Multiple scales approach to the gas-piston non-equilibrium thermodynamics

D Chiuchiù and G Gubbiotti

1 Dipartimento di Fisica e Geologia, NiPS Lab, Università degli Studi di Perugia, Perugia PG, Italy
2 Dipartimento di Matematica e Fisica and Sezione INFN di Roma Tre, Università degli Studi Roma tre, Roma, Italy
E-mail: davide.chiuchiu@nipslab.org and gubbiotti@mat.uniroma3.it

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Abstract. The non-equilibrium thermodynamics of a gas inside a piston is a conceptually simple problem where analytic results are rare. For example, it is hard to find in the literature analytic formulas that describe the heat exchanged with the reservoir when the system either relaxes to equilibrium or is compressed over a finite time. In this paper we derive this kind of analytic formula. To achieve this result, we take the equations derived by Cerino et al (2015 Phys. Rev. E 91 032128) describing the dynamic evolution of a gas-piston system, we cast them in a dimensionless form, and we solve the dimensionless equations with the multiple scales expansion method. With the approximated solutions we obtained, we express in a closed form the heat exchanged by the gas-piston system with the reservoir for a large class of relevant non-equilibrium situations.

Keywords: rarefied gases dynamics, exact results
1. Introduction

One of the irrefutable facts in learning thermodynamics and statistical mechanics is that, when facing the two topics for the first time, one has to get skilled in solving problems that involve gasses and pistons. This is due to historical reasons. One is that thermodynamics was initially developed to understand the phenomenology of steam engines. One other is that the first model where the macroscopic properties of a body were linked to its internal constituents is the kinetic theory of gasses by Maxwell and Boltzmann. More than one hundred years have now passed from the work of Maxwell and Boltzmann and thermodynamics and statistical mechanics have widely changed: from theories that describe equilibrium conditions only, they are slowly but steadily evolving to include non-equilibrium frameworks [2, 7, 15, 29, 30] and results [8, 14, 16, 20, 28]. However, gasses and pistons have never stopped playing a prominent role in this evolution as their non-equilibrium behaviour is not yet fully understood. As a proof, Elliot Lieb stated that he would like to see solved the adiabatic piston problem [24]. It can be formulated as follows: we take an insulating canister with two gasses inside that are separated by a perfectly insulating moving piston. If the two gasses are initially at different pressures and temperatures, how is the equilibrium condition approached? It turns out that it is easy to answer this question qualitatively [17], while quantitative answers are still a very active field of research [9, 10, 18]. Curiously enough, the adiabatic piston problem is very similar to another that is not that much investigated, although it involves a conceptually simpler system and it’s more relevant for applications. We consider the simplest thermodynamic machine: a perfect gas enclosed by a cylindrical canister with a movable piston and in contact with a heat reservoir (figure 1). This system is simpler than the adiabatic piston as (1) only a single perfect gas is involved and (2) gas-reservoir heat exchanges are easily modelled microscopically [31]. Additionally, this device can be described by a limited set of macroscopic variables: the piston position \( x \), the gas internal temperature \( T \), the reservoir temperature \( T_r \),
and the external force $\overline{F}$ applied on the piston. Among those variables, $\overline{F}$ and $\overline{T}_b$ can be changed according to some external time dependent protocol, while $\overline{x}$ and $\overline{T}$ evolve as a consequence. It is worth noting that in the adiabatic piston problem, the force exerted by the piston can never be considered as an external known function, as it is an internal variable.

One possible way to describe the time evolution of $\overline{x}$ and $\overline{T}$ is given in [11] through the following gas-piston equations (GPE)

$$\begin{align}
\dot{x} + \frac{\overline{F}}{M} - \frac{\nu N}{\nu + 1} \frac{1}{\overline{x}} \text{erfc}\left(\sqrt{\frac{m}{2\overline{T}}} \frac{x}{\overline{T}}\right) (\dot{x}^2 + \frac{T}{m}) \\
+ \frac{\nu N}{\nu + 1} \exp\left(-\frac{m\overline{x}^2}{2\overline{T}}\right) \frac{\overline{x}}{\pi \overline{m}} = 0, \tag{1a}
\end{align}$$

$$\begin{align}
\dot{T} + \frac{2\dot{x}^2 + \overline{T}(1 - 2\nu)}{\overline{x}(\nu + 1)^2} \text{erfc}\left(\sqrt{\frac{m}{2\overline{T}}} \frac{x}{\overline{T}}\right) + \frac{2\overline{T}}{\pi \overline{m}} \frac{\overline{T} - \overline{T}_b}{\overline{x}} \\
+ \frac{2}{\overline{x}(\nu + 1)^2} \sqrt{\frac{2m\overline{T}}{\pi}} \left(\frac{2\nu T}{m} - \dot{x}^2\right) \exp\left(-\frac{m\overline{x}^2}{2\overline{T}}\right) = 0, \tag{1b}
\end{align}$$

where the upper dots denote time derivatives, $M$ is the mass of the piston, $m$ is the mass of a single gas molecule, $N$ is the total number of gas molecules, $\nu = m/M$ and erfc(·) is the complementary error function. Without going into too much detail, these macroscopic equations are obtained by averaging microscopic properties with the aid of heavy assumptions. The first one is that the piston and each gas particle undergoes elastic collisions, so work is the energy exchanged in this way. The second assumption is that the velocity of a gas particle is randomly changed according to the Maxwell–Boltzmann distribution of the reservoir when reservoir-gas particle collisions occur [31]. Heat is no more than the change in energy of the gas due to this collision mechanism. The third assumption is that the gas distribution is always Maxwellian, although gas-reservoir and gas-piston collisions change the gas temperature $\overline{T}$ over time. From a physical point of view, this third assumption rules out any shock-wave propagation, making the gas an efficient macroscopic dissipative medium [9].

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$^3$ An anonymous referee pointed out that our equation (1a) and the original equation (A1) in [10] do not have the same last term. This discrepancy is due to a confirmed misprint [1] in [11].

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Figure 1. A schematic representation of the gas canister with the macroscopic variables required to describe its dynamic.
If the solution of equation (1) is known, it is possible to compute the total energy of the system $E$ [11, 21], the work $W$ performed on the piston [29, 30] and the heat $Q$ exchanged with the reservoir as:

$$E = M \frac{\dot{x}^2}{2} + F \, \dot{x} + \frac{N \dot{T}}{2}, \quad (2a)$$

$$W(t_i, t_f) = \int_{t_i}^{t_f} F \, \dot{x} \, dt, \quad (2b)$$

$$Q(t_i, t_f) = - \int_{t_i}^{t_f} \left[ F \, \ddot{x} + M \ddot{x} \, \ddot{x} + \frac{N \dot{T}}{2} \right] \, dt. \quad (2c)$$

It is worth noting that $F \, \dot{x}$ is considered as a part of the total energy of the system. As a consequence, work must be defined as equation (2b) [21].

