DUALITY FOR PARTIAL GROUP ACTIONS

CHRISTIAN LOMP

ABSTRACT. Given a finite group $G$ acting as automorphisms on a ring $A$, the skew group ring $A \ast G$ is an important tool for studying the structure of $G$-stable ideals of $A$. The ring $A \ast G$ is $G$-graded, i.e. $G$ coacts on $A \ast G$. The Cohen-Montgomery duality says that the smash product $A \ast G \# k[G]^*$ of $A \ast G$ with the dual group ring $k[G]^*$ is isomorphic to the full matrix ring $M_n(A)$ over $A$, where $n$ is the order of $G$. In this note we show how much of the Cohen-Montgomery duality carries over to partial group actions in the sense of R.Exel. In particular we show that the smash product $(A \ast_\alpha G) \# k[G]^*$ of the partial skew group ring $A \ast_\alpha G$ and $k[G]^*$ is isomorphic to a direct product of the form $K \times e M_n(A)e$ where $e$ is a certain idempotent of $M_n(A)$ and $K$ is a subalgebra of $(A \ast_\alpha G) \# k[G]^*$. Moreover $A \ast_\alpha G$ is shown to be isomorphic to a separable subalgebra of $e M_n(A)e$. We also look at duality for infinite partial group actions and for partial Hopf actions.

1. Introduction

Let $k$ be a commutative unital ring and $A$ a unital $k$-algebra. Given a finite group $G$ acting as $k$-linear automorphisms on $A$, Cohen and Montgomery showed in [3] that the smash product $A \ast G \# k[G]^*$ of the skew group ring $A \ast G$ and the dual group ring $k[G]^*$ is isomorphic to the full matrix ring $M_n(A)$ over $A$, where $n$ is the order of $G$.

R.Exel introduced in [6] the notion of a partial group action on a $k$-algebra: $G$ acts partially on $A$ by a family $\{\alpha_g : D_{g^{-1}} \to D_g\}_{g \in G}$ if for all $g \in G$, $D_g$ is an ideal of $A$ and $\alpha_g$ is an isomorphism of $k$-algebras such that for all $g, h \in G$:

(i) $D_e = A$ and $\alpha_e$ is the identity map of $A$;
(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

The partial skew group ring of $A$ and $G$ is defined to be the projective left $A$-module $A \ast_\alpha G = \bigoplus_{g \in G} D_g$ with multiplication

$(a \circ g)(b \circ h) = \alpha_g(\alpha_{g^{-1}}(a)b) \circ gh$

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Lemma 1.1. With the notation above we have that

(1) $\beta_g$ are $k$-algebra endomorphisms of $\mathcal{A}$ for all $g \in G$, i.e.

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \forall a, b \in \mathcal{A}.$$ 

(2) $g \cdot (h \cdot a) = ((gh) \cdot a)1_g$ for all $g, h \in G$ and $a \in \mathcal{A}$.

(3) $(g \cdot a)b = g \cdot (a(g^{-1} \cdot b))$ for all $a, b \in \mathcal{A}$ and $g \in G$.

Proof. (1) follows since the $\alpha_g$ are algebra homomorphisms and the idempotents $1_g$ are central, i.e. for all $a, b \in \mathcal{A}$:

$$\beta_g(ab) = \alpha_g(ab1_{g^{-1}}) = \alpha_g(a1_{g^{-1}}b1_{g^{-1}}) = \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}) = \beta_g(a)\beta_g(b).$$

(2) follows from [5] 2.1(ii):

$$\alpha_g(\alpha_h(a1_{h^{-1}})1_{g^{-1}}) = \alpha_{gh}(a1_{h^{-1}g^{-1}}1_g)$$

what expressed by $\beta$ yields the statement of (2).

(3) Using (1), (2) and the fact that $\beta_e = id$ and that the image of $\beta_g$ is $D_g = A1_g$ we have that

$$g \cdot (a(g^{-1} \cdot b)) = (g \cdot a)(g \cdot (g^{-1} \cdot b)) = (g \cdot a)b1_g = (g \cdot a)b.$$

Obviously we also have $g \cdot 1 = \alpha_g(1_{g^{-1}}1_g) = 1_g$ and $g \cdot (g^{-1} \cdot a) = ((gg^{-1}) \cdot a)1_g = a1_g$

for all $a \in \mathcal{A}$ and $g \in G$ using property (2). Moreover using the fact that $\alpha_g$ is bijective and $1_g$ central we have for all $a \in \mathcal{A}$ and $g \in G$ that $g \cdot a = 0$ if and only if $a \in \mathcal{A}(1-1_g)$.

