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Guarding a Non-Maneuverable Translating Line with an Attached Defender

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Abstract

In this paper we consider a target-guarding differential game where the defender must protect a linearly translating line segment by intercepting an attacker who tries to reach it. In contrast to common target-guarding problems, we assume that the defender is attached to the target and moves along. This assumption affects the defender’s maximum speed in an inertial frame, which depends on the target’s direction of motion. Zero-sum differential game of degree for both the attacker-win and defender-win scenarios are studied. The payoff is defined as the distance between the two agents at the time of game termination. We derive the equilibrium strategies and the Value function by leveraging the solution for the infinite-length target scenario. The zero-level set of this Value function provides the barrier surface that divides the state space into defender-win and attacker-win regions. We present simulation results to demonstrate the theoretical results.

Keywords: differential games, pursuit-evasion, target-guarding.
1 Introduction

Pursuit-evasion games (PEG) are a class of differential games in which an agent (i.e., pursuer/defender) attempts to capture another agent (i.e., evader/attacker) who seeks to avoid or delay the capture. This paper focuses on a particular variant of PEG that involves an asset/target that must be guarded. In both civilian and military defense applications [1] such a scenario is highly relevant.

Target-attacker-defender games (TADG) study situations where the attacker seeks to reach the target without being intercepted by the defender. In the literature, targets are typically modeled as points or agents that are stationary [6] and guarded by the defenders. Alternatively, the target can be a non-stationary agent which cooperates with the defender by either actively evading the attacker or by rendezvousing with the defender [7, 8, 9]. The defender wins the game either by intercepting the attacker [1, 10, 11, 12] or by rendezvousing with the target [13].

A related class of PEGs is target guarding (TG), which was introduced by Isaacs [14]. In TG games, the target is a region rather than a point, which renders rendezvous-type strategies ineffective for the defender. Several variants of TG exist, including reach-avoid games [15, 16, 17] and coastline guarding or border-defense problems [18, 19, 20]. These works have extended the problem to multi-agent scenarios and considered various geometric settings; however, it is generally assumed that agents have simple motions and are free to move within a planar space.

In this study, we are interested in TG scenarios where the defender is constrained to move only along the perimeter of the target. Similar works have been previously studied as perimeter-defense games [21, 22, 23]. Unlike standard TG, these papers assume that the defender cannot pass through the target region. Therefore, the defender must move around the perimeter in order to reach the attacker, thereby affecting the dynamics and thus the capturability. Different variants have been studied with differential game techniques [21] and with geometric approaches [22, 23]; however, these studies are based on stationary target regions.

In this paper, we consider a target that translates on a plane. As an initial step towards a more realistic scenario, a non-maneuverable target with no rotational motion is studied. The attacker moves freely and tries to reach the target while avoiding the defender. However, the defender is constrained to move only on the linear target. In the inertial frame, the defender is dragged in the direction of the target’s motion, but the attacker is not affected by the motion of the target. In this context, there is a connection to the work presented in [24, 25], where PEG is played in a flow field; however, the results in [24, 25] do not extend naturally to TG objectives considered in this paper. Moreover, the flow field affects only one of the two agents in this paper.

The main contributions of this paper are: (i) the characterization of the barrier surface that separates the state space into defender-win and attacker-win regions; and (ii) the equilibrium strategies and the Value function in each
regions. By allowing the target to translate in an arbitrary direction, this paper generalizes the result in [26] which assumes that the target can only translate in $x$-direction. In addition, we provide the solution to both the attacker-win and defender-win scenarios, where the latter was missing in [26].

2 Problem Formulation

This section formulates the translating line guarding game on a plane for one defender and one attacker shown in Fig. 1. The inertial frame $\mathcal{I} = (O, e_x, e_y)$ is defined by the origin $O$, and the basis vectors $e_x$ and $e_y$. The positions of the agents in $\mathcal{I}$ are denoted as $x_i = [x_i, y_i]^T \in \mathbb{R}^2$, where $i = \{A, D, T\}$ represents the attacker, defender, and target, respectively. The target, $T$, is a line segment and it is aligned with $e_x$. The length of $T$ is $L$, and so the endpoints are given by $x_T$ and $x_T + Le_x$.

