QUANTUM GROUP-TWISTED TENSOR PRODUCTS OF C*-ALGEBRAS

RALF MEYER, SUTANU ROY, AND STANISŁAW LECH WORONOWICZ

Abstract. We put two C*-algebras together in a noncommutative tensor product using quantum group coactions on them and a bicharacter relating the two quantum groups that act. We describe this twisted tensor product in two equivalent ways, based on certain pairs of quantum group representations and based on covariant Hilbert space representations, respectively. We establish basic properties of the twisted tensor product and study some examples.

1. Introduction

Several important constructions put together two C*-algebras in a kind of tensor product where the tensor factors do not commute. For instance, a noncommutative two-torus is obtained in this way from two copies of \( C(T) \). The reduced crossed product \( A \rtimes_\alpha G \) for a continuous action \( \alpha : G \to \operatorname{Aut}(A) \) of a locally compact group \( G \) combines \( A \) and the reduced group C*-algebra of \( G \). Such crossed products also exist for locally compact quantum groups. Another example is the skew-commutative tensor product for \( \mathbb{Z}/2 \)-graded C*-algebras, which is defined so that the odd elements anticommute.

We shall construct twisted tensor products using quantum group coactions on the tensor factors. The examples mentioned above are special cases of our theory. Our construction is closely related to one by Vaes [15], studied in more detail by Nest and Voigt [12]; it is more general because we allow two different quantum groups to act on the tensor factors and do not need Haar weights on quantum groups. Moreover, we provide two different constructions of the noncommutative tensor product and use them to prove many formal properties.

Our twisted tensor product uses the following data: two C*-quantum groups \( G = (A, \Delta_A) \) and \( H = (B, \Delta_B) \) (in the sense of [14]); a bicharacter \( \chi \in \mathcal{U}(\hat{A} \otimes \hat{B}) \); and two C*-algebras \( C \) and \( D \) with continuous coactions \( \gamma : C \to C \otimes A \) and \( \delta : D \to D \otimes B \) of \( G \) and \( H \), respectively. Then we define a C*-algebra

\[
C 
\otimes_{\chi} D = (C, \gamma) \boxtimes_{\chi} (D, \delta)
\]

with nondegenerate *-homomorphisms

\[
C \overset{\iota_C}{\to} M(C \boxtimes_{\chi} D) \overset{\iota_D}{\leftarrow} D
\]

such that \( \iota_C(C) \cdot \iota_D(D) \) is linearly dense in \( C \boxtimes_{\chi} D \). We briefly call \( (C \boxtimes_{\chi} D, \iota_C, \iota_D) \) a crossed product of \( C \) and \( D \).

We now give several examples.

First the trivial, commutative case. If \( \chi = 1 \) or if \( \gamma \) or \( \delta \) is trivial, then \( C \boxtimes_{\chi} D \) is the minimal C*-tensor product with the usual maps \( \iota_C \) and \( \iota_D \).

---

2010 Mathematics Subject Classification. 81R50 (46L05 46L55).

Key words and phrases. C*-algebra, tensor product, crossed product, Heisenberg pair.

Supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Research Training Group 1493 and the Institutional Strategy of the University of Göttingen, and by the Alexander von Humboldt-Stiftung and the National Science Centre (NCN) Grant no. 2011/01/B/ST1/05011.
Secondly, let \( A = B = C^*(\mathbb{Z}/2) \). Then \( C \) and \( D \) are \( \mathbb{Z}/2 \)-graded \( C^* \)-algebras. Let \( \chi \) be the unique non-trivial bicharacter in \( \hat{A} \otimes \hat{B} = C(\mathbb{Z}/2 \times \mathbb{Z}/2) \), defined by \( \chi(a, b) := a \cdot b \) for \( a, b \in \mathbb{Z}/2 = \{ \pm 1 \} \). Then \( C \boxtimes_{\chi} D \) is the (spatial) skew-commutative tensor product of \( C \) and \( D \).

Thirdly, let \( H = \hat{G} \) be the reduced dual of \( G \) and let \( \chi = W^A \in \mathcal{U}(\hat{A} \otimes A) \) be the reduced bicharacter; here we identify the bidual of \( G \) with \( G \). If \( G \) has a Haar weight then our construction is equivalent to one by Nest and Voigt [12]. In particular, for \( D = \hat{A} \) and \( \delta = \Delta_A \), \( C \boxtimes_{W^A} \hat{A} \) is the reduced crossed product for the coaction \( \gamma \).

Finally, let \( A = B = C(\mathbb{T}^n) \), so that coactions of \( G = \hat{H} \) are actions of the \( n \)-torus group \( \mathbb{T}^n \), and let \( C = D = C(\mathbb{T}^n) \) with \( \gamma = \delta = \Delta_A \), corresponding to the translation action of \( \mathbb{T}^n \) on itself. A bicharacter \( \chi \in \mathcal{U}(\hat{A} \otimes \hat{B}) \) is equivalent to a map \( \chi: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T} \) that is multiplicative in both variables. Thus \( \chi((a_n), (b_n)) = \prod_{i,j=1}^n \chi_i^{a_i b_i} \) for some \( (\lambda_{ij})_{1 \leq i, j \leq n} \in \mathbb{T} \). The resulting tensor product \( C(\mathbb{T}^n) \boxtimes_{\chi} C(\mathbb{T}^n) \) is generated by \( 2n \) unitaries \( U_1, \ldots, U_n \) and \( V_1, \ldots, V_n \), with the following commutation relations. First, the \( U_i \) and the \( V_i \) commute among themselves, so that they generate two copies of \( C(\mathbb{T}^n) \). Secondly, \( V_i U_j = \lambda_{ij} U_j V_i \) for all \( 1 \leq i, j \leq n \). We get all noncommutative \( 2n \)-tori in this way.

Now we describe two constructions of \( C \boxtimes_{\chi} D \).

The first one uses a pair of representations \( (\alpha, \beta): A \to \mathbb{B}(H), B \to \mathbb{B}(K) \) on the same Hilbert space. This yields embeddings \( \iota_C := ((id_C \otimes \alpha) \circ \gamma)_{13}: C \to \mathcal{M}(C \otimes D \otimes \mathbb{K}(H)) \) and \( \iota_D := ((id \otimes \beta) \circ \delta)_{23}: D \to \mathcal{M}(C \otimes D \otimes \mathbb{K}(K)) \). We let \( C \boxtimes_{\chi} D \) be the closed linear span of \( \iota_C(C) \cdot \iota_D(D) \) for a suitable choice of \( \alpha \) and \( \beta \). We call suitable pairs \( (\alpha, \beta) \) \( \chi \)-Heisenberg pairs. The closed linear span of \( \iota_C(C) \cdot \iota_D(D) \) is a \( C^* \)-algebra, and different \( \chi \)-Heisenberg pairs \( (\alpha, \beta) \) yield equivalent crossed products.

The definition of a \( \chi \)-Heisenberg pair generalises the Weyl form of the canonical commutation relations (see Example 5.4). It is also a variant of the usual pentagon equation for multiplicative unitaries (see Example 5.5). In terms of the reduced bicharacters \( W^A \in \mathcal{U}(\hat{A} \otimes A) \) and \( W^B \in \mathcal{U}(\hat{B} \otimes B) \), the pair \( (\alpha, \beta) \) is a \( \chi \)-Heisenberg pair if \( W^A_{10} W^B_{23} = W^B_{23} W^A_{10} \chi_{12} \) in \( \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(H)) \); here \( W^A_{10} \) means that we apply \( id \otimes id \otimes \alpha \) to \( W^A_{13} \).

Our second approach uses covariant representations of \( (C, \gamma) \) and \( (D, \delta) \) on Hilbert spaces \( H \) and \( K \). These contain corepresentations of \( A \) and \( B \), which allow us to turn \( \chi \) into a unitary operator \( Z \) on \( H \otimes K \). Assuming that the representations of \( C \) and \( D \) are faithful, we show that we get a faithful representation \( C \boxtimes_{\chi} D \to \mathbb{B}(H \otimes K) \), mapping \( C \ni c \to c \otimes 1 \) and \( D \ni d \to Z(1 \otimes d)Z^* \).

We also establish functoriality properties of \( \boxtimes \). These say that a pair of equivariant “maps” \( f: C \to C', g: D \to D' \) induces a “map” \( C \boxtimes_{\chi} D \to C' \boxtimes_{\chi} D' \). “Maps” could mean, among others, morphisms, -homomorphisms, completely positive contractions, or \( C^* \)-correspondences. We also examine when \( f \boxtimes_{\chi} g \) is injective or surjective. Functoriality for \( C^* \)-correspondences also shows that \( C \boxtimes_{\chi} D \) is Morita–Rieffel equivalent to \( C' \boxtimes_{\chi} D' \) if \( C, C', D, D' \) are equivariantly Morita–Rieffel equivalent.

Related to Morita–Rieffel equivalence, we show that:

\[
(C, \gamma) \boxtimes_{\chi} (D, \delta) \cong (C, \gamma') \boxtimes_{\chi} (D, \delta')
\]

if \( \gamma' = Ad_{u_\gamma} \circ \gamma \) and \( \delta' = Ad_{u_\delta} \circ \delta \) for cocycles \( u_\gamma \in \mathcal{U}(C \otimes A) \) and \( u_\delta \in \mathcal{U}(D \otimes B) \). This generalises the well-known isomorphism between the reduced crossed products for an inner action and the trivial action.

Finally, we consider the examples mentioned above. If \( A \) and \( B \) are Abelian groups, then we identify \( C \boxtimes_{\chi} D \) with a Rieffel deformation of the ordinary tensor product.
product $C \otimes D$. We identify $(C, \gamma) \boxtimes_{W^*} (\hat{A}, \hat{\Delta}_A)$ with the reduced crossed product for the coaction $\gamma$ on $C$.

The dual coaction on reduced crossed products is an instance of the functoriality of $\boxtimes$. The coaction $\hat{\Delta}_A: \hat{A} \to \hat{A} \otimes \hat{A}$ is a $\hat{G}$-equivariant map if $\hat{A} \otimes \hat{A}$ carries the right $\hat{G}$-coaction $\text{id}_{\hat{A}} \otimes \hat{\Delta}_A$. By functoriality of $\boxtimes$, it induces a morphism $(C, \gamma) \boxtimes_{W^*} (\hat{A}, \hat{\Delta}_A) \to (C, \gamma) \boxtimes_{W^*} (\hat{\Delta}_A) \equiv \hat{A} \otimes ((C, \gamma) \boxtimes_{W^*} (\hat{A}, \hat{\Delta}_A)).$

This is a continuous left $A$-coaction on $(C, \gamma) \boxtimes_{W^*} (\hat{A}, \hat{\Delta}_A)$. It is equivalent to the dual coaction on the reduced crossed product.

In a purely algebraic setting, noncommutative tensor products of two algebras $A$ and $B$ may be studied using a commutation map $R: B \otimes A \to A \otimes B$, such that $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := a_1 \cdot R(b_1 \otimes a_2) \cdot b_2 \in A \otimes B$; the properties needed for this to be associative are worked out in [3]. Even more general twisting procedures work quite well algebraically, see [9]. The closest precursor of our construction $\boxtimes$ is [10, Corollary 9.2.13], which defines a noncommutative tensor product for $H$-comodule algebras over a quasitriangular Hopf algebra $H$. In a $C^*$-algebra context, Ruy Exel [4] treats some examples of noncommutative tensor products using commutation maps. Mostly, however, commutation maps or explicit formulas for the product do not help to construct $C^*$-algebras. First, $R(b \otimes a) = b \cdot a$ need not be a finite linear combination of products $a_i \cdot b_i$; secondly, we do not yet know the $C^*$-norm on $A \otimes B$ in which we could approximate $b \cdot a$ by finite sums of products $a_i \cdot b_i$; thirdly, the choice of a $C^*$-norm for this convergence would already impose some subtle constraints on the possible twisted multiplications (recall, for instance, that the spectral radius of a self-adjoint element in a $C^*$-algebra is equal to its norm). This article addresses the analytic difficulties in defining noncommutative $C^*$-algebra tensor products.

2. Preliminaries

For two norm-closed subsets $X$ and $Y$ of a $C^*$-algebra, let

$$X \cdot Y := \{xy : x \in X, y \in Y\}^{\text{CLS}},$$

where CLS stands for the closed linear span.

For a $C^*$-algebra $A$, let $\mathcal{M}(A)$ be its multiplier algebra and let $U(A)$ be the group of unitary multipliers of $A$. A unitary $U \in U(A)$ defines an automorphism $Ad_U: A \to A, a \mapsto UaU^*$. Let $\mathfrak{C}^*\text{alg}$ be the category of $C^*$-algebras with nondegenerate $^*$-homomorphisms $\varphi: A \to \mathcal{M}(B)$ as morphisms $A \to B$; let $\text{Mor}(A, B)$ denote this set of morphisms.

Let $\mathcal{H}$ be a Hilbert space. A representation of a $C^*$-algebra $A$ is a nondegenerate $^*$-homomorphism $A \to \mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H}) = \mathcal{M}(\mathcal{K}(\mathcal{H}))$ and the nondegeneracy conditions $A \cdot \mathcal{K}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ and $A \cdot \mathcal{H} = \mathcal{H}$ are equivalent, this is the same as a morphism from $A$ to $\mathcal{K}(\mathcal{H})$.

We write $\Sigma$ for the tensor flip $\mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$, $x \otimes y \mapsto y \otimes x$, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We write $\sigma$ for the tensor flip isomorphism $A \otimes B \to B \otimes A$ for two $C^*$-algebras $A$ and $B$.

2.1. Crossed tensor products.

**Definition 2.1** (compare [16]). Let $A$, $B$, $C$ be $C^*$-algebras, $\alpha \in \text{Mor}(A, C)$ and $\beta \in \text{Mor}(B, C)$. If $\alpha(A) \cdot \beta(B) = C$, then we call $(C, \alpha, \beta)$ a crossed product or crossed tensor product of $A$ and $B$.

**Example 2.2.** The spatial tensor product $C = A \otimes B$ of two $C^*$-algebras with $\alpha(a) = a \otimes 1_B$ and $\beta(b) = 1_A \otimes b$ is the simplest example of a crossed product. Here $1_A \in \mathcal{M}(A)$ and $1_B \in \mathcal{M}(B)$. 
Let \( \alpha \) and \( \beta \) be (nondegenerate) representations of \( A \) and \( B \) on the same Hilbert space \( \mathcal{H} \) such that \( \alpha(A) \cdot \beta(B) \) and \( \beta(B) \cdot \alpha(A) \) are the same subspace of \( \mathcal{B}(\mathcal{H}) \). Then \( C := \alpha(A) \cdot \beta(B) \) is a \( C^* \)-algebra, \( \alpha \in \text{Mor}(A,C) \) and \( \beta \in \text{Mor}(B,C) \). Thus \( C \) is a crossed product of \( A \) and \( B \). This suggests that crossed products are defined by some commutation relations between \( \alpha \) and \( \beta \). Analytic difficulties may, however, prevent us from writing down such commutation relations explicitly (see our discussion at the end of the introduction).

**Definition 2.3.** An equivalence between two crossed products \( C_1 = \alpha_1(A) \cdot \beta_1(B) \) and \( C_2 = \alpha_2(A) \cdot \beta_2(B) \) of \( A \) and \( B \) is an isomorphism \( \varphi: C_1 \to C_2 \) with \( \varphi \circ \alpha_1 = \alpha_2 \) and \( \varphi \circ \beta_1 = \beta_2 \).

Any faithful morphism \( \varphi \in \text{Mor}(C_1,C_2) \) with \( \varphi \circ \alpha_1 = \alpha_2 \) and \( \varphi \circ \beta_1 = \beta_2 \) satisfies \( \varphi(C_1) = C_2 \) and hence is an equivalence of crossed products.

**Example 2.4.** Let \( C = \alpha(A) \cdot \beta(B) \) be a crossed product and \( U \in \mathcal{U}(C) \). Then \( (C, \alpha, \beta) \simeq (C, \text{Ad}_U \circ \alpha, \text{Ad}_U \circ \beta) \).

### 2.2. Multiplicative unitaries and quantum groups.

**Definition 2.5 (2).** Let \( \mathcal{H} \) be a Hilbert space. A unitary \( \mathcal{W} \in \mathcal{U}((\mathcal{H} \otimes \mathcal{H})) \) is multiplicative if it satisfies the pentagon equation

\[
\mathcal{W}_{23} \mathcal{W}_{12} = \mathcal{W}_{12} \mathcal{W}_{13} \mathcal{W}_{23} \quad \text{in} \quad \mathcal{U}((\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H}).
\]

Technical assumptions such as manageability (17) or, more generally, modularity (13) are needed in order to construct \( C^* \)-algebras out of a multiplicative unitary.

**Theorem 2.7 (13,14,17).** Let \( \mathcal{H} \) be a separable Hilbert space and \( \mathcal{W} \in \mathcal{U}((\mathcal{H} \otimes \mathcal{H})) \) a modular multiplicative unitary. Let

\[
\mathcal{A} := \{(\omega \otimes \text{id}) \mathcal{W} : \omega \in \mathcal{B}((\mathcal{H})_+)\}^{\text{CLS}},
\]

\[
\hat{\mathcal{A}} := \{(\text{id} \otimes \omega) \mathcal{W} : \omega \in \mathcal{B}((\mathcal{H})_+)\}^{\text{CLS}}.
\]

1. \( \mathcal{A} \) and \( \hat{\mathcal{A}} \) are separable, nondegenerate \( C^* \)-subalgebras of \( \mathcal{B}((\mathcal{H})_+) \).
2. \( \mathcal{W} \in \mathcal{U}(\hat{\mathcal{A}} \otimes \mathcal{A}) \subseteq \mathcal{U}((\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H}) \). We write \( \mathcal{W}^A \) for \( \mathcal{W} \) viewed as a unitary multiplier of \( \hat{\mathcal{A}} \otimes \mathcal{A} \) and call it reduced bicharacter.
3. There is a unique \( \Delta_A \in \text{Mor}(A, A \otimes A) \) such that

\[
(\text{id}_A \otimes \Delta_A) \mathcal{W}^A = \mathcal{W}^A_{12} \mathcal{W}^A_{13} \quad \text{in} \quad \mathcal{U}(\hat{\mathcal{A}} \otimes \mathcal{A} \otimes \mathcal{A});
\]

it is coassociative:

\[
(\Delta_A \otimes \text{id}_A) \circ \Delta_A = (\Delta_A \otimes \Delta_A) \circ \Delta_A,
\]

and satisfies the Podleś condition

\[
\Delta_A(A) \cdot (1_A \otimes A) = A \otimes A = (A \otimes 1_A) \cdot \Delta_A(A).
\]

4. There is a unique ultraweakly continuous, linear anti-automorphism \( R_A \) of \( A \) with

\[
\Delta_A \circ R_A = \sigma \circ (R_A \otimes R_A) \circ \Delta_A,
\]

where \( \sigma(x \otimes y) = y \otimes x \). It satisfies \( R_A^* = \text{id}_A \).

