Time response of a scalar dynamical system with multiple delays via Lambert W functions

Shuo-Tsung Chen\textsuperscript{a}, Shun-Pin Hsu\textsuperscript{b}, Huang-Nan Huang\textsuperscript{a,∗}, Bin-Yan Yang\textsuperscript{b}

\textsuperscript{a}Department of Applied Mathematics, Tunghai University, Taichung 40704, Taiwan
\textsuperscript{b}Department of Electrical Engineering, National Chung-Hsing University, Taichung 40704, Taiwan

Abstract

In this work, we establish the response of scalar systems with multiple discrete delays based on the Laplace transform. The time response function is expressed as the sum of infinite series of exponentials acting on eigenvalues inside countable branches of the Lambert W functions. Eigenvalues in each branch of Lambert W function are computed by a numerical iteration. Numerical examples are presented to illustrate the results obtained.

Keywords: delay system, Lambert W function, multiply delays, time response

2010 MSC: 34A45, 34K06, 34K07, 34K35, 44A10

1. Introduction

Dynamical systems with delays play an important role in modeling natural processes which include the influence of past effects for a better description of the evolution. In classical physics, life sciences, physiology, engineering system, neural networks, epidemiology, and economics, realistic models must take in account the time-delays due to the finite propagation speed to determine the future evolution. Many examples of real phenomena with delay effects can be found in \textsuperscript{[1, 2, 3, 4, 5, 6, 7, 8]}. Furthermore of importance in applications, the delay system is, in general, described by delay differential equations which have several distinct mathematical properties of ordinary and partial differential equations, and also provides them with a purely mathematical interest.

In this work, we consider the differential equation with discrete delays. Let \( N \in \mathbb{N}, a, b, h, x_0 \in \mathbb{R}, \) not all zero \( a_{jd} \in \mathbb{R}, 1 \leq j \leq N, \) and \( \phi \) is a continuous preshape function defined on \([-Nh, 0]\). Consider the following delay differential
equation:
\[
\dot{x}(t) = a x(t) + \sum_{j=1}^{N} a_j x(t - jh) + b u(t),
\]
\[
x(0) = x_0, \quad (DDE)
\]
\[
x(\tau) = \phi(\tau), \quad -Nh \leq \tau < 0
\]

We allow the discontinuity of the preshape function \(\phi\) at \(t = 0\), i.e., \(\phi(0)\) is not necessary equal to \(x_0\). We want to find analytically the formula to describe the effect of input function \(u(t)\) to the system’s output \(x(t)\) instead of just the output due to a specified input via numerical method, like \texttt{dde23} in Matlab. For those delay systems with different lengths of delay-time, it can also be transferred into the form given by (DDE). Thus our result can be applied without any difficulty.

For the single delay differential equation, i.e., set \(N = 1\), \(h = 1\), \(a = 0\), and \(b = 0\) into (DDE):
\[
\dot{x}(t) = a_1 x(t - 1).
\]

Refer to the paper by Coreless et. al., 1996 [9], it is suggested by using Lambert W function to describe the corresponding solution as
\[
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{W_k(a_1)t}
\]
for any reasonable choice of \(c_k\) (i.e. such that the sum makes sense). Here \(W_k\) denotes the \(k\)-th branch of Lambert W function.

Introduced by Lambert and Euler in 1700’s, the Lambert W function is defined to be any function \(W(x)\) satisfies
\[
W(x)e^{W(x)} = x \quad (1.1)
\]
Taking the logarithms on both sides of (1.1) leads to
\[
\log W(x) = \log(x) - W(x)
\]
after some arrangement, and it is obviously that there are infinite many solution due to the property of logarithmic functions in the complex plane. When \(x \in \mathbb{R}\) and \(-1/e \leq x < 0\), there are two possible real values of \(W(x)\) (see Figure 1a).

Denote the branch satisfying \(-1 \leq W(x)\) by \(W_0(x)\), or just \(W(x)\) when there is no confusion, and the branch satisfying \(W(x) \leq -1\) by \(W_{-1}(x)\). The \(W(x)\) is referred to as the \textit{principal branch} of the Lambert W function. When \(x\) is complex, we use \(W_k(x)\) to denote the \(k\)-th branch of the Lambert W function whose ranges are shown in Figure 1b. We can verify that \(W_k(z) = W_{-k}(z)\) for all integers \(k\). Applications of Lambert W function, we refer to Coreless et. al, 1996 [9], and references there in.

Alternative, consider a single delay system, i.e., \(N = 1\) in (DDE) and the differential equation becomes
\[
\dot{x}(t) = a x(t) + a_1 x(t - h) + b u(t).
\]
Asl and Ulsoy, 2003 [10] and Yi et. al., 2010 [11] use Lambert W function to represent the time response function of this equation and its extension to systems of differential equations, i.e., real numbers $a$ and $a_{1d}$ turn to be real matrices $A$ and $A_{1d}$ of size $n \times n$. And only a single delay is considered in these study.

In this work, we consider for a general $N \in \mathbb{N}$, i.e., multiple discrete delays. Firstly, Laplace transform is applied to the equation (DDE) and state transition function is then obtained. The associated characteristic equation with the exponential terms is iteratively solved via Lambert W function with infinite countable branches. Then the transformed equation is expanded in partial fractional expansion of the system’s eigenvalues, and we can then obtain the analytical expression of time response as the sum of serial combination of exponential functions acting on different branches of the corresponding Lambert W function. Numerical examples are provided for illustrative purpose.

This article is organized as follows. In Section 2 we use the Laplace transform to obtain the transformed (DDE) in the frequency domain and then find the associated time response via state transition function. Section 3 gives a brief introduction of Lambert W functions and then use it to express the state transition function. The eigenvalues of the system is then solved iteratively in terms of Lambert W functions. Numerical examples are given in section 4 for illustrative purpose. Some concluding remark is presented in last section.

2. Time response function by Laplace transform

2.1. Laplace transform

Let $F(s) = \mathcal{L}[f(\cdot);s] = \int_0^\infty f(t)e^{-st}dt$ denote the Laplace transform of an exponential order function $f(\cdot)$ defined on $[-Nh, \infty)$ for some $N \in \mathbb{N}$. Let $X(s)$
and \( U(s) \) be the Laplace transform of the \( x(\cdot) \) and \( u(\cdot) \), respectively. And we can also define
\[
\Phi_j(s) \triangleq \int_{-jh}^{0} \phi(\tau) e^{-s\tau} d\tau \tag{2.1}
\]
to be the Laplace transform of the preshape function for some \( k \in \mathbb{N} \).

