Deltoid tangents with evenly distributed orientations and random tilings

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Abstract. We study the construction of substitution tilings of the plane based on certain simplicial configurations of tangents of the deltoid with evenly distributed orientations. The random tiling ensembles are obtained as a result of tile rearrangements in the substitution rules associated to edge flips.

Mathematics Subject Classification (2010). Primary 52C20; Secondary 52C30, 52C23, 60D05.

Keywords. substitution tilings, random tilings, configurations of lines, simplicial arrangements.

1. Introduction

Tiling theory developed from several directions in the last half century. Interested in problems of logic, Wang [21] and Berger [1] produced in the 1960s non periodic tilings of the plane with a large number of basic geometric shapes. Later Penrose generated a non periodic planar tiling by using a set of two basic tiles [18]. The discovery of quasicrystals by Shechtman in the early 1980s [20] stimulated the study of tilings with non crystallographic symmetries, because certain tilings underlie the atomic structure of quasicrystals. Two main approaches to modelling the structure of quasicrystals are the construction of cut and project sets, also called model sets, and the generation of substitution tilings. Much progress has been done in the field of deterministic substitution tilings during the last decades, but not in the construction of random tilings. We will consider the problem of the generation of tilings by means of both deterministic and random substitutions.

In 1996 Nischke and Danzer provided a construction of substitution tilings based on the system of certain $d$ tangents of the deltoid, when $d$ is odd, higher than 5 and non divisible by three [17]. In recent papers we have shown that the sets of deltoid tangents with $d$ evenly distributed orientations are images under a certain map of periodic billiard trajectories inside the fundamental region of the affine Weyl group of the root system $A_2$ [8]. In some cases the arrangements of deltoid tangents are simplicial, namely, all their bounded cells are triangles. Several sets of simplicial arrangements corresponding to $d$ odd and non divisible by 3 were used in [17] to produce tilings, although the same sets were studied before with a different purpose [10]. In this paper we study a construction of tilings by using deltoid tangents having any $d > 4$ evenly distributed orientations and we use them to investigate the generation of random tilings for $d$ even.

The paper is organised as follows. In Section 2 we study several properties of the simplicial arrangements of deltoid tangents with $d$ evenly distributed orientations. They contain the prototiles, or elementary triangles, and scaled copies of them necessary for the derivation of the inflation or substitution rules. The existence of deterministic substitution rules producing planar tilings is analysed in Section 3. The relationship between algebraic numbers and mathematical quasicrystals was established first by Meyer in [15, 16] and later in [2]. Pisot-Vijayaraghavan (PV) numbers are real algebraic
2. Simplicial arrangements of deltoid tangents with evenly distributed orientations

We consider the lines \( L_{d,\kappa,\nu}(x, y) = 0, d = 5, 6, ..., \kappa \in \{-2, 0, 2\}, \nu \in I(\kappa) \), having parametric equations \( x = x(t), y = y(t) \), where

\[
x(t) + \sqrt{-1}y(t) := e^{-2\pi \frac{3\nu - \kappa}{3d} \sqrt{-1}} + te^{2\pi \frac{3\nu - \kappa}{3d} \sqrt{-1}}, t \in \mathbb{R},
\]

with \( I(\kappa) = \{0, 1, 2, ..., d - 1\} \), for \( \kappa \in \{-2, 0\} \) and \( I(2) = \{0, -1, -2, ..., -d + 1\} \). The construction is based on the configurations of \( d \) lines

\[
A_d^{(\kappa)} := \{L_{d,\kappa,\nu}(x, y) = 0\}_{\nu \in I(\kappa)}
\]

which are defined \( \forall d \) if \( \kappa = 0 \) and for \( d = 3q, q = 2, 3, ... \) when \( \kappa = \mp 2 \). The configuration corresponding to \( d = 14 \) is represented in Fig.1, together with a superimposed heptagon which will be necessary in Section 4 for the construction of random tilings.

The lines are tangents of the deltoid \( D \), which has parametric equation

\[
z(\varphi) = 2e^{\varphi \sqrt{-1}} + e^{-2\varphi \sqrt{-1}}, 0 \leq \varphi \leq 2\pi
\]

If \( \varphi = \frac{3\nu - \kappa}{3d} \pi \) then the tangency point of \( L_{\varphi}(x, y) := y + (\cos2\varphi - x)\tan\varphi + \sin2\varphi = 0 \) to \( D \) is \( z(-2\varphi) \) and it intersects \( D \) also in \( z(\varphi) \) and \( z(\varphi + \pi) \). We define the segments

\[
G(\varphi) := z(\varphi); z(\varphi + \pi)
\]

and we write \( z_{\nu}, G_{\nu, d}, s_{\nu} \) instead of \( z(\frac{3(\nu - \kappa)\pi}{3d}), G(\frac{3(\nu - \kappa)\pi}{3d}), \sin(\frac{\nu\pi}{d}) \) respectively. The length of the segment \( G_{\nu, d}^{(\kappa)} \) (the index \( \nu \) is taken mod \( d \)) is \( |G_{\nu, d}^{(\kappa)}| = 4 \) and the tangency point of \( G_{\nu, d}^{(\kappa)} \) at the deltoid \( D \) is \( z_{2(\nu - \kappa) + \kappa} \).

We use the following well-known results about congruences:

**Proposition 2.1.** If \( \gcd(a, d) = 1 \), then the linear congruence \( ax \equiv b \pmod{d} \) has exactly one solution modulo \( d \).

By using Euclid’s algorithm it can be shown that if \( \gcd(a, d) = m \) then there are two integers \( s \) and \( t \) such that \( m = as + dt \) (Bézout’s identity). Therefore when \( m = 1 \), the solution of \( ax \equiv b \pmod{d} \) is \( x = bs \).

**Proposition 2.2.** If \( \gcd(a, d) = m \), then the congruence \( ax \equiv b \pmod{d} \) has a solution iff \( m|b \). In that case there are exactly \( m \) solutions modulo \( d \) which can be written as \( x_1, x_1 + d_1, ..., x_1 + (m - 1)d_1 \), where \( d = md_1 \) and \( x_1 \) is the solution of the congruence \( a_1 x \equiv b_1 \pmod{d_1} \), \( a = ma_1 \), \( b = mb_1 \).
If \( G \) is an elementary triangle.

Proof. Let \( G \) be a segment with length \( 4l \) and \( \kappa \) the number of vertices with multiplicity (number of segments incident to a vertex) \( j \). We note that the cohomology of tiling spaces). Other cases have been treated in [6] and references therein.

Definition 2.4. A triangle \( \Delta \) is said to be elementary if \( \lambda, \mu, \nu \) are \( 4 \) times \( \kappa \) with angles \((\kappa - \nu \pm \mu) \frac{\pi}{2}, (\nu - \lambda) \frac{\pi}{2}, (\lambda + \mu) \frac{\pi}{2}\) and \( |\sin(\kappa - \nu)\mid \cdot \sin(\nu - \chi) \) (2.5)

Proof. The point \( G(\varphi) \cap G(\psi) \) is

\[
p(\varphi, \psi) := (3 - 4\sin^2 \varphi - 4\sin^2 \psi + 2\sin \varphi \cos \psi \sin \psi), -4 \sin \varphi \sin \psi (\sin \varphi + \psi)\]

and

\[
|p(\varphi, \psi) - p(\varphi, \chi)| = 4|\sin(\varphi + \chi + \psi)|\sin(\varphi - \chi) \quad (2.7)
\]

A consequence of Eq. (2.7) is that if \( \varphi + \chi + \psi \equiv 0 \mod \pi \) then \( G(\varphi) \cap G(\chi) \cap G(\psi) \neq \emptyset \).

Lemma 2.5. If \( 0 \leq \varphi < \chi < \psi < \pi \) and \( \psi := \varphi + \chi + \psi \) then

\[
|G(\varphi) \cap G(\psi) - G(\varphi) \cap G(\chi)| = 4|\sin \varphi | \cdot \sin(\psi - \chi)
\]

Definition 2.4. A triangle \( \Delta_\kappa(\lambda, \mu, \nu) \) formed by \( G_{\lambda,d}, G_{\mu,d}, G_{\nu,d} \) is said to be elementary if \( \lambda + \mu + \nu \equiv \kappa \pm 1 \mod d \), \( \kappa \in \{ -2, 0, 2 \} \).

Lemma 2.5. If \( 0 \leq \varphi < \chi < \psi < \pi \) and \( \omega := \varphi + \chi + \psi \) then

\[
|G(\varphi) \cap G(\psi)| = 4|\sin(\varphi + \chi + \psi)|\sin(\varphi - \chi)
\]

Proof. If \( \lambda + \mu + \nu \equiv \kappa \mod d \) then \( G_{\lambda,d}, G_{\mu,d}, G_{\nu,d} \) are not concurrent and form a triangle \( \Delta_d(\lambda, \mu, \nu) \) with angles \((\nu - \mu) \frac{\pi}{2}, (\mu - \lambda) \frac{\pi}{2}, (\lambda + \mu) \frac{\pi}{2}\).

