Maximum Eigenmode Relaying with statistical Channel State Information at the Relay

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Abstract—Optimal precoding in the relay is investigated to maximize ergodic capacity of a multiple antenna relay channel. The source and the relay nodes are equipped with multiple antennas and the destination with a single antenna. It is assumed that the channel covariance matrices of the relay’s receive and transmit channels are available to the relay, and optimal precoding at the relay is investigated. It is shown that the optimal transmission from the relay should be conducted in the direction of the eigenvectors of the transmit-channel covariance matrix. Then, we derive the necessary and sufficient conditions under which the relay transmission only from the strongest eigenvector achieves capacity; this method is called Maximum Eigenmode Relaying (MER).

I. INTRODUCTION AND RELATED WORK

Cooperative communication has been rather active field of research in the recent past (e.g. [1]–[4]). It was first shown by van der Meulen in [3] that cooperation can enhance the transmission rate of a communication system. Later on, substantial work was carried out, investigating the effect of cooperation in various types of communication systems. The first works concentrated on communication nodes with single antennas and several relaying protocols were proposed of which two important ones are (non-regenerative) AF (Amplify-and-Forward) relaying and (regenerative) DF (Decode-and-Forward) relaying. In this paper, we will focus on non-regenerative relaying techniques.

Multiple antenna systems are well known to boost the Shannon capacity of a communication system (see e.g. [5]). A natural setup is the combination of cooperative communications with multiple antennas, which is at the heart of current research in the field. Assuming multiple antennas at the relay, one of the major tasks is to design a suitable relaying protocol. However, depending on the system or the desired performance criterion, different “optimal” relaying protocols can exist: for instance, the non-regenerative relaying protocols, e.g. in [7]–[9], are designed to minimize Mean Square Error (MSE) but other relaying protocols, e.g. in [10]–[13], are assumed to maximize Shannon capacity. Moreover, depending on the available channel information in the relay, the “optimal” relaying protocol can be different.

A. Related Work

A noncoherent cooperative system was investigated in [12]. The source and the relay nodes are equipped with multiple antennas and the destination node with single antenna. It was assumed that the source and the relay have access to the covariance matrix of the channels. Another assumption was that the antennas in the source are correlated but no correlation was assumed in the relay. Using that knowledge, the optimal transmit direction in the source and the relay was derived. Assume \( Q \) is transmit covariance matrix in the source with spectral decomposition \( Q = U_{Q} \Lambda_{Q} U_{Q}^{H} \) and \( \Sigma \) is the correlation matrix of the source, with spectral decomposition \( \Sigma = U_{\Sigma} \Lambda_{\Sigma} U_{\Sigma}^{H} \). It was proved that the optimal \( Q \) must be in the form of \( Q^{o} = U_{\Sigma} \Lambda_{Q^{o}} U_{\Sigma}^{H} ; \text{i.e. } U_{Q^{o}} = U_{\Sigma} \) and \( \Lambda_{Q^{o}} \) is in descending order which needs to be solved numerically. It was also proved that the optimal gain matrix in the relay is a weighted identity matrix which depends on \( n_{R} \), \( \Sigma \) and \( Q \). Furthermore, the necessary and sufficient condition for the optimality of beamforming in the source was derived as

\[
\gamma \lambda^{\Sigma}_{2} \leq \frac{(1 + \gamma \lambda^{\Sigma}_{1})D(1 + \gamma \lambda^{\Sigma}_{1})}{D(1 + \gamma \lambda^{\Sigma}_{1}) + A(\gamma \lambda^{\Sigma}_{1}) - 1} - 1
\]

where \( D(1 + \gamma \lambda^{\Sigma}_{1}) \) is a function given in [12 Eq. 16] and \( A(\gamma \lambda^{\Sigma}_{1}) = \mathbb{E}\{(1 + z)e^{1+z}f(0, z)\} \) where \( \mathbb{E}\{ \cdot \} \) represents expectation operation. Note that the evaluation of the optimality of beamforming in [1] requires computationally expensive Monte Carlo simulations or numerical integrations due to \( A(\gamma \lambda^{\Sigma}_{1}) \).

