THE SECOND DESCENT OBSTRUCTION AND GERBES

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Abstract. We extend the notion of rational points and cohomological obstructions on varieties to categories fibred in groupoids. We also establish the generalized theory of descent by torsors. Then we interpret the obstruction given by the second cohomology of abelian sheaves in terms of categorical points of gerbes, analogue to descent by torsors. As an application, we construct composite obstructions and show that they are not larger than the descent obstruction. We also propose some new kinds of obstructions.

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1. Introduction

The existence of rational points is a fundamental arithmetic property of varieties. One of the most common methods to consider is the local-global principle. When it fails, one considers various obstructions to it, such as Brauer-Manin obstruction and the descent obstruction. In this section, we review the classical descent set and descent by torsors. Then we introduce our results on the generalization of points and obstructions, and descent by gerbes.

1.1. Rational points on varieties and the local-global principle. Let $X$ be a variety over a number field $k$, we write $X(-)$ for the functor $\text{Hom}_k(-, X)$. The rational points of $X$ is the set $X(k)$, which is contained in the ad\`elic points $X(\mathbb{A}_k)$. We say that the local-global principle holds if $X(\mathbb{A}_k) \neq \emptyset$ implies $X(k) \neq \emptyset$. 

Date: September 16, 2020.
2000 Mathematics Subject Classification. Primary 14G05, 18D30; secondary 18G50, 18F20.
Key words and phrases. descent obstruction, torsors, second cohomology, fibred categories, gerbes.
This work was supported by National Natural Science Foundation of China (Grant No. 11701552).
1.2. The $F$-obstruction to the local-global principle. Let $F : (\text{Sch}/k)^{\circ} \to \text{Set}$ be a contravariant functor from the category of $k$-schemes to the category of sets. For a $k$-scheme $T$ and $A \in F(X)$, the evaluation of a $T$-point $T \to X \in X(T)$ at $A$ is defined to be the image of $A$ under the pull-back map $F(x) : F(X) \to F(T)$ induced by $x$, denoted by $A(x)$. We have an obvious commutative diagram

$$
\begin{array}{ccc}
X(k) & \longrightarrow & X(A_k) \\
\downarrow A(-) & & \downarrow A(-) \\
F(k) & \longrightarrow & F(A_k)
\end{array}
$$

Define the obstruction (set) given by $A$ to be $X(A_k)^A = \{ x \in X(A_k) | A(x) \in \text{im}(F(k) \to F(A_k)) \}$. Then we have $\emptyset \subseteq X(k) \subseteq X(A_k)^A \subseteq X(A_k)$, giving a constraint on the locus of rational points in adèlic points. Putting

$$
X(A_k)^F = X(A_k)^{F(X)} = \bigcap_{A \in F(X)} X(A_k)^A,
$$

called the $F$-set, which also yields the inclusion $X(k) \subseteq X(A_k)^F$. See Poonen [17, 8.1.1].

1.3. Definition. We say that the $F$-obstruction to the local-global principle is the only one if $X(A_k)^F \neq \emptyset$ implies $X(k) \neq \emptyset$.

We say that there is an $F$-obstruction to the local-global principle if $X(A_k) \neq \emptyset$ but $X(A_k)^F = \emptyset$ (a priori $X(k) = \emptyset$).

1.4. Examples. We are mainly interested in cohomological obstructions, namely, ones where $F$ is taken to be a cohomological functor.

Let $F = \text{Br} = H^2_{\text{ét}}(-, \mathbb{G}_m)$ be the cohomological Brauer-Grothendieck group (cf. Grothendieck [7, 8, 9]). Thus we obtain $X(A_k)^{\text{Br}, X}$, the Brauer-Manin set (see, e.g., Skorobogatov [18]).

Another example is that $F = H^1_{\text{fppf}}(-, G)$, the obstruction given by the first Čech cohomology, where $G$ is an affine $k$-group. In general, $H^1_{\text{fppf}}(X, G)$ is a pointed set, which is isomorphic to $H_{\text{fppf}}^1(X, G)$ if $G$ is commutative, and further to $H_{\text{ét}}^1(X, G)$ since $G$ is smooth. The classical descent set is given by

$$
(1.5) \quad X(A_k)^{\text{cdesc}} = \bigcap_{\text{all affine } k\text{-group } G} X(A_k)^{H^1_{\text{fppf}}(X, G)}.
$$

1.6. Remark. In Section 3 we shall generalize the classical descent set from varieties to fibred categories. To avoid confusion, here we use cdesc instead of desc to indicate that it is the “classical” descent set.

The descent theory was established by Colliot-Thélène and Sansuc [4] for tori and Skorobogatov [20] for groups of multiplicative type. Harari and Skorobogatov [10, 12, 11] studied the descent obstruction for general algebraic groups and compared it with Brauer-Manin obstruction. One of the results is the well-known inclusion $X(A_k)^{\text{cdesc}} \subseteq X(A_k)^{\text{Br}}$ for regular, quasi-projective $k$-variety $X$ (see, e.g., [17, Prop. 8.5.3]).

The classical descent by torsors says that

$$
X(A_k)^f = \bigcup_{\sigma \in H^1(k, G)} f^\sigma(Y^\sigma(A_k))
$$

where $G$ is the results is the well-known inclusion $X(A_k)^{\text{cdesc}} \subseteq X(A_k)^{\text{Br}}$ for regular, quasi-projective $k$-variety $X$ (see, e.g., [17, Prop. 8.5.3]).
where $f : Y \to X$ is a $G$-torsor over $X$, representing the class $[Y] \in \check{H}^1_{fppf}(X, G)$. Then we may define various of composite (or iterated) obstruction sets $X(A_k)^{\text{cdesc, ob}}$ between $X(k)$ and $X(A_k)^{\text{cdesc}}$, such as

$$X(A_k)^{\text{cdesc, cdesc}} = \bigcap_{\text{all affine } G \text{ and all torsor } f:Y \to X} \bigcup_{\sigma \in \check{H}^1(k,G)} f^\sigma(Y^\sigma(A_k)^{\text{cdesc}})$$

and

$$X(A_k)^{\text{et, Br}} = \bigcap_{\text{all finite } G \text{ and all torsor } f:Y \to X} \bigcup_{\sigma \in \check{H}^1(k,G)} f^\sigma(Y^\sigma(A_k)^{\text{Br}}).$$

By works of Stoll, Skorobogatov, Demarche, Poonen, Xu and Cao, it is known that for smooth, quasi-projective, geometrically integral $k$-variety $X$, $X(A_k)^{\text{cdesc}} = X(A_k)^{\text{et, Br}}$. In particular, $X(A_k)^{\text{cdesc, cdesc}} = X(A_k)^{\text{cdesc}}$. However we do not know whether $X(A_k)^{\text{2-desc}} \subseteq X(A_k)^{\text{cdesc}}$ holds.

The key point to go further is to use gerbes to interpret the obstruction given by second cohomology, which is analogue to classical descent by torsors. Recall that a gerbe is a kind of categories fibred in groupoids. In fact, all these obstructions on $X(A_k)$ can be generalized to that on $\mathcal{X}(A)$, where $\mathcal{X}$ and $\mathcal{A}$ are categories over $S$ fibred in groupoids, $S = \text{Sch}/S$ is the category of $S$-schemes. See Section 2.

Assume further that $\mathcal{A}$ is a category over $S$ fibred in sets. Then we show that (descent by gerbes)

$$\mathcal{X}(\mathcal{A})^f = \bigcup_{\sigma \in \check{H}^2(S, \mathcal{G})} f^\sigma(Y^\sigma(\mathcal{A}))$$

where $f : \mathcal{Y} \to \mathcal{X}_{et}$ is any fixed gerbe over $\mathcal{X}_{et}$ bounded by an abelian sheaf $\mathcal{G}$ on $S_{et}$, representing the class $[\mathcal{Y}] \in \check{H}^2(\mathcal{X}, \mathcal{G})$. See 4.2. Then we have

$$\mathcal{X}(\mathcal{A})^{\text{2-desc}} = \bigcap_{\mathcal{G} \text{ on } S_{et} \text{ and all gerbes } f:Y \to \mathcal{X}_{et}} \bigcup_{\sigma \in \check{H}^2(S, \mathcal{G})} f^\sigma(Y^\sigma(\mathcal{A})).$$

This leads to the composite obstruction

$$\mathcal{X}(\mathcal{A})^{\text{2-desc, ob}} = \bigcap_{\mathcal{G} \text{ on } S_{et} \text{ and all gerbes } f:Y \to \mathcal{X}_{et}} \bigcup_{\sigma \in \check{H}^2(S, \mathcal{G})} f^\sigma(Y^\sigma(\mathcal{A})^{\text{ob}}).$$

For example, let $\text{ob} = \text{desc}$, we obtain $\mathcal{X}(\mathcal{A})^{\text{2-dd}} := \mathcal{X}(\mathcal{A})^{\text{2-desc, desc}}$. Then let $\text{ob} = 2$-$\text{dd}$, we obtain $\mathcal{X}(\mathcal{A})^{2\text{-desc}, 2\text{-dd}}$. Let $\text{ob} = (2\text{-desc}, 2\text{-dd})$, we obtain $\mathcal{X}(\mathcal{A})^{2\text{-desc}, 2\text{-desc}, 2\text{-dd}}$, and so on. In general, taking into account the generalized descent by torsors developed in Section 3, we shall show in 3.13 that $X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{cdesc}}$, and in 3.14 that

$$\mathcal{X}(S) \subseteq \cdots \subseteq \mathcal{X}(\mathcal{A})^{\delta, \delta', \delta, 2\text{-dd}} \subseteq \mathcal{X}(\mathcal{A})^{\delta', \delta, 2\text{-dd}} \subseteq \mathcal{X}(\mathcal{A})^{2\text{-dd}} \subseteq \mathcal{X}(\mathcal{A})^{\delta} \subseteq \mathcal{X}(\mathcal{A}).$$
where $\delta, \delta', \delta'', \cdots \in \{\text{desc}, 2\text{-desc}\}$.

1.8. The paper is organized as follows. Section 2 gives a framework that extends the classical notion of rational points and cohomological obstructions on varieties to categories fibred in groupoids. In Section 3 we reformulate the classical descent theory in the context of categories fibred in groupoids. Section 4 is devoted to define the second abelian descent obstructions on categories fibred in groupoids. In this section we extend the classical notion of rational points and cohomological obstructions on varieties to categories fibred in groupoids.

2. Generalization on classical points and obstructions

In this section we extend the classical notion of rational points and cohomological obstructions on varieties to categories fibred in groupoids.

2.1. Fibred categories. If there is no other confusion, we write $C \subseteq D$ (resp. $V \in D$) if $C$ is a full subcategory of $D$ (resp. $V \in \text{Ob}(D)$). Without extra explanation, fibred categories mean categories fibred in groupoids. Let $C$ be a category. Then we have $\text{Fib}/C$, the 2-category of fibred categories over $C$. Recall that in $\text{Fib}/C$, all 2-morphisms in $\text{Fib}/C$ are 2-isomorphisms and 2-fibred products exist. Let $\mathcal{T}$ and $\mathcal{T}'$ be two fibred categories over $C$. Then the category $\text{Hom}_{\text{Fib}/C}(\mathcal{T}', \mathcal{T})$ in $\text{Fib}/C$ consists of, by definition, 1-morphisms $\mathcal{T}' \to \mathcal{T}$ as objects and 2-morphisms between them as morphisms. We also recall some basic definitions and properties.

2.2. Definition. Let $\mathcal{Y} \xrightarrow{f} \mathcal{X}'$ be two 1-morphisms in a 2-category $\mathcal{C}$. We say that $g$ is a quasi-section (resp. section) of $f$ if the diagram

$$
\begin{array}{c}
\mathcal{Y} \xrightarrow{f} \mathcal{X}' \\
\downarrow \quad \quad \downarrow \text{id} \\
\mathcal{Y} \xrightarrow{g} \mathcal{X}
\end{array}
$$

2-commutes (resp. commutes), that is, $f \circ g$ is 2-isomorphic (resp. equal) to the identity.

We say that $f$ and $g$ define an equivalence (resp. isomorphism) if they are quasi-sections (resp. sections) of each other. If such $f$ and $g$ exist, $\mathcal{Y}$ and $\mathcal{X}$ are equivalent (resp. isomorphic), denoted by $\mathcal{Y} \cong \mathcal{X}$ (resp. $\mathcal{Y} \Rightarrow \mathcal{X}$).

2.3. Proposition. For an arbitrary category $C$, we have the following functors

$$
\begin{array}{ccc}
C & \xleftarrow{c_C} & \text{PSh}(C) & \xrightarrow{\eta_C} & \text{Fib}_{\text{Set}}/C & \xleftarrow{\text{Fib/}C}
\end{array}
$$

where $c_C(U) = h_U = \text{Hom}_C(-, U)$ is the Yoneda embedding, $\text{PSh}(C)$ is the category of presheaves on $C$, $\eta_C$ is an equivalence, $\text{Fib}_{\text{Set}}/C$ is the category of categories over $C$ fibred in sets, and that:

(i) For $\mathcal{F} \in \text{PSh}(C)$, the category $\eta_C(\mathcal{F}) = C_{\mathcal{F}}$ is defined as follows. The objects are pairs $(U, T)$ where $U \in C$ and $T \in \mathcal{F}(U)$, and

$$
\text{Hom}_{C_{\mathcal{F}}}((U', T'), (U, T)) = \{ f \in \text{Hom}_C(U', U) \mid \mathcal{F}(f)(T) = T' \},
$$

and the structure functor $C_{\mathcal{F}} \to C$ is the projection to the first component. The quasi-inverse of $\eta_C$ is defined to be $\eta_C^{-1}(T)(U) = T_U$, the fiber category over $U$, and $\eta_C^{-1}(T)(f) = f^*$ where $f : U' \to U$ and $f^* : T_U \to T_{U'}$ is the pull-back functor in $\mathcal{T}$. 

(ii) Any $\mathcal{T} \in \text{Fib}/\mathcal{C}$ is equivalent to $\eta_{\mathcal{K}}(\mathcal{F})$ for some $\mathcal{F}$ (unique up to an isomorphism) if and only if $\mathcal{T}$ is fibred in setoids. Then we call $\mathcal{T}$ is represented by presheaf, or is represented by $\mathcal{F}$.

