Static spherical vacuum solutions in the bumblebee gravity model

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The bumblebee gravity model is a vector-tensor theory of gravitation where the vector field nonminimally couples to the Ricci tensor. By investigating the vacuum field equations with spherical symmetry, we find two families of black-hole (BH) solutions in this model: one has a vanishing radial component of the vector field and the other has a vanishing radial component of the Ricci tensor. When the coupling between the vector field and the Ricci tensor is set to zero, the first family becomes the Reissner-Nordström solution while the second family degenerates to the Schwarzschild solution with the vector field being zero. General numerical solutions in both families are obtained for nonzero coupling between the vector field and the Ricci tensor. Besides BH solutions, we also reveal the existence of solutions that have a nonvanishing $tt$-component of the metric on the supposed event horizon where the $rr$-component of the metric diverges while the curvature scalars are finite. These solutions are not supported by existing observations but present certain properties that are of academic interest. We conclude the study by putting the BH solutions into tests against the Solar-System observations and the images of supermassive BHs.

I. INTRODUCTION

Einstein’s theory of general relativity (GR) and Maxwell’s theory of electromagnetism (EM) are the two representative classical field theories in physics. A straightforward combination of them, i.e., the Einstein-Maxwell theory, provides the simplest unification of the gravitational interaction and the EM interaction at the classical level, and serves as the prototype of vector-tensor theories in which vector fields are employed to complement the gravitational interaction described conventionally by the metric tensor. The action of the Einstein-Maxwell theory reads

$$ S = \int d^4 x \sqrt{-g} \left( \frac{1}{2k} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) + S_m, \quad (1) $$

where $g$ is the determinant of the metric $g_{\mu\nu}$, the constant $\kappa$ is $8\pi G$ with $G$ being the gravitational constant, $R$ is the Ricci scalar, $F_{\mu\nu} := D_{\mu} A_{\nu} - D_{\nu} A_{\mu}$ is the field strength of the vector field $A_{\mu}$ with $D_{\mu}$ being the covariant derivative, and $S_m$ represents the action of matter. Without identifying the vector field as the EM vector potential, the action in Eq. (1) was first used by Will and Nordtvedt to illustrate possible preferred-frame effects in gravity [1]. The point is that if the vector field $A_{\mu}$ possesses a nonzero background configuration, then the spacetime is anisotropic and hence has preferred frames.

Preferred frames violate Lorentz symmetry. The motivation of such a violation from string theory and the implications of it in low-energy effective field theories were first studied by Kostelecký and Samuel [2, 3]. A generic framework, called the Standard-Model Extension (SME), has been developed to systematically incorporate all possible Lorentz-violating couplings into the actions of the Standard Model of particle physics and GR [4–10], and to calculate a number of predictions that can be tested in modern high-precision experiments [11, 12]. The primary Lorentz-violating coupling in the gravitational sector of the SME takes the form $s_{\mu\nu} R_{\mu\nu}$, where $s_{\mu\nu}$ is a tensor field possessing a nonzero background configuration and $R_{\mu\nu}$ is the Ricci tensor. Because the background configuration of $s_{\mu\nu}$ defines preferred frames in general, this coupling violates Lorentz symmetry. The SME focuses on general properties and consequences caused by the coupling term, but asks little about the dynamics of the field $s_{\mu\nu}$. To fill the gap between the SME and specific gravitational theories having the Lorentz-violating coupling, the action in Eq. (1) was generalized to [6, 13]

$$ S = \int d^4 x \sqrt{-g} \left( \frac{1}{2k} R + \frac{\xi}{2k} B^{\mu} B_{\mu} R_{\mu\nu} - \frac{1}{4} B^{\rho\sigma} B_{\rho\sigma} - V \right) + S_m, \quad (2) $$

where $B^{\mu}$ is a dynamical vector field sometimes called the bumblebee field, and the generalized vector-tensor theory (2) is called the bumblebee gravity model [6].

Apart from replacing the vector field $A^{\mu}$ by $B^{\mu}$ (correspondingly $F_{\mu\nu}$ by $B_{\mu\nu} := D_{\mu} B_{\nu} - D_{\nu} B_{\mu}$), the important changes are adding (i) the coupling term $B^{\mu} B_{\mu} R_{\mu\nu}$ controlled by the constant $\xi$ to resemble the SME term $s^{\mu\nu} R_{\mu\nu}$, and (ii) the potential $V$ that gives its extremum when the bumblebee vector field $B^{\mu}$ acquires a certain background configuration. By considering the linearized, perturbative bumblebee model where the deviation of the metric from the Minkowski metric $\eta_{\mu\nu}$ and the deviation of the bumblebee vector field from its background configuration are small, conclusions made in the general framework of the SME are verified by and contrasted with results in the bumblebee model [6, 13–18].

Simply speaking, the bumblebee model is an essential example theory for studying Lorentz violation in gravity as its specified action provides information on the dynamics of the auxiliary field that breaks Lorentz symmetry, which is otherwise not contained in the general framework of the SME. But it is more than that. In fact, the action in Eq. (2) without the potential $V$ has been studied by Hellings and Nordtvedt [19]...
as an alternative to GR in the context of the parametrized post-Newtonian (PPN) formalism as well as the cosmological solutions. At the end of their work, they brought up the proposal of identifying the vector field as the EM vector potential. A similar possibility was suggested by Bluhm and Kostelecký [14] in the framework of the SME, where EM waves are the Nabut-Goldstone modes of the bumblebee vector field when Lorentz symmetry is spontaneously broken due to a background bumblebee field resulting from the potential \( V \). The idea of replacing the Einstein-Maxwell theory with the bumblebee model for unifying gravity and the EM theory is attractive.

Compared with the Einstein-Maxwell theory, the bumblebee model is expected to produce new and maybe even eccentric phenomena when the vector field interacts with gravity through the coupling term \( B^\mu B^\nu R_{\mu\nu} \). The gravitational field is optimally to be strong for such phenomena to be detectable. Therefore, strong-field solutions without applying the perturbation approach, as was done in the SME or the PPN formalism, should be considered. Black holes (BHs) are ideal strong-field systems to study, for not only the absence of matter simplifies the field equations, but also they have been detected with both gravitational waves (GWs) [20–22] and EM waves [23, 24] so that unprecedented tests can be performed using the rapidly developing technology of multimessenger astronomy [25].

In the bumblebee model, an analytical solution that is very close to the Schwarzschild spacetime with the bumblebee field having only a nonvanishing radial component has been reported by Casana et al. [26]. In our work, we substantially extend the study and find spherical solutions with a nonvanishing temporal component of the bumblebee field. In fact, as suggested by the EM-like kinetic term in the action, the radial component of the bumblebee field is nondynamic, and can be eliminated from the final set of equations. It turns out there are two families of solutions: one solved from a group of two second-order ordinary differential equations (ODEs), and one solved from a group of three second-order ODEs. The presence of the temporal component of the bumblebee field is vital for both families of solutions as it alters the behavior of the metric near the BH event horizon radically. Most amazingly, it makes solutions with a nonvanishing \( g_{tt} \) at the event horizon possible.

We find that it is necessary to distinguish the BH solutions where \( g_{tt} \) vanishes at the event horizon from the solutions having a nonvanishing \( g_{tt} \) at the event horizon, for geodesics in the spacetime of the latter solutions remarkably bounce back on the event horizon. For this reason, those solutions with a nonvanishing \( g_{tt} \) at the event horizon are called compact hills (CHs) in our work. The bouncing-back behavior of geodesics has no experimental or observational evidence in gravity phenomena yet. Nevertheless, academically interesting features of the CH solutions are discussed, including the GW echoes [27].

For the BH solutions, we put them into tests against Solar-System observations and the shadow images of supermassive BHs achieved by the Event Horizon Telescope (EHT) Collaboration. As the two families of solutions have different numbers of free parameters, constraints on their parameter spaces are different. In one family, the BHs are characterized by two parameters, namely the mass and the bumblebee charge, so bounds on the bumblebee charge of the Sun and of the supermassive BHs are obtained. In the other family, the BHs have five free parameters where three of them are in the metric functions and two of them are in the bumblebee field. The Solar-System observations happen to only depend on the metric parameters for the solutions in this family, so bounds on the metric parameters are obtained while leaving the two bumblebee parameters unconstrained. The parameter space excluded in our work is limited, especially for the second family of solutions. Future study on the GWs in the spacetime of the BH solutions might yield tighter or complementary constraints.

The organization of this paper is as follows. We start with setting up the equations in Sec. II A. Then an analytical solution is presented in Sec. II B. General numerical solutions are discussed in Sec. II C, and a detailed discussion about the CH solutions is made in Sec. II D. In Sec. III A, the BH solutions are tested by considering Solar-System observations, while in Sec. III B, the test is done by considering the results from the shadow images of supermassive BHs. Finally, we summarize our findings and give a brief outlook for the directions worthy of further study in Sec. IV. The appendices list equations to supplement the main text.

In the remaining of the paper, equations are written in the geometrized unit system where \( G = c = 1 \). The sign convention of the metric is \((- (+, +, +))\).

