Topological Interpretation of Function Spaces Stable under a General Operation

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Abstract

Function (linear) spaces $E$ on which some arbitrary real or complex continuous function operates (in other words, $E$ is stable under the unary operation defined by this function) were studied by K. de Leeuw, Y. Katznelson, Y. Sternfeld and Y. Weit, showing, e.g. that for a complex function space $E$ consisting of bounded functions, containing the constants and closed w.r.t. the sup-norm, if some non-affine continuous function operates on $E$ then every analytic function operates on $E$: $E$ is an algebra, and if further some non-analytic continuous function (say, complex conjugation) operates on $E$ then any continuous function does so: $E$ is a self-adjoint algebra.

In this note we approach this issue from a different point of view. Our scalars are any field $K$, and the functions are defined on an abstract set $I$, with both $K$ and $I$, at the outset, without a topology. It turns out that the very fact that the function space $E$ is stable w.r.t. a non-affine operation (no continuity required) allows it to induce a topology on $I$, while the space of functions which operate on it induces topologies on $K^n$, $n = 1, 2, \ldots$. The topologized $I$ is used to prove a density property for such $E$. Also, these topologies allow us, in some cases, to investigate general “homomorphisms” between function spaces w.r.t. to non-affine operations under which they are stable.

0 Definitions

Let $K$ be a field. We refer to any $f : K^n \to K$ as an operation and say that it is additively affine if it can be written as a sum of a constant and a homomorphism of the additive groups.

Let $I$ be a set, and let $E \subset K^I$ be a (linear) subspace (a function space). We call $E$ unital if it contain the constant functions, which we always assume. Let $f : K^n \to K$ be an operation. We say that $E$ is stable w.r.t. the operation $f$ or that $f$ operates on $E$, if for any $\xi_1, \ldots, \xi_n \in E$ also $t \in I \mapsto f (\xi_1(t), \ldots, \xi_n(t))$ is in $E$. We say that $E$ is stable w.r.t. a set of functions (operations) if it is stable w.r.t. each member.

We shall say that $E$ separates points in $I$ if for any different $t_1, t_2 \in I$ and any $a_1, a_2 \in K$ \( \exists \) a $\xi \in E$ with $\xi(t_1) = a_1, \xi(t_2) = a_2$. Equivalently, for no different $t_1$ and $t_2$, $\xi(t_1)$ and $\xi(t_2)$ are proportional for the $\xi \in E$. 

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1 Function spaces stable under a non-additively-affine operation induce a topology

We try to make $E$ define a topology on $I$. Call a subset $S \subset I$ closed if for any $t \in I \setminus S \exists$ a $\xi \in E$ vanishing on $S$ but not at $t$. It is clear that $\emptyset$ and $I$ are closed and that any intersection of closed sets is closed (if $t \notin \cap S_\alpha$ then $t \notin S_\alpha$ for some $\alpha$ and a $\xi$ as above for $S_\alpha$ will serve also for $\cap S_\alpha$). Closed sets can also be defined as sets of solutions of (possibly infinite) sets of equations of the form $\xi(t) = 0$ with $\xi \in E$.

**Definition 1** Suppose the subspace $E \subset K^I$ is unital, i.e. contains the constants. We say that $E$ is topology-inducing if the union of any two closed subsets of $I$ (defined as above) is closed, so one has a topology on $I$, which we call the topology induced by $E$.

Note that if $E$ is topology-inducing and separates points, then the induced topology is $T_1$ (i.e. every singleton is closed).

Note also that a general subspace need not be topology-inducing. For example, for the hyperplane

$$E = \left\{ \xi \in K^{\{1, \ldots, n\}} \mid \sum_{i=1}^{n} \lambda_i \xi_i = 0 \right\}$$

with all $\lambda_i \neq 0$ and $\sum_i \lambda_i = 0$ (making $E$ unital), the closed sets are the subsets $S \subset \{1, \ldots, n\}$ whose number of elements is different from $n - 1$.

Of course, for $I = K^n$, $E = \{\text{the polynomials}\}$ is topology-inducing and we obtain the Zariski topology, while for $K = \mathbb{R}$ (the reals), $I$ a Hausdorff compact topological space and $E = \{\text{the continuous functions}\}$ we obtain the original topology on $I$.

