ON A RELATIVISTIC BGK MODEL FOR POLYATOMIC GASES NEAR EQUILIBRIUM

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Abstract. Recently, a novel relativistic polyatomic BGK model was suggested by Pennisi and Ruggeri [J. of Phys. Conf. Series, 1035, (2018)] to overcome drawbacks of the Anderson-Witting model and Marle model. In this paper, we prove the unique existence and asymptotic behavior of classical solutions to the relativistic polyatomic BGK model when the initial data is sufficiently close to a global equilibrium.

1. Introduction

In the classical kinetic theory of gases, the BGK relaxation operator \[ v \] has been successfully used in place of the Boltzmann collision operator, yielding satisfactory simulation of the Boltzmann flows at much lower numerical cost. The relativistic generalization of the BGK approximation was first made by Marle [43, 42] and successively by Anderson and Witting [1]. The Marle model is an extension of the classical BGK model in the Eckart frame [12, 18], and the Anderson-Witting model obtains such extension using the Landau-Lifshitz frame [12, 39].

These models have been widely employed for various relativistic problems [13, 14, 19, 20, 28, 33], but several drawbacks were also recognized in the literature. For the Marle model, the relaxation time becomes unbounded for particles with zero rest mass [12]. The problem for the Anderson-Witting model is that the Landau-Lifshitz frame was established on the assumption that some of high order non-equilibrium quantities are negligible near equilibrium [1, 12], which inevitably leads to some loss of consistency.

Starting from these considerations, Pennisi and Ruggeri proposed a variant of Anderson-Witting model in the Eckart frame both for monatomic and polyatomic gas, and proved that the conservation laws of particle number and energy-momentum are satisfied and the H-theorem holds [44] (see also [48]). In the case of polyatomic gas, the Cauchy problem for the relativistic BGK model reads:

\[
\partial_t F + \hat{p} \cdot \nabla_x F = \frac{U_{\mu} p^{\mu}}{c \tau p^0} \left\{ 1 - p^\mu q_\mu \frac{1 + \beta mc^2}{bmc^2} \right\} F_E - F,
\]

where \( F(x^\alpha, p^\beta, I) \) is the momentum distribution function representing the number density of relativistic particles at the phase point \((x^\alpha, p^\beta)\) \((\alpha, \beta = 0, 1, 2, 3)\) with the microscopic internal energy \( I \in \mathbb{R}^+ \) that takes into account the energy due to the internal degrees of freedom of the particles. Here \( x^\alpha = (ct, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \) is the space-time coordinate, and \( p^\beta = (\sqrt{(mc)^2 + |p|}, p) \in \mathbb{R}^+ \times \mathbb{R}^3 \) is the four-momentum. Greek indices run from 0 to 3 and the repeated indices are assumed to be summed over their whole range; \( c \) is the light velocity, \( \hat{p} := cp/p^0 \) is the normalized momentum, \( m \) and \( \tau \) are respectively the mass and the relaxation time in the rest frame where the momentum of particles is zero. The macroscopic quantity \( b \) is given in (1.9). Throughout this paper, the metric tensor \( g_{\alpha\beta} \) and its inverse \( g^{\alpha\beta} \) are given by

\[
g_{\alpha\beta} = g^{\alpha\beta} = \text{diag}(1, -1, -1, -1)
\]

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and we use the raising and lowering indices as
\[ g_{\alpha \mu} p^\mu = p_\alpha, \quad g^{\alpha \mu} p_\mu = p^\alpha, \]
which implies \( p_\alpha = (p^0, -p) \). The Minkowski inner product is defined by
\[ p^\mu q_\mu = p^0 q^0 - \sum_{i=1}^{3} p^i q^i. \]

To present the macroscopic fields of \( F \), we define the particle-particle flux \( V^\mu \) and energy-momentum tensor \( T^{\mu \nu} \) by
\[ V^\mu = m c \int_{\mathbb{R}^3} \int_{0}^{\infty} p^\mu F \phi(I) \frac{dp}{p^0}, \quad T^{\mu \nu} = \frac{1}{mc} \int_{\mathbb{R}^3} \int_{0}^{\infty} p^\mu p^\nu F (mc^2 + I) \phi(I) \frac{dp}{p^0}. \]

Here \( \phi(I) \geq 0 \) is the state density of the internal mode such that \( \phi(I) \, dI \) represents the number of the internal states of a molecule having the internal energy between \( I \) and \( I + dI \) which can take various forms according to the physical context. For example, the following form of \( \phi(I) \) is employed in
\[ \phi(I) = I^{(f^i - 2)/2} \]
to get the correct classical limit of internal energy of polyatomic gas. Here \( f^i \geq 0 \) is the internal degrees of freedom due to the internal motion of molecules. In this paper, instead of choosing the specific form of \( \phi(I) \), we develop an existence theory for (1.1) that is valid for a general class of \( \phi(I) \) satisfying a certain condition which covers all the physically relevant cases (see (1.11)).

Going back to (1.2), we introduce the decomposition of \( V^\mu \) and \( T^{\mu \nu} \):
\[ V^\mu = nm U^\mu, \]
\[ T^{\mu \nu} = \sigma^{(\mu \nu)} + (p + \Pi) h^{\mu \nu} + \frac{1}{c^2} (q^{\mu} U^\nu + q^{\nu} U^\mu) + \frac{e}{c^2} U^\mu U^\nu, \]
which is called the Eckart frame [12] [18]. In (1.3), \( n m = c^{-1} \sqrt{V^\mu V^\mu} \) denotes the number density, \( U^\mu = (\sqrt{c^2 + |U|^2}, U) \) the Eckart four-velocity, \( p \) the pressure, \( \Pi \) the dynamical pressure, \( h^{\mu \nu} = -g^{\mu \nu} + \frac{1}{c^2} U^\mu U^\nu \) the projection tensor, \( \sigma^{\mu \nu} = T^{\alpha \beta} \left( h^{\mu \alpha} h^{\nu \beta} - \frac{1}{3} h^{\mu \nu} h_{\alpha \beta} \right) \) the viscous deviatoric stress, \( c \) the energy, and \( q^{\mu} = -h^{\nu}_{\alpha} U^\beta T^{\alpha \beta} \) the heat flux. We recall that only 14 field variables in (1.3) are independent due to the constraints:
\[ U_\alpha U^\alpha = c^2, \quad U_\alpha q^\alpha = 0, \quad U_\alpha \sigma^{<\alpha \beta>} = 0, \quad g_{\alpha \beta} \sigma^{<\alpha \beta>} = 0. \]

The macroscopic fields that appear frequently in this paper are defined as a suitable moment of \( F \) in the following manner.
\[ n^2 = \left( \int_{\mathbb{R}^3} \int_{0}^{\infty} F \phi(I) \, dI \, dp \right)^2 - \sum_{i=1}^{3} \left( \int_{\mathbb{R}^3} \int_{0}^{\infty} p^i F \phi(I) \, dI \, dp \right)^2, \]
\[ U^\mu = \frac{c}{n} \int_{\mathbb{R}^3} \int_{0}^{\infty} p^\mu F \phi(I) \, dI \, dp, \]
\[ e = \frac{1}{c} \int_{\mathbb{R}^3} \int_{0}^{\infty} (U^\mu p_\mu)^2 F \left( 1 + \frac{I}{mc^2} \right) \phi(I) \, dI \, dp, \]
\[ q^\mu = \frac{c}{U} \int_{\mathbb{R}^3} \int_{0}^{\infty} p^\mu (U^\nu p_\nu) F \left( 1 + \frac{I}{mc^2} \right) \phi(I) \, dI \, dp, \]
\[ -\frac{1}{c^2} U^\mu \int_{\mathbb{R}^3} \int_{0}^{\infty} (U^\nu p_\nu)^2 F \left( 1 + \frac{I}{mc^2} \right) \phi(I) \, dI \, dp. \]

The equilibrium distribution function \( F_E \) of (1.4) reads [44]:
\[ F_E = \exp \left\{ -1 + \frac{m}{k_B T} g_r - \left( 1 + \frac{I}{mc^2} \right) \frac{1}{k_B T} U^\mu p_\mu \right\} \]
We note that (1.4) reduces to the well known Jüttner distribution function \[38\] in the monatomic case. The equilibrium temperature \(T\) is determined by the following nonlinear relation:

\[
\overline{c}(T) \equiv \frac{\int_{\mathbb{R}^3} \int_0^\infty \sqrt{1 + \frac{|p|^2}{c^2}} (mc^2 + I) \phi(I) dI dp}{\int_{\mathbb{R}^3} \int_0^\infty e^{-\frac{(mc^2 + I)}{c^2}} \phi(I) dI dp} = \frac{e}{n},
\]

which is required for (1.1) to satisfy the conservation laws. With such \(T\), the relativistic chemical potential \(g_r\) is defined by the following equation

\[
ge^{-1 + \frac{mc^2}{2kB} T} = \frac{n}{(mc)^3 \int_{\mathbb{R}^3} \int_0^\infty e^{-\frac{(mc^2 + I)}{c^2}} \phi(I) dI dp}.
\]

Here \(kB\) is the Boltzmann constant. The uniqueness of the equilibrium temperature \(T\) will be considered later (see Proposition 2.1). Then, \(F_E\) satisfies

\[
U_\mu \int_{\mathbb{R}^3} \int_0^\infty p^\mu \left\{ \left( 1 - p^\mu q_\mu \right) \frac{1 + \frac{T}{mc^2}}{bmc^2} \right\} F_E - F = 0,
\]

\[
U_\mu \int_{\mathbb{R}^3} \int_0^\infty p^\mu p^\nu \left\{ \left( 1 - p^\mu q_\mu \right) \frac{1 + \frac{T}{mc^2}}{bmc^2} \right\} F_E - F \left( 1 + \frac{T}{mc^2} \right) \phi(I) dI dp = 0,
\]

so that the following conservation laws for \(V^\mu\) and \(T^{\mu\nu}\) hold true

\[
\partial_\mu V^\mu = 0, \quad \partial_\mu T^{\mu\nu} = 0.
\]

Finally, the macroscopic quantity \(b\) of (1.1) is defined as

\[
b = \frac{nmc^2}{\gamma^2} \left( \int_0^\infty K_2(\gamma^*) \phi(I) dI \right)^{-1} \int_0^\infty K_3(\gamma^*) \phi(I) dI,
\]

where \(\gamma\) and \(\gamma^*\) denote

\[
\gamma = \frac{mc^2}{kB T}, \quad \gamma^* = \gamma \left( 1 + \frac{T}{mc^2} \right)
\]

and \(K_n\) is the modified Bessel function of the second kind:

\[
K_n(\gamma) = \int_0^\infty \cosh(nr)e^{-\gamma \cosh(r)} dr.
\]

1.1. **Main result.** The aim of this paper is to study the global in time existence and asymptotic behavior of classical solutions to the Pennisi-Ruggeri model (1.1) when the initial data starts sufficiently close to a global equilibrium. For this, we decompose the solution \(F\) around the global equilibrium:

\[
F = F_E^0 + f \sqrt{F_E^0}.
\]

Here \(f\) is the perturbation part, and \(F_E^0\) is the global equilibrium defined by

\[
F_E^0 \equiv F_E(g_{r0}, 0, T_0; p) = \exp \left\{ -1 + \frac{m g_{r0}}{kB T_0} - \left( 1 + \frac{T}{mc^2} \right) \frac{cp_0}{kB T_0} \right\}
\]

where \(g_{r0}\) and \(T_0\) are positive constants. Inserting (1.10), then (1.1) can be rewritten as

\[
\partial_t f + \hat{p} \cdot \nabla_x f = \frac{1}{r} \left( L(f) + \Gamma(f) \right),
\]

\[
f_0(x, p) = f(0, x, p)
\]

where \(L\) is the linearized operator and \(\Gamma\) is the nonlinear perturbation whose definition can be found in (3.3), Lemma 3.2 and Proposition 3.1. The initial perturbation \(f_0\) is given by \(F_0 = F_E^0 + f_0 \sqrt{F_E^0}\). We now define the notations to state our main result.
We define the weighted $L^2$ inner product:
\[
\langle f, g \rangle_{L^2_{p,x}} = \int_{\mathbb{R}^3} \int_0^\infty f(p, I)g(p, I)\phi(I) \, dI \, dp,
\]
\[
\langle f, g \rangle_{L^2_{x,p,z}} = \int_{\mathbb{R}^3} \int_0^\infty f(x, p, I)g(x, p, I)\phi(I) \, dI \, dp \, dx
\]
and the corresponding norms:
\[
\|f\|_{L^2_{p,z}}^2 = \int_{\mathbb{R}^3} \int_0^\infty |f(p, I)|^2 \phi(I) \, dI \, dp,
\]
\[
\|f\|_{L^2_{x,p,z}}^2 = \int_{\mathbb{R}^3} \int_0^\infty |f(x, p, I)|^2 \phi(I) \, dI \, dp \, dx.
\]

We define the operator $\Lambda s$ ($s \in \mathbb{R}$) by
\[
\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi)e^{2\pi i x \cdot \xi} \, d\xi
\]
where $\hat{f}$ is the Fourier transformation of $f$.

We denote $H^s_x$ to be the homogeneous space endowed with the norm:
\[
\|f\|_{H^s_x} := \|\Lambda^s f\|_{L^2_x} = \left\| |\xi|^s \hat{f}(\xi) \right\|_{L^2_x}
\]
where $\| \cdot \|_{L^2}$ and $\| \cdot \|_{L^2_x}$ are the usual $L^2$-norm.

We use the notation $L^2_{p,x}H^s_x$ to denote
\[
\|f\|_{L^2_{p,x}H^s_x} = \|\|f\|_{H^s_x}\|_{L^2_{p,x}}
\]
where $\| \cdot \|_{H^s}$ is the usual Sobolev norm.

We define the energy functional $E$ and dissipation rate $D$ by
\[
E(f)(t) = \sum_{|\alpha| + |\beta| \leq N} \|\partial^\alpha \partial^\beta f\|_{L^2_{p,x}}^2,
\]
\[
D(f)(t) = \|\{I - P\}f\|_{L^2_{p,x}}^2 + \sum_{1 \leq |\alpha| + |\beta| \leq N} \|\partial^\alpha \partial^\beta f\|_{L^2_{p,x}}^2,
\]
where the multi-index $\alpha = [\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$ are used to denote
\[
\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}, \quad \partial^\beta = \partial_0^{\beta_0}\partial_1^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}.
\]

We also define the energy functional and dissipation rate for spatial derivatives by
\[
E_N(f)(t) = \sum_{0 \leq k \leq N} \|\nabla^k f\|_{L^2_{p,x}}^2,
\]
\[
D_N(f)(t) = \|\|f\|_{L^2_{p,x}}^2 + \sum_{1 \leq k \leq N} \|\nabla^k f\|_{L^2_{p,x}}^2.
\]

Then, our main result is as follows.

**Theorem 1.1.** Let $N \geq 3$ be an integer. Assume that the state density $\phi$ satisfies
\[
(1.11) \quad \int_{\mathbb{R}^3} \int_0^\infty P(p^0, I)e^{-C(1+|p^0|^{\frac{4}{3}})}p^0\phi(I) \, dI \, dp < \infty
\]
for any positive constant $C$ and arbitrary polynomial $P$ of $p^0$ and $I$. Then, there exists a positive constant $\delta$ such that if $E(f_0) < \delta$, \[(1.11)\] admits a unique global in time solution such that the energy functional is uniformly bounded:
\[
E_N(f)(t) + \int_0^t D_N(f)(s) \, ds \leq CE_N(f_0).
\]

If further $f_0 \in L^2_{p,x}H^{-s}_x$ for some $s \in [0, 3/2)$, then
(1) The negative Sobolev norm is uniformly bounded:
\[ \| \Lambda^{-s} f(t) \|_{L^2_{x,p}} \leq C_0. \]

(2) The solution converges to the global equilibrium with algebraic decay rate:
\[ \sum_{\ell \leq k \leq N} \| \nabla^k f(t) \|_{L^2_{x,p}} \leq C(1 + t)^{-\frac{N-\ell}{2}} \quad \text{for } -s \leq \ell \leq N - 1. \]

(3) The microscopic part decays faster by 1/2:
\[ \| \nabla^k (I - P) f(t) \|_{L^2_{x,p}} \leq C(1 + t)^{-\frac{N-\ell}{2}} \quad \text{for } -s \leq \ell \leq N - 2. \]

Remark 1.2. The choice of state density \( \phi(I) \) to guarantee the correct classical limit is not unique. For example, the following choices of \( \phi(I) \)
\[ T^{(f'-2)/2} \quad \text{or} \quad e^{-b \frac{I}{mc^2}} \left( 1 + \frac{I}{mc^2} \right)^r T^{(f'-2)/2} \quad \text{with} \quad b \geq 0, \ r > 0 \]
lead to the correct classical limit (See [9]).

Unlike the classical BGK models [3, 29], the equilibrium temperature of [111] is determined through the nonlinear relation [101] due to the relativistic nature of the equilibrium distribution function \( F_E \). For rigorous analysis, therefore, it must be first analyzed whether or not the relation [101] provides the unique equilibrium temperature as a moment of the solution \( F \). That is, any existence problem for [111] must be understood as the problem of solving the coupled system of [111] and [16]. In the case of relativistic models for a monatomic gas, such solvability problem was addressed in [3] for the Marle model, and [30] for the Anderson-Witting model. In [3, 30], there is clever manipulations of the modified Bessel functions of the second kind that was crucially used to show the monotonicity property of \( \tilde{e} \), and it plays an important role in proving the one-to-one correspondence between \( T \) and \( e/n \). However, in the case of a polyatomic gas, similar line of argument using the modified Bessel functions does not work due to the presence of the state density of the internal mode \( \phi(I) \) which can takes various forms (See Remark [12]). In view of these difficulties, we derive the following identity
\[ (\tilde{e})'(T) = \frac{1}{k_BT^2} \int_{\mathbb{R}^N} \int_0^\infty \left\{ cp^0 \left( 1 + \frac{T}{mc^2} \right) - \frac{e}{n} \right\}^2 F_E(g_r, 0, T) \phi(I) d\mathcal{I} dp \]
to investigate the monotonicity property of \( \tilde{e} \) in a different way (See Proposition [24]). Since the number density \( n \) and the energy \( e \) are strictly positive for sufficiently small \( E(f)(t) \), the above relation implies that \( \tilde{e}(T) \) is strictly increasing on \( T \in (0, \infty) \), which enables us to solve the solvability problem of \( T \). We mention that since the relativistic BGK model [111] does not guarantee the positivity of solutions, the smallness condition of \( E(f)(t) \) was required to preserve the sign of \( n \) and \( e \).