2. Grading of the partial skew group ring

The partial skew group ring is the projective left $\mathcal{A}$-module $\mathcal{A} \rtimes_\alpha G = \bigoplus_{g \in G} D_g$. We will write an element of $\mathcal{A} \rtimes_\alpha G$ as a finite sum of elements $\sum_{g \in G} a_g \alpha g$ where $a_g \in D_g = A1_g$ and $\alpha g$ is a placeholder for the $g$-th component. $\mathcal{A} \rtimes_\alpha G$ becomes an associative $k$-algebra by the product:

$$(a\alpha g)(b\alpha h) = \alpha_g(\alpha_{g^{-1}}(a)b)\alpha gh$$
for all $g, h \in G$ and $a \in D_g$ and $b \in D_h$. Using our $\cdot$-notation we see easily

$$(a \circ g)(b \circ h) = a(g \cdot b) \circ gh.$$ 

The algebra $A \star_\alpha G$ is naturally $G$-graded where the homogeneous elements are those in $\{ D_g \}_{g \in G}$, i.e. $D_g D_h \subseteq D_{gh}$ by definition of the multiplication in $A \star_\alpha G$. Thus $A \star_\alpha G$ becomes a $k[G]$-comodule algebra. Note that the $G$-grading is strong, in the sense that $D_g D_h = D_{gh}$ if and only if $D_g = A$ for all $g \in G$, i.e. the $G$-action is global (since if $D_g D_h = D_{gh}$ for all $g, h \in G$, then

$$A 1_g 1_{g^{-1}} = D_g D_{g^{-1}} = D_{gg^{-1}} = D_e = A,$$

thus $1_g$ is an invertible central idempotent and hence equals $1$, i.e. $D_g = A$). Known results on graded rings can be applied to the $G$-grading of $A \star_\alpha G$.

3. Duality for partial actions of finite groups

Assume $G$ to be finite, then $k[G]^*$ becomes a Hopf algebra with projective basis $p_g \in k[G]^*$ where $p_g(h) = \delta_{g,h}$ for all $g, h \in H$. The multiplication is defined as $p_g \star p_h = \delta_{g,h} p_g$ and the identity element of $k[G]^*$ is $1 = \sum_{h \in H} p_h$. Now $A \star_\alpha G$ becomes a $k[G]^*$-module algebra by

$$p_h \triangleright (a \circ g) = \delta_{g,h} a \circ g$$

for all $g, h \in G$ and $a_g \in D_g$. The multiplication of the smash product $(A \star_\alpha G) \# k[G]^*$ is defined as

$$(a \circ g \# p_h)(b \circ h \# p_l) = \sum_{s \in G} (a \circ g)[p_s \triangleright (b \circ k)] \# p_{s^{-1}h} \# p_l = (a \circ g)(b \circ k) \# p_{k^{-1}h} \# p_l = a(g \cdot b) \circ gh \# \delta_{h,kl} p_l.$$ 

The identity element of $B = A \star_\alpha G \# k[G]^*$ is $\sum_{h \in G} 1 \circ e \# p_h$. In the case of global actions Cohen and Montgomery proved in [3] that $A \star G \# k[G]^* \cong M_n(A)$ where $n = |G|$ and $M_n(A)$ denotes the ring of $n \times n$-matrices over $A$. We will index the matrices of $M_n(A)$ by elements of $G$ and denote by $E_{g,h}$ the elementary matrix that has the value $1$ in the $g$-th row and the $h$-th column and zero elsewhere.

**Proposition 3.1.** Let $G$ be a finite group of $n$ elements, acting partially on an $k$-algebra $A$ and consider the $k$-algebra $B = (A \star_\alpha G) \# k[G]^*$. The map

$$\Phi : B \longrightarrow M_n(A)$$

with

$$\sum_{g, h} a_{g,h} \circ g \# p_h \mapsto \sum_{g, h} h^{-1} \cdot (g^{-1} \cdot a_{g,h}) E_{gh,h}$$

is a $k$-algebra homomorphism.
Proof. First note that for any \( g, h, k \in G \) and \( a \in D_g, b \in D_h \) we have, using Lemma 2(2) in the 2nd, 4th and 6th line and Lemma 1.1(1) in the 3rd line:

\[
\begin{align*}
k^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b))) &= k^{-1} \cdot (((gh)^{-1} \cdot a)((gh)^{-1} \cdot (g \cdot b))) \\
&= k^{-1} \cdot (((gh)^{-1} \cdot a)(h^{-1} \cdot b)1_{(gh)^{-1}}) \\
&= [k^{-1} \cdot ((gh)^{-1} \cdot a)][k^{-1} \cdot (h^{-1} \cdot b)] \\
&= ((gk)^{-1} \cdot a)((hk)^{-1} \cdot b)1_{k^{-1}} \\
&= ((gk)^{-1} \cdot a)(1_{hk^{-1}})((hk)^{-1} \cdot b)1_{k^{-1}} \\
&= ((hk)^{-1} \cdot (g^{-1} \cdot a))(k^{-1} \cdot (h^{-1} \cdot b))
\end{align*}
\]

