SOME REMARKS ON FREE ARRANGEMENTS

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Abstract. We exhibit a particular free subarrangement of a certain restriction of the Weyl arrangement of type $E_7$ and use it to give an affirmative answer to a recent conjecture by T. Abe on the nature of additionally free and stair-free arrangements.

1. Introduction

The interplay between algebraic and combinatorial structures of hyperplane arrangements has been a driving force in the study of the subject for a long time. At the very heart of these investigations lies Terao’s Conjecture 1.1 which asserts that the algebraic property of freeness of an arrangement is determined by purely combinatorial data.

Conjecture 1.1 ([OT92, Conj. 4.138]). For a fixed field, freeness of the arrangement $\mathcal{A}$ only depends on its lattice $L(\mathcal{A})$, i.e. is combinatorial.

In his recent papers [Abe17] and [Abe18], T. Abe shows that all free arrangements that obey Terao’s Addition-Deletion Theorem 2.3 are indeed combinatorial. In [Abe18], he introduced a new class of free arrangements, so called stair-free arrangements $\mathcal{SF}$ (Definition 2.9). Its significance lies in the fact that Terao’s Conjecture 1.1 is still valid within $\mathcal{SF}$ ([Abe18, Thm. 4.3]). To date this is the largest known class of free arrangements with this property. This class encompasses the class of divisionally free arrangements $\mathcal{DF}$ (Definition 2.7) and the class of additionally free arrangements $\mathcal{AF}$ (Definition 2.8), [Abe18, Thm. 4.3].

The class of divisionally free arrangements $\mathcal{DF}$ in turn contains the class of inductively free arrangements $\mathcal{IF}$ (Definition 2.5), cf. [Abe16, Thm. 1.6]. The following confirms a conjecture of Abe, [Abe18, Conj. 4.4], which resolves the containment relations among these classes of free arrangements.

Theorem 1.2. With the notation from above, we have

(i) $\mathcal{IF} \subsetneq \mathcal{AF}$;
(ii) $\mathcal{DF} \not\supsetneq \mathcal{AF}$;
(iii) $\mathcal{DF} \cup \mathcal{AF} \subsetneq \mathcal{SF}$.

In §3 we exhibit a subarrangement $\mathcal{D}$ of the rank 5 restriction of type $(E_7, A_1^2)$ of the Weyl arrangement of type $E_7$ which is additionally free but not inductively free and which at the

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same time is not divisionally free; so parts (i) and (ii) of Theorem 1.2 follow. It turns out that \( \mathcal{D} \) is the restriction of a subarrangement \( \mathcal{B} \) of the Weyl arrangement of type \( E_7 \) which also shares these features. These two arrangements are the only instances known to us with these properties. Each of \( \mathcal{B} \) and \( \mathcal{D} \) is obtained from an inductively free arrangement by deleting a single hyperplane. Moreover, \( \mathcal{D} \) is also crucially involved in our construction of an example in \( \mathcal{SF} \setminus (\mathcal{DF} \cup \mathcal{AF}) \) which gives part (iii) of Theorem 1.2. It is quite remarkable that a particular subarrangement of a restriction of the Weyl arrangement of type \( E_7 \) provides the basis for all the statements in Theorem 1.2.

Free arrangements are compatible with the product construction for arrangements, \([\text{OT92}, \text{Prop. 4.28}]\). It is easy to show that this is also the case for Abe’s new classes \( \mathcal{AF} \) and \( \mathcal{SF} \), see Proposition 2.11.

In addition, we show that \( \mathcal{AF} \) is not closed under taking restrictions, see \( \S 3.3 \). In turn \( \mathcal{DF} \) is not closed under taking localizations, see \([\text{Rö18}, \text{Ex. 2.16}]\). Consequently, the larger class \( \mathcal{SF} \) is not closed under these operations either.

In our final section we address another conjecture of Abe, \([\text{Abe18}, \text{Conj. 3.5(2)}]\), which states that if the characteristic polynomials \( \chi(\mathcal{A}, t) \) and \( \chi(\mathcal{A}', t) \) of \( \mathcal{A} \) and a deletion \( \mathcal{A}' \) of \( \mathcal{A} \) factor over \( \mathbb{Z} \) and share all but one root, then both \( \mathcal{A} \) and \( \mathcal{A}' \) are free. While this is true in dimension at most 3, thanks to \([\text{Abe16}, \text{Thm. 1.1}]\), in Example 4.1, we give a counterexample to this conjecture in dimension 4. Specifically, we present a triple of arrangements \( (\mathcal{A}, \mathcal{A}', \mathcal{A}'') \) with the property that none of its members is free but each of their characteristic polynomials factors over \( \mathbb{Z} \) and \( \chi(\mathcal{A}'', t) \) divides both \( \chi(\mathcal{A}', t) \) and \( \chi(\mathcal{A}, t) \). We end with a general construction for examples of this kind in Example 4.2.

### 2. Preliminaries

#### 2.1. Hyperplane Arrangements

Let \( \mathbb{K} \) be a field and let \( V = \mathbb{K}^\ell \). By a hyperplane arrangement in \( V \) we mean a finite set \( \mathcal{A} \) of hyperplanes in \( V \). Such an arrangement is denoted \( (\mathcal{A}, V) \) or simply \( \mathcal{A} \). If \( \dim V = \ell \) we call \( \mathcal{A} \) an \( \ell \)-arrangement. The number of elements in \( \mathcal{A} \) is given by \( |\mathcal{A}| \). The empty \( \ell \)-arrangement is denoted by \( \Phi_\ell \).