Table 1. Vertex multiplicities in \( G_{\mu,d}^{(\kappa)} \), with \( l = 1, 2, 3, \ldots \)

| \( d \) | \( q \) | \( \kappa \) | \( \mu \) | \( v_2(G_{\mu,d}^{(\kappa)}) \) | \( v_3(G_{\mu,d}^{(\kappa)}) \) |
|---|---|---|---|---|---|
| \( 2q + 1 \neq 3l + 1 \) | 0 | 0 | 0 | \( \frac{d}{3} + 1 \) | \( \frac{d}{3} - 3 \) |
| \( 2q + 1 \neq 3l + 1 \) | 0 | \( \neq 0 \) | 2 | \( \frac{d}{3} - 1 \) | \( \frac{d}{3} - 3 \) |
| \( 2q + 1 \neq 3l + 1 \) | 0 | \( \in \{ 0, 2l + 1, 4l + 2 \} \) | 2 | \( \frac{d}{3} + 1 \) | \( \frac{d}{3} - 3 \) |
| \( 2q + 1 \neq 3l + 1 \) | 0 | \( \notin \{ 0, 2l + 1, 4l + 2 \} \) | \( \frac{d}{3} - 2 \) | \( \frac{d}{3} - 3 \) |
| \( 2q \neq 3l \) | 0 | \( \in I_{\text{odd}} \cup \{ 0 \} \) | 1 | \( \frac{d}{3} - 2 \) | \( \frac{d}{3} - 3 \) |
| \( 2q \neq 3l \) | 0 | \( \in I_{\text{even}} \cup \{ 0 \} \) | 3 | \( \frac{d}{3} - 2 \) | \( \frac{d}{3} - 3 \) |
| \( 3q \neq 2l \) | \( \neq 2 \) | \( \in I_{\text{odd}} \cup \{ 0 \} \) | 3 | \( \frac{d}{3} - 2 \) | \( \frac{d}{3} - 3 \) |
| \( 3q \neq 2l \) | \( \neq 2 \) | \( \in I_{\text{even}} \cup \{ 0 \} \) | \( \frac{d}{3} - 2 \) | \( \frac{d}{3} - 3 \) |

In what follows let \( d > 4 \). The case \( d = 5 \) has been studied in [7], where we have obtained the minimal first cohomology for a quasicrystal tiling space (see [19] for an introduction to the study of the cohomology of tiling spaces). Other cases have been treated in [6] and references therein.

The tilings we want to study have triangles as basic building blocks. Their edges lie on segments \( G(\varphi), G(\psi) \) and \( G(\chi) \).

Table 1. Vertex multiplicities in \( G_{\mu,d}^{(\kappa)} \), with \( l = 1, 2, 3, \ldots \)
Lemma 2.6. Each set $\{G^{(\kappa)}_{\mu,d}\}_{\mu \in I(\kappa)}$ makes up a triangular pattern $G^{(\kappa)}_{\Delta,d}$ inside $D$. Every interior vertex is shared by exactly six elementary triangles. The number of vertices with multiplicity 2 and 3 in each $G^{(\kappa)}_{\mu,d}$ is included in Table 1.

Proof. For $\mu \neq 0$ the segment $G^{(0)}_{2\mu,d}$ intersects $G^{(0)}_{\mu,d}$ at the tangency point of $G^{(0)}_{\mu,d}$ to $D$, denoted by $x^D_{\mu}$, and $G^{(0)}_{\mu,d}$ intersects $G^{(0)}_{\mu,d}$ at the tangency point $x^D_{\mu}$ of $G^{(0)}_{\mu,d}$ to $D$, where $\nu$ is a solution to the equation $2\nu \equiv (\mu, d, \mod d)$.

1) $d = 2q + 1, \kappa = 0, q \neq 3l + 1$. The solution is $\nu = q\mu$. If $\mu \neq 0$ there is no $\nu \neq q\mu, -2\mu (\mod d)$ such that $\lambda + \mu + \nu \equiv 0 (\mod d)$ with $\lambda, \mu, \nu$ pairwise distinct, therefore $v_2(G^{(0)}_{\mu,d}) = 2, v_3(G^{(0)}_{\mu,d}) = \frac{d-2}{2}$. When $\mu = 0$ we have $x^D_{\mu} = x^D_{\mu}$, hence all the vertices in $G^{(0)}_{0,d}$ have multiplicity 3 and $v_2(G^{(0)}_{0,d}) = 0, v_3(G^{(0)}_{0,d}) = \frac{d-1}{2}$.

2) $d = 2q + 1, \kappa = 0, q = 3l + 1, l = 1, 2, 3, ...$. These cases are similar to the cases discussed in 1) except that $x^D_{\nu} = x^D_{\mu}$, when $\mu = 0, 2l + 1, 4l + 2$, and we have $v_2(G^{(0)}_{\mu,d}) = 0, v_3(G^{(0)}_{\mu,d}) = \frac{d-1}{2}$ if $\mu = 0, 2l + 1, 4l + 2$ and $v_2(G^{(0)}_{\mu,d}) = 2, v_3(G^{(0)}_{\mu,d}) = \frac{d-3}{2}$ otherwise.

3) $d = 2q, \kappa = 0, q = 2l + 1, l = 2, 3, 4, ...$. If $q \neq 6p, p = 1, 2, 3, ...$ then for $\mu \in I_{even}$ there are two different solutions: $\nu = d - \frac{\mu}{2}, \frac{d-\mu}{2}$. Hence $v_2(G^{(0)}_{\mu,d}) = 3, v_3(G^{(0)}_{\mu,d}) = \frac{d-4}{2}$. If $\mu = 0$ there is only one solution which is $\nu = \frac{d}{2}$. For $\mu = 1, 3, 5, ... d - 1$ there is no solution distinct from $\mu$ and only $G^{(0)}_{-2,\mu,d}$ intersects $G^{(0)}_{\mu,d}$ at the tangency point of $G^{(0)}_{\mu,d}$ to $D$. Therefore $v_2(G^{(0)}_{\mu,d}) = 1, v_3(G^{(0)}_{\mu,d}) = \frac{d-2}{2}$ for $\mu \in I_{odd} \cup \{0\}$.

The remaining cases are treated along the same lines and are given in Table 1. In all cases all vertices of multiplicity 2 are on $D$ and all interior vertices have multiplicity 3, hence they are shared by exactly six triangles. □

Lemma 2.7. Every segment $G^{(\kappa)}_{\mu,d}$ is cut by the other segments into pieces forming the following sequences:

1) $d = 2d + 1, q \neq 3l + 1, \kappa = 0$: $(S_n)_{n \in I_{odd}, \forall \mu \neq 0}$, and $(S_n)_{n \in I_{odd} \setminus \{1\}}$ for $\mu = 0$.

2) $d = 2q + 1, q = 3l + 1$:

2.1) $\kappa = 0$: $(S_n)_{n \in I_{odd}, \forall \mu \neq 0}$, and $(S_n)_{n \in I_{odd} \setminus \{1\}}$ for $\mu = 0$.

2.2) $\kappa = \pm 2$: $(S_n)_{n \in I_{odd}, \forall \mu}$.

3) $d = 2q, q = 3l$:

3.1) $\kappa = 0$: $(S_n)_{n \in I_{odd}}$ if $\mu$ is even with $\mu \notin \{0, 2l + 1, 4l + 2\}$ and $(S_n)_{n \in I_{odd} \setminus \{1\}}$ if $\mu \in \{0, 2l + 1, 4l + 2\}$.

3.2) $\kappa = \pm 2$: $(S_n)_{n \in I_{odd}}$ if $\mu$ is even or 0.

3.3) $\kappa = 0, \pm 2$: $(S_n)_{n \in even}$ if $|\mu|$ is odd.

4) $d = 2q, q \neq 3l$: $(S_n)_{n \in I_{odd} \setminus \{1\}}$ for $\mu$ even, $(S_n)_{n \in even}$ for $\mu$ odd, and $(S_n)_{n \in I_{odd} \setminus \{1\}}$ for $\mu = 0$.

Proof. We write $p_{\mu,q\mu}$ instead of $p(\frac{\mu \pi}{d}, \frac{q \pi}{d})$. For $d = 2q + 1, \kappa = 0, q \neq 3l + 1$ we take $p_{\mu,q\mu}$ as starting point in $G^{(0)}_{\mu,d}$ for $\mu \neq 0$, which corresponds to the left most point in Fig 2.c. This is a multiplicity 2 point, and we have a number of consecutive multiplicity 3 points $p_{\mu,q\mu+1} = p_{\mu,q\mu-1}, p_{\mu,q\mu+2} = p_{\mu,q\mu-2}$, ... $p_{\mu,q\mu+n-1} = p_{\mu,q\mu+n+1}$ until we reach a value of $n$ such that either $\mu q - n = n$ or $\mu q + n = n$, namely, either $\mu q + n = -2\mu$ or $\mu q - n = -2\mu$ which corresponds to a point of multiplicity 2 (the second point from the left in Fig 2.c). The remaining vertex points in $G^{(0)}_{\mu,d}$ are $p_{\mu,q\mu+n+1} = p_{\mu,q\mu-n-1}, p_{\mu,q\mu+n+2} = p_{\mu,q\mu-n-2}$, ... $p_{\mu,q\mu+q} = p_{\mu,q\mu-q}$ and have multiplicity 3. Now $|p_{\mu,q\mu+1} - p_{\mu,q\mu+q+1}| = 4S_2 - 1$ and we get the sequence $(S_n)_{n \in I_{odd} \setminus \{1\}}$.

On $G^{(0)}_{0,d}$ all the points have multiplicity 3 and the sequence of points, starting from the nearest point to the intersection of $G^{(0)}_{0,d}$ with the deltoid cusp (Fig 2a, right) is $p_{0,d-1} = p_{0,1}, p_{0,d-2} = ...$
Figure 1. The configuration of lines $A_{14}^{(0)}$. The segments $G_{\mu,14}^{(0)}$, for $\mu = 0, 1, ..., 13$, lie on the lines $L_{14,0,\mu}(x, y) = 0$ which are labelled by $\mu$. The heptagon associated with the random tilings is included.

Figure 2. Schematic representation of the cases with $v_2(G_{\mu,d}^{(\kappa)}) = 0, 1, 2, 3$. The curved fragments represent parts of the deltoid in the neighbourhood of points where the line $L_{d,\kappa,\mu}$ either intersects transversally, passes through one of the deltoid cusps, or is tangent to $D$.

