Boundary value problem for a higher order equation with changing time direction

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Abstract. In this paper, using the stationary Galerkin method, we establish the unique regular solvability of boundary value problem for a higher order equation with changing time direction. We estimate the rate of convergence of the stationary Galerkin method, it expresses in terms of eigenvalues of the spectral problem for a quasi-elliptic equation.

1. Introduction

Boundary value problems for a higher order nonclassical equations of mathematical physics with changing time direction were studied, for example, in [1–11]. In those works a functional method, the Galerkin method and the regularization method were applied. In this paper we consider the application of the stationary Galerkin method to the boundary value problem for a higher order equation with changing time direction. The statement of this boundary value problem is different from the first boundary value problem [1].

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, 

\[ Q = \Omega \times (0, T), \quad \Gamma = \gamma \times (0, T), \quad T = \text{const} > 0; \quad \Omega_t = \Omega \times \{t\}, \quad 0 \leq t \leq T. \]

In the cylindrical domain $Q$ we consider the equation

\[ Lu = \sum_{i=1}^{2s+1} k_i(x, t)D_i^t u - \Delta u + c(x)u = f(x, t). \] (1)

Assume that the coefficients of equation (1) are infinitely differentiable in $\overline{Q}$. We introduce the sets

\[ S_0^\pm = \{(x, 0) : x \in \Omega, (-1)^s k_{2s+1}(x, 0) \geq 0\}, \quad S_T^\pm = \{(x, T) : x \in \Omega, (-1)^s k_{2s+1}(x, T) \geq 0\}. \]

**Problem 1** Find a solution of equation (1) in $Q$ such that

\[ u|_\Gamma = 0; \] (2)

\[ D_i^t u|_{t=0} = 0, \quad i = 0, s - 1; \quad D_i^s u|_{S_0^+} = 0; \quad D_i^s u|_{S_T^-} = 0; \quad D_i^t u|_{t=T} = 0, \quad i = s + 1, 2s. \] (3)
The sign of the coefficient \( k_{2s+1}(x,t) \) is an arbitrary in \( Q \). Hence the class of equations (1) includes the elliptic-parabolic equations, equations with changing time direction and other equations [1]. In [4], using the nonstationary Galerkin method and regularization method, existence and uniqueness of a regular solution of the Problem 1 were established. In this paper, using the stationary Galerkin method, we establish the unique regular solvability of the Problem 1. We estimate the rate of convergence of the approximate solutions of this problem. Note that the Problem 1 for \( k_{2s+1} = \pm(-1)^s \) was proposed in [12].

2. Derivation of the a priori estimates

We denote by \( W^{m,s}_2(Q) \) the anisotropic Sobolev space equipped with the norm
\[
\|u\|_{m,s}^2 = \int_0^T \left( \|u\|_m^2 + \|D_t^su\|_0^2 \right) dt,
\]
where \( \|\cdot\|_m \) is the norm of the Sobolev space \( W^m_2(\Omega) \) [13]; \( \|\cdot\|_0 = \|\cdot\| \) is the norm in \( L^2(Q) \).

Let \( C_L \) be the class of functions in \( W^{2,2s+1}_2(Q) \) that satisfy boundary conditions (2), (3). In the paper [4] the following assertion was proved.

**Lemma 1** [4] Assume that the coefficient \( c(x) > 0 \) is sufficiently large and the conditions
\[
(-1)^s[2k_{2s} - (2s + 1)k_{2s+1,t}] \geq \delta > 0; \quad (-1)^sk_{2s+1}(x,T) > 0.
\]
are satisfied. Then the following estimate holds for every function \( u(x,t) \in C_L \):
\[
(Lu, u) \geq C_1\|u\|^2_{3,s}, \quad C_1 = \text{const} > 0,
\]
where \((.,.)\) is inner product in \( L^2(Q) \).

**Corollary 1** Let the conditions of Lemma 1 are satisfied. Then the theorem of the uniqueness of a regular solution for the Problem 1 holds.