By \( L(\mathcal{A}) \) we denote the set of all nonempty intersections of elements of \( \mathcal{A} \), \([\text{OT92}, \text{Def. 1.12}]\). For \( X \in L(\mathcal{A}) \), we have two associated arrangements, firstly the subarrangement \( \mathcal{A}_X := \{ H \in \mathcal{A} \mid X \subseteq H \} \subseteq \mathcal{A} \) of \( \mathcal{A} \) and secondly, the restriction of \( \mathcal{A} \) to \( X \), \( (\mathcal{A}_X, X) \), where \( \mathcal{A}_X := \{ X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X \} \), \([\text{OT92}, \text{Def. 1.13}]\). Note that \( V \) belongs to \( L(\mathcal{A}) \) as the intersection of the empty collection of hyperplanes and \( \mathcal{A}^V = \mathcal{A} \).

If \( 0 \in H \) for each \( H \) in \( \mathcal{A} \), then \( \mathcal{A} \) is called central. We only consider central arrangements.

Let \( H \in \mathcal{A} \) (for \( \mathcal{A} \neq \Phi_\ell \)) and define \( \mathcal{A}' := \mathcal{A} \setminus \{ H \} \), and \( \mathcal{A}'' := \mathcal{A}^H \). Then \( (\mathcal{A}, \mathcal{A}', \mathcal{A}'') \) is a triple of arrangements, \([\text{OT92}, \text{Def. 1.14}]\).

The product \( \mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2) \) of two arrangements \( (\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2) \) is defined by

\[
\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2 = \{ H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1 \} \cup \{ V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2 \},
\]

see \([\text{OT92}, \text{Def. 2.13}]\). In particular, \( |\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| \). For \( H = H_1 \oplus V_2 \in \mathcal{A} \), we have

\[
\mathcal{A}^H = \mathcal{A}_1^{H_1} \times \mathcal{A}_2.
\]
The characteristic polynomial $\chi(\mathcal{A}, t) \in \mathbb{Z}[t]$ of $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X)t^{\dim X},$$

where $\mu$ is the Möbius function of $L(\mathcal{A})$, see [OT92, Def. 2.52].

If $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a product, then, thanks to [OT92, Lem. 2.50],

$$(2.2) \quad \chi(\mathcal{A}, t) = \chi(\mathcal{A}_1, t) \cdot \chi(\mathcal{A}_2, t).$$

2.2. Free Arrangements. Let $S = S(V^*)$ be the symmetric algebra of the dual space $V^*$ of $V$. If $\mathcal{A}$ is an arrangement in $V$, then for every $H \in \mathcal{A}$ we may fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. We call $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$ the defining polynomial of $\mathcal{A}$. The module of $\mathcal{A}$-derivations is the $S$-submodule of $\text{Der}(S)$, the $S$-module of $\mathbb{K}$-derivations of $S$, defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$ 

The arrangement $\mathcal{A}$ is said to be free if $D(\mathcal{A})$ is a free $S$-module.

If $\mathcal{A}$ is a free $\ell$-arrangement, then $D(\mathcal{A})$ admits an $S$-basis of $\ell$ homogeneous derivations $\theta_1, \ldots, \theta_\ell$, by [OT92, Prop. 4.18]. While such a homogeneous $S$-basis of $D(\mathcal{A})$ need not be unique, the multiset consisting of the polynomial degrees of the $\theta_i$ is unique. They are called the exponents of the free arrangement $\mathcal{A}$ and are denoted by $\text{exp } \mathcal{A} := \{\text{pdeg } \theta_1, \ldots, \text{pdeg } \theta_\ell\}$.

Terao’s basic Addition-Deletion Theorem plays a key role in the study of free arrangements.

**Theorem 2.3** ([Ter80], [OT92, Thm. 4.51]). Suppose $\mathcal{A} \neq \Phi_\ell$ and let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Then any two of the following statements imply the third:

(i) $\mathcal{A}$ is free with $\text{exp } \mathcal{A} = \{b_1, \ldots, b_\ell-1, b_\ell\}$;

(ii) $\mathcal{A}'$ is free with $\text{exp } \mathcal{A}' = \{b_1, \ldots, b_\ell-1, b_\ell - 1\}$;

(iii) $\mathcal{A}''$ is free with $\text{exp } \mathcal{A}'' = \{b_1, \ldots, b_{\ell-1}\}$.

The following is Terao’s celebrated Factorization Theorem for free arrangements.

**Theorem 2.4** ([OT92, Thm. 4.137]). If $\mathcal{A}$ is free with $\text{exp } \mathcal{A} = \{b_1, \ldots, b_\ell\}$, then

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - b_i).$$

2.3. Inductively Free Arrangements. An iterative application of the addition part of Theorem 2.3 leads to the class of inductively free arrangements.

**Definition 2.5** ([OT92, Def. 4.53]). The class $\mathcal{IF}$ of inductively free arrangements is the smallest class of arrangements subject to

(i) $\Phi_\ell \in \mathcal{IF}$ for each $\ell \geq 0$;

(ii) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that both $\mathcal{A}'$ and $\mathcal{A}''$ belong to $\mathcal{IF}$, and $\text{exp } \mathcal{A}'' \subseteq \text{exp } \mathcal{A}'$, then $\mathcal{A}$ also belongs to $\mathcal{IF}$.