$p_{0,2}, \ldots, p_{0,d-q} = p_{0,q}$. Having in mind $|p_{0,d-n} - p_{0,d-n-1}| = 4s_{2n+1}s_1$, $n = 1, 2, \ldots, q - 1$, we obtain the sequence $(S_n)_{n \in I_{odd}\setminus\{1\}}$.

For $d = 2q + 1, q = 3l + 1$ the segments with $\mu \in \{0, 2l + 1, 4l + 2\}$ are analogous to the segment with $\mu = 0$ when $q \neq 3l + 1$ and those with $\mu \notin \{0, 2l + 1, 4l + 2\}$ are like $\mu \neq 0$. The remaining cases can be analysed in a similar way having in mind the results of Lemma 2.6 summarised in Table 1. □
3. Substitution tilings of the plane

Definition 3.1. A tiling of an $n$-dimensional space $S$ is a decomposition of $S$ into a countable number of $n$-cells $T$, called prototiles, such that $S$ is the union of tiles and distinct tiles have non-intersecting interiors. A collection of tiles, any two of which intersect only in their boundaries, is called a patch.

A substitution rule determines how to replace each prototile by a patch of tiles. Iteration of the substitution rules gives in the limit a substitution tiling of $S$. In this work $S$ is the real plane and the prototiles are triangles (see [11], [14] for recent related work done with computer assistance where the prototiles are triangles and rhombuses respectively).

In [17] the prototile edges are marked with arrows. Prototiles having the same shape but different arrow decorations have different substitution rules. We use a procedure to distinguish prototiles having the same shape that we call interior decoration, or ID for short. If two arrangements $X_d, X_{2d}$ satisfy $X_d \subset X_{2d}$ then an elementary triangle $t_{2d}$ in $X_{2d}$ is inscribed in an elementary triangle $t_d$ of $X_d$ which is a scaled copy of $t_{2d}$. We decorate the interior of $t_d$ with the inscribed copy $t_{2d}$ and the decorated prototile is $T = t_d \cup t_{2d}$ (see [3], Fig.2, p.105). The set of prototiles $T$ is denoted by $F_d$. From the definitions we have $G_{\lambda,d}^{(0)} = G_{2\lambda,2d}^{(0)}, G_{\lambda,d}^{(\pm 2)} = G_{2\lambda \mp 2, 2d}^{(2)}, G_{\lambda,d}^{(2)} = G_{2\lambda, 2-2d}^{(2)}$, therefore $A_d^{(\pm)} \subset A_{2d}^{(-\pm)}$.

The elementary triangle $\Delta_d^{(\pm)}(\lambda, \mu, \nu)$ in $G_{\Delta,d}^{(\pm)}$ is congruent to $\Delta_d^{(\pm)}(2\lambda, 2\mu, 2\nu)$ in $G_{\Delta,d}^{(\pm 2)}$. If $\sigma_d - \kappa = \pm 2$ then the elementary triangle of $G_{\Delta,d}^{(\pm)}$ having its vertices on the edges in $\Delta_d^{(\pm)}(2\lambda, 2\mu, 2\nu)$ is $\Delta_{2d}^{(\pm)}(2\lambda \mp 2, 2\mu \mp 2, 2\nu \mp 1)$, which on the other hand is a congruent copy of $\Delta_{2d}^{(\pm)}(2\lambda, 2\mu, 2\nu)$ scaled by $\frac{1}{2d}$. Now each edge of $\Delta_d^{(\pm)}(\lambda, \mu, \nu)$ is subdivided into two sections defined by the vertex of the inscribed triangle $\Delta_{2d}^{(\pm)}(2\lambda \mp 2, 2\mu \mp 2, 2\nu \mp 1)$ lying on the edge. The sequences of sections on the edges of $\Delta_{2d}^{(\pm)}(2\lambda, 2\mu, 2\nu)$ lying on the segments and determining the ID are the following:

$G_{2\lambda,2d}^{(\pm)}: (S_{2\lambda-2\mu \mp 2, 2\nu-2\mu \pm 1}; G_{2\mu,2d}^{(\pm)}: (S_{2\lambda-2\mu \mp 1+2d, S_{2\lambda-2\nu \pm 1+2d}); G_{2\nu,2d}^{(\pm)}: (S_{2\mu-2\lambda \mp 1}, S_{2\mu-2\lambda \pm 1})$ for $0 \leq 2\lambda < 2\mu < 2\nu < 2d; 2\lambda + 2\mu + 2\nu - \kappa = \pm 2$.

Lemma 3.2. Given integers $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma \equiv 0 \pmod{d}$, $\alpha \leq \beta \leq \gamma$, $\sigma_d - \kappa \equiv \pm 1 \pmod{d}$ and $l = 1, 2, 3, \ldots$, we have the following properties:

1) $G_{\Delta,d}^{(0)}, d = 2q + 1, q \neq 3l + 1$, or $d = 2q, q \neq 3l$. The number of triangles in $G_{\Delta,d}^{(0)}$ with angles $\frac{\alpha \pi}{d}, \frac{\beta \pi}{d}, \frac{\gamma \pi}{d}$ and the given $\sigma_d$ equals one, if $\alpha, \beta, \gamma$ are not pairwise distinct (the triangle then is isosceles), and equals two otherwise, and the two triangles then differ with respect to the ID.

2) $G_{\Delta,d}^{(\pm)}, d = 3q$. There are no isosceles elementary triangles. The non isosceles elementary triangles appear provided $\beta - \alpha$ is not a multiple of 3. There are three congruent copies of each elementary triangle but they have the same ID and must be considered as a single prototile.

3) $G_{\Delta,d}^{(0)} \cup G_{\Delta,d}^{(\pm 2)} \cup G_{\Delta,d}^{(2)}, d = 3q$. There are isosceles elementary triangles and one of them is equilateral. There are two types of non isosceles elementary triangles. The number of non isosceles elementary triangles with angles $\frac{\alpha \pi}{d}, \frac{\beta \pi}{d}, \frac{\gamma \pi}{d}$ and a given $\sigma_d$ equals one, if $\beta - \alpha$ is not a multiple of 3 and $|\sigma_d| = 1$, and equals two if $\beta - \alpha$ is a multiple of 3 and $|\sigma_d| = 3$, and the two triangles then differ with respect to the ID.

Proof. 1.1) $G_{\Delta,d}^{(0)}, d = 2q + 1, q \neq 3l + 1$. Any triangle with angles $\frac{\alpha \pi}{d}, \frac{\beta \pi}{d}, \frac{\gamma \pi}{d}$ is either $\Delta_d^{(\pm)}(\mu - \beta, \mu, \mu + \alpha)$ or $\Delta_d^{(\pm)}(\beta - \mu, \mu + \beta)$. In this case $d \not\equiv 0 \pmod{3}$, therefore by Prop. 2.2 there is no equilateral triangle and we may assume $\gamma \neq \alpha, \beta$. We look for solutions to the congruences $\mu \equiv \beta + \mu + \mu \pm \alpha \equiv \sigma_d \pmod{d}$.

If $\alpha = \beta$ (isosceles triangle) then $3\mu \equiv \sigma_d \pmod{d}$. If an elementary isosceles triangle has sides $S_a, S_b, S_b$ we now prove that the side $S_a$ can be in $G_{\mu,d}^{(0)}$ for only two different values of $\mu$. We have to look for solutions of $3\mu \equiv \pm 1 \pmod{d}$ and, according to Prop. 2.2 there is exactly one solution in each case. The unique solution to $3\mu \equiv 1 \pmod{d}$ is $\mu = 2l$ for $d = 6l - 1$ and $\mu = 4l + 1$ for $d = 6l + 1$. The solution to $3\mu \equiv -1 \pmod{d}$ is $\mu = 4l - 1$ for $d = 6l - 1$ and $\mu = 2l$ for $d = 6l + 1$. 

By Lemma 2.6 the number of subdivisions of $G_{d}^{(0)}$ in these cases is $q$, and we have $q$ different isosceles triangles: $\Delta_{d}^{(0)}(\mu - \alpha, \mu, \mu + \alpha)$, $\alpha = 1, 2, ..., q$. The triangle $\Delta_{d}^{(0)}(-2l - \alpha, -2l, -2l + \alpha)$ in $G_{-2l,d}^{(0)}$ is a congruent copy of $\Delta_{d}^{(0)}(2l - \alpha, 2l, 2l + \alpha)$ in $G_{2l,d}^{(0)}$ rotated by $\frac{(2l+1)\pi}{d}$ for $d = 6l \pm 1$. But it differs on the ID because they have values of $\sigma_d$ with opposite signs: the sequences of sections on the edges are reversed.

If $\alpha \neq \beta$, the equation $3x \equiv b \pmod{d}$ has one solution because $gcd(3,d) = 1$ therefore we have one solution for $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$ and another one for $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$. For $d = 6l + 1$, given a solution of $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$ with $\sigma_d = -1$ corresponding to $\Delta_{d}^{(0)}(\mu_l - \beta, \mu_1, \mu_1 + \alpha)$ then $\mu_1 = \mu_1 + 2l + 1$ is a solution of $3\bar{\mu}_1 \equiv 1 - \alpha + \beta \pmod{d}$ and $\Delta_{d}^{(0)}(\mu_1 - \beta + 2l + 1, \mu_1 + 2l + 1, \mu_1 + \alpha + 2l + 1)$ is a congruent copy of $\Delta_{d}^{(0)}(\mu_1 - \beta, \mu_1, \mu_1 + \alpha)$ but with different ID. Also if $\Delta_{d}^{(0)}(\mu_2 - \alpha, \mu_2, \mu_2 + \beta)$ corresponds to the solution of $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$ with $\sigma_d = -1$ then we have its congruent triangle $\Delta_{d}^{(0)}(\mu_2 - \alpha + 2l + 1, \mu_2 + 2l + 1, \mu_2 + \beta + 2l + 1)$ with different ID. For $d = 6l - 1$ if $\Delta_{d}^{(0)}(\mu_1 - \beta, \mu_1, \mu_1 + \alpha)$ is linked with the solution of $3\mu_1 \equiv \sigma_d + \alpha - \beta \pmod{d}$ with $\sigma_d = 1$, then we have $\Delta_{d}^{(0)}(\mu_1 - \beta + 2l - 1, \mu_1 + 2l - 1, \mu_1 + \alpha + 2l - 1)$ and if $\Delta_{d}^{(0)}(\mu_2 - \alpha, \mu_2, \mu_2 + \beta)$ corresponds to the solution of $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$ with $\sigma_d = 1$, then its congruent triangle is $\Delta_{d}^{(0)}(\mu_2 - \alpha + 2l - 1, \mu_2 + 2l - 1, \mu_2 + \beta + 2l - 1)$.

