RATIONAL FORMALITY OF MAPPING SPACES

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Abstract
Let $X$ and $Y$ be finite nilpotent CW complexes with dimension of $X$ less than the connectivity of $Y$. Generalizing results of Vigué-Poirrier and Yamaguchi, we prove that the mapping space $\text{Map}(X,Y)$ is rationally formal if and only if $Y$ has the rational homotopy type of a finite product of odd dimensional spheres.

1. Introduction
Let $X$ and $Y$ be connected spaces that have the rational homotopy type of finite CW complexes. We denote by $n$ the maximum integer $q$ such that $H^q(X;\mathbb{Q}) \neq 0$. In this text we consider mapping spaces $\text{Map}(X,Y)$ satisfying the following hypotheses (H).

\[ H \begin{cases} 
(i) \ X \text{ and } Y \text{ are not rationally contractible,} \\
(ii) \ There \ exists \ n \geq 1 \ such \ that \ H^n(X;\mathbb{Q}) \neq 0, \ H^q(X;\mathbb{Q}) = 0 \text{ if } q > n, \text{ and } Y \text{ is } n\text{-connected} 
\end{cases} \]

Under those hypotheses, $\text{Map}(X,Y)$ is a nilpotent space and its rational homotopy is described by Haefliger [6] and Brown and Szczarba [1].

Our main interest here is to understand when $\text{Map}(X,Y)$ is a (rationally) formal space. Formality is important in rational homotopy. If a space is formal then its rational homotopy type is completely determined by its rational cohomology. More precisely a nilpotent space $Z$ is formal if its Sullivan minimal model is quasi-isomorphic to the differential graded algebra $(\mathcal{H}^*(Z;\mathbb{Q}),0)$. Many spaces coming from geometry are formal. Among formal spaces we find the spheres, the projective spaces, the products of Eilenberg-MacLane spaces, the compact Kähler manifolds ([2]), and the $(p-1)$-connected compact manifolds, $p \geq 2$, of dimension $\leq 4p-2$ [8].

The formality of mapping spaces has been the subject of previous works. In [3], N. Dupont and M. Vigué-Poirrier prove that when $\mathcal{H}^*(Y;\mathbb{Q})$ is finitely generated, then $\text{Map}(S^1,Y)$ is formal if and only if $Y$ is rationally a product of Eilenberg-MacLane spaces. In [14] T. Yamaguchi proves that when $Y$ is elliptic, the formality of $\text{Map}(X,Y)$ implies that $Y$ is rationally a product of odd dimensional spheres. In [13] M. Vigué-Poirrier proves that if $\text{Map}(X,Y)$ is formal and if the Hurewicz map $\pi_q(X) \otimes \mathbb{Q} \to H_q(X;\mathbb{Q})$ is nonzero in some odd degree $q$, then $Y$ has the homotopy type of a product of Eilenberg-MacLane spaces. When $Y$ is a finite complex, we prove here that the hypothesis on the Hurewicz map is not necessary.

Theorem 1. Under the above hypotheses (H), $\text{Map}(X,Y)$ is formal if and only if $Y$ has the rational homotopy type of a product of odd dimensional spheres.

As an important step in the proof of Theorem 1 we prove

Theorem 2. If $\dim Y = N$, then the Hurewicz map

$$\pi_q(\text{Map}(X,Y)) \otimes \mathbb{Q} \to H_q(\text{Map}(X,Y);\mathbb{Q})$$

is zero for $q > N$. 

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2. Rational homotopy

The theory of minimal models originates in the works of Sullivan [10] and Quillen [9]. For recall a graded algebra $A$ is graded commutative if $ab = (-1)^{|a||b|}ba$ for homogeneous elements $a$ and $b$. A graded commutative algebra $A$ is free on a graded vector space $V$, $A = \bigwedge V$, if $A$ is the quotient of the tensor algebra $TV$ by the ideal generated by the elements $xy - (-1)^{|x||y|}yx, x, y \in V$. A (Sullivan) minimal algebra is a graded commutative differential algebra of the form $(\bigwedge V, d)$ where $V$ admits a basis $v_i$ indexed by a well ordered set $I$ with $d(v_i) \in \bigwedge (v_j, j < i)$. Now if $(A, d)$ is a graded commutative differential algebra whose cohomology is connected and finite type, there is a unique (up to isomorphism) minimal algebra $(\bigwedge V, d)$ with a quasi-isomorphism $\varphi : (\bigwedge V, d) \to (A, d)$. The differential graded algebra $(\bigwedge V, d)$ is then called the (Sullivan) minimal model of $(A, d)$.

