Remark on Koide’s $Z_3$-symmetric parametrization of quark masses

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Abstract

The charged lepton masses may be parametrized in a $Z_3$-symmetric language appropriate to the discussions of Koide’s formula. The phase parameter $\delta_L$ appearing in this parametrization is experimentally indistinguishable from $2/9$. We analyse Koide’s parametrization for the up ($U$) and down ($D$) quarks and argue that the data are suggestive of the low-energy values $\delta_U = \delta_L/3 = 2/27$ and $\delta_D = 2\delta_L/3 = 4/27$.

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Although the Standard Model is extremely successful, various questions concerning elementary particles cannot be answered within it. Among them, notwithstanding the recent discovery at CERN, there is still the issue of particle masses. This problem seems to be intimately related to the appearance of three generations of leptons and quarks. Since the understanding of these issues may require completely new ideas, phenomenological identification of regularities observed in the pattern of particle masses and mixings is of crucial importance. It may provide us with analogues of Balmer and Rydberg’s formulae and should hopefully lead us to a genuinely new physics.

1. One of the most interesting of such regularities was discovered by Koide \[1\] (for a brief review see \[2\]). It reads:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{1 + k_L^2}{3},$$  \(\text{(1)}\)

with \(k_L\) equal exactly 1.

In fact, if one plugs into the above formula the central values of experimental electron and muon masses \[3\]:

\[
\begin{align*}
    m_e(\text{exp}) &= 0.510998928(11) \text{ MeV}, \\
    m_\mu(\text{exp}) &= 105.65836715(35) \text{ MeV},
\end{align*}
\]

\(\text{(2)}\)

one finds (with \(k_L = 1\)) that the larger of the solutions of Eq. (1) is

$$m_\tau = 1776.9689 \text{ MeV},$$  \(\text{(3)}\)

to be compared with the experimental \(\tau\) mass

$$m_\tau(\text{exp}) = 1776.82 \pm 0.16 \text{ MeV}.$$  \(\text{(4)}\)

Discussions of this success of Koide’s formula are often formulated in a \(Z_3\)-symmetric language by parametrizing the masses of any three given fermions \(f_1, f_2, f_3\) as \[4\] \[5\]:

$$\sqrt{m_j} = \sqrt{M_f} \left(1 + \sqrt{2} k_f \cos \left(\frac{2\pi j}{3} + \delta_f\right)\right), \quad (j = 1, 2, 3).$$  \(\text{(5)}\)

In general, with appropriately chosen three parameters (here: the overall mass scale \(M_f\) and the pattern parameters \(k_f\) and \(\delta_f\)) one can fit any three-particle spectrum, of course. The above choice of parametrization is, however, particularly suited to Koide’s formula, since the latter is then independent of parameter \(\delta_f\), as specified in Eq. (1). That the resulting formula works then for \(k_L\) equal exactly 1 (or almost 1 with very high precision) is truly amazing.
2. The peculiar feature of formula (1) is that it works at a low energy scale and not at some high mass scale (like \( M_Z \) or \( 10^{16} \) GeV) \[^6, 7, 8\]. For example, taking the values of charged lepton masses as appropriate at the scale of \( M_Z \) (\( m_e = 0.486755106 \) MeV, \( m_\mu = 102.740394 \) MeV, \( m_\tau = 1746.56 \) MeV) one finds that \( k_L(M_Z) = 1.00188 \), i.e. it deviates from unity quite significantly. If that value of \( k_L \) worked for physical masses it would remove much of the excitement Koide’s formula generates.

Attempts have been made to apply Koide’s formula to quarks and neutrinos. The general conclusion is that the formula does not work there (i.e. \( k_f \neq 1 \)). Specifically, using the quark mass values as appropriate at \( \mu = 2 \) GeV, for the down quarks (\( D \)) one obtains the values of \( k_D \) around 1.08, while for the up quarks (\( U \)) one gets \( k_U \) around 1.25 \[^7, 9, 10\] (with the mathematically allowed region \( 0 \leq k_f \leq \sqrt{2} \)). Furthermore, for neutrinos \( \nu \) one estimates directly from experiment that \( k_\nu \leq 0.81 \) \[^10\]. If a higher energy scale \( \mu \) is taken, the agreement deteriorates further (at the \( M_Z \) mass scale one gets \( k_D = 1.12 \) and \( k_U = 1.29 \)).

3. Given the success of Koide’s parametrization \[^5\] with \( k_f = 1 \) for charged leptons (\( f = L \)) and its failure for other fundamental fermions (\( f = U, D, \nu \)), one should perhaps look in a different direction. In fact, parametrization \[^5\] reveals another miracle in the charged lepton sector. Namely, using the experimental values of charged lepton masses one can determine the value of the phase parameter \( \delta_L \).

From Eq. \((5)\) one finds

\[
\frac{1}{\sqrt{2}}(\sqrt{m_2} - \sqrt{m_1}) = \sqrt{M_f} k_f \sqrt{3} \sin \delta_f, \tag{6}
\]

\[
\frac{1}{\sqrt{6}}(2\sqrt{m_3} - \sqrt{m_2} - \sqrt{m_1}) = \sqrt{M_f} k_f \sqrt{3} \cos \delta_f. \tag{7}
\]

Since \( \delta_f \) is free we may assume \( m_1 \leq m_2 \leq m_3 \) without any loss of generality. Then, independently of the value of \( k_L \), one gets

\[
\tan \delta_L = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}}. \tag{8}
\]