We consider the case
\[
(-1)^sk_{2s+1}(x,0) > 0, \quad (-1)^sk_{2s+1}(x,T) > 0.
\]
In this case the boundary conditions (3) has the form:
\[
D^i_tu|_{t=0} = 0, \quad i = 0, s; \quad D^i_tu|_{t=T} = 0, \quad i = s + 1, 2s.
\]

Consider the spectral problem:
\[
Z\varphi \equiv (-1)^{s+1}D^{2s+2}_t\varphi - \Delta\varphi = \lambda\varphi, \quad (4)
\]
\[
\varphi|_t = 0; \quad D_t^i\varphi|_{t=0} = 0, \quad i = 0, s; \quad D_t^i\varphi|_{t=T} = 0, \quad i = s + 1, 2s + 1 \quad (5)
\]

Let \( \hat{W}^{2,2s+2}_2(Q) \) be a closure of the class of functions in \( C^\infty(\overline{Q}) \), that satisfied the boundary conditions (5), in the norm of the space \( W^{2,2s+2}_2(Q) \). Denote by \( \hat{W}^{1,s+1}_2(Q) \) the closure of \( \hat{W}^{2,2s+2}_2(Q) \) in the norm of the space \( W^{1,s+1}_2(Q) \). We obtain the equality
\[
(Z\varphi, \psi) = (\varphi, Z\psi), \quad \forall \varphi, \psi \in \hat{W}^{2,2s+2}_2(Q),
\]
and the estimate

\[(Z \varphi, \varphi) = \int_Q [(D_t^{s+1} \varphi)^2 + \sum_{i=1}^{n} \varphi_i^2] dQ \geq C_2 \| \varphi \|^2_{1,s+1}, \quad C_2 = \text{const} > 0.\]

The spectral problem (4), (5) has the eigenvalues \( \lambda_k \) such that \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) and \( \lambda_k \to +\infty \) as \( k \to \infty \). And their corresponding eigenfunctions \( \varphi_k(x,t) \) form an orthogonal basis in \( \dot{W}^{1,s+1}_2(Q) \) and an orthonormal basis in \( L_2(Q) \). Then for any function \( u(x,t) \in \dot{W}^{1,s+1}_2(Q) \) there is a Fourier series expansion

\[u(x,t) = \sum_{k=1}^{\infty} c_k \varphi_k(x,t), \quad c_k = (u, \varphi_k),\]

(6)

The series (6) converges in \( \dot{W}^{1,s+1}_2(Q) \). In other side, by the second basic inequality for the Laplas operator [12] the following inequality holds

\[\| u \|_{2,2s+2} \leq C_3 \| Zu \|, \quad C_3 > 0, \quad \forall u \in \dot{W}^{2,2s+2}_2(Q).\]

By the last inequality, the eigenfunctions of the spectral problem (6) \( \varphi_k(x,t) \in \dot{W}^{2,2s+2}_2(Q) \).

**Lemma 2** For \( \forall u(x,t) \in \dot{W}^{2,2s+2}_2(Q) \) the series (6) converges in \( \dot{W}^{2,2s+2}_2(Q) \).

**Proof.** For \( u(x,t) \in \dot{W}^{2,2s+2}_2(Q) \) the norm \( \| u \|_{2,2s+2} \) is equivalent to new norm \( |u| = \| Zu + u \| \) with the inner product \( [u,v] = (Zu+u, Zv+v), \quad u, v \in \dot{W}^{2,2s+2}_2(Q), \) also \( [\varphi_k, \varphi_l] = (\lambda_k + 1)^2 \delta_{k,l} \).

