Introduction

More than forty years ago, Igusa proved in [Ig] a “Fundamental Lemma”, according to which, theta constants give an explicit generically injective map from some modular varieties related to the symplectic group $\text{Sp}(n, \mathbb{R})$ into the projective space. The embedded varieties satisfy some quartic relations, the so called Riemann’s relations. We expect that similar results hold for orthogonal instead of symplectic groups, where one should use typical “orthogonal constructions” instead of theta series.

Some years ago, Borcherds described in [Bo1] two methods for constructing modular forms on modular varieties related to the orthogonal group $O(2, n)$. They are the so called Borcherds’ additive and multiplicative lifting. The multiplicative lifting has been used by Borcherds himself and other authors to construct modular forms with known vanishing locus and interesting properties. The additive lifting has been used to construct explicit maps from some modular varieties related to $O(2, 4)$, $O(2, 6)$, $O(2, 8)$, $O(2, 10)$ and also for some unitary groups as $U(1, 4)$ and $U(1, 5)$. cf. [FH], [AF], [Fr2], [Fr3], [FS], [Ko1], [Ko2].

In this paper we try a more systematic treatment in certain level 2 cases: We are mainly interested in even lattices $L$ of signature $(2, n)$, such that the discriminant $L'/L$ is a vector space over the field of two elements and such the the induced quadratic form $q : L'/L \to \mathbb{F}_2$ is of even type. This means that the dimension $2m$ of $L'/L$ is even and that $q$ is of the form

$$q(x) = x_1x_2 + \cdots + x_{2m-1}x_{2m}$$

with respect to a suitable basis. This implies also that $n \equiv 2 \mod 8$. To every such finite geometry $(\mathbb{P}_2^{2m}, q)$ we attach a certain graded algebra $\mathcal{R}_m/\mathcal{I}_m$. We expect that this algebra is related to the structure of certain modular varieties. There are three cases where this can be verified. The first one has been investigated by Kondo’s in connection with his research about the moduli.
space of Enriques surfaces. The lattice here is \( U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8) \) and in this case \( m = 5 \). In section 11 we describe in some detail this case again, giving the connection with our results and correcting some error in Kondo’s paper.

Another case, related to the configuration space of 8 points in the projective line, appears in [Ko2] and [Koi]. It is related to the lattice \( U \oplus \sqrt{2}U \oplus (-D_4) \oplus (-E_8) \), i.e. we have \( m = 3 \) in this case.

A third member is studied in this paper, namely the lattice \( \sqrt{2}(U \oplus U \oplus (-E_8)) \). Here \( m = 6 \). In all three cases the ideal \( \mathcal{I}_m \subset \mathcal{R}_m \) occurs. The modular variety, which we investigate, is a covering of Kondo’s variety (modular variety of Enriques surfaces), which can be treated in a similar manner. But we have a bigger symmetry group \( O(F_{12}) \) instead of \( O(F_{10}) \) with the effect that some difficulties of Kondo’s approach disappear.

The ideal \( \mathcal{I}_m \) is defined by quadratic relations in a polynomial ring \( \mathcal{R}_m \). Kondo considers instead of this only quartic relations. In [Ko2] and [Koi] has been explained in the case \( m = 3 \) that the quartic relations are consequences of quadratic relations. This is true for arbitrary \( m \), hence also in Kondo’s \( O(2,10) \)-case.

### 1. Lattices

A lattice \( L \) is a free abelian group together with a symmetric non degenerated bilinear quadratic form

\[
q_L : L \rightarrow \mathbb{Q}.
\]

The associated bilinear form is \( (x,y) = q_L(x,y) - q_L(x) - q_L(y) \). We extend \( (x,y) \) to a \( \mathbb{R} \)-bilinear from on \( V = L \otimes \mathbb{R} \) and moreover to a \( \mathbb{C} \)-bilinear form on \( L \otimes \mathbb{C} \) and we use the notation \( q_L(x) = (x,x)/2 \) also in this situation.

The norm of a vector \( a \in V \) is \( (a,a) = 2q_L(a) \). The lattice is called even if \( q_L(a) \) is integral for all \( a \). The dual lattice \( L' \subset L \otimes \mathbb{Z} \) consists of all elements \( a \) such that \( (a,x) \) is integral for all \( x \in L \). The lattice is called unimodular if \( L = L' \). We recall that an even unimodular lattice of signature \( (m,n) \) exists if and only if \( m - n \) is divisible by 8 and that it is determined up to isomorphism in this case. This lattice is denoted by \( \Pi_{m,n} \). We use the usual notation \( U = \Pi_{1,1} \).

In this paper we will treat mainly even lattices, such that \( 2L' \subset L \). Let us assume that \( V \) is of signature \( (2,n) \). We recall that there is a unique subgroup of index two \( O^+(V) \subset O(V) \), which contains all reflections

\[
x \mapsto x - \frac{(a,x)}{q_L(a)}a
\]

for vectors \( a \in V \) with negative \( q_L(a) \). We also consider the integral orthogonal group \( O(L) \) and

\[
O^+(L) = O(L) \cap O^+(V).
\]
Since $L$ is even, $q_L : L' \to \mathbb{Q}$ factors through a map

$$\bar{q}_L : L'/L \longrightarrow \mathbb{Q}/\mathbb{Z},$$

which is called the discriminant of $(L, q_L)$.

We denote by $\text{Aut}(A)$ the group of automorphisms of an abelian group and by

$$\text{O}(L'/L) = \text{O}(L'/L, \bar{q}_L) \subset \text{Aut}(L'/L)$$

the subgroup which preserves $\bar{q}_L$. The kernel of the map $\text{O}(L) \longrightarrow \text{O}(L'/L)$ is called the discriminant kernel. We define

$$\Gamma_L = \ker(\text{O}^+(L) \longrightarrow \text{O}(L'/L)).$$

Let $\mathcal{H}$ be the bounded symmetric domain of type IV associated to $\text{O}(V)$, then we shall consider the modular variety $\mathcal{H}/\Gamma_L$.

### 2. Finite geometries and related projective varieties

We consider $\mathbb{F}_2^{2m}$ equipped with the quadratic form

$$q(x) = x_1x_2 + \cdots + x_{2m-1}x_{2m}$$

and the associated bilinear form

$$(x, y) = q(x + y) - q(x) - q(y).$$

The orthogonal group (group of all automorphisms which preserve $q$) is denoted by $\text{O}(\mathbb{F}_2^{2m})$. Its order is

$$2^{m(m-1)+1}(2^m - 1) \prod_{i=1}^{m-1} (2^{2i} - 1).$$

The number of isotropic elements is $2^m(2^m + 1)$ and the number of anisotropic elements is $2^m(2^m - 1)$. We will need the following refined information:

**2.1 Lemma.** 1) Let $a \in \mathbb{F}_2^{2m}$ be a non-zero isotropic element. The number of all non-zero isotropic elements $b$ with $(a, b) = 0$ is

$$2^{m-1}(2^{m-1} + 1) - 1$$

and the number of all anisotropic elements $b$ with $(a, b) = 0$ is

$$2^{m-1}(2^{m-1} - 1).$$
The number of all isotropic elements with \((a, b) \neq 0\) is
\[2^{2(m-1)}.\]
and the number anisotropic elements \(b\) with \((a, b) \neq 0\) is also
\[2^{2(m-1)}.\]

2) Let \(a \in \mathbb{F}_2^{2m}\) be an anisotropic element.
The number of all non-zero isotropic elements \(b\) with \((a, b) = 0\) is
\[2^{2(m-1)} - 1\]
and the number of all anisotropic elements \(b\) with \((a, b) = 0\) is
\[2^{2(m-1)}.\]

The number of all isotropic elements with \((a, b) \neq 0\) is
\[2^m - 2^{m-1} + 1.\]
and the number anisotropic elements \(b\) with \((a, b) \neq 0\) is also
\[2^{m-1}(2^{m-1} - 1).\]

A sub-space \(A \subset \mathbb{F}_2^{2m}\) is called totally isotropic if all elements are isotropic. The maximal totally isotropic spaces are of dimension \(m\). They form one orbit under the orthogonal group.

2.2 Lemma. Let \(A \subset \mathbb{F}_2^{2m}\) be a totally isotropic subspace of dimension \(m - 2\). There are precisely 6 maximal totally isotropic spaces (of dimension \(m\)) which contain \(A\).

Proof. We decompose \(\mathbb{F}_2^{2m} = \mathbb{F}_2^{2m-4} \oplus \mathbb{F}_2^4\). Let \(A \subset \mathbb{F}_2^{2m-4}\) be a maximal totally isotropic subspace. Then \(A\) can be consider as a \((m-2)\)-dimensional totally isotropic space in \(\mathbb{F}_2^{2m}\). The maximal totally isotropic subspaces which contain it are of the form \(A \oplus B\) where \(B \subset \mathbb{F}_2^4\) is a two dimensional totally isotropic subspace of \(\mathbb{F}_2^4\). It is easy to check that there are 6 such spaces. \(\square\)

We list the non-zero elements of these subspaces \(B\):

\[
\begin{align*}
I_1 &: (1,0,0,0) (0,0,1,0) (1,0,1,0) \\
I_2 &: (1,0,0,0) (0,0,0,1) (1,0,0,1) \\
I_3 &: (0,1,0,0) (0,0,0,1) (0,1,0,1) \\
I_4 &: (0,1,0,0) (0,0,1,0) (0,1,1,0) \\
I_5 &: (1,0,0,1) (0,1,1,0) (1,1,1,1) \\
I_6 &: (1,0,1,0) (0,1,0,1) (1,1,1,1)
\end{align*}
\]

This table shows:
2.3 Lemma. Let \( A \subset \mathbb{F}_2^{2m} \) be a totally isotropic subspace of dimension \( m-2 \). The space which is generated by the characteristic functions of the maximal (=\( m \)-dimensional) totally isotropic subspaces containing \( A \) has dimension 5. If \( I_1, \ldots, I_6 \) denotes these subspaces in a suitable ordering and \( \chi_1, \ldots, \chi_6 \) are their characteristic functions then one has the relation

\[
\chi_1 + \chi_3 + \chi_5 = \chi_2 + \chi_4 + \chi_6
\]

and furthermore the nine spaces \( I_i \cap I_j \) with odd \( i \) and even \( j \) are \( (m-1) \)-dimensional.

Proof. This is a calculation inside \( \mathbb{F}_2^4 \) which can be done by means of the above table.

Recall that the dimension of a maximal totally isotropic subspace of \( \mathbb{F}_2^{2m} \) is \( m \). A typical example is the subspace defined by \( x_2 = x_4 = \cdots = x_{2m} = 0 \). We consider now \( (m-1) \)-dimensional totally isotropic subspaces \( N \subset \mathbb{F}_2^{2m} \), for example that one which is defined by the additional condition \( x_{2m-1} = 0 \).

2.4 Lemma. Let \( N \subset \mathbb{F}_2^{2m} \) be a \( (m-1) \)-dimensional totally isotropic subspace. There exist exactly two cosets different from \( N \), which contain only isotropic vectors and one coset which contains only anisotropic vectors.

Proof. Because we have a transitive action of the orthogonal group on the set of \( (m-1) \)-dimensional totally isotropic subspaces, it is sufficient to treat the above example. In this case the relevant orbits are represented vectors of the form \((1,0,1,0,1,0,1,0,1,0,*,*)\).

2.5 Definition. A star is a set of \( 2^{m-1} \) anisotropic vectors of \( \mathbb{F}_2^{2m} \) which form a coset of a \( (m-1) \)-dimensional totally isotropic space.

This isotropic space is determined by the star. A simple calculation shows:

2.6 Lemma. Let \( M \subset \mathbb{F}_2^{2m} \) be star with the underlying \( (m-1) \)-dimensional totally isotropic subspace \( N \subset \mathbb{F}_2^{2m} \). There exist precisely two \( m \)-dimensional (i.e. maximal) totally isotropic subspaces which contain \( N \). The difference of their characteristic functions is an element of \( \mathbb{C}[\mathbb{F}_2^{2m}] \) with the following property. It changes the sign if one applies the reflections \( x \mapsto x + (a,x)a \) along the \( 2^{m-1} \) anisotropic vectors \( a \) of the associated star.

Next we consider all \( (m-1) \)-dimensional totally isotropic subspaces which contain our \( (m-2) \)-dimensional standard space \( A \). They are of the form \( A + \mathbb{F}_2 \alpha \), where \( \alpha \) is a non-zero isotropic. Hence there are nine such spaces. We list them and the corresponding star (=unique orbit of this space which contains only anisotropic elements): We only list the \( \mathbb{F}_2^4 \)-components. The first \( 2m-4 \) components vary arbitrarily in \( A \).
Recall that every \((m-1)\)-dimensional totally isotropic space \(B\) is associated to a star, which is the unique coset consisting of anistropic elements. There are two \(m\)-dimensional totally isotropic spaces which contain \(B\).

