Improved Algorithms for Maintaining DFS Tree in Undirected Graphs

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Abstract

Depth first search (DFS) tree is one of the most well-known data structures for designing efficient graph algorithms. Given an undirected graph $G = (V, E)$ with $n$ vertices and $m$ edges, the textbook algorithm takes $O(n + m)$ time to construct a DFS tree. In this paper, we study the problem of maintaining a DFS tree when the graph is undergoing updates. Formally, we show the following results.

- Incremental DFS tree:
  Given any arbitrary online sequence of edge or vertex insertions, we can report a DFS tree in $O(n)$ worst case time per operation, and $O\left(\min\{m \log n + n \log^2 n, n^2\}\right)$ preprocessing time.

- Fault tolerant DFS tree:
  There exists a data structure of size $O(m \log n)$, such that given any set $F$ of failed vertices or edges, it can report a DFS tree of the graph $G \setminus F$ in $O(n|F| \log^2 n)$ time.

- Fully dynamic DFS tree:
  Given any arbitrary online sequence of edge or vertex updates, we can report a DFS tree in $O(\sqrt{nm} \log^{1.5} n)$ worst case time per operation.

- Offline Incremental DFS tree:
  There exists a data structure with construction time $O\left(\min\{m \log n + n \log^2 n, n^2\}\right)$ such that given any set $I$ of inserted vertices or edges, a DFS tree of the graph $G + I$ (denotes the graph obtained applying the insertions in $I$ to $G$) can be reported in $O(n + |I|)$ time.

Our result for incremental DFS tree improves the previous $O(n \log^3 n)$ worst case update time by Baswana et al. [1], and matches the trivial $\Omega(n)$ lower bound when it is required to explicitly output the DFS tree. Our other results also improve the corresponding state-of-the-art results by Baswana et al. [1] as well.

Our work builds on the framework of the breakthrough work by Baswana et al. [1], together with a novel use of a tree-partition lemma by Duan and Zhang [8], and the celebrated fractional cascading technique by Chazelle and Guibas [5, 6].

1 Introduction

Depth First Search (DFS) is one of the most renowned graph traversal techniques. After Tarjan’s seminal work [20], it demonstrates its power by leading to efficient algorithms to many fundamental graph problems, like biconnected components, strongly connected components, topological sorting, bipartite matching, dominators in directed graph and planarity testing.

Real world applications often deal with graphs that keep changing with time. Therefore it is natural to study the dynamic version of graph problems, where there is an online sequence of updates on the graph, and the algorithm aims to maintain the solution of the studied graph problem efficiently after seeing each update. Clearly, we want the update time to be much smaller than that of the best static algorithm for that problem. The last two decades have witnessed a surge of research in this area, like connectivity [9, 11, 12, 13], reachability [17, 19], shortest path [7, 18], bipartite matching [3, 14], and min-cut [21].
A dynamic graph algorithm is said to be fully dynamic if it handles both insertion and deletion. A partial
dynamic graph algorithm is said to be incremental or decremental if it handles only insertion or deletion
updates respectively. In this paper, we study the problem of maintaining a DFS tree efficiently in several
dynamic settings.

1.1 Previous works on dynamic DFS

Despite the significant role of DFS tree in static algorithms, there is little progress on maintaining a DFS
tree in the dynamic setting.

Many previous works focus on the total time of the algorithm for any arbitrary updates. Franciosa
et al. [10] designed an incremental algorithm for maintaining a DFS tree in a DAG from a given source,
with $O(mn)$ total time for any arbitrary sequence of edge insertions; Baswana and Choudhary [2] designed
a decremental algorithm for maintaining a DFS tree in DAG with expected $O(mn \log n)$ total time. For
undirected graphs, Baswana and Khan [4] designed a decremental algorithm for maintaining a DFS tree
with $O(n^2)$ total time.

These algorithms used to be the only results known for the dynamic DFS tree problem. However, none of
these existing algorithms, despite that they are designed for only a partially dynamic environment, achieves a
worst case bound of $o(m)$ on the update time.

That barrier is overcame in the recent breakthrough work of Baswana et al. [1], they provide a fully
dynamic algorithm with worst case $O(\sqrt{mn} \log^{2.5} n)$ update time, and an incremental algorithm with worst
case $O(n \log^3 n)$ update time for the undirected case. Due to the rich information in a DFS tree, their results
directly imply faster worst case fully dynamic algorithms for subgraph connectivity, biconnectivity and 2-edge
connectivity.

Their results suggest a promising way to further improve the worst case update time for those fully
dynamic algorithms by designing better dynamic algorithms for maintaining a DFS tree.

1.2 Our results

In this paper, following the approach of [1], we improve the update time for all the problems that are
studied in [1] by using a better data structure, a novel tree-partition lemma by Duan and Zhang [8] and the
fractional-cascading technique by Chazelle and Guibas [5, 6].

For any set $U$ of updates (insertion/deletion of a vertex/an edge), we let $G + U$ denote the graph obtained
by applying the updates in $U$ to the graph $G$. Our results build on the following main theorems.

**Theorem 1.1.** An undirected graph can be preprocessed to build a data structure of $O(m \log n)$ size, such
that given a set $U$ of $k \leq n$ updates, it can report a DFS tree of $G + U$ in $O(nk \log^2 n)$ time.

Moreover, if we further restrict that $U$ consists of only insertions, there is a data structure with $O(m \log n)$
size, and can be built in $O(\min\{m \log n + n \log^2 n, n^2\})$ time, such that given a set $U$ of $k$ updates, a DFS
tree of $G + U$ can be reported in $O(n + k)$ time.

Armed with the above theorem, and following the approach in [1], we establish the following results for
maintaining a DFS tree in an undirected graph.

1. **Fault Tolerant DFS tree:**
   Given a set of $k$ failed vertices or edges, a DFS tree of the resulted graph can be reported in $O(nk \log^2 n)$
time.

2. **Fully Dynamic DFS tree:**
   Given a sequence of online edge/vertex updates, a DFS tree can be maintained in $O(\sqrt{mn} \log^{1.5} n)$
worst case time per update.

3. **Incremental DFS tree:**
   Given a sequence of online edge/vertex insertions, a DFS tree can be maintained in $O(n)$ worst case
time per insertion.
In the same way as in [1], our results imply algorithms for dynamic subgraph connectivity, biconnectivity and 2-edge connectivity, with \(O(\sqrt{nm \log^{1.5} n})\) worst case update time and \(O(1)\) query time, improving the \(O(\sqrt{nm \log^{2.5} n})\) update time in [1] by a \(\log n\) factor.

