Minimal representations, spherical vectors, and exceptional theta series

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Abstract: Theta series for exceptional groups have been suggested as a possible description of the eleven-dimensional quantum supermembrane. We present explicit formulae for these automorphic forms whenever the underlying Lie group $G$ is split (or complex) and simply laced. Specifically, we review and construct explicitly the minimal representation of $G$, generalizing the Schrödinger representation of symplectic groups. We compute the spherical vector in this representation, i.e. the wave function invariant under the maximal compact subgroup, which plays the role of the summand in the automorphic theta series. We also determine the spherical vector over the complex field. We outline how the spherical vector over the $p$-adic number fields provides the summation measure in the theta series, postponing its determination to a sequel of this work. The simplicity of our result is suggestive of a new Born-Infeld-like description of the membrane where U-duality is realized non-linearly. Our results may also be used in constructing quantum mechanical systems with spectrum generating symmetries.

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1. Introduction

Despite considerable insights afforded by dualities, the fundamental degrees of freedom of M-theory remain elusive. Recently the rôle of the eleven-dimensional supermembrane has been tested \[1\] in an attempt to rederive toroidally compactified, M-theoretic, supersymmetric four-graviton scattering amplitudes at order $R^4$. These
amplitudes are known independently on the basis of supersymmetry and duality, to be given by an Eisenstein series of the U-duality group \( \mathbb{U} \) (see \( \mathbb{G} \) for a review), but still lack a finite microscopic derivation (see however \( \mathbb{H} \) for a discussion of perturbative computations in eleven-dimensional supergravity). In analogy with the string one-loop computation, a one-loop membrane amplitude was constructed as the integral of a modular invariant partition function on the fundamental domain of a membrane modular group \( GL(3, \mathbb{Z}) \). The action of a membrane instanton configuration with given winding numbers is given by the Polyakov action and as a working hypothesis the summation measure was taken to be unity. A comparison to the exact result showed that the mass spectrum and the instanton saddle points were correctly reproduced by this ansatz, but the spectrum multiplicities and instanton summation measure were incorrect \(^3\). The proposed partition function was therefore not U-duality invariant. However, a general method to construct invariant partition functions was outlined: exceptional theta series should provide the correct partition function for the BPS membrane on torii.

While theta series for symplectic groups are very common both in mathematics, e.g., in the study of Riemann surfaces, and physics where they arise as partition functions of free theories, their generalization to other groups is not as well understood. One difficulty is that group invariance requires a generalization of the standard Poisson resummation formula (i.e., Gaussian integration) to cubic characters (i.e., “Airy” integration). This scenario is clearly well adapted to the membrane situation, where the Wess-Zumino interaction is cubic in the brane winding numbers. Since theta series reside at the heart of many problems in the theory of automorphic forms, it would be very desirable from both physical and mathematical viewpoints, to have explicit expressions for them.

As outlined in \( \mathbb{I} \), the construction of theta series for a simple non-compact group \( G \) requires three main ingredients: (i) An irreducible representation of the group in an appropriate space of functions. In the symplectic case, this is simply the Weil representation of the Heisenberg algebra \( [p_i, x^j] = -i \delta^j_i \), which gives rise to the Schrödinger representation of \( Sp(n, \mathbb{R}) \). (ii) A special function \( f \), known as the spherical vector, which is invariant under the maximal compact subgroup \( K \) of \( G \). This generalizes the Gaussian character \( e^{2\pi i (x^i)^2} \) appearing in the symplectic theta series. (iii) A distribution \( \delta \) invariant under an arithmetic subgroup \( G(\mathbb{Z}) \subset G \) generalizing the sum with unit weight over integers \( x^i \in \mathbb{Z} \) of the symplectic case.

As for step (i), one observes that for any simple Lie algebra \( \mathfrak{g} \) there exists a unique non-zero minimal conjugacy class \( \mathcal{O} \subset \mathfrak{g} \). This nilpotent orbit carries the standard Kirillov-Kostant symplectic form, whose quantization furnishes a representation of \( G \) on the Hilbert space of wave functions on a Lagrangian submanifold of \( \mathcal{O} \). Its quantization relies heavily on the existence, discovered by Joseph \( \mathbb{J} \), of a unique

\(^3\)See \( \mathbb{K} \) for a very recent discussion of the membrane summation measure.
completely prime two-sided ideal $J$ of the enveloping algebra $U(\mathcal{G})$ whose characteristic variety coincides with $\mathcal{O} \cup \{0\}$. The obtained representation is minimal, in the sense that its Gelfand-Kirillov dimension is smallest among all representations, being equal to half the dimension of $\mathcal{O}$. The minimal representation exists not only for the split real group $G(\mathbb{R})$, but also for the group $G(F)$ for arbitrary local field $F$ as long as $\mathcal{G}$ is any simply-laced split Lie algebra. In the case when $\mathcal{G}$ is of the type $D_n$, the minimal representation can be realized using Howe’s theory of dual pairs \cite{12}. The general construction was described in \cite{10} and \cite{11}, the latter of which we will closely follow in this work. Step (ii) is the main subject of the present paper; we will obtain the spherical vector for all groups $G(\mathbb{R})$ of $A, D, E$ type in the split real form, using techniques from Eisenstein series ($A_n$), dual pairs ($D_n$) and PDE’s ($E_6, 7, 8$). A simple generalization will also provide the spherical vector for the complex group $G(\mathbb{C})$. As we will see, step (iii) amounts to solving step (ii) over all $p$-adic number fields $\mathbb{Q}_p$ instead of the reals. Our methods will allow us to obtain the $p$-adic spherical vector for $A$ and $D$ groups. The exceptional case requires more powerful techniques, and will be treated in a sequel to this paper \cite{13}.

While this paper is mostly concerned with the mathematical construction of exceptional theta series, a few words about the physical implications of our results are in order. First and foremost, we find that a membrane partition function invariant under both the modular group $GL(3, \mathbb{Z})$ and the U-duality group $E_d(\mathbb{Z})$ cannot be constructed by summing over the $3d$ membrane winding numbers alone (which confirms the findings of \cite{1}). Indeed, the dimension of the minimal representation of the smallest simple group $G$ containing $SL(3, \mathbb{R}) \times E_d$ is always bigger than $3d$. Second, we find that the minimal representation of $G$ has a structure quite reminiscent of the membrane, but (in the simplest $d=3$ case) necessitates two new quantum numbers, which would be very interesting to understand from the point of view of the quantum membrane. In fact, the form of the spherical vector in this representation, displayed in equations (4.41) and (4.52) below, is very suggestive of a Born-Infeld-like formulation of the membrane, which would then exhibit a hidden dynamical $E_{d+2}(\mathbb{Z})$ symmetry. A more complete physical analysis of these results in the context of the eleven-dimensional supermembrane will appear elsewhere. In addition, our minimal representation provides the quantized phase space for quantum mechanical systems with dynamical non-compact symmetries, which may find a use in M-theory or other contexts. By choosing one of the compact generators as the Hamiltonian, one may construct integrable quantum mechanical systems with a spectrum-generating exceptional symmetry, and the spherical vector we constructed would then give the ground state wave function.

The organization of this paper is as follows: In Section 2, we use the $SL(2)$ case as a simple example to introduce the main technology. In Section 3, we review the construction of the minimal representation for simply-laced groups. Section 4 contains the new results of this paper; real and complex spherical vectors for all $A, D, E$ groups
(the main formulae may be found in equations (4.18), (1.23), (1.43), (1.53), (1.69), (1.84) and (4.88)). We close in Section 5 with a preliminary discussion of the physics interpretation of our formulae. Miscellaneous group theoretical data is gathered in the Appendix.

2. \textit{Sl}(2) revisited

As an introduction to our techniques, let us consider two familiar examples of automorphic forms for \textit{Sl}(2, \mathbb{Z}).

2.1 Symplectic theta series

Our first example is the standard Jacobi theta series

$$
\theta(\tau) = \tau^{1/4} \sum_{m \in \mathbb{Z}} e^{i \pi \tau m^2} = \sum_{m \in \mathbb{Z}} f_\tau(m),
$$

where we inserted a power of \( \tau_2 \) to cancel the modular weight. As is well known, this series is an holomorphic modular form of \textit{Sl}(2, \mathbb{Z}) up to a system of phases. The invariance under the generator \( T : \tau \rightarrow \tau + 2 \) is manifest, while the transformation under \( S : \tau \rightarrow -1/\tau \) yielding

$$
\theta(-1/\tau) = \sqrt{i} \ \theta(\tau),
$$

follows from the Poisson resummation formula,

$$
\sum_{m \in \mathbb{Z}} f(m) = \sum_{p \in \mathbb{Z}} \tilde{f}(p), \quad \tilde{f}(p) \equiv \int dx f(x) e^{2\pi ipx},
$$

applied to the Gaussian kernel \( f_\tau(x) \). A better understanding of the mechanism behind the invariance of the theta series (2.1) can be gained (see e.g., [14]) by rewriting it as

$$
\theta(\tau) = \langle \delta, \rho(g_\tau) \cdot f \rangle.
$$

In this symbolic form, \( \rho \) is a representation of the double cover \( \hat{G} \) of \textit{Sl}(2, \mathbb{R}) in the space \( \mathcal{S} \) of Schwartz functions of one variable; \( g_\tau = \left( \begin{array}{cc} 1 & \tau_1 \\ 0 & \tau_2 \end{array} \right) / \sqrt{\tau_2} \) is an element of \( G = \textit{Sl}(2, \mathbb{R}) \) parameterizing the coset \( U(1) \backslash \textit{Sl}(2, \mathbb{R}) \) in the Iwasawa gauge; \( f(x) = e^{-x^2/2} \) is the spherical vector of the representation \( \rho \), \textit{i.e.} an element of \( \mathcal{S} \) which is an eigen-vector of the preimage \( \hat{U} \subset \hat{G} \) of the maximal compact subgroup \( K = U(1) \) of \( G \) corresponding to the basic character of \( \hat{U} \); finally, \( \delta_\mathcal{S}(x) = \sum_{m \in \mathbb{Z}} \delta(x - m) \) is a distribution in the dual space of \( \mathcal{S} \), invariant under the action of \( \textit{Sl}(2, \mathbb{Z}) \). [The inner product \( \langle \cdot, \cdot \rangle \) is just integration \( \int dx \).] The invariance of \( \theta(\tau) \) then follows trivially from the covariance of the various pieces in (2.4).
More explicitly, $\rho$ is the so-called metaplectic representation

$$
\rho \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) : \phi(x) \rightarrow e^{i\pi t x^2} \phi(x),
$$

(2.5)

$$
\rho \left( \begin{array}{cc} e^{-t} & 0 \\ 0 & e^t \end{array} \right) : \phi(x) \rightarrow e^{t/2} \phi(e^t x),
$$

(2.6)

$$
\rho \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) : \phi(x) \rightarrow e^{i\pi/4} \tilde{\phi}(-x).
$$

(2.7)

acting on a function $\phi \in S$. It is easily checked that the defining relation $(ST)^3 = 1$ holds modulo a phase, and that the generators $S$ and $T$ leave the distribution $\delta$ invariant. Linearizing (2.3) and (2.6) yields generators for the positive root and Cartan elements

$$
E_+ = i\pi x^2, \quad H = \frac{1}{2} (x \partial_x + \partial_x x),
$$

(2.8)

while the negative root follows by a Weyl reflection

$$
E_- = -\rho(S) \cdot E_+ \cdot \rho(S^{-1}) = \frac{i}{4\pi} \partial^2_x,
$$

(2.9)

and we have the $SL(2, \mathbb{R})$ algebra,

$$
[H, E_{\pm}] = \pm 2 E_{\pm}, \quad H = [E_+, E_-].
$$

(2.10)

In this representation, there does not exist a spherical vector strictly speaking, since the compact generator $E_+ - E_-$ (recognized as the Hamiltonian of the harmonic oscillator) does not admit a state with zero eigenvalue. The lowest state has eigenvalue $i/2$, and plays the role of the spherical vector in (2.4),

$$
(E_+ - E_-) f = \frac{i}{2} f, \quad f(x) = e^{-\pi x^2}.
$$

(2.11)

Its invariance (up to a phase) under the compact $K$ guarantees that the theta series (2.4) depends only on $\tau \in K \backslash G$ (up to a phase). In particular, the $S$ generator, corresponds to the rotation by an angle $\pi$ inside $K$, and therefore leaves $f$ invariant. This is the statement that the Gaussian kernel $f$ is invariant under Fourier transformation, and lies at the heart of the automorphic invariance of the theta series (2.1). The construction holds, in fact, for any symplectic group $Sp(n, \mathbb{Z})$ (with $Sp(1) = Sl(2)$), and leads to the well known Jacobi-Siegel theta functions,

$$
\theta_{Sp(n, \mathbb{Z})} = \sum_{(m^i) \in \mathbb{Z}^n} e^{i\pi m^i \tau_{ij} m^j}.
$$

(2.12)

This corresponds to the minimal representation

$$
E_{ij} = \frac{i}{2} x^i x^j, \quad E_{ij} = \frac{i}{2} \partial_i \partial_j, \quad H^i_j = (x^i \partial_j + \partial_j x^i)/2
$$

(2.13)
of $Sp(n, \mathbb{R})$, with algebra
\[ [E^{ij}, E_{kl}] = \frac{1}{4} \left( \delta^i_l H^j_k + \delta^j_l H^i_k + \delta^j_k H^i_l + \delta^i_k H^j_l \right), \tag{2.14} \]
acting on the Schwartz space of functions of $n$ variables $x_i$ (see e.g., [9]).

2.2 Eisenstein series and spherical vector

Our second example is the non-holomorphic Eisenstein series (see e.g., [15, 4])
\[ E_s(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \left( \frac{\tau_2}{m+n\tau_2} \right)^s, \tag{2.15} \]
which is a function on upper half plane $U(1) \setminus Sl(2, \mathbb{R})$ parameterized by $\tau$ and is invariant under the right action of $Sl(2, \mathbb{Z})$ given by $\tau \rightarrow (a \tau + b)/(c \tau + d)$. This action can be compensated by a linear one on the vector $(m, n)$ and the Eisenstein series can therefore be rewritten in the symbolic form (2.4), where now $\delta = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \delta(x - m) \delta(y - n)$ and $\rho$ is the linear representation
\[ \rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \phi(x, y) \rightarrow \phi(ax + by, cx + dy) \tag{2.16} \]
corresponding to the infinitesimal generators
\[ E_+ = x\partial_y, \quad E_- = y\partial_x, \quad H = x\partial_x - y\partial_y, \tag{2.17} \]
generating the $Sl(2)$ algebra (2.10). The spherical vector $f(x, y) = (x^2 + y^2)^{-s}$ of the representation $\rho$ is clearly invariant under the maximal compact subgroup $U(1) \subset Sl(2)$ generated by $E_+ - E_-$. In this case, it is not unique (any function of $x^2 + y^2$ is $U(1)$ invariant) because the linear action (2.10) on functions of two variables is reducible. An irreducible representation in a single variable, known as the first principal series, is obtained by restricting to homogeneous, even functions of degree $2s$
\[ \phi(x, y) = \lambda^{2s} \phi(\lambda x, \lambda y) \tag{2.18} \]
and setting $y = 1$ (say)
\[ \phi(x) \equiv \phi(x, y) \big|_{y=1}. \tag{2.19} \]
The representation $\rho$ induces an irreducible one
\[ \tilde{\rho}_s \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \phi(x) \rightarrow \phi(x + t), \tag{2.20} \]
\[ \tilde{\rho}_s \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} : \phi(x) \rightarrow e^{-2st} \phi(e^{-2t}x), \tag{2.21} \]
\[ \tilde{\rho}_s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \phi(x) \rightarrow x^{-2s} \phi(-1/x). \tag{2.22} \]
with spherical vector

\[ f_s = (x^2 + 1)^{-s} \]  

(2.23)

An equivalent representation can be obtained by Fourier transforming the variable \( x \). In terms of the Eisenstein series (2.15), this amounts to performing a Poisson resummation on \( m \),

\[
\mathcal{E}_{2,s}^{SL(2,\mathbb{Z})} = 2 \zeta(2s) \tau_2^s + \frac{2\sqrt{\pi} \tau_2^{1-s} \Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s)} \\
+ \frac{2\pi^s \sqrt{\tau_2}}{\Gamma(s)} \sum_{m \neq 0} \sum_{n \neq 0} \frac{|m|^s}{n} \frac{|n|^{s-1/2}}{K_{s-1/2} (2\pi |mn| \tau_2)} e^{-2\pi i mn \tau_1}.
\]  

(2.24)

Using instead the summation variable \( N = mn \), this can be rewritten as

\[
\mathcal{E}_{2,s}^{SL(2,\mathbb{Z})} = 2 \zeta(2s) \tau_2^s + \frac{2\sqrt{\pi} \tau_2^{1-s} \Gamma(s-1/2) \zeta(2s-1)}{\Gamma(s)} \\
+ \frac{2\pi^s \sqrt{\tau_2}}{\Gamma(s)} \sum_{N \in \mathbb{Z}^*} \mu_s(N) N^{s-1/2} K_{s-1/2} (2\pi \tau_2 N) e^{2\pi i \tau_1 N},
\]  

(2.25)

where the summation measure of the bulk term can be expressed in terms of the number-theoretic quantity

\[
\mu_s(N) = \sum_{n | N} n^{-2s+1}.
\]  

(2.26)

Indeed, disregarding for now the first two degenerate terms, we see that the Eisenstein series can again be written as in (2.4), where the summation measure is

\[
\delta_s(y) = \sum_{N \in \mathbb{Z}^*} \mu_s(N) \delta(y - N),
\]  

(2.27)

and the one-dimensional representation \( \rho_s \) acting as

\[
\rho_s \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \phi(y) \rightarrow e^{-it} \phi(y),
\]  

(2.28)

\[
\rho_s \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} : \phi(y) \rightarrow e^{2(s-1)t} \phi(e^{2t} y),
\]  

(2.29)

is generated by

\[
E_+ = iy, \quad E_- = i(y \partial_y + 2 - 2s) \partial_y, \quad H = 2y \partial_y + 2 - 2s.
\]  

(2.30)

Note that this minimal representation has a parameter \( s \), and is distinct from the one in (2.8,2.9). It is, of course, intertwined with the representation (2.21)-(2.22) by Fourier transform. The function

\[
f_s = y^{s-1/2} K_{s-1/2}(y)
\]  

(2.31)

can be easily checked to be annihilated by the compact generator \( K = E_+ - E_- = -i(y \partial_y^2 + (2 - 2s) \partial_y - y) \), and therefore is a spherical vector of the representation (2.30). At each value of \( s \), it is unique if one requires that it vanishes as \( y \rightarrow \infty \).
2.3 Summation measure, \( p \)-adic fields and degenerate contributions.

