TROPICAL CONVEXITY

MIKE DEVELIN AND BERND STURMFELS

Abstract. The notions of convexity and convex polytopes are introduced in the setting of tropical geometry. Combinatorial types of tropical polytopes are shown to be in bijection with regular triangulations of products of two simplices. Applications to phylogenetic trees are discussed.

1. Introduction

The tropical semiring \((\mathbb{R}, \oplus, \odot)\) is the set of real numbers with the arithmetic operations of tropical addition, which is taking the minimum of two numbers, and tropical multiplication, which is ordinary addition. Thus the two arithmetic operations are defined as follows:

\[ a \oplus b := \min(a, b) \quad \text{and} \quad a \odot b := a + b. \]

The \(n\)-dimensional space \(\mathbb{R}^n\) is a semimodule over the tropical semiring, with tropical addition \((x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) = (x_1 \oplus y_1, \ldots, x_n \oplus y_n)\), and tropical scalar multiplication \(c \odot (x_1, x_2, \ldots, x_n) = (c \odot x_1, c \odot x_2, \ldots, c \odot x_n)\).

The semiring \((\mathbb{R}, \oplus, \odot)\) and its semimodule \(\mathbb{R}^n\) obey the usual distributive and associative laws.

The purpose of this paper is to propose a tropical theory of convex polytopes. Convexity in arbitrary idempotent semimodules was introduced by Cohen, Gaubert and Quadrat [3] and Litvinov, Maslov and Shpiz [13]. Some of our results (such as Theorem 23 and Propositions 20 and 21) are known in a different guise in idempotent analysis. Our objective is to provide a combinatorial approach to convexity in the tropical semiring which is consistent with the recent developments in tropical algebraic geometry (see [15], [18], [20]). The connection to tropical methods in representation theory (see [12], [16]) is less clear and deserves further study.

There are many notions of discrete convexity in the computational geometry literature, but none of them seems to be quite like tropical convexity. For instance, the notion of directional convexity studied by Matoušek [14] has similar features but it is different and much harder to compute with.

A subset \(S\) of \(\mathbb{R}^n\) is called tropical convex if the set \(S\) contains the point \(a \odot x \oplus b \odot y\) for all \(x, y \in S\) and all \(a, b \in \mathbb{R}\). The tropical convex hull of a given subset \(V \subseteq \mathbb{R}^n\) is the smallest tropically convex subset of \(\mathbb{R}^n\) which contains \(V\). We shall see in Proposition 4 that the tropical convex hull of \(V\) coincides with the set of all tropical linear combinations

\begin{equation}
(a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus \cdots \oplus a_r \odot v_r), \quad \text{where} \ v_1, \ldots, v_r \in V \text{ and } a_1, \ldots, a_r \in \mathbb{R}.
\end{equation}

Any tropically convex subset \(S\) of \(\mathbb{R}^n\) is closed under tropical scalar multiplication, \(\mathbb{R} \odot S \subseteq S\). In other words, if \(x \in S\) then \(x + \lambda(1, \ldots, 1) \in S\) for all \(\lambda \in \mathbb{R}\). We will therefore identify the tropically convex set \(S\) with its image in the \((n - 1)\)-dimensional tropical projective space

\[ \mathbb{T}P^{n-1} = \mathbb{R}^n/(1, \ldots, 1)\mathbb{R}. \]

Basic properties of (tropically) convex subsets in \(\mathbb{T}P^{n-1}\) will be presented in Section 2. In Section 3 we introduce tropical polytopes and study their combinatorial structure. A tropical polytope is the tropical convex hull of a finite subset \(V\) in \(\mathbb{T}P^{n-1}\). Every tropical polytope is a finite union of
convex polytopes in the usual sense: given a set \( V = \{v_1, \ldots, v_n\} \), their convex hull has a natural decomposition as a polyhedral complex, which we call the tropical complex generated by \( V \). The following main result will be proved in Section 4:

**Theorem 1.** The combinatorial types of tropical complexes generated by a set of \( r \) vertices in \( \mathbb{T}^{n-1} \) are in natural bijection with the regular polyhedral subdivisions of the product of two simplices \( \Delta_{n-1} \times \Delta_{r-1} \).

This implies a remarkable duality between tropical \((n-1)\)-polytopes with \( r \) vertices and tropical \((r-1)\)-polytopes with \( n \) vertices. Another consequence of Theorem 1 is a formula for the \( f \)-vector of a generic tropical complex. In Section 5 we discuss applications of tropical convexity to phylogenetic analysis, extending known results on injective hulls of finite metric spaces (cf. [7], [8], [9] and [20]).

### 2. Tropically convex sets

We begin with two pictures of tropical convex sets in the tropical plane \( \mathbb{T}^2 \). A point \((x_1, x_2, x_3) \in \mathbb{T}^2 \) is represented by drawing the point with coordinates \((x_2 - x_1, x_3 - x_1) \) in the plane of the paper. The triangle on the left hand side in Figure 1 is tropically convex, but it is not a tropical polytope because it is not the tropical convex hull of finitely many points. The thick edges indicate two tropical line segments. The picture on the right hand side is a tropical triangle, namely, it is the tropical convex hull of the three points \((0,0,1)\), \((0,2,0)\) and \((0,-1,-2)\) in the tropical plane \( \mathbb{T}^2 \). The thick edges represent the tropical segments connecting any two of these three points.

![Figure 1. Tropical convex sets and tropical line segments in \( \mathbb{T}^2 \).](image)

We next show that tropical convex sets enjoy many of the features of ordinary convex sets.

**Theorem 2.** The intersection of two tropically convex sets in \( \mathbb{R}^n \) or in \( \mathbb{T}^{n-1} \) is tropically convex. The projection of a tropically convex set onto a coordinate hyperplane is tropically convex. The ordinary hyperplane \( \{x_i - x_j = l\} \) is tropically convex, and the projection map from this hyperplane to \( \mathbb{R}^{n-1} \) given by eliminating \( x_i \) is an isomorphism of tropical semimodules. Tropical convex sets are contractible spaces. The Cartesian product of two tropically convex sets is tropically convex.

**Proof.** We prove the statements in the order given. If \( S \) and \( T \) are tropically convex, then for any two points \( x, y \in S \cap T \), both \( S \) and \( T \) contain the tropical line segment between \( x \) and \( y \), and consequently so does \( S \cap T \). Therefore \( S \cap T \) is tropically convex by definition.

Suppose \( S \) is a tropically convex set in \( \mathbb{R}^n \). We wish to show that the image of \( S \) under the coordinate projection \( \phi : \mathbb{R}^n \to \mathbb{R}^{n-1}, (x_1, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n) \) is a tropically convex subset
of $\mathbb{R}^{n-1}$. If $x, y \in S$ then we have the obvious identity

$$\phi(c \odot x \oplus d \odot y) = c \odot \phi(x) \oplus d \odot \phi(y).$$

This means that $\phi$ is a homomorphism of tropical semimodules. Therefore, if $S$ contains the tropical line segment between $x$ and $y$, then $\phi(S)$ contains the tropical line segment between $\phi(x)$ and $\phi(y)$ and hence is tropically convex. The same holds for the induced map $\phi : \mathbb{TP}^{n-1} \to \mathbb{TP}^{n-2}$.

Most ordinary hyperplanes in $\mathbb{R}^n$ are not tropically convex, but we are claiming that hyperplanes of the special form $x_i - x_j = k$ are tropically convex. If $x$ and $y$ lie in that hyperplane then $x_i - y_i = x_j - y_j$. This last equation implies the following identity for any real numbers $c, d \in \mathbb{R}$:

$$(c \odot x \oplus d \odot y)_i - (c \odot x \oplus d \odot y)_j = \min(x_i + c, y_i + d) - \min(x_j + c, y_j + d) = k.$$

Hence the tropical line segment between $x$ and $y$ also lies in the hyperplane $\{x_i - x_j = k\}$.

Consider the map from $\{x_i - x_j = k\}$ to $\mathbb{R}^{n-1}$ given by deleting the $i$-th coordinate. This map is injective: if two points differ in the $x_i$ coordinate they must also differ in the $x_j$ coordinate. It is clearly surjective because we can recover an $i$-th coordinate by setting $x_i = x_j + k$. Hence this map is an isomorphism of $\mathbb{R}$-vector spaces and it is also an isomorphism of $(\mathbb{R}, \oplus, \odot)$-semimodules.

