Abstract

We compare the description of the M-theory form fields via cohomotopy versus that via integral cohomology. The conditions for lifting the former to the latter are identified using obstruction theory in the form of Postnikov towers, where torsion plays a central role. A subset of these conditions are shown to correspond compatibly to existing consistency conditions, while the rest are new and point to further consistency requirements for M-theory. Bringing in the geometry leads to a differential refinement of the Postnikov tower, which should be of independent interest. This provides another confirmation that cohomotopy is the proper generalized cohomology theory to describe these fields.

1 Introduction

One of the main problems that would shed some light on M-theory is the precise nature of the C-field in the theory. Earlier literature viewed the C-field as a cocycle in cohomology, or a higher gauge field, with some extra structure (see [DFM03] [AJ04] [Sa10] [SSS12] [FSS14a] [FSS14b]). More recently, homotopy theory has been used to describe the dynamics of the C-field in M-theory, leading to the proposal in [Sa13] that it is quantized in cohomotopy cohomology theory $\pi^*$ [Bo36] [Sp49]. Later it was shown that cohomotopy captures the fields in M-theory very nicely, for the dynamics in the rational approximation [FSS13] [FSS15] [FSS16a] [FSS16b] [BSS18] [HSS19]; see [FSS19] for a review.

At a first approximation, ignoring torsion, we have that rational cohomology is essentially equivalent to rational cohomotopy in the same degree. For the stable case, for degree four corresponding to $G_4$, we have an isomorphism

$$H^4(Y^{11}; \mathbb{Q}) \cong \pi_4^*(Y^{11}) \otimes \mathbb{Q}. \quad (1)$$
In the \textit{unstable} case, schematically, we have
\begin{equation}
\text{Rational cohomotopy} \leftrightarrow \text{Rational cohomology + trivialization of the cup square.} \quad (2)
\end{equation}
In this case we do not have an isomorphism; for example for $Y^{11} = S^7 \times \mathbb{R}^4$, we have $H^4(S^7 \times \mathbb{R}^4; \mathbb{Q}) = 0$, while $\pi^4(S^7 \times \mathbb{R}^4) \otimes \mathbb{Q} \cong \mathbb{Q}$. Hence, it might seem like there is nothing to be gained here by bringing in (co)homotopy theory. Nevertheless, somewhat surprisingly, placing the problem in homotopy theory, even rationally, has brought in interesting structures beyond just this (as indicated above). This interesting rational structure come from the unstable case. Rationally and stably $S^4_\mathbb{Q}$ is just the Eilenberg-MacLane space $K(\mathbb{Q}, 4)$ so there is not much new to say. On the other hand, \textit{integrally} and \textit{stably} we do see new effects, which is what we highlight here.

More remarkably, going beyond the rational approximation, the cancellation of the main anomalies of M-theory follows naturally from cohomotopy. It was shown in \cite{FSS19b,FSS19c} that that C-field charge quantization in twisted cohomotopy implies various fundamental anomaly cancellation and quantization conditions, with similar effects for D-branes and orientifolds \cite{BSS18,SS19a}. This led to the formulation of:

\begin{center}
\textbf{Hypothesis II. The C-field is charge-quantized in cohomotopy theory, even non-rationally.}
\end{center}

Rational cohomotopy of spacetime $Y$ is given by homotopy classes of maps to the rational 4-sphere, $[Y, S^4_\mathbb{Q}]$, while cohomotopy deals with maps to the standard 4-sphere, equipped with the usual subspace topology, $[Y, S^4]$. Since we are a priori given the former, we ask for a natural lift to the integral level, and whether this would indeed give us the latter. This amounts to giving the \mathbb{Z}-form for the Sullivan algebra associated with the rational homotopy type $S^4_\mathbb{Q}$, as in \cite[p. 246]{FOT08}. Furthermore, we need to know whether or not the result is indeed a finite-dimensional space. This would give us a topological space of the same rational homotopy type of $S^4_\mathbb{Q}$. As we are ultimately interested in differential refinements, we need to have this as a finite-dimensional manifold, i.e., the smooth 4-sphere with its standard differentiable structure. These generally follow via the result of Sullivan on realizability; see \cite[Theorem A]{Su74} \cite{Su71} \cite{Su05}.

If we start with the rational 4-sphere $S^4_\mathbb{Q}$, then how can we lift it to an “integral” space? We need to ‘supply the missing’ torsion information that was killed upon the reverse process of rationalization. We would get a space $S_\mathbb{Z}$. What is this? There could be many spaces whose rationalizations coincide; in fact infinitely many, measured by the Mislin genus \cite{Mi71}. This is the case even if the spaces coincide when localized at every prime $p$, not just the trivial prime corresponding to rationalization. The extended genus \cite{Hi88} of a space $X$ is the set of homotopy types $[Y]$ of nilpotent CW-spaces $Y$ which are locally homotopy equivalent to $X$ at each prime. The Mislin genus \cite{Mi71} of $X$ is defined to be the subset of the extended genus consisting of those $[Y]$s of finite type. While the Mislin genus can be finite \cite{Wil76}, the extended genus is always infinite for nontrivial homotopy types \cite{Mc94} \cite{Mc96}. The corresponding spaces might also not be finite-dimensional or of finite type; in fact most of them will not be (see \cite{Wil76}).

However, the actual 4-sphere $S^4$ stands out as not only the most natural but the finite-dimensional one.

\begin{equation}
X \cdots S^4 \cdots Y
\end{equation}

Aiming for $S^4$ itself, one has to go through the fibration $S^4_+ \rightarrow S^4 \rightarrow S^4_\mathbb{Q}$, where $S^4_+$ is the pure torsion part (see, e.g., \cite{MS17}). This is the source of the torsion obstructions we will identify. Now that we have explained what is involved in moving beyond the rational approximation, we also found that the 4-sphere itself is the most natural. Hence we will adopt this perspective henceforth and consider a lift of the form

\begin{equation}
Y \rightarrow \bullet \rightarrow S^4
\end{equation}
With $S^4$ as the proper lift of the rational homotopy type, the natural question then becomes: before throwing in additional structure, such as differential refinements, how different is the description via (twisted) cohomotopy from the description via (twisted/shifted) integral cohomology? To answer this, we would like to start with integral cohomology as describing the (shifted/twisted) C-field and then transition to a description in terms of cohomotopy. By representability, this amounts to lifting

$$\begin{array}{c}
\text{Nonlinear} \\
\text{prequantum}
\end{array} \downarrow \downarrow \downarrow \begin{array}{c}
Y \\
\text{Linear} \\
\text{quantum}
\end{array} \rightarrow K(\mathbb{Z}, 4)
$$

The map from the 4-sphere of Eilenberg-MacLane space $K(\mathbb{Z}, 4)$ assembles, upon taking homotopy classes, into the integral cohomology $H^4(S^4; \mathbb{Z})$ generated by a fundamental class.

It turns out that such maps to spheres are quite involved, but they can be seen to arise via an infinite number of intermediate maps, nonetheless packaged nicely in terms of other Eilenberg-MacLane spaces. The series of approximations of a space by Eilenberg-MacLane spaces, and starting with one, assemble into its Postnikov tower. The successive liftings from one level to the other is governed by obstruction theory. The 4-sphere admits a Postnikov tower of principal fibrations by virtue of it being simply connected (see [MP12]). Note that the Postnikov tower for odd spheres is easier to deal with; see [FF16] Sec. 27.4, while even spheres are much more involved.

Even when we adopt the 4-sphere and start going through the Postnikov tower and identify the obstructions, there remains a question of what happens beyond the stage seen by spacetime, beyond which there are infinite number of layers of the sphere, since the number of nontrivial homotopy groups is countably infinite. That is, we ask: which spaces look like the 4-sphere through the eyes of the 4th Postnikov section functor? This can be answered using the notion of a Postnikov genus [MS17]. Unlike the case of odd sphere, this is quite involved for even spheres, including the 4-sphere.

Notwithstanding the above subtleties, overall what we have is a description of the form

$$\text{C-field in (twisted) } \pi^4(Y^{11}) \iff \text{C-field in (twisted) } H^4(Y^{11}; \mathbb{Z}) + \text{nontrivial conditions.}$$

Explicitly then, one of the main goals of this paper is to unpack and describe these nontrivial conditions arising from obstruction theory and highlight what they correspond to on the physics side.

**Differential refinement.** Ultimately we would like refine the topological lift (3) to a geometric lift at the level of smooth stacks of the form

$$\begin{array}{c}
\text{Differential cohomotopy,} \\
\text{prequantum and geometric}
\end{array} \downarrow \downarrow \downarrow \begin{array}{c}
\hat{S}^4 \\
\text{Differential cocycle,} \\
\text{quantum and geometric}
\end{array} \rightarrow B^3 U(1)_V
$$

where $\hat{S}^4$ is the differential refinement of the 4-sphere (see [FSS15][FSS16][SS19b] for complimentary approaches) and $B^3 U(1)_V$ is the smooth stack of 3-bundles with connections (see [FSS12][SSS12][FSS14c][FSS15][Sc13]). This would require a differential refinement of the Postnikov tower which uses refinement of cohomology operations, primary [GS18a] and secondary [GS17a].

In between full non-abelian cohomotopy and abelian ordinary cohomology sits *stable cohomotopy*, represented not by actual spheres, but by their stabilization to the sphere spectrum. There is a description of the C-field in each one of these flavors (see [FSS19b][BSS18]):

| Cohomology theory | Rational cohomology | Integral cohomology | Stable cohomotopy | Non-abelian cohomotopy |
|-------------------|---------------------|---------------------|------------------|------------------------|
| Cocycle           | $G_4$               | $\hat{G}_4$         | $\Sigma^\infty c$ | $c$                    |
We will consider both stable and unstable forms of cohomotopy, with more emphasis on the latter. We will also not make a distinction in notation, and use $G_4$ uniformly.

All the conditions we will encounter are torsion in the topological case (in the stable range, at least), with further contribution from refinement of integral classes in the differential case. Note that the M-theory fields have been considered from the point of view of Morava K-theory $K(n)$ [SW15] [SY17], which sits somewhat in between rational cohomology $H^*(Y^{11}; \mathbb{Q}) \cong K(0)(Y^{11})$ and mod $p$ cohomology $H^*(Y^{11}; \mathbb{Z}_p) \cong K(\infty)(Y^{11})$.

The paper is organized as follows. We consider the systematic comparison between the cohomology and cohomotopy treatments of the C-field in §2. First we consider $\mathbb{Z}_2$ coefficients in §2.1 the obstructions classes for which we identify in §2.2 and then consider $\mathbb{Z}_3$ and $\mathbb{Z}_5$ coefficients in §2.3. Putting all together gives us the Postnikov tower with $\mathbb{Z}$ coefficients in §2.4. These stable considerations are then extended to unstable 4-cohomotopy in §2.5. Then we describe physical manifestations and corresponding examples in §2.6. From the topological case we move to differential refinements in §3.1 where we first describe differential cohomotopy in §3.1 then characterize the torsion obstructions in differential cohomology in §3.2. This allows us to compare differential cohomotopy and differential cohomology in §3.3 which serves as a refinement of the topological description in §2 and in which we also provide examples and main applications to M-theory.

2 Cohomological interpretation of cohomotopy: $K(\mathbb{Z}, 4)$ vs. $S^4$

We consider the comparison between degree four integral cohomology $H^4(Y; \mathbb{Z})$, given by maps from $Y$ to the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$, and degree four cohomotopy $\pi^4(Y)$, given by maps to the 4-sphere $S^4$, as in diagram (4). To that end, we will provide a description of cohomotopy via integral and mod $p$ cohomology, together with corresponding cohomology operations leading to conditions on the fields. We will start with mod $p$ coefficients, for $p \in \{2, 3, 5\}$, and then assemble into integral coefficients.

2.1 $\mathbb{Z}_2$ coefficients

We will use obstruction theory, one dimension at a time, in the range of dimensions relevant for M-theory, and extensively applying the constructions and presentation of the Postnikov tower in [MT08]. Since degree $n$ cohomotopy of spaces of dimension less than $n$ is trivial by $n$-connectedness of the target sphere (see [Wh78]) we will start with dimension four.

**Dimension 4:** The space $K(\mathbb{Z}, 4)$ has the same cohomology and homotopy groups as $S^4$ up to dimension 4, as $H^4(S^4; \mathbb{Z}) \cong \mathbb{Z} \cong H^4(K(\mathbb{Z}, 4); \mathbb{Z})$, which in fact holds with any coefficients. This means that if $Y$ has dimension 4 then the two descriptions agree. In fact, the Hopf degree theorem (see [Ko93] IX (5.8))) in our case states that the 4th cohomotopy classes $[Y \xrightarrow{\sim} S^4] \in \pi^4(Y)$ of $Y$ are in bijection with the degree $\deg(c) \in \mathbb{Z}$ of the representing functions, hence that there is a bijection

$$\mathcal{A} : \pi^4(X) \cong [Y, S^4] \xrightarrow{S^4 \to K(\mathbb{Z}, 4) \cong H^4(K(\mathbb{Z}, 4); \mathbb{Z}) \cong [Y, K(\mathbb{Z}, 4)] \cong \mathbb{Z}} H^4(Y; \mathbb{Z})$$

from the 4th cohomotopy to the 4th integral cohomology. This map $\mathcal{A}$ is given by $\mathcal{A}(f) = f^*([S^4]^*)$, with $[S^4] \in H_4(Y; \mathbb{Z})$ the fundamental homology class of $S^4$ and $[S^4]^*$ is its dual. We can form factorization $X \to S^4 \xrightarrow{k} S^4$ as maps from $X$ to $S^4$ of degree $k \geq 1$. Then $[k] = k_{t_4}$ in $\pi_4(S^4) \cong \mathbb{Z}$, where $t_4 = [1]$ is the class of the identity map. In this (and the next) dimension $f^* : \pi^4(S^4) \to \pi^4(Y)$ is a homomorphism and we have $\pi^4(S^4) \cong \pi_4(S^4)$ as groups, then $[k \circ f] = f^* [k] = df^* (t_4) = k[f]$ in $\pi^4(Y)$. See, e.g., [Mi97][OR09].