Let us extend the preshape function \( \phi(\cdot) \) to the whole real line as \( \phi_e(\cdot) \):
\[
\phi_e(t) = \begin{cases} 
0, & t > 0 \\
\phi(t), & t \in [-Nh,0] \\
0, & t < -Nh,
\end{cases}
\]
Then its Laplace transform with \( 1 \leq j \leq N \) is computed as following:
\[
L[\phi_e(\cdot-jh);s] = \int_{0}^{\infty} \phi_e(t-jh) e^{-st} dt = \int_{-jh}^{\infty} \phi_e(\tau) e^{-s\tau} e^{-jsh} d\tau
\]
\[
= e^{-jsh} \int_{-jh}^{0} \phi(\tau) e^{-s\tau} d\tau = e^{-jsh} \Phi_j(s)
\]
and it follows that
\[
\int_{0}^{\infty} x(t-jh) e^{-st} dt
\]
\[
= e^{-jsh} \int_{-jh}^{0} \phi(\tau) e^{-s\tau} d\tau + e^{-jsh} \int_{0}^{\infty} x(\tau) e^{-s\tau} d\tau
\]
\[
= e^{-jsh} \Phi_j(s) + e^{-jsh} X(s) \tag{2.2}
\]

The corresponding Laplace transform of \( \text{[DDE]} \) is given by
\[
sX(s) - x_0 = aX(s) + \sum_{j=1}^{N} \left( a_je^{-jsh} X(s) + a_je^{-jsh} \Phi_j(s) \right) + bU(s)
\]
which implies
\[
\left( s - a - \sum_{j=1}^{N} a_je^{-jsh} \right) X(s) = x_0 + \sum_{j=1}^{N} a_je^{-jsh} \Phi_j(s) + bU(s) \tag{2.3}
\]
Thus,
\[
X(s) = \frac{1}{\Delta(s)} \left( x_0 + \sum_{j=1}^{N} a_je^{-jsh} \Phi_j(s) + bU(s) \right) \tag{2.4a}
\]
where
\[
\Delta(s) = s - a - \sum_{j=1}^{N} a_je^{-jsh}. \tag{2.4b}
\]
where \( \Phi_j(s) \) is defined in \( \text{[2.1]} \). Equation (2.3) gives us the Laplace transform version of the \( \text{[DDE]} \).
2.2. State transition function

How to find the inverse Laplace transform of (2.3)? Let

\[ \mathcal{L}[\Psi(t); s] \triangleq \frac{1}{\Delta(s)} \]  (2.5)

which \( \Psi(t) \) denotes the corresponding state transition function of a DDE whose definition is described by

**Definition 2.1.** A function \( \Psi(t) \) is called the fundamental solution or state transition function of a DDE if it satisfies the following conditions:

\[
\frac{d}{dt}\Psi(t) = a\Psi(t) + \sum_{j=1}^{N} a_{jd}\Psi(t - jh), \quad t > 0
\]

\[
\Psi(0) = 1, \Psi(t) = 0, \quad t < 0.
\]

It is obvious that the state transition function in Definition 2.1 must also satisfy

\[
\frac{d}{dt}\Psi(t) = \begin{cases} 
a\Psi(t), & 0 < t \leq h \\
a\Psi(t) + a_{1d}\Psi(t - h), & h < t \leq 2h \\
a\Psi(t) + a_{1d}\Psi(t - h) + a_{2d}\Psi(t - 2h), & 2h < t \leq 3h \\
\vdots
\end{cases}
\]

\[
\Psi(0) = 1, \Psi(t) = 0, \quad t < 0.
\]

This definition of fundamental solution is the same as Gu et al. [12, check (1.11)] and is also much simpler than the one proposed by Yi et al. 2010 [11, check (2.27)] when \( N = 1 \) and Krisztin and Vas, 2011 [13] for general \( N \) but no \( a \) terms.

Before we step into finding the inverse Laplace transform, consider the integral given by the lemma:

**Lemma 2.2.** For \( 1 \leq j \leq N \), the following relationship holds:

\[
\mathcal{L} \left[ \int_{0}^{t} \Psi(t - \tau)a_{jd}\phi_{e}(\tau - jh)d\tau; s \right] = \frac{e^{-sjh}a_{jd}\Phi(s)}{\Delta(s)}
\]

PROOF. Note that by changing of variable, we have

\[
\int_{0}^{t} \Psi(t - \tau)a_{jd}\phi_{e}(\tau - jh)d\tau = \int_{-jh}^{t-jh} \Psi(t - \xi - jh)a_{jd}\phi_{e}(\xi)d\xi
\]
For a fixed \(1 \leq j \leq N\), we let

\[
\mathcal{L} \left[ \int_0^t \Psi(t-\tau) a_{jd}\phi_e(\tau - jh)d\tau; s \right]
= \int_0^\infty \left[ \int_0^t \Psi(t-\tau) a_{jd}\phi_e(\tau - jh)d\tau \right] e^{-st}dt
\]

and switching the order of integration leads to

\[
\int_0^\infty \int_0^\infty \Psi(t-\tau) a_{jd}\phi_e(\tau - jh)e^{-st}d\tau dt
= \int_0^\infty \int_0^\infty \Psi(t-\tau)e^{-st}dt a_{jd}\phi_e(\tau - jh)d\tau
= \int_0^\infty \int_0^\infty \Psi(t-\tau)e^{-s\xi}d\xi e^{-st}a_{jd}\phi_e(\tau - jh)d\tau
= \int_0^\infty \frac{1}{\Delta(s)} a_{jd}\phi_e(\tau - jh)e^{-st}d\tau
= \frac{a_{jd}e^{-jsh}}{\Delta(s)} \int_{-jh}^\infty \phi_e(\xi)e^{-s\xi}d\xi
= \frac{a_{jd}e^{-jsh}}{\Delta(s)} \int_{-jh}^{0} \phi_e(\xi)e^{-s\xi}d\xi
= \frac{a_{jd}e^{-jsh} \Phi_j(s)}{\Delta(s)}
\]

### 2.3. Time response function

Lemma 2.2 gives us the time domain correspondence of the second term in (2.4a). Thus the time response of \(DDE\) for \(t \geq 0\) is then obtained by finding the time-domain function corresponding to equation (2.4a), i.e.,

\[
x(t) = \Psi(t)x_0 + \sum_{j=1}^N \int_0^t \Psi(t-\tau) a_{jd}\phi_e(\tau - jh)d\tau + \int_0^t \Psi(t-\tau)bu(\tau)d\tau
= \Psi(t)x_0 + \sum_{j=1}^N \int_{-jh}^\min(0,t-jh) \Psi(t-\tau - jh)a_{jd}\phi(\tau)d\tau + \int_0^t \Psi(t-\tau)bu(\tau)d\tau
\]

