GEODESIC CURVES ON QUANTIZED MANIFOLDS

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Abstract

A general definition of the curves and geodesics associated with a given connection on a quantized manifold is given. In the particular case of the functional quantization we define geodesics in the same way as in the classical case and we will show that the two definitions are compatibles. As an example we examine our results for the quantum Manin plane.
0. Introduction

Recently there have been so many works on noncommutative geometry and the related topics by physicists and mathematicians (A. Connes 1986, 1995, Dubois-Violette 1994, Mourad 1994, Wess & Zumino 1990). In this paper following the above works we give the definition of geodesics on quantized manifolds and find them for the particular case of Manin plane.

1. Linear Connections

We apply the general definition of a linear connection in noncommutative geometry proposed by A. Connes [1] in order to generalize the notion of geodesic curves on quantized manifolds.

First we recall the definition of a linear connection in the commutative case. If \( M \) is a smooth manifold and \( C^\infty(M) \) the algebra of smooth functions on \( M \), then a linear connection on a smooth complex vector bundle \( E \) on \( M \) is given by the \( C \)-linear map

\[
\nabla : \Gamma(E) \to \Omega^1(M) \otimes_{C^\infty(M)} \Gamma(E)
\]

where \( \Gamma(E) \) is the finitely generated projective \( C^\infty(M) \)-module of sections of \( E \) and \( \Omega^1(M) \) is the \( C^\infty(M) \)-module of first order differential forms on \( M \). The linear map \( \nabla \) satisfies the following Leibniz rule

\[
\nabla(f\xi) = df \otimes \xi + f\nabla(\xi)
\]  \hspace{1cm} (1)

for each \( f \in C^\infty(M) \) and \( \xi \in \Gamma(E) \).

The fact that \( C^\infty(M) \) is a commutative algebra allows us to use the relation \( \nabla(f\xi) = \)

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\( \nabla(\xi f) \) to write (1) as

\[
\nabla(\xi f) = \sigma(\xi \otimes df) + (\nabla \xi)f
\]

where \( \sigma \) is the permutation operator.

Thanks to the well known equivalence \( E \leftrightarrow \Gamma(E) \), between the category of smooth vector bundles on \( M \) and the category of finitely generated projective \( C^\infty(M) \)-modules, each linear connection can be considered on finitely generated projective modules. The latter is suitable enough to be generalized to the noncommutative case.

In noncommutative geometry the role of \( C^\infty(M) \) is replaced by a noncommutative associative algebra \( \mathcal{A} \) and \( \Gamma(E) \) is replaced by a finitely generated projective \( \mathcal{A} \)-module \( P \). A linear connection in general case can now be defined on \( P \). For our purpose we take \( P = \Omega^1(\mathcal{A}) \), the \( \mathcal{A} \)-module of first order differential forms on \( \mathcal{A} \) and a linear connection on \( \Omega(\mathcal{A}) \) is a linear map

\[
\nabla : \Omega(\mathcal{A}) \to \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \Omega(\mathcal{A})
\]

satisfying (1) for \( f \in \mathcal{A} \) and \( \xi \in \Omega(\mathcal{A}) \).

In noncommutative case in general \( \nabla(f\xi) \neq \nabla(\xi f) \) and so we impose relation (2) for a suitable substitution for \( \sigma \). From now on \( I \subset R \) is an interval.

2. Geodesic Curves in Commutative Geometry

To each linear connection in commutative geometry, there correspond geodesic curves. In what follows we try to give an equivalent definition for this concept appropriate for the quantized contexts.

On a smooth manifold \( M \) with a fixed linear connection \( \nabla_M \) and associated christofel
symbols $\Gamma^k_{ij}$, the geodesic curves are defined locally as the solutions of the following differential equation

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0.$$  \hspace{1cm} (3)

Direct calculation shows that the condition for a parametrized curve $\gamma$ on $(M, \nabla_M)$ to be a geodesic curve, is equivalent to the relation

$$(\gamma^* \otimes \gamma^*) \circ \nabla_M = \nabla_R \circ \gamma^*$$  \hspace{1cm} (4)

where $\nabla_R$ is the extension of the standard linear connection $\delta(f) = df$ given by $\nabla_R(f \xi) = df \otimes \xi$ for $f \in C^\infty(I)$ and $\xi \in \Omega^1(I)$.

This relation can be used to generalize geodesics in quantized case.

3. Curves and Geodesics on Quantized Manifolds

Let $A$ be a quantization of the commutative algebra $B$. More precisely $A$ is a noncommutative algebra over a subring of the formal power series in $\lambda$, having the character $\psi$ and $\phi : A \to B$ is a $\psi$-homomorphism i.e. $\varphi(\lambda f) = \psi(\lambda) \varphi(f)$. More details can be found in [2,3].

