Random Quantum Random Walk as a Maximal Entanglement Generator

Rafael Vieira,† Edgard P. M. Amorim,‡ and Gustavo Rigolin

†Departamento de Física, Universidade do Estado de Santa Catarina, 89219-710, Joinville, SC, Brazil
‡Departamento de Física, Universidade Federal de São Carlos, 13565-905, São Carlos, SP, Brazil

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We show that the entanglement between the internal (spin) and external (position) degrees of freedom of a qubit in a random (disordered) one-dimensional discrete time quantum random walk (QRW) achieves its maximal possible value asymptotically in the number of steps, outperforming the entanglement attained using ordered QRW. The disorder is modeled by introducing an extra random aspect to QRW, a classical coin that randomly dictates which quantum coin drives the system’s time evolution. We also show that maximal entanglement is achieved independent of the initial state of the walker, study the number of steps the system must move to get infinitesimally close to its asymptotic limit, and propose two experiments where these ideas can be tested.

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I. INTRODUCTION

Imagine a one-dimensional lattice of regularly spaced points, where each point corresponds to the positions a classical particle can be found. Let us assume this particle moves right or left according to the result of a coin tossing game. The probability of obtaining heads (moves right) is \( p \) and of getting tails (moves left) is \( 1 - p \). This process is known as the one-dimensional discrete time classical random walk (CRW) [1] and we call it unbiased or symmetric if we deal with a fair coin (\( p = 1/2 \)) and biased otherwise.

Consider now that we have a qubit, a quantum particle that in addition to its external degrees of freedom (position and momentum) has a spin-1/2-like internal one (two level system). This internal degree of freedom can be the spin of an electron, the polarization of photons or the ground and excited states of an atom. For the sake of argument we assume the qubit is initially with spin up (\( |\uparrow\rangle \)) and at the origin (\( |j = 0\rangle \)) of the previous lattice. Now, whether it moves left or right is decided by the tossing of a “quantum coin”, where probability amplitudes rather than probabilities play the major role [2]. We first apply a unitary operation \( C \) (our quantum coin) acting only on the qubit’s internal degree of freedom, leaving it in an arbitrary superposition of spin up and down. We then apply another unitary operation \( S \) that correlates the displacement of the qubit to its internal degree of freedom. It moves right if the spin state at a given site is up and left otherwise. In this way we entangle the internal and external degrees of freedom of the system. Successive applications of the previous procedure lead to the discrete time evolution (displacement) of the qubit. After a certain number of steps the square of the absolute value of each amplitude gives the probability of seeing the particle with a certain spin state at a given position. This is what we call the one-dimensional discrete time quantum random walk (QRW) [2] [3].

The key difference between CRW and QRW is the superposition principle of quantum mechanics, a feature that is obviously lacking in CRW. The application of \( C \) followed by the displacement operator \( S \) at each step generates a cat-like state among all possible positions of the particle, setting the stage for interference effects to take place. The interference among the probability amplitudes manifests itself producing a position probability distribution \( P(j) \) drastically different from the classical one. Indeed, \( P(j) \) for the unbiased CRW is always peaked about the initial position and drops off exponentially with the distance (Gaussian distribution). Also, its variance \( \sigma^2 \) is proportional to the number \( n \) of steps (coins flipped). This is the diffusive behavior. For the unbiased QRW, however, \( P(j) \) is roughly uniform as we move away from the origin, having peaks far from it. Moreover, depending on the initial spin state we can have one peak at the left, or at the right, or two symmetrical peaks [3], and \( \sigma^2 \propto n^2 \), a quadratic gain (ballistic behavior) in the propagation of the particle when compared to CRW. Furthermore, due to the SU(2) structure of \( C \), we have an infinity number of coins to choose while classically there is only one coin.

Both CRW and QRW, in the one or higher dimensional versions, have many important applications. The former is employed from the modeling of biological processes [4] to polymer physics [5] and to the study of nonequilibrium processes [6]. The latter, still in its infancy, has proved to give important insights into the implementation of quantum search algorithms [7], the understanding of the physics of photosynthesis [8], the construction of a universal quantum computer [9], and the generation of entangled states for systems with one [10] and more walkers [11]. The entanglement generation in systems with one walker is our main focus here.

In the previous and many other applications of QRW one assumes that \( C \) is the same during all steps of the walk. What would happen, though, if noise, disorder, or fluctuations change \( C \) from one step to the other? What
would happen if $C$ changes randomly between two possible coins? A naive guess would suggest that all features of QRW may be washed out by such a process. Indeed, it is known that some typical features of QRW, such as $P(j)$ and $\sigma^2$, change in such random processes and approach the classical case \[13\]. However, so far no systematic study along this line was done for the entanglement content of the walker.

Our main goal here is to investigate such extra random aspect on a quantum walk (QW) and analyze whether or not it is detrimental to its entanglement generation capacity. And our main finding is, surprisingly, that the opposite from the naive guess occurs when it comes to entanglement generation using a disordered QW. We show that the entanglement, a genuine quantum feature, between the internal and external degrees of freedom of the walker is enhanced when $C$ changes from one step to the other in a truly random way. We also show that we achieve, asymptotically in the number of steps, a maximally entangled state. Moreover, we show that this effect is independent of the initial condition, contrary to standard entanglement generation schemes that rely critically on the initial state of the system and never achieve maximal entanglement \[11\]. It is worth mentioning that this initial state independence has important practical consequences and shows that the entanglement generation scheme here presented is robust against imperfections in the preparation of the initial state.