From the experimental values of Eqs \((2, 4)\) one then calculates:

\[
\delta_L = 0.2222324, \tag{9}
\]

which is extremely close to \( \delta_L = 2/9 \) \[^11, 12\]. After inverting formula \(^{8}\) and assuming \( \delta_L = 2/9 \) one can predict the value of \( \tau \) mass, given the experimental masses of electron and muon. The result is:

\[
m_\tau = 1776.9664 \text{ MeV}. \tag{10}
\]

This is as good a prediction of the tauon mass as that given by the original Koide’s formula \(^{3}\). The two predictions of Eqs \((3, 10)\) are mutually incompatible, but either
of them leads to an excellent prediction for $m_\tau$. They could be made consistent with each other by allowing extremely tiny departures of either $\delta_L$ from $2/9$ or $k_L$ from 1. Keeping in mind the violation of Koide’s formula for quarks and neutrinos, one should perhaps try the $\delta_L = 2/9$ alternative, for example by maintaining the values of all $\delta_f$ equal to $2/9$ or $k_L$ from 1. Then, he constructs a $\mathbb{Z}_3$-symmetric model which modifies (in a calculable way) the value of $M_q$ for each quark $q_j$ separately (hence $M_q \rightarrow M_{q,j}$). However, then the Koide parametrization of Eq.(5) ceases to be valid.

4. On the other hand, a different route possible may be taken which keeps parametrization (5) intact. Namely, one could refrain for the time being from the discussions of Koide-like formulas (with $k_f \neq 1$). Instead, one might try to analyze the issue of $\delta_f$ in more detail. After all, the assumption of $\delta_L = 2/9$ yields as good a prediction for the tauon mass as the assumption of $k_L = 1$.

Thus, the idea is to analyze what the experimental values of quark masses tell us about the $\delta_f$ parameters for $f = U, D$. For this simple exercise we take the following typical set of the values of experimental masses at $\mu = 2$ GeV (in MeV):

\[
\begin{align*}
  m_u &= 1.7 - 3.3, \\
  m_c &= 1270^{+70}_{-90}, \\
  m_t &= 172000 \pm 1600, \\
  m_d &= 4.1 - 5.8, \\
  m_s &= 101^{+29}_{-21}, \\
  m_b &= 4190^{+180}_{-60}.
\end{align*}
\] (11)

For the discussion of $\delta_f$ we introduce $z_k = \sqrt{m_k/m_3}$ (assuming $m_1 < m_2 < m_3$). Thus

\[
\delta_f = \arctan \left( \frac{\sqrt{3} \left( z_2 - z_1 \right)}{2 - z_2 - z_1} \right). \tag{12}
\]

Fig. 1 shows a contour plot of $\delta_f(z_1,z_2)$ and the corresponding approximate positions of $(z_1,z_2)$ (together with their errors) as calculated from Eq. (11) for the up quarks (marked $U$) and the down quarks (marked $D$). For comparison with the lepton case the position of $\delta_L$ is also shown (marked with a dot $L$). Slanted solid lines represent constant $\delta$ (= 0, 1/27, 2/27, ..., 2/9, ...). It can be seen that the observed value of $\delta_U$ is consistent with $\delta_U = \delta_L/3 = 2/27$. Thanks to a huge top quark mass, the errors are quite small here. For the down sector the mass hierarchy is not as strong as in the up sector, and therefore the corresponding errors are much larger.

We have to remember, however, that at $\mu = M_Z$ both $k_L$ and $\delta_L$ deviate from their ‘perfect’ values of 1 and 2/9, by about 0.2 % and 0.5 % respectively (the

\[\text{Hence, due to the above mentioned incompatibility, Rosen cannot simultaneously describe the electron and muon masses with maximal precision as required by the data.}\]
deviation from $2/9$ is virtually indiscernible in the scale used for the presentation of Fig. 1). Apparently, we should be interested in the values of current quark masses at the low energy scale and not at $\mu = 2$ GeV as in Eq. (11). For the up quark sector, thanks to the huge mass of the top quark, such a change cannot significantly modify the position of the corresponding $U$ point in Fig. 1. This is not the case for the down quarks. In order to show what happens there, we assumed that $\kappa = m_s/m_d$ is fixed at the value of $\kappa = 20.4$ as obtained at $\mu = 2$ GeV from Eq. (11) and as also valid at low energies when extracted from $\pi$ and $K$ masses (see e.g. [13]). The corresponding relation between $z_1$ and $z_2$ is marked in Fig. 1 with a dashed line. Two points along this line, corresponding to $m_s = 130$ and 160 MeV, are shown there as well. We observe that at the expected low-energy-scale value of the strange quark mass (i.e. for $m_s$ around 160 MeV) the value of $\delta_D$ appears to be close to $2\delta_L/3 = 4/27$. The obtained low-energy-scale values of $\delta_U$, $\delta_D$ and $\delta_L$ are therefore suggestive of a nice (even if only fairly approximate) symmetry between the lepton and quark sectors, with the values of $I_3 = -1/2$ particle phases $\delta_L, \delta_D$ depending on weak hypercharge $Y$ and given by

$$\delta(I_3 = -1/2, Y) = \frac{1}{9}(1 + |Y|),$$

(13)

and the $I_3 = +1/2$ particle phases $\delta_\nu, \delta_U$ expected to be given by

$$\delta(I_3 = +1/2, Y) = \frac{1}{9}(1 - |Y|),$$

(14)
together forming an equally-spaced set and satisfying a lepton-quark sum rule \( \delta_L + \delta_\nu = \delta_U + \delta_D \).

Obviously, due to the possible Majorana mass term, the observed masses of neutrinos do not have to realize the pattern \( m_1 = m_2 < m_3 \) suggested by \( \delta(I_3 = +1/2, |Y| = 1) = 0 \).

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