We denote by $\mathcal{IF}_\ell$ the subclass of $\ell$-arrangements in $\mathcal{IF}$. 
2.4. Divisionally Free Arrangements. First we recall the key result from [Abe16].

**Theorem 2.6** ([Abe16, Thm. 1.1]). Let $\mathcal{A} \neq \Phi_\ell$. Suppose there is a hyperplane $H$ in $\mathcal{A}$ such that the restriction $\mathcal{A}^H$ is free and that $\chi(\mathcal{A}^H, t)$ divides $\chi(\mathcal{A}, t)$. Then $\mathcal{A}$ is free.

Theorem 2.6 can be viewed as a strengthening of the addition part of Theorem 2.3. An iterative application leads to the class $\mathcal{DF}$.

**Definition 2.7** ([Abe16, Def. 1.5]). An arrangement $\mathcal{A}$ is called divisionally free if either $\ell \leq 2$, $\mathcal{A} = \Phi_\ell$, or else there is a flag of subspaces $X_i$ of rank $i$ in $L(\mathcal{A})$

$$X_0 = V \supset X_1 \supset X_2 \supset \cdots \supset X_{\ell-2}$$

so that $\chi(\mathcal{A}^{X_i}, t)$ divides $\chi(\mathcal{A}^{X_{i-1}}, t)$, for $i = 1, \ldots, \ell - 2$. Denote this class by $\mathcal{DF}$.

We denote by $\mathcal{DF}_\ell$ the subclass of $\ell$-arrangements in $\mathcal{DF}$.

In [Abe16, Thms. 1.3 and 1.6], Abe observed that $\mathcal{IF} \subset \mathcal{DF}$ (the reflection arrangement of the complex reflection group $G_{31}$ is divisionally free but not inductively free), each $\mathcal{A}$ in $\mathcal{DF}$ is free and Terao’s Conjecture 1.1 is valid in $\mathcal{DF}$.

2.5. Additionally Free and Stair-Free Arrangements. Using the addition part of Theorem 2.3, it is natural to consider the following class.

**Definition 2.8** ([Abe18, Def. 1.6]). An arrangement $\mathcal{A}$ is called additionally free if there is a filtration

$$\Phi_\ell = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n = \mathcal{A},$$

of $\mathcal{A}$, where each $\mathcal{A}_i$ is free with $|\mathcal{A}_i| = i$ and $|\mathcal{A}| = n$. Denote this class by $\mathcal{AF}$.

We denote by $\mathcal{AF}_\ell$ the subclass of $\ell$-arrangements in $\mathcal{AF}$.

The members of $\mathcal{AF}$ are constructed by means of the addition part of Theorem 2.3. In particular, each member of $\mathcal{AF}$ is free. Clearly, $\mathcal{IF} \subset \mathcal{AF}$. In [Abe18, Thm. 1.8], Abe showed that Terao’s conjecture is still valid within $\mathcal{AF}$.

Combining the procedures of addition from Theorem 2.3 and the construction of freeness from Theorem 2.6, we obtain the following new natural class.

**Definition 2.9** ([Abe18, Def. 4.2]). An arrangement $\mathcal{A}$ is called stair-free if $\mathcal{A}$ is build up from some empty arrangement by consecutive applications of addition from Theorem 2.3 or an extension along Theorem 2.6. Denote this class by $\mathcal{SF}$.

We denote by $\mathcal{SF}_\ell$ the subclass of $\ell$-arrangements in $\mathcal{SF}$.

The significance of this new class $\mathcal{SF}$ stems from the following result.

**Theorem 2.10** ([Abe18, Thm. 4.3]). With the notation from above, we have

(i) every member of $\mathcal{SF}$ is free;

(ii) $\mathcal{IF} \subset \mathcal{DF} \cup \mathcal{AF} \subset \mathcal{SF}$;

(iii) Terao’s Conjecture 1.1 is valid within $\mathcal{SF}$. 
Each of the classes of free, inductively free and divisionally free arrangements is compatible with the product construction for arrangements, cf. [OT92, Prop. 4.28], [HR15, Prop. 2.10], [Rö18, Prop. 2.9]. Next we observe that this also holds for the classes \( \mathcal{AF} \) and \( \mathcal{SF} \).

**Proposition 2.11.** Let \( (\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2) \) be two arrangements. Then \( \mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2) \) is stair-free (resp. additionally free) if and only if both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are stair-free (resp. additionally free).

**Proof.** First suppose that both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are stair-free. We claim that then so is \( \mathcal{A} \). We argue via induction on \(|\mathcal{A}|\). If both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are empty, there is nothing to show. So suppose that \(|\mathcal{A}| \geq 1 \) and that the claim holds for any product of stair-free arrangements with fewer than \(|\mathcal{A}|\) hyperplanes. Without loss of generality there exists an \( H_1 \) in \( \mathcal{A}_1 \) such that either \( \mathcal{A}_1 \setminus \{H_1\} \) or \( \mathcal{A}_1^{H_1} \) still belongs to \( \mathcal{SF} \). Consequently, in the first instance \((\mathcal{A}_1 \setminus \{H_1\}) \times \mathcal{A}_2 \in \mathcal{SF} \), by induction. Moreover, as both \( \mathcal{A}_1 \) and \( \mathcal{A}_1 \setminus \{H_1\} \) are free, it follows from the strong form of the restriction part of Theorem 2.3 ([OT92, Thm. 4.46]) that also the restriction \( \mathcal{A}_1^{H_1} \) is free and \( \exp(\mathcal{A}_1^{H_1}) \subset \exp(\mathcal{A}_1 \setminus \{H_1\}) \). Consequently, setting \( H := H_1 \oplus V_2 \in \mathcal{A} \) and using (2.1) and [OT92, Prop. 4.28], we have