(iii) Any $\mathcal{T} \in \text{Fib}/\mathcal{C}$ is equivalent to $\eta_{\mathcal{K}C}(U)$ for some $U$ (unique up to an isomorphism) if and only if $\mathcal{T}$ is fibred in setoids and that the presheaf $U \mapsto \text{Ob}(\mathcal{T}_U)/\equiv$ is representable. Then we call $\mathcal{T}$ representable, or is represented by $U$.

Proof. See [22, Tag 02Y2, Tag 0045, Tag 02Y3].

2.5. Remark. Here are some remarks related to $\text{Fib}_{\mathcal{C}}/\mathcal{C}$.

(i) If $U \in \mathcal{C}$, then in [2.4] the corresponding presheaf of sets $h_U$ represents an object $\eta_{\mathcal{K}C}(U)$ in $\text{Fib}_{\mathcal{C}}/\mathcal{C}$, it is the localized category $\mathcal{C}/U$ [22, Tag 0044], still denoted by $T$.

(ii) In [2.2] let $\mathcal{C} = \text{Fib}/\mathcal{C}$, if in addition $X \in \text{Fib}_{\mathcal{C}}/\mathcal{C}$, i.e., there is an isomorphism $X \cong \eta_{\mathcal{K}}(\mathcal{F})$ for some $\mathcal{F} \in \text{PSh}(\mathcal{C})$, then all quasi-sections of $f$ are sections of $f$ since $\text{Hom}_{\text{Fib}_{\mathcal{C}}/\mathcal{C}}(X, A)$ is a discrete category.

2.6. Pull-back of a fibred category. Let $u : \mathcal{T}' \to \mathcal{T}$ be a 1-morphism in $\text{Fib}/\mathcal{C}$ and $f : \mathcal{Y} \to \mathcal{T}$ be a fibred category over $\mathcal{T}$. In particular, $\mathcal{Y} \in \text{Fib}/\mathcal{C}$ via compositing with the structure functor $\mathcal{T} \to \mathcal{C}$ [22, Tag 09WW]. We shall always assume that $f' : u^*\mathcal{Y} \to \mathcal{T}'$ is a fibred category, and call it the pull-back of $\mathcal{Y}$ under $u$.

2.7. Definition. Consider the base change of $\mathcal{Y}$ by $u$, i.e., the 2-fibred product $u^*\mathcal{Y} = \mathcal{Y} \times_{\mathcal{T}} \mathcal{T}'$ indicated in the following 2-cartesian diagram in $\text{Fib}/\mathcal{C}$

$$
\begin{array}{ccc}
\mathcal{Y} & \rightarrow & \mathcal{T}' \\
\downarrow^u & & \downarrow^f \\
\mathcal{Y} \times_{\mathcal{T}} \mathcal{T}' & \rightarrow & \mathcal{T}
\end{array}
$$

Then $f' : u^*\mathcal{Y} \to \mathcal{T}'$ is unique up to an equivalence and there is a choice making it a fibred category over $\mathcal{T}'$ [22, Tag 003Q, Tag 06N7]. We shall always assume that $f' : u^*\mathcal{Y} \to \mathcal{T}'$ is a fibred category, and call it the pull-back of $\mathcal{Y}$ under $u$.

2.8. Lemma. With notations in 2.7, assume further that $\mathcal{T}' \in \text{Fib}_{\mathcal{C}}/\mathcal{C}$. Let $\mathcal{Y}' = u^*\mathcal{Y}, \mathcal{T}' \in \mathcal{T}'$ and $T = u(T') \in \mathcal{T}$, then there is an equivalence in $\text{Cat}/\mathcal{T}$ (the $(2, 1)$-category of categories over $\mathcal{T}$) between the fiber categories $\mathcal{Y}'_T \cong \mathcal{Y}_T$, which is, up to equivalences, induced by $u'$.

Proof. We now working in $\text{Cat}/\mathcal{T}$. Consider the following 2-commutative diagram in $\text{Fib}/\mathcal{C}$

$$
\begin{array}{ccc}
\text{Pt} \times_{\mathcal{T}} \mathcal{Y}' & \rightarrow & \mathcal{Y}' \\
\downarrow & & \downarrow^f \\
\text{Pt} \times_{\mathcal{T}} \mathcal{Y} & \rightarrow & \mathcal{Y}
\end{array}
$$

where $\text{Pt} = \{\ast\}$ is the singleton category, and the arrow labeled with $T'$ (resp. $T$) is the functor sending $\ast$ to $T'$ (resp. $T$). We see that all squares are 2-cartesian and hence there is an equivalence $\text{Pt} \times_{\mathcal{T}} \mathcal{Y}' \cong \text{Pt} \times_{\mathcal{T}} \mathcal{Y}$.

Note that for any $\mathcal{R} \in \text{Cat}/\mathcal{T}$, $\text{Hom}_{\text{Cat}/\mathcal{T}}(\mathcal{R}, \mathcal{T})$ is a discrete category, and so is $\text{Hom}_{\text{Cat}/\mathcal{T}}(\mathcal{R}, \mathcal{T}')$ since $\mathcal{T}'$ is in $\text{Fib}_{\mathcal{C}}/\mathcal{C}$ hence in $\text{Fib}_{\mathcal{C}}/\mathcal{T}$. Thus by the universal property of 2-fibred product, we have equivalences $\mathcal{Y}'_T \cong \text{Pt} \times_{\mathcal{T}} \mathcal{Y}$ and $\mathcal{Y}'_T \cong \text{Pt} \times_{\mathcal{T}} \mathcal{Y}'$. It follows that $\mathcal{Y}'_T \cong \mathcal{Y}_T$ and one checks that this is induced by $u'$, up to equivalences. \qed
2.9. Definition (Points). For any \( \mathcal{X}, \mathcal{T} \in \text{Fib}/C \), we call \( \mathcal{X}(\mathcal{T}) = \text{Hom}_{\text{Fib}/C}(\mathcal{T}, \mathcal{X}) \) the category of \( \mathcal{T} \)-points in \( \mathcal{X} \). In particular, \( \mathcal{X}(C) \) is called rational points of \( \mathcal{X} \).

2.10. From now on, we work with \( \text{Fib}/S \), where \( S = \text{Sch}/S \) is the category of \( S \)-schemes, and \( S \) is a fixed scheme. We also fix a fibred category \( q : \mathcal{A} \rightarrow S \) in \( \text{Fib}/S \). Let \( p : \mathcal{X} \rightarrow S \) be another fibred category. By abusing notations, we also use \( \mathcal{X}(S) \) for the essential image of the natural functor \( \text{Hom}_{\text{Fib}/S}(q, \mathcal{X}) : \mathcal{X}(S) \rightarrow \mathcal{X}(\mathcal{A}) \) induced by \( q \). For example, we have \( \emptyset \subseteq \mathcal{X}(S) \subseteq \mathcal{X}(\mathcal{A}) \) (for the meaning of inclusions, see [2.1]). Then we are interested in full subcategories between \( \mathcal{X}(S) \subseteq \mathcal{X}(\mathcal{A}) \) and the “local-global principle”, which means that \( \mathcal{X}(\mathcal{A}) \neq \emptyset \) implies \( \mathcal{X}(S) \neq \emptyset \).

2.11. Remark. To recover the classical problem of the local-global principle to rational points on varieties over a field \( k \), let \( \mathcal{X} \) (resp. \( \mathcal{T} \)) be represented by \( X \) (resp. \( T \)). Then [22, Tag 04SF]

\[
\text{Hom}_{\text{Fib}/S}(\mathcal{T}, \mathcal{X})/2\text{-isomorphisms} = \text{Hom}_S(T, X).
\]

Thus in this situation one may say that the only morphisms in \( \mathcal{X}(\mathcal{T}) = \text{Hom}_{\text{Fib}/S}(\mathcal{T}, \mathcal{X}) \) are identities.

Another way to explain this is that when \( X \in S, \mathcal{X} = X \in \text{Fib}_{\text{Set}}/S, \mathcal{X}(\mathcal{T}) = \text{Hom}_{\text{Fib}/S}(\mathcal{T}, \mathcal{X}) \) is a discrete category.

Then let \( S = \text{Spec} k, \mathcal{X} = X \) a variety over \( k \) and \( \mathcal{A} \rightarrow S \) be \( \text{Spec} A_k \rightarrow \text{Spec} k \) induced by the inclusion \( k \subseteq A_k \), and then we have \( \mathcal{X}(\mathcal{A}) = X(A_k) \) and \( \mathcal{X}(S) = X(k) \).

2.12. Definition. A category \( \mathcal{X}(\mathcal{A})' \) is an obstruction category of \( \mathcal{X}(\mathcal{A}) \), if it is a full subcategory satisfying

(a) \( \mathcal{X}(S) \subseteq \mathcal{X}(\mathcal{A})' \subseteq \mathcal{X}(\mathcal{A}) \) and

(b) if \( x, y \in \mathcal{X}(\mathcal{A}) \) are isomorphic (i.e., 2-isomorphic in \( \text{Fib}/S \)), then \( x \in \mathcal{X}(\mathcal{A})' \) if and only if \( y \in \mathcal{X}(\mathcal{A})' \).

2.13. Definition. Let \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{T} \) be fibred categories over \( S \) and \( f : \mathcal{Y} \rightarrow \mathcal{X} \) a 1-morphism. Let \( \mathcal{Y}(\mathcal{T})' \subseteq \mathcal{Y}(\mathcal{T}) \) be a full subcategory.

(a) Denote by \( f(\mathcal{Y}(\mathcal{T})') \) the essential image of the functor \( \text{Hom}_{\text{Fib}/S}(\mathcal{T}, f) : \mathcal{Y}(\mathcal{T})' \rightarrow \mathcal{X}(\mathcal{T}) \).

(b) We say that a point \( x \in \mathcal{X}(\mathcal{T}) \) can be lifted to a point in \( \mathcal{Y}(\mathcal{T})' \) if \( x \in f(\mathcal{Y}(\mathcal{T})') \). This is equivalent to say that there exists \( y \in \mathcal{Y}(\mathcal{T})' \) such that \( x = f(y) = f \circ y \) are 2-isomorphic, that is, the following diagram is 2-commutative

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\
\downarrow y & & \downarrow x \\
\mathcal{T} & \xrightarrow{f} & \mathcal{T}
\end{array}
\]

In this case, \( y \) is called a lift of \( x \) to \( \mathcal{Y}(\mathcal{T})' \).

The next lemma gives a condition describing whether a point \( x \in \mathcal{X}(\mathcal{T}) \) can be lifted to \( \mathcal{Y}(\mathcal{T}) \).

2.14. Lemma. With notations in [2.13] let \( x \in \mathcal{X}(\mathcal{T}) \) be arbitrary and \( \mathcal{Y}_x = \mathcal{Y} \times_{\mathcal{X}(\mathcal{T})} \mathcal{T} \) the 2-fibred product in \( \text{Fib}/S \). Then \( x \in f(\mathcal{Y}(\mathcal{T})) \) if and only if \( \mathcal{Y}_x \rightarrow \mathcal{T} \) has a quasi-section.
Proof. By properties of a 2-fibred product, in the following 2-commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\text{id}} & T \\
\downarrow y & \downarrow y & \downarrow y \\
\mathcal{Y} & \xrightarrow{f} & X \\
\end{array}
\]
given any lift \( y \in f(\mathcal{X}(T)) \), there is, up to a 2-isomorphism, a unique quasi-section \( s \) of \( \mathcal{Y}_x \to T \) making the whole diagram 2-commutes. Thus compositing with \( \mathcal{Y}_x \to \mathcal{Y} \) yields an one-to-one correspondence

\[\{\text{quasi-sections of } \mathcal{Y}_x \to T\} / \text{2-isomorphisms} \sim f(\mathcal{X}(T)) / \text{2-isomorphisms}.\]

The proof is complete. \( \square \)

2.15. The \( F \)-obstruction. Now we generalize the \( F \)-obstruction in \( \text{[2.12]} \) defined on varieties to that on fibred categories. Keep notations in \( \text{[2.10]} \).

2.16. Definition. Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor from a 2-category to a category. We say that \( F \) is stable if for any 1-morphisms \( f \) and \( g \) of \( \mathcal{C} \) that is 2-isomorphic, we have \( F(f) = F(g) \).

2.17. Definition. Let \( F : (\text{Fib}/S)^\circ \to \text{Set} \) be a stable functor. Let \( \mathcal{T} \) be a fibred category over \( S \) and \( A \in F(\mathcal{X}) \). The evaluation of a \( \mathcal{T} \)-point \( x \in \mathcal{X}(T) = \text{Hom}_{\text{Fib}/S}(T, \mathcal{X}) \) at \( A \) is defined to be the image of \( A \) under the pull-back map \( F(x) : F(\mathcal{X}) \to F(\mathcal{T}) \) induced by \( x \), denoted by \( A(x) \).

Then forgetting morphisms in \( \mathcal{X}(S) \) and \( \mathcal{X}(A) \), we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}(S) & \xrightarrow{A(-)} & \mathcal{X}(A) \\
\downarrow A(-) & & \downarrow A(-) \\
F(S) & \xrightarrow{A(-)} & F(A) \\
\end{array}
\]

2.18. Definition. The obstruction given by \( A \) is the full subcategory \( \mathcal{X}(A)^F \) of \( \mathcal{X}(A) \) whose objects are characterized by

\[\mathcal{X}(A)^F = \{x \in \mathcal{X}(A) \mid A(x) \in \text{im}(F(S) \to F(A))\} \]

The \( F \)-category is the full subcategory \( \mathcal{X}(A)^F \) whose objects are characterized by

\[\mathcal{X}(A)^F = \bigcap_{A \in F(\mathcal{X})} \mathcal{X}(A)^A,\]

also denoted by \( \mathcal{X}(A)^{F(\mathcal{X})} \).

