Central charges of aperiodic holographic tensor network models

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Central to the AdS/CFT correspondence is a precise relationship between the curvature of an anti-de Sitter (AdS) spacetime and the central charge of the dual conformal field theory (CFT) on its boundary. Recent years have witnessed a significant interest in devising toy models that capture aspects of the AdS/CFT correspondence in a concrete fashion. In this work, we introduce a quantitative picture that enables novel analytical computations: Specifically, we identify upper bounds on the central charges of boundary states for regular hyperbolic bulk tilings, a geometry underlying many tensor network-based AdS/CFT toy models. Relating these central charges to the curvature of the embedding space, we analytically derive discrete analogues to the continuum Brown-Henneaux formula. We also compute the exact central charges of a class of Majorana dimer models exhibiting holographic quantum error correction, which are proven to saturate the generic bounds in the limit of large curvature. The renormalization group description of these states is shown to be analogous to the well-known strong disorder renormalization group, thus giving the first example of an exact quantum error correcting code that gives rise to genuinely critical systems. These systems exhibit a large range of fractional central charges, tunable by the choice of bulk tiling. Our approach thus provides a precise physical interpretation of tensor network models on regular hyperbolic geometries and establishes quantitative connections to a wide range of existing models.

I. INTRODUCTION

Years before the formulation of the holographic principle, J. D. Brown and M. Henneaux noticed a peculiar property of anti-de Sitter (AdS) spacetime, a solution to Einstein’s equation with constant negative curvature: At its asymptotic boundary, the symmetry group $SO(2,2)$ of 2+1-dimensional AdS$_3$ spacetime turns into the 2-dimensional conformal group, with an effective central charge depending on the curvature of the AdS bulk. Rather than a mathematical coincidence, the AdS/CFT correspondence [1] propelled this observation to the cornerstone of a holographic duality between gravity in d+2-dimensional AdS$_{d+2}$ spacetime and a conformal field theory (CFT) on its d+1-dimensional boundary, with an equivalent action describing both sides of the duality [2]. A key motivation for the holographic principle was the discovery of black hole entropy scaling with its horizon area, rather than its volume [3, 4]. The Bekenstein-Hawking entropy formula

$$S_{BH} = \frac{A_{hor}}{4G}, \quad (1)$$

where $A_{hor}$ is the horizon area and $G$ the gravitational constant, has a surprising generalization in the context of AdS/CFT: The entanglement entropy $S_A$ [5] of a boundary region $A$ follows the Ryu-Takayanagi (RT) formula [6]

$$S_A = \frac{|\gamma_A|}{4G}, \quad (2)$$

where $|\gamma_A|$ is the area of an extremal surface $\gamma_A$ in the bulk whose boundary $\partial \gamma_A$ matches the boundary $\partial A$. In 2+1 dimensions, $\gamma_A$ is simply a geodesic curve and $|\gamma_A|$ its length. Both formulae (1) and (2) suggest an encoding of information in Planckian pieces of area of size $\sim G = l_p^2$ (in 3+1 bulk dimensions).

Recent years have added a new ingredient to this discussion that has stimulated a further line of research: While AdS/CFT is generally used in its continuum formulation, a number of discretized models based on tensor networks [5, 7–10] have been constructed, capturing key aspects of the continuum theory while being easier to grasp in rigorous and numerical analysis. Importantly, the multi-scale entanglement renormalization ansatz (MERA) [11], a tensor network that well approximates critical boundary states, was quickly identified as a possible realization of discrete holography [12, 13]. However, the bulk geometry of the MERA cannot be directly related to an AdS time-slice [14,16]. Regular hyperbolic tessellations based on $k$-gons at each corner vertex, denoted by the Schläfli symbol $\{n,k\}$, can be naturally embedded into a time slice metric and have been the basis of several discrete holographic models [17,23]. These models feature many aspects of the AdS/CFT correspondence, particularly elucidating the deep connection between holography and notions of quantum error correction [12,24].

In this work, we report significant progress in describing the AdS/CFT correspondence in terms of discrete modelling and with tensor networks. Building on a machinery of inflation rules and Majorana dimer models, we present an approach that allows to perform analytical computations that seemed out of reach before. Specifically, we derive new bounds on central charges that generalize the continuum Brown-Henneaux formula [25]. Moving further, we are in the position to exactly compute central charges for a class of Majorana dimer models – that at the same time are stabilizer models – that has been identified as featuring holographic quantum error correction. Specifically, we prove that the hyperbolic pentagon code (HyPeC) [17] saturates the central charge bounds for tilings with large negative curvature, a result we further extend to block perfect generalizations of the HyPeC. We argue that these dimer models are a discrete approximation of a CFT,
with the inflation rules of the tiling providing a local renormalization group (RG) transformation. This establishes a relation between our fermionic models and strongly disordered systems with aperiodic symmetries from which a rigorous RG interpretation arises. Our approach thus produces a precise physical interpretation of tensor network models on regular hyperbolic geometries and introduces a machinery to provide rigorous and at the same time quantitative connections to a wide range of existing holographic models.

II. CENTRAL CHARGES AND CURVATURE

A typical metric representation of AdS$_3$ spacetime is given by global AdS coordinates,

$$ ds^2 = -(1 + r^2/\ell^2)dt^2 + \frac{\alpha^2 dr^2}{\alpha^2 + r^2} + r^2 d\phi^2 , $$

where $\alpha$ is the AdS radius. The scalar curvature or Ricci scalar $R$ of AdS$_d$ spacetime with $d = 2 + 1$ dimensions is given by

$$ R = -\frac{d(d-1)}{\alpha^2} = -\frac{\alpha}{\ell^2} . $$

This geometry corresponds to a negative cosmological constant $\Lambda = -1/\alpha^2$. We consider now a timeslice of AdS$_3$ with a deformation in the centre and a boundary cutoff shown as a dashed curve. In the asymptotic region towards the boundary, the shape of $\gamma_A$ is independent of bulk deformations.

Compare this with the entanglement entropy of a conformal field theory for a small subsystem ($\Delta \phi \ll 2\pi$), given by

$$ S_A = \frac{c}{3} \log \left( \frac{2\ell}{\Delta \phi \epsilon} \sin \frac{\Delta \phi}{2} \right) \approx \frac{c}{3} \log \frac{\ell}{\epsilon} , $$

where $\epsilon$ denotes the lattice spacing and $c$ is the central charge of the CFT. Assuming that the RT prescription holds, we recover the Brown-Henneaux formula $[25]$

$$ c = \frac{3\alpha}{2G} . $$

III. DISCRETE TENSOR NETWORK MODELS

Recent models of AdS/CFT often rely on discretizations of hyperbolic space in terms of a tensor network $[5,7-10]$. Starting with a central tensor, the network is iteratively inflated, i.e., layers of tensors are added through contraction. Each iteration yields a discrete analogue to a radial cutoff in the continuum model. A discretized boundary region $A$ of a tiling embedded into the Poincaré disk will generally not follow a constant radius $\rho$ and have a larger geodesic length $\ell$ than a comparable boundary at a fixed radial cutoff in the continuum model. Indeed, for a regular bulk tiling of $n$-gons, the boundary exhibits a quasi-regular symmetry $[26]$ with self-similar geometric features.

