Block Spin Effective Action
for 4d SU(2) Finite Temperature
Lattice Gauge Theory

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Abstract
The Svetitsky–Yaffe conjecture for finite temperature 4d SU(2) lattice gauge theory is confirmed by observing matching of block spin effective actions of the gauge model with those of the 3d Ising model. The effective action for the gauge model is defined by blocking the signs of the Polyakov loops with the majority rule. To compute it numerically, we apply a variant of the IMCRG method of Gupta and Cordery.

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1 Introduction

Block spin renormalization group [1] has become an important tool in the qualitative and quantitative understanding of critical phenomena in classical statistical mechanics and Euclidean quantum field theory. As a basic ingredient, it introduces effective Hamiltonians (actions in field theoretic language) which govern block spin degrees of freedom. The block spins are determined from the original degrees of freedom by averaging them over blocks.

In principle, renormalization group (RG) solves the problems posed by critical or nearly critical statistical systems. Under iterated application of the block transformation, either the correlation length in the system becomes small, or the flow of effective Hamiltonians eventually reaches a fixed point which determines the universal properties of the system.

A major drawback of the RG approach stems from the fact that effective Hamiltonians in general contain an infinite number of couplings, in contrast to the original Hamiltonians one starts from, which usually contain only a small number of interaction terms. The proliferation of couplings has a number of consequences. It is, e.g., not always clear how a certain truncation to a finite number of couplings affects the physical results. Furthermore, even if one relies on a certain truncation scheme, it might be tedious to explicitly compute the effective couplings.

This might be an explanation why the task of explicit computation of block spin effective actions has not received very much attention in the literature. See, however, e.g., Refs. [2] and [3].

Svetitsky and Yaffe [4] have conjectured that a (continuous) deconfinement transition of a \((d+1)\)-dimensional finite temperature lattice gauge theory should be in the same universality class as the phase transition of a corresponding \(d\)-dimensional spin system. This spin system should have the center of the gauge group as a global symmetry group.

The Svetitsky–Yaffe conjecture for \(SU(2)\) gauge theory at finite temperature offers the possibility of an interesting application of the block spin renormalization group. First, it has never been rigorously proved that this model belongs to the Ising universality class. On the other hand, the conjecture has been checked several times by comparison of Monte Carlo (MC) estimates for the critical indices (which were found in good agreement [4]), as well as with a mean field like analytical approach (which gives also predictions for \(SU(N)\) deconfinement temperatures [4]). However, so far there have been no numerical attempts to explicitly compute the effective action.
for the Polyakov loops and compare it with that of the Ising model.

With this article, we intend to fill this gap. We will demonstrate that actions for the degrees of freedom relevant for the deconfinement transition can well be computed by Monte Carlo. Comparing them with the corresponding actions for the Ising model we are able to confirm the validity of the Svetitsky–Yaffe conjecture in a very fundamental way.

The article is organized as follows: In Section 2 we introduce the notations for finite temperature lattice gauge theory and recall the Svetitsky–Yaffe conjecture. In Section 3 we introduce the block spin renormalization group. Section 4 explains the idea of flow matching, and the application of Improved Monte Carlo Renormalization Group (IMCRG) \cite{1} to $SU(2)$ lattice gauge theory is described in some detail. In Section 5 we discuss some details of our Monte Carlo methods and present the results. Conclusions follow.

## 2 Finite Temperature Lattice Gauge Theory

Let us briefly review the formulation of finite temperature gauge theory on a lattice (see for instance Ref. \cite{3}).

Consider an $SU(N)$ gauge system on a $(d+1)$–dimensional hypercubic lattice of size $L^d \cdot N_T$, where $L$ and $N_T$ are the spatial and temporal extensions, respectively, in units of the lattice spacing $a$.

