SQUEEZING ANDFINITE DIMENSIONALITY OF COCYCLE 
ATTRACTORS FOR 2D STOCHASTIC NAVIER-STOKES 
eQUATION WITH NON-AUTONOMOUS FORCING 

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Dedicated to the memory of professor Igor Chueshov

Abstract. In this paper, we study the squeezing property and finite dimensionality of cocycle attractors for non-autonomous dynamical systems (NRDS). We show that the generalized random cocycle squeezing property (RCSP) is a sufficient condition to prove a determining modes result and the finite dimensionality of invariant non-autonomous random sets, where the upper bound of the dimension is uniform for all components of the invariant set. We also prove that the RCSP can imply the pullback flattening property in uniformly convex Banach space so that could also contribute to establish the asymptotic compactness of the system. The cocycle attractor for 2D Navier-Stokes equation with additive white noise and translation bounded non-autonomous forcing is studied as an application.

1. Introduction. Non-autonomous random dynamical systems (NRDS for short) are introduced to study evolution equations driven by time-dependent forcings and perturbed by stochastic noises, see for instance [27, 50, 51, 28, 29]. Because of the non-autonomous and stochastic features, the dynamical behavior of NRDS is relatively more complicated and deserves more efforts to go into than deterministic and autonomous random dynamical systems.

Global attractor is a useful objective to learn the dynamical behavior of a dynamical system. For NRDS there are considerably many publications on various random attractors, see for instance [50, 52, 31] for pullback attractors, [28, 29] for cocycle attractors and most recently [29] for uniform attractors. In addition to the

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attracting property, the compactness and then the finite-dimensionality are inter-
esting and significant properties, making attractors more important for studying
the dynamical system, see for instance Robinson [45] and references therein. Igor
Chueshov gave us nice and important results, both in the deterministic and the ran-
case, in all the main items where the theory of infinite-dimensional dynamical
systems has been developed in the last four decades: existence of attractors and its
characterization (see [15, 2, 24, 11, 21]), dependence of the asymptotic behaviour on
a finite number of degrees of freedom, including determining modes and squeezing
property ([14, 17, 23]), finite fractal dimensionality of attractors ([16, 20]), invariant
manifolds ([12, 3]) or inertial manifolds ([37, 22]). Applications to important models
of PDEs also focused an important part of his work (see, for instance, [13, 19, 18]).
Science in these areas of research is really grateful on all his efforts and such a
great work, which has been and it is so helpful to a big number of collaborators and
researches in the field.

In this paper, we generalize the idea of squeezing property to a random co-
cycle squeezing property (RCSP for short) for NRDS, by which we further study the
finite dimensionality of invariant sets, including cocycle attractors of NRDS. The
squeezing property was firstly introduced by Foias & Temam [36] in the context of
the Navier-Stokes equations, and then applied to many other classes of dissipative
partial differential equation [33, 47, 44]. For autonomous RDS, Debussche [32] gen-
eralizes the method of [36], and the ideas of [32] motivated Flandoli and Langa [35]
to define the “random squeezing property”. As will be shown latter, the generalized
squeezing property could imply the determining modes and finite dimensionality for
cocycle attractors of NRDS, and, as the squeezing implies asymptotic compactness
of the system, it could also contribute to obtain the compactness and existence of
the cocycle attractor.

What is worth a note is that the finite dimensionality of cocycle attractors ob-
tained in this paper does hold uniformly for all components of the attractor, that is,
the upper bound of the dimension is uniform for all components. Applying to a 2D
Navier-Stokes equation we find that the uniformity of the bound is supported by
the translation-boundedness of the external force of the system ([10, 8, 9]), which is
known often a sufficient condition for the existence of a random uniform attractor
[29].

We carry out this paper as follows. In Section 2, we introduce preliminary
definitions to NRDS. In Section 3, we extend the idea of squeezing property to
NRDS, and prove that it is a sufficient condition to prove a determining modes
results and finite dimensionality of random non-autonomous random invariant set.
In Section 4, we apply our theoretical results to a non-autonomous stochastic 2D
Navier-Stokes equation.

2. Non-autonomous random dynamical systems. In this section, we first re-
call some general concepts related to NRDS and their cocycle attractors.

2.1. Preliminaries. In this part, we recall from [1, 28, 50] some basic definitions
which will be used throughout this paper. Let \((X, \|\cdot\|)\) be a separable Banach
space, and denote the Hausdorff semi-metric between sets in \(X\) by

\[
\text{dist}(A, B) = \sup_{a \in A} \text{dist}(a, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|, \quad \forall A, B \in 2^X \setminus \emptyset.
\]

For any set \(B \subset X\), we denote by \(\|B\| := \sup_{x \in B} \|x\|\).
Let $\Sigma$ be a topological space. Most generally, we do not require compactness or boundedness (under some metric) on $\Sigma$ unless otherwise stated, but we always assume a group $\{\theta_t\}_{t \in \mathbb{R}}$ of operators acting on $\Sigma$ satisfying

- $\theta_0 =$ identity operator on $\Sigma$;
- $\theta_t \Sigma = \Sigma, \ \forall t \in \mathbb{R}$;
- $\theta_s \circ \theta_t = \theta_{t+s}, \ \forall t, s \in \mathbb{R}$;
- $\sigma \mapsto \theta_t \sigma$ is continuous for each $t$ fixed.

In applications, space $\Sigma$ refers to the symbol space of an evolution equation. A popular example is the space $\Sigma = (g(\cdot + s) : s \in \mathbb{R})$, instead of $g(\cdot)$. A time-dependent function $g$ taking values in some metric space, and $\theta_t$ is the translation operator $\theta_t g = g(\cdot + t)$.

Throughout this paper, for any metric space $M$ we denote by $\mathcal{B}(M)$ the Borel sigma-algebra of $M$.

To define non-autonomous random dynamical systems studied in this paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, which need not be $\mathbb{P}$-complete, endowed also with a flow $\{\vartheta_t\}_{t \in \mathbb{R}}$ satisfying

- $\vartheta_0 =$ identity operator on $\Omega$;
- $\vartheta_t \Omega = \Omega; \ \forall t \in \mathbb{R}$;
- $\vartheta_s \circ \vartheta_t = \vartheta_{t+s}, \ \forall t, s \in \mathbb{R}$;
- $(t, \omega) \mapsto \vartheta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable;
- $\mathbb{P}(\vartheta_t F) = \mathbb{P}(F), \ \forall t \in \mathbb{R}, F \in \mathcal{F}$.

The two groups $\{\theta_t\}_{t \in \mathbb{R}}$ and $\{\vartheta_t\}_{t \in \mathbb{R}}$ acting respectively on $\Sigma$ and $\Omega$ are called base flows or parametric dynamical systems. For the ease of notations, we often use $\theta$ (or $\vartheta$), instead of $\theta_t$ (or $\vartheta_t$), when describing universal properties hold for each $t \in \mathbb{R}$.

**Definition 2.1.** A non-autonomous random dynamical system (NRDS for short) on $X$ with base flows $\{\theta_t\}_{t \in \mathbb{R}}$ and $\{\vartheta_t\}_{t \in \mathbb{R}}$ is defined as a mapping $\phi(t, \omega, \sigma, x) : \mathbb{R}^+ \times \Omega \times \Sigma \times X \mapsto X$ satisfying

1. $\phi$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\Sigma) \times \mathcal{B}(X))$-measurable;
2. $\phi(0, \omega, \sigma, \cdot)$ is the identity on $X$ for each $\omega$ and $\sigma$ fixed;
3. it holds the cocycle property

$$\phi(t + s, \omega, \sigma, x) = \phi(t, \vartheta_s \omega, \theta_t \sigma) \circ \phi(s, \omega, \sigma, x), \ \forall t, s \in \mathbb{R}^+, \omega \in \Omega, \sigma \in \Sigma.$$  

Moreover, an NRDS $\phi$ is said to be continuous if the mapping $\phi(t, \omega, \sigma, \cdot)$ is continuous for each $t$, $\omega$, $\sigma$ fixed.

Next we define the so-called non-autonomous random set, which plays a central role in the study of NRDS.

**Definition 2.2.** A set-valued mapping $D : \Omega \times \Sigma \mapsto 2^X \setminus \emptyset$, $(\omega, \sigma) \mapsto D_\sigma(\omega)$ is called a non-autonomous random set in $X$ if it is measurable in the sense that the mapping $\omega \mapsto \text{dist}(x, D_\omega(\sigma))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for each $x \in X$ and $\sigma \in \Sigma$. If each its image $D_\sigma(\omega)$ is closed (or bounded, compact, etc.) in $X$, then $D$ is called a closed (or bounded, compact, etc.) non-autonomous random set in $X$.

Given two non-autonomous random sets $D^1, D^2$, write $D^1 \subset D^2$ if $D^2_\sigma(\omega) \subset D^1_\sigma(\omega)$ for all $\sigma \in \Sigma, \omega \in \Omega$, and we then say $D_1$ is smaller than $D_2$.

**Definition 2.3.** A non-autonomous random set $D$ in $X$ is called tempered if, for any $\epsilon > 0$,

$$\lim_{t \to \infty} e^{-\epsilon t} \|D_{\theta_t \sigma}(\vartheta_t \omega)\| = 0, \ \forall \sigma \in \Sigma, \omega \in \Omega.$$
In the following, denote by $\mathcal{D}$ the class of the tempered non-autonomous random set in $X$. Then the universe $\mathcal{D}$ is inclusion-closed, i.e., if $D \in \mathcal{D}$ then each random set smaller than $D$ belongs to $\mathcal{D}$.

Before the definition of $\mathcal{D}$-random cocycle attractors let us recall invariant, pullback attracting and pullback absorbing sets.

**Definition 2.4.** Given an NRDS $\phi$, a non-autonomous random set $K = \{K_\sigma(\cdot)\}_{\sigma \in \Sigma}$ is called positively invariant under $\phi$ if for each $\sigma \in \Sigma$ and $\omega \in \Omega$,

$$\phi(t,\omega,\sigma,K_\sigma(\omega)) \subseteq K_{\theta_t\sigma}(\theta_t\omega), \quad \forall t \geq 0,$$

while it is called invariant under $\phi$ if for each $\sigma \in \Sigma$ and $\omega \in \Omega$,

$$\phi(t,\omega,\sigma,K_\sigma(\omega)) = K_{\theta_t\sigma}(\theta_t\omega), \quad \forall t \geq 0.$$

**Definition 2.5.** Given an NRDS $\phi$, a non-autonomous random set $K = \{K_\sigma(\cdot)\}_{\sigma \in \Sigma}$ is called $\mathcal{D}$-pullback attracting under $\phi$ if for each $D \in \mathcal{D}$,

$$\lim_{t \to \infty} \text{dist}(\phi(t,\omega,\sigma,D_{\theta^{-t}\sigma}(\theta^{-t}\omega)),K_\sigma(\omega)) = 0, \quad \forall \sigma \in \Sigma, \omega \in \Omega, \quad (2.1)$$

while it is called $\mathcal{D}$-pullback absorbing if for each $D \in \mathcal{D}$, $\sigma \in \Sigma$ and $\omega \in \Omega$, there exists a time $T_D(\omega,\sigma) > 0$ such that

$$\phi(t,\omega,\sigma,D_{\theta^{-t}\sigma}(\theta^{-t}\omega)) \subset K_\sigma(\omega), \quad \forall t \geq T_D(\omega,\sigma). \quad (2.2)$$

**Definition 2.6.** A non-autonomous random set $A = \{A_\sigma(\cdot)\}_{\sigma \in \Sigma} \in \mathcal{D}$ is called a $\mathcal{D}$-random cocycle attractor (or $\mathcal{D}$-random pullback attractor) of the NRDS $\phi$ if

1. $A$ is compact;
2. $A$ is $\mathcal{D}$-pullback attracting;
3. $A$ is invariant under $\phi$.

Note that for cocycle attractors of autonomous random dynamical systems, the pathwise pullback attraction implies a forward attraction in probability property, see e.g. [43]. Such a connection between pullback and forward attraction was generalized to random uniform attractors for non-autonomous random dynamical systems by Cui & Langa [29]. However, for cocycle attractors defined above has no corresponding property, i.e., the pullback attraction (2.1) does not imply the forward attraction in probability that

$$\lim_{t \to \infty} \mathbb{P}\{\omega \in \Omega : \text{dist}(\phi(t,\omega,\sigma,D_\sigma(\omega)),K_{\theta_t\sigma}(\theta_t\omega)) > \varepsilon\} = 0, \quad \forall \varepsilon > 0.$$ 

In fact, even for deterministic non-autonomous dynamical systems, pullback attraction and forward attraction have no general relationship [41, 39, 4]. But we have the following.

