THE GLOBAL DIMENSION OF SCHUR ALGEBRAS FOR GL₂ AND GL₃

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Abstract. We first define the notion of good filtration dimension and Weyl filtration dimension in a quasi-hereditary algebra. We calculate these dimensions explicitly for all irreducible modules in \( \text{SL}_2 \) and \( \text{SL}_3 \). We use these to show that the global dimension of a Schur algebra for \( \text{GL}_2 \) and \( \text{GL}_3 \) is twice the good filtration dimension. To do this for \( \text{SL}_3 \), we give an explicit filtration of the modules \( \nabla(\lambda) \) by modules of the form \( \nabla(\mu)^P \otimes L(\nu) \) where \( \mu \) is a dominant weight and \( \nu \) is \( p \)-restricted.

Introduction and Background

The global dimension of a \( q \)-Schur algebra \( S_q(n, r) \) has been determined when \( r \leq n \). This was calculated by Totaro [21] (for the classical case) and Donkin [11] (for the quantum case). In general the global dimension of \( S_q(n, r) \) is not known, although we do have upper bounds for this value, (see [21] for more details). In this paper we will calculate explicitly the global dimension of \( S(n, r) \) for \( n = 2 \) and \( n = 3 \). We also find the global dimension of \( S_q(2, r) \).

We first briefly review some of the notation and definitions that we will use in this paper. The reader is referred to [11] and [17] for further information. We let \( G = \text{SL}_n(k) \) where \( k \) is an algebraically closed field of characteristic \( p \) and \( F : G \to G \) the corresponding Frobenius morphism.

We let \( G_1 \) be the first Frobenius kernel. Let \( T \) be a maximal torus of \( G \), \( W \) the corresponding Weyl group and \( B \supseteq T \) a Borel subgroup. Let \( X = X(T) \) be the weight lattice and let \( X^+ \) be the set of dominant weights. Let \( X_1 \) be the set of \( p \)-restricted dominant weights and \( A_0 \) the set of weights in the interior of the fundamental alcove.

For \( \lambda \in X^+ \), let \( k_\lambda \) be the one-dimensional module for \( B \) which has weight \( \lambda \). We define \( \nabla(\lambda) = \text{Ind}_B^G(k_\lambda) \). This module has character given by Weyl’s character formula and has simple socle \( L(\lambda) \), the irreducible \( G \)-module of highest weight \( \lambda \).

We define \( S(n, r) \) to be the Schur algebra corresponding to \( \text{GL}_n \) and \( \Lambda^+(n, r) \) to be the set of partitions of \( r \) with at most \( n \) parts. Modules for \( S(n, r) \) are naturally polynomial modules for \( \text{GL}_n \) which are homogeneous of degree \( r \). An irreducible module for \( S(n, r) \) is also an irreducible module for \( \text{GL}_n \) and correspond to elements of \( \Lambda^+(n, r) \). We write \( L(a_1, \cdots, a_n) \) for the irreducible module of highest weight \( (a_1, \cdots, a_n) \in \Lambda^+(n, r) \). More details can be found in [14] chapter 2.

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We have a natural correspondence between weights for $S(n,r)$-modules and $\text{SL}_n$-modules given by
\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mapsto (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n). \]

The usual partial ordering on $\text{SL}_n$-weights is equivalent by the above correspondence to the dominance ordering on partitions.

The category of rational $G$-modules has enough injectives and so we may define $\text{Ext}^*(\cdot, \cdot)$ as usual by using injective resolutions. We have a canonical isomorphism for all $S(n,r)$-modules [8, 2.2d]
\[ \text{Ext}^i_{S(n,r)}(V,W) \cong \text{Ext}^i_{\text{GL}_n}(V,W) \cong \text{Ext}^i_{\text{SL}_n}(V,W). \]

Thus we may do all Ext calculations in $\text{mod}(G)$.

1. **Quasi-hereditary Algebras**

We start with the definition of a highest weight category given by Cline, Parshall and Scott [1].

**Definition 1.1.** Let $k$ be a field, $S$ a finite dimensional algebra over $k$, $\Lambda$ an indexing set for the isomorphism classes of simple $S$-modules with a correspondence $\lambda \leftrightarrow L(\lambda)$, and $\leq$ a partial order on $\Lambda$. We say $(S, \Lambda)$ is a highest weight category if and only if, for all $\lambda \in \Lambda$ there is a left $S$-module $\nabla(\lambda)$, called the costandard module, such that:

(i) there exists an injection $\phi_{\lambda} : L(\lambda) \to \nabla(\lambda)$, and the composition factors, $L(\mu)$ of the cokernel satisfy $\mu < \lambda$

(ii) the indecomposable injective hull, $I(\lambda)$, of $L(\lambda)$ contains $\nabla(\lambda)$ via the injection $\psi_{\lambda} : \nabla(\lambda) \to I(\lambda)$ and the cokernel of $\psi_{\lambda}$ is filtered by modules $\nabla(\mu)$ with $\mu > \lambda$.

Dually we have $\Delta(\lambda)$ as standard modules by replacing injective by projective and cokernel by kernel in the above definition. Thus the standard modules $\Delta(\lambda)$ have simple head $L(\lambda)$. (We will assume that $S$ is Schurian in the sense that $\text{End}_S(L) = k$ for all simple modules $L$.)

A highest weight category is equivalent to the module category for a quasi-hereditary algebra although we will not show this here. See [18] or [11, Appendix] for details.

We say $X \in \text{mod}(S)$ has a good filtration if it has a filtration
\[ 0 = X_0 \subset X_1 \subset \cdots \subset X_i = X \]
with quotients $X_j/X_{j-1}$ isomorphic to $\nabla(\mu_j)$ for some $\mu_j \in \Lambda$. We denote the class of $S$-modules with good filtration $F(\nabla)$, and dually the class of modules filtered by $\Delta(\mu)$’s as $F(\Delta)$. We will say that $X \in F(\Delta)$ has a Weyl filtration. The multiplicity of $\nabla(\mu)$ in a filtration of $X \in F(\nabla)$ is independent of the filtration chosen. This multiplicity is denoted by $(X : \nabla(\mu))$ and the composition multiplicity of $L(\mu)$ in $X \in \text{mod}(S)$ is denoted by $[X : L(\mu)]$.

We state some of the results that we will need from [11 Appendix A].
Proposition 1.2.  
(i) Let $X \in \text{mod}(S)$ and $\lambda \in \Lambda$. If $\text{Ext}^1(X, \nabla(\lambda)) \neq 0$ then $X$ has a composition factor $L(\mu)$ with $\mu > \lambda$.

(ii) For $X \in \mathcal{F}(\Delta)$, $Y \in \mathcal{F}(\nabla)$ and $i > 0$, we have $\text{Ext}^i(X, Y) = 0$.

(iii) Suppose $\text{Ext}^1(\Delta(\mu), M) = 0$ for all $\nu \in \Lambda$ then $M \in \mathcal{F}(\nabla)$.

(iv) Let $X \in \mathcal{F}(\nabla)$ (resp. $X \in \mathcal{F}(\Delta)$) and $Y$ a direct summand of $X$ then $Y \in \mathcal{F}(\nabla)$ (resp. $Y \in \mathcal{F}(\Delta)$).

Proof. See [11, A2.2].

2. GOOD FILTRATION, WEYL AND GLOBAL DIMENSIONS

In this section $S$ is a quasi-hereditary algebra with poset $(\Lambda, \leq)$.

Suppose $X \in \text{mod}(S)$. We can resolve $X$ by modules $M_i \in \mathcal{F}(\nabla)$ as follows

$0 \to X \to M_0 \to M_1 \to \ldots \to M_d \to 0$.

We call such a resolution a good resolution for $X$. Good resolutions exist for all $S$-modules as an injective resolution is also a good resolution.

Definition 2.1. Let $X \in \text{mod}(S)$. We say $X$ has good filtration dimension $d$, denoted $\text{gfd}(X) = d$, if the following two equivalent conditions hold:

(i) $0 \to X \to M_0 \to M_1 \to \ldots \to M_d \to 0$ is a resolution for $X$ with $M_i \in \mathcal{F}(\nabla)$, of shortest possible length.

(ii) $\text{Ext}^i(\Delta(\lambda), X) = 0$ for all $i > d$ and all $\lambda \in \Lambda$, but there exists $\lambda \in \Lambda$ such that $\text{Ext}^d(\Delta(\lambda), X) \neq 0$.

Proof. See [13, proposition 3.4].

Similarly we have the dual notion of the Weyl filtration dimension of $M$ which we will denote $\text{wfd}(M)$.

Lemma 2.2. Given $S$-modules $M$ and $N$, we have

$\text{Ext}^i(N, M) = 0$ for $i > \text{wfd}(N) + \text{gfd}(M)$.

Proof. We proceed by induction on $\text{wfd}(N) + \text{gfd}(M)$. If either of $\text{wfd}(N) = 0$ or $\text{gfd}(M) = 0$ then we are done by the definition of good filtration dimension and Weyl filtration dimension, so we may assume that both $\text{wfd}(N)$ and $\text{gfd}(M)$ are non-zero. Now we can embed $M$ in $A \in \mathcal{F}(\nabla)$ with quotient $B$. The exact sequence gives us:

$\cdots \to \text{Ext}^{i-1}(N, B) \to \text{Ext}^i(N, M) \to \text{Ext}^i(N, A) \to \cdots$.

Now $\text{Ext}^i(N, A) = 0$ if $i > \text{wfd}(N)$. Also $\text{Ext}^{i-1}(N, B) = 0$ as $i-1 > \text{wfd}(N) + \text{gfd}(B)$ by induction. ($B$ has strictly smaller good filtration dimension than $M$). Hence $\text{Ext}^i(N, M) = 0$. □
Definition 2.3. Let \( g = \sup \{ \text{gfd}(X) \mid X \in \text{mod}(S) \} \). We say \( S \) has good filtration dimension \( g \) and denote this by \( \text{gfd}(S) = g \). Let \( w = \sup \{ \text{wfd}(X) \mid X \in \text{mod}(S) \} \). We say \( S \) has Weyl filtration dimension \( w \) and denote this by \( \text{wfd}(S) = w \).

Remark 2.4. In general \( \text{gfd}(S) \) is not the good filtration dimension of \( S \) when considered as its own left (or right) module. Similar remarks apply to \( \text{wfd}(S) \). We will only use \( \text{gfd}(S) \) and \( \text{wfd}(S) \) in the sense that they are defined above.

For a finite dimensional \( k \)-algebra \( S \), the injective dimension of an \( S \)-module \( M \), is the length of a shortest possible injective resolution and is denoted by \( \text{inj}(M) \). Equivalently we have

\[
\text{inj}(M) = \sup \{ d \mid \text{Ext}^d(N, M) \neq 0 \text{ for } N \in \text{mod}(S) \}.
\]

The global dimension of \( S \) is the supremum of all the injective dimensions for \( S \)-modules, and is denoted by \( \text{glob}(S) \). This is equivalent to

\[
\text{glob}(S) = \sup \{ d \mid \text{Ext}^d(N, M) \neq 0 \text{ for } N, M \in \text{mod}(S) \}.
\]

Corollary 2.5. The global dimension of \( S \) has an upper bound of \( \text{wfd}(S) + \text{gfd}(S) \).

Now suppose \( S \) is a quasi-hereditary algebra with contravariant duality preserving simples. That is there exists an involutory, contravariant functor \( \circ : \text{mod}(S) \to \text{mod}(S) \) such that \( \Delta(\lambda) \circ \cong \nabla(\lambda) \) (and \( \text{Ext}^i(M, N) \cong \text{Ext}^i(N^\circ, M^\circ) \)). We will usually shorten this and say \( S \) has a simple preserving duality.

Remark 2.6. It is clear (given the equivalences in the definition for the good filtration dimension) that for \( S \) with simple preserving duality and \( M \) an \( S \)-module we have \( \text{wfd}(M) = \text{gfd}(M^\circ) \). We will use this without further comment.

Schur algebras are quasi-hereditary with poset \( \Lambda^+(n, r) \) ordered by dominance [8, 2.2h]. The costandard modules correspond to the \( \text{GL}_n \)-modules \( \nabla(\lambda) \). Schur algebras also have a simple preserving duality see [11] remark (ii) following Lemma 4.1.3. The rest of this paper is devoted to calculating \( \text{gfd}(S_q(2, r)) \), \( \text{glob}(S_q(2, r)) \), \( \text{gfd}(S(3, r)) \) and \( \text{glob}(S(3, r)) \) explicitly.

In all these cases we need the following technical lemma.

Lemma 2.7. Suppose we have a short exact sequence of \( S \)-modules

\[
0 \to A \to B \to C \to 0.
\]

(i) If \( \text{gfd}(B) > \text{gfd}(C) \) then \( \text{gfd}(A) = \text{gfd}(B) \). Furthermore if \( \text{gfd}(B) > \text{gfd}(C) + 1 \) then for all \( S \)-modules \( M \) we have

\[
\text{Ext}^{\text{wfd}(M) + \text{gfd}(A)}(M, A) \cong \text{Ext}^{\text{wfd}(M) + \text{gfd}(B)}(M, B).
\]
(ii) If \( \text{gfd}(B) < \text{gfd}(C) \) then \( \text{gfd}(A) = \text{gfd}(C) + 1 \). Furthermore for all \( S \)-modules \( M \) we have

\[
\text{Ext}^{\text{wfd}(M)+\text{gfd}(A)}(M, A) \cong \text{Ext}^{\text{wfd}(M)+\text{gfd}(C)}(M, C).
\]

Proof. We consider (i), the proof of (ii) is similar. The long exact sequence corresponding to the short exact sequence gives us

\[
\text{gfd}(B) \leq \max\{\text{gfd}(A), \text{gfd}(C)\} \text{ and } \text{gfd}(A) \leq \max\{\text{gfd}(B), \text{gfd}(C) + 1\}.
\]

So if \( \text{gfd}(B) > \text{gfd}(C) \) then \( \text{gfd}(B) = \text{gfd}(A) \). If furthermore \( \text{gfd}(B) > \text{gfd}(C) + 1 \) then for all \( S \)-modules \( M \) we have

\[
\text{Ext}^{\text{wfd}(M)+\text{gfd}(A)}(M, A) \cong \text{Ext}^{\text{wfd}(M)+\text{gfd}(B)}(M, B)
\]

using Lemma 2.2 and the long exact sequence. \( \square \)

Lemma 2.8. Suppose \( S \) is a quasi-hereditary algebra with simple preserving duality \( ^o \) and \( \lambda \in \Lambda \) with \( \text{gfd}(L(\lambda)) = 1 \). Then \( \text{Ext}^2(L(\lambda), L(\lambda)) \cong \text{Hom}(Q^o, Q) \not\equiv 0 \), where \( Q = \nabla(\lambda)/L(\lambda) \).