If $p(d, n)$ denotes the number of partitions of $d$ into exactly $n$ positive integers then $p(d - \binom{n}{2}, n)$ is the number of ways of writing $d$ as a sum of $n$ different positive integers $\mathbb{N}$. The number of elementary non isosceles triangles with $\sigma_d = 1$ or $\sigma_d = -1$ is equal to $2p(d - \binom{3}{2}, 3) = 2\left[\frac{(d-3)^2}{12}\right]$, where $\left[\frac{\cdot}{\cdot}\right]$ is the nearest integer function. The total number of elementary triangles in $G_{\Delta,d}^{(0)}$ is therefore $d - 1 + 4\left[\frac{(d-3)^2}{12}\right]$ for $d = 6l \pm 1$.

1.2) $G_{\Delta,d}^{(0)}(d = 2q, q \neq 3l$. This case is analogous to 1.1). If $\alpha = \beta$ then the unique solution to $3\mu \equiv 1 \pmod{d}$ is $\mu = 2l + 1$ for $d = 6l + 2$ and $\mu = 4l + 3$ for $d = 6l + 4$. The solution to $3\mu \equiv -1 \pmod{d}$ is $\mu = 4l + 1$ for $d = 6l + 2$ and $\mu = 2l + 1$ for $d = 6l + 4$. By Lemma 2.6 the number of subdivisions of $G_{d}^{(0)}$ in these cases is $q - 1$, and we have $q - 1$ different isosceles triangles: $\Delta_{d}^{(0)}(\mu - \alpha, \mu, \mu + \alpha), \alpha = 1, 2, ..., q - 1$.

If $\alpha \neq \beta$, the equation $3x \equiv b \pmod{d}$ has one solution. For $d = 6l + 2$, given a solution of $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$ with $\sigma_d = -1$ corresponding to $\Delta_{d}^{(0)}(\mu_l - \beta, \mu_1, \mu_1 + \alpha)$ then $\mu_1 = \mu_1 + 2l$ is a solution of $3\bar{\mu}_1 \equiv 1 - \alpha + \beta \pmod{d}$ and $\Delta_{d}^{(0)}(\mu_1 - \beta - 2l, \mu_1 - 2l, \mu_1 + \alpha - 2l)$ is a congruent copy of $\Delta_{d}^{(0)}(\mu_l - \beta, \mu_1, \mu_1 + \alpha)$ but with different ID. Also if $\Delta_{d}^{(0)}(\mu_2 - \alpha, \mu_2, \mu_2 + \beta)$ corresponds to the solution of $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$ with $\sigma_d = 1$ we have its congruent triangle $\Delta_{d}^{(0)}(\mu_2 - \alpha + 2l + 1, \mu_2 + 2l + 1, \mu_2 + \beta + 2l + 1)$ with different ID. For $d = 6l + 4$ if $\Delta_{d}^{(0)}(\mu_l - \beta, \mu_1, \mu_1 + \alpha)$ is related to the solution of $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$ with $\sigma_d = -1$, then we have $\Delta_{d}^{(0)}(\mu_1 - \beta + 2l + 1, \mu_1 + 2l + 1, \mu_1 + \alpha + 2l + 2)$ and if $\Delta_{d}^{(0)}(\mu_2 - \alpha, \mu_2, \mu_2 + \beta)$ is associated to the solution of $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$ with $\sigma_d = -1$ its congruent triangle is $\Delta_{d}^{(0)}(\mu_2 - \alpha - 2l - 2, \mu_2 + 2l - 2, \mu_2 + \beta - 2l - 2)$.

The total number of elementary triangles in $G_{\Delta,d}^{(0)}$ is therefore $d - 2 + 4\left[\frac{(d-3)^2}{12}\right]$.

2) $G_{\Delta,d}^{(0)}(d = 3q)$. Now $gcd(3,d) = 3$ then $3\mu \equiv \pm 1 \pmod{d}$ has no solution and we do not have elementary isosceles triangles. For $\alpha \neq \beta$ if $\sigma_d = \pm 1$ then $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$ has solution, according to Prop.2.2, when $\alpha - \beta \in A = \{-\sigma_d \pm 3n\}_{n=0,1,2,...}$ and $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$ has solution when $\alpha + \beta \in B = \{\sigma_d \pm 3n\}_{n=0,1,2,...}$. In both cases we have 3 solutions when $\beta - \alpha$ is not divisible by 3: $\Delta_{d}^{(0)}(\lambda + nq, \mu + nq, \nu + nq), n = 0, 1, 2$ which correspond to the same prototile (they have the same ID) rotated by $\frac{2\pi n}{3}$.
For $d = 3q$, the number of solutions of $\beta - \alpha = 3m$, $m = 1, 2, 3, \ldots$ is $q - 2m - 1$, therefore the total number of elementary triangles in $G_{\Delta, q}^{(0)}$ is $6(\left\lceil \frac{(d - 3)^2}{12} \right\rceil - l(l - 1))$ for $q = 2l + 1$ and $6(\left\lceil \frac{(d - 3)^2}{12} \right\rceil - (l - 1)^2)$ for $q = 2l$.

3) $G_{\Delta, d}^{(0)} \cup G_{\Delta, d}^{(-2)} \cup G_{\Delta, d}^{(2)}$, $d = 3q$. The case $\kappa = 0$ has already been analysed. For $\kappa = -2$ the elementary triangles have $\sigma_d \in \{-3, -1\}$. If $\alpha = \beta$ then $3\mu \equiv -1 \pmod{d}$ has no solution. If $q = 2l$, then $\mu = 2l - 1 + n, n = 0, q, 2q$ are the solutions to $3\mu \equiv -3 \pmod{d}$ and have the same ID. According to Lemma 2.6 when $\mu \in I_{odd}$, the number of subdivisions of the segment $G_{\mu, d}^{(-2)}$ is $3l - 1$. The triangles with an edge on $G_{\mu, d}^{(-2)}$ are $\Delta_d^{(-2)}(\mu - \alpha, \mu, \mu + \alpha), \alpha = 1, 2, \ldots 3l - 1$ and the equilateral, which corresponds to $\alpha = 2l$, has its sides on $G_{\mu, d}^{(-2)}$, $\mu = 2l - 1 + n, n = 0, q, 2q$, therefore the number of isosceles triangles is $1 + 3(3l - 1) = 9l - 5$. For $q = 2l + 1$, the solutions to $3\mu \equiv -3 \pmod{d}$ are $\mu = 2l + n, n = 0, q, 2q$, the segment $G_{\mu, d}^{(-2)}$ is subdivided into $3l + 1$ segments and the number of isosceles triangles is $9l - 1$. The case $\kappa = 2$ can be studied in a similar way having in mind that $\sigma_d \in \{1, 3\}$.

For $\alpha \neq \beta$ and $\kappa = -2$, the equation $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$, with $\sigma_d = -1$ has solution if $\beta - \alpha \in \{3m + 1\}_{m=1,2,3,\ldots}$ and $3\mu_2 \equiv \sigma_d + \alpha - \beta \pmod{d}$ when $\beta - \alpha \in \{3m\}_{m=1,2,3,\ldots}$. In both cases we have 3 solutions which correspond to the same prototile rotated by $\frac{2m\pi}{3}, n = 0, 1, 2$. For $\sigma_d = -3$ the equations $3\mu_1 \equiv \sigma_d - \alpha + \beta \pmod{d}$ have solutions when $\beta - \alpha \in \{3m\}_{m=1,2,3,\ldots}$ but now for each solution of $\mu_1$ linked to $\Delta_d^{(-2)}(\mu_1 - \alpha, \mu_1, \mu_1 + \alpha)$ we have a solution of $3\mu_1 \equiv 3 - \alpha + \beta \pmod{d}$ in $G_{\mu, d}^{(2)}$ corresponding to a congruent triangle $\Delta_d^{(2)}(\mu_1 - \alpha, \mu_1, \mu_1 + \alpha)$ having a different ID, and the same occurs for $\mu_2$. The number of non isosceles elementary triangles with two different ID is, up to mirror reflection, $3l(l - 1)$ for $q = 2l + 1$ and $3l(l - 2)$ for $q = 2l$.

