Some Open Problems Concerning Orthogonal Polynomials on Fractals and Related Questions

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1 Background and notation

1.1 Chebyshev and orthogonal polynomials

Let $K \subset \mathbb{C}$ be a compact set containing infinitely many points. We use $\| \cdot \|_{L^\infty(K)}$ to denote the sup-norm on $K$, $\mathcal{M}_n$ is the set of all monic polynomials of degree $n$. The polynomial $T_{n,K}$ that minimizes $\|Q_n\|_{L^\infty(K)}$ for $Q_n \in \mathcal{M}_n$ is called the $n$-th Chebyshev polynomial on $K$.

Let the logarithmic capacity $\text{Cap}(K)$ be positive. Then we define the $n$-th Widom factor for $K$ by

$$W_n(K) := \frac{\|T_{n,K}\|_{L^\infty(K)}}{\text{Cap}(K)^n}.$$

In what follows we consider unit Borel measures $\mu$ with non-polar compact support $\text{supp}(\mu)$ in $\mathbb{C}$. The $n$-th monic orthogonal polynomial $P_n(z;\mu) = z^n + \ldots$ associated with $\mu$ has the property

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)}^2 = \inf_{Q_n \in \mathcal{M}_n} \int |Q_n(z)|^2 d\mu(z),$$

where $\| \cdot \|_{L^2(\mu)}$ is the norm in $L^2(\mu)$. Then the $n$-th Widom-Hilbert factor for $\mu$ is

$$W_n^2(\mu) := \frac{\|P_n(\cdot;\mu)\|_{L^2(\mu)}}{(\text{Cap}(\text{supp}(\mu)))^n}.$$

If $\text{supp}(\mu) \subset \mathbb{R}$ then a three-term recurrence relation

$$xP_n(x;\mu) = P_{n+1}(x;\mu) + b_{n+1}P_n(x;\mu) + a_n^2 P_{n-1}(x;\mu)$$

is valid for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The initial conditions $P_{-1}(x;\mu) \equiv 0$ and $P_0(x;\mu) \equiv 1$ generate two bounded sequences $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ of recurrence coefficients associated with $\mu$. Here, $a_n > 0, b_n \in \mathbb{R}$ for $n \in \mathbb{N}$ and

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)} = a_1 \cdots a_n.$$

A bounded two sided $\mathbb{C}$-valued sequence $(d_n)_{n=-\infty}^{\infty}$ is called almost periodic if the set $\{(d_{n+k})_{n=-\infty}^{\infty} : k \in \mathbb{Z}\}$ is precompact in $l^\infty(\mathbb{Z})$. A one sided sequence $(c_n)_{n=1}^{\infty}$ is called almost periodic if it is the restriction of a two sided almost periodic sequence.
A sequence \((e_n)_{n=1}^{\infty}\) is called asymptotically almost periodic if there is an almost periodic sequence \((e'_n)_{n=1}^{\infty}\) such that \(|e_n - e'_n| \to 0\) as \(n \to 0\).

A class of Parreau-Widom sets plays a special role in the recent theory of orthogonal and Chebyshev polynomials. Let \(K\) be a non-polar compact set and \(g_{\mathbb{C} \setminus K}\) denote the Green function for \(\mathbb{C} \setminus K\) with a pole at infinity. Suppose \(K\) is regular with respect to the Dirichlet problem, so the set \(\mathcal{C}\) of critical points of \(g_{\mathbb{C} \setminus K}\) is at most countable. Then \(K\) is said to be a Parreau-Widom set if \(\sum_{c \in \mathcal{C}} g_{\mathbb{C} \setminus K}(c) < \infty\). Parreau-Widom sets on \(\mathbb{R}\) have positive Lebesgue measure. For different aspects of such sets, see [9, 16, 24].

A class of regular in the sense of Stahl-Totik measures can be defined by the following condition

\[
\lim_{n \to \infty} W_n(\mu)^{1/n} = 1.
\]

For a measure \(\mu\) supported on \(\mathbb{R}\) we use the Lebesgue decomposition of \(\mu\) with respect to the Lebesgue measure:

\[
d\mu(x) = f(x)dx + d\mu_s(x).
\]

Following [10], let us define the Szegő class \(\text{Sz}(K)\) of measures on a given Parreau-Widom set \(K \subset \mathbb{R}\). Let \(\mu_K\) be the equilibrium measure on \(K\). By \(\text{ess supp}(\cdot)\) we denote the essential support of the measure, that is the set of accumulation points of the support. We have \(\text{Cap}(\text{supp}(\mu)) = \text{Cap}(\text{ess supp}(\mu))\), see Section 1 of [22]. A measure \(\mu\) is in the Szegő class of \(K\) if

(i) \(\text{ess supp}(\mu) = K\).

