Equation of state for dark energy in $f(T)$ gravity

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Abstract

We study the cosmological evolutions of the equation of state for dark energy $w_{DE}$ in the exponential and logarithmic as well as their combination $f(T)$ theories. We show that the crossing of the phantom divide line of $w_{DE} = -1$ can be realized in the combined $f(T)$ theory even though it cannot be in the exponential or logarithmic $f(T)$ theory. In particular, the crossing is from $w_{DE} > -1$ to $w_{DE} < -1$, in the opposite manner from $f(R)$ gravity models. We also demonstrate that this feature is favored by the recent observational data.

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I. INTRODUCTION

Cosmic observations from Supernovae Ia (SNe Ia) [1], cosmic microwave background (CMB) radiation [2–4], large scale structure (LSS) [5], baryon acoustic oscillations (BAO) [6], and weak lensing [7] have implied that the expansion of the universe is currently accelerating. This is one of the most important issues in modern physics. Approaches to account for the late time cosmic acceleration fall into two representative categories: One is to introduce "dark energy" in the right-hand side of the Einstein equation in the framework of general relativity (for a review on dark energy, see [8]). The other is to modify the left-hand side of the Einstein equation, called as a modified gravitational theory, e.g., $f(R)$ gravity [9–11].

As another possible way to examine gravity beyond general relativity, one could use the Weitzenböck connection, which has no curvature but torsion, rather than the curvature defined by the Levi-Civita connection. Such an approach is referred to “teleparallelism” (see, e.g., [12–15]), which was also taken by Einstein [16]. To explain the late time acceleration of the universe, the teleparallel Lagrangian density described by the torsion scalar $T$ has been extended to a function of $T$ [17, 18]. This idea is equivalent to the concept of $f(R)$ gravity, in which the Ricci scalar $R$ in the Einstein-Hilbert action is promoted to a function of $R$.

Recently, $f(T)$ gravity has been extensively studied in the literature [20–25]. In this paper, we explicitly examine the cosmological evolution in the exponential $f(T)$ theory [18, 23] in more detail with the analysis method in Ref. [26]. In particular, we study the equation of state ($w_{\text{DE}}$) and energy density ($\rho_{\text{DE}}$) for dark energy. The recent cosmological observational data [27] seems to imply a dynamical dark energy of equation of state with the crossing of the phantom divide line $w_{\text{DE}} = -1$ from the non-phantom phase to phantom phase as the redshift $z$ decreases in the near past. However, we illustrate that the universe with the exponential $f(T)$ theory always stays in the non-phantom (quintessence) phase or the phantom one, and hence the crossing of the phantom divide cannot be realized [23]. It is interesting to mention that such an exponential type as $f(R)$ gravity models has been investigated in Refs. [28–30]. We also present a logarithmic $f(T)$ theory and show that it has a similar feature as the exponential one. Our motivation in this paper is to build up a realistic $f(T)$ theory in which the same behavior of the crossing of the phantom divide

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1 Models based on modified teleparallel gravity for inflation have been investigated in Ref. [10].
as indicated by the data can be achieved. For this purpose, we will construct a combined $f(T)$ theory with both logarithmic and exponential terms. Furthermore, we examine the observational constraints on the combined $f(T)$ theory by using the recent observational data of SNe Ia, BAO and CMB. We note that two $f(T)$ models with the crossing of the phantom divide have also been proposed in Ref. [22].

The paper is organized as follows. In Sec. II, we introduce $f(T)$ theory and derive the gravitational field equations. In Sec. III, we investigate the cosmological evolutions in the exponential and logarithmic $f(T)$ theories. In Sec. IV, we propose a model which combines the two theories in Sec. III to achieve the crossing of the phantom divide. In Sec. V, we explore the observational constraints on the combined model in Sec. IV. Finally, conclusions are given in Sec. VI.

II. THE COSMOLOGICAL FORMULAE IN $f(T)$ GRAVITY

In the teleparallelism, orthonormal tetrad components $e_A(x^\mu)$ are used, where an index $A$ runs over 0, 1, 2, 3 for the tangent space at each point $x^\mu$ of the manifold. Their relation to the metric $g^{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu ,$$  

where $\mu$ and $\nu$ are coordinate indices on the manifold and also run over 0, 1, 2, 3, and $e^\mu_A$ forms the tangent vector of the manifold.

The torsion $T^\rho_{\mu\nu}$ and contorsion $K^{\mu\nu}_{\rho}$ tensors are defined by

$$T^\rho_{\mu\nu} \equiv e^\rho_A \left( \partial_\mu e^A_\nu - \partial_\nu e^A_\mu \right) ,$$

$$K^{\mu\nu}_{\rho} \equiv -\frac{1}{2} \left( T^{\mu\nu}_{\rho} - T^{\nu\mu}_{\rho} - T^{\rho}_{\mu\nu} \right) .$$

Instead of the Ricci scalar $R$ for the Lagrangian density in general relativity, the teleparallel Lagrangian density is described by the torsion scalar $T^{\mu\nu\rho}$, defined as

$$T^{\mu\nu\rho} \equiv S^{\mu\nu\rho} ,$$

where

$$S^{\mu\nu\rho} \equiv \frac{1}{2} \left( K^{\mu\nu}_{\rho} + \delta^\mu_\rho T^{\alpha\nu}_{\alpha} - \delta^\nu_\rho T^{\alpha\mu}_{\alpha} \right) .$$

Consequently, the modified teleparallel action for $f(T)$ theory is given by

$$I = \frac{1}{16\pi G} \int d^4x |e| (T + f(T)) ,$$
where $|e| = \det (e^A_\mu) = \sqrt{-g}$.

