Performance of PPM Multipath Synchronization in the Limit of Large Bandwidth

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Abstract

The acquisition, or synchronization, of the multipath profile for an ultrawideband pulse position modulation (PPM) communication system is considered. Synchronization is critical for the proper operation of PPM based systems. For the multipath channel, it is assumed that channel gains are known, but path delays are unknown. In the limit of large bandwidth, it is assumed that the number of paths, \( L \), grows. The delay spread of the channel, \( M \), is proportional to the bandwidth. The rate of growth of \( L \) versus \( M \) determines whether synchronization can occur. It is shown that if \( \frac{L}{\sqrt{M}} \rightarrow 0 \), then the maximum likelihood synchronizer cannot acquire any of the paths and alternatively if \( \frac{L}{M} \rightarrow 0 \), the maximum likelihood synchronizer is guaranteed to miss at least one path.

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1 Introduction

We consider the asymptotic performance of maximum likelihood synchronization schemes for pulse position modulations (PPM) in the limit of large bandwidth. This study is motivated by recent interest in ultra wideband (UWB) signaling schemes for radio communications. While significant diversity is achievable given the large amount of multipath in UWB channels, methods for harnessing such diversity when the channel is unknown remain a challenge. In fact, the fundamental limits of UWB signaling in the presence of channel uncertainty have not been fully established. In this work, we focus on the uncertainty in determining the multipath profile (including leading delay) of the UWB channel. For the purposes of this work, such uncertainty is equivalent to uncertainty in synchronization. In particular, knowledge of channel path delays is critical for the operation of pulse position modulation systems over large bandwidths. Timing errors as small as fractions of nanoseconds can seriously degrade system performance as reported in [4, 8]. Furthermore, [10] suggests that threshold-based ultra wideband (UWB) synchronization for PPM does not perform well even in asymptotically high SNR; further implying that the performance of pragmatic synchronization could limit their UWB potential. We have shown in [5] that in the limit of large bandwidth, threshold based synchronizers cannot achieve synchronization.
For fixed bandwidth spread spectrum systems (see [1] and references therein), it has been observed that detectors are more sensitive to mismatch in delay information versus other channel parameters. Information theoretic analysis of spread-spectrum systems [6] shows that the scenario of unknown path gains with known delay locations achieves almost the same throughput as that of complete channel knowledge (gains, delays) in the limit of large bandwidth. In contrast, for unknown channel parameters, the throughput diminishes in the limit of large bandwidth if the number of channel paths increase faster than a logarithmic rate on the bandwidth.

This work shows that pulse position modulation is very sensitive to the knowledge of path delays. In fact, even a mild increase (with bandwidth) of the number of identically and independently distributed paths composing a channel will cause a PPM system to fail in the limit of large bandwidth. The rate of increase dictates system performance. This paper is organized as follows: Section 2 describes the transmitted signal, channel model and received signal. The main result for maximum likelihood synchronization is provided in Section 3. Discussion of the new result with respect to our prior work is given in Section 4.

2 Signal Model

2.1 The Transmitted Signal

We consider pulse position modulation (PPM), where the transmitted signal can be written as

\[ x(t) = \sum_{n=-\infty}^{\infty} p \left( t - nT_s - \frac{1}{W} b[n] \right) \]

\[ p(t) = \begin{cases} \frac{E}{\theta} & t \in \left[ 0, \frac{T_s}{N} \right) \\ 0 & \text{else} \end{cases} \]

The symbol duration is given by \( T_s \) and the number of pulse positions is dictated by the transmission bandwidth \( W \), i.e. \( N = WT_s \). The data symbol is denoted \( b[n] \in \{0, 1, \ldots, N - 1\} \). \( E \) is the average transmitted energy per symbol, that is bandwidth independent, and \( \theta \) is a flash parameter to be explained shortly. Thus, in a symbol duration, there is a single rectangular pulse of duration \( \frac{T_s}{N} \). Our goal is to investigate performance of such a PPM system as the transmission bandwidth increases. We shall assume that the symbol duration does not diminish; however, due the use of flash signaling [9], the information rate will not grow without bound. With the use of flash signaling, transmission is bursty and communication occurs over a fraction \( \theta \) of the total communication period. The flash parameter \( \theta \) is known at the receiver and furthermore, the receiver is aware of the on-periods of communication. The transmission frame corresponds to a coherence period of the channel and as such, one out of every \( \frac{1}{\theta} \) coherence periods is employed for transmission (an on-period).

