Relativistic Einstein-Podolsky-Rosen correlation and Bell’s inequality

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Abstract

We formulate the Einstein-Podolsky-Rosen (EPR) gedankenexperiment within the framework of relativistic quantum theory to analyze a situation in which measurements are performed by moving observers. We point out that under certain conditions the perfect anti-correlation of an EPR pair of spins in the same direction is deteriorated in the moving observers’ frame due to the Wigner rotation, and show that the degree of the violation of Bell’s inequality prima facie decreases with increasing the velocity of the observers if the directions of the measurement are fixed. However, this does not imply a breakdown of non-local correlation since the perfect anti-correlation is maintained in appropriately chosen different directions. We must take account of this relativistic effect in utilizing in moving frames the EPR correlation and the violation of Bell’s inequality for quantum communication.

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1 Introduction

In the context of physical reality, Einstein, Podolsky, and Rosen (EPR) \[1\] proposed a gedankenexperiment. Bohm’s version \[2\] of this experiment is described as follows: Suppose that two spin-1/2 particles with total spin zero are moving in opposite directions and that their spin components are measured by two observers. The spin component of each particle is measured in some direction by each observer. If the two measurements are performed in the same direction, the outcomes are perfectly anti-correlated in whichever direction and at however remote place they are performed. This remarkable property is known as the EPR correlation. The reason for the perfect anti-correlation in any direction is that the spin-singlet state is isotropic and has no preferred direction.
In this paper, we consider Bohm’s version of the EPR experiment in the framework of relativistic quantum theory and discuss an effect of observer’s motion on the EPR correlation. A number of articles [3, 4, 5, 6, 7, 8, 9] have discussed the case of a moving observer in connection with the instantaneous reduction of the state caused by the measurement in relativistic quantum theory. However, we shall not consider such a problem and concentrate on the symmetry transformation of the spin correlation under the Lorentz transformation.

To address the issue of how the spin of a particle is seen from a moving observer, we need not resort to the explicit dynamics of the particle such as the Dirac equation, but only to a group-theoretical approach [10, 11]. This is analogous to the fact that the transformation law of the spin under the spatial rotation is derived only from knowledge of the spatial rotation group $SO(3)$. In relativistic theory, the relevant group is the Poincaré group $ISO(1, 3)$ which contains $SO(3)$ as a subgroup. For a massive particle (such as an electron) and a massless particle (such as a photon), transformation laws under the Lorentz transformation are derived from the unitary representations of the Poincaré group. Note that massless particles are entirely different from massive particles in relativistic theory since massless particles move at the speed of light and thus have no rest frame. In the massive case, the Lorentz transformation rotates the spin of a particle, known as the Wigner rotation [12]. The angle of this rotation depends not only on the Lorentz transformation but also on the momentum of the particle. On the other hand, in the massless case, the Lorentz transformation rotates the plane of polarization.

Recently, the Wigner rotation has seen a remarkable resurgence of interest in the context of the entanglement or the EPR correlation in the relativistic regime [13, 14, 15, 16, 17]. Peres, Scudo, and Terno [13] have shown that the spin entropy for a single particle is not Lorentz invariant if the particle is not in the momentum eigenstate, because spin is entangled with momentum by the Wigner rotation. Gingrich and Adami [16] have shown that entanglement between the spins of two particles is carried over to the entanglement between the momenta of the particles by the Wigner rotation, even though the entanglement of the entire system is Lorentz invariant. Alsing and Milburn [14] have considered entanglement in one of the Bell states between the spins of two particles moving in opposite directions with definite momenta, and shown that it is Lorentz invariant since the Wigner rotation is a local unitary operation. Instead of the state vector in the Hilbert space,
they have used a 4-component Dirac spinor or a polarization vector in favor of quantum field theory. Independently of their research, the authors [15] considered a similar situation but discussed the EPR correlation rather than the entanglement using the spin-singlet state in terms of the state vector. (The entanglement is independent of the basis for the measurement, but the correlation depends on it.) Later, Ahn et al. [17] also calculated the same situation with all the Bell states and obtained a conclusion on the EPR correlation that contradicts our result. They concluded that the Wigner rotation could cause “a counter example for the nonlocality of the EPR paradox”.

This paper is an extended and comprehensive version of Ref. [15], discussing a possible experiment and relevance to quantum communications. We first consider two massive particles in the spin-singlet state moving in opposite directions with definite momenta, and two observers moving in the same direction at the same velocity with respect to the laboratory frame. By applying the Wigner rotation, we show that the measurement results are not perfectly anti-correlated when the two observers measure the spins in the same direction in their common rest frame. Seen from the moving observers, the anti-correlation in the same direction decreases unlike the non-relativistic case. In particular, the perfect anti-correlation in every direction is not maintained in all inertial frames. The special directions along which the perfect anti-correlation is not maintained are specified by the motion of the observers and by that of the particles. We also examine the violation of Bell’s inequality [18, 19] in this case and show that the motion of the observers prima facie decreases the degree of violation of Bell’s inequality if the directions of the measurement are fixed. We extend these considerations to the massless case and obtain qualitatively similar but quantitatively different results.

At first sight, the above results appear to contradict the nonlocality inherent in quantum mechanics. One might think that the quantum correlation breaks down and that local hidden variable theories are restored in the relativistic regime. However, this is not the case, of course. We explicitly show that the perfect anti-correlation is maintained in different directions and Bell’s inequality for an appropriate set of observables is still maximally violated. Note that the entanglement is preserved under the Lorentz transformation [14]. We emphasize that our aim is not to discuss the foundation of quantum mechanics but to explore effects of the relative motion between the sender and receiver in quantum communications that utilize the EPR correlation and the violation of Bell’s inequality.

This paper is organized as follows: In order to make this paper self-
contained, Sections 2 and 3 review the Lorentz transformation laws for a massive particle and for a massless particle, respectively. Section 4 formulates relativistic EPR experiments both for massive particles and for massless particles, and applies the transformation laws to them. Section 5 analyzes Bell’s inequality in the relativistic EPR experiments. Section 6 summarizes our results, and discusses an experimental possibility and relevance to quantum information. Appendix summarizes the Poincaré group and our conventions.

2 Massive Particle

In this section, we follow Ref. [11] to discuss the one particle states for a massive particle which furnish an irreducible representation of the Poincaré group. We then calculate the transformation law explicitly in a specific example.

First, we consider the rest frame of the massive particle, since the spin is most conveniently identified in the rest frame. The four-momentum of the particle with mass $M$ then becomes the rest momentum, $k^\mu = (Mc, 0, 0, 0)$.

In this frame, the state $|k, \sigma\rangle_{\text{rest}}$ is specified in terms of the eigenvalues of the Hamiltonian $H$, the momentum operator $\vec{P}$, and the $z$-component of the total angular momentum operator $\vec{J}$ as

\begin{align*}
H|k, \sigma\rangle_{\text{rest}} &= Mc^2|k, \sigma\rangle_{\text{rest}}, \\
\vec{P}|k, \sigma\rangle_{\text{rest}} &= 0, \\
J^3|k, \sigma\rangle_{\text{rest}} &= \sigma \hbar |k, \sigma\rangle_{\text{rest}}.
\end{align*}

For a spatial rotation $R^\mu_\nu$, there is a corresponding unitary operator $U(R)$ which can be represented by a $(2j + 1)$-dimensional unitary matrix $D^{(j)}(R)$,

\begin{equation}
U(R) |k, \sigma\rangle_{\text{rest}} = \sum_{\sigma'} D^{(j)}_{\sigma' \sigma}(R) |k, \sigma'\rangle_{\text{rest}},
\end{equation}

where \(j\) is an integer or a half-integer and \(-j \leq \sigma \leq j\). Note that the momentum on the right-hand side is also the rest momentum $k^\mu$ since the spatial rotation group $SO(3)$ leaves $k^\mu$ invariant. That is, the particle remains at rest if the frame is rotated spatially. The total angular momentum is equal
to the spin in the rest frame because the orbital angular momentum is absent. Therefore, \( j \) is the spin of the particle and \( \sigma \) its \( z \)-component.