Thus we showed:

\[
(1) \quad k^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b))) = ((hk)^{-1} \cdot (g^{-1} \cdot a))(k^{-1} \cdot (h^{-1} \cdot b))
\]

For any \( a \circ g \# p_h, b \circ k \# p_l \in (A \ast \alpha G) \# k[G]^\ast \) we have, using equation \( (\Pi) \):

\[
\begin{align*}
\Phi((a \circ g \# p_h)(b \circ k \# p_l)) &= \Phi(a(g \cdot b) \circ g \# k \circ \delta_{h,kl}p_l) \\
&= l^{-1} \cdot ((gk)^{-1} \cdot (a(g \cdot b)))E_{gk,l} \delta_{h,kl} \\
&= ((kl)^{-1} \cdot (g^{-1} \cdot a))(l^{-1} \cdot (k^{-1} \cdot b))E_{gh,h,k}E_{kl,l} \delta_{h,kl} \\
&= (h^{-1} \cdot (g^{-1} \cdot a))E_{gh,h}(l^{-1} \cdot (k^{-1} \cdot b))E_{kl,l} \\
&= \Phi(a \circ g \# p_h)\Phi(b \circ k \# p_l)
\end{align*}
\]

Hence \( \Phi \) is an algebra homomorphism. \( \square \)

Note that \( \Phi \) restricted to \( A \ast \alpha G \) is injective, i.e. \( A \ast \alpha G \) can be considered a subalgebra of \( M_n(A) \). In general \( \text{Ker}(\Phi) \) is non-trivial, unless the partial action is a global action.

**Proposition 3.2.** \( \text{Ker}(\Phi) = \bigoplus_{g,h \in G} A(1 - 1_{gh})1_g \circ g \# p_h. \)

*Proof.* Suppose \( \gamma = \sum_{g,h} a_{g,h} \circ g \# p_h \in \text{Ker}(\Phi) \), then \( h^{-1} \cdot (g^{-1} \cdot a_{g,h}) = 0 \) for all \( g, h \in G \). Thus \( (g^{-1} \cdot a_{g,h}) \in A(1 - 1_h) \cap D_{g^{-1}} = A(1 - 1_h)1_{g^{-1}}. \) Hence

\[
a_{g,h} = g \cdot (g^{-1} \cdot a_{g,h}) \in Ag \cdot (1 - 1_h) = A(1_g - 1_g1_{gh}),
\]

i.e. \( \gamma \in \bigoplus_{g,h} A(1 - 1_{gh})1_g \circ g \# p_h. \) The other inclusion follows because

\[
\begin{align*}
\Phi((g \cdot (1 - 1_h)) \circ g \# p_h) &= h^{-1} \cdot (g^{-1} \cdot (g \cdot (1 - 1_h)))E_{gh,h} = h^{-1} \cdot ((1 - 1_h)1_g)E_{gh,h} = 0.
\end{align*}
\]

\( \square \)

Note that the inclusion of \( A \ast \alpha G \) into \( (A \ast \alpha G) \# k[G]^\ast \) is given by \( a \circ g \mapsto \sum_{h \in G} a \circ g \# p_h \) for all \( g \in G \) and \( a \in D_g \). If \( \sum_{h \in G} a \circ g \# p_h \in \text{Ker}(\Phi) \), then \( a \in A(1 - 1_{gh})1_g \) for all \( h \in G \). In particular for \( h = e \) we have \( a \in A(1 - 1_g)1_g = 0. \) Hence \( \Phi \) restricted to \( A \ast \alpha G \) is injective.
Theorem 3.4.

Proof. The kernel of \( \Phi \) is an ideal and a direct summand of \( B \); see this we first show that the left \( (e) \) this shows that \( \text{Im}(\Phi) \) consists of all matrices of the given form and hence is equal to \( I \times I \). For any \( g, h \in G \) such that by definition of partial group action we have

\[
\Phi(a_{gh})_1 = (a_{gh})_1 = (a_{gh}^{-1})_1 E_{gh,h} = (b_{r,s} l_{r-1} l_{s-1})_{r,s \in G}
\]

with \( b_{r,s} = r^{-1} \cdot a_{rs^{-1},s} \) for all \( r, s \in G \).

Proposition 3.3. The image of \( \Phi \) consists of all matrices of the form \( (b_{gh})_1 g^{-1} h^{-1} \) for any matrix \( (b_{gh})_1 \) of elements of \( A \). In particular \( \text{Im}(\Phi) = e M_n(A)e \), where \( e \) is the idempotent \( \sum_{g \in G} 1_{g^{-1} E_{g,g}} \).

