Tensor rank problem in statistical high-dimensional data and quantum information theory: their comparisons on the methods and the results

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Abstract—Quantum communication is concerned with the complexity of entanglement of a state and statistical data analysis is concerned with the complexity of a model. It is a common key word for both community is tracing the same target and that the methods used are slightly different. Two different methods, the range criterion method from quantum communication and the determinant polynomial method, are shown as an examples.

Keywords: tensor rank, quantum information, quantum entangled states, SLOCC, 3-slices, range criterion

1 Introduction

Quantum communication are strongly interested in entanglement and classification of entangled states. Multipartite states are elements, that is, tensors, in some tensor product space of the Hilbert space for each site. On the other hand, in statistical data analysis, a "tensor" means a multi-array high dimensional data, which are recently used in various field. Statistical data analysis are strongly concerned with the data complexity and the model complexity. They are expressed by "rank" and "maximal rank" respectively. Clearly fixing bases in each component Hilbert space, tensors are expressed its coefficient numerical tensors. Classification of entanglement under SLOCC is easily seen to be a classification of the coefficient tensors under mode products. Classification of entanglement or determination of maximal rank is now completed in both community, (in statistical community, it is solved from a view point of Jordan form, but somewhat differently in quantum communication) and the research interest is moved to the classification of 3 party entanglement, that is, coefficient tensors with 3 dimensions, $N_A \geq N_B \geq N_C$. The problem is known to be notoriously difficult even for $N_c = 3$, for general. In section 2 we review some terminology in both tensor rank and quantum information theory. In section 3 we introduce the equivalent states under SLOCC, as well as some related open problems arising in quantum information and computation. In section 4 we define a determinant polynomial(see Sakata-Sumi-Miyazaki(Abstract Book, ISI 2009,Durban)) and show its usefulness as a equivalence criterion. In section 5 we develop the technique of range criterion and use it to distinguish inequivalent multipartite states under SLOCC. In section 6 we develop the classification of pure states for the case $3 \times 3 \times 5$, with the introduction of the classification of the $2 \times m \times n$ case by Lin Chen et all(2006) [6]. For the sake of lack of space, all are written as possible as concisely.

2 Terminologies

In this section we briefly review basic terminologies of quantum information theory and tensor rank, for both physicists and non-physicists. We are especially concerned with the precise mathematical definition for various physical terminologies.
2.1 Qubit and quNit

A quantum state with two levels is called a qubit. A qubit is mathematically represented as a vector in a 2 dimensional complex Hilbert space $H$. There is a standard basis in $H$, denoted by

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1)$$

These are denoted by $|0\rangle$ and $|1\rangle$ conventionally. Therefore, stating precisely, a 1-qubit state is a vector $x$ in $H$, such that

$$x = \alpha |0\rangle + \beta |1\rangle \quad (2)$$

with the constraint $|\alpha|^2 + |\beta|^2 = 1$.

Similarly, for a general $N$, a Hilbert space of $N$ dimensional complex Hilbert space is a state space of $N$-level quantum states, or quNits. The state is of the form

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots + \alpha_{N-1} |N-1\rangle \quad (3)$$

with the restriction $|\alpha_0|^2 + \cdots + |\alpha_{N-1}|^2 = 1$. Here, we choose the bases as $|0\rangle = (1, 0, 0, \ldots)^T$, $|1\rangle = (0, 1, 0, \ldots)^T, \ldots, |n-1\rangle = (0, 0, \ldots, 1)^T$. The set of bases is frequently used in quantum information and we call it computational bases for simplicity. Any state can be expressed by the computational bases. Of course, we can also choose different sets of bases to decompose the state $|\psi\rangle$. It can be realized totally by skills in linear algebra.