We assume the flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric,

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2,$$

where $a(t)$ is the scale factor. In this space-time, $g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2)$ and therefore the tetrad components $e^A_\mu = (1, a, a, a)$ yield the exact value of torsion scalar $T = -6H^2$, where $H = \dot{a}/a$ is the Hubble parameter. We use units of $k_B = c = \hbar = 1$ and the gravitational constant $G = M_{\text{Pl}}^{-2}$ with the Planck mass of $M_{\text{Pl}} = 1.2 \times 10^{19}$ GeV.

It follows from the variation principle that in the flat FLRW background, the modified Friedmann equations are given by \[17, 18\]

$$H^2 = \frac{8\pi G}{3}\rho_M - \frac{f}{6} - 2H^2 f_T,$$

$$\left(H^2\right)' = \frac{16\pi GP_M + 6H^2 + f + 12H^2 f_T}{24H^2 f_{TT} - 2 - 2f_T},$$

where a prime denotes a derivative with respect to $\ln a$, $f_T \equiv d^2f(T)/dT^2$, and $\rho_M$ and $P_M$ are the energy density and pressure of all perfect fluids of generic matter, respectively.

By comparing the above modified Friedmann equations \[2.8\] and \[2.9\] with the ordinary ones in general relativity:

$$H^2 = \frac{8\pi G}{3}(\rho_M + \rho_{\text{DE}}),$$

$$\left(H^2\right)' = -8\pi G (\rho_M + P_M + \rho_{\text{DE}} + P_{\text{DE}}),$$

the energy density and pressure of the effective dark energy can be described by

$$\rho_{\text{DE}} = \frac{1}{16\pi G} (-f + 2T f_T),$$

$$P_{\text{DE}} = \frac{1}{16\pi G} \frac{f - T f_T + 2T^2 f_{TT}}{1 + f_T + 2T f_{TT}}.$$

The equation of state for dark energy is defined as \[18\]

$$w_{\text{DE}} \equiv \frac{P_{\text{DE}}}{\rho_{\text{DE}}} = -1 + \frac{T' f_T + 2T f_{TT}}{3T f/T - 2f_T} = -\frac{f/T - f_T + 2T f_{TT}}{(1 + f_T + 2T f_{TT})(f/T - 2f_T)}.$$

Since we are interested in the late time universe, we consider only non-relativistic matter (cold dark matter and baryon) with $\rho_M = \rho_m$ and $P_M = P_m = 0$, where $\rho_m$ and $P_m$ are the energy density and pressure of non-relativistic matter, respectively. Consequently, from
Eqs. (2.8) and (2.9) it can be shown that the effective dark energy satisfies the continuity equation

\[ \frac{d\rho_{\text{DE}}}{dN} \equiv \rho_{\text{DE}}' = -3 (1 + w_{\text{DE}}) \rho_{\text{DE}}, \]  

where \( N \equiv \ln a. \)

## III. COSMOLOGICAL EVOLUTION IN THE EXPONENTIAL AND LOGARITHMIC \( f(T) \) THEORIES

To analyze the cosmological evolution of the equation of state for dark energy \( w_{\text{DE}} \) in \( f(T) \) theory, we define a dimensionless variable [26]:

\[ y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{\text{DE}}}{\rho_{m}^{(0)}}, \]  

where

\[ \bar{m}^2 \equiv \frac{8\pi G \rho_{m}^{(0)}}{3}, \]

and \( \rho_{m}^{(0)} = \rho_{m}(z = 0) \) is the current density parameter of non-relativistic matter with the redshift \( z \equiv 1/a - 1. \) Using Eq. (2.15), we obtain the evolution equation of the universe as follows:

\[ y_H' = -3 (1 + w_{\text{DE}}) y_H. \]  

Note that \( w_{\text{DE}} \) is a function of \( T, \) while \( T \) is a function of \( H^2. \) Moreover, it follows from Eq. (3.1) that \( H^2 = \bar{m}^2 (y_H + a^{-3}). \)

### A. Exponential \( f(T) \) theory

We consider the exponential \( f(T) \) theory in Ref. [18], given by

\[ f(T) = \alpha T \left( 1 - e^{pT_0/T} \right) \]  

with

\[ \alpha = - \frac{1 - \Omega_{m}^{(0)}}{1 - (1 - 2p) e^p}, \]

where \( p \) is a constant with \( p = 0 \) corresponding to the \( \Lambda \text{CDM} \) model and \( T_0 = T(z = 0) \) is the current torsion. Here, \( \Omega_{m}^{(0)} \equiv \rho_{m}^{(0)}/\rho_{\text{crit}}^{(0)} \) where \( \rho_{m}^{(0)} \) is the energy density of non-relativistic matter at the present time and \( \rho_{\text{crit}}^{(0)} = 3H_0^2/(8\pi G) \) is the critical density with
FIG. 1: $w_{\text{DE}}$ as a function of the redshift $z$ for $|p| = 0.1$ (solid line), 0.01 (dashed line), 0.001 (dash-dotted line) and $\Omega_m^{(0)} = 0.26$ in the exponential $f(T)$ theory, where the left and right panels are for $p > 0$ and $p < 0$, respectively.

$H_0$ being the current Hubble parameter. We note that $\alpha$ in Eq. (3.5) has been derived from $\Omega_{\text{DE}}^{(0)} \equiv \rho_{\text{DE}}^{(0)}/\rho_{\text{crit}}^{(0)} = 1 - \Omega_m^{(0)} = -\alpha [1 - (1 - 2p) e^p]$ by using Eq. (2.12) with $T = T_0$ and $\rho_{\text{DE}}^{(0)} = \rho_{\text{DE}}(z = 0)$. We remark that the theory in Eq. (3.4) contains only one parameter $p$ if the value of $\Omega_m^{(0)}$ is given. The values of other dimensionless quantities at $z = 0$ can be calculated by using $p$ and $\Omega_m^{(0)}$, e.g., $\bar{m}^2/T_0 = \left(8\pi G \rho_m^{(0)}/3\right)/(-6H_0^2) = -\Omega_m^{(0)}/6$ and $y_H(z = 0) = H_0^2/\bar{m}^2 - 1 = 1/\Omega_m^{(0)} - 1$.

