Comments on the M Theory Approach to N=1 SQCD and Brane Dynamics

A. Brandhuber\textsuperscript{a}, N. Itzhaki\textsuperscript{a}, V. Kaplunovsky\textsuperscript{b,1}, J. Sonnenschein\textsuperscript{a}\textsuperscript{1} and S. Yankielowicz\textsuperscript{a,1}

\textsuperscript{a}School of Physics and Astronomy
Beverly and Raymond-Sackler Faculty of Exact Sciences
Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

and

\textsuperscript{b}Theory Group, Dept. of Physics, University of Texas
Austin, TX 78712, USA

Abstract

We use the M theory approach of Witten to investigate N=1 SU($N_c$) SQCD with $N_f$ flavors. We reproduce the field theoretical results and identify in M theory the gluino condensate and the eigenvalues of the meson matrix. This approach allows us to identify the constant piece of the effective field theory coupling from which the coefficient of the one-loop $\beta$-function can be identified. By studying the area of the M-theory five-brane we investigate the stability of type IIA brane configurations. We prove that in a supersymmetric setup there is no force between static D4-branes that end on NS five-branes. The force in the case that there is a relative velocity between the branes is computed. We show that at the regions of intersecting IIA branes the curvature of the M theory five-brane is singular in the type IIA limit.

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1 Introduction

By now it is clear that D-branes of string theory are an important tool in constructing and unraveling the non-perturbative physics of supersymmetric gauge theories. These gauge theories live on the world volume of the corresponding D-brane. The interplay between the brane dynamics and the gauge theory dynamics has proven to give important results and insight into the non-perturbative physics of both of them. Following the work of Hanany and Witten [1] who considered a particularly useful construction giving rise to $N=4$ supersymmetric theories in 2+1 dimensions many other works generalized it to other dimensions [2, 3, 4, 5] and three dimensional theories with four supercharges [6]. In particular, the construction was generalized to four-dimensional $N=1$ SQCD in [2] where also the brane origin of Seiberg’s duality was revealed in [7]. This approach was further explored in refs. [8, 9, 10, 11, 12] and generalized to all classical gauge groups [11, 13, 14]. The same class of field theories can also be studied by wrapping D branes on cycles of Calabi-Yau manifolds [15, 16, 17, 18, 19, 20, 21].

The basic set-up of the brane configuration giving rise to supersymmetric SU($N_c$) SQCD involves two NS fivebranes with $N_c$ D4 branes stretched between them. Having one of the dimensions bounded between the NS fivebranes the world-volume of the D4 branes is effectively 3+1 dimensional. One can also include D6 branes to account for matter degrees of freedom (flavors). Alternatively, the flavors can be introduced via semi-infinite D4 branes emanating from the two NS branes.

In a recent paper Witten [22] has given an M theory interpretation for the brane configuration associated with $N=2$ SU($N_c$) supersymmetric gauge theories in four dimensions. The NS fivebranes and the D4 branes are generated in the $R_{10} \to 0$ limit, in which type IIA theory is recovered, from a single M theory fivebrane (M5). The world volume of the M5 brane is $\mathbb{R}^{1,3} \times \Sigma$, where $\Sigma$ is embedded holomorphically in a $\mathbb{R}^{3} \times S^{1}$ part of eleven dimensions. To incorporate D6 branes $\mathbb{R}^{3} \times S^{1}$ has to be replaced by a multi-Taub-NUT space. This approach provides a geometrical picture of the corresponding four-dimensional supersymmetric quantum gauge theory. In particular the beta function in the case of $N=2$ theories acquires a geometrical interpretation. This construction has been generalized to all classical gauge groups in [23, 24]. This approach has also been studied in [25].

In the present paper, by analyzing the M theory five-brane we derive known non-perturbative results of 4d $N=1$ SQCD and investigate the stability of the type IIA brane configurations that admit 4d $N=2$ and $N=1$ supersymmetric gauge theories.

