Genus-2 Jacobians with torsion points of large order

Everett W. Howe

Abstract
We produce new explicit examples of genus-2 curves over the rational numbers whose Jacobian varieties have rational torsion points of large order. In particular, we produce a family of genus-2 curves over $\mathbb{Q}$ whose Jacobians have a rational point of order 48, parameterized by a rank-2 elliptic curve over $\mathbb{Q}$, and we exhibit a single genus-2 curve over $\mathbb{Q}$ whose Jacobian has a rational point of order 70, the largest order known. We also give new examples of genus-2 Jacobians with rational points of order 27, 28, and 39.

Most of our examples are produced by ‘gluing’ two elliptic curves together along their $n$-torsion subgroups, where $n$ is either 2 or 3. The 2-gluing examples arise from techniques developed by the author in joint work with Leprévost and Poonen 15 years ago. The 3-gluing examples are made possible by an algorithm for explicit 3-gluing over non-algebraically closed fields recently developed by the author in joint work with Bröker, Lauter, and Stevenhagen.

1. Introduction
In the late 1970s, Mazur [14–16] determined the fifteen groups that can appear as the group of rational torsion points on an elliptic curve over $\mathbb{Q}$. It is not known at present whether or not there are only finitely many groups (up to isomorphism) that occur as the rational torsion subgroups of Jacobians of genus-2 curves over $\mathbb{Q}$. Over the past 25 years, researchers have searched for genus-2 curves over $\mathbb{Q}$ whose Jacobians have torsion points of large order, and have found or constructed examples of curves whose Jacobians have rational torsion points of order $n$, for $1 \leq n \leq 30$, $32 \leq n \leq 36$, and $n \in \{39, 40, 45, 48, 60, 63\}$ (see [3, 4, 6, 9–13, 18–20]). In fact, there are infinite families of genus-2 Jacobians over $\mathbb{Q}$ with rational torsion points of order $n$ for $1 \leq n \leq 26$ and $n \in \{30, 32, 35, 40, 45, 60\}$; this can be seen from the references cited above, except for $n \in \{14, 16, 18, 22, 26\}$. For these values of $n$, the existence of infinite families was proved by Leprévost in an unpublished preprint. Leprévost obtained Jacobians with torsion points of order 14, 18, 22, and 26 by specializing families of genus-2 curves $y^2 = f$ with torsion points of order 7, 9, 11, and 13 so that the polynomial $f$ splits in a way that ensures the existence of a rational 2-torsion point, and he obtained an infinite family of curves with torsion points of order 16 by using the methods of [11].

In this paper, we present a genus-2 curve over $\mathbb{Q}$ whose Jacobian has a rational torsion point of order 70, the largest order yet discovered. We also exhibit five genus-2 curves whose Jacobians have a rational torsion point of order 28; previously, only two such curves were known [19, Theorem 4, p. 288], [21, Theorem 3, p. 320]. As we explain in Section 2, we obtain these curves by ‘gluing’ two elliptic curves together along their 3-torsion subgroup, using formulas from [1, Appendix].

We also show that there is an infinite family of genus-2 curves over $\mathbb{Q}$ whose Jacobians have a rational torsion point of order 48, the second-largest order for which an infinite family of curves is known. This family is produced by using the methods of [6], and we present it in Section 3.
Finally, by conducting a naïve search of genus-2 curves given by equations with small coefficients, we find four new examples of genus-2 curves whose Jacobians have rational torsion points of large order: three Jacobians that have a rational point of order 27 and one with a rational point of order 39. We present these curves in Section 4.

2. Torsion points of order 28 and 70

Let $E_1$ and $E_2$ be elliptic curves over $\mathbb{Q}$, and suppose that there is an isomorphism $\psi: E_1[3] \rightarrow E_2[3]$ of the 3-torsion subgroup-schemes of $E_1$ and $E_2$ that is an anti-isometry with respect to the Weil pairings on $E_1[3]$ and $E_2[3]$. Let $G$ be the graph of $\psi$, so that $G$ is a maximal isotropic subgroup of $(E_1 \times E_2)[3]$ with respect to the product of the Weil pairings. Let $A$ be the abelian surface $(E_1 \times E_2)/G$ and let $\varphi: E_1 \times E_2 \rightarrow A$ be the natural isogeny. Then there is a commutative diagram

$$
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{3} & E_1 \times E_2 \\
\downarrow \varphi & & \uparrow \varphi \\
A & \xrightarrow{\lambda} & \hat{A}
\end{array}
$$

Here the top arrow is the multiplication-by-3 map and $\hat{A}$ is the dual abelian surface of $A$. The existence of the isogeny $\lambda: A \rightarrow \hat{A}$ follows from the fact that $G$ is a maximal isotropic subgroup of the 3-torsion of $E_1 \times E_2$ (see [17, Proposition 16.8, p. 135]). By considering the degrees of the other maps in the diagram, we see that $\lambda$ is an isomorphism; furthermore, it is a polarization. Thus, $(A, \lambda)$ is a principally polarized abelian surface, so it is the Jacobian of a possibly singular curve $C$. A result of Kani [7, Theorem 3, p. 95] shows that $C$ will be singular if and only if $\psi$ is the restriction to $E_1[3]$ of a 2-isogeny $E_1 \rightarrow E_2$.