A Euclidean quantum field theory at finite temperature is obtained if one compactifies the (imaginary) temporal direction, keeping infinite the spatial directions. In a finite lattice formulation one therefore assumes $L \gg N_T$. The compactification length is proportional to the inverse physical temperature $T$

$$N_T \cdot a = \frac{1}{T}.$$  \hspace{1cm} (1)

Denote with $U_\mu(n)$ the $SU(N)$ group element belonging to the link with origin in the site $n \equiv (x, t)$ and pointing in the $\mu$–direction. The usual Wilson action reads

$$S_g(U) = \beta \sum_P (N - \text{Re} \text{Tr}U_P) ,$$ \hspace{1cm} (2)

$$\beta = \frac{2N}{g^2} a^{d-1} ,$$ \hspace{1cm} (3)

where $U_P$ is the product of the group elements around the plaquette $P$. The

\[2\]
The partition function is given by
\[ Z = \int \prod_{n,\mu} dU(\mu)(n) \exp \left[ -S_g(U) \right]. \] (4)

Because of the periodicity in the temporal direction, the system is also invariant under a global \( Z_N \) symmetry, i.e. the center of the gauge group: its spontaneous symmetry breaking at a finite temperature \( T_c \) is the signal of the deconfinement transition.

The Polyakov loop is an order parameter for the finite temperature deconfinement transition. It is the trace of the ordered product of all timelike links with the same space coordinate, wrapping in the time direction
\[ \mathcal{L}(x) = \text{Tr} \left( \prod_{t=1}^{N_T} U_0(x, t) \right). \] (5)

It is a non–trivial observable from a topological point of view: its vacuum expectation value is not invariant under \( Z_N \) transformations. It is zero in the confining phase, while it acquires a finite value in the deconfined phase.

According to the 15 years old Svetitsky–Yaffe conjecture [4], integrating out the space–like degrees of freedom one should obtain an effective action for the Polyakov loops which is short ranged and has the center of \( SU(N) \) as a global symmetry group.

Thus, given a \( d \)-dimensional classical spin system with the same symmetry properties, undergoing a continuous phase transition, the \( (d+1) \)-dimensional quantum gauge model is expected to be in its universality class if the deconfinement transition is a continuous one and the effective Hamiltonian has good locality properties.

This applies in particular to the 4–dimensional \( SU(2) \) gauge model which should belong to the 3d Ising universality class.

### 3 Block Spin Renormalization Group

To define the block spin transformation, consider a magnetic system consisting of spins \( \sigma \) on the sites of a \( d \)-dimensional lattice, defined by a Hamiltonian \( H \) and a set of couplings \( \{K\} \),

\[ H(\sigma) = - \sum_{\alpha} K_{\alpha} S_{\alpha}(\sigma). \] (6)
The partition function reads

\[ Z = \sum_{\{\sigma\}} \exp[-H(\sigma)] . \]  

(7)

The “operators” \( S_\alpha(\sigma) \) are in general all possible products of spins compatible with the symmetry of the Hamiltonian. Explicit examples will be given below.

A block spin transformation maps the fine \( L^d \) lattice onto the block lattice of size \( L'_d \), where \( L = L_B L' \). This is achieved by averaging the original spins over cubical blocks of side length \( L_B \) according to a certain rule.

The Hamiltonian \( H' \) of the block spins \( \{\mu\} \) assigned to the sites of the block lattice is defined by

\[ \exp[-H'(\mu)] = \sum_{\{\sigma\}} P(\mu, \sigma) \exp[-H(\sigma)] , \]  

where \( P \) encodes the mapping from the fine to the coarse lattice. It obeys

\[ P(\mu, \sigma) \geq 0 \quad \text{and} \quad \sum_{\{\mu\}} P(\mu, \sigma) = 1 . \]  

(8)

(9)

The latter property ensures that the partition function remains unchanged,

\[ Z = \sum_{\{\mu\}} \exp[-H'(\mu)] . \]  

(10)

In this work we use the majority rule prescription (i.e. the \( \mu \) spins take values plus or minus one)

\[ P(\mu, \sigma) = \left( \prod_{x'} \right)^{(\text{blocks})} \frac{1}{2} \left[ 1 + \mu_{x'} \text{sign} \sum_{x \in x'} \sigma_x \right] . \]  

(11)

The sign function \( \text{sign}(x) \) in Eq. (11) is defined such that it vanishes for \( x = 0 \). This ensures that in case of a zero sum of spins inside a block a positive (negative) \( \mu_{x'} \) is selected with probability one half.