Section 2 will be devoted to the M theory approach to $N=1$ SU($N_c$) SQCD
with \( N_f \) flavors. In our formulation the matter will be generically massive and represented by \( N_f \) semi-infinite D4 branes. We shall show that the M theory approach correctly reproduces the known non-perturbative field theoretical results associated with the structure of the vacuum. In particular, we will identify within the M theory approach the parameters associated with the gluino condensate and with the meson field matrix eigenvalues. We will establish relations between these parameters and the masses of the quark hypermultiplets which were previously derived within the effective potential approach by Taylor, Veneziano and Yankielowicz [28]. In section 3 we will discuss the stability of the brane configurations. We shall prove a general theorem showing that as long as we are dealing with \( \Sigma \) which is holomorphically embedded in \( \mathbb{R}^{2n} \) (or \( \mathbb{R}^{2n-d} \times T^d \)) any parameter which may appear is a real modulus i.e. corresponds to a flat direction of the superpotential. Thus the classical potential which corresponds to the minimal area cannot depend on it. This will amount to no static force. As an example we will demonstrate this general result in one case (relevant to \( N = 2 \) or \( N = 1 \) supersymmetric gauge theories) involving D4-branes and NS fivebranes. In section 4 we show that between D4-branes with relative velocity \( v \) there is a force that depends on \( v^2 \). In the field theoretical language it corresponds to evaluating the Kähler function (while the static force is related to the superpotential). Another feature of the metric on M5 that will be investigated in section 5 is the curvature. We will show that at the “point” where D4-branes end on NS five-branes there is a curvature singularity in the type IIA limit.

While finishing this paper we have received a preprint by Hori, Ooguri and Oz [26] who consider \( N=1 \) SQCD as a flow from \( N=2 \) SQCD. They also introduced D6 branes and considered the massless as well as the massive case. Furthermore, a preprint by Witten appeared [27] which applies the M theory approach to discuss chiral symmetry breaking, confinement and domain walls in \( N=1 \) SYM.

## 2 M-theory description of \( N=1 \) SQCD

Pure \( N=1 \) supersymmetric Yang Mills theory is described by a brane configuration involving NS, NS’ and \( N_c \) D4 branes stretched between them. In the original set-up of [2] the world volumes associated with each of the branes are \((x_0, x_1, x_2, x_3, x_4, x_5)\) for the NS branes, \((x_0, x_1, x_2, x_3, x_8, x_9)\) for the NS’ branes, and \((x_0, x_1, x_2, x_3, x_6)\) for the D4 branes.

The 4d space-time \( \mathbb{R}^{1,3} \) with coordinates \((x_0, x_1, x_2, x_3)\) will not be important in what follows and we will “ignore” it. The world-volume of the M theory five-brane is taken to be \( \mathbb{R}^{1,3} \times \Sigma \) [22], where \( \Sigma \) is a complex Riemann
surface. A 4d $N = 2$ supersymmetry was shown to be associated with the embedding of $\Sigma$ in $Q \simeq \mathbb{R}^3 \times S_1$ which is equipped with a complex structure in terms of

$$v = x_4 + ix_5, \quad t = \exp\left(-\left(x_6 + ix_{10}\right)/R_{10}\right),$$

(1)

where $x_{10}$ is the coordinate along the eleventh compactified dimension $S_1$ of radius $R_{10}$ and the remaining coordinates $x_7, x_8$ and $x_9$ are held constant.

The M theory five-brane setup which admits only $N = 1$ supersymmetry in $\mathbb{R}_{1,3}$ requires the embedding of $\Sigma$ in $Q \simeq \mathbb{R}^5 \times S_1$ with a global complex structure in terms of $v, t$ and $u = x_8 + ix_9$. The latter condition implies that $v(u), u(t)$ and $v(t)$ are all holomorphic functions.

We start with the classical configuration where the NS and the NS’ brane are orthogonal in the $(x_4, x_5, x_8, x_9)$ space. The coordinate $v$ is associated with the NS fivebrane and the coordinate $u$ with the NS’ fivebrane. In M-theory terms, quantum effects are manifested via bending of all the branes. Thus, the NS brane spans a finite range of $u$ while the NS’ spans a finite range of $v$. However, only the NS brane extends into the asymptotic $v \to \infty$ region. Therefore, the holomorphic functions $u(v)$ and $t(v)$ are single valued for large $v$ which in turn implies single valuedness of $u(v)$ and $t(v)$ throughout the complex $v$ plane. Similarly, only the NS’ brane exists in the asymptotic $u \to \infty$ region and hence the functions $v(u)$ and $t(u)$ should also be single valued. In particular, the holomorphic map between $v$ and $u$ is bijective; all such maps are of the form $u = (av + b)/(cv + d)$ with $a, b, c, d \in \mathbb{C}$ and $bc \neq 0$ to ensure the correct asymptotic behavior. Hence, after suitable constant shifts of the complex $u$ and $v$ coordinates, the embedding of the M theory fivebrane into $\mathbb{R}_{4,5,8,9}^4$ is described as

$$uv = S = \text{const}.$$  

(2)