II. STATIC SPHERICAL SOLUTIONS IN VACUUM

A. Field equations

The field equations are obtained by taking variations with respect to the metric and the bumblebee field in Eq. (2). Under the assumption that the bumblebee field does not couple to conventional matter, the field equations can be written as

\[
G_{\mu\nu} = \kappa (T_m)_{\mu\nu} + \kappa (T_B)_{\mu\nu},
\]

\[
D^\mu B_{\mu\nu} - 2B_\nu \frac{dV}{d(B^2 B_1)} + \frac{\tilde{\xi}}{\kappa} B^\mu R_{\mu\nu} = 0,
\]

where \((T_m)_{\mu\nu}\) is the energy-momentum tensor for conventional matter, and the contribution of the bumblebee field to the Einstein equations is

\[
(T_B)_{\mu\nu} = \frac{\tilde{\xi}}{2\kappa} \left[ g_{\mu\nu} B^\rho B^\sigma R_{\rho\sigma} - 2B_\rho B_\sigma R_{\mu\nu} - 2B_\rho B_\sigma R_{\mu\nu} + \Box_{\mu\nu} (B_\rho B_\sigma) - g_{\mu\nu} D_\alpha D_\beta (B^\rho B^\sigma) + D_\alpha D_\beta (B^\rho B_\sigma) + D_\alpha D_\beta (B_\rho B^\sigma) \right] + B_{\mu\rho} B_{\nu}\sigma - g_{\mu\nu} \left\{ \frac{1}{4} B^\rho B_\rho + V \right\} + 2B_\mu B_\nu \frac{dV}{d(B^2 B_1)}.
\]

The d’Alembertian in the curved spacetime is \( \Box_{\mu\nu} := g^{\rho\sigma} D_\rho D_\sigma \).

Note that the potential \( V \) has been assumed to be a function of the scalar product \( B^\mu B_\mu \).

We are interested in nontrivial background configurations of the bumblebee field that are compatible with vacuum,
namely \((T_{m})_{\mu\nu} = 0\). The potential \(V\) is supposed to take extrema at these background configurations. Denoting \(B_\mu = b_\mu\) as a background bumblebee field, we have

\[
\left. \frac{dV}{d(B^\mu B_\mu)} \right|_{B_\mu = b_\mu} = 0. \tag{5}
\]

In addition, the value of \(V\) at \(B_\mu = b_\mu\) is equivalent to a cosmological constant. We drop it in the current study of BH solutions. Therefore, the vacuum field equations are simplified to

\[
G_{\mu\nu} = \kappa (T_\mu)_\nu, \tag{6}
\]

\[
D^\mu b_{\mu\nu} + \frac{\xi}{\kappa} b^2 R_{\mu\nu} = 0,
\]

where \(b_{\mu\nu} = D_\mu b_\nu - D_\nu b_\mu\), and

\[
(T_\mu)_\nu = \frac{\xi}{\kappa} \left[ g_{\mu\nu} b^\rho b^\lambda R_{\rho\lambda} - 2 b_\rho b_\lambda R_{\rho\mu\nu} - 2 b_\rho b_\mu R_{\rho\lambda\nu} - \Box g (b_\mu b_\nu) - 2 b_{\rho\mu} D_\nu (b^\rho b_\lambda) + D_\nu (b^\rho b_\rho b_\lambda) \right] + \frac{4}{\kappa} g_{\mu\nu} b^\rho b_\rho b_{\lambda\lambda}.
\]

Focusing on static spherical solutions, we use the metric ansatz

\[
ds^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

and assume the background bumblebee field to be \(b_\mu = (b_t, b_r, 0, 0)\), where the unknowns, \(\mu, \nu, b_t, \) and \(b_r\), are functions of \(r\). Then the Einstein field equations are nonvanishing for the diagonal components as well as the \(tr\)-component, and the vector field equation has nonvanishing \(t\)-component and \(r\)-component. It turns out that the \(tr\)-component of the Einstein field equations is guaranteed by the \(r\)-component of the vector field equation, leaving 5 equations for the 4 unknowns, \(\mu, \nu, b_t, \) and \(b_r\). So one of the equations depends on the others. These equations are displayed in Appendix A.

### B. An analytical solution

Inspired by the solution obtained in Ref. [26], we checked

\[
\nu = \frac{1}{2} \ln \left( 1 - \frac{2M}{r} \right),
\]

\[
\mu = \mu_0 - \frac{1}{2} \ln \left( 1 - \frac{2M}{r} \right),
\]

\[
b_t = \lambda_0 + \frac{\lambda_1}{r},
\]

against Eqs. (A1) and (A2), or equivalently Eq. (6) with the static spherical ansatz, to find that as long as

\[
b_t^2 = e^{2\mu_0} \left[ \frac{1}{\xi} \left( e^{2\mu} - 1 \right) r - \frac{\kappa \lambda_1^2}{r - 2M} - \frac{\kappa \lambda_2}{3M(r - 2M)} \right] + \lambda_1^2 (2 - M) + 6\lambda_0 \lambda_1 M r + 6\lambda_1^2 M^2 r \]

\[
+ \frac{\lambda_1^2 (2 - M) + 6\lambda_0 \lambda_1 M r + 6\lambda_1^2 M^2 r}{3M(r - 2M)^2}.
\]

Eq. (9) is a solution to the field equations. It is an analytical solution characterized by 4 integral constants, \(\mu_0, M, \lambda_0\) and \(\lambda_1\). The 2-parameter solution obtained in Ref. [26] simply corresponds to \(\lambda_0 = 0\). We also notice that the 3-parameter solution corresponding to \(\mu_0 = 0\) has been previously obtained in Ref. [28].

The following two observations can be made based upon the solution given by Eqs. (9) and (10).

1. The Schwarzschild metric corresponding to \(\mu_0 = 0\) is no longer necessarily accompanied by a vanishing bumblebee field. Among the valid choices of \(\lambda_0\) and \(\lambda_1\) that give nonnegative \(b_t^2\), an intriguing choice is \(\lambda_1 = -2M\lambda_0\), which simplifies Eq. (10) to

\[
b_t^2 = \frac{2}{3} \left( 1 - 2\kappa \right) \frac{\lambda_0^2 M}{r - 2M},
\]

for \(\mu_0 = 0\). As we require \(b_t^2 \geq 0\) outside the event horizon, this choice of \(\lambda_1\) can only be made for \(\xi \geq 2\kappa\). For the special case of \(\xi = 2\kappa\), the Schwarzschild metric is a solution to the field equations together with a nontrivial bumblebee field that has only a temporal component \(b_t \propto 1 - 2M/r\).

2. The Minkowski metric is recovered with \(\mu_0 = 0\) and \(M = 0\). The limit of Eq. (10) exists if \(\lambda_1 \propto \sqrt{M}\) while \(M \to 0\). That is to say, the Minkowski metric can be accompanied by a nontrivial bumblebee field that has a radial component

\[
b_r^2 = \frac{1}{3} \left( 2 - \frac{\kappa}{\xi} \right) \frac{C}{r},
\]

where \(C\) being the ratio \(\lambda_1^2/M\) has the dimension of length, and might represent the length scale of possible Lorentz violation in an underlying theory.

### C. General numerical solutions

General solutions to Eqs. (A1) and (A2) are obtained numerically in two cases enumerated in Appendix A according to the two factors in the \(r\)-component of the vector equation: (i) \(b_t = 0\); (ii) \(R_{tr} = 0\). For the first case, the system of equations reduce to two coupled second-order ODEs, indicating a family of solutions determined by 4 parameters. For the second case, the system of equations reduce to three coupled second-order ODEs, indicating a family of solutions determined by 6 parameters. Both families of solutions turn out to have the asymptotic expansions

\[
\nu = \sum_{n=1}^{\infty} \frac{y_n}{r^n},
\]

\[
\mu = \sum_{n=0}^{\infty} \frac{\mu_n}{r^n},
\]
After substituting them into the ODEs to find the recurrence relations for the expansion coefficients \(\nu_r, \mu_r, \text{ and } \lambda_r\), these expansions can be utilized to assign the initial values to start the numerical integrations from large enough \(r\). Note that a constant term in the expansion of \(\nu\) is unnecessary as it amounts to a change of the time coordinate. Therefore, the number of parameters characterizing the solutions is actually three for the first family \((b_r = 0)\) and five for the second family \((R_r = 0)\).

The recurrence relations for the first few coefficients are given in Appendix B. The conclusion is that for the case of \(b_r = 0\), the solutions must have \(\mu_0 = 0\) and are fixed once the three free coefficients \(\mu_1, \lambda_0\), and \(\lambda_1\) are specified, and that for the case of \(R_r = 0\), there are 5 free coefficients, \(\mu_0, \mu_1, \mu_2, \lambda_0\), and \(\lambda_1\), describing the solutions. We point out that the analytical solution given by Eqs. (9) and (10) belongs to the second case and is recovered when taking \(\mu_2 = \mu_1^2 = M^2\).

Before further discussing the numerical solutions, let us relate the parameters \(\mu_1\) and \(\lambda_1\) to the mass and the bumblebee charge of the spacetime. With \(\mu_0 = 0\), the Arnowitt-Deser-Misner (ADM) mass of the spacetime can be calculated and it is \(\mu_1\) [29]. However, the definition of the ADM mass does not apply to the solutions with \(\mu_0 \neq 0\), when the metric is not asymptotically flat. Therefore we adopt the Komar mass as a measure of the spacetime mass when \(\mu_0 \neq 0\). The Komar mass is defined by the conserved current associated with the time-translation Killing vector \(\mathcal{K}^\mu = (1, 0, 0, 0)\). The conserved current is [30]

\[
J_M^\mu := K_v R_v^\mu = D_v D^\mu \mathcal{K}^v,
\]

where the curvature identity \([D_v, D_w]K^\alpha = R^\alpha_{\delta\mu
u}K^\delta\) and the Killing equation \(D_v K_w + D_w K_v = 0\) have been used. The Komar mass then can be calculated as

\[
M_K := -\frac{1}{4\pi} \int d^3x \sqrt{-g} J_M^\mu
= -\frac{1}{4\pi} \lim_{r \rightarrow \infty} \int d\theta d\phi \sqrt{-g} D^\mu \mathcal{K}^\mu
= e^{-i\mu_0} \mu_1.
\]

It is evident that the Komar mass reduces to the ADM mass when \(\mu_0 = 0\). The bumblebee charge can be defined in a similar way. Instead of using a Killing vector, the current

\[
J_Q^\mu := \frac{\xi}{k} b_v R_v^\mu = -D_v b^{\mu v}
\]

is automatically conserved as \(b_{\mu v}\) is antisymmetric. Therefore the bumblebee charge having the dimension of length can be defined as

\[
Q := -\frac{1}{4\pi} \sqrt{\frac{k}{2}} \int d^3x \sqrt{-g} J_Q^\mu
= -\frac{1}{4\pi} \sqrt{\frac{k}{2}} \lim_{r \rightarrow \infty} \int d\theta d\phi \sqrt{-g} b^{\mu v}
= \sqrt{\frac{k}{2}} e^{-i\mu_0} \lambda_1,
\]

where the factor \(\sqrt{k/2}\) has been chosen so that the Reissner-Nordström metric in Eq. (B3) can be recovered when \(\xi = 0\).