In order to explore these ideas we introduce a walker that combines the features of both the classical and quantum ones in a single formalism. It has two random ingredients, one of which is a classical coin similar to that of CRW. This coin dictates which quantum coin (the other random ingredient) will be used at each step of the walk. This is the essence of this walker and the presence of these two different random aspects, one classical and another quantum, leads us to call it a random quantum random walk (RQRW) process. It will be shown that CRW and QRW are two particular cases of RQRW.

II. MATHEMATICAL FORMALISM

We now pass to the introduction of the mathematical formalism of RQRW. First we note that the Hilbert space of RQRW is $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$, where $\mathcal{H}_C$ is a two-dimensional complex vector space associated to the spin states $\{|\uparrow\rangle, |\downarrow\rangle\}$ and $\mathcal{H}_P$ is an infinite but countable complex Hilbert space spanned by all integers. Its base is represented by the kets $|j\rangle$, $j \in \mathbb{Z}$, and they denote the position of the qubit on the lattice. With this notation we write an arbitrary initial state of the qubit (walker) as $|\Psi(0)\rangle = \sum_j (a(j,0)|\uparrow\rangle \otimes |j\rangle + b(j,0)|\downarrow\rangle \otimes |j\rangle)$, with $\sum_j |a(j,0)|^2 + |b(j,0)|^2 = 1$ being the normalization condition and $j$ running over all integers. This initial state could be entangled or not (separable). The time $t$ is “discrete” and it denotes the steps of the walker. In a $n$-step process the time changes from $t = 0$ to $t = n$ in increments of one and the walker’s state is $|\Psi(n)\rangle = U(n) \cdots U(1)|\Psi(0)\rangle = T \prod_{t=1}^n U(t)|\Psi(0)\rangle$, where $T$ denotes a time-ordered product, and

$$U(t) = S(C(t) \otimes 1_P).$$

(1)

Here $1_P$ is the identity operator that acts on the space $\mathcal{H}_P$, $C(t)$ the time-dependent quantum coin that changes at each step according to predefined rules that may or may not depend on $|\Psi(t-1)\rangle$, and $S$ the conditional displacement operator. The operator $S$ moves the qubit at site $j$ to the site $j + 1$ if its spin is up and to the site $j - 1$ if its spin is down and, using the present notation, it reads $S = \sum_j (|\uparrow\rangle \otimes |j + 1\rangle + |\downarrow\rangle \otimes |j - 1\rangle \langle j|$).

An arbitrary $C(t)$ is given by the most general way of writing an $SU(2)$ unitary transformation. Up to an irrelevant global phase we have $C(t) = c_{\uparrow\uparrow}(t) |\uparrow\rangle \langle \uparrow| + c_{\downarrow\downarrow}(t) |\downarrow\rangle \langle \downarrow| + c_{\uparrow\downarrow}(t) |\uparrow\rangle \langle \downarrow| + c_{\downarrow\uparrow}(t) |\downarrow\rangle \langle \uparrow|$, with $c_{\uparrow\uparrow}(t) = \sqrt{q(t)} \exp(i \phi(t))$, $c_{\downarrow\downarrow}(t) = \sqrt{1 - q(t)} \exp(i \phi(t))$, and $c_{\uparrow\downarrow}(t) = -\sqrt{q(t)} \exp(i \theta(t))$. Here $0 \leq q(t) \leq 1$, $0 \leq \theta(t) \leq \pi$, and $0 \leq \phi(t) \leq 2\pi$. The first parameter controls the bias of $C(t)$. For $q(t) = 1/2$ the coin creates an equal superposition of the spin states when acting on either $|\uparrow\rangle$ or $|\downarrow\rangle$ and an unbalanced one for $q(t) \neq 1/2$. The last two parameters control the relative phase between the two states in the superposition. Note that we are exploring the full $SU(2)$ structure of $C(t)$ with its three independent parameters, which makes it more general than the ones in \[13\]. Time-dependent walkers were also explored in \[14\], where instead of $C$, $S$ was made time-dependent, and in \[15\].

Looking at $C(t)$ we see that it changes at each step by how $q(t)$, $\theta(t)$, and $\phi(t)$ change with time. If they are constant in time we recover QRW. Moreover, assume the system’s initial state is $\rho$ with $|\psi\rangle$ an arbitrary spin state and let the initial coin be such that $C(0)|\psi\rangle = |\uparrow\rangle$. This can always be achieved since $C(t)$ is an arbitrary $SU(2)$ rotation. Using Eq. 1 we see that after the first step the particle moves right. Now, for $t \geq 2$ let $C(t)$ be chosen between two choices according to the result of a classical coin tossing in the following way. If one gets heads $C(t)$ is chosen such that the particle moves right and if one gets tails it is chosen such that the particle moves left. The first case is achieved by choosing $C(t) = 1_C$ ($\sigma_1$) if the spin state is $|\uparrow\rangle$ ($|\downarrow\rangle$) and the second one by choosing $C(t) = 1_C$ ($\sigma_1$) if the spin state is $|\downarrow\rangle$ ($|\uparrow\rangle$), where $\sigma_1$ is the spin flip operator ($q(t) = \theta(t) = \phi(t) = 0$). It is not difficult to see that this is an exact simulation of CRW and a proof that it is a particular case of RQRW.