The RT formula applies to tensor networks as well: Given a boundary region $A$ of the tensor network (i.e., a set of boundary indices or edges of the equivalent tiling), we can define $\gamma_A$ as the shortest cut through the network from the two boundary points. Assuming constant bond dimension $\chi$ across the network, the entanglement entropy is bounded by

$$ S_A \leq \frac{|\gamma_A|}{s} \log \chi , $$

where $|\gamma_A|$ and $s$ are the geodesic lengths of the shortest cut and of each individual edge, respectively, the latter assumed to be constant. To approximate holographic states, we seek tensors that saturate $[10]$ for any boundary region $A$. 

FIG. 1. Continuous and discretized geodesic $\gamma_A$ in the Poincaré disk with a deformation in the centre and a boundary cutoff shown as a dashed curve. In the asymptotic region towards the boundary, the shape of $\gamma_A$ is independent of bulk deformations.
However, the Brown-Henneaux formula does not trivially generalize to discrete tilings, as the discretization breaks the continuous AdS symmetries from which it is derived. The resulting discretized CFT exhibits new symmetries whose exact form depends on the choice of discretization. Specifically, different regular tilings result in a different relative growth of the boundary region length $\ell = |A|$ and the geodesic length $|\gamma_A|$ under inflation. Let us consider regular $\{n,k\}$ tilings with $k$ $n$-gon tiles at each vertex. If the sum of inner angles of each $n$-gon is smaller than $(n-2)\pi$, i.e., when $1/n + 1/k < 1/2$, the geometry is hyperbolic. A regular hyperbolic tiling can be naturally embedded into the Poincaré disk. We consider vertex inflation, whereby each inflation step consists of filling each open vertex with tiles. First consider the $n = 3$ case, the triangular hyperbolic tiling, whose vertex inflation is shown in Fig. 2 (left). We start with a single triangle with three vertices, each of which has two neighbours. The first inflation step gives each vertex $k-2$ additional neighbouring vertices, two of which are shared with its previous neighbours. Thus, the inflation step adds $k-3$ new vertices for each old one. After the first inflation step, all boundary vertices have either three or four neighbours, two of which are other boundary vertices. Denoting vertices with two, three, and four neighbours with the letters $a$, $b$, and $c$, respectively, this pattern is summarized in the inflation rule

$$a \mapsto b^{k-4}c, \quad b \mapsto b^{k-5}c, \quad c \mapsto b^{k-6}c,$$

(11)

where we encode the boundary vertices as a string of $a$, $b$ and $c$, $a^k$ denoting $k$ repetitions of $a$. The inflation rule for any hyperbolic $\{n,k\}$ tiling produces a quasi-regular sequence exhibiting self-similarity: After sufficiently many inflation steps, any starting sequence will lead to a sequence with the same distribution of letters. In this steady state the relative frequency of letters is given by the largest eigenvalue of the substitution matrix $M$, where $M_{i,j}$ is the number of $j$ vertices resulting from applying the inflation rule on an $i$ vertex. For the $\{3,k\}$ tiling, it is given by

$$M = \begin{pmatrix} 0 & k-4 & 1 \\ 0 & k-5 & 1 \\ 0 & k-6 & 1 \end{pmatrix}. \quad (12)$$

Here the rows and columns correspond to $(a, b, c)$ vertices. The largest eigenvalue of $M$, 

$$\lambda = \frac{1}{2} \left( \sqrt{k^2 - 8k + 12 + k - 4} \right), \quad (13)$$

is the scaling factor of the sequence (and sufficiently large subsystems thereof) in the steady state, i.e., after many inflation steps. The scaling of discrete geodesics can also be computed: Coarse-graining a subsystem $A$ of the sequence by a deflation step maps the two vertices that bound $A$ (and a few of its neighbours) onto two vertices at a lower inflation layer. For the $\{3,7\}$ tiling, this corresponds to removing two edges from the geodesic $\gamma_A$, one on either end. Thus, the average difference in entanglement entropy between both layers, denoted as $\Delta S_A$, is bounded by $2 \log \chi$. Relating this to Fig. 2, we obtain

$$c_{\{3,k\}} = \frac{3 \Delta S_A}{\log \chi} \leq \frac{6 \log \chi}{\log \left( \sqrt{k^2 - 6k + 12} + k - 4 \right)} \equiv \frac{c_{\max}}{\max_{3 \leq k \leq 6}} \quad (14)$$

Generalizing this result to arbitrary hyperbolic $\{n,k\}$ tilings leads to further complications. For the $\{4,k\}$ tiling (Fig. 2, right), the vertex inflation rule is

$$a \mapsto b(ab)^{k-3}, \quad b \mapsto b(ab)^{k-4}.$$

(15)

Again a and $b$ denote vertices with two and three neighbours up to a given inflation layer. The substitution matrix and its largest eigenvalues are found to be

$$M = \begin{pmatrix} k-3 & k-2 \\ k-4 & k-3 \end{pmatrix}, \quad \lambda = \sqrt{k^2 - 6k + 8 + k - 3}.$$

(16)

Unfortunately, the change of geodesic length under deflation now depends on the vertices involved: As we can see in Fig. 2 (right), the deflation $a \leftrightarrow a$ still only involves moving along one edge, but the deflation $a \leftrightarrow b$ involves two. To determine the average change in geodesic length per deflation step, we first compute the left and right eigenvectors of $M$ for the eigenvalue $\lambda$, given by

$$\bar{\ell} = \left( \begin{pmatrix} \sqrt{8 - 6k + k^2} \\ k-2 \end{pmatrix} \right), \quad \mathcal{F} = \left( \begin{pmatrix} \sqrt{8 - 6k + k^2} \\ k-4 \end{pmatrix} \right).$$

(17)

When divided by their total sum, the components of $\bar{\ell}$ give the relative frequencies $P(a)$ and $P(b)$ of $a$ and $b$ vertices in the steady state. This is not a probabilistic process: the relative frequencies can be captured on the formal level, however, by a discrete Markov chain. In this sense, we now wish to compute the probability of a deflation step $i \leftrightarrow j$. Each vertex type corresponds to a state with transition probabilities to other states under a deflation step. After sufficiently many steps, the probability of reaching any given state becomes independent of the starting point. While $M_{i,j} \propto P(i \rightarrow j|i)$ is the (relative) transition probability of reaching a $j$ vertex from an $i$ one, we can construct the deflation matrix $D$ giving the probability of the reverse process,

$$D_{i,j} = P(i \leftrightarrow j|i) = \frac{P(i \rightarrow j|i)P(i)}{\sum_k P(k \rightarrow j|i)P(k)} = \frac{M_{i,j}I_i}{\sum_k M_{k,j}I_k} = \frac{M_{i,j}I_i}{\lambda I_j}. \quad (18)$$

FIG. 2. Vertex inflation of a $\{3,7\}$ (left) and $\{4,5\}$ tiling (right), with vertices labeled by type and each inflation layer colour-coded.
The eigenvector \( \vec{p} \) of \( D \) with eigenvalue 1 now encodes the average probability of reaching each vertex type through deflation. We find \( p_i = l_i r_i \), as

\[
\sum_j D_{i,j} p_j = \sum_j \frac{M_{i,j} l_i r_j}{\lambda} = l_i r_i = p_i .
\]  

