LEVI LAPLACIANS AND INSTANTONS ON MANIFOLDS

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Dedicated to the memory of Alexander A. Belyaev

Abstract: The equivalence of the anti-selfduality Yang-Mills equations on the 4-dimensional orientable Riemannian manifold and Laplace equations for some infinite dimensional Laplacians is proved. A class of modified Levi Laplacians parameterized by the choice of a curve in the group \( SO(4) \) is introduced. It is shown that a connection is an instanton (a solution of the anti-selfduality Yang-Mills equations) if and only if the parallel transport generalized by this connection is a solution of the Laplace equations for some three modified Levy Laplacians from this class.

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Introduction

One of the main causes of the interest in infinite dimensional differential operators of the Levy type is their connection with the Yang-Mills fields. The Yang-Mills equations on a connection \( A \) in the vector bundle over \( d \)-dimensional orientable Riemannian manifold \( M \) are

\[
D_{A}^* F = 0,
\]

where \( F = dA + A \wedge A \) is the curvature of \( A \) and \( D_{A}^* \) is the adjoint operator to the exterior covariant derivative. The Yang-Mills fields are solutions of the Yang-Mills equations. The parallel transport \( U \) generated by the connection \( A \) can be considered as a section in some vector bundle over the Hilbert manifold of \( H^1 \)-curves with the fixed origin in \( M \) (if \( M = \mathbb{R}^d \) the parallel transport can be considered as an operator-valued function on the Hilbert space of \( H^1 \)-curves with the origin at zero). The theorem proved by Accardi, Gibilisco and Volkovich in [3] states that the connection \( A \) in a vector bundle over \( \mathbb{R}^d \) satisfies the Yang-Mills equations if and only if

\[
\Delta_{AGV}^{L} U = 0,
\]

where \( \Delta_{AGV}^{L} \) is some infinite dimensional Laplacian. This Laplacian was defined by analogy with the famous Levy Laplacian \( \Delta_{L} \) (see [23]) and was also called the same name. Accardi-Gibilisco-Volkovich theorem was generalized for Riemannian manifolds by Leandre and Volovich in [22].

In the case \( d = 4 \), the Hodge dual \( * \) transforms 2-forms on \( M \) into 2-forms. So it is possible to consider the selfduality equations

\[
F = *F
\]

or anti-selfduality equations

\[
F = -*F
\]
on a connection $A$. A connection is called an instanton or an anti-instanton if it is a solution of equations (3) or (2) respectively (see [25]). Any connection $A$ satisfies the Bianchi identities $D_A F = 0$. Due to $D_A^* = - * D_A^*$, instantons and anti-instantons are solutions of the Yang–Mills equations (1). In the current paper, the family of modified Levy Laplacians is introduced. It is shown that the connection satisfies the anti-selfduality Yang–Mills equations on a 4-dimensional orientable Riemannian manifold if and only if the parallel transport satisfies the Laplace equations for three operators from this family. In the fact, the problem of the description of instantons in the terms of the parallel transport and the Levy Laplacians was stated by Accardi in [1]. So, in the current paper, this problem is solved for the Riemannian case. For the flat case it was solved by author in [29].

The following scheme from [12] can be useful for the definition of differential operators particularly for the definition of the modified Levy Laplacians. Let $E$ be a real normed vector space and $E^*$ be its conjugate. Let $\mathcal{L}(E,E^*)$ be the space of all linear continuous operators from $E$ to $E^*$. If $f \in C^2(E,\mathbb{R})$, then $f'(x) \in E^*$ and $f''(x) \in \mathcal{L}(E,E^*)$ for any $x \in E$. Let $S: \text{dom} S \to \mathbb{R}$ be a linear functional and $\text{dom} S \subset \mathcal{L}(E,E^*)$. The functional $S$ defines the second order differential operator $D^{2,S}$ by the formula

$$D^{2,S}f(x) = S(f''(x)).$$

(4)

If we choose $E = \mathbb{R}^d$ and $S = tr$, then $D^{2,tr}$ is the Laplace operator $\Delta$. The original Levy Laplacian on the functions on $L_2[0,1]$ was introduced by P. Levy (see [23]). It can be defined as the second order differential operator $D^{2,trL}$ associated with the so called the Levy trace $tr_L$ (see [1]). The Levy trace is a linear functional defined in the following way. Let $\mathcal{K}_{\text{comp}}$ be the ring of compact operators on $L_2[0,1]$ and $\mathcal{A}_{\text{mult}}$ be the algebra of operators of multiplication on functions from $L_\infty[0,1]$. Let $\mathcal{A} = \mathcal{K}_{\text{comp}} \oplus \mathcal{A}_{\text{mult}}$. We will identify $h \in L_\infty[0,1]$ with the operator of multiplication on $h$. Then the Lévy trace $tr_L$ is a linear functional on $\mathcal{A}$ defined by

$$tr_L K = 0,$$

(5)

if $K \in \mathcal{K}_{\text{comp}}$, and by

$$tr_L h = \int_0^1 h(t)dt,$$

(6)

if $h \in L_\infty[0,1]$.

The Levy Laplacian $\Delta_{L}^{AGV}$ that was introduced in the papers [2,3] by Accardi, Gibilisco and Volovich can be associated with the linear functional $tr_L^{AGV}$ on some special space of bilinear forms on $H^1([0,1],\mathbb{R}^d)$ (see also [35]). This linear functional is more complicated analogue of the Levy trace $tr_L$ and we will call it the same name. The tangent bundle over the Hilbert manifold of the $H^1$-curves with the fixed origin in the Riemannian manifold $M$ is trivial. It allows to transfer the scheme of the definition of the second order differential operators to the space of sections in the vector bundle over this Hilbert manifold. In this case, the linear functional $tr_L^{AGV}$ defines the Levy Laplacian that was used by Leandre and Volovich in [22].

Any smooth curve $W \in C^1([0,1],SO(4))$ defines an orthogonal operator in $L_2([0,1],\mathbb{R}^4)$ by pointwise left multiplication. The subspace $H^1([0,1],\mathbb{R}^4) \subset L_2([0,1],\mathbb{R}^4)$

1This linear functional defines a singular quantum state and is well studied (see for example [24, 27]).
is invariant under the action of $W$. The modified Levy trace associated with $W \in C^1([0,1], SO(4))$ acts on bilinear form $K$ on $H^1([0,1], \mathbb{R}^4)$ by formula

$$tr_L W K = tr_L AGV W^* KW.$$ 

If $W$ is not constant, the modified Levy trace $tr_L W$ does not coincide with the Levy trace $tr_L AGV$. So the Levy trace has not some properties of an usual trace. The modified Levy Laplacian $\Delta_L W$ is the second order derivative operator associated with $tr_L W$.