Proof. We have a short exact sequence

\[
0 \to Q^o \to \Delta(\lambda) \to L(\lambda) \to 0.
\]

Dimension shifting gives us \( \text{Ext}^2(L(\lambda), L(\lambda)) \cong \text{Ext}^1(Q^o, L(\lambda)) \). Now \( \text{Hom}(Q^o, L(\lambda)) \cong \text{Hom}(Q^o, \nabla(\lambda)) \equiv 0 \) as \( L(\lambda) \) is not a composition factor of \( Q \). So another dimension shift gives us \( \text{Ext}^1(Q^o, L(\lambda)) \cong \text{Hom}(Q^o, Q) \). We can map simple modules in the head of \( Q^o \) to the socle of \( Q \), so \( \text{Hom}(Q^o, Q) \not\equiv 0 \) and we are done. \( \square \)

3. The SL_2 Case

In this section we focus on SL_2. We will identify a block of a Schur algebra with the highest weights of the irreducibles that belong to the block.

Suppose \( \lambda \in X^+ \) and write \( \lambda = p\lambda_1 + \lambda_0 \) with \( \lambda_1 \in X^+ \) and \( \lambda_0 \in X_1 \). We define \( \nabla_p(\lambda) = \nabla(\lambda_1)^F \otimes L(\lambda_0) \), and \( \Delta_p(\lambda) = \Delta(\lambda_1)^F \otimes L(\lambda_0) \). It is clear that \( \nabla_p(\lambda) = (\Delta_p(\lambda))^o \). These modules will play an important role in what follows. We define \( \text{hw}(M) \) to be the set of weights \( \lambda \in X^+ \) with \( L(\lambda) \) a composition factor of \( M \) such that there is no \( \mu \in X^+ \) with \( \mu > \lambda \) and \( L(\mu) \) a composition factor of \( M \).

We now note a sequence, first remarked upon by Jantzen in [16] remark 2 following theorem 3.8] and proved by Xanthopoulous in [23] proposition 6.1.1] which we will use repeatedly in this section.

Lemma 3.1. We have for \( r \geq 1 \) and \( 0 \leq a \leq p - 2 \) a short exact sequence

\[
0 \to \nabla(a) \otimes \nabla(r)^F \to \nabla(pr + a) \to \nabla(p - a - 2) \otimes \nabla(r - 1)^F \to 0.
\]

If \( a = p - 1 \) then \( \nabla(a) \otimes \nabla(r)^F \cong \nabla(pr + a) \).
We write \( r = r_0 + pr_1 \) where \( 0 \leq r_0 < p \). If \( \lambda = (r) \) then we define \( g(\lambda) = r_1 \).

We denote the block of \( S(n, r) \) containing \((a_1, a_2, \ldots, a_n) \in \Lambda^+(n, r) \) by \( B(a_1, a_2, \ldots, a_n) \). For \( n = 2 \), \( B(a_1, a_2) \) is totally ordered. A block \( B(a_1, a_2) \) is defined to be primitive if \( a_1 - a_2 \neq -1 \) (mod \( p \)). We say a weight \( \lambda = (r) \in X^+ \) is primitive if \( r_0 \neq p - 1 \).

**Lemma 3.2.** Suppose \( \lambda \) is primitive and \( Q \) is the quotient \( \nabla_p(\lambda)/L(\lambda) \). Then \( g(\text{hw}(Q)) \leq g(\lambda) - 2 \) with \( g(\lambda) \) as defined above.

**Proof.** We write \( \lambda = r = pr_1 + r_0 \) and \( r_1 = pr'_1 + r'_0 \) with \( 0 \leq r_0 < p \) and \( 0 \leq r'_0 < p \). Then \( L(\lambda) \cong L(r_1)^F \otimes L(r_0) \) by Steinberg’s tensor product theorem and so \( Q \cong (Q')^F \otimes L(r_0) \) where \( Q' \) is the quotient \( \nabla(r_1)/L(r_1) \). It is clear using \([11] \) that \([\nabla(r_1): L(pr'_1 - r'_0 - 2)] \neq 0 \). We have \([Q' : L(r_1)] = 0 \). Also the first weight in the \( G \)-block of \( r_1 \) smaller than \( r_1 \) is \( (pr'_1 - r'_0 - 2) \) using \([9] \) main theorem. Hence \( \text{hw}(Q') = (pr'_1 - r'_0 - 2) \). Thus \( \text{hw}(Q) = p(pr'_1 - r'_0 - 2) + (r_0) = (p(r_1 - 2r'_0 - 2) + r_0) \). Thus \( g(\text{hw}(Q)) = r_1 - 2r'_0 - 2 \leq g(\lambda) - 2 \) as required. \( \square \)

The next Theorem shows that \( \text{gfd}(L(\lambda)) = g(\lambda) \). The following Lemma forms part of the inductive step.

**Lemma 3.3.** Let \( \lambda \in X^+ \) and suppose \( g(\mu) = \text{gfd}(L(\mu)) = \text{gfd}(\nabla_p(\mu)) \) for all \( \mu < \lambda \) and \( \text{gfd}(\nabla_p(\lambda)) = g(\lambda) \). Then \( \text{gfd}(L(\lambda)) = \text{gfd}(\nabla_p(\lambda)) \). Furthermore

\[
\text{Ext}^2_{\text{gfd}(L(\lambda))}(L(\lambda), L(\lambda)) \cong \text{Ext}^2_{\text{gfd}(\nabla_p(\lambda))}(\Delta_p(\lambda), \nabla_p(\lambda)).
\]

**Proof.** We have a short exact sequence

\[
0 \to L(\lambda) \to \nabla_p(\lambda) \to Q \to 0.
\]

Lemma \([8, 2]\) gives us \( \text{gfd}(\nabla_p(\lambda)) > \text{gfd}(Q) + 1 \), as \( \text{gfd}(Q) \) is at most the maximum of the good filtration dimensions of its composition factors. The Lemma now follows by Lemma \([2, 3] \) part (i) (applied twice). \( \square \)

**Theorem 3.4.** Suppose \( r = r_0 + pr_1 \) with \( 0 \leq r_0 \leq p - 2 \). Then

\[
\text{gfd}(\nabla(r_1)^F \otimes L(r_0)) = \text{gfd}(L(r)) = r_1
\]

and

\[
\text{Ext}^{2r_1}_{L(r)}(L(r), L(r)) \cong \text{Ext}^{2r_1}(\Delta_p(r), \nabla_p(r)) \cong k.
\]

**Proof.** We proceed by induction on \( r_1 \). For \( r_1 = 0 \) we have \( L(r_0) = \nabla(r_0) \) and so \( \text{gfd}(L(r_0)) = 0 \).

For \( r_1 = 1 \) we have \( L(r) = \nabla(1)^F \otimes \nabla(r_0) \) and a non-split short exact sequence

\[
0 \to L(r) \to \nabla(r) \to \nabla(p - r_0 - 2) \to 0.
\]

Thus \( \text{gfd}(L(r)) = 1 \) by Lemma \([2, 3] \). This Lemma also gives

\[
\text{Ext}^2(L(r), L(r)) \cong \text{Hom}(L(p - r_0 - 2), L(p - r_0 - 2)) \cong k.
\]
We now suppose that \( r_1 \geq 2 \). Since by induction \( \text{gfd}(\nabla(r_1-1)^F \otimes L(p-r_0-2)) = r_1 - 1 \geq 1 \), applying Lemma 3.1 to the short exact sequence from Lemma 3.1 we have \( \text{gfd}(\nabla(r_1)^F \otimes L(r_0)) = r_1 \).

By Lemma 3.3 and induction we have \( \text{gfd}(L(r)) = \text{gfd}(\nabla_p(r)) = r_1 \).

We also have \( \text{Ext}^{2r_1}(\Delta_p(r), \nabla_p(r)) \cong \text{Ext}^{2r_1-2}(\Delta_p(pr_1-r_0-2), \nabla_p(pr_1-r_0-2)) \) by Lemma 3.3, part (ii) and Lemma 3.1. But by induction this is isomorphic to \( k \). Lemma 3.3 then completes the proof.

**Corollary 3.5.** Suppose \( r \) is not primitive and write \( r = p^d r_1 + p^d - 1 \) for some \( d \in \mathbb{N}^+ \) with \( r_1 \not\equiv -1 \) (mod \( p \)). Then \( \text{gfd}(L(r)) = \text{gfd}(L(r_1)) \) and

\[
\text{Ext}^{2\text{gfd}(L(r))}(L(r), L(r)) \cong k.
\]

**Proof.** We have by [2] section 4, theorem 6 that \( B(r, 0) \) is Morita equivalent to \( B(r_1, 0) \) in \( S(2, r_1) \) with \( r_1 \) primitive and the result follows.

**Corollary 3.6.** Given \( (a_1, a_2) \in \Lambda^+(2, r) \), we write \( a_1 - a_2 = p^d c_1 + p^d c_0 + p^d - 1 \) with \( d \in \mathbb{N} \) and \( 0 \leq c_0 \leq p - 2 \). Then \( \text{gfd}(L(a_1, a_2)) = c_1 \) where \( L(a_1, a_2) \) is the irreducible module of highest weight \( (a_1, a_2) \) for \( S(2, r) \). Moreover

\[
\text{Ext}^{2c_1}_{S(2, r)}(L(a_1, a_2), L(a_1, a_2)) \cong k.
\]

**Proof.** Now Theorem 3.4 and Corollary 3.5 give us

\[
\text{Ext}^i_{S(2, r)}(\Delta(b_1, b_2), L(a_1, a_2)) \cong \text{Ext}^i_{S}(\Delta(b_1 - b_2), L(a_1 - a_2)) \cong 0
\]

if \( i > c_1 \) and so we have \( \text{gfd}(L(a_1, a_2)) \leq c_1 \). We also have

\[
\text{Ext}^{2c_1}_{S(2, r)}(L(a_1, a_2), L(a_1, a_2)) \cong \text{Ext}^{2c_1}_{S}(L(a_1 - a_2), L(a_1 - a_2)) \cong k.
\]

Thus Lemma 2.2 gives us \( \text{wfd}(L(a_1, a_2)) + \text{gfd}(L(a_1, a_2)) \geq 2c_1 \), But \( \text{wfd}(L(a_1, a_2)) = \text{gfd}(L(a_1, a_2)) \), and so we have \( \text{gfd}(L(a_1, a_2)) = c_1 \), as required.

**Theorem 3.7.** The global dimension of \( S(2, r) \) is twice its good filtration dimension and is given as follows:

\[
\text{glob}(S(2, r)) = \begin{cases} r, & \text{if } r \text{ is even} \\ 2\left\lfloor \frac{r}{2} \right\rfloor, & \text{if } r \text{ is odd} \end{cases}
\]

\[
\text{glob}(S(2, r)) = 2\left\lfloor \frac{r}{2} \right\rfloor.
\]

**Proof.** If \( r \) is even then \( \Lambda^+(2, r) \) consists of all partitions \((a, b)\) of \( r \) whose difference \( a - b \) are all even numbers less than \( r \). If \( r \) is odd then \( \Lambda^+(2, r) \) consists of all partitions \((a, b)\) of \( r \) whose difference \( a - b \) are all odd numbers less than \( r \).
We let $S = S(2, r)$. Using Corollary 2.6 and simple preserving duality we know $\text{glob}(S) \leq 2 \text{gfd}(S)$. We also have

$$\text{gfd}(S) = \max \{ \text{gfd}(L(a_1, a_2)) \mid (a_1, a_2) \in \Lambda^+(2, r) \}.$$ 

But Corollary 4.6 gives us

$$\text{glob}(S) \geq 2 \max \{ \text{gfd}(L(a_1, a_2)) \mid (a_1, a_2) \in \Lambda^+(2, r) \}.$$ 

Hence $\text{glob}(S) = 2 \text{gfd}(S)$.

Using the main theorem in [9] we have for $p = 2$ and $r$ even that $\mathcal{B}(r, 0) = \Lambda^+(2, r)$ is primitive so $\text{glob}(S) = 2 \frac{r}{2} = r$, as all other simple modules corresponding to weights in $\Lambda^+(2, r)$ have smaller or equal good filtration dimension to that of $L(r, 0)$. For $p = 2$ and $r$ odd, $\mathcal{B}(r, 0)$ is not primitive. We write $r = 2^d r_1 + 2^d - 1$ for some $d \in \mathbb{N}$ and $r_1$ even. If $d = 1$ then all other simples modules corresponding to weights in $\Lambda^+(2, r)$ have smaller or equal good filtration dimension to that of $L(r, 0)$, so $\text{glob}(S) = r_1 = 2 \lfloor \frac{r}{p} \rfloor$. If $d \geq 2$ then we consider $r' = r - 2$ so $(r' + 1, 1) \in \Lambda^+(2, r)$. We have $r' = 2(2^{d-1}(r_1 - 2)) + 1$ with $2^{d-1}(r_1 - 2)$ even, so $\text{glob}(\mathcal{B}(r' + 1, 1)) = \text{glob}(\mathcal{B}(r', 0)) = 2^{d-1}(r_1 - 2) = 2 \lfloor \frac{r}{2} \rfloor = 2 \lfloor \frac{r'}{2} \rfloor$.