A \( C^* \)-quantum group is a \( C^* \)-bialgebra \( \mathcal{G} = (A, \Delta_A) \) constructed from a modular multiplicative unitary. We do not need Haar weights.
The dual multiplicative unitary is \( \hat{\mathcal{W}} := \Sigma \hat{\mathcal{W}}^* \hat{\mathcal{W}} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H}) \), where \( \Sigma(x \otimes y) = y \otimes x \). It is modular or manageable if \( \hat{\mathcal{W}} \) is. The C*-quantum group \( \hat{\mathcal{G}} = (\hat{A}, \hat{\Delta}) \) generated by \( \hat{\mathcal{W}} \) is the dual of \( \mathcal{G} \). Its comultiplication is characterised by

\[
(\hat{\Delta}_A \otimes \text{id}_A)W^A = W^A_{21}W^A_{13} \quad \text{in} \; \mathcal{U}(\hat{A} \otimes \hat{A} \otimes A).
\]

Let \( \mathcal{G} = (A, \Delta_A) \) be a C*-quantum group.

**Definition 2.15.** A continuous (right) coaction of \( \mathcal{G} \) on a C*-algebra \( A \) is a morphism \( \gamma: C \rightarrow C \otimes A \) with the following properties:

1. \( \gamma \) is injective;
2. \( \gamma \) is a comodule structure, that is, \((\text{id}_C \otimes \Delta_A)\gamma = (\gamma \otimes \text{id}_A)\gamma:

\[
\begin{array}{ccc}
C & \xrightarrow{\gamma} & C \otimes A \\
\downarrow{\gamma} & & \downarrow{\gamma \otimes \text{id}_A} \\
C \otimes A & \xrightarrow{\text{id}_C \otimes \Delta_A} & C \otimes A \otimes A
\end{array}
\]

(2.16)

3. \( \gamma \) satisfies the Podleś condition \( \gamma(C) \cdot (1_C \otimes A) = C \otimes A \).

We call \((C, \gamma)\) a \( \mathcal{G}\text{-C}^*\text{-algebra} \). We often drop \( \gamma \) from our notation.

A morphism \( f: C \rightarrow D \) between two \( \mathcal{G}\text{-C}^*\text{-algebras} \) \((C, \gamma)\) and \((D, \delta)\) is \( \mathcal{G}\text{-equivariant} \) if \( \delta \circ f = (f \otimes \text{id}_A) \circ \gamma \). Let \( \text{Mor}^\mathcal{G}(C, D) \) be the set of \( \mathcal{G}\text{-equivariant morphisms} \) from \( C \) to \( D \). Let \( \mathfrak{C}\text{alg}(\mathcal{G}) \) be the category with \( \mathcal{G}\text{-C}^*\text{-algebras} \) as objects and \( \mathcal{G}\text{-equivariant morphisms} \) as arrows.

**Example 2.17.** The trivial \( \mathcal{G}\text{-coaction} \) on a C*-algebra \( C \) is \( \tau: C \rightarrow C \otimes A, \; c \mapsto c \otimes 1_A \). It is always continuous. Theorem 2.7.3 implies that \( \Delta_A: A \rightarrow A \otimes A \) is a \( \mathcal{G}\text{-coaction} \) on \( A \) for any C*-quantum group \( \mathcal{G} = (A, \Delta_A) \). More generally, \( \text{id}_C \otimes \Delta_A: C \otimes A \rightarrow C \otimes A \otimes A \) is a continuous \( \mathcal{G}\text{-coaction} \) on \( C \otimes A \) for any C*-algebra \( C \). The following lemma says that any continuous coaction may be embedded into one of this form.

**Lemma 2.18.** Let \( C \) be a C*-algebra and \( D \) a C*-subalgebra of \( \mathcal{M}(C \otimes A) \) with

\[
(\text{id}_C \otimes \Delta_A)(D) \cdot (1_C \otimes A) = D \otimes A.
\]

Then \( D \) with the coaction \( \delta := (\text{id}_C \otimes \Delta_A)|_D: D \rightarrow D \otimes A \) is a \( \mathcal{G}\text{-C}^*\text{-algebra} \), and the embedding \( D \rightarrow \mathcal{M}(C \otimes A) \) is a \( \mathcal{G}\text{-equivariant morphism} \).

Every \( \mathcal{G}\text{-C}^*\text{-algebra} \) is isomorphic to one of this form.

**Proof.** Equation (2.19) implies that \( \text{id}_C \otimes \Delta_A \) maps \( D \) into \( \mathcal{M}(D \otimes A) \) as claimed.

The coaction \( \delta \) is injective and coassociative because \( \text{id}_C \otimes \Delta_A \) is, and (2.19) is the Podleś condition for \( \delta \). Thus \( \delta \) is a continuous \( \mathcal{G}\text{-coaction} \). The embedding is equivariant by construction.

Now let \( (C, \gamma) \) be a \( \mathcal{G}\text{-C}^*\text{-algebra} \). Let \( D := \gamma(C) \subseteq \mathcal{M}(C \otimes A) \). The comodule property (2.16) and the Podleś condition for \( \gamma \) imply that \( D \) satisfies (2.19):

\[
(\text{id}_C \otimes \Delta_A)\gamma(C) \cdot (1_C \otimes A) = (\gamma \otimes \text{id}_A)(\gamma(C) \cdot (1_C \otimes A))
\]

\[
= (\gamma \otimes \text{id}_A)(C \otimes A) = \gamma(C) \otimes A.
\]

The isomorphism \( \gamma: C \rightarrow D \) is \( \mathcal{G}\text{-equivariant} \) by the comodule property (2.16) of \( \gamma \).

**Definition 2.20.** A (right) corepresentation of \( \mathcal{G} \) on a Hilbert space \( \mathcal{H} \) is a unitary \( U \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A) \) with

\[
(\text{id}_{\mathbb{K}(\mathcal{H})} \otimes \Delta_A)U = U_{12}U_{13} \quad \text{in} \; \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A \otimes A).
\]
Definition 2.22. A covariant representation of \((C, \gamma, A)\) on a Hilbert space \(\mathcal{H}\) is a pair consisting of a corepresentation \(U \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)\) and a representation \(\varphi : C \to \mathcal{B}(\mathcal{H})\) that satisfy the covariance condition
\[
(\varphi \otimes \text{id}_A) \circ \gamma(c) = U(\varphi(c) \otimes 1_A)U^* \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)
\]
for all \(c \in C\). A covariant representation is called faithful if \(\varphi\) is faithful.

2.3. Universal quantum groups. The universal quantum group \(G^u := (A^u, \Delta_A^u)\) associated to \(G = (A, \Delta_A)\) is introduced in [II]. By construction, it comes with a reducing map \(A^u \to A\) and a universal bicharacter \(W^A \in \mathcal{U}(\hat{A} \otimes A^u)\). This may also be characterised as the unique bicharacter in \(\mathcal{U}(\hat{A} \otimes A^u)\) that lifts \(W^A \in \mathcal{U}(\hat{A} \otimes A)\) in the sense that \(\text{id}_A \otimes \Lambda(W^A) = W^A\).

Similarly, there are unique bicharacters in \(\mathcal{U}(\hat{A}^u \otimes A)\) and \(\mathcal{U}(\hat{A}^u \otimes A^u)\) that lift \(W^A \in \mathcal{U}(\hat{A} \otimes A)\); the latter is constructed in [II].

The universality of \(W^A \in \mathcal{U}(\hat{A}^u \otimes A)\) implies that for any corepresentation \(U^H\) of \(G\) on a Hilbert space (or Hilbert module) \(\mathcal{H}\), there is a unique representation \(\rho : \hat{A}^u \to \mathcal{B}(\mathcal{H})\) with \((\rho \otimes \text{id}_A)W^A = U^H\).

2.4. Bicharacters as quantum group morphisms. Let \(G = (A, \Delta_A)\) and \(H = (B, \Delta_B)\) be C\(^*\)-quantum groups. Let \(\hat{G} = (\hat{A}, \hat{\Delta}_A)\) and \(\hat{H} = (\hat{B}, \hat{\Delta}_B)\) be their duals.

Definition 2.24 ([II, Definition 16]). A bicharacter from \(G\) to \(\hat{H}\) is a unitary \(\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})\) with
\[
(\hat{\Delta}_A \otimes \text{id}_B)\chi = \chi_{23} \chi_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes \hat{B}),
\]
\[
(\text{id}_A \otimes \hat{\Delta}_B)\chi = \chi_{12} \chi_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \hat{B}).
\]

Bicharacters in \(\mathcal{U}(\hat{A} \otimes \hat{B})\) are interpreted as quantum group morphisms from \(G\) to \(H\) in [II]. We shall use bicharacters in \(\mathcal{U}(\hat{A} \otimes \hat{B})\) throughout and rewrite some definitions in [II] in this setting.

Definition 2.27. A right quantum group morphism from \(G\) to \(\hat{H}\) is a morphism \(\Delta_R : A \to A \otimes \hat{B}\) such that the following diagrams commute:
\[
\begin{array}{ccc}
A & \xrightarrow{\Delta_R} & A \otimes \hat{B} \\
\downarrow{\Delta_A} & & \downarrow{\Delta_A \otimes \text{id}_B} \\
A \otimes A & \xrightarrow{\text{id}_A \otimes \Delta_R} & A \otimes A \otimes \hat{B}
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\Delta_R} & A \otimes \hat{B} \\
\downarrow{\text{id}_A \otimes \hat{\Delta}_B} & & \downarrow{\text{id}_A \otimes \hat{\Delta}_B} \\
A \otimes \hat{B} & \xrightarrow{\Delta_R \otimes \text{id}_\hat{B}} & A \otimes \hat{B} \otimes \hat{B}
\end{array}
\]

The following theorem summarises some of the main results of [II].

Theorem 2.29. There are natural bijections between the following sets:
1. bicharacters \(\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})\) from \(G\) to \(\hat{H}\);
2. bicharacters \(\tilde{\chi} \in \mathcal{U}(\hat{B} \otimes \hat{A})\) from \(H\) to \(\hat{G}\);
3. right quantum group homomorphisms \(\Delta_R : A \to A \otimes \hat{B}\);
4. functors \(F : \mathcal{C}^*\text{alg}(\hat{G}) \to \mathcal{C}^*\text{alg}(\hat{H})\) with \(\text{For}_H \circ F = \text{For}_G\) for the forgetful functor \(\text{For}_G : \mathcal{C}^*\text{alg}(\hat{G}) \to \mathcal{C}^*\text{alg}\);
5. Hopf*-homomorphisms \(f : A^u \to \hat{B}^u\) between universal quantum groups;
6. bicharacters \(\chi^u \in \mathcal{U}(\hat{A}^u \otimes \hat{B}^u)\).

The first bijection maps a bicharacter \(\chi\) to
\[
\tilde{\chi} := \sigma(\chi^*).
\]
A bicharacter $\chi$ and a right quantum group homomorphism $\Delta_R$ determine each other uniquely via
\begin{equation}
(id_A \otimes \Delta_R)(W^A) = W^A_{12} \chi_{12}.
\end{equation}

The functor $F$ associated to $\Delta_R$ is the unique one that maps $(A, \Delta_A)$ to $(A, \Delta_R)$. In general, $F$ maps a continuous $\mathbb{G}$-coaction $\gamma: C \to C \otimes A$ to the unique $\mathbb{H}$-coaction $\delta: C \to C \otimes \hat{B}$ for which the following diagram commutes:
\begin{equation}
\begin{array}{ccc}
C & \xrightarrow{\gamma} & C \otimes A \\
\downarrow{\delta} & & \downarrow{id_C \otimes \Delta_R} \\
C \otimes \hat{B} & \xrightarrow{\gamma \otimes id_{\hat{B}}} & C \otimes A \otimes \hat{B}
\end{array}
\end{equation}

The bicharacter in $\mathcal{U}(\hat{A} \otimes \hat{B})$ associated to a Hopf $^*$-homomorphism $f: A^u \to \hat{B}^u$ is $\chi := (id_A \otimes \Delta_{\hat{B}})(W^A)$, where $W^A \in \mathcal{U}(\hat{A} \otimes A^u)$ is the unique bicharacter lifting $W^A \in \mathcal{U}(\hat{A} \otimes A)$ and $\Delta_{\hat{B}}: \hat{B}^u \to \hat{B}$ is the reducing map.

3. Heisenberg pairs and twisted tensor products

This section introduces Heisenberg and anti-Heisenberg pairs and uses them to construct our noncommutative tensor product, after establishing properties of Heisenberg pairs necessary for that purpose.

Let $\mathbb{G} = (A, \Delta_A)$ and $\mathbb{H} = (B, \Delta_B)$ be $C^*$-quantum groups. Let $W^A \in \mathcal{U}(\hat{A} \otimes A)$ and $W^B \in \mathcal{U}(\hat{B} \otimes B)$ be their reduced bicharacters. Let $\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})$ be a bicharacter from $A$ to $\hat{B}$. Heisenberg pairs and anti-Heisenberg pairs are pairs of representations $(\alpha, \beta)$ of $\hat{A}$ and $\hat{B}$ on the same Hilbert space $\mathcal{H}$ that satisfy suitable commutation relation.

We use these pairs to define twisted tensor products $C \boxtimes_D D$ in Section 3.2. A crucial technical point is to show that a pair of representations of $C$ and $D$ generates a crossed product $C^*$-algebra. Here the commutativity result in Section 3.1 is crucial. In addition, we construct examples of $\chi$-Heisenberg and $\chi$-anti-Heisenberg pairs, thus proving their existence, and characterise them in equivalent ways.

**Definition 3.1.** A pair of representations $\alpha: A \to \mathcal{B}(\mathcal{H}), \beta: B \to \mathcal{B}(\mathcal{H})$ is called a $\chi$-Heisenberg pair or briefly Heisenberg pair if
\begin{equation}
W^A_{1\alpha} W^B_{2\beta} = W^B_{2\beta} W^A_{1\alpha} \chi_{12} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}));
\end{equation}
here $W^A_{1\alpha} := ((id_A \otimes \alpha)W^A)_{13}$ and $W^B_{2\beta} := ((id_B \otimes \beta)W^B)_{23}$. It is called a $\chi$-anti-Heisenberg pair or briefly anti-Heisenberg pair if
\begin{equation}
W^B_{2\beta} W^A_{1\alpha} = \chi_{12} W^A_{1\alpha} W^B_{2\beta} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H})),
\end{equation}
with similar conventions as above.

We name these pairs after Heisenberg because of the following example:

**Example 3.4.** Let $A = B = C^*(\mathbb{R})$ be the group $\mathbb{R}$ viewed as a quantum group, and let $\chi \in \mathcal{U}(\hat{A} \otimes \hat{B}) \cong C(\mathbb{R} \times \mathbb{R}, \mathbb{T})$ be the standard bicharacter $(s, t) \mapsto \exp(i st)$. A pair of representations of $\hat{A}$ is a pair of unitary one-parameter groups $(U_1(s), U_2(t))_{s,t \in \mathbb{R}}$. Equation (3.2) is equivalent to the canonical commutation relation in the Weyl form:
\[
U_2(t)U_1(s) = \exp(-ist)U_1(s)U_2(t) \quad \text{for all } s, t \in \mathbb{R}.
\]

The case where $\mathbb{H} = \hat{\mathbb{G}}$ and $\chi = W^A \in \mathcal{U}(\hat{A} \otimes A)$ is the reduced bicharacter of $\mathbb{G}$ is particularly interesting:
Then the following are equivalent:

Theorem 2.7. Therefore, if

Example

The transposition is a linear, involutive anti-automorphism

Lemma 3.11. Let

Let

the condition of being a Heisenberg pair in terms of

\( \hat{\Delta} \)

quantum group homomorphism

Since

is a

bicharacter in

Definition 3.5. A \( W^A \)-Heisenberg or \( W^A \)-anti-Heisenberg pair is also called a \( G \)-Heisenberg pair or \( G \)-anti-Heisenberg pair, respectively.

Lemma 3.6. A pair of representations \((\pi, \hat{\pi})\) of \( A \) and \( \hat{A} \) on \( H \) is a \( G \)-Heisenberg pair if and only if

\[ W^A_{\pm \alpha} W^A_{1\pi} = W^A_{1\pi} W^A_{\pm \alpha} \quad \text{in } U(\hat{A} \otimes \mathbb{K}(H) \otimes A). \]

It is a \( G \)-anti-Heisenberg pair if and only if

\[ W^A_{1\pi} W^A_{\pm \alpha} = W^A_{\pm \alpha} W^A_{1\pi} \quad \text{in } U(\hat{A} \otimes \mathbb{K}(H) \otimes A). \]

Proof. Let \( \pi \) and \( \hat{\pi} \) be representations of \( A \) and \( \hat{A} \) on \( H \) satisfying (3.7). When we apply \( \sigma_{23} \) to both sides of (3.7) we get

\[ (\hat{W}^A_{\pm \alpha})^* W^A_{1\pi} = W^A_{1\pi} (\hat{W}^A_{\pm \alpha})^* \quad \text{in } U(\hat{A} \otimes A \otimes \mathbb{K}(H)). \]

This is equivalent to \( W^A_{1\pi} \hat{W}^A_{\pm \alpha} = \hat{W}^A_{\pm \alpha} W^A_{1\pi} \), which is (3.2) for \( \hat{B} = A, \chi = W^A, \alpha = \pi \) and \( \beta = \hat{\pi} \). This computation may be reversed as well.

The computation for anti-Heisenberg pairs is similar.

Example 3.9. Let \( W \in U(H \otimes H) \) be a modular multiplicative unitary generating \( G \). Thus there are faithful representations \( \pi: A \to B(H) \) and \( \hat{\pi}: \hat{A} \to B(H) \) with \( W = (\hat{\pi} \otimes \pi)(W^A) \). They form a \( G \)-Heisenberg pair: the condition (3.7) is equivalent to the pentagon equation (2.6) for \( W \). Conversely, a pair \((\pi, \hat{\pi})\) of faithful representations is a \( G \)-Heisenberg pair if and only if \((\hat{\pi} \otimes \pi)(W^A) \) is a multiplicative unitary.

Let \( H \) be the conjugate Hilbert space to the Hilbert space \( H \). The transpose of an operator \( x \in B(H) \) is the operator \( x^T \in B(\overline{H}) \) defined by \( x^T(\xi) := \overline{x^*} \xi \) for all \( \xi \in H \). The transposition is a linear, involutive anti-automorphism \( B(H) \to B(\overline{H}) \). The unitary antipode \( R_A: A \to A \) is also a linear, involutive anti-automorphism (see Theorem 2.4). Therefore, if \( \alpha: A \to B(H) \) and \( \beta: B \to B(\overline{H}) \) are representations, then so are

\[ \tilde{\alpha}: A \to B(\overline{H}), \quad a \mapsto (R_A(a))^T, \]

\[ \tilde{\beta}: B \to B(\overline{H}), \quad b \mapsto (R_B(b))^T. \]

Lemma 3.10. The pair \((\alpha, \beta)\) is Heisenberg if and only if \((\tilde{\alpha}, \tilde{\beta})\) is anti-Heisenberg.