Note the distinction between flashy transmission and PPM modulation. For regular data transmission, the receiver must detect which one of the \( N = WT_s \) pulse positions has been employed in each symbol; in contrast, with flashy transmission, the receiver is synchronized to the on-periods of communication. We note that if \( \theta \) is quite small, then the transmitter is predominantly silent.

The fraction of time utilized for transmission may decrease as the bandwidth \( W \) increases, but it cannot do so too fast. In order to maintain a positive (non-diminishing) data-rate, the parameter \( \theta \) must be large enough so that \( \theta \log W \) does not diminish. The reasoning for this is straightforward: \( \log_2 WT_s \) bits are transmitted per symbol; however, only a fraction of the
coherence periods are employed and thus the data rate is proportional to $\theta \log_2 WT_s$. The requirement on $\theta$ can be written as:

$$\theta \geq \frac{k_1}{\log(Wk_2)}$$

with fixed $k_1, k_2$ that are independent of the bandwidth.

Several features of our setup should be underscored. The first is that there is no limit imposed on the number of PPM positions that are employed for data signaling. Thus, a guard time can be implemented by limiting the positions employed. Second, we emphasize the employment of a lower bounded symbol time, where the lower bound does not depend on the signal bandwidth. We do not consider schemes where the symbol time diminishes with bandwidth. Thus, the number of bits that can be transmitted in a single coherence period depends logarithmically on the bandwidth. Note that systems that use a guard period between symbols, that depends on the channel path delays, have a natural lower bound on their symbol time.

### 2.2 The Channel and Received Signal

We assume an tapped delay line model for the channel $h(t)$, thus

$$h(t) = \sum_{l=1}^{L} g_l \delta \left( t - \frac{d_l}{W} \right)$$

where the channel gains are given by $g_l$ and we assume that $\sum g_l^2 = 1$; $\delta(\cdot)$ denotes the Kronecker delta function and $d_i$ represent the path delays which are assumed non-negative integers between 1 and $M$. For simplicity of exposition we shall assume a uniform profile for the path gains and therefore $g_l = \frac{1}{\sqrt{L}}$ for $l = 1, \ldots, L$. The maximal possible number of resolvable paths is given by $M = WT_d$, where $T_d$ represents the maximum delay of the channel, thus the actual number of paths $L$ must satisfy $L \leq M$. Recent wideband channel propagation measurements suggest that the number of channel paths grows sub-linearly with bandwidth $W$, possibly satisfying

$$\lim_{W \to \infty} L = \infty \quad \text{and} \quad \lim_{W \to \infty} \frac{L}{W} = 0$$

Given $M$ possible values of the path delays, we assume that the realizations of the path delays are uniformly distributed over $(\frac{M!}{L!(M-L)!})$ possibilities. The channel model is of the block-type: the channel is fixed over the channel coherence time $T_c$; channel realizations at different coherence periods are statistically independent.

The received signal is given by,

$$y(t) = h(t) \otimes x(t) + z(t) = \sum_{l=1}^{L} g_l x \left( t - \frac{d_l}{W} \right) + z(t),$$

where $z(t)$ is a zero-mean, white Gaussian noise process.

At the receiver, the received signal is matched filtered with the pulse shape and sampled at
yielding the following discrete time equivalent signal:

\[
Y_i = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} X_{i-l} + Z_i
\]  

(2)

\[
X_i = \begin{cases} \sqrt{\frac{\varepsilon}{\theta}} & \text{if } \exists n : i \div N = n \\
0 & \text{and } i \mod N = b[n] \\
\end{cases}
\]

\(i \div N\) signifies the largest integer \(k\) such that \(kN \leq i\). The signal \(X_i\) is zero-valued except at
the positions corresponding to the transmitted PPM pulse; recall that \(N = WT_s\) is the number of possible positions. The amplitude of the signal at the non-zero position is normalized so the
noise samples \(\{Z_i\}\) are zero-mean with unit variance.

In order to assess the challenges of synchronization of PPM in multipath, we analyze a further simplified system that operates under two additional conditions. We assume a sufficiently
large guard time, \(T_d\), to ensure no intersymbol interference resulting in an effectively longer coherence time \(T_c = \frac{L+T_d}{T_s} T_c\). And we assume knowledge of the PPM symbols, essentially
assuming training information. The receiver sums over all the symbols per coherence period before it begins processing. Given that we show the failure to synchronize for an optimal de-
tector under these idealized conditions, we effectively make statements about more practical
systems as well.