Proof. We saw already that an element of the image of \( \Phi \) is of the given form. Note that by definition of partial group action we have

\[
D_g \cap D_{gh} = a_g(D_{g^{-1}} \cap D_h)
\]

for all \( g, h \in G \). Hence also

\[
D_{g^{-1}} \cap D_{h^{-1}} = a_{g^{-1}}(D_g \cap D_{gh^{-1}})
\]

holds for all \( g, h \in G \). Thus for all \( b \in A \) there exists \( a \in A \) such that

\[
b_{1_{g^{-1} h^{-1}}} = a_{g^{-1}}(a_{1_{g^{-1} h^{-1}}}) = g^{-1} \cdot (a_{1_{g^{-1} h^{-1}}}).
\]

This implies that

\[
\Phi(a_{1_{g^{-1} h^{-1}}} g^{-1} h^{-1} p_{gh}) = h^{-1} \cdot (h g^{-1}) \cdot (a_{1_{g^{-1} h^{-1}}}) E_{g,h}
\]

\[
= g^{-1} \cdot (a_{1_{g^{-1} h^{-1}}}) E_{g,h}
\]

Hence given any matrix \( (b_{gh})_1 \) there are elements \( a_{gh} \) such that

\[
\Phi \left( \sum_{g,h} a_{gh} g^{-1} h^{-1} p_{gh} \right) = \sum_{g,h} b_{gh} g^{-1} h^{-1} E_{g,h} = (b_{gh} g^{-1} h^{-1} g h)_{g,h \in G}.
\]

This shows that \( \text{Im}(\Phi) \) consists of all matrices of the given form and hence is equal to \( e M_n(A)e \). Note that \( e \) is the image of the identity element of \( B \).

The last Propositions yield our main result in this section

Theorem 3.4. \( (A \otimes G) \otimes k[G]^* \simeq \text{Ker}(\Phi) \times e M_n(A)e \).

Proof. The kernel of \( \Phi \) is an ideal and a direct summand of \( B = (A \otimes G) \otimes k[G]^* \). To see this we first show that the left \( A \)-module \( I = \bigoplus_{g,h \in G} A \cdot \delta_{g,h,1} \cdot g^{-1} h^{-1} p_{gh} \) is a two-sided ideal of \( B \). For any \( x \cdot k \cdot p_{1} \in B \) and \( a_{1_{gh}} g^{-1} h^{-1} p_{gh} \in I \) we have

\[
(a_{1_{gh}} g^{-1} h^{-1} p_{gh}) (b_{k} p_{1}) = a_{1_{gh}} g^{-1} \cdot b_{k} \cdot p_{1} = a_{1_{gh}} g^{-1} h^{-1} p_{gh} \cdot k \cdot p_{1} \in I.
\]

Since \( I \otimes \text{Ker}(\Phi) = B \) and both direct summands are two-sided ideals we have \( B = I \times \text{Ker}(\Phi) \) (ring direct product). Moreover \( \Phi(I) = e M_n(A)e = \text{Im}(\Phi) \). This implies \( B \simeq \text{Ker}(\Phi) \times e M_n(A)e \).
Note that \( \Phi \) embedds \( \mathcal{A} \ast_a G \) into the Pierce corner \( eM_n(\mathcal{A})e \).

**Corollary 3.5.** \( \mathcal{A} \ast_a G \) is isomorphic to a separable subalgebra of \( eM_n(\mathcal{A})e \).

**Proof.** Recall that the subalgebra \( \mathcal{A} \ast_a G \) sits into \( \mathcal{B} \) by \( a \circ g \mapsto \sum_{h \in G} a \circ g \# p_h \). The right action of \( \mathcal{A} \ast_a G \) on \( \mathcal{B} \) is given by

\[
(x \circ k \# p_l) \cdot a \circ g = (x \circ k \# p_l) \left( \sum_{h \in G} a \circ g \# p_h \right) = (x \circ k)(a \circ g) \# p_{g^{-1} l}
\]

The left action is given by

\[
a \circ g \cdot (x \circ k \# p_l) = \left( \sum_{h \in G} a \circ g \# p_h \right) (x \circ k \# p_l) = (a \circ g)(x \circ k) \# p_l
\]

The element

\[
f = \sum_{g \in G} \circ e \# p_g \otimes \circ e \# p_g \in \mathcal{B} \otimes_{\mathcal{A} \ast_a G} \mathcal{B}
\]

is \( \mathcal{A} \ast_a G \)-centralising, i.e. for all \( a \circ h \in \mathcal{A} \ast_a G \) we have

\[
f \circ a \circ h = \sum_{g \in G} \circ e \# p_g \otimes a \circ h \# p_{h^{-1} g} = \sum_{g \in G} a \circ h \# p_{h^{-1} g} \otimes \circ e \# p_{h^{-1} g} = a \circ h \circ f
\]

Since also \( \mu(f) = \circ e \# \sum_{g \in G} p_g = 1_B \) we have that \( f \) is a separability idempotent for \( \mathcal{B} \) over \( \mathcal{A} \ast_a G \). Hence \( eM_n(\mathcal{A})e \simeq \Phi(\mathcal{B}) \) is separable over \( \Phi(\mathcal{A} \ast_a G) \simeq \mathcal{A} \ast_a G \).