While the spherical vector can be easily obtained by solving a linear differential equation, the distribution \( \delta \) invariant under the discrete subgroup \( SL(2, \mathbb{Z}) \) appears to be more mysterious. In fact, it has a simple interpretation in terms of \( p \)-adic number fields, as we now explain.

The simplest instance arises for the \( \theta \) series (2.1) itself which can be rewritten (at the origin \( \tau = i \)) as a sum over principal adeles

\[
\theta(\tau = i) = \sum_{x \in \mathbb{Q}} \exp(-\pi x^2) \prod_{p \text{ prime}} \gamma_p(x)
\]

where \( \gamma_p(x) \) is 1 on the \( p \)-adic integers and 0 elsewhere. The real spherical vector is the Gaussian and the function \( \gamma_p(x) \) is its \( p \)-adic analog; just like the real Gaussian it is invariant under \( p \)-adic Fourier transform (the review [16] provides an introduction to \( p \)-adic numbers and integration theory for physicists). Hence \( \gamma_p(x) \) is the \( p \)-adic spherical vector of the representation (2.5), and we have thus obtained an “adelic” formula for the unit weight summation measure.

To take a less trivial case, consider the summation measure (2.26) appearing in the distribution \( \delta \) in (2.27). It can also be rewritten as an infinite product over primes,

\[
\sum_{N} \mu_s(N) = \sum_{x \in \mathbb{Q}} \prod_{p \text{ prime}} f_p(x), \quad f_p(x) = \gamma_p(x) \frac{1 - p^{-2s+1}|x|^2_{p^{s-1}}}{1 - p^{-2s+1}}, \quad \text{(2.33)}
\]

where \( |x|_p \) is the \( p \)-adic norm of \( N \) (if \( N \) is integer, \( |N| = p^{-k} \) where \( k \) is the largest integer such that \( p^k \) divides \( N \)). Just as above, \( f_p(x) \) can in fact be interpreted as the \( p \)-adic spherical vector of the representation (2.29). To convince oneself of this fact, one may take the \( p \)-adic Fourier transform of \( f_p \), and find

\[
\tilde{f}_p(u) = (1 - p^{-2s})^{-1} \max(|u|_p, 1)^{-2s}. \quad \text{(2.34)}
\]

This is indeed invariant under \( u \rightarrow -1/u \), and therefore is a spherical vector for the representation (2.20)\(^4\). It is in fact identical to the real spherical vector (2.15), upon replacing the orthogonal real norm \( ||(x,1)||^2 \equiv x^2 + 1 \) by the \( p \)-adic norm \( ||(x,1)||_p \equiv \max(|x|_p, 1) \). This suggests that the \( p \)-adic spherical vector is simply related to the real spherical vector by changing from orthogonal to \( p \)-adic norms and Bessel functions to “\( p \)-adic Bessel” functions. We shall not pursue this line further here, referring to [13] for a rigorous derivation.

Finally, we should say a word about the first two power terms in (2.25). As seen from the above Poisson resummation, these two terms can viewed as the regulated\(^4\)

\(^4\)One may also check that the product of \( \tilde{f}_p(u) \) over all \( p \) reproduces the correct summation measure in the Eisenstein series (2.15) upon using the summation variable \( u = m/n \).
value of the spherical vector $f(x)$ at $x = 0$. Unfortunately, we do not know of a direct way to extract them from $f(x)$ alone; an unsatisfactory method is to deduce them by imposing invariance of (2.25) under the generator $S$.

2.4 Generalization to $Sl(n, \mathbb{Z})$

The construction of the minimal representation of $Sl(2, \mathbb{R})$ above can be easily generalized to any $Sl(n)$ by starting with the $Sl(n, \mathbb{Z})$ Eisenstein series in the fundamental representation,

$$\mathcal{E}_{n; s}^{Sl(n, \mathbb{Z})} = \sum_{m^I \in \mathbb{Z}^n \setminus \{0\}} [m^I g_{IJ} m^J]^{-s}$$

and Poisson resumming one integer, $m^1 \equiv m$ say. In the language of $\mathbb{F}$, this amounts to the small radius expansion in one direction and we find

$$\mathcal{E}_{n; s}^{Sl(n, \mathbb{Z})} = 2\zeta(2s) R^{-2s} + \frac{\sqrt{\pi}}{R} \Gamma(s - 1/2) \sum_{m^I \in \mathbb{Z}^{n-1} \setminus \{0\}} [m^I \hat{g}_{ij} m^j]^{-s+1/2}$$

$$+ \frac{2\pi^s}{\Gamma(s) R^{s+1/2}} \sum_{m^I \in \mathbb{Z}^{n-1} \setminus \{0\}} \frac{m^2}{m^I \hat{g}_{ij} m^j} \left(\frac{s-1/2}{2} K_{s-1/2} \left(\frac{2\pi |m^I \hat{g}_{ij} m^j|}{R \sqrt{m^I \hat{g}_{ij} m^j}}\right) e^{-2\pi i m^I A_i}\right).$$

We have decomposed the $n$-dimensional metric $g_{IJ}$ parameterizing $SO(n, \mathbb{R}) \setminus Sl(n, \mathbb{R})$ into an $n - 1$ dimensional metric $\hat{g}_{ij} = g_{ij} - \frac{1}{R^2} A_i A_j$, the radius of the $n$-th direction $R = g^{1/2}$ and the off-diagonal metric $A_i = g_{1i}/g_{11}$. We now have an $n - 1$ dimensional representation of $Sl(n)$ on $n - 1$ variables $x^i$ with $Sl(n - 1)$ realized linearly. The infinitesimal generators corresponding to positive and negative roots are given by

$$E^i_+ = ix^i \quad E^-_i = i(x^j \partial_j + 2 - 2s) \partial_i,$$

$$E^i_+ j = x^i \partial_j \quad E^-_i j = x^j \partial_i \quad (i > j),$$

with Cartan elements following by commutation. This is the minimal representation of $Sl(n, \mathbb{R})$, generalizing the $Sl(2, \mathbb{R})$ case in (2.30). Note that this minimal representation again has a continuous parameter $s$. For other groups than $A_n$, the minimal representation will in fact be unique. For $A_n$, the above representation is unitary when $\text{Re}(s) = n/4$. The spherical vector is easily read off from (2.30), evaluated at the origin $\hat{g}_{ij} = g_{ij} = \delta_{ij}$, $R = 1$ (rescaling $x^i \rightarrow x^i/(2\pi)$)

$$f_{A_n, s} = \mathcal{K}_{s-1/2} \left(\sqrt{(x^i)^2}\right) = \mathcal{K}_{s-1/2}(||(x^1, \ldots, x^{n-1})||),$$

where $\mathcal{K}_t(x) \equiv x^{-t} K_t(x)$ ($K_t$ is the modified Bessel function of the second kind) and the Euclidean norm $||(x_1, x_2, \ldots)|| \equiv \sqrt{x_1^2 + x_2^2 + \cdots}$. This spherical vector is indeed annihilated by the compact generators following from (2.37). The $p$-adic
spherical vector in the representation corresponding to (2.37) may be obtained from the summation measure in (2.36) by the method as outlined in Section 2.3. The result is

$$f_p(x^1, \ldots, x^{n-1}) = \gamma_p(x^1) \cdots \gamma_p(x^{n-1}) \frac{1 - p^{-s} ||(x^1, \ldots, x^{n-1})||_p}{1 - p^{-s}}.$$  \hfill (2.39)

Again, this may be obtained from the real spherical vector (2.33) by replacing the Euclidean norm by the $p$-adic one along with $K_s \to K_{p,s}(x) = (1 - p^{-s} x)/(1 - p^{-s})$.

### 3. Minimal representation for simply laced Lie groups

The minimal representation we have described for $Sl(n, \mathbb{R})$ has been generalized in [11] to the case of simply-laced groups $G(F)$ for arbitrary local field $F$. In this Section, we shall review the construction of [11], and make it fully explicit.

#### 3.1 Nilpotent orbit and canonical polarization

The minimal representation can be understood as the quantization of the smallest co-adjoint orbit in $G$. In order to construct this minimal orbit, one observes that all simple Lie algebras have an essentially unique 5-grading (see e.g., [18])

$$G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$$ \hfill (3.1)

by the charge under the Cartan generator $H_\omega$ associated to the highest root $E_\omega$ (for a given choice of Cartan subalgebra and system of simple roots $\alpha_i$). The spaces $G_{\pm 2}$ have dimension 1 and are generated by the highest and lowest root $E_{\pm \omega}$ respectively. $G_1$ contains only positive roots, and $G_0$ contains all Cartan generators as well as the remaining positive roots and the corresponding negative ones; $G_{-k}$ is obtained from $G_k$ by mapping all positive roots to minus themselves. The grading (3.1) can also be obtained by obtaining by branching the adjoint representation of $G$ into the maximal subgroup $Sl(2) \times H$, where $Sl(2)$ is generated by $(E_\omega, H_\omega, E_{-\omega})$ and $H$ is the maximal subgroup of $G$ commuting with $Sl(2)$ (explicit decompositions are shown in Table 4 for all simply-laced groups):

$$G \supset Sl(2) \times H$$

$$adj_G = (3, 1) \oplus (2, R) \oplus (1, adj_H) \hfill (3.2)$$

$$= 1 \oplus R \oplus [1 \oplus adj_H] \oplus R \oplus 1$$

In particular, $G_1$ and $G_{-1}$ transform as a (possibly reducible) representation $R$ of $H$, with a symplectic reality condition so that $(2, R)$ is real. The set $\mathbb{C}H_\omega \oplus G_1 \oplus \mathbb{C}E_\omega$ is the coadjoint orbit of the highest root $E_\omega$, namely the minimal orbit $O$ we are seeking. Since the highest root generator $E_\omega$ is nilpotent, this is in fact a nilpotent
orbit. As any coadjoint orbit, it carries a standard Kirillov-Kostant symplectic form, and its restriction to $G_1$ is the symplectic form providing the reality condition just mentioned. The nilpotent orbit can also be understood as the coset $P \backslash G$ where $P$ is the parabolic subgroup generated by $G_{-2} \oplus G_{-1} \oplus (G_0 \setminus \{H_\omega\})$. The group $G$ acts on $\mathcal{O}$ by right multiplication on the coset $P \backslash G$, and therefore on the functions on $\mathcal{O}$.

The minimal representation can be obtained by quantizing the orbit $\mathcal{O}$, i.e. by replacing functions on the symplectic manifold $\mathcal{O}$ by operators on the Hilbert space of sections of a line bundle on a Lagrangian submanifold of $\mathcal{O}$. In more mundane terms, we need to choose a polarization, i.e. a set of positions and momenta among the coordinates of $\mathcal{O}$. For this, note that, as a consequence of the grading, the subspace $G_1 \oplus G_2$ forms an Heisenberg algebra

$$[E_{\alpha_1}, E_{\alpha_2}] = (\alpha_1, \alpha_2) E_\omega, \quad \alpha_1, \alpha_2 \in G_1,$$

where $(\cdot, \cdot)$ is the symplectic form. A standard polarization can be constructing by picking in $G_1$ the simple root $\beta_0$ to which the affine root attaches on the extended Dynkin diagram. The positive roots in $G_1$ then split into roots that have an inner product $\langle \alpha, \beta_0 \rangle$ with $\beta_0$ equal to 1 (we denote them $\beta_i$), −1 (denoted $\gamma_i = \omega - \beta_i$), 2 ($\beta_0$ itself), or 0 (denoted $\gamma_0 = \omega - \beta_0$). We choose as position operators $E_{\gamma_0}, E_{\gamma_i}$ and $E_\omega$:

$$E_\omega = iy, \quad E_{\gamma_i} = ix_i, \quad i = 0, \ldots, d - 1$$

acting on a space of functions of the variables $y, x_i$. The conjugate momenta are then represented as derivative operators,

$$E_{\beta_i} = y \partial_i \quad i = 0, \ldots, d - 1$$

The expression for the remaining momentum-like generator $H_\omega$ will be determinated below, but could be obtained at this stage by computing the Kirillov-Kostant symplectic form on $P \backslash G$.

To summarize our notations at that stage, the 5-grading (3.1) therefore corresponds to the decomposition

$$G_2 = \{E_\omega\},$$
$$G_1 = \{(E_{\beta_i}, E_{\gamma_i})\}$$
$$G_0 = \{E_{-\alpha_j}, H_{\alpha_k}, E_{\alpha_j}\}$$
$$G_{-1} = \{(E_{-\beta_i}, E_{-\gamma_i})\}$$
$$G_{-2} = \{E_{-\omega}\}$$

where $i = 0, \ldots, d - 1 = \dim(R)/2 - 1, j = 1, \ldots, (\dim(H) - \text{rank}(G) + 1)/2$ and $H_{\alpha_k}$ are the Cartan generators of the simple roots with $k = 1, \ldots, \text{rank}(G)$.

5For $\mathfrak{sl}(n)$, the affine root attaches to two roots $\alpha_1$ and $\alpha_{n-1}$. We choose $\beta_0 = \alpha_1$. 
\[ Sl(n) \supset Sl(2) \times Sl(n-2) \times \mathbb{R}^+ \]
\[ adj = (3,1,0) \oplus [(2,n-2,1) \oplus (2,n-2,-1)] \oplus (1,adj,0) \]
\[ = 1 \oplus 2(n-2) \oplus [1 \oplus adj] \oplus 2(n-2) \oplus 1 \]

\[ SO(2n) \supset Sl(2) \times Sl(2) \times SO(2n-4) \]
\[ adj = (3,1,1) \oplus (2,2,2n-4) \oplus (1,3,1) \oplus (1,1,adj) \]
\[ = 1 \oplus (2,2n-4) \oplus [1 \oplus adj] \oplus (2,2n-4) \oplus 1 \]

\[ E_6 \supset Sl(2) \times Sl(6) \]
\[ 78 = (3,1) \oplus (2,20) \oplus (1,35) \]
\[ = 1 \oplus 20 \oplus [1 \oplus 35] \oplus 20 \oplus 1 \]

\[ E_7 \supset Sl(2) \times SO(6,6) \]
\[ 133 = (3,1) \oplus (2,32) \oplus (1,66) \]
\[ = 1 \oplus 32 \oplus [1 \oplus 66] \oplus 32 \oplus 1 \]

\[ E_8 \supset Sl(2) \times E_7 \]
\[ 248 = (3,1) \oplus (2,56) \oplus (1,133) \]
\[ = 1 \oplus 56 \oplus [1 \oplus 133] \oplus 56 \oplus 1 \]

**Table 1:** Five-graded decomposition for simply laced simple groups.

### 3.2 Induced representation and Weyl generators

Having represented the Heisenberg subalgebra on a space of functions of \(d + 1\) variables \((y, x_i = 0, \ldots, d-1)\), it remains to extend this representation to all generators in \(G\). This can be done by unitary induction from the parabolic subgroup \(P\). Rather than taking this approach, we prefer to generate the missing generators using the unbroken symmetry under \(H\) and Weyl generators.

As a first step, it is useful to note that the choice of polarization parameter \(\Pi\) is invariant under a subalgebra \(H_0 \subset H\) acting linearly on \((x_{i=1,\ldots,d-1})\) while leaving \((y, x_0)\) invariant. For the \(D\) and \(E\) groups, \(H_0\) is the subalgebra generated by the simple roots which are not attached to \(\beta_0\) in the Dynkin diagram of \(G\), whilst for the \(A\) series, that by the simple roots attached to neither \(\beta_0\) nor the root at the other end of the Dynkin diagram. The subalgebras \(H_0\) are listed in Table 2.

In order to extend the action of \(H_0\) and the Heisenberg subalgebra to the rest of \(G\), we introduce the action of two Weyl generators \(S\) and \(A\). The first, \(S\), exchanges the momenta \(\beta_i\) with the positions \(\gamma_i\) for all \(i = 0, \ldots, d-1\) and is therefore achieved by Fourier transformation in the Heisenberg coordinates \(x_i = 0, \ldots, d-1\),

\[(Sf)(y, x_0, \ldots, x_{d-1}) = \int \prod_{i=0}^{d-1} \frac{dp_i}{(2\pi)^{d/2}} f(y, p_0, \ldots, p_d) e^{\frac{i}{2} \sum_{i=0}^{d-1} p_i x_i} \tag{3.7} \]
Table 2: Dimension of minimal representation, linearly realized subgroup $H_0 \subset H \subset G$, representation of $G_1^*$ under $H_0$, and associated cubic invariant $I_3$.

| $G$      | dim | $H_0$             | $G_1^*$ | $I_3$ |
|----------|-----|-------------------|---------|-------|
| $Sl(n)$  | $n-1$ | $Sl(n-3)$ | $[n-3]$ | 0     |
| $SO(n,n)$| $2n-3$ | $SO(n-3,n-3)$ | $1 \oplus [2n-6]$ | $x_1(\sum x_{2i} x_{2i+1})$ |
| $E_6$    | 11  | $Sl(3) \times Sl(3)$ | $(3,3)$ | det $|\operatorname{det}|$ |
| $E_7$    | 17  | $Sl(6)$          | 15      | Pf |
| $E_8$    | 29  | $E_6$            | 27      | $27^{0,3}|_1$ |

It also sends all $\alpha_i$ to $-\alpha_i$, while leaving $\omega$ invariant,

$$SE_{\alpha_i} S^{-1} = E_{-\alpha_i}, \quad SE_{\omega} S^{-1} = E_{\omega}.$$ (3.8)

The second generator $A$ is the Weyl reflection with respect to the root $\beta_0$. It maps $\beta_0$ to minus itself, $\gamma_0$ to $\omega$, and all $\beta_i$ to the roots $\alpha_j$ that were not in $H_0$. All roots in $H_0$ are invariant under $A$, and so are all $\gamma_i=1,\ldots,d-1$. In order to write the action of $A$, we need to introduce an $H_0$-invariant cubic form on $G_1^*$,

$$I_3 = \sum_{i<j<k} c(i,j,k)x_ix_jx_k$$ (3.9)

where the sum extends over all $i,j,k = 1,\ldots,d-1$ such that $\beta_i + \beta_j + \beta_k = \beta_0 + \omega$. The sign $c(i,j,k)$ is given by [11]

$$c(i,j,k) = (-)^{B(\beta_i,\beta_j)+B(\beta_i,\beta_k)+B(\beta_j,\beta_k)+B(\beta_0,\omega)+1}$$ (3.10)

where $B(\alpha,\beta)$ is the adjacency matrix (namely a bilinear form such that $\langle \alpha, \beta \rangle = B(\alpha, \beta) + B(\beta, \alpha)$). The cubic invariant $I_3$ in (3.9) is unique, except for the case of $G = Sl(n)$ where there is none, and is listed in the last column of Table 2. The action of $A$ is given in terms of $I_3$ by

$$(Af)(y,x_0,x_1,\ldots,x_{d-1}) = e^{-\frac{iI_3}{\alpha_0^2}}f(-x_0,y,x_1,\ldots,x_{d-1}).$$ (3.11)

One may check that the generators $A$ and $S$ satisfy the relation

$$(AS)^3 = (SA)^3$$ (3.12)

in the Weyl group. In fact, as in the symplectic case where the relation $(ST)^3$ was equivalent to the invariance of the Gaussian character $e^{ix^2}$ under Fourier transform, the relation (3.12) amounts to the invariance of the cubic character $x_0^\alpha(I_3)^{\beta}e^{I_3/x_0}$ under Fourier transform over all $x_{i=0,\ldots,d-1}$. This invariance can be easily checked in
the stationary phase approximation, and holds exactly for particular values of the exponents $\alpha, \beta$. In fact, the minimal representation yields all cubic forms $I_3$ such that $e^{i I_3/x_0}$ is invariant.