The dynamics of the attacker in $\mathcal{I}$ are

$$\dot{x}_A = \begin{bmatrix} \dot{x}_A \\ \dot{y}_A \end{bmatrix} = v_A \begin{bmatrix} \cos \phi_A \\ \sin \phi_A \end{bmatrix},$$

(1)

where $\phi_A \in [-\pi, \pi]$ is the attacker’s control (i.e., its heading angle), and $v_A$ is its speed given as part of the game parameters. The target moves at a constant velocity

$$\dot{x}_T = \begin{bmatrix} \dot{x}_T \\ \dot{y}_T \end{bmatrix} = v_T \begin{bmatrix} \cos \phi_T \\ \sin \phi_T \end{bmatrix},$$

(2)

where $v_T$ and $\phi_T$ are the game parameters known to the players. The defender is assumed to be “attached” to the target, and can move in the $x$-direction relative to the target:

$$\dot{x}_D = \begin{bmatrix} \dot{x}_D \\ \dot{y}_D \end{bmatrix} = v_T \begin{bmatrix} \cos \phi_T \\ \sin \phi_T \end{bmatrix} + \begin{bmatrix} \omega_D \\ 0 \end{bmatrix},$$

(3)

Fig. 1: Illustration of translating line guarding problem for one defender and one attacker in target frame and in the inertial frame.
Guarding a Non-Maneuverable Translating Line with an Attached Defender

where $\omega_D \in [-1,1]$, is the defender’s control. Since the defender is attached to the target, its states must satisfy $x_D \in [x_T, x_T + L e_x]$. Consequently, $\omega_D \geq 0$ when $x_D = x_T$, and $\omega_D \leq 0$ when $x_D = x_T + L e_x$.

For convenience, we perform our analysis in the translating target frame $B = (T, e_x, e_y)$ attached to the leftmost point of the target. Let $\hat{x}_{i}, i = \{A, D\}$ denote the agents’ positions in $B$, where $\hat{y}_D = 0$. Letting $\hat{x} = [\hat{x}_D, \hat{x}_A, \hat{y}_A]^{\top}$ represent the stacked state and using (1)–(3) yields

$$f(\hat{x}, \omega_D, \phi_A) = \dot{\hat{x}} = \begin{bmatrix} \dot{\hat{x}}_D \\ \dot{\hat{x}}_A \\ \dot{\hat{y}}_A \end{bmatrix} = \begin{bmatrix} \omega_D \\ v_A \cos \phi_A - v_T \cos \phi_T \\ v_A \sin \phi_A - v_T \sin \phi_T \end{bmatrix}.$$ (4)

Assume the following on the agents’ speeds:

A1) The attacker is faster than the target, $v_A > v_T$.
A2) The defender can outrun the attacker in $e_x$ direction: i.e., $v_A < 1 - |v_T \cos \phi_T|$.

Assumption (A1) avoids the degenerate case where the attacker is too slow to reach the target even if there was no defender. Assumption (A2) ensures that once the defender aligns itself with the attacker (i.e., $\hat{x}_D = \hat{x}_A$), it has sufficient control authority to maintain that alignment regardless of the attacker’s control (as long as $\hat{x}_A \in [0, L]$). For this paper, we consider the game of kind as the question of whether the attacker can reach the target or if the defender can prevent it. The barrier surface that provides the answer to this question will be obtained by solving a related game of degree. The terminal conditions and the payoff functions that define the game of degree will be provided separately for the attacker-win and defender-win scenarios.

3 Attacker-win scenario

In this section, we are concerned with the game of degree when the attacker is able to reach the target (i.e., drive $\hat{y}_A \to 0$), with a nonzero miss-distance from the defender. The initial condition of the system lies inside the attacker-win region (i.e., $\hat{x} \in R_A$), and we use subscript $a$ to refer to the game of degree in this region.

We consider a zero-sum differential game with the following payoff that describes the miss-distance

$$J_a(\tilde{x}_0, \omega_D, \phi_A) = \Phi_a(\tilde{x}_f) = |\hat{x}_A(t_f) - \hat{x}_D(t_f)|,$$ (5)

where $\tilde{x}_f := \tilde{x}(t_f)$ and $t_f$ represent the terminal time. Here the defender is the minimizing player who seeks to minimize the miss-distance, and the attacker is the maximizing player whose goal is to maximize it. If an equilibrium exists,
the value function is defined as

\[
V_a(\hat{x}_0) = \min_{\omega_D} \max_{\phi_A} J_a = \max_{\phi_A} \min_{\omega_D} J_a. \tag{6}
\]

The equilibrium strategies \(\omega_D^*, \phi_A^*\) satisfy the following saddle-point condition:

\[
J_a(\cdot, \omega_D^*, \phi_A^*) \leq J_a(\cdot, \omega_D, \phi_A^*) \leq J_a(\cdot, \omega_D^*, \phi_A^*). \tag{7}
\]

The terminal constraint is given by

\[
\psi_a(\hat{x}_f) = \hat{y}_A(t_f) = 0. \tag{8}
\]

Thus, the terminal surface is defined by the set of states satisfying (8):

\[
S_{T_a} = \{ \hat{x} \mid \hat{y}_A = 0 \text{ and, } \hat{x}_A \in [0, L] \}. \tag{9}
\]

We will derive \(V_a\) and the corresponding equilibrium strategies in the following sections.

