Heavy Top Quark From Fritzsch Mass Matrices

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Abstract

It is shown, contrary to common belief, that the Fritzsch ansatz for the quark mass matrices admits a heavy top quark. With the ansatz prescribed at the supersymmetric grand unified (GUT) scale, one finds that the top quark may be as heavy as 145 GeV, provided that \( \tan \beta \) (the ratio of the vacuum expectation values of the two higgs doublets) \( \gg 1 \). Within a non-supersymmetric GUT framework with two (one) light higgs doublets, the corresponding approximate upper bound on the top mass is 120 (90) GeV. Our results are based on a general one-loop renormalization group analysis of the quark masses and mixing angles and are readily applied to alternative mass matrix ansätze.

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The standard $SU(3) \times SU(2) \times U(1)$ gauge model predicts the existence of a sixth (top) quark whose mass is known experimentally to be heavier than 91 GeV.\(^1\) Analysis based on electroweak radiative corrections may favor a moderately heavy top quark, say below 160 GeV, with a central value possibly around the 120-140 GeV mark.\(^2\) Other independent sources of information, such as recovering the $b$ quark mass in the range $4.25 \pm 0.1$ GeV within a supersymmetric grand unification framework (SUSY GUTs) with unified third generation Yukawa couplings, also lead to a top mass in the above range.\(^3\)

A simple approach for incorporating the observed hierarchy of quark masses and their mixings was suggested a long time ago by Fritzsch.\(^4\) It prescribes a form for the mass matrices which has a certain amount of predictive power. Based on considerations of the $V_{cb}$ element of the Cabibbo–Kobayashi–Maskawa (CKM) matrix, it has been argued that if the top quark mass turns out to be significantly heavier than 90 GeV, then the simplest version of the Fritzsch ansatz is excluded.\(^4,5\)

The main purpose of this note is to point out that the Fritzsch ansatz actually permits the top quark to be much heavier, provided it is considered within the framework of grand unification involving more than one ‘light’ higgs doublet. Perhaps the most prominent example of this is provided by supersymmetric grand unification. An important new parameter is $\tan \beta$, the well known ratio of the vacuum expectation values of the two higgs doublets that supersymmetry mandates. It turns out that with $\tan \beta$ sufficiently greater than unity, the grand unified supersymmetric version of the Fritzsch ansatz permits the top quark to be as heavy as 145 GeV. A similar result
holds within a non-supersymmetric GUT framework provided that there are two ‘light’ higgs doublets. In this case the top quark mass can be as high as 120 GeV. For completeness, we also present the results for a single higgs doublet and confirm that the top mass cannot exceed by much 90 GeV.

Our analysis is based on the one-loop renormalization group evolution of the quark masses and mixing angles. The results are quite general and can be used to test alternative mass matrix ans"atze as well. As an additional application, we derive lower limits on the top mass in the standard, two-higgs and supersymmetric models assuming that the relation $|V_{cb}^0| = \sqrt{m_c^0/m_t^0}$ holds at the unification scale.

The Fritzsch ansatz for the quark and charged lepton mass matrices takes the following form at the GUT (or some other appropriate superheavy) scale:

$$M_{u,d,\ell} = P_{u,d,\ell} \begin{pmatrix} 0 & a_{u,d,\ell} & 0 \\ a_{u,d,\ell} & 0 & b_{u,d,\ell} \\ 0 & b_{u,d,\ell} & c_{u,d,\ell} \end{pmatrix} Q_{u,d,\ell}$$

(1)

where $P_{u,d,\ell}$ and $Q_{u,d,\ell}$ denote diagonal phase matrices and $a, b, c$ are real (positive) quantities. Among the various phases contained in $P$ and $Q$, only two are relevant for quark mixing, we denote them by $\psi$ and $\phi$. The ansatz predicts the CKM matrix elements in terms of the quark mass ratios and these two phase parameters. In order to incorporate the fermion mass hierarchy, one finds that $c \gg b \gg a$. From the mass eigenvalues in the quark sector in particular,

$$c_u \simeq m_t^0, \quad b_u \simeq (m_c^0 m_t^0)^{\frac{1}{2}}, \quad a_u \simeq (m_u^0 m_c^0)^{\frac{1}{2}}$$

$$c_d \simeq m_b^0, \quad b_d \simeq (m_s^0 m_b^0)^{\frac{1}{2}}, \quad a_d \simeq (m_d^0 m_s^0)^{\frac{1}{2}}$$

(2)
The superscripts are to emphasize that the ansatz (1) is prescribed at some superheavy scale. [As a consequence of grand unification the parameters $a_d,b_d,c_d$ of the down sector typically are related to $a_\ell,b_\ell,c_\ell$ of the lepton sector.]