Back to the numerical results, as we integrate the system of ODEs from a large radius to \(r = 0\), there exist three types of solutions. The one that we are interested in has \(g_{tt} = e^{2\mu}\) diverging at a certain radius \(r = r_h\). The other two types that we will ignore are solutions of naked singularity and solutions with \(g_{tt} = -e^{2\nu}\) diverging at a finite radius. Concentrating on the interesting solutions that have \(g_{tt}\) diverging at a finite radius \(r = r_h\), the BH solutions are selected with two extra conditions: (i) vanishing \(g_{tt}\) at \(r_h\) and (ii) finite curvature scalars at \(r_h\). It turns out that the finiteness of the curvature scalars at \(r_h\) is trivially satisfied by all the solutions with \(g_{tt}\) diverging at \(r_h\). But among these solutions, it turns out that not all of them have \(g_{tt} = 0\) at \(r_h\). In fact, for the solutions in the first family \((b_r = 0)\), most of them have nonvanishing \(g_{tt}\) at \(r_h\).

To comprehend what happens near \(r_h\), we seek expansions of the variables in terms of \(\delta := r - r_h\). It is after a careful investigation of the numerical solutions that we find the variables taking the following expansions near \(r_h\),

\[
g_{tt} = -e^{2\nu} = -\sum_{n=0}^{\infty} N_{1n} \delta^n - \sqrt{\delta} \sum_{n=0}^{\infty} N_{2n} \delta^n,
\]

\[
g_{rr} = e^{2\nu} = 1 - \sum_{n=0}^{\infty} M_{1n} \delta^n + \frac{1}{\sqrt{\delta}} \sum_{n=0}^{\infty} M_{2n} \delta^n,
\]

\[
b_r = \sum_{n=0}^{\infty} L_{1n} \delta^n + \sqrt{\delta} \sum_{n=0}^{\infty} L_{2n} \delta^n,
\]

where \([N_{11}, N_{21}, M_{11}, M_{21}, L_{11}, L_{21}]\) is the set of expansion coefficients. The recurrence relations can be obtained by substituting the expansions into the ODEs and setting each order of \(\delta\) to zero. The recurrence relations for the first few coefficients are given in Appendix C. The most important fact we learn is that the coefficient \(N_{10}\) is closely related to the coefficients for the terms involving \(\sqrt{\delta}\) in the following way. If \(N_{10} = 0\), then \(N_{20} = M_{20} = L_{20} = 0\), and vice versa.

The solutions with \(g_{tt}\) diverging at a finite radius \(r_h\), namely those can be expanded in the form of Eq. (18), thus can be divided into two categories according to whether \(N_{10}\) is zero. The BH solutions have \(N_{10} = 0\), while the solutions with \(N_{10} \neq 0\), for their unexpected property of bouncing geodesics at \(r_h\), which will be discussed in Sec. II D, will be dubbed CHs in the remaining of the paper. Note that both the BH solutions and the CH solutions have finite curvature scalars at \(r_h\) (see Appendix D).

Figures 1 and 2 show how \(r_h\) depends on the asymptotic parameters \(\lambda_0\) and \(\lambda_1\). We take the asymptotic parameter \(\mu_1\) as the length unit in the numerical calculations. For the first family of solutions \((b_r = 0)\), a solution is obtained once the values of the coupling constant \(\xi\) and the asymptotic parameters \(\lambda_0\) and \(\lambda_1\) are specified properly. Sufficient amount of solutions obtained by varying \(\lambda_0\) and \(\lambda_1\) then generate the contour plots of \(r_h\) on the \(\lambda_0\)\(-\lambda_1\) plane as shown in Fig. 1. For the second family of solutions \((R_r = 0)\), with \(\mu_1\) being the unit, a solution is obtained when the asymptotic parameters \(\mu_0, \mu_2, \lambda_0, \lambda_1\), as well as the coupling constant \(\xi\) are specified properly. To generate Fig. 2, we have taken \(\mu_0 = 0\) and two representative
FIG. 1. Contour plots for $r_s$ on the $\lambda_0$-$\lambda_1$ plane for the solutions in the first family ($b_r = 0$). The asymptotic parameter $\mu_1$ is used as the length unit in the geometrized unit system. The blank region in the first plot consists of uninteresting solutions, namely naked singularities and solutions with singular $g_{tt}$. Notice that when $\xi = 2k$, metric solutions are identical given $\lambda_0$ and $\lambda_1$ on a line $\lambda_1 = -2\lambda_0 + \text{constant}$. The line $\lambda_1 = -2\lambda_0$ gives the Schwarzschild metric specially.

FIG. 2. Contour plots for $r_s$ on the $\lambda_0$-$\lambda_1$ plane for the solutions in the second family ($R_{tr} = 0$). We have taken $\mu_0 = 0$ and used $\mu_1$ as the length unit in the geometrized unit system. The blank regions in the two plots in the first row consist of naked singularities. The red lines are boundaries of the inequality (B7), which is a requirement of $b_r^2 \geq 0$ at infinity. Among the 5 plots with the red lines, the region for the inequality to hold contains the origin $\lambda_0 = \lambda_1 = 0$ only for the second plot in the first row.

values for $\mu_2$. Solutions with nonzero but small $\mu_0$ turn out to produce similar shapes of the contour plots, while solutions with $\mu_0$ deviating from zero significantly are uninteresting to us as $\mu_0$ has been constrained to be smaller than $10^{-12}$ using the advance of perihelia of the Solar-System planets in Ref. [26].

Figures 3 and 4 show how $N_{10}$ depends on $\lambda_0$ and $\lambda_1$, using the same solutions producing Figs. 1 and 2. The values of the asymptotic parameters $\lambda_0$ and $\lambda_1$ for BH solutions and for CH solutions are conveniently revealed in Figs. 3 and 4. For the first family of solutions, the requirement of $N_{10} = 0$ turns out to force $L_{10}$ to vanish, which acts as an extra constraint on the asymptotic parameters $\lambda_0$ and $\lambda_1$. Therefore, the BH solutions in the first family only exist along a single line on the $\lambda_0$-$\lambda_1$ plane for a given $\xi$, leaving the majority of the valid $\lambda_0$ and $\lambda_1$ to generate the CH solutions characterized by $N_{10} \neq 0$. For the second family of solutions, the requirement of $N_{10} = 0$ forces no extra constraint on $\lambda_0$ and $\lambda_1$, so the BH solutions exist on the two-dimensional plane of $\lambda_0$ and $\lambda_1$. In fact, as shown in Fig. 4, there is no CH solution for any values of $\lambda_0$ and $\lambda_1$ as long as one takes $\mu_2 \leq \mu_1^2$.

Comparing the BH solutions in the first family ($b_r = 0$) with the charged BH solution in GR, i.e., the Reissner-Nordström metric, is worthwhile. The asymptotic expansions of the metric functions $\mu$ and $\nu$ given by the recurrence relations in Eq. (B1) include the Reissner-Nordström metric as a special case for $\xi = 0$, when the bumblebee model recovers the Einstein-Maxwell theory. This suggests that the bumblebee BH solutions in the first family, characterized by the two parameters, $M = \mu_1$ and $Q = \sqrt{k/2} \lambda_1$, can be viewed as a gen-
FIG. 3. Contour plots for $N_{10} = -g_{tt}(r_h)$ on the $\lambda_0-\lambda_1$ plane for the solutions in the first family ($b_r = 0$). The BH solutions, corresponding to $N_{10} = 0$, are located along the brown lines. The colored contours with $N_{10} > 0$ are solutions dubbed “CHs” in our work.

FIG. 4. Contour plots for $N_{10} = -g_{tt}(r_h)$ on the $\lambda_0-\lambda_1$ plane for the solutions in the second family ($R_{pp} = 0$). The BH solutions, corresponding to $N_{10} = 0$, form the regions colored in brown. We have checked that our numerical integrations have errors less than $5 \times 10^{-5}$. The contours in other colors represent solutions with $N_{10} > 0$. These solutions, dubbed “CHs”, do not exist for $\xi = 2\kappa$ or when $\mu_2 \leq \mu_1^2$. The red lines have the same meaning as in Fig. 2.

eralization of the Reissner-Nordström metric. A plot of $r_h$ versus $Q$ in unit of $\mu_1$ as shown in Fig. 5 demonstrates how the family of solutions change with the coupling constant $\xi$. We would like to direct the readers’ attention to two special values of $\xi$, namely $\xi = 0$ and $\xi = 2\kappa$. When $\xi < 0$, the maximal charge is less than $M$, and when $\xi > 0$, the maximal charge is greater than $M$. When $\xi < 2\kappa$, the maximal radius of the event horizon is $2M$ (Schwarzschild metric), and when $\xi > 2\kappa$, the maximal radius of the event horizon is greater than $2M$. We notice that when $\xi > 2\kappa$ or $\xi < 0$ there exist two black-hole solutions with the same $r_h$ but different $Q$. The two black holes are certainly observationally distinguishable as the difference in $Q$ makes their metric functions different. In fact, when we calculate the shadow of the black holes in Sec. III B, we see that they have different shadow radii (see Fig. 10). A perhaps more interesting but puzzling observation is that when $\xi = \kappa$, $r_h$ has two values given $Q$ near its maximum. When this happens, we suspect that the black hole with the smaller $r_h$ might be unstable. Further study on quasi-normal modes of the black holes found here will shed light on this point.

