THE REPRESENTATION THEORY OF GENERALIZED HYPEROCTAHEDRAL GROUPS

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Abstract. We give an explicit decomposition of $\text{Ind}(1)_{S_{2n}B_n}^{S_{2n}}$, following Barbasch and Vogan \[1\]. We define two natural generalizations of $B_n$, and extend the proof in \[1\] to recursively compute these decompositions. Although the decompositions do not appear to follow a simple pattern, we prove enough of their structure to show that they are almost never multiplicity-free.

1. Introduction

Let $B_n \subseteq S_{2n}$ be the hyperoctahedral group; that is, the stabilizer of $\sigma = (12)(34)\ldots(2n-1,2n)$ in $S_{2n}$. Barbasch and Vogan \[1\] showed that the induced representation $\text{Ind}_{S_{2n}}^{S_{2n}}(1)$ decomposes into a sum of Specht modules $\bigoplus_{\lambda} S^\lambda$, one for each $\lambda \vdash 2n$ such that each $\lambda_i$ is even.

We define two subgroups $C_{m,n}$ and $D_{m,n}$ of $S_{mn}$, each of which is a natural generalization of $B_n$. Let $C_{m,n} = S_n \wr S_m$ and $D_{m,n} = S_n \wr C_m$. Here $C_m \subseteq S_m$ is the cyclic group generated by an $m$-cycle. When $m = 2$, $C_{m,n} = D_{m,n} = B_n$.

These generalizations arise naturally when using symmetry to reduce the dimension of semidefinite programs in combinatorial optimization. The $S_{2n}$-module $\text{Ind}_{S_{2n}}^{S_{2n}}(1)$ is naturally isomorphic to the vector space of perfect matchings on $K_{2n}$. Decomposing this vector space into irreducible representations corresponds to a block diagonalization of the semidefinite program underlying the theta body for these matchings, an approximation based on sums of squares.

Similarly, the $S_{mn}$-module $\text{Ind}_{C_{m,n}}^{S_{mn}}(1)$ is naturally isomorphic to the vector space of perfect $m$-uniform hypermatchings on the $m$-uniform complete hypergraph $K_{mn}^{(m)}$. Likewise, $\text{Ind}_{D_{m,n}}^{S_{mn}}(1)$ is naturally isomorphic to the vector space of decompositions of the vertex set $[mn]$ of the complete graph $K_{mn}$ into $n$ disjoint $m$-cycles. Decomposing these into irreducible representations would allow symmetry reduction of the corresponding combinatorial optimization problems.

We generalize Barbasch and Vogan’s proof to recursively describe the decomposition of both $\text{Ind}_{C_{m,n}}^{S_{mn}}(1)$ and $\text{Ind}_{D_{m,n}}^{S_{mn}}(1)$. We do not believe that a simple pattern for the decomposition exists for $m > 2$ in either case. However, we are able to establish enough of the structure of $\text{Ind}_{C_{3,n}}^{S_{3n}}(1)$ to show that, unlike the case $m = 2$, the irreducible representations are not multiplicity-free for $n \geq 5$.

The structure of this paper is as follows. In Section 2, we give a method for determining $\text{Ind}_{C_{m,n}}^{S_{mn}}(1)$ from $\text{Ind}_{C_{m,n-1}}^{S_{mn-1}}(1)$. In Section 3, we produce an explicit linear isomorphism corresponding to the $m = 2$ case. In Section 4, we prove that $\text{Ind}_{C_{m,n}}^{S_{mn}}(1)$ is not multiplicity-free for $n \geq 5$. 

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2. Recursive construction

We generalize the induction step in the proof of Barbasch and Vogan to the cases of \( C_{m,n} \) and \( D_{m,n} \). First, we will recall the main ingredients in the case of \( B_n \).