**2.7 Remark.** Let \(\psi_i\) be the characteristic functions of the nine stars above. They satisfy the relations

\[
\begin{align*}
\psi_1 + \psi_3 &= \psi_8 + \psi_9 & \psi_1 + \psi_6 &= \psi_2 + \psi_8 & \psi_3 + \psi_5 &= \psi_4 + \psi_6 \\
\psi_1 + \psi_4 &= \psi_7 + \psi_9 & \psi_2 + \psi_3 &= \psi_6 + \psi_9 & \psi_3 + \psi_7 &= \psi_4 + \psi_8 \\
\psi_1 + \psi_5 &= \psi_2 + \psi_7 & \psi_3 + \psi_4 &= \psi_5 + \psi_9 & \psi_5 + \psi_8 &= \psi_6 + \psi_7
\end{align*}
\]

This relations are related to certain quadratic relations of characteristic functions.

**2.8 Lemma.** Let \(A \subset \mathbb{F}_2^{2m}\) be an totally isotropic subspace of dimension \(m-2\) and \(I_1, \ldots, I_6\) the maximal totally isotropic subspaces in a suitable ordering. Denote by \(\chi_1, \ldots, \chi_6: \mathbb{F}_2^m \to \mathbb{C}\) their characteristic functions. There are the quadratic relations

\[
\begin{align*}
(\chi_1 - \chi_2)(\chi_1 - \chi_4) &= (\chi_3 - \chi_6)(\chi_5 - \chi_6) \\
(\chi_1 - \chi_2)(\chi_3 - \chi_2) &= (\chi_5 - \chi_4)(\chi_5 - \chi_6) \\
(\chi_1 - \chi_2)(\chi_1 - \chi_6) &= (\chi_3 - \chi_4)(\chi_5 - \chi_4) \\
(\chi_1 - \chi_2)(\chi_5 - \chi_2) &= (\chi_3 - \chi_4)(\chi_3 - \chi_6) \\
(\chi_3 - \chi_4)(\chi_1 - \chi_4) &= (\chi_5 - \chi_2)(\chi_5 - \chi_6) \\
(\chi_3 - \chi_4)(\chi_3 - \chi_2) &= (\chi_1 - \chi_6)(\chi_5 - \chi_6) \\
(\chi_1 - \chi_4)(\chi_1 - \chi_6) &= (\chi_3 - \chi_2)(\chi_5 - \chi_2) \\
(\chi_1 - \chi_4)(\chi_5 - \chi_4) &= (\chi_3 - \chi_2)(\chi_3 - \chi_6) \\
(\chi_1 - \chi_6)(\chi_3 - \chi_6) &= (\chi_5 - \chi_2)(\chi_5 - \chi_4)
\end{align*}
\]

The proofs of 2.7 and 2.8 are trivial.
There are many other relations between characteristic functions. But the mentioned here will improve to be important, because in some cases they lead to relations between certain modular forms.

We remark that the number of \((m - 2)\)-dimensional totally isotropic spaces in \(\mathbb{F}_2^{2m}\) is

\[
\frac{5}{2^m + 1} \cdot \prod_{i=1}^{m-2} \frac{(2^{m+1-i} - 1)}{(2^{m-1-i} - 1)}
\]

Fortunately the 9 quadratic relations collapse to one if one takes the linear relation 2.3 into account. This gives us:

**2.9 Theorem.** Let \(A\) be a \((m - 2)\)-dimensional totally isotropic space. Let \(P\) be the five dimensional space which is generated by the characteristic functions of maximal totally isotropic spaces containing \(A\) (see 2.3). There is a distinguished element \(\pm R_A \in \text{Sym}^2(P)\) which for the standard \(A\) can be read off from 2.8 and is defined for general \(A\) in an obvious way (using decompositions of a union of two stars as union of two other stars). This element is non-zero and contained in the kernel of

\[
\text{Sym}^2(P) \longrightarrow \mathbb{C}[\mathbb{F}_2^{2m}].
\]

**A homogenous ideal**

For each maximal totally isotropic subspace \(V \subset \mathbb{F}_2^{2m}\) we consider a variable \(X_A\) and we consider the polynomial ring \(\mathbb{C}[\ldots X_A \ldots]\) in all these variables. We are interested in the subring \(R_m := \mathbb{C}[\ldots X_A - X_B \ldots]\) which is generated by differences \(X_A - X_B\). It is sufficient to take a fixed \(A\) and arbitrary \(B\). Hence \(R_m\) is a polynomial ring in one variable less. For the generation of \(R_m\) it is also sufficient to take those with \(\dim(A \cap B) = m - 1\). Replacing the characteristic function by the corresponding variable, we obtain from 2.3 and 2.8 a finite set of linear and quadratic relations of the form \(X_1 + X_3 + X_5 - X_2 - X_4 - X_6\) (s. 2.3) and from elements of the form \((X_1 - X_2)(X_1 - X_4) - (X_3 - X_6)(X_5 - X_6)\) as explained in 2.8. They are contained in \(R_m\) and we can consider the ideal

\[
I_m \subset R_m
\]

generated by them. We believe that this ideal is of interest and related to modular varieties. This paper contains a contribution in the case \(m = 6\). There are related investigations in the cases \(m = 3\) and \(m = 5\), cf. [Ko1] and [Ko2]. One knows

\[
\dim(R_m/I_m) = \begin{cases} 1 & \text{if } m = 1 \\ 3 & \text{if } m = 2 \\ 6 & \text{if } m = 3 \end{cases}
\]
Unfortunately not much more is known about this ideal. A result which has an application to modular varieties is that in the ideal certain quartic relations can be found: To explain this we consider an totally isotropic subspace $A \subset \mathbb{F}_2^{2m}$ of dimension $m - 3$, where we assume $m \geq 3$. Let $B$ be a coset which consists of anisotropic elements only. There are precisely 15 stars containing $B$. These 15 stars belong to 30 maximal totally isotropic spaces. We describe them concretely. For this we can assume $\mathbb{F}_2^{2m} = \mathbb{F}_2^{2(m-3)} \oplus \mathbb{F}_2^{6}$, $A$ is the standard totally isotropic space in $\mathbb{F}_2^{2(m-3)}$ defined by $x_2 = x_4 = \cdots = x_{2m-6} = 0$ and $B$ is the coset $A + (0, \ldots, 0, 1, 1, 0, 0, 0, 0)$. Then the thirty spaces are of the form $A \oplus X$, where $X$ is a maximal totally isotropic subspace in $\mathbb{F}_2^{6}$. We list the 30 spaces giving a basis in each case:

\begin{align*}
1 & \quad (0, 0, 0, 0, 0, 1)(0, 0, 1, 0, 0)(0, 1, 0, 0, 0) \\
2 & \quad (0, 0, 0, 0, 0, 1)(0, 0, 1, 0, 0)(1, 0, 0, 0, 0) \\
3 & \quad (0, 0, 0, 0, 0, 1)(0, 1, 0, 0, 0)(0, 1, 0, 0, 0) \\
4 & \quad (0, 0, 0, 0, 0, 1)(0, 1, 0, 0, 0)(1, 0, 0, 0, 0) \\
5 & \quad (0, 0, 0, 0, 0, 1)(1, 0, 1, 0, 0)(1, 0, 0, 0, 0) \\
6 & \quad (0, 0, 0, 0, 0, 1)(0, 1, 1, 0, 0)(1, 0, 0, 0, 0) \\
7 & \quad (0, 0, 0, 0, 0, 1)(0, 0, 1, 0, 0)(0, 1, 0, 0, 0) \\
8 & \quad (0, 0, 0, 0, 0, 1)(0, 0, 1, 0, 0)(0, 1, 0, 0, 0) \\
9 & \quad (0, 0, 0, 0, 0, 1)(0, 1, 0, 0, 0)(0, 1, 0, 0, 0) \\
10 & \quad (0, 0, 0, 0, 1, 0)(0, 1, 0, 0, 0)(1, 0, 0, 0, 0) \\
11 & \quad (0, 0, 0, 1, 0, 0)(0, 1, 1, 0, 0)(1, 0, 1, 0, 0) \\
12 & \quad (0, 0, 0, 1, 0, 0)(0, 1, 0, 0, 0)(1, 0, 0, 1, 0) \\
13 & \quad (0, 0, 0, 1, 0, 0)(0, 1, 0, 0, 0)(1, 0, 0, 1, 0) \\
14 & \quad (0, 0, 0, 1, 0, 0)(0, 1, 0, 0, 0)(1, 0, 0, 0, 1) \\
15 & \quad (0, 0, 0, 1, 0, 0)(0, 1, 0, 0, 0)(0, 0, 0, 0, 0)
\end{align*}

\[ (X_{29} - X_{30})(X_{17} - X_{18})(X_{27} - X_{28})(X_{7} - X_{8}) + (X_{11} - X_{12})(X_{13} - X_{14})(X_{15} - X_{16})(X_{21} - X_{22}) \]

is contained in the ideal $\mathcal{I}_m$.

We proceed as in 2.2 , We decompose $\mathbb{F}_2^{2m} = \mathbb{F}_2^{2m-6} \oplus \mathbb{F}_2^{6}$. Let $C \in \mathbb{F}_2^{2m-6}$ be a maximal totally isotropic subspace. Then $C$ can be considered as a $m - 3$ dimensional totally isotropic space in $\mathbb{F}_2^{2m}$. The maximal totally isotropic subspaces which contain it are of the form $C \oplus B$ where $B \subset \mathbb{F}_2^{6}$ is a three dimensional totally isotropic subspace of $\mathbb{F}_2^{6}$. So we reduced ourselves to the study of totally isotropic subspaces in $\mathbb{F}_2^{6}$. In this case we obtain 420 quartic relations, cf [Ko2] , 36 quadratic relations and 36 linear relations between the 105 stars. Using the computer algebra system SINGULAR, we got that there are 14 linear independent quadratic relations and the generated ideal contains the quartics. A similar calculation seems to be performed in [Koi].
3. Orthogonal modular forms

In the rest of this text, $L$ denotes an even lattice of signature $(2, n)$. We consider the subset of $L \otimes \mathbb{Z} \mathbb{C}$ defined by

\[(z, z) = 0, \quad (z, \bar{z}) > 0.\]

It is an analytic manifold, which consists of two connected components, which can be exchanged by the map $z \mapsto \bar{z}$. We denote by $\mathcal{H}_n$ one of the components and by $\mathcal{H}_n$ its image in the complex projective space $P(L \otimes \mathbb{Z} \mathbb{C})$. There is a subgroup of index two $O^+(V)$ of the orthogonal group $O(V)$ ($V = L \otimes \mathbb{Z} \mathbb{R}$) which preserves the two connected components. This group acts on $\mathcal{H}_n$ holomorphically. For any $z = x + iy \subset \mathcal{H}_n$ we consider the vector space

\[W = \mathbb{R}x + \mathbb{R}iy \in V := L \otimes \mathbb{Z} \mathbb{R}.\]

This is a positive definite subspace of $V$. Moreover the space $W$ depends only on the image of $z$ in $\mathcal{H}_n$. This defines a bijection between $\mathcal{H}_n$ and the set of two dimensional positive definite subspaces of $V$.

The full group $O(V)$ acts on the set of two-dimensional positive definite subspaces. Using the bijection above, on obtains an action of the whole $O(V)$ (not only $O^+(V)$) on $\mathcal{H}_n$. But this action is non-holomorphic for $g \in O(V) - O^+(V)$. It is given by the real analytic map $z \mapsto g(z)$.

We recall the notion of an (holomorphic or meromorphic) orthogonal modular form of weight $k \in \mathbb{Z}$ with respect to a subgroup $\Gamma \subset O^+(L)$ of finite index and with respect to a character $v : \Gamma \to \mathbb{C}^\bullet$. It is a (holomorphic or meromorphic) function $f : \mathcal{H}_n \to \mathbb{C}$ with the properties

\[f(\gamma z) = v(\gamma) f(z),\]
\[f(tz) = t^{-k} f(z).\]

A condition at the cusps has to be added. This condition is in most cases automatically satisfied, for example for $n \geq 3$. The vector space of holomorphic forms, which are regular at the cusps, is denoted by $[\Gamma, k, v]$ and by $[\Gamma, k]$ when $v$ is trivial. These are finite dimensional spaces and moreover, the algebra

\[A(\Gamma) = \bigoplus_{k \in \mathbb{Z}} [\Gamma, k]\]

is finitely generated.

Let $\text{Mp}(2, \mathbb{Z})$ be the metaplectic cover of $\text{SL}(2, \mathbb{Z})$. The elements of $\text{Mp}(2, \mathbb{Z})$ are pairs $(M, \sqrt{ct + d})$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, and $\sqrt{ct + d}$
denotes a holomorphic root on $H$ of $c\tau + d$. It is well known that $Mp(2, \mathbb{Z})$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}, \text{ Re } \sqrt{\tau} > 0.$$  

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i$ is the standard generator of the center of $Mp(2, \mathbb{Z})$.