The group $SO(4)$ is not simple and has the normal subgroups $S^3_L \cong SU(2)$ and $S^3_R \cong SU(2)$. The Lie algebra $so(4) = Lie(S^3_L) \oplus Lie(S^3_R)$ and this corresponds to decomposition of the space of 2-forms into the direct sum of the space of self-dual and anti-selfdual 2-forms. In the paper, we show that a connection $A$ is an instanton (antinstanton) on a 4-dimensional orientable Riemannian manifold if and only if $\Delta_L W U = 0$ for any $W \in C^1([0,1], S^3_L)$ (for any $W \in C^1([0,1], S^3_R)$). Let $\{e_1, e_2, e_3\}$ be a some basis of the Lie algebra of $S^3_L$. Let $W_i(t) = e^{it}$ for $i \in \{1, 2, 3\}$. We prove that it is sufficient to check $\Delta_L W_i U_{1,0} = 0$ to show that the connection $A$ is an instanton.

The modified Levy Laplacians were introduced in the work [29] by author, where only the flat case and instantons over $\mathbb{R}^4$ were considered. In that case, it is possible to use only one Laplace equation for some modified Levy Laplacian instead of three of them. In [29], the sufficient conditions on a smooth curve $W \in C^1([0,1], S^3_R)$ that the equality $\Delta_L W U = 0$ implies that the connection $A$ is an instanton were found. In the proof, the fact that $\mathbb{R}^4$ is not compact was essentially used. So it is the open question whether it is possible to transfer the result of [29] for an arbitrary 4-dimensional orientable Riemannian manifold. The simple Abelian case for a 4-dimensional orientable Riemannian manifold was considered in [31].

Another approach to the definition of the Levy Laplacian is to define it as the Cesaro mean of the second order directional derivatives along the vectors of some orthonormal basis (see [23, 20]). This approach can be also useful in the connection with the Yang-Mills equations (see [28, 30, 32, 35, 36]) and instantons (see [29, 34]). Different approaches to the Yang-Mills equations based on the parallel transport but not based on the Levy Laplacian were used in [17, 15, 13, 14, 8, 9]. Particularly, instantons were studied in [13, 14, 8].

The paper is organized as follows. In Sec. 1 we give preliminary information about the Yang-Mills equations and instantons on 4-dimensional orientable Riemannian manifolds. In Sec. 2 we give preliminary information about the parallel transport. We consider it as a section in some infinite-dimensional vector bundle over the Hilbert manifold of $H^1$-curves with the fixed origin. In Sec. 3 we transfer the scheme of the definition of the second order derivative operators on the space of sections in this infinite dimensional bundle. In Sec. 4 we define the modified Levy trace as the result of the action of the curve from $C^1([0,1], SO(4))$ on the Levy trace. We define the modified Levy Laplacian as the second order derivative operator associated with the modified Levy trace. In Sec. 5 we find the value of the modified Levy Laplacian on the parallel transport. In Sec. 6 we prove the main theorem on the equivalence of the self-duality Yang-Mills equations and the Laplace equations for the modified Levy Laplacians.
1 Instantons on manifold

In the paper, all manifolds are finite or infinite dimensional Hilbert manifolds. In the infinite-dimensional case, all derivatives are understood in the Frechet sense and the symbol $d_X$ will denote the derivative in the direction $X$. For information about the infinite dimensional geometry see \[21\] \[18\] \[19\].

Let $M$ be a smooth orientable Riemannian 4-dimensional manifold. Let $g$ denote the Riemannian metric on $M$. We will raise and lower indices using this metric and we will sum over repeated indices. Let $G \subseteq SU(N)$ be a closed Lie group and $\text{Lie}(G) \subseteq su(N)$ be its Lie algebra. Let $E = E(C^N, \pi, M, G)$ be a vector bundle over $M$ with the projection $\pi: E \to M$, the fiber $C^N$ and the structure group $G$. We will denote the fiber $\pi^{-1}(x) \cong C^N$ over $x \in M$ by the symbol $E_x$. Let $P$ be the principle bundle over $M$ associated with $E$. Let $ad(P) = \text{Lie}(G) \times_G M$ and $autP = G \times_G M$ be the adjoint and automorphism bundles of $P$ respectively (the fiber of $adP$ is isomorphic to $\text{Lie}(G)$ and the fiber of $autP$ is isomorphic to $G$).

A connection $A$ in the vector bundle $E$ is a smooth section in $\Lambda^1 \otimes adP$. (The symbol $\Lambda^p$ denotes the bundle of exterior $p$-forms.) If $W_a$ is an open subset of $M$ and $\psi_a: \pi^{-1}(W_a) = W_a \times C^N$ is a local trivialization of $E$, then, in this local trivialization, the connection $A$ is a smooth $\text{Lie}(G)$-valued 1-form $A^a(x) = A^a(x) dx^\mu = \psi_a A(x) \psi_a^{-1}$ on $W_a$. Let $\psi_a: \pi^{-1}(W_a) \cong W_a \times C^N$ and $\psi_b: \pi^{-1}(W_b) \cong W_b \times C^N$ be two local trivializations of $E$ and $\psi_{ab}: W_a \cap W_b \to G$ be the transition function, i.e. it is the function such that $\psi_{a} \circ \psi_{b}^{-1}(x, \xi) = (x, \psi_{ab}(x,\xi))$ for all $(x, \xi) \in (W_a \cap W_b) \times C^N$. If $x \in W_a \cap W_b$, then

$$A^b(x) = \psi_{ab}^{-1}(x) A^a(x) \psi_{ab}(x) + \psi_{ab}^{-1}(x) d\psi_{ab}(x). \quad (7)$$

The curvature $F$ of the connection $A$ is a smooth section in $\Lambda^2 \otimes adP$ such that, in the local trivialization, it has the form $F^a(x) = \sum_{\mu \lt \nu} F^a_{\mu \nu}(x) dx^\mu \wedge dx^\nu = \psi_a F(x) \psi_a^{-1}$, where $F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + [A^a_\mu, A^a_\nu]$. If $x \in W_a \cap W_b$, then

$$F^b(x) = \psi_{ab}^{-1}(x) F^a(x) \psi_{ab}(x). \quad (8)$$

If $\phi$ is a smooth section in $adP$, its covariant derivative is defined as $$\nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi].$$

Also the following holds

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \phi = [F_{\mu \nu}, \phi].$$

Let $D_A: C^\infty(M, \Lambda^p \otimes adP) \to C^\infty(M, \Lambda^{p+1} \otimes adP)$ be the operator of the exterior covariant derivative. It is determined by its action on forms $\alpha \otimes \phi$, where $\alpha$ is a real $p$-form and $\phi$ is a section in $adP$, by the formula

$$D_A(\alpha \otimes \phi) = d\alpha \otimes \phi + (-1)^p \alpha \otimes \nabla \phi,$$

where $d$ denotes the operator of the usual exterior derivative. Let $D^*_A: C^\infty(M, \Lambda^{p+1} \otimes adP) \to C^\infty(M, \Lambda^p \otimes adP)$ be a formally adjoint to the operator $D_A$. We have $D^*_A = - * D_A *$, where $*$ is the Hodge star on the manifold $M$. 