More recent results were obtained in [13] where an optimal relay precoding was investigated for a system with correlated antennas at the relay. It was assumed that the relay has access to full CSI of \( H_{1} \) but only to the covariance matrix of \( H_{2} \), i.e. to \( \mathbf{R}_{k} \) with spectral decomposition \( \mathbf{R}_{k} = U_{k} \Lambda_{k} U_{k}^{H} \). In [13], [14], it was proved that the optimal relay precoding matrix, \( \mathbf{F}^{o} \), is in the form of \( \mathbf{F}^{o} = U_{k} \Lambda_{k} F^{o}_{H_{1}} \), where \( V^{o}_{H_{1}} \) is the unitary matrix with columns as the eigenvectors of \( H_{1}^{H} H_{1} \). Along with the numerical methods to derive \( \Lambda_{k} \) in [13], the optimality of beamforming was considered only for the asymptotic case of high transmit SNR. The relay beamforming was found to be optimal if following inequality holds:

\[
\left( \frac{\sigma^{2}}{f_{k}^{R}_{\Lambda}} \right)^{n_{R}} e^{\frac{\sigma^{2}}{f_{k}^{R}_{\Lambda}}} \Gamma(1 - n_{D}, \frac{\sigma^{2}}{f_{k}^{R}_{\Lambda}}) \times \left( \frac{f_{k}^{R}_{\Lambda} \lambda^{\Sigma}_{2}}{\sigma^{2}} + 1 \right) + \frac{f_{k}^{R}_{\Lambda} \lambda^{\Sigma}_{2}}{\sigma^{2}} (n_{D} - 1) \leq 1
\]

Details can be found in [13] Sec. III-C.

In this paper, we consider a MIMO cooperative system with multiple antennas at the relay in which the relay has access to the covariance matrices of preceding and following channels. Moreover, the relay antennas are assumed to be correlated. With partial channel knowledge (the covariance matrices) available to the relay, a capacity maximizing relaying method will be introduced; the exact system model and the desired performance criterion will be discussed in forthcoming sections.

The paper is organized as follows: In Section II the system model is explained and two problems are stated that are the main subject of this paper. Section III deals with designing optimal precoding in the relay in order to maximise capacity. In Section IV the necessary and sufficient conditions for the optimality of MER are investigated. In Section V some asymptotic results and their usefulness in practical systems are considered. In Section VI numerical results are presented and finally, some conclusion remarks are explained in Section VII.

II. SYSTEM MODEL AND PROBLEM STATEMENT

A. Notation

Matrices are represented by boldface upper cases \((H)\). Column and row vectors are denoted by boldface lower cases \((h)\), and \( h_{i} \) indicates the \( i \)-th element of \( h \). The superscript \(^{H}\) stands for Hermitian transposition. We refer to the identity matrix by \( I \). The expectation operation is indicated by \( \mathbb{E}\{ \cdot \} \) and \( f_{X}(z) \) is reserved for probability density functions (pdf); \( \Lambda_{\Sigma} \) represents a diagonal matrix...
C. Problem Statement

with elements organized in descending order and \( \lambda_i^G \) denotes the 
i-th diagonal element of \( \Sigma \). For simplicity of notation, \((\lambda_i^G)^2\) is abbreviated by \( \lambda_i^2 \). The trace of a matrix is denoted by \( \text{Tr}(\cdot) \).

B. System Model

A dual hop, half duplex non-coherent MIMO communication system is considered in this paper. A source node with \( n_s \) antennas communicates with a single-antenna destination node only via a relay node that is equipped with \( n_R \) antennas (that are used for both reception and transmission). It is assumed that a direct link between the source and the destination is not available. The half duplex constraint is accomplished by time sharing between the source and the relay; i.e., each transmission period is divided into two time slots: the source transmits during the first time slot and the relay during the second one. The relay remains silent during the source transmission and vice versa. It is assumed that the source does not have access to any statistical or deterministic channel state information (CSI). The signal received at the relay \( (y_R) \) due to the source transmission is given by
\[
y_R = H_1 x + w_R
\]
where the \( n_s \times n_R \) matrix \( H_1 \) represents channel between the source and the relay (below, only the statistics of \( H_1 \) are assumed to be known at the relay). With \( P_s \) the power constraint of the source, the column vector \( x \) is the signal transmitted from the source with \( Q = E(x x^H) = \frac{P_s}{n_s} I_{n_s} \) and the column vector \( w_R \) represents the receiver noise in the relay with elements independently drawn from a complex Gaussian random variable with variance \( N_0 \). The relay multiplies \( y_R \) with gain matrix \( F \) and forwards it to the destination. Then, the received signal at the destination is
\[
y_D = h_2 F y_R + w_D
\]
where the row vector \( h_2 \) indicates the channel between the relay and the destination; \( w_D \) represents the receiver noise at the destination. For simplicity, we assume that \( w_D \) is statistically equivalent to \( w_R \) and that both noise processes have unit-variance, i.e., \( N_0 = 1 \); the latter choice is no extra restriction, as the ratios of transmit powers and noise powers determine performance, and we are still free to choose \( P_s \) and \( P_R \) arbitrarily.