**Proposition 2.7.** For any $\mathcal{D}$-pullback attracting set $K = \{K_\sigma(\cdot)\}_{\sigma \in \Sigma}$ it holds that for each $D \in \mathcal{D}$ and $\sigma \in \Sigma$

$$\lim_{t \to \infty} \mathbb{P}\{\omega \in \Omega : \text{dist}(\phi(t,\omega,\sigma,D_{\theta^{-t}\sigma}(\theta^{-t}\omega)),K_\sigma(\theta_t\omega)) > \varepsilon\} = 0, \quad \forall \varepsilon > 0.$$

**Proof.** For each $D \in \mathcal{D}$ and $\sigma \in \Sigma$ fixed and given $\varepsilon > 0$, by the pullback attraction of $K$ we have

$$\lim_{t \to \infty} \mathbb{P}\{\omega \in \Omega : \text{dist}(\phi(t,\omega,\sigma,D_{\theta^{-t}\sigma}(\theta^{-t}\omega)),K_\sigma(\omega)) > \varepsilon\} = 0.$$
Notice that \( \vartheta_t \) is \( \mathbb{P} \)-preserving and \( \Omega \) is invariant under \( \vartheta_t \) for any \( t > 0 \), so we have

\[
\mathbb{P}\left\{ \omega \in \Omega : \text{dist}\left(\phi(t, \omega, \vartheta^{-t} \sigma, D_{\vartheta^{-t} \sigma}(\omega)), K_{\sigma}(\vartheta_t \omega)\right) > \varepsilon \right\} \\
= \mathbb{P}\left\{ \vartheta_t \omega \in \Omega : \text{dist}\left(\phi(t, \omega, \vartheta^{-t} \sigma, D_{\vartheta^{-t} \sigma}(\omega)), K_{\sigma}(\vartheta_t \omega)\right) > \varepsilon \right\} \\
= \mathbb{P}\left\{ \tilde{\omega} \in \Omega : \text{dist}\left(\phi(t, \vartheta^{-t} \tilde{\omega}, \vartheta^{-t} \sigma, D_{\vartheta^{-t} \sigma}(\vartheta^{-t} \tilde{\omega})), K_{\sigma}(\tilde{\omega})\right) > \varepsilon \right\} \\
\xrightarrow{t \to \infty} 0.
\]

The proof is complete. \( \Box \)

For each non-empty non-autonomous random set \( D \) and \( \sigma \in \Sigma \), we define the random omega-limit set \( \mathcal{W}(\cdot, \sigma, D) \) of \( D \) driven by \( \sigma \) by

\[
\mathcal{W}(\omega, \sigma, D) = \bigcap_{s \geq 0} \bigcup_{t \geq s} \phi(t, \vartheta^{-t} \omega, \vartheta^{-t} \sigma, D_{\vartheta^{-t} \sigma}(\vartheta^{-t} \omega)), \quad \forall \omega \in \Omega.
\]

Omega-limit sets are important in attractor theory. It is straightforward to have the following lemma.

**Lemma 2.8.** For each \( \omega \in \Omega \), \( \sigma \in \Sigma \) and \( D \in \mathcal{D} \), \( y \in \mathcal{W}(\omega, \sigma, D) \) if and only if there exist sequences \( t_n \to \infty \) and \( x_n \in D_{\vartheta^{-t_n} \sigma}(\vartheta^{-t_n} \omega) \) such that \( \phi(t_n, \vartheta^{-t_n} \omega, \vartheta^{-t_n} \sigma, x_n) \to y \).

**Definition 2.9.** An NRDS \( \phi \) on \( X \) is called \( \mathcal{D} \)-pullback asymptotically compact if for each \( \omega \in \Omega \), \( \sigma \in \Sigma \), \( D \in \mathcal{D} \) and any sequences \( t_n \to \infty \) and \( x_n \in D_{\vartheta^{-t_n} \sigma}(\vartheta^{-t_n} \omega) \) the set \( \{\phi(t_n, \vartheta^{-t_n} \omega, \vartheta^{-t_n} \sigma, x_n)\}_{n \in \mathbb{N}} \) is precompact in \( X \).

For \( \mathcal{D} \)-random cocycle attractors, Wang [50, 51] studied the existence and characterization by complete trajectories. The following existence result is well known.

**Theorem 2.10.** [50, 51] Suppose \( \phi \) is a continuous NRDS with a compact \( \mathcal{D} \)-pullback attracting set \( K \) and a closed \( \mathcal{D} \)-pullback absorbing set \( B \in \mathcal{D} \). Then \( \phi \) has a unique \( \mathcal{D} \)-random cocycle attractor \( \mathcal{A} \in \mathcal{D} \) given by

\[
\mathcal{A}(\omega) = \mathcal{W}(\omega, \sigma, B).
\]

Observe that in the above theorem the set \( K \) does not necessarily belong to the class \( \mathcal{D} \).

### 2.2. Several alternative dynamical compactnesses.

Theorem 2.10 shows a direct relationship between attractors and compact attracting sets. However, the existence of a compact attracting set is often the open problem. Hence, there are other dynamical compactnesses in the literature, such as asymptotic compactness, pullback omega-limit compactness, flattening and squeezing properties [42, 38, 30, 25], which ensures the omega-limit set of a \( \mathcal{D} \)-absorbing set is a compact \( \mathcal{D} \)-attracting set. Now we introduce analogous concepts in the context of NRDS, and show that these dynamical compactness of an NRDS will ensure the omega-limit set of a \( \mathcal{D} \)-absorbing set to be a compact \( \mathcal{D} \)-attracting set.

**Definition 2.11.** An NRDS \( \phi \) on \( X \) is called \( \mathcal{D} \)-pullback flattening if for each \( \varepsilon > 0 \), \( \omega \in \Omega \), \( \sigma \in \Sigma \) and \( B \in \mathcal{D} \), there exists a \( T_0 = T_0(\varepsilon, \omega, \sigma, B) > 0 \) and a finite-dimensional subspace \( X_\varepsilon \) of \( X \) such that

(i) \( \cup_{t \geq T_0} \phi(t, \vartheta^{-t} \omega, \vartheta^{-t} \sigma, B_{\vartheta^{-t} \sigma}(\vartheta^{-t} \omega)) \) is bounded, and

(ii) \( \left\| (I - P) \left( \cup_{t \geq T_0} \phi(t, \vartheta^{-t} \omega, \vartheta^{-t} \sigma, B_{\vartheta^{-t} \sigma}(\vartheta^{-t} \omega)) \right) \right\| < \varepsilon \),

where \( P : X \to X_\varepsilon \) is a bounded projection.
Definition 2.12. An NRDS $\phi$ on $X$ is called $D$-pullback omega-limit compact if for each $\epsilon > 0$, $\omega \in \Omega$, $\sigma \in \Omega$ and $B \in D$, there exists a $T_1 = T_1(\epsilon, \omega, \sigma, B) > 0$ such that

$$\kappa \left( \bigcup_{t \geq T_1} \phi(t, \theta_t \omega, \theta_t \sigma, B_{\theta_t \sigma(\theta_t \omega)}) \right) < \epsilon,$$

where $\kappa$ denotes the Kuratowski measure [42] of noncompactness of sets defined as

$$\kappa(B) = \inf \{ \delta > 0 : B \text{ has a finite cover by balls of } X \text{ of diameter less than } \delta \}.$$

The following theorem shows the equivalence between these concepts.

Theorem 2.13. Suppose that $X$ is a uniformly convex Banach space (particularly, a Hilbert space). The following dynamical compactness properties of an NRDS $\phi$ on $X$ are equivalent:

(i) $D$-pullback flattening;
(ii) $D$-pullback omega-limit compactness;
(iii) $D$-pullback asymptotically compactness,

where the uniformly convex property of $X$ is only for the relation (iii) $\Rightarrow$ (i). Moreover, each of these dynamical compactnesses implies that the omega-limit set $W(\omega, \sigma, B)$ of a $D$-pullback absorbing $B \in D$ is a compact $D$-pullback attracting set.

Proof. Similar to [38, Theorems 4.5 & 4.6], also see [25, Section 2].

The above theorem implies that these dynamical compactnesses could replace the requirement of a compact $D$-attracting set in Theorem 2.10, since they are stronger under suitable conditions. Let us study the squeezing property in the next section.

3. The random cocycle squeezing property. In this section, we introduce the definition of squeezing property for NRDS. We prove that it is a sufficient condition to prove a determining modes results and finite dimensionality of non-autonomous random invariant set, also it can imply the pullback flattening in uniformly convex Banach space.

Definition 3.1 (Random cocycle squeezing property). Suppose $\phi$ is an NRDS on $X$ with a non-autonomous random invariant set $B \in D$. We say that $\phi$ satisfies the random cocycle squeezing property (RCSP for short) on $B$ if there exists $\delta \in (0, 1/2)$, a $m$-dimensional orthogonal projector $P : X \rightarrow PX$ ($\dim PX = m$) and a random variable $C : \Omega \rightarrow \mathbb{R}$ with finite expectation $\mathbb{E}(C(\omega)) < \ln(1/2\delta)$, such that for each $\omega \in \Omega$ and $\sigma \in \Sigma$

$$\|P(\phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y))\| \leq e^{\int_0^1 C(\theta_s \omega) \, ds} \|x - y\|$$

and

$$\|Q(\phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y))\| \leq \delta e^{\int_0^1 C(\theta_s \omega) \, ds} \|x - y\|,$$

for all $x, y \in B_{\sigma}(\omega)$, where $Q = I - P$.

Remark 3.2. It is important to note that, the RCSP gives an alternative on the one-step (in time) behaviour of the solutions associated to the NRDS: either the low modes control the behavior of the high modes, that is,

$$\|Q(\phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y))\| \leq \|P(\phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y))\|$$
or the one-time evolution produces a flattening effect on the trajectories, that is,
\[ \| \phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y) \| \leq 2\delta e^{\int_0^s C(\theta, \omega) \, ds} \| x - y \|. \]

Indeed, if
\[ \| Q(\phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y)) \| > \| P(\phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y)) \|, \]
then, by (3.2) and triangle inequality, we obtain that
\[ \| \phi(1, \omega, \sigma, x) - \phi(1, \omega, \sigma, y) \| \leq 2\delta e^{\int_0^s C(\theta, \omega) \, ds} \| x - y \|. \]
The flattening on the one-time evolution of the trajectories is due to the condition on the expectation of \( C(\omega) \), which is very important in order to prove the property of determining modes.

**Theorem 3.3 (Determining modes).** Suppose that a continuous NRDS \( \phi \) on \( X \) is Lipschitz in the variable \( x \in X \) uniformly in \( t \in [0, 1] \), over the \( D \)-random cocycle attractor \( A \), that is, for each \( \omega \in \Omega, \sigma \in \Sigma \) and any \( t \in [0, 1] \) there exists \( L(\omega) > 0 \) such that
\[ \| \phi(t, \omega, \sigma, x) - \phi(t, \omega, \sigma, y) \| \leq L(\omega) \| x - y \|, \quad \forall x, y \in A_\sigma(\omega). \tag{3.3} \]
with
\[ \lim_{m \to \infty} \frac{1}{m} \ln L(\theta_m \omega) = 0. \tag{3.4} \]

Suppose that \( \phi \) satisfies the RCSP on \( A \). Then, for each \( \omega \in \Omega \) and \( \sigma \in \Sigma \) the following result on determining modes is true: Let \( k \in \mathbb{R}^+ \) be a number satisfying
\[ \mathbb{E}(C(\omega)) < k < \ln(1/2\delta), \]
and let \( x, y \in A_\sigma(\omega) \) be two points for which
\[ \lim_{t \to +\infty} e^{k t} \| P(\phi(t, \omega, \sigma, x) - \phi(t, \omega, \sigma, y)) \| = 0. \tag{3.5} \]
Then
\[ \lim_{t \to +\infty} e^{\tilde{k} t} \| \phi(t, \omega, \sigma, x) - \phi(t, \omega, \sigma, y) \| = 0, \]
for \( 0 < \tilde{k} < k - \mathbb{E}(C(\omega)) \).