(ii) \(\int_K \log f(x) d\mu_K(x) > -\infty\). (Szegő condition)

(iii) the isolated points \(\{x_n\}\) of \(\text{supp}(\mu)\) satisfy \(\sum_n g_{\mathbb{C} \setminus K}(x_n) < \infty\).

By Theorem 2 in [10] and its proof, (ii) can be replaced by one of the following conditions:

(ii') \(\limsup_{n \to \infty} W_n^2(\mu) > 0\). (Widom condition)

(ii'') \(\liminf_{n \to \infty} W_n^2(\mu) > 0\). (Widom condition 2)

One can show that any \(\mu \in \text{Sz}(K)\) is regular in the sense of Stahl-Totik.

1.2 Generalized Julia sets and \(K(\gamma)\)

Let \((f_n)_{n=1}^{\infty}\) be a sequence of rational functions with \(\deg f_n \geq 2\) in \(\overline{\mathbb{C}}\) and \(F_n := f_n \circ f_{n-1} \circ \ldots \circ f_1\). The domain of normality for \((F_n)_{n=1}^{\infty}\) in the sense of Montel is called the Fatou set for \((f_n)\). The complement of the Fatou set in \(\overline{\mathbb{C}}\) is called the Julia set for \((f_n)\). We denote them by \(F_{(f_n)}\) and \(J_{(f_n)}\) respectively. These sets were considered first
in [12]. In particular, if \( f_n = f \) for some fixed rational function \( f \) for all \( n \) then \( F(f) \) and \( J(f) \) are used instead. To distinguish this last case, the word autonomous is used in the literature.

Suppose \( f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j \) where \( d_n \geq 2 \) and \( a_{n,d_n} \neq 0 \) for all \( n \in \mathbb{N} \). Following [8], we say that \( (f_n) \) is a regular polynomial sequence (write \( f_n \in \mathcal{R} \)) if positive constants \( A_1, A_2, A_3 \) exist such that for all \( n \in \mathbb{N} \) we have the following three conditions:

\[
\begin{align*}
|a_{n,d_n}| &\geq A_1 \\
|a_{n,j}| &\leq A_2 |a_{n,d_n}| \text{ for } j = 0, 1, \ldots, d_n - 1 \\
\log |a_{n,d_n}| &\leq A_3 \cdot d_n
\end{align*}
\]

For such polynomial sequences, by [8], \( J(f_n) \) is a regular compact set in \( \mathbb{C} \). In addition, \( \text{Cap}(J(f_n)) > 0 \) and \( J(f_n) \) is the boundary of

\[
\mathcal{A}(f_n)(\infty) := \{ z \in \mathbb{C} : F_n(z) \text{ goes locally uniformly to } \infty \}.
\]

The following construction is from [13]. Let \( \gamma := (\gamma_k)_{k=1}^{\infty} \) be a sequence provided that \( 0 < \gamma_k < 1/4 \) holds for all \( k \in \mathbb{N} \) and \( \gamma_0 := 1 \). Let \( f_1(z) = 2z(z - 1)/\gamma_1 + 1 \) and \( f_n(z) = \frac{1}{2\gamma_n}(z^{2^n} - 1) + 1 \) for \( n > 1 \). Then \( K(\gamma) := \bigcap_{s=1}^{\infty} F_s^{-1}([-1, 1]) \) is a Cantor set on \( \mathbb{R} \). Furthermore, \( F_s^{-1}([-1, 1]) \subset F_t^{-1}([-1, 1]) \subset [0, 1] \) whenever \( s > t \).

Also we use an expanded version of this set. For a sequence \( \gamma \) as above, let \( f_n(z) = \frac{1}{2\gamma_n}(z^{2^n} - 1) + 1 \) for \( n \in \mathbb{N} \). Then \( K_1(\gamma) := \bigcap_{s=1}^{\infty} F_s^{-1}([-1, 1]) \subset [-1, 1] \) and \( F_s^{-1}([-1, 1]) \subset F_t^{-1}([-1, 1]) \subset [-1, 1] \) provided that \( s > t \). It is a Cantor set. If there is a \( c \) with \( 0 < c < \gamma_k \) for all \( k \) then \( (f_n) \in \mathcal{R} \) and \( J(f_n) = K_1(\gamma) \), see [5]. If \( \gamma_1 = \ldots = \gamma_k \) for all \( k \in \mathbb{N} \) then \( K_1(\gamma) \) is an autonomous polynomial Julia set.