The Yang–Mills action functional has the form

\[ S_{YM}(A) = -\frac{1}{2} \int_M \text{tr}(F_{\mu\nu}(x) F^{\mu\nu}(x)) \text{Vol}(dx), \]  

(9)

where Vol is the Riemannian volume measure on the manifold \( M \). The Yang–Mills equations on a connection \( A \) have the form

\[ (D_A^* F) = 0. \]  

(10)

In local coordinates, we have

\[ (D_A^* F)_{\nu} = -\nabla^\mu F_{\mu\nu}, \]

and

\[ \nabla_{\lambda} F_{\mu\nu} = \partial_{\lambda} F_{\mu\nu} + [A_{\lambda}, F_{\mu\nu}] - F_{\mu\kappa} \Gamma_{\lambda \nu}^{\kappa} - F_{\kappa\nu} \Gamma_{\lambda \mu}^{\kappa}, \]

where \( \Gamma_{\lambda \nu}^{\kappa} \) are the Christoffel symbols of the Levy-Civita connection on \( M \). The Yang–Mills equations are the Euler-Lagrange equations for the Yang–Mills action functional (9).

The Hodge star acts on the curvature in the following way. If \( \varepsilon_{\mu \nu \lambda \kappa} \) is the Levi-Civita symbol, then \( (\ast F)_{\mu\nu} = \sqrt{\frac{1}{\det g}} \varepsilon_{\mu \nu \lambda \kappa} F^{\lambda \kappa} \). The selfduality (anti-selfduality equations) are following equations on the connection \( A \):

\[ F = \ast F \quad (F = - \ast F). \]  

(11)

Let \( F_- = F - \ast F \) and \( F_+ = F + \ast F \) be anti-selfdual and selfdual parts of the curvature \( F \) respectively. The selfduality (anti-selfduality equations) can been rewritten

\[ F_- = 0 \quad (F_+ = 0). \]  

(12)

If a connection is a solution of the self-duality equations or the antiself-duality equations than it is called the antiinstanton or the instanton respectively. The instantons and the antiinstantons are local extrema of the Yang–Mills action functional (9).

The gauge transform is a smooth section in \( \text{Aut}P \). Such a section \( \psi \) acts on the connection by the formula

\[ A \rightarrow A' = \psi^{-1} A \psi + \psi^{-1} d\psi \]  

(13)

and on the curvature by the formula

\[ F \rightarrow F' = \psi^{-1} F \psi \]  

(14)

The Lagrange function of (9), the Yang–Mills equations (10), the self-duality equations and the antself-duality equations are invariant under the action of gauge transform. The moduli space of instantons is the factor space of all instantons with the respect to the gauge equivalence. The moduli space of instantons over \( \mathbb{R}^4 \) was described in [10]. The moduli space of instantons over a 4-dimensional oriented Riemannian compact manifold was described in [26]. If the intersection form on the manifold is positive then on this manifold there exist solutions of the self-dual Yang–Mills equations and the moduli space of instantons is a 5-dimensional manifold (see also [16]). In the case of the self-dual base manifold instantons were described in [11]. The review on the gauge fields and the instantons can been found in [25].
2 Parallel transport

For any sub-interval $I \subset [0,1]$ let the symbol $H^1(I, \mathbb{R}^4)$ denote the space of all $H^1$-functions on $I$ with values in $\mathbb{R}^4$. It is the Hilbert space with scalar product
\[(h_1, h_2)_1 = \int_I (h_1(t), h_2(t))_{\mathbb{R}^4} dt + \int_I (\dot{h}_1(t), \dot{h}_2(t))_{\mathbb{R}^4} dt.\]

Let $H^1_0 = \{ h \in H^1([0,1], \mathbb{R}^4) : h(0) = 0 \}$ and $H^1_{0,0} = \{ h \in H^1_0 : h(1) = 0 \}$.

The curve $\gamma : [0,1] \to M$ on the manifold $M$ is called $H^1$-curve, if $\phi_a \circ \gamma \mid I \in H^1(I, \mathbb{R}^4)$ for any interval $I \subset [0,1]$ and for any coordinate chart $(\phi_a, W_a)$ of the manifold $M$ such that $\gamma(I) \subset W_a$. Let $\Omega$ be the set of all $H^1$-curves in $M$. If $m \in M$ let $\Omega_m = \{ \gamma \in \Omega : \gamma(0) = m \}$. So $\Omega_m$ is the set of all $H^1$-curves in $M$ with the origin at $m \in M$. The sets $\Omega$ and $\Omega_m$ can be endowed with the structure of an infinite dimensional Hilbert manifold (see [15, 18, 19, 36]).

Let $E$ and $E_m$ be the vector bundles over $\Omega$ and $\Omega_m$ respectively, which fiber over $\gamma \in \Omega$ (over $\gamma \in \Omega_m$) is the space $\mathcal{L}(E_\gamma(0), E_\gamma(1))$ of all linear mappings from $E_\gamma(0)$ to $E_\gamma(1)$. The parallel transport generated by the connection $A$ in $E$ can be considered as a section in $E$. Let $\psi_a : \pi^{-1}(W_a) \cong W_a \times \mathbb{C}^N$ be a local trivialization of the vector bundle $E$. For $\gamma \in \Omega$ such that $\gamma([s_0, t_0]) \subset W_a$ let $U^a_{t,s}$, where $s_0 \leq s \leq t \leq t_0$, be a solution of the system of differential equations
\[
\begin{align*}
\frac{d}{dt} U^a_{t,s}(\gamma) &= -A^a_\mu(\gamma(t)) \dot{\gamma}^\mu(t) U^a_{t,s}(\gamma) \\
\frac{d}{ds} U^a_{t,s}(\gamma) &= U^a_{t,s}(\gamma) A^a_\mu(\gamma(s)) \dot{\gamma}^\mu(s)
\end{align*}
\] (15)

Then $U_{t_0,s_0}(\gamma) = \psi^{-1}_a U^a_{t_0,s_0}(\gamma) \psi_a$ is the parallel transport along the restriction of $\gamma$ on $[s_0, t_0]$. If $\gamma([s_0, t_0]) \subset W_a \cap W_b$, then equality (16) implies that
\[
U^a_{t_0,s_0}(\gamma) = \psi_{ab}(\gamma(t_0)) U^b_{t_0,s_0}(\gamma) \psi_{ba}(\gamma(s_0)).
\] (16)

For an arbitrary $\gamma \in \Omega$ we can consider a partition $s_0 = t_1 \leq t_2 \leq \ldots \leq t_n = t_0$ such that $\gamma([t_i, t_{i+1}]) \subset W_a$ and a family of local trivializations $\psi_a : \pi^{-1}(W_a) \cong W_a \times \mathbb{C}^N$ of the vector bundle $E$. Let
\[
U^a_{t_0,s_0}(\gamma) = U^a_{t_n,t_{n-1}}(\gamma) \psi_a U^a_{t_{n-1},t_{n-2}}(\gamma) \ldots U^a_{t_2,t_1}(\gamma) \psi_a U^a_{t_1,t_0}(\gamma).
\] (17)

then $U_{t_0,s_0}(\gamma) = \psi^{-1}_a U^a_{t_0,s_0}(\gamma) \psi_a$. The parallel transport along $\gamma$ is $U_{1,0}(\gamma)$. By (16), the definition of parallel transport does not depend on the choice of the partition and the choice of the family of trivializations.