Let $S$ be a tropically convex set in $\mathbb{R}^n$ or $\mathbb{TP}^{n-1}$. Consider the family of hyperplanes $H_l = \{x_1 - x_2 = l\}$ for $l \in \mathbb{R}$. We know that the intersection $S \cap H_l$ is tropically convex, and isomorphic to its (convex) image under the map deleting the first coordinate. This image is contractible by induction on the dimension $n$ of the ambient space. Therefore, $S \cap H_l$ is contractible. The result then follows from the topological result that if $S$ is connected, which all tropically convex sets obviously are, and if $S \cap H_l$ is contractible for each $l$, then $S$ itself is also contractible.

Suppose that $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ are tropically convex. Our last assertion states that $S \times T$ is a tropically convex subset of $\mathbb{R}^{n+m}$. Take any $(x, y)$ and $(x', y')$ in $S \times T$ and $c, d \in \mathbb{R}$. Then

$$c \odot (x, y) \oplus d \odot (x', y') = (c \odot x \oplus d \odot x', c \odot y \oplus d \odot y')$$

lies in $S \times T$ since $S$ and $T$ are tropically convex.

We next give a more precise description of what tropical line segments look like.

**Proposition 3.** The tropical line segment between two points $x$ and $y$ in $\mathbb{TP}^{n-1}$ is the concatenation of at most $n - 1$ ordinary line segments. The slope of each line segment is a zero-one vector.

**Proof.** After relabeling the coordinates of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we may assume $y_1 - x_1 \leq y_2 - x_2 \leq \cdots \leq y_n - x_n$.

The following points lie in the given order on the tropical segment between $x$ and $y$:

- $x = (y_1 - x_1) \odot x \oplus y = (y_1, y_1 - x_1 + x_2, y_1 - x_1 + x_3, \ldots, y_1 - x_1 + x_{n-1}, y_1 - x_1 + x_n)$
- $(y_2 - x_2) \odot x \oplus y = (y_1, y_2, y_2 - x_2 + x_3, \ldots, y_2 - x_2 + x_{n-1}, y_2 - x_2 + x_n)$
- $(y_3 - x_3) \odot x \oplus y = (y_1, y_2, y_3, \ldots, y_3 - x_3 + x_{n-1}, y_3 - x_3 + x_n)$
- \ldots
- $(y_{n-1} - x_{n-1}) \odot x \oplus y = (y_1, y_2, y_3, \ldots, y_{n-1}, y_{n-1} - x_{n-1} + x_n)$
- $y = (y_n - x_n) \odot x \oplus y = (y_1, y_2, y_3, \ldots, y_{n-1}, y_n)$.

Between any two consecutive points, the tropical line segment agrees with the ordinary line segment, which has slope $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$. Hence the tropical line segment between $x$ and $y$ is the concatenation of at most $n - 1$ ordinary line segments, one for each strict inequality in (2). \(\square\)

This description of tropical segments shows an important feature of tropical polytopes: their edges use a limited set of directions. The following result characterizes the *tropical convex hull.*
Proposition 6. Tropical hyperplanes in $\mathbb{T}\mathbb{P}^{n-1}$ are tropically convex.

Proof. Let $x = \bigoplus_{i=1}^{r} a_i \odot v_i$ be the point in (1). If $r \leq 2$ then $x$ is clearly in the tropical convex hull of $V$. If $r > 2$ then we write $x = a_1 \odot v_1 \oplus \bigoplus_{i=2}^{r} a_i \odot v_i$. The parenthesized vector lies the tropical convex hull, by induction on $r$, and hence so does $x$. For the converse, consider any two tropical linear combinations $x = \bigoplus_{i=1}^{r} c_i \odot v_i$ and $y = \bigoplus_{j=1}^{s} d_j \odot v_i$. By the distributive law, $a \odot x \oplus b \odot y$ is also a tropical linear combination of $v_1, \ldots, v_r \in V$. Hence the set of all tropical linear combinations of $V$ is tropically convex, so it contains the tropical convex hull of $V$. □

If $V$ is a finite subset of $\mathbb{T}\mathbb{P}^{n-1}$ then $\text{tconv}(V)$ is a tropical polytope. In Figure 2 we see three small examples of tropical polytopes. The first and second are tropical convex hulls of three points in $\mathbb{T}\mathbb{P}^2$. The third tropical polytope lies in $\mathbb{T}\mathbb{P}^3$ and is the union of three squares.

One of the basic results in the usual theory of convex polytopes is Carathéodory’s theorem. This theorem holds in the tropical setting.

Proposition 5 (Tropical Carathéodory’s Theorem). If $x$ is in the tropical convex hull of a set of $r$ points $v_i$ in $\mathbb{T}\mathbb{P}^{n-1}$, then $x$ is in the tropical convex hull of at most $n$ of them.

Proof. Let $x = \bigoplus_{i=1}^{r} a_i \odot v_i$ and suppose $r > n$. For each coordinate $j \in \{1, \ldots, n\}$, there exists an index $i \in \{1, \ldots, r\}$ such that $x_j = c_i + v_{ij}$. Take a subset $I$ of $\{1, \ldots, r\}$ composed of one such $i$ for each $j$. Then we also have $x = \bigoplus_{i \in I} a_i \odot v_i$, where $\#(I) \leq n$. □

The basic theory of tropical linear subspaces in $\mathbb{T}\mathbb{P}^{n-1}$ was developed in [18] and [20]. Recall that the tropical hyperplane defined by a tropical linear form $a_1 \odot x_1 \oplus a_2 \odot x_2 \oplus \cdots \oplus a_n \odot x_n$ consists of all points $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{T}\mathbb{P}^{n-1}$ such that the following holds (in ordinary arithmetic):

$$a_i + x_i = a_j + x_j = \min \{a_k + x_k : k = 1, \ldots, n\} \quad \text{for some indices } i \neq j.$$  

Just like in ordinary geometry, hyperplanes are convex sets:

Proposition 6. Tropical hyperplanes in $\mathbb{T}\mathbb{P}^{n-1}$ are tropically convex.
that \( \lambda \)

Corollary 7. Tropical planes in \( \mathbb{T}P^{n-1} \) are tropically convex.

A theorem in classical geometry states that every point outside a closed convex set can be separated from the convex set by a hyperplane. The same statement holds in tropical geometry. This follows from the results in [3]. Some caution is needed, however, since the definition of hyperplane in [3] differs from our definition of hyperplane, as explained in [18]. In our definition, a tropical hyperplane is a fan which divides \( \mathbb{T}P^{n-1} \) into \( n \) convex cones, each of which is also tropically convex. Rather than stating the most general separation theorem, we will now focus our attention on tropical polytopes, in which case the separation theorem is the Farkas Lemma stated in the next section.

3. Tropical polytopes and cell complexes

Throughout this section we fix a finite subset \( V = \{ v_1, v_2, \ldots, v_r \} \) of tropical projective space \( \mathbb{T}P^{n-1} \). Here \( v_i = (v_{i1}, v_{i2}, \ldots, v_{in}) \). Our goal is to study the tropical polytope \( P = \text{tconv}(V) \). We begin by describing the natural cell decomposition of \( \mathbb{T}P^{n-1} \) induced by the fixed finite subset \( V \).

Let \( x \) be any point in \( \mathbb{T}P^{n-1} \). The type of \( x \) relative to \( V \) is the ordered \( n \)-tuple \( (S_1, \ldots, S_n) \) of subsets \( S_j \subseteq \{1, 2, \ldots, r\} \) which is defined as follows: An index \( i \) is in \( S_j \) if

\[
\min(v_{ij} - x_j) = \min(v_{i1} - x_1, v_{i2} - x_2, \ldots, v_{in} - x_n).
\]

Equivalently, if we set \( \lambda_i = \min\{ \lambda \in \mathbb{R} : \lambda \odot v_i \oplus x = x \} \) then \( S_j \) is the set of all indices \( i \) such that \( \lambda_i \odot v_i \) and \( x \) have the same \( j \)-th coordinate. We say that an \( n \)-tuple of indices \( S = (S_1, \ldots, S_n) \) is a type if it arises in this manner. Note that every \( i \) must be in some \( S_j \).

Example 8. Let \( r = n = 3, v_1 = (0,0,2), v_2 = (0,2,0) \) and \( v_3 = (0,1,-2) \). There are 30 possible types as \( x \) ranges over the plane \( \mathbb{T}P^2 \). The corresponding cell decomposition has six convex regions (one bounded, five unbounded), 15 edges (6 bounded, 9 unbounded) and 6 vertices. For instance, the point \( x = (0,1,-1) \) has type \( (x) = \{(2), (1), (3)\} \) and its cell is a bounded pentagon. The point \( x' = (0,0,0) \) has type \( (x') = \{(1,2), (1), (2,3)\} \) and its cell is one of the six vertices. The point \( x'' = (0,0,-3) \) has type \( (x'') = \{(1,2,3), (1), \emptyset\} \) and its cell is an unbounded edge.

