On the Partial Hyperbolicity of Robustly Transitive Sets with Singularities

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Abstract
Homoclinic tangencies and singular hyperbolicity are involved in the Palis conjecture for vector fields. Typical three dimensional vector fields are well understood by recent works. We study the dynamics of higher dimensional vector fields that are away from homoclinic tangencies. More precisely, we prove that for any dimensional vector field that is away from homoclinic tangencies, all singularities contained in its robustly transitive singular set are all hyperbolic and have the same index. Moreover, the robustly transitive set is $C^1$-generically partially hyperbolic if the vector field cannot be accumulated by ones with a homoclinic tangency.

Keywords Partial hyperbolicity · Robust transitivity · Vector field · Singularity · Homoclinic tangency

1 Introduction

Palis has proposed a sequence of conjectures for understanding typical dynamics. There are lots of progresses of Palis’ projects for diffeomorphisms. It is well known that the suspensions of diffeomorphisms give phenomena of flows. However, Lorenz-like attractors, which are sometimes called the “butterfly phenomena”, cannot be presented by the suspensions of diffeomorphisms. For vector fields, by Palis’ program, one has to consider homoclinic tan-
gencies of a regular hyperbolic periodic orbit, singular hyperbolic vector fields which contain hyperbolic ones, etc. It has been proved recently in [10] for the three dimensional case, any typical vector field, which cannot be accumulated by ones with a homoclinic tangency, is globally singular hyperbolic.

However, for the higher dimensional case, there are very few work studying vector fields away from ones with a homoclinic tangency. Meanwhile, there are a sequence of works studying diffeomorphisms that are away from ones with a homoclinic tangencies. The theory of vector fields gives a different picture from the theory of diffeomorphisms because the vector field may contain a non-isolated singularities.

Let $M$ be a compact $C^\infty$ Riemannian manifold without boundary. Denote by $\mathcal{X}^1(M)$ the set of all $C^1$ vector fields of $M$ endowed with the $C^1$ topology. Denote by $\varphi_t = \varphi_t^X$ the flow generated by a vector field $X \in \mathcal{X}^1(M)$. Recall that a point $x \in M$ is a singularity of $X$ if $X(x) = 0$; and an orbit $\text{Orb}(x)$ (under the flow $\varphi_t$) is periodic (or closed) if it is diffeomorphic to a circle $S^1$. The singularities and periodic orbits are the simplest invariant sets of a flow. A singularity or a periodic point is called a critical point. But sometimes, the singularities and the periodic orbits will produce essentially different dynamics such as in the hyperbolic settings. Let $< X >$ denote the subspace generated by the vector field $X$. Let $\Phi_t = \Phi_t^X$ be the tangent flow generated by $\Phi_t = d\varphi_t^X : TM \to TM$. An invariant set $\Lambda \subset M$ is called hyperbolic for $X$ if $T_\Lambda M$ has a $\Phi_t$-invariant continuous splitting $E^s \oplus < X > \oplus E^u$ such that for some constants $C > 1$, $\lambda > 0$,

$$\|\Phi_t|_{E^s}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|\Phi_{-t}|_{E^u}\| \leq Ce^{\lambda t}$$

for all $x \in \Lambda$ and $t > 0$. The index or the stable index of $\Lambda$ is defined to be the dimension of $E^s$ if $\dim E^s$ is constant. Denote by $\text{Ind}(\Lambda)$ the index of $\Lambda$. It is well known that in a compact transitive hyperbolic set $\Lambda$, the singularities and periodic orbits cannot co-exist. However, there are very famous robust examples as the Lorenz attractor showing that singularities and regular periodic orbits do co-exist in a robustly transitive attractor. This kind of examples leads dynamists to study compact invariant sets with similar properties in an abstract way.

Denote by $\text{Sing}(X)$ the set of all singularities and $\text{Per}(X)$ the set of all periodic points of $\varphi_t^X$. A singularity $\sigma$ is hyperbolic if $\{\sigma\}$ is a hyperbolic set. A periodic orbit $P \subset \text{Per}(X)$ is hyperbolic if $P$ is a hyperbolic set. For a hyperbolic singularity or a hyperbolic periodic point $x$, the index of $x$ is the index of the hyperbolic set $\text{Orb}(x)$ and denote it by $\text{Ind}(x)$.

Denote by $W^s(P)$ and $W^u(P)$ the stable and unstable manifolds for a hyperbolic periodic orbit $P$. We say that $X$ has a homoclinic tangency associated to hyperbolic periodic orbit $P$ if there exists a non-transverse intersection $x \in W^s(P) \cap W^u(P)$, that is, $\text{Orb}(x) \cup P$ is not a hyperbolic set. This is related to the very famous Newhouse phenomena [22–24].

We would like to emphasize the homoclinic tangencies here is for regular periodic orbits, but not for singularities.

We are interested in characterizing dynamics of vector fields that are away from homoclinic tangencies. For diffeomorphisms, higher-dimensional systems away from homoclinic tangencies were studied in lots of works. One of these works is [9] by Crovisier, Sambarino and Yang. They have proved generic diffeomorphisms away from homoclinic tangencies are partially hyperbolic. The extension of [9] to non-singular flows might be regarded to be parallel. In this work, we want to do some extension of [9] to singular flows. However, it seems very difficult to do a full extension of [9] to singular chain-recurrent classes. We put Conjecture 1 in Sect. 5 to characterize the global dynamics of generic vector fields away from ones with a homoclinic tangency. Now, we will concentrate on a class of sets, which are called robustly transitive sets with singularities. In the three-dimensional case, the Lorenz
attractor mentioned above is one prototype of this kind of sets. In higher-dimensional case, we do not know enough about this kind of sets.

In a series of articles, the robust transitivity of $C^1$ vector fields is studied. In [21], Morales, Pacifico and Pujals proved that every robustly transitive singular set for a 3-dimensional flow must be partially hyperbolic. A similar results in higher dimensional manifolds is got in Li, Gan and Wen [16], the authors proved that under a strong homogenous condition, a robustly transitive set for a star flow must be partially hyperbolic. Some subsequent works along this direction can be seen in Metzger and Morales [20] and Zhu, Gan and Wen [34].

A compact invariant set $\Lambda$ of $X$ is transitive if there is $x \in \Lambda$ such that $\Lambda = \omega(x)$. A transitive set $\Lambda$ is said to be non-trivial if it is neither a singularity nor a periodic orbit. A compact invariant set $\Lambda$ of $X$ is isolated or locally maximal if there is an open neighborhood $U$ of $\Lambda$ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} \varphi^X_t(U).$$

Here the neighborhood $U$ is called an isolated neighborhood of $\Lambda$.

**Definition 1.1** A compact locally maximal invariant set $\Lambda$ of $\varphi^X_t$ is robustly transitive if there are a neighborhood $U \subset X(M)$ of $\Lambda$ and a neighborhood $U \subset M$ of $\Lambda$ such that for any $Y \in U$,

$$\Lambda_Y = \bigcap_{t \in \mathbb{R}} \varphi^Y_t(U)$$

is a non-trivial transitive set.

In this article, we firstly characterize the singularities in a robustly transitive set for a vector field which is far away from homoclinic tangencies.

**Theorem A** If $\Lambda$ is a robustly transitive set of $X \in X^1(M)$, then either all the singularities in $\Lambda$ are hyperbolic and have the same index, or $X$ can be accumulated by vector fields with a homoclinic tangency.

Note that the assumption “robust transitivity” cannot be relaxed. A recent work of da Luz [11] gave an example: under the star assumption, there is robustly chain-transitive chain recurrent classes with singularities of different indices. A way to read Theorem A is: under the assumption “far away from tangencies”, all singularities in any robustly transitive set have the same index.

Recall that singularities and periodic orbits in a hyperbolic invariant set should be separated. Hence the singularities and hyperbolic periodic orbits can not co-exist in a hyperbolic robustly transitive set. But in general, hyperbolic singularities and hyperbolic periodic orbits can co-exist in a robustly transitive set. The famous geometric Lorenz attractor is an example of this phenomenon. To describe such phenomena, we need to relax the notions of hyperbolicity.

**Definition 1.2** Let $\Lambda$ be a compact invariant set of $\varphi^X_t$. We say that $\Lambda$ is partially hyperbolic if there is a $\Phi_t$-invariant splitting $T_\Lambda M = E \oplus F$ with constants $C \geq 1$ and $\lambda > 0$ such that the following properties are satisfied:

1. $T_\Lambda M = E \oplus F$ is a $(C, \lambda)$-dominated splitting for the tangent flow $\Phi_t$, i.e.,

$$\|\Phi_{t}|_{E_x}\| \cdot \|\Phi_{-t}|_{\Phi_t(F_x)}\| \leq Ce^{-\lambda t}$$

for any $x \in \Lambda$ and $t \geq 0$,
2. There is a dichotomy as following: either $E$ is contracting, that is, $\|\Phi_t|_{E_x}\| \leq Ce^{-\lambda t}$ for any $x \in \Lambda$ and $t \geq 0$; or $F$ is expanding, that is, $\|\Phi_{-t}|_{E_x}\| \leq Ce^{-\lambda t}$ for any $x \in \Lambda$ and $t \geq 0$.

In the article, we give a characterization of robustly transitive set for $C^1$-generic vector fields far away from homoclinic tangencies. Recall that $\mathcal{R} \subset \mathcal{X}^1(M)$ is called a residual set if it contains a countable intersection of countably many open and dense subset of $\mathcal{X}^1(M)$. A property for vector fields is said to be generic if it is satisfied for a system in a residual set of $\mathcal{X}^1(M)$.

**Theorem B** There is a residual set $\mathcal{R} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{R}$, if $X$ is far away from homoclinic tangencies, then every robustly transitive set $\Lambda$ of $X$ is partially hyperbolic.

Bonatti and da Luz [2] introduced a new notion “multi-singular hyperbolicity” to characterize higher-dimensional star flows with singularities. The property that is “far away from homoclinic tangencies” is generally weaker than the condition “star”. So generally we can not image very strong hyperbolicity like “singular hyperbolicity” or “multi-singular hyperbolicity” under the assumption “far away from homoclinic tangencies”.

By the techniques of this paper, one can improve the statement of the main theorem in [34] a little bit. Recall that a compact invariant set $\Lambda$ of a vector field $X \in \mathcal{X}^1(M)$ is star if there exist a neighborhood $U \subset \mathcal{X}^1(M)$ of $X$ and a neighborhood $U \subset M$ of $\Lambda$ such that for any $Y \in U$ and any periodic orbit $P$ of $Y$ containing in $U$ is hyperbolic. And we say $\Lambda$ is strongly homogenous if there exist a neighborhood $U \subset \mathcal{X}^1(M)$ of $X$ and a neighborhood $U \subset M$ of $\Lambda$ such that for any $Y \in U$, all periodic orbit $P$ of $Y$ contained in $U$ have the same index. One can see that strong homogeneity implies star condition automatically. The converse is true under an additional assumption that $\Lambda$ contains a regular periodic orbit by [16, Lemma 1.6]. Here we can get a following conclusion.

**Theorem C** If $\Lambda$ is a robustly transitive set, and $\Lambda$ is star, then every singularity in $\Lambda$ is hyperbolic and $\Lambda$ is strongly homogenous.

Note that we do not have to assume that any singularity in the robustly transitive set is hyperbolic and we can get the strong homogeneity from the star property.

**Further remarks** In the long preparation of this work, we noticed a paper by A. da Luz [12]. We were also told [12] is one part of her thesis. She tried to study robustly chain-transitive sets and use the notion “singular volume partial hyperbolicity” to characterize robustly chain-transitive sets. In this paper, we try to study robustly transitive set that are away from homoclinic tangencies, and we use the notion “partial hyperbolicity” and “multi-singular partial hyperbolicity”. The notions in this work and in [12] are slightly different. We would say the philosophies are similar. These works are inspired by several papers for diffeomorphisms. We give a partial list of papers from diffeomorphisms: [4,5,9]. As we mentioned, this work is mainly inspired by [9]. [12] may mainly be inspired by [4].

[34] studied robustly transitive sets under the “strongly homogeneous” condition, which is stronger than “star” by definition. We can extend the result of [34] to the star case because the theory has been developed. So we give Theorem C.

One of the main ingredients for singular star flow is the “blow-up” started by the work of Li, Gan and Wen [16]. Bonatti and da Luz [2] used this idea to give a very good description of star flows by introducing cocycles. In the proof of this work, we also use lots of extended linear Poincaré flow as a tool. The usage may be slightly different from [2,12,16].

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1 We think that Theorem C is folklore now. However, it is worth to clarify in the literature.
2 Preliminary

Connecting lemma and Mañé’s ergodic closing lemma for flows In the article, we need the following version of $C^1$ connecting lemma from [32] to perturb the system. As usual, denote by $B(\Lambda, \varepsilon)$ the $\varepsilon$ neighborhood of a closed set $\Lambda$ in $M$.

Lemma 2.1 ([32], connecting lemma) Let $X \in \mathcal{X}^1(M)$ and $z \in M$ be neither singularity nor periodic point. Then for any $C^1$ neighborhood $U \subset \mathcal{X}^1(M)$ of $X$, there are constants $L > 1, T > 1$ and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ and any two points $x, y$ outside the tube $\Delta = \cup_{t \in [0,T]} B(\varphi_t^X(z), \delta)$, if the positive $\varphi_t^X$-orbit of $x$ and the negative $\varphi_t^X$ orbit of $y$ both hit $B(z, \delta/L)$, then there exists $Y \in U$ with $Y = X$ outside $\Delta$ such that $y$ is on the positive $\varphi_t^X$-orbit of $x$. Moreover, the resulted $\varphi_t^Y$-orbit segment from $x$ to $y$ meets $B(z, \delta)$.

Let $\Lambda$ be a compact invariant set of $X$, if there exist vector fields $Y_n$ such that $\Lambda$ is the Hausdorff limit of $P_n$ as $n \to \infty$, then we call $\Lambda$ is a periodic limit. As an application of the connecting lemma, we can have the following result, whose proof is omitted.

Lemma 2.2 Any transitive set is a period limit.

Another important perturbation technique is the ergodic closing lemma first given by Mañé in [19]. Recall that a point $x \in M \setminus \text{Sing}(X)$ is called a well closable point of $X$ if for any $C^1$ neighborhood $U$ of $X$, there is $T > 0$ such that for any $\delta > 0$, there exist $Y \in U$ and a periodic point $z \in M$ of $Y$ with periodic $\tau$ such that the following are satisfied:

1. $d(\varphi_t^X(x), \varphi_t^Y(z)) < \delta$ for all $0 \leq t \leq \tau$,
2. $X = Y$ on $M \setminus B_\delta(\varphi_{[-T,0]}^X(x))$ where $B_\delta(\varphi_{[-T,0]}^X(x))$ denotes the $\delta$ neighborhood of the arc $\varphi_{[-T,0]}^X(x)$.

Denoted by $\Sigma(X)$ be the set of the well closable point of $X$. Here we use the following version of ergodic closing lemma first proved in [29].

Proposition 2.3 ([29]) For every $X \in \mathcal{X}^1(M)$ and every Borel probability measure $\mu$ which is invariant by the flow $\varphi_t$, one has $\mu(\Sigma(X)) = 1$.

Linear Poincaré flow and its extension We will introduce the linear Poincaré flow proposed firstly by Liao and the extended linear Poincaré flow firstly by Li-Gan-Wen [16]. For any regular point $x \in M \setminus \text{Sing}(X)$, denote the normal space of $X$ at $x$ by

$$N_x = \langle X(x) \rangle\perp = \{v \in T_x M | v \perp X(x)\}.$$ 

Denote the normal bundle of $X$ by

$$N = N(X) = \bigcup_{x \in M \setminus \text{Sing}(X)} N_x.$$ 

Let $\pi : TN \setminus \text{Sing}(X) M \to N$ be the canonical projection.

The linear Poincaré flow $\psi_t = \psi_t^X : N \to N$ of $X$ is then defined to be the orthogonal projection of $\Phi_t|_N$ to $N$, i.e.,

$$\psi_t(v) = \pi \circ \Phi_t(v), \quad v \in N_x.$$ 

\(^2\) This can be in fact deduced by Pugh’s closing lemma [25].
One can compactify the linear Poincaré flow to be the extended linear Poincaré flow. Denote by $SM = \{ e \in TM : \| e \| = 1 \}$ the unit sphere bundle of $M$ and $\rho : SM \to M$ the canonical bundle projection. The tangent flow $\Phi_t$ thus induces a flow
\[
\Phi_t^\# : SM \to SM \\
\Phi_t^\#(e) = \Phi_t(e)/\|\Phi_t(e)\|.
\]
For each $e \in SM$, denote by
\[
N_e = \{ v \in T_{\rho(e)}M : v \perp e \}
\]
the normal space of $e$. Let
\[
N_{SM} = \bigcup_{e \in SM} N_e.
\]
One can define the following “linear Poincaré flow”:
\[
\tilde{\psi}_t : N_{SM} \to N_{SM}
\]
by
\[
\tilde{\psi}_t(v) = \pi_{\Phi_t^\#(e)} \circ \Phi_t(v),
\]
where $\pi_{\Phi_t^\#(e)}$ is the orthogonal projection from $T_{\rho(\Phi_t^\#(e))}M$ to $N_{\Phi_t^\#(e)}$.

When we take $e = X(x)/\|X(x)\|$ for some regular point $x$, then $N_e = N_X(X)$ and
\[
\tilde{\psi}_t|_{N_e} = \psi_t|_{N_X}.
\]
By this reason, Li-Gan-Wen [16] have defined the compatification of $\psi_t$ on $N_{SM}$ to be the extended linear Poincaré flow, which is also denoted by $\tilde{\psi}_t$.

In other words, the extended linear Poincaré flow $\tilde{\psi}_t$ over the subset
\[
\{ X(x)/\|X(x)\| : x \in M \setminus Sing(X) \}
\]
of $SM$ can be identified with the usual linear Poincaré flow $\psi_t$ over $M \setminus Sing(X)$.

Let $\Lambda \subset M$ be a compact invariant set of $X$. Denote by
\[
\tilde{\Lambda} = \overline{\{ X(x)/\|X(x)\| : x \in \Lambda \setminus Sing(X) \}},
\]
where the closure is taken in $SM$. The set $\tilde{\Lambda}$ is compact and $\Phi_t^\#$-invariant. Due to the parallel property of vector fields near regular points, at every $x \in \Lambda \setminus Sing(X)$, $\tilde{\Lambda}$ gives a single unit vector $X(x)/\|X(x)\|$. Thus in a sense $\tilde{\Lambda}$ is a “compatification” of $\Lambda \setminus Sing(X)$.

For a continuous function $h : \Lambda \setminus Sing(X) \to \mathbb{R}$, for any sequence $\{x_n\} \subset \Lambda \setminus Sing(X)$ converging to $x$, if $X(x_n)/\|X(x_n)\|$ and $[h(x_n)]$ are also converging to some limits, then $h$ can be extended to be a continuous function $\tilde{h} : \tilde{\Lambda} \to \mathbb{R}$.