The parallel transport has the following properties:

1. The mapping $\Omega \ni \gamma \to U_{1,0}(\gamma)$ is a $C^\infty$-smooth section in the vector bundle $E$ (for the proof of smoothness see [17, 15]).

2. The parallel transport does not depend on the choice of parametrization of the curve. Let $\sigma : [0,1] \to [0,1]$ be a non-decreasing piecewise $C^1$-smooth function such that $\sigma(0) = 0$ and $\sigma(1) = 1$. Then
\[
U_{\sigma(t),\sigma(s)}(\gamma) = U_{t,s}(\gamma \circ \sigma)
\] (18)

for any $\gamma \in \Omega$. 
3. For any $\gamma \in \Omega$ the parallel transport satisfies the multiplicative property:

$$U_{t,s}(\gamma)U_{s,r}(\gamma) = U_{t,r}(\gamma) \text{ for } r \leq s \leq t.$$  

(19)

4. If the restriction of $\gamma \in \Omega$ on $[s,t]$ is constant, then

$$U_{t,s}(\gamma) \equiv Id.$$  

(20)

3 Second order directional operators

The mapping $X: [0, 1] \to TM$ such that $X(t) \in T_{\gamma(t)}M$ for any $t \in [0, 1]$ is a vector field along $\gamma \in \Omega_m$, i.e. it is a section in the pullback bundle $\gamma^* TM$. If $X$ is an absolutely continuous field along $\gamma \in \Omega$, its covariant derivative $\nabla X$ with respect to the Levi-Civita connection is the field along $\gamma$ defined by

$$\nabla X(t) = \dot{X}(t) + \Gamma(\gamma(t)) (X(t), \dot{\gamma}(t)),$$  

(21)

where $(\Gamma(x)(X,Y))^\mu = \Gamma^\mu_{\lambda
u}(x) X^\lambda Y^\nu$ in local coordinates. Let $Q(\gamma)$ denote the parallel transport generated by the Levi-Civita connection along the curve $\gamma$. Then

$$\nabla X(t) = Q_{t,0}(\gamma) \frac{d}{dt} (Q_{t,0}(\gamma)^{-1} X(t)).$$

The symbol $H^1_{\gamma}(TM)$ denotes the Hilbert space of all $H^1$-fields $X$ along $\gamma$ such that $X(0) = 0$. The scalar product on this space is defined by the formula

$$G_1(X,Y) = \int_0^1 g(X(t), Y(t)) dt + \int_0^1 g(\nabla X(t), \nabla Y(t)) dt.$$  

(22)

We can identify the Hilbert spaces $H^1([0, 1], \mathbb{R}^4)$ and $H^1([0, 1], T_m M)$. Let $\{Z_1, \ldots, Z_4\}$ be an orthonormal basis in $T_m M$. We identify

$$H^1([0, 1], \mathbb{R}^4) \ni h(\cdot) = (h^\mu(\cdot)) \leftrightarrow Z_\mu h^\mu(\cdot) \in H^1([0, 1], T_m M)$$

Due to (21), for any $\gamma \in \Omega_m$ the Levy-Civita connection generates the canonical isometrical isomorphism between $H^1_0$ and $H^1_{\gamma}(TM)$, which action on $h \in H^1_0$ we will denote by $\tilde{h}$. This isomorphism acts by the formula

$$\tilde{h}(\gamma; t) = Q_{t,0}(\gamma) h(t) = Z_\mu(\gamma, t) h^\mu(t),$$  

(23)

where $Z_i(\gamma, t) = Q_{t,0}(\gamma) Z_i$ for $i \in \{1, 2, 3, 4\}$. Sometimes we will miss in the notation the dependence of the infinite dimensional field on $\gamma$. Let $H^1_{\delta}$ be the vector bundle over $\Omega_m$ which fiber over $\gamma \in \Omega_m$ is $H^1_{\gamma}(TM)$. Let $H^1_{0,0}$ denote the sub-bundle of $H^1_{\delta}$ such that the fiber of $H^1_{0,0}$ over $\gamma \in \Omega_m$ is the space $\{X \in H^1_{\gamma}(TM) : X(1) = 0\}$. The vector bundle $H^1_{\delta}$ is the tangent bundle over $\Omega_m$. Due to isomorphism (23), this bundle is trivial. For any smooth section $f$ in $\mathcal{E}_m$ there exists the section $\tilde{D} f$ in $\mathcal{E}_m \otimes H^1_{0,0} \cong H^1_{0,0}$ such that $d_{\tilde{h}} f(\gamma) = \langle \tilde{D} f(\gamma), h \rangle$ for any $h \in H^1_{0,0}$. Also there exists the section $D^2 f$ in $\mathcal{E}_m \otimes \mathcal{E}(H^1_{0,0}, H^1_{0,0})$ such that $d_{\tilde{h}_1} D^2 f(\gamma), h_2 \rangle = \langle \tilde{D} f(\gamma) h_1, h_2 >$ for any $h_1, h_2 \in H^1_{0,0}$.

The scheme of the definition of the second order differential operator can been transferred to the case of manifold in the following way.
Definition 1. Let $S$ be a linear functional on $\text{dom} \mathcal{S} \subset \mathcal{L}(H_{1,0}^1, H_{0,0}^1)$. The domain of the second order differential operator $D^{2,S}$ associated with $S$ is the space of all smooth sections $f$ in $\mathcal{E}_m$ such that $\tilde{D}^2 f(\gamma) \in \text{dom} \mathcal{S}$ for all $\gamma \in \Omega_m$. The second order differential operator $D^{2,S}$ acts on $f$ by the formula

$$D^{2,S}(f) = \tilde{S}(\tilde{D}^2 f(\gamma)).$$

Remark 1. We use the space $H_{1,0}^1$ instead of $H_{0,1}^1$ for the definition of the second order derivative operator because the directional derivative $d_X f$ of the section $f \in \mathcal{E}_m$ is covariant if and only if $X(1) = 0$ (see [22]).

## 4 Levy trace and Levy Laplacian

Let $T_{AGV}^2$ be the space of all continuous bilinear real-valued functionals on $H_{0,0}^1 \times H_{0,0}^1$ that have the form

$$Q(u, v) = \int_0^1 \int_0^1 Q^V(t, s) < u(t), v(s) > dt ds +$$

$$+ \int_0^1 Q^L(t) < u(t), v(t) > dt +$$

$$+ \frac{1}{2} \int_0^1 Q^S(t) < \dot{u}(t), v(t) > dt + \frac{1}{2} \int_0^1 Q^S(t) < \dot{v}(t), u(t) > dt,$$

(24)

where $Q^V \in L_2([0, 1] \times [0, 1], T^2(\mathbb{R}^4))$, $Q^L \in L_1([0, 1], \text{Sym}^2(\mathbb{R}^4))$, $Q^S \in L_\infty([0, 1], \Lambda^2(\mathbb{R}^4))$; where $T^2(\mathbb{R}^4)$, $\text{Sym}^2(\mathbb{R}^4)$ and $\Lambda^2(\mathbb{R}^4)$ are the spaces of all tensors, all symmetrical tensors and all antisymmetrical tensors of type $(0, 2)$ on $\mathbb{R}^4$ respectively.