Our main concern here is the following asymptotic relation which is a consequence of eq. (1):

$$|V_{cb}^0| = \left| \frac{m_s^0}{m_b^0} - e^{i\phi} \frac{m_d^0}{m_t^0} \right|. \quad (3)$$

The implications of (3) at low energies are evaluated by studying the evolutions of the gauge and Yukawa couplings. In the absence of Yukawa couplings that are comparable to or larger than the gauge couplings, relation (3) would be essentially unrenormalized, and would require that the top mass not exceed 90 GeV. However, since the top quark mass is known to be greater than 91 GeV, and indeed may be significantly larger than this, its Yukawa coupling cannot be ignored. Moreover, models in which the up and down type fermions obtain masses from couplings to distinct higgs doublets allow for large intrinsic Yukawa couplings $h_b$ and $h_\tau$. We will exploit, in particular, the fact that if $\tan \beta$ (the ratio of the two doublet vev’s which provide masses to up type and down type fermions) is sufficiently large, then $h_b(h_\tau)$ can even exceed $h_t$! As a consequence, in contrast to the single light higgs doublet case, relation (3) it turns out can lead to a low energy prediction for $|V_{cb}|$ which is compatible with the existence of a heavy top quark.

The Yukawa sector relevant for our discussion has the generic form
\[ \mathcal{L}_Y = \bar{q}_L H_u \phi_u u_R + \bar{q}_L H_d \phi_d d_R + \bar{\ell}_L H_\ell \phi_\ell e_R + h.c. \quad (4) \]

where $H_u, H_d, H_\ell$ denote the $3 \times 3$ Yukawa coupling matrices for the up quarks, down quarks and charged leptons. Recall that in the standard model, $\phi_u \equiv \tilde{\phi}_d$. The one loop evolution equations for the Yukawa matrices take the form ($t \equiv \ln(\mu/M_Z)$):

\[
16\pi^2 \frac{dH_u}{dt} = \left[ Tr(3H_u H_u^\dagger + 3aH_d H_d^\dagger + aH_\ell H_\ell^\dagger) \right. \\
+ \left. \frac{3}{2}(bH_u H_u^\dagger + cH_d H_d^\dagger) - G_U \right] H_u \\
16\pi^2 \frac{dH_d}{dt} = \left[ Tr(3aH_u H_u^\dagger + 3H_d H_d^\dagger + H_\ell H_\ell^\dagger) \right. \\
+ \left. \frac{3}{2}(bH_d H_d^\dagger + cH_u H_u^\dagger) - G_D \right] H_d \\
16\pi^2 \frac{dH_\ell}{dt} = \left[ Tr(3aH_u H_u^\dagger + 3H_d H_d^\dagger + H_\ell H_\ell^\dagger) \right. \\
+ \left. \frac{3}{2}bH_\ell H_\ell^\dagger - G_E \right] H_\ell \quad (5)
\]

For the three low energy models under discussion the coefficients $a, b, c$ are given by

\[
(a, b, c) = \begin{cases} 
(0, 2, \frac{2}{3}) & \text{MSSM (Minimal SUSY)} \\
(0, 1, \frac{1}{3}) & \text{Two higgs doublets (non-SUSY)} \\
(1, 1, -1) & \text{SM (Standard Model)}
\end{cases} \quad (6)
\]

Also, the quantities $G_U, G_D$ and $G_E$ for the SUSY (non-SUSY) case are respectively:
\[ G_U = \frac{13}{9} g_1^2 + 3 g_2^2 + \frac{16}{3} g_3^2 \]
\[ G_D = \frac{7}{9} g_1^2 + 3 g_2^2 + \frac{10}{3} g_3^2 \]  \hspace{1cm} (MSSM) \hspace{1cm} (7)
\[ G_E = 3 g_1^2 + 3 g_2^2 ; \]

\[ G_U = \frac{17}{12} g_1^2 + \frac{9}{4} g_2^2 + 8 g_3^2 \]
\[ G_D = \frac{5}{12} g_1^2 + \frac{9}{4} g_2^2 + 8 g_3^2 \]  \hspace{1cm} (Two higgs ; SM) \hspace{1cm} (8)
\[ G_E = \frac{15}{4} g_1^2 + \frac{9}{4} g_2^2 . \]