1.2. Brief history. The mathematical research on the relativistic BGK model was initiated in 2012 by Bellouquid et al [3] for the Marle model, where the unique determination of equilibrium variables, asymptotic limits and linearization problem were addressed. Afterward, Bellouquid et al [4] proved the existence and asymptotic behavior of solutions for the Marle model when the initial data starts close to the global equilibrium. Recently, Hwang and Yun [32] established the existence and uniqueness of stationary solutions to the boundary value problem for the Marle model in a finite interval. The weak solutions were covered by Calvo et al [8]. In the case of the Anderson-Witting model, the unique determination of equilibrium variables, and the existence and asymptotic behavior of near-equilibrium solutions were addressed in [30]. The unique existence of stationary solutions to the Anderson-Witting model in a slab was studied in [31].

For the relativistic Boltzmann equation, much more have been established. We refer to [6, 15, 17] for the local existence and linearized solution, [21, 22, 26, 53, 52] for the global existence and asymptotic behavior of near-equilibrium solutions, and [16, 37, 38] for the existence with large data. The spatially
homogeneous case was addressed in [41, 10, 11, 45, 46, 49]. The regularizing effect of the collision operator has been studied in [2, 33, 35]. The propagation of the uniform upper bound was established in [34]. We refer to [7, 34] for the Newtonian limit and [50] for the hydrodynamic limit. For the results on the relativistic theories of continuum for rarefied gases and its connections with the kinetic theory see for example [11, 10, 11, 45, 46, 49].

This paper is organized as follows. In Section 2, the unique determination of the equilibrium variable $T$ is discussed. In Section 3, we study the linearization of the relativistic BGK model (1.1). In Section 4, we provide estimates for the macroscopic fields and nonlinear perturbation. Section 5 is devoted to the proof of Theorem 1.1.

2. UNIQUE DETERMINATION OF THE EQUILIBRIUM TEMPERATURE $T$

We recall from (1.6) that $T$ is determined through the following relation

$$
\bar{c}(T) = \frac{1}{\int_{\mathbb{R}^3} \int_0^\infty e^{-(mc^2 + T)} \frac{1}{1 + |p|^2} \left((mc^2 + T) \phi(I) dI dp\right)} = \frac{c}{n}.
$$

In this section, formal calculations are first presented to show that the relativistic BGK model satisfies the conservation laws (1.8) if the above relation admits a unique $T$. And then we prove that when $E(f)(t)$ is small enough, $T$ indeed can be uniquely determined. The following lemma will be used later to simplify the integral of $F_E$.

**Lemma 2.1.** [54] For $U^\mu = (\sqrt{c^2 + |U|^2}, U)$, define $\Lambda$ by

$$
\Lambda = \begin{bmatrix}
  c^{-1}U_0 & -c^{-1}U^1 & -c^{-1}U^2 & -c^{-1}U^3 \\
  -U^1 & 1 + (U^0 - 1) \frac{U^1 U^2}{|U|^2} & (U^0 - 1) \frac{U^1 U^2}{|U|^2} & (U^0 - 1) \frac{U^1 U^3}{|U|^2} \\
  -U^2 & (U^0 - 1) \frac{U^2 U^3}{|U|^2} & 1 + (U^0 - 1) \frac{U^2 U^3}{|U|^2} & (U^0 - 1) \frac{U^2 U^3}{|U|^2} \\
  -U^3 & (U^0 - 1) \frac{U^3 U^1}{|U|^2} & (U^0 - 1) \frac{U^3 U^2}{|U|^2} & 1 + (U^0 - 1) \frac{U^3 U^1}{|U|^2}
\end{bmatrix}.
$$

Then $\Lambda$ transforms $U^\mu$ into the local rest frame $(c, 0, 0, 0)$.

**Proof.** The proof that $\Lambda$ is the Lorentz transformation can be found in [54]. The identity $\Lambda U^\mu = (c, 0, 0, 0)$ can be verified by an explicit computation:

$$
\Lambda U^\mu = \begin{bmatrix}
  c^{-1}(U^0)^2 - c^{-1}(U^1)^2 - c^{-1}(U^2)^2 - c^{-1}(U^3)^2 \\
  -U^0 U^1 + U^1 + \frac{(U^0 - 1) U^1 U^2}{|U|^2} |U|^2 \\
  -U^0 U^2 + U^2 + \frac{(U^0 - 1) U^2 U^3}{|U|^2} |U|^2 \\
  -U^0 U^3 + U^3 + \frac{(U^0 - 1) U^3 U^1}{|U|^2} |U|^2
\end{bmatrix} = \begin{bmatrix}
  c^2 + |U|^2 - |U|^2 \\
  -U^0 U^1 + U^1 + (U^0 - 1) U^1 \\
  -U^0 U^2 + U^2 + (U^0 - 1) U^2 \\
  -U^0 U^3 + U^3 + (U^0 - 1) U^3
\end{bmatrix} = \begin{bmatrix}
  c \\\n  0 \\
  0 \\
  0
\end{bmatrix}.
$$

**Lemma 2.2.** Assume (1.6) admits a unique $T$, then $F_E$ satisfies (1.7).

**Remark 2.3.** The solvability problem of $T$ in (1.6) is addressed in Proposition 2.3.

**Proof.** We write (1.7) in terms of the macroscopic fields using the Eckart frame (1.3):

$$
\frac{1}{c} c^\mu \int_{\mathbb{R}^3} \int_0^\infty p^\mu \left(1 - p^\nu q_\alpha - \frac{T}{b m c^2}\right) F_E(\phi(I) dI dp) \frac{dp}{p^\mu} = \frac{1}{mc^2} U^\mu U^\mu = n,
$$

$$
\frac{1}{c} U^\mu \int_{\mathbb{R}^3} \int_0^\infty p^\mu p^\nu \left(1 - p^\eta q_\alpha - \frac{T}{b m c^2}\right) F_E(1 + \frac{T}{mc^2}) \phi(I) dI dp = U^\mu T^{\mu \nu} = q^\nu + c U^\nu.
$$
By the change of variables $P^\mu := \Lambda p^\mu$ using Lemma 2.1, we have
\[
\begin{align*}
\int_{\mathbb{R}^3} \int_0^\infty p^\mu F_E \phi(I) \, dI \frac{dp}{p^0} &= e^{-1 + \frac{e}{m \beta}} \int_{\mathbb{R}^3} \int_0^\infty (\Lambda^{-1} P^\mu) e^{-(1 + \frac{p^0}{c^2}) \frac{mc}{p^0}} (\Lambda P^\mu) \phi(I) \, dI \frac{dP}{P^0} \\
&= e^{-1 + \frac{e}{m \beta}} \Lambda^{-1} \left( \int_{\mathbb{R}^3} \int_0^\infty e^{-(1 + \frac{p^0}{c^2}) \frac{mc}{p^0}} \phi(I) \, dI \frac{dP}{P^0} \right) \Xi(c, 0, 0, 0) \\
&= \frac{1}{c} e^{-1 + \frac{e}{m \beta}} \left( \int_{\mathbb{R}^3} \int_0^\infty e^{-(1 + \frac{p^0}{c^2}) \frac{mc}{p^0}} \phi(I) \, dI \frac{dP}{P^0} \right) \Lambda^{-1} \Xi(c, 0, 0, 0)
\end{align*}
\]
(2.2)

and
\[
\begin{align*}
U_\mu &\int_{\mathbb{R}^3} \int_0^\infty p^{\mu} p^{\nu} p^{\alpha} F_E \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \frac{dp}{p^0} \\
&= e^{-1 + \frac{e}{m \beta}} \left( \int_{\mathbb{R}^3} \int_0^\infty \{(\Lambda_{\mu} \nu)(\Lambda^\nu \nu)\} (\Lambda^{-1} P^\mu) e^{-(1 + \frac{p^0}{c^2}) \frac{mc}{p^0}} (\Lambda P^\mu) \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \frac{dP}{P^0} \right) \Xi(c, 0, 0, 0) \\
&= e^{-1 + \frac{e}{m \beta}} \left( \int_{\mathbb{R}^3} \int_0^\infty p^{\mu} e^{-(1 + \frac{p^0}{c^2}) \frac{mc}{p^0}} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \frac{dP}{P^0} \right) U^\nu
\end{align*}
\]
(2.3)

where we used the fact that (1) Lorentz inner product and the volume element $dp/p^0$ are invariant under $\Lambda$, and (2) $P^\mu$ takes the form of $(\sqrt{(mc)^2 + |P|^2}, P)$. Then, it follows from (2.3) and the decomposition of third moment (14):
\[
\int_{\mathbb{R}^3} \int_0^\infty p^{\mu} p^{\nu} p^{\alpha} F_E \left( 1 + \frac{T}{mc^2} \right)^2 \phi(I) \, dI \frac{dp}{p^0} = \frac{m}{c} \{a U^\alpha U^\mu U^\nu + b (h^{\alpha \mu} U^\nu + h^{\nu \nu} U^\mu + h^{\nu \nu} U^\alpha)\}
\]

that the second terms on the l.h.s of (2.3) are calculated respectively as follows
\[
\begin{align*}
-\frac{1}{c} U_\mu &\int_{\mathbb{R}^3} \int_0^\infty p^{\mu} p^{\nu} p^{\alpha} \frac{1 + \frac{T}{mc^2}}{bmc^2} F_E \phi(I) \, dI \frac{dp}{p^0} \\
&= -\frac{1}{bmc^2} q_\alpha \int_{\mathbb{R}^3} \int_0^\infty p^{\alpha} U_\mu p^{\mu} \left( 1 + \frac{T}{mc^2} \right) F_E \phi(I) \, dI \frac{dp}{p^0} \\
&= -\frac{1}{bmc^2} e^{-1 + \frac{e}{m \beta}} \left( \int_{\mathbb{R}^3} \int_0^\infty p^{\alpha} e^{-(1 + \frac{p^0}{c^2}) \frac{mc}{p^0}} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \frac{dp}{p^0} \right) q_\alpha U^\alpha \\
&= 0
\end{align*}
\]
(2.4)

and
\[
\begin{align*}
-c U_\mu &\int_{\mathbb{R}^3} \int_0^\infty p^{\mu} p^{\nu} p^{\alpha} \frac{1 + \frac{T}{mc^2}}{bmc^2} F_E \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \frac{dp}{p^0} \\
&= -\frac{1}{b c^2} U_\mu q_\alpha \{a U^\alpha U^\mu U^\nu + b (h^{\alpha \mu} U^\nu + h^{\nu \nu} U^\mu + h^{\nu \nu} U^\alpha)\} \\
&= -\frac{1}{c^2} U_\mu q_\alpha \{h^{\nu \nu} U^\mu\} \\
&= q^\nu.
\end{align*}
\]
(2.5)

Here we used the fact that (1) $U_\mu U^\mu = c^2$, and (2) $h^{\alpha \mu}, h^{\nu \nu}$ and $q^\mu$ are orthogonal to $U_\mu$ in the following sense
\[
U_\mu h^{\nu \nu} = U_\mu \left( -g^{\mu \nu} + \frac{1}{c^2} U^\mu U^\nu \right) = -U^\nu + U^\mu = 0, \quad U_\mu q^\mu = -U_\mu h^{\alpha \mu} U_\beta T^{\alpha \beta} = 0.
\]
Finally, we go back to (2.1) with (2.2)–(2.5) to obtain
\[ e^{-1 + \frac{m}{\sqrt{P}} \frac{\Delta}{n}} \int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{x}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp = n, \]
\[ ce^{-1 + \frac{m}{\sqrt{P}} \frac{\Delta}{n}} \left\{ \int_{\mathbb{R}^3} \int_0^\infty p^0 e^{-\left(1 + \frac{x}{\Delta} \right) \frac{P}{e^2}} \left(1 + \frac{1}{mc^2}\right) \phi(\mathcal{I}) \, d\mathcal{I} \right\} U'' = eU''. \]
Using the change of variables \( \frac{rn}{mc} \rightarrow p \), we get
\[ e = \frac{\int_{\mathbb{R}^3} \int_0^\infty \sqrt{1 + |p|^2} e^{-\left(1 + \frac{r}{\Delta} \right) \frac{P}{e^2}} \left(1 + \frac{1}{mc^2}\right) \phi(\mathcal{I}) \, d\mathcal{I} \, dp}{\int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{r}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp}, \]
\[ e^{-1 + \frac{m}{\sqrt{P}} \frac{\Delta}{n}} = \frac{n}{(mc)^3 \int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{r}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp}, \]
which gives the desired result. \( \square \)

The following lemma provides information about the ranges of \( n \) and \( e/n \) when \( E(f)(t) \) is small enough.

**Lemma 2.4.** Suppose \( E(f)(t) \) is sufficiently small. Then we have
\[ |n - 1| + \left| \frac{e}{n} - \bar{c}(T_0) \right| \leq C \sqrt{E(f)(t)}. \]

**Proof.** By Lemma 1.2 and the Soboelv embedding \( H^2(\mathbb{R}^2_x) \subseteq L^\infty(\mathbb{R}^2_x) \), we have
\[ |n - 1| + \left| \frac{e}{n} - \bar{c}(T_0) \right| \leq C \| f \|_{L^2_{T,x}} \leq C \| f \|_{L^\infty_{T,x}} \leq C \| f \|_{L^2_{T,x}} \leq C \sqrt{E(f)(t)}. \]

Note that the above result is independent of the determination of \( T \) since Lemma 1.2 is established by the definition of \( n \) and \( e/n \) given in 1.2. \( \square \)

We are now ready to prove that (1.6) determines a unique \( T \) in the near-equilibrium regime.

**Proposition 1.** Suppose \( E(f)(t) \) is sufficiently small. Then \( T \) can be uniquely determined by the relation (1.6). Thus \( T \) is written as
\[ T = (\bar{c})^{-1} \left( \frac{e}{n} \right). \]

**Proof.** We observe that
\[ \{ \bar{c} \}'(T) = \frac{1}{k_B T^2} \left\{ \int_{\mathbb{R}^3} \int_0^\infty (1 + |p|^2) e^{-\left(1 + \frac{x}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp \right\} \]
\[ = \frac{1}{k_B T^2} \left\{ \int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{x}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp \right\} \]
\[ = \frac{1}{k_B T^2} \left\{ \int_{\mathbb{R}^3} \int_0^\infty (1 + |p|^2) e^{-\left(1 + \frac{x}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp \right\} - \left( \frac{e}{n} \right)^2 \right\}. \]
Using the change of variables \( p \rightarrow \frac{p}{mc} \), we find
\[ \{ \bar{c} \}'(T) = \frac{1}{k_B T^2} \left\{ e \int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{x}{\Delta} \right) \frac{P}{e^2}} \phi(\mathcal{I}) \, d\mathcal{I} \, dp \right\} \]
\[ = \frac{1}{k_B T^2} \left\{ \int_{\mathbb{R}^3} \int_0^\infty \left\{ c_p \left(1 + \frac{I}{mc^2} \right) \right\}^2 F_E(g_r, 0, T) \phi(\mathcal{I}) \, d\mathcal{I} \, dp \right\} \]
where \( F_E(g_r, 0, T) \) denotes

\[
F_E(g_r, 0, T) = e^{-1 + \frac{mc}{r} - \left(1 + \frac{x}{mc^2}\right) \frac{p^0}{n}}
\]

\[
= \frac{n}{\int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{x}{mc^2}\right) \frac{p^0}{n}} \phi(I) dI dp}
\]

We also observe that

\[
2 \left(\frac{e}{n}\right)^2 - \left(\frac{e}{n}\right)^2 = \frac{2e}{n} \int_{\mathbb{R}^3} \int_0^\infty \sqrt{1 + |p|^2} e^{-\left(1 + \frac{x}{mc^2}\right) \frac{p^0}{n}} \sqrt{1 + |p|^2} (mc^2 + I) \phi(I) dI dp
\]

\[
\int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{x}{mc^2}\right) \frac{p^0}{n}} \phi(I) dI dp
\]

(2.7)

\[
= \int_{\mathbb{R}^3} \int_0^\infty \frac{2e}{n} \int_0^\infty \phi(I) dI dp
\]

\[
= \int_{\mathbb{R}^3} \int_0^\infty \frac{2e}{n} \int_0^\infty \phi(I) dI dp
\]

\[
= \frac{1}{n} \int_{\mathbb{R}^3} \int_0^\infty \left\{ \frac{2e}{n} \int_0^\infty \phi(I) dI dp \right\}
\]

\[
= \frac{1}{n} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \phi(I) dI dp
\]

\[
= \frac{1}{n} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \phi(I) dI dp
\]

\[
= \frac{1}{n} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \phi(I) dI dp
\]

Combining (2.6) and (2.7), we have

(2.8)

\[
\left\{ \frac{e}{n} \right\}^2 = \frac{1}{k_B nT^2} \int_{\mathbb{R}^3} \int_0^\infty \left\{ \frac{2e}{n} \int_0^\infty \phi(I) dI dp \right\}^2
\]

\[
= \frac{1}{k_B nT^2} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \phi(I) dI dp
\]

\[
= \frac{1}{k_B nT^2} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \phi(I) dI dp
\]

\[
= \frac{1}{k_B nT^2} \int_{\mathbb{R}^3} \int_0^\infty \int_0^\infty \phi(I) dI dp
\]

Since Lemma 2.4 says that \( n \) is positive for sufficiently small \( E(f(t)) \), (2.8) implies that \( \tilde{c}(T) \) is a strictly increasing function. Furthermore, \( \tilde{c}(T) \) is continuous on \( T \in (0, \infty) \) under the assumption (1.1). So, there exists a positive constant \( \delta_0 \) such that

\[
[\tilde{c}(T_0) - \delta_0, \tilde{c}(T_0) + \delta_0] \subseteq \text{Range}(\tilde{c}(T)).
\]

If \( E(f(t)) \leq \delta_0 \), then we have from Lemma 2.4 that

\[
\tilde{c}(T_0) - \delta_0 \leq \frac{e}{n} \leq \tilde{c}(T_0) + \delta_0
\]

which implies that the range of \( e/n \) is included in the range of \( \tilde{c}(T) \) for sufficiently small \( E(f(t)) \). Therefore, there exists a one-to-one correspondence between \( T \) and \( e/n \) providing

\[
T = (\tilde{c})^{-1} \left( \frac{e}{n} \right).
\]

\[ \square \]

3. Linearization

In this section, the linearization of (1.1) is discussed when the solution is sufficiently close to the global equilibrium. First, we provide computations of \( F_E^0 \) that will be used often later.