The block Hamiltonian \( H' \) can be expressed in terms of operators \( S'_\alpha(\mu) \), defined on the block lattice,

\[ H'(\mu) = - \sum_{\alpha} K'_\alpha S'_\alpha(\mu) . \]  

(12)
In the case of the 4–dimensional $SU(2)$ gauge model, the 3d effective action for the signs of the Polyakov loops shares (by definition) the $Z_2$ symmetry with the 3d Ising model. To define this action we assign to each Polyakov loop its sign

$$\sigma_x(U) = \text{sign} \ L(x).$$

(13)

Then we block the $\sigma$–spins with the majority rule to obtain Ising type block $\mu$–spins.

It follows that, similarly to Eq. (8), the effective Hamiltonian $H'$ for the finite temperature gauge system is given by

$$\exp[-H'(\mu)] = \int DU \ P(\mu, U) \ \exp[-S_g(U)],$$

(14)

with

$$P(\mu, U) = \prod_{x'} \frac{1}{2} \left[ 1 + \mu_{x'} \ \text{sign} \ \sum_{x \in x'} \sigma_x(U) \right].$$

(15)

Other procedures of blocking, like first averaging the Polyakov loops inside the blocks and then take as Ising type spin its sign, could also be employed.

We close this section by defining a renormalization group flow. A natural way to do it would be to fix a block length, e.g., $L_B = 2$, and then iterate the transformation (15). We do not stick to this definition here. Instead we define the flow by just increasing the block size $L_B$. This allows us to compute Hamiltonians not only for scales $2^n$, but also for arbitrary scales $L_B$, with $L_B$ integer.

4 Monte Carlo Renormalization Group for Polyakov Loops

In this section we first recall the RG matching idea. Then we show how to apply the Improved Monte Carlo Renormalization Group (IMCRG) method by Gupta and Cordery [7] to 4d $SU(2)$ gauge theory at finite temperature.

4.1 Matching of RG Trajectories

In the infinite–dimensional space of couplings $\{K\}$, a renormalization group transformation $R$ can be looked at as a mapping of the original bare Hamiltonian $H$ onto a new Hamiltonian $H' = R(H)$, defined by the couplings $\{K'\}$. 
Figure 1: Matching of the RG trajectories of two critical models A and B in the neighbourhood of an RG fixed point.

The RG flow obtained under iterated RG transformations will eventually end in a fixed point \( \{ K^* \} \), defined through \( H^* = R(H^*) \). The critical surface is identified by all flows connected to the fixed point in this way.

The RG matching method is based on the assumption that different physical systems, belonging to the same universality class, will follow RG flows which originate from different “bare couplings” on the critical surface and eventually match in a neighbourhood of the common fixed point. Of course, a matching close to a non–trivial fixed point will only take place if both models under consideration are at criticality. A matching thus confirms both universality and allows to check for criticality. This matching method has been successfully applied in the context of spin models, see e.g. [9]. The feasibility of matching different critical flows by means of MC methods mainly relies on the assumption that the different trajectories come close to each other, i.e. approximately match, before the fixed point is actually reached, c.f. Figure [1]. As we shall see, this condition is met for our models.

### 4.2 Effective Couplings from IMCRG

The Improved Monte Carlo Renormalization Group method [7] allows to compute effective actions for Ising type block spins\(^1\)

\(^1\)A generalization to systems with continuous block variables is not straightforward.
The main idea is to avoid simulations of the original partition function. Instead, consider a *modified* system defined through

\[
Z_c = \sum_{\{\mu\}} \exp \left[ -H'(\mu) + \bar{H}(\mu) \right]
\]

where

\[\bar{H}(\mu) = -\sum_{\alpha} \bar{K}'_\alpha S'_\alpha(\mu)\] (17)

is a *guess* for \(H'(\mu)\).

This system can be simulated once \(H\) and \(\bar{H}\) are given. Note the plus sign in front of \(\bar{H}\) in Eq. (16).