The gauge couplings \( g_i \) (above) obey the standard one loop renormalization group equations:

\[ 8\pi^2 \frac{d g_i^2}{dt} = b_i g_i^4, \quad i = 1, 2, 3 \]  \hspace{1cm} (9)

where

\[ (b_1, b_2, b_3) = (11, 1, -3) \]  \hspace{1cm} MSSM
\[ = (7, 3, -7) \]  \hspace{1cm} Two higgs \hspace{1cm} (10)
\[ = (\frac{11}{6}, -\frac{19}{6}, -7) \]  \hspace{1cm} SM

Note that we are not using the \( SU(5) \) normalization for the hypercharge coupling \( g_1 \). From eq. (5), one can compute the evolution equations for the eigenvalues of the Yukawa coupling matrices.\(^7\,^8\)

\[ 16\pi^2 \frac{d g_i}{dt} = g_i \left[ 3 \sum_{j=u,c,t} g_j^2 + 3a \sum_{\beta=d,s,b} g_\beta^2 + a \sum_{b=e,\mu,\tau} g_b^2 - G_U \right] \]
\[ + \frac{3}{2} b g_i^2 + \frac{3}{2} c \sum_{\beta = d, s, b} g_{i\beta}^2 | V_{i\beta} |^2 \]

\[ 16\pi^2 \frac{d g_a}{d t} = g_a \left[ 3a \sum_{j = u, c, t} g_j^2 + 3 \sum_{\beta = d, s, b} g_{\beta}^2 + \sum_{b = e, \mu, \tau} g_b^2 - G_D \right. \]
\[ + \frac{3}{2} b g_a^2 + \frac{3}{2} c \sum_{j = u, c, t} g_j^2 | V_{ja} |^2 \]

\[ 16\pi^2 \frac{d g_{\alpha}}{d t} = g_{\alpha} \left[ 3a \sum_{j = u, c, t} g_j^2 + 3 \sum_{\beta = d, s, b} g_{\beta}^2 + \sum_{b = e, \mu, \tau} g_b^2 - G_E \right. \]
\[ + \frac{3}{2} b g_{\alpha}^2 \right] \tag{11} \]

where \( i = (u, c, t), \alpha = (d, s, b), a = (e, \mu, \tau) \).

We will also need the evolution equations for the elements of the CKM matrix: \(^7,^8\)

\[ 16\pi^2 \frac{d}{d t} | V_{i\alpha} |^2 = 3c \left[ \sum_{j \neq i} \sum_{\beta = d, s, b} \frac{g_i^2 + g_j^2}{g_i^2 - g_j^2} g_{i\beta}^2 Re \left( V_{i\beta} V_{j\alpha}^* V_{j\alpha} V_{i\alpha}^* \right) \right. \]
\[ + \sum_{\beta \neq \alpha} \sum_{j = u, c, t} \frac{g_{\alpha}^2 + g_{\beta}^2}{g_{\alpha}^2 - g_{\beta}^2} g_j^2 Re \left( V_{j\beta} V_{j\alpha} V_{i\beta} V_{i\alpha}^* \right) \right] \tag{12} \]

The above expressions simplify considerably if we exploit the hierarchy in the Yukawa couplings \((g_b \gg g_s \gg g_d, \text{etc})\) and in the CKM matrix elements. If only the leading terms are kept, one obtains the following approximate expressions for the evolution of the various mass ratios and the mixing angles:

\[ 16\pi^2 \frac{d}{d t} \left( \frac{m_{\alpha}}{m_b} \right) = -\frac{3}{2} \left( \frac{m_{\alpha}}{m_b} \right) (b g_b^2 + c g_t^2), \quad \alpha = d, s \]

\[ 16\pi^2 \frac{d}{d t} \left( \frac{m_i}{m_t} \right) = -\frac{3}{2} \left( \frac{m_i}{m_t} \right) (b g_t^2 + c g_b^2), \quad i = u, c \]

\[ 16\pi^2 \frac{d}{d t} \left( \frac{m_d}{m_s} \right) = -\frac{3}{2} \left( \frac{m_d}{m_s} \right) (b g_s^2 + c g_c^2 + c g_t^2 | V_{is} |^2) \]
\[ 16\pi^2 \frac{d}{dt} \left( \frac{m_u}{m_c} \right) = -\frac{3}{2} \left( \frac{m_u}{m_c} \right) \left( bg_c^2 + cg_s^2 + cg_b^2 \left| V_{cb} \right|^2 \right) \]

