TRAVELING WAVE SOLUTIONS WITH CONVEX DOMAINS FOR A FREE BOUNDARY PROBLEM

HARUNORI MONOBE
Meiji Institute of Mathematical Sciences, Meiji University
4-21-1 Nakano, Nakano-ku, Tokyo, 164-8525, Japan

HIROKAZU NINOMIYA
School of Interdisciplinary Mathematical Sciences, Meiji University
4-21-1 Nakano, Nakano-ku, Tokyo, 164-8525, Japan

Abstract. In this paper, a free boundary problem related to cell motility is discussed. This free boundary problem consists of an interface equation for the domain evolution and a parabolic equation governing actin concentration in the domain. In [10], the existence of traveling wave solutions with disk-shaped domains were shown in a special situation where a polymerization rate is specified. In this paper, by relaxing the condition for the polymerization rate, the previous result is extended to the existence of traveling wave solutions with convex domains.

1. Introduction. Amoeba movement is a kind of cell crawling observed in many cells like amoeboid, white blood cells and keratocyte. From recent research results, it is known that two proteins actin and myosin in a cell have a key role on crawling. Crawling mainly consists of three stages: protrusion, adhesion and retraction. Protrusion involves the extension of cellular membranes at the cell front. The cell front interacts with the extracellular matrix (ECM) by forming adhesions that are connected to the actin network. Translocation of the cell body is followed by retraction of the rear of the cell. Actin filaments are essential for all stages. Actin filaments (F-actin) can grow by adding actin monomers (G-actin) at the ends. It is called polymerization. The rate of growth is faster at the plus end than at the minus end. Since most of the filaments have their plus ends close to the membrane, actin polymerization pushes forward the cell membrane. Thus cells extend thin protrusions at the leading edge and elsewhere on their surface. Adhesion occurs along a dense meshwork of actin filaments. Rear retraction requires disassembly of the adhesions at the trailing edge. The cells are propelled forward by repeating these stages. Based on this mechanism, many researchers proposed mathematical models to understand cell motility [5, 6, 7].

In this paper, we consider a two-dimensional model proposed by Tamiki Umeda to study how the distributions of F-actin and G-actin affect the cytoskeleton of a cell. We denote a thin two dimensional single cell by Ω(t) and let the density
of F-actin inside be denoted by \( u = u(x,y,t) \). At the same time we assume that G-actin distributes uniformly in \( \Omega(t) \) and denote it by \( U = U(t) \).

Since the total amount of G-actin monomers in the cell does not change, we have

\[
\int \int_{\Omega(t)} u(x,t) \, dx \, dy + U(t) = C_0,
\]

where \( C_0 \) is a positive constant describing the total amount of G-actin. On the boundary \( \partial \Omega(t) \), let \( n = n(x,y,t) \) be its unit outward normal vector, \( V = V(x,y,t) \) its normal velocity and \( \kappa = \kappa(x,y,t) \) its inward curvature. Our model is a free boundary problem with unknowns \( \Omega(t) \subset \mathbb{R}^2 \), \( u \) and \( U \) governed by

\[
\begin{aligned}
\begin{cases}
  u_t = d \Delta u + k_1 U - u + k_2 & \text{in } Q := \bigcup_{t > 0} \Omega(t) \times \{t\}, \\
  u = 1 + A \kappa & \text{on } \Gamma := \bigcup_{t > 0} \partial \Omega(t) \times \{t\}, \\
  V = \gamma U - 1 - A \kappa & \text{on } \Gamma, \\
  u = \phi \geq k_2 & \text{in } \Omega(0) \times \{t = 0\},
\end{cases}
\end{aligned}
\]

where \( u_t \) represents the derivative of \( u \) with respect to \( t \), and \( d, k_1, k_2, A \) are positive constants. By an appropriate choice of variables \( x, t, u \) and \( U \), we can normalize the coefficient of \( u \) in the first equation of (1) and the constant of both boundary conditions.