\( \square \)

4. **Trivial Partial Actions**

The easiest example of partial actions arise from (central) idempotents in a \( k \)-algebra \( \mathcal{A} \). Suppose that \( \mathcal{A} \) admits a non-zero central idempotent, i.e. there exist algebras \( R, S \) such that \( \mathcal{A} = R \times S \) as algebras. For any group \( G \) set \( D_g = R \times 0 \) and \( \alpha_g = id_{D_g} \) for all \( g \neq e \) and \( D_e = \mathcal{A} \) and \( \alpha_e = id_\mathcal{A} \). Then \( \{ \alpha_g \mid g \in G \} \) is a partial action of \( G \) on \( \mathcal{A} \). The partial skew group ring turns out to be \( \mathcal{A} \ast_a G \simeq R[G] \times S \), where \( R[G] \) denotes the group ring of \( R \) and \( G \). Note that \( 0 \times S \) is in the zero-componente of the \( G \)-grading on \( \mathcal{A} \ast_a G \). If \( G \) is finite, say of order \( n \), then a short calculation (using Cohen-Montgomery duality and Theorem 3.4) shows that \( \mathcal{B} = (\mathcal{A} \ast_a G) \# k[G] \) is isomorphic to \( M_n(R) \times S^n \) where \( S^n \) denotes the direct product of \( n \) copies of \( S \). Depending on the rings \( R \) and \( S \), \( \mathcal{B} \) might or might not be Morita equivalent to \( \mathcal{A} \). For instance if \( R = S = F \) is a field, then any progenerator \( P \) for \( \mathcal{A} \) has the form \( F^k \times F^m \) for numbers \( k, m \geq 1 \). Thus \( \text{End}_k(P) \simeq M_k(F) \times M_m(F) \), whose center is isomorphic to \( F^2 = \mathcal{A} \). On the other hand \( \mathcal{B} = (\mathcal{A} \ast_a G) \# k[G] \simeq M_n(F) \times F^n \) has center \( F^{n+1} \), i.e. \( \mathcal{B} \) will be Morita equivalent to \( \mathcal{A} \) if and only if \( G \) is trivial.

On the other hand, there are algebras which satisfy (as algebras) \( \mathcal{A}^n \simeq \mathcal{A} \simeq M_n(\mathcal{A}) \) for any \( n \). To give an example, let \( R \) be the ring of sequences of elements of a field \( k \), i.e. \( R = k^N \). The function \( \chi \) with \( \chi(2n) = 1 \) and \( \chi(2n + 1) = 0 \) for all \( n \) defines an idempotent of \( R \). The map \( \Psi : \chi R \to R \) with \( \Psi(\chi f)(n) = f(2n) \) is a ring isomorphism.
Analogously we can show that \((1 - \chi)R \simeq R\). Hence \(R^2 \simeq R\). Now take \(A = \text{End}_k(F)\), where \(F = R^{(\mathbb{N})}\) denotes the countable infinite free \(R\)-module. Using again \(\chi\) we have that

\[
A = (\chi A) \times ((1 - \chi) A) \simeq A \times A \simeq \cdots \simeq A^n
\]

for any \(n \geq 2\). Moreover for any partition of \(\mathbb{N}\) into \(n\) infinite disjoint subsets \(\Lambda_1, \ldots, \Lambda_n\), we have that

\[
F = R^{(\mathbb{N})} \simeq R^{(\Lambda_1)} \oplus \cdots \oplus R^{(\Lambda_n)} \simeq F^n.
\]

Hence \(A = \text{End}_k(F) \simeq \text{End}_k(F^n) \simeq M_n(A)\). Applying the double skew group ring construction again we conclude that

\[
B = (A \ast_\alpha G) \# k[G] \simeq M_n(\chi A) \times ((1 - \chi) A)^n \simeq A \times A \simeq A.
\]

5. Infinite partial group action

Following Quinn [3] we define \(\Phi\) in case of \(G\) being infinite as a map from \(A \ast_\alpha G\) to the ring of row and column finite matrices. Let \(M_G(A)\) be the subring of \(\text{End}_k(A[\alpha])\) consisting of row and column finite matrices \((a_{g,h})_{g,h \in G}\) indexed by elements of \(G\) with entries in \(A\), i.e. for any \(g \in G\) the sets \(\{a_{gh} | h \in G\}\) and \(\{a_{hg} | h \in G\}\) are finite. Let \(E_{g,h}\) be, as above, those matrices that are 1 in the \((g,h)\)th component and zero elsewhere. Note that \(E_{g,h}E_{r,s} = \delta_{h,r}E_{g,s}\). Then define \(\Phi : A \ast_\alpha G \rightarrow M_G(A)\) by

\[
a \omega g \rightarrow \sum_{h \in G} h^{-1} \cdot (g^{-1} \cdot a) E_{gh,h}
\]

for any \(a \omega g \in A \ast_\alpha G\). Note that the (infinite) sum on the right side makes sense in \(M_G(A)\). As above one checks that \(\Phi\) is an algebra homomorphism.