The following assertion is immediate

**Proposition 1** If $E$ is topology-inducing on $I$ and $I' \subset I$, then the space of all the restrictions of members of $E$ to $I'$ is topology-inducing and induces on $I'$ the relative topology from the topology that $E$ induces on $I$.

**QED**

**Remark 1** Suppose $E$ topology-inducing. Note that if one takes in $K$ the $T_1$ topology where the closed sets are just the finite ones and the whole space, then all members of $E$ will be continuous, and for this topology in $K^I$ is (mock!) “completely regular” – for any closed set and a point not in it there is a continuous function (even a member of $E$) vanishing on the closed set but not at the point.

**Theorem 1** Suppose the subspace $E \subset K^I$ contains the constants (i.e. is unital) and is stable w.r.t. some $f : K^n \to K$ which is not additively affine. Then $E$ is topology-inducing on $I$. 
Proof Let $S,T \subset I$ be closed and we wish to prove $S \cup T$ closed. Let $t_0 \in I \setminus (S \cup T)$. We have $\xi, \eta \in E$ such that $\xi$ vanishes on $S$ and $\eta$ on $T$ while both are different from 0 at $t_0$. We have to show that there exists a function in $E$ vanishing on $S \cup T$ but not at $t_0$.

Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in K^n$. Then

\[
f(a\xi) := [t \mapsto f(a_1\xi(t), \ldots, a_n\xi(t))] \in E
\]

and similarly

\[
f(a\xi + b\eta) \in E.
\]

From $\xi|_S = 0$ and $\eta|_T = 0$ it follows immediately that

\[
f(a\xi + b\eta) = f(a\xi) + f(b\eta) - f(0)1 \quad \text{on } S \cup T
\]

where 1 is the constant function 1.

The difference of the two sides of (1) vanishes on $S \cup T$. If it does not vanish at $t_0$, we are done. If it does vanish we have:

\[
f(a\xi + b\eta)(t_0) = f(a\xi)(t_0) + f(b\eta)(t_0) - f(0)
\]

that is

\[
f(a\xi(t_0) + b\eta(t_0)) = f(a\xi(t_0)) + f(b\eta(t_0)) - f(0)
\]

Note that $\xi(t_0) \neq 0, \eta(t_0) \neq 0$. Now let $x, y \in E^n$. Taking $a = [\xi(t_0)]^{-1}x, b = [\eta(t_0)]^{-1}y$, one gets:

\[
f(x + y) = f(x) + f(y) - f(0) \quad x, y \in E^n,
\]

making $f$ additively affine, a contradiction.

QED

2 A density property

For topology-inducing function spaces, in particular for those unital and stable for a non-additively-affine operation, we have the following density property:

Theorem 2 Let $E \subset K^I$ be separating points and topology-inducing. If $I$ is finite then $E$ is the whole $K^I$. Consequently for general $I$ for any finite subset $I' \subset I$ all functions on $I'$ are obtained as restrictions of members of $E$ (density property).

Proof The second assertion follows from the first and from Prop. 1. If $I$ is finite, the topology induced, being $T_1$, must be discrete. In particular for any $i \in I \setminus \{i\}$ is closed hence $E$ contains a function vanishing outside $i$ but not in $i$, thus contains the standard basis and therefore is the whole $K^I$.

QED
3 The spaces of operating functions, operating semigroups

Let $E$ be a unital function space stable w.r.t. some non-additively-affine operation $f : K^n \to K$. For $n = 1, 2, \ldots$ we have the spaces $G_n \subset K^K$ of the functions from $K^n$ to $K$ which operate on $E$. Clearly, any member of $G_m$ operates on $G_n$ for any $m, n$, so they induce topologies on $K^n$, $n = 1, 2, \ldots$, for which all members of the $G_m$ or $E$ are continuous and also all “vector-functions” whose “components” are in $G_m$ or $E$ are continuous. Note that the topology on $K^{m+n}$ is finer than the product topology $K^m \times K^n$, so the continuity of vector-functions is not a consequence of the continuity of each component.