Recall that there is a unitary representation $\rho_L$ of $Mp(2, \mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$.

$$\rho_L(T) = (e^{2\pi i q_L(\alpha)})_{\alpha \in L'/L} $$

(diagonal matrix)

$$\rho_L(S) = \frac{\sqrt{1}}{\sqrt{|L'/L|}} (e^{-2\pi i (\alpha,\beta)})_{\alpha,\beta \in L'/L'}.$$  

This representation is the Weil representation attached to the finite quadratic form $(L'/L, \bar{q}_L)$.

We recall the notion of an elliptic modular form with respect to a finite dimensional representation $\varrho : Mp(2, \mathbb{Z}) \rightarrow GL(W)$. Let $k \in (1/2)\mathbb{Z}$ and $f : H \rightarrow W$ be a holomorphic function. Then $f$ is called modular form of weight $k$ with respect to $\varrho$ if

$$f(M\tau) = \sqrt{c\tau + d}^{2k} \varrho(M, \sqrt{c\tau + d}) f(\tau)$$

for all $(M, \sqrt{c\tau + d}) \in Mp(2, \mathbb{Z})$ and if $f$ is holomorphic at $i\infty$. We denote the space of all these modular forms by $[SL(2, \mathbb{Z}), k, \varrho]$. A form is called a cusp form if it vanishes at $\infty$. The space $[SL(2, \mathbb{Z}), k, \varrho]$ decomposes into the direct sum of the subspace of cusp forms and into the space of Eisenstein series which can be defined as orthogonal complement with respect to the Petersson scalar product. An Eisenstein series is determined by its constant Fourier coefficient.

The additive lift

In [Bo1], Borcherds defined for integral $k + n/2$ a linear map

$$[SL(2, \mathbb{Z}), k, \varrho_L] \rightarrow \Gamma_L, [k + n/2 - 1],$$

which generalizes constructions of Saito-Kurokawa, Shimura, Maass, Gritsenko, Oda et.al. We are interested in this construction in the case $k = 0$. Modular forms of weight zero are constant, hence we get for even $n$ a map

$$\mathbb{C}[L'/L]^{SL(2, \mathbb{Z})} \rightarrow \Gamma_L, n/2 - 1].$$
These orthogonal modular forms are the simplest examples of modular forms of several variables. The weight \( n/2 - 1 \) is the so-called singular weight. Every modular form of weight \( 0 < k < n/2 - 1 \) vanishes. Modular forms of singular weight which are cusp-forms vanish identically*).

There is a remarkable special case. Assume that \( L \) is even and selfdual. Then the Weil representation is the trivial one-dimensional representation and we get a linear map \( \mathbb{C} \to [O^+(L), n/2 - 1] \). The image of 1 is a very distinguished modular form. Borcherd’s formula for the Fourier coefficients show that it is different from 0 and that it agrees with a modular form which has already considered earlier by Gritsenko and in a special case also by Krieg. We call it here Gritsenko’s singular modular form.

We recall from [Fr2] some facts about the Baily Borel compactification of \( \mathcal{H}_n/\Gamma \) for a subgroup of finite index \( \Gamma \subset O^+(L) \). There are one- and zero-dimensional boundary components which correspond to the \( \Gamma \)-orbits of two and one-dimensional totally isotropic subspaces of \( L \otimes \mathbb{Q} \). We recall that the value of an orthogonal modular form \( F \in [\Gamma, k] \) at a non-zero isotropic element \( \alpha \in L \otimes \mathbb{Q} \) can be defined. One chooses \( \beta \in L \otimes \mathbb{Q} \) such that \( (\alpha, \beta) = 1 \) and defines

\[
F(\alpha) : \lim_{\mathop{\text{Im}} \tau \to \infty} F(\tau \alpha + 2i(\alpha, \beta)\beta).
\]

It is easy to check that this limit exists (using the Fourier expansion of \( F \) as described in [Bo1]) and that this limit is independent of the choice of \( \beta \) and that it depends only on the \( \Gamma \)-orbit of \( \alpha \). One has \( F(t\alpha) = t^{-k}F(\alpha) \).

Representing the zero-dimensional boundary components by primitive elements \( \alpha \in L' \) we obtain:

When the weight \( k \) is even, \( F(\alpha) \) (for \( \alpha \in L' \) primitive isotropic) can be considered as function on the set of zero-dimensional boundary points of \( \mathcal{H}_n/\Gamma \).

Additive lifts of constants have a basic property:

3.1 Lemma. Let \( F \) be a singular modular form, which is the additive lift of a constant elliptic modular form. When all values \( F(\alpha), \alpha \in L' \), primitive and isotropic, are zero, then \( F \) is identically zero.

This follows from Borcherds’ description of the Fourier expansion of \( F \) [Bo1], theorem 14.3. The formulae there show that the „restriction“ of \( F \) to the one-dimensional boundary components are elliptic Eisenstein series, which hence are orthogonal to cusp-forms. Eisenstein series vanish identically if they vanish at all cusps. We don’t want to give more details. We only mention that Borcherds constructed examples of non-trivial singular modular forms which

* The theory of singular modular forms is well established in the Siegel case. For the orthogonal case there seems to be no good reference at the moment. The proofs for the basic facts are the same.
vanish at all zero dimensional cusps. So these cannot be additive lifts. A systematic treatment of them is a major unsolved problem.

Because the value at a cusp appears as constant Fourier coefficient, we can use Borcherds formula from [Bo1], theorem 14.3 to compute the values of additive lifts at the cusps. (In the first line of the final formula in 5. one has to replace \(c_{\delta z}(0)\) by \(c_{\delta z/N}(0)\).) We only reproduce the formula in a very special case. Let \(B(n)\) be the \(n\)-th Bernoulli number.

3.2 Proposition. Assume that \(n \equiv 2 \mod 4\) and that \(2L' \subset L\). Let \(F\) be the additive lift of \(C \in \mathbb{C}[L'/L]^{\text{SL}(2,\mathbb{Z})}\). Then \(F(\alpha)\) at a primitive isotropic element \(\alpha \in L'\) is

\[-\frac{B(n/2 - 1)}{n - 2}C(0)\]  
if \(\alpha \in L\)

and otherwise

\[-\frac{B(n/2 - 1)}{n - 2}(C(0) + C(\alpha)(1 - 2^{n/2 - 1}))\]

We immediately obtain:

3.3 Proposition. Assume that \(n \equiv 2 \mod 4\), \(n > 2\) and that \(2L' \subset L\). Assume furthermore that a linear subspace \(H \subset \mathbb{C}[L'/L]^{\text{SL}(2,\mathbb{Z})}\) with the following property is given: All elements of \(H\) vanish at the zero element of \(L'/L\). Then there exists a non-zero constant \(\gamma = \gamma(n)\) such that for any \(C \in H\) with corresponding additive lift \(F = FC\) and every primitive isotropic \(\alpha \in L'\) which is not contained in \(L\) the formula

\[F(\alpha) = \gamma \cdot C(\alpha)\]

holds.

Corollary. If in addition every non zero element \(\bar{\alpha} \in L'/L\) with \(\bar{q}_L(\bar{\alpha}) = 0\) can be represented by an isotropic element of \(L'\), then the additive lift

\[H \rightarrow [\Gamma, n/2 - 1]\]

is injective.

Let \(\alpha \in L'\) be an element of the dual lattice and \(m < 0\) a negative integer. The Heegner divisor \(H(\alpha, m) \subset \mathcal{H}_n\) is the union of all

\[v^\perp \cap \mathcal{H}_n\]

\((v^\perp\) orthogonal complement of \(v\) in \(P(V \otimes_R \mathbb{C})\)),

where \(v\) runs through all elements in \(L'\) with

\[v \equiv \alpha \mod L\]  
and  
\[q_L(v) = m.\]
We consider $H(\alpha, m)$ as a divisor on $\mathcal{H}_n$ by attaching multiplicity 1 to all components. We have $H(\alpha, m) = H(-\alpha, m)$, more precisely, this divisor depends only on the image of $\alpha$ in $(L'/L)/\pm 1$.

The existence of an orthogonal modular forms whose zero divisor is a given Heegner divisor is regulated by Borcherds’ space of obstructions

$$[\text{SL}(2, \mathbb{Z}), n/2 + 1, \delta_L].$$

**3.4 Theorem.** Assume $n > 2$. Then the space of obstructions contains an Eisenstein series $\sum b_\alpha(m)e^{2\pi im\tau}$ with constant Fourier coefficient

$$b_\alpha(0) = \begin{cases} -1/2 & \text{if } \alpha = 0, \\ 0 & \text{else.} \end{cases}$$

A finite linear combination

$$\sum_{\alpha \in (L'/L)/\pm 1, m < 0} C(\alpha, m)H(\alpha, m) \quad (C(\alpha, m) \in \mathbb{Z})$$

is the divisor of a meromorphic modular form $F$ of weight

$$k = \sum_{m \in \mathbb{Z}, \alpha \in L'/L} b_\alpha(m)C(\alpha, -m)$$

if this number is integral and if for every cusp form $f$ in the space of obstructions,

$$f_\alpha(\tau) = \sum_{m \in \mathbb{Q}} a_\alpha(m)\exp(2\pi im\tau),$$

the relation

$$\sum_{m < 0, \alpha \in L'/L} a_\alpha(-m)C(\alpha, m) = 0$$

holds.

(The assumption that $k$ is integral can be omitted if one introduces modular forms of non-necessarily integral weight).
4. Level-two-cases

We want to consider level-two situations which means that we assume $2L' \subset L$. Then $L'/L$ is a vector space over the field $\mathbb{F}_2$ of two elements. We use the embedding

$$\mathbb{F}_2 \rightarrow \mathbb{Q}/\mathbb{Z}, \quad 0 \mapsto 0 + \mathbb{Z}, \quad 1 \mapsto 1/2 + \mathbb{Z}.$$

The quadratic form $\bar{q}_L : L'/L \rightarrow \mathbb{Q}/\mathbb{Z}$ then can be considered (as usual in the theory of quadratic forms over finite fields) as function

$$\bar{q}_L : L'/L \rightarrow \mathbb{F}_2.$$

For the rest of this paper we make the

4.1 Assumption. The lattice $L$ is even and has signature $(2,n)$. It has the further property $2L' \subset L$. The $\mathbb{F}_2$-vector space $L'/L$ admits a basis such that

$$\bar{q}_L(x) := x_1x_2 + \cdots + x_{2m-1}x_{2m} \quad (2m = \dim(L'/L)).$$

(This means that the finite quadratic form is of “even type”.)

It is well-known that the signature $n - 2 \mod 8$ is determined by the associated finite quadratic form. In our case it follows

$$n \equiv 2 \mod 8.$$

The Weil representation on $\mathbb{C}[\mathbb{F}_2^m]$ factors through $\text{SL}(2,\mathbb{Z}/2\mathbb{Z})$. It is a real representation with the action of $\varrho_L(S)$ and

$$\varrho_L(T) = ((-1)^{q(\alpha)})_{\alpha \in \mathbb{F}_2^m} \quad \text{(diagonal matrix)}.$$

We compute the character of this representation. Recall that $\text{SL}(2,\mathbb{Z}/2\mathbb{Z})$ is isomorphic to the symmetric group $S_3$. Hence it has three conjugacy classes, which are characterized by their orders 1,2,3. Representatives are the matrices $E,S,ST$ (where $E$ denotes the unit matrix). The traces are given by

$$\text{trace}(\varrho(E)) = 2^{2m}$$
$$\text{trace}(\varrho(T)) = 2^{m-1}(2^m + 1) - 2^{m-1}(2^m - 1) = 2^m$$
$$\text{trace}(\varrho(ST)) = \frac{1}{2^m}(2^{m-1}(2^m + 1) - 2^{m-1}(2^m - 1)) = 1$$

The character table is

|     | $E$ | $T$ | $ST$ |
|-----|-----|-----|------|
| 1   | 1   | 1   | 1    |
| 1   | -1  | 1   |      |
| 2   | 0   | -1  |      |

We obtain
4.2 Lemma. We have

\[ \dim \mathbb{C}[\mathbb{F}_2^{2m}]^{\text{SL}(2,\mathbb{Z})} = 2^{m-1} + \frac{1}{3}2^{2m-1} + \frac{1}{3}. \]

In the case \( m = 6 \) this is 715.