The general time evolution can be obtained applying $U(t)$, Eq. 1, to an arbitrary state at time $t - 1$. This leads to $|\Psi(t)\rangle = U(t)|\Psi(t - 1)\rangle = \sum_j (a(j,t)|\uparrow\rangle \langle j| + b(j,t)|\downarrow\rangle \langle j|)$, where

$$a(j,t) = c_{\uparrow\uparrow}(t)a(j - 1, t - 1) + c_{\downarrow\downarrow}(t)b(j - 1, t - 1),$$

$$b(j,t) = c_{\uparrow\downarrow}(t)a(j + 1, t - 1) + c_{\downarrow\uparrow}(t)b(j + 1, t - 1).$$

(2)

We will focus here on two types of RQRW (see Fig. 1). The first one deals with only two quantum coins,
$C_1$ and $C_2$. At each step of the walk the decision to use $C_1$ or $C_2$ is made by the result of a classical coin. If we get heads at step $t$ we use $C_1$ and if we get tails we use $C_2$. A typical realization of this walk looks like $|\Psi(t)\rangle = U_1(t)U_2(t-1)\cdots U_2(3)U_1(2)U_1(1)|\Psi(0)\rangle$, where $U_j(t) = S(C_j \otimes 1_P)$, $j = 1, 2$. We call this process a $RQRW_2$, with the subindex denoting that our choices are made randomly between two quantum coins.

![FIG. 1: (color online) (a) The dashed line represents a possible realization of CRW, where no superposition occurs. The solid curves represent probability amplitudes for $RQRW_2$, where only two $C(t)$ are allowed (red and blue discs). (b) Schematic view for $RQRW_\infty$, where $C(t)$ are chosen randomly from uniform continuous distributions of quantum coins (at each step a different color/coin is used). Note that at each step all coins/colors are the same (dynamical disorder).](image)

In the second $RQRW$ we have an infinite number of $C(t)$ to choose at each step. The independent parameters of $C(t)$, namely, $q(t)$, $\theta(t)$, and $\varphi(t)$, are chosen from continuous uniform distributions spanning the range of their allowed values. Note that we can have a walk where either one, or two or all parameters change at each step. We call such walks $RQRW_\infty$.

III. ENTANGLEMENT

Since $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ is pure we quantify the entanglement between the internal and external degrees of freedom by the von Neumann entropy of the partially reduced state $\rho_C(t) = Tr_P(\rho(t)) = -Tr(\rho_C(t) \log_2 \rho_C(t))$, with $Tr_P(\cdot)$ being the trace over the position degrees of freedom. $S_E$ is $0$ for separable states and $1$ for maximally entangled ones. Since $\rho_C(t) = \alpha(t)|\uparrow\rangle|\uparrow\rangle + \beta(t)|\downarrow\rangle|\downarrow\rangle + \gamma(t)|\downarrow\rangle|\uparrow\rangle + \gamma^*(t)|\uparrow\rangle|\downarrow\rangle$, with $\alpha(t) = \sum_j a(j, t)\beta(t) = \sum_j b(j, t)^2$, $\gamma(t) = \sum_j a(j, t)b^*(j, t)$, and $z^*$ is the complex conjugate of $z$, we have $S_E(\rho(t)) = -\lambda_+|\log_2 \lambda_+ - 1| - \lambda_-|\log_2 \lambda_- - 1|$, with $\lambda_\pm = (1/2 \pm \sqrt{1 - 4 - \alpha(t)(1 - \alpha(t)) + |\gamma(t)|^2})$ being the eigenvalues of $\rho_C(t)$.

IV. RESULTS

We start studying two typical representatives of $RQRW$. The first one is $RQRW_2$ with $C_1$ being the Hadamard ($H$) coin ($q(t) = 1/2, \theta(t) = \varphi(t) = 0$) and $C_2$ the Fourier/Kempe ($F$) coin ($q(t) = 1/2, \theta(t) = \varphi(t) = \pi/2$). These are unbiased quantum coins ($q(t) = 1/2$) generating equal superpositions of the spin states when acting on either $|\uparrow\rangle$ or $|\downarrow\rangle$. Note that the latter coin introduces a $\pi/2$ relative phase between $|\uparrow\rangle$ and $|\downarrow\rangle$.

The other walk is $RQRW_\infty$, where at each step the values of $q(t), \theta(t)$ and $\varphi(t)$ are chosen randomly from three distinct continuous uniform distributions.

![FIG. 2: (color online) $\langle S_E \rangle$ was computed averaging over 16,384 initial conditions of the following form, $|\Psi(0)\rangle = (\cos \alpha_0|\uparrow\rangle + e^{i\beta_0}\sin \alpha_0|\downarrow\rangle) \otimes (\cos \alpha_p - 1) + e^{i\beta_p}\sin \alpha_p|1\rangle$, where $\alpha_{s,p} \in [0, \pi]$ and $\beta_{s,p} \in [0, 2\pi]$. The first realization used the initial condition $(\alpha_s, \beta_s, \alpha_p, \beta_p) = (0, 0, 0, 0)$ and the subsequent ones all quadruples of points in independent increments of $0.4$ until $\alpha_{s,p} = \pi$ and $\beta_{s,p} = 2\pi$. We worked with a 400-step walk. The blue/square curve represents $RQRW_\infty$, the red/circle curve $RQRW_2$, and the green/solid one the Hadamard $QRW$. The left inset shows the average probability distribution ($\langle P(j) \rangle$) to find the qubit at a given place after 400 steps for each one of the walks and the right one shows the average dispersion ($\langle \sqrt{\tau^2} \rangle$). The black/dashed curves represent the expected results for CRW. Note that $\langle P(j) \rangle$ for $QRW$ are similar to CRW with $RQRW_\infty$ being more localized than $RQRW_2$. The two spikes on $\langle P(j) \rangle$ for $RQRW_2$ is due to the fact that it is built on the Hadamard and Fourier coins, where these spikes are a common trend.