Based on previous discussion, the solution of delay differential equation can be expressed in terms of fundamental matrix \(\Psi(t)\) is given in the following theorem.
Theorem 2.3. Under appropriate conditions, the solution of the DDE for \( t \geq 0 \) is given by the following equation:

\[
x(t) = \Psi(t)x_0 + \sum_{j=1}^{N} \int_{t-jh}^{\min\{0,t-jh\}} \Psi(t-\xi-jh)a_{1d}\phi(\xi)d\xi
\]

\[+ \int_{0}^{t} \Psi(t-\tau)bu(\tau)d\tau \tag{2.7}\]

PROOF. For simplicity of the algebraic operation in the proof, we set \( N = 1 \), i.e., single delay case, to explore the basic idea. For more general \( N \), a similar procedure is applicable. Taking the derivative to (2.7) gives us the following equation:

\[
\dot{x}(t) = \Psi(t)x_0 + bu(t) + \int_{0}^{t} \Psi(t-\tau)bu(\tau)d\tau
\]

\[+ \begin{cases}
\int_{-h}^{0} \dot{\Psi}(t-\xi-h)a_{1d}\phi(\xi)d\xi & \text{if } t > h \\
a_{1d}\phi(t-h) + \int_{-h}^{t-h} \dot{\Psi}(t-\xi-h)a_{1d}\phi(\xi)d\xi & \text{if } t \leq h
\end{cases} \tag{2.8}\]

Since \( \xi : -h \to \min\{0,t-h\} \) implies \( t - \xi - h : t \to \max\{0,t-h\} \), then

\[
\frac{d}{dt} \Psi(t-\xi-h) = \begin{cases}
a\Psi(t-\xi-h), 0 < t - \xi - h \leq h \text{ (or } t - 2h \leq \xi < \min\{0,t-h\}) \\
a\Psi(t-\xi-h) + a_{1d}\Psi(t-\xi-2h), h < t - \xi - h \leq t \text{ (or } -h < \xi \leq t - 2h) 
\end{cases}
\]

And similarly, as \( \tau : 0 \to t \) implies \( t - \tau : t \to 0 \), then

\[
\frac{d}{dt} \Psi(t-\tau) = \begin{cases}
a\Psi(t-\tau), 0 < t - \tau \leq h \text{ (or } t - h \leq \tau < t) \\
a\Psi(t-\tau) + a_{1d}\Psi(t-\tau-h), h < t - \tau \leq t \text{ (or } 0 \leq \tau < t-h) 
\end{cases}
\]

Thus

\[
\int_{-h}^{\min\{0,t-h\}} \dot{\Psi}(t-\xi-h)a_{1d}\phi(\xi)d\xi
\]

\[= \begin{cases}
\int_{-h}^{t-2h} [a\Psi(t-\xi-h) + a_{1d}\Psi(t-\xi-2h)]a_{1d}\phi(\xi)d\xi & \text{if } t > h \\
\int_{-h}^{0} a\Psi(t-\xi-h)a_{1d}\phi(\xi)d\xi + \int_{-h}^{t-2h} a\Psi(t-\xi-h)a_{1d}\phi(\xi)d\xi & \text{if } t \leq h
\end{cases}
\]

and

\[
\int_{0}^{t} \dot{\Psi}(t-\tau)bu(\tau)d\tau = \begin{cases}
\int_{0}^{t-h} [a\Psi(t-\tau) + a_{1d}\Psi(t-\tau-h)]bu(\tau)d\tau & \text{if } t > h \\
\int_{0}^{t} a\Psi(t-\tau)bu(\tau)d\tau & \text{if } t \leq h
\end{cases}
\]

Therefore, the substitution of above derivative terms into (2.8) lead to the following equation: when \( t > h \)

\[
\dot{x}(t) = a\Psi(t)x_0 + a_{1d}\Psi(t-h)x_0 + \int_{-h}^{t-2h} [a\Psi(t-\xi-h) + a_{1d}\Psi(t-\xi-2h)]a_{1d}\phi(\xi)d\xi
\]

7
where

\[ x \text{ when } t \leq 0 \]

Similarly, we can prove for the case with \( 2 \leq i \leq N \).

### 3. Approximate Solution via Lambert W Functions

#### 3.1. Lambert W function

Introduced by Lambert and Euler in 1700’s, the Lambert W function is defined to be any function \( W(x) \) satisfies

\[ W(x)e^{W(x)} = x \]  

As suggested by Coreless et. al., 1996 [9] and Als and Ulsoy, 2003 [10], the solution may be expressed in terms of infinite number of branches of the Lambert W function. We begin from (2.3):

\[ X(s) = \frac{1}{\Delta(s)} \left( x_0 + \sum_{j=1}^{N} a_{jd}e^{-jsb} \Phi_j(s) + bU(s) \right) \]  

where

\[ \Delta(s) = s - a - \sum_{j=1}^{N} a_{jd}e^{-jsb}. \]
And the equation
\[ \Delta(s) = 0 \]
is called the characteristic equation of the system. The zeros of the characteristic equation are the eigenvalues of the generator of the strong continuous semigroup defined by the solution operators \([13]\). To find its zeros is corresponding to solving the equation
\[
s - a - \sum_{j=1}^{N} a_j d e^{-jsh} = 0
\]
which can be re-expressed as
\[
(s - a) he^{(s-a)h} = \left( \sum_{j=1}^{N} a_j d e^{-jsh} \right) he^{(s-a)h} = \sum_{j=1}^{N} a_j d h e^{-jah} e^{-(j-1)(s-a)h}.
\]
Suppose \(s_{j,k}\) is the root of the characteristic equation, i.e., \(\Delta(s_{j,k}) = 0\), \(s_{j,k}\) must also satisfy the Lambert W equation
\[
s_{j,k} = \frac{1}{h} W_k \left( \sum_{j=1}^{N} a_j d h e^{-jah} e^{-(j-1)(s_{j,k}-a)h} \right) + a, \quad 1 \leq j \leq N
\]
where \(W_k(z)\) denotes the \(k\)-th branches of Lambert W function, which is a one-one function for \(z \neq 0 \in \mathbb{C}\) (Note \(W_0(0) = 0\) is in the principal branch), thus \(s_{j,k} \neq s_{i,\ell}\) when \(k \neq \ell \in \mathbb{Z}\) for any \(1 \leq i, j \leq N\). It is obviously true that the conjugate of the solution of \(\Delta(s) = 0\) is also another solution whose proof is given below.