The system with partition function \(Z_c\) is *non–critical*, even in the case of a critical Hamiltonian \(H(\sigma)\). Consider the expectation values

\[
< S'_\alpha >_c = \frac{1}{Z_c} \sum_{\{\mu\}} S'_\alpha e^{\left[ -H'(\mu) + \bar{H}(\mu) \right]} \tag{18}
\]

\[
= \frac{1}{Z_c} \sum_{\{\sigma\}} \sum_{\{\mu\}} S'_\alpha P(\mu, \sigma) e^{\left[ -H(\sigma) + \bar{H}(\mu) \right]}, \tag{19}
\]

If the guess is exact, i.e.,

\[\bar{H}(\mu) = H'(\mu),\] (20)

the block spins \(\mu_x\) completely decouple and fluctuate independently. In other words, the system is non–critical and the correlations in \(Z_c\) are bounded by the block size \(L_B\). The correlations functions are then known exactly,

\[
< S'_\alpha >_o = 0,
\]

\[
< S'_\alpha S'_\beta >_o = n_\alpha \delta_{\alpha\beta},\] (21)

where \(n_\alpha\) are trivial multiplicity factors.

Let us assume that \(\bar{H}(\mu)\) is close to \(H'(\mu)\). Then a first order expansion gives

\[
< S'_\alpha >_c = n_\alpha (K'_\alpha - \bar{K}'_\alpha) + O \left( (K'_\alpha - \bar{K}'_\alpha)^2 \right). \tag{22}
\]

Solving this equation for \(K'_\alpha\) allows to improve the guess \(\bar{K}'_\alpha\). Usually a few iterations

\[
\bar{K}'_\alpha \rightarrow \bar{K}'_\alpha + n_\alpha^{-1} < S'_\alpha >_c, \tag{23}
\]
where the expectation values are determined by simulation of the system (16), are sufficient to determine $H'$ to a good precision.

To apply the IMCRG procedure to the $SU(2)$ gauge system, one has to simulate the partition function

$$Z_c = \sum \{\mu\} \exp \left[ -\mathcal{H}'(\mu) + \bar{\mathcal{H}}(\mu) \right]$$

$$= \sum \{\mu\} \int DU \, P(\mu, U) \exp \left[ -S_g(U) + \bar{\mathcal{H}}(\mu) \right]. \quad (24)$$

Remember that the $\mu$-variables are defined as the majority rule block spins of the signs of the Polyakov loops. It is straightforward to design a MC procedure for the updating of this system. Details will be given in the next section.

## 5 Monte Carlo Simulations

We applied the IMCRG method to three different systems: the 3d standard Ising model (with nearest-neighbour coupling), the 3d "I_3" model, which includes also a third, cube-diagonal neighbour coupling [10], and the 4d $SU(2)$ pure gauge model. We simulated the system defined by Eq. (16) for the spin models and by Eq. (24) for the $SU(2)$ gauge model.

For the updating of the Ising model we used a Metropolis algorithm: A single spin $\sigma_x$ is proposed to be flipped. It is checked whether this update leads to a flip of the block spin $\mu_{x'}$, with $x \in x'$. The total change of energy $\Delta \mathcal{H}(\sigma) - \Delta \bar{\mathcal{H}}(\mu)$ is then computed and used in the usual Metropolis acceptance/rejectance step.

In case of the $SU(2)$ model, only the temporal links couple to the compensating block Hamiltonian. The space-like links are updated using the incomplete Kennedy–Pendleton heat bath sweep [11] supplied with a number of overrelaxation sweeps. For temporal links one employs again a Metropolis procedure: A proposed change of a link matrix leads to a change of the Polyakov loop $\mathcal{L}(x)$ of which it is member. If the sign $\sigma_x$ changes, the block spin $\mu_{x'}$ in turn might flip and give rise to a change of $\bar{\mathcal{H}}(\mu)$. The relevant energy change for the Metropolis step is $\Delta S_g(U) - \Delta \bar{\mathcal{H}}(\mu)$.

In practical calculations one has to truncate the interactions in $H'$ and $\bar{\mathcal{H}}$. We chose to include in the ansatz eight 2–point couplings and six 4–point couplings. The 2–point couplings can be labelled by specifying the relative
position of the interacting spins (up to obvious symmetries): Our couplings 
$K_1 \ldots K_8$ then correspond to $001, 011, 111, 002, 012, 112, 022, 122$. The 
4-point couplings $K_9 \ldots K_{14}$ are defined in an obvious way through Figure 2. 
The corresponding interaction terms in the effective Hamiltonian are denoted 
by $S'_\alpha$, $\alpha = 1 \ldots 14$.