In this paper we present equation (1) in a new dimensionless form, which allows to easily take the thermodynamic limits

$$\nu \to 0, \quad N \to \infty, \quad (3)$$

and to isolate

$$\varepsilon = \lim_{\nu \to 0, N \to \infty} \sqrt{\nu N}, \quad (4)$$

as the only relevant free parameter [18]. Assuming that $\varepsilon$ is small and that the external force is slowly-varying over time, we linearise equation (1) and then proceed to an asymptotic expansion using the multiple scales method. This way we find the approximated solution of the linearised equations, which contains all the relevant physical behaviors of the system. Furthermore we use the obtained solutions to build closed form expressions for the heat exchanged with the reservoir for two relevant non-equilibrium transformations, namely the relaxation toward equilibrium and the isothermal compression of a gas realised in a finite time. In the next section we give an account of the multiple scales method and then proceed with the outline presented above.

2. Short review of the multiple scales method

The history of the multiple scales method dates back to the 18th century from the works of by Lindstedt [25] and Poincaré [27], and was developed in its modern form in [13, 23]. The core of the method is to find asymptotic approximated solutions to a differential equation when the standard perturbation theory produces secular terms. During the years the multiple scales method has proven to be very useful in the construction of approximate solutions of differential equations and is now included in every textbook on perturbation theory [3, 19, 22, 26]. Such a powerful method has also found

4 The sign conventions for $Q$ and $W$ are the same used in [29], i.e. $Q > 0$ for heat given to the reservoir and $W > 0$ for work applied on the system.
applications in fields which seem not to have any correlation with such problems, for example in the theory of integrable systems [4–6, 32].

The key feature that allows the elimination of the secular terms is the introduction of fast-scale variables and slow-scale variables in a way that the dependence on the slow-scale variables will prevent the secularities. To be more precise, suppose that we are in the case of an ordinary differential equation with respect to the independent variable \( t \) and a single dependent variable \( x = x(t) \):

\[
E_t \left( t, x, \dot{x}, \ddot{x}, \ldots, \frac{(n)}{x} \right) = 0, \tag{5}
\]

where the \( \varepsilon \) subscript means that we have some dependence on a small parameter \( \varepsilon \). We now assume that \( x \) has an asymptotic series of the form:

\[
x(t) = \sum_{i=0}^{N+M} \varepsilon^i x_i(t_0, t_1, t_2, \ldots, t_N) + \mathcal{O}(\varepsilon^{N+M+1}), \tag{6}
\]

truncated at some positive integer \( N + M \), with \( M \geq 0 \). On the right-hand side of (6) the dependence on the time variable \( t \) appears through the so-called scales\(^5\) \( t_i = t_i(t, \varepsilon) \). Intuitively, the scales isolate different behaviors inside equation (5), e.g. in the damped harmonic oscillator they separate oscillations from the amplitude suppression. The number of scales to be introduced depends on the desired asymptotic approximation order: the expansion is guaranteed to be asymptotic until

\[
t_N(t, \varepsilon) = \mathcal{O}(1) \tag{7}
\]

is satisfied. The number of scales also sets the approximation error, in the sense that the maximum discrepancy from the complete solution

\[
\max_{t \in [0, t_{\max}]} \left| x(t) - \sum_{i=0}^{N+M-1} \varepsilon^i x_i(t_0, t_1, t_2, \ldots, t_N) \right| \tag{8}
\]

is \( \mathcal{O}(\varepsilon^{N+M}) \), where \( t_{\max} \) is the time such that the condition (7) holds.

The mathematical structure of the scales is the most delicate point in the whole expansion method: it involves the knowledge of the equation (5) structure, and the constraint that they must be non-decreasing functions of \( t \) which satisfy the condition:

\[
\lim_{\varepsilon \to 0} \frac{t_{i+1}(t, \varepsilon)}{t_i(t, \varepsilon)} = 0, \quad \forall i = 0, 1, \ldots, N - 1. \tag{9}
\]

We note that \( N \) has to be sufficiently high, not just to give a longer asymptotic range of validity of the expansion, but also to capture the behaviour of the system.

The substitution (6) can be extended to all the derivatives of \( x \) by differentiation, or more operatively by substituting

\[
\frac{d}{dt} \to \sum_{i=0}^{N} \frac{\partial}{\partial t} \frac{\partial}{\partial t_i}. \tag{10}
\]

\(^5\) If \( t \) is a time variable, the scales are the characteristic time scales of \( x \). Similarly, if \( t \) is a length variable, the scales are the characteristic length scales of \( x \)

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5
Substituting equation (6) and all its derivatives in equation (5), eventually expanding in Taylor series with respect to $\varepsilon$, we obtain a polynomial in $\varepsilon$ which is identically equal to zero. We can then separately set to zero all the coefficients of $\varepsilon$-powers and obtain a system of $N + M + 1$ partial differential equations. If the scales are correctly chosen, the $\varepsilon^0$-equation will contain $x_0$ only and will depend just on $t_0$. This will give rise to a solution depending on arbitrary functions of the remaining scales $t_1, \ldots, t_N$. Substituting $x_0$ into the $\varepsilon^1$-equation we use these arbitrary functions to prevent the birth of the secular terms in $x_1$. Solving iteratively for the remaining $x_i$ one finally writes down the $N + M$ terms of the wanted expansion (6). In the case of high-order expansions ($N > 1$) sometimes the previous iterative method is not sufficient to completely specify the terms of the asymptotic series. In these cases the strategy of the suppression of the order mixing is adopted: it consists in eliminating from the $\varepsilon^{i+1}$-equation all the contributions coming from the arbitrary functions in the lower-order solutions $x_i$, $x_{i-1}$, etc. This increases the accuracy of the first $i$ terms by reducing the amounts of corrective terms in $x_{i+1}$ [19].