This then captures the essence of cohomotopy for spacetimes with only a 4-dimensional manifold piece $X^4$ being topologically nontrivial.
**Dimension 5:** At this first stage we consider the cohomology group in degree five, $H^5(K(\mathbb{Z}, 4); \mathbb{Z}_2) = 0$, so there is no obstruction in dimension five, which means that if $Y$ is a 5-dimensional spacetime, then every degree 4 class lifts on a five manifold, but not uniquely. The two descriptions still agree in the sense that there are no obstructions coming from cohomotopy beyond what we have in cohomology.

However, here an interesting effect occurs, analogous to the 4-dimensional case. For $Y$ of dimension at most five we can use the results of Pontrjagin and Steenrod (see, e.g., [Ba89, Theorem 16.9]). For $u \in H^4(S^4; \mathbb{Z})$ a generator, the degree map $\deg : [Y, S^4] \to H^4(Y; \mathbb{Z})$ with $\deg(F) = F^*(u)$ is surjective with inverse image $\deg^{-1}(x_4) \cong H^5(Y; \mathbb{Z}_2)/\text{Sq}^2 H^3(Y; \mathbb{Z})$.

This places conditions on the cohomology of $Y$ and will be relevant in the second obstruction in §2.2 and in Remark 2.7 and Remark 2.14.

**Dimension 6:** At this stage we enter the stable range. The cohomology group in degree six, $H^6(K(\mathbb{Z}, 4); \mathbb{Z}_2) \cong \mathbb{Z}_2$, generated by $\text{Sq}^2 t_4$, where $t_4 \in H^4(K(\mathbb{Z}, 4); \mathbb{Z}) \cong \mathbb{Z}$ is the fundamental class acted upon by the Steenrod square $\text{Sq}^2 : K(\mathbb{Z}, 4) \to K(\mathbb{Z}_2, 6)$. The first stage of the Postnikov tower is the pullback along $\text{Sq}^2 \rho_2$ of the path-loop fibration of the codomain, i.e.,

$$
\begin{align*}
K(\mathbb{Z}_2, 5) = \Omega K(\mathbb{Z}_2, 6) & \quad \longrightarrow \quad X_1 & \quad PK(\mathbb{Z}_2, 6) \\
K(\mathbb{Z}, 4) & \quad \longrightarrow \quad \text{Sq}^2 \rho_2 & \quad K(\mathbb{Z}_2, 6)
\end{align*}
$$

with $X_1$ being a better approximation to $S^4$ than $K(\mathbb{Z}, 4)$ is. Indeed, by definition the class $\text{Sq}^2 \rho_2 \in H^6(K(\mathbb{Z}, 4); \mathbb{Z}_2)$ has been killed on $X_1$. The fundamental class $t_5$ of the fiber $K(\mathbb{Z}_2, 5)$ transgresses to $\text{Sq}^2 \rho_2 t_4$. The cohomology of $X_1$ can be calculated via the Serre long exact sequence

$$
H^*(K(\mathbb{Z}_2, 5); \mathbb{Z}_2) \longrightarrow H^*(X_1; \mathbb{Z}_2) \longrightarrow H^*(K(\mathbb{Z}, 4); \mathbb{Z}_2).
$$

The transgression is given by $\tau(t_5) = \text{Sq}^2 \rho_2 t_4$, which gives in particular that $H^6(X_1; \mathbb{Z}_2) = 0$ (using the Adem relation $\text{Sq}^1 \text{Sq}^2 = \text{Sq}^3$).

Here we have another interesting effect, which is the last dimension in our case where cohomotopy is a group as opposed to just a set. The set $[Y, S^4]$ has a natural group structure if $Y$ has dimension at most 6. This follows from the fact that $S^4$ and the loop space of its suspension $\Omega \Sigma S^4 \sim \Omega S^5$ have the same homotopy 7-type, by the Freudenthal suspension theorem – see [MT08, Chapter 14] and [Wh78].

**Dimension 7:** The next step is to kill $H^7(X_1; \mathbb{Z}_2)$, obtaining a fiber sequence $F_2 \to X_2 \to X_1$, with $X_2$ having cohomology in dimension 7, so the same homotopy groups as $S^4$ in dimension 6.

$$
\begin{align*}
K(\mathbb{Z}_2, 6) = \Omega K(\mathbb{Z}_2, 7) & \quad \longrightarrow \quad X_2 & \quad PK(\mathbb{Z}_2, 7) \\
X_1 & \quad \longrightarrow \quad K(\mathbb{Z}_2, 7)
\end{align*}
$$

The transgression vanishes on $\text{Sq}^2 t_5 \in H^7(K(\mathbb{Z}_2, 5); \mathbb{Z}_2)$ and one gets $H^7(X_2; \mathbb{Z}_2) \cong \mathbb{Z}_2$ generated by a class $\alpha_7$ such that

$$
i^* (\alpha_7) = \text{Sq}^2 t_5,
$$

where $i^* : H^7(X_1; \mathbb{Z}_2) \longrightarrow H^7(K(\mathbb{Z}_2, 5); \mathbb{Z}_2)$. Indeed, if $H^7(X_2; \mathbb{Z}_2) = 0$ then $X_2$ is an improvement over $X_1$ as an approximation to the 4-sphere $S^4$. 

5
**Dimensions 8, 9, 10:** At this level, we consider \(H^8(X_1; \mathbb{Z}_2)\). Here there is a class \(p^*(\text{Sq}^4t_4)\) and also a class \(\beta_8\) such that \(i^*(\beta_8) = \text{Sq}^3t_5\), which has an indeterminacy since \(i^*(p^*(\text{Sq}^4t_4)) = 0\), but the identification does not depend on this choice. We need to kill the class \(\text{Sq}^4t_4 \in H^8(X_2; \mathbb{Z}_2)\). This requires working out the Bockstein relations at this level, which is done in [MT08, Ch. 12]. The procedure is to map \(X_2\) into \(K(\mathbb{Z}_8, 8)\) by a map corresponding to a class which reduces to \(\text{Sq}^4t_4\) (mod 2). This leads to the fibration

\[
K(\mathbb{Z}_8, 7) = \Omega K(\mathbb{Z}_8, 8) \rightarrow X_3 \quad PK(\mathbb{Z}_8, 8) \quad \downarrow \\
X_2 \quad \text{“Sq}^4\text{t}_4\text{”} \rightarrow K(\mathbb{Z}_8, 8).
\]

Note that this process kills not only \(H^8\) but also \(H^9\) and \(H^{10}\).

**Dimension 11:** Next we must kill the class \(p_1 \in H^{11}(X_3; \mathbb{Z}_2)\), where \(i^*p_1 = \text{Sq}^4t_7\), which is the obstruction for the next fibration

\[
K(\mathbb{Z}_2, 10) = \Omega K(\mathbb{Z}_2, 11) \rightarrow X_4 \quad PK(\mathbb{Z}_2, 11) \quad \downarrow \\
X_3 \quad P_{11} \rightarrow K(\mathbb{Z}_2, 11).
\]

**Dimension 12:** The next step is to kill \(H^{12}(X_4; \mathbb{Z}_2)\) by using a map

\[
X_4 \quad \text{“Sq}^4\text{t}_4\text{”} \rightarrow K(\mathbb{Z}_{16}, 12),
\]

where this class has mod 2 reduction equal to \(\text{Sq}^8t_4\). But of course, \(\text{Sq}^8t_4 = 0\) is automatic, from which we can infer at least that “\(\text{Sq}^8\text{t}_4\)” is a multiple of 2 times the generator. Due to the dimension of spacetime, the obstruction at this level is irrelevant for the lifting of \(G_4\). However, we will see a twelve manifold appear when analyzing the Chern-Simons term in the M-theory action and then this condition will become relevant.

### 2.2 Identifying the obstruction classes

We identify the relevant mod 2 classes above via transgressions. Note that in some cases (i.e. \(\mathbb{Z}_8\) and \(\mathbb{Z}_{16}\), we need to pass to lifts of the corresponding mod 2 classes (see [MT08 Ch. 12]):

- **The transgressions: Universally** We consider \(\tau : H^j(\text{fiber}; \mathbb{Z}_2) \rightarrow H^{j+1}(\text{base}; \mathbb{Z}_2)\).
  - \(j = 4\) We have \(t_4 \in H^4(K(\mathbb{Z}, 4); \mathbb{Z}_2)\).
  - \(j = 5\) We have the class \(t_5 \in H^5(K(\mathbb{Z}_2, 5); \mathbb{Z}_2)\). Here, under transgression \(H^5(K(\mathbb{Z}_2, 5); \mathbb{Z}_2) \rightarrow H^6(K(\mathbb{Z}, 4); \mathbb{Z}_2)\), we have \(\tau(t_5) = \text{Sq}^2t_4\).
- **The transgressions: From \((S^4)_1\)** We consider \(\tau : H^j(\text{fiber}; \mathbb{Z}_2) \rightarrow H^{j+1}((S^4)_1; \mathbb{Z}_2)\).
  - \(j = 8\) We have a class \(\text{Sq}^4t_4 \in H^8((S^4)_1; \mathbb{Z}_2)\) that survives.
- **The transgressions: From \((S^4)_2\)** We consider \(\tau : H^j(\text{fiber}; \mathbb{Z}_2) \rightarrow H^{j+1}((S^4)_2; \mathbb{Z}_2)\).
  - \(j = 8\) We have a class \(\text{Sq}^4t_4 \in H^8((S^4)_2; \mathbb{Z}_2)\) that survives.
- **The transgressions: From \((S^4)_3\)** We consider \(\tau : H^j(\text{fiber}; \mathbb{Z}_2) \rightarrow H^{j+1}((S^4)_3; \mathbb{Z}_2)\).
  - \(j = 10\) We have the class \(t_{10} \in H^{10}(K(\mathbb{Z}_2, 10); \mathbb{Z}_2)\). Under transgression \(H^{10}(K(\mathbb{Z}_2, 10); \mathbb{Z}_2) \rightarrow H^{11}((S^4)_3; \mathbb{Z}_2)\), we have \(\tau(t_{10}) = p_{11}(t_4)\).
- **The transgressions: From \((S^4)_4\)** We consider \(\tau : H^j(\text{fiber}; \mathbb{Z}_2) \rightarrow H^{j+1}((S^4)_4; \mathbb{Z}_2)\).
We have the class $\iota_{11} \in H^{11}(K(\mathbb{Z}_{16}, 11); \mathbb{Z}_2)$. Under the transgression $H^{11}(K(\mathbb{Z}_{16}, 11); \mathbb{Z}_2) \xrightarrow{\tau} H^{12}((S^4)_4; \mathbb{Z}_2)$, we have $\tau(\iota_{11}) = Sq^8 \iota_4 = 0$.

**Lemma 2.1 (2-primary Postnikov tower of $S^4$).** Overall, we have the Postnikov tower

\[
\begin{array}{ccccccc}
\vdots & & \rightarrow & & & & \rightarrow \\
K(\mathbb{Z}_2, 10) & & \rightarrow & & (S^4)_4 & & K(\mathbb{Z}_{16}, 12) \\
K(\mathbb{Z}_8, 7) & & \rightarrow & & (S^4)_3 & & K(\mathbb{Z}_2, 11) \\
K(\mathbb{Z}_2, 6) & & \rightarrow & & (S^4)_2 & & K(\mathbb{Z}_8, 8) \\
K(\mathbb{Z}_2, 5) & & \rightarrow & & (S^4)_1 & & K(\mathbb{Z}_2, 7) \\
\mathbb{Z} & & \rightarrow & & (S^4)_0 = K(\mathbb{Z}, 4) & & Sq^2_4 K(\mathbb{Z}_2, 6) \\
\end{array}
\]

This shows that schematically, as in the Introduction,

"Cohomotopy in deg 4 = Integral 4-cohomology + four obstructions".

The main point is to identify the four obstructions above with conditions arising from M-theory and provide interpretations for them. This is done after pulling back to spacetime $Y$, where the fundamental class $\iota_4$ pulls back to the field

$$G_4 - \frac{1}{2} \lambda =: \tilde{G}_4 = f^* \iota_4$$

where $\lambda = \frac{1}{2} p_1$ is the first Spin characteristic class of the lifted tangent bundle to spacetime.

**(i) The first obstruction.** The first obstruction is

$$Sq^2 \tilde{G}_4 \overset{1}{\in} H^6(Y; \mathbb{Z}_2).$$

Indeed, it was shown in [FSS19b] that this follows from anomaly cancellation in M-theory. Note that it is stronger than the obstruction given by the 3rd Steenrod square $Sq^3$, as the former is a condition for KO-theory while the latter is for K-theory.

**(ii) The second obstruction.** Overall, the second obstruction is

$$f^* (\alpha_7) \overset{1}{\in} H^7(Y; \mathbb{Z}_2)$$

where $\alpha_7$ is a secondary operation, restricting fiberwise to $Sq^2 \iota_5$. In particular, this means that if $G_4$ vanishes in cohomology, we have the more relatable condition

$$f^* (i^* \alpha_7) = f^* (Sq^2 \iota_5) = Sq^2 f^* (\iota_5) \overset{1}{\in} H^7(Y; \mathbb{Z}_2).$$
In this case, we impose the condition $\text{Sq}^2 f^*(t_5) = 0$. At this stage we note that in the current formulation there are no fields of degree five in M-theory. Hence we will instead impose the natural condition

$$f^*(t_5) =: G_5 = 0 \in H^7(Y; \mathbb{Z}_2).$$

Note that rationally we could consider the possibility that $G_5$ as being $*_{11} C_6$, the Hodge dual of the potential for $G_7$. However, there is no natural degree four potential, except if we view $G_4$ as such, but this is closed, hence such a candidate $G_5$ would vanish even as a form. Another possibility is that in the presence of M5-branes, the Bianchi identity $dG_4 = 0$ is violated by a delta function supported on the M5-brane worldvolume. The latter can be viewed as giving rise to a degree five class, but it does not satisfy the right condition, which is torsion. We will get back to this in Remark 2.7 and Remark 2.14.