### 3.1 Infinite Length Target

As a building block towards the complete solution, this section assumes that the target length is infinite. The system dynamics, payoff, and terminal constraint remains the same as stated in (4), (5), and (8) respectively. The terminal surface for the infinite target is given by

\[
S_{T_a, \text{inf}} = \{ \hat{x} \mid \hat{y}_A = 0 \}. \tag{10}
\]

#### 3.1.1 First Order Necessary Conditions for Optimality

This section presents the optimal strategies for the defender and the attacker for \(\hat{x} \in \mathcal{R}_A\). First order necessary conditions \[14\] are used to derive the equilibrium strategies for the players. The solution approach involves defining and optimizing a function known as Hamiltonian. The Hamiltonian for the differential game (4) is given by

\[
H_a(\hat{x}, \omega_D, \phi_A, \sigma, t) = l(\hat{x}, \omega_D, \phi_A, t) + \sigma^\top (t)f(\hat{x}, \omega_D, \phi_A, t) \\
= \sigma_{\hat{x}D} \omega_D + \sigma_{\hat{x}A} v_A \cos \phi_A - \sigma_{\hat{x}T} v_T \cos \phi_T \\
+ \sigma_{\hat{y}A} v_A \sin \phi_A - \sigma_{\hat{y}T} v_T \sin \phi_T, \tag{11}
\]

where the integral cost \(l(\cdot)\) is 0 in our problem, and \(\sigma := [\sigma_{\hat{x}D}, \sigma_{\hat{x}A}, \sigma_{\hat{y}A}]^\top\), is the adjoint vector. Notice that the Hamiltonian in (11) is a separable function of the controls \(\omega_D\) and \(\phi_A\), and thus Isaacs’ condition \[14\] , \[28\] holds:

\[
\min_{\omega_D} \max_{\phi_A} H_a = \max_{\phi_A} \min_{\omega_D} H_a. \tag{12}
\]
The equilibrium adjoint dynamics are given by

\[
\dot{\sigma} = \frac{\partial H_a}{\partial \hat{x}} = [0, 0, 0].
\] (13)

The terminal adjoint values are obtained from the transversality condition [29]:

\[
\sigma^T(t_f) = \frac{\partial \Phi_a}{\partial \hat{x}} + \eta \frac{\partial \psi_a}{\partial \hat{x}} = [-\lambda, \lambda, \eta],
\] (14)

where \( \lambda := \text{sgn}(\hat{x}_A - \hat{x}_D) \) and \( \eta \) is Lagrange multiplier vector [27]. Therefore, with (13) and (14), the following holds:

\[
\sigma(t) = [-\lambda, \lambda, \eta]^T, \quad \forall t \in [t_0, t_f].
\] (15)

The terminal Hamiltonian satisfies

\[
H_a(t_f) = -\frac{\partial \Phi_a}{\partial t_f} - \eta \frac{\partial \phi_a}{\partial t_f} = 0,
\] (16)

and \( \frac{dH_a}{dt} = 0 \), therefore, \( H_a(t) = 0 \) for all \( t \in [t_0, t_f] \).

The equilibrium control actions of the attacker and the defender maximize and minimize (11) respectively: \( H^*_a = \max_{\phi_A} \min_{\omega_D} H_a \). For the saddle point solution of the problem, we have

\[
\omega^*_D = \arg\min_{\omega_D} H_a = \arg\min_{\omega_D}(\sigma \hat{x}_D, \omega_D) = -\text{sgn}(\sigma \hat{x}_D) = \lambda,
\] (17)

\[
\phi^*_A = \arg\max_{\phi_A} H_a = \arg\max_{\phi_A}(\sigma \hat{x}_A v_A \cos \phi_A + \sigma \hat{y}_A v_A \sin \phi_A).
\] (18)

Solving (18), we have

\[
\cos \phi^*_A = \frac{\sigma \hat{x}_A}{\sqrt{\sigma^2 \hat{x}_A + \sigma^2 \hat{y}_A}} = \frac{\lambda}{\sqrt{\eta^2 + 1}},
\] (19)

\[
\sin \phi^*_A = \frac{\sigma \hat{y}_A}{\sqrt{\sigma^2 \hat{x}_A + \sigma^2 \hat{y}_A}} = \frac{\eta}{\sqrt{\eta^2 + 1}}.
\] (20)

Substituting the equilibrium controls, (17), (19) and (20), into the Hamiltonian, (11), and evaluating at \( t_f \) gives

\[
H^*_a(t_f) = 0 = v_A \sqrt{1 + \eta^2} - v_T (\eta \sin \phi_T + \lambda \cos \phi_T) - 1.
\] (21)
Solving (21) gives

\[ \eta = \frac{ab \pm v_A \sqrt{a^2 + b^2 - v_T^2}}{v_A^2 - a^2}, \tag{22} \]

where \( a := v_T \sin \phi_T \), and \( b := (1 + \lambda v_T \cos \phi_T) \).