We next consider the general frame, where the four-momentum of the particle is

\[
p^\mu = (\sqrt{|\vec{p}|^2 + M^2c^2}, p^1, p^2, p^3).
\]  

We obtain this momentum by performing a standard Lorentz transformation \( L(p)_{\mu\nu} \) on the rest momentum \( (2.1) \),

\[
p^\mu = L(p)_{\mu\nu} k^\nu.
\]  

An explicit form of \( L(p)_{\mu\nu} \) is written as

\[
L(p)^0_0 = \gamma, \\
L(p)^0_i = L(p)_i^0 = p^i/Mc, \\
L(p)^i_k = \delta_{ik} + (\gamma - 1) p^i p^k/|\vec{p}|^2,
\]

where

\[
\gamma = \sqrt{|\vec{p}|^2 + M^2c^2}/Mc.
\]  

Note that the inverse matrix \( L^{-1}(p)_{\mu\nu} \) of \( L(p)_{\mu\nu} \) can be obtained by replacing \( p^i \) by \(-p^i\), i.e., \( L^{-1}(p^0, p^i)_{\mu\nu} = L(p^0, -p^i)_{\mu\nu} \).

Using the unitary operator \( U(L(p)) \) corresponding to \( L(p)_{\mu\nu} \), we define the state in the general frame as

\[
|p, \sigma\rangle \equiv U(L(p)) |k, \sigma\rangle_{\text{rest}}.
\]  

Note that

\[
H|p, \sigma\rangle = \sqrt{|\vec{p}|^2c^2 + M^2c^4} |p, \sigma\rangle, \\
\vec{p}|p, \sigma\rangle = \vec{p}|p, \sigma\rangle,
\]

where \( \vec{p} = (p^1, p^2, p^3) \), but that

\[
J^3|p, \sigma\rangle \neq \sigma \hbar |p, \sigma\rangle.
\]

This is because \( J^3 \) now contains a contribution from the orbital motion. Since we have

\[
\begin{align*}
[J^3, P^1] &= i\hbar P^2 \neq 0, \\
[J^3, P^2] &= -i\hbar P^1 \neq 0,
\end{align*}
\]
\[ |p, \sigma \rangle \] cannot be a simultaneous eigenstate of \( J^3 \) and \( \vec{P} \). Unlike in the rest frame, the label \( \sigma \) denotes not the eigenvalue of \( J^3 \) but the \( z \)-component of the spin.

The 3-dimensional delta function \( \delta^3(\vec{p}' - \vec{p}) \) is not Lorentz invariant. Instead,

\[
\sqrt{\vec{p}}^2 + M^2 c^2 \delta^3(\vec{p}' - \vec{p})
\]

is an invariant delta function with the mass shell condition,

\[
p^0 = \sqrt{|\vec{p}|^2 + M^2 c^2}.
\]

The normalization of the state is thus chosen to be

\[
\langle p', \sigma' | p, \sigma \rangle = \left( \frac{\sqrt{|\vec{p}|^2 + M^2 c^2}}{M c} \right) \delta^3(\vec{p}' - \vec{p}) \delta_{\sigma' \sigma}.
\]

The state defined in Eq. (2.10) transforms under the Lorentz transformation \( \Lambda^\mu_\nu \) as

\[
U(\Lambda) |p, \sigma \rangle = U(\Lambda) U(\Lambda p)) |k, \sigma \rangle^{\text{rest}} = U(\Lambda p)) U(W(\Lambda, p)) |k, \sigma \rangle^{\text{rest}},
\]

where

\[
W(\Lambda, p)^\mu_\nu = \left[ L^{-1}(\Lambda p) \Lambda L(p) \right]^\mu_\nu
\]

is called Wigner rotation. It follows from Eq. (2.7) that the Wigner rotation leaves the rest four-momentum (2.1) invariant,

\[
W(\Lambda, p)^\mu_\nu k^\nu = k^\mu.
\]

This means that the Wigner rotation is an element of the spatial rotation group \( SO(3) \). We can view this Wigner rotation as follows: we perform the Lorentz transformation \( L(p)^\mu_\nu \) on the rest frame to obtain a moving frame 1, and then operate the Lorentz transformation \( \Lambda^\mu_\nu \) to obtain another moving frame 2, as depicted by Fig. [i]. We return to the rest frame by further performing the Lorentz transformation \( L^{-1}(\Lambda p)^\mu_\nu \) on the moving frame 2. However, the resultant rest frame is different from the original one by the Wigner rotation as indicated by Eq. (2.18).
Therefore, if the spin of the particle is $j$ as in Eq. (2.5), we find that
\[ U(\Lambda) |p, \sigma\rangle = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda, p)) |\Lambda p, \sigma'\rangle. \tag{2.20} \]

Note that the spin part is transformed in a manner depending on the momentum $p^\mu$, in general. This dependence plays a crucial role in later discussions on the EPR correlation. On the other hand, the transformation of the spin part under the spatial rotation is independent of $p^\mu$. In fact, when $\Lambda^\mu_\nu$ is a spatial rotation $R^\mu_\nu$, the Wigner rotation $W(R, p)^\mu_\nu$ reduces to $R^\mu_\nu$ itself for all $p^\mu$. That is, under spatial rotations, the state $|p, \sigma\rangle$ transforms in the same way as in non-relativistic quantum mechanics,
\[ U(R) |p, \sigma\rangle = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(R) |Rp, \sigma'\rangle. \tag{2.21} \]

Let us find the transformation law (2.20) in a specific example. Suppose that in a laboratory frame the massive particle is moving along the $x$-axis with four-momentum
\[ p^\mu = (Mc \cosh \xi, Mc \sinh \xi, 0, 0), \tag{2.22} \]
where the rapidity $\xi$ is related to the velocity of the particle $v = dx/dt$ by
\[ \frac{v}{c} = \tanh \xi. \tag{2.23} \]
The standard Lorentz transformation \((2.7)\) becomes

\[
L(p)_{\mu \nu} = B_x(\xi)_{\mu \nu},
\]

where \(B_x(\xi)_{\mu \nu}\) is defined by

\[
B_x(\xi)_{\mu \nu} = \begin{pmatrix}
\cosh \xi & \sinh \xi & 0 & 0 \\
\sinh \xi & \cosh \xi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(2.25)

We then introduce an observer whose velocity \(V\) is given in the laboratory frame by

\[
\frac{V}{c} = \tanh \chi.
\]

(2.26)

From the rotational symmetry about the \(x\)-axis, it suffices to assume that the observer is moving in the direction \((\cos \theta, 0, \sin \theta)\) on the \(xz\)-plane. The rest frame of the observer is obtained by performing a Lorentz transformation

\[
\Lambda_{\mu \nu} = \left[ R_y(\theta)B_x^{-1}(\chi)R_y^{-1}(\theta) \right]_{\mu \nu}
\]

(2.27)

on the laboratory frame. (Note that the rotation \(R_y^{-1}(\theta)_{\mu \nu}\) brings the direction \((\cos \theta, 0, \sin \theta)\) into the \(x\)-axis.) In this frame, the observer describes the state \(|p, \sigma\rangle\) as \(U(\Lambda)|p, \sigma\rangle\).

A straightforward calculation shows that the Wigner rotation \((2.18)\) reduces to a rotation about the \(y\)-axis,

\[
W(\Lambda, p)_{\mu \nu} = R_y^{-1}(\delta)_{\mu \nu},
\]

(2.28)

where

\[
\cos \delta = \frac{A - B \cos \theta + C \cos^2 \theta}{D - B \cos \theta},
\]

(2.29)

\[
\sin \delta = \frac{B \sin \theta - C \sin \theta \cos \theta}{D - B \cos \theta},
\]

(2.30)

with

\[
A = \cosh \xi + \cosh \chi,
\]

(2.31)

\[
B = \sinh \xi \sinh \chi,
\]

(2.32)

\[
C = (\cosh \xi - 1)(\cosh \chi - 1),
\]

(2.33)

\[
D = \cosh \xi \cosh \chi + 1.
\]

(2.34)
The transformation law (2.20) thus becomes

$$U(\Lambda) |p, \sigma\rangle = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma} (R^{-1}_y(\delta)) |\Lambda p, \sigma'\rangle.$$  \hspace{1cm} (2.35)

This means that the spin is rotated about the $y$-axis through the angle $\delta$ in the observer’s rest frame. In general, it is rotated within the plane spanned by the motion of the particle and that of the observer.