Proof. Let \((\alpha, \beta)\) be a pair of representations. We have \((R_A \otimes R_B)\chi = \chi\) for any bicharacter in \( U(A \otimes \hat{B}) \) by [11, Proposition 3.10]. In particular, this applies to \( W^A \) and \( W^B \). Since \( R_A \otimes R_B \otimes T \) is antimultiplicative, we get

\[ W^B_{2\beta} W^A_{1\alpha} = (R_A \otimes R_B \otimes T)(W^A_{1\alpha} W^B_{2\beta}), \]

\[ \chi_{12} W^A_{1\alpha} W^B_{2\beta} = (R_A \otimes R_B \otimes T)(W^B_{2\beta} W^A_{1\alpha} \chi_{12}). \]

Since \( R_A \otimes R_B \otimes T \) is bijective, we see that \((\alpha, \beta)\) is a Heisenberg pair if and only if \((\tilde{\alpha}, \tilde{\beta})\) is an anti-Heisenberg pair.

Thus Heisenberg pairs and anti-Heisenberg pairs are essentially equivalent.

Recall that a bicharacter \( \chi \) yields a dual bicharacter \( \hat{\chi} \in U(\hat{B} \otimes \hat{A}) \) and a right quantum group homomorphism \( \Delta_R: A \to A \otimes \hat{B} \) by Theorem 2.29. Similarly, \( \hat{\chi} \) yields a right quantum group homomorphism \( \hat{\Delta}_R: B \to B \otimes \hat{A} \). We reformulate the condition of being a Heisenberg pair in terms of \( \hat{\chi}, \Delta_R \) and \( \hat{\Delta}_R \), respectively:

Lemma 3.11. Let \( \alpha \) and \( \beta \) be representations of \( A \) and \( B \) on a Hilbert space \( H \). Then the following are equivalent:

1. \((\alpha, \beta)\) is a Heisenberg pair;
2. \((\beta, \alpha)\) is a \( \hat{\chi} \)-Heisenberg pair;
Proof. (1) $\iff$ (2): (1) is equivalent to
\[ W_{1\alpha}^A W_{2\beta}^B \chi_{12} = W_{2\beta}^B W_{1\alpha}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H})) \]
by (3.2). Applying $\sigma_{12}$ gives
\[ (3.12) \quad W_{2\alpha}^A W_{1\beta}^B \chi_{12} = W_{1\beta}^B W_{2\alpha}^A \quad \text{in } \mathcal{U}(\hat{B} \otimes \hat{A} \otimes \mathbb{K}(\mathcal{H})) \]
which is equivalent to $(\beta, \alpha)$ being a $\chi$-Heisenberg pair. Thus (1) $\iff$ (2).

(1) $\iff$ (3): Let $(\alpha, \beta)$ be a Heisenberg pair. The following computation takes place in $\mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes \hat{B})$:
\[
(id_{\hat{A}} \otimes \alpha \otimes id_{\hat{B}})(id_{\hat{A}} \otimes \Delta_R)W^A = W_{1\alpha}^A \chi_{12} = \sigma_{23}(W_{1\alpha}^A) = \sigma_{23}((W_{2\beta}^B)^* W_{1\alpha}^A) = (\overline{W_{\beta}^B})_i (\overline{W_{\beta}^B})_j^*; 
\]
the first equality uses (2.31); the second equality is obvious; the third equality uses (3.2); and the last equality uses (3.14). Since $\{(\omega \otimes id_{\hat{A}})W^A : \omega \in \hat{A}^\prime\}$ is linearly dense in $A$, slicing the first leg of the first and the last expression in the above equation shows that (1)$\iff$(3)

Conversely, applying $id_{\hat{A}} \otimes \alpha \otimes id_{\hat{B}}$ on both sides of (2.31) and using (3), we get
\[ W_{1\alpha}^A \chi_{13} = (id_{\hat{A}} \otimes (\alpha \otimes id_{\hat{B}})\Delta_R)W^A = (\overline{W_{\beta}^B})_i (\overline{W_{\beta}^B})_j^* \quad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes \hat{B}); \]
applying $\sigma_{23}$ to this gives (3.2). Thus (3)$\iff$(1).

To prove (2)$\iff$(4) argue as in the proof that (1)$\iff$(3).

\[\Box\]

Lemma 3.13. Let $(\pi, \hat{\pi})$ and $(\eta, \hat{\eta})$ be $G$- and $H$-Heisenberg pairs on Hilbert spaces $\mathcal{H}_{\pi}$ and $\mathcal{H}_{\eta}$ respectively. Then the pair of representations $(\alpha, \beta)$ of $A$ and $B$ on $\mathcal{H}_{\pi} \otimes \mathcal{H}_{\eta}$ defined by $\alpha(a) := (\pi \otimes \hat{\eta})\Delta_R(a)$ and $\beta(b) := 1_{\mathcal{H}_{\pi}} \otimes \eta(b)$ is a $\chi$-Heisenberg pair.

Proof. First we check the following equation:
\[ (3.14) \quad \chi_{1\theta} W_{2\eta}^B = W_{2\eta}^B \chi_{1\theta} \chi_{12} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}_{\eta})). \]
The coaction $\hat{B} \to \hat{B} \otimes \hat{B}$ associated to the reduced bicharacter $W^B$ is the usual comultiplication $\Delta_B$. Hence
\[
(\overline{W_{\eta}^B})_i \chi_{1\theta} (\overline{W_{\eta}^B})_j^* = (id_{\hat{A}} \otimes \hat{\eta} \otimes id_{\hat{B}})(id \otimes \Delta_B) \chi = (id_{\hat{A}} \otimes \hat{\eta} \otimes id_{\hat{B}})(\chi_{12} \chi_{13}) = \chi_{1\theta} \chi_{13}
\]
in $\mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}_{\eta}) \otimes \hat{B})$ because of Lemma 3.11(4) and the bicharacter property of $\chi$. When we flip the last two legs, we turn $\overline{W_{\eta}^B}$ into $(W_{2\eta}^B)^*$. Rearranging then gives (3.14).

Now we can check that $(\alpha, \beta)$ is a Heisenberg pair. The following computation takes place in $\mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}_{\pi}) \otimes \mathbb{K}(\mathcal{H}_{\eta}))$:
\[
W_{1\alpha}^A W_{2\beta}^B = W_{1\alpha}^A W_{2\beta}^B \chi_{1\theta} \chi_{12} = W_{2\beta}^B W_{1\alpha}^A W_{1\theta}^A \chi_{1\theta} \chi_{12} = W_{2\beta}^B W_{1\alpha}^A \chi_{1\theta} \chi_{12}
\]
the first equality uses the definitions of $\alpha$ and $\beta$ and (3.14); the second equality uses (3.14); the third equality uses that $W_{1\tau}^A$ and $W_{2\eta}^B$ commute; and the fourth equality uses the definitions of $\alpha$ and $\beta$ again.

\[\Box\]
3.1. Commutativity and Heisenberg pairs. Locality principles in quantum field theory always require commutation relations of the simplest possible form $xy = yx$. Our noncommutative tensor products are based on more complicated commutation relations. We get ordinary commutativity, however, if we put a Heisenberg and an anti-Heisenberg pair together. This is crucial for our noncommutative tensor product to exist.

Proposition 3.15. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces; let $\alpha$ and $\beta$ be representations of $A$ and $B$ on $\mathcal{H}$, respectively; and let $\bar{\alpha}$ and $\bar{\beta}$ be representations of $A$ and $B$ on $\mathcal{K}$, respectively. Then the following are equivalent:

(1) the representations $(\alpha \otimes \bar{\alpha})\Delta_A$ and $(\beta \otimes \bar{\beta})\Delta_B$ of $A$ and $B$ on $\mathcal{H} \otimes \mathcal{K}$ commute, that is, for any $a \in A$ and $b \in B$, we have

$$[(\alpha \otimes \bar{\alpha})\Delta_A(a), (\beta \otimes \bar{\beta})\Delta_B(b)] = 0;$$

(2) there is a bicharacter $\chi \in \mathcal{U}(\hat{\mathcal{A}} \otimes \hat{\mathcal{B}})$ such that $(\alpha, \beta)$ is a $\chi$-Heisenberg pair and $(\bar{\alpha}, \bar{\beta})$ is a $\chi$-anti-Heisenberg pair.

Proof. Equation (3.16) is equivalent to

$$W_{1\alpha}^A W_{1\beta}^B W_{2\beta}^B W_{2\beta}^B = W_{2\beta}^B W_{2\beta}^B W_{1\alpha}^A W_{1\alpha}^A \quad \text{in } \mathcal{U}(\hat{\mathcal{A}} \otimes \hat{\mathcal{B}} \otimes \mathcal{K}(\mathcal{H} \otimes \mathcal{K}(\mathcal{K}))$$

because of (2.10) and (2.8) for $W^A$ and $W^B$. Commuting $W_{1\alpha}^A$ with $W_{2\beta}^B$ and $W_{2\beta}^B$ with $W_{1\alpha}^A$, Equation (3.16) becomes

$$W_{1\alpha}^A (W_{2\beta}^B)^* W_{1\alpha}^A W_{2\beta}^B = W_{2\beta}^B W_{1\alpha}^A (W_{2\beta}^B)^* (W_{1\alpha}^A)^*.$$

If $\chi \in \mathcal{U}(\hat{\mathcal{A}} \otimes \hat{\mathcal{B}})$ is a bicharacter, $(\alpha, \beta)$ a Heisenberg pair and $(\bar{\alpha}, \bar{\beta})$ an anti-Heisenberg pair, then (3.17) follows, and hence (3.16).

It remains to show (2) assuming (3.17). Let $\chi$ be the unitary in (3.17). It belongs to $1 \otimes \hat{\mathcal{A}} \otimes \hat{\mathcal{B}} \otimes 1$ because its first definition has $1_{\mathcal{K}}$ in the fourth leg and its second definition has $1_{\mathcal{H}}$ in the third leg.

We check that $\chi$ is a bicharacter in both legs. First we check $\Delta A \otimes \text{id}_B$:

$$(\Delta A \otimes \text{id}_B)\chi = (\Delta A \otimes \text{id}_B)(W_{3\beta}^B W_{1\alpha}^A (W_{2\beta}^B)^* (W_{1\alpha}^A)^*) = W_{3\beta}^B W_{2\alpha}^A W_{1\alpha}^A (W_{2\beta}^B)^* (W_{1\alpha}^A)^* = \chi_{23} W_{2\alpha}^A \chi_{13} (W_{1\alpha}^A)^* $$

the first and third equality use that $\chi$ is the right hand side of (3.17); the second equality uses (2.25) for $W^A$; and the last equality uses that $W_{2\alpha}^A$ and $\chi_{13}$ commute. A similar computation using that $\chi$ is the left hand side in (3.17) yields $\chi_{12} \chi_{13}$; thus $\chi \in \mathcal{U}(\hat{\mathcal{A}} \otimes \hat{\mathcal{B}})$ is a bicharacter. Finally, (3.17) says that $(\alpha, \beta)$ is a $\chi$-Heisenberg pair and that $(\bar{\alpha}, \bar{\beta})$ is a $\chi$-anti-Heisenberg pair.

3.2. Twisted tensor products via Heisenberg pairs. Let $G = (A, \Delta_A)$ and $H = (B, \Delta_B)$ be C*-quantum groups, let $\chi \in \mathcal{U}(\hat{\mathcal{A}} \otimes \hat{\mathcal{B}})$ be a bicharacter, let $(C, \gamma)$ be a $G$-C*-algebra, and let $(D, \delta)$ be an $\mathbb{H}^\times$-C*-algebra. Let $(\alpha, \beta)$ be a $\chi$-Heisenberg pair on some Hilbert space $\mathcal{H}$.

Using this data, we now construct a crossed product $(C \boxtimes \chi D, \iota_C, \iota_D)$ of $C$ and $D$ in the sense of Definition 2.1. A more precise notation is

$$C \boxtimes \chi D = (C, \gamma) \boxtimes_{\chi} (D, \delta).$$

There is no need to mention $(\alpha, \beta)$ in our notation because all Heisenberg pairs give equivalent crossed products; we will prove this in Section 4. Our definition is based
Then applying a state

\[ \tau_C : C \to C \otimes D \otimes \mathbb{K}(\mathcal{H}), \quad c \mapsto (\text{id}_C \otimes \alpha) \gamma(c)_{13}, \]

\[ \tau_D : D \to C \otimes D \otimes \mathbb{K}(\mathcal{H}), \quad d \mapsto (\text{id}_D \otimes \beta) \delta(d)_{23}. \]

**Lemma 3.18.** Let \( X \subseteq C \) and \( Y \subseteq D \) be closed subspaces with

\[ \gamma(X) \cdot (1_C \otimes A) = X \otimes A \quad \text{and} \quad \delta(Y) \cdot (1_D \otimes B) = Y \otimes B. \]

Then \( \tau_C(X) \cdot \tau_D(Y) = \tau_D(Y) \cdot \tau_C(X) \) in \( \mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H})). \)

**Proof.** Let \((\bar{\alpha}, \bar{\beta})\) be a \( \chi \)-anti-Heisenberg pair on a Hilbert space \( \mathcal{K} \). The definition of \( \tau_C \) and the comodule property \( (2.10) \) for \( \gamma \) yield

\[ (\tau_C \otimes \bar{\alpha}) \gamma = ((\text{id}_C \otimes \alpha \otimes \bar{\alpha})(\gamma \otimes \text{id}_A) \gamma)_{134} = ((\text{id}_C \otimes (\alpha \otimes \bar{\alpha}) \Delta_A) \gamma)_{134}; \]

Similarly,

\[ (\tau_D \otimes \bar{\beta}) \delta = ((\text{id}_D \otimes (\beta \otimes \bar{\beta}) \Delta_B) \delta)_{234}. \]

Now Proposition \( 3.15 \) yields

\[ (\tau_C \otimes \bar{\alpha}) \gamma(c) \cdot (\tau_D \otimes \bar{\beta}) \delta(d) = (\tau_D \otimes \bar{\beta}) \delta(d) \cdot (\tau_C \otimes \bar{\alpha}) \gamma(c) \]

for all \( c \in C, \; d \in D. \)

Since \( \bar{\alpha}(A) \cdot \mathbb{K}(\mathcal{K}) = \mathbb{K}(\mathcal{K}) \), our assumption \( \gamma(X) \cdot (1_C \otimes A) = X \otimes A \) gives

\[ ((\tau_C \otimes \bar{\alpha}) \gamma(X)) \cdot \mathbb{K}(\mathcal{K})_4 = (\tau_C \otimes \bar{\alpha})(\gamma(X) \cdot (1_C \otimes A)) \cdot \mathbb{K}(\mathcal{K})_4 \]

\[ = (\tau_C(X) \otimes \bar{\alpha}(A)) \cdot \mathbb{K}(\mathcal{K})_4 = \tau_C(X) \otimes \mathbb{K}(\mathcal{K}). \]

Similarly, \( \bar{\beta}(B) \cdot \mathbb{K}(\mathcal{K}) = \mathbb{K}(\mathcal{K}) \) and \( \delta(Y) \cdot (1_D \otimes B) = Y \otimes B \) give

\[ ((\tau_D \otimes \bar{\beta}) \delta(Y)) \cdot \mathbb{K}(\mathcal{K})_4 = \tau_D(Y) \otimes \mathbb{K}(\mathcal{K}). \]

Equation \( (3.19) \) gives

\[ (\tau_C \otimes \bar{\alpha}) \gamma(X) \cdot (\tau_D \otimes \bar{\beta}) \delta(Y) = (\tau_D \otimes \bar{\beta}) \delta(Y) \cdot (\tau_C \otimes \bar{\alpha}) \gamma(X). \]

Multiplying this equation on the right with \( 1_C \otimes D \otimes \mathcal{H} \otimes \mathbb{K}(\mathcal{K}) \) and using the computations above to simplify, we get

\[ (\tau_C(X) \cdot \tau_D(Y)) \otimes \mathbb{K}(\mathcal{K}) = (\tau_D(X) \cdot \tau_C(Y)) \otimes \mathbb{K}(\mathcal{K}). \]

Applying a state \( \omega \) on \( \mathbb{K}(\mathcal{K}) \) to this equation gives \( \tau_C(X) \cdot \tau_D(Y) = \tau_D(X) \cdot \tau_C(Y) \) as desired. \( \square \)

**Lemma 3.20.** \( \tau_C(C) \cdot \tau_D(D) = \tau_D(D) \cdot \tau_C(C) \) in \( \mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H})). \)

**Proof.** Since our coactions satisfy the Podleś conditions, this is the special case \( X = C \) and \( Y = D \) of Lemma 3.18. \( \square \)

Lemma 3.20 and the discussion after Definition 2.1 imply that

\[ C \boxtimes_{\chi} D := \tau_C(C) \cdot \tau_D(D) \]

is a \( C^* \)-algebra, that \( \tau_C \) and \( \tau_D \) are morphisms from \( C \) and \( D \) to \( C \boxtimes_{\chi} D \), respectively, and that \( (C \boxtimes_{\chi} D, \tau_C, \tau_D) \) is a crossed product of \( C \) and \( D \).

The following observation is useful to study slice maps on \( C \boxtimes_{\chi} D \).

**Lemma 3.21.** In the situation of Lemma 3.18

\[ (3.22) \quad \tau_C(X) \cdot \tau_D(Y) \cdot \mathbb{K}(\mathcal{H})_3 = X \otimes Y \otimes \mathbb{K}(\mathcal{H}), \]

where the right hand side means the closed linear span of \( x \otimes y \otimes z \) with \( x \in X \), \( y \in Y \), \( z \in \mathbb{K}(\mathcal{H}) \). In particular, \( (C \boxtimes_{\chi} D) \cdot \mathbb{K}(\mathcal{H})_3 = C \otimes D \otimes \mathbb{K}(\mathcal{H}). \)
Proof. Since $\mathbb{K}(\mathcal{H}) = \beta(B) \cdot \mathbb{K}(\mathcal{H})$, we may compute

$$\iota_{D}(Y) \cdot \mathbb{K}(\mathcal{H})_{3} = ((\text{id}_{D} \otimes \beta)(\delta(Y) \cdot (1_{D} \otimes B)))_{23} \mathbb{K}(\mathcal{H})_{3} = (Y \otimes \beta(B) \cdot \mathbb{K}(\mathcal{H}))_{23} \mathbb{K}(\mathcal{H})_{3} \cdot Y_{2}.$$  

Here $Y_{2}$ and $\mathbb{K}(\mathcal{H})_{3}$ mean $Y$ and $\mathbb{K}(\mathcal{H})$ in the second and third leg, respectively. A similar computation for $\iota_{C}(X)$ using $\mathbb{K}(\mathcal{H}) = \alpha(A) \cdot \mathbb{K}(\mathcal{H})$ now gives (3.22). □

Example 3.23. Assume that the coaction $\gamma$ is trivial. Then $\gamma(c)_{13} = c \otimes 1 \otimes 1$, so that $C \boxtimes_{\chi} D \cong C \otimes D$, embedded into $\mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$ via $\text{id}_{C} \otimes (\text{id}_{D} \otimes \beta)$. We also get $C \boxtimes_{\chi} D \cong C \otimes D$ if $\delta$ is trivial.