If \( \dot{y}_A < 0 \), the attacker must move to the positive \( y \) direction to reach the target, which implies \( \sin \phi_A^* > 0 \). Based on this observation and (20), we know \( \eta > 0 \), and therefore the + sign in (22) will be used. Likewise the – sign will be used when \( \dot{y}_A > 0 \).

### 3.1.2 Solution Characteristics

The retrograde equilibrium kinematics [14] can be obtained by substituting the equilibrium controls, (17), (19) and (20), along with the adjoints into (4) which yields

\[
\begin{align*}
\dot{x}_A^* &= \frac{\lambda v_A}{\eta^2 + 1} - v_T \cos \phi_T, \\
\dot{y}_A^* &= \frac{\eta v_A}{\eta^2 + 1} - v_T \sin \phi_T,
\end{align*}
\tag{23}
\]

with boundary condition, \( \dot{y}_A(t_f) = 0 \).

Let \([X,Y]\) denote the relative position of the attacker with respect to the defender, i.e.,

\[ [X,Y] := [x_A - x_D, y_A - y_D] = [\dot{x}_A - \dot{x}_D, \dot{y}_A]. \tag{24} \]

Note that we have \( \lambda = \text{sgn}(X) \). Differentiating \( X \) and \( Y \) with respect to \( t \), and manipulating the equations we have

\[
m(\phi_A^*, \omega_D^*) := \frac{dY}{dX} = \frac{v_A \sin \phi_A - v_T \sin \phi_T}{v_A \cos \phi_A - v_T \cos \phi_T - \omega_D}. \tag{25} \]

Since both \( \phi_A^* \) and \( \omega_D^* \) are constant, the equilibrium trajectories of the system in the \( XY \)-plane are given by straight lines:

\[ Y = m^* X + C, \tag{26} \]

where \( m^* = m(\phi_A^*, \omega_D^*) \).

The red solid lines in Fig. 2 present the equilibrium trajectories for \( \dot{x} \in \mathcal{R}_A \).

It can be seen that the terminal payoff in (5) is determined by the \( X \) intercept of the state trajectory, which we denote by \( X_f \). The black solid lines indicate the critical case in which the attacker reaches the target at the time of capture with zero miss-distance (i.e., \( X_f = 0 \)). Beyond this critical case, the region shown in blue is the defender-win region, which will be discussed in Sec. 4.

**Theorem 1 (Infinite-Length Target)** Consider the game of degree with payoff given in (5), and suppose the target is infinitely long. Then the equilibrium state feedback control strategies are given by (17), (19) and (20). Moreover, the Value of the game is

\[ V_a = \text{sgn}(X) \left( X - \frac{Y}{m^*} \right), \tag{27} \]
Guarding a Non-Maneuverable Translating Line with an Attached Defender

Fig. 2: Equilibrium trajectories of the relative position vector \([X, Y]\) in infinite-length target scenario. The color indicates the Value of the game. The following parameters are used: \(v_A = 0.7\), \(v_T = 0.2\) and \(\phi_T = 2\pi/3\).

where \(m^* = m(\phi_A^*, \omega_D^*)\) with the expression given in (25).

Proof The players’ strategies are derived using the first order necessary condition for optimality. As discussed with Fig. 2, the Value is given by the \(X\) intercept of the equilibrium trajectory. More specifically, the miss-distance is \(X_f > 0\) if the game starts in the positive \(X\) region, whereas it is \(-X_f > 0\) if the game starts in the negative \(X\) region. For a given initial condition \([X_0, Y_0]\), we have

\[
C = Y_0 - m^* X_0. \tag{28}
\]

Substituting \(C\) back into the equation and solving for the \(X\) intercept gives:

\[
X_f = X_0 - \frac{Y_0}{m^*}. \tag{29}
\]

This completes the proof that (27) provides the Value of the game.

3.2 Finite Length Target

In the original problem, the endpoints of the target become important consideration. Notice that there is always one endpoint that is relevant to the game: i.e., the one that the attacker may be able to reach without crossing \(X = 0\). We denote this endpoint as \(\hat{x}_E = [\hat{x}_E, 0]^\top\), where

\[
\hat{x}_E = (1 + \text{sgn}(X)) \frac{L}{2}. \tag{30}
\]

The defender strategy will remain the same since it only depends on the relative position of the players, \(X\). However, the attacker’s heading from Theorem 1 is valid only if it intersects with the finite target. Let \(\hat{x}_B = [\hat{x}_B, 0]^\top\) denote the point on the \(\hat{x}\) axis that the attacker reaches following equilibrium strategies stated in Theorem 1:

\[
\hat{x}_B := \hat{x}_A - \frac{\hat{y}_A}{m_B}, \quad \text{where } m_B := \frac{v_A \sin \phi_A^* - v_T \sin \phi_T}{v_A \cos \phi_A^* - v_T \cos \phi_T}. \tag{31}
\]
Guarding a Non-Maneuverable Translating Line with an Attached Defender

Fig. 3: Attacker strategy in target frame and inertial frame. When the strategy in Theorem 1 fails to intercept the target, the attacker employs a heading to reach the endpoint of the target.