Our first application of types is the following separation theorem.

Proposition 9 (Tropical Farkas Lemma). For all \( x \in \mathbb{T}P^{n-1} \), exactly one of the following is true.

(i) the point \( x \) is in the tropical polytope \( P = \text{tconv}(V) \), or
(ii) there exists a tropical hyperplane which separates \( x \) from \( P \).
The separation statement in part (ii) means the following: if the hyperplane is given by (4) and \( a_k + x_k = \min(a_1 + x_1, \ldots, a_n + x_n) \) then \( a_k + y_k > \min(a_1 + y_1, \ldots, a_n + y_n) \) for all \( y \in P \).

**Proof.** Consider any point \( x \in \mathbb{T}^{P^{n-1}} \), with \( \text{type}(x) = (S_1, \ldots, S_n) \), and let \( \lambda_i = \min\{\lambda \in \mathbb{R} : \lambda \odot v_i \odot x = x\} \) as before. We define
\[
\pi_V(x) = \lambda_1 \odot v_1 \oplus \lambda_2 \odot v_2 \oplus \cdots \oplus \lambda_r \odot v_r.
\]
There are two cases: either \( \pi_V(x) = x \) or \( \pi_V(x) \neq x \). The first case implies (i). Since (i) and (ii) clearly cannot occur at the same time, it suffices to prove that the second case implies (ii).

Suppose that \( \pi_V(x) \neq x \). Then \( S_k \) is empty for some index \( k \in \{1, \ldots, n\} \). This means that \( v_{ik} + \lambda_i - x_k > 0 \) for \( i = 1, 2, \ldots, r \). Let \( \varepsilon > 0 \) be smaller than any of these \( r \) positive reals. We now choose our separating tropical hyperplane \( \pi \) as follows:
\[
a_k := -x_k - \varepsilon \quad \text{and} \quad a_j := -x_j \quad \text{for } j \in \{1, \ldots, n\}\setminus\{k\}.
\]
This certainly satisfies \( a_k + x_k = \min(a_1 + x_1, \ldots, a_n + x_n) \). Now, consider any point \( y = \bigoplus_{i=1}^{r} c_i \odot v_i \) in \( \text{tconv}(V) \). Pick any \( m \) such that \( y_k = c_m + v_{mk} \). By definition of the \( \lambda_i \), we have \( x_k \leq \lambda_m + v_{mk} \) for all \( k \), and there exists some \( j \) with \( x_j = \lambda_m + v_{mj} \). These equations and inequalities imply
\[
a_k + y_k = a_k + c_m + v_{mk} = c_m + v_{mk} - x_k - \varepsilon > c_m - \lambda_m
\]
\[
= c_m + v_{mj} - x_j \geq y_j - x_j = a_j + y_j \geq \min(a_1 + y_1, \ldots, a_n + y_n).
\]
Therefore, the hyperplane defined by (4) separates \( x \) from \( P \) as desired. \( \square \)

The construction in (4) defines a map \( \pi_V : \mathbb{T}^{P^{n-1}} \rightarrow P \) whose restriction to \( P \) is the identity. This map is the tropical version of the **nearest point map** onto a closed convex set in ordinary geometry. Such maps were studied in [3] for convex subsets in arbitrary idempotent semimodules.

If \( S = (S_1, \ldots, S_n) \) and \( T = (T_1, \ldots, T_n) \) are \( n \)-tuples of subsets of \( \{1, 2, \ldots, r\} \), then we write \( S \subseteq T \) if \( S_j \subseteq T_j \) for \( j = 1, \ldots, n \). We also consider the set of all points whose type contains \( S \):
\[
X_S := \{ x \in \mathbb{T}^{P^{n-1}} : S \subseteq \text{type}(x) \}.
\]

**Lemma 10.** The set \( X_S \) is a closed convex polyhedron (in the usual sense). More precisely,
\[
X_S = \{ x \in \mathbb{T}^{P^{n-1}} : x_k - x_j \leq v_{ik} - v_{ij} \quad \text{for all } j, k \in \{1, \ldots, n\} \text{ such that } i \in S_j \}.
\]

**Proof.** Let \( x \in \mathbb{T}^{P^{n-1}} \) and \( T = \text{type}(x) \). First, suppose \( x \) is in \( X_S \). Then \( S \subseteq T \). For every \( i, j, k \) such that \( i \in S_j \), we also have \( i \in T_j \), and so by definition we have \( v_{ij} - x_j \leq v_{ik} - x_k \), or \( x_k - x_j \leq v_{ik} - v_{ij} \). Hence \( x \) lies in the set on the right hand side of (4). For the proof of the reverse inclusion, suppose that \( x \) lies in the right hand side of (6). Then, for all \( i, j \) with \( i \in S_j \), and for all \( k \), we have \( v_{ij} - x_j \leq v_{ik} - x_k \). This means that \( v_{ij} - x_j = \min(v_{i1} - x_1, \ldots, v_{in} - x_n) \) and hence \( i \in T_j \). Consequently, for all \( j \), we have \( S_j \subseteq T_j \), and so \( x \in X_S \). \( \square \)

As an example for Lemma 10 we consider the region \( X_{(2,1,3)} \) in the tropical convex hull of \( v_1 = (0, 0, 2) \), \( v_2 = (0, 2, 0) \), and \( v_3 = (0, 1, -2) \). This region is defined by six linear inequalities, one of which is redundant, as depicted in Figure 3. Lemma 10 has the following immediate corollaries.

**Corollary 11.** The intersection \( X_S \cap X_T \) is equal to the polyhedron \( X_{S \cup T} \).

**Proof.** The inequalities defining \( X_{S \cup T} \) are precisely the union of the inequalities defining \( X_S \) and \( X_T \), and points satisfying these inequalities are precisely those in \( X_S \cap X_T \). \( \square \)

**Corollary 12.** The polyhedron \( X_S \) is bounded if and only if \( S_j \neq \emptyset \) for all \( j = 1, 2, \ldots, n \).
Since containment, i.e. when 

\begin{equation}
\{ x_k - x_j \leq v_{ik} - v_{ij} \mid k \in \{1, \ldots, n\} \}.
\end{equation}

By assumption, \( i \) is in some \( S_m \). We claim that \( X_T \) is the face of \( S \) defined by the equality

\begin{equation}
x_m - x_j = v_{im} - v_{ij}.
\end{equation}

Since \( X_S \) satisfies the inequality \( x_j - x_m \leq v_{ij} - v_{im} \), \( \Box \) defines a face \( F \) of \( S \). The inequality \( x_m - x_j \leq v_{im} - v_{ij} \) is in the set \( \Box \), so \( \Box \) is valid on \( X_T \) and \( X_T \subseteq F \). However, any point in \( F \), being in \( X_S \), satisfies \( x_k - x_m \leq v_{ik} - v_{im} \) for all \( k \in \{1, \ldots, n\} \). Adding \( \Box \) to these inequalities proves that the inequalities \( \Box \) are valid on \( F \), and hence \( F \subseteq X_T \). So \( X_T = F \) as desired.

By the discussion in the proof of the first part, prescribing equality in the facet-defining inequality \( x_k - x_j \leq v_{ik} - v_{ij} \) yields \( X_T \), where \( T_k = S_k \cup \{i\} \) and \( T_j = S_j \) for \( j \neq k \). Therefore, all facets of \( X_S \) can be obtained as regions \( X_T \), and it follows recursively that all faces of \( X_S \) are of this form. □

**Corollary 14.** Suppose that \( S = (S_1, \ldots, S_n) \) is an \( n \)-tuple of indices satisfying \( S_1 \cup \cdots \cup S_n = \{1, \ldots, r\} \). Then \( X_S \) is equal to \( X_T \) for some type \( T \).
Theorem 15. The collection of convex polyhedra $X_S$, where $S$ ranges over all types, defines a cell decomposition $C_V$ of $\mathbb{TP}^{n-1}$. The tropical polytope $P = \operatorname{tconv}(V)$ equals the union of all bounded cells $X_S$ in this decomposition.

