Performance of data networks with random links

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Abstract

We investigate simplified models of computer data networks and examine how the introduction of additional random links influences the performance of these networks. In general, the impact of additional random links on the performance of the network strongly depends on the routing algorithm used in the network. Significant performance gains can be achieved if the routing is based on “geometrical distance” or shortest path reduced table routing. With shortest path full table routing degradation of performance is observed.

Keywords: computer data networks, routing algorithms, network topology, network performance, congestion

1. Introduction

Models of computer data networks have attracted much attention in recent years. Generally, these models assume either regular topology of the network, in the form of a square lattice [1, 2, 7] or a binary Cayley tree [11], or random graph topology [5]. On the other hand, it has been recently demonstrated that many technological, biological, and social networks are neither completely regular nor completely random, being somewhere between these two

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extremes \cite{3}. Regular lattices rewired to introduce small amount of random connections, termed “small world networks”, offer many advantages over purely regular or purely random topologies. In particular, models of dynamical systems based on “small world” lattices can often exhibit enhanced signal propagation capabilities, as observed in epidemic models or multi-player prisoners dilemma games played on such lattices \cite{3}.

The purpose of this work is to investigate if a similar effect can be achieved in simple data network models, although our approach is different than the approach taken in \cite{3}. We do not rewire the network, but rather examine how introduction of additional random links influences its performance. This is motivated by the question whether introduction of additional links can help to decongest an existing network.

2. Network Models Definitions

The purpose of the network is to transmit messages from points of origin to destination points. In our model, we will assume that the entire message is contained in a single “capsule” of information, which, by analogy to packet-switching networks, will be simply called a packet. In a real packet-switching network, a single packet carries the information “payload”, and some additional information related to the internal structure of the network. We will ignore the information “payload” entirely, and assume that the packet carries only two pieces of information: time of its creation and the destination address.

Our simulated network consists of a number of interconnected nodes. Each node can perform two functions: of a hosts, meaning that it can generate and receive messages, and of a router (message processor), meaning that it can store and forward messages. Packets are created and moved according to a discrete time parallel algorithm. The structure of the considered networks and the update algorithm will be described in subsections which follow.

2.1. Connection Topologies

We will consider two types of network connection topologies: a two-dimensional square lattice \( \mathcal{L} = \mathcal{L}(L) \) and a two-dimensional square lattice \( \mathcal{L} \) with additional links added randomly, denoted by \( \mathcal{L}_l = \mathcal{L}_l(L) \). The value of the subscript \( l \) gives a number of an extra links in a network and the value of \( L \) gives a number of nodes in the horizontal and vertical direction of the lattice \( \mathcal{L} \). The lattices \( \mathcal{L} \) and \( \mathcal{L}_l \) with periodic boundary conditions will be denoted by \( \mathcal{L}_p \) and \( \mathcal{L}_l^p \), respectively, and with non-periodic boundary conditions by \( \mathcal{L}_p^{np} \) and \( \mathcal{L}_l^{np} \), respectively. Hence, with this notation, \( \mathcal{L}_0^p = \mathcal{L}_p \) and \( \mathcal{L}_0^{np} = \mathcal{L}^{np} \). Most of our simulations will be performed on lattices with periodic boundary conditions.

The network hosts and routers are located at nodes of the lattice \( \mathcal{L} \). The position of each node on a lattice \( \mathcal{L} \) is described by a discrete space variable \( \mathbf{r} \), such that

\[
\mathbf{r} = ic_x + jc_y,
\]

where \( c_x, c_y \) are Cartesian unit vectors, and \( i, j = 1, \ldots, L \). For each node \( \mathbf{r} \) we denoted by \( C(\mathbf{r}) \) the set of all nodes directly connected with the node \( \mathbf{r} \). Hence, for each \( \mathbf{r} \in \mathcal{L}_p \), the set \( C(\mathbf{r}) \) is of the form