**Lemma 3.1.** The following identities hold:

(1) \[
\int_{\mathbb{R}^3} \int_0^\infty \left( p^0 \right)^2 F_E^0 \left( 1 + \frac{T}{mc^2} \right) \phi(I) dI dp = \frac{k_B T_0}{c}.
\]
Proof of (2): It can be obtained in a similar way to (1) as follows
\[
\frac{\mc^2 b_0}{k_B T_0} = \frac{e}{n} + k_B T.
\]
Here \( b_0 \) denotes
\[
b_0 = \frac{mc^2}{\gamma_0^2} \left( \int_0^{\infty} \frac{K_2(\gamma_0^*)}{\gamma_0^*} \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \right)^{-1} \int_0^{\infty} K_3(\gamma_0^*) \phi(\mathcal{I}) \mathrm{d}\mathcal{I}
\]
with
\[
\gamma_0 = \frac{mc^2}{k_B T_0}, \quad \gamma_0^* = \gamma_0 \left( 1 + \frac{T}{mc^2} \right).
\]
Proof. For reader’s convenience, we record the definition of \( F_E^0 \):
\[
F_E^0 = \exp \left\{ -1 + \frac{mc^2}{k_B T_0} \left( 1 + \frac{T}{mc^2} \right) \right\}
\]
where
\[
e^{-1 + \frac{mc^2}{k_B T_0} \frac{\phi}{\gamma_0^2}} = \frac{1}{(mc)^3 \int_0^{\infty} e^{-(mc^2 + \mathcal{I})} \frac{1}{\sqrt{1 + |p|^2}} \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \mathrm{d}p}
\]
- Proof of (1): It follows from the change of variables \( \frac{mc}{p} \rightarrow p \) that
\[
\int_{\mathbb{R}^3} \int_0^{\infty} (p^2)^2 F_E^0 \left( 1 + \frac{T}{mc^2} \right) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{p^0} = e^{-1 + \frac{mc^2}{k_B T_0} \frac{\phi}{\gamma_0^2}} \int_{\mathbb{R}^3} \int_0^{\infty} (p^2)^2 e^{-\left(1 + \frac{T}{mc^2}\right) \frac{\phi}{\gamma_0^2}} \left( 1 + \frac{T}{mc^2} \right) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{p^0}
\]
\[
= \frac{m^3 c^2 e^{-1 + \frac{mc^2}{k_B T_0} \frac{\phi}{\gamma_0^2}}}{3c} \int_{\mathbb{R}^3} \int_0^{\infty} e^{-(mc^2 + \mathcal{I})} \frac{1}{\sqrt{1 + |p|^2}} (mc^2 + \mathcal{I}) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{\sqrt{1 + |p|^2}}
\]
Using spherical coordinates and integration by parts, we have
\[
\int_{\mathbb{R}^3} \int_0^{\infty} (p^2)^2 F_E^0 \left( 1 + \frac{T}{mc^2} \right) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{p^0} = \frac{1}{3c} \int_{\mathbb{R}^3} \int_0^{\infty} |p|^2 e^{-\left(1 + \frac{T}{mc^2}\right) \frac{\phi}{\gamma_0^2}} (mc^2 + \mathcal{I}) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{\sqrt{1 + |p|^2}}
\]
\[
= \frac{1}{3c} \int_0^{\infty} \int_0^{\infty} e^{-\left(1 + \frac{T}{mc^2}\right) \frac{\phi}{\gamma_0^2}} (mc^2 + \mathcal{I}) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{\sqrt{1 + |\mathcal{I}|^2}}
\]
\[
= \frac{k_B T_0}{c}
\]
- Proof of (2): It can be obtained in a similar way to (1) as follows
\[
\int_{\mathbb{R}^3} \int_0^{\infty} (p^0)^2 F_E^0 \left( 1 + \frac{T}{mc^2} \right) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \frac{dp}{p^0} = e^{-1 + \frac{mc^2}{k_B T_0} \frac{\phi}{\gamma_0^2}} \int_{\mathbb{R}^3} \int_0^{\infty} p^0 e^{-\left(1 + \frac{T}{mc^2}\right) \frac{\phi}{\gamma_0^2}} (1 + \frac{T}{mc^2}) \phi(\mathcal{I}) \mathrm{d}\mathcal{I} \mathrm{d}p
\]
Proof of (3): We first introduce another representation of the modified Bessel functions of the second kind:

\[ K_2(\gamma) = \int_0^\infty \frac{2r^2 + 1}{\sqrt{1 + r^2}} e^{-\gamma \sqrt{1 + r^2}} \, dr, \quad K_3(\gamma) = \int_0^\infty (4r^2 + 1)e^{-\gamma \sqrt{1 + r^2}} \, dr. \]

Using this, one can rewrite \( e^{-1 + \frac{mc}{\gamma E}} \) as

\[
e^{-1 + \frac{mc}{\gamma E}} = \frac{1}{4\pi (mc)^3} \left( \int_0^\infty \int_0^\infty r^2 e^{-\left(1 + \frac{mc}{\gamma E}\right)\gamma \sqrt{1 + r^2}} \phi(I) \, dI \, dr \right)^{-1}
\]

\[
= \frac{1}{4\pi (mc)^3} \left( \int_0^\infty \int_0^\infty 2r^2 + 1 \frac{1}{\sqrt{1 + r^2}} e^{-\gamma \sqrt{1 + r^2}} \phi(I) \, dI \, dr \right)^{-1}
\]

\[
= \frac{1}{4\pi (mc)^3} \left( \int_0^\infty \int_0^\infty \int \frac{K_2(\gamma^0)}{\gamma^0} \phi(I) \, dI \, d\gamma \right)^{-1}.
\]

Using spherical coordinates and change of variables \( \frac{mc}{\gamma E} \rightarrow p \), we have from (3.1) that

\[
\int_{\mathbb{R}^3} \int_0^\infty \left( p' \right)^2 F_0 \left( 1 + \frac{T}{mc^2} \right)^2 \phi(I) \, dI \, dp
\]

\[
= \frac{(mc)^2}{3} \left( \int_0^\infty \frac{K_2(\gamma^0)}{\gamma^0} \phi(I) \, dI \right)^{-1} \int_0^\infty \int_0^\infty \int \frac{K_3(\gamma^0)}{\gamma^0} \phi(I) \, dI \, d\gamma
\]

which gives the desired result.

**Proof of (4):** Recall from (1.6) that

\[
e = \int_{\mathbb{R}^3} \int_0^\infty \sqrt{1 + |p|^2} e^{-\left(1 + \frac{mc}{\gamma E}\right)\sqrt{1 + |p|^2}} \left( mc^2 + I \right) \phi(I) \, dI \, dp
\]

\[
= \int_{\mathbb{R}^3} \int_0^\infty e^{-\left(1 + \frac{mc}{\gamma E}\right)\sqrt{1 + |p|^2}} \left( mc^2 + I \right) \phi(I) \, dI \, dp
\]

By integration by parts twice, (3.2) becomes

\[
= \frac{(mc)^2}{3} \left( \int_0^\infty \frac{K_2(\gamma^0)}{\gamma^0} \phi(I) \, dI \right)^{-1} \int_0^\infty \int_0^\infty \int \frac{K_3(\gamma^0)}{\gamma^0} \phi(I) \, dI \, d\gamma
\]

By integration by parts twice, (3.2) becomes

\[
= \frac{(mc)^2}{3} \left( \int_0^\infty \frac{K_2(\gamma^0)}{\gamma^0} \phi(I) \, dI \right)^{-1} \int_0^\infty \int_0^\infty \int \frac{K_3(\gamma^0)}{\gamma^0} \phi(I) \, dI \, d\gamma
\]

which gives the desired result.
By (3.1) and integration by parts, one finds

\[
e_1 = mc^2 \left( \int_0^\infty \frac{K_2(\gamma^*)}{\gamma^*} \phi(I) dI \right)^{-1} \int_0^\infty \int_0^\infty r^2 \sqrt{1 + r^2} e^{-\left(1 + \frac{I}{mc^2}\right)} \gamma \sqrt{1 + r^2} \left(1 + \frac{I}{mc^2}\right) \phi(I) dIdr
\]

\[
= mc^2 \gamma \left( \int_0^\infty \frac{K_2(\gamma^*)}{\gamma^*} \phi(I) dI \right)^{-1} \int_0^\infty \int_0^\infty (3r^2 + 1) e^{-\left(1 + \frac{I}{mc^2}\right)} \gamma \sqrt{1 + r^2} \phi(I) dIdr
\]

\[
= mc^2 \gamma \left( \int_0^\infty \frac{K_2(\gamma^*)}{\gamma^*} \phi(I) dI \right)^{-1} \left( \int_0^\infty K_3(\gamma^*) \phi(I) dI - \int_0^\infty \int_0^\infty r^2 e^{-\left(1 + \frac{I}{mc^2}\right)} \gamma \sqrt{1 + r^2} \phi(I) dIdr \right)
\]

\[
= \frac{\gamma b}{n} - \frac{mc^2}{\gamma}
\]

which gives the desired result. Since (5) can be obtained in the same manner as in (4), we omit it. \(\square\)

3.1. Linearization of (1.1). Define \(e_i (i = 1, \cdots, 5)\) by

\[
e_1 = \sqrt{F_E^0}, \quad e_{2,3,4} = \sqrt{\frac{1}{b_0m} \left(1 + \frac{I}{mc^2}\right)} \rho \sqrt{F_E^0},
\]

\[
e_5 = \sqrt{\frac{1}{k_B T_0^2 \{\varepsilon\} (T_0)}} \left\{ c p_0 \left(1 + \frac{I}{mc^2}\right) - \bar{e}(T_0) \right\} \sqrt{F_E^0}
\]

and the projection operator \(P(f)\) by

\[
(3.3) \quad P(f) = \sum_{i=1}^5 \langle f, e_i \rangle_{L_{p,t}} e_i.
\]

Then, the equilibrium distribution function \(F_E\) given in (1.5) is linearized as follows.

**Lemma 3.2.** Suppose \(E(f)(t)\) is sufficiently small. We then have

\[
\left(1 - p^\mu q_{r\mu} \frac{1 + \frac{I}{mc^2}}{bmc^2} \right) F_E - F_E^0 = \left( P(f) + \sum_{i=1}^4 \Gamma_i(f) \right) \sqrt{F_E^0}.
\]

Here the nonlinear perturbations \(\Gamma_i(f) (i = 1, \cdots, 4)\) are given by

\[
\Gamma_1(f) = \frac{\Psi_1}{2} \left(1 - \frac{\Psi^2}{2(2 + \Psi + 2\sqrt{1 + \Psi})} \right) \sqrt{F_E^0},
\]

\[
\Gamma_2(f) = \frac{1}{\{\varepsilon\} (T_0)} \frac{1}{k_B T_0^2} \left\{ c p_0 \left(1 + \frac{I}{mc^2}\right) - \bar{e}(T_0) \right\} \sqrt{F_E^0}
\]

\[
\times \left\{ c \left(1 - \int_{\mathbb{R}^3} f \sqrt{F_E^0} \phi(I) dI dp \right) \int_{\mathbb{R}^3} \int_0^\infty \{2 c p_0 \Phi + \Phi^2\} F \left(1 + \frac{I}{mc^2}\right) \phi(I) dI dp \right\}
\]

\[
- \frac{1}{c} \left( \Psi_1^2 + \Psi^3 - 3 \Psi^2 \right) \int_{\mathbb{R}^3} \int_0^\infty \left( U_{\mu\nu} p_{\mu} \right)^2 F \left(1 + \frac{I}{mc^2}\right) \phi(I) dI dp \right\}
\]

\[
- c \int_{\mathbb{R}^3} \int_0^\infty f \sqrt{F_E^0} \phi(I) dI dp \int_{\mathbb{R}^3} \int_0^\infty p_0 f \sqrt{F_E^0} \left(1 + \frac{I}{mc^2}\right) \phi(I) dI dp \right\}
\]

\[
\Gamma_3(f) = -c \frac{1 + \frac{I}{mc^2}}{k_B T_0} \left\{ \frac{2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})} \right\} \int_{\mathbb{R}^3} \int_0^\infty p f \sqrt{F_E^0} \phi(I) dI dp \right\}
\]

\[
+ \frac{1 + \frac{I}{mc^2}}{b_0mc^2} \Gamma_3(f) \cdot p \sqrt{F_E^0} p_0 q_{r\mu} \left(1 + \frac{I}{mc^2}\right) \sqrt{F_E^0}
\]

\[
\Gamma_4(f) = \frac{1}{\sqrt{F_E^0}} \int_0^1 \left(1 - \theta \right) \left( -(n - 1, U, \frac{e}{n} - \bar{e}(T_0)), q_{r\mu} \right) D^2 \tilde{F}(\theta) \left( n - 1, U, \frac{e}{n} - \bar{e}(T_0), q_{r\mu} \right) d\theta,
\]
where $\Gamma_3^*(f)$ denotes

$$
\Gamma_3^*(f) = -c^2 \sum_{i=1}^{3} \int_{\mathbb{R}^3} \int_{0}^{\infty} p^i f \sqrt{F_E^0_0} d\mathbb{L} \frac{dp}{p^0} \int_{\mathbb{R}^3} \int_{0}^{\infty} pp^i f \sqrt{F_E} \left(1 + \frac{T}{mc^2}\right) \phi(\mathbb{I}) d\mathbb{L} \frac{dp}{p^0} \\
+ c \int_{\mathbb{R}^3} \int_{0}^{\infty} p\Phi_1 \left(1 + \frac{T}{mc^2}\right) \phi(\mathbb{I}) d\mathbb{L} \frac{dp}{p^0} - \int_{\mathbb{R}^3} \int_{0}^{\infty} pp f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} \frac{dp}{p^0} \\
\times \int_{\mathbb{R}^3} \int_{0}^{\infty} \left\{ c^2 p^0 f \sqrt{F_E^0_0} + \frac{1}{p^0} \left(2cp^0\Phi + \Phi^2\right) F \right\} \left(1 + \frac{T}{mc^2}\right) \phi(\mathbb{I}) d\mathbb{L} dp \\
+ \left\{ \Psi + \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})} \right\} \int_{\mathbb{R}^3} \int_{0}^{\infty} p f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} \frac{dp}{p^0} \\
\times \int_{\mathbb{R}^3} \int_{0}^{\infty} \left(U^\nu p_\nu\right)^2 F \left(1 + \frac{T}{mc^2}\right) \phi(\mathbb{I}) d\mathbb{L} \frac{dp}{p^0},
$$

and $\Psi, \Psi_1, \Phi$ and $\Phi_1$ are defined as

$$
\Psi = 2 \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} dp + \left( \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} dp \right)^2 \\
\Psi_1 = \left( \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} dp \right)^2 - 3 \sum_{i=1}^{3} \left( \int_{\mathbb{R}^3} \int_{0}^{\infty} p^i f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} dp \right)^2, \\
\Phi = U_\mu p^\mu - cp^0, \\
\Phi_1 = U_\mu p^\mu - cp^0 + c \sum_{i=1}^{3} p^i \int_{\mathbb{R}^3} \int_{0}^{\infty} p^i f \sqrt{F_E^0_0} \phi(\mathbb{I}) d\mathbb{L} \frac{dp}{p^0}.
$$

Proof. We consider the transitional macroscopic fields between $F$ and $F_E^0$:

$$
\left( n_\theta, U_\theta, \frac{c}{n} \theta, q_\theta^0 \right) = \theta \left( n, U, \frac{c}{n}, q^0 \right) + (1 - \theta)(1, 0, \bar{c}(T_\theta), 0),
$$

and define $\bar{F}(\theta)$ and $F(\theta)$ by

$$
\bar{F}(\theta) = \left(1 - p^\mu q_\theta \mu \frac{1 + \frac{T}{mc^2}}{b_\theta mc^2}\right) e^{-1 + \frac{mc}{T_\theta \gamma_\theta}} \frac{1 + \frac{T}{mc^2}}{b_\theta T_\theta} U_\theta^\nu p_\nu, \\
F(\theta) = e^{-1 + \frac{mc}{T_\theta \gamma_\theta}} \left(1 - \frac{mc}{T_\theta \gamma_\theta}\right) \frac{1 + \frac{T}{mc^2}}{b_\theta T_\theta} U_\theta^\nu p_\nu.
$$

Here $g_\theta, T_\theta$ and $b_\theta$ are given by

$$
e^{-1 + \frac{mc}{T_\theta \gamma_\theta}} \frac{g_\theta b_\theta}{n_\theta} = \frac{n_\theta}{(mc)^3 \int_{\mathbb{R}^3} \int_{0}^{\infty} \frac{1 + \frac{T}{mc^2}}{b_\theta mc^2} \phi(\mathbb{I}) d\mathbb{L} dp}, \\
b_\theta = \frac{n_\theta mc^2}{\gamma_\theta^2} \left( \int_{0}^{\infty} K_2(\gamma_\theta^*) \phi(\mathbb{I}) d\mathbb{L} \right)^{-1} \int_{0}^{\infty} K_3(\gamma_\theta^*) \phi(\mathbb{I}) d\mathbb{I}
$$

with

$$
\gamma_\theta = \frac{mc}{k_B T_\theta}, \\
\gamma_\theta^* = \gamma_\theta + \left(1 + \frac{T}{mc^2}\right).
$$