Proof. Since each point has a type, it is clear that the union of the $X_S$ is equal to $\mathbb{TP}^{n-1}$. By Corollary 13 the faces of $X_S$ are equal to $X_U$ for $S \subseteq U$, and by Corollary 14 $X_U = X_W$ for some type $W$, and hence $X_U$ is in our collection. The only thing remaining to check is that this collection defines a cell decomposition is that $X_S \cap X_T$ is a face of both $X_S$ and $X_T$, but $X_S \cap X_T = X_{S \cap T}$ by Corollary 11 and $X_{S \cup T}$ is a face of $X_S$ and $X_T$ by Corollary 13.

For the second assertion consider any point $x \in \mathbb{TP}^{n-1}$ and let $S = \text{type}(x)$. We have seen in the proof of the Tropical Farkas Lemma (Proposition 9) that $x$ lies in $P$ if and only if no $S_j$ is empty. By Corollary 12 this is equivalent to the polyhedron $X_S$ being bounded. □

The collection of bounded cells $X_S$ is referred to as the tropical complex generated by $V$; thus, Theorem 12 states that this provides a polyhedral decomposition of the polytope $P = \operatorname{tconv}(V)$. Different sets $V$ may have the same tropical polytope as their convex hull, but generate different tropical complexes; the decomposition of a tropical polytope depends on the chosen generating set, although we will see later (Proposition 21) that there is a unique minimal generating set.

Here is a nice geometric construction of the cell decomposition $C_V$ of $\mathbb{TP}^{n-1}$ induced by $V = \{v_1, \ldots, v_r\}$. Let $F$ be the fan in $\mathbb{TP}^{n-1}$ defined by the tropical hyperplane $\langle \rangle$ with $a_1 = \cdots = a_n = 0$. Two vectors $x$ and $y$ lie in the same relatively open cone of the fan $F$ if and only if $\{ j : x_j = \min(x_1, \ldots, x_n) \} = \{ j : y_j = \min(y_1, \ldots, y_n) \}$. If we translate the negative of $F$ by the vector $v_i$ then we get a new fan which we denote by $v_i - F$. Two vectors $x$ and $y$ lie in the same relatively open cone of the fan $v_i - F$ if and only if $\{ j : x_j - v_{ij} = \max(x_1 - v_{i1}, \ldots, x_n - v_{in}) \} = \{ j : y_j - v_{ij} = \max(y_1 - v_{i1}, \ldots, y_n - v_{in}) \}$.

Proposition 16. The cell decomposition $C_V$ is the common refinement of the $r$ fans $v_i - F$.

Proof. We need to show that the cells of this common refinement are precisely the convex polyhedra $X_S$. Take a point $x$, with $T = \text{type}(x)$ and define $S_x = (S_{z1}, \ldots, S_{zn})$ by letting $i \in S_{xj}$ whenever

$$x_j - v_{ij} = \max(x_1 - v_{i1}, \ldots, x_n - v_{in}).$$

Two points $x$ and $y$ are in the relative interior of the same cell of the common refinement if and only if they are in the same relatively open cone of each fan; this is tantamount to saying that $S_x = S_y$. However, we claim that $S_x = T$. Indeed, taking the negative of both sides of (9) yields exactly the condition for $i$ being in $T_j$, by the definition of type. Consequently, the condition for two points having the same type is the same as the condition for the two points being in the relative interior of the same chamber of the common refinement of the fans $v_1 - F, v_2 - F, \ldots, v_r - F$. □

An example of this construction is shown for our usual example, where $v_1 = (0,0,2)$, $v_2 = (0,2,0)$, and $v_3 = (0,1,-2)$, in Figure 4.

The next few results provide additional information about the polyhedron $X_S$. Let $G_S$ denote the undirected graph with vertices $\{1, \ldots, n\}$, where $\{j,k\}$ is an edge if and only if $S_j \cap S_k \neq \emptyset$.

Proposition 17. The dimension $d$ of the polyhedron $X_S$ is one less than the number of connected components of $G_S$, and $X_S$ is affinely and tropically isomorphic to some polyhedron $X_T$ in $\mathbb{TP}^d$. 

is at most a polytope, equal to the tropical convex hull of its vertices. The number of vertices of the polytope $X_S$ is at most $(\binom{n-2}{n-1})$, and this bound is tight for all positive integers $n$.

**Proof.** The proof is by induction on $n$. Suppose we have $i \in S_j \cap S_k$. Then $X_S$ satisfies the linear equation $x_k - x_j = c$ where $c = v_{ik} - v_{ij}$. Eliminating the variable $x_k$ (projecting onto $\mathbb{TP}^{n-2}$), $X_S$ is affinely and tropically isomorphic to $X_T$ where the type $T$ is defined by $T_r = S_r$ for $r \neq j$ and $T_j = S_j \cup S_k$. The region $X_T$ exists in the cell decomposition of $\mathbb{TP}^{n-2}$ induced by the vectors $w_1, \ldots, w_n$ with $w_{ir} = v_{ir}$ for $r \neq j$, and $w_{ij} = \max(v_{ij}, v_{ik} - c)$. The graph $G_T$ is obtained from the graph $G_S$ by contracting the edge $\{j, k\}$, and thus has the same number of connected components.

This induction on $n$ reduces us to the case where all of the $S_j$ are pairwise disjoint. We must show that $X_S$ has dimension $n - 1$. Suppose not. Then $X_S$ lies in $\mathbb{TP}^{n-1}$ but has dimension less than $n - 1$. Therefore, one of the inequalities in (8) holds with equality, say $x_k - x_j = v_{ik} - v_{ij}$ for all $x \in X_S$. The inequality “≤” implies $i \in S_j$ and the inequality “≥” implies $i \in S_k$. Hence $S_j$ and $S_k$ are not disjoint, a contradiction.

The following proposition can be regarded as a converse to Lemma 10.

**Proposition 18.** Let $R$ be any polytope in $\mathbb{TP}^{n-1}$ defined by inequalities of the form $x_k - x_j \leq c_{jk}$. Then $R$ arises as a cell $X_S$ in the decomposition $C_V$ of $\mathbb{TP}^{n-1}$ defined by some set $V = \{v_1, \ldots, v_n\}$.

**Proof.** Define the vectors $v_i$ to have coordinates $v_{ij} = c_{ij}$ for $i \neq j$, and $v_{ii} = 0$. (If $c_{ij}$ did not appear in the given inequality presentation then simply take it to be a very large positive number.) Then by Lemma 10 the polytope in $\mathbb{TP}^{n-1}$ defined by the inequalities $x_k - x_j \leq c_{jk}$ is precisely the unique cell of type $(1, 2, \ldots, n)$ in the tropical convex hull of $\{v_1, \ldots, v_n\}$.

The region $X_S$ is a polytope both in the ordinary sense and in the tropical sense.

**Proposition 19.** Every bounded cell $X_S$ in the tropical complex generated by $V$ is itself a tropical polytope, equal to the tropical convex hull of its vertices. The number of vertices of the polytope $X_S$ is at most $(\binom{n-2}{n-1})$, and this bound is tight for all positive integers $n$.

**Proof.** By Proposition 17 if $X_S$ has dimension $d$, it is affinely and tropically isomorphic to a region in the convex hull of a set of points in $\mathbb{TP}^d$, so it suffices to consider the full-dimensional case.
The inequality presentation of Lemma 10 demonstrates that $X_S$ is tropically convex for all $S$, since if two points satisfy an inequality of that form, so does any tropical linear combination thereof. Therefore, it suffices to show that $X_S$ is contained in the tropical convex hull of its vertices.

The proof is by induction on the dimension of $X_S$. All proper faces of $X_S$ are polytopes $X_T$ of lower dimension, and by induction are contained in the tropical convex hull of their vertices. These vertices are a subset of the vertices of $X_S$, and so this face is in the tropical convex hull.

Take any point $x = (x_1, \ldots, x_n)$ in the interior of $X_S$. Since $X_S$ has dimension $n$, we can travel in any direction from $x$ while remaining in $X_S$. Let us travel in the $(1,0,\ldots,0)$ direction until we hit the boundary, to obtain points $y_1 = (x_1 + b, x_2, \ldots, x_n)$ and $y_2 = (x_1 - c, x_2, \ldots, x_n)$ in the boundary of $X_S$. These points are contained in the tropical convex hull by the inductive hypothesis, which means that $x = y_1 \oplus c \odot y_2$ is also, completing the proof of the first assertion.