In order to investigate the dependence of the asymptotic behavior of $S_E$ on initial conditions, we run several thousands numerical experiments, each of which with a different initial condition. Each realization of the walk gives at step $t$ a value for $S_E$ and in Fig. 2 we show the average values of $S_E$ over all realizations at each step $t$.

As can be seen from Fig. 2 the average entanglement $\langle S_E \rangle$ approaches the maximal value possible ($S_E = 1$) for both $RQRW$ cases after a few hundreds steps. For comparison, we show the usual $QRW$ with a Hadamard coin, where clearly $\langle S_E \rangle \neq 1$ asymptotically. Indeed, for the ordered case the asymptotic value of $S_E$ is highly sensitive to the initial conditions and the set of initial states giving high values of $S_E$ is not dense. An important example is the Hadamard walk, where it can be shown [11] that the asymptotic values of $S_E$ continuously oscillate between $S_E = 0.661$ and $S_E = 0.979$ as we cover a set of initial conditions similar to the ones in Fig. 2.
Now, since $S_E$ is bounded from above by one, $\langle S_E \rangle \to 1$ implies that for RQRW the set of initial states in which $S_E \to 1$ asymptotically is dense. In other words, this suggests that in the asymptotic limit $S_E \to 1$ for any initial condition. This leads us to state and prove in Appendix A the following theorem, which justifies the last assertion.

**Theorem.** In the asymptotic limit and for any initial condition, $S_E \to 1$ if the quantum coin acting on the walker at each step is a random $SU(2)$ unitary operator.

To gain further insights into the asymptotic limit of $S_E$ we run another set of numerical experiments for the three walks described in Fig. 2, but now going up to 1000 steps and also counting the number of initial conditions leading to high values of $S_E$. Looking at Fig. 3 it is clear that $S_E \to 1$ for RQRW$_\infty$ and RQRW$_2$ while the Hadamard QRW asymptotic entanglement is highly sensitive to the initial conditions.

![FIG. 3](image-url)  
*FIG. 3: (color online) $\langle S_E \rangle$ was got averaging over 2,016 localized initial conditions given as $|\Psi(0)\rangle = (\cos \alpha_\iota |\uparrow\rangle + e^{i\beta_\iota} \sin \alpha_\iota |\downarrow\rangle) \otimes |0\rangle$, where $\alpha_\iota, \beta_\iota \in [0, \pi]$ and $\beta_\iota \in [0, 2\pi]$. The first realization used the initial condition $(\alpha_\iota, \beta_\iota) = (0, 0)$ and the next ones all pairs of points in independent increments of 0.1 until $\alpha_\iota = \pi$ and $\beta_\iota = 2\pi$. We worked with a 1000-step walk. The blue/square curve represents RQRW$_\infty$, the red/circle curve RQRW$_2$, and the green/solid one the Hadamard QRW. The inset shows the rate of initial conditions leading to $S_E > 0.95$ at step 1000. Note that for RQRW (middle/blue and right/red bars) almost 100% of the initial conditions lead to $S_E > 0.97$ while for QRW (left/green bar) this occurs for less than 10%.*

We have also investigated numerically other RQRW, some of them not covered by the theorem, and how much disorder we must have to achieve $S_E \to 1$ asymptotically. As can be seen in Fig. 4 weak disorder is sufficient to generate highly entangled states for arbitrary initial conditions in a variety of RQRW. In Appendix B we numerically study other interesting RQRW where maximal entanglement is achieved and give further details about the probability distribution of the walker and its dispersion properties. We have also investigated in Appendix B highly non-local initial conditions (Gaussian distributions) and studied how the asymptotic limit $S_E \to 1$ is approached.

**V. EXPERIMENTAL PROPOSALS**

Current technology allows one to implement in at least two ways the previous walks. The first one is based on passive optical elements where the internal degree of freedom of the walker is the polarization of a photon and the position/external one is mapped to different arrival times of the photon at the photodetector (time bins). The random coin operator $C(t)$ is implemented using a half-wave plate (HWP) and a phase-shifter while the conditional displacement operator $S$ by two polarizing beam-splitters (PBS) and a fiber delay line where one polarization follows a longer optical path. Also, arbitrary initial conditions are simply generated by a quarter-wave plate (QWP) and HWP. An important feature of the scheme given in [17] is its scalability with the number of steps. Indeed, the techniques of optical feedback loop allow the implementation of a many-step walk using a few optical elements. In [17] the authors have already implemented static and dynamical disordered walks of 28 steps.

The second way also uses photons as walkers but it is based on integrated photonics, where a disordered walk is built on integrated waveguide circuits, providing...
perfect phase stability. By using state-of-the-art femtosecond laser writing techniques, the authors in [18] were able to wrought an array of interferometers in a glass that reproduces the dynamics of RQRW. In integrated waveguide circuits PBS is a directional coupler and phase shifts are implemented writing circuits with different length/deformation.

To test the ideas here presented we need to measure the entanglement of the walker, which is obtained if we know the coin state \( \rho_C(t) \). But \( \rho_C(t) \) can be experimentally measured by slightly changing the two schemes outlined above. Indeed, since a general photon polarization state is written as \( \rho_C(t) = 1_C + \sum_j r_j \sigma_j \), with \( \sigma_j \) being Pauli matrices, we can determine \( \rho_C(t) \) if we measure \( r_j \). But this is achieved by measuring the average polarization of the photon in the vertical/horizontal axis \( (r_3) \), in the \( \pm 45^\circ \) axis \( (r_1) \), and the average right/left circular polarization \( (r_2) \) [19]. These measurements can be easily implemented by properly arranging HWP and QWP before the photon passes a PBS with photodetectors at each one of its arms. In Appendix C we show that just a few steps are enough to have different predictions for the behavior of \( S_E \) if we work with either RQRW\(_2\) or RQRW.