Lemma 3.24. Let $C_{0}$ and $D_{0}$ be $C^{*}$-algebras and equip $C_{0} \otimes C$ and $D_{0} \otimes D$ with the coactions $\text{id}_{C_{0}} \otimes \gamma$ and $\text{id}_{D_{0}} \otimes \delta$, respectively. Then

$$(3.25) \quad (C_{0} \otimes C) \boxtimes_{\chi} (D_{0} \otimes D) = C_{0} \otimes D_{0} \otimes (C \boxtimes_{\chi} D).$$

Proof. The maps $\iota_{C_{0} \otimes C}$ and $\iota_{D_{0} \otimes D}$ are $\text{id}_{C_{0}} \otimes \iota_{C}$ and $\text{id}_{D_{0}} \otimes \iota_{D}$, respectively. □

4. Hilbert space representation of the twisted tensor product

Let $\mathbb{G} = (A, \Delta_{A})$, $\mathbb{H} = (B, \Delta_{B})$, $\chi \in U(\hat{A} \otimes \hat{B})$, $(C, \gamma)$ and $(D, \delta)$ be as before, so that the twisted tensor product $C \boxtimes_{\chi} D$ is defined. We are going to construct a faithful Hilbert space representation of $C \boxtimes_{\chi} D$ using covariant Hilbert space representations of $(C, \gamma)$ and $(D, \delta)$. This yields an alternative definition of $C \boxtimes_{\chi} D$ and shows that $C \boxtimes_{\chi} D$ does not depend on the Heisenberg pair used in its construction.

Our new construction uses faithful covariant representations $(\varphi, U^{\mathbb{G}})$ of $(C, \gamma, A)$ and $(\psi, U^{\mathbb{K}})$ of $(D, \delta, B)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. (We will show below that such faithful covariant representations always exist.)

The bicharacter $\chi$ and the corepresentations provide a unitary operator $Z$ on $\mathcal{H} \otimes \mathcal{K}$ as follows:

Theorem 4.1. Let $U^{\mathbb{G}} \in U(\mathcal{H} \otimes A)$ and $U^{\mathbb{K}} \in U(\mathcal{K} \otimes B)$ be corepresentations of $\mathbb{G}$ and $\mathbb{H}$, respectively. Then there is a unique unitary $Z \in U(\mathcal{H} \otimes \mathcal{K})$ that satisfies

$$(4.2) \quad U^{\mathbb{G}}_{10}U^{\mathbb{K}}_{20}Z_{12} = U^{\mathbb{K}}_{20}U^{\mathbb{G}}_{10} \quad \text{in} U(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L})$$

for any $\chi$-Heisenberg pair $(\alpha, \beta)$ on any Hilbert space $\mathcal{L}$.

With this unitary $Z$, define representations $\varphi_{1}$ and $\tilde{\psi}_{2}$ of $C$ and $D$ on $\mathcal{H} \otimes \mathcal{K}$ by

$$\varphi_{1}(c) := \varphi(c) \otimes 1_{\mathcal{K}}, \quad \tilde{\psi}_{2}(d) := Z(1_{\mathcal{H}} \otimes \psi(d))Z^{*}.$$  

Theorem 4.3. Let $(\varphi, U^{\mathbb{G}})$ and $(\psi, U^{\mathbb{K}})$ be faithful covariant representations of $(C, \gamma, A)$ and $(D, \delta, B)$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Construct $\varphi_{1}$ and $\tilde{\psi}_{2}$ as above. Then there is a unique faithful representation $\rho: C \boxtimes_{\chi} D \to B(\mathcal{H} \otimes \mathcal{K})$ with $\rho \circ \iota_{C} = \varphi_{1}$ and $\rho \circ \iota_{D} = \tilde{\psi}_{2}$.

Example 4.4. If $\chi = 1$, then we may take $Z = 1$. Thus $\tilde{\psi}_{2} = \psi_{2}$ and the crossed product is simply the minimal tensor product $C \otimes D$.

In the rest of this section, we prove the claims above and use the main theorem to show that the twisted tensor product does not depend on auxiliary choices.

First we construct faithful covariant representations:

Example 4.5. Let $\varphi_{0}: C \to B(\mathcal{H}_{0})$ be any faithful Hilbert space representation. Let $(\pi, \hat{\pi})$ be a faithful $\mathbb{G}$-Heisenberg pair on a Hilbert space $\mathcal{H}_{\pi}$; this exists because of Example 3.9. Let $\mathcal{H} := \mathcal{H}_{0} \otimes \mathcal{H}_{\pi}$ and identify $\mathbb{K}(\mathcal{H}) \cong \mathbb{K}(\mathcal{H}_{0}) \otimes \mathbb{K}(\mathcal{H}_{\pi})$. The
unitary $U := 1_{\mathcal{H}_0} \otimes W^A_{23} \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$ is a corepresentation; since $\varphi_0$, $\pi$ and $\gamma$ are faithful morphisms, $\varphi := (\varphi_0 \otimes \pi) \circ \gamma: C \to \mathbb{B}(\mathcal{H})$ is a faithful representation. The following computation in $\mathcal{M}(C \otimes \mathbb{K}(\mathcal{H}_z) \otimes A)$ implies the covariance condition for $(\varphi, U)$:

$$(\text{id}_C \otimes \pi)\gamma(c)\gamma(c) = (\text{id}_C \otimes (\pi \otimes \text{id}_A))\Delta_A(c)\gamma(c) = (W^A_{23})(\gamma(c) \otimes 1_A)(W^{A^*}_{23})^*$$

for all $c \in C$, where we used (4.6) and Lemma 3.1 with $B = \hat{A}$ and $\Delta_R = \Delta_A$.

Now we prove Theorem 4.1. The uniqueness of $Z$ is clear from

$$Z_{12} = (U_{12})^*(U_{13}^*)^*U_{23}^*U_{12}^R.$$ 

Existence means that the operator on the right acts identically on the third leg and does not depend on the Heisenberg pair. The quickest way to prove this uses universal quantum groups to turn corepresentations into representations. Let $\rho_1: \hat{A}^u \to \mathbb{B}(\mathcal{H})$ and $\rho_2: \hat{B}^u \to \mathbb{B}(\mathcal{K})$ be the unique representations with $(\rho_1 \otimes \text{id}_A)\hat{W}^A = U^H$ and $(\rho_2 \otimes \text{id}_B)\hat{W}^B = U^K$ (see Section 2.3).

The bicharacter $\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})$ lifts uniquely to a bicharacter $\chi^u \in \mathcal{U}(\hat{A}^u \otimes \hat{B}^u)$ by Theorem 2.29. We claim that

$$(4.6) \quad Z := (\rho_1 \otimes \rho_2)(\chi^u)^* \in \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$$

verifies (1.2) (for any $\chi$-Heisenberg pair $(\alpha, \beta)$). (Our formulation of Theorem 4.1 highlights the property of the operator $(\rho_1 \otimes \rho_2)(\chi^u)^*$ that is crucial for the proof of Theorem 4.3 and it avoids universal quantum groups.)

We will actually prove

$$(4.7) \quad \overline{W}_{1a}^A \overline{W}_{23}^B = W_{1a}^B W_{12}^A \chi_{12}^a \quad \text{in } \mathcal{U}(\hat{A}^u \otimes \hat{B}^u \otimes \mathbb{K}(\mathcal{L}))$$

for any $\chi$-Heisenberg pair $(\alpha, \beta)$. Applying $\rho_1$ and $\rho_2$ to the first two legs then gives (4.2) because $(\rho_1 \otimes \text{id}_A)\hat{W}^A = U^H$ and $(\rho_2 \otimes \text{id}_B)\hat{W}^B = U^K$.

When we apply the reducing morphisms $\Lambda_A: \hat{A}^u \to \hat{A}$ and $\Lambda_B: \hat{B}^u \to \hat{B}$ to the first two legs in (4.7), we get $W_{1a}^A W_{23}^B = W_{23}^B W_{1a}^A \chi_{12}^a$, which is exactly the definition of a Heisenberg pair (see Definition 4.1). A routine computation shows that

$$T := (\overline{W}_{1a}^A)^*(\overline{W}_{23}^B)^* \overline{W}_{12}^A \overline{W}_{1a}^B \in \mathcal{U}(\hat{A}^u \otimes \hat{B}^u \otimes \mathbb{K}(\mathcal{L}))$$

is a character in the first two legs, that is, $(\hat{\Delta}_A^u \otimes \text{id}_{\hat{B}^u} \otimes \text{id}_{\mathcal{L}})T = T_{234}T_{134}$ and $(\text{id}_{\hat{A}^u} \otimes \hat{\Delta}_B^u \otimes \text{id}_{\mathcal{L}})T = T_{124}T_{134}$. Thus $T$ and $\chi_{12}^a$ are two bicharacters in $\mathcal{U}(\hat{A}^u \otimes \hat{B}^u \otimes \mathbb{K}(\mathcal{L}))$ that both lift the bicharacter $\chi_{12}^a$ in $\mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{L}))$. Using Lemma 4.6 twice, we get that any such bicharacter has a unique lifting. Thus $T = \chi_{12}^a$ as asserted. This finishes the proof of Theorem 4.1.

Now we come to the proof of Theorem 4.3. The Hilbert space representation

$$\varphi \otimes \psi \otimes \text{id}: C \otimes D \otimes \mathbb{K}(\mathcal{L}) \to \mathbb{B}(\mathcal{H} \otimes \mathbb{K} \otimes \mathcal{L})$$

is faithful because $\varphi$ and $\psi$ are faithful. Hence the pair of representations

$$(\varphi \otimes \alpha)\gamma_{13}: C \to \mathbb{B}(\mathcal{H} \otimes \mathbb{K} \otimes \mathcal{L})$$

$$(\psi \otimes \beta)\delta_{23}: D \to \mathbb{B}(\mathbb{K} \otimes \mathbb{K} \otimes \mathcal{L})$$

of $C$ and $D$ gives a faithful representation of $C \boxtimes_a D$; that is, there is a unique faithful representation of $C \boxtimes_a D$ that gives the above two representations when composed with $\iota_C$ and $\iota_D$.

**Lemma 4.8.** The pair of representations $(\varphi_1, Ad_{Z_{\alpha}} \circ \psi_2)$ of $(C, D)$ on $\mathbb{K} \otimes \mathbb{K} \otimes \mathcal{L}$ is unitarily equivalent to the pair $((\varphi \otimes \alpha)\gamma_{13}, (\psi \otimes \beta)\delta_{23})$ on the same Hilbert space through conjugation by the unitary $U_{1a}^H U_{23}^K$. 


Proof. We must prove
\[ U_{10}^H U_{23}^K (\phi(c) \otimes 1_C \otimes 1_L) (U_{23}^K)^* (U_{10}^H)^* = (\phi \otimes \alpha) \gamma_{13}(c), \]
\[ U_{10}^H U_{23}^K Z_{12}(1_H \otimes \psi(d) \otimes 1_L) Z_{12}^* (U_{23}^K)^* (U_{10}^H)^* = (\psi \otimes \beta) \delta_{23}(d) \]
for all \( c \in C, \ d \in D \). To check the first equality, we use first that \( U_{23}^K \) commutes with \( \phi(c) \) because both act on different legs, and secondly the covariance condition (4.23) for \((\phi, U^H)\) with \( \alpha \) applied to the leg \( A \):
\[ U_{10}^H U_{23}^K (\phi(c) \otimes 1_C \otimes 1_L) (U_{23}^K)^* (U_{10}^H)^* = U_{10}^H (\phi(c) \otimes 1_C \otimes 1_L) (U_{10}^H)^* = (\phi \otimes \alpha) \gamma_{13}(c). \]
A similar computation gives the second equality:
\[ U_{10}^H U_{23}^K Z_{12}(1_H \otimes \psi(d) \otimes 1_L) Z_{12}^* (U_{23}^K)^* (U_{10}^H)^* = U_{10}^K U_{10}^H \psi(d) (U_{23}^K)^* (U_{10}^H)^* \]
\[ = U_{23}^K \psi(d) (U_{23}^K)^* = (\psi \otimes \beta) \delta_{23}(d); \]
we first use (4.23); secondly, that \( U_{10}^H \) and \( \psi(d) \) act in different legs to commute them; and thirdly the covariance condition (2.23) for \((\psi, U^K)\) with \( \beta \) applied to the leg \( B \).

We remarked above that the pair of representations \((\phi \otimes \alpha) \gamma_{13}, (\psi \otimes \beta) \delta_{23}\) generates a faithful representation of \( C \boxtimes D \). Lemma 4.8 shows that this representation is unitarily equivalent to another representation that restricts to \( \varphi_1 \otimes 1_L \) and \( \text{Ad}_{Z_{12}} \circ \psi_2 = \psi_2 \otimes 1_L \) on \( C \) and \( D \), respectively. The latter representation is \( \rho \otimes 1_L \) for a faithful representation of \( C \boxtimes D \) on \( H \otimes K \). This is the faithful representation whose existence is asserted in Theorem 4.3. Uniqueness is clear because \( C \boxtimes D = \iota_C(C) \cdot \iota_D(D) \). This finishes the proof of Theorem 4.3.

**Theorem 4.9.** In the notation of Theorem 4.3, the subspace
\[ C \boxtimes D := \varphi_1(C) \cdot \psi_2(D) \subseteq B(H \otimes K) \]
is a \( C^* \)-subalgebra and \( (C \boxtimes D, \varphi_1, \tilde{\psi}_2) \) is a crossed product of \( C \) and \( D \). Up to equivalence of crossed products, it does not depend on \((\phi, U^H)\) and \((\psi, U^K)\).

The crossed product \((C \boxtimes D, \iota_C, \iota_D)\) is equivalent to \((C \boxtimes D, \varphi_1, \tilde{\psi}_2)\) and, up to equivalence of crossed products, does not depend on the Heisenberg pair \((\alpha, \beta)\).

Proof. Since \( C \boxtimes D = \rho(C \boxtimes D) \) and \( \rho \circ \iota_C = \varphi_1, \rho \circ \iota_D = \tilde{\psi}_2 \), by Theorem 4.3 \( C \boxtimes D \) is a \( C^* \)-algebra, \( (C \boxtimes D, \varphi_1, \tilde{\psi}_2) \) is a crossed product of \( C \) and \( D \), and it is equivalent to the crossed product \((C \boxtimes D, \iota_C, \iota_D)\).

Since the unitary \( Z \) is the same for all Heisenberg pairs \((\alpha, \beta)\), the crossed product \((C \boxtimes D, \varphi_1, \tilde{\psi}_2)\) does not depend on \((\alpha, \beta)\); hence up to equivalence \((C \boxtimes D, \iota_C, \iota_D)\) does not depend on \((\alpha, \beta)\). And since \((C \boxtimes D, \iota_C, \iota_D)\) does not depend on \((\phi, U^H)\) and \((\psi, U^K)\), neither does \((C \boxtimes D, \varphi_1, \tilde{\psi}_2)\), up to equivalence. \( \square \)

As a special case of Theorem 4.9, the usual spatial tensor product \( C \otimes D \) does not depend on the chosen faithful representations of \( C \) and \( D \). But we have not reproved this classical result. Rather, we have reduced analogous statements for noncommutative tensor products to this case by embedding the latter into commutative tensor products with more factors.

5. Properties of the twisted tensor product

In this section, we establish several functoriality properties of the twisted tensor product. We also discuss exactness for equivariantly semi-split extensions and invariance under Morita–Rieffel equivalence, which gives a result about cocycle conjugacy.

We begin with an easy symmetry property:
Proposition 5.1. The crossed products \( (C \otimes \phi, D, \delta) \) and \( (D \otimes \psi, \gamma) \) are canonically isomorphic.

Proof. Let \((U^H, \phi)\) and \((U^K, \psi)\) be faithful covariant representations of \(C\) and \(D\) on Hilbert spaces \(H\) and \(K\), respectively. Theorem 4.3 yields
\[
(C \otimes \phi) = (D \otimes \psi)
\]
with \(C \otimes H \subseteq B(H \otimes K),\ D \otimes C \subseteq B(K \otimes H)\); here \(\psi_1(d) := (\psi(d) \otimes 1)\) and \(\psi_2(c) := \tilde{Z}(1_K \otimes \psi(c))\). The pair of representations \((\varphi, \psi)\) of \((C, D)\) on \(H \otimes K\) is unitarily equivalent to the pair of representations \((\tilde{\varphi}, \tilde{\psi})\) on \(K \otimes H\) via the unitary \(\tilde{Z}\).

5.1. Functoriality for quantum group morphisms. Let \(G = (A, \Delta_A), H = (B, \Delta_B), G_2 = (A_2, \Delta_{A_2})\) and \(H_2 = (B_2, \Delta_{B_2})\) be quantum groups. Let \(f : G \rightarrow G_2\) and \(g : H \rightarrow H_2\) be quantum group morphisms in the sense of the several equivalent descriptions in Theorem 4.2.

Let \(\chi_2 \in \mathcal{U}(\hat{A}_2 \otimes \hat{B}_2)\) be a bicharacter. We may view \(\chi_2\) as a quantum group morphism \(\hat{\chi}_2 : G_2 \rightarrow \hat{H}_2\). Composing this with the given quantum group morphisms \(f : G \rightarrow G_2\) and the dual \(\hat{g} : \hat{H}_2 \rightarrow \hat{H}\), we get a quantum group morphism \(\hat{\chi} := \hat{g} \circ \chi_2 \circ f : G \rightarrow \hat{H},\) which we view as a bicharacter \(\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})\).

Let \((C, \gamma)\) and \((D, \delta)\) be a \(G\)-C*-algebra and an \(H\)-C*-algebra, respectively. The description of \(f\) in Theorem 2.23(4) is a functor between the categories of \(G\)- and \(G_2\)-C*-algebras that does not change the underlying C*-algebras. In particular, this functor maps \(\gamma\) to a continuous \(G_2\)-coaction \(\gamma_2 : C \rightarrow C \otimes A_2\) on \(C\). Similarly, \(g\) maps \(\delta\) to a continuous \(H_2\)-coaction \(\delta_2 : D \rightarrow D \otimes B_2\) on \(D\).

