“Regular element” global SLAE of the finite element method when simulating electromagnetic processes of electric devices

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Abstract. The article is devoted to mathematical and simulation modeling of electromagnetic processes of electric devices (ED). The development of industry, transport, power facilities, agriculture, and household services, health care leads to creation of electrical devices and systems for various functional purposes. It complicates the design (geometry) of electromagnetic and magnetic systems, improves magnetic and electrical characteristics of ED materials. The development of new software and hardware allows solving mixed (connected) tasks of simulating various physical processes in ED, electrical and electromechanical systems. However, general-purpose software complicates its engineering assimilation and implementation into production. The stage of ED numerical simulation modeling based on the finite element method when their investigating and designing is offered to be reduced and cheapened by changing the process of developing a global system of linear algebraic equations (SLAE) out of the system of elemental equations. The purpose of the work is to develop an algorithm to form global SLAE based on recurrent relations for its coefficients. The study subject is quasi-variable and magneto static vector models of ED electromagnetic processes, electro technical and electromechanical systems. Tasks are: developing of a "regular element", obtaining recurrent relations for global SLAE coefficients. Research methods are: numerical calculations based on the Galerkin’s projection-grid method along with the finite element method (FEM). Results are: a seven-point “regular element” for a regular triangulation network from triangular finite elements (FE) of arbitrary shape and rectangular triangles has been developed; recurrent relations for global SLAE coefficients have been determined; a possibility of obtaining engineering techniques for investigating ED with minimal computer's dynamic memory expenses has been shown on the example of calculating a linear induction motor (LIM).

1. Introduction
ED operation both as in system and as a separate energy-transforming unit, such as electric motors of various designs, electric drives, transformers, iron separators, switching and measuring equipment, etc., depends on various factors. However, it is the electromagnetic field power action that determines the electrical device operation, as it has an electromagnet or a permanent magnet as the main assembly, therefore much attention is paid to simulating electromagnetic processes there. Knowing the field distribution in the device, its differential and integral parameters and power characteristics in both static and dynamic modes can be calculated. And it, in turn, allows determining of ED energy, technical, economic and design parameters and characteristics [1, 2]. Research of electromagnetic processes, parameters and characteristics of electrical devices are carried out, mainly with known software packages, that is 2D and 3D simulation of ED with open core [3], mixed simulation of mechanical and electrical oscillations of induction motors [4], analytical simulation of linear induction machines (linear induction motors) [5, 6], FEM-simulation for determining losses in windings of electrical machines [7].

At present, a systematic approach to the analysis of various problems, especially technical, technological, production, design and engineering in the development of mixed or connected models of devices and systems connecting nodes and subsystems of various physical nature is being applied. The use of high-performance computer technology and its development to increase resource capacity and speed operation allows solving the listed problems at a high theoretical and mathematical level [8]. ED electromagnetic processes research and simulation is carried out based on numerical
projection-grid methods, mainly the finite element method [9, 10]. It is the basis of many modern software packages ANSYS, ELCUT, COMSOL etc [11, 12]. Due to the software packages versatility it is hard for the engineering staff to use them in production and it also leads to rather high cost, especially when replacing with a higher package version. That is why, the author suggests another, simplified approach [1] in the finite element method to form global SLAE out of elemental algebraic systems based on the triangulation net "regular element".

2. Problem statement. ED vector models
Electromagnetic or field processes occurring in electrical devices are described with two-dimensional and three-dimensional numerical vector models based on Maxwell's equations in partial derivatives [1, 2]. Maxwell differential equations system along with boundary conditions at outer boundaries and the conjugation conditions on the inner boundaries of domains with various magnetic and electrical properties, initial conditions is ED electromagnetic field simulation and allows to determine vectors values any time any point; \( \vec{E} \) is an electric field vector, \( \vec{H} \) is a magnetic field vector, \( \vec{B} \) is a magnetic induction vector for given initial conditions.

1. ED quasistationary or quasi-variable vector model
In determining quasistationary distribution of the electromagnetic field in the linear (the relative material permeability is \( \mu = \text{const} \) medium of investigated ED when there is no free electricity, with a sufficiently slow change in the time of currents and fields (quasi-variable or sinusoidal with frequency 50 Hz) Maxwell’s equations take the form:

\[
\begin{align*}
\text{rot } \vec{H} &= \vec{J}_{st} + \vec{J}_{pr}, \\
\text{rot } \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\
\text{div } \vec{B} &= 0, \\
\vec{B} &= \mu \mu_0 \vec{H}, \\
\vec{J}_{pr} &= \gamma(\vec{E} + \overline{v} \times \vec{B}).
\end{align*}
\]

where \( \vec{J}_{st} \) is an extraneous currents density vector; \( \vec{J}_{pr} \) is a vector of conduction current density caused in the conducting medium by electromagnetic field change in time and this medium movement there with velocity \( \overline{v} \); \( \mu_0 \) is a magnetic constant; \( \gamma \) is specific electrical conductivity.

