Classical Aspects of a Distributional 3+1 Foam Model

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Abstract

To understand the fabric of spacetime from a classical perspective we can search for inspiration in Wheeler's layman's analogy of the Quantum foam using an ocean surface observed from an aeroplane at different heights. From a 3+1 spacetime perspective, the ocean surface then represents the hypersurfaces of spacetime at different times. Therefore, the ocean surface observed from a great height, when it appears smooth, homogenous, isotropic and flat, represents an observation of a hypersurface in the current universe. When we get closer and see the waves, then we are moving backwards in time, and when we start to see waves breaking and coarse foam-like structures emerge, then we are observing hypersurfaces at the Planck scale. This structure is called the Quantum foam and to understand it from a classical perspective - would it then not make sense to create a model that mimics Wheeler’s analogy? Here a 3+1 spacetime model that achieves this is introduced. The model has a constant lapse and a shift vector that is the fundamental solution to the linearised wave operator. The existence of a sequence of compactly supported shift vectors that converge to the fundamental solution is used to understand the distributional aspects of the model as the formal limit of a sequence of 3+1 smooth spacetimes. It is shown that there exists a positive integer such that for all elements in the sequence with a greater index value than this integer and for any Eulerian observer the shift vector increases more rapidly than any polynomial and the volume expansion is more rapid than a polynomial in all directions. The same conclusion remains valid for the trace of the extrinsic curvature. Nonetheless, it is shown, no matter how volatile the extrinsic curvature is for these elements there also exists elements in the other end of the sequence where the extrinsic curvature is negligible and the spacetime flat.

Keywords: Quantum foam characteristics, 3+1 distributional spacetime, dynamic shift vector, index scaled lapse
Introduction

The standard cosmology models for the observable universe presented in undergraduate or graduate textbooks on General Relativity or Quantum Fields in Curved Spacetime are homogeneous and isotropic solutions to Einstein’s field equations, see e.g. [1],[2] or [3]. These are models where spacetime is foliated by a one-parameter family of hypersurfaces such that for each instant of time and for any two points on the corresponding hypersurface there exists an isometry of the metric field such that the points are indistinguishable. Moreover, the property that the universe is isotropic is a statement about rotational symmetry in the hypersurfaces for the timelike observers whose tangents necessarily are orthogonal to each hypersurface. The flow of time in any such model is decomposed by a lapse function and a shift vector that are not considered dynamical since the view is that they only provide information about the time evolution and the shift of the space coordinates between neighbouring hypersurfaces in the foliated spacetime.

But is it really possible that the universe remains homogeneous and isotropic as we trace time backwards into the high frequency epoch of the early universe? Is it not more likely that the early universe would be aligned with Wheeler’s Quantum foam and its “wiggly business” [4]?

To understand the meaning of this question, let us use Wheeler’s ocean surface analogy [4]. In this analogy the ocean surface is observed from an aeroplane but at different heights. An interpretation of this, from a purely classical and phenomenological albeit theoretical perspective of a 3+1 model, is that the ocean, when observed from different heights, represents hypersurfaces at different times in the foliated universe. Therefore, the ocean surface observed from a great height when it appears smooth, flat and homogeneous is an observation of the current hypersurfaces in our universe. When we get closer to the ocean and see the waves then we are moving backwards in time, and as we move even further closer and start to see waves breaking and a wiggly and coarse foam like structure, then we are getting closer to the physics at the Planck scale and the beginning of time. It is this foamy and hence bubbly structure that is called the Quantum foam and to understand it - would it then not make sense at least from a speculative aspect to create a basic classical phenomenological model of the foam that mimics this specific interpretation of Wheeler’s analogy?

Before we proceed and in order to avoid any misunderstandings it must here be strongly emphasised that the Quantum foam inherently should be quantum and hence as a field possible an algebra valued distribution if we were the assume the validity of an extension of the Wightman-Gårding axioms into a non-linear context, see e.g. [5] for details about the axioms in Minkowski spacetime. In this paper we will not enter into any further quantitative discussions about the operator or algebra valued distributional aspects of the Quantum foam field.