Then we have

\[\emptyset \subseteq \mathcal{X}(S) \subseteq \mathcal{X}(A)^F \subseteq \mathcal{X}(A)^A \subseteq \mathcal{X}(A),\]

and \( \text{[2.12]} \) holds. In particular, both \( \mathcal{X}(A)^F \) and \( \mathcal{X}(A)^A \) are obstruction categories.
2.19. Cohomology for fibred categories. For any fibred category $\mathcal{T}$ over $\mathcal{S}$, denote by $\mathcal{T}_{\text{ét}}$ (resp. $\mathcal{T}_{\text{fppf}}$) the site with topology inherited from the big étale site $(\text{Sch}/\mathcal{S})_{\text{ét}}$ (resp. the big fppf site $(\text{Sch}/\mathcal{S})_{\text{fppf}}$) [22, Tag 067N, Tag 06TP]. Let $\tau \in \{\text{ét}, \text{fppf}\}$. Then we have the cohomology functors $H^i_{\tau}(\mathcal{T}, -) := H^i(\mathcal{T}_{\tau}, -)$ on abelian sheaves in $\text{Ab}(\mathcal{T}_{\tau})$. See [22, Tag 075E]. For $T \in \mathcal{S}$, the corresponding $T \in \text{Fib}/\mathcal{S}$ (see 2.5 (i)) has a final object $T$. Thus by [22, Tag 075F], $H^i_{\tau}(T, -)$ recovers the $\tau$-cohomology for schemes over $\mathcal{S}$. In particular, we have $H^i_{\tau}(\mathcal{S}, -) = H^i_{\tau}(\mathcal{S}, -)$, agrees with the $\tau$-cohomology for the scheme $\mathcal{S}$.

2.20. Pull-back of a sheaf. Let $f : \mathcal{T}' \to \mathcal{T}$ be a 1-morphism in $\text{Fib}/\mathcal{S}$. Then compositing with $f$ gives the functor $f^* : \text{PSh}(\mathcal{T}_\tau) \to \text{PSh}(\mathcal{T}'_{\tau})$, which has a right adjoint $f_*$. Both $f^*$ and $f_*$ send $\tau$-sheaves to $\tau$-sheaves, that is, we have a morphism of topoi $(f^*, f_*) : \text{Sh}(\mathcal{T}'_{\tau}) \to \text{Sh}(\mathcal{T}_\tau)$, which is also a morphism with $\text{Sh}$ replaced by $\text{Ab}$ (or $\text{SGrp}$ the category of sheaves of groups) and is compatible with the classical case when $\mathcal{T}$ and $\mathcal{T}'$ are represented by $\mathcal{S}$-schemes. See [22, Tag 067S, Tag 075J].

2.21. Now using the notion of $F$-obstruction 2.15, we generalize the cohomological obstructions in 1.4. Let $f : \mathcal{T}' \to \mathcal{T}$ be a 1-morphism in $\text{Fib}/\mathcal{S}$ and $\mathcal{F} \in \text{Ab}(\mathcal{T}_\tau)$. Then we have an obvious map $\Gamma(\mathcal{T}, -) \to \Gamma(\mathcal{T}', f^* -)$, functorial in $\mathcal{F}$. Thus it induces the morphism of right derived functors $R^i \Gamma(\mathcal{T}, -) \to R^i \Gamma(\mathcal{T}', f^* -)$ since $f^*$ is exact. Taking cohomology, we obtain

\[ f^* : H^i_{\tau}(\mathcal{T}, -) \to H^i_{\tau}(\mathcal{T}', f^* -). \]

2.23. Lemma. With above notations, let

\[ \begin{aligned} \begin{array}{ccc} \mathcal{T}' & \xrightarrow{f} & \mathcal{T} \\ \downarrow{\xi} & & \downarrow{g} \\ \mathcal{S} & \xleftarrow{\alpha} & \mathcal{S} \end{array} \end{aligned} \]

be a 2-isomorphism in $\text{Fib}/\mathcal{S}$.

(i) We have the following commutative diagram of functors

\[ \begin{array}{ccc} H^i_{\tau}(\mathcal{T}', f^* -) & \xrightarrow{f^*} & H^i_{\tau}(\mathcal{T}, f^* -) \\ \downarrow{\xi_*} & & \downarrow{\xi_*} \\ H^i_{\tau}(\mathcal{T}', g^* -) & \xrightarrow{g^*} & H^i_{\tau}(\mathcal{T}, g^* -) \end{array} \]

where $\xi_*$ is induced by $\xi$.

(ii) For all $\mathcal{G} \in \text{Ab}(\mathcal{S}_\tau)$, let $\mathcal{F} = \alpha^* \mathcal{G} \in \text{Ab}(\mathcal{T}_\tau)$ be the pull-back sheaf. Then we have $f^* \mathcal{F} = g^* \mathcal{F} \in \text{Ab}(\mathcal{T}_\tau)$, and $\xi_*(\mathcal{F})$ is the identity in $\mathcal{F}$.
Let \( Y \) be the commutative diagram

\[
\begin{array}{ccc}
\Gamma(T, f^*) & \rightarrow & \Gamma(T, f^*) \\
\downarrow & & \downarrow \\
\Gamma(T', g^*) & \rightarrow & \Gamma(T', g^*)
\end{array}
\]

For \( ii \), note that \( \xi \) induces the identities \( f^* \mathcal{F} = f^* \mathcal{A} = d^* \mathcal{A} = g^* \mathcal{F} \), and the result follows. 

2.24. Cohomological obstructions. Now fix \( \mathcal{G} \in \text{Ab}(\mathcal{S}) \) and let \( \mathcal{G}_T = a^* \mathcal{G} \) be the pull-back of \( \mathcal{G} \) to \( T \). By \( 2.23 \) \( ii \) we have \( \mathcal{G}_T = f^* \mathcal{G}_T \), and the pull-back map \( 2.22 \) becomes

\[
(2.25)
\]

which makes \( H^1(\mathcal{T}, \mathcal{G}_T) : (\text{Fib}/\mathcal{S})^0 \rightarrow \text{Set} \) a functor. Since \( \mathcal{G} \in \text{Ab}(\mathcal{S}) \), by \( 2.23 \) \( ii \) again we have \( f^* = g^* \) for any 2-isomorphism \( f \Rightarrow g \), which is to say that the functor \( H^1(\mathcal{T}, \mathcal{G}_T) : (\text{Fib}/\mathcal{S})^0 \rightarrow \text{Set} \) is stable. From now on, if there is no other confusion, we simply write \( \mathcal{G} \in \text{Ab}(\mathcal{T}) \) for \( \mathcal{G}_T \). Thus we can take \( F = H^1(\mathcal{T}, \mathcal{G}) = H^1(\mathcal{T}, \mathcal{G}_T) \) in \( 2.15 \) and obtain \( \chi(A)^{H^1(\mathcal{T}, \mathcal{G})} \), the \( H^1(\mathcal{T}, \mathcal{G}) \)-category.

2.26. Example (Brauer-Manin obstruction). The cohomological Brauer-Grothendieck group is also defined for any fibred category \( \mathcal{T} \) over \( \mathcal{S} \). It is the étale cohomology group \( \text{Br} := H^2_{\text{ét}}(\mathcal{T}, \mathcal{G}_{\mathcal{m}, \mathcal{T}}) \), where \( \mathcal{G}_{\mathcal{m}, \mathcal{T}} \) is the pull-back sheaf of \( \mathcal{G}_{\mathcal{m}, \mathcal{S}} \) to \( \mathcal{T} \). When \( \mathcal{T} \) is a scheme, this agrees with the cohomological Brauer-Grothendieck group. The Brauer group of a stack is considered in, for example, Bertolin and Galluzzi [1, Thm. 3.4], and Zahnd [24, 4.3]. Let \( F = \text{Br} = H^2_{\text{ét}}(-, \mathcal{G}_{\mathcal{m}}) \), we obtain the Brauer-Manin category \( \chi(A)^{\text{Br}} \).

3. The generalized descent obstruction and descent by torsors

In order to generalize the classical descent theory from varieties to fibred categories, we need to reformulate everything in details. Since groups are not necessarily commutative, we cannot use cohomology functors on abelian sheaves discussed in \( 2.19 \) and shall do in a different manner. Let us first recall the

3.1. Definition (Torsors). Let \( C \) be a site and \( \text{Sh}(C) \) the category of sheaves of sets on \( C \).

(a) Let \( \mathcal{U} \in \text{Sh}(C) \). A \( \mathcal{U} \)-group is a group object in the localized category \( \text{Sh}(C)/\mathcal{U} \). Let \( \mathcal{A} \) be a \( \mathcal{U} \)-group. An object \( \mathcal{V} \in \text{Sh}(C)/\mathcal{U} \) is a (right) \( \mathcal{U} \)-object over \( \mathcal{U} \) if it is acted by \( \mathcal{A} \) on the right.

(b) Let \( \mathcal{V} \) be a \( \mathcal{U} \)-object over \( \mathcal{U} \). Then \( \mathcal{V} \) is a (right) \( \mathcal{U} \)-torsor over \( \mathcal{U} \) if the following two conditions hold:

(i) the morphism \( \mathcal{V} \rightarrow \mathcal{U} \) is an epimorphism, and

(ii) the morphism

\[
\mathcal{V} \times_{\mathcal{U}} \mathcal{A} \rightarrow \mathcal{V} \times_{\mathcal{U}} \mathcal{V}
\]

\[
(y, g) \mapsto (y, yg)
\]

is an isomorphism.

Morphisms of \( \mathcal{U} \)-torsors over \( \mathcal{U} \) are morphisms in \( \text{Sh}(C)/\mathcal{U} \) compatible with the action of \( \mathcal{A} \). Let \( \mathcal{V} \) be a \( \mathcal{U} \)-torsor over \( \mathcal{U} \). If \( \mathcal{V} \rightarrow \mathcal{U} \) has a section, we call \( \mathcal{V} \) trivial, which is equivalent to say that \( \mathcal{V} \cong \mathcal{U} \), where \( \mathcal{U} \) is the torsor \( \mathcal{U} \) acted on by itself on the right. Left torsors are defined in a similar way.
(c) For \( U \in \mathcal{C} \), we write \( U = h_U^\# \) for the sheafification of the presheaf \( h_U = \text{Hom}_{\mathcal{C}}(-, U) \in \text{PSh}(\mathcal{C}) \).

Then for any \( \mathcal{G} \in \text{SGrp}(\mathcal{C}) \), by a \( \mathcal{G} \)-torsor over \( U \) we mean a \( \mathcal{G} \)-torsor over \( U \). Denote by \( \text{Tors}(\mathcal{C}/U, \mathcal{G}) \) the set of \( \mathcal{G} \)-torsors over \( U \).

(d) Let \( e \in \text{Sh}(\mathcal{C}) \) be a final object of the topos \( \text{Sh}(\mathcal{C}) \) and \( \mathcal{G} \) a \( e \)-group, which is to say that \( \mathcal{G} \) is an object of \( \text{SGrp}(\mathcal{C}) \). A \( \mathcal{G} \)-torsor over \( C \) is a \( \mathcal{G} \)-torsor over \( e \). In particular, it is in \( \text{Sh}(\mathcal{C}) \). Denote by \( \text{Tors}(\mathcal{C}, \mathcal{G}) \) the set of \( \mathcal{G} \)-torsors over \( C \).

(e) Denote by \( \text{H}^1(\mathcal{C}/U, \mathcal{G}) \) (resp. \( \text{H}^1(\mathcal{C}, \mathcal{G}) \)) the set of isomorphism classes in \( \text{Tors}(\mathcal{C}/U, \mathcal{G}) \) (resp. \( \text{Tors}(\mathcal{C}, \mathcal{G}) \)). All trivial torsors form a class, called the neutral element, which makes \( \text{H}^1(\mathcal{C}/U, \mathcal{G}) \) (resp. \( \text{H}^1(\mathcal{C}, \mathcal{G}) \)) a pointed set.

3.2. Torsors as fibred categories. Let \( \tau \in \{ \text{ét}, \text{fppf} \} \) and \( \mathcal{T} \in \text{Fib}/\mathcal{S} \). Recall that \( \mathcal{T}_\tau \) is with the inherited topology from \( \mathcal{S}_\tau = \mathcal{S}(\text{Sch}/\mathcal{S})_\tau \). Let \( \mathcal{G} \in \text{SGrp}(\mathcal{T}_\tau) \) and \( \mathcal{Y} \in \text{Tors}(\mathcal{T}_\tau, \mathcal{G}) \). As presheaf, \( \mathcal{Y} \) is embedded in \( \text{Fib}/\mathcal{T} \) by the functor \( \eta_{\mathcal{T}} \) in (2.3). Denote by \( f : \mathcal{Y} \to \mathcal{T} \), the image \( \eta_{\mathcal{T}}(\mathcal{Y}) \in \text{Fib}/\mathcal{T} \), called the fibred category associated to \( \mathcal{Y} \), and we simply write \( f : \mathcal{Y} \to \mathcal{T} \in \text{Tors}(\mathcal{T}_\tau, \mathcal{G}) \) to indicate it. This is justified by the fact that \( \eta_{\mathcal{T}} : \text{PSh}(\mathcal{T}) \xrightarrow{\sim} \text{Fib}_\text{set}/\mathcal{T} \) is an equivalence (2.3). Keep in mind that \( \mathcal{Y} \to \mathcal{T} \to \mathcal{S} \in \text{Fib}/\mathcal{S} \) and the topology inherited from \( \mathcal{T}_\tau \) agrees with \( \mathcal{Y}_\tau \) (inherited directly from \( \mathcal{S}_\tau \)).

3.3. Classify torsors. With notations in 3.2 let \( T \in \mathcal{T} \). Then the first (non-abelian) Čech cohomology \( \check{H}^1(\mathcal{T}/T, \mathcal{G}) \) is defined to be \( \lim_{\mathcal{U}_T} \check{H}^1(\mathcal{U}/T, \mathcal{G}) \) (where \( \mathcal{U} \) runs over all covering of \( T \) in \( \mathcal{T}_\tau \)) and there is a one-to-one correspondence (c.f. Giraud [3] III.3.6.5 (5)) of pointed sets

\[
\text{(3.4)} \quad \check{H}^1(\mathcal{T}/T, \mathcal{G}) \xrightarrow{\sim} \check{H}^1(\mathcal{T}/T, \mathcal{G}).
\]

If \( \mathcal{T} \) has a final object, then \( \check{H}^1(\mathcal{T}/T, \mathcal{G}) \) runs over all covering of \( T \) in \( \mathcal{T}_\tau \) and there is a one-to-one correspondence (c.f. Giraud [3] III.3.6.5 (5)) of pointed sets

\[
\text{(3.5)} \quad \text{H}^1(\mathcal{T}_\tau, \mathcal{G}) \xrightarrow{\sim} \check{H}^1(\mathcal{T}/T, \mathcal{G}).
\]

3.6. The pull-back of a torsor. From now on, we shall write \( \text{H}^1(\mathcal{T}, \mathcal{G}) \to \text{H}^1(T, u^*\mathcal{G}) \). Let \( \mathcal{G} \in \text{SGrp}(\mathcal{T}_\tau) \) and \( \mathcal{Y} \in \text{Tors}(\mathcal{T}_\tau, \mathcal{G}) \). For any 1-morphism \( u : \mathcal{T}' \to \mathcal{T} \), the pull-back sheaf \( u^*\mathcal{Y} \in \text{Sh}(\mathcal{T}) \) is a \( u^*\mathcal{G} \)-torsor over \( \mathcal{T}' \), called the pull-back of \( \mathcal{Y} \) under \( u \). Thus we have a well-defined map

\[
\text{(3.7)} \quad u^* : \check{H}^1(\mathcal{T}, \mathcal{G}) \to \check{H}^1(\mathcal{T}', u^*\mathcal{G})
\]

\[
\text{(3.8)} \quad [\mathcal{Y}] \mapsto [u^*\mathcal{Y}].
\]

The pull-back of a torsor to a category fibred in sets agrees with that of the fibred category associated to it. To be precise, we have the following

3.9. Lemma. With notations in 3.6 suppose that \( \mathcal{T}' \in \text{Fib}_{\text{set}}/\mathcal{S} \). Then the pull-back functor \( u^* \) commutes with \( \eta \) in (2.4). In other words, there is an equivalence in \( \text{Fib}/\mathcal{T}' \)

\[
\alpha : \eta_{\mathcal{T}'}(u^*\mathcal{Y}) \xrightarrow{\sim} u^*\eta_{\mathcal{T}}(\mathcal{Y}),
\]

where the second \( u^* \) is defined by (2.7).