5.1 Reduction of Critical Slowing Down

A merit of the IMCRG method is that block spin observables are (nearly) 
decorrelated and the critical slowing down problem is less severe than in 
standard simulations.

In Figure 3 we show scatterplots (MC time history) of measurements of 
the nearest neighbour block spin 2-point function. The comparison is be-
tween a simulation of the pure gauge system (without IMCRG compensation 
on the block level) and the system defined through Eq. (24). The plot clearly 
shows that the IMCRG type simulation suffers much less from critical slowing 
down.

Analogous observations were made in case of the Ising model simulations 
with the compensating block Hamiltonian switched on. It is the reduction
Figure 3: Comparing scatter plots of the nearest neighbour block correlation function observable in $SU(2)$ simulations with $N_T = 2$. The standard simulations (no compensation on block level) are shown on the left, the IMCRG simulations on the right.
of critical slowing down obtained from the compensation even on moderate lattice sizes which enabled us to obtain reasonable results with moderate CPU expense.

### 5.2 Matching of the Two Ising Models

We started by comparing the RG flow of the two Ising models. In Table 1 we summarize some parameters of the MC simulations. We made simulations on lattices consisting of $8^3$ blocks of size $L_B$ at the infinite volume critical couplings $\beta = 0.2216544$ \[12\] for the ordinary Ising and $(\beta_1, \beta_3) = (0.128003, 0.051201)$ \[10\] for the $I_3$ model. At each RG step (fixed $L_B$ value) we made usually two, three or four IMCRG iterations in order to have the guesses of the effective couplings converge to reasonable precision. The number of sweeps in each run ranged from $\sim 10^5$ for the largest lattice sizes to $\sim 6 \cdot 10^6$ for the smallest ones.

As the final error of the estimate for an effective coupling, we took the maximum of the statistical error and the last change of guess in the IMCRG procedure. In Figure 4 we show the results for the flow of the two leading couplings $K_1'$ and $K_2'$, with increasing block size $L_B$. To achieve matching, the block sizes $L_B$ of the $I_3$ model have been rescaled by a factor of $\lambda = 0.59$. A rescaling is always needed to obtain matching. The reason for this is that the two models have a different distance to “travel” before they meet on a common trajectory, c.f. Figure 1.

The figure shows that the two flows collapse nicely on a single trajectory, indicating that they are approaching a common fixed point. This happens also with the other 12 couplings not shown in the plot. It turns out that the approach to the fixed point can be well fitted by a power law,

$$K_\alpha(L_B) = K_\alpha^* + a_\alpha \cdot L_B^{-\rho}.$$  \hspace{1cm} (25)
Figure 4: Flows of nearest and second nearest neighbour couplings in the standard (diamonds) and the $I_3$ (bars) Ising model with increasing block size $L_B$. In order to obtain matching, the block sizes of the $I_3$ model were rescaled by a factor $\lambda = 0.59$. The dotted lines are fits of the flows with a power law.
If the flows of two models obey such a law, with the same fixed point $K_\alpha^*$ and exponent $\rho$, but different “amplitudes” $a$ and $a'$, the rescale factor $\lambda$ to obtain matching is

$$
\lambda = \left( \frac{a'}{a} \right)^{\frac{1}{\rho}}.
$$

(26)

We always used as a reference trajectory the flow of the standard Ising model and rescaled the block sizes of the other models by an appropriate factor.

The results of our various power law fits of the Ising model are summarized in Table 2. We fitted the models separately, checking also the effect of discarding the effective couplings for the smallest block sizes. That the fixed point value and the exponent of the two models coincide is confirmed by a