For measures, we have the following observations:

- for any measure $\mu$ supported on $\Lambda$, if $\mu(Sing(X)) = 0$, then one can lift $\mu$ to be $\tilde{\mu}$ supported on $\tilde{\Lambda}$ in a natural way. $\tilde{\mu}$ is said to be the transgression of $\mu$.
- for any invariant ergodic measure $\tilde{\mu}$ of $\Phi_t^\#$ supported on $\tilde{\Lambda}$, if $\tilde{\mu}(S_{Sing(X)}M) = 0$, then $\tilde{\mu}$ can be projected to be an ergodic measure $\mu$ supported on $\Lambda$ in a unique way with the property that $\mu(Sing(X)) = 0$.

Furthermore, $\mu$ and $\tilde{\mu}$ have the same Lyapunov exponents.

Note that we have a following easy lemma for transitive sets.
Lemma 2.4 Let \( \Lambda \) be a non-trivial transitive set of \( X \) and \( \sigma \in \Lambda \) be a hyperbolic singularity. Then \( (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset \) and \( (W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset \).

Proof By the stable manifold theorem we know that there is \( r \geq 0 \) such that the local stable manifold of \( \sigma \) can be characterized by

\[
W^s(\sigma) = \{ y \in M : d(\psi^t(\sigma), \sigma) \leq r, \forall t \geq 0 \}.
\]

Since \( \Lambda \) is transitive, there is a point \( x \in \Lambda \) such that \( \Lambda = \omega(x) \). Now we can take a sequence of \( x_n \) in the positive orbit of \( x \) with \( s_n > 0 \) such that \( d(x_n, x) \to 0 \) and \( d(\psi^{-s_n}(x_n), \sigma) = r \) and \( d(\psi^t(x_n), \sigma) \to r \) for all \( t \in (-s_n, 0) \). By the choice of the sequence \( x_n \) and \( s_n \), one can see that \( s_n \to +\infty \).

By the stable manifold theorem we know that there is \( y \in \omega(x) \). Let \( \psi^t \) be an accumulated point of \( \psi^{-s_n}(x_n) \), then we know that \( d(y, \sigma) = r \) and \( d(\psi^t(y), \sigma) \leq r \) for all \( t \geq 0 \). Hence \( y \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \). Similarly, we can get that \( W^u(\sigma) \setminus \{\sigma\} \cap \Lambda \neq \emptyset \). \( \square \)

So we have the following lemma for \( \tilde{\Lambda} \).

Lemma 2.5 Let \( \Lambda \) be a non-trivial transitive set of \( X \) and \( \sigma \in \Lambda \) be a hyperbolic singularity with hyperbolic splitting \( T_\sigma M = E^s_\sigma \oplus E^u_\sigma \). Then we have \( \tilde{\Lambda} \cap E^s_\sigma \neq \emptyset \) and \( \tilde{\Lambda} \cap E^u_\sigma \neq \emptyset \).

Proof By Lemma 2.4 we know that there is \( x \in \Lambda \cap W^s(\sigma) \setminus \{\sigma\} \), and then \( X(x)/\|X(x)\| \in \tilde{\Lambda} \). Take a sequence \( t_n \to +\infty \) such that \( \Phi_{t_n}(X(x)/\|X(x)\|) \) accumulates to some unit vector \( e \), hence \( e \in \tilde{\Lambda} \cap E^s_\sigma \). Similarly we have \( \tilde{\Lambda} \cap E^u_\sigma \neq \emptyset \). \( \square \)

If \( \Lambda \) is a robustly transitive set of \( X \) with a neighborhood \( U \) as in definition, then the following extension \( B(\Lambda) \) of \( \Lambda \) in \( SM \) will be more suitable.

These compactifications started from Li-Gan-Wen [16]. For recent results, one can see [2]. There are also compactifications for non-linear dynamics in [15] and [10].

Here

\[
B(\Lambda) = \{ e \in SM : \rho(e) \in \Lambda, \exists X_n \to X, \text{ and } p_n \in \text{Per}(X_n) \,
\text{ such that } X_n(p_n) \subset U, \text{ s.t. } \frac{X_n(p_n)}{\|X_n(p_n)\|} \to e \}.
\]

The set \( B(\Lambda) \) is also a compact invariant set of \( \Phi^\#: \Phi_{t_n} \).

Lemma 2.6 If \( \Lambda \) is transitive, then \( \tilde{\Lambda} \subset B(\Lambda) \).

Proof Fix \( e \in \tilde{\Lambda} \). Given any \( \varepsilon > 0 \), there is a regular point \( x \in \Lambda \) such that \( X(x)/\|X(x)\| \) is \( \varepsilon/2 \)-close to \( e \). By Lemma 2.2, \( \Lambda \) is a periodic limit. So there is a sequence of vector fields \( \{X_n\} \) such that each \( X_n \) has a periodic point \( p_n \) with the property \( \lim_{n \to \infty} X_n(p_n) = x \). By the fact that \( x \) is a regular point of \( X \), one has that

\[
\lim_{n \to \infty} \frac{X(p_n)}{\|X(p_n)\|} = \frac{X(x)}{\|X(x)\|}.
\]

This implies that \( X(x)/\|X(x)\| \) is contained in \( B(\Lambda) \). So \( e \) is \( \varepsilon/2 \)-close to \( B(\Lambda) \). By the fact that \( B(\Lambda) \) is compact, one has that \( e \in B(\Lambda) \). \( \square \)

Dominated splittings Similar to the dominated splitting with respect to tangent flow on a \( \psi_t \) invariant set, we can define the dominated splitting with respect to the linear Poincaré flow. Let \( \Lambda \) be a compact invariant set of \( \psi_t \). We say that a \( \psi_t \) invariant splitting \( N_{A\setminus \text{Sing}(X)} = N_1 \oplus N_2 \) is an \( l \)-dominated splitting (with respect to \( \psi_t \)) or bundle \( N_1 \) is \( l \)-dominated by bundle \( N_2 \) if

\[
\|\psi_t|_{N_1(x)}\| \cdot \|\psi_{-t}|_{N_2(\psi_t(x))}\| \leq 1/2
\]
for all \( x \in \Lambda \) and all \( t \geq l \). If there is an positive integer \( i \) such that \( \dim N_1(x) = i \) for all \( x \in \Lambda \setminus \text{Sing}(X) \), then we say the dominated splitting is homogenous and \( i \) is the index of the dominated splitting. The dominated splitting for the extended linear Poincaré flow \( \psi_t \) on a \( \Phi_t \) invariant set can be defined similarly.

We remark here that an equivalent definition for dominated splitting can be given as following. Usually, a \( \psi_t \)-invariant bundle \( N_1 \) is dominated by a \( \psi_t \)-invariant bundle \( N_2 \) on an invariant set \( \Lambda \) means that there exist constants \( C \geq 1, \lambda > 0 \) such that

\[
\|\psi_t|_{N_1(x)}\| \cdot \|\psi_t^{-1}|_{N_2(\psi_t(x))}\| \leq C e^{-\lambda t}
\]

for any \( x \in \Lambda \setminus \text{Sing}(X) \) and \( t \geq 0 \). We say an invariant splitting \( N_1 \oplus N_2 \oplus \cdots \oplus N_k \) is dominated means that \( N_i \) is dominated by \( N_{i+1} \) for every \( i = 1, 2, \cdots, k-1 \). It is known that \( N_1 \oplus N_2 \oplus \cdots \oplus N_k \) is a dominated splitting if \( (N_1 \oplus \cdots \oplus N_l) \oplus (N_{l+1} \oplus \cdots \oplus N_k) \) is a dominated splitting for all \( l = 1, 2, \cdots, k-1 \).

For the relationship between the dominated splittings for the linear Poincaré flow and the extended linear Poincaré flow, we have the following proposition.

**Proposition 2.7** ([16]) If \( \Lambda \) admits a dominated splitting \( N_\Lambda \setminus \text{Sing}(X) = N_1 \oplus N_2 \) with respect to the linear Poincaré flow, then one has a dominated splitting \( N_\Lambda = N_1 \oplus N_2 \) with respect to the extended linear Poincaré flow \( \tilde{\psi}_t^X \) such that \( N_1(e) = N_1(\rho(e)) \) and \( N_2(e) = N_2(\rho(e)) \) for all \( e \in \Lambda \) with \( \rho(e) \in \Lambda \setminus \text{Sing}(X) \).

For more discussion on the extended linear Poincaré flow and the dominated splitting, one can see Sects. 2 and 3 of [16].

**Estimation on the periodic orbits** Here we collect some known results from [30,31] for the systems far away from homoclinic tangencies. Let \( \overline{HT} \subset C^1(M) \) be the closure of the set of vector fields which has a homoclinic tangency. Let \( P \) be a hyperbolic periodic orbit of a \( C^1 \) vector field \( Y \) with period \( \pi(P) \). Then for any point \( p \in P \), the normal space \( N_p \) of \( Y \) at \( p \) can be split into \( N_p^s \oplus N_p^u \) where \( N_p^s \) is the sum of eigenspaces related to the eigenvalues of \( \psi^Y_{\pi(P)}|_{N_p^s} \) with modulus less than 1, \( N_p^u \) is the sum of eigenspaces related to the eigenvalues of \( \psi^Y_{\pi(P)}|_{N_p^u} \) with modulus greater than 1.

**Proposition 2.8** ([30]) Let \( X \in C^1(M) \setminus \overline{HT} \), then there exist a \( C^1 \) neighborhood \( U \) of \( X \) and constant \( l > 0 \) such that for any \( Y \in U \) and any hyperbolic periodic orbit \( P \) of \( Y \), the splitting \( N_P = N^s(P) \oplus N^u(P) \) is an \( l \)-dominated splitting with respect to the linear Poincaré flow \( \psi_t^Y \).

Actually, in [30] it is proved that once \( X \) is far away from homoclinic tangencies, then there exist a neighborhood \( U \subset C^1(M) \) and a constant \( a > 0 \) such that for any \( Y \in U \) and any periodic point \( p \) of \( Y \) with period \( \pi(p) \), there exists at most one eigenvalue of \( \psi^Y_{\pi(P)}|_{N_p^s} \) with modulus in \( (e^{-a\pi(P)}, e^{a\pi(P)}) \). Then for any periodic orbit \( P \) of \( Y \in U \), \( N_P \) can be split into \( N_{ss}^s \oplus N^c \oplus N_{uu}^u \) where three bundles are the sum of eigenspaces associated to eigenvalues with modulus in \( (0, e^{-a\pi(P)}) \), \( (e^{-a\pi(P)}, e^{a\pi(P)}) \) and \( [e^{a\pi(P)}, +\infty) \) respectively. The following proposition is proved in [30,31]. Wen stated his theorems for diffeomorphisms in [30,31]. Under the help of the Franks’ Lemma for flows [5, Theorem A.1], one can adapt Wen’s result for flows.

**Proposition 2.9** ([30,31]) Let \( X \in C^1(M) \setminus \overline{HT} \), then there exist a \( C^1 \) neighborhood \( U \) of \( X \) and constants \( C > 1, l > 0 \) and \( \eta > 0 \) such that for any \( Y \in U \) and any hyperbolic periodic orbit \( P \) of \( Y \), the following are satisfied:
1. \( N^c \) has at most dimension one,
2. the splitting \( N^{ss} \oplus N^c \oplus N^{uu} \) are \( l \)-dominated,
3. for any point \( p \in P \), we have
   \[
   \prod_{i=0}^{[\pi(P)/l-1]} \| \psi_i^Y|_{N^{ss}(\varphi_i^Y(p))} \| < Ce^{-\eta\pi(P)},
   \]
   \[
   \prod_{i=0}^{[\pi(P)/l-1]} \| \psi_i^Y|_{N^{uu}(\varphi_i^Y(p))} \| < Ce^{-\eta\pi(P)}.
   \]

**Generic results** For any two hyperbolic periodic orbits \( P_1 \) and \( P_2 \) of a \( C^1 \) vector field \( X \), for any open set \( U \), we say \( P_1 \) is homoclinic related to \( P_2 \) in \( U \) if the stable manifold of \( P_1 \) and the unstable manifold of \( P_2 \) have a transverse intersection point whose orbit is in \( U \), and the stable manifold of \( P_2 \) and the unstable manifold of \( P_1 \) have a transverse intersection point whose orbit is in \( U \). Let \( P \) and a neighborhood \( U \) of \( P \) be fixed, we call
   \[ H(P, U) = \{ x \in P' : P' \subset U, P' \text{ is homoclinic related to } P \text{ in } U \} \]
the relative homoclinic class of \( P \) in \( U \). As usual, a \( C^1 \) vector field \( X \) is called a Kupka-Smale system if every critical point of \( X \) is hyperbolic, and for any two critical orbit \( O_1 \) and \( O_2 \) of \( X \), the stable manifold of \( O_1 \) and the unstable manifold of \( O_2 \) intersect transversely. Given \( \delta > 0 \), a sequence \( \{ (x_i, t_i) : x_i \in M, t_i \geq 1 \} \) is called a \( \delta \)-pseudo orbit if \( d(\varphi_i(x_j), x_{i+1}) < \delta \) for any \( i \). We say that a compact invariant set \( \Lambda \) is chain transitive if for any \( x, y \in \Lambda \) and any \( \delta > 0 \), there is a \( \delta \)-pseudo orbit \( \{ (x_i, t_i) \}_{i=1}^n (n > 1) \) with all \( x_i \in \Lambda \) such that \( x_1 = x \) and \( x_n = y \). A compact invariant set \( \Lambda \) is said to be robustly chain transitive if there exist a neighborhood \( \mathcal{U} \subset X^1(M) \) of \( X \) and a neighborhood \( U \subset M \) of \( \Lambda \) such that for every \( Y \in U \),
   \[ \Lambda_Y = \bigcap_{i \in \mathbb{R}} \varphi_i^Y(U) \]
is chain transitive.

Here we collect some generic properties of \( C^1 \) vector fields. Recall a subset \( \mathcal{R} \subset X^1(M) \) is called residual if it contains an intersection of countably many open and dense subsets of \( X^1(M) \) and a property is called a generic property if it holds in a residual set. Recall the definition of index in Sect. 1.

**Proposition 2.10** There is a residual set \( \mathcal{R} \subset X^1(M) \) such that for any \( X \in \mathcal{R} \), the following properties are satisfied:

1. \( X \) is Kupka-Smale.
2. For any isolated transitive set \( \Lambda \) with isolated neighborhood \( U \), if \( \Lambda \) contains a periodic orbit \( P \), then \( \Lambda \) equals to the relative homoclinic class \( H(P, U) \).
3. Let \( \Lambda \) be a compact invariant set of \( X \). If there is a sequence \( X_n \rightarrow X \) and periodic orbits \( P_n \) of \( X_n \) of index \( i \) such that \( \lim_{n \rightarrow \infty} P_n = \Lambda \) in the Hausdorff topology, then there exists a sequence of periodic orbits \( Q^X_n \) of index \( i \) of \( X \) itself such that \( \lim_{n \rightarrow \infty} Q^X_n = \Lambda \) in the Hausdorff topology.
4. For two open sets \( U, V \) satisfying \( \overline{U} \subset V \), if there are two hyperbolic periodic orbits \( P_1, P_2 \subset U \) with \( \text{Ind}(P_1) < \text{Ind}(P_2) \) such that for the relative homoclinic classes, one has \( H(P_1, U) = H(P_2, U) \), then for any \( i \in [\text{Ind}(P_1), \text{Ind}(P_2)] \), there is a hyperbolic periodic orbit \( P \) of index \( i \) such that \( P_1, P_2 \subset H(P, V) \).
5. Every chain transitive set of \( X \) is a periodic limit.
6. An isolated chain transitive set $\Lambda$ of $X$ is robustly chain transitive.

Item 1 is from the classical Kupka-Smale theorem. Item 2 comes from a standard application of connecting lemma, one can see [13] for a proof. Item 3 is from the fact that hyperbolic periodic orbits are persistent and a standard generic argument, one can see [31] for a proof. Item 4 is a local version of the main result of [1]. Item 5 is one of the main results of [8]. Item 6 is a localized version of [3, Corollaire 1.13] Saddle value and Shilnikov bifurcation

Let $\sigma$ be a hyperbolic singularity of $X$. Assume that the eigenvalues of $DX(\sigma)$ can be arranged by the following:

$$Re(\lambda_m) < \cdots < Re(\lambda_1) < 0 < Re(\eta_1) < \cdots < Re(\eta_n).$$

Denote by $I(\sigma) = I(\sigma, X) = Re(\lambda_1) + Re(\eta_1)$. $I(\sigma)$ is said to be the saddle value as in [28,34]. From the Shilnikov bifurcation theory, we have the following proposition.

**Proposition 2.11** ([28]) Let $X \in \mathcal{X}^1(M)$ and $\sigma$ be a singularity of $X$ with saddle value $I(\sigma) < 0$. If there is a homoclinic orbit $\Gamma$ of $\sigma$, then there exists an arbitrary small perturbation $Y$ of $X$ such that $Y$ has a periodic orbit of index $\text{Ind}(\sigma)$, which is close to the homoclinic orbit $\Gamma$.

One can change the sign of the saddle value under some conditions by small perturbations.

**Lemma 2.12** Given a $C^1$ vector field $X$, for any neighborhood $U$ of $X$, there exist a neighborhood $V$ of $X$ and $\delta > 0$ such that for any hyperbolic singularity $\sigma$ of $Z \in V$ with $|I(\sigma, Z)| < \delta$ and any neighborhood $U \subset M$ of $\sigma$, there is $Y \in U$ such that $I(\sigma_Y, Y) < 0$ and $Z(x) = Y(x)$ for any $x \in M \setminus U$.

**Proof** Let $b : [0, +\infty) \to [0, 1]$ be the bump function with $b(x) = 1$ for $x \in [0, 1/3]$, $b(x) = 0$ for $x \geq 1$ and $0 \leq b'(x) < 4$ for all $x \in [0, +\infty)$.

Given a neighborhood $U$ of $X$, there are a neighborhood $V$ of $X$ and $\epsilon > 0$ such that for any $Z \in V$, any vector field $Y$ which is $\epsilon$ $C^1$-close to $Z$ is contained in $U$. Take $\delta = \epsilon/10$.

Now we consider a vector field $Z \in V$ and a hyperbolic singularity $\sigma$ with $I(\sigma, Z) \in [0, \delta)$, one can have $Y \in U$ such that $I(\sigma_Y, Y) < 0$. Now we give the construction of $Y$. In a local chart of $\sigma$, for which we identity $\sigma = 0 \in \mathbb{R}^d$, for $a \in (-\epsilon/5, 0)$, for any arbitrarily small $r > 0$, the expression of $Y$ is

$$Y(y) = Z(y) + a \cdot b\left(\frac{\|y\|}{r}\right)y, \forall y \in B_r(\sigma).$$

We have the following facts:

- $I(\sigma_Y, Y) = I(\sigma, Z) + 2a$. Thus, there is some $a$ such that $\sigma_Y$ and $\sigma$ have the same index, and $I(\sigma_Y, Y) < 0$.
- The perturbation can be chosen in arbitrarily small neighborhood of $\sigma$ since $r$ can be chosen arbitrarily small independent of $a$ and $\epsilon$.

The proof is now complete.

3 The Periodic Orbits in Transitive Sets

In this section we will discuss the dominated splitting along periodic orbits on normal bundle, then extend it to $\Lambda$. Let $X \in \mathcal{X}^1(M)$, a sequence $(P_n, Y_n)$ is called an $i$-fundamental sequence
of $X$ if $Y_n$ converges $X$ in $C^1$ topology, the periodic orbit $P_n$ of $Y_n$ have index $i$ and converge in the Hausdorff metric. The Hausdorff limit $\Gamma$ of $P_n$ is called an $i$-periodic limit of $X$. Note that every $i$-periodic limit is a compact invariant set of $X$.

