CONVERGENCE ANALYSIS OF AN ACCURATE AND EFFICIENT METHOD FOR NONLINEAR MAXWELL’S EQUATIONS

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Abstract. In this paper, we investigate an accurate and efficient method for nonlinear Maxwell’s equation. DG method and Crank-Nicolson scheme are employed for spatial and time discretization, respectively. A semi-explicit extrapolation technique is adopted for the linearization of the nonlinear term. Since the proposed scheme is semi-implicit, only a linear system needs to be solved at each time step. Optimal convergent order of $O(\tau^{2} + h^{p+\frac{1}{2}})$ is proved under time step size condition $\tau \leq h^{d/4}$. Finally, 2D and 3D numerical examples are provided to validate the theoretical convergence rate.

1. Introduction. In this paper, we consider nonlinear Maxwell’s equation:

\begin{align}
\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} + \sigma(x, |\mathbf{E}|) \mathbf{E} &= 0, \quad \text{in } \Omega \times (0, T), \\
\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} &= 0, \quad \text{in } \Omega \times (0, T),
\end{align}

with initial and boundary conditions

\begin{align}
\mathbf{E}(x, 0) &= \mathbf{E}_0(x), \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x), \quad \text{in } \Omega \times (0, T), \\
\mathbf{n} \times \mathbf{E} &= 0, \quad \text{on } \partial\Omega \times (0, T).
\end{align}

Here $\Omega$ is a bounded, convex, simply-connected domain in $\mathbb{R}^d (d = 2, 3)$, $\mathbf{E}(x, t)$, $\mathbf{H}(x, t)$ represent the electric and magnetic fields, $\mathbf{n}$ is the outward normal vector on $\partial\Omega$, and $\epsilon_0$ and $\mu_0$ stand for the permittivity and the magnetic permeability, respectively. In addition, the electric conductivity $\sigma(x, s) \in C^{1,1}(\Omega, [0, \infty))$ is a real valued function. This nonlinear problem occurs in applications where the electric conductivity switches on eddy currents \cite{14}, e.g., microwave heating. A power law

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for the conductivity have been used to modeling the type-II superconductors [17] and modelling the charge-density wave state of NbSe3 [9].

The well poseness of the system (1a)–(1b) have been investigated in [5–7, 26]. The existence of the weak solution for a nonlinear function $J(E) = \sigma(|E|)E$, with $\sigma(s)$ monotonically increasing is shown in [7, 26]. In [6], the authors presented the existence and uniqueness of the system (1a)–(1b) when $J(E) = \sigma(|E|)E$, with $\sigma(s)$ monotonically increasing. In [5], the authors proved the existence of the weak solution for $J(E) = \sigma(|E|)E$, with $\sigma(s) = |s|^\alpha$.

Recently, finite element methods have been applied to solve nonlinear Maxwell equation (cf. [5,6,15,18,19]). Numerical schemes with backward Euler discretization in time and mixed conforming finite elements in space were discussed for nonlinear conductivity problems in [5,6]. A fully-discrete $(T, \psi) - \psi_e$ finite element decoupled scheme was developed to solve time-dependent eddy current problem with multiple-connected conductors [2]. Subsequently, an improved convergence rate analysis was presented in [15]. For monotonically increasing conductivity $\sigma(s)$, the existence of weak solution of system (1a)-(1b) has been proved in [7]. Under some restriction of time step $\tau$, a third order linearized BDF scheme together with finite element was developed and analyzed [25]. Very recently, in [12] the authors development a nonlinear FDTD method for Kerr-type nonlinear media.

The discontinuous Galerkin (DG) method was introduced by Reed and Hill in 1973 [16]. It has many nice features, e.g., applicability for non-conforming mesh, high-order accuracy, flexibility in handling material interface and high parallelizability. We refer to the survey paper [4], the monograph [11] and references therein for more details about DG method. As an important application, DG methods have been used to solve Maxwell’s equation in free space [3] and dispersive media [13,21–24] and dispersive media [8,13,21–24]. In recently years, the DG method had been widely used to solve the nonlinear partial differential equations, e.g., the Allen-Cahn equations [10], the Cahn-Hilliard equation [20].