3. Adimensionalization and expansion of the piston-gas equations

Going back to the main aim of this paper, we introduce a dimensionless version of equation (1), namely

$$\ddot{x} + F + \text{erfc}\left(\frac{\varepsilon \dot{x}}{\sqrt{2T}}\right)\varepsilon^2\dot{x}^2 + T + \exp\left(-\frac{\varepsilon^2\dot{x}^2}{2T}\right)\frac{\dot{x}}{x}\sqrt{\frac{2T}{\pi}} \varepsilon = 0, \quad (11a)$$

$$\dot{T} - 2\text{erfc}\left(\frac{\varepsilon \dot{x}}{\sqrt{2T}}\right)\frac{\dot{x}}{x}(\varepsilon^2\dot{x}^2 + T) + \sqrt{\frac{2T}{\pi}} \frac{T - T_b}{\varepsilon x} - 2\sqrt{\frac{2T}{\pi}} \frac{\dot{x}}{x} \exp\left(-\frac{\varepsilon^2\dot{x}^2}{2T}\right) = 0. \quad (11b)$$

To obtain this expression we have introduced in equation (1) the dimensionless quantities $x$, $T$, $F$, $T_b$ and the dimensionless time $t$ via

$$\bar{x} = x \cdot \frac{T_{br}N}{F_r(1 + \nu)g(\nu)} \quad (12a)$$

$$\bar{T} = T \cdot \frac{T_{br}}{g(\nu)} \quad (12b)$$

$$\bar{F} = F \cdot F_r \quad (12c)$$

$$\bar{T_b} = T_b \cdot T_{br} \quad (12d)$$

$$\bar{t} = \frac{t}{F_r} \sqrt{\frac{mN T_{br}}{\nu(1 + \nu)g(\nu)}} \quad (12e)$$

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where
\[ g(\nu) = \frac{1 + 6\nu + \nu^2}{(1 + \nu)^2} \]  
and \( F_r(T_{br}) \) is an arbitrary force (temperature) reference value. Similarly, we write the dimensionless \( E, W, \) and \( Q \) densities:
\[
E = \frac{E}{NT_{br}} = x^2 + \frac{T}{2},
\]  
\[
W(t_i, t_f) = \frac{W}{NT_{br}} = \int_{t_i}^{t_f} \dot{F}x \, dt,
\]  
\[
Q(t_i, t_f) = \frac{Q}{NT_{br}} = -\int_{t_i}^{t_f} \left[ F\ddot{x} + \dot{x}\ddot{x} + \frac{1}{2} \dot{T} \right] \, dt.
\]

It is important to note that in equations (11) and (14) the thermodynamic limits equation (3) are already taken, provided that the gas-piston mass ratio \( \varepsilon \) defined in equation (4) is finite. We also note that (11) and (14) show an important property first observed in the related adiabatic piston problem [18]: the sole knowledge of \( \varepsilon \) is sufficient to describe the general features of the system, while the remaining parameters appear as scaling factors. Following again [18] we also treat \( \varepsilon \) as a small perturbation parameter.

We now restrict ourselves to the case where the reservoir temperature is constant over time, while \( F \) evolves according to a given slow protocol. W.l.o.g, we take \( T_b = 1 \). \( F \) can be considered slow if
\[
\frac{\partial F}{\partial t} \propto \varepsilon^g \quad \text{for} \quad g \geq 1.
\]  
The simplest way to satisfy equation (15) is to take \( F = F(\varepsilon t) \), which we assume from now on.

Since \( F \) varies slowly, the equilibrium point of equation (11), namely
\[
x_{eq} = \frac{1}{F}, \quad \dot{x}_{eq} = 0, \quad T_{eq} = 1,
\]  
is also slowly varying over time. Since we are interested in the results of thermodynamic relevance, we are allowed to linearise equation (11) around equation (16), which yields
\[
\ddot{x} + F^2(\varepsilon t) x + 2\sqrt{\frac{2}{\pi}} \varepsilon F(\varepsilon t) \dot{x} - F(\varepsilon t) T = 0,
\]  
\[
\ddot{T} + 2 F(\varepsilon t) \dot{T} + \sqrt{\frac{2}{\pi}} \frac{F(\varepsilon t)}{\varepsilon} (T - 1) = 0.
\]

We call equation (17) the linearised dimensionless gas piston equations (LDGPE).
Being the LDPGE linear, we could expect an exact analytic solution. However, Computer Algebra Systems like Macsima or Maple show that analytic solutions of the LDGPE are of no practical use or even impossible to express because they are too strongly dependent on the \(\epsilon\) value and on the functional form of \(F\).

The LDGPE equations (17) are a system of a second-order equation coupled with a first-order one. However, we show that it can be treated as a single third-order equation. We solve equation (17a) for \(T\):

\[
T = \frac{\\dot{\varepsilon}}{F(\varepsilon t)} + \varepsilon F(\varepsilon t)x + 2 \frac{2}{\sqrt{\pi}} \varepsilon \hat{\varepsilon}
\]

and then insert it into (17b) to obtain:

\[
\frac{\dddot{\varepsilon}}{F(\varepsilon t)} + \left[ \frac{2 \sqrt{2} \varepsilon}{\sqrt{\pi}} + \frac{\sqrt{2}}{\sqrt{\pi} \varepsilon} - \frac{F'(\varepsilon t)\varepsilon}{F^2(\varepsilon t)} \right] \ddot{\varepsilon} + \left[ 3F(\varepsilon t) + \frac{4F(\varepsilon t)}{\pi} \right] \dot{\varepsilon} + \left[ F'(\varepsilon t) + \frac{\sqrt{2} F^2(\varepsilon t)}{\sqrt{\pi} \varepsilon} \right] x - \frac{\sqrt{2} F(\varepsilon t)}{\sqrt{\pi} \varepsilon} = 0,
\]

where \(\dot{}\) stands for the differentiation with respect to the argument\(^6\).

Since the force is slowly varying and we do not want to treat just a particular case, we are naturally led to consider not a standard perturbative approach\(^7\), but the multiple scales one.