**Lemma 3.1** Assume that $\Lambda$ is a non-trivial transitive set. If $\Lambda$ contains a hyperbolic singularity $\sigma$ with saddle value $I(\sigma) \leq 0$, then $\Lambda$ contains an $\text{Ind}(\sigma)$-periodic limit. Symmetrically, if $\Lambda$ contains a hyperbolic singularity $\sigma$ with saddle value $I(\sigma) \geq 0$, then $\Lambda$ contains an $(\text{Ind}(\sigma) - 1)$-periodic limit.

**Proof** We only prove the lemma in the case of $I(\sigma) \leq 0$ since the case of $I(\sigma) \geq 0$ can be treated as the case that $I(\sigma) \leq 0$ after we reverse the vector field. Since $\Lambda$ is non-trivial and transitive, by Lemma 2.4, we can find $y_1 \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda$ and $y_2 \in (W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda$.

Given any neighborhood $U \subset X^1(M)$ of $X$ and any neighborhood $U \subset M$ of $\Lambda$, we will see that there exists $Y \in U$ such that $Y$ has a homoclinic orbit associated to $\sigma$ in $U$. This is a standard application of connecting lemma. Let $U$ and $B$ be given neighborhoods. Then we can choose $L_1, T_1, \delta_{0,1}$ and $L_2, T_2, \delta_{0,2}$ be the triples of constants given in Lemma 2.1 that $\text{Hausdorff limit}$ of $\{\{X_n\}\}$ is transitive, there exists $x \in \Lambda$ such that $\text{Hausdorff limit}$ of $\{X_n\}$ converges to $y_1$ and $y_2$. Then we can take $\delta_1 < \delta_{0,1}$ small enough such that the tube $T_1 = \bigcup_{t \in [0, L_1]} B(\varphi^X_t(y_1), \delta_1) \subset U$ and the positive orbit of $y_1$ does not touch $\Delta_1$ once it leaves $\Delta_1$. Similarly, we can take $\delta_2 < \delta_{0,2}$ small enough such that the tube $T_2 = \bigcup_{t \in [0, L_2]} B(\varphi^X_t(y_2), \delta_2) \subset U$ and the negative orbit of $y_2$ does not touch $\Delta_2$ once it leaves $\Delta_2$. Without loss of generality, we can assume that $y_1, y_2$ in different orbits and $\Delta_1 \cap \Delta_2 = \emptyset$. Since $\Lambda$ is transitive, there exists $x \in \Lambda$ such that $\Delta = \omega(x)$. So we can find orbit segment $\varphi_{[t_1, t_2]}(x)$ in $\Lambda$ such that $\varphi_{t_1}(x) \in B_{L_2/L_1}(y_2)$ and $\varphi_{t_2}(x) \in B_{L_1/L_1}(y_1)$. Let $t_1 < t_0 < t_2$ be chosen such that $\varphi_{t_0}(x) \notin (\Delta_1 \cup \Delta_2)$. Then we can apply the connecting lemma in $\Delta_1$ and $\Delta_2$ such that there is $Y \in U$ such that $\varphi^X_{t_0}(x)$ is in the positive orbit of $y_2$ and the negative orbit of $y_1$ with respect to the flow $\varphi^Y_t$. Now we get a homoclinic orbit in $U$ associated to $\sigma$.

With an additional perturbation if necessary, we can assume that $I(\sigma) < 0$ with respect to $Y$. Then by Proposition 2.11, we can find a vector field $Z$ arbitrarily close to $Y$ such that $Z$ has a periodic orbit of index $\text{Ind}(\sigma)$ in $U$. So for any neighborhood $U$ of $X$ and any neighborhood $U$ of $\Lambda$, we can find $Z \in U$ with a periodic orbit of index $\text{Ind}(\sigma)$ in $U$. Hence we can find a sequence $Z_n \to X$ with periodic orbit $P_n$ of index $\text{Ind}(\sigma)$ such that the Hausdorff limit of $\{P_n\}$ is contained in $\Lambda$. This ends the proof of the lemma.

**Lemma 3.2** Assume that $X$ is away from homoclinic tangencies. If $\Lambda$ is an $i$-periodic limit, then

- $\mathcal{N}_{\Lambda \setminus \text{Sing}(X)}$ admits a dominated splitting of index $i$ with respect to $\psi_t$.
- $\Lambda$ admits a dominated splitting of index $i$ with respect to $\tilde{\psi}_t$.

**Proof** We assume that $X$ is far away from homoclinic tangencies. Denote by

$$B^i(\Lambda) = \{e \in SM : \rho(e) \in \Lambda, \exists X_n \to X, \text{ and } p_n \in \text{Per}(X_n),$$

$$\text{Ind}(p_n) = i, \text{ Orbx}_n(p_n) \subset U, \text{ such that } \frac{X_n(p_n)}{\|X_n(p_n)\|} \to e\}.$$ 

By the assumption that $\Lambda$ is an $i$-periodic limit, then the set

$$\{e = \frac{X(x)}{\|X(x)\|} : x \in \Lambda \setminus \text{Sing}(X)\} \subset B^i(\Lambda).$$

This implies $\tilde{\Lambda} \subset B^i(\Lambda)$. Let $e \in B^i(\Lambda)$ and $\{p_n\}$ be a sequence of hyperbolic periodic point of $X_n$ of index $i$ such that $X_n \to X, X_n(p_n)/\|X_n(p_n)\| \to e$ as $n \to \infty$. Denoted
by $P_n$ the orbit of $p_n$ of $X_n$. Then from Proposition~2.8, we know that for $n$ large, the individual hyperbolic splittings $N_{P_n} = N^s(P_n) \oplus N^u(p_n)$ of the usual linear Poincaré flow $\psi_t$ on $P_n$, put together, are $l$-dominated splitting. Now take limits $N^s(e) = \lim_{n \to \infty} N^s(p_n)$ and $N^u(e) = \lim_{n \to \infty} N^u(p_n)$ (here we take a subsequence such that the limits exist if necessary). This gives a splitting $N_e = N^s(e) \oplus N^u(e)$ at every $e \in B^l(\Lambda)$ with the property

$$\frac{\|\tilde{\psi}_t|_{N^s(e)}\|}{m(\psi_t|_{N^s(e)})} \leq 1/2$$

$$\frac{\|\tilde{\psi}_{-t}|_{N^u(e)}\|}{m(\psi_t|_{N^u(e)})} \leq 1/2$$

for all $t \geq 1$ where $m(A)$ denotes the mininorm of a linear map $A$. By the uniqueness of dominated splitting we know that $N_e = N^s(e) \oplus N^u(e)$ is uniquely determined and is $\tilde{\psi}_t$-variant hence we get a dominated splitting

$$N_{B^l(\Lambda)} = N^s \oplus N^u$$

of index $i$ on $B^l(\Lambda)$ with respect to $\tilde{\psi}_t$. In particular, $\tilde{\Lambda}$ have a dominated splitting of index $i$ with respect to $\tilde{\psi}_t$ and then $\Lambda \setminus \text{Sing}(X)$ have a dominated splitting of index $i$ with respect to $\psi_t$. \hfill \Box

**Lemma 3.3** There is a residual set $\mathcal{R} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{R} \setminus \mathcal{T\mathcal{T}}$ and any isolated transitive set $\Lambda$ of $X$, if $\Lambda$ contains an $i$-periodic limit, then there are a $C^1$ neighborhood $\mathcal{U}$ of $X$, a neighborhood $U$ of $\Lambda$ and a constant $l > 0$ such that for any periodic orbit $P \subset U$ of $Y \in \mathcal{U}$, $P$ admits an $l$-dominated splitting of index $i$ in the normal bundle w.r.t. the linear Poincaré flow.

**Proof** Let $\mathcal{R}$ be the residual set chosen in Proposition~2.10. Let $X \in \mathcal{R} \setminus \mathcal{T\mathcal{T}}$ and $\Lambda$ be an isolated transitive set of $X$. Note here that for a locally maximal invariant set of a $C^1$-generic vector field, transitivity is equivalent to chain transitivity by Item 3 and Item 5 of Proposition 2.10. Thus $\Lambda$ is robustly chain transitive by Item 6 of Proposition 2.10. This is a consequence of the connecting lemma of pseudo orbits by Bonatti and Crovisier [3].

Let $\mathcal{U}_1 \subset \mathcal{X}^1(M)$ and $U \subset M$ be the neighborhood of $X$ and $\Lambda$ such that for any $Y \in \mathcal{U}_1$, the maximal invariant set $\Lambda_Y$ in $U$ is chain transitive. Let $\mathcal{U}_2$ be the neighborhood of $X$ and constant $l$ be get from Proposition 2.8 such that for any $Y \in \mathcal{U}_2$ and any periodic orbit $P$ of $Y$, the hyperbolic splitting $N_Q = N^s(Q) \oplus N^u(Q)$ along $Q$ is an $l$-dominated splitting. Since $X$ contains an $i$-periodic limit we know that $X$ contains a hyperbolic periodic orbit $Q_X$ of index $i$ from Item 3 of Proposition 2.10. Then we can take a neighborhood $\mathcal{U}_3$ of $X$ such that for any $Y \in \mathcal{U}_3$, there is a hyperbolic periodic orbit $Q_Y$ (close to $Q_X$) of index $i$ contained in $U$. We will prove that $U = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$, $U$ and $l$ satisfy the request of the lemma.

Let $Y \in \mathcal{U}$ and $P$ be a periodic orbit of $Y$ contained in $U$. Since $X$ is far away from homoclinic tangencies, by Lemma 3.2, we just need to prove that $P$ is contained in an $i$-periodic limit of $Y$, then we will know that $P$ admits an $l$-dominated splitting of index $i$ in the normal bundle w.r.t. the linear Poincaré flow although the index of $P$ may not be $i$. After a perturbation, we can assume $P$ is a hyperbolic periodic orbit for $Y_1$ where $Y_1$ is arbitrarily close to $Y$. After another arbitrarily small perturbation, we can find $Z \in \mathcal{R}$ arbitrarily close to $Y_1$. By Item 2 of Proposition 2.10 we know that $P_Z \subset \Lambda_Z = H(Q_Z, U)$ where $P_Z$. $Q_Z$ is the continuation of $P$, $Q_X$ w.r.t. $Z$, then we know that for any $\varepsilon > 0$, there exists a periodic orbit $Q'$ of $Z$ of index $i$ such that $P_Z \subset B(Q', \varepsilon)$. Hence we can find a sequence of $Z_n$ with periodic orbits $Q_n'$ of $Z_n$ such that $Z_n \to Y$ and $P$ is contained in the Hausdorff limit of $Q'_n$. This proves that $P$ is contained in an $i$-periodic limit of $Y$. This ends the proof of the lemma. \hfill \Box
4 The Singularities in Robustly Transitive Sets

In this section, we will focus on the singularities in an isolated transitive set with dominated splittings for the linear Poincaré flow. At the end of this section, we prove Theorem A.

Lemma 4.1 Let $\Lambda$ be a non-trivial transitive set. Assume that $\Lambda$ contains a hyperbolic singularity $\sigma$ with hyperbolic splitting $E^s_\sigma + E^u_\sigma$, and

- $\Lambda \setminus \text{Sing}(X)$ admits a dominated splitting of index $i$ in the normal bundle w.r.t. the linear Poincaré flow $\psi^X_t$;
- $\text{Ind}(\sigma) > i$.

Then $T_\sigma M$ admits a dominated splitting $T_\sigma M = E^{ss}_\sigma + E^{cs}_\sigma + E^u_\sigma$ w.r.t. the tangent flow $\Phi^X_t$, where $E^{ss}_\sigma$ is strongly contracting and $\dim E^{ss}_\sigma = i$.

Proof Assume $\Lambda \setminus \text{Sing}(X)$ admits a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)}M = N^{cs} + N^{cu}$ with $\dim(N^{cs}) = i$ w.r.t. the linear Poincaré flow $\psi^X_t$. Then we can extend the dominated splitting to be $\tilde{\Lambda} = N^{cs} (\tilde{\Lambda}) \oplus N^{cu} (\tilde{\Lambda})$ w.r.t. the extended linear Poincaré flow $\tilde{\psi}_t$ as in Proposition 2.7.

By Lemma 2.5 we know that there exists an element $e \in \tilde{\Lambda} \cap E^u_\sigma$. Denote by $\pi_e$ the orthogonal projection from $T_\sigma M$ to $N_e$. Let $E^{ss}_\sigma = \pi_e^{-1}(N^{cs}(e)) \cap E^s_\sigma$ and $E^{cs}_\sigma = \pi_e^{-1}(N^{cu}(e)) \cap E^s_\sigma$. We are going to prove that $E^{ss}_\sigma$ and $E^{cs}_\sigma$ are independent of the choice of $e$.

Since $e \in E^u_\sigma$, we know that $\pi_e(E^u_\sigma)$ is isomorphic to $E^u_\sigma$, then $E^{ss}_\sigma \oplus E^{cs}_\sigma = E^s_\sigma$. Since $\pi_e(E^{ss}_\sigma) = N^{cs}(e)$, we have that $\dim(E^{ss}_\sigma) = i$. Note that $\Phi_t(\pi_e^{-1}(N^{cs}(e)) = \pi_{\Phi_t(e)}^{-1}(N^{cs}(\Phi^t_e(e)))$. Hence $\Phi_t(E^{ss}_\sigma) = \pi_{\Phi_t(e)}^{-1}(N^{cs}(\Phi^t_e(e))) \oplus E^{ss}_\sigma$ by the fact that $E^s_\sigma$ is invariant for $\Phi_t$. Similarly, we have $\Phi_t(E^{cs}_\sigma) = \pi_{\Phi_t(e)}^{-1}(N^{cu}(\Phi^t_e(e))) \oplus E^{cs}_\sigma$.

Now we proceed to prove that $E^{ss}_\sigma \oplus E^{cs}_\sigma$ is a dominated splitting with respect to tangent flow $\Phi_t$. Since $N^{cs} + N^{cu}$ is a dominated splitting w.r.t. the extended linear Poincaré flow $\tilde{\psi}_t$, there exist $C, \lambda > 0$ such that for any unit vectors $u \in N^{cs}(e)$ and $v \in N^{cu}(e)$,

$$\frac{|\tilde{\psi}_t(u)|}{|\tilde{\psi}_t(v)|} < Ce^{-\lambda t}$$

for any $t > 0$. Note that $\Phi^t_{\Phi_t(e)} \in E^u_\sigma$ for any $t \in \mathbb{R}$, hence the angle between $\Phi^t_{\Phi_t(e)}$ and $E^u_\sigma$ has a positive lower bound. Thus there is a constant $K > 1$ depending on the angle between $E^s_\sigma$ and $E^u_\sigma$ such that the orthogonal projection $\pi_{\Phi^t_{\Phi_t(e)}} : E^s_\sigma \to N_{\Phi^t_{\Phi_t(e)}}$ has norm $\|\pi_{\Phi^t_{\Phi_t(e)}}^{-1}\| \leq K$ for all $t \in \mathbb{R}$. Then for any unit vectors $u_{ss} \in E^{ss}_\sigma$ and $u_{cs} \in E^{cs}_\sigma$, we have

$$\frac{|\Phi_t(u_{ss})|}{|\Phi_t(u_{cs})|} = \frac{|\pi_{\Phi^t_{\Phi_t(e)}}^{-1}(\psi_t(\pi_e(u_{ss})))|}{|\pi_{\Phi^t_{\Phi_t(e)}}^{-1}(\psi_t(\pi_e(u_{cs})))|} \leq K \frac{|\psi_t(\pi_e(u_{ss}))|}{|\psi_t(\pi_e(u_{cs}))|}$$

$$< KCe^{-\lambda t}\frac{|\pi_e(u_{ss})|}{|\pi_e(u_{cs})|} \leq KCe^{-\lambda t}\frac{1}{|\pi_e(u_{cs})|} < K^2Ce^{-\lambda t} \tag{1}$$

for all $t > 0$. Similarly, we have

$$\frac{|\Phi_{-t}(u_{ss})|}{|\Phi_{-t}(u_{ss})|} < K^2Ce^{-\lambda t} \tag{2}$$

for all $t > 0$. Hence $E^s_\sigma = E^{ss}_\sigma \oplus E^{cs}_\sigma$ is an invariant splitting and then is a dominated splitting from (1) and (2). From the uniqueness of dominated splitting, we can see that the splitting is independent of the choice of $e$. This ends the proof of Lemma 4.1. □
In the following lemmas we choose $\mathcal{R}$ to be the residual set given in Proposition 2.10 and Lemma 3.3. As a direct consequence of Lemma 4.1, we have the following lemma.

**Lemma 4.2** Let $X \in (\chi^1(M) \setminus \overline{HT}) \cap \mathcal{R}$ and $\Lambda$ be an isolated transitive set of $X$. Assume that $\Lambda$ contains an $i$-periodic limit. If $\sigma$ is a hyperbolic singularity in $\Lambda$, then we have the following cases:

- when $\text{Ind}(\sigma) > i$, there is a dominated splitting $T_{\sigma} M = E_{\sigma}^{ss} \oplus E_{\sigma}^{cs} \oplus E_{\sigma}^{uu}$ for the tangent flow with $\dim E_{\sigma}^{ss} = i$ and $E_{\sigma}^{uu}$ is the unstable space of $\sigma$.
- when $\text{Ind}(\sigma) \leq i$, there is a dominated splitting $T_{\sigma} M = E_{\sigma}^{s} \oplus E_{\sigma}^{cu} \oplus E_{\sigma}^{uu}$ for the tangent flow with $\dim E_{\sigma}^{uu} = \dim M - 1 - i$ and $E_{\sigma}^{s}$ is the stable space of $\sigma$.

**Proof** Since $\Lambda$ contains an $i$-periodic limit, from Item 3 of Proposition 2.10 we know that $\Lambda$ contains a periodic orbit $P$ of index $i$ and then $\Lambda$ is the relative homoclinic class of $P$ by the Item 2 of Proposition 2.10. Hence $\Lambda$ is an $i$-periodic limit, then by Lemma 3.2 we know that there is a dominated splitting of index $i$ on $\Lambda \setminus \text{Sing}(X)$ with respect to the linear Poincaré flow. From Lemma 4.1, for any given hyperbolic singularity $\sigma \in \Lambda$ with $\text{Ind}(\sigma) > i$, we know that there is a dominated splitting $E_{\sigma}^{ss} \oplus E_{\sigma}^{cs} \oplus E_{\sigma}^{uu}$ with respect to the tangent flow $\Phi_{t}$ with $\dim E_{\sigma}^{ss} = i$ and $E_{\sigma}^{ss} \oplus E_{\sigma}^{cs}$ is the stable space of $\sigma$. Symmetrically, for any hyperbolic singularity $\sigma \in \Lambda$ with $\text{Ind}(\sigma) \leq i$, there is a dominated splitting $E_{\sigma}^{s} \oplus E_{\sigma}^{cu} \oplus E_{\sigma}^{uu}$ with respect to tangent flow $\Phi_{t}$ with $\dim E_{\sigma}^{uu} = \dim M - 1 - i$. □

**Lemma 4.3** Let $X \in \chi^1(M)$. Assume that $\Lambda$ is a non-trivial transitive set and contains a singularity $\sigma$. Suppose that we have the following properties:

- $T_{\sigma} M = E_{\sigma}^{ss} \oplus E_{\sigma}^{c} \oplus E_{\sigma}^{uu}$ is a partially hyperbolic splitting on $\sigma$ w.r.t. the tangent flow.
- There are $i \in \{\dim E_{\sigma}^{ss}, \dim E_{\sigma}^{cs} + \dim E_{\sigma}^{cu} - 1\}, l > 0$, and a $C^{l}$-neighborhood $U$ of $X$ such that every periodic orbit of $Y$ close to $\Lambda$ admits an $l$-dominated splitting of index $i$ w.r.t. the linear Poincaré flow $\psi^Y_t$.
- For the corresponding strong unstable manifold $W_{uu}(\sigma)$, one has $W_{uu}(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$.

