Norm Inequalities Related to Heinz and Logarithmic Means

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Abstract

In this paper, we got some refinements of the norm inequalities related to the Heinz mean and logarithmic mean.

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1 Introduction

There are several means that interpolate between the geometric and arithmetic means. For instance, the Heinz mean $H_t(a, b)$, defined by

$$H_t(a, b) = \frac{a^{1-t}b^t + a^t b^{1-t}}{2} \quad \text{for} \quad 0 \leq t \leq 1.$$ 

In 1993, Bhatia-Davis [2] obtained that if $A$, $B$ and $X$ are $n \times n$ matrices with $A$, $B$ positive semidefinite, then for every unitarily invariant norm $\|\cdot\|$, 

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \frac{1}{2} \|A^{1-t}XB^t + A^t XB^{1-t}\| \leq \frac{1}{2} \|AX + XB\|.$$  \hspace{1cm} (1.1)

The logarithmic mean $L(a, b)$, defined by

$$L(a, b) = \frac{a - b}{\log a - \log b} = \int_0^1 a^t b^{1-t} dt,$$

also interpolates the geometric and arithmetic means. In 1999, Hiai-Kosaki [3] proved the following inequality

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \left\| \int_0^1 A^v XB^{1-v} dv \right\| \leq \frac{1}{2} \|AX + XB\|.$$  \hspace{1cm} (1.2)
Moreover, in 2006, Drissi [4] proved that the following Heinz-logarithmic inequality
\[
\left\| A^{1-t}XB^t + A^tXB^{1-t} \right\| \leq 2 \left\| \int_0^1 A^vXB^{1-v}dv \right\|
\] (1.3)
holds for \( \frac{1}{4} \leq t \leq \frac{3}{4} \).

A complex-valued function \( \varphi \) on \( \mathbb{R} \) is said to be positive definite if the matrix \([\varphi(x_i - x_j)]\) is positive semidefinite for all choices of real numbers \( x_1, x_2, \ldots, x_n \), and \( n = 1, 2, \ldots \). Set \( M(a, b) \) and \( N(a, b) \) are two symmetric homogeneous means on \((0, \infty) \times (0, \infty)\). \( M \) is said to strongly dominate \( N \), denoted by \( M \ll N \), if and only if the matrix
\[
\begin{bmatrix}
M(\lambda_1, \lambda_j) \\
N(\lambda_1, \lambda_j)
\end{bmatrix}_{i,j=1,\ldots,n}
\]
is positive semidefinite for any size \( n \) and \( \lambda_1, \ldots, \lambda_n > 0 \). Drissi [4] proved that for \( a, b \geq 0 \), \( H_t(a, b) \ll L(a, b) \) if and only if \( \frac{1}{4} \leq t \leq \frac{3}{4} \). In general, the inequality \( M \ll N \) could be stronger than the Löwner’s order inequality \( M \leq N \).

For more operator or norm inequalities related to the Heinz mean and logarithmic mean we refer the readers to [7–9] and the references therein.

2 Main results

Lemma 2.1. For \( \sinh x \) and \( \cosh x \), we have

(i) If \( |\beta| > |\alpha| > 0 \), then the function \( \frac{\cosh \alpha x}{\cosh \beta x} \) is positive definite.

(ii) If \( |\beta| > |\alpha| > 0 \) with \( \alpha, \beta \) the same sign, then \( \frac{\sinh \alpha x}{\sinh \beta x} \) is positive definite.

(iii) If \( \beta > 0 \) and \( |\alpha| < \beta/2 \), then \( \frac{\beta x \cosh \alpha x}{\sinh \beta x} \) is positive definite.

Proof. We follow a similar argument as in Chapter 5 of [1]. From the product representations in p. 147-148 of [1],
\[
\frac{\sinh x}{x} = \prod_{k=1}^{\infty} \left( 1 + \frac{x^2}{k^2 \pi^2} \right), \quad \cosh x = \prod_{k=0}^{\infty} \left( 1 + \frac{4x^2}{(2k + 1)^2 \pi^2} \right), \tag{2.1}
\]
we have
\[
\frac{\sinh \alpha x}{\sinh \beta x} = \frac{\alpha}{\beta} \prod_{k=1}^{\infty} \frac{1 + \alpha^2 x^2/k^2 \pi^2}{1 + \beta^2 x^2/k^2 \pi^2}, \quad \frac{\cosh \alpha x}{\cosh \beta x} = \prod_{k=0}^{\infty} \frac{1 + 4\alpha^2 x^2/(2k + 1)^2 \pi^2}{1 + 4\beta^2 x^2/(2k + 1)^2 \pi^2}. \tag{2.2}
\]
Each factor in the product is of the form
\[
\frac{1 + a^2 x^2}{1 + b^2 x^2} = \frac{a^2}{b^2} + \frac{1 - a^2/b^2}{1 + b^2 x^2}, \quad b^2 > a^2.
\]
Since \( 1/(1+b^2 x^2) \) is positive definite [1, 5.2.7], it follows that the functions \( \cosh \alpha x / \cosh \beta x \) is positive definite for \( |\beta| > |\alpha| > 0 \), and \( \sinh \alpha x / \sinh \beta x \) is positive definite for \( |\beta| > |\alpha| > 0 \) with \( \alpha, \beta \) the same sign.
Since
\[
\frac{\beta x \cosh \alpha x}{\sinh \beta x} = \frac{\beta x}{\sinh \frac{2}{\beta} x} \cdot \frac{\cosh \alpha x}{\cosh \frac{2}{\beta} x}
\]
and \(x/\sinh x\) is positive definite [1, 5.2.9], it follows from the above argument that when \(\frac{\beta}{2} > |\alpha|\) the function \(\frac{\beta x \cosh \alpha x}{\sinh \beta x}\) is positive definite. \(\square\)