\[
C(\mathbf{r}) = \{ \mathbf{r} - c_x, \mathbf{r} + c_x, \mathbf{r} - c_y, \mathbf{r} + c_y \}.
\]
In this case, the node \( r \) is connected with its four nearest neighbours. However, for lattices with non-periodic boundary conditions \( \mathcal{L}^{np} \) or square lattices with additional links added randomly \( \mathcal{L}^p_i \) and \( \mathcal{L}^{np}_i \) the form of the set \( C(r) \) can be different, for some nodes \( r \), from the one in (2). For example, for nodes \( r \) on the boundary of a lattice \( \mathcal{L}^{np} \) the set \( C(r) \) can contain two or three elements only, depending on where a node \( r \) is located on the boundary. In the case of a lattice \( \mathcal{L}^p_i \) or \( \mathcal{L}^{np}_i \) the set \( C(r) \) can contain many non-nearest neighbours nodes depending on a number of additional links which originate from the node \( r \).

The extra links are constructed using the following procedure. We first select randomly a node \( r_1 \) on a square lattice \( \mathcal{L}^p \) or \( \mathcal{L}^{np} \). Next, we select randomly another node \( r_2 \), different from the node \( r_1 \), and connect these two nodes with direct communication link. By repeating this procedure independently \( l \) times we obtain a lattice \( \mathcal{L}^p_l \) or \( \mathcal{L}^{np}_l \), respectively, with additional \( l \) random links. It can happen that the nodes \( r_1 \) and \( r_2 \) can be selected again to form a new link. Hence, in the network there can be several links connecting directly the same nodes. We want to emphasize that all the connections in our models are static, during the simulation period they do not change. Additional random links are added before the simulation starts, and remain unchanged.

In the networks considered here, each node maintains a queue of unlimited length where the arriving packets are stored. The number of packets in the queue at a node \( r \) at time step \( k \) will be denoted by \( n(r, k) \), while the total number of packets in the system at time step \( k \) will be denoted by \( N(k) \),

\[
N(k) = \sum_{r \in \mathcal{L}} n(r, k).
\] (3)

Packets stored in queues, at individual lattice nodes, must be delivered to their destination addresses. To assess how far a given packet is from its destination, we introduce the concept of distance between nodes. Depending on a network connection topology we will use three metric functions to compute the distance between nodes \( r_1 = (i_1, j_1) \) and \( r_2 = (i_2, j_2) \). Namely, we will use

1. for lattices with non-periodic boundary conditions “Manhattan” metric

\[
d_M(r_1, r_2) = |i_2 - i_1| + |j_2 - j_1|,
\] (4)

2. for lattices with periodic boundary conditions periodic “Manhattan” metric

\[
d_{PM}(r_1, r_2) = L - \left| \frac{|i_2 - i_1|}{L} - \frac{L}{2} \right| - \left| \frac{|j_2 - j_1|}{L} - \frac{L}{2} \right|,
\] (5)

3. and regardless of boundary conditions the “shortest path” metric \( d_{SP}(r_1, r_2) \) defined as the number of links in the shortest path joining \( r_1 \) and \( r_2 \). By “shortest” we mean the path with the smallest number of links.

Metric \( d_{PM} \) will be used on lattices \( \mathcal{L}^p \), while \( d_M \) will be used on lattices \( \mathcal{L}^{np} \). Note that on a square lattice with no extra links \( d_{SP}(r_1, r_2) = d_M(r_1, r_2) \), and \( d_{SP}(r_1, r_2) = d_{PM}(r_1, r_2) \) for lattices with non-periodic and periodic boundary conditions, respectively. Furthermore, for each \( r \in \mathcal{L}^p_i \), where \( \kappa = p \) or \( np \) and \( l \in \{0, 1, ...\} \)
\[ C(r) = \{ x \in L^c_r : d_{SP}(x, r) = 1 \} . \] (6)

When irregularities such as extra links are present, \( d_{SP}(r_1, r_2) \) can be computed using one of the well known algorithms. In our simulations, we used shortest path backward tree algorithm [8].

### 2.2. Routing Algorithms

The dynamics of the networks are governed by the parallel update algorithms shown Figure 1, similar to the algorithm used in [7]. We start with an empty queue at each node, and with discrete time clock \( k \) set to zero. Then, the following actions are performed in sequel:

1. At each node, independently of the others, a packet is created with probability \( \lambda \). Its destination address is randomly selected among all other nodes in the network with uniform probability distribution. The newly created packet is placed at the end of the queue.