To calculate quasistationary vector model (1)-(5) vector magnetic potential \( \vec{A} \) is used, its rotor is \( \text{rot } \vec{A} = \vec{B} \). Considering Coulomb’s calibration test div \( \vec{A} = 0 \) an equation of quasi-variable electromagnetic field with respect to vector potential \( \vec{A} \) for slow moving medium is obtained

\[
\frac{1}{\mu} \nabla^2 \vec{A} = -\mu_0 \vec{J} + \mu_0 \gamma \frac{\partial \vec{A}}{\partial t} - \mu_0 \gamma \overline{v} \times \text{rot } \vec{A},
\]

and for the medium fixed in space (\( \overline{v} = 0 \))

\[
\frac{1}{\mu} \nabla^2 \vec{A} = -\mu_0 \vec{J} + \mu_0 \gamma \frac{\partial \vec{A}}{\partial t},
\]

where \( \vec{J} = \vec{J}_{st} \).

Equations (6) and (7) are parabolic and have one solution if initial conditions are set and boundary (marginal) conditions are described. These equations are solved with numerical methods, and it allows determining the distribution of the vector potential \( \vec{A} \) in the modeling domain \( \mathcal{V} \) in time period \( t \), and then it allows determining the distribution of vectors \( \vec{H} \) and \( \vec{E} \); and induction \( \vec{B} \) of electromagnetic field, as well as ED power, technical and economic parameters and characteristics.

2. ED magnetostatic vector model
For some electrical devices when there is no time-varying current supply Maxwell’s equations have form
\[ \text{rot} \, \vec{H} = \vec{J}, \]
\[ \text{div} \, \vec{B} = 0, \]
\[ \vec{B} = \mu \mu_0 \vec{H}. \]

For linear medium, these equations are transformed to an elliptic equation or Laplace-Poisson’s equation with respect to the vector potential
\[ \frac{1}{\mu} \nabla^2 \vec{A} = -\mu_0 \vec{J}. \] (8)

In linear sectionally-homogeneous medium equation (8) with boundary conditions and conjugation conditions on inner boundaries of V domain is a magnetostatic vector model or a boundary problem to be solved with numerical projection-grid.

Quasistationary and magnetostatic vector models are used to study electromagnetic processes occurring in electric energy converters, such as electromagnetic and magnetic iron separators, magnetic field concentrators of various design versions, an electromagnetic motor, a linear induction pump (LIP).

3. Numerical projection-grid method or finite elements method for solving quasistationary electromagnetic boundary problems

Numerical calculation methods are used to study quasistationary magnetic fields and power characteristics of electrical devices with a complex geometry of the modeling domain and dissimilar physical properties. ED numerical models have inner boundaries of sectionally-homogeneous medium and some of them are mobile, as well as domains with distributed density of extraneous winding currents and secondary elements with eddy currents. The solution of differential equations in partial derivatives for their models is carried out with the projection-grid method or the finite element method as projection methods modifications (Ritz, Galerkin etc.) [1, 8]. Projection methods allow approximating the solution of a differential equation by a finite linear combination of basis (trial) functions. The form of a basis function and a criterion for calculating coefficients of a linear combination determine the projection method.

If a linear combination
\[ \vec{A} = \sum_{m=1}^{q} \varphi_m A_m \]
is an approximate solution of equation \( L_f(A) = f \), where \( L_f \) is a differential operator, then according to Galerkin coefficients of a given linear combination are determined out of the orthogonality condition residual to basis functions \( N_m \), i.e.
\[ \int_V N_m (L_f(\vec{A}) - f) \, dV = 0, \] (9)

where \( V \) is a modeling domain; \( m = 1, ..., q \). In case when \( \varphi_m = N_m \), the result obtained by Galerkin’s method coincides with the result of solution \( L_f(A) = f \) at variational approach [8].

