Satellites of the Broucke-Hadjidemetriou-Hénon family of periodic unequal-mass three-body orbits

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Abstract

The Broucke-Hadjidemetriou-Hénon’s (BHH) orbits are a family of periodic orbits of the three-body system with the simplest topological free group word \( a \), while the BHH satellites have free group words \( a^k \) \((k > 1)\), where \( k \) is the topological exponent. Janković and Dmitrasinović [Phy. Rev. Lett. 116, 064301 (2016)] reported 57 new BHH satellites with equal mass and found that at a fixed energy the relationship between the angular momentum \((L)\) and the topologically rescaled period \((T/k)\) is the same for both of the BHH orbits \((k = 1)\) and the BHH satellites \((k > 1)\). In this letter, we report 419,743 new BHH orbits \((k = 1)\) and 179,253 new BHH satellites \((k > 1)\) of the three-body system with unequal mass, which have never been reported, to the best of our knowledge. Among these newly-found 598,996 BBH orbits and satellites, about 33.5\% (i.e., 200,686) are linearly stable and thus many among them might be observed in practice. Besides, we discover that, for the three-body system with unequal mass at a fixed energy, the relationship between the angular momentum \((L)\) and topologically rescaled period \((T/k)\) of the BHH satellites \((k > 1)\) is different from that of the BHH orbits \((k = 1)\).

I. INTRODUCTION

The three-body problem can be traced back to Newton in 1680s, but is still an open question in astrophysics and mathematics today, mainly because it is not an integrable system \([1]\) and besides has the sensitivity dependence on initial condition (SDIC) \([2]\) that broke a new field of scientific research, i.e. chaos. Even today the three-body problem is still one of central issues for scientists \([3]\). Especially, periodic orbits of triple system play an important role since they are “the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible”, as pointed out by Poincaré \([2]\). However, since the famous three-body problem was first put forward, only three families of periodic orbits were found in about three hundred years: (1) the Lagrange-Euler family discovered by Lagrange and Euler in the 18th century; (2) the Broucke-Hadjidemetriou-Hénon (BHH) family \([4]\); (3) the figure-eight family, discovered numerically by Moore \([7]\) in 1993 and then proofed by Chenciner & Montgomery \([8]\) in 2000, until 2013 when Šuvakov and Dmitrasinović \([10]\) numerically found 13 distinct periodic orbits of the three-body system with equal mass. In recent years, numerically searching for periodic orbits of the three-body system has been received much attention \([10, 13]\). Šuvakov \([10]\) reported the satellites of the figure-eight periodic orbit with equal mass. Especially, more than six hundred new families of periodic orbits of equal-mass three-body system were found by Li and Liao \([11]\) using a new numerical strategy, namely the clean numerical simulation (CNS) \([14]\) that can give the convergent/reliable numerical solution of chaotic systems in a long enough duration. Li et al. \([12]\) further used the CNS to obtain more than one thousand new families of periodic orbits of three-body system with two equal-mass bodies. All of these greatly enrich our knowledge of the famous three-body problem.

With the topological classification method \([17]\), the BHH orbits have the simplest topology (free group word \( w = a \)), while the BHH satellites have more free group words \( w = a^k \), where \( k \) is the topological exponent. Recently, Janković and Dmitrasinović \([15]\) reported 57 BHH satellites \((k > 1)\) with \textit{equal} mass. Especially, it was found \([15]\) that the relationship between the scale-invariant angular momentum \((L)\) and the topologically rescaled period \((T/k)\) is the \textit{same} for both of the BHH orbits \((k = 1)\) and satellites \((k > 1)\). However, all of these BHH orbits and satellites have \textit{equal} mass. The BHH satellites with \textit{unequal} mass have never been reported yet, to the best of our knowledge.