Actin filament grows from the polymerization of G-actin (subunit) and the growth depends on the concentration of G-actin. Actin polymerization is reversible process and subunit loss in actin filament is called depolymerization. The loss depends on the density of actin filament. The terms \( k_1 U \) and \( -u + k_2 \) in the right-hand side of the first equation of (1) represent the actin addition and the loss of G-actin in actin filament, respectively. Here \( k_1 \) is the polymerization rate and \( k_2 \) is the lowest density of F-actin. Because protrusion is mainly resulted by polymerization of G-actin and depolymerization of F-actin on the boundary, we assume that the growth of region \( \Omega(t) \) are determined by the difference between polymerization and depolymerization. Namely, \( V = \gamma U - u \) on \( \Gamma \) where \( \gamma \) is a polymerization rate on the boundary. The distribution of \( \gamma \) plays an important role in determining the direction of the cell movement. The first boundary condition \( u = 1 + A \kappa \) is derived from the balance with the surface tension at the cell boundary, where \( A \) is the surface tension of the cytoskeleton. Using this condition, we automatically obtain the third equation of (1). The activity of polymerization rate is dependent on various factors, e.g., chemotaxis, the chemical reaction in cells and the shape of the cell and so on. However, the mechanism of activity is not clearly understood yet. In this model, it is assumed that the activity \( \gamma = \gamma(n) \) is a given positive function depending only on \( n \). For more details, see [9] and [10].

The first author [9] considered the behavior of a pair \( (u, \Omega(t)) \) of (1) with radially symmetric initial data, where \( \gamma \) is a positive constant. Also the authors [10] confirmed the existence of stationary solutions with disk-shaped domains. These results imply that, if \( \gamma \) is a positive constant, then there exist at least two stationary solutions with disk-shaped domains in [11], and it is expected that one stationary solution with a large disk domain is stable and the other with a small disk domain is unstable.
At the same time the authors [10] showed the existence of traveling wave solutions, where \( \gamma \) satisfies a condition
\[
\gamma(n) = \gamma(\theta) = \eta + \xi \cos \theta.
\]
Here \( \theta \) stands for the angle between \( n \) and the unit vector \( e_1 = (1, 0) \). As a result, it was shown that there exist at least two disk-shaped traveling wave solutions for (1). Note that the solutions move along the direction \( e_1 \) where \( \gamma \) attains its maximum. The speed of these two traveling waves solutions differ depending on the size of domains, but these solutions have a common property that \( u(x, t) \) concentrates on the boundary \( \partial \Omega(t) \), in particular, on the leading edge. However this result still not be enough to describe keratocyte. In fact, fish keratocyte moves for long distance keeping the half-moon shape. Thus it is important to examine the existence of traveling wave solutions with non-disk-shaped domain and more general convex domain. In this paper, we will give a sufficient condition of \( \gamma \) for (1) to have a traveling wave solution with a symmetric and convex domain.