Now we can define two strategic regions for the attacker-win game as follows:

- **$S_{1a}$**: $\hat{x} \in R_A$ and the strategy stated in Theorem 1 is still valid for finite-length target case, given that the following condition holds:

  $$\hat{x}_B \in [0, L].$$  \hfill (32)

- **$S_0$**: $\hat{x} \in R_A$, however, (32) does not hold.

In $S_0$, the attacker must sacrifice the separation with the defender at $t_f$ and pick an aim point that actually intercepts the target. The aim point that achieves the least deviation from optimal heading, $\phi^*_A$, is the endpoint $\hat{x}_E$ as shown in Fig. 3.

The attacker’s heading angle in target frame for it to hit the endpoint $\hat{x}_E$, is given by

$$\left[ \cos \hat{\phi}^*_A, \sin \hat{\phi}^*_A \right] = \left[ \frac{\hat{x}_E - \hat{x}_A}{\|\hat{x}_E - \hat{x}_A\|}, \frac{-\hat{y}_A}{\|\hat{x}_E - \hat{x}_A\|} \right].$$  \hfill (33)

Note that we use the superscript $^*$ to denote the optimal strategies for the finite-length case. Using the law of cosines we obtain

$$v^2_A = v^2_T + \hat{v}^2_A + 2v_T \hat{v}_A \cos(\hat{\phi}^*_A - \phi_T).$$  \hfill (34)

Solving for $\hat{v}_A$ yields

$$\hat{v}_A = -v_T(\cos \hat{\phi}^*_A \cos \phi_T + \sin \hat{\phi}^*_A \sin \phi_T)$$

$$+ \sqrt{v^2_A - v^2_T \left( \sin \hat{\phi}^*_A \cos \phi_T - \cos \hat{\phi}^*_A \sin \phi_T \right)^2}. \hfill (35)$$

\footnote{Note that there is no incentive for the attacker to go around the endpoint and approach the target from the positive side, i.e., enter $\hat{y}_A > 0$ region because the attacker cannot improve the miss-distance as long as defender plays optimally.}
Now we are ready to state the main theorem.

**Theorem 2 (Finite-Length Target)** The equilibrium state feedback control strategy for the defender remains the same as stated in Theorem 1. The equilibrium state feedback strategy for the attacker is given in Theorem 1 if (32) holds; otherwise, it is given by

\[
\begin{bmatrix}
\cos \phi_a^*, \sin \phi_a^*
\end{bmatrix} = \frac{1}{v_A} \begin{bmatrix}
\dot{v}_a^* \cos \phi_a^* + v_T \cos \phi_T, \\
\dot{v}_a^* \sin \phi_a^* + v_T \sin \phi_T
\end{bmatrix},
\]

where \(\cos \dot{\phi}_a^*\), \(\sin \dot{\phi}_a^*\), and \(\dot{v}_a^*\) are given by (33) and (35) respectively. The Value of the game is given by the expression in (27), but with the slope \(m^* = m(\phi_a^*, \omega_D^*)\) when (32) does not hold.

The barrier surface is given by the zero-level set of the Value function (27):

\[S_B = \{\hat{x} \mid V_a = 0\},\]

which separates the state space into attacker-win and defender-win regions, respectively,

\[\mathcal{R}_A = \{\hat{x} \mid V_a > 0\}, \quad \mathcal{R}_D = \{\hat{x} \mid V_a \leq 0\}.\]

See Fig. 5 for the illustration of the barrier surface. The closed form expression for the barrier surface will be discussed in Sec. 5.

## 4 DEFENDER-WIN SCENARIO

In this section, we consider a game of degree for initial states in the defender-win region (i.e., \(\hat{x} \in \mathcal{R}_D\)). We use the subscript \(d\) to refer to the game in this region. We consider the following payoff function:

\[J_d(\hat{x}_0, \omega_D, \phi_a) = \Phi_d(\hat{x}_f) = -\sqrt{X_f^2 + Y_f^2},\]

which is the negative of the distance between the attacker and the defender at terminal time. The negative sign is used to maintain the convention that the attacker (resp. defender) is the minimizer (resp. maximizer).