In [10] Sullivan associated to each nilpotent space $Z$ a graded commutative differential algebra of rational polynomials forms on $Z$, $AP_L(Z)$, that is a rational replacement of the algebra of de Rham forms on a manifold. The minimal model $(\bigwedge V, d)$ of $AP_L(Z)$ is then called the minimal model of $Z$. More generally a model of $Z$ is a graded commutative differential algebra quasi-isomorphic to its minimal model. For more details we refer to [10], [4] and [5].

A space $X$ is called (rationally) formal if its minimal model, $(\bigwedge V, d)$, is quasi-isomorphic to its cohomology with differential $0$, $\psi : (\bigwedge V, d) \to (H^*(X; \mathbb{Q}), 0)$.

A formal space $X$ admits a minimal model equipped with a bigradation on $V, V = \bigoplus_{p \geq 0, q \geq 1} V_{pq}$ such that $d(V_p^q) \subset (\bigwedge V)^{q+1}_{p-1}$, and such that the bigradation induced on the homology satisfies $H^q_{q+1} = 0$ for $p \neq 0$. This model has been constructed by Halperin and Stasheff in [7], and is called the bigraded model of $X$. We will use this model for the proof of Theorem 2.

A nilpotent space $X$ is called (rationally) elliptic if $\pi_*(X) \otimes \mathbb{Q}$ and $H^*(X; \mathbb{Q})$ are finite dimensional vector spaces. To be elliptic for a space $X$ is a very restrictive condition. For instance $H^*(X; \mathbb{Q})$ satisfies Poincaré duality and $\pi_*(X) \otimes \mathbb{Q}$ is zero for $q \geq 2 \cdot \dim X$. A nilpotent space $X$ is called (rationally) hyperbolic if $\pi_*(X) \otimes \mathbb{Q}$ is infinite dimensional and $H^*(X; \mathbb{Q})$ finite dimensional. The homotopy groups of elliptic and hyperbolic spaces have a completely different behavior. For instance, for an hyperbolic space $X$, the sequence $\sum_{i \leq q} \dim \pi_i(X) \otimes \mathbb{Q}$ has an exponential growth (4).

In [6], Haefliger gives a process to construct a minimal model for $\text{Map}(X, Y)$. With the hypotheses (H) of the Introduction, suppose that $(\bigwedge W, d)$ is the Sullivan minimal model of $X$. Denote by $S \subset (\bigwedge W)^n$ a supplement of the subvector space generated by the cocycles. Then $I = (\bigwedge W)^{>n} \oplus S$ is an acyclic differential graded ideal, and the quotient $(A, d) = (\bigwedge W/I, d)$ is a finite dimensional model for $X$. We denote by $(B, d)$ the dual coalgebra. Let $(a_i), i = 0, \ldots, p$ be a graded basis for $A$ with $a_0 = 1$ and denote by $\overline{a_i}$ the dual basis for $B$.

Denote also by $(\bigwedge V, d)$ the minimal model of $Y$. We define a morphism of graded algebras $\varphi : \bigwedge V \to A \otimes (\bigwedge B \otimes V)$ by putting $\varphi(v) = \sum_i a_i \otimes (\overline{a_i} \otimes v)$. In [6] Haefliger proves that there is a unique differential $D$ on $\bigwedge (B \otimes V)$ making $\varphi : (\bigwedge V, d) \to (A, d) \otimes (\bigwedge (B \otimes V), D)$ a morphism of differential graded algebras. Then $(\bigwedge (B \otimes V), D)$ is a model for $\text{Map}(X, Y)$ and $\varphi$ is a model for the evaluation map $\text{Map}(X, Y) \times X \to Y$. In particular, (12), the rational homotopy groups of $\text{Map}(X, Y)$ are given by $\pi_q(\text{Map}(X, Y)) \otimes \mathbb{Q} = \oplus_i [H_i(X; \mathbb{Q}) \otimes \pi_{q+i}(Y) \otimes \mathbb{Q}]$.