Theorem 5.2. In the situation above, the crossed products \(C \otimes_\chi (D, \gamma_2)\) and \((C, \gamma) \otimes_\chi (D, \delta)\) of \(C\) and \(D\) are equivalent.

Proof. Let \((\varphi, U^H)\) be a \(G\)-covariant representation of \((C, \gamma)\) on \(H\) and let \((\psi, U^K)\) be an \(H\)-covariant representation of \((D, \delta)\) on \(K\).

The quantum group morphism \(f\) turns \(U^H\) into a corepresentation \(U_2^H\) of \(G_2\) on \(H\). This is asserted in [11] Proposition 6.5. Since the quick proof given in [11] only works for corepresentations that induce a continuous coaction on \(B(K)\), which is not automatic, we give a different proof here using universal quantum groups.

We may view the quantum group morphism \(f\) as a Hopf *-homomorphism \(\hat{f} : \hat{A}_2^\text{u} \rightarrow \hat{A}_2^\text{v}\) between the duals of the associated universal C*-algebras by Theorem 2.23. By the universal property, \(U^H\) is equivalent to a representation of \(A^\text{u}\) on \(H\). Composing this with \(\hat{f}\) gives a representation of \(A^\text{u}_2\), which is equivalent to the desired corepresentation \(U^H_2\) of \(G_2\) on \(H\).

This operation on the level of corepresentations is compatible with the map \(\gamma \mapsto \gamma_2\) on coactions in the sense that \((\varphi, U^H_2)\) is a \(G_2\)-covariant representation of \((C, \gamma_2)\).

Similarly, \(g\) turns \(U^K\) into a corepresentation \(U^K_2\) of \(H_2\) on \(K\), and \((\psi, U^K_2)\) is a covariant representation of \((D, \delta_2)\).

The bicharacters \(\chi \in \mathcal{U}(\hat{A} \otimes \hat{B})\) and \(\chi_2 \in \mathcal{U}(\hat{A}_2 \otimes \hat{B}_2)\) lift uniquely to bicharacters \(\chi^u \in \mathcal{U}(\hat{A}_2^\text{u} \otimes \hat{B})\) and \(\chi^u_2 \in \mathcal{U}(\hat{A}_2^\text{u} \otimes \hat{B}_2^\text{u})\) by Theorem 2.23. The bijection between bicharacters and quantum group morphisms is defined in such a way that \(\chi^u = (f \otimes \hat{g})(\chi^u_2)\). Equation 4.13 then shows that the unitaries \(Z\) on \(H \otimes K\) that are used to construct the twisted tensor products with respect to \(\chi\) and \(\chi_2\) are the same.

Now Theorem 4.3 yields the desired equivalence of crossed products because both are faithfully represented by the same C*-algebra \(\varphi(C) \cdot Z \psi(D)Z^*\) on \(H \otimes K\). □
The following special cases of Theorem 5.2 are particularly noteworthy.

Example 5.3. Let $G_2 = \mathcal{H}, H_2 = \mathcal{H}$, let $g = \text{id}: \mathcal{H} \to \mathcal{H}$ and let $f: G \to G_2 = \hat{\mathcal{H}}$ be the bicharacter $\chi$ itself. Let $\chi_2 = \hat{W}^B$ be the reduced bicharacter of $\hat{\mathcal{H}}$. Then

$$(C, \gamma) \boxtimes_\chi (D, \delta) \cong (C, \gamma_2) \boxtimes_{\hat{W}^B} (D, \delta),$$

where $\gamma_2: C \to C \otimes \hat{\mathcal{H}}$ is the $\hat{\mathcal{H}}$-coaction associated to $\gamma$ by the quantum group morphism $f$ corresponding to $\chi$.

This is a special case of Theorem 5.2 because the bicharacter $\hat{W}^B$ describes the identity morphism on the quantum group $\hat{\mathcal{H}}$. The composition of this with $f$ gives again $f$, so that the bicharacter $\chi$ that we get from $\chi_2 = \hat{W}^B$ by the above construction is indeed the given one.

Example 5.4. Let $G_2 = G, H_2 = \hat{G}$, let $f = \text{id}: G_2 \to G$ and let $g: \mathcal{H} \to \mathcal{H}_2 = \hat{G}$ be the dual of the morphism $G \to \mathcal{H}_2$ associated to the bicharacter $\chi$. Let $\chi_2 = W^A$ be the reduced bicharacter of $\hat{G}$. Then

$$(C, \gamma) \boxtimes_\chi (D, \delta) \cong (C, \gamma_2) \boxtimes_{W^A} (D, \delta_2),$$

where $\delta_2: D \to D \otimes \hat{\mathcal{H}}$ is the $\hat{\mathcal{H}}$-coaction associated to $\delta$ by the quantum group morphism $g$.

The last example reduces the twisted tensor product $\boxtimes_\chi$ for an arbitrary bicharacter to the special case $H = \hat{G}$ and $\chi = W^A$.

### 5.2. Functoriality for various kinds of maps

$C^*$-algebras may be turned into a category using several types of maps:

- morphisms (nondegenerate $\ast$-homomorphisms $C_1 \to M(C_2)$);
- proper morphisms (nondegenerate $\ast$-homomorphisms $C_1 \to C_2$);
- possibly degenerate $\ast$-homomorphisms $C_1 \to C_2$;
- completely positive maps $C_1 \to C_2$;
- completely positive contractions $C_1 \to C_2$;
- completely contractive maps $C_1 \to C_2$;
- completely bounded maps $C_1 \to C_2$.

It is well known that the minimal tensor product is functorial for such maps; that is, two “maps” $f: C_1 \to C_2$ and $g: D_1 \to D_2$ induce a “map” $f \otimes g: C_1 \otimes D_1 \to C_2 \otimes D_2$, which is determined by $(f \otimes g)(c \otimes d) := f(c) \otimes g(d)$. We claim that the tensor product $\boxtimes_\chi$ is also functorial for all these kinds of “maps” in the following sense.

Lemma 5.5. Let $f: (C_1, \gamma_1) \to (C_2, \gamma_2)$ and $g: (D_1, \delta_1) \to (D_2, \delta_2)$ be a “maps” that are $G$- and $H$-equivariant, respectively. Then there is a unique “map”

$$f \boxtimes_\chi g: C_1 \boxtimes_\chi D_1 \to C_2 \boxtimes_\chi D_2, \quad \iota_{C_1}(c) \cdot \iota_{D_1}(d) \mapsto \iota_{C_2}(f(c)) \cdot \iota_{D_2}(g(d)),$$

and $(f, g) \mapsto f \boxtimes_\chi g$ is a bifunctor.

The notion of equivariance for possibly degenerate $\ast$-homomorphisms or completely bounded maps is defined as in [11, Definition 1.8]. The multiplier algebra is not functorial for such maps, but the comultiplication morphism $\gamma_1: C_1 \to M(C_1 \otimes A)$ takes values in the smaller algebra $\hat{M}(C_1 \otimes A) = \hat{M}_A(C_1 \otimes A) := \{x \in M(C_1 \otimes A) \mid x \cdot (1_C \otimes A) \cup (1_C \otimes A) \cdot x \subseteq C_1 \otimes A\}$.

We write a subscript on $\hat{M}$ to avoid ambiguities: $\hat{M}(C \otimes D \otimes A)$ could mean either $\hat{M}_A(C \otimes D \otimes A)$ or $\hat{M}_{D \otimes A}(C \otimes D \otimes A)$.

A completely bounded map $f: C_1 \to C_2$ induces a completely bounded, $\hat{M}(D)$-bilinear map

$$f \otimes \text{id}_D: C_1 \otimes D \to C_2 \otimes D.$$
The map \( f \otimes \text{id}_D \) is completely positive or completely contractive if \( f \) is, and a \(^*\)-homomorphism if \( f \) is. Any \( D \)-bilinear “map” \( h \colon C_1 \otimes D \to C_2 \otimes D \) extends uniquely to a \( M(D) \)-bilinear “map” \( h \colon M(C_1 \otimes D) \to M(C_2 \otimes D) \) for \( x \in M(C_1 \otimes D) \), there is a unique \( h(x) \in M(C_2 \otimes D) \) with \( h(x) \cdot (1 \otimes d) = h(x) \cdot (1 \otimes d) \) and \((1 \otimes d) \cdot h(x) = h((1 \otimes d) \cdot x)\) for all \( d \in D \) because \( h \) is \( D \)-linear. On \( M(C_1 \otimes D) \), this definition only works if \( h \) is \( C_1 \otimes D \)-linear, which is a serious restriction. After having constructed the extension, we write \( f \otimes \text{id}_D \) for the unique extension of \( f \otimes \text{id}_D \) to \( M \) in order not to change our formulas. We use this extension to make sense of the equivariance condition \( \delta \circ f = (f \otimes \text{id}_A) \circ \gamma \) for “maps.”

**Proof of Lemma 5.5.** The uniqueness and hence the functoriality is clear because the linear span of \( \iota_{C_1}(c) \cdot \iota_{D_1}(d) \) with \( c \in C_1, d \in D_1 \) is dense in \( C_1 \boxtimes \chi D_1 \) and all types of “maps” we consider are bounded linear.

We remarked above that ordinary minimal \( C^* \)-tensor products are functorial for “maps,” that is, there is a well-defined \( \mathbb{B}(\mathcal{H}) \)-linear “map”

\[
(f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})}) : C_1 \otimes D_1 \otimes \mathbb{K}(\mathcal{H}) \to C_2 \otimes D_2 \otimes \mathbb{K}(\mathcal{H}).
\]

We may extend it to a “map”

\[
(f,g)_\cdot : \hat{M}_{\mathbb{K}(\mathcal{H})}(C_1 \otimes D_1 \otimes \mathbb{K}(\mathcal{H})) \to \hat{M}_{\mathbb{K}(\mathcal{H})}(C_2 \otimes D_2 \otimes \mathbb{K}(\mathcal{H})),
\]

Lemma [5.2] implies \( C_1 \boxtimes \chi D_1 \subseteq \hat{M}_{\mathbb{K}(\mathcal{H})}(C_1 \otimes D_1 \otimes \mathbb{K}(\mathcal{H})) \) for \( i = 1, 2 \).

We claim that the “map” \((f,g)_\cdot \) sends \( \iota_{C_1}(c) \cdot \iota_{D_1}(d) \) to \( \iota_{C_2}(f(c)) \cdot \iota_{D_2}(g(d)) \); hence it restricts to a “map” \( f \boxtimes \chi g : C_1 \boxtimes \chi D_1 \to C_2 \boxtimes \chi D_2 \) with the required property. To prove this claim, we look at two cases separately.

First let \( f \) and \( g \) be equivariant morphisms; then \( f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})} \) is a morphism, hence it extends to a \(^*\)-homomorphism between multiplier algebras. Since \( f \) and \( g \) are equivariant, this canonical extension maps \( \gamma_1(c)_{1\tau} = \gamma_2(c)_{2\tau} \) and \( \delta_1(c)_{2\tau} = \delta_2(c)_{2\tau} \). Hence it maps \( \iota_{C_1}(c) \iota_{D_1}(d) \) to \( \iota_{C_2}(c) \iota_{D_2}(g(d)) \) as needed. If \( f \) and \( g \) are proper morphisms, then \( \iota_{C_1}(f(c)) \cdot \iota_{D_1}(g(d)) \in C_1 \boxtimes \chi D_1 \) for all \( c \in C_1, d \in D_1 \), so that \( f \boxtimes \chi g \) is a proper morphism as well.

Now let \( f \) and \( g \) be completely bounded maps; this contains the remaining types as special cases. By definition,

\[
(f \otimes g \otimes \text{id})(c \otimes 1_D \otimes x) \cdot (f \otimes g \otimes \text{id})(d \otimes 1_D \otimes y) = f(c) \otimes g(d) \otimes x \cdot y
= (f \otimes g \otimes \text{id})(c \otimes d \otimes x \cdot y)
\]

for all \( c \in C_1, d \in D_1, x,y \in \mathbb{K}(\mathcal{H}) \). Since \( f \otimes g \) is bounded linear, this implies the partial multiplicativity \((f \otimes g \otimes \text{id})(x \cdot y) = (f \otimes \text{id})(x) \cdot (g \otimes \text{id})(y) \) if \( x \in M(C_1 \otimes \mathbb{K}(\mathcal{H})), y \in M(C_2 \otimes \mathbb{K}(\mathcal{H})) \). In particular,

\[
(f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})})(\iota_{C_1}(c) \cdot \iota_{D_1}(d)) = (f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})})(\iota_{C_1}(c)) \cdot (f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})})(\iota_{D_1}(d))
\]

for all \( c \in C_1, d \in D_1 \). Finally, the equivariance of \( f \) and \( g \) shows that the right hand side is \( \iota_{C_2}(f(c)) \cdot \iota_{D_2}(g(d)) \).

**Proposition 5.6.** If \( f \) and \( g \) are injective morphisms or \(^*\)-homomorphisms, then so is \( f \boxtimes \chi g \), and vice versa.

If \( f \) and \( g \) are surjective \(^*\)-homomorphisms, then so is \( f \boxtimes \chi g \), and vice versa.

Hence \( f \boxtimes \chi g \) is bijective if and only if both \( f \) and \( g \) are bijective.

**Proof.** If \( f \) and \( g \) are injective, so is \( f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})} \); hence its extension to multipliers is injective, and so is the restriction to \( C_1 \boxtimes \chi D_1 \). Conversely, \((f \boxtimes \chi g)(\iota_{C_1}(c) \iota_{D_1}(d)) \) vanishes if \( f(c) = 0 \) or \( g(d) = 0 \); hence \( f \) and \( g \) are injective if \( f \boxtimes \chi g \) is.

If \( f \) and \( g \) are surjective, then elements of the form \((f \boxtimes \chi g)(\iota_{C_1}(c) \iota_{D_1}(d)) = \iota_{C_2}(f(c)) \iota_{D_2}(g(d)) \) are linearly dense in \( C_2 \boxtimes \chi D_2 \). Hence \( f \boxtimes \chi g \) is surjective as
well. Conversely, suppose that \( f \boxtimes \gamma \) is surjective. Then
\[
\iota C_2(f(C_1)) \cdot \iota D_2(g(D_1)) \cdot \mathcal{K}(H)_3 = (C_2 \boxtimes f(D_1)) \cdot \mathcal{K}(H)_3 = C_2 \otimes D_2 \otimes \mathcal{K}(H)
\]
d by Lemma \[2,21\]. We also have \( \iota C_2(f(C_1)) \cdot \iota D_2(g(D_1)) \cdot \mathcal{K}(H)_3 \subseteq f(C_1) \otimes g(D_1) \otimes \mathcal{K}(H) \). Applying slice maps to \( C_2 \) and \( D_2 \), we get \( f(C_1) = C_2 \) and \( g(D_1) = D_2 \). \( \square \)

Now we apply Proposition \[5.6\] to the equivariant embeddings \( \gamma : C \to C \otimes A \) and \( \delta : D \to D \otimes B \) provided in Lemma \[2,18\] to get an embedding
\[
(C, \gamma) \boxtimes (D, \delta) \to C \otimes D \otimes (A, \Delta) \boxtimes (B, \Delta).
\]
Thus we may describe \( (C, \gamma) \boxtimes (D, \delta) \) as the crossed product generated by the embeddings \( (\text{id} \otimes \iota A) \gamma_{13} \) of \( C \) and \( (\text{id} \otimes \iota B) \delta_{23} \) of \( D \) into \( C \otimes D \otimes (A, \Delta) \boxtimes (B, \Delta) \).

This description is particularly useful if we know \( (A, \Delta) \boxtimes (B, \Delta) \) more explicitly.

### 5.3. Exactness on equivariantly semi-split extensions.

**Proposition 5.7.** The functor \( \boxtimes \chi \) \( D \) maps an extension \( C_1 \to C_2 \to C_3 \) of \( G \)-C\(^*\)-algebras with a \( G \)-equivariant completely bounded section to an extension of \( C\(^*\)-algebras with a completely bounded section. If the section \( C_3 \to C_2 \) is an equivariant *-homomorphism, completely positive or completely contractive, then so is the induced section \( C_3 \boxtimes \chi D \to C_2 \boxtimes \chi D \). Analogous statements hold for the functor \( C \boxtimes \chi \).

**Proof.** We have \( C_1 \oplus C_3 \cong C_2 \) in the additive category of \( G \)-equivariant completely bounded maps, using the inclusion map \( C_1 \to C_2 \) and the section \( s : C_3 \to C_2 \).

Since \( \boxtimes \chi \) \( D \) is an additive functor, this implies \( C_1 \boxtimes \chi D \oplus C_3 \boxtimes \chi D \cong C_2 \boxtimes \chi D \) in the category of completely bounded maps. Thus
\[
C_1 \boxtimes \chi D \to C_2 \boxtimes \chi D \to C_3 \boxtimes \chi D
\]
is an extension of \( C\(^*\)-algebras with \( s \boxtimes \chi \) \text{id}_D \) as a completely bounded linear section. This section is again a \(^*\)-homomorphism, completely contractive, or completely positive if \( s \) is so. \( \square \)

The functor \( \boxtimes \chi \) \( D \) cannot be exact for arbitrary extensions because this already fails for the commutative minimal tensor product. An \( H \)-C\(^*\)-algebra \( D \) deserves to be called “exact” if \( \boxtimes \chi \) \( D \) is an exact functor for all \( G \) and all bicharacters \( \chi \in \mathcal{U}(\hat{A} \otimes \hat{B}) \). We plan the study this notion in future work.

### 5.4. Functoriality for correspondences.

Next we want to show that \( \boxtimes \chi \) is functorial for equivariant correspondences. Recall that a correspondence between two \( C\(^*\)-algebras \( C_1 \) and \( C_2 \) is a Hilbert \( C_2 \)-module \( E \) with a nondegenerate left \( C_1 \)-action (by adjointable operators). In this section, we assume familiarity with Hilbert modules, see \[8\].

We want to show that a \( G \)-equivariant correspondence \( E : C_1 \to C_2 \) and an \( H \)-equivariant correspondence \( F : D_1 \to D_2 \) induce a correspondence
\[
E \boxtimes \chi F : C_1 \boxtimes \chi D_1 \to C_2 \boxtimes \chi D_2
\]
with suitable functoriality properties, including compatibility with the composition of correspondences: given further equivariant correspondences \( E_2 : C_2 \to C_3 \) and \( F_2 : D_2 \to D_3 \), there is a natural isomorphism of correspondences
\[
(E \otimes_{C_1} E_2) \boxtimes (F \otimes_{C_2} F_2) \cong (E \boxtimes \chi F) \otimes_{C_2 \boxtimes \chi D_2} (E_2 \boxtimes \chi F_2).
\]
This also implies that if \( E \) and \( F \) are equivariant Morita–Rieffel equivalences (that is, full Hilbert bimodules), then \( E \boxtimes \chi F \) is a Morita–Rieffel equivalence.