We end the general discussion on the solutions by tabulating several representative numerical solutions for further reference. Table I lists 4 BH solutions and 4 CH solutions in the first family. Table II lists 8 BH solutions and 3 CH solutions in the second family. To have a brief picture about how different from the Schwarzschild metric these solutions are, the metric functions, as well as the scalar $b^2 := \eta^{\mu\nu}b_\mu b_\nu$ and the Ricci scalar $R$, are plotted in Fig. 6, for several of the representative solutions.
geodesics bounce back on the event horizon, which is the rea-
gulations at large radius. But near the event horizon, the non-
expansions with the BH solutions, so they mimic the BH so-

| Solution | $\xi$ | Asymptotic parameters | Event horizon behavior |
|----------|-------|------------------------|-----------------------|
|          | $\lambda_0$ | $\lambda_1$ | $r_h$ | $-g_{tt}$ | $g_{rr}$ | $b_t$ |
| BH-Ia  | $-\kappa$ | 0.1389 | -0.1974 | 1.719 | 0.4913 $\delta$ | $\frac{1.50}{d}$ | 0.1254 $\delta$ |
| CH-Ia  | $-\kappa$ | 0.2 | -0.25 | 1.079 | 0.04675 - 0.05157 $\sqrt{d}$ | $\frac{0.6751}{d}$ - $\frac{0.05772}{\sqrt{d}}$ | $-0.07031 + 0.07584 \sqrt{d}$ |
| BH-Ib  | $\kappa$ | 0.2568 | -0.3637 | 0.6168 | 0.6754 $\delta$ | $\frac{0.8299}{d}$ | 0.2454 $\delta$ |
| CH-Ib  | $\kappa$ | 0.2 | -0.25 | 2.624 | 0.1839 + 0.2205 $\sqrt{d}$ | $\frac{0.4535}{d}$ + $\frac{1.080}{\sqrt{d}}$ | 0.09549 + 0.03323 $\sqrt{d}$ |
| BH-Ic  | $2\kappa$ | $\lambda_0$ | -2$\lambda_0$ | 2 | $\frac{1}{2}$ $\delta$ | $\frac{3}{2}$ | $\frac{\lambda_0}{d}$ $\delta$ |
| CH-Ic  | $2\kappa$ | 0.2 | -0.25 | 3.073 | 0.1742 + 0.2380 $\sqrt{d}$ | $\frac{0.7683}{d}$ + $\frac{1.050}{\sqrt{d}}$ | 0.08396 + 0.04761 $\sqrt{d}$ |
| BH-Id  | $3\kappa$ | 0.2055 | -0.3813 | 2.168 | 0.4878 $\delta$ | $\frac{1.419}{d}$ | 0.1112 $\delta$ |
| CH-Id  | $3\kappa$ | 0.2 | -0.25 | 2.880 | 0.1324 + 0.2289 $\sqrt{d}$ | $\frac{0.8226}{d}$ + $\frac{1.901}{\sqrt{d}}$ | 0.06732 + 0.05483 $\sqrt{d}$ |

|TABLE I. Representative solutions in the first family ($b_t = 0$). The asymptotic parameter $\mu_1$ is used as the length unit in the geometrized unit system. In the columns of “Event horizon behavior”, the leading terms of the expansions in Eq. (18) are given.|

D. The name “compact hill” and its geodesics

The CH solutions share the same form of the asymptotic expansions with the BH solutions, so they mimic the BH solutions at large radius. But near the event horizon, the non-vanishing $g_{tt}$ causes a distinctive feature for them. For CHs, geodesics bounce back on the event horizon, which is the reason for us to name them “compact hill”. The very terminology “event horizon” actually seems improper as this feature of the CH solutions is radically different from the event horizon of BHs. However, in the sense that the null hypersurface $r = r_h$ of the CHs prevents geodesics outside of it from going in while the event horizon of BHs prevents geodesics inside from going out, we still refer $r = r_h$ as the event horizon of
the CHs.

Now to show that geodesics bounce back on the event horizon of the CHs, one first finds that under the metric in Eq. (8) the radial components of the four-velocity and the four-acceleration can be written as

\[
\left(\frac{dr}{d\lambda}\right)^2 = e^{-2\mu} \left( e^{2\nu} - \frac{l^2}{r^2} + k \right),
\]

\[
\frac{d^2r}{d\lambda^2} = e^{-2\mu} \left[ -l^2 e^{2\nu} (\mu' + \nu') + \frac{l^2}{r^2} (\mu' + \frac{1}{r}) - k\mu' \right],
\]

where the prime denotes the derivative with respect to \( r \), and the integral constants \( \epsilon \) and \( l \) are

\[
\epsilon = e^{2\nu} \frac{dt}{d\lambda},
\]

\[
l = r^2 \frac{d\phi}{d\lambda},
\]

while the integral constant \( k \) takes \(-1, 0, \) and \( 1, \) for timelike, lightlike, and spacelike geodesics as \( \lambda \) is the suitable affine parameter in each case. With \( g_{tt} = -e^{2\nu} \) being nonzero on the event horizon, we immediately see \( dr/d\lambda \to 0 \) on the event horizon. Moreover, using the expansions in Eq. (18) we find near the event horizon

\[
\left(\frac{dr}{d\lambda}\right)^2 \to \frac{r - r_h}{M_{10}} \left( \frac{e^2}{N_{10}} - \frac{l^2}{r_h^2} + k \right),
\]

\[
\frac{d^2r}{d\lambda^2} \to \frac{1}{2M_{10}} \left( \frac{e^2}{N_{10}} - \frac{l^2}{r_h^2} + k \right).
\]

Because \( M_{10} \) is positive, the combination in the parentheses must be nonnegative for \( (dr/d\lambda)^2 \geq 0 \). Therefore near the event horizon the radial acceleration \( d^2r/d\lambda^2 \) is nonnegative. Together with the fact that the radial velocity \( dr/d\lambda \) approaches zero on the event horizon, we conclude that geodesics bounce back there.

The gravitational force of a CH exerting on a test mass then can be thought as repulsive near the event horizon while attractive far away. As far as we know there is completely no evidence for this kind of gravitational interaction in experiments and astrophysical observations. Anyway, the effect is academically interesting. The simplest scenario is that a test mass is released at rest and then oscillates between the initial position and the event horizon. More generally, bound orbits exist near the event horizon. The repulsive nature near the event horizon ensures that even the lightlike circular orbit at the radius determined by \( \nu' = 1/r \) is stable. Figure 7 shows example orbits bouncing back on the event horizon of a CH.

Classical trajectories bounce back on the event horizon, suggesting that GWs echo, which is a genuine effect that can be searched for in GW observations \([31, 32]\). It turns out that the potentials for waves are finite on the event horizon so waves are not perfectly reflected. What marvelous is that the potentials become complex inside the event horizon, and it is well known in quantum mechanics that complex potentials cause absorption of waves (e.g., see Refs. \([33, 34]\)).

Before we approximately demonstrate the reflection on the event horizon of a CH using a scalar wave, let us admit that not only the potentials for waves but also the Ricci scalar becomes complex inside the event horizon of CHs. The origin of their imaginary parts can be seen from Eq. (18), where \( \sqrt{\delta} \) becomes imaginary when \( r < r_h \). Though it first appears to us that the CH solutions are unphysical due to the complex Ricci scalar, a second thought about the absorption property related to the imaginary parts of the potentials strikes us the idea that a spacetime region with a complex Ricci scalar might be where information is lost. The idea needs to be further explored, but that is beyond the scope of the present work. In the remaining of this section, we only try to use a brute but manageable approximation to illustrate the reflection of a scalar wave on the event horizon of a CH.

Starting with the Klein-Gordon equation

\[
0 = \Box \Phi = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( g^{\mu\nu} \sqrt{-g} \partial_{\nu} \Phi \right),
\]

and using the spherical metric in Eq. (8) as well as the spherical wave ansatz for the scalar field \( \Phi = e^{i\omega t} \Psi(r) \), one obtains an equation for \( \Psi(r) \)

\[
\Psi'' + \left( \nu' - \mu' + \frac{2}{r} \right) \Psi' + \sigma^2 e^{2(\mu-\nu)} \Psi = 0.
\]

The changes of variables

\[
z = r^{\nu'},
\]

\[
\frac{dz}{dr} = e^{i\omega - \nu'}
\]

put Eq. (23) into the standard form \([35]\)

\[
\frac{d^2z}{dz^2} + (\sigma^2 - V_z) z = 0,
\]

with the potential for the scalar wave being

\[
V_z = e^{2(\nu-\mu)} \frac{\nu' - \mu'}{r}.
\]
Fully solving the scattering problem of Eq. (25) is beyond the scope of the present work. We shall focus on the behavior of the wave near the event horizon, where the potential \( V \) and the derivative \( d r_e / d r \) have the expansions

\[
V_s = \frac{1}{2r_e} \frac{N_{10}}{M_{10}} \left[ 1 + \frac{3}{2} \left( \frac{N_{20}}{N_{10}} - \frac{M_{20}}{M_{10}} \right) \sqrt{\delta} + \ldots \right],
\]
\[
dr_e = \sqrt{\frac{M_{10}}{N_{10} \sqrt{\delta}}} \left[ 1 - \frac{1}{2} \left( \frac{N_{20}}{N_{10}} - \frac{M_{20}}{M_{10}} \right) \sqrt{\delta} + \ldots \right].
\] (27)

Besides being complex inside the event horizon, \( V_s \) is zero on the event horizon while Eq. (27) is not. Then, the tortoise radius \( r_e \) goes to negative infinity on the event horizon of BHs, but now that \( dr_e / d r \) is only divergent as \( 1 / \sqrt{\delta} \) for the CHs we find \( r_e \) proportional to \( \sqrt{\delta} \) plus an integral constant.