Now we reduce (1) to an elliptic problem by use of a moving coordinate. Let \( z = x - ct \), where \( c \) is positive. Suppose that \( u(x, y, t) \) and \( \Omega(t) \) are represented by \( \tilde{u}(z, y) \) and \( \tilde{\Omega} \) as follows:
\[
u(x, y, t) = \tilde{u}(z, y), \quad \Omega(t) = \{(x, y) \mid (x - ct, y) \in \tilde{\Omega}\},
\]
where \( \tilde{\Omega} \) stands for \( \Omega(0) \). Then the first equation of (1) is rewritten as
\[
d\Delta_{z,y} \tilde{u} + c\tilde{u}_z + k_1 \tilde{U} - \tilde{u} + k_2 = 0 \quad \text{in} \ \tilde{\Omega},
\]
where
\[
\Delta_{z,y} := \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}, \quad \tilde{U} := C_0 - \int_{\tilde{\Omega}} \tilde{u} \, dz \, dy.
\]
Let \( s \) and \( \theta \) be the arc length along \( \partial \tilde{\Omega} \) and the angle between the \( x \)-axis and \( \tilde{n} \), respectively. Then the outward normal vector \( \tilde{n} \), the curvature \( \tilde{k} \) and the outward normal velocity \( \tilde{V} \) of \( \tilde{\Omega} \) are represented by
\[
\tilde{n} = (\cos \theta(s), \sin \theta(s)), \quad \tilde{k} = \theta_s, \quad \tilde{V} = \nu \cdot \tilde{n} = c \cos \theta,
\]
respectively. Thus \( \gamma(\tilde{n}) \) is represented by a function with respect to \( \theta \). From now on, we use the notation \( \gamma(\theta) \) instead of \( \gamma(\tilde{n}) \). For simplicity of notation, we remove the tilde symbol from \( \tilde{u}, \tilde{U} \), and use \( x \) instead of \( z \). Then a triple \( (u, \Omega, c) \) satisfies
\[
\begin{cases}
d\Delta u + cu_x + k_1 U - u + k_2 = 0 & \text{in} \ \Omega, \\
u = 1 + A\theta & \text{on} \ \partial \Omega, \\
\cos \theta = \gamma(\theta) U - 1 - A\theta & \text{on} \ \partial \Omega,
\end{cases}
\]
where
\[
U = C_0 - \int_{\Omega} u \, dx \, dy.
\]

**Definition 1.1.** We call \((u, \Omega, c)\) a traveling wave solution of (1) if \((u, \Omega, c)\) satisfies (3) and belongs to \((C^2(\Omega) \cap C^0(\tilde{\Omega})) \times C^1 \times \mathbb{R})\).

Now we prepare some notation and impose some assumptions on \( \gamma \). We employ \( \gamma_{\min} \) and \( \gamma_{\max} \) to denote the minimum and the maximum of \( \gamma \) on \([0, 2\pi] \), respectively.

**Assumption 1.2.** Assume that \( \gamma = \gamma(\theta) \) satisfies the following properties (a)–(c):
\[
\begin{align*}
(a) & \quad \gamma \in C^2([0, 2\pi]), \\
(b) & \quad \gamma(\theta) = \gamma(2\pi - \theta) \quad \text{in} \ [0, \pi], \\
\end{align*}
\]
For example, if \( \gamma \) is monotone decreasing in \((0, \pi)\) with the properties \( \gamma(\pi) > 0 \) and \( \gamma(\theta) = \gamma(2\pi - \theta) \) for \( \pi < \theta < 2\pi \), then it is easily checked that \( \gamma \) enjoys the conditions \((a)-(c)\). We remark that the function \( \gamma(\theta) = \eta \cos \theta + \xi \) \( (\xi > \eta \geq 0) \), which was analyzed in \([10]\), is a typical example.

We introduce our main result:

**Theorem 1.3.** Assume that \( k_1, k_2, C_0 \) satisfy

\[
\begin{align*}
\gamma_{\min} C_0 - 1 + k_2 < 0, \\
\gamma_{\min} C_0 - 1 > 0.
\end{align*}
\]

Then there is a positive constant \( A_0 \) such that, for any \( A \in (0, A_0) \), there exist a traveling wave solution \((u, \Omega, c)\) of \((1)\) with the following properties:

\((P_1)\) The domain \( \Omega \) is convex,

\((P_2)\) For any \((x, y) \in \Omega\), it holds that

\[
k_2 \leq u < 1 + \max_{\partial \Omega} \theta_s, \quad U > 0,
\]

\((P_3)\) The speed \( c \) of traveling wave solution is positive.