It is straightforward to show that if $C$ is a genus-2 curve over $\mathbb{Q}$ whose polarized Jacobian is $(3,3)$-isogenous over $\mathbb{Q}$ to a product of two elliptic curves $E_1$ and $E_2$ over $\mathbb{Q}$ with the product polarization, then $C$ can be obtained from this construction for some anti-isometry $\psi: E_1[3] \rightarrow E_2[3]$; the argument is an easy variant of the proof of [5, Lemma 7, p. 1684].

Suppose that $E_1$ and $E_2$ have rational torsion points of order $N_1$ and $N_2$, respectively, and suppose that $C$ is a curve whose Jacobian is $(3,3)$-isogenous to $E_1 \times E_2$. If $N_1$ and $N_2$ are coprime to one another and neither is divisible by 3, then $\text{Jac} C$ has a torsion point of order $N_1 N_2$. If $N_1$ and $N_2$ are both divisible by 3 and if $(N_1/3, N_2/3) = 1$, then $\text{Jac} C$ will have a torsion point of order $N_1 N_2/3$.

By Mazur’s theorem, the possible values of $N_1$ and $N_2$ for elliptic curves over $\mathbb{Q}$ are the integers from 1 to 12, excluding 11. The only $(N_1, N_2)$ pairs that will possibly give us new orders of torsion in genus-2 Jacobians, or orders for which we have only finitely many examples, are $(4,7), (7,8)$, and $(7,10)$.

Algorithm 5.4 of [1] (which we will refer to as the ‘3-gluing algorithm’) takes as input a pair of elliptic curves $E_1$ and $E_2$ over a base field $k$, and outputs the list of all of the genus-2 curves $C$ over $k$ whose Jacobians are $(3,3)$-isogenous (over $k$) to the product $E_1 \times E_2$. As we have noted, such curves $C$ will exist only when there is an anti-isometry between the group schemes $E_1[3]$ and $E_2[3]$. The existence of such an anti-isometry implies that the mod-$3$ Galois representations attached to $E_1$ and $E_2$ are isomorphic, so before we apply the 3-gluing algorithm to a pair of elliptic curves over $\mathbb{Q}$, it makes sense to first check, for several primes $\ell$ of good reduction, that the mod-$\ell$ reductions of the two curves have traces of Frobenius that are congruent modulo 3.

For our pool of candidate elliptic curves, we combined two databases of curves over $\mathbb{Q}$: Cremona’s database [2] of all elliptic curves of conductor at most 339,999, and the Stein–Watkins database [22] of certain elliptic curves of conductor at most $10^8$ and certain elliptic curves of prime conductor at most $10^{10}$. For the pairs $(N_1, N_2)$ of interest to us, we went
through the combined databases and made a list of the curves with $N_1$-torsion points and a list of the curves with $N_2$-torsion points. Then, for every $E_1$ in the first list and $E_2$ in the second list, we used the BHLS 3-gluing algorithm to try to glue $E_1$ to $E_2$ along their 3-torsion subgroups. Our results follow. (We also tried using elliptic curves with large torsion subgroups produced by specializing the universal elliptic curves with $N$-torsion. We used the models for the universal curves given in [6, Table 3, p. 219], which are based on Kubert’s curves [8, Table 3, p. 217], and we let the parameter $t$ run through all rational numbers of height at most 1000. This additional pool of elliptic curves did not lead us to any further examples.)

2.1. **Torsion points of order 4 and 7**

For $N_1 = 4$ and $N_2 = 7$, we found five pairs of curves that we could glue together.