Proof. Let \( \mathcal{Y} = \eta_{\mathcal{T}}(\mathcal{Y}) = \mathcal{T}_{\mathcal{G}}, \mathcal{Y}' = u^*\mathcal{Y} \) and \( \mathcal{Y}'' = \eta_{\mathcal{T}'}(u^*\mathcal{Y}) = \mathcal{T}_{u^*\mathcal{G}} \). By the constructions of \( \mathcal{T}_{\mathcal{G}} \) and \( \mathcal{T}_{u^*\mathcal{G}} \) in (2.3), and the definition of \( u^*\mathcal{Y} \) (see (2.20)), we may define a 1-morphism \( u^* : \mathcal{Y}'' \to \mathcal{Y} \).
exists a 1-morphism $\alpha$ actually a 1-morphism in $F$

the identity in $F$

where $f$ be a 2-isomorphism in $F$

where $\xi$ suffices to show that $T_{\text{tors}}$ and $Y$

Remark.

3.10. The proof is complete. □

PSh

3.12. Let $H$ which makes $u$ (3.11) becomes (3.7) becomes

Now we assume that $G \in \text{SGrp}(S_r)$ (whose pull-back to any $T_r \in \text{Fib}/S$ is also denoted by $G$), then (3.7) becomes

(3.11) $u^* : H^1_r(T, G) \to H^1_r(T', G)$,

which makes $H^1_r(-, G) : (\text{Fib}/S)^o \to \text{Set}$ a functor.

3.12. Let $G \in \text{SGrp}(S_r)$. Since in general, $T \in \text{Fib}/S$ need not to have a final object, we can not choose $F$ in 2.13 to be the first Cech cohomology in 3.3 Instead we shall use $H^1_r(-, G)$ directly. First we need to verify that

3.13. Lemma. If $G \in \text{SGrp}(S_r)$, then the functor $H^1_r(-, G) : (\text{Fib}/S)^o \to \text{Set}$ is stable.

Proof. Let

be a 2-isomorphism in $\text{Fib}/S$. Then we have the following commutative diagram of functors

$$
\begin{array}{ccc}
H^1_r(T, f^* -) & \xrightarrow{\xi} & H^1_r(T', g^* -) \\
\downarrow f^* & & \downarrow g^* \\
H^1_r(T', f^* -) & \xrightarrow{[f^*G]} & H^1_r(T', g^* -) \\
\end{array}
$$

where $\xi$, is induced by $\xi$. Then the same argument as in the proof of 3.13 shows that $\xi$ induces the identity $f^*G = g^*G \in \text{SGrp}(T_r')$. It follows that $\xi$ induces the isomorphism $f^*G \to g^*G$ in $\text{Tors}(T_r', G)$, and hence the identity on $H^1_r(T', G)$. The proof is complete. □
Then we have $\mathcal{X}(A)^{H^1_{\text{fppf}}(X)}$, the $H^1_{\text{fppf}}(-, \mathcal{G})$-category, and the following

3.14. **Definition.** Define the descent category to be the full subcategory $\mathcal{X}(A)^{\text{desc}}$ of $\mathcal{X}(A)$ whose objects are characterized by

$$\mathcal{X}(A)^{\text{desc}} = \bigcap_{\mathcal{G} \in \text{SGrp}(\mathcal{S}_{\text{fppf}})} \mathcal{X}(A)^{H^1_{\text{fppf}}(X, \mathcal{G})}.$$ 

Clearly, $\mathcal{X}(A)^{\text{desc}}$ is an obstruction category.

3.15. We shall first compare $\text{desc}$ with $\text{cdesc}$ (1.5). For $T \in \text{Fib} / \mathcal{S}$ represented by $T$, $T$ is a final object of $\mathcal{T}$. Recall that in 3.3 we write $H^1_{\text{fppf}}(T, \mathcal{G}) = H^1_{\text{fppf}}(T / T, \mathcal{G})$. Suppose that $\mathcal{G} \in \text{SGrp}(\mathcal{S}_{\tau})$. Then by (3.5) and 3.13, $\mathcal{H}^1_{\text{fppf}}(T, \mathcal{G})$ is viewed as a stable functor on the full sub 2-category of representable objects (see 2.3 (iii)) in $\text{Fib} / \mathcal{S}$. Thus $\mathcal{X}(A)^{H^1_{\text{fppf}}(X, \mathcal{G})}$ is defined if $\mathcal{X}$ and $\mathcal{A}$ are representable.

3.16. **Proposition.** With notations above, let $\mathcal{G} \in \text{SGrp}(\mathcal{S}_{\tau})$. If $\mathcal{X}$ and $\mathcal{A}$ are representable, then $\mathcal{X}(A)^{H^1_{\text{fppf}}(X, \mathcal{G})} = \mathcal{X}(\mathcal{A})^{H^1_{\text{fppf}}(X, \mathcal{G})} = \mathcal{X}(A)^{H^1_{\text{fppf}}(X, \mathcal{G})}$, where the second equality holds if in addition $\mathcal{G} \in \text{Ab}(\mathcal{S}_{\tau})$.

In particular, in the classical case where $X$ is a variety over a number field $k$, the set $X(A_k)^{\text{desc}}$ defined by (3.14) is contained in the classical descent set defined by (1.5), i.e.

$$X(A_k)^{\text{desc}} \subseteq X(A_k)^{\text{cdesc}}.$$ 

**Proof.** Now all of $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{S}$ are representable. Since $H^1_{\text{fppf}}(-, \mathcal{G})$ is identified with $H^1_{\text{fppf}}(-, \mathcal{G})$ via (3.5), the resulting obstruction categories are the same. This shows the first equality. The second equality is clear since if $\mathcal{G} \in \text{Ab}(\mathcal{S}_{\tau})$, $H^1_{\text{fppf}}(\mathcal{X}, \mathcal{G}) = H^1_{\text{fppf}}(\mathcal{X}, \mathcal{G})$.

In the classical case, let $\mathcal{S} = \text{Spec} k$, $X \in \mathcal{S}$ and $\mathcal{X} = X \in \text{Fib}_{\text{Set}} / \mathcal{S}$. Note that in (1.5), $G$ runs over all affine $k$-groups, viewed as a sheaf $\mathcal{G} \in \text{SGrp}(\mathcal{S}_{\text{fppf}})$. Thus the inclusion follows. 

3.17. **The obstruction given by torsors.** We now extend the classical theory of descent by torsors. Recall that a $\tau$-object $\mathcal{G} \in \text{Tors}(\mathcal{S}_{\tau}, \mathcal{G})$ is mapped by the functor $\eta_{\mathcal{T}}$ to $f : \mathcal{Y} \rightarrow \mathcal{T} \in \text{Fib} / \mathcal{S}$ associated to it (see 3.2), which represents a class $A = [\mathcal{G}] \in H^1_{\text{fppf}}(\mathcal{T}, \mathcal{G})$. We write $\mathcal{X}(A)^{f} = \mathcal{X}(A)^{A}$, the obstruction category given by $A$.

3.18. **Proposition.** Suppose that $A \in \text{Fib}_{\text{Set}} / \mathcal{S}$. Let $\mathcal{G} \in \text{SGrp}(\mathcal{S}_{\tau})$ and $f : \mathcal{Y} \rightarrow \mathcal{X} \in \text{Tors}(\mathcal{X}_{\tau}, \mathcal{G})$. Then $\text{Ob}(\mathcal{X}(A)^{f})$ is characterized by

$$\mathcal{X}(A)^{f} = \bigcup_{\sigma \in H^1_{\text{fppf}}(\mathcal{S}, \mathcal{G})} f^{\sigma}(\mathcal{Y}^{\sigma}(A)),$$ 

where $f^{\sigma} : \mathcal{Y}^{\sigma} \rightarrow \mathcal{X} \in \text{Tors}(\mathcal{X}_{\tau}, \mathcal{G}^{\sigma})$ is the twist of $\mathcal{Y}$ by $\sigma$ (see (3.28)).

This leads to the following

3.19. **Corollary.** Suppose that $A \in \text{Fib}_{\text{Set}} / \mathcal{S}$. Then we have

$$\mathcal{X}(A)^{\text{desc}} = \bigcap_{\mathcal{G} \in \text{SGrp}(\mathcal{S}_{\text{fppf}})} \bigcup_{\text{f : \mathcal{Y} \rightarrow \mathcal{X} \in \text{Tors}(\mathcal{X}_{\text{fppf}}, \mathcal{G})}} f^{\sigma}(\mathcal{Y}^{\sigma}(A)).$$ 

**Proof.** Since $H^1_{\text{fppf}}(\mathcal{X}, \mathcal{G})$ is the pointed set of isomorphism classes of $\text{Tors}(\mathcal{X}_{\text{fppf}}, \mathcal{G})$ (see 3.17), the result follows immediately from (3.18). 

$\square$
3.20. **Corollary.** With assumptions and notations in 3.18, we have

\[ \mathcal{X}(S) = \bigcup_{\sigma \in H^1_{\tau}(S, \mathcal{F})} f^\sigma(\mathcal{Y}(S)). \]

**Proof.** Take \( A = S \) in 3.18 and note that \( \mathcal{X}(S)^f = \mathcal{X}(S) \). Then the result follows. \( \square \)

The proof of 3.18 will be given in 3.30. Note that the assumption that \( A \in \text{Fib}_{\text{Set}/S} \) holds automatically in the classical case where \( X \) is a \( k \)-variety.

3.21. **Bitorsors and the inverse torsor.** Let \( T \in \text{Fib}/S \) and \( \mathcal{G}, \mathcal{H} \in \text{SGrp}(T_\tau) \). For \( \mathcal{Y} \in \text{Tors}(T_\tau, \mathcal{G}) \), if \( \mathcal{H} \) acts on \( \mathcal{Y} \) on the left such that \( \mathcal{Y} \) is also a left \( \mathcal{H} \)-torsor over \( T_\tau \), \( \mathcal{Y} \) is called a \( \mathcal{H} \)-\( \mathcal{G} \)-bitorsor.

Now let \( \mathcal{G} \) and \( \mathcal{H} \) act on \( \mathcal{Y} \) by \( \mathcal{G} \times \mathcal{Y} \times \mathcal{H} \to \mathcal{Y} \)

\[(g, y, h) \mapsto h^{-1}yg^{-1}.\]

Then \( \mathcal{Y} \) becomes a \( \mathcal{G}-\mathcal{H} \)-bitorsor, called the **inverse torsor** of \( \mathcal{Y} \), denoted by \( \mathcal{Y}^\circ \). Of course, we have \( \mathcal{Y}^\circ \in \text{Tors}(T_\tau, \mathcal{H}) \).

In particular, for arbitrary \( \mathcal{Y} \in \text{Tors}(T_\tau, \mathcal{G}) \), let \( \text{ad}(\mathcal{Y}) = \text{Aut}_\mathcal{G}(\mathcal{Y}) \) so that it acts \( \mathcal{Y} \) on the left. Then \( \mathcal{Y} \) becomes an \( \text{ad}(\mathcal{Y}) \)-\( \mathcal{G} \)-bitorsor, so that \( \mathcal{Y}^\circ \in \text{Tors}(T_\tau, \text{ad}(\mathcal{Y})) \). See [6, III.1.5] for more details.

3.22. **The contracted product of torsors.** Let \( \mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{SGrp}(T_\tau) \). Let \( \mathcal{Y} \) (resp. \( \mathcal{Z} \)) be a \( \mathcal{F}-\mathcal{G} \)- (resp. \( \mathcal{G}-\mathcal{H} \))-bitorsor. The **contracted product** [6, III.1.3] of \( \mathcal{Y} \) and \( \mathcal{Z} \) is the quotient of \( \mathcal{Y} \times \mathcal{Z} \) by the diagonal operation of \( \mathcal{G} \)

\[ (y, z, g) \mapsto (yg, g^{-1}z), \]

which exists and is unique up to an isomorphism. It is a \( \mathcal{F}-\mathcal{H} \)-bitorsor over \( T_\tau \), denoted by \( \mathcal{Y} \mathcal{G} \times \mathcal{Z} \).