This theorem implies the existence of a cell with a convex shape which moves with a constant speed. According to the migration of the cell, the density of F-action is high near the cell membrane and is low in the mid-region of the cell. The assumption \((4)\) implies that the depolymerization is stronger than polymerization inside the cell and it corresponds to the latter issue. The assumption \((5)\) can be interpreted as follows. The portion of the interface where \( \theta \in [-\pi/2, \pi/2] \) is called a front. When a cell migrates, the front must move forward. Since \( \kappa \geq 0 \) if the cell is convex, \( \gamma U - 1 > 0 \) is required on the front. The assumption \((5)\) corresponds to this condition. Thus the assumptions \((4)\) and \((5)\) are reasonable for cell migration, though they may be weaken.

**Theorem 1.4.** Assume that \( k_1, k_2, C_0, \gamma_{\min} \) and \( A \in (0, A_0) \) satisfy \((4)\), \((5)\) and

\[
4(\gamma_{\min} C_0 - 1) \left( \frac{\gamma_{\max}}{\gamma_{\min}} - 1 \right)^2 < \pi \gamma_{\min} A^2.
\]

Then there exist at least two traveling wave solutions of \((1)\) with properties \((P_1)-(P_3)\).

Note that \((6)\) is fulfilled if the function \( \gamma \) is close to a constant. This implies that the domain \( \Omega \) for traveling wave solution is close to a disk-shaped domain. Moreover, from the previous result \([9]\), it is expected that one stationary solution with a large convex domain is stable and the other with a small convex domain is unstable.

Choi et al. \([1, 2, 3]\) also considered mathematical models, proposed by Mogilner and Verzi \([8]\), describing crawling nematode sperm cell. In particular, Choi and Lui \([3]\) extended one-dimensional model proposed by \([8]\) to a two-dimensional model. Their model consists of a parabolic equation and an interface equation. The parabolic equation includes a nonlinear diffusion \( \Delta (u^m) \) \( (m \geq 1) \) with homogeneous Dirichlet boundary conditions. The evolution equation of the interface \( \partial \Omega(t) \) is composed of a mean curvature flow equation with an anisotropy. They succeeded in constructing a traveling wave with a convex domain for the model. On the other
hand, as seen in (1), our problem also consists of a parabolic equation and an interface equation. The parabolic equation of (1) includes a linear diffusion $\Delta u$ and a non-local term $U$ with Dirichlet boundary condition depending on the curvature of $\partial \Omega(t)$. Moreover the evolution equation for the interface $\partial \Omega(t)$ in our model involves the nonlocal term $U$ as well. As a result, there exist at least two traveling wave solutions of (1).

This paper is organized as follows: In Section 2, we consider the existence of solutions for (3) with $\lambda$ instead of $U$. First, we construct a convex domain $\Omega(\lambda)$ which satisfies the boundary condition

$$c \cos \theta = \gamma(\theta) \lambda - 1 - A \theta_s.$$  

(7)

Second, the existence of solution for the elliptic boundary value problem is considered:

$$\begin{align*}
\begin{cases}
d\Delta u + cu_x + k_1 \lambda - u + k_2 = 0 & \text{in } \Omega(\lambda), \\
u = 1 + A \theta_s & \text{on } \partial \Omega(\lambda)
\end{cases}
\end{align*}$$  

where $\Omega(\lambda)$ is obtained by (7). As a result, it is proved the existence of solutions $(u(x, y; \lambda), \Omega(\lambda), c(\lambda))$ for the system composed of (7) and (8). Finally, we find $\lambda$ which satisfies

$$\lambda = C_0 - \iint_{\Omega(\lambda)} u(x, y; \lambda) \, dx \, dy.$$  

To obtain such $\lambda$, we give some estimate, independent of the velocity $c$, for $L^\infty$-norm of $u$. In conclusion, we find at least two traveling waves for (1).

2. A priori estimate. In this section, we construct auxiliary solutions of (3). In other words, we consider a family of solutions $(u(\lambda), \Omega(\lambda), c(\lambda))$ of

$$\begin{align*}
\begin{cases}
d\Delta u + cu_x + k_1 \lambda - u + k_2 = 0 & \text{in } \Omega, \\
u = 1 + A \theta_s & \text{on } \partial \Omega, \\
c \cos \theta = \gamma(\theta) \lambda - 1 - A \theta_s & \text{on } \partial \Omega,
\end{cases}
\end{align*}$$  

(9)

for a given constant $\lambda > 0$.