For $p \geq 3$ and $r$ primitive then $\text{glob}(S) = 2 r_1 = 2 \lfloor \frac{r}{2} \rfloor$, as all other simple modules corresponding to weights in $\Lambda^+(2, r)$ have smaller or equal good filtration dimension to that of $L(r, 0)$. If $r$ is not primitive then $r = p^d r_1 + p^d - 1$ for some $d \in \mathbb{N}$ and $r_1 \equiv -1 \pmod{p}$. Consider $r' = r - 2$, then $(r' + 1, 1) \in \Lambda^+(2, r)$ is primitive and so $\text{glob}(S) = 2 \lfloor \frac{r}{p} \rfloor = 2 \lfloor \frac{r'}{p} \rfloor$.

We now consider the quantum version of the Schur algebra $S_q(n, r)$ with $0 \neq q \in k$. This is a deformation of the classical Schur algebra with parameter $q$. See the introduction of [11] for the basic properties of $S_q(n, r)$. When $q = 1$ then $S_q(n, r)$ is just the classical Schur algebra. If $q$ is not a root of unity then $S_q(n, r)$ is semi-simple. We will consider the case where $q$ is a primitive $l$th root of unity with $l \geq 2$.

We now show that the argument above generalises to the quantum case. To do this we need the appropriate quantum versions of the results used above. We will be using the Dipper-Donkin quantum group $q$-$\text{GL}_n$ defined in [9].

Now we know that $S_q(n, r)$ is quasi-hereditary with poset $\Lambda^+(n, r)$ by [10] section 4, (6)]. We also have the property for all $S_q(n, r)$-modules $V$ and $W$ that

$$\text{Ext}^i_{S_q(n, r)}(V, W) \cong \text{Ext}^i_{q$-$\text{GL}_n}(V, W)$$

by [10] section 4, (5)]. The blocks of $S_q(n, r)$ were determined in [2]. We also know that all blocks of $S_q(n, r)$ are Morita equivalent to a block of $S_q(n, r')$ with $r'$ primitive by [5] lemma 6.10]. We have a quantum Frobenius morphism $F : q$-$\text{GL}_n \to \text{GL}_n(k)$. Some of the other basic properties of $q$-$\text{GL}_n$-modules appears in [11] chapter 3 including a proof of the quantum version of Steinberg’s
tensor product theorem. Suppose we write $\lambda = \ell \lambda_1 + \lambda_0$ with $\lambda_0$ $l$-restricted and $\lambda_1$ dominant. We define $\nabla_l(\lambda) = \nabla(\lambda_1)^F \otimes L(\lambda_0)$, where $\nabla(\lambda_1)$ is the classical module in characteristic $p$.

We now let $n = 2$. The generalisation of Lemma 3.1 appears in [3, Proposition 3.1]. Moreover all it does is to replace $p$ with $l$. Thus Lemmas 3.2 and 3.3 will carry through unchanged. Theorem 3.4 and Corollary 3.6 now generalise to give:

**Theorem 3.8.** Suppose $r = r_0 + lr_1$ with $0 \leq r_0 \leq l - 2$. Then
$$\text{gfd}(\nabla_l(r_1,0)^F \otimes L(r_0,0)) = \text{gfd}(L(r,0)) = r_1.$$

and
$$\text{Ext}^{2r_1}(L(r,0), L(r,0)) \cong \text{Ext}^{2r_1}(\Delta_l(r,0), \nabla_l(r,0)) \cong k.$$

Also we can use [5, lemma 6.10] and the classical result 3.4 to give the good filtration dimension of $L(\lambda)$ for $\lambda$ non-primitive, so we have the following Corollary.

**Corollary 3.9.** Suppose $r$ is not primitive so $r = lp^{d-1}r_1 + lp^{d-1} - 1$ for some $d \in \mathbb{N}^+$ and $r_1 \not\equiv -1 \pmod{p}$. Then $\text{gfd}(L(r,0)) = \text{gfd}(\bar{L}(r_1,0)) = \left\lfloor \frac{r}{p} \right\rfloor$ and
$$\text{Ext}^{2\text{gfd}(L(r,0))}(L(r,0), L(r,0)) \cong k$$

where $\bar{L}(r_1,0)$ is the irreducible $S(2,r_1)$-module of highest weight $(r_1,0)$.

We now have the following theorem for the quantum case.

**Theorem 3.10.** The global dimension of $S_q(2,r)$ is twice its good filtration dimension and is given as follows:

$$\text{glob}(S_q(2,r)) = \begin{cases} r & \text{if } r \text{ is even} \\ 2\left\lfloor \frac{r}{2} \right\rfloor & \text{if } r \text{ is odd} \end{cases}$$

for $l \geq 3$

$$\text{glob}(S_q(2,r)) = 2\left\lfloor \frac{r}{2} \right\rfloor.$$

*Proof. The argument is very similar to that of Theorem 3.7.*

4. SOME FILTRATIONS FOR $\text{SL}_3$

In this section we obtain a filtration of the modules $\nabla(\lambda)$ for $\text{SL}_3$ by modules of the form $\nabla(\lambda)^F \otimes L(\mu)$ with $\lambda \in X^+$ and $\mu \in X_1$. We call such a filtration a $p$-filtration. Throughout this section $G = \text{SL}_3$. We start by proving some Lemmas about extensions between two modules $\nabla_p(\lambda)$ and $\nabla_p(\mu)$.

Given a rational $G$-module $V$, we have a five term exact sequence,

$$0 \rightarrow H^1(G/G_1, V^{G_1}) \rightarrow H^1(G, V) \rightarrow H^1(G_1, V)^{G/G_1} \rightarrow H^2(G/G_1, V^{G_1}) \rightarrow H^2(G, V).$$
This is the Lyndon-Hochschild-Serre sequence for $G$ and $G_1$. A $G$-module $W$ which is trivial as a $G_1$-module, is of the form $V^F$ for some $G$-module $V$ which is unique up to isomorphism. We define $W^{(-1)} = V$. If $W$ and $V$ are $G$-modules then $\text{Ext}_{G_1}^1(W, V)$ has a natural structure as a $G$-module. Moreover when $W$ and $V$ are finite dimensional we have,

$$\text{Ext}_{G_1}^1(W, V \otimes Y^F) \cong \text{Ext}_{G_1}^1(W, V) \otimes Y^F$$

as $G$-modules. If $H = G$ or $G_1$ then we have

$$\text{Ext}_H^1(W, V) \cong \text{Ext}_H^1(V^*, W^*) \cong \text{Ext}_H^1(k, W^* \otimes V) \cong H^1(H, W^* \otimes V)$$

where * is the ordinary dual. We have $\nabla(a, b)^* \cong \Delta(b, a)$. We also note that $(V^F)^{G/G_1} \cong V^G$, and $H^i(G/G_1, V^F) \cong H^i(G, V)$.

In the following sections we make repeated use of a Proposition proved in the PhD thesis of Yehia. We reproduce his results here for the convenience of the reader.

**Proposition 4.1.** The non-zero $\text{Ext}_{G_1}^1(L(\alpha), L(\beta))$ for $\alpha, \beta \in X_1$ are given by the following tables.

(i) $\text{For } (r, s) \in X_1 \text{ with } r + s = p - 2, \text{ we have}$

$$\begin{array}{c|ccc}
\alpha \downarrow, \beta \rightarrow & (r, s) & (p - 1, r) & (s, p - 1) \\
(r, s) & 0 & \nabla(0, 1)^F & \nabla(1, 0)^F \\
(p - 1, r) & \nabla(1, 0)^F & 0 & 0 \\
(s, p - 1) & \nabla(0, 1)^F & 0 & 0 \\
\end{array}$$

(ii) $\text{For } (r, s) \in A_0 \text{ and } p \geq 5, \text{ we have}$

$$\begin{array}{c|ccc}
\alpha \downarrow, \beta \rightarrow & (p - s - 2, p - r - 2) & (r + s + 1, p - s - 2) & (p - r - 2, r + s + 1) \\
(r, s) & k & \nabla(0, 1)^F & \nabla(1, 0)^F \\
\end{array}$$

$$\begin{array}{c|ccc}
\alpha \downarrow, \beta \rightarrow & (r, s) & (s, p - r - s - 3) & (p - r - s - 3, r) \\
(p - s - 2, p - r - 2) & k & \nabla(0, 1)^F & \nabla(1, 0)^F \\
\end{array}$$

If $p = 3$ then all the entries in the two tables above are replaced by $k \oplus \nabla(0, 1)^F \oplus \nabla(1, 0)^F$.

**Proof.** See [24, proposition 3.3.2].

**Lemma 4.2.** For $\lambda, \mu \in X^+$ and $\alpha, \beta \in X_1$ we have,

$$\text{Ext}_{G}^1(\nabla(\lambda)^F \otimes L(\alpha), \nabla(\mu)^F \otimes L(\beta)) \cong \begin{cases} 
\text{Ext}_{G}^1(\nabla(\lambda), \nabla(\mu)) & \text{if } \alpha = \beta, \\
\text{Hom}_G \left(\nabla(\lambda), \nabla(\mu) \otimes \text{Ext}_{G_1}^1(L(\alpha), L(\beta))^{(-1)}\right) & \text{otherwise.}
\end{cases}$$
Lemma 4.3. Let \((a - 1, b + 1), (a, b - 1) \in X^+\) and \((r, s) \in X_1\), then

\[
\text{Ext}^1_G(\nabla(a - 1, b + 1)^F \otimes L(r, s), \nabla(a, b - 1)^F \otimes L(r, s)) \\
\cong \left\{ \begin{array}{ll}
0 & \text{if } b \not\equiv -1 \pmod{p} \\
k & \text{if } b \equiv -1 \pmod{p}.
\end{array} \right.
\]
Proof. Lemma 4.2 gives us
\[
\operatorname{Ext}^1_G((a - 1, b + 1)^F \otimes L(r, s), \nabla (a, b - 1)^F \otimes L(r, s)) 
\cong \operatorname{Ext}^1((a - 1, b + 1), \nabla (a, b - 1)).
\]
Now \((a - 1, b + 1)\) and \((a, b - 1)\) differ by a single root so we may apply the result of [12] (4.3 and (3.6)] and we are done. 

Similarly we have:

**Lemma 4.4.** Let \((a + 1, b - 1), (a - 1, b) \in X^+\) and \((r, s) \in X_1\), then
\[
\operatorname{Ext}^1_G\left((a + 1, b - 1)^F \otimes L(r, s), (a - 1, b)^F \otimes L(r, s)\right) \cong \begin{cases} 
0 & \text{if } a \not\equiv -1 \pmod{p} \\
 k & \text{if } a \equiv -1 \pmod{p}.
\end{cases}
\]

We also have:

**Lemma 4.5.** Suppose \(\alpha, \beta \in X_1\) with \(\alpha \not\equiv \beta\), and \(\alpha\) and \(\beta\) in the same alcove. Then for all \(\lambda, \mu \in X^+\) we have
\[
\operatorname{Ext}^1((\nabla(\lambda)^F \otimes L(\alpha), (\nabla(\mu)^F \otimes L(\beta)) \cong 0.
\]

Proof. By Proposition 4.1 we have
\[
\operatorname{Ext}^1_G\left(L(\alpha), L(\beta)\right) \cong 0
\]
and the result follows by Lemma 4.2. 

In the main theorem in this section we use repeatedly the fact that the modules \(\nabla(\lambda)\) have both simple socle and simple head for \(\lambda\) dominant. Jantzen proved that \(\nabla(\lambda)\) has simple head for \(\text{SL}_3\) when \(p > 3\). The following four Lemmas extend this result to \(p = 2\) and 3. In these Lemmas \(\rho = (1, 1)\), which is half the sum of all positive roots for the Weyl group, \(w_0\) is the longest word in the Weyl group and \(\text{St} = L(p - 1, p - 1)\) is the Steinberg module. We also have a \(G\)-module \(Q_1(\mu)\) which when considered as a \(u_1\)-\(T\)-module is the injective hull of \(L(\mu)\) for \(\mu \in X_1\).

**Lemma 4.6.** For all \(\lambda, \mu \in X^+\) the module \(\nabla(\lambda) \otimes \nabla(\mu)\) has a good filtration. Moreover the \(\nabla(\nu)\) which appear as quotients in this filtration are given by Brauer’s character formula.

Proof. A proof of the property that \(\nabla(\lambda) \otimes \nabla(\mu)\) has a good filtration, for type \(A_n\), is given in [22]. It is proved for most other cases in [7]. The general proof is given in [20]. For a version of Brauer’s character formula see [17] II, lemma 5.8].

**Corollary 4.7.** \(\text{St}\) is a direct summand of \(\text{St} \otimes \text{St}\).

Proof. All the quotients \(\nabla(\nu) = \nabla(\nu_1, \nu_2)\) which appear in a good filtration of \(\text{St} \otimes \text{St}\) have \(0 \leq \nu_i \leq 2p - 2\). Also \((0, 0)\) is a weight in \(\text{St}\). Hence using Brauer’s character formula, \(\text{St} = \nabla(p - 1, p - 1)\) appears as a quotient in a good filtration of \(\text{St} \otimes \text{St}\), but all other weights that appear are not.
linked to \((p - 1, p - 1)\). (The first such possible weight is \((2p - 1, 2p - 1)\).) Thus \(St\) is a direct summand of \(St \otimes St\) as required.