### 3.3 Example: minimal representation of $D_4$

Using the Weyl generators (3.7) and (3.11), we can now compute the action of $E_\alpha$ in the minimal representation for all positive and negative roots and in turn obtain the Cartan generators through $[E_\alpha, E_{-\alpha}] = \alpha \cdot H$. As an illustration, we display the $SO(4,4)$ case in detail. The data for other groups are tabulated in the Appendix. The Dynkin diagram of $D_4$ is

![Dynkin diagram of $D_4$](image)

where we have indicated the standard labeling as well as one more convenient for our purposes; the construction will be symmetric under permutations of $(\alpha_1, \alpha_2, \alpha_3)$ and hence under $SO(4,4)$ triality. The positive roots graded by their height along $\beta_0$ are

$$
\begin{align*}
\alpha_1 &= (1,0,0,0) = A(\beta_1) \\
\alpha_2 &= (0,0,1,0) = A(\beta_2) \\
\alpha_3 &= (0,0,0,1) = A(\beta_3)
\end{align*}
$$

(3.13)

$$
\begin{align*}
\beta_0 &= (0,1,0,0) \\
\gamma_0 &= (1,1,1,1) \\
\beta_1 &= (1,1,0,0) \\
\gamma_1 &= (0,1,1,1) \\
\beta_2 &= (0,1,1,0) \\
\gamma_2 &= (1,1,0,1) \\
\beta_3 &= (0,1,0,1) \\
\gamma_3 &= (1,1,1,0)
\end{align*}
$$

(3.14)

$$
\omega = (1,2,1,1) = A(\gamma_0).
$$

(3.15)

We start with the generators

$$
\begin{align*}
E_{\beta_0} &= y \partial_0 \\
E_{\beta_1} &= y \partial_1 \\
E_{\beta_2} &= y \partial_2 \\
E_{\beta_3} &= y \partial_3 \\
E_\omega &= iy.
\end{align*}
$$

(3.16)

(3.17)

---

\footnote{For $D_4$, the invariance of the function $(1/|x_0|) e^{i x_1 x_2 x_3 / x_0}$ under Fourier transform of all 4 variables can be checked explicitly by performing the (delta function) integrals over $x_1, x_2, x_0, x_3$ in that order.}
The cubic form \((3.9)\) reduces to \(I_3 = x_1x_2x_3\). The Weyl generator \(A\) (acting by conjugation) yields the generators for the remaining simple roots \(E_{\alpha_i}\) upon which we act with \(S\) to obtain \(E_{-\alpha_i}\),

\[
E_{\alpha_1} = -x_0\partial_1 - \frac{ix_1x_3}{y}, \quad E_{-\alpha_1} = x_1\partial_1 + iy\partial_3
\]

\[
E_{\alpha_2} = -x_0\partial_2 - \frac{ix_1x_3}{y}, \quad E_{-\alpha_2} = x_2\partial_1 + iy\partial_3
\]

\[
E_{\alpha_3} = -x_0\partial_3 - \frac{ix_1x_3}{y}, \quad E_{-\alpha_3} = x_3\partial_1 + iy\partial_2.
\]

A further application of \(A\) on \((E_{\beta_0}, E_{-\alpha_i})\) yields \((E_{-\beta_0}, -E_{-\alpha_i})\) upon which \(S\) produces the \((E_{-\gamma_0}, -E_{-\gamma_i})\). Penultimately we may act with \(A\) on \(E_{-\gamma_0}\) to produce the lowest root \(E_{-\omega}\),

\[
E_{-\beta_0} = -x_0\partial + \frac{ix_1x_2x_3}{y^2}
\]

\[
E_{-\beta_1} = x_1\partial + \frac{x_1}{y}(1 + x_2\partial_2 + x_3\partial_3) - ix_0\partial_2\partial_3
\]

\[
E_{-\gamma_0} = 3i\partial_0 + iy\partial_0 - y\partial_1\partial_3 + i(x_0\partial_0 + x_1\partial_1 + x_2\partial_2 + x_3\partial_3)\partial_0
\]

\[
E_{-\gamma_1} = iy\partial_1\partial + i(2 + x_0\partial_0 + x_1\partial_1)\partial_1 - \frac{x_2x_3}{y}\partial_0
\]

\[
E_{-\omega} = 3i\partial + iy\partial^2 + \frac{i}{y} + ix_0\partial_0\partial + \frac{x_1x_2x_3}{y^2}\partial_0 + \frac{i}{y}(x_1x_2\partial_1\partial_2 + x_3x_1\partial_3\partial_1 + x_2x_3\partial_2\partial_3)
\]

\[
+ i(x_1\partial_1 + x_2\partial_2 + x_3\partial_3)(\partial + \frac{1}{y}) + x_0\partial_1\partial_2\partial_3,
\]

as well as cyclic permutations of \((1,2,3)\), denoting \(\partial \equiv \partial_y\). Finally, commutators produce the Cartan generators,

\[
H_{\beta_0} = -y\partial + x_0\partial_0
\]

\[
H_{\alpha_1} = -1 - x_0\partial_0 + x_1\partial_1 - x_2\partial_2 - x_3\partial_3
\]

\[
H_{\alpha_2} = -1 - x_0\partial_0 - x_1\partial_1 + x_2\partial_2 - x_3\partial_3
\]

\[
H_{\alpha_3} = -1 - x_0\partial_0 - x_1\partial_1 - x_2\partial_2 + x_3\partial_3,
\]

where \(H_{\alpha} \equiv \alpha \cdot H = [E_{\alpha}, E_{-\alpha}]\) is the Cartan generator along the simple root \(\alpha\). Note that the Cartan generator corresponding to the highest root \(\omega\) has a simple form,

\[
H_{\omega} = [E_{\omega}, E_{-\omega}] = -3 - 2y\partial - x_0\partial_0 - x_1\partial_1 - x_2\partial_2 - x_3\partial_3
\]

and therefore acts by a uniform rescaling on all \(x_i\), and a double rescaling on \(y\). This agrees with the fact that \(H_{\omega}\) is the grading operator in \((3.1)\). The form of this expression holds therefore for all groups (save for the non-universal constant term \(-3\) above). This is also true of the expression for \(H_{\beta_0}\). The generators for the positive and negative simple roots and Cartan elements for all simply-laced groups, computed following the same procedure, are given in the Appendix. All other roots
can be obtained by commuting

$$[E_\alpha, E_\beta] = \begin{cases} \pm E_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ 0 & \text{otherwise} \end{cases} \quad (3.22)$$

Finally, to specify our conventions for positive and negative roots, we record that the quadratic Casimir operator for $D_4$ is

$$C = \sum_i H_i C^{ij} H_j - \sum_k \mathcal{E}_{\alpha_k} + \sum_l \left(\mathcal{E}_{\beta_l} + \mathcal{E}_{\gamma_l}\right) - \mathcal{E}_{\beta_0} - \mathcal{E}_{\gamma_0} - \mathcal{E}_\omega, \quad (3.23)$$

The same formula may be applied to other groups as well. Here $C^{ij}$ is the inverse Cartan matrix and $\mathcal{E}_\alpha \equiv E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha$. Evaluated on the minimal representation above, we have $C = -8$, in agreement with irreducibility. Note however that in contrast to ordinary representations, the center of the minimal representation (Joseph’s ideal) is much larger: e.g., any quadratic polynomial in the Cartan generators $H_i$ can be supplemented with a linear combination of $\mathcal{E}_\alpha$ operators to make a scalar element.

To summarize, we have obtained a unitary irreducible representation of any simply-laced split group $G$ by quantizing the action of $G$ on its minimal nilpotent orbit (the classical limit can be obtained from our formulæ for the generators by replacing the derivative operators $i(\partial, \partial_i)$ by momenta $(p, p_i)$ conjugate to the coordinates $(y, x_i)$ and dropping the “normal ordering” terms; this yields the Hamiltonians for the generators of $G$ on the nilpotent orbit with symplectic form $dp \wedge dy + \sum dp_i \wedge dx_i$). Choosing one of the generators as the Hamiltonian gives a dynamical system with a spectrum generating symmetry $G$. This generalizes the $A_1$ case corresponding to conformally invariant quantum mechanics [26].

4. Spherical vectors for $D_n$ and $E_{6,7,8}$ Lie groups

With the explicit minimal representation for all simply-laced groups at hand, we focus our attention on the spherical vector; a function $f(y, x_i)$ annihilated by all compact generators in $G$. This is our main result and is a central building block for the construction of theta series for all groups.

4.1 From symplectic to orthogonal

One way of obtaining the symplectic vector is to solve the differential equation $(E_\alpha \pm E_{-\alpha}) f = 0$ for all roots $\alpha$ (the sign is chosen so that the generator is compact). It is sufficient to solve these equations for $\alpha$ a simple root only, since all other equations can be obtained by commutation. This still sounds like a formidable task, even though we shall in fact be able to carry it out later on for exceptional groups. For now however, we would like to take an alternate approach, well suited to orthogonal groups. The spherical vector we shall obtain will turn out to generalize quite simply to exceptional groups as well.
The main observation is that there is a maximal embedding of $SO(n,n,\mathbb{R}) \times Sl(2,\mathbb{R})$ in $Sp(2n,\mathbb{R})$. The minimal representation of $Sp(2n,\mathbb{R})$ has dimension $2n$, and is also a representation of $SO(n,n,\mathbb{R})$, albeit reducible. By considering functions invariant under $Sl(2,\mathbb{R})$ however, we can reduce it to a $2n-3$ dimensional representation, which is the dimension of the minimal representation. In this way we thus obtain a representation equivalent to the one described in Section 3. In order to obtain the spherical vector in that representation, we just need to integrate over the second factor in the decomposition

$$SO(n,n,\mathbb{R}) \times Sl(2,\mathbb{R}) \subset Sp(2n,\mathbb{R}) \cup (2n,\mathbb{R}) \quad (4.1)$$

to get a function on the first space.

This procedure is familiar to string theorists since it gives precisely the one-loop result for half-BPS amplitudes. Indeed, the partition function of the worldsheet winding modes on a torus $T^n$ is a theta series for the symplectic group $Sp(2n,\mathbb{R})$, restricted to the subspace (4.1) of the moduli space. It can be written in a form which makes the modular symmetry $Sl(2,\mathbb{Z})$ manifest,

$$\theta_{Sp} = V \sum_{m^i,n^i} \exp \left( -\pi \frac{(m^i + n^i \tau) g_{ij} (m^j + n^j \bar{\tau})}{\tau_2} + 2\pi i m^i B_{ij} n^j \right). \quad (4.2)$$

where we recognize a sum weighted by the Polyakov action for classical toroidal strings winding around $T^n$ with volume $V = \sqrt{\det g_{ij}}$ via $X^i(\sigma_1, \sigma_2) = m^i \sigma_1 + n^i \sigma_2$; or with manifest $SO(n,n,\mathbb{Z})$ target space symmetry,

$$\theta_{Sp} = \tau_2^{n/2} \sum_{m_i,n_i} \exp \left( -\pi \tau_2 \left[ (m_i + B_{ik} n^k) g^{ij} (m_j + B_{jl} n^l) + n_i g_{ij} n^j \right] + 2\pi i \tau_1 m_i n^i \right). \quad (4.3)$$

In this form, we recognize the contribution of states with momentum $m_i$ and winding $n^i$ in the Schwinger representation, with a BPS constraint $m_i n^i = 0$. The two representations are related by Poisson resummation over all Kaluza–Klein modes $m^i \leftrightarrow m_i$. The one-loop amplitude is obtained by integrating this theta series over the fundamental domain of the upper half-plane $U(1) \backslash Sl(2)$ parameterized by the worldsheet modulus $\tau$:

$$\theta_{SO(n,n)}(g_{ij}, B_{ij}) = 2\pi \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \theta_{Sp}(\tau, \bar{\tau}; g_{ij}, B_{ij}) \, . \quad (4.4)$$

The result is an automorphic form under the T-duality group $SO(n,n,\mathbb{Z})$. Its expansion at large volume using the methods described, e.g., in [6], reads

$$\theta_{SO(n,n)} = \frac{2\pi^2}{3} V + 2V \sum_{m^i \neq 0} \frac{1}{m^i g_{ij} m^j} + 4\pi V \sum_{(m^i,n^i)/Sl(2)} e^{-2\pi \sqrt{(m^i)^2 + 2\pi i m^i B_{ij}}} \frac{1}{\sqrt{(m^i)^2}} \, . \quad (4.5)$$
and exhibits a sum of power-suppressed contributions, together with worldsheet instantons. The double sum runs over integer vectors \((m^i, n^i)\) modulo the linear action of \(SL(2, \mathbb{Z})\). The worldsheet instantons however depend only on the \(SL(2)\) invariant combination \(m^{ij} = m^i n^j - m^j n^i\) (with \((m^{ij})^2 \equiv \frac{1}{4} m^{ij} g_{ik} g_{jl} m^{kl}\)), so they can be rewritten as

\[
\sum_{m^{ij} \text{ rank 2}} \mu(m^{ij}) \frac{e^{-2\pi \sqrt{(m^{ij})^2 + 2\pi i m^{ij} B_{ij}}}}{\sqrt{(m^{ij})^2}} \quad (4.6)
\]

where the measure factor \(\mu(m^{ij}) = \sum_{n|m^{ij}} n\) accounts for the Jacobian factor between variables \((m^i, n^i)\) and \(m^{ij}\). We thus have a representation of \(SO(n, n, \mathbb{R})\) on a space of rank 2 antisymmetric matrices \(m^{ij}\). The dimension of this space is precisely \(2n - 3\) and ought therefore be equivalent to the minimal representation described in Section 3. We can also read off the real spherical vector immediately by going to the origin \((g_{ij} = \delta_{ij}, B_{ij} = 0)\) of the moduli space,

\[
\tilde{f}_{D_4} = \frac{e^{-2\pi \sqrt{(m^{ij})^2}}}{\sqrt{(m^{ij})^2}} \quad (4.7)
\]

The \(p\)-adic spherical vector can also be extracted from the summation measure \(\mu(m^{ij})\) in the same way as in (2.33), and reads

\[
\tilde{f}_p = \gamma_p(m^{ij}) \frac{1 - p \|[m^{ij}]\|_p}{1 - p} . \quad (4.8)
\]

**D₄ spherical vector in the standard minimal representation.**

Having found the spherical vector in this “string inspired” representation, we now would like to map it to the standard minimal representation, with the aim of generalizing it to exceptional groups. For this we need to find the linear operator that intertwines between the two representations and let it act on the spherical vector (4.7). For simplicity, we will describe the \(SO(4, 4)\) case only, since the method generalizes easily to higher \(n\). In this case, the constraint that \(m^{ij}\) of the “string-inspired” representation has rank 2 is

\[
\epsilon_{ijkl} m^{ij} m^{kl} = 0 , \quad (4.9)
\]

which describes a quadratic cone in \(\mathbb{R}^6\).

Firstly, consider the operator acting by multiplication by \(m^{ij}\) (corresponding to shifting \(B_{ij}\) by a constant) in the representation (4.9). These shifts make a 6-dimensional Abelian subalgebra of the Borel subgroup of \(SO(4, 4)\) (i.e., the group generated by the positive roots). We can identify six commuting generators by choosing those for roots with height one in the direction of, for example, \(\alpha_3\), namely \((E_{\alpha_3}, E_{\beta_3}, E_{\gamma_1}, E_{\gamma_2}, E_{\gamma_3}, E_\omega)\). Since these operators commute, we can diagonalize
them simultaneously and we call their eigenvalues $i(m^{43}, m^{24}, m^{14}, m^{23}, m^{13}, m^{12})$. Using the expressions in (3.16)-(3.19) for the generators, we find a common eigenstate

$$\psi_{m^{ij}} = \delta(y - m^{12}) \delta(x_0 - m^{13}) \delta(x_1 - m^{14}) \delta(x_2 - m^{23}) e^{im^{24} x_3},$$

but only if the eigenvalues are related by

$$m^{43} = -\frac{m^{14} m^{23}}{m^{12}} - \frac{m^{13} m^{24}}{m^{12}}.$$  

(4.11)

This is the same as (4.9), providing the rationale for our identification. Therefore the two representations are intertwined by Fourier transformation in a single variable $x_3$,

$$\tilde{f}(m^{ij}) = \int dy dx_0 dx_3 \psi_{m^{ij}}(y, x_0, x) f(y, x_0, x)$$

$$= \int dx_3 \exp(im^{24} x_3/m^{12}) f(m^{12}, m^{13}, m^{14}, m^{23}, x_3).$$

(4.12)

where $x$ stands for $(x_1, x_2, x_3)$. Conversely, we have

$$f(y, x_0, x) = \int dm^{24}/y e^{-2\pi i m^{24} x_3/y} \tilde{f}(y, x_0, x_1, x_2, m_{24}, x_1 x_2 + x_0 m^{24})$$

$$= \int dm^{24} dm^{43} e^{-2\pi i m^{24} x_3/y} \delta(x_1 x_2 + x_0 m^{24} + y m^{43}) \tilde{f}(y, x_0, x_1, x_2, m^{24}, m^{43}),$$

where $\tilde{f}_{m^{ij}} \equiv \tilde{f}(m^{12}, m^{13}, m^{14}, m^{23}, m^{24}, m^{43})$.