Now, we define
\[
L_s(a, b) = \frac{a^{1-s}b^s - a^s b^{1-s}}{(1 - 2s)(\log a - \log b)} = \frac{1}{1 - 2s} \int_s^{1-s} a^v b^{1-v} dv
\]
for \(a, b > 0\), and \(0 \leq s < \frac{1}{2}\). When \(s = 0\), it is the logarithmic mean. So we can call it the

generalized logarithmic mean. And we also have \(\lim_{s \to \frac{1}{2}} L_s(a, b) = a^\frac{1}{2} b^\frac{1}{2}\).

**Theorem 2.2.** For Heinz mean and the generalized logarithmic mean, we have

(i) If \(0 \leq s < 1/2\), and \(|1 - 2t| < \frac{2a}{1 - 2s}\), then \(H_t(a, b) \ll L_s(a, b)\).

(ii) If \(|1 - 2t| < |1 - 2s|\), then \(H_t(a, b) \ll H_s(a, b)\).

(iii) If \(0 \leq s < t < 1/2\), or \(1 \geq s > t > 1/2\), then \(L_t(a, b) \ll L_s(a, b)\).

**Proof.** By definition, \(H_t(a, b) \ll L_s(a, b)\) if
\[
[y_{i,j}] = \left[ \begin{array}{c}
H_t(\lambda_i, \lambda_j) \\
L_s(\lambda_i, \lambda_j)
\end{array} \right]_{i,j=1,...,n}
\]
is positive semidefinite. Set \(\lambda_i = e^{x_i}\) and \(\lambda_j = e^{x_j}\), with \(x_i, x_j \in \mathbb{R}\). Then
\[
y_{i,j} = (1 - 2s) \frac{(x_i - x_j)(e^{(1-2t)(x_i-x_j)} + e^{(1-2t)(x_j-x_i)})}{e^{(1-2s)(x_i-x_j)} - e^{(1-2s)(x_j-x_i)}}
\]
Thus the matrix \([y_{i,j}]\) is congruent to one with entries
\[
\frac{\beta(x_i-x_j) \cosh(\alpha(x_i-x_j))}{\sinh(\beta(x_i-x_j))},
\]
where \(\alpha = 1 - 2t, \beta = 1 - 2s\). Hence \([y_{i,j}]\) is positive semidefinite if and only if the function \(\frac{\beta x \cosh \alpha x}{\sinh \beta x}\) is positive definite, which by lemma 2.1 is correct.

Similarly, we have \(H_t(a, b) \ll H_s(a, b)\) if \(\frac{\cosh \alpha x}{\cosh \beta x}\) is positive definite, and \(L_t(a, b) \ll L_s(a, b)\) if \(\frac{\sinh \alpha x}{\sinh \beta x}\) is positive definite. \(\square\)

**Theorem 2.3.** Let \(A, B\) be any positive matrices. Then for any matrix \(X\) and for \(0 \leq s < 1/2\) and \(|1 - 2t| < (1 - 2s)/2\), we have
\[
\left\| A^{1-t}XB^t + A^tXB^{1-t} \right\| \leq \frac{2}{1 - 2s} \left\| \int_s^{1-s} A^v XB^{1-v} dv \right\|.
\]

(2.3)
Theorem 2.4. Let $1 \leq s < t$. The first inequality of (1.3) in the paper, where $A = B$, now replacing $B$ with $X$. A well-known result related to the Schur multiplier norm [6, Theorem 5.5.18, 5.5.19] says that if $Y$ is any positive semidefinite matrix, then for all matrix $A$ and for every unitarily invariant norm, by Theorem 2.2, $Y$ is positive semidefinite. Applying (2.4), we have

$$
\|Y \circ X\| \leq \max_{i} y_{ii} \|X\|
$$

for every unitarily invariant norm. By Theorem 2.2, $Y$ is positive semidefinite. Applying (2.4), we have

$$
\|A^{1-t}XA^t + A^tXA^{1-t}\| \leq \frac{2}{1 - 2s} \left\| \int_{s}^{1-s} A^t XA^{1-t} dv \right\|.
$$

(2.5)

Now replacing $A$ and $X$ in the inequality (2.5) by the 2 by 2 matrices \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) and \( \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \). This gives the desired inequality (2.3).