**Lemma 2.1.** As homogeneous spaces, \( S_{2n}/B_n \cong S_{2n−1}/(B_n \cap S_{2n−1}) = S_{2n−1}/B_{n−1} \).

**Proof.** The second equality follows from \( B_n \cap S_{2n−1} = B_{n−1} \). For the first, define a map \( \phi : S_{2n}/B_n \to S_{2n−1}/B_{n−1} \) by \( \phi(gB_n) = (gB_n) \cap S_{2n−1} \). When defining \( \phi \), choosing the coset representative \( g \in S_{2n−1} \) shows that \( \phi \) is well-defined. It’s straightforward to check that the \( S_{2n−1} \) action commutes with \( \phi \). □

The next step lets us determine \( \text{Ind}_{B_n}^{S_{2n}}(1) \) by considering its restriction to \( S_{2n−1} \). Although in general a representation is not uniquely determined by its restriction to a subgroup, we will see that in this case there is enough extra information to determine the decomposition.

**Lemma 2.2.** The following recursive rule holds:

\[
\text{Res}_{S_{2n−1}}^{S_{2n}}(\text{Ind}_{B_n}^{S_{2n}}(1)) = \text{Ind}_{B_{n−1}}^{S_{2n−1}}(\text{Ind}_{B_{n−1}}^{S_{2n−2}}(1)).
\]

**Proof.** By Lemma 2.1 \( \text{Res}_{S_{2n−1}}^{S_{2n}}(\text{Ind}_{B_n}^{S_{2n}}(1)) = S_{2n−1}/B_{n−1} \). But this is just a restatement of the definition of \( \text{Ind}_{B_{n−1}}^{S_{2n−1}}(1) = \text{Ind}_{B_{n−1}}^{S_{2n−2}}(\text{Ind}_{B_{n−1}}^{S_{2n−2}}(1)) \). □

We will use the original result of Barbasch and Vogan in Section 3 so we prove it here for completeness. Here we say a partition \( \lambda \vdash 2n \) is even if each of its parts \( \lambda_i \) is even.

**Theorem 2.3.** The decomposition of \( \text{Ind}_{B_n}^{S_{2n}}(1) \) into irreducibles is

\[
\text{Ind}_{B_n}^{S_{2n}}(1) \cong \bigoplus_{\lambda \vdash 2n, \text{\lambda is even}} S^\lambda.
\]

**Proof.** This is true for \( n = 1 \). We use induction. Assume \( \text{Ind}_{B_{n−1}}^{S_{2n−2}}(1) \) has the described decomposition. We use Lemma 2.1. By the branching rule, \( \text{Ind}_{S_{2n−2}}^{S_{2n−1}}(\text{Ind}_{B_{n−1}}^{S_{2n−2}}(1)) \) contains each \( \mu \vdash 2n−1 \) having exactly one odd part, and each such \( \mu \) appears once. Suppose \( \text{Ind}_{B_n}^{S_{2n}}(1) \) contains \( \lambda \) with at least three rows and at least two odd parts. Then the restriction of \( \lambda \) contains a \( \mu \) with at least two odd parts; thus these \( \lambda \) do not occur. To rule out \( \lambda = (\lambda_1, \lambda_2) \) with \( \lambda_1 \) and \( \lambda_2 \) odd, note that \( (2n) \) occurs in \( \text{Ind}_{B_n}^{S_{2n}}(1) \) by Frobenius reciprocity. Therefore \( (2n−1, 1) \) can’t occur in \( \text{Ind}_{B_n}^{S_{2n}}(1) \), as it would contribute a second copy of \( (2n−1) \) to \( \text{Ind}_{S_{2n−2}}^{S_{2n−1}}(\text{Ind}_{B_{n−1}}^{S_{2n−2}}(1)) \). An induction on \( i \) shows that \( (2n−i, i) \) occurs in \( \text{Ind}_{B_n}^{S_{2n}}(1) \) if and only if \( i \) is even.