**Proof.** Because of the Lipschitz condition of the NRDS, it is enough to prove that the result is true in the discrete case, that is
\[ \lim_{m \to +\infty} e^{k m} \| \phi(m, \omega, \sigma, x) - \phi(m, \omega, \sigma, y) \| = 0, \]
since for any \( t = m + s, s \in [0, 1] \) by the cocycle property we have
\[
e^{k t} \| \phi(t, \omega, \sigma, x) - \phi(t, \omega, \sigma, y) \|
= e^{k m} e^{\tilde{k} s} \| \phi(m + s, \omega, \sigma, x) - \phi(m + s, \omega, \sigma, y) \|
= e^{k m} e^{\tilde{k} s} \| \phi(s, \theta_m \omega, \theta_m \sigma, \phi(m, \omega, \sigma, x)) - \phi(s, \theta_m \omega, \theta_m \sigma, \phi(m, \omega, \sigma, y)) \|
\leq e^{\tilde{k} m} \mathbb{E}(C(\omega)) \| \phi(m, \omega, \sigma, x) - \phi(m, \omega, \sigma, y) \|.
\]
We can now write for \( \epsilon_0 > 0 \) small enough
\[
e^{k m} \mathbb{E}(C(\omega)) \| \phi(m, \omega, \sigma, x) - \phi(m, \omega, \sigma, y) \|
= e^{(k + \epsilon_0) m} e^{-\epsilon_0 m} \mathbb{E}(C(\omega)) \| \phi(m, \omega, \sigma, x) - \phi(m, \omega, \sigma, y) \|.
\]
and for $e^{-\epsilon m}L(\vartheta_m,\omega)$ we have that from (3.4) and for all $m$ large enough and $\epsilon > 0$ small enough such that $\epsilon - \epsilon_0 < 0$

$$e^{-\epsilon_0 m} \leq e^{(\epsilon - \epsilon_0)m},$$

and this implies

$$e^{k^*} e^{k m} L(\vartheta_m,\omega) \|\phi(m,\omega,\sigma,x) - \phi(m,\omega,\sigma,y)\| \xrightarrow{m \to \infty} 0. \tag{3.6}$$

Going back to the discrete case, we argue by contradiction (see [35]). Suppose that there exist $\epsilon > 0$ and a sequence $m_j \to \infty$ such that

$$e^{k m_j} \|\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y)\| \geq \epsilon > 0, \quad \forall j \in \mathbb{N}. \tag{3.7}$$

Denote $Q = I - P$, then there exist $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have

$$e^{k m_j} \|Q(\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y))\|
\geq e^{k m_j} \|P(\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y))\|. \tag{3.8}$$

Indeed, if not, by (3.5) there exists a subsequence of $\{m_j\}_{j \in \mathbb{N}}$, relabeled $\{m_j\}_{j \in \mathbb{N}}$ again, such that

$$e^{k m_j} \|\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y)\|
\leq 2 e^{k m_j} \|P(\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y))\| \xrightarrow{j \to \infty} 0,$$

which contradicts (3.7). Thus, for each $j$ fixed, from (3.8) and RCSP it follows

$$\|\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y)\|
\leq \tilde{C}(\vartheta_{m_j-1\omega}) \|\phi(m_j-1,\omega,\sigma,x) - \phi(m_j-1,\omega,\sigma,y)\|,$$

where and in the sequel

$$\tilde{C}(\omega) = 2\delta e^{\int_0^1 \mathcal{C}(\vartheta,\omega) \, ds}, \quad \forall \omega \in \Omega.$$

Now consider $\phi(m_j-1,\omega,\sigma,x)$ and $\phi(m_j-1,\omega,\sigma,y)$. Again, by the RCSP, either they satisfy

$$\|Q(\phi(m_j-1,\omega,\sigma,x) - \phi(m_j-1,\omega,\sigma,y))\|
\leq \|P(\phi(m_j-1,\omega,\sigma,x) - \phi(m_j-1,\omega,\sigma,y))\|$$

or

$$\|\phi(m_j-1,\omega,\sigma,x) - \phi(m_j-1,\omega,\sigma,y)\|
\leq \tilde{C}(\vartheta_{m_j-2\omega}) \|\phi(m_j-2,\omega,\sigma,x) - \phi(m_j-2,\omega,\sigma,y)\|.$$

Continue this way until we reach $M_j$ with either $M_j = m_j$ or for which ($M_j < m_j$)

$$\|Q(\phi(m_j-M_j,\omega,\sigma,x) - \phi(m_j-M_j,\omega,\sigma,y))\|
\leq \|P(\phi(m_j-M_j,\omega,\sigma,x) - \phi(m_j-M_j,\omega,\sigma,y))\|.$$

Then applying the RCSP $M_j$ times, we have

$$\|\phi(m_j,\omega,\sigma,x) - \phi(m_j,\omega,\sigma,y)\|
\leq \sqrt{2\tilde{C}(\vartheta_{m_j-1\omega}) \cdots \tilde{C}(\vartheta_{m_j-M_j\omega})} \|P(\phi(m_j-M_j,\omega,\sigma,x) - \phi(m_j-M_j,\omega,\sigma,y))\|
= \sqrt{2(2\delta)^{M_j} e^{\int_{m_j-M_j}^{m_j} \mathcal{C}(\vartheta,\omega) \, ds}} \|P(\phi(m_j-M_j,\omega,\sigma,x) - \phi(m_j-M_j,\omega,\sigma,y))\|. \tag{3.9}$$
Notice that we now have two possibilities for the sequence \( \{ M_j \}_{j \in \mathbb{N}} \).

**Case 1.** Suppose that \( \{ M_j \}_{j \in \mathbb{N}} \) is bounded, that is

\[
\sup_{j \in \mathbb{N}} M_j = M < \infty.
\]

Then, multiplying both sides in (3.9) by \( e^{km_j} \) and taking into account that \( 2\delta < 1 \) we get

\[
e^{km_j} \| \phi(m_j, \omega, \sigma, x) - \phi(m_j, \omega, \sigma, y) \| \\
\leq \sqrt{2} e^{km_j} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j} \sup_{1 \leq m \leq M} \left\| \mathcal{P}(\phi(m_j - m, \omega, \sigma, x) - \phi(m_j - m, \omega, \sigma, y)) \right\| \\
\leq \sqrt{2} e^{-(k - \tilde{k})m_j} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j} e^{kM} \sup_{1 \leq m \leq M} e^{k(m_j - m)} \times \left\| P (\phi(m_j - m, \omega, \sigma, x) - \phi(m_j - m, \omega, \sigma, y)) \right\|.
\]

On the other hand, by the ergodicity of the shift \( \vartheta \), we have that

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathcal{C}(\vartheta, \omega) \, ds = \mathbb{E}(\mathcal{C}(\omega)).
\]

So, for all \( \epsilon > 0 \) there exists a \( j_0 \in \mathbb{N} \) such that

\[
\int_0^{m_j} \mathcal{C}(\vartheta, \omega) \, ds \leq (\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j, \quad \forall j \geq j_0.
\]

Thus, from (3.10) we have

\[
e^{km_j} \| \phi(m_j, \omega, \sigma, x) - \phi(m_j, \omega, \sigma, y) \| \\
\leq \sqrt{2} e^{-(k - \tilde{k})m_j} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j} e^{kM} \sup_{1 \leq m \leq M} e^{k(m_j - m)} \times \left\| P (\phi(m_j - m, \omega, \sigma, x) - \phi(m_j - m, \omega, \sigma, y)) \right\|.
\]

But as, for \( \epsilon \) small enough \( \tilde{k} < k - (\epsilon + \mathbb{E}(\mathcal{C}(\omega))) \), that is, \( k - \tilde{k} > (\epsilon + \mathbb{E}(\mathcal{C}(\omega))) \) and by (3.5) we have exponential convergence (with exponent \( k \)) of the first \( m \) modes of the solutions, it is clear that the last expression tends to zero when \( j \) goes to infinity, which contradicts (3.7).

**Case 2.** Suppose that there exists a subsequence of \( M_j \), we denote again by \( M_j \), such that \( M_j \to \infty \). In this situation we have again two possibilities:

**Case 2.1.** Suppose that \( \lim_{j \to \infty} (m_j - M_j) \) is bounded. Then, from (3.9) and (3.11)

\[
\| \phi(m_j, \omega, \sigma, x) - \phi(m_j, \omega, \sigma, y) \| \\
\leq \sqrt{2} (2\delta)^{m_j} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j} \mathbb{E}(\mathcal{C}(\omega))^m \| P (\phi(m_j - M_j, \omega, \sigma, x) - \phi(m_j - M_j, \omega, \sigma, y)) \| \\
\leq \sqrt{2} (2\delta)^{m_j} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j} K,
\]

where \( K \) is a bound for the last term in (3.13). Then, multiplying both sides in (3.13) by \( e^{km_j} \), we have

\[
e^{km_j} \| \phi(m_j, \omega, \sigma, x) - \phi(m_j, \omega, \sigma, y) \| \\
\leq \sqrt{2} (2\delta)^{m_j} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega))) m_j} K \\
= \sqrt{2} e^{(\epsilon + \mathbb{E}(\mathcal{C}(\omega)) + \tilde{k} - \ln(1/2\delta)) m_j} K.
\]
Taking into account that
\[ \epsilon + \mathbb{E}(C(\omega)) + \tilde{k} < k < \ln(1/2\delta) \]
for \( \epsilon > 0 \) small enough, we obtain the convergence to zero of the right expression in (3.14).

**Case 2.2.** If \( \lim_{j \to \infty} (m_j - M_j) = \infty \) then by (3.9)
\[
e^{km_j} \| \phi(m_j, \omega, \sigma, x) - \phi(m_j, \omega, \sigma, y) \| \\
\leq \sqrt{2}(2\delta)^{M_j} e^{km_j} e^{(e+\mathbb{E}(C(\omega)))m_j} \| P(\phi(m_j - M_j, \omega, \sigma, x) - \phi(m_j - M_j, \omega, \sigma, y)) \| \\
= \sqrt{2} e^{(e+\mathbb{E}(C(\omega)))(k-\ln(1/2\delta))} m_j e^{k(m_j - M_j)} \\
\times \| P(\phi(m_j - M_j, \omega, \sigma, x) - \phi(m_j - M_j, \omega, \sigma, y)) \|
\]
and note that this last expression tends to zero by the conditions we have for the constants \( \tilde{k} \) and \( k \).

Now we give a sufficient condition for the finite dimensionality of non-autonomous random invariant sets.

**Lemma 3.4.** [32, Lemma 1.2] Let \( E \) be a euclidean space of dimension \( m \), then a ball of radius \( r > 0 \) can be covered by \( n \) balls of radius \( R > 0 \) with
\[
n \leq \left( \frac{r \sqrt{m}}{R} + 1 \right)^m.
\]

**Theorem 3.5.** Suppose a continuous NRDS \( \phi \) on \( X \) has an invariant set \( B \in \mathcal{D} \). Suppose further that there exists a tempered random variable \( R(\omega) \) such that \( \sup_{\sigma \in \Sigma} \| B_{\sigma}(\omega) \| \leq R(\omega) \). If \( \phi \) satisfies the RCSP on \( B \), then there exists an absolute constant \( b < \infty \) such that
\[
d_f(B_{\sigma}(\omega)) < b, \quad \forall \omega \in \Omega, \sigma \in \Sigma,
\]
where
\[
d_f(K) := \lim_{\epsilon \to 0} \sup_{K} \frac{\ln N_\epsilon(K)}{\ln(1/\epsilon)}
\]
denotes the fractal dimensionality of a compact set \( K \subset X \), \( N_\epsilon(K) \) the minimal number of balls of radius \( \epsilon \) necessary to cover \( K \).