The \(\varepsilon\)-perturbation is singular, i.e. if \(\varepsilon \to 0\) then (19) ceases to be a third-order equation, but collapses into a second-order one. Indeed, from (19) it is clear that as \(\varepsilon \to 0\) the dominant term is given by:

\[
\dddot{\varepsilon} + F^2(\varepsilon t)x = F(\varepsilon t)
\]

which is just a harmonic oscillator with a slowly varying frequency \(\omega(t) = F(\varepsilon t)\) and slow forcing \(\mathcal{F}(t) = F(\varepsilon t)\). To avoid the singularity we just make an \(\varepsilon\)-scaling in \(x\) and \(t\), with undetermined coefficients, i.e. \(x(t) = \varepsilon^\alpha X(\varepsilon^\beta t)\). Substituting into (19) we found that we have to impose \(\alpha = 0\) and \(\beta = -1\); putting \(\tau = t / \varepsilon\) we obtain:

\[
\frac{1}{F(\varepsilon^2 \tau)} \frac{d^2 X}{d\tau^2} + \sqrt{\frac{2}{\pi}} \frac{d^2 X}{d\tau^2} + \varepsilon^2 \left\{ \left[ 2 \sqrt{\frac{2}{\pi}} - \frac{F'(\varepsilon^2 \tau)}{F^2(\varepsilon^2 \tau)} \right] \frac{d^2 X}{d\tau^2} + \left( 3 + \frac{4}{\pi} \right) F(\varepsilon^2 \tau) \frac{dX}{d\tau} \right. \\
+ \sqrt{\frac{2}{\pi}} F^2(\varepsilon^2 \tau) X - \sqrt{\frac{2}{\pi}} F(\varepsilon^2 \tau) \right\} + \varepsilon^4 F'(\varepsilon^2 \tau) X = 0. 
\]

As can be easily seen now we have no singularity as \(\varepsilon \to 0\) and the dominant term is now a third-order equation.

The next step is to introduce the time scales. It is easy to see that if we choose the trivial time scales

\[
\tau = (\tau_0, \tau_1, \tau_2, \ldots) = (\tau, \varepsilon \tau, \varepsilon^2 \tau, \ldots)
\]

\(^6\) In general \(G(j) = \frac{d^j u}{dx^j} \bigg|_{x=0}\) where \(\mu\) is a dummy variable.

\(^7\) Expanding in the standard way requires to express \(F(\varepsilon t)\) as a Taylor series, thus losing generality.
we end up with an asymptotic expansion that is identically zero, meaning that this choice is not correct. However, we can use the trivial time scales when the force $F(\varepsilon^2 \tau) \equiv 1$. To construct the time scales in the general setting $F(\varepsilon^2 \tau) \neq 1$, we search a change of variables $\tau \equiv F(\varepsilon^2 \tau) \approx \varepsilon \tau_0(\tau)$ such that the $\varepsilon^0$ term in equation (19) reduces to the $\varepsilon^0$ term for the $F(\varepsilon^2 \tau) \equiv 1$ case. Performing this change of variables we obtain:

$$X''(\tau_0(\tau))(\tau_0'(\tau))^2 + 3X''(\tau_0(\tau))\tau_0''(\tau) + X'(\tau_0(\tau))\tau_0''(\tau) + O(\varepsilon^2) = 0.$$  \hspace{1cm} (23)

Since the $\varepsilon^0$ term for $F(\varepsilon^2 \tau) \equiv 1$ is

$$\varepsilon^0 : \frac{d^3 X}{d\tau^3} + \sqrt{\frac{2}{\pi}} \frac{d^2 X}{d\tau^2} = 0$$

we see that we have:

$$\tau_0 = \int_0^\tau F(\varepsilon^2 \chi) \, d\chi.$$  \hspace{1cm} (24)

For the second scale we repeat the same procedure, as $\tau_1 = \varepsilon \tau$ implies an identically zero asymptotic behaviour. Performing again this procedure we see that the second scale is $\tau_2 = \varepsilon \tau_0$. As the third scale we can safely choose $\tau_2 = \varepsilon^2 \tau$. In the end we have the following three scales:

$$\tau = (\tau_0, \tau_1, \tau_2) \left( \int_0^\tau F(\varepsilon^2 \chi) \, d\chi, \varepsilon \int_0^\tau F(\varepsilon^2 \chi) \, d\chi, \varepsilon^2 \tau \right).$$  \hspace{1cm} (26)

It is worth noting that two different time scales faster that the external driving time scale $\varepsilon \tau$ are required for a full description of the system. Since the time scales of the multiple scales method are each associated with a different physical phenomenon, equation (26) gives us a rigorous proof that $F = F(\varepsilon t)$ is indeed a slow force if compared with the remaining fundamental time scales of the system. We will address to which phenomena $\tau_0$ and $\tau_1$ are related later in this paper. We note that the condition (9) is satisfied by the scales (26), but the requirement for them to be non-decreasing functions imposes some restrictions on $F$. We note that if function $F$ is always positive, then this requirement is automatically satisfied. Many cases of physical interests satisfies this positivity requirement, and these will be discussed later in this paper.

Now we introduce the truncated expansion:

$$X(t) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + O(\varepsilon^3),$$

and substitute it into (19) with $\tau = (\tau_0, \tau_1, \tau_2)$. Taking the coefficients with respect to $\varepsilon$ to be zero, we found the following equations up to $\varepsilon^2$:

$$\varepsilon^0 : \frac{\partial^3 X_0}{\partial \tau_0^3} + \sqrt{\frac{2}{\pi}} \frac{\partial^2 X_0}{\partial \tau_0^2} = 0,$$  \hspace{1cm} (28a)

$$\varepsilon^1 : \frac{\partial^3 X_1}{\partial \tau_0^3} + \sqrt{\frac{2}{\pi}} \frac{\partial^2 X_1}{\partial \tau_0^2} + 2 \sqrt{\frac{2}{\pi}} \frac{\partial^2 X_0}{\partial \tau_0^2} \frac{\partial^2 \tau_0}{\partial \tau_0^2} + 3 \frac{\partial^3 X_0}{\partial \tau_0^2 \partial \tau_0} = 0.$$  \hspace{1cm} (28b)
It is very easy to see that the solutions to these equations are \textit{weakly secular} in the sense that, except in some notable cases, the convergence of expansion is ensured by the presence of exponentially decreasing functions. Therefore, we adopt the strategy of the suppression of order mixing: we use the arbitrary functions in \( X_0, \ldots, X_i \) to eliminate as much as possible the presence of \( X_0, \ldots, X_i \) in the equations for \( X_{i+1}, \ldots, X_{N+M} \). At this point we notice that to give a \textit{complete characterisation} to the function \( X_0, X_1 \) and \( X_2 \) it is not possible to just use the three equations above, but one must add terms up to \( \varepsilon^6 \). We omit the further three equations and all the calculations, since they are very long, but in fact trivial. The results of the calculations, once written in the original variable time scales

\[
t(\varepsilon) = \left\{ \frac{1}{\varepsilon} \int_0^t F(\varepsilon \chi) d\chi, \int_0^t F(\varepsilon \chi) d\chi, \varepsilon t \right\}
\]  