In the fact, the space of bilinear functionals $T_{AGV}^2$ was considered in the paper [3] by Accardi, Gibilisco and Volovich (see also [7]). The kernel $Q^V(\cdot, \cdot)$ is called the Volterra kernel, the $Q^L(\cdot)$ is the Levy kernel and $Q^S(\cdot)$ is the singular kernel. These kernels are defined in a unique way (see [3]).

Definition 2. The Levy trace $\text{tr}^L_{AGV}$ acts on $Q \in T_{AGV}^2$ by the formula

$$\text{tr}^L_{AGV} Q = \int_0^1 \text{tr} Q^L(t) dt.$$

Definition 3. The Levy Laplacian $\Delta^L_{AGV}$ is the second order differential operator $D^{2,\text{tr}^L_{AGV}}$ associated with the Levy trace $\text{tr}^L_{AGV}$.

This operator was introduced by Accardi, Gibilisco and Volovich in [3] for the flat case and by Leandre and Volovich in [22] for the case of Riemannian manifold.

Example 1. Let $f_1 \in \mathcal{C}^\infty(M, \mathbb{R})$. Let $L_{f_1} : \Omega_m \to \mathbb{R}$ be defined by:

$$L_{f_1}(\gamma) = \int_0^1 f_1(\gamma(t)) dt.$$

Then,

$$\Delta^L_{AGV} L_{f_1}(\gamma) = \int_0^1 \Delta_{(M,g)} f_1(\gamma(t)) dt,$$

where $\Delta_{(M,g)}$ is the Laplace-Beltrami operator on the manifold $M$. 
Let us introduce the modification of the Levy trace that is connected with instantons. Let $W \in C^1([0, 1], SO(4))$. We can consider it as an orthogonal operator on $L_2([0, 1], \mathbb{R}^4)$ defined by

$$(Wu)(t) = W(t)u(t).$$

The space $H^1_0([0, 1], \mathbb{R}^4)$ is invariant under the action of $W$.

**Definition 4.** The modified Levy trace is a linear functional $\text{tr}^W_L$ on $T^2_{\text{AGV}}$ defined by

$$\text{tr}^W_L Q = \text{tr}^{AGV}_L (W^*QW).$$

**Definition 5.** The modified Levy Laplacian $\Delta^W_L$ associated with the curve $W \in C^1([0, 1], SO(4))$ is the second order differential operator $D^{2,*\text{tr}^W_L}$.

The groups $S^3_L$ and $S^3_R$ are the normal subgroups of $SO(4)$ that consists of real matrices of the form

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

(25)

respectively, where $a^2 + b^2 + c^2 + d^2 = 1$. The Lie algebras $\text{Lie}(S^3_L)$ and $\text{Lie}(S^3_R)$ of the Lie groups $S^3_L$ and $S^3_R$ consist of real matrices of the form

$$\begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -d & c \\ c & d & 0 & -b \\ d & -c & b & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & d & -c \\ c & -d & 0 & b \\ d & c & -b & 0 \end{pmatrix}$$

respectively. So it holds that

$$so(4) = \text{Lie}(S^3_L) \oplus \text{Lie}(S^3_R)$$

and

$$\text{Lie}(S^3_L) \cong \text{Lie}(S^3_R) \cong so(3).$$

Let the symbols $P_L$ and $P_R$ denote the orthogonal projections in $so(4)$ on the subalgebras $\text{Lie}(S^3_L)$ and $\text{Lie}(S^3_R)$ respectively.

Let formally define the action of the Hodge star on $Q^S(t)$:

$$*Q^S_{\mu\nu}(t) = \frac{1}{2} \sum_{\lambda=1}^4 \sum_{\kappa=1}^4 \epsilon_{\mu\nu\lambda\kappa} Q^S_{\lambda\kappa}(t).$$

Let $Q^S_+(t) = \frac{1}{2}(Q^S(t) + *Q^S(t))$ and $Q^S_-(t) = \frac{1}{2}(Q^S(t) - *Q^S(t))$. Due to the fact that $Q^S(t)$ is anti symmetric, it can be considered as an element from the algebra $so(4)$. Let $Q^S_+(t) = P_L(Q^S(t))$ and $Q^S_-(t) = P_R(Q^S(t))$.

If $W \in C^1([0, 1], SO(4))$ let $L_W(t) = W^{-1}(t)\dot{W}(t)$. Then $L_W$ is continuous curve in $so(4)$:

$$L^*_W(t) = -L_W(t).$$

(26)
Proposition 1. Let $W \in C^1([0,1], SO(4))$. It holds

$$tr^W Q = \int_0^1 tr Q^L(t) dt - \int_0^1 tr (P_L(L_W(t)) Q^S_+(t)) dt - \int_0^1 tr (P_R(L_W(t)) Q^S_-(t)) dt$$  \hspace{1cm} (27)

Proof. We have

$$Q(Wu,Wv) = \int_0^1 \int_0^1 Q^V(t,s) < W(t)u(t), W(s)v(s) > dt ds + \int_0^1 Q^L(t) < W(t)u(t), W(t)v(t) > dt + \frac{1}{2} \int_0^1 Q^S(t) < W(t)\dot{u}(t) + \dot{W}(t)u(t), W(t)v(t) > dt + \frac{1}{2} \int_0^1 Q^S(t) < W(t)\dot{v}(t) + \dot{W}(t)v(t), W(t)u(t) > dt.$$  \hspace{1cm} (28)

Using direct calculations, we get that $Q_W := W^* Q W \in T^2_{AGV}$ and

$$Q_W(u,v) = Q(Wu,Wv) = \int_0^1 \int_0^1 Q^V_W(t,s) < u(t), v(s) > dt ds + \int_0^1 Q^L_W(t) < u(t), v(t) > dt + \frac{1}{2} \int_0^1 Q^S_W(t) < \dot{u}(t), v(t) > dt + \frac{1}{2} \int_0^1 Q^S_W(t) < \dot{v}(t), u(t) > dt,$$  \hspace{1cm} (29)

where the Volterra kernel of $Q_W$ has the form

$Q^V_W(t,s) = W^*(t)Q^V(t,s)W(s);$  

the Levy kernel of $Q_W$ has the form

$Q^L_W(t) = W^*(t)Q^L(t)W(t) + \frac{1}{2} W^*(t)L^*_W(t) Q^S(t) W(t) - \frac{1}{2} W^*(t)Q^S(t) L_W(t) W(t)$  \hspace{1cm} (30)

and the singular kernel of $Q_S$ has the form

$Q^S_W(t) = W^*(t)Q^S(t)W(t).$  \hspace{1cm} (31)