On \( \mathbb{C}[\mathbb{F}_2^{2m}] \) also \( \text{O}(\mathbb{F}_2^{2m}) \) acts and this action commutes with the Weil representation. Hence the group \( \text{SL}(2,\mathbb{Z}/2\mathbb{Z}) \) also acts on the space of \( \text{O}(\mathbb{F}_2^{2m}) \)-invariants

\[ \mathbb{C}[\mathbb{F}_2^{2m}]^{\text{O}(\mathbb{F}_2^{2m})}. \]

This space is three dimensional and generated by three elements \( E_0, E_+, E_- \). Here \( E_0 \) means the characteristic function of the zero element, \( E_+ \) is the characteristic function of the set of non-zero isotropic elements and \( E_- \) is the characteristic function of the non isotropic elements. Using this basis, we get a representation

\[ \tilde{\varrho} : \text{SL}(2,\mathbb{Z}/2\mathbb{Z}) \to \text{GL}(3,\mathbb{C}). \]

From 2.1 we obtain:
1) The matrix \( \tilde{\varrho}(T) \) is the diagonal matrix with the entries 1, 1, -1.
2) The diagonal entries of \( \tilde{\varrho}(S) \) are

\[ \frac{1}{2^m}, \frac{1}{2} - \frac{1}{2^m}, \frac{1}{2}. \]

We obtain

\[ \text{trace}\tilde{\varrho}(E) = 3, \quad \text{trace}\tilde{\varrho}(T) = 1, \quad \text{trace}\tilde{\varrho}(ST) = 0. \]

A glance at the character table shows:

4.3 Lemma. The space

\[ \mathbb{C}[\mathbb{F}_2^{2m}]^{\text{O}(\mathbb{F}_2^{2m})} \]

decomposes under the Weil representation into an irreducible two dimensional and a trivial one-dimensional representation. The space

\[ \mathbb{C}[\mathbb{F}_2^{2m}]^{\text{O}(\mathbb{F}_2^{2m}) \times \text{SL}(2,\mathbb{Z})} \]

is one dimensional. It is generated by the function which assigns \( (2^{m-1}+1) \) to the zero element, 1 to all non-zero isotropic and 0 to the anisotropic elements of \( \mathbb{F}_2^{2m} \).
4.4 Proposition. The additive lift

\[ \mathbb{C}[[F_2^m]]^{\text{SL}(2, \mathbb{Z})} \longrightarrow [\Gamma_L, n/2 - 1] \]

is injective.

Proof. We decompose the space \( \mathbb{C}[[F_2^m]]^{\text{SL}(2, \mathbb{Z})} \). We consider the subspace \( H \) of all functions \( C \), which satisfy the linear equation

\[ C(0) = 0. \]

The full invariant element is not contained in \( H \). Hence \( \mathbb{C}[[F_2^m]]^{\text{SL}(2, \mathbb{Z})} \) is the direct sum of the one-dimensional invariant space and \( H \). Since 3.3 applies, we obtain that the additive lift restricted to \( H \) is injective. The image contains no full invariant form. But the image of the invariant element is a full invariant form, which is non-zero. Hence we have injectivity on the whole. \( \square \)

It is possible to construct explicit elements of \( \mathbb{C}[[F_2^{12}]]^{\text{SL}(2, \mathbb{Z})} \). The following lemma is well-known, [Ko1], [Sch].

4.5 Lemma. The characteristic function of a maximal totally isotropic subspace of \( F_2^m \) is \( \text{SL}(2, \mathbb{Z}) \)-invariant.

The sum of all these characteristic functions is invariant under the full modular group. Hence we obtain:

4.6 Lemma. Assume that \( \mathbb{C}[[F_2^m]]^{\text{SL}(2, \mathbb{Z})} \) decomposes under \( \text{O}(F_2^m) \) into the sum of a one-dimensional and an irreducible subspace \( H \). Then the space \( \mathbb{C}[[F_2^m]]^{\text{SL}(2, \mathbb{Z})} \) is spanned by the characteristic functions of maximal totally isotropic subspaces.

Remark. The irreducibility of \( H \) can be checked for \( m \leq 6 \).

There is another important invariant element, which belongs to an arbitrary non-zero isotropic element:

4.7 Remark. Let \( a \) be a non-zero isotropic element. Let \( A \) be the set of all maximal totally isotropic subspaces which contain \( a \) and \( B \) the complementary set. Denote by \( \chi_A \) the characteristic function of a maximal totally isotropic subspace. Consider the element

\[ \sum_{A \in A} \chi_A - 2^{m-2} \sum_{B \in B} \chi_B. \]

It is a multiple of the function that assumes the value \( 2^{m-2} \) on the null vector, \( -2^{m-2} \) on the vector \( a \), 1 on the isotropic vectors \( b \) with \( (a, b) = 1 \) and 0 on the remaining vectors.
We omit the easy proof. In the Special case \( a = (1, 0, \ldots, 0) \) we get
\[
\begin{cases}
2^{m-2} & \text{for } x = (0, 0, \ldots, 0) \\
-2^{m-2} & \text{for } x = (1, 0, \ldots, 0) \\
1 & \text{for isotropic } x \text{ with } x_2 = 1 \\
0 & \text{else}
\end{cases}
\]

Now we consider the even and self dual lattice \( \Pi_{2,n} \), \( n \equiv 2 \mod 8 \). The discriminant group is zero. To obtain a non-trivial discriminant group we rescale this lattice and consider
\[
L = \sqrt{2} \Pi_{2,n}, \quad L' = L/2.
\]

Then \( L'/L \) is a vector space of dimension \( 2+n \) over the field \( \mathbb{F}_2 \) of two elements. We can choose an isomorphism \( \mathbb{F}_2^{2+n} \to L'/L \) such that the quadratic form is
\[
q(x) := x_1x_2 + \cdots + x_{2m-1}x_{2m}, \quad m = 1 + n/2.
\]

Let us assume \( n = 10 \), the group \( \text{O(12)} \) contains a simple subgroup of index two. The character table of this group is known. For example the command “CharTable(O12+(2).2)” of the computer algebra system GAP gives it. The representations of lowest dimensions have dimension 1, 651 and 714 and they are unique in these cases. Hence we obtain:

**4.8 Proposition.** The additive lift of \( \mathbb{C}[\mathbb{F}_2^{12}]^{\text{SL}(2,\mathbb{Z})} \) to \( [\text{O}^+(\Pi_{2,10}[2],4)] \) is injective and decomposes into the one-dimension trivial and the unique 714-dimensional irreducible representation.

## 5. Some Borcherds’ products

We first investigate Borcherds’ products for the full modular group \( \text{O}^+(\Pi_{2,n}) \), \( n \equiv 2 \mod 8 \). The space of obstructions is the space of usual elliptic modular forms of weight \( n/2 + 1 \). The cases \( n = 10, 18, 26 \) are of particular interest because there are no elliptic cusp forms of weight 6, 10 and 14. It is known that the group \( \text{O}^+(\Pi_{2,n}) \) acts transitively on the set of all primitive vectors of a given norm. We study the two lowest cases, vectors of norm \( (x,x) = -2 \) and \( (x,x) = -4 \). Before we continue, we modify slightly our notation for the Heegner divisors. We denote by \( H(m) \) the set of all elements of \( \mathcal{H}_n \), which are orthogonal to some vector \( x \in \Pi_{2,n} \) with \( (x,x) = m \). For \( \alpha \in \mathbb{F}_2^{2+n} \) we denote by \( H_\alpha(m) \) the set of all points of \( \mathcal{H}_n \), which are orthogonal to some \( x \in \Pi_{2,n} \) with \( (x,x) = m \) and \( x \equiv \alpha \mod 2 \). Hence we have
\[
H(m) = \bigcup_{\alpha \in \mathbb{F}_2^{2+n}} H_\alpha(m).
\]
With the notations of section 3 we have $H(m) = H(0, 2m)$ where the underlying lattice for the definition of $H(0, m)$ is $\Pi_{2,n}$. If we identify $\alpha$ with an element of $L'/L$ where $L = \sqrt{2} \Pi_{2,n}$ we get because of the rescaling factor

$$H_\alpha(m) = H(\alpha, 2m).$$

The Heegner divisors $H(-1)$ and $H(-2)$ are irreducible in $\mathcal{H}_n/O^+(\Pi_{2,n})$. This follows from the fact that two primitive vectors of $\Pi_{2,n}$ of the same norm are equivalent mod $O^+(\Pi_{2,n})$ and that elements of norm $-2$ and $-4$ are automatically primitive. We compute the weights of the modular forms with this divisor. They are regulated by the Eisenstein series of weight 6, 10 and 14. Their Fourier expansion, normalized such the constant coefficient is $-1/2$, starts as follows:

- **weight 6:** $-\frac{1}{2} + 252q + 8316q^2 + \cdots$
- **weight 10:** $-\frac{1}{2} + 132q + 67716q^2 + \cdots$
- **weight 14:** $-\frac{1}{2} + 12q + 98316q^2 + \cdots$

5.1 Lemma. *In the cases $n = 10$, 18, 26 there exists a modular form on the full group $O^+(\Pi_{2,n})$ whose zero divisor is the irreducible divisor $H(-1)$. The weights are 252, 132, 12. There also exists a modular forms whose zero divisor is the irreducible divisor $H(-2)$. The weights are 8316, 67716, 98316.*

We now investigate the level two case.

5.2 Lemma. *The divisor $H_\alpha(-1)$ is not empty if and only if $\alpha \in \mathbb{F}_2^m$ is anisotropic. In this case it is irreducible in $\mathcal{H}_n/O^+(\Pi_{2,n})[2]$. Hence $H(-1)$ considered in $\mathcal{H}_n/O^+(\Pi_{2,n})[2]$ has $2^{m-1}(2^m - 1)$ irreducible components.\*\*\*Lmm\*\*

The divisor $H_\alpha(-2)$ is not empty if and only if $\alpha \in \mathbb{F}_2^m$ is a non zero isotropic. In this case it is irreducible in $\mathcal{H}_n/O^+(\Pi_{2,n})[2]$. Hence $H(-2)$ considered in $\mathcal{H}_n/O^+(\Pi_{2,n})[2]$ has $2^{m-1}(2^m + 1) - 1$ irreducible components.*\*\*Lhc\*\*

We are interested in the case that the modular form, which belongs to $H(-1)$ or $H(-2)$, splits into a product of forms with irreducible divisors $H_\alpha(-1)$ or $H_\alpha(-2)$. But then the number of components should divide the weight of this modular form. There is one distinguished case where this happens. In the case $\Pi_{2,10}$ and the divisor $H(-2)$ the weight has been computed as 8316 (s. 6.1) and the number of components is 2079. One has 8316/2079 = 4. Thus we are lead to

5.3 Proposition. *For every non-zero isotropic $\alpha \in \mathbb{F}_2^{12}$ (their number is 2079) there exists a modular form on the congruence group of level two $O^+(\Pi_{2,10})[2]$ with divisor $H_\alpha(-2)$. The weight of this form is 4.* \*\*Pfs\*\*
§6. Projection of groups

Proof. We want to apply Borcherd’s obstruction theory. For this purpose we have to compute the space of elliptic cusp forms of weight 6 with respect to the Weil representation on $\mathbb{C}[\mathbb{F}_2^{12}]$. The Weil representation is trivial on the principal congruence subgroup of level 2 of the elliptic modular group. The space of cusp forms of weight 6 of this group has dimension one, a generating element is $\eta^{12}$, where $\eta$ denotes the Dedekind $\eta$-function. The expansion of $\eta^{12}$ is of the form

$$\eta(\tau)^{12} = \sum_{n>0} a_n e^{2\pi i n\tau}, \quad a_n \neq 0 \implies n \in \frac{1}{2} + \mathbb{Z}.$$ 

Hence no $a_n \neq 0$ with integral $n$ occurs. We see that actually all $H_\alpha(m)$ with even negative $m$ (and non zero isotropic $\alpha$) are divisors of modular forms.

We shall consider particular additive lifts of the elements from $\mathbb{C}[\mathbb{F}_2^{12}]^{SL(2,\mathbb{Z})}$ and identify them with Borcherds products.

5.4 Proposition. Let $M \subset \mathbb{F}_2^{12}$ be a star (2.5). The additive lift of the corresponding Weil invariant (2.6) is a modular form in $[O^+(H_{2,10})][2,4]$ whose zero divisor consists precisely of the 32 Heegner divisors $H_\alpha(-1)$ with $\alpha \in M$.

Proof. Since the additive lift changes its sign under the $2^5$ reflections, there must be zeros along the $2^5$ Heegner divisors. We have to show that there are no other zeros. For this one takes the product with respect to all $a$ and divides by a suitable power with the form of weight 252 from 6.1. The result is a form of weight 0 which has to be constant. ⊓⊔

For similar statements cf [Ko1] and [Ko2].

5.5 Definition. Let $M$ be a star. The divisor consisting of the 32 Heegner divisors $H_\alpha(-1)$ with $\alpha \in M$ is called the associated star divisor.

We consider now the additive lift of the Weil invariant, which has been attached to an arbitrary non zero isotropic element in 4.7. We will show that this is also a Borcherds product.

5.6 Theorem. Let $\alpha \in \mathbb{F}_2^{12}$ be a non-zero isotropic vector. The Borcherds product with divisor $H_\alpha(-2)$ as described in 6.3 is contained in the additive lift space. It is (up to a constant factor) the additive lift of the Weil invariant defined in 4.7.