Taking the distinctive features of Eq. (27) into consideration, we reckon that the step potential \( V_{\text{step}} \) might be used to approximate the effective potential \( V_s \) near the event horizon. In Eq. (28), the real constant \( V_{s1} \) and the complex constant \( V_{s2} + i \Gamma_2 \) can be thought as certain average values of \( V_s \) right outside and inside the event horizon. The tortoise radius, which also becomes complex inside the event horizon, has been set to zero at the event horizon. With \( V_s \) abruptly approximated by \( V_{\text{step}} \), the scattering problem of Eq. (25) is analytically solvable. Assuming no waves coming out of the center at \( r = 0 \), the plane wave ansatz

\[
z = \begin{cases} 
C_+ e^{ik_1r_e} + C_- e^{-ik_1r_e} \quad & \text{Re}(r_e) \geq 0 \\
D_+ e^{ik_2r_e} \quad & \text{Re}(r_e) < 0
\end{cases}
\] (29)

satisfies the equation when

\[
k_1 = \sqrt{\sigma^2 - V_{s1}},
\]
\[
k_2 = \sqrt{\sigma^2 - V_{s2} - \Gamma_2^2}.
\] (30)

The continuity of \( z \) and \( dz / dr_e \) at \( r_e = 0 \) fixes the ratio between the amplitudes of the reflected and the incident waves, which is

\[
\frac{C_-}{C_+} = \frac{k_1 - k_2}{k_1 + k_2}.
\] (31)

The reflection rate is then

\[
R = \left| \frac{C_-}{C_+} \right|^2 = \frac{k_1^2 + |k_2|^2 - 2k_1 \text{Re}(k_2)}{k_1^2 + |k_2|^2 + 2k_1 \text{Re}(k_2)},
\] (32)

where we have assumed the energy of the incident wave \( \sigma^2 \) to be greater than \( V_{s1} \) so that \( k_1 \) is real. If we define \( \theta_2 \in [-\pi, \pi) \) by

\[
\cos \theta_2 = \frac{V_{s2}}{\sqrt{(\sigma^2 - V_{s2})^2 + \Gamma_2^2}},
\]
the same incident velocities in the Schwarzschild spacetime that fall
comparison. In the upper panel, the dotted trajectories are the orbits with
Schwarzschild radius, \( r_\text{S} \), is used as the length unit. The black circles with a radius
1073 in both panels indicate the event horizon of the CH. The
\( \Gamma \) terms the reflection rate when the fre-
\[ \sigma^2 > \frac{3V_{s1} - V_{s2}}{2}. \] (36)
Because we have required \( \sigma^2 > V_{s1} \), Eq. (36) is always satis-
fied if \( V_{s1} \geq V_{s2} \). If \( V_{s1} < V_{s2} \), then the imaginary poten-
tial contributes negatively to the reflection rate when the fre-
quency of the incident wave is lower than the value in Eq. (36).

III. OBSERVATIONAL TESTS

The exotic solutions of CHs are not supported by current ob-
servations in the strong-field regime such as the GW detec-
tions and the imagies of the supermassive black holes. We con-
sider testing the BH solutions in the bumblebee model against observations in this section.

A. Classical tests

Classical tests of the weak-field metric in the bumblebee model using observations in the Solar System was first done by Hellings and Nordtvedt [19] within the PPN framework. It is equivalent to testing the metric asymptotic expansion in Eq. (13) with \( \mu_0 = 0 \). In Appendix E, we transform the metric expansion in Eq. (13) to the harmonic coordinates so that the PPN parameters \( \beta \) and \( \gamma \) [36] can be directly read off. Existing constraints on the PPN parameters can thus be translated to constrain the free coefficients in the asymptotic expansions in Eq. (13). This is an easy way to obtain constraints on the free parameters in our solutions. But to clearly see how the parameters in our solutions enter the classical effects and thus obtaining tailored constraints for them, we reckon that it is worth leaving the convenient PPN framework but calculating predictions directly using Eq. (13) to compare with observations.

Considering three classical tests of GR, the advance of perihelion, the deflection of light, and the Shapiro time delay, we find that the effect of the advance of perihelion is sensitive to the coefficients \( \mu_0, \mu_1, v_1 \) and \( v_2 \) while the effects of the other two are only sensitive to \( \mu_0, \mu_1 \) and \( v_1 \).\(^1\) The calculations are

\(^1\) The other classical test of GR, the gravitational redshift, only tests the equivalence principle [37], and it is left out of the current study because the bumblebee gravity model has already satisfied the Einstein equivalence principle.
done in the weak-field regime where the ratio of the central mass to the typical length involved is small so that the results are kept to the linear order of this ratio. The results for the deflection of light and the Shapiro time delay are the same as those derived in Ref. [26], where tight constraints have been set on the corresponding parameter \( l = e^{\nu_1} - 1 \). We focus on the new modification introduced by the coefficient \( \nu_2 \) in the effect of the advance of perihelion.

For a bound timelike geodesic with perihelion separation \( r_{\text{min}} \) and aphelion separation \( r_{\text{max}} \), the advance of perihelion per orbit is given by

\[
\delta \phi = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{d\phi}{dr} dr - 2\pi.
\]

Using Eqs. (19) and (20) to calculate \( dr/d\phi \) and substituting the expansions for \( \mu \) and \( \nu \) in Eq. (13) into it, up to the leading order of \( \mu_1/r_{\text{min}} \), the advance of perihelion turns out to be

\[
\delta \phi \approx 2\pi \left( e^{\nu_0} - 1 \right) + \pi e^{\nu_0} \left( \mu_1 - \nu_1 \right) \left( \frac{r_{\text{min}} + r_{\text{max}}}{r_{\text{min}} r_{\text{max}}} \right).
\]

Because there are two families of solutions with different recurrence relations for the coefficients in the asymptotic expansions, we discuss them separately.

For the family of solutions with \( b_r = 0 \), the first few recurrence relations are given in Eq. (B1) with three free parameters, \( \mu_1, \lambda_0 \), and \( \lambda_1 \). Using the recurrence relations, Eq. (38) becomes

\[
\delta \phi \approx \pi \left( \frac{-2 \xi \lambda_1^2 + 2 \xi \lambda_0 \lambda_1 + (\xi - \nu) \lambda_1^2}{4 \xi (1 - \nu) \lambda_0^2 - 4} \right) \times \mu_1 \left( \frac{r_{\text{min}} + r_{\text{max}}}{r_{\text{min}} r_{\text{max}}} \right),
\]

where \( \lambda_1 = \lambda_1/\mu_1 \). The recurrence relations in Eq. (B1) are for both BH solutions and CH solutions. If we restrict our attention to the BH solutions in this family, we should keep in mind that there is only one free parameter among \( \lambda_0 \) and \( \lambda_1 \). It is clear that Eq. (39) reproduces the well-known result

\[
\delta \phi_{\text{GR}} \approx \frac{3\pi \mu_1 (r_{\text{min}} + r_{\text{max}})}{r_{\text{min}} r_{\text{max}}}
\]

for the Schwarzschild metric when \( \lambda_0 = \lambda_1 = 0 \). In addition, the bumblebee model recovers the Einstein-Maxwell theory when \( \xi = 0 \), so Eq. (39) gives the result for the Reissner-Nordström metric when \( \xi = 0 \) with \( \lambda_1 \) set to \( Q/\sqrt{\kappa} \).

For the family of solutions with \( R_r = 0 \), the first few recurrence relations are given in Eq. (B4) with 5 free parameters, \( \mu_0, \mu_1, \mu_2, A_0 \), and \( \lambda_1 \). Using the recurrence relations, Eq. (38) becomes

\[
\delta \phi \approx 2\pi \left( e^{\nu_0} - 1 \right) + e^{\nu_0} (7 + 2\tilde{\mu}_2) \frac{\pi \mu_1 (r_{\text{min}} + r_{\text{max}})}{3r_{\text{min}} r_{\text{max}}},
\]

where \( \tilde{\mu}_2 = \mu_2/\mu_1^2 \). We see that the bumblebee parameters \( \lambda_0 \) and \( \lambda_1 \) do not affect the advance of perihelion at the leading order of \( \mu_1/r_{\text{min}} \) in this family. Setting \( \mu_2 = 1 \), Eq. (41) reproduces the result in Ref. [26] which in turn goes back to the Schwarzschild result if taking \( \mu_0 = 0 \).

Pitjeva and Pitjev [38] fitted the advances of perihelia of 6 planets using standard gravity described by GR. We collect their fitting residuals as well as the orbital parameters in Table III. We use the larger between a fitting residual and its uncertainty as an upper bound for \( |\delta \phi - \delta \phi_{\text{GR}}| \). Then constraints on \( \lambda_0 \) and \( \lambda_1 \) for the first family of solutions can be set using Eq. (39), and constraints on \( \mu_0 \) and \( \mu_2 \) can be set for the second family of solutions using Eq. (41). The results are shown in Figs. 8 and 9. As expected, parameters in both solution families are severely constrained so that the metric outside the Sun is reasonably the Schwarzschild solution to a good approximation. Notice that the constraints on \( \mu_0 \) in Fig. 9 are consistent with the results in Ref. [26]. In the future, a global fit of data directly with expressions given by the bumblebee gravity model is desirable to further eliminate degeneracy among orbital parameters and theory parameters.

Though both pointing to the Schwarzschild solution, we feel that the constraints for the two families of solutions have different interpretations. For the first family of solutions, the constraints are on the bumblebee parameters \( \lambda_0 \) and \( \lambda_1 \), meaning that the bumblebee charge of the Sun must be very small if not vanishing. Naively, different objects can carry different amount of bumblebee charge, so these constraints are not necessarily valid for other types of astrophysical objects, like BHs or neutron stars. For the second family of solutions, the constraints are directly on the metric parameters \( \mu_0 \) and \( \mu_2 \) regardless of the bumblebee parameters \( \lambda_0 \) and \( \lambda_1 \). So they are supposed to be general and valid for all types of astrophysical objects.