In Fig. 1 we depict the equation of state for dark energy $w_{\text{DE}}$ in Eq. (2.14) as a function of the redshift $z$ for $|p| = 0.1, 0.01, 0.001$. We have used $\Omega_m^{(0)} = 0.26$ [4] in all numerical calculations. From Fig. 1 we see that $w_{\text{DE}}$ does not cross the phantom divide line $w_{\text{DE}} = -1$ in the exponential $f(T)$ theory [18, 23]. In particular, for $p < 0$ the universe always stays in the non-phantom (quintessence) phase ($w_{\text{DE}} > -1$), whereas for $p > 0$ it in the phantom phase ($w_{\text{DE}} < -1$). The present values of $w_{\text{DE}}$ are $w_{\text{DE}}(z = 0) = -1.03, -1.003, -1.0003, -0.954, -0.996$ and $-0.999$ for $p = 0.1, 0.01, 0.001, p = -0.1, -0.01$ and $-0.001$, respectively. The larger $|p|$ is, the larger the deviation of the exponential $f(T)$ theory from the $\Lambda$CDM model is. We note that in solving Eq. (3.3) numerically, we have taken the initial conditions at $z = 0$ as $y_H(z = 0) = 2.8$. 

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FIG. 2: $w_{\text{DE}}$ as a function of $|T/T_0|$, where the thin solid line shows $w_{\text{DE}} = -1$ (cosmological constant). Legend is the same as Fig. 1.

FIG. 3: $|T/T_0|$ as a function of the redshift $z$ for $|p| = 0.1$.

We display $w_{\text{DE}}$ as a function of $|T/T_0|$ in Fig. 2 where the left and right panels are for $p > 0$ and $p < 0$, respectively. In Fig. 3 we show the cosmological evolution of $|T/T_0|$ as a function of the redshift $z$ for $|p| = 0.1$. The qualitative behaviors of $|T/T_0|$ for $|p| = 0.01$ and 0.001 are similar to those for $|p| = 0.1$. In addition, the cosmological evolution of $\rho_{\text{DE}}^* \equiv \rho_{\text{DE}} / \rho_{\text{DE}}(0)$ as a function of the redshift $z$ is shown in Fig. 4.

From Fig. 2 we find that the crossing of the phantom divide occurs around $|T/T_0| \approx 0.7 < 1$. This means that the crossing cannot be realized in the past and far future, unlike those in $f(R)$ models [29]. Explicitly, by denoting $|T/T_0|_{\text{cross}}$ being the crossing point, we have $|T/T_0|_{\text{cross}} \simeq 0.772, 0.744, 0.740, 0.689, 0.736$, and 0.740 for $p = 0.1, 0.01, 0.001, -0.1,$
FIG. 4: $\rho_{\text{DE}}^{\ast} \equiv \rho_{\text{DE}}/\rho_{\text{DE}}^{(0)}$ as a function of the redshift $z$, where the thin solid line shows $w_{\text{DE}} = -1$ (cosmological constant). Legend is the same as Fig. 1.

$-0.01$, and $-0.001$, respectively. Clearly, the universe reaches the line $w_{\text{DE}} = -1$ without a crossing even in the very far future ($z \simeq -1$).

In the high $z$ regime, the universe is at the matter-dominated stage and therefore, $|T| = 6H^2 = 16\pi G (\rho_m + \rho_{\text{DE}})$ decreases with time. Around the present time ($z = 0$), $\rho_m$ is smaller than $\rho_{\text{DE}}$ and dark energy becomes dominant over matter. Fig. 3 illustrates that $|T/T_0|$ decreases monotonously with $z$. At the infinite future, $|T/T_0|_{\text{cross}}$ is reached. However, the universe is in either the phantom phase ($p > 0$) or non-phantom phase ($p < 0$) without the crossing. Since by that time the contribution from non-relativistic matter and radiation is supposed to be negligible, such an evolution pattern of $|T/T_0|$ is driven by the $f(T)$ term.

In the future ($z < 0$), $\rho_{\text{DE}}$ arrives at the crossing point and finally becomes constant as shown in Fig. 4.

The right-hand side of the first equality in Eq. (2.14) implies that when $w_{\text{DE}}$ crosses the phantom divide line of $w_{\text{DE}} = -1$, the term $T' (f_T + 2T f_{TT}) / [3T (f/T - 2f_T)]$ has to flip the sign. Since the sign of $T$ is fixed to be negative due to $T = -6H^2$, the sufficient and necessary conditions for the crossing of the phantom divide is that the sign of the combination $T' (f_T + 2T f_{TT}) / (f/T - 2f_T)$ changes in the cosmological evolution. Note that the sign of $\dot{H}$ is opposite to $T'$ since $T' = -12HH' = -12\dot{H}$. When the energy density of dark energy becomes perfectly dominant over those of non-relativistic matter and radiation, one has $w_{\text{DE}} \approx w_{\text{eff}} \equiv -1 - 2\dot{H}/(3H^2) = P_{\text{tot}}/\rho_{\text{tot}}$, where $w_{\text{eff}}$ is the effective equation of state for the universe [9], and $\rho_{\text{tot}} \equiv \rho_{\text{DE}} + \rho_m + \rho_r$ and $P_{\text{tot}} \equiv P_{\text{DE}} + P_r$ are the total energy density and...
pressure of the universe, with $\rho$ and $P$ being the energy density and pressure of radiation, respectively. For $\dot{H} < 0$ ($> 0$), $w_{\text{eff}} > -1$ ($<-1$), representing the non-phantom (phantom) phase, while $w_{\text{eff}} = -1$ for $\dot{H} = 0$, corresponding to the cosmological constant. Clearly, the physical reason why the crossing of the phantom divide appears is that the sign of $\dot{H}$ changes from negative (the non-phantom phase) to positive (the phantom phase) due to the dominance of dark energy over non-relativistic matter and radiation.