These $G_n$ are an instance of what we call an operating semigroup defined as a sequence of linear subspaces $G_n \subset K^K$, $n = 1, 2, \ldots$ such that $G_n$ contains the constants and the $n$ coordinate functions $K^n \to K$ and every member of $G_m$ operates on $G_n$ for each $m, n$. The minimal operating semigroup is the sequence of the spaces of affine (i.e. constants + linear functionals). A possibly bigger one consists of all additively affine functions. If some $G_n$ contains a non-additively-affine function, the operating semigroup will induce topologies on the $K^n$ as above.

If $G_n$ and $G'_n$ are operating semigroups, so is the sequence $G_n \cap G'_n$. Therefore if $F$ is any set of functions $f : K^n \to K$ (possibly with different $n$ for different $f$’s), then there is a minimal operating semigroup containing $F$ which we call the operating semigroup generated by $F$ and denote by $G(F)$. Since the functions operating on a function space form an operating semigroup, any function space stable under $F$ is stable under $G(F)$.

4 Examples

Proposition 2 Let $G_n$ be an operating semigroup $G_n$ which contains a non-additively-affine function. Then (i)⇒(ii)⇒(iii):

(i) For all $n$, the topology induced on $K^n$ is the product topology of the one induced on $K$.

(ii) The topology induced on $K^2$ is the product topology of the one induced on $K$.

(iii) The topology induced on $K$ is Hausdorff.

Proof (i)⇒(ii) is trivial, and (ii)⇒(iii) follows from the fact that the diagonal in $K^2$ is always closed, being defined by a linear equation.

QED

Note that in general (iii) does not imply (i). As an example take $K = \mathbb{R}(u)$ – the field of real rational functions on one variable, and $G_n$ – the set of functions that are continuous on every bounded $\mathbb{R}$-finite-dimensional set into a bounded $\mathbb{R}$-finite-dimensional set. This is an operating semigroup. The topology it induces on $K^n$ is the finest topology that gives the usual topology on any $\mathbb{R}$-finite-dimensional subspace. It is Hausdorff, and as is well-known, (i) is violated.
Proposition 3 Let $G_n$ be an operating semigroup $G_n$ which contains a non-additively-affine function. Then (i)$\Rightarrow$(ii):

(i) The topology induced on $K$ is not Hausdorff.

(ii) For all $n$, any two non-empty open sets, w.r.t. the topology induced on $K^n$, intersect.

Proof Firstly, if $n > 1$ and $\exists$ two non-intersecting non-empty open sets $U, V$ in $K^n$ then $\exists$ two such sets in $K^{(n-1)}$. Indeed, suppose not. Then any $K^{(n-1)}$-section of $K^n$ does not intersect either $U$ or $V$, which means that projections of $U$ and $V$ on the first coordinate are disjoint, thus, being open (as union of all parallel-to-1st-coordinate sections – we don’t need product topology), one of them is empty hence either $U$ or $V$ is empty.

Therefore it suffices to prove that any two non-empty open sets in $K$ intersect. But if not, then some $x \neq y$ in $K$ would have disjoint neighborhoods, and since the topology is invariant w.r.t. all invertible affine transformations of $K$, every two points would have disjoint neighborhoods, i.e. the topology is Hausdorff.

QED

The case $K = \mathbb{R}$ (the real numbers) or $\mathbb{C}$ (the complex numbers) is studied, in the context of continuous operations, in [LK], [S-a], [S-ds] and [SW]. As is shown there, for $K = \mathbb{C}$, if $E$ consists of bounded functions and is closed w.r.t. the sup-norm, then if some non-affine continuous function operates on $E$ then every analytic function operates on $E$: $E$ is an algebra, and if further some non-analytic continuous function (say, complex conjugation) operates on $E$ then any continuous function does so: $E$ is a self-adjoint algebra.\(^1\) Thus in the latter case $G_m, m = 1, 2, \ldots$, the spaces of functions that operate on $E$, contain all continuous functions, and the topologies that they induce on $C^m$ are finer than the usual complex topologies.

They can be strictly finer. Indeed, if $I$ is infinite and $E$ is the space of all $\mathbb{C}$-valued bounded functions, then $G_m$ is the set of all $f : \mathbb{C}^m \rightarrow \mathbb{C}$ that map every bounded sequence to a bounded sequence, equivalently are bounded on every bounded set. The topology induced on $\mathbb{C}^n$ is the discrete one. If instead $E$ is the space of all $\mathbb{C}$-valued functions on $I$ then $G_m$ is the space of all functions $\mathbb{C}^m \rightarrow \mathbb{C}$, while the topology induced on $\mathbb{C}^m$ is again the discrete one.

