Irreversibility, Information and Randomness in Quantum Measurements

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Abstract

Irreversibility in quantum measurements is considered from the point of quantum information theory. For that purpose the information transfer between the measured object Ω and measuring system Ω is analyzed. It’s found that due to the principal constraints of quantum-mechanical origin, the information about the purity of Ω state isn’t transferred to Ω during the measurement of arbitrary Ω observable V. Consequently Ω can’t discriminate the pure and mixed Ω ensembles with the same V̅. As the result, the random outcomes should be detected by Ω in V measurement for Ω pure ensemble of V eigenstate superposition. It’s shown that the outcome probabilities obey to Born rule. The influence of Ω decoherence by its environment is studied, however the account of its effects doesn’t change these results principally.

1 Introduction

Quantum mechanics (QM) after more than 90 years of its development achieved the tremendous success in the description of nature. However, its foundations are still disputed extensively and seems to contain some unsettled questions [1, 2]. The most famous and oldest of them is State Collapse or Measurement problem [1, 2, 3]. In its essence, the experimental measurement of pure quantum state parameters result in the random outcomes with probabilities described by Born rule, such situation seems to contradict to fundamental QM linearity [2, 3]. This effect called also the irreversibility of 2-nd kind supposes that Ω interaction with measuring device is nonunitary and irreversible, whereas in all other situations quantum evolution is known to be unitary.

In our paper this problem will be considered mainly within the framework of information theory [1, 3]. Really, the measurement of physical parameters, characterizing an arbitrary object Ω, includes the transfer of information about Ω state to the information system Ω (Observer) which processes and memorizes it [1, 3]. Correspondingly, in information theory any measuring system (MS) can be considered as
the information channel connecting $\Omega$ and $O$ [3, 4, 5]. Such approach, in principle, can have important implications for the theory of quantum measurements. In particular, if some constraints on the information transfer via such channel exist, they can influence the information available for $O$ during the measurement of $\Omega$ state and distort the measurement results. Quantum information studies have shown earlier that such quantum constraints are really exist in typical information channels and define the channel capacity [4, 5]. However, until now the influence of this and other information-theoretical effects on the outcomes of quantum measurements wasn’t analyzed thoroughly. Basing on these premises, in our previous papers the measurement of $\Omega$ observable $V$ by $O$ was analyzed for simple MS model [6, 7]. It was shown that under simple assumptions an arbitrary $\Omega$ observable $V$ can be effectively measured by $O$, whereas the information characterizing the purity of $\Omega$ state isn’t transferred to $O$ at all. As the result, $O$ can’t discriminate the pure and mixed $\Omega$ ensembles with the same $\hat{V}$. For individual events such information losses result in the appearance of random $O$ outcomes in the measurement of $V$ eigenstate superposition [7, 8].

Here the information-theoretical approach of system self-description will be applied to the analysis of information acquisition by $O$ and signal recognition [3, 9]. The formalism of restrictive maps in MS Hilbert space will be used for the calculation of information transfer from $\Omega$ to $O$ during the measurement [10]. Its application permits to drop the majority of ad hoc assumptions used earlier in our papers. It will be shown also that beside the information losses in MS, the initial $\Omega$ information about purity is by itself principally incomplete and so insufficient for pure/mixed state discrimination in a single event. Basing on these results, the model-independent restriction on state purity information will be obtained. It supposes that the randomness is generic for standard quantum measurements and so QM Reduction postulate is excessive in QM formalism [8]. The considered effect isn’t related directly to MS decoherence by its environment, yet its account performed in our paper enlarges the considered information losses and stabilizes the final $O$ states [11].

Plainly, the most detailed measurement study should include the analysis of individual events. In quantum case the individual quantum states are exploited for that purpose [1, 2]. We shall consider here only such individual states which can be prepared ‘event by event’ by experimentalist in the idealized scheme of state preparation [2]. For the finite-dimensional system $S$ they are the pure states (rays) in $S$ Hilbert space $\mathcal{H}$. The statistical or ensemble states are described by the density matrix $\rho$, i.e. the positive trace one operator on $\mathcal{H}$. If the individual procedure of ensemble preparation is used, such ensemble can admit more detailed description in form of gemenge [1], its example is given below. Only those QM observables, which correspond to the linear, self-adjoint operators, will be used in our formalism; POV generalization of QM observables is unimportant for that.

In information theory the outcome of $\Omega$ measurement by $O$ in event $n$ is described by the array of real or discrete parameters called the information pattern (IP) $J(n) = \{e_1, \ldots, e_l\}$ [12]. For example, if $O$ measures the spin projection and momentum of some particle, then the resulting IP is: $J(n) = \{S_z, P_x, P_y, P_z\}$. The difference between two signals for $O$ is reflected by the difference of their IPs, its typical measure is: $dJ = \sum |e_i^1 - e_i^2|$ [12]. Usually, it is admitted implicitly that $O$
physical structure permits to perform such operations, which result in the recognition of incoming signals, yet for quantum O models studied here this is the additional assumption.