The total number of elementary triangles in $G_{\Delta, 3q}^{(-2)}$ or in $G_{\Delta, 3q}^{(2)}$ is $9l + 1 + 3\left(\left\lceil \frac{(d - 3)^2}{12} \right\rceil - (l - 1)^2\right) + 6l(l - 1)$ for $q = 2l + 1$ and $9l - 5 + 3\left(\left\lceil \frac{(d - 3)^2}{12} \right\rceil - (l - 1)^2\right) + 6(l - 1)^2$ for $q = 2l$.

\[
\begin{array}{c}
\text{Figure 3. } \text{The elementary triangles } F := \Delta_{14}^{(0)}(4, 10, 13), \hat{F} := \Delta_{14}^{(0)}(0, 5, 8) \text{ and } G := \Delta_{14}^{(0)}(0, 4, 9) \text{ (from left to right) with the ID. } F \text{ and } \hat{F} \text{ have the same edge subdivision.}
\end{array}
\]

Example. We denote by $\hat{X}$ an elementary triangle which is the mirror image of the non-isosceles $X$ but with edges having the same subdivision (Fig.3). For $d = 14$ the elementary triangles with $\sigma_{14} = -1$ are the following:

\[
\begin{array}{c}
A := \Delta_{14}^{(0)}(0, 1, 12), \hat{A} := \Delta_{14}^{(0)}(3, 4, 6), B := \Delta_{14}^{(0)}(2, 12, 13), \hat{B} := \Delta_{14}^{(0)}(2, 5, 6), C := \Delta_{14}^{(0)}(0, 2, 11), \\
\hat{C} := \Delta_{14}^{(0)}(2, 4, 7), D := \Delta_{14}^{(0)}(3, 11, 13), \hat{D} := \Delta_{14}^{(0)}(1, 5, 7), E := \Delta_{14}^{(0)}(0, 3, 10), \hat{E} := \Delta_{14}^{(0)}(1, 4, 8), \\
F := \Delta_{14}^{(0)}(4, 10, 13), \hat{F} := \Delta_{14}^{(0)}(0, 5, 8), G := \Delta_{14}^{(0)}(0, 4, 9), H := \Delta_{14}^{(0)}(5, 9, 13), I := \Delta_{14}^{(0)}(6, 8, 13), \\
\hat{I} := \Delta_{14}^{(0)}(5, 10, 12), J := \Delta_{14}^{(0)}(6, 6, 7), \hat{J} := \Delta_{14}^{(0)}(4, 11, 12), K := \Delta_{14}^{(0)}(1, 2, 10), \hat{K} := \Delta_{14}^{(0)}(2, 3, 8), \\
L := \Delta_{14}^{(0)}(1, 3, 9), M := \Delta_{14}^{(0)}(6, 9, 12), N := \Delta_{14}^{(0)}(7, 8, 12), \hat{N} := \Delta_{14}^{(0)}(6, 10, 11), O := \Delta_{14}^{(0)}(7, 9, 11), \\
P := \Delta_{14}^{(0)}(8, 9, 10)
\end{array}
\]
Lemma 3.3. The edges of the inflated triangles with inflation factor \( t_{d,l} \) are subdivided according to the sequences \( (S_{l-j+1}, S_{l-j+3}, \ldots, S_{l+j-1}) \) if \( 1 \leq j \leq l \) and \( (S_{j-l+1}, S_{j-l+3}, \ldots, S_{j+l-1}) \) if \( l + 1 \leq j \leq q = \lceil \frac{d-1}{2} \rceil \).

Proof. We first prove the trigonometric identities \([7][9][6]\)

\[
\frac{s_l}{s_1} s_j = \sum_{k=0}^{j-1} s_{l-j+2k+1}, 1 \leq j \leq l; \tag{3.1}
\]

\[
\frac{s_l}{s_1} s_j = \sum_{k=0}^{l-1} s_{j-l+2k+1}, l + 1 \leq j \leq q \tag{3.2}
\]

Eq. (3.1) can be obtained by induction on \( j \). For \( j = 1 \) it is an identity. For \( j = 2 \) we have to prove

\[
s_l s_2 = s_1 (s_{l-1} + s_{l+1}) \tag{3.3}
\]

which is a consequence of the Mollweide’s formula and the law of sines. Now assuming it is true for \( j - 2 \) then it is true for \( j \) because \( \frac{s_l}{s_1} (s_j - s_{j-2}) = s_{l+j-1} + s_{l-j+1} \). On the other hand, for \( j = 1, 2, \ldots, q \), the identity \( \frac{s_l}{s_1} s_j = \sum_{k=0}^{l-1} s_{j-l+2k+1} \) can be obtained also by induction on \( j \). After cancelling terms like \( s_0 + s_{-n} \), we get it for \( j = 1 \). For \( j = 2 \) it is given by eq.(3.1). Now assuming it is true for \( j - 2 \) then it is true for \( j \) because \( \frac{s_l}{s_1} (s_j - s_{j-2}) = s_{j+l-1} - s_{j-l-1} \).

Therefore we have

\[
t_{d,l} \cdot 4s_1 s_j = 4s_1 (s_{l-j+1} + s_{l-j+3} + \ldots s_{l+j-1}), 1 \leq j \leq l \tag{3.4}
\]

and

\[
t_{d,l} \cdot 4s_1 s_j = 4s_1 (s_{j-l+1} + s_{j-l+3} + \ldots s_{j+l-1}), l + 1 \leq j \leq q \tag{3.5}
\]

We get a sequence \( (S_{l-j+1}, S_{l-j+3}, \ldots, S_{l+j-1}) \) if \( 1 \leq j \leq l \) and \( (S_{j-l+1}, S_{j-l+3}, \ldots, S_{j+l-1}) \) if \( l + 1 \leq j \leq q \), and both represent a unique interval on every \( G_\mu \).

Lemma 3.4. If two triangles with the same ID and with \( \sigma_d - \kappa \neq \pm 1 \) (mod \( d \)) are congruent they are dissected in the same way.

Proof. 1.1) \( G_{d, \Delta}^{(0)}, d = 2q + 1, q \neq 3l + 1, l = 1, 2, 3, \ldots \) If \( \alpha = \beta \) then the solution to \( 3\mu \equiv \sigma_d \) (mod \( d \)), \( \sigma_d = 2, 3, \ldots, q \) is \( \mu = 2\sigma_d \) for \( d = 6l - 1 \) and \( \mu = (4l + 1)\sigma_d \) for \( d = 6l + 1 \). If \( \alpha \neq \beta \), given a solution to \( 3\mu_1 \equiv \sigma_d - \alpha + \beta \) (mod \( d \)) then \( \mu_1 = \mu_1 - (2l + 1)\sigma_d \) is a solution of \( 3\mu_1 \equiv -\sigma_d - \alpha + \beta \) (mod \( d \)) for \( d = 6l + 1 \) and \( \mu_1 = \mu_1 + (2l - 1)\sigma_d \) for \( d = 6l - 1 \). We have the same results by replacing \( \mu_1 \) by \( \mu_2 \) if we analyse \( 3\mu_2 \equiv \sigma_d + \alpha - \beta \).

1.2) \( G_{d, \Delta}^{(0)}, d = 2q, q \neq 3l, l = 1, 2, 3, \ldots \) If \( \alpha = \beta \) then the solution to \( 3\mu \equiv \sigma_d \) (mod \( d \)), \( \sigma_d = 2, 3, \ldots, q \) is \( \mu = (2l + 1)\sigma_d \) for \( d = 6l + 2 \) and \( \mu = (4l + 3)\sigma_d \) for \( d = 6l + 4 \). If \( \alpha \neq \beta \), given a solution to \( 3\mu_1 \equiv \sigma_d - \alpha + \beta \) (mod \( d \)) then \( \mu_1 = \mu_1 + 2l\sigma_d \) is a solution of \( 3\mu_1 \equiv -\sigma_d - \alpha + \beta \) (mod \( d \)) for \( d = 6l + 2 \) and \( \mu_1 = \mu_1 - (2l + 2)\sigma_d \) for \( d = 6l + 4 \). We have the same results if we analyse \( 3\mu_2 \equiv \sigma_d + \alpha - \beta \) (mod \( d \)).

When \( d \) is not divisible by 3, \( \lambda + \mu + \nu = -1 \) and \( n \) is a solution to \( 3n \equiv p + 1 \) (mod \( d \)) then

\[
t_{d,p} \cdot \Delta_{d}^{(0)}(\pm \lambda, \pm \mu, \pm \nu) \cong \Delta_{d}^{(0)}(\pm \lambda \pm n, \pm \mu \pm n, \pm \nu \pm n) \tag{3.6}
\]

2) \( G_{d, \Delta}^{(0)}, d = 3q, q = 2l + 1 \) or \( q = 2l, l = 1, 2, 3, \ldots \) Now there are inflated elementary triangles in \( G_{d, \Delta}^{(0)} \) only when \( \sigma_d \) is not a multiple of 3. If \( \sigma_d > 1 \) the only values giving solutions are \( \sigma_d \in \{ \pm 1 + 3n \}_{n=1,2,\ldots} \) and we will not have tilings with inflation factors \( t_{d,p} \) with \( p = \sigma_d - \kappa = \sigma_d \) a multiple of 3 and prototypes contained in \( G_{d, \Delta}^{(0)} \). However there are non elementary triangles which are not inflated copies of the elementary ones. For instance the non-elementary triangle \( \Delta_{d}^{(0)}(2, 5, 8) \) has \( \sigma_d = 6 \) and it is one of the three inflated copies (in this case the inflation factor is \( t_{9,3} \)) of the elementary
equilateral triangle appearing in $\mathcal{G}_{\Delta,9}^{(\pm 2)}$ (the other two appear in $\mathcal{G}_{\Delta,9}^{(\pm 2)}$ , see 3) in this Lemma). The values $\sigma_d = \pm 1 + 3n$ ($n = 1$ for $d = 9$) correspond to the inflated copies of the elementary triangles appearing in $\mathcal{G}_{\Delta,9}^{(0)}$.