Quantum group coactions on Hilbert modules are defined by Baaj and Skandalis in \[1\] Definition 2.2], but without considering Podles’ continuity condition. Therefore, we add one condition to our definition.
Definition 5.8. A $\mathbb{G}$-equivariant Hilbert module over a $\mathbb{G}$-$C^*$-algebra $(C, \gamma)$ is a Hilbert $C$-module $\mathcal{E}$ with a coaction $\epsilon : \mathcal{E} \to \tilde{\mathcal{M}}(\mathcal{E} \otimes A)$ with the following properties:

1. $\epsilon(\xi)\gamma(c) = \epsilon(\xi c)$ for $\xi \in \mathcal{E}$, $c \in C$;
2. $\gamma(\langle \xi, \eta \rangle c) = \langle \epsilon(\xi), \epsilon(\eta) \rangle \tilde{\mathcal{M}}(C \otimes A)$;
3. $\epsilon(\mathcal{E}) \cdot (1 \otimes A) = \mathcal{E} \otimes A$;
4. $(1 \otimes A) \cdot \epsilon(\mathcal{E}) = \mathcal{E} \otimes A$;
5. $(\epsilon \circ \text{id}_A) \epsilon = (\text{id}_\mathcal{E} \otimes \epsilon) \epsilon$.

Here

$$\tilde{\mathcal{M}}(\mathcal{E} \otimes A) := \{ T \in \mathcal{B}(C \otimes A, \mathcal{E} \otimes A) \mid (1_\mathcal{E} \otimes A) T \cup T(1_C \otimes A) \subseteq \mathcal{E} \otimes A \},$$

where $\mathcal{B}$ means adjointable operators between Hilbert modules. Condition (5) uses canonical extensions of $\epsilon \otimes \text{id}_A$ and $\text{id}_A \otimes \epsilon$ to maps

$$\mathcal{B}(C \otimes A, \mathcal{E} \otimes A) \to \mathcal{B}(C \otimes A \otimes A, \mathcal{E} \otimes A \otimes A),$$

which are described in [1] Remarque 2.5]. The map $\epsilon$ is automatically norm-isometric by [1] Proposition 2.4.

Since $\gamma$ satisfies the Podleś condition and $\epsilon(\mathcal{E}) = \epsilon(\mathcal{E}) \cdot \gamma(C)$, our condition (3) is equivalent to $\epsilon(\mathcal{E}) \cdot (C \otimes A) = \mathcal{E} \otimes A$. Thus the conditions in [1] Definition 2.2 are equivalent to our conditions (1)--(3) and (5).

Remark 5.9. Our definition and the one by Baaj and Skandalis give the same definition for Hilbert bimodules and hence the same notion of equivariant Morita–Rieffel equivalence (provided the $C^*$-algebras involved carry continuous coactions).

A Hilbert bimodule between $C_1$ and $C_2$ is both a right Hilbert $C_2$-module and a left Hilbert $C_1$-module, such that the left and right module structures commute and the inner products satisfy $\langle \xi, \eta \rangle_{C_1} \cdot \zeta = \langle \xi, \eta, \zeta \rangle_{C_2}$ for all $\xi, \eta, \zeta \in \mathcal{E}$.

The left and right Hilbert module structures both give the same multiplier space $\tilde{\mathcal{M}}(\mathcal{E} \otimes A)$ because $\mathbb{K}(\mathcal{H} \otimes A)$ maps $\mathcal{M}(\mathcal{E} \otimes A)$ into $\mathcal{E} \otimes A$.

A $\mathbb{G}$-equivariant Hilbert bimodule is a Hilbert bimodule with a $\mathbb{G}$-coaction $\epsilon : \mathcal{E} \to \tilde{\mathcal{M}}(\mathcal{E} \otimes A)$ that satisfies conditions (1)--(5) both for the left and the right Hilbert module structure. There is, however, some duplication here. Condition (5) is the same for the left and right Hilbert module structure, and the change between left and right exchanges conditions (3) and (4). Thus if both Hilbert module structures satisfy conditions (1)--(3) and (5), then they both satisfy (1)--(5). Hence the definitions here and in [1] give the same notion of Hilbert bimodule.

The linking algebra associated to a Hilbert $C$-module $\mathcal{E}$ is the algebra of compact operators on $C \otimes \mathcal{E}$ with its block decomposition into $\mathbb{K}(C, C) \cong C$, $\mathbb{K}(C, \mathcal{E}) \cong \mathcal{E}$, $\mathbb{K}(\mathcal{E}, C) \cong \mathcal{E}^*$ and $\mathbb{K}(\mathcal{E}, \mathcal{E}) = \mathbb{K}(\mathcal{E})$. A $\mathbb{G}$-coaction on $\mathcal{E}$ induces a coaction $\gamma' : \mathbb{K}(C \otimes \mathcal{E}) \to \mathbb{K}(C \otimes \mathcal{E}) \otimes A$ that is compatible with this block decomposition by [1] Proposition 2.7]; $\gamma' \epsilon$ restricts to $\epsilon$ and $\gamma$ on the blocks $E$ and $C$ in $\mathbb{K}(C \otimes \mathcal{E})$. Under the assumptions in [1], this coaction need not satisfy the Podleś condition, even if $\gamma$ does. Our additional condition (4) ensures this because it is equivalent to $\gamma'(\mathcal{E}^*) \cdot (1 \otimes A) = \mathcal{E}^* \otimes A$, and this implies $\gamma'(\mathbb{K}(\mathcal{E})) \cdot (1 \otimes A) = \mathbb{K}(\mathcal{E}) \otimes A$ because $\mathbb{K}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^*$. Condition (3) and the continuity of $\gamma$ give $\gamma'(\mathcal{E}) \cdot (1 \otimes A) = \mathcal{E} \otimes A$ and $\gamma'(\mathcal{E}) \cdot (1 \otimes A) = \mathcal{E} \otimes A$.

Proposition 5.10. Let $\mathcal{E}$ be a $\mathbb{G}$-equivariant Hilbert module over $(C, \gamma)$ and let $\mathcal{F}$ be a $\mathbb{G}$-equivariant Hilbert module over $(D, \delta)$. Let $C' := \mathbb{K}(C \otimes \mathcal{E})$ and $D' := \mathbb{K}(D \otimes \mathcal{F})$ with the induced continuous coactions $\gamma'$ and $\delta'$. Choose a $\chi$-Heisenberg pair $(\alpha, \beta)$ and view $C \boxtimes_{\chi} D$ and $\mathbb{K}(\mathcal{E}) \boxtimes_{\chi} \mathbb{K}(\mathcal{F})$ as $C^*$-subalgebras of $\mathcal{M}(C' \otimes D' \otimes \mathbb{K}(\mathcal{H}))$. Then

$$\mathcal{E} \boxtimes_{\chi} \mathcal{F} := \iota_{C'}(\mathcal{E}) \cdot \iota_{D'}(\mathcal{F}) = \iota_{D'}(\mathcal{F}) \cdot \iota_{C'}(\mathcal{E})$$
is a Hilbert module over $C \boxtimes \chi D$, where the right $C \boxtimes \chi D$-module structure is the multiplication in $\mathcal{M}(C' \otimes D' \otimes \mathbb{K}(\mathcal{H}))$, and the $C \boxtimes \chi D$-valued inner product is $\langle \xi, \eta \rangle := \xi^* \cdot \eta$. Furthermore,

$$K(C \boxtimes \chi F) \cong K(\mathcal{E}) \boxtimes K(F).$$

**Proof.** All this follows from Lemma 3.18. Lemma 3.18 for $X = E, Y = F$ asserts $\iota_C(\xi) \cdot \iota_D(C) = \iota_C(D) \cdot \iota_C(\xi)$. To check that $E \boxtimes \chi F$ is closed under right multiplication under $C \boxtimes \chi D$, we compute

$$\begin{align*}
\langle E \boxtimes \chi F \rangle \cdot (C \boxtimes \chi D) &= \iota_C(\xi) \cdot \iota_D(D) \cdot \iota_C(\xi) \\
&= \iota_C(D) \cdot \iota_C(\xi) \\
&= \iota_C(D) \cdot \iota_C(\xi) = E \boxtimes \chi F.
\end{align*}$$

Similar computations give

$$\begin{align*}
\langle E \boxtimes \chi F \rangle^* \cdot (E \boxtimes \chi F) &\subseteq C \boxtimes \chi D, \\
\langle E \boxtimes \chi F \rangle \cdot (E \boxtimes \chi F)^* &\subseteq K(C) \boxtimes K(D).
\end{align*}$$

The first line completes the proof that $E \boxtimes \chi F$ is a Hilbert module over $C \boxtimes \chi D$. The second line says that $K(E \boxtimes \chi F) \cong K(E) \boxtimes K(F)$. □

Let $E_1$ and $E_2$ be $G$-equivariant Hilbert modules over $C$ and let $S : E_1 \to E_2$ be an adjointable operator. We want to construct an induced adjointable operator

$$S \boxtimes \chi \operatorname{id}_F : E_1 \boxtimes \chi F \to E_2 \boxtimes \chi F.$$ 

We may view $S$ as an adjointable operator on $E := E_1 \oplus E_2$ that vanishes on $E_2$ and has image contained in $E_2$. There is a canonical unital *-homomorphism

$$\mathcal{B}(E) \cong M(K(E)) \to M(K(C) \boxtimes K(F)) \cong M(K(E \boxtimes \chi F)) \cong \mathcal{B}(E \boxtimes \chi F).$$

We apply it to $S$ and then notice that the resulting operator is the extension by zero of an adjointable operator $E_1 \boxtimes \chi F \to E_2 \boxtimes \chi F$. This defines $S \boxtimes \chi \operatorname{id}_F$. The map $S \mapsto S \boxtimes \chi \operatorname{id}_F$ is a unital, strictly continuous *-homomorphism.

A similar construction turns an adjointable operator $T : F_1 \to F_2$ between $H$-equivariant Hilbert $D$-modules $F_1$ and $F_2$ into an adjointable operators

$$\operatorname{id}_E \boxtimes \chi T : E \boxtimes \chi F_1 \to E \boxtimes \chi F_2.$$ 

The adjointable operators $S \boxtimes \chi \operatorname{id}_F$ and $\operatorname{id}_E \boxtimes \chi T$ usually do not commute with each other, so that $S \boxtimes \chi T$ is not defined unambiguously. However, if $S$ and $T$ are both equivariant and unitary, then they are compatible with all structure that is used to define $E_1 \boxtimes \chi F_1$ and $E_2 \boxtimes \chi F_2$ and hence must induce an isomorphism

$$S \boxtimes \chi T : E_1 \boxtimes \chi F_1 \to E_2 \boxtimes \chi F_2.$$ 

Indeed, the following lemma shows that $S \boxtimes \chi \operatorname{id}_F$ and $\operatorname{id}_E \boxtimes \chi T$ commute whenever $S$ or $T$ is equivariant.

**Lemma 5.11.** Let $x \in M(C)$ and $y \in M(D)$ and assume that $x$ is $G$-invariant or that $y$ is $H$-invariant, that is, $\gamma(x) = x \otimes 1$ or $\delta(y) = y \otimes 1$. Then $[\iota_C(x), \iota_D(y)] = 0$ in $C \boxtimes \chi D$.

**Proof.** If $x$ is $G$-invariant, then $\iota_C(x) = x \otimes 1 \otimes 1$ in $M(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$. This commutes with $\iota_D(y) \in M(D \otimes \mathbb{K}(\mathcal{H}))$ because it lives in a different leg. The argument for $H$-invariant $y$ is the same. □

Now we turn from Hilbert modules to correspondences.

A $G$-equivariant correspondence from $C_1$ to $C_2$ is a $G$-equivariant Hilbert module $\mathcal{E}$ over $C_2$ with a nondegenerate representation of $C_1$, that is, with a morphism $f : C_1 \to \mathcal{B}(\mathcal{E}) = M(K(\mathcal{E}))$. Usually, we do not mention $f$ and instead equip $\mathcal{E}$ with
the left $C_1$-module structure given by $f$; thus a correspondence is a bimodule with a $C_2$-valued right inner product and a $C$-coaction with suitable properties.

Let $E$ with $f: C_1 \to \mathcal{B}(E)$ be a $G$-equivariant correspondence from $C_1$ to $C_2$ and let $F$ with $g: D_1 \to \mathcal{B}(F)$ be an $H$-equivariant correspondence from $D_1$ to $D_2$. Then we get a Hilbert module $C_{\chi} \otimes D_1 \to \mathcal{M}(K(E) \otimes K(F)) \cong \mathcal{M}(K(E \otimes F)) \cong \mathcal{B}(E \otimes F)$ by Proposition 5.10. This gives a correspondence from $C_{\chi} \otimes D_1$ to $C_{\chi} \otimes D_2$.

Let $S: E_1 \to E_2$ and $T: F_1 \to F_2$ be isomorphisms of equivariant correspondences (that is, equivariant unitaries commuting with the left module structures). Then

$$S \otimes T: E_1 \otimes_{\chi} F_1 \to E_2 \otimes_{\chi} F_2$$

is an isomorphism of correspondences. Thus our construction descends to isomorphism classes of correspondences.

Next we consider the composition of correspondences. Let $C_i$ for $i = 1, 2, 3$ be $G$-$C^*$-algebras and let $D_i$ for $i = 1, 2, 3$ be $H$-$C^*$-algebras; let $E_1$ be a $G$-equivariant correspondence from $C_1$ to $C_2$, let $E_2$ be a $G$-equivariant correspondence from $C_2$ to $C_3$, let $F_1$ be an $H$-equivariant correspondence from $D_1$ to $D_2$, and let $F_2$ be an $H$-equivariant correspondence from $D_2$ to $D_3$. The composite correspondences $E_1 \otimes C_2 E_2$ and $F_1 \otimes_D F_2$ are again equivariant (see [1] Proposition 2.10, our extra continuity condition is easily checked).

**Lemma 5.12.** There is a natural isomorphism of correspondences

$$(E_1 \otimes C_2 E_2) \otimes_{\chi} (F_1 \otimes_D F_2) \cong (E_1 \otimes_{\chi} F_1) \otimes_{C_2 \otimes_D} (E_2 \otimes_{\chi} F_2).$$

**Proof.** Let us assume that $C_i$ is a subalgebra of $\mathcal{B}(E_i)$ for $i = 1, 2$ (we can make these representations faithful by taking direct sums with suitable correspondences, and then argue in the end that the result remains true without these additional summands). The direct sum

$$E' := C_3 \oplus E_2 \oplus (E_1 \otimes C_2 E_2)$$

is a $G$-equivariant Hilbert $C_{\chi}$-module on which the $G$-$C^*$-algebras $C_i$ for $i = 1, 2, 3$ and the $G$-equivariant Hilbert modules $E_1$, $E_2$ and $E_1 \otimes C_2 E_2$ act by adjointable operators. Namely, $C_1$ acts by the given left action on $E_1$ and by zero on the other summands; $C_2$ acts by the given left action on $E_2$ and by zero on the other summands; $C_3$ acts on itself by left multiplication and by zero on the other summands; $E_2$ and $E_1 \otimes C_2 E_2$ act by the isomorphisms $E_2 \cong K(C_3, E_2)$ and $E_1 \otimes C_2 E_2 \cong K(C_3, E_1 \otimes C_2 E_2)$ on $C_3$ and by zero on the other summands; $E_1$ acts on $E_2$ by the map

$$E_1 \to \mathcal{B}(E_2, E_1 \otimes C_2 E_2), \quad \xi \mapsto T_\xi,$$

with $T_\xi(\eta) := \xi \otimes \eta$ for all $\eta \in E_2$, $\xi \in E_1$, and $E_1$ acts by zero on the other summands.

These representations are nicely compatible in the following sense: bimodule structures on our Hilbert modules are always represented by composition of adjointable operators, and inner products are always represented by $\langle x, y \rangle := x^* \circ y$. Hence they extend to representations of the linking algebras $K(C_2 \oplus E_1)$ of $E_1$, $K(C_3 \oplus E_2)$ of $E_2$, and $K(C_3 \oplus E_1 \otimes C_2 E_2)$ of $E_1 \otimes C_2 E_2$.

Let us assume similarly that $D_i \subseteq \mathcal{B}(F_i)$ for $i = 1, 2$, and let us embed the $H$-$C^*$-algebras $D_i$ for $i = 1, 2, 3$ and the $H$-equivariant Hilbert modules $F_1$, $F_2$ and $F_1 \otimes_D F_2$ in a similar fashion into $\mathcal{B}(F')$ with

$$F' := D_3 \oplus F_2 \oplus (F_1 \otimes_D F_2).$$
The tensor products $C_1 \boxtimes \chi_{\mathcal{D}_1}, \mathcal{E} \boxtimes \chi_{\mathcal{F}_1}$ and $(\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2) \boxtimes \chi_{(\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2)}$ are all embedded into the multiplier algebra of $
abla(E') \boxtimes \nabla(F') \cong \nabla(E' \boxtimes \chi_{\mathcal{F}'})$ by Proposition 5.8 and Proposition 5.10.

The construction in Proposition 5.10 also shows that, in this representation, the bimodule structures on the Hilbert modules $\mathcal{E} \boxtimes \chi_{\mathcal{F}_1}$ and $(\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2) \boxtimes \chi_{(\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2)}$ are given by composition, and the inner products by $b$-module structures on the Hilbert modules $\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2$.

Proof. The morphism $\gamma_{\mathcal{F}_1}: \mathcal{F}_1 \to \mathcal{F}_1$ is a coaction by a cocycle. For all $x \in \mathcal{E}$, we may change a coaction by a cocycle: $u_{12}(\gamma \otimes \text{id}_A)u = (\text{id}_C \otimes \Delta_A)u$ in $\mathcal{U}(C \otimes A \otimes A)$.

We can only treat cocycles that satisfy an extra Podleś condition:

**Lemma 5.15.** Let $u \in \mathcal{U}(\mathcal{E} \otimes \mathcal{A})$ be a $\gamma$-cocycle. Define a morphism $\gamma_u := \text{Ad}_u \circ \gamma: C \to \mathcal{A}$. This is a continuous coaction of $\mathcal{G}$ if and only if

$$\gamma(C) \cdot u^* \cdot (1_C \otimes \mathcal{A}) = C \otimes \mathcal{A}.$$  

**Proof.** The morphism $\gamma_u$ is faithful because $\gamma$ is. We check that it is a comodule structure:

$$(\text{id}_C \otimes \Delta_A)(u \gamma(c)u^*) = u_{12}(\gamma \otimes \text{id}_A)(u \gamma(c))u^* u_{12}^* = (\gamma_u \otimes \text{id}_A) \gamma_u(c)$$

for all $c \in C$; the first equality uses (5.14) and (2.10) for $\gamma$; the second equality again uses (5.14) for all $c \in C$.