**Lemma 4.8.** If \(\lambda \in X^+\) and \(\mu \in X_1\) then \(\Delta(\lambda)^F \otimes Q_1(p\rho + w_0(\mu + \rho))\) has a Weyl filtration.

*Proof.* This is proved for \(p > 3\) in [16, theorem 5.6]. Now [24, lemma 3.1.3] shows for \(p = 2\) and 3 that \(Q_1(p\rho + w_0(\mu + \rho))\) is a \(G\)-direct summand of \(L(\mu) \otimes St\). Hence \(\Delta(\lambda)^F \otimes Q_1(p\rho + w_0(\mu + \rho))\) is a direct summand of \(\Delta(\lambda)^F \otimes L(\mu) \otimes St \otimes St\). But this last module has a Weyl filtration. First \(\Delta(\lambda)^F \otimes St \cong \Delta(p\rho + w_0(\mu + \rho))\), also \(L(\mu) \otimes St\) has a Weyl filtration by [24, lemma 3.1.2]. Hence \(\Delta(\lambda)^F \otimes L(\mu) \otimes St \otimes St\) has a Weyl filtration and we are done.

**Corollary 4.9.** Suppose \(\lambda \in X^+\) and \(\mu \in X_1\) then \(\Delta(p(\lambda + \rho) - \rho + \mu)\) embeds in \(\Delta(\lambda)^F \otimes Q_1(p\rho + w_0(\mu + \rho))\).

*Proof.* This follows using characters as in the proof of [16, corollary 5.7].

**Lemma 4.10.** For all \(\lambda \in X^+\) and \(\mu \in X_1\) we have

\[
\text{soc}_G \Delta(p(\lambda + \rho) + \mu - \rho) \cong (\text{soc}_G \Delta(\lambda))^F \otimes L(p\rho + w_0(\mu + \rho)).
\]

*Proof.* This follows as in Jantzen [16, theorem 6.2] using Corollary 4.9 to remove the restriction on \(p\).

**Proposition 4.11.** For all \(\lambda \in X^+\) the module \(\Delta(\lambda)\) has simple socle.

*Proof.* We can use 4.10 to remove the restriction on \(p\) in the argument of Jantzen in [16, 6.9], the result then follows.

We will now give explicitly the \(p\)-filtrations of \(\nabla(\lambda)\). Further information about these filtrations (for \(p > 3\)) appears in [19, chapter 2, lemma 2 and following]. In the following diagrams the sections of a filtration of \(\nabla(\lambda)\) are written as modules with lines connecting them. Sections that have a non-trivial extension between them that appear in \(\nabla(\lambda)\) are joined by a line. Sections that only have trivial extensions appearing between them are not joined by a line. Sections that embed in \(\nabla(\lambda)\) are at the bottom of the diagram. The sections that occur above it, are above these sections, and so on. In what follows if \(\lambda_1\) is not dominant but one of its parts is \(-1\) then we take \(\nabla_p(\lambda)\) to be the zero module. In other words, it does not appear as a section in a \(p\)-filtration.

**Theorem 4.12.** Each \(\nabla(\lambda)\) has a \(p\)-filtration. This filtration takes the following form:

(i) Suppose \(\lambda = p(a, b) + (p - 1, p - 1)\) with \((a, b) \in X^+\). Then

\[
\nabla(\lambda) = \nabla(a, b)^F \otimes L(p - 1, p - 1).
\]
(ii) Suppose $\lambda = p(a, b) + (p - 1, r)$ with $(a, b) \in X^+$ and $(p - 1, r) \in X_1$. If we set $s = p - r - 2$ then for $a \equiv -1 \pmod{p}$, the module $\nabla(\lambda)$ has filtration

$$
\nabla(a, b - 1)^F \otimes L(s, p - 1) \downarrow \nabla(a, b)^F \otimes L(p - 1, r)
$$

while for $a \not\equiv -1 \pmod{p}$, $\nabla(\lambda)$ has filtration

$$
\nabla(a, b - 1)^F \otimes L(s, p - 1) \downarrow \nabla(a + 1, b - 1)^F \otimes L(r, s) \downarrow \nabla(a - 1, b)^F \otimes L(r, s) \downarrow \nabla(a, b)^F \otimes L(p - 1, r).
$$

(iii) Suppose $\lambda = p(a, b) + (s, p - 1)$ with $(a, b) \in X^+$ and $(s, p - 1) \in X_1$. If we set $r = p - s - 2$ then for $b \equiv -1 \pmod{p}$, the module $\nabla(\lambda)$ has filtration

$$
\nabla(a - 1, b)^F \otimes L(p - 1, r) \downarrow \nabla(a, b)^F \otimes L(s, p - 1)
$$

while for $b \not\equiv -1 \pmod{p}$, $\nabla(\lambda)$ has filtration

$$
\nabla(a - 1, b)^F \otimes L(p - 1, r) \downarrow \nabla(a - 1, b + 1)^F \otimes L(r, s) \downarrow \nabla(a, b - 1)^F \otimes L(r, s) \downarrow \nabla(a, b)^F \otimes L(s, p - 1).
$$

(iv) Suppose $\lambda = p(a, b) + (r, s)$ with $(a, b) \in X^+$, $(r, s) \in X_1$ and $r + s = p - 2$. Then the module $\nabla(\lambda)$ has filtration

$$
\nabla(a - 1, b - 1)^F \otimes L(r, s) \downarrow \nabla(a - 1, b - 1)^F \otimes L(p - 1, r) \downarrow \nabla(a - 1, b)^F \otimes L(s, p - 1) \downarrow \nabla(a, b)^F \otimes L(r, s).
$$
(v) Suppose $\lambda = p(a,0) + (r,s)$ with $a \geq 1$ and $(r,s) \in A_0$ then the module $\nabla(\lambda)$ has filtration

\[
\nabla(a-2,0)^F \otimes L(s,p-r-s-3) \\
\nabla(a-1,0)^F \otimes L(p-r-2,r+s+1) \\
\nabla(a,0)^F \otimes L(r,s).
\]

(vi) Suppose $\lambda = p(0,b) + (r,s)$ with $b \geq 1$ and $(r,s) \in A_0$. Then the module $\nabla(\lambda)$ has filtration

\[
\nabla(0,b-2)^F \otimes L(p-r-s-3,r) \\
\nabla(0,b-1)^F \otimes L(r+s+1,p-s-2) \\
\nabla(0,b)^F \otimes L(r,s).
\]

(vii) Suppose $\lambda = p(a,b) + (r,s)$ with $(a,b) \in X^+$, $a \geq 1$, and $(r,s) \in A_0$. We let

\[
\mu_1 = \lambda, \quad \mu_2 = (pa + r + s + 1, pb - s - 2), \\
\mu_3 = (pa + p - r - s - 3, pb - 2p + r), \quad \mu_4 = (pa - r - 2, pb + r + s + 1), \\
\mu_5 = (pa - 2p + s, pb + p - r - s - 3), \quad \mu_6 = (pa + s, pb - r - s - 3), \\
\mu_7 = (pa - p + r, pb - p + s), \quad \mu_8 = (pa - r - s - 3, pb + r), \\
\mu_9 = (pa - s - 2, pb - r - 2).
\]

These weights are depicted in Figure 1 (a), where the number corresponds to the subscript of $\mu$.

![Figure 1. Diagram showing weights for $\lambda$ inside (a) a lower alcove and (b) an upper alcove](image-url)
Then for \( a \) and \( b \equiv 0 \pmod{p} \), \( \nabla(\lambda) \) has filtration

\[
\begin{array}{c}
\nabla_p(\mu_9) \\
\downarrow \\
\nabla_p(\mu_6) \quad \quad \quad \quad \quad \nabla_p(\mu_7) \quad \quad \quad \quad \quad \nabla_p(\mu_8) \\
\downarrow \\
\nabla_p(\mu_5) \\
\quad \quad \quad \quad \quad \nabla_p(\mu_4) \quad \quad \quad \quad \quad \nabla_p(\mu_2) \\
\end{array}
\]

\[
\nabla_p(\mu_1).
\]

For \( a \neq 0 \pmod{p} \) there is no extension of \( \nabla_p(\mu_5) \) by \( \nabla_p(\mu_6) \). For \( b \neq 0 \pmod{p} \) there is no extension of \( \nabla_p(\mu_3) \) by \( \nabla_p(\mu_8) \). So for \( a \) and \( b \neq 0 \pmod{p} \) we have:

\[
\begin{array}{c}
\nabla_p(\mu_9) \\
\downarrow \\
\nabla_p(\mu_6) \quad \quad \quad \quad \quad \nabla_p(\mu_7) \quad \quad \quad \quad \quad \nabla_p(\mu_8) \\
\downarrow \\
\nabla_p(\mu_5) \\
\quad \quad \quad \quad \quad \nabla_p(\mu_4) \quad \quad \quad \quad \quad \nabla_p(\mu_2) \\
\end{array}
\]

and similarly for the other cases for \( a \) and \( b \).

(viii) Suppose \( \lambda = p(a,b) + (p - s - 2, p - r - 2) \) with \( (a,b) \in X^+ \), and \( (r,s) \in A_0 \). We let

\[
\begin{align*}
\mu_1 &= (pa - p + s, pb + 2p - r - s - 3), \\
\mu_2 &= (pa - r - 2, pb + r + s + 1), \\
\mu_3 &= (pa - p + r, pb - p + s), \\
\mu_4 &= \lambda, \\
\mu_5 &= (pa - r - s - 3, pb + r), \\
\mu_6 &= (pa + 2p - r - s - 3, pb - p + r), \\
\mu_7 &= (pa + s, pb - r - s - 3), \\
\mu_8 &= (pa + r, pb + s), \\
\mu_9 &= (pa + r + s + 1, pb - s - 2).
\end{align*}
\]

These weights are depicted in Figure 1 (b). Then for \( a \) and \( b \equiv -1 \pmod{p} \), \( \nabla(\lambda) \) has filtration

\[
\begin{array}{c}
\nabla_p(\mu_3) \\
\nabla_p(\mu_2) \quad \quad \quad \quad \quad \nabla_p(\mu_9) \\
\nabla_p(\mu_1) \\
\nabla_p(\mu_7) \\
\nabla_p(\mu_4) \\
\end{array}
\]
For $a \not\equiv -1 \pmod{p}$ there is no extension of $\nabla_p(\mu_5)$ by $\nabla_p(\mu_6)$. For $b \not\equiv -1 \pmod{p}$ there is no extension of $\nabla_p(\mu_7)$ by $\nabla_p(\mu_1)$. So for $a$ and $b \not\equiv -1 \pmod{p}$ we have:

\[
\begin{array}{c}
\nabla_p(\mu_3) \\
\nabla_p(\mu_2) \\
\nabla_p(\mu_1) \\
\nabla_p(\mu_7) \\
\nabla_p(\mu_8) \\
\nabla_p(\mu_5) \\
\nabla_p(\mu_6) \\
\nabla_p(\mu_4)
\end{array}
\]

and similarly for the other cases for $a$ and $b$.

**Proof.** In \[15\], the structures of certain $u_1B$-modules $\hat{Z}(\lambda)$ are calculated. We invert these structures to get the corresponding structure diagrams for the $G_1B$-modules $\hat{Z}'_1(\lambda) = \text{Ind}^G_{G_1B}(k\lambda)$ as defined in \[17\, II, 9.1\]. We will apply \[17\, II, proposition 9.11\] to these diagrams to produce a $p$-filtration of $\nabla(\lambda)$. By the remarks of Jantzen \[16\, 3.13\] such filtrations exist for all $\nabla(\lambda)$ with $\lambda$ dominant and can be obtained in this way.

In all these filtrations there can only be one $\nabla_p(\mu)$ occurring as a sub-module of $\nabla(\lambda)$, namely $\nabla_p(\lambda)$, as these both have the same simple $G$-socle $L(\lambda)$. Any module $\nabla_p(\nu)$ appearing above $\nabla_p(\lambda)$ must have an extension with $\nabla_p(\lambda)$ and this extension has simple socle. There also can be only one $\nabla_p(\mu)$ which occurs at the top of $\nabla(\lambda)$, as $\nabla(\lambda)$ has simple $G$-head by Proposition \[4.11\]. This module $\nabla_p(\mu)$ then must have an extension with all $\nabla_p(\nu)$ appearing below it and this extension must have simple head.

Case (i). This is clear as $L(p-1, p-1)$ is the Steinberg module.

Case (iii) (and dually case (ii)). By inverting the diagram in \[15\, theorem 5.2\] and using the translation principle we have the $G_1B$ composition series for $\hat{Z}'_1(\lambda)$, which by applying \[17\, II, proposition 9.11\] gives us the $p$-filtration of $\nabla(\lambda)$ as in the statement of the Lemma above for $b \equiv -1 \pmod{p}$. Lemma \[4.3\] gives us the required filtrations when $b \not\equiv -1 \pmod{p}$.

We now show that the structure obtained so far does not refine further for $b \equiv -1 \pmod{p}$.