\[
\exp(\mathcal{A}^H) = \exp(\mathcal{A}_1^{H_1}), \exp(\mathcal{A}_2) \subset \exp(\mathcal{A}_1 \setminus \{H_1\}), \exp(\mathcal{A}_2) = \exp(\mathcal{A} \setminus \{H\}),
\]

and so by the addition part of Theorem 2.3, \( \mathcal{A} \) also belongs to \( \mathcal{SF} \).

In the second instance when \( \mathcal{A}_1^{H_1} \) still belongs to \( \mathcal{SF} \), we have that \( \chi(\mathcal{A}_1^{H_1}, t) \) divides \( \chi(\mathcal{A}_1, t) \). For \( H = H_1 \oplus V_2 \in \mathcal{A} \), we see that \( \mathcal{A}^H = \mathcal{A}_1^{H_1} \times \mathcal{A}_2 \) is a product of stair-free arrangements with \(|\mathcal{A}^H| < |\mathcal{A}|\), by (2.1). So, by our induction hypothesis, \( \mathcal{A}^H \) is stair-free. In addition, since \( \chi(\mathcal{A}^H, t) = \chi(\mathcal{A}_1^{H_1}, t) \cdot \chi(\mathcal{A}_2, t) \) divides \( \chi(\mathcal{A}_1, t) \cdot \chi(\mathcal{A}_2, t) = \chi(\mathcal{A}, t) \), cf. (2.2), we infer that \( \mathcal{A} \) belongs to \( \mathcal{SF} \), by Theorem 2.6.

Conversely, suppose that \( \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \) belongs to \( \mathcal{SF} \). We claim that then both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) also belong to \( \mathcal{SF} \). Again we argue by induction on \(|\mathcal{A}|\). If both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are empty, there is nothing to show. So suppose that \(|\mathcal{A}| \geq 1 \) and that the claim holds for any product in \( \mathcal{SF} \) with fewer than \(|\mathcal{A}|\) hyperplanes. Without loss of generality we may assume that there is an \( H = H_1 \oplus V_2 \in \mathcal{A} \) such that either \( \mathcal{A} \setminus \{H\} = \mathcal{A}_1 \setminus \{H_1\} \times \mathcal{A}_2 \) or \( \mathcal{A}^H \) belongs to \( \mathcal{SF} \) and \( \chi(\mathcal{A}^H, t) \) divides \( \chi(\mathcal{A}, t) \) in the second instance. Thus in the first case, by our induction hypothesis, both \( \mathcal{A}_1 \setminus \{H_1\} \) and \( \mathcal{A}_2 \) also belong to \( \mathcal{SF} \). Once again, by the strong form of the restriction part of Theorem 2.3 ([OT92, Thm. 4.46]) also the restriction \( \mathcal{A}^H \) is free and \( \exp(\mathcal{A}^H) \subset \exp(\mathcal{A} \setminus \{H\}) \). Thus it follows from [OT92, Prop. 4.28] that

\[
\{\exp(\mathcal{A}_1^{H_1}), \exp(\mathcal{A}_2)\} = \exp(\mathcal{A}^H) \subset \exp(\mathcal{A} \setminus \{H\}) = \{\exp(\mathcal{A}_1 \setminus \{H_1\}), \exp(\mathcal{A}_2)\},
\]

and so \( \exp(\mathcal{A}_1^{H_1}) \subset \exp(\mathcal{A}_1 \setminus \{H_1\}) \). Therefore, by Theorem 2.3, \( \mathcal{A}_1 \) also belongs to \( \mathcal{SF} \).

Now consider the second case when \( \mathcal{A}^H \) belongs to \( \mathcal{SF} \) and \( \chi(\mathcal{A}^H, t) \) divides \( \chi(\mathcal{A}, t) \). Since \(|\mathcal{A}^H| < |\mathcal{A}|\) and \( \mathcal{A}^H = \mathcal{A}_1^{H_1} \times \mathcal{A}_2 \), both \( \mathcal{A}_1^{H_1} \) and \( \mathcal{A}_2 \) belong to \( \mathcal{SF} \), by our induction hypothesis. Moreover, since \( \chi(\mathcal{A}^H, t) = \chi(\mathcal{A}_1^{H_1}, t) \cdot \chi(\mathcal{A}_2, t) \) and \( \chi(\mathcal{A}, t) = \chi(\mathcal{A}_1, t) \cdot \chi(\mathcal{A}_2, t) \), cf. (2.2), it follows that \( \chi(\mathcal{A}_1^{H_1}, t) \) divides \( \chi(\mathcal{A}_1, t) \). Therefore, also \( \mathcal{A}_1 \) belongs to \( \mathcal{SF} \).