Finally, consider a \( \lambda \) with at least three even odd rows. Each \( \mu \) obtained by deleting a box from \( \lambda \) occurs in \( \text{Ind}_{S_{2n−2}}^{S_{2n−1}}(\text{Ind}_{B_{n−1}}^{S_{2n−2}}(1)) \) exactly once, and a single copy of \( \lambda \) in \( \text{Ind}_{B_n}^{S_{2n}}(1) \) is the only way remaining to account for these \( \mu \). □

We now generalize Lemmas 2.1 and 2.2 to the cases of \( C_{m,n} \) and \( D_{m,n} \).

**Lemma 2.4.** The following two recursive rules hold:

\[
\begin{align*}
\text{Res}_{S_{2m−1}}^{S_{2m}}(\text{Ind}_{C_{m,n}}^{S_{2m}}(1)) &= \text{Ind}_{S_{(m−1)m}}^{S_{m−1} \times S_{m−1}}(\text{Ind}_{C_{m,n−1}}^{S_{m−1}}(1) \otimes 1), \\
\text{Res}_{S_{2m−1}}^{S_{2m}}(\text{Ind}_{D_{m,n}}^{S_{2m}}(1)) &= \text{Ind}_{S_{(m−1)m}}^{S_{m−1} \times S_{m−1}}(\text{Ind}_{D_{m,n−1}}^{S_{m−1}}(1)).
\end{align*}
\]
Proof. The proof is a straightforward generalization of Lemmas 2.1 and 2.2. Observe that 
\( C_{m,n} \cap S_{mn-1} = C_{m,n-1} \times S_{m-1} \) and that \( D_{m,n} \cap S_{mn-1} = D_{m,n-1} \). We then have that 
\( S_{mn}/C_{m,n} \cong S_{mn-1}/(C_{m,n-1} \times S_{m-1}) \) and \( S_{mn}/D_{m,n} \cong S_{mn-1}/D_{m,n-1} \). The results follow. □

If we know the decomposition of \( \text{Ind}_{C_{m,n}}^{S_{m,n}}(1) \) into irreducibles, we can use Lemma 2.4 and 
Pieri’s rule to decompose \( \text{Ind}_{C_{m,n+1}}^{S_{m,n+1}}(1) \) into irreducibles. The same is true for \( \text{Ind}_{D_{m,n}}^{S_{m,n}}(1) \) and 
\( \text{Ind}_{D_{m,n+1}}^{S_{m,n+1}}(1) \), except that we use the branching rule. See Table 1 for some results for \( m = 3 \) 
and small \( n \).

3. Explicit isomorphism for \( m = 2 \)

Recall that a **matching** in a graph is a set of disjoint edges; we say a matching is a \( k \)-
matching if it consists of \( k \) edges. Take \( S \) to be the set of \( n \)-matchings in \( K_{2n} \); these are also 
named as perfect matchings. If we let \( S_{2n} \) permute the vertices of \( K_{2n} \), then \( S \) is an \( S_{2n} \)-set and 
\( \mathbb{C}[S] \) an \( S_{2n} \)-module. Note that \( S_{2n} \) acts transitively on \( S \).

Fix the matching \( s = 12|34|\cdots|2n-1,2n \). Then the stabilizer of \( s \) in \( S_{2n} \) is exactly 
\( B_n \) as defined in Section 1. Then \( \text{Ind}_{B_n}^{S_{2n}}(1) \cong \mathbb{C}[S] \) as \( S_{2n} \)-modules. We give an explicit 
decomposition of \( \mathbb{C}[S] \) into irreducibles; i.e., we provide a concrete linear map from each 
summand \( S^\lambda \to \mathbb{C}[S] \). Note that the decomposition is determined up to isomorphism by 
Theorem 2.3. Our contribution here is to give an effectively computable isomorphism.

**Lemma 3.1.** Let \( S \) be the set of \( k \)-matchings in \( K_{2k} \). Then \( \mathbb{C}[S] \cong \bigoplus \lambda S^\lambda \), where the direct 
sum is over all partitions \( \lambda \) of \( 2k \) consisting of even parts. The multiplicity of each \( S^\lambda \) is 1.