\[ 16\pi^2 \frac{d}{dt} |V_{i\alpha}| = -\frac{3}{2} c |V_{i\alpha}| \left( g_t^2 + g_b^2 \left| V_{ib} \right|^2 - \left| V_{ib} \right|^2 \right) \]

\[ 16\pi^2 \frac{d}{dt} |V_{us}| = -\frac{3}{2} c |V_{us}| \left( g_c^2 + g_s^2 + g_t^2 \left| V_{td} \right|^2 - \left| V_{td} \right|^2 \right) \]

\[ 16\pi^2 \frac{d}{dt} |V_{cd}| = -\frac{3}{2} c |V_{cd}| \left( g_c^2 + g_s^2 + g_b^2 \left| V_{ub} \right|^2 - \left| V_{ub} \right|^2 \right) \]

Several comments are in order:

1. The quantities \( |V_{ub}|, |V_{cb}|, |V_{td}| \) and \( |V_{ts}| \) have identical evolutions.

2. The evolutions of \( |V_{us}| \) and \( |V_{cd}| \) involve second family Yukawa couplings and consequently are negligible.

3. \( \frac{m_d}{m_b} \) and \( \frac{m_s}{m_b} \) have identical evolutions. Similarly for \( \frac{m_u}{m_t} \) and \( \frac{m_c}{m_t} \).

4. The evolutions of \( \frac{m_d}{m_s} \) and \( \frac{m_u}{m_c} \) can be ignored.

5. The CP violating parameter \( J \) is not independent, its evolution can be obtained from the running of the square of \( |V_{cb}| \), for example.

Using eq. (13) the asymptotic expression (3) can be recast in the form

\[ \eta_{cb} \left| V_{cb} \right| = \left| \eta_{cb} \right| \sqrt{\frac{m_s}{m_b}} - e^{i\phi} \eta_{ct} \sqrt{\frac{m_c}{m_t}} \]

(14)

where \( |V_{cb}| \) stands for the CKM angle at the weak scale, and the \( \eta \)'s are the respective running factors, \( \eta_{cb} = |V_{cb}^0|/|V_{cb}| \), \( \eta_{sb} = (m_s/m_b)/(m_s/m_b) \), etc. To compute these \( \eta \) factors, one has to solve numerically the evolution equations.
in eq. (13). The running of $g_t, g_b, g_\tau$ are obtained from eq. (11), neglecting the light two families. As input parameters we take

$$
\begin{align*}
\alpha_1(M_Z) &= 0.01013 \\
\alpha_2(M_Z) &= 0.03322 \\
\alpha_3(M_Z) &= 0.115.
\end{align*}
$$

For the superheavy scale $M_X$ above which the asymptotic relations are valid we take

$$
M_X = 10^{16} \text{ GeV (SUSY)}
$$

$$
= 10^{14} \text{ GeV (non-SUSY)}.
$$

For the running quark masses we use the values of ref. (9). In particular, $m_s(1 \text{ GeV}) = 175 \pm 55 \text{ MeV}, m_c(m_c) = 1.27 \pm 0.05 \text{ GeV}$ and $m_b(m_b) = 4.25 \pm 0.1 \text{ GeV}$.

To get a feeling for the effect running has on the prediction for $|V_{cb}|$, let us first define the functions

$$
f_{t,b} = \exp \left[ \frac{1}{16\pi^2} \int_0^{\ln(M_X/M_Z)} g_{t,b}^2(\tau) d\tau \right].
$$

Note that $f_{t,b} \geq 1$. Let us first consider the supersymmetric case. For small $\tan\beta$, $g_b$ can be neglected since $g_b \ll g_t$. The renormalized prediction for $|V_{cb}|$ in terms of the mass ratios at the weak scale is then

$$
|V_{cb}| = f_t^{\frac{1}{2}} \sqrt{\frac{m_s}{m_b}} - e^{i\phi} f_t^{-\frac{1}{2}} \sqrt{\frac{m_c}{m_t}}.
$$

Since $f_t \geq 1$, we see that the disagreement of $|V_{cb}|$ with observations is more prominent for small $\tan\beta$. This feature persists as $\tan\beta$ is increased, until it
approaches $m_t/m_b$. For $\tan \beta = m_t/m_b$, $g_t \simeq g_b$ at all scales since both couplings track the same evolution equations (except for small corrections from the hypercharge and the $\tau$ Yukawa couplings). To a good approximation, we see that the renormalized $|V_{cb}|$ obeys

$$|V_{cb}| = \left| \sqrt{\frac{m_s}{m_b}} - e^{i\phi} \sqrt{\frac{m_c}{m_t}} \right|.$$  