The statement for \( \mathcal{AF} \) of the proposition follows from just parts of the argument above. \( \square \)

**Remark 2.12.** Clearly, \( \mathcal{AF} \) is closed under taking localizations. For, freeness is closed under taking localizations ([OT92, Thm. 4.37]), so that a free chain of a member of \( \mathcal{AF} \) descends
to a free chain of any of its localizations by removing redundant hyperplanes. In contrast, by [Rö18, Ex. 2.16], $\mathcal{DF}$ is not closed under taking localizations, thus neither is $\mathcal{SF}$.

We close this section by discussing the reflection arrangements that belong to $\mathcal{SF}$.

**Example 2.13.** Let $W$ be an irreducible unitary reflection group with $\mathcal{A}(W)$ its reflection arrangement consisting of the hyperplanes associated with reflections in $W$, see [OT92, §6]. Then $\mathcal{A}(W)$ belongs to $\mathcal{SF}$ if and only if either $\mathcal{A}(W)$ is inductively free or else $W = G_{31}$. For, by Theorem 2.10 and [Abe16, Thm. 1.6], each inductively free reflection arrangement and also $\mathcal{A}(G_{31})$ belongs to $\mathcal{SF}$. In contrast, none of the remaining irreducible, non-inductively free reflection arrangements $\mathcal{A}(W)$ belongs to $\mathcal{SF}$. Indeed, for any such $\mathcal{A}(W)$ and any choice of hyperplane, $\mathcal{A}(W)'$ fails to be free. Moreover, by [HR15, Cor. 1.3, Cor. 2.18], $\exp \mathcal{A}(W)' \not\subseteq \exp \mathcal{A}(W)$, so that $\chi(\mathcal{A}(W)', t)$ does not divide $\chi(\mathcal{A}(W), t)$. So $\mathcal{A}(W) \not\in \mathcal{SF}$, by Definition 2.9.

3. **Proof of Theorem 1.2**

3.1. Observe that in dimension 3 we have $\mathcal{SF}_3 = \mathcal{DF}_3 \cup \mathcal{AF}_3 = \mathcal{DF}_3 = \mathcal{IF}_3$. For, since every rank 2 arrangement is already inductively free, the result follows from [OT92, Thm. 4.46, Prop. 4.52]. Consequently, examples to demonstrate the statements claimed in Theorem 1.2 can only occur in dimension at least 4.

3.2. We first consider the inductively free arrangement $\mathcal{A}$ of rank 7 consisting of 32 hyperplanes, with induction table given in Table 1 below.

Here $\mathcal{A}$ is realized as a subarrangement of the Weyl arrangement $\mathcal{A}(E_7)$ of the Weyl group of type $E_7$. The $x_1, \ldots, x_7$ represent the simple roots according to the labeling in [Bou68, Planche VI]. One checks that the resulting arrangement is inductively free with exponents $\exp \mathcal{A} = \{1, 5, 5, 5, 5, 5, 6\}$. Of course, this entails checking inductive freeness of all rank 6 restrictions in Table 1 and again their restrictions, etc. In particular, if we remove the last hyperplane in the inductive chain, $\ker(x_1)$, then $\mathcal{A}'$ is still inductively free with $\exp \mathcal{A}' = \{1, 4, 5, 5, 5, 5, 6\}$.

However, if instead we remove the penultimate hyperplane from $\mathcal{A}$ in the chain below, $\ker(x_3 + x_4)$, then the resulting arrangement, say $\mathcal{B}$, while still additionally free with exponents $\exp \mathcal{B} = \{1, 5, 5, 5, 5, 5\}$, is no longer inductively free, as no restriction $\mathcal{B}''$ with matching exponents $\{1, 5, 5, 5, 5\}$ is inductively free, see Table 2. Indeed, up to isomorphism there are only two restrictions $\mathcal{B}''$ with matching set of exponents $\{1, 5, 5, 5, 5\}$, then there is only one such further restriction of rank 5 up to isomorphism. This is the arrangement $\mathcal{D}$ which we are going to examine in §3.3. There we show that $\mathcal{D}$ is not inductively free which in turn shows that $\mathcal{B}$ is not inductively free either. This in particular then implies that $\mathcal{B} \in \mathcal{AF} \setminus \mathcal{IF}$.

If we further remove $\ker(x_1)$ from $\mathcal{B}$, the resulting arrangement $\mathcal{B}$ is of course inductively free again, by Table 1, as it coincides with $\mathcal{A} \setminus \{\ker(x_1), \ker(x_3 + x_4)\}$. So the non-inductively
free arrangement $\mathcal{B}$ is tightly sandwiched between the inductively free arrangements $\mathcal{B}'$ and $\mathcal{A}$.

3.3. Next, we consider the restriction $\mathcal{C} := \mathcal{A}^Z$ of $\mathcal{A}$, where $Z$ is the intersection of the hyperplanes $H_1 := \ker(x_1)$ and $H' := \ker(x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + x_6)$. Then $\mathcal{C}$ is a subarrangement of the restriction $\mathcal{A}(E_7)^Z$ of $\mathcal{A}(E_7)$ which is of type $(E_7, A_1^2)$. 