### 4.1 Infinite Length target

Similar to the attacker-win case, we start by looking into the infinite-length target case. Here the terminal condition is \(\hat{x}_A = \hat{x}_D\). If the target length is infinite, the payoff function in (39) reduces to

\[\phi_d(t_f) = -|Y_f|, \quad \text{where } Y_f = y_A(t_f) - y_D(t_f).\]

The terminal constraint is given by

\[\psi_d(\hat{x}_f) = \hat{x}_D(t_f) - \hat{x}_A(t_f) = 0.\]
Thus the terminal surface is defined by

$$S_{T_d, int} = \{ \hat{x} \mid \hat{x}_D(t_f) = \hat{x}_A(t_f) \}. \quad (42)$$

The following theorem shows that the strategies remain the same as in the attacker-win scenario for infinite-length target.

**Theorem 3 (Infinite-Length Target)** The equilibrium state feedback control strategies for the attacker and the defender remain the same as stated in Theorem 1, and the Value function is given by

$$V_d = \text{sgn}(Y) \left( m^* X - Y \right), \quad (43)$$

where $m^* = m(\phi^*_A, \omega^*_D)$ is given in (25).

**Proof** This proof is based on the substitution of the proposed equilibrium strategies and Value function into the Hamiltonian-Jacobi-Isaacs (HJI) equation:

$$\min_{\omega_D} \max_{\phi_A} \left\{ l(\cdot) + \partial V_d / \partial t + V_\hat{x} \cdot f(\cdot) \right\} = 0, \quad (44)$$

where the omitted function arguments are $l(\cdot) = (\hat{x}, \omega_D, \phi_A)$, $V_\hat{x}$ is the vector $[\partial V_d / \partial \hat{x}_D, \partial V_d / \partial \hat{x}_A, \partial V_d / \partial \hat{y}_A]$, and $l$ represents an integral cost component. First, note that the cost, (40), has no integral component, and thus $l = 0$. Also the proposed Value function, (43), is not an explicit function of time and thus $\partial V_d / \partial t = 0$.

The vector $V_\hat{x}$ is obtained by differentiating (43) with respect to each state:

$$V_\hat{x} = \text{sgn}(Y)[-m^*, m^*, -1]. \quad (45)$$

The (forward) equilibrium dynamics, $f$, are given by the negative of (23). Substituting all of these expressions into (44) gives

$$\min_{\omega_D} \max_{\phi_A} \left\{ \frac{\partial V_d}{\partial \hat{x}_D} \hat{x}_D + \frac{\partial V_d}{\partial \hat{x}_A} \hat{x}_A + \frac{\partial V_d}{\partial \hat{y}_A} \hat{y}_A \right\} = -m^*(v_A \cos \phi_T^* - v_T \cos \phi_T - \lambda) + v_A \sin \phi_T^* - v_T \sin \phi_T = 0.$$

Thus the proposed Value function is continuous and continuously differentiable, and it satisfies the HJI hyperbolic PDE. \hfill \Box

### 4.2 Finite-Length target

In this section we provide the equilibrium strategies for the finite length target for $\hat{x} \in \mathcal{R}_D$. The defender strategy will remain the same for the finite length target. However, the defender is limited to move within the line segment. Therefore, the terminal surface is defined by

$$S_{T_d} = \{ \hat{x} \mid \hat{x}_D(t_{f_1}) = \hat{x}_A(t_{f_1}) \text{ or } \hat{x}_D(t_{f_2}) = \hat{x}_E \}, \quad (46)$$

where, $t_{f_1}$ and $t_{f_2}$ are given by the time when attacker aligns with the defender, or the defender reaches the endpoint, respectively. The endpoint $\hat{x}_E$ is defined
Guarding a Non-Maneuverable Translating Line with an Attached Defender

in (30). Note that \( \hat{x}_D(t_{f_2}) = \hat{x}_E \) is part of the terminal surface since once the defender reaches the desired endpoint, the attacker will no longer be able to reach the target without satisfying (42). Thus the terminal time is

\[
    t_f = \min\{t_{f_1}, t_{f_2}\}. \tag{47}
\]

Let \( x_A^*(t_{f_2}) = [x_A^*(t_{f_2}), y_A^*(t_{f_2})]^T \) denote the point that the attacker will reach at \( t_{f_2} \) following the strategy stated in Theorem 3:

\[
    x_A^*(t_{f_2}) = x_A + v_A \cos \phi^*_A \cdot t_{f_2}. \tag{48}
\]

Also let us define a segment on \( x \) axis bounded by \( x_D \) and \( x_E(t_{f_2}) \) as follows:

\[
    \mathcal{X} := (x_D, x_E(t_{f_2})) \text{ or } (x_E(t_{f_2}), x_D). \tag{49}
\]

Now we can define three strategic regions for the defender-win game based on the location of \( x_A \) and \( x_A^*(t_{f_2}) \) with respect to \( \mathcal{X} \) as follows (also see Fig. 5):

- **\( S_1d \):** \( \hat{x} \in \mathcal{R}_D, x_A \in \mathcal{X} \), and the strategy stated in Theorem 3 is still valid for finite-length target case, given that the following condition holds:

\[
    x_A^*(t_{f_2}) \in \mathcal{X}, \tag{50}
\]

- **\( S_2 \):** \( \hat{x} \in \mathcal{R}_D \) and \( x_A \in \mathcal{X} \), however (50) does not hold.