This formula is natural in $X$ and $Y$.

3. Proof of Theorem 1.

In [11] Thom computes the rational homotopy type of $\text{Map}(X, K(\mathbb{Q}, r))$ when $\dim X < r$. He proves that the mapping space is a product of Eilenberg-MacLane spaces, $\text{Map}(X, K(\mathbb{Q}, r)) = \prod_i K(H_i(X; \mathbb{Q}), r-i)$.
Since odd dimensional spheres are rationally Eilenberg-MacLane spaces, it follows that if $Y$ has the rational homotopy type of a product of odd dimensional spheres, then $\text{Map}(X, Y)$ is formal.

Suppose now that $\text{Map}(X, Y)$ is formal. Since any retract of a formal space is formal, $Y$ is formal. By Theorem 2, the image of the Hurewicz map for $\text{Map}(X, Y)$ is finite dimensional. Recall that for a formal space, the cohomology is generated by classes that evaluate non trivially on the image of the Hurewicz map. Therefore the algebra $H^*(\text{Map}(X, Y); \mathbb{Q})$ is finitely generated.

The square of an even dimensional generator $x_i$ of $H^*(\text{Map}(X, Y); \mathbb{Q})$ gives a map $\text{Map}(X, Y) \to K(\mathbb{Q}, 2r_i)$, $r_i = 2|x_i|$. We denote by $\theta$ the product of those maps,

$$\theta : \text{Map}(X, Y) \to \prod_i K(\mathbb{Q}, 2r_i).$$

We do not suppose that $x_i^2 \neq 0$. In fact if $x_i^2 = 0$ for all $i$, then $\theta$ is homotopically trivial but this has no effect on our argument. The pullback along $\theta$ of the product of the principal fibrations $K(\mathbb{Q}, 2r_i - 1) \to PK(\mathbb{Q}, 2r_i) \to K(\mathbb{Q}, 2r_i)$ is a fibration

$$\prod_i K(\mathbb{Q}, 2r_i - 1) \to E \to \text{Map}(X, Y).$$

By construction the rational cohomology of $E$ is finite dimensional, and so the rational category of $E$ is also finite.

Now from the definition of the dimension of $X$, there is a cofibration $X' \to X$ such that $H_n(q; \mathbb{Q})$ is surjective. The restriction to $X'$ induces a map $\text{Map}(X, Y) \to \text{Map}(X', Y)$ whose homotopy fiber is the injection

$$j : \Omega^n Y = \text{Map}_*(S^n, Y) \to \text{Map}(X, Y).$$

From the naturality of the formula for the rational homotopy groups of a mapping space, we deduce that $\pi_*(j) \otimes \mathbb{Q}$ is injective. Denote now $E'$ the pullback of $E \to \text{Map}(X, Y)$ along $j$,

$$\prod_i K(\mathbb{Q}, 2r_i - 1) \to \prod_i K(\mathbb{Q}, 2r_i - 1)$$

$$\downarrow$$

$$E' \xrightarrow{j} E$$

$$\downarrow$$

$$\Omega^n Y \xrightarrow{j} \text{Map}(X, Y)$$

Since $\pi_*(j) \otimes \mathbb{Q}$ is injective, it follows from the mapping theorem that the rational category of $E'$ is finite. In particular the cup length of $E'$ is finite.

Now the rational cohomology of $\Omega^n Y$ is the free commutative graded algebra on the graded vector space $S_*$, with $S_n = \pi_{n+q}(Y) \otimes \mathbb{Q}$. Therefore if $Y$ is hyperbolic, $H^*(E'; \mathbb{Q})$ will contain a free commutative graded algebra on an infinite number of generators, and in particular its cup length is infinite. It follows that $Y$ is elliptic. To end the proof we only apply Yamaguchi result that asserts that when $Y$ is elliptic, and $\text{Map}(X, Y)$ is formal, then $Y$ has the rational homotopy type of a finite product of odd dimensional spheres.