### Table 1. Induction table for the subarrangement $\mathcal{A}$ of $\mathcal{A}(E_7)$.

| $\exp \mathcal{A}'$ | $\alpha_H$ | $\exp \mathcal{A}''$ |
|---------------------|-------------|------------------|
| 0, 0, 0, 0, 0, 0    | $x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + x_6 + x_7$ | 0, 0, 0, 0, 0, 0 |
| 0, 0, 0, 0, 0, 1    | $x_1 + 2x_2 + 2x_3 + 3x_4 + 2x_5 + x_6 + x_7$ | 0, 0, 0, 0, 1 |
| 0, 0, 0, 0, 1, 1    | $x_2 + x_4$ | 0, 0, 0, 0, 1 |
| 0, 0, 0, 0, 1, 2    | $x_4$ | 0, 0, 0, 1, 2 |
| 0, 0, 0, 1, 1, 2    | $x_2$ | 0, 0, 0, 1, 2 |
| 0, 0, 0, 1, 1, 2, 2 | $x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 + 2x_6 + x_7$ | 0, 0, 0, 1, 2, 3 |
| 0, 0, 0, 1, 1, 2, 3 | $x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 + 2x_6 + x_7$ | 0, 0, 0, 1, 2, 3 |
| 0, 0, 0, 1, 2, 2, 3 | $x_5 + x_6$ | 0, 0, 1, 2, 3 |
| 0, 0, 0, 1, 2, 3, 3 | $x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 + 2x_6 + x_7$ | 0, 0, 1, 3, 3 |
| 0, 0, 0, 1, 3, 3, 3 | $x_4 + x_5 + x_6$ | 0, 0, 1, 3, 3 |
| 0, 0, 0, 1, 3, 3, 4 | $x_2 + x_4 + x_5 + x_6$ | 0, 0, 1, 3, 3 |
| 0, 0, 0, 1, 3, 3, 5 | $x_3 + x_4 + x_5$ | 0, 0, 1, 3, 3, 5 |
| 0, 0, 0, 1, 3, 3, 5 | $x_2 + x_3 + 2x_4 + x_5$ | 0, 0, 1, 3, 3, 5 |
| 0, 0, 0, 1, 3, 3, 5 | $x_2 + x_3 + 2x_4 + x_5$ | 0, 0, 1, 3, 3, 5 |
| 0, 0, 0, 1, 3, 3, 5 | $x_2 + x_3 + 2x_4 + x_5$ | 0, 0, 1, 3, 3, 5 |
| 0, 0, 0, 1, 3, 4, 5 | $x_1 + x_2 + x_3 + 2x_4 + 2x_5 + 2x_6 + x_7$ | 0, 0, 1, 3, 4, 5 |
| 0, 0, 0, 1, 3, 4, 5 | $x_2 + x_3 + 2x_4 + x_5 + x_6$ | 0, 1, 3, 4, 5 |
| 0, 0, 0, 1, 3, 4, 5 | $x_1 + x_2 + x_3 + 2x_4 + 2x_5 + x_6 + x_7$ | 0, 1, 3, 4, 5 |
| 0, 0, 0, 1, 3, 4, 5 | $x_2 + x_3 + x_4 + x_5 + x_6$ | 0, 1, 3, 4, 5 |
| 0, 0, 0, 1, 3, 4, 5 | $x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 + 2x_6 + x_7$ | 0, 1, 4, 4, 4, 5 |
| 0, 0, 0, 1, 3, 4, 5 | $x_1 + x_2 + x_3 + 2x_4 + 2x_5 + x_6$ | 1, 4, 4, 4, 5 |
| 1, 1, 4, 4, 4, 5 | $x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7$ | 1, 4, 4, 4, 5 |
| 1, 2, 4, 4, 4, 5 | $x_1 + x_2 + x_3 + x_4 + x_5$ | 1, 4, 4, 4, 5 |
| 1, 3, 4, 4, 4, 5 | $x_1 + x_3 + x_4 + x_5$ | 1, 4, 4, 4, 5 |
| 1, 4, 4, 4, 4, 5 | $x_3$ | 1, 4, 4, 4, 5 |
| 1, 4, 4, 4, 4, 5 | $x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7$ | 1, 4, 4, 4, 5 |
| 1, 4, 4, 4, 5, 5 | $x_6$ | 1, 4, 5, 5, 5, 5 |
| 1, 4, 4, 4, 5, 5 | $x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + x_6$ | 1, 4, 5, 5, 5, 5 |
| 1, 4, 5, 5, 5, 5 | $x_3 + x_4$ | 1, 4, 5, 5, 5, 5 |
| 1, 4, 5, 5, 5, 5 | $x_1$ | 1, 5, 5, 5, 5, 6 |
| 1, 5, 5, 5, 5, 6 | $\ldots$ | \ldots |

### Table 2. Chain of hyperplanes for $\mathcal{B}$ in $\mathcal{A}F$.

| $\exp \mathcal{A}'$ | $\alpha_H$ | $\exp \mathcal{A}''$ |
|---------------------|-------------|------------------|
| 1, 4, 4, 4, 5, 5, 5 | $x_6$ | 1, 4, 4, 5, 5, 5 |
| 1, 4, 4, 5, 5, 5, 5 | $x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + x_6$ | 1, 4, 5, 5, 5, 5 |
| 1, 4, 5, 5, 5, 5, 5 | $x_1$ | 1, 5, 5, 5, 5, 5 |
| 1, 5, 5, 5, 5, 5, 5 | $\ldots$ | \ldots |
One checks that $\mathcal{C}$ is again inductively free with exponents $\exp \mathcal{C} = \{1, 5, 5, 5, 6\}$. An induction table for $\mathcal{C}$ is given in Table 3. In particular, if we remove the last hyperplane in the inductive chain, $\ker(x_4)$, then $\mathcal{C}'$ is still inductively free with $\exp \mathcal{C}' = \{1, 4, 5, 5, 6\}$.