The DG method for spatial variables is usually combined with fully implicit discretization to avoid the excessive restriction. However, the fully implicit schemes are more difficult in efficient implementation. Meanwhile, if using fully implicit method, a iterative method is need to solve the coupled systems of nonlinear equations per time step. If using explicit methods, will impose a severe time step restriction. Hence, it is important to design accurate and efficient numerical method for nonlinear Maxwell’s equations.

The main aim of this paper is to develop some efficient and effective numerical schemes to solve the Maxwell’s equations (1a)-(1b). We expect that our schemes can combine the following two desired properties, i.e., (i) accurate (second order in time); (ii) easy to implement and efficient (only need to solve some fully linear equations at each time step). To this end, we consider the Crank-Nicolson time discretization and an extrapolation is used to discretize the nonlinear term. Meanwhile, spatial discretization is the discontinuous Galerkin method. Since our method is semi-implicit, only linear system is needed to solve at each time step. Convergent rate of $O(\tau^2 + h^{p+1/2})$ is proved under time step size restriction $\tau \leq h^{d/4}$. This restriction is due to the low regularity of the Maxwell’s equations and the inverse estimate. Theoretical results are validated by both 2D and 3D numerical examples.

The rest of this paper is organized as follows. We state fully discrete scheme and the corresponding priori error estimate in Section 2. Detailed proof of the main
result is given in Section 3. In Section 4, both 2D and 3D numerical examples are presented to support our theoretical analysis.

**Notation.** In this paper, standard notations for Sobolev spaces (e.g., $L^2(\Omega)$, $H^k(\Omega)$ for $k \in \mathbb{N}$, etc.) and the associated norms and seminorms will be adopted. Plain and bold fonts are used for scalars and vectors, respectively. We denote by $C$ a positive number independent of the mesh size, polynomial order, and time-steps.

**2. Numerical scheme.** Let $\mathcal{T}_h$ be a quasi-uniform mesh of $\Omega$. For an element $K \in \mathcal{T}_h$, denote by $h_K = \text{diam}(K)$ the diameter of $K$. Let $h = \max_{K \in \mathcal{T}_h} h_K$ be the mesh size of $\mathcal{T}_h$. Denote by $e$ a face in $\mathcal{T}_h$, $\mathcal{E}_T$ the collection of all interior faces, $\mathcal{E}_D$ the collection of all boundary faces. For an element $K$, the outward unit normal is denoted by $n_K$ and the polynomial space in $K$ of degree at most $k$ is denoted by $P^k(K)$. Define the DG finite element space:

$$U_h = \{ v \in (L^2(\Omega))^3 : v|_K \in (P^k(K))^3, \ K \in \mathcal{T}_h \}.$$ 

Denote the standard Sobolev space in domain $D$ by $H^p(D)$ and corresponding norm by $\| \cdot \|_p$. When $p = 0$, $H^0(D)$ is just $L^2(D)$. In this paper, we also use broken Sobolev spaces $H^p(\mathcal{T}_h)$ with broken $L^2$ inner products

$$(u,v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (u,v)_K, \quad (u,v) = \sum_{K \in \mathcal{T}_h} (u,v)_K, \quad \langle u,v \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (u,v)_{\partial K}.$$ 

and standard DG notations $\{ \cdot \}$, $\| \cdot \|$ for average and jump across element boundary (cf. [23]).

Testing (1a)–(1b) by $v, w$ and integrating by parts on $K$ yield

$$\epsilon_0 \left( \frac{\partial E}{\partial t}, v \right)_K - \langle H, \nabla \times v \rangle_K - \langle n_K \times H, v \rangle_{\partial K} + \langle \sigma(x)|E| E, v \rangle_K = 0,$$  

$$\mu_0 \left( \frac{\partial H}{\partial t}, w \right)_K + \langle E, \nabla \times w \rangle_K + \langle n_K \times E, w \rangle_{\partial K} = 0.$$  

It is known that the choice of the numerical flux is very important in the definition of DG scheme. For this purpose, we need to introduce some notations first. Let $e$ be an interior face belonging to the element $K$. Set

$$v^{\text{int}}(x) = \lim_{\delta \to 0^-} v(x + \delta n_K), \quad v^{\text{ext}}(x) = \lim_{\delta \to 0^+} v(x + \delta n_K), \quad \forall x \in e.$$ 