2. At each node, one packet (or none, if the local queue is empty) is picked up from the top of the queue and forwarded to one of its neighboring sites according to one of the routing algorithms to be described below. Upon arrival, the packet is placed at the end of the appropriate queue. If several packets arrive to a given node at the same time, then they are placed at the end of the queue in a random order. When a packet arrives to its destination node, it is immediately destroyed.

3. \( k \) is incremented by 1.

This sequence of events, which constitutes a single time step update, is then repeated arbitrary number of times. The state of the network is observed after sub-step 3 (clock increase), but before sub-step 1 (creation of new packets). In order to explain the routing algorithms mentioned in sub-step 2, we will first describe one of its simplified versions.

Let us assume that we measure distance using some metric \( d \), where \( d \) could be any of the previously defined metrics \( d_M, d_{PM}, \) or \( d_{SP} \). To decide where to forward a packet located at a node \( r \) with the destination address \( r_d \), two steps are performed:

1. From sites directly connected to \( r \), we select sites which are closest to the destination \( r_d \) of the packet. More formally, we construct a set \( A_\infty(r) \) such that

\[
A_\infty(r) = \{ a \in C(r) : d(a, r_d) = \min_{x \in C(r)} d(x, r_d) \} \] (7)

2. From \( A_\infty(r) \), we select a site which has the smallest queue size. If there are several such sites, then we select one of them randomly with uniform probability distribution. The packet is forwarded to this site. Using a formal notation again, we could say that the packet is forwarded to a site selected randomly and uniformly from elements of a set \( B_\infty(r) \) defined as

\[
B_\infty(r) = \{ a \in A_\infty(r) : n(a, k) = \min_{x \in A_\infty(r)} n(x, k) \}. \] (8)
To summarize, the routing algorithm $R_\infty$ described above sends the packet to a site which is closest to the destination (in the sense of the metric $d$), and if there are several such sites, then it selects from them the one with the smallest queue. If there is still more than one such node, random selection takes place. It is clear that each packet routed according to the algorithm $R_\infty$ will travel to its destination taking the shortest possible path (shortest in the sense of the metric $d$, not necessarily in terms of the number of time steps required to reach the destination). In real networks, this does not always happen. In order to allow packets to take alternative routes, not necessarily shortest path routes, we will introduce a small modification to the routing algorithm $R_\infty$ described above.

The modified algorithm $R_m$, for each node $r$, will use instead of the set $A_\infty(r)$ a set $A_m(r)$ defined as follows. In the construction of the set $A_m(r)$ instead of minimizing distance to the destination $d(x,r_d)$, as it was done in (7), we will minimize $\Theta_m(d(x,r_d))$, where

$$\Theta_m(y) = \begin{cases} y, & \text{if } y < m, \\ m, & \text{otherwise}, \end{cases}$$ (9)

for a given integer $m$. Thus, the definition of the set $A_m(r)$ is

$$A_m(r) = \{ a \in C(r) : \Theta_m(d(a,r_d)) = \min_{x \in C(r)} \Theta_m(d(x,r_d)) \}$$ (10)

The above modification is equivalent to saying that nodes which are further than $m$ distance units from the destination are treated by the routing algorithm as if they were exactly $m$ units away from the destination. If a packet is at a node $r$ such that all nodes directly linked with $r$ are further than $m$ units from its destination, then the packet will be forwarded to a site selected randomly and uniformly from the subset of $C(r)$ containing the nodes with the smallest queue size in the set $C(r)$. It can happen that the selected site can be further away from the destination than the node $r$.

Therefore, introduction of the cutoff parameter $m$ adds more randomness to the network dynamics. One could also say that the destination attracts packets, but this attractive interaction has a finite range $m$: packets further away than $m$ units from the destination are not being attracted.