For the case of the spin-1/2 particle, we have

$$D^{(1/2)}_{\sigma'\sigma} (R^{-1}_y(\delta)) = \exp \left( -i \frac{\sigma_y}{2} \delta \right)$$

$$= \begin{pmatrix} \cos \frac{\delta}{2} - \sin \frac{\delta}{2} \\ \sin \frac{\delta}{2} \cos \frac{\delta}{2} \end{pmatrix},$$ \hspace{1cm} (2.36)

where $\sigma_y$ is the Pauli matrix. The transformation law for the spin-1/2 particle is thus given by

$$U(\Lambda) |p, \uparrow\rangle = \cos \frac{\delta}{2} |\Lambda p, \uparrow\rangle + \sin \frac{\delta}{2} |\Lambda p, \downarrow\rangle,$$ \hspace{1cm} (2.37)

$$U(\Lambda) |p, \downarrow\rangle = -\sin \frac{\delta}{2} |\Lambda p, \uparrow\rangle + \cos \frac{\delta}{2} |\Lambda p, \downarrow\rangle,$$ \hspace{1cm} (2.38)

where $\uparrow = +1/2$ and $\downarrow = -1/2$.

If the observer is moving in the same direction as the particle (i.e. $\theta = 0$), we find that the Wigner angle $\delta$ is zero, and therefore, the spin part is unchanged by this transformation. From now on, we concentrate on a situation in which the observer is moving in the $z$ direction (i.e. $\theta = \pi/2$), as illustrated in Fig. 2. In this case, the Wigner angle $\delta$ is given by (see Fig. 3)

$$\tan \delta = \frac{\sinh \xi \sinh \chi}{\cosh \xi + \cosh \chi}.$$ \hspace{1cm} (2.39)

When either $\xi = 0$ or $\chi = 0$, we find that $\delta$ is 0. That is, when either the particle or the observer is at rest, the direction of the spin is unchanged. In the limit of $\xi \to 0$ and $\chi \to 0$, the Wigner angle $\delta$ becomes $\xi \chi / 2$. On the other hand, in the opposite case where $\xi \to \infty$ and $\chi \to \infty$, the Wigner angle $\delta$ becomes $\pi / 2$. This is the maximal effect, where the spin, which in the laboratory frame points in the $z$ direction, tilts into the $x$ direction in the observer’s rest frame.
Figure 2: The orthogonal observer.

Figure 3: The Wigner angle $\delta$ for the orthogonal observer as a function of $V/c = \tanh \chi$ and $v/c = \tanh \xi$. 

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A physical picture of the Wigner rotation is as follows: The Lorentz transformation $\Lambda^{\mu}_{\nu}$, from the laboratory frame to the observer’s frame, rotates the direction of the momentum into

$$\vec{\Lambda}p = (Mc \sinh \xi, 0, -Mc \cosh \xi \sinh \chi),$$

(2.40)
as depicted in Fig. 2, where the angle of this rotation, $\delta_p$, is given by

$$\tan \delta_p = \frac{|(\Lambda p)_3|}{|(\Lambda p)_1|} = \frac{\sinh \chi}{\tanh \xi}. \quad (2.41)$$

The spin is dragged by this rotation since the spin is coupled to the momentum in relativistic quantum theory. Note that the angle of the spin rotation is less than or equal to that of the momentum rotation, $\delta \leq \delta_p$.

In non-relativistic quantum theory, the Galilean transformation rotates the momentum by the angle $\tan \delta_p = V/v$ but does not rotate the spin since the spin is not coupled to the momentum. In fact, it is easy to see that $\delta \to 0$ in the non-relativistic limit $c \to \infty$ since this limit is equivalent to the limit $\xi \to 0$ and $\chi \to 0$.

We can also understand the Wigner rotation in terms of the unitary operator. It is the difference between $\Lambda L(\vec{p})^{\mu}_{\nu}$ and $L(\Lambda \vec{p})^{\mu}_{\nu}$ that gives rise to the Wigner rotation (2.18), as illustrated in Fig. 1. These are not equal, even though both of them bring the momentum $k^{\mu}$ to $\Lambda \vec{p}^{\mu}$. In the limit of $\xi \to 0$ and $\chi \to 0$,

$$U(\Lambda)U(L(\vec{p})) = e^{-\frac{i}{\hbar}K^3 \chi} e^{\frac{i}{\hbar}K^1 \xi},$$

(2.42)

$$U(L(\Lambda \vec{p})) \simeq e^{\frac{i}{\hbar}(\varepsilon - K^3 \chi + K^1 \xi)},$$

(2.43)

where $\vec{K}$ is the boost operator (A.20). Using the Baker-Campbell-Hausdorff formula and the commutation relation $[K^1, K^3] = i\hbar J^2$, we obtain

$$U(\Lambda)U(L(\vec{p})) \simeq U(L(\Lambda \vec{p})) e^{-\frac{i}{\hbar}J^2 \frac{\chi}{2}},$$

(2.44)

This means that $U(\Lambda)|p,\sigma\rangle$ is different from $|\Lambda p,\sigma\rangle$ by the rotation of the spin about the $y$-axis, because

$$U(\Lambda)|p,\sigma\rangle = U(\Lambda)U(L(p)) |k,\sigma\rangle^{\text{rest}}$$

$$\simeq U(L(\Lambda \vec{p})) e^{-\frac{i}{\hbar}J^2 \frac{\chi}{2}} |k,\sigma\rangle^{\text{rest}},$$

(2.45)

but

$$|\Lambda p,\sigma\rangle = U(L(\Lambda \vec{p})) |k,\sigma\rangle^{\text{rest}}.$$

(2.46)
3 Massless Particle

We again follow Ref. [11] to discuss the one particle states for a massless particle and calculate the transformation law in a specific example. The main difference from the massive case is that massless particles cannot be at rest in any reference frame. It is convenient to consider a standard frame in which the four-momentum becomes a standard momentum

\[ k^\mu = (\kappa, 0, 0, \kappa), \quad (3.1) \]

with \( \kappa (> 0) \) held fixed. In this frame, the state \( |k, \sigma\rangle_{\text{std}} \) is specified in terms of the eigenvalues of the Hamiltonian \( H \), the momentum operator \( \vec{P} \), and the \( z \)-component of the total angular momentum operator \( \vec{J} \) as

\[
\begin{align*}
H|k, \sigma\rangle_{\text{std}} &= \kappa|k, \sigma\rangle_{\text{std}}, \\
P^1|k, \sigma\rangle_{\text{std}} &= P^2|k, \sigma\rangle_{\text{std}} = 0, \\
P^3|k, \sigma\rangle_{\text{std}} &= \kappa|k, \sigma\rangle_{\text{std}}, \\
J^3|k, \sigma\rangle_{\text{std}} &= \sigma\hbar|k, \sigma\rangle_{\text{std}}.
\end{align*}
\]

(3.2) \quad (3.3) \quad (3.4) \quad (3.5)

Note that \( \sigma \) means the helicity, that is, the component of angular momentum in the direction of motion.