All what we have to show is that the additive lift vanishes along $H_\alpha(-2)$. The proof of this is quite involved because we have now reflection with fixes this divisor. We postpone the proof to the next section.
6. Projection of groups

Let $L$ be an even lattice of signature $(2,n)$ and $M \subset L$ a sublattice of signature $(2,m)$. If the connected components $\mathcal{H}_n, \mathcal{H}_m$ have been properly chosen then there is a natural inclusion $\tilde{\mathcal{H}}_m \subset \mathcal{H}_n$, which induces an inclusion of the corresponding half planes $\tilde{\mathcal{H}}_m \subset \mathcal{H}_n$. Let $\Gamma \subset O^+(L)$ a subgroup of finite index. We consider the subgroup of all $g \in \Gamma$ with $g(M) = M$. There is a natural homomorphism of this subgroup into $O(M)$ and it is easy to see that this subgroup —let’s denote it by $\Gamma'$— is contained in $O^+(M)$. We call this group the projected group. If $v, v'$ are compatible characters of $\Gamma, \Gamma'$ there is a natural restriction map

$$[\Gamma, k, v] \rightarrow [\Gamma', k, v'].$$

6.1 Theorem (Kneser). Assume that $L$ is a lattice of index $(2,m)$, $m \geq 4$, which has Witt-rank two (which means that $L \times \mathbb{Z} \mathbb{Q}$ contains a totally isotropic subspace of dimension two). Assume furthermore that $L$ contains a vector of norm $-2$ and that the following two conditions hold:

a) There exists a sublattice $L_1$ of rank $\geq 5$ whose discriminant is not divisible by 3.

b) There exists a sublattice of rank $\geq 6$ with odd discriminant.

Then the discriminant kernel $\Gamma_L$ is generated by reflections $x \mapsto x + (a, x)a$ along norm $-2$ vectors $a$.

The lattice $D_m$ consists of all $x \in \mathbb{Q}^m$ such that $x_1 + \cdots + x_m$ is even. The quadratic form is

$$q(x) = \frac{1}{2}(x_1^2 + \cdots + x_m^2).$$

We apply Knesers result 7.1 to the lattice $\mathbb{Z}^4 \times (-D_m)$ where $m \geq 3$. The lattices $A_1$ and $A_2$ can be embedded into $D(m)$, $m \geq 3$. Hence we can take $L_1 = \mathbb{Z}^4 \times (-A_1)$ (discriminant 2) and $L_2 = \mathbb{Z}^4 \times (-A_2)$ (discriminant 3).

6.2 Lemma. The discriminant kernels $\Gamma_L$ are generated by reflections along vectors of norm $-2$ in the cases

$$L = L_m = \mathbb{Z}^4 \times (-D_m) \quad \text{with} \quad m \geq 3.$$
A vanishing result

It may happen that a Heegner divisor is the divisor of a Borcherds product. In this case the weight of this product is given by a certain Fourier coefficient of an elliptic Eisenstein series. This Fourier coefficient is defined even if the Heegner divisor is not a Borcherds product. We denote this number the virtual weight of the Heegner divisor. We refer to the paper [BK] of Bruinier and Kuss for the definition and computation of these numbers. We want to use the following non-trivial result of Bruinier:

6.3 Proposition (Bruinier). Let $L$ be an even lattice of signature $(2, m)$ with $m \geq 4$ and $f$ a non vanishing (entire) modular form of weight $k$ on the discriminant kernel $\Gamma_L$. Assume that $f$ vanishes along a certain Heegner divisor $H$ (and maybe also on another other divisor). Then $k$ is greater or equal than the virtual weight of $H$.

Now we have the tools for the
Proof of 6.6. We take the realization $\Pi_{2,10} = \mathbb{Z}^4 \times (-E_8)$ and we choose a concrete isomorphism $\Pi_{2,10}/2\Pi_{2,10} \cong \mathbb{F}_2^{12}$. We consider a vector $a$ of norm -4. It can be taken inside $-E_8$. It defines a non-zero isotropic vector $\alpha$ in $\mathbb{F}_2^{12}$. This vector defines a Weil invariant (s. 4.7). We denote by $f$ the additive lift of it. This is modular form with trivial character under the stabilizer $\Gamma$ of $\alpha$ inside $O^+(\Pi_{2,10})$. From its concrete description in connection with 3.3 follows that it vanishes at all cusps which are defined by isotropic elements orthogonal to $a$.

All what we have to show is that $f$ vanishes along the orthogonal complement of $a$. The orthogonal complement of a a norm 4 vector inside $E_8$ is isomorphic to the root lattice $D_7$. Generally inside $D_{m+1}$ sits $D_m$ as orthogonal complement of a root. Hence we can consider the chain

$$\mathbb{Z}^4 \times (-D_2) \subset \cdots \subset \mathbb{Z}^4 \times (-D_7) \subset \mathbb{Z}^4 \times (-E_8).$$

We denote the corresponding half planes by $H_4 \subset \cdots \subset H_{10}$. The claim is that $f$ vanishes on the nine dimensional $H_9$. This will be done inductively starting form $H_4$.

The case $D_2$ is exceptional, one can not apply Kneser’s theorem. This lattice is generated by two orthogonal vectors $(1, \pm 1)$ of norm -2. Hence it is isomorphic to $A_1 \times A_1$. From the explicit description of the automorphism groups [CS] follows that every automorphism of $D_2$ extends to an automorphism of $D_7$ and this extends to an automorphism of $E_8$ which fixes $a$. The orthogonal group $O^+(\mathbb{Z}^4 \times -(A_1 \times A_1))$ is isomorphic to the extended Hermitian modular group of degree two of the Gauss number field [FH]. This group is generated by Eichler transformations and by the automorphisms of $A_1 \times A_1$. Eichler transformations of course extend to $O^+(\mathbb{Z}^4 \times (-E_8))$. Hence we obtain that the projected group of $\Gamma$ is the full $O^+(\mathbb{Z}^4 \times (-D_2))$ which can be identified with
the extended Hermitian modular group. From the structure theorem [Fr3] one can see every symmetric cusp form of weight 4 vanishes.

The next step is to prove that \( f \) vanishes on \( H_5 \). It is a modular form with respect to a certain projected group \( \Gamma_5 \). Now we can apply Kneser’s result to prove that this group contains the discriminant kernel of \( \mathbb{Z}^4 \times (-D_5) \). The image of \( H_4 \) is contained in the Heegner divisor which is defined by a root. But this Heegner divisor is irreducible. This follows from the following well-known fact:

6.4 Lemma. Let \( L \) be an even lattice of which contains a copy of \( \Pi_{2,2} \). Then any two primitive vectors \( a, b \in L' \) with the same image in \( L'/L \) are equivalent mod \( \Gamma_L \).

We have seen that the image of \( H_4 \) inside \( H_5/\Gamma_5 \) is a Heegner divisor and that \( f \) vanishes along it. Now we can apply Brunier’s result 7.3. A concrete calculation with coefficients of Eisenstein series gives the virtual weight 9. Because this exceeds the actual weight 4, the form \( f \) must vanish on \( H_5 \). In the same way we prove the vanishing inductively on \( H_6, \ldots, H_9 \). The corresponding virtual weights are 8, 7, 6, 5. This completes the proof of 6.6.

□

7. Relations

In this section we shall consider algebraic relations between modular forms that are additive liftings.

Since the additive lifting is a linear map, the linear relations between the characteristic functions described in 2.3 of course imply the same relations between the corresponding modular forms. But for the algebraic relations this is not clear. We only can see that an algebraic relation between modular forms implies the same relation between their values at the cusps. If we are inside the space \( H \) (defined by \( C'(0) = 0 \) then 3.3 shows that the input functions \( C \in (\mathbb{F}_2^{2m})^{SL(2,\mathbb{Z})} \) (considered as functions on \( \mathbb{F}_2^m \)) satisfy the same relation. Hence there is a chance that the quadratic relations in \( (\mathbb{F}_2^{2m})^{SL(2,\mathbb{Z})} \), which have been described in 2.8 also hold for the corresponding modular forms. We don’t know, whether this is always true, but it is at least true when there are modular forms whose zero set are star divisors. This is the case when \( n = 10 \) and \( m = 3, 5, 6 \).

7.1 Lemma. Let \( A \subset \mathbb{F}_2^{12} \) be a 4-dimensional totally isotropic subspace over and \( I_1, \ldots, I_6 \) the maximal totally isotropic subspaces containing \( A \) and in suitable ordering (as described in 2.8.) Denote by \( f_1, \ldots, f_6 \) the additive lifts
of their characteristic functions. There are the quadratic relations

\[(f_1 - f_2)(f_1 - f_4) = (f_3 - f_6)(f_5 - f_6)\]
\[(f_1 - f_2)(f_3 - f_2) = (f_5 - f_4)(f_5 - f_6)\]
\[(f_1 - f_2)(f_1 - f_6) = (f_3 - f_4)(f_5 - f_4)\]
\[(f_1 - f_2)(f_5 - f_2) = (f_3 - f_4)(f_3 - f_6)\]
\[(f_3 - f_4)(f_1 - f_4) = (f_5 - f_2)(f_5 - f_6)\]
\[(f_3 - f_4)(f_3 - f_2) = (f_1 - f_6)(f_5 - f_6)\]
\[(f_1 - f_4)(f_1 - f_6) = (f_3 - f_2)(f_5 - f_2)\]
\[(f_1 - f_4)(f_5 - f_4) = (f_3 - f_2)(f_3 - f_6)\]
\[(f_1 - f_6)(f_3 - f_6) = (f_5 - f_2)(f_5 - f_4)\]

These relations are induced by decompositions of a union of two disjoint stars as union of two other stars as for example $O_1 \cup O_3 = O_8 \cup O_9$.

**Proof.** The point is that the divisors of both sides agree by 6.4, because they are the unions of the corresponding star divisors. Hence both sides are equal up to a constant factor. The normalizing factor follows from 2.8. \[\Box\]

We recall that the 9 quadratic relations collapse to one if one takes the linear relation 2.3 into account.

In section 2 we introduced a certain ideal $I_6$ generated by linear and quadratic relations in the polynomial ring $\mathbb{C}[\ldots X_A - X_B \ldots]$, where $X_A$ are formal variables attached to maximal totally isotropic subspaces. We can summarize

**7.2 Proposition.** There is a natural homomorphism

\[
\mathcal{R}_6/I_6 \longrightarrow A(O^+(\Pi_{2,10})[2]),
\]

**8. Embedded lattices**

Let us assume that the $L$ is a sublattice of finite index of another even lattice $M$. Then we have

\[L \subset M \subset M' \subset L'.\]

The orthogonal groups $O^+(M)$ and $O^+(L)$ are usually not contained in each other but for the discriminant groups one has

\[\Gamma_L \subset \Gamma_M.\]
To prove this one considers some element \( g \in \Gamma_L \) and an element \( x \in M' \). Because \( x \) is contained also in \( L' \) we have \( g(x) - x \in L \subseteq M \). This shows \( g(x) \in M' \). Hence \( g \) defines an automorphism of \( M' \) and then also for \( M = M'' \). From \( g(x) - x \in M \) we see \( g \in \Gamma_M \).

Following Scheithauer we define an injective map
\[
\Psi : \mathbb{C}[M'/M]^{SL(2,\mathbb{Z})} \to \mathbb{C}[L'/L]^{SL(2,\mathbb{Z})}.
\]
as follows: We extend a function \( M'/M \to \mathbb{C} \) by zero to a function \( L'/M \) (i.e. the values are zero on the complement \( L'/M - M'/M \)) and then compose the result with the natural projection \( L'/L \to L'/M \). Scheithauer [Sch] proved that Weil invariants go to Weil invariants.

8.1 Lemma. The diagram
\[
\begin{array}{ccc}
\mathbb{C}[M'/M]^{SL(2,\mathbb{Z})} & \longrightarrow & \mathbb{C}[L'/L]^{SL(2,\mathbb{Z})} \\
\downarrow & & \downarrow \\
[\Gamma_M, n/2 - 1] & \subset & [\Gamma_L, n/2 - 1]
\end{array}
\]
commutes.