Let us instead return to the quest of attempting to construct a classical 3+1 spacetime model to mimic the interpretation of the foam. The 3+1 model that will be used is based on the assumption that the shift vector is dynamical but where the lapse is constant and scales the proper time in relation to the coordinate time. The shift vector is taken to be the unique fundamental solution to a linearised and homogeneous wave equation that is zero at the beginning of time but with a particle velocity that equals the Dirac measure. In this way the necessary ‘kinematics’ is provided since a shift vector that is the fundamental solution to the wave equation and hence a distribution should be sufficient to generate a coarse and wiggly foam-like shifting in the hypersurfaces. Furthermore, a basic theorem from distribution theory states that for a given distribution there exists a sequence of compactly supported test functions such that the sequence converges to the given distribution as distributions, see e.g. [6] or [7] for proof of this statement. Thus, a guaranteed existence of a sequence of smooth and compactly supported shift vector fields that converge as distributions to the fundamental solution provide a mechanism to observe the hypersurfaces at different heights (times) simply by introducing a corresponding sequence of 3+1 spacetimes each with a smooth and compactly supported test shift vector from the sequence of smooth and compactly supported shift vector fields with the corresponding constant lapse that is the reciprocal of the index. Thus each class member represents a proper period that for any observer orthogonal to the hypersurfaces equals the coordinate time scaled by the sequence index. Using these elements and distribution theory as a tool we should then be able to build a qualitatively and classic understanding of the different epochs closer and further away from the inherently quantum and algebra valued distributional nature of the Quantum foam.

To be more specific it will be shown that there exists a positive integer such that for all index values and hence elements in the sequence where the index value is greater than this positive integer and for any Eulerian observer in these elements will the shift vector in proper time increase more rapidly than any polynomial. The same conclusion remains valid for the expansion of the volume element for any Eulerian observer, whose worldline is a geodesic since the lapse is the reciprocal of the sequence index. It is also shown that for the same elements, the rate of change of the trace of the extrinsic curvature is more rapid than any polynomial. Nevertheless, it is shown, no matter how volatile
the extrinsic curvature is for these elements there also exists elements in the other end of the spectrum of the indices such that the extrinsic curvature is negligible and the spacetime flat. Thus the class has flat spacetime elements but exhibits inflationary features as the sequence index increases and the sequence converges as distributions to the fundamental solution. Conversely, looking at this from an evolution perspective and hence moving forward in time from the apex of the supporting forward boundary of the light cone there is a period of a rapidly increasing and coarse expansion with a gravitational drag that eventually disappears as the spacetime becomes flat.

Before this section is closed we need to address “the elephant in the room” namely that the shift vector in the class of spacetimes converges to a linear distribution supported in the boundary of the light cone where the apex of the cone at the beginning of time is a point of singular support and hence a linear combination of the Dirac measure and its derivatives. This then implies that the metric field depending on such a shift vector is inconsistent with Einstein’s field equations. Or to put it precisely, the existence of a smooth sequence that converges to the fundamental solution as distributions is restricted to a linear theory but general relativity is inherently a non-linear theory. Therefore, general relativity requires non-linear operations to be well-defined and this is problematic in relation to linear distribution theory since the product of two arbitrary distributions is undefined. To handle this we would perhaps be forced to use a non-linear distribution theory based on e.g. Colombeau algebra [8] or it might be possible that micro local analysis will provide for an alternative path forward. Nonetheless, the limit, that is a measure in the form of a linear combination of the Dirac measure and its derivatives scales to zero in proper time by the sequence index in such a way that it has a zero limit as the index goes to infinity. To understand this, imagine an observer following the flow backwards in time. If such an observer would perform any measurements of the fabric of spacetime and hence necessarily average over the natural volume element due to that any measurement device would have a size. Then in proper time would the observer conclude that the fabric scales towards zero as a consequence of the lapse reciprocal relation to the sequence index. That the classical spacetimes scales to zero as we trace time backwards is necessarily a requirement for an quantum theory of the universe to exist since a universe without any observer would not have any separation between the system and the observer that is performing the measurement, see Haag [5] for more details on measurement and on the Heisenberg cut or separation between the observed system and the observer.

Perhaps and somewhat remarkable, this signifies that the potential issue with a metric field where the shift vector in the limit is a distribution is not that problematic and that we should be able to measure the consequences of the “immeasurable” and eventual algebra valued Quantum foam with a smooth sequence of globally hyperbolic 3+1 spacetimes. This also captures the essence of applied distribution theory. That is, to measure any physical quantity we must necessarily use a class of measurement devices (test functions) where each class member has a well-defined size and hence non-zero support. In consequence, since any measurement device occupies a non-zero volume element it follows that any measurement necessarily is a functional (average) on the space of the test functions over the volume element or a sequence of test functions that converge as distributions to the physical quantity.

