Counting numerical semigroups

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Abstract. We are interested in formulas for the number of elements in certain classes of numerical semigroups

Key words. Numerical semigroup (symmetric, pseudo-symmetric, of maximal embedding dimension), Apéry set, genus, polyhedral cone, lattice point, quasi-polynomial, Ehrhart’s theorem, generating function, lattice path.

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1. Introduction

For an integer \( p \geq 3 \) let \( \mathcal{H}_p \) be the set of all numerical semigroups containing \( p \). Using the notion of Apéry set one can construct a bijective map from \( \mathcal{H}_p \) onto the set of all lattice points of a certain polyhedral cone \( C_p \subset \mathbb{R}^{p-1} \) of dimension \( p - 1 \) ([Ku],[RGGB]). The lattice points in the interior \( C_p^0 \) of \( C_p \) are in one-to-one correspondence with the \( H \in \mathcal{H}_p \) of maximal embedding dimension \( p \).

Apéry’s characterization of symmetric semigroups [A] allows to show that the symmetric \( H \in \mathcal{H}_p \) are mapped onto the lattice points of certain \( \lfloor \frac{p}{2} \rfloor \)-dimensional closed faces of \( C_p \). Here we describe also the distribution of the lattice points corresponding to the pseudo-symmetric semigroups (Proposition 3.2). Further semigroups correspond to the intersection of \( C_p \) or its faces with hyperplanes, hence with the lattice points in polyhedrons, and their number can be expressed by quasi-polynomials (Theorem 4.2).

This is the case for the \( H \in \mathcal{H}_p \) with fixed genus \( g \). Also the semigroups \( H \in \mathcal{H}_p \) containing another number \( q \) which is prime to \( p \) can be described using hyperplane sections of \( C_p \) (Example 4.1b)). We denote this set of numerical semigroups by \( \mathcal{H}_{pq} \).

We want to study the degree, the leading term and a quasi-period of the involved Ehrhart quasi-polynomials. For the semigroups \( H \in \mathcal{H}_{pq} \) the leading term is constant and gives therefore an asymptotic estimate for \( q \to \infty \) of the number of the \( H \in \mathcal{H}_{pq} \) (Proposition 5.2). Similarly for the \( H \in \mathcal{H}_p \) of maximal embedding dimension \( p \) and the symmetric \( H \in \mathcal{H}_{pq} \) (Propositions 4.5 and 4.6).
The following figure illustrates the situation in the simplest case $p = 3$.

![Diagram](image)

**Figure 1:** Geographical distribution of the $H \in \mathcal{S}_3$.

The symmetric semigroups (here=complete intersections) belong to the edges of $C_3$ (see Proposition 3.1), those with maximal embedding dimension 3 (here=almost complete intersections) to the interior of $C_3$. The living space of the pseudo-symmetric semigroups are the dashed lines $X_1 = 2X_2$ and $X_2 = 2X_1 - 1$ (see Proposition 3.2) where the origin (which corresponds to $H = \mathbb{N}$) has to be excluded. That of the semigroups with genus $g = 7$ is the dotted line $X_1 + X_2 = 7$. Parallels to this line correspond to semigroups with fixed genus. The figure shows, for example, that for given $g \in \mathbb{N}$ there is exactly one symmetric (complete intersection) semigroup in $\mathcal{S}_3$ of genus $g$ if $g \not\equiv 2 \mod 3$, and no such semigroup if $g \equiv 2 \mod 3$. Similarly there is exactly one pseudo-symmetric $H \in \mathcal{S}_3$ of genus $g > 0$ if $g \not\equiv 1 \mod 3$, and no such semigroup if $g \equiv 1 \mod 3$. The lattice points of $C_3$ with $X$-coordinate $\leq i$ (with $Y$-coordinate $\leq i$) correspond to the $H \in \mathcal{S}_3$ with $1 + 3i \in H$ (with $2 + 3i \in H$), see Example 4.1b).

Using another visualization of the $H \in \mathcal{S}_{pq}$ by lattice paths in the plane ([KKW],[KW]) recursion formulas for the number of semigroups $H \in \mathcal{S}_{pq}$ which interest us can be derived, see Section 5. Finally in Section 6 explicit formulas
are given for $p \leq 5$. For related counting problems, see [Ka], [BGP].

2. The polyhedral cone $C_p$ and its faces

We recall some facts about $C_p$ which are relevant for us. Details can be found in [Ku] or [RGGB]. For $H \in \mathcal{S}_p$ let $\{h_1, \ldots, h_{p-1}\} = \text{Ap}(H, p)$ be the Apéry set of $H$ with respect to $p$, that is, $h_i$ is the smallest element of $H$ in $i + p\mathbb{N}$ for $i = 1, \ldots, p-1$. Write $h_i = i + \mu_i p$. The semigroup $H$ is uniquely determined by $(\mu_1, \ldots, \mu_{p-1})$ since $H = \langle p, h_1, \ldots, h_{p-1} \rangle$. The points $(\mu_1, \ldots, \mu_{p-1}) \in \mathbb{N}^{p-1}$ are the solutions in $\mathbb{N}^{p-1}$ of the system of linear inequalities

$$\begin{cases} X_i + X_j \geq X_{i+j} & (i + j < p) \\ X_i + X_j \geq X_{i+j-p} - 1 & (i + j > p) \end{cases}$$

Let $C_p$ be the solution set of (1) in $\mathbb{R}^{p-1}$. This is a polyhedral cone with vertex $v := (-\frac{1}{p}, -\frac{2}{p}, \ldots, -\frac{p-1}{p})$ and

$$\mu : \mathcal{S}_p \to C_p \cap \mathbb{N}^{p-1} \quad (H \mapsto (\mu_1, \ldots, \mu_{p-1}))$$

is a bijection of $\mathcal{S}_p$ onto the set of lattice points of $C_p$. For $H \in \mathcal{S}_p$ we call $\mu(H)$ the lattice point associated to $H$, and for $P = \mu(H)$ we say that $H$ is the semigroup belonging to the point $P$. By a face of $C_p$ we always understand an open face. Its dimension is the dimension of the smallest affine space containing it. We consider the interior $C_p^0$ of $C_p$ and the vertex $v$ also as faces.

We have $C_p^0 \cap \mathbb{N}^{p-1} = (1, \ldots, 1) + C_p \cap \mathbb{N}^{p-1}$ ([Ku], 1.4d). $C_p$ and $C_p^0$ have dimension $p - 1$. The facets (1-codimensional faces) correspond bijectively to the hyperplanes

$$\begin{cases} E_{ij} : X_i + X_j - X_{i+j} = 0 & (i + j < p) \\ E_{ij} : X_i + X_j - X_{i+j-p} + 1 = 0 & (i + j > p) \end{cases}$$

Hence for even $p$ there are $\frac{(p-1)^2 - 1}{2}$ facets, for odd $p$ their number is $\frac{(p-1)^2}{2}$.

The translation of $C_p$ by the vector $-v$ leads to a polyhedral cone $C_p^* := -v + C_p$ in $\mathbb{R}_+^{p-1}$ with vertex at the origin. It is the solution set of the system $X_i + X_j \geq X_{i+j}$ ($i + j \neq p$ with indices reduced modulo $p$). Therefore $C_p^* \subset C_p$. On each face of $C_p^*$ there are lattice points. This is true in particular for its edges (1-dimensional faces). On each such edge there is a primitive lattice point $\delta$, that is a point with relatively prime integral coordinates, and all other lattice points on the edge are integral multiples of $\delta$. The set of all these $\delta$ is called the canonical system of representatives for the edges of $C_p^*$.