For the second assertion, we consider the convex hull of all differences of unit vectors, $e_i - e_j$. This is a lattice polytope of dimension $n-1$ and normalized volume $\frac{(2n-2)}{n-1}$. To see this, we observe that this polytope is tiled by $n$ copies of the convex hull of the origin and the $\binom{n}{2}$ vectors $e_i - e_j$ with $i < j$. The other $n-1$ copies are gotten by cyclic permutation of the coordinates. This latter polytope was studied by Gel’fand, Graev and Postnikov, who showed in [4, Theorem 2.3 (2)] that the normalized volume of this polytope equals the Catalan number $\frac{1}{n} \frac{(2n-2)}{n-1}$.

We conclude that every complete fan whose rays are among the vectors $e_i - e_j$ has at most $\binom{2n-2}{n-1}$ maximal cones. This applies in particular to the normal fan of $X_S$, hence $X_S$ has at most $\frac{(2n-2)}{n-1}$ vertices. Since the configuration $\{e_i - e_j\}$ is unimodular, the bound is tight whenever the fan is simplicial and uses all the rays $e_i - e_j$. \hfill $\square$

We close this section with two more results about arbitrary tropical polytopes in $\mathbb{T}P^n$.

**Proposition 20.** If $P$ and $Q$ are tropical polytopes in $\mathbb{T}P^{n-1}$ then $P \cap Q$ is also a tropical polytope.

*Proof.* Since $P$ and $Q$ are both tropically convex, $P \cap Q$ must also be. Consequently, if we can find a finite set of points in $P \cap Q$ whose convex hull contains all of $P \cap Q$, we will be done. By Theorem 15 $P$ and $Q$ are the finite unions of bounded cells $\{X_S\}$ and $\{X_T\}$ respectively, so $P \cap Q$ is the finite union of the cells $X_S \cap X_T$. Consider any $X_S \cap X_T$. Using Lemma 10 to obtain the inequality representations of $X_S$ and $X_T$, we see that this region is of the form dictated by Proposition 18 and therefore obtainable as a cell $X_W$ in some tropical complex. By Proposition 19 $X_W$ is itself a tropical polytope, and we can therefore find a finite set whose convex hull is equal to $X_S \cap X_T$. Taking the union of these sets over all choices of $S$ and $T$ then gives us the desired set of points whose convex hull contains all of $P \cap Q$. \hfill $\square$

**Proposition 21.** Let $P \subset \mathbb{T}P^{n-1}$ be a tropical polytope. Then there exists a unique minimal set $V$ such that $P = \text{tconv}(V)$.

*Proof.* Suppose that $P$ has two minimal generating sets, $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_r\}$. Write each element of $W$ as $w_i = \oplus_{j=1}^m c_{ij} \odot v_j$. We claim that $V \subseteq W$. Consider $v_1 \in V$ and write

$$v_1 = \bigoplus_{i=1}^r d_i \odot w_i = \bigoplus_{j=1}^m f_j \odot v_j \quad \text{where } f_j = \min_i(d_i + c_{ij}).$$

If the term $f_1 \odot v_1$ does not minimize any coordinate in the right-hand side of (10), then $v_1$ is a linear combination of $v_2, \ldots, v_m$, contradicting the minimality of $V$. However, if $f_1 \odot v_1$ minimizes any coordinate in this expression, it must minimize all of them, since $(v_1)_j - (v_1)_k = (f_1 \odot v_1)_j - (f_1 \odot v_1)_k$. In this case we get $v_1 = f_1 \odot v_1$, or $f_1 = 0$. Pick any $i$ for which $f_1 = d_i + c_{ij}$; we claim that $w_i = c_{i1} \odot v_1$. Indeed, if any other term in $w_i = \oplus_{j=1}^m c_{ij} \odot v_j$ contributed nontrivially to $w_i$, that
term would also contribute to the expression on the right-hand side of (10), which is a contradiction. Consequently, \( V \subseteq W \), which means \( V = W \) since both sets are minimal by hypothesis.

Like many of the results presented in this section, Propositions 20 and 21 parallel results on ordinary polytopes. We have already mentioned the tropical analogues of the Farkas Lemma and of Carathéodory’s Theorem (Propositions 5 and 9); Proposition 17 is analogous to the result that a polytope \( P \subseteq \mathbb{R}^n \) of dimension \( d \) is affinely isomorphic to some \( Q \subseteq \mathbb{R}^d \). Proposition 19 hints at a duality between an inequality representation and a vertex representation of a tropical polytope; this duality has been studied in greater detail by Michael Joswig [11].

4. Subdividing products of simplices

Every set \( V = \{v_1, \ldots, v_r\} \) of \( r \) points in \( \mathbb{T}P^{n-1} \) begets a tropical polytope \( P = tconv(V) \) equipped with a cell decomposition into the tropical complex generated by \( V \). Each cell of this tropical complex is labelled by its type, which is an \( n \)-vector of finite subsets of \( \{1, \ldots, r\} \). Two configurations (and their corresponding tropical complexes) \( V \) and \( W \) have the same combinatorial type if the types occurring in their tropical complexes are identical; note that by Lemma 13 this implies that the face posets of these polyhedral complexes are isomorphic.

With the definition in the previous paragraph, the statement of Theorem 4 has now finally been made precise. We will prove this correspondence between tropical complexes and subdivisions of products of simplices by constructing the polyhedral complex \( C_P \) in a higher-dimensional space.

Let \( W \) denote the \((r + n - 1)\)-dimensional real vector space \( \mathbb{R}^{r+n}/(1, \ldots, 1,-1,\ldots,-1) \). The natural coordinates on \( W \) are denoted \( (y, z) = (y_1, \ldots, y_r, z_1, \ldots, z_n) \). As before, we fix an ordered subset \( V = \{v_1, \ldots, v_r\} \) of \( \mathbb{T}P^{n-1} \) where \( v_i = (v_{i1}, \ldots, v_{in}) \). This defines the unbounded polyhedron

\[
(11) \quad \mathcal{P}_V = \{(y, z) \in W : y_i + z_j \leq v_{ij} \text{ for all } i \in \{1, \ldots, r\} \text{ and } j \in \{1, \ldots, n\}\}.
\]

**Lemma 22.** There is a piecewise-linear isomorphism between the tropical complex generated by \( V \) and the complex of bounded faces of the \((r + n - 1)\)-dimensional polyhedron \( \mathcal{P}_V \). The image of a cell \( X_S \) of \( C_P \) under this isomorphism is the bounded face \( \{y_i + z_j = v_{ij} : i \in S_j\} \) of the polyhedron \( \mathcal{P}_V \). That bounded face maps isomorphically to \( X_S \) via projection onto the \( z \)-coordinates.

**Proof.** Let \( F \) be a bounded face of \( \mathcal{P}_V \), and define \( S_j \) via \( i \in S_j \) if \( y_i + z_j = v_{ij} \) is valid on all of \( F \). If some \( y_i \) or \( z_j \) appears in no equality, then we can subtract arbitrary positive multiples of that basis vector to obtain elements of \( F \), contradicting the assumption that \( F \) is bounded. Therefore, each \( i \) must appear in some \( S_j \), and each \( S_j \) must be nonempty.

Since every \( y_i \) appears in some equality, given a specific \( z \) in the projection of \( F \) onto the \( z \)-coordinates, there exists a unique \( y \) for which \( (y, z) \in F \), so this projection is an affine isomorphism from \( F \) to its image. We need to show that this image is equal to \( X_S \).

Let \( z \) be a point in the image of this projection, coming from a point \((y, z)\) in the relative interior of \( F \). We claim that \( z \in X_S \). Indeed, looking at the \( j \)th coordinate of \( z \), we find

\[
(12) \quad -y_i + v_{ij} \geq z_j \quad \text{for all } i,
\]

\[
(13) \quad -y_i + v_{ij} = z_j \quad \text{for } i \in S_j.
\]

The defining inequalities of \( X_S \) are \( x_j - x_k \leq v_{ij} - v_{ik} \) with \( i \in S_j \). Subtracting the inequality \(-y_i + v_{ik} \geq z_k \) from the equality in (13) yields that this inequality is valid on \( z \) as well. Therefore, \( z \in X_S \). Similar reasoning shows that \( S = \text{type}(z) \). We note that the relations (12) and (13) can
be rewritten elegantly in terms of the tropical product of a row vector and a matrix:

\[(14)\quad z = (-y) \odot V = \bigoplus_{i=1}^{r} (-y_i) \odot v_i.\]

For the reverse inclusion, suppose that \( z \in X_S \). We define \( y = V \odot (-z) \). This means that

\[(15)\quad y_i = \min(v_{i1} - z_1, v_{i2} - z_2, \ldots, v_{in} - z_n).\]

We claim that \((y, z) \in F\). Indeed, we certainly have \( y_i + z_j \leq v_{ij} \) for all \( i \) and \( j \), so \((y, z) \in \mathcal{P}_V\). Furthermore, when \( i \in S_j \), we know that \( v_{ij} - z_j \) achieves the minimum in the right-hand side of (15), so that \( v_{ij} - z_j = y_i \) and \( y_i + z_j = v_{ij} \) is satisfied. Consequently, \((y, z) \in F\) as desired.