Since $u \in \mathcal{U}(\mathcal{E} \otimes \mathcal{A})$ we have $u(\mathcal{E} \otimes \mathcal{A}) = C \otimes \mathcal{A}$. Hence (5.16) is equivalent to the Podleś condition $u \gamma(C)u^* \cdot (1 \otimes \mathcal{A}) = C \otimes \mathcal{A}$ for $\gamma_u$. \hfill $\Box$

The following result generalises [2 Proposition 7.6].

**Theorem 5.17.** Let $u$ be a $\gamma$-cocycle and let $v$ be a $\delta$-cocycle. Assume both satisfy the Podleś condition (5.10). Define the coactions $\gamma_u$ and $\delta_v$ as above. Then

$$(\mathcal{E}, \gamma) \boxtimes \chi_{(D, \delta)} \cong (\mathcal{E}, \gamma_u) \boxtimes \chi_{(D, \delta_v)}.$$ 

This isomorphism is not one of crossed products, that is, it is not compatible with the embeddings of $\mathcal{C}$ and $D$.

**Proof.** Let $\mathcal{E}$ be $C$ viewed as a Hilbert module over itself. Define the coaction $\epsilon: \mathcal{E} \to \mathcal{M}(\mathcal{E} \otimes \mathcal{A})$ by $\epsilon(c) := u \cdot \gamma(c)$. We claim that this gives a $\mathcal{G}$-equivariant Hilbert $C$-module. Conditions (1)–(3) in Definition 5.5 are immediate. Condition (4) is equivalent to (5.10) by taking adjoints, and (5) is equivalent to the cocycle condition (5.13).

Since $\mathcal{E} = C$ as a Hilbert module, the left multiplication action of $C$ gives an isomorphism $C \cong \nabla(\mathcal{E})$. The induced $\mathcal{G}$-coaction on $\nabla(\mathcal{H})$ is, however, not equivalent to $\gamma$ but to $\gamma_u$: $\gamma_u(c_1) \cdot \epsilon(c_2) = \epsilon(c_1 c_2)$ for all $c_1, c_2 \in C$. 


Similarly, let \( F \) be \( D \) viewed as a Hilbert module over itself, with the \( H \)-coaction \( \varphi: F \to \mathcal{M}(F \otimes B), d \mapsto v \cdot \delta(d) \). Then \( \mathbb{K}(F) \cong D \) with induced coaction \( \delta_\varphi \). Now Proposition 5.14 gives
\[
(C, \gamma_u) \boxtimes \chi (D, \gamma_u) \cong \mathbb{K}(\mathcal{E} \boxtimes \chi F).
\]
The identity maps \( C \to \mathcal{E} \) and \( D \to F \) are (non-equivariant) unitary operators. They give unitary operators
\[
(C, \gamma) \boxtimes \chi (D, \delta) \to (\mathcal{E}, \epsilon) \boxtimes \chi (D, \delta) \to (\mathcal{E}, \epsilon) \boxtimes \chi (F, \varphi)
\]
of Hilbert \( (C, \gamma) \boxtimes \chi (D, \delta) \)-modules. Conjugating by this unitary gives a \( C^* \)-algebra isomorphism
\[
(C, \gamma) \boxtimes \chi (D, \delta) \cong \mathbb{K}(C \boxtimes \chi D) \to \mathbb{K}(\mathcal{E} \boxtimes \chi F).
\]
Now compose this with the isomorphism \( \mathbb{K}(\mathcal{E} \boxtimes \chi F) \cong (C, \gamma_u) \boxtimes \chi (D, \gamma_u) \).

We describe the isomorphism above more explicitly. To simplify notation, we treat only \( u \) and assume \( v = 1 \). The linking algebra for \( \mathcal{E} \) is \( M_2(C) \) with the \( \mathbb{G} \)-coaction
\[
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \mapsto \begin{pmatrix} \gamma(c_{11}) u^* & \gamma(c_{12}) u^* \\ u\gamma(c_{21}) & u\gamma(c_{22}) \end{pmatrix}.
\]
The upper left and lower right corners are \( (C, \gamma) \) and \( (C, \gamma_u) \), respectively. Thus \( (C, \gamma) \boxtimes \chi (D, \delta) \) and \( (C, \gamma_u) \boxtimes \chi (D, \delta) \) are subalgebras of \( M_2(C) \boxtimes D \).

Conjugation by the partial isometry \( s = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and its adjoint gives isomorphisms between the two corners \( C \subseteq M_2(C) \). The strictly continuous extension of \( \iota_{M_2(C)} \) maps \( s \) to a partial isometry in \( M_2(C) \boxtimes D \). Conjugation by this partial isometry and its adjoint restricts to isomorphisms between \( (C, \gamma) \boxtimes \chi (D, \delta) \) and \( (C, \gamma_u) \boxtimes \chi (D, \delta) \).

Call a continuous coaction \textit{inner} if it is a cocycle-twist of the trivial coaction.

**Corollary 5.18.** The crossed product \( (C, \gamma) \boxtimes \chi (D, \delta) \) is isomorphic to \( C \otimes D \) if \( \gamma \) or \( \delta \) is inner.

**Proof.** Let \( u \in \mathcal{M}(C \otimes A) \) be a cocycle for the trivial coaction \( \tau(c) := c \otimes 1 \) and let \( \gamma = \tau_u \). The cocycle \( u \) satisfies (5.10) by Lemma 5.15. Now Theorem 5.17 and Example 5.16 give \( (C, \gamma) \boxtimes \chi D \cong (C, \tau) \boxtimes \chi D \cong C \otimes D \). A similar proof works if \( \delta \) is inner. \( \square \)

**Example 5.19.** Let \( U^H \) and \( U^K \) be corepresentations of \( A \) and \( B \) on Hilbert spaces \( H \) and \( K \). These are cocycles for the trivial action on \( \mathbb{K}(H) \). Assume (5.10) to get continuous coactions on \( \mathbb{K}(H) \) and \( \mathbb{K}(K) \). Then
\[
\mathbb{K}(H) \boxtimes \chi \mathbb{K}(K) \cong \mathbb{K}(H) \otimes \mathbb{K}(K) \cong \mathbb{K}(H \otimes K).
\]

This explains the Hilbert space realisation of \( C \boxtimes \chi D \) in Theorem 4.3 in the case where the corepresentations \( U^H \) and \( U^K \) used there satisfy the technical condition (5.10). Then we get a faithful morphism \( C \boxtimes \chi D \to \mathbb{K}(H) \boxtimes \chi \mathbb{K}(K) \) from Proposition 5.6. When we identify \( \mathbb{K}(H) \boxtimes \chi \mathbb{K}(K) \cong \mathbb{K}(H \otimes K) \) as above, we get a faithful representation of \( C \boxtimes \chi D \) on \( H \otimes K \).

### 6. Examples of twisted tensor products

We show in Section 5.11 that the skew-commutative tensor product of \( \mathbb{Z}/2 \)-graded \( C^* \)-algebras is a special case of our theory.

In Section 5.2 we consider the case where both \( A \) and \( B \) are duals of locally compact groups; in particular, this covers the case where \( A \) and \( B \) are locally compact Abelian groups. Here we understand bicharacters in a classical way, and we show that \( C \boxtimes \chi D \) for any bicharacter is a Rieffel deformation of \( C \otimes D \).
Theorem 6.2. \(\mathrm{covariant\ representation}\) \(H \in i,j\) \(\overline{\mathbb{H}}\) over all non-zero elements coaction on a crossed product using the functoriality of \(\mathbb{H}\).

Let \(\mathbb{H} = \mathbb{C}^*(\mathbb{Z}/2)\) with the usual comultiplication. Thus a \(\mathbb{G}\)-coaction on a \(\mathbb{C}^*\)-algebra \(C\) is a \(\mathbb{Z}/2\)-grading; a decomposition \(C = C_0 \oplus C_1\) into involutive, closed, linear subspaces \(C_0\) and \(C_1\) of even and odd elements such that

\[ C_1 \cdot C_j = C_{i+j \mod 2}, \quad C_i^* = C_i. \]

Equivalently, \(\alpha'(c_0 + c_1) := c_0 - c_1\) for \(c_i \in C\) defines an involutive \(*\)-automorphism of \(C\).

The *skew-commutative tensor product* of two \(\mathbb{Z}/2\)-graded \(\mathbb{C}^*\)-algebras \(C\) and \(D\) is defined in [5, §2.6] by imposing the commutation relation that \(e \in C\) and \(d \in D\) anti-commute if both are odd, and commute if one of them is even. This leads to the \(*\)-algebra structure

\[
(c_1 \circ d_1) \cdot (c_2 \circ d_2) := (-1)^{\deg(c_2) \cdot \deg(d_1)} c_1 c_2 \circ d_1 d_2, \\
(c \circ d)^* := (-1)^{\deg(c) \cdot \deg(d)} c^* \circ d^*
\]

on the algebraic tensor product \(C \circ D\) of \(C\) and \(D\). The skew-commutative \(\mathbb{C}^*\)-tensor product \(C \circ D\) is the completion of the \(*\)-algebra \(C \circ D\) in the \(\mathbb{C}^*\)-norm

\[
\|x\| := \sup \frac{(\rho \cdot \lambda)(y^* \cdot x^* \cdot x \cdot y)}{(\rho \cdot \lambda)(y^* \cdot y)}
\]

over all non-zero elements \(y \in C \circ D\) and all even states \(\rho \in C^*\) and \(\lambda \in D^*\) (even means that \(\rho\) and \(\lambda\) vanish on \(C_1\) and \(D_1\), respectively); here the products and adjoints have respect to the \(*\)-algebra structure on \(C \circ D\).

The obvious formulas define morphisms \(\iota_C : C \to C \circ D\) and \(\iota_D : D \to C \circ D\), so that \(C \circ D\) is a crossed product of \(C\) and \(D\). We want to show that \(C \circ D \cong C \mathbb{B}_\chi D\) for a suitable bicharacter \(\chi \in \mathcal{U}(\hat{A} \circ \hat{A})\).

The dual \(\hat{\mathbb{G}}\) is the group \(\mathbb{Z}/2\), so that \(\hat{A} \circ \hat{B} \cong \mathbb{C}(\mathbb{Z}/2 \times \mathbb{Z}/2)\) and a bicharacter \(\chi\) is a bicharacter \(\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{T}\) in a more classical sense. The unique non-trivial bicharacter is defined by \(\chi(1,1) = -1\) and \(\chi(i,j) = 1\) if \(i = 0\) or \(j = 0\).

**Theorem 6.2.** Let \(C\) and \(D\) be \(\mathbb{Z}/2\)-graded \(\mathbb{C}^*\)-algebras and let \(\chi\) be the non-trivial bicharacter in \(C(\mathbb{Z}/2 \times \mathbb{Z}/2)\). Then the crossed product \((C \mathbb{B}_\chi D, \iota_C, \iota_D)\) of \(C\) and \(D\) is naturally isomorphic to their skew-commutative tensor product.

**Proof.** A covariant representation of \(C\) is given by a \(\mathbb{Z}/2\)-graded Hilbert space \(\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1\) and a representation \(\varphi : C \to \mathbb{B}(\mathcal{H})\) with \(\varphi(c_i)(\mathcal{H}_j) \subseteq \mathcal{H}_{i+j}\) for all \(i, j \in \mathbb{Z}/2\). We choose such a faithful covariant representation of \(A\) and a faithful covariant representation \(\psi : D \to \mathbb{B}(\mathcal{K})\) on a \(\mathbb{Z}/2\)-graded Hilbert space \(\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1\).

Since \(\hat{A}^a = \hat{A}\), the unitary \(Z\) that is used in the Hilbert space description of \(C \mathbb{B}_\chi D\) is described most easily by [14,10]. This gives \(Z(\xi \otimes \eta) = -\xi \otimes \eta\) if \(\xi \in \mathcal{H}_1\) and \(\eta \in \mathcal{K}_1\), and \(Z(\xi \otimes \eta) = \xi \otimes \eta\) if \(\xi \in \mathcal{H}_0\) or \(\eta \in \mathcal{K}_0\). Thus \(\Sigma Z : \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}\) is the braiding operator from the Koszul sign rule. The representations \(\varphi_1\) and \(\tilde{\psi}_2\) in Theorem 6.2 are

\[
\varphi_1(c)(\xi \otimes \eta) = (\varphi(c)\xi) \otimes \eta, \quad \tilde{\psi}_2(d)(\xi \otimes \eta) = (-1)^{\deg(d) \cdot \deg(\xi)} \xi \otimes \psi(d)\eta,
\]

as expected from the Koszul sign rule. It remains to show that this pair of representations of \(C\) and \(D\) yields a faithful representation of the skew-commutative tensor product \(C \circ D\). It is clear that we get a \(*\)-representation of \(C \circ D\).

We must show that, for any \(x \in C \circ D\), its operator norm on \(\mathcal{H} \otimes \mathcal{K}\) is equal to the norm defined in 6.3. The GNS-representation for an even state \(\rho : C \to \mathbb{C}\) on the Hilbert space \(L^2(C, \rho)\) is a covariant representation if we let \(L^2(C, \rho)_1\) be the
Abelianisations

The quotient maps $\varphi: \phi$ The resulting representations variables. This makes it easy to list all bicharacters for two given Abelian locally commutative tensor product $G$ compact groups and $\hat{H}$ inherits an action of $\Gamma := \hat{\Gamma}$; hence we may without loss of generality assume that $G$ and $H$ are Abelian.

For instance, there is an isomorphism

$$C^*(\mathbb{Z}/2) \boxtimes \chi C^*(\mathbb{Z}/2) \cong \mathcal{M}_2(C),$$

mapping the generators of the two copies of the group $\mathbb{Z}/2$ to the anti-commuting involutions

$$g_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ Combining this computation with the discussion after Proposition 5.6 gives an alternative description of the skew-commutative tensor product: it is the crossed product generated by the embeddings of $G$ and $D$ into $\mathcal{M}_2(C \otimes D)$, mapping $c \mapsto c \otimes 1 \otimes 1$ for even $c \in C$, $c \mapsto c \otimes 1 \otimes g_1$ for odd $c \in C$, $d \mapsto 1 \otimes d \otimes 1$ for even $d \in D$, $d \mapsto 1 \otimes d \otimes g_2$ for odd $d \in D$.

6.2. General group coactions. Now we consider the case where $A = C^*_r(G)$ and $B = C^*_r(H)$ for two locally compact groups $G$ and $H$ with the usual comultiplications. Thus coactions of $G$ and $H$ are coactions of these groups $G$ and $H$ in the usual sense. We are going to identify $C \boxtimes \chi D$ with a Rieffel deformation of the commutative tensor product $C \otimes D$ in the sense of [6]. To begin with, we reduce to the case where both $G$ and $H$ are Abelian.

The dual quantum groups are the groups $G$ and $H$, respectively. Since $\hat{G} \otimes \hat{H} = C_0(G \times H)$, a bicharacter $\chi \in \mathcal{U}(\hat{G} \otimes \hat{H})$ is a bicharacter $\chi: G \times H \to \mathbb{T}$ in the classical sense. Since $\mathbb{T}$ is commutative, $\chi(g, h)$ vanishes if $g$ or $h$ is a commutator. Hence $\chi$ descends to a continuous biadditive map $\chi': G^{ab} \times H^{ab} \to \mathbb{T}$ on the Abelianisations $G^{ab}$ and $H^{ab}$, giving us a bicharacter $\chi^{ab} \in \mathcal{U}(C_0(G^{ab} \times H^{ab}))$. The quotient maps $G \to G^{ab}$ and $H \to H^{ab}$ are quantum group morphisms. They allow us to turn the given coactions of $G$ and $H$ on $C$ and $D$ into coactions of $G^{ab}$ and $H^{ab}$, respectively. Theorem 5.2 shows $C \boxtimes \chi D = C \boxtimes \chi^{ab} D$, where the right hand side uses only the induced coactions of $G^{ab}$ and $H^{ab}$. Hence we may without loss of generality assume that $G$ and $H$ are Abelian locally compact groups.

Let $\tilde{G}$ and $\tilde{H}$ be their Pontryagin duals. We may also view a bicharacter as a continuous group homomorphism $G \to \tilde{H}$ or $H \to \tilde{G}$ by fixing one of the two variables. This makes it easy to list all bicharacters for two given Abelian locally compact groups $G$ and $H$. Coactions of $G$ and $H$ are equivalent to actions of $\tilde{G}$ and $\tilde{H}$, respectively. Thus $C$ and $D$ carry actions of $\tilde{G}$ and $\tilde{H}$, respectively. The commutative tensor product

$$E := C \otimes D$$

inherits an action of $\Gamma := \hat{G} \times \hat{H}$. The bicharacter $\chi: G \times H \to \mathbb{T}$ yields a bicharacter $\Psi: \hat{G} \times \hat{H} \to \mathbb{T}$, $\Psi((g_1, h_1), (g_2, h_2)) := \chi(g_2, h_1)^{-1}$. 
Any bicharacter is also a two-cocycle, which may be used as a deformation parameter for Rieffel deformations. Here we define Rieffel deformations following Kasprzak [2] using crossed products and Landstad theory.

**Theorem 6.3.** \( C \boxtimes_\chi D \) is naturally isomorphic to the Rieffel deformation of \( E \) with respect to \( \Psi \).

**Proof.** Pick faithful representations of \( C \times \hat{G} \) and \( D \times \hat{H} \) on Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively. These give faithful covariant representations of \( C \) and \( D \), which we use to represent \( C \boxtimes_\chi D \) faithfully on \( \mathcal{H} \otimes \mathcal{K} \). They also generate a faithful representation of

\[
E \times \Gamma \cong (C \times \hat{G}) \otimes (D \times \hat{H})
\]
on \( \mathcal{H} \otimes \mathcal{K} \). The description of the operator \( Z \) in [4.16] shows that \( Z \in \mathcal{M}(E \times \Gamma) \).

Thus \( C \boxtimes_\chi D \) is contained in \( \mathcal{M} \).\( E \times \Gamma \), generated by the canonical embeddings of \( C \) and \( D \), with the latter twisted by \( \text{Ad}_Z \).

The Rieffel deformation \( E^\Psi \) of \( E \) with respect to \( \Psi \) is described in [6] as a subalgebra of \( \mathcal{M}(E \times \Gamma) \) as well. We will use formal properties of \( E^\Psi \) to deduce that \( E^\Psi = C \boxtimes_\chi D \) as \( C^* \)-subalgebras of \( E \times \Gamma \).

Since \( E = (C \otimes 1) \cdot (1 \otimes D) \), [7, Lemma 3.4] yields (6.4)

\[
E^\Psi = (C \otimes 1)^\Psi \cdot (1 \otimes D)^\Psi;
\]

Here we let \( \Gamma \) act on \( C \) and \( D \) by letting \( \hat{H} \) act trivially on \( C \) and \( \hat{G} \) trivially on \( D \).