The symmetry of this system is no longer the spatial rotation group \( SO(3) \) since the particle is moving in the \( z \) direction. The symmetry of the standard momentum \( k^\mu \) is \( ISO(2) \), whose general element \( T^\mu_\nu \) can be decomposed into

\[
T(\alpha, \beta, \gamma)^\mu_\nu = [S(\alpha, \beta)R_z(\gamma)]^\mu_\nu,
\]

(3.6)

where \( R_z(\gamma)^\mu_\nu \) is the rotation about the \( z \)-axis through the angle \( \gamma \) and

\[
S(\alpha, \beta)^\mu_\nu = \begin{pmatrix}
1 + \zeta & \alpha & \beta & -\zeta \\
\alpha & 1 & 0 & -\alpha \\
\beta & 0 & 1 & -\beta \\
\zeta & \alpha & \beta & 1 - \zeta
\end{pmatrix},
\]

(3.7)

with \( \zeta = (\alpha^2 + \beta^2)/2 \) being the “translation” by the vector \( (\alpha, \beta) \) in the sense that

\[
[S(\bar{\alpha}, \bar{\beta})S(\alpha, \beta)]^\mu_\nu = S(\bar{\alpha} + \alpha, \bar{\beta} + \beta)^\mu_\nu.
\]

(3.8)
The unitary operator corresponding to $T_{\mu \nu}$ is given by

$$ U(T(\alpha, \beta, \gamma)) = U(S(\alpha, \beta))U(R_z(\gamma)) = e^{i\alpha A + i\beta B}e^{iJ^3 \gamma}, \quad (3.9) $$

where $A = J^2 + K^1$ and $B = -J^1 + K^2$. However, in the case of the photon, the eigenvalues of $A$ and $B$ are both zero. We thus have

$$ U(T(\alpha, \beta, \gamma)) |k, \sigma\rangle_{\text{std}} = e^{i\gamma \sigma} |k, \sigma\rangle_{\text{std}}, \quad (3.10) $$

for the transformation $T_{\mu \nu}$ belonging to $ISO(2)$.

We next consider the general frame, where the four-momentum of the particle is given by

$$ p^\mu = (|\vec{p}|, p^1, p^2, p^3). \quad (3.11) $$

We obtain this momentum by performing a standard Lorentz transformation $L(p)^\mu_{\nu}$ on the standard momentum $(3.1)$,

$$ p^\mu = L(p)^\mu_{\nu} k^\nu, \quad (3.12) $$

where

$$ L(p)^\mu_{\nu} = \left[R_z^{-1}(\varphi)R_y^{-1}(\vartheta)B_z(\ln(|\vec{p}|/\kappa))\right]^\mu_{\nu}, \quad (3.13) $$

and

$$ \hat{p} = \vec{p}/|\vec{p}| = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \quad (3.14) $$

with $0 \leq \vartheta \leq \pi$ and $0 \leq \varphi < 2\pi$. Note that the rotation $[R_z^{-1}(\varphi)R_y^{-1}(\vartheta)]^\mu_{\nu}$ brings the $z$-axis into the direction of $\hat{p}$.

Using the unitary operator corresponding to $L(p)^\mu_{\nu}$, we define the state in the general frame as

$$ |p, \sigma\rangle \equiv U(L(p)) |k, \sigma\rangle_{\text{std}}. \quad (3.15) $$

Note again that

$$ H|p, \sigma\rangle = |\vec{p}|c|p, \sigma\rangle, \quad (3.16) $$

$$ \vec{P}|p, \sigma\rangle = \vec{p}|p, \sigma\rangle, \quad (3.17) $$

but that

$$ J^3|p, \sigma\rangle \neq \sigma \hbar |p, \sigma\rangle. \quad (3.18) $$
As in the case of a massive particle, \( \lvert p, \sigma \rangle \) cannot be a simultaneous eigenstate of \( J^3 \) and \( \vec{P} \) due to the motion on the \( xy \)-plane. The label \( \sigma \) denotes not the eigenvalue of \( J^3 \) but the helicity. In fact, since

\[
\left[ U(R_y(\vartheta)) U(R_z(\varphi)) \right] (\vec{J} \cdot \hat{p}) \left[ U(R_y(\vartheta)) U(R_z(\varphi)) \right]^{-1}
\]

is equal to \( J^3 \) and \([J^3, K^3] = 0\), we obtain

\[
\vec{J} \cdot \hat{p} \lvert p, \sigma \rangle = \sigma \hbar \lvert p, \sigma \rangle.
\]

(3.19)

We normalize the state by the invariant delta function \((2.14)\) with \( M = 0 \),

\[
\langle p', \sigma' \lvert p, \sigma \rangle = \left( \frac{|\vec{p}|}{\kappa} \right) \delta^3(\vec{p}' - \vec{p}) \delta_{\sigma' \sigma}.
\]

(3.20)

The state defined in Eq. \((3.15)\) transforms under the Lorentz transformation \( \Lambda_{\mu \nu} \) as

\[
U(\Lambda) \lvert p, \sigma \rangle = U(L(\Lambda p)) U(W(\Lambda, p)) \lvert k, \sigma \rangle^{\text{std}},
\]

(3.21)

where

\[
W(\Lambda, p)^{\mu \nu}_\nu = \left[ L^{-1}(\Lambda p) \Lambda L(p) \right]^{\mu \nu}_\nu.
\]

(3.22)

By the definition \((3.12)\), this transformation does not change the standard momentum \((3.1)\),

\[
W(\Lambda, p)^{\mu}_\nu k^\nu = k^\mu.
\]

(3.23)

That is, \( W(\Lambda, p)^{\mu}_\nu \) is an element of \( ISO(2) \). Therefore, from Eq. \((3.10)\), we find that

\[
U(\Lambda) \lvert p, \sigma \rangle = e^{i\gamma(\Lambda, p) \sigma} \lvert \Lambda p, \sigma \rangle,
\]

(3.24)

where \( \gamma(\Lambda, p) \) is given from the decomposition \((3.9)\) of \( W(\Lambda, p)^{\mu}_\nu \). Note that the helicity is Lorentz invariant since the Lorentz transformation changes only the phase factor of the state.

In the case of the photon, the helicity \( \sigma \) is either +1 or −1. The states \( \lvert p, \pm 1 \rangle \) are circularly polarized states. The transformations of these states are trivial except for the helicity-dependent phase factor. However, in linearly polarized bases,

\[
\lvert p; \zeta \rangle_\pm \equiv \frac{1}{\sqrt{2}} \left[ e^{i\zeta} \lvert p, +1 \rangle \pm e^{-i\zeta} \lvert p, -1 \rangle \right],
\]

(3.25)
the transformation law becomes

\[ U(\Lambda) |p; \zeta\rangle_\pm = |\Lambda p; \zeta + \gamma(\Lambda, p)\rangle_\pm. \] (3.26)

This means that the plane of polarization is rotated by the angle \( \gamma(\Lambda, p) \) due to the Lorentz transformation.

Here we explicitly calculate the transformation law in a specific example. Suppose that in the laboratory frame the particle is moving along the \( x \)-axis with a four-momentum,

\[ p^\mu_\pm = (\kappa e^\xi, \pm \kappa e^\xi, 0, 0). \] (3.27)

Note that we cannot obtain the particle moving in the opposite direction by \( \xi \to -\xi \) unlike the massive particle. This is because massless particles move at the speed of light and thus their motion cannot be reversed. The standard Lorentz transformation (3.12) becomes

\[ L(p_+)^\mu_\nu = \left[ R_y^{-1}(\pi/2)B_z(\xi) \right]^\mu_\nu, \] \( (3.28) \)

for \( p_+ \) and

\[ L(p_-)^\mu_\nu = \left[ R_x^{-1}(\phi)R_y(\theta)B_z(\xi) \right]^\mu_\nu, \] \( (3.29) \)

for \( p_- \). Note the range of the angles below Eq. (3.14). We then introduce an observer moving in the general direction \((\cos \theta, \sin \theta \sin \phi, \sin \theta \cos \phi)\) with the velocity \( V = c \tanh \chi \). The corresponding Lorentz transformation is given by

\[ \Lambda^\mu_\nu = \left[ R_x(\phi)R_y(\theta)B_z^{-1}(\chi)R_y^{-1}(\phi)B_x^{-1}(\theta) \right]^\mu_\nu. \] (3.30)

(Note that the rotation \( [R_y^{-1}(\theta)R_x^{-1}(\phi)]^\mu_\nu \) brings the direction of the observer’s motion into the \( x \)-axis.) The state \( |p, \sigma\rangle \) in the laboratory frame is described by \( U(\Lambda)|p, \sigma\rangle \) in the observer’s rest frame.