Then $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.

**Proof** The proof is very similar to the proof of [16, Lemma 4.3]. For the completeness, we give the idea here. Without loss of generality, we assume that $E_{\sigma}^{ss}, E_{\sigma}^{c}, E_{\sigma}^{uu}$ are mutually orthogonal. Suppose on the contrary that $(W^{ss}(\sigma) \cap \Lambda) \setminus \{\sigma\} \neq \emptyset$. By using the Connecting Lemma (Lemma 2.1), we can take a vector field $Y$ which is arbitrarily close to $X$ such that $Y$ admits a strong homoclinic connection $\Gamma \subset W^{ss}(\sigma) \cap W^{uu}(\sigma)$. By another small perturbation, we can assume that the dynamics near $\sigma$ is linear, then we have, the local strong stable manifold $W^{ss}(\sigma) = \exp(E_{\sigma}^{ss}(\delta))$ and the local strong unstable manifold $W^{uu}(\sigma) = \exp(E_{\sigma}^{uu}(\delta))$ for some small $\delta$, where $E_{\sigma}^{ss}(\delta)$ denote the disc in $E_{\sigma}^{i}$ with radius $\delta$ for $i = ss, uu$. Then we can get a vector field $Z$ arbitrarily close to $Y$ with the following properties: (a). $Z = Y$ in a neighborhood of $\sigma$, (b). $Z$ has a periodic orbit $P$ arbitrarily close to $\Gamma$ and $P$ has a segment located in $\exp(E_{\sigma}^{ss} \oplus E_{\sigma}^{uu})(\delta)$ for arbitrarily small $\delta$ (See the proof of [16, Lemma 4.3] for the details of the perturbations).

Let $X_n \to X$ be a sequence of the perturbations given by the above way and $P_n$ be the periodic orbit of $X_n$. Let $\Gamma$ be the Hausdorff limit of $P_n$. From the second assumption of the Lemma, we have an $l$-dominated splitting along the periodic orbit $P_n$ with respect to the linear Poincaré flow of index $i$. Hence there is a dominated splitting of index $i$ on $\Gamma \setminus \text{Sing}(X)$ with respect to the linear Poincaré flow and therefore there is a dominated splitting of index $i$ on $\Gamma$ for the extended linear Poincaré flow. Denote by $N_{\Gamma} = N_{\Gamma}^{cs} \oplus N_{\Gamma}^{cu}$ the dominated splitting of index $i$ with respect to the extended linear Poincaré flow which generated by the
l-dominated splitting along the periodic orbits $P_n$. By the assumptions of the Lemma, we know that $\dim(N^{cs}(e)) \geq \dim(E^{ss})$ and $\dim(N^{cu}(e)) \geq \dim(E^{uu})$ for all $e \in \tilde{K}$.

**Claim** We have that

1. $E^{uu}_\sigma \subset N^{cu}(e)$ for any $e \in \tilde{K} \cap E^{ss}$;
2. $E^{ss}_\sigma \subset N^{cs}(e)$ for any $e \in \tilde{K} \cap E^{uu}$.

**The proof of Claim** We just prove item 1. The item 2 can be proved similarly. For any $e \in \tilde{K} \cap E^{ss}$, we have two splitting $N_e = N^{cs}(e) \oplus N^{cu}(e)$ and $N_e = (E^{ss}_\sigma \oplus E^{cu}_\sigma) \cap N_e \ominus E^{uu}_\sigma$. Denote by $\Delta^{cs}_e = (E^{ss}_\sigma \oplus E^{cu}_\sigma) \cap N_e$ and $\Delta^{cu}_e = E^{uu}_\sigma$. This gives a $\tilde{\psi}_t$-invariant splitting $\tilde{N} \cap E^{ss}_\sigma$ (here we use the assumption $E^{ss}_\sigma$ is orthogonal to $E^{uu}_\sigma$). We will see that $N^{ss}_\sigma \cap E^{ss}_\sigma = \Delta^{cs}_e \oplus \Delta^{cu}_e$ is a dominated splitting with respect to the extended linear Poincaré flow $\tilde{\psi}_t = \tilde{\psi}^X_t$. Since $(E^{ss}_\sigma \oplus E^{cu}_\sigma) \oplus E^{uu}_\sigma$ is a dominated splitting for $\Phi_t$, there exist $C \geq 1$ and $\lambda > 0$ such that for any unit vectors $u \in (E^{ss}_\sigma \oplus E^{cu}_\sigma)$ and $v \in E^{uu}_\sigma$, one has

$$\frac{|\Phi_t(u)|}{|\Phi_t(v)|} < Ce^{-\lambda t}$$

for all $t > 0$. For any $e \in \tilde{K} \cap E^{ss}$ and any unit vectors $u \in \Delta^{cs}_e$ and $v \in \Delta^{cu}_e$, we have

$$\frac{|\tilde{\psi}_t(u)|}{|\tilde{\psi}_t(v)|} = \frac{|\tilde{\psi}_t(u)|}{|\tilde{\psi}_t(v)|} \leq \frac{|\Phi_t(u)|}{|\Phi_t(v)|} < Ce^{-\lambda t}$$

for all $t > 0$. This proves that $\Delta^{cs}_e \oplus \Delta^{cu}_e$ is a dominated splitting for $\tilde{\psi}_t$. By the uniqueness of dominated splitting and the fact that $\dim(N^{cu}_e) \geq \dim(\Delta^{cu}_e)$ we have $E^{uu}_\sigma = \Delta^{cu}_e \subset N^{cu}(e)$ for any $e \in \tilde{K} \cap E^{ss}$. This ends the proof of Claim. \hfill $\square$

Now let us continue the proof of Lemma 4.3. For every $P_n$, one can choose a point $p_n \in P_n \cap \exp_e(E^{ss}_\sigma \oplus E^{uu}_\sigma)$ such that $X_{e_n}(p_n) = v^{ss}_n + v^{uu}_n$ where $v^{ss}_n \in E^{ss}_\sigma$, $v^{uu}_n \in E^{uu}_\sigma$ and $|v^{ss}_n| = |v^{uu}_n|$. By choosing subsequences, we can assume that $X_{e_n}(p_n) \|X_{e_n}(p_n)\| \to 0$ accumulate to some point $e \in \tilde{K}$. By the choice of $p_n$ we know that $e \in E^{ss}_\sigma \oplus E^{uu}_\sigma$ and $e$ has a splitting $e = v^{ss}_n + v^{uu}_n$ such that $v^{ss}_n \in E^{ss}_\sigma$, $v^{uu}_n \in E^{uu}_\sigma$ and $|v^{ss}_n| = |v^{uu}_n|$. Denote by $e_t = \Phi^t(e)$. One has that $e_t \to E^{uu}_\sigma$ as $t \to +\infty$ and $e_t \to E^{ss}_\sigma$ as $t \to -\infty$. Let $u = v^{ss} - v^{uu} \in N_e$ and $u_t = \tilde{\psi}_t^X(u)$. A direct computation shows that

$$u_t = \Phi_t(u) - \langle \Phi_t(u), e_t \rangle e_t = \Phi_t(v^{ss} - v^{uu}) - \frac{\|\Phi_t(v^{ss})\|^2 - \|\Phi_t(v^{uu})\|^2}{\|\Phi_t(v^{ss})\|^2 + \|\Phi_t(v^{uu})\|^2} \Phi_t(v^{ss} + v^{uu})$$

$$= 2 \frac{\|\Phi_t(v^{ss})\|^2 \Phi_t(v^{ss}) - \|\Phi_t(v^{ss})\|^2 \Phi_t(v^{uu})}{\|\Phi_t(v^{ss})\|^2 + \|\Phi_t(v^{uu})\|^2} := u^{ss}_t + u^{uu}_t$$

where $u^{ss}_t \in E^{ss}_\sigma$ and $u^{uu}_t \in E^{uu}_\sigma$. By the domination, $\|u^{uu}_t\|/\|u^{ss}_t\| \to 0$ as $t \to +\infty$, hence $u_t/\|u_t\| \to E^{ss}_\sigma$ as $t \to +\infty$. Similarly we have $u_t/\|u_t\| \to E^{uu}_\sigma$ as $t \to -\infty$.

Let $N_e = N^{cs}(e) \oplus N^{cu}(e)$ be the dominated splitting of index $i$ generated by the dominated splitting along $P_n$. Then we have two cases as following.

**Case 1:** $u \in N^{cs}(e)$. In this case we take a sequence $t_n \to -\infty$ such that $e_{t_n}$ converges to some $e'$ and $u_{t_n}/\|u_{t_n}\|$ converges to some $u'$. By the discussion in the previous paragraph, we know that $e' \in \tilde{K} \cap E^{ss}_\sigma$ and $u' \in E^{uu}_\sigma$. By the invariance and continuous of subbundle $N^{cs}$ we know $u' \in N^{cs}(e')$. On the other hand we have $u' \in E^{uu}_\sigma \subset N^{cu}(e')$ as $e' \in \tilde{K} \cap E^{ss}_\sigma$ from the claim. We get a contradiction, hence this case can not happen.
Case 2: \( u \notin N^{cs}(e) \). In this case we take a sequence \( t_n \to +\infty \) such that \( e_{t_n} \) converges to some \( e' \) and \( u_{t_n}/\|u_{t_n}\| \) converges to some \( u' \). By the discussion in the previous paragraph, we know that \( e' \in \Gamma \cap E^{uu}_\sigma \) and \( u' \in E^{ss}_\sigma \). By the fact that \( N^{cs} \oplus N^{cu} \) is a dominated splitting and \( u \notin N^{cs}(e) \) we know that \( u' \in N^{cu}(e') \). This is also a contradiction with \( E^{ss}_\sigma \subset N^{cs}(e') \) for any \( e' \in \Gamma \cap E^{uu}_\sigma \).

Hence both two cases can not happen. The assumption \( W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset \) can not be true. This ends the proof of Lemma 4.3.

Now let us consider a vector field \( X \in (\mathcal{X}^1(M) \setminus \overline{HT}) \cap \mathcal{R} \). Assume \( \Lambda \) is an isolated transitive set of \( X \) and \( \sigma \in \Lambda \) is a singularity. If \( \Lambda \) contains an \( i \)-periodic limit, then by Lemma 4.2 we know that if \( \text{Ind}(\sigma) > i \) then there is a partially splitting \( T_\sigma M = E^{ss}_\sigma \oplus E^{cs}_\sigma \oplus E^{u}_\sigma \) with \( \dim E^{ss}_\sigma = i \), if \( \text{Ind}(\sigma) \leq i \) then there is a partially hyperbolic splitting \( T_\sigma M = E^{s}_\sigma \oplus E^{cu}_\sigma \oplus E^{uu}_\sigma \) with \( \dim E^{ss}_\sigma = \dim M - 1 - i \). Denote by \( W^{ss}(\sigma) \) and \( W^{uu}(\sigma) \) the strong stable and unstable manifolds correspond to \( E^{ss}_\sigma \) and \( E^{uu}_\sigma \) in these two cases.

\begin{lemma}
There is a residual set \( \mathcal{R} \) with the following properties. Let \( X \in (\mathcal{X}^1(M) \setminus \overline{HT}) \cap \mathcal{R} \) and \( \Lambda \) be an isolated transitive set of \( X \). Assume that \( \Lambda \) contains an \( i \)-periodic limit. If \( \sigma \) is a hyperbolic singularity in \( \Lambda \), then we have the following cases:

- when \( \text{Ind}(\sigma) > i \), one has \( (W^{ss}(\sigma) \setminus \{\sigma\}) \cap \Lambda = \emptyset \).
- when \( \text{Ind}(\sigma) \leq i \), one has \( (W^{uu}(\sigma) \setminus \{\sigma\}) \cap \Lambda = \emptyset \).
\end{lemma}

\begin{proof}
This is a direct corollary of Lemma 4.3. Let \( \sigma \in \Lambda \) be a hyperbolic singularity with \( \text{Ind}(\sigma) > i \). Since \( \Lambda \) is a transitive set, from Lemma 2.4 we have \( (W^{ss}(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset \). Lemma 3.3 tells us that there is a \( C^1 \)-neighborhood \( \mathcal{U} \) of \( X \) such that every periodic orbit of \( Y \) close to \( \Lambda \) admits an \( i \)-dominated splitting of index \( i \) w.r.t. the linear Poincaré flow \( \psi^Y_t \). Note that \( i = \dim E^{ss}_\sigma \) here. Then we can get that \( (W^{ss}(\sigma) \setminus \{\sigma\}) \cap \Lambda = \emptyset \) directly by Lemma 4.3. Similarly we can see that if \( \text{Ind}(\sigma) \leq i \), then \( (W^{ss}(\sigma) \setminus \{\sigma\}) \cap \Lambda = \emptyset \).
\end{proof}

\begin{lemma}
There is a residual set \( \mathcal{R} \) with the following properties. Let \( X \in (\mathcal{X}^1(M) \setminus \overline{HT}) \cap \mathcal{R} \) and \( \Lambda \) be an isolated transitive set of \( X \). Assume that \( \Lambda \) contains an \( i \)-periodic limit. If \( \sigma \) is a hyperbolic singularity in \( \Lambda \), then we have the following cases:

- when \( \text{Ind}(\sigma) > i \), one has \( B(\Lambda) \cap T_\sigma M \subset E^{cs}_\sigma \oplus E^{cu}_\sigma \).
- when \( \text{Ind}(\sigma) \leq i \), one has \( B(\Lambda) \cap T_\sigma M \subset E^{ss}_\sigma \oplus E^{cu}_\sigma \).
\end{lemma}

\begin{proof}
The proof is similar to the proof of \([16, \text{Lemma 4.4}]\). We give the sketch of proof here for completeness. Let \( \sigma \in \Lambda \) be a hyperbolic singularity with \( \text{Ind}(\sigma) > i \). Once we have \( e \in B(\Lambda) \setminus (E^{cs}_\sigma \oplus E^{cu}_\sigma) \), then one can find an accumulated point of \( \Phi_t^X(e) \) in \( E^{ss}_\sigma \), hence in \( B(\Lambda) \cap E^{ss}_\sigma \). To prove \( B(\Lambda) \cap T_\sigma M \subset (E^{cs}_\sigma \oplus E^{cu}_\sigma) \), one just need to show that \( B(\Lambda) \cap E^{ss}_\sigma = \emptyset \). Now we assume on the contrary, that is, there exists \( e \in B(\Lambda) \cap E^{ss}_\sigma \). This means there are a sequence \( X_n \to X \) and periodic points \( p_n \) of \( X_n \) whose orbit contain in a neighborhood \( U \) (the isolation neighborhood of \( \Lambda \)) such that \( X_n(p_n)/\|X_n(p_n)\| \to e \) as \( n \to \infty \).

We can get a contradiction by the following way. Firstly, we can continuously extend the splitting \( E^{ss}_\sigma \oplus (E^{cs}_\sigma \oplus E^{cu}_\sigma) \) to a neighborhood of \( \sigma \), and then put a cone field \( C^{cu} \) in a neighborhood of \( \sigma \) by

\[
C^{cu}_\sigma(x) = \{ v = v^{ss} + v^{cu} \in T_x M : v^{ss} \in E^{ss}_\sigma, v^{cu} \in E^{cs}_\sigma \oplus E^{cu}_\sigma, \|v^{ss}\| < \|v^{cu}\| \}
\]

at every \( x \) close to \( \sigma \). Note that \( E^{ss}_\sigma \oplus (E^{cs}_\sigma \oplus E^{cu}_\sigma) \) is a dominated splitting for \( \Phi_t \), and the flow \( \Phi^X_t \) generated by \( Y \) will be close to \( \Phi_t^X \) in a small neighborhood of \( \sigma \) once \( Y \) close to \( X \) enough in some local chart. Then we can take a neighborhood \( \mathcal{U} \) of \( X \) and a neighborhood \( V(\sigma) \) of...
\(\sigma\) and \(T > 0\) such that for any \(Y \in \mathcal{U}\), once there is an orbit segment \(\varphi_{[0,T]}^Y(x)\) of \(Y\) contained in \(V(\sigma)\) with \(t > T\), then \(\Phi^Y_t(C^{cu}(x)) \subset C^{cu}(\varphi^Y_t(x))\) where \(\Phi^Y_t\) denote the flow and tangent flow generated by \(Y\) as usual. Since \(p_n \to \sigma\) and \(\sigma\) is a hyperbolic fixed point, if necessary, take a subsequence of \(p_n\), one can find a sequence \(q_n = \varphi_{t_n}(p_n)\) with \(t_n \to -\infty\) such that \(q_n\) converges to some point \(q\) in \(W^s(\sigma)\) as \(n \to \infty\). By applying Lemma 4.4 we have that \((W^{ss}(\sigma) \setminus \{\sigma\}) \cap \Lambda = \emptyset\), then \(q \notin W^{ss}(\sigma)\). Thus \(X(q)\) will accumulate in \(E^{cs}\) as \(t \to +\infty\). Now we can choose \(T_1 > 0\) big enough such that \(X(q)\) in \(C^{cu}_{E}(q)\). Then we have \(X_n(\varphi^{X_n}_{T_1}(q)) \in C^{cu}_{E}(q)\) for \(n\) large enough. By the fact that \(t_n \to -\infty\), we have \(-t_n > T_1 + T\) for \(n\) large enough. Then we have

\[X_n(p_n) = X_n(\varphi^{X_n}_{-t_n}(q_n)) = \Phi^{X_n}_{-t_n-T_1}(X_n(\varphi^{X_n}_{T_1}(q_n))) \in C^{cu}(p_n)\]

for \(n\) large enough. Hence \(X_n(p_n)/\|X_n(p_n)\|\) can not converge to any vector in \(E^{ss}\). We have a contradiction. This proves that if \(\text{Ind}(\sigma) > i\), then \(B(\Lambda) \cap T_{\sigma}M \subset E^{cs}_{\sigma} \oplus E^{uu}_{\sigma}\). Similar proof holds for the case of \(\text{Ind}(\sigma) \leq i\). \(\square\)

In the following lemma, we assume that \(X\) is locally linearizable near a singularity \(\sigma\), that is, there is a small neighborhood \(V\) of \(\sigma\) and a diffeomorphism \(\alpha: V \to \mathbb{R}^n\) such that \(\alpha(\sigma) = 0\) and \(D_\alpha X(p) = A(\alpha(p))\) for every \(p \in V\) where \(A\) is a linear vector field on \(\mathbb{R}^n\).