In the pure SU$(N_c)$ SYM theory the NS fivebrane has $N_c$ D4 branes attached to its left side (in the $x_6$ direction) while the NS’ fivebrane has $N_c$ D4 branes attached to its right side. Hence, according to Witten [22]

$$t - P(v) = 0, \quad Q(u)t - A = 0$$

(3)

where $A$ is a normalization constant and

$$P(v) = v^{N_c} + p_1 v^{N_c-1} + \ldots + p_{N_c},$$

$$Q(v) = u^{N_c} + q_1 u^{N_c-1} + \ldots + q_{N_c},$$

(4)

are some polynomials of degree $N_c$. Note that both equations (3) are linear in $t$ in accordance with the single-valuedness of the holomorphic functions $t(v)$ and $t(u)$. 

3
Consistency of eqs. (2) and (3) requires
\[ P(v)Q(u = S/v) \equiv A, \quad \forall v \in \mathbb{C} \quad (5) \]
Note that this is not an equation for \( v \) (that would select just a point on the curve \( \Sigma \)) but a set of conditions for the parameters \( p_1, \ldots, p_{N_c}, q_1, \ldots, q_{N_c} \) and \( A \) that would make (3) an identity with respect to \( v \). The unique solution of this system is
\[ P(v) = v^{N_c}, \quad Q(u) = u^{N_c}, \quad S^{N_c} = A, \quad (6) \]
which tells us that there are \( N_c \) discrete vacua related by a \( \mathbb{Z}_{N_c} \) symmetry – a well known result in SYM [29, 30]. In all these vacua the \( N_c \) D4 branes sit on top of each other.

Next, we would like to add matter to the SYM theory. In terms of the string/M-theory setup, this can be done by attaching semi-infinite D4-branes to the NS or NS' branes and/or inserting D6-branes between NS and NS'.

We are interested in massive quarks with generic masses \( m_1, \ldots, m_{N_f} \) and use the simpler semi-infinite D4-brane setup; the massless case is more involved and is (presumably) more tractable in the D6-brane setup.

Without loss of generality, we attach all the semi-infinite D4-branes to the same NS brane. Thus, following ref. [22], we replace eq. (3) with
\[ R(v)t - P(v) = 0, \quad Q(u)t - A = 0. \quad (7) \]
where \( P(v) \) and \( Q(v) \) are polynomials of degree \( N_c \), (cf. eq. (4)) while
\[ R(v) = \prod_{i=1}^{N_f} (v - m_i). \quad (8) \]
In particular, for \( N_f = 0 \), \( R(v) = 1 \) and we recover eq. (3).

By a suitable rescaling of the variable \( t \) we can always set the leading terms of the polynomials \( P, Q \) and \( R \) to \( v^{N_c}, u^{N_c} \) and \( v^{N_f} \) respectively. This leaves us with one normalization factor \( A \), which we shall now identify with \( \Lambda_{QCD}^{3N_c-N_f} \) for the following reasons: Consider the geometric symmetry \( U(1)_{A5} \otimes U(1)_{89} \) of the M theory associated with complex rotations of the NS and NS' fivebranes. The charges of the coordinates \( v, u \) and \( t \) and various parameters of the curve \( \Sigma \) are summarized in Table 1 (The charges of \( u \) and \( v \) are obvious, the rest follows from eqs. (2), (7) and (8).)

Now consider the SQCD, which has two broken abelian symmetries \( U(1)_A \otimes U(1)_R \) (one linear combination of \( U(1)_A \) and \( U(1)_R \) is broken by the quark masses while the other is anomalous). Table 2 below summarizes the charges
of various physical quantities under these symmetries, as well as under two linear combinations $U(1)\phi$ and $U(1)\psi$ leaving invariant the squarks $\phi_L$ and the quarks $\psi_L$, respectively. Note that $\Lambda^{3N_c-N_f}_{\text{QCD}} \sim \exp(2\pi i \tau) = -\frac{8\pi^2}{g^2} + i\theta$ is charged under anomalous symmetries; indeed, its charge is precisely the anomaly of the symmetry.