By Lemma 3.1, $b_\theta$ can be also expressed in the following manner

$$
b_\theta = \frac{k_B n_\theta T_\theta \bar{c}(T_\theta)}{mc^2} + \frac{n_\theta (k_B T_\theta)^2}{mc^2}.
$$
Thus $\bar{F}(\theta)$ can be regarded as a function of $n_\theta, U_\theta, (c/n)_\theta$ and $q_\theta^0$, so it follows from the Taylor expansion that

$$
\left(1 - p^\mu q_\mu - \frac{T}{bmc^2}\right) F_E - F_E^0
$$

$$
= \bar{F}(1) - \bar{F}(0)
$$

$$
= \frac{\partial \bar{F}}{\partial n_\theta} \bigg|_{\theta=0} \frac{\partial n_\theta}{\partial \theta} + \nabla_{U_\theta} \bar{F} \bigg|_{\theta=0} \frac{\partial U_\theta}{\partial \theta} + \frac{\partial \bar{F}}{\partial (c/n)_\theta} \bigg|_{\theta=0} \frac{\partial (c/n)_\theta}{\partial \theta} + \frac{\partial \bar{F}}{\partial q_\theta^0} \bigg|_{\theta=0} \frac{\partial q_\theta^0}{\partial \theta} + \nabla_{q_\theta} \bar{F} \bigg|_{\theta=0} \frac{\partial q_\theta}{\partial \theta}
$$

$$
+ \int_0^1 (1 - \theta) (n - 1, U, T - T_0, q^\mu) D^2 \bar{F}(\theta) (n - 1, U, T - T_0, q^\mu)^T \, d\theta
$$

$$
= (n - 1)F_E^0 + \frac{1}{\{e\}_3} (T_0) \frac{1}{kBT_0^2} \left( \frac{e}{n} - \bar{c}(T_0) \right) \left\{ c^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right\} F_E^0 + \frac{1 + \frac{T}{mc^2}}{kBT_0} U \cdot pF_E^0
$$

$$
- p^\mu q_\mu + \frac{1 + \frac{T}{mc^2}}{b_0mc^2} F_E^0 + \int_0^1 (1 - \theta) \left( n - 1, U, \frac{e}{n} - \bar{c}(T_0), q^\mu \right) D^2 \bar{F}(\theta) \left( n - 1, U, \frac{e}{n} - \bar{c}(T_0), q^\mu \right)^T \, d\theta.
$$

In the last identity, we used the simple calculations

$$
\frac{\partial \bar{F}}{\partial n_\theta} \bigg|_{\theta=0} = F_E^0, \quad \nabla_{U_\theta} \bar{F} \bigg|_{\theta=0} = \frac{1 + \frac{T}{mc^2}}{kBT_0} pF_E^0,
$$

$$
\frac{\partial}{\partial (c/n)_\theta} \bar{F} \bigg|_{\theta=0} = \frac{1}{\{e\}_3} (T_0) \frac{1}{kBT_0^2} \left\{ c^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right\} F_E^0,
$$

$$
\nabla_{q_\theta} \bar{F} \bigg|_{\theta=0} = \frac{1 + \frac{T}{mc^2}}{b_0mc^2} F_E^0.
$$

Now we use the notations $I_i$ ($i = 1, \ldots, 4$) to denote the first four terms on the last identity, and we decompose them into the linear part and the nonlinear part.

- Decomposition of $I_1$: Inserting $F = F_E^0 + f\sqrt{F_E^0}$ into a definition of $n$, one finds

$$
n = \left\{ \left( \int_{\mathbb{R}^3} \int_0^\infty F\phi(I) \, dI \, dp \right)^2 - \sum_{i=1}^3 \left( \int_{\mathbb{R}^3} \int_0^\infty p^i F\phi(I) \, dI \, dp \right)^2 \right\}^{\frac{1}{2}}
$$

$$
= \left\{ 1 + 2 \int_{\mathbb{R}^3} \int_0^\infty f\sqrt{F_E^0}\phi(I) \, dI \, dp + \left( \int_{\mathbb{R}^3} \int_0^\infty f\sqrt{F_E^0}\phi(I) \, dI \, dp \right)^2 \right. - \sum_{i=1}^3 \left( \int_{\mathbb{R}^3} \int_0^\infty p^i f\sqrt{F_E^0}\phi(I) \, dI \, dp \right)^2 \right\}^{\frac{1}{2}}
$$

$$
= \sqrt{1 + \Psi}
$$

which, together with the following identity [3]:

$$
\sqrt{1 + \Psi} = 1 + \frac{\Psi}{2} - \frac{\Psi^2}{2(2 + \Psi + 2\sqrt{1 + \Psi})}
$$

(3.6)

gives

$$
n - 1 = \frac{\Psi}{2} - \frac{\Psi^2}{2(2 + \Psi + 2\sqrt{1 + \Psi})}.
$$
Therefore,

\[
I_1 = (n - 1) F^0_E \\
= \left( \Psi - \frac{\Psi^2}{2(2 + \Psi + 2\sqrt{1 + \Psi})} \right) F^0_E \\
= \left( \int_{\mathbb{R}^3} \int_0^\infty \sqrt{F^0_E} \phi(I) dI dp + \frac{\Psi_1}{2} - \frac{\Psi^2}{2(2 + \Psi + 2\sqrt{1 + \Psi})} \right) F^0_E \\
= \left\{ \int_{\mathbb{R}^3} \int_0^\infty \sqrt{F^0_E} \phi(I) dI dp \right\} F^0_E + \Gamma_1(f) \sqrt{F^0_E}.
\]

\[\text{\bullet \ Decomposition \ of \ } I_2: \text{ \ Considering \ (3.5) \ and \ the \ following \ identity \ [3]:}\]

\[
\frac{1}{\sqrt{1 + \Psi}} = 1 - \Psi - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})},
\]

one can see that

\[
\frac{1}{n} = 1 - \Psi - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})}
\]

\[
\text{(3.7)}
\]

\[
= 1 - \int_{\mathbb{R}^3} \int_0^\infty \sqrt{F^0_E} \phi(I) dI dp - \Psi_1 \frac{1}{2} - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})}.
\]

We have from (3.3) that

\[
(U^\mu p_\mu)^2 = (cp^0)^2 + 2cp^0\Phi + \Phi^2.
\]

By (5.1) and (5.3), \(e/n - \bar{c}(T_0)\) is decomposed as

\[
\frac{e}{n} - \bar{c}(T_0) = \frac{1}{nc} \int_{\mathbb{R}^3} \int_0^\infty (U^\mu p_\mu)^2 F \left( 1 + \frac{I}{mc^2} \right) \phi(I) dI dp - \bar{c}(T_0)
\]

\[
= \frac{1}{c} \left\{ 1 - \int_{\mathbb{R}^3} \int_0^\infty \sqrt{F^0_E} \phi(I) dI dp - \frac{\Psi_1}{2} - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})} \right\}
\]

\[
\times \int_{\mathbb{R}^3} \int_0^\infty \left\{ (cp^0)^2 + 2cp^0\Phi + \Phi^2 \right\} F \left( 1 + \frac{I}{mc^2} \right) \phi(I) dI dp - \bar{c}(T_0)
\]

\[
\equiv \left( 1 - \int_{\mathbb{R}^3} \int_0^\infty \sqrt{F^0_E} \phi(I) dI dp \right) \int_{\mathbb{R}^3} \int_0^\infty c(p^0)^2 F \left( 1 + \frac{I}{mc^2} \right) \phi(I) dI dp - \bar{c}(T_0)
\]

\[
+ R_{I_2},
\]

where \(R_{I_2}\) denotes

\[
R_{I_2} = \frac{1}{c} \left( 1 - \int_{\mathbb{R}^3} \int_0^\infty \sqrt{F^0_E} \phi(I) dI dp \right) \int_{\mathbb{R}^3} \int_0^\infty \left\{ 2cp^0\Phi + \Phi^2 \right\} F \left( 1 + \frac{I}{mc^2} \right) \phi(I) dI dp - \bar{c}(T_0)
\]

\[
- \frac{1}{c} \left( \frac{\Psi_1}{2} + \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})} \right) \int_{\mathbb{R}^3} \int_0^\infty (U^\mu p_\mu)^2 F \left( 1 + \frac{I}{mc^2} \right) \phi(I) dI dp.
\]

Here \(R_{I_2}\) is nonlinear with respect to \(f\) that can be shown by the following identity

\[
\Phi = \sqrt{c^2 + |U|^2} p^0 - U \cdot p - cp^0
\]

\[
= \left( \frac{|U|^2}{2c} - \frac{|U|^4}{2c(2c^2 + |U|^2 + 2c\sqrt{1 + |U|^2})} \right) p^0 - U \cdot p.
\]
Using Lemma 3.4 (2), the first two terms on the last identity of (3.9) reduce to

\begin{equation}
(1 - \int_{R^3} \int_0^\infty f \sqrt{F_E^0} \phi(I) \, dI \, dp) \int_{R^3} \int_0^\infty c p^0 \left( F_E^0 + f \sqrt{F_E^0} \right) \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp - \bar{c}(T_0)
\end{equation}

\begin{align*}
&= \left( 1 - \int_{R^3} \int_0^\infty f \sqrt{F_E^0} \phi(I) \, dI \, dp \right) \left( \bar{c}(T_0) + \int_{R^3} \int_0^\infty c p^0 f \sqrt{F_E^0} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \right) - \bar{c}(T_0) \\
&= \int_{R^3} \int_0^\infty \left\{ c p^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right\} f \sqrt{F_E^0} \phi(I) \, dI \, dp \\
&\quad - c \int_{R^3} \int_0^\infty f \sqrt{F_E^0} \phi(I) \, dI \, dp \int_{R^3} \int_0^\infty p^0 f \sqrt{F_E^0} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp.
\end{align*}

Now we go back to (3.9) with (3.10) to get

\begin{equation}
\frac{e}{n} - \bar{c}(T_0) = \int_{R^3} \int_0^\infty \left\{ c p^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right\} f \sqrt{F_E^0} \phi(I) \, dI \, dp \\
- c \int_{R^3} \int_0^\infty f \sqrt{F_E^0} \phi(I) \, dI \, dp \int_{R^3} \int_0^\infty p^0 f \sqrt{F_E^0} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp + R_{I_2},
\end{equation}

which leads to

\begin{align*}
I_2 &= \frac{1}{\bar{c}'(T_0) \frac{1}{k_B T_0}} \left( \frac{e}{n} - \bar{c}(T_0) \right) \left( c p^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right) F_E^0 \\
&= \frac{1}{\bar{c}'(T_0) \frac{1}{k_B T_0}} \int_{R^3} \int_0^\infty \left( c p^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right) f \sqrt{F_E^0} \phi(I) \, dI \, dp \\
&\quad \times \left( c p^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right) F_E^0 + \Gamma_2(f) \sqrt{F_E^0}.
\end{align*}

• Decomposition of $I_3$: By (3.7), $U$ is written as

\begin{equation}
U = \frac{c}{n} \int_{R^3} \int_0^\infty p F \phi(I) \, dI \, \frac{dp}{p^0}
= c \left\{ 1 - \frac{\Psi}{2} - \frac{\Psi^3 - 3 \Psi^2}{2(2 + \Psi - \Psi^2 + 2 \sqrt{1 + 1 + \Psi})} \right\} \int_{R^3} \int_0^\infty p f \sqrt{F_E^0} \phi(I) \, dI \, \frac{dp}{p^0}.
\end{equation}

This directly leads to the linearization of $I_3$ as follows

\begin{equation}
I_3 = \frac{1 + \frac{T}{mc^2}}{k_B T_0} U \cdot p F_E^0 \\
= c \frac{1 + \frac{T}{mc^2}}{k_B T_0} \int_{R^3} \int_0^\infty p f \sqrt{F_E^0} \phi(I) \, dI \, \frac{dp}{p^0} \cdot p F_E^0 \\
- c \frac{1 + \frac{T}{mc^2}}{k_B T_0} \left\{ \frac{\Psi}{2} + \frac{\Psi^3 - 3 \Psi^2}{2(2 + \Psi - \Psi^2 + 2 \sqrt{1 + 1 + \Psi})} \right\} \int_{R^3} \int_0^\infty p f \sqrt{F_E^0} \phi(I) \, dI \, \frac{dp}{p^0} \cdot p F_E^0
\end{equation}

which will be combined with the result of $I_4$.

• Decomposition of $I_4$: Recall from (1.4) that the heat flux $q$ is defined by

\begin{equation}
q = c \int_{R^3} \int_0^\infty p (U^\nu \nu) F \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, \frac{dp}{p^0} \\
- \frac{1}{c} U \int_{R^3} \int_0^\infty (U^\nu \nu)^2 F \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, \frac{dp}{p^0}.
\end{equation}
Inserting (3.14) into the first term of (3.14), one finds

\[ c \int_{\mathbb{R}^3} \int_0^\infty p(U^\nu p_\nu)F \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ = c^2 \int_{\mathbb{R}^3} \int_0^\infty p \left( F_E^0 + f \sqrt{F_E^0} \right) \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ - c^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} \int_0^\infty p_i^\nu \sqrt{F_E^0} \phi(I) \, dI \, dp \int_{\mathbb{R}^3} \int_0^\infty p_i^\nu \left( F_E^0 + f \sqrt{F_E^0} \right) \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ = \int_{\mathbb{R}^3} \int_0^\infty p^\nu \Phi \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

In the last identity, we used the spherical symmetry of $F_E^0$ and Lemma 3.1 (1) so that

\[ \sum_{i=1}^3 \int_{\mathbb{R}^3} \int_0^\infty p_i^\nu \sqrt{F_E^0} \phi(I) \, dI \, dp \int_{\mathbb{R}^3} \int_0^\infty p_i^\nu \left( F_E^0 + f \sqrt{F_E^0} \right) \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ = \frac{1}{3} \int_{\mathbb{R}^3} \int_0^\infty p \sqrt{F_E^0} \phi(I) \, dI \, dp \int_{\mathbb{R}^3} \int_0^\infty \frac{\left| p \right|^2 F_E^0 \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \}

To deal with the second term of (3.14), we use Lemma 3.1 (2) and (3.8) to obtain

\[ \frac{1}{c} U \int_{\mathbb{R}^3} \int_0^\infty \left( U^\nu p_\nu \right)^2 F \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ = \frac{1}{c} U \int_{\mathbb{R}^3} \int_0^\infty \left\{ (cp_\nu)^2 + 2cp_\nu \Phi + \Phi^2 \right\} \left( F_E^0 + f \sqrt{F_E^0} \right) \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ = \tilde{e}(T_0) U + U \int_{\mathbb{R}^3} \int_0^\infty \left\{ cp_\nu \sqrt{F_E^0} + \frac{1}{cp_\nu} \left( 2cp_\nu \Phi + \Phi^2 \right) F \right\} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp, \]

which, together with (3.12) gives

\[ \frac{1}{c} U \int_{\mathbb{R}^3} \int_0^\infty \left( U^\nu p_\nu \right)^2 F \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ = \tilde{e}(T_0) U \int_{\mathbb{R}^3} \int_0^\infty p \sqrt{F_E^0} \phi(I) \, dI \, dp + \int_{\mathbb{R}^3} \int_0^\infty p \sqrt{F_E^0} \phi(I) \, dI \, dp \]

\[ \times \int_{\mathbb{R}^3} \int_0^\infty \left\{ (cp_\nu)^2 + \frac{1}{cp_\nu} \left( 2cp_\nu \Phi + \Phi^2 \right) F \right\} \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \]

\[ - \left\{ \frac{\Psi}{2} + \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})} \right\} \int_{\mathbb{R}^3} \int_0^\infty p \sqrt{F_E^0} \phi(I) \, dI \, dp \]

\[ \times \int_{\mathbb{R}^3} \int_0^\infty \left( U^\nu p_\nu \right)^2 F \left( 1 + \frac{T}{mc^2} \right) \phi(I) \, dI \, dp \].
We go back to (3.14) with (3.15) and (3.16) to get
\[ p \cdot q \left( 1 + \frac{I}{mc^2} \right) F_E^0 \]
\[ = \frac{1 + \frac{I}{mc^2}}{b_0 m} \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \left( 1 + \frac{I}{mc^2} \right) \phi(t) dI dp \cdot p F_E^0 \]
\[ (3.17) - \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} (k_B T_0 + \tilde{c}(T_0)) \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \phi(t) dI dp \frac{dp}{p^0} \cdot p F_E^0 + \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} \Gamma_3^h(f) \cdot p F_E^0 \]
\[ = \frac{1 + \frac{I}{mc^2}}{b_0 m} \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \left( 1 + \frac{I}{mc^2} \right) \phi(t) dI dp \cdot p F_E^0 \]
\[ - \frac{1 + \frac{I}{mc^2}}{b_0 m} \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \phi(t) dI dp \frac{dp}{p^0} \cdot p F_E^0 \]
where we used Lemma 3.1 (5). We combine (3.13) and (3.17) to conclude
\[ I_3 + I_4 = \frac{1 + \frac{I}{mc^2}}{k_B T_0} U \cdot p F_E^0 + p \cdot q \left( \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} F_E^0 - p^0 q^0 \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} F_E^0 \right) \]
\[ = \frac{1 + \frac{I}{mc^2}}{k_B T_0} \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \phi(t) dI dp \frac{dp}{p^0} \cdot p F_E^0 + p \cdot q \left( \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} F_E^0 - p^0 q^0 \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} F_E^0 \right) \]
\[ - \frac{1 + \frac{I}{mc^2}}{k_B T_0} \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \phi(t) dI dp \frac{dp}{p^0} \cdot p F_E^0 \]
\[ = \frac{1 + \frac{I}{mc^2}}{b_0 m} \int_\mathbb{R}^3 \int_0^\infty p f \sqrt{F_E^0} \left( 1 + \frac{I}{mc^2} \right) \phi(t) dI dp \cdot p F_E^0 + \Gamma_3(f) \sqrt{F_E^0} \]
In the last identity, \( p^0 q^0 \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} F_E^0 \) was absorbed into \( \Gamma_3(f) \sqrt{F_E^0} \) since \( q^0 \) is nonlinear with respect to \( f \). Finally, letting
\[ \Gamma_4(f) = \frac{1}{2 \sqrt{F_E^0}} \int_0^1 (1 - \theta) \left( n - 1, U, \frac{E}{n} - \tilde{c}(T_0), q^0 \right) \frac{D^2 \tilde{F}(\theta)}{D^2 \tilde{F}(\theta)} \left( n - 1, U, \frac{E}{n} - \tilde{c}(T_0), q^0 \right) d\theta \]
completes the proof.