B. BH images

Advances of perihelia of the planets in the Solar System set stringent constraints on the parameters of the static spherical solutions in the bumblebee gravity model. But the results are doubly universal for the first family of solutions where the constraints are on the bumblebee parameters \( \lambda_0 \) and \( \lambda_1 \). While the Sun has tiny or zero bumblebee charge, more compact objects like BHs may still possess a considerable amount of it. So we consider to put bounds on the bumblebee charge carried by the supermassive BHs whose shadow images have been recently taken by the EHT Collaboration [23, 24].

The mathematical shape of the shadow of a BH depends on the mass and the spin of it. When extracting from observations, the precise shape is also influenced by the observing resolution as well as the structure of the surrounding matter that emits EM waves. The mass of the BH determines the size of the shadow, while other elements contribute corrections typically of order unity [23]. For our purpose of setting preliminary bounds on the bumblebee charge, it is sufficient to consider the simplest model where a circular shadow is created by a static spherical BH. In this case, the radius of the shadow is the impact parameter of a lightlike geodesic infinitely approaching the lightlike circular orbit, i.e., the light ring. In the static spherical spacetime described by Eq. (8),
TABLE III. Orbital parameters of 6 Solar-System planets and the fitting residuals of their perihelion advances.*

| Planet   | Perihelion (10^6 km) | Aphelion (10^6 km) | Period (days) | Perihelion advance residual (mas yr⁻¹) |
|----------|---------------------|--------------------|--------------|---------------------------------------|
| Mercury  | 46.00               | 69.82              | 87.97        | −0.020 ± 0.030                        |
| Venus    | 107.5               | 108.9              | 224.7        | 0.026 ± 0.016                         |
| Earth    | 147.1               | 152.1              | 365.2        | 0.0019 ± 0.0019                       |
| Mars     | 206.7               | 249.3              | 687.0        | −0.00020 ± 0.00037                    |
| Jupiter  | 740.6               | 816.4              | 4331         | 0.59 ± 0.28                           |
| Saturn   | 1358                | 1507               | 10747        | −0.0032 ± 0.0047                      |

* Orbital parameters are taken from [https://nssdc.gsfc.nasa.gov/planetary/factsheet/](https://nssdc.gsfc.nasa.gov/planetary/factsheet/).

FIG. 8. Constraints on the bumblebee parameters \(\lambda_0\) and \(\lambda_1\) for the first family of solutions \((b_r = 0)\) from the fitting residuals of perihelion advances in Table III. The metric parameter \(\mu_1\) is used as the unit. In these plots, BHs exist along the black lines passing through the origin in the \(\lambda_0\)-\(\lambda_1\) plane (the brown lines in Fig. 3). Note that Jupiter and Saturn give constraints too loose to shown in the plots and they are omitted.

FIG. 9. Constraints on the metric parameters \(\mu_0\) and \(\mu_2\) for the second family of solutions \((R_r = 0)\) from the fitting residuals of perihelion advances in Table III. The metric parameter \(\mu_1\) is used as the unit. Note that Jupiter and Saturn give constraints too loose to shown here and they are omitted.

Denoting the distance from the observer to the BH as \(D\), the angular diameter of the shadow is

\[
d = \frac{2\sigma_r}{D} = 2\theta_g \frac{\sigma_r}{M},
\]

where \(M = \mu_1\) is the mass of the BH and \(\theta_g = M/D\). Adopting the observational results

\[
d = 42 \pm 3 \mu\text{as},
\]

\[
\theta_g = 3.62 \pm 0.60 \mu\text{as}
\]

according to Refs. [23, 39] for the supermassive BH in M87, we get

\[
\frac{\sigma_r}{M} = 5.80 \pm 1.05.
\]

Adopting the observational results\(^2\)

\[
d = 51.8 \pm 2.3 \mu\text{as},
\]

\[
\theta_g = 5.02 \pm 0.20 \mu\text{as}
\]

according to Refs. [24, 40, 41] for the supermassive BH in the Milky Way, we get

\[
\frac{\sigma_r}{M} = 5.16 \pm 0.31.
\]

\(^2\) The average of the results from the Very Large Telescope Interferometer (VLTI) and Keck is used as the value for \(\theta_g\), and their difference is used as the uncertainty.
in the first family, the number of free parameters reduces from three to two, namely the mass and the charge, while in the second family there are still 5 free parameters. The BH solutions in both families have their merits. In the first family, the solutions recover the Reissner-Nordström metric when \( \xi = 0 \). In this sense, they are the expected BH solutions when the coupling between the Ricci tensor and the vector field is added to the Einstein-Maxwell theory. In the second family, there is an analytical solution given by Eqs. (9) and (10), which deserves a mention because by setting \( \mu_0 = 0 \) the Schwarzschild metric is recovered together with a nontrivial bumblebee background field. It is amusing to notice the special case of \( \xi = 2 \), for both families give the Schwarzschild metric together with a bumblebee background field that has only a temporal component proportional to \( 1 - 2M/r \).

In Sec. III, the BH solutions are tested using Solar-System observations and the images of the supermassive BHs. For the solutions in the first family, constraints on the bumblebee charge carried by the central objects are obtained. The upper bound for the ratio of the bumblebee charge to the mass is at the order of \( 10^{-3} \) for the Sun while it roughly varies from 0.7 to 3.5 depending on the value of \( \xi \) for the supermassive BHs. We need to point out that the bumblebee charge is completely unconstrained when \( \xi = 2 \), as the metric is exactly Schwarzschild in this case. For the solutions in the second family, stringent constraints on the metric parameters are obtained, suggesting that if static spherical objects are described by the solutions in this family, the spacetime is most likely to be Schwarzschild but can be accompanied by a nontrivial bumblebee background field.

In conclusion, as a generalization of the Einstein-Maxwell theory and an important example for Lorentz-violating gravitational theories, the bumblebee gravity model in Eq. (2) indeed contains richer solutions. Testing these solutions in experiments and observations is undoubtedly a worthy subject for fundamental physics. On the one hand, the bumblebee background field showing up from nowhere in the vacuum solutions defines special local frames so that local Lorentz symmetry is broken. Therefore testing whether there is any such background field is testing the fundamental symmetry of our Universe. On the other hand, if the bumblebee field is regarded as the EM vector potential, testing the solutions in the bumblebee model will tell us whether the EM field only minimally couples with gravity as in the Einstein-Maxwell theory or the nonminimal coupling between the Ricci tensor and the vector field exists. The tests done in this work have not excluded the possibility of a bumblebee background field though the metric is constrained to be close to the Schwarzschild metric in the spherically symmetric situation. The next step is

\[ M = \frac{1}{2} \frac{\lambda}{V} \]

\[ r = \frac{2GM}{r} \]

\[ \xi = \kappa \]

\[ \xi = 2\kappa \text{ (Schwarzschild metric)} \]

\[ \xi = 3\kappa \]

\[ \xi = 0 \text{ (RN metric)} \]
naturally to put the solutions into test against the GW observations. Future works on extending the spherical BH solutions to rotating BHs, calculating the quasinormal modes of the BHs, as well as solving the two-body problem in the PPN framework, are directions that need to be explored.

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Appendix A: Field equations under the static spherical ansatz

Denoting $E_{\mu\nu} = G_{\mu\nu} - \kappa (T_\mu)_{\nu}$, with the static spherical ansatz the nonvanishing Einstein field equations are

\begin{align}
0 &= E_{tt} = \frac{e^{2r-2\mu}}{r^2} \left[ (1 - 2r\mu' - e^{2\mu}) \left( b''_t - b'_t \left( \mu' + 4\nu' - \frac{2}{r} \right) + b_t \left( 2\mu'\nu' - \frac{4}{r} \nu' - 2\nu'' + \nu^2 \right) \right) \right] + \left( \frac{\kappa}{2} - \xi \right) e^{2\mu} b_t'^2 + \xi e^{2\mu - 4\mu} b''_t - b'_t \left( 5\mu' - \frac{4}{r} \mu' + b_t \left( -\mu'' + 3\mu'^2 - 6 \frac{6}{r} \mu' + \frac{1}{r^2} \right) \right) b_t + \xi e^{2\mu - 4\mu} b_t'^2, \\
0 &= E_{rr} = \frac{E_{\phi\phi}}{\sin^2 \theta} = r^2 e^{-2\mu} \left( \mu' + \frac{1}{r} \mu' - \frac{1}{r} \nu' - \nu'' - \nu^2 \right) + r^2 e^{-2\mu} \left( b'_t \left( \nu' + \frac{1}{r} \nu' - \nu'' + \nu^2 \right) + b_t \left( 3\mu' \left( \mu' - \nu' - \frac{1}{r} \right) + \nu'' - \mu'' + \nu^2 + \frac{\nu'}{r} \right) \right) b_t, \\
0 &= E_{\phi \phi} = \xi e^{-2\mu} \left( \mu' + \frac{1}{r} \mu' - \nu'' - \nu^2 \right) b_t, \tag{A1}
\end{align}

and the vector field equation gives

\begin{align}
0 &= b''_t - b'_t \left( \mu' + \nu' - \frac{2}{r} \right) + \xi b_t \left( \mu' - \frac{2}{r} \nu' - \nu'' - \nu^2 \right), \\
0 &= \xi \left( \mu' + \frac{1}{r} \mu' - \nu'' - \nu^2 \right) b_t, \tag{A2}
\end{align}

where the prime indicates the derivative with respect to $r$.

Noticing $R_{rr} = \left( \mu' + 2\mu'/r - \nu'' - \nu^2 \right)$, the second equation in Eq. (A2) leads to two possibilities:

1. $b_t = 0$, in which case $\mu$ and $\mu'$ can be eliminated from the equations, so we arrive at a system of two second-order ODEs for $\nu$ and $b_t$;
2. $R_{rr} = 0$, in which case we find that $b_t$ can be solved in terms of other variables,

\begin{align}
b_t^2 = \frac{e^{2\mu - 2\nu} \left( (2\kappa - \xi) r \nu' + \kappa - 2\xi \right) b_t'^2 + 6\xi r b_t b'_t - 2\xi \left( \xi - 1 \right) (r \mu' \nu' + 2\mu + 2\nu') + \left( \frac{\xi}{\kappa} + 1 \right) r \nu^2 \right)^2 b_t^2}{r \mu' \nu' + 2\mu' - \nu^2 + \nu',}
\end{align}

so the system consists of three second-order ODEs for $\mu$, $\nu$ and $b_t$.