A quick comparison of Tables 1 and 2 immediately suggests that the $Q\psi$ should be identified with $Q_{89}$ while $Q\phi$ should be identified with $Q_{45}$. Given this charge identification, we see that the $A$ parameter of the M theory (cf. eq. (7)) has the same charge as $\Lambda^{3N_c-N_f}_{\text{QCD}}$ parameter of the field theory.

Furthermore, $\log A$ is associated with a constant (i.e. it is $u$ and $v$-independent) term in the $x_6$ distance between the NS and the NS' fivebranes. A constant shift in $x_6$ amounts to a real rescaling in $t$ which, according to eq. (7), amounts to a rescaling of $A$ and therefore a shift in $\log A$. Similarly in SQCD there is also a $\log \Lambda^{3N_c-N_f}_{\text{SQCD}}$ term in the field dependent effective coupling $8\pi^2/g^2$. Since the $x_6$ distance between the NS and NS' fivebranes corresponds to $8\pi^2/g^2$ in the field theory, we have yet another reason to identify

$$A = \Lambda^{3N_c-N_f}_{\text{SQCD}}. \quad (9)$$

Equation (7) implies

$$t = \frac{P(v)}{R(v)} = \frac{A}{Q(u)} \quad (10)$$

and hence

$$AR(v) = P(v)Q(u = S/v). \quad (11)$$

Again, eq. (11) should hold identically for every $v$, which implies a system of
equations for $S$ and the coefficients of the polynomials $P_{N_c}(v)$ and $Q_{N_c}(u)$. (The coefficients of $R_{N_f}(v)$ are already determined via the masses – see eq. (8)). The unique solution is:

\[
P = v^{N_c},
\]

\[
Q = u^{N_c-N_f} \prod_{j=1}^{N_f} (u - S/m_j),
\]

\[
S^{N_c} = \Lambda^{3N_c-N_f} \prod_{j=1}^{N_f} (-m_j).
\]

By comparison, SQCD gives rise to an effective superpotential [28]

\[
W_{\text{eff}}(S, M) = \text{tr}(mM) + S \left( \log \left( \frac{S^{N_c-N_f} \det M}{\Lambda^{3N_c-N_f}} \right) - (N_c - N_f) \right),
\]

where $S = 16\pi^2\langle \text{tr}\lambda^\alpha\lambda^\beta \rangle$ is the gaugino condensate and $M$ is the meson matrix $\langle \phi_i\tilde{\phi}_j \rangle$ of the quark-anti-quark condensates. When all the quarks are massive, SQCD has the same Witten index as pure SYM, namely $N_c$ and hence should have $N_c$ discrete vacua. Indeed, extremizing the superpotential (14) with respect to $S$ and $M_{ij}$ we obtain

\[
S^{N_c} = \Lambda^{3N_c-N_f} \det(-m),
\]

\[
M = -Sm^{-1}.
\]

(In the decoupling limit where all masses are large the first equation reproduces the well-known relation of $N=1$ pure SYM $S^{N_c} = \Lambda_{\text{eff}}^{3N_c} \equiv \Lambda^{3N_c-N_f} \det(-m)$)

Thus, $S$ of the M theory (eq. 3) is identified with the gaugino condensate while the non-zero roots of $Q(u)$ are identified as the eigenvalues of the “meson” matrix. The physical origin of this correspondence needs further investigation.

3 Stability of brane configurations from M theory

When a multi-brane configuration of string theory is used to elucidate the vacuum structure of a field theory in four dimensions, the positions of various branes in $x^4, \ldots, x^{10}$ should be static, i.e., independent on $x^0, \ldots, x^3$; consistency of such a configuration requires static balance of forces exerted by the branes upon each other. In the M theory description, the web of
multiple D4, NS, NS’, etc., branes becomes a single convoluted five-brane of the form $\mathbb{R}^{1,3}_{0,1,2,3} \otimes \Sigma$ where $\Sigma$ is a two-dimensional surface embedded in the space spanned by the $x^4, \ldots, x^{10}$. (This space may be curved as the parallel D6 branes of the IIA theory translate into a multi-Taub-NUT metric for the coordinates $x^{4,5,6,10}$.) The potential energy of this fivebrane (or rather the energy density in four dimensions) is simply the brane tension $T_5$ times the area of the surface $\Sigma$ and the requirement of the static force balance means simply that that area — as a function of the shape of the $\Sigma$ — should be at a local minimum.

