Tensor-based derivation of standard vector identities

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Abstract. Vector algebra is a powerful and needful tool for Physics but unfortunately, due to lack of mathematical skills, it becomes misleading for first undergraduate courses of science and engineering studies. Standard vector identities are usually proved using Cartesian components or geometrical arguments, accordingly. Instead, this work presents a new teaching strategy in order to derive symbolically vector identities without analytical expansions in components, either explicitly or using indicial notation. This strategy is mainly based on the correspondence between three-dimensional vectors and skew-symmetric second-rank tensors. Hence, the derivations are performed from skew tensors and dyadic products, rather than cross products. Some examples of skew-symmetric tensors in Physics are illustrated.

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1. Introduction

Vector analysis [1] plays a key role in many branches of Physics: Mechanics, Fluid dynamics, Electromagnetism theory, .., because it is a powerful mathematical tool that can express physical laws in invariant forms. Hence, learning of vector skills must be a priority goal for science and engineering students of undergraduate courses [2]. However, understanding of vectors often becomes intricate [3] due to the underlying mathematics, which can even hide the meaning of the involved physical quantities [4]. Common pitfalls are originated by the lack of mathematical resources for deriving vector identities.

In undergraduate physics courses, the standard identities of introductory vector algebra are mostly proved either from geometrical arguments [5] or analytically using rectangular Cartesian components, and at best from the indicial notation [1, 6]. Unlike the analytical proofs, geometrical derivations are performed regardless of the coordinate system. Instead, the demonstrations based on the indicial notation are more elegant and compact although they require to handle complex symbolic expressions, without any physical insight into the problem at hand.

We present an alternate approach of vector identity derivation based on the use of tensors and dyadic products rather than cross products. Tensor algebra using matrix format [7] become less cumbersome than indicial notation and further, the operations involving second-order tensors are readily understood as transformations of vectors.

Hereafter, only for illustrative purposes, just first- and second-rank Cartesian tensors are considered, i.e. the three-dimensional space is Euclidean. Hence, the contravariant and covariant components are identical to one another because the metric tensor and conjugate metric tensor are equal to the identity matrix. Nevertheless, the derivations compiled in this text are equally valid for other metrics with minor modifications.

2. Dyadics

Aside from the well-known dot product (particular case of the inner product), a dyadic is formed by the outer or direct product of two vectors. The dyadic between the vectors \( \vec{a} \) and \( \vec{b} \) produces the following second-order tensor [7] of nine components:

\[
(\vec{a}\vec{b})_{ij} \triangleq a_i b_j
\]

with \( i, j = 1, 2, 3 \) and where \( a_i \) and \( b_j \) are the respective Cartesian components of both operating vectors. Unlike the inner product or contraction, symbolized by a point, and the double inner product, symbolized by colon, no specific symbol is employed for the dyadic product.

Since an arbitrary vector can be expressed as a linear combination of the unit vector basis \( \{\hat{e}_i\}_{i=1,2,3} \), an arbitrary dyadic can be written into components from the concerning unit dyads \( \{\hat{e}_i\hat{e}_j\}_{i,j=1,2,3} \) as follows:

\[
\vec{a}\vec{b} = (\vec{a}\vec{b})_{ij} \hat{e}_i\hat{e}_j
\]
where the summation convention is in effect for the repeated indices \([1]\). If the unit vectors \(\hat{e}_i\) are mutually orthogonal, a special dyadic called the identical dyadic arises:

\[
1 = \hat{e}_i \hat{e}_i
\]

where the summation convention is again invoked. This quantity is the second-order identity tensor of three-dimensional space.

