1}$) has more than $2d^+(v)/k$ out-neighbours of the same colour. At the end of this process we obtain $C_r$, which is the desired $2/k$-majority $L$-colouring of $V(G)$.

\hfill \square

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