**Theorem 2.1.** The Jacobian of each of the following genus-2 curves over $\mathbb{Q}$ has a rational torsion point of order 28:

- $C_{28,1}: y^2 + (x^2 + x)y = x^6 + 3x^5 + 5x^4 - 4x^2 - 10x + 4$,
- $C_{28,2}: y^2 + (x^2 + x)y = x^6 + 3x^5 + 3x^4 + 13x^3 - 6x^2 + 18x$,
- $C_{28,3}: y^2 + (x^2 + x)y = 4x^6 - 2x^5 + 18x^4 + 3x^3 + 13x^2 + 23x - 11$,
- $C_{28,4}: y^2 + (x^2 + x)y = 28320768x^6 + 167100960x^5 + 213557586x^4 - 35302844x^3 + 154134546x^2 - 155174208x + 40064896$,
- $C_{28,5}: y^2 + (x^2 + x)y = 25x^6 - 455x^5 + 1675x^4 + 2494x^3 + 570x^2 - 1210x$.

**Proof.** Table 1 lists five pairs of elliptic curves over $\mathbb{Q}$. The first curve in each pair has a rational torsion point of order 4, and the second of each pair has a rational torsion point of order 7; torsion points of these orders are listed in the fourth column of the table. Applying the 3-gluing algorithm to the $i$th pair and applying Magma’s ReducedMinimalWeierstrassModel to the output gives us the genus-2 curve $C_{28,i}$ listed in the statement of the theorem.

**Remark 2.2.** In fact, Magma’s TorsionSubgroup command shows that the Jacobian of each of these curves has no rational torsion other than that generated by the point of order 28.

| #  | Cremona label | Equation                          | Torsion point   |
|----|---------------|-----------------------------------|----------------|}
| 1  | 182a1         | $y^2 = x^3 + 1122741x + 310814982$ | (891, 44928)   |
| 2  | 26b1          | $y^2 = x^3 - 3483x + 121014$      | (27, 216)      |
| 2  | 294c1         | $y^2 = x^3 - 255339x - 109668762$ | (1659, 63504)  |
| 2  | 294b2         | $y^2 = x^3 - 182763x + 31201254$  | (219, 1296)    |
| 3  | 490h1         | $y^2 = x^3 + 146853x + 34506486$  | (215, 10584)   |
| 3  | 490k2         | $y^2 = x^3 + 1190133x + 257487174$| (−117, 10800)  |
| 4  | 1518s1        | $y^2 = x^3 - 215054379x + 2013507848358$ | (9579, 912384) |
| 4  | 858k1         | $y^2 = x^3 - 7483623723x + 249446508217254$ | (48459, 769824) |
| 5  | 193930c1      | $y^2 = x^3 - 8212844907x - 260196865770906$ | (−51237, 5108400) |
| 5  | 4730k1        | $y^2 = x^3 - 7234611147x + 236852477159814$ | (48843, 118800) |

**Table 1. Pairs of elliptic curves that can be glued along their 3-torsion.**
Remark 2.3. As we noted, a result of Kani shows that an anti-isometry $\psi : E_1[3] \to E_2[3]$ between the 3-torsion of two elliptic curves gives rise (via the construction sketched earlier) to a non-singular curve if and only if $\psi$ is not the restriction to $E_1[3]$ of a 2-isogeny from $E_1$ to $E_2$. Therefore, there are two ways for a prime $p$ to be a prime of bad reduction for a curve over $\mathbb{Q}$ produced in this way: First, $p$ may be a prime of bad reduction for $E_1$ or $E_2$; and second, $p$ may be a prime such that the reduction of $\psi$ modulo $p$ is the restriction to $E_{1,p}[3]$ of a 2-isogeny $E_{1,p} \to E_{2,p}$ of the elliptic curves modulo $p$. For example, the curve $C_{28,4}$ has bad reduction at 439 for the latter reason. The bad reduction of this curve at 439 makes the comparatively large coefficients of its reduced model more understandable.

2.2. Torsion points of order 7 and 8
For $N_1 = 7$ and $N_2 = 8$, we found no pairs of elliptic curves from the Cremona and Stein–Watkins databases that we could glue together.

2.3. Torsion points of order 7 and 10
For $N_1 = 7$ and $N_2 = 10$, we found exactly one pair of elliptic curves from the Cremona and Stein–Watkins databases that we could glue together.

Theorem 2.4. Let $C_{70}$ be the genus-2 curve
\[ y^2 + (2x^3 - 3x^2 - 41x + 110)y = x^3 - 51x^2 + 425x + 179 \]
over $\mathbb{Q}$. The Jacobian of $C_{70}$ has a rational torsion point of order 70.