**Lemma 4.6** Assume that \(\Lambda\) is a compact invariant set of \(X\) and there are a partially hyperbolic splitting \(T_\Lambda M = E^{ss}_{\sigma} \oplus E^c_{\sigma} \oplus E^{uu}_{\sigma}\) w.r.t. the tangent flow and a dominated splitting \(N_{\Lambda \setminus \text{Sing}(\alpha)} = N^{cs}_{\sigma} \oplus N^{cu}_{\sigma}\) w.r.t. the linear Poincaré flow with \(\dim N^{cs}_{\sigma} = \dim E^{ss}_{\sigma}\). If \(W^{ss}(\sigma) \cap \Lambda = \{\sigma\}\), then for any \(y \in W^{uu}(\sigma) \setminus \{\sigma\} \cap \Lambda\) and any sequence \(y_n \in (\Lambda \setminus \{y\}) \cap \exp_y(N_y)\) with \(\lim_{n \to \infty} y_n = y\), the set of accumulation points of

\[
\left\{ \begin{array}{l}
\exp_y^{-1} y_n \\
\| \exp_y^{-1} y_n \|
\end{array} \right\}
\]

is contained in \(N^{cu}_{\sigma}\).

**Proof** The idea of this lemma is from [34, Lemma 3.3]. For any \(x \in \Lambda \setminus \text{Sing}(X)\), define

\[
D(x) = \{ e \in T_x M : \exists x_n \in \Lambda, \text{s.t.,} \frac{\exp_x^{-1}(x_n)}{\| \exp_x^{-1}(x_n) \|} \to e \text{ as } n \to \infty \},
\]

\[
D_N(x) = D(x) \cap N_x.
\]

Then from [34, Lemma 3.1] we know that \(D(x)\) is \(\Phi^x_t\) invariant. From Lemma 3.2 of [34] we know that \(D_N(x)\) is \(\psi^x_t\) invariant where \(\psi^x_t\) is defined by \(\psi^x_t(v) = \psi_t(v)/\|\psi_t(v)\|\) for all unit vector \(v \in N^{ss}_{\sigma} \cap \Lambda\). The lemma here means that for any \(y \in W^{uu}(\sigma) \setminus \{\sigma\} \cap \Lambda\), we have \(D_N(y) \subset N^{cu}_{\sigma}\).

Without loss of generality, we assume that \(E^{ss}_{\sigma}, E^c_{\sigma}\) and \(E^{uu}_{\sigma}\) are mutually orthogonal. The dominated splitting \(N_{\Lambda \setminus \text{Sing}(\sigma)} = N^{cs}_{\sigma} \oplus N^{cu}_{\sigma}\) for the linear Poincaré flow \(\psi_t\) can be extended to be a dominated splitting \(N_{\Lambda} = N^{cs}_{\sigma} \oplus N^{cu}_{\sigma}\) for the extended linear Poincaré flow. For any \(y \in W^{uu}(\sigma) \setminus \{\sigma\} \cap \Lambda\) and any accumulation point \(e\) of \([X(\psi_t(y))/\|X(\psi_t(y))\| : t < 0]\), one has that \(e \in E^{uu}_{\sigma}\). From the claim in the proof of Lemma 4.3, we know that \(N^{cs}_{\sigma} = E^{ss}_{\sigma}\) by the assumption \(\dim N^{cs}_{\sigma} = \dim E^{ss}_{\sigma}\).

By the assumption of locally linearizable of \(X\) near \(\sigma\), taking the local chart \((V, \alpha)\), ignoring the chart map \(\alpha\), we can assume \(\sigma = 0 \in \mathbb{R}^n = E^{ss}_{\sigma} \oplus E^c_{\sigma} \oplus E^{uu}_{\sigma}\), \(X(x) = (A^{ss}(x^{ss}), A^c(x^c), A^{uu}(x^{uu}))\) where \(x = x^{ss} + x^c + x^{uu}\) and \(x^{ss} \in E^{ss}_{\sigma}, x^c \in E^c_{\sigma}, x^{uu} \in E^{uu}_{\sigma}\),
\[ A_{ss} = A |_{E_{ss}^s}, A^c = A |_{E_{ss}^c}, A^{uu} = A |_{E_{ss}^{uu}}. \] Without loss of generality, we assume \( \alpha(V) \) contains a box
\[ \{x^{ss} + x^c + x^{uu} | x^i \in E_i^i, \|x^i\| \leq 2 ; i = ss, c, uu \}. \]
and \( y = y^{uu} \in W_{ss}^{uu} \) with \( \|y^{uu}\| = 1 \). We can check that \( N_{y}^{uu} = N_y \cap (E_y^{c} \oplus E_y^{uu}) \) where \( E_y^{c} \) and \( E_y^{uu} \) are the natural translation of \( E_{ss}^c \) and \( E_{ss}^{uu} \) respectively. Otherwise, if we have a vector \( v \in N_{y}^{uu} \) such that \( v = v^{ss} + v^{cu} \), \( 0 \neq v^{ss} \in E_y^{ss} \) and \( v^{cu} \in E_y^{c} \oplus E_y^{uu} \), then a direct computation shows that the directions of \( \psi_t(v) \) will tend to the bundle \( E_y^{ss} \) as \( t \to -\infty \), this contradicts with \( N_{y}^{ss} = E_y^{ss} \) for any accumulation point \( \epsilon \in \{X(\varphi_\epsilon(y))/\|X(\varphi_\epsilon(y))\| : t < 0\} \).

Now we suppose that \( D_N(y) \subset N_{y}^{cu} \) does not hold. This means there is a vector \( u \in D_N(y) \setminus N_{y}^{cu} \) with \( u = u^{ss} + u^{cu} \) where \( 0 \neq u^{ss} \in E_y^{ss} \) and \( u^{cu} \in E_y^{c} \oplus E_y^{uu} \). Then there exists a sequence \( y_n \to y \) as \( n \to \infty \) such that
\[ (y_n - y)/\|y_n - y\| = u_n^{ss} + u_n^{cu} \to u, \]
where \( u_n^{ss} \in E_y^{ss} \) and \( u_n^{cu} \in E_y^{c} \oplus E_y^{uu} \). Write \( a_n = \|y_n - y\| \), then \( y_n = y + a_n(u_n^{ss} + u_n^{cu}) \). Since \( u^{ss} \neq 0 \), we have \( u_n^{ss} \neq 0 \) for \( n \) large enough, then we can take \( t_n < 0 \) such that \( \|\text{exp}(t_n A)(a_n u_n^{ss})\| = 1 \). By the facts that \( a_n \to 0 \) and \( \text{exp}(t_n A)|_{E_y^{ss}} \) is contracting, we can see that \( t_n \to -\infty \). By taking a subsequence if necessary, we can assume that \( \|\text{exp}(t_n A)(a_n u_n^{ss})\| \to z^{ss} \) as \( n \to \infty \). Then by the fact that \( E_y^{ss} \oplus E_y^{c} \oplus E_y^{uu} \) is a dominated splitting for \( \text{exp}(t_n A) \), we have \( \text{exp}(t_n A)(y_n) \to z^{ss} \) as \( n \to \infty \). Now we find a point \( z^{ss} \in \Lambda \cap (W_{ss}^{ss} \setminus \{\sigma\}) \). This is a contradiction. \( \square \)

**Lemma 4.7** Assume that \( \Lambda \) is a robustly transitive set, which contains two hyperbolic singularities \( \sigma_1 \) and \( \sigma_2 \) such that \( \text{Ind}(\sigma_1) < \text{Ind}(\sigma_2) \). Then for any \( C^1 \) neighborhood \( \mathcal{U} \) of \( Y \), there is \( Y \in \mathcal{U} \) which has a hyperbolic periodic orbit \( P \subset \Lambda_Y \) such that \( \text{Ind}(P) \in [\text{Ind}(\sigma_1), \text{Ind}(\sigma_2) - 1] \).

**Proof** If \( I(\sigma_1) \leq 0 \), then by Lemma 3.1 we can find \( Y \) arbitrarily close to \( X \) such that \( Y \) has a hyperbolic periodic orbit \( P \) with \( \text{Ind}(P) = \text{Ind}(\sigma_1) \). So the lemma is true in this case. Similarly, in the case of \( I(\sigma_2) \geq 0 \) the lemma is also true.

Now we can assume that \( I(\sigma_2) < 0 \) and \( I(\sigma_1) > 0 \). Firstly, after applying Lemma 3.1 we can get a periodic orbit of index \( \text{Ind}(\sigma_1) - 1 \) after an arbitrarily small perturbation. Since \( \Lambda \) is robustly transitive, apply Lemma 3.1 again for \( \sigma_2 \), we can get another periodic orbit of index \( \text{Ind}(\sigma_2) \) after another arbitrarily small perturbation. Thus for any neighborhood \( \mathcal{U} \) of \( X \), there is \( Z \in \mathcal{U} \) such that \( \Lambda_Z \) has hyperbolic periodic orbits \( P_1 \) of index \( \text{Ind}(\sigma_1) - 1 \) and \( P_2 \) of index \( \text{Ind}(\sigma_2) \). Since \( Z \) can be accumulated by generic vector fields, one can assume that \( Z \) is in the residual set \( \mathcal{R} \) in Proposition 2.10. Thus for an isolated neighborhood \( U \), one has \( H(P_1, U) = H(P_2, U) = \Lambda_Z \) by Item 2 of Proposition 2.10. Then by Item 4 of Proposition 2.10, we get the conclusion. \( \square \)

To prove Theorem A, we prove the following theorem under some generic assumptions. **Theorem A’** There is a residual set \( \mathcal{R} \) such that for any \( X \in \mathcal{R} \), if \( \Lambda \) is a robustly transitive set of \( X \times \mathcal{X} (M) \), then either all the singularities in \( \Lambda \) have the same index, or \( X \) can be accumulated by vector fields with a homoclinic tangency.

**Proof** We prove the theorem by contradiction. Assume \( X \in \mathcal{R} \setminus \mathcal{H} \) and \( \Lambda \) is a robustly transitive set of \( X \), and there exist two singularities \( \sigma_1 \) and \( \sigma_2 \) in \( \Lambda \) with different indices. Without loss of generality, we can assume \( \text{Ind}(\sigma_1) < \text{Ind}(\sigma_2) \). By Lemma 4.7, there is \( i \in [\text{Ind}(\sigma_1), \text{Ind}(\sigma_2) - 1] \) such that for any neighborhood \( \mathcal{U} \) of \( X \), there exist \( Y \in \mathcal{U} \) and a hyperbolic periodic orbit \( P \) of \( Y \) of index \( i \) such that \( P \subset \Lambda_Y \). As a corollary, \( \Lambda \) contains an
i-periodic limit. Note that a robustly transitive set is automatically an isolated transitive set. From Lemma 4.2 we have

- $T_{\sigma_2}M$ admits a dominated splitting $E^s_{\sigma_2} \oplus E^c_{\sigma_2} \oplus E^u_{\sigma_2}$ w.r.t. the tangent flow, where $E^s_{\sigma_2}$ is strongly contracting and $\dim E^s_{\sigma_2} = \text{Ind}_{\sigma_2}$.
- $T_{\sigma_1}M$ admits a dominated splitting $E^s_{\sigma_1} \oplus E^c_{\sigma_1} \oplus E^u_{\sigma_1}$ w.r.t. the tangent flow, where $E^c_{\sigma_1}$ is strongly expanding and $\dim E^c_{\sigma_1} = \dim M - i - 1$.

Now by Lemma 4.4, we have that $W^s(\sigma_2) \cap \Lambda = \{\sigma_2\}$ and $W^u(\sigma_1) \cap \Lambda = \{\sigma_1\}$. Since the local strong stable and unstable manifolds of hyperbolic critical elements vary continuously and the robust transitive set $\Lambda_Y$ varies lower continuously on $Y$, there is a $C^1$ neighborhood $U_0$ of $X$ such that for any $Y \in U_0$, one has $W^s(\sigma_2,Y) \cap \Lambda_Y = \{\sigma_2,Y\}$ and $W^u(\sigma_1,Y) \cap \Lambda_Y = \{\sigma_1,Y\}$ where $\sigma_1,Y$ and $\sigma_2,Y$ are the continuations of $\sigma_1, \sigma_2$ with respect to $Y$.

For any $C^1$ neighborhood $U \subset U_0$ of $X$, by using the connecting lemma (Lemma 2.1), there is $Y \in U$ such that $W^s(\sigma_1) \cap W^u(\sigma_2) \neq \emptyset$. With another arbitrarily small perturbation if necessary, we can assume that $Y$ is locally linearizable at $\sigma_1$ and $\sigma_2$. We have the following properties.

- By Lemmas 2.2 and 3.3, we have that $\Lambda_Y \setminus \text{Sing}(Y)$ admits a dominated splitting $N^c \oplus N^u$ of index $i$ in the normal bundle w.r.t. the linear Poincaré flow.
- Take $y \in W^s(\sigma_1) \cap W^u(\sigma_2)$. Since $\Lambda_Y$ is transitive, there is a sequence of points $\{y_n\} \subset \Lambda_Y$ with $y_n \neq y$ such that $\lim_{n \to \infty} y_n = y$ and $\omega(y_n) = \Lambda_Y$ for each $n$. Without loss of generality, we can assume that $y_n \in \exp_y(N_y)$ since $y$ is a regular point.

By Lemma 4.6, the non-empty set $D_N(y)$ of the accumulation points of

$$\left\{ \frac{\exp_y^{-1}y_n}{\|\exp_y^{-1}y_n\|} \right\}$$

has the following properties:

- The fact that $y \in W^u(\sigma_2)$ implies $D_N(y) \subset N^{cu}(y)$.
- The fact that $y \in W^s(\sigma_1)$ implies $D_N(y) \subset N^{cs}(y)$.

This gives a contradiction and ends the proof of Theorem A'.

**The proof of Theorem A** First we notice that if $X$ is a vector field which is far away from ones with a homoclinic tangency, and if $\Lambda$ is a robustly transitive set, then all the hyperbolic singularities in $\Lambda$ should have the same index. This is because otherwise, there is a vector field $Y \in \mathcal{R}$ which is close to $X$ such that $\Lambda_Y$ is a robustly transitive set with singularities of different indices. Then by using Theorem A’, we get a contradiction.

Now if $\Lambda$ is a robustly transitive set which contains a non-hyperbolic singularity $\sigma$. Without loss of generality, one can assume that

- either, $DX(\sigma)$ has only one eigenvalue whose real part is zero,
- or, $DX(\sigma)$ has two eigenvalues whose real parts are zero, which are conjugate.

In the first case, by a simple perturbation, we can have singularities with different indices in the robustly transitive set, which gives a contradiction.

Now we consider the second case. After a small perturbation, we can get a vector field $Z \in \mathcal{R}$ (here $\mathcal{R}$ is given as in Lemma 4.2) that $\sigma$ is a hyperbolic singularity of $Z$ whose saddle value is larger than 0 and $T_{\sigma}M = E^s \oplus E^c \oplus E^u$, where $\dim E^s = 2$ and corresponds the complex eigenvalues. By Lemma 3.1, $\Lambda_Z$ contains an $(\text{Ind}(\sigma) - 1)$-periodic limit. Then by Lemma 4.2, $E^c$ should be divided into a dominated splitting, thus a contradiction. This ends the proof of Theorem A.
We do some preparations for proving Theorem C.

**Lemma 4.8** If $\Lambda$ is a robustly transitive set, and $\Lambda$ is star, then there are a neighborhood $U$ of $\Lambda$ and a neighborhood $\mathcal{U}$ of $X$ such that $Y \in \mathcal{U}$ is far away from homoclinic tangencies in $U$.

**Proof** A homoclinic tangency is in fact a minimally non-hyperbolic set of simple type in [14]. Recall that a non-singular compact invariant set $\Gamma$ is said to be a *minimally non-hyperbolic set* if $\Gamma$ is not hyperbolic, but any proper compact invariant subset of $\Gamma$ is hyperbolic. A minimally non-hyperbolic set $\Gamma$ is of *simple* type, if there is a point $x \in \Gamma$ such that

- $\omega(x)$ and $\alpha(x)$ are all proper subsets of $\Gamma$;
- $N(x) \neq N^+(x) \oplus N^-(x)$,

where

$$N^\pm(x) = \{ v \in N(x) : \lim_{t \to \pm \infty} \| \psi^t(v) \| = 0 \}.$$

By [14, Lemma 6.1], if $\Lambda$ is star, then there are a neighborhood $U$ of $\Lambda$ and a neighborhood $\mathcal{U}$ of $X$ such that $Y \in \mathcal{U}$ has no minimally non-hyperbolic set of simple type in $U$. Hence $Y \in \mathcal{U}$ has no homoclinic tangencies in $U$. □

**The proof of Theorem C** By Lemma 4.8, if $\Lambda$ is star, then there are a neighborhood $U$ of $\Lambda$ and a neighborhood $\mathcal{U}$ of $X$ such that $X$ is far away from homoclinic tangencies in $U$. Thus by Theorem A we can see that if $\Lambda$ is robustly transitive and star, then all singularity in $\Lambda$ should be hyperbolic and have a common index (note here in the proof of Theorem A we just need consider the orbit near $\Lambda$).

It remains to show that $X$ is strongly homogenous.

**Claim** For any singularity $\sigma \in \Lambda$, the saddle value of $\sigma$ can not be zero.

**Proof** Assume on the contrary that there exists $\sigma \in \Lambda$ with $I(\sigma) = 0$.

Now we choose a neighborhood $\mathcal{U}$ of $X$ such that $\Lambda$ has its robust continuation and star in $\mathcal{U}$. For this $\mathcal{U}$, one can find $\delta > 0$ by Lemma 2.12.

By Lemma 3.1 we can find a perturbation $Y_1$ close to $X$ such that

- $Y_1$ contains a hyperbolic periodic orbit $P$ with $\text{Ind}(P) = \text{Ind}(\sigma) - 1$.
- $I(\sigma_{Y_1}, Y_1)$ is $\delta$-close to $I(\sigma, X) = 0$, i.e., $|I(\sigma_{Y_1}, Y_1)| < \delta$.

For $Y_1$, we can make another perturbation in a small neighborhood of $\sigma$ by apply Lemma 2.12 to get $Y_2 \in \mathcal{U}$ such that $P$ is still a periodic orbit of $Y_2$ with $\text{Ind}(P) = \text{Ind}(\sigma) - 1$ but the saddle value $I(\sigma) < 0$. By applying Lemma 3.1 again, we can get a vector field $Y$ arbitrarily close to $Y_2$ with a periodic orbit $Q$ of index $\text{Ind}(\sigma)$. If the perturbation is small enough, the continuation of $P$ exists with index $\text{Ind}(\sigma) - 1$. Thus we get a contradiction to the star condition by [16, Lemma 1.6]. We complete the proof of the Claim. □

Without loss of generality, now we assume that there exists a singularity $\sigma \in \Lambda$ with $I(\sigma) > 0$. Then we can take a neighborhood $\mathcal{V} \subset \mathcal{U}$ such that for any $Z \in \mathcal{V}$,

- the continuation $\sigma_Z$ of $\sigma$ with respect to $Z$ has the positive saddle value;
- the set $\Lambda_Y$ is star for any $Y \in \mathcal{V}$.