Then the average and tangential jump of $v$ on any interior face $e$ are defined as follows:

$$\| v \| = \frac{v^{\text{int}} + v^{\text{ext}}}{2}, \quad \| v \|_T = n_K \times v^{\text{int}}(x) - n_K \times v^{\text{ext}}(x).$$ 

For a boundary face $e \subset \mathcal{E}_D$, which is belong to the element $K$, we define

$$v^{\text{int}}(x) = v^{\text{int}}(x), \quad \forall x \in e.$$ 

Then, we can define semi-discrete discontinuous Galerkin scheme for (1a)–(1b) as: Find $E_h, H_h \in U_h$, such that

$$\epsilon_0 \left( \frac{\partial E_h}{\partial t}, v_h \right) - \langle H_h, \nabla \times v_h \rangle - \langle n_K \times H_h, v_h \rangle_{\partial \mathcal{T}_h} + \langle \sigma(x)|E_h| E_h, v_h \rangle = 0,$$  

$$\mu_0 \left( \frac{\partial H_h}{\partial t}, w_h \right) + \langle E_h, \nabla \times w_h \rangle + \langle n_K \times E_h, w_h \rangle_{\partial \mathcal{T}_h} = 0,$$  

where $\mathcal{T}_h$ is a face in $\mathcal{T}_h$. Set $K \in \mathcal{T}_h$, we define

$$v^{\text{int}}(x) = v^{\text{int}}(x), \quad \forall x \in e.$$ 

Then, we can define semi-discrete discontinuous Galerkin scheme for (1a)–(1b) as: Find $E_h, H_h \in U_h$, such that
for all $v_h, w_h \in U_h$. Here we adopt the well-known upwinding numerical fluxes:

$$
\mathbf{n}_K \times \mathbf{H}_h = \mathbf{n}_K \times (\|\mathbf{H}_h\| + \frac{1}{2Z} \|\mathbf{E}_h\|), \quad \mathbf{n}_K \times \mathbf{E}_h = \mathbf{n}_K \times (\|\mathbf{E}_h\| - \frac{Z}{2} \|\mathbf{H}_h\|)
$$
on an interior face $e = \partial K \cap \mathcal{E}_I$ and

$$
\mathbf{n}_K \times \mathbf{H}_h = \mathbf{n} \times \mathbf{H}^{\text{int}}_h + \frac{1}{Z} \mathbf{n}_K \times \mathbf{E}^{\text{int}}_h, \quad \mathbf{n}_K \times \mathbf{E}_h = \mathbf{0}
$$
on a boundary face $e = \partial K \cap \mathcal{E}_D$, where $Z = \sqrt{\frac{\mu_0}{\epsilon_0}}$ denotes the impedance of the medium. Apparently the adopted numerical fluxes are consistent with $\mathbf{n}_K \times \mathbf{E}$, $\mathbf{n}_K \times \mathbf{H}$.

Consider time interval $[0, T]$ with equiv-spaced division $I_n = [t_{n-1}, t_n]$, $t_n = n\tau$, $n = 1, 2, \cdots, N$. Denote by $\tau = T/N$ the time step size. Given any time discrete function $\{u^n(x)\}_{n=0}^N$ define

$$
\delta_t u^n = \frac{u^n - u^{n-1}}{\tau}, \quad u^n = \frac{u^n + u^{n-1}}{2}, \quad n = 1, \cdots, N,
$$

and

$$
\bar{u}^n = \frac{1}{2}(3u^{n-1} - u^{n-2}), \quad n = 2, \cdots, N.
$$
The fully discrete DG scheme for (1a)-(2b) is defined as: find $\mathbf{E}^n_h, \mathbf{H}^n_h \in U_h$ such that

$$
\epsilon_0(\delta_t \mathbf{E}^n_h, v_h) - \mathcal{B}_H(\mathbf{H}^n_h, v_h) + (\sigma(x, |\mathbf{E}^n_h|)\mathbf{E}^n_h, v_h) = 0, \quad (5a)
$$

$$
\mu_0(\delta_t \mathbf{H}^n_h, w_h) + \mathcal{B}_E(\mathbf{E}^n_h, w_h) = 0, \quad (5b)
$$

for all $v_h, w_h \in U_h$, $j = 1, 2, \cdots, N$, where $\bar{E}^1_h = (E^{1,*}_h + E^0_h)/2$ and $(E^{1,*}_h, H^{1,*}_h)$ satisfies