2 Model of Quantum Measurements

In our model MS consists of measured object S, detector D and the information system O. D and O supposedly can be treated as the quantum objects and MS, as the whole, is described by the quantum state $\Psi_{MS}(t)$ relative to some external observer or reference frame (RF) O’. S is taken to be the particle with the spin $\frac{1}{2}$ and its z projection $S_z$ is measured. Its u, d eigenstates are denoted $|S_{1,2}\rangle$, so that their superposition has the form:

$$\Phi_S = a_1|S_1\rangle + a_2|S_2\rangle$$

To compare the measurements of similar pure and mixed S ensembles, $|S_i\rangle$ ensemble with the same $\bar{S}_z$ will be used. From its preparation such mixed S ensemble can be described as the gemenge:

$$W^* = \{|S_i\rangle, P_i\}$$

where $P_i = |a_i|^2$ are $|S_i\rangle$ ensemble probabilities [1]; its density matrix denoted $\rho_S$.

Normally, D amplifies S amplitude to the level accessible for O processing, here for the simplicity it just doubles it. D pure states are described by Dirac vectors in two-dimensional Hilbert space $H_D$. Its basis is constituted by $|D_{1,2}\rangle$ eigenstates of $\Lambda$ 'pointer' observable with eigenvalues $\lambda_{1,2}$. The initial D state is:

$$|D_0\rangle = \frac{1}{\sqrt{2}}(|D_1\rangle + |D_2\rangle)$$

It is supposed that S, D interaction starts at $t_0$ and finishes effectively at some $t_1 = t_0 + \tau$. For Zurek S, D hamiltonian $H_{S,D}$ with the parameters tuned optimally for given $\tau$ value, the measurement of S eigenstate $|S_{1,2}\rangle$ induces the final product state [11]:

$$\Psi_{C,1,2} = |S_{1,2}\rangle|D_{1,2}\rangle$$

From the linearity of Schroedinger evolution it follows that for $a_{1,2} \neq 0$ such S, D interaction will result in S, D entangled final state [11]:

$$\Psi_{S,D} = \sum_{i=1}^{2} a_i |S_i\rangle|D_i\rangle$$

for initial $|D_0\rangle$, $\Phi_S$.

If $a_{1,2} \neq 0$, then D separate state $\Delta_D$ also can be formally defined, however, due to S, D entanglement, D and S properties can’t be completely factorized. It admitted usually that $\Delta_D$ coincides with D reduced state, i.e. the partial trace over 'external' DFs, which for $\Psi_{S,D}$ is equal to:

$$R^e_D = \sum_{i=1}^{2} |a_i|^2 |D_i\rangle\langle D_i|$$
in terms of density matrixes. This expression is rather obvious for $D$ statistical state, but for the individual state such definition seems to be controversial, first of all, because $R^o_D$ isn’t the pure state in $H_D$. This formal difficulty, in fact, is unimportant, because in all calculations $\Psi_{S,D}$ can be used in place of $\Delta_D$. The proper ansatz for the separate states will be discussed below, until then it will be no need to use it.

Our model of information system $O$ assumes that it’s the quantum object and analogously to $D$, its states are defined on two-dimensional Hilbert space $H_O$. Before the measurement starts $O$ initial state is equal to:

$$|O_0\rangle = \frac{1}{\sqrt{2}}(|O_1\rangle + |O_2\rangle)$$  \hspace{1cm} (7)

where $|O_{1,2}\rangle$ are eigenstates of $O$ 'internal pointer' observable $Q$ with eigenvalues $q_{1,2}$. $D$, $O$ interaction starts at some $t_2 > t_1$ and finishes at $t_3 = t_2 + \tau$, during this time interval the information about $D$ state is transferred to $O$. $D$, $O$ interaction is also described by Zurek hamiltonian $H_{D,O}$ with the same parameters as $H_{S,D}$. Under this conditions the incoming $\Psi_{S,D}$, $|O_0\rangle$ states will evolve into:

$$\Psi_{MS} = \sum_{i=1}^{2} a_i |S_i\rangle |D_i\rangle |O_i\rangle$$  \hspace{1cm} (8)

relative to external RF $O'$. Such triple decomposition is unique and in this sense defines $|O_{1,2}\rangle$ as the preferred basis (PB) of $O$ states [13]. In principle, $O$ can include other degrees of freedom (DFs) which participate in information processing, yet we shall suppose that during $D$, $O$ interaction they don’t interact with $O$ DFs described by $H_O$. The effects of MS decoherence by the environment will be considered separately, because our main results don’t depend on them directly.