For the sake of the four-dimensional supersymmetry, we would like $\Sigma$ to be a holomorphic complex curve in a Kähler space. That is, the metric of the embedding space should be Kähler with respect to three complex coordinates such as

$$v = x^4 + ix^5, \quad u = x^8 + ix^9, \quad s = x^6 + ix^{10} \quad (16)$$

and flat with respect to the remaining real coordinate $x^7$ while the surface $\Sigma$ is spanned by

$$x^7 = \text{const}, \quad v = v(z), \quad u = u(z), \quad s = s(z) \quad (17)$$

for some holomorphic functions $v(z), u(z)$ and $s(z)$ of the $\Sigma$’ coordinate $z$. It is well known from the theory of holomorphic world-sheet instantons that all such surfaces automatically satisfy the local minimal-area variational equations. In other words, supersymmetry implies holomorphy which in turn implies that the functions are automatically harmonic. But the converse is not true in general. A solution satisfying the minimal-area equations can correspond to a non-supersymmetric configuration. Examples of brane configurations in type IIA string theory without supersymmetry can be found in [9], related non-supersymmetric brane configurations in M theory have recently been studied in [27].

Unfortunately, the surfaces we are interested in have non-compact features describing the asymptotic NS branes and semi-infinite D4 branes and hence infinite total areas. Consequently, the boundary contributions to possible finite variations of the infinite area become a non-trivial problem of regularization and boundary conditions.

We are now going to show that a properly regularized area is independent on any moduli of the surface $\Sigma$ that does not affect its asymptotic behavior in any of the non-compact directions. Such moduli — for example, those describing the positions of finite D4 branes connecting two parallel NS branes — have truly flat potential energies and the corresponding vacua of the four-dimensional field theory remains exactly degenerate in the M theory. In the IIA terms, even though the D4 branes bend the NS branes upon which they
terminate, the resulting net force between the D4 branes remains exactly zero.

To understand the proper regularization procedure for the infinite or semi-infinite asymptotic regions of \( \Sigma \), consider a classical mechanical system involving a semi-infinite string under tension. In order to correctly account for the force of the string’s tension, the infinite potential energy of the string should be regularized by replacing the infinite terminus of the string with one at \textit{fixed location} at some finite but very large distance from the other end of the string. Likewise, the potential energy of an infinite membrane with tension should be regularized by considering a large but finite membrane attached to a fixed boundary very far away.

Clearly, the same prescription applies to the \( \Sigma \) surfaces of the M theory: A tube describing a semi-infinite D4 brane extended to \( x^6 \rightarrow \pm \infty \) should be terminated at large but fixed \( |s + \bar{s}| = \lambda_s \) while an asymptotic NS brane extending to \( x^{4,5} \rightarrow \infty \) should be terminated at large but fixed \( |v| = \lambda_v \); in \( N = 1 \) configurations, the NS’ branes should be likewise terminated at a large fixed \( |u| = \lambda_u \). Note that this cutoff does not violate the \( U(1)_{45} \times U(1)_{89} \) symmetry of the NS and NS’ or the cyclical nature of the \( x^{10} \) coordinate.

As an example, consider an \( SU(N_c)^K \) theory with \( N = 2 \) SUSY of ref.[22]; for simplicity we take same \( N_c \) for all the gauge groups. The brane setup consists of \( K + 1 \) parallel NS branes connected by D4 branes; there are \( N \) D4 segments between each pair of adjacent NS branes and also \( N \) semi-infinite D4 branes attached to the first and to the last NS branes; there are no D6 branes. In the M theory, the embedding space is flat \( \mathbb{R}^6 \times S^1 \) and the surface \( \Sigma \) is described by

\[
\begin{align*}
  u &= 0, \\
  t^{K+1} P^{(K+1)}(v) + t^K P^{(K)}(v) + \cdots + t^0 P^{(0)}(v) &= 0 
\end{align*}
\]