In relation to this discussion about the field equations we should also say something about the energy conditions for the sequence. This we can do here immediately since it turns out that Alcubierre’s warp drive [9], or rather that a warp bubble with a constant velocity is a special case of this model if we extend the model to the space that the shift vectors is dense in, that is to shift vectors in the space of rapidly decreasing functions and its dual space of tempered distributions. The warp bubble is then a representation of the model presented here and it has been shown (as expected) that it violates the weak energy condition [10]. Also, the null energy condition is violated [10]. Given this, we conclude that the model will manifest at least the same violations even though the severity will vary due to a lapse function's reciprocal relation to the propagation number.

Returning to the issue that the shift vector usually is assumed to be a non-dynamical quantity, then it should be noted that there is an ongoing change in this perception and there exist a number studies, see [9], [11], [12] for models and a summary of models where the view is that the shift vector is dynamical. Also, I am working on a semi-classical analysis for a massless scalar field on an explicit class of spacetimes with a dynamic shift vector field [13].

Finally, this work is closed with a discussion and a summary of the results.

**Construction of a 3+1 distributional valued foam model**

In this section is the construction of the classical but distributional valued 3+1 model presented. Formally the model is a 3+1 spacetime with a linear shift vector field that is the fundamental solution to a linearised wave equation and hence a distribution on the hypersurfaces such that it is zero at the beginning of time and with a particle velocity that
equals the Dirac measure. To make sense of this distributional valued 3+1 spacetime we use that any given linear distribution can be approximated by a sequence of smooth and compactly supported test functions that converge to the distribution as distributions, see [6] or [7]. Thus, if a vector field is the fundamental solution to a linearised wave equation then the solution is the convergence limit of a sequence of smooth and compactly supported vector fields. The distributional valued spacetime can hence be understood as a formal convergence limit of a sequence of smooth 3+1 spacetimes, each with a metric field that depends on a smooth and compactly supported shift vector field that is a solution to the wave equation. This is the definition that will be applied here. This limit will have zero support in the apex of a light cone and hence it follows from a basic structure theorem, see [6] or theorem 2.3.4 in [7], in distribution theory that the limit necessarily has the form of a linear combination of the Dirac measure and its derivatives. However, if it would be possible for a freely falling Eulerian observer to observe this limit following the flow backward in proper time and perform measurements then the observer should notice that the linear combination of the Dirac measures and its derivatives is scaled to zero by the sequence index and hence that the limit arises almost out of nothing.

It is shown that there exists a positive integer value for the index in the sequence such that for all elements in the sequence with a greater index and for any freely falling Eulerian observer in these elements that the shift vector in proper time increases more rapidly than any polynomial. In consequence the same conclusion remains valid for the volume element expansion for any freely falling Eulerian observer since the lapse that is the reciprocal of the sequence index is constant. It is also shown that for the same elements the rate of change of the trace of the extrinsic curvature is more rapid than any polynomial. Nevertheless, it is shown, no matter how volatile the extrinsic curvature is for these elements, there also exists elements in the other end of the spectrum of the indices such that the extrinsic curvature is negligible and the spacetime flat.

Before the model is constructed let us recall some basic definitions and results from topology, differential geometry and distribution theory that will be needed in the construction. Here the presentation will follow standard texts on general relativity and distribution theory in [1], [2], [3] and [6] and [7].

To start, any manifold \( M \) in a spacetime, that is a pair \((M, g)\) where \( g \) is a metric field on \( M \), is a 4-dimensional Hausdorff topological space with a \( C^\infty \) structure. This means that \( M \) is covered by countable basis \( \{O_i\} \) and that for each point \( p \in M \) in the countable basis there exists a neighbourhood of \( p \) that is homeomorphic to an open subset in \( \mathbb{R}^4 \). That is, for each open subset \( O_i \) in the cover there exists a homeomorphism

\[
\kappa_i: O_i \rightarrow U_i \subset \mathbb{R}^4
\]

where \( U_i \) is an open subset in \( \mathbb{R}^4 \). Moreover, if the intersection of any two homeomorphisms \( \kappa_i \) and \( \kappa_j \) is not empty then the transformation between \( \kappa_i \) and \( \kappa_j \) is homeomorphic and infinitely and continuously differentiable. The homeomorphism is interchangeable in the literature called a chart or local coordinate system, see e.g., [2], [7]. Here the homeomorphism will be called a local coordinate system.

The 3+1 spacetime model that will be used throughout the article is a globally hyperbolic spacetime that hence can be sliced or foliated by a one-parameter family of hypersurfaces. Recall that a hypersurface in a manifold \( M \) is the image \( \Sigma = \phi(\hat{\Sigma}) \) of a homeomorphic embedding \( \phi: \hat{\Sigma} \rightarrow M \) where \( \hat{\Sigma} \) is a 3-dimensional manifold, see e.g. [1], [2] or [12].