Consider

\[
\text{Ext}^1(\nabla(a-1,b+1)^F \otimes L(r,s), \nabla(a,b) \otimes L(s,p-1)) \\
\cong \text{Hom}_G(\nabla(a-1,b+1), \nabla(a,b) \otimes \nabla(1,0)) \\
\cong \text{Hom}_G(\nabla(a-1,b+1) \otimes \nabla(0,1), \nabla(a,b)) \\
\cong k
\]
where the first isomorphism follows by Lemma \[102\] and Proposition \[103\]. The last isomorphism follows as \(\nabla(a - 1, b + 1) \otimes \nabla(0, 1)\) has good filtration

\[
\begin{aligned}
\nabla(a - 1, b + 2) \\
\nabla(a, b) \\
\nabla(a - 2, b + 1)
\end{aligned}
\]

by Lemma \[104\] and \((a, b)\) and \((a - 1, b + 2)\) are not linked for \(b \equiv -1 \pmod{p}\). Let \(E\) be the unique (non-split) extension represented by the Ext group above. We show that \(E\) does not have simple \(G\)-socle and so does not appear in \(\nabla(pa + s, pb + p - 1)\). From the long exact sequence associated to the short exact sequence for \(E\) we have

\[
0 \to \text{Hom}_{G_1}(L(r, s), E) \to \text{Hom}_{G_1}(L(r, s), \nabla(a - 1, b + 1)^F \otimes L(r, s)) \\
\to \text{Ext}_{G_1}^1(L(r, s), \nabla(a, b)^F \otimes L(s, p - 1)) \to \text{Ext}_{G_1}^1(L(r, s), E) \to 0
\]

As

\[
\text{Ext}_{G_1}^1(L(r, s), \nabla(a - 1, b + 1)^F \otimes L(r, s)) \\
\cong \nabla(a - 1, b + 1)^F \otimes \text{Ext}_{G_1}^1(L(r, s), L(r, s)) \cong 0
\]

by Lemma \[105\] and Proposition \[106\] and

\[
\text{Hom}_{G_1}(L(r, s), \nabla(a, b)^F \otimes L(s, p - 1)) \cong 0
\]

We also have

\[
\text{Ext}_{G_1}^1(L(r, s), \nabla(a, b)^F \otimes L(s, p - 1)) \cong \nabla(a, b)^F \otimes \text{Ext}_{G_1}^1(L(r, s), L(s, p - 1)) \cong \nabla(a, b)^F \otimes \nabla(1, 0)^F
\]

by Proposition \[107\].

Hence we have an exact sequence

\[
0 \to \text{Hom}_{G_1}(L(r, s), E) \to \nabla(a - 1, b + 1)^F \\
\xrightarrow{\phi} \nabla(a, b)^F \otimes \nabla(1, 0)^F \to \text{Ext}_{G_1}^1(L(r, s), E) \to 0
\]

If \(\phi\) is injective then \(\text{Hom}_{G_1}(L(r, s), E) \cong 0\). We claim that this is not the case.

Now

\(\phi\) injective \(\Rightarrow \nabla(a - 1, b + 1)^F\) embeds in \(\nabla(a, b)^F \otimes \nabla(1, 0)^F\)

\(\Leftrightarrow \nabla(a - 1, b + 1)\) embeds in \(\nabla(a, b) \otimes \nabla(1, 0)\)

but \(\nabla(a - 1, b + 1)\) cannot embed in \(\nabla(a, b) \otimes \nabla(1, 0)\) as

\[
k \cong \text{Hom}_G(\nabla(a - 1, b + 1), \nabla(a, b - 1)) \\
\subseteq \text{Hom}_G(\nabla(a - 1, b + 1), \nabla(a, b) \otimes \nabla(1, 0)) \\
\cong \text{Hom}_G(\nabla(a - 1, b + 1) \otimes \nabla(0, 1), \nabla(a, b)) \cong k,
\]

where the first isomorphism follows as \((a - 1, b + 1)\) and \((a, b - 1)\) differ by a single reflection \[108\] II, corollary 6.24\] and the last Hom group we calculated above. However homomorphisms in the first Hom group are clearly not 1-1. Hence \(\phi\) cannot be injective. Thus we have \(\text{Hom}_{G_1}(L(r, s), E) \neq 0\).
Now using \(16, 2.2 (1)\)

\[
\text{soc}_G(E) = \text{soc}_G\left(\text{Hom}_{G_1}(L(s, p - 1), E) \otimes L(s, p - 1)\right) \\
\oplus \text{soc}_G\left(\text{Hom}_{G_1}(L(r, s), E) \otimes L(r, s)\right) \\
= \text{soc}_G\left(\nabla(a, b)^F \otimes L(s, p - 1)\right) \\
\oplus \text{soc}_G\left(\text{Hom}_{G_1}(L(r, s), E) \otimes L(r, s)\right)
\]

Both terms are non-zero. Hence \(E\) does not have simple \(G\)-socle.

Case (iv). We write \(\lambda = p(a, b) + (r, s)\) with \(r + s = p - 2\), and \((a, b) \in X^+\). By inverting the diagram in \(15\) theorem 3.3] and using the translation principle we see that \(\hat{X}_1(\lambda)\) has \(G_1T\) composition series:

\[
\tilde{L}_1(p(a - 1, b - 1) + (r, s)) \\
\tilde{L}_1(p(a, b - 1) + (p - 1, r)) \quad \tilde{L}_1(p(a - 1, b) + (s, p - 1)) \\
\tilde{L}_1(p(a, b) + (r, s)).
\]

So we need to show that

\[
\text{Ext}^1(\nabla(a, b - 1)^F \otimes L(p - 1, r), \nabla(a - 1, b)^F \otimes L(s, p - 1)) \\
\cong \text{Ext}^1(\nabla(a, b - 1)^F \otimes L(s, p - 1), \nabla(a, b - 1)^F \otimes L(p - 1, r)) \\
\cong 0.
\]

But \(\text{Ext}^1_{G_1}(L(s, p - 1), L(p - 1, r)) \cong \text{Ext}^1_{G_1}(L(p - 1, r), L(s, p - 1)) \cong 0\) by Proposition \(4.1\) and so by Lemma \(4.2\) we are done.

Case (v) (and dually case (vi)). We know that this filtration is correct on the level of characters using \(15\) theorem 3.1] and the results of \(15\). We also know that \(\nabla_p(\lambda)\) embeds in \(\nabla(\lambda)\). Further, for \(p \neq 3\)

\[
\text{Ext}^1_{G_1}(L(s, p - r - s - 3), L(r, s)) = 0 \quad \text{by Proposition } 4.1
\]

so by Lemma \(4.2\) we have

\[
\text{Ext}^1_{G_1}(\nabla(a, 0)^F \otimes L(s, p - r - s - 3), \nabla(a, 0)^F \otimes L(r, s)) \cong 0.
\]

For \(p = 3\), \((s, p - r - s - 3) = (r, s) = (0, 0)\) and we then have

\[
\text{Ext}^1_{G_1}(\nabla(a - 2, 0)^F \otimes L(s, p - r - s - 3), \nabla(a, 0)^F \otimes L(r, s)) \\
\cong \text{Ext}^1_{G_1}(\nabla(a - 2, 0), \nabla(a, 0)) \cong 0
\]

using Lemma \(4.2\) and observing that \((a - 2, 0)\) is not comparable to \((a, 0)\) in the usual ordering of \(G\)-weights. Hence we get the filtration as in the statement of the Proposition.

Case (viii). By inverting the diagram in \(15\) theorem 5.3] we get the \(G_1B\) structure of \(\hat{X}_1(\lambda)\) with \(\lambda = (p - s - 2, p - r - 2) + p(a, b)\) and \((r, s) \in A_0\). So applying \(17\) II, proposition 9.11] we get a filtration as stated for the case \(a + b \equiv -1 \pmod{p}\).
If \( p \neq 3 \), then by Lemma 4.2 none of the \( \nabla_p(\mu_i) \) which appear above \( \nabla_p(\lambda) \) can have non-split extensions by \( \nabla_p(\mu_3) \). Also \( \nabla_p(\lambda) \) does not have an extension by \( \nabla_p(\mu_2) \) nor by \( \nabla_p(\mu_9) \). Now consider \( p = 3 \). We write \( \mu_i = p\mu_{i1} + \mu_{i0} \). For \( i = 1, 5, 6, 7 \) and 8 Lemma 4.2 gives us

\[
\text{Ext}^1(\nabla_p(\mu_3), \nabla_p(\mu_i)) \cong \text{Ext}^1(\nabla(\mu_{31}), \nabla(\mu_{1i})) \cong 0
\]

as none of the \( \mu_{1i} \) are less than \( \mu_{31} \).

We now take \( p \geq 3 \) again. We need to show that the filtration simplifies for \( a \) (or \( b \)) \( \not\equiv -1 \pmod{p} \), and that it doesn’t simplify for \( a \) (or \( b \)) \( \equiv -1 \pmod{p} \).

Lemmas 4.3 and 4.4 give us the filtration as stated in the Theorem, part (viii) when \( a \) (or \( b \)) \( \not\equiv 1 \pmod{p} \).

Now the same argument as for case (iii) (with \( p \geq 5 \)) shows that both the non-split extensions \( E_1 \) and \( E_2 \) defined via:

\[
0 \to \nabla_p(\lambda) \to E_1 \to \nabla_p(\mu_1) \to 0 \\
0 \to \nabla_p(\lambda) \to E_2 \to \nabla_p(\mu_6) \to 0
\]
do not have simple socle and so the filtration does not refine any further for \( a \) (or \( b \)) \( \equiv -1 \pmod{p} \).

If \( p = 3 \) then the argument used in case (iii) still carries through, as when we remove the Frobenius twist then block considerations also allow us to remove the other direct summands.

Case (vii). We get the \( u_1-B \)-filtration of \( \tilde{Z}(\lambda) \) by taking its dual and then applying [15, theorem 5.3]. We then invert this structure to get the \( G_1B \) structure of \( \tilde{Z}'_1(\lambda) \). Then applying [17, II, proposition 9.11] gives us the corresponding filtration of \( \nabla(\lambda) \). The same argument as before shows that there is no non-split extension of \( \nabla_p(\lambda) \) by \( \nabla_p(\mu_i) \) for \( i = 3, 5, 6, 7 \) and 8, and also that there is no non-split extension of \( \nabla_p(\mu_i) \) by \( \nabla_p(\mu_9) \) for \( i = 2 \) and 4. The extensions of \( \nabla_p(\mu_2) \) by \( \nabla_p(\mu_9) \) and of \( \nabla_p(\mu_3) \) by \( \nabla_p(\mu_8) \) are split as in Case (vii) for \( a \) or \( b \) \( \not\equiv 0 \pmod{p} \) using Lemmas 4.3 and 4.4.

Now suppose \( a \equiv 0 \pmod{p} \). (The case with \( b \equiv 0 \pmod{p} \) is similar.) Consider the extension \( E \) defined via the short exact sequence

\[
0 \to \nabla_p(\mu_5) \to E \to \nabla_p(\mu_9) \to 0.
\]

Such a non-split extension \( E \), exists and is unique, since,

\[
\text{Ext}^1_{G_1}(\nabla_p(\mu_9), \nabla_p(\mu_5)) \cong \text{Hom}_{G_1}(\nabla(a-1,b-1), \nabla(a-2,b) \otimes \nabla(0,1))
\]

using Lemma 4.2 and Proposition 4.4. But this is isomorphic to \( k \), as the costandard modules \((\nabla(a-2,b+1), \nabla(a-1,b-1)\) and \( \nabla(a-3,b) \) appearing in a good filtration of \( \nabla(a-2,b) \otimes \nabla(0,1) \) are not linked for \( a \equiv 0 \pmod{p} \). We have a long exact sequence:

\[
0 \to \text{Hom}_{G_1}(\nabla(\mu_9), L(s,p-r-s-3)) \to \text{Hom}_{G_1}(E, L(s,p-r-s-3)) \\
\to \text{Hom}_{G_1}(\nabla(\mu_9), L(s,p-r-s-3)) \to \text{Ext}^1_{G_1}(\nabla(\mu_9), L(s,p-r-s-3)) \\
\to \text{Ext}^1_{G_1}(E, L(s,p-r-s-3)) \to \text{Ext}^1_{G_1}(\nabla(\mu_9), L(s,p-r-s-3)).
\]
The first Hom group and the last Ext group are both $0$. The third Hom group is known and the first Ext group is known using Proposition 4.1. So we have, for $p \geq 5$, (for $p = 3$ the $\nabla(0, 1)^F$ would be replaced by $k \oplus \nabla(1, 0)^F \oplus \nabla(0, 1)^F$):

$$0 \to \text{Hom}_{G_1}(E, L(s, p - r - s - 3)) \to \Delta(b, a - 2)^F$$

$$\overset{\phi}{\to} \Delta(b - 1, a - 1)^F \otimes \nabla(0, 1)^F \to \text{Ext}^1_{G_1}(E, L(s, p - r - s - 3))$$

We show that $\phi$ is not injective. Now

$$\phi \in \text{Hom}_{G/G_1}(\Delta(b, a - 2)^F, \Delta(b - 1, a - 1)^F \otimes \nabla(0, 1)^F)$$

$$\cong \text{Hom}_G(\Delta(b, a - 2), \Delta(b - 1, a - 1) \otimes \Delta(0, 1))$$

$$\cong \text{Hom}_G(\nabla(a - 1, b - 1), \nabla(a - 2, b) \otimes \nabla(0, 1)) \cong k.$$ 

Also,

$$\phi \text{ injective } \iff \Delta(b, a - 2)^F \text{ embeds in } \Delta(b - 1, a - 1)^F \otimes \Delta(0, 1)^F$$

$$\iff \Delta(b, a - 2) \text{ embeds in } \Delta(b - 1, a - 1) \otimes \Delta(0, 1)$$

$$\iff \nabla(a - 2, b) \text{ is a quotient of } \nabla(a - 1, b - 1) \otimes \nabla(1, 0).$$

However, $\nabla(a - 2, b)$ cannot be quotient of $\nabla(a - 1, b - 1) \otimes \nabla(1, 0)$ as

$$k \cong \text{Hom}_G(\nabla(a - 1, b - 1), \nabla(a - 2, b))$$

$$\subseteq \text{Hom}_G(\nabla(a - 1, b - 1) \otimes \nabla(1, 0), \nabla(a - 2, b))$$

$$\cong \text{Hom}_G(\nabla(a - 1, b - 1), \nabla(a - 2, b) \otimes \nabla(0, 1)) \cong k,$$

where the first isomorphism follows as $(a, b - 1)$ and $(a - 2, b)$ differ by a single reflection \cite{17} II, corollary 6.24. But homomorphisms in the first Hom group are clearly not onto. Hence $\phi$ cannot be injective. and so we have that $\text{Hom}_{G_1}(E, L(s, p - r - s - 3))$ is non-zero. By a similar argument to that before, (using the head functor $\text{hd}$, in place of the socle functor $\text{soc}$), we have that $E$ does not have simple $G$-head and so it does not appear in $\nabla(\lambda)$ which has simple head using Lemma 4.10. The argument for $p = 3$ is essentially the same. As soon as we remove the Frobenius twists ($F$) from the modules then block considerations allow us to remove the extra summands that appear. \hfill \square

5. Good filtration dimensions and global dimensions for $\text{SL}_3$.

In this section we calculate the good filtration dimension of $\nabla_p(\lambda)$. There are two cases to consider, one where $\lambda$ is on a wall and the other where $\lambda$ is inside an alcove. We consider this latter case first.