A straightforward calculation shows that

\[ W(\Lambda, p_\pm)^\mu_\nu = \left( \begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & -\sin \varepsilon_\pm & \cos \varepsilon_\pm & * \\ * & * & * & * \end{array} \right), \] (3.31)
where the *’s denote unimportant components, and

\[
\begin{align*}
\cos \varepsilon_\pm &= \frac{D_\pm}{\sqrt{D_\pm^2 + E_\pm^2}}, \\
\sin \varepsilon_\pm &= \frac{E_\pm}{\sqrt{D_\pm^2 + E_\pm^2}},
\end{align*}
\]

with

\[
\begin{align*}
D_\pm &= 1 \mp \sinh \chi \cos \theta \\
&\quad + (\cosh \chi - 1)(\cos^2 \theta + \sin^2 \theta \sin^2 \phi), \\
E_\pm &= \pm (\cosh \chi - 1) \sin^2 \theta \sin \phi \cos \phi.
\end{align*}
\]

Then, it is easy to see that \(\gamma(\Lambda, p_\pm) = \varepsilon_\pm\) via the decomposition (3.6). The transformation law (3.24) thus becomes

\[
U(\Lambda) |p_\pm, \sigma\rangle = e^{i\varepsilon_\pm \sigma} |\Lambda p_\pm, \sigma\rangle.
\]

In terms of the linearly polarized bases (3.25), this gives

\[
U(\Lambda) |p_\pm; \zeta\rangle_\pm = |\Lambda p_\pm; \zeta + \varepsilon_\pm\rangle_\pm.
\]

Note that \(\varepsilon_\pm\) depends neither on the standard momentum \(\kappa\) nor the momentum of the particle characterized by parameter \(\xi\). On the other hand, the rotational symmetry about the \(x\)-axis is apparently broken since \(\varepsilon_\pm\) depends on \(\phi\). This is because the \(z\)-axis is a special axis that labels the states in the massless case. A different choice of the \(z\)-axis leads to a redefinition of the phase factor of the state.

If the observer is moving in the same direction as the particle (i.e. \(\theta = 0\)), the angles \(\varepsilon_\pm\) are zero. If the observer is moving in the \(z\) direction (i.e. \(\theta = \pi/2\) and \(\phi = 0\)), we again find that \(\varepsilon_\pm\) are zero, unlike the massive case. This is a consequence of the fact that the state is labeled in such a manner that the \(z\)-axis is a special axis. Actually, in a situation in which the observer is moving in the direction \((0, \sin \phi, \cos \phi)\) on the \(yz\)-plane, \(\varepsilon_\pm\) take non-trivial values given by

\[
\tan \varepsilon_\pm \equiv \pm \tan \varepsilon = \pm \frac{(\cosh \chi - 1) \sin \phi \cos \phi}{1 + (\cosh \chi - 1) \sin^2 \phi}.
\]

The value of \(\varepsilon\) as a function of \(\chi\) and \(\phi\) is shown in Fig. 4. If \(\chi = 0\) or \(\phi = 0\),

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Figure 4: The rotation angle $\epsilon$ of the plane of polarization for the orthogonal observer as a function of $V/c = \tanh \chi$ and $\phi$.

$\pi/2$, $\pi$, $3\pi/2$, $\epsilon$ is zero. When $\chi \to \infty$, we obtain

$$
\epsilon = \begin{cases} 
\frac{\pi}{2} - \phi & (0 < \phi < \pi) \\
\frac{3\pi}{2} - \phi & (\pi < \phi < 2\pi)
\end{cases}.
$$

(3.39)

In this limit, there are discontinuities at $\phi = 0, \pi$ due to the singular behavior,

$$
\lim_{\phi \to 0, \pi} \lim_{\chi \to \infty} \epsilon \neq \lim_{\chi \to \infty} \lim_{\phi \to 0, \pi} \epsilon.
$$

(3.40)

For a given $\chi$, the maximum of $\epsilon$ is attained at

$$
\phi = \frac{1}{2} \arccos \left( \tanh^2 \frac{\chi}{2} \right).
$$

(3.41)

In terms of the unitary operator, we can understand the rotation of the plane of polarization as follows: When $\theta = \pi/2$ and $\chi$ is small, the Lorentz transformation $\Lambda_{\mu \nu}$ becomes

$$
U(\Lambda) = e^{-\frac{i}{\hbar} (K^2 \sin \phi + K^3 \cos \phi) \chi}
\approx e^{-\frac{i}{\hbar} K^3 \chi \sin \phi} e^{\frac{i}{\hbar} K^2 \chi \sin \phi} e^{\frac{i}{\hbar} \frac{1}{4} J^1 \chi^2 \sin 2\phi}.
$$

(3.42)

The last factor rotates the momentum $p^\mu$ pointing in the $x$ direction through an angle $\chi^2 \sin 2\phi/4$ about the $x$-axis and gives rise to the phase factor. The other factors contribute only to $U(S(\alpha, \beta))$, which is irrelevant in the case of the photon.
4 EPR Correlation

In non-relativistic quantum mechanics, the EPR correlation is often discussed using the EPR state (or one of the Bell states),

\[ |\phi\rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle \right], \tag{4.1} \]

which is the spin-singlet state of two spin-1/2 particles. This state is also referred to as an entangled state of two particles. Moreover, this state has an important property of the perfect anti-correlation in any direction of space. The eigenstates of spin in the direction \( \hat{n} \), denoted by \( |\uparrow\{\hat{n}\}\rangle \) and \( |\downarrow\{\hat{n}\}\rangle \), are connected with \( |\uparrow\rangle \) and \( |\downarrow\rangle \) by

\[
|\uparrow\rangle = \alpha |\uparrow\{\hat{n}\}\rangle + \beta |\downarrow\{\hat{n}\}\rangle, \tag{4.2} \\
|\downarrow\rangle = \gamma |\uparrow\{\hat{n}\}\rangle + \delta |\downarrow\{\hat{n}\}\rangle, \tag{4.3} 
\]

where \( \alpha, \beta, \gamma, \) and \( \delta \) are the components of a unitary matrix. Since \( \alpha \delta - \beta \gamma \) is the determinant of the unitary matrix which is equal to unity except for an overall phase factor, the state \( |\phi\rangle \) can be written as

\[
|\phi\rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow\{\hat{n}\}\rangle|\downarrow\{\hat{n}\}\rangle - |\downarrow\{\hat{n}\}\rangle|\uparrow\{\hat{n}\}\rangle \right]. \tag{4.4} 
\]

This means that, if the spins of both particles are measured in the same direction \( \hat{n} \), the results are always anti-correlated, irrespective of the choice of \( \hat{n} \). This property comes from the fact that the spin-singlet state is isotropic and has no preferred direction.

We note that other Bell states do not possess such a property. For example, a spin-triplet state

\[
|\tilde{\phi}\rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle \right] \tag{4.5}
\]

is also an entangled state. If the spins of both particles are measured in the \( z \) direction, the results are always correlated (rather than anti-correlated). However, in terms of the eigenstates of the \( y \)-component, the state \( |\tilde{\phi}\rangle \) is rewritten as

\[
|\tilde{\phi}\rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow\{y\}\rangle|\downarrow\{y\}\rangle + |\downarrow\{y\}\rangle|\uparrow\{y\}\rangle \right]. \tag{4.6} 
\]
This means that the measurements of the \( y \)-component are always anti-correlated. Moreover, the measurements of the spin in the direction \( \hat{m} = (0, 1/\sqrt{2}, 1/\sqrt{2}) \) are completely random because

\[
|\tilde{\phi}\rangle = \frac{1}{2} \left[ i |\uparrow \{\hat{m}\}\rangle |\downarrow \{\hat{m}\}\rangle + i |\downarrow \{\hat{m}\}\rangle |\uparrow \{\hat{m}\}\rangle + |\uparrow \{\hat{m}\}\rangle |\uparrow \{\hat{m}\}\rangle + |\downarrow \{\hat{m}\}\rangle |\downarrow \{\hat{m}\}\rangle \right].
\]

(4.7)

Although the entanglement is independent of the basis, the correlation depends on the basis for the measurement.

The spin-singlet state is invariant under the spatial rotation. However, it is not invariant under the Lorentz transformation. This is because the Poincaré group \( ISO(1,3) \) is larger than the spatial rotation group \( SO(3) \). Thus, in some moving frames, the spin-singlet state is mixed with the spin-triplet states by the Lorentz transformation, losing the property of anti-correlation.