(19)

We normalize \( \vec{p} \) so that \( \sum_i p_i = 1 \). If an inflation step \( i \mapsto j \) adds \( E_{i,j} \) edges to a geodesic ending at an \( i \) vertex, i.e., adding \( E_{i,j} \log \chi \) to the entanglement bounded by the cut, then the average entanglement entropy loss per deflation step is given by

\[
S_A \leq \sum_{i,j} D_{i,j} E_{i,j} p_j \log \chi = \frac{1}{\lambda} \sum_{i,j} M_{i,j} E_{i,j} l_i r_i \log \chi.
\]  

(20)

We thus call \( E \) the entanglement matrix. The central charge bound for the hyperbolic \( \{ n, k \} \) tiling thus becomes

\[
C_{\{n, k\}} \leq C_{\{n, k\}}^{\max} = \frac{6 \sum_{i,j} M_{i,j} E_{i,j} l_i r_i \log \chi}{\lambda \log \lambda}.
\]  

(21)

For the \( \{ 4, k \} \) case, the entanglement matrix is simply

\[
E = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},
\]  

(22)

which yields a central charge bound

\[
C_{\{4, k\}}^{\max} = \frac{9 \log \chi}{\log (\sqrt{k^2 - 6k + 8 + k - 3})}.
\]  

(23)

Eq. (21) can be used to derive central charge bounds for arbitrary \( \{ n, k \} \) tilings. For \( n, k > 3 \), the inflation rule is given by

\[
a \mapsto a^{-4} b (a^{-3} b)^k , \quad b \mapsto a^{-4} b (a^{-3} b)^k ,
\]  

(24)

while for \( k = 3 \) we require three letters \( a, b, c \), where \( c \) denotes a vertex to the right of a \( b \)-type vertex, leading to

\[
a \mapsto c a^{-5} b , \quad b \mapsto c a^{-6} b , \quad c \mapsto \emptyset.
\]  

(25)

Here \( \emptyset \) is the empty set, i.e., the letter disappears. While (24) and (25) reproduce the quasi-regular sequences resulting from vertex inflation, these forms are not sufficient to describe the propagation of geodesics for \( n > 4 \). This requires distinguishing vertices by the graph distance of their neighbouring vertices to the centre, which determines which paths from one inflation layer to the next correspond to discretized radial geodesics. As in the continuous case, where we studied radial geodesics in an asymptotically AdS geometry, our tiling can be non-regular in the centre; only the tiling structure near the boundary of the Poincaré disk is relevant to the central charge of the boundary state. The maximum central charges resulting from the full calculation for an arbitrary \( \{ n, k \} \) tiling are summarized in Tab. 1. The corresponding inflation rules and matrices \( M \) and \( E \) are given in the Appendix.

FIG. 3. Sketch of a \( \{ 5, 4 \} \) tiling in the Poincaré disk with three reference points and one edge marked.

FIG. 4. Central charge bounds and AdS radii for \( \{ n, k \} \) tilings, with the continuum Brown-Henneaux formula for \( G = d/4 \log \chi \) shown as a dashed line. The data series start at \( k = 7 \) for \( n = 3, k = 5 \) for \( n = 4 \), and \( k = 4 \) for both \( n = 5 \) and \( n = 6 \) (first data point of each series in the upper-right corner).

IV. CURVATURE OF REGULAR TILINGS

An \( \{ n, k \} \) tiling embedded into the Poincaré disk is constructed of identical \( n \)-gons with an angle of \( 2\pi/k \) at each corner (see Fig. 3). The geodesic length \( P_1 P_2 = s \) between two points \( P_1 \) and \( P_2 \) of the tiling determines the length between all other points in the tiling. The parameters \( n \) and \( k \) further fix the angles \( \beta = \angle(O P_1, O P_2) = 2\pi/n \) and \( \gamma = \angle(P_1 P_2, P_1 O) = \angle(P_2 O, P_2 P_1) = \pi/k \). The hyperbolic law of cosines then states that

\[
\cos \beta = \cos^2 \gamma + \sin^2 \gamma \ cosh \frac{s}{\alpha}.
\]  

(26)

Note that this form of the law of cosines holds for a Gaussian curvature \( K = R/2 = -1/\alpha^2 \) of the time-slice metric. Using this relation we can now express the AdS_3 radius in terms of the tiling parameters as

\[
\frac{s}{\alpha} = 2 \arccosh \left( \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right) = 2 \log \left( \frac{2k}{\pi} \cos \frac{\pi}{n} \right) + O(k^{-2}) .
\]  

(27)

Thus, \( s/\alpha \) diverges logarithmically in the large \( k \) limit. Note that the hyperbolic area \( A = \alpha^2(n-2n/k-2) \) is finite in this limit.

We can now directly relate the previously derived bounds on central charges \( c \) to the AdS radius \( \alpha \) of the correspond-
ing AdS geometry, with the results for various choices of $n$ shown in Fig. 4. These bounds can be compared to the continuum Brown-Henneaux prescription [9], with the gravitational constant $G$ fixed through the RT formula: The length of a discretized minimal geodesic $\gamma_A$ corresponding to a boundary region $A$ can be written as $|\gamma_A| = N_s$, where $N$ is the number of edges that $\gamma_A$ consists of (note that $N \to \infty$ in the asymptotic limit). As each edge contributes log $\chi$ to $S_A$, we find

$$S_A = \frac{|\gamma_A|}{4G} = \frac{N_s}{4G} = N \log \chi .$$

We can thus rewrite [9] as

$$\epsilon_{\text{max}} = \frac{6\alpha \log \chi}{s} .$$

Comparing this to the behaviour of boundary states of $\{n,k\}$ tilings in Fig. 4 we find that these bounds are always above [29]. This implies that tensor networks with the same bulk curvature and entanglement entropy growth as a continuum model can always be constructed by choosing appropriate tensors. Furthermore, we find a linear regime at large $k$ in all tilings with the slope depending on $n$. For example,

$$\lim_{k \to \infty} \frac{\epsilon_{\text{max}}(3,k)}{\alpha(3,k)} \log \chi = 12 , \quad \lim_{k \to \infty} \frac{\epsilon_{\text{max}}(4,k)}{\alpha(4,k)} \log \chi = 18 .$$

The general coefficients are given in Tab. 1. Note that they are significantly larger than the continuum value at small curvature, and increase monotonically with $n$. At small $k$, a second linear regime appears, with a slope much closer to the Brown-Henneaux form, e.g.

$$\frac{\epsilon_{\text{max}}(3,7) - \epsilon_{\text{max}}(3,8)}{\alpha(3,7) - \alpha(3,8)} \approx 6.38 \frac{\log \chi}{s} .$$

As a tiling of lower curvature is a better approximation of a continuum geometry, a result closer to the BH formula is not unexpected; however, fixing $n$ while varying $k$ appears to produce a central charge shift relative to the BH result that remains constant for a large range of $k$, even as the curvature increases significantly.