Thus the topology of the manifold \( M \) is \( \mathbb{R} \times \Sigma \) where \( \Sigma \) is any Cauchy surface. In more details, since a globally hyperbolic spacetime is causally stable, meaning that no closed time-like curves exists, there exist a regular function \( f: M \rightarrow \mathbb{R} \) such that each level surface of constant \( f \) is a Cauchy surface:

\[
\Sigma_t = \{ p \in M : f(p) = t \} .
\]

This generates a one-parameter family of Cauchy surfaces as time \( t \) goes on. Therefore,

\[
M = \bigcup_{t \in \mathbb{R}} \Sigma_t .
\]

This notation will be abused and for the union we will instead write

\[
\Sigma_t \equiv \bigcup_{t \in \mathbb{R}} \Sigma_t
\]

whenever it is needed.

Now let us endow the manifold \( M \) with a structure of distributions by making use of the local coordinate systems and convolution as follows. For each coordinate system \( \kappa_i: O_i \rightarrow U_i \) let \( C^\infty(U_i) \) be the set of infinitely differentiable compactly supported test functions compactly supported on \( U_i \). This space is denoted \( \mathcal{D}(U_i) \) going forward. If for each such coordinate system \( \kappa_i \) we have a local distribution \( u_{\kappa_i} \in \mathcal{D}'(U_i) \) such that

\[
u_{\kappa_j} = (\kappa_i \circ \kappa_j^{-1})^* u_{\kappa_i}
\]

then we say that \( u_{\kappa_i} \) is a distribution \( u_i \) in \( O_i \subset M \). The set of all such distributions is denoted \( \mathcal{D}'(M) \).
Next, we will make use of the notion of cut-off test functions from distribution theory as they are defined implicitly in theorem 1 below since they are used to construct sequences of test functions that converge to a given distribution as distributions, theorem 2 below.

**Theorem 1** If $U$ is in an open set in $\mathbb{R}^n$ and $K$ is a compact subset of $U$ then there exists a compactly supported positive cut-off test function $\phi$ in $U$ such that $0 \leq \phi \leq 1$ and where $\phi = 1$ in a neighbourhood of $K$. For a proof see reference [6] or theorem 1.4.1 in reference [7].

The following theorem will turn out to be crucial in the construction of the model:

**Theorem 2** For any distribution $u \in \mathcal{D}'(U)$, there exists a sequence $(\phi_k)_{k \in \mathbb{N}}$ of test functions in $\mathcal{D}(U)$ such that $\phi_k \to u$ as distributions. See theorem 2.2.1 in reference [6] or 4.1.5 in reference [7] for a proof.

This ends the presentation of the preliminaries needed for the rest of the section and we are now ready to start the construction of a classical 3+1 foam model with a dynamical shift vector field and will begin with the notion of a distributional valued foam shift vector field. The main results are theorem 4-6 and corollary 1.

**Definition 1** Foam shift vector field $\beta$

Let $\beta$ be shift vector on any hypersurface $\Sigma_t$ with components $\beta^i$ where $i = 1, 2, 3$ with $\beta^i \in \mathcal{D}'(\Sigma_t)$ for each $i$ and such that each $\beta^i$ is the fundamental solution to the linearised wave equation on $\Sigma_t$. That is,

$$\Delta \beta - \frac{1}{N^2} \partial_t^2 \beta = 0 \quad (7)$$

with the initial conditions $\beta(x,0) = 0$ and $\partial_t \beta(x,t)|_{t=0} = \delta(x), x \in \Sigma_t$ and where $N$ is a constant lapse. Then $\beta$ is said to be a foam shift vector field.

**Remark 1:**

In local Cartesian coordinates is (7) expressed as

$$\frac{\partial^2 \beta^i}{\partial x^2} + \frac{\partial^2 \beta^i}{\partial y^2} + \frac{\partial^2 \beta^i}{\partial z^2} = \frac{1}{N^2} \frac{\partial^2 \beta^i}{\partial t^2}, i = x, y, z. \quad (8)$$

Furthermore and to reinforce, with the fundamental solution to the linearised wave equation we mean a unique distribution $\beta$ that is parametrised by time such that it is solution to the linearised wave equation (7). The existence and uniqueness of the solution and that its support is in the boundary of the forward (backward) light cone is guaranteed by theorem 6.2.3 in reference [7].

The linearisation, that is the linear part of the wave equation with constant lapse in a foliated 3+1 spacetime is clearly an approximation but required to avoid any problems having products of distributions that are not well-defined.