In this letter, we investigate the BHH orbits \((k = 1)\) and satellites \((k > 1)\) with \textit{unequal} mass. In § 2, the basic ideas of the numerical continuation method are briefly introduced. In § 3, we numerically search for the BHH orbits and satellites with unequal mass, and show the corresponding results. Brief conclusions are given in § 4.
orthogonal to the line determined by the three bodies: \( \dot{\mathbf{r}} \) for the BHH family of periodic orbits: 

\[
\dot{\mathbf{r}} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + m_3 \mathbf{v}_3
\]

Newtonian gravitational constant \( G \) and the initial velocities are \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 = -(m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2)/m_3 \), and the corresponding initial positions are \((x_1, 0), (1, 0)\) and \((0, 0)\).

II. THE INITIAL CONFIGURATION AND NUMERICAL METHOD

Let us consider a three-body system in the Newtonian gravitational field. Without loss of generality, let the Newtonian gravitational constant \( G \) be 1. As shown in Figure 1, the three bodies have collinear initial configuration for the BHH family of periodic orbits: \( \mathbf{r}_1(0) = (x_1, 0) \), \( \mathbf{r}_2(0) = (x_2, 0) \), \( \mathbf{r}_3(0) = (x_3, 0) \), and their initial velocities are orthogonal to the line determined by the three bodies: \( \dot{\mathbf{r}}_1(0) = (0, v_1) \), \( \dot{\mathbf{r}}_2(0) = (0, v_2) \), \( \dot{\mathbf{r}}_3(0) = (0, v_3) \).

Due to the homogeneity of the potential field of the three-body system, there is a scaling law: \( \dot{\mathbf{r}}' = \alpha \dot{\mathbf{r}}, \mathbf{v}' = \mathbf{v}/\sqrt{\alpha} \), \( t' = \alpha^{3/2} t \), energy \( E' = E/\alpha \) and angular momentum \( L' = \sqrt{\alpha}L \). The known periodic orbits of the BHH family and their satellites with equal mass \( m_1 = m_2 = m_3 \) have zero total momentum (i.e., \( m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 + m_3 \dot{\mathbf{r}}_3 = 0 \)). Using the scaling law, we can transform the initial conditions of known periodic orbits of the BHH family and their satellites to the initial positions

\[
\mathbf{r}_1(0) = (x_1, 0), \quad \mathbf{r}_2(0) = (1, 0), \quad \mathbf{r}_3(0) = (0, 0),
\]

and the initial velocities

\[
\dot{\mathbf{r}}_1(0) = (0, v_1), \quad \dot{\mathbf{r}}_2(0) = (0, v_2), \quad \dot{\mathbf{r}}_3(0) = \left(0, -\frac{m_1 v_1 + m_2 v_2}{m_3}\right).
\]

We use the numerical continuation method \[19\] to gain the BHH orbits \( k = 1 \) and their satellites \( k > 1 \) with unequal mass. Briefly speaking, the numerical continuation method can be used to gain periodic solutions of the nonlinear differential system

\[
\mathbf{u} = F(\mathbf{u}, \lambda),
\]

where \( \lambda \) a physical parameter, called “natural parameter”. Assume that \( \mathbf{u}_0 \) is a solution at a natural parameter \( \lambda = \lambda_0 \). Using the solution \( \mathbf{u}_0 \) as an initial guess, a new periodic orbit \( \mathbf{u}' \) can be obtained at a new natural parameter \( \lambda = \lambda_0 + \Delta \lambda \) through the Newton-Raphson method \[20, 21\] and the clean numerical simulation (CNS) \[14, 16\] if the increment \( \Delta \lambda \) is small enough to make sure iterations convergence. The CNS is a numerical strategy to obtain reliable numerical simulation of chaotic systems in a given time of interval. The CNS is based on an arbitrary high order Taylor series method \[22, 23\] and the multiple precision arithmetic \[24\], plus a convergence check using an additional computation with even smaller numerical error.

Note that all of the known BHH orbits \( k = 1 \) and satellites \( k > 1 \) are “relative periodic orbits”: after a period, these relative periodic orbits will return to initial conditions in a rotating frame of reference. So, there is an individual rotation angle \( \theta \) for each relative periodic orbit.