(iii) The third obstruction. The third obstruction is

$$f^*(("\text{Sq}^4 t_4") = 0 \in H^8(Y; \mathbb{Z}_8).$$

Note that, by construction, this implies also that (upon mod 2 reduction)

$$f^*(\text{Sq}^4 t_4) = \text{Sq}^4 f^*(t_4) = \text{Sq}^4 \tilde{G}_4 = \tilde{G}_4 \cup \tilde{G}_4 = 0 \in H^8(Y; \mathbb{Z}_2).$$

This captures the anomaly given by trivializing the cup product of $G_4$, at least mod 2. This is the affect in the stable setting, but (as we will see) the unstable obstruction implies the stronger condition that the cup product vanish even integrally. Recall that rationally we have the equation of motion

$$d \ast G_4^{\text{form}} = \frac{1}{2} G_4^{\text{form}} \wedge G_4^{\text{form}}.$$ 

At the level of cohomology classes (with torsion), we have that the cup product of the class $G_4$ of $G_4^{\text{form}}$ with itself is zero. This of course implies immediately that the mod 2 reduction also vanish

$$\text{Sq}^4 \tilde{G}_4 = \tilde{G}_4 \cup \tilde{G}_4 = 0.$$ 

What about the coefficients being $\mathbb{Z}_8$ rather than $\mathbb{Z}_2$? We first argue that $\mathbb{Z}_8$ coefficients are somewhat natural to appear in this context. We consider the fields reduced modulo 4, for instance the shift in the field is given by $\frac{1}{4} \lambda$, where $\lambda = \frac{1}{2} p_1$ the Spin characteristic class arising from the first Pontrjagin class $p_1$ being even in the cohomology of $B\text{Spin}$. If we start with an oriented – rather than a Spin – setting, then we are considering modding out $p_1$ by 4. Hence it makes sense to consider a corresponding class $x_4 \in H^4(Y^{11}; \mathbb{Z}_4)$, given by mod 4 reduction, or in the lift to the bounding manifold $Z^{12}$. There is an operation $\mathbb{P}_2 : H^4(Y^{11}; \mathbb{Z}_4) \rightarrow H^8(Y^{11}; \mathbb{Z}_8)$ called the Pontrjagin square operation. It has the property that

$$\mathbb{P}_2 \rho_2(x_4) = x_4,
\rho_4 \mathbb{P}_2(x_4) = x_4,$$

where $\rho_2$ and $\rho_4$ are the mod 2 and 4 reductions, respectively. Then in fact

$$\rho_2 \mathbb{P}_2(x_4) = \rho_2(x_4)^2 = \text{Sq}^4 \rho_2(x_4).$$

This works for any degree 4 class, not just the reduction of $p_1$. Hence the operation $[\mathbb{P}_2 \rho_4 t_4]$ indeed gives a mod 8 lift of $\text{Sq}^4 t_4$.

---

1We are tempted to identify this operation explicitly as "$\text{Sq}^4 t_4$", but unfortunately the Pontrjagin square is an unstable operation, while "$\text{Sq}^4 t_4$" is stable. Thus at best there is a stable operation which reduces to the Pontrjagin square in the given degree.
(iv) The fourth obstruction. The fourth obstruction is
\[ f^*(P_{11}) = 0, \]
where \( P_{11} \) is a class which fiberwise restricts to \( Sq^1 t_7 \). The Universal Coefficient Theorem gives
\[ 0 \rightarrow \text{Ext}^1_{\mathbb{Z}_2}(H_{11}(Y;\mathbb{Z}_2),\mathbb{Z}_2) \rightarrow H^{11}(Y;\mathbb{Z}_2) \rightarrow \text{Hom}(H_{11}(Y;\mathbb{Z}_2),\mathbb{Z}_2) \rightarrow 0. \]
Since \( \mathbb{Z}_2 \) is a field, the Ext term vanishes and we have the isomorphism \( H^{11}(Y;\mathbb{Z}_2) \cong H_{11}(Y;\mathbb{Z}_2) \). If \( Y \) is non-orientable, then this group is trivial, so that \( P_{11} \) has no effect. However, if \( Y \) is orientable, then \( H_{11}(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2 \), so that we get a detectable effect for M-theory on orientable spacetimes.

Remark 2.2 (Obstructions as \( n \)-ary constraints). The 2-primary Postnikov resolution of the sphere can also be organized into primary, secondary, etc. obstructions, in the sense of cohomology operations (see \([LT72]\)). For our case of the 4-sphere, up to degree eight, it looks as follows (necessarily mixing dimensions):

\[
\begin{array}{c}
K(\mathbb{Z}_2, 7) \xrightarrow{j_3} (S^4)_3 \\
K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 7) \xrightarrow{j_2} (S^4)_2 \xrightarrow{\beta_4} K(\mathbb{Z}_2, 8) \\
K(\mathbb{Z}_2, 5) \times K(\mathbb{Z}_2, 7) \xrightarrow{j_1} (S^4)_1 \xrightarrow{(\alpha_3, \alpha_4)} K(\mathbb{Z}_2, 7) \times K(\mathbb{Z}_2, 8) \\
\end{array}
\]

(i) Primary obstruction: The Steenrod squares are primary cohomology operations,

(ii) Secondary obstruction: The classes \((\alpha_3, \alpha_4)\) represent secondary cohomology operations,

\[ j_1^* \alpha_3 = Sq^2 t_5 \otimes 1, \]
\[ j_1^* \alpha_4 = Sq^2 Sq^1 t_5 \otimes 1 + 1 \otimes Sq^1 t_7. \]

(iii) Tertiary obstruction: The class \( \beta_4 \) represents a tertiary cohomology operations

\[ j_2^* \beta_4 = Sq^2 t_6 \otimes 1 + 1 \otimes Sq^1 t_7. \]

2.3 \( \mathbb{Z}_3 \) and \( \mathbb{Z}_5 \) coefficients

The main contribution to the Postnikov tower of \( S^4 \) is from the prime \( p = 2 \) as we saw above. However, the primes 3 and 5 also contribute, albeit to a lesser extent. The structure of homotopy groups of spheres give some immediate consequences for the Postnikov tower at different primes 3 and 5. In particular, the Serre spectral sequence implies that at the second stage we have an isomorphism

\[ H^*((S^4)_2;\mathbb{Z}_3) \cong H^*((S^4)_1;\mathbb{Z}_3) \cong H^*(K(\mathbb{Z}, 4);\mathbb{Z}_3), \]

and similarly at the prime 5 we have an isomorphism

\[ H^*((S^4)_4;\mathbb{Z}_5) \cong H^*(K(\mathbb{Z}, 4);\mathbb{Z}_5). \]

For \( p \) odd, the structure of mod \( p \) cohomology rings of Eilenberg-MacLane spaces was determined by Cartan \([Ca54]\) and Serre \([Se51]\) in terms of admissible monomials of Steenrod reduced powers and the Bockstein (see
also \cite[Lecture 30]{FF16}). This has been recast by Tamanoi \cite[Sec. 5.2]{Ta99} using the Milnor basis of the dual Steenrod algebra, giving explicit polynomial generators for $H^*(K(\mathbb{Z},n);\mathbb{Z}_p)$. Using this identification, we have

$$H^8(K(\mathbb{Z},4);\mathbb{Z}_3) \cong \mathbb{Z}_3(\mathcal{P}_3^1 t_4), \quad H^{12}(K(\mathbb{Z},4);\mathbb{Z}_3) \cong \mathbb{Z}_3((\rho_3 t_4)^3), \quad H^{12}(K(\mathbb{Z},4);\mathbb{Z}_5) \cong \mathbb{Z}_5(\mathcal{P}_5^1 t_4).$$

Hence we will get conditions on the vanishing of the pullback of the classes

$$\mathcal{P}_3^1 t_4, \quad (\rho_3 t_4)^3, \quad \mathcal{P}_5^1 t_4.$$

**Remark 2.3** (Interpretation). Mod 3 reductions are shown to play a prominent role in topological considerations in M-theory \cite{Sa08}, where similar conditions, including $\mathcal{P}_3^1 t_4 G_4 = 0$, have been highlighted in the context of Spin K-theory.

### 2.4 $\mathbb{Z}$ coefficients

Using our discussion in \S 2.1 we can assemble the tower integrally in the desired range. To do this, we observe that by killing all cohomology classes in $H^{n+*}(K(\mathbb{Z}, n+1);\mathbb{Z}_p)$, for fixed $n \gg 1$ and for each prime $p$, we can utilize \cite[Theorem 4, Ch. 10]{MT08} to construct a space $(S^4)_4$ for which there exists a map $f : (S^4)_4 \to S^4$ inducing an isomorphism on each $p$-component of $\pi_{n+1}$. Since the homotopy groups of $S^4$ in this range are all finitely generated and torsion, this will imply that $f$ is actually 12-connected. Overall, we have the following:

**Lemma 2.4** (Integral Postnikov tower for $S^4$). The tower takes the form

\[
\begin{array}{cccccccc}
S^4 & \rightarrow & K(\mathbb{Z}, 5) & \rightarrow & (S^4)_1 & \rightarrow & \cdots & \rightarrow & (S^4)_{11} \\
& & \mathcal{P}_2^1 t_4 \rightarrow & K(\mathbb{Z}, 10) & \rightarrow & (S^4)_2 & \rightarrow & \cdots & \rightarrow & (S^4)_{12} \\
& & & K(\mathbb{Z}, 11) & \rightarrow & (S^4)_3 & \rightarrow & \cdots & \rightarrow & (S^4)_{13} \\
& & & & K(\mathbb{Z}, 5) & \rightarrow & \cdots & \rightarrow & (S^4)_{17} & \rightarrow & (S^4)_{18} \\
& & & & & K(\mathbb{Z}, 2, 7) & \rightarrow & \cdots & \rightarrow & (S^4)_{22} & \rightarrow & (S^4)_{23} \\
& & & & & & K(\mathbb{Z}, 2, 11)
\end{array}
\]

Note that at the top level the three conditions vanish necessarily on $Y^{11}$, for dimension reasons.

**Low-dimensional obstructions.** The obstructions in degree 6 and 7 are identified with the obstructions at the prime 2, given in \S 2.2.

**The tertiary obstruction.** Here we have again the prime 2 obstructions, identified in \S 2.2 but also a new condition which occurs at the prime 3. Namely, we have the condition

$$\mathcal{P}_3^1 (G_4) = 0 \in H^8(Y;\mathbb{Z}_3).$$

As indicated above, this is compatible with \cite{Sa08}, where a similar condition was proposed using Spin K-theory.

**The quaternary obstructions.** This is identified as the obstruction class $P_{11}$ at the prime 2, as in \S 2.2.
The quinary obstructions. These obstructions necessarily vanish on $Y^{11}$. However we will consider a closed $12$-manifold $Z^{12}$ in analyzing the congruences of the Chern-Simons term in the M-theory action. In this case, the three conditions

\[ \text{"Sq}^8 t_4 \equiv 0, \quad t_4^3 \equiv 0, \quad \mathcal{P}_4 t_4 \equiv 0 \]

are nontrivial. We will see that the second obstruction gives exactly the mod $3$ congruences in the M-theory action discussed in [Wit97].

Summarizing, we have the following.

Proposition 2.5 (Cohomotopy vs. integral cohomology). Let $Y^{11}$ be an $11$-dimensional (smooth) manifold. Then a class $c \in H^4(Y^{11}; \mathbb{Z})$ lifts to a class $\tilde{c} \in \pi^4(Y^{11})$ if and only if the following conditions are satisfied.

(i) $\text{Sq}^2(c) \equiv 0 \mod 2$, \quad $\mathcal{P}_4(c) \equiv 0 \mod 3$

(ii) There is a lift $c': Y^{11} \to (S^4)^1$ of $c$ such that $\alpha_5(c') \equiv 0 \mod 2$

(iii) There is a further lift $c'' : Y^{11} \to (S^4)^2$ such that $\beta_8(c'') \equiv 0 \mod 8$. In particular, upon mod $2$-reduction, we have $\text{Sq}^4(c) = c^2 \equiv 0 \mod 2$.

(iv) There is a further lift $c'''$ of $c''$ such that $P_{11}(c''') \equiv 0 \mod 2$. In particular upon mod $2$ reduction, we have the tautological relation $\text{Sq}^8(c) \equiv 0 \mod 2$.

Much of the information in the above proposition is $2$-torsion. We now directly apply this to our field.

Proposition 2.6 (Cohomotopy vs. cohomology for the C-field). Consider the M-theory (shifted) C-field $\tilde{G}_4$ as an integral cohomology class in degree four. Then if $G_4$ lifts to a cohomotopy class $\mathcal{G}_4 \in \pi^4(Y^{11})$ the following obstructions necessarily vanish

(i) $\text{Sq}^2 \tilde{G}_4 = 0 \in H^8(Y^{11}; \mathbb{Z}_2)$.

(ii) $\mathcal{P}_4(\tilde{G}_4) = 0 \in H^8(Y^{11}; \mathbb{Z}_3)$.

(iii) $\text{Sq}^4 \tilde{G}_4 = \tilde{G}_4 \cup \tilde{G}_4 = 0 \in H^8(Y^{11}; \mathbb{Z}_2)$.