where \( t = \exp(-s/R_{10}) \) and each of the \( P^{(K+1)}(v), \ldots, P^{(0)}(v) \) is a polynomial in \( v \) of degree \( N_c \); their coefficients are moduli of the \( \Sigma \) encoding the positions of all the NS branes and the D4 brane segments. Since the embedding space is flat, the induced metric on \( \Sigma \) is

\[
h_{zz} dz d\bar{z} = dv(z) d\bar{v}(\bar{z}) + ds(z) d\bar{s}(\bar{z})
\]

and hence the area of \( \Sigma \) is simply

\[
\int \int dz d\bar{z} h_{zz}(z, \bar{z}) = \int \int dv d\bar{v} + \int \int ds d\bar{s}
\]

and the only question is the precise region of integration.

In the absence of regularization, \( \Sigma \) covers the complex \( v \) plane \( K + 1 \) times and the complex \( t \) plane \( N_c \) times, which corresponds to a \( N_c \)-fold cover of
the cylinder spanned by the $s$ coordinate. The regularization of $\Sigma$ results in an $(K+1)$-fold cover of the radius-$\lambda_v$ circle in the $v$ plane with a few tiny holes cut out where $s + \bar{s}$ goes to $\pm \infty$ i.e., $t \to 0$ or $t \to \infty$. Such holes are located at zeros of the polynomials $P^{(0)}(v)$ and $P^{(K+1)}(v)$ and have sizes of the order $O(e^{-\lambda_s/2R_{10}})$. Although the exact sizes and locations of such holes depend on the moduli of the curve, the holes are so small that they can be simply ignored altogether while performing the $\int \int dv \, \bar{v}$ integral. As the result, for large $\lambda_s$, the first integral on the right hand side of eq. 20 becomes $\pi (K + 1) \lambda_v^2$ independent on any moduli of the surface. Similarly, for the $s$ coordinate we have an $N_c$ cover of the cylinder of radius $R_{10}$ and fixed height $\lambda_s$ with $K + 1$ small holes where $v \to \infty$. Again, the sizes and locations of the holes depend on the moduli of $\Sigma$, but in the large $\lambda_v$ limit the holes are negligibly small regardless of the moduli and the second integral on the right hand side of eq. 20 become $2\pi N_c R_{10} \lambda_s$ independent on any moduli.

More complicated brane configurations and corresponding $\Sigma$ surfaces can be analyzed in a similar way, with similar results. The conclusion is that the area of properly regularized $\Sigma$ is indeed independent on any moduli, the corresponding vacua of the field theory remain exactly degenerate and there are no net forces between the D4 branes of the IIA theory.

4 Dynamical force between moving branes

In this section we use M theory configurations to study the dependence of the force on the relative velocity. Forces between slowly moving infinite D-branes were studied in details in \cite{31, 32, 33}. The force is proportional to $v^{4}_{\text{rel}}$ (where $v_{\text{rel}}$ is the relative velocity) when preserving sixteen of the super-charges and proportional to $v^{2}_{\text{rel}}$ when preserving eight of the super-charges. The configurations that we consider in this section preserve eight of the super-charges. Therefore, the force should be proportional to $v^{2}_{\text{rel}}$.

4.1 Strings on D2-branes

To simplify the discussion we start with an analogous type IIA configuration of parallel D2 branes joined by fundamental strings. This setup is described in M theory by a single M2 brane. The curve of such a configuration is identical to the curve of the corresponding configuration of fourbranes between parallel fivebranes.

Our starting point is the Lagrangian of the M2 brane

$$\mathcal{L} = -T_2 \sqrt{-\det \, \bar{h}},$$

(21)
where \( h_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}, \alpha, \beta = 0, 1, 2 \). Choosing \( \xi_0 = x_0 \) the Hamiltonian is

\[
H = T_2 \int d\xi_1 d\xi_2 \frac{\partial_1 X^\mu \partial_1 X_\mu \partial_2 X_\nu \partial_2 X_\nu - (\partial_1 X_\nu \partial_2 X_\nu)^2}{\sqrt{-\det h}}.
\] (22)