To see how the kernel (4.7) translates into the standard minimal representation we must compute the Fourier transform (4.13). For that purpose, it is convenient to take the integral representation

$$\tilde{f}(m^{ij}) = \int dt \frac{t^{3/2}}{t^{1/2}} \exp(-\pi t - \pi t m^{ij})$$

(4.14)

along with the standard one for the Dirac delta function of the constraint. Hence, the action of the intertwining operator on the string-inspired spherical vector may be written as

$$f = \int_{0}^{\infty} dt \frac{t^{1/2}}{t^{1/2}} \int dm^{24} dm^{43} e^{-\pi t - \pi t (m^{ij})^2 - 2\pi i m^{ij} x_3 + \theta (x_1 x_2 + x_0 m^{24} + y m^{43}) - 2\pi i x_3 x_3/y} m^{12} y, m^{13} = x_0, m^{14} = x_1, m^{23} = x_2.$$  

(4.15)

The integrals over $m^{24}, m^{43}$ are Gaussian and yield

$$f = \int dt \frac{t^{3/2}}{t^{3/2}} \int d\theta e^{-\pi (y^2 + x_0^2 + x_1^2) - \frac{\theta}{2} (1 + (\theta y)^2 + (\theta x_0 - \frac{t}{\theta} x_1^2)^2) - 2\pi i \theta x_1 x_2}.$$  

(4.16)
The integral over $\theta$ is again Gaussian, and the $t$ integral is of Bessel type so all integrals can be computed explicitly. The saddle point yields a classical action at

$$S = 2\pi \sqrt{(y^2 + x_0^2 + x_1^2)(y^2 + x_0^2 + x_2^2)(y^2 + x_0^2 + x_3^2)} - 2\pi i \frac{x_0 x_1 x_2 x_3}{y(y^2 + x_0^2)}. \quad (4.17)$$

Taking into account the measure factor, we find that in the standard representation, the kernel becomes (rescaling all variables $(y, x_0, x_1, x_2, x_3)$ by $1/(2\pi)$)

$$f_{D_4} = \frac{4\pi}{\sqrt{y^2 + x_0^2}} K_0 \left( \frac{\sqrt{(y^2 + x_0^2 + x_1^2)(y^2 + x_0^2 + x_2^2)(y^2 + x_0^2 + x_3^2)}}{y^2 + x_0^2} \right) e^{-i \frac{x_0 x_1 x_2 x_3}{y(y^2 + x_0^2)}}. \quad (4.18)$$

This expression is the prototype of the spherical vectors that we will obtain later on, and therefore deserves several comments:

(i) It is invariant under permutations of $(x_1, x_2, x_3)$, i.e., under $SO(4,4)$ triality which was manifest in the standard representation but not at all in the string-inspired one. In fact, on the basis of Heterotic/type II duality, it was found that the one-loop string amplitude (4.4) for $n = 4$ would have to be invariant under triality [21] (see also [22] for a related observation). Therefore our triality invariant result gives strong support to non-perturbative Heterotic/type II duality.

(ii) The spherical vector (4.18) could also have been derived by solving the differential equations for $K$-invariance. As we will show for exceptional cases later, the system of PDE’s reduces to a single differential equation for a single function of the variable $S_1$,

$$S_1 = \frac{\sqrt{(y^2 + x_0^2 + x_1^2)(y^2 + x_0^2 + x_2^2)(y^2 + x_0^2 + x_3^2)}}{y^2 + x_0^2}. \quad (4.19)$$

This is equation is a linear second order differential equation of Bessel type, for which (4.18) is the only solution with exponential decrease at infinity (i.e., $S_1 \to \infty$). The same phenomenon will also hold for exceptional groups, except that the variable $S_1$ will be a more complicated function of the coordinates $(y, x_i)$ (but reducing to the same form (4.19) for particular configurations of the variables $x_i$), and that the order of the Bessel function will be different.

(iii) The phase exp$(-iS_2)$, where $S_2$ is the imaginary part of the classical action

$$S_2 = \frac{x_0 I_3}{y(y^2 + x_0^2)}, \quad (4.20)$$

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is precisely such that the spherical vector is invariant under the Weyl generator $A$ in (3.11). Indeed, this follows from the trivial identity
\[
\frac{I_3}{x_0 y} - \frac{y I_3}{x_0 (x_0^2 + y^2)} = \frac{x_0 I_3}{y(x_0^2 + y^2)}.
\] (4.21)

Defining
\[
f(y, x_i) \equiv g(y, x_i) e^{-i x_0 I_3 y (y^2 + x_0^2)}
\] (4.22)
we see that the invariance under $A$ requires $g$ to be symmetric under $(y, x_0) \rightarrow (-x_0, y)$. In fact, the invariance under the compact generator $E_{\beta_0} + E_{-\beta_0}$ requires $g$ to depend on $(y, x_0)$ through $y^2 + x_0^2$ only since it acts on the function $g(y, x_i)$ as the rotation operator $y \partial_0 - x_0 \partial$. This will hold for all simply-laced groups.

(iv) In the limit $(y, x_0) \rightarrow 0$, due to the asymptotic behavior $K_s(y \rightarrow \infty) \sim e^{-y \sqrt{\pi/(2y)}}$, the spherical vector takes a much simpler form
\[
f_{D4} \sim \frac{1}{\sqrt{|x_1 x_2 x_3|}} e^{-\frac{|x_1 x_2 x_3|}{y^2}} \quad \text{or} \quad \frac{1}{\sqrt{|x_1 x_2 x_3|}} e^{-\frac{|x_1 x_2 x_3|}{y^2}}
\] (4.23)
depending of the sign of $x_1 x_2 x_3$, where $z = y + ix_0$. We recognize the same kernel as in the definition of the Weyl generator $A$ in (3.11). The spherical vector (4.18) can therefore be thought of as a Fourier-invariant non-linear (physicists would say “Born-Infeld”) completion of the Fourier invariant kernel in (3.11).

(v) The result for the spherical vector (4.18) may be rewritten more compactly in terms of the Euclidean norm $|| (x_1, x_2, \ldots) || \equiv \sqrt{x_1^2 + x_2^2 + \cdots}$ as
\[
f = \frac{1}{R} K_0 \left( || (X, \nabla_X (I_3/R)) || \right) \exp(-i \frac{x_0 I_3}{y R^2}),
\] (4.24)
where $R = ||(y, x_0)||$, $X \equiv (y, x_0, x_1, x_2, x_3)$ and $\nabla_X (I_3/R)$ denotes the gradient of $I_3/R$ with respect to the $X$ coordinates.

(vi) The $p$-adic spherical vector in the string inspired representation can be read off from the large volume expansion (4.3),
\[
f_p(m^{ij}, m \wedge m = 0) = \left( \prod_{i<j} \gamma_p(m^{ij}) \right) \frac{1 - p ||(m^{ij})||_p}{1 - p}.
\] (4.25)
The corresponding spherical vector in the triality invariant representation can be obtained via intertwining by $p$-adic Fourier transform. We leave the details of this computation to [13], and simply mention that it takes the same form as (4.23), upon replacing the Euclidean norm with the $p$-adic norm $|| (x_1, x_2, \ldots) || = \max(|x_1|_p, |x_2|_p, \ldots)$, and $K_0$ by a simple algebraic function.
**$D_n$ spherical vector in the standard minimal representation.**

Before moving on to exceptional groups, let us note that the same manipulation can be performed for higher $SO(n, n)$ groups. In the string representation there are $n(n - 1)/2$ variables $m^{ij}$ subject to constraints

$$\epsilon_{i_1\ldots i_d} m^{ijkl} m^{ij} m^{kl} = 0 \iff m^{ij} m^{kl} = 0.$$  \hspace{1cm} (4.26)

Of these, only $(n - 2)(n - 3)/2$ namely $m^{12} m^{34} = 0$ (say), are independent, so the dimension of the minimal representation is

$$\frac{n(n - 1)}{2} - \frac{(n - 2)(n - 3)}{2} = 2n - 3$$  \hspace{1cm} (4.27)

as given in Table 2.

The intertwining operator is a Fourier transform on $n - 3$ variables, and its action on the spherical vector (4.7) can be computed using the same manipulations as before. We quote

$$f_{D_n} = \left( \frac{y^2 + x_0^2 + x_1^2}{(y^2 + x_0^2)^2 + (y^2 + x_0^2)P + Q^2} \right)^{\frac{n-4}{4}} \frac{K_{n-4}(S_1) e^{-i S_2}}{\sqrt{y^2 + x_0^2}},$$

where

$$S_1 = \frac{(y^2 + x_0^2)^3 + (y^2 + x_0^2)^2 I_2 + (y^2 + x_0^2) (I_2^2 - I_4)/2 + (I_3)^2}{y^2 + x_0^2},$$

$$S_2 = \frac{x_0 I_3}{y(y^2 + x_0^2)}$$

and

$$P = \sum_{j=2}^{2n-5} x_j^2, \quad Q = \sum_{i=1}^{n-3} (-)^{i+1} x_{2i} x_{2i+1},$$

$$I_2 = x_1^2 + P, \quad I_3 = x_1 Q, \quad I_4 = x_1^4 + P^2 - 2Q^2.$$  \hspace{1cm} (4.31)

In contrast to the $n = 4$ case, for $n > 4$ the spherical vector must be invariant under the maximal compact subgroup $K_0 = SO(n - 3) \times SO(n - 3)$ of the linearly realized $H_0 = SO(n - 3, n - 3, \mathbb{R})$. This is indeed the case of our result, since $P$ and $Q$ are the $K_0$-invariant square norms of the $SO(n - 3, n - 3)$-vector $(x_2, \ldots, x_{2n-5})$ ($Q$ is even $H_0$ invariant). Using this symmetry we can choose all $x_{i>3}$ to vanish. In this case the classical action, $S = S_1 + i S_2$ reduces to the $D_4$ case (4.19, 4.20). As for $D_4$, we can express the $D_n$ spherical vector more compactly as

$$f_{D_n} = \frac{1}{R} \left( \frac{||\langle y, x_0, x_1 \rangle \rangle}{R} \right)^{n-4} K_{n-4} \left( \frac{(||\langle X, \nabla_X (I_3/R) \rangle \rangle)}{R} \right) \exp \left( -i \frac{x_0 I_3}{y R^2} \right).$$

$$\hspace{1cm} (4.32)$$
Here $K_t(x) \equiv x^{-t}K_t(x)$, $X \equiv (y, x_0, x_1, \ldots, x_{2n-5})$ and $R \equiv ||(y, x_0)||$. The form (4.29) of the argument of the Bessel function in terms of the three $K_0$-invariants $I_2, I_3, I_4$, will apply to the exceptional groups as well as well as the overall form (4.32).

As in the $D_4$ case, the $p$-adic spherical vector in the string inspired representation can be read off from the large volume expansion of the symplectic theta series. The spherical vector in the “standard” minimal representation could therefore be obtained by $p$-adic Fourier transform.

4.2 $E_6$

In the case of exceptional groups, we unfortunately do not have a string-inspired representation which we could use to obtain the spherical vector. In fact, it is the other way around, since we are aiming at a “membrane-inspired” representation for exceptional theta series! Our only remaining line of attack is therefore to find an explicit solution of the differential equations $(E_\alpha \pm E_{-\alpha})f = 0$ determining the spherical vector.

For this, let us recall that (i) once the phase factor in (4.22) is factored out, the dependence of $f$ on $(y, x_0)$ is through $(y^2 + x_0^2)$ only, and (ii) that the spherical vector has to be invariant under the maximal compact subgroup $K_0$ of $H_0$, which is linearly realized on $(x_1, \ldots, x_d)$. Our first task, therefore, is to determine the invariants of $(x_1, \ldots, x_d)$ under $K_0$.

In the $E_6$ case, from Table 2 the variables $(x_1, \ldots, x_9)$ transform in a $(3, 3)$ representation of $H_0 = SL(3) \times SL(3)$. Using the $K_0$ transformations implied by the explicit expressions for the roots given in the Appendix, we can assign the 9 variables to a $3 \times 3$ matrix

$$Z = \begin{pmatrix} x_1 & x_3 & x_6 \\ x_2 & x_5 & x_9 \\ x_4 & x_7 & x_8 \end{pmatrix}, \quad (4.33)$$

on which $SL(3) \times SL(3)$ act linearly by left and right multiplication respectively. An independent set of invariants under the maximal compact subgroup $K_0 = SO(3) \times SO(3)$ is given by the quadratic, cubic and quartic combinations

$$I_2 = \text{Tr}(Z^t Z), \quad I_3 = -\det(Z), \quad I_4 = \text{Tr}(Z^t ZZ^t Z), \quad (4.34)$$

In fact, $I_3$ is our familiar cubic form, invariant under the whole of $H_0$ and not only its maximal compact subgroup. Note also that higher traces are algebraically related to the ones above.

Now, given that the spherical vector has to be invariant under $K_0$, we can work in a frame where $Z$ is diagonal keeping $(x_1, x_5, x_8)$ as the only non-vanishing entries. The invariants then reduce to

$$I_2 = x_1^2 + x_5^2 + x_8^2, \quad I_3 = -x_1 x_5 x_8, \quad I_4 = x_1^4 + x_5^4 + x_8^4. \quad (4.35)$$
Let us now consider the equation \((E_{\beta_1} - E_{-\beta_1})f = 0\). The negative root generator \(E_{-\beta_1}\) can be obtained by commuting the negative roots given in the appendix, and reads
\[
E_{-\beta_1} = x_1 \partial + ix_0(\partial_5 \partial_8 - \partial_7 \partial_9) + \frac{2}{y} x_1 + \frac{1}{y} (x_1 (x_5 \partial_5 + x_7 \partial_7 + x_8 \partial_8 + x_9 \partial_9) - x_2 x_3 \partial_5 - x_2 x_6 \partial_9 - x_3 x_4 \partial_7 - x_4 x_6 \partial_8).
\] (4.36)

Using the ansatz (4.22) and setting all \(x_{i=0,\ldots,9}\) but \((x_1, x_5, x_8)\) to zero at the end, we get a first order differential equation
\[
x_1 (x_5 \partial_5 + x_8 \partial_8 - 2) g + y^2 (\partial_1 + 2x_1 \partial_y) g = 0,
\] (4.37)

which is solved by \(g(y^2, x_1, x_5, x_8) = \frac{1}{y^2} h(y^2 + x_1^2, \frac{x_5}{y}, \frac{x_8}{y})\). Demanding invariance under the compact generators of \(\beta_5\) and \(\beta_8\) requires the same equation to hold for permutations of \(x_1, x_5, x_8\) so the only possibility is
\[
g(y^2, x_1, x_5, x_8) = \frac{1}{y^2} h \left( \frac{\sqrt{(y^2 + x_1^2)(y^2 + x_5^2)(y^2 + x_8^2)}}{y^2} \right) .
\] (4.38)

The argument of \(h\) is easily recognizable as the universal form \(S_1\) in (4.19) and we can restore the dependence on all variables using \(K_0\) invariance; first in our particular frame we write
\[
S_1 = \frac{\sqrt{(y^2 + x_1^2)(y^2 + x_5^2)(y^2 + x_8^2)}}{y^2} = \frac{\sqrt{y^6 + y^4 (x_1^2 + x_5^2 + x_8^2) + y^2 (x_1^4 + x_5^4 + x_8^4) + (x_1 x_5 x_8)^6}}{y^2} .
\] (4.39)

Then using the relations (4.33) for the invariants \(I_2, I_3\) and \(I_4\), and recalling that the dependence on \((y, x_0)\) is through the norm \(y^2 + x_0^2\), we find that \(h\) has to depend on the coordinates \((y, x_0, \ldots, x_9)\) through the combination
\[
S_1 = \frac{\sqrt{(y^2 + x_0^2)^3 + (y^2 + x_0^2) I_2 + (y^2 + x_0^2)(I_2^2 - I_4)/2 + I_3^2}}{(y^2 + x_0^2)} .
\] (4.40)

In fact, this expression can be rewritten in a much more concise way as
\[
S_1 = \frac{\sqrt{\det(Z Z^t + |z|^2 I_3)}}{|z|^2} ,
\] (4.41)

where \(z = y + ix_0\) and \(I_3\) is the \(3 \times 3\) identity matrix. This expression is manifestly invariant under \(SO(3) \times SO(3) \times SO(2) \subset K\).

Finally, the \((E_{\alpha_3} - E_{-\alpha_3})f = 0\) equation reduces to
\[
h'' + \frac{2}{S_1} h' - h = 0.
\] (4.42)
This is a Bessel-type equation, solved by \( h = K_{1/2}(S_1)/\sqrt{S_1} \propto e^{-S_1}/S_1 \). Altogether, we have thus found an explicit expression for the \( E_6 \) spherical vector in the minimal representation,

\[
f_{E_6} = \frac{e^{-S_1-i\frac{x_0 I_3}{y(x_0^2+y^2)}}}{(y^2+x_0^2)\sqrt{S_1}} K_{1/2}(S_1)e^{-i\frac{x_0 I_3}{y(x_0^2+y^2)}} \quad (4.43)
\]

As in the \( D_4 \) case, this expression simplifies greatly in the limit \( |z| \to 0 \),

\[
f_{E_6} \sim e^{-\frac{|\det Z|}{y^2}} \quad (4.44)
\]

or its complex conjugate, depending on the sign of \( \det(Z) \).