Due to $W(t)W^*(t) = W^*(t)W(t) = Id$ and equality \[30\], we have

$$tr(Q^L_W(t)) =$$

$$= tr(W^*(t)Q^L(t)W(t)) + \frac{1}{2} tr(W^*(t)L^*_W(t) Q^S(t) W(t)) - \frac{1}{2} (W^*(t)Q^S(t) L_W(t) W(t)) =$$

$$= tr(W(t)W^*(t)Q^L(t)) + \frac{1}{2} tr(W(t)W^*(t)L^*_W(t) Q^S(t)) - \frac{1}{2} (W(t)W^*(t)Q^S(t) L_W(t)) =$$

$$= trQ^L(t) - tr(L_W(t)Q^S(t)) =$$

$$= trQ^L(t) - tr(P_L(L_W(t))Q^S_+(t)) - tr(P_R(L_W(t))Q^S_-(t)).$$  \hspace{1cm} (32)

The last equality holds due to $tr(P_L(L_W(t))Q^S_+(t)) = tr(P_R(L_W(t))Q^S_-(t)) = 0$. Equality \[32\] implies the statement of the proposition. \hfill $\Box$
5 Value of Levy Laplacian on parallel transport

The first derivative of the parallel transport is well-known.

**Proposition 2.** The first derivative of the parallel transport has the form

\[
d_X U_{t_2,t_1}(\gamma) = - \int_{t_1}^{t_2} U_{t_2,t}(\gamma) F(\gamma(t)) \lesssim X(t), \dot{\gamma}(t) > U_{t_1}(\gamma) dt - A(\gamma(t_2)) X(t_2) U_{t_2,t_1}(\gamma) + U_{t_2,t_1}(\gamma) A(\gamma(t_1)) X(t_1). \quad (33)
\]

**Proof.** For the proof see [15] (see also [17 36]).

**Remark 2.** If \(X(t_1) = X(t_2) = 0\), formula (33) has an interpretation as Non-Abelean Stokes formula (see [6, 17]). If \(h \in H^1_{0,0}\), the first derivative of the parallel transport has the form

\[
\tilde{D} U_{1,0}(\gamma) h = d_{h_2} U_{1,0}(\gamma) = - \int_0^1 U_{1,t}(\gamma) F(\gamma(t)) \lesssim \tilde{h}(\gamma,t), \dot{\gamma}(t) > U_{1,0}(\gamma) dt =
\]

\[
= - \int_0^1 U_{1,t}(\gamma) F(\gamma(t)) \lesssim Z_{\mu}(\gamma,t), \dot{\gamma}(t) > h^\mu(t) U_{1,0}(\gamma) dt. \quad (34)
\]

Let \(h_1, h_2 \in H^1_{0,0}\). The second derivative of the parallel transport has the form

\[
< \tilde{D}^2 U_{1,0}(\gamma) h_1, h_2 > = d_{h_2} \tilde{D} U_{1,0}(\gamma) h_2 =
\]

\[
= - \int_0^1 \int_0^s U_{1,t}(\gamma) F(\gamma(t)) \lesssim \dot{\gamma}(t), \tilde{h}_1(\gamma,t) > \times
\]

\[
\times U_{t,s}(\gamma) F(\gamma(s)) < \dot{\gamma}(s), \tilde{h}_2(\gamma,s) > U_{s,0}(\gamma) ds dt -
\]

\[
- \int_0^1 \int_0^s U_{1,s}(\gamma) F(\gamma(s)) < \dot{\gamma}(s), \tilde{h}_2(\gamma,s) > \times
\]

\[
\times U_{s,t}(\gamma) F(\gamma(t)) < \dot{\gamma}(t), \tilde{h}_1(\gamma,t) > U_{t,0}(\gamma) dt ds -
\]

\[
- \int_0^1 U_{1,t}(\gamma) F(\gamma(t)) \lesssim \tilde{h}_1(\gamma,t), \frac{d}{dt} \tilde{h}_2(\gamma,t) > U_{1,0}(\gamma) dt -
\]

\[
- \int_0^1 U_{1,t}(\gamma) F(\gamma(t)) < d_{h_2} \tilde{h}_1(\gamma,t), \dot{\gamma}(t) > U_{t,0}(\gamma) dt -
\]

\[
- \int_0^1 U_{1,t}(\gamma) \partial \tilde{h}_2(t) F(\gamma(t)) < \tilde{h}_1(\gamma,t), \dot{\gamma}(t) > U_{t,0}(\gamma) dt -
\]

\[
- \int_0^1 U_{1,t}(\gamma)[A(\gamma(t)) \tilde{h}_2(\gamma,t), F(\gamma(t)) < \tilde{h}_1(\gamma,t), \dot{\gamma}(t) >] U_{t,0}(\gamma) dt. \quad (35)
\]

Note that

\[
\frac{d}{dt} \tilde{h}_2(\gamma,t) = \frac{d}{dt} (Q_{t,0}(\gamma) h_2(t)) = \left( \frac{d}{dt} Q_{t,0}(\gamma) \right) h_2(t) + Q_{t,0}(\gamma) \dot{h}_2(t) =
\]