| $\alpha$, model | $L_B \geq$ | $K_\alpha^*$ | $a_\alpha$ | $\rho$ | $\chi^2$/dof |
|-----------------|--------------|---------------|-----------|-------|--------------|
| 001, Ising      | 3            | 0.1990(5)     | -0.077(4) | 1.37(7)| 0.99         |
| 001, Ising      | 5            | 0.1977(7)     | -0.15(7)  | 1.9(3) | 0.01         |
| 001, $I_3$      | 3            | 0.1981(4)     | -0.240(8) | 1.60(4)| 0.50         |
| 001, $I_3$      | 5            | 0.1975(5)     | -0.33(9)  | 1.80(14)| 0.09         |
| 001, combined   | 7            | 0.1979(6)     | -0.10(3)  | 1.67(20)| 0.17         |
| 001, combined   | 7            |               | -0.27(9)  |       |              |
| 011, Ising      | 5            | 0.0224(1)     | 0.023(2)  | 1.67 fix| 0.94         |
| 011, $I_3$      | 5            | 0.0225(2)     | 0.055(6)  | 1.67 fix| 0.27         |
| 111, Ising      | 5            | 0.0013(1)     | 0.015(2)  | 1.67 fix| 0.15         |
| 111, $I_3$      | 5            | 0.0013(1)     | 0.029(2)  | 1.67 fix| 1.12         |
| 002, Ising      | 5            | -0.0202(1)    | 0.047(2)  | 1.67 fix| 1.53         |
| 002, $I_3$      | 5            | -0.0201(3)    | 0.059(8)  | 1.67 fix| 0.38         |
| 9, Ising        | 5            | 0.00210(4)    | -0.006(1) | 1.67 fix| 0.10         |
| 9, $I_3$        | 5            | 0.00215(10)   | -0.005(3) | 1.67 fix| 0.17         |

Table 2: Fit results for the Ising model flows for a number of 2–point couplings and for the largest 4–point coupling $K_9$. The fits were done with Eq. (25). The second column gives the minimum block sizes that were used in the fit. $K_\alpha^*$ are the estimates for the fixed point values. A “fix” after a parameter means that the value was kept fixed during the fitting procedure.
common fit of the two Ising flows, where in the fit function only the amplitudes $a_\alpha$ were allowed to depend on the model. This yields the results quoted in the last two lines of the first block of the table. We then fixed $\rho = 1.67$ and fitted the flows of the non–leading couplings with two parameters (fixed point value and amplitude of power law correction). We found a very nice agreement of the resulting fixed point values for all couplings.

The value of the exponent $\rho$ turns out to be too big to be identified with the first correction to scaling exponent $\omega \approx 0.8$. We expected that $\omega$ should be the leading exponent. A possible explanation of the present observation is the following: The amplitude of a power term with $\omega$ as exponent is too small to be detected within our precision. The exponent $\rho$ with its relatively large amplitudes is due to the presence of a redundant operator of the particular blocking scheme we used.

### 5.3 Matching of the 4d SU(2) with the Ising Models

We then turned to the 4d $SU(2)$ gauge model at finite temperature. Informations on MC simulations made in this case are given in Table 3. We made MC simulations for the $N_T = 1$ and $N_T = 2$ cases on lattices consisting of $6^3$ blocks of size $L_B$.

As the critical deconfinement transition value we used the gauge couplings $\beta_c = 0.8730(2)$ \cite{13} for $N_T = 1$. For $N_T = 2$ we studied a neighbourhood of the critical value $\beta_c = 1.880(3)$ \cite{14} (see Table 3).

For the $SU(2)$ model statistic has necessarily been reduced compared to the Ising models. For $N_T = 2$ measurements ranged from $\sim 10^4$ for the

| $N_T$ | $\beta$  | $L_B$  | $L^3 \cdot N_T$ |
|------|---------|-------|-----------------|
| 1    | 0.8730  | 2,3,4,5,6,7 | $42^3 \cdot 1$ |
| 2    | 1.871   | 3,4,5,6 | $36^3 \cdot 2$ |
| 2    | 1.874   | 3,4,5,6 | $36^3 \cdot 2$ |
| 2    | 1.877   | 2,3,4,5,6 | $36^3 \cdot 2$ |
| 2    | 1.880   | 2,3,4,5,6 | $36^3 \cdot 2$ |

Table 3: Lattice sizes and values of $\beta$ used for the 4d $SU(2)$ model at finite temperature. In the last column the maximum lattice size is shown. For all runs we used $L' = 6$. 


Figure 5: Flow of the nearest neighbour coupling in the effective action for $N_T = 1$ lattice gauge theory with gauge coupling $\beta = 0.8730$ (squares). Also shown is the standard Ising (diamonds) and and $I_3$ model (bars). The rescaling factor of the gauge block size with respect to the standard Ising scale is $\lambda = 0.61$.

largest sizes up to $\sim 5 \cdot 10^5$ for the smaller ones.