It is also possible to relate various values of the cutoff parameter $m$ to different types of routing schemes used in real packet-switching networks. Assume that each node $r$ maintains a table containing all possible values of $d(x,r_d)$, for all possible destinations $r_d$ and all nodes $x \in C(r)$, and that packets are routed according to this table by selecting nodes minimizing distance, measured in the metric $d$, travelled by a packet from its origin to its destination. Such a routing scheme is called table-driven routing [8] and it is equivalent to the routing algorithm $R_\infty$. In this case, construction of the set $A_\infty(r)$ would require looking up appropriate entries in the stored table.

Let us now define $D_{max}$ to be the largest possible distance between two nodes in the network. When $m < D_{max}$, then for a given $x$, we need to store values of $d(x,r_d)$ only for nodes $r_d$ which are less than $m$ units of distance away – for all other nodes distance does not matter, since it will be treated as $m$ by the routing algorithm. Hence, at each node $r$ the routing table to be stored is smaller than in the case when $m = D_{max}$. The routing scheme based on this smaller routing table is called the reduced table routing algorithm [8] and it is
equivalent to the routing algorithm $R_m$. In the case when $m = D_{\text{max}}$ the routing algorithm $R_m = R_\infty$.

Finally, when $m = 1$, the distances between hosts and destinations are not considered in the routing process of packets. Therefore, there is no need to store any table of possible paths at nodes of the network. This case corresponds to the table-free routing algorithm [8] in which packets are routed randomly. Hence, this algorithm can send packets on circuitous and long routes to their destinations. The analysis of this routing algorithm has been done in [4], where some analytical results are also presented. At present such results are not available for routing algorithms with $m > 1$.

3. Performance of networks with square lattice connection topology $(\mathcal{L}_0^p, d_{PM}), (\mathcal{L}_0^{np}, d_M)$

In order to assess the performance of a network, graphs of delay as a function of presented load are frequently used in network performance literature [9]. In our case, delay $\tau$ will be defined as the number of time steps elapsed from the creation of a packet to its delivery to the destination address. We will also use average delay $\overline{\tau}(k)$, where the average is taken over all packets delivered to their destination from the beginning of the simulation ($k = 0$) up to time $k$. Probability of a packet creation $\lambda$ will be used as a measure of a presented load.

3.1. Full table routing

We assume that the network topology is a square lattice $\mathcal{L}_0^p(L)$ with $d_{PM}$ metric and the network routing algorithm is the full table routing algorithm $R_m$, with $m = D_{\text{max}}$, i.e. $R_m = R_\infty$. Figure 2a shows graphs of the average delay $\overline{\tau}(k)$ versus presented load $\lambda$, as measured during simulation performed on a lattice $\mathcal{L}_0^p(50)$. The three curves shown there correspond to different times. It is clear that beyond a certain critical value of $\lambda = \lambda_c$, the average delay drastically increases. Moreover, the average delay grows with time, which suggests that for $\lambda > \lambda_c$ there is no equilibrium state. In fact, when $\lambda > \lambda_c$, a typical queue size and consequently, the number of packets in the system $N(k)$, grows without bounds, as shown in Figure 2b and Figure 3.

It is possible to find an approximate value of the critical load $\lambda_c$ by the following argument. For $\lambda < \lambda_c$, the system reaches steady state, and in the steady state the number of packets created per unit time (given by $L^2\lambda$) must be equal to the number of packets delivered per unit time. Since the average time spent in the system by a packet is $\overline{\tau}(k)$, we can reasonably assume that $N(k)/\overline{\tau}(k)$ packets are delivered to their destinations per unit time, hence

$$\frac{N(k)}{\overline{\tau}(k)} = L^2\lambda. \quad (11)$$

This relationship, known as Little’s law in queuing theory [3], holds only below the critical point, as shown in Figure 4.