Let $m(H)$ denote the multiplicity and edim($H$) the embedding dimension of a numerical semigroup $H$. The lattice points in the interior of $C_p$ correspond to the $H \in \mathcal{S}_p$ with $\text{edim}(H) = m(H) = p$ ([Ku], 2.4b), called in modern language the semigroups of maximal embedding dimension $p$.

Let $S$ be a $d$-dimensional face of $C_p$, $S^*$ the face of $C_p^*$ parallel to $S$ and $S$ resp. $S^*$ the topological closures of these faces. The generating function
(Hilbert series) for the lattice points on $S$ is the formal power series

$$H_S(X_1, \ldots, X_{p-1}) = \sum_{\mu \in S \cap \mathbb{N}^{p-1}} X^\mu.$$ 

The generating functions $H_{\overline{S}}$ for $\overline{S}$, $H_p$ for $C_p$ and $H^0_p$ for the interior $C^0_p$ of $C_p$ are defined accordingly.

2.1. Remark. ([Ku], 2.1) $H^0_p(X_1, \ldots, X_{p-1}) = X_1 \cdots X_{p-1} H_p(X_1, \ldots, X_{p-1}).$

Let \{\delta_1, \ldots, \delta_r\} be the canonical system of representatives for the edges of $C^*_p$ and $\overline{S}^*$. Using slack variables $T_{ij}$ the system (1) of Section 1 becomes a linear system of equations

$$X_i + X_j - X_{i+j} - T_{ij} = 0 \quad (i + j < p)$$

$$X_i + X_j - X_{i+j-p} - T_{ij} = -1 \quad (i + j > p).$$

To it the results of Stanley [S1] Chap.I can be applied. One obtains

2.2. Proposition. $H_{\overline{S}}$ can be written in the form

$$H_{\overline{S}} = \frac{F_{\overline{S}}}{\prod_{i=1}^t (1 - X^{\delta_i})}, \quad F_{\overline{S}} \in \mathbb{Z}[X_1, \ldots, X_{p-1}]$$

Similarly $H_p$ has the form

$$H_p = \frac{F_p}{\prod_{i=1}^t (1 - X^{\delta_i})}, \quad F_p \in \mathbb{Z}[X_1, \ldots, X_{p-1}].$$

Further $\deg X_i H_{\overline{S}} \leq -1$ and $\deg X_i H_p \leq -1$ ($i = 1, \ldots, p - 1$).

See [Ku], 3.2 and 3.3 for details. The proof of [Ku], 3.2 contains the false statement that $R_{S^*} := \mathbb{C} \{X^{\mu} \mid \mu \in S^* \cap \mathbb{N}^{p-1}\}$ as an algebra over $\mathbb{C}$ is generated by $X^{\delta_1}, \ldots, X^{\delta_r}$. However the algebra is a finitely generated module over $A := \mathbb{C}[X^{\delta_1}, \ldots, X^{\delta_r}]$. Then also the module $M_{\overline{S}} := \bigoplus_{\mu \in S^* \cap \mathbb{N}^{p-1}} \mathbb{C} \cdot X^\mu$ is a finitely generated $A$-module, which is what is actually used in the proof.

3. Symmetric and pseudo-symmetric semigroups

Apéry’s characterization of symmetric semigroups in terms of their Apéry sets ([A]) allows to describe the lattice points in $C_p$ belonging to such semigroups. Let

$$t := \begin{cases} p - 1, & p \text{ odd} \\ \frac{p}{2}, & p \text{ even} \end{cases}.$$

3.1. Proposition ([Ku], 2.9) There are $t$ faces $S_1, \ldots, S_t$ of dimension $\left\lfloor \frac{p}{2} \right\rfloor$ such that any symmetric $H \in \delta_p$ belongs to exactly one $S_j$, and all lattice points on the $S_j$ correspond to symmetric $H \in \delta_p$. 

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In particular $\tilde{S}_j \cap \tilde{S}_k \cap \mathbb{N}^{p-1} = \emptyset$ for $j \neq k$. According to Proposition 2.2 the generating function for the symmetric semigroups

$$H_{\text{sym}}(X_1, \ldots, X_{p-1}) := \sum_{\mu \in \cup_{j=1}^p \tilde{S}_j \cap \mathbb{N}^{p-1}} X^\mu = \sum_{j=1}^p H_{\tilde{S}_j}(X_1, \ldots, X_{p-1})$$

is a rational function.

For pseudo-symmetric semigroups the situation is more complicated. Remember that a numerical semigroup $H$ of genus $g(H)$ and Frobenius number $F(H)$ is pseudo-symmetric if $2g(H) = F(H) + 2$.

Let $\mathfrak{S}_{p-1}$ be the permutation group of $\{1, \ldots, p-1\}$ and $\mathfrak{S}^*_p$ the set of all $\sigma \in \mathfrak{S}^*_p$ such that

$$\sigma(i) + \sigma(p - 2 - i) \equiv \sigma(p - 2) \mod p \text{ (i = 1, \ldots, p - 3)}$$

$$2\sigma(p - 1) \equiv \sigma(p - 2) \mod p.$$ 

For $\sigma \in \mathfrak{S}^*_p$ let $L_\sigma$ be the affine subspace of $\mathbb{R}^{p-1}$ defined by the linear equations

$$(1) \quad X_{\sigma(i)} + X_{\sigma(p - 2 - i)} = \begin{cases} X_{\sigma(p - 2)} & \text{if } \sigma(i) + \sigma(p - 2 - i) = \sigma(p - 2) \\ X_{\sigma(p - 2)} - 1 & \text{if } \sigma(i) + \sigma(p - 2 - i) = \sigma(p - 2) + p \end{cases}$$

(i = 1, \ldots, p - 3) and

$$2X_{\sigma(p - 1)} = X_{\sigma(p - 2)} + \begin{cases} 1 & \text{if } 2\sigma(p - 1) = \sigma(p - 2) \\ 0 & \text{if } 2\sigma(p - 1) = \sigma(p - 2) + p \end{cases}.\tag{2}$$

This space has dimension $\lfloor \frac{p-1}{2} \rfloor$.

The following Proposition is based on [RG], 4.15.

**3.2. Proposition.** a) For $\sigma \in \mathfrak{S}^*_p$ each lattice point $(\mu_{\sigma(1)}, \ldots, \mu_{\sigma(p-1)}) \neq 0$ of $L_\sigma \cap C_p$ belongs to a pseudo-symmetric semigroup.

b) For each pseudo-symmetric $H \in \mathcal{H}_p$ there exists a $\sigma \in \mathfrak{S}^*_p$ such that $\mu(H) \in L_\sigma \cap C_p$.

c) For $p > 3$ the $L_\sigma \cap C_p$ are contained in the boundary of $C_p$.

d) $L_{id} \cap C_p = \bar{S} \cap H$ where $\bar{S}$ is the closure of a face $S$ of $C_p$ of dimension $\lfloor \frac{p-1}{2} \rfloor$ containing 0 and $H$ a hyperplane through 0 defining a facet of $C^*_p$.