2 Preliminaries

Let \(G = (V, E)\) denote the original graph, \(T\) a corresponding DFS tree, and \(U\) a set of inserted vertices and edges. We introduce necessary notations.

- \(T(x)\): The subtree of \(T\) rooted at \(x\).
- \(\text{path}(x, y)\): The path from \(x\) to \(y\) in \(T\).
- \(\text{par}(v)\): The parent of \(v\) in \(T\).
- \(N(x)\): The adjacency list of \(x\) in \(G\).
- \(L(x)\): The reduced adjacency list for vertex \(x\), maintained and harnessed during the main procedure.
- \(T^*\): The newly generated DFS tree.
- \(\text{par}^*(v)\): The parent of \(v\) in \(T^*\).
- \(\text{dep}(v)\): The depth of \(v\) in \(T\). Particularly \(\text{dep}(r) = 0\) where \(r\) is the root of \(T\).
- \(\text{deg}(v)\): The number of children of \(v\) in \(T\).

Also, we say a subtree \(T(x)\) is hanging from a path \(p\) if the root \(x\) of \(T(x)\) is a child of some vertex on \(p\), but \(x\) is not on \(p\) itself.

The following lemma, which is first observed in [1], will be used to reduce the size of candidate edges in our algorithm.

Lemma 2.1 ([1]). Let \(T^*\) be a partially constructed DFS tree, \(v\) the current vertex being visited, \(w\) an (not necessarily proper) ancestor of \(v\), and \(C\) a connected component of the subgraph induced by unvisited vertices. If there are two edges \(e\) and \(e'\) from \(C\) incident on \(v\) and \(w\), then it is sufficient to consider only \(e\) during the rest of the DFS traversal.

Lemma 2.1 is the central lemma for constructing the DFS tree, following which comes the query subroutine \(Q\). Let \(x\) and \(y\) be two vertices, \(T(w)\) be a subtree hanging from \(\text{path}(x, y)\). Among all edges from \(T(w)\) that are incident to \(\text{path}(x, y)\) in \(G\), \(Q(T(w), x, y)\) returns an edge that is nearest to \(x\). If no such edge exists, \(Q(T(w), x, y)\) returns \text{NULL}.

\(Q(T(w), x, y)\) plays a central role when we change the root of a subtree of the DFS tree. Consider, say, we are to change the root of \(T(x)\) to \(y\), where \(T\) is a DFS tree of \(G\). Starting the DFS from \(x\), we go through \(\text{path}(x, y)\) first, and all hanging subtrees subsequently. For a hanging subtree \(T(w)\), by Lemma 2.1 it suffices to consider the edge \(Q(T(w), x, y)\) only.

In [1], in order to solve the Fault Tolerant DFS tree problem and the Fully Dynamic DFS tree problem, the algorithm also needs another query subroutine \(Q(w, x, y)\). Among all edges from \(w\) to \(\text{path}(x, y)\) in \(G\), \(Q(w, x, y)\) returns an edge that is nearest to \(x\). If no such edge exists, \(Q(w, x, y)\) returns \text{NULL}.

In Section 3 and Section 7, we build efficient data structures to answer queries of these forms. Our algorithms for dynamic DFS tree rely largely on these data structures.

3 Answering any query in \(O(\log n)\) time

We show in this section that the query \(Q\) can be reduced efficiently to range successor query (see, e.g., [15], for the definition of range successor query), and show how to answer the range successor query, and thus any individual query \(Q\) in time \(O(\log n)\).

3
3.1 Answering $Q(T(w), x, y)$

To deal with a query $Q(T(w), x, y)$, first note that since $T$ is a DFS tree, all edges not in $T$ but in the original graph $G$ must be ancestor-descendant edges. Querying edges between $T(w)$ and $\text{path}(x, y)$ where $x$ is an ancestor of $y$ and $T(w)$ is hanging from $\text{path}(x, y)$ is therefore tantamount to querying edges between $T(w)$ and $\text{path}(x, \text{par}(w))$, i.e., $Q(T(w), x, y) = Q(T(w), x, \text{par}(w))$. From now on, we will consider queries of the latter form only.

Consider the DFS sequence of $T$, where the $i$-th element is the $i$-th vertex reached during the DFS on $T$. Note that every subtree $T(w)$ corresponds to an interval in the DFS sequence. Denote the index of vertex $v$ in the DFS sequence by $\text{first}(v)$, and the index of the last vertex in $T(v)$ by $\text{last}(v)$. During the preprocessing, we build a 2D point set $S$. For each edge $(u, v) \in E$, we add a point $p = (\text{first}(u), \text{first}(v))$ into $S$. Notice that for each point $p \in S$, there exists exactly one edge $(u, v)$ corresponding to $p$.

For each query $Q(T(w), x, \text{par}(w))$, we first query the point with minimum $x$-coordinate lying in the rectangle $\Omega = [\text{first}(x), \text{first}(w) - 1] \times [\text{first}(w), \text{last}(w)]$. If no such point exists, we return NULL for $Q(T(w), x, \text{par}(w))$. Otherwise we return the edge corresponds to the point with minimum $x$-coordinate.

Now we show that our method will always return a feasible solution to $Q(T(w), x, \text{par}(w))$.

- If our method returns NULL, $Q(T(w), x, \text{par}(w))$ must equal NULL. Otherwise, suppose $Q(T(w), x, \text{par}(w)) = (u, v)$. We notice that $(\text{first}(u), \text{first}(v))$ is in $\Omega$, which means our method will not return NULL in that case.

- If our method does not return NULL, denote $(u', v')$ to be the edge returned by our method. We can deduce from the query rectangle that $u' \in T(x) \setminus T(w)$ and $v' \in T(w)$. Thus, $Q(T(w), x, \text{par}(w))$ \neq NULL. Suppose $Q(T(w), x, \text{par}(w)) = (u, v)$. Notice that $(\text{first}(u), \text{first}(v))$ is in $\Omega$, which means $\text{first}(u') \leq \text{first}(u)$. If $u' = u$, then our method returns a feasible solution. Otherwise, from the fact that $\text{first}(u') < \text{first}(u)$, we know that $u'$ is an ancestor of $u$, which contradicts the definition of $Q(T(w), x, \text{par}(w))$.

3.2 Answering $Q(w, x, y)$

Here we keep using the notations $\text{first}$ and $\text{last}$ introduced in Section 3.1. During the preprocessing, for each vertex $u \in V$, we maintain a 2D point set $N_p(u)$. For each vertex $v \in N(u)$, we insert a point $(\text{first}(v), \text{last}(v))$ into $N_p(u)$.