**Proof.** Fix an even \( \lambda \). We will define a map \( f : M^\lambda \to \mathbb{C}[S] \). For a single-row tabloid \( R \), let 
\( f(R) \) be the sum of all matchings in \( R \). For a tabloid \( T \) with rows \( R_i \), let \( f(T) = \prod_i f(R_i) \); we interpret the product of disjoint matchings as their union. For example:

\[
f \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 \\
\end{array} \right) = f(12|34)f([34])
\]

\[
= (12|34) + 13|24 + 14|23
\]

\[
= (12|34 + 13|24 + 14|23) \cdot (56)
\]

\[
= 12|34|56 + 13|24|56 + 14|23|56
\]

Extend by linearity to \( M^\lambda \). This is a map of \( S_n \)-modules, so its restriction to \( S^\lambda \) is either 0 
or an isomorphism.

Let \( t \) be the standard tableau with entries in increasing order. We will show \( f(e_i) \neq 0 \). 
\( f(\{t\}) \) contains the term \( m = 12|34|\cdots|2k-1,2k \). If \( \pm \pi\{t\} \) is another term in \( e_t \) such 
that \( f(\pi\{t\}) \) also contains \( m \), then \( 2i \) and \( 2i - 1 \) must be in the same row of \( \pi\{t\} \) for all 
\( i \). But using column group operations, this is only possible if we switch \( 2i \) with \( 2j \) and 
\( 2i - 1 \) with \( 2j - 1 \). Therefore \( \pi \) is a product of an even number of disjoint transpositions, 
and in particular, \( \text{sign}(\pi) = 1 \). So \( f(\{t\}) \) appears with positive sign in \( f(e_t) \), and therefore 
\( f(e_t) \neq 0 \).

The proof is completed by noting that, per Theorem 2.3, we have accounted for each 
irreducible representation that appears. □
4. Multiplicities occur for \( m = 3, n \geq 5 \)

The recursion rules established in Lemma 2.4 can be used to compute the decompositions of \( \text{Ind}^{S_{mn}}_{C_{m,n}}(1) \) and \( \text{Ind}^{S_{mn}}_{D_{m,n}}(1) \) for small values of \( m \) and \( n \); see Table I at the end of this section.

As discussed above, in all cases we computed, there was a unique solution to the recursion containing a copy of the trivial representation. However, unlike the case \( m = 2 \), there does not seem to be any simple pattern to the decomposition. In particular, the decompositions are not multiplicity-free after the first few values of \( n \).

In this section, we consider \( V_n := \text{Ind}^{S_{3n}}_{C_{3,n}}(1) \), and determine enough of the structure of \( V_n \) to show that for \( n \geq 5 \), \( V_n \) is not multiplicity-free. We accomplish this by considering partition patterns. A partition pattern \( \lambda = (\ast, \lambda_1, \ldots, \lambda_k) \) represents any partition of length \( k + 1 \) whose second through last parts equal \( \lambda \). We also abandon tuple notation and simply concatenate digits, as all our entries are at most 9. For instance, the partition pattern 42 represents the partition \((n-6, 4, 2)\) for any \( n \). As a special case, we let 0 denote the pattern \( \emptyset \), representing the partition \((n)\) for any \( n \).

For any partition pattern \( \lambda \), let \( \text{mult}(\lambda, n) \) be the multiplicity of \( S^\lambda \) in \( V_n \). Also let \( \text{mult}(\lambda, n^-) \) be the multiplicity of \( S^\lambda \) in \( \text{Res}^{S_{3n}}_{S_{3n-1}}(V_n) = \text{Ind}^{S_{3n-1}}_{S_{3n-3}}(V_{n-1}) \). It is also convenient to refer to \( V_n \) and \( \text{Res}^{S_{3n}}_{S_{3n-1}}(V_n) \) as level \( n \) and level \( n^- \), respectively.