**Proposition 5.1.** Let \(G\) be any group acting partially on \(A\). Then \(A \ast_\alpha G\) is isomorphic to a subalgebra of \(eM_G(A)e\) where \(M_G(A)\) denotes the ring of row and column finite matrices indexed by elements of \(G\) and with entries in \(A\). The element \(e\) is the idempotent \(\sum_{g \in G} 1_{g^{-1}} E_{gg}\).

**Proof.** For all \(a \omega g, b \omega h \in A \ast_\alpha G\) we have using equation (II) in the 4th line:

\[
\Phi(a \omega g)\Phi(b \omega h) = \left( \sum_{k \in G} k^{-1} \cdot (g^{-1} \cdot a) E_{gk,k} \right) \left( \sum_{l \in G} l^{-1} \cdot (h^{-1} \cdot b) E_{hl,l} \right)
\]

\[
= \sum_{k,l \in G} (k^{-1} \cdot (g^{-1} \cdot a)) (l^{-1} \cdot (h^{-1} \cdot b)) E_{gk,k} E_{hl,l}
\]

\[
= \sum_{l \in G} ((hl)^{-1} \cdot (g^{-1} \cdot a)) (l^{-1} \cdot (h^{-1} \cdot b)) E_{ghl,l}
\]

\[
= \sum_{l \in G} l^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot b))) E_{ghl,l}
\]

\[
= \Phi(a(g \cdot b) \omega gh)
\]

\[
= \Phi((a \omega g)(b \omega h))
\]
Hence $\Phi$ is an algebra homomorphism. Since 
\[ \Phi(aog) = 0 \Leftrightarrow (\forall h \in G) : h^{-1} \cdot (g^{-1} \cdot a) = 0 \Rightarrow g \cdot (g^{-1} \cdot a) = a1_g = 0 \Rightarrow a = 0, \]
we have that $\Phi$ is injective. Moreover $\Phi(aog) \in eM_G(A)e$ as above.

6. Partial Hopf action

In [2] Caenepeel and Janssen defined the notion of a partial Hopf action as follows: Let $H$ be a Hopf algebra, with comultiplication $\Delta$, counit $\epsilon$ and antipode $S$, and let $A$ be a $k$-algebra such that there exists a $k$-linear map
\[ \cdot : H \otimes A \rightarrow A \]
sending $h \otimes a \mapsto h \cdot a$. The action $\cdot$ is called a partial Hopf action if for all $h,g \in H$ and $a,b \in A$:

1. $h \cdot (ab) = \sum(h)(h_1 \cdot a)(h_2 \cdot b)$;
2. $1_H \cdot a = a$;
3. $h \cdot (g \cdot a) = \sum(h)(h_1 \cdot 1)((h_2g) \cdot a)$;

Let $H$ be a Hopf algebra which is finitely generated and projective as $k$-module with dual basis $\{(b_i,p_i) \in H \times H^* | 1 \leq i \leq n\}$. Then there exist structure constants $c_{k,l}^i$ and $m_{k,l}^i$ in $k$ such that $\Delta(b_i) = \sum_{k,l=1}^n c_{k,l}^i b_k \otimes b_l$ and $b_kb_l = \sum_{i=1}^n m_{k,l}^i b_i$ for all $1 \leq i, k, l \leq n$. It is well-known that $H^*$ becomes a Hopf algebra with comultiplication and multiplication defined on the generators $\{p_i | 1 \leq i \leq n\}$ as follows: $\Delta_{H^*}(p_i) = \sum_{k,l=1}^n m_{k,l}^i p_k \otimes p_l$ and $p_k * p_l = \sum_{i=1}^n c_{k,l}^i p_i$. The counit of $H^*$ is given by $\epsilon_{H^*}(f) = f(1)$.

Recall that $H^*$ acts on $H$ from the left by $f \rightarrow h = \sum_{(h)} h_1 f(h_2)$, such that the smash product $H \#, H^*$ can be considered whose multiplication is given by
\[ (h \#: f)(k \#: g) = \sum_{(f)} h(f_1 \rightarrow k) \#: f_2 * g \]
for all $h, k \in H$ and $f, g \in H^*$. The smash product yields a left module action on $H$, i.e. an algebra homomorphism
\[ \lambda : H \# H^* \rightarrow \text{End}_k(H) \quad h \# f \mapsto [k \mapsto h(f \rightarrow k)]. \]

The smash product $H \# H^*$ is sometimes called the Heisenberg double of $H$ and in case $H$ is free of finite rank isomorphic to $\text{End}_k(H)$ (see [7 9.4.3]).