**Lemma 3.1.** Suppose \(\Delta(S) = 0\) for some complex number, then \(\Delta(\bar{S}) = 0\) also holds where \(\bar{S}\) is the complex conjugate of \(S\).

**Proof.** Let \(S = \alpha + i\beta\) be a solution of \([3.2]\) where \(\alpha, \beta \in \mathbb{R}\) and be substituted into \([3.2]\):
\[
0 = (\alpha + i\beta) - a - \sum_{j=1}^{N} a_j d e^{-j(\alpha+i\beta)h}
\]
\[
= \alpha + i\beta - a - \sum_{j=1}^{N} a_j d e^{-jah} (\cos j\beta h - i \sin j\beta h)
\]
\[
= \alpha - a - \sum_{j=1}^{N} a_j d e^{-jah} \cos j\beta h + i \left( \beta + \sum_{j=1}^{N} a_j d e^{-jah} \sin j\beta h \right)
\]
thus
\[ \alpha = a + \sum_{j=1}^{N} a_{jd}e^{-jah} \cos j\beta h, \quad \beta = -\sum_{j=1}^{N} a_{jd}e^{-jah} \sin j\beta h. \] (3.4)

And taking the conjugate of right hand side of (3.2) gives us
\[
(\alpha + i\beta) - a - \sum_{j=1}^{N} a_{jd} e^{-j(\alpha+i\beta)h} = 0
\]
by using the relationship (3.4). Hence \( \bar{S} \) is also a solution.

**Remark 1:** From this lemma, when \( s_{j,k} \), \( k > 0 \) is a solution, then \( s_{j,-k} = \bar{s}_{j,k} \) will be also a solution. When \( k = 0 \) and \( s_{j,0} \) is a real number, then there are only one eigenvalue, i.e., \( s_{1,0} = s_{2,0} = \cdots = s_{N,0} \), corresponding to this principle branch \( W_0 \). Otherwise when \( s_{j,0} \) is a complex number, then \( s_{1,0}, s_{2,0}, \ldots, s_{N,0} \) will becomes one complex number plus its complex conjugate.

**Remark 2:** For simplicity, we can renumber the roots \( s_{j,k}, 1 \leq j \leq N \) and \( k \in \mathbb{Z} \), of \( \Delta(s) = 0 \) as \( S_n \), with \( n \in \mathbb{Z} \) and \( S_0 = s_{1,0} \) being the number with largest real part. If \( S_0 \) is a real number, then \( S_{-n} = \bar{S}_n \) for \( n > 0 \) and otherwise \( S_{-(n+1)} = \bar{S}_n \) for \( n \geq 0 \). Thus \( S_n \) with \( n > 0 \) are located inside the upper half complex plane.

**Remark 3:** Once \( S_0 \) is determined, then the stability of the linear delay-equation (DDE) can be easy determined by check its real part to be positive or not (i.e., when \( S_0 \) has negative real part, then the corresponding system is stable).

We have use the following property in the discussion of Remarks 2 and 3,

**Lemma 3.2.** [12] Lemma 3] For arbitrary \( z \in \mathbb{C} \),
\[
\max\{\Re(W_k(z))|k = 0, \pm 1, \pm 2 \ldots\} = \Re(W_0(z))
\]
is satisfied.

### 3.2. Time response function

Based on previous discussion, we then can write
\[
\frac{1}{\Delta(s)} = \sum_{n=-\infty}^{\infty} \frac{C_n}{s - S_n}
\]
and the coefficient \( C_k \) is then determined by the following Lemma:
Lemma 3.3. Suppose \( a, a_jd \) with \( 1 \leq j \leq N \), and \( h \in \mathbb{R} \) then for any \( s \in \mathbb{C} \)

\[
\frac{1}{\Delta(s)} = \frac{1}{s - a - \sum_{j=1}^{N} a_jd e^{-jsh}} = \sum_{n=-\infty}^{\infty} \frac{C_n}{s - S_n} \tag{3.5a}
\]

where

\[
C_n = \frac{1}{1 + \sum_{j=1}^{N} ja_jdhe^{-jS_nh}}, \quad S_n = \frac{1}{h} W_k \left( \sum_{j=1}^{N} a_jdhe^{-jnh} e^{-(j-1)(S_n-a)h} \right) + a \tag{3.5b}
\]

where \( W_k \) is the branches where \( S_n \) belongs to.

**Proof.** Suppose \( S_n \) is given by (3.5b), then we have

\[
\Delta(S_n) = S_n - a - \sum_{j=1}^{N} a_jd e^{-jS_nh} = 0,
\]

i.e., \( S_n \) is the solution of the characteristic equation for all \( n \in \mathbb{Z} \). Let \( s = S_\ell \) with any \( \ell \in \mathbb{Z} \) and consider

\[
\sum_{k=-\infty}^{\infty} \left( S_\ell - a - \sum_{j=1}^{N} a_jd e^{-jS_kh} \right) \frac{C_k}{S_\ell - S_k} = \sum_{k=-\infty}^{\infty} S_\ell - a - \sum_{j=1}^{N} a_jd e^{-jS_kh} \frac{1}{1 + \sum_{j=1}^{N} ja_jdhe^{-jS_kh}} = \left( \lim_{s \to S_\ell} s - a - \sum_{j=1}^{N} a_jd e^{-js} \right) \frac{1}{1 + \sum_{j=1}^{N} ja_jdhe^{-jS_kh}} = 1.
\]

Next consider \( s \neq S_k \) for all \( k \in \mathbb{Z} \), and pick one \( \ell \in \mathbb{Z} \) then \( S_\ell \) is a simple pole of \( \Psi(s) \) and the associated Laurent’s expansion of \( \Psi(s) \) about \( s = S_\ell \) is given by

\[
\Psi(s) = \frac{a_{-1}}{s - S_\ell} + a_0 + a_1(s - S_\ell) + a_2(s - S_\ell)^2 + \cdots \tag{3.6}
\]

where \( a_{-1} \) is the residue of \( \Psi(s) \), i.e.,

\[
a_{-1} = \lim_{s \to S_\ell} (s - S_\ell) \Psi(s) = C_\ell.
\]

Since for \( k \neq \ell \),

\[
\frac{1}{S_k - s} = \frac{1}{S_k - S_\ell - (s - S_\ell)}
\]
\[
\frac{1}{S_k - S_\ell} \left( 1 - \frac{s - S_k}{S_k - S_\ell} \right) = \frac{1}{S_k - S_\ell} \left[ 1 + \frac{s - S_\ell}{S_k - S_\ell} + \frac{(s - S_\ell)^2}{(S_k - S_\ell)^2} + \cdots \right]
\]
\[
= \sum_{m=0}^\infty \frac{1}{(S_k - S_\ell)^{m+1}} (s - S_\ell)^m
\]