### 3 Measurements and Information Acquisition

In information theory the most general and mathematically powerful approach to the measurements is introduced by the formalism of system self-description [3, 9]. To illustrate its meaning, let’s consider some information system $O$, which measures the parameters of arbitrary object $\Omega$. Then, $O$ can be considered formally as the subsystem of ‘complete’ system $\Xi = \{\Omega, O\}$, the set of $\Xi$ states denoted $N_T$, so that $N_O \subseteq N_T$ where $N_O$ is the set of $O$ states. When $\Omega$, $O$ interaction is finished, $\Xi$ will be in some final state $\Gamma$, which for the effective measuring set-up would be correlated with the initial $\Omega$ state $\varphi_{in}$. Therefore, the measurement of $\Omega$ state by $O$ in this approach is equivalent to $\Gamma$ measurement by $\Xi$ subsystem $O$. Hence it can be described as the mapping of $\Xi$ set $N_T$ to its subset $N_O$, i.e. to itself, so such process can be called $\Xi$ measurement from inside [10]. The restrictive map $M_O \Gamma \rightarrow R$ describes the restriction of $\Xi$ state to $O$, by the slight abuse of definitions, $R$ called also $O$ restricted state. The inverse map $M_{O}\overline{R} \rightarrow \Gamma$ is called the inference map. In practice, the information acquisition by $O$ always correlates with the change of its internal state, correspondingly, in our approach $R$ should be correlated with $\Gamma$. For
example, if $O$ is the atom, $R$ can be the state of its electronic shells, their excitations would ’record’ the incoming $\Omega$ signals.

The important property of $\Xi$ restrictions is formulated by Breuer Theorem: if for two arbitrary $\Xi$ states $\Gamma, \Gamma'$ their restrictions $R, R'$ coincide, then for $O$ this $\Xi$ states are indistinguishable [9]. It follows that for any nontrivial $\Xi, O$ at least one pair of such $\Xi$ states exist. For classical systems the incompleteness of $\Xi$ description by $O$ has the obvious reason: $O$ is only the part of $\Xi$, but it should describe its own state $R$ plus the state of $\Xi$ ’residual’, hence $N_T$ mapping to $N_O$ can’t be unambiguous [3, 9].

Constructing the quantum self-description formalism, we shall follow standard QM axiomatic, if no direct contradictions to it would appear. $O$ is $R$ internal state, so let’s suppose for the start that it defined on $\mathcal{H}_O$ only, below this assumption will be reconsidered.

It’s reasonable to study first the restrictions of $\Xi$ statistical states $\Gamma_{\text{st}}$ which derivation is more simple. The expectation values of all $O$ observables should be the same both for given $\Gamma_{\text{st}}$ and its $O$ restriction $R_{\text{st}}$ [8, 10]. Then, from QM correspondence between the set of such expectation values and statistical states it follows that the only consistent solution for $R_{\text{st}}$ is the partial trace of $\Xi$ state over $\Omega$ DFs, i.e. is $O$ reduced state. For our MS model $\Omega$ formally corresponds not to $S$, but to $S, D$ subsystem, the statistical restriction corresponding to MS state of (8) is equal to:

$$R_{\text{st}} = Tr_\Omega \rho_{\text{MS}} = \sum |a_i|^2 |O_i\rangle\langle O_i|$$  (9)

where $\rho_{\text{MS}} = |\Psi_{\text{MS}}\rangle\langle \Psi_{\text{MS}}|$.

Let’s start the study of individual $\Xi$ restrictions from the situations when $\Xi$ final state is the tensor product of $\Omega$, $O$ states. For our MS model they appear in the measurements of $S_z$ eigenstates $|S_i\rangle$. In this case, the final MS state is equal to:

$$\Psi_i = |s_i\rangle \otimes |D_i\rangle \otimes |O_i\rangle$$  (10)

Plainly, due to the factorization of $S, D$ and $O$ states, defined on their own Hilbert spaces, $\Psi_i$ restriction to $O$ is given by: $\xi_i = |O_i\rangle$. Really, $\Psi_i$ is $Q$ eigenstate with eigenvalue $q_i$, which is MS real (objective) property, thus $R_O$ possesses the same real property. Yet it means that $R_O$ is $Q$ eigenstate with the same eigenvalue, but the only such $O$ state is $|O_i\rangle$. Since $Q$ eigenvalues $q_i$ are $O$ real properties [1], the difference between the restricted states $|O_i\rangle$ is also objective. Therefore, it’s plausible to admit that in such measurement from inside $O$ can identify them as the different states characterized by IP $J^O = q_i$, it can be called the minimal recognition assumption (MRA). Due to such unambiguous correspondence with particular IPs, $|O_{1,2}\rangle$ constitute the ’recognition’ basis, the comparison with it will help to derive the measurement outcomes for other MS states.