$$
\epsilon_0\frac{(E^{1,*}_h - E^0_h)}{\tau}, v_h) - \mathcal{B}_H\left(\frac{H^{1,*}_h + H^0_h}{2}, v_h\right) + \left(\sigma(x, |E^0_h|)\frac{E^{1,*}_h + E^0_h}{2}, v\right) = 0, \quad (6a)
$$

$$
\mu_0\frac{(H^{1,*}_h - H^0_h)}{\tau} w_h) + \mathcal{B}_E\left(\frac{E^{1,*}_h + E^0_h}{2}, w_h\right) = 0, \quad (6b)
$$

with

$$
\mathcal{B}_u(u, v) = (u, \nabla \times v)_{T_h} + (\nabla \times u, v)_{\partial T_h}.
$$

With an explicit treatment for the nonlinear term, another slightly different semi-implicit DG scheme can be defined as

$$
\epsilon_0(\delta_t \mathbf{E}^n_h, v_h) - \mathcal{B}_H(\mathbf{H}^{n-\frac{1}{2}}_h, v_h) + (\sigma(|\mathbf{E}^n_h|)\mathbf{E}^n_h, v_h) = 0, \quad (7a)
$$

$$
\mu_0(\delta_t \mathbf{H}^n_h, w_h) + \mathcal{B}_E(\mathbf{E}^{n-\frac{1}{2}}_h, w_h) = 0, \quad (7b)
$$

for all $v_h, w_h \in U_h$, where $(\mathbf{E}^{1}, \mathbf{H}^{1})$ is defined in (6a)-(6b). The discretizations for initial conditions (2a) are given by

$$
\mathbf{E}^0_h = \mathbf{E}_0(x, 0) = \mathbb{P}_h \mathbf{E}_0(x), \quad \mathbf{H}^0_h = \mathbf{H}_h(x, 0) = \mathbb{P}_h \mathbf{H}_0(x), \quad (8)
$$

where $\mathbb{P}_h : H^{k+1}(\Omega) \rightarrow U_h$ is $L^2$-projection operator defined as

$$
\int_K \mathbb{P}_h uvdx = \int_K uvdx, \quad \forall v \in P^k(K), \quad \forall K \in \mathcal{T}_h. \quad (9)
$$

In this paper, we focus on the DG scheme (5a)-(6b). The analysis can be easily extended to the DG scheme (7a)-(7b).
Let $E^n = E(\cdot, n\tau), H^n = H(\cdot, n\tau)$ be the exact solution of (1a)-(2b) at time $t_n = n\tau$. The main error estimate is summarized as follow.

**Theorem 2.1.** Suppose that $E^n, H^n$ be the solution of problem (5a)-(5b). $E, H$ be the solution of (1a)-(1b). Suppose that $E \in C^3((H^{p+1}(\Omega))^3 \cap (L^\infty(\Omega))^3, (0, T)), H \in C^3((H^{p+1}(\Omega))^3, (0, T))$. Then

$$
\| \frac{\tau^2}{2} \| E^n - E^n_h \|_0 + \| \frac{\tau^2}{2} \| H^n - H^n_h \|_0 \leq C(\tau^2 + h^{p+\frac{1}{2}}).
$$

for all $\tau \leq C h^\frac{2}{3}$.

3. **Proof of the main result.** In this section, we proceed the proof for the error estimate presented in Theorem 2.1. Firstly, let us introduce some fundamental approximation lemmas.

**Lemma 3.1.** Suppose $u \in H^{p+1}(K)$, then

$$
\| u - P_h u \|_{0, K} \leq C h^{p+1} |u|_{p+1, K}, \quad \| u - P_h u \|_{0, \partial K} \leq C h^{p+1/2} |u|_{p+1, K}.
$$

For the simplicity of analysis for the nonlinear term $\sigma(\sigma, s) \in C^{1,1}(\Omega, [0, \infty))$, define

$$
\sigma_M(s) = \sup_{x \in \Omega} \max_{\xi \in [0, s]} |\sigma(x, \xi)|, \quad \sigma_M'(s) = \sup_{x \in \Omega} \max_{\xi \in [0, s]} |\partial_\xi \sigma(x, \xi)|.
$$

Then we have $\sigma_M(\|E\|_\infty), \sigma_M'(\|E\|_\infty) < \infty$, if $E \in (L^\infty(\Omega))^3$.