In the following proposition, we present the linearization of (1.1).

**Proposition 3.1.** Suppose \( E(f)(t) \) is sufficiently small. Then, (1.1) can be linearized with respect to the perturbation \( f \) as follows

\[ \partial_t f + \hat{p} \cdot \nabla_x f = \frac{1}{r} \left( L(f) + \Gamma(f) \right), \]
\[ f_0(x, p) = f(0, x, p), \]
where the linearized operator \( L(f) \) and the nonlinear perturbation \( \Gamma(f) \) are defined as

\[ L(f) = P(f) - f, \]
\[ \Gamma(f) = \frac{U_p p^\mu}{c p^0} \sum_{i=1}^4 \Gamma_i(f) + \frac{P(f) - f}{c p^0} \Phi \]
respectively.

**Proof.** Inserting \( F = F_E^0 + f \sqrt{F_E^0} \) into (1.1), one finds

\[ \partial_t f + \hat{p} \cdot \nabla_x f = \frac{U_p p^\mu}{c r p^0} \frac{1}{\sqrt{F_E^0}} \left( 1 - p^\mu q^\mu \frac{1 + \frac{I}{mc^2}}{b_0 mc^2} \right) F_E - F_E^0 - f \sqrt{F_E^0} \].
Proof. Since it is straightforward by the definition of \(3.2\).

3.2. Analysis of the linearized operator \(L\). Let \(N\) be a five dimensional space spanned by

\[
\left\{ \sqrt{F_E^0}, \left(1 + \frac{T}{mc^2}\right)p^\mu \sqrt{F_E^0} \right\}.
\]

**Lemma 3.3.** \(P\) is an orthonormal projection from \(L^2_{\mu,i}(\mathbb{R}^3)\) onto \(N\).

**Proof.** It is enough to show that \(\{e_i\} \ (i = 1, \cdots, 5)\) forms an orthonormal basis with respect to the inner product \(\langle \cdot, \cdot \rangle_{L^2_{\mu,i}}\).

1. \(\|e_1\|^2_{L^2_{\mu,i}} = 1\): By a definition of \(F_E^0\), it is straightforward that

\[
\langle e_1, e_1 \rangle_{L^2_{\mu,i}} = \int_{\mathbb{R}^3} \int_0^\infty F_E^0 \phi(I) dI dp = \frac{1}{(mc)^3} \int_{\mathbb{R}^3} \int_0^\infty e^{-(1 + \frac{T}{mc^2}) \frac{\beta^0}{m} \phi(I)} dI dp = 1.
\]

2. \(\|e_2\|^2_{L^2_{\mu,i}} = 1\): Inserting \((n, 0, T) = (1, 0, T_0)\) into (2.8), one finds\n
\[
\{\bar{e}\}'(T_0) = \int \int \left\{ \frac{e^0}{kn} \left(1 + \frac{T}{mc^2}\right) - \bar{e}(T_0) \right\}^2 F_E^0 \phi(I) dI dp.
\]

where we used \(e \big|_{T=T_0} = \bar{e}(T)|_{T=T_0} = \bar{e}(T_0)\).

Using this, we have

\[
\langle e_3, e_3 \rangle_{L^2_{\mu,i}} = \int \int \left\{ \frac{e^0}{kn} \left(1 + \frac{T}{mc^2}\right) - \bar{e}(T_0) \right\}^2 F_E^0 \phi(I) dI dp = 1.
\]

3. \(\|e_5\|^2_{L^2_{\mu,i}} = 0\): Since the orthogonality can be proved in the same manner, we omit it.

**Proposition 3.2.** The linearized operator \(L\) satisfies the following properties:

1. \(\text{Ker}(L) = N\).
2. \(L\) is dissipative in the following sense:

\[
\langle L(f), f \rangle_{L^2_{\mu,i}} = -\|\{I - P\}(f)\|^2_{L^2_{\mu,i}} \leq 0.
\]

**Proof.** Since it is straightforward by the definition of \(L\), we omit the proof.
4. Estimates for macroscopic fields and nonlinear perturbations

To deal with the macroscopic fields, we first estimate $\Psi, \Psi_1, \Phi$ and $\Phi_1$ whose definitions are given in (3.4). For brevity, we set $\tau = 1$ and use the generic constant $C$ which can change line by line but does not affect the proof of Theorem 1.1.

Lemma 4.1. Suppose $E(f)(t)$ is sufficiently small. Then $\Psi, \Psi_1, \Phi$ and $\Phi_1$ satisfy

\[(1) \quad |\partial^\alpha \Psi| + |\partial^\alpha \Phi| \leq C p^0 \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}},\]

\[(2) \quad |\partial^\alpha \Psi_1| + |\partial^\alpha \Phi_1| \leq C p^0 \sqrt{E(f)(t)} \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}}.\]

Proof. Estimates of $\Psi$ and $\Psi_1$: Recalling (3.4), we see that

\[
\begin{align*}
\partial^\alpha \Psi &= 2 \int_{\mathbb{R}^3} \int_0^\infty \partial^\alpha f \sqrt{F^0_E \phi(I)} dI dp + \partial^\beta \left\{ \int_{\mathbb{R}^3} \int_0^\infty f \sqrt{F^0_E \phi(I)} dI dp \right\}^2 \\
&\quad - 3 \sum_{i=1}^3 \partial^\alpha \left\{ \int_{\mathbb{R}^3} \int_0^\infty p^i f \sqrt{F^0_E \phi(I)} dI dp \right\}^2,
\end{align*}
\]

and it follows from Hölder’s inequality that

\[
|\partial^\alpha \Psi| \leq C \left( \|\partial^\beta f\|_{L^2_{p,x}} + \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}} \|\partial^{\alpha - \alpha_1} f\|_{L^2_{p,x}} \right).
\]

Applying the Sobolev embedding $H^2 \subseteq L^\infty$ to lower order terms of (4.1), we get

\[
|\partial^\alpha \Psi| \leq C \left( \|\partial^\beta f\|_{L^2_{p,x}} + \sqrt{E(f)(t)} \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}} \right) \leq C \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}}
\]

for sufficiently small $E(f)(t)$. In the same manner, one can have

\[
|\partial^\alpha \Psi_1| \leq C \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}} \|\partial^{\alpha - \alpha_1} f\|_{L^2_{p,x}}
\]

\[
\leq C \sqrt{E(f)(t)} \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}}.
\]

Estimates of $\Phi$: Observe from (3.4) and (5.3) that

\[
\Phi = \frac{c}{n} p^\mu \int_{\mathbb{R}^3} \int_0^\infty p_\mu F^0_E \phi(I) dI dp - cp^0.
\]

Inserting $F = F^0_E + f \sqrt{F^0_E}$, then (4.4) becomes

\[
\begin{align*}
\Phi &= \frac{c}{n} p^\mu \int_{\mathbb{R}^3} \int_0^\infty p_\mu F^0_E \phi(I) dI dp + \frac{c}{n} p^\mu \int_{\mathbb{R}^3} \int_0^\infty p_\mu f \sqrt{F^0_E \phi(I)} dI dp - cp^0 \\
&= \frac{c}{n} \left( p^0 + p^\mu \int_{\mathbb{R}^3} \int_0^\infty p_\mu \sqrt{F^0_E \phi(I)} dI dp \right) - cp^0,
\end{align*}
\]

where we used

\[
\int_{\mathbb{R}^3} \int_0^\infty F^0_E \phi(I) dI dp = 1, \quad \int_{\mathbb{R}^3} \int_0^\infty p F^0_E \phi(I) dI dp = 0.
\]
Now we use (4.1) to expand 1/n as
\[
\Phi = \left(1 - \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^x(\mathcal{I})} d\mathcal{I} dp \right) \frac{\Psi_1}{2} - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})}
\]
\[
\times \left( c p^0 + c p^\mu \int_{\mathbb{R}^3} \int_{0}^{\infty} p_\mu f \sqrt{F_E^x(\mathcal{I})} d\mathcal{I} \frac{dp}{p^0} \right) - cp^0
\]
\[
= cp^\mu \int_{\mathbb{R}^3} \int_{0}^{\infty} p_\mu f \sqrt{F_E^x(\mathcal{I})} d\mathcal{I} \frac{dp}{p^0} + \left( cp^0 + c p^\mu \int_{\mathbb{R}^3} \int_{0}^{\infty} p_\mu f \sqrt{F_E^0(\mathcal{I})} d\mathcal{I} \frac{dp}{p^0} \right)
\]
\[
\times \left( - \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^0(\mathcal{I})} d\mathcal{I} dp \right) \frac{\Psi_1}{2} - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})}.
\]
\[(4.5)\]

Since it follows from (4.2), (4.3) and the Sobolev embedding \(H_x^2 \subset L_x^\infty\) that
\[
|\Psi| \leq C\|f\|_{L_x^2}, \quad |\Psi_1| \leq C\|f\|_{L_x^2}
\]
for sufficiently small \(E(f)(t)\), the last identity of (4.5) leads to
\[
|\Phi| \leq Cp^0 \|f\|_{L_x^2}.
\]
For \(\alpha \neq 0\), we have from (4.5) that
\[
\partial^\alpha \Phi = cp^\mu \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^x(\mathcal{I})} d\mathcal{I} \frac{dp}{p^0} + \sum_{|\alpha_x| \leq |\alpha|} C_{\alpha_x} \partial^{\alpha_x} \left\{ cp^0 + cp^\mu \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^0(\mathcal{I})} d\mathcal{I} \frac{dp}{p^0} \right\}
\]
\[
\times \partial^{\alpha - \alpha_x} \left\{ - \int_{\mathbb{R}^3} \int_{0}^{\infty} f \sqrt{F_E^0(\mathcal{I})} d\mathcal{I} dp \right\} \frac{\Psi_1}{2} - \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})}.
\]
Using Hölder’s inequality and (1.9), we have
\[
|\partial^\alpha \Phi| \leq Cp^0 \|\partial^\alpha f\|_{L_x^2} + Cp^0 \sum_{|\alpha_x| \leq |\alpha|} \left( 1 + \|\partial^{\alpha_x} f\|_{L_x^2} \right) \left( \|\partial^{\alpha - \alpha_x} f\|_{L_x^2} \right)
\]
\[
+ \sqrt{E(f)(t)} \sum_{|\alpha| \leq |\alpha - \alpha_1|} \|\partial^{\alpha_x} f\|_{L_x^2} + \left( \frac{P(\Psi, \ldots, \partial^{\alpha - \alpha_1} \Psi, 1 + \Psi)}{M (2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi}, \sqrt{1 + \Psi})} \right)
\]
\[(4.6)\]
where \(P\) and \(M\) denote the homogeneous generic polynomial and monomial respectively. Using (1.9) and the Sobolev embedding \(H_x^2 \subset L_x^\infty\), the last term on the r.h.s of (4.7) can be estimated explicitly as
\[
|\frac{P(\Psi, \ldots, \partial^{\alpha - \alpha_1} \Psi, 1 + \Psi)}{M (2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi}, \sqrt{1 + \Psi})} | \leq C \sum_{|\alpha_x| \leq |\alpha - \alpha_1|} \|\partial^{\alpha_x} f\|_{L_x^2}.
\]
Inserting (4.5) into (4.7), one finds
\[
|\partial^\alpha \Phi| \leq Cp^0 \sum_{|\alpha_x| \leq |\alpha|} \|\partial^{\alpha_x} f\|_{L_x^2}.
\]
(4.8)

**Estimates of \(\Phi_1\):** In the same manner as in the case of \(\Phi\), one can have
\[
|\partial^\alpha \Phi_1| \leq Cp^0 \sqrt{E(f)(t)} \sum_{|\alpha| \leq |\alpha|} \|\partial^{\alpha_x} f\|_{L_x^2}.
\]
(4.9) As a result, (1) is obtained by (4.1) and (4.9), and combining (4.3) and (4.10) gives (2).

**Lemma 4.2.** Suppose \(E(f)(t)\) is sufficiently small. Then we have
\[
(1) \quad |\partial^\alpha \{ n - 1 \} + |\partial^\alpha U | + |\partial^\alpha \left( \frac{c}{n} - \bar{e}(T_0) \right) | \leq C \sum_{|\alpha| \leq |\alpha|} \|\partial^{\alpha_x} f\|_{L_x^2}.
\]
(4.10)
\[
(2) \quad |\partial^\alpha \{ U^0 - c \}| \leq C \sqrt{E(f)(t)} \sum_{|\alpha| \leq |\alpha|} \|\partial^{\alpha_x} f\|_{L_x^2}.
\]
Proof. • $\partial^\alpha \{n - 1\}$ : It follows from (3.5), (3.6) and (4.10) that
\begin{equation}
|n - 1| = \left| \frac{\Psi}{2} - \frac{\Psi^2}{2(2 + \Psi + 2\sqrt{1 + \Psi})} \right| \leq C\|f\|_{L^2_{p,x}}
\end{equation}
for sufficiently small $E(f)(t)$. For $\alpha \neq 0$, it takes the form of
$$\partial^\alpha \{n - 1\} = \partial^\alpha \left\{ \sqrt{1 + \Psi} \right\} = \frac{P(\Psi, \cdots, \partial^\alpha \Psi, \sqrt{1 + \Psi})}{M(\sqrt{1 + \Psi})}$$
which can be estimated explicitly by using (4.10) as follows
$$|\partial^\alpha \{n - 1\}| \leq C \sum_{|\alpha_1| \leq |\alpha|} \|\partial^\alpha_1 f\|_{L^2_{p,x}}$$
for sufficiently small $E(f)(t)$.
• $\partial^\alpha U$ : Observe that
$$|U| = \left| \frac{1}{n} \int_{\mathbb{R}^3} \int_0^\infty p F \phi(I) dI dp \right| = \frac{1}{n} \int_{\mathbb{R}^3} \int_0^\infty p f \sqrt{F_{E}^0(\phi(I))} dI dp \leq C\frac{\|f\|_{L^2_{p,x}}}{n}.$$ 
Using (4.11) and the Sobolev embedding $H^2_{x} \subseteq L^\infty_{x}$, this leads to
$$|U| \leq \frac{C}{1 - C\sqrt{E(f)(t)}} \|f\|_{L^2_{p,x}} \leq C\|f\|_{L^2_{p,x}}$$
for sufficiently small $E(f)(t)$. For $\alpha \neq 0$, it follows from (5.7), (1.3) and (4.10) that
$$\left| \partial^\alpha U \right| = \sum_{|\alpha_1| \leq |\alpha|} C_{\alpha_1} \partial^\alpha_1 \left\{ \frac{1}{n} \right\} \int_{\mathbb{R}^3} \int_0^\infty p \partial^{\alpha - \alpha_1} f \sqrt{F_{E}^0(\phi(I))} dI dp \leq C \left( 1 + \|f\|_{L^2_{p,x}} \right) \|\partial^\alpha f\|_{L^2_{p,x}} + C \sum_{0 < |\alpha_1| \leq |\alpha|} \|\partial^{\alpha - \alpha_1} f\|_{L^2_{p,x}}$$
$$\times \left( \|\partial^\alpha_1 f\|_{L^2_{p,x}} + \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_2}_1 f\|_{L^2_{p,x}} + \left| \frac{P(\Psi, \cdots, \partial^\alpha \Psi, \sqrt{1 + \Psi})}{M(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi}, \sqrt{1 + \Psi})} \right| \right)$$
for sufficiently small $E(f)(t)$. Using (4.8) and the Sobolev embedding $H^2(\mathbb{R}^3) \subseteq L^\infty(\mathbb{R}^3)$, we have
$$|\partial^\alpha U| \leq C \sum_{|\alpha_1| \leq |\alpha|} \|\partial^\alpha_1 f\|_{L^2_{p,x}}.$$ 
• $\partial^\alpha \left\{ \frac{e}{n} - \bar{e}(T_0) \right\}$ : Recall from (3.11) that
$$\frac{e}{n} - \bar{e}(T_0) = \int_{\mathbb{R}^3} \int_0^\infty \left\{ c p^0 \left( \frac{T}{mc^2} \right) - \bar{e}(T_0) \right\} f \sqrt{F_{E}^0(\phi(I))} dI dp$$
$$- c \int_{\mathbb{R}^3} \int_0^\infty f \sqrt{F_{E}^0(\phi(I))} dI dp \int_{\mathbb{R}^3} \int_0^\infty p^0 f \sqrt{F_{E}^0 \left( \frac{T}{mc^2} \right)} \phi(I) dI dp$$
$$- \frac{1}{c} \left( \Psi + \frac{\Psi^3 - 3\Psi^2}{2(2 + \Psi - \Psi^2 + 2\sqrt{1 + \Psi})} \right) \int_{\mathbb{R}^3} \int_0^\infty (U_p I_p)^2 F \left( \frac{T}{mc^2} \right) dI dp$$
$$+ \frac{1}{c} \left( 1 - \int_{\mathbb{R}^3} \int_0^\infty f \sqrt{F_{E}^0(\phi(I))} dI dp \right) \int_{\mathbb{R}^3} \int_0^\infty \left\{ 2c p^0 \bar{\Phi} + \bar{\Phi}^2 \right\} F \left( \frac{T}{mc^2} \right) dI dp.$$
This can be handled in the same manner as in the previous cases but it requires tedious computations, so we omit the proof for brevity.
\[ \partial^\alpha \{ U^0 - c \} : \text{By (3.6), } U^0 - c \text{ can be rewritten as} \]
\[ U^0 - c = \sqrt{c^2 + |U|^2} - c = \frac{|U|^2}{2c^2} - \frac{|U|^4}{2(2c^4 + c^2|U|^2 + 2c^3\sqrt{c^2 + |U|^2})} \]

Then it follows from the previous result of \( U \) that
\[ |U^0 - c| = \frac{|U|^2}{2c^2} - \frac{|U|^4}{2(2c^4 + c^2|U|^2 + 2c^3\sqrt{c^2 + |U|^2})} \leq C \sqrt{f(L_2)} \frac{\|f\|_{L_2}}{f(L_2)} \]

for sufficiently small \( E(f)(t) \). Similarly, one can have
\[ |\partial^\alpha \{ U^0 - c \}| = \frac{|U|^2}{2c^2} - \frac{|U|^4}{2(2c^4 + c^2|U|^2 + 2c^3\sqrt{c^2 + |U|^2})} \leq C \sum_{|\alpha_1| \leq |\alpha|} |\partial^{\alpha_1} U||\partial^{\alpha - \alpha_1} U| \leq C \sqrt{E(f)(t)} \frac{\|f\|_{L_2}}{f(L_2)} \]

In the following lemma, we present the Hessian matrix \( D^2 \hat{F}(\theta) \) with respect to \( n_0, U_\theta, (e/n)_\theta, q_\theta^0 \) to clarify the explicit form of the nonlinear perturbation \( \Gamma_4 \) given in Lemma 3.2.