Proof. The proof follows from the definition of the theta lift [Bo1]. We explain it briefly for the lattice \( M \). We consider a point \( \tau \) form the usual upper half plane and a \( z \in \mathcal{H}_n \). Let \( A \) be the two dimensional positive definite subspace of \( V = M \otimes \mathbb{R} \) and \( B \) its orthogonal complement. We decompose an arbitrary \( v \in V \) as \( v = v_A + v_B \) with \( v_A \in A \) and \( v_B \in B \). The Siegel theta function corresponding to a coset \( M + \alpha \) form \( M'/M \) is defined as
\[
\theta_{M+\alpha}(\tau, z) = \sum_{\lambda \in M + \alpha} e^{\pi i (\tau(\lambda_A, \lambda_A) + \bar{\tau}(\lambda_B, \lambda_B))}.
\]
This can be considered as a function with values in \( \mathbb{C}[M'/M] \). Using a standard pairing on \( \mathbb{C}[M'/M] \) we can pair the Siegel theta function with an element \( C \in \mathbb{C}[M'/M]^{SL(2,\mathbb{Z})} \) and integrate then along \( \tau \) (with respect to an invariant volume element). The result essentially is the additive lift of \( C \). Now we compare the Siegel theta series with respect to the two lattices \( L \subseteq M \). For \( \alpha \in M' \) one obviously has
\[
\theta_{M+\alpha} = \sum_{\beta \in M/L} \theta_{L+\alpha + \beta}.
\]
From this follows 5.1 immediatly.

8.2 Remark. Let \( M \) be an even lattice and \( L \subseteq M \) a sub-lattice of finite index. The image of \( \mathbb{C}[M'/M]^{SL(2,\mathbb{Z})} \) under the map \( \Psi \) consists of all elements \( C \in \mathbb{C}[L'/L]^{SL(2,\mathbb{Z})} \), which are periodic under \( M/L \).
Proof. Let $C \in \mathbb{C}[L'/L]^{\text{SL}(2, \mathbb{Z})}$. Then

$$C(\alpha) = \frac{\sqrt{1}^{n-2}}{\sqrt{|L'/L|}} \sum_{\beta \in L'/L} C(\beta) e^{-2\pi i (\alpha, \beta)}.$$ 

If $C$ is periodic we can write

$$C(\alpha) = \frac{\sqrt{1}^{n-2}}{\sqrt{|L'/L|}} \sum_{\beta \in L'/M} C(\beta) \sum_{\gamma \in M/L} e^{-2\pi i (\alpha, \beta + \gamma)}.$$ 

The inner sum is zero if $\alpha$ is not contained in $M'$. Hence we can consider $C$ as a function on $M'/M$ and it is clear that this function is invariant under $\text{SL}(2, \mathbb{Z})$. \hfill \qed 

9. Distinguished points

The aim of this and next section is to prove the generic injectivity of the map from the modular variety $\mathcal{H}_{10}/O^+(\Pi_{2,10})[2]$ to the projective space of dimension 714. We recall some basic facts from [Ni] about primitive lattices.

Let $(L, (\cdot, \cdot))$ be an even lattice. There is a natural isomorphism

$$L' \cong \text{Hom}_\mathbb{Z}(L, \mathbb{Z}).$$

Let $M \subset L$ be a sublattice. Then there is a natural map $L' \to M'$. The lattice $M$ is called primitive inside $L$ if $L/M$ is free. Then the sequence

$$0 \to M \to L \to L/M \to 0$$

splits and $L' \to M'$ is surjective. We mention that $M$ is primitive inside $L$ if and only if

$$\mathbb{Q}M \cap L = M.$$ 

A lattice $M$ is called maximal if there is no even over-lattice $M \subset \tilde{M}, M \neq \tilde{M}$, of the same rank. We see that maximal lattices $M$ are primitive in every even over-lattice $M \subset L$. Lattices with square free determinant are maximal.

Recall that for a quadratic space of signature $(2, n)$, $n > 0$, the group $O^+(V)$ is generated by reflections along vectors of negative norm. Let $V = V_1 \oplus V_2$ be an orthogonal decomposition into a positive definite $V_1$ and a negative definite $V_2$. An automorphism of $V$ which fixes this decomposition is contained in $O^+(V)$ if and only of the determinant on $V_1$ is positive.

Again we consider an even unimodular lattice $L$ and a primitive sublattice $M \subset L$. We ask whether every primitive sublattice $N \subset L$ which is isometric
to $M$ can be obtained from $M$ by applying an automorphism of $L$. For this we have to consider the orthogonal complements $M^\perp$ and $N^\perp$ inside $L$. We know that there discriminants agree. Hence they define the same genus. We now make the rather strong assumption that $M^\perp$ is unique in its genus. Then $M^\perp$ and $N^\perp$ are isometric lattices. We choose isometries
\[ M \sim \rightarrow N \quad \text{and} \quad M^\perp \sim \rightarrow N^\perp. \]

They define an isometric isomorphism
\[ M + M^\perp \sim \rightarrow N + N^\perp. \]

This isomorphism extends to $L$ if and only if the glue groups agree. Recall that the glue group of $M + M^\perp$ in $L$ is of the form
\[ A = \{ (a, \sigma a), \quad a \in M' / M \}, \]
where $\sigma : M' / M \rightarrow (M')^\perp / M^\perp$ is an isomorphism (changing the sign of the quadratic form). We assume that every isometric automorphism of $(M')^\perp / M^\perp$ extends to an element of $O(M)$. Then we can obtain by a suitable choice of the isomorphism $M^\perp \rightarrow N^\perp$ that the glue groups match. We see:

**9.1 Remark (Nikulin).** Let $L$ be a unimodular lattice and $M \subset L$ a primitive sublattice such that the lattice $M^\perp \subset L$ is unique in its genus. Assume that every isometry of the discriminant group of $M^\perp$ extends to an isometry of $M^\perp$. Then the group $O(L)$ acts transitively on the set of isometric embeddings $M \hookrightarrow L$.

This applies to the following situation. Let $L = \Pi_{2,10}$ and $M = A_2$. We can take the realization $L = \Pi_{2,2} \times (-E_8)$ and realize $A_2$ inside $\Pi_{2,2}$, for example as the lattice generated by $(1,1,0,0;0,\ldots,0)$ and $(1,0,1,1;0,\ldots,0)$. We see that a realization of $A_2$ in $L$ exists such that the orthogonal complement is $-(A_2 \times E_8)$. From Kneser’s classification of definite lattices of determinant $\leq 3$ up to dimension 17 follows that this lattice is unique in its genus. The discriminant is isomorphic to $\mathbb{Z} / 3\mathbb{Z}$. The non trivial automorphism $x \mapsto -x$ extends. Hence 9.1 applies. We can even more replace $O(L)$ by $O^+(L)$ because $A_2$ admits an automorphism of determinant -1.

**9.2 Lemma.** There is precisely one $O^+(L)$-orbit of sublattices $M \subset L = \Pi_{2,10}$ isometric to $A_2$.

We want to determine the subgroup of $O^+(L)$ which fixes a given $A_2$. We start with the full group $O(L)$. This group acts on the set of two dimensional positive definite subspaces in a natural way. The automorphism group of $A_2$ has order 12. By a result of Eichler one knows
\[ O(A_2 \times E_8) \cong O(A_2) \times O(E_8). \]

The order of $O(E_8)$ is 696 729 600. We get
\[ \#O(A_2 \times (-A_2 \times E_8)) = 12 \cdot 12 \cdot 696 729 600 = 2^{18} \cdot 3^7 \cdot 5^2 \cdot 7. \]

Only half of this group survives in $O(L)$ because of the matching condition.
9.3 Lemma. The subgroup of $O(\Pi_{2,10})$, which stabilizes a given $A_2$, has order $2^{17} \cdot 3^6 \cdot 5^2 \cdot 7$.

Next we consider the congruence group

$$O(\Pi_{2,10})[2] := \text{kernel}(O(\Pi_{2,10}) \rightarrow O(F_{2}^{12})).$$

The number of all $O(\Pi_{2,10})[2]$-orbits of $A_2$-sublattices is the index $[O(F_{2}^{12}) : H]$, where $H$ denotes the image of the stabilizer described in 9.3. To get the order of $H$ we have to know the order of the kernel of the homomorphism

$$O(A_2 \times (-A_2 \times E_8)) \rightarrow O(F_{2}^{12}).$$

When $M$ is a primitive sublattice of an even lattice $N$, one has $M \cap 2N = 2M$. This shows that in the kernel there are only signs changes of the occurring $A_2$, $-A_2$ and $-E_8$. Hence the kernel has order 8. Because of the gluing condition only order 4 survives. From 9.3 we obtain $\#H = 2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$. Hence the number of $O(\Pi_{2,10})[2]$-orbits of $A_2$-sublattices is $2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 17 \cdot 31$. Let $M$ be a lattice which is isomorphic to $A_2 \times E_8$, then there is a unique sublattice of $M$ which is isomorphic to $A_2$ and whose orthogonal complement is isomorphic to $E_8$. This follows from the mentioned result of Eichler. Hence an $A_2$-lattice inside $\Pi_{2,10}$ defines a unique sublattice isomorphic to $-A_2$. There images in $F_{2}^{12}$ define an ordered pair $(A, B)$ of orthogonal hyperbolic planes. Every pair occurs because the orthogonal group acts transitively on them. The number of these pairs is easy to compute, one obtains $2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 17 \cdot 31$, the same number as above. Hence we obtain:

9.4 Lemma. The $O(\Pi_{2,10})[2]$-orbits of sublattices of $\Pi_{2,10}$ isomorphic to $A_2$ are in one-to-one correspondence to the ordered pairs $(A, B)$ of orthogonal hyperbolic planes in $F_{2}^{12}$. Their number is $2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 17 \cdot 31$.

The situation for $O^+(\Pi_{2,10})[2]$ is slightly different. The reason is that an automorphism which stabilizes an embedded $A_2$ is in $O^+$ if and only if it has positive determinant on $A_2$. Hence the number of $O^+(\Pi_{2,10})[2]$-orbits is the double of the number given in 9.4. This can be understood as follows:

9.5 Definition. Two $O^+(\Pi_{2,10})[2]$-classes of sublattices of $\Pi_{2,10}$ of type $A_2$ are called companions, if there exists an element $g \in O(\Pi_{2,10})[2]$, which is not contained in $O^+(\Pi_{2,10})[2]$ such that $g$ maps one class to the other.

As an example we realize $\Pi_{2,10}$ as $\Pi_{2,2} \times (-E_8)$. We consider the lattice generated by $(1, 1, 0, 0; 0, \ldots, 0)$ and $(1, 0, 1, 1; 0, \ldots, 0)$. The automorphism which changes the signs of the first two components is not in $O^+$. Hence the companion class is represented by the lattice with basis $(-1, -1, 0, 0; 0, \ldots, 0)$ and $(-1, 0, 1, 1; 0, \ldots, 0)$. 

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9.6 Remark. Companions are equivalent mod $O(\Pi_{2,10})[2]$ and also equivalent mod $O^{+}(\Pi_{2,10})$ but not equivalent mod $O^{+}(\Pi_{2,10})[2]$.

Our method also shows that the stabilizer of a lattice of type $A_2$ (embedded in $\Pi_{2,10}$) inside $O^{+}(\Pi_{2,10})[2]$ only consists of the identity and its negative. We also can restate our results in terms of modular varieties:

9.7 Proposition. The set of points of type $A_2$ in 

$$\mathcal{H}_{10}/\Gamma[2] \quad (\Gamma[2] = O^{+}(\Pi_{2,10})[2])$$

consists of $2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 17 \cdot 31$ pairs of companions. The map $\mathcal{H}_{10} \to \mathcal{H}_{10}/\Gamma[2]$ is locally biholomorphic at the $A_2$-points. Especially they define smooth points in $\mathcal{H}_{10}/\Gamma[2]$.

10. Mapping to the projective space

We have to consider the Baily-Borel compactification $\mathcal{H}_{10}/\Gamma[2]$, which is an algebraic variety [BB]. We want to use a basis $F_0, \ldots, F_{713}$ of the 714-dimensional irreducible part of the additive lift space to define a regular map from it to $P^{713}(\mathbb{C})$. For this we need

10.1 Lemma. The modular forms of the 714-dimensional space of additive lifts don't have common zeros in $\mathcal{H}_{10}/\Gamma[2]$.

Proof. The more involved part is the interior. Here the argument is the same as in Kondo’s case and we omit it. (Actually it can be obtained as a consequence of Kondo’s case using 8.3.) For the boundary one can argue as follows. The irreducible components of the boundary are modular curves corresponding to $SL(2, \mathbb{Z})[2]$. The restriction of the 714-dimensional space contains a form of weight 4, which is not invariant under the full modular group. The space of all elliptic modular forms is three-dimensional and generated by $\vartheta^8_1, \vartheta^8_2, \vartheta^8_3$, where $\vartheta_1, \vartheta_2, \vartheta_3$ denote the Jacobi theta constants with respect to the characteristics $(0,0), (1,0), (0,1)$. Recall that $\vartheta_1^4 = \vartheta_2^4 + \vartheta_3^4$. The space generated by the the three eighth powers splits under $SL(2, \mathbb{Z})$ into a one dimensional space generated by $f = \vartheta^8_1 + \vartheta^8_2 + \vartheta^8_3$ and an irreducible two dimensional space generated by $\vartheta^8_1 - \vartheta^8_2, \vartheta^8_1 - \vartheta^8_3$ and $\vartheta^8_2 - \vartheta^8_3$. This space must be in the restriction of the 714-dimensional space. But this two dimensional space has no joint zero. This follows from the fact that $f^2$ is in the second symmetric power of this two dimensional space (consider the sum of the $(\vartheta^8_1 - \vartheta^8_j)^2$).