Consider now the individual measurements of $S$ mixed ensemble $W^s$ of (2). By preparation, this is probabilistic mixture of $|S_{1,2}\rangle$ states, for each of them $S, D$ and $D, O$ interactions results in appearance of orthogonal MS states $\Psi_i$. Thus, such MS ensemble is described by the gemenge $W^{MS} = \{\Psi_i, P_i\}$. The corresponding individual MS state is random, i.e. it can change from event to event:

$$\Upsilon_{\text{MS}} = \Psi_1. \text{or.} \Psi_2$$  (11)
where the frequencies of $\Psi_{1,2}$ appearance are described by the same probabilities $P_{1,2}$. $\Psi_{1,2}$ restrictions were obtained above, so $O$ restriction of such random MS state is equal to:

$$R_{\text{mix}} = |O_1\rangle . \text{or. } |O_2\rangle$$  \hspace{1cm} (12)

Each $|O_i\rangle$ appears with the corresponding probability $P_i$, so that the ensemble of $O$ restricted states described by the gemenge $W^O = \{|O_i\rangle, P_i\}$ with density matrix:

$$\rho = \sum P_i |O_i\rangle \langle O_i|$$  \hspace{1cm} (13)

For nonfactorized individual $\Xi$ states Breuer assumed phenomenologically that, analogously to the statistical states, their restrictions are equal to $O$ reduced states $[9, 10]$. For $\Psi_{MS}$ of (8) it gives:

$$R_B = Tr_{\Omega} \rho_{MS} = \sum |a_i|^2 |O_i\rangle \langle O_i|$$  \hspace{1cm} (14)

Plainly, this ansatz excludes beforehand any kind of stochastic behavior for MS restriction. The resulting $R_B$ differs from $R_{\text{mix}}$ of (12) which describes the restriction of corresponding mixed MS ensemble $W^{MS}$. It supposes that, in principle, $O$ can discriminate the individual pure/mixed MS states ‘from inside’, because the condition of Breuer theorem is violated. Yet it will be shown below that the analysis of individual measurements permits to derive the MS restrictions to $O$ unambiguously without any ad hoc assumptions, yet these results will disagree with the former conclusion. Note also that even for this simple ansatz the inference map $M_O$ is ambiguous: all MS states of (8) with the same $|a_{1,2}|$ has the same restriction $R_B$ of (14), so it’s not possible, in principle, to choose just one of them from the knowledge of $R_B$ only.

4 Discrimination of Individual States

As was shown, the measurement of $S_z$ eigenstates $|S_{1,2}\rangle$ produces final MS states $\Psi_{1,2}$, which $O$ restrictions $\xi_{1,2}$ are equal to $|O_{1,2}\rangle$; MRA claims that they are identified by $O$ as IP $J^O = q_{1,2}^O$. Let’s calculate in this framework $O$ restricted state which appears in MS measurement of $|S_{1,2}\rangle$ superposition (1). In this case MS final state of (8) $\Psi_{MS} \neq \Psi_{1,2}$, but by itself, the formal difference of two MS individual states is the necessary but not sufficient condition for their discrimination by $O$. In addition, such MS measurement from inside should permit $O$ to detect the difference between the restrictions of those MS states to $O$. Previously, in MRA ansatz $J^O_s$ was formally expressed as $e^1 = q_i$ but, in principle, it can include more parameters $e^2, ..., e^m$, whose values are identical for $\xi_{1,2}$. According to Boolean logic, if for $O$ MS restriction $\xi_s$ differs from $\xi_{1,2}$, then $\xi_s$ can be identified by $O$ in the event of measurement as the different set of real parameters, i.e. IP $J^O_s \neq J^O_{1,2}$. Therefore, $J^O_s, J^O_{1,2}$ should include at least one parameter $e^j$, which value $g_0$ for $\xi_s$ is different from its values $g_{1,2}$ for $\xi_{1,2}$. In QM framework such $e^j$ should be some MS observable $G$ to which corresponds the linear, self-adjoint operator $\hat{G}$. In this case $\xi_s, \xi_{1,2}$ will be $G$ eigenstates with the eigenvalues $g_{0,1,2}$; so $O$ would discriminate $\xi_s$ from $\xi_{1,2}$, if $g_0 \neq g_{1,2}$. It was supposed earlier that MS restrictions to $O$ are defined on $\mathcal{H}_O$, so it follows that $G$ should
belong to the set (algebra) of \( O \) observables \( \mathcal{U}_O \). In our MS model \( \mathcal{U}_O \) is equivalent to observable algebra of spin \( \frac{1}{2} \) object, so any \( O \) nontrivial observable can be expressed as [2]:

\[
A = d_0Q + d_1Q^x + d_2Q^y
\]

(15)

where arbitrary real \( d_i \) coefficients are normalized to \( \sum d_i^2 = 1 \). \( O \) observables \( Q^{x,y} \) are conjugated to \( Q \) and obey the standard commutation relations: \([Q, Q^{x,y}] = i\hbar \beta Q^{y,x}\)

where \( \beta = 1 \) for \( Q^x \) commutator, \( \beta = -1 \) for \( Q^y \) one. \( \xi_{1,2} \) exhaust the spectra of \( Q \) eigenstates and so \( G \neq Q \). In the same time, considering the equation \( \hat{A}\xi_i = v_i \xi_i \) for real \( v_i \), it follows that \( \xi_i \) can’t be the eigenstate of any other \( A \neq Q \), hence there is no \( O \) observable \( G \) which can satisfy to all our demands simultaneously.