(29)

become:

\[
x(t) = \frac{1}{F(\varepsilon t)} + \frac{K_1}{F(\varepsilon t)} e^\frac{1}{\varepsilon} \left( \sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}} \right) \int_0^t F(\varepsilon \chi) d\chi
\]

\[+ e^{-\varepsilon \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}} \right)} \frac{F''(\varepsilon t)}{F(\varepsilon t)} \left[ \left( C_1 \sin \left( \int_0^t F(\varepsilon \chi) d\chi + C_2 \cos \left( \int_0^t F(\varepsilon \chi) d\chi \right) \right) \right] \]

\[+ \varepsilon \left\{ e^{-\varepsilon \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}} \right)} \frac{F''(\varepsilon t)}{F(\varepsilon t)} \left[ \left( C_3 + C_2 \Theta(\varepsilon t) \right) \sin \left( \int_0^t F(\varepsilon \chi) d\chi \right) \right] \right. \]

\[+ (C_4 - C_2 \Theta(\varepsilon t)) \cos \left( \int_0^t F(\varepsilon \chi) d\chi \right) \right) + \frac{K_2}{F(\varepsilon t)} e^\frac{1}{\varepsilon} \left( \sqrt{\frac{2}{\pi}} - \sqrt{\frac{2}{\pi}} \right) \int_0^t F(\varepsilon \chi) d\chi \}
\]

\[+ \varepsilon^2 x_2(t; \{C_i, K_0\}) + O(\varepsilon^3), \]

(30)

where:

\[\Theta(s) = \int_0^s \left[ \left( \frac{1}{\pi} + 1 - \frac{\pi}{4} \right) F(s) + \frac{1}{2} \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}} \right) \frac{F''(s)}{F(s)} \right] ds
\]

\[+ \frac{1}{4} \frac{F''(s)}{F(s)} - \frac{3}{8} \frac{(F'')^2(s)}{F(s)} \]  

(31)

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The $x_2$ part of the function is not displayed in its generality since it is very long and cumbersome, but we note that the writing $x_2 = x_2(t, \{C_i, K_3\})$ means that $x_2$ has a parametric dependence on the $C_i$, $i = 1, 2, 3, 4, 5, 6$ and $K_3$, which are the constants of integration. In the next sections, while discussing some particular cases of $F$, we show the specific forms it assumes.

We can construct a general formula for the expression of the asymptotic series for $T$, substituting equation (30) into equation (18):

$$
T(t) = \frac{F(\varepsilon t)}{\varepsilon^2} \frac{\partial^2 x_0}{\partial \tau_0^2} + \frac{F(\varepsilon t)}{\varepsilon} \left( \frac{\partial^2 x_0}{\partial \tau_2^2} + 2 \frac{\partial^2 x_0}{\partial \tau_0 \partial h} \right) + F(\varepsilon t) x_0
$$

$$
+ 2 \sqrt{\frac{2}{\pi}} F(\varepsilon t) \frac{\partial x_0}{\partial t_0} + 2 \frac{\partial^2 x_0}{\partial t_0 \partial t_2} + F(\varepsilon t) \frac{\partial^2 x_0}{\partial t_0^2}
$$

$$
+ 2 F(\varepsilon t) \frac{\partial^2 x_1}{\partial t_0 \partial h} + F(\varepsilon t) \frac{\partial^2 x_0}{\partial t_1^2} + \frac{F'(\varepsilon t)}{F(\varepsilon t)} \frac{\partial x_0}{\partial t_0} + O(\varepsilon).
$$

We have then that the error on $T$ is of order $\varepsilon$ since to have a better estimate on it, the knowledge of the $x_3$ term is necessary.

A particularly interesting case arises when all the constants of integration are taken to be zero, $C_i = K_j = 0$, which, being the system linear, corresponds to the case when the initial condition is trivial and the system evolves according to the external forcing. Since the system is, in general, non-autonomous this is the dynamical equilibrium and is given by:

$$
x_{d, eq}(t) = \frac{1}{F(\varepsilon t)} + \varepsilon^2 \left[ \frac{F''(\varepsilon t)}{F^4(\varepsilon t)} - 2 \left( \frac{F''(\varepsilon t)}{F^4(\varepsilon t)} \right)^2 + \left( 2 \sqrt{\frac{2}{\pi}} + \sqrt{2\pi} \right) \frac{F'(\varepsilon t)}{F^3(\varepsilon t)} \right] + O(\varepsilon^4).
$$

Note that, as was previously known [11], the $\varepsilon^0$ contribution to the dynamical equilibrium solution is the same as that of the underlying forced harmonic oscillator (20). The first correction is then second order, while the next one will be at fourth.

Upon differentiation with respect to $t$ from (33) we obtain the modified equilibrium conditions for $\dot{x}$ and $T$ (the latter by using (32)). For the dynamical equilibrium $T$ we obtain from (32) the following very simple expression:

$$
T_{d, eq}(t) = 1 + \varepsilon^2 \sqrt{2\pi} \frac{F'(\varepsilon t)}{F^3(\varepsilon t)} + O(\varepsilon^4).
$$

We remark that upon substituting $F \equiv 1$ the dynamical equilibrium reduces exactly to the usual equilibrium condition $x = 1$, $\dot{x} = 0$, and $T = 1$. We also note that equation (33) and (34) can be derived as a standard perturbation expansion assuming $x(t) = x_0(\varepsilon t) + \varepsilon x_1(\varepsilon t) + \varepsilon^2 x_2(\varepsilon t) + \varepsilon^3 x_3(\varepsilon t) + O(\varepsilon^4)$.