First of all, using the known BHH orbits \( k = 1 \) and satellites \( k > 1 \) with equal mass \( m_1 = m_2 = m_3 = 1 \) as initial guesses and \( m_1 \) as a natural parameter of the continuation method, we obtain new periodic orbits with various \( m_1 \) by continually correcting the initial conditions \( x_1, v_1, v_2, T \) and the rotation angle \( \theta \). Then, using these periodic solutions with \( m_1 \neq 1, m_2 = m_3 = 1 \) as initial guesses and \( m_2 \) as a natural parameter of the continuation method, we similarly gain periodic orbits for different values of \( m_2 \). In this way, we can obtain the corresponding BHH (relative periodic) orbits \( k = 1 \) and satellites \( k > 1 \) with unequal mass \( m_1 \neq m_2 \neq m_3 = 1 \).

FIG. 1. (color online.) The initial configuration of the three-body system. Here \( m_1, m_2 \) and \( m_3 \) denote the mass of body-1, body-2 and body-3, respectively. The corresponding initial positions are \((x_1, 0)\), \((1, 0)\) and \((0, 0)\).
FIG. 2. (color online.) The stable BHH satellites \((k > 1)\) of the three-body system with unequal mass in a rotating system. Blue line: body-1; red line: body-2; black line: body-3. The corresponding physical parameters are given in Table I.

TABLE I. Initial conditions and periods \(T\) of some BHH satellites of three-body system with unequal mass in case of \(r_1(0) = (x_1, 0), r_2(0) = (1, 0), r_3(0) = (0, 0), \dot{r}_1(0) = (0, v_1), \dot{r}_2(0) = (0, v_2), \dot{r}_3(0) = (0, -(m_1v_1 + m_2v_2)/m_3)\). Here \(m_i, x_i\) and \(v_i\) are the mass, initial position and velocity of the \(i\)th body, \(\theta\) is the rotation angle of relative periodic orbits, and \(k\) is the topological power of periodic orbits, respectively.

| No. | \(m_1\) | \(m_2\) | \(m_3\) | \(x_1\) | \(v_1\) | \(v_2\) | \(T\) | \(\theta\) | \(k\) |
|-----|--------|--------|--------|--------|--------|--------|------|-------|------|
| (a) | 0.44   | 0.87   | 1      | -1.21199294811702 | -0.992232134619392 | -0.51302429825505 | 9.1758282973000 | 0.500325594634030 | 3    |
| (b) | 0.1    | 0.2    | 1      | -2.5903883768724  | -0.619538016547757 | -0.865730420457027 | 23.1822105206534 | 0.110299604356735 | 5    |
| (c) | 0.64   | 0.36   | 1      | -5.3430377956375  | -0.320961649402539 | -0.737471717803478 | 36.3965789125352 | 0.085296971469750 | 7    |
| (d) | 0.4    | 0.7    | 1      | -6.19095126372906 | -0.39020805609860 | -0.703472163111217 | 40.8464178849168 | 0.1079279415387 | 9    |
| (e) | 0.6    | 0.8    | 1      | -9.25316717310893 | -0.25548701086519 | -0.68551074764106 | 60.335581589708 | 0.08698368647642 | 13   |
| (f) | 0.82   | 0.9    | 1      | -25.5854144048497 | -0.0912009292998 | -0.673916187504190 | 204.731904275836 | 0.05623258939249 | 48   |

III. RESULTS

Starting from the 16 known Broucke’s periodic orbits \((k = 1\) with equal mass) \([4]\), the 45 known Hénon’s periodic orbits \((k = 1\) with equal mass) \([6]\) and the 58 known BHH satellites \((k > 1\) with equal mass) \([18]\), we obtain 124780, 294963 and 179253 new periodic orbits of the three-body system with unequal mass for \(m_1 \in [0.1, 1), \ m_2 \in [0.1, 1)\) and \(m_3 = 1\), respectively. Totally, we gain 419,743 new BHH orbits \((k = 1\) with unequal mass) and 179,253 new BHH satellites \((k > 1\) with unequal mass). Note that all of them are retrograde, say, the binary system and the third body move in opposite direction. The corresponding initial conditions, the periods and the rotation angles of these new BBH orbits and satellites with unequal mass are given in the supplementary data. The return distance of these periodic orbits and satellites satisfies