3) $\mathcal{G}_{\Delta,d}^{(0)} \cup \mathcal{G}_{\Delta,d}^{(-2)} \cup \mathcal{G}_{\Delta,d}^{(2)}$, $d = 3q$. For $\kappa = -2$ and $\alpha = \beta$ we have 3 solutions to $3\mu \equiv \sigma_d \pmod{d}$ when $\sigma_d \in \{-3+3n\}_{n \in \{0,1,2,...,q-1\}}$. The non elementary triangles $\Delta_d^{(-2)}(n - 1 - \alpha, n - 1, n - 1 + \alpha), n > 0, \alpha = 1, 2, ..., m$, with $m = 3l - 1, 3l + 1$ for $q = 2l, 2l + 1$ respectively, correspond to inflation factors $t_{d,p}$ with $p = \sigma_d - \kappa = \sigma_d + 2$ not a multiple of 3. The remaining two solutions for $\mu$ are $n - 1 + t_q, t = 1, 2$ and give congruent copies rotated by $\frac{2\pi}{3}$, except for the equilateral triangle. The inflated triangles with $t_{d,p}, p = 3n, n = 1, 2, ..., l$ are in $\mathcal{G}_{\Delta,d}^{(0)}$, because we have solutions to $3\mu \equiv \sigma_d = d - 3n \pmod{d}$ like $\Delta_d^{(0)}(-n, q - n, 2q - n)$. For $\alpha \neq \beta$ the distribution of inflated triangles in $\mathcal{G}_{\Delta,d}^{(\kappa)}$ can be analysed in a similar way. We have seen in Lemma 3.3 that there are two types of non-isosceles elementary triangles that we call $\Delta_{N1}, \Delta_{N2}$ such that $\Delta_{N2}$ is contained only in $\mathcal{G}_{\Delta,d}^{(-2)} \cup \mathcal{G}_{\Delta,d}^{(2)}$ and there are two $\Delta_{N2}$ differing only on their ID. The inflated copy of any $\Delta_{N1}$ is contained in $\mathcal{G}_{\Delta,d}^{(0)}$ when the inflation factor is $t_{d,p}$ with $p$ not a multiple of 3 (see 2) in this Lemma), and it is contained in $\mathcal{G}_{\Delta,d}^{(-2)} \cup \mathcal{G}_{\Delta,d}^{(2)}$ otherwise. On the other hand the inflated copy of any $\Delta_{N2}$ is contained in $\mathcal{G}_{\Delta,d}^{(0)}$ when the inflation factor is $t_{d,p}$ with $p$ a multiple of 3 and in $\mathcal{G}_{\Delta,d}^{(-2)} \cup \mathcal{G}_{\Delta,d}^{(2)}$ otherwise.

For $d = 3q$ and $1 \leq p < q$ with $p$ non divisible by 3, there is a solution to the congruences $3n \equiv p + 1 \pmod{d}, 3n \equiv p - 1 \pmod{d}$ and if $\lambda + \mu + \nu = \kappa \pm 1$ then

$$t_{d,p} : \Delta_d^{(\kappa)}(\lambda, \mu, \nu) \cong \Delta_d^{(\kappa)}(\lambda + n, \mu + n, \nu + n)$$

(3.7)

If $p$ is divisible by 3, then there is no solution to $3n \equiv p \pm 1 \pmod{d}$. For $\lambda + \mu + \nu = -3$ we have:

$$t_{d,3} : \Delta_d^{(-2)}(\lambda, \mu, \nu) \cong \Delta_d^{(0)}(\lambda, \mu, \nu), \forall q$$

(3.8)

and

$$t_{d,3m} : \Delta_d^{(-2)}(\lambda, \mu, \nu) \cong \Delta_d^{(0)}(\lambda + m + 1, \mu + m + 1, \nu + m + 1), m = 2, 3, ..., \left\lfloor \frac{q}{2} \right\rfloor, q > 3$$

(3.9)

For $\lambda + \mu + \nu = 1$ and $d = 3q$, $q = 2, 3$:

$$t_{d,3} : \Delta_d^{(0)}(\lambda, \mu, \nu) \cong \Delta_d^{(-2)}(\lambda + q - 2, \mu + q - 2, \nu + q - 2)$$

(3.10)

and for $d = 3q, q > 3$

$$t_{d,3m} : \Delta_d^{(0)}(\lambda, \mu, \nu) \cong \Delta_d^{(-2)}(\lambda + m - 1, \mu + m - 1, \nu + m - 1), m = 1, 2, ..., \left\lfloor \frac{q}{2} \right\rfloor$$

(3.11)

The corresponding inflated copies of $\tilde{\Delta} = \Delta_d^{(0)}(-\lambda, -\mu, -\nu)$ appear in $\mathcal{G}_{\Delta,d}^{(2)}$.

Both $\Delta = \Delta_{2d}^{(0)}(2\lambda, 2\mu, 2\nu), 2\lambda + 2\mu + 2\nu = 2$ and $\tilde{\Delta} = \Delta_{2d}^{(0)}(-2\lambda, -2\mu, -2\nu)$ are included in $\mathcal{G}_{\Delta,2d}^{(0)}$ and are mirror images with respect to $0_{\Delta,2d}$. When $\kappa \neq 0$ if $\Delta = \Delta_{2d}^{(\kappa)}(2\lambda, 2\mu, 2\nu) \subset \mathcal{G}_{\Delta,2d}^{(\kappa)}$ $2\lambda + 2\mu + 2\nu = \kappa = 2$ then $\tilde{\Delta} = \Delta_{2d}^{(-\kappa)}(-2\lambda, -2\mu, -2\nu) \subset \mathcal{G}_{\Delta,2d}^{(-\kappa)}$. There are exactly two triangles with the same $\sigma_d$ and the same ID for all non-isosceles triangles in $\mathcal{G}_{\Delta,d}^{(\kappa)}$ for $d$ non-divisible by 3 and one for $d$ divisible by 3. Every triangle can be reflected in $0_{\Delta,d}$, they have to be mirror images of each other and hence are dissected in the same way. For the tilings with prototiles in $\mathcal{G}_{\Delta,d}^{(0)} \cup \mathcal{G}_{\Delta,d}^{(-2)} \cup \mathcal{G}_{\Delta,d}^{(2)}$ the triangles contained in $\mathcal{G}_{\Delta,d}^{(\kappa)}$ appear reflected in $\mathcal{G}_{\Delta,d}^{(-\kappa)}$ and are also dissected in the same way. $\square$

An edge with length $4\sin(\frac{2\pi}{d})\sin(\frac{\pi}{d})$ and subdivision $(S_{a\pm 1}, S_{a\pm 1})$ induced by the ID of a prototile (we choose for instance anticlockwise orientation) can be represented by a letter $W_0^\beta$ or $W_0^{-1}$ provided that $S_{a\pm 1} < S_{a\pm 1}$ or $S_{a\pm 1} > S_{a\pm 1}$ respectively, and by $W_0^\beta$ if $S_{a-1} = S_{a+1}$. We can get face-to-face tilings with the property that the vertices of the inscribed triangles on the prototiles edges match.
We have seen that if there is an elementary triangle $\Delta := \Delta(W_{i_1}, W_{i_2}, W_{i_3})$, $i, j, k \in \{-1, 0, 1\}$ then there exists $\tilde{\Delta} := \Delta(W_{i^1}, W_{i^2}, W_{i^3})$. The mirror image of a word $w = W_i W_j W_k$ is denoted by $\text{Mir}(w)$, the projection of $W_i$ into $W$ by $P(W_i)$, and $\rho$ is the map $\rho(W_{i_1} W_{i_2} ... W_{i_n}) = W_{-i_1} W_{-i_2} ... W_{-i_n}$. We choose the tile dissections in such a way that the inflation rules for the edges of $\Delta$ and $\tilde{\Delta}$ satisfy

$$\phi(W^i) = \text{Mir}(\rho(\phi(W^{-i}))) \quad (3.12)$$

The common edges of two adjacent triangles are represented by $W_i, W^{-i}$. Therefore the fact that the tilings are face-to-face is equivalent to $P(\phi^n(W^i)) = \text{Mir}(P(\phi^n(W^{-i})))$, which is a consequence of Eq. (3.12).

**Example.** The prototiles $F, \hat{F}$ in Fig.3 have edges $W_3^{-1}, W_5^{-1}, W_6^{-1}$ corresponding to their subdivisions $(S_7, S_9), (S_{11}, S_9), (S_{13}, S_{11})$ respectively. The edges of $G$ are $W_4^{-1}, W_5^{-1}, W_6^{-1}$ with subdivisions $(S_9, S_7), (S_{11}, S_9), (S_{11}, S_9)$. A solution to $3n \equiv p + 1 \pmod{14}$ for $p = 3$ is $n = 6$. An edge inflation rule corresponding to $\iota_{14,3}$ is

$$\phi_+(W^i_1) = W_3^{-1}, \phi_+(W^i_2) = W_4^{-1} W_2^{-1}, \phi_+(W^i_3) = W_5^{-1} W_3^{-1} W_1^{-1}, \phi_+(W^i_4) = W_6^{-1} W_4^{-1} W_2^{-1},$$

$$\phi_+(W^i_5) = W_7^2 W_5^{-1} W_3^{-1}, \phi_+(W^i_6) = W_6 W_6^{-1} W_4^{-1}, \phi_+(W^i_7) = W_5 W_7 W_5^{-1}$$

In this case the words $\phi_+(W^i_{14})$ are not palindromic therefore we can define another edge inflation rule by $\phi_-(W^i) := \phi_+(W^{-i})$.

**Figure 4.** The segments $G_{\mu,14}^{(0)}$ for $\mu = 4, 10, 13$ determine the elementary triangle $F := \Delta_{14}^{(0)}(4, 10, 13)$. A dissection of $\iota_{14,3} \Delta_{14}^{(0)}(4, 10, 13) \equiv \Delta_{14}^{(0)}(10, 2, 5)$ is shown on the right.