While this spherical vector has been constructed in the standard representation presented in Section 3, other choices of polarization can be relevant in certain applications, and yield different expressions for the spherical vector. In the \( E_6 \) case, there is another interesting polarization, where the linearly realized group is \( Sl(5) \) rather than \( Sl(3) \times Sl(3) \). By looking at the root lattice displayed in the appendix, it is easy to see that this representation can be reached by performing a Fourier transform on the coordinates \((x_6, x_9, x_8)\). This breaks one of the \( Sl(3) \) factors, but the other unbroken factor gets enlarged to an \( Sl(5) \), under which the 10 new coordinates transform as an antisymmetric matrix

\[
X = \begin{pmatrix} 0 & -p_8 & p_9 & x_1 & x_3 \\ 0 & -p_6 & x_2 & x_5 \\ 0 & x_4 & x_7 \\ a/s & 0 & x_0 \\ 0 & & & & \end{pmatrix}
\quad (4.45)
\]

where \((p_6, p_8, p_9)\) are the momenta conjugate to \((x_6, x_8, x_9)\). The spherical vector in this representation is obtained by Fourier transform on \((x_6, x_8, x_9)\),

\[
\tilde{f}_{E_6} = \int \frac{dx_6 \, dx_8 \, dx_9}{y^{3/2}} f_{E_6} e^{2i(p_6 x_6 + p_8 x_8 + p_9 x_9)/y} \quad (4.46)
\]

Remarkably, the integral can still be computed by the same method as in Section 4.1, and yields the simple result

\[
\tilde{f}_{E_6} = \frac{1}{\sqrt{y J_4}} K_1 \left( \frac{1}{y} \sqrt{J_4} \right)
\quad (4.47)
\]

where \( J_4 \) is the polynomial of order 4,

\[
J_4 = y^4 - \frac{y^2}{2} \text{Tr}(X^2) + \frac{1}{8} \left( (\text{Tr}X^2)^2 - 2\text{Tr}X^4 \right)
\quad (4.48)
\]

manifestly invariant under the maximal compact subgroup \( SO(5) \subset Sl(5) \). Note that \( x_0 \) is now unified with the other \( x_i \) coordinates, and that the phase has disappeared.
4.3 $E_7$

The same strategy presented for $E_6$ above yields the $E_7$ and $E_8$ spherical vectors. In the case of $E_7$, the minimal representation has dimension 17, and is realized on a space of functions of $(y, x_0, \ldots, x_{15})$. The linearly realized subgroup is $H_0 = SL(6, \mathbb{R})$, with maximal compact subgroup $K_0 = SO(6, \mathbb{R})$. The coordinates $(x_1, \ldots, x_{15})$ transform in the adjoint representation of $H_0$. Using the explicit expression for the roots given in the Appendix, we can fit them into an antisymmetric matrix

\[
Z = \begin{pmatrix}
0 & -x_1 & x_2 & -x_4 & -x_6 & x_9 \\
0 & x_3 & -x_5 & -x_8 & x_{12} & 0 \\
0 & x_7 & x_{11} & -x_{15} & 0 & x_{13} \\
0 & -x_{14} & x_{13} & 0 & x_{10} \\
a/s & 0 & x_{10} & 0 & 0 & 0
\end{pmatrix}
\]

(4.49)

The independent invariants of $Z$ under the adjoint action of $SO(6, \mathbb{R})$ are the three Casimir operators of $SO(6) \sim Sl(4)$, i.e.

\[
I_2 = -\frac{1}{2} \text{Tr}(Z^2), \quad I_3 = -\text{Pf}Z, \quad I_4 = \frac{1}{2} \text{Tr}(Z^4)
\]

(4.50)

As for $E_6$, $I_3$ is in fact invariant under the full $H_0$, and is the cubic form that enters the expression of the Weyl generator $A$ in (3.11). Using the action of $K_0$, we can skew-diagonalize $Z$, and set all coordinates but $x_1, x_7, x_{10}$ to zero. The invariants then reduce to the simple symmetric combinations

\[
I_2 = x_1^2 + x_7^2 + x_{10}^2, \quad I_3 = -x_1x_7x_{10}, \quad I_4 = x_1^4 + x_7^4 + x_{10}^4.
\]

(4.51)

Looking at the action of $E_{\beta_{1,5,8}}$, we again find that the spherical vector must take the form $f = h(S_1)e^{-\frac{\text{Pf}Z}{y^2 + x_0^2}}/(y^2 + x_0^2)^{3/2}$, with $S_1$ the usual form in (4.40). As in the $E_6$ case, it can be written more compactly as

\[
S_1 = \frac{\sqrt{\det(Z + |z|\mathbb{I}_6)}}{|z|^2},
\]

(4.52)

where again $z = y + ix_0$.

The equation $(E_{\alpha_2} + E_{-\alpha_2})f = 0$ now requires $h'' + \frac{3}{S_1}h' - h = 0$, hence $h = K_1(S_1)/S_1$. The $E_7$ spherical vector is therefore given by

\[
f_{E_7} = \frac{K_1(S_1)}{(y^2 + x_0^2)^{3/2}S_1}e^{-\frac{\text{Pf}Z}{y^2 + x_0^2}}
\]

(4.53)

with $S_1$ as in (4.40). In the limit $|z| \to 0$, this reduces to

\[
f_{E_7} \sim e^{\frac{\text{Pf}Z}{y^2 + x_0^2}}
\]

(4.54)
or its complex conjugate, depending on the sign of $\text{Pf}Z$.

As in the $E_6$ case, we can find the spherical vector for other polarizations as well. A particularly interesting one is obtained by Fourier transform on the last column of the matrix $Z$ in (4.44), which, as examination of the root lattice shows, yields a representation with an $SO(5,5)$ group acting linearly. The 16 coordinates now transform as a spinor of $SO(5,5)$, or as $1 + 10 + 5$ in terms of its $Sl(5)$ subgroup,

$$x_0, \quad X = \begin{pmatrix} 0 & -x_1 & x_2 & -x_4 & -x_6 \\ 0 & x_3 & -x_5 & -x_8 \\ 0 & x_7 & x_{11} \\ a/s & 0 & -x_{14} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} p_9 \\ p_{12} \\ p_{13} \\ p_10 \\ -p_{15} \end{pmatrix}. \tag{4.55}$$

Again, the Fourier transform of the spherical vector (4.53) can be computed using the same method as in Section 4.1, and yields a simple form

$$\tilde{f}_{E_7} = \frac{y^{3/2}}{J_4^{5/4}} K_{3/2} \left( \frac{1}{y} \sqrt{J_4} \right) \tag{4.56}$$

where $J_4$ is a $SO(5) \times SO(5)$ invariant polynomial of degree 4,

$$J_4 = y^4 + y^2 \left( x_0^2 + Y^t Y - \frac{1}{2} \text{Tr}X^2 \right)$$

$$+ \left( x_0^2 Y^t Y - \frac{1}{4} (\text{Tr}X^4 - \frac{1}{2} (\text{Tr}X^2)^2) - 2 x_0 X \wedge X \wedge Y \right) \tag{4.57}$$

and the last term denotes the contraction with the five-dimensional Levi-Civita tensor.

### 4.4 $E_8$

Finally, in the $E_8$ case, the minimal representation has dimension 29, and is realized on a space of functions of $(y, x_0, \ldots, x_{27})$. $E_6$ is linearly realized, and acts on $(x_1, \ldots, x_{27})$ in the 27 representation. Its maximal compact subgroup is $USp(8)$, under which the $(x_1, \ldots, x_{27})$ transform as an antitraceless antisymmetric representation. It is somewhat awkward to fit the 27 coordinates into an such a matrix, nevertheless we can easily find their transformation under the $Sl(3) \times Sl(3) \times Sl(3)$ subgroup of $H_0 = E_6$. We have the branching rule

$$E_6 \supset Sl(3) \times Sl(3) \times Sl(3)$$

$$27 = (3,3,1) \oplus (3,1,3) \oplus (1,3,3) \tag{4.58}$$

so that the $x_i$ can be assigned to three $3 \times 3$ matrices

$$U_{31} = - \begin{pmatrix} x_{10} & x_{11} & x_{13} \\ x_{12} & x_{14} & x_{16} \\ x_{15} & x_{17} & x_{20} \end{pmatrix}, \quad V_{12} = \begin{pmatrix} x_7 & x_9 & x_{18} \\ -x_6 & -x_8 & x_{21} \\ x_4 & x_5 & x_{24} \end{pmatrix}, \quad W_{23} = \begin{pmatrix} x_{27} - x_{25} & x_{22} \\ x_{26} - x_{23} & x_{19} \\ x_3 & x_2 & x_1 \end{pmatrix}. \tag{4.59}$$
acted upon from the left and from the right by the $\text{Sl}(3)$ factors denoted in subscript. The maximal subgroup $K_0 = USp(8)$ of $H_0 = E_6$ branches itself into $SO(3) \times SO(3) \times SO(3)$, where the three $SU(2)$ are generated by $(K_{\alpha_1}, K_{\alpha_3}, K_{\alpha_5})$, $(-K_{\alpha_2}, K_{\alpha_0}, K_{\alpha_3})$ and $(K_{\alpha_5}, K_{\alpha_6}, K_{\alpha_1})$ where $K_\alpha \equiv E_\alpha + E_{-\alpha}$, respectively. The invariants under $K_0$ can be constructed out of the $SO(3) \times SO(3) \times SO(3)$ invariants by requiring invariance under the extra $K_{\alpha_4}$ compact generator, and read

$$I_2 = \text{Tr}(U^tU) + \text{Tr}(V^tV) + \text{Tr}(W^tW) \quad (4.60)$$

$$I_3 = \text{Tr}(UVW) - (\det(U) + \det(V) + \det(W)) \quad (4.61)$$

$$I_4 = \text{Tr}(UU^tUU^t) + \text{Tr}(VV^tVV^t) + \text{Tr}(WW^tWW^t) - 2(\text{Tr}(UVW^tU^t) + \text{Tr}(VWW^tV^t) + \text{Tr}(WWU^tW^t))$$

$$+ 2(\text{Tr}(U^tU)\text{Tr}(V^tV) + \text{Tr}(V^tV)\text{Tr}(W^tW) + \text{Tr}(W^tW)\text{Tr}(U^tU))$$

$$+ 4(\det(W)\text{Tr}(UVW^t) + \det(U)\text{Tr}(VWU^t) + \det(V)\text{Tr}(WUV^t)) \quad (4.62)$$

Equivalently, we can make the $\text{Sl}(6) \times \text{Sl}(2)$ subgroup of $H_0$ manifest, by arranging the $(15, 1) + (6, 2)$ $x_i$’s into an antisymmetric $6 \times 6$ matrix and a doublet of 6-vectors,

$$Z = \begin{pmatrix}
0 & x_5 & x_8 & x_{10} & x_{12} & x_{15} \\
0 & x_9 & x_{11} & x_{14} & x_{17} & \vdots \\
0 & x_{13} & x_{16} & x_{20} & \vdots & \vdots \\
0 & x_{19} & x_{23} & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\overline{a/s} & 0 & x_{26} & \vdots & \vdots & \vdots \\
\end{pmatrix}, \quad Y_1 = \begin{pmatrix}
x_7 \\
x_4 \\
x_3 \\
x_2 \\
x_1 \\
\end{pmatrix}, \quad Y_2 = \begin{pmatrix}
x_{18} \\
x_{24} \\
x_{25} \\
x_{21} \\
x_{22} \\
\end{pmatrix} \quad (4.63)$$

In this notation, the $K_0$-invariants can be rewritten more concisely as

$$I_2 = -\text{Tr}(Z^2)/2 + \text{Tr}(Y_1Y_1^t) \quad (4.64)$$

$$I_3 = \text{Pf}(Z) + \text{Tr}(Y_1^tZY_2) \quad (4.65)$$

$$I_4 = \frac{1}{2}\text{Tr}(Z^4) + \text{Tr}((Y_1Y_1^t)^2) + 2\text{Tr}Y_1^tZ^2Y_1$$

$$- (\text{Tr}Z^2)(\text{Tr}Y_1Y_1^t) + \frac{1}{2}\epsilon_{ijklmn}Z_{ij}Z_{kl}Y_1^mY_1^n. \quad (4.66)$$

Using an $USp(8)$ rotation, we can set all $x_i$’s to zero except e.g., $x_1, x_{20}, x_{24}$. The invariants above then reduce to

$$I_2 = x_1^2 + x_{20}^2 + x_{24}^2, \quad I_3 = -x_1x_{20}x_{24}, \quad I_4 = x_1^4 + x_{20}^4 + x_{24}^4. \quad (4.67)$$

The $\beta_{1,20,24}$ equations require the ansatz

$$f = (y^2 + x_0^2)^{-5/2}h(S_1)e^{-i\frac{x_1y_1}{y(x_0^2+y^2)}} \quad (4.68)$$
while the \( \alpha_7 \) equation gives \( h'' + \frac{2}{S_1} h' - h = 0 \) and hence \( h = K_2(S_1)/S_1^2 \). The \( E_8 \) spherical vector is therefore

\[
f_{E_8} = \frac{K_2(S_1)}{(y^2 + x_0^2)^{5/2} S_1^2} e^{-i \frac{x_0 I_3}{y(x_0^2 + y^2)}}
\]  

(4.69)

with \( S_1 \) as in (4.40). Again, the real part of the action \( S_1 \) can be more compactly written as

\[
S_1 = \frac{1}{|z|^2} \left[ \det(Z + |z|L_6) + |z|^4 \text{Tr}(Y_i Y_i^\dagger) \right. \\
+ |z|^2 \left( 2 \text{det}(Y_\alpha Y_\beta^\dagger) + \text{Tr}(Y_i Y_i^\dagger) \text{Tr}Z^2 - \frac{1}{2} Z \wedge Z \wedge Y_1 \wedge Y_2 \right) \\
\left. + \text{Tr}(Y_i^\dagger Z Y_2) \right) 2 \text{Pf}(Z) + \text{Tr}(Y_1^\dagger Z Y_2) \right]
\]  

(4.70)

As for \( E_6 \) and \( E_7 \), another interesting representation can be obtained by Fourier transforming on the 13 coordinates \((x_0, Y_1, Y_2)\) (or, equivalently, under the 15 coordinates in \( X \))\(^7\). In this polarization, the linearly realized symmetry group is enlarged to \( SL(8) \), and the 28 coordinates \( x_0, \ldots, x_{27} \) transform as an antisymmetric matrix

\[
X = \begin{pmatrix}
0 & x_5 & x_8 & x_{10} & x_{12} & x_{15} & p_7 & p_{18} \\
0 & x_9 & x_{11} & x_{14} & x_{17} & -p_6 & p_{21} \\
0 & x_{13} & x_{16} & x_{20} & p_4 & p_{24} \\
0 & x_9 & x_{23} & -p_3 & -p_{27} \\
0 & x_{26} & p_2 & p_{25} \\
0 & p_1 & -p_{22} \\
\end{pmatrix}
\]  

(4.71)

It would be interesting to find the spherical vector in this representation.

**Summary.** The general form of the spherical invariant for \( E_{6,7,8} \) in the standard minimal representation is

\[
f_{E_n} = \frac{\mathcal{K}_{s/2}(S_1)}{(y^2 + x_0^2)^{(s+1)/2}} e^{-i \frac{x_0 I_3}{y(x_0^2 + y^2)}}
\]

\[
= \frac{1}{R^{s+1}} \mathcal{K}_{s/2} \left( \| (X, \nabla_X \left( \frac{I_3}{R} \right) ) \| \right) \exp \left( -i \frac{x_0 I_3}{y R^2} \right),
\]  

(4.72)

where \( \mathcal{K}_e \) is expressed in terms of the standard Bessel function as \( \mathcal{K}_e(x) \equiv x^{-\ell} K_\ell(x) \) and the “classical action” \( S_1 \) is given in terms of the quadratic, cubic and quartic invariants \( I_{2,3,4} \) by

\[
S_1 = \frac{\sqrt{(y^2 + x_0^2)^3 + (y^2 + x_0^2)I_2 + (y^2 + x_0^2)(I_2^2 - I_4)/2 + I_3^2}}{y^2 + x_0^2} = \| (X, \nabla_X \left( \frac{I_3}{R} \right) ) \|,
\]  

(4.73)

\(^7\)This representation has also been constructed independently in [22].
where \( X = (y, x_0, x_i) \), \( R = ||(y, x_0)|| \) and \( s = 1, 2, 4 \) for \( n = 6, 7, 8 \) respectively. The parameter \( s \) can be identified with the dimension of the field \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) entering in the alternate construction of the minimal representations of \( E_{6,7,8} \) through Jordan algebras in [11]. We have also found alternate representations for \( E_6 \) and \( E_7 \), where \( Sl(5) \) and \( SO(5,5) \) act linearly, respectively; the spherical vectors in this representation can be found in (4.47), (4.50).

### 4.5 Complex spherical vectors

For completeness, we discuss the case of a complex group, for which our methods also allow us to derive the spherical vector. The complex group \( G(\mathbb{C}) \) can be obtained by complexifying its split real form \( G(\mathbb{R}) \), i.e., adjoining to the real generators \( E_i \) of \( G(\mathbb{R}) \) a set of “imaginary” generators \( E'_i \) such that

\[
[T_i, T_j] = c_{ijk}T_k, \quad [T_i, T'_j] = c_{ijk}T'_k, \quad [T'_i, T'_j] = -c_{ijk}T_k .
\]  

(4.74)

Equivalently, one can introduce the holomorphic and anti-holomorphic generators

\[
T_i = T_i + iT'_i, \quad \overline{T}_i = T_i - iT'_i ,
\]

satisfying

\[
[T_i, T_j] = c_{ijk}\overline{T}_k, \quad [T_i, \overline{T}_j] = 0 , \quad [\overline{T}_i, \overline{T}_j] = c_{ijk}T_k .
\]  

(4.75)

We stress that the holomorphic generators \( T_i \) are identical in form to the original \( T_i \) except that the variables are now complex, while the generators \( \overline{T}_i \) are obtained by replacing all variables by their complex conjugates. Dividing the generators into Cartan ones \( H_\alpha \) and those associated with simple roots \( E_{\pm \alpha} \), the maximal compact subgroup \( K \subset G(\mathbb{C}) \) is generated by

\[
E_\alpha \pm \overline{E}_{-\alpha}, \quad E_\alpha \pm \overline{E}_{-\alpha} \quad \text{and} \quad H_\alpha - \overline{H}_\alpha .
\]  

(4.76)

[The choice of sign again depends on the conventions for positive and negative roots.] \( K \) is simply the real compact group of the same type as \( G \). The simplest example is \( Sl(2, \mathbb{C}) \) with maximally compact subgroup \( SU(2) \). The spherical vectors in the complexified metaplectic and Eisenstein representations (see (2.5), (2.6), (2.7) and (2.20), (2.21), (2.22) respectively) are

\[
f_{\text{meta}} = e^{-xy}, \quad f_{\text{Eis}} = |x|^{1-\nu}K_{\nu-1}(2|x|) .
\]  

(4.77)

For the groups \( D_4, E_6, E_7 \) and \( E_8 \), the form of the complex spherical vector in the standard minimal representation is uniform. Again, the requirement that \( f \) be annihilated by the linearly realized compact subgroup allows us to reduce the problem to one in five complex variables \((y, x_0, x_1, x_2, x_3)(\text{for } E_{6,7,8} \text{ we have renamed the remaining } x_i \text{'s for simplicity and will relabel corresponding roots accordingly})\). The compact Cartan generators \( H_\alpha - \overline{H}_\alpha \) imply that the all dependence is through the complex modulus or the ratio \( x_1x_2x_3/(yx_0) \), namely \( f =
The compact generators of the root attached to the affine one

\[
E_{\beta_0} + \overline{E}_{-\beta_0} = y\partial_0 - \bar{x}_0\bar{\partial} + \frac{iI_3}{y^2}, \quad \overline{E}_{-\beta_0} - E_{\beta_0} = \bar{y}\bar{\partial}_0 - x_0\partial + \frac{iI_3}{\bar{y}} \tag{4.78}
\]

[where \(T_3 \equiv \bar{x}_1\bar{x}_2\bar{x}_3 + \cdots\)] imply the phase factor

\[
f = \exp(-iS_2) g(||(y, x_0)||, |x_1|, |x_2|, |x_3|), \tag{4.79}
\]

with

\[
S_2 \equiv \frac{1}{|y|^2 + |x_0|^2} \left( \frac{x_0I_3}{y} + \frac{x_0\bar{I}_3}{\bar{y}} \right). \tag{4.80}
\]

We can drop the \(x_0\) dependence of the function \(g\) and reinstate it at the end of the calculation, so we look at the root \(\beta_1\) at \(x_0 = 0\) (and drop non-diagonal terms in the \(x_i\)'s),

\[
E_{\beta_1} - \overline{E}_{-\beta_1} = y\partial_1 - \bar{x}_1\bar{\partial} - \frac{\bar{x}_1}{y} \left( s + 1 + \bar{x}_2\bar{\partial}_2 + \bar{x}_3\bar{\partial}_3 \right) \tag{4.81}
\]

where the constant \(s = 0\) for \(D_4\) and \(s = 1, 2, 4\) for \(E_{6,7,8}\). This implies (at \(x_0 = 0\)) that

\[
g = |y|^{-2(s+1)} h \left( 2\sqrt{(|y|^2 + |x_1|^2)(|y|^2 + |x_2|^2)(|y|^2 + |x_3|^2)} \right). \tag{4.82}
\]

Finally, we examine the root \(\alpha_1\) at \(x_0 = 0\) with diagonal \(x_i\)'s

\[
E_{\alpha_1} + \overline{E}_{-\alpha_1} = -\frac{i\bar{x}_2x_3}{y} + \bar{x}_1\bar{\partial}_0 + i\bar{y} \left( \partial_2\bar{\partial}_3 + [s \text{ off-diagonal double derivatives}] \right). \tag{4.83}
\]

Note that we may not neglect \(x_0\) or off-diagonal \(x_i\) derivatives even though these variables are set to zero at the end of the calculation. In turn the function \(h\) satisfies the ordinary, Bessel-type, differential equation \(xh'' + (2s + 1)h' + xh = 0\) (to verify that the \(s\) off-diagonal double derivative terms in (1.83) produce the coefficient \(2s\) requires knowledge of the invariants of the linearly realized compact subgroup described below).