The $K'_1$ leading coupling result for $N_T = 1$ is shown in Figure 3. Also shown is the Ising flow. Notice that here and in the figures which follow the Ising flows are the same as in Figure 4, which also means that the fit lines plotted are those obtained from the Ising data. Fits were made again according to Eq. (25). Omitting the $L_B = 2$ value, they gave results consistent with the fits of the Ising data, both for the exponent and the asymptotic values, however with bigger errors due the lower statistics. The rescaling of the gauge block sizes with respect to the standard Ising scale used in this case is $\lambda = 0.61$. 

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Figure 6: Flows of four different effective couplings for $N_T = 2$ $SU(2)$ lattice gauge theory at $\beta = 1.877$, matching with the two Ising models (bars, diamonds, and fit lines). The block sizes of the gauge model are rescaled by a factor $\lambda = 0.65$ with respect to the standard Ising scale.
| α | $L_B = 3$ | $L_B = 4$ | $L_B = 5$ | $L_B = 6$ |
|---|---|---|---|---|
| 001 | 0.15962(94) [50] | 0.17442(52) [34] | 0.18333(70) [27] | 0.18800(92) [48] |
| 011 | 0.02918(57) [118] | 0.02800(32) [4] | 0.02689(42) [25] | 0.02500(57) [58] |
| 111 | 0.00968(73) [103] | 0.00432(35) [24] | 0.00357(45) [68] | 0.00300(61) [80] |
| 002 | −0.00728(94) [177] | −0.01282(41) [62] | −0.01514(49) [77] | −0.01691(69) [139] |
| 012 | −0.00277(46) [70] | −0.00444(21) [12] | −0.00500(25) [56] | −0.00505(38) [35] |
| 112 | −0.00115(42) [62] | −0.00177(21) [5] | −0.00179(26) [59] | −0.00164(35) [53] |
| 022 | 0.00042(63) [131] | −0.00041(29) [59] | −0.00041(39) [14] | −0.00077(49) [77] |
| 122 | −0.00030(52) [33] | 0.00012(21) [42] | −0.00043(30) [55] | 0.00046(36) [110] |
| 9 | 0.00085(34) [31] | 0.00146(17) [20] | 0.00182(21) [48] | 0.00212(28) [45] |
| 10 | 0.00005(20) [13] | 0.00014(10) [20] | 0.00014(12) [11] | 0.00020(16) [3] |
| 11 | 0.00026(55) [32] | −0.00096(28) [70] | 0.00006(33) [23] | −0.00040(43) [8] |
| 12 | −0.00013(18) [6] | −0.00013(10) [9] | −0.00013(12) [11] | −0.00030(16) [10] |
| 13 | −0.00020(36) [9] | −0.00020(20) [15] | −0.00029(23) [41] | 0.00004(31) [66] |
| 14 | 0.00012(59) [4] | 0.00012(33) [5] | 0.00012(39) [11] | 0.00033(53) [95] |

stat  $12 \cdot 10^3$  $27 \cdot 10^3$  $25 \cdot 10^3$  $20 \cdot 10^3$

Table 4: Values of the effective couplings for the 4d SU(2) model at $N_T = 2$, $\beta = 1.877$ for different block size $L_B$. In the bottom row the statistics of the last IMCRG iteration (fixed $L_B$) is given. Statistical errors are given in parenthesis. Square brackets contain the change of the coupling $\Delta K'_\alpha$ in the last IMCRG iteration.

The flows of the effective couplings for $N_T = 2$ at $\beta = 1.877$ are given in Figure 6. One can see a clear matching with the two Ising models (bars and diamonds) and the fit lines for the first two couplings. The $L_B$'s of the $SU(2)$ gauge model were rescaled in this case by a factor of $\lambda = 0.65$.

We found that, among the different gauge couplings used for $N_T = 2$, the supposed critical coupling $\beta = 1.880$ is actually ruled out, no matter which value of the rescaling parameter $\lambda$ is chosen. This can be clearly seen in Figure 7 where the flows of the NN coupling in the effective action are shown for the four different gauge couplings $\beta = 1.880, 1.877, 1.874, 1.871$ and compared with the fitted curve of the Ising model.