\[
d = \sqrt{\sum_{i=1}^{3} ((r_i(T) - r_i(0))^2 + (\dot{r}_i(T) - \dot{r}_i(0))^2)} < 10^{-10}
\]

in a rotating frame of reference, where \(T\) is the period. Note that Broucke and Boggs \([25]\) gave dozens of the BHH orbits with unequal mass (their ratios of mass are different from ours), but neither have any BHH satellites with
FIG. 3. (color online.) The angular momentum \( (L) \) versus the topological rescaled period \( (T/k) \) for BHH periodic orbits and their satellites at fixed energy \( E = -1/2 \) with different mass: (a) \( m_1 = m_2 = m_3 = 1 \); (b) \( m_1 = 0.7, m_2 = 0.9, m_3 = 1 \); (c) \( m_1 = 0.5, m_2 = 0.8, m_3 = 1 \); (d) \( m_1 = 0.4, m_2 = 0.7, m_3 = 1 \).

unequal mass been reported, to the best of our knowledge. It should be emphasized that, among our newly-found 598,996 BHH orbits and satellites with unequal mass, there are 151,925 stable BHH orbits \( (k = 1) \) and 48,761 stable BHH satellites \( (k > 1) \). In other words, about 33.5% of the new BBH orbits and satellites with unequal mass (i.e., 200,686) are stable, and thus many among them might be observed in practice. The stability of these periodic orbits and satellites is marked by “S” in the supplementary data. Six new BHH satellites with unequal mass are shown in Figure 2. All of the six orbits are linearly stable. Their initial conditions, periods and topological powers are listed in Table I.

With rescaling to the same energy \( E = -1/2 \), Janković and Dmitrašinović \[8\] found that, in case of equal mass, the relationship between the scale-invariant angular momentum \( (L) \) and the topologically rescaled period \( (T/k) \) is the same for both of the BHH orbits \( (k = 1) \) and satellites \( (k > 1) \), as shown in Figure 3(a), where \( k \) is the topological exponent of periodic orbits. However, for our newly-found periodic orbits with unequal masses (at the same energy \( E = -1/2 \)), it is found that the relationship between the scale-invariant angular momentum \( (L) \) and period \( (T/k) \) of the BHH satellites \( (k > 1) \) is different from that of the BHH orbits \( (k = 1) \), as illustrated in Figure 3(b)-(d). It suggests that the relationship between the scale-invariant angular momentum \( (L) \) and topologically rescaled period
(T/k) of the BHH orbits (k = 1) and satellites (k > 1) in general cases of unequal masses \( m_1 \neq m_2 \neq m_3 \) should be more complicated than that in the case of the equal mass \( m_1 = m_2 = m_3 \).

IV. CONCLUSION

The BHH orbits are a family of periodic orbits of the three-body system with the simplest topological free group word \( a \), while the BHH satellites have free group words \( a^k \) (\( k > 1 \)), where \( k \) is the topological exponent. In this paper, starting from the 16 known Broucke’s periodic orbits (\( k = 1 \) with equal mass) [4], the 45 known Hénon’s periodic orbits (\( k = 1 \) with equal mass) [18], we found 419,743 new BHH orbits (\( k = 1 \)) and 179,253 new BHH satellites (\( k > 1 \)) for three-body system with unequal mass, which have never been reported, to the best of our knowledge. Among these newly-found 598,996 BHH orbits and satellites of three-body system with unequal mass, about 33.5% (i.e., 200,686) are linearly stable and thus many among them might be observed in practice.

For the three-body system with equal mass at a fixed energy \( E = -1/2 \), it was reported [18] that the relationship between the angular momentum (\( L \)) and topological period (\( T/k \)) of the BHH satellites (\( k > 1 \)) is the same as that of the BHH orbits (\( k = 1 \)). However, this does not hold for the three-body system with unequal mass, as reported in this letter.

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