Proof. Let $E_1$ and $E_2$ be the following two elliptic curves:
\begin{align*}
(858k1) & \quad y^2 = x^3 - 7483623723x + 249446508217254 \\
(66c2) & \quad y^2 = x^3 + 149013x + 25726950.
\end{align*}
Then $(48459,769824)$ is a torsion point of order 7 on $E_1$, and $(147,7128)$ is a torsion point of order 10 on $E_2$. The 3-gluing algorithm, applied to these curves, gives a single genus-2 curve as its output. We obtain the equation for $C_{70}$ given in the theorem by applying Magma’s ReducedMinimalWeierstrassModel function to the curve produced by the algorithm, and then shifting $y$ by polynomials in $x$, and shifting $x$ by constants, in order to reduce the size of coefficients. □

Remark 2.5. Let $P_1$ and $P_2$ be the two points at infinity on $C_{70}$. One can check that the divisor $(2,17) + (4,23) - P_1 - P_2$ represents a point of order 70 on the Jacobian of $C_{70}$.

Remark 2.6. Magma’s TorsionSubgroup command shows that the Jacobian has no rational torsion points other than the multiples of this 70-torsion point.

3. Torsion points of order 48
In this section, we use the techniques of [6] to produce a family of genus-2 curves over $\mathbb{Q}$, parameterized by an elliptic curve over $\mathbb{Q}$ of rank 2, whose members all have Jacobians with a rational torsion point of order 48. Prior to this work, only two examples of such curves had appeared in the literature [20, Theorem 3, p. 643].

First we construct a 1-parameter family of genus-2 curves whose Jacobians have a rational torsion point of order 24. The construction works over any field whose characteristic is neither
2, 3, nor 7. For the remainder of this section, we fix such a field $K$, and we let $\overline{K}$ denote a separable closure of $K$.

**Theorem 3.1.** Let $s$ be an element of the field $K$, and set
\[
\begin{align*}
c_4 &= -31(s^4 + 42s^2 - (32\,200/93)s - 147), \\
c_2 &= 2^8(s^8 + 84s^6 - (3472/3)s^5 + 1470s^4 - 48\,608s^3 + 53\,508s^2 + 170\,128s + 21\,609), \\
c_0 &= (2^{20} \cdot 7/3)s(s^2 + 7)^3(s^2 + 63), \\
d &= s^4 + 42s^2 + (1736/3)s - 147.
\end{align*}
\] Suppose $c_0d \neq 0$. Then the equation
\[dy^2 = x^6 + c_4x^4 + c_2x^2 + c_0\]
defines a non-singular genus-2 curve $C_{24}^s$ over $K$, and the Jacobian of $C_{24}^s$ has a $K$-rational torsion point of order 24.

**Proof.** Let $g$ be the polynomial $x^3 - 31x^2 + 256x$, so that the roots of $g$ in $\overline{K}$ are
\[
\beta_1 = 0, \quad \beta_2 = (31 - 3r)/2, \quad \text{and} \quad \beta_3 = (31 + 3r)/2,
\]
where $r \in \overline{K}$ is a square root of $-7$. Let $F$ be the elliptic curve $y^2 = g$ over $K$. Note that $Q = (32, 96)$ is a torsion point of order 8 on $F$, and that $4Q = (0, 0)$.

Let $s$ be as in the statement of the theorem. Set
\[
a = \frac{-8(s^4 + 42s^2 - 147)}{(s^2 + 63)^2} \quad \text{and} \quad b = \frac{16(s^2 + 7)^3}{(s^2 + 63)^3},
\]
and let $f = x(x^2 + ax + b)$, so that the roots of $f$ in $\overline{K}$ are
\[
\alpha_1 = 0, \quad \alpha_2 = \frac{4(s + r)^3(s - 3r)}{(s^2 + 63)^2}, \quad \text{and} \quad \alpha_3 = \frac{4(s - r)^3(s + 3r)}{(s^2 + 63)^2}.
\]
The assumption that $c_0d \neq 0$ shows that $f$ is separable. Let $E$ be the elliptic curve $y^2 = f$, and note that
\[
P = \left(\frac{4(s^2 + 7)}{s^2 + 63}, \frac{224(s^2 + 7)}{(s^2 + 63)^2}\right)
\]
is a torsion point of order 6 on $E$, and that $3P = (0, 0)$.

Let $\psi : E[2] \to F[2]$ be the Galois-module isomorphism that sends $(\alpha_i, 0)$ to $(\beta_i, 0)$, for $i = 1, 2, 3$. We check that the condition that $d \neq 0$ shows that $\psi$ is not the restriction to $E[2]$ of an isogeny $E \to F$.

Proposition 4 (p. 324) of [6] shows how to glue $E$ and $F$ together along their 2-torsion subgroups using $\psi$ to get a genus-2 curve $C$ whose Jacobian is isomorphic to $(E \times F)/G$, where $G$ is the graph of $\psi$. The formulas in [6, Proposition 4] show that $C$ is given by an equation $y^2 = h$, for an explicit sextic polynomial $h \in K[x]$. If we take that model for $C$ and replace $x$ and $y$ with
\[
\frac{x}{2^3 \cdot (s^2 + 63)} \quad \text{and} \quad \frac{2^{16} \cdot 3^3 \cdot 7^3 \cdot s(s^2 + 7)^3 d^2 y}{(s^2 + 63)^{11}},
\]
respectively, we wind up with the equation for $C_{24}^s$ in the statement of the theorem.