\[
= \Gamma(\gamma(t)) < \dot{\gamma}(t), \tilde{h}_2(\gamma,t) > + Q_{t,0}(\gamma) \dot{h}_2(t) \quad (36)
\]
and
\[ d_{\tilde{h}_2} (\tilde{h}_1 (\gamma; t)) = d_{\tilde{h}_2} Q_{t,0} (\gamma) h_1 (t) = \]
\[ = - \int_0^t Q_{t,s} (\gamma) R (\gamma(s)) < Q_{s,0} (\gamma) h_1 (t), \tilde{h}_2 (\gamma; s), \dot{\gamma} (s) > ds - \]
\[ - \Gamma (\gamma(t)) < \tilde{h}_1 (\gamma; t), \tilde{h}_2 (\gamma; t) >, \] (37)
where \( R = (R^\lambda_{\mu\nu\kappa}) \) is a Riemannian curvature tensor. Let
\[ k^s (\gamma; t, s) < h_1 (t), h_2 (s) >= Q_{t,s} (\gamma) R (\gamma(s)) < Q_{s,0} (\gamma) h_1 (t), \tilde{h}_2 (\gamma; s), \dot{\gamma} (s) >. \]
After we group the terms, we get
\[ < \tilde{D}^2 U_{1,0} (\gamma) h_1, h_2 > = \]
\[ = \int_0^1 \int_0^1 K^V (\gamma; s, t) < h_1 (s), h_2 (t) > dt ds - \]
\[ - \int_0^1 U_{1,t} (\gamma) \nabla_{\tilde{h}_2 (t)} F (\gamma(t)) < \tilde{h}_1 (\gamma, t), \dot{\gamma} (t) > U_{t,0} (\gamma) dt - \]
\[ - \int_0^1 U_{1,t} (\gamma) F (\gamma(t)) < \tilde{h}_1 (\gamma, t), \tilde{h}_2 (\gamma, t) > U_{t,0} (\gamma) dt, \] (38)
where
\[ K^V (\gamma; t, s) < h_1 (t), h_2 (s) >= \]
\[ = \begin{cases} U_{1,t} (\gamma) F (\gamma(t)) < \dot{\gamma} (t), \tilde{h}_1 (\gamma, t) > U_{t,s} (\gamma) F (\gamma(s)) < \dot{\gamma} (s), \tilde{h}_2 (\gamma, s) > U_{s,0} (\gamma) + \\ + U_{1,t} (\gamma) F (\gamma(t)) < k^s (\gamma; t, s) < h_1 (t), h_2 (s) >, \dot{\gamma} (t) > U_{t,0} (\gamma), \text{ if } t \geq s \\ U_{1,s} (\gamma) F (\gamma(s)) < \dot{\gamma} (s), \tilde{h}_2 (\gamma, s) > U_{s,t} (\gamma) F (\gamma(t)) < \dot{\gamma} (t), \tilde{h}_1 (\gamma, t) > U_{t,0} (\gamma), \text{ if } t < s. \end{cases} \]
(39)
It is possible to transform the last two term in (38) by integrating the expression
\[ \frac{1}{2} \int_0^1 U_{1,t} (\gamma) F (\gamma(t)) < \tilde{h}_1 (\gamma, t), \tilde{h}_2 (\gamma, t) > U_{t,0} (\gamma) dt \]
by parts and using the Bianchi identities
\[ \nabla_{\tilde{h}_2 (t)} F (\gamma(t)) < \tilde{h}_1 (t), \dot{\gamma} (t) > + \nabla_{\tilde{h}_1 (t)} F (\gamma(t)) < \dot{\gamma} (t), \tilde{h}_2 (t) > + \]
\[ + \nabla_{\dot{\gamma} (t)} F (\gamma(t)) < \tilde{h}_2 (t), \tilde{h}_1 (t) >= 0. \]
Due to \( h_1 (0) = h_1 (1) = h_2 (0) = h_2 (1) = 0 \), we have
\[ < \tilde{D}^2 U_{1,0} (\gamma) h_1, h_2 >= \]
\[ = \int_0^1 \int_0^1 K^V (\gamma; t, s) < h_1 (t), h_2 (s) > dt ds + \]
\[ + \int_0^1 K^L (\gamma, t) < h_1 (t), h_2 (t) > dt + \]
\[ + \frac{1}{2} \int_0^1 K^S (\gamma; t) < \dot{h}_1 (t), h_2 (t) > dt + \frac{1}{2} \int_0^1 K^S (\gamma; t) < \dot{h}_2 (t), h_1 (t) > dt, \] (40)
where the Levy kernel $K^L$ and the singular kernel $K^S$ have the form

$$K^L_{\mu \nu}(\gamma; t) = \frac{1}{2} U_{1,t}(\gamma)(-\nabla_{Z_{\mu}(\gamma,t)} F(\gamma(t)) < Z_{\nu}(\gamma,t), \dot{\gamma}(t) > - \nabla_{Z_{\nu}(\gamma,t)} F(\gamma(t)) < Z_{\mu}(\gamma,t), \dot{\gamma}(t) > ) U_{t,0}(\gamma),$$

and

$$K^S_{\mu \nu}(\gamma; t) = U_{1,t}(\gamma) F(\gamma(t)) < Z_{\mu}(\gamma,t), Z_{\nu}(\gamma,t) > U_{t,0}(\gamma)$$

respectively in the orthonormal basis \{Z_1(\gamma,t), Z_2(\gamma,t), Z_3(\gamma,t), Z_4(\gamma,t)\}.

Bellow, if $\gamma \in \Omega_m$ and $K(\gamma, \cdot)$ is a section in the pullback bundle $\gamma^*\Lambda^2 \otimes \text{ad} P$, the symbol $K(\gamma, t)$ means that we consider $K(\gamma, t)$ in the orthonormal basis \{Z_1(\gamma,t), \ldots, Z_4(\gamma,t)\}. So $K_{\mu \nu}(\gamma, t) = K(\gamma, t) < Z_{\mu}(\gamma,t), Z_{\nu}(\gamma,t) >$ is antisymmetrical matrix which elements are $su(N)$-matrices.

So we have proved the following theorem.

**Theorem 1.** The value of the modified Levy Laplacian $\Delta_L^W$ on the parallel transport is

$$\Delta_L^W U_{1,0}(\gamma) = - \int_0^1 U_{1,t}(\gamma) D_A^* F(\gamma(t)) \dot{\gamma}(t) U_{t,0}(\gamma) dt - \int_0^1 U_{1,t}(\gamma) \text{tr}(L_W(t) F(\gamma(t))) U_{t,0}(\gamma) dt = - \int_0^1 U_{1,t}(\gamma) D_A^* F(\gamma(t)) \dot{\gamma}(t) U_{t,0}(\gamma) dt - \int_0^1 U_{1,t}(\gamma) \text{tr}(P_L(L_W(t)) F_+(\gamma(t))) U_{t,0}(\gamma) dt - \int_0^1 U_{1,t}(\gamma) \text{tr}(P_R(L_W(t)) F_-(\gamma(t))) U_{t,0}(\gamma) dt. \quad (41)$$

If $F_+ = 0$, then $D_A^* F$. In this case, the first and the second terms in the right side of (41) are equal to zero. So, if additionally $W \in C^1([0,1], S^3_R)$, we have $P_R(L_W(t)) = 0$ and, hence,

$$\Delta_L^W U_{1,0} = 0.$$

In the following section, we will prove that the converse statement is true in some sense.

### 6 Main theorem

In the beginning, we prove auxiliary lemmas.

**Lemma 1.** If the parallel transport $U_{1,0}$ is a solution of the equation

$$\Delta_L^W U_{1,0} = 0,$$

then the connection $A$ is a solution of the Yang-Mills equations:

$$D_A^* F = 0.$$
Proof. For any $\gamma \in \Omega_m$ let $\gamma^r \in \Omega_m$ be defined as

$$\gamma^r(t) = \begin{cases} 
\gamma(t), & \text{if } t \leq r, \\
\gamma(r), & \text{if } t > r.
\end{cases}$$

(42)

Then the properties of the parallel transport (18), (19) and (20) imply

$$\Delta^W_{L}U_{1,0}(\gamma^r) = -\int_0^r U_{r,t}(\gamma)D^*_AF(\gamma(t))\dot{\gamma}(t)U_{t,0}(\gamma)dt - \int_0^r U_{r,t}(\gamma)tr(L_W(t)F(\gamma(t)))U_{t,0}(\gamma)dt - \int_0^1 tr(L_W(t)F(\gamma(r)))dtU_{r,0}(\gamma).$$