**Definition 3.5.** Given $d$ and $p$, for every prototile $T \in \mathcal{F}_d$ we define the inflation rule $\Phi_{d,p,+}(T)$ by dissecting $\iota_{d,p} \cdot T$ according to Eqs. (3.6)-(3.12) and

$$\Phi_{d,p,-}(T) := \Phi_{d,p,+}(\tilde{T}), \Phi_{d,p,-}(\tilde{T}) := \Phi_{d,p,+(\tilde{T}), \Phi_{d,p,-}(\tilde{T}) := \Phi_{d,p,+(\tilde{T})}$$

if $T$ is non-isosceles and

$$\Phi_{d,p,-}(T) := \Phi_{d,p,+}(\tilde{T}), \Phi_{d,p,-}(\tilde{T}) := \Phi_{d,p,+(T)}$$

otherwise, where $\tilde{T}$ is related to $T$ through a reflection, including the ID in the reflection.
The substitution rules $\Phi_{14,3}$ determined by Eq. (3.6) and is given by the interior of $\Delta_{14}^{(0)}(10,2,5)$ in $G_{14,14}$, which is shown in Fig.4 (right). The substitution rules $\Phi_{14,3,+(A)}$ are

$$
\Phi_{14,3,+(A)} = \hat{A} \cup \tilde{O} \cup \tilde{B} \cup \tilde{N}, \Phi_{14,3,+(\hat{A})} = P \cup \hat{A} \cup O \cup \tilde{C}, \Phi_{14,3,+(B)} = \tilde{P} \cup \hat{A} \cup \tilde{O} \cup \hat{C} \cup \tilde{M}
$$

$$
\Phi_{14,3,+(\hat{B})} = \tilde{B} \cup O \cup \tilde{C} \cup \tilde{N} \cup \tilde{K}, \Phi_{14,3,+(C)} = \tilde{O} \cup \tilde{B} \cup \tilde{N} \cup \tilde{D} \cup \tilde{N} \cup \tilde{M} \cup \tilde{C} \cup \tilde{N},
$$

$$
\Phi_{14,3,+(\hat{C})} = \tilde{A} \cup O \cup \tilde{C} \cup M \cup \tilde{E} \cup \tilde{D} \cup N \cup \tilde{B}, \Phi_{14,3,+(\hat{D})} = \tilde{O} \cup \tilde{C} \cup \tilde{M} \cup \tilde{E} \cup \tilde{H} \cup \tilde{I} \cup \tilde{K} \cup \tilde{N},
$$

$$
\Phi_{14,3,+(\hat{E})} = \tilde{C} \cup \tilde{N} \cup \tilde{K} \cup \tilde{I} \cup \tilde{L} \cup \tilde{E} \cup \tilde{M} \cup \tilde{C} \cup \tilde{D}, \Phi_{14,3,+(\hat{F})} = \tilde{F} \cup \tilde{D} \cup \tilde{I} \cup \tilde{F} \cup \tilde{E} \cup \tilde{H} \cup \tilde{G} \cup \tilde{F}, \Phi_{14,3,+(\hat{G})} = \tilde{G} \cup \tilde{F} \cup \tilde{H} \cup \tilde{E} \cup \tilde{H} \cup \tilde{G} \cup \tilde{G} \cup \tilde{F},
$$

$$
\Phi_{14,3,+(H)} = \tilde{D} \cup \tilde{E} \cup \tilde{F} \cup \tilde{G} \cup \tilde{H} \cup \tilde{G} \cup \tilde{F} \cup \tilde{E} \cup \tilde{L}, \Phi_{14,3,+(I)} = \tilde{C} \cup \tilde{D} \cup \tilde{E} \cup \tilde{F} \cup \tilde{G} \cup \tilde{L} \cup \tilde{J} \cup \tilde{K},
$$

$$
\Phi_{14,3,+(J)} = \tilde{I} \cup \tilde{L} \cup \tilde{J} \cup \tilde{K} \cup \tilde{F} \cup \tilde{E} \cup \tilde{D} \cup \tilde{E}, \Phi_{14,3,+(K)} = \tilde{M} \cup \tilde{D} \cup \tilde{I} \cup \tilde{J} \cup \tilde{K}, \Phi_{14,3,+(L)} = \tilde{N} \cup \tilde{K} \cup \tilde{I} \cup \tilde{L} \cup \tilde{F}, \Phi_{14,3,+(N)} = \tilde{A} \cup \tilde{B} \cup \tilde{C} \cup \tilde{D} \cup \tilde{A} \cup \tilde{B} \cup \tilde{C}, \Phi_{14,3,+(P)} = \tilde{A} \cup \tilde{B}
$$

We have only given the prototile content in the substitutions rules. A more precise description can be given in terms of formal languages as in $5,7$ and references within. The substitutions $\Phi_{14,3,+(T)}$ for $T = F, \tilde{F}, G$ are represented in Fig.5.

**Definition 3.6.** By $S(F_d; \Phi_{d,p,\epsilon})$ ($\epsilon = \pm$) we denote the set of all tilings $\mathcal{P}$ of the entire plane, where every patch of $\mathcal{P}$ is congruent to some patch in some $\Phi_{d,p,\epsilon}(T)$, with $T \in F_d$.  

![Figure 5. Substitution rules for $F, \tilde{F}, G$ corresponding to $\iota_{14,3}$.](image1)

![Figure 6. Vertex configurations with rotational symmetry appearing in some examples of tilings with inflation factors $\iota_{5,2}, \iota_{8,3}, \iota_{9,4}, \iota_{10,3}, \iota^2_{12,2}$.](image2)
As a consequence of the previous results we have

**Theorem 3.7.** Assume \( d \in \mathbb{N}, d > 4, p = 2, 3, \ldots \lfloor \frac{d}{2} \rfloor, \epsilon = \pm \). The set \( S(F_d; \Phi_{d,p,\epsilon}) \) is non empty. It consists of tilings of the plane by tiles being congruent to members of \( F_d \).

A vertex configuration is a patch where all the tiles have a vertex in common. In previous papers we have studied several particular cases of the construction presented here (see a more exhaustive list of references in [8]). In Fig.6 we show vertex configurations with rotational symmetry appearing in the studied examples of tilings with inflation factors \( \iota_{5,2}, \iota_{8,3}, \iota_{9,4}, \iota_{10,3}, \iota_{12,2} \), which are PV numbers.

**Example.** For \( d = 14 \) we represent in Figs.7, 8 some patches of tilings with inflation factors \( \iota_{14,3} \) and \( \iota_{14,5} \). A fragment of \( \Phi_{14,3,+,\epsilon}(G) \) is shown in Fig.7 (left) and one of \( \Phi_{14,3,,-}(G) \) in the same figure (right). In Fig.8 we can see fragments corresponding to \( \Phi_{14,5,+,\epsilon}(G) \) and \( \Phi_{14,5,,-}(G) \) (\( \iota_{14,5} \) is also a PV number [6]).

![Figure 7. A fragment of \( \Phi_{14,3,+,\epsilon}(G) \) (left) and \( \Phi_{14,3,,-}(G) \) (right).](image)

![Figure 8. Fragments of \( \Phi_{14,5,+,\epsilon}(G) \) (left) and \( \Phi_{14,5,,-}(G) \) (right).](image)

### 4. Random tilings for \( d = 2q \)

Multiple different substitution rules defined on the same set of prototiles have been studied in [12][17][6][11]. The method of composition of inflation rules, also called multisubstitutions, consists in applying the same inflation rule to each tile in a given inflation step. In Fig.9 we can see parts of
(Φ_{14,3,+} Φ_{14,3,-})^3(G) and (Φ_{14,5,+} Φ_{14,5,-})^2(G). Also composition of substitution rules corresponding to different inflation factors are possible as shown in Fig. 10 for (Φ_{14,3,+} Φ_{14,5,+})^2(G) and Fig. 11 for (Φ_{14,5,-} Φ_{14,3,+})^2(G), where two different magnifications are used for the same fragment (\( \tau_{14,3} \tau_{14,5} \) is a PV number \([6]\)). In this section we treat the problem of the generation of random tilings by means of prototile rearrangements.
Assume Theorem 4.4.

Given a set of substitution rules $\Phi$ denoted by $\Phi_{d,R}$, we denote by $\mathcal{F}_d$ all prototile sets such that one can apply to each prototile and at each inflation step different substitution rules. We construct, in the interior of $\mathcal{G}_{\Delta,2q}^{(\kappa)}$, the regular polygon with vertices $\{p_{c,q+1}\} c=0,1,2,...,q-1$ [7].

Its edges have length $4s_1 s_{q-1}$ and some of them may lie on a $G_{\mu,2q}^{(\kappa)}$ (see Fig.1 for $q = 7$). The edge with vertices $p_{c,q+1}$, $p_{c+1,q+1}$ lie in some $G_{\mu,2q}^{(\kappa)}$ for $\kappa = 0$ when $3c \equiv q \mod 2q$ or $3c \equiv q - 2 \mod 2q$, namely $(c,\mu) \in \{(l,l+1),(2l,5l+1)\}$ if $q = 3l + 1$ and $(c,\mu) \in \{(l,l),(2l+1,5l+4)\}$ if $q = 3l + 2$. There is no solution for $c$ when $q = 3l$ if $\kappa = 0$. For $\kappa = -2$, $c$ must be a solution of $3c \equiv q - 2 \mod 2q$ or $3c \equiv q - 3 \mod 2q$, therefore $(c,\mu) \in \{(l,l-1),(3l-1,3l),(5l-1,5l)\}$ if $q = 3l$.

The sides not lying on a $G_{\mu,2q}^{(\kappa)}$ define tile rearrangements in the substitution rules [6] associated to edge flips $S_i \rightarrow S_{i-1}$. The set of substitution rules $\Phi_{d,R} := \{\Phi_{d,1},\Phi_{d,2},...\Phi_{d,k}\}$ is obtained by considering all possible combinations of tile rearrangements in the inflated prototiles.