### 4.1 Massive Case

We now formulate a relativistic EPR experiment for massive particles. Suppose that a pair of spin-1/2 particles with total spin zero are moving away from each other along the \( x \)-axis, as illustrated in Fig. 5. This system is described in the laboratory frame by a relativistic EPR state

\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |p_+\rangle |p_-\rangle + |p_+\rangle |p_-\rangle \right],
\]

(4.8)

where

\[
p_\mu = (Mc \cosh \xi, \pm Mc \sinh \xi, 0, 0).
\]

(4.9)
Figure 6: The relativistic EPR experiment in the observers’ frame.

Note here that we explicitly specify the momentum of each particle since the Wigner rotation depends on it as in Eq. (2.20). We then assume that two observers who perform the measurements of the spins are both moving in the $z$ direction at the same velocity (2.26) with respect to the laboratory frame. (See Fig. 5.)

In a frame in which the two observers are at rest, the observers describe the relativistic EPR state (4.8) as $U(\Lambda)|\psi\rangle$. Using the transformation formulas (2.37) and (2.38), we find that

\[
U(\Lambda)|\psi\rangle = \frac{1}{\sqrt{2}} \left[ \cos \delta \left( |\Lambda p_+\uparrow\rangle|\Lambda p_-\downarrow\rangle - |\Lambda p_+\downarrow\rangle|\Lambda p_-\uparrow\rangle \right) \\
+ \sin \delta \left( |\Lambda p_+\uparrow\rangle|\Lambda p_-\uparrow\rangle + |\Lambda p_+\downarrow\rangle|\Lambda p_-\downarrow\rangle \right) \right],
\]

(4.10)

where $\delta$ is given by Eq. (2.39). Note that the spin of the second particle is rotated through the angle $-\delta$ since it is moving in the negative $x$-direction. The same unitary transformation thus acts on each spin in different ways. The situation is depicted in Fig. 6.

From Eq. (4.10), we can see that the spin-singlet state is mixed with the spin-triplet state which has the same form as Eq. (4.5). Thus, in the observers’ frame, even if the spins are measured in the $z$ direction, the results are not always anti-correlated. In the extreme case of $\xi \to \infty$ and $\chi \to \infty$, they are perfectly correlated rather than anti-correlated. We here emphasize that the $z$ direction in the observers’ frame is also the $z$ direction in the laboratory frame, since the observers are moving in the $z$ direction. More generally, the directions that are parallel in the laboratory frame re-
main parallel in the observers’ frame. Nevertheless, the results of the spin measurements in the same direction are not always anti-correlated in the observers’ frame, unlike in the non-relativistic case. (The measurements of the spin $y$-component remain anti-correlated for any $\xi$ and $\chi$, as Eq. (4.6).) We thus conclude that the perfect anti-correlation in an arbitrary direction is not maintained in all inertial frames.

One might think that this result contradicts the nonlocality that is regarded as an inherent property of quantum mechanics. For example, after the first observer have measured the spin of the first particle of an EPR pair in the $z$ direction and obtained a result $\uparrow$, the state of the second particle becomes
\[
\cos \delta |\Lambda p_-, \downarrow\rangle + \sin \delta |\Lambda p_-, \uparrow\rangle. \tag{4.11}
\]
In this state, the spin $z$-component does not have a definite value. One might consider that this contradicts the EPR argument [17]. However, the spin component in the direction $(-\sin 2\delta, 0, \cos 2\delta)$ have the definite value $\downarrow$. The above contradiction is therefore superficial and the perfect EPR correlation is maintained in different directions. Note that, unlike the correlation that we discuss here, the entanglement is invariant under the Lorentz transformation because the Wigner rotation is a local unitary transformation [14].

4.2 Massless Case

We next formulate a relativistic EPR experiment for massless particles with helicity $\pm 1$. Since the Lorentz transformation of the circularly polarized states $|p, \pm 1\rangle$ are trivial as in Eq. (3.24), one might think that there is no effect due to the Lorentz transformation. However, in the linearly polarized basis (3.23), the plane of polarization is rotated by the Lorentz transformation. We thus consider a relativistic EPR state
\[
|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |p_+; \zeta\rangle_+ |p_-; \zeta\rangle_- - |p_+; \zeta\rangle_- |p_-; \zeta\rangle_+ \right], \tag{4.12}
\]
where $p_\pm$ is defined by Eq. (3.27). Note that the measurements of the polarization with respect to the same angle are always anti-correlated, irrespective of the angle. Both of the two observers are assumed to move in the direction $(0, \sin \phi, \cos \phi)$ on the $yz$-plane at the velocity $V = c \tanh \chi$.

In the common rest frame of the observers, the relativistic EPR state
(4.12) is seen as \( U(\Lambda)|\psi\rangle \). Using the transformation law (3.37), we find that
\[
U(\Lambda)|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |\Lambda p_+; \zeta + \varepsilon\rangle_+ |\Lambda p_-; \zeta - \varepsilon\rangle_- 
- |\Lambda p_+; \zeta + \varepsilon\rangle_- |\Lambda p_-; \zeta - \varepsilon\rangle_+ \right],
\] (4.13)
where \( \varepsilon \) is given by Eq. (3.38). Thus, the measurements of the polarization with respect to the same angle are not always anti-correlated in this frame.

## 5 Bell’s Inequality

Suppose that one particle is performed a measurement either \( Q \) or \( R \), and the other particle is performed a measurement either \( S \) or \( T \). Each measurement produces an outcome +1 or −1. Then, any local realistic theory predicts that the following Bell’s inequality holds [18, 19, 20]:
\[
E(QS) + E(RS) + E(RT) - E(QT) \leq 2,
\] (5.1)
where \( E(QS) \) denotes the expectation value of \( QS \), etc. However, in quantum theory, this inequality may be violated. For example, if we choose the observables as
\[
Q = \sigma_z, \\
R = \sigma_y, \\
S = \frac{-\sigma_y - \sigma_z}{\sqrt{2}}, \\
T = \frac{-\sigma_y + \sigma_z}{\sqrt{2}},
\] (5.2)
where \( \sigma_y \) and \( \sigma_z \) are the Pauli matrices, then for the spin-singlet state (4.1), we obtain
\[
\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}.
\] (5.3)
This is the case of the maximum violation of Bell’s inequality due to the perfect anti-correlation of the spin-singlet state.

In the preceding section, we have shown that the property of the anti-correlation is not maintained if the measurements are performed by the moving observers. Thus, it is interesting to see its influence on the violation of this Bell’s inequality. (By introducing moving observers, Bell-type theorems for Lorentz-invariant realistic theories have been discussed in Refs. [21, 22]. However, they have no use for the Lorentz transformation of the state.)

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5.1 Massive Case

To read out the spin component, we first define spin operators

\[
\sigma_x(p) = \frac{1}{C(p)} \left[ \left| p, \uparrow \right> \left< p, \downarrow \right| + \left| p, \downarrow \right> \left< p, \uparrow \right| \right],
\]
(5.4)

\[
\sigma_y(p) = \frac{1}{C(p)} \left[ -i \left| p, \uparrow \right> \left< p, \downarrow \right| + i \left| p, \downarrow \right> \left< p, \uparrow \right| \right],
\]
(5.5)

\[
\sigma_z(p) = \frac{1}{C(p)} \left[ \left| p, \uparrow \right> \left< p, \uparrow \right| - \left| p, \downarrow \right> \left< p, \downarrow \right| \right],
\]
(5.6)

where \( C(p) \) is a normalization factor,

\[
C(p) = \langle p, \uparrow | p, \uparrow \rangle = \langle p, \downarrow | p, \downarrow \rangle.
\]
(5.7)

(Note that \( C(p) \) would diverge to infinity. However, by using a wave packet rather than the momentum eigenstate, the divergence can be removed, in practice.) These operators behave as the Pauli matrices for the states \( |p, \uparrow \rangle \) and \( |p, \downarrow \rangle \). For example,

\[
\sigma_z(p) |p, \uparrow \rangle = |p, \uparrow \rangle,
\]
\[
\sigma_z(p) |p, \downarrow \rangle = -|p, \downarrow \rangle,
\]
\[
\sigma_x(p) |p, \uparrow \rangle = |p, \downarrow \rangle.
\]