We now investigate why the crossing of the phantom divide cannot occur in the exponential $f(T)$ theory in Eq. (3.4) for $p \leq 1$. From Eq. (3.4), we obtain

$$f_T = \alpha \left( 1 - e^{pT_0/T} + \frac{pT_0}{T} e^{pT_0/T} \right),$$  \hspace{1cm} (3.6)

$$f_{TT} = -\alpha \left( \frac{pT_0}{T} \right)^2 \frac{1}{T} e^{pT_0/T}. \hspace{1cm} (3.7)$$

To illustrate our results, we only concentrate on the limits of $0 < p \ll 1$ and $X \equiv pT_0/T \ll 1$. In this case, $T_0/T \lesssim 1$, which corresponds to the region from the far past to the near future. Consequently, Eqs. (3.6), (3.8) and (3.10) are approximately expressed as

$$f_T \approx -\alpha \left( X + \frac{X^2}{2} \right), \quad f_T \approx \frac{\alpha X^2}{2}, \quad T f_{TT} \approx -\alpha X^2.$$  \hspace{1cm} (3.8)

By using these equations, we find

$$\frac{f}{T} - 2f_T \approx -\alpha X \left( 1 + \frac{3X}{2} \right), \hspace{1cm} (3.9)$$

$$f_T + 2T f_{TT} \approx -\frac{3\alpha X^2}{2}. \hspace{1cm} (3.10)$$

From Eqs. (3.9) and (3.10), we see that the signs of both $f/T - 2f_T$ and $f_T + 2T f_{TT}$ do not change. Note that the sign of $T'$ is also unchanged because $|T/T_0|$ decreases monotonously with $z$ until $|T/T_0|_{\text{min}}$ as shown in Fig. 3. As a result, the crossing of the phantom divide cannot be realized in the exponential $f(T)$ theory.

In Fig. 4, we illustrate the fractional densities of dark energy ($\Omega_{\text{DE}} \equiv \rho_{\text{DE}}/\rho_{\text{crit}}^{(0)}$), non-relativistic matter ($\Omega_m \equiv \rho_m/\rho_{\text{crit}}^{(0)}$) and radiation ($\Omega_r \equiv \rho_r/\rho_{\text{crit}}^{(0)}$) as functions of the redshift $z$ for $|p| = 0.1$. The cosmological evolutions of $\Omega_{\text{DE}}$, $\Omega_m$ and $\Omega_r$ for $|p| = 0.01$ and $0.001$ are similar to those for $|p| = 0.1$. In order to analyze not only non-relativistic matter and dark energy but also radiation, we have included the contribution from radiation in Eq. (3.1) and used the following variable [26]:

$$y_H = \frac{H^2}{m^2} - a^{-3} - \chi a^{-4}, \hspace{1cm} (3.11)$$
FIG. 5: $\Omega_{\text{DE}}$ (dashed line), $\Omega_m$ (solid line) and $\Omega_r$ (dash-dotted line) as functions of the redshift $z$ for $|p| = 0.1$. Legend is the same as Fig. 1.

with

$$\chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4},$$  \hspace{1cm} (3.12)

where $\rho_r^{(0)}$ is the energy density of radiation at the present time.

In the high $z$ regime ($z \gtrsim 3225$), the universe is at the radiation-dominated stage ($\Omega_r \gg \Omega_{\text{DE}}$, $\Omega_r > \Omega_m$). As $z$ decreases, the universe enters the matter-dominated stage ($\Omega_m > \Omega_{\text{DE}}$, $\Omega_m \gg \Omega_r$). After that, eventually dark energy becomes dominant over matter for $z < z_{\text{DE}}$, where $z_{\text{DE}}$ is the crossover point in which $\Omega_{\text{DE}} = \Omega_m$. Explicitly, we have $z_{\text{DE}} = 0.40$ and 0.44 for $p = 0.1$ and $-0.1$, respectively. At the present time ($z = 0$), as an initial condition we have used $(\Omega_{\text{DE}}^{(0)}, \Omega_m^{(0)}, \Omega_r^{(0)}) = (0.74, 0.26, 8.1 \times 10^{-5})$. Note that $z_{\text{DE}} = 0.42$ in the $\Lambda$CDM model. As a result, the current accelerated expansion of the universe following the radiation-dominated and matter-dominated stages can be achieved in the exponential $f(T)$ theory.

B. Logarithmic $f(T)$ theory

In this subsection, we examine a logarithmic $f(T)$ theory, given by

$$f(T) = \beta T_0 \left( \frac{qT_0}{T} \right)^{-1/2} \ln \left( \frac{qT_0}{T} \right),$$  \hspace{1cm} (3.13)

with

$$\beta \equiv \frac{1 - \Omega_m^{(0)}}{2q^{-1/2}},$$  \hspace{1cm} (3.14)
where \( q \) is a positive constant. We note that the theory in Eq. (3.13) contains only one parameter \( q \) if the value of \( \Omega_m^{(0)} \) is obtained in the same way as the exponential \( f(T) \) theory.