It follows immediately that the two complexes are isomorphic: if \( F \) is a face corresponding to \( X_S \) and \( G \) is a face corresponding to \( X_T \), where \( S \) and \( T \) are both types, then \( X_S \) is a face of \( X_T \) if and only if \( T \subseteq S \). However, by the discussion above, this is equivalent to saying that the equalities \( G \) satisfies (which correspond to \( T \)) are a subset of the equalities \( F \) satisfies (which correspond to \( S \)); this is true if and only if \( F \) is a face of \( G \). So \( X_S \) is a face of \( X_T \) if and only if \( F \) is a face of \( G \), which implies the isomorphism of complexes. \(\square\)

The boundary complex of the polyhedron \( \mathcal{P}_V \) is polar to the regular subdivision of the product of simplices \( \Delta_{r-1} \times \Delta_{n-1} \) defined by the weights \( v_{ij} \). We denote this regular polyhedral subdivision by \( (\partial \mathcal{P}_V)^* \). Explicitly, a subset of vertices \((e_i, e_j)\) of \( \Delta_{r-1} \times \Delta_{n-1} \) forms a cell of \( (\partial \mathcal{P}_V)^* \) if and only if the equations \( y_i + z_j = v_{ij} \) indexed by these vertices specify a face of the polyhedron \( \mathcal{P}_V \). We refer to the book of De Loera, Rambau and Santos [5] for basics on polyhedral subdivisions.

We now present the proof of the result stated in the introduction.

**Proof of Theorem 1.** The poset of bounded faces of \( \mathcal{P}_V \) is antisomorphic to the poset of interior cells of the subdivision \((\partial \mathcal{P}_V)^* \) of \( \Delta_{r-1} \times \Delta_{n-1} \). Since every full-dimensional cell of \( (\partial \mathcal{P}_V)^* \) is interior, the subdivision is uniquely determined by its interior cells. In other words, the combinatorial type of \( \mathcal{P}_V \) is uniquely determined by the lists of facets containing each bounded face of \( \mathcal{P}_V \). These lists are precisely the types of regions in \( \mathcal{C}_P \) by Lemma 22. This completes the proof. \(\square\)

Theorem 1 which establishes a bijection between the tropical complexes generated by \( r \) points in \( \mathbb{T}P^{n-1} \) and the regular subdivisions of a product of simplices \( \Delta_{r-1} \times \Delta_{n-1} \), has many striking consequences. First of all, we can pick off the types present in a tropical complex simply by looking at the cells present in the corresponding regular subdivision. In particular, if we have an interior cell \( M \), the corresponding type appearing in the tropical complex is defined via \( S_j = \{ i \in [n]\mid (j, i) \in M \} \).

It is worth noting that via the Cayley Trick [19], Theorem 1 is equivalent to saying that tropical complexes generated by \( r \) points in \( \mathbb{T}P^{n-1} \) are in bijection with the regular mixed subdivisions of the dilated simplex \( r\Delta^{n-1} \). This connection is expanded upon and employed in a paper with Francisco Santos [6]. Another astonishing consequence of Theorem 1 is the identification of the row span and column span of a matrix. This result can also be derived from [9], Theorem 42.

**Theorem 23.** Given any matrix \( M \in \mathbb{R}^{r \times n} \), the tropical complex generated by its column vectors is isomorphic to the tropical complex generated by its row vectors. This isomorphism is gotten by restricting the piecewise linear maps \( \mathbb{R}^n \to \mathbb{R}^r \), \( z \mapsto M \odot (-z) \) and \( \mathbb{R}^r \to \mathbb{R}^n \), \( y \mapsto (-y) \odot M \).

**Proof.** By Theorem 1, the matrix \( M \) corresponds via the polyhedron \( \mathcal{P}_M \) to a regular subdivision of \( \Delta_{r-1} \times \Delta_{n-1} \), and the complex of interior faces of this regular subdivision is combinatorially isomorphic to both the tropical complex generated by its row vectors, which are \( r \) points in \( \mathbb{T}P^{n-1} \), and the tropical complex generated by its column vectors, which are \( n \) points in \( \mathbb{T}P^{r-1} \). Furthermore, Lemma 22 tells us that the cell in \( \mathcal{P}_M \) is affinely isomorphic to its corresponding cell in both
tropical complexes. Finally, in the proof of Lemma \[22\] we showed that the point \((y, z)\) in a bounded face \(F\) of \(P_M\) satisfies \(y = M \odot (-z)\) and \(z = (-y) \odot M\). This point projects to \(y\) and \(z\), and so the piecewise-linear isomorphism mapping these two complexes to each other is defined by the stated maps.

The common tropical complex of these two tropical polytopes is given by the complex of bounded faces of the common polyhedron \(P_M\), which lives in a space of dimension \(r + n - 1\); the tropical polytopes are unfoldings of this complex into dimensions \(r - 1\) and \(n - 1\). Theorem \[23\] also gives a natural bijection between the combinatorial types of tropical convex hulls of \(r\) points in \(\mathbb{T}P^{n-1}\) and the combinatorial types of tropical convex hulls of \(n\) points in \(\mathbb{T}P^{r-1}\), incidentally proving that there are the same number of each. This duality statement extends a similar statement in \[3\].

Figure 5 shows the dual of the convex hull of \(\{(0,0,2), (0,2,0), (0,1,-2)\}\), also a tropical triangle (here \(r = n = 3\)). For instance, we compute:

\[
\begin{pmatrix}
0 & 0 & 2 \\
0 & 2 & 0 \\
0 & 1 & -2
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
-2
\end{pmatrix}
=
\begin{pmatrix}
0 \\
-2 \\
-4
\end{pmatrix}.
\]

This point is the image of the point \((0,0,2)\) under this duality map. Note that duality does not preserve the generating set; the polytope on the right is the convex hull of points \(\{F, D, B\}\), while the polytope on the left is the convex hull of points \(\{F, A, C\}\). This is necessary, of course, since in general a polytope with \(r\) vertices is mapped to a polytope with \(n\) vertices, and \(r\) need not equal \(n\) as it does in our example.

We now discuss the generic case when the subdivision \((\partial P_V)^*\) is a regular triangulation of \(\Delta_{r-1} \times \Delta_{n-1}\). We refer to \[18\, \S\, 5\] for the geometric interpretation of the tropical determinant.

**Proposition 24.** For a configuration \(V\) of \(r\) points in \(\mathbb{T}P^{n-1}\) with \(r \geq n\) the following are equivalent:

1. The regular subdivision \((\partial P_V)^*\) is a triangulation of \(\Delta_{r-1} \times \Delta_{n-1}\).
2. No \(k\) of the points in \(V\) have projections onto a \(k\)-dimensional coordinate subspace which lie in a tropical hyperplane, for any \(2 \leq k \leq n\).
3. No \(k \times k\)-submatrix of the \(r \times n\)-matrix \((v_{ij})\) is tropically singular, i.e. has vanishing tropical determinant, for any \(2 \leq k \leq n\).
The original triangulation as well. 

Consider any index \( f \) and hence equidecomposable, so all of these induced triangulations have the same \( k \)-vector. The number of faces of dimension \( k \) appears in some such inequality, and thus it will have a cycle if and only if it has at least \( r + n \) edges. Consequently, \( \mathcal{P}_V \) is not simple if and only there exists some \( G_v \) with a cycle.