We assume spatial correlation only at the relay, which can be due to unobstructed relay node or space limits at the relay which force antennas to be closely located. Justifications to assume transceivers with spatial correlation can be found in [13], [16]. The correlation matrix in the relay is represented by \( \Sigma \) with spectral decomposition \( \Sigma = U_\Sigma \Lambda_\Sigma U_\Sigma^H \), where \( U_\Sigma \) is a unitary matrix with its columns the eigenvectors corresponding to \( \Sigma \), and \( \Lambda_\Sigma \) is diagonal matrix with the eigenvalues of \( \Sigma \) in decreasing order. Note that we assume i.i.d. channels for \( H_1 \) and \( h_2 \). Therefore, the channel matrices \( H_1 \) and \( h_2 \) can be written using the Kronecker model as
\[
H_1 = \Sigma^{\frac{1}{2}} H_{1,sw}
\]
\[
h_2 = h_{2,w} \Sigma^{\frac{1}{2}}
\]
where \( H_{1,sw} \) and \( h_{2,w} \) are i.i.d., zero mean, unit variance complex Gaussian random variables, independent of each other; \( \Sigma^{\frac{1}{2}} \) denotes a matrix formed by the square roots of the elements in the matrix \( \Sigma \).

C. Problem Statement

With [4], the ergodic capacity of the system is defined as
\[
C = \frac{1}{2} \max_{Q=\Lambda_\Sigma \Lambda_\Sigma^H} \mathbb{E} \left\{ C(H_1, h_2, F) \right\}
\]
where the relay gain matrix \( F \) is to be designed to maximise \( \mathbb{E} \left\{ C(H_1, h_2, F) \right\} \), when the expectation operation is carried out over \( H_1 \) and \( h_2 \). Assuming \( Q = \frac{P_s}{n_s} I_{n_s} \) (this means equal transmit power from each antenna is chosen in the source, because no channel knowledge is available there), the channel capacity \( C(H_1, h_2, F) \) for given channel matrices is
\[
C(H_1, h_2, F) = \log \left( 1 + \frac{P_R}{n_R} h_2 F H_1 H_1^H F^H h_2^H \right)
\]
(8)
where \( N_0 (1 + h_2^2 F H_1 H_1^H F^H h_2^H) \) is the total equivalent noise power which remains constant per coherence time due to a block fading assumption we impose.

Two major problems are addressed below:

Problem 1: Solving the optimization problem in [7], in order to find the optimal precoder \( F^* \) in the relay that maximizes mutual information between the transmit signal from the source and the received signal at the destination, given that the correlation matrix \( \Sigma \) as well as the unit variance of elements of i.i.d \( H_{1,sw} \) and \( h_{2,w} \) are available at the relay.

Problem 2: Obtaining necessary and sufficient conditions based on \( \Sigma \), so that maximum eigenmode relaying (MER) is optimal.

III. OPTIMAL PRECODING IN THE RELAY

In this section, Problem 1 from Section II-C is addressed. However, before we continue to define \( F^* \), a closer look at the power constraint of the relay in [7] is provided. The power constraint of the relay is given by
\[
\mathbb{E} \{ \| F y_R \|^2 \} = \mathbb{E} \{ \text{Tr}(F y_R y_R^H F^H) \} \leq P_R.
\]
(9)
One can write \( \mathbb{E} \{ F y_R y_R^H F^H \} = \text{Tr}(F F^H y_R y_R^H) \). Note that \( F \) is a positive gain matrix; therefore, \( G = F^H F \) is a positive symmetric matrix. One can also choose the gain matrix \( F \) to be symmetric (i.e. \( F = G^\frac{1}{2} \)), which follows from substituting (5) and (6) in (4) where \( F = G^\frac{1}{2} \). The problem of finding \( F^* \) will be replaced with finding \( G^* \). By substituting (5) and (6) in (9) and applying spectral decomposition of \( \Sigma \), \( G \) will be further simplified to
\[
\mathbb{E} \{ \text{Tr}(G y_R y_R^H) \} = P_R \text{Tr}(G U_\Sigma \Lambda_\Sigma^H U_\Sigma^H) + N_0 \text{Tr}(G) \leq P_R.
\]
(10)
By combining (10) and (7), we have
\[
C = \frac{1}{2} \max_{\text{\text{Q}}=\Lambda_\Sigma \Lambda_\Sigma^H} \mathbb{E} \{ C(H_1, h_2, G) \}.
\]
(11)
Assuming (3) and (6), applying spectral decomposition according to \( \Sigma = U_\Sigma \Lambda_\Sigma U_\Sigma^H \) and considering that the statistics of \( H_1 \) and \( h_2 \) do not change by a multiplication with a unitary matrix, the “instantaneous” capacity \( C \) for given channels \( H_1, h_2 \) is obtained as
\[
C(H_1, h_2, G) = \log \left( 1 + \frac{\sqrt{h_{2,w}^2} G \frac{1}{2} H_{1,sw} H_{1,sw}^H G^\frac{1}{2} h_{2,w}^2}{n_s (1 + h_{2,w}^2 G \frac{1}{2} h_{2,w}^2)} \right)
\]
(12)
where \( \gamma = P_S / N_0 \) and
\[
G^\frac{1}{2} = \Lambda_\Sigma^\frac{1}{2} U_\Sigma^H G^\frac{1}{2} U_\Sigma \Lambda_\Sigma^\frac{1}{2}.
\]
(13)
Then, the relay power constraint in [10] can be written as
\[
P_R \text{Tr}(\Lambda_\Sigma \hat{G}) + N_0 \text{Tr}(\Lambda_\Sigma \hat{G}) \leq P_R.
\]
(14)
On the other hand, applying the spectral decomposition \( \hat{G} = U_{\hat{G}} \Lambda_{\hat{G}} U_{\hat{G}}^H \) in (12) and considering that the statistics of a random matrix do not change by multiplying with a unitary matrix, so we
obtain
\[ C(H_1, h_2, \Lambda_G) = \log \left(1 + \frac{\gamma h_{2w} \Lambda^\frac{1}{2} \Lambda_G^\frac{1}{2} H_{1w}^H \Lambda_G^\frac{1}{2} h_{2w}}{n_S (1 + h_{2w} \Lambda_G h_{2w}^H)} \right). \] (15)