3  Maximum Likelihood Synchronization

Recall that the position of the PPM symbol is known; however, the initial delay and multipath
profile are unknown. The synchronization problem can be posed as a multiple hypothesis
testing problem for which there are \(\binom{M}{L}\) hypotheses given a delay spread of \(M = WT_d\) and
\(L\) non-zero channel taps. The received signal under each hypothesis can be written as, (recall
(2)),

\[
Y|H_i = [Y_0, Y_1, \cdots Y_{M-1}]^T = \sqrt{\frac{\varepsilon}{L\theta}} s_i + Z
\]  

(3)

\[
s_i = \begin{bmatrix} 1,1,0,\cdots,1,0, \\
L 1's over M positions \\
\end{bmatrix} T \sim \mathcal{N}(0, I)
\]  

(4)

The optimal detector for such a scenario is a simple correlator:

\[
\hat{i} = \arg \max_i s_i^T Y
\]  

(5)

However, given the form of \(s_i\), we can see the following equivalence. Let \(\tilde{Y}_1, \tilde{Y}_2, \cdots, \tilde{Y}_L\) be
the \(L\) largest components of \(Y_s\). Then,

\[
\max s_i^T Y = \sum_{j=1}^{L} \tilde{Y}_j
\]  

(6)

Thus the maximum likelihood detector is equivalent to determining the multipath locations by
selecting the \(L\) positions with the \(L\) largest signal values. With this perspective of the maximum
likelihood detector, we can develop a method for evaluating the likelihood of an error through order statistics. The statistics of signal positions and noise positions are Gaussian and given by,

\[ Y_i | \text{path location} \sim \mathcal{N}\left(\sqrt{\frac{E}{\theta L}}, 1\right) \]  

(7)

\[ Y_i | \text{noise only position} \sim \mathcal{N}(0, 1) \]

Given that we know the PPM symbol, the observation vector is of length \( M \) and of the \( M \) possible positions, \( L \) correspond to the transmitted signal. The remaining \( M - L \) correspond to noise.

We present the main theorem of the work:

**Theorem 1.** Consider \( M \) independent, Gaussian random variables with the following distributions,

\[ Y_i \sim \mathcal{N}\left(\sqrt{k \log \frac{M}{L}}, 1\right), \quad i = 1, 2, \ldots, L \]  

(8)

\[ W_i \sim \mathcal{N}(0, 1), \quad i = 1, 2, \ldots, M - L \]  

(9)

\( k \) is a constant which does not depend on \( L \) or \( M \); \( L \leq M \). We order the \( Y_i \) such that \( B_1 = \max Y_i \) and \( B_L = \min Y_i \); similarly, \( S_1 = \max W_i \) and \( S_{M-L} = \min W_i \). Then,

\[
\lim_{M,L \to \infty} P[S_L > B_1] = 1 \quad \text{if} \quad \frac{L}{\sqrt{M}} \to 0
\]  

(10)

\[
\lim_{M,L \to \infty} P[S_1 > B_L] = 1 \quad \text{if} \quad \frac{L}{M} \to 0
\]  

(11)

If \( \frac{L}{\sqrt{M}} \to 0 \), then the maximum likelihood detector will always detect noise variables and none of the correct paths will be detected. On the other hand if a faster growth rate on the multipath exists, \( \frac{L}{M} \to 0 \), the maximum likelihood detector is guaranteed to miss at least one of the path locations. In the limit of large \( L \), missing one path is insignificant; however, we have two limits on bounds on performance.

To provide intuition about our result consider a detector that randomly selects positions, where the selection does not depend on the amplitudes. If we assume a uniform selection of \( L \) variables out of \( M \), this selection will include, on average, \( L^2/M \) of the \( L \) big variables. To see this, look at an example with \( L = 1/2M \). The probability that each big variable is chosen is 1/2, so on average the number of big variables chosen is 1/2\( L = L^2/M \). With \( L \geq c\sqrt{M} \), the average number of big variables chosen this way does not diminish as \( M \) increases. The optimal detector performs better than a random choice, so it detects at least \( L^2/M \) of the \( L \) big variables. This is the reason for the condition \( L/\sqrt{M} \to 0 \).