Due to a result of Hilbert we obtain from 10.1: The map

$$\mathcal{H}_{10}/\Gamma[2] \longrightarrow P^{713}(\mathbb{C})$$
defined by a basis of the additive lift space is a finite regular map. We denote by
\[ A(\Gamma[2]) = \sum_{r=0}^{\infty}[\Gamma[2], r] \]
the graded algebra of modular forms (similarly for other groups) and by
\[ B(\Gamma[2]) = \mathbb{C}[F_0, \ldots, F_{714}] \subset A(\Gamma[2]) \]
the subring which is generated by the 715-dimensional additive lift space. Be aware that we include the invariant form and not only the 714-dimensional part.

10.2 Lemma. The finite map
\[ \mathcal{H}_{10}/\Gamma[2] = \text{proj}(A(\Gamma[2])) \rightarrow \text{proj}(B(\Gamma[2])) \quad (\hookrightarrow P^{714}(\mathbb{C})) \]
is locally biholomorphic at the \( A_2 \)-points.

Proof. Let \( A_2 \times (-A_2 \times E_8) \hookrightarrow \Pi_{2,10} \) an embedding which describes an \( A_2 \)-point. In \(- (A_2 \times E_8) \) exist 10 linearly independent vectors of norm -4. We know that there exist ten modular forms in the additive lift space whose zero divisor consists of the corresponding Heegner divisors. These define locally at the point a coordinate frame.

We want to proof that the map in 10.2 is generically injective. We follow closely Kondo ([Ko1], section 6). First of all we consider for a two-dimensional positive definite subspace \( W \subset \Pi_{2,10} \otimes \mathbb{Z} \mathbb{R} \) the set of all norm -2 vectors in \( \Pi_{2,10} \), which are orthogonal to \( W \). The image of this set in \( \mathbb{F}_{12}^2 \) is denoted by
\[ \Delta_W := \{ \alpha \in \mathbb{F}_{12}^2; \alpha \text{ image of a norm -2 vector } a \in \Pi_{2,10} \text{ orthogonal to } W \}. \]
If \( W \) is an \( A_2 \)-point (i.e. the vector space generated by an \( A_2 \)-sublattice of \( \Pi_{2,10} \)), the set \( \Delta_W \) contains 123 elements. This can be seen if one takes the realization
\[ \Pi_{2,10} = \Pi_{2,2} \times (-E_8) \]
and for \( W \) the space generated by \((1,1,0,0;0,\ldots,0) \) and \((1,0,1,1;0,\ldots,0) \). The orthogonal norm -2 vectors are vectors of norm -2 orthogonal to them are \( \pm (0,0,-1,1;0), \pm (1,-1,0;0,\ldots,0) \), \( \pm (1,-1,0,-1;0) \) and \( (0,0,0,0,x) \) with \( x \) a root of \(-E_8\). This description also shows:

10.3 Remark. Let \( \varrho \) be an \( A_2 \)-point and \( A, B \subset \mathbb{F}_{12}^2 \) the corresponding pair (9.4). The set \( \Delta_\varrho \) consists of two types:

First type: Three points inside \( A \oplus B \).
Second type: 120 points orthogonal to \( A \oplus B \).

The set \( \Delta_W \) only depends on the \( \Gamma[2] \)-equivalence class of \( W \). Hence we can define the set \( \Delta_\varrho \) also for points \( \varrho \in \mathcal{H}_{10}/\Gamma[2] \):

\[ \Delta_\varrho := \{ \alpha \in \mathbb{F}_{12}^2; \alpha \text{ image of a norm -2 vector } a \in \Pi_{2,10} \text{ orthogonal to } W \}. \]
10.4 Lemma. Let \( \varrho, \varrho' \in \mathcal{H}_{10}/\Gamma[2] \) be two points. Assume that one of them is an \( A_2 \)-points and furthermore that \( \Delta_\varrho = \Delta_{\varrho'} \). Then the two points are equal or companions.

The following two lemmas are similar to Kondo’s 6.4 and 6.5 in [Ko1]:

10.5 Lemma. Let \( \Delta = \Delta_\varrho \) for a \( A_2 \)-point \( \varrho \). Assume that \( \alpha \in \mathbb{F}_2^{12} \) is a non-isotropic element, which is not contained in \( \Delta \). Then there exists a star, which contains \( \alpha \) and has empty intersection with \( \Delta \).

This is a finite problem which can be checked by calculation. One can assume that \( \varrho \) is our standard point. There is a group isomorphic to \( S_3 \times O(E_8/2E_8) \) which stabilizes \( \Delta \) and also its complement. The complement consists of 5 orbits, which can be represented by

\[
(0, 0, 1, 1; 0, 1, 0, 0, 0) \quad (1, 1, 0, 0; 0, 0, 0, 0, 0) \\
(1, 1, 0, 0; 0, 1, 0, 0, 0) \quad (0, 0, 0, 1; 1, 1, 0, 0, 0) \\
(0, 1, 0, 0; 1, 1, 0, 0, 0)
\]

In the first four cases we take the following star: The underlying totally isotropic space is spanned by

\[
(1, 1, 1, 0, 1, 0, 0, 0, 0) \quad (1, 1, 0, 1; 1, 1, 0, 0, 0) \\
(0, 0, 0, 0, 0, 1, 0, 0, 0) \quad (0, 0, 0, 0, 0, 0, 1, 0, 0) \\
(0, 0, 0, 0, 0, 0, 0, 1, 0)
\]

The star is the coset which contains \( (1, 1, 0, \ldots, 0) \). In the last case, we consider the even totally isotropic space spanned by

\[
(1, 1, 1, 1, 0, 1, 0, 0, 0) \quad (0, 1, 1, 1, 1, 1, 0, 0, 0) \\
(0, 0, 0, 0, 0, 1, 0, 0, 0) \quad (0, 0, 0, 0, 0, 0, 1, 0, 0) \\
(0, 0, 0, 0, 0, 0, 0, 1, 0)
\]

The star is the coset which contains \( (0, 0, 1, 1; 0, 0, \ldots, 0) \).

\[ \square \]

10.6 Lemma. Let \( \Delta = \Delta_\varrho \) for a \( A_2 \)-point \( \varrho \). Assume that \( \alpha \in \Delta \) is a non-isotropic element.

a) Assume that \( \alpha \) is of the first type (see 10.4). There exists a star \( S \) such that \( S \cap \Delta = \{ \alpha \} \).

b) Assume that \( \alpha, \beta \in \Delta \) are two orthogonal elements of the second type. There exists a star \( S \) such that \( S \cap \Delta = \{ \alpha, \beta \} \).

Proof. Again we assume that \( \varrho \) is the standard point. Let \( \alpha \) be of first type. Up to the action of the group \( S_3 \) we can take \( \alpha = (0, 0, 1, 1; 0, \ldots, 0) \). We can take the star

\[
(*, 0, 1, 1, *, 0, *, 0, *, 0, *, 0).
\]

Let \( \alpha, \beta \in \Delta \) be orthogonal elements of second type. Using the action of \( O(E_8/2E_8) \), we can assume

\[
\alpha = (0, 0, 0, 0; 1, 1, 0, \ldots, 0), \quad \beta = (0, 0, 0, 0; 1, 1, 0, 0, 0, 1, 0).
\]
In this case we can take the star $\alpha + M$ where $M$ is the totally isotropic space spanned by the vectors

\begin{align*}
(1, 0, 0, 0; 0, 0, 1, 0, 0, 0, 0, 0), & \quad (0, 1, 0, 0; 0, 0, 1, 0, 0, 0, 0, 0), \\
(0, 0, 1, 0; 0, 0, 0, 0, 1, 0, 0, 0), & \quad (0, 0, 0, 1; 0, 0, 0, 0, 1, 0, 0, 0), \\
(0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 1, 0). & \quad (0, 0, 0, 0; 0, 0, 0, 0, 0, 1, 0). \quad \square
\end{align*}

Before we continue we recall that a non-isotropic element $\alpha \in \mathbb{F}_2^{12}$ defines a certain Heegner divisor $H_\alpha(-1)$ in $\mathcal{H}_{10}/\Gamma[2]$. It is the fixed point set of the reflection corresponding to a norm -2 representative $a \in \Pi_{2,10}$. Hence we call this divisor a mirror. We also notice that a point $q$ is contained in this mirror if and only if $\alpha \in \Delta_q$. If $S \subset \mathbb{F}_2^{12}$ is a star, we can consider the star divisor which consists of the corresponding 32 mirrors. The star divisor of a star $S \subset \mathbb{F}_2^{12}$ contains $q$ if and only of $S \cap \Delta_q$ is not empty.

**10.7 Lemma.** Let $q \in \mathcal{H}_{10}/\Gamma[2]$ be an $A_2$-point and let $q'$ be an arbitrary point which is different from $q$ and its companion. Then there exists a star which contains one of the two points but not both.

**Proof.** We assume that the two points cannot be separated. If there exists an $\alpha \in \Delta_{q'} - \Delta_q$ we find by 10.6 a star which contains $q'$ but not $q$. This shows $\Delta_{q'} \subset \Delta_q$. Let be $M = \Delta_q - \Delta_{q'}$. Every $\alpha \in M$ must be of second kind, because otherwise we could use 10.7, a) to separate the two points. In $M$ two elements cannot be orthogonal, because otherwise we could separate the two points using 10.7, b). The same argument shows that every element of second type which is orthogonal to an element of $M$ is contained in $\Delta_{q'}$. Before we continue we fix a notation:

An $E_8$-root system in $E_8/2E_8$ is a system of anisotropic vectors $\alpha_1, \ldots, \alpha_8$ with the property that

$$ (\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = \cdots (\alpha_7, \alpha_8) = 1 \quad \text{and} \quad (\alpha_5, \alpha_8) = 1. $$

**Claim.** Let $\mathcal{M} \subset E_8/2E_8$ be a subset containing only anisotropic elements, such that any two elements of $\mathcal{M}$ are not orthogonal. Then the complement of $\mathcal{M}$ contains an $E_8$-root system.

**Proof of the claim.** The cases where $\mathcal{M}$ is empty or contains only one element is trivial. Hence $\mathcal{M}$ contains at least two elements $\alpha, \beta$. We use the standard realization $E_8/2E_8 = \mathbb{F}_2^8$ with $q(x) = x_1x_2 + \cdots + x_7x_8$ and can then assume

$$ \alpha = (1, 1, 0, 0, 0, 0, 0), \quad \beta = (0, 1, 1, 0, 1, 0, 1). $$

The system

\begin{align*}
(1, 1, 1, 0, 0, 0, 0, 0) & \quad (0, 0, 1, 1, 1, 0, 0, 0) & \quad (0, 0, 0, 0, 1, 1, 1, 0) & \quad (0, 0, 0, 0, 0, 0, 1, 1) \\
(1, 1, 0, 0, 0, 0, 1, 0) & \quad (0, 1, 0, 1, 1, 1, 0, 0) & \quad (0, 1, 0, 1, 0, 1, 1, 1) & \quad (1, 0, 1, 1, 0, 0, 0, 0), \quad (0, 0, 1, 1, 0, 1, 1, 1)
\end{align*}

is an $E_8$-root system. The first five are orthogonal to $\alpha$ the rest to $\beta$. Hence they are all in the complement. \quad \square
Proof of 10.8 continued. We want to apply this to the following situation. As above a representative of $\varrho$ belongs to a decomposition $\Pi_{2,10} = \Pi_{2,2} \times (-E_8)$. The set $\mathcal{M} = \Delta_\varrho - \Delta_{\varrho'}$ belongs to the part $E_8/2E_8$ in this decomposition (second type). From the above follows that $\Delta_{\varrho'}$ contains an $E_8$-root system. Together with the three elements of the first type we obtain: The set $\Delta_{\varrho'}$ contains an $A_2 \times E_8$-root system in an obvious sense. Now we have to recall the definition of $\Delta_{\varrho'}$. It is the image of all elements of norm -2 contained in a ten-dimensional negative definite sublattice of $\Pi_{2,10}$. From the classification of root lattices by their Dynkin diagram we see that the lattice generated by these roots is a copy of $-(A_2 \times E_8)$. This means that $\varrho'$ is an $A_2$-point. Hence $\Delta_{\varrho'}$ contains 123 elements and we get $\Delta_\varrho = \Delta_{\varrho'}$. Now 10.5 completes the proof of 10.8. 