### V. STRONG DISORDER RENORMALIZATION

After having established the previous bounds on entanglement entropy asymptotics, we will consider cases when the central charge can be calculated exactly. Interestingly, the method that allows for such an exact calculation is deeply related to a very early approach to real-space renormalization group transformations that were originally introduced in Ref. [27] and later in Ref. [28], to study the ground-states, low-energy excitations and spatio-temporal correlations of random quantum spin chains. This technique, called the strong disorder renormalization group (SDRG) [29] has recently again gained considerable attention due to its role in studying many-body localization [30], quantum critical Floquet dynamics [31] and models with highly area-law breaking ground states (rainbow states) [32], see Ref. [33] and reference therein for recent development.

We now describe the basic results of SDRG on some aperiodic singlet models that share the quasi-regular symmetries of the boundary states described previously. One example is given by the Fibonacci XXZ chain, that is defined by the Hamiltonian

$$H = \sum_i J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z) ,$$

where the $S_i^\alpha$’s ($\alpha = x, y, z$) are spin-$\frac{1}{2}$ operators, the site-dependent couplings $J_0 > J_a > 0$ are modulated according to the aperiodic Fibonacci sequence, that is obtained by the inflation rule

$$a \mapsto ababa , \quad b \mapsto aba .$$

The SDRG procedure predicts that for this aperiodic Hamiltonian the ground state (in the large system limit) is characterized by fully entangled pairs of sites [34] [35]. For example, inflating the letter $b$ twice leads to a Hamiltonian with the
ground-state given by
\[ ababaabababa \]
where each double-line denotes a singlet bond. The entanglement entropy of a subregion \( A \) of such a singlet state is simply computed by counting the number of singlets connecting it to its complement \( A^c \). For example, in the state
\[ A \]
we find five singlets passing through the cuts between \( A \) and \( A^c \), resulting in an entanglement entropy \( S_A = 5 \log 2 \).

Applying the SDRG procedure for this model [34], it follows that one can systematically obtain the ground state corresponding to the Hamiltonian after inflating the letter \( b \) \( n \)-times by iterating the inverse of the renormalization steps, giving rise to the inflation rules
\[ a \rightarrow a \quad \text{and} \quad b \rightarrow a b a b a. \]

In the next section, we give concrete examples of aperiodic models that can be embedded into the regular bulk geometries considered previously, and we will generalize this method to calculate the central charge analytically. Distinct from usual singlet models, we will consider fractionalized fermionic modes. Also, the entangled pairs in these models exhibit crossing, requiring a new approach to computing their entanglement entropies.

### VI. Majorana Dimer Models

An efficiently contractible class of tensor networks with a holographic interpretation is given by Majorana dimer states [23]: This versatile class of states corresponds to the intersection of stabilizer and free fermionic states; as part of the latter, they can also be efficiently described by matchgate tensor networks [19]. In particular, the hyperbolic pentagon code (HyPeC), a toy model of holographic quantum error correction [17], can be expressed in this form. Explicitly, the logical qubit that the HyPeC associates with each tile in a \( \{5, 4\} \) bulk geometry is spanned by the two logical basis states
\[ |0\rangle_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad |1\rangle_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]

In this visualization, each edge of a pentagon tile is identified with two Majorana modes, with each arrow \( j \rightarrow k \) between two modes \( i \) and \( j \) corresponding to a term \( \gamma_j \gamma_k \) in the stabilizer Hamiltonian, expressed in terms of Majorana operators \( \gamma_j \). The orientation of each arrow relative to the mode ordering gives it an associated dimer parity \( p_{j,k} \), with \( p_{j,k} = +1 \) for \( j < k \) (blue) and \( p_{j,k} = -1 \) for \( j > k \) (orange). The transformation to spin operators, in which the HyPeC has originally been formulated, is performed by a standard Jordan-Wigner transformation.

The Majorana dimer picture has the great advantage that contracting states of the form (38) is equivalent to simply pairing up dimers along the contracted edges and multiplying their dimer parities [23]. While general logical states are formed from superpositions of 0 and 1, orthogonality conditions between the contracted states ensure that two-point correlation functions still exhibit a dimer structure, i.e., vanishing correlations between Majorana modes unconnected by dimers. Furthermore, computing the entanglement entropy \( S_A \) of a connected subsystem \( A \) of a Majorana dimer state (or contraction thereof) reduces to simply counting the dimers between \( A \) and its complement \( A^c \), each contributing \( \log(2)/2 \) to \( S_A \). Beyond the HyPeC, whose logical states on each tile are represented by perfect tensors that maximally entangle each possible subsystem \( A \) with the remaining sites, Majorana dimer states also represent block perfect tensors, where this condition is relaxed to only hold for connected subsystems. A suitable pair of logical eigenstates \( 0_n \) and \( 1_n \) can be found for any
The new dimers added at each step are drawn in a lighter colour, while the ones that are extended from the previous layer are drawn darker. As we are interested in entanglement properties, the dimer parities (which we previously colour-coded) are irrelevant here. Note that each inflated dimer configuration contains two open dimers on either end of the sequence that connect to the previous and following sequence. The full dimer configuration in the \( \{5, 4\} \) tiling is shown in Fig. 5 in the Poincaré disk projection along with the dimers at the first three inflation layers. When starting from the central pentagon, the initial sequence is given by \( a_1 a_1 a_1 a_1 a_1 \).

Having associated geometrical features on the vertex-inflated tiling with a dimer configuration, we can now calculate the entanglement entropy that each inflation step produces. As before, we first consider a deflation or coarse-graining step that removes dimers and thus, entanglement entropy. Consider how a cut (green line) changes throughout a deflation step:

\begin{align*}
    a_1 &\rightarrow a_1 b_1 a_2 a_1 b_2, & (41) \\
    a_2 &\rightarrow a_2 b_1 a_2 a_1 b_2, & (42) \\
    b_1 &\rightarrow a_1 b_1, & (43) \\
    b_2 &\rightarrow a_1 b_2. & (44)
\end{align*}

The green number counts the dimers that pass through the cut to the right to it. From these diagrams, we now construct the substitution and entanglement matrices \( M \) and \( E \) that describe the Markov process underlying vertex inflation. While \( M \) is constructed as before, the entries of \( E \) are now composed of half the difference in dimer cuts between two inflation layers for a given substitution, as each dimer carries \( \log(2)/2 \) entanglement. We thus find

\begin{equation}
    M = \begin{pmatrix}
        2 & 1 & 1 & 1 \\
        1 & 2 & 1 & 1 \\
        1 & 0 & 1 & 0 \\
        1 & 0 & 0 & 1
    \end{pmatrix}, \\
    E = \begin{pmatrix}
        3/2 & 2 & 1 & 1 \\
        3/2 & 1 & 2 & 1 \\
        1 & 0 & 1 & 0 \\
        2 & 0 & 0 & 1
    \end{pmatrix}. 
\end{equation}

Using (21), which now becomes an equality rather than an upper bound, this leads to an effective central charge

\begin{equation}
    c_{(5,4)}^d = \frac{9 \log 2}{\log (\sqrt{3} + 2)} \approx 4.74, 
\end{equation}

or
where the superscript denotes our dimer construction. As in the aperiodic singlet case, this central charge describes the growth of entanglement entropies of sufficiently large boundary regions \( A \) with their length \(|A|\). Note that this value depends on the choice of inflation rule: For example, inflation via edges rather than vertices over the same dimer model leads to a smaller value for the central charge \([23]\).