That is, the relation (3) is essentially unrenormalized and the top quark cannot be heavy. This prompts us to consider values of $\tan \beta$ even larger than $m_t/m_b$, in which case better agreement is possible. Indeed, if we take the extreme limit $g_b \gg g_t$, then in terms of $f_b$ defined in eq. (17), we see that

$$|V_{cb}| = \left| f_b^{-\frac{1}{2}} \sqrt{\frac{m_s}{m_b}} - e^{i\phi} f_b^{\frac{1}{2}} \sqrt{\frac{m_c}{m_t}} \right|.$$  

Note that the first term in eq. (20) has a reduction factor whereas the second term is enhanced, precisely what one needs to bring the prediction for $|V_{cb}|$ down and allow for a heavy top quark. These features are indeed borne out by the actual numerical computation to which we now turn.

In fig. 1a, we plot the behavior of the ratios $g_t(M_X)/g_t(M_Z)$, $g_b(M_X)/g_b(M_Z)$ and $g_\tau(M_X)/g_\tau(M_Z)$ as functions of $m_t$ for $\tan \beta = 3$. Throughout this paper, we shall mean by $m_t$ the running mass $m_t(m_t)$. The pole mass is related to the running mass via

$$m_t^{\text{pole}} = m_t(m_t) \left[ 1 + \frac{4}{3} \frac{\alpha_3}{\pi} \right].$$  

Fig. 1b shows the dependence of the functions $f(M_X)/f(M_Z)$ for $f = |V_{cb}|$, $|m_s/m_b|$ and $|m_c/m_t|$. The behavior of $|V_{ub}|$, $|V_{td}|$ and $|V_{ts}|$ are identical to that of $|V_{cb}|$. Similarly, $|m_d/m_b|$ runs as $|m_s/m_b|$ and $|m_u/m_t|$ as
$|m_c/m_t|$. The running factor for the CP parameter $J$ is obtained from the square of $|V_{cb}|$. Since $\tan \beta$ is small, the evolution of $|m_s/m_b|$ coincides with that of $|V_{cb}|$ which can be seen from eq. (13). In figures 2a and 2b, we plot the same quantities for the case of $\tan \beta = m_t/m_b$. Note that the running of $g_t$ and $g_b$ are almost identical. Finally, in figures 3a and 3b we plot these functions for $\tan \beta = 60$. This value of $\tan \beta$ corresponds to the infra–red fixed point solution for $g_b$ and $g_\tau$.

From figures 1–3, one can compute the running factors $\eta$’s entering in (14). It is clear from the discussions above as well as from figures 1–3 that unless $\tan \beta$ is larger than $m_t/m_b$, the situation for $|V_{cb}|$ is worse than the case with no running. However, for sufficiently large $\tan \beta$, $|V_{cb}|$ is in the experimentally allowed range$^{10}$ of $0.034 \leq |V_{cb}| \leq 0.054$. We give in Table 1 the respective running factors for $|V_{cb}|$, $(m_s/m_b)$ and $(m_c/m_t)$ for $\tan \beta = 60$ and also list the lowest allowed value of $|V_{cb}|$. To arrive at the latter, we choose $m_s(1 \text{ GeV}) = 120 \text{ MeV}$, $m_b(m_b) = 4.35 \text{ GeV}$, $m_c(m_c) = 1.32 \text{ GeV}$ and set the phase $\phi = 0$. We use two–loop QCD renormalization group equations to extrapolate the light quark masses from low energies to $M_Z$. There is an upper limit of about 145 GeV on the top mass in this case coming from the requirement that $g_b$ and $g_\tau$ should remain perturbatively small in the entire range from $M_Z$ to $M_X$. One sees that for all values of $m_t$ up to 145 GeV, $|V_{cb}|$ is in the experimentally allowed range.

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The running factors for $|V_{cb}|$, $(m_s/m_b)$ and $(m_c/m_t)$ in the supersymmetric model with $\tan\beta = 60$ as functions of $m_t$.