The first case that we wish to study is a configuration of two strings attached to one D2-brane from opposite sides. The relevant curve is

\[
t(v - c/2) - v - c/2 = 0,
\] (23)

where \( c(x_0) = c_0 + v_{\text{rel}}x_0 \). Thus the relative distance between the fourbranes is \( c \) and the relative velocity is \( v_{\text{rel}} \). To first order in \( v_{\text{rel}}^2 \) and \( R_{10}^2/c^2 \) one gets from eqs.(22, 23)

\[
H = -\frac{\pi}{2} T_2 v_{\text{rel}}^2 R_{10}^2 \log(\lambda_t^2 \lambda_v/c^2),
\] (24)

where \( \lambda_t \) is the cutoff at the \( t \) plane, \( 1/\lambda_t < |t| < \lambda_t \), and \( \lambda_v \) is the cutoff in the \( v \)-plane, \( |v| < \lambda_v \). The force between the ending points of the fourbranes is attractive

\[
F = \frac{\pi v_{\text{rel}}^2 R_{10}^2 T_2}{c}.
\] (25)

Now, let us consider two strings which end on the same side of a 2D-brane. The curve is given by

\[
t - \frac{v^2 - c^2/4}{a^2} = 0,
\] (26)

where again \( c(x_0) = c_0 + v_{\text{rel}}x_0 \). To first order in \( v_{\text{rel}}^2 \) and \( R_{10}^2/c^2 \) one finds

\[
H = -\frac{\pi}{2} T_2 v_{\text{rel}}^2 R_{10}^2 \log\left(\frac{\lambda_t c^2}{a^2}\right).
\] (27)

Note that \( \lambda_v \) does not appear in this expression unlike in eq.(24). The derivative with respect to \( c \) gives a repulsive force

\[
F = -\frac{\pi v_{\text{rel}}^2 R_{10}^2 T_2}{c}.
\] (28)

### 4.2 D4-branes on NS fivebranes

The only difference between a configuration of fourbranes ending on fivebranes and the corresponding configuration of strings ending on D2-branes is that the ends of the fourbranes depend on \( x_0, x_1, x_2 \) and \( x_3 \). The generalization of the result of the previous subsection yields when the two fourbranes are on the same side of the fivebrane

\[
H = -2\pi T_5 R_{10}^2 \int d^3 x (\partial_t c)^2 \log\left(\frac{\lambda_t c^2}{a^2}\right),
\] (29)
where \( i = 0, 1, 2, 3 \) and \( c = c(x_0, x_1, x_2, x_3) \). Since \( c \) plays the role of the expectation value of the scalar field in the four-dimensional theory this Hamiltonian should be related to the kinetic energy in the four-dimensional theory. To make this relation more precise we need to consider two NS fivebranes joined by two D4-branes. The relevant curve is

\[
t^2 + \frac{v^2 - c^2/4}{a^2}t + 1 = 0, \tag{30}
\]

where \( c \) is the distance between the fourbranes. When \( c^2/a^2 \gg 1 \) the distance between the fivebranes at \( v = 0 \) is

\[
\Delta x_6 = 4R_{10} \log\left(\frac{c}{a}\right). \tag{31}
\]

This distance is related to the coupling of the gauge theory on the four branes

\[
\frac{R_{10}}{g^2} = \Delta x_6. \tag{32}
\]

When \( c/a \to \infty \) we recover at the semi-infinite result. This means that \( \lambda_t \) (which cannot appear in the absence of semi-infinite 4-branes) should be replaced with \( c^2/a^2 \) (as eq.(31) implies). Hence in this limit eq.(29) yields

\[
H \propto \frac{1}{g^2} \int d^3x (\partial_i c)^2. \tag{33}
\]

As expected the computation of the dynamical force gives a first order approximation to the (non-holomorphic) Kähler function. We recall that the static force is connected to the holomorphic superpotential. Clearly, our general discussion in section 3 relies heavily on holomorphicity and does not apply to the Kähler function.

5 The curvature

The curve of the M theory five-brane is smooth and invariant under rescaling of \( R_{10} \). Thus, the curve is not a useful tool to pass to the type IIA brane configurations. Instead the embedding metric on the five-brane, and in particular the corresponding curvature, can be used to approach the non smooth nature of the intersections of branes. In this section we show that the ends of the D4-branes on the NS fivebranes form curvature singularities in the type IIA limit.