For each query $Q(w, x, y)$, without loss of generality, we assume $x$ is an ancestor of $y$. We first query the point with minimum $x$-coordinate in the rectangle $\Omega = [\text{first}(x), \text{first}(y)] \times [\text{last}(y), \text{last}(x)]$. If no such point exists, we return NULL for $Q(w, x, y)$. Otherwise we return the edge $(w, w')$, where $w'$ is the vertex corresponding to the point with minimum $x$-coordinate in $\Omega$.

Now we show that our method will always return a feasible solution to $Q(T(w), x, \text{par}(w))$.

- If our method returns NULL, $Q(w, x, y)$ must equal NULL. Otherwise, suppose $Q(w, x, y) = (w, w')$. We notice that $(\text{first}(w), \text{last}(w'))$ is in $\Omega$, as $w'$ is a vertex on $\text{path}(x, y)$. Thus, our method will not return NULL in that case.

- If our method does not return NULL, denote $(w, w')$ to be the edge returned by our method. We can deduce from the query rectangle that $w'$ is on $\text{path}(x, y)$. Thus, $Q(w, x, y) \neq NULL$. Suppose $Q(w, x, y) = (w, w^*)$. Notice that $(\text{first}(w^*), \text{last}(w^*))$ is in $\Omega$, which means $\text{first}(w') \leq \text{first}(w^*)$. If $w' = w^*$, then our method returns a feasible solution. Otherwise, from the fact that $\text{first}(w') < \text{first}(w^*)$, we know that $w'$ is an ancestor of $w$, which contradicts the definition of $Q(w, x, y)$.

Clearly, finding the point with minimum $x$-coordinate lying in a rectangle $[x_1, x_2] \times [y_1, y_2]$ can be efficiently reduced to 2D range successor query. Notice that $|S| = m$, by using the standard range tree structure [16] on the point set $S$, we can answer queries of 2D range successor query in $O(\log m) = O(\log n)$ time per query, and $O(m \log m) = O(m \log n)$ time and space to preprocess. Moreover, if there are $k$ deletions in advance, then these $k$ deletions can be handled in $O(k \log m)$ time.
4 Fault tolerant DFS tree and fully dynamic DFS tree

We show in this section, that the structure presented in Section 3 directly implies the first half of Theorem 1.1, which also implies two of our main results for fault tolerant DFS tree and fully dynamic DFS tree.

The following lemma has been implicitly established in [1].

Lemma 4.1 ([1]). Suppose there is a data structure $D$, where

- $D$ can be built in $O(f)$ time.
- $D$ answers any query $Q$ in $O(g)$ time.
- $k$ edge deletions from $D$ can be done in $O(kh)$ time.

Then,

- There is an algorithm exploiting $D$, which takes $O(f)$ time for preprocessing, and reports a new DFS tree upon $k$ vertex or edge updates in $O(nk \log n \cdot g + k \cdot h)$ time.
- There is an algorithm exploiting $D$, which takes $O(f)$ time for preprocessing, and maintains a DFS tree fully dynamically in $O\left(\sqrt{f \cdot (gn \log n + h)}\right)$ worst case time per update.

With the data structure introduced in Section 3 and the powerful Lemma 4.1, we are ready for two main results.

Theorem 4.1 (Fault tolerant DFS tree). Given a set of $k$ failed vertices or edges and any vertex $u$, a DFS tree rooted at $u$ in the resulted graph can be reported in $O(nk \log^2 n)$ time.

Theorem 4.2 (Fully dynamic DFS tree). Given a sequence of online edge/vertex updates, a DFS tree can be maintained in $O(\sqrt{mn \log^{1.5} n})$ worst case time per update.

Proof. As shown in Section 3, there is a data structure $D$, where

- $D$ can be built in $O(m \log n)$ time.
- $D$ answers any query $Q$ in $O(\log n)$ time.
- $k$ edge deletions from $D$ can be done in $O(k \log m)$ time.

Theorem 4.1 and 4.2 directly follows from Lemma 4.1 by letting $f = m \log n$, $g = \log n$ and $h = \log m$.

5 Main procedure for incremental DFS tree

In this section, we prove the second half of Theorem 1.1. We first consider the offline version of incremental DFS tree, for which we present a procedure $\text{IncrementalMain}$, drastically simplifying the one introduced in [1]. The time needed by $\text{IncrementalMain}$ is $O(n + |U|)$ given that we can answer all the queries in $O(n)$ time in
Algorithm 1: IncrementalMain

**Data:** a DFS tree $T$ of $G$, the preprocessed data structure $D$, the set of insertions $U$

**Result:** a DFS tree $T^*$ of $G + U$

1. Add each inserted vertex $v$ into $T$, set $\text{par}(v) = r$;
2. Initialize $L(v)$ to be $\emptyset$ for each $v$;
3. Add each inserted edge $(u, v)$ to $L(u)$ and $L(v)$;
4. Call DFS($v$);

Algorithm 2: DFS

**Data:** a DFS tree $T$ of $G$, the preprocessed data structure $D$, the entering vertex $v$

**Result:** a partial DFS tree $T$.

1. Let $u = v$;
2. while par($u$) is not visited do
   1. Let $u = \text{par}(u)$;
3. Mark path($u, v$) to be visited;
4. Let $\tilde{w}_1, \ldots, \tilde{w}_t = \text{path}(u, v)$;
5. for $i \in [t]$ do
   1. if $i \neq t$ then
      1. Let $\text{par}^+(\tilde{w}_i) = \tilde{w}_{i+1}$;
      2. for child $x$ of $\tilde{w}_i$ in $T$ except $\tilde{w}_{i+1}$ do
         1. Let $(y, z) = Q(T(x), u, v)$, where $y \in \text{path}(u, v)$;
         2. Add $z$ into $L(y)$;
   1. for $i \in [t]$ do
      1. if $x$ is not visited then
         1. Let $\text{par}^+(x) = \tilde{w}_i$;
         2. Call DFS($x$);

In IncrementalMain, we first attach each inserted vertex to the super root $r$, and pretend it has been there since the very beginning. What left to be considered are merely edge insertions. All inserted edges are added into the reduced adjacency lists of corresponding vertices. We then use DFS to traverse the graph starting from $r$ based on $T$, $L$ and $D$, and build the new DFS tree while traversing the entire graph and updating the reduced adjacency lists.