Appendix B: Asymptotic expansions of the variables

For the case of $b_t = 0$, substituting the expansions in Eq. (13) into Eqs. (A1) and (A2), we find the following re-
For the case of $R_{rr} = 0$, substituting the expansions in Eq. (13) together with Eq. (A3) into Eqs. (A1) and (A2), we find the following recurrence relations

\[ v_1 = -\mu_1, \]
\[ v_2 = \frac{1}{3} (\mu_1^2 + 2\mu_2), \]
\[ \lambda_2 = \frac{2e^{\mu_0}}{3\kappa} (\mu_1^2 - \mu_2), \]
\[ \ldots, \]  

(B4)

where 5 coefficients, $\mu_0$, $\mu_1$, $\mu_2$, $\lambda_0$ and $\lambda_1$, are unfixed. In this case, the system of equations can be reduced to three second-order ODEs, so there are 6 free coefficients in the solutions taking into account the trivial coefficient $v_0$. The asymptotic expansion of $b_r$ can also be obtained. It is

\[ b_r^2 = \frac{e^{2\mu_0} (e^{\mu_0} - 1)}{\xi} + \frac{e^{2\mu_0}}{9\xi \mu_1} \left( \frac{2\xi}{\kappa} \right) \left( (\kappa + 4\xi) \mu_1^2 + (8\kappa - 4\xi) \mu_2 \right) \lambda_0^2 + 18\xi \mu_1 \lambda_0 a_1 + 3(2\xi - \kappa) \lambda_1^2 \]
\[ + 2 \left( 15e^{2\mu_0} - 9 \right) \mu_1^2 - 12e^{2\mu_0} \mu_2 \frac{1}{r} + \ldots. \]  

(B5)

For $b_r^2 \geq 0$ at $r \to \infty$, it implies a constraint

\[ e^{\mu_0} - 1 \leq 0. \]  

(B6)

If $\mu_0 = 0$, then there is a constraint

\[ 2 \left[ \left( 1 + \frac{4\xi}{\kappa} \right) \mu_1^2 + \left( 8 - \frac{4\xi}{\kappa} \right) \mu_2 \right] \lambda_0^2 + 18\xi \mu_1 a_0 a_1 + 3(2\xi - \kappa) \lambda_1^2 \geq 0. \]  

(B7)

We point out that this family of solutions includes the analytical solution given by Eqs. (9) and (10). It is recovered by taking $\mu_2 = \mu_1^2$.

Appendix C: Behavior of the variables near the event horizon

For the case of $b_r = 0$, substituting the expansions in Eq. (18) into Eqs. (A1) and (A2), we find the following three equations relating the 7 parameters \( \{r_h, N_{10}, N_{20}, M_{10}, M_{20}, L_{10}, L_{20} \} \) if $N_{10} \neq 0$,

\[ 0 = -16 M_{10} N_{10}^3 + \xi N_{20} L_{10} L_{20} r_h^2 - 2\xi N_{10} N_{20} L_{10} L_{20} r_h^2 + 2\kappa N_{10} L_{20}^2 r_h^2, \]
\[ 0 = 16\kappa M_{10} N_{10}^4 - 16\kappa N_{10}^4 r_h - 8\xi (\xi - 2\kappa) N_{10}^3 L_{10} L_{20} r_h^2 + 3\kappa N_{10} N_{20} L_{10} L_{20} r_h^2 + \xi^2 (\xi - 2\kappa) N_{10}^2 L_{10}^2 r_h^2 - 6\kappa^2 N_{20}^2 N_{10} L_{10} L_{20} L_{20}^2 r_h^2, \]
\[ 0 = 64 M_{10} M_{20} N_{10}^5 L_{10} + 176 M_{10}^2 N_{10}^4 N_{20} L_{10} + 16\xi (\xi - 2\kappa) M_{10} N_{10}^3 N_{20} L_{10}^3 - 128 M_{10} N_{10}^2 N_{10} L_{10} L_{20} M_{20}-16 M_{20} N_{20}^3 L_{10} M_{10} M_{20} + 128 M_{10} L_{10} N_{10} L_{20}^2 r_h, \]
\[ -176 M_{10} N_{10}^4 L_{10} L_{20} r_h - 8\xi (\xi - 2\kappa) M_{10} N_{10}^3 L_{10} L_{20} r_h - 56\xi (\xi - 2\kappa) M_{10} N_{10}^2 N_{20} L_{10} L_{20} r_h + 128 M_{10} N_{10}^3 L_{20}^2 r_h. \]
From Eq. (C1), three of the parameters can be solved in terms of the other 4 parameters. Other higher-order coefficients are also fixed by their recurrence relations once the 4 free parameters are specified. If \( N_{10} = 0 \), then substituting the expansions in Eq. (18) into Eqs. (A1) and (A2) simply leads to \( N_{2n} = M_{2n} = L_{2n} = 0 \) and \( L_{10} = 0 \). The recurrence relations are much simplified with only three free parameters. Taking the three free parameters as \( N_{11}, L_{11} \) and \( r_h \), the recurrence relations for the first few coefficients are

\[
M_{10} = r_h + \left( \frac{\kappa^2}{4} - \frac{\xi}{16} \right) \frac{L_{10}^2}{N_{11}},
\]

\[
M_{11} = \left( 1 - \frac{\kappa^2}{4} - \frac{\xi}{16} + \frac{13 \xi^2}{16} - \frac{3 \xi^3}{16} \right) \frac{L_{11}^2}{N_{11}} + \left( \frac{3 \kappa^2 \xi}{8} - \frac{9 \kappa \xi^2}{16} + \frac{9 \xi^3}{32} - \frac{3 \xi^4}{64} \right) \frac{L_{10}^2}{N_{11}},
\]

\[
M_{12} = \left( \frac{\kappa^2}{16} - \frac{\xi}{16} + \frac{13 \xi^2}{16} - \frac{3 \xi^3}{16} \right) \frac{L_{12}^2}{N_{12}} + \left( \frac{\kappa^2}{16} - \frac{\xi}{16} + \frac{13 \xi^2}{16} - \frac{3 \xi^3}{16} \right) \frac{L_{10}^2}{N_{12}} - \left( \frac{9 \kappa^2 \xi}{16} - \frac{9 \kappa \xi^2}{16} + \frac{9 \xi^3}{32} - \frac{3 \xi^4}{64} \right) \frac{L_{11}^2}{N_{12}},
\]

\[
\frac{N_{12}}{r_h} = \frac{N_{11}}{N_{12}} + \left( \frac{\kappa^2}{4} - \frac{\xi}{4} + \frac{\xi^3}{16} \right) \frac{L_{11}^2}{N_{12}} + \left( \frac{\kappa^2}{4} - \frac{\xi}{4} + \frac{\xi^3}{16} \right) \frac{L_{10}^2}{N_{12}},
\]

\[
L_{12} = \frac{L_{12}}{r_h} + \left( \frac{\kappa^2}{4} - \frac{\xi}{4} + \frac{\xi^3}{16} \right) \frac{L_{10}^2}{N_{12}},
\]

Note that for \( \xi \neq 2\kappa \), the Schwarzschild metric is recovered only when \( L_{11} = 0 \) and therefore \( b_1 = 0 \). For \( \xi = 2\kappa \), the Schwarzschild metric can exist together with a nonzero \( b_1 \).

For the case of \( R_{rr} = 0 \), using Eq. (A3) to eliminate \( b_r \) and then substituting the expansions in Eq. (18) into Eqs. (A1) and (A2), we find the following equation relating the 7 parameters \( \{ r_h, N_{10}, N_{20}, M_{10}, M_{20}, L_{10}, L_{20} \} \) if \( N_{10} \neq 0 \),

\[
0 = 384 \kappa M_{10}^2 N_{10}^3 - 36 \kappa M_{10} N_{20}^2 L_{20}^2 r_h + 12 \kappa M_{10} N_{20}^2 L_{20}^2 r_h - 8 \kappa (2\xi - \kappa) M_{10} N_{20}^2 L_{20}^2 r_h^2 - 48 \kappa^2 M_{10} N_{20}^2 L_{20}^2 r_h^2
\]

\[
\quad + 16 \kappa (2\xi - \kappa) M_{10} N_{20}^2 L_{20}^2 r_h^3 - 3 \kappa (\xi - 2\kappa) M_{20} N_{20}^2 L_{20}^2 r_h^3 + \kappa (\xi - 2\kappa) M_{10} N_{20}^2 L_{20}^2 r_h^3.
\]

From Eq. (C3), one of the parameters can be solved in terms of the other 6 parameters. Other higher-order coefficients are then fixed by their recurrence relations once the 6 free parameters are specified. Similar to the first case, if \( N_{10} = 0 \) then recurrence relations give \( N_{2n} = M_{2n} = L_{2n} = 0 \), but in this case \( L_{10} \) is no longer fixed to zero so there are still 6 free parameters. Taking the 6 free parameters as \( N_{11}, M_{10}, M_{11}, L_{10}, L_{11} \) and \( r_h \), the recurrence relations for the first few coefficients are

\[
M_{12} = \frac{M_{10} - M_{11} r_h}{\kappa - 2\xi} \left[ \frac{10(13\kappa - 8\xi)}{27 r_h^2} - \frac{10 \kappa M_{10} N_{11}}{3 \xi L_{10} r_h^2} + \frac{13(2\xi - \kappa)}{27} \right] - \frac{M_{11}}{M_{10} r_h} - \frac{40(\xi - 2\kappa)}{L_{10} r_h} - \frac{10 \kappa (\xi - 2\kappa)}{9} \frac{L_{11}^2}{L_{10}^2}.
\]
Note that the analytical solution in Eqs. (9) and (10) corresponds to taking

\[ r_h = 2M, \]
\[ N_{10} = 0, \]
\[ N_{11} = 1 \]
\[ M_{10} = 2Me^{2\mu}, \]
\[ M_{11} = e^{2\mu}, \]