If there is a cycle, without loss of generality it reads \( y_1, z_1, y_2, z_2, \ldots, y_k, z_k \). Consider the submatrix \( M \) of \( (v_{ij}) \) given by \( 1 \leq i \leq k \) and \( 1 \leq j \leq k \). We have \( y_1 + z_1 = M_{11}, y_2 + z_2 = M_{22} \), and so on, and also \( z_1 + y_2 = M_{12}, \ldots, z_k + y_1 = M_{k1} \). Adding up all of these equalities yields \( y_1 + \cdots + y_k + z_1 + \cdots + z_k = M_{11} + \cdots + M_{kk} = M_{12} + \cdots + M_{k1} \). But consider any permutation \( \sigma \) in the symmetric group \( S_k \). Since we have \( M_{i\sigma(i)} = v_{i\sigma(i)} \geq y_i + z_{\sigma(i)} \), we have \( \sum M_{i\sigma(i)} \geq x_1 + \cdots + x_k + y_1 + \cdots + y_k \). Consequently, the permutations equal to the identity and to \((12\cdots k)\) simultaneously minimize the determinant of the minor \( M \). This logic is reversible, proving the equivalence of (1) and (3).

If the \( r \) points of \( V \) are in general position, the tropical complex they generate is called a generic tropical complex. These polyhedral complexes are then polar to the complexes of interior faces of regular triangulations of \( \Delta_{r-1} \times \Delta_{n-1} \).

### Corollary 25

All tropical complexes generated by \( r \) points in general position in \( \mathbb{T}^P_x \) have the same \( f \)-vector. Specifically, the number of faces of dimension \( k \) is equal to the multinomial coefficient

\[
\binom{r + n - k - 2}{r - k - 1, n - k - 1, k} = \frac{(r + n - k - 2)!}{(r - k - 1)! \cdot (n - k - 1)! \cdot k!}
\]

### Proof

By Proposition 24, these objects are in bijection with regular triangulations of \( P = \Delta_{r-1} \times \Delta_{n-1} \). The polytope \( P \) is equidecomposable [1], meaning that all of its triangulations have the same \( f \)-vector. The number of faces of dimension \( k \) of the tropical complex generated by given \( r \) points is equal to the number of interior faces of codimension \( k \) in the corresponding triangulation. Since all triangulations of all products of simplices have the same \( f \)-vector, they must also have the same interior \( f \)-vector, which can be computed by taking the \( f \)-vector and subtracting off the \( f \)-vectors of the induced triangulations on the proper faces of \( P \). These proper faces are all products of simplices and hence equidecomposable, so all of these induced triangulations have \( f \)-vectors independent of the original triangulation as well.

To compute this number, we therefore need only compute it for one tropical complex. Let the vectors \( v_i, 1 \leq i \leq r \), be given by \( v_i = (i, 2i, \ldots, ni) \). By Theorem 10 to count the faces of dimension \( k \) in this tropical complex, we enumerate the existing types with \( k \) degrees of freedom. Consider any index \( i \). We claim that for any \( x \) in the tropical convex hull of \( \{v_i\} \), the set \( \{j \mid i \in S_j\} \) is an interval \( I_i \), and that if \( i < m \), the intervals \( I_m \) and \( I_i \) meet in at most one point, which in that case is the largest element of \( I_m \) and the smallest element of \( I_i \).

Suppose we have \( i \in S_j \) and \( m \in S_l \) with \( i < m \). Then we have by definition \( v_{ij} - x_j \leq v_{il} - x_l \) and \( v_{ml} - x_l \leq v_{mj} - x_j \). Adding these inequalities yields \( v_{ij} + v_{ml} \leq v_{il} + v_{mj} \), or \( ij + ml \leq il + mj \). Since \( i < m \), it follows that we must have \( l \leq j \). Therefore, we can never have \( i \in S_j \) and \( m \in S_l \) with \( i < m \) and \( j < l \). The claim follows immediately, since the \( I_i \) cover \( [1, n] \).
The number of degrees of freedom of an interval set \((I_1, \ldots, I_r)\) is easily seen to be the number of \(i\) for which \(I_i\) and \(I_{i+1}\) are disjoint. Given this, it follows from a simple combinatorial counting argument that the number of interval sets with \(k\) degrees of freedom is the multinomial coefficient given above. Finally, a representative for every interval set is given by \(x_{j} = x_{j+1} - c_{j}\), where if \(S_j\) and \(S_{j+1}\) have an element \(i\) in common (they can have at most one), \(c_{j} = i\), and if not then \(c_{j} = (\min(S_j) + \max(S_{j+1}))/2\). Therefore, each interval set is in fact a valid type, and our enumeration is complete. \(\square\)

**Corollary 26.** The number of combinatorially distinct generic tropical complexes generated by \(r\) points in \(\mathbb{T}P^{n-1}\) equals the number of distinct regular triangulations of \(\Delta_{r-1} \times \Delta_{n-1}\). The number of respective symmetry classes under the natural action of the product of symmetric groups \(G = S_r \times S_n\) on both spaces is also the same.

The symmetries in the group \(G\) correspond to a natural action on \(\Delta_{r-1} \times \Delta_{n-1}\) given by permuting the vertices of the two component simplices; the symmetries in the symmetric group \(S_r\) correspond to permuting the points in a tropical polytope, while those in the symmetric group \(S_n\) correspond to permuting the coordinates. (These are dual, as per Corollary 23.) The number of symmetry classes of regular triangulations of the polytope \(\Delta_{r-1} \times \Delta_{n-1}\) is computable via Jörg Rambau’s TOPCOM \cite{17} for small \(r\) and \(n\):

|   | 2   | 3   |
|---|-----|-----|
| 2 | 5   | 35  |
| 3 | 35  | 7,955 |
| 4 | 530 |     |
| 5 | 13,631 |   |

For example, the \((2,3)\) entry of the table divulges that there are 35 symmetry classes of regular triangulations of \(\Delta_2 \times \Delta_3\). These correspond to the 35 combinatorial types of four-point configurations in \(\mathbb{T}P^2\), or to the 35 combinatorial types of three-point configurations in \(\mathbb{T}P^3\). These 35 configurations (with the tropical complexes they generate) are shown in Figure 6; the labeling corresponds to Rambau’s labeling (see \cite{17}) of the regular triangulations of \(\Delta_3 \times \Delta_2\).

### 5. Phylogenetic Analysis Using Tropical Polytopes

A fundamental problem in bioinformatics is the reconstruction of phylogenetic trees from approximate distance data. In this section we show how tropical convexity might help provide new algorithmic tools for this problem. Our approach augments the results in \cite{20} \& \cite{4} and it provides a tropical interpretation of the work on \(T\)-theory by Andreas Dress and his collaborators \cite{7, 8, 9}.

Consider a symmetric \(n \times n\)-matrix \(D = (d_{ij})\) whose entries \(d_{ij}\) are non-negative real numbers and whose diagonal entries \(d_{ii}\) are all zero. We say that \(D\) is a (finite) metric if the triangle inequality \(d_{ij} \leq d_{ik} + d_{jk}\) holds for all indices \(i, j, k\). Our starting point is the following easy observation:

**Proposition 27.** The symmetric matrix \(D\) is a metric if and only if all principal \(3 \times 3\)-minors of the negated symmetric matrix \(-D = (-d_{ij})\) are tropically singular.

**Proof.** Both properties involve only three points, so we may assume \(n = 3\), in which case

\[
-D = \begin{pmatrix}
0 & -d_{12} & -d_{13} \\
-d_{12} & 0 & -d_{23} \\
-d_{13} & -d_{23} & 0
\end{pmatrix}.
\]

The tropical determinant of this matrix is the minimum of the six expressions

\[
0, -2d_{12}, -2d_{13}, -2d_{23}, -d_{12} - d_{13} - d_{23}\text{ and } -d_{12} - d_{13} - d_{23}.
\]
Figure 6. The 35 symmetry classes of tropical complexes generated by four points in $\mathbb{TP}^2$. 
This minimum is attained twice if and only if it is attained by the last two (identical) expressions, which occurs if and only if the three triangle inequalities are satisfied.

In what follows we assume that $D = (d_{ij})$ is a metric. Let $P_D$ denote the tropical convex hull in $\mathbb{T}P^{n-1}$ of the $n$ row vectors (or column vectors) of the negated matrix $-D = (-d_{ij})$. Proposition 27 tells us that the tropical polytope $P_D$ is always one-dimensional for $n = 3$.

The finite metric $D = (d_{ij})$ is said to be a tree metric if there exists a weighted tree $T$ with $n$ leaves such that $d_{ij}$ denotes the distance between the $i$-th leaf and the $j$-th leaf along the unique path between these leaves in $T$. The next theorem characterizes tree metrics among all metrics by the dimension of the tropical polytope $P_D$. It is the tropical interpretation of results that are quite classical and well-known in the phylogenetics literature.