**Proof.** a) Let $h_{\sigma(i)} := \sigma(i) + \mu_{\sigma(i)}p \ (i = 1, \ldots, p - 1)$. Then $\{h_{\sigma(1)}, \ldots, h_{\sigma(p-1)}\}$ is the Apéry set of a semigroup $H \in \mathcal{S}_p$ and $\sum_{i=1}^{p-1} h_{\sigma(i)} = \binom{p}{2} + p \sum_{i=1}^{p-1} \mu_{\sigma(i)} = \binom{p}{2} + g(H)p$.

From the equations defining $L_\sigma$ we obtain

$$h_{\sigma(i)} + h_{\sigma(p - 2 - i)} = h_{\sigma(p - 2)} \ (i = 1, \ldots, p - 3)$$

$$2h_{\sigma(p - 1)} = h_{\sigma(p - 2)} + p.$$
Obviously \( h_{\sigma(i)} < h_{\sigma(p-2)} \) for \( i = 1, \ldots, p - 3 \), hence \( h_{\sigma(p-2)} \geq p - 2 \), and then by the last equation \( h_{\sigma(p-1)} \geq p - 1 \) and \( h_{\sigma(p-2)} - h_{\sigma(p-1)} = h_{\sigma(p-1)} - p \geq -1 \). In case \( h_{\sigma(p-1)} > h_{\sigma(p-2)} \) we would have \( h_{\sigma(p-1)} = p - 1, \mu_{\sigma(p-1)} = 0 \). But then \( \mu_i = 0 \) for \( i = 1, \ldots, p - 1 \), contrary to the assumption.

Thus \( h_{\sigma(p-2)} \) is the maximal element of the Apéry set of \( H \), i.e. \( h_{\sigma(p-2)} = F(H) + p \) with the Frobenius number \( F(H) \) of \( H \). Adding the above equations gives

\[
2 \sum_{i=1}^{p-1} h_{\sigma(i)} = ph_{\sigma(p-2)} + p = p(F(H) + p) + p
\]

that is

\[
2g(H) = F(H) + 2
\]

which is equivalent to \( H \) being pseudo-symmetric.

b) If \( H \) is pseudo-symmetric, then \( F(H) \) is an even number. By [RG], 4.15 there is a permutation \( \sigma \in \mathfrak{S}_{p-1}^{\ast} \) such that \( \{h_{\sigma(1)}, \ldots, h_{\sigma(p-1)}\} \) is the Apéry set of \( H \) and

\[
h_{\sigma(i)} + h_{\sigma(p-2-i)} = h_{\sigma(p-2)} \quad (i = 1, \ldots, p - 3)
\]

\[
h_{\sigma(p-2)} = F(H) + p, h_{\sigma(p-1)} = \frac{F(H)}{2} + p
\]

in particular

\[
2h_{\sigma(p-1)} = F(H) + 2p = h_{\sigma(p-2)} + p.
\]

With \( h_{\sigma(i)} := \sigma(i) + \mu_{\sigma(i)}p \) \( (i = 1, \ldots, p - 1) \) it follows that \( (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(p-1)}) \in L_\sigma \cap C_p \).

c) For \( p > 3 \) the equations (1) show that \( L_\sigma \) is contained in a hyperplane which defines a facet of \( C_p \), hence \( L_\sigma \cap C_p \) belongs to the boundary of \( C_p \).

d) \( \sigma = id \) belongs to \( \mathfrak{S}_{p-1}^{\ast} \) and in this case \( L_\sigma \) is the intersection of the hyperplanes

\[
H_i : X_i + X_{p-2-i} = X_{p-2} \quad (i = 1, \ldots, p - 3), \quad H : 2X_{p-1} = X_{p-2}.
\]

The \( H_i \) define facets of \( C_p \) containing the origin, and \( H \) a facet of \( C_p^\ast \). We have

\[
L_{id} \cap C_p = \cap_{i=1}^{p-3} (H_i \cap C_p) \cap H
\]

and \( \bar{S} := \cap_{i=1}^{p-3} (H_i \cap C_p) \) is the closure of a face \( S \) of \( C_p \).

The point \( P := (1, \ldots, 1, 2, 1) \) is in \( L_{id} \cap C_p \) and in the interior of the half-spaces \( X_i + X_j \geq X_{i+j} (i + j \neq p - 2) \). Therefore an open neighborhood of \( P \) in the \( \lfloor \frac{p-3}{2} \rfloor \)-dimensional affine space \( L_{id} \) is contained in \( L_{id} \cap C_p \). Hence the face \( S \) must have dimension \( \lfloor \frac{p-3}{2} \rfloor \).

3.3. Corollary. ([RG], 4.26). For \( p > 3 \) pseudo-symmetric semigroups of \( \Omega_p \) have embedding dimension \( < p \).

Proof. Otherwise the lattice points of such semigroups would be in the interior of \( C_p \), contradicting 3.2c). □
3.4. Example. In general $L_{id} \cap C_p$ is not a polyhedral cone. If $p = 7$, then $L_{id}$ is a 3-space, in which $L_{id} \cap C_7$ is as shown in the next figure:

![Figure 2](image)

4. Intersections of $C_p$ and its faces with hyperplanes

Certain classes of numerical semigroups correspond to the lattice points in the intersection of $C_p$ or some of its faces with hyperplanes.

4.1. Examples.

a) If $\mu(H) = (\mu_1, \ldots, \mu_{p-1})$ for $H \in \mathcal{H}_p$, then $g(H) = \sum_{i=1}^{p-1} \mu_i$ is the genus of $H$. Thus if $H_g$ is the hyperplane $\sum_{i=1}^{p-1} X_i = g$, then the lattice points in $C_p \cap H_g$ are in one-to-one correspondence with the semigroups in $\mathcal{H}_p$ of genus $g$ and those of $C_0 \cap H_g$ with the $H \in \mathcal{H}_p$ of genus $g$ and maximal embedding dimension $p$.

b) Given $q = i + np$ ($i < p$, $i$ and $p$ coprime) the lattice points in the intersection of $C_p$ with the hyperplane $X_i = n$ correspond bijectively to the $H \in \mathcal{H}_{pq}$ such that $q \in \text{Ap}(H, p)$, and the lattice points in the intersection of $C_p$ with the half-space $X_i \leq n$ to all $H \in \mathcal{H}_{pq}$. In fact, if $\mu(H) = (\mu_1, \ldots, \mu_{p-1})$ and $\mu_i \leq n$, then $i + \mu_i p \in H$ implies that also $q \in H$.

More generally, let $\alpha \in \mathbb{N}^{p-1} \setminus \{0\}$ be a primitive lattice point, that is, if $\alpha = (\alpha_1, \ldots, \alpha_{p-1})$, then $\alpha_1, \ldots, \alpha_{p-1}$ are relatively prime. For $x \in \mathbb{R}^{p-1}$ let $\alpha \cdot x$ denote the scalar product of $\alpha$ and $x$, and $H_n = H_n(\alpha) := \{x \in \mathbb{R}^{p-1} | \alpha \cdot x = n\}$ for $n \in \mathbb{N}$. We assume that no edge of $C_p^*$ is contained in the hyperplane $H_0$. This condition is satisfied in the Examples 4.1. If in 4.1b) there would be an edge vector $\delta = (\mu_1, \ldots, \mu_{i-1}, 0, \mu_{i+1}, \ldots, \mu_{p-1})$ there would be infinitely many lattice points on the edge of $C_p^*$ determined by $\delta$. These correspond to semigroups $H \in \mathcal{H}$ with $i \in H$. Since $i$ is prime to $p$ there exist only finitely many such $H$, a contradiction.