Note that the analytical solution in Eqs. (9) and (10) corresponds to taking

\[ r_h = 2M, \]
\[ N_{10} = 0, \]
\[ N_{11} = 1 \]
\[ M_{10} = 2Me^{2\mu}, \]
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\[ N_{10} = 0, \]
\[ N_{11} = 1 \]
\[ M_{10} = 2Me^{2\mu}, \]
\[ M_{11} = e^{2\mu}, \]

Note that the analytical solution in Eqs. (9) and (10) corresponds to taking

\[ r_h = 2M, \]
\[ N_{10} = 0, \]
\[ N_{11} = 1 \]
\[ M_{10} = 2Me^{2\mu}, \]
\[ M_{11} = e^{2\mu}, \]

Note that the analytical solution in Eqs. (9) and (10) corresponds to taking

\[ r_h = 2M, \]
\[ N_{10} = 0, \]
\[ N_{11} = 1 \]
\[ M_{10} = 2Me^{2\mu}, \]
\[ M_{11} = e^{2\mu}, \]

Note that the analytical solution in Eqs. (9) and (10) corresponds to taking

\[ r_h = 2M, \]
\[ N_{10} = 0, \]
\[ N_{11} = 1 \]
\[ M_{10} = 2Me^{2\mu}, \]
\[ M_{11} = e^{2\mu}, \]
when $\delta$ approaches zero. The recurrence relations for the first family of solutions do not bring any simplification for the results in Eq. (D5). The recurrence relations for the second family of solutions simplify to

$$ R_{\alpha\beta}|_{r_0} = \frac{2}{r_0^2} - \frac{2}{r_0 M_{10}}, $$

$$ R_{\alpha\beta} R_{\alpha\beta}|_{r_0} = \frac{2}{r_0^2} - \frac{4}{r_0^2 M_{10}^2} + \frac{2}{r_0 M_{10}} r_{10}, $$

$$ R_{\alpha\beta} R_{\alpha\beta} R_{\alpha\beta}|_{r_0} = \frac{4}{r_0^2} + \frac{8}{r_0^2 M_{10}^2}, $$

which recovers the results for the Schwarzschild metric by taking $M_{10} = r_0$. With $N_{10} = 0$, we have $b^2|_{r_0} = 0$ for the first family of solutions, and

$$ b^2|_{r_0} = \frac{M_{10}}{\frac{8}{3} r_{10}} - \frac{(4 \xi + 7 \kappa) L_{10}^2}{9 \kappa r_{10} N_{11}} + \frac{2(2 \xi - \kappa) M_{11} L_{10}^2}{9 \kappa M_{10} N_{11}} + \frac{(\xi - 2 \kappa) r_{10} L_{11}^2}{3 \kappa N_{11}} - \frac{1}{\xi}, $$

for the second family of solutions.

**Appendix E: Asymptotic expansion of the metric in the harmonic coordinates**

The PPN metric expansion is constructed in the harmonic coordinates. For static spherical spacetime, it is [36]

$$ g_{00} = -1 + \frac{2M}{r} - 2\beta \left( \frac{M}{r} \right)^2 + ..., $$

$$ g_{ij} = \left( 1 + 2\gamma \frac{M}{r} \right) \delta_{ij} + ..., $$

(E1)

where $r = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean length of the position vector in the harmonic coordinates, and $\beta$ and $\gamma$ are the PPN parameters that turn out to be $2 + v_2/\mu_1^2$ and 1 and to match the expansion in Eq. (13) when $\mu_0 = 0$. In the following derivation, we show the match by transforming the metric expansion in Eq. (13) to the harmonic coordinates.

The transformation includes two steps: (i) from the coordinates $(t, r, \theta, \phi)$ to $(\bar{r}, \bar{\theta}, \bar{\phi}, \bar{\phi})$, and (ii) then to the harmonic coordinates $(\bar{r}, \bar{\theta}, \bar{\phi})$. Denoting

$$ g_{\bar{r} \bar{r}} = e^{2\nu} \left( \frac{dr}{d\bar{r}} \right)^2, $$

(E2)

the metric components in the harmonic coordinates are

$$ g_{\bar{r} \bar{r}} = -e^{2\nu}, $$

$$ g_{\bar{r} \bar{\theta}} = \frac{\partial \bar{r}}{\partial \bar{r}} \frac{\partial \bar{\theta}}{\partial \bar{r}} g_{\bar{r} \bar{r}} + \frac{\partial \bar{\theta}}{\partial \bar{r}} \frac{\partial \bar{\theta}}{\partial \bar{r}} \bar{r}^2 + \frac{\partial \bar{\phi}}{\partial \bar{r}} \frac{\partial \bar{\phi}}{\partial \bar{r}} \bar{r}^2 \sin^2 \theta, $$

(E3)

where the Jacobian matrix for the transformation between $(\bar{r}, \bar{\theta}, \bar{\phi})$ and $(\bar{x}, \bar{y}, \bar{z})$ has the usual elements

$$ \begin{pmatrix}
\frac{\partial \bar{r}}{\partial \bar{x}} & \frac{\partial \bar{r}}{\partial \bar{y}} & \frac{\partial \bar{r}}{\partial \bar{z}} \\
\frac{\partial \bar{r}}{\partial \bar{x}} & \frac{\partial \bar{r}}{\partial \bar{y}} & \frac{\partial \bar{r}}{\partial \bar{z}} \\
\frac{\partial \bar{r}}{\partial \bar{x}} & \frac{\partial \bar{r}}{\partial \bar{y}} & \frac{\partial \bar{r}}{\partial \bar{z}}
\end{pmatrix} = \begin{pmatrix}
\sin \theta \cos \phi & -\sin \theta \sin \phi & -\cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{pmatrix}, $$

(E4)

Calculating the inverse metric $\bar{g}^{\alpha\beta}$ and the Christoffel symbols $\Gamma_{\alpha\beta}^{\gamma}$ in the harmonic coordinates, we find that the harmonic condition

$$ \bar{g}^{\alpha\beta} \Gamma_{\alpha\beta}^{\gamma} = 0 $$

(E5)

simplifies to an ODE for $\bar{r}$ as a function of $r$,

$$ \bar{r}'' + \left( \nu - \mu' + \frac{2}{r} \right) \bar{r}' - 2 \mu' \frac{\bar{r}^2}{r^2} = 0, $$

(E6)

where the prime denotes the derivative with respect to $r$. With $\mu$ and $\nu$ expressed as the expansions in Eq. (13), Eq. (E6) has the following asymptotic solutions

$$ \bar{r} = r \left( 1 + \frac{c_1}{r} + \frac{c_2}{r^2} + ... \right), $$

(E7)

where the constant $s$ and the coefficients $c_1, c_2, ...$, can be determined by substituting the solution back into Eq. (E6). We find the indicial equation for $s$ to be

$$ s^2 + s - 2 e^{2\mu_0} = 0, $$

(E8)

and the coefficients to be

$$ c_1 = \frac{s(\mu_1 - v_1) - 4 e^{2\mu_0} \mu_1}{2 s}, $$

$$ c_2 = e^{2\mu_0} \left[ \frac{4 e^{2\mu_0} \mu_1^2 + (1 - 4 s) \mu_1^2 + (2 s - 1) \mu_1 v_1 - 2 s t_2}{s(2 s - 1)} \right] + \left( s - 1 \right) (\mu_1 - v_1)^2 + 4 s (\mu_2 - v_2) \right] \frac{4(2 s - 1)}{s(2 s - 1)}, $$

(E9)

Note that the two roots of Eq. (E8) give two independent solutions whose linear combinations produce general solutions to Eq. (E6). But for our purpose, it is sensible to have $s = 1$ when $\mu_0 = 0$, suggesting to take the special solution corresponding to the root

$$ s = -1 + \sqrt{1 + 8 e^{2\mu_0}} $$

(E10)

To eliminate $r$ in Eq. (E3) so that it can be compared with Eq. (E1), the inverse of Eq. (E7) is required. We find

$$ r = u \left( 1 - \frac{c_1}{s} - \frac{c_2}{s} \right) - \frac{c_2}{s} + \left( s - \frac{c_1^2}{s} + \frac{c_2^2}{s} \right) \left( \frac{1}{u^2} \right) + ... $$

(E11)

where $u = \bar{r}^{1/\nu}$. Focusing on the transformation when $\mu_0 = 0$, we have $s = 1$ and Eq. (E11) is simplified to

$$ r = \bar{r} \left( 1 + \frac{\mu_1}{\bar{r}} + \frac{\mu_2 + v_2}{\bar{r}^2} + ... \right), $$

(E12)

where the relation $v_1 = -\mu_1$, which is valid for both families of solutions, has also been used. Using Eq. (E12) to eliminate $r$ in Eq. (E3) then leads to

$$ \bar{g}_{00} = -1 + \frac{2 \mu_1}{\bar{r}} - \frac{4 \mu_1^2 + 2 v_2}{\bar{r}^2} + ..., $$

$$ \bar{g}_{ij} = \left( 1 + \frac{2 \mu_1}{\bar{r}} \right) \delta_{ij} + ..., $$

(E13)
With $\mu_1 = M$, we see
\[ \beta = 2 + \frac{\nu_1}{\mu_1^2}, \quad \gamma = 1. \] (E14)

Using the recurrence relations for $\nu_2$ in Eqs. (B1) and (B4), $\beta$ can be related to the free coefficients in each family of solutions.

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