### 3.1 Outline of Proof

The proof follows from a simple observation of the events leading to an error for the maximum likelihood detector. That is, \( P_e^{ML} = P[\bigcup_i A_i] \), where the \( A_i \) are error events where one or more noise positions are members of the set of \( L \) largest values. Thus, \( P_e^{ML} \geq P[A_i] \) for any
i. We consider two particular error events: one path is incorrectly detected and all paths are correctly detected. In the context of the ordered random variables we can view these events as:

\[ P_{e_{ML}} \geq \max_i P[A_i] \] which corresponds to a single path error – this is the most likely error to occur and

\[ P_{e_{ML}} \geq \min_i P[A_i] \] which corresponds to all paths being incorrectly detected – this is the least likely error to occur. However, for both events, we shall show that for reasonable growth rates on \( L \) versus \( M \), the probability of these two events is unity in the limits of large \( L \) and \( M \). As \( M \) is proportional to the bandwidth of the system, we ultimately obtain a large bandwidth result. We shall consider order statistics on the signal and noise variables. We shall show that the mean of the relevant noise variable dominates over the mean of the relevant signal variable. For \( \frac{L}{\sqrt{M}} \to 0 \), the maximum likelihood detector finds none of the signal positions, leading us to the conclusion that under these conditions, no detector can synchronize.

### 3.2 Useful Formulas

From Cramér’s book [2], Section 28.6, page 376, we have the mean and variance of the order statistics of Gaussian variables. We order \( G \) identically and independently distributed Gaussians with mean \( m \) and variance \( \sigma^2 \). The \( \nu \)th variable from the top has mean

\[
E_{\nu,G} = m + \sigma \left( \sqrt{2 \ln G} - \frac{\ln \ln G + \ln 4\pi + 2 (S_1(\nu) - C)}{2 \sqrt{2 \ln G}} + O \left( \frac{1}{\ln G} \right) \right) \tag{12}
\]

and variance

\[
\text{var}_{\nu,G} = \frac{\sigma^2}{2 \ln G} \left( \frac{\pi^2}{6} - S_2(\nu) \right) + O \left( \frac{1}{\ln^2 G} \right) \tag{13}
\]

where \( C \approx 0.5772 \) is Euler’s constant, and for \( \nu > 1 \)

\[
S_1(\nu) = 1 + \frac{1}{2} + \cdots + \frac{1}{\nu - 1} \tag{14}
\]

\[
S_2(\nu) = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(\nu - 1)^2} \tag{15}
\]

For \( \nu = 1 \) (the highest of the \( G \) variables), we have from [3], Section 21-4, page 278:

\[
S_1(1) = S_2(1) = 0 \tag{16}
\]

### 3.3 The \( L \)th Largest Noise Variable

We show that the \( L \)th largest of the noise variables equals \( \sqrt{2 \ln (M - L)} - \frac{\ln L}{\sqrt{2 \ln (M - L)}} \) in the limit of large \( M, L \). We use (12) and (13) with \( G = M - L \) random variables, and consider the \( \nu = L \) largest variable, i.e. there are \( L - 1 \) variables larger than the one we investigate. We recall that \( m = 0 \) and \( \sigma = 1 \), which yields the following mean for the variable of interest,

\[
E_{L:\!M-L} = \sqrt{2 \ln (M - L)} - \frac{\ln (M - L) + \ln 4\pi + 2 (S_1(L) - C)}{2 \sqrt{2 \ln (M - L)}} + O \left( \frac{1}{\ln (M - L)} \right) \tag{17}
\]

and variance

\[
\text{var}_{L:\!M-L} = \frac{1}{2 \ln (M - L)} \left( \frac{\pi^2}{6} - S_2(L) \right) + O \left( \frac{1}{\ln^2 (M - L)} \right) \tag{18}
\]
where \( C \approx 0.5772 \) and

\[
S_1(L) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{L-1}  \tag{19}
\]

\[
S_2(L) = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(L-1)^2}  \tag{20}
\]

We observe that \( S_2(L) \) is finite for any \( L \), thus \( \lim_{L,M} \frac{1}{M} \cdot \text{var}_{L:M} = 0 \). Therefore, in the limit of large \( M \) and \( L \), the \( L \)th largest variable approaches a constant which is equivalent to the mean.