| $\exp \mathcal{A}'$ | $\alpha_H$ | $\exp \mathcal{A}''$ |
|---------------------|-------------|---------------------|
| 0, 0, 0, 0, 0       | $x_2$       | 0, 0, 0, 0          |
| 0, 0, 0, 0, 1       | $x_1 + x_3 - x_5$ | 0, 0, 0, 1          |
| 0, 0, 0, 1, 1       | $2x_1 + x_2 + x_3$ | 0, 0, 1, 1          |
| 0, 0, 1, 1, 1       | $2x_1 + x_2 + 2x_3 + x_4 - x_5$ | 0, 1, 1, 1          |
| 0, 1, 1, 1, 1       | $x_5$       | 1, 1, 1, 1          |
| 1, 1, 1, 1, 1       | $x_1 + x_3$ | 1, 1, 1, 1          |
| 1, 1, 1, 1, 2       | $x_2 + x_5$ | 1, 1, 1, 2          |
| 1, 1, 2, 2, 2       | $2x_1 + x_2 + 2x_3 + x_4$ | 1, 1, 2, 2          |
| 1, 2, 2, 2, 2       | $2x_1 + x_3 - x_5$ | 1, 2, 2, 2          |
| 1, 2, 2, 2, 3       | $2x_1 + 2x_2 + 2x_3 + x_4$ | 1, 2, 2, 3          |
| 1, 2, 3, 3, 3       | $x_2 + x_3 + x_4$ | 1, 2, 3, 3          |
| 1, 3, 3, 3, 3       | $x_1 + x_2 + x_3$ | 1, 3, 3, 3          |
| 1, 3, 3, 3, 4       | $x_1$       | 1, 3, 3, 4          |
| 1, 3, 3, 4, 4       | $x_1 + x_3 + x_4$ | 1, 3, 4, 4          |
| 1, 3, 4, 4, 4       | $2x_1 + x_2 + x_3 - x_5$ | 1, 4, 4, 4          |
| 1, 4, 4, 4, 4       | $x_2 + x_3 + x_4 + x_5$ | 1, 4, 4, 4          |
| 1, 4, 4, 4, 5       | $x_1 - x_5$ | 1, 4, 4, 5          |
| 1, 4, 4, 5, 5       | $x_1 - x_4 - x_5$ | 1, 4, 5, 5          |
| 1, 4, 5, 5, 5       | $x_1 + x_2$ | 1, 4, 5, 5          |
| 1, 4, 5, 5, 6       | $x_4$       | 1, 5, 5, 6          |
| 1, 5, 5, 5, 6       |             |                     |

Table 3. Induction table for the rank 5 arrangement $\mathcal{C}$.

However, if instead we remove the penultimate hyperplane from $\mathcal{C}$ in the chain in Table 3, $\ker(x_1 + x_2)$, then the resulting arrangement, say $\mathcal{D}$, while still additionally free with exponents $\exp \mathcal{D} = \{1, 5, 5, 5\}$, is no longer inductively free, as no restriction $\mathcal{D}''$ with matching exponents $\{1, 5, 5, 5\}$ is inductively free, see Table 4. Up to isomorphism there is only one restriction $\mathcal{D}'' \cong \mathcal{D}^{\ker x_4}$ with $\exp \mathcal{D}'' = \{1, 5, 5, 5\}$. While this restriction is necessarily free, it is no longer additionally free (and so clearly not inductively free). For any choice of hyperplane in $\mathcal{D}''$, the resulting deletion even if free does not have matching exponents $\{1, 4, 5, 5\}$. In particular, we have $\mathcal{D} \in \mathcal{AF} \setminus \mathcal{IF}$. This in particular proves Theorem 1.2(i). In addition this also shows that $\mathcal{AF}$ is not closed under taking restrictions.

If we further remove $\ker(x_4)$ from $\mathcal{D}$, the resulting arrangement $\mathcal{D}'$ is of course inductively free again, by Table 3, as it coincides with $\mathcal{C} \setminus \{\ker(x_4), \ker(x_1 + x_2)\}$. So the non-inductively free arrangement $\mathcal{D}$ is sandwiched between the inductively free arrangements $\mathcal{D}'$ and $\mathcal{C}$. 
| exp $\mathcal{A}$ | $\alpha_H$ | exp $\mathcal{A}$'' |
|----------------|---------|----------------|
| ...            | ...     | ...            |
| 1, 4, 4, 5, 5  | $x_1 - x_4 - x_5$ | 1, 4, 5, 5 |
| 1, 4, 5, 5, 5  | $x_4$     | 1, 5, 5, 5   |
| 1, 5, 5, 5, 5  |          |               |

Table 4. Chain of hyperplanes for $D$ in $\mathcal{A}\mathcal{F}$.

Moreover, one checks that $D$ is not divisionally free. For, there is no restriction of $D''$ with exponents $\{1, 5, 5\}$ and so the characteristic polynomial of any such restriction does not divide the characteristic polynomial of $D''$. In particular, this proves Theorem 1.2(ii).

As a subarrangement of $C$, also $D$ is a subarrangement of the restriction of $\mathcal{A}(E_7)$ of type $(E_7, A_2)$. Explicitly, $D$ is obtained from the arrangement $B$ as the restriction $D = B^X$, where $X := \ker(x_1) \cap \ker(x_6)$. The properties of $D$ we have established imply that $B$ above also satisfies the conditions in Theorem 1.2(i) and (ii). These are the only examples known to us which satisfy these properties.