The inner product can be applied between vectors and second-order tensors as well, like a matrix product keeping their own properties. Thus, dyadics hold the following properties (derivation not shown):

\[
\begin{align*}
\vec{a} \vec{b} &= (\vec{b} \vec{a})^t \\
(\vec{c} \vec{a}) \cdot \vec{b} &= (\vec{a} \cdot \vec{b}) \vec{c} \\
\vec{c} \cdot (\vec{a} \vec{b}) &= (\vec{c} \cdot \vec{a}) \vec{b} \\
(\vec{a} \vec{b}) \cdot (\vec{c} \vec{d}) &= (\vec{b} \cdot \vec{c}) (\vec{a} \vec{d})
\end{align*}
\]

where the superscript \(t\) stands for the matrix transpose. Note that even though the vector transpose is represented by a \(1 \times 3\) matrix instead of the conventional \(3 \times 1\) matrix, the vector after transposition remains identical, i.e. \(\vec{a} \equiv (\vec{a})^t\). By default, vectors at left-hand side in an inner product are transpose.

Although it is not used in this paper, the trace of \(\vec{a} \vec{b}\), i.e. the sum of their diagonal components, is indeed the concerning dot product:

\[
\text{trace}(\vec{a} \vec{b}) = \vec{a} \cdot \vec{b}
\]

In fact, the trace of \(\vec{a} \vec{b}\) can be expressed in terms of double inner product as \(\frac{1}{3} \vec{a} \vec{b} : 1\).

### 3. Skew-symmetric tensor associated to a vector

In vector algebra \([8]\), the skew-symmetric tensor \(\Omega_{\vec{a}}\) of rank two associated to a vector \(\vec{a}\) is defined by:

\[
(\Omega_{\vec{a}})_{ij} \triangleq -\varepsilon_{ijk}a_k
\]

where \(\varepsilon_{ijk}\) stands for the Levi-Civita symbol \([7]\), also referred to as \(\varepsilon\)-permutation symbol, and where all indices have the range 1, 2, 3. The index \(k\) is the dummy summation index according to the summation convention. The epsilon symbol \(\varepsilon_{ijk}\) holds the following rules:

\[
\begin{align*}
\varepsilon_{123} &= \varepsilon_{231} = \varepsilon_{321} = 1 \\
\varepsilon_{123} &= -\varepsilon_{213} = -\varepsilon_{132} \\
\varepsilon_{ijk} &= 0, \text{ otherwise}
\end{align*}
\]
There is an additional relation known as epsilon-delta identity:

\[ \varepsilon_{mni} \varepsilon_{ijk} = \delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj} \]  

(5)

where \( \delta_{ij} \) is the Kronecker delta (\( ij \)-component of the second-order identity tensor) and the summation is performed over the \( i \) index. Indeed, the epsilon symbol and the Kronecker delta are both numerical tensors which have fixed components in every coordinate system. As the identity tensor, \( \mathbf{1} \), can be generated from the summation of the unit dyads \( \{ \hat{e}_i \}_{i=1,2,3} \), the epsilon symbol can be accordingly found from the following triple scalar product:

\[ \varepsilon_{ijk} = (\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k \]

where the cross product is symbolized by \( \times \). From the anti-cyclic rule of \( \varepsilon_{ijk} \) and the definition (4), it is straightforwardly shown that the tensor \( \Omega_a \) is anti-symmetric:

\[ \Omega^t_a = -\Omega_a \]  

(6)

and this can be readily illustrated from the matrix form of \( \Omega_a \):

\[ \Omega_a = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \]

Also, \( \tilde{a} \) is called the (Hodge) dual vector of the skew-symmetric tensor \( \Omega_a \). Hence, for instance, the magnetic field tensor in Electrodynamics \[9\] is indeed the skew-symmetric tensor associated to the magnetic field vector.