We will show that for every $Y \in \mathcal{V}$, the periodic orbit of $Y$ in $U$ should have index $\text{Ind}(\sigma) - 1$. We will prove by contradiction and assume that there exists $Y \in \mathcal{V}$ such that $Y$ has a hyperbolic periodic orbit $Q$ such that the index of $Q$ is not $\text{Ind}(\sigma) - 1$. One has that
5 Multisingular Partial Hyperbolicity

For compact invariant sets with singularities, we would like to study the weak form of hyperbolicity. However, it is not always reasonable to consider the tangent flow:

- By the suspension of a robust transitive diffeomorphism by Bonatti-Viana [7], there is a robustly transitive vector field without any dominated splitting of the tangent flow.
- By a recent example of da Luz [11], even for 5-dimensional vector field, there are star hyperbolicity. However, it is not always reasonable to consider the tangent flow:

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Thus, we need more general notion to understand vector fields that are away from homoclinic tangencies. Following the idea of multi-singular hyperbolicity by Bonatti-da Luz [2], here we give a notion of \textit{multisingular partially hyperbolicity}. Firstly we give the notion of reparametrizing cocycle following [2]. Let $\Lambda$ be a compact invariant set of vector field $X$. Let $\sigma \in \Lambda$ be a hyperbolic singularity of $X$, we say a positive continuous function $h : (\Lambda \setminus \text{Sing}(X)) \times \mathbb{R} \to \mathbb{R}^+$ is a \textit{pragmatical cocycle} (associated to $\sigma$) if $h(t + s, x) = h(t, x) \cdot h(s, \varphi_t(x))$ for all $s, t \in \mathbb{R}$ and $x \in \Lambda \setminus \text{Sing}(X)$ and there is an isolated neighborhood $V_\sigma$ of $\sigma$ such that the following conditions are satisfied:

1. if $x$ and $\varphi_t(x)$ are both in $V_\sigma$, then $h(x, t) = \|\Phi_t\|_{<X(x)>}$;
2. if $x$ and $\varphi_t(x)$ are both outside $V_\sigma$, then $h(x, t) = 1$.

One can see [2] for the existence of a pragmatic cocycle associated to $\sigma$. A \textit{reparametrizing cocycle} is a finite product of powers of pragmatic cocycles. As usual, we use $h_t(x)$ to denote $h(x, t)$ for a reparametrizing cocycle.

\textbf{Definition 5.1} Let $\Lambda$ be a compact invariant set. We say that $\Lambda$ is \textit{multisingular partially hyperbolic}, if $N_{\Lambda \setminus \text{Sing}(X)}$ has a dominated splitting $N^{cs} \oplus N_1 \oplus N^{cu}$ w.r.t. the linear Poincaré flow, and there exist two reparametrizing cocycles $h^t_1 : \Lambda \setminus \text{Sing}(X) \to \mathbb{R}^+$ and $h^t_2 : \Lambda \setminus \text{Sing}(X) \to \mathbb{R}^+$ such that,

1. $h^{t_1}_1 \cdot \psi_t|_{N^{cs}}$ is uniformly contracting, that is, there are constants $C > 0$ and $\lambda > 0$ such that for any regular point $x \in \Lambda$ and any $t > 0$, one has

   $\|h^{t_1}_1(x) \cdot \psi_t|_{N^{cs}(x)}\| \leq Ce^{-\lambda t};$

2. $h^{t_2}_1 \cdot \psi_t|_{N^{cu}}$ is uniformly expanding, that is, there are constants $C > 0$ and $\lambda > 0$ such that for any regular point $x \in \Lambda$ and any $t > 0$, one has

   $\|h^{t_2}_1(x) \cdot \psi_{-t}|_{N^{cu}(x)}\| \leq Ce^{-\lambda t}.$
Remark Here we do not use the extended linear Poincaré flow from [16] and another version from [2]. The advantage by using extended linear Poincaré flow in [2] is that the notion is robust under perturbation.

Note that da Luz [12] introduced a notion called “singular volume partial hyperbolicity”. In the definition of singular volume partial hyperbolicity, the volumes of the extremal bundles are uniformly contracted or expanded over cocycles. In the definition of multisingular partial hyperbolicity, the extremal bundles themselves are uniformly contracting or expanding over cocycles.

Here we verify that every non-trivial isolated transitive set is multisingular partially hyperbolic for a $C^1$ generic vector field away from ones with a homoclinic tangency.

**Theorem 5.2** For any $X \in \mathcal{R}$ which is far away from ones with a homoclinic tangency, every isolated non-trivial transitive set of $X$ admits a multisingular partially hyperbolic splitting $N^{cs} \oplus N_1 \oplus \cdots \oplus N_k \oplus N^{cu}$, where $\dim N_i = 1$ for any $1 \leq i \leq k$.

Before we give the proof of Theorem 5.2, we take some preparations. In the following, we will always assume $X \in \mathcal{R}$ and $\Lambda$ is an isolated non-trivial transitive set of $X$.

Since $\Lambda$ is transitive, applying Lemma 2.2, $\Lambda$ is the Hausdorff limit of some hyperbolic periodic orbit of some index. Assume that $\alpha$ and $\beta$ be the minimal and maximal number among the indices. That is,

\[
\alpha = \min \{i : \exists Y_n \text{ and periodic orbit } \gamma_n \text{ of } Y_n, \text{ s.t. } Y_n \rightarrow Y, \text{ Ind}(\gamma_n) = i \text{ and } \gamma_n \text{ converges into } \Lambda\},
\]

\[
\beta = \max \{i : \exists Y_n \text{ and periodic orbit } \gamma_n \text{ of } Y_n, \text{ s.t. } Y_n \rightarrow Y, \text{ Ind}(\gamma_n) = i \text{ and } \gamma_n \text{ converges into } \Lambda\}.
\]

Because $\Lambda$ is nontrivial, $\Lambda$ contains no sinks, then from Item 3 of Proposition 2.10 we know that $\Lambda$ contains no 0-periodic limit, hence we have $\alpha > 0$. Symmetrically, we have $\beta < \dim M - 1$.

**Lemma 5.3** There is a residual set $\mathcal{R} \subset \mathcal{X}^1 (M)$ with the following properties. Let $X \in \mathcal{R} \setminus \overline{HT}$ and $\Lambda$ be an isolated non-trivial transitive set of $X$ and $\alpha, \beta$ be given as above. Then for any integer $i \in [\alpha, \beta]$, there is a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)} = N^1 \oplus N^2$ (w.r.t. $\psi_t$) of index $i$. As a consequence, we have a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)} = N^{cs} \oplus N_1 \oplus \cdots \oplus N_{\beta-\alpha} \oplus N^{cu}$ with $\dim N^{cs} = \alpha$ and $\dim N^{cu} = d - 1 - \beta$ and $\dim N_i = 1$ for any $1 \leq i \leq \beta - \alpha$.

**Proof** Note that $\Lambda$ is isolated with an isolated neighborhood $U$, this implies that if there is a periodic orbit in $U$, then this periodic orbit is contained in $\Lambda$. Thus by Item 3 of Proposition 2.10 we know that $\Lambda$ contains a periodic orbit $P_1$ of index $\alpha$ and contains a periodic orbit $P_2$ of index $\beta$. Then by Item 4 of Proposition 2.10 we know that for any $i \in [\alpha, \beta]$, there exists a periodic orbit $P$ of index $i$ in $\Lambda$. Then by Item 2 of Proposition 2.10 we know that $\Lambda = H(P, U)$ hence an $i$-periodic limit. Thus we know that for any $i \in [\alpha, \beta]$, $\Lambda$ is an $i$-periodic limit. By Lemma 3.2, since $X$ is far away from vector fields with a homoclinic tangency, we know that for any $i \in [\alpha, \beta]$, there is a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)} = N^1 \oplus N^2$ of index $i$. As a consequence we have a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)} = N^{cs} \oplus N_1 \oplus \cdots \oplus N_{\beta-\alpha} \oplus N^{cu}$ with $\dim N^{cs} = \alpha$ and $\dim N^{cu} = d - 1 - \beta$ and $\dim N_i = 1$ for any $1 \leq i \leq k$. □

**Lemma 5.4** There is a residual set $\mathcal{R} \subset \mathcal{X}^1 (M)$ with the following properties. Let $X \in \mathcal{R} \setminus \overline{HT}$ and $\Lambda$ be an isolated non-trivial transitive set of $X$ and $N^{cs}$, $N^{cu}$ be the sub-bundles...
given in Lemma 5.3. Then there exist constants $l' > 0, \eta' > 0$ such that for any periodic point $p \in \Lambda$ of $X$, one has
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \psi_{l'} |_{N^{cs}(\phi_{il'}(p))} \| < -\eta'
\]
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \psi_{-l'} |_{N^{cu}(\phi_{-il'}(p))} \| < -\eta'
\]

**Proof** Let $P$ be a periodic orbit of $X$ in $\Lambda$ with hyperbolic splitting $N^s(P) \oplus N^u(P)$. By the choice of $\alpha$ we know that $\text{Ind}(P) \geq \alpha$, then we have $N^{cs}(P) \subset N^s(P)$. By Proposition 2.9, we can take $\eta > 0$ and $N_0 > 0$ such that for any periodic orbit $P$ of $X$ with $\pi(P) \geq N_0$, one has
\[
\prod_{i=0}^{[\pi(P)/l]-1} \| \psi_l |_{N^{cs}(\phi_{il}(p))} \| < e^{-\eta [\pi(P)/l]},
\]
for any $p \in P$.

From the above estimation we know that for any periodic point $p \in \Lambda$ of $X$ with $\pi(p) \geq N_0$, one has
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \psi_{l} |_{N^{cs}(\phi_{il}(p))} \| < -\eta.
\]
This property still holds once we replace $l$ by one of its multiples. Hence we just need to find a positive integer $k$ and a constant $\eta' > 0$ such that for any periodic point $p \in \Lambda$ of $X$ with $\pi(p) \leq N_0$, one has
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \psi_{kl} |_{N^{cs}(\phi_{kil}(p))} \| < -\eta'.
\]

Let
\[
P_{N_0}(\Lambda) = \{ p \in \Lambda : p \text{ is a periodic point with } \pi(p) \leq N_0 \}.
\]
Then $P_{N_0}(\Lambda)$ is a close subset of $\Lambda$ and hence a compact subset. For any $p \in P_{N_0}(\Lambda)$, since $\psi_l |_{N^{cs}(\text{Orb}(p))}$ is contracting (by the choice of $\alpha$), we can find $k(p)$ such that
\[
\| \psi_{k(p)}l |_{N^{cs}(p)} \| \leq \frac{1}{2}.
\]
By the compactness of $P_{N_0}(\Lambda)$, we can find constants $\eta' > 0$ and $C > 1$ such that
\[
\| \psi_{kl} |_{N^{cs}(p)} \| < Ce^{-k\eta'}
\]
for all $p \in P_{N_0}(\Lambda)$ and $k \geq 1$. Then we can find a large $k$ such that
\[
\| \psi_{kl} |_{N^{cs}(p)} \| < e^{-k\eta'}
\]
for all $p \in P_{N_0}(\Lambda)$. Then we can see that $l' = kl$ and $\eta'$ satisfy the requirements of the lemma. Symmetrically, we can take $l'$ and $\eta'$ such that the estimation for the bundle $N^{cu}$ is satisfied. \[\Box\]
To simplify the notations, we still use \( l, \eta \) to denote \( l', \eta' \) in the above lemma since we can choose \( l', \eta' \) keep all properties of \( l, \eta \) we have listed in Proposition 2.9.

Now we analyse the singularities in \( \Lambda \).

**Lemma 5.5** There is a residual set \( \mathcal{R} \subset X^1(M) \) with the following properties. Let \( X \in \mathcal{R} \) and \( \Lambda \) be an isolated non-trivial transitive set of \( X \) and \( \alpha, \beta \) be given as above. Then for any singularity \( \sigma \in \Lambda \), we have \( \alpha \leq \text{Ind}(\sigma) \leq \beta + 1 \). Furthermore, if \( \text{Ind}(\sigma) = \alpha \) then \( I(\sigma) < 0 \); if \( \text{Ind}(\sigma) = \beta + 1 \), then \( I(\sigma) > 0 \).

**Proof** This is a direct consequence of Lemma 3.1. Let \( \sigma \in \Lambda \) be a singularity. By Lemma 3.1 we know that \( \Lambda \) contains an \( \text{Ind}(\sigma) \)-periodic limit in the case of \( \text{Ind}(\sigma) \leq 0 \) and contains an \( (\text{Ind}(\sigma) - 1) \)-periodic limit in the case of \( \text{Ind}(\sigma) \geq 0 \). By the choice of \( \alpha \) we know that \( \text{Ind}(\sigma) \geq \alpha \) and if \( \text{Ind}(\sigma) = \alpha \) then \( I(\sigma) < 0 \). Symmetrically we have \( \text{Ind}(\sigma) \leq \beta + 1 \) and if \( \text{Ind}(\sigma) = \beta + 1 \) then \( I(\sigma) > 0 \) by the choice of \( \beta \). This ends the proof of the lemma. \( \square \)

Recall that \( \tilde{\Lambda} \) denotes the closure of \( \{X(x)/\|X(x)\|: x \in \Lambda \setminus \text{Sing}(X)\} \) in \( SM \). Thus, when we use \( \sim \) over a set in \( M \), we mean the lift of the set in \( SM \). Given a singularity \( \sigma \in \Lambda \), denote by \( \tilde{\Lambda}_\sigma = \tilde{\Lambda} \cap T_\sigma M \). Clearly \( \tilde{\Lambda}_\sigma \subset S_\sigma M \). Since both \( \tilde{\Lambda} \) and \( S_\sigma M \) are \( \Phi^t \)-invariant, we know \( \tilde{\Lambda}_\sigma \) is invariant under \( \Phi^t \).

**Lemma 5.6** There is a residual set \( \mathcal{R} \subset X^1(M) \) with the following properties. Let \( X \in \mathcal{R} \setminus HT \) and \( \Lambda \) be an isolated non-trivial transitive set of \( X \) and \( \alpha, \beta \) be given as above.

Then for any singularity \( \sigma \in \Lambda \), there is a partially hyperbolic splitting \( T_\sigma M = E_{\sigma}^{ss} \oplus E_{\sigma}^{s} \oplus E_{\sigma}^{u} \oplus \cdots \oplus E_{\sigma}^{\beta_1+1-\alpha} \oplus E_{\sigma}^{uu} \) (w.r.t. \( \Phi_t \)), where \( E_{\sigma}^{ss} \) is contracting with \( \dim E_{\sigma}^{ss} = \alpha \), \( E_{\sigma}^{uu} \) is expanding with \( \dim E_{\sigma}^{uu} = \dim M - \beta - 1 \) and \( \dim E_{\sigma}^{i} = 1 \) for all \( i = 1, 2, \ldots, \beta + 1 - \alpha \).

Moreover, for \( \tilde{\Lambda}_\sigma \subset S_\sigma M \), one has that
1. if \( \text{Ind}(\sigma) = \alpha \) then \( \tilde{\Lambda}_\sigma \subset E_{\sigma}^{ss} \oplus E_{\sigma}^{s} \),
2. if \( \text{Ind}(\sigma) > \alpha \) then \( \tilde{\Lambda}_\sigma \cap E_{\sigma}^{ss} = \emptyset \).

**Proof** Let \( \sigma \in \Lambda \) be a singularity with a hyperbolic splitting \( E_{\sigma}^{s} \oplus E_{\sigma}^{u} \). By Lemma 5.5 we know that \( \text{Ind}(\sigma) \in [\alpha, \beta + 1] \). Since for all \( i \in [\alpha, \beta] \), \( \Lambda \) contains \( i \)-periodic limit, by Lemma 4.2 we know that (1) for any \( \alpha \leq i < \text{Ind}(\sigma) \), there exists a dominated splitting \( T_\sigma M = E \oplus F \) of index \( i \) with \( E \subset E_{\sigma}^{si} \); (2) for any \( \text{Ind}(\sigma) < i \leq \beta + 1 \), there exists a dominated splitting \( T_\sigma M = E \oplus F \) of index \( i \) with \( F \subset E_{\sigma}^{ui} \). Note that \( E_{\sigma}^{s} \oplus E_{\sigma}^{u} \) is already a dominated splitting of index \( \text{Ind}(\sigma) \). Hence for all \( \alpha \leq i \leq \beta + 1 \), there is a dominated splitting \( T_\sigma M = E \oplus F \) of index \( i \), then we have a dominated splitting \( T_\sigma M = E_{\sigma}^{ss} \oplus E_{\sigma}^{s} \oplus E_{\sigma}^{ui} \oplus \cdots \oplus E_{\sigma}^{\beta+1-\alpha} \oplus E_{\sigma}^{uu} \) with the properties: (1) \( E_{\sigma}^{ss} \subset E_{\sigma}^{s} \) and \( E_{\sigma}^{uu} \subset E_{\sigma}^{u} \); (2) \( \dim E_{\sigma}^{ss} = \alpha \) and \( \dim E_{\sigma}^{uu} = \dim M - \beta - 1 \); (3) \( \dim E_{\sigma}^{i} = 1 \) for all \( i = 1, 2, \ldots, \beta + 1 - \alpha \).

Let \( \sigma \in \Lambda \) be a singularity of \( X \). If \( \text{Ind}(\sigma) = \alpha \), then by the fact that \( \Lambda \) contains an \( \alpha \)-periodic limit we know \( \tilde{\Lambda} \subset B(\Lambda) \subset E_{\sigma}^{ss} \oplus E_{\sigma}^{s} \) from Lemmas 2.6 and 4.5. If \( \text{Ind}(\sigma) > \alpha \), also by the fact that \( \Lambda \) contains an \( \alpha \)-periodic limit we know that \( \tilde{\Lambda} \subset B(\Lambda) \subset E_{\sigma}^{s} \oplus E_{\sigma}^{s} \oplus \cdots \oplus E_{\sigma}^{\beta+1-\alpha} \oplus E_{\sigma}^{uu} \), hence \( \tilde{\Lambda} \cap E_{\sigma}^{ss} = \emptyset \). This ends the proof of the lemma. \( \square \)

By Proposition 2.7 we know that the dominated splitting \( N_{\Lambda \setminus \text{Sing}(X)} = N^{cs} \oplus N_1 \oplus \cdots \oplus N_{\beta-\alpha} \oplus N^{cu} \) on \( \Lambda \setminus \text{Sing}(X) \) (w.r.t. \( \psi_t \)) can be extended to be a dominated splitting \( N_{\tilde{\Lambda}} = N^{cs} \oplus N_1 \oplus \cdots \oplus N_{\beta-\alpha} \oplus N^{cu} \) on \( \tilde{\Lambda} \) with respect to the extended linear Poincaré flow \( \psi_t \).

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Lemma 5.7 There is a residual set \( \mathcal{R} \subset X^1(M) \) with the following properties. Let \( X \in \mathcal{R} \setminus HT \) and \( \Lambda \) be an isolated non-trivial transitive set of \( X \) and \( \sigma \) be a singularity in \( \Lambda \). If \( \text{Ind}(\sigma) = \alpha \), then there exist \( C > 1, \lambda > 0 \) such that
\[
\| \Phi_t(e) \| \cdot \| \tilde{\psi}_t|_{N^{cs}(e)} \| < Ce^{-\lambda t}
\]
for all \( e \in \tilde{\Lambda}_\sigma \) and \( t > 0 \). If \( \text{Ind}(\sigma) > \alpha \), then there exist \( C > 1, \lambda > 0 \) such that
\[
\| \Phi_t(e) \|^{-1} \cdot \| \tilde{\psi}_t|_{N^{cs}(e)} \| < Ce^{-\lambda t}
\]
for all \( e \in \tilde{\Lambda}_\sigma \) and \( t > 0 \).