VI. CONCLUSION

We defined the random quantum random walk (RQRW), a discrete time quantum random walk scheme whose unitary evolution at each step is chosen randomly using a two-sided (or infinitely-sided) classical coin. We showed that both the usual classical and quantum random walks are particular cases of RQRW. We then studied its entanglement generation capacity. We showed that RQRW creates maximally entangled sates in the asymptotic limit for several types of dynamical disorder (random time evolution), contrary to the ordered QRW. Furthermore, and surprisingly, we proved that RQRW entanglement creation capabilities is independent of the initial condition of the walker, another property in contrast to ordered QRW.

Finally, we would like to point out that our findings naturally lead to new important questions. For example, what is the interplay between order/disorder and entanglement creation for two- or three-dimensional walkers? What would happen to the entanglement for static disorder \( \chi, [17, 18, 21, 22] \)? Can the previous results be adapted to the case of two or more walkers to improve the creation of bipartite and multipartite entanglement, respectively, only among the internal degrees of freedom? We believe investigations along these lines may bring other unexpected and intriguing results and foster the development of new entanglement generation protocols.

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Appendix A: Proof of Theorem

We want to prove the following theorem:

Theorem. In the asymptotic limit and for any initial condition, \( S_E \to 1 \) if the quantum coin acting on the walker at each step is a random \( SU(2) \) unitary operator.

Before we start, we must clarify what we mean by “the asymptotic limit”. It is the time \( t_\text{as} \) after which the density matrix \( \rho(t) = |\Psi(t)\rangle \langle \Psi(t) | \) describing the system assumes a stationary value. Here \( |\Psi(t)\rangle = \sum_j (a(j, t)|\uparrow\rangle|j\rangle + b(j, t)|\downarrow\rangle|j\rangle) \). This implies that all observables no longer change with time. Mathematically, this means that for any \( t \geq t_\text{as} \), we have \( \rho(t+1) = \rho(t) \). Therefore, since \( \rho_C(t) = Tr_P(\rho(t)) \) we also have \( \rho_C(t+1) = \rho_C(t) \) for \( t \geq t_\text{as} \). But since

\[
\rho_C(t) = \begin{pmatrix} \alpha(t) & \gamma(t) \\ \gamma^*(t) & \beta(t) \end{pmatrix},
\]

where \( \alpha(t) = \sum_j |a(j, t)|^2, \ \beta(t) = \sum_j |b(j, t)|^2 \), and \( \gamma(t) = \sum_j a(j, t)b^*(j, t) \), we have in the asymptotic limit \( \alpha(t+1) = \alpha(t), \ \beta(t+1) = \beta(t) \), and \( \gamma(t+1) = \gamma(t) \).

For any initial condition, the time evolution of the system for a given RQRW is given by

\[
a(j, t+1) = c_{\uparrow\uparrow}(t+1)a(j-1, t) + c_{\downarrow\uparrow}(t+1)b(j-1, t),
b(j, t+1) = c_{\uparrow\downarrow}(t+1)a(j+1, t) + c_{\downarrow\downarrow}(t+1)b(j+1, t),
\]

(A1)

where different RQRW’s are obtained changing the way \( c_{kl}(t), k, l = \uparrow, \downarrow \), evolves with time.

Using Eq. (A1) we have

\[
\alpha(t+1) = \sum_j a(j, t+1)a^*(j, t+1)
\]

\[
= |c_{\uparrow\uparrow}(t+1)|^2 \alpha(t) + |c_{\downarrow\uparrow}(t+1)|^2 \beta(t) + 2\text{Re}[c_{\uparrow\uparrow}(t+1)c_{\downarrow\uparrow}^*(t+1)\gamma(t)],
\]

(A2)

where \( \text{Re}[z] \) is the real part of the number \( z \). Now, employing the unitarity of the coin \( |c_{\uparrow\uparrow}(t+1)|^2 + |c_{\downarrow\uparrow}(t+1)|^2 = 1 \) and the normalization condition \( \alpha(t) + \beta(t) = 1 \) we have

\[
\alpha(t+1) = |c_{\downarrow\downarrow}(t+1)|^2 + (1 - 2|c_{\downarrow\uparrow}(t+1)|^2) \alpha(t) + 2\text{Re}[c_{\uparrow\uparrow}(t+1)c_{\downarrow\uparrow}^*(t+1)\gamma(t)].
\]

(A3)
Finally, using the assumption that we are in the asymptotic limit \( \alpha(t+1) = \alpha(t) \) we obtain
\[
\alpha(t) = \frac{1}{2} + \text{Re}\left[ \frac{c_{t+1}(t + 1)}{c_{t+1}(t + 1)} \gamma(t) \right]. \tag{A4}
\]

Similarly, starting with \( \beta(t+1) \) in Eq. (A2) we get
\[
\beta(t) = \frac{1}{2} + \text{Re}\left[ \frac{c_{t+1}(t + 1)}{c_{t+1}(t + 1)} \gamma(t) \right]. \tag{A5}
\]