Let $R$ be the point $(2P, Q)$ on $E \times F$. The smallest positive integer $n$ such that $nR$ lies in the kernel $G$ of the natural map $E \times F \to \text{Jac} \, C_{24}^s$ is $n = 24$, so the image of $R$ in $(\text{Jac} \, C_{24}^s)(K)$ is a point of order 24. \hfill $\Box$

Next we show that for some values of $s$, the point of order 24 on $(\text{Jac} \, C_{24}^s)(K)$ constructed at the end of the preceding proof is the double of a point of order 48.
Theorem 3.2. Let $s$ be an element of the field $K$ such that the quantity $c_{0d}$ from Theorem 3.1 is non-zero. Let $D$ be the elliptic curve $y^2 = x^3 + 14x^2 + 196x$ over $K$. Suppose that there is a non-zero point $(z, w) \in D(K)$ such that
\[ s = -21 \frac{(z^2 + 196)(z^2 + 56z + 196) + 32(z + 14)(z - 14)w}{z^4 - 896z^3 - 24696z^2 - 175616z + 38416}. \] (3.1)

Then there is a $K$-rational point of order 48 on the Jacobian of the curve $C_{24}$ from Theorem 3.1.

Remark 3.3. In the case $K = \mathbb{Q}$, the Mordell–Weil group of $D$ has rank 2; it is generated by the 2-torsion point $P_1 = (0, 0)$ and the independent points $P_2 = (7, -49)$ and $P_3 = (16, -104)$ of infinite order. The right-hand side of equation (3.1) is a degree-16 function on $D$, so each $s \in \mathbb{Q}$ arises from at most sixteen points of $D(Q)$. It follows that there are infinitely many genus-2 curves over $\mathbb{Q}$ whose Jacobians have a rational torsion point of order 48.

Proof of Theorem 3.2. Let notation be as in the proof of Theorem 3.1, and let $A$ be the abelian surface $E \times F$. The point $(3P, Q)$ on $A$ maps to a point of order 8 on $J = \text{Jac} C_{24}$. Proposition 12 (p. 338) of [6] gives conditions under which non-rational points on $A$ will map to rational points of $J$. In particular, we can use the proposition to determine when there is a rational point of $J$ whose double is the image of $(3P, Q)$ in $J$; if such a point exists, then $J(K)$ will have a point of order 16, and hence also a point of order 48.

Proposition 12 of [6] deals with the Galois cohomology groups $H^1(G_K, E[2])$ and $H^1(G_K, F[2])$. As is summarized in [6, Section 3.7], the group $H^1(G_K, E[2])$ can be identified with the kernel of the norm map
\[ L^*_f / L^* \rightarrow K^*/K^* \]
where $L_f$ is the $K$-algebra $K[T]/f(T)$. Likewise, $H^1(G_K, F[2])$ can be identified with the kernel of the norm
\[ L^*_g / L^* \rightarrow K^*/K^* \]
The polynomials $f$ and $g$ both have linear factors over $K$, and their other roots involve the square root $r$ of $-7$, so both of these kernels are isomorphic to $L^*/L^{*2}$, where $L = K[T]/(T^2 + 7)$. Let $\rho$ denote the image of $T$ in $L$. If $K$ does not contain $r$, then there is an isomorphism $L \cong K(r)$ that sends $\rho$ to $r$. If $K$ does contain $r$, then there is an isomorphism $L \cong K \times K$ that sends $\rho$ to $(r, -r)$.

Let $A$ and $B$ be the elements
\[ \frac{4(s + \rho)^3(s - 3\rho)}{(s^2 + 63)^2} \quad \text{and} \quad \frac{31 - 3\rho}{2} \]
of $L$, corresponding to the roots $\alpha_2$ and $\beta_2$ of $f$ and $g$. Then the map
\[ \iota: E(K)/2E(K) \rightarrow L^*/L^{*2} \]
from [6, Proposition 12] is given by sending the class of a finite point $(x, y)$ of $E(K)$ to the class of $x - A$ in $L^*/L^{*2}$, provided that $x \neq \alpha_2, \alpha_3$. Likewise, the map
\[ \iota': F(K)/2F(K) \rightarrow L^*/L^{*2} \]
sends the class of $(x, y)$ in $F(K)$ to the class of $x - B$, provided that $x \neq \beta_2, \beta_3$.