Proof Assume \( \sigma \in \Lambda \) is a singularity with \( \text{Ind}(\sigma) = \alpha \). From Lemma 5.6 we know that \( \tilde{\Lambda}_\sigma \subset E^{cs}_\sigma \oplus E^u_\sigma \). Denote by \( F^{cs}_\sigma = E^{cs}_\sigma \oplus E^1_\sigma \) and \( F^u_\sigma = E^2_\sigma \oplus \cdots \oplus E^{\beta+1-\alpha}_\sigma \oplus E^{uu}_\sigma \). Let \( \pi_e \) be the orthogonal projection from \( T_\sigma M \) to \( N_\sigma e \) for any \( e \in \tilde{\Lambda}_\sigma \).

Claim \( N^{cs}(e) = \pi_e(F^{cs}_\sigma) \) for all \( e \in \tilde{\Lambda}_\sigma \).

Proof of the Claim We define two sub-bundles \( \Delta^{cs}, \Delta^{cu} \subset N^{cs}_{\tilde{\Lambda}_\sigma} \) by letting
\[
\Delta^{cs}(e) = \pi_e(F^{cs}_\sigma), \quad \Delta^{cu}(e) = \pi_e(F^u_\sigma), \quad \forall e \in \tilde{\Lambda}_\sigma.
\]
By the fact that \( T_\sigma M = F^{cs}_\sigma \oplus F^u_\sigma \) we know that \( N^{cs}_{\tilde{\Lambda}_\sigma} = \Delta^{cs} \oplus \Delta^{cu} \). Since \( F^{cs}_\sigma \) and \( F^u_\sigma \) are invariant by \( \Phi_t \), we can get that \( \Delta^{cs} \) and \( \Delta^{cu} \) are invariant by \( \tilde{\psi}_t \), that is,
\[
\tilde{\psi}_t(\Delta^{cs}(e)) = \Delta^{cs}(\tilde{\psi}_t(e)), \quad \tilde{\psi}_t(\Delta^{cu}(e)) = \Delta^{cu}(\tilde{\psi}_t(e)), \quad \forall e \in \tilde{\Lambda}_\sigma, \ t \in \mathbb{R}.
\]
Since \( e \in F^{cs}_\sigma \) we know that there is a lower bound for the angle between \( e \in \tilde{\Lambda}_\sigma \) and \( F^u_\sigma \), hence there is \( K > 1 \) such that \( m(\pi_e|_{F^u_\sigma}) \geq K^{-1} \) for all \( e \in \tilde{\Lambda}_\sigma \). For any \( e \in \tilde{\Lambda}_\sigma \) and any unit vectors \( u \in \Delta^{cs}(e), v \in \Delta^{cu}(e) \), letting \( v = \pi_e(v') \) where \( v' \in F^u_\sigma \), we have
\[
\frac{\| \tilde{\psi}_t(u) \|}{\| \tilde{\psi}_t(v) \|} \leq \frac{\| \Phi_t(u) \|}{\| \Phi_t(v) \|} = \frac{\| \pi_{\Phi_t(u)}(\tilde{\psi}_t(v')) \|}{\| \tilde{\psi}_t(v') \|} \leq K \frac{\| \Phi_t(u) \|}{m(\pi_e|_{F^u_\sigma})}.
\]
Note that \( F^{cs}_\sigma \oplus F^u_\sigma \) is a dominated splitting of index \( \alpha + 1 \). From the above estimation we can see that \( \Delta^{cs} \oplus \Delta^{cu} \) is a dominated splitting. Since \( e \in F^{cs}_\sigma \), we know that \( \text{dim} \Delta^{cs} = \dim F^{cs}_\sigma = \alpha \). Note that \( N^{cs} \oplus (N_1 \oplus \cdots \oplus N_{\beta+1-\alpha} \oplus N^{cu}) \) is also a dominated splitting of index \( \alpha \). We can see that \( \Delta^{cs} = N^{cs} \) by the uniqueness of dominated splitting. Hence we have \( N^{cs}(e) = \pi_e(F^{cs}_\sigma) \) for all \( e \in \tilde{\Lambda}_\sigma \).

Note that \( E^{cs}_\sigma \) is the stable space of \( \sigma \) and \( E^1_\sigma \) is the eigenspace associated to the weakest unstable eigenvalue of \( \Phi_1 \). From Lemma 5.5 we know that
\[
\lim_{t \to +\infty} \frac{1}{t} \log \| (\wedge^2 \Phi_t|_{F^{cs}_\sigma}) \| = I(\sigma) < 0.
\]
Hence there are \( C > 1, \lambda > 0 \) such that
\[
\| (\wedge^2 \Phi_t|_{F^{cs}_\sigma}) \| < Ce^{-\lambda t}
\]
for all \( t > 0 \). For any \( e \in \tilde{\Lambda}_\sigma \) and any unit vector \( u \in N^{cs}(e) \), we have
\[
\| \Phi_t(e) \| \cdot \| \tilde{\psi}_t(u) \| = \| \Phi_t(e) \wedge \Phi_t(u) \| = \| (\wedge^2 \Phi_t(e \wedge u)) \| < Ce^{-\lambda t}
\]
for all \( t > 0 \). Hence we have
for all $e \in \tilde{\Lambda}_\sigma$ and $t > 0$.

Now let us assume $\sigma \in \Lambda$ is a singularity with $\text{Ind}(\sigma) > \alpha$. In this case we know that $\tilde{\Lambda}_\sigma \cap E^{ss}_{\sigma} = \emptyset$. Similar as in above, by the uniqueness of dominated splitting, we can prove that $N^{cs}(e) = \pi_e(E^{ss}_\sigma)$ for any $e \in \tilde{\Lambda}_\sigma$, where $\pi_e$ denotes the orthogonal projection from $T_e M$ to $N_e$. Note that $\tilde{\Lambda}_\sigma \subset E^t_\sigma \oplus E^{\alpha+1}_\sigma \oplus E^{uu}_\sigma$, we have a constant $K' > 1$ such that $m(\pi_e|_{E^{ss}_\sigma}) > K' - 1$ for all $e \in \tilde{\Lambda}_\sigma$. Then by the fact that $E^{ss}_\sigma \oplus (E^t_\sigma \oplus E^{\alpha+1}_\sigma \oplus E^{uu}_\sigma)$ is a dominated splitting we know that there are $C_0 > 1$ and $\lambda > 0$ such that for any $u \in E^t_\sigma$ and $v \in E^{\alpha+1}_\sigma \oplus E^{uu}_\sigma$ and $t > 0$, one has

$$\|\Phi_t(u)\| \leq C_0 e^{-\lambda t}.$$ 

Hence for any $e \in \tilde{\Lambda}_\sigma$ and any unit vector $u \in N^{cs}(e)$ and $t > 0$, letting $u = \pi_e(u')$ where $u' \in E^{ss}_\sigma$, we have

$$\frac{\|\tilde{\psi}_t(u')\|}{\|\Phi_t(e)\|} = \frac{\|\pi_{\Phi_t(e)}(\Phi_t(u'))\|}{\|\Phi_t(e)\|} \leq \frac{\|\Phi_t(u')\|}{\|\Phi_t(e)\|} \leq K'C_0 e^{-\lambda t}.$$ 

Let $C = K'C_0$, then for any $e \in \tilde{\Lambda}_\sigma$ and $t > 0$, we have

$$\|\Phi_t(e)\|^{-1} \cdot \|\tilde{\psi}_t|_{N^{cs}(e)}\| < Ce^{\lambda t}.$$ 

This ends the proof of Lemma 5.7. \[ \square \]

No we proceed to prove Theorem 5.2.

**Proof of Theorem 5.2** To prove $\Lambda$ is multisingular partially hyperbolic, we will find two reparemtrizing cocycles $h^*_t$ and $h^*_t$ such that $h^*_t \cdot \psi_t|_{N^{cs}}$ is contracting and $h^*_t \cdot \tilde{\psi}_t|_{N^{cs}}$ is expanding. Here we construct $h^*_t$, $h^*_t$ as following. Let $S$ be the set of singularities of $X$ contained in $\Lambda$ and denote by

$$S_1 = \{\sigma \in \Lambda : \sigma \text{ is a singularity of index } \alpha\},$$

$$S_2 = \{\sigma \in \Lambda : \sigma \text{ is a singularity of index } \beta + 1\}.$$ 

Since $X$ is a Kupka-Smale system we know that $X$ contains only finitely many singularities. Write

$$S_1 = \{\sigma_1, \sigma_2, \cdots, \sigma_k\},$$

$$S_2 = \{\sigma_1', \sigma_2', \cdots, \sigma_l'\}.$$ 

Now we choose $c > 0$ small enough such that for any $\sigma \in S$, the connected component $V_\sigma$ of $\{x \in M : \|X(x)\| < c\}$ containing $\sigma$ is an isolated neighborhood for $\sigma$. Now we define a positive continuous functions $k^\sigma : \Lambda \setminus \text{Sing}(X) \rightarrow \mathbb{R}$ associated to $\sigma$ by the following way:

$$k^\sigma(x) = \|X(x)\|, \text{ when } x \in V_\sigma; \quad k^\sigma(x) = c, \text{ when } x \notin V_\sigma;$$

Then define a pragmatic cocycle $h^\sigma_t$ associated to the singularity $\sigma$ by

$$h^\sigma_t(x) = \frac{k^\sigma(\psi_t(x))}{k^\sigma(x)}, \forall x \in \Lambda \setminus \text{Sing}(x), t \in \mathbb{R};$$

\[ \square \]
Now we can define two reparametrizing cocycles $h^s_t$, $h^u_t$ by letting

$$h^s_t(x) = \left( \prod_{\sigma \in S_1} h^s_\sigma(x) \right) \cdot \left( \prod_{\sigma \in S \setminus S_1} h^c_\sigma(x) \right)^{-1}, \quad \forall x \in \Lambda \setminus \text{Sing}(X), \ t \in \mathbb{R};$$

$$h^u_t(x) = \left( \prod_{\sigma \in S_2} h^s_\sigma(x) \right) \cdot \left( \prod_{\sigma \in S \setminus S_2} h^c_\sigma(x) \right)^{-1}, \quad \forall x \in \Lambda \setminus \text{Sing}(X), \ t \in \mathbb{R}.$$

We will prove that $h^s_t \cdot \psi_t |_{NC^s}$ is contracting. Note that for any singularity $\sigma \in \Lambda$, the pragmatic cocycle $h^c_\sigma$ associated to $\sigma$ gives automatically a cocycle $\tilde{h}^c_\sigma$ on $\{X(x)/\|X(x)\| : x \in \Lambda \setminus \text{Sing}(X)\}$ (with respect to the flow $\Phi^t_\sigma$) by letting

$$\tilde{h}^c_\sigma(X(x)/\|X(x)\|) = h^c_\sigma(x), \quad \forall x \in \Lambda \setminus \text{Sing}(X).$$

$\tilde{h}^c_\sigma$ can be continuously extended to $\tilde{\Lambda}$ as following:

$$\tilde{h}^c_\sigma(e) = \|\Phi_t(e)\|, \quad \forall e \in \tilde{\Lambda}_\sigma,$$

$$\tilde{h}^c_\sigma(e) = 1, \quad \forall e \in \tilde{\Lambda}_{\sigma'},$$

where $\sigma' \in \text{Sing}(X)$, $\sigma' \neq \sigma$.

Hence the reparametrizing cocycle $h^s_t : \Lambda \setminus \text{Sing}(X) \rightarrow \mathbb{R}$ can be extend to be a continuous cocycle $\tilde{h}^s_t : \tilde{\Lambda} \rightarrow \mathbb{R}$ with the following properties:

1. $\tilde{h}^s_t(e) = h^s_t(x)$, if $\rho(e) = x \in \Lambda \setminus \text{Sing}(X)$;
2. $\tilde{h}^s_t(e) = \|\Phi_t(e)\|$, if $\rho(e) \in S_1$;
3. $\tilde{h}^s_t(e) = \|\Phi_t(e)\|^{-1}$, if $\rho(e) \in S \setminus S_1$.

To prove $h^s_t \cdot \psi_t |_{NC^s}$ is contracting, we just need to prove that $\tilde{h}^s_t \cdot \tilde{\psi}_t |_{NC^s}$ is contracting. We will achieve this by contradiction.

From Proposition 2.9, Lemmas 5.4 and 5.7, we can choose constants $l$, $\eta$ with the following properties:

(A1) there exist a neighborhood $U$ of $X$ and a neighborhood $U_1$ of $\Lambda$ and a constant $N_0 > 0$ such that for any $Y \in U$ and any periodic orbit $P$ of $Y$ in $U$ with $\pi(P) > N_0$, one has

$$\prod_{i=0}^{[\pi(P)/l-1]} \|\psi^Y_P |_{NC^s(\psi^Y_P (p))}\| < e^{-\eta[\pi(P)/l]},$$

for all $p \in P$;

(A2) for any periodic point $p \in \Lambda$ of $X$, one has

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|\psi^X_P |_{NC^s(\psi^X_P (p))}\| < -\eta;$$

(A3) for any singularity $\sigma \in \Lambda$ with $\text{Ind}(\sigma) = \alpha$, one has

$$\tilde{h}^s_t(e) \cdot \|\tilde{\psi}_t |_{NC^s(e)}\| = \|\Phi_t(e)\| \cdot \|\tilde{\psi}_t |_{NC^s(e)}\| < e^{-\eta l}, \quad \forall e \in \tilde{\Lambda}_\sigma;$$

for any singularity $\sigma \in \Lambda$ with $\text{Ind}(\sigma) > \alpha$, one has

$$\tilde{h}^s_t(e) \cdot \|\tilde{\psi}_t |_{NC^s(e)}\| = \|\Phi_t(e)\|^{-1} \cdot \|\tilde{\psi}_t |_{NC^s(e)}\| < e^{-\eta l}, \quad \forall e \in \tilde{\Lambda}_\sigma.$$
Assume on the contrary that \( \tilde{h}_t \cdot \tilde{\psi}_t \mid_{N^{cs}} \) is not contracting, then there exists \( e \in \tilde{\Lambda} \) such that
\[
\prod_{i=0}^{n-1} \tilde{h}_t(\Phi_{it}^\#(e)) \cdot \| \tilde{\psi}_t \mid_{N^{cs}(\Phi_{it}^\#(e))} \| \geq 1, \quad \forall n = 1, 2, \ldots ,
\]
thus by the continuity of \( \tilde{h}_t \cdot \| \tilde{\psi}_t \mid_{N^{cs}} \| \), we can find a \( \Phi_{it}^\# \)-ergodic measure \( \mu \) supported on \( \tilde{\Lambda} \) such that
\[
\int_{\tilde{\Lambda}} (\log(\tilde{h}_t(e)) + \log \| \tilde{\psi}_t \mid_{N^{cs}(e)} \|) d\mu \geq 0.
\]
From the condition (A3) of \( l, \eta \), we know that \( \mu \) can not be supported on \( \bigcup_{\sigma \in S} \tilde{\Lambda}_{\sigma} \). Then we have
\[
\int_{\Lambda \setminus \text{Sing}(X)} (\log(h_t(x)) + \log \| \psi_t \mid_{N^{cs}(x)} \|) d\rho_\ast \mu \geq 0,
\]
where \( \rho_\ast \mu \) denoted the \( \varphi_l \)-ergodic measure on \( \Lambda \) given by \( \rho_\ast \mu(\cdot) = \mu(\varphi_1^{-1}(\cdot)) \). Note that for \( \rho_\ast \mu \)-a.e. point \( x \), we have
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(h_t^i(\varphi_i(x))) = \lim_{n \to +\infty} \frac{1}{n} (\log(h_{nl}^i(x)))
\]
\[
= \lim_{n \to +\infty} \frac{1}{n} \left[ \sum_{\sigma \in S_1} (\log(k^\sigma(\varphi_{nl}(x))) - \log(k(x)))
\right.
\]
\[
- \sum_{\sigma \in S_2} (\log(k^\sigma(\varphi_{nl}(x))) - \log(k(x))) \right] = 0.
\]
then we have
\[
\int_{\Lambda \setminus \text{Sing}(X)} \log(h_t^i(x)) d\rho_\ast \mu = 0
\]
by the Birkhoff’s ergodic theorem. Hence we have
\[
\int_{\Lambda \setminus \text{Sing}(X)} \log \| \psi_t \mid_{N^{cs}(x)} \| d\rho_\ast \mu \geq 0.
\]
Then we can take a well closable point \( a \) which is also a generic point of \( \rho_\ast \mu \) by the ergodic closing lemma (Proposition 2.3). By the Birkhoff’s ergodic theorem, we have
\[
\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| \psi_t \mid_{N^{cs}(\varphi_i(a))} \| \geq 0.
\]
From condition (A2) of \( l, \eta \), we know that \( a \) can not be a periodic point of \( X \). By the choice that \( a \) is also a well closable point, we can find a sequence of \( Y_n \to X \) with a periodic point \( p_n \) such that the orbit of \( p_n \) (under \( Y_n \) respectively) close to the orbit segment of \( \varphi_{[0,\pi(p_n)]}(a) \).
Since \( a \) is not periodic, we have \( \pi(p_n) \to \infty \) as \( n \to \infty \).