The new DFS tree is built in a recursive fashion. Every time we enter an untouched subtree, say $T(u)$, from vertex $v \in T(u)$, we change the root of $T(u)$ to $v$ and go through path$(v, u)$, visit all vertices on the path and cut off all subtrees hanging from it. By Lemma 2.1, it suffices to ignore all other edges but just keep the edge returned by $Q(T(u), u, v)$. Denote $(x, y)$ to be the edge returned by $Q(T(w), u, v)$ where $x \in \text{path}(u, v)$, we then add $y$ into $L(x)$. After finding an appropriate entering edge for each hanging subtree, we proceed each vertex $v \in \text{path}(u, v)$ in ascending order of depth. For every unvisited $w \in L(v)$, we set $\text{par}^+(w) = v$, and recursively call DFS($w$).

**Theorem 5.1.** *IncrementalMain* correctly reports a feasible DFS tree $T^*$ of graph $G + U$.

**Proof.** We argue that in a single call DFS($v$), where $u$ is the highest unvisited ancestor of $v$, every unvisited (at the moment being enumerated) subtree $T(u)$ hanging from path$(u, v)$, as well as every vertex on path$(u, v)$ except $v$, will be assigned an appropriate parent such that these parent-child relationships constitute a DFS tree of $G$ at the termination of IncrementalMain. When the traversal reaches $v$, the entire $T(u)$ is untouched, or else $u$ would have been marked by a previous visit to some vertex in $T(u)$. We could therefore choose to go through path$(v, u)$ to reach $u$ first. By Lemma 2.1, if a subtree $T(w)$ is reached from some vertex on path$(u, v)$, it suffices to consider only the edge $Q(T(w), u, v)$. After adding the query results of all hanging subtrees into the adjacency lists of vertices on path$(u, v)$, every hanging subtree visited from some vertex $x$ on
Theorem 5.2. IncrementalMain

As a subroutine of our scheme for answering queries, we briefly recapitulate the scintillating idea of fractional $2 \leq \sqrt{2n^2}$ integers. ∀ $O$ be $O$ which is greater than or equal to $x$

Definition 6.1

Lemma 6.1 (Tree partition structure, Lemma 5.1 in [8]). Given a rooted tree $T$ and any integer $k$ such that $2 \leq k \leq n = |V(T)|$, there exists a subset of vertices $M \subseteq V(T)$, $|M| \leq 3k - 5$, such that after removing all vertices in $M$, the tree $T$ is partitioned into sub-trees of size at most $n/k$. We call every $v \in M$ an $M$-marked vertex, and $M$ a marked set. Also, such $M$ can be computed in $O(n \log k)$ time.

6 Useful tools

6.1 Tree partition structure

In order to answer queries efficiently, we need the tree partition structure. The following lemma is first given in [8].

Lemma 6.1 (Tree partition structure, Lemma 5.1 in [8]). Given a rooted tree $T$ and any integer $k$ such that $2 \leq k \leq n = |V(T)|$, there exists a subset of vertices $M \subseteq V(T)$, $|M| \leq 3k - 5$, such that after removing all vertices in $M$, the tree $T$ is partitioned into sub-trees of size at most $n/k$. We call every $v \in M$ an $M$-marked vertex, and $M$ a marked set. Also, such $M$ can be computed in $O(n \log k)$ time.

6.2 Fractional cascading: an overview

As a subroutine of our scheme for answering queries, we briefly recapitulate the scintillating idea of fractional cascading propounded by Chazelle and Guibas [5, 6].

Consider such a problem:

Definition 6.1 (Multiple Successor Problem). There are $k$ sorted arrays $\{A_i\}_{i \in [k]}$ of integers, each of size $m_i = |A_i|$. Let $m = \sum_{i \in [k]} m_i$. Given an integer $x$, we are to find a minimum element in each array $A_i$ which is greater than or equal to $x$, i.e., the successor of $x$ in $A_i$.

A naive way to solve the Multiple Successor Problem is to apply binary search for each $A_i$, which takes $O(\sum_{i \in [k]} \log m_i)$ time. When $k = \sqrt{m}$ and $m_i = m/k$, the time needed by naive binary search turns out to be $O(\sqrt{m} \log m)$. Can we do better, say, in $O(\log m + k)$ time?

The answer, of course, is yes. We build $k$ arrays $\{B_i\}_{i \in [k]}$ one by one. i.e., we build $B_{i+1}$ after building $B_i$ through $B_i$ first. Let $B^\text{even}_i$ denote the sorted array consisting of elements in $B_i$ with even subscripts, i.e., $\forall j \in [\lfloor m_i/2 \rfloor]$, $B^\text{even}_i[j] = B_i[2j]$. $B_{i+1}$ is obtained by merging $A_{i+1}$ and $B^\text{even}_i$, retaining the order of integers.

Let $a(x)_i$ and $b(x)_i$ denote respectively the successor of $x$ in $A_i$ and $B_i$. Note that whenever we know $b(x)_i$:

- The successor of $x$ in $A_i$, $a(x)_i$, can be found by looking for the element which is on the right side of $b(x)_i$ and nearest to $b(x)_i$ (instead of $B_{i-1}$ when $i > 1$).
• Also, for \( i > 1 \), when we know \( b(x) \), already, \( b(x)_{i-1} \) can be found by “tracing” the nearest element(s) in \( B_i \) that comes from \( B_{i-1} \) (instead of \( A_i \)), and checking \( O(1) \) neighbors of them in \( B_{i-1} \).

If we build pointers for each element \( t \) in \( B_i \) pointing to its “hometown” (i.e., \( A_i \) if \( t \) comes from \( A_i \), and \( B_{i-1} \) otherwise) and the nearest neighbors of \( t \) with a different “identity” (i.e., elements from \( B_{i-1} \) if \( t \) comes from \( A_i \), and those from \( A_i \) otherwise), the two tracing operations above can both be done in \( O(1) \) time.

When answering successors of \( x \) in all arrays \( \{A_i\} \), first we do a single binary search, getting the successor of \( x \) in \( B_k \), \( b(x)_k \). Then we get \( a(x)_k \) and \( b(x)_{k-1} \) using \( b(x)_k \) in \( O(1) \) time. Recursively we can find successors of \( x \) in all \( \{A_i\} \) in time \( O(\log m + k) \).

7 Dealing with queries in IncrementalMain

First we present the PathQuery scheme and later the PreprocessAll scheme, achieving \( O(m \log n + n \log^2 n) \) and \( O(n^2) \) preprocessing time respectively, and \( O(n) \) query time in total for a call of IncrementalMain. We note again, that an \( O(n) \) total query time is possible for incremental updates (but very likely not for fully dynamic updates), since we no longer need to remove edges from the data structure once it has been built.