Random substitution tilings are characterised by the fact that one can apply at each inflation step different substitution rules to each tile [3] [11].

**Definition 4.3.** Given a set of substitution rules $\Phi_{d,R} := \{\Phi_{d,1},\Phi_{d,2},...\Phi_{d,k}\}$, we define the random substitution tiling ensemble corresponding to a prototile set $\mathcal{F}$, denoted by $\mathcal{R}_s(\mathcal{F};\Phi_{d,R})$, by considering all possible combinations of tile rearrangements in the inflated prototiles. The inflation factor associated with $\mathcal{R}_s(\mathcal{F};\Phi_{d,R})$ is $t_d a_q$.

**Theorem 4.4.** Assume $d = 2q \in \mathbb{N}, q > 2$. The sets $\mathcal{R}_s(\mathcal{F};\Phi_{d,p,e})$ and $\mathcal{R}_s(\mathcal{F};\Phi_{d,R})$, with prototiles congruent to members of $\mathcal{F}_d$, are non-empty. The inflation factor associated with $\mathcal{R}_s(\mathcal{F};\Phi_{d,R})$ is $t_d a_q$.

**Proof.** 1) $\kappa = 0, q = 3l + 1$: each pair of consecutive polygon vertices $p_{l+a,4l+a+1},p_{l+a+1,4l+a+2}$ define a quadrilateral $\Box$ in $\mathcal{G}_{\Delta,2q}^{(\kappa)}$ which is the union of two adjacent prototiles:

$$
\Box_{6l+2}(l+a,4l+a+1;l+a+1,4l+a+2) = \Delta_{6l+2}^{(\kappa)}(l+a,4l+a+1,l-2a) \cup \Delta_{6l+2}^{(\kappa)}(l+a+1,4l+a+2,l-2a)
$$

We introduce edge flips consisting in replacing the common edge of the prototiles by the polygon edge when $a \in \{1,2,...,l-1,l+1,...,3l\}$ which produces a prototile rearrangement. We denote by $\Delta'$ a triangle not belonging to $\mathcal{G}_{\Delta,2q}^{(\kappa)}$ but congruent to some prototile and by $\mu'$ the index of a segment parallel to $G_{\mu,2q}^{(\kappa)}$ not belonging to $\mathcal{G}_{\Delta,2q}^{(\kappa)}$. After the edge flip we have

$$
\Box_{6l+2}^{(\kappa)}(l+a,4l+a+1;l+a+1,4l+a+2) = \Delta'(l+a,4l+a+2,(l+4a+1)') \cup \Delta'(4l+a+1,l+a+1,(l+4a+1)')
$$

with $\Delta'(l+a,4l+a+2,(l+4a+1)') \cong \Delta_{6l+2}^{(\kappa)}(3l-a,3l+2a+1,6l-a+2)$ and $\Delta'(4l+a+1,l+a+1,(l+4a+1)') \cong \Delta_{6l+2}^{(\kappa)}(3l-a+1,3l+2a+1,6l-a+1)$.

2) $\kappa = 0, q = 3l + 2$: the vertices $p_{l+a,4l+a+2},p_{l+a+1,4l+a+3}$ define a quadrilateral :

$$
\Box_{6l+4}^{(\kappa)}(l+a,4l+a+2;l+a+1,4l+a+3) = \Delta_{6l+4}^{(\kappa)}(l+a,4l+a+2,l-2a+1) \cup \Delta_{6l+4}^{(\kappa)}(l+a+1,4l+a+3,l-2a+1)
$$

and the edge flip for $a \in \{1,2,...,l-1,l+1,...,3l\}$ gives

$$
\Box_{6l+4}^{(\kappa)}(l+a,4l+a+2;l+a+1,4l+a+3) = \Delta'(l+a,4l+a+3,(l+4a)') \cup \Delta'(l+a+1,4l+a+2,(l+4a)')
$$

with $\Delta'(l+a,4l+a+3,(l+4a)') \cong \Delta_{6l+4}^{(\kappa)}(3l-a+2,-a+1,3l+2a+2)$ and $\Delta'(l+a+1,4l+a+2,(l+4a)') \cong \Delta_{6l+4}^{(\kappa)}(3l-a+3,-a,3l+2a+2)$.

3) $q = 3l$. 

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3.1) $\kappa = 0$: the vertices $p_{l+a,4l+a}, p_{l+a+1,4l+a+1}$ define a quadrilateral

$\square_{6l}^{(0)}(l+a, 4l+a; l+a+1, 4l+a+1) = \Delta_{6l}^{(0)}(l+a, 4l+a, l-2a-1) \cup \Delta_{6l}^{(0)}(l+a+1, 4l+a+1, l-2a-1)$

An edge flip for $a \in \{1, 2, ..., 3l-1\}$ gives

$\square_{6l}^{(0)}(l+a, 4l+a; l+a+1, 4l+a+1) = \Delta'(4l+a, l+a+1, (l+4a+2)) \cup \Delta'(4l+a, l+a+1, (l+4a+2))'$

In this case we do not have congruent prototiles $\Delta_{6l}^{(0)}(\lambda, \mu, \nu)$ inside $G_{\Delta_{6l}}^{(0)}$, with, for instance, $\sigma_{6l} = 1$, because $3\mu \equiv -3a+1 \pmod{6l}$ has no solution. They are congruent to elementary triangles with $\sigma_{6l} = -3$ which belong to $G_{\Delta_{6l}}^{(-2)}$. $\Delta'(4l+a, l+a+1, (l+4a+2)) \cong \Delta_{6l}^{(-2)}(3l-a-2, 6l-a-1, 3l+2a)$ and $\Delta'(4l+a, l+a+1, (l+4a+2))' \cong \Delta_{6l}^{(-2)}(2l-a-2, 5l-a-1, 5l+2a)$.  

3.2) $\kappa = \pm 2$: in $G_{\Delta_{6l}}^{(-2)}$ the vertices $p_{l+a-1,4l+a-1}, p_{l+a,4l+a}$ define a quadrilateral

$\square_{6l}^{(-2)}(l+a-1, 4l+a-1; l+a, 4l+a) = \Delta_{6l}^{(-2)}(l+a-1, 4l+a-1, l-2a-1) \cup \Delta_{6l}^{(-2)}(l+a, 4l+a, l-2a-1)$

An edge flip for $a \in \{1, 2, ..., l-1, l+1, ..., 2l-1, 2l+1, ..., 3l-1\}$ gives

$\square_{6l}^{(-2)}(l+a-1, 4l+a-1; l+a, 4l+a) = \Delta'(l+a-1, 4l+a, (l+4a)) \cup \Delta'(l+a, 4l+a-1, (l+4a))'$

with $\Delta'(l+a-1, 4l+a, (l+4a))' \cong \Delta_{6l}^{(-2)}(3l-a-1, 6l-a-1, 3l+2a)$ and $\Delta'(l+a, 4l+a-1, (l+4a))' \cong \Delta_{6l}^{(-2)}(5l-a, 2l-a-1, 5l+2a)$. If we change the signs we get a similar result for the edge flips in $G_{\Delta_{6l}}^{(2)}$.

In $R_r(F_d; \Phi_{d,R})$, where the tilings are obtained by tile rearrangements in a deterministic tiling, we have random tilings associated with all inflation factors. For $R_a(F_d; \Phi_{d,R})$, with $d = 2q$, if the inflation factor is $t_{d,q}$, then in the sequence

$$(S_q-j+1, S_q-j+3, ..., S_q+j-1), \; 1 \leq j \leq q$$

we have

$$S_q+j-1 = S_{d-(q+j-1)} = S_q-j+1, S_q+j-3 = S_{d-(q+j-3)} = S_q-j+3, ...$$

and therefore the sequence is palindromic. Now the ID is not needed to get face to face substitution tilings [6]. In order to produce face to face tilings with other inflation factors, combinations of several substitutions must be considered (see [8], Figs.6,7 for $d = 5, 9$).

Example. The tile rearrangements for $d = 14$ are obtained having in mind that $E \cup \tilde{E} \cong M \cup H, I \cup \tilde{J} \cong L \cup K, \tilde{E} \cup \tilde{J} \cong F \cup D$ (Fig.12 (left)) and their mirror reflections, where we consider $M, H, L, K, F, D$ without the ID, namely, as members of $F_d$. In order to illustrate how to get $R_r(F_{14}; \Phi_{14,5,+})$ we have shown in Fig.12 (right) the prototiles $E, \tilde{E}, J, \tilde{J}, \tilde{I}$ in $F_{14,5,+}^3(G)$ with different grey levels. The edge flips corresponding to the tile rearrangements are represented by a thicker segment, as in Fig.1.

It is possible to describe the random tilings in terms of formal grammars along the lines of [7]. In that approach the elements of the alphabet are letters representing the prototiles and the set of production rules $H$ in the language would correspond in this case to the set of substitution rules $\Phi_{d,R}$, having in mind the orientation of the prototiles. For non-deterministic structures one has to introduce a function $\pi : H \rightarrow (0, 1]$, called the probability distribution, which maps the set of production rules into the set of production probabilities. A question that deserves further study is the role played by the choice of $\pi$ in the analysis of diffraction patterns, phase transitions and other properties of the associated structures (see also [4] for a recent application of weighted context-free grammars in a different domain).
Figure 12. Tile rearrangements $E \cup \tilde{E} \rightarrow M \cup H$, $I \cup \tilde{I} \rightarrow L \cup K$, $\tilde{E} \cup \tilde{I} \rightarrow F \cup D$ for $d = 14$ (left). In $\Phi_{14,5,+}^2(G)$ the edge flips are marked with a thicker segment (right).

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