We begin with the simplest example, one D4-brane ending on one NS brane. The corresponding curve is

\[
v t = 1. \tag{34}
\]
The induced metric is

$$ds^2 = dv d\bar{v} (1 + \frac{R^2_{10}}{|v|^2}),$$  \hspace{1cm} (35)$$

and the curvature is given by

$$R = 2 \frac{|v|^2 R^2_{10}}{(|v|^2 + R^2_{10})^3}. \hspace{1cm} (36)$$

The curvature vanishes far on the NS brane ($v \to \infty$) and far on the D4-brane ($v \to 0$). The maximal value of the curvature is obtained in the region where $|v|^2 \approx R^2_{10}$

$$R_{\text{max}} \approx \frac{1}{R^2_{10}}. \hspace{1cm} (37)$$

Let us consider now two D4-branes attached to one NS brane. As long as the distance between them is larger than $R_{10}$ we expect to get the same qualitative result. Namely, that the maximal curvature is at the ends of the fourbranes. More interesting is the case when the distance between the D4-branes is smaller than $R_{10}$. Consider first two semi-infinite D4-branes attached to one NS brane from opposite sides. Using eq.(23) we find that the metric is

$$ds^2 = dv d\bar{v} \left(1 + \frac{c^2 R^2_{10}}{|v - c/2|^2|v + c/2|^2}\right), \hspace{1cm} (38)$$

and the curvature is

$$R = \frac{8|v + c/2|^2|v - c/2|^2 |v|^2 c^2 R^2_{10}}{(|v + c/2|^2|v - c/2|^2 + c^2 R^2_{10})^3}. \hspace{1cm} (39)$$

Again far on the branes the curvature vanishes. The vanishing of the curvature at $v = 0$ is due to the parity symmetry of the curve. When $c \gg R_{10}$ the maximal value of the curvature is $R_{\text{max}} \approx 1/R^2_{10}$. When $c \ll R_{10}$ the maximal value of the curvature is

$$R_{\text{max}} \approx \frac{c^2}{R^4_{10}} \ll 1/R^2_{10}. \hspace{1cm} (40)$$

If the D4-branes are at the same side we can use eq.(28) to find that the metric is

$$ds^2 = dv d\bar{v} \left(1 + \frac{R^2}{|v^2 - c^2/4|^2}\right), \hspace{1cm} (41)$$

and the curvature is

$$R = -2R^2_{10} \frac{|v^2 - c^2/4|^2 |v^2 + c^2/4|^2}{(|v^2 - c^2/4|^2 + R^2_{10} |v|^2)^3}. \hspace{1cm} (42)$$
Again when \( c \gg R_{10} \) the maximal value of the curvature is of the order of \( 1/R_{10}^2 \). But unlike the previous case when \( c \ll R_{10} \) the maximal value of the curvature is

\[
R_{\text{max}} \approx \frac{32R_{10}^2}{c^4} \gg \frac{1}{R_{10}^2}
\]  

at \( v = 0 \).

Now let us consider two NS fivebranes which are connected by one D4-brane. The relevant curve is

\[
t^2 + \frac{v}{a} + 1 = 0,
\]

(44)

The metric is

\[
ds^2 = dv\bar{d}v(1 + \frac{R_{10}^2}{|v^2 - a^2|}),
\]

(45)

and the curvature is

\[
R = 2R_{10}^2 \frac{|v|^2}{|v^2 - a^2| + R_{10}^2}.
\]

(46)

When \( a \to 0 \) the distance between the NS branes diverges (at finite \( v \)) and hence we find that the maximal curvature is, \( \approx 1/R_{10}^2 \). When \( a \gg R_{10} \) the maximal curvature is \( R_{\text{max}} \approx \frac{a^2}{R_{10}^2} \gg \frac{1}{R_{10}^2} \) at \( v = a \). Notice that like in the previous case the curvature is not bounded by \( 1/R_{10}^2 \) but can be arbitrary large.

To summarize, when the distances between the branes are much larger then \( R_{10} \) the maximal curvature on \( \Sigma \) is of the order of \( 1/R_{10}^2 \), and hence diverges in the type \( II_A \) limit.

However, when the distances between the branes are smaller then \( R_{10} \) the curvature is not bounded by \( 1/R_{10}^2 \) and can be arbitrary large.

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