The Levi-Civita symbol also appears in the definition of the cross product of \( \vec{a} \) and \( \vec{b} \) \[1\]:

\[ (\vec{a} \times \vec{b})_i \triangleq \varepsilon_{ijk} a_j b_k \]  

(7)

then, from the definition (4) and the anti-cyclic rule of the epsilon symbol, it is possible rewritten the cross product in terms of the concerning skew-symmetric tensor (4) as:

\[ (\vec{a} \times \vec{b})_i = (\Omega_{\tilde{a}})_{ik} b_k \]  

(8)

or in vector notation as:

\[ \vec{a} \times \vec{b} = \Omega_{\tilde{a}} \cdot \vec{b} \]  

(9)

A cross product typically returns a (true) vector or polar vector. More exactly, the cross product \( [9] \) is a vector if either \( \vec{a} \) or \( \vec{b} \) (but not both) are pseudovectors. Otherwise, \( \vec{a} \times \vec{b} \) is a pseudovector \[10\]. Then, it is worthy to mention that the tensor \( \Omega_{\tilde{a}} \) will be a relative tensor or pseudotensor if the vector \( \vec{a} \) is axial and otherwise, it will be an absolute tensor if the vector \( \vec{a} \) is polar.

In addition to the skew-symmetry \[9\], the tensor \( \Omega_{\tilde{a}} \) holds the following properties (derivation not shown):

- \( \Omega_{\alpha \tilde{a}} = \alpha \Omega_{\tilde{a}} \)
- \( \Omega_{\tilde{a} + \tilde{b}} = \Omega_{\tilde{a}} + \Omega_{\tilde{b}} \)
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- \( \Omega_{\vec{a}} \cdot \vec{a} = 0 \)
- \( \Omega_{\vec{b}} \cdot \vec{a} = \vec{b} \cdot \Omega_{\vec{a}} \)
- \( \Omega_{\vec{a}} \cdot \Omega_{\vec{b}} = \vec{b} \vec{a} - (\vec{a} \cdot \vec{b}) \mathbf{1} \)
- \( \Omega_{\vec{a} \times \vec{b}} = \vec{b} \vec{a} - \vec{a} \vec{b} = \Omega_{\vec{a}} \cdot \Omega_{\vec{b}} - \Omega_{\vec{b}} \cdot \Omega_{\vec{a}} \)

where \( \alpha \) is a scalar. These properties can be straightforwardly proved using index notation and the above-mentioned rules of the Levi-Civita symbol. Thus, the epsilon-delta identity (5) draws to the last two properties, which are very helpful for the derivations compiled in section 4. In particular, these other properties are also very useful:

- \( \Omega_{-\vec{a}} = \Omega_{\vec{a}}^t \)
- \( \Omega_{\vec{a}} \cdot \Omega_{\vec{b}} = (\Omega_{\vec{b}} \cdot \Omega_{\vec{a}})^t \)
- \( \Omega_{\vec{a}} \cdot \vec{b} = -\vec{b} \cdot \Omega_{\vec{a}} \)
- \( \Omega_{\vec{e}} = \hat{e} \hat{e} - 1 \)
- \( \Omega_{\vec{e}}^2 = -\hat{\Omega}_{\vec{e}} \)

where \( \hat{e} \) is a vector of unit length. Due to Eq. (3), \( \Omega_{\vec{e}}^2 \) is equal to the second-order identity tensor of the two-dimensional space (plane) with normal unit \( \hat{e} \).

4. Standard vector identities

Next, the most useful vector identities are demonstrated from the concerning dyadics (1) and skew-symmetric tensors (4). The above-listed properties, the associative rule of matrix product and the matrix transposition rules are used accordingly.

- Cyclic permutation of the scalar triple product:

\[
\left( \vec{a} \times \vec{b} \right) \cdot \vec{c} = (\vec{a} \cdot \Omega_{\vec{b}}) \cdot \vec{c} = \vec{a} \cdot (\Omega_{\vec{b}} \cdot \vec{c}) = \vec{a} \cdot \left( \vec{b} \times \vec{c} \right) = (\vec{b} \cdot \Omega_{-\vec{a}}) \cdot \vec{c} = \vec{b} \cdot (\Omega_{-\vec{a}} \cdot \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})
\]

From these identities, the orthogonality between \( \vec{a} \times \vec{b} \) and each vector can be readily illustrated:

\[
\left( \vec{a} \times \vec{b} \right) \cdot \vec{a} = 0
\]