It is now a matter of reinstating the dependence on the remaining variables; orchestrating the above results we find the complex spherical vector

\[
f = \frac{K_s(S_1)}{(|y|^2 + |x_0|^2)^{s+1}} \exp \left( -i \frac{x_0I_3}{|y|^2 + |x_0|^2} \left( \frac{x_0\bar{I}_3}{\bar{y}} \right) \right), \tag{4.84}
\]
where $s = 0, 1, 2, 4$ for $D_4$, $E_6, 7, 8$, respectively, and $K_t(x) \equiv x^{-t}K_t(x)$. The action is

$$S_1 = 2 \sqrt{(|y|^2 + |x_0|^2)^3 + (|y|^2 + |x_0|^2)^2 I_2 + (|y|^2 + |x_0|^2)(I_2^2 - I_4)/2 + |I_3|^2} \over |y|^2 + |x_0|^2}, \quad (4.85)$$

and $I_2$, $I_3$, $\mathcal{T}_3$ and $I_4$ are the invariants of the linearly realized subgroup. For $E_6$ and $E_7$ they are subsumed by the elegant formulae

$$S_{E_6}^{1} = \frac{\sqrt{\det(ZZ^\dagger + (|y|^2 + |x_0|^2)I_3)}}{|y|^2 + |x_0|^2}, \quad S_{E_7}^{1} = \left(\frac{\det(ZZ^\dagger + (|y|^2 + |x_0|^2)I_6)}{|y|^2 + |x_0|^2}\right)^{1/4}. \quad (4.86)$$

The matrices $Z$ are the same as (4.33) and (4.49) for complex variables; the cubic invariant $I_3$ is holomorphic and takes the same expression as in the real case. The quadratic and quartic invariants $I_2$ and $I_4$ have the same form as in the real case with hermitian conjugation replacing transposition. Finally, we note that a rewriting in terms of the norm $||(x_1, x_2, \ldots)|| \equiv \sqrt{|x_1|^2 + |x_2|^2 + \cdots}$ and $R \equiv ||(y, x_0)||$ also holds

$$f = \frac{1}{R^{2(s+1)}} K_n(2 ||(X, R \nabla_X( I_3 R^2)||) \exp\left(-i \frac{\bar{x}_0 I_3}{R^2} + \frac{x_0 \mathcal{T}_3}{y}\right)). \quad (4.87)$$

Similar generalizations of the spherical vectors for complexifications of the groups $A_n$ and $D_n$ may also be obtained using exactly the methods presented above, the former is trivial, while for the $D_n$ case we find

$$f_{D_n} = \frac{1}{R^2} \left(\frac{||(y, x_0, x_1)||}{R}\right)^{n-4} K_{n-4} \left(2 ||(X, R \nabla_X( I_3 R^2)||) \exp\left(-i \frac{\bar{x}_0 I_3}{R^2} + \frac{x_0 \mathcal{T}_3}{y}\right)\right). \quad (4.88)$$

5. Discussion

The main object of this paper was the derivation of the spherical vector for the minimal representation of real simply laced groups in the split real form. Our results are displayed in (4.28) and (4.72) above. This spherical vector is an essential component in the construction of automorphic theta series for exceptional groups. It has been proposed that such objects would provide the partition function for the winding modes of the quantum eleven-dimensional supermembrane in M-theory. Although the physical interpretation of the results obtained herein will be discussed elsewhere, we close with some comments, both mathematical and physical:

(i) We have obtained the spherical vector over the real field. As we have explained in Section 2, the spherical vectors over the $p$-adic fields $\mathbb{Q}_p$ are also important, since
their product gives the summation measure when constructing a theta series. Unfortunately in the $p$-adic case, we cannot rely on partial differential equations anymore. One may however require both the spherical vector and its $A$ and $S$ transforms to have support on the $p$-adic integers, which together with invariance under the linearly realized maximal compact group $K_0(\mathbb{Z}_p)$ should fix it uniquely. Nonetheless the presentation of the real spherical vectors in terms of norms does suggest a natural generalization to the $p$-adic case.

(ii) We have left the space of functions of $(y, x_i)$ unspecified. In order for the generators $S$ and $A$ to be well-defined, it should consist of functions in the Schwartz space, as well as their images. The main issue is the regularity at the origin. A proper understanding of this issue would provide the degenerate contributions to the theta series which we have overlooked in this paper.

(iii) We have only discussed the minimal representation for simply-laced groups. For non-simply laced cases, some differences arise: for $G_2$ and $F_4$, the minimal representation does not contain any singlet under reduction to the maximal compact subgroup $K \subset G$, therefore there is no spherical vector, but a multiplet of such vectors transforming into each other \[24\]. It would be interesting to find the wave function for the lowest such multiplet. For $B_{n \geq 3}$, there simply does not exist a representation of the same dimension as that of the smallest nilpotent subalgebra \[24\]. For the symplectic case $C_n$, the spherical vector is not annihilated by the compact generators, but has a non-vanishing eigenvalue.

(iv) From a physical viewpoint, the simplicity of our results is very encouraging. Specifically, in the $E_6$ case, we have found a representation on $2 + 9$ variables, such that $9$ of them transform as a bifundamental under $Sl(3) \times Sl(3)$ in $E_6$: these variables should be identified with the winding numbers of a membrane on $T^3$, with the two $Sl(3)$ factors being the worldvolume and target space modular groups, respectively. The target space $Sl(2)$ U-duality group, corresponding to the modular transformations of the modulus $\tau = C_{123} + iV_3$, would then be realized through Fourier transform (i.e., Poisson resummation). It would be very interesting to understand the physical meaning of the two extra quantum numbers $(y, x_0)$ that we found necessary for realizing the symmetry. It would also be important to understand if the Born-Infeld-like action (4.41) for the zero-modes of the membrane can be generalized to include fluctuations, and yield a new description of the membrane where U-duality is non-linearly realized as a dynamical symmetry. For $E_7$, we have constructed a representation in terms of $2 + 15$ variables, with an action (4.52) suggestive of a Born-Infeld-like action on a six dimensional worldvolume, whose interpretation is unclear. For $E_8$, we have a representation on $2 + 27$ variables, where $27$ of them transform linearly under $E_6$. This is the appropriate number of charges for M-theory on $T^6$, so the spherical vector (4.69) should correspond to the membrane (or, perhaps more appropriately, the five-brane) partition function on $T^6$ in the Schwinger representation, where the sum over BPS states is apparent. The modular group $Sl(3)$
should then be realized by Poisson resummation. It would be very interesting to find a representation where \(Sl(3)\) acts linearly, and identify it with the membrane action. The intertwining operator between the two representations would then realize membrane/five-brane duality. Finally, the representations we have displayed can be used to construct quantum mechanical systems with spectrum-generating exceptional symmetries, much in the spirit of conformal quantum mechanics [26]. It would be interesting to see if they can play a role in string theory.

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**A. Group theory data**

In this appendix, we supply the list of positive roots for all simply-laced groups, graded by their charge under the affine Cartan generator \(H_\omega\). The grade-one subspace is presented in the standard polarization, as explained in the text below equation (3.3), and the action of the Weyl reflection \(A\) w.r.t. to \(\beta_0\) is indicated. The explicit expressions for the cubic invariant \(I_3\), the Cartan generators and the (positive and negative) simple roots in the minimal representation are listed. The expressions for the grade 1 and 2 positive roots are given in (3.4), and repeated here for convenience,

\[
E_{\beta_i} = y \partial_i, \quad E_{\gamma_i} = ix_i, \quad E_\omega = iy, \quad i = 0, \ldots, d - 1. \tag{A.1}
\]

The Cartan generators for the simple root \(\beta_0\) and the affine root \(\omega\) take also universal forms,

\[
H_{\beta_0} = -y \partial + x_0 \partial_0 \tag{A.2}
\]

\[
H_\omega = -\nu - 2y \partial - \sum_{i=0}^{d-1} x_i \partial_i \tag{A.3}
\]

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up to the “normal ordering constant” $\nu = (n - 1), 6, 9, 15$ for $D_n, E_6, E_7, E_8$, respectively. More compact expressions can be obtained by making manifest the covariance under the linearly realized group $H_0$.

A.1 $A_n$

Dynkin diagram:

```
1 2 3 ... n
β₀ α₁ α₂ ... α_{n-1}
```

Positive roots:

\[
\begin{align*}
\alpha_1 &= (0, 1, 0, \ldots, 0, 0) = A(\beta_1) \\
\alpha_2 &= (0, 0, 1, \ldots, 0, 0) \\
\vdots &\quad (0, 0, 0, \ddots, 0, 0) \\
\alpha_{n-2} &= (0, 0, 0, \ldots, 1, 0) \\
\alpha_{n-1} &= (0, 1, 0, \ldots, 0) = A(\beta_2) \\
\alpha_n &= (0, 0, 1, 1, 0, 0) \\
\vdots &\quad (0, 0, 0, \ddots, 0) \\
\alpha_{2n-5} &= (0, 0, \ldots, 1, 1, 0) \\
\vdots \\
\alpha_{(n-1)(n-2)/2} &= (0, 1, \ldots, 1, 1, 0) = A(\beta_{n-2})
\end{align*}
\]

\[
\begin{align*}
\beta_0 &= (1, 0, 0, 0, 0) & \gamma_0 &= (0, 1, \ldots, 1, 1) \\
\beta_1 &= (1, 1, 0, 0, 0) & \gamma_1 &= (0, 0, 1, \ldots, 1) \\
\vdots &\quad (i, \ddots, \ddots, 0, 0) & \vdots &\quad (0, 0, \ldots, 1) \\
\beta_{n-2} &= (1, 1, \ldots, 1, 0) & \gamma_{n-2} &= (0, 0, \ldots, 0, 1) \\
\omega &= (1, 1, \ldots, 1, 1) = A(\gamma_0)
\end{align*}
\]

Cartan generators ($\nu = (n + 1)/2$ in the standard minimal rep):

\[
\begin{align*}
H_{\beta_0} &= -y \partial + x_0 \partial_0 \\
H_{\alpha_1} &= -x_0 \partial_0 + x_1 \partial_1 \\
H_{\alpha_2} &= -x_1 \partial_1 + x_2 \partial_2 \\
\vdots \\
H_{\alpha_{n-2}} &= -x_{n-3} \partial_{n-3} + x_{n-2} \partial_{n-2} \\
H_{\gamma_{n-2}} &= -\nu - y \partial - x_0 \partial_0 - \cdots - x_{n-3} \partial_{n-3} - 2x_{n-2} \partial_{n-2}
\end{align*}
\]
Simple roots:

\[
\begin{align*}
E_{\alpha_1} &= x_0 \partial_1, & E_{-\alpha_1} &= x_1 \partial_0 \\
E_{\alpha_2} &= x_1 \partial_2, & E_{-\alpha_2} &= x_2 \partial_1 \\
&\vdots & & \vdots \\
E_{\alpha_{n-2}} &= x_{n-3} \partial_{n-2}, & E_{-\alpha_{n-2}} &= x_{n-2} \partial_{n-3}
\end{align*}
\]

A.2 $D_5$

Dynkin diagram:

```
+---+---+---+---+
| 4 | α3 |
+---+---+---+---+
| 1 | α1 | β0 | α2 | α4 |
```

Positive roots:

\[
\begin{align*}
\alpha_1 &= (1, 0, 0, 0, 0) = A(β_1) \\
\alpha_2 &= (0, 0, 1, 0, 0) = A(β_2) \\
\alpha_3 &= (0, 0, 0, 1, 0) = A(α_3) \\
\alpha_4 &= (0, 0, 0, 0, 1) = A(α_4) \\
\alpha_5 &= (0, 0, 1, 1, 0) = A(β_4) \\
\alpha_6 &= (0, 0, 1, 0, 1) = A(β_5) \\
\alpha_7 &= (0, 0, 1, 1, 1) = A(β_3)
\end{align*}
\]

\[
\begin{align*}
β_0 &= (0, 1, 0, 0, 0) & γ_0 &= (1, 1, 2, 1, 1) \\
β_1 &= (1, 1, 0, 0, 0) & γ_1 &= (0, 1, 2, 1, 1) \\
β_2 &= (0, 1, 1, 0, 0) & γ_2 &= (1, 1, 1, 1, 1) \\
β_3 &= (0, 1, 1, 1, 1) & γ_3 &= (1, 1, 1, 0, 0) \\
β_4 &= (0, 1, 1, 1, 0) & γ_4 &= (1, 1, 1, 0, 1) \\
β_5 &= (0, 1, 1, 0, 1) & γ_5 &= (1, 1, 1, 1, 0)
\end{align*}
\]

\[
ω = (1, 2, 2, 1, 1) = A(γ_0)
\]

Cubic form:

\[
I_3 = x_1(x_2x_3 - x_4x_5)
\]

Cartan generators:

\[
\begin{align*}
H_{β_0} &= -y \partial + x_0 \partial_0 \\
H_{α_1} &= -2 - x_0 \partial_0 + x_1 \partial_1 - x_2 \partial_2 - x_3 \partial_3 - x_4 \partial_4 - x_5 \partial_5 \\
H_{α_2} &= -1 - x_0 \partial_0 - x_1 \partial_1 + x_2 \partial_2 - x_3 \partial_3 \\
H_{α_3} &= -x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 - x_5 \partial_5 \\
H_{α_4} &= -x_2 \partial_2 + x_3 \partial_3 - x_4 \partial_4 + x_5 \partial_5
\end{align*}
\]
Simple roots:

\[ E_{\alpha_1} = -x_0 \partial_1 - i(x_2 x_3 - x_4 x_5)/y \]
\[ E_{\alpha_2} = -x_0 \partial_2 - i x_1 x_3/y \]
\[ E_{\alpha_3} = x_2 \partial_1 + x_5 \partial_3 \]
\[ E_{\alpha_4} = -x_2 \partial_5 - x_4 \partial_3 \]
\[ E_{-\alpha_1} = x_1 \partial_0 + i y (\partial_2 \partial_3 - \partial_4 \partial_5) \]
\[ E_{-\alpha_2} = x_2 \partial_0 + i y \partial_1 \partial_3 \]
\[ E_{-\alpha_3} = -x_3 \partial_5 - x_4 \partial_2 \]
\[ E_{-\alpha_4} = x_3 \partial_4 + x_5 \partial_2 \]

A.3 \( E_6 \)

Dynkin diagram:

```
1  2  3  4  5  6  
\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5
```

Positive roots:

\[ \alpha_1 = (1, 0, 0, 0, 0, 0) = A(\alpha_1) \]
\[ \alpha_2 = (0, 0, 1, 0, 0, 0) = A(\alpha_2) \]
\[ \alpha_3 = (0, 0, 0, 1, 0, 0) = A(\beta_1) \]
\[ \alpha_4 = (0, 0, 0, 0, 1, 0) = A(\alpha_4) \]
\[ \alpha_5 = (0, 0, 0, 0, 0, 1) = A(\alpha_5) \]
\[ \alpha_6 = (1, 0, 1, 0, 0, 0) = A(\alpha_6) \]
\[ \alpha_7 = (0, 0, 1, 1, 0, 0) = A(\beta_2) \]
\[ \alpha_8 = (0, 0, 0, 1, 1, 0) = A(\beta_3) \]
\[ \alpha_9 = (0, 0, 0, 0, 1, 1) = A(\alpha_9) \]
\[ \alpha_{10} = (1, 0, 1, 1, 0, 0) = A(\beta_4) \]
\[ \alpha_{11} = (0, 0, 1, 1, 1, 0) = A(\beta_5) \]
\[ \alpha_{12} = (0, 0, 0, 1, 1, 1) = A(\beta_6) \]
\[ \alpha_{13} = (1, 0, 1, 1, 1, 0) = A(\beta_7) \]
\[ \alpha_{14} = (0, 0, 1, 1, 1, 1) = A(\beta_8) \]
\[ \alpha_{15} = (1, 0, 1, 1, 1, 1) = A(\beta_8) \]

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\[ \beta_0 = (0, 1, 0, 0, 0, 0) \quad \gamma_0 = (1, 1, 2, 3, 2, 1) \\
\beta_1 = (0, 1, 0, 1, 0, 0) \quad \gamma_1 = (1, 1, 2, 2, 2, 1) \\
\beta_2 = (0, 1, 1, 1, 0, 0) \quad \gamma_2 = (1, 1, 1, 2, 2, 1) \\
\beta_3 = (0, 1, 0, 1, 1, 0) \quad \gamma_3 = (1, 1, 2, 2, 1, 1) \\
\beta_4 = (1, 1, 1, 1, 0, 0) \quad \gamma_4 = (0, 1, 1, 2, 2, 1) \\
\beta_5 = (0, 1, 1, 1, 1, 0) \quad \gamma_5 = (1, 1, 1, 2, 1, 1) \\
\beta_6 = (0, 1, 0, 1, 1, 0) \quad \gamma_6 = (0, 1, 2, 2, 1, 0) \\
\beta_7 = (1, 1, 1, 1, 1, 0) \quad \gamma_7 = (0, 1, 1, 2, 1, 1) \\
\beta_8 = (1, 1, 1, 1, 1, 1) \quad \gamma_8 = (0, 1, 1, 2, 1, 0) \\
\beta_9 = (0, 1, 1, 1, 1, 1) \quad \gamma_9 = (1, 1, 1, 2, 1, 1) \\
\]

\[ \omega = (1, 2, 2, 3, 2, 1) = A(\gamma_0) \]