The deformation procedure in [6] uses the unitaries

\[
U_{g,h} \in C(G \times H, \mathbb{T}), \quad U_{g,h}(g_1,h_1) := \Psi((g_1,h_1),(g,h)) = \chi(g,h)^{-1}
\]
for \((g,h),(g_1,h_1) \in G \times H = \hat{G}\). Since \( \hat{H} \) acts trivially on \( C \), we have \( C \times \Gamma \cong (C \times \hat{G}) \otimes C^*(\hat{H}) \).

In this case, the Rieffel deformation does nothing, that is, \( C^\Psi = C \) as subalgebras of \( \mathcal{M}(C \times \Gamma) \). Thus \( (C \otimes 1)^\Psi \) in (6.3) is represented on \( \mathcal{H} \otimes \mathcal{K} \) by \( \phi_1(C) \).

Now define another two-cocycle on \( \hat{G} \) by \( \Psi'((g_1,h_1),(g_2,h_2)) := \chi(g_1,h_2) \); the Rieffel deformation for \( \Psi' \) involves the unitaries

\[
U'_{g,h}(g_1,h_1) := \Psi'((g_1,h_1),(g,h)) = \chi(g_1,h),
\]
which are mapped to central elements in \( D \times \Gamma \).

Therefore, \( D^\Psi = D \) as subalgebras of \( \mathcal{M}(D \times \Gamma) \).

The two-cocycles \( \Psi \) and \( \Psi' \) are cohomologous: let \( f(g,h) = \chi(g,h)^{-1} \), then

\[
\partial f((g_1,h_1),(g_2,h_2)) := \frac{f(g_1g_2,h_1h_2)}{f(g_1,h_1)f(g_2,h_2)} = \chi(g_2,h_1)^{-1}\chi(g_1,h_2)^{-1};
\]

thus \( \partial f \cdot \Psi' = \Psi \). Now [6] Lemmas 3.4 and 3.5 yield

\[
D^\Psi = (D^\Psi)^\partial f = D^Df^* = fDf^*
\]
as \( C^* \)-subalgebras of \( D \times \Gamma \). Here \( f \) is viewed as a unitary element of \( C^*(\Gamma) \subseteq D \times \Gamma \).

Equation (1.3) shows that the representation \( D \times \Gamma \to E \times \Gamma \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \) maps \( f \) to \( Z \). Thus (1 \otimes D)^\Psi = 1 \otimes fDf^* in (6.4) is represented on \( \mathcal{H} \otimes \mathcal{K} \) by \( \tilde{\psi}_2(D) \). Finally, [6.4] becomes \( E^\Psi = \phi_1(C) \cdot \tilde{\psi}_2(D) \cong C \boxtimes_\chi D \) as desired. \( \square \)

6.3. **Crossed products.** Consider the special case where \( \mathbb{H} = \hat{G}, \chi = W^A \in \mathcal{U}(\hat{A} \otimes A), D = \hat{A}, \delta = \Delta_A; \hat{A} \to \hat{A} \otimes \hat{A} \). We claim that \( (C,\gamma) \boxtimes_{W^A} (\hat{A},\Delta_A) \) is the reduced crossed product of \( (C,\gamma) \). More precisely, the reduced crossed product \( C \rtimes_\gamma \hat{A} \) comes equipped with canonical morphisms \( \iota_C: C \to C \rtimes_\gamma \hat{A} \) and \( \iota_A: \hat{A} \to C \rtimes_\gamma \hat{A} \), such that \( (C \rtimes_\gamma \hat{A},\iota_C,\iota_A) \) is a crossed product in the sense of Definition 2.3.

We claim that this is equivalent to \( (C,\gamma) \boxtimes_{W^A} (\hat{A},\Delta_A) \) as a crossed product.

Let \((\pi,\hat{\pi})\) be a \( G \)-Heisenberg pair on the Hilbert space \( \mathcal{H} \) of the special form in Example 3.3 that is, \( W_{\pi,\hat{\pi}} = \mathcal{W} \) is a multiplicative unitary generating \( G \).
We now describe the dual coaction in this way, using the functoriality of the reduced crossed product $C \rtimes_r \hat{A}$ as the crossed product generated by the representations $\gamma_1$ and $\hat{\pi}_2$. This definition is standard for locally compact quantum groups with Haar weights, where $\pi$ and $\hat{\pi}$ are taken as the regular representations. For general $C^*$-quantum groups, Theorem 6.5 says that $C \rtimes_r \hat{A}$ does not depend on the choice of $\mathbb{W}$ and that there is an isomorphism of crossed products

$$(C, \gamma) \boxtimes_{\mathbb{W}} (\hat{A}, \hat{\Delta}_A) \cong C \rtimes_r \hat{A}.$$  

**Proof of Theorem 6.5** Since $\gamma_1 \otimes \text{id}_{K(H)} : C \otimes K(H) \to C \otimes K(H) \otimes K(H)$ is a faithful morphism, the pair of representations $(\gamma_1 \otimes \hat{\pi})$ generates a faithful representation of $C \boxtimes \hat{A}$ if and only if the pair $((\gamma_1 \otimes \text{id}_{K(H)}) \circ \gamma_1, (\gamma_1 \otimes \text{id}_{K(H)}) \circ \hat{\pi}_2)$ does so. We have $(\gamma_1 \otimes \text{id}_{K(H)}) \circ \hat{\pi}_2 = \hat{\pi}_2(\hat{a}) = \hat{\pi}_2(\hat{a})$ and $(\gamma_1 \otimes \text{id}_{K(H)}) \gamma_1(c) = (\gamma \otimes \text{id}_{A}) \gamma(c)_1 = (\text{id} \otimes \Delta_A) \gamma(c) = \mathbb{W}(\text{id} \otimes \pi) \gamma(c) 1_2 \mathbb{W}^*$. Let $\Sigma_{23}$ be the coordinate flip. Conjugating both representations by the same unitary $\Sigma_{23} \mathbb{W}^*$ gives an unitarily equivalent pair of representations. Hence we may further replace $\gamma_1$ and $\hat{\pi}_2$ by the representations $c \mapsto (\text{id} \otimes \pi) \gamma(c)_{13}$ of $C$ and $\hat{a} \mapsto \Sigma_{23} \mathbb{W}^* \hat{\pi}(\hat{a})$ of $\hat{A}$; here we use the standard description in terms of $\mathbb{W}$.

Thus we arrive at the pair of representations $(\text{id}_{C} \otimes \hat{\pi} \otimes \text{id}_{K(H)}) \gamma_2$ and $(\text{id}_{C} \otimes \hat{\pi} \otimes \text{id}_{K(H)}) \gamma_2$ with $\gamma_2 = \gamma_1$ and $\gamma_2 = (\hat{\Delta}_A)_{12}$ in $C \otimes \hat{A} \otimes K(H)$. Since $\hat{\pi}$ is faithful, this pair is equivalent to $(\gamma_2, \gamma_2)$. Since this pair defines the crossed product $C \boxtimes \hat{A}$, we see that $(\gamma_1, \hat{\pi}_2)$ generates an equivalent crossed product as claimed. □

Viewing the reduced crossed product as a special case of $\boxtimes$ gives us more freedom because we may also tensor $(C, \gamma)$ with other $\hat{G}$-$C^*$-algebras and use functoriality. We now describe the dual coaction in this way, using the functoriality of $\boxtimes$.

The comultiplication $\hat{\Delta} : \hat{A} \to \hat{A} \otimes \hat{A}$ is $\hat{G}$-equivariant if $\hat{G}$ coacts on $\hat{A} \otimes \hat{A}$ by $\text{id} \otimes \hat{\Delta} : \hat{A} \otimes \hat{A} \to \hat{A} \otimes \hat{A} \otimes \hat{A}$. By the functoriality of $\boxtimes$, this equivariant morphism induces a morphism $\hat{\delta} : C \rtimes_r \hat{A} \cong C \boxtimes \hat{A} \to C \boxtimes (\hat{A} \otimes \hat{A}) \cong \hat{A} \otimes (C \boxtimes \hat{A}) \cong \hat{A} \otimes (C \rtimes_r \hat{A})$; here we use Lemma 6.6 in the second variable to pull out the first factor $\hat{A}$.

**Lemma 6.6.** The map $\hat{\delta} : C \rtimes_r \hat{A} \to \hat{A} \otimes (C \rtimes_r \hat{A})$ is a continuous left $\hat{G}$-coaction.

**Proof.** The comodule property of $\hat{\delta}$ follows from the coassociativity of $\hat{\Delta}$ and the functoriality of $\boxtimes$. The map $\hat{\delta}$ is faithful by Proposition 5.6. The Podleś condition for $\hat{\delta}$ follows because $(\hat{A} \otimes 1) \Delta(\hat{A}) = \hat{A} \otimes \hat{A}$: apply $\iota_{\hat{A} \otimes \hat{A}}$ to this equality. □

This coaction is uniquely determined by the conditions $\hat{\delta}(\iota_C(c)) = 1 \otimes \iota_{C}(c)$ and $\hat{\delta}(\iota_{\hat{A}}(\hat{a})) = (\iota_{\hat{A}} \otimes \iota_{\hat{A}}) \Delta$. The same conditions characterise the dual coaction. Thus we have indeed constructed the dual coaction.

The functoriality of $\boxtimes$ in the first variable gives us the usual functoriality of reduced crossed products.

General tensor products $C \boxtimes_{\mathbb{W}} (D, \delta)$ are closely related to the crossed product through Lemma 2.1 in the second variable to pull out the first factor $\hat{A}$. Theorem 6.5 says that $C \rtimes_r \hat{A}$ does not depend on the choice of $\mathbb{W}$ and that there is an isomorphism of crossed products

$$(C, \gamma) \boxtimes_{\mathbb{W}} (\hat{A}, \hat{\Delta}_A) \cong C \rtimes_r \hat{A}.$$
for the coaction \( \text{id}_D \otimes \Delta \) on \( D \otimes \hat{A} \). By Proposition 3.4 and Lemma 3.21, this induces a faithful morphism

\[
C \otimes_{W} D \rightarrow C \otimes_{W} (D \otimes \hat{A}) \cong D \otimes (C \otimes_{W} \hat{A}) \cong D \otimes (C \times_{\rho} \hat{A}).
\]

Now we consider once again the general situation of two quantum groups \( \mathbb{G} = (A, \Delta_A) \) and \( \mathbb{H} = (B, \Delta_B) \) and a bicharacter \( \chi \in \mathcal{U}(\hat{A} \otimes \hat{B}) \).

**Theorem 6.7.** View \( \hat{A} \otimes \hat{B} \) as a subalgebra of \((C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B})\) via \( \iota_{\hat{A}} \otimes \iota_{\hat{B}} \) and use this to view \( \chi \) as a multiplier of \((C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B})\). The embeddings

\[
(i_C)_1: C \rightarrow (C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B}), \quad c \mapsto \iota_C(c) \otimes 1,
\]

\[
Ad_{\chi} \circ (i_D)_2: D \rightarrow (C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B}), \quad d \mapsto \chi^*(1 \otimes \iota_D(d)) \chi,
\]

induce a faithful morphism

\[
C \otimes_{\chi} D \rightarrow (C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B}).
\]

**Proof.** Choose faithful representation \( \varphi_0: C \rightarrow \mathcal{B}(K_0) \). Let \( (\pi, \hat{\pi}) \) be a \( \mathbb{G} \)-Heisenberg pair as in Example 3.4 acting on a Hilbert space \( \mathcal{H}_\pi \). Let \( \mathcal{H} := \mathcal{H}_0 \otimes \mathcal{H}_\pi \). Then we get a faithful representation \( \varphi : C \otimes \text{id} \) of \( C \otimes \mathcal{K}(\mathcal{H}_\pi) \) on \( \mathcal{H} \). This restricts to a faithful representation \( \varphi' : C \times_{\rho} \hat{A} \subseteq C \otimes \mathcal{K}(\mathcal{H}_\pi) \rightarrow \mathcal{B}(\mathcal{H}) \), where we realise \( C \times_{\rho} \hat{A} \) as in Theorem 6.3.

We compare this with the construction of a covariant representation of \((C, \gamma)\) in Example 3.4. We see that this covariant representation consists of \( \rho \circ \iota_C \): \( C \rightarrow \mathcal{B}(\mathcal{H}) \) and \( W^A_{\rho, A, 1, 2} \in \mathcal{U}(\mathcal{K}(\mathcal{H}_\pi) \otimes A) \). Furthermore, the representation of \( \hat{A}^u \) used later in the proof of Theorem 4.4 is \( \rho \circ \hat{\lambda} \) for the reducing morphism \( \lambda: \hat{A}^u \rightarrow \hat{A} \). (Actually, any representation of \( C \times_{\rho} \hat{A} \) gives a covariant representation of \( (C, \gamma) \) in a similar way.)

Now do the same things for \((D, \delta)\): let \( \psi_0 : D \rightarrow \mathcal{B}(K_0) \) be a faithful representation; choose an \( \mathbb{H} \)-Heisenberg pair \((\rho, \hat{\rho})\) as in Example 3.4 acting on a Hilbert space \( \mathcal{K}_\rho \); let \( \mathcal{K} := \mathcal{K}_0 \otimes \mathcal{K}_\rho \); let \( \psi' \) be the resulting faithful representation of \( D \times_{\pi} \hat{B} \) on \( \mathcal{K} \); construct a covariant representation of \((D, \delta)\) on \( \mathcal{K} \) as in Example 4.4.

Theorem 4.3 gives a faithful representation of \( C \otimes_{\chi} D \) on \( \mathcal{H} \otimes \mathcal{K} \), generated by the representations \( \varphi_1 \) and \( Ad_{\psi} \varphi_2 \). By construction, we also get a faithful representation \( \varphi' \otimes \psi' \) of \((C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B}) \) on \( \mathcal{H} \otimes \mathcal{K} \). The description of \( Z \) in 4.3 yields \( Z = \langle \varphi' \otimes \psi' \rangle(\chi^*) \). Hence the representations \( \varphi_1 \) and \( Ad_{\psi} \varphi_2 \) both factor through the embedding \( \varphi' \otimes \psi' \) and the maps \( (i_C)_1 \) and \( Ad_{\chi} \circ (i_D)_2 \) in the statement of the theorem. We thus get a faithful morphism \( C \otimes_{\chi} D \rightarrow (C \times_{\rho} \hat{A}) \otimes (D \times_{\pi} \hat{B}) \) restricting to \((i_C)_1 \) and \( Ad_{\chi} \circ (i_D)_2 \) on \( C \) and \( D \), respectively.

For instance, in the situation of Section 6.4, this realises the skew-commutative tensor product \( C \otimes D \) as a subalgebra of \((C \times \mathbb{Z}/2) \otimes (D \times \mathbb{Z}/2)\).

**References**

[1] Saad Baaj and Georges Skandalis, \( C^* \)-algebras de Hopf et théorie de Kasparov équivariante, K-Theory 2 (1989), no. 6, 683–721. doi: 10.1007/BF00053842 MR 1010978

[2] , Unitaires multiplicatifs et dualité pour les produits croisés de \( C^* \)-algèbres, Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 4, 425–488, available at http://www.numdam.org/item?id=AENS_1993_4_26_4_425_0 MR 1235438

[3] Alfons van Daele and S. van Keer, The Yang–Baxter and pentagon equation, Compositio Math. 91 (1994), no. 2, 201–221, available at http://www.numdam.org/item?id=CM_1994__91_2_201_0 MR 1273649

[4] Ruy Exel, Blends and algebras, C. R. Math. Rep. Acad. Sci. Canada (2013), accepted.

[5] Gennadi G. Kasparov, The operator \( K \)-functor and extensions of \( C^* \)-algebras, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 3, 571–636, 719, available at http://mi.mathnet.ru/izv/1739 English transl., Math. USSR-Izv. 16 (1981), no. 3, 513–572 (1981), doi: 10.1070/IM1981v016n03ABEH001320 MR 582160
Paweł Kasprzak, *Rieffel deformation via crossed products*, J. Funct. Anal. 257 (2009), no. 5, 1288–1332. doi: 10.1016/j.jfa.2009.05.013 [MR 2541270]

---

[6] Paweł Kasprzak, *Rieffel deformation of group coactions*, Comm. Math. Phys. 300 (2010), no. 3, 741–763. doi: 10.1007/s00220-010-1093-9 [MR 2736961]

---

[7] E. Christopher Lance, *Hilbert C*-modules*, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, 1995. MR 1325694

---

[8] Javier López Peña, Florin Panaite, and Freddy Van Oystaeyen, *General twisting of algebras*, Adv. Math. 212 (2007), no. 1, 315–337. doi: 10.1016/j.aim.2006.10.003 [MR 2319771]

---

[9] Shahn Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995. doi: 10.1017/CBO9780511613104 [MR 1381692]

---

[10] Ralf Meyer, Sutanu Roy, and Stanisław Lech Woronowicz, *Homomorphisms of quantum groups*, Münster J. Math. 5 (2012), 1–24, available at [http://wwmath.uni-muenster.de/mjm/vol_5/mjm_vol_5_01.pdf](http://wwmath.uni-muenster.de/mjm/vol_5/mjm_vol_5_01.pdf)

---

[11] Ryszard Nest and Christian Voigt, *Equivariant Poincaré duality for quantum group actions*, J. Funct. Anal. 258 (2010), no. 5, 1466–1503. doi: 10.1016/j.jfa.2009.10.015 [MR 2566309]

---

[12] Piotr M. Sołtan and Stanisław Lech Woronowicz, *A remark on manageable multiplicative unitaries*, Lett. Math. Phys. 57 (2001), no. 3, 239–252. doi: 10.1023/A:1012230629865 [MR 1862455]

---

[13] Stefaan Vaes, *A new approach to induction and imprimitivity results*, J. Funct. Anal. 229 (2005), no. 2, 317–374. doi: 10.1016/j.jfa.2004.11.016 [MR 2182592]

---

[14] Stanisław Lech Woronowicz, *An example of a braided locally compact group*, Quantum Groups: Formalism and Applications, XXX Karpacz Winter School (Karpacz, 1994), PWN, Warsaw, 1995, pp. 155–171. MR 1647968

---

[15] Stanisław Lech Woronowicz, *From multiplicative unitaries to quantum groups II*, J. Funct. Anal. 252 (2007), no. 1, 42–67. doi: 10.1016/j.jfa.2007.07.006 [MR 2357350]

---

[16] Stanisław Lech Woronowicz, *From multiplicative unitaries to quantum groups*, Internat. J. Math. 7 (1996), no. 1, 127–149. doi: 10.1142/S0129167X96000086 [MR 1369068]

---

E-mail address: rameyer@uni-math.gwdg.de

E-mail address: sutanu@uni-math.gwdg.de

E-mail address: Stanislaw.Woronowicz@fuw.edu.pl

Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany

---

E-mail address: sutanu@uni-math.gwdg.de

E-mail address: Stanislaw.Woronowicz@fuw.edu.pl

---

Instytut Matematyki, Uniwersytet w Białymstoku, and, Katedra Metod Matematycznych Fizyki, Wydział Fizyki, Uniwersytet Warszawski, Hoża 74, 00-682 Warszawa, Poland