2.2 Quantum multipartite states

The state $|\psi\rangle$ actually describes a local system in physics. Here the local system can be referred to as one particle, photon, atom, etc. We can describe a few local systems by using the direct (Kronecker) product of their states. For example, suppose a few states $|\psi_i\rangle$, $i = 1, 2, \ldots, m$ describe the systems $A_i$, $i = 1, 2, \ldots, m$, respectively, then the total system $A_1 A_2 \ldots A_m$ is in the state

$$|\Psi\rangle_{A_1 A_2 \ldots A_m} = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_m\rangle. \quad (4)$$

We call it the product multipartite (bipartite for $m = 2$) quantum state. By the language of tensor rank (see next subsection), we say the state $|\Psi\rangle_{A_1 A_2 \ldots A_m}$ has tensor rank 1. However, there are multipartite states which cannot be expressed in Eq. (4). For example, the 2 $\otimes$ 2 bipartite state

$$|\psi\rangle = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (5)$$

For convenience we will use $|i\rangle \otimes |j\rangle = |ij\rangle$ when there is no confusion. The above state can then be written as $|\psi\rangle = |00\rangle + |11\rangle$. It’s easy to check $|\psi\rangle$ has tensor rank 2; namely we cannot write that $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$. As another example, every trilinear form in algebraic complexity stands for a tripartite quantum state, such as the following state

$$|\Psi\rangle_{ABC} = |001\rangle + |010\rangle + |100\rangle. \quad (6)$$

This is a typical multipartite state in quantum information and its tensor rank is 3.

Generally, a multipartite state in the space $H = H_1 \otimes \cdots \otimes H_m$ can be written in terms of the computational bases as follows

$$|\Psi\rangle_{A_1 A_2 \ldots A_m} = \sum_{i_1, i_2, \ldots, i_m} a_{i_1, i_2, \ldots, i_m} |i_1, i_2, \ldots, i_m\rangle, \quad (7)$$

where the normalization condition keeps

$$\sum_{i_1, i_2, \ldots, i_m} |a_{i_1, i_2, \ldots, i_m}|^2 = 1.$$ 

A multipartite state is called an entangled state when it’s not of product state in $H$ namely

**Corollary 1**: A multipartite pure state is entangled if and only if it has tensor rank larger than 1.

Multipartite states are fundamental resources for a cross-disciplinary field between quantum physics and information theory—quantum information and computation theory, which has developed very fast since 1990s. There have been a rapidly increasing number of papers contributed to this field because of the novel ideas and great potential of application and methods to realize physical and informational tasks. Besides the physics, the contributors are also from many related areas like mathematics, chemistry, biology, computer science, engineering and so on. In recent years, many theoretical plans in quantum information theory have been realized in experiments. In what follows, we will introduce the concept of tensor rank and find out its relation to quantum information.

2.3 Tensor rank of multipartite states

Tensor rank is an important concept in data processing, and more generally in computer science, and has been used in many branches of science [1]. Expressed in the language of quantum information, the tensor rank $R(\Psi)$ of a multipartite pure state $|\Psi\rangle$ in the space $H = H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_m}$ is the minimal number $R$ of product states $|\pi_j\rangle = \bigotimes_{k=1}^n |\phi_{jk}\rangle$ with $|\phi_{jk}\rangle \in H_{A_k}$, $j = 1, \ldots, R$, $k = 1, \ldots, n$, whose superposition forms the state $|\Psi\rangle$; that is, $|\Psi\rangle = \sum_{j=1}^R \bigotimes_{k=1}^n |\phi_{jk}\rangle$, or equivalently,

$$|\Psi\rangle \in \text{span}\{ |\pi_1\rangle, \ldots, |\pi_R\rangle \}. \quad (8)$$

For example, consider two well-known three-qubit states, the Greenberger-Horne-Zeilinger (GHZ) state and the W state,

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

$$|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \quad (9)$$

We need at least two product states to form the GHZ state. So the tensor rank of $|\text{GHZ}\rangle$ is two. Similarly, one can show that the tensor rank of $|\text{W}\rangle$ is three. Generally, there have been techniques for calculating or estimating the tensor rank of multipartite states in algebraic computational complexity [2]. Note that for
bipartite systems, the tensor rank is identical to the well-known Schmidt rank of a state.