Thus, only \( \xi_{1,2} \) states can be unambiguously discriminated by \( O \) in such MS measurement from inside, there is no IP \( J_O \neq J_{O1,2} \) which can be correctly ascribed to \( \xi_s \). Since any alternative outcomes for MS measurement by \( O \) are supposedly impossible in our formalism, in particular, ‘undefined’ or ‘incomparable’ outcome, the only consistent \( J_O \) ansatz is equal to:

\[
J_O = q_1 \text{ or } q_2.
\]

As the result, \( O \) can’t distinguish \( \xi_s \) and \( \xi_i \) states and \( O \) restriction of \( \Psi_{MS} \) of (8) is equivalent to:

\[
\xi_s = \xi_{1,2}
\]

(16)
i.e. it’s equivalent to \( R_{mix} \) of (12). It suppose that MS restrictive map \( M_O \) is stochastic, and because of it, the corresponding inference map \( M_O^{-1} \) is ambiguous.

Summing up, it means that the ensemble of \( O \) restricted states \( \xi_s \) is described by the gemenge \( W^a = \{|\Omega_i\rangle, P'_i\} \) where probabilities \( P'_i \) should be calculated. Remind that \( W^a \) statistical state \( R_{st} \) is described by density matrix (9), from that the relation \( P'_i = |a_i|^2 \) follows. Thus the probabilities in such subjective gemenge obey to Born rule. Note that Born rule for outcome probabilities isn’t self-obvious in QM formalism, it should be independently derived in any new theory of measurement [1, 15].

### 5 Measurement Correlations and Joint Observables

In our calculations it was supposed that \( O \) restrictions of MS states can be discriminated by \( O \) observables only which seems quite reasonable. Yet to be safe, let’s relax this condition and check full MS observable algebra in search of observables which can discriminate the pure and mixed MS ensembles described above. As follows from the properties of statistical restrictions considered in sect. 3, if the restriction \( \xi_s \) of some state \( \Psi_a \) is the eigenstate of some observable \( \Lambda_a \), then \( \Psi_a \) is also \( \Lambda_a \) eigenstate [14]. In this framework MS states can be used in search of suitable observable \( G \), for which the following relations should be fulfilled simultaneously:

\[
\hat{G}\Psi_{MS} = g_0\Psi_{MS}
\]

\[
\hat{G}\Psi_{1,2} = g_{1,2}\Psi_{1,2}
\]

(17)
for \( g_0 \neq g_{1,2} \). However, in our MS model \( \Psi_{MS} = \sum a_i \Psi_i \), and the substitution of second equation into the first one gives: \( g_0 = g_1 = g_2 \). Hence no MS observable \( G \) possesses the different eigenvalues for \( \Psi_{MS} \) of (8) and \( \Psi_{1,2} \) of (10), so even all MS observables would be available for \( O \) measurement from inside it will not permit \( O \) to discriminate such MS states. Only MS observables corresponding to the nonlinear operators can reveal the difference between \( \Psi_{MS} \) and \( \Psi_{1,2} \) restrictions to \( O \), but their measurability contradicts to standard QM axiomatic. Note that if only \( O \) observables \( G \) are considered, then our previous results of (15) are reproduced by this ansatz.

These calculations for the measurement from inside are applicable to arbitrary MS observable, let’s compare them with its measurement by some external RF \( O' \). It’s argued often that if it can be shown that MS final state after \( S \) measurement is pure for external RF \( O' \), then it excludes the possibility of its random outcomes for \( O \). As was shown, this supposedly is untrue for MS measurement from inside, yet such reasoning results in frequent confusions, so it’s instructive to consider it here. For MS final states the difference between their pure and mixed ensembles with the same \( \tilde{S}_z \) can be revealed by interference term (IT) observable which are the joint \( S,D,O \) observables [1]. Such IT can’t be measured by \( O \) ‘from inside’ for described MS set-up, which tuned to the optimal \( \tilde{S}_z \) measurement. IT general ansatz is rather complicated, here only symmetric IT will be exploited:

\[
B = |O_1\rangle\langle O_2|D_1\rangle\langle D_2||S_1\rangle\langle S_2| + |O_2\rangle\langle O_1|D_2\rangle\langle D_1||S_2\rangle\langle S_1| \tag{18}
\]