For the routing algorithm $R_m$, with $m = D_{\text{max}}$, when the number of packets in the network is small, an individual packet is always routed in such a way that it follows the shortest path to its destination avoiding all occupied nodes. This means that for small
$N(k)$, the average packet delay is approximately equal to average distance from the packet’s origin to its destination, which will be called "free packet" delay $\tau_0$

$$\tau_0 = \frac{1}{L^4} \sum_{r_1, r_2} d_{PM}(r_1, r_2)$$  \hspace{1cm} (12)

After some algebra, this leads to

$$\tau_0 = \frac{1}{L^4} \sum_{i_1, i_2, j_1, j_2=0}^{L-1} \left\{ L - \left| i_2 - i_1 \right| - \frac{L}{2} - \left| j_2 - j_1 \right| - \frac{L}{2} \right\} = \frac{L}{2}$$  \hspace{1cm} (13)

Obviously, when the load increases, at some point the number of packets in the network will be so large that it would not be possible to find a route to a destination completely avoiding other packets. Assuming that packets are approximately uniformly distributed over the entire lattice, this will happen when all sites are occupied, i.e. when $N(k) = L^2$. Using (11) this gives an estimate of $\lambda_c$:

$$\lambda_c = \frac{1}{\tau_0}$$  \hspace{1cm} (14)

For $L_0^p(50)$ we obtain $\tau_0 = 25$ and $\lambda_c = 0.04$, in good agreement with the value obtained from simulations $\lambda_c = 0.039 \pm 0.001$.

Quite similar calculations can be performed for a square lattice with non-periodic boundary $L_0^{np}(L)$. In this case,

$$\tau_0 = \frac{1}{L^4} \sum_{r_1, r_2} d_M(r_1, r_2) = \frac{2L^2}{3} - \frac{1}{L} \approx \frac{2}{3}L,$$  \hspace{1cm} (15)

yielding $\lambda_c = 0.03$ for $L_0^{np}(50)$. The measured value of $\lambda_c$ for $L_0^{np}(50)$ is $0.020 \pm 0.001$, i.e., much lower. The discrepancy is mainly due to the fact that for the lattices $L_0^{np}(L)$ packets are not uniformly distributed on the lattice, having a tendency to cluster at the center. Consequently, jamming occurs earlier than one would expect assuming uniform distribution of packets.

### 3.2. Partial table routing

Decrease in value of the cutoff parameter $m$ has a profound effect on the critical load. Smaller $m$ means that packets which are located further than $m$ links from their destination move with a high degree of randomness, and as a result, their average delay is larger. This increase of the delay can be also seen in a plot of a single packet delay as a function of $m$ (Figure 3). While values of $m$ close to $D_{max}$ do not significantly change $\tau_0$, values of $m$ close to 1 result in an increase of $\tau_0$ by up to two orders of magnitude.

### 4. Performance of networks based on square lattices ($L_0^p, d_{PM}$), ($L_q^p, d_M$) with additional $l$ random links

7
4.1. Full table routing

Let us now consider the network dynamics governed by the routing algorithm $R_\infty$ taking place on lattices $L^p_l$ and $L^{np}_l$ which in addition to normal nearest neighbor connections, feature $l$ additional links, where $l > 0$. Figure 6 shows how addition of random links changes the graph of delay vs. load for both non-periodic and periodic case. We are still using $d_M$ and $d_{PM}$ metric for $L^{np}_l$ and $L^p_l$ lattice, respectively, which means that the distance between two points $r_1$ and $r_2$ is still computed using $d_M(r_1, r_2)$ or $d_{PM}(r_1, r_2)$ metric, respectively, even if $r_1$ and $r_2$ are directly connected by some extra link.

As expected, addition of extra links improves performance of the network, shifting the critical point $\lambda_c$ to the right (see Figures 6 and 7). This means that the network can carry more load without experiencing congestion. Performance improvement is more pronounced for lattices with non-periodic boundaries, as shown in Figure 7. For example, by adding 100 random links to $50 \times 50$ lattice, which increases total number of links by 2%, we increase the critical load by over 25%. Increasing the number of links by 8% doubles the critical load. This can be attributed to the fact that some packets can bypass congested central area by using “shortcuts”, and their delay decreases not only because they have shorter distance to travel, but also because they have avoided congestion. In the case of lattices with periodic boundaries, packets are more uniformly spread even in the presence of additional random links. Thus, the performance improvement is only caused by the decrease in the distance traveled, but not by bypassing congestion, since congestions are also uniformly spread in the case of lattices with periodic boundaries.

In the remainder of this article, we will focus our discussion on lattices with periodic boundaries only.