In this case, we can use the tensor rank to characterize a given quantum state. We will list a few applications and problems in quantum information by using the results in tensor rank.

3 Equivalent states under stochastic local operations and classical communications (SLOCC)

In quantum physics, physical operators can be denoted as some square matrix, which may be either nonsingular or singular. The general LOCC operation or a completely positive (CP) map on a state has the form

$$\epsilon(\rho) = \sum_i A_i \otimes B_i \rho A_i^\dagger \otimes B_i^\dagger.$$  \hfill (10)

Notice it actually denotes separable operations, which includes the set of LOCC operations. The reason is that the standard form of LOCC is very complicated and not easily handled, so we usually use the expression in Eq. (10) In particular, two states $\rho_1, \rho_2$ are one-way equivalent under stochastic LOCC (SLOCC) when there is some CP map $\epsilon$ such that $\epsilon(\rho_1) = \rho_2$ or $\epsilon(\rho_2) = \rho_1$. If both of them are correct (maybe by virtue of two CP maps), the two states are equivalent under SLOCC. The reason we use "stochastic" is that the map may be enforced with a probability larger than 0 but smaller than 1. It is one of the most usual restrictions in quantum information theory.

As the equivalence problem is quite general and extensive, we only deal with pure states in this talk. An important problem in quantum information is to determine whether two pure multipartite states are one-way equivalent or not under SLOCC. Mathematically, two $n$ partite states $|\psi\rangle, |\varphi\rangle$ are one-way equivalent if and only if there are some local operators $A_1, \ldots, A_n$ such that

$$|\psi\rangle_{1,\ldots,n} = A_1 \otimes \ldots \otimes A_n |\varphi\rangle_{1,\ldots,n}. \hfill (11)$$

Moreover, $|\psi\rangle, |\varphi\rangle$ are equivalent under SLOCC when the local operators are non-singular square matrix. That is, we can also write

$$|\varphi\rangle_{1,\ldots,n} = A_1^{-1} \otimes \ldots \otimes A_n^{-1} |\psi\rangle_{1,\ldots,n}. \hfill (12)$$

For example, two states $|00\rangle + |11\rangle$ and $|01\rangle - |10\rangle$ are equivalent because

$$|00\rangle + |11\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (|01\rangle - |10\rangle). \hfill (13)$$

So the point of deciding two equivalent states is to find out the direct product of nonsingular matrices that link them. However, this is usually difficult, even for the tripartite system (c.f. next paragraph). On the other hand, tensor rank can provide some useful tool in this context.

Lemma 2 Tensor rank of multipartite states cannot be increased under SLOCC.  \hfill \Box

Lemma 3 If $|\psi\rangle$ is one-way equivalent to $|\varphi\rangle$, then the tensor rank of $|\psi\rangle$ is not smaller than $|\varphi\rangle$.  \hfill \Box

Corollary 4 Two equivalent multipartite states have identical tensor rank.

So tensor rank provides a necessary condition for two equivalent states. For example, the following two tripartite states cannot be equivalent:

$$|\varphi\rangle_{ABC} = |000\rangle + |111\rangle + |222\rangle, \hfill (14)$$

and

$$|\psi\rangle_{ABC} = |012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle. \hfill (15)$$

The reason is that the tensor rank of them are 3 and 4 respectively, see Atkinson-Lloyd’s paper in 1979 for details of Eq. (13). So the calculation or estimation of two states may imply the equivalence of them or not. Notice the two states are not one-way equivalent either. The reason is that operators without full rank will reduce the local rank of a multipartite state, while the local rank of both states here are three.