Since our starting hypothesis was that the original DGPE are in linear regime, it is particularly useful to express the initial conditions of the system not as generic, but as deviations from the dynamical equilibrium (33). Using (34) and (18) valued at $t = 0$ we find the following values for the near equilibrium initial conditions for equation (19):

\[\text{We remark that to compute the complete expansion we needed six terms.}\]
x(0) = x_0 + \frac{1}{F(0)} + \varepsilon^2 \left[ \frac{F'(0)}{F^4(0)} - 2 \frac{(F')^2(0)}{F^5(0)} + \left( 2 \sqrt{\frac{2}{\pi}} + \sqrt{2\pi} \right) \frac{F''(0)}{F^3(0)} \right] + \mathcal{O}(\varepsilon^4) \quad (35a)

\dot{x}(0) = \dot{x}_0 - \frac{\varepsilon F'(0)}{F^2(0)} + \mathcal{O}(\varepsilon^3) \quad (35b)

\ddot{x}(0) = 1 + T_0 - x_0 - \frac{1}{F(0)} + \mathcal{O}(\varepsilon). \quad (35c)

Here, \(x_0, \dot{x}_0\) and \(T_0\) are taken to be \(\mathcal{O}(1)\) deviations from equilibrium equation (33). This gives us the following values for the constants of integration:

\[
\begin{align*}
C_1 &= \frac{\dot{x}_0}{\sqrt{F(0)}} \\
C_2 &= x_0 \sqrt{F(0)} \\
C_3 &= \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{\pi}} \right) \sqrt{F(0)} x_0 + \frac{1}{2} \frac{F'(0)}{F^3/2(0)} x_0 \\
&\quad + \frac{1}{2} \frac{\sqrt{2\pi}}{F^3/2(0)} (1 + T_0 - x_0) - \frac{1}{2} \frac{\sqrt{2\pi}}{F^5/2(0)} \\
C_5 &= \frac{\pi x_0}{2 F^3/2(0)} - \frac{1}{2} \frac{\pi}{F^3/2(0)} - \frac{1}{2} \pi \sqrt{F(0)} x_0 + \frac{1}{2} \frac{\pi}{F^5/2(0)} - \frac{1}{2} \frac{\pi T_0}{F^3/2(0)} \\
K_3 &= \frac{\pi}{2 F(0)} + \frac{T_0 \pi}{2 F(0)} - \frac{\pi}{2 F^2(0)} + \frac{1}{2} x_0 \pi F(0) - \frac{\pi x_0}{2 F(0)} \quad (36)
\end{align*}
\]

whereas \(K_1 = K_2 = C_4 = 0\) and \(C_6\) is left unspecified, meaning that it can be safely put to zero. We note that the initial conditions are satisfied exactly at \(x(0), \) up to order \(\mathcal{O}(\varepsilon^2)\) at \(\dot{x}(0)\) and up to order \(\mathcal{O}(\varepsilon)\) at \(\ddot{x}(0)\). This is not an accident of our system, but is a standard feature of the multiple scales approach [22]. The fact that \(K_1 = K_2 = 0\) is not surprising. From equation (32) it is quite clear that the first two orders \(\varepsilon^{-2}\) and \(\varepsilon^{-1}\) must vanish to get as an initial condition \(1 + T_0 = \mathcal{O}(1)\).

Without the need for an explicit form of \(F\) we can now give an intuitive meaning to the the three scales we introduced. The \(t_0\)-scale is the fastest one and characterises an exponential-relaxation of the system toward the equilibrium position given by (33). We notice from \(K_1 = K_2 = 0\) that this scale appears in \(x\) as an \(\varepsilon^2\) term, giving very little contribution. We also note that, due to the presence of second-order derivatives in equation (32) the \(t_0\) scale appears in \(T\) as the leading order term. This means that the temperature of the gas can have deviations from equilibrium as large as \(\varepsilon^{-2}\) and still rapidly converge to the equilibrium value. The \(t_1\)-scale is the one at which the oscillations of the system are established. It is worth noting that even if the system possesses a clear dissipative behaviour, the basic frequency of the system is unaltered adding the third scale, meaning that, if any correction in the basic frequency exists, then it should be at least of order \(\varepsilon^2 t\). In the \(t_2\)-scale the exponential suppression of the oscillatory terms appears; this means that the oscillations are slowly modulated. Overall we see, under suitable assumption on the smallness of \(\varepsilon\), that the approximated solution tends to the dynamical
equilibrium solution (33) as $t \to \infty$, which is coherent with a transient-like behaviour. We note that using only two time scales would have resulted in missing the modulation of the oscillation, leading to an erroneous result from both the physical and mathematical point of view. The previous considerations give us an \textit{a posteriori} justification of the choice of using only three time scales, since all the above features describe well the behaviour of the system from both a numerical and a physical point of view.

As a final recall on terminology we call from now on $x_{ap}$ the truncated part of the expansion for $x$ at order $\varepsilon^3$ given by (30), and we call $T_{ap}$ the truncated part of the expansion for $T$ at order $\varepsilon$. The next two sections are devoted to two particular examples of thermodynamic relevance, which we will use to test the quality of $x_{ap}$ as an approximated solution of the LDGPE and to derive new closed form expression of $Q$.

4. Relaxation to equilibrium

The first case under study is the one in which

$$F = 1 \quad \forall t \geq 0$$

with initial conditions

$$x(0) = 1 + x_0$$
$$\dot{x}(0) = \dot{x}_0$$
$$T(0) = 1 + T_0.$$  \hspace{1cm} (38)

This simple case encompasses all the situations where the gas and the piston relax from a given non-equilibrium condition $\{1 + x_0, \dot{x}_0, 1 + T_0\}$ to the thermodynamic equilibrium $\{x_{eq} = 1, \dot{x}_{eq} = 0, T_{eq} = 1\}$. Substituting equation (37) in equation (30) and equation (29), and then imposing that equation (36) must hold, we obtain an approximated expression for the piston position

$$x_{ap}(t) = 1 + \exp\left(-\varepsilon t(\pi + 2)\right)\left[ C_1 \sin(t) + C_2 \cos(t) \right.\right.$$  
$$+ \left( -\frac{\varepsilon^3}{2} (\varepsilon t^2 - 2\eta) C_1 + \theta t C_2 \varepsilon^2 + C_3 \varepsilon \right) \sin(t)$$
$$+ \left( -\frac{\theta^2 t^2}{2} C_2 \varepsilon^4 + (C_2 \eta t - C_5 \theta) \varepsilon^3 + (C_5 - C_1 t \theta) \varepsilon^2 \right) \cos(t) \right]$$
$$+ K_0 \varepsilon^2 \exp\left(\frac{2}{\pi} \frac{t}{\varepsilon} (\pi \varepsilon^2 - 1)\right)$$

with

$$\eta = -\frac{\sqrt{2\pi}}{4} (\pi + 4)$$
$$\theta = -\frac{(\pi^2 - 4\pi - 4)}{4\pi}$$  \hspace{1cm} (40)
and nonzero integration constants
\[ C_1 = \dot{x}_0 \]
\[ C_2 = x_0 \]
\[ C_3 = \frac{(\pi T_0 + \pi x_0 + 2x_0)}{\sqrt{2\pi}} \]
\[ C_5 = -\frac{\pi T_0}{2} \]
\[ K_3 = \frac{\pi T_0}{2}. \] (41)