To further investigate the mean, we calculate a simple approximation for \( S_1(L) \), for large \( L \):

\[
S_1(L) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{L-1} \approx \int_1^L \frac{1}{x} \, dx = \ln x \bigg|_1^L = \ln L \tag{21}
\]

\[
E_{L:M-L} \approx \sqrt{2 \ln (M-L)} - \frac{\ln \ln (M-L) + \ln 4\pi + 2(\ln L - C)}{2\sqrt{2 \ln (M-L)}} + O \left( \frac{1}{\ln (M-L)} \right) \tag{22}
\]

\[
\approx \sqrt{2 \ln (M-L)} - \frac{\ln L}{\sqrt{2 \ln (M-L)}} \tag{23}
\]

Observe that \( \lim_{M,L} E_{L:M-L} = \infty \), furthermore, it is straightforward to show that \( \sum_{M=L}^\infty \text{var}_{L:M-L} = \infty \). Thus, the \( L \)th largest variable does not converge to a limit in the mean square sense (see e.g. [11]) despite the fact that \( \lim_{M,L} \text{var}_{L:M-L} = 0 \). Thus, we can only conclude that the random sequence of interest converges in distribution.

### 3.4 The Largest Signal Variable

We show that the largest of the signal variables approaches its mean value, \( \sqrt{\frac{k \log M}{L}} + \sqrt{2 \ln L} \), in the limit of large \( M, L \), if \( L^2 < M \). We use Equations \( \{12\} \) and \( \{13\} \) again, with \( G = L \) random variables, and examine the largest variable, that is \( \nu = 1 \). Recall that \( m = \sqrt{\frac{k \log M}{L}} \) and \( \sigma = 1 \), yielding the mean,

\[
E_{1:L} = \sqrt{\frac{k \log M}{L}} + \sqrt{2 \ln L} - \frac{\ln \ln L + \ln 4\pi - 2C}{2\sqrt{2 \ln L}} + O \left( \frac{1}{\ln L} \right) \tag{24}
\]

\[
> \sqrt{\frac{k \log M}{L}} + \sqrt{2 \ln L} \tag{25}
\]

and variance

\[
\text{var}_{1:L} = \frac{1}{2 \ln L} \frac{\pi^2}{6} + O \left( \frac{1}{\ln^2 L} \right) \tag{26}
\]

where \( C \approx 0.5772 \). As in the previous case, \( \lim_{M,L} E_{1:L} = \infty \), and \( \lim_{M,L} \text{var}_{1:L} = 0 \).
3.5 Conditions for Dominance

We seek to determine the conditions for which \( E_{L,M-L} > E_{1:L} \). Thus, the desired strict inequality is given below,

\[
\sqrt{2 \ln (M - L)} - \frac{\ln L}{\sqrt{2 \ln (M - L)}} \gg \sqrt{k \log \frac{M}{L}} + \sqrt{2 \ln L}
\]

(27)

\[
\sqrt{2 \ln (M - L)} \gg \sqrt{k \log \frac{M}{L}} + \sqrt{2 \ln L} + \frac{\ln L}{\sqrt{2 \ln (M - L)}}
\]

(28)

To assess this comparison, we compare each of the three terms on the right hand side of (28) with three terms of a decomposition of the left hand side. That is, let,

\[
\sqrt{2 \ln (M - L)} = (\alpha + \beta + \gamma) \sqrt{2 \ln (M - L)} \text{ where } \alpha + \beta + \gamma = 1
\]

(29)

\( \alpha, \beta \) and \( \gamma \) are positive constants which are not functions of \( M \) and \( L \). We next execute three comparisons. For the first term of (28), we square both sides of the inequality of interest to achieve,

\[
\alpha^2 2 \ln (M - L) \gg \frac{k \log M}{L}
\]

(30)

\[
\alpha^2 2 \left( \ln M + \ln \left(1 - \frac{L}{M}\right)\right) \gg \frac{k \log M}{L}
\]

(31)

The last equation is an inequality for any positive \( \alpha, L, M \to \infty \) and \( \frac{L}{M} \to 0 \). Consider the second term of (28), we also square both sides and yield,

\[
\beta^2 \ln (M - L) \gg \ln L
\]

(32)

\[
(M - L)^{\beta^2} \gg L
\]

(33)

\[
M^{\beta^2} \left(1 - \frac{L}{M}\right)^{\beta^2} \gg L
\]

(34)

The final equation is an equality as long as \( \beta^2 > 0.5 \), due to the constraint of small \( L \), that is \( L^2 < M \). Examining the final term of (28):

\[
\gamma \sqrt{2 \ln (M - L)} \gg \frac{\ln L}{\sqrt{2 \ln (M - L)}}
\]

(36)

\[
\gamma 2 \ln (M - L) \gg \ln L
\]

(37)

\[
(M - L)^{2\gamma} \gg L
\]

(38)

\[
M^{2\gamma} \gg L
\]

(39)

We achieve in equality when \( 2\gamma > \frac{1}{2} \), again due to the relationship between \( M \) and \( L \).