Instead of a \(\{5, 4\}\) tiling, we can also consider a general \(\{5, k\}\) tiling with \(k > 3\), using the same perfect tensors on each tile. This corresponds to an inflation rule

\[
\begin{align*}
 a_1 &\mapsto a_1 b_1 (a_2 a_1 b_2)^{k-3}, & b_1 &\mapsto a_1 b_1 (a_2 a_1 b_2)^{k-4}, \\
 a_2 &\mapsto a_2 b_1 (a_2 a_1 b_2)^{k-3}, & b_2 &\mapsto a_1 b_2 (a_2 a_1 b_2)^{k-4}.
\end{align*}
\]

The substitution and entanglement matrices then take the more general form

\[
M = \begin{pmatrix}
  k - 2 & k - 3 & 1 & k - 3 \\
  k - 3 & k - 2 & 1 & k - 3 \\
  k - 3 & k - 4 & 1 & k - 4 \\
  k - 3 & k - 4 & 0 & k - 3
\end{pmatrix},
\]

and

\[
E = \begin{pmatrix}
  \frac{1+2(k-3)}{1+k-3} & \frac{2}{1+k-3} & \frac{1}{1+k-3} & 1 \\
  \frac{1+2(k-4)}{1+k-4} & \frac{2}{1+k-4} & \frac{1}{1+k-4} & 1 \\
  \frac{1+2(k-3)}{1+k-3} & 1 & 1 & 1 \\
  \frac{1+2(k-4)}{1+k-4} & 1 & 1 & 1
\end{pmatrix}.
\]

This leads us to the central charge

\[
e_{c}^{d_{\{5, k\}}} = \frac{\left(\frac{2}{m-3} + 10\right) \log 2}{\log \left(\frac{1}{2} \left(9k^2 - 48k + 60 + 3k - 8\right)\right)}.
\]

Note that this model corresponds to a bond dimension \(\chi = 2\), hence the \(\log 2\) term in the numerator. Considering the large \(k\) limit, we find

\[
e_{c}^{d_{\{5, k\}}} = \frac{10 \log 2}{\log (3k - 8)} + O(k^{-1}),
\]

which is exactly the same limit as the geodesic bound on central charges (Tab. I). As shown in Fig. 6 this saturation occurs quickly as \(k\) is increased. We can further extend this approach to the block perfect tensors associated with \(\{4m + 1, k\}\) tilings, whose central charges are computed in the Appendix B. In particular, consider the \(m = 2\) and \(m = 3\) case of nonagon and tridecagon codes, also shown in Fig. 6. As in the HyPeC case, the geodesic bound is saturated at large \(k\), with a slope at small \(k\) similar to the Brown-Henneaux value. Explicitly, at large \(n\) and \(k\) both the central charge bound and the exact Majorana dimer value scale as

\[
e_{c}^{d_{\{4m + 1, k\}}} = c_{\max}^{d_{\{4m + 1, k\}}} = \frac{(6m + 2) \log \chi}{\log (4mk - 8m - k)} + O(m^{-1}),
\]

with a bond dimension \(\chi = 2\) for the dimer model. Thus, we conclude that for tilings with high curvature (large \(n\) and \(k\)), our class of hyperbolic block perfect codes based on Majorana dimers produce maximal entanglement for any connected boundary regions \(A\). This is equivalent to saying that residual bulk regions become negligible in this limit, with a maximal flow of entanglement through the minimal cut \(\gamma_A\).

VII. DISCUSSION

In this work we have studied the entanglement entropy scaling of boundary states of generic hyperbolic tensor networks based on regular tilings. This has allowed us to derive a maximal central charge \(c_{\max}\) that such boundary states can possess, with a saturation of this bound corresponding to maximal entanglement through the bulk for any connected boundary region. We have then related \(c_{\max}\) to the radius of curvature \(\alpha\) of the metric into which the tiling is embedded, leading to a discrete analogue of the continuum Brown-Henneaux (BH) formula, where we have identified the gravitational constant \(G\) via the Ryu-Takayanagi (RT) prescription. We find that these bounds are always above the continuum value, i.e., that bulk entanglement through a regular hyperbolic tensor network can be as large as through a continuum AdS time-slice. We have further identified two distinctly different regimes: At large AdS radius \(\alpha\) and central charge \(c_{\max}\), we find an approximate relationship

\[
c_{\max} \approx c_0 + 6 \frac{\alpha \log \chi}{s},
\]

where \(s\) is the geodesic length of each edge in the tiling and \(\chi\) the bond dimension of the tensor network embedded into the tiling. The constant \(c_0\) depends on the \(n\)-gon tiling and increases with \(n\). In this limit, where the RT identification of \(G\) is expected to hold, the central charge increases with a constant offset compared to the BH formula. At small \(\alpha\) and \(c\), we identify a linear relationship

\[
c_{\max} \approx f \frac{\alpha \log \chi}{s},
\]

where the tiling-dependent constant \(f\) increases with \(n\), taking its lowest value \(f = 12\) for triangular tilings \((n = 3)\).
Furthermore, we find a specific holographic tensor network model that saturates these bounds: The hyperbolic pentagon code (HyPeC), which reproduces quantum error correction as expected in AdS/CFT. This model as well as its generalizations can be expressed in the fermionic language of Majorana dimers, which allows an exact treatment of its entanglement structure in terms of paired Majorana modes. Despite being an interacting model, its two-point correlations also follow the dimer structure. Using this picture, we showed how successively larger contractions of the tensor network produce a disordered renormalization group flow. This allows us to endow a class of models of holographic quantum error correction with the notion of a discretized conformal field theory with aperiodic structure. The exact central charges resulting from this physical CFT interpretation were derived and shown to saturate to $c^{\text{max}}$ at large curvature.

Our approach greatly advances the understanding of boundary states of holographic tensor network models, with bounds on central charges for any model based on a regular bulk geometry, which includes the HaPPY holographic codes [17], block perfect CSS codes [21], holographic codes on ideal regular tilings [20], hyper-invariant tensor networks [18], random tensor networks on fixed backgrounds [37], and $p$-adic AdS/CFT models [38], whose Bruhat-Tits tree is identified with a regular tiling [39]. We have also shown that the formulation of quantum error correcting codes in terms of Majorana dimers is essential for understanding their boundary states and RG flow. While the Majorana dimer states exhibiting strong disorder renormalization are non-interacting, strikingly, their use as a code basis in a quantum error correction code such as the HyPeC generally leads to interacting boundary states that can be captured by our approach. Our results thus show that entanglement renormalization of CFTs can be performed with tensor network approaches other than the MERA, realizing geometries that can be more naturally embedded into an AdS bulk geometry. Understanding their discrete symmetries will be crucial for the development of more powerful tensor network models of AdS/CFT.