(iv) If $G_4^{\text{form}} = 0$ and $dC^3_3 = 0$, so $C_3$ can be lifted to an integral class $\tilde{C}_3$, then we also have $\text{Sq}^3 \text{Sq}^1 \tilde{C}_3 = 0 \in H^7(Y^{11}; \mathbb{Z}_2)$.

(v) If $dC^3_3 = G_4^{\text{form}} \wedge G_4^{\text{form}} = 0$ and $G_7^{\text{form}}$ can be lifted to an integral class $\tilde{G}_7$, then we also have the condition $\text{Sq}^4 \tilde{G}_7 = 0 \in H^11(Y^{11}; \mathbb{Z}_2)$.

Proof. The first three conditions are immediate consequences of Proposition 2.5. By stability of Steenrod squares, applying the based loop functor to the mapping

\[ \text{Sq}^2 t_4 : K(\mathbb{Z}, 4) \to K(\mathbb{Z}_2, 6) \]

gives $\text{Sq}^2 t_3 : K(\mathbb{Z}, 3) \to K(\mathbb{Z}_2, 5)$. Trivializing $G_4 = 0$ by $\tilde{C}_3$, we get a choice of lift of $\tilde{G}_4 = 0$ to the fiber $K(\mathbb{Z}_2, 5)$, given by $\text{Sq}^2 \tilde{C}_3$. Then in this case the obstruction class is

\[ i^* \alpha_7(\text{Sq}^2 \tilde{C}_3) = \text{Sq}^2 \text{Sq}^2 \tilde{C}_3 = \text{Sq}^3 \text{Sq}^1 \tilde{C}_3, \]

where we have used the Adem relation $\text{Sq}^2 \text{Sq}^2 = \text{Sq}^3 \text{Sq}^1$. The last identification follows by a similar argument, using that the obstruction class is $\text{Sq}^4 t_4 = t_4 \cup t_4$, and that $P_{11}$ restricts fiberwise to $\text{Sq}^4 t_7$. □

Proposition 2.5 also has some immediate striking consequences. In particular, Proposition 2.6 implies that even if $G_4 = 0$, there are still obstructions to lifting the C-field to a cohomotopy class. Thus, quantization in cohomotopy seems to uncover extremely subtle quantization conditions on the C-field.
Remark 2.7 (Cohomotopy first contribution to the C-field). We highlight that even if $\tilde{G}_4 = 0$, there are still obstructions to lifting the C-field to a class in cohomotopy. In particular, we have a mysterious degree 5 class $\eta \in H^5(Y^{11}; \mathbb{Z}_2)$ which transgresses to $Sq^2 \tilde{G}_4$. By construction of the transgression, this class can be interpreted concretely as follows. Fix a map $Sq^2: K(\mathbb{Z}, 4) \to K(\mathbb{Z}_2, 6)$ representing the Steenrod square $Sq^2$. Since $\tilde{G}_4$ vanishes in integral cohomology, we have a global trivialization $\delta \tilde{C}_3 = \tilde{G}_4$ of $\tilde{G}_4$ as an integral cochain, which gives rise to a trivialization $Sq^2 \tilde{C}_3$ of $Sq^2 \tilde{G}_4$ in $\mathbb{Z}_2$-cohomology, by naturality. Let us fix another trivialization $\delta \varepsilon = Sq^2 \tilde{G}_4$ in cochains with $\mathbb{Z}_2$-coefficients. Setting $\eta := Sq^2 \tilde{G}_4$, we have
\[ \delta(\eta) = \delta(Sq^2 \tilde{C}_3 - \varepsilon) = Sq^2 \tilde{G}_4 - Sq^2 \tilde{G}_4 = 0,\]
so that $\eta$ indeed represents a degree 5 cocycle in $\mathbb{Z}_2$ cohomology, which may be generally nonvanishing. Note that there is a degree five class associated with the C-field, namely the fifth integral Steifel-Whitney class $W_5$ (see [DFM03]), but it is different from this class.

Remark 2.8 (Congruences for the M-theory action via cohomotopy). Another interesting effect occurs when considering the Chern-Simons term in the M-theory action
\[ \frac{1}{8} \int_{Y^{11}} C_3 \wedge G_4 \wedge G_4. \]
As usual, since $C_3$ may not be globally defined in general, one may consider $Y^{11}$ as the boundary of a 12-manifold $Z^{12}$ and analyzes the globally well defined term
\[ \frac{1}{8} \int_{Z^{12}} G_4 \wedge G_4 \wedge G_4 \] (7)
on $Z^{12}$. To show that the integral is independent of the choice of $Z^{12}$, one considers another $Z'^{12}$ with boundary $Y^{11}$ and integrates over the closed manifold $Q = Z'^{12} \sqcup Z^{12}$. However, as remarked in [Wit96], the usual quantization law of $G_4$ does not give rise to a well defined Chern-Simons action, as (7) might fail to be integral by a factor of 6.

Note that our obstruction theory works just as well for a closed 12-manifold $Z^{12}$. In this case, the obstruction at the top stage of the tower gives the condition
\[ \tilde{G}_4^3 \equiv 0 \mod 3. \]
This is in addition to condition (iii) in Corollary 2.6 which states that $\tilde{G}_4^2 = Sq^4(\tilde{G}_4) \equiv 0 \mod 2$. These two conditions together imply the divisibility by 6 condition on $\tilde{G}_4^3$. The crucial distinction is that our congruences are obtained without reference to $E_8$-gauge theory. An alternative formulation of the congruence via a proposed higher form of index theory is given in [Sa05a] [Sa05b].

2.5 Obstructions for unstable 4-cohomotopy

So far, our work has been limited to the stable range of cohomotopy in degree 4. In part, this is due to the fact that the obstruction theory in the unstable case is considerably more complicated. Moreover, working out the $k$-invariants in the Postnikov tower unstably does not yield information which can be directly compared with existing literature: there are many secondary and tertiary obstructions, which arise as classes defined modulo some ambiguity, but are not familiar primary obstructions or Massey products. Nevertheless, we highlight the following.

Remark 2.9 (Quaternionic Hopf fibration). One exception occurs in degree 8, where we have a $k$-invariant coming from the quaternionic Hopf fibration, and takes the form
\[ k : (S^4)^2 \longrightarrow K(\mathbb{Z}_{12}, 8) \times K(\mathbb{Z}, 8). \]
Mapping out of $Y^{11}$, we identify the projection to the factor $K(\mathbb{Z}, 8)$ as $\phi_2^*(\tilde{G}_4)$, where $\phi_2 : (S^4)^2 \to K(\mathbb{Z}, 4)$ is the map at the second stage. Killing the $k$-invariant at this stage corresponds to a choice of trivialization $\delta \tilde{C}_7 = \tilde{G}_4$.

\[ \text{From a complimentary point of view, this case is discussed in detail in [FSS19a] [FSS19b].} \]

$^2$Note that divisibility by 2 is not immediate, but can be deduced using the same argument in [Wit97] p. 12], still without reference to an $E_8$-theory.
Lemma 2.10 (Unstable Postnikov tower of $S^4$). Overall, the Postnikov tower takes the following form

\[
\begin{array}{c}
\vdots \\
K(\mathbb{Z}_{15}, 11) \rightarrow (S^4)_7 \\
K(\mathbb{Z}_{24} \times \mathbb{Z}_3, 10) \rightarrow (S^4)_6 \rightarrow K(\mathbb{Z}_{15}, 12) \\
K(\mathbb{Z}_2 \times \mathbb{Z}_2, 9) \rightarrow (S^4)_5 \rightarrow K(\mathbb{Z}_{24} \times \mathbb{Z}_3, 11) \\
K(\mathbb{Z}_2 \times \mathbb{Z}_2, 8) \rightarrow (S^4)_4 \rightarrow K(\mathbb{Z}_2 \times \mathbb{Z}_2, 10) \\
K(\mathbb{Z}_{12}, 7) \times K(\mathbb{Z}, 7) \rightarrow (S^4)_3 \rightarrow K(\mathbb{Z}_2 \times \mathbb{Z}_2, 9) \\
K(\mathbb{Z}_2, 6) \rightarrow (S^4)_2 \rightarrow K(\mathbb{Z}_{12}, 8) \times K(\mathbb{Z}, 8) \\
K(\mathbb{Z}_2, 5) \rightarrow (S^4)_1 \rightarrow K(\mathbb{Z}_2, 7) \\
(S^4)_0 \rightarrow K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}/2, 6)
\end{array}
\]

where we have identified the first few obstructions.

Proof. This follows directly from the identification of the homotopy groups of $S^4$ in the relevant degrees and assembling them into the tower one degree at a time. See [T], for a tabulation. □

2.6 Physical manifestations and examples

Most of the obstructions for lifting cohomology classes to cohomotopy are torsion obstructions, as we have seen. Given that the fields $G_4$ and $G_7$ are classes which appear in cohomology with real coefficients, it is natural to wonder how torsion obstructions could impose constraints on these classes. For instance, we saw that there is an obstruction $\alpha_7$ which acts on $\mathbb{Z}_2$-classes in degree 5. At first glance, this might seem awkward since no fields of degree 5 in M-theory — see also Remark 2.7 — In this section we offer further physical interpretations of this and similar obstructions.

Many of the anomaly cancellation conditions present in the M-theory literature require an integral lift of a real cohomology class.

Remark 2.11 (The anomaly in the partition function). Quantization in cohomotopy yields the condition $\text{Sq}^2(\widetilde{G}_4) = 0$ for some integral lift of $\mathbb{G}_4$. As highlighted in [FSS99], this immediately implies the vanishing of the DMW anomaly $\text{Sq}^3(\widetilde{G}_4) = 0$. From the obstruction theory for $S^4$, we have an exact sequence of pointed sets (cf. relation (6))

\[
0 \rightarrow H^5(Y^{11}; \mathbb{Z}_2) / \text{Sq}^2 H^3(Y^{11}; \mathbb{Z}) \stackrel{j}{\rightarrow} [Y^{11}, (S^4)_1] \rightarrow \{ x \in H^4(Y^{11}; \mathbb{Z}); \text{Sq}^2(x) = 0 \} \rightarrow 0
\]

(8)

In [DMW00] it was shown that the phase of the partition function for the $C$-field on $X^{10} \times S^1$ is $\pm 1$, depending on the the vanishing of a function $f: H^4(X^{10}; \mathbb{Z}) \rightarrow \mathbb{Z}_2$. Now $f$ is not linear, but obeys the relation

\[
f(a + b) = f(a) + f(b) + \int_X a \cup \text{Sq}^2(b)
\]

(9)
and \( f(a) = 0 \) when \( a = 0 \). In their notation, \( a \) is a choice of integral lift of the \( G_4 \). The discussion in [DMW00, Section 6.2] notes that if \( f \) were linear, then the contribution of \( a \) to the partition function should vanish unless \( f(a + c) = f(a) \) with \( c \) torsion (i.e., \( f \) should not actually depend on the choice of integral lift). However, the last term on the right of (9) prevent \( f \) from being linear. To circumvent this issue, the authors consider the subset \( L' \) of all torsion \( c \in H^4(Y^{11}; \mathbb{Z}) \) such that \( \sum c'^2 = 0 \) and analyze the nonvanishing conditions of the phase

\[
\sum_{c \in L'} (-1)^{f(a + c)}.
\]  

(10)

What is interesting is that the condition that \( c \) lift to cohomotopy already forces \( c \) to be in \( L' \), by the exact sequence (8). In fact, the calculation using the torsion pairing in [DMW00, p. 42] also shows that the condition on \( a \) becomes that (after possible modification by a torsion class) \( Sq_2^3(a) = 0 \). It follows that the fields which contribute to the phase (10) are just the field which lift to the first Postnikov stage in cohomotopy.

**Remark 2.12** (Mod 2 invariant and geometric submanifolds). There is another mod 2 invariant which can be defined using cohomotopy. Recall that by Pontrjagin-Thom theory, \([Y^{11}, S^4]\) can be identified with framed bordism classes of 7-dimensional submanifolds. Let \( M \) be a 7-dimensional submanifold defined by a map \( Y^{11} \to S^4 \) and let \( \phi : M \times \mathbb{R}^4 \to N \) be the framing of the normal bundle. Then a choice of volume form \( \omega \) on \( Y^{11} \) naturally gives rise to a volume form \( \omega_\phi \) on \( M \) by contracting out the four unit normal vector fields, defined via \( \phi \). Moreover, if \( \omega \) is integral on \( Y^{11} \), then so is \( \omega_\phi \). This gives an assignment

\[
[Y^{11}, S^4] = \{([M], \phi), M \subset Y^{11}\} \to \int_M \omega_\phi \mod 2 \in \mathbb{Z}_2.
\]

This assignment is additive with respect to disjoint union and defines a group homomorphism and gives the parity of the volume of \( M \). We will come back to this in Remark 3.4.

**Remark 2.13** (Lifts of integral cohomology classes to \( K(O) \)-theory). As we saw above, in order to read the condition (see also [PSS19b])

\[
Sq^2(G_4) = 0
\]

(11)

properly, one needs to choose an integral lift \( \tilde{G}_4 \) of \( G_4 \) and there is no canonical way to do this. For the analogous case of \( Sq^2(F_4) \), where \( F_4 \) is the Ramond-Ramond (RR) field from which \( G_4 \) is lifted from \( X^{10} \) to \( Y^{11} = X^{10} \times S^1 \), this is interpreted as a condition on an integral lift of \( F_4 \) in order that it lift to \( K \)-theory [DMW00] (see [GS19d] for an extensive discussion of such lifts). This indicates that the partition function of the RR fields is sensitive to the choice of integral lift of \( F_4 \) (in addition to other degrees as well). The condition at hand (11) provides an analogous sensitivity to the integral lift \( \tilde{G}_4 \) of \( G_4 \) as well as to lifting to \( KO \)-theory instead of \( K \)-theory.