4.2. Partial table routing

When a routing algorithm $R_m$, with $m < D_{max}$, is used, additional random links can significantly increase critical load, and the relative performance gain is much larger than in the case of the full table routing algorithm. Figure 8a,b shows the relative change of the critical load, defined by

$$\frac{\Delta \lambda_c}{\lambda_c} = \frac{\lambda_c(m, l) - \lambda_c(m, 0)}{\lambda_c(m, 0)},$$

where $\lambda_c(m, l)$ denotes the critical load at a given $m$ and $l$, for two different values of $m$, $m = 50$ and $m = 20$. One can immediately notice that the impact of additional links on performance of the network is much stronger in the case of partial table routing ($m = 20$) than in the case of full table routing ($m = 50$). For example, about 50 extra links are sufficient to double the critical load corresponding to $m = 20$, while the same number of links has almost negligible impact on the critical load when $m = 50$. 

8
5. Performance of networks based on square lattice \((L^p_l, d_{SP})\) with additional \(l\) random links and \(d_{SP}\) metric

As stated before, for a square lattice without additional links, metric \(d_{SP}\) is identical to \(d_M\) or \(d_{PM}\) metric. This is no longer true for a square lattice with additional random links. Routing based on \(d_{SP}\) metric fully utilizes shortcuts provided by additional links, significantly decreasing “free packet” delay \(\tau_0\). One would expect that a decrease in “free packet” delay will decrease also average delay, as it was in the case of networks with \(d_M\) and \(d_{PM}\) metric. In reality, we observe just opposite effect (Figure 8).

5.1. Full table routing

Figure 8c shows how the critical load \(\lambda_c(m, l)\) changes when additional random links are introduced. This is shown for the network dynamics governed by \(R_\infty\), i.e. the full table routing algorithm with \(m = D_{max}\), on a square lattice with periodic boundaries. One can clearly see that if the number of additional random links \(l\) is below some critical value \(l_c(m)\) the critical load \(\lambda_c(D_{max}, l)\) is actually smaller than \(\lambda_c(D_{max}, 0)\), in spite of increased connectivity between nodes of the network. The performance of the network is at its worst when just a few additional random links are added. However, it improves with the increase of a number \(l\) of additional random links and at some critical value \(l_c(m)\) it becomes the same as of the network without any additional random link. When the number \(l\) of additional random links is greater than \(l_c(m)\) an improvement in the network performance is observed. For the network \(L^p_{500}(50)\) the critical load \(\lambda_c(50, 500)\) is almost equal to the critical load \(\lambda_c(50, 0)\) of the network \(L^p_{0}(50)\), i.e. \(\lambda_c(50, 500) \approx \lambda_c(50, 0)\), and the improvement of the \(L^p_{l}(50)\) network performance is observed for \(l\) greater than 500.

This rather unexpected phenomenon can be understood as follows. When additional links are introduced, and their number is less than \(l_c(m)\), they provide a shortcut between distant parts of the network. Since packets are forwarded to their destinations via the shortest path, it often happens that one link serves as a shortcut for many packets from the neighborhood. One could say that additional links “attract” most of the traffic and quickly become congested, even though sites which are not close to extra links are almost empty. This is well illustrated in Figure 3, which shows snapshots of dynamics of the network with \(R_\infty\) routing algorithm, 50 × 50 nodes and periodic boundary conditions. The presented load is \(\lambda(50, 0) = 0.025\), just below the critical value \(\lambda_c(50, 0) = 0.028\). If there are no additional random links the network dynamics remains in the steady state, as is showed by the left column of Figure 3. The number of packets in the network fluctuates slightly over time, but remains at the same level: at \(k = 100\) there are 2106 packets in the network, while at \(k = 1000\) there are 2127 packets. When additional 100 random links are introduced, which is less than \(l_c(50)\), keeping all other network parameters unchanged, the network dynamics enters the congested phase, as it is illustrated by the right column of Figure 3. The number of packets in the network increases rapidly over time from 2238 packets in the network at \(k = 100\), to 14990 packets at \(k = 1000\). Congestions occur mainly at inputs and exists from the extra links. At these nodes the queue sizes are substantially larger than in other nodes of the network. This is illustrated by the dark spots in the figures of the right column of
5.2. Partial table routing