Then
\[
\sum_{k=-\infty}^{\infty} \frac{C_k}{s - S_k} = \frac{C_\ell}{s - S_\ell} + \sum_{k=-\infty}^{\infty} \frac{C_k}{s - S_k}
\]
\[
= \frac{C_\ell}{s - S_\ell} + \sum_{k=-\infty}^{\infty} \frac{(-C_k)}{S_k - s}
\]
\[
= \frac{C_\ell}{s - S_\ell} + \sum_{k=-\infty}^{\infty} \frac{(-C_k)}{S_k - S_\ell} \sum_{m=0}^{\infty} \frac{1}{(S_k - S_\ell)^{m+1}} (s - S_\ell)^m
\]
\[
= \frac{C_\ell}{s - S_\ell} + \sum_{m=0}^{\infty} \left[ \sum_{k=-\infty}^{\infty} \frac{-C_k}{(S_k - S_\ell)^{m+1}} \right] (s - S_\ell)^m
\]

Thus the series
\[
\frac{C_\ell}{s - S_\ell} + \sum_{m=0}^{\infty} \left[ \sum_{k=-\infty}^{\infty} \frac{-C_k}{(S_k - S_\ell)^{m+1}} \right] (s - S_\ell)^m
\] (3.7)

is also a Laurent series of \( \Psi(s) \) about \( s = S_\ell \). By the uniqueness of Laurent’s series of a function about its simple pole, we must have (3.6) is the same as (3.7), i.e., for any \( s \) not equal any \( S_k, k \in \mathbb{Z} \),
\[
\frac{1}{s - a - \sum_{j=1}^{N} a_j e^{-jS_k}} = \sum_{k=-\infty}^{\infty} \frac{C_k}{s - S_k}
\]

where
\[
C_k = \lim_{s \to S_k} (s - S_k) \Psi(s)
\]
\[
= \lim_{s \to S_k} \frac{(s - S_k)}{s - a - \sum_{j=1}^{N} a_j e^{-jS_k}}
\]
\[
= \frac{1}{1 + \sum_{j=1}^{N} j a_j \mu e^{-jS_k}}
\]

This concludes our proof.
By Lemma 3.3, the state transition function $\Psi(t)$ can be expressed as

$$\Psi(t) = \sum_{k=-\infty}^{\infty} C_k e^{S_k t}$$  \hfill (3.8)

where $C_k$ and $S_k$ are given in (3.5b). This equation is of the form firstly proposed by Bellman and Cooke, 1963 [15]. Similarly, the second terms in (2.4a) can also be expressed by

$$\Psi(s) = \sum_{j=1}^{N} a_j d e^{-j s h} \Phi_j(s) = \sum_{k=-\infty}^{\infty} C^I_k \frac{s}{s - S_k}$$

and the coefficient $C^I_k$ is then determined by

$$C^I_k = \lim_{s \to S_k} (s - S_k) \Psi(s) \sum_{j=1}^{N} a_j d e^{-j S_k h} \Phi_j(s)$$

i.e.,

$$C^I_k = \frac{\sum_{j=1}^{N} a_j d e^{-j S_k h} \Phi_j(S_k)}{1 + \sum_{j=1}^{N} a_j d e^{-j S_k h}}$$  \hfill (3.9)

Therefore the corresponding Laplace transform of (DDE) becomes

$$X(s) = \sum_{k=-\infty}^{\infty} \frac{C_k}{s - S_k} x_0 + \sum_{k=-\infty}^{\infty} \frac{C^I_k}{s - S_k} + L[\Psi(t) * bu(t); s]$$

and its time domain response is then given by

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{S_k t} x_0 + \sum_{k=-\infty}^{\infty} C^I_k e^{S_k t} + \int_0^t \sum_{k=-\infty}^{\infty} C_k e^{S_k (t - \tau)} bu(\tau) d\tau$$  \hfill (3.10)

which is the characterization of (2.7) in Theorem 2.3. It is noted that when $N = 1$, (3.10) is equal to the formula given by Yi et. al. [11, (2.37)].

Since there are infinite-countable branches of Lambert W function, we can approximate the time response with $2N + 1$ (or $2N$) branches of the Lambert W function for application, i.e.,

$$x(t) \approx \sum_{k=-M}^{M} C_k e^{S_k t} x_0 + \sum_{k=-M}^{M} C^I_k e^{S_k t} + \int_0^t \sum_{k=-M}^{M} C_k e^{S_k (t - \tau)} bu(\tau) d\tau$$  \hfill (3.11)

where $M \in \mathbb{N}$. We note that the upper limit of summation may be equal to $M - 1$ depending on the value of $S_0$. If $S_0$ is real, then the upper limit is $M$ instead of $M - 1$ and in this case when $S_{-1} = \overline{S_0}$. 
3.3. Computation of $S_n$

The only left problem is how to compute the value of $s_{j,k}$ (or equivalently, $S_n$) for $k \in \mathbb{Z}$ to satisfy (3.3). When $N = 1$, i.e., for a single delay DDE,

$$S_k = s_{1,k} = \frac{1}{h} W_k(a_{1d}he^{-ah}) + a$$  \hspace{1cm} (3.12)

i.e., $S_k$ is equal to the $k$-th branch of the Lambert W function acting on the real number $a_{1d}he^{-ah}$.

When $N > 1$, we use the `fsolve` command in Matlab to compute $s_{j,k}$ as the solution of the following equation expressed by Lambert W functions:

$$(s - a)h - W_k \left( \sum_{j=1}^{N} a_{jd}he^{-jah}e^{-(j-1)(s-a)h} \right) = 0.$$  \hspace{1cm} (3.13)

with initial guess based on (3.12), for example we can choose:

$$s_{j,k}^{(0)} = \frac{1}{h} W_k(a_{1d}he^{-ah}) + a + i(j - 2) \begin{cases} \pi, & k = 0, \\ 2\pi, & k > 0, \end{cases}$$  \hspace{1cm} (3.14)

when $N = 3$, i.e., by adjusting the imaginary part of $s_{j,k}^{(0)}$ such that the `fsolve` command in Matlab converges. Since the roots are complex pair if it is a complex number, we only need to perform the computation for $k \geq 0$. The following two examples demonstrate the computation of $S_n$ for two and three delays cases, respectively.