4. Proof of Theorem 2

Denote by $(\wedge V, d)$ the bigraded model for $Y$ and by $(A, d)$ a connected finite dimensional model for $X$. Connected means that $A^0 = \mathbb{Q}$. Denote as above by $a_i$, an homogeneous basis of $A$, and by $\overline{a_i}$ the dual basis for $B = \text{Hom}(A, \mathbb{Q})$. We write also $B_+ = \text{Hom}(A^+, \mathbb{Q})$.

Recall now that a model for the evaluation map $X \times \text{Map}(X, Y) \to Y$ is given by the morphism

$$\varphi : (\wedge V, d) \to (A, d) \otimes (\wedge (B \otimes V), D),$$

defined by $\varphi(v) = \sum_i a_i \otimes (\overline{a_i} \otimes v)$.

We consider the differential ideal $I = \wedge V \otimes \wedge^2(B_+ \otimes V)$, and we denote by $\pi : (\wedge (B \otimes V), D) \to (\wedge (B \otimes V)/I, D)$ the quotient map. In $\wedge (B \otimes V)/I$ the equation $\pi \circ \varphi \circ d = (d \otimes 1 + 1 \otimes D) \circ \pi \circ \varphi$ gives for each $v \in V$ the equation

$$\sum_i da_i \otimes (\overline{a_i} \otimes v) + \sum_i (-1)^{a_i} a_i \otimes D(\overline{a_i} \otimes v) = 1 \otimes dv + \sum_{a_i \in A^+} a_i \otimes \theta_i(v),$$
where $\theta_i$ is the derivation of $\wedge V \otimes (\wedge (B \otimes V)$ defined by $\theta_i(v) = \pi_i \otimes v$ and $\theta_i(B \otimes V) = 0$.

To go further we specialize the basis of $A^+$. We denote by $\{y_i\}$ a basis of $d(A)$, by $\{e_j\}$ a set of cocycles such that $\{y_i, e_j\}$ is a basis of the cocycles in $A$. Finally we choose elements $x_i$ with $d(x_i) = y_i$. A basis of $A$ is then given by $1$ and the elements $x_i, y_i$ and $e_j$. Denote then by $\psi_j, \psi'_j$ and $\psi''_j$ the derivations $\theta$ associated respectively to $e_j, x_i$ and $y_i$. Then we have

$$\overline{D}(\pi_i \otimes v) = (-1)^{|e_i|} \psi_i(v), \quad \overline{D}(\eta_i \otimes v) = (-1)^{|y_i|} \psi'_i(v),$$

$$\overline{D}(\eta_i \otimes v) = (-1)^{|y_i|} \psi''_i(v) - (\pi_i \otimes v).$$

it follows that the complex $(\wedge (B \otimes V)/I, \overline{D})$ decomposes into a direct sum

$$(\wedge (B \otimes V)/I, \overline{D}) = \wedge V \oplus (\wedge q C_j) \oplus D,$$

with $C_j = (\pi_j \otimes V) \otimes \wedge V$,

and where $D$ is the ideal generated by the $\pi_i \otimes v$ and $\eta_i \otimes v$.

Consider now in $(\wedge (B \otimes V), D)$ a cocycle $\alpha$ of the form

$$\alpha = \sum_j \pi_j \otimes v_j + \sum_i \pi_i \otimes u_i + \sum_i \eta_i \otimes w_i + \omega$$

where $\omega$ is a decomposable element. Looking at the linear term of $D(\alpha)$ we obtain that $\sum_i \eta_i \otimes w_i = 0$. We can replace $\alpha$ by $\alpha + D(\sum_i (-1)^{|y_i|} \eta_i \otimes u_i)$ to cancel the linear part $\sum_i \pi_i \otimes u_i$. We can thus suppose that $\alpha$ has the form

$$\alpha = \sum_j \pi_j \otimes v_j + \omega$$

where $\omega$ is a decomposable element.