In Fig. 6, we draw \( w_{\text{DE}} \) as functions of \( z \) (left panel) and \(|T/T_0|\) (right panel) for \( q = 1 \) and \( \Omega_m^{(0)} = 0.26 \) in the logarithmic \( f(T) \) theory in Eq. (3.13). From Fig. 6, we observe that \( w_{\text{DE}} \) does not cross the phantom divide line \( w_{\text{DE}} = -1 \), similar to the exponential theory. In this logarithmic theory, the universe is always in the non-phantom phase (\( w_{\text{DE}} > -1 \)) and \(|T/T_0|\) decreases monotonously with \( z \). The present value of \( w_{\text{DE}} \) is \( w_{\text{DE}}(z = 0) = -0.79 \) for \( q = 1 \). The right figure in Fig. 6 indicates that the crossing of the phantom divide occurs at \( |T/T_0|_{\text{cross}} = 0.547600 \) and the minimum of \(|T/T_0|\) is \(|T/T_0|_{\text{min}} = 0.547600 \). Thus, the universe does not cross the line of \( w_{\text{DE}} = -1 \) even in the very far future (\( z \simeq -1 \)).

We remark that \( w_{\text{DE}} \) is qualitatively independent of the value of \( q \). This is because in the logarithmic \( f(T) \) theory, \( w_{\text{DE}} \) and \( \rho_{\text{DE}} \) can be expressed as

\[
w_{\text{DE}} = -\frac{1}{2 - \left(1 - \Omega_m^{(0)}\right) \left(T/T_0\right)^{-1/2}} \tag{3.15}
\]

and

\[
\rho_{\text{DE}} = -\frac{T}{16\pi G} \left(1 - \Omega_m^{(0)}\right) \left(\frac{T}{T_0}\right)^{-1/2}, \tag{3.16}
\]

respectively. Similar to the exponential \( f(T) \) theory, it is easy to show that the crossing of the phantom divide cannot occur in the logarithmic \( f(T) \) theory.
As shown in the previous section, the universe always stays in the phantom and non-phantom phases in the exponential \((p > 0)\) and logarithmic \(f(T)\) theories, respectively. In this section, we investigate the cosmological evolution in a combined \(f(T)\) theory with both logarithmic and exponential terms and check if the crossing of the phantom divide can happen in this combined theory. The explicit form of the theory is given by

\[
f(T) = \gamma \left[ T_0 \left( \frac{uT_0}{T} \right)^{-1/2} \ln \left( \frac{uT_0}{T} \right) - T \left( 1 - e^{uT_0/T} \right) \right]
\]

with

\[
\gamma \equiv \frac{1 - \Omega_m^{(0)}}{2u^{-1/2} + [1 - (1 - 2u)e^u]},
\]

where \(u\) is a constant. Here, we have taken \(p = q = u > 0\) to simplify our discussions when we combine Eqs. (3.4) and (3.13). We note that the model in Eq. (4.1) contains only one parameter \(u\) if one has the value of \(\Omega_m^{(0)}\) like the exponential and logarithmic models.

In Fig. 7 we plot the equation of state for dark energy \(w_{DE}\) as a function of the redshift \(z\) for \(u = 1, 0.8\) and \(0.5\) and \(\Omega_m^{(0)} = 0.26\) in the combined \(f(T)\) theory in Eq. (4.1). As seen from Fig. 7 it is clear that \(w_{DE}\) can cross \(w_{DE} = -1\) at \(z = z_{cross}\). The universe evolves from the non-phantom phase \((w_{DE} > -1)\) to the phantom one \((w_{DE} < -1)\) and finally approaches \(w_{DE} = -1\). The present values of \(w_{DE}\) are \(w_{DE}(z = 0) = -1.1, -1.1\) and \(-0.94\) for \(u = 1, 0.8\) and \(0.5\), respectively. The crossing values of \(z_{cross}\) are 0.70 and 0.36 for \(u = 1\) and 0.8, respectively. It is interesting to note that such a crossing behavior is consistent with the recent cosmological observational data [27], but it is opposite to those in the viable \(f(R)\) theories [29]. We note that for \(u \leq 0.5\), the universe asymptotically approaches \(w_{DE} = -1\) without crossing it even in the very far future. The cosmological evolution of \(w_{DE}\) as a function of \(|T/T_0|\) is shown in Fig. 8 while that of \(|T/T_0|\) as a function of \(z\) for \(u = 1\) is depicted in Fig. 9. In addition, \(\rho_{DE}^* \equiv \rho_{DE}/\rho_{DE}^{(0)}\) as a function of \(z\) is given in Fig. 10.
FIG. 7: $w_{\text{DE}}$ as a function of the redshift $z$ for $u = 1$ (solid line), 0.8 (dashed line), 0.5 (dash-dotted line) and $\Omega_{m}^{(0)} = 0.26$ in the combined $f(T)$ theory with both logarithmic and exponential terms in Eq. (4.1), where the thin solid line shows $w_{\text{DE}} = -1$ (cosmological constant).

FIG. 8: $w_{\text{DE}}$ as a function of $|T/T_0|$. Legend is the same as Fig. 7.

We now demonstrate that the crossing of the phantom divide can occur in the combined $f(T)$ theory in Eq. (4.1). In this theory, both signs of $T$ and $T'$ are unchanged like the
FIG. 9: $|T/T_0|$ as a function of the redshift $z$ for $u = 1$. The range of $-0.95 \leq z \leq -0.40$ is magnified in the small window. Legend is the same as Fig. 7.