Let $\bar{S}$ be the topological closure of a $d$-dimensional face $S$ of $C_p$ and let $P_{\alpha,n}$
resp. $P_{α,n}^S$ be the intersection of the hyperplane $H_n$ with the cone $C_p$ resp. $S$. For $n ≥ 1$ the sets $P_{α,n}$ and $P_{α,n}^S$ are rational polytopes of dimension $p - 2$ resp. $d - 1$. We are interested in the numbers $f_α(n)$ resp. $f_α^S(n)$ of lattice points in $P_{α,n}$ and $P_{α,n}^S$ and in the number $f_α^0(n)$ of lattice points in $C_p^* \cap P_{α,n}$.

4.2. Theorem. Let $δ_1, \ldots, δ_r$ be the canonical system of representatives for the edges of $C_p^*$ and $δ_1, \ldots, δ_r$ the edge vectors contained in $S^*$ where $S^*$ is the face of $C_p^*$ parallel to $S$. If $S$ contains a lattice point $u ∈ \mathbb{N}^{p−1}$, then $f_α^S$ is a quasi-polynomial of degree $d − 1$ with non-negative leading coefficient. The least common multiple of $\{α · δ_i\}_{i=1,\ldots, r}$ is a quasi-period of $f_α^S$. In particular $f_α$ is a quasi-polynomial of degree $p − 2$. Moreover for $n ≥ \sum_{i=1}^{p−1} α_i$

$$f_α^0(n) = f_α(n - \sum_{i=1}^{p−1} α_i).$$

Remember that a function $f : \mathbb{N} → \mathbb{C}, f \neq 0$ with generating function $H_f(T) := \sum_{n=0}^{∞} f(n)T^n$ is called a quasi-polynomial of degree $d$ and quasi-period $N > 0$ if $f$ is of the form

$$f(n) = c_d(n)n^d + c_{d−1}(n)n^{d−1} + \cdots + c_0(n) \quad (n ∈ \mathbb{N})$$

where the $c_i : \mathbb{N} → \mathbb{C}$ are periodic functions with integral period $N$ and $c_d(n)$ does not vanish identically. Equivalently, $f$ is a quasi-polynomial if there exist an integer $N > 0$ and polynomials $f_0, \ldots, f_{N−1}$ such that

$$f(n) = f_i(n) \text{ if } n \equiv i \text{ mod } N.$$

If there exists an integer $N > 0$ and polynomials $P(T), Q(T) ∈ \mathbb{C}[T] \setminus \{0\}$ with $\deg P(T) < \deg Q(T)$ so that

$$H_f(T) = \frac{P(T)}{Q(T)}$$

and $α^N = 1$ for each zero $α$ of $Q$, then $f$ is a quasi-polynomial with quasi-period $N$. Its degree is one less than the maximum pole order of the rational function $\frac{P(T)}{Q(T)}$ ([S2],4.4.1). $f ≡ 0$ is also considered as a quasi-polynomial.

Proof of Theorem 4.2. At first we show that $f_α^S$ is a quasi-polynomial. Since $H_{α}(α)$ does not contain an edge of $S^*$ the denominator $\prod_{i=1}^{r}(1 − X^{δ_i})$ of the generating function $H_S$ (see Proposition 2.2) does not vanish if we replace the $X_j$ by $T^{αj}$ with a variable $T$. Then

$$H_S(T^{α_1}, \ldots, T^{α_p−1}) = \frac{F_S(T^{α_1}, \ldots, T^{α_p−1})}{\prod_{i=1}^{r}(1 − T^{αδ_i})} = \sum_{n=0}^{∞} f_α^S(n)T^n.$$

Set $P(T) := F_S(T^{α_1}, \ldots, T^{α_p−1})$ and $Q(T) := \prod_{i=1}^{r}(1 − T^{αδ_i})$. Since $\deg X_j, H_S ≤ −1$ we have $\deg P < \deg Q$. Hence $f_α^S(n)$ is a quasi-polynomial with quasi-period as stated. Its leading coefficient is non-negative since $f_α^S(n) ∈ \mathbb{N}$ for all $n ∈ \mathbb{N}$. 

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Now we show that \( f_S^\delta \) has degree \( d - 1 \). By assumption \( \tilde{S} \) contains a lattice
point \( u \in \mathbb{N}^{p-1} \). On the line through \( u \) and the vertex \( v \) of \( C_p \) there is a lattice
point \( w \) not contained in \( S \), hence \( -w \in \mathbb{N}^{p-1} \) and
\[
(1) \quad u + \overline{S} \subset \tilde{S} \subset w + \overline{S}.
\]
Let \( P_{\alpha,n}^* := \tilde{S} \cap H_n \) for \( n \in \mathbb{N}_+ \). Then \( P_{\alpha,t}^* \) is a convex polytope of dimension
\( d - 1 \) and \( P_{\alpha,n}^* = n \cdot P_{\alpha,t}^* \). To this situation a theorem of Ehrhart ([S2],4.6.8)
can be applied. It states that if \( i(P_{\alpha,n}^*) \) is the number of lattice points in \( P_{\alpha,n}^* \),
then this function of \( n \) is a quasi-polynomial of degree \( d - 1 \).

In order to show this also for \( f_S^\delta \) observe that (1) implies that for \( n \geq l := \alpha \cdot u \)
and \( k := \alpha \cdot w \) we have
\[
u + P_{\alpha,n-l}^* \subset \tilde{S} \cap H_n \subset w + P_{\alpha,n-k}^*.
\]
Thus for large \( n \) the quasi-polynomial \( f_S^\delta \) is trapped by two quasi-polynomials
of degree \( d - 1 \), hence it has also the degree \( d - 1 \).

The formula for \( f_n^0 \) follows from Remark 2.1 after substituting \( X_i = T^{\alpha_i} \) \( (i = 1, \ldots, p - 1) \)
and expanding into power series in \( T \). \( \square \)

In the situation of Example 4.1a) let \( G(p,g) \) be the number of \( H \in \mathcal{H}_p \) with
genus \( g \). Then with \( \alpha = (1, \ldots, 1) \) we have \( G(p,g) = f_\alpha(n) \), and Theorem
4.2 tells us that \( G(p,g) \), as a function of \( g \), is a quasi-polynomial of degree
\( p - 2 \) with non-negative leading term. Moreover the least common multiple of
\( \{ \alpha \cdot \delta_i \}_{i=1}^{r} \) is a quasi-period of \( G(p,g) \).

\[
G^0(p,g) := f_\alpha^0(g) \quad \text{is the number of } H \in \mathcal{H}_p \quad \text{having maximal embedding}
\]
dimension \( p \) and genus \( g \). For \( g \geq p - 1 \) we have
\[
(2) \quad G^0(p,g) = G(p,g - (p - 1)).
\]

If \( G_{sym}(p,g) \) is the the number of symmetric \( H \in \mathcal{H}_p \) with genus \( g \), then Proposition 3.1 and Theorem 4.2 imply that \( G_{sym}(p,g) \) is a quasi-polynomial of degree
\( \lfloor \tfrac{d}{2} \rfloor - 1 \).