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Appendix A: Geodesic inflation

In order to build inflation rules for regular tilings that inflate vertices on radial geodesics, we need to label vertices by their graph distance \(d\) to the centre of the tiling, or depth. For an \(\{n, k\}\) tiling with \(n > 5\) and \(k > 3\), we first distinguish between \(a\) vertices, which have two neighbouring vertices (up to the given inflation layer), and \(b\) vertices, which have three. Within the sequence of vertices at a given layer, we consider the depths \(d_L\) and \(d_R\) of the neighbours to the left and right with respect to the depth \(d\) of a given vertex. For \(b\) vertices, \((d_L, d_R) = (d + 1, d + 1)\). However, we need to distinguish five types of \(a\) vertices, listed in Table I. For even \(n = 2m\), only \(a_1\) to \(a_3\) appear, leading to an inflation rule

\[
\begin{align*}
  a_1 & \mapsto a_3^{m-3} b (a_2^{m-2} a_4 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  a_2 & \mapsto a_3^{m-3} b (a_2^{m-2} a_4 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  a_3 & \mapsto a_3^{m-3} b (a_2^{m-2} a_4 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  b & \mapsto a_3^{m-3} b (a_2^{m-2} a_4 a_3^{m-1} b) k^{-4} a_2^{m-3} a_4.
\end{align*}
\]

(A1) (A2) (A3) (A4) (A5)

and a corresponding substitution matrix

\[
M_{2m,k} = \begin{pmatrix}
  k - 2 & k(m - 2) - 2m + 3 & k(m - 2) - 2m + 3 & k - 2 \\
  k - 2 & (k - 2)(m - 2) & k(m - 2) - 2m + 3 & k - 2 \\
  k - 2 & k(m - 2) - 2m + 3 & (k - 2)(m - 2) & k - 2 \\
  k - 3 & (k - 3)(m - 2) & (k - 3)(m - 2) & k - 3
\end{pmatrix}.
\]

(A6)

The edge increase from inflation onto a \(b\) vertex is always 1, and increases with distance from the nearest \(b\) vertex. This is summarized in the entanglement matrix

\[
E_{2m,k} = \begin{pmatrix}
  m + \frac{1}{2 - k} & -2m^2 + k(m - 2)(m + 1) + 6 & -2m^2 + k(m - 2)(m + 1) + 6 \\
  m & 2k(m - 2) - 4m + 6 & 2k(m - 2) - 4m + 6 \\
  m + \frac{1}{2 - k} & -2m^2 + k(m - 2)(m + 1) + 6 & -2m^2 + k(m - 2)(m + 1) + 6 \\
  m & 2k(m - 2) - 4m + 6 & 2k(m - 2) - 4m + 6
\end{pmatrix}.
\]

(A7)

Applying (21) leads to the central charge bound

\[
c_{2m,k} \leq c_{\text{max}}^{2m,k} = \frac{3(m + 1) \log \chi}{\log (k(m - 1) + \sqrt{(k - 2)(m - 1)((k - 2)m - k) - 2m + 1})},
\]

(A8)

where \(\chi\) is the bond dimension of the underlying tensor network embedded into the \(\{2m, k\}\) tiling. For odd \(n = 2m + 1\), the inflation rule is more complicated and includes all five types of \(a\) vertices,

\[
\begin{align*}
  a_1 & \mapsto a_5 a_3^{m-3} b (a_2^{m-1} a_4 a_5 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  a_2 & \mapsto a_5 a_3^{m-3} b (a_2^{m-1} a_4 a_5 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  a_3 & \mapsto a_5 a_3^{m-3} b (a_2^{m-1} a_4 a_5 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  a_4 & \mapsto a_3^{m-3} b (a_2^{m-1} a_4 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  a_5 & \mapsto a_3^{m-3} b (a_2^{m-1} a_4 a_3^{m-1} b) k^{-3} a_2^{m-3} a_4, \\
  b & \mapsto a_3^{m-3} b (a_2^{m-1} a_4 a_3^{m-1} b) k^{-4} a_2^{m-3} a_4.
\end{align*}
\]

(A9) (A10) (A11) (A12) (A13) (A14)

\[
\begin{array}{c|c|c|c|c|c|c|c}
\text{Type} & a_1 & a_2 & a_3 & a_4 & a_5 & b = b_1 & b_2 & b_3 \\
\hline
\text{d}_L & d - 1 & d - 1 & d + 1 & d & d & d + 1 & d + 1 & d \\
\text{d}_R & d - 1 & d + 1 & d - 1 & d & d & d + 1 & d & d + 1
\end{array}
\]

TABLE II. Relative depth of vertex neighbours to the left and right of a given vertex with depth \(d\).
This leads to a substitution matrix

$$M_{(2m+1,k)} = \begin{pmatrix}
0 & k(m-2) - 2m + 3 & k(m-2) - 2m + 3 & k - 2 & k - 2 & k - 2 \\
0 & (k-2)(m-2) & k(m-2) - 2m + 3 & k - 2 & k - 2 & k - 2 \\
0 & k(m-2) - 2m + 3 & (k-2)(m-2) & k - 2 & k - 2 & k - 2 \\
1 & (k-2)(m-2) & k(m-2) - 2m + 3 & k - 3 & k - 2 & k - 2 \\
0 & k(m-2) - 2m + 3 & (k-2)(m-2) & k - 2 & k - 3 & k - 2 \\
0 & (k-3)(m-2) & (k-3)(m-2) & k - 3 & k - 3 & k - 3
\end{pmatrix}.$$  \hspace{1cm} (A15)

The entanglement matrix is given by

$$E_{(2m+1,k)} = \begin{pmatrix}
0 & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{m+1}{2} & \frac{m+1}{2} & 1 \\
0 & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{m+1}{2} & \frac{m+1}{2} & 1 \\
m & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{m+1}{2} & \frac{m+1}{2} & 1 \\
0 & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{m+1}{2} & \frac{m+1}{2} & 1 \\
0 & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{m+1}{2} & \frac{m+1}{2} & 1 \\
0 & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{-2m^2 + k(m-2)(m)+6}{2k(m-2)-4m+6} & \frac{m+1}{2} & \frac{m+1}{2} & 1
\end{pmatrix}.$$  \hspace{1cm} (A16)

The resulting central charge bound is

$$c_{\text{max}}^{(2m+1,k)} = \frac{3\left(m - \frac{1}{2(m-2)} + \frac{3}{2}\right) \log \chi}{\log \frac{2km+\sqrt{(-2km+k+4m)^2-4k-4m}}{2}}.$$  \hspace{1cm} (A17)

Note that for large $n$, (A8) and (A17) lead to the same asymptotic behavior,

$$c_{\text{max}}^{(n,k)} = \frac{(6+n) \log \chi}{2 \log (2 - 2k + (k-2)n)} + O\left(n^{-1}\right).$$  \hspace{1cm} (A18)

For \{n,3\} tilings (hyperbolic for $n > 6$), we also need to distinguish between even and odd $n$. In the case $n = 2m$, we find the inflation rule

$$a_1 \mapsto a_3^{m-3} b a_2^{m-3} a_1,$$  \hspace{1cm} (A19)

$$a_2 \mapsto a_3^{m-3} b a_2^{m-2} a_1,$$  \hspace{1cm} (A20)

$$a_3 \mapsto a_3^{m-2} b a_2^{m-3} a_1,$$  \hspace{1cm} (A21)

$$b \mapsto \emptyset.$$  \hspace{1cm} (A22)

and the substitution and entanglement matrices

$$M_{(2m,3)} = \begin{pmatrix}
1 & m - 3 & m - 3 & 1 \\
1 & m - 2 & m - 3 & 1 \\
1 & m - 3 & m - 2 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad E_{(2m,3)} = \begin{pmatrix}
m & m + 1 & m & 0 \\
m & m + 1 & m & 0 \\
m & m + 1 & m & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$  \hspace{1cm} (A23)