So far we considered the one-copy state. The question will be more interesting when we study the case of many copies. Given $m$ copies of $|\psi\rangle$ and $n$ copies of $|\varphi\rangle$, we may ask whether its possible to convert $|\psi\rangle^\otimes m$ into $|\varphi\rangle^\otimes n$ by SLOCC. Here, the tensor product state forms a new state in the way that the corresponding parties are combined together and form a new party in the new state, i.e.,

$$|\psi\rangle^\otimes n_{A_1\ldots A_n} = |\psi\rangle_{A_1^\prime\ldots A_1^\prime} \otimes \ldots \otimes |\psi\rangle_{A_m^\prime\ldots A_m^\prime} = |\psi\rangle_{A_1^\prime\ldots A_m^\prime A_1\ldots A_m}, \hfill (16)$$

where the new state is still in $n$ parties each of which contains $A_i^\prime, A_i, i = 1, \ldots, n$.

For example, we recall that the $W$ state $|001\rangle + |100\rangle$ has tensor rank 3 and the GHZ state $|000\rangle + |111\rangle$ has tensor rank 2. So it’s impossible to make them equivalent under SLOCC. However, authors have found that we can realize the one-way equivalence of them when a few copies are available \footnote{As GHZ states, $|GHZ\rangle^\otimes 3$ can be transformed into $|W\rangle^\otimes 2$ by SLOCC. (The authors deduced that the tensor rank of two copies of W state is not bigger than 8). It implies we may realize more equivalence by using of multi-copy states. In a latest paper \footnote{Sakata-Sumii-Miyazaki(Abstract Book, ISI 2009,Durban).}, authors have proved that the tensor rank of two-copies of W states analytically equals 7, which is a surprising fact. It’s going to be an interesting problem to address the tensor rank of $|W\rangle^\otimes n$.}

4 A necessary condition by determinant polynomials of SLOCC-equivalence of tensors with 3-slices

In this section we introduce a necessary condition of deciding the equivalent multipartite states under SLOCC, by using the so-called 3-slices’ determinant polynomials developed by Sakata-Sumii-Miyazaki(1989, Durban)
Let matrices $A, B$ and $C$ be $n \times n$ matrices, and we call a triple $(A, B, C)$ as a $n \times n \times 3$ tensor and denote by $T = A : B : C$. Then each matrices $A$, $B$ and $C$ is called an $i$-th slice of the tensor $T$, $i = 1, 2, 3$, respectively. From a data analytic point of view, for a tensor $T$, we are mainly concerned with its tensor rank, denoted by $\text{rank}(T)$ which is the smallest number of rank 1 tensors by which the tensor is expressed as a sum (notice it coincides with the concept in Section 2.3). Therefor tensor rank is an index of complexity of data.

For a tensor $T = A : B : C$ we consider the two types of transformation,

Type 1 $(A, B, C) \rightarrow (PAQ, PBQ, PCQ)$ with nonsingular matrices $P$ and $Q$

Type 2 $(A, B, C) \rightarrow (g_{11}A + g_{21}B + g_{31}C; g_{12}A + g_{22}B + g_{32}C; g_{13}A + g_{23}B + g_{33}C)$, where $G = (g_{ij})$ is a nonsingular matrix.

**Definition 5** Two tensor $T_1$ and $T_2$ are said to be equivalent if $T_1$ and $T_2$ are inter-convertible by a sequence of transformations of type 1 and type 2.