We also define the expectation value of an operator \( O \) as

\[
\langle O \rangle \equiv\frac{\langle \psi | O | \psi \rangle}{\langle \psi | \psi \rangle}.
\]
(5.8)

We consider the same situation as in Fig. 6 and choose the observables to be measured as

\[
Q = \sigma_z(\Lambda p_+),
\]

\[
R = \sigma_y(\Lambda p_+),
\]

\[
S = -\frac{\sigma_y(\Lambda p_-) - \sigma_z(\Lambda p_-)}{\sqrt{2}},
\]

\[
T = -\frac{\sigma_y(\Lambda p_-) + \sigma_z(\Lambda p_-)}{\sqrt{2}},
\]
(5.9)

where \( Q \) and \( R \) are the spin \( z- \) and \( y- \) components of the first particle, and \( S \) and \( T \) are the spin components in the directions \((0, -1/\sqrt{2}, -1/\sqrt{2})\) and
(0, −1/√2, 1/√2) of the second particle. This set of the observables gives rise to the maximum violation of Bell’s inequality in non-relativistic quantum theory. We, however, obtain in relativistic quantum theory,

\[ \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2} \cos^2 \delta, \]

(5.10)

for the relativistic EPR state (4.10) in the moving observers’ frame. If either the particle or the observer is at rest, the relativistic EPR state violates Bell’s inequality (5.1). The amount of violation *prima facie* decreases as their velocities increase [23]. In the extreme case of \( \xi \to \infty \) and \( \chi \to \infty \), the right-hand side of Eq. (5.10) becomes zero.

One might think that this means a restoration of local realistic theories. However, this conclusion is not correct since different sets of observables or different states still violate Bell’s inequality. For example, if we rotate the directions of the measurements in accordance with the Wigner rotation by the replacement

\[ \sigma_z(\Lambda p_+) \to \sigma_z(\Lambda p_+) \cos \delta + \sigma_x(\Lambda p_+) \sin \delta \]

(5.11)

for the first particle and by the replacement

\[ \sigma_z(\Lambda p_-) \to \sigma_z(\Lambda p_-) \cos \delta - \sigma_x(\Lambda p_-) \sin \delta \]

(5.12)

for the second particle, Bell’s inequality turns out to be maximally violated,

\[ \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}. \]

(5.13)

Instead, if we can use the spin-singlet state in the observers’ frame (rather than the laboratory frame),

\[ \frac{1}{\sqrt{2}} \left[ |\Lambda p_+, \uparrow \rangle |\Lambda p_-, \downarrow \rangle - |\Lambda p_+, \downarrow \rangle |\Lambda p_-, \uparrow \rangle \right], \]

(5.14)

Bell’s inequality is again maximally violated. An apparent decrease in the degree of violation of Bell’s inequality results from the fact that the Lorentz transformation rotates the directions of the spins in a different manner; however, since the rotations are local transformations, they preserve the perfect anti-correlation in appropriately chosen different direction.
5.2 Massless Case

We define the polarization operator with respect to the angle $\zeta$,

$$ P(p, \zeta) = \frac{1}{C(p)} \left[ e^{-2i\zeta} |p, -\rangle \langle p, +| + e^{2i\zeta} |p, +\rangle \langle p, -| \right], \quad (5.15) $$

where $C(p)$ is again the normalization factor,

$$ C(p) = \langle p, +| p, + \rangle = \langle p, -| p, - \rangle. \quad (5.16) $$

This operator reads out the polarization with respect to the angle $\zeta$,

$$ P(p, \zeta) |p; \zeta\rangle_\pm = \pm |p; \zeta\rangle_\pm. \quad (5.17) $$

We then choose the observables to be measured as

$$ Q = P(\Lambda p_+, 0), $$
$$ R = P\left(\Lambda p_+, \frac{\pi}{4}\right), $$
$$ S = P\left(\Lambda p_-, -\frac{3\pi}{8}\right), $$
$$ T = P\left(\Lambda p_-, -\frac{\pi}{8}\right), \quad (5.18) $$

under the same situation as in the preceding section. The maximal violation of the inequality is achieved by this set of the observables in non-relativistic quantum theory. However, in relativistic quantum theory, we obtain

$$ \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2} \cos 4\epsilon, \quad (5.19) $$

for the relativistic EPR state (4.13) in the observers’ frame. The right-hand side vanishes if $\epsilon = \pm \pi/8$ or $\pm 3\pi/8$. This is achieved for $\phi = (2n + 1)\pi/8$ ($n = 0, 1, \cdots, 7$) in the limit of $\chi \to \infty$ as can be seen from Eq. (3.39). The right-hand side can also be negative, unlike in the massive case.

Instead, if we choose the observables in accordance with the rotation of the plane of polarization,

$$ Q = P(\Lambda p_+, \epsilon), $$
$$ R = P\left(\Lambda p_+, \frac{\pi}{4} + \epsilon\right), $$
$$ S = P\left(\Lambda p_-, -\frac{3\pi}{8} - \epsilon\right), $$
$$ T = P\left(\Lambda p_-, -\frac{\pi}{8} - \epsilon\right), \quad (5.20) $$
or if we use the state

$$\frac{1}{\sqrt{2}} \left[ |\Lambda p_+; \zeta_+ \rangle |\Lambda p_-; \zeta_- \rangle - |\Lambda p_+; \zeta_- \rangle |\Lambda p_-; \zeta_+ \rangle \right],$$

(5.21)

Bell’s inequality is maximally violated.

6 Summary and Discussion

In the present paper, we have formulated the EPR gedankenexperiment within relativistic quantum theory. Using the Poincaré group and its representation, we have shown that the perfect anti-correlation in the same direction deteriorates if the measurements are performed by moving observers. The perfect anti-correlation in all the directions cannot be maintained in all inertial frames. Note that this effect is not generated by the Galilean transformation in non-relativistic quantum theory. The spin-singlet state in the laboratory frame is mixed with the spin-triplet state in the observers’ frame, because the Poincaré group ISO(1, 3) is larger than the spatial rotation group SO(3). Moreover, the degree of the violation of Bell’s inequality prima facie decreases with increasing the velocity of the observers. To observe the perfect anti-correlation and the maximal violation of Bell’s inequality, we have to perform the measurements in appropriately chosen different directions.

We now discuss a relevance to quantum information. The entangled state are often utilized in quantum communication, such as quantum teleportation [24] and quantum cryptography [25, 26]. Since the entanglement is preserved under the Lorentz transformation [14], the entangled state can be utilized even by moving observers. Nevertheless, this does not mean that the quantum communication succeeds without conditions. We have to rotate the direction of the measurement depending on the relative motion between the EPR source and the observers in order to utilize the perfect anti-correlation and the maximal violation of Bell’s inequality. If the observers unfortunately do not know the relative motion, the accuracy of the quantum communication is limited. For example, suppose that one observer wants to teleport an unknown state to the other observer by quantum teleportation using the EPR state provided, a posteriori, by the EPR source. We further assume that they do not know their velocities each other. In non-relativistic quantum theory, the teleportation will perfectly succeed even in such a situation. On the other hand, in relativistic quantum theory, the fidelity of the teleportation is reduced due to the interference by the Wigner rotation. Although
this interference may be small in practice, it is interesting to recognize in principle what kind of ignorance fundamentally limits the accuracy of the quantum communication in various situations.

We finally explore an experimental possibility to detect the above effect. While it seems not feasible to move detectors (such as the Stern-Gerlach apparatus) at relativistic velocities, we may move the EPR source. By special relativity, moving detectors are equivalent to a moving EPR source (see Fig. 3). It would be feasible to move the EPR source at relativistic velocities by the accelerator or the cosmic ray. For example, we may utilize a spinless particle accelerated to a relativistic velocity by the accelerator, which decays into a pair of spin-1/2 particles moving in opposite directions in the rest frame of the original particle. The velocity $V$ corresponds to that of the original particle in the laboratory frame and the velocity $v$ corresponds to that of the resultant particles in the original particle’s frame, caused by the decay process. Using rest detectors, we could see a reduction in the anticorrelation in the same direction. The locations of the detectors are decided by the angle of the momentum rotation (3.11). While there are many technological challenges, we should be able to observe the relativistic effect by this kind of experiment.