Proposition 12 of [6] says that the image of $(3P, Q)$ in $J(K)$ will be the double of a point in $J(K)$ if $\iota(3P) = \iota'(Q)$ in $L^*/L^{*2}$. Since $\iota(3P)$ is the class of $0 - A$ mod squares, and $\iota'(Q)$ is the class of $32 - B$ mod squares, we would like to check whether $A(B - 32)$ is a square in $L$.

We compute that $A(B - 32) = \ell^2 m$, where
\[ \ell = \frac{(1 - \rho)(s + \rho)^2}{s^2 + 63} \quad \text{and} \quad m = \frac{3(5 - \rho)(s - 3\rho)}{2(s + \rho)}, \]
so \( A(B - 32) \) is a square if and only if \( m \) is a square. But using the expression for \( s \) in terms of \( z \) and \( w \) from the statement of the theorem, we find that \( m = n^2 \), where

\[
n = 6 \frac{(7 + 11\rho)w - 4s^2 + 784}{8z^2 + 7(1 - 3\rho)z + 1568}.
\]

Therefore, there is a \( K \)-rational 48-torsion point on \( J \).

\[\square\]

Remark 3.4. One can check that the function on \( D \) given by the right-hand side of equation (3.1) is invariant under translation by the 2-torsion point \( P_1 = (0,0) \).

Corollary 3.5. The Jacobian of each of the following genus-2 curves over \( \mathbb{Q} \) has a rational torsion point of order 48:

\[
y^2 + (x^2 + x)y = x^6 - 3x^5 - 5x^4 + 14x^3 + 8x^2 - 16x,
y^2 + (x^2 + x)y = x^6 - x^5 + 5x^4 - 11x^3 + 10x^2 - 6x + 2,
y^2 + (x^2 + x)y = 1217x^6 - 3651x^5 + 15717859x^4 - 31429634x^3
\]

\[
+ 60403483004x^2 - 60387768796x + 80875050306064.
\]

Proof. These are reduced models of the curves \( C_{24}^{-21}, C_{24}^3 \), and \( C_{24}^{21/31} \), which come from the values of \( s \) obtained as in Theorem 3.2 from the points \( P_1, P_2, \) and \( P_3 \) of \( D(\mathbb{Q}) \) defined in Remark 3.3. \[\square\]

Remark 3.6. The 1-parameter family of curves in Theorem 3.1 was obtained by gluing a fixed elliptic curve with an 8-torsion point to a family of elliptic curves with a 6-torsion point. One can also construct a more general family of examples, as follows.

Given elements \( t \) and \( u \) of the field \( K \), let \( E \) be the elliptic curve \( F_6^t \) with a rational 6-torsion point \( P \) defined in [6, pp. 320–322], and let \( F \) be the curve \( F_8^u \) with a rational 8-torsion point \( Q \).

(This requires that \( t \) and \( u \) avoid a certain finite set of values, and for this discussion we tacitly assume that all such exceptional values are excluded.) Then \( E \) and \( F \) can be glued together along their 2-torsion subgroups if their discriminants are equal to one another, up to squares. According to [6, Table 6, p. 321], this will be the case if there is a non-zero \( v \in \mathbb{K} \) such that

\[
(9t + 1)/(t + 1) = (2u^2 - 1)v^2.
\]

Since \( u \) and \( v \) then determine \( t \), this gives us a family of genus-2 curves with a rational point of order 48 on their Jacobians, parameterized by an open subset \( V \) of the \((u,v)\)-plane.

Following the same argument as in the proof of Theorem 3.2, we see that the 24-torsion point on the Jacobian corresponding to a given \((u,v)\) pair will be the double of a rational point of order 48 if and only if a certain element of the quadratic algebra \( L \) determined by the discriminants of \( E \) and \( F \) is a square. This condition can be rephrased as saying that a given \((u,v)\) pair gives a curve with a 48-torsion point on its Jacobian if and only if \((u,v)\) is the image of a rational point under a map \( U \rightarrow V \) from a surface \( U \) to \( V \).

The equations we derived for the surface \( U \) are lengthier than we would like to present here. We looked for curves of small genus on \( U \), and were able to find a number of curves of genus 1, but none of genus 0. The family given in Theorem 3.2 corresponds to the fiber of \( U \) over the line \( u = 1/3 \) in \( V \). (The variable \( s \) in Theorem 3.1 is then \( 7v/3 \).)