**Remark 6** Tensor rank is invariant under both type of transformations.

Subsequently, we are going to find out the connection between a $n \otimes n \otimes 3$ state and its corresponding 3-slices tensor. Let us consider two $n \otimes n \otimes 3$ states

$$
|\Psi\rangle = \sum_{k=0}^{2} \left( \sum_{i,j=0}^{n-1} a_{ijk}|i,j\rangle \right) |k\rangle
$$

and

$$
|\Phi\rangle = \sum_{k=0}^{2} \left( \sum_{i,j=0}^{n-1} b_{ijk}|i,j\rangle \right) |k\rangle.
$$

According to the definition of SLOCC equivalence in Eq. 17 there should be three nonsingular matrices $A, B, C$ such that $|\Psi\rangle = A \otimes B \otimes C|\Phi\rangle$, namely

$$
A \otimes B \left( \sum_{i,j=0}^{n-1} a_{ijk}|i,j\rangle \right) = \sum_{k=0}^{2} \left( \sum_{i,j=0}^{n-1} b_{ijk}|i,j\rangle \right),
$$

where $G = [g_{ij}]^{3 \times 3}$ is nonsingular. If we perform the partial rotation on system B by column row $|i\rangle \rightarrow \langle i|$ and define two tensors

$$
T_1 = \sum_{i,j=0}^{n-1} a_{ij0}|i\rangle \langle j| : \sum_{i,j=0}^{n-1} a_{ijs}|i\rangle \langle j| : \sum_{i,j=0}^{n-1} a_{ijs}|i\rangle \langle j|,
$$

and

$$
T_2 = \sum_{i,j=0}^{n-1} b_{ij0}|i\rangle \langle j| : \sum_{i,j=0}^{n-1} b_{ij1}|i\rangle \langle j| : \sum_{i,j=0}^{n-1} b_{ij2}|i\rangle \langle j|.
$$

Then Eq. 19 states that tensors $T_1$ and $T_2$ are equivalent in terms of definition 5. In this sense, the equivalence of two tensors will immediately cause the equivalence of two tripartite states under SLOCC.

However, it should be noted that tensor rank does not determine equivalent class, and that it is merely an invariant under SLOCC. If $T_1$ and $T_2$ are equivalent under SLOCC, the rank of them are the same. Thus the rank is used as a test function of equivalence, that is, if the ranks are not identical, the two states are not equivalent. So, tensor rank is a necessary condition for equivalence. However, it is too coarse as an index of equivalence classes. In fact, the number of tensor rank is extremely less than the number of equivalent classes under SLOCC in general.

In this paper we propose a new index of equivalence under SLOCC, which is more fine than tensor rank as an index of equivalence. The index is the polynomial determinant of a tensor. For tensor $T = A : B : C$ we define the polynomial by

$$
f(x, y, z) = \det(xA + yB + zC),
$$

which we call the determinant polynomial of the tensor $T$. By the results in Sakata-Sumi-Miyazaki(Abstract Book, ISI 2009, Durban), we have

**Theorem 7** Two tensors $T_1$ and $T_2$ are equivalent only if, based on the two monic polynomials obtained from both determinant polynomials, the algebraic equation about a nonsingular matrix $G$ defined by

$$
f_2(x) = f_1(xG^t),
$$

has at least one solution $G$.

**Example 8** (Sakata-Sumi-Miyazaki(Abstract Book, ISI 2009, Durban)) We use theorem 7 to derive the equivalence of two $3 \times 3 \times 3$ tensors

$$
T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},
$$

and

$$
T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Let $f_1$ and $f_2$ be the determinant polynomials of $T_1$ and $T_2$ respectively. Then we have

$$
f_1(x, y, z) = xyz,
$$

and

$$
f_2(x, y, z) = (x + y)(y + z)(z + x).
$$

So, by the transformation

$$
x \rightarrow (x-y+z)/2, \quad y \rightarrow (x+y-z)/2, \quad z \rightarrow (-x+y+z)/2,
$$

we have $f_2 \rightarrow f_1$. So the equivalence of $T_1$ and $T_2$ is not denied. Carefully looking, we see that they are equivalent. In fact, after subtracting $B_2$ from $A_2$ and adding $C_2$, dividing $A_2$ by 2, $A_2$ becomes $A_1$. Then by subtracting $A_2$ from $C_2$, $C_2$ becomes $C_1$. Finally by subtracting $C_2$ from $B_2$, $B_2$ becomes $B_1$. Thus, $T_1$ and $T_2$ are equivalent.
5 Classification of SLOCC equivalent states based on range criterion

Classification of multipartite entangled states is an important topic in quantum information. There have been many papers in this topic over past years [9]. Especially the classification of K partite ($K \geq 3$) entangled states under SLOCC is known to be very difficult. First we introduce the necessary and sufficient condition for equivalence under SLOCC which appeared in the paper of Lin Chen et al.[6]. To explain our method, we need to build more basic concepts in quantum information.

