Weak type interpolation for noncommutative maximal operators

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1. Classical weak type interpolation

2. Noncommutative Doob maximal inequality

3. Weak type interpolation for noncommutative maximal operators

4. Doob maximal inequality in noncommutative symmetric spaces
Let \((\mathcal{A}, \Sigma, \nu)\) be a \(\sigma\)-finite measure space. Let \(S(\mathcal{A})\) be the space of measurable functions on \(\mathcal{A}\).
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One is often interested in showing that \(T\) is bounded on a certain function space on \(\mathcal{A}\).
Doob’s maximal inequality states that for any increasing sequence of conditional expectations and any \(1 < p \leq \infty\),

\[
\left\| \sup_{n \geq 1} \mathbb{E}_n(f) \right\|_{L^p} \leq c_p \left\| f \right\|_{L^p}.
\]
A sublinear operator $T$ is of weak type $(p, p)$ if for any $f \in L^p$,

$$d(v; Tf)^{\frac{1}{p}} \leq Cv^{-1}\|f\|_{L^p} \quad (v > 0).$$  (1)
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Here $d(v; g) = \nu(x \in \mathcal{A} : |g(x)| > v)$ denotes the distribution function of a measurable function $g$. 

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The maximal operator $T f = \sup_{n \geq 1} |E_n(f)|$ is of weak types $(1, 1)$ and $(\infty, \infty)$. There are examples of restricted weak type $(1, 1)$ operators which are not weak type $(1, 1)$. 
Theorem (Marcinkiewicz ’39, Zygmund ’56)

Let $1 \leq p < q \leq \infty$. If $T$ is of weak types $(p, p)$ and $(q, q)$, then $T$ is bounded on $L^r$, for any $p < r < q$. 
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$$d(v; x) = \tau(\lambda_{(v, \infty)}(|x|)) \quad (v \geq 0),$$

where $\lambda(|x|)$ is the spectral measure of $|x|$. We say that $x$ is $\tau$-measurable if $d(v; x) < \infty$ for some $v > 0$. 

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This supremum is in general not well-defined. In fact, there are $2 \times 2$ matrices $x_1, x_2, x_3$ such that there is no $2 \times 2$ matrix $x$ satisfying

$$\langle x \xi, \xi \rangle = \max_{i=1,2,3} \langle x_i \xi, \xi \rangle \quad (\xi \in \mathbb{C}^2).$$
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However, we can still define the $L^p$-norm of $T$. 
Let $L^p(M; l^\infty)$ be the space of all sequences $(x_n)_{n \geq 1}$ in $L^p(M)$ for which there exist $u, v \in L^{2p}(M)$ and a bounded sequence $(y_n)_{n \geq 1}$ in $\mathcal{M}$ such that

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Define

$$\|(x_n)_{n\geq 1}\|_{L^p(\mathcal{M}; l^\infty)} = \inf\left\{ \|u\|_{L^{2p}(\mathcal{M})} \sup_{n\geq 1} \|y_n\|_\infty \|v\|_{L^{2p}(\mathcal{M})} \right\}.$$
Let \( L^p(\mathcal{M}; l^\infty) \) be the space of all sequences \((x_n)_{n \geq 1}\) in \( L^p(\mathcal{M}) \) for which there exist \( u, v \in L^{2p}(\mathcal{M}) \) and a bounded sequence \((y_n)_{n \geq 1}\) in \( \mathcal{M} \) such that
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\]
If \( x_n \geq 0 \) for all \( n \geq 1 \), then
\[
\|(x_n)_{n \geq 1}\|_{L^p(\mathcal{M}; l^\infty)} = \inf \{ \|a\|_{L^p(\mathcal{M})} : x_n \leq a \text{ for all } n \geq 1 \}.
\]
Theorem (Junge, ’02)

Let \((\mathcal{E}_n)_{n \geq 1}\) be an increasing sequence of conditional expectations in \(\mathcal{M}\). For any \(1 < p \leq \infty\),

\[
\left\| \left(\mathcal{E}_n(x)\right)_{n \geq 1} \right\|_{L^p(\mathcal{M};l^{\infty})} \leq C_p \left\| x \right\|_{L^p(\mathcal{M})}.
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It was later noted by Junge and Xu that \(C_p\) is of order \((p - 1)^{-2}\), in contrast with the classical case.
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**Definition**

$T$ is of *weak type* $(r, r)$ if there is a constant $C_r > 0$ such that for any $x \in L^r(\mathcal{M})_+$ and any $\theta > 0$, there exists a projection $e^{(\theta)} = e_x^{(\theta)}$ satisfying, for all $n \geq 1$,
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\sigma(1 - e^{(\theta)}) \leq (C_r \theta^{-1})^r \|x\|_{L^r(\mathcal{M})}^r \quad \text{and} \quad e^{(\theta)} T_n(x) e^{(\theta)} \leq \theta.
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In the commutative case, if $T = \sup_{n \geq 1} T_n$ is of (restricted) weak type, then we can take $e_x^{(\theta)} = 1\{T(x) \leq \theta\}$. 
Definition (Continued)