**Theorem 28.** For a given finite metric $D = (d_{ij})$ the following conditions are equivalent:

1. $D$ is a tree metric,
2. the tropical polytope $P_D$ has dimension one,
3. all $4 \times 4$-minors of the matrix $-D$ are tropically singular,
4. all principal $4 \times 4$-minors of the matrix $-D$ are tropically singular,
5. For any choice of four indices $i, j, k, l \in \{1, 2, \ldots, n\}$, the maximum of the three numbers $d_{ij} + d_{kl}, d_{ik} + d_{jl}$ and $d_{il} + d_{kj}$ is attained at least twice.

**Proof.** The condition (5) is the familiar **Four Point Condition** for tree metrics. The equivalence of (1) and (5) is a classical result due to various authors, including Buneman [2] and Zaretsky [21]. See equation (B3) on page 57 in [7].

Suppose that the condition (5) holds. By the discussion in [20], this means that $-D$ is a point in the tropical Grassmannian of lines, in symbols $-D \in \text{Gr}(2, n) \subset \mathbb{T}P^{n-1}(2)$. By [20] Theorem 3.8, the point $-D$ corresponds to a tropical line $L_D$ in $\mathbb{T}P^{n-1}$. The $n$ distinguished points whose coordinates are the rows of $-D$ lie on the line $L_D$. By Corollary 7 it follows that their tropical convex hull $P_D$ is contained in $L_D$. This means that $P_D$ has dimension one, that is, (2) holds.

Suppose that (2) holds. Then the tropical rank of the matrix $-D$ is equal to two, by [6, Theorem 4.2]. This means that all $r \times r$-minors of $-D$ are tropically singular for $r \geq 3$. The case $r = 4$ is precisely the statement (3).

Obviously, the condition (3) implies the condition (4). What remains is to prove the implication from (4) to (5). For this we note that the tropical determinant of the $4 \times 4$-matrix

$$
\begin{pmatrix}
0 & -d_{12} & -d_{13} & -d_{14} \\
-d_{12} & 0 & -d_{23} & -d_{24} \\
-d_{13} & -d_{23} & 0 & -d_{34} \\
-d_{14} & -d_{24} & -d_{34} & 0
\end{pmatrix}
$$

equals twice the minimum of $-d_{12} - d_{34}, -d_{13} - d_{24}$ and $-d_{14} - d_{23}$. (It’s the tropicalization of a $4 \times 4$-Pfaffian). The matrix is tropically singular if and only if the minimum is attained twice. □

If the five equivalent conditions of Theorem 28 are satisfied then the metric tree $T$ coincides with the one-dimensional tropical polytope $P_D$. To make sense of this statement, we regard tropical projective space $\mathbb{T}P^{n-1}$ as a metric space with respect to the infinity norm induced from $\mathbb{R}^n$,

$$
||x - y|| = \max \{ |x_i + y_j - x_j - y_i| : 1 \leq i < j \leq n \},
$$

and we note that the finite metric $D$ embeds isometrically into $P_D$ via the rows of $-\frac{1}{2}D$:

$$
i \mapsto \frac{1}{2} \cdot (-d_{i1}, -d_{i2}, -d_{i3}, \ldots, -d_{in}) \quad \text{for } i = 1, 2, \ldots, n
$$
We learned from [8] that the tropical polytope $P_D$ first appeared in the 1964 paper [10] by John Isbell. For the proof of the following result we assume familiarity with results from [7] and [8].

**Theorem 29.** The tropical polytope $P_D$ equals Isbell’s injective hull of the metric $D$.

**Proof.** According to Lemma [22] the tropical polytope $P_D$ is the bounded complex of the following unbounded polyhedron in the $(2n-1)$-dimensional space $W = \mathbb{R}^{2n}/\mathbb{R}(1, \ldots, 1, -1, \ldots, -1)$:

$$P_D := \{ (y, z) \in W : y_i + z_j \leq -d_{ij} \text{ for all } 1 \leq i, j \leq n \}.$$

Dress et al. [7] showed that the injective hull $T(D)$ of the finite metric $D$ coincides with the complex of bounded faces of the following $n$-dimensional unbounded polyhedron:

$$Q_D := \{ x \in \mathbb{R}^n : x_i + x_j \geq d_{ij} \text{ for all } 1 \leq i, j \leq n \}.$$

What we need to show is that the two polyhedra have the same bounded complex.

The metric $D$ satisfies the tropical matrix identity $-D = D \circ (-D)$, because $-d_{ij} = \min_k(d_{ik} - d_{kj})$. This implies that any column vector $y$ of $-D$ satisfies $y = (-y) \circ (-D)$.

Consider any vertex $(y, z)$ of $P_D$. Then $y$ is a column vector of $-D$. Equation (14) implies $z = (-y) \circ (-D) = y$. Hence every vertex of $P_D$ lies in the subspace defined by $y = z$, and so does the complex of bounded faces of $P_D$. Therefore the linear map $(y, z) \mapsto -y$ induces an isomorphism between the bounded complex of $P_D$ and the bounded complex of $Q_D$. 

**Theorem 28** specifies an involution on the set of all tropical complexes. We are interested in the fixed points of this canonical involution. A necessary condition is that $r = n$ and $V$ is a symmetric matrix. The previous result and its proof can be reinterpreted as follows:

**Corollary 30.** A tropical complex $P$ is pointwise fixed under the canonical involution (on the set of all tropical complexes) if and only if $P$ is the injective hull of a metric on $\{1, 2, \ldots, n\}$.

**Proof.** In order for $P$ to be fixed under the canonical involution, it is necessary that $n = d$. Hence we can write $P = tconv(-D)$ for some non-negative square matrix $D$. Now, $P$ is fixed under the involution if and only if the identity $-D = D \circ (-D)$ holds. This identity is equivalent to $D$ being a metric. 

Dress, Huber and Moulton [7] emphasize that the tropical polytope $P_D$ records many important invariants of a given finite metric $D$. For instance, the dimension of $P_D$ gives information about how far the metric is from being a tree metric. In practical biological applications of phylogenetic reconstruction, the distances $d_{ij}$ are not known exactly, and $P_D$ appears to contain many of the various trees which are found by existing software for phylogenetic reconstruction.

The dimension of the tropical complex $P_D = tconv(-D)$ can be characterized combinatorially by tropicalizing the sub-Pfaffians of a skew-symmetric $n \times n$-matrix. The tropical Pfaffians of format $4 \times 4$ specify the four point condition (5) in Theorem [28] while the tropical sub-Pfaffians of format $6 \times 6$ specify the six-point condition which is discussed in [7] page 25. The combinatorial study of $k$-compatible split systems can be interpreted in the setting of tropical algebraic geometry (cf. [15], [18], [20]) as the study of the $k$-th secant variety in the Grassmannian $\text{Gr}(2, n) \subset TP^{n-1}(2)$.

Tropical convexity provides a convenient language to study numerous extensions of the classical problem of tree reconstruction. As an example, imagine the following scenario, which would correspond to the Grassmannian of planes in $TP^{n-1}$, denoted $\text{Gr}(3, n)$.

Suppose there are $n$ taxa, labeled $1, 2, \ldots, n$, and rather than having a distance for any pair $i, j$, we are now given a proximity measure $d_{ijk}$ for any triple $i, j, k \in \{1, 2, \ldots, n\}$. We can then construct a tropical polytope by taking the tropical convex hull of $\binom{n}{2}$ points as follows:

$$P = tconv\{ (-d_{i1}, -d_{i2}, -d_{i3}, \ldots, -d_{in}) \in TP^{n-1} : 1 \leq i < j \leq n \}.$$
Under certain hypotheses, the tropical polytope $P$ can be realized as the complex of bounded faces of the polyhedron in $\mathbb{R}^n$ defined by the inequalities $x_i + x_j + x_k \geq d_{ijk}$. It provides a polyhedral model for the tree-like nature of the data $(d_{ijk})$. The case of most interest is when $P$ is two-dimensional in which case it plays the role of a two-dimensional phylogenetic tree.

The construction of this particular tropical polytope $P$ was pioneered by Dress and Terhalle in the important paper [9]. There they discuss valued matroids, which are essentially the points on the tropical Grassmannian of [20], and they call $P$ the tight span of a valued matroid. We share their view that these tropical polytopes constitute a promising tool for phylogenetic analysis.

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Mike Develin, American Institute of Mathematics, 360 Portage Ave., Palo Alto, CA 94306-2244, USA
E-mail address: develin@post.harvard.edu

Bernd Sturmfels, Department of Mathematics, University of California, Berkeley, CA 94720, USA
E-mail address: bernd@math.berkeley.edu