By careful inspecting (12) and (15), the conclusion is that
\[ E(C(H_1, h_2, \Lambda_G)) = E(C(H_1, h_2, \Lambda_G)). \] (16)

One interpretation of (16) is that \( C(H_1, h_2, \Lambda_G) \) in (12) will be maximized by choosing \( G \) to be a diagonal matrix, i.e. \( \hat{G} = \Lambda_G^{-1} \), that is equivalent to choosing \( U_G = I \). Considering that the spectral decomposition of \( G \) and \( \hat{G} \) in (13) are \( G = U_G \Lambda_G U_G^H \) and \( \hat{G} = U_G \Lambda_G \hat{U}_G \), respectively, then choosing \( U_G = I \) in (13) dictates that \( \hat{U}_G = U_G \) must hold.

So far, it is proved that \( \Lambda_G \) can achieve the same capacity as \( \hat{G} \). However, for a complete proof, it must be made sure that by choosing \( G = \Lambda_G \), the power constraint of the relay defined in (14) is not violated. It is clear from (13) that the power constraint of the relay depends on \( G \) only via two terms: \( \text{Tr}(\Lambda_G^{-1} G) \) and \( \text{Tr}(\Lambda_G^{-2} \hat{G}) \). In what follows, we will prove that by choosing \( G = \Lambda_G \), both terms in the relay power constraint will be minimized, hence, fulfilling the constraint.

**Lemma:** Let \( \hat{G} \) be an arbitrary positive symmetric matrix with spectral decomposition \( G = U_G \Lambda_G U_G^H \). Let \( \Lambda_G \) be a diagonal matrix with the elements on the diagonal in descending order and \( k > 0 \). Then
\[ \text{Tr}(\Lambda_G^{-k} \Lambda_G) \leq \text{Tr}(\Lambda_G^{-k} \hat{G}). \] (17)

**Proof:** See [17] Theorems 1-3 for a detailed proof for \( k = 1 \). The proof can easily be extended to arbitrary \( k \). \( \blacksquare \)

By applying (17) to (14), the power constraint will be fulfilled; that is, by choosing \( U_G = \hat{U}_G \), or equivalently \( G = \Lambda_G = \hat{G} \), it is assured that
\[ P_k \text{Tr}(\Lambda_G^{-1} \Lambda_G) + N_0 \text{Tr}(\Lambda_G^{-2} \Lambda_G) \leq P_k \text{Tr}(\Lambda_G^{-1} \hat{G}) + N_0 \text{Tr}(\Lambda_G^{-2} \hat{G}) \leq P_k. \] (18)

By completing the proof for the power constraint of the relay, the answer to **Problem 1** in Section III is accomplished. Therefore, the transmission from the relay should be conducted in the direction of the eigenvectors of the correlation matrix \( \Sigma \). However, the power per eigenvector \( \Lambda_G \) must be determined using numerical methods (see e.g. [18] for a single hop system model).

So far, we assumed equal correlation matrix \( \Sigma \) in \( H_1 \) and \( h_2 \) which implies that the correlation is a consequence of space limit in the relay node, however, when the correlation occurs due to unobstructed relay node, the correlation matrices corresponding to \( H_1 \) and \( h_2 \) can be different. Nevertheless, derivation of \( F \) will be much different. Let assume that \( H_1 = \Sigma_1^\frac{1}{2} H_{1w} \) and \( h_2 = h_{2w} \Sigma_2^\frac{1}{2} \).

Then, applying a similar method as explained in this section, one can prove that the optimal relaying matrix is
\[ F = U_{S_2} A U_{S_1}^H \] (19)
where \( U_{S_i}, i \in \{1,2\} \) is a unitary matrix corresponding to the eigenvectors of \( \Sigma_i \) and \( \Lambda \) is a diagonal matrix with its elements in decreasing order which is to be calculated numerically.