**Proof.** For notational convenience, we will sometimes write \( \phi(1, \omega, \sigma, x) \) as \( \phi(1, \omega, \sigma) \) \( x \). Let \( \omega \in \Omega \) and \( \sigma \in \Sigma \), by hypothesis there exists a \( u_0 \in B_{\sigma}(\omega) \) such that
\[
B_{\sigma}(\omega) \subseteq B(u_0, R(\omega)), \quad (3.15)
\]
where \( B(x, r) \) denotes the closed ball centered at \( x \) with radius \( r \). For all \( u \in B_{\sigma}(\omega) = B_{\sigma}(\omega) \cap B(u_0, R(\omega)) \), by RCSP we have for some \( \delta \in (0, 1/2) \) that
\[
\| P(\phi(1, \omega, \sigma)u - \phi(1, \omega, \sigma)u_0) \| \leq e^{f_0} C(\theta, \omega) \| P(u) \|
\]
and
\[
\| Q(\phi(1, \omega, \sigma)u - \phi(1, \omega, \sigma)u_0) \| \leq \delta e^{f_0} C(\theta, \omega) \| P(u) \|
\]
Since \( \dim PX = m \), by Lemma 3.4, we can find \( y_1^1, \ldots, y_k^0 \in PX \) such that
\[
B_{PX}(P\phi(1, \omega, \sigma)u_0, e^{f_0} C(\theta, \omega) \| R(\omega)) \subseteq \bigcup_{j=1}^{k_0} B_{PX}(y_j^1, \delta e^{f_0} C(\theta, \omega) \| R(\omega)),
\]
with

\[ k_0 \leq \left( \frac{\sqrt{m}}{\delta} + 1 \right)^m, \]

where \( B_P X(v, r) \) denotes the ball in \( PX \) of radius \( r \) and center \( v \). Without loss of generality, we take \( k_0 \) as the maximal number no less than \( (\sqrt{m}/\delta + 1)^m \), i.e.,

\[ k_0 := \left\lfloor \left( \frac{\sqrt{m}}{\delta} + 1 \right)^m \right\rfloor. \]  \hfill (3.16)

For \( j = 1, \ldots, k_0 \), set

\[ x_j^1 = y_j^1 + Q_0(1, \sigma, \omega) u_0. \]

Then there exists a \( j \) such that

\[ \| \phi(1, \omega, \sigma) u - x_j^1 \| \leq \| P \phi(1, \omega, \sigma) u - y_j^1 \| + \| Q_0(1, \omega, \sigma) u - Q_0(1, \omega, \sigma) u_0 \| \]

\[ \leq \delta e^{f_0^1 C(\omega)} ds R(\omega) + \delta e^{f_0^1 C(\omega)} ds R(\omega) \]  \hfill (3.17)

Thus, by (3.17) we have

\[ \phi(1, \omega, \sigma)(B_\sigma(\omega) \cap B(u_0, R(\omega))) \subseteq \bigcup_{j=1}^{k_0} B(x_j^1, 2\delta e^{f_0^1 C(\omega)} ds R(\omega)). \]  \hfill (3.18)

Noticing that \( k_0 \) is independent of \( \omega \) and \( \sigma \), in a same way for each \( n \in \mathbb{N} \) we have that

\[ \phi(1, \theta_n \omega, \theta_n \sigma) B_{\theta_n \sigma}(\theta_n \omega) \]

\[ = \phi(1, \theta_n \omega, \theta_n \sigma)(B_{\theta_n \sigma}(\theta_n \omega) \cap B(u_0, R(\theta_n \omega))) \]  \hfill (3.19)

can be covered by \( k_0 \) balls of radius

\[ 2\delta e^{f_{n+1}^0 C(\omega)} ds R(\theta_n \omega) = 2\delta e^{f_{n+1}^0 C(\omega)} ds R(\theta_n \omega). \]

Hence, applying iteratively the covers above, we obtain that the set

\[ \phi(n, \theta_n \omega, \theta_n \sigma) B_{\theta_n \sigma}(\theta_n \omega) \]

\[ = \phi(1, \theta_1 \omega, \theta_1 \sigma) \circ \phi(1, \theta_2 \omega, \theta_2 \sigma) \circ \cdots \circ \phi(1, \theta_n \omega, \theta_n \sigma) B_{\theta_n \sigma}(\theta_n \omega) \]

can be covered by \( k_0^n \) balls of radius given by (3.20) as well. Moreover, by the ergodicity of \( \theta_t \) and \( R(\omega) \) being tempered, we obtain, for \( n \) large enough, that

\[ (2\delta)^n e^{f_{n}^0 C(\omega)} ds R(\theta_n \omega) \]

\[ \leq (2\delta)^n e^{(E(C(\omega)) + \gamma)n} R(\theta_n \omega) \]

\[ = e^{(E(C(\omega)) + \gamma + \rho - \ln(1/2\delta))n} e^{(-\rho/2)n} R(\theta_n \omega) \]

\[ \leq e^{(-\rho/2)n}, \]  \hfill (3.21)

where \( \gamma \) and \( \rho \) are taken such that

\[ E(C(\omega)) + \gamma + \rho < \ln(1/2\delta). \]
Theorem 3.6. Suppose that a continuous NRDS \( \phi \) on \( X \) has a positively invariant \( D \)-pullback absorbing set \( B \in D \). Suppose further that there exists a tempered random variable \( R(\omega) \) such that \( \sup_{\sigma \in \Sigma} \| B_\sigma(\omega) \| \leq R(\omega) \). If \( \phi \) satisfies the RCSP, then is \( D \)-pullback omega-limit compact and hence has a \( D \)-random cocycle attractor. In addition, if \( X \) is a uniformly convex Banach space, then \( \phi \) is \( D \)-pullback flattening.

Proof. Following the proof of Theorem 3.5 before (3.20) we have that, for each \( n \in \mathbb{N} \), the set

\[
\phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma)B_{\theta_{-n}\sigma}(\vartheta_{-n}\omega)
\]

can be covered by \( k_0^n \) balls of radius

\[
(2\delta)^n e^{\int_0^\rho C(\theta_{-n}\omega) \, ds} R(\vartheta_{-n}\omega).
\]

(3.22)

By (3.21), given \( \epsilon > 0 \) we can find an \( n_0 \in \mathbb{N} \) such that

\[
(2\delta)^n e^{\int_0^\rho C(\theta_{-n}\omega) \, ds} R(\vartheta_{-n}\omega) \leq e^{(-\rho/2)n} < \epsilon,
\]

so that each \( \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma)B_{\theta_{-n}\sigma}(\vartheta_{-n}\omega) \) with \( n \geq n_0 \) can be covered by a finite number of balls of radius less than \( \epsilon \).

But for \( t = n_0 + \tau, \tau \geq 0 \), by the positive invariance of \( B_\sigma(\omega) \) and the cocycle property

\[
\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma)B_{\theta_{-t}\sigma}(\vartheta_{-t}\omega) = \phi(n_0, \vartheta_{-n_0}\omega, \theta_{-n_0}\sigma) \circ \phi(\tau, \vartheta_{-\tau}\omega, \theta_{-\tau}\sigma)B_{\theta_{-\tau}\sigma}(\vartheta_{-\tau}\omega)
\]

\[
\subseteq \phi(n, \vartheta_{-n}\omega, \theta_{-n}\sigma)B_{\theta_{-n_0}\sigma}(\vartheta_{-n_0}\omega),
\]

(3.23)

for all \( t \geq n_0 \). Thus

\[
\kappa \left( \bigcup_{t \geq n_0} \phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma)B_{\theta_{-t}\sigma}(\vartheta_{-t}\omega) \right) \leq \kappa \left( \phi(n, \vartheta_{-n}\omega, \theta_{-n_0}\sigma)B_{\theta_{-n_0}\sigma}(\vartheta_{-n_0}\omega) \right) < \epsilon,
\]

so the NRDS is thus \( D \)-pullback omega-limit compact. Finally, if \( X \) is a uniformly convex Banach space, then Theorem 2.13 implies that the NRDS is \( D \)-pullback flattening.

\[\square\]

4. Applications to 2D Navier-Stokes equations. In this section we study a 2D Navier-Stokes equation as an example to apply our theoretical analysis. First, let us introduce translation compact/bounded functions which are known important in the study of uniform attractors [10, 8, 9, 29].
4.1. Translation bounded external forcing and the symbol space. Let \( \mathcal{O} \) be an open bounded set of \( \mathbb{R}^2 \) with smooth boundary \( \partial \mathcal{O} \). Take some \( g(t, x) \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathcal{O}))^2) \), and define the symbol space \( \Sigma \) as the closed hull \( \mathcal{H}(g) \) of \( g \) in \( L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathcal{O}))^2) \) under the local weak convergence topology, i.e., \( \Sigma = \mathcal{H}(g) = \{ g(\cdot + s) : s \in \mathbb{R} \} \) in which \( \sigma_n \to \sigma \) in \( \Sigma \) if and only if for any bounded \( (t_1, t_2) \subset \mathbb{R} \),

\[
\int_{t_1}^{t_2} \langle v(s), \sigma_n(s) - \sigma(s) \rangle_{(L^2(\mathcal{O}))^2} \, ds \to 0, \quad \forall v \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathcal{O}))^2).
\]

Define a flow \( \{ \theta_s \}_{s \in \mathbb{R}} \) on \( \Sigma \) by

\[
\theta_s \sigma(\cdot) := \sigma(\cdot + s).
\]

**Definition 4.1.** A function \( g(t, x) \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathcal{O}))^2) \) is called translation bounded if

\[
\eta(g) := \sup_{\tau \in \mathbb{R}} \int_{t-1}^{t} |g(s)|^2 \, ds < \infty. \tag{4.1}
\]

For translation bounded functions we have the following proposition.

**Proposition 4.2.** [29, Propositions 6.1 & 6.2]. Suppose \( g(t, x) \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathcal{O}))^2) \) is translation bounded. Then

(i) \( g \) is translation compact, i.e., the symbol space \( \Sigma \) defined above is compact;

(ii) \( \Sigma \) is invariant under \( \theta_t \);

(iii) any function \( \sigma \in \mathcal{H}(g) \) is translation bounded, and \( \eta(\sigma) \leq \eta(g) \);

(iv) for any positive constant \( \varepsilon \) it holds

\[
\sup_{\sigma \in \Sigma} \int_{-\infty}^{0} e^{\varepsilon s} |\sigma(s)|^2 \, ds \leq \frac{\eta(g)}{1 - e^{-\varepsilon}}. \tag{4.2}
\]

From now on, we fix \( g(t, x) \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathcal{O}))^2) \) to be translation bounded. From the above proposition it is clear that the family \( \{ \theta_t \}_{t \in \mathbb{R}} \) is a well-defined base flow on \( \Sigma \) compact. Moreover, similar to [29, Section 6.1] one can see that \( \Sigma \) is in fact a compact Polish metric space, and the mapping \( t \mapsto \theta_t \sigma \) is \( (\mathbb{R}, \Sigma) \)-continuous.

4.2. Preliminaries on 2D Navier-Stokes equation. Set Banach spaces \((H, |.|)\) and \((V, ||.|)\) by

\[
H = \{ \varphi \in (L^2(\mathcal{O}))^2 : \nabla \cdot \varphi = 0, n \cdot \varphi = 0 \text{ on } \partial \mathcal{O} \},
\]

\[
V = \{ \varphi \in (H^1_0(\mathcal{O}))^2 : \nabla \cdot \varphi = 0 \},
\]

respectively, where \( n \) is the outward normal (for further details see Temam [46]).

We consider the two-dimensional stochastic Navier-Stokes equation on \( \mathcal{O} \) with translation bounded external forcing and scalar additive noise. This equation reads

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \sigma(t) + \psi \frac{\partial W(t)}{\partial t}, \\
\nabla \cdot u = 0,
\end{cases} \tag{4.3}
\]

endowed with initial-boundary value condition

\[
\begin{cases}
u(t, x)|_{t=0} = u_0(x), \\
u(t, x)|_{\partial \mathcal{O}} = 0,
\end{cases} \tag{4.4}
\]

where \( \nu > 0 \) is a constant, \( \sigma \in \Sigma \) and \( \Sigma \) is the previously defined symbol space defined as the hull of a translation bounded function \( g \). The term \( W(t) \) is a scalar Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\) specified later.
Define the operator of Stokes $A : D(A) \subset H \to H$ as $Au = -\mathcal{P} \Delta u$, where $\mathcal{P}$ is the orthogonal projection in $(L^2(\Omega))^2$ over $H$ and $D(A) = (H^2(\Omega))^2 \cap V$. The operator $A$ is a self-adjoint positive operator in $H$ with compact inverse (see Temam [47]), it is known by spectral theory that there exists a sequence $\{\lambda_j\}_{j=1}^\infty$ satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and a sequence of vectors $\{e_j\}_{j=1}^\infty \subset D(A)$, which is orthonormal in $H$ and such that

$$Ae_j = \lambda_j e_j, \quad j = 1, 2, \ldots.$$

Moreover, define the bilinear operator $B$ as

$$\langle B(u, v), w \rangle = \int_\Omega w(x) \cdot (u \cdot \nabla)v \, dx$$

$$= \sum_{i,j=1}^2 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad \forall u \in H, v, w \in H.$$ By the incompressibility condition we have

$$\langle B(u, v), v \rangle = 0, \quad \langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad \forall u, v, w \in (H^1_0(\Omega))^2.$$ We assume that

$$\psi \in (W^{1,\infty}(\Omega))^2 \cap D(A).$$

With these preliminaries, equation (4.3) is written in the following abstract form

$$\frac{du}{dt} + \nu Au + B(u, u) = \sigma(t) + \psi \, dW(t).$$

To describe the probability space that will be used in this paper, we write

$$\Omega = \{ \omega \in C(\mathbb{R}, H) : \omega(0) = 0 \}.$$ Let $\mathcal{F}$ be the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$, and $\mathbb{P}$ be the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Define a group $\{\theta_t\}_{t \in \mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in \Omega.$$ Inspired by [26], for some $\alpha > 0$ (specified later by (4.22)), we consider

$$z(\omega) = -\int_{-\infty}^0 e^{\alpha \tau}dW(\tau), \quad \omega \in \Omega.$$ Then $z(\omega)$ is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$dz(\theta_t \omega) + \alpha z(\theta_t \omega) dt = dW(t).$$