**Example 3.4.** For the two delays ($N = 2$) system with unit delay $h = 1$, we consider the following two cases:

**Case (a)** Given parameters $a = -1$, $a_{1d} = -1$, $a_{2d} = -1/2$. The characteristic equation in terms of the $k$-th branch of Lambert W function is given by:

$$s + 1 - W_k \left( -e^{-\frac{1}{2}e^{1-s}} \right) = 0$$

with initial guess

$$s_{j,k}^{(0)} = W_k(-e) - 1 - i(j - 1) \begin{cases} \pi, & k = 0, \\ 2\pi, & k > 0. \end{cases}$$

For $k = 0$ the initial guess is $s_{j,0}^{(0)} = -0.60502 + 1.78819 - (j - 1)\pi$ and Matlab’s `fsolve` command gives us

$$S_0 = s_{1,0} \approx -0.27495 + 1.47517i, \quad S_{-1} = s_{2,0} \approx -0.27495 - 1.47517i$$

by taking 4 and 3 iterations to achieve the iteration error within $10^{-10}$ and $S_0$ is a complex number. Similarly, the central 10 values of $S_n$'s are computed and listed below:

$$S_0 = s_{1,0}, S_{-1} = s_{2,0} \approx -0.27495 \pm 1.47517i,$$
\[ S_1 = s_{1,1}, S_{-2} = s_{1,-1} \approx -1.14682 \pm 7.24009i, \]
\[ S_2 = s_{2,1}, S_{-3} = s_{2,-1} \approx -1.27049 \pm 3.64513i, \]
\[ S_3 = s_{1,2}, S_{-4} = s_{1,-2} \approx -1.50747 \pm 13.4657i, \]
\[ S_4 = s_{2,2}, S_{-5} = s_{2,-2} \approx -1.66428 \pm 10.0261i. \]

And the distribution of roots of the characteristic equation for branches \( n = -5, \ldots, 5 \) are shown in Fig. 2(a).

**Case (b)** Given parameters become \( a = -1, a_{1d} = 1/2, a_{2d} = 1/4 \). The characteristic equation is then given by:

\[ s + 1 - W_k \left( \frac{1}{2}e + \frac{1}{4}e^{1-s} \right) = 0 \]

with same initial guess:

\[ s^{(0)}_j = W_k \left( \frac{1}{2}e \right) - 1 - i(j - 1) \begin{cases} \pi, & k = 0, \\ 2\pi, & k > 0. \end{cases} \]

The central 11 values of \( S_n \)’s are computed by using \texttt{fsolve} with at most 7 iterations to achieve the iteration error within \( 10^{-10} \) and are listed below:

\[ S_0 = s_{1,0} \approx -0.11929, \]
\[ S_1 = s_{1,1}, S_{-1} = s_{1,-1} \approx -1.36927 \pm 2.51759i, \]
\[ S_2 = s_{2,1}, S_{-2} = s_{2,-1} \approx -1.37965 \pm 5.30446i, \]
\[ S_3 = s_{1,2}, S_{-3} = s_{1,-2} \approx -1.82137 \pm 11.63900i, \]
\[ S_4 = s_{2,2}, S_{-4} = s_{2,-2} \approx -1.89204 \pm 8.71328i, \]
\[ S_5 = s_{1,3}, S_{-5} = s_{1,-3} \approx -2.05668 \pm 17.94914i, \]
\[ S_6 = s_{2,3}, S_{-6} = s_{1,-3} \approx -2.13589 \pm 14.97910i. \]

And the distribution of roots for branches \( k = -5, \ldots, 5 \) are shown in Fig. 2(b).

As shown in this Fig. 2 it is evident that there are two roots in each branches of Lambert W function and roots behave in the manner with almost constant slopes of change. Since the value \( S_0 \) of both systems has negative real part, these systems are stable.

**Example 3.5.** For the three delays (\( N = 3 \)) equation with given parameters \( a = -1, a_{1d} = 1/2, a_{2d} = -1, a_{3d} = -1 \), and the delay time \( h = 1 \), the equation for calculate the characteristic equation in terms of the \( k \)-th branch of Lambert W function is given by:

\[ s + 1 - W_k \left( \frac{1}{2}e - e^{1-s} - e^{1-2s} \right) = 0 \]
(a) $a = -1$, $a_1 d = -1$, and $a_2 d = -1/2$

(b) $a = -1$, $a_1 d = 1/2$, and $a_2 d = 1/4$

Figure 2: Roots of the characteristic equation for $k = -5, \ldots, 5$.

with initial guess

$$s^{(0)}_j = W_k (\frac{1}{2} e) - 1 + i(j - 2) \begin{cases} \pi/2, & k = 0, \\ 3\pi, & k > 0. \end{cases}$$

The central 14 values of $S_n$’s are computed by using *fsolve* with at most 11 iterations to achieve the iteration error within $10^{-10}$ which are listed below:

\[
\begin{align*}
S_0 &= s_{2,0}, S_{-1} = s_{1,0} \approx 0.08945 \pm 0.86785i, \\
S_1 &= s_{2,1}, S_{-2} = s_{2,1} \approx -0.41586 \pm 4.56389i, \\
S_2 &= s_{3,1}, S_{-3} = s_{3,1} \approx -0.52264 \pm 6.86847i, \\
S_3 &= s_{1,1}, S_{-4} = s_{1,1} \approx -0.62749 \pm 2.84865i, \\
S_4 &= s_{2,2}, S_{-5} = s_{2,2} \approx -0.74691 \pm 10.85396i, \\
S_5 &= s_{3,2}, S_{-6} = s_{3,2} \approx -0.75648 \pm 13.13363i, \\
S_6 &= s_{1,2}, S_{-5} = s_{1,2} \approx -0.89966 \pm 9.02658i.
\end{align*}
\]

And the distribution of roots for branches $k = -5, \ldots, 5$ are shown in Fig. 3.

As shown in this Fig. 3, it is evident that there are three roots in each branches of Lambert W function except for $k = 0$ branch which has two roots inside. And since $\text{Re}(S_0) > 0$, this system is unstable.

4. Illustrative Examples

In this section, examples with single, two and three delays cases are presented to illustrate the application of our method to compute their corresponding time responses. Since there are some references deal with the time response of single
delay system, we just present the basic information to construct the approximate solution. But for two-delay system, more detail discussion will be addressed.