FIG. 10: $\rho_{DE}^* \equiv \rho_{DE}/\rho_{DE}^{(0)}$ as a function of the redshift $z$. Legend is the same as Fig. 7.

exponential $f(T)$ theory. From Eq. (4.1), we obtain

$$f_T = -\frac{\gamma}{u} \sqrt{\frac{uT_0}{T}} \left( 1 - \frac{1}{2} \ln \left( \frac{uT_0}{T} \right) \right) - \gamma \left( 1 - e^{uT_0/T} + \frac{uT_0}{T} e^{uT_0/T} \right),$$

$$f_{TT} = -\frac{\gamma}{4u^2T_0} \left( \frac{uT_0}{T} \right)^{3/2} \ln \left( \frac{uT_0}{T} \right) + \frac{\gamma}{uT_0} \left( \frac{uT_0}{T} \right)^{3} e^{uT_0/T},$$

(4.3)
leading to

\[ \frac{f}{T} - 2f_T \approx \gamma \sqrt{\frac{u T_0}{T}} \left[ \frac{2}{u} + \sqrt{\frac{u T_0}{T}} \left( 1 + \frac{3}{2} \frac{u T_0}{T} \right) \right], \]

\[ f_T + 2T f_{TT} \approx -\gamma \sqrt{\frac{u T_0}{T}} \left[ \frac{1}{u} - \frac{3}{2} \left( \frac{u T_0}{T} \right)^{3/2} \right], \quad (4.4) \]

where we have only concentrated on the case of \( u T_0 / T < 1 \) to simplify our discussion. From Eq. (4.4), we see that the sign of \( f/T - 2f_T \) is fixed, while that of \( f_T + 2T f_{TT} \) can change when \( u T_0 / T \) becomes larger or smaller than the value of \( u T_0 / T = [2 / (3u)]^{2/3} \). Hence, the sufficient and necessary conditions for the crossing of the phantom divide in the combined \( f(T) \) theory can be satisfied.

In Fig. 11 we demonstrate the fractional densities of dark energy (\( \Omega_{DE} \equiv \rho_{DE} / \rho_{\text{crit}}^{(0)} \)), non-relativistic matter (\( \Omega_m \equiv \rho_m / \rho_{\text{crit}}^{(0)} \)) and radiation (\( \Omega_r \equiv \rho_r / \rho_{\text{crit}}^{(0)} \)) as functions of \( z \) for \( u = 1 \). The evolutions of \( |T/T_0|, \Omega_{DE}, \Omega_m \) and \( \Omega_r \) for \( u = 0.8 \) and \( u = 0.5 \) are similar to those for \( u = 1 \). In the high \( z \) regime (\( z \gtrsim 3225 \)), the universe is at the radiation-dominated stage (\( \Omega_r \gg \Omega_{DE}, \Omega_r > \Omega_m \)). As \( z \) decreases, the matter-dominated stage (\( \Omega_m > \Omega_{DE}, \Omega_m \gg \Omega_r \)) follows, and eventually dark energy becomes dominant over matter for \( z < z_{DE} \). Explicitly, we have \( z_{DE} = 0.36, 0.40 \) and 0.49 for \( u = 1, 0.8 \) and 0.5, respectively. As a consequence,
after the radiation and matter-dominated stages, the current accelerated expansion of the universe can be realized in the combined $f(T)$ theory in Eq. (4.1), similar to that in the exponential $f(T)$ theory.

Finally, we remark that since the gravitational field equation in $f(R)$ gravity has the fourth-order nature, it is possible to reproduce an arbitrary background expansion history, provided that we do not consider the stability issue. On the other hand, in the modified Gauss-Bonnet or $f(G)$ gravities, the ΛCDM background expansion history cannot be achieved all the way down to the far future. In $f(T)$ gravity, there exists a similar restriction. Physically, in such modified gravity theories the modified background expansion rate $H$ is in general a function of $H$ itself and $\dot{H}$, which means that arbitrary expansion histories are not guaranteed to be realizable.

V. OBSERVATIONAL CONSTRAINTS ON THE COMBINED $f(T)$ THEORY

In Sec. IV, we have constructed the combined $f(T)$ theory with both logarithmic and exponential terms to realize the crossing of the phantom divide. Since it is difficult to obtain the information on the form of $f(T)$ from more fundamental theories, it is important to examine whether this theory is compatible with the recent observational data. In this section we demonstrate how well the combined $f(T)$ theory can fit the observational data. We examine the constraints on the model parameter $u$ and the current fractional non-relativistic matter density $\Omega_m^{(0)}$ by taking the $\chi^2$ method for the recent observational data. We use type Ia supernovae (SNe Ia) data from the Supernova Cosmology Project (SCP) Union2 compilation, the baryon acoustic oscillations (BAO) data from the Two-Degree Field Galaxy Redshift Survey (2dFGRS) and the Sloan Digital Sky Survey Data Release 7 (SDSS DR7), and the cosmic microwave background (CMB) radiation data from Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations. The detailed analysis method for the observational data of SNe Ia, BAO and CMB is summarized in Appendix A.

We plot the contours of the 68.27% (1σ), 95.45% (2σ) and 99.73% (3σ) confidence levels (CL) in the $(\Omega_m^{(0)}, u)$ plane from the SNe Ia, BAO and CMB data for the combined
FIG. 12: Contours of 68.27% (1σ), 95.45% (2σ) and 99.73% (3σ) confidence levels in the \((\Omega_m^0, u)\) plane from SNe Ia, BAO and CMB data for the combined \(f(T)\) theory, where the plus sign depicts the best-fit point.

TABLE I: The best-fit values of \(u, \Omega_m^0, h\) and \(\chi^2_{\text{min}}\) for the combined \(f(T)\) and \(\Lambda CDM\) models.

| Model  | \(u\)  | \(\Omega_m^0\) | \(h\)  | \(\chi^2_{\text{min}}\) |
|--------|--------|----------------|--------|-------------------|
| \(f(T)\) | 0.829  | 0.282         | 0.691  | 544.56            |
| \(\Lambda CDM\) | 0.275  | 0.707         |        | 545.23            |

\(f(T)\) theory in Fig. 12 In this figure, we have fitted three parameters of \(u, \Omega_m^0\) and \(h \equiv H_0/100/[\text{km sec}^{-1}\text{Mpc}^{-1}]\) [30]. The model parameter \(u\) is constrained to 0.6 < \(u\) < 1.13 (95.45% CL) whereas 0.255 < \(\Omega_m^0\) < 0.312 (95.45% CL), which is consistent with the current observations. The best-fit value, i.e., the minimum \(\chi^2\), in the parameter space is \(\chi^2_{\text{min}} = 544.56\) with \(u = 0.829\), \(\Omega_m^0 = 0.282\) and \(h = 0.691\). In Table I, we present the best-fit values of \(u, \Omega_m^0\) and \(h\), and \(\chi^2_{\text{min}}\) for the combined \(f(T)\) theory along with those for the \(\Lambda CDM\) model. From Table I we see that \(\chi^2_{\text{min}}\) of the combined \(f(T)\) theory is slightly smaller than that of the \(\Lambda CDM\) model. This implies that the combined \(f(T)\) theory can fit the observational data well.