4. Curves with small coefficients

In the literature, one finds several examples of genus-2 curves over \( \mathbb{Q} \) with torsion points of large order on their Jacobians and with models whose defining equations have coefficients of very
Table 2. Examples of genus-2 curves with torsion points of large order.

| Order | Equation                                           | Reference          |
|-------|----------------------------------------------------|--------------------|
| 27    | \(y^2 + (6x^3 + 3x^2 + 3x - 2)y = 6x\)            | [12, Théorème 1.2.1]|
|       | \(y^2 + (x^3 - 2x + 1)y = x^3\)                  | New                |
|       | \(y^2 + (2x^3 + 3x^2 - 3x + 2)y = 6x^3 + 6\)     | New                |
|       | \(y^2 + (6x^3 + 9x^2 + 6x - 1)y = -3x^2\)        | New                |
| 28    | \(y^2 + (3x^3 + 2x^2 + 1)y = -x^2 - x\)           | [19, Theorem 4]    |
|       | \(y^2 + (2x^3 - 3x^2 + 3x + 4)y = 4x\)            | [21]*Theorem 3     |
| 29    | \(y^2 + (2x^3 - 2x^2 - x + 1)y = x\)              | [12, Théorème 1.2.1]|
| 33    | \(y^2 + (3x^3 + 9x^2 + x + 2)y = -8x\)           | [20, Corollary 1]  |
| 34    | \(y^2 + (6x^3 + 5x^2 - 4)y = -4x^2\)             | [3, ‘N = 34’]      |
| 36    | \(y^2 + (6x^3 - 3x^2 - x + 2)y = 3x^3 - 4x^2 + 2x\) | [20, Theorem 2]    |
| 39    | \(y^2 + (x^3 + 3x - 1)y = 3x^3\)                 | [21]*Theorem 3     |
|       | \(y^2 + (6x^3 + 6x^2 - 7x - 9)y = -2x - 2\)      | New                |

small height. Inspired by these examples, we searched through a number of families of curves with small-height coefficients in search of further examples. We found no new orders of torsion points, but we did find some new curves, as well as some small models of curves already in the literature.

We had the most success when searching for curves of the form

\[y^2 + (a_3 x^3 + a_2 x^2 + a_1 x + a_0)y = b_2 x^2 + b_1 x + b_0.\]

(\footnote{Note that every genus-2 curve with a rational non-Weierstrass point has a model of this form.} ) For this family, we let the coefficients run through the integers from \(-10\) through 10. (By changing the signs of \(x\) and \(y\), we could assume that \(a_3\) was positive and \(a_2\) non-negative.)

We limited our search to torsion orders for which there is not a known infinite family of curves with torsion points of that order; let us call such orders interesting. For most curves, we could quickly show that the curve’s Jacobian had no interesting rational torsion by looking at the number of points on the Jacobians of the reductions of the curve modulo several small primes of good reduction. For curves whose reductions did allow for the existence of torsion points of interesting order, we used Magma’s TorsionSubgroup command to compute the actual torsion subgroup of the Jacobian of the curve.

Table 2 gives the curves we found, the order of the torsion point of largest order on the Jacobian, and, if applicable, a reference to where the curve has appeared previously in the literature. In some of the examples, we give a model where there is an \(x^3\) term on the right-hand side of the curve’s equation, because allowing that term reduced the coefficient size or the number of non-zero coefficients.

With help from Reinier Bröker, we also searched specifically for genus-2 curves over \(\mathbb{Q}\) with rational 31-torsion points on their Jacobian. We searched through all curves of the form \(y^2 = f\), where \(f \in \mathbb{Z}[x]\) is a quintic or sextic with all coefficients bounded in absolute value by 20. We also searched through all genus-2 curves of the form \(y^2 + gy = f\), where \(g\) and \(f\) are polynomials in \(\mathbb{Z}[x]\) of degree at most 3 and 6, respectively, and with all coefficients bounded in absolute value by 5. We found no examples.

References

1. R. Bröker, E. W. Howe, K. E. Lauter and P. Stevenhagen, ‘Genus-2 curves and Jacobians with a given number of points’, LMS J. Comput. Math., Preprint, 2014, arXiv:1403.6911 [math.NT].
2. J. Cremona, ‘Elliptic curve data’, retrieved 8 May 2014, http://homepages.warwick.ac.uk/staff/J.E.Cremona/ftp/data/INDEX.html.
3. N. D. Elkies, ‘Curves of genus 2 over Q whose Jacobians are absolutely simple abelian surfaces with torsion points of high order’, retrieved 1 May 2014, http://www.math.harvard.edu/~elkies/g2_tors.html.

4. E. V. Flynn, ‘Large rational torsion on abelian varieties’, J. Number Theory 36 (1990) 257–265, doi: 10.1016/0022-314X(90)90089-A.