In the last column is listed the lowest allowed value of $|V_{cb}|$.

| $m_t$ | $\eta_{cb}$ | $\eta_{sb}$ | $\eta_{ct}$ | $|V_{cb}|_{\min}$ |
|------|-------------|-------------|-------------|-----------------|
| 80   | 0.790       | 0.510       | 0.763       | 0.023           |
| 90   | 0.781       | 0.499       | 0.746       | 0.029           |
| 100  | 0.770       | 0.485       | 0.727       | 0.034           |
| 110  | 0.757       | 0.467       | 0.703       | 0.038           |
| 120  | 0.740       | 0.445       | 0.675       | 0.042           |
| 130  | 0.718       | 0.415       | 0.639       | 0.045           |
| 140  | 0.684       | 0.371       | 0.591       | 0.046           |
| 145  | 0.660       | 0.338       | 0.559       | 0.045           |

**Table 1.**

The precise values of $m_t$, $|V_{cb}|$ etc. will depend somewhat on the input value of $\alpha_3(M_Z)$. For larger values of $\alpha_3$, the top quark can be heavier by a few GeV. For e.g., with $\alpha_3(M_Z) = 0.12$, the top mass is raised by about 5 GeV. Most of the dependence arises from changes in extrapolating the light quark masses from 1 GeV to $M_Z$.

In figures 4, 5 and 6, we plot the same quantities for the two higgs model for $\tan\beta = (3, m_t/m_b, 70)$. The behavior of the various functions is identical to the case of SUSY, except that due to the smaller beta function coefficients, the variation is not as pronounced. In Table 2, we list the running factors corresponding to $\tan\beta = 70$ (the infra-red fixed point for $g_b$ and $g_\tau$) and conclude that $m_t$ as large as 120 GeV is allowed.
Table 2.

Running factors and lowest allowed $|V_{cb}|$ in the two higgs model with $\tan\beta = 70$.

For completeness, we also display in figures 7a and 7b the variation of the relevant quantities in the standard model. Table 3 shows the $\eta$ factors from which it is clear that $m_t$ cannot much exceed 90 GeV. This feature can also be understood qualitatively in terms of the function $f_t$ of eq. (17). The renormalized relation for $|V_{cb}|$ in the standard model case is

$$|V_{cb}| = f_t^{-\frac{3}{4}} \left| \sqrt{\frac{m_s}{m_b}} - e^{i\phi} f_t^{-\frac{3}{2}} \sqrt{\frac{m_c}{m_t}} \right|. \quad (22)$$

Although there is an overall suppression factor, the second term also becomes small and so the cancellation between the two terms becomes inefficient.
### Table 3.

Running factors and $|V_{cb}|_{\text{min}}$ for the case of standard model.

| $m_t$ | $\eta_{cb}$ | $\eta_{sb}$ | $\eta_{ct}$ | $|V_{cb}|_{\text{min}}$ |
|-------|-------------|-------------|-------------|-----------------|
| 80    | 1.020       | 1.020       | 0.981       | 0.049           |
| 90    | 1.026       | 1.026       | 0.975       | 0.054           |
| 100   | 1.033       | 1.032       | 0.969       | 0.059           |
| 110   | 1.040       | 1.040       | 0.961       | 0.064           |
| 120   | 1.049       | 1.049       | 0.953       | 0.067           |

Turning now to some other predictions of the Fritzsch ansatz, the CKM elements $|V_{us}|$, $|V_{ub}|$ and $|V_{td}|$ have the asymptotic forms

\[
|V_{us}^0| = \sqrt{\frac{m^0_d}{m^0_s} - e^{i\psi} \frac{m^0_u}{m^0_c}} \\
|V_{ub}^0| = \sqrt{\frac{m^0_s}{m^0_b} \left( \frac{m^0_d}{m^0_b} + e^{i\psi} \frac{m^0_u}{m^0_c} \left( \frac{m^0_d}{m^0_b} - e^{i\phi} \frac{m^0_u}{m^0_c} \right) \right)} \\
|V_{td}^0| = \sqrt{\frac{m^0_c}{m^0_t} \left( \frac{m^0_u}{m^0_t} + e^{i\psi} \frac{m^0_d}{m^0_s} \left( \frac{m^0_u}{m^0_t} - e^{i\phi} \frac{m^0_d}{m^0_s} \right) \right)} .
\]