In addition to the general properties of the scales inherited from equation (30), equation (39) shows two interesting properties. The first one is given by its general structure: being equation (39) made of a constant plus two decaying functions, it thermalises and describes well the relaxation to equilibrium. The second property is that we are now able to address the physical phenomena to which \( t_0 (\tau_0) \) and \( t_1 (\tau_1) \) are related. As a matter of fact, \( \tau_1 \) gives the suppression mechanism related to the mechanical damping the gas acts on the piston with a characteristic time of \( \frac{\sqrt{2\pi}}{\varepsilon x + 2} \). On the other hand, \( \tau_0 \) gives a second suppression mechanism emerging from the indirect coupling of the piston with the reservoir with a characteristic time of \( \frac{\varepsilon}{\pi^2 - 1} \sqrt{2} \). The fact that the piston-reservoir interaction is indirect is shown by the fact that this effect is \( \mathcal{O}(\varepsilon^2) \) in \( x \), whereas in \( T \) (where the gas-reservoir contact is direct) this effect is \( \mathcal{O}(\varepsilon^0) \). This feature is not surprising, as the the temperature reservoir appears explicitly only in equation (17b) and not in equation (17a), but we are now able to describe quantitatively this phenomena.

To test the quality of \( x_{ap} \) as a solution we note that the exponents of equation (39) converge only if
\[ 0 < \varepsilon < \pi^{-\frac{1}{2}}, \] (42)
which gives a more rigorous meaning to the small \( \varepsilon \) assumption. We then compute the differences
\[ \Delta x = \max_{t \in [0,1]}(|x - x_{ap}|) \]
\[ \Delta T = \max_{t \in [0,1]}(|T - T_{ap}|) \] (43)

between the approximated and the numerical solutions of equation (17) as functions of \( \varepsilon \) and one free initial condition, while the remaining two are the equilibrium values (e.g. \( x_0 \) free, \( x_0 = 0 \) and \( T_0 = 0 \)). Figures 2 and 3 show that, for a wide range of \( \varepsilon \) values and initial conditions, \( \Delta x \) is \( \mathcal{O}(\varepsilon^3) \) while \( \Delta T \) is \( \mathcal{O}(\varepsilon) \), which is consistent with the general properties of equations (30) and (32).

We now compute the heat produced while the system relaxes to equilibrium. Substituting equation (37) and \( T_b = 1 \) in equation (14c), gives
\[ Q^{rel}(0, t) = x(0) - x(t) + \frac{x^2(0) - x^2(t)}{2} + \frac{T(0) - T(t)}{2} \] (44)
Using now that $x(t) = x_{ap}(t) + \mathcal{O}(\varepsilon^3)$ and that $T(t) = T_{ap}(t) + \mathcal{O}(\varepsilon)$, we obtain the approximate expression for the heat

$$Q_{ap}^{rel}(0, t) = (1 + x_0 - x_{ap}(t)) + \frac{x_0^2 - x_{ap}^2(t)}{2} + \frac{1 + T_0 - T_{ap}(t)}{2} + \mathcal{O}(\varepsilon), \quad (45)$$

where we evaluated $x(0) = x_{ap}(0)$ and $T(0) = T_{ap}(0)$ instead of using equation (38). This expression describes the heat exchanged with the reservoir as a relaxation process takes place and is fully analytic: numerical evaluation becomes a trivial task, while using it for some formal calculations is likely to allow results to be expressed in a closed form.

Figure 2. Plots of $\log_\varepsilon(\Delta x)$ as a function of $\varepsilon$, $x_0$, $\dot{x}_0$, $T_0$. Subplot a: $x_0$ is varied with $\dot{x}_0 = 0$ and $T_0 = 0$. Subplot b: $\dot{x}_0$ is varied with $x_0 = 0$ and $T_0 = 0$. Subplot c: $T_0$ is varied with $x_0 = 0$ and $\dot{x}_0 = 0$. 

\[ \text{doi:10.1088/1742-5468/2016/05/053110} \]
5. Compression in a finite time

The second example we consider is the case in which the system, initially at equilibrium, undergoes a linear increase of the external force over a finite time and then relaxes to the new equilibrium condition. This encompasses all the isothermal compressions occurring in a finite time. W.l.o.g. this is modelled by taking

\[
F = \begin{cases} 
1 & \text{for } t < 0 \\
1 + fae t & \text{for } 0 \leq t \leq \frac{1}{ae} \\
1 + f & \text{for } t > \frac{1}{ae}.
\end{cases}
\]  

(46)
with initial conditions
\begin{align*}
x(0) &= 1 \\
\dot{x}(0) &= 0 \\
T(0) &= 1.
\end{align*} 
(47)

The additional parameters appearing in equation (46) are the amount of force $f$ by which $F$ is increased and the constant $a$, which fixes the time-span $1/(a\varepsilon)$. Substituting equation (46) in equation (30) and then imposing that equation (47) must hold, we obtain after a long but straightforward calculation that, for $0 \leq t \leq 1/(a\varepsilon)$,
\begin{align*}
x_{ap}(t) &= \frac{1}{1 + fa\varepsilon t} + \varepsilon^2 \left( -\frac{2a^2f^2}{(1 + fa\varepsilon t)^5} + \sqrt{\frac{2}{\pi}} \frac{af(\pi + 2)}{(1 + fa\varepsilon t)^3} \right),
\end{align*} 
(48)

while for $t \geq 1/(a\varepsilon)$, the system relaxes and $x_{ap}(t)$ is given by an adjusted version of equation (39) such that $x(\infty) = 1/(f + 1)$, $\dot{x}(\infty) = 0$ and $T(\infty) = 1$. We stress out that equation (48) is valid up to order $O(\varepsilon^3)$ because it satisfies equilibrium boundary conditions. Consequently, the corresponding $T_{ap}$ is valid up to $O(\varepsilon^3)$.