Invoking again that we are in the asymptotic limit, \( \alpha(t+1) = \alpha(t) \), we have after inserting Eq. (A3) in the previous relation,
\[
\text{Re}\left[ \left( \frac{c_{t+1}(t + 2)}{c_{t+1}(t + 2)} - \frac{c_{t+1}(t + 1)}{c_{t+1}(t + 1)} \right) \gamma(t) \right] = 0, \tag{A6}
\]
where we have used that \( \gamma(t+1) = \gamma(t) \) to arrive at the last equality. Since we are dealing with random \( SU(2) \) unitary matrices, the term inside the parenthesis above is a random complex number \( z(t) = x(t) + i y(t) \), with \( x(t) \) and \( y(t) \) random reals. Writing \( \gamma(t) = \text{Re}[\gamma(t)] + i \text{Im}[\gamma(t)] \), Eq. (A6) implies
\[
x(t)\text{Re}[\gamma(t)] - y(t)\text{Im}[\gamma(t)] = 0. \tag{A7}
\]

We can repeat the previous argument starting with \( \alpha(t + 2) = \alpha(t) \). Using the asymptotic assumption that \( \gamma(t + 2) = \gamma(t + 1) = \gamma(t) \), for any \( t \geq t_s \) leads to
\[
x(t + 1)\text{Re}[\gamma(t)] - y(t + 1)\text{Im}[\gamma(t)] = 0. \tag{A8}
\]

Now, Eqs. (A7) and (A8) constitute a homogeneous system of linear equations on the variables \( \text{Re}[\gamma(t)] \) and \( \text{Im}[\gamma(t)] \). We can only achieve a non-trivial solution if the determinant of its coefficients are zero, i.e., if \( x(t)y(t + 1) - y(t)x(t + 1) = 0 \). But since \( x(t), y(t), x(t + 1), \) and \( y(t + 1) \) are four completely random real numbers, \( x(t)y(t + 1) \neq y(t)x(t + 1) \). Hence, Eqs. (A7) and (A8) imply
\[
\text{Re}[\gamma(t)] = \text{Im}[\gamma(t)] = 0 \rightarrow \gamma(t) = 0, \tag{A9}
\]
and consequently (see Eqs. (A4) and (A5))
\[
\alpha(t) = \beta(t) = 1/2. \tag{A10}
\]

Finally, inserting the asymptotic values of \( \alpha(t), \beta(t) \), and \( \gamma(t) \) just computed into the expression for \( \rho_C(t) \) we get
\[
\rho_C(t) = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},
\]
which immediately leads to the maximal allowed value for the entanglement, \( S_E = -\text{Tr}[\rho_C(t) \log_2 \rho_C(t)] = 1. \)

Remark 1: Actually, after we achieve the asymptotic limit (\( t \geq t_s \)) we have an arbitrary number of equations similar to Eqs. (A7) and (A8). Each additional step gives another one, all of which must be zero. So, even in the rare situation where the random reals \( x(t), y(t), x(t + 1), y(t + 1) \) non-trivially solve Eqs. (A7) and (A8), we will always find a quadruple of reals \( x(\tau), y(\tau), x(\tau + 1), x(\tau + 1) \), \( \tau \geq t_s \), where they will not solve the two equations and, therefore, the arguments leading to the proof must apply.

Remark 2: The previous proof also applies whenever we have a quantum coin \( C(t) \) with at least \( \theta(t) \) random; and also with only \( q(t) \) random and with \( \theta(t) \) not zero or a multiple of \( \pi/2 \). To see that, let us rewrite here the quantum coin,
\[
C(t) = \begin{pmatrix} c_{t+1}(t) & c_{t+1}(t) \\ c_{t+1}(t) & c_{t+1}(t) \end{pmatrix}, \tag{A11}
\]
with \( c_{t+1}(t) = \sqrt{q(t)} \), \( c_{t+1}(t) = \sqrt{1 - q(t)}e^{i\theta(t)} \), \( c_{t+1}(t) = \sqrt{1 - q(t)}e^{i\varphi(t)} \), and \( c_{t+1}(t) = -\sqrt{q(t)}e^{i(\theta(t) + \varphi(t))} \). Here \( 0 \leq q(t) \leq 1, 0 \leq \theta(t) \leq \pi \), and \( 0 \leq \varphi(t) \leq 2\pi \).

If we take the case where at least \( \theta(t) \) changes randomly with time, with \( \varphi(t) \) or \( q(t) \) changing or not, we see that \( c_{t+1}(t)/c_{t+1}(t) = \sqrt{q(t)}/(1 - q(t))e^{-i\theta(t)} \) is a random complex number due to the randomness of \( \theta(t) \). Therefore, Eq. (A4) has the same properties as in the original proof and implies Eqs. (A6). Eqs. (A4) and (A5) have complex numbers multiplying \( \gamma(t) \) at each step. By this simple argument we have proved that in the asymptotic limit \( S_E \rightarrow 1 \) for four cases: all parameters randomly changing with time (the original proof), \( \theta(t) \) and \( \varphi(t) \) random with \( q(t) \) fixed, \( \theta(t) \) and \( q(t) \) random with \( \varphi(t) \) fixed, and \( \theta(t) \) random with both \( \varphi(t) \) and \( q(t) \) fixed.

For \( q(t) \) random and \( \theta(t) \) fixed but not zero or a multiple of \( \pi/2 \) we have \( c_{t+1}(t)/c_{t+1}(t) = \sqrt{q(t)}/(1 - q(t))e^{-i\theta(t)} \), which due to the randomness of \( q(t) \) is a random complex number and the proof follows. Note that if \( \theta(t) = 0, \theta(t) = \pi/2 \), or \( \theta(t) = \pi \), the proof cannot be carried out to its completion. In the first and third cases \( c_{t+1}(t)/c_{t+1}(t) \) is a random real and following the steps of the proof leads only to \( \text{Re}[\gamma(t)] = 0 \); nothing can be said about the imaginary part of \( \gamma(t) \). And in the second case \( c_{t+1}(t)/c_{t+1}(t) \) is a random pure imaginary leading to \( \text{Im}[\gamma(t)] = 0 \), while nothing can be said about the real part.