**Remark 2.14** (Purely cohomotopic contribution). We give an instance where cohomotopy gives a contribution even when the corresponding cohomology is trivial (complimenting Remark 2.7). The choice of generator of \( H^4(S^4; \mathbb{Z}) \) defines a map \( S^4 \to K(\mathbb{Z}, 4) \) and hence a homotopy fibration sequence

\[
K(\mathbb{Z}, 3) \to F \to S^4 \to K(\mathbb{Z}, 4),
\]

with \( F \) the homotopy fiber. This gives an exact sequence of pointed sets

\[
H^3(Y^{11}; \mathbb{Z}) \to [Y^{11}, F] \to [Y^{11}, S^4] \to H^4(Y^{11}; \mathbb{Z}).
\]

\footnote{Actually the weaker condition \( Sq^1(c) = 0 \) is considered, but the discussion works equally well if we pass to this smaller class.}

\footnote{In [DMW00], \( Sq^3 \) is used, but the same discussion works with \( Sq^2 \) by letting \( M \) be \( H^6(X^{10}; \mathbb{Z}_2) \) and using the cup product pairing

\[
\int_{X^{10}} (-) \cup (-) : L' \times M \to \mathbb{Z}_2
\]

directly instead of the induced torsion pairing.}
If \( H^4(Y^{11}; \mathbb{Z}) = 0 = H^3(Y^{11}; \mathbb{Z}) \) then we get a bijection \( [Y^{11}, F] = [Y^{11}, S^4] \). We know that, by definition, \( \pi_i(F) = 0 \) for \( i \leq 4 \), while \( \pi_5(F) \cong \pi_5(S^4) \cong \mathbb{Z}_2 \). Hence, by cellular approximation, we get

\[
[Y^{11}, F] \cong [Y^{11}, K(\mathbb{Z}_2, 5)] \cong H^5(Y^{11}; \mathbb{Z}_2).
\]

Therefore, in this case we get that degree 5 cohomotopy gives a contribution to cohomology in higher degree, in this case degree five, \( |[Y^{11}, S^5]| = |H^5(Y^{11}; \mathbb{Z}_2)| \). See also Remark 2.7 for an interpretation.

**Examples 2.15** (Flux compactification spaces). We consider the following examples, involving Anti-de Sitter space \( AdS_n \). This space is homotopically essentially trivial aside from the fundamental group. In order stay away from matters related to insisting the action of the fundamental group to be nice (e.g., nilpotent) will assume simply-connectedness, which will ensure the homotopy techniques can be safely used. This then can be arranged by taking the double cover \( \widetilde{AdS}_n \) of \( AdS_n \).

(i) \( \widetilde{AdS}_7 \times \mathbb{R}P^4 \): This example is important in considering M-theory on an orientifold \([\text{Wi96, Ho99}]\). The internal space \( \mathbb{R}P^4 \) is obtained by attaching a 4-cell to \( \mathbb{R}P^3 \) by the Hopf map \( f_3 : S^3 \to \mathbb{R}P^3 \) which identifies the antipodal points. Collapsing the subspace \( \mathbb{R}P^3 \subset \mathbb{R}P^4 \) to a point yields a map \( q_4 : \mathbb{R}P^4 \to S^4 \). This gives rise to an element \( \eta_4 \in \pi^4(\mathbb{R}P^4) \). Then, from \([\text{We70}]\), we have \( \pi^4(\mathbb{R}P^4) \cong \mathbb{Z}_2 \) with generator \( \eta_4 \). Comparing with integral cohomology, \( H^4(\mathbb{R}P^4; \mathbb{Z}) \cong \mathbb{Z}_2 \) with generator \( \eta_4 \), indeed shows that cohomotopy detects more.

(ii) \( \widetilde{AdS}_4 \times \mathbb{C}P^2 \times T^2 \): This example is important in supersymmetry without supersymmetry \([\text{DL98}]\) and T-duality \([\text{BEM04}]\). We will again take the covering space of the AdS factor. Furthermore, note that the \( T^2 \) factor does not contribute to cohomotopy due to dimension reasons. The complex projective space \( \mathbb{C}P^2 \) is obtained by attaching a 4-cell to \( \mathbb{C}P^1 = S^2 \) by the Hopf map \( f_1 : S^3 \to \mathbb{C}P^1 \), which is also the Hopf map \( \eta_2 \) above. Collapsing \( \mathbb{C}P^1 = S^2 \subset \mathbb{C}P^2 \) to a point yields a map \( q_2 : \mathbb{C}P^2 \to S^4 \). Then, \([\text{We70}]\), \( \pi^4(\mathbb{C}P^2) \cong \mathbb{Z} \) with generator \( q_2 \). Comparing to cohomology, we have \( H^4(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z} \), so that in this case, the two coincide, so that no new information is supplied by cohomotopy.

(iii) \( \widetilde{AdS}_7 \times \mathbb{C}P^2 \): The example is similar to the previous. Passing again to the simply connected cover of \( \widetilde{AdS}_7 \), the only nontrivial contribution again comes from \( \pi^4(\mathbb{C}P^2) \cong \mathbb{Z} \), again with no extra contribution.

(iv) \( \widetilde{AdS}_4 \times \mathbb{R}P^5 \times T^2 \): It follows from \([\text{We70}]\) that \( \pi^4(\mathbb{R}P^5) \) is cyclic or order 4, i.e., either \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), with generator \( \eta_4 q_5 \), where \( \eta_4 : S^5 \to S^4 \) is the 2-fold iteration of the Hopf map \( \eta_2 : S^3 \to S^2 \), and \( q_5 \) is defined analogously to \( q_4 \) from above. On the other hand, \( H^4(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2 \), so that there is further contribution from cohomotopy, either as an extra \( \mathbb{Z}_2 \) or as a \( \mathbb{Z}_4 \) vs. \( \mathbb{Z}_2 \).

(v) \( \widetilde{AdS}_4 \times \mathbb{C}P^3 \times S^1 \): This example is also important in the phenomenon of supersymmetry without supersymmetry. Let \( i_2 : \mathbb{C}P^2 \hookrightarrow \mathbb{C}P^3 \) denote the inclusion and \( 2 : S^4 \to S^4 \) a map of degree 2. Then, again invoking \([\text{We70}]\),

\[
\pi^4(\mathbb{C}P^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2
\]

where the generator of \( \mathbb{Z} \) is \( \alpha_3 \) where \( \alpha_3 i_2 \simeq 2q_2 \) and the generator of \( \mathbb{Z}_2 \) is \( \eta_4 \eta_3 q_3 \). Comparing to cohomology, we have \( H^4(\mathbb{C}P^3; \mathbb{Z}) \cong \mathbb{Z} \), so that there is an extra contribution of \( \mathbb{Z}_2 \) present in cohomotopy.

We have seen that in several backgrounds there is an extra torsion contribution from cohomotopy over integral cohomology. This is an interesting effect that deserves further investigation, to which we hope to get back elsewhere.

**Examples 2.16** (Quaternionic and octonionic projective planes). Similarly for \( \mathbb{H}P^2 \) and \( \mathbb{O}P^2 \), we have the following, again making use of some of the constructions in \([\text{We70}]\).

(i) For \( \mathbb{H}P^2 \): consider the Puppe sequence or the mapping cone sequence of the quaternionic Hopf fibration

\[
S^7 \xrightarrow{h_\mathbb{H}} S^4 \xrightarrow{p} \mathbb{H}P^2 \xrightarrow{q} S^8 \xrightarrow{\Sigma h_\mathbb{H}} S^5 \rightarrow \ldots
\]

Now apply the 2-fold suspension \( \Sigma^2 h_\mathbb{H} \). This gives

\[
S^9 \xrightarrow{\Sigma^2 h_\mathbb{H}} S^6 \xrightarrow{\Sigma^2 p} \Sigma^2 \mathbb{H}P^2 \xrightarrow{\Sigma^2 q} S^{10} \xrightarrow{\Sigma^2 h_\mathbb{H}} S^7 \rightarrow \ldots
\]
Taking the cohomotopy groups gives the long exact sequence
\[
\pi^6(S^9) \xrightarrow{(\Sigma^2 h_4)^*} \pi^6(S^6) \xrightarrow{(\Sigma^2 p)^*} \pi^6(\Sigma^2 \mathbb{H} P^2) \xrightarrow{(\Sigma^2 q)^*} \pi^6(S^{10}) \xrightarrow{(\Sigma^2 h_4)^*} \pi^6(S^7) \xrightarrow{\ldots}
\]
Now \(\pi^6(S^9) \cong \pi_0(S^6) \cong \mathbb{Z}_{24}, \pi^6(S^6) \cong \pi_6(S^6) \cong \mathbb{Z},\) and \(\pi^6(S^{10}) = 0,\) so we have a sequence \(\mathbb{Z}_{24} \to \mathbb{Z} \to A \to 0,\) which gives \(\pi^6(\Sigma^2 \mathbb{H} P^2) \cong A \cong \mathbb{Z}.\) Therefore,
\[
\pi^4(\mathbb{H} P^2) \cong \mathbb{Z}.
\] (12)

As in the complex case, this agrees with cohomology, \(H^4(\mathbb{H} P^2; \mathbb{Z}) \cong \mathbb{Z},\) and hence no new contribution,

(ii) For \(\mathbb{O} P^2:\) In the octonionic case we have a cofiber sequences \(S^{15} \to S^8 \to \mathbb{O} P^2 \to S^{16} \to S^9,\) which (after suspending 4-times) yields
\[
\pi_{10}(S^8) \to \pi^8(S^8) \to \pi^8(\Sigma^4 \mathbb{O} P^2) \to \pi_{20}(S^8).
\]
Identifying low-dimensional homotopy groups of spheres gives the exact sequence \(\mathbb{Z}_{1008} \to \mathbb{Z} \to \pi^4(\mathbb{O} P^2) \to 0,\) so that
\[
\pi^4(\mathbb{O} P^2) \cong \mathbb{Z}.
\]

While this is similar to the complex and quaternionic cases, the comparison to to cohomology is different, in that we have \(H^4(\mathbb{O} P^2; \mathbb{Z}) = 0,\) signaling a new effect. However, the dimension takes us outside those of critical M-theory and string theory, but are nonetheless very interesting for the bosonic case (see [Sa09a][Sa09b]). The effects in these examples of projective spaces also deserve further investigation.

### 3 Differential refinements: \(B^3 U(1)_\nabla\) vs. \(\hat{S}^4\)

#### 3.1 Differential cohomotopy

Here we expand on the discussion of differential cohomotopy in [FSS15]. As with any differential refinement, differential cohomotopy involves an interplay between topological information on smooth manifolds and the geometric information of differential forms via a general de Rham type theorem. The basic ingredients for this general machinery can be found in [FSS12][SSS12][FSS14c] and our discussion here will assume familiarity with these ingredients. We encourage the reader to consult these references for more details as needed.

Let \(s^4\) be the Lie 7-algebra whose corresponding Chevallay-Eilenberg algebra is the exterior algebra on generators \(g_4\) and \(g_7\) with relations
\[
dg_4 = 0, \quad dg_7 = g_4 \wedge g_4.
\]
As a de Rham model for flat 1-forms with values in \(S^4\) we take the sheaf on the site of Cartesian spaces given by the assignment
\[
\Omega^1_H(-; s^4) : U \mapsto \text{hom}_{dgAlg}(CE(s^4), \Omega^*(U)),
\]
for each Cartesian space \(U \cong \mathbb{R}^n.\) Here the morphisms in the set on the right are taken in differentially graded commutative algebras. The homotopy type of \(\Omega^1_H(-; s^4)\) can be computed via the Sullivan construction as the \(\mathbb{R}\)-local 4-sphere, which we denote \(S^4_\mathbb{R}\). Then pulling back along the canonical map \(S^4 \to S^4_\mathbb{R},\) we get a smooth stack
\[
\begin{array}{ccc}
\hat{S}^4 & \to & \Omega^1_H(-; s^4) \\
\downarrow & & \downarrow \\
S^4 & \to & S^4_\mathbb{R}.
\end{array}
\]

This construction is essentially the same as the familiar construction of a rational space in rational homotopy theory, but over the field \(k = \mathbb{R}.\) Note however, that we have taken smooth forms instead of polynomial forms. That this agrees with the usual Sullivan construction follows readily from the fact that \(A^*_p(\Delta^4) \hookrightarrow \Omega^*(\Delta^4)\) is a quasi-isomorphism of complexes.

Note that the diagram evidently involves both spaces and smooth stacks. Whenever such diagrams appear, we are implicitly embedding the space as a stack via the constant stack functor \(\delta.\)
We have the following natural definition.

**Definition 3.1** (Differential unstable cohomotopy). For a smooth manifold $X$, let $i(X)$ denote its embedding as a smooth stacks via its sheaf of smooth plots. Then the differential cohomotopy of $X$ in degree 4 is defined as the pointed set

$$ \tilde{\pi}_4^i(X) := \pi_0 \text{Map}(i(X), \tilde{S}^4) $$

where the maps on the right are those of smooth stacks.

This gives a geometric model for *unstable cohomotopy*, but we will also need a geometric model for *stable cohomotopy*. Stably, $S^4$ has only torsion groups in higher degrees and hence the canonical map $S^4 \to K(\mathbb{R}, 4)$ is a stable $\mathbb{R}$-local equivalence. Geometrically, the realification if modeled by closed 4-forms $\Omega_4^4(-)$. Stable differential cohomotopy in degree 4 fits into a pullback square

$$
\begin{array}{ccc}
\Sigma^\infty S^4 & \to & H\big(\tau^{\leq 0}\Omega^4(\cdot)\big) \\
\downarrow & & \downarrow \\
\Sigma^\infty S^4 & \to & \Sigma^4 H\mathbb{R}.
\end{array}
$$

where $\Omega^4(\cdot)$ denotes the de Rham complex, shifted so that $\Omega^4$ is in degree zero, and $\tau^{\leq 0}$ truncates the complex in degree zero so that the complex is concentrated in negative degrees. The functor $H$ denotes the Eilenberg-Maclane functor (see e.g. [Sh07]) which turns a chain complex into a spectrum.