In the situation of Example 4.1b) we can apply Theorem 4.2 to the functions
\( f_S^\delta \) where \( \epsilon_i \) is the i-th unit vector and \( \tilde{S} \) the closure of a face of \( C_p \)
containing a lattice point. Let \( g_{\epsilon_i}(n) := \sum_{j=0}^{n} f_{\epsilon_i}(j) \) and \( g_{\epsilon_i}^0(n) := \sum_{j=0}^{n} f_{\epsilon_i}^0(j) \), that is, the
number of \( H \in \mathcal{H}_p \) (of maximal embedding dimension \( p \)) containing also \( i + np \).
In order to apply Theorem 4.2 also to \( g_{\epsilon_i} \) and \( g_{\epsilon_i}^0 \) we need the following facts.

For functions \( f : \mathbb{N} \to \mathbb{C} \) we consider the operators
\[
E : f(n) \to f(n + 1) \quad \text{(shift)}
\]
\[
\Delta : f(n) \to f(n + 1) - f(n) \quad \text{(difference)}
\]
\[
\Sigma : f(n) \to \sum_{i=0}^{n} f(i) \quad \text{(sum)}.
\]
It is easy to see that \( f \) is a quasi-polynomial if and only if this is the case for
\( Ef, \Delta f \) or \( \Sigma f \). Moreover \( \deg Ef = \deg f \).
4.4. Lemma. Let \( f : \mathbb{N} \to \mathbb{R}, f \neq 0 \) be a quasi-polynomial of degree \( d \).

a) If \( f \) is increasing, then \( c_d(n) \) is constant.

b) If \( f \geq 0 \), then \( \deg\Sigma f = d + 1 \).

Proof. a) Let \( N \) be a quasi-period of \( f = \sum_{k=0}^{d} c_k(n) \) and \( f_0, \ldots, f_{N-1} \) the polynomials with \( f(n) = f_i(n) \) for \( i = 0, \ldots, N-1 \). Then \( c_d(i) \) is the coefficient of \( t^d \) in \( f_i(t) \) for \( i = 0, \ldots, N-1 \). With \( k \in \mathbb{N}_+ \) and \( i \in \{0, \ldots, N-1\} \) we have

\[
 f_0(kN) = f(kN) \leq f(i+kN) = f_i(i+kN) \leq f(N+kN) = f_0((k+1)N).
\]

If follows that

\[
 c_d(0) = \lim_{k \to \infty} \frac{f_0(kN)}{(kN)^d} \leq c_d(i) = \lim_{k \to \infty} \frac{f_i(i+kN)}{(kN)^d} \leq \lim_{k \to \infty} \frac{f_0((k+1)N)}{(kN)^d} = \frac{c_d(0)}{k}^{d} = c_d(0),
\]

hence \( c_d(0) = c_d(i) \) and \( c_d \) is constant.

b) If \( c_d(n) = c_d \) is constant, then for \( n \equiv i \mod d \)

\[
 f(n+1) - f(n) = c_d(n+1)^d - c_d n^d + g
\]

with a quasi-polynomial \( g \) of degree \( \leq d - 1 \). Hence \( \deg(\Delta f) \leq d - 1 \) for an increasing \( f \). As \( \Sigma f \) is increasing for \( f \geq 0 \) and \( \Delta \Sigma f = Ef \) we have

\[
 \deg \Sigma f \geq \deg(\Sigma f) + 1 = \deg Ef + 1 = d + 1.
\]

Since \( f \geq 0 \) there exists \( k > 0 \) so that for \( n > k \) all polynomial functions \( f_j(n) \) are increasing. Moreover there exists \( i \in \{0, \ldots, N-1\} \) such that \( f_i(n) \geq f_j(n) \) for all \( j \) and \( n > k \), if \( k \) is sufficiently large. Then for \( n > k \)

\[
 f(0) + f(1) + \cdots + f(n) \leq f(0) + \cdots + f(k) + (n-k)f_i(n) := h(n)
\]

where \( h(n) \) is a polynomial of degree \( \leq d + 1 \). Consequently \( \deg \Sigma f \leq d + 1 \), and b) follows. \( \square \)

Theorem 4.2 and Lemma 4.4 imply

4.5. Proposition. Let \( i \in \{1, \ldots, p-1\} \) be prime to \( p \) and \( N(p, i + np) := g_e(n) \) resp. \( \text{Medim}(p, i + np) := g^0_e(n) \) the number of \( H \in \mathcal{H}_p \) (of maximal embedding dimension \( p \)) with \( i + np \in H \). Then the functions \( N(p, i + np) \) and \( \text{Medim}(p, i + np) \) of the variable \( n \) can be expressed as quasi-polynomials of degree \( p - 1 \) having the same highest coefficient, which is independent of \( n \). More precisely for each \( q > p \) which is prime to \( p \)

\[
 \text{Medim}(p, q) = N(p, q - p).
\]

For the last formula note that \( f_e(0) = 0, f_e(n) = f_e(n-1) \) for \( n \geq 1 \) by 4.2, hence \( \text{Medim}(p, i + np) = g^0_e(n) = \sum_{k=0}^{n} f^0_e(k) = \sum_{k=0}^{n-1} f_e(k) = g_e(n - 1) = N(p, i + (n-1)p) \).
Similarly as in Proposition 4.5, if $H(p, g)$ denotes the number of $H \in \mathcal{H}_p$ of genus $\leq g$, then $H(p, g)$ is a quasi-polynomial of $g$ with degree $p - 1$ and constant highest coefficient.

As in Proposition 3.1 let $\bar{S}_1, \ldots, \bar{S}_t$ be the closures of the faces of dimension $\lfloor \frac{p}{2} \rfloor$ of $C_p$ whose lattice points correspond to the symmetric $H \in \mathcal{H}_p$. Let $\text{Sym}(p, i + np)$ be the number of symmetric $H \in \mathcal{H}_p$ containing $i + np$. Application of Theorem 4.2 and Lemma 4.4 to $\Sigma f_{e_{i_j}}$ ($j = 1, \ldots, t$) yields

4.6. Proposition. $\text{Sym}(p, i + np)$ is, as a function of $n$, a quasi-polynomial of degree $\lfloor \frac{p}{2} \rfloor$ whose leading coefficient is independent of $n$.

5. Asymptotic estimates and recursion formulas

For an integer $q$ which is prime to $p$ let $\mathcal{H}_{pq}$ be the set of all $H \in \mathcal{H}_p$ with $q \in H$. We are interested in the functions $N(p, q), \text{Sym}(p, q)$ and $\text{Psym}(p, q)$ where $N(p, q)$ is the number of elements of $\mathcal{H}_{pq}$, $\text{Sym}(p, q)$ resp. $\text{Psym}(p, q)$ the number of symmetric resp. pseudo-symmetric $H \in \mathcal{H}_{pq}$.

As in [KKW] and [KW] we associate with each $H \in \mathcal{H}_{pq}$ ($q > p$) a certain lattice path in the plane which is contained in the triangle $\Delta_0$ bounded by the line $g_0 : p(X + 1) + q(Y + 1) = pq$ and the coordinate axes, starts on the $Y$-axis, ends on the $X$-axis and has only right or downward steps. In the following the word ”lattice path” always means such a path. The set of all lattice paths for given $p, q$ is called the $(p, q)$-system.

The lattice path belonging to $H \in \mathcal{H}_{pq}$ is constructed as follows. A semigroup $H \in \mathcal{H}_{pq}$ is obtained from $H_{pq} = < p, q >$ by closing some of its gaps. Each such gap $\gamma$ can be written

(1) $\gamma = pq - (a + 1)p - (b + 1)q$

with a unique $(a, b) \in \Delta_0$. The set $L_H$ of all these $(a, b)$ is bounded by a lattice path and the coordinate axes. It is by definition the path associated to $H$. 