**Example 1** Opposite moving Heaviside disturbances

In one dimension we have that a fundamental solution is

$$\beta(t,x) = \frac{1}{2} \left( H(x + t) - H(x - t) \right) \quad (9)$$

where $H$ is the Heaviside function.

Clearly,

$$\beta(0,x) = 0 \quad (10)$$

and since the derivative of the Heaviside function is a Dirac measure

$$\partial_t \beta(t,x)|_{t=0} = \delta(x). \quad (11)$$

Given the foam shift vector field then there exist a smooth and a compactly supported sequence of shift vector fields that converge as distributions to the shift vector in definition 1, see theorem 2. This sequence will now be introduced.

**Definition 2** Smooth foam shift vector field $\beta_k$

Let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of compactly supported shift vector fields $\beta_k \in \mathcal{D}(\mathbb{R})$ such that each $\beta_k = (\beta_k^1, \beta_k^2, \beta_k^3)$ is a smooth solution to the wave equation (7),

$$\Delta \beta_k - \frac{1}{N_k^2} \partial_t^2 \beta_k = 0. \quad (12)$$

If the sequence converge as distributions to the Quantum foam vector field $\beta$ in definition 1 then it is said to be a sequence of foam shift vector fields and an element $\beta_k$ is said to be a smooth Quantum foam shift vector field with a propagation index $k$ and an associated constant lapse $N_k$.

**Remark 1:**

The index $k$ is the sequence index and is not an index that is summed over if it occurs twice.

**Example 2** The logistic representation

In 1+1 dimension we have that one representation of a Quantum foam shift vector field sequence can be constructed from a sigmoid with suitable support:

$$\beta_k(x,t) = \frac{1}{2} \left( \tanh(v_k(x + t)) - \tanh(v_k(x - t)) \right), \quad (13)$$

$$v_k = ky \quad (14)$$

where $v_k$ is the indexed frequency and $v$ is said to be the background frequency.

**Remark 2:**

Here it is worthwhile to notice the resemblance of the shift vector (13) with Alcubierre’s warp drive shift vector [9] when the velocity is constant.

Furthermore, to get some insight into the indexed frequency, $v_k$ (14),

$$v_k = kv,$$
we introduce a regular function $K_\Sigma: \Sigma_\tau \rightarrow \mathbb{R}$. Then a local level surface for each hypersurface $\Sigma_\tau$ can be defined using the map

$$K_\Sigma = K_\tau \circ k^{-1}.$$  

Hence, in local coordinates we have

$$K_\Sigma(t, x, y, z) = c_t$$

where $c_t$ is the height at a given global time $t$. Consequently we have the local level surface

$$loc(\Sigma_\tau) = K^{-1}_\Sigma(c_t).$$

To continue for a freely falling observer whose worldline is orthogonal to the hypersurfaces then the proper period is $dt = dt/k$ when the shift has completed a period and hence when then the height $c_t$ has decreased with one unit. As a plane wave and linear approximation we then can define the indexed frequency by

$$v_k = \partial_t K_\Sigma = \frac{k}{dt} = kv$$

where

$$v = \frac{1}{dt}$$

is said to be the background frequency.

Next, the class of smooth Quantum foam spacetimes is given by definition 3.

**Definition 3** Class of smooth foam models

Let $(\beta_k)_{k \in \mathbb{N}}$ be a sequence of foam shift vector fields as in definition 2 and $(g_{\beta_k})_{k \in \mathbb{N}}$ a sequence of metric fields such that

$$g_{\beta_k} = -N_k^{-2}dt^2 + \eta_{ij}(dx^i + \beta_k^i dt)(dx^j + \beta_k^j dt)$$

where

$$\eta = diag(1, 1, 1)$$

and

$$N_k = \frac{1}{k}k^2 = k_1^2 + k_2^2 + k_3^2, k_1, k_2, k_3 \in \mathbb{N}$$

where $N_k$ is a constant lapse function. Then the sequence

$$\left(\left(\Sigma_{\tau/k}, g_{\beta_k}\right)\right)_{k \in \mathbb{N}}$$

of smooth 3+1 spacetimes is said to be a class of smooth and classical foam models and an element $(\Sigma_{\tau/k}, g_{\beta_k})$ is said to be an classical foam model with a propagation index $k$.