5.1 Density matrix and reduced density matrix of a state

**Definition 9** Let $H$ be $N_1$ dimensional complex Hilbert space and $|x\rangle$ be an element of it, that is, a state. Then

$$\rho = |x\rangle \langle x|$$

is called as the density matrix of the state $|x\rangle$.

**Proposition 10**

$$Tr(\rho) = ||x||^2 = 1.$$ (30)

**Definition 11** Assume that $|x_1\rangle, \cdots, |x_K\rangle$ are a collection of $K$ pure states and $p_1, \ldots, p_K (\sum_{i=1}^{K} p_i = 1)$ is a probability. Then the mixed state density matrix is defined as

$$\rho = \sum_{i=1}^{K} p_i |x_i\rangle \langle x_i|.$$ (31)

Properties of density matrices are listed below.

1. A pure state is defined as both a vector and a density matrix, however, a mixed state is defined only through a density matrix, that is, through a set of pure states and probabilities.

2. A quantum state always corresponds to a density operator one-to-one. When two pure states are proportional, e.g., $|0\rangle$ and $e^{i\alpha}|0\rangle$, they stand for the same state, for the global phase $e^{i\alpha}$ does not lead to difference in physics. For the case of mixed state which is always a matrix, there will not be global phase otherwise it’s not positive semidefinite. For instance, if $\rho$ is a density matrix, $e^{i\alpha}\rho$ is not legal by the definition of density operators.

**Definition 12** Partial trace is a method of obtaining the reduced density matrix, that is, the marginal density matrix, of Alice and Bob respectively. Alice’s reduced density matrix and Bob’s reduced density matrix are obtained respectively by

$$Tr_{AB}^{A}\langle \Phi | \Phi = \sum_{j} (I \otimes (j)) \langle \Phi |((I \otimes |j\rangle)$$ (32)

and

$$Tr_{AC}^{B}| \Phi = \sum_{i} (|i\rangle \otimes I) \langle \Phi |((i \otimes I).$$ (33)

where $\{|i\rangle\}$ and $\{|j\rangle\}$ are orthonormal bases of $H_A$ and $H_B$, respectively.

**Remark 13** Partial trace is also extendedly defined for joint mixture states by linearity. We also denote reduced density matrix as reduced density in this paper.

**Definition 14** Let $|\Phi\rangle \in H_A \otimes H_B \otimes H_C$ be the joint state of Alice, Bob and Chan. Then the density matrix of the joint state is $|\Phi\rangle \langle \Phi|$ and Alice’s reduced density is defined by

$$Tr_{BC}^{C}| \Phi = \sum_{j} (I \otimes (j) \otimes (k)) \langle \Phi |((I \otimes |j\rangle \otimes |k\rangle)$$ (34)

where $\{|j\rangle\}$ and $\{|k\rangle\}$ are orthonormal bases of $H_B$ and $H_C$.