Claim There is \( N_1 > 0 \), such that for any \( n > N_1 \), we have
\[
\frac{1}{[\pi(p_n)/l]} \sum_{i=0}^{[\pi(p_n)/l-1]} \log \| \psi_t^Y \mid_{N^{cs}(\varphi_{it}^Y(p_n))} \| > -\eta.
\]
Proof of the Claim 1  Take $K > 0$ be a uniform bound of $\log \| \psi^Y_l | N^{cs}_x \|$ on $x \in \Lambda Y \setminus Sing(Y)$, that is, for any $Y \in \mathcal{U}$ and any $x \in \Lambda Y \setminus Sing(Y)$, one has

$$| \log \| \psi^Y_l | N^{cs}_x \| | < K.$$ 

Since $\rho_* \mu(Sing(X) \cap \Lambda) = 0$, we can take an open neighborhood $V$ of $Sing(X) \cap \Lambda$ such that $\rho_* \mu(\partial V) = 0$ and $\rho_* \mu(V) < \frac{n}{6K}$. Since $a$ is a generic point of $\rho_* \mu$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{\psi^Y_l(a)} \to \rho_* \mu$$

in the weak$^*$-topology, where $\delta_x$ denotes the atom measure at $x$. Denote $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{\psi^Y_l(a)}$ by $\mu_n$ for all $n \in \mathbb{N}$. Then we can find $N_0 > 0$ such that for any $n > N_0$, we have $\mu_n(V) < \frac{n}{6K}$. This implies

$$\sum_{0 \leq i < n : \psi^Y_l(a) \in V} \frac{n}{\eta} < \frac{\eta}{6K}$$

for all $n \geq N_0$. From the estimation (5) we can also take $N_0$ with the property

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \| \psi^Y_l | N^{cs}_x \| > -\frac{\eta}{3}$$

for all $n > N_0$. By the continuity of $\psi_l$ and bundle $N^{cs}$ we can take $N_1 > 0$ and $\delta > 0$ such that for any $n > N_1$ and any $x \in \Lambda \setminus V$ and $y \in \Lambda Y_n \setminus V$, if $d(x, y) < \delta$, then

$$| \log \| \psi^Y_l | N^{cs}_x \| - \log \| \psi^Y_l | N^{cs}_y \| | < \frac{\eta}{3}.$$ 

Without loss of generality, we can assume that for any $n > N_1$, we have $[\pi(p_n)/l - 1] > N_0$ and

$$d(\psi^Y_l(a), \psi^Y_l(p_n)) < \delta$$

for all $i = 0, 1, \cdots, [\pi(p_n)/l - 1]$. For shortness of notations, denote $[\pi(p_n)/l]$ by $k_n$. Then we have

$$\frac{1}{k_n} \sum_{i=0}^{k_n-1} \log \| \psi^Y_l | N^{cs}_x (\psi^Y_l(p_n)) \|$$

$$= \frac{1}{k_n} \left( \sum_{0 \leq i < k_n, \psi^Y_l(a) \in V} \log \| \psi^Y_l | N^{cs}_x (\psi^Y_l(p_n)) \| + \sum_{0 \leq i < k_n, \psi^Y_l(a) \notin V} \log \| \psi^Y_l | N^{cs}_x (\psi^Y_l(p_n)) \| \right)$$

$$\geq \frac{1}{k_n} \left( \sum_{i=0}^{k_n-1} \log \| \psi^Y_l | N^{cs}_x (\psi^Y_l(a)) \| - \sum_{0 \leq i < k_n, \psi^Y_l(a) \notin V} 2K - \sum_{0 \leq i < k_n, \psi^Y_l(a) \notin V} \frac{\eta}{3} \right)$$

$$< - \frac{\eta}{3} - \frac{\sum_{0 \leq i < n : \psi^Y_l(a) \in V} \eta}{k_n} \cdot 2K - \frac{\eta}{3} \geq -\eta.$$ 

for all $n > N_1$. This ends the proof of the claim \(\square\)

This claim contradicts with the condition (A1) on the periodic orbits of $Y$ close to $X$. This proves that $h^*_l \cdot \psi^l | N^{cs}$ is contracting. Similarly we can prove that $h^*_l \cdot \psi^l | N^{cs}$ is expanding. This ends the proof of Theorem 5.2. \(\square\)
Addendum  It was pointed out by the referee, the definition of multisingular partially hyperbolic in Theorem 5.2 may not be robust. By using the blow-up by Bonatti-da Luz, it is possible to define another hyperbolic notion that is robust. For a compact invariant set \( \Lambda \), one defines the extended set \( E(\Lambda) = E(\Lambda, X) \) in \( S_\Lambda M \) by the following way: a unit vector \( u \in E(\Lambda) \) if and only if for any \( \epsilon > 0 \), there are a vector field \( Y \) that is \( \epsilon \)-close to \( X \), a point \( y \in M \) such that \( y \) is a regular point of \( Y \), \( d(\varphi^t_Y(y), \Lambda) < \epsilon \) for any \( t \in \mathbb{R} \) and \( d(Y(y)/\|Y(y)\|, u) < \epsilon \). An extended normal bundle \( \tilde{N} \) can be attached on \( E(\Lambda) \) and an extended linear Poincaré flow \( \tilde{\psi} \) can be defined on \( \tilde{N} \). Similarly one can define the pragmatical cocycles associated to singularities and reparametrizing cocyles on \( E(\Lambda) \times \mathbb{R} \). A compact invariant set \( \Lambda \) is said to be extended multisingular partially hyperbolic if the extended normal bundle \( \tilde{N}_E(\Lambda) \) has a dominated splitting \( \tilde{N}^{cs} \oplus \tilde{N}_1 \oplus \tilde{N}^{cu} \) w.r.t. the extended linear Poincaré flow \( \tilde{\psi}_t \), and there exist two reparametrizing cocycles \( h^s_t : E(\Lambda) \rightarrow \mathbb{R}^+ \) and \( h^u_t : E(\Lambda) \rightarrow \mathbb{R}^+ \) such that, \( h^s_t : \tilde{\psi}_t|\tilde{N}^{cs} \) is uniformly contracting and \( h^u_t : \tilde{\psi}_t|\tilde{N}^{cu} \) is uniformly expanding.

One can know that such a splitting is robust and in Theorem 5.2, it is possible to prove for generic vector field away from ones with a homoclinic tangency, every isolated non-trivial transitive set admits an extended multisingular partially hyperbolic splitting. But it will involve more technics and it is far from the main topic of this paper because now we try to define some notion of hyperbolicity without using the blow-ups.

At the ending of this section, we propose a conjecture here. Similar to the definition of hyperbolic or partially hyperbolic diffeomorphisms, we can say a vector field \( X \) is multisingular partially hyperbolic if the chain-recurrence set of \( X \) can be split into finitely many compact invariant sets such that each one admits a multisingular partially hyperbolic splitting, whose center bundle can be split into one-dimensional dominated bundles for the linear Poincaré flow. Inspired by [9], one can even have the following conjecture:

**Conjecture 1** Any vector field can be either accumulated by ones with a homoclinic tangency, or accumulated by a multisingular partially hyperbolic vector field.

Note that from the results of da Luz, hyperbolicity and homoclinic tangencies may not be good dichotomy for higher dimensional singular flows. So in the above conjecture, we put the notion “multisingular partially hyperbolic” and homoclinic tangencies as a dichotomy. Even multisingular partial hyperbolicity may not be robust, we still wonder whether the above conjecture is true as a first step to understand global dynamics for vector fields away from ones with a homoclinic tangency.

### 6 Partial Hyperbolicity

In this section we will prove Theorem B. We give a name for a special case of multisingular partially hyperbolicity before we give the proof of Theorem B.

**Definition 6.1** Let \( \Lambda \) be a compact invariant set. We say that \( \Lambda \) is singular partially hyperbolic, if \( N_{\Lambda \setminus \text{Sing}(X)} \) has a dominated splitting \( N^{cs} \oplus N_1 \oplus N^{cu} \) w.r.t. the linear Poincaré flow, and if one of the following cases occur:

1. There are constants \( C > 0 \) and \( \lambda > 0 \) such that for any regular point \( x \in \Lambda \) and any \( t > 0 \), one has

\[
\frac{\|\Phi_t|_{N^{cs}(x)}\|}{\|\Phi_t|_{<X(x)>}\|} \leq Ce^{-\lambda t}, \quad \frac{\|\Phi_{-t}|_{N^{cu}(x)}\|}{\|\Phi_{-t}|_{<X(x)>}\|} \leq Ce^{-\lambda t}.
\]
2. There are constants $C > 0$ and $\lambda > 0$ such that for any regular point $x \in \Lambda$ and any $t > 0$, one has
\[
\frac{\|\psi_t|N^{cs}(x)\|}{\|\Phi_t|_{\langle X(x)\rangle}\|} \leq Ce^{-\lambda t}, \quad \frac{\|\psi_{-t}|N^{cu}(x)\|}{\|\Phi_{-t}|_{\langle X(x)\rangle}\|} \leq Ce^{-\lambda t}.
\]

3. There are constants $C > 0$ and $\lambda > 0$ such that for any regular point $x \in \Lambda$ and any $t > 0$, one has
\[
\frac{\|\psi_t|N^{cs}(x)\|}{\|\Phi_t|_{\langle X(x)\rangle}\|} \leq Ce^{-\lambda t}, \quad \frac{\|\psi_{-t}|N^{cu}(x)\|}{\|\Phi_{-t}|_{\langle X(x)\rangle}\|} \leq Ce^{-\lambda t}.
\]

**Remark** (1) Liao [18] defined the rescaled linear Poincaré flow $\psi_t^*$, which is defined for any regular point $x$ and any vector $v \in N_x$ by the following way:
\[
\psi_t^*(v) = \frac{\psi_t(v)}{\|\psi_t|_{\langle X(x)\rangle}\|} = \frac{|X(x)|\psi_t(v)}{|X(\varphi_t(x))|}.
\]
Thus the condition $\frac{\|\psi_t|N^{cs}(x)\|}{\|\Phi_t|_{\langle X(x)\rangle}\|} \leq Ce^{-\lambda t}$ is equivalent to say that $N^{cs}$ is contracting w.r.t. $\psi_t^*$.

(2) The notion was inspired by singular hyperbolicity. It is proved in [33] that if $\Lambda$ is a transitive set and every singularity in $\Lambda$ is hyperbolic, and $\Lambda$ admits a singular-partially hyperbolic splitting without neutral sub-bundles, then $\Lambda$ is singular hyperbolic.

One can verify the singular partially hyperbolic splitting for a nontrivial isolated transitive set with homogenous index on singularities when we consider a $C^1$-generic vector field away from ones with a homoclinic tangency.

**Theorem 6.2** Let $X \in \mathcal{X}^1(M) \setminus \overline{HT} \cap \mathcal{R}$ and $\Lambda$ be a nontrivial isolated transitive set of $X$. If all singularities in $\Lambda$ have the same index, then $\Lambda$ admit a singular-partially hyperbolic splitting $N^{cs} \oplus N_1 \oplus \cdots \oplus N_k \oplus N^{cu}$, where $\dim N_i = 1$ for any $1 \leq i \leq k$.

**Proof** In Theorem 5.2 we have proved that there is a mutisingular partially hyperbolic splitting $N_{\Lambda \setminus \text{Sing}}(x) = N^{cs} \oplus N_1 \oplus \cdots \oplus N_k \oplus N^{cu}$ with two cocycles $h_t^x, h_t^\sigma$ such that $h_t^x \cdot \psi_t|_{N^{cs}}$ is contracting and $h_t^\sigma \cdot \psi_t|_{N^{cu}}$ is expanding. Under the additional assumption that all singularities in $\Lambda$ have the same index, we will prove that this splitting is a singular partially hyperbolic splitting satisfying Theorem 6.2.

Without loss of generality, we assume that there exists at least one singularity in $\Lambda$. Then we have three cases: (1) $\alpha < \text{Ind}(\sigma) < \beta + 1$ for all $\sigma \in \Lambda \setminus \text{Sing}(X)$; (2) $\alpha < \text{Ind}(\sigma) = \beta + 1$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$; (3) $\alpha = \text{Ind}(\sigma) < \beta + 1$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$, where $\alpha, \beta$ is the minimal and maximal number of
\[
\{i : \Lambda \text{ contains an } i \text{ - periodic limit}\}.
\]

**Claim** If $\alpha < \text{Ind}(\sigma) < \beta + 1$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$, then Condition 1 of Definition 6.1 will be satisfied.

**Proof of the Claim** Let us recall the constructions of $h_t^x, h_t^\sigma$. We know that
\[
h_t^x(x) = \left(\prod_{\sigma \in S} h_t^\sigma\right)^{-1} = h_t^\sigma(x)
\]

where $h_t^\sigma$ is the pragmatic cocycle associated to $\sigma$ and $S = \Lambda \setminus \text{Sing}(X)$. Here $h_t^\sigma$ was defined by the following way: we defined a positive continuous function $k_\sigma : \Lambda \setminus \text{Sing}(x) \to \mathbb{R}$
such that \( k^\sigma(x) = \|X(x)\| \) for \( x \in V(\sigma) \) and \( k^\sigma(x) = c \) for \( x \notin V(\sigma) \) where \( V(\sigma) \) is an isolated neighborhood of \( \sigma \) and \( c \) is a positive constant, then set \( h_1^\sigma(x) = \frac{k^\sigma(x)}{k^\sigma(x)} \). Now we define a positive continuous function \( k : \Lambda \setminus Sing(x) \to \mathbb{R} \) by letting

\[
k(x) = \|X(x)\|, \quad \forall x \in \cup_{\sigma \in S} V(\sigma); \quad k(x) = c, \quad \forall x \notin \cup_{\sigma \in S} V(\sigma).
\]

By a direct computation we know that

\[
h_1^t(x) = \left( \frac{k(\varphi_t(x))}{k(x)} \right)^{-1} = h_t^\sigma(x), \quad \forall x \in \Lambda \setminus Sing(x), t \in \mathbb{R}.
\]

There exist two positive constants

\[
c_1 = \frac{\max\{\|X(x)\| : x \in \Lambda \setminus (\cup_{\sigma \in S} V(\sigma))\}}{c}, \quad c_2 = \frac{c}{\min\{\|X(x)\| : x \in \Lambda \setminus (\cup_{\sigma \in S} V(\sigma))\}}
\]

such that for any \( x \in \Lambda \setminus Sing(X) \), one has

\[
c_2\|X(x)\| \leq k(x) \leq c_1\|X(x)\|.
\]

So we have

\[
c_1^{-1}c_2\|\Phi_t|_{<X(x)>}\| = c_1^{-1}c_2\frac{\|X(\varphi_t(x))\|}{\|X(x)\|} \leq h_t^\sigma(x)
\]

\[
h_t^\sigma(x) \leq c_1c_2^{-1}\frac{\|X(\varphi_t(x))\|}{\|X(x)\|} = c_1c_2^{-1}\|\Phi_t|_{<X(x)>}\|.
\]

From \( h_t^\sigma(x) \cdot \psi_t|_{N^{cs}} \) and \( h_t^\sigma(x) \cdot \psi_t|_{N^{cu}} \) is contracting and expanding respectively we know that there exist \( C > 1, \lambda > 0 \) such that for any \( x \in \Lambda \setminus Sing(X) \) and any \( t > 0 \),

\[
h_t^\sigma(x) \cdot \|\psi_t|_{N^{cs}(x)}\| \leq Ce^{-\lambda t} \quad \text{and} \quad h_t^\sigma(x) \cdot \|\psi_t|_{N^{cu}(x)}\| \leq Ce^{-\lambda t}.
\]

Thus we have

\[
\frac{\|\psi_t|_{N^{cs}(x)}\|}{\|\Phi_t|_{<X(x)>}\|} \leq c_1c_2^{-1}Ce^{-\lambda t}, \quad \frac{\|\psi_t|_{N^{cu}(x)}\|}{\|\Phi_t|_{<X(x)>}\|} \leq c_1c_2^{-1}Ce^{-\lambda t}.
\]

This proves that Condition 1 of Definition 6.1 is satisfied.

Similarly, in the case of \( \alpha < \text{Ind}(\sigma) = \beta + 1 \) for all \( \sigma \in \Lambda \cap Sing(X) \) we can take a positive continuous map \( k : \Lambda \setminus Sing(X) \to \mathbb{R} \) defined by

\[
k(x) = \|X(x)\|, \quad \forall x \in \cup_{\sigma \in S} V(\sigma); \quad k(x) = c, \quad \forall x \notin \cup_{\sigma \in S} V(\sigma),
\]

such that

\[
h_1^\sigma(x) = \frac{k(\varphi_t(x))}{k(x)} \quad \text{and} \quad h_t^\sigma(x) = \left( \frac{k(\varphi_t(x))}{k(x)} \right)^{-1}
\]

for all \( x \in \Lambda \setminus Sing(X) \) and \( t \in \mathbb{R} \), where \( V(\sigma) \) are isolated neighborhoods for the hyperbolic singularities \( \sigma \in \Lambda \). Then by the fact that \( h_t^\sigma(x) \cdot \psi_t|_{N^{cs}} \) is contracting and \( h_t^\sigma(x) \cdot \psi_t|_{N^{cu}} \) is expanding we can find \( C > 1, \lambda > 0 \) such that

\[
\frac{\|\psi_t|_{N^{cs}(x)}\|}{\|\Phi_t|_{<X(x)>}\|} \leq Ce^{-\lambda t}, \quad \|\Phi_t|_{<X(x)>}\| \cdot \|\psi_t|_{N^{cu}(x)}\| \leq Ce^{-\lambda t}
\]

for all \( x \in \Lambda \setminus Sing(X) \) and \( t > 0 \). This proves that Condition 2 in Definition 6.1 is satisfied in this case.
Similarly, in the case of $\alpha = \text{Ind}(\sigma) < \beta + 1$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$, Condition 3 in Definition 6.1 will be satisfied. Note that we already have $\dim N_i = 1$ for any $1 \leq i \leq k$. This proves that in both cases the splitting $N^{cs} \oplus N_1 \oplus \cdots \oplus N_k \oplus N^{cu}$ is the singular partially hyperbolic splitting of the theorem. This ends the proof of Theorem 6.2.

Here we present a criterion on partially hyperbolic splitting whose proof is in [33]. See also [15, Sect. 2.4].

**Lemma 6.3** Assume $\Lambda$ is a nontrivial transitive set and contains no nonhyperbolic singularities. If there is a dominated splitting $N_{\Lambda \setminus \text{Sing}(X)} = N^{cs} \oplus N^{cu}$ w.r.t. $\psi$, with constants $C > 1, \lambda > 0$ such that

$$\|\Phi_t|_{\langle X(\cdot)\rangle}\|^{-1} \cdot \|\psi_t|_{N^{cs}(\cdot)}\| \leq C e^{-\lambda t}$$

for all $x \in \Lambda \setminus \text{Sing}(X)$ and $t > 0$, then there is a partially hyperbolic splitting $T_{\Lambda} M = E^{ss} \oplus F$ w.r.t. $\Phi_t$ with $\dim E^{ss} = \dim N^{cs}$.

We prove the following proposition which will imply Theorem B.

**Proposition 6.4** Let $X \in (X^1(M) \setminus \overline{HT}) \cap R$ and $\Lambda$ be a nontrivial isolated transitive set of $X$. If all singularities in $\Lambda$ have the same index and there is a singularity $\sigma$ whose saddle value $I(\sigma) \geq 0$, then $\Lambda$ admits a partially hyperbolic splitting $T_{\Lambda} M = E^{ss} \oplus F$ w.r.t. the tangent flow, where $E^{ss}$ is contracting under $\Phi_t$.

**Proof** By Theorem 6.2 we know that under the assumptions, there exists a singular partially hyperbolic splitting $N^{cs} \oplus N_1 \oplus \cdots \oplus N_k \oplus N^{cu}$. Note here we have $I(\sigma) \geq 0$ for some singularities $\sigma \in \Lambda$. By Lemma 3.1 we know that either $\alpha < \text{Ind}(\sigma) < \beta + 1$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$ or $\alpha < \text{Ind}(\sigma) = \beta + 1$ for all $\sigma \in \Lambda \cap \text{Sing}(X)$. In both cases we have

$$\|\psi_t|_{N^{cs}(\cdot)}\| \leq C e^{-\lambda t},$$

for any $x \in \Lambda \setminus \text{Sing}(X)$ and $t > 0$ by the proof of Theorem 6.2. By Lemma 6.3 we know that $\Lambda$ admits a partially hyperbolic splitting $T_{\Lambda} M = E^{ss} \oplus F$ w.r.t. $\Phi_t$ such that $\dim E^{ss} = \dim N^{cs}$. This completes the proof.

**The proof of Theorem B** Let $X \in R \cap (X^1(M) \setminus \overline{HT})$ and $\Lambda$ be a robustly transitive set of $X$. By Theorem A, we know that all singularities in $\Lambda$ are hyperbolic and have the same index. If there is a singularity $\sigma \in \Lambda$ with $I(\sigma) \geq 0$ then by Proposition 6.4 we know that $\Lambda$ is partially hyperbolic with a splitting $E^{ss} \oplus E^{cu}$ where $E^{ss}$ is uniformly contracting. If there is a singularity $\sigma \in \Lambda$ with $I(\sigma) \leq 0$, then by Proposition 6.4 we know that $\Lambda$ is partially hyperbolic with a splitting $E^{cs} \oplus E^{uu}$ where $E^{uu}$ is uniformly expanding. This proves Theorem B.

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