- Vector triple product expansion (or Lagrange’s formula):

\[
\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \cdot \Omega_{\vec{b} \times \vec{c}} = \vec{a} \cdot (\vec{c} \vec{b} - \vec{b} \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}
\]

also known as the acb minus abc rule. Using this identity, any vector can be expressed as linear combination of two mutually perpendicular vectors according to an arbitrary direction, \( \hat{e} \):

\[
\vec{a} = (\vec{a} \cdot \hat{e}) \hat{e} + \hat{e} \times (\vec{a} \times \hat{e})
\]
Furthermore, Eq. (12) might be used directly in many below-mentioned identities, thereby it is one of the most important vector identities.

- Jacobi’s identity:
  \[
  \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{a} \cdot \Omega_{\vec{b} \times \vec{c}} + \vec{b} \cdot \Omega_{\vec{c} \times \vec{a}} + \vec{c} \cdot \Omega_{\vec{a} \times \vec{b}} = \vec{a} \cdot (\vec{c} \vec{b} - \vec{b} \vec{c}) + \vec{c} \cdot (\vec{b} \vec{a} - \vec{a} \vec{b}) = \vec{0}
  \]
  (14)
i.e. the sum of all the cyclic permutations of the vector double product comes to zero.

- Dot product of two cross products:
  \[
  (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\Omega_{\vec{a} \times \vec{b}}) \cdot (\Omega_{\vec{c} \times \vec{d}}) = \vec{b} \cdot (\Omega_{\vec{c} \times \vec{a}}) \cdot \vec{d} - \vec{c} \cdot (\Omega_{\vec{b} \times \vec{a}}) \cdot \vec{d}
  \]
  \[
  = \vec{b} \cdot ((\vec{a} \cdot \vec{c}) \cdot \vec{d} - (\vec{a} \cdot \vec{d}) \cdot \vec{c})
  \]
  \[
  = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{d}) \vec{c}
  \]
  (15)

This identity and the next one afford a simple means of deducing the formulae of Spherical Trigonometry. From Eq. (15), it is also derived the well-known identity:

\[
(\vec{a} \times \vec{b})^2 = (\vec{a} \cdot \vec{a}) (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2
\]
(16)
which geometrical interpretation is the Pythagorean theorem.

- Cross product of two cross products:
  \[
  (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \Omega_{\vec{a} \times \vec{b}} \cdot (\Omega_{\vec{c} \times \vec{d}}) = (\vec{b} \vec{a} - \vec{a} \vec{b}) \cdot (\vec{c} \vec{d} - \vec{d} \vec{c})
  \]
  \[
  = (\vec{b} \vec{a} \vec{c} - \vec{a} \vec{b} \vec{c}) - (\vec{a} \vec{b} \vec{d} - \vec{a} \vec{d} \vec{b})
  \]
  \[
  = (\vec{a} \cdot (\Omega_{\vec{c} \times \vec{d}})) \vec{b} - (\vec{b} \cdot (\Omega_{\vec{c} \times \vec{d}})) \vec{a}
  \]
  \[
  = (\vec{a} \cdot (\vec{c} \vec{d} - \vec{d} \vec{c})) \vec{b} - (\vec{b} \cdot (\vec{c} \vec{d} - \vec{d} \vec{c})) \vec{a}
  \]
  (17)

\[
(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \cdot \Omega_{\vec{b} \times \vec{d}}) - (\vec{b} \cdot \Omega_{\vec{a} \times \vec{d}}) \vec{c} - (\vec{c} \cdot \Omega_{\vec{a} \times \vec{d}}) \vec{d}
\]
(18)