**Deal with all queries in \( O(n \log n) \) time.** We have shown in Section 3 that there exists a data structure which answers each individual query in time \( O(\log n) \), with preprocessing time \( O(m \log n) \). With that we can easily answer all queries in a single run of IncrementalMain in \( O(n \log n) \) time.

We note that \( O(n \log n) \) total time is already a notable improvement over the former state-of-the-art [1]. However, to achieve the main goal of this paper, i.e., to attain a scheme which updates the DFS tree upon each insertion in worst case \( O(n) \) time, we have to answer all queries in \( O(n) \) time.

**Deal with all queries in \( O(n) \) time.** To further improve the query time to \( O(n) \), observe that in IncrementalMain (and DFS), a bunch of queries \( \{Q(T(w_i), x, y)\} \) are always made simultaneously, where \( \{T(w_i)\} \) is the set of subtrees hanging from \( \text{path}(x, y) \). We may therefore answer all queries for a path in one pass, instead of answering them one by one. In doing so, we will inevitably confront two types of hard queries.

First consider an example where the original DFS tree \( T \) is a chain \( L \) such that \( a_1 \) is the root of \( L \) and for \( 1 \leq i \leq n-1, a_{i+1} \) is the unique child of \( a_i \). When we invoke DFS\((a_1)\) on \( L \), \( \text{path}(u, v) \) is the single node \( a_1 \). Thus, we will call \( Q(T(a_2), a_1, a_1) \) and add the returned edge into \( L(a_1) \). Supposing there are no back-edges in this graph, the answer of \( Q(T(a_2), a_1, a_1) \) will be the edge \( (a_1, a_2) \). Therefore, we will recursively call the DFS\((a_2)\) on the chain \( (a_2, a_n) \). Following further steps of DFS, we can see that we will call the query \( Q(T(w), x, y) \) for \( \Omega(N) \) times. In Section 7.2 we show that we can deal with this example in linear time. The idea is to answer queries involving short paths in constant time. For instance, in the example shown above, \( \text{path}(u, v) \) always has constant length. We show that when the length of \( \text{path}(u, v) \) is smaller than \( 2 \log n \), it is affordable to preprocess all the answers to queries of this kind in \( O(n \log^2 n) \) time and \( O(n \log n) \) space.

![Figure 1: An example](image1.png)

The second example we considered is given as Figure 1. In this tree, the original root is \( r \). Suppose the distance between \( r \) and \( r' \) is \( n/2 \). When we invoke DFS\((r')\), \( \text{path}(u, v) \) the path from \( r \) to \( r' \). Thus, we will call \( T(a_1, r, r'), T(a_2, r, r'), \ldots, T(a_{n-2}, r, r') \), which means we make \( \Omega(N) \) queries. In Section 7.2 we show that we can also deal with this example in linear time. The main idea is using fractional cascading to answer
all queries \( Q(T(w), x, y) \) with a fixed path \((u, v)\), for all subtrees \( T(w) \) with small size, by using affordable time. In the example shown above, all subtrees cut off path \((u, v)\) have constant size and thus the total time complexity for this example is \( O(n) \).

Surprisingly, we finally show in Section 7.3 that, by combining the two techniques mentioned above, it is enough to answer all queries \( Q(T(w), x, y) \) in linear time and thus prove Lemma 5.1.

Another approach to answer all queries in linear time is by using a look-up table. When the PreprocessAll scheme is applied, things are much simpler. We merely preprocess and store answers for all \( n^2 \) possible queries, and answer each in \( O(1) \) time. As we show in Section 7.4 the preprocessing can be done in \( O(n^2) \) time.

7.1 Answering for short paths in \( O(1) \) time

For paths not longer than 2 log \( n \), we preprocess and store the answer for each query. We simply enumerate every pair \((x, w)\), where \( x \) is an ancestor of \( w \) and the distance between them is at most 2 log \( n \), and store the answer \( Q(T(w), x, \text{par}(w)) \). This part of preprocessing can be done in \( O(n \log^2 n) \) time given we can answer any query in \( O(n \log n) \) time. For further queries, we can directly use the look-up table to answer queries \( Q(T(w), x, y) \) when the distance between \( x \) and \( w \) is at most 2 log \( n \).

7.2 Answering for small subtrees on long paths

We present a scheme to answer for small subtrees, of size not exceeding log \( n \), on long paths, of length greater than 2 log \( n \).

**Preprocessing.** In the preprocessing, we first build the tree partition structure stated in Lemma 6.1 out of \( T \) with \( k = n / \log n \). As mentioned in Lemma 6.1, the tree partition structure can be built in \( O(n \log k) = O(n \log n) \) time. Let \( M_{\text{init}} \) denote the vertex set marked by tree partition. We build a reduced marked set \( M \) from \( M_{\text{init}} \) consisting only of roots of subtrees with notable sizes. For each vertex \( v \in M_{\text{init}} \), if the size of subtree \( T(v) > \log n \), we add \( v \) into \( M \). Thus, \( |M| \leq |M_{\text{init}}| = O(k) = O(n / \log n) \). For each \( v \in M \), we maintain a set \( F_v \) of arrays. Each element of \( F_v \) is an array of edges, or more specifically, an adjacency list of a vertex in \( G \).

For each vertex \( v \in V \), we find the \( M \)-marked ancestor \( \text{anc}(v) \) of \( v \) with a maximum depth. We add the array consisting of all edges incident to \( v \), i.e., the adjacency list \( N(v) \), into \( F_{\text{anc}(v)} \). (Keep in mind that \( F_{\text{anc}(v)} \) is a set of adjacency lists.) If no such \( \text{anc}(v) \) exists, we ignore this step for \( v \).

For each \( M \)-marked vertex \( v \in M \), we build a fractional cascading structure \( \text{cas}_v \) based on the set \( F_v \) of arrays, as mentioned in Section 6.2. In \( \text{cas}_v \), we sort each array of edges in ascending order of the depth of the ancestor vertex, i.e., the higher end of the edge. (Recall that in a DFS tree, all non-tree edges are ancestor-descendant edges.) As mentioned in Section 6.2 if \( F_v \) contains \( \text{size}_v \) arrays, for a vertex \( u \), a query that requires a descendant (not necessarily proper) of \( u \) with a minimum depth in each array in \( F_v \) can be answered in \( O(\text{size}_v + \log n) \) total time. This finishes our preprocessing.