The contracted product is associative, i.e., if given \( E \in \text{SGrp}(T_\tau) \) and an \( E \)-\( \mathcal{F} \)-bitorsor \( \mathcal{X} \), we have a canonical isomorphism of \( E \)-\( \mathcal{H} \)-bitorsors [6, III.1.3.5]

\[(\mathcal{Y} \mathcal{G} \times \mathcal{Z}) \mathcal{E} \cong \mathcal{X} \mathcal{E} \mathcal{G} \times \mathcal{E} \mathcal{Z}, \]

Also it commutes with the pull-back, that is, for 1-morphism \( u : \mathcal{T}' \to \mathcal{T} \), we have an isomorphism of \( u^* \mathcal{F} \)-\( u^* \mathcal{H} \)-bitorsors

\[(3.24) \quad u^*(\mathcal{Y} \mathcal{G} \times \mathcal{Z}) \cong u^* \mathcal{Y} \mathcal{G} \times u^* \mathcal{Z}. \]

3.25. **Lemma.** Let \( \mathcal{Y} \) be a \( \mathcal{H}-\mathcal{G} \)-bitorsor. Then we have isomorphisms

\[ \mathcal{Y} \mathcal{G} \cong \mathcal{Y}, \quad \mathcal{H} \mathcal{G} \cong \mathcal{Y} \quad (\text{as } \mathcal{H}-\mathcal{G} \text{-bitorsors}), \]

\[ \mathcal{Y} \mathcal{H} \cong \mathcal{Y} \mathcal{G} \mathcal{H} \cong \mathcal{Y} \quad \text{(as } \mathcal{H} \text{-bitorsors}), \]

\[ \mathcal{Y} \mathcal{H} \cong \mathcal{Y} \mathcal{G} \mathcal{H} \cong \mathcal{Y} \quad \text{(as } \mathcal{G} \text{-bitorsors}), \]

where \( \mathcal{G} \) and \( \mathcal{H} \) are the bitorsors by the obvious actions.

**Proof.** See [6, III.1.3.1.3 and III.1.6.5]. \( \square \)
Let \( \mathcal{I} \in \text{SGrp}(\mathcal{S}) \) (whose pull-back to any \( \mathcal{T}_r \in \text{Fib} / \mathcal{S} \) is also denoted by \( \mathcal{I} \)) and \( \sigma \in \check{H}_1^1(\mathcal{S}, \mathcal{I}) \approx H_1^1(\mathcal{S}, \mathcal{I}) \) (see (3.3)). Let

\[
\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{g} & \mathcal{T} \\
\downarrow{a'} & & \downarrow{a} \\
\mathcal{S} & \xrightarrow{q} & \mathcal{A}
\end{array}
\]

be a 1-morphism in \( \text{Fib} / \mathcal{S} \). Then we have

(3.27)
\[
g^* \circ a^* = a'^*
\]

for the pull-back maps defined by (3.11).

3.28. Definition (Twist of a torsor). With notations in 3.26 let \( \sigma \in \check{H}_1^1(\mathcal{S}, \mathcal{I}) \approx H_1^1(\mathcal{S}, \mathcal{I}) \) and \( \mathcal{X} \in \text{Tors}(\mathcal{S}, \mathcal{I}) \) represents \( \sigma \). For any \( \mathcal{Y} \in \text{Tors}(\mathcal{T}_r, \mathcal{I}) \) with \( \eta_\mathcal{T}(\mathcal{Y}) = (f : \mathcal{Y} \to \mathcal{T}) \) associated to it, we know the class \( [\mathcal{Y} \times (a^* \mathcal{X})] \) is independent of the choice of \( \mathcal{X} \) (see 3.22). A twist of \( \mathcal{Y} \) by \( \sigma \in \check{H}_1^1(\mathcal{S}, \mathcal{I}) \) is any representative of the class \( [\mathcal{Y} \times (a^* \mathcal{X})] \), denoted by \( \mathcal{Y}^\sigma \in \text{Tors}(\mathcal{T}_r, \mathcal{I}) \), where \( \mathcal{Y}^\sigma \in \text{SGrp}(\mathcal{T}_r) \), depending on \( \mathcal{X} \), is in the isomorphism class \( [a^* \text{ad}(\mathcal{X})] \) (independent of \( \mathcal{X} \)) of groups. We also write \( f^* : \mathcal{Y}^\sigma \to \mathcal{T} \) for \( \eta_\mathcal{T}(\mathcal{Y}^\sigma) \) associated to \( \mathcal{Y}^\sigma \) (see 3.2).

3.29. Lemma. With the above notations, we have \( g^* (\mathcal{Y}^\sigma) \equiv \mathcal{Y}^\sigma_r \) in \( \text{SGrp}(\mathcal{T}_r) \) (this suggests us to write \( \mathcal{Y}^\sigma \) for \( \mathcal{Y}^\sigma_r \) for any \( \mathcal{T} \in \text{Fib} / \mathcal{S} \) if no confusion caused) and \( (g^* \mathcal{Y})^\sigma \equiv g^*(\mathcal{Y}^\sigma) \) in \( \text{Tors}(\mathcal{T}_r, \mathcal{I}) \).

Proof. Let \( \mathcal{X} \in \text{Tors}(\mathcal{S}, \mathcal{I}) \) represents \( \sigma \in \check{H}_1^1(\mathcal{S}, \mathcal{I}) \). Then by (3.21) and 3.28

\[
\mathcal{Y}^\sigma_r \equiv a^* \text{ad}(\mathcal{X}) = (g^* \circ a^*) \text{ad}(\mathcal{X}) \equiv g^* \mathcal{Y}^\sigma
\]

and the first statement follows.

For the second statement, by (3.24), (3.27) and (3.28) we have

\[
(g^* \mathcal{Y})^\sigma = g^* \mathcal{Y} \times (a^* \mathcal{X}) = g^* \mathcal{Y} \times g^*(a^* \mathcal{X}) = g^*(\mathcal{Y} \times (a^* \mathcal{X})) = g^*(\mathcal{Y}^\sigma).
\]

The proof is complete.

3.30. Proof of 3.18. Proposition. Let \( \mathcal{Y} \in \text{Tors}(\mathcal{X}, \mathcal{I}) \) whose image under \( \eta_\mathcal{X} \) is \( f : \mathcal{Y} \to \mathcal{X} \). By definition, \( x \in \mathcal{X}(\mathcal{A})^r \) means that the evaluation of \( [\mathcal{Y}] \in \check{H}_1^1(\mathcal{X}, \mathcal{I}) \) at \( x \) comes from \( H_1^1(\mathcal{S}, \mathcal{I}) \), that is, there exists \( \sigma \in \check{H}_1^1(\mathcal{S}, \mathcal{I}) \approx H_1^1(\mathcal{S}, \mathcal{I}) \) such that

(3.31)
\[
[\mathcal{Y}] (x) = q^* \sigma \in H_1^1(\mathcal{X}, \mathcal{I}).
\]

Let \( \mathcal{X} \in \text{Tors}(\mathcal{S}, \mathcal{I}) \) represents \( \sigma \). Then (3.31) means that \( x^* \mathcal{Y} \equiv q^* \mathcal{X} \) in \( \text{Tors}(\mathcal{A}, \mathcal{I}) \), which, by 3.28, is to say that

\[
(x^* \mathcal{Y})^\sigma := x^* \mathcal{Y} \times (q^* \mathcal{X}) \equiv q^* \mathcal{X} \times (q^* \mathcal{X})
\]

in \( \text{Tors}(\mathcal{A}, \mathcal{I}) \) (note the notation for \( \mathcal{Y}^\sigma \) in 3.29). By 3.25, \( q^* \mathcal{X} \times (q^* \mathcal{X}) \equiv (\mathcal{Y}^\sigma)_r \in \text{Tors}(\mathcal{A}, \mathcal{I}) \). On the other hand, by 3.29, \( x^* (\mathcal{Y}^\sigma) \equiv (x^* \mathcal{Y})^\sigma \). It follows that (3.31) is equivalent to that \( x^* (\mathcal{Y}^\sigma) \in \text{Tors}(\mathcal{A}, \mathcal{I}) \) is trivial, i.e., \( x^* (\mathcal{Y}^\sigma) \to e \) has a section, where \( e \) is a final object of \( \text{Sh}(\mathcal{A}) \). Note that \( \mathcal{A} \in \text{Fib}_{\text{Set}} / \mathcal{S} \), by 3.4, there is an equivalence

(3.32)
\[
\eta_\mathcal{A}(x^* (\mathcal{Y}^\sigma)) \approx (\mathcal{Y}^\sigma)_x,
\]
where \((Y^\sigma)_x := x^*(Y^\sigma) \in \text{Fib}/A\). Thus that \(x^*(Y^\sigma) \to e\) has a section
\[
\Leftrightarrow \eta_A(x^*(Y^\sigma)) \to A \text{ has a section in } \text{Fib}/A \quad (\eta_A \text{ is an equivalence}),
\]
\[
\Leftrightarrow (Y^\sigma)_x \to A \text{ has a section in } \text{Fib}/A \quad (\text{by } \ref{lem:second-coh-2}),
\]
\[
\Leftrightarrow (Y^\sigma)_x \to A \text{ has a quasi-section in } \text{Fib}/S \quad (\text{by } \ref{prop:second-coh-3}),
\]
\[
\Leftrightarrow x \in f^*(Y^\sigma(A)) \quad (\text{by } \ref{def:second-coh-4}).
\]
The proof is complete. \(\square\)

4. The second descent obstruction and descent by gerbes

We now turn back to abelian sheaves. Under the discussion in \[\ref{lem:second-coh-1}\] and \[\ref{prop:second-coh-2}\] we define the second (abelian) descent obstruction on a fibred category and formulate the idea of descent by gerbes.

4.1. The obstruction given by second cohomology. Let \(\tau \in \{\text{ét}, \text{fppf}\}, \mathcal{G} \in \text{Ab}(S_\tau)\) (whose pull-back to any \(\mathcal{T}_\tau\) is also denoted by \(\mathcal{G}\)) and \(F = H^2_{\text{ét}}(-, \mathcal{G})\), as in \[\ref{def:second-coh-2}\]. We have \(\mathcal{X}(A)^{H^2_{\text{ét}}(X, \mathcal{G})}\), the \(H^2_{\text{ét}}(X, \mathcal{G})\)-category. Recall that elements of \(H^2_{\text{ét}}(X, \mathcal{G})\) are in one-to-one correspondence with those in \(H^2(X, \mathcal{G})\), the \(\mathcal{G}\)-equivalence classes of gerbes over \(X_{\tau}\) bounded by \(Y\) (see \[\ref{lem:second-coh-3}\] and \[\ref{lem:second-coh-4}\] below). Thus any \(A \in H^2_{\text{ét}}(X, \mathcal{G})\) is represented by a gerbe \(f : Y \to X \in \text{Gerb}(X_{\tau}, \mathcal{G})\). That is, in \[\ref{def:second-coh-2}\], the class \([Y]\) maps to \(A\). Then we have \(\mathcal{X}(A)^f := \mathcal{X}(A)^A\), the obstruction given by \(f\).

4.2. Definition. Define the second descent category to be the full subcategory \(\mathcal{X}(A)^{2-\text{desc}}\) of \(\mathcal{X}(A)\) whose objects are characterized by
\[
\mathcal{X}(A)^{2-\text{desc}} = \bigcap_{\mathcal{G} \in \text{Ab}(S_{\text{ét}})} \mathcal{X}(A)^{H^2_{\text{ét}}(X, \mathcal{G})}.
\]

It is clear that \(\mathcal{X}(A)^{2-\text{desc}}\) is an obstruction category satisfying that \(\mathcal{X}(A)^{2-\text{desc}} \subseteq \mathcal{X}(A)^{H^2_{\text{ét}}(X, \mathcal{G})}\) for all \(\mathcal{G} \in \text{Ab}(S_{\text{ét}})\). In particular, we have \(\mathcal{X}(A)^{2-\text{desc}} \subseteq \mathcal{X}(A)^{\text{Br}}\) since \(\mathbb{G}_m, S\) represents a sheaf in \(\text{Ab}(S_{\text{ét}})\). Unfortunately, we do not know the relation between \(\mathcal{X}(A)^{2-\text{desc}}\) and \(\mathcal{X}(A)^{\text{desc}}\). The goal of the rest of this section is to obtain full subcategories between \(\mathcal{X}(S)\) and \(\mathcal{X}(A)^{\text{desc}}\).

We now recall the

4.3. Definition. Let \(C\) be a site.

(a) A stack in groupoid over \(C\) is a fibred category \(\mathcal{Y} \to C\) such that for any covering \(\mathcal{U} \in C\), the functor \(\mathcal{Y}_\mathcal{U} \to DD(\mathcal{U})\) which associates to an object its canonical descent datum, is an equivalence \[\ref{lem:second-coh-2} \text{Tag 026E}]. Note that if \(\mathcal{C} = \mathcal{T}_\tau\) where \(\mathcal{T} \in \text{Fib}/S\), then we also have that \(\mathcal{Y} \in \text{Fib}/S\) and the topology inherited from \(\mathcal{T}_\tau\) agrees with \(\mathcal{Y}_\tau\) (inherited directly from \(S_\tau\)). In this text, by a stack we always mean a stack in groupoid.

(b) A stack \(\mathcal{Y} \to C\) is a gerbe over \(C\) if
\[
\begin{align*}
(i) & \text{ for any } U \in C, \text{ there exists a covering } \{U_i \to U\} \in C \text{ such that } \mathcal{Y}_{U_i} \text{ is nonempty, and} \\
(ii) & \text{ for any } U \in C \text{ and } x, y \in \mathcal{Y}_U, \text{ there exists a covering } \{U_i \to U\} \in C \text{ such that } x|_{U_i} \cong y|_{U_i} \text{ in } \mathcal{Y}_{U_i}.
\end{align*}
\]
If \(\mathcal{Y} \to C\) has a section, we call \(\mathcal{Y}\) trivial.

(c) If a gerbe \(\mathcal{Y} \to C\) is bounded by a \(\mathcal{G} \in \text{Ab}(C)\) \[\ref{def:second-coh-2} \text{IV.2.2}\], we denote it by \(\mathcal{Y} \overset{\mathcal{G}}{\rightarrow} C\). One may define \(\mathcal{G}\)-equivalence between them \[\ref{def:second-coh-2} \text{IV.2.2} \text{.7}\]. Denote by \(\text{Gerb}(C, \mathcal{G})\) the set of gerbes over \(C\) bounded by \(\mathcal{G}\), whose quotient by \(\mathcal{G}\)-equivalence is denoted by \(H^2(C, \mathcal{G})\). In this section, we always consider gerbes bounded by abelian sheaves \(\mathcal{G} \in \text{Ab}(C)\). Then all trivial gerbes form a class, called the neutral element, which makes \(H^2(C, \mathcal{G})\) a pointed set.
If \( C = T_{\tau} \), \( T \in \text{Fib}/S \), then as in [m], we have \((\mathcal{Y} \to T \to S) \in \text{Fib}/S\), and the topology inherited from \( T_{\tau} \) agrees with \( \mathcal{Y}_{\tau} \).

4.4. **Remark.** If \( \mathcal{G} \) is not commutative, there may exists more than one neutral class.