**Lemma 5.1.** Let $\lambda = p(a, b) + (r, s)$ with $(r, s) \in A_0$, $(a, b) \in X^+$ and $b \geq 1$. We take $M_\lambda$ be the submodule of $\nabla(\lambda)$ with $p$-filtration labelled by $\mu_1$ through to $\mu_3$ in the diagram of Proposition 4.12 part (vii). (If $b = 1$ we take $\nabla(a, b - 2) = 0$.) Then $M_\lambda$ has good resolution

$$0 \to M_\lambda \to \nabla(\lambda_0) \to \nabla(\lambda_1) \to \cdots \to \nabla(\lambda_n) \to 0$$
where \( \lambda_{2i} = p(a - 2i, b + i) + (r, s) \) and \( \lambda_{2i+1} = p(a - 2i - 1, b + i) + (p - r - 2, r + s + 1) \) for \( 0 \leq i \leq \left\lfloor \frac{a}{2} \right\rfloor \).

Similarly, let \( \mu = p(a, b) + (p - r - 2, r + s + 1) \) with \( (p - r - 2, r + s + 1) \) in the upper alcove of \( X_1 \) and \( (a, b) \in X^+ \). We take \( M_\mu \) be the submodule of \( \nabla(\mu) \) with \( p \)-filtration labelled by \( \mu_4 \) through to \( \mu_9 \) in the diagram of Proposition 4.12 part (viii). Then \( M_\mu \) has good resolution

\[
0 \rightarrow M_\mu \rightarrow \nabla(\mu_0) \rightarrow \nabla(\mu_1) \rightarrow \cdots \rightarrow \nabla(\mu_a) \rightarrow 0
\]

where \( \mu_{2i} = p(a - 2i, b + i) + (p - r - 2, r + s + 1) \) and \( \mu_{2i+1} = p(a - 2i - 1, b + i + 1) + (r, s) \) for \( 0 \leq i \leq \left\lfloor \frac{a}{2} \right\rfloor \).

Proof. We proceed by induction on \( a \). If \( a = 0 \) then \( M_\lambda = \nabla(\lambda) \) and \( M_\mu = \nabla(\mu) \) by Proposition 4.12 and we are done. For \( a \geq 1 \) it is sufficient to show the existence of two short exact sequences

\[
0 \rightarrow M_\lambda \rightarrow \nabla(\lambda) \rightarrow M_{\lambda_1} \rightarrow 0
\]

\[
0 \rightarrow M_\mu \rightarrow \nabla(\mu) \rightarrow M_{\mu_1} \rightarrow 0.
\]

Now \( \text{Hom}(\nabla(\lambda), \nabla(\lambda_1)) \cong k \), as \( \lambda \) and \( \lambda_1 \) satisfy the conditions in [17, II, corollary 6.24]. So we need to show that \( M_{\lambda_1} \) is the image of this unique homomorphism \( \phi \) (as suggested by the labelling of the \( \mu \)'s in Proposition 4.12) and that \( M_{\lambda} \) is contained in the kernel of \( \phi \). Now \( \nabla(\lambda_1) \) has simple socle \( L(\lambda_1) \) which appears just once right above \( M_{\lambda} \) in \( \nabla(\lambda) \). (As all \( \nabla_p(\mu) \) which occur above \( \nabla_p(\lambda_1) \) have \( \mu < \lambda_1 \).) Hence \( M_{\lambda_1} \) is the image of \( \phi \) and we are done. The second sequence is similar. \( \square \)

Remark 5.2. We have identified the images of the maps in the resolution (1) of Jantzen [16, 3.12, remark 2].

Corollary 5.3. For \( M_\lambda \) and \( M_\mu \) as defined above we have \( \text{gfd}(M_\lambda) = \text{gfd}(M_\mu) = a \). Furthermore for \( \tau \in X^+ \) we have

\[
\text{Ext}^a(\Delta(\tau), M_\lambda) \cong \text{Hom}(\Delta(\tau), \nabla(\lambda_0)) \cong \delta_{\tau_{\lambda_0}} k
\]

and

\[
\text{Ext}^a(\Delta(\tau), M_\mu) \cong \text{Hom}(\Delta(\tau), \nabla(\mu_0)) \cong \delta_{\tau_{\mu_0}} k
\]

Proof. The result follows by dimension shifting and by noting that \( \text{Hom}(\Delta(\tau), \nabla(\lambda_{a-1})) \) is zero if \( \text{Hom}(\Delta(\tau), \nabla(\lambda_a)) \) is non-zero. The argument for \( M_\mu \) is similar. \( \square \)

Suppose \( \lambda = p(a, b) + \nu \) and \( \nu \neq (p - 1, p - 1) \). We define \( g(\lambda) \) as follows

\[
g(\lambda) = \begin{cases} 
2(a + b) & \text{if } \nu \text{ is inside a lower alcove} \\
2(a + b) + 1 & \text{if } \nu \text{ is inside an upper alcove} \\
a + b & \text{if } \nu \text{ lies on a wall.}
\end{cases}
\]

We will eventually show that \( \text{gfd}(L(\lambda)) = g(\lambda) \).
Lemma 5.4. If \( \mu < \lambda \) and \( \mu \in \mathcal{B}(\lambda) \) then \( g(\mu) \leq g(\lambda) \).

Proof. Suppose \( \mu = (c, d) < \lambda = (a, b) \). If we think of \( \lambda \) as the \( S(3, a + 2b) \) weight \((a + b, b, 0)\) then \( \mu \) can be thought of the \( S(3, a + 2b) \) weight \((c + d + e, d + e, e)\) for an appropriate \( e \in \mathbb{N} \). Since \( \mu < \lambda \) we have \((c + d + e, d + e, e) < (a + b, b, 0)\) in the dominance ordering on partitions. But this corresponds to ‘falling boxes’ in the corresponding Young diagram. Consequently it is clear that \( g(\lambda) \) cannot be increased by moving boxes further down the diagram, hence the result. \( \square \)

Lemma 5.5. Suppose \( \lambda \) is primitive and define \( Q \) to be the quotient \( \nabla_p(\lambda)/L(\lambda) \). Then

\[
g(\text{hw}(Q)) \leq g(\lambda) - 1.
\]

Proof. We write \( \lambda = p(a, b) + \nu \). We define \( \bar{\lambda} = a + b \). By assumption \( \nu \neq (p - 1, p - 1) \). We have \( L(\lambda) \cong L(a, b)^p \otimes L(\nu) \) by Steinberg’s tensor product theorem and so \( Q \cong R^p \otimes L(\nu) \) where \( R \) is the quotient \( \nabla(a, b)/L(a, b) \). If we write \((a, b) = p(a', b') + \nu'\) with \( \nu' \) \( p \)-restricted, then it is clear using \( \bar{\lambda} \) and induction that

\[
\text{hw}(R) = \begin{cases}
  p\text{hw}(\nabla(a', b')/L(a', b')) + \nu' & \text{if } \nu' = (p - 1, p - 1) \\
  p(a' + 1, b' - 1) + (r, p - r - 2) & \text{if } \nu' = (p - 1, r) \\
  p(a' - 1, b' + 1) + (p - s - 2, s) & \text{if } \nu' = (s, p - 1) \\
  \{p(a', b' - 1) + (p - 1, r), & \text{if } \nu' = (r, s) \text{ with } r + s = p - 2 \\
  p(a' - 1, b') + (s, p - 1)\} & \text{if } \nu' = (r, s) \in A_0 \\
  \{p(a', b' - 1) + (r + s + 1, p - s - 2), & \text{if } \nu' = (r, s) \in A_0 \\
  p(a' - 1, b') + (p - r - 2, r + s + 1)\} & \text{if } \nu' = (p - s - 2, p - r - 2) \\
  \{p(a', b') + (r, s), & \text{if } \nu' = (p - s - 2, p - r - 2) \\
  p(a' - 1, b' + 1) + (s, p - r - 3), & \text{inside an upper alcove.} \\
  p(a' + 1, b' - 1) + (p - r - 3, r)\} & \text{inside an upper alcove.}
\end{cases}
\]

Since \( \text{hw}(Q) = p(\text{hw}(R)) + \nu \) we have

\[
\bar{\lambda}(\text{hw}(Q)) = \begin{cases}
  \bar{\lambda}(p^2 \text{hw}(\nabla(a', b')/L(a', b')) + 2p - 2 & \text{if } \nu' = (p - 1, p - 1) \\
  p(a' + b') + p - 2 & \text{if } \nu' = (p - 1, r) \text{ or } \nu' = (s, p - 1) \\
  \{p(a' + b') + r - 1, p(a' + b') + s - 1\} & \text{if } \nu' = (r, s) \text{ with } r + s = p - 2 \\
  \{p(a' + b') + r - 1, p(a' + b') + s - 1\} & \text{if } \nu' = (r, s) \in A_0 \\
  \{p(a' + b') + r + s, & \text{if } \nu' = (p - s - 2, p - r - 2) \\
  p(a' + b') + p - r - 3, & \text{inside an upper alcove.} \\
  p(a' + b') + p - s - 3\} & \text{inside an upper alcove.}
\end{cases}
\]
Now if $\nu' = (c, d)$ then $\bar{g}(\lambda) = p(a' + b') + c + d$ and so we have

$$
\bar{g}(\text{hw}(Q)) = \begin{cases} 
\bar{g}(p^2 \text{hw}(R')) + 2p - 2 & \text{if } \nu' = (p - 1, p - 1) \\
\bar{g}(\lambda) - (r + 1) & \text{if } \nu' = (p - 1, r) \\
\bar{g}(\lambda) - (s + 1) & \text{if } \nu' = (s, p - 1) \\
\{\bar{g}(\lambda) - (s + 1), \bar{g}(\lambda) - (r + 1)\} & \text{if } \nu' = (r, s) \text{ with } r + s = p - 2 \\
\{\bar{g}(\lambda) - (p - r - 2), \bar{g}(\lambda) - (p - 2)\} & \text{if } \nu' = (p - s - 2, p - r - 2) \\
\bar{g}(\lambda) - (p - s - 1), \bar{g}(\lambda) - (p - r - 1) & \text{inside an upper alcove}
\end{cases}
$$

where $R' = \nabla(a', b')/L(a', b')$. We write $(a', b') = p(a'' + b'') + \nu''$. Now $\bar{g}(\text{hw}(R')) \leq a'' + b''$. So if $\nu' = (p - 1, p - 1)$ we have $\bar{g}(\text{hw}(Q)) \leq p^2(a'' + b'') + 2p - 2 < p(a' + b') + 2p - 2 = \bar{g}(\lambda)$ provided $\nu'' \neq 0$. If $\nu'' = 0$ then $\bar{g}(R') = a'' + b'' - 1$ and so we have $\bar{g}(\text{hw}(Q)) = p^2(a'' + b'') - p^2 + 2p - 2 < p(a' + b') + 2p - 2 = \bar{g}(\lambda)$. Thus the result follows for $\nu' = (p - 1, p - 1)$. Since $0 \leq r, s \leq p - 2$ and $r + s < p - 2$ inside an alcove the result follows for all other $\nu'$.

The next Proposition shows that $gfd(L(\lambda)) = \bar{g}(\lambda)$ for $\lambda$ inside an alcove. The following Lemma forms part of the inductive step.

**Lemma 5.6.** Let $\lambda \in X^+$ and suppose $g(\mu) = gfd(L(\mu)) = gfd(\nabla_p(\mu))$ for all $\mu < \lambda$ with $\mu \in B(\lambda)$ and $gfd(\nabla_p(\lambda)) = \bar{g}(\lambda)$. Then $gfd(L(\lambda)) = gfd(\nabla_p(\lambda))$.

**Proof.** We have a short exact sequence

$$
0 \to L(\lambda) \to \nabla_p(\lambda) \to Q \to 0.
$$

Lemma 5.4 gives us $gfd(\nabla_p(\lambda)) > gfd(Q)$, since all the composition factors of $Q$ have weights with smaller good filtration dimension, and the result follows by Lemma 2.7 part (i).

**Theorem 5.7.** Let $\nu = (r, s)$ be a weight inside the fundamental alcove and $\bar{\nu} = (p - s - 2, p - r - 2)$ its reflection in the upper alcove. Then we have,

$$
gfd(L(p(a, b) + \nu)) = gfd(\nabla(a, b)^F \otimes L(\nu)) = 2(a + b)
$$

and

$$
gfd(L(p(a, b) + \bar{\nu})) = gfd(\nabla(a, b)^F \otimes L(\bar{\nu})) = 2(a + b) + 1.
$$

Moreover

$$
\text{Ext}^4(a + b)(\Delta(a, b)^F \otimes L(\nu), \nabla(a, b)^F \otimes L(\nu)) \neq 0
$$
and

$$
\text{Ext}^4(a + b + 2)(\Delta(a, b)^F \otimes L(\bar{\nu}), \nabla(a, b)^F \otimes L(\bar{\nu})) \neq 0.
$$

**Proof.** We proceed by induction on $a + b$. For $a + b = 0$ we have $L(\nu) = \nabla(\nu) = \Delta(\nu)$ and

$$
gfd(L(\nu)) = gfd(\nabla(\nu)) = 0.
$$
Also we have a non-split short exact sequence for $L(\bar{\nu})$, namely

$$0 \to L(\bar{\nu}) \to \nabla(\bar{\nu}) \to L(\nu) \to 0.$$ 

Hence we have $\text{gfd}(L(\bar{\nu})) = 1$ and $\text{Ext}^2(L(\bar{\nu}), L(\bar{\nu})) \neq 0$ by Lemma 2.8.