Being measured by external \( O' \) via its interaction with \( S,D,O \), for arbitrary \( B \) it gives \( \tilde{B} = 0 \) for \( \Psi_{1,2} \) probabilistic mixture \( W^{MS} \), but for some MS states \( \Psi_{MS} \) of (8) one obtains that \( \tilde{B} \neq 0 \). In particular, for symmetric \( S \) state \( \Phi_s \) of (1) with \( a_{1,2} = \frac{1}{\sqrt{2}} \), the resulting \( \Psi_{MS} \) of (8) is \( B \) eigenstate with eigenvalue \( b_1 = 1 \). \( B \) possesses also two other eigenvalues \( b_{0,2} \), of them only \( b_2 = -1 \) is important in this case. The probability \( P_B(b_{1,2}) = .5 \) for \( \Psi_{1,2} \) mixture with \( \tilde{S}_z = 0 \), the intersection of its \( b \) probability distribution with the one for \( \Psi_{MS} \) results in their overlap \( K_b = .5 \). Hence, in accordance with our previous conclusions, the pure/mixed MS states with the same \( \tilde{S}_z \) can be discriminated even by external \( O' \) only statistically, there is no MS observable which can discriminate them ‘event by event’. It demonstrates that the operational difference between such MS states is relatively small. The properties of other MS ITs are similar to the symmetric one, but the difference between pure and mixed MS ensembles is less pronounced. The joint \( S,D,O \) observables posses the similar properties, in particular, symmetric IT \( B_{S,D} \) can be obtained from (18), if to remove all \( O \) terms. It can be measured by \( O \) via the simultaneous interaction with \( S \) and \( D \).

The proposed theory admits that in general the same MS pure state can look stochastic for \( O \) measuring it from inside, but in the same time can evolve linearly relative to some external \( O' \). It was shown earlier that such situation by itself doesn’t lead to any experimentally observed inconsistency for the results of measurements which can be performed by \( O \) and \( O' \) [7, 8]. The consistent description of this situation can be given by the formalism of unitary nonequivalent representations [14]. In particular, in our model MS restricted states are defined on \( \mathcal{H}_C \), which is the subspace of MS Hilbert space \( \mathcal{H}_O \). Correspondingly, the transformation of MS states
between $O$ and $O'$ will be nonunitary, i.e. there is no unitary operator $\hat{U}$, for which $\hat{H}_c=\hat{U}\hat{H}_O\hat{U}^{-1}$. Yet only if such $\hat{U}$ exists, MS state, which is pure for $O'$, would be also with necessity the pure state for $O$ [2].

It’s well known that the decoherence of pure states by its environment $E$ is the important effect in quantum measurements [11]. In the simplified calculations its formalism permits to suppose that MS, $E$ start to interact only at the final stage of $S$ measurement. If $D, O$ interact with $E$ only at $t > t_3$, for the typical decoherence hamiltonian it follows that $\Psi_{MS}$ of (8) will evolve into MS, $E$ final state:

$$\Psi_{MS,E} = \sum_{i=1}^{2} a_i |S_i\rangle|D_i\rangle|O_i\rangle \prod_j |E_j\rangle$$

where $E_j$ are $E$ elements, $N_E$ is $E_j$ total number. If an arbitrary $O$ pure state $\Psi_O$ is prepared, it will also decohere in a very short time into the analogous $|O_i\rangle$ combinations entangled with $E$. Thus, of all pure $O$ states, only $|O_i\rangle$ states are stable relative to $E$ decoherence. Hence it advocates the choice of such states as $O$ preferred basis, since in such environment $O$ simply can’t percept and memorize any other $O$ pure state during any sizable time interval. $D, O$ decoherence by $E$ makes the considered discrimination of pure and mixed final MS states by external $O'$ quite complicated, but the analysis of corresponding ITs show that their main properties don’t change principally [7, 8].

6 Information Incompleteness and Measurements

Now we shall demonstrate that our results for $S_z$ measurement from inside by $O$ can be obtained avoiding the direct use of self-description formalism or, at least, its most sophisticated part. In particular, it will be argued that the incompleteness of information carried by individual $S$ state stipulates the randomness in $S_z$ measurement by $O$, and so this effect is in some sense is objective and observer-independent. For this study it’s worth to have the statistical estimate of state discrimination in the measurement of particular observable. Such statistical measure for two finite-dimensional states $\rho_{1,2}$ and observable $\Lambda$ can be described as the coincidence rate (overlap) of their $\lambda_i$ eigenvalue distributions [5]:

$$K(\Lambda) = \sum_i [w_1(\lambda_i)w_2(\lambda_i)]^{\frac{1}{2}}$$

here $w_{1,2}(\lambda_i) = Tr\rho_{1,2}\Pi(\lambda_i)$ where $\Pi(\lambda_i)$ is the orthogonal projector on $\lambda_i$. In particular, the difference between the pure and mixed $S$ states is indicated by $S$ observables, which expectation value is sensitive to the presence of the component interference. For the regarded $S$ pure/mixed states with the same $\bar{S}_z$ they are $S_{x,y}$ linear forms. For example, if $\frac{a_1}{a_2}$ is real, the maximal distinction reveals $S_x$ observable, for which $|\bar{S}_x| = |a_1||a_2|$ for the pure states and $\bar{S}_x = 0$ for the mixture. In this case, their overlap

$$K(S_x) = 1 - |a_1||a_2|$$
For the arbitrary \( a_{1,2} \) the maximal discrimination of pure and mixed \( S \) states gives the expectation value of observable:

\[
S_\gamma = S_x \cos \gamma + S_y \sin \gamma
\]  