**Definition 3.2** (Differential stable cohomotopy). Let $X$ be a smooth manifold with $i(X)$ its associated smooth stack. The *stable* differential cohomotopy group of $X$ is defined as

$$ \tilde{\pi}_4^i(X) := \pi_0 \text{Map}(i(X); (\Sigma^\infty S^4)_0). $$

where the subscript 0 denotes the degree zero component of the sheaf of spectra $\Sigma^\infty S^4$.

Ultimately, we will be most interested in the above unstable version of differential cohomotopy. However, the stable version will be useful as an approximation and is topologically easier to analyze (as we have seen in §2).

**Geometric meaning of cocycles.** We now discuss a geometric interpretation for cocycles in differential cohomotopy. More precisely, we address what type of geometric data a differential cocycle $\hat{c} : M \to \tilde{S}^4$ classifies.

**Definition 3.3** (Geometric cohomotopy cocycles). If $X$ is a smooth manifold, a morphism $\hat{c} : X \to \tilde{S}^4$ can be identified with a triple $(c, h, \omega)$ where

(i) $c : X \to S^4$ is a cocycle in ordinary cohomotopy,

(ii) $\omega : CE(s^4) \to \Omega^4(X)$ is a DGA morphism, determined by specifying forms $\omega_4$ and $\omega_7$ on $M$ satisfying $d\omega_7 = \omega_2^2$ and $d\omega_4 = 0$,

(iii) $h$ is a homotopy interpolating between the rational cocycle represented by the form data and the rationalization of the classifying map $c : X \to S^4$. Thus, $h$ exhibits a sort of de Rham theorem for cohomotopy.

**Remark 3.4** (Relation to the Pontrjagin-Thom (PT) construction).

(i) Recall from Remark 2.12 that by the PT construction, a mapping $c : X \to S^4$ classifies a bordism class of framed codimension 4 submanifolds of $X$. This correspondence realizes the codimension 4 submanifold $M$ as the preimage of a fixed regular value on $S^4$ and maps the closure of a tubular neighborhood of $M$ in $X$ onto $S^4$ via the given framing of the normal bundle $\mathcal{N} \cong \mathbb{R}^4 \times M \to S^4 \times M \xrightarrow{pr} S^4$.

(ii) Hence, the cocycle $\hat{c}$ gives in particular the data of a codimension 4 submanifold $M \subset X$. It also gives a choice of fiberwise volume form $\omega_4 = c^*g_4$ of the trivial sphere bundle, where $g_4 \in CE(s^4)$ is identified with a choice of volume form for the sphere $S^4$. 

---

17
Much more could be said about the geometric model provided by the Pontrjagin-Thom equivalence, but this falls outside the scope of the present paper. We only include this brief discussion to provide some conceptual geometric intuition.

In view of geometric interpretation via volume forms, we can introduce dynamics by throwing in a radius as a parameter, viewed as the breathing mode (see, e.g., [LS01]).

3.2 Torsion obstructions in differential cohomology

We saw in §2.4 that the Postnikov tower for the 4-sphere has many $k$-invariants which are torsion classes. For our physics applications, the tower must be refined to obtain an obstruction theory for lifting cohomotopy classes to the differential refinement of cohomotopy and it is not completely clear how to deal with such obstructions in the refinement. Indeed, the obstruction theory for differential refinements is obtained by Chern-Weil form representatives of the $k$-invariants, and one requires these forms to trivialize when the topological obstructions vanish (the choice lift through the next stage in the tower gives rise to the trivialization). Since Chern-Weil theory is not available for torsion classes, we need to find an alternative method for the differential refinement.

Recall that the moduli stack of circle $n$-bundles with connection fits into a homotopy pullback diagram [FSS12] [SSS12] [FSS14c]

\[
\begin{array}{ccc}
\mathcal{B} \mathcal{U}(1)_\mathbb{V} & \xrightarrow{R} & \Omega^0_{\text{cl}} \\
\downarrow I & & \downarrow \\
K(\mathbb{Z}, n+1) & \xrightarrow{k} & \Omega^0_{\text{cl}}^{n+1}
\end{array}
\]

where $\Omega^0_{\text{cl}}^{n+1}$ is obtained by applying the Dold-Kan functor to the sheaf of positively graded chain complexes

\[\Omega^0_{\text{cl}}^{n+1} := \Gamma(\ldots \xrightarrow{0} \Omega^0 \xrightarrow{0} \ldots \xrightarrow{0} \Omega^0_{\text{cl}}^{n+1})\]

and $K(\mathbb{Z}, n+1) \to \Omega^0_{\text{cl}}^{n+1}$ is induced by the inclusion $\mathbb{Z} \hookrightarrow \Omega^0$.

**Remark 3.5** (Integral lifts of differential forms). Consider a map $\hat{k} : \mathcal{B} \mathcal{U}(1)_\mathbb{V} \to K(\mathbb{Z}_p, m)$. Since $K(\mathbb{Z}_p, m)$ is a geometrically discrete (i.e., a constant stack), the map $\hat{k}$ factors through the corresponding topological realization of the domain as

\[\hat{k} : \mathcal{B} \mathcal{U}(1)_\mathbb{V} \xrightarrow{I} K(\mathbb{Z}, n+1) \xrightarrow{k} K(\mathbb{Z}_p, m)\]

By the pasting law for pullbacks, we have iterative fiber products

\[\begin{array}{ccc}
\hat{F} & \xrightarrow{F} & * \\
\downarrow & & \\
\mathcal{B} \mathcal{U}(1)_\mathbb{V} & \xrightarrow{I} & K(\mathbb{Z}, n+1) \\
\downarrow & & \downarrow k \\
\Omega^0_{\text{cl}}^{n+1} & \xrightarrow{k} & K(\mathbb{Z}_p, m) \\
\downarrow & & \\
\Omega^0_{\text{cl}}^{n+1} & \xrightarrow{\text{cl}} & \Omega^0_{\text{cl}}^{n+1}
\end{array}\]

The smooth stack $\Omega^0_{\text{cl}}^{n+1}$ represents cohomology with $\mathbb{R}$ coefficients in degree $n+1$. In fact, the canonical inclusion

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{d} & 0 \\
\downarrow & & \\
\Omega^0 & \xrightarrow{d} & \Omega^1 \\
\downarrow & & \downarrow \\
\Omega^0_{\text{cl}}^{n+1} & \xrightarrow{d} & \Omega^1_{\text{cl}}^{n+1}
\end{array}
\]

is a quasi-isomorphism of sheaves of complexes, inducing an isomorphism $H^n(X; \mathbb{R}) \cong H^0(X; \mathbb{R}[n+1]) \cong H^0(X; \Omega^0_{\text{cl}}^{n+1})$. 

18
For each fixed manifold $X$ mapping to the diagram, this naturally gives rise to a map

$$[X, \hat{F}] \longrightarrow \Omega^{n+1}_{cl}(X) \times_{H^{n+1}(X; \mathbb{R})} [X, F].$$

The group on the right can be identified with differential forms whose de Rham class is in the image of the composite

$$[M, F] \longrightarrow H^{n+1}(M; \mathbb{Z}) \longrightarrow H^{n+1}(M; \mathbb{R}).$$

Such conditions can be realized as conditions on the possible integral lifts of differential forms.

**Remark 3.6** (From differential forms to torsion constraints). From the above discussion, we see that if we take the usual fiber at Postnikov stages with torsion $k$-invariants, then differential forms still detect this information. More precisely, passing to the fiber leads to more constrained quantization conditions on the differential forms. This is precisely what is needed for our applications and we will treat torsion obstructions in this manner.

**Example 3.7** (Constraints associated with reduction of coefficients). Let us take $k$ to be the mod $p$ reduction $k = \rho_p : K(\mathbb{Z}, n) \to K(\mathbb{Z}_p, n + 1)$. Then $F$ is easily seen to be $K(\mathbb{Z}, n + 1)$ and the canonical map out of the fiber is

$$x_p : K(\mathbb{Z}, n + 1) \longrightarrow K(\mathbb{Z}, n + 1).$$

Hence, classes in $[M, \hat{F}]$ give rise to closed forms which, when paired with cycles gives an integer divisible by $p$. Such divisibility conditions, in the context of describing fields via K(O)-theory, are discussed extensively in [GS19c] [GS19d].

**Example 3.8** (Obstructions via refinement of cohomology operations). Consider the refinement of the Steenrod square $\tilde{Sq}^2$, given by the composition [GST18a]

$$\tilde{Sq} : B^m U(1)_\mathbb{C} \longrightarrow K(\mathbb{Z}, m) \longrightarrow K(\mathbb{Z}_2, m) \longrightarrow K(\mathbb{Z}_2, m + 2).$$

This is almost, but not quite, the differential refinement of $\tilde{Sq}^3$ discussed in [GS18a]. The two become the same after including $\tilde{Sq}$ into differential cohomology via the map

$$K(\mathbb{Z}_2, m + 2) \longrightarrow K(U(1), m + 2) \simeq B^{m+2} U(1)_{\text{flat}} \longleftarrow B^{m+2} U(1).$$

induced by the inclusion $\mathbb{Z}_2 \hookrightarrow U(1)$ via the 2-roots of unity. Let $K := \ker (\tilde{Sq}^2 \rho_2 : H^m(M; \mathbb{Z}) \to H^{m+2}(M; \mathbb{Z}_2))$. Then classes in $[Y, \hat{F}]$ give rise to forms admitting integral lifts which are in the image of $K \hookrightarrow H^m(Y; \mathbb{Z})$. For the field $G_4$ in spacetime $Y$, we take $m = 3$, so that differential cohomotopy classes $[Y, \hat{F}]$ are given by 4-forms $G^\text{form}_4$ admitting integral images in the image of $\ker (\tilde{Sq}^2 \rho_2 : H^4(M; \mathbb{Z}) \to H^6(M; \mathbb{Z}_2)) \hookrightarrow H^4(Y; \mathbb{Z})$.

### 3.3 Differential cohomotopy vs. differential cohomology

In this section, we refine the Postnikov tower (see Lemma 2.4) to the setting of differential cohomology. Our strategy for building the Postnikov tower for $\tilde{S}^4$ stems from the basic observation that we can split this construction into the following three more elementary constructions.

(i) The Postnikov tower in the opposite category of DGCA’s.

(ii) The ordinary Postnikov tower in spaces.

(iii) The Postnikov tower in spaces localized at $\mathbb{R}$.

It turns out that the process of differential refinement is compatible (in a certain sense) with the Postnikov construction. Before proving that this is the case, we begin with some preliminary remarks. First, from the equivalence of Sullivan algebras and simply connected rational spaces $X$ (the key properties of the Sullivan construction are proved in [FHT01] Section 15), it follows immediately that the Sullivan construction sends forms built from the
wedge products of elements in degree \( \leq k \), i.e. \( \Lambda V^{\leq k} \), to the \( k \)th Postnikov section of the rational space \( X_{\mathbb{R}} \). The geometric realization of the flat \( \Lambda V^{\leq k} \)-valued 1-forms \( \Omega^1_\Pi(-; \Lambda V^{\leq k}) \) is a presentation for the Sullivan construction. As a consequence, we have a canonical map

\[
\eta : \Omega^1_\Pi(-; \Lambda V^{\leq k}) \to (X_{\mathbb{R}})_k
\]

which is induced by the unit of the adjunction \( \Pi \dashv \text{disc} \), where \( \Pi \) denotes geometric realization and \( \text{disc} \) denotes the locally constant stack functor (see [Sc13] for details on these adjoint functors, and [FSS19] for a gentle review).

Localization at \( \mathbb{R} \) is also compatible with the Postnikov process. Indeed, it follows from [FHT01, Theorem 15.8] (see also [He, Theorem 2.2]) that \( \mathbb{R} \)-localization preserves homotopy fibers of spaces of rational finite type. Since the induced map on homotopy groups just tensors with \( \mathbb{R} \), the localization map also gives a well-defined map between the \( k \)th stage of corresponding Postnikov systems

\[
L_\mathbb{R} : (X)_k \to (X_{\mathbb{R}})_k.
\]

Although there is a well-defined notion of Postnikov tower which is intrinsic to smooth stacks, this tower does not give quite the right information when passing to the differential refinement. We would really like a tower which converges to the refinement \( \hat{X} \) and which is compatible with the pullback property of \( \hat{X} \). Motivated by this, we introduce the notion of the differential Postnikov tower.

**Definition 3.9 (Differential Postnikov systems).** Let \( X \) be a simply connected space of rational finite type and let \( (\Lambda V, d) \) be a Sullivan model for \( X_{\mathbb{R}} \). Consider the canonical pullback diagram of smooth stacks

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\eta} & \Omega^1_\Pi(-; \Lambda V) \\
\downarrow & & \downarrow \\
X & \xrightarrow{L_\mathbb{R}} & X_{\mathbb{R}}
\end{array}
\]

where the \( \eta \)'s appearing are the respective components of the unit of the adjunction \( \Pi \dashv \text{disc} \) and \( L_{\mathbb{R}} \) is the localization at \( \mathbb{R} \). A differential Postnikov system for \( \hat{X} \) is sequence of smooth stacks

\[
(\hat{X})_k \to (\hat{X})_{k-1} \to \ldots \to (\hat{X})_0,
\]

such that for each \( k \), \( (\hat{X})_k \) fits into a Cartesian square

\[
\begin{array}{ccc}
(\hat{X})_k & \xrightarrow{\eta} & \Omega^1_\Pi(-; \Lambda V^{\leq k}) \\
\downarrow & & \downarrow \\
(X)_k & \xrightarrow{L_\mathbb{R}} & (X_{\mathbb{R}})_k
\end{array}
\]

with each vertex representing the corresponding \( k \)th Postnikov section (in spaces, rational spaces, and DGCA’s), and the maps (13) are universal maps induced by pullback.