Now suppose $a + b \geq 1$. Let $\lambda = p(a, b) + \nu$ and $M_\lambda$ be as in Lemma 5.1. We first show $\text{gfd}(\nabla_p(\lambda)) = g(\lambda)$.

Case (i): Suppose $b \geq 2$. We have a short exact sequence

$$0 \to \nabla(a, b)^F \otimes L(r, s) \to M_\lambda \to Q \to 0$$

(3)

where $Q$ is the (unique) extension

$$0 \to \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2) \to Q \to \nabla(a, b - 2)^F \otimes L(p - r - s - 3, r) \to 0.$$  

(4)

By induction we have $\text{gfd}(\nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2)) = 2(a + b) - 1$ and $\text{gfd}(\nabla(a, b - 2)^F \otimes L(p - r - s - 3, r)) = 2(a + b) - 4$. Thus we have $\text{gfd}(Q) = 2(a + b) - 1$ by Lemma 2.7. We also have $\text{gfd}(Q) > \text{gfd}(M_\lambda) = a$ by Corollary 5.3, so we have $\text{gfd}(\nabla(\lambda)) = 2(a + b)$ by Lemma 2.7 as required. Lemma 2.8 also gives us

$$\text{Ext}^{4(a+b)}(\Delta_p(\lambda), \nabla_p(\lambda)) \cong \text{Ext}^{4(a+b)-1}(\Delta_p(\lambda), Q)$$

$$\cong \text{Ext}^{4(a+b)-1}(\Delta_p(\lambda), \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2))$$

$$\cong \text{Ext}^{4(a+b)-2}(Q^*, \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2))$$

$$\cong \text{Ext}^{4(a+b)-2}(\Delta(a, b - 1)^F \otimes L(r + s + 1, p - s - 2), \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2))$$

where the first isomorphism follows using sequence (3), the second from (4), the third from the dual of (3) and the fourth from the dual of (4). This last Ext group is non-zero by induction.

Case (ii): Suppose $b = 1$. The argument above simplifies as $Q = \nabla(a, 0)^F \otimes L(r + s + 1, p - s - 2)$ and so by induction we have $\text{gfd}(Q) = 2(a + 1) - 1 > \text{gfd}(M_\lambda) = a$. Hence $\text{gfd}(\nabla_p(\lambda)) = 2(a + 1)$ and

$$\text{Ext}^{4(a+1)}(\Delta_p(\lambda), \nabla_p(\lambda)) \neq 0$$

as required.

Case (iii): Suppose $b = 0$. We have a short exact sequence

$$0 \to \nabla(a, 0)^F \otimes L(r, s) \to \nabla(\lambda) \to Q \to 0$$

where $Q$ is the (unique) extension

$$0 \to \nabla(a - 1, 0)^F \otimes L(p - r - 2, r + s + 1) \to Q \to \nabla(a - 2, 0)^F \otimes L(s, p - r - s - 3) \to 0.$$ 

By induction we have $\text{gfd}(\nabla(a - 1, 0)^F \otimes L(r + s + 1, p - s - 2)) = 2a - 1$ and $\text{gfd}(\nabla(a - 2, 0)^F \otimes L(s, p - r - s - 3)) = 2a - 4$. A similar argument to Case (i) yields the required result.

Now let $\mu = p(a, b) + \bar{\nu}$ and $M_\mu$ be as in Lemma 5.1. We have a short exact sequence

$$0 \to \nabla_p(\mu) \to M_\mu \to Q \to 0.$$
Thus \( \text{gfd}(\nabla_p(\mu)) = \text{gfd}(Q) + 1 \) provided \( \text{gfd}(Q) > \text{gfd}(M_{\mu}) = a \). We have that \( \text{gfd}(Q) \leq 2(a + b) \) by induction, so \( \text{gfd}(\nabla_p(\mu)) \leq 2(a + b) + 1 \). We need to show that both these bounds are attained.

Suppose \( b \geq 1 \). Define \( R \) via the short exact sequence

\[
0 \rightarrow R \rightarrow Q \rightarrow \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2) \rightarrow 0.
\]

Here (using the notation of Proposition \( \text{Proposition} \)), it is clear by induction and using the \( p \)-filtration of \( R \) that for \( M \) a \( G \)-module we have

\[
\text{Ext}^{\text{wfd}(M)+2(a+b)}(M, R) \cong \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla_p(\mu_b)) \oplus \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla_p(\mu_a))
\]

since the other \( \nabla_p(\mu_i) \) that appear in \( R \) have good filtration dimension equal to \( 2(a + b) - 2 \) and so cannot contribute to this \( \text{Ext} \) group. We have a direct sum since there is no extension appearing between \( \nabla_p(\mu_b) \) and \( \nabla_p(\mu_s) \).

The long exact sequence gives us

\[
\text{Ext}^{\text{wfd}(M)+2(a+b)-1}(M, \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2))
\rightarrow \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, R) \rightarrow \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, Q) \rightarrow 0.
\]

But the middle \( \text{Ext} \) group is as above. Also

\[
\text{Ext}^{\text{wfd}(M)+2(a+b)-1}(M, \nabla(a, b - 1)^F \otimes L(r + s + 1, p - s - 2))
\cong \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla(a, b)^F \otimes L(r, s))
\]

using the case above. Thus we have

\[
\text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla(a, b)^F \otimes L(r, s))
\rightarrow \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla(a, b)^F \otimes L(r, s))
\oplus \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla(a + 1, b - 1)^F \otimes L(p - r - s - 3, r))
\rightarrow \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, Q) \rightarrow 0.
\]

Hence \( \text{Ext}^{\text{wfd}(M)+2(a+b)}(M, Q) \) will be non-zero if

\[
\text{Ext}^{\text{wfd}(M)+2(a+b)}(M, \nabla(a + 1, b - 1)^F \otimes L(p - r - s - 3, r)) \neq 0.
\]

We know by induction that \( \text{gfd}(\nabla(a + 1, b - 1)^F \otimes L(p - r - s - 3, r)) = 2(a + b) \) and \( \text{gfd}(Q) \leq 2(a + b) \). If we take \( M \in \mathcal{F}(\Delta) \) (so \( \text{wfd}(M) = 0 \)) with the last \( \text{Ext} \) group being non-zero then we have \( \text{Ext}^{2(a+b)}(M, Q) \) is non-zero and so \( \text{gfd}(Q) = 2(a + b) \).

Now

\[
\text{Ext}^{4(a+b)+2}(\Delta_p(\mu), \nabla_p(\mu)) \cong \text{Ext}^{4(a+b)}(Q^\circ, Q)
\]
as \( \text{gfd}(M_{\mu}) = a < 2(a + b) - 1 \) for \( b \geq 1 \). We let \( \lambda = p(a + 1, b - 1) + (p - r - s - 3, r) \). We also have

\[
\text{Ext}^{4(a+b)}(Q^\circ, \nabla_p(\lambda)) \cong \text{Ext}^{4(a+b)}(\Delta_p(\lambda), Q).
\]
Now $\Ext^{4(a+b)}(\Delta_p(\lambda), \nabla_p(\lambda)) \neq 0$ by induction. Hence

$$\Ext^{4(a+b)}(\Delta_p(\lambda), Q) \cong \Ext^{4(a+b)}(Q^r, \nabla_p(\lambda)) \neq 0$$

and so $\Ext^{4(a+b)}(Q^r, Q)$ is non-zero, using $M = Q^r$ in the sequence above. This implies that $\gfd(\nabla_p(\mu)) = \gfd(Q) + 1 = 2(a + b) + 1$ and $\Ext^{4(a+b)+2}(\Delta_p(\mu), \nabla_p(\mu))$ is non-zero as required.

If $b = 0$ then

$$Q \cong (\nabla(a - 1, 0)^F \otimes L(p - r - s - 3, r)) \oplus (\nabla(a, 0)^F \otimes L(r, s)).$$

Hence $\gfd(Q) = 2a$ by induction and $\gfd(\nabla_p(\mu)) = 2a + 1$. Also

$$\Ext^{4a+2}(\Delta_p(\mu), \nabla_p(\mu)) \cong \Ext^4(Q^r, Q)$$

$$\cong \Ext^4(\Delta(a, 0)^F \otimes L(r, s), \nabla(a, 0)^F \otimes L(r, s))$$

which is non-zero by induction.

In all cases for both $\lambda$ and $\mu$ as defined above we have by Lemma 5.6 that $\gfd(\Lambda(\lambda)) = \gfd(\nabla_p(\lambda))$ and $\gfd(M_\mu) = \gfd(\nabla_p(\mu))$. This completes the induction. □

We now consider the case where $\lambda$ lies on a wall but is not a Steinberg weight.

**Lemma 5.8.** Let $\lambda = p(a, b) + (s, p - 1)$ with $(a, b) \in X^+$ and $0 \leq s \leq p - 2$. We define $M_\lambda$ to be the (unique up to equivalence) non-split extension

$$0 \to \nabla(a, b)^F \otimes L(s, p - 1) \to M_\lambda \to \nabla(a, b - 1)^F \otimes L(r, s) \to 0$$

with $r + s = p - 2$. Then $M_\lambda$ has good resolution

$$0 \to M_\lambda \to \nabla(\lambda_0) \to \nabla(\lambda_1) \to \cdots \to \nabla(\lambda_a) \to 0$$

where $\lambda_{2i} = p(a - 2i, b + i) + (s, p - 1)$ and $\lambda_{2i+1} = p(a - 2i - 1, b + i + 1) + (r, s)$ for $i$ an integer between 0 and $\lfloor \frac{a}{2} \rfloor$.

Similarly, let $\mu = p(a, b) + (r, s)$ with $(a, b) \in X^+$ and $r + s = p - 2$. We define $M_\mu$ to be the (unique up to equivalence) non-split extension

$$0 \to \nabla(a, b)^F \otimes L(r, s) \to M_\mu \to \nabla(a, b - 1)^F \otimes L(p - 1, r) \to 0.$$

Then $M_\mu$ has good resolution

$$0 \to M_\mu \to \nabla(\mu_0) \to \nabla(\mu_1) \to \cdots \to \nabla(\mu_a) \to 0$$

where $\mu_{2i} = p(a - 2i, b + i) + (r, s)$ and $\mu_{2i+1} = p(a - 2i - 1, b + i) + (p - 1, r)$ for $i$ an integer between 0 and $\lfloor \frac{a}{2} \rfloor$.

**Proof.** Similar to that of Lemma 5.4 □
Corollary 5.9. For $M_{\lambda}$ and $M_{\mu}$ as defined above we have $\text{gfd}(M_{\lambda}) = \text{gfd}(M_{\mu}) = a$. Furthermore for $\tau \in X^+$ we have
\[
\text{Ext}^a(\Delta(\tau), M_{\lambda}) \cong \text{Hom}(\Delta(\tau), \nabla(\lambda_{n})) \cong \delta_{\tau,\lambda_{n}} k
\]
and
\[
\text{Ext}^a(\Delta(\tau), M_{\mu}) \cong \text{Hom}(\Delta(\tau), \nabla(\mu_{n})) \cong \delta_{\tau,\mu_{n}} k.
\]

The next Proposition shows that $\text{gfd}(L(\lambda)) = g(\lambda)$ for $\lambda$ lying on a wall.

Theorem 5.10. Let $\nu$ be a non-Steinberg weight on a wall. (So $\nu$ is in a primitive block.) Then we have
\[
\text{gfd}(L(p(a, b) + \nu)) = \text{gfd}(\nabla(a, b)^{F} \otimes L(\nu)) = a + b.
\]
Furthermore
\[
\text{Ext}^{2(a+b)}(\Delta(a, b)^{F} \otimes L(\nu), \nabla(a, b)^{F} \otimes L(\nu)) \neq 0.
\]

Proof. We proceed by induction on $a + b$. For $a + b = 0$ we have $L(\nu) = \nabla(\nu) = \Delta(\nu)$ and $\text{gfd}(L(\nu)) = \text{gfd}(\nabla(\nu)) = 0$.

For $a + b = 1$, consider $a = 1$, $b = 0$ (the other case being similar).

Case (i): $\nu = (r, s)$ with $r + s = p - 2$. Here we have $L(p + r, s) \cong \nabla(1, 0)^{F} \otimes L(r, s)$. We have a non-split short exact sequence [24, lemma 3.2.4 (iv)]
\[
0 \to L(p + r, s) \to \nabla(p + r, s) \to \nabla(s, p - 1) \to 0,
\]
so $\text{gfd}(L(p + r, s)) = 1$ and $\text{Ext}^{2}(L(p + r, s), L(p + r, s)) \neq 0$ by Lemma 2.8.

Case (ii): $\nu = (p - 1, r)$. This case follows exactly as in Case (i).

Case (iii): $\nu = (s, p - 1)$. Here $L(p + s, p - 1) \cong \nabla(1, 0)^{F} \otimes L(p - 1, r)$. We have a non-split short exact sequence [24, lemma 3.2.4 (ii) and (vii)]
\[
0 \to L(p + s, p - 1) \to \nabla(p + s, p - 1) \to \nabla(r, p + s) \to 0
\]
so $\text{gfd}(L(p + s, p - 1)) = 1$ and $\text{Ext}^{2}(L(p + s, p - 1), L(p + s, p - 1)) \neq 0$ by Lemma 2.8.