4.1. Single delay case

Consider the following DDE:
\[
\dot{x}(t) = -x(t) - x(t-1) + \cos t, \quad t > 0
\]
\[
x(0) = x_0 = 1, \quad x(t) = \phi(t) = 1, \quad t \in [-1, 0)
\]
The corresponding eigenvalue $S_k$ is computed by the Lambert W function $S_k = W_k(-e) - 1$ and $C_k = 1/(1 - e^{-S_k})$ for $k = -3, -2, \ldots, 2$. The state transition function is then given by
\[
\Psi(t) = \sum_{k=-\infty}^{\infty} C_k e^{-S_k t} \approx \sum_{k=-3}^{2} C_k e^{-S_k t}
\]
Also, since
\[
\Phi_1(s) = \int_{-1}^{0} \phi(t) e^{-st} dt = \frac{e^s - 1}{s}
\]
and
\[
C_k^t = (1 - C_k) \frac{\Phi_1(S_k)}{h}
\]
and thus the total response of this DDE is then given by
\[
x(t) \approx \sum_{k=-3}^{2} (C_k x_0 + C_k^t e^{-S_k t}) + \sum_{k=-3}^{2} C_k \int_{0}^{t} e^{-S_k (t-\tau)} \cos(\tau) d\tau
\]
and the first part of \( x(t) \) is called the initial condition response. The corresponding values for \( S_k \) and \( C_k \) are listed below:

\[
S_{0,-1} \approx -0.6050 \pm 1.7882i, \quad S_{1,-2} \approx -2.0528 \pm 7.7184i, \quad S_{2,-3} \approx -2.6474 \pm 14.0202i; \quad C_{0,-1} \approx 0.2712 \mp 0.3477i, \quad C_{1,-2} \approx -0.008867 \mp 0.1296i, \quad C_{2,-3} \approx -0.003286 \mp 0.07117i.
\]

The initial condition and total responses of this single delay system are given in Fig. 4.

![Time responses of the single delay system with \( a = -1 \) and \( a_{1d} = -1 \).](image)

**Figure 4:** Time responses of the single delay system with \( a = -1 \) and \( a_{1d} = -1 \).

### 4.2 Two-delay case

Consider the following differential equation with two delays:

\[
\dot{x}(t) = -x(t) - x(t-1) - \frac{1}{2} x(t-2) + \cos t, \quad t > 0
\]

\[
x(0) = x_0 = 1, \quad x(t) = \phi(t) = 1, \quad t \in [-2, 0)
\]

The corresponding eigenvalues \( S_n \) are computed in **Case (a)** of Example 3.4 in previous section. Since

\[
\Phi_1(s) = \int_{-1}^{0} \phi(t)e^{-st}dt = \frac{e^s - 1}{s}, \quad \Phi_2(s) = \int_{-2}^{0} \phi(t)e^{-st}dt = \frac{e^{2s} - 1}{s},
\]

thus coefficients for time responses are given by

\[
C_n = \frac{1}{(1 - e^{-S_n} - \frac{1}{2} e^{-2S_n})}
\]

\[
C'_n = C_n \left( -e^{-S_n} \Phi_1(S_n) - \frac{1}{2} e^{-2S_n} \Phi_2(S_n) \right)
\]
and their values are described in Table 1. The state transition function is then approximated by

\[ \Psi(t) = \sum_{n=-\infty}^{\infty} C_n^N e^{-S_n t} \approx \sum_{n=-11}^{n=10} C_n^N e^{-S_n t} \]

The first three branches (i.e., \( k = 0, 1, 2 \)) contribution to the initial condition response are shown in Fig. 5(a). It is obviously, the mode shape due to the branch \( k = 0 \) is similar to the solution obtained by \texttt{dde23} of Matlab command which is referred here as exact solution. And if we increase the number of branches in approximate the time response, then the maximal absolute error between computed solution and the one obtained by \texttt{dde23} is shown in Fig. 5(b). This figure shows as the number of branches increases the error is reduced as well with slow rate. Therefore the total response of this two-delay system is then given by

\[ x(t) \approx \sum_{n=-11}^{10} (C_n^N + C_n^IN) e^{-S_n t} + \sum_{n=-11}^{10} C_n^N \int_0^t e^{-S_n (t-\tau)} \cos(\tau) d\tau \]

and the first term gives the initial condition response. The initial condition response and total response of this system are given in Fig. 6 which are very close to the responses produced by \texttt{dde23}.

![Figure 5: Modal effect of initial responses due to branches of Lambert W function](image-url)
Figure 6: Time Responses of the two-delay systems with $a = -1$, $a_{1d} = -1$, and $a_{2d} = -1/2$. 

(a) Initial response  
(b) Total response
Table 1: Eigenvalues $S_n$ with associated coefficients $C_n$ and $C_n^{I}$ for response computation

| $S_n$ | $C_n$ | $C_n^{I}$ |
|-------|-------|-----------|
| $S_0 = s_{1,0}, S_{-1} = s_{2,0} \approx -0.27495 \pm 1.47517i$ | $C_0, C_{-1} \approx 0.27626 \pm 0.17588i$ | $C_0^{I}, C_{-1}^{I} \approx 0.096135 \pm 0.57465i$ |
| $S_1 = s_{1,1}, S_{-2} = s_{1,-1} \approx -1.14682 \pm 7.24009i$, | $C_1, C_{-2} \approx 0.017012 \mp 0.080385i$ | $C_1^{I}, C_{-2}^{I} \approx 0.010973 \pm 0.081826i$ |
| $S_2 = s_{2,1}, S_{-3} = s_{2,-1} \approx -1.27049 \pm 3.64513i$ | $C_2, C_{-3} \approx -0.030157 \mp 0.10209i$ | $C_2^{I}, C_{-3}^{I} \approx 0.086161 \pm 0.061884i$ |
| $S_3 = s_{1,2}, S_{-4} = s_{1,-2} \approx -1.50747 \pm 13.4657i$ | $C_3, C_{-4} \approx 0.0050331 \mp 0.042141i$ | $C_3^{I}, C_{-4}^{I} \approx 0.002791 \pm 0.042198i$ |
| $S_4 = s_{2,2}, S_{-5} = s_{2,-2} \approx -1.66428 \pm 10.0261i$ | $C_4, C_{-5} \approx -0.0084729 \mp 0.0041680i$ | $C_4^{I}, C_{-5}^{I} \approx 0.008246 \pm 0.037945i$ |
| $S_5 = s_{1,3}, S_{-6} = s_{1,-3} \approx -1.77287 \pm 19.72659i$ | $C_5, C_{-6} \approx 0.0025181 \mp 0.028227i$ | $C_5^{I}, C_{-6}^{I} \approx 0.0010598 \pm 0.028234i$ |
| $S_6 = s_{2,3}, S_{-7} = s_{2,-3} \approx -1.87170 \pm 16.34717i$ | $C_6, C_{-7} \approx -0.0043406 \mp 0.026636i$ | $C_6^{I}, C_{-7}^{I} \approx 0.008265 \pm 0.025521i$ |
| $S_7 = s_{1,4}, S_{-8} = s_{1,-4} \approx -1.87589 \pm 25.99734i$ | $C_7, C_{-8} \approx 0.0015665 \mp 0.021147i$ | $C_7^{I}, C_{-8}^{I} \approx 0.00046741 \pm 0.021151i$ |
| $S_8 = s_{2,4}, S_{-9} = s_{2,-4} \approx -2.01488 \pm 22.65195i$ | $C_8, C_{-9} \approx -0.0027376 \mp 0.0196560194i$ | $C_8^{I}, C_{-9}^{I} \approx 0.0048633 \pm 0.019165i$ |
| $S_9 = s_{1,5}, S_{-10} = s_{1,-5} \approx -1.99436 \pm 32.27231i$ | $C_9, C_{-10} \approx 0.0010945 \mp 0.016880i$ | $C_9^{I}, C_{-10}^{I} \approx 0.00021337 \pm 0.016884i$ |
| $S_{10} = s_{2,5}, S_{-11} = s_{2,-5} \approx -2.124656 \pm 28.94953i$ | $C_{10}, C_{-11} \approx -0.0019236 \mp 0.015603i$ | $C_{10}^{I}, C_{-11}^{I} \approx 0.0032516 \pm 0.015339i$ |
4.3. Three-delay case