As an analogue to descent by torsors (see 3.18), we give

4.5. **Theorem** (Descent by gerbes). Suppose that \( A \in \text{Fib}_{\text{Set}}/S \). Let \( \mathcal{G} \in \text{Ab}(S_{\tau}) \) and \( f : \mathcal{Y} \to X \in \text{Gerb}(X_{\tau}, \mathcal{G}) \). Then \( \text{Ob}(X(A)^f) \) is characterized by

\[
X(A)^f = \bigcup_{\sigma \in \text{H}^2(S_{\tau}, \mathcal{G})} f^\sigma(Y^\sigma(A)),
\]

where \( f^\sigma : Y^\sigma \to X \in \text{Gerb}(X_{\tau}, \mathcal{G}) \) is the twist of \( Y \) by \( \sigma \) (see 4.18).

The proof will be given in 4.20.

4.6. **Corollary.** Suppose that \( A \in \text{Fib}_{\text{Set}}/S \). Then we have

\[
X(A)^{2\text{-desc}} = \bigcup_{\mathcal{G} \in \text{Ab}(S_{\tau})} \bigcup_{\sigma \in \text{H}^2(S_{\tau}, \mathcal{G})} f^\sigma(Y^\sigma(A)).
\]

**Proof.** This follows immediately from 4.1 and 4.5. \( \square \)

4.7. **Corollary.** With assumptions and notations in 4.5, we have

\[
X(S) = \bigcup_{\sigma \in \text{H}^2(S_{\tau}, \mathcal{G})} f^\sigma(Y^\sigma(S)).
\]

**Proof.** Take \( A = S \) in 4.5 and note that \( X(S)^f = X(S) \). Then the result follows. \( \square \)

4.8. **Classify gerbes.** We recall notations in 2.19. Let \( \tau \in \{\text{et}, \text{fppf}\} \) and \( T \in \text{Fib}/S \). Recall that \( T_{\tau} \) is with the inherited topology from \( S_{\tau} = (\text{Sch}/S)_{\tau} \). From now on, we shall write \( \text{H}^2(T_{\tau}, -) \) for \( \text{H}^2(T_{\tau}, -) \). For \( \mathcal{G} \in \text{Ab}(T_{\tau}) \), there is a one-to-one correspondence (c.f. [1] IV.3.4.2 (i))

\[
\text{H}^2(T_{\tau}, \mathcal{G}) \xrightarrow{\sim} \text{H}^2(T, \mathcal{G}),
\]

which maps the neutral element to 0.

4.10. **The pull-back of a gerbe.** Let \( \tau \in \{\text{et}, \text{fppf}\} \) and \( \mathcal{G} \in \text{Ab}(S_{\tau}) \). For any \( f : \mathcal{Y} \to T \in \text{Gerb}(T_{\tau}, \mathcal{G}) \) and 1-morphism \( u : T' \to T \), we have the

4.11. **Definition** (Pull-back of a gerbe). The pull-back of the gerbe \( \mathcal{Y} \) under \( u \), denoted by \( u^* \mathcal{Y} \), is the pull-back \( u^* \mathcal{Y} \) as fibred categories (see 2.7), indicated in the following 2-cartesian diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{f} & T' \\
\downarrow u' & & \downarrow u \\
\mathcal{Y} & \xrightarrow{u} & T
\end{array}
\]

Up to an equivalence, it is a gerbe over \( T'_{\tau} \) bounded by \( \mathcal{G} \) whose structure functor is \( f' \) [22, Tag 06P2, Tag 06P3], and the class \([u^* \mathcal{Y}]\) depends only on the class \([\mathcal{Y}]\).
Thus we have a well-defined map

\[ u^* : H^2_\tau(T, \mathcal{G}) \rightarrow H^2_\tau(T', \mathcal{G}) \]

\[ [Y] \mapsto [u^*Y]. \]

Under the correspondence (4.9) we have the following commutative diagram

(4.12)

\[
\begin{array}{ccc}
H^2_\tau(T, \mathcal{G}) & \xrightarrow{u^*} & H^2_\tau(T', \mathcal{G}) \\
\downarrow & & \downarrow \\
H^2_\tau(T, \mathcal{G}) & \xrightarrow{u^*} & H^2_\tau(T', \mathcal{G})
\end{array}
\]

where the lower map is defined by (2.25). See [6, V.1.5.3].

4.13. **The contracted product of gerbes.** For \( \mathcal{G} \in \text{Ab}(S_\tau) \) and \( \mathcal{Y}, \mathcal{Z} \in \text{Gerb}(T_\tau, \mathcal{G}) \), the *contraction product* [6, IV.2.4] of \( \mathcal{Y} \) and \( \mathcal{Z} \) exists and is unique up to a \( \mathcal{G} \)-equivalence. It is also a gerbe over \( T_\tau \) bounded by \( G \), denoted by \( \mathcal{Y} \times \mathcal{Z} \). This gives a well-defined pairing

\[ (\cdot, \cdot)_c : H^2_\tau(T, \mathcal{G}) \times H^2_\tau(T, \mathcal{G}) \rightarrow H^2_\tau(T, \mathcal{G}) \]

\[ ([Y], [Z]) \mapsto [Y \times Z] \]

which fits in to the following commutative diagram

(4.14)

\[
\begin{array}{ccc}
H^2_\tau(T, \mathcal{G}) \times H^2_\tau(T, \mathcal{G}) & \xrightarrow{(\cdot, \cdot)_c} & H^2_\tau(T, \mathcal{G}) \\
\downarrow & & \downarrow \\
H^2_\tau(T, \mathcal{G}) \times H^2_\tau(T, \mathcal{G}) & \xrightarrow{(\cdot, \cdot)_c} & H^2_\tau(T, \mathcal{G})
\end{array}
\]

where the vertical maps come from the correspondence (4.9) and the lower pairing is the addition law of the abelian group \( H^2_\tau(T, \mathcal{G}) \). See [6, IV.3.3.2 (i), IV.3.4.2 (i)].

4.15. **Lemma.** Let \( u : T' \rightarrow T \) be a 1-morphism. Then we have a commutative diagram

\[
\begin{array}{ccc}
H^2_\tau(T, \mathcal{G}) \times H^2_\tau(T, \mathcal{G}) & \xrightarrow{(\cdot, \cdot)_c} & H^2_\tau(T, \mathcal{G}) \\
\downarrow & & \downarrow \\
H^2_\tau(T', \mathcal{G}) \times H^2_\tau(T', \mathcal{G}) & \xrightarrow{(\cdot, \cdot)_c} & H^2_\tau(T', \mathcal{G})
\end{array}
\]

\[ u^* \]

\[ u'^* \]

\[ u^* \]

\[ u'^* \]

\[ (\cdot, \cdot)_c : H^2_\tau(T, \mathcal{G}) \times H^2_\tau(T, \mathcal{G}) \rightarrow H^2_\tau(T, \mathcal{G}) \]

\[ ([Y], [Z]) \mapsto [Y \times Z] \]

*Proof.* The diagram clearly comes from (4.12), (4.14) and the fact that \( u^* : H^2_\tau(T, \mathcal{G}) \rightarrow H^2_\tau(T', \mathcal{G}) \)

is a homomorphism. \( \square \)

4.16. Let \( \sigma \in H^1(S, \mathcal{G}) \) and

\[
\begin{array}{ccc}
T' & \xrightarrow{g} & T \\
\downarrow & & \downarrow \\
S & \xrightarrow{a} & T
\end{array}
\]

be a 1-morphism in \( \text{Fib}/S \). Then we have

(4.17)

\[ g^* \circ a^* = a'^* \]

for the pull-back maps defined by (2.25).
4.18. **Definition** (Twist of a gerbe). With notations in 4.16 let \( \sigma \in H^2_2(S, \mathcal{G}) \), and \( Z \to S \in \text{Gerb}(S, \mathcal{G}) \) represents \( \sigma \). For any \( f : \mathcal{Y} \to T \in \text{Gerb}(T, \mathcal{G}) \), we know the class \([\mathcal{Y} \times a^*Z] \) is independent of the choice of \( Z \). A twist of \( \mathcal{Y} \) by \( \sigma \in H^2_2(S, \mathcal{G}) \) is any representative of the class \([\mathcal{Y} \times a^*Z] \), denoted by \( f^\sigma : \mathcal{Y}^\sigma \to T \in \text{Gerb}(T, \mathcal{G}) \).

4.19. **Corollary.** With the above notations, we have

\[ (g^*\mathcal{Y})^\sigma = [g^*(\mathcal{Y}^\sigma)]. \]

**Proof.** Considering 4.17 and 4.18 one knows that this is a special case of 4.15

4.20. **Proof of 4.19 Theorem.** By definition, \( x \in \mathcal{X}(A)^f \) means that the evaluation of \([\mathcal{Y}] \in H^2_2(\mathcal{X}, \mathcal{G}) \) at \( x \) comes from \( H^2_2(S, \mathcal{G}) \), that is, there exists \( \sigma \in H^2_2(S, \mathcal{G}) \) such that

\[ (\mathcal{Y})(x) = -q^*\sigma \in H^2_2(A, \mathcal{G}). \]

By 4.19 we have \([\mathcal{Y}_x]) = [(\mathcal{Y}^\sigma)_x] \), where \( \mathcal{Y}_x = x^*\mathcal{Y} \) and \((\mathcal{Y}^\sigma)_x = x^*(\mathcal{Y}^\sigma) \) are pull-backs (see 4.11). Since the lower pairing in 4.14 is in line with the addition law of the cohomology group, we also have

\[ [\mathcal{Y}](x) + q^*\sigma = [\mathcal{Y}_x] + q^*\sigma = [(\mathcal{Y}^\sigma)_x]. \]

Then 4.21 is equivalent to that \([\mathcal{Y}^\sigma]) = 0 \in H^2_2(A, \mathcal{G}) \), i.e., \((\mathcal{Y}^\sigma)_x \) is a trivial gerbe over \( A \), which by definition means that \((\mathcal{Y}^\sigma)_x \to A \) has a section in \( \text{Fib}/A \), i.e., has a quasi-section in \( \text{Fib}/S \) since \( A \in \text{Fib}_{\text{Set}}/S \) (see 2.5 (ii)). By 2.13 the last condition is to say that \( x \in f^\sigma(\mathcal{Y}^\sigma(A)) \). The proof is complete.

5. **The composite obstruction using second descent**

We now use the second descent and theory of descent by gerbes developed in Section 4 to give some full subcategories between \( \mathcal{X}(S) \) and \( \mathcal{X}(A)^\text{desc} \).

5.1. **Composite obstruction categories.**

5.2. **Definition.** Let \( \text{ob} \) be a map sending each \( \mathcal{X} \in \text{Fib}/S \) to an obstruction category (see 2.12) of \( \mathcal{X}(A) \), called an obstruction map. Define the trivial obstruction map to be the constant map sending each \( \mathcal{X} \in \text{Fib}/S \) to \( \mathcal{X}(A) \).

We say that \( \text{ob} \) is functorial if for any 1-morphism \( f : \mathcal{Y} \to \mathcal{X} \) in \( \text{Fib}/S \), we have \( f(\mathcal{X}(A)^\text{ob}) \subseteq \mathcal{X}(A)^\text{ob} \) as a full subcategory.

5.3. **Example.** With the above notations, if \( \text{ob} = F \) in 2.17 then it is functorial since \( F \) is stable and we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}(S) & \xrightarrow{\text{Hom}_{\text{Fib}/S}(S,f)} & \mathcal{Y}(A) \\
\downarrow B(-) & & \downarrow \text{Hom}_{\text{Fib}/S}(A,f) \\
\mathcal{X}(S) & \xrightarrow{\text{A}(-)} & \mathcal{X}(A) \\
\downarrow A(-) & & \downarrow A(-) \\
F(S) & \xrightarrow{F(-)} & F(A)
\end{array}
\]

for all \( A \in F(\mathcal{X}) \) and \( B = F(f)(A) \).

5.4. **Proposition.** Let \( \text{ob} \) be a functorial obstruction map.
(i) Let $\mathcal{X}' \xrightarrow{\phi} \mathcal{X}$ be an equivalence in $\text{Fib}/S$. Then $f(\mathcal{X}'(A)^{\text{ob}}) = \mathcal{X}(A)^{\text{ob}}$ and $\mathcal{X}'(A)^{\text{ob}} = g(\mathcal{X}(A)^{\text{ob}})$.

(ii) Let $\mathcal{Y}' \xrightarrow{\theta} \mathcal{Y}$ be a 2-commutative diagram of 1-morphisms in $\text{Fib}/S$. Then $f'(\mathcal{Y}'(A)^{\text{ob}}) \subseteq f(\mathcal{Y}(A)^{\text{ob}})$. In particular, if $\theta$ is an equivalence over $\mathcal{X}$, (for example, $\mathcal{Y}'$ and $\mathcal{Y}$ are gerbes such that $[\mathcal{Y}'] = [\mathcal{Y}] \in H^2_2(\mathcal{X}, \mathcal{G})$), then $f'(\mathcal{Y}'(A)^{\text{ob}}) = f(\mathcal{Y}(A)^{\text{ob}})$.

Proof. Since $\text{ob}$ is functorial, for (i), we have $\mathcal{X}'(A)^{\text{ob}} = g(f(\mathcal{X}'(A)^{\text{ob}})) \subseteq g(\mathcal{X}(A)^{\text{ob}}) \subseteq \mathcal{X}'(A)^{\text{ob}}$. Thus $\mathcal{X}'(A)^{\text{ob}} = g(\mathcal{X}(A)^{\text{ob}})$ and the other equality is similar.

For (ii), the same reason yields $\theta(\mathcal{Y}'(A)^{\text{ob}}) \subseteq \mathcal{Y}(A)^{\text{ob}}$. Then we have $f'(\mathcal{Y}'(A)^{\text{ob}}) = f(\theta(\mathcal{Y}'(A)^{\text{ob}})) \subseteq f(\mathcal{Y}(A)^{\text{ob}})$.