At this point we can compute equation (43) as functions of $\varepsilon$, $a$ and $f$ to investigate the quality of equation (48). Since this parametric study does not yield results strikingly different from the ones we obtained for the relaxation case, we rather test the goodness of $x_{ap}$ by looking at the heat produced during the gas compression. If we neglect heat exchanges at intermediate times, the net heat produced by this thermodynamic transformation is obtained by substitution of $T_{br} = 1$ and equation (46) into equation (14c). After some simple calculations we obtain
\begin{align*}
Q_{ap}^{\text{lin}}(a) &= Q_{ap}^{\text{lin}}(0, \infty, a) \\
&= x(0) - 1 + \frac{\dot{x}^2(0)}{2} + \frac{T(0) - 1}{2} + fa\varepsilon \int_0^{\frac{1}{a\varepsilon}} x(t)dt,
\end{align*} 
(49)

where we dropped $t_i = 0$ and $tf = \infty$ for compactness and the new $a$ dependence is to remind that the compression occurs for $t \in [0, 1/(a\varepsilon)]$. Substituting here equation (48) gives the approximated expression for the heat$^9$
\begin{align*}
Q_{ap}^{\text{lin}}(a) &= \ln(1 + f) - \frac{2(fa\varepsilon)^2 \left[ (1 + f)^2 - \frac{1}{2} \right] \left[ (1 + f) + \frac{1}{2} \right]}{(1 + f)^4} \\
&\quad + \frac{2fa\varepsilon^2 \sqrt{\frac{2}{\pi}} \left[ (\pi + \frac{3}{2})(1 + f)^2 - \frac{\pi + 2}{4} \right]}{(1 + f)^2} + O(\varepsilon^3).
\end{align*} 
(50)

This formula is our main thermodynamic result: it estimates the heat produced by a perfect gas under the action of a finite-time compression. One interesting property of equation (50) is that in the limit of quasi-static transformations, i.e. $a \to 0$,
\begin{align*}
\lim_{a \to 0} Q_{ap}^{\text{lin}}(a) &= \ln(1 + f).
\end{align*} 
(51)

$^9$ As in the previous case, $x(0) = x_{ap}(0)$ and $T(0) = T_{ap}(0)$. 

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This is exactly the value prescribed by the Clausius theorem. As a consequence, the remaining terms of equation (50) are contributions coming to the fact that the system is driven in a finite time. To have physical meaning, such contributions must be positive. This is a non-trivial requirement. However, the multiple scales method can be applied only if

\[ f > 0 \]
\[ a \lesssim \frac{1}{f} \]  

and, within these constrains, this positivity requirement is always satisfied.

We note that equation (52) is not the only validity constrain because the worst protocol with the functional form of equation (46) is the one where the compression is instantaneous. This corresponds to take the limit \( a \to \infty \) in equation (14c) with equation (47) and yields the upper bound \( Q^{\text{lin}}(a) \leq f \). Therefore, we must have that \( Q^{\text{lin}}(a) \in [\ln(1 + f), f] \). Since equation (50) has an evaluation error of the order of \( \varepsilon^2 \), if the inequality

\[ f - \ln(1 + f) \gg \varepsilon^2 \]  

is not satisfied, the estimation error on equation (50) is bigger than the energy region we want to investigate. Any result obtained with \( Q^{\text{lin}}(a) \) is then of no practical use. We therefore conclude that the validity region of equation (50) is given by equations (52) and (53).

We conclude this section with a numerical study of equation (50): we compute the difference

\[ \Delta Q = |Q^{\text{lin}}(a) - Q_{\text{ap}}^{\text{lin}}(a)| \]  

where

\[ a > \frac{1}{f} \]

and

\[ f \lesssim \frac{1}{\varepsilon^2} \]

This region corresponds to the regime where the estimation error is small enough to have physical meaning.
as functions of $a$ and $f$ for a given $\varepsilon$ value. Figures 4 and 5 show the result obtained for $\varepsilon = 0.1$ and $\varepsilon = 0.01$. From the picture it clearly appears that $\Delta Q$ is $O(\varepsilon^3)$ within the constraints defined by equations (52) and (53). This proves that equation (50) is a good analytical expression of the heat produced during a finite time compression of a perfect gas.

6. Conclusions and future developments

In this paper we focused on the LDGPE derived in [11] to describe the dynamical evolution of a gas enclosed by a piston and in contact with a heat reservoir. In particular, we showed that the LDGPE has an approximated analytic solution when the temperature of the reservoir is fixed and the force applied on the piston varies according to a general force protocol. To derive this result we used the multiple scales expansion method. Although this is valid only within some constraints, we can use the approximated solutions to describe the thermodynamics of fundamental non-equilibrium processes. Our main result is that we are now able to compute in a closed form the heat produced when the gas-piston system (1) relaxes toward equilibrium, and (2) undergoes an isothermal compression in a finite time. As the derivation of analytic heat relations through the multiple scales method turned out to be quite straightforward, we believe that this perturbative technique could be useful in understanding finite time thermodynamics.

An issue that deserves future investigations is the following: we already stated that equation (50) is valid when $a \lesssim 1/f$. Nonetheless, it is clear that an instantaneous compression ($a = \infty$) produces heat only because the system relaxes toward the new equilibrium condition. It is not hard to prove by means of equation (45) that $\dot{Q}^{\text{lin}}(\infty) = Q_{\text{eq}}^{\text{rel}}(0, \infty) = f$. We thus have that equation (30) allows to simultaneously describe the heat produced by an isothermal compression for both $a \lesssim 1/f$ and $a \gg 1/f$. This gives us a hint that there could be a way to access the missing $a$ region by means of some well-aimed multiple scales expansion. One other issue that can be treated with

\textbf{Figure 5.} Same as in figure 4 but for $\varepsilon = 0.01$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Same as in figure 4 but for $\varepsilon = 0.01$.}
\end{figure}
the multiple scales method is the study of the resonance of equation (17). Since the system has a unitary characteristic frequency, this problem can be efficiently addressed with the multiple scales method [3, 19, 22] by choosing

$$F(t) = 1 + \varepsilon \sin((1 + \varepsilon)t),$$

(55)

which is small in amplitude but not slow anymore. We also note that our results are a direct consequence of linearity of the LDGPE. However, we are also interested to derive a multiple scales expansion for the DGPE. It is clear to the reader that this problem is far more difficult from the linear one because (1) the system does not reduce to a single equation and (2) \(x\) and \(T\) need different scalings to avoid singularities.

We conclude by noting that if we restrict ourselves to dynamical equilibrium solutions we proved that it is possible to compute the heat exchanged when the reservoir temperature is also slowly varying over time [12]. The price to pay is that we lose the characterisation of transient behaviours. However, these can be accessed by a full multiple scales method, where also \(T_0 = T_0(\varepsilon t)\).

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