It is known from [1, 11, 34] that there exists a $\theta$-invariant subset $\hat{\Omega} \subset \Omega$ of full measure such that $z(\theta_t \omega)$ is continuous in $t$ for every $\omega \in \hat{\Omega}$ and

$$\mathbb{E} \left( e^{\int_{-\infty}^t |z(\theta_s \omega)|^2 \, ds} \right) \leq e^{\frac{\nu}{2} t}, \forall s \in \mathbb{R}, \forall \alpha^2 \geq e^2 > 0, \forall t \geq 0,$$

$$\mathbb{E} (|z(\theta_s \omega)|^r) = \frac{\Gamma \left( \frac{1+r}{2} \right)}{\sqrt{\pi} \alpha^r}, \quad \forall r > 0, \forall s \in \mathbb{R},$$

$$\lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{t} = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t z(\theta_s \omega) \, ds = 0,$$

$$\lim_{t \to \infty} e^{-at} |z(\theta^{-t} \omega)| = 0, \quad \forall a > 0,$$

where $\Gamma$ is the Gamma function. Hereafter, we will not distinguish $\hat{\Omega}$ and $\Omega$. 
Consider the following deterministic problem with random coefficients
\[
\begin{cases}
\frac{dv}{dt} + \nu Av + B(v + z(\partial_t \omega) \psi, v + z(\partial_t \omega) \psi) = \sigma(t) + \alpha z(\partial_t \omega) \psi - \nu z(\partial_t \omega) A \psi, \\
\nabla \cdot v = 0,
\end{cases}
\]
with the initial-boundary condition
\[
\begin{cases}
v(t, x)|_{t=0} = v_0(x), \\
v(t, x)|_{\partial \Omega} = 0.
\end{cases}
\]

The following result is standard, see [10, Chapter VI.1] and references therein.

**Lemma 4.3.** For each \( \omega \in \Omega, \sigma \in \Sigma, \) and \( v_0 \in H, \) problem (4.12) and (4.13) has a unique solution \( v(t, \omega, \sigma, v_0) \in C(\mathbb{R}^+; H) \cap L^2_{loc}(\mathbb{R}^+; V) \) and \( \partial_t v \in L^2_{loc}(\mathbb{R}^+; V') \). Moreover, the mapping \( v(t, \omega, \sigma, \cdot) \) is \( (\Sigma \times H, H) \)-continuous.

Now define an NRDS \( \phi: \mathbb{R}^+ \times \Omega \times \Sigma \times H \to H \) for the stochastic problem (4.3). Given \( t \geq 0, \omega \in \Omega \) and \( \sigma \in \Sigma \) and \( v_0 \in H \), set
\[
\phi(t, \omega, \sigma, u_0) = u(t, \omega, \sigma, u_0) := v(t, \omega, \sigma, u_0 - z(\omega) \psi) + z(\partial_t \omega) \psi,
\]
where \( v \) is the solution of (4.12) and (4.13). Then by (4.7) and \( \psi \in D(A) \) we see that \( u \) is the solution of (4.3) and (4.4). Moreover, the mapping \( \phi \) defines a continuous NRDS.

In the following, we study tempered cocycle attractors for the Navier-Stokes equation.

### 4.3. Uniform Estimates of Solutions

In this part we establish uniform estimates for solutions of the Navier-Stokes equation.

**Lemma 4.4.** For each \( D \in \mathcal{D}, \omega \in \Omega \) and \( \sigma \in \Sigma \), there exists a time \( T = T(D, \omega, \sigma) > 0 \) and a tempered random variable \( R_1(\omega) \) (independent of \( \sigma \)) such that the solution \( v \) of (4.12) with \( v_0 \in D \) satisfies
\[
|v(t, \partial_{-i} \omega, \theta_{-1} \sigma, v_0)|^2 \leq R_1(\omega), \quad \forall t \geq T.
\]
Moreover, for any \( p \geq 1 \) with
\[
\alpha \geq (2pc_0)^{2/3},
\]
the expectation \( \mathbb{E}(R_1(\omega)^p) < \infty \).

**Proof.** It follows from (4.12) that
\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \nu \|v\|^2 \leq |\langle B(v + z(\partial_t \omega) \psi, v + z(\partial_t \omega) \psi), v \rangle | + \| \sigma(t) \| \|v\| \\
\quad + \alpha |z(\partial_t \omega) \| \psi \| \|v\| + \nu \|z(\partial_t \omega)\| \|v\| \|\psi\| \|v\|.
\]
Since \( \langle B(\xi, \eta), \eta \rangle = 0 \) and \( \psi \in (W^{1, \infty}(\Omega))^2 \), we have
\[
|\langle B(v + z(\partial_t \omega) \psi, v + z(\partial_t \omega) \psi), v \rangle | = |\langle B(v + z(\partial_t \omega) \psi, z(\partial_t \omega) \psi), v \rangle |
\leq \| \nabla \psi \|_{(L^\infty)^2} |z(\partial_t \omega) \| \|v\|^2
\quad + |\psi\| \|\nabla \psi\|_{(L^\infty)^2} |z(\partial_t \omega) \| \|v\|^2.
\]
Denote by \( c_0 = (|\psi| + 1) \|\nabla \psi\|_{(L^\infty)^2} \). Then
\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \nu \|v\|^2 \leq c_0 |z(\partial_t \omega)\| |v|^2 + c_0 |z(\partial_t \omega)\|^2 |v| + |\sigma(t)| \|v\|
\quad + \alpha c_0 |z(\partial_t \omega)\| |v| + c |z(\partial_t \omega)\| \|v\|,
\]
(4.17)
where and hereafter $c$ denotes a positive constant which depends only on $\psi$ and $\nu$ and may change its value when necessary. By Young’s inequality and Poincaré’s inequality $\|v\| \geq \lambda_1 |v|$ for some $\lambda_1 > 0$ we have

$$\frac{d}{dt}|v|^2 + \nu |v|^2 \leq c_0 |z(\theta_t \omega)||v|^2 + c |z(\theta_t \omega)|^4 + c |\sigma(t)|^2 + c,$$  \hspace{1cm} (4.18)

and then

$$\frac{d}{dt}|v|^2 + \nu \lambda_1 |v|^2 \leq c_0 |z(\theta_t \omega)||v|^2 + c |z(\theta_t \omega)|^4 + c |\sigma(t)|^2 + c.$$  \hspace{1cm} (4.19)

Applying Gronwall’s technique to (4.19) we have

$$|v(t, \omega, \sigma, v_0)|^2 \leq e^\int_0^t (-\nu \lambda_1 + c_0 |z(\theta_{r-t} \omega)|) \, dr |v_0|^2$$

$$+ c \int_0^t e^\int_r^t (-\nu \lambda_1 + c_0 |z(\theta_{s-t} \omega)|) \, dr (|z(\theta_s \omega)|^4 + |\sigma(s)|^2 + 1) \, ds.$$  \hspace{1cm} (4.20)

Replacing $\omega$ by $\theta_{-t} \omega$ and $\sigma$ by $\theta_{-t} \sigma$, respectively, we have

$$|v(t, \theta_{-t} \omega, \theta_{-t} \sigma, v_0)|^2 \leq e^\int_0^t (-\nu \lambda_1 + c_0 |z(\theta_{r-t} \omega)|) \, dr |v_0|^2$$

$$+ c \int_0^t e^\int_r^t (-\nu \lambda_1 + c_0 |z(\theta_{s-t} \omega)|) \, dr (|z(\theta_s \omega)|^4 + |\sigma(s)|^2 + 1) \, ds.$$  \hspace{1cm} (4.21)

Since the process $|z(\theta_t \omega)|$ is stationary and ergodic, by the ergodic theorem we see that

$$\frac{1}{s} \int_{-s}^0 |z(\theta_t \omega)| \, dt \to E(|z(\omega)|) \quad \text{as} \quad s \to \infty.$$  \hspace{1cm} (4.22)

This means that there exists an $s_0(\omega) > 0$ such that

$$\frac{1}{s} \int_{-s}^0 |z(\theta_t \omega)| \, dt \leq 2E(|z(\omega)|) \leq \frac{2}{\sqrt{2\alpha}} \quad \forall s \geq s_0(\omega).$$  \hspace{1cm} (4.23)

Let $\alpha > 0$ be large enough such that

$$\frac{2}{\sqrt{2\alpha}} \leq \frac{\nu \lambda_1}{2c_0},$$  \hspace{1cm} (4.24)

so that (4.21) implies

$$e^\int_0^t (-\nu \lambda_1 + c_0 |z(\theta_{r-t} \omega)|) \, dr \leq e^{-\frac{\nu \lambda_1}{2c_0} s} \quad \forall s \geq s_0(\omega).$$  \hspace{1cm} (4.25)

Hence, since $v_0 \in D_{\theta_{-t} \sigma}(\theta_{-t} \omega)$ and $D$ is tempered, it follows from (4.23) and (4.20) that there exists a $T = T(D, \omega, \sigma) > 0$ such that

$$|v(t, \theta_{-t} \omega, \theta_{-t} \sigma, v_0)|^2$$

$$\leq 1 + c \int_{-\infty}^0 e^\int_r^t (-\nu \lambda_1 + c_0 |z(\theta_{s-t} \omega)|) \, dr (|z(\theta_s \omega)|^4 + |\sigma(s)|^2 + 1) \, ds, \quad \forall t \geq T.$$  \hspace{1cm} (4.26)

Now, consider the $\sigma$-dependent term involved in (4.24). This term reads

$$\int_{-\infty}^0 e^\int_r^t (-\nu \lambda_1 + c_0 |z(\theta_{s-t} \omega)|) \, dr |\sigma(s)|^2 \, ds = \int_{-s_0}^0 + \int_{-\infty}^{-s_0},$$
which has been split into two parts at \( s_0 = s_0(\omega) > 0 \) given by (4.21). By (4.23) and Proposition 4.2 (iv), the two parts are bounded respectively by

\[
\int_{-s_0}^{0} e^{\int_{s_0}^{0} (-\nu_1 + c_0 |\vartheta(s)|) \, ds} |\sigma(s)|^2 \, ds
\]

\[
\leq e^{\int_{s_0}^{0} \left(-\frac{3\nu_1}{2} + c_0 |\vartheta(s)|\right) \, ds} \int_{-s_0}^{0} e^{\int_{s_0}^{0} \frac{\nu_1}{t} \, ds} |\sigma(s)|^2 \, ds
\]

\[
\leq e^{\int_{s_0}^{0} \frac{\nu_1}{t} \, ds} \int_{-s_0}^{0} e^{\frac{\nu_1}{t} \, ds} \eta(g) \frac{1}{1 - e^{-\frac{\nu_1}{t}}}
\]

and

\[
\int_{-\infty}^{-s_0} e^{\int_{s_0}^{0} (-\nu_1 + c_0 |\vartheta(s)|) \, ds} |\sigma(s)|^2 \, ds \leq \int_{-\infty}^{-s_0} e^{\frac{\nu_1}{t} \, ds} |\sigma(s)|^2 \, ds \leq \frac{\eta(g)}{1 - e^{-\frac{\nu_1}{t}}},
\]

where \( \eta(g) \) is a positive constant given by (4.1). Hence, if we set

\[
\rho = \frac{\eta(g)}{1 - e^{-\frac{\nu_1}{t}}} + \frac{\eta(g)}{1 - e^{-\frac{\nu_1}{t}}},
\]

then \( \rho \) is a constant such that

\[
\sup_{\sigma \in \Sigma} \int_{-\infty}^{0} e^{\int_{t}^{0} (-\nu_1 + c_0 |\vartheta(s)|) \, ds} |\sigma(s)|^2 \, ds \leq \rho.
\]