The proof is complete. \qed

5.5. Definition. For a functorial obstruction map $\text{ob}$, we define the full subcategory $\mathcal{X}(A)^{\text{desc,ob}}$ and $\mathcal{X}(A)^{2\text{-desc,ob}}$ whose objects are characterized by

\begin{align}
\mathcal{X}(A)^{\text{desc,ob}} &= \bigcap_{\mathfrak{G} \in \text{Grp}(\mathcal{S}_{\text{fppf}})} \bigcup_{\mathfrak{f}: \mathcal{Y} \to \mathcal{X} \in \text{Tors}(\mathcal{X}_{\text{fppf}}, \mathfrak{G})} f^\sigma(\mathcal{Y}^\sigma(A)^{\text{ob}}), \\
\mathcal{X}(A)^{2\text{-desc,ob}} &= \bigcap_{\mathfrak{G} \in \text{Ab}(\mathcal{S}_{\text{fppf}})} \bigcup_{\mathfrak{f}: \mathcal{Y} \to \mathcal{X} \in \text{Gerbi}(\mathcal{X}_{\text{fppf}}, \mathfrak{G})} f^\sigma(\mathcal{Y}^\sigma(A)^{\text{ob}}).
\end{align}

5.8. Remark. These are well-defined. More precisely, by 5.4 (iii), $f^\sigma(\mathcal{Y}^\sigma(A)^{\text{ob}})$ is independent of the choice of the twisted gerbe (resp. torsor) $f^\sigma : \mathcal{Y}^\sigma \not\sim \mathcal{X}$ in its isomorphism class.

In particular, taking $\text{ob}$ to be the trivial obstruction map, we see that the characterization 4.6 (resp. 3.19) is independent of the choice of the twisted gerbe (resp. torsor).

5.9. Theorem. Suppose that $A \in \text{FibSet}/S$. Let $\text{ob}$ be functorial and $\delta \in \{\text{desc,2-desc}\}$.

(i) The category $\mathcal{X}(A)^{\delta,\text{ob}}$ is an obstruction category and $(\delta, \text{ob})$ is also functorial. In particular, $\delta$ is functorial.

(ii) The objects of the category $\mathcal{X}(A)^{\delta,\text{ob}}$ satisfying

$$\mathcal{X}(A)^{\delta,\text{ob}} \subseteq \mathcal{X}(A)^{\delta} \cap \mathcal{X}(A)^{\text{ob}}$$

Proof. Since each $\mathcal{Y}^\sigma(A)^{\text{ob}}$ is an obstruction category, by 4.6 and 5.7 (resp. 3.19 and 3.20), we have

$$\mathcal{X}(S) \subseteq \mathcal{X}(A)^{\delta,\text{ob}} \subseteq \mathcal{X}(A)^{\delta} \subseteq \mathcal{X}(A).$$

In particular, $\mathcal{X}(A)^{\delta,\text{ob}}$ is an obstruction category. Since $\text{ob}$ is functorial, $f^\sigma(\mathcal{Y}^\sigma(A)^{\text{ob}}) \subseteq \mathcal{X}(A)^{\text{ob}}$. Thus (ii) is correct since $\mathcal{X}(A)^{\delta,\text{ob}}$ is an intersection of unions of $f^\sigma(\mathcal{Y}^\sigma(A)^{\text{ob}})$.

Next, we show that $(\delta, \text{ob})$ is functorial. For any 1-morphism $g : \mathcal{X}' \to \mathcal{X}$ in $\text{Fib}/S$, we want to verify that

\begin{equation}
\label{equation5.10}
g(\mathcal{X}'(A)^{\delta,\text{ob}}) \subseteq \mathcal{X}(A)^{\delta,\text{ob}}.
\end{equation}
(1) $\delta = 2$-desc. Let $f : \mathcal{Y} \to \mathcal{X}_{\text{et}} \in \text{Gerb}(\mathcal{X}_{\text{et}}, \mathcal{G})$. We have the pull-back of $\mathcal{Y}$ under $g$ (see \ref{4.11}) $f' : \mathcal{Y}' := g^* \mathcal{Y} \to \mathcal{X}' \in \text{Gerb}(\mathcal{X}_{\text{et}}, \mathcal{G})$. Let $\sigma \in H^2_{\text{et}}(S, \mathcal{G})$ and $f^\sigma : \mathcal{Y}^\sigma \to \mathcal{X} \in \text{Gerb}(\mathcal{X}_{\text{et}}, \mathcal{G})$ be the twist (see \ref{4.18}). The pull-back $\mathcal{Y}'' := g^*(\mathcal{Y}^\sigma)$ fits into the following diagram of 2-fibred product in $\text{Fib}/S$

$$
\begin{array}{ccc}
\mathcal{Y}'' & \xrightarrow{f''} & \mathcal{X}' \\
\downarrow{g''} & & \downarrow{g} \\
\mathcal{Y} & \xrightarrow{f'} & \mathcal{X}
\end{array}
$$

Since $\text{ob}$ is functorial, we have

$$g(f''(\mathcal{Y}''(\mathcal{A})^{\text{ob}})) = f^\sigma(g''(\mathcal{Y}''(\mathcal{A})^{\text{ob}})) \subseteq f^\sigma(\mathcal{Y}^\sigma(\mathcal{A})^{\text{ob}}).$$

On the other hand, by \ref{4.19} we have

$$[\mathcal{Y}'''] = [g^*(\mathcal{Y}^\sigma)] = [(g^*\mathcal{Y})^\sigma] = [\mathcal{Y}']'.$$

Thus by \ref{5.4 11}, $f''(\mathcal{Y}''(\mathcal{A})^{\text{ob}}) = f''(\mathcal{Y}'(\mathcal{A})^{\text{ob}})$. It follows that

$$g\left(\bigcup_{\sigma \in H^2_{\text{et}}(S, \mathcal{G})} f^\sigma(\mathcal{Y}''(\mathcal{A})^{\text{ob}})\right) = g\left(\bigcup_{\sigma \in H^2_{\text{et}}(S, \mathcal{G})} f''(\mathcal{Y}''(\mathcal{A})^{\text{ob}})\right) \subseteq \bigcup_{\sigma \in H^2_{\text{et}}(S, \mathcal{G})} f^\sigma(\mathcal{Y}^\sigma(\mathcal{A})^{\text{ob}}).$$

If $f : \mathcal{Y} \to \mathcal{X}$ runs over $\text{Gerb}(\mathcal{X}_{\text{et}}, \mathcal{G})$, then the resulting $f' : \mathcal{Y}' \to \mathcal{X}'$ forms a subset of $\text{Gerb}(\mathcal{X}'_{\text{et}}, \mathcal{G})$. This proves \ref{5.10}, i.e., (2-desc, ob) is functorial.

(2) $\delta = \text{desc}$. The argument for this case is the same as (1), except a few points. Let $\mathcal{Y} \in \text{Tors}(\mathcal{X}_{\text{ppf}}, \mathcal{G})$ and $f : \mathcal{Y} \to \mathcal{X} \in \text{Fib}/\mathcal{X}$ be associated to it (see \ref{3.2}). We have the pull-back $g^* \mathcal{Y} \in \text{Tors}(\mathcal{X}_{\text{ppf}}, \mathcal{G})$ (see \ref{3.6}) and its associated $f' : \mathcal{Y}' \to \mathcal{X}' \in \text{Fib}/\mathcal{X}'$. Let $\sigma \in H^2_{\text{ppf}}(S, \mathcal{G})$ and

$$f^\sigma : \mathcal{Y}^\sigma \to \mathcal{X} \in \text{Tors}(\mathcal{X}_{\text{ppf}}, \mathcal{G}^\sigma),$$

$$f'^\sigma : \mathcal{Y}'^\sigma \to \mathcal{X}' \in \text{Tors}(\mathcal{X}'_{\text{ppf}}, \mathcal{G}^\sigma),$$

be the corresponding twists (see \ref{3.28}). By \ref{3.29} we have an isomorphism over $\mathcal{X}'$

$$
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{\beta} & \tilde{\mathcal{Y}}' \\
\downarrow{f'^\sigma} & & \downarrow{f'^\sigma} \\
\mathcal{X}' & \xrightarrow{\tilde{h}_\sigma} & \mathcal{X}'
\end{array}
$$

where $(\tilde{h}_\sigma : \tilde{\mathcal{Y}}' \to \mathcal{X}') := \tilde{\eta}_{\mathcal{X}'}(g^*(\mathcal{Y}'^\sigma)) \in \text{Fib}/\mathcal{S}$. On the other hand, the pull-back $\mathcal{Y}''' := g^*(\mathcal{Y}^\sigma)$ (see \ref{2.7}) fits into the following 2-cartesian diagram in $\text{Fib}/\mathcal{S}$

$$
\begin{array}{ccc}
\mathcal{Y}''' & \xrightarrow{f'''} & \mathcal{X}' \\
\downarrow{g''} & & \downarrow{g} \\
\mathcal{Y} & \xrightarrow{f'} & \mathcal{X}
\end{array}
$$

Since $\text{ob}$ is functorial, we have

$$g(f''(\mathcal{Y}'''(\mathcal{A})^{\text{ob}})) = f^\sigma(g''(\mathcal{Y}'''(\mathcal{A})^{\text{ob}})) \subseteq f^\sigma(\mathcal{Y}^\sigma(\mathcal{A})^{\text{ob}}).$$
We know from the proof of 3.9 that there exists a 2-commutative diagram in \( \text{Fib}/S \):

\[
\begin{array}{ccc}
\hat{Y}_\sigma & \xrightarrow{\alpha} & Y''_\sigma \\
\downarrow \hat{h}_\sigma & & \downarrow f''_\sigma \\
X' & \xrightarrow{f'} & Y''
\end{array}
\]

Thus apply 5.4 (ii) to \( \beta \) and \( \alpha \), we have

\[
f'^\sigma(Y'^\sigma(A)^{\text{ob}}) = \hat{h}_\sigma(\hat{Y}_\sigma(A)^{\text{ob}}) \subseteq f''_\sigma(Y''_\sigma(A)^{\text{ob}}).
\]

It follows that

\[
g(\bigcup_{\sigma \in H^1_{\text{fpf}}(S, \mathcal{G})} f'^\sigma(Y'^\sigma(A)^{\text{ob}})) \subseteq \bigcup_{\sigma \in H^1_{\text{fpf}}(S, \mathcal{G})} f''_\sigma(Y''_\sigma(A)^{\text{ob}}).
\]

If \( f : Y \to X \) runs over \( \text{Tors}(X'_{\text{fpf}}, \mathcal{G}) \), then the resulting \( f' : Y' \to X' \) forms a subset of \( \text{Tors}(X'_{\text{fpf}}, \mathcal{G}) \). This proves (5.10), i.e., \( (\text{desc}, \text{ob}) \) is functorial.

Finally, taking \( \text{ob} \) be the trivial obstruction map, then \( \delta = (\delta, \text{ob}) \) is functorial. The proof is complete. \( \square \)

5.11. **Obstruction categories contained in the descent ones.** Since for varieties there is no smaller obstruction set than the descent set is discovered. We only focus on obstruction categories not larger than \( X(A)^{\text{desc}} \).

5.12. **Example.** Suppose that \( A \in \text{Fib}_{\text{Set}}/S \).

(i) By 5.9 (i), desc is functorial. Let \( \text{ob} = \text{desc} \) in (5.7). Then we have an obstruction category

\[
X(A)^{2\text{-dd}} = X(A)^{2\text{-desc}, \text{desc}} = \bigcap_{\sigma \in H^1_{\text{fpf}}(S, \mathcal{G})} \bigcup_{\sigma \in H^1_{\text{fpf}}(S, \mathcal{G})} f^\sigma(Y^\sigma(A)^{\text{ob}}).
\]

(ii) Then by 5.9 (i) again, (2-dd) is also functorial. Thus let \( \text{ob} = (\text{2-dd}) \) in 5.5, we obtain \( X(A)^{\text{desc}, 2\text{-dd}} \) and \( X(A)^{\text{2-desc}, 2\text{-dd}} \).

(iii) Still by 5.9 (i), \( (\delta, 2\text{-dd}) \) is functorial, where \( \delta \in \{ \text{desc}, 2\text{-desc} \} \). Thus let \( \text{ob} = (\delta, 2\text{-dd}) \) and \( \delta' \in \{ \text{desc}, 2\text{-desc} \} \). Then one may define \( (\delta', \delta, 2\text{-dd}) \), and so on.

5.13. **Proposition.** Suppose that \( A \in \text{Fib}_{\text{Set}}/S \). Let \( \delta, \delta_i \in \{ \text{desc}, 2\text{-desc} \}, i \in \{1, 2, \ldots \} \) and define \( X(A)^{\delta, 2\text{-dd}} = X(A)^{\delta_i, \delta_{i-1}, \ldots, \delta_1, 2\text{-dd}} \). Then we have

\[
X(S) \subseteq \cdots \subseteq X(A)^{\delta + 1, 2\text{-dd}} \subseteq X(A)^{\delta, 2\text{-dd}} \subseteq \cdots \subseteq X(A)^{\delta_1, 2\text{-dd}} \subseteq X(A)^{2\text{-dd}} \subseteq X(A)^{\delta} \subseteq X(A),
\]

and that all obstruction maps above are functorial.

**Proof.** The inclusions (as full subcategories) follows from 5.9 (i), and the functoriality from 5.9 (ii).

5.14. **Remark.** Combined with 5.10, we note that in the classical case where \( X \) is a \( k \)-variety, each obstruction map appeared in 5.13 yields an obstruction set contained in the classical descent set \( X(A_k)^{\text{desc}} \).
6. Higher descent obstruction and derived obstruction

We propose some new obstructions in this section and discuss some of their properties.

6.1. Higher descent categories. In 2.24 we obtain \( \mathcal{X}(\mathcal{A})^{H_i(\mathcal{X}, \mathcal{G})} \), the \( H_i(\mathcal{X}, \mathcal{G}) \)-category, where \( \mathcal{G} \in \text{Ab}(\mathcal{S}) \) and \( \tau \in \{ \text{ét}, \text{fppf} \} \).

6.2. Definition. For \( i \geq 0 \), define the \( i \)-th \( \tau \)-descent category to be the full subcategory \( \mathcal{X}(\mathcal{A})^{i\text{-desc}_\tau} \) of \( \mathcal{X}(\mathcal{A}) \) whose objects are characterized by

\[
\mathcal{X}(\mathcal{A})^{i\text{-desc}_\tau} = \bigcap_{\mathcal{G} \in \text{Ab}(\mathcal{S})} \mathcal{X}(\mathcal{A})^{H_i(\mathcal{X}, \mathcal{G})}.
\]

It is clear that \( 2\text{-desc} = 2\text{-desc}_\text{ét} \) and for desc we have the following.

6.3. Proposition. If \( \mathcal{X} \) and \( \mathcal{A} \) is representable, then \( \mathcal{X}(\mathcal{A})^{\text{desc}} \subseteq \mathcal{X}(\mathcal{A})^{1\text{-desc}_{\text{fppf}}} \).