**Remark 15** $Tr_{AC}^{B}, Tr_{AC}^{B}, Tr_{AB}^{C}, Tr_{AB}^{C}$ and $Tr_{AC}^{B}$ is also defined similarly.

5.2 Local rank

The local rank of a multipartite state means the ranks of reduced density operators; For example, the state $|000\rangle + |111\rangle$ has local rank 2, 2, 2; and the state $|000\rangle + (|1\rangle + |2\rangle)(|3\rangle + |4\rangle + |7\rangle)(|5\rangle + |6\rangle - |0\rangle)$ still has local ranks 2, 2, 2. In fact the latter has the reduced density operator $|0\rangle \langle 0| + a(|1\rangle + |2\rangle)(|1\rangle + |2\rangle), |0\rangle \langle 0| + b(|3\rangle + |4\rangle + |7\rangle)(|3\rangle + |4\rangle + |7\rangle), \text{ and } |0\rangle \langle 0| + c(|5\rangle + |6\rangle - |0\rangle)(|5\rangle + |6\rangle - |0\rangle)$, for each system respectively, where $a,b,c$ are non-zero constants that can be calculated. Evidently, they all have rank 2; and this is the meaning of local ranks.

5.3 Range criterion (Lin Chen et al., [6])

**Definition 16** Let $\rho = |\Psi\rangle \langle \Psi |$ be a joint density given by a pure state $|\Psi\rangle \in H_A \otimes H_B \otimes H_C$. Then, for clarification, the marginal density for each party are denoted by $\rho_A, \rho_B, \rho_C$, respectively.

**Theorem 17** (Range criterion). Let $\Psi_{ABC}$ and $\Phi_{ABC}$ are two pure states in $H_A \otimes H_B \otimes H_C$ such that $\Psi_{ABC} = V_A \otimes V_B \otimes V_C \Phi_{ABC}$. Let $S_1 = \{|x\rangle_{BC} \in \mathcal{R}(\rho_{ABC})\}$ and $S_2 = \{|x\rangle_{BC} \in \mathcal{V}_B \otimes V_C \mathcal{R}(\rho_{ABC})\}$ Then

1. all local ranks are equal
2. $S_1 = S_2$

This is a necessary and sufficient condition for equivalence of both states.

**Corollary 18** If two pure states of a multipartite system are equivalent under SLOCC, the number of linearly independent product states in the range of the adjoint reduced density matrices of each party of them must be equal.

**Example 19** Consider the well-known GHZ and W state. One can check that there are two product states $|000\rangle, |111\rangle$ in $\mathcal{R}(\rho_{GHZ}^{BC})$, while there is only one $|000\rangle$ in $\mathcal{R}(\rho_{W}^{BC})$. So we conclude that GHZ and W states are not equivalent under SLOCC.
6 Classification of $2 \times M \times N$ states and application to $3 \times 3 \times 5$ states

By using the range criterion in theorem 17, we already developed a technique to classify inequivalent states of $2 \times M \times N$ system as follows.

Theorem 20 (Lin-Chen et al [7]) we have

$$|\Psi\rangle_{2 \times M \times N} \sim$$

$$(|\Omega_0\rangle \equiv (a|0\rangle + b|1\rangle)\langle M-1,N-1| + |\Psi\rangle_{2 \times (M-1)\times (N-1)} ) ,$$

where $|\chi\rangle \equiv |\Omega_0\rangle + |0,M-1\rangle |\chi\rangle , b \neq 0,$

$|\Omega_1\rangle \equiv |0,M-1,N-1\rangle + |1,M-1,N-2\rangle + |\Psi\rangle_{2 \times (M-1)\times (N-2)} ,$$

$|\Omega_2\rangle \equiv |\Omega_0\rangle + |0,M-1\rangle |\chi\rangle , b \neq 0,$

$|\Omega_3\rangle \equiv |\Omega_0\rangle + |1,M-1\rangle |\chi\rangle , a \neq 0.$

Here $|\chi\rangle = \sum_{i=0}^{N-2} a_i |i\rangle$ as a random state. The condition $a \neq 0$ or $b \neq 0$ keeps $|\Omega_2\rangle$ and $|\Omega_3\rangle$ not becoming $|\Omega_0\rangle$.