Now any $\psi$-homomorphism

$$\gamma^* : A \to C^\infty(I)$$

is called a curve on the quantized algebra $A$.

We say that a curve $\gamma : A \to C^\infty(I)$ is maximal if for each interval $J \subset R$ with $I \subset J$ and each curve $\gamma' : A \to C^\infty(J)$, the relation $i^* \circ \gamma' = \gamma$ implies $\gamma = \gamma'$, where $i$ is the inclusion map.

In the following $x^i$ are linearly independent generators of the algebra $A$. Given a connection $\nabla$ on $\Omega^1(A)$ the geodesic curves are defined to be those curves $\gamma^*$ on $A$ satisfying the
relation (4) with the same symbol $\gamma^*$ as the extension of $\gamma^*$ to $\Omega(\mathcal{A})$ by $\gamma^*(df) = d\gamma^*(f)$ for $f \in \mathcal{A}$.

As in the commutative case we find that the relation (4) is equivalent to the equation

$$\frac{d^2\gamma^*(x^k)}{dt^2} + \gamma^*(\Gamma^k_{ij}) \frac{d\gamma^*(x^i)}{dt} \frac{d\gamma^*(x^j)}{dt} = 0$$

(5)

where $\Gamma^k_{ij} \in \mathcal{A}$ are the associated christofel symbols for $\nabla$. We emphasize that by knowing the christofel symbols $\gamma^*(\Gamma^k_{ij})$ is determined in terms of $\gamma^*(x^i), \gamma^*(x^j)$. Therefore this equation is the usual second order differential equation in terms of $\gamma^*(x^i).$ Since according to the functional quantization [3], the quantized manifold $M$ is the set of its points, so curves on these manifolds have parametrized representations as in the classical case. In the case of smooth functional quantization, to each such parametrized curve $\gamma : I \to M$ there corresponds a $\psi$-homomorphism

$$\gamma^* : \mathcal{A} \to C^\infty(I)$$

given by $\gamma^*(f) = f \circ \gamma|_{\lambda = \lambda_0}.$ In this case we can define the geodesic curves locally as the solutions of the equation

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$$

(6)

with the associated christofel symbols for $\nabla$ on $\Omega(\mathcal{A})$ as functions of $t.$ It is seen that the solutions of this equation $\gamma(t) = (x^i(t))_i$ are related to the solutions of the equation (5) by $\gamma^*(f) = f \circ \gamma|_{\lambda = \lambda_0}.$ So the two definitions are compatible.

4. The Quantum Plane

In this section we use our definitions of curves and geodesics for the case of quantum plane. The algebra of differential forms on the quantum plane $\Omega = \Omega^0 \oplus \Omega^1 \oplus \Omega^2$ has
generators \(x, y, dx, dy\) satisfying the commutation relations

\[
xy = qyx, \quad xdx = q^2 dxx, \quad ydy = q^2 dyy, \quad dydx + qdxdy = 0,
\]

\[
xdy = qdyx + (q^2 - 1) dxy, \quad dx dx = dy dy = 0.
\]

A linear connection \(\nabla\) on this plane compatible with the above relations is given by [4] as

\[
\nabla(dx^i) = \mu^4 x^i \theta \otimes \theta
\]

where \(x^1 = x, x^2 = y\) and \(\theta = xdy - qdyx\). The parameter \(\mu\) is such that in the classical limit \(q \to 1\) tends to zero.

The associated Christoffel symbols are found to be

\[
\Gamma^1_{11} = -\mu^4 xy^2, \quad \Gamma^1_{12} = \mu^4 q^{-1} x^2 y, \quad \Gamma^1_{21} = \mu^4 x^2 y,
\]

\[
\Gamma^1_{22} = -\mu^4 q^{-1} x^3, \quad \Gamma^2_{11} = -\mu^4 qy^2, \quad \Gamma^2_{12} = \mu^4 q^{-2} xy^2,
\]

\[
\Gamma^2_{21} = \mu^4 q^{-1} xy^2, \quad \Gamma^2_{22} = \mu^4 q^{-3} x^2 y.
\]

Since in this case \(\gamma^*(\Gamma^k_{ij}) = 0\) the equation (5) becomes

\[
\frac{d^2 \gamma^*(x^i)}{dt^2} = 0
\]

which has the solutions

\[
\gamma^*(x^i)(t) = a^i t + b^i \tag{7}
\]

where \(a^i, b^i\) are numbers.

Since we can consider the Manin plane as the functional quantization of \(\mathbb{R}^2\), by using the above Christoffel symbols we see that the solutions of (4) are of the following form \(x^i(t) = a^i_0(q) + a^i_1(q)t + \mu^4 \varphi(t)\), where \(\varphi \in C^\infty(\mathbb{R})\). Clearly these solutions satisfy (7).
References

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