Then, we can derive the following estimate:

**Lemma 3.2.** Suppose that $u \in C^3(0, T, (L^2(\Omega))^3)$. Then we have

$$
\| (\delta_t u^n - \partial_t u^{n-\frac{1}{2}}, v)_{T_h} \| \leq C \tau^2 \| u_{ttt}^{n-\frac{1}{2}} \|_0 \| v \|_0,
$$

$$
\| (\nabla \times \nabla u^n - \nabla \times u^{n-\frac{1}{2}}, v)_{T_h} \| \leq C \tau^2 \| u_{ttt}^{n-\frac{1}{2}} \|_0 \| v \|_0.
$$

**Proof.** Using Taylor formulation, we have

$$
u^n = u^{n-\frac{1}{2}} + u^{n-\frac{1}{2}} + \frac{u^{n-\frac{1}{2}}}{2!} (\frac{\tau}{2})^2 + \frac{u^{n-\frac{1}{2}}}{3!} (\frac{\tau}{2})^3 + O(\tau^4), \quad (10a)
$$

$$
u^{n-1} = u^{n-\frac{1}{2}} + u^{n-\frac{1}{2}} (\frac{-\tau}{2}) + \frac{u^{n-\frac{1}{2}}}{2!} (\frac{-\tau}{2})^2 + \frac{u^{n-\frac{1}{2}}}{3!} (\frac{-\tau}{2})^3 + O(\tau^4). \quad (10b)
$$

Hence, using the definition of $\delta_t u^n$, we have

$$
\| \mu_0 (\delta_t u^n - \partial_t u^{n-\frac{1}{2}}) \| \leq C \tau^2 \| u_{ttt}^{n-\frac{1}{2}} \|.
$$

Following almost the same strategy, we can obtain the other estimate. 

In the following, we will consider the truncation error caused by the nonlinear term.

**Lemma 3.3.** Suppose that $E \in C^2([0, T], (H^{p+1}(\Omega))^3 \cap (L^\infty(\Omega))^3)$. Then we have

$$
\left| \sigma(\sigma, |E^n|)E^n - \sigma(\sigma, |E^{n-\frac{1}{2}}|)E^{n-\frac{1}{2}}, v)_{T_h} \right|
\leq C \tau^2 \left( \sigma_M (\|E^n\|_\infty) + \sigma_M'(\|E^n\|_\infty + \|E^{n-\frac{1}{2}}\|_\infty) \right) \| E_{ttt}^{n-\frac{1}{2}} \|_0 \| v \|_0,
$$

$$
\left| \sigma(\sigma, |E^0|) \frac{E^1 + E^0}{2} - \sigma(\sigma, |E^{\frac{1}{2}}|)E^{\frac{1}{2}}, v)_{T_h} \right|
\leq C \tau \sigma_M (\|E^0\|_\infty + \|E^{\frac{1}{2}}\|_\infty) \| E_{ttt}^{\frac{1}{2}} \|_0 + C \tau^2 \sigma_M (\|E^0\|_\infty) \| E_{ttt}^{\frac{1}{2}} \|_0 \| v \|_0.
$$
Proof. Direct calculation leads to
\[
\sigma(x,|E^n|)|E^n| - \sigma(x,|E^{n-\frac{1}{2}}|)|E^{n-\frac{1}{2}}
=\sigma(x,|\tilde{E}^n|)(|E^n| - |E^{n-\frac{1}{2}}|) + \left(\sigma(x,|\tilde{E}^n|) - \sigma(x,|E^{n-\frac{1}{2}}|)\right)|E^{n-\frac{1}{2}},
\]
\[
\sigma(x,|E^0|)|\frac{E^1 + E^0}{2} - \sigma(x,|E^\frac{1}{2}|)|E^\frac{1}{2}
=\sigma(x,|E^0|)(|\frac{E^1 + E^0}{2} - E^\frac{1}{2}|) + \left(\sigma(x,|E^0|) - \sigma(x,|E^\frac{1}{2}|)\right)|E^\frac{1}{2}.
\]
Using Taylor formulation, we have
\[
\left|\sigma(x,|\tilde{E}^n|) - \sigma(x,|E^{n-\frac{1}{2}}|)\right| = \left|\sigma'(x,\rho_n)(|\tilde{E}^n| - |E^{n-\frac{1}{2}}|)\right|
\leq \left|\sigma'(x,\rho_n)(|\tilde{E}^n| - |E^{n-\frac{1}{2}}|)\right|,
\]
where \(\rho_n \leq |\tilde{E}^n| + |E^{n-\frac{1}{2}}| \leq C\), and
\[
\left(\sigma(x,|E^0|) - \sigma(x,|E^\frac{1}{2}|)\right) = \left|\sigma'(x,\zeta_1)(|E^0| - |E^\frac{1}{2}|)\right| \leq \left|\sigma'(x,\zeta_1)(|E^0| - |E^\frac{1}{2}|)\right|,
\]
where \(\zeta_1 \leq |E^0| + |E^\frac{1}{2}| \leq C\).