Such equivalence relation shows that the lower rank classes of the entangled states can be used to generate the higher rank classes of the true entangled states for any $2 \times M \times N$ system, called as "Low-to-High Rank Generating Mode" or LHRGM for short. So the corollary and the range criterion of the theorem 1 provide a systematic method to classify all kinds of true tripartite entangled states in the $2 \times M \times N$ system under SLOCC [6].

In particular, we can use the classification of $2 \times M \times N$ states with $M \leq 3, N \leq 6$ in [6] to derive the tensor rank of $3 \times 3 \times 5$ states, which is either 6 or 7 by Atkinson et al [2].

By using a tensor product of three nonsingular matrices on the system, every $3 \otimes 3 \otimes 5$ state can be written as

$$|\Psi\rangle_{ABC} = |\psi\rangle |0\rangle |0\rangle |1\rangle |3\rangle |2\rangle |\gamma\rangle ,$$

where $|\psi\rangle$ is a $2 \otimes n \otimes p$ state with $n \leq 3, 2 \leq p \leq 5$. So it’s possible to infer the tensor rank of $|\Psi\rangle_{ABC}$ by simplifying the expression in Eq. (35) while which is already classified in [6]. For convenience, we list the inequivalent states derived thereof, where readers can check the details. At present we are still studying this problem and new results will be reported later.

### Systems’ ranks | Inequivalent states under SLOCC
|-------|-----------------|
| $2 \times 3 \times 6$ | $|000\rangle + |011\rangle + |022\rangle + |103\rangle$ |
|       | $+ |114\rangle + |125\rangle$ ; |
| $2 \times 3 \times 5$ | $|024\rangle + |000\rangle + |011\rangle$ |
|       | $+ |102\rangle + |113\rangle$ ; |
|       | $|024\rangle + |121\rangle + |000\rangle + |011\rangle$ |
|       | $+ |102\rangle + |113\rangle$ ; |
| $2 \times 3 \times 4$ | $|123\rangle + |012\rangle + |000\rangle + |101\rangle$ |
|       | $|023\rangle + |012\rangle + |000\rangle + |101\rangle$ ; |
|       | $|123\rangle + |012\rangle + |110\rangle + |000\rangle + |101\rangle$ ; |
|       | $|223\rangle + |122\rangle + |012\rangle + |000\rangle + |101\rangle$ ; |
|       | $|223\rangle + |122\rangle + |012\rangle + |110\rangle$ |
|       | $+ |000\rangle + |101\rangle ;$ |
| $2 \times 3 \times 3$ | $|000\rangle + |111\rangle + |022\rangle$ |
|       | $|000\rangle + |111\rangle + |022\rangle + |122\rangle$ |
|       | $|010\rangle + |001\rangle + |112\rangle + |121\rangle$ |
|       | $|100\rangle + |010\rangle + |001\rangle + |112\rangle + |121\rangle$ |
|       | $|100\rangle + |010\rangle + |001\rangle + |222\rangle$ |
|       | $|100\rangle + |010\rangle + |001\rangle + |222\rangle$ ; |
| $2 \times 3 \times 2$ | $|000\rangle + |011\rangle + |121\rangle$ ; |

7 Conclusions

In this paper we investigated a few problems between tensor rank and quantum information theory. We proposed the 3-sliced tensor to address the SLOCC-equivalence problem. We also have shown that the range criterion can help find out inequivalent states and thus tensors, which is likely to help compute the tensor rank of $3 \times 3 \times 5$ tensors.

The open problems could be that how to find out the sufficient condition by using 3-sliced tensors. Besides, it’s also interesting to compute the multi-copies of W states, which is an important resource in quantum information, in both theory and experiment. Nevertheless, we only said some marginal applications of tensor rank to quantum information due to the restriction of space, as we already proposed many other connections arising from quantum information (e.g., entanglement measure [7], separability problem, construction of positive partial transpose entangled states and so on). We will propose more results based on the connections between these two rapidly developing fields.

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