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Appendix

A Poincaré Group

We summarize here the Poincaré group as it serves as a basic mathematical tool. The Poincaré group is the symmetry of the Minkowski spacetime which is a set of coordinate transformations that leave the world length,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2,$$  \hspace{1cm} (A.1)
invariant. This set includes translations, rotations, and boosts (or Lorentz transformations). For notational simplicity, we adopt the following conventions:

- The spacetime coordinates \((ct, x, y, z)\) are denoted as \((x^0, x^1, x^2, x^3)\).
- Greek letters \(\mu, \nu, \cdots\) represent the spacetime indices 0, 1, 2, 3.
- Latin letters \(i, j, k, \cdots\) represent the spatial indices 1, 2, 3.
- 3-dimensional vectors \((a^1, a^2, a^3)\) are denoted as \(\vec{a}\).
- 4-dimensional contravariant vectors \((a^0, a^1, a^2, a^3)\) are denoted as \(a^\mu\).
- 4-dimensional covariant vectors \((-a^0, a^1, a^2, a^3)\) are denoted as \(a_\mu\).
- Repeated indices are to be summed.

The temporal and spatial translations are written together as

\[
x^\mu \rightarrow x'^\mu = x^\mu + a^\mu,
\]

where \(a^\mu\) is a constant vector.

The spatial rotation belongs to the \(SO(3)\) subgroup of the Poincaré group. The rotation about a unit vector \(\hat{n}\) through an angle \(\theta\) is written as

\[
x^\mu \rightarrow x'^\mu = R_{\hat{n}}(\theta)_{\mu}^{\nu} x^\nu,
\]

where \(R_{\hat{n}}(\theta)_{\mu}^{\nu}\) is an orthogonal matrix that acts non-trivially only on the spatial coordinates. For example, the rotations about the \(z\)-axis and about the \(y\)-axis are described by

\[
R_z(\theta)_{\mu}^{\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

and

\[
R_y(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & -\sin \theta \\
0 & 0 & 1 & 0 \\
0 & \sin \theta & 0 & \cos \theta
\end{pmatrix},
\]
The boost is the Lorentz transformation in a narrow sense. The boost in the direction of a unit vector \( \hat{n} \) with the rapidity \( \xi \) is written as

\[
x^\mu \rightarrow x'^\mu = B_{\hat{n}}(\xi)^\mu_\nu x^\nu,
\]

where \( B_{\hat{n}}(\xi)^\mu_\nu \) is a pseudo-orthogonal matrix, in the sense that

\[
[B_{\hat{n}}(\xi)]^T \eta B_{\hat{n}}(\xi) = \eta, \quad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

and acts on both temporal and spatial components. For example, the boost along the \( x \)-axis is described by

\[
B_x(\xi)^\mu_\nu = \begin{pmatrix} \cosh \xi & \sinh \xi & 0 & 0 \\ \sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Note that, if the rapidity \( \xi \) is set as

\[
\tanh \xi = -\frac{v}{c},
\]

\( B_x(\xi)^\mu_\nu \) in Eq. (A.8) reduces to the well-known formula for the Lorentz transformation,

\[
\begin{pmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & -\frac{v/c}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Similarly, the boost along the \( z \)-axis is described by

\[
B_z(\xi)^\mu_\nu = \begin{pmatrix} \cosh \xi & 0 & 0 & \sinh \xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \xi & 0 & 0 & \cosh \xi \end{pmatrix}.
\]

The set of all boosts does not constitutes a subgroup of the Poincaré group because it is not closed.
The Lorentz transformation in a wider sense, which we shall denote as \( \Lambda_{\mu\nu} \), describes a combination of rotation and boost. It consists of all the matrices that satisfy the pseudo-orthogonal condition,

\[
\Lambda^T \eta \Lambda = \eta.
\]  

(A.12)

The set of all such transformations constitutes a subgroup of the Poincaré group, known as the Lorentz group \( SO(1,3) \).

In quantum theory, the symmetry transformation is represented by a unitary operator which acts on the Hilbert space. Since relativistic quantum theory means quantum theory with the Poincaré symmetry, we have unitary operators corresponding to all the Poincaré transformations.

The unitary operator corresponding to the translation \( a^\mu = (a^0, \vec{a}) \) is given by

\[
U(a) = \exp \left[ \frac{i}{\hbar} \left( \frac{Ha^0}{c} - \vec{P} \cdot \vec{a} \right) \right],
\]

(A.13)

where \( H \) and \( \vec{P} \) are the Hamiltonian and momentum operators which are combined to form a four-momentum operator, \( \mathcal{P}^\mu = (H/c, \vec{P}) \). Equation (A.13) can thus be written as

\[
U(a) = \exp \left( -\frac{i}{\hbar} \mathcal{P}^\mu a_\mu \right).
\]

(A.14)

For a particle with mass \( M \) and velocity \( \vec{v} = d\vec{x}/dt \), the four-momentum is defined by

\[
p^\mu = \begin{pmatrix} \frac{Mc}{\sqrt{1 - |\vec{v}/c|^2}} & M\vec{v} \\ \sqrt{1 - |\vec{v}/c|^2} & \sqrt{1 - |\vec{v}/c|^2} \end{pmatrix}.
\]

(A.15)

Since this four-momentum satisfies the basic relation in special relativity,

\[
p_\mu p^\mu = -(p^0)^2 + |\vec{p}|^2 = -M^2 c^2,
\]

(A.16)

the Hamiltonian and momentum operators must satisfy

\[
\mathcal{P}_\mu \mathcal{P}^\mu = -\frac{H^2}{c^2} + |\vec{P}|^2 = -M^2 c^2.
\]

(A.17)

On the other hand, for a massless particle with momentum \( \vec{p} \) (and energy \( E = |\vec{p}|c \)), the four-momentum is defined by

\[
p^\mu = (|\vec{p}|, \vec{p}).
\]

(A.18)
Thus, the Hamiltonian and momentum operators satisfy Eq. (A.17) with $M = 0$.

The unitary operators corresponding to the rotation $R_n(\theta)_{\mu\nu}$ and the boost $B_n(\xi)_{\mu\nu}$ are given by

$$U(R_n(\theta)) = \exp \left( \frac{i}{\hbar} \vec{J} \cdot \hat{n} \theta \right), \quad (A.19)$$

$$U(B_n(\xi)) = \exp \left( \frac{i}{\hbar} \vec{K} \cdot \hat{n} \xi \right), \quad (A.20)$$

where $\vec{J}$ is the total angular momentum operator and $\vec{K}$ is the boost operator. (In classical mechanics, these correspond to $x^i p^j - x^j p^i$ and $x^i p^0 - x^0 p^i$, respectively.) The operators $\vec{J}$ and $\vec{K}$ are combined to form an antisymmetric tensor of rank 2, $J^{\mu\nu} (= -J^{\nu\mu})$, where

$$(J^{23}, J^{31}, J^{12}) = \vec{J},$$

$$(J^{10}, J^{20}, J^{30}) = \vec{K}.$$ 

Since the explicit forms of these operators are unnecessary for our present purpose, we simply view Eqs. (A.19) and (A.20) as the definitions of $\vec{J}$ and $\vec{K}$.

In order for these unitary operators to represent the Poincaré group, the generators $H$, $\vec{P}$, $\vec{J}$, and $\vec{K}$ must satisfy the commutation relations (11),

$$[J^i, J^j] = i\hbar \epsilon_{ijk} J^k,$$

$$[\vec{J}, \vec{K}] = i\hbar \epsilon_{ijk} K^k,$$

$$[K^i, J^j] = -i\hbar \epsilon_{ijk} J^k,$$

$$[J^i, P^j] = i\hbar \epsilon_{ijk} P^k,$$

$$[K^i, P^j] = \frac{i\hbar}{c} H \delta_{ij},$$

$$[K^i, H] = i\hbar c P^i,$$

$$[J^i, H] = 0,$$

$$[P^i, H] = 0.$$ \hspace{0.5cm} (A.21)

When $\vec{K} = 0$, we obtain the usual commutation relations of ISO(3) which consists of spatial translations and rotations.
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