Now suppose that $a + b \geq 2$. We let $\lambda = p(a, b) + \nu$. We will only consider the case with $\nu = (r, s)$ or $(s, p - 1)$. The other case with $\nu = (p - 1, r)$ is exactly dual to the case of $\nu = (s, p - 1)$. We define $\mu$ by
\[
\mu = \begin{cases} 
  p(a, b - 1) + (p - 1, r) & \text{if } \nu = (r, s) \\
  p(a, b - 1) + (r, s) & \text{if } \nu = (s, p - 1)
\end{cases}
\]
where $r + s$ equals $p - 2$. We have a short exact sequence
\[
0 \to \nabla_{p}(\lambda) \to M_{\lambda} \to \nabla_{p}(\mu) \to 0
\]
(5) with $M_{\lambda}$ as defined in Lemma 5.8.
Case (i): $b = 0$. Here $\nabla_p(\mu) = 0$ so $\nabla_p(\lambda) = M_\lambda$. Corollary 5.8 gives $\text{gfd}(\nabla_p(\lambda)) = a$ and
\[
\text{Ext}^{2a}(\Delta_p(\lambda), \nabla_p(\lambda)) \cong \text{Hom}(\Delta(\lambda_\mu), \nabla(\lambda_\mu)) \cong k
\]
where $\lambda_\mu$ is defined as in Lemma 5.8 and using the good resolution for $M_\lambda$ and its $\omega$-dual.

Case (ii): $b = 1$. We know by Case (i) that $\text{gfd}(\nabla_p(\mu)) = a$ and Corollary 5.9 give $\text{gfd}(M_\lambda) = a$. Hence Lemma 2.7 applied to sequence (5) gives $\text{gfd}(\nabla_p(\lambda)) \leq a+1$. Using Case (i) and Corollary 5.9 we have the commutative diagram
\[
\begin{array}{cccccc}
\text{Ext}^a(\Delta(\tau), M_\lambda) & \rightarrow & \text{Ext}^a(\Delta(\tau), \nabla_p(\mu)) & \rightarrow & \text{Ext}^{a+1}(\Delta(\tau), \nabla_p(\lambda)) & \rightarrow 0 \\
\delta_{\tau\lambda_a} k & \downarrow & \delta_{\tau\mu_a} k & \downarrow & \text{Ext}^{a+1}(\Delta(\tau), \nabla_p(\lambda)) & \rightarrow 0
\end{array}
\]
But $\lambda_a \neq \mu_a$ and hence we have $\text{gfd}(\nabla_p(\lambda)) = a + 1$.

We now wish to show that $\text{Ext}^{2a+2}(\Delta_p(\lambda), \nabla_p(\lambda))$ is non-zero. We have just shown that $\text{wfd}(\Delta_p(\lambda)) = \text{gfd}(\nabla_p(\lambda)) = a + 1$. Also by Case (i) we have $\text{wfd}(\Delta_p(\mu)) = \text{gfd}(\nabla_p(\mu)) = a$. Hence, using Lemma 2.7 the long exact sequence from sequence (6) gives us
\[
\text{Ext}^{2a+1}(\Delta_p(\lambda), M_\lambda) \rightarrow \text{Ext}^{2a+1}(\Delta_p(\lambda), \nabla_p(\mu)) \rightarrow \text{Ext}^{2a+2}(\Delta_p(\lambda), \nabla_p(\lambda)) \rightarrow 0
\]
Now $\text{Ext}^{2a+1}(\Delta_p(\lambda), M_\lambda) \cong \text{Ext}^{a+2}(\Delta_p(\lambda), M_{\lambda_a-1})$ using the good resolution for $M_\lambda$ from Lemma 5.8. We have the following short exact sequence for $M_{\lambda_a-1}$
\[
0 \rightarrow M_{\lambda_a-1} \rightarrow \nabla(\lambda_{a-1}) \rightarrow \nabla(\lambda_a) \rightarrow 0
\]
so the long exact sequence gives us
\[
\text{Ext}^{a+1}(\Delta_p(\lambda), \nabla(\lambda_a)) \rightarrow \text{Ext}^{a+2}(\Delta_p(\lambda), M_{\lambda_a-1}) \rightarrow 0
\]
as $\text{wfd}(\Delta_p(\lambda)) = a + 1$. But $\text{Ext}^{a+1}(\Delta_p(\lambda), \nabla(\lambda_a)) \cong \delta_{\lambda_a} k = 0$. Hence $\text{Ext}^{a+2}(\Delta_p(\lambda), M_{\lambda_a-1}) \cong 0$. Sequence (6) then gives us
\[
\text{Ext}^{2a+2}(\Delta_p(\lambda), \nabla_p(\lambda)) \cong \text{Ext}^{2a+1}(\Delta_p(\lambda), \nabla_p(\mu)) \cong \text{Ext}^{2a+1}(\Delta_p(\mu), \nabla_p(\lambda)).
\]
We now need to show that
\[
\text{Ext}^{2a+1}(\Delta_p(\mu), \nabla_p(\lambda)) \cong \text{Ext}^{2a}(\Delta_p(\mu), \nabla_p(\mu))
\]
where the last Ext group is non-zero by induction. But we may repeat the argument above in one less degree using $\Delta_p(\mu)$ in place of $\Delta_p(\lambda)$ to get the required isomorphism.

Case (iii): $b \geq 2$. Here $\text{gfd}(\nabla_p(\mu)) = a + b - 1 > \text{gfd}(M) = a$ so Lemma 2.5 gives us $\text{gfd}(\nabla_p(\lambda)) = a + b$, and
\[
\text{Ext}^{2(a+b)}(\Delta_p(\lambda), \nabla_p(\lambda)) \cong \text{Ext}^{2(a+b-1)}(\Delta_p(\mu), \nabla_p(\lambda)) \cong \text{Ext}^{2(a+b)-2}(\Delta_p(\mu), \nabla_p(\mu)) \neq 0
\]
by induction.
Lemma [5.10] then completes the proof. \qed

We now consider the case where \( \lambda \) is not primitive.

**Corollary 5.11.** Suppose \( \lambda \) is a dominant weight and \( \lambda = p^d \lambda_1 + (p^d - 1, p^d - 1) \) for some \( d \in \mathbb{N} \) and \( \lambda_1 \in X^+ \). Then \( \text{gfd}(L(\lambda)) = \text{gfd}(L(\lambda_1)) \) and

\[
\text{Ext}^2_{\text{gfd}(\lambda)}(L(\lambda), L(\lambda)) \neq 0.
\]

**Proof.** Suppose \( \lambda = (a, b) \), and \( \lambda_1 = (a_1, b_1) \). We have by [9, section 4, theorem] that \( \mathcal{B}(a+b, b, 0) \) is Morita equivalent to \( \mathcal{B}(a_1+b_1, b_1, 0) \) in \( S(2, r_1) \) with \( r_1 = a_1 + 2b_1 \). But \( \lambda_1 \) is primitive and the result follows by Theorems 5.7 and 5.10 \( \Box \)

**Corollary 5.12.** Given \( (a_1, a_2, a_3) \in \Lambda^+(3, r) \), we let \( (a_1-a_2, a_2-a_3) = p^d \lambda + (p^d - 1, p^d - 1) \) with \( \lambda \in X^+ \) and \( d \in \mathbb{N} \). We also let \( L(a_1, a_2, a_3) \) be the irreducible module of highest weight \((a_1, a_2, a_3)\) for \( S(3, r) \). Then

\[
\text{gfd}(L(a_1, a_2, a_3)) = g(\lambda).
\]

Moreover

\[
\text{Ext}^2_{S(3, r)}(L(a_1, a_2, a_3), L(a_1, a_2, a_3)) \neq 0.
\]

**Proof.** We let \( g = g(\lambda) \). Now Theorems 5.7 and 5.10 and Corollary 5.11 give us

\[
\text{Ext}^i_{S(3, r)}(\Delta(b_1, b_2, b_3), L(a_1, a_2, a_3)) \cong \text{Ext}^i_G(\Delta(b_1 - b_2, b_2 - b_3, L(a_1 - a_2, a_2 - a_3)) \cong 0
\]

if \( i > g \) and so we have \( \text{gfd}(L(a_1, a_2, a_3)) \leq g \). We also have

\[
\text{Ext}^2_{S(3, r)}(L(a_1, a_2, a_3), L(a_1, a_2, a_3)) \cong \text{Ext}^2_G(L(a_1 - a_2, a_2 - a_3), L(a_1 - a_2, a_2 - a_3)) \neq 0.
\]

Now Lemma 5.2 gives \( \text{wfd}(L(a_1, a_2, a_3)) + \text{gfd}(L(a_1, a_2, a_3)) \geq 2g \). But \( \text{wfd}(L(a_1, a_2, a_3)) = \text{gfd}(L(a_1, a_2, a_3)) \), and so we have \( \text{gfd}(L(a_1, a_2, a_3)) = g \), as required. \( \Box \)

**Theorem 5.13.** The global dimension of \( S(3, r) \) is twice its good filtration dimension and is given as follows

for \( p = 2 \)

\[
\text{glob}(S(3, r)) = 2\left\lfloor \frac{r}{3} \right\rfloor
\]

for \( p = 3 \)

\[
\text{glob}(S(3, r)) = \begin{cases} 
4\left\lfloor \frac{r}{3} \right\rfloor & \text{if } r \equiv 0 \pmod{3} \\
2\left\lfloor \frac{r}{p} \right\rfloor & \text{if } r \not\equiv 0 \pmod{3}
\end{cases}
\]

for \( p \geq 5 \)

\[
\text{glob}(S(3, r)) = 4\left\lfloor \frac{r}{p} \right\rfloor.
\]
Proof. Using Corollary 5.12 the same argument as in the second paragraph of the proof of Theorem 5.9 gives us \( \text{glob}(S(3, r)) = 2 \text{gfd}(S(3, r)) \).

For \( \text{SL}_3 \), \( (r, 0) \) is always primitive. Lemmas 5.2 and 2.2 mean we need only consider weights in \( S(3, r) \) which are maximal in their block in determining \( \text{gfd}(S(3, r)) \).

Case (i): \( p = 2 \). Here all the corresponding \( \text{SL}_3 \) weights lie on walls. Also all the irreducible modules corresponding to weights in \( \Lambda^+(3, r) \) have smaller or equal good filtration dimension to \( (r, 0) \), so \( \text{glob}(S(3, r)) = 2 \text{gfd}(L(r, 0, 0)) = 2 \lfloor \frac{r}{2} \rfloor \).

Case (ii): \( p = 3 \). If \( r \equiv 0 \mod 3 \) then \( (r, 0) \) lies inside a lower alcove and \( \text{glob}(S(3, r)) = 4 \lfloor \frac{r}{3} \rfloor \).

If \( r \equiv 1 \) or \( 2 \mod 3 \) then \( (r, 0) \) lies on a wall. We claim all the \( \text{SL}_3 \) weights corresponding to partitions in \( \Lambda^+(3, r) \) lie on a wall and then all irreducible modules corresponding to weights in \( \Lambda^+(3, r) \) will have smaller or equal good filtration dimension. If \( (a, b) \) lies inside an alcove then \( (a, b) \equiv (0, 0) \) or \( (2, 2) \mod 3 \). So in both cases \( a + 2b \equiv 0 \mod p \). For \( (a, b) \) to come from a partition in \( \Lambda^+(3, r) \) we need \( (a + b + c, b + c) \in \Lambda^+(3, r) \) for some \( c \in \mathbb{N} \) with \( a + 2b + 3c = r \). But we have \( a + 2b \equiv 0 \mod 3 \) so \( r \equiv 0 \mod 3 \), and this is a contradiction to our assumption on \( r \). Thus all the \( \text{SL}_3 \) weights corresponding to partitions in \( \Lambda^+(3, r) \) lie on a wall and hence \( \text{glob}(S(3, r)) = 2 \lfloor \frac{r}{3} \rfloor \).

Case (iii): \( p \geq 5 \). If \( (r, 0) \) is not on a wall then \( \text{glob}(S(3, r)) = 4 \lfloor \frac{r}{p} \rfloor \). If \( (r, 0) \) is on a wall then \( r \equiv -1 \) or \( -2 \mod p \). But then \( \Lambda^+(3, r) \) contains \( (r - 2, 1, 1) \) and this corresponds to the \( \text{SL}_3 \) weight \( (r - 3, 0) \) which lies inside an alcove. Hence \( \text{glob}(S(3, r)) = 4 \lfloor \frac{r - 3}{p} \rfloor = 4 \lfloor \frac{r}{p} \rfloor \). \( \square \)

We now consider the quantum case with \( n = 3 \) and \( l \geq 2 \). The cohomological theory of quantum groups and their \( q \)-Schur algebras can be found in [10]. We need the appropriate generalisation of the \( p \)-filtration of \( \nabla(\lambda) \) in Theorem 4.12 for which we need the generalisation of Proposition 4.1. We expect that this would replace \( p \) by \( l \) in all cases. Once we have a \( G_1B \) composition series of \( \hat{Z}_1(\lambda) \) then [11] proposition 5.2] would give us the required \( l \)-filtration of \( \nabla(\lambda) \). This would then give a quantum version of the \( M_\lambda \)'s used extensively in Section 5 and then all the Theorems in this section would generalise. We expect that the \( G_1B \) composition series of \( \hat{Z}_1(\lambda) \) will have the same weights in terms of relative alcoves as in the classical case. This would then give us the quantum version of Theorem 5.13 with \( p \) replaced by \( l \) and 5 replaced by 4.

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