**Lemma 4.3.** We have
\[ D^2_{n_0, U_\theta, (e/n)_\theta, q_\theta^0} \hat{F}(\theta) = QF(\theta), \]
where \( Q \) is a \( 9 \times 9 \) symmetric matrix whose elements are given by
\[ Q_{1,1} = 0, \quad Q_{1,1+i} = -\frac{1 + \frac{e}{mc}}{k_B n_0 T_0} \left( \frac{U_\theta}{U_\theta} p^0 - p \right), \]
\[ Q_{1,5} = \frac{1}{k_B n_0 T_0} \left\{ \frac{k_B T_\theta^2}{mc^2} \frac{1}{\langle \vec{e}(\vec{T}_\theta) \rangle} \left( \frac{k_B T_\theta^2}{mc^2} \frac{1}{\langle \vec{e}(\vec{T}_\theta) \rangle} + \frac{1 + \frac{e}{mc}}{k_B T_\theta} \right) U_\theta p^0 \right\}, \quad Q_{1,6} = 0, \quad Q_{1,6+i} = 0, \]
\[ Q_{1+i,1+j} = \left( 1 - \frac{e}{mc} \right) \frac{1 + \frac{e}{mc}}{k_B T_\theta} \left\{ \left( 1 - \frac{e}{mc} \right) \frac{1 + \frac{e}{mc}}{k_B T_\theta} U_\theta p^0 \right\}, \quad Q_{1+i,6+j} = -\left( 1 + \frac{e}{mc} \right) \frac{1 + \frac{e}{mc}}{k_B T_\theta} \left( \frac{U_\theta}{U_\theta} p^0 - p \right) p^j, \]
\[ Q_{1+i,6} = \left( 1 - \frac{e}{mc} \right) \frac{1 + \frac{e}{mc}}{k_B T_\theta} \left( \frac{U_\theta}{U_\theta} p^0 - p \right) p^0, \quad Q_{1+i,6+j} = -\left( 1 + \frac{e}{mc} \right) \frac{1 + \frac{e}{mc}}{k_B T_\theta} \left( \frac{U_\theta}{U_\theta} p^0 - p \right) p^j, \]
\[ Q_{5,5} = \frac{1 - \frac{e}{mc}}{k_B T_\theta} \left\{ \frac{mc^2 \bar{c}(\vec{T}_\theta)}{\langle \bar{c}(\vec{T}_\theta) \rangle^2} + \frac{1 + \frac{e}{mc}}{k_B T_\theta} \right\} + \frac{p^\mu p_\mu \left( 1 + \frac{e}{mc} \right)}{b_\theta^2 \langle \bar{c}(\vec{T}_\theta) \rangle^2} U_\theta p^0 \left( \frac{2 \bar{c}(\vec{T}_\theta)}{mc^2} + \frac{k_B T_\theta^2}{mc^2} \right) \]
\[ + \frac{k_B T_\theta^2}{mc^2} \frac{1 + \frac{e}{mc}}{k_B T_\theta} \left( \frac{2 \bar{c}(\vec{T}_\theta)}{mc^2} + \frac{k_B T_\theta^2}{mc^2} \right) \]
\[ - \frac{2k_B T_\theta^2}{mc^2} \frac{1 + \frac{e}{mc}}{k_B T_\theta} \left( \frac{2 \bar{c}(\vec{T}_\theta)}{mc^2} + \frac{k_B T_\theta^2}{mc^2} \right) \]
\[ Q_{5,6} = -\frac{1 + \frac{\tau}{m c^2}}{b_0 b_T b_\theta} \rho_0 \left\{ \frac{-\varepsilon(T_0)}{mc^2} + \left( 1 + \frac{I}{mc^2} \right) \frac{U_{0b} \rho_0}{mc^2} - \frac{k_B T_0^2}{mc^2} \frac{\partial T_0}{b_\theta} \right\}, \]
\[ Q_{5,6+i} = -\frac{1 + \frac{\tau}{m c^2}}{b_0 b_T b_\theta} \rho_0 \left\{ \frac{-\varepsilon(T_0)}{mc^2} + \left( 1 + \frac{I}{mc^2} \right) \frac{U_{0b} \rho_0}{mc^2} - \frac{k_B T_0^2}{mc^2} \frac{\partial T_0}{b_\theta} \right\}, \]
\[ Q_{6,6} = 0, \quad Q_{6,6+i} = 0, \quad Q_{6,6+i+j} = 0. \]

for \( i, j = 1, 2, 3. \)

**Proof.** The proof is straightforward. We omit it. \[ \square \]

We are now ready to deal with the nonlinear perturbations.

**Lemma 4.4.** Suppose \( E(f)(t) \) is sufficiently small. Then we have

\begin{align*}
(1) & \quad \left| \int_{0}^{\infty} \partial^\alpha \Gamma(f) g(p, I) \phi(I) dI dp \right| \leq C \sum_{|\alpha| \leq |\alpha|} \| \partial^{\alpha_1} f \|_{L^2_{p,x}} \| \partial^{\alpha_2} f \|_{L^2_{p,x}} \| g \|_{L^2_{p,x}}, \\
(2) & \quad \left\| \partial^\alpha \Gamma(f) \right\|_{L^2_{p,x}} \leq C \sum_{|\alpha| \leq |\alpha|} \| \partial^{\alpha_1} f \|_{L^2_{p,x}} \| \partial^{\alpha_2} f \|_{L^2_{p,x}}.
\end{align*}

**Proof.** Proof of (1): Recall from Proposition 4.11 that

\[ \Gamma(f) = \frac{U_{0b} \rho_0}{cp^0} \sum_{i=1}^{4} \Gamma_i(f) + \frac{P(f) - f}{cp^0} \Phi. \]

To avoid the repetition, we only prove the most complicated term:

\[ \frac{U_{0b} \rho_0}{cp^0} \Gamma_4(f) = \frac{U_{0b} \rho_0}{cp^0} \frac{1}{\sqrt{F_E}} \int_{0}^{1} (1 - \theta) \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) D^2 \bar{F}(\theta) \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) d\theta \]

since the other terms can be handled in the same manner. For this, we use the following notation

\[ (y_1, \cdots, y_9) := \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) \]

to denote

\[ \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) D^2_{n, U, e/n, \varepsilon(T_0)} F(\theta) \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) = \sum_{i,j=1}^{9} y_i y_j Q_{ij} F(\theta) \]

where \( Q_{ij} (i, j = 1, \cdots, 9) \) was given in Lemma 4.3. We observe that

\[ \partial^\alpha \left\{ \frac{1}{\sqrt{F_E}} \int_{0}^{1} (1 - \theta) \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) D^2 \bar{F}(\theta) \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) d\theta \right\} \]

\begin{align*}
&= \sum_{i,j=1}^{9} \sum_{|\alpha_1| + \cdots + |\alpha_4| = |\alpha|} \partial^{\alpha_1} y_i \partial^{\alpha_2} y_j \left( \sum_{|\beta_1| + \cdots + |\beta_3| = \beta} \int_{0}^{1} (1 - \theta) \partial^{\alpha_3} Q_{ij} \partial^{\alpha_4} F(\theta) d\theta \right) \partial_{\beta_3} \left\{ \frac{1}{\sqrt{F_E}} \right\}.
\end{align*}

Here \( \partial^{\alpha_1} y_i \partial^{\alpha_2} y_j \) is bounded from above by Lemma 4.2

\[ |\partial^{\alpha_1} y_i \partial^{\alpha_2} y_j| \leq C \sum_{|\alpha_1| \leq |\alpha|} \| \partial^{\alpha_1} f \|_{L^2_{p,x}} \sum_{|\alpha_2| \leq |\alpha|} \| \partial^{\alpha_2} f \|_{L^2_{p,x}}. \]

Note from Lemma 4.11 and Lemma 4.2 that derivatives of the macroscopic fields \( n, U, e/n \) and \( q^\mu \) are dominated by the \( L^2_{p,x} \) norm of the derivative of \( f \), and with the aid of the Sobolev embedding \( H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) \), one can see that for \( |\alpha| \leq N - 2 \),

\[ \partial^\alpha \left( n - 1, U, \frac{e}{n} - \varepsilon(T_0), q^\mu \right) \approx (0, 0, 0, 0) \]
when $E(f)(t)$ is small enough. From this observation, we see that $\partial_{\beta_1}^{\alpha_3} Q_{ij}$ is well-defined and can be estimated as

\[
|\partial_{\beta_1}^{\alpha_3} Q_{ij}| \leq C(p^0)^3 \left(1 + \frac{T}{mc^2}\right) \left(1 + \|f\|_{L^2_{p,x}}^2\right), \quad \text{if } |\alpha_3| = 0
\]  
(4.14)

\[
|\partial_{\beta_1}^{\alpha_3} Q_{ij}| \leq C(p^0)^3 \left(1 + \frac{T}{mc^2}\right) \sum_{|\alpha_j| \leq |\alpha_3|} \|\partial^{\alpha_j} f\|_{L^2_{p,x}}, \quad \text{otherwise}
\]

for sufficiently small $E(f)(t)$. For the same reason, one can have

\[
|\partial_{\beta_2}^{\alpha_4} F(\theta)| \leq C(p^0)^{|\alpha_4|} \left(1 + \frac{T}{mc^2}\right) \frac{1}{\sqrt{F_E}} F(\theta).
\]  
(4.15)

By a definition of $F_E^0$, one finds

\[
\left| \partial_{\beta_3} \left\{ \frac{1}{\sqrt{F_E}} \right\} \right| \leq C \left(1 + \frac{T}{mc^2}\right) \frac{|\beta_3|}{\sqrt{F_E}}
\]

which, together with (4.15) gives

\[
\left| \partial_{\beta_2}^{\alpha_4} F(\theta) \partial_{\beta_3} \left\{ \frac{1}{\sqrt{F_E}} \right\} \right| \leq C(p^0)^{|\alpha_4|} \left(1 + \frac{T}{mc^2}\right) \frac{|\alpha_4| + |\beta_2| + |\beta_3|}{\sqrt{F_E}} F(\theta)
\]

\[
\leq |\mathbb{P}(p^0, I)| e^{-C'(1 + \frac{T}{mc^2}) p^0}
\]

(4.16)

for sufficiently small $E(f)(t)$. Here $C'$ is the positive constant defined as

\[
e^{-\frac{T}{mc^2}} \sqrt{T} \left(1 + \frac{T}{mc^2}\right) \leq e^{-\frac{T}{mc^2}} \sqrt{T} \left(1 + \frac{T}{mc^2}\right) \leq e^{-\frac{2T}{mc^2}} \sqrt{T} \left(1 + \frac{T}{mc^2}\right) = e^{-C'(1 + \frac{T}{mc^2}) p^0}
\]

where we assumed that $E(f)(t)$ is small enough to satisfy

\[
\min \frac{\sqrt{c^2 + |U_\theta|^2} - |U_\theta|}{T_0} > \frac{c}{2T_0}
\]

to ensure the positivity of $C'$. Combining (4.12), (4.14) and (4.16), and applying the Sobolev embedding $H^2(\mathbb{R}^3_+) \subseteq L^\infty(\mathbb{R}^3_+)$ to the terms having lower derivative order, we get

\[
\partial_{\beta}^{\alpha} \Gamma_4(f) = \partial_{\beta}^{\alpha} \left\{ \frac{1}{\sqrt{F_E}} \int_{0}^{1} \left(1 - \theta\right) \left(n - 1, U, \theta, e - \tilde{e}(T_0), q^\mu\right) D^2 \tilde{F}(\theta) \left(n - 1, U, \theta, e - \tilde{e}(T_0), q^\mu\right)^T d\theta \right\}
\]

\[
\leq |\mathbb{P}(p^0, I)| \sum_{|\alpha'| \leq |\alpha|} \|\partial^{\alpha'} f\|_{L^2_{p,x}} \|\partial^{\alpha - \alpha'} f\|_{L^2_{p,x}} \leq C' \left(1 + \frac{T}{mc^2}\right) p^0
\]

which, together with (1.11) and Lemma 4.2 gives the desired result as follows

\[
\left| \int_{\mathbb{R}^3} \int_{0}^{\infty} \partial_{\beta}^{\alpha} \left\{ \frac{U_p p^\mu}{\sqrt{p^0}} \Gamma_4(f) \right\} g(p, I) \phi(I) dI dp \right|
\]

\[
\leq C \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}} \|\partial^{\alpha - \alpha_1} f\|_{L^2_{p,x}} \int_{\mathbb{R}^3} \int_{0}^{\infty} |\mathbb{P}(p^0, I)| e^{-C'(1 + \frac{T}{mc^2}) p^0} |g(p, I)| \phi(I) dI dp
\]

\[
\leq C \sum_{|\alpha_1| \leq |\alpha|} \|\partial^{\alpha_1} f\|_{L^2_{p,x}} \|\partial^{\alpha - \alpha_1} f\|_{L^2_{p,x}} \|g\|_{L^2_{p,x}}.
\]
Proof of (2): Since the proof is the same as (1), we omit it.

The following lemma is necessary to prove the uniqueness of solutions.

Lemma 4.5. Assume $\tilde{F} := F_E^0 + \tilde{f}\sqrt{F_E^0}$ is another solution of (1.1). For sufficiently small $E(f)(t)$ and $E(f)(t)$, we then have

$$\left| \int_{R^3} \int_{R^3} \int_0^\infty \left\{ \Gamma(f) - \Gamma(\tilde{f}) \right\} (f - \tilde{f}) \phi(I) dI dp dx \right| \leq C\|f - \tilde{f}\|^2_{L^2_{p,x}}.$$

**Proof.** Since it can be proved in the same manner as in Lemma 4.4, we omit it.

Lemma 4.6. Suppose $E(f)(t)$ is sufficiently small. Then we have

$$\int_{R^3} \int_{R^3} \int_0^\infty \partial^\alpha \Gamma(f) \partial^\alpha P(f) \phi(I) dI dp dx = 0.$$

**Proof.** Recall from Proposition 3.2 that

$$\{L(f) + \Gamma(f)\} \sqrt{F_E^0} = \frac{U_p}{c^p} \left( \left( 1 - p^\alpha q^\mu \frac{1 + \frac{T}{mc^2}}{bmc^2} \right) F_E - F \right) = \frac{1}{p^\alpha} Q.$$

We then have from Proposition 3.2 (1) that

$$\int_{R^3} \int_{R^3} \int_0^\infty \partial^\alpha \Gamma(f) \partial^\alpha P(f) \phi(I) dI dp dx$$

$$= \int_{R^3} \int_{R^3} \int_0^\infty \partial^\alpha \left\{ \frac{1}{p^\alpha \sqrt{F_E^0}} Q - L(f) \right\} \partial^\alpha P(f) \phi(I) dI dp dx$$

$$= \int_{R^3} \int_{R^3} \int_0^\infty \frac{1}{p^\alpha \sqrt{F_E^0}} \partial^\alpha Q \partial^\alpha f \phi(I) dI dp dx - \langle L(\partial^\alpha f), P(\partial^\alpha f) \rangle_{L^2_{p,x}}$$

$$= \int_{R^3} \int_{R^3} \int_0^\infty \partial^\alpha Q \frac{P(\partial^\alpha f)}{\sqrt{F_E^0}} \phi(I) dI dp dx.$$

Here $P(f)$ is given in (3.3) as

$$P(f) = a(t,x) \sqrt{F_E^0} + b(t,x) \cdot \left( 1 + \frac{T}{mc^2} \right) p \sqrt{F_E^0} + c(t,x) \left\{ c^p \left( 1 + \frac{T}{mc^2} \right) - c(T_0) \right\} \sqrt{F_E^0}$$

where

$$a(t,x) = \int_{R^3} \int_0^\infty f \sqrt{F_E^0} \phi(I) dI dp, \quad b(t,x) = \frac{1}{b_0m} \int_{R^3} \int_0^\infty pf \sqrt{F_E^0} \left( 1 + \frac{T}{mc^2} \right) \phi(I) dI dp,$$

$$c(t,x) = \frac{1}{k_B T_0^2 \{ e \}^3 (T_0)} \int_{R^3} \int_0^\infty \left\{ c^p \left( 1 + \frac{T}{mc^2} \right) - c(T_0) \right\} f \sqrt{F_E^0} \phi(I) dI dp.$$

Inserting (4.13) into (4.14), one finds

$$\int_{R^3} \int_{R^3} \int_0^\infty \partial^\alpha Q \frac{P(\partial^\alpha f)}{\sqrt{F_E^0}} \phi(I) dI dp dx$$

$$= \int_{R^3} \left\{ \partial^\alpha a(t,x) - c(T_0) \partial^\alpha c(t,x) \right\} \partial^\alpha \left\{ \int_{R^3} \int_0^\infty Q \phi(I) dI dp \right\} dx$$

$$+ \int_{R^3} \partial^\alpha b(t,x) \cdot \partial^\alpha \left\{ \int_{R^3} \int_0^\infty pQ \left( 1 + \frac{T}{mc^2} \right) \phi(I) dI dp \right\} dx$$

$$+ c \int_{R^3} \partial^\alpha c(t,x) \partial^\alpha \left\{ \int_{R^3} \int_0^\infty p^\alpha Q \left( 1 + \frac{T}{mc^2} \right) \phi(I) dI dp \right\} dx$$

which, combined with (1.1) gives the desired result.
5. Proof of theorem 1.1

5.1. Local in time existence. Using Lemma 4.2 and Lemma 4.3, the local in time existence and uniqueness of solutions to (1.1) can be obtained by the standard argument [24, 25].