**Example 3** A 1+1 toy Heaviside Universe

In 1+1 dimension we can use the sigmoid sequence in example 2 to construct a foam spacetime with an early inflation period but that becomes flat as time goes on:

$$g_{\beta_k} = -\frac{1}{k^2}dt^2 + (dx + \beta_k^l dt)(dx + \beta_k^l dt)$$

where from (13)

$$\beta_k^l(x, t) = \frac{1}{2}(\tanh(v_k(x + t)) - \tanh(v_k(x - t)))$$

$$v_k = kv, k \in \mathbb{N}.$$  

It is straightforward to extend this example to more realistic 3+1 spacetime with a bump surface that is a compactly supported test function. Also, an illustration of a bumpy shift vector that gives an idea of a single bubble or foam like structure of the foam is provided in figure 1:

*Figure 1. A classical Quantum foam illustration with a bumpy and bubbly shift vector element.*

We are now ready for the definition of a classical 3+1 Quantum foam at distances less than the Planck scale and hence to provide meaning to the notion of the initial hypersurface.

**Definition 4** Planck scale 3+1 Quantum foam spacetime

Given the class of Quantum foam spacetimes, then we define an initial Planck scale Quantum foam spacetime by the formal limit

$$\left(\Sigma_{0, g_{\beta}}\right) = \lim_{k \to \infty} \left(\Sigma_{\tau/k}, g_{\beta_k}\right).$$  

**Remark 3:**

It is important to notice that there do exist many possible classes of smooth spacetimes that converge to this initial hypersurface and metric field. In terms of measurement this is a statement of the existence of different measurement devices that can be used to measure the physical quantity. Nevertheless, observations in our current universe should uniquely determine a class of spacetimes that fits the observations.

**Remark 4:**

In definition 4 the limit is as distributions for the shift vector and hence a distribution with zero support and it then follows from a structure theorem in distribution theory, see reference [6] or theorem 2.3.4 in reference [7], that the distribution necessarily is a linear combination of the Dirac measure and its distributional derivatives. However, the natural volume form will introduce a scaling to zero of this limit and hence an observer tracing time backwards would see a fabric of spacetime that arises out of almost nothing.

It will now be proved that the sequence has elements when the spacetime expands more rapidly than any polynomial.

**Theorem 4** There exist a positive integer $k_0$ in the class of classical foam spacetimes in definition 3 such that for all
elements with an index value \( k > k_0 \) will the shift vector field in proper time increase more rapidly than any polynomial for any observer with a relative velocity \( \beta_k^i \) to the lines of constant space coordinates that for any \( k \) is given by
\[
\frac{dx^i}{dt} = -\beta_k^i, \ i = 1,2,3 \tag{28}
\]

**Proof**

Using the metric field (20) it follows that the proper time period for an observer with the relative velocity \( \beta_k^i \) to the lines of constant space coordinates is
\[
d\tau = \frac{dt}{k}. \tag{29}
\]

The shift vector \( \beta_k \) is constructed from cut-off functions and converge as distributions to a linear distribution, it then follows that \( \text{supp}(\beta_k) \rightarrow \{0\} \) as \( k \rightarrow \infty \) but since \( \beta_k \) is a smooth and compactly supported test function, that is \( \beta_k \in C^\infty_0(\Sigma_{t/k}) \) it follows that there exist a positive integer \( k_0 \in \mathbb{N} \) and hence a proper period \( d\tau_k = dt/k_0 \) such that for all \( k > k_0 \) and hence all periods \( d\tau < d\tau_k \) is \( \beta_k \) increasing from 0 to 1 more rapidly than any polynomial \( \Box \).

This statement can be made more comprehensible by considering the volume element expansion for the observer in theorem 4 and hence any Eulerian observer that follows the flow in the cosmology in the very same proper period:

**Theorem 5** The volume element expansion for any freely falling Eulerian observer in the spacetimes in theorem 4 is more rapid than any polynomial.

**Proof**

We can without any loss of generality consider an observer moving with a 4-velocity \( U_k \) given by
\[
U_k^a = n_k^a \tag{30}
\]
where \( n_k \) is defined as the lapse scaled gradient of the regular function (2) that is
\[
n_{k\alpha} = -N_k \nabla_\alpha t = (-N_k, 0, 0, 0) \tag{31}
\]
and hence \( n_{k\alpha} \) is normal to any hypersurface \( \Sigma_{t/k} \). Consequently, we have
\[
n_k^a = \left( \frac{1}{N_k}, -\frac{\beta_1^1}{N_k}, -\frac{\beta_2^2}{N_k}, -\frac{\beta_3^3}{N_k} \right) \tag{32}
\]
and therefore the worldline of the timelike observer is orthogonal to any hypersurface. Moreover, the observer is falling freely, and hence following the flow of evolution of the spacetime, since the acceleration \( A_k \) is zero:
\[
A_k^a = U_k^c \nabla_c U_k^a = g^{ab} U_k^c \nabla_c (-N_k \nabla_\alpha t) = g^{ab} U_k^c \frac{1}{2} g^{cd} \nabla_d (U_k^c U_k^d) = 0. \tag{33}
\]
The lapse of proper time \( d\tau \) for any two neighbouring events for the observer can as we have already seen be determined using the metric (20) and that the relative velocity \( \beta^i \) for the Eulerian observer and the lines of constant space coordinates for any \( k \) is given by
\[
\frac{dx^i}{dt} = -\beta_k^i, \ i = 1,2,3 \tag{34}
\]
we then get that
\[
d\tau = N_k dt = \frac{dt}{k}. \tag{35}
\]
Thus, the proper time for any Eulerian observer equals the coordinate time scaled by the constant propagation index. Here we remark that this relation between the proper time and the coordinate time has already been used in the discussion of the indexed frequency (14) without any detailed motivation except in an implicit assumption of a freely falling observer.