Cubic form:

\[ I_3 = -x_1 x_5 x_8 + x_1 x_7 x_9 + x_2 x_3 x_8 - x_2 x_6 x_7 - x_3 x_4 x_9 + x_4 x_5 x_6 \]

Cartan generators:

\[ H_{\beta_0} = -y \partial + x_0 \partial_0 \]
\[ H_{\alpha_1} = -x_2 \partial_2 + x_3 \partial_4 - x_5 \partial_5 + x_7 \partial_7 + x_8 \partial_8 - x_9 \partial_9 \]
\[ H_{\alpha_2} = -x_1 \partial_1 + x_2 \partial_2 - x_3 \partial_3 + x_5 \partial_5 - x_6 \partial_6 + x_9 \partial_9 \]
\[ H_{\alpha_3} = -2 - x_0 \partial_0 + x_1 \partial_1 - x_5 \partial_5 - x_7 \partial_7 - x_8 \partial_8 - x_9 \partial_9 \]
\[ H_{\alpha_4} = -x_1 \partial_1 - x_2 \partial_2 + x_3 \partial_3 - x_4 \partial_4 + x_5 \partial_5 + x_7 \partial_7 \]
\[ H_{\alpha_5} = -x_3 \partial_3 - x_5 \partial_5 + x_6 \partial_6 - x_7 \partial_7 + x_8 \partial_8 + x_9 \partial_9 \]

Simple roots:

\[ E_{\alpha_1} = -x_2 \partial_4 - x_5 \partial_7 - x_9 \partial_9 \]
\[ E_{\alpha_2} = -x_1 \partial_2 - x_7 \partial_5 - x_6 \partial_9 \]
\[ E_{\alpha_3} = -x_0 \partial_1 + i(x_5 x_8 - x_7 x_9)/y \]
\[ E_{\alpha_4} = -x_1 \partial_3 - x_2 \partial_5 - x_4 \partial_7 \]
\[ E_{\alpha_5} = -x_3 \partial_6 - x_5 \partial_9 - x_7 \partial_8 \]
\[ E_{-\alpha_1} = x_4 \partial_2 + x_7 \partial_5 + x_8 \partial_9 \]
\[ E_{-\alpha_2} = x_2 \partial_1 + x_5 \partial_3 + x_9 \partial_6 \]
\[ E_{-\alpha_3} = x_1 \partial_0 - iy(\partial_5 \partial_8 - \partial_7 \partial_9) \]
\[ E_{-\alpha_4} = x_3 \partial_1 + x_5 \partial_2 + x_7 \partial_4 \]
\[ E_{-\alpha_5} = x_6 \partial_3 + x_8 \partial_7 + x_9 \partial_5 \]
A.4 $E_7$

Dynkin diagram:

![Dynkin Diagram](image)

Positive roots:

\[
\begin{align*}
\alpha_1 &= (0, 1, 0, 0, 0, 0, 0) = A(\alpha_1) \\
\alpha_2 &= (0, 0, 1, 0, 0, 0) = A(\beta_1) \\
\alpha_3 &= (0, 0, 0, 1, 0, 0, 0) = A(\alpha_3) \\
\alpha_4 &= (0, 0, 0, 1, 0, 0, 0) = A(\alpha_4) \\
\alpha_5 &= (0, 0, 0, 0, 1, 0, 0) = A(\alpha_5) \\
\alpha_6 &= (0, 0, 0, 0, 0, 1) = A(\alpha_6) \\
\alpha_7 &= (0, 1, 0, 1, 0, 0, 0) = A(\alpha_7) \\
\alpha_8 &= (0, 0, 1, 1, 0, 0, 0) = A(\beta_2) \\
\alpha_9 &= (0, 0, 0, 1, 1, 0, 0) = A(\alpha_9) \\
\alpha_{10} &= (0, 0, 0, 1, 1, 0) = A(\alpha_{10}) \\
\alpha_{11} &= (0, 0, 0, 1, 0, 1, 1) = A(\alpha_{11}) \\
\alpha_{12} &= (0, 1, 1, 0, 0, 0, 0) = A(\beta_3) \\
\alpha_{13} &= (0, 1, 0, 1, 1, 0, 0) = A(\alpha_{13}) \\
\alpha_{14} &= (0, 0, 1, 1, 1, 0, 0) = A(\beta_4) \\
\alpha_{15} &= (0, 0, 0, 1, 1, 1, 0) = A(\alpha_{15}) \\
\alpha_{16} &= (0, 0, 0, 1, 1, 1, 1) = A(\alpha_{16}) \\
\alpha_{17} &= (0, 1, 1, 1, 1, 0, 0) = A(\beta_5) \\
\alpha_{18} &= (0, 1, 0, 1, 1, 1, 0) = A(\alpha_{18}) \\
\alpha_{19} &= (0, 0, 1, 1, 1, 1, 0) = A(\beta_6) \\
\alpha_{20} &= (0, 0, 1, 1, 1, 1, 1) = A(\alpha_{20}) \\
\alpha_{21} &= (0, 1, 1, 2, 1, 0, 0) = A(\beta_7) \\
\alpha_{22} &= (0, 1, 1, 1, 1, 1, 0) = A(\beta_8) \\
\alpha_{23} &= (0, 1, 0, 1, 1, 1, 1) = A(\alpha_{23}) \\
\alpha_{24} &= (0, 0, 1, 1, 1, 1, 1) = A(\beta_9) \\
\alpha_{25} &= (0, 1, 1, 2, 1, 1, 0) = A(\beta_{11}) \\
\alpha_{26} &= (0, 1, 1, 1, 1, 1, 1) = A(\beta_{12}) \\
\alpha_{27} &= (0, 1, 1, 2, 2, 1, 0) = A(\beta_{14}) \\
\alpha_{28} &= (0, 1, 1, 2, 1, 1, 1) = A(\beta_{13}) \\
\alpha_{29} &= (0, 1, 1, 2, 2, 1, 1) = A(\beta_{15}) \\
\alpha_{30} &= (0, 1, 1, 2, 2, 2, 1) = A(\beta_{16})
\end{align*}
\]
\[ \beta_0 = (1,0,0,0,0,0,0) \quad \gamma_0 = (1,2,3,4,3,2,1) \]
\[ \beta_1 = (1,0,1,0,0,0,0) \quad \gamma_1 = (1,2,2,4,3,2,1) \]
\[ \beta_2 = (1,0,1,1,0,0,0) \quad \gamma_2 = (1,2,2,3,3,2,1) \]
\[ \beta_3 = (1,1,1,0,0,0,0) \quad \gamma_3 = (1,2,3,3,2,1) \]
\[ \beta_4 = (1,0,1,1,1,0,0) \quad \gamma_4 = (1,2,2,3,2,2,1) \]
\[ \beta_5 = (1,1,1,1,0,0,0) \quad \gamma_5 = (1,2,3,2,2,1) \]
\[ \beta_6 = (1,0,1,1,1,1,0) \quad \gamma_6 = (1,2,2,3,2,1,1) \]
\[ \beta_7 = (1,1,1,2,1,0,0) \quad \gamma_7 = (1,2,2,2,2,1) \]
\[ \beta_8 = (1,1,1,1,1,1,0) \quad \gamma_8 = (1,2,3,2,2,1,1) \]
\[ \beta_9 = (1,0,1,1,1,1,1) \quad \gamma_9 = (1,2,3,2,1,0) \]
\[ \beta_{10} = (1,1,1,1,2,2,1) \quad \gamma_{10} = (1,2,2,1,0,0) \]
\[ \beta_{11} = (1,1,1,2,1,1,0) \quad \gamma_{11} = (1,2,2,2,1,1) \]
\[ \beta_{12} = (1,1,1,1,1,1,1) \quad \gamma_{12} = (1,2,3,2,1,1) \]
\[ \beta_{13} = (1,1,1,2,2,1,1) \quad \gamma_{13} = (1,2,2,1,1,0) \]
\[ \beta_{14} = (1,1,1,2,2,1,0) \quad \gamma_{14} = (1,2,2,2,1,1,1) \]
\[ \beta_{15} = (1,1,1,2,1,1,1) \quad \gamma_{15} = (1,2,2,2,1,0) \]

\[ \omega = (2,2,3,4,3,2,1) = A(\gamma_0) \]

Cubic form:

\[ I_3 = -x_1x_7x_{10} + x_1x_{11}x_{13} - x_1x_{14}x_{15} + x_2x_5x_{10} - x_2x_8x_{13} + x_2x_{12}x_{14} - x_3x_4x_{10} + x_3x_6x_{13} \]
\[ -x_3x_9x_{14} + x_4x_8x_{15} - x_4x_{11}x_{12} - x_5x_6x_{15} + x_5x_9x_{11} + x_6x_7x_{12} - x_7x_8x_9 \]

Cartan generators:

\[ H_{\beta_0} = -y \partial + x_0 \partial_0 \]
\[ H_{\alpha_1} = -x_2 \partial_2 + x_3 \partial_3 - x_4 \partial_4 + x_5 \partial_5 - x_6 \partial_6 + x_8 \partial_8 - x_9 \partial_9 + x_{12} \partial_{12} \]
\[ H_{\alpha_2} = -3 - x_6 \partial_0 + x_1 \partial_1 - x_7 \partial_7 - x_{10} \partial_{10} - x_{11} \partial_{11} - x_{13} \partial_{13} - x_{14} \partial_{14} - x_{15} \partial_{15} \]
\[ H_{\alpha_3} = -x_1 \partial_1 + x_2 \partial_2 - x_5 \partial_5 + x_7 \partial_7 - x_8 \partial_8 + x_{11} \partial_{11} - x_{12} \partial_{12} + x_{15} \partial_{15} \]
\[ H_{\alpha_4} = -x_2 \partial_2 - x_3 \partial_3 + x_4 \partial_4 + x_5 \partial_5 - x_{11} \partial_{11} + x_{13} \partial_{13} + x_{14} \partial_{14} - x_{15} \partial_{15} \]
\[ H_{\alpha_5} = -x_4 \partial_4 - x_5 \partial_5 + x_6 \partial_6 - x_7 \partial_7 + x_8 \partial_8 + x_{10} \partial_{10} + x_{11} \partial_{11} - x_{13} \partial_{13} \]
\[ H_{\alpha_6} = -x_6 \partial_6 - x_8 \partial_8 + x_9 \partial_9 - x_{11} \partial_{11} + x_{12} \partial_{12} + x_{13} \partial_{13} - x_{14} \partial_{14} + x_{15} \partial_{15} \]
Simple roots:

\[ E_{\alpha_1} = x_2 \partial_3 + x_4 \partial_5 + x_6 \partial_8 + x_9 \partial_{12} \]
\[ E_{\alpha_2} = -x_0 \partial_1 + i(x_7 x_{10} - x_{11} x_{13} + x_{14} x_{15})/y \]
\[ E_{\alpha_3} = x_1 \partial_2 + x_5 \partial_7 + x_8 \partial_{11} + x_{12} \partial_{15} \]
\[ E_{\alpha_4} = x_2 \partial_4 + x_3 \partial_5 + x_{11} \partial_{14} + x_{15} \partial_{13} \]
\[ E_{\alpha_5} = x_4 \partial_6 + x_5 \partial_8 + x_7 \partial_{11} + x_{13} \partial_{10} \]
\[ E_{\alpha_6} = x_6 \partial_9 + x_8 \partial_{12} + x_{11} \partial_{15} + x_{14} \partial_{13} \]
\[ E_{-\alpha_1} = -x_3 \partial_2 - x_5 \partial_4 - x_8 \partial_6 - x_{12} \partial_9 \]
\[ E_{-\alpha_2} = x_1 \partial_0 - iy(\partial_7 \partial_{10} - \partial_{11} \partial_{13} + \partial_{14} \partial_{15}) \]
\[ E_{-\alpha_3} = -x_2 \partial_1 - x_7 \partial_5 - x_{11} \partial_8 - x_{15} \partial_{12} \]
\[ E_{-\alpha_4} = -x_4 \partial_2 - x_5 \partial_3 - x_{13} \partial_{15} - x_{14} \partial_{11} \]
\[ E_{-\alpha_5} = -x_6 \partial_4 - x_8 \partial_5 - x_{10} \partial_{13} - x_{11} \partial_7 \]
\[ E_{-\alpha_6} = -x_9 \partial_6 - x_{12} \partial_8 - x_{13} \partial_{14} - x_{15} \partial_{11} \]

A.5 \ E_8

Dynkin diagram:
Positive roots:

\[
\begin{align*}
\alpha_1 &= \left(1, 0, 0, 0, 0, 0, 0, 0\right) = A(\alpha_1) \\
\alpha_2 &= \left(0, 1, 0, 0, 0, 0, 0, 0\right) = A(\alpha_2) \\
\alpha_3 &= \left(0, 0, 1, 0, 0, 0, 0, 0\right) = A(\alpha_3) \\
\alpha_4 &= \left(0, 0, 0, 1, 0, 0, 0, 0\right) = A(\alpha_4) \\
\alpha_5 &= \left(0, 0, 0, 0, 1, 0, 0, 0\right) = A(\alpha_5) \\
\alpha_6 &= \left(0, 0, 0, 0, 0, 1, 0, 0\right) = A(\alpha_6) \\
\alpha_7 &= \left(0, 0, 0, 0, 0, 0, 1, 0\right) = A(\beta_1) \\
\alpha_8 &= \left(1, 0, 1, 0, 0, 0, 0, 0\right) = A(\alpha_8) \\
\alpha_9 &= \left(0, 1, 0, 1, 0, 0, 0, 0\right) = A(\alpha_9) \\
\alpha_{10} &= \left(0, 0, 1, 1, 0, 0, 0, 0\right) = A(\alpha_{10}) \\
\alpha_{11} &= \left(0, 0, 0, 1, 1, 0, 0, 0\right) = A(\alpha_{11}) \\
\alpha_{12} &= \left(0, 0, 0, 0, 1, 1, 0, 0\right) = A(\alpha_{12}) \\
\alpha_{13} &= \left(0, 0, 0, 0, 0, 1, 1, 0\right) = A(\beta_2) \\
\alpha_{14} &= \left(1, 0, 1, 1, 0, 0, 0, 0\right) = A(\alpha_{14}) \\
\alpha_{15} &= \left(0, 1, 1, 1, 0, 0, 0, 0\right) = A(\alpha_{15}) \\
\alpha_{16} &= \left(0, 1, 0, 1, 1, 0, 0, 0\right) = A(\alpha_{16}) \\
\alpha_{17} &= \left(0, 0, 1, 1, 1, 0, 0, 0\right) = A(\alpha_{17}) \\
\alpha_{18} &= \left(0, 0, 0, 1, 1, 1, 0, 0\right) = A(\alpha_{18}) \\
\alpha_{19} &= \left(0, 0, 0, 0, 1, 1, 1, 0\right) = A(\beta_3) \\
\alpha_{20} &= \left(1, 1, 1, 1, 0, 0, 0, 0\right) = A(\alpha_{20}) \\
\alpha_{21} &= \left(1, 0, 1, 1, 1, 0, 0, 0\right) = A(\alpha_{21}) \\
\alpha_{22} &= \left(0, 1, 1, 1, 1, 0, 0, 0\right) = A(\alpha_{22}) \\
\alpha_{23} &= \left(0, 1, 0, 1, 1, 1, 0, 0\right) = A(\alpha_{23}) \\
\alpha_{24} &= \left(0, 0, 1, 1, 1, 1, 0, 0\right) = A(\alpha_{24}) \\
\alpha_{25} &= \left(0, 0, 0, 1, 1, 1, 1, 0\right) = A(\beta_4) \\
\alpha_{26} &= \left(1, 1, 1, 1, 1, 0, 0, 0\right) = A(\alpha_{26}) \\
\alpha_{27} &= \left(1, 0, 1, 1, 1, 1, 0, 0\right) = A(\alpha_{27}) \\
\alpha_{28} &= \left(0, 1, 1, 2, 1, 0, 0, 0\right) = A(\alpha_{28}) \\
\alpha_{29} &= \left(0, 1, 1, 1, 1, 1, 0, 0\right) = A(\alpha_{29}) \\
\alpha_{30} &= \left(0, 1, 0, 1, 1, 1, 1, 0\right) = A(\beta_5) \\
\alpha_{31} &= \left(0, 0, 1, 1, 1, 1, 1, 0\right) = A(\beta_6) \\
\alpha_{32} &= \left(1, 1, 1, 2, 1, 0, 0, 0\right) = A(\alpha_{32}) \\
\alpha_{33} &= \left(1, 1, 1, 1, 1, 1, 0, 0\right) = A(\alpha_{33}) \\
\alpha_{34} &= \left(1, 0, 1, 1, 1, 1, 1, 0\right) = A(\beta_7) \\
\alpha_{35} &= \left(0, 1, 1, 2, 1, 1, 0, 0\right) = A(\alpha_{35}) \\
\alpha_{36} &= \left(0, 1, 1, 1, 1, 1, 1, 0\right) = A(\beta_8) \\
\alpha_{37} &= \left(1, 1, 2, 2, 1, 0, 0, 0\right) = A(\alpha_{37}) \\
\alpha_{38} &= \left(1, 1, 1, 2, 1, 1, 0, 0\right) = A(\alpha_{38}) \\
\alpha_{39} &= \left(1, 1, 1, 1, 1, 1, 1, 0\right) = A(\beta_9)
\end{align*}
\]
\[
\begin{align*}
\alpha_{40} &= (0,1,1,2,2,1,0,0) = A(\alpha_{40}) \\
\alpha_{41} &= (0,1,1,2,1,1,1,0) = A(\beta_{10}) \\
\alpha_{42} &= (1,1,2,2,1,1,0,0) = A(\alpha_{42}) \\
\alpha_{43} &= (1,1,1,2,1,0,0,0) = A(\alpha_{43}) \\
\alpha_{44} &= (1,1,1,2,1,1,1,0) = A(\beta_{11}) \\
\alpha_{45} &= (0,1,1,2,1,1,1,0) = A(\beta_{12}) \\
\alpha_{46} &= (1,1,2,2,2,1,0,0) = A(\alpha_{46}) \\
\alpha_{47} &= (1,1,2,2,1,1,1,0) = A(\beta_{13}) \\
\alpha_{48} &= (1,1,2,2,1,1,1,0) = A(\beta_{14}) \\
\alpha_{49} &= (0,1,1,2,2,2,1,0) = A(\beta_{15}) \\
\alpha_{50} &= (1,1,2,3,2,1,0,0) = A(\alpha_{50}) \\
\alpha_{51} &= (1,1,2,2,2,1,1,0) = A(\beta_{16}) \\
\alpha_{52} &= (1,1,2,2,2,2,1,0) = A(\beta_{17}) \\
\alpha_{53} &= (1,2,2,3,2,1,0,0) = A(\alpha_{53}) \\
\alpha_{54} &= (1,1,2,3,2,1,1,0) = A(\beta_{19}) \\
\alpha_{55} &= (1,1,2,2,2,2,1,0) = A(\beta_{20}) \\
\alpha_{56} &= (1,2,2,3,2,1,1,0) = A(\beta_{22}) \\
\alpha_{57} &= (1,1,2,3,2,2,1,0) = A(\beta_{23}) \\
\alpha_{58} &= (1,2,2,3,2,2,1,0) = A(\beta_{25}) \\
\alpha_{59} &= (1,1,2,3,3,2,1,0) = A(\beta_{26}) \\
\alpha_{60} &= (1,2,2,3,3,2,1,0) = A(\beta_{27}) \\
\alpha_{61} &= (1,2,2,4,3,2,1,0) = A(\beta_{24}) \\
\alpha_{62} &= (1,2,3,4,3,2,1,0) = A(\beta_{21}) \\
\alpha_{63} &= (2,2,3,4,3,2,1,0) = A(\beta_{18})
\end{align*}
\]