Summarizing the conditions, we need to determine a set of \( \alpha, \beta \) and \( \gamma \) such that, (a) \( \alpha + \beta + \gamma = 1 \) (b) \( \alpha \) is any positive constant (c) \( \beta > \sqrt{\frac{1}{2}} \) and (d) \( \gamma > \frac{1}{4} \). However, these conditions can always be met, thus for the appropriate growth rates on \( L \) and \( M \left(\frac{L}{\sqrt{M}} \to 0\right) \), the mean of the \( L^{th} \) largest noise variable dominates the mean of the largest signal variable.

We assume the required conditions above. We form a new random variable \( D_{M,L} = S_L - B_1 \). As the signal and noise variables are independent, we can determine that \( \mathbb{E}[D_{M,L}] = \)
\[ E_{L:M-L} - E_{1:L} \quad \text{and} \quad \text{var} \left[ D_{M,L} \right] = \text{var}_{L:M-L} + \text{var}_{1:L}. \] Furthermore, \( \lim_{M,L \to \infty} E \left[ D_{M,L} \right] = \infty \) and \( \lim_{M,L \to \infty} \text{var} \left[ D_{M,L} \right] = 0. \)

From the Chebyshev inequality we can show
\[ \lim_{M,L \to \infty} P \left[ |D_{M,L} - E \left[ D_{M,L} \right]| < \epsilon \right] \geq 1 - \lim_{M,L \to \infty} \frac{\text{var} \left[ D_{M,L} \right]}{\epsilon^2} = 1 \quad (40) \]

In the limit of large \( M \) and \( L \), this difference variable approaches its mean with probability 1; recall that this mean value is infinity. Thus, the limiting distribution of the difference variable, \( D_{M,L} \) is a delta function. Using Fatou’s theorem which enables the interchange of limits and integration, we determine that \( \lim_{M,L \to \infty} P \left[ D_{M,L} > 0 \right] = \lim_{M,L \to \infty} P \left[ S_L > B_1 \right] = 1. \)

The proof for the comparison of \( S_L \) and \( B_1 \) is similar, we note that the key statistics/approximations are given as follows for the largest noise variable:
\[ E_{1:M-L} \gtrsim \sqrt{2 \ln(M-L)} \quad (41) \]
\[ \text{var}_{1:M-L} = \frac{\pi}{12 \ln(M-L)} + O \left( \frac{1}{\ln^2(M-L)} \right) \quad (42) \]

and for the smallest signal variable,
\[ E_{L:L} \approx \sqrt{k \log \frac{M}{L}} + \sqrt{\frac{\ln L}{2}} \quad (43) \]
\[ \text{var}_{L:L} = \frac{1}{2 \ln L} \left( \frac{\pi^2}{6} - S_2(L) \right) + O \left( \frac{1}{\ln^2 L} \right) \quad (44) \]

Thus, the above arguments prove the theorem of the work. We note that the corollary statement is that if the growth rate of the number of paths, \( L \) versus the delay spread \( M \) is such that \( \frac{L}{M} \to 0 \), then the maximum likelihood multipath detector cannot synchronize any path and if \( \frac{L}{M} \to 0 \), at least one path will be incorrectly detected. Two features of practical UWB channels and systems have not been considered herein, but are under current investigation. The first is that we have bounds on the relative growth rates of multipath to delay spread for the two extreme cases: no paths synchronized and all but one path synchronized. Of interest is to determine what ratio of paths is necessary. Finally, the multipath profile considered is for equal energy paths. In reality, the path energy appears to decay from the first path.