If the modeling domain is divided into finite elements (FE), where the required function \( A \) is approximated with polynomial, and finite functions are used as basis functions, that become zero everywhere except FE, then transformation of integral equations system (9) results to SLAE with a rarefied banded structure. The numerical solution of equation \( L_f(A) = f \) by Galerkin’s method along with FEM has positive qualities of variational, projective and finite difference methods. This article considers electric motors and power converters of electromagnetic energy with a planar magnetic system; therefore, a plane quasistationary electromagnetic field in a Cartesian coordinate system is investigated, and the modeling domain \( V \) is presented with \( S \) domain and \( x \) and \( y \) coordinates.
4. “Regular element” and algorithm of forming global SLAE

Distribution of ED quasistationary plane-parallel magnetic field is described by the Laplace-Poisson’s equation system relatively to the vector magnetic potential, and also with homogeneous and inhomogeneous Neumann’s or Dirichlet’s boundary conditions at the outer boundary of the model. The solution of this problem is carried out by Galerkin’s method - FEM, the sectionally-homogeneous modeling domain S of the electrical device is divided by a grid with q nodes per p triangular simplex-elements, vector magnetic potential is represented on every one as follows:

$$\tilde{A}_R = (N_i A_j + N_j A_k + N_k A_i) R = \{N_m\} R \{A_m\} R,$$

where \{A_m\} R is a column matrix of the magnetic vector potential in the computing net nodes, \{N_m\} R is row matrix of basis functions \(N_m = (a_m + b_m x + c_m y) / (2 S_R)\), and coefficients \(a_m, b_m, c_m\) are determined with coordinates of three nodes \(m = 1, j, k\) of element \(R\) [9, 10]. The elemental SLAE is ordered with respect to q nodes of the computing triangulation net by summing the corresponding element equations (10), and thereby a global SLAE is formed

$$[U] [A] = [F],$$

where \([U]\) is a banded matrix of coefficients \(u_{ms}\) and \(m = 1, ..., q; s = 1, ..., q; [A]\) is a column matrix of nodal values of vector magnetic potential \(A_m\) to be determined; \([F]\) is a column matrix of transient components of global SLAE.

The algorithm for obtaining matrix coefficients \([U]\) is simplified by using a regular triangulation net, and it automates SLAE formation, therefore minimizing the costs of the heap memory. The global system of equations (11) is solved with a direct or iterative method.

"Regular element" (figure 1), is a geometric figure, there is a calculating node in the center, it is surrounded with \(p_n\) finite elements, for example, six, i.e. \(p_6 = 6\). A triangular element of arbitrary shape (figure 1, a) or a rectangular triangle (figure 1, b) is selected in Cartesian coordinate system as FE. FE number can be arbitrary in a "regular element". The selected seven-point pattern is the most convenient, as for FE in rectangular triangles form a part of coefficients \(u_{ms}\) is equal to zero.

![Figure 1. "Regular element" for a regular triangulation net out of triangles (FE) of arbitrary form (a) and of rectangular triangles (b)](image)

The simulation domain is divided with the same or similar "regular elements". For nodes of the net consisting of "regular elements", the global SLAE equations are written down according to a single computing scheme.

As an example of the approach, the process of research and design of a linear induction motor (LIM) was considered [1, 5, 6]. Its technical-economic and power parameters and characteristics analysis was
carried out and it was based on the calculation of vector models of a quasi-variable electromagnetic field for complex variables that simulate values varying in time with a sinusoidal frequency 50 Hz. LIM mathematical model is two-dimensional in the Cartesian coordinate system, it has an ideally laminated inductor \((\mu_r = \infty)\). Secondary medium with specific electrical conductivity \(\gamma\) moves with speed \(v = i v\) along a working channel with height \(\Delta\). The model is symmetric about the \(x\) axis, so only its upper part is considered. The current load is a sinusoidal time-varying extraneous current and is given with a system of pulsatory currents in \(q_1\) slots on the inductor surface with a current density vector that has only \(z\)-component, i.e.

\[
J_n = k J_n e^{-j\omega t},
\]

where \(I\) is amplitude of extraneous current; \(S_n\) is slot space; \(w\) is a number of turns in the slot; \(\Psi_R\) is current phase; \(n = 1, 2, ..., q_1\).