### 1.3 Hausdorff measure

A function \( h : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is called a dimension function if it is increasing, continuous and \( h(0) = 0 \). Given a set \( E \subset \mathbb{C} \), its \( h \)-Hausdorff measure is defined as

\[
\Lambda_h(E) = \liminf_{\delta \to 0} \left\{ \sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j < \delta \right\},
\]

where \( B(z, r) \) is the open ball of radius \( r \) centered at \( z \). For a dimension function \( h \), a set \( K \subset \mathbb{C} \) is an \( h \)-set if \( 0 < \Lambda_h(K) < \infty \). To denote the Hausdorff measure for \( h(t) = t^\alpha \), \( \Lambda_\alpha \) is used. Hausdorff dimension of \( K \) is defined as \( \inf \{ \alpha > 0 : \Lambda_\alpha(K) = 0 \} \).

### 2 Smoothness of Green functions and Markov Factors

The next set of problems is concerned with the smoothness properties of the Green function \( g_{\mathbb{C}\setminus K} \) near compact set \( K \) and related questions. We suppose that \( K \) is regular with respect to the Dirichlet problem, so the function \( g_{\mathbb{C}\setminus K} \) is continuous throughout \( \mathbb{C} \). The next problem was posed in [13].
**Problem 1.** Given modulus of continuity $\omega$, find a compact set $K$ such that the modulus of continuity $\omega(g_{C\setminus K}, \cdot)$ is similar to $\omega$.

Here, one can consider similarity as coincidence of moduli on some null sequence or in the sense of weak equivalence: $\exists C_1, C_2$ such that

$$C_1 \omega(\delta) \leq \omega(g_{C\setminus K}, \delta) \leq C_2 \omega(\delta)$$

for sufficiently small positive $\delta$.

We guess that a set $K(\gamma)$ from [13] is a candidate for the desired $K$ provided a suitable choice of the parameters. We recall that, for many moduli of continuity, the corresponding Green’s functions were given in [13], whereas the characterization of optimal smoothness for $g_{C\setminus K(\gamma)}$ is presented in [[5], Th.6.3].

A stronger version of the problem is about pointwise estimation of the Green function:

**Problem 2.** Given modulus of continuity $\omega$, find a compact set $K$ such that

$$C_1 \omega(\delta) \leq g_{C\setminus K}(z) \leq C_2 \omega(\delta)$$

for $\delta = dist(z, K) \leq \delta_0$, where $C_1, C_2$ and $\delta_0$ do not depend on $z$.

In the most important case we get a problem about “two-sided Hölder” Green function, which was posed by A. Volberg on his seminar (quoted with permission):

**Problem 3.** Find a compact set $K$ on the line such that for some $\alpha > 0$ and constants $C_1, C_2$, if $\delta = dist(z, K)$ is small enough then

$$C_1 \delta^\alpha \leq g_{C\setminus K}(z) \leq C_2 \delta^\alpha. \quad (1)$$

Clearly, a closed analytic curve gives a solution for sets on the plane.

If $K \subset \mathbb{R}$ satisfies (1), then $K$ is of Cantor-type. Indeed, if interior of $K$ (with respect to $\mathbb{R}$) is not empty, let $(a, b) \subset K$, then $g_{C\setminus K}$ has $\text{Lip } 1$ behavior near the point $(a + b)/2$. On the other hand, near endpoints of $K$ the function $g_{C\setminus K}$ cannot be better than $\text{Lip } 1/2$.

By the Bernstein-Walsh inequality, smoothness properties of Green functions are closely related with a character of maximal growth of polynomials outside the corresponding compact sets, which, in turn, allows to evaluate Markov’s factors for the sets. Recall that, for a fixed $n \in \mathbb{N}$ and (infinite) compact set $K$, the $n$–th Markov factor $M_n(K)$ is the norm of operator of differentiation in the space of holomorphic polynomials $P_n$ with the uniform norm on $K$. In particular, the Hölder smoothness (the right inequality in (1)) implies Markov’s property of the set $K$ (a polynomial growth rate of $M_n(K)$). The problem about inverse implication (see e.g [20]) has attracted attention of many researches.
By W. Pleśniak [20], any Markov set $K \subset \mathbb{R}^d$ has the extension property ($EP$), which means that there exists a continuous linear extension operator from the space of Whitney functions $\mathcal{E}(K)$ to the space of infinitely differentiable functions on $\mathbb{R}^d$. We guess that there is some extremal growth rate of $M_n$ which implies the lack of $EP$. Recently it was shown in [15] that there is no complete characterization of $EP$ in terms of growth rate of Markov’s factors. Namely, two sets were presented, $K_1$ with $EP$ and $K_2$ without it, such that $M_n(K_1)$ grows essentially faster than $M_n(K_2)$ as $n \to \infty$. Thus there exists non-empty zone of uncertainty where the growth rate of $M_n(K)$ is not related with $EP$ of the set $K$.

