Ricci Curvature and Gauss Maps of Minimal Submanifolds

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Abstract

We present conditions on the Ricci curvature for complete, oriented, minimal submanifolds of Euclidean space, as well as the standard unit sphere, when the Gauss maps are bounded embeddings.
1 Introduction

A fundamental approach to the study of minimal submanifolds was inspired by Bernstein, whose theorem relates properties about the Gauss map of a minimal surface to information about the minimal surface itself. Herein we investigate upper bounds on the Ricci curvature of minimal submanifolds of Euclidean space and the standard unit sphere when the Gauss map is a bounded embedding. A bounded embedding is an embedding whose image $I$ in the oriented Grassmann manifold $\tilde{G}$ is bounded with respect to the metric on $I$ induced from the canonical metric on $\tilde{G}$. We prove the following two theorems.

**Theorem A.** Let $M$ be a complete, oriented, minimal submanifold of $R^k$ and suppose the Gauss map is a bounded embedding. Then $\text{Ric} < 0$ and

$$\sup \text{Ric} = 0$$

**Theorem B.** Let $M^m$ be a complete, oriented, minimal submanifold of a unit sphere and suppose the second Gauss map is a bounded embedding. Then $\text{Ric} < m - 1$. Furthermore, $M$ is compact if and only if

$$\sup \text{Ric} < m - 1$$

Bounds for the Ricci curvature of complete, oriented, minimal hypersurfaces of the unit sphere have been obtained by Hasanis and Vlachos [4]. A pinching theorem for minimal submanifolds of a sphere having positive Ricci curvature was proved by Ejiri [1].

2 The First Gauss Map

The oriented Grassmannian $\tilde{G}_{m,n}$ is the Riemannian manifold of oriented $m$-planes in $R^{m+n}$ with canonical metric $g_c$ determined as follows. Choose any two points $P, Q \in \tilde{G}_{m,n}$ and let $\xi_1, \ldots, \xi_m$ and $\zeta_1, \ldots, \zeta_m$ be oriented, orthonormal bases of $P$ and $Q$, respectively. Form the $m \times m$ matrix $\alpha$ by $\alpha_{ij} = \langle \xi_i, \zeta_j \rangle$ and let $A$ be the product of $\alpha$ with its transpose: $A := \alpha\alpha^T$. $A$ is non-negative and symmetric with eigenvalues $\lambda_1^2, \ldots, \lambda_m^2$, say, where $0 \leq \lambda_i \leq 1$. Define $d_c : \tilde{G}_{m,n} \times \tilde{G}_{m,n} \to [0, \sqrt{m\pi}/2]$ by

$$d_c(P, Q) := \sqrt{\sum_{i=1}^{m} \arccos^2 \lambda_i}$$
$d_c$ is a local - but not global - distance function; for instance, if $P$ and $Q$ define the same $m$-plane, but with opposite orientations, then $d_c(P, Q) = 0$. The metric $g_c$ is generated by $d_c$: 

$$g_c(X, X) := \frac{d}{dt}d_c(x(0), x(t))|_{t=0}$$

where $x = x(t)$ is any smooth path in $\bar{G}_{m,n}$ with $\dot{x}(0) = X$.

For our purposes it will be convenient to consider another metric $g_s$ on $\bar{G}_{m,n}$, which is related to spherical geometry. Put 

$$d_s(P, Q) := \arccos \left( \prod_{i=1}^{m} \lambda_i \right)$$

and let $g_s$ be generated by $d_s$. It shall be shown below that $g_s$ is, in fact, a well-defined Riemannian metric. We will prove that $d_s \leq d_c$.

**Lemma 1** $g_s \leq g_c$

Next, we consider a natural realization of the metric $g_s$. The vector space $\Lambda^m R^k$, where $m \leq k$, possesses a canonical inner-product induced from the standard inner-product on $R^k$:

$$<\xi_1 \wedge \cdots \wedge \xi_m, \zeta_1 \wedge \cdots \wedge \zeta_m>_{\Lambda^m R^k} := \det <\xi_i, \zeta_j>_{R^k}$$

The unit sphere in $\Lambda^m R^k$ shall be denoted $S^\mu$. Consider the submanifold $H$ of $S^\mu$ consisting of all elements of the form $\xi_1 \wedge \cdots \wedge \xi_m \in \Lambda^m R^k$. There is a canonical diffeomorphism $\rho : H \rightarrow \bar{G}_{m,n}$ defined by 

$$\rho(\xi_1 \wedge \cdots \wedge \xi_m) := (\text{span}\{\xi_1, ..., \xi_m\}, [\xi_1 \wedge \cdots \wedge \xi_m])$$

where $[\xi_1 \wedge \cdots \wedge \xi_m]$ denotes the orientation class of $\xi_1 \wedge \cdots \wedge \xi_m$.