For $\beta = 1.880$ the system is definitely in the deconfined phase, whereas the flow for $\beta = 1.871$ moves away towards the high temperature fixed point (confinement phase). The effective coupling values for the best matching
Figure 7: Flows of the NN coupling in the effective action for $N_T = 2$ $4d \text{SU}(2)$ at four different gauge couplings $\beta = 1.880, 1.877, 1.874, 1.871$ (triangles, diagonal crosses, squares and stars respectively). Also shown is the Ising flow (diamonds, bars and fitted curve). The rescaling of the gauge block sizes with respect to the standard Ising scale is $\lambda = 0.65$. 
Table 5: Comparison of effective critical couplings for different sizes of the coarse lattice. The example shown is the ordinary Ising, with $L_B = 9$. In the bottom line the statistics is given.

| $\alpha$ | $L' = 8$            | $L' = 6$            |
|----------|---------------------|---------------------|
| 001      | 0.19532(20)         | 0.19529(60)         |
| 011      | 0.02307(6)          | 0.02315(13)         |
| 111      | 0.00172(14)         | 0.00192(51)         |
| 002      | −0.01899(7)         | −0.01946(16)        |
| 012      | −0.00551(5)         | −0.00558(11)        |
| 112      | −0.00168(3)         | −0.00168(11)        |
| 022      | −0.00002(19)        | 0.00002(13)         |
| 122      | 0.00010(3)          | 0.00024(29)         |
| $K'_9$   | 0.00195(2)          | 0.00191(9)          |
| $K'_10$  | 0.00024(2)          | 0.00024(4)          |
| $K'_{11}$| −0.00050(3)         | −0.00049(11)        |
| $K'_{12}$| −0.00010(1)         | −0.00007(9)         |
| $K'_{13}$| −0.00014(8)         | −0.00010(8)         |
| $K'_{14}$| −0.00003(7)         | −0.00025(29)        |

| stat    | $1.3 \cdot 10^6$   | $3 \cdot 10^5$      |

trajectory of $\beta = 1.877$ are given in Table 4. The less significant values, at $L_B = 2$, have been omitted. We explicitly reported the statistical errors and the $\Delta K'_\alpha$ variations in the last IMCRG iteration (the latter in square brackets).

Let us conclude this analysis with two remarks.

First, it is worthwhile to stress that weak finite size effects are present within this approach: Comparing the $6^3$ block lattice of the gauge model with the $8^3$ of the Ising model should not give sizable systematic errors within our precision.

As a check, in Table 5 the effective coupling values of the ordinary Ising model are reported for two different block lattice sizes, $L' = 8$ and $L' = 6$. The result confirms that all couplings are consistent within errors.

Finally, let us notice that within our statistic the $\beta = 1.874$ flow can also be made compatible with the Ising trajectory: A better resolution to
discriminate between the two beta values would have required to extend the MC analysis to bigger block sizes $L_B$, of course with much more CPU time consuming. Even though, using the block sizes at our disposal the corresponding fit is not as good as that of the $\beta = 1.877$ value.

Therefore we assume the latter as the critical coupling value for $N_T = 2$, consistently (within one standard deviation) with Ref. [14].

6 Conclusion and Outlook

The discussion of MC results shows that the Svetitsky–Yaffe conjecture is confirmed in a very fundamental way by observing matching of the $SU(2)$ RG trajectory with that of the Ising model.

At the same time, we showed that IMCRG works well as a method to compute the effective action of Ising type degrees of freedom in a genuine non–Ising model like $4d$ finite temperature $SU(2)$ gauge theory.

Notice also that this kind of calculations could be done on workstations, with relatively small computer resources.

An extension to $N_T$ greater than two would be interesting but more expensive. The reason is that with increasing temporal size the small $L_B$ actions move farther away from the fixed points, i.e. they need to be blocked more in order to come close to the reference Ising flows. This observation is in agreement with the fact that also in more standard approaches, e.g. via the Binder cumulant, the spatial size of the lattice has to be increased very much with increasing $N_T$.

Finally, it would be of interest to check this approach with different blocking prescriptions. The rate of approaching the RG fixed point is in fact very sensitive to the blocking rule used and a faster convergence can in principle be obtained using a more sophisticated blocking scheme than the majority rule.

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