$T$ is of strong type $(r, r)$ if

$$\| (T_n(x))_{n \geq 1} \|_{L^r(\mathcal{N}; l^\infty)} \leq C_r \| x \|_{L^r(\mathcal{M})}.$$
Definition (Continued)

$T$ is of strong type $(r, r)$ if

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Cuculescu showed in '71 that any increasing sequence of conditional expectations is of weak type $(1, 1)$. 
Theorem (Junge & Xu, ’07)

Let $1 \leq p < q \leq \infty$. If $T = (T_n)_{n \geq 1}$ is a sequence of positive, subadditive maps and $T$ is of weak type $(p, p)$ and strong type $(q, q)$.
Theorem (Junge & Xu, '07)

Let $1 \leq p < q \leq \infty$. If $T = (T_n)_{n \geq 1}$ is a sequence of positive, subadditive maps and $T$ is of weak type $(p, p)$ and strong type $(q, q)$, then for any $p < r < q$,

$$\|(T_n(x))_{n \geq 1}\|_{L^r(\mathcal{N}; l^\infty)} \lesssim C_p^{1-\theta} C_q^{\theta} \left( \frac{rp}{r-p} \right)^2 \|x\|_{L^r(\mathcal{M})},$$

where $\theta$ is chosen such that $\frac{1}{r} = \frac{1}{p} - \theta + \frac{\theta}{q}$. This result should be regarded as a generalization of Marcinkiewicz' theorem. Together with Cuculescu's result, it yields Doob's maximal inequality with constant of optimal order.
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Let $1 \leq p < q \leq \infty$. If $T = (T_n)_{n \geq 1}$ is a sequence of positive, subadditive maps and $T$ is of weak type $(p, p)$ and strong type $(q, q)$, then for any $p < r < q$,

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For a measurable operator $x$ we denote by $\mu(x)$ the *decreasing rearrangement* of $x$,

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We say that $x$ is *submajorized* by $y$, notation $x \prec \prec y$, if

$$\int_0^t \mu_s(x) \, ds \leq \int_0^t \mu_s(y) \, ds \quad (t \geq 0).$$
For $0 < p < q < \infty$ define Calderón’s operator by

$$S_{p,q}f(t) = t^{-\frac{1}{p}} \int_{0}^{t} s^\frac{1}{p} f(s) \frac{ds}{s} + t^{-\frac{1}{q}} \int_{t}^{\infty} s^\frac{1}{q} f(s) \frac{ds}{s}.$$
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**Theorem (Calderón, ’66)**

*T is of restricted weak types $(p, p)$ and $(q, q)$ if and only if*

$$\mu_t(T1_A) \lesssim_{p,q} \left( S_{p,q} \mu(1_A) \right)(t) \quad (t > 0)$$

*for all measurable sets $A$.**
Theorem (D., ’12)

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where

$$K_{p, p', q, q'} = O((p' - p)^{-1}) \text{ as } p' \downarrow p, \quad O((q - q')^{-1}) \text{ as } q' \uparrow q.$$
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$$\|T^n(x)\|_{L^q(N; l^\infty)} \lesssim \max\{C_p, C_q\} (rp^{-1} + rq^{-1}) \|x\|_{L^r(M)}.$$
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This result should be regarded as a generalization of the Stein-Weiss interpolation theorem. It implies Doob's maximal inequality with constant of optimal order.
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\]
It is called fully symmetric if moreover,
\[
f \in S(\mathbb{R}_+), \quad g \in E, \quad f \lll g \Rightarrow f \in E \quad \text{and} \quad \|f\|_E \leq \|g\|_E.
\]
A normed linear subspace $E$ of $S(\mathbb{R}_+)$ is called a symmetric space on $\mathbb{R}_+$ if it is complete and if

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Examples: $L^p$-spaces, Lorentz spaces, Orlicz spaces,...
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We can define an associated noncommutative version by

$$E(\mathcal{M}, \tau) := \{x \in S(\tau) : \|\mu(x)\|_E < \infty\}.$$ 

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We let $E(\mathcal{M}; l^\infty)_+$ denote the set of all sequences $x = (x_n)_{n \geq 1}$ in $E(\mathcal{M})_+$ for which there exists an $a \in E(\mathcal{M})_+$ such that $x_n \leq a$ for all $n \geq 1$. For these elements we set

$$\|x\|_{E(\mathcal{M}; l^\infty)} := \inf \{\|a\|_{E(\mathcal{M})} : \ x_n \leq a \text{ for all } n \geq 1\}.$$
For any $0 < a < \infty$ we define the dilation operator $D_a$ on $S(\mathbb{R}_+)$ 

$$(D_a f)(s) = f(as) \quad (s \in \mathbb{R}_+).$$
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**Theorem (Boyd, ’69)**

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As a consequence, all sublinear operators of restricted weak types $(p, p)$ and $(q, q)$ are bounded on $E$ if $p < p_E \leq q_E < q$. 