**Problem 4.** Characterize the growth rates of Markov’s factors that define the boundaries of the zone of uncertainty for the extension property.

### 3 Orthogonal polynomials

One of the most interesting problems concerning orthogonal polynomials on Cantor sets on $\mathbb{R}$ is the character of periodicity of recurrence coefficients. It was conjectured in p. 123 of [7] that if $f$ is a non-linear polynomial such that $J(f)$ is a totally disconnected subset of $\mathbb{R}$ then the recurrence coefficients for $\mu_{J(f)}$ are all almost periodic. This is still an open problem. In [6], the authors conjectured that the recurrence coefficients for $\mu_{K(\gamma)}$ are asymptotically almost periodic for any $\gamma$. It may be hoped that a more general and slightly weaker version of Bellissard’s conjecture can be valid.

**Problem 5.** Let $(f_n)$ be a regular polynomial sequence such that $J(f_n)$ is a Cantor-type subset of the real line. Prove that the recurrence coefficients for $\mu_{J(f_n)}$ are asymptotically almost periodic.

For a measure $\mu$ which is supported on $\mathbb{R}$, let $Z_n(\mu) := \{x : P_n(x; \mu) = 0\}$. We define $U_n(\mu)$ by

$$U_n(\mu) := \inf_{x, x' \in Z_n(\mu)} |x - x'|.$$

In [18] Krüger and Simon gave a lower bound for $U_n(\mu)$ where $\mu$ is the Cantor-Lebesgue measure of the (translated and scaled) Cantor ternary set. In [17], it was shown that Markov’s inequality and spacing of the zeros of orthogonal polynomials are somewhat related.

Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ and $n \in \mathbb{N}$ with $n > 1$ be given and define $\delta_k = \gamma_0 \cdots \gamma_k$ for all $k \in \mathbb{N}_0$. Let $s$ be the integer satisfying $2^{s-1} \leq n < 2^s$. By [2],

$$\frac{\pi^2}{4} \cdot \delta_{s+2} \leq U_n(\mu_{K(\gamma)}) \leq \frac{\pi^2}{4} \cdot \delta_{s-2}$$

holds. In particular, if there is a number $c$ such that $0 < c < \gamma_k < 1/4$ holds for all $k \in \mathbb{N}$ then, by [2], we have

$$c^2 \cdot \delta_s \leq U_n(\mu_{K(\gamma)}) \leq \frac{\pi^2}{4c^2} \cdot \delta_s. \quad (2)$$
By [13], at least for small sets $K(\gamma)$, we have $M_2(\gamma) \sim 2/\delta$, where the symbol $\sim$ means the strong equivalence.

**Problem 6.** Let $K$ be a non-polar compact subset of $\mathbb{R}$. Is there a general relation between the zero spacing of orthogonal polynomials for $\mu_K$ and smoothness of $g_{\mathbb{C}\setminus K}$? Is there a relation between the zero spacing of $\mu_K$ and the Markov factors?

As mentioned in section 1, the Szegö condition and the Widom condition are equivalent for Parreau-Widom sets. Let $K$ be a Parreau-Widom set. Let $\mu$ be a measure such that $\text{ess supp}(\mu) = K$ and $\{x_n\}$ of $\text{supp}(\mu)$ satisfy $\sum_n g_{\mathbb{C}\setminus K}(x_n) < \infty$. Then, as it is discussed in Section 6 of [4], the Szegö condition is equivalent to the condition

$$\int_K \log(d\mu/d\mu_K) d\mu_K(x) > -\infty.$$  \hspace{1cm} (3)

This condition is also equivalent to the Widom condition under these assumptions.

It was shown in [1] that $\inf_{n \in \mathbb{N}} W_n(\mu_K) \geq 1$ for non-polar compact $K \subset \mathbb{R}$. Thus the Szegö condition in the form (3) and the Widom condition are related on arbitrary non-polar sets.

**Problem 7.** Let $K$ be a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Let $\mu$ be a measure such that $\text{ess supp}(\mu) = K$. Assume that the isolated points $\{x_n\}$ of $\text{supp}(\mu)$ satisfy $\sum_n g_{\mathbb{C}\setminus K}(x_n) < \infty$. If the condition (3) is valid for $\mu$ is it necessarily true that the Widom condition or the Widom condition 2 holds? Conversely, does the Widom condition imply (3)?