**IV. OPTIMALITY OF MAXIMUM EIGENMODE RELAYING**

In this Section, **Problem 2** from Section III-C is addressed. The focus of this section is to derive necessary and sufficient conditions under which MER is the optimal transmission method from the relay.

By the results obtained in Section III and considering that \( \Lambda_G = \Lambda_G^2 \), the ergodic capacity in (7) can be rewritten as
\[ C = \frac{1}{2} \max \frac{Q^n t}{n_S} \mathbb{E}\{C(H_{1w}, h_{2w}, \Lambda_G)\} \] (20)
where the expectation-operation is carried out over \( H_{1w} \) and \( h_{2w} \) with
\[ C(H_{1w}, h_{2w}, \Lambda_G) = \log \left(1 + \frac{\gamma h_{2w} \Lambda^\frac{1}{2} \Lambda_G^\frac{1}{2} H_{1w}^H \Lambda_G^\frac{1}{2} h_{2w}}{n_S (1 + h_{2w} \Lambda_G h_{2w}^H)} \right). \] (21)

Then \( C(H_{1w}, h_{2w}, \Lambda_G) \) can be simplified according to
\[ C(H_{1w}, h_{2w}, \Lambda_G) = \log \left(1 + \frac{\gamma \sum_{i=1}^{n_S} | h_{2w} \Lambda^\frac{1}{2} \Lambda_G \Sigma^{-1} h_{1w,i} |^2}{n_S (1 + h_{2w} \Lambda_G h_{2w}^H)} \right) \] (22)
where \( h_{1w,i} \) represents the \( i \)-th column of \( H_{1w} \). One can write the numerator
\[ \frac{1}{n_S} \sum_{i=1}^{n_S} | h_{2w} \Lambda^\frac{1}{2} \Lambda_G \Sigma^{-1} h_{1w,i} |^2 = \sum_{j=1}^{n_R} \lambda_j^G \lambda_j^{S_2} X_j \] (23)
and the denominator
\[ h_{2w} \Lambda_G \Sigma_{S_2}^{-1} h_{2w}^H = \sum_{j=1}^{n_R} \lambda_j^G \lambda_j^{S_2} X_j \] (24)
where \( X_j = | h_{2w,j} |^2 \) is an exponential random variable with unit mean, i.e. \( f_{X_j}(t) = e^{-t} \), and \( Y = \frac{1}{n_S} \sum_{i=1}^{n_S} | h_{1w,i} |^2 \) is the sum of \( n_S \) i.i.d exponential random variables with parameter \( n_S \). Indeed, \( Y \) has an Erlang-distribution with rate and shape equal to \( n_S \), i.e. \( f_Y(t) = (t/n_S - 1)e^{-t/(n_S - 1)} \). The proof for the identity (23) is omitted due to lack of space but a similar proof can be found in [12] Eq. (71)-(72)]. Plugging in (23) and (24) into (22) will simplify to
\[ C(H_{1w}, h_{2w}, \Lambda_G) = \log \left(1 + \frac{\gamma Y \sum_{j=1}^{n_R} \lambda_j^G \lambda_j^{S_2} X_j}{1 + \sum_{j=1}^{n_R} \lambda_j^G \lambda_j^{S_2} X_j} \right) \] (25)
\[ = \log \left(1 + \sum_{j=1}^{n_R} \lambda_j^G \lambda_j^{S_2} (1 + \gamma Y) X_j \right) - \log \left(1 + \sum_{j=1}^{n_R} \lambda_j^G \lambda_j^{S_2} X_j \right). \] (26)

**A. Necessary Condition**

Now we turn the attention to the question under which condition MER is optimal; i.e. the condition under which \( \lambda_j^G = 0 \) for \( j \geq 2 \) is the capacity achieving transmission method with
\[ \lambda_j^G = \frac{P_k}{(1 + \lambda_j^G P_k)} \] (26)
which is calculated by substituting \( \lambda_j^G = 0 \) for \( j \geq 2 \) in (18).