The volume expansion \( \theta \) for the observer, that is a scalar, is given by
\[
\theta = \nabla_a U_k^a = -\frac{1}{N_k} \partial_t \beta_1^k. \tag{36}
\]
Here and again the last equality follows from that the lapse is constant. Notice also that the expansion is a linear sum of partial derivatives of the shift vector field.

Finally, using the same arguments as in theorem 4 for (36) and hence on the distributional partial derivative and for each direction of the shift vector proves the theorem \( \Box \).

Notice that that there exist proper periods when the rate of change of the trace of the extrinsic curvature \( K_k \) is faster than any polynomial. This follows from theorem 5 since
\[
\text{Tr}(K_k) = -\frac{1}{N_k} \nabla_a U_k^a. \tag{37}
\]
Thus we have the following corollary:

**Corollary 1** There rate of change of the trace of the extrinsic curvature \( K_k \) is more rapid than any polynomial for all proper periods \( d\tau < d\tau_k \).

In relation to the freely falling Eulerian observers we need check that the shift vector is consistent in the sense that the period of faster than polynomial expansion is coherent with the existence of proper periods when the relative speed locally between a coordinate observer that remains at fixed space coordinates and the Eulerian observer (that is shifting in relation to the fixed space observer and following the flow as time goes on) should be faster than the speed of light. That is, that the spacetimes exhibits a coordinate observer horizon effect. Thus, using the metric (20) and that the coordinate observer remains at the same space coordinates coordinates, then it follows that the coordinate observer has the 4-velocity:
\[
V_k^a = \left( \frac{1}{\sqrt{N_k^2 - \beta_k^2}}, 0, 0, 0 \right). \tag{38}
\]
The relative and local speed $u$ can now be determined from
\[ \gamma(u) = -U_{k\alpha}V^\alpha_k = \frac{1}{1 - \frac{\beta_k^2}{N_k^2}} \]  
where $\gamma$ is the Lorentz factor and where the negative sign is due to the metric signature. If the relative speed $u$ locally is greater than the speed of light then from (39) it puts the following constraint on the length of the shift vector
\[ \beta_k > \frac{1}{k}. \]  

We have seen (theorem 4) that the shift vector satisfies this and in fact it will remain true for all elements until the shift decrease from its plateau and hence the spacetime elements flattens and any drag effect comes to halt as we will see that it does in theorem 7. Thus the shift vector is consistent with a coordinate observer horizon effect.

**Remark 5:**
At this point it is worthwhile to mention that the results from Alcubierre’s warp drive [9] (when the bubble has a constant speed) can be reproduced by this model and one can therefore argue that the bubble in [9] is simply a manifestation of a foam bubble. In fact any warp drive model that is constructed from compactly supported test functions that is a solution to the wave equation (7) is a representation of the model presented here. Furthermore, the compactly supported test functions is dense in the space of rapidly decreasing test functions. The dual to this space is the space of tempered distributions. Thus, by closing the space and hence by including the limit points we include the Alcubierre warp bubble [9]. However, this comes with a price since tempered distributions are somewhat “nicer and less rough” than the distributions on the space of compactly supported test functions, for more details on tempered distributions see [6] or [7].

It will now be proved that there exists elements in the sequence where each spacetime is homogeneous, isotropic and flat in the low wavenumber limit.

**Theorem 6** The class of spacetimes is homogeneous, isotropic and flat in the propagation index limit $k_i \rightarrow 1, i = 1,2,3$.