The first relation involves only the first two family masses and is therefore essentially unrenormalized. The phase $\psi$ should be near $\pi/2$ for agreement with the Cabibbo angle. From the second relation in (23), one can easily write down the renormalized value of $|V_{ub}|$. It turns out that for $\tan\beta = 60$ and $m_t = 130$ $GeV$ in the SUSY model, the magnitude of the first term is only about 15% of the second term. Given that the phase $\psi \simeq \pi/2$ to a good
approximation, the first term can be neglected, which leads to the weak scale relation

\[ \frac{|V_{ub}|}{|V_{cb}|} \simeq \sqrt{\frac{m_u}{m_c}} \simeq 0.06 \], \quad (24)

in good agreement with the recent charmless B decay data. The same conclusion also follows for the two higgs doublet model. As for \(|V_{td}|\), again one finds that the first term is negligible compared to the second term, so that at the weak scale we have

\[ \frac{|V_{td}|}{|V_{cb}|} \simeq \sqrt{\frac{m_d}{m_s}} \simeq 0.2 . \] \quad (25)

Finally, the renormalized value of the CP violating parameter \(J\) at the weak scale is given by (for \(\psi = \pi/2, \phi = 0\))

\[ J \simeq \sqrt{\frac{m_d}{m_s}} \frac{m_u}{m_c} |V_{cb}|^2 \simeq 3 \times 10^{-5} , \quad (26)\]

consistent with observations.

In the lepton sector, the Fritzsch ansatz can lead to two successful mass predictions provided that \(a_d = a_l, c_d = c_l\) in eq. (1). Such relations arise naturally in GUT’s if the elements \(a, c\) arise from a higgs \(\mathbf{5}\) of \(SU(5)\) or a higgs \(\mathbf{10}\) of \(SO(10)\). Note that \(b_d\) and \(b_l\) should be independent. The two asymptotic mass relations are

\[ m_b^0 - m_s^0 + m_d^0 = m_r^0 - m_\mu^0 + m_e^0 ; \quad m_d^0 m_s^0 m_b^0 = m_e^0 m_\mu^0 m_r^0 . \] \quad (27)

The first relation would lead to a prediction of \(m_b(m_b) = 4.2\) GeV in the SUSY model with \(m_t = 130\) GeV and \(\tan\beta = 60\), which is in good agreement with the value derived in ref. (9). The second relation would lead to
$m_d(1 \text{ GeV}) \simeq 7 \text{ MeV}$ (if $m_s \simeq 140 \text{ MeV}$), also in good agreement with observations.

Since our analysis has been quite general, it is readily applicable to other mass ansätze as well. One particularly interesting case which has recently attracted a fair amount of attention is the asymptotic relation\textsuperscript{11,12,13}

$$|V_{cb}^0| = \sqrt{\frac{m_c^0}{m_t^0}}$$

(28)

The renormalized relation in this case can be written down as

$$\eta_{cb}|V_{cb}| = \eta_{ct} \sqrt{\frac{m_c}{m_t}}$$

(29)

In Table 4, we list $\eta_{cb}$, $\eta_{ct}$ and the corresponding renormalized values of $|V_{cb}|$ for the case of the standard model. It is clear that there is a lower limit of about 170 GeV on the top quark mass in this case.

| $m_t$  | $\eta_{cb}$ | $\eta_{ct}$ | $|V_{cb}|$ |
|-------|-------------|-------------|-----------|
| 100   | 1.033       | 0.969       | 0.081     |
| 130   | 1.060       | 0.944       | 0.068     |
| 160   | 1.104       | 0.906       | 0.058     |
| 170   | 1.124       | 0.890       | 0.054     |
| 180   | 1.149       | 0.870       | 0.051     |
| 190   | 1.180       | 0.847       | 0.048     |
| 200   | 1.221       | 0.819       | 0.044     |

Table 4.

Test of the asymptotic relation $|V_{cb}^0| = \sqrt{m_c^0/m_t^0}$ in the standard model. $m_c(m_c) = 1.32 \text{ GeV}$ is used.
How does this asymptotic relation fare in the supersymmetric and two higgs models? In the SUSY model, the value of $|V_{cb}|$ at weak scale can be written down as

$$|V_{cb}| = f_t^{-\frac{3}{2}} f_b^{\frac{1}{2}} \sqrt{\frac{m_c}{m_t}}.$$  

(30)

Since both $f_t$ and $f_b$ are greater than unity, it is clear that smaller values of $\tan \beta$ will give better agreement. A similar conclusion holds for the two higgs model as well. In Tables 5 and 6, we list the $\eta$'s and values of $|V_{cb}|$ as functions of $m_t$ in the SUSY model and in the two higgs model for $\tan \beta = 3$. In the SUSY model, the top quark mass should be close to the infra–red fixed point value of about 185 GeV. In the two higgs model, we see that there is no acceptable solution with $m_t \leq 200$ GeV and $\tan \beta = 3$. Larger values of $\tan \beta$ are not displayed, since the agreement is worse.