- **\( S_3 \):** \( \hat{x} \in \mathcal{R}_D \) and \( x_A \notin \mathcal{X} \).

The time it takes for the defender to reach the endpoint is given by

\[
    t_{f_2} = \|\hat{x}_E - \hat{x}_D\| |\omega^*_D| = \|\hat{x}_E - \hat{x}_D\|. \tag{51}
\]

The coordinates of \( x_E(t_{f_2}) \) are given by the following:

\[
    x_E(t_{f_2}) = x_E + v_T \cos \phi_T \cdot t_{f_2},
\]

\[
    y_E(t_{f_2}) = y_E + v_T \sin \phi_T \cdot t_{f_2}. \tag{52}
\]

If (50) holds, then the attacker cannot improve its payoff by unilaterally deviating from the equilibrium strategy stated in Theorem 3, thus the game ends at \( t_{f_1} \) by satisfying (41). On the other hand, if (50) does not hold, and if the attacker uses the strategy in Theorem 3, (41) will not be satisfied because the defender will reach the endpoint before it aligns with the attacker. In this case, the attacker can choose an alternate heading to minimize the distance from the target endpoint at final time.\(^2\) Specifically, the attacker will seek to align with the defender at time \( t_{f_2} \) by deviating least amount from the

\(^2\)The suboptimality of the strategy in Theorem 3 for \( \hat{x}_A \in S_2 \) is illustrated in Fig. 4.
optimal strategy given by Theorem 3. We define this alignment point, \( x_S \), of the attacker and the defender at \( t_{f_2} \) as follows:

\[
x_S(t_{f_2}) = x_E(t_{f_2}),
\]
\[
y_S(t_{f_2}) = \begin{cases} 
\min\{y_1, y_2\}, & \text{if } Y > 0 \text{ and,} \\
\max\{y_1, y_2\}, & \text{if } Y < 0.
\end{cases}
\] (53)

Here \( y_1 \) and \( y_2 \) are given by

\[
y_1 = y_A + \sqrt{r_A^2 - (x_A - x_A(t_{f_2}))^2}, \text{ and}
\]
\[
y_2 = y_A - \sqrt{r_A^2 - (x_A - x_A(t_{f_2}))^2}.
\] (54)

where \( r_A = v_A \cdot t_{f_2} \), is the distance traveled by the attacker by the time defender reach the endpoint \( x_E(t_{f_2}) \), and \([x_A, y_A]^T\) is the attacker’s initial position shown in Fig. 4.

**Theorem 4** (Finite-Length Target) The equilibrium state feedback control strategy for the defender remains the same as stated in Theorem 3, and the equilibrium state feedback control strategy for the attacker is given in Theorem 3 if \( \hat{x} \in S_{1d} \), and otherwise

\[
[\cos \phi_A^*, \sin \phi_A^*] = \begin{cases} 
\frac{x_S(t_{f_2}) - x_A}{\|x_S(t_{f_2}) - x_A\|}, \frac{y_A - y_S(t_{f_2})}{\|y_A - y_S(t_{f_2})\|}, & \text{if } \hat{x} \in S_2, \\
\frac{x_S(t_{f_2}) - x_A}{\|x_S(t_{f_2}) - x_A\|}, \frac{y_A - y_S(t_{f_2})}{\|y_A - y_S(t_{f_2})\|}, & \text{if } \hat{x} \in S_3.
\end{cases}
\] (55)

where \( x_E(t_{f_2}) \) and \( x_S(t_{f_2}) \) are given by (52) and (53), respectively. If \( \hat{x} \in S_2 \cup S_3 \), the Value function is given by

\[
V_d = -\left\{ \left( X + (v_A \cos \phi_A^* - v_T \cos \phi_T - \omega_D^*) \cdot t_{f_2} \right)^2 \right\}
\]
\begin{equation}
+ \left( Y + (v_A \sin \phi^*_A - v_T \sin \phi_T) \cdot t_{f_2} \right)^2 \right)^{1/2}, \quad (56)
\end{equation}
otherwise, it is given in (43).