The performance of a network changes from the one described above when the value of the cutoff parameter \( m \) is less than \( D_{\text{max}} \), i.e. \( m < D_{\text{max}} \). For the routing algorithm \( R_{20} \) applied on the lattice \( \mathcal{L}_p(50) \) for various values of the parameter \( l \) the performance of the network is shown on Figure 8d. From this figure we observe that the critical value \( l_c(m) \) below which the performance of the network with the routing algorithm \( R_{20} \) is worse than the performance of the network without random links added is rather low. This value is lower than the corresponding value when \( R_{\infty} \) routing algorithm has been used. For example, adding more than about 15 links increases the critical load \( \lambda_c \). Adding about 50 links, just a 2% increment in the number of links, increases the critical load by 100%! The performance of the network improves significantly further with the increase of the number of random links.

The explanation of this behavior is straightforward. As we have already mentioned, for a regular square lattice without random links added, the values of the critical load \( \lambda_c(m,0) \) strongly decrease with decrease of the cutoff parameter \( m \). This results from the fact that packets which are further away than \( m \) units from their destinations are not being attracted to the destinations and travel randomly through the network. However, addition of random links significantly decreases the average distances between network nodes in the metric \( d_{\text{SP}} \). Therefore, it does not matter what is the exact value of the cutoff parameter \( m \). Most of the time distances in the metric \( d_{\text{SP}} \) are way below \( m \) and packets can be attracted to their destinations much faster. This attraction increases with the increase in the number of additional random links. Hence, when additional links are present, the critical load \( \lambda_c(m,l) \) is not very much dependent on \( m \), unless \( m \) is very small and increases with increase in value of the number \( l \) of the random links added.

Let \( \lambda_c(m,l) \) be the critical load of a network with the cutoff parameter \( m \) and with \( l \) extra links added. Let \( m_1 = D_{\text{max}} \) and let \( m_2 \) be smaller than \( m_1 \), but not too close to 1, for example, \( m_1 = 50, \) \( m_2 = 20 \), as in Figure 8. The performance of the networks with the routing algorithms \( R_{m_1} \) and \( R_{m_2} \), as in the Figure 8 can be summarized as follows

\[
\begin{align*}
    l_c(m_2) &< l_c(m_1), \\
    \lambda_c(m_2,0) &< \lambda_c(m_1,0), \\
    \lambda_c(m_i,l) &< \lambda_c(m_i,0),
\end{align*}
\]

for \( i = 1,2 \) when \( l < l_c(m_i) \) and

\[
\lambda_c(m_i,0) < \lambda_c(m_i,l),
\]

for \( i = 1,2 \) when \( l > l_c(m_i) \), and for sufficiently large \( l \)

\[
\lambda_c(m_1,l) \approx \lambda_c(m_2,l).
\]

Figure 10 shows how the introduction of 100 additional links to a network with 50 \times 50 nodes, routing algorithm \( R_{20} \) and presented load \( \lambda = 0.008 \) (which is above the critical load for \( l = 0 \)), affects the network dynamics. When \( l = 0 \), the number of packets in the network
grows with time, from 1248 at $k = 100$ to 6750 at $k = 1000$, indicating that the system is in the congested state. The left column of Figure 11 shows that the queue sizes grow almost uniformly over all nodes of the network. Hence, the congestion is distributed uniformly over all nodes of the network. However, when $l = 100$ additional random links are introduced, the right column of Figure 11 shows that congestion is eliminated. The number of packets in the network remains almost steady and it fluctuates around 230 (Figure 11). If an occasional small congestion occurs near the entrance to one of the shortcuts, it quickly disappears. For example, dark square visible in the right column of Figure 11 at $k = 100$ is not visible at $k = 1000$, demonstrating that local congestions are not permanent.