We also have to take boundary points into account. Recall that there are one-dimensional and zero dimensional boundary components. Each one-dimensional boundary component corresponds to a two dimensional totally isotropic space $B$. It belongs to the group $\text{SL}(2, \mathbb{Z})[2]$ and the three cusp classes of this group correspond to the three non-zero isotropics of $B$. Let $\alpha \in F_{12}^2$ be an anisotropic element, which is orthogonal to $B$. Then the three zero-dimensional boundary points are contained in the closure of the Heegner divisor $H_\alpha(-1)$. The number of these $\alpha$ is 480. Let $\varrho$ now be any $A_2$-point. As we know it is contained in 123 Heegner divisors $H_\beta(-1)$. Hence there exists an $\alpha$ such that the closure of $H_\alpha(-1)$ contains the three cusps but not $\varrho$. Now we use 10.6 and obtain a star modular form $f_S$ with the following property: It doesn’t vanish at $\varrho$ but it vanishes in the three cusps in consideration. Now we use that every cusp from of weight 4 for the group $\text{SL}(2, \mathbb{Z})[2]$ vanishes. Hence we obtain:

10.8 Lemma. Let $\varrho$ be an $A_2$-point. For every one-dimensional boundary component there exist a star modular form which vanishes along this boundary component but does not vanish at $\varrho$.

From 10.8, and 10.2 we know that the map $\overline{\mathcal{H}_{10}/\Gamma[2]} \to \text{proj}(B(\Gamma[2]))$ has covering degree at most two. We want to show that it is one. Because we cannot separate companions directly by our methods, we have to take a small detour. We take invariants under the group $O(\mathbb{F}_2^{12})$ to go down to the full modular group $\Gamma = O^+(\Pi_{2,10})$. We claim this finite group acts faithfully on $B(\Gamma[2])$. This follows from the fact that $O(\mathbb{F}_2^{12})$ has a simple subgroup of index 2 (the kernel of the so-called Dixon invariant) and that this subgroup cannot act trivially on $B(\Gamma[2])$. Set

$$B(\Gamma) := B(\Gamma[2])^{O(\mathbb{F}_2^{12})}.$$ 

We consider the diagram

$$
\begin{array}{ccc}
\overline{\mathcal{H}_{10}/\Gamma[2]} & \to & \text{proj}(B(\Gamma[2])) \\
\downarrow & & \downarrow \\
\overline{\mathcal{H}_{10}/\Gamma} & \to & \text{proj}(B(\Gamma))
\end{array}
$$
Since both vertical arrows have the same covering degree, it is sufficient to show that the second row has covering degree one. But this follows from

**10.9 Lemma.** The finite map

\[ \overline{\mathcal{H}_{10}/\Gamma} = \text{proj}(A(\Gamma)) \longrightarrow \text{proj}(B(\Gamma)) \]

is locally biholomorphic at the \( A_2 \)-point. The \( A_2 \)-point is unique in its fibre.

This follows from our considerations about the level two case and from the fact that companions have the same image in level 1.

**10.10 Theorem.** The map

\[ \overline{\mathcal{H}_{10}/\Gamma[2]} \rightarrow \text{proj}(B(\Gamma[2])) \]

is everywhere regular finite and birational. The ring of modular forms of weight divisible by 4

\[ A^{(4)}(\Gamma[2]) := \bigoplus_{r=0}^{\infty} [\Gamma[2], 4r] \]

is the normalization of the ring \( B(\Gamma[2]) \) (the ring generated by the 715 dimensional additive lift space). Using 8.2 we get a finite map

\[ \overline{\mathcal{H}_{10}/\mathcal{O}^+(\Pi_{2,10})[2]} \longrightarrow \text{proj}(\mathcal{R}_6/\mathcal{I}_6). \]

### 11. Enriques surfaces, Kondo’s approach

Denote by \( U \) the unimodular lattice \( \mathbb{Z} \times \mathbb{Z} \) with quadratic form \((x, x) = 2x_1x_2\). Kondo investigated in [Ko1] the case of the lattice

\[ M = U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8). \]

We recall that thus case is related to the moduli space of marked Enriques surfaces, i.e. Enriques surfaces with a choice of level 2 structure of the Picard lattice.

Our lattice

\[ L = \sqrt{2}\Pi_{2,10} \cong \sqrt{2}U \oplus \sqrt{2}U + \oplus(-\sqrt{2}E_8) \]
can be embedded into Kondo’s lattice by means of
\[ \sqrt{2}U \rightarrow U, \quad \sqrt{2}(x_1, x_2) \mapsto (x_1, 2x_2). \]

The image consists of all integral pairs \((x_1, x_2)\) with even \(x_2\). The dual of this lattice is \((1/2)\mathbb{Z} \times \mathbb{Z}\). Hence \(M' \subset L'\) corresponds to
\[ \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{\sqrt{2}}\mathbb{Z} \oplus \frac{1}{\sqrt{2}}\mathbb{Z} \oplus \frac{1}{\sqrt{2}}E_8 \subset \frac{1}{2}\mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{\sqrt{2}}\mathbb{Z} \oplus \frac{1}{\sqrt{2}}\mathbb{Z} \oplus -\frac{1}{\sqrt{2}}E_8. \]

Assume that \(x = (x_1, x_2, x_3, x_4, y_3, y_4)\) is a primitive element of \(M'\), which is not primitive in \(L'\). Then \(x_3 = 2y_3, x_4 = 2y_4\) must be even and \(x = 2y\) with \(y \in E_8\). This gives \(x = (x_1, x_2, y_3, y_4, y_3\sqrt{2}, y_4\sqrt{2}, y\sqrt{2})\). Hence \(x\) is contained in \(M\). This shows that we can apply 5.1. Kondo proved that \(\mathbb{C}[M'/M]^{SL(2,\mathbb{Z})}\) is the direct sum of a full invariant one-dimensional space and a 186-dimensional space \(H\) which is irreducible under \(O(M'/M) = \mathbb{F}_2^3\). The elements \(C \in H\) satisfy \(C(0) = 0\). Hence the additive lift space of \(H\) — which is of dimension 186 by Kondo — appears as a subspace of our 714-dimensional space:

11.1 Proposition. The embedding of \(L = \sqrt{2}U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8) \cong \mathbb{KU} \sqrt{2}I_{2,10}\) into \(M = U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8)\) defines an embedding of Kondo’s 186-dimensional space of modular forms of weight four into our 714-dimensional space.

The main result Kondo’s paper is that the 186-dimensional space defines birational map \(\psi\) from this moduli space, i.e. \(\mathcal{H}_{10}/\Gamma_M\) onto its image in \(P^{185}(\mathbb{C})\).

Unfortunately the proof is not correct, in fact he proves that the map \(\psi\) is holomorphic onto \(\mathcal{H}_{10}/\Gamma_M\) and is locally biholomorphic at a special point. But his map does not extend to \(\overline{\mathcal{H}_{10}/\Gamma_M}\).

However Kondo’s argument works with a small modification. In fact we know that the additive lifting from \(\mathbb{C}[\mathbb{F}_2^{10}]^{SL(2,\mathbb{Z})}\) to \([\Gamma_M, 4]\) in injective. Thus we get another modular form that is relative to \(O^+(M)\), so also in this case we add the full invariant form to the 186 dimensional space of modular forms.

Using the results of proposition 3.2, it can be easily checked that this form does not vanishes along those 0-dimensional points that are the base locus for the map \(\psi\). In fact these are points parametrized by primitive isotropic elements of \(M'\) that are also in \(M\).

We denote by
\[ A(\Gamma_M) = \sum_{r=0}^{\infty}[\Gamma_M, r] \]
the graded algebra of modular forms and by
\[ B(\Gamma_M) = \mathbb{C}[G_0, \ldots, G_{186}] \subset A(\Gamma_M) \]
the subring which is generated by the 187-dimensional additive lift space. The correct modification of Kondo’s statement is:
11.2 Theorem. The map
\[ H_{10}/\Gamma_M \to \text{proj}(B(\Gamma_M)) \quad (\hookrightarrow P^{186}(\mathbb{C})) \]
is everywhere regular finite and birational. The ring of modular forms of weight divisible by 4
\[ A^{(4)}(\Gamma_M) := \bigoplus_{r=0}^{\infty} [\Gamma_M, 4r] \]
is the normalization of the ring \( B(\Gamma_M) \) (the ring generated by the 187 dimensional additive lift space).

We consider now \([\Gamma_M, r]\) as subspace of \([\Gamma_L, r]\) using this concrete imbedding. The image of \( M \) in \( L'/L \cong \mathbb{F}_2^{12} \) is a one dimensional subspace. We consider two six-dimensional (maximal) totally isotropic subspaces of \( L'/L \) which contain the image of \( M \). The difference of their characteristic functions defines a modular form in \( [\Gamma_L, 4] \). Let now \( A \) be a four dimensional totally isotropic subspace of \( L'/L \) which contains the image of \( M \). From 2.9 we obtain quadratic relations between modular forms inside Kondo’s space \( [\Gamma_M, 4] \).

We recall that in its papers Kondo’s defined some quartic relations. We don’t want to go into the details of Kondo’s paper and mention just that Kondo’s quartic relations are those we defined in 2.10.

11.3 Proposition. Kondo’s quartic relations are a consequence of our quadratic relations.

Proof. This follows from 2.10. More precisely, the model which Kondo describes in [Ko1] is nothing else but \( \text{proj}(\mathcal{R}_5/\mathcal{I}_5) \). \( \square \)

12. Relation to a result of Koike

There is another interesting case, which belongs to the lattice \( N = U \oplus \sqrt{2}U \oplus (-D_4) \oplus (-D_4) \). In this case we have \( N'/N \cong \mathbb{F}_2^6 \). Obviously Kondo’s lattice \( M \) can be embedded into \( N \). Hence we obtain embeddings \( \Gamma_L \subset \Gamma_M \subset \Gamma_N \). The space \( \mathbb{C}[N'/N]^{SL(2,\mathbb{Z})} \) splits into a full invariant one-dimensional and a 14-dimensional irreducible space under \( S_8 \cong \text{O}(\mathbb{F}_2^6) \). The additive lift of the 14-dimensional space appears as 14-dimensional subspace of Kondo’s 185-dimensional space and hence of our 714-dimensional space. We get a homomorphism
\[ \mathcal{R}_3/\mathcal{I}_3 \longrightarrow A(\Gamma_N). \]

Computer algebra shows that the dimension of \( \mathcal{R}_3/\mathcal{I}_3 \) is 6 (the dimension of the associated projective variety is 5). The lattice \( N \) admits a structure as
Hermitian lattice of signature $(1,5)$ over the ring of Gauß integers (s. [Ko2]). This defines a certain 5-ball $B_5$ inside $\mathcal{H}_{10}$. The corresponding ball quotient is the configuration space $X(2,8)$ of 8 points in the projective line [MY]. We obtain a rational map

$$X(2,8) \longrightarrow \text{proj}(R_3/I_3).$$

Koike [Koi] proved that this is a biholomorphic map. So we recover Koike’s observation that certain quartic relations are consequences of quadratic ones and even more that all these relation live already in the ten-dimensional space $\mathcal{H}_{10}/\Gamma_L$.

13. Final Remark

in 10.9 we used the full 715-dimensional additive lift space. But the relations which we described play in the 714-dimensional irreducible part. It may be true that the invariant form is not necessary, because it may be possible that the second symmetric power of the 714- and 715-dimensional space is the same. We don’t know whether this is true or not.

The invariant form in the 715-dimensional space possible being superfluous does not detain it to be in some sense very basically. To explain this we consider a rational orthogonal transformation $g \in O^+(\Pi_{2,10} \otimes \mathbb{Z} \mathbb{Q})$ with the property

$$gO^+(\Pi_{2,10}[2])g^{-1} \subset O^+(\Pi_{2,10}).$$

Then one can transform $f$ with $g$ to obtain a form on $O^+(\Pi_{2,10})[2]$. To give a simple example we realize $\Pi_{2,10}$ as

$$\Pi_{2,10} = \Pi_{1,1} \times \Pi_{1,9}.$$ 

We take for $g$ the orthogonal transformation

$$(a, b, C) \mapsto (a/2, 2b, C).$$

To understand it better we consider the standard embedding of $\mathcal{H}_{10}$ into $\Pi_{1,9} \otimes \mathbb{C}$: The modular form $f$ is determined by the function

$$F(Z) := f(1, *, Z),$$

where the star is taken such that the entry is in the zero-quadric. Now the effect on $F$ is given by the transformation $F(Z) \mapsto F(2Z)$. It is possible to compute the values at the cusps and to prove in this way that the transformed form is contained in the 715-dimensional additive lift space. This gives:
13.1 Proposition. The 715-dimensional additive lift space \((\mathcal{O}^+_{\Pi_{2,10}})[2,4]\) in the space of modular forms \([\mathcal{O}^+_{\Pi_{2,10}}][2,4]\) is generated as \(\mathcal{O}(\mathbb{F}_2^{10})\)-module by the single form \(F(2Z)\), where \(F(Z)\) denotes the full invariant form, expressed in standard coordinates.

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