On the other hand, using Taylor formulation leads to
\[
|\tilde{E}^n - E^{n-\frac{1}{2}}| \leq Cr^2|E^{n-\frac{1}{2}}|^2, \quad |\tilde{E}^n - E^{n-\frac{1}{2}}| \leq Cr^2|E^{n-\frac{1}{2}}|^2, \quad |E^0 - E^\frac{1}{2}| \leq Cr|E^\frac{1}{2}|.
\]

The combination of the above three inequalities and Cauchy-Swarz inequality, we get the desired results. \(\square\)

To prove the error estimate of the fully discrete scheme, we need the following lemma.

**Lemma 3.4.** Suppose \(E \in C^2([0,T], (H^{p+1}(\Omega))^3 \cap (L^\infty(\Omega))^3)\) and \(\tilde{E}^1 = \frac{E^1 + E^0}{2}\). For all \(v \in (L^2(\Omega))^3\), we define
\[
A_n(v) = (\sigma(x,|E^n|)|E^n|, v)_{T_n} - (\sigma(x,|P_h E^n|)|P_h E^n|, v)_{T_n},
\]
\[
\tilde{A}_1(v) = (\sigma(x,|E^0|)|\frac{E^1 + E^0}{2}|, v)_{T_n} - (\sigma(x,|P_h E^0|)|P_h \frac{E^1 + E^0}{2}|, v)_{T_n},
\]
then
\[
|A_n(v)| \leq Ch^{p+1}(\|E^n\|_{p+1} + \|E^n\|_{p+1})\|v\|_0,
\]
\[
|\tilde{A}_1(v)| \leq Ch^{p+1}\|E^0\|_{p+1}\|v\|_0.
\]

**Proof.** By introducing intermediate terms, we have
\[
A_n(v) = (|\sigma(x,|E^n|) - \sigma(x,|P_h E^n|)|E^n|, v) + (\sigma(x,|P_h E^n|)(E^n - P_h E^n), v).
\]
Since \(E \in L^\infty\), we have
\[
\left|(\sigma(x,|P_h E^n|)(E^n - P_h E^n), v)\right| \leq |\sigma_M(\|P_h E^n\|_\infty)\|E^n - P_h E^n\|_0||v||_0
\leq Ch^{p+1}\|E^n\|_{p+1}\|v\|_0.
\]
For the other term, we apply mean value theorem. That is
\[
\left|(\sigma(x,|\tilde{E}^n|) - \sigma(x,|P_h E^n|)|E^n|, v)\right| = \left|(\sigma'(x,\xi)(E^n - |P_h E^n|)|E^n|, v)\right|
\leq |\sigma'_M(\xi)||E^n|\|E^n - P_h E^n\|_0||v||_0 \leq Ch^{p+1}\|E^n\|_{p+1}\|v\|_0,
\]
where \( \xi \leq \zeta = \| \tilde{E}_n \|_\infty + \| P_h \tilde{E}_n \|_\infty \) are bounded quantities depends on \( \| E \|_\infty \).

The combination of the above inequalities complete the estimate of \( A_n(v) \). The estimate of \( A_1(v) \) can be proved by the same analysis.