For the described LIM model in rectangular Cartesian coordinates and considering the complex nature of changes in time of the vector potential, the current density vector, quasi-variable magnetic field vectors

\[
A = k \dot{A} e^{j\omega t}, \quad \dot{E} = -j \frac{\partial \dot{A}}{\partial t} = -j \omega \dot{A},
\]

scalar analogue (6) for the \(z\)-component of the magnetic vector potential can be written down

\[
\frac{\partial^2 \dot{A}}{\partial x^2} + \frac{\partial^2 \dot{A}}{\partial y^2} - \mu_0 \gamma v \frac{\partial \dot{A}}{\partial x} - j \mu_0 \gamma \omega \dot{A} = - \mu_0 \sum_{m=1}^{q_1} \dot{J}_n.
\]

To solve equation (12) with Galerkin’s projection-grid method - FEM, the computing domain is divided into \(p\) triangular simplex-elements; the function \(\dot{A}\) is approximated with the polynomial

\[
\tilde{A}_R = (N_i A_i + N_j A_j + N_k A_k)_{R} = \{N_m \}_R \{ A_m \}_R,
\]

where \(\{A_m\}_R = \{A_m/R\} + j\{A_m/\}_R\) nodal values column vector of complex values of magnetic vector potential \(A\).

The system of linear algebraic equations (13) is transformed into global \(q\) nodes with respect to the computing net. Each node corresponds to two equations from the global system, for example, for node 4 (figure 1), they have the form

\[
\sum_{m=1}^{7} (w_{4m} A'_m - v_{4m} A''_m) = -\sum_{R=1}^{6} \tilde{F}_R', \quad \sum_{m=1}^{7} (v_{4m} A'_m + w_{4m} A''_m) = -\sum_{R=1}^{6} \tilde{F}_R''.
\]

Coefficients \(w_{4d}\) and \(v_{4d}\) are determined with the recurrence relations:

\[
w_{4d} = \frac{1}{4\mu_0 S_1}(b^{(1)}_4 c^{(1)}_4 + c^{(1)}_4 c^{(1)}_4) + p'/2 b^{(1)}_4/6 + p''/2 b^{(2)}_4/6 + \frac{1}{4\mu_0 S_2}(b^{(2)}_4 c^{(2)}_4 + c^{(2)}_4 c^{(2)}_4);
\]

\[
v_{4d} = (d'/2 S_1 + d''/2 S_2)/12.
\]

Real and imaginary components of transient components in (14) for the elements simulating the stator winding slots are calculated as follows:

\[
\tilde{F}' = -\mu_0 IwS_R \cos \psi_R/(3S_n),
\]

\[
\tilde{F}'' = -\mu_0 IwS_R \sin \psi_R/(3S_n).
\]

Coefficients for indeterminate \(A'_m\) and \(A''_m\) in SLAE (14) for the "regular element" with the same rectangular triangles (figure 1, a) are simplified:

\[
w_{4d} = -I/4 - k_1 p' h/6; \quad w_{4d} = -I/4 + k_1 p' h/6;
\]
$$w_{34} = - v' - p' h/6; \quad w_{54} = - v' + p' h/6;$$

$$w_{24} = - p' h/6; \quad w_{64} = p' h/6; \quad w_{44} = 2\sqrt{v'} + 2 v';$$

$$v_{14} = v_{24} = v_{34} = v_{54} = v_{64} = v_{74} = k_1 q' S/12; \quad v_{44} = k_1 q' S/2,$$

where $k_1 = 1$ for nodes on the boundary "air-secondary medium" and $k_1 = 2$ for secondary medium nodes.

Global SLAE is solved with iterative or direct methods. Then power and technical and economic parameters and characteristics can be calculated.

5. Discussion

A seven-point "regular element" for a regular triangulation net out of triangular finite elements (FE) of arbitrary shape and rectangular triangles has been developed; recurrence relations for global SLAE coefficients has been derived, possibility to obtain engineering methods for ED investigating with minimal costs of heap memory of computation, time has been presented on the example of linear induction motor (LIM) calculation. It simplifies the process of numerical simulation and research both ED and electrical and electromechanical systems, i.e. to reduce the cost of these devices engineering.

The improvement of the algorithm to form global SLAE coefficients based on the recurrence relations obtained for the "regular element" in the numerical method FEM while studying two-dimensional electromagnetic fields is aimed at the use of a three-dimensional "regular element" based on a FE parallelepiped-type for the calculation three-dimensional electromagnetic fields of ED.

6. Conclusion

The use of "regular elements" in calculation of ED magnetic field with Galerkin’s method - FEM allows passing the stage of forming elemental equations and start forming global SLAE based on recurrence relations. Thus, the problem of storage and transformation (primarily for direct methods of solving SLAE) of the global matrix in the computer memory, especially for a PC, is simplified. It is important when developing software for engineering design systems for electrical devices and drives when mixed or linked problems for technical systems with subsystems of a diverse physical nature: electric, magnetic, electromechanical, thermal, magneto hydrodynamic, are to be solved.

7. References

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