In order to evaluate the necessary and sufficient condition on the optimality of MER, we assume \( P = \sum_{j=1}^{n_R} \lambda_j^G \). Assume that the power \( P - p \) is allocated to the dominant eigenvector of \( G \), i.e. \( \lambda_1^G = P - p \), and the power \( p \) is allocated to the remaining eigenvectors of \( G \), i.e. \( p = \sum_{j=2}^{n_R} \lambda_j^G \). It is clear that if MER is optimal, then \( \partial C(p)/\partial p \bigg|_{p=0} \leq 0 \) which provides the necessary condition for the optimality of MER and \( \partial^2 C(p)/\partial p^2 \bigg|_{p=0} \leq 0 \) which confirms the sufficiency. It can be proved that \( \partial C(p)/\partial p \bigg|_{p=0} \) will be maximized if \( \lambda_j^G = p = \lambda_j^G = 0 \) for \( j > 2 \); similar discussions are provided in [12], [17], [19] and, hence, we do not repeat it here. Therefore, given the assumption \( \lambda_j^G = P - p \) and \( \lambda_j^G = p \), (25) will be further
simplified to
\[ C(H_{1w}, h_{2w}, A_G) = \log \left( 1 + (P - p)\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1 + p\lambda_2^{\Sigma_2}(1 + \gamma Y)X_2 \right) \]
\[ - \log \left( 1 + (P - p)\lambda_2^{\Sigma_2}X_1 + p\lambda_2^{\Sigma_2}X_2 \right) \]
and so, \( \frac{\partial C(p)}{\partial p} \bigg|_{p=0} \) from (20) and (27) equals
\[
\frac{\partial}{\partial p} E(C(H_{1w}, h_{2w}, A_G)) \bigg|_{p=0} = E\left\{ \frac{\lambda_2^{\Sigma_2}(1 + \gamma Y)X_2}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} - E\left\{ \frac{\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} \]
and
\[
E\left\{ \frac{\lambda_2^{\Sigma_2}X_2 - \lambda_2^{\Sigma_2}X_1}{1 + P\lambda_2^{\Sigma_2}X_1} \right\}. \quad (28)\]
The first expectation in (28) can be written as
\[
E\left\{ \frac{\lambda_2^{\Sigma_2}(1 + \gamma Y)X_2}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} = \lambda_2^{\Sigma_2}E\left\{ \frac{(1 + \gamma Y)}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} \quad (29) \]
which follows from the fact that \( X_2 \) is independent of \( X_1 \) and that \( E\{X_2\} = 1 \). By simple manipulations, the second expectation in (28) equals
\[
E\left\{ \frac{\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} = \frac{1}{P} - \frac{1}{P} \frac{1}{P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \quad (30) \]
and the last expectation in (28) can be written as
\[
E\left\{ \frac{\lambda_2^{\Sigma_2}X_2 - \lambda_2^{\Sigma_2}X_1}{1 + P\lambda_2^{\Sigma_2}X_1} \right\} = - \frac{1}{P} + \frac{e^{\frac{1}{P\lambda_2^{\Sigma_2}}} \Gamma(0, \frac{1}{P\lambda_2^{\Sigma_2}})}{(P^2\lambda_2^{\Sigma_2})^2} \cdot \quad (31) \]
where \( \Gamma(0, z) \) is the incomplete Gamma function [20] Sec. 6.5]. By combining (20), (28), (29), (30) and (31), in order to compute \( \frac{\partial C(p)}{\partial p} \bigg|_{p=0} \leq 0 \), we have
\[
\frac{\partial C(p)}{\partial p} \bigg|_{p=0} = E\left\{ \frac{\lambda_2^{\Sigma_2}(1 + \gamma Y) + 1/P}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} - \frac{e^{\frac{1}{P\lambda_2^{\Sigma_2}}} \Gamma(0, \frac{1}{P\lambda_2^{\Sigma_2}})}{(P^2\lambda_2^{\Sigma_2})^2} \leq 0 \quad (32) \]
from which by elementary operations the following constraint will be obtained for the MER to be optimal:
\[
\lambda_2^{\Sigma_2} \leq \frac{e^{\frac{1}{P\lambda_2^{\Sigma_2}}} \Gamma(0, \frac{1}{P\lambda_2^{\Sigma_2}}) - P\lambda_2^{\Sigma_2} E\{\frac{1}{Z}\}}{P^2\lambda_2^{\Sigma_2} E\{\frac{1 + \gamma Y}{Z}\} - P e^{\frac{1}{P\lambda_2^{\Sigma_2}}} \Gamma(0, \frac{1}{P\lambda_2^{\Sigma_2}})} \quad (33) \]
where \( Z = 1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1 \). Note that \( E\{1/Z\} \) and \( E\{(1 + \gamma Y)/Z\} \) in (33) are non-trivial and do not seem to have closed-form solutions. However, they can be calculated either using monte carlo simulations or by numerical integration of
\[
E\left\{ \frac{1}{Z} \right\} = \int_0^\infty A(t)e^{A(t)} \Gamma(0, A(t)) \, dt \quad (34) \]
\[
E\left\{ \frac{1 + \gamma Y}{Z} \right\} = \frac{1}{P\lambda_2^{\Sigma_2}} \int_0^\infty e^{A(t)} \Gamma(0, A(t)) \, dt \quad (35) \]
with
\[
A(t) = \frac{1}{P\lambda_2^{\Sigma_2}(1 + \gamma t)}. \quad (36) \]
The proof of (34) and (35) follows from the definition of the expectation operation and is omitted due to lack of space.