6. Conclusion

We found that the impact of additional random links on the performance of the network strongly depends on the routing scheme used in the network. Critical load of a network can be notably improved if the routing is based on a “geometrical distance”. Adding small number of additional links can decrease the average delay and shift the transition to the jammed phase toward higher load values. This, in general, is not true for routing schemes based on the “shortest path metric”. In this case, if the number of additional links is small, one can actually observe degradation of performance: many packets attempt to utilize shortcuts introduced by additional links, causing congestion which in effect pushes the network to a jammed phase. Reduced table routing can, to some extent, eliminate this problem. If packets located further than $m$ links from the destination are routed randomly (other packets taking the shortest possible path), performance gains obtained by adding sufficient number of extra links can be quite significant.

In order to relate our findings to data network protocols used in practice, more research is clearly needed. In particular, congestion control mechanisms built into protocols such as TCP/IP will certainly affect phenomena reported here. This issue, as well as other possible modifications of the model, is currently under investigation. Furthermore, the authors believe that some of the issues raised in [11], related to self-similar traffic modeling and analysis, and performance modeling of modern high-speed networks can be addressed by the methodology of this paper.

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Figure 1: Network update algorithm. Symbol $Q(r)$ denotes the queue at node $r$. 
Figure 2: (a) Average lifetime of a packet $\tau(k)$ as a function of $\lambda$, for $L_0^p(50)$ with $d_{PM}$ and $m = D_{max}$ after $k = 1000$ ($\circ$), $k = 1500$ ($\bullet$), and $k = 2000$ ($\star$) iterations. (b) Number of packets in the system $N(k)$ as a function of $\lambda$ after $k = 1000$ ($\circ$), $k = 1500$ ($\bullet$), and $k = 2000$ ($\star$) iterations.
Figure 3: Number of packets in the system $N(k)$ for subcritical and supercritical values of $\lambda$ ($\lambda = 0.035$ and $\lambda = 0.042$, respectively). $L_p^0(50)$ with $d_{PM}$ metric and $m = D_{\text{max}}$.

Figure 4: Verification of Little’s law for a lattice $L_p^0(50)$ with $d_{PM}$ and $m = D_{\text{max}}$, at $k = 1500$. Continuous line corresponds to $L^2(\lambda) = \lambda$. 
Figure 5: Free packet delay $\tau_0$ as a function of $m$ for a lattice $L_{10}^{p}(50)$ with $d_{PM}$ metric.
Figure 6: (a) Average lifetime of a packet $\tau(k)$ as a function of $\lambda$ for $k = 1500$ and for $\mathcal{L}^{np}_i(50)$ lattice with $d_M$ metric and $m = D_{\text{max}}$, and with the number of extra random links $l = 0$ (○), $l = 100$ (●), $l = 200$ (⋆), and $l = 400$ (+). (b) The same plot for $\mathcal{L}^p_i(50)$ lattice with $d_{PM}$ metric and $m = D_{\text{max}}$. 
Figure 7: Critical load $\lambda_c$ as a function of a number of extra links $l$ for the lattice $50 \times 50$ with periodic (○) and non-periodic (●) boundaries, using $d_{PM}$ and $d_M$ metric, respectively. In both cases, $m = D_{max}$. 
Figure 8: Change in critical load for a lattice $\mathcal{L}_0^0(50)$ using metrics $d_{PM}$ and $d_{SP}$ with $m = 20$ and $m = 50$. Vertical axis corresponds to $(\lambda_c(m, l) - \lambda_c(m, 0))/\lambda_c(m, 0)$, where $\lambda_c(m, l)$ is a critical load at a given $m$ and $l$. 
Figure 9: Comparison of dynamics of the network \( \mathcal{L}_l^p(50) \) with \( d_{SP} \) and \( R_\infty \) algorithm for \( l = 0 \) (left column) and \( l = 100 \) (right column). Queue sizes are represented as shades of gray, from the highest queue size of 20 or more represented by black color to the empty queue represented by white color. In order to preserve clarity, additional links are shown for \( k = 0 \) only.
Figure 10: Comparison of dynamics of the network $\mathcal{L}_t^p(50)$ with $d_{SP}$ and R$_{20}$ algorithm for $l = 0$ (left column) and $l = 100$ (right column). Queue sizes are represented as shades of gray, from the highest queue size of 20 or more represented by black color to the empty queue represented by white color. In order to preserve clarity, additional links are shown for $k = 0$ only.