This yields a maximum central charge

$$c_{\text{max}}^{(2m,3)} = \frac{3(m+1) \log \chi}{\log \left(\sqrt{m^2 - 4m + 3} + m - 2\right)}.$$  \hspace{1cm} (A24)

For odd $n = 2m + 1$, inflation again involves $a_3$ to $a_5$:

$$a_1 \mapsto a_5 a_3^{m-3} b a_2^{m-3} a_4,$$  \hspace{1cm} (A25)

$$a_2 \mapsto a_5 a_3^{m-3} b a_2^{m-2} a_4,$$  \hspace{1cm} (A26)

$$a_3 \mapsto a_5 a_3^{m-2} b a_2^{m-3} a_4,$$  \hspace{1cm} (A27)

$$a_4 \mapsto a_5 a_3^{m-3} b a_2^{m-2} a_1,$$  \hspace{1cm} (A28)

$$a_5 \mapsto a_3^{m-2} b a_2^{m-3} a_4,$$  \hspace{1cm} (A29)

$$b \mapsto \emptyset.$$  \hspace{1cm} (A30)
This corresponds to

\[
M_{(2m+1,3)} = \begin{pmatrix}
0 & m-3 & m-3 & 1 & 1 & 1 \\
0 & m-2 & m-3 & 1 & 1 & 1 \\
0 & m-3 & m-2 & 1 & 1 & 1 \\
1 & m-2 & m-3 & 0 & 1 & 1 \\
0 & m-3 & m-2 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{A31}
\]

\[
E_{(2m+1,3)} = \begin{pmatrix}
0 & m & m & m & m-1 & 1 & 1 \\
0 & m+1 & m & m & m & m-1 & 1 \\
0 & m & m+1 & m & m & m-1 & 1 \\
m & m & m & m & m & m-1 & 1 \\
0 & m & m+1 & m & 0 & m-1 & 1 \\
0 & m & m & m+1 & m & m-1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{A32}
\]

and gives a central charge bound of

\[
c_{\{2m+1,3\}}^{\text{max}} = 3 \left( m - \frac{1}{4m-2} + \frac{3}{7} \right) \log \chi \left( \log \sqrt{4m^2 - 12m + 5 + 2m - 3} \right). \tag{A33}
\]

Note that even though the inflation rules are different, the bounds \[A24\] and \[A33\] agree with the generic \{n, k\} bounds \[A8\] and \[A17\] derived earlier. Similarly, the \{n, k\} inflation rules for \(n = 4\) and \(n = 5\) are special, as well, but lead to the same bounds. The \(n = 4\) case was already covered in the main text. For \(n = 5\), we need to split up \(b\) vertices into three categories \(b_1\), \(b_2\), and \(b_3\). For \(n = 5\), we find the inflation rules

\[
a_1 \mapsto b_3(a_2a_3b_1)^{k-4}a_2a_3b_2, \tag{A34}
\]

\[
a_2 \mapsto b_3(a_2a_3b_1)^{k-4}a_2a_3b_1a_1, \tag{A35}
\]

\[
a_3 \mapsto b_1(a_2a_3b_1)^{k-4}a_2a_3b_2, \tag{A36}
\]

\[
b_1 \mapsto a_3b_1(a_2a_3b_1)^{k-4}a_2, \tag{A37}
\]

\[
b_2 \mapsto a_3b_1(a_2a_3b_1)^{k-4}a_1, \tag{A38}
\]

\[
b_3 \mapsto b_1(a_2a_3b_1)^{k-4}a_2, \tag{A39}
\]

leading to substitution and entanglement matrices

\[
M_{\{5,k\}} = \begin{pmatrix}
0 & k-3 & k-3 & k-4 & 1 & 1 \\
1 & k-3 & k-3 & k-3 & 0 & 1 \\
0 & k-3 & k-3 & k-3 & 1 & 0 \\
0 & k-3 & k-3 & k-3 & 0 & 0 \\
1 & k-4 & k-3 & k-3 & 0 & 0 \\
0 & k-3 & k-4 & k-3 & 0 & 0
\end{pmatrix}, \quad E_{\{5,k\}} = \begin{pmatrix}
0 & 2 & 2 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 0 & 1 \\
0 & 2 & 2 & 1 & 1 & 0 \\
0 & 2 & 2 & 1 & 0 & 0 \\
2 & 2 & 2 & 1 & 0 & 0 \\
0 & 2 & 2 & 1 & 0 & 0
\end{pmatrix}. \tag{A40}
\]

This yields the expected maximum central charge

\[
c_{\{5,k\}}^{\text{max}} = \frac{10 \log \chi}{\log \sqrt{9k^2 - 48k + 60 + 3k - 8}}. \tag{A41}
\]
Appendix B: Majorana dimer polygon models

We can construct block perfect Majorana dimer models for an \( \{n, k\} \) tiling for \( n = 4m + 1 \), \( m \in \mathbb{N} \). The \( n = 5 \) case is simply the HyPeC model considered in the main text. For \( n = 9 \) and more complex polygons, we have to distinguish two cases: If \( k = 3 \), the inflation rule requires five different types of letters, while only four are needed in the \( k > 3 \) case. The inflation rule for the \( \{9, 3\} \) tiling follows from (25) and is given by the following dimer substitutions:

(B1) \[ a_1 \mapsto \begin{array}{c} \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \end{array}, \quad a_2 \mapsto \begin{array}{c} \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \end{array}, \quad b \mapsto \begin{array}{c} \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \end{array}. \]

(B2) \[ c \mapsto \begin{array}{c} \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \end{array}, \quad \text{a1} \mapsto \begin{array}{c} \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \end{array}, \quad \text{a2} \mapsto \begin{array}{c} \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \end{array}. \]

The inflation rule for the letters \( b \) and \( c \) has been combined for the sake of simplicity. The entanglement change under deflation depends on the cut and is given by

(B3) \[ a_1 \mapsto \begin{array}{c} \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \end{array}, \quad a_2 \mapsto \begin{array}{c} \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \end{array}, \quad b \mapsto \begin{array}{c} \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \\ \text{b} \end{array}. \]

(B4) \[ c \mapsto \begin{array}{c} \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \\ \text{c} \end{array}, \quad \text{a1} \mapsto \begin{array}{c} \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \\ \text{a1} \end{array}, \quad \text{a2} \mapsto \begin{array}{c} \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \\ \text{a2} \end{array}. \]

The substitution and entanglement matrices follow accordingly,

\[ M = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 3 & 2 & 1 & 2 \\ 3 & 5 & 2 & 1 & 1 \\ 4 & 4 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

which leads to a central charge

\[ c_d^{ \{9, 3\} } = \frac{16 \log \frac{2}{2+5} }{\log \frac{2\sqrt{7}+5}{2}} \approx 7.08. \]  