**B. Sufficient Condition**

By deriving the necessary condition for MER to be optimal, it can be proved that the necessary condition is also a sufficient condition for MER to be optimal. It will be proved by showing that \( \partial^2 C(p)/\partial p^2 \leq 0 \) for arbitrary \( p \in [0, P] \).

For deriving \( \partial^2 C(p)/\partial p^2 \), from (20) and (27) we have
\[
\frac{\partial^2}{\partial p^2} E\{C(H_{1w}, h_{2w}, A_G)\} \bigg|_{p=0} = - P \left( \frac{\lambda_2^{\Sigma_2} X_1 - \lambda_2^{\Sigma_2} X_2}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right)^2 \left( 2 + \gamma Y + 2P\lambda_2^{\Sigma_2}X_1(1 + \gamma Y) \right) \quad (37) \]
It is clear from (37) that \( \partial^2 C(p)/\partial p^2 \leq 0 \) for every \( p \), and so, the necessary condition for the optimality of MER derived in (33) is sufficient as well.

**V. DISCUSSION**

In the previous section, a necessary and sufficient condition were derived which show the optimality region of MER. As it is clear from (32) or (33), there is need to perform Monte Carlo simulations or numerical integrations to validate the optimality of MER. However, computationally expensive Monte Carlo simulations and also numerical integration methods are not a good choice in practical real-time communication systems; therefore, in the sequel, we will investigate two simplified approaches which lead to closed form solutions.

We first derive a lower bound for the necessary and the sufficient conditions for the optimality of MER which are a direct result of Jensen’s inequality. Consequently, if the lower bound condition confirms the optimality of MER, there is no need for the Monte Carlo simulations and, hence, MER can be used as a capacity-achieving method.

Then we investigate the effect of the source antenna array on the optimality of MER. That is done by considering large antenna arrays in the source using the central limit theorem. Novel, closed form necessary and sufficient conditions are derived for the optimality of MER which, according to the simulations, turn out to very tightly approximate the results for arbitrary numbers of antennas in the source.

**A. A Lower Bound on the Optimality of Maximum Eigenmode Relaying**

From previous discussions, it is clear that \( \partial C(p)/\partial p < 0 \bigg|_{p=0} \) defines the condition under which MER is optimal. We will exploit it to derive necessary and sufficient conditions which specify a lower bound on the optimality of MER. From careful inspection of (32) it is clear that
\[
\lambda_2^{\Sigma_2}(1 + \gamma Y) + 1/P \quad (38) \]
which equals the expression inside the expectation operation in (32) On the other hand, according to Jensen’s inequality who have \( E\{g(X)\} \geq g(E\{X\}) \) for a convex function \( g(x) \). Let us assume \( g(X) = 1/X \), which is a convex function for \( X > 0 \). Therefore, by applying Jensen’s inequality to the expectation operation in (32), we have
\[
E\left\{ \frac{\lambda_2^{\Sigma_2}(1 + \gamma Y) + 1/P}{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1} \right\} > \frac{1}{E\{1 + P\lambda_2^{\Sigma_2}(1 + \gamma Y)X_1\}} \quad (39) \]
with
\[ \mathbb{E}\left\{ \frac{1 + P\lambda_1^{S_2}(1 + \gamma Y)X_1}{\lambda_1^{S_2}(1 + \gamma Y) + 1/P} \right\} = \frac{P\left(\lambda_1^{S_2}(1 + P\lambda_1^{S_2}) + n_s(\lambda_1^{S_2} - \lambda_2^{S_2})e^{D}E_{n_S+1}(D)\right)}{\lambda_2^{S_2}(1 + P\lambda_2^{S_2})} < \frac{1}{\lambda_2^{S_2}} \left(1 + P\lambda_2^{S_2}\right)e^{\gamma Y}(1 + P\lambda_2^{S_2})\Gamma(0, \frac{1}{P\lambda_2^{S_2}}). \] (40)

where \( D = n_s(1 + P\lambda_2^{S_2})/(\gamma P\lambda_2^{S_2}) \) and \( E_m(z) \) the Exponential Integral function [20 Sec. 5.1]. Therefore, the following expression is defined as a lower bound which specifies that MER is optimal:
\[ \frac{P\lambda_1^{S_2}\lambda_2^{S_2}(1 + P\lambda_1^{S_2}) + n_s(\lambda_1^{S_2} - \lambda_2^{S_2})e^{D}E_{n_S+1}(D)}{\lambda_2^{S_2}(1 + P\lambda_2^{S_2})} < e^{-\gamma Y}(1 + P\lambda_2^{S_2})\Gamma(0, \frac{1}{P\lambda_2^{S_2}}). \] (41)

\[ \text{B. Large Antenna Array in the Source} \]