Let $d_H$ be the restriction of the distance function $d_{S^\mu}$ on the sphere $S^\mu$ to 

$$\mathcal{D} := \{(u, v) \in H \times H : <u, v>_{\Lambda^m R^k} \text{ is non-negative}\}$$

For $(p, q) \in \mathcal{D}$,

$$d_H(p, q) = d_{S^\mu}(p, q) = \arccos <p, q>_{\Lambda^m R^k} = d_s(\rho(p), \rho(q))$$

Consequently, the metrics $h$ and $g_s$ generated by $d_H$ and $d_s$, respectively, are related by a pull-back: $h = \rho^* g_s$. 
Lemma 2 \( h \) is the induced metric on \( H \), regarded as a submanifold of the Euclidean space \( \Lambda^m \mathbb{R}^k \).

Corollary 3 \( h \) and \( g_s \) are well-defined and positive definite. Moreover, \( \rho \) is an isometry of Riemannian manifolds \((H, h)\) and \((\tilde{G}_{m,n}, g_s)\).

Let \( M \) be an \( m \)-dimensional, oriented submanifold of \( \mathbb{R}^k \) and let \( Z_1, \ldots, Z_m \) be a local, oriented, orthonormal frame for \( M \). Define the map \( \phi : M \to H \subseteq S^m \) by

\[
\phi := Z_1 \wedge \cdots \wedge Z_m
\]

\( \rho \circ \phi : M \to \tilde{G}_{m,n} \) is the (first) Gauss map.

The differential of \( \phi \) is

\[
\phi_* (X) = \sum_{i=1}^{m} Z_1 \wedge \cdots \wedge Z_{i-1} \wedge B(X, Z_i) \wedge Z_{i+1} \wedge \cdots \wedge Z_m
\]

Define the homomorphism of vector bundles \( B : TM \to \text{Hom}(TM, TM^\perp) \) by \( B(X)(Y) := B(X, Y) \). We may extend the domain of definition of \( B \) by requiring it to act as a derivation on tensor products. Then the above equation may be expressed succinctly as

\[
\phi_*(X) = B(X)\phi
\]

(1)

In what follows, \( \text{Ric} \) designates the Ricci curvature of \( M \).

Lemma 4

\[
\phi^* (h)(X, Y) = <B(X, Y), tr B> - \text{Ric}(X, Y)
\]

This leads to a characterization of the minimal submanifolds of \( \mathbb{R}^k \).

Corollary 5 \( M \) is a minimal submanifold of \( \mathbb{R}^k \) if and only if

\[
\phi^* (h) = - \text{Ric}
\]

Theorem 6 Let \( M \) be a complete, oriented, minimal submanifold of \( \mathbb{R}^k \) and suppose the Gauss map is a bounded embedding. Then \( \text{Ric} < 0 \) and

\[
\sup \text{Ric} = 0
\]
3 The Second Gauss Map

We suppose that $M$ is an oriented submanifold of $N$, which is an oriented submanifold of $R^k$. $B_{M \subseteq N}$ (resp. $B_{N \subseteq E}$) shall denote the second fundamental form of $M$ (resp. $N$) viewed as a submanifold of $N$ (resp. $R^k$). The discussion below proceeds in a manner similar to the previous section and so will not be as detailed.

Let $Z_1,\ldots,Z_m,V_1,\ldots,V_n$ be a local, oriented, orthonormal frame for $N$, where $Z_1,\ldots,Z_m$ is a local oriented frame for $M$, and put $r := k - n$. Define the map
\[ \psi : M \to S^\nu \] by
\[ \psi = V_1 \wedge \cdots \wedge V_n \]
where $S^\nu$ is the unit sphere in $\Lambda^n R^k$.

$H$ shall be the subset of $S^\nu$ consisting of elements of the form $\xi_1 \wedge \cdots \wedge \xi_n$ and $h$ shall denote the metric on $H$ induced from $\Lambda^n R^k$. The diffeomorphism $\rho : H \to \tilde{G}_{n,r}$ defined by
\[ \rho(\xi_1 \wedge \cdots \wedge \xi_n) := (\text{span}\{\xi_1,\ldots,\xi_n\}, [\xi_1 \wedge \cdots \wedge \xi_n]) \]
is an isometry of Riemannian manifolds $(H,h)$ and $(\tilde{G}_{n,r},g_s)$. $\rho \circ \psi : M \to \tilde{G}_{n,r}$ is the second Gauss map.

Lemma 7
\[
\psi^*(h)(X,Y) = <B_{M \subseteq N}(X,Y), \text{tr} B_{M \subseteq N}> - \text{Ric}_M(X,Y) + \sum_{i=1}^n <B_{N \subseteq E}(X,Y), B_{N \subseteq E}(V_i,V_i)> + \sum_{i=1}^m <R_N(X,Z_i)Y, Z_i> - \sum_{i=1}^n <R_N(X,V_i)Y, V_i>
\]

Corollary 8 If $M^m$ is a minimal submanifold of $S^{k-1}$, the unit hypersphere of $R^k$ then
\[ \psi^*(h) = (m-1) <,>_M - \text{Ric} \]

The proof of the theorem below is essentially contained in the proof of Theorem 6.

Theorem 9 Let $M^m$ be a complete, oriented, minimal submanifold of a unit sphere and suppose the second Gauss map is a bounded embedding. Then $\text{Ric} < m - 1$. Furthermore, $M$ is compact if and only if
\[ \sup \text{Ric} < m - 1 \]
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