Therefore, by (4.24) and (4.25) we obtain

\[
|v(t, \vartheta_{t}, \vartheta_{t}, \vartheta_{t}, \vartheta_{t})|^2 \leq c + c \int_{-\infty}^{0} e^{\int_{t}^{0} (-\nu_1 + c_0 |\vartheta(s)|) \, ds} (|\vartheta(s)|^4 + 1) \, ds =: R_1(\omega), \quad \forall t \geq T.
\]

By Hölder’s inequality we have

\[
E(R_1(\omega)^p) \leq c(p)^p + c(p) E \left( \left( \int_{-\infty}^{0} e^{\int_{t}^{0} (-\nu_1 + c_0 |\vartheta(s)|) \, ds} (|\vartheta(s)|^4 + 1) \, ds \right)^p \right)
\]

\[
\leq c(p)^p + c(p) \left( \int_{-\infty}^{+\infty} e^{-\frac{\nu_1}{p} |\vartheta(s)|^4} \, ds \right)^{p-1} \times
\]

\[
\times E \left( \left( \int_{-\infty}^{+\infty} e^{\frac{\nu_1}{p} |\vartheta(s)|^4} \, ds \right)^p \right)
\]

\[
= c(p)^p + c(p) \left( \left( \frac{2(p-1)}{2p \nu_1} \right)^{p-1} \int_{-\infty}^{+\infty} e^{-\frac{\nu_1}{p} |\vartheta(s)|^4} \, ds \right)^p \times E \left( \left( \int_{-\infty}^{+\infty} e^{\frac{\nu_1}{p} |\vartheta(s)|^4} \, ds \right)^p \right)
\]

\[
\leq c(p)^p + \tilde{c}(p) \int_{-\infty}^{+\infty} e^{-\frac{\nu_1}{p} |\vartheta(s)|^4} \left( E \left( \int_{-\infty}^{+\infty} e^{2pc_0 |\vartheta(s)|^4} \, ds \right)^{\frac{1}{2}} \right)^2 \left( E \left( \int_{-\infty}^{+\infty} e^{2pc_0 |\vartheta(s)|^4} \, ds \right) \right)^{\frac{1}{2}} \, ds.
\]

(4.27)
By (4.11), (4.9) and (4.33) it follows that
\[ E((|z(\theta_\omega)|^4 + 1)^{2p}) \leq c_1(p)E(|z(\theta_\omega)|^{8p}) + c_1(p) \]
\[ = c_1(p) \frac{\Gamma\left(\frac{1+8p}{2}\right)}{\sqrt{\pi} \alpha^{4p}} + c_1(p), \]  
(4.28)
and
\[ E\left(e^{2pc_0 \int_{t_0}^t |z(\theta_\omega)| \, dr}\right) = E\left(e^{2pc_0 \int_{t_0}^{t+\infty} |z(\theta_\omega)| \, dr}\right) \leq e^{2pc_0 \alpha s}, \quad \forall s \geq 0. \]  
(4.29)
Using (4.28) and (4.29) in (4.27) we obtain
\[ E(R_1(\omega)^p) \leq c(p)^p + c(p)c_1(p) \left(\frac{\Gamma\left(\frac{1+8p}{2}\right)}{\sqrt{\pi} \alpha^{4p}}\right)^{1/2} \int_0^{+\infty} e^{\frac{p}{2}}(\frac{2\sqrt{\alpha}}{\alpha^2} - \nu \lambda_1)s \, ds. \]  
(4.30)
Since \( \alpha > 0 \) satisfies (4.22), then
\[ \frac{2c_0}{\sqrt{\alpha}} - \nu \lambda_1 < 0. \]  
(4.31)
Thus, by (4.30) and (4.31) we conclude that
\[ E(R_1(\omega)^p) \leq c(p)^p + c(p)c_1(p) \left(\frac{\Gamma\left(\frac{1+8p}{2}\right)}{\sqrt{\pi} \alpha^{4p}}\right)^{1/2} \left(\frac{2\sqrt{\alpha}}{\alpha^2} - \nu \lambda_1\right) < \infty. \]
The proof is complete.

**Lemma 4.5.** For each \( D \in \mathcal{D}, \omega \in \Omega \) and \( \sigma \in \Sigma \), there exist a time \( T = T(D, \omega, \sigma) > 0 \) given by Lemma 4.4 and a tempered random variable \( R_2(\omega) \) (independent of \( \sigma \)) such that the solution \( v \) of (4.12) with \( v_0 \in D \) satisfies
\[ \int_{t-1}^t ||v(s, \theta_{-t\omega}, \theta_{-t}\sigma, v_0)||^2 \, ds \leq R_2(\omega), \quad \forall t \geq T + 1. \]  
(4.32)
Moreover, for any \( p \geq 1 \) with \( \alpha \geq (4pc_0)^{2/3} \),
\[ \text{the expectation } E(R_2(\omega)^p) < \infty. \]  
(4.33)

**Proof.** Integrating (4.18) over \((t-1, t)\) we obtain
\[ \int_{t-1}^t ||v(s, \omega, \sigma, v_0)||^2 \, ds \leq c \int_{t-1}^t |z(\theta_\omega)||v(s, \omega, \sigma, v_0)||^2 \, ds \]
\[ + c \int_{t-1}^t (|z(\theta_\omega)|^4 + |\sigma(s)|^2 + 1) \, ds + |v(t - 1, \omega, \sigma, v_0)|^2. \]
Replacing \( \omega \) by \( \theta_{-t}\omega \) and \( \sigma \) by \( \theta_{-t}\sigma \), we have that
\[ \int_{t-1}^t ||v(s, \theta_{-t}\omega, \theta_{-t}\sigma, v_0)||^2 \, ds \leq c \int_{t-1}^t |z(\theta_{-t}\omega)||v(s, \theta_{-t}\omega, \theta_{-t}\sigma, v_0)||^2 \, ds \]
\[ + c \int_{t-1}^t (|z(\theta_{-t}\omega)|^4 + |\sigma(s - t)|^2) \, ds + |v(t - 1, \theta_{-t}\omega, \theta_{-t}\sigma, v_0)|^2 + c. \]
For any \( t \geq T + 1 \), by Lemma 4.4, we conclude that
\[ \int_{t-1}^t ||v(s, \theta_{-t}\omega, \theta_{-t}\sigma, v_0)||^2 \, ds \leq R_2(\omega) \]
with
\[ R_2(\omega) := e^{\alpha} \frac{\eta(g)}{1 - e^{-\alpha}} + c \int_{-1}^{0} |z(\vartheta_s \omega)| R_1(\vartheta_s \omega) \, ds + c \int_{-1}^{0} |z(\vartheta_s \omega)|^4 \, ds + R_1(\vartheta_{-1} \omega), \]
where we have used the relation \( \int_{-1}^{0} |\sigma(s)|^2 \, ds \leq e^{\alpha} \int_{-1}^{0} e^{\alpha s} |\sigma(s)|^2 \, ds \) and Proposition 4.2 (iv). By Hölder’s inequality we have
\[
\mathbb{E}(R_2(\omega)^p) \leq c(p) e^{p\alpha} \frac{\eta(g)^p}{(1 - e^{-\alpha})^p} + c(p) \mathbb{E}\left( \left( \int_{-1}^{0} |z(\vartheta_s \omega)| R_1(\vartheta_s \omega) \, ds \right)^p \right) \\
+ c(p) \mathbb{E}\left( \left( \int_{-1}^{0} |z(\vartheta_s \omega)|^4 \, ds \right)^p \right) + c(p) \mathbb{E}(R_1(\vartheta_{-1} \omega)^p) \\
\leq c(p) e^{p\alpha} \frac{\eta(g)^p}{(1 - e^{-\alpha})^p} + c(p) \int_{-1}^{0} \left( \mathbb{E}(|z(\vartheta_s \omega)|^{2p}) \right)^{\frac{1}{2}} \left( \mathbb{E}(R_1(\vartheta_s \omega)^{2p}) \right)^{\frac{1}{2}} \, ds \\
+ c(p) \int_{-1}^{0} \mathbb{E}(|z(\vartheta_s \omega)|^{4p}) \, ds + c(p) \mathbb{E}(R_1(\vartheta_{-1} \omega)^p). \tag{4.35}
\]
Using the condition (4.33) and Lemma 4.4 we obtain
\[ \mathbb{E}(R_1(\omega)^{2p}) < \infty. \]
Since \( \vartheta_t \) is measure-preserving and ergodic on \( (\Omega, \mathcal{F}, P) \), by Birkhoff ergodic Theorem (see \cite{49}) we have that
\[ \mathbb{E}(R_1(\vartheta_s \omega)^{2p}) = \mathbb{E}(R_1(\omega)^{2p}) < \infty, \quad \text{(independent of } s \text{ and } \omega). \tag{4.36} \]
Now, by (4.9) it follows that
\[ \mathbb{E}(|z(\vartheta_s \omega)|^{2p}) = c_3(p) < \infty, \quad \text{(independent of } s \text{ and } \omega), \tag{4.37} \]
and
\[ \mathbb{E}(|z(\vartheta_s \omega)|^{4p}) = c_4(p) < \infty, \quad \text{(independent of } s \text{ and } \omega). \tag{4.38} \]
Thus, applying (4.36)-(4.38) to (4.35), we conclude that
\[ \mathbb{E}(R_2(\omega)^p) < \infty. \]
The proof is complete. \( \square \)

**Lemma 4.6.** For each \( D \in \mathcal{D}, \omega \in \Omega \) and \( \sigma \in \Sigma \), there exists a time \( T = T(D, \omega, \sigma) > 0 \) given by Lemma 4.4 and a tempered random variable \( R_3(\omega) \) (independent of \( \sigma \)) such that the solution \( v \) of (4.12) with \( v_0 \in D \) satisfies
\[ \|v(t, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, v_0)\|^2 \leq R_3(\omega), \quad \forall t \geq T + 1. \tag{4.39} \]
Moreover, for any \( p \geq 1 \) satisfying (4.33) the expectation \( \mathbb{E}(R_3(\omega)^p) < \infty. \)

**Proof.** The vorticity field \( \xi = \text{rot} u = \partial_{x_2} u_1 - \partial_{x_1} u_2 \), satisfies the equation
\[ \frac{\partial \xi}{\partial t} - \nu \Delta \xi + (u \cdot \nabla) \xi = \text{rot} \sigma + \text{rot} \psi \frac{\partial W(t)}{\partial t} \]
(we have \( \text{rot}(u \cdot \nabla)u) = (u \cdot \nabla) \xi \) using the condition \( \text{div} u = 0 \), with initial-boundary value condition (4.4). It is possible to prove that the seminorm \( \|\phi\|_{H^1(\mathcal{O})} \) is equivalent to the classical norm \( \|\phi\|_{H^1(\mathcal{O})} \).
Replacing $\omega$ by elementary inequalities, we get

$$
\frac{d}{dt} \tilde{v} - \nu \Delta \tilde{v} + (u \cdot \nabla) \tilde{v} = \rot \sigma + \alpha z(\theta \omega) \rot \psi - z(\theta \omega)(u \cdot \nabla) \rot \psi + \nu z(\theta \omega) \Delta \rot \psi.
$$

(4.40)

Using the boundary conditions, it follows that

$$
\frac{d}{dt} \| \tilde{v}(t, \omega, \sigma, v_0) \|_{L^2(\Omega)}^2 + 2 \nu \| \tilde{v}(t, \omega, \sigma, v_0) \|_{H^1(\Omega)}^2 = 2(\rot \sigma + \alpha z(\theta \omega) \rot \psi - z(\theta \omega)(u \cdot \nabla) \rot \psi + \nu z(\theta \omega) \Delta \rot \psi, \tilde{v})
\leq \nu \| \tilde{v}(t, \omega, \sigma, v_0) \|_{H^1(\Omega)}^2 + c(\| \rot \sigma \|_{H^{-1}(\Omega)}^2 + |z(\theta \omega)|^2 |\rot \psi|_{H^{-1}(\Omega)}^2)
+ |z(\theta \omega)|^2 (v, \omega, \sigma, v_0 \cdot \nabla) \rot \psi|_{H^{-1}(\Omega)}^2
+ |z(\theta \omega)|^2 |\Delta \rot \psi|_{H^{-1}(\Omega)}^2).
$$