**Proposition 3.10 (Compatibility of differential refinement with Postnikov construction).** Let \( X \) be a simply connected space of rational finite type and let \( (\Lambda V, d) \) be a Sullivan model for \( X_{\mathbb{R}} \). Consider the canonical pullback diagram of smooth stacks

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\eta} & \Omega^1_\Pi(-; \Lambda V) \\
\downarrow & & \downarrow \\
X & \xrightarrow{L_\mathbb{R}} & X_{\mathbb{R}}
\end{array}
\]

be a differential Postnikov system for \( \hat{X} \). Then the system satisfies the following properties:

(i) \( \lim_{k \to \infty} \hat{X}_k \simeq \hat{X} \).

(ii) For \( \pi_{n+1}(X) \) a torsion group, the map \( \hat{X}_{n+1} \to \hat{X}_n \) has fiber \( K(\pi_{n+1}(X), n+1) \) and is classified by the \( k \)-invariant

\[
\hat{X}_n \xrightarrow{\eta} X_n \to K(\pi_{n+1}(X), n+2).
\]
(iii) For $\pi_{n+1}(X)$ free of rank $m$, the map $\hat{X}_{n+1} \to \hat{X}_n$ has fiber $K(\pi_{n+1}(X), n+1)$ and fits into a pullback diagram of the form

$$\begin{array}{ccc}
\hat{X}_{n+1} & \rightarrow & \prod_{i=1}^{m} \Omega^n \\
\downarrow & & \downarrow \prod_{i=1}^{m} a \\
\hat{X}_n & \rightarrow & \prod_{i=1}^{m} B^{n+1}U(1) \nu .
\end{array}$$

(14)

Here $a : \Omega^n \rightarrow B^{n+1}U(1)\nu$ is part of the data of the differential refinement, whose curvature gives the exterior derivative. The bottom map in (14) refines the topological $k$-invariant and the $k$-invariant of DGCA's.

**Proof.** Using basic properties of filtered colimits of spaces, it is straightforward to verify that $\mathbb{R}$-localization preserves filtered colimits of simply-connected spaces. Hence, from our observations above, $L_\mathbb{R}$ sends the full Postnikov system for $X$ to a corresponding system for $X_\mathbb{R}$. Since finite limits (and in particular pullbacks) commute with filtered colimits, it then follows that indeed

$$\lim_{k \to \infty} (\hat{X})_k \simeq \hat{X} .$$

The claim about the torsion $k$-invariants follows by observing that, by commutativity of limits, the map classifying the extension is the pullback of the corresponding classifying maps in the topological, rational, and DGCA case. Since $\pi_{n+1}(X)$ is torsion, the rationalization kills the topological $k$-invariant and the refinement of the $k$-invariant at this stage collapses to the purely topological case, as claimed.

Finally, for the non-torsion case, we let $\{v_i\}_{i=1}^{m}$ be a basis for $\pi_{n+1}(X) \otimes \mathbb{R}$. Then, in DGCA's, the extension is classified by the pushout diagram

$$\begin{array}{ccc}
\Lambda V^{\leq n+1} & \leftarrow & \Lambda (w_1, w_2, \ldots, w_m, dw_1, dw_2, \ldots dw_m) \\
\downarrow \varphi & & \downarrow \phi \\
\Lambda V^{\leq n} & \leftarrow & \mathbb{R}[v_1, v_2, \ldots, v_m]
\end{array}$$

where the bottom map $\varphi$ dualizes to the classifying map and the right vertical arrow $\phi$ is defined by sending $v_i \mapsto dw_i$. Applying the GCA homomorphism to forms, $\text{hom}_{\text{dgca}}(-; \Omega^*)$, gives a corresponding pullback diagram

$$\begin{array}{ccc}
\Omega^* (\cdot; \Lambda V^{\leq n+1}) & \rightarrow & \prod_{i=1}^{m} \Omega^n \\
\downarrow & & \downarrow \prod_{i=1}^{m} d \\
\Omega^* (\cdot; \Lambda V^{\leq n}) & \rightarrow & \prod_{i=1}^{m} \Omega^{n+1} \Omega^* \Omega^* \cl .
\end{array}$$

This diagram then geometrically realizes to a homotopy pullback square

$$\begin{array}{ccc}
(X_\mathbb{R})_{n+1} & \rightarrow & * \\
\downarrow & & \downarrow \\
(X_\mathbb{R})_n & \rightarrow & \prod_{i=1}^{m} K(\mathbb{R}, n+1) .
\end{array}$$

(15)

Since $\pi_{n+1}(X)$ is given to be free of rank $m$, we also have a homotopy fiber sequence

$$\begin{array}{ccc}
(X)_{n+1} & \rightarrow & * \\
\downarrow & & \downarrow \\
(X)_{n} & \rightarrow & \prod_{i=1}^{m} K(\mathbb{Z}, n+1) .
\end{array}$$

Since $\pi_{n+1}(X)$ is given to be free of rank $m$, we also have a homotopy fiber sequence

$$\begin{array}{ccc}
(X)_{n+1} & \rightarrow & * \\
\downarrow & & \downarrow \\
(X)_{n} & \rightarrow & \prod_{i=1}^{m} K(\mathbb{Z}, n+1) .
\end{array}$$
which $\mathbb{R}$-localizes to the fiber sequence (15) above. Finally, the $k$-invariant in the differential Postnikov tower is just the pullback of the corresponding $k$-invariants in spaces and $\mathbb{R}$-local spaces and $B^{n+1}U(1)_\nabla$ fits into the iterated Cartesian squares

$$
\Omega^d \rightarrow B^{n+1}U(1)_\nabla \rightarrow \Omega_{cl}^{n+1}
$$

Using these two last facts, the $k$-invariant indeed refines to take values in the claimed product and $\hat{X}_{n+1}$ fits into the desired pullback. □

**Remark 3.11** (Extension of the 4-sphere algebra and quaternionic Hopf fibration). The only nontrivial extension in the Postnikov approximation to the Sullivan algebra $CE(s^4)$ occurs in degree $n = 5$, where we get a pushout diagram

$$
CE(s^4) \leftarrow \Lambda(g_7, dg_7)
\mathbb{R}[g_4] \leftarrow \mathbb{R}[g_8]
$$

in which the bottom map sends $g_8 \mapsto g_4^2$ and the right map sends $g_8 \mapsto dg_7$. Rationally, this level corresponds to the quaternionic Hopf fibration generating $\pi_7(S^4) \otimes \mathbb{R}$ (see [FSS19b] [FSS19c]). See also Remark 2.9.

Let us recall that the Deligne-Beilinson cup product gives a cup product structure in differential cohomology and uniquely refines (up to homotopy) the wedge product of forms and the cup product in integral cohomology (see [FSS14c] [FSS15] [Sc13]). These considerations, along with Proposition 3.10, immediately imply the following.

**Proposition 3.12** (Refinement vs. Postnikov for the 4-sphere). The $n$th section of the differential Postnikov tower takes the form

$$
(S^4)_{\mathbb{R}}^n \rightarrow \Omega_{cl}^1(\cdot; (s^4)^{\leq n})
$$

(i) As $n \rightarrow \infty$, we have $\lim_{k \rightarrow \infty} (\hat{S}^4)_k = \hat{S}^4$.

(ii) Moreover, for $\pi_{n+1}(X)$ torsion, the $k$-invariant at the $n$th stage of the Postnikov system for $S^4$ refines to a $k$-invariant for $\hat{S}^4$ via the canonical map

$$
(S^4)_{\mathbb{R}}^n \xrightarrow{\eta} (S^4)^n \xrightarrow{k} K(\pi_{n+1}(X), n+2),
$$

(iii) while for $\pi_7(S^4) \cong \mathbb{Z} \times \mathbb{Z}_{12}$ the $k$-invariant takes the form

$$
(S^4)_{\mathbb{R}}^3 \rightarrow \Omega^7 \rightarrow K(\mathbb{Z}_{12}, 8) \times B^7U(1)_\nabla,
$$

where the projection of the $k$-invariant to the second factor is the Deligne-Beilinson cup product $\hat{G}_4 \cup_{DB} \hat{G}_4$.

Proposition 3.12 gives a complete characterization of the obstruction theory for $S^4$ in the differential setting. The $k$-invariants are either purely topological, in the torsion case, or are differential refinements of the topological $k$-invariants in the free case. As usual, to consider structures on spacetime $Y$, we pull back these universal classes and obstruction and evaluate on $Y$. 

22
Proposition 3.13. [Differential refinement of Postnikov tower of the sphere] The full differential refinement of the Postnikov tower for \(S^4\) takes the following form

\[
\begin{align*}
K(\mathbb{Z}_{15}, 11) & \longrightarrow (\tilde{S}^4)_7 \\
K(\mathbb{Z}_{24} \times \mathbb{Z}_3, 10) & \longrightarrow (\tilde{S}^4)_6 \longrightarrow K(\mathbb{Z}_{15}, 12) \\
K(\mathbb{Z}_2 \times \mathbb{Z}_2, 9) & \longrightarrow (\tilde{S}^4)_5 \longrightarrow K(\mathbb{Z}_{24} \times \mathbb{Z}_3, 11) \\
K(\mathbb{Z}_2 \times \mathbb{Z}_2, 8) & \longrightarrow (\tilde{S}^4)_4 \longrightarrow K(\mathbb{Z}_2 \times \mathbb{Z}_2, 10) \\
K(\mathbb{Z}_{12}, 7) \times K(\mathbb{Z}, 7) & \longrightarrow (\tilde{S}^4)_3 \longrightarrow K(\mathbb{Z}_2 \times \mathbb{Z}_2, 9) \\
K(\mathbb{Z}_2, 6) & \longrightarrow (\tilde{S}^4)_2 \stackrel{(\gamma, \tilde{\gamma})}{\longrightarrow} K(\mathbb{Z}_{12}, 8) \times \mathbb{B}^7U(1)_{\mathcal{V}} \\
K(\mathbb{Z}_2, 5) & \longrightarrow (\tilde{S}^4)_1 \stackrel{\alpha_5 I}{\longrightarrow} K(\mathbb{Z}_2, 7) \\
(\tilde{S}^4)_0 = \mathbb{B}^3U(1)_{\mathcal{V}} & \stackrel{\text{Sq}^2 \rho I}{\longrightarrow} K(\mathbb{Z}/2, 6)
\end{align*}
\]

where we have identified the first few obstructions.

Remark 3.14 (The obstruction in M-theory via higher bundles with connections). Note that locally the Deligne-Beilinson cup product in M-theory \(\hat{G}_4 \cup_{\text{DB}} \hat{G}_4\) gives a 7-bundle with connection form locally given by \(C_3 \wedge G_4\) \cite{FSS14a,FSS14b,FSS14c,FSS15}. From the identification of the \(k\)-invariant at the second stage in Proposition 3.13 (the Deligne-Beilinson square), it follows that to lift past the 2nd stage in the Postnikov tower for \(S^4\), this connection must be globally defined. Explicitly, in terms of differential cohomology, we have

\[
a(C_3 \wedge G_4) = \hat{G}_4 \cup_{\text{DB}} \hat{G}_4,
\]

where \(a : \Omega^7(Y^{11}) \rightarrow \hat{H}^8(Y^{11})\) is the canonical map.

Remark 3.15 (The stable case). The above has been the treatment in the unstable case, and the discussion goes through essentially verbatim in the stable setting, with minor modifications. The only nontrivial modification is to replace DGCAs with the correct algebraic model for \(\mathbb{R}\)-local spectra. The \(\mathbb{R}\)-localization of a spectrum is simply given by smashing with the real Eilenberg-MacLane spectrum \(H\mathbb{R}\). By the work of Shipley \cite{Sh07}, these are equivalent to just differentially graded vector spaces. Since there are no non-trivial rational obstructions for \(S^4\) (stably), these effects are not seen and we will not spell out these details. We simply note that the first two properties of Proposition 3.12 hold equally well in the stable setting. This gives rise to the following proposition.

Proposition 3.16 (Differential cohomotopy vs. differential cohomology). Let \(Y^{11}\) be an 11-dimensional smooth manifold. Let \(I : \hat{H}^*(Y^{11}) \rightarrow H^*(Y^{11}; \mathbb{Z})\) be the canonical map relating differential cohomology and integral cohomology. Then a class \(\hat{a} \in \hat{H}^4(Y^{11})\) lifts to a class \(\hat{b} \in \hat{\pi}_5(Y^{11})\) if and only if the following conditions are satisfied.

(i) \(\text{Sq}^2 I(\hat{a}) \equiv 0 \mod 2, \quad \mathcal{P}^3_1 I(\hat{a}) \equiv 0 \mod 3\).

(ii) There is a lift \(\hat{a}' : Y^{11} \rightarrow (\tilde{S}^4)_1\) of \(\hat{a}\) such that \(\alpha_7 I(\hat{a}') \equiv 0 \mod 2\).

\(^{8}\)Note that the unit map \(S \rightarrow H\mathbb{R}\) induces an equivalence \(S\mathbb{R} \simeq H\mathbb{R}\).
There is a further lift \( \hat{a}'' : Y^{11} \rightarrow (\Sigma^4, 2) \) such that \( \beta_8 I(\hat{a}'') \equiv 0 \mod 8 \). In particular, upon mod 2 reduction, we have \( \hat{a}' \equiv 0 \mod 2 \).