First we focus on the second boundary condition, that is,

$$c \cos \theta = \gamma(\theta) \lambda - 1 - A \theta_s.$$  

(10)

Let $(x(s), y(s))$ be a point of $\partial \Omega$. Due to (2), we have

$$(x_s, y_s) = (- \sin \theta, \cos \theta)$$  

(11)

where $x_s$ and $y_s$ represent the derivatives of $x(s)$ and $y(s)$ with respect to $s$, respectively. Hence (10) and (11) lead the following initial value problem:

$$\begin{align*}
\begin{cases}
\theta_s = \frac{1}{A} (\gamma(\theta) \lambda - 1 - c \cos \theta) & \text{in } (0, L), \\
y_s = \cos \theta & \text{in } (0, L), \\
y(0) = 0, \ \theta(0) = 0,
\end{cases}
\end{align*}$$  

(12)

where $L$ is a positive constant specified later. As mentioned above, we are looking for a traveling wave solution with a symmetric domain $\Omega$. Hence we demand that the solution $y$ and $\theta$ of (12) satisfy boundary conditions

$$y(L) = 0, \ \theta(L) = \pi.$$  

(13)

which means that $L$ is a half circumference of $\Omega$. Now we show the existence of $(L, c, \theta, y)$ satisfying (12) and (13).
Lemma 2.1. Let \( \lambda \) satisfy \( \lambda > 1/\gamma_{\min} \). Then there exists a unique solution \((L, c, \theta, y)\) satisfying (12) and (13). Moreover it holds that \( \theta_s > 0 \) in \([0, L]\).

Proof. First we confirm the condition of \( c \) such that the solution \( \theta(s) \) in (12) satisfies the boundary condition \( \theta(L) = \pi \) in (13). Let \( c \) be a positive constant. It is easily seen that \( \gamma(\theta) \lambda - 1 - c \cos \theta \) is Lipschitz continuous and uniformly bounded in \([0, \pi]\). By the standard existence and uniqueness theorems of ODE, there exists a unique solution \((\theta, y)\) of (12) for any \( L > 0 \), because \( \gamma(\theta) \lambda - 1 - c \cos \theta \) is bounded. Since \( \lambda > 1/\gamma_{\min} \), we can take a constant \( c^* > 0 \) such that

\[
\min_{0 < \theta \leq \pi/2} \{ \gamma(\theta) \lambda - 1 - c^* \cos \theta \} = 0, \\
\min_{0 < \theta \leq \pi} \{ \gamma(\theta) \lambda - 1 - \cos \theta \} > 0 \quad \text{for any} \quad c \in (0, c^*).
\]

If \( c \in (0, c^*) \), \( \theta_s \) is strictly positive for \( \theta \in [0, \pi] \). Then there exists a constant \( L = L(c) \) such that \( \theta(L) = \pi \).

Next we will show that there exists a unique constant \( c \in (0, c^*) \) satisfying \( y(L(c)) = 0 \). Let \( Y(c) \) be defined by

\[
Y(c) := y(\pi) - y(0) = \int_0^\pi \frac{A \cos \theta}{\gamma(\theta) \lambda - 1 - c \cos \theta} \, d\theta.
\]

By the assumption (c), we obtain

\[
Y(0) = \int_0^{\pi/2} \frac{A(\gamma(\pi - \theta) - \gamma(\theta)) \lambda \cos \theta}{(\gamma(\theta) \lambda - 1)(\gamma(\pi - \theta) \lambda - 1)} \, d\theta < 0.
\]

On the other hand, set

\[
\theta_{\min}^* := \min \{ \theta \in [0, \pi/2] \mid \gamma(\theta) \lambda - 1 - c^* \cos \theta = 0 \}.
\]