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We use the notation \((P_0, P_1, \ldots, P_m)\) for lattice paths where \(P_0\) is the point where it starts, \(P_m\) the point where it ends, and the other \(P_i\) are the points where after a right step a downward step follows. As is seen from (1) a downward step in the lattice path of \(H\) means for the corresponding \(\gamma\) an addition of \(q\), a right step a subtraction of \(p\). If \((a, b)\) is a point of the lattice path, then the corresponding \(\gamma\) is the smallest element of \(H\) in the residue class of \(- (b + 1) q \mod p\), hence an element of \(\text{Ap}(H, p)\), and different \(b\) give different elements of the Apéry set. Thus the lattice path is given by the Apéry set and conversely also determines this set. This also indicates the relation to the points of the polyhedral cone \(C_p\). To \(H = H_{pq}\) we may associate the empty path and the empty set \(L\).

Not every lattice path as above belongs to a semigroup. For an arbitrary lattice path let \(L\) be the set of lattice points in the area bounded by the path and the coordinate axes. In order that \(L = L_H\) with an \(H \in \mathcal{H}_{pq}\) the following conditions must be satisfied ([KKW])

a) For \((a, b), (a', b') \in L\) with \(a + a' \geq q - 1\) also \((a + a' - q + 1, b + b' + 1)\) must be in \(L\).

b) For \((a, b), (a', b') \in L\) with \(b + b' \geq p - 1\) also \((a + a' + 1, b + b' - p + 1)\) must be in \(L\).

Lattice paths whose corresponding set \(L\) satisfied a) and b) were called \textit{admissible}, and their number was denoted by \(L(p, q)\).

\textbf{5.1. Lemma.} \(L(p, q)\) as a function of \(p\) and of \(q\) is increasing.

\textbf{Proof.} Let \((\tilde{p}, \tilde{q})\) be another pair of relatively prime integers with \(\tilde{p} \geq p, \tilde{q} \geq q\). Let \(\Delta_0\) be the triangle bounded by the line \(\tilde{p}(X + 1) + \tilde{q}(Y + 1) = \tilde{p}\tilde{q}\) and the axes. Clearly \(\Delta_0 \subset \Delta_0\) so that any lattice path in \(\Delta_0\) is also one in \(\Delta_0\). Let \(L \subset \Delta_0\) be the set of lattice points corresponding to it. If it satisfies the admissibility
conditions a) and b) above, then they are also satisfied in the \((\bar{p}, \bar{q})\)-system: Let 
\((a, b), (a', b') \in L\) and \(a + a' \geq q - 1\). If 
\((a + a' - q + 1, b + b' + 1) \in L\), then also 
\((a + a' - \bar{q} + 1, b + b' + 1) = (a + a' - q + 1, b + b' + 1) - (\bar{q} - q, 0) \in L\). If 
b + b' \geq p - 1 the proof is analogous. □

If \(q = i + np\) and \(g_{e_i}\) is the quasi-polynomial studied in Section 4 we have

(2) \[ N(p, q) = L(p, q) + 1 = g_{e_i}(n) \]

where the 1 comes from \(H_{pq}\) or the empty lattice path. Proposition 4.5 and
Lemma 5.1 imply

5.2. Proposition. \(\lim_{q \to \infty} \frac{N(p, q)}{q^{p-1}}\) and \(\lim_{q \to \infty} \frac{\text{Medim}(p,q)}{q^{p-1}}\) exist and are equal.

The above limits give asymptotic estimates of how many \(H \in \mathcal{H}_{pq}\) (of maximal
embedding dimension \(p\)) exist. With a somewhat different approach it is shown in [HW] that also for an arbitrary \(q\) which is prime to \(p\) the function \(N(p, q)\) is a quasi-polynomial in \(q\) of degree \(p - 1\) and estimates of its (constant)
highest coefficients are given, i.e. of the above limits.

It was shown in [KW] that the admissible lattice paths starting at the point
\((0, p - 2)\) are in one-to-one correspondence with the semigroups of \(\mathcal{H}_{p,q-p}\) (if
\(q - p < p\) exchange \(p\) and \(q - p\)). Thus we have the recursion formula

(3) \[ N(p, q) = N_{pq} + N(p, q - p) + 1 \quad (q > p) \]

where \(N_{pq}\) is the number of admissible lattice paths starting at \((0, b)\) with \(b \in \mathbb{N}, b \leq p - 3\).

Also for the numbers \(\text{Sym}(p, q)\) resp. \(\text{Psym}(p, q)\) of symmetric (pseudosymmetric) \(H \in \mathcal{H}_{pq}\) one has recursion formulas

(4) \[ \text{Sym}(p, q) = S_{pq} + \text{Sym}(p, q - p) + 1 \quad (q > p) \]

(5) \[ \text{Psym}(p, q) = P_{pq} + \text{Psym}(p, q - p) \quad (q > p) \]

where \(S_{pq}\) (resp. \(P_{pq}\)) is the number of admissible lattice paths starting at
a point \((0, b)\) with \(b \leq p - 3\) and defining a symmetric (pseudo-symmetric) semigroup. We shall use the formulas (3)-(5) in the next section to derive explicit recursion formulas for \(N(p, q), \text{Sym}(p, q)\) and \(\text{Psym}(p, q)\) in the cases \(p = 3, p = 4\).
6. Examples

The function $G(p, g)$ tells us in how many ways we can remove $g$ numbers from $\mathbb{N}$, where 0 and $p$ are not removed, so that the remaining set is additively closed. In [Ku], Appendix C various generating functions for $\alpha = (1, \ldots, 1)$ and $p \leq 5$ are listed. The following explicit formulas can be easily derived:

$$G(3, g) = \left\lfloor \frac{g}{3} \right\rfloor + 1$$

$$G(4, g) = \frac{1}{12} g^2 + \frac{1}{2} g + \begin{cases} 
1 & \text{if } g \equiv 0 \Mod 6 \\
\frac{5}{12} & \text{if } g \equiv 1 \Mod 6 \\
\frac{2}{3} & \text{if } g \equiv 2 \Mod 6 \\
\frac{3}{4} & \text{if } g \equiv 3 \Mod 6 \\
\frac{4}{3} & \text{if } g \equiv 4 \Mod 6 \\
\frac{5}{12} & \text{if } g \equiv 5 \Mod 6 
\end{cases} = \left\lfloor \frac{1}{12} g^2 + \frac{1}{2} g \right\rfloor + 1$$

$$G_{\text{sym}}(3, g) = \begin{cases} 
1 & \text{if } g \not\equiv 2 \Mod 3 \\
0 & \text{if } g \equiv 2 \Mod 3 
\end{cases}$$

For $p = 3$ see also Fig.1. For $G^0(p, g)$ see formula (2) of Section 4.