Analogously we have a right action of $H^*$ on $H$ by $h \leftarrow f = \sum_{(h)} h_2 f(h_1)$ for all $f \in H^*$ and $h \in H$, turning $H$ into a right $H^*$-module algebra. The smash product $H^* \# H$ yields a right module action on $H$, i.e. an algebra anti-homomorphism
\[ \rho : H^* \# H \rightarrow \text{End}_k(H) \quad f \# h \mapsto [k \mapsto (k \leftarrow f)h] \]
As in [7 9.4.10] one shows that for all $h,k \in H$ and $f,g \in H^*$:
\[ \lambda(h \# f)\rho(g \# 1) = \sum_{(g)} \rho(g_2 \# 1)\lambda((h \leftarrow S(g_1)) \# f) \]
Now assume that $H$ acts partially on $A$, then the map $\Delta_A : A \rightarrow A \otimes H^*$ with

$$\Delta(a)_A = \sum_{i=1}^{n} (b_i \cdot a) \otimes p_i$$

for all $a \in A$ defines a partially coaction. The map $\Delta_A$ satisfies:

$$\Delta_A(ab) = \Delta_A(a)\Delta_A(b)$$

$$(1 \otimes \epsilon_H^*)\Delta_A(a) = id_A(a)$$

$$(\Delta_A \otimes 1)\Delta_A(a) = (\Delta_A(1) \otimes 1)(1 \otimes \Delta_H^*)\Delta_A(a)$$

The last equation shows that in general this coaction does not make $A$ into a right $H$-comodule. It can be deduced using the structure constants and property (3) from above

$$\sum_{i,j=1}^{n} b_j \cdot (b_i \cdot a) \otimes p_j \otimes p_i = \sum_{i,j,k,l,r=1}^{n} c_{i,j}^k m_{l,i}^r (b_k \cdot 1)(b_r \cdot a) \otimes p_j \otimes p_i$$

$$= \sum_{i,k,l,r=1}^{n} m_{l,i}^r (b_k \cdot 1)(b_r \cdot a) \otimes p_k p_l \otimes p_i$$

$$= \sum_{k,r=1}^{n} (b_k \cdot 1)(b_r \cdot a) \otimes (p_k \otimes 1)\Delta_H^*(p_r)$$

$$= \left( \sum_{k=1}^{n} (b_k \cdot 1) \otimes p_k \right) \left( \sum_{r=1}^{n} (b_r \cdot a) \otimes \Delta(p_r) \right)$$

With the above notation we define a homomorphism $\phi : A \rightarrow A \otimes \text{End}_k(H)$ by

$$\phi(a) = \sum_{i=1}^{n} (b_i \cdot a) \otimes \rho(S^{-1}(p_i)\#1).$$
Then $\phi$ is an algebra homomorphism, because

$$\phi(ab) = \sum_{i=1}^{n} (b_i \cdot (ab) \otimes \rho(S^{-1}(p_i)\#1)$$

$$= \sum_{i=1}^{n} ((b_i)_1 \cdot a) \otimes ((b_i)_2 \cdot b) \otimes \rho(S^{-1}(p_i)\#1)$$

$$= \sum_{k,l} (b_k \cdot a) \otimes (b_l \cdot b) \otimes \rho(S^{-1}(p_k)S^{-1}(p_l)\#1)$$

$$= \sum_{k,l} (b_k \cdot a) \otimes (b_l \cdot b) \otimes \rho(S^{-1}(p_k)\#1)\rho(S^{-1}(p_l)\#1)$$

$$= \phi(a)\phi(b).$$

where we use in the line before the last the fact that $\rho$ is an anti-homomorphism.

The partial smash product of $A$ and $H$ is defined as a certain submodule of $A \otimes H$. On $A \otimes H$ we define a new (associative) multiplication by

$$(a \otimes h)(b \otimes g) := \sum_{(h)} a(h_1 \cdot b) \otimes h_2 g.$$  

for all $a, b \in A$, $h, g \in H$. Note that $A \otimes H$ is naturally an $A$-bimodule given by

$$x(a \otimes h)y = (x \otimes 1)(a \otimes h)(y \otimes 1) = \sum_{(h)} xa(h_1 \cdot y) \otimes h_2$$