Then by the Parseval’s equality, we obtain

\[\| \sum_{k=1}^{\infty} (u, \varphi_k) \varphi_k \| = \sum_{k=1}^{\infty} (u, \varphi_k)^2 (\lambda_k + 1)^2.\]

(7)

The equalities hold

\[ (u, \varphi_k) = \frac{1}{\lambda_k} (u, Z\varphi_k) = \frac{1}{\lambda_k} (Zu, \varphi_k).\]

Estimate the series (7) by the following series

\[(1 + \frac{1}{\lambda_1})^2 \sum_{k=1}^{\infty} (Zu, \varphi_k)^2 = (1 + \frac{1}{\lambda_1})^2 \| Zu \|^2 < \infty,
\]

it yields the convergence of (7). Hence it follows that (6) converges in \( \dot{W}^{2,2s+2}_2(Q) \). The Lemma 2 is proved.

Construct an approximate solution \( u^N(x,t) \) of the boundary value problem (1)-(3) of the form

\[u^N(x,t) = \sum_{k=1}^{N} c_k^N \varphi_k(x,t),\]

where \( c_k^N \) are defined as the solution of the following system of the linear algebraic equations:

\[(Lu^N, \varphi_l) = (f, \varphi_l), \quad l = 1, N.\]

(8)

The unique solvability of the system (8) follows from the Lemma 1.
Lemma 3 Assume that the coefficient \( c(x) > 0 \) is sufficiently large and conditions
\[
(-1)^s[2k_{2s} - (2s + 1)k_{2s+1,t}] \geq \delta > 0, \quad (-1)^s k_{2s+1}(x, T) > 0, \quad f(x, t) \in L_2(Q),
\]
are satisfied. Then the following estimate holds
\[
\|u^N\|_{1,s} \leq C_3 \|f\|, \quad C_3 = \text{const} > 0.
\]  
(9)

Proof. Multiplying each equation in (8) by \( c_i^N \) and taking the sum with respect to \( l \) from 1 to \( N \), we get
\[
(Lu^N, u^N) = (f, u^N).
\] 
(10)
Based on Lemma 1 and Holder’s inequality, from (10) we find:
\[
C_1 \|u^N\|_{1,s}^2 \leq \|f\| \cdot \|u^N\|. 
\]
From the last inequality follows the estimate (9).

Lemma 4 Let assumptions of Lemma 3 and
\[
(-1)^s[2k_{2s} + k_{2s+1,t}] \geq \delta > 0, \quad (-1)^s k_{2s+1}(x, 0) > 0, \quad f(x, t) \in W^{0,1}_2(Q).
\]
hold. Let \( f(x, 0) = 0 \) almost everywhere in \( \Omega \). Then the following estimate holds:
\[
\|u^N\|_{2s+1} \leq C_4 \|f\|_{0,1}, \quad C_4 = \text{const} > 0.
\]  
(11)

Proof. Multiplying (8) by \( \lambda_i c_i^N \) and taking the sum with respect to \( l \) from 1 to \( N \), we get
\[
(Lu^N, (-1)^{s+1} D_t^{2s+2} u^N - \Delta u^N) = (f, (-1)^{s+1} D_t^{2s+2} u^N - \Delta u^N).
\]  
(12)
Integrating by parts and taking into account (5), we obtain \( (v \equiv u^N) \):
\[
(Lv, (-1)^{s+1} D_t^{2s+2} v - \Delta v) = \int_Q \left\{ \frac{1}{2} (-1)^s [2k_{2s} + k_{2s+1,t}] (D_t^{2s+1} v)^2 + \right. \\
\frac{1}{2} (-1)^s [2k_{2s} - (2s + 1)k_{2s+1,t}] \sum_{i=1}^n (D_t^i v_x)^2 + \sum_{i=1}^n (D_t^{s+1} v_x)^2 + (\Delta v)^2 + c(x) (D_t^{s+1} v)^2 + \sum_{i=1}^n v_x^2 |] dQ + \int_Q D_t^{2s+1} v \sum_{j=1}^{2s} a_j(x, t) D_t^j v dQ + \right. \\
\left. \sum_{j=1}^{2s+1} \int_Q D_t^j v \sum_{i=1}^n k_{j,xi} v_x dQ + \int_Q v \sum_{i=1}^n c_{xi} v_x dQ + \\
\int_Q \sum_{j=0}^{s-1} \beta_j \sum_{i=1}^n D_t^i v_x D_t^j v_x dQ + \frac{(-1)^s}{2} \int_{\Omega_T} k_{2s+1} \sum_{i=1}^n (D_t^i v_x)^2 dx + \\
\int_{\Omega_T} \sum_{j=0}^{s-1} \gamma_j \sum_{i=1}^n D_t^i v_x D_t^j v_x dx + \frac{(-1)^s}{2} \int_{\Omega_0} k_{2s+1} (D_t^{2s+1} v)^2 dx + \right.
\]
Theorem 2 Let assumptions of Theorem 1 be satisfied and \( s \geq 1 \). Then for the approximate solutions \( u^N(x, t) \) following estimate holds:

\[
\|u - u^N\|_{1, s} \leq C_5\|f\|_{0, 1}\lambda_{N+1}^{-1/4},
\]

where \( u(x, t) \) is an exact solution to the boundary value problem (1)-(3), and \( C_5 = \text{const} > 0 \) does not depend on \( N \).

Proof. For the regular solution \( u(x, t) \) of boundary value problem (1)-(3) equalities (16) are satisfied. The Fourier series (6) is true for \( u(x, t) \). Let \( H_N \) be the linear span of \( \varphi_1, \ldots, \varphi_N \), and let \( P_N \) be the projection operator from \( L_2(Q) \) onto \( H_N \).
From (8) and (16) we have

\[(Lu^N, \eta) = (f, \eta), \quad (Lu, \eta) = (f, \eta), \quad \forall \eta \in H_N.\]

Hence with \(\eta = \omega - u^N, \quad \omega \in H_N,\) we find

\[(L(u - u^N), \omega - u^N) = 0\]
or

\[(L(u - u^N), u - u^N) = (L(u - u^N), u - \omega).\]

By the estimate of Lemma 1 we obtain

\[C_1 \|u - u^N\|^2_{1,s} \leq \|f - Lu^N\| : \|u - \omega\|.\]

Hence

\[\|u - u^N\|^2_{1,s} \leq C_6 \|f\|_{0,1} \cdot \|u - \omega\|, \quad C_6 = \text{const} > 0, \quad (16)\]

for \(\forall \omega \in H_N.\)

The following equality holds:

\[(-1)^{s+1} D^2_{s+2} u^N - \Delta u^N = \sum_{k=1}^{N} c_k \lambda_k \varphi_k.\]

From this follows an estimate

\[\sum_{k=1}^{N} c_k^2 \lambda_k = \int_Q \left(\left(D^+_{s+1} u^N\right)^2 + \sum_{i=1}^{n} (u_{x_i}^N)^2\right) dQ \leq C_7 \|f\|^2_{0,1}, \quad C_7 = \text{const} > 0. \quad (17)\]

Passing to the limit in (18) as \(N \to \infty,\) we obtain

\[\sum_{k=1}^{\infty} c_k^2 \lambda_k = \int_Q \left(\left(D^+_{s+1} u\right)^2 + \sum_{i=1}^{n} (u_{x_i}^2)^2\right) dQ \leq C_7 \|f\|^2_{0,1}. \quad (18)\]

By (19) the following relation holds:

\[\|u - P_N u\|^2 = \|\sum_{k=N+1}^\infty c_k \varphi_k\|^2 = \sum_{k=N+1}^\infty c_k^2 \lambda_k \leq \sum_{k=N+1}^\infty \lambda_k^{-1} \sum_{k=N+1}^\infty c_k^2 \lambda_k \leq C_7 \|f\|_{0,1}^2 \lambda^{-1}_{N+1}. \quad (19)\]

Setting \(\omega = P_N u\) in the inequality (17) and using the inequality (20), we estimate the rate of convergence of the stationary Galerkin method. Theorem is proved.

**Remark 1** In the case \(s = 0\) the boundary value problem (1)-(3) is a first boundary value problem, and in this case Problem 1 was studied in [14–18].

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