- Other identities:
  \[
  ((\vec{a} \times \vec{b}) \times \vec{b}) \cdot \vec{a} = (\Omega_{\vec{a} \times \vec{b}}) \cdot \vec{b} = (\vec{b} \vec{a} - \vec{a} \vec{b}) \cdot \vec{b} + \vec{a}
  \]
  \[
  = (\vec{a} \cdot \vec{b}) \vec{b} - (\vec{b} \cdot \vec{b}) \vec{a} + \vec{a}
  \]
  \[
  = - (\vec{a} \times \vec{b})^2
  \]
(19)
The identity \((16)\) was invoked for this result. However, \(\left(\vec{a} \times \vec{b}\right) \times \vec{b} = 0\) is readily found from Eq. \((11)\). Likewise, in the next two expressions, the identity \((11)\) was again applied accordingly:

\[
\left(\left(\vec{a} \times \vec{b}\right) \times \vec{b}\right) \times \vec{b} = \Omega \left(\vec{a} \times \vec{b}\right) \times \vec{b} \cdot \vec{b} = \Omega \left(\vec{a} \times \vec{b}\right) \cdot \vec{b} = \left(\vec{b} \cdot \vec{b}\right) \left(\vec{a} \times \vec{b}\right)
\]

(20)

\[
\left(\left(\vec{a} \times \vec{b}\right) \times \vec{b}\right) \times \vec{b} = \Omega \left(\vec{a} \times \vec{b}\right) \times \vec{b} = \Omega \left(\vec{b} \cdot \vec{b}\right) \left(\vec{a} \times \vec{b}\right)
\]

(21)

5. Skew-symmetric tensors in Physics

The substitution of physical pseudovectors (such as angular velocity or magnetic field) with skew-symmetric tensors \((4)\) provides an alternate to cross product. This notation is much easier to work and allows to understand the vector operations in terms of rotations \((11)\). In fact, an arbitrary vector \(\vec{a}\), which rotates an angle \(\theta\) about an axis along the unit vector \(\hat{e}\) \((12)\), is expressed by:

\[
\vec{a}^* = (1 + ((1 - \cos \theta) \Omega \hat{e} + \sin \theta \hat{1}) \cdot \vec{a})
\]

(22)

5.1. Rotating systems

A system rotates with constant angular velocity, \(\vec{\omega}\), relative to a rest frame. The time variation of a unit vector \(\hat{e}\) fixed to the rotating system \([13, 14]\) is given by:

\[
\frac{d\hat{e}}{dt} = \vec{\omega} \times \hat{e} = \Omega \hat{e} \cdot \hat{e}
\]

(23)

where the tensor \(\Omega \hat{e}\) acts as a rotation operator. Also, if a point particle linked to the rotating system is moving at linear velocity, \(\vec{v}\), regarding to the rest frame, then it undergoes the following Coriolis’ acceleration \([13, 14]\):

\[
\vec{a}_{\text{Coriolis}} = 2\vec{\omega} \times \vec{v} = \Omega_{2\hat{e}} \cdot \vec{v}
\]

(24)

where now the rotated vector is \(\vec{v}\).

5.2. Rotation dynamics of rigid body motion

A system of \(N\)–point particles of mass \(m_i\) and position vector \(\vec{r}_i\) relative to a rest reference frame, describes a pure rotation with angular velocity \(\vec{\omega}\) and acceleration \(\vec{a}\) \([13]\).

- **Kinematics**
  The linear velocity and acceleration of the \(i\)-particle are rewritten as:

\[
\vec{v}_i = \vec{\omega} \times \vec{r}_i = \Omega_{2\hat{e}} \cdot \vec{r}_i
\]

(25)

\[
\vec{a}_i = \vec{a} \times \vec{r}_i + \vec{\omega} \times (\vec{\omega} \times \vec{r}_i) = (\Omega \vec{a} + \Omega_{2\hat{e}}) \cdot \vec{r}_i
\]

(26)
• Inertia tensor

The inertia tensor relative to the rest coordinate system is given by:

\[ I \equiv \sum_{i=1}^{N} m_i \left( r_i^2 1 - \vec{r}_i \vec{r}_i \right) = \sum_{i=1}^{N} m_i \vec{\Omega} \cdot \vec{r}_i \vec{r}_i \]  

(27)

• Steiner’s theorem

The inertia tensor relative to a second rest frame is:

\[ I^* = I - \left( \sum_{i=1}^{N} m_i \right) \vec{\Omega} \vec{r}_c \cdot \vec{\Omega} - \vec{r}_c \]  

(28)

where \( \vec{r}_c \) is the position vector of the system center regarding to the initial rest coordinate system.