Remark 2 To quote another example, let $K$ be any ordered field, and suppose $E$ is a lattice with pointwise lattice operations, which can be expressed as: the function $a \mapsto a_+ := \max(a, 0)$ from $K \rightarrow K$ operates on $E$. In this case the topology induced on $K$ is finer than the order topology, hence is Hausdorff, and of course the topologies on $K^n$ are finer than the products of the order-topology.

\(^1\)This is proved by showing that if the function space $E$ is stable w.r.t. some continuous function $h$, then, being stable also w.r.t. the shifts of $h$ hence w.r.t. its convolution with, say, smooth functions (so one may assume $h$ smooth), one can make a derivative-like limiting process which shows that if $h$ is not affine then $E$ is stable w.r.t. the square function, hence is an algebra, and if $h$ is not analytic – w.r.t. complex conjugation, hence is self-adjoint.
5 Homomorphisms w.r.t. non-affine operations

Fix a set $F$ of functions $f : K^n \to K$ (each possibly with different $n$), such that $F$ contains a non-additively-affine function. We have the category of unital function spaces $E \subset K^I$ stable under $F$ (equivalently, stable under $G(F)$, hence we may assume at the outset that $F$ is an operating semigroup as defined above), with unital $F$-homomorphisms, i.e. $K$-linear mappings $\varphi$ which commute with the action of every $f \in F$ and preserve $1$ (thus preserve all constants).

A unital $F$-homomorphism from a $E \subset K^I$ to $E' \subset K^{I'}$, both unital and stable under $F$, has a graph, which may be viewed as a unital function space $\varphi \subset K^{I \coprod I'}$, stable under $F$. The function spaces $E$, $E'$ and $\varphi$ induce topologies on $I$, $I'$ and $I \coprod I'$, resp. $E$ coincides with the space of restrictions to $I$ of members of $\varphi$, therefore the topology on $I$ is that of a subspace of $I \coprod I'$. Moreover, the only member of $\varphi$ vanishing on $I$ is identically 0. Consequently $I$ is dense in $I \coprod I'$. The topology of $I'$ as a subspace of $I \coprod I'$ is that induced by the image of $\varphi$ and is coarser than that induced by $E'$.

Thus we have

**Corollary 4** Let $\varphi$ be a unital $F$-homomorphism from an $E \subset K^I$ to an $E' \subset K^{I'}$, both unital and stable under $F$, and suppose $F$ contains a non-additively-affine member. Then for any $n = 1, 2, \ldots$ and $S \subset K^n$ closed w.r.t. the topology induced by $G(F)_n$, if $\xi_1, \ldots, \xi_n \in E$ assume together values in $S$ on $I$ then so do $\varphi(\xi_1), \ldots, \varphi(\xi_n)$ on $I'$.

(Of course no such thing holds for general linear mappings between function spaces!)

For instance, by Remark 2, we obtain a neat proof of the following fact:

**Corollary 5** For any ordered field $K$, and for any lattice-homomorphism $\varphi$, preserving the constants, between two $K$-valued function spaces $E$ and $E'$ which are lattices w.r.t. pointwise lattice operations, the image in $K^n$ of $(\varphi(\xi_1), \ldots, \varphi(\xi_n))$, $\xi_1, \ldots, \xi_n \in E$ is contained in the closure (w.r.t. the order-topology) of the image of $(\xi_1, \ldots, \xi_n)$.

When the topology induced by $G(F)$ is Hausdorff we have:

**Corollary 6** Suppose $F$ is a set of functions $K^n \to K$ (possibly with different $n$ for different functions), such that $F$ contains a non-additively-affine function and the topology that $G(F)$ induces on $K$ is Hausdorff. Then any unital $F$-homomorphism can be described as follows: take a unital $F$-stable function space $E \subset K^I$. It induces a topology on $I$. Embed $I$ as a dense subset of some topological space $I \coprod I'$, such that each $\xi \in E$ has an extension (necessarily unique!) to a function $\xi : I \coprod I' \to K$ continuous when $K$ is endowed with the topology induced by $G(F)$, and map $\xi$ to $\xi|_I$.

References

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