**Answer Queries.** Each time we invoke the DFS subroutine with the entering vertex being \( v \), let \( u \) denote the highest unvisited ancestor of \( v \). We are to answer a bunch of queries of the form \( Q(T(w), u, v) \) for subtrees hanging from path \((u, v)\). Let \( (w_1, w_2, \ldots, w_t) = \text{path}(u, v) \) be the path from \( u \) to \( v \), where \( t > 2 \log n \), since we consider long paths only in this section. (See the description of DFS for more details.)

First note that there exists some \( p \leq \log n + 1 \) such that \( w_p \in M \). Suppose not, denoting the vertex set \( \{w_1, \ldots, w_{\log n + 1}\} \cap M_{\text{init}} \) by \( S \), there are two seemingly possible cases:

- \( S = \emptyset \). This is impossible, because \( \{w_1, \ldots, w_{\log n + 1}\} \) forms a subtree of size \( \log n + 1 \), contradicting Lemma 6.1.
- \( \forall s \in S, T(s) \leq \log n \). This is also impossible, because \( \forall s \in S, \{w_{\log n + 1}, \ldots, w_1\} \subseteq T(s) \), such that \( |T(s)| \geq t - \log n > \log n \).

For each vertex \( w \in M \cap \{w_1, w_2, \ldots, w_t\} \), for each adjacency list \( L \in F_w \), we find an edge whose higher end is a descendant (not necessarily proper) of \( u \) with a minimum depth in \( L \) using the fractional cascading structure \( \text{cas}_u \).

Now for each query \( Q(T(x), u, v) \) in DFS where the size of the subtree \( T(x) \) is at most \( \log n \), if \( \text{par}(x) \in \{w_1, \ldots, w_{\log n}\} \), the query \( Q(T(x), u, v) \), being equivalent to \( Q(T(x), u, \text{par}(x)) \), can be answered by the method to answer queries involving short paths described in Section 7.1 in constant time.
For a small subtree \( T(x) \) where \( \text{par}(x) \in \{ w_{\log n+1}, \ldots, w_i \} \). Let vertex \( w \) be the nearest \( M \)-marked ancestor of \( x \) on \( \text{path}(u, v) \), whose existence is guaranteed. Since \( M \cap T(x) = \emptyset \), for any vertex \( x' \in T(x) \), the adjacency list \( N(x') \) is in \( \mathcal{F}_w \). That means, we have already found an edge in \( N(x') \) whose higher end is a descendant of \( u \) with a minimum depth in former steps. \( Q(T(x), u, v) \) can then be answered by aggregating results from adjacency lists of all \( x' \in T(x) \) in \( O(|T(x)|) \) time.

### 7.3 The overall \textit{PathQuery} scheme

With all the ingredients at our hands, we present the overall \textit{PathQuery} scheme, and prove that it answers all queries in a single call of \textit{IncrementalMain} in \( O(n) \) time.

For a query involving path \( p = \text{path}(x, y) \), we answer all queries in the following manner:

- If \( p \) is a short path, i.e., \( |p| \leq 2 \log n \), we answer all subtree queries using the method described in Section 7.1, i.e., by looking up in the preprocessed table.
- If \( p \) is a long path, i.e., \( |p| > 2 \log n \), we answer all small subtrees using the method described in Section 7.2 and for all large subtrees using directly the data structure described in Section 3.

We now analyze the query and preprocessing time of the \textit{PathQuery} scheme. First we prove the total cost generated by small subtrees on long paths is \( O(n) \).

#### Lemma 7.1. The total cost due to queries involving small subtrees on long paths is \( O(n) \).

\[ \text{Proof.} \] Recall that as described in Section 7.2, the time needed to answer queries involving small subtrees hanging from any long path \( p = \text{path}(u, v) \) consists of four parts:

- Answering for small subtrees hanging close to the upper end of the path. For such a subtree \( T(w) \), since \( |\text{path}(u, \text{par}(w))| \leq \log n \), \( Q(T(w), u, v) \) can be answered in \( O(1) \) time. Note that a subtree can only appear in at most one such query. This part of cost is therefore \( O(n) \).
- Initial binary search at \( M \)-marked vertices. Note that a binary search is exerted at an \( M \)-marked vertex \( v \) only if \( v \) is on some long path being queried. Once such a query occurs, \( v \) will no longer appear in later queries. I.e., an \( M \)-marked vertex can bear a binary search at most once. Since \( M = \left( \frac{n}{\log n} \right) \), this part of cost is \( O(\log m) \cdot O \left( \frac{n}{\log n} \right) = O(n) \).
- Cascading procedure at \( M \)-marked vertices. Similar to a binary search, a cascading procedure is exerted for \( v \in M \) at most once. Recall that the cascading procedure for \( v \) requires \( O(\text{size}_v) \) time. When preprocessing, any vertex \( u \in T \) gives its adjacency list to at most one \( M \)-marked vertex \( v \), contributing \( 1 \) to \( \text{size}_v \). This part of cost is subsequently \( O \left( \sum_{v \in M} \text{size}_v \right) = O(n) \).
- Aggregating results for each small subtree. The aggregating costs \( O(|T(w)|) \) time for a small subtree \( T(w) \). Since each subtree can be involved in at most one long path query, and all such subtrees are disjoint, this part of time is \( O(n) \).

We conclude that the total cost to answer for small subtrees on long paths is \( O(n) \).

Then we bound the number of “heavy” queries. I.e., those require \( O(\log n) \) time to answer each.

#### Lemma 7.2. The number of heavy queries is \( O \left( \frac{n}{\log n} \right) \).

\[ \text{Proof.} \] What we want, obviously, is the number of subtrees with sizes larger than \( \log n \) cut from long paths with lengths larger than \( \log n \). For simplicity, we consider a relaxed setting instead of bounding the size directly.

We build a “splitting tree” \( T_s \) based on \textit{PathQuery} calls occurred in \textit{IncrementalMain}. Every time \textit{PathQuery} is called, a path is picked, and all hanging subtrees are cut off. For each \textit{PathQuery}(\( u_i, v_i \)), if \( |T(u_i)| \geq \log n \), we map \( T(u_i) \) to a vertex in the splitting tree, and let all hanging subtrees \( T(w_i) \) whose size is larger than \( \log n \) be its children. Also, \( \text{path}(u_i, v_i) \) is considered a child of \( T(u_i) \) if its length is larger than \( \log n \). Let
\(d_s(v)\) denote the number of children of \(v \in T_s\). Observe that whenever the \text{PathQuery} call generating \(v \in T_s\) involves any heavy queries, \(d_s(v) \geq 2\), and the number of heavy queries involved is at most \(d_s(v) - 1\). Let \(I_s\) be the set of inner vertices, and \(L_s\) be that of leaves. Notice that \(\sum_{v \in I_s} (d_s(v) - 1) = |L_s| - 1\). Therefore, the total number of heavy queries is \(O(|L_s|)\).