\textbf{Proof} If $\mathbf{x} \in \mathcal{R}_D$, the attacker cannot reach the target using the strategy in equilibrium. Therefore, the attacker seeks to minimize the distance at the terminal time. If $\mathbf{x} \in \mathcal{S}_2$ (resp. $\mathbf{x} \in \mathcal{S}_3$) the closest point from the target at the final time is given by $\mathbf{x}_E(t_{f_2})$ (resp. $\mathbf{x}_S(t_{f_2})$).

The relative position of the attacker to the defender in $x$ and $y$ direction at $t_{f_2}$ is given by
\begin{align}
X_f &= X - (v_A \cos \phi^*_A - v_T \cos \phi_T - \omega_D) \cdot t_{f_2}; \\
Y_f &= Y + (v_A \sin \phi^*_A - v_T \sin \phi_T) \cdot t_{f_2}.
\end{align}
Thus at the final time, the distance between the players is given by (56).

\section{Game of Kind}

Following our previous analyses in Sec. 3 and Sec. 4, Theorem 1-4 provides the equilibrium strategies and Value function for the game of degree. Figure 5 shows the attacker and defender-win regions along with different strategic regions based on the equilibrium attacker strategy. The terminal surface from the defender’s position segregates the state space into two regions, and the defender’s strategy depends on which side the attacker resides in.

In Fig. 5, the barrier surface for the game of kind is indicated by the black line which divides the state space into defender-win and attacker-win regions. It is composed of two sections: a linear section and a circular section. The linear section is given by
\begin{equation}
\mathcal{S}_{B,\text{linear}} := \{Y = mX\}.
\end{equation}
Fig. 6: Example of attacker-win case, game state at time $t_0$ and $t_f$ with equilibrium strategies.

The circular section is denoted by $S_{B,\text{circular}}$, whose center is at $x_C = [x_c, y_c]^\top$:

\begin{align*}
x_c &= x_E + v_T \cos \phi_T \cdot t_{f_2}, \\
y_c &= y_E + v_T \sin \phi_T \cdot t_{f_2},
\end{align*}

and the radius is $r_c := v_A \cdot t_{f_2}$. The transition between the circular and linear part occurs at critical points where $\hat{x}_B = 0$ or $\hat{x}_B = L$ (recall condition (32)).

6 SIMULATIONS

In this section, attacker-win and defender-win scenarios are illustrated for the following parameters: $v_A = 0.7$, $v_T = 0.2$, $\phi_T = 2\pi/3$ and $L = 1$.\(^3\) For all the following examples, $x_D(t_0) = [0.4, 0.0]$, and the target frame coincides with the inertial frame at $t_0$.

In Fig. 6, $x_A(t_0) = [0.75, 0.25]^\top \in S_0$, and $x_0 \in R_A$. This initial condition gives $V_a = 0.2$. Under the equilibrium strategies on both players, the attacker stays on this level set throughout the game and reaches the endpoint of the target with $J_a = 0.2$.

In Fig. 7, $x_A(t_0) = [0.05, 0.50]^\top \in S_2$, and $x_0 \in R_D$. At final time, $t_f$, attacker reaches the alignment point and defender reaches the endpoint of the target. The Value of the game is, $V_d = -0.167$. The negative Value indicates the defender win case as oppose to positive Value for the attacker win game.

Figure 8 illustrates a scenario in which the attacker loses the game starting from a winning position when it employs a sub-optimal strategy. At time $t_0$, $x_A = [0.85, 0.48]^\top \in S_0$, and $x_0 \in R_A$. In equilibrium, the attacker will seek the endpoint of the target and wins the game. However, in this example attacker employs a sub-optimal strategy (i.e., moves straight toward the target). As a result, the state shifted inside the defender-win region at some time $t_f \geq t > t_0$ and the game ends with capture. The figure also depicts the

\[^3\]The animated version of the simulations can be found online at https://youtu.be/WJUvbmYj3AU.
Fig. 7: Example of defender-win case at initial and final state of the game with equilibrium strategies.

Fig. 8: Example of defender-win case where defender wins against a naive attacker that employs a sub-optimal strategy.

potential Value of the game under equilibrium strategy (+0.036) at $t_0$, and the actual payoff (−0.037) at $t_f$.

7 CONCLUSIONS

In this paper, we address the problem of defending a non-maneuverable translating target. By determining players’ equilibrium strategies and the Value of the game for an infinite-length target, we were able to leverage those results to the original problem with a finite-length target. As a solution to the game of kind, we provide expressions of the barrier surface both in the numerical form and in an analytical form. In addition, we provided examples of defender-win and attacker-win scenarios using optimal strategies, and we examined how unilateral deviation from these strategies would affect the outcome of the game. Future works may include more practical shapes of the targets involving multiple defenders and attackers. Information structure and dynamics of the players can be adapted to fit real-world situations.
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