(21)

where \( \gamma \) is the \( \psi \) quantum phase between \( |s_{1,2} \rangle \) components. The value of \( r_p = 2|\bar{S}_\gamma| \), which lays between 0 and 1, can be chosen as \( S \) purity rate. These estimates indicate that even the incoming pure and mixed \( S \) states with the same \( \bar{S}_z \) differ, in fact, only statistically with the minimal overlap 50%, but not on 'event by event' basis. Hence in such case the purity can be measured consistently only for \( S \) ensemble and not for individual state. Let’s consider the final state \( \Psi_{S,D} \) of (5) and corresponding mixed ensemble induced by gemenge \( W^s \) of (2) with the same \( \bar{S}_z \). For \( D \) pointer observable \( \Lambda \) and all \( D \) observables conjugated to it, their overlap between pure and mixed states \( K(\Lambda), K(\Lambda_{x,y,\gamma}) \) is equal to 1 for arbitrary \( a_{1,2} \); here \( \Lambda_{x,y,\gamma} \) are defined by the analogy with \( S \) observables of (21). Hence even statistically no information about \( S \) state purity is transferred to \( D \) via MS information channel, which is tuned to the optimal \( S_z \) value measurement. The similar results were obtained for information transfer via quantum channels [5].

In this framework let’s consider the information content of individual \( S \) state. Plainly, \( S_z \) eigenstate \( |S_i \rangle \) transfers 1 bit of information in \( S_z \) measurement corresponding to the choice of two possible \( S_z \) values \( \pm \frac{1}{2} \). Correspondingly, the overlap (20) of such states \( K(S_z) = 0 \). In this vein let’s calculate the amount of information about \( S \) state purity \( I_p \) for symmetric \( \Phi_S \) with \( a_{1,2} = \frac{1}{\sqrt{2}} \). As was shown above, the minimal overlap with the corresponding \( |S_i \rangle \) mixture \( W^s \) of (2) is dispatched by \( S_x \) observable and gives \( K(S_x) = .5 \) for two such ensembles. In this case one can conclude that for such \( S \) states the information \( I_p \) is described only by 'half-bit' of information per event at MS input. Hence even if \( O \) in place of \( S_z \) would measure \( S_z, S \) purity can’t be defined in a single event. The same or lesser \( I_p \) value can be obtained for arbitrary \( \Phi_S \) if \( O \) knows the phase \( \gamma \) of (21). If it’s unknown for \( O \) then \( I_p \) at MS input will be formally two time less, i.e. less than \( \frac{1}{4} \) bit. These results demonstrate that the amount of information about purity carried by the individual \( S \) state is principally insufficient for discrimination of pure and mixed \( S \) ensembles on 'event by event' basis, because the operational difference between the pure and mixed states is too small for that. Plainly, no \( S \) quantum interaction with other objects can enlarge its amount [5]. For individual states the difference between such pure and mixed \( S \) states is described only by the observables related to nonlinear operators. For standard QM observables such purity information can be extracted only from the simultaneous joint measurements of large ensembles [1, 15].

Let’s study how such incomplete information about \( S \) purity is transferred to \( D \) and then to \( O \) and what is its effect. Remind that in our MS model after some time moment \( t_1 \) \( S \) and \( D \) stop to interact, whereas \( D \) and \( O \) start to interact at \( t_2 > t_1 \), so that even for arbitrary \( H_{D,O} \) interaction \( O \) can measure directly only \( D \) observables. Really, at \( t > t_2 \) the object \( S \) can be miles away from \( D \) and \( O \), in this case \( S \) and \( S,D \) observables surely will be unavailable for \( O \) directly. Thus, one can regard MS as the information channel, which transfers first \( S \) signal to \( D \), and after that \( D \) signal
to $O$. In MS model with $H_{D,O}$ exploited here the measurement of $S_z$ eigenstates $|s_i⟩$ induces factorized MS state $Ψ_i$ of (10); in this case $O$ interacts with $D$ separate state $Δ_D = |D_{1,2}⟩$ and, as was shown, $O$ percepts it as IPs $J_{1,2}^{O} = q_{1,2}$.