Then there exists a constant \( K \) with the property

\[
|\gamma(\theta) \lambda - 1 - c \cos \theta| \leq K|\theta_{\min}^* - \theta|^p
\]

in a neighborhood of \( \theta_{\min}^* \) with \( p \geq 2 \). This result shows that \( Y(c) > 0 \) near \( c = c^* \). Note that \( Y(c) \) is a monotone increasing function with respect to \( c \) in \((0, c^*)\). Therefore there exists a unique constant \( c \) satisfying \( Y(c) = 0 \). \( \square \)

By Lemma 2.1 for each \( \lambda > 1/\gamma_{\min} \), we obtain a domain \( \Omega(\lambda) \) composed of a boundary

\[
\partial \Omega(\lambda) = \{(x(s), y(s)) \mid s \in [0, L]\} \cup \{(x(s), -y(s)) \mid s \in (0, L]\},
\]

where \((x(s), y(s))\) is derived from \((L, c, \theta, y)\) satisfying (12) and (13). Here we remark that \( \partial \Omega(\lambda) \in C^2 \).

Next we consider an elliptic problem derived from (9) as follows:

\[
\begin{cases}
d \Delta u + cu_x + k_1 \lambda - u + k_2 = 0 & \text{in} \ (\Omega(\lambda),) \\
u = 1 + Ak_1 & \text{on} \ \partial \Omega(\lambda),
\end{cases}
\]

where \( k(x(s), y(s)) = \theta_s(s) \) is a curvature of \( \partial \Omega(\lambda) \) at \((x(s), y(s))\).

Lemma 2.2. Let \( k_1, k_2, C_0 \) and \( \gamma_{\min} \) satisfy conditions (1) and (5). Suppose that \( C_0 \geq \lambda > 1/\gamma_{\min} \) additionally. Then (14) has a unique solution \( u \in C^2(\Omega(\lambda)) \cap C(\overline{\Omega(\lambda)}) \) such that

\[
k_2 < u(x, y) < 1 + A \max_{s \in [0, L]} \theta_s
\]

for any \((x, y) \in \Omega(\lambda)\).
Proof. By the standard argument in [4], there exists a unique classical solution \( u \) of (14). Hence we only need to examine the boundedness of \( u \). The proof is based on standard arguments of the maximum principle. Suppose that there exists a point \((x_*, y_*) \in \Omega(\lambda)\) such that
\[
\min_{(x,y) \in \Omega(\lambda)} u(x, y) = u(x_*, y_*) \leq k_2.
\]
Then, at the point \((x_*, y_*)\),
\[
0 = d\Delta u(x_*, y_*) + cu_x(x_*, y_*) + k_1 \lambda - u(x_*, y_*) + k_2 > 0,
\]
which implies a contradiction. It follows that \( u > k_2 \) in \( \Omega(\lambda) \).

On the other hand, we suppose that there exists a point \((x_*, y_*)\) such that
\[
\max_{(x,y) \in \Omega(\lambda)} u(x, y) = u(x_*, y_*) \geq 1 + A \max_{s \in [0,L]} \theta_s.
\]
Then it follows from (4) and Lemma 2.1 that
\[
0 = d\Delta u(x_*, y_*) + cu_x(x_*, y_*) + k_1 \lambda - u(x_*, y_*) + k_2 \leq k_1 \lambda - (1 + A \max_{s \in [0,L]} \theta_s) + k_2 \leq k_1 C_0 - 1 + k_2 < 0.
\]
Therefore we have the desired estimate.

For \( C_0 \geq \lambda > 1/\gamma_{\min} \), Lemmas 2.1 and 2.2 ensure the existence of a unique solution for (9). From now on, \((u(x, y; \lambda), \Omega(\lambda), c(\lambda))\) denote the solution of (9).