Assume that $\gamma \in C^1([0, 1], M)$. Let us introduce the function $J \in C^1([0, 1], \mathcal{L}(E_m, E_{\gamma(1)}))$ in the following way:

$$J(r) = U_{1,r}(\gamma)(\Delta^W_{L}U_{1,0}(\gamma^r)) = -\int_0^r U_{1,t}(\gamma)D^*_AF(\gamma(t))\dot{\gamma}(t)U_{t,0}(\gamma)dt - \int_0^r U_{1,t}(\gamma)tr(L_W(t)F(\gamma(t)))U_{t,0}(\gamma)dt - U_{1,r}(\gamma)\int_0^1 tr(L_W(t)F(\gamma(r)))dtU_{r,0}(\gamma).$$

Due to

$$\frac{d}{dr}U_{1,r}(\gamma)F(\gamma(r))U_{r,0}(\gamma) = U_{1,r}(\gamma)\nabla_{\dot{\gamma}(r)}F(\gamma(r))U_{r,0}(\gamma),$$

we have

$$J'(r) = -U_{1,r}(\gamma)D^*_AF(\gamma(t))\dot{\gamma}(t)U_{r,0}(\gamma) + U_{1,r}(\gamma)\int_0^1 tr(L_W(t)\nabla_{\dot{\gamma}(r)}F(\gamma(r)))dtU_{r,0}(\gamma).$$

The equality $\Delta^W_{L}U_{1,0} = 0$ implies $J \equiv 0$ and, hence, $J' \equiv 0$. Then

$$J'(1) = -D^*_AF(\gamma(1))\dot{\gamma}(1)U_{1,0}(\gamma) = 0.$$

Hence, for any $\gamma \in C^1([0, 1], M)$ with the origin at $m$ we have

$$D^*_AF(\gamma(1))\dot{\gamma}(1) = 0.$$

Taking suitable curves $\gamma$, we find that the connection $A$ is a solution of the Yang–Mills equations on $M$. \qed

Lemma 2. Let $a \in so(4)$ and $W_a(t) = e^{at}$. The parallel transport $U_{1,0}$ is a solution of the Laplace equation for the modified Levy Laplacian

$$\Delta^W_{L}aU_{1,0} = 0,$$

if and only if the connection $A$ is a solution of the Yang-Mills equations and for any $\gamma \in \Omega_m$ and $r \in [0, 1]$ the following holds

$$tr(aF(\gamma(r))) = 0.$$
Proof. Let $\Delta_{L}^{W_{a}}U_{1,0} = 0$. Consider an arbitrary $\gamma \in \Omega_{m}$. Let $\gamma_{r} \in \Omega_{m}$ be defined as (12) in the proof of Lemma 1. For $\varepsilon > 0$ let us introduce $\gamma_{r,\varepsilon} \in \Omega_{x}$ by the following way

$$\gamma_{r,\varepsilon}(t) = \begin{cases} \gamma(rt/\varepsilon), & \text{if } 0 < t \leq \varepsilon, \\ \gamma(r), & \text{if } \varepsilon < t \leq 1. \end{cases}$$

Then the properties of the parallel transport (17) and (19) imply

$$\Delta_{L}^{W_{a}}U_{1,0}(\gamma_{r,\varepsilon}) = \int_{\varepsilon}^{1} tr(aF(\gamma_{r}(1))) dt U_{1,0}(\gamma_{r}) + \int_{0}^{\varepsilon} U_{1,\frac{\varepsilon}{t}}(\gamma_{r}) tr(aF(\gamma_{r}(t/\varepsilon))) U_{\frac{\varepsilon}{t},0}(\gamma_{r}) dt = 0. \quad (43)$$

Let us introduce the notations

$$I_{1}(\varepsilon) = U_{1,0}^{-1}(\gamma_{r}) \int_{\varepsilon}^{1} tr(aF(\gamma_{r}(1))) dt U_{1,0}(\gamma_{r})$$

and

$$I_{2}(\varepsilon) = \int_{0}^{\varepsilon} U_{1,\frac{\varepsilon}{t}}^{-1}(\gamma_{r}) tr(aF(\gamma_{r}(t/\varepsilon))) U_{\frac{\varepsilon}{t},0}(\gamma_{r}) dt.$$ 

Due to equality (43), we have

$$I_{1}(\varepsilon) = -I_{2}(\varepsilon).$$

Let $\| \cdot \|$ be a standard norm on $so(4)$. The mapping $[0, 1] \ni r \rightarrow \|tr(aF(\gamma_{x}(r)))\|$ is continuous. Hence, there exists $C > 0$, such that

$$\sup_{r \in [0, 1]} \|tr(aF(\gamma_{x}(r)))\| \leq C.$$

So we have the estimate

$$\|I_{1}(\varepsilon)\| = \|I_{2}(\varepsilon)\| \leq \int_{\varepsilon}^{1} \|tr(aF(\gamma_{x}(t/\varepsilon)))\| dt \leq C\varepsilon. \quad (44)$$

From this estimate it follows that

$$U_{1,0}^{-1}(\gamma_{r}) \int_{0}^{1} tr(aF(\gamma_{r}(1))) dt U_{1,0}(\gamma_{r}) = \lim_{\varepsilon \to 0} I_{1}(\varepsilon) = 0.$$

Hence,

$$tr(aF(\gamma(r))) = tr(aF(\gamma_{r}(1))) = \int_{0}^{1} tr(aF(\gamma_{r}(1))) dt = 0. \quad (45)$$

The other side of the statement of the lemma is trivial.

Theorem 2. Let $\{e_{1}, e_{2}, e_{3}\}$ be a basis of the Lie algebra $\text{Lie}(S^{3}_{L})$ (Lie algebra $\text{Lie}(S^{3}_{R})$). Let $W_{i}(t) = e^{t e_{i}}$ for $i \in \{1, 2, 3\}$. The following two assertions are equivalent:

1. a connection $A$ is a solution of anti-selfduality equations (selfduality) equations (12);
2. $\Delta^{W_i}_{L} U_{1,0} = 0$ for $i \in \{1, 2, 3\}$.

Proof. Let $\{e_1, e_2, e_3\}$ be a some basis of $\text{Lie}(S^3_L)$ and let $\Delta^{W_i}_{L} U_{1,0} = 0$ for $i \in \{1, 2, 3\}$ Then Lemma 2 implies that for any $\gamma \in \Omega_m$ and $r \in [0, 1]$ the following holds

$$\text{tr}(e_i F(\gamma(r))) = \text{tr}(P_L(e_i)F_+(\gamma(r))) + \text{tr}(P_R(e_i)F_-(\gamma(r))) = \text{tr}(e_i F_+(\gamma(r))) = 0$$

for $i \in \{1, 2, 3\}$. Then for any $\gamma \in \Omega_m$ and $r \in [0, 1]$ we have $F_+(\gamma(r)) = 0$. Hence, $A$ is an instanton. The other side of the statement of the theorem is trivial. \qed

7 Conclusion

In the paper, we introduced a class of the modified Lévy Laplacians parameterized by the choice of a curve in the group $SO(4)$ on the infinite dimensional manifold. We showed that it is possible to choose three Laplacians from this class such that a connection on the 4-dimensional orientable Riemannian manifold is an instanton if and only if the parallel transport associated with this connection is a solution of the Laplace equations for these Laplacians.

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