Thus,

$$
\| \tilde{v}(t, \omega, \sigma, v_0) \|_{L^2(\Omega)}^2
\leq \| \tilde{v}(s, \omega, \sigma, v_0) \|_{L^2(\Omega)}^2 + c \int_t^s (|\rot \sigma|_{H^{-1}(\Omega)}^2 + |z(\theta \omega)|^2 |\rot \psi|_{H^{-1}(\Omega)}^2)
+ |z(\theta \omega)|^2 (v, \omega, \sigma, v_0 \cdot \nabla) \rot \psi|_{H^{-1}(\Omega)}^2
+ |z(\theta \omega)|^2 |\Delta \rot \psi|_{H^{-1}(\Omega)}^2) ds.
$$

Integrating the above relation with respect to $s$ over $(t - 1, t)$ and using some elementary inequalities, we get

$$
\| \tilde{v}(t, \omega, \sigma, v_0) \|_{L^2(\Omega)}^2
\leq \int_{t-1}^t \| \tilde{v}(s, \omega, \sigma, v_0) \|_{L^2(\Omega)}^2 ds
+ c \int_{t-1}^t \left( |\sigma(s)|^2 + |z(\theta s \omega)|^2 |\psi|_{W^{1, \infty}}^2 + |z(\theta s \omega)|^2 \| v(s, \omega, \sigma, v_0) \|_{W^{1, \infty}}^2
+ |z(\theta s \omega)|^4 |\psi|_{W^{1, \infty}}^2 + |z(\theta s \omega)|^2 |\rot \psi|_{H^1(\Omega)}^2 \right) ds.
$$

Replacing $\omega$ by $\theta^{-1} \omega$ and $\sigma$ by $\theta^{-1} \sigma$, we have that

$$
\| \tilde{v}(t, \theta^{-1} \omega, \theta^{-1} \sigma, v_0) \|_{L^2(\Omega)}^2
\leq \int_{t-1}^t \| \tilde{v}(s, \theta^{-1} \omega, \theta^{-1} \sigma, v_0) \|_{L^2(\Omega)}^2 ds
+ c \int_{t-1}^t \left( |\sigma(s - t)|^2 + |z(\theta^{-1} \omega)(s)|^2 |\psi|_{W^{1, \infty}}^2
+ |z(\theta^{-1} \omega)(s)|^2 \| v(s, \theta^{-1} \omega, \theta^{-1} \sigma, v_0) \|_{W^{1, \infty}}^2
+ |z(\theta^{-1} \omega)(s)|^4 |\psi|_{W^{1, \infty}}^2 + |z(\theta^{-1} \omega)(s)|^2 |\rot \psi|_{H^1(\Omega)}^2 \right) ds.
$$

(4.41)

As $|\rot \psi|_{L^2(\Omega)}$ is equivalent to the classical norm $\| \phi \|_{H^1(\Omega)}$, there exists constants $c_1 > 0$ and $c_2 > 0$ such that

$$
c_1 \| v(t, \theta^{-1} \omega, \theta^{-1} \sigma, v_0) \|_{W^{1, \infty}}^2 \leq c_2 \int_{t-1}^t \| v(s, \theta^{-1} \omega, \theta^{-1} \sigma, v_0) \|_{W^{1, \infty}}^2 ds.
$$
For any $t \geq T + 1$, by Lemmas 4.4 and 4.5, we conclude that
$$
\|v(t, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, v_0)\|^2 \leq R_3(\omega),
$$
with
$$
R_3(\omega) := \frac{c_2}{c_1} R_2(\omega) + e^{\alpha} \frac{c}{c_1} \frac{\eta(g)}{1 - e^{-\alpha}}
+ \frac{c}{c_1} \int_{-1}^{0} \left( |z(\vartheta_s \omega)|^2 |\psi|^2 + |z(\vartheta_s \omega)|^2 R_1(\vartheta_s \omega) |\psi|^2 \right) ds
+ |z(\vartheta_s \omega)|^2 \|\text{rot}\psi\|^2_{H^1(\mathcal{O})} \right) \right) ds. \tag{4.43}
$$
where we have used the relation $\int_{-1}^{0} |\sigma(s)|^2 ds \leq e^{\alpha} \int_{-1}^{0} e^{\alpha s} |\sigma(s)|^2 ds$ and Proposition 4.2 (iv). It is clear that from definition of $R_3(\omega)$ and Lemmas 4.4 and 4.5, we have for any $p \geq 1$ satisfying (4.33) that $\mathbb{E}(R_3(\omega)^p) < \infty$. The proof is complete. $\square$

By (4.14) and Lemmas 4.4 and 4.6 we have the following estimate for solutions (4.14).

**Corollary 4.6.1.** For each $D \in \mathcal{D}$, $\omega \in \Omega$ and $\sigma \in \Sigma$ there exists a time $T = T(D, \omega, \sigma) > 1$ such that the solution $u(t, \omega)$ of the stochastic Navier-Stokes equation (4.5) with $u_0 \in D$ satisfies
$$
|u(t, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, u_0)|^2 \leq c R_1(\omega) + c |z(\omega)|^2,
\|u(t, \vartheta_{-t} \omega, \vartheta_{-t} \sigma, u_0)\|^2 \leq c R_3(\omega) + c |z(\omega)|^2,
$$
for all $t \geq T$, where $R_1(\omega)$ is a tempered random variable given by (4.26), $R_3(\omega)$ is the random variable given by (4.43) and $c$ is a positive constant.

4.4. **Tempered cocycle attractors.** For each $\omega \in \Omega$ and $\sigma \in \Sigma$, define
$$
B(\omega) = \{ u \in H : |u|^2 \leq c R_1(\omega) + c |z(\omega)|^2 \},
K(\omega) = \{ u \in V : \|u\|^2 \leq c R_3(\omega) + c |z(\omega)|^2 \}. \tag{4.44}
$$

The Corollary 4.6.1 indicates that $B$ and $K$ are both $D$-pullback absorbing sets.

**Theorem 4.7.** The NRDS $\phi$ generated by the stochastic Navier-Stokes equation (4.14) with translation bounded forcing has a $D$-random cocycle attractor $A = \{ A_\sigma(\cdot) \}_{\sigma \in \Sigma}$ in $H$ given by
$$
A_\sigma(\omega) = W(\omega, \sigma, B), \quad \forall \sigma \in \Sigma, \, \omega \in \Omega. \tag{4.45}
$$
Moreover, there exists a random variable $R_4(\omega)$ such that
$$
\sup_{\sigma \in \Sigma} |A_\sigma(\omega)|^2 = \sup_{\sigma \in \Sigma} \sup_{u_0 \in A_\sigma(\omega)} |u_0|^2 \leq R_4(\omega), \tag{4.46}
$$
and for any $p \geq 1$ satisfying (4.33) the expectation
$$
\mathbb{E}(R_4(\omega)^p) < \infty. \tag{4.47}
$$
Proof. The existence of cocycle attractor is proved in [28]. Since the \( \mathcal{D} \)-random cocycle attractor \( \mathcal{A} \) is smaller than the \( \mathcal{D} \)-pullback absorbing set \( K \), then \( \bigcup_{\sigma \in \Sigma} \mathcal{A}_{\sigma}(\omega) \subseteq K(\omega) \). Thus, by definition of \( K \) we have
\[
\sup_{\sigma \in \Sigma} |A_{\sigma}(\omega)|^2 \leq cR_\mathcal{A}(\omega) + c|\omega|^2 := R_4(\omega).
\]
It is clear that from definition of \( R_4(\omega) \) and Lemma 4.6, for any \( p \geq 1 \) satisfying (4.33)
\[
\mathbb{E}(R_4(\omega)^p) < \infty.
\] (4.48)
The proof is complete. \( \square \)

4.5. Random cocycle squeezing property. In this part, we show that the NRDS generated by the Navier-Stokes equation satisfies the RCSP on the \( \mathcal{D} \)-random cocycle attractor \( \mathcal{A} \).

**Theorem 4.8.** Suppose that
\[
\alpha \geq (8c_0)^{2/3}.
\] (4.49)
Then the NRDS \( \phi \) generated by the stochastic Navier-Stokes equation (4.14) with translation bounded forcing satisfies the RCSP on the \( \mathcal{D} \)-random cocycle attractor \( \mathcal{A} \).
Proof. Let \( u \) and \( v \) two solutions of (4.5), then
\[
\frac{d(u - v)}{dt} + \nu A(u - v) + B(u, u - v) + B(u - v, v) = 0.
\] (4.50)
Denote by \( P \) the orthogonal projector onto the subspace of \( H \) spanned by the first \( m \) eigenfunctions associated with the Stokes operator \( A \) and \( Q = I - P \). Taking the inner product of (4.50) with \( Q(u - v) \), we obtain
\[
\frac{1}{2} \frac{d}{dt} |Q(u - v)|^2 + \nu |Q(u - v)|^2 \\
\leq (B(u, u - v), Q(u - v)) + |(B(u - v, v), Q(u - v))| \\
\leq \|u\|_{L^4(O)^2} \|Q(u - v)\| \|u - v\|_{L^4(O)^2} + \|v\|_{L^4(O)^2} \|Q(u - v)\| \|u - v\|_{L^4(O)^2} \\
= \|u\|_{L^4(O)^2} \|Q(u - v)\| \|u - v\|_{L^4(O)^2} \\
\leq \frac{\nu}{2} |Q(u - v)|^2 + c \left( \|u\|^2_{L^4(O)^2} + \|v\|^2_{L^4(O)^2} \right) \|u - v\|^2_{L^4(O)^2},
\]
and thus
\[
\frac{d}{dt} |Q(u - v)|^2 + \nu |Q(u - v)|^2 \\
\leq c \left( \|u\|^2_{L^4(O)^2} + \|v\|^2_{L^4(O)^2} \right) \|u - v\|^2_{L^4(O)^2}.
\]
Since \( H_0^1(O) \hookrightarrow L^4(O) \) for \( n = 2 \) and \( \|Q(u - v)|^2 \geq \lambda_{m+1} |Q(u - v)|^2 \), it follows that
\[
\frac{d}{dt} |Q(u - v)|^2 + \nu \lambda_{m+1} |Q(u - v)|^2 \\
\leq c \left( \|u\|^2 + \|v\|^2 \right) \|u - v\|^2.
\]
By Gronwall lemma, we have
\[
|Q(u(t) - v(t))|^2 \leq \int_0^t e^{-\nu \lambda_{m+1}(t-s)} \left( \|u(s)|^2 + \|v(s)|^2 \right) \|u(s) - v(s)|^2 \, ds \\
+ e^{-\nu \lambda_{m+1}t} |Q(u_0 - v_0)|^2.
\] (4.51)
Now, we estimate the norm $\|u - v\|^2$ by $|\text{rot}(u - v)|^2$ and the latter by $|u_0 - v_0|^2$. Taking the inner product of (4.50) with $u - v$ in $H$, we get

$$\frac{1}{2} \frac{d}{dt} |u - v|^2 + \nu \|u - v\|^2 \leq |(B(u - v), u - v)| \leq \|u - v\|^2 \|u\| \|v\|.$$  

Using the following inequality (see Temam [47])

$$\|u\|_{L^4(\mathcal{O})} \leq c \|u\|^2_{H^1_0(\mathcal{O})}, \quad \forall u \in H^1_0(\mathcal{O}),$$  

(4.52)

and Young’s inequality we have

$$\frac{1}{2} \frac{d}{dt} |u - v|^2 + \nu \|u - v\|^2 \leq c |u - v| \|u - v\| \|v\| \leq \frac{\nu}{2} |u - v|^2 + c |u - v|^2 \|v\|^2.$$  

Hence,

$$\frac{d}{dt} |u - v|^2 + \nu \|u - v\|^2 \leq c |u - v|^2 \|v\|^2.$$  

(4.53)

By Gronwall’s lemma, we obtain

$$|u(t) - v(t)|^2 \leq e^{\int_0^t c \|v(s)\|^2 ds} |u_0 - v_0|^2.$$  

(4.54)

Thus integrating (4.53) with respect to $t$ in the interval $(t - 1, t)$, with $t \geq 1$ and using (4.54), we have that

$$\nu \int_{t-1}^t \|u(s) - v(s)\|^2 ds \leq |u(t-1) - v(t-1)|^2 + c \int_{t-1}^t \|v(s)\|^2 |u(s) - v(s)|^2 ds \leq \left( e^{\int_0^t c \|v(r)\|^2 dr} + c \int_{t-1}^t \|v(s)\|^2 e^{\int_0^t c \|v(r)\|^2 dr} ds \right) |u_0 - v_0|^2 \leq e^{\int_0^t c \|v(r)\|^2 dr} \left( 1 + c \int_{t-1}^t \|v(s)\|^2 ds \right) |u_0 - v_0|^2.$$  