It is interesting to investigate the effect of the dimension of the antenna array on the optimality of MER. Note that as we focus on the optimality of maximum eigenmode transmission from the relay; therefore we only consider the effect of large antenna arrays on the optimality of MER. Assuming \( n_S \gg 1 \), the random variable \( Y \sim E(n_S, \lambda_S) \) can be approximated by a Gaussian distribution with mean 1 and variance 1/n_S, i.e., approximately, \( Y \sim N(1, 1/n_S) \). Therefore, as \( n_S \rightarrow \infty \), we obtain \( Y \rightarrow 1 \). By substituting \( Y = 1 \) in (33), we have \( 1 + \gamma Y = 1 + \gamma \). Hence, \( \mathbb{E}\{1/Z\} \) in (33) can be written in closed form as
\[ \mathbb{E}\left\{ \frac{1}{Z} \right\} = A_1e^{A_1}\Gamma(0, A_1) \] (42)
where \( A_1 = 1/P\lambda_1^{S_2}(1 + \gamma) \) is obtained by substituting \( t = 1 \) in (36). Then, the necessary and sufficient conditions for the optimality of MER, i.e. \( \partial C(p)/\partial p \mid_{p=0} \leq 0 \) in (32) or (33) for large \( n_S \), after some manipulation lead to the following closed-form constraint:
\[ \lambda_2^{S_2} \leq \frac{1}{\frac{1}{\lambda_2^{S_2}}\Gamma(0, \frac{1}{P\lambda_2^{S_2}}) - P\lambda_1^{S_2}A_1e^{A_1}\Gamma(0, A_1)} \] (43)

\[ \text{1The proof of the expectation operation in (40) is omitted due to space limit, however, it can be validated using symbolic tools, e.g. Mathematica.} \]

\[ \text{Fig. 1. MER optimality region for (}P_s, P_r)\text{ pairs for various correlations } \Sigma. \]

\[ \text{Fig. 2. Mutual information between input and output of the relay system with } n_S = 2 = n_R \text{ and MER. Given } P_s = 10 \text{dB}, \text{ the } P_k \text{ corresponding to solid lines illustrates the region where MER achieves capacity, while dashed lines show the region where MER can not achieve capacity.} \]

\[ \text{It is clear from (43) that with large antenna array in the source, the optimality of MER depends on the source transmission only via } P_s \text{ and it is independent of } H_1. \]

\[ \text{It will be shown, by numerical simulations in the next Section, that the closed-form constraint in (43) approximates (33) with high accuracy.} \]

\[ \text{VI. NUMERICAL RESULTS} \]

Computer simulations are provided to verify the analytical results derived in the previous sections. We assume that the relay is equipped with two antennas, but various numbers of antennas in the source are evaluated. As explained in Section II we assume i.i.d \( H_{1,w} \) and \( h_{2,w} \), where the elements are circularly symmetric complex Gaussian random variables with zero mean and unit variance (block Rayleigh fading assumption). The noise power in the relay and the destination is assumed to be \( N_0 = 1 \). Fig. 1 illustrates the \( (P_s, P_r) \) pairs for which MER is optimal. The optimality region of MER is calculated for \( \rho = 0.3 \) and 0.5 where \( \rho \) is defined as the inter-antenna correlation or the off-diagonal elements of the correlation matrix \( \Sigma \). It is clear from Fig. 1 that by increasing \( \rho \), the optimality region of MER increases, too. Another interesting observation from Fig. 1 is that the optimality region of MER is almost independent of the number of antennas in the source node. The figure illustrates the optimality region using the expression derived in (33) for \( n_S = 2 \) and 4 which require numerical integrations and also using the closed form expression derived in (43) when \( n_S \rightarrow \infty \) it explicitly shows that regardless of \( n_S \), the optimality regions of MER coincide with very little difference. Fig. 2 shows the mutual information between the input and output of the specified MIMO relay channel. Note that the solid lines achieve capacity using MER but the dashed lines do not achieve capacity and, hence, MER is not optimal in this part of the curve.

\[ \text{VII. CONCLUSION} \]

A dual hop cooperative system was investigated in this paper. The source and the relay nodes are equipped with multiple antennas and the destination with single antenna. Optimal precoding matrix in the
relay was derived. It was shown that the optimal transmission from the relay should be conducted in the direction of the eigenvectors of the transmit-channel covariance matrix. Then, a necessary and sufficient condition was derived, under which, the relay transmission only from the strongest eigenvector achieves capacity; this method of transmission was called Maximum Eigenmode Relaying. The exact result contains two integrals which need to be solved numerically. Moreover, a closed form lower bound was derived for the optimality region of MER. We further investigate to evaluate the effect of the source antenna array on the optimality of MER and derived closed form expression when the source is equipped with infinite antennas. The simulation results show that MER optimal region with infinite antennas in the source coincides with MER optimal region when the source is equipped with much lower number of antennas with inappreciable difference.

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