**Proof**
This follows immediately from that the shift vector field
\[ \beta_k \rightarrow 0 \text{ as } k_i \rightarrow 1, i = 1,2,3 \]  
and hence that
\[ g_{\beta_k} = -\frac{1}{k^2} dt^2 + \eta_{ij}(dx^i + \beta_k^i dt)(dx^j + \beta_k^j dt) \rightarrow \]
\[ g = -\frac{1}{3} dt^2 + \eta_{ij} dx^i dx^j. \]  

**Remark 6:**
A consequence of this is that independently of whatever volatility that is exhibited in the extrinsic curvature of the spacetime and in a neighbourhood of the Planck region, it follows that the volatility will average out as time goes on.

**Discussion**
In this final section I will summarise the developed classical and phenomenological framework that has been introduced. Finally, I would also like to point out future directions of research.

This framework has been inspired from an interview with the late Professor Wheeler [4] and his use of the ocean surface analogy to describe the Quantum foam. A description I have interpreted from the perspective of a foliated universe with the ocean surface as the hypersurfaces. I then concluded that it would be reasonable to assume that the shift vector, tangent to the hypersurfaces, should not be an unchanging and static quantity but rather a dynamic entity with wave-like properties in relation to the fixed space coordinates and not necessarily smooth. If this is true then assuming that the shift vector is the fundamental solution to the linear part of the wave equation in a $3+1$ spacetime is reasonable since not only do we allow for wiggliness and coarse hypersurfaces but we also maintain time as a parameter. In this study I have avoided working directly with this fundamental solution since it is a distribution with support in the boundary of the light cone and with singular support at the apex at the beginning of time. Instead I have used the existence of a sequence of well-behaved smooth test functions that converge to the solution as distributions to define a class of smooth spacetimes. This means that each spacetime in the class is a globally hyperbolic spacetime, but it could well be that these spacetimes will stack up and generate a Cauchy horizon as we move closer to the beginning of time. But it is not clear if this really is a problem given the distributional limit and hence that the Planck scale Quantum foam spacetime will be an almost measure zero set since the natural volume element scales the measure to zero. Thus the sequence of smooth spacetimes should be a valid way to model the evolution of spacetime. To continue, the framework that has been introduced here, to create a classical and hence phenomenological model of a Quantum foam that inherently is quantum and hence a field that likely should be an algebra valued distribution if the Wightman and Gårding axioms extends to the fabric of spacetime is not only based Wheeler’s description but also on a view that the correspondence limit of the distributional Quantum nature of the foam necessarily should be represented by a sequence of globally hyperbolic spacetimes. This representation by
causally stable spacetimes with no closed time like curves, that can be foliated by the hyper surfaces of a regular function with a height that is the time, see e.g. [2] for a proof that globally hyperbolic spacetimes admits a foliation, strongly suggests that the model has more value than just being a toy model for my interpretation of the Quantum foam and that has been used to understand how any Eulerian observer following the flow of the spacetime would experience an almost arbitrarily rough and wiggly shift of the relative velocity towards any fixed coordinate observer. Or to express it slightly differently and hence from another aspect:

– if the shift vector is the fundamental solution to the wave equation then there is a wide class of causally reasonable solutions to Einstein’s field equations that has a natural inflation mechanism and a mechanism such that the spacetime eventually becomes flat without any drag effects and the question is if we should judge them all as toy models for a classic interpretation of the distributional aspects of the Quantum foam?

Nonetheless, this work has shown that it is possible to construct a self-consistent classical and phenomenological model for a conjectured hidden and in its nature operator or algebra valued distribution that represents the quantum and Planck scale foam using a metric field that is wiggly in the sense that the linear vector shift field is the fundamental solution to the linearised wave equation but where linear distribution theory allows us to use a sequence of smooth and globally hyperbolic spacetime to gain some understanding of the foam. Specifically, it has been shown that there exists a positive integer in the sequence such that for all elements in the sequence with a greater index value than this integer and for any freely falling Eulerian observer in these elements that the shift vector in proper time increases more rapidly than any polynomial. Also it has been shown that the same conclusion remains valid for the volume element expansion for any freely falling Eulerian observer since the lapse that is the reciprocal of the sequence index is constant. Furthermore, it has been shown that for the same elements the rate of change of the trace of the extrinsic curvature is more rapid than any polynomial. To continue it has been proved that no matter how volatile the extrinsic curvature is for these elements, there also exists elements in the other end of the spectrum of the indices such that the extrinsic curvature is negligible and the spacetime flat.

Finally, I am working on the analysis for a quantum field theory for a massless scalar field on an explicit 3+1 model where the fundamental solution is a time-scaled Dirac measure. The motivation behind that study is to understand the effects the volatility has on the scalar field in relation to regions that would be causally disconnected if it were not for the inflation mechanism inherent in the Quantum foam cosmology [13].

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