Consider the following differential equation with three delays:

\[ \dot{x}(t) = -x(t) + \frac{1}{2} x(t-1) - x(t-2) - x(t-3) + \cos t, \ t > 0 \]
\[ x(0) = x_0 = 1, \ x(t) = \phi(t) = 1, \ t \in [-3, 0) \]

The corresponding central 14 eigenvalues \( S_n \) is computed in Example 3.5 of previous section. Since

\[ \Phi_j(s) = \int_{-j}^{0} \phi(t)e^{-st}dt = \frac{e^{js} - 1}{s}, \quad 1 \leq j \leq 3, \]

thus coefficients for time responses are given by

\[ C_n = \frac{1}{1 + \frac{1}{2}e^{-S_n} - e^{-2S_n} - e^{-2S_n}}, \]
\[ C_n^I = C_n \left( \frac{1}{2}e^{-S_n} \Phi_1(S_n) - e^{-2S_n} \Phi_2(S_n) - e^{-3S_n} \Phi_3(S_n) \right) \]

and their values are omitted here. We use eigenvalues corresponding to the branches \( k = 0, \pm1, \ldots, \pm5 \) of Lambert W function for approximation (i.e., we use 32 eigenvalues in total), and thus the state transition function is approximated by

\[ \Psi(t) = \sum_{n=-\infty}^{\infty} C_n^N e^{-S_n t} \approx \sum_{n=-11}^{n=10} C_n^N e^{-S_n t} \]

and the total response of this two-delay system is then given by

\[ x(t) \approx \sum_{n=-15}^{n=15} (C_n^N + C_n^I N) e^{-S_n t} + \sum_{n=-15}^{n=15} C_n^N \int_{0}^{t} e^{-S_n (t-\tau)} \cos(\tau)d\tau \]

with the first term for the initial condition response. The initial condition response and total response of this system are given in Fig. 7 which are almost distinguishable to the responses generated by dde23.

5. Conclusion

Lambert W function has been widely applied in the research of science and engineering problems. In this paper, we construct the time response of scalar systems with multiple discrete delays based on the Laplace transform. We first establish the state transition function as the sum of infinite series of exponentials acting on eigenvalues inside countable branches of the Lambert W functions. And the time response function is then obtained in terms of initial condition, preshape function, and the convolution between state transition function and input function. Eigenvalues in each branch of Lambert W function are
Figure 7: Time Responses of the three-delay systems with $a = -1$, $a_{1d} = 1/2$, $a_{2d} = -1$, and $a_{3d} = -1$.

computed by a numerical iteration. Once the eigenvalues are obtained, the approximation of time responses is calculated. Numerical examples are presented to conform the successful application of Lambert W function in establishing the time response of multi-delay systems. Our main contribution is to construct an analytical formula in representing the time response which can act as the concise foundation of designing the feedback controller for multi-delay systems.

Acknowledgments

This research was partially supported by grant MOST 104-2115-M-029-004. (H.-N. Huang).

References

[1] V. Kolmanovskii, A. Myshkis; *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[2] V. Kolmanovskii, A. Myshkis; *Applied Theory of Functional Differential Equations*, Kluwer Academic Press Publisher, Dordrect, 1992.

[3] S. A. Gourley, Y. Kuang; A stage structured predator-prey model and its dependence on maturation delay and death rate, *J. Math. Biol. 49*(2004), 188–200.

[4] Y. Kuang; *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.

[5] J. D. Murray; *Mathematical Biology: An Introduction*, Springer-Verlag, 3rd Ed., New York, 2002
[6] R. W. Shonkwiler, J. Herod; *Mathematical Biology, an Introduction with Maple and Matlab*, 2nd Ed., Springer, San Francisco, 2009.

[7] H. Smith; *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer, New York, 2011.

[8] S. P. Travis; A one-dimensional two-body problem of classical electrodynamics, *J. Appl. Math.* 28(1975), 611–632.

[9] R. M. Coreless, G. H. Gonnet, D. E. G. Hare, D. J. Jerey, D. E. Knuth, On the Lambert W function, *Adv. Comput. Math.* 5(1996), 329–359.

[10] F. M. Asl, A. G. Ulsoy, Analysis of a system of linear delay differential equations, *Journal of Dynamic Systems, Measurement, and Control* 125(2003), 215–223.

[11] S. Yi, P. W. Nelson, A. G. Ulsoy, *Time-Delay Systems: Analysis and Control Using the Lambert W Function*, World Scientific, Singapore, 2010.

[12] K. Gu, V. L. Kharitonov, J. Chen, *Stability of Time-Delay Systems*, Birkh"{a}user, Boston, 2003.

[13] T. Krisztin, G. Vas, On the fundamental solution of linear delay differential equations with multiple delays, *Electronic Journal of Qualitative Theory of Differential Equations* 2011, No.36, 1-28.

[14] H. Shinozaki, T. Mori, Robust stability analysis of linear time-delay systems by Lambert W function: Some extreme point results, *Automatica*, 43 (2006) 1791-1799.

[15] R. Bellman, K. L. Cooke, *Differential Difference Equations*, Academic Press, New York, 1963.