We remark that for around the best-fit value of the model parameter \(u \approx 0.8\), \(w_{\text{DE}}\) can cross the phantom divide line of \(w_{\text{DE}} = -1\) from the non-phantom phase \((w_{\text{DE}} > -1)\) to the
phantom one \((w_{\text{DE}} < -1)\) in the combined \(f(T)\) theory as shown in Fig. 7 and therefore the observational data may favor the crossing of the phantom divide. This consequence agrees with the one in Ref. [22].

VI. CONCLUSIONS

We have investigated the cosmological evolution in the exponential \(f(T)\) theory, which is one of the simplest \(f(T)\) theories as it contains only one model parameter \(p\) along with the current fractional density of non-relativistic matter \(\Omega_{m}^{(0)}\). We have explicitly shown that the phase of the universe depends on the sign of the parameter \(p\), i.e., for \(p < 0(>0)\) the universe is always in the non-phantom (phantom) phase without the crossing of the phantom divide. We have presented another simplest \(f(T)\) model, which is the logarithmic type. Similar to the exponential model, the logarithmic one has only one free parameter \(q\), and it does not allow the crossing of the phantom divide. To realize the crossing of the phantom divide, we have constructed a \(f(T)\) theory by combining the logarithmic and exponential terms. In particular, we have shown that the crossing in the combined \(f(T)\) theory is from \(w_{\text{DE}} > -1\) to \(w_{\text{DE}} < -1\), which is opposite to the typical manner in \(f(R)\) gravity models. Furthermore, we have also illustrated that this combined theory is consistent with the recent observational data of SNe Ia, BAO and CMB.

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Appendix A: Analysis method for the observational data

In this appendix, we explain the analysis method [30] for the observational data of type Ia supernovae (SNe Ia), baryon acoustic oscillations (BAO) and cosmic microwave background (CMB) radiation (for more detailed explanations on the data analysis, see, e.g., [37]).

1. Type Ia Supernovae (SNe Ia)

The information on the luminosity distance $D_L$ as a function of the redshift $z$ is given by SNe Ia observations. The theoretical distance modulus $\mu_{\text{th}}$ is defined by

$$\mu_{\text{th}}(z_i) \equiv 5 \log_{10} D_L(z_i) + \mu_0, \quad (A1)$$

where $\mu_0 \equiv 42.38 - 5 \log_{10} h$ with $h \equiv H_0/100/(\text{km sec}^{-1} \text{Mpc}^{-1})$ [36]. The Hubble-free luminosity distance for the flat universe is described as

$$D_L(z) = (1 + z) \int_0^z \frac{dz'}{E(z')}, \quad (A2)$$

where $E(z) \equiv H(z)/H_0$ with

$$H(z) = H_0 \sqrt{\Omega_m(0)} (1 + z)^3 + \Omega_r(0)(1 + z)^4 + \Omega_{\text{DE}}(0)(1 + z)^{3(1+w_{\text{DE}})}.$$

Here, $\Omega_r(0) = \Omega_{\gamma}(0)(1 + 0.2271 N_{\text{eff}})$, where $\Omega_{\gamma}(0)$ is the present fractional photon energy density and $N_{\text{eff}} = 3.04$ is the effective number of neutrino species [4]. We note that $H(z)$ is evaluated by using numerical solutions of Eq. (3.3).

The $\chi^2$ of the SNe Ia data is given by

$$\chi^2_{\text{SN}} = \sum_i \left[ \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i)}{\sigma_i^2} \right]^2, \quad (A4)$$

where $\mu_{\text{obs}}$ is the observed value of the distance modulus. In what follows, subscripts “th” and “obs” denote the theoretically predicted and observed values, respectively. $\chi^2_{\text{SN}}$ should be minimized with respect to $\mu_0$, which relates to the absolute magnitude, because the absolute magnitude of SNe Ia is not known. $\chi^2_{\text{SN}}$ in Eq. (A4) is expanded to be [38]

$$\chi^2_{\text{SN}} = A - 2\mu_0 B + \mu_0^2 C, \quad (A5)$$
with
\[ A = \sum_i \left[ \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0)}{\sigma_i^2} \right]^2, \quad B = \sum_i \left[ \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0)}{\sigma_i^2} \right], \quad C = \sum_i \frac{1}{\sigma_i^2}. \]  

(A6)

The minimum of \( \chi^2_{\text{SN}} \) with respect to \( \mu_0 \) is expressed as
\[ \chi^2_{\text{SN}} = A - \frac{B^2}{C}. \]  

(A7)

In our analysis, we take Eq. (A7) for the \( \chi^2 \) minimization and use the Supernova Cosmology Project (SCP) Union2 compilation, which contains 557 supernovae [34], ranging from \( z = 0.015 \) to \( z = 1.4 \).