The partial smash product is defined to be $A\#H = (A \otimes H)1_A$ and is spanned by the elements of the form $\sum_{(h)} a(h_1 \cdot 1_A) \otimes h_2$ for all $a \in A, h \in H$. The partial smash product becomes naturally a right $H$-comodule algebra by

$$\rho = 1 \otimes \Delta : A \otimes H \to A \otimes H \otimes H, \ a \otimes h \mapsto \sum_{(h)} a \otimes h_1 \otimes h_2$$

and for all $(a \otimes h)1_A \in A\#H$ we have

$$\rho((a \otimes h)1_A) = \sum_{(h)} a(h_1 \cdot 1_A) \otimes h_2 \otimes h_3,$$

making $A\#H$ into a right $H$-comodule algebra. Moreover $A\#H$ becomes a left $H^*$-module algebra, where the action is defined by

$$f \triangleright ((a\#h)1_A) = \sum_{(h)} (a(h_1 \cdot 1_A)\#(f \to h_2) = (a\#(f \to h))1_A,$$
for all \( f \in H^*, h \in H, a \in A \). The classical Blattner-Montgomery duality \([1]\) says that the double smash product \( A \# H \# H^* \) is isomorphic to \( M_n(A) \) where \( n \) is the rank of \( H \) over \( k \).

**Lemma 6.1.** Let \( \psi : H \# H^* \to A \otimes \text{End}_k(H) \) be the map defined by \( h \# f \mapsto \lambda(h \# f) \) for all \( h \in H, f \in H^* \). Then for all \( a \in A, h \in H, f \in H^* \) we have

\[
\phi(1) \psi(h \# f) \phi(a) = \sum_{(h)} \phi(h_1 \cdot a) \psi(h_2 \# f).
\]

**Proof.** Let \( a \in A, h \in H, f \in H^* \).

\[
\sum_{(h)} \phi(h_1 \cdot a) \psi(h_2 \# f) = \sum_{(h),i} p_i(h_1) \phi(b_i \cdot a) \psi(h_2 \# f)
\]

\[
= \sum_{i,j} b_j \cdot (b_i \cdot a) \otimes \rho(S^{-1}(p_j) \# 1) \lambda(h \leftarrow p_i \# f)
\]

\[
= \sum_{k,r} (b_k \cdot 1)(b_r \cdot a) \otimes \rho(S^{-1}(p_k(p_r)_1) \# 1) \lambda(h \leftarrow (p_r)_2 \# f)
\]

\[
= \sum_{k,r} (b_k \cdot 1)(b_r \cdot a) \otimes \rho(S^{-1}(p_k) \# 1) \rho(S^{-1}(p_r)_1) \lambda(h \leftarrow (p_r)_2 \# f)
\]

\[
= \phi(1) \sum_r (b_r \cdot a) \otimes \rho(S^{-1}(p_r)_2 \# 1) \lambda(h \leftarrow (S^{-1}(p_r)_1) \# f)
\]

\[
= \phi(1) \sum_r (b_r \cdot a) \otimes \lambda(h \# f) \rho(S^{-1}(p_r) \# 1)
\]

\[
= \phi(1) \psi(h \# f) \phi(a)
\]

where we use equation (2) in the third line from below. \( \square \)

**Theorem 6.2.** Suppose that \( H \) is a Hopf algebra, finitely generated projective over \( k \), which partially actions on \( A \). Then \( \Phi : A \otimes H \# H^* \to A \otimes \text{End}_k(H) \) with

\[
a \otimes h \# f \mapsto \phi(a) \psi(h \# f)
\]

is an algebra homomorphism. The image of the restriction to \( A \# H \# H^* \) lies inside \( e(A \otimes \text{End}_k(H)) e \) where \( e \) is the idempotent

\[
e = \sum_{i=1}^{n} (b_i \cdot 1) \otimes \rho(S^{-1}(p_i) \otimes 1).
\]
Proof. For any $a, b \in \mathcal{A}$, $h, k \in H$ and $f, g \in H^*$ we have
\[
\Phi(a \otimes h \# f)\Phi(b \otimes k \# g)) = \phi(a)\psi(h \# f)\phi(b)\psi(k \# g) \\
= \phi(a)\phi(1)\psi(h \# f)\phi(b)\psi(k \# g) \\
= \sum_{(h)} \phi(a)\phi(h_1 \cdot b)\psi(h_2 \# f)\psi(k \# g) \\
= \sum_{(h, f)} \phi(a(h_1 \cdot b))\psi(h_2(f_1 \to k) \# f_2 \star g) \\
= \Phi \left( \sum_{(h, f)} a(h_1 \cdot b) \otimes h_2(f_1 \to k) \# f_2 \star g \right) \\
= \Phi \left( (a \otimes h \# f)(b \otimes k \# g) \right).
\]
Hence $\Phi$ is an algebra homomorphism. Since the image of the identity $1 = 1_{\mathcal{A}} \# 1_H \# 1_{H^*}$ of $\mathcal{A} \# H \# H^*$ under the map $\Phi$ is $e$, $e$ is an idempotent. Moreover
\[
\Phi(\gamma) = \Phi(1\gamma 1) \in e(\mathcal{A} \otimes \text{End}_k(H))e,
\]
for all $\gamma \in \mathcal{A} \# H \# H^*$.

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