It was proved in [11] that if $K$ is a Parreau-Widom set which is a subset of $\mathbb{R}$ then $(W_n(K))_{n=1}^{\infty}$ is bounded above. On the other hand, $(W_n(K))_{n=1}^{\infty}$ is unbounded for some Cantor-type sets, see e.g. [14].

**Problem 8.** Is it possible to find a regular non-polar compact subset $K$ of $\mathbb{R}$ which is not Parreau-Widom but $(W_n(K))_{n=1}^{\infty}$ is bounded? If $K$ has zero Lebesgue measure then is it true that $(W_n(K))_{n=1}^{\infty}$ is bounded? We can ask the same problems if we replace $(W_n(K))_{n=1}^{\infty}$ by $(W_n^2(\mu_K))_{n=1}^{\infty}$ above.

Let $T_N$ be a real polynomial of degree $N$ with $N \geq 2$ such that it has $N$ real and simple zeros $x_1 < \cdots < x_n$ and $N - 1$ critical points $y_1 < \cdots < y_{n-1}$ with $|T_N(y_i)| \geq 1$ for each $i \in \{1, \ldots, N-1\}$. We call such a polynomial admissible. If $K = T_N^{-1}([-1, 1])$ for an admissible polynomial $T_N$ then $K$ is called a $T$-set. The following result is well known, see e.g. [23].

**Theorem 1.** Let $K = \cup_{j=1}^n [\alpha_j, \beta_j]$ be a disjoint union of $n$ intervals such that $\alpha_1$ is the leftmost end point. Then $K$ is a $T$-set if and only if $\mu_K([\alpha_1, c])$ is in $\mathbb{Q}$ for all $c \in \mathbb{R} \setminus K$. 
For $K(\gamma)$, it is known that $\mu_{K(\gamma)}([0,c]) \in \mathbb{Q}$ if $c \in \mathbb{R} \setminus K(\gamma)$, see Section 4 in [2].

**Problem 9.** Let $K$ be a regular non-polar compact subset of $\mathbb{R}$ and $\alpha$ be the leftmost end point of $K$. Let $\mu_K([\alpha,c]) \in \mathbb{Q}$ for all $c \in \mathbb{R} \setminus K$. What can we say about $K$? Is it necessarily a polynomial generalized Julia set? Does this imply that there is a sequence of admissible polynomials $(f_n)_{n=1}^{\infty}$ such that $(F_n^{-1}[-1,1])_{n=1}^{\infty}$ is a decreasing sequence of sets such that $K = \cap_{n=1}^{\infty} F_n^{-1}[-1,1]$?

## 4 Hausdorff measures

It is valid for a wide class of Cantor sets that the equilibrium measure and the corresponding Hausdorff measure on this set are mutually singular, see e.g. [19].

Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $0 < \gamma_k < 1/32$ satisfy $\sum_{k=1}^{\infty} \gamma_k < \infty$. This implies that $K(\gamma)$ has Hausdorff dimension 0. In [3], the authors constructed a dimension function $h_\gamma$ that makes $K(\gamma)$ an $h$-set. Provided also that $K(\gamma)$ is not polar it was shown that there is a $C > 0$ such that for any Borel set $B$, $C^{-1} \cdot \mu_K(\gamma)(B) < \Lambda_h(B) < C \cdot \mu_K(\gamma)(B)$ and in particular the equilibrium measure and $\Lambda_h$ restricted to $K(\gamma)$ are mutually absolutely continuous. In [15], it was shown by the authors that indeed these two measures coincide. To the best of our knowledge, this is the first example of a subset of $\mathbb{R}$ such that the equilibrium measure is a Hausdorff measure restricted to the set.

**Problem 10.** Let $K$ be a non-polar compact subset of $\mathbb{R}$ such that $\mu_K$ is equal to a Hausdorff measure restricted to $K$. Is it necessarily true that the Hausdorff dimension of $K$ is 0?

Hausdorff dimension of a unit Borel measure $\mu$ supported on $\mathbb{C}$ is defined by $\dim(\mu) := \inf\{\text{HD}(K) : \mu(K) = 1\}$ where $\text{HD}(\cdot)$ denotes Hausdorff dimension of the given set. For polynomial Julia sets which are totally disconnected there is a formula for $\dim(\mu_{J(f)})$, see e.g. p. 23 in [19] and p.176-177 in [21].

**Problem 11.** Is it possible to find simple formulas for $\dim \left( \mu_{J(f_n)} \right)$ where $(f_n)$ is a regular polynomial sequence?

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