We further observe that if we label the last three hyperplanes in the chain for $B$ in Table 2 by $H_1 := \ker(x_1)$, $H' := \ker(x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + x_6)$, and $H_6 := \ker(x_6)$, then these hyperplanes $H$ are precisely the ones so that the restriction $B^H$ has got the required exponents $\{1, 5, 5, 5, 5\}$ to satisfy the deletion part of Theorem 2.3 for $B$. Further, for $Y := H_1 \cap H' \cap H_6$ we have $B_Y = \{H_1, H', H_6\}$ and $B_Y = D''$. This implies that also $B$ fails to be divisionally free. For, a divisional chain as in Definition 2.7 necessarily has to pass through a restriction isomorphic to $D''$.

3.4. Thanks to [Abe16, Thm. 1.6], the reflection arrangement $\mathcal{A}(G_{31})$ of the complex reflection group $G_{31}$ is divisionally free but it is not additionally free, see the proof of [HR15, Lem. 3.5]. Thus $\mathcal{A}(G_{31})$ belongs to $DF$ but not to $AF$. Since $D$ above belongs to $AF$ but not to $DF$, it follows from Proposition 2.11 and [Rü18, Prop. 2.9] that
\[
E := D \times A(G_{31})
\]

neither belongs to $DF$, nor to $AF$, but at the same time $E$ is still stair-free, i.e.
\[
E \in SF \setminus (DF \cup AF)
\]

and so Theorem 1.2(iii) follows. It would be interesting to know of an irreducible example in $SF \setminus (DF \cup AF)$.

3.5. The facts that $A$ and $C$ above are inductively free and that both $B$ and $D$ are still additionally free were checked by computational means. Likewise the fact that $D$ is not divisionally free was checked with the aid of a computer.

4. Non-free triples of arrangements

In this section we discuss counterexamples to another conjecture of Abe, [Abe18, Conj. 3.5(2)]. Specifically, here we provide an example of a triple of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ with the
property that none of them is free but each of their characteristic polynomials factors over \( \mathbb{Z} \) and \( \chi(\mathcal{A}'', t) \) divides both \( \chi(\mathcal{A}', t) \) and \( \chi(\mathcal{A}, t) \) so that the latter two polynomials share all but one root.

**Example 4.1.** Let \( w, x, y, z \) be indeterminates over \( \mathbb{Q} \) and let \( \mathcal{A} \) be the arrangement in \( \mathbb{Q}^4 \) with 11 hyperplanes given by
\[
Q(\mathcal{A}) = wxyz(x + y)(x + z)(x - z)(y - z)(y + z)(x + y - z)(w + x - y).
\]

It is easy to check that for \( H = \ker(x + y - z) \), we have
\[
\chi(\mathcal{A}, t) = (t - 1)(t - 3)^2(t - 4),
\]
\[
\chi(\mathcal{A}', t) = (t - 1)(t - 3)^3,
\]
and
\[
\chi(\mathcal{A}'', t) = (t - 1)(t - 3)^2.
\]

Although the factorization over \( \mathbb{Z} \) of each of these polynomials is consistent with Terao’s Factorization Theorem 2.4, none of the arrangements in the triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\) is free.

In the following we provide a general construction to generate counterexamples to [Abe18, Conj. 3.5(2)] with an arbitrary number of hyperplanes in any dimension at least 3.

**Example 4.2.** Let \( \mathcal{B} \) be a fixed non-free arrangement in dimension \( \ell \geq 3 \) over \( \mathbb{Q} \) with the property that its characteristic polynomial factors over \( \mathbb{Z} \), e.g. take the arrangement \( \mathcal{A}''' \) from Example 4.1. Without loss we may assume that \( \ker x \in \mathcal{B} \), where \( x \) is a coordinate of \( \mathbb{Q}^{\ell+1} \).

Now view \( \mathcal{B} \) as an arrangement in \( \mathbb{Q}^{\ell+1} \) and let \( z \) be the new coordinate. Fix an integer \( m \geq 0 \) and define \( \mathcal{A} \) in \( \mathbb{Q}^{\ell+1} \) by adding the hyperplanes \( \ker(z), \ker(x-z), \ker(2x-z), \ldots, \ker(mx-z) \) to \( \mathcal{B} \). Consider the triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\) with respect to \( \ker(mx-z) \). Then we have \( \mathcal{A}'' \cong \mathcal{B} \).

By induction on \( m \) and the fact that \( \chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t) \) ([OT92, Cor. 2.57]), we obtain
\[
\chi(\mathcal{A}, t) = \chi(\mathcal{B}, t)(t - m - 1),
\]
\[
\chi(\mathcal{A}', t) = \chi(\mathcal{B}, t)(t - m),
\]
and
\[
\chi(\mathcal{A}'', t) = \chi(\mathcal{B}, t).
\]

Still none of the arrangements in the triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\) is free. For, the localization of \( \mathcal{A} \) at the center of \( \mathcal{B} \) in \( L(\mathcal{A}) \) is isomorphic to \( \mathcal{B} \) and thus is not free, thus neither is \( \mathcal{A} \), by [OT92, Thm. 4.37]; likewise for \( \mathcal{A}' \).

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