Proposition 5.1. Let \( N \geq 3 \) and \( F_0 = F_{E0} + \sqrt{F_{E0}} f_0 \) be positive. Then there exist \( M_0 > 0 \) and \( T_* > 0 \) such that if \( T_* \leq \frac{M_0}{F_0} \) and \( E(f_0) \leq \frac{M_0}{F_0} \), there is a unique solution \( F(x, p, t) \) to (1.1) such that the energy functional is continuous in \([0, T_*)\) and uniformly bounded:

\[
\sup_{0 \leq t \leq T_*} E(f(t)) \leq M_0.
\]

5.2. Global in time existence. Recall from Lemma 3.3 that \( P(f) \) is an orthonormal projection defined by

\[
P(f) = \left\{ a(t, x) + b(t, x) \cdot \left( 1 + \frac{T}{mc^2} \right) p + c(t, x) \left( cp^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right) \right\} \sqrt{F_0}
\]

where

\[
a(t, x) = \int_{\mathbb{R}^3} \int_0^\infty f \sqrt{F_{E0}} \phi(I) dI dp,
\]

\[
b(t, x) = \frac{1}{\rho_0 m} \int_{\mathbb{R}^3} \int_0^\infty p f \sqrt{F_{E0}} \left( 1 + \frac{T}{mc^2} \right) \phi(I) dI dp,
\]

\[
c(t, x) = \frac{1}{k_B T_0^2} \left\{ \int_{\mathbb{R}^3} \int_0^\infty \left\{ cp^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right\} f \sqrt{F_{E0}} \phi(I) dI dp \right\}.
\]

Decompose \( f \) as

\[
f = P(f) + \{ I - P \}(f)
\]

and insert (5.2) into (5.1) to see that

\[
\{ \partial_t + \hat{\rho} \cdot \nabla_x \} P(f) = \left\{ -\partial_t - \hat{\rho} \cdot \nabla_x + L \right\} \{ I - P \}(f) + \Gamma(f)
\]

where we used \( L[P(f)] = 0 \) (see Proposition 3.2). Observe that

\[
\{ \partial_t + \hat{\rho} \cdot \nabla_x \} P(f)
\]

\[
= \left\{ \partial_t + \hat{\rho} \cdot \nabla_x \right\} \left\{ a(t, x) + b(t, x) \cdot \left( 1 + \frac{T}{mc^2} \right) p + c(t, x) \left( cp^0 \left( 1 + \frac{T}{mc^2} \right) - \bar{c}(T_0) \right) \right\} \sqrt{F_0}
\]

\[
= \left\{ \partial_t \left( a(t, x) - \bar{c}(T_0)c(t, x) \right) + c \sum_{i=1}^3 \partial_x_i \left( a(t, x) - \bar{c}(T_0)c(t, x) \right) \frac{p_i}{p^0} + \sum_{i=1}^3 \left( \partial_x_i b_1(t, x) + c^2 \partial_x_x c(t, x) \right) \right\} \sqrt{F_0}
\]

\[
\times \left( 1 + \frac{T}{mc^2} \right) p^i + c \sum_{j=1}^3 \sum_{i=1}^3 \partial_x_x b_j(t, x) \left( 1 + \frac{T}{mc^2} \right) \frac{p_ip_j}{p^0} + c \partial_x c(t, x) \left( 1 + \frac{T}{mc^2} \right) \frac{p^0}{p^0} \right\} \sqrt{F_0}
\]

\[
:= \partial_t \tilde{a}(t, x)e_{a_0} + c \sum_{i=1}^3 \partial_x_i \tilde{a}(t, x)e_{a_i} + \sum_{i=1}^3 \left( \partial_x_i b_1(t, x) + c \partial_x_x c(t, x) \right) e_{b_{c_i}} + c \sum_{j=1}^3 \sum_{i=1}^3 \partial_x_x b_j(t, x)e_{i_j}
\]

\[
+ \partial_t \tilde{c}(t, x)e_c
\]

where

\[
\tilde{a}(t, x) := a(t, x) - \bar{c}(T_0)c(t, x), \quad \tilde{c}(t, x) := cc(t, x)
\]

and \( e_{a_0}, \ldots, e_c \) denote

\[
\{ e_{a_0}, e_{a_1}, e_{b_{c_i}}, e_{i_j}, e_c \} = \left\{ 1, \frac{p^i}{p^0}, \left( 1 + \frac{T}{mc^2} \right) p^i, \left( 1 + \frac{T}{mc^2} \right) \frac{p_ip_j}{p^0}, \left( 1 + \frac{T}{mc^2} \right) \frac{p^0}{p^0} \right\} \sqrt{F_0}
\]

for \( 1 \leq i, j \leq 3 \). In terms of the basis (5.3), (5.4) leads to the following relation, that is often called a micro-macro system:

Lemma 5.1. Let \( l_{a_0} \cdots l_c \) and \( h_{a_0}, \ldots, h_c \) denote the inner product of \( \{ I - P \}(f) \) and \( h(f) \) with the corresponding basis (5.4). Then we have
Lemma 5.2. The following relations hold

1. \( \partial_t \hat{u}(t, x) = l_{a_0} + h_{a_0} \).
2. \( \partial_t \hat{c}(t, x) = l_c + h_c \).
3. \( \partial_t b_i(t, x) + c \partial_x \hat{c}(t, x) = l_{b_c} + h_{b_c} \).
4. \( \partial_t x_i \hat{u}(t, x) = l_{a_1} + h_{a_1} \).
5. \( c(1 - \delta_{ij}) \partial_x b_i(t, x) + c \partial_x b_i(t, x) = l_{ij} + h_{ij} \).

Proof. Since the proof is straightforward, we omit it. \( \square \)

Using the same line of the argument as in [24, Theorem 5.4], one finds

(5.5) \[ \sum_{0 < |a| \leq N} \langle L(\partial^a f), \partial^a f \rangle_{L^2_{x,p,I}} \leq -\delta \sum_{0 < |a| \leq N} \|\partial^a f\|_{L^2_{x,p,I}}^2 - C \frac{d}{dt} \int_{\mathbb{R}^3} (\nabla_x \cdot b(t, x)) c(t, x) \, dx. \]

To extend the local in time solution to the global one, we take inner product of (3.18) with \( f \) and use Proposition 3.2 (2), Lemma 4.4 and Lemma 4.6 to obtain

\[ \frac{d}{dt} \|\partial^\alpha f\|_{L^2_{x,p,I}}^2 + \delta_1 \|f\|_{L^2_{x,p,I}}^2 \leq C \sqrt{E(f)(t)} D(t) \]

Applying \( \partial^\beta_\lambda \) to (3.18), taking \( L^2_{x,p,I} \) inner product with \( \partial^\beta_\lambda f \), and employing Lemma 4.5 and (5.5), we have

\[ \frac{d}{dt} \|\partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 + \delta_2 \|\partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 \leq C \sqrt{E(f)(t)} D(t) \]

for \( \alpha \neq 0 \), \( \beta = 0 \), and

\[ \frac{d}{dt} \|\partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 + \delta_3 \|\partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 \leq C \sum_{|\beta| < |\alpha|} \left( \|\partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 + \|\nabla_x \partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 \right) + C \sqrt{E(f)(t)} D(t) \]

for \( \alpha, \beta \neq 0 \). Combining above estimates, we obtain the following energy estimate [24]:

\[ \frac{d}{dt} \left\{ C_1 \|f\|_{L^2_{x,p,I}}^2 + \sum_{0 < |a| + |\beta| \leq N} C_{|\beta|} \|\partial^\beta_\lambda f\|_{L^2_{x,p,I}}^2 - C_2 \int_{\mathbb{R}^3} (\nabla_x \cdot b(t, x)) c(t, x) \, dx \right\} + \delta_N D(t) \]

\[ \leq C \sqrt{E(f)(t)} D(t) \]

for some positive constants \( C_1, C_{|\beta|}, C_2 \) and \( \delta_N \). Then, the standard continuity argument [24] gives the global in time existence of solutions satisfying

\[ E_N(f)(t) + \int_0^t \mathcal{D}_N(f)(s) \, ds \leq CE_N(f_0). \]

5.3. Proof of the asymptotic behaviors (Theorem 1.1 (1)–(3)). We start with the derivation of the local conservation laws for the linearized relativistic BGK model (3.18).

Lemma 5.2. The following relations hold

\[ \partial_t a(t, x) + k_B T_0 \nabla_x \cdot b(t, x) = \left\langle -\hat{p} \cdot \nabla_x \{ I - P \} (f) + \frac{1}{\tau} \Gamma(f), \sqrt{F_E^0} \right\rangle_{L^2_{p,I}} \],

\[ \partial_t b(t, x) + \frac{k_B T_0}{b_0 m} \nabla_x \left\{ a(t, x) + k_B T_0 c(t, x) \right\} \]

\[ = \frac{1}{b_0} \left\langle -\hat{p} \cdot \nabla_x \{ I - P \} (f) + \frac{1}{\tau} \Gamma(f), \left( 1 + \frac{T}{mc^2} \right) p \sqrt{F_E^0} \right\rangle_{L^2_{p,I}} \],

\[ \partial_t c(t, x) + \frac{k_B}{\{\varepsilon\}'(T_0)} \nabla_x \cdot b(t, x) \]

\[ = \frac{1}{k_B T_0 \{\varepsilon\}'(T_0)} \left( -\hat{p} \cdot \nabla_x \{ I - P \} (f) + \frac{1}{\tau} \Gamma(f), \left\{ cp^0 \left( 1 + \frac{T}{mc^2} \right) - \varepsilon(T_0) \right\} \sqrt{F_E^0} \right\rangle_{L^2_{p,I}} \].
Proof. We rewrite (5.18) as

\[
\partial_t f + \hat{p} \cdot \nabla_x P(f) = -\hat{p} \cdot \nabla_x \{ I - P \}(f) + \frac{1}{\tau} (L(f) + \Gamma(f))
\]

Multiplying (5.19) by

\[
\sqrt{F^0_E} \phi(I), \quad \frac{1}{b_0 m} \left( 1 + \frac{T}{m c^2} \right) p \sqrt{F^0_E} \phi(I), \quad \frac{1}{k_B T_0^2} \frac{1}{(\hat{c})^t(T_0)} \left\{ cp^0 \left( 1 + \frac{T}{m c^2} \right) - \bar{e}(T_0) \right\} \sqrt{F^0_E} \phi(I),
\]

and integrating over \( p, I \in \mathbb{R}^3 \times \mathbb{R}^+ \), one finds

\[
\partial_t a(t, x) + \left\langle \hat{p} \cdot \nabla_x P(f), \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = \left\langle -\hat{p} \cdot \nabla_x \{ I - P \}(f) + \frac{1}{\tau} \Gamma(f), \sqrt{F^0_E} \right\rangle_{L_{p,I}^2},
\]

\[
\partial_t b(t, x) + \frac{1}{b_0 m} \left\langle \hat{p} \cdot \nabla_x P(f), \left( 1 + \frac{T}{m c^2} \right) p \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = \frac{1}{b_0 m} \left\langle -\hat{p} \cdot \nabla_x \{ I - P \}(f) + \frac{1}{\tau} \Gamma(f), \left( 1 + \frac{T}{m c^2} \right) p \sqrt{F^0_E} \right\rangle_{L_{p,I}^2},
\]

\[
\partial_t c(t, x) + \frac{1}{k_B T_0^2} \frac{1}{(\hat{c})^t(T_0)} \left\langle \hat{p} \cdot \nabla_x P(f), \left\{ cp^0 \left( 1 + \frac{T}{m c^2} \right) - \bar{e}(T_0) \right\} \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = \frac{1}{k_B T_0^2} \frac{1}{(\hat{c})^t(T_0)} \left\langle -\hat{p} \cdot \nabla_x \{ I - P \}(f) + \frac{1}{\tau} \Gamma(f), \left\{ cp^0 \left( 1 + \frac{T}{m c^2} \right) - \bar{e}(T_0) \right\} \sqrt{F^0_E} \right\rangle_{L_{p,I}^2}.
\]

Claim that

1. \( \left\langle \hat{p} \cdot \nabla_x P(f), \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = k_B T_0 \nabla_x \cdot b(t, x) \),

2. \( \left\langle \hat{p} \cdot \nabla_x P(f), \left( 1 + \frac{T}{m c^2} \right) p \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = k_B T_0 \nabla_x \{ a(t, x) + k_B T_0 c(t, x) \} \),

3. \( \left\langle \hat{p} \cdot \nabla_x P(f), \left\{ cp^0 \left( 1 + \frac{T}{m c^2} \right) - \bar{e}(T_0) \right\} \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = (k_B T_0)^2 \nabla_x \cdot b(t, x) \),

which completes the proof.

- Proof of (1): Observe from (5.1) that

\[
\left\langle \hat{p} \cdot \nabla_x P(f), \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = \sum_{i=1}^{3} \int_{\mathbb{R}^3} \int_{0}^{\infty} \frac{cp^i}{p^0} \partial_{x_i} P(f) \sqrt{F^0_E} \phi(I) \, dI \, dp
\]

\[
= c \sum_{i=1}^{3} \int_{\mathbb{R}^3} \int_{0}^{\infty} \frac{p^i}{p^0} \partial_{x_i} \left\{ \phi(t, x) + b(t, x) \cdot \left( 1 + \frac{T}{m c^2} \right) p + c(t, x) \left( cp^0 \left( 1 + \frac{T}{m c^2} \right) - \bar{e}(T_0) \right) \right\}
\]

\[
	imes F^0_E \phi(I) \, dI \, dp.
\]

By the spherical symmetry of \( F^0_E \), (5.7) becomes

\[
\left\langle \hat{p} \cdot \nabla_x P(f), \sqrt{F^0_E} \right\rangle_{L_{p,I}^2} = c \sum_{i=1}^{3} \partial_{x_i} b_i(t, x) \int_{\mathbb{R}^3} \int_{0}^{\infty} \frac{(p^i)^2}{p^0} F^0_E \left( 1 + \frac{T}{m c^2} \right) \phi(I) \, dI \, dp
\]

which, combined with Lemma 3.3 (1) gives the proof of (1).

- Proof of (2): In the same way as the proof of (2), one finds

\[
\left\langle \hat{p} \cdot \nabla_x P(f), \left( 1 + \frac{T}{m c^2} \right) p^i \sqrt{F^0_E} \right\rangle_{L_{p,I}^2}
\]
Here 6.1 of [23]:

for \( j = 1, 2, 3 \). Using the Lemma 3.1 (2) and (5), we then have

\[
\left\langle \hat{p} \cdot \nabla_x P, \left( 1 + \frac{T}{mc^2} \right) p^i \sqrt{F_{E}^0} \right\rangle_{L^2_{p,x}} \leq C \left( \left\| \nabla^k \{ Iure P \} f \right\|_{L^2_{p,x}} + \sum_{|\alpha| \leq k} \left\| \nabla^{|\alpha|} f \right\|_{L^2_{p,x}} \right)
\]

where \( G_k(t) \) denotes

\[
\sum_{|\alpha| = k} \int_{\mathbb{R}^3} \langle \{ Iure P \} \partial^\alpha f, \epsilon_a(p, \mathcal{I}) \rangle_{L^2_{p,x}} \cdot \nabla_x \partial^\alpha a(t, x) + \langle \{ Iure P \} \partial^\alpha f, \epsilon_c(p, \mathcal{I}) \rangle_{L^2_{p,x}} \cdot \nabla_x \partial^\alpha c(t, x) dx
\]

and \( \epsilon_a, \epsilon_b, \) and \( \epsilon_c \) are linear combinations of the basis (5.4).