On the other hand, every vertex in \(T_s\) corresponds to a subtree or a path in \(T\) whose size is larger than \(\log n\). For two different leaves of \(T_s\), the corresponding subtrees/paths are disjoint, from which we conclude \(|L_s| < \frac{n}{\log n}\). The engendered total cost hence will not exceed \(O\left(\frac{n}{\log n} \cdot \log n\right) = O(n)\).

Finally consider the cost begotten by small paths.

**Lemma 7.3.** The total cost of small path queries is \(O(n)\).

**Proof.** For a small path \(p\), the cost caused when cutting all subtrees hanging from \(p\) off is the number of hanging subtrees. Again, a vertex \(w\) can only be cut off once as the root of a subtree (i.e., \(T(w)\)). This part of cost is obviously bounded by \(O(|T|) = O(n)\).

Last but not least, the preprocessing time is bounded by \(O(m \log n + n \log^2 n)\).

**Lemma 7.4.** The preprocessing time of the \text{PathQuery} scheme is \(O(m \log n + n \log^2 n)\).

**Proof.** As shown in Section 3, the time needed to build the structure to answer any query in \(O(\log n)\) time is \(O(m \log n)\). As shown in Section 7.2, the preprocessing time for small subtree queries involving long paths is \(O(m \log n)\). When preprocessing answers for short path queries, we enumerate all \(n \log n\) possible queries, calculate and store the answer using \(O(n \log n)\) time for each of them. Altogether the preprocessing time for the \text{PathQuery} scheme is \(O(m \log n + n \log^2 n)\).

### 7.4 The PreprocessAll scheme

As will be shown in Section 8 for efficient deamortization, we want the preprocessing time of the data structure to be \(O(n^2)\). That means, when \(m \log n = \omega(n^2)\), simply applying the \text{PathQuery} scheme would not suffice for our purpose. Instead, we apply the \text{PreprocessAll} scheme, where answers for all possible queries are preprocessed, and answered directly by looking up.

For preprocessing, we enumerate each subtree \(T(w)\), fix the lower end of the path to be \(v = \text{par}(w)\), let the upper end \(u\) grow upward from \(v\) by 1 vertex at a time, and calculate \(Q(T(w), u, v)\) incrementally, thereby getting all answers involving subtree \(T(w)\) in \(O(n)\) time. When the path grows, we must check whether there is an edge from \(T(w)\) to the new upper end \(u\) in \(O(1)\) time, for which we build an array (based the DFS sequence of \(T\)) for each vertex, insert an 1 into the appropriate array for each edge, and apply the standard prefix summation trick to check whether there is an 1 in the range corresponding to \(T(w)\).

To be precise, let \(A_u : [n] \rightarrow \{0, 1\}\) denote the array for vertex \(u\). Recall that \(\text{first}(v)\) denotes the index of vertex \(v\) in the DFS sequence, and \(\text{last}(v)\) the index of the last vertex in \(T(v)\). For a vertex \(u\), we set \(A_u[\text{first}(v)]\) to be 1 if and only if there is an edge \((u, v)\) where \(u\) is the higher end. Now say, we have the answer to \(Q(T(w), u, v)\) already, and want to get \(Q(T(w), u', v)\) in \(O(1)\) time, where \(u' = \text{par}(u)\). If there is an edge between \(T(w)\) and \(u'\), then it will be the answer. Or else the answer to \(Q(T(w), u', v)\) will be the same as to \(Q(T(w), u, v)\). In order to know whether there is an edge between \(T(w)\) and \(u'\), we check the range \([\text{first}(w), \text{last}(w)]\) in \(A_u\), and see if there is an 1 in \(O(1)\) time using the prefix summation trick.

**Lemma 7.5.** The preprocessing time and query time for the \text{PreprocessAll} scheme are \(O(n^2)\) and \(O(1)\) respectively.

**Proof.** The array \(A_u\) and its prefix sum can be computed for each vertex \(u\) in total time \(O(n^2)\). For each subtree \(T(w)\), we go up the path from \(w\) to the root \(r\), and spend \(O(1)\) time for each vertex \(u\) on \(\text{path}(r, w)\) to get the answer to \(Q(T(w), u, \text{par}(w))\). There are at most \(n\) vertices on \(\text{path}(r, w)\), so the time needed for a single subtree is \(O(n)\), and that needed for all subtrees is \(n \cdot O(n) = O(n^2)\) in total. On the other hand, for each query, we simply look it up and answer in \(O(1)\) time. We conclude that the preprocessing time and query time for the \text{PreprocessAll} scheme are \(O(n^2)\) and \(O(1)\) respectively.
8 Deamortization

In this section, we extend \texttt{IncrementalMain} to handle the online version of Incremental DFS Tree Problem. The data structure $D$, as shown above, can be constructed in $O(m \log n + n \log^2 n)$ time using the \texttt{PathQuery} scheme, and in $O(n^2)$ time using the \texttt{PreprocessAll} scheme. Following \cite{1}, when insertions arrive online, and a DFS tree is demanded after every single insertion, we deamortize \texttt{IncrementalMain} to attain $O(n)$ worst case query time. I.e., We constantly rebuild $D$ as insertions arrive, and always use the newest version of $D$ to answer each query after insertion. The point is, the construction of $D$ is distributed to $n$ time slots, so we need only $O(n)$ time to build $D$ after each insertion. The following lemma was introduced in \cite{1}.

**Lemma 8.1.** (Lemma 6.1 in \cite{1}) Let $D$ be a data structure that can be used to report the solution of a graph problem after a set of $U$ updates on an input graph $G$. If $D$ can be built in $O(f)$ time and the solution for graph $G + U$ can be reported in $O(h + |U| \times g)$ time, then $D$ can be used to report the solution after every update in worst case $O(\sqrt{|f|g} + h)$ update time, given that $\sqrt{|f|g} \leq n$.

**Theorem 8.1.** There exists an incremental algorithm for maintaining a DFS tree in an undirected graph that uses $O(\min\{m \log n + n \log^2 n, n^2\})$ preprocessing time and can report a DFS tree after each edge insertion in the worst case $O(n)$ time.