We will first determine the multiplicities of certain \( S^\lambda \) in \( V_n \). Then, we will use this structure to show that \( V_n \) is not multiplicity-free for \( n \geq 5 \).

**Lemma 4.1.** The following \( \lambda \) have multiplicity 1 in all levels \( n \geq 5 \): 0, 2, 3, 4, 22, 5, 41, 32. The following \( \lambda \) do not appear in any level: 1, 21, 31, 221, 311, 411.

**Proof.** It is easy to check that this holds for \( n = 5 \); see Table I. Assuming by induction that the given decomposition holds for \( n - 1 \), we get a partial list of multiplicities at level \( n^- \):

| \( \lambda \) | 0 | 1 | 11 | 2 | 21 | 3 | 111 | 4 | 31 | 22 | 211 | 1111 | 5 | 41 | 32 | 311 | 221 |
|---|---|---|---|---|---|---|-----|---|---|---|-----|-------|---|---|---|-----|---|
| \text{mult}(\lambda, n^-) | 1 | 1 | 0 | 2 | 1 | 2 | 0 | 3 | 2 | 2 | 0 | 0 | 3 | 3 | 3 | 0 | 1 |

It is then straightforward to check that the given decomposition for level \( n \) is the only way to recover these multiplicities at level \( n^- \).

**Theorem 4.2.** All levels \( V_n \) for \( n \geq 5 \) have multiplicities.

**Proof.** By Lemma 4.1, it follows that \( \text{mult}(51, n) + \text{mult}(42, n) = 2 \) for all \( n \). By considering the relevant children at level \( n - 1 \), we can see that \( \text{mult}(51, n^-) + \text{mult}(42, n^-) = 9 \). Therefore, one of \( \text{mult}(51, n^-) \), \( \text{mult}(42, n^-) \geq 5 \). But since each of 51 and 42 has four parents at level \( n \), we must have multiplicities at level \( n \).

**References**

[1] Dan Barbasch and David Vogan. Weyl group representations and nilpotent orbits. In *Representation theory of reductive groups (Park City, Utah, 1982)*, volume 40 of *Progr. Math.*, pages 21–33. Birkhäuser Boston, Boston, MA, 1983.
Table 1. The decomposition of $\text{Ind}_{C_{m,n}}^{S_{m,n}}(1)$ into irreducible representations for $m = 3$ and small $n$. Note that $n = 5$ is the first to contain multiplicities.

| $n$ | $m$ | $\text{Ind}_{C_{m,n}}^{S_{m,n}}(1)$ |
|-----|-----|----------------------------------|
| 2   | 3   | $[4, 2], [6]$                   |
| 3   | 3   | $[4, 4, 1], [5, 2, 2], [6, 3], [7, 2], [9]$ |
| 4   | 3   | $[4, 4, 4], [5, 4, 2, 1], [6, 2, 2, 2], [6, 4, 2], [6, 6], [7, 3, 2], [7, 4, 1], [8, 2, 2], [8, 4], [9, 3], [10, 2], [12]$ |
| 5   | 3   | $[5, 4, 4, 2], [5, 5, 3, 1, 1], [6, 4, 2, 2, 1], [6, 4, 4, 1], [6, 5, 2, 2], [6, 6, 3], [7, 2, 2, 2, 2], [7, 4, 2, 2], [7, 4, 3, 1], [7, 4, 4], [7, 5, 2, 1], [7, 6, 2], [8, 3, 2, 2], [8, 4, 2, 1], [8, 4, 3], [8, 5, 2], [8, 6, 1], [9, 2, 2, 2], [9, 4, 2], [9, 4, 2], [9, 6], [10, 3, 2], [10, 4, 1], [10, 5], [11, 2, 2], [11, 4], [12, 3], [13, 2], [15]$ |

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