Proof. This immediately follows from 6.2 and 3.16.

6.4. Proposition. For \( i \geq 1 \) and \( \tau \in \{ \text{ét}, \text{fppf} \} \), we have

\[
\mathcal{X}(\mathcal{A})^{i\text{-desc}_\tau} \subseteq \mathcal{X}(\mathcal{A})^{(i+1)\text{-desc}_\tau}.
\]

Proof. Let \( \mathcal{G} \in \text{Ab}(\mathcal{S}) \), and \( \mathcal{F} \in C(\mathcal{S})_{\geq 0} \) be an injective resolution of \( \mathcal{G} \) where \( C(\mathcal{S}) \) is the category of complexes of sheaves in \( \text{Ab}(\mathcal{S}) \). Then we have a short exact sequence

\[
0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0,
\]

where \( \mathcal{F} = \mathcal{F}^0 \) and \( \mathcal{G} = \tau_{\leq 1} \mathcal{F}^\bullet \). Since the pull-back of a sheaf is an exact functor and sends acyclic sheaves to acyclic ones, it follows that for each \( i \geq 1 \), we have a natural isomorphism

\[
\delta^i : H^i(\mathcal{X}, \mathcal{F}) \simeq H^{i+1}(\mathcal{X}, \mathcal{G}) : (\text{Fib}/\mathcal{S})^\circ \to \text{Ab}.
\]

Then by 6.5 below we have \( \mathcal{X}^{H^i(\mathcal{X}, \mathcal{F})} = \mathcal{X}(\mathcal{A})^{H^{i+1}(\mathcal{X}, \mathcal{G})} \). Thus let \( \mathcal{G} \) runs over all objects in \( \text{Ab}(\mathcal{S}) \), and we obtain the desired inclusion.

6.5. Lemma. Let \( F, G : (\text{Fib}/\mathcal{S})^\circ \to \text{Set} \) be two stable functors and \( \xi : F \to G \) a natural transformation. For \( A \in F(\mathcal{X}) \) and \( B = \xi(\mathcal{A}) \), we have \( \mathcal{X}(\mathcal{A})^A \subseteq \mathcal{X}(\mathcal{A})^B \).

In particular, if \( \xi \) is a natural isomorphism, \( \mathcal{X}(\mathcal{A})^F = \mathcal{X}(\mathcal{A})^G \).

Proof. The result clearly follows from the commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}(\mathcal{S}) & \xrightarrow{A(-)} & \mathcal{X}(\mathcal{A}) \\
\downarrow B(-) & & \downarrow A(-) \\
\mathcal{G}(\mathcal{S}) & \xrightarrow{\xi_A} & \mathcal{G}(\mathcal{A})
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{F}(\mathcal{S}) & \xrightarrow{\xi} & \mathcal{F}(\mathcal{A}) \\
\downarrow B(-) & & \downarrow A(-) \\
\mathcal{G}(\mathcal{S}) & \xrightarrow{\xi_A} & \mathcal{G}(\mathcal{A})
\end{array}
\]
6.6. Besides the obstructions constructed from functors (see \[\ref{2.15}\]), Harpaz and Schlank \[\ref{13}\] developed a new kind of obstruction using étale homotopy theory (\textit{homotopy obstruction}), and established the connection with some existing obstructions such as \( Br, (\text{ét}, Br) \) and desc. One consequence is the following

6.7. **Theorem** (\[\ref{13}\], Thm. 9.147). Let \( X \) and \( Y \) be smooth geometrically connected varieties over a number field \( k \). Then we have

\[
(X \times_k Y)(\mathbb{A}_k)^{\text{ét}, Br} = X(\mathbb{A}_k)^{\text{ét}, Br} \times Y(\mathbb{A}_k)^{\text{ét}, Br}.
\]

This result ensured the commutativity of Brauer-Manin sets with taking product of two varieties. The same result with \((\text{ét}, Br)\) replaced by \( Br \) was given by Skorobogatov and Zarhin \[\ref{21}\] for smooth projective geometrically integral varieties and the author \[\ref{15}\] for open varieties. Next we propose another new kind of obstructions for fibred categories, given by pseudofunctors. In particular we define the derived obstruction which, under mild assumptions, also has good behavior under a product.

6.8. **Obstructions given by pseudofunctors.** Let \( \mathbb{F} : \text{Fib}/S \to \text{Tri} \) be a pseudofunctor to the 2-category of triangulated categories. Then we have the following diagram of functors

\[
\begin{array}{ccc}
\mathcal{X}(S) & \xrightarrow{\text{Hom}_{\text{Fib}/S}(q, \mathcal{X})} & \mathcal{X}(A) \\
\downarrow F & & \downarrow F \\
\text{Hom}_{\text{Tri}}(\mathbb{F}(S), \mathbb{F}(\mathcal{X})) & \xrightarrow{\varphi_{\mathbb{F}}} & \text{Hom}_{\text{Tri}}(\mathbb{F}(A), \mathbb{F}(\mathcal{X}))
\end{array}
\]

where \( \varphi_{\mathbb{F}} = \text{Hom}_{\text{Tri}}(\mathbb{F}(q), \mathbb{F}(\mathcal{X})) \). Note that this diagram is not necessarily 2-commutative in \( \text{Cat} \), but at least, by the definition of pseudofunctors, for each \( z \in \mathcal{X}(S) \), there is an isomorphism of functors

\[
(6.9) \quad \mathbb{F} \circ \text{Hom}_{\text{Fib}/S}(q, \mathcal{X})(z) = \mathbb{F}(z \circ q) \xrightarrow{\sim} \varphi_{\mathbb{F}} \circ \mathbb{F} = \mathbb{F}(z) \circ \mathbb{F}(q).
\]

This is enough for making the following

6.10. **Definition.** The \( \mathbb{F} \)-category is the full subcategory \( \mathcal{X}(A)^{\mathbb{F}} \) of \( \mathcal{X}(A) \) whose objects are characterized by

\[
\mathcal{X}(A)^{\mathbb{F}} = \{ x \in \mathcal{X}(A) \mid \mathbb{F}(x) \text{ is in the essential image of } \varphi_{\mathbb{F}} \}.
\]

By (6.9), one checks that \( \mathcal{X}(A)^{\mathbb{F}} \) is also an obstruction category (see \[\ref{2.12}\]).

6.11. **The derived obstruction.** With notations in \[\ref{6.8}\], let \( \Lambda \) be a ring. For any \( \mathcal{T} \in \text{Fib}/S \), denote by \( D(\mathcal{T}, \Lambda) \) the derived category of complexes of sheaves of \( \Lambda \)-modules on \( T_{\text{ét}} \). Let \( \mathbb{F} \) be the pseudofunctor defined by

\[
\mathcal{T} \mapsto D(\mathcal{T}, \Lambda), \quad f \mapsto Rf_{**}, \quad \xi \mapsto R\xi_{**}.
\]

Then we define the \textit{derived (obstruction)} category to be the category defined by \( \mathbb{F} \) given above, denoted by \( \mathcal{X}(A)^{\text{der}} \). In other words, \( x \in \mathcal{X}(A)^{\text{der}} \) if and only if there exist \( \theta \in \text{Hom}_{\text{Tri}}(D(S, \Lambda), D(\mathcal{X}, \Lambda)) \) and isomorphism of functors \( R\xi_{**} \xrightarrow{\sim} \theta R\eta_{**} \).

By an argument analogue to \[\ref{5.3}\] we know that \( \mathcal{X}(A)^{\mathbb{F}} \), and in particular \( \mathcal{X}(A)^{\text{der}} \) are functorial obstruction categories. Thus we have the following analogue of \[\ref{5.13}\].
6.12. Proposition. Suppose that $A \in \text{Fib}_{\text{Set}}/S$ and $F$ as in (6.8). Let $\delta_i \in \{\text{desc, 2-desc}\}, i \in \{1, 2, \ldots \}$ and define $\mathcal{X}(A)^{\delta_i,F} = \mathcal{X}(A)^{\delta_i,\delta_{i-1},\ldots,\delta_1,F}$. Then we have

$$\mathcal{X}(S) \subseteq \cdots \subseteq \mathcal{X}(A)^{\delta_1,F} \subseteq \mathcal{X}(A)^{\delta_2,F} \subseteq \cdots \subseteq \mathcal{X}(A)^{\delta_k,F} \subseteq \mathcal{X}(A),$$

and that all obstruction maps above are functorial.

Proof. The same as 5.13

6.13. Now we work with special objects in Fib/$S$, Deligne-Mumford $S$-stacks. Under some additional assumptions, and using the develop by Zheng [25] of six operations and Lefschetz-Verdier formula for Deligne-Mumford stacks, we show the commutativity of the derived obstruction categories with taking 2-fibred product of two Deligne-Mumford $S$-stacks. Here we adopt the definition in [25, Convention 1.1] for Deligne-Mumford stacks.

Following Illusie and Zheng we first introduce the

6.14. Definition ([14, Def. 3.8]). Let $m$ be an integer. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks is of $m$-prime inertia if for every algebraic closed field $\Omega$ and every point $x \in \mathcal{X}(\Omega)$, the order of the group

$$\text{Aut}_{\mathcal{X}}(x) \xrightarrow{\sim} \ker(\text{Aut}_{\mathcal{X}}(x) \to \text{Aut}_{\mathcal{Y}}(y))$$

is prime to $m$, where $y = f \circ x \in \mathcal{Y}(\Omega)$.

As an example, if $\mathcal{X}$ is representable, then $\text{Aut}_{\mathcal{X}}(x)$ is trivial for any $x$ and hence $f$ is always of $m$-prime inertia for any $m$.

Morphisms of $m$-prime inertia are closed under composition and base change.

Now suppose that $S$ is a spectrum of a field. Let $\Lambda$ be a Noetherian ring annihilated by an integer $m$ invertible on $S$. Let $D_*(\mathcal{T}, \Lambda)^-$ be the triangulated subcategory of $D(\mathcal{T}, \Lambda)$ consisting of complexes bounded above with constructible (see [25, Def. 3.1]) cohomology sheaves. We assume that all Deligne-Mumford $S$-stacks are $m$-prime inertia, separated and of finite-type. Let $\text{der}^-_c$ be the corresponding obstruction map as in (6.11) with $D(-, \Lambda)$ replaced by $D_*(\mathcal{T}, \Lambda)^-$. This makes sense since for any 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford $S$-stacks, $Rf_*$ sends $D_*(\mathcal{X}, \Lambda)$ to $D_*(\mathcal{Y}, \Lambda)$ ([25, Prop. 3.16]).

Since $\text{der}^-_c$ also is functorial, the projections $p_{\mathcal{X}_i} : \mathcal{X}_1 \times_S \mathcal{X}_2 \to \mathcal{X}_i, i = 1, 2$, induce a map on isomorphism classes of objects

$$p^{\text{der}^-_c} : (\mathcal{X}_1 \times_S \mathcal{X}_2)(\mathcal{A})^{\text{der}^-_c} \to \mathcal{X}_1(\mathcal{A})^{\text{der}^-_c} \times \mathcal{X}_2(\mathcal{A})^{\text{der}^-_c}.$$  

6.16. Theorem. The map $p^{\text{der}^-_c}$ in (6.15) is a bijection.

Proof. By the universal property, given $x_i \in \mathcal{X}_i(\mathcal{A}), i = 1, 2$, there is a unique 1-morphism $(x_1, x_2)$, up to a 2-isomorphism, making the following diagram 2-commutes

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\phi_1, \phi_2} & \mathcal{X}_1 \\
\downarrow & & \downarrow p_{\mathcal{X}_1} \\
\mathcal{X}_1 \times_S \mathcal{X}_2 & \xrightarrow{p_{\mathcal{X}_2}} & \mathcal{X}_2 \\
\downarrow & & \downarrow p_{\mathcal{X}_2} \\
\mathcal{X}_1 & \xrightarrow{p_{\mathcal{X}_1}} & S
\end{array}$$
Assume that $x_i \in \mathcal{X}_i(A)^{\text{der}}$, $i = 1, 2$, that is, there exist $\theta_i \in \text{Hom}_{\mathcal{G}l}(D_c(S, \Lambda)^{-}, D_c(A_i, \Lambda)^{-})$ and isomorphisms of functors
\begin{equation}
R_{x_i} \sim \theta_i Rq_*.
\end{equation}
Then it suffices to show that $(x_1, x_2) \in (A_1 \times_S A_2)(A)^{\text{der}}$. For $M_i \in D(T, \Lambda)$, $i = 1, 2$, we write $M_1 \boxtimes M_2 = p_1^* M_1 \otimes_{\Lambda}^L p_2^* M_2$, where $p_1^*: T_1 \times_S T_2 \to T_i$ is the projection. Also let $x : A \times_S A \to A_1 \times_S A_2$ be induced by $x_i$, $i = 1, 2$. By Künneth formula for Deligne-Mumford stacks \cite[Prop. 8.1]{A} we have isomorphisms of functors on $D_c(A, \Lambda)^{-} \times D_c(A, \Lambda)^{-}$
\begin{align*}
R(x_1, x_2)_* (\sim \otimes_{\Lambda}^L) \sim R_x R \Delta_{A/S, *}(\sim \otimes_{\Lambda}^L) \sim R_x (\sim \boxtimes_{\Lambda}^L) \sim R_x 1_* - \boxtimes_{\Lambda}^L R x_2_* - .
\end{align*}
Then by (6.17) and the isomorphism $\sim \otimes_{\Lambda}^L$ we have
\begin{align*}
R(x_1, x_2)_* \sim R(x_1, x_2)_* (\sim \otimes_{\Lambda}^L) \sim \theta_1 Rq_* - \boxtimes_{\Lambda}^L \theta_2 Rq_*, \Lambda = \theta Rq_* -
\end{align*}
on $D_c(A, \Lambda)^{-}$ where
\begin{align*}
\theta = \theta_1 - \boxtimes_{\Lambda}^L \theta_2 Rq_* \in \text{Hom}_{\mathcal{G}l}(D_c(S, \Lambda)^{-}, D_c(A_1 \times_S A_2, \Lambda)^{-}).
\end{align*}
Thus $(x_1, x_2) \in (A_1 \times_S A_2)(A)^{\text{der}}$ and the proof is complete. \hfill \Box

Acknowledgment

The author would like to thank Junchao Shentu for helpful discussions.

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