Consider now $S_z$ measurement for the incoming $|S_i⟩$ probabilistic mixture (gemenge) $W_s$ of (2) with some $S_z$. When $S,D$ interaction is finished at $t_1$, then $S,D$ ensemble becomes the mixture (gemenge) of $Ψ_{1,2}^{C}$ of (4) with $Q = S_z$. As was shown above, at $t > t_3$ when $D$ measurement by $O$ is finished, its result will be perceived by $O$ as $J_1^{O}$ or $J_2^{O}$ with the probabilities $P_{1,2}$. In the same vein, consider the possible outcome for pure $S$ state $Φ_s$ of (1) with the same $S_z$. In this framework at the final stage of $S_z$ measurement, $D$ separate state $Δ_D$ interacts with $O$, which results in appearance of some $O$ IP $J_1^{O}$, which can either coincide with one of the basic $O$ IPs $J_{1,2}^{O}$ or differ from them. As was shown in sect. 5, $Ψ_{S,D}$ of (5) and $Ψ_{1,2}^{C}$ can’t be the nondegenerate eigenstates of the same $D$ observable $G$. Thus, even if $O$ can measure all $D$ observables simultaneously, it wouldn’t permit $O$ to detect the difference between such pure and mixed $S$ ensembles; so for such pure ensemble $O$ would perceive in the individual events IP: $J_1^{O} = J_1^{O} . or J_2^{O}$ with probabilities $P_{1,2}$ correspondingly, their values are defined by $S_z$. Really the opposite result, i.e the observation by $O$ of such difference would mean that $O$ can measure $D$ observable which corresponds to the nonlinear operator, but this contradicts to standard QM. The obtained results don’t mean that in the pure case the separate $D$ state $Δ_D$ of (6) is the objective probabilistic mixture of $|Δ_D⟩$, rather $Δ_D$ can be characterized as their ‘weak’ superposition stipulated by the entanglement of $S,D$ states. In this framework $R_D$ of (6) can be regarded as the symbolic expression of this difference. Yet the complete description of $D$ properties is performed only by $S,D$ state as the whole, so that some of them are described by the nonlocal $S,D$ observables. However, no measurement performed on $D$ only can reveal the difference of $Ψ_{S,D}$ from the corresponding $Ψ_{S,D}^t$ mixture. Only the measurement of some joint $S,D$ observables, like $B^{S,D}$ can reveal it, but also only statistically [1].

These results indicate that in this set-up $D$ state doesn’t contain the information about $S$ purity and so it principally can’t be dispatched to $O$. This conclusion doesn’t change even if to admit that $O$ can measure the joint $S,D$ observables. If the information about $S$ purity isn’t transferred by $D$, then $O$ functioning by itself plays the minimal role in the appearance of the outcome randomness. The only feature which $O$ should possess is the proper discrimination of $|D_{1,2}⟩$ states as $J_{1,2}^{O}$. Those semi-qualitative arguments aren’t sufficient for the consistent proof of measurement randomness without exploit of self-description formalism. Yet they evidence that such randomness can be the consequence of fundamental information incompleteness of individual quantum states and so can be regarded as observer-independent.

7 Discussion

The presented calculations, in our opinion, reveal the origin of the principal randomness of quantum measurements. As follows from our considerations, the structure of QM Observable Algebra which includes only the linear, self-adjoint operators corresponds to Boolean logics of signal recognition. In our case its operands correspond
to IP set \( \{J^O\} \) [12]. In particular, it excludes the simultaneous \( O \) registration of two opposite outcomes of measurement, which is the essence of 'Schroedinger cat' paradox. It permits to suppose that the independent Reduction postulate is unnecessary in QM. It follows that all the measurement features can be deduced from QM axiom which postulates QM observables to be the linear, self-adjoint operators and settles the relation between their eigenstates and the outcomes of corresponding measurements. In this approach the randomness in quantum measurements is related to the incompleteness and undecidability aspects of information theory, their studies were initiated by notorious Gödel theorem [3]. The considered phenomenon can be called the subjective collapse of quantum state, because MS as the whole is in the pure state throughout the measurement relative to external observer \( O' \). Correspondingly, the considered effect represents also the subjective irreversibility, induced by the incompleteness of \( O \) information about MS state and itself.

These considerations are closely related to the question whether this theory is applicable to human observer \( O \), in particular, whether in this case IP \( J^O \) can describe the true \( O \) 'impressions' concerned with the measurements' outcomes? This is open problem, but at the microscopic level the human brain, as the dynamical system, should obey QM laws as any other object, so we don’t see any serious reasons to make the exceptions [3, 7]. In our model the detection of eigenstate \( |D_i\rangle \) by \( O \) can be associated with the excitation of some \( O \) internal levels. This process is similar to the excitation of brain molecules during the acquisition of external signal. In this vein MRA used in our approach looks reasonable and consistent. Note also that in our theory the brain or any other processor \( O \) plays only the passive role of signal receiver. The real effect of information loss which stipulates the outcome randomness occurs 'on the way' when the quantum signal passes through MS information channel. Hence the observer’s consciousness, in principle, can’t have any relation to it.

We conclude that standard Schroedinger QM formalism together with the information-theoretical considerations permit to derive the 'subjective' collapse of pure states without implementation of independent Reduction postulate into QM axiomatic. In our approach the main sources of randomness are the principal constraints on the transfer of information in \( S \to O \) information channel and the incompleteness of information about \( S \) purity carried by individual \( S \) state. This 'lost' information characterizes the purity of \( S \) state, because of its loss, \( O \) can’t discriminate the pure and mixed \( S \) states. As the result of this information incompleteness, the randomness of measurement outcomes appear, the probabilities of \( O \) outcomes obeys to Born rule.

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