To complete the proof of Theorem 1.3, we confirm the existence of \( \lambda \) satisfying
\[
\lambda = C_0 - \int_{\Omega(\lambda)} u(x, y; \lambda) \, dx \, dy.
\]
We introduce the function \( H \) by
\[
H(\lambda; A) := C_0 - \int_{\Omega(\lambda)} u(x, y; \lambda) \, dx - \lambda.
\]
When \( \lambda = C_0 \), it holds that \( H(C_0; A) < 0 \) from the positivity of \( u(x, y; \lambda) \). It suffices to show the existence of \( \lambda_0 \in (1/\gamma_{\min}, C_0) \) satisfying \( H(\lambda_0; A) > 0 \) for a small \( A > 0 \). To obtain this inequality, we prepare the following two lemmas which give us a priori estimate of \( \Omega(\lambda) \):

**Lemma 2.3.** Let \((u(x, y; \lambda), \Omega(\lambda), c(\lambda))\) be a solution of (9). Then it holds that
\[
A^2 \pi \frac{(\gamma_{\max} + \gamma(0)) \lambda - 2}{\gamma_{\min} \lambda - 1} < |\Omega(\lambda)| < \frac{A^2 \pi}{(\gamma_{\min} \lambda - 1)^2}.
\]

**Proof.** The integrating the third equation of (9) with respect to \( s \) yields that
\[
0 = \int_{\partial \Omega(\lambda)} \{ \gamma(\theta(s)) \lambda - 1 \} \, ds - 2\pi A,
\]
which implies
\[
2\pi A \geq (\gamma_{\min} \lambda - 1) |\partial \Omega(\lambda)|. \tag{15}
\]
In addition, it follows from the isoperimetric inequality that
\[
4\pi |\Omega(\lambda)| \leq |\partial \Omega(\lambda)|^2. \tag{16}
\]
Combining (15) and (16) yields
\[ |\Omega(\lambda)| \leq \frac{|\partial \Omega(\lambda)|^2}{4\pi} \leq \frac{A^2 \pi}{(\gamma_{\min} \lambda - 1)^2}. \]

The convexity of \( \Omega(\lambda) \) implies that the constant \( c \) satisfies
\[ \gamma(0) \lambda - 1 - c > 0. \]

Since \( |\Omega(\lambda)| \) is larger than the area of a disk with a radius \( \kappa_{\text{max}} := \max_{0 \leq s \leq L} \theta_s(s) \), we obtain
\[ |\Omega(\lambda)| \geq \pi \left( \max_{0 \leq s \leq L} \theta_s \right)^{-2} \geq \frac{A^2 \pi}{(\gamma_{\max} \lambda - 1 + c)^2} \geq \frac{A^2 \pi}{((\gamma_{\max} + \gamma(0)) \lambda - 2)^2}. \]

Therefore we obtain the desire estimate. \( \Box \)

**Lemma 2.4.** Let \((u(x, y; \lambda), \Omega(\lambda), c(\lambda))\) be a solution of (9). Then it holds that
\[ \iint_{\Omega(\lambda)} u(x, y; \lambda) \, dx \, dy < A^2 \frac{2 \gamma_{\max} \lambda - 1}{(\gamma_{\min} \lambda - 1)^2}. \]

**Proof.** From Lemma 2.2, it follows that
\[ \iint_{\Omega(\lambda)} u(x, y; \lambda) \, dx \, dy \leq |\Omega(\lambda)| \max_{\Omega(\lambda)} u < \left( 1 + A \max_{0 \leq s \leq L} \theta_s \right) \frac{A^2 \pi}{(\gamma_{\min} \lambda - 1)^2}. \]

On the other hand, we obtain that
\[ \max_{0 \leq s \leq 2\pi} \theta_s < \frac{2}{A} (\gamma_{\max} \lambda - 1). \]

Here we used the property of convexity of \( \Omega(\lambda) \) and Assumption 1.2(c). Therefore we have the desired inequality. \( \Box \)

**Lemma 2.5.** Let \((u(x, y; \lambda), \Omega(\lambda), c(\lambda))\) be a solution of (9). Then it holds that
\[ \iint_{\Omega(\lambda)} u(x, y; \lambda) \, dx \, dy > \frac{k_2 \pi A^2}{((\gamma_{\max} + \gamma(0)) \lambda - 2)^2}. \]

This lemma immediately follows from Lemmas 2.2 and 2.3.