• Inertia momentum with respect to an axis in the direction \( \hat{e} \)

\[ I_{\hat{e}} \equiv \sum_{i=1}^{N} m_i (\hat{e} \times \vec{r}_i)^2 = \sum_{i=1}^{N} m_i (\hat{e} \cdot \vec{\Omega}_i) \cdot (\vec{\Omega} - \vec{r}_i \cdot \hat{e}) = \hat{e} \cdot I \cdot \hat{e} \]  

(29)

• Angular momentum

\[ \vec{L} \equiv \sum_{i=1}^{N} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_{i=1}^{N} m_i \vec{\Omega}_i \cdot (\vec{\Omega} - \vec{r}_i \cdot \vec{\omega}) = \sum_{i=1}^{N} m_i (\vec{\Omega}_i \cdot \vec{r}_i \cdot \vec{\omega}) = I \cdot \vec{\omega} \]  

(30)

• Rotation kinetic energy

\[ E_k \equiv \frac{1}{2} \sum_{i=1}^{N} m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2} \sum_{i=1}^{N} m_i (\vec{\omega} \cdot \vec{\Omega}_i) \cdot (\vec{\Omega} - \vec{r}_i \cdot \vec{\omega}) = \frac{1}{2} \vec{\omega} \cdot I \cdot \vec{\omega} \]  

(31)

5.3. Electric quadrupole

The quadrupole moment tensor of a system of point electric charges \( \{q_i\}_{i=1..N} \) can be expressed as:

\[ Q \equiv \sum_{i=1}^{N} q_i \left( 3\vec{r}_i \vec{r}_i - r_i^2 1 \right) = 3 \sum_{i=1}^{N} q_i \vec{\Omega}_i^2 + 2 \left( \sum_{i=1}^{N} q_i r_i^2 \right) 1 \]  

(32)

This second-rank tensor is traceless.

5.4. Vector field identities

Vector field identities can be also derived using the skew-symmetric tensor associated to the differential vector operator \( \nabla \) rather than the curl \[15\], as follows:

\[ \nabla \times \vec{a} = \vec{\Omega}_\nabla \cdot \vec{a} \]  

(33)

However, special care must be taken in tensor calculus because the order of elements is important:

• \( (\nabla \vec{a})^t \neq \vec{a} \nabla \)

• \( \vec{a} \cdot \vec{\Omega}_\nabla \neq -\vec{\Omega}_\nabla \cdot \vec{a} \)
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- \((\Omega \vec{a} \cdot \Omega \nabla)^t \neq \Omega \nabla \cdot \Omega \vec{a}\)
- \((\vec{a} \nabla) \cdot \vec{b} = (\nabla \cdot \vec{b}) \vec{a}\)
- \((\nabla \vec{a})^t \cdot \vec{b} = (\vec{b} \cdot \nabla) \vec{a}\)

Thus, the most relevant properties are:

- \(\Omega \nabla \cdot \Omega \vec{a} = (\nabla \vec{a})^t - (\nabla \cdot \vec{a}) \mathbf{1}\)
- \(\Omega \vec{a} \cdot \Omega \nabla = (\vec{a} \nabla)^t - \mathbf{1} (\vec{a} \cdot \nabla)\)
- \(\Omega \nabla \cdot (\Omega \vec{a} \cdot \vec{b}) = (\Omega \nabla \cdot \Omega \vec{a}) \cdot \vec{b} + (\Omega \vec{a} \cdot \Omega \nabla)^t \cdot \vec{b}\)
- \(\nabla \cdot (\Omega \vec{a} \cdot \vec{b}) = (\Omega \vec{b} \cdot \nabla) \cdot \vec{a} + (\Omega \vec{a} \cdot \nabla)^t \cdot \vec{b}\)

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