Remark 3: The structure of the proof does not allow us to reach any conclusion when only \( \varphi(t) \) changes. Noting that \( c_{t+1}(t)/c_{t+1}(t) = \sqrt{q(t)}/(1 - q(t))e^{-i\theta(t)} \) and \( c_{t+1}(t)/c_{t+1}(t) = -\sqrt{q(t)}/(1 - q(t))e^{i\theta(t)} \), we see that they are both complex constants if only \( \varphi(t) \) changes. Thus, since we do not have random complex numbers multiplying \( \gamma(t) \) in either Eq. (A4) or (A5), the proof cannot follow.

Remark 4: Although the proof here cannot be extended to some particular cases that do not explore the full \( SU(2) \) structure of the coin \( C(t) \), numerical simulations (see the main text and Sec. 3) suggest that if we have at least one of the three independent parameters of \( C(t) \) random, we obtain \( S_E \rightarrow 1 \) asymptotically. Also, we found no analytical proof that the binary randomness
of the balanced or unbalanced $RQRW_2$ leads to maximal entanglement asymptotically. However, extensive numerical analysis showed that this is true (see main text and Sec. B).

Remark 5: Finally, the proof does not tell us when the system will approach the asymptotic limit. It may happen for a few hundreds steps or we may need several thousands or more steps. As we show in Sec. B the more delocalized in position the initial condition the more steps we need.

Appendix B: More numerical results

1. Several random initial conditions

Here we give more details about RQRW’s depicted in Fig. 4 of the main text. In addition to the average entanglement $\langle S_E \rangle$ given in the main text, we now present the rate at which highly entangled states are generated, the average probability distribution for finding the qubit at a given position, and the dispersion for these RQRW’s.

The first thing worth mentioning is the robustness of these RQRW’s to generate highly entangled states, even if we have random coins very close to the Hadamard coin, i.e., even for not too big deviations from the values of $q(t) = 1/2$ and $\theta(t) = \varphi(t) = 0$ that characterize the H coin. Also, $RQRW_\infty$ (panel b) of Fig. 5 has higher generation rates of highly entangled states than the other two $RQRW_\infty$ (see panels b) of Figs. 6 and 7. And if we look at Fig. 5b, we see that $RQRW_2$ with H and F coins is the most efficient entanglement generator. It can produce highly entangled states for almost any initial condition within just a few hundreds steps, even for a very tiny deviation from the Fourier QRW ($p = 0.05$).

We also note that the average probability distribution $\langle P(j) \rangle$ and the dispersion possess a wide range of behaviors. However, there is a common trend for all RQRW’s: the more we deviate from a fixed coin, the more it approaches the classical case. Also, among all $RQRW_\infty$ cases, the one in Fig. 5 gives the smallest peaks far from the origin (Fig. 5c) while the other two cases exhibit the greatest symmetric peaks away from it. Furthermore, the weaker the disorder the greater the peaks. This is expected since for weaker disorder we are approaching the fixed coin case, where these peaks are a common trend.

2. Delocalized Gaussian initial conditions

We now study the ability of RQRW’s against the standard QRW to entangle the external and internal degrees of freedoms of the walker for highly non-local/delocalized initial conditions in position. We will work with $RQRW_\infty$, with $q(t), \theta(t)$, and $\varphi(t)$ completely random, and the Hadamard QRW.

The several initial positions of the qubit are given by

$$\psi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}},$$

where $\sigma^2$ is the variance. The global initial state is there-
We work with the discretized and normalized version of the Gaussian distribution. In what follows we will be working with two types of initial spin states, $|\uparrow\rangle$ and $|\downarrow\rangle$, and several Gaussians with dispersions ($\sigma$) as given in Fig. 9.

Therefore,

$$\Psi(0) = |\xi\rangle \otimes \sum_{j=-\infty}^{\infty} \psi(j)|j\rangle,$$

where we work with the discretized and normalized version of the Gaussian distribution. In what follows we will be working with two types of initial spin states, $|\uparrow\rangle$ = $(|\uparrow\rangle + i|\downarrow\rangle)/\sqrt{2}$ and $|\xi_2\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$.

In Fig. 10 we plot the entanglement for both walks starting with the previous initial conditions. It is clear that the asymptotic entanglement of the Hadamard QRW is highly sensitive to initial conditions and many do not lead to maximal entanglement. For $RQRW_{\infty}$, we see the independence on the initial state for the asymptotic value of entanglement, although the rate at which it is approached depends on the broadness of the initial probability distribution for the position of the particle. Indeed, as we show in Fig. 11 for $\sigma = 10$ the walker needs to travel about 6000 steps to achieve $\langle S_E \rangle > 0.9$. For just 1500 steps, we have $\langle S_E \rangle \approx 0.8$.

Finally, in Figs. 12 and 13 we show the probability distribution after 1500 steps for walkers starting with the initial conditions given in Fig. 9 for the Hadamard QRW and $RQRW_{\infty}$, respectively.

**Appendix C: Experimental predictions**

The two experiments 17, 18 on which our experimental proposal is built achieve so far a few tens of steps. In particular, in 17 RQRW’s with 28 steps were implemented. Our goal here is, therefore, to show that with just tens of steps we already have different predictions for the entanglement $S_E$ whether we implement the standard QRW or RQRW.