**Proof.** As shown in Section \cite{1}, there is a data structure $D$ which can be built in $O(n^2)$ time and the solution for graph $G + U$ can be reported in $O(n + |U|)$ time. Consider $f$, $g$, and $h$ described in Lemma \cite{1}. Here let $f = n^3$, $g = 1$, and $h = n$. Clearly $\sqrt{|f|g} = n \leq n$. By Lemma \cite{1}, $D$ can be used to report the solution after every update in worst case $O(\sqrt{|f|g} + h) = O(n)$ update time. When preprocessing, we choose the better scheme between \texttt{PathQuery} and \texttt{PreprocessAll}. The preprocessing time is hence $O(\min\{m \log n + n \log^2 n, n^2\})$.

Let $q_1, q_2, \ldots$ be the online insertions and corresponding queries demanding a new DFS tree (or just time slots for short), and $G_i = (V_i, E_i)$ be the graph after the $i$-th insertion. Let $n_i = |V_i|$ and $m_i = |E_i|$. To be precise, we build $D_0$ when preprocessing, and use $D_0$ to answer $q_1, \ldots, q_{2n}$. Meanwhile, we build $D_1$ based on $G_n$ as we process $q_{n+1}, \ldots, q_{2n}$. After the first $2n$ time slots, for $i \geq 1$, we use $D_i$ to answer queries $q_{(i+1)n+1}, \ldots, q_{(i+2)n}$, and build $D_{i+1}$ based on $G_{(i+1)n}$ at the same time. Whenever a new $D$ is finished, we discard the old version.

As $m_i$ may increase as insertions happen, we always choose the better scheme between \texttt{PathQuery} and \texttt{PreprocessAll} (or simply switch to \texttt{PreprocessAll} after preprocessing) to build $D$. Particularly, when building $D_0$, we choose the \texttt{PathQuery} scheme if $m \log n \leq n^2$, and the \texttt{PreprocessAll} scheme otherwise. At each time slot, we spend $O(n)$ time to build the new $D$ for future use. The $O(n^2)$ building time can thusly be distributed into $n$ time slots, and the running time of \texttt{IncrementalMain} is bounded by $O(n + U_i) = O(n)$ where $U_i$ is the “increment” of the graph from the version corresponding to the latest $D$, i.e., $G_i = G_{i\lfloor \frac{\log n}{2} \rfloor} + U_i$. The overall preprocessing time is $O(\min\{m \log n + n \log^2 n, n^2\})$, and the worst case query time is $O(n)$.

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**References**

[1] Surender Baswana, Shreejit Ray Chaudhury, Keerti Choudhary, and Shahbaz Khan. Dynamic dfs in undirected graphs: breaking the $O(m)$ barrier. In Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 730–739. SIAM, 2016.

[2] Surender Baswana and Keerti Choudhary. On dynamic DFS tree in directed graphs. In International Symposium on Mathematical Foundations of Computer Science (MFCS), pages 102–114. Springer, 2015.

[3] Surender Baswana, Manoj Gupta, and Sandeep Sen. Fully dynamic maximal matching in $O(\log n)$ update time. In Foundations of Computer Science (FOCS), 2011 IEEE 52nd Annual Symposium on, pages 383–392. IEEE, 2011.
[4] Surender Baswana and Shahbaz Khan. Incremental algorithm for maintaining DFS tree for undirected graphs. In *International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 138–149. Springer, 2014.

[5] Bernard Chazelle and Leonidas J Guibas. Fractional cascading: I. a data structuring technique. *Algorithmica*, 1(1-4):133–162, 1986.

[6] Bernard Chazelle and Leonidas J Guibas. Fractional cascading: II. applications. *Algorithmica*, 1(1-4):163–191, 1986.

[7] Camil Demetrescu and Giuseppe F Italiano. A new approach to dynamic all pairs shortest paths. *Journal of the ACM (JACM)*, 51(6):968–992, 2004.

[8] Ran Duan and Tianyi Zhang. Improved distance sensitivity oracles via tree partitioning. *arXiv preprint arXiv:1605.04491*, 2016.

[9] David Eppstein, Zvi Galil, Giuseppe F Italiano, and Amnon Nissenzweig. Sparsification technique for speeding up dynamic graph algorithms. *Journal of the ACM (JACM)*, 44(5):669–696, 1997.

[10] Paolo G Franciosa, Giorgio Gambosi, and Umberto Nanni. The incremental maintenance of a depth-first-search tree in directed acyclic graphs. *Information processing letters*, 61(2):113–120, 1997.

[11] Monika R Henzinger and Valerie King. Randomized fully dynamic graph algorithms with polylogarithmic time per operation. *Journal of the ACM (JACM)*, 46(4):502–516, 1999.

[12] Jacob Holm, Kristian De Lichtenberg, and Mikkel Thorup. Poly-logarithmic deterministic fully-dynamic algorithms for connectivity, minimum spanning tree, 2-edge, and biconnectivity. *Journal of the ACM (JACM)*, 48(4):723–760, 2001.

[13] Bruce M Kapron, Valerie King, and Ben Mountjoy. Dynamic graph connectivity in polylogarithmic worst case time. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1131–1142. Society for Industrial and Applied Mathematics, 2013.

[14] Ofer Neiman and Shay Solomon. Simple deterministic algorithms for fully dynamic maximal matching. *ACM Transactions on Algorithms (TALG)*, 12(1):7, 2016.

[15] Yakov Nekrich and Gonzalo Navarro. Sorted range reporting. In *Scandinavian Workshop on Algorithm Theory (SWAT)*, pages 271–282. Springer, 2012.

[16] Franco P Preparata and Michael Shamos. *Computational geometry: an introduction*. Springer Science & Business Media, 2012.

[17] Liam Roditty and Uri Zwick. Improved dynamic reachability algorithms for directed graphs. *SIAM Journal on Computing*, 37(5):1455–1471, 2008.

[18] Liam Roditty and Uri Zwick. Dynamic approximate all-pairs shortest paths in undirected graphs. *SIAM Journal on Computing*, 41(3):670–683, 2012.

[19] Piotr Sankowski. Dynamic transitive closure via dynamic matrix inverse. In *Foundations of Computer Science (FOCS), 2004. Proceedings. 45th Annual IEEE Symposium on*, pages 509–517. IEEE, 2004.

[20] Robert Tarjan. Depth-first search and linear graph algorithms. *SIAM journal on computing*, 1(2):146–160, 1972.

[21] Mikkel Thorup. Fully-dynamic min-cut. In *Proceedings of the thirty-third annual ACM symposium on Theory of computing (STOC)*, pages 224–230. ACM, 2001.