(4.55)

Let us put $\xi_u = \text{rot} u$ and $\xi_v = \text{rot} v$, we have

$$\frac{d}{dt}(\xi_u - \xi_v) + ((u - v) \cdot \nabla) \xi_u + (v \cdot \nabla)(\xi_u - \xi_v) = \nu \Delta (\xi_u - \xi_v),$$  

(4.56)

Taking the inner product of (4.56) with $\xi_u - \xi_v$ in $H$ and using Agmon inequality, see Temam [47] we have that

$$\frac{1}{2} \frac{d}{dt} |\xi_u - \xi_v|^2 + \nu \|\xi_u - \xi_v\|^2 \leq \nu \|\xi_u - \xi_v\|^2 + c |\xi_u|^4 |\xi_u - \xi_v|^2,$$

and then

$$\frac{d}{dt} |\xi_u - \xi_v|^2 + \nu \|\xi_u - \xi_v\|^2 \leq c |\xi_u|^4 |\xi_u - \xi_v|^2.$$  

By Gronwall’s lemma we get

$$|\xi_u(t) - \xi_v(t)|^2 \leq e^{\int_0^t c |\xi_u(r)|^4 dr} |\xi_u(s) - \xi_v(s)|^2, \quad \forall t \geq s.$$  

Integrating the above relation with respect to $s$ over $(t - 1, t)$ with $t \geq 1$ we obtain

$$|\xi_u(t) - \xi_v(t)|^2 \leq e^{\int_{t-1}^t c |\xi_u(r)|^4 dr} \int_{t-1}^t |\xi_u(s) - \xi_v(s)|^2 ds.$$  

(4.57)
For all $t \geq 1$, by (4.55) and (4.57), it follows that
\[
|\xi_u(t) - \xi_v(t)|^2 \leq e^{f_{t-1}^t c|\xi_u(r)|^4 dr} \int_{t-1}^t |\xi_u(s) - \xi_v(s)|^2 \, ds \\
\leq ce^{f_{t-1}^t c|\xi_u(r)|^4 dr} \left( 1 + \int_{t-1}^t \|v(s)\|^2 \, ds \right) e^{f_{t-1}^t c\|v(s)\|^2 \, ds} |u_0 - v_0|^2 \\
\leq c \left( 1 + \int_{t-1}^t \|v(s)\|^2 \, ds \right) e^{f_{t-1}^t c(\|u(r)\|^4 + \|v(r)\|^2) \, dr} |u_0 - v_0|^2. \tag{4.58}
\]

Using this inequality in (4.51) we have that
\[
|Q(u(t) - v(t))|^2 \leq e^{-\nu \lambda_{m+1} t} |u_0 - v_0|^2 + c \int_0^t e^{-\nu \lambda_{m+1} (t-s)} \left( \|u(s)\|^2 + \|v(s)\|^2 \right) \times \\
\times \left( 1 + \int_{t-1}^t \|v(r)\|^2 \, dr \right) e^{f_{t-1}^t c(\|u(r)\|^4 + \|v(r)\|^2) \, dr} |u_0 - v_0|^2 \, ds. \tag{4.59}
\]

By (4.54), for each $\omega \in \Omega$, $\sigma \in \Sigma$ and any $u_0, v_0 \in A_{\sigma}(\omega)$ we have
\[
|P(\phi(t, \omega, \sigma, u_0) - \phi(t, \omega, \sigma, v_0))|^2 \leq e^{f_{t-1}^t c\|v(s)\|^2 \, ds} |u_0 - v_0|^2.
\]

By the invariance of $A$ we know $v(s, \omega, \sigma, v_0) = \phi(s, \omega, \sigma, v_0) \in A_{\sigma, \sigma}(\vartheta_\sigma \omega)$ and by (4.46) we conclude that
\[
|P(\phi(t, \omega, \sigma, u_0) - \phi(t, \omega, \sigma, v_0))| \leq e^{f_{t-1}^t \vartheta R_4(\vartheta_\sigma \omega) \, ds} |u_0 - v_0|. \tag{4.60}
\]

Now, by (4.59) and (4.46) we have
\[
|Q(u(t) - v(t))|^2 \leq e^{-\nu \lambda_{m+1} t} |u_0 - v_0|^2 \times \\
+ \int_0^t e^{-\nu \lambda_{m+1} (t-s)} R_4(\vartheta_\sigma \omega) \, ds \int_{t-1}^t c \left( 1 + R_4(\vartheta_\sigma \omega) \right) \, ds \times \\
\times e^{f_{t-1}^t (R_4(\vartheta_\sigma \omega)^2 + R_4(\vartheta_\sigma \omega)) \, ds} |u_0 - v_0|^2 \\
\leq e^{-\nu \lambda_{m+1} t} |u_0 - v_0|^2 \times \\
+ c \left( \int_0^t e^{-2\nu \lambda_{m+1} (t-s)} \, ds \right)^{1/2} \left( \int_0^t R_4(\vartheta_\sigma \omega)^2 \, ds \right)^{1/2} \times \\
\times \int_0^t \left( 1 + R_4(\vartheta_\sigma \omega) \right) ds e^{f_{t-1}^t (R_4(\vartheta_\sigma \omega)^2 + R_4(\vartheta_\sigma \omega)) \, ds} |u_0 - v_0|^2 \\
\leq e^{-\nu \lambda_{m+1} t} |u_0 - v_0|^2 + \frac{c}{\sqrt{2\nu \lambda_{m+1}}} \left( \int_0^t R_4(\vartheta_\sigma \omega)^2 \, ds \right)^{1/2} \times \\
\times \int_0^t \left( 1 + R_4(\vartheta_\sigma \omega) \right) ds e^{f_{t-1}^t (R_4(\vartheta_\sigma \omega)^2 + R_4(\vartheta_\sigma \omega)) \, ds} |u_0 - v_0|^2. \tag{4.61}
\]

Since $\sqrt{x} \leq e^x$ for all $x \geq 0$, from (4.61) it follows that
\[
|Q(\phi(t, \omega, \sigma, u_0) - \phi(t, \omega, \sigma, u_0))|^2 \leq \left( e^{-\nu \lambda_{m+1} t} + \frac{1}{\sqrt{2\nu \lambda_{m+1}}} \right) e^{f_{t-1}^t c(R_4(\vartheta_\sigma \omega)^2 + R_4(\vartheta_\sigma \omega) + 1) \, ds} |u_0 - v_0|^2,
\]
and hence,

\[ |Q(\phi(t, \omega, \sigma, u_0) - \phi(t, \omega, \sigma, u_0))| \leq \left( e^{-\frac{\nu \lambda_{m+1}}{2} t} + \frac{1}{(2\nu \lambda_{m+1})^{1/4}} \right) e^{\int_0^t \frac{1}{2} (R_4(\theta, \omega)^2 + R_4(\theta, \omega) + 1) ds|u_0 - v_0|}. \]

(4.62)

We now set the random variable

\[ C(\omega) = \frac{c}{2}(R_4(\omega)^2 + R_4(\omega) + 1). \]

For each \( \sigma \in \Sigma \) and \( \omega \in \Omega \) and any \( u_0, v_0 \in A_\sigma(\omega) \). It follows from (4.60) and (4.62) with \( t = 1 \) that

\[ |P(\phi(1, \omega, \sigma, u_0) - \phi(1, \omega, \sigma, v_0))| \leq e^{\int_0^1 C(\theta, \omega) ds|u_0 - v_0|} \]

(4.63)

and

\[ |Q(\phi(1, \omega, \sigma, u_0) - \phi(1, \omega, \sigma, v_0))| \leq \left( e^{-\frac{\nu \lambda_{m+1}}{2} t} + \frac{1}{(2\nu \lambda_{m+1})^{1/4}} \right) e^{\int_0^1 C(\theta, \omega) ds|u_0 - v_0|}. \]

(4.64)

Using (4.47) we conclude that

\[ \mathbb{E}(C(\omega)) < \infty. \]

We choose \( m \) large enough so that

\[ \delta := \left( e^{-\frac{\nu \lambda_{m+1}}{2}} + \frac{1}{(2\nu \lambda_{m+1})^{1/4}} \right) < \frac{1}{2} \]

(4.65)

and

\[ \mathbb{E}(C(\omega)) < \ln(1/2\delta). \]

The proof is complete.

4.6. Determining modes and finite dimensionality of the random cocycle attractor. In this part, we show a determining modes result and finite dimensionality of the \( \mathcal{D} \)-random cocycle attractor for the Navier-Stokes equation. The finite dimensionality of the attractor (kernel sections) for non-autonomous Navier-Stokes equations and other models goes back to the works of Chepyzhov and Vishik (see, for instance, [8, 9, 5, 48, 7]). Note that our method does not give the better upper bounds on the dimension of attractors for the non-autonomous Navier-Stokes, as the Lyapunov method (see, for instance, [6, 10, 40]).

**Theorem 4.9 (Determining modes).** Suppose that \( \alpha > 0 \) satisfy (4.49). Let \( k \in \mathbb{R}^+ \) satisfying

\[ \mathbb{E}(C(\omega)) < k < \ln(1/2\delta), \]

(4.66)

and for each \( \omega \in \Omega, \sigma \in \Sigma \) let \( u_0, v_0 \in A_\sigma(\omega) \) be two points for which

\[ \lim_{t \to \infty} e^{kt}|P(\phi(t, \omega, \sigma, u_0) - \phi(t, \omega, \sigma, v_0))| = 0. \]

Then

\[ \lim_{t \to \infty} e^{kt}|\phi(t, \omega, \sigma, u_0) - \phi(t, \omega, \sigma, v_0)| = 0, \]

for \( 0 < \hat{k} < k - \mathbb{E}(C(\omega)) \).
Proof. By the invariance of $\mathcal{A}$ we know $v(s, \omega, \sigma, v_0) = \phi(s, \omega, \sigma, v_0) \in \mathcal{A}_{\theta, \sigma}(\theta_s \omega)$ and by (4.46) and (4.54) we conclude that
\[
|\phi(t, \sigma, \omega, u_0) - \phi(t, \sigma, \omega, v_0)| \leq e^{\int_0^t \frac{1}{2} R_4(\theta_s \omega) \, ds} |u_0 - v_0|, \quad \forall t \in [0, 1].
\]
We set
\[
L(\omega) = e^{\int_0^1 \frac{1}{2} R_4(\theta_s \omega) \, ds},
\]
then $L$ satisfies (3.4). Indeed, by the ergodicity of the shift $\vartheta_t$ for a general measurable random variable $C_0(\omega)$ we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t C_0(\vartheta_s \omega) \, ds = E(C_0(\omega)), \quad (4.67)
\]
and thus,
\[
\lim_{m \to \infty} \frac{1}{m} \ln(L(\vartheta_m \omega)) = \lim_{m \to \infty} \frac{1}{m} \int_0^m \frac{1}{2} R_4(\vartheta_s \omega) \, ds
\]
\[
= \lim_{m \to \infty} \frac{m+1}{m} \int_0^{m+1} \frac{1}{2} R_4(\vartheta_s \omega) \, ds
\]
\[
- \lim_{m \to \infty} \frac{1}{m} \int_0^m \frac{1}{2} R_4(\vartheta_s \omega) \, ds
\]
\[
= E\left(\frac{e}{2} R_4(\omega)\right) - E\left(\frac{e}{2} R_4(\omega)\right) = 0.
\]
So the result follows from Theorem 3.3. \qed

Theorem 4.10. Suppose that $\alpha > 0$ satisfy (4.49). Then the $D$-random cocycle attractor $A = \{A_\sigma(\cdot)\}_{\sigma \in \Sigma}$ given in Theorem 4.7 for the NRDS $\phi$ generated by the stochastic Navier-Stokes equation (4.14) with translation bounded forcing has finite fractal dimension satisfying that for some absolute constant $b < \infty$,
\[
d_f(A_\sigma(\omega)) < b, \quad \forall \omega \in \Omega, \sigma \in \Sigma.
\]
Proof. It is clear that, from definition of $B(\omega)$, cf. (4.44), Lemma 4.4 and minimal property of $\mathcal{A}$ that there exists a tempered random variable $R(\omega)$ such that
\[
\sup_{\sigma \in \Sigma} |A_\sigma(\omega)| \leq R(\omega)\]
Then the result follows from Theorem 3.5. \qed

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