We can generalize this result to tilings at higher \( n = 4m+1 \), which correspond to an inflation rule

\[ a_1 \mapsto ca_2^{2m-3}a_1a_3^{2m-3}b, \quad b \mapsto ca_2^{2m-3}a_1a_3^{2m-3}b, \quad c \mapsto \emptyset. \]  

\[ a_2 \mapsto ca_2^{2m-2}a_1a_3^{2m-3}b, \quad a_3 \mapsto ca_2^{2m-3}a_1a_3^{2m-2}b. \]

The matrices \( M \) and \( E \) then take the form

\[ M = \begin{pmatrix} 2 & 2m-3 & 2m-3 & 1 & 1 \\ 1 & 2m-2 & 2m-3 & 1 & 1 \\ 1 & 2m-3 & 2m-2 & 1 & 1 \\ 1 & 2m-3 & 2m-3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 2m-1 & m+1 & m & 1 & 2 \\ 2m-1 & m+\frac{1}{2} & m & 1 & 1 \\ 2m & m+2 & m+\frac{1}{2} & 1 & 3 \\ 2m & m+2 & m+1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
From this we find the central charge
\[
c^d_{(4m+1,3)} = \left( \frac{6m + \frac{3}{10-8m} + \frac{9}{2} \log 2}{\log \sqrt{16m^2 - 24m + 3 + 4m - 3}} \right).
\] (B11)

Now consider the cases \( n = 4m+1, k > 3 \), which correspond to the inflation rules
\[
a_1 \mapsto a_2^{2m-2}a_1a_3^{2m-3}b \left( a_2^{2m-1}a_1a_3^{2m-2}b \right)^{k-3},
\] (B12)
\[
a_2 \mapsto a_2^{2m-1}a_1a_3^{2m-3}b \left( a_2^{2m-1}a_1a_3^{2m-2}b \right)^{k-3},
\] (B13)
\[
a_3 \mapsto a_2^{2m-2}a_1a_3^{2m-2}b \left( a_2^{2m-1}a_1a_3^{2m-2}b \right)^{k-3},
\] (B14)
\[
b \mapsto a_2^{2m-2}a_1a_3^{2m-2}b \left( a_2^{2m-1}a_1a_3^{2m-2}b \right)^{k-4}.
\] (B15)

We explicitly compute the \( \{9,4\} \) tiling, which can be expressed by the dimer inflation rules

\[
\begin{align*}
a_1 & \mapsto a_2, a_2, a_1, a_2, b, a_2, a_2, a_1, a_2, b, \\
& \quad \cdots,
\end{align*}
\] (B16)
\[
\begin{align*}
a_2 & \mapsto a_2, a_2, a_1, a_2, b, a_2, a_2, a_1, a_2, b, \\
& \quad \cdots,
\end{align*}
\] (B17)

Under deflation, the letters correspond to the following cuts:

\[
\begin{align*}
a_1 & \leftrightarrow a_2, a_2, a_1, a_2, b, a_2, a_2, a_1, a_2, b, \\
& \quad \cdots,
\end{align*}
\] (B18)
\[
\begin{align*}
a_2 & \leftrightarrow a_2, a_2, a_1, a_2, b, a_2, a_2, a_1, a_2, b, \\
& \quad \cdots,
\end{align*}
\] (B19)
From this we construct the entanglement and substitution matrices

\[ M = \begin{pmatrix} 3 & 5 & 3 & 2 \\ 2 & 6 & 3 & 2 \\ 2 & 5 & 4 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 10 & 14 & 7 & 1 \\ 2 & 5 & 1 & 1 \\ 4 & 16 & 5 & 1 \\ 4 & 7 & 2 & 1 \end{pmatrix}. \] (B20)

We then find the central charge

\[ c_{(9,4)}^d = \frac{81 \log 2}{5 \log (\sqrt{35} + 6)} \approx 4.53 . \] (B21)

For arbitrary \( k \), we find

\[ M = \begin{pmatrix} k-1 & 3k-7 & 2k-5 & k-2 \\ k-2 & 3(k-2) & 2k-5 & k-2 \\ k-2 & 3k-7 & 2(k-2) & k-2 \\ k-3 & 3k-10 & 2(k-3) & k-3 \end{pmatrix}, \quad E = \begin{pmatrix} \frac{4k-6}{k-1} & \frac{22-9k}{k-2} & \frac{13-5k}{k-2} & 1 \\ \frac{k-1}{k-2} & \frac{7-3k}{k-2} & \frac{5-2k}{k-2} & 1 \\ \frac{k-2}{k-2} & \frac{7-3k}{k-2} & \frac{5-2k}{k-2} & 1 \\ \frac{k-3}{k-2} & \frac{29-9k}{10-3k} & \frac{5}{2} & 1 \end{pmatrix}. \] (B22)

leading to

\[ c_{(9,k)}^d = \frac{619k-49 \log 2}{\log (\sqrt{19k^2-224k+252}+7k-16)}. \] (B23)

Generalizing even further to arbitrary \( n = 4m+1 \) yields the matrices

\[ M = \begin{pmatrix} k-1 & -4m + k(2m-1) + 1 & 2k(m-1) - 4m + 3 & k-2 \\ k-2 & (k-2)(2m-1) & 2k(m-1) - 4m + 3 & k-2 \\ k-2 & -4m + k(2m-1) + 1 & 2(k-2)(m-1) & k-2 \\ k-3 & -6m + k(2m-1) + 2 & 2(k-3)(m-1) & k-3 \end{pmatrix}, \quad E = \begin{pmatrix} 2m - \frac{2}{k-1} & \frac{2m + 1}{2-k} & \frac{2m}{2(k-3)m^2 + (k-3)m + 1} & 1 \\ \frac{2m}{2(k-2)m^2 + (k-2)m - k} & \frac{k-3}{k-2} + m & \frac{2m}{2(k-3)m^2 + (k-3)m + 1} & m + \frac{1}{2} \\ \frac{2m}{k-2(k-1)m + 1} & \frac{k-3}{k-2} + m & \frac{2m}{2(k-3)m^2 + (k-3)m + 1} & m + \frac{1}{2} \\ \frac{2m}{k-2(k-1)m + 1} & \frac{k-3}{k-2} + m & \frac{2m}{2(k-3)m^2 + (k-3)m + 1} & m + \frac{1}{2} \end{pmatrix}. \] (B24, B25)

Finally, the central charge for the \( \{4m+1,k\} \) (block) perfect Majorana dimer model for \( m \geq 1, k \geq 4 \) follows as

\[ c_{(4m+1,k)}^d = \frac{6 \left( -\frac{3k}{4}m + k + 6m + 1 + m \right) \log 2}{\log 4km + \sqrt{(-4km + k + 8m)^2 - 1^2}}. \] (B26)

In the large \( k \) limit the central charge behaves as

\[ c_{(4m+1,k)}^d \approx \frac{6 \left( \frac{4m^2 + 2m - 1}{4m-1} \right) \log 2}{\log \left( -1 + 4m \right) k - 8m} + O \left( k^{-1} \right). \] (B27)