Proposition 3.17 (Differential cohomotopy vs. differential cohomology for the C-field). Consider the differentially refined M-theory (shifted) C-field \( \hat{G}_4 \) as an integral cohomology class in degree four. Then if \( \hat{G}_4 \) lifts to a cohomotopy class \( \gamma_4 \in \tilde{H}^4(Y^{11}) \) the following obstructions necessarily vanish

1. \( \hat{a} \equiv 0 \mod 12 \). We will use this exact sequence to compute some examples.

Remark 2.7 holds equally in the differential case and is closely related to the condition \( \hat{a} \equiv 0 \mod 2 \).

Remark 2.8, we can also obtain the mod 3 congruence by considering the top obstruction on a closed 12-manifold.

Remark 2.9 (Differential cohomotopy first contribution to the C-field). The interpretation of the degree 5 class in Remark 2.7 holds equally in the differential case and is closely related to the condition \( \hat{a} \equiv 0 \mod 2 \).

Example 3.20 (Differential cohomotopy of flux compactification spaces). We consider the differential cohomotopy of the spacetime backgrounds computed in Examples 2.15. First observe that, by the general machinery of differential refinements of generalized cohomology (see [GS17] [GS19c]), we have a long exact sequence in stable cohomotopy

\[ \cdots \rightarrow \pi^3(X) \xrightarrow{\deg} \Omega^3(X) \rightarrow \tilde{\pi}_4^3(X) \rightarrow \pi_4^4(X) \rightarrow \cdots. \]

We will use this exact sequence to compute some examples.

(i) \( \text{AdS}_7 \times \mathbb{R}P^4 \): Here we observe that the cofiber sequence \( \mathbb{R}P^3 \rightarrow \mathbb{R}P^4 \xrightarrow{q_4} S^4 \). This gives rise to an exact sequence in cohomology

\[ \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow H^4(S^4; \mathbb{Z}) \rightarrow H^4(\mathbb{R}P^4; \mathbb{Z}) \rightarrow 0 \]

from which we learn that the pullback of the fundamental class of \( S^4 \) by \( q_4 \) is the generator of \( H^4(\mathbb{R}P^4; \mathbb{Z}) \). This gives an isomorphism

\[ \pi_4(\mathbb{R}P^4) \cong H^4(\mathbb{R}P^4; \mathbb{Z}), \quad q_4 \mapsto q_4^4. \]

Now from the cofiber sequence above, we also compute \( \pi_3(\mathbb{R}P^4) \cong 0 \). We therefore have a short exact sequence

\[ 0 \rightarrow \Omega^3(\mathbb{R}P^4) \rightarrow \tilde{\pi}_4^3(\mathbb{R}P^4) \rightarrow \pi_4^4(\mathbb{R}P^4) \cong \mathbb{Z}_2 \rightarrow 0. \]

The Five Lemma produces an isomorphism \( \tilde{\pi}_4^3(\mathbb{R}P^4) \cong \tilde{H}_4(\mathbb{R}P^4) \). Using the fact that AdS_7 is topologically trivial, this also implies an isomorphism \( \tilde{\pi}_4^3(\text{AdS}_7 \times \mathbb{R}P^4) \cong \tilde{H}_4(\text{AdS}_7 \times \mathbb{R}P^4) \).
(ii) $\text{AdS}_4 \times \mathbb{C}P^2$: Here we recall that, in the topological case, $\pi^4(\mathbb{C}P^2) \cong \mathbb{Z}$ with generator $[q_2]$. The ‘realification’ map $\mathbb{Z} \cong \pi^4(\mathbb{C}P^2) \rightarrow \pi^4(\mathbb{C}P^2) \otimes \mathbb{R} \cong \mathbb{R}$ is the canonical inclusion. It is easy to check that pullback by $q_2 : \mathbb{C}P^2 \rightarrow S^4$ induces an isomorphism on $H^4$. Hence, $\Omega^4(\mathbb{C}P^2) \rightarrow \pi^4(\mathbb{C}P^2) \otimes \mathbb{R}$ maps a closed form $\omega_4$, generating $H^4(\mathbb{C}P^2; \mathbb{R}) \cong \mathbb{R}$ to the generator $[q_2]$. Using the Hopf fibration, one can show that $\pi^3(\mathbb{C}P^2) \cong 0$. In this case, these considerations lead to a short exact sequence

$$0 \rightarrow \Omega^3(\mathbb{C}P^2) \rightarrow \tilde{\pi}_4^h(\mathbb{C}P^2) \rightarrow \pi^4(\mathbb{C}P^2) \rightarrow 0,$$

and the Five Lemma produces an isomorphism $\tilde{\pi}_4(\mathbb{C}P^2) \cong \hat{H}^4(\mathbb{C}P^2)$. Using the fact that $\text{AdS}_4$ is topologically trivial, this also implies an isomorphism $\tilde{\pi}_4(\text{AdS}_4 \times \mathbb{C}P^2) \cong \hat{H}^4(\text{AdS}_4 \times \mathbb{C}P^2)$.

(iii) $\text{AdS}_4 \times \mathbb{R}P^5 \times T^2$: In this case, $T^2$ does not contribute to $\pi^4$ or $\pi^3$ topologically (as in Examples 2.15). Then the same argument as in part (ii) above gives

$$\tilde{\pi}_4^h(\text{AdS}_4 \times \mathbb{C}P^2 \times T^2) \cong \hat{H}^4(\text{AdS}_4 \times \mathbb{C}P^2 \times T^2).$$

(iv) $\text{AdS}_4 \times \mathbb{R}P^5 \times T^2$. As noted in part (iv) of Examples 2.15, $\pi^4(\mathbb{R}P^5)$ is order 4, either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, while $H^4(\mathbb{R}P^5; \mathbb{Z}) \cong \mathbb{Z}_2$. From [We70], $\pi^3(\mathbb{R}P^5)$ is finite. We therefore have a short exact sequence

$$0 \rightarrow \Omega^3(\mathbb{R}P^5) \rightarrow \tilde{\pi}_4(\mathbb{R}P^5) \rightarrow \pi^4(\mathbb{R}P^5) \rightarrow 0.$$

Since $\pi^4(\mathbb{R}P^5)$ is generated by $q_5 \eta_4$, with $\eta_4 : S^5 \rightarrow S^4$ the two-fold suspension of the Hopf map, the induced map on $H^4$ necessarily vanishes. Hence, in this case, differential cohomotopy yields considerably different information than ordinary differential cohomology.

References

[AJ04] P. Aschieri and B. Jurco, *Gerbes, M5-Brane Anomalies and E8 Gauge Theory*, J. High energy Phys. **0410** (2004), 068, [arXiv:hep-th/0409200].

[Ba89] H. J. Baues, *Algebraic homotopy*, Cambridge University Press, Cambridge, 1989.

[Bo36] K. Borsuk, *Sur les groupes des classes de transformations continues*, C.R. Acad. Sci. Paris **202** (1936), 1400-1403.

[BEM04] P. Bouwknegt, J. Evslin, and V. Mathai, *T-duality: Topology change from H-flux*, Commun. Math. Phys. **249** (2004), 383-415, [arXiv:hep-th/0306062].

[BSS18] V. Braunack-Mayer, H. Sati, and U. Schreiber, *Gauge enhancement for Super M-branes via Parameterized stable homotopy theory*, Comm. Math. Phys. **371** (2019), 197-265, [doi:10.1007/s00220-019-03441-4], [arXiv:1805.05987][hep-th].

[BSS18] S. Burton, H. Sati, and U Schreiber, *Lift of fractional D-brane charge to equivariant Cohomotopy theory*, [arXiv:1812.09679][math.RT].

[Ca54] H. Cartan, *Sur les groupes d’Eilenberg-MacLane I, II*, Proc. Nat. Acad. Sci. USA **40** (1954), 467-471 and 704-707.

[DFM03] E. Diaconescu, D. S. Freed, and G. Moore, *The M-theory 3-form and E8 gauge theory*, Elliptic Cohomology, 44-88, Cambridge University Press, 2007, [arXiv:hep-th/0312069].

[DMW00] D. Diaconescu, G. Moore, and E. Witten, *E8-gauge theory and a derivation of K-theory from M-theory*, Adv. Theor. Math. Phys. **6** (2003), 1031–1134, [arXiv:hep-th/0005090].

[DLP98] M. J. Duff, H. Lu, and C. N. Pope, *AdS$_5$ × S$^5$ untwisted*, Nucl. Phys. **B532** (1998), 181-209, [arXiv:hep-th/9803061].

[FHT01] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics 205, Springer-Verlag, New York, 2001.

[FOT08] Y. Félix, J. Oprea, and D. Tanré *Algebraic Models in Geometry*, Oxford University Press, 2008.
D. Fiorenza, H. Sati, and U. Schreiber, *Super Lie n-algebra extensions, higher WZW models, and super p-branes with tensor multiplet fields*, Intern. J. Geom. Meth. Mod. Phys. 12 (2015) 1550018, [arXiv:1308.5264](https://arxiv.org/abs/1308.5264).

D. Fiorenza, H. Sati, and U. Schreiber, *The E_8 moduli 3-stack of the C-field*, Commun. Math. Phys. 333 (2015), 117-151, [arXiv:1202.2455](https://arxiv.org/abs/1202.2455).

D. Fiorenza, H. Sati, and U. Schreiber, *Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory*, Adv. Theor. Math. Phys. 18 (2014), 229 - 321, [arXiv:1201.5277](https://arxiv.org/abs/1201.5277).

D. Fiorenza, H. Sati, and U. Schreiber, *Extended higher cup-product Chern-Simons theories*, J. Geom. Phys. 74 (2013), 130-163, [arXiv:1207.5449](https://arxiv.org/abs/1207.5449).

D. Fiorenza, H. Sati, and U. Schreiber, *The WZW term of the M5-brane and differential cohomotopy*, J. Math. Phys. 56 (2015), 102301, [arXiv:1506.07557](https://arxiv.org/abs/1506.07557).

D. Fiorenza, H. Sati, and U. Schreiber, *Rational sphere valued supercocycles in M-theory and type IIA string theory*, J. Geom. Phys. 114 (2017) 91-108, [arXiv:1606.03206](https://arxiv.org/abs/1606.03206).

D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M-Theory anomaly cancellation on 8-manifolds*, [arXiv:1904.10207](https://arxiv.org/abs/1904.10207)

D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies level quantization of the full 6d Wess-Zumino term of the M5-brane*, [arXiv:1906.07417](https://arxiv.org/abs/1906.07417)

D. Fiorenza, U. Schreiber, and J. Stasheff, *ˇCech cocycles for differential characteristic classes – an infinity-Lie theoretic construction*, Adv. Th. Math. Phys. 16 (2012) 149, [arXiv:1011.4735](https://arxiv.org/abs/1011.4735).

A. Fomenko and D. Fuchs, *Homotopical topology*, 2nd ed., Graduate Texts in Mathematics 273, Springer, 2016.

D. Grady, H. Sati, and U. Schreiber, *Massey products in differential cohomology via stacks*, J. Homotopy Relat. Struct. 13 (2017), 169-223, [arXiv:1510.06366](https://arxiv.org/abs/1510.06366)

D. Grady and H. Sati, *Twisted differential generalized cohomology theories and their Atiyah-Hirzebruch spectral sequence*, Alg. Geom. Topol. 19 (2019), 2899-2960, [arXiv:1711.06650](https://arxiv.org/abs/1711.06650).

D. Grady and H. Sati, *Differential KO-theory: constructions, computations and applications*, [arXiv:1809.07059](https://arxiv.org/abs/1809.07059)

D. Grady and H. Sati, *Ramond-Ramond fields and twisted differential K-theory*, [https://arxiv.org/abs/1903.08843](https://arxiv.org/abs/1903.08843).

K. Hess, *Rational homotopy theory: a brief introduction* In: Interactions Between Homotopy Theory and Algebra. Contemp. Math. 436, pp. 175-202, [arXiv:math.AT/0604626](https://arxiv.org/abs/math.AT/0604626).

P. Hilton, *On the extended genus*, Acta Math. Sinica 4 (1988), 372–382.

K. Hori, *Consistency Conditions for Fivebrane in M Theory on \( \mathbb{R}^5 / \mathbb{Z}_2 \) Orbifold*, Nucl. Phys. B539 (1999), 35-78, [arXiv:hep-th/9805141](https://arxiv.org/abs/hep-th/9805141).

J. Huerta, H. Sati, and U. Schreiber, *Real ADE-equivariant (co)homotopy and Super M-branes*, Commun. Math. Phys. 371 (2019), 425-524, [arXiv:1805.05987](https://arxiv.org/abs/1805.05987).
[Ta99] H. Tamanoi, *Q-subalgebras, Milnor basis, and cohomology of Eilenberg-MacLane spaces*, J. Pure Appl. Algebra 137 (1999), 153–198.

[To62] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Princeton University Press, Princeton, NJ, 1962.

[We70] R. W. West, *Some cohomotopy of projective space*, Indiana Univ. Math. J. 20 (1970/1971), 807-827.

[Wh78] G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, Berlin, 1978.

[Wil76] C. W. Wilkerson, *Applications of minimal simplicial groups*, Topology 15 (1976), 111-130.

[Wit96] E. Witten, *Five-branes and M-Theory On An Orbifold*, Nucl. Phys. B463 (1996), 383-397, [hep-th/9512219].

[Wit97] E. Witten, *On Flux Quantization In M-Theory And The Effective Action*, J. Geom. Phys. 22 (1997), 1-13, [arXiv:hep-th/9609122].

Daniel Grady, *Department of Mathematics, Texas Tech University, Lubbock, TX 79409, USA.*

Hisham Sati, *Mathematics, Division of Science, New York University Abu Dhabi, UAE.*