When $s = 0$, we get Drissi’s result (1.3). Moreover, when $s = 0, t = 1/2$, we get the first inequality of (1.2).

Theorem 2.4. Let $A, B$ be any positive matrices. Then for any matrix $X$ and for $0 \leq s < 1/2$, we have

$$
\left\| \int_{s}^{1-s} A^t XB^{1-t} dv \right\| \leq \frac{1 - 2s}{2} \|A^{1-s} XB^s + A^s XB^{1-s}\|.
$$

(2.6)

Proof. Suppose $A$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. Then we have

$$
\int_{s}^{1-s} A^tXA^{1-t} dv = Y \circ (A^{1-s}XA^s + A^sXA^{1-s}),
$$

where $Y$ is the matrix with entries

$$
y_{i,j} = \frac{\lambda_i^{1-s}\lambda_j^s - \lambda_i^s\lambda_j^{1-s}}{\log \lambda_i - \log \lambda_j} = \frac{(1 - 2s) L_s(\lambda_i, \lambda_j)}{2H_s(\lambda_i, \lambda_j)}.
$$

By a similar argument as in the proof of Theorem 2.2, we know that $Y$ is positive definite if and only if

$$
\beta \sinh \beta x = \frac{\beta \tanh \beta x}{2 \beta x \cosh \beta x} = \frac{2 \beta x \tanh \beta x}{\beta x}
$$
is positive definite for $\beta = 1 - 2s > 0$, because $\tanh x/x$ is positive definite (see Bhatia [1, 5.2.11]). Thus Applying (2.4) we have

$$\left\| \int_{s}^{1-s} A^v X A^{1-v} \, dv \right\| \leq \frac{1 - 2s}{2} \left\| A^{1-s} X A^s + A^s X A^{1-s} \right\|. \tag{2.7}$$

Hence the desired result follows. \hfill \Box

When $s = 0$, we get the second inequality of (1.2).

**Theorem 2.5.** Let $A, B$ be any positive matrices. Then for any matrix $X$ and for $|1-2t| < |1-2s|$ with $1 - 2t, 1 - 2s$ the same sign, we have

$$\left\| A^{1-t} X B^t - A^t X B^{1-t} \right\| \leq \frac{1 - 2t}{1 - 2s} \left\| A^{1-s} X B^s - A^s X B^{1-s} \right\|. \tag{2.8}$$

**Proof.** Suppose $A$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. Then we have

$$A^{1-t} X A^t - A^t X A^{1-t} = Y \circ (A^{1-s} X A^s - A^s X A^{1-s}),$$

where $Y$ is the matrix with entries

$$y_{i,j} = \frac{\lambda_i^{1-t} \lambda_j^t - \lambda_i^t \lambda_j^{1-t}}{\lambda_i^{1-s} \lambda_j^s - \lambda_i^s \lambda_j^{1-s}}.$$

Put $\lambda_i = e^{x_i}$ and $\lambda_j = e^{x_j}$, with $x_i, x_j \in \mathbb{R}$. Then $Y$ is congruent to the matrix with entries

$$\frac{\sinh(\alpha \frac{x_i - x_j}{2})}{\sinh(\beta \frac{x_i - x_j}{2})}$$

where $\alpha = 1 - 2t, \beta = 1 - 2s$. Since $(\sinh x/x)/(\sinh \beta x)$ is positive for $|\beta| > |\alpha| > 0$ with $\alpha, \beta$ the same sign, it follows that $Y$ is positive definite. Applying the inequality (2.4), we have

$$\left\| A^{1-t} X A^t - A^t X A^{1-t} \right\| \leq \frac{1 - 2t}{1 - 2s} \left\| A^{1-s} X A^s - A^s X A^{1-s} \right\|. \tag{2.9}$$

Hence the desired result follows. \hfill \Box

Set $s = 0$ and $s = 1$, and combine the conclusions, we have the following inequality proved by Bhatia-Davis [2],

**Corollary 2.6.** Let $A, B$ be any positive matrices. Then for any matrix $X$ and for $0 \leq t \leq 1$ we have

$$\left\| A^{1-t} X B^t - A^t X B^{1-t} \right\| \leq |1 - 2t| \left\| AX - XB \right\|. \tag{2.10}$$

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