2. Baryon Acoustic Oscillations (BAO)

The distance ratio of \( d_z \equiv r_s(z_d)/D_V(z) \) is measured by the observation of BAO. Here, \( D_V \) is the volume-averaged distance, \( r_s \) is the comoving sound horizon and \( z_d \) is the redshift at the drag epoch [35]. The volume-averaged distance \( D_V(z) \) is defined as [4]
\[ D_V(z) \equiv \left[ (1 + z)^2 D_A^2(z) \frac{z}{H(z)} \right]^{1/3}, \]  

(A8)

where \( D_A(z) \) is the proper angular diameter distance for the flat universe, defined by
\[ D_A(z) \equiv \frac{1}{1 + z} \int_0^z \frac{dz'}{H(z')} . \]  

(A9)

The comoving sound horizon \( r_s(z) \) is given by
\[ r_s(z) = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{a^2 H(z') = 1/a - 1}} \sqrt{1 + \left( \frac{3 \Omega_b^{(0)} / 4 \Omega_\gamma^{(0)}}{a^2} \right)} , \]  

(A10)

where \( \Omega_b^{(0)} = 2.2765 \times 10^{-2} h^{-2} \) and \( \Omega_\gamma^{(0)} = 2.469 \times 10^{-5} h^{-2} \) are the current values of baryon and photon density parameters, respectively [4]. The fitting formula for \( z_d \) is given by [39]
\[ z_d = \frac{1291(\Omega_m^{(0)} h^2)^{0.251}}{1 + 0.659(\Omega_m^{(0)} h^2)^{0.828}} \left[ 1 + b_1 \left( \Omega_b^{(0)} h^2 \right)^{0.2} \right], \]  

(A11)

with
\[ b_1 = 0.313(\Omega_m^{(0)} h^2)^{-0.419} \left[ 1 + 0.607 (\Omega_m^{(0)} h^2)^{0.674} \right], \quad b_2 = 0.238 (\Omega_m^{(0)} h^2)^{0.223}. \]  

(A12)
The typical value of $z_d$ is about 1021 for $\Omega_{m}^{(0)} = 0.276$ and $h = 0.705$.

According to the BAO data from the Two-Degree Field Galaxy Redshift Survey (2dF-GRS) and the Sloan Digital Sky Survey Data Release 7 (SDSS DR7) [35], the distance ratio $d_z$ at two redshifts $z = 0.2$ and $z = 0.35$ is measured to be $d_{z=0.2}^{\text{obs}} = 0.1905 \pm 0.0061$ and $d_{z=0.35}^{\text{obs}} = 0.1097 \pm 0.0036$ with the inverse covariance matrix:

$$C_{\text{BAO}}^{-1} = \begin{pmatrix} 30124 & -17227 \\ -17227 & 86977 \end{pmatrix}.$$ (A13)

The $\chi^2$ for the BAO data is described as

$$\chi^2_{\text{BAO}} = \left( x_{i,\text{BAO}} - x_{i,\text{BAO}}^{\text{obs}} \right) \left( C_{\text{BAO}}^{-1} \right)_{ij} \left( x_{j,\text{BAO}} - x_{j,\text{BAO}}^{\text{obs}} \right),$$ (A14)

where $x_{i,\text{BAO}} \equiv (d_{0.2}, d_{0.35})$.

### 3. Cosmic Microwave Background (CMB) radiation

The observational data of CMB are sensitive to the distance to the decoupling epoch $z_* \equiv [3]$. Hence, by using these data we obtain constraints on the model in the high redshift regime ($z \sim 1000$).

The acoustic scale $l_A$ and the shift parameter $\mathcal{R} \equiv [40]$ are defined by

$$l_A(z_*) \equiv (1 + z_*) \frac{\pi D_A(z_*)}{r_s(z_*)},$$ (A15)

$$\mathcal{R}(z_*) \equiv \sqrt{\Omega_m^{(0)} H_0 (1 + z_*) D_A(z_*)},$$ (A16)

where $z_*$ is the redshift of the decoupling epoch, given by [41]

$$z_* = 1048 \left[ 1 + 0.00124 \left( \Omega_b^{(0)} h^2 \right)^{-0.738} \right] \left[ 1 + g_1 \left( \Omega_m^{(0)} h^2 \right)^{0.238} \right],$$ (A17)

with

$$g_1 = \frac{0.0783 \left( \Omega_b^{(0)} h^2 \right)^{-0.238}}{1 + 39.5 \left( \Omega_b^{(0)} h^2 \right)^{0.763}}, \quad g_2 = \frac{0.560}{1 + 21.1 \left( \Omega_b^{(0)} h^2 \right)^{1.81}}.$$ (A18)

We use the data from Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations [4] on CMB.

The $\chi^2$ of the CMB data is

$$\chi^2_{\text{CMB}} = \left( x_{i,\text{CMB}} - x_{i,\text{CMB}}^{\text{obs}} \right) \left( C_{\text{CMB}}^{-1} \right)_{ij} \left( x_{j,\text{CMB}} - x_{j,\text{CMB}}^{\text{obs}} \right),$$ (A19)
where \( x_{i, \text{CMB}} \equiv ( l_A(z_*) , R(z_*) , z_*) \) and \( C_{\text{CMB}}^{-1} \) is the inverse covariance matrix. The data from WMAP7 observations \(^4\) lead to \( l_A(z_*) = 302.09 \), \( R(z_*) = 1.725 \) and \( z_* = 1091.3 \) with the inverse covariance matrix:

\[
C_{\text{CMB}}^{-1} = \begin{pmatrix}
2.305 & 29.698 & -1.333 \\
29.698 & 6825.27 & -113.180 \\
-1.333 & -113.180 & 3.414
\end{pmatrix} .
\] (A20)

Consequently, the \( \chi^2 \) of all the observational data is given by

\[
\chi^2 = \tilde{\chi}^2_{\text{SN}} + \chi^2_{\text{BAO}} + \chi^2_{\text{CMB}} .
\] (A21)

We note that in our fitting procedure, we take the simple \( \chi^2 \) method rather than the Markov chain Monte Carlo (MCMC) approach such as CosmoMC \(^4\).

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