In $\wedge (B \otimes V)/I$, $\alpha$ decomposes into a sum of cocycles, $\alpha = \sum_i \alpha_i$ with $\alpha_i \in C_i$. Let fix some $i$. We write $r = |e_i|$ and $\pi = (\pi_i \otimes v)$. We denote $\psi = \pi \otimes \wedge V$. Then the component $C_i$ is isomorphic to $(\wedge V \otimes D, \overline{D})$ and $\overline{D}$ is equipped with an isomorphism of degree $-r$,

$$s : V^q \to V^{q-r}.$$ We extend $s$ in a derivation of $\wedge V \otimes \wedge V$ by $s(\psi) = 0$, and the differential $\overline{D}$ satisfies $\overline{D}(\pi) = (-1)^r s d(v)$.

Write $\alpha_i = \pi + \omega$, where $\omega \in \psi \otimes \wedge^+ V$. We show that in that case $v$ is a cocycle. If this is true for any $i$, this implies that the map

$$\rho_q : H^q((\wedge V \otimes \wedge (B \otimes V))/\wedge^{q+2} (V \otimes (B \otimes V)), D)$$

is zero in degrees $q \geq \dim Y$. Since $\rho_q$ is the dual of the Hurewicz map $h_q : \pi_q(\text{Map}(X, Y)) \otimes \mathbb{Q} \to H_q(\text{Map}(X, Y); \mathbb{Q})$, this implies the result.

We now follow the lines of the proof given for $r = 1$ by Dupont and Vigué-Poirrier in [3]. Write $\wedge V = \wedge V^{\text{even}} \otimes \wedge V^{\text{odd}}$, and denote by $(x_i)_{i \in I}$ a graded basis of $V^{\text{even}} \oplus V^{\text{odd}}$. We denote by $\partial_{x_i}$ the derivation of degree $-|x_i|$ defined by

$$\frac{\partial}{\partial x_i}(x_j) = 1 \quad \text{and} \quad \frac{\partial}{\partial x_i}(x_j) = 0, \text{ } i \neq j.$$ If $v \in V^q_p$, we denote $\ell(v) = p + q$. This is a new gradation, and for any element $P$ of $\wedge V$, we have

$$\ell(P) = \sum_i \ell(x_i) x_i \frac{\partial}{\partial x_i}(P).$$

The lower gradation on $V$ extends to $\psi$. If $v \in V^q_p$, then $s(v) \in \psi^{q-r}_p$. The differential $\overline{D}$ is compatible with this double gradation,

$$\overline{D} : (\wedge V \otimes \psi)^q \to (\wedge V \otimes \psi)^{q+1}.$$ Write $P = \overline{D} x_i, P_i = \overline{D} x_i$ and $\omega = \sum \pi_i a_i$ with $x_i \in V, a_i \in \wedge^+ V$. Then

$$0 = \overline{D} \psi + \sum \overline{D}(\pi_i a_i) = (-1)^r \left( s(P) + \sum s(P_i) a_i \right) + \sum (-1)^{|\pi_i|} \pi_i \cdot \overline{D}(a_i)$$
\[
= (-1)^r \left( \sum_i \pi_i \frac{\partial P}{\partial x_i} + \sum_{ij} \pi_i \frac{\partial P_j}{\partial x_i} a_j \right) + \sum_i (\pi_i x_i \cdot \overline{D} (a_i)).
\]

Therefore
\[
\frac{\partial P}{\partial x_i} = -(-1)^{|x_i|} \overline{D} a_i - \sum_j \frac{\partial P_j}{\partial x_i} a_j,
\]
and
\[
\ell(P)P = \sum_i \ell(x_i) x_i \frac{\partial P}{\partial x_i} = - \left( \sum_{ij} \ell(x_i) x_i \frac{\partial P_j}{\partial x_i} a_j + \sum_i (\pi_i x_i \cdot \overline{D} a_i) \right)
\]
\[
= - \left( \sum_i \ell(P_i) a_i + \sum_i (-1)^{x_i} \ell(x_i) x_i \overline{D} a_i \right) = -\overline{D} \left( \sum_i \ell(x_i) x_i a_i \right).
\]

This implies that
\[
v + \sum_i \frac{\ell(x_i)}{\ell(x)} x_i a_i
\]
is a cocycle. In particular, \(v \in V_0\) is a cocycle. This ends the proof of theorem 2.

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