3. **Proof of theorems.** \( \Box \)[Proof of Theorem 1.3] As mentioned above, our goal is to confirm the existence of roots \( \lambda \) of
\[ H(\lambda; A) = 0 \quad (17) \]
for a small constant \( A > 0 \). By Lemmas 2.4 and 2.5, we have
\[ U_1(\lambda; A) < C_0 - \iint_{\Omega(\lambda)} u(x, y; \lambda) \, dx \, dy < U_2(\lambda; A) \]
for \( \lambda > 1/\gamma_{\min} \), where
\[
U_1(\lambda; A) = C_0 - A^2 \frac{2 \gamma_{\max} \lambda - 1}{(\gamma_{\min} \lambda - 1)^2}, \\
U_2(\lambda; A) = C_0 - \frac{k_2 \pi A^2}{((\gamma_{\max} + \gamma(0)) \lambda - 2)^2}.
\]
Note that $C_0 > 1/\gamma_{\min}$ by [5]. If $\lambda \in (1/\gamma_{\min}, C_0)$, then $U_1(\lambda; A)$ converges to $C_0$ as $A$ tends to zero. Thus there is a positive constant $A_0$ such that there exists a unique constant $\lambda_0 \in (1/\gamma_{\min}, C_0)$ satisfying $H(\lambda_0; A_0) = 0$. By the monotonicity of $U_1(\lambda_0; A)$ with respect to $A$, we have, for any $A \in (0, A_0)$,

$$H(\lambda_0; A) > U_1(\lambda_0; A) - \lambda_0 > 0.$$  

Furthermore we obtain

$$H(C_0; A) = -\int_{\Omega(C_0)} u(x, y; C_0) \, dx < 0 \quad (19)$$

for any $A > 0$. Since these two estimates (18) and (19) are maintained for any $A \in (0, A_0)$, there is at least a constant $\lambda_1 \in (\lambda_0, C_0)$ satisfying (17). Thus we complete the proof of Theorem 1.3.

**Proof of Theorem 1.4.** In Theorem 1.3 we showed the existence of a root $\lambda_1 \in (\lambda_0, C_0)$ of (17). Here we confirm that there exists a constant $\lambda \in (1/\gamma_{\min}, \lambda_0)$ satisfying (17). Assume that $A \in (0, A_0)$ and [6]. Substituting $\lambda = 1/\gamma_{\min}$ in $U_2(\lambda; A)$, we have

$$H(1/\gamma_{\min}; A) < U_2(1/\gamma_{\min}; A) - \frac{1}{\gamma_{\min}}$$

$$< C_0 - \frac{k_2 \pi A^2}{4(\gamma_{\max}/\gamma_{\min} - 1)^2} - \frac{1}{\gamma_{\min}}$$

$$< \frac{1}{\gamma_{\min}} \left( \gamma_{\min} C_0 - \frac{k_2 \pi \gamma_{\min} A^2}{4(\gamma_{\max}/\gamma_{\min} - 1)^2} - 1 \right)$$

$$< 0.$$  

This implies that there is at least a constant $\lambda_2 \in (1/\gamma_{\min}, \lambda_0)$ satisfying (17). By Lemmas 2.1 and 2.2 we define $(u_j, \Omega_j, c_j) = (u(x, y; \lambda_j), \Omega(\lambda_j), c(\lambda_j))$ as the solution of (9) corresponding $\lambda_j$. Since $\lambda_2 < \lambda_1$, the solution $(u_2, \Omega_2, c_2)$ is different from $(u_1, \Omega_1, c_1)$.

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E-mail address: te12001@meiji.ac.jp
E-mail address: hirokazu.ninomiya@gmail.com