| $g$ | $G(3, g)$ | $G^0(3, g)$ | $G_{\text{sym}}(3, g)$ | $G(4, g)$ | $G^0(4, g)$ | $G_{\text{sym}}(4, g)$ |
|-----|-----------|-------------|------------------------|-----------|-------------|------------------------|
| 0   | 1         | 0           | 1                      | 1         | 0           | 1                      |
| 1   | 1         | 0           | 1                      | 1         | 0           | 1                      |
| 2   | 1         | 1           | 0                      | 2         | 0           | 1                      |
| 3   | 2         | 1           | 1                      | 3         | 1           | 2                      |
| 4   | 2         | 1           | 1                      | 4         | 1           | 2                      |
| 5   | 2         | 2           | 0                      | 5         | 2           | 2                      |
| 6   | 3         | 2           | 1                      | 7         | 3           | 3                      |
| 7   | 3         | 2           | 1                      | 8         | 4           | 3                      |
| 8   | 3         | 3           | 0                      | 10        | 5           | 3                      |

The following formulas were communicated to us by H. Knebl:

$$G(5, g) = \frac{1}{135} g^3 + \frac{4}{45} g^2 + R(i)$$

$$G_{\text{sym}}(5, g) = \begin{cases} 
\frac{1}{9} g + S(i) & \text{if } g \not\equiv 3 \Mod 5 \\
0 & \text{if } g \equiv 3 \Mod 5 
\end{cases}$$

with $R(i)$ and $S(i)$, depending on $i \equiv g \Mod 30$, as in the table below.
In [Ka], table 1 an extensive list with values of the function $S(m, g)$ counting the semigroups with multiplicity $m$ and genus $g$ is given. If $m$ is a prime number, then $S(m, g) = G(m, g)$ for large $g$, since there are only finitely many $H \in S_m$ of multiplicity $< m$.

In the following the recursion formulas (3)-(5) from Section 5 will be used.

1) For $p = 3$, in order to determine $N_{3q}, S_{3q}$ and $P_{3q}$, we have only to consider admissible lattice paths on the $X$-axis. These are the paths ending at $(j, 0)$ with $j \leq \left\lfloor \frac{q}{2} \right\rfloor - 1$. They correspond to the semigroups $H_j = \langle 3, q, 2q - 3(j + 1) \rangle$ where $H_j$ is symmetric if and only if $q$ is even and $j = \frac{q}{2} - 1$ and $H_j$ is pseudo-symmetric if and only if $j = 0$ or $q$ is odd and $j = \frac{q-1}{2} - 1$ ([KW], Example 2.5b). Thus $N_{3q} = \left\lfloor \frac{q}{2} \right\rfloor$ and

\[(1) \quad N(3, q) = N(3, q - 3) + \left\lfloor \frac{q}{2} \right\rfloor + 1 \quad (q > 3),\]
and for \( q \geq 4 \)

\[
(2) \quad \text{Sym}(3, q) = \text{Sym}(3, q - 3) + \begin{cases} 2 & \text{if } q \text{ is even} \\ 1 & \text{if } q \text{ is odd} \end{cases}
\]

\[
(3) \quad \text{Psym}(3, q) = \text{Psym}(3, q - 3) + \begin{cases} 1 & \text{if } q \text{ is even} \\ 2 & \text{if } q \text{ is odd} \end{cases}.
\]

Clearly \( \text{Sym}(3, 1) = 1 \) and \( \text{Sym}(3, 2) = 2 \), further \( \text{Psym}(3, 1) = \text{Psym}(3, 2) = 0 \).

For \( \text{Medim}(p, q) \), thanks to Proposition 4.5, no further discussion is necessary.

| \( q \) | \( N(3, q) \) | \( \text{Medim}(3, q) \) | \( \text{Sym}(3, q) \) | \( \text{Psym}(3, q) \) |
|----|---|---|---|---|
| 1  | 1  | 0  | 1  | 0  |
| 2  | 2  | 0  | 2  | 0  |
| 4  | 4  | 1  | 3  | 1  |
| 5  | 5  | 2  | 3  | 2  |
| 7  | 8  | 4  | 4  | 3  |
| 8  | 10 | 5  | 5  | 3  |
| 10 | 14 | 8  | 6  | 4  |
| 11 | 16 | 10 | 6  | 5  |
| 13 | 21 | 14 | 7  | 6  |
| 14 | 24 | 16 | 8  | 6  |

The numbers in the columns can also be found by using figure 1. By what was said there we have \( N(3, q) = \text{Medim}(3, q) + \text{Sym}(3, q) \), and by Proposition 4.5 \( \text{Medim}(3, q) = N(3, q - 3) \).

6.1. **Proposition.** a) \( \lim_{q \to \infty} \frac{N(3, q)}{q^2} = \lim_{q \to \infty} \frac{\text{Medim}(3, q)}{q^2} = \frac{1}{12} \).

b) \( \lim_{q \to \infty} \frac{\text{Sym}(3, q)}{q} = \lim_{q \to \infty} \frac{\text{Psym}(3, q)}{q} = \frac{1}{2} \).

**Proof.** a) By [KW], Example 3.5 we have \( N(3, q) = \lfloor \frac{q^2}{12} + \frac{q}{2} \rfloor + 1 \) which implies a). Alternately, from the recursion formula (1) follows for \( q \geq 7 \) that

\[
N(3, q) - N(3, q - 6) = \lfloor \frac{q}{2} \rfloor + \lfloor \frac{q - 3}{2} \rfloor + 2 = q
\]

\[
N(3, q - 6(i - 1)) - N(3, q - 6i) = q - 6(i - 1) \quad (i \geq 1)
\]

hence

\[
N(3, q) - N(3, q - 6\lfloor \frac{q}{6} \rfloor) = \lfloor \frac{q}{6} \rfloor q - 6 \sum_{i=0}^{\lfloor \frac{q}{6} \rfloor - 1} i = \frac{q^2}{12} + \frac{q}{2} - 3(\lfloor \frac{q}{6} \rfloor - \lfloor \frac{q}{6} \rfloor + 1)(\frac{q}{6} - \lfloor \frac{q}{6} \rfloor)
\]

which also gives a).
b) By (2) and (3) for \( q \geq 7 \)

\[
\text{Sym}(3, q) - \text{Sym}(3, q - 6) = 3 = \text{Psym}(3, q) - \text{Psym}(3, q - 6),
\]

hence

\[
\text{Sym}(3, q) - \text{Sym}(3, q - 6) + \left\lfloor \frac{q}{6} \right\rfloor = 3 + \left\lfloor \frac{q}{6} \right\rfloor = \text{Psym}(3, q) - \text{Psym}(3, q - 6) + \left\lfloor \frac{q}{6} \right\rfloor,
\]

from which b) follows. \( \square \)

II) For \( p = 4 \), in order to determine \( N_{4q}, S_{4q} \) and \( P_{4q} \), we have to investigate the lattice paths starting at \((0, 1)\) or \((0, 0)\) and the corresponding semigroups.

By [KW], 2.6 there are \( \frac{(2+q')}{2} - 1 = \frac{1}{6}(q^2 + 4q + 3) - 1 (q' = \frac{q - 1}{2}) \) lattice paths in the rectangle \( R \) with the corners \((0, 0), (0, 1), (q' - 1, 1), (q' - 1, 0) \) and all are admissible. To find \( N_{4q} \) we have also to determine the admissible paths starting at \( P_0 = (0, 1) \) and ending at a point \((q' + i, 0)\) \((i = 0, \ldots, q - 2 - \left\lfloor \frac{q}{4} \right\rfloor - q'\).