5. E. W. Howe and K. E. Lauter, ‘Improved upper bounds for the number of points on curves over finite fields’, Ann. Institut. Fourier (Grenoble) 53 (2003) 1677–1737, doi: 10.5802/aif.1990.

6. E. W. Howe, F. Leprévost and B. Poonen, ‘Large torsion subgroups of split Jacobians of curves of genus two or three’, Forum Math. 12 (2000) 315–364, doi: 10.1515/form.2000.008.

7. E. Kani, ‘The number of curves of genus two with elliptic differentials’, J. reine angew. Math. 485 (1997) 93–121, doi: 10.1515/crll.1997.485.93.

8. D. S. Kubert, ‘Universal bounds on the torsion of elliptic curves’, Proc. London Math. Soc. (3) 33 (1976) 193–237, doi: 10.1112/plms/s3-33.2.193.

9. F. Leprévost, ‘Famille de courbes de genre 2 munies d’une classe de diviseurs rationnels d’ordre 13’, C. R. Acad. Sci. Paris Sér. I Math. 313 (1991) 451–454, http://gallica.bnf.fr/ark:/12148/bpt6k57325582/f455.image.

10. F. Leprévost, ‘Familles de courbes de genre 2 munies d’une classe de diviseurs rationnels d’ordre 15, 17, 19 ou 21’, C. R. Acad. Sci. Paris Sér. I Math. 313 (1991) 771–774, http://gallica.bnf.fr/ark:/12148/bpt6k57325582/f775.image.

11. F. Leprévost, ‘Torsion sur des familles de courbes de genre g’, Manuscripta Math. 75 (1992) 303–326, doi: 10.1007/BF02567087.

12. F. Leprévost, ‘Jacobiennes de certaines courbes de genre 2: torsion et simplicité’, J. Théor. Nombres Bordeaux 7 (1995) 283–306, Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993), http://www.emis.de/journals/JTNB/1995-1/jtnb7-1.html.

13. F. Leprévost, ‘Sur une conjecture sur les points de torsion rationnels des jacobiennes de courbes’, J. reine angew. Math. 473 (1996) 59–68, doi: 10.1515/crll.1996.473.59.

14. B. Mazur, ‘Rational points on modular curves’, Modular functions of one variable, V, Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976 (eds J.-P. Serre and D. B. Zagier), Lecture Notes in Mathematics 601 (Springer, Berlin, 1977) 107–148, doi: 10.1007/BFb0063947.

15. B. Mazur, ‘Modular curves and the Eisenstein ideal’, Inst. Hautes Études Sci. Publ. Math. 47 (1977/78) 33–186, doi: 10.1007/BF02684339.

16. B. Mazur, ‘Rational isogenies of prime degree (with an appendix by D. Goldfeld)’, Invent. Math. 44 (1978) 129–162, doi: 10.1007/BF01390348.

17. J. S. Milne, ‘Abelian varieties’, Arithmetic geometry, Storrs, Conn. 1984 (eds G. Cornell and J. H. Silverman; Springer, New York, 1986) 103–150, doi: 10.1007/978-1-4613-8655-1.5.

18. H. Ogawa, ‘Curves of genus 2 with a rational torsion divisor of order 23’, Proc. Japan Acad. Ser. A Math. Sci. 70 (1994) 295–298, http://projecteuclid.org/euclid.pja/1195510899.

19. V. P. Platonov and M. M. Petrunin, ‘New orders of torsion points in Jacobians of curves of genus 2 over the rational number field’, Dokl. Math. 85 (2012) 286–288, doi: 10.1134/S1064562412020330. Translated from Dokl. Akad. Nauk 443 (2012) 664–667.

20. V. P. Platonov and M. M. Petrunin, ‘On the torsion problem in Jacobians of curves of genus 2 over the rational number field’, Dokl. Math. 86 (2012) 642–643, doi: 10.1134/S1064562412050304. Translated from Dokl. Akad. Nauk 446 (2012) 263–264.

21. V. P. Platonov, V. S. Zhgun and M. M. Petrunin, ‘On the simplicity of Jacobians for curves of genus 2 over the rational number field containing torsion points of large orders’, Dokl. Math. 87 (2013) 318–321, doi: 10.1134/S1064562413030216. Translated from Dokl. Akad. Nauk 450 (2013) 385–388.

22. W. A. Stein and M. Watkins, ‘A database of elliptic curves—first report’, Algorithmic number theory, Sydney, 2002 (eds C. Fieker and D. R. Kohel), Lecture Notes in Computer Science 2309 (Springer, Berlin, 2002) 267–275, doi: 10.1007/3-540-45455-1_22.