To build our experimental proposal, we started with the localized initial condition $\Psi(0) = (\cos \alpha_s |\uparrow\rangle + e^{i\beta_s} \sin \alpha_s |\downarrow\rangle) \otimes |0\rangle$ and searched in increments of 0.1 for the pair of points $(\alpha_s, \beta_s)$ giving the lowest $S_E$ for the Hadamard QRW at step 28. We found $(\alpha_s, \beta_s) = (2.7, \pi)$ with $S_E = 0.645$. By keeping $\alpha_s = 2.7$ and changing $\beta_s$ we get several different values for $S_E$ at step 28 for the Hadamard QRW: $(\alpha_s, \beta_s) = (2.7, \pm \pi) \rightarrow S_E = 0.645$, $(\alpha_s, \beta_s) = (2.7, 0) \rightarrow S_E = 0.983$, and $(\alpha_s, \beta_s) = (2.7, \pm \pi/2) \rightarrow S_E = 0.869$. All these initial conditions can be easily prepared with half (HWP) and quarter wave plates (QWP).
Then, with the initial condition giving the lowest $S_E$ for the Hadamard QRW, we implemented 10,000 numerical experiments using the balanced $RQRW_2$ with the Hadamard (H) and Fourier/Kempe (F) coins, search-

![Figure 10](image1.png)

**FIG. 10:** (color online) Upper panel: The solid curves represent the entanglement for Gaussian initial conditions with spin state $|\xi_1\rangle$ and the dashed ones with $|\xi_2\rangle$. We are working with the Hadamard QRW. Note that the asymptotic entanglement is extremely sensitive to initial conditions, where we can see it approaching a wide range of possible values. Lower panel: The same initial conditions as above but now we work with $RQRW_\infty$. The solid and dashed curves are almost the same, so we employ up triangles to the cases starting with $|\xi_1\rangle$ and down triangles to those starting with $|\xi_2\rangle$. We now see that all cases approach the maximal entanglement value $<S_E> = 1$, although the greater the initial dispersion $\sigma$ the slower the rate at each the maximal value is approached. For $RQRW_\infty$ we implemented 500 different realizations and plotted the average entanglement.

![Figure 11](image2.png)

**FIG. 11:** (color online) We implement 100 realizations of $RQRW_\infty$ up to 6000 steps. Up triangles refer to $<S_E>$ for the cases starting with $|\xi_1\rangle$ and down triangles to $|\xi_2\rangle$. Now we obtain $<S_E> > 0.9$ for a Gaussian with $\sigma = 10$.

![Figure 12](image3.png)

**FIG. 12:** (color online) The upper panel shows the probability distributions $P(j)$ after 1500 steps for initial conditions given by Gaussians with dispersion $\sigma$ and initial spin state $|\xi_1\rangle$ while the lower panel shows the cases with spin state $|\xi_2\rangle$. In both cases we have the Hadamard QRW. Note that $|\xi_1\rangle$ produces symmetrical peaks while $|\xi_2\rangle$ asymmetrical ones. The wider the initial conditions the wider the peaks.

![Figure 13](image4.png)

**FIG. 13:** (color online) The upper panel shows the average probability distributions $\langle P(j) \rangle$ over 500 realizations after 1500 steps for initial conditions given by Gaussians with dispersion $\sigma$ and initial spin state $|\xi_1\rangle$. The lower panel shows the cases with spin state $|\xi_2\rangle$. In both cases we have $RQRW_\infty$. Now all spin initial conditions give similar centralized peaks. The dashed line shows the expected classical $P(j)$.
giving \( S_E > 0.97 \). In contrast, the Hadamard QRW has only one initial condition giving \( S_E > 0.93 \) and two of them giving \( S_E < 0.65 \). This huge contrast can be experimentally detected with current day technology.

The sequence of H’s and F’s leading to such predictions is (with time flowing from left to right),

\[
HHFHFFFHFFHFHFFHHFHFHFH,
\]

and it can be implemented in \[17\] by adjusting the phase-shifter before the passage of the photon to the HWP that generates the standard coin and in \[18\] by writing the integrated waveguide circuit with the correct optical path differences between successive directional couplers (the equivalent of polarizing beam splitters).

In Fig. 14 we show the entanglement time evolution for both the Hadamard \( QRW \) and \( RQRW_2 \) for all the previous five initial conditions.

The same analysis can be carried out to other RQRW’s in order to find a sequence of random coins that clearly gives different predictions for RQRW and QRW within just a few steps. In the asymptotic limit, of course, any random sequence will do.

FIG. 14: (color online) In the top panel we have the entanglement evolution for the Hadamard \( QRW \) and in the bottom panel its evolution for the \( RQRW_2 \) with a sequence of H and F coins as given in the text. Note that the entanglement at step 28 for all initial conditions clusters around 0.9 for \( RQRW_2 \) while it appreciably differs for different initial conditions for the Hadamard \( QRW \). Also, note that for QRW the entanglement evolves oscillating periodically about its asymptotic value. For \( RQRW_2 \) this periodicity is lost.

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[21] Note: After completion of this work we became aware of Ref. [20], where the author showed numerically that dynamic disorder may be better than static disorder to enhance the entanglement. He worked, however, with only one initial condition and only with $RQRW_\infty$ where $q(t), \theta(t)$, and $\phi(t)$ are fully random. A systematic study for several initial conditions and other random coins is needed to check if this fact is not a peculiarity of that initial condition and that particular RQRW.