Such a lattice path is admissible if and only if it contains \((2i, 1)\) and avoids \((q' - i, 1)\). Thus it is of the form \((P_0, P_1, P_2)\) with \( P_1 = (j, 1), P_2 = (q' + i, 0)\) \((2i \leq j < q' - i)\) and defines the semigroup

\[
G_{ij} := 4, q, 2q - 4(j + 1), 3q - 4(q' + i + 1) > .
\]

Given \( i \) there are \( q' - 3i \) such paths, the number of possible downward steps, where \( i \leq \left\lfloor \frac{q}{4} \right\rfloor = \left\lfloor \frac{q - 1}{6} \right\rfloor \). Note that the last condition implies that

\[
i \leq q - 2 - \left\lfloor \frac{q}{4} \right\rfloor - q'
\]

so that we stay below the line \( g_0 \). Altogether there are

\[
\sum_{i=0}^{\left\lfloor \frac{q - 1}{6} \right\rfloor} (q' - 3i) = \left(\left\lfloor \frac{q - 1}{6} \right\rfloor + 1\right)\frac{q - 1}{2} - \frac{3}{2}\left(\left\lfloor \frac{q - 1}{6} \right\rfloor + 1\right)\frac{q - 1}{2}
\]

such lattice paths (semigroups \( G_{ij} \)), and therefore for \( q > 4 \)

\[
N(4, q) = \frac{1}{8}(q^2 + 4q + 3) + (\left\lfloor \frac{q - 1}{6} \right\rfloor + 1)\frac{q - 1}{2} - \frac{3}{2}\left(\left\lfloor \frac{q - 1}{6} \right\rfloor + 1\right)\frac{q - 1}{2} + N(4, q - 4).
\]

Clearly \( N(4, 1) = 1, N(4, 3) = 4 \).

In order to determine the numbers \( S_{4q} \) and \( P_{4q} \) we first check which of the semigroups \( G_{ij} \) are symmetric resp. pseudo-symmetric. This is done by considering their genus and Frobenius number.

\( G_{ij} \) is obtained from \( H_{4q} \) by closing \( q' + i + j + 2 \) of its gaps, hence \( G_{ij} \) has genus

\[
g(G_{ij}) = \frac{3}{2}(q - 1) - (q' + i + j + 2) = 2q' - (i + j + 2).
\]

The parallel \( 4(X + 1) + q(Y + 1) = 3q + 4 \) to \( g_0 \) through \((0, 2)\) passes also through \((\frac{q}{4}, 1)\) and cuts the \( X \)-axis at \((\frac{q}{4}, 0)\). If \( j \geq \left\lfloor \frac{q}{4} \right\rfloor \), then by [KW], 3.1 the point \((0, 2)\) corresponds to the Frobenius number of \( G_{ij} \), hence

\[
F(G_{ij}) = 3(q - 1) - 1 - 2q = q - 4 = 2q' - 3
\]
and

\[ 2g(G_{ij}) - F(G_{ij}) - 1 = 2(q' - i - j - 1), \]

an even number. Therefore \( G_{ij} \) is not pseudo-symmetric, and symmetric if and only if \( j = q' - i - 1 \). Furthermore \( q' - i - 1 \geq \left\lceil \frac{q}{4} \right\rceil \) by (4). Thus \( \left\lceil \frac{q-1}{6} \right\rceil + 1 \) of the \( G_{ij} \) are symmetric for \( q \not\equiv 1 \mod 6 \), and \( \left\lceil \frac{q-1}{6} \right\rceil \) are symmetric for \( q \equiv 1 \mod 6 \).

In case \( j < \left\lfloor \frac{q}{4} \right\rfloor \) the point \((j+1, 1)\) corresponds to \( F(G_{ij}) \), hence

\[ F(G_{ij}) = 3(q - 1) - 1 - 4(j + 1) - q = 4q' - 4j - 6 \]

and

\[ 2g(G_{ij}) - F(G_{ij}) - 1 = 2j - 2i + 1. \]

In this case \( G_{ij} \) is not symmetric, and pseudo-symmetric only for \( j = i \). Since \( j \geq 2i \) this implies \( i = j = 0 \).

According to [KW] 2.8, of the semigroups \( \tilde{G}_{ij} \) corresponding to lattice paths in the rectangle \( R \) exactly \( q' \) are symmetric and just one is pseudo-symmetric. Altogether for \( q \geq 7 \)

\[ S_{4q} = \begin{cases} \left\lceil \frac{q-1}{6} \right\rceil + \frac{q-1}{2} & \text{if } q \equiv 1 \mod 6 \\ \left\lceil \frac{q-1}{6} \right\rceil + \frac{q-1}{2} + 1 & \text{if } q \not\equiv 1 \mod 6 \end{cases} \]

and \( P_{4q} = 2 \). Thus

\[ \text{Sym}(4, q) = \text{Sym}(4, q-4) + 1 + \begin{cases} \left\lceil \frac{q-1}{6} \right\rceil + \frac{q-1}{2} & \text{if } q \equiv 1 \mod 6 \\ \left\lceil \frac{q-1}{6} \right\rceil + \frac{q-1}{2} + 1 & \text{if } q \not\equiv 1 \mod 6 \end{cases} \]

\[ \text{Psym}(4, q) = \text{Psym}(4, q-4) + 2. \]

It is easy to see that \( \text{Sym}(4, 3) = 3, \text{Sym}(4, 5) = 5, \text{Psym}(4, 3) = 1, \text{Psym}(4, 5) = 2 \).

| \( q \) | \( N(4, q) \) | \( \text{Medim}(4, q) \) | \( \text{Sym}(4, q) \) | \( \text{Psym}(4, q) \) |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 3 | 4 | 0 | 3 | 1 |
| 5 | 9 | 1 | 5 | 2 |
| 7 | 17 | 4 | 8 | 3 |
| 9 | 29 | 9 | 11 | 4 |
| 11 | 45 | 17 | 15 | 5 |
| 13 | 66 | 29 | 19 | 6 |
| 15 | 93 | 45 | 25 | 7 |

6.2. Proposition. \( \lim_{q \to \infty} \frac{N(4, q)}{q^4} = \lim_{q \to \infty} \frac{\text{Medim}(4, q)}{q^4} = \frac{1}{12} \).

Proof. We know by 5.2 that the limits exists and are equal. Therefore it is enough to consider \( q \) of the form \( q = 4m + 1 \) \((m > 0)\). By formula (5) we obtain for \( q \geq 5 \)

\[ N(4, q) - N(4, q - 4) = \frac{1}{8}(q^2 + 4q + 3) + \left(\left\lceil \frac{q-1}{6} \right\rceil + 1\right)\left(\frac{q-1}{2} - \frac{3}{2}\left\lceil \frac{q-1}{6} \right\rceil\right) \]
\[
= \frac{8}{3} m^2 + g(m)
\]
with a quasi-polynomial \( g \) of degree 1, \( g(0) = 1 \). Therefore with the sum operator \( \Sigma \)
\[
N(4, q) = \frac{8}{3} \sum_{j=1}^{m} j^2 + (\Sigma g)(m).
\]
By Fibonacci’s formula (Liber abaci 1202) \( \sum_{j=1}^{m} j^2 = \frac{1}{6} m(m + 1)(2m + 1) \) this implies
\[
N(4, q) = \frac{8}{9} m^3 + h(m) = \frac{1}{72} (q - 1)^3 + h(m)
\]
with a quasi-polynomial \( h \) of degree 2, and the assertion about the limits follows.

\[\square\]

\( N(4, q) \) and \( N(5, q) \) have been explicitly computed by H. Knebl (see [HW], Example 4.3).

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