\[
\begin{align*}
\beta_0 &= (0,0,0,0,0,0,0,1) & \gamma_0 &= (2,3,4,6,5,4,3,1) \\
\beta_1 &= (0,0,0,0,0,0,1,1) & \gamma_1 &= (2,3,4,6,5,4,2,1) \\
\beta_2 &= (0,0,0,0,0,1,1,1) & \gamma_2 &= (2,3,4,6,5,3,2,1) \\
\beta_3 &= (0,0,0,0,1,1,1,1) & \gamma_3 &= (2,3,4,6,4,3,2,1) \\
\beta_4 &= (0,0,0,1,1,1,1,1) & \gamma_4 &= (2,3,4,5,4,3,2,1) \\
\beta_5 &= (0,1,0,1,1,1,1,1) & \gamma_5 &= (2,2,4,5,4,3,2,1) \\
\beta_6 &= (0,0,1,1,1,1,1,1) & \gamma_6 &= (2,3,3,5,4,3,2,1) \\
\beta_7 &= (1,0,1,1,1,1,1,1) & \gamma_7 &= (1,3,3,5,4,3,2,1) \\
\beta_8 &= (0,1,1,1,1,1,1,1) & \gamma_8 &= (2,2,3,5,4,3,2,1) \\
\beta_9 &= (1,1,1,1,1,1,1,1) & \gamma_9 &= (1,2,3,5,4,3,2,1) \\
\beta_{10} &= (0,1,1,2,1,1,1,1) & \gamma_{10} &= (2,2,3,4,4,3,2,1) \\
\beta_{11} &= (1,1,1,2,1,1,1,1) & \gamma_{11} &= (1,2,3,4,4,3,2,1) \\
\beta_{12} &= (0,1,1,2,2,1,1,1) & \gamma_{12} &= (2,2,3,4,3,3,2,1) \\
\beta_{13} &= (1,1,2,2,1,1,1,1) & \gamma_{13} &= (1,2,2,4,4,3,2,1) \\
\beta_{14} &= (1,1,1,2,2,1,1,1) & \gamma_{14} &= (1,2,3,4,3,3,2,1)
\end{align*}
\]
Cubic form:

\[
\begin{align*}
\beta_{15} &= (0, 1, 1, 2, 2, 2, 1, 1) & \gamma_{15} &= (2, 2, 3, 4, 3, 2, 2, 1) \\
\beta_{16} &= (1, 1, 2, 2, 2, 1, 1, 1) & \gamma_{16} &= (1, 2, 2, 4, 3, 3, 2, 1) \\
\beta_{17} &= (1, 1, 1, 2, 2, 2, 1, 1) & \gamma_{17} &= (1, 2, 3, 4, 3, 2, 2, 1) \\
\beta_{18} &= (2, 2, 3, 4, 3, 2, 1, 1) & \gamma_{18} &= (0, 1, 1, 2, 2, 2, 2, 1) \\
\beta_{19} &= (1, 1, 2, 3, 2, 1, 1, 1) & \gamma_{19} &= (1, 2, 2, 3, 3, 3, 2, 1) \\
\beta_{20} &= (1, 1, 2, 2, 2, 2, 1, 1) & \gamma_{20} &= (1, 2, 2, 4, 3, 2, 2, 1) \\
\beta_{21} &= (1, 2, 3, 4, 3, 2, 1, 1) & \gamma_{21} &= (1, 1, 1, 2, 2, 2, 2, 1) \\
\beta_{22} &= (1, 2, 3, 2, 1, 1, 1) & \gamma_{22} &= (1, 1, 2, 3, 3, 3, 2, 1) \\
\beta_{23} &= (1, 1, 2, 3, 2, 1, 1, 1) & \gamma_{23} &= (1, 2, 2, 3, 3, 3, 2, 1) \\
\beta_{24} &= (1, 2, 2, 4, 3, 2, 1, 1) & \gamma_{24} &= (1, 1, 2, 2, 2, 2, 2, 1) \\
\beta_{25} &= (1, 2, 2, 3, 2, 1, 1, 1) & \gamma_{25} &= (1, 1, 2, 3, 3, 3, 2, 1) \\
\beta_{26} &= (1, 1, 2, 3, 2, 1, 1) & \gamma_{26} &= (1, 2, 2, 3, 2, 2, 2, 1) \\
\beta_{27} &= (1, 2, 2, 3, 3, 2, 1, 1) & \gamma_{27} &= (1, 1, 2, 3, 2, 2, 2, 1) \\
\end{align*}
\]

\[
\omega = (2, 3, 4, 6, 5, 4, 3, 2) = A(\gamma_0)
\]

Cubic form:

\[
I_3 = x_1x_{15}x_{18} + x_1x_{17}x_{21} + x_1x_{20}x_{24} - x_1x_{23}x_{27} + x_1x_{25}x_{26} + x_2x_{12}x_{18} + x_2x_{14}x_{21} + x_2x_{16}x_{24} \\
- x_2x_{19}x_{27} + x_2x_{22}x_{26} + x_3x_{10}x_{18} + x_3x_{11}x_{21} + x_3x_{13}x_{24} - x_3x_{19}x_{25} + x_3x_{22}x_{23} + x_4x_{8}x_{18} \\
+ x_4x_{9}x_{21} + x_4x_{13}x_{27} - x_4x_{16}x_{25} + x_4x_{20}x_{22} - x_5x_{6}x_{18} - x_5x_{7}x_{21} + x_5x_{13}x_{26} - x_5x_{16}x_{23} \\
+ x_5x_{19}x_{20} + x_6x_{9}x_{24} - x_6x_{11}x_{27} + x_6x_{14}x_{25} - x_6x_{17}x_{22} - x_7x_{8}x_{24} + x_7x_{10}x_{27} - x_7x_{12}x_{25} \\
+ x_7x_{15}x_{22} - x_8x_{11}x_{26} + x_8x_{14}x_{23} - x_8x_{17}x_{19} + x_9x_{10}x_{26} - x_9x_{12}x_{23} + x_9x_{15}x_{19} - x_9x_{14}x_{20} \\
+ x_{10}x_{16}x_{17} + x_{11}x_{12}x_{20} - x_{11}x_{15}x_{16} - x_{12}x_{13}x_{17}
\]

Cartan generators:

\[
H_{\beta_0} = -y\partial + x_9\partial_0 \\
H_{\alpha_1} = -x_6\partial_6 + x_7\partial_7 - x_8\partial_8 + x_9\partial_9 - x_{10}\partial_{10} + x_{11}\partial_{11} - x_{12}\partial_{12} + x_{14}\partial_{14} - x_{15}\partial_{15} + x_{17}\partial_{17} \\
+ x_{18}\partial_{18} - x_{21}\partial_{21} \\
H_{\alpha_2} = -x_{14}\partial_{14} + x_5\partial_5 - x_6\partial_6 - x_7\partial_7 + x_8\partial_8 + x_9\partial_9 - x_{19}\partial_{19} + x_{22}\partial_{22} - x_{23}\partial_{23} + x_{25}\partial_{25} \\
- x_{26}\partial_{26} + x_{27}\partial_{27}
\]
\[
H_{\alpha_3} = -x_4 \partial_4 - x_5 \partial_5 + x_6 \partial_6 + x_8 \partial_8 - x_{11} \partial_{11} + x_{13} \partial_{13} - x_{14} \partial_{14} + x_{16} \partial_{16} - x_{17} \partial_{17} + x_{20} \partial_{20} \\
+ x_{21} \partial_{21} - x_{23} \partial_{24} \\
H_{\alpha_4} = -x_3 \partial_3 + x_4 \partial_4 - x_8 \partial_8 - x_9 \partial_9 + x_{10} \partial_{10} + x_{11} \partial_{11} - x_{16} \partial_{16} + x_{19} \partial_{19} - x_{26} \partial_{20} + x_{23} \partial_{23} \\
+ x_{24} \partial_{24} - x_{27} \partial_{27} \\
H_{\alpha_5} = -x_2 \partial_2 + x_3 \partial_3 - x_{10} \partial_{10} - x_{11} \partial_{11} + x_{12} \partial_{12} - x_{13} \partial_{13} + x_{14} \partial_{14} + x_{16} \partial_{16} - x_{23} \partial_{23} - x_{25} \partial_{25} \\
+ x_{26} \partial_{26} + x_{27} \partial_{27} \\
H_{\alpha_6} = -x_1 \partial_1 + x_2 \partial_2 - x_{12} \partial_{12} - x_{14} \partial_{14} + x_{15} \partial_{15} - x_{16} \partial_{16} + x_{17} \partial_{17} - x_{19} \partial_{19} + x_{20} \partial_{20} - x_{22} \partial_{22} \\
+ x_{23} \partial_{23} + x_{25} \partial_{25} \\
H_{\alpha_7} = -5 - x_0 \partial_0 + x_1 \partial_1 - x_{15} \partial_{15} - x_{17} \partial_{17} - x_{18} \partial_{18} - x_{20} \partial_{20} - x_{21} \partial_{21} - x_{23} \partial_{23} - x_{24} \partial_{24} - x_{25} \partial_{25} \\
- x_{26} \partial_{26} - x_{27} \partial_{27} \\
\]

Simple roots:

\[
E_{\alpha_1} = -x_6 \partial_7 - x_8 \partial_9 - x_{10} \partial_{11} - x_{12} \partial_{14} - x_{15} \partial_{17} + x_{21} \partial_{18} \\
E_{\alpha_2} = -x_4 \partial_5 - x_6 \partial_8 - x_7 \partial_9 + x_{19} \partial_{22} + x_{23} \partial_{25} + x_{26} \partial_{27} \\
E_{\alpha_3} = -x_4 \partial_6 - x_5 \partial_8 - x_{11} \partial_{13} - x_{14} \partial_{16} - x_{17} \partial_{20} + x_{24} \partial_{21} \\
E_{\alpha_4} = -x_3 \partial_4 + x_8 \partial_{10} + x_9 \partial_{11} + x_{16} \partial_{19} + x_{20} \partial_{23} + x_{27} \partial_{24} \\
E_{\alpha_5} = -x_2 \partial_3 + x_{10} \partial_{12} + x_{11} \partial_{14} + x_{13} \partial_{16} + x_{23} \partial_{26} + x_{25} \partial_{27} \\
E_{\alpha_6} = -x_1 \partial_2 + x_{12} \partial_{15} + x_{14} \partial_{17} + x_{16} \partial_{20} + x_{19} \partial_{23} + x_{22} \partial_{25} \\
E_{\alpha_7} = -x_9 \partial_1 + i(-x_{15} x_{18} - x_{17} x_{21} - x_{20} x_{24} + x_{23} x_{27} - x_{25} x_{26})/y \\
E_{-\alpha_1} = x_7 \partial_6 + x_9 \partial_8 + x_{11} \partial_{10} + x_{14} \partial_{12} + x_{17} \partial_{15} - x_{18} \partial_{21} \\
E_{-\alpha_2} = x_5 \partial_4 + x_8 \partial_6 + x_9 \partial_7 - x_{22} \partial_{19} - x_{25} \partial_{23} - x_{27} \partial_{26} \\
E_{-\alpha_3} = x_6 \partial_4 + x_8 \partial_5 + x_{13} \partial_{11} + x_{16} \partial_{14} + x_{20} \partial_{17} - x_{21} \partial_{24} \\
E_{-\alpha_4} = x_4 \partial_3 - x_{10} \partial_8 - x_{11} \partial_9 - x_{19} \partial_{16} - x_{23} \partial_{20} - x_{24} \partial_{27} \\
E_{-\alpha_5} = x_3 \partial_2 - x_{12} \partial_{10} - x_{14} \partial_{11} - x_{16} \partial_{13} - x_{26} \partial_{23} - x_{27} \partial_{25} \\
E_{-\alpha_6} = x_2 \partial_1 - x_{15} \partial_{12} - x_{17} \partial_{14} - x_{20} \partial_{16} - x_{23} \partial_{19} - x_{25} \partial_{22} \\
E_{-\alpha_7} = x_1 \partial_0 + i y (\partial_{15} \partial_{18} + \partial_{17} \partial_{21} + \partial_{20} \partial_{24} - \partial_{23} \partial_{27} + \partial_{25} \partial_{26})
\]
References

[1] B. Pioline, H. Nicolai, J. Plefka and A. Waldron, “$R^4$ couplings, the fundamental membrane and exceptional theta correspondences,” JHEP 0103, 036 (2001), hep-th/0102123.

[2] M. B. Green and M. Gutperle, “Effects of D instantons,” Nucl. Phys. B498 (1997) 195–227, hep-th/9701093.

[3] E. Kiritsis and B. Pioline, “On $R^4$ threshold corrections in IIB string theory and $(p,q)$ string instantons,” Nucl. Phys. B508 (1997) 509–534, hep-th/9707018; B. Pioline and E. Kiritsis, “U-duality and D-brane combinatorics,” Phys. Lett. B418 (1998) 61, hep-th/9710078.

[4] B. Pioline, “A note on non-perturbative $R^4$ couplings,” Phys. Lett. B 431, 73 (1998) hep-th/9804023; M. B. Green and S. Sethi, “Supersymmetry constraints on type IIB supergravity,” Phys. Rev. D 59, 046006 (1999) hep-th/9808061.

[5] N. A. Obers and B. Pioline, “Eisenstein series and string thresholds,” Commun. Math. Phys. 209 (2000) 275, hep-th/9903113; N. A. Obers and B. Pioline, “Eisenstein series in string theory,” Class. Quant. Grav. 17 (2000) 1215, hep-th/9910115.

[6] N. A. Obers and B. Pioline, “U-duality and M-theory,” Phys. Rept. 318 (1999) 113, hep-th/9809039; N. A. Obers and B. Pioline, “U-duality and M-theory, an algebraic approach,” hep-th/9812139.

[7] M. B. Green, M. Gutperle and P. Vanhove, “One loop in eleven dimensions,” Phys. Lett. B 409, 177 (1997) hep-th/9706175; M. B. Green, H. h. Kwon and P. Vanhove, Phys. Rev. D 61, 104010 (2000) hep-th/9910055.

[8] F. Sugino and P. Vanhove, “U-duality from matrix membrane partition function,” hep-th/0107145.

[9] A. Joseph, “Minimal realizations and spectrum generating algebras,” Comm. Math. Phys. 36 (1974) 325; “The minimal orbit in a simple Lie algebra and its associated maximal ideal,” Ann. Scient. Ecole Normale Sup. 4ème série 9 (1976) 1-30.

[10] R. Brylinski and B. Kostant, “Minimal representations of $E_6$, $E_7$, and $E_8$ and the generalized Capelli identity”, Proc. Nat. Acad. Sci. U.S.A. 91 (1994) 2469; “Minimal representations, geometric quantization, and unitarity”, Proc. Nat. Acad. Sci. U.S.A. 91 (1994) 6026; “Lagrangian models of minimal representations of $E_6$, $E_7$ and $E_8$”, in Functional Analysis on the Eve of the 21st century, Progress in Math., Birkhäuser (1995).

[11] D. Kazhdan and G. Savin, “The smallest representation of simply laced groups,” Israel Math. Conf. Proceedings, Piatetski-Shapiro Festschrift 2 (1990) 209-223.
[12] R. Howe, “θ-series and invariant theory”, Proceedings of Symposia in Pure Mathematics XXXIII, AMS, Providence, RI, 1979.

[13] D. Kazhdan and A. Polishchuk, “Spherical vector in the minimal representation of a simply-laced p-adic group”, to appear.

[14] G. Lion, M. Vergne, “The Weil representation, Maslov index and theta series”, Progress in Mathematics 6, Birkhäuser, 1980; D. Mumford, “Tata lectures on Theta III”, Progress in Mathematics, Birkhäuser, 1991.

[15] A. Terras, “Harmonic Analysis on Symmetric Spaces and Applications II”, Springer-Verlag, New-York, 1985.

[16] L. Brekke and P.G.O. Freund, “p-adic numbers in physics”, Phys. Rep. 233 (1993) 1.

[17] I. Gelfand, M. Graev and I. Pjatetskii-Shapiro, “Representation theory and automorphic functions”, Saunders, London, 1969.

[18] M. Gunaydin, K. Koepsell and H. Nicolai, “Conformal and quasiconformal realizations of exceptional Lie groups,” Commun. Math. Phys. 221 (2001) 57 hep-th/0008063.

[19] D. Kazhdan, “The minimal representation of $D_4$”, in Operator algebras, Unitary representations, enveloping algebras and invariant theories, A. Connes, M. Duflo, A. Joseph, R. Rentschler eds., Progress in Mathematics 92, Birkhäuser Boston (1990) 125.

[20] P. Etingof, D. Kazhdan, and A. Polishchuk, “When is the Fourier transform of an elementary function elementary?”, math.AG/0003009.

[21] E. Kiritsis, N. A. Obers and B. Pioline, “Heterotic/type II triality and instantons on K3,” JHEP 0001, 029 (2000), hep-th/0001083.

[22] W. Nahm and K. Wendland, “A hiker’s guide to K3: Aspects of N = (4,4) superconformal field theory with central charge c = 6,” Commun. Math. Phys. 216, 85 (2001), hep-th/9912067.

[23] M. Gunaydin, K. Koepsell and H. Nicolai, “The Minimal Unitary Representation of $E_{8(8)}$”, hep-th/0109005.

[24] D. Vogan, ”The unitary dual of $G_2$”, Invent. Math. 116 (1994), 677-791.

[25] D. Vogan, “Singular unitary representations,” Lecture Notes in Math. 880, Springer Verlag.

[26] V. de Alfaro, S. Fubini and G. Furlan, “Conformal Invariance In Quantum Mechanics,” Nuovo Cim. A 34, 569 (1976).

[27] J.A.M. Vermaseren, “New Features of Form”, math-ph/0010025.