Sjoerd Dirksen  
Noncommutative weak type interpolation
Theorem (D., ’12)

Let $1 \leq p < q \leq \infty$ and let $E$ be a fully symmetric space on $\mathbb{R}_+$. 

Improves earlier result of Bekjan, Chen and Osekowski (’12), who prove this for $T$ of weak type $(p, p)$ and $(\infty, \infty)$ with suboptimal interpolation constant.
Theorem (D., ’12)

Let $1 \leq p < q \leq \infty$ and let $E$ be a fully symmetric space on $\mathbb{R}_+$. Let $(T_n)_{n \geq 1}$ be a sequence of order preserving, sublinear maps which is of restricted weak types $(p, p)$ and $(q, q)$. 

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$$\left\| (T_n(x))_{n \geq 1} \right\|_{E(\mathcal{N}; l_{\infty})} \leq \max\{C_p, C_q\} K_{p, p', q, q'} \left\| S_{p', q'} \right\|_{E \rightarrow E} \left\| x \right\|_{E(\mathcal{M})}.$$
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Corollary

Let \( \mathcal{M} \) be a semi-finite von Neumann algebra and let \((\mathcal{E}_n)_{n \geq 1}\) be an increasing sequence of conditional expectations in \( \mathcal{M} \). If \( E \) is a symmetric space on \( \mathbb{R}_+ \) with \( p_E > 1 \), then there is a constant \( C_E \) depending only on \( E \) such that
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$$

If $p > 1$ then $C_{L^p}$ is of optimal order $O((p - 1)^{-2})$ as $p \downarrow 1$. 

Sjoerd Dirksen
Noncommutative weak type interpolation
Lemma (Special case: projections)

Fix $0 < p < q \leq \infty$. Let $(T_n)_{n \geq 1}$ be a sequence of positive maps of restricted weak types $(p, p)$ and $(q, q)$. 
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Lemma (Special case: projections)

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$$T_n(f) \leq a \quad (n \geq 1)$$

and

$$\mu_t(a) \leq \max \{C_p, C_q\} \ K_{p',p',q,q'} \ S_{p',q'} \mu(f)(t) \quad (t > 0).$$
Proof of main result.

Consider dyadic discretization \( \hat{x} = \sum_{j \in \mathbb{Z}} 2^{j+1} \lambda(2^j, 2^{j+1}) (x) \).
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\hat{x} = \sum_{k \in \mathbb{Z}} 2^k \lambda_{(2^k, \infty)}(x)
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\]

Apply lemma to $\lambda_{(2^k,\infty)}(x)$ to find a majorant $a_k$. Define

\[
a = \sum_{k \in \mathbb{Z}} 2^k a_k.
\]
Proof, continued.

\[ \mu(a) \ll \sum_{k \in \mathbb{Z}} 2^k \mu(a_k) \]
\[ \leq \max\{C_p, C_q\} K_{p',q',q'} \sum_{k \in \mathbb{Z}} 2^k S_{p',q'} \mu(f_k) \]
\[ \leq \max\{C_p, C_q\} K_{p',q',q'} S_{p',q'} \mu(\hat{x}) \]
\[ \leq 2 \max\{C_p, C_q\} K_{p',q',q'} S_{p',q'} \mu(x). \]
Proof of the lemma.

For $\theta > 1$ fix a projection $e_{q}^{(\theta)}$ and for $0 < \theta \leq 1$ we pick $e_{p}^{(\theta)}$ such that

$$\sigma(1 - e^{(\theta)}) \leq (C_{r}\theta^{-1})^{r}\tau(f) \quad \text{and} \quad e^{(\theta)}T_{n}(f)e^{(\theta)} \leq \theta.$$ 

for $r = q$ and $r = p$. 
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for $r = q$ and $r = p$. For every $k \in \mathbb{Z}$ define

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e_k = \left( \bigwedge_{l \geq k} e_{q}^{(2^l)} \right) (k > 0), \quad e_k = \left( \bigwedge_{0 \leq l \leq k} e_{p}^{(2^l)} \right) \wedge \left( \bigwedge_{l \geq 0} e_{q}^{(2^l)} \right) (k \leq 0)$$
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Set $d_k = e_k - e_{k-1}$. Define
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$$

Set $d_k = e_k - e_{k-1}$. Define

$$
a = K_{p,p',q,q'} \left( \sum_{-\infty < k \leq 0} 2^{(k-1)p/p'} d_k + \sum_{k > 0} 2^{(k-1)q/q'} d_k \right).
$$