**Proof.** For \( |\alpha| \leq N - 1 \), we have from Lemma 5.1 (5) that

\[
c \Delta \partial^\alpha b_i(t, x) = \partial^\alpha \sum_{j \neq i} [- \partial_j (l_{jj} + h_{jj}) + \partial_j (l_{ij} + h_{ij})] + \partial_i \partial^\alpha (l_{ii} + h_{ii}).
\]

Here \( l_{a1}, \ldots, l_{c} \) take the following form:

\[
\langle - \partial_i \{ Iure P \} \partial^\alpha f, \epsilon(p, \mathcal{I}) \rangle_{L^2_{p,x}} - \langle \{ \hat{p} \cdot \nabla_x - L \} \{ Iure P \} \partial^\alpha f, \epsilon(p, \mathcal{I}) \rangle_{L^2_{p,x}}
\]

Now we prove the key estimate of this subsection which is a relativistic generalization of Lemma 6.1 of [23]:

**Lemma 5.3.** Let \( N \geq 3 \). Then for \( k = 0, \ldots, N - 1 \), we have

\[
d \frac{d}{dt} G_k + \left\| \nabla^{k+1} P f \right\|_{L^2_{p,x}}^2 \leq C \left( \left\| \nabla^k \{ Iure P \} f \right\|_{L^2_{p,x}}^2 + \sum_{|\alpha| \leq k} \left\| \nabla^{|\alpha|} f \right\|_{L^2_{p,x}} \right)
\]

where \( G_k(t) \) denotes

\[
\sum_{|\alpha| = k} \int_{\mathbb{R}^3} \langle \{ Iure P \} \partial^\alpha f, \epsilon_a(p, \mathcal{I}) \rangle_{L^2_{p,x}} \cdot \nabla_x \partial^\alpha a(t, x) + \langle \{ Iure P \} \partial^\alpha f, \epsilon_c(p, \mathcal{I}) \rangle_{L^2_{p,x}} \cdot \nabla_x \partial^\alpha c(t, x) dx
\]

and \( \epsilon_a, \epsilon_b, \) and \( \epsilon_c \) are linear combinations of the basis (5.4).
where \( \epsilon(p, I) \) is a suitable linear combination of the basis \([5.4]\). We then have
\[
\| \nabla \partial^\alpha b \|^2_{L^2_x} \leq \int_{\mathbb{R}^3} \langle \partial_t \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \cdot \partial^\alpha b(t,x) \, dx \\
(5.8)
+ C \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \left\| \nabla_x \partial^\alpha b \right\|_{L^2_x} \\
+ \left( \| \{ I - P \} \nabla_x \partial^\alpha f \|_{L^2_{p,x}}^2 + \| \{ I - P \} \partial^\alpha f \|_{L^2_{p,x}}^2 \right) \| \nabla_x \partial^\alpha b \|_{L^2_x}^2
\]
where we used Lemma \([5.4]\) to obtain
\[
\| \partial^\alpha h_{i,j} \|_{L^2_x} = \| \langle \partial^\alpha \Gamma(f), \epsilon_{ij} \rangle \|_{L^2_{p,x}} \|_{L^2_x} \leq C \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \left\| \nabla_x \partial^\alpha b \right\|_{L^2_x}.
\]
Here the first term on the r.h.s of \((5.8)\) can be written as
\[
\int_{\mathbb{R}^3} \langle \partial_t \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \cdot \partial^\alpha b(t,x) \, dx \\
= \frac{d}{dt} \int_{\mathbb{R}^3} \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \cdot \partial^\alpha b(t,x) \, dx \\
- \int_{\mathbb{R}^3} \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \cdot \partial_t \partial^\alpha b(t,x) \, dx \\
= -\frac{d}{dt} \int_{\mathbb{R}^3} \langle \{ I - P \} \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \nabla_x \cdot \partial^\alpha b(t,x) \, dx \\
- \int_{\mathbb{R}^3} \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \cdot \partial_t \partial^\alpha b(t,x) \, dx,
\]
which, combined with Lemma \([5.2,2)\) leads to
\[
\int_{\mathbb{R}^3} \langle \partial_t \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \cdot \partial^\alpha b(t,x) \, dx \\
(5.9)
\leq -\frac{d}{dt} \int_{\mathbb{R}^3} \langle \{ I - P \} \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \nabla_x \cdot \partial^\alpha b(t,x) \, dx + C \| \{ I - P \} \nabla_x \partial^\alpha f \|_{L^2_{p,x}}^2 \\
+ \epsilon \| \nabla_x \partial^\alpha a + \nabla_x \partial^\alpha c \|_{L^2_x}^2 + \epsilon \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \left\| \nabla_x \partial^\alpha b \right\|_{L^2_x}^2.
\]
Go back to \((5.8)\) with \((5.9)\) to get
\[
\| \nabla \partial^\alpha b \|^2_{L^2_x} \leq -\frac{d}{dt} \int_{\mathbb{R}^3} \langle \{ I - P \} \partial^\alpha f, \epsilon_b(p, I) \rangle_{L^2_{p,x}} \nabla_x \cdot \partial^\alpha b(t,x) \, dx \\
(5.10)
+ C \left( \| \{ I - P \} \nabla_x \partial^\alpha f \|_{L^2_{p,x}}^2 + \| \{ I - P \} \partial^\alpha f \|_{L^2_{p,x}}^2 \right) \\
+ \epsilon_1 \left( \| \nabla_x \partial^\alpha a \|_{L^2_x}^2 + \| \nabla_x \partial^\alpha c \|_{L^2_x}^2 + \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \| \nabla_x \partial^\alpha b \|_{L^2_x}^2 \right).
\]
In a similar way, it follows from Lemma \([5.1,4)\) and Lemma \([5.2,1)\) that
\[
\| \nabla_x \partial^\alpha a \|_{L^2_x}^2 \leq -\frac{d}{dt} \int_{\mathbb{R}^3} \langle \{ I - P \} \partial^\alpha f, \epsilon_a(p, I) \rangle_{L^2_{p,x}} \nabla_x \partial^\alpha a(t,x) \, dx \\
(5.11)
+ C \left( \| \{ I - P \} \nabla_x \partial^\alpha f \|_{L^2_{p,x}}^2 + \| \{ I - P \} \partial^\alpha f \|_{L^2_{p,x}}^2 \right) \\
+ \epsilon_2 \left( \| \nabla_x \partial^\alpha b \|_{L^2_x}^2 + \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \| \nabla_x \partial^\alpha b \|_{L^2_x}^2 \right).
Also, we have from Lemma 5.1.1 that

\[ \| \nabla^\alpha \tilde{c} \|^2_{L^2_t} \leq - \int_{\mathbb{R}^3} \partial_t \partial^\alpha b(t, x) \cdot \nabla_x \partial^\alpha \tilde{c}(t, x) + \langle \partial \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_c(p, I) \rangle_{L^2_{p,x}} \partial^\alpha \tilde{c}(t, x) \, dx \]

(5.12)

\[ + C \left( \| \{ I - P \} \nabla_x \partial^\alpha f \|^2_{L^2_{p,x}} + \| \{ I - P \} \partial^\alpha f \|^2_{L^2_{p,x}} \right) \]

\[ + C \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}}. \]

Using integration by parts and Lemma 5.2.3, the first term on the r.h.s of (5.12) can be estimated as

\[- \int_{\mathbb{R}^3} \partial_t \partial^\alpha b(t, x) \cdot \nabla_x \partial^\alpha \tilde{c}(t, x) + \langle \partial \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_c(p, I) \rangle_{L^2_{p,x}} \partial^\alpha \tilde{c}(t, x) \, dx \]

\[ = - \frac{d}{dt} \left\{ \int_{\mathbb{R}^3} \partial^\alpha b(t, x) \cdot \nabla_x \partial^\alpha \tilde{c}(t, x) - \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_c(p, I) \rangle_{L^2_{p,x}} \partial^\alpha \tilde{c}(t, x) \, dx \right\} \]

\[ - \int_{\mathbb{R}^3} \nabla_x \cdot \partial^\alpha b(t, x) \partial_t \partial^\alpha \tilde{c}(t, x) - \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_c(p, I) \rangle_{L^2_{p,x}} \partial_t \partial^\alpha \tilde{c}(t, x) \, dx \]

\[ \leq - \frac{d}{dt} \left\{ \int_{\mathbb{R}^3} \partial^\alpha b(t, x) \cdot \nabla_x \partial^\alpha \tilde{c}(t, x) + \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_c(p, I) \rangle_{L^2_{p,x}} \nabla_x \partial^\alpha \tilde{c}(t, x) \, dx \right\} \]

\[ + C' \| \nabla_x \partial^\alpha b \|^2_{L^2_t} + C \left( \| \{ I - P \} \nabla_x \partial^\alpha f \|^2_{L^2_{p,x}} + \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \right)^2 \].

which, together with (5.12) gives

(5.13)

\[ \| \nabla^\alpha \tilde{c} \|^2_{L^2_t} \]

\[ \leq - \frac{d}{dt} \left\{ \int_{\mathbb{R}^3} \partial^\alpha b(t, x) \cdot \nabla_x \partial^\alpha \tilde{c}(t, x) + \langle \{ I - P \} \nabla_x \partial^\alpha f, \epsilon_c(p, I) \rangle_{L^2_{p,x}} \nabla_x \partial^\alpha \tilde{c}(t, x) \, dx \right\} + C' \| \nabla_x \partial^\alpha b \|^2_{L^2_t} \]

\[ + C \left( \| \{ I - P \} \nabla_x \partial^\alpha f \|^2_{L^2_{p,x}} + \| \{ I - P \} \partial^\alpha f \|^2_{L^2_{p,x}} + \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \right)^2 \].

For sufficiently small \( \varepsilon_1 \) and \( \varepsilon_2 \) satisfying \( C' \varepsilon_1 \ll 1 \), combining (5.10), (5.11) and (5.13) gives the desired result for spatial derivative. We now employ the case of temporal derivative \( d/dt \). Recall from Lemma 5.2.1 that

\[ \partial_t \partial^\alpha a(t, x) = -k_B T_0 \nabla_x \cdot \partial^\alpha b(t, x) + \left\{ -\hat{p} \cdot \nabla \{ I - P \} (\partial^\alpha f) + \partial^\alpha h, \sqrt{F^0} \right\}_{L^2_{p,x}} \]

\[ \leq -k_B T_0 \nabla_x \cdot \partial^\alpha b(t, x) + C \| \nabla_x \{ I - P \} \partial^\alpha f \|^2_{L^2_{p,x}} \]

\[ + C \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}}. \]

Taking inner product with \( \partial_t \partial^\alpha a(t, x) \), the above estimate leads to

\[ \| \partial_t \partial^\alpha a \|^2_{L^2_t} \leq C \left( \| \nabla_x \partial^\alpha b \|^2_{L^2_t} + \| \nabla_x \{ I - P \} \partial^\alpha f \|^2_{L^2_{p,x}} + C \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \right)^2 \]

\[ + \varepsilon \| \partial_t \partial^\alpha a \|^2_{L^2_t} \]

yielding

(5.14)

\[ \| \partial_t \partial^\alpha a \|^2_{L^2_t} \leq C \left( \| \nabla_x \partial^\alpha b \|^2_{L^2_t} + \| \nabla_x \{ I - P \} \partial^\alpha f \|^2_{L^2_{p,x}} + C \sum_{|\alpha_1| \leq |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_{p,x}} \left\| \partial^{\alpha - \alpha_1} f \right\|_{L^2_{p,x}} \right)^2 \]
for sufficiently \( c \). In the same manner, one can have from Lemma (2.2), (3) that

\[
\| \partial_t \partial^a b \|_{L^2_x}^2 \leq C \left( \| \nabla_x \partial^a a \|_{L^2_x}^2 + \| \nabla_x \partial^a c \|_{L^2_x}^2 + \| \nabla_x \{ I - P \} \partial^a f \|_{L^2_{x,p,z}}^2 \right),
\]

\[\text{(5.15)}\]

and

\[
\| \partial_t \partial^a c \|_{L^2_x}^2 \leq C \left( \| \nabla_x \partial^a b \|_{L^2_x}^2 + \| \nabla_x \{ I - P \} \partial^a f \|_{L^2_{x,p,z}}^2 \right)
\]

\[\text{(5.16)}\]

Combining (5.14)–(5.16) and results for the spatial derivative completes the proof.

Now we are ready to prove the rest of Theorem (1). Since the argument is similar as in Section 4, we only present a sketch of the proof of Theorem (1.1) (2)–(3) for brevity.

- **Proof of Theorem (1.1) (2):** Let \( 0 \leq \ell \leq N - 1 \). Applying \( \nabla^k \) to (6.34), taking the \( L^2_{x,p,z} \) inner product with \( \nabla^k f \), and using Proposition (5.2) Lemmas (4.4) and (4.9) we have

\[
\frac{1}{2} \frac{d}{dt} \sum_{\ell \leq k \leq N-1} \| \nabla^k f \|_{L^2_{x,p,z}}^2 + \sum_{\ell \leq k \leq N-1} \| \nabla^k \{ I - P \} f \|_{L^2_{x,p,z}}^2
\]

\[\text{(5.17)}\]

Using the Sobolev type inequalities and Minkowski’s inequality, the right-hand side of (5.17) can be bounded from above as

\[
\sum_{|\alpha| \leq k} \left\| \nabla^{|\alpha|} f \right\|^2_{L^2_{p,z}} \leq C \delta^2 \left( \| \nabla^{k+1} f \|_{L^2_{x,p,z}}^2 + \| \nabla^k \{ I - P \} f \|_{L^2_{x,p,z}}^2 \right)
\]

for \( k = 0, \cdots, N - 1 \), and

\[
\sum_{|\alpha| \leq k} \left\| \nabla^{|\alpha|} f \right\|^2_{L^2_{p,z}} \leq C \delta^2 \| \nabla^{N} f \|_{L^2_{p,z}}^2
\]

(5.19)

for \( k = N \) respectively. Combining (5.17)–(5.19), one can see that

\[
\frac{d}{dt} \sum_{\ell \leq k \leq N} \| \nabla^k f \|_{L^2_{x,p,z}}^2 + C \sum_{\ell \leq k \leq N} \| \nabla^k \{ I - P \} f \|_{L^2_{x,p,z}}^2
\]

\[\text{(5.20)}\]

On the other hand, it follows from Lemma (5.3) Sobolev interpolation and Minkowski’s inequality that

\[
\frac{d}{dt} \eta \sum_{\ell \leq k \leq N-1} G_k + \eta \sum_{\ell+1 \leq k \leq N} \| \nabla^k P f \|_{L^2_{x,p,z}}^2
\]

\[\leq C \eta \sum_{\ell \leq k \leq N-1} \| \nabla^k \{ I - P \} f \|_{L^2_{x,p,z}}^2 + C \eta \sum_{\ell \leq k \leq N-1} \sum_{|\alpha| \leq k} \left\| \nabla^{|\alpha|} f \right\|_{L^2_{p,z}} \| \nabla^{k-|\alpha|} f \|_{L^2_{p,z}}^2
\]

\[\leq C \eta \sum_{\ell \leq k \leq N} \| \nabla^k \{ I - P \} f \|_{L^2_{x,p,z}}^2.
\]
This, together with (5.20) yields that for sufficiently small \( \eta \) and \( \delta \),
\[
\frac{d}{dt} E_\ell(t) + \| \nabla^\ell (I - P) f \|^2_{L^2_{t,x,p}} + \sum_{\ell + 1 \leq k \leq N} \| \nabla^k f \|^2_{L^2_{t,x,p}} \leq 0,
\]
where \( E_\ell(t) \) denotes
\[
E_\ell(t) = \sum_{\ell \leq k \leq N} \| \nabla^k f \|^2_{L^2_{t,x,p}} + \eta \sum_{\ell \leq k \leq N-1} G_k.
\]
Using the following Sobolev interpolation (for details, please see [27, Lemma A.4]):
\[
\| \nabla^\ell f \|_{L^2} \leq C \| \nabla^{\ell+1} f \|_{L^2} \| \Lambda^{-\theta} f \|_{L^2},
\]
where \( \theta = \frac{1}{\ell + 1 + s} \) \( s, \ell \geq 0 \), we have from (5.21) and Theorem 1.1 (1) that
\[
\frac{d}{dt} E_\ell(t) + C_0 \left( \sum_{\ell + 1 \leq k \leq N} \| \nabla^{k-1} f \|^2_{L^2_{t,x,p}} \right)^{1+\frac{1}{1+s}} \leq 0.
\]
Since \( E_\ell(t) \) is equivalent to \( \sum_{\ell \leq k \leq N} \| \nabla^k f \|^2_{L^2_{t,x,p}} \) for sufficiently small \( \eta \), this gives the desired result.

- **Proof of Theorem 1.1 (3):** Applying \( \{ I - P \} \) to (3.18), and using Proposition (3.2) and Lemma 4.6, one finds
\[
\partial_t \{ I - P \} f + \hat{\rho} \cdot \nabla_x \{ I - P \} f + L \{ I - P \} f = \Gamma(f) - \hat{\rho} \cdot \nabla_x Pf + P(\hat{\rho} \cdot \nabla_x f).
\]
Applying \( \nabla^k \) \( (k = 0, \ldots, N - 2) \) and taking the \( L^2_{t,x,p} \) inner product with \( \nabla^k \{ I - P \} f \), this leads to
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} + \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} \leq \langle \nabla^k \Gamma(f), \nabla^k \{ I - P \} f \rangle_{L^2_{t,x,p}} - \langle \hat{\rho} \cdot \nabla_x P \nabla^k f - P(\hat{\rho} \cdot \nabla_x \nabla^k f), \nabla^k \{ I - P \} f \rangle_{L^2_{t,x,p}},
\]
which, combined with Lemma 4.4 gives that for small \( \varepsilon \),
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} + \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} \leq C_\varepsilon \left( \sum_{|\alpha_1| \leq k} \| \nabla^{|\alpha_1|} f \|_{L^2_{t,x,p}} \| \nabla^{k-|\alpha_1|} f \|_{L^2_{t,x,p}} \|^2_{L^2_{t,x,p}} + \| \nabla^{k+1} f \|^2_{L^2_{t,x,p}} \right) + \varepsilon \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}}.
\]
On the other hand, note from [27, Lemma 4.4] that using the Sobolev type inequalities and Minkowski’s inequality, one can have
\[
\sum_{|\alpha_1| \leq k} \| \nabla^{|\alpha_1|} f \|_{L^2_{t,x,p}} \| \nabla^{k-|\alpha_1|} f \|_{L^2_{t,x,p}} \|^2_{L^2_{t,x,p}} \leq C \delta^2 \left( \| \nabla^{k+1} f \|^2_{L^2_{t,x,p}} + \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} \right).
\]
This, together with (5.22) gives that for sufficiently small \( \varepsilon \) and \( \delta \),
\[
\frac{d}{dt} \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} + \| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} \leq C \| \nabla^{k+1} f \|^2_{L^2_{t,x,p}}.
\]
For \( k = 1, \ldots, N - 2 \), applying the Gronwall inequality to (5.23) and using Theorem 1.1 (2) with \( \ell = k + 1 \), one finds
\[
\| \nabla^k \{ I - P \} f \|^2_{L^2_{t,x,p}} \leq e^{-t} \| \nabla^k \{ I - P \} f_0 \|^2_{L^2_{t,x,p}} + C \int_0^t e^{-(t-s)} \| \nabla^{k+1} f(s) \|^2_{L^2_{t,x,p}} ds \leq C_0 (1 + t)^{-(k+1+s)}
\]
which, together with the interpolation gives the desired result for \(-s < k \leq N - 2\).

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