4 Relationship to Prior Results

In \[5\], we provided two results. The first was that there did not exist a threshold for which position by position threshold detection could achieve arbitrarily small probability of error for multipath synchronization in the limit of large bandwidth. The second result determined the conditions under which maximum likelihood synchronization could achieve an arbitrarily small probability of error in the limit of large bandwidth. The conditions for the maximum likelihood detector from \[5\] are:
\[ L < \sqrt{\frac{E \log(Wk_2)}{k_3 \log 2)WT_d}} \]

where, \( k_2, k_3 \) are constants which are independent of \( L \) and \( W \). Recall that \( W \) is the bandwidth and \( T_d \) is the delay spread. As \( M \) is proportional to the bandwidth, the conditions above imply
that we want the behavior of \( L \sim o \left( \sqrt{\log \frac{M}{M}} \right) \). Thus if the rate of growth on \( L \) is as just noted, we have \( \lim_{M \to \infty} L \approx \lim_{M \to \infty} \sqrt{\log \frac{M}{M}} = 0 \). Thus, the required condition is that the number of multipath actually diminish with increasing bandwidth. This is equivalent to the energy of the multipath profile concentrating itself in proportionally fewer and fewer components. Recall that the mean of the unordered signal variables is given by \( \sqrt{k \log \frac{M}{M}} \), thus as there are fewer and fewer paths, the energy of the non-zero paths increases making them more “detectable”. We contrast this result to result of the current work. Herein, if \( \lim_{M,L \to \infty} \frac{L}{\sqrt{M}} = 0 \), the optimal detector cannot synchronize. In this case, the number of paths does grow without bound and thus the energy in each non-zero path is decreasing to zero, simultaneously, the number of noise positions is dominating the number of signal positions. As such, the mean of the larger noise positions is increasing, while the mean of the smallest signal position is in fact decreasing, leading to the large likelihood of an error.

5 Conclusions

In this work, we have considered the problem of channel synchronization for PPM modulation for ultrawideband communication systems. Even under idealized conditions of no intersymbol interference and perfect knowledge of transmitted symbols and channel gains, the optimal synchronizer fails when the rate of growth of the number of multipath grows too slowly relative the bandwidth. We have shown that for \( \frac{L}{\sqrt{M}} \to 0 \), the maximum likelihood synchronizer fails to capture any of the paths and for \( \frac{L}{M} \to 0 \), the maximum likelihood synchronizer is guaranteed to miss at least one path. Ongoing research is considering the effect of the profile of the gains – equal energy paths are not consistent with experimental channel data as well as the necessary growth rates when a fraction of the multiple paths need to be properly synchronized.

References

[1] L.-C. Chu and U. Mitra. Performance analysis of the improved MMSE multi-user receiver for mismatched delay channels. IEEE Trans. on Comm., 46(10):1369–1380, October 1998.

[2] Harald Cramér. Mathematical Methods of Statistics. Princeton University Press, 1946.

[3] Norman L. Johnson and Samuel Kotz. Continuous Univariate Distributions - 1. Houghton Mifflin Company, 1970.

[4] W. M. Lovelace and J. K. Townsend. The effects of timing jitter on the performance of impulse radio. IEEE JSAC, 20(9):1646 – 1651, Dec. 2002.

[5] D. Porrat and U. Mitra. On Synchronization of Wideband Impulsive Systems in Multipath. In Proc. IEEE ISIT 2005, September 2005.

[6] Dana Porrat, David Tse, and Serban Nacu. Channel uncertainty in ultra wideband communication systems. In Preparation, available at http://wireless.stanford.edu/dporrat/ChannelUncertainty.pdf.

\(^1\)This is little-o notation. That is, \( f(n) = o(g(n)) \) if \( \exists \ k \in \mathbb{N}, f(n) < cg(n) \ \forall \ n \geq k \).
[7] Leslie Rusch, Cliff Prettie, David Cheung, Qinghua Li, and Minnie Ho. Characterization of UWB propagation from 2 to 8 GHz in a residential environment. IEEE Journal on Selected Areas in Communications.

[8] Z. Tian and G.B. Giannakis. BER sensitivity to mis-timing in correlation-based UWB receivers. Proc. IEEE Globecom, 2:441–445, Dec. 2003. San Francisco, US.

[9] Sergio Verdú. Spectral efficiency in the wideband regime. IEEE Transactions on Information Theory, 48(6):1319–1343, June 2002.

[10] S. Vijayakumaran, T. F. Wong, and S. Aedudodla. On the Asymptotic Performance of Threshold-based Acquisition Systems in Multipath Fading Channels. In Proc. IEEE Information Theory Workshop, October 2004.

[11] Yannis Viniotis. Probability and Random Processes. McGraw Hill, Boston, MA, 1998.