| $m_t$ | $\eta_{cb}$ | $\eta_{ct}$ | $|V_{cb}|$ |
|------|------------|------------|----------|
| 100  | 0.972      | 0.919      | 0.084    |
| 130  | 0.946      | 0.847      | 0.072    |
| 160  | 0.895      | 0.717      | 0.063    |
| 170  | 0.862      | 0.641      | 0.060    |
| 180  | 0.800      | 0.511      | 0.057    |
| 185  | 0.708      | 0.356      | 0.052    |

Table 5.

Test of the asymptotic relation $|V_{cb}^0| = \sqrt{m_c^0/m_t^0}$

in the SUSY model with $\tan \beta = 3$. 

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To summarize, the Fritzsch ansatz for the quark mass matrices permits the top quark to be in the mass range suggested by the recent analysis of precision electroweak data as well as by other independent estimates. However, it requires that the parameter $\tan \beta$ be considerably greater than unity, even exceeding $m_t/m_b$. It would be interesting to see whether this can be reconciled with the scenario of radiative electroweak breaking in the SUSY model which usually requires that $\tan \beta \lesssim m_t/m_b$. The ansatz also predicts values for the CKM matrix elements $|V_{us}|$, $|V_{ub}|$, $|V_{td}|$ and the CP violating parameter $J$ that are in agreement with observations.

| $m_t$ | $\eta_{cb}$ | $\eta_{ct}$ | $|V_{cb}|$ |
|-------|-------------|-------------|----------|
| 100   | 0.988       | 0.965       | 0.084    |
| 130   | 0.978       | 0.936       | 0.074    |
| 160   | 0.963       | 0.892       | 0.066    |
| 170   | 0.955       | 0.872       | 0.064    |
| 180   | 0.947       | 0.848       | 0.061    |
| 190   | 0.935       | 0.818       | 0.059    |
| 200   | 0.921       | 0.780       | 0.057    |

Table 6.

Test of the asymptotic relation $|V_{cb}^0| = \sqrt{m_c^0/m_t^0}$
in the two higgs model with $\tan \beta = 3$. 

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Figure Captions

Fig. 1a: Plot of \( g_t(M_X)/g_t(M_Z) \) (solid), \( g_b(M_X)/g_b(M_Z) \) (dashed) and \( g_r(M_X)/g_r(M_Z) \) (dot-dash) as functions of \( m_t(m_t) \) in the supersymmetric model with tan\( \beta \) = 3.

Fig. 1b: The running factors \( f(M_X)/f(M_Z) \) for \( f = |V_{cb}| \) (solid), \( |m_s/m_b| \) (dashed) and \( |m_c/m_t| \) (dot–dash) versus \( m_t \). The running factors for \( |V_{ub}|, |V_{td}| \) and \( |V_{ts}| \) are identical to that of \( |V_{cb}| \). Similarly, \( |m_d/m_b| \) runs as \( |m_s/m_b| \) and \( |m_u/m_t| \) as \( |m_c/m_t| \).

Fig. 2a: Same as in fig. 1a, except that tan\( \beta \) = \( m_t/m_b \).

Fig. 2b: Same as in fig. 1b, except that tan\( \beta \) = \( m_t/m_b \).

Fig. 3a: Same as in fig. 1a, except that tan\( \beta \) = 60.

Fig. 3b: Same as in fig. 1b, except that tan\( \beta \) = 60.

Fig. 4a: The running factors for the two higgs (non–SUSY) model with tan\( \beta \) = 3. Notation same as in fig. 1a.

Fig. 4b: Running factors for the two higgs (non–SUSY) model with tan\( \beta \) = 3, notation same as in fig. 1b.

Fig. 5a: Same as in fig. 4a, except that tan\( \beta \) = \( m_t/m_b \).

Fig. 5b: Same as in fig. 4b, except that tan\( \beta \) = \( m_t/m_b \).

Fig. 6a: Same as in fig. 4a, except that tan\( \beta \) = 70.

Fig. 6b: Same as in fig. 4b, except that tan\( \beta \) = 70.

Fig. 7a: The running factors for the minimal standard model in the same notation as fig. 1a.

Fig. 7b: Running of the mixing angles and mass ratios in the standard model, notation same as in fig. 1b.