Then we turn to prove our main theorem:

**Proof.** Let

\[
\begin{align*}
e^n_n - E^n_h = R^n_h - \theta^n, & \quad e^n_h = H^n - H^n_h = R^n_h - \theta^n, \\
\end{align*}
\]

where

\[
\begin{align*}
R^n_h = E^n - P_h E^n, \quad \theta^n = E^n_h - P_h E^n, R^n_h = H^n - P_h H^n, \quad \theta^n_h = H^n_h - P_h H^n.
\end{align*}
\]

Obviously, the model problem (1a)-(1b) satisfies

\[
\begin{align*}
\epsilon_0 \partial_t E^n - \frac{1}{2} - \nabla \times H^n + \sigma(x, |E^n|) E^n - \sigma(x, |E^n_h|) E^n_h + v_h, \\
\mu_0 \partial_t H^n - \frac{1}{2} + \nabla \times E^n = 0,
\end{align*}
\]

for all \( n = 1, 2, \ldots, N \).

Let \( \tilde{E}_h = E^n + E^n_h \). Multiplying (12) by \( v_h, w_h \) and subtracting (5a)-(5b), we obtain the following error equation

\[
\begin{align*}
\epsilon_0 (\delta_t e^n_h, v_h)_T + (\delta_t E^n_h, v_h)_T + B_H(e^n_h, v_h) = \left( \sum_{i=1}^3 S_i, v_h \right), \quad (13a)
\end{align*}
\]

\[
\begin{align*}
\mu_0 (\delta_t e^n_h, w_h)_T + B_E(e^n_h, w_h) = (S_4 + S_5, w_h), \quad (13b)
\end{align*}
\]

where

\[
\begin{align*}
S_1 = -\nabla \times H^n + \nabla \times H^n - \frac{1}{2}, & \quad S_2 = \sigma(x, |\tilde{E}_n|) \tilde{E}_n - \sigma(x, |\tilde{E}_n_h|) \tilde{E}_n_h, \\
S_3 = \epsilon_0 (\delta_t E^n - \delta_t E^n_h), & \quad S_4 = \mu_0 (\delta_t H^n - \delta_t H^n_h), \quad S_5 = \nabla \times E^n - \nabla \times E^n - \frac{1}{2}.
\end{align*}
\]

Taking \( v_h = \tau \tilde{E}_h, w_h = \tau \tilde{E}_h \) in (13a)-(13b), and using the orthogonality of \( L^2 \) projection gives

\[
\begin{align*}
\epsilon_0 (\delta_t \theta^n, \tau \tilde{E}_h)_T + (\sigma(x, |\tilde{E}_n|) \tilde{E}_n - \sigma(x, |\tilde{E}_n_h|) \tilde{E}_n_h, \tau \tilde{E}_h)_T \\
- B_H(e^n_h, \tau \tilde{E}_h) - \left( \sum_{i=1}^3 S_i, \tau \tilde{E}_h \right), \quad (14a)
\end{align*}
\]

\[
\begin{align*}
\mu_0 (\delta_t \theta^n_h, \tau \tilde{E}_h)_T = B_E(e^n_h, \tau \tilde{E}_h) - (S_4 + S_5, \tau \tilde{E}_h). \quad (14b)
\end{align*}
\]

Summing the above equalities together we obtain

\[
\begin{align*}
\epsilon_0 (\| \theta^n \|_\infty^2 - \| \theta^n - \frac{1}{2} \|_\infty^2 + \mu_0 (\| \theta^n_h \|_\infty^2 - \| \theta^n_h - \frac{1}{2} \|_\infty^2) = \sum_{i=1}^3 Err_i, \quad (15)
\end{align*}
\]

where three parts of the error are given by

\[
\begin{align*}
Err_1 = - \left( \sum_{i=1}^3 S_i, \tau \tilde{E}_h \right) - (S_4 + S_5, \tau \tilde{E}_h), & \quad Err_2 = B_H(e^n_h, \tau \tilde{E}_h) - B_H(e^n_h, \tau \tilde{E}_h), \\
Err_3 = (\sigma(x, |\tilde{E}_n|) \tilde{E}_n - \sigma(x, |\tilde{E}_n_h|) \tilde{E}_n_h, \tau \tilde{E}_h)_T.
\end{align*}
\]
Using the approximation results in lemma 3.2, lemma 3.3 and Young’s inequality, we arrive at
\[ |\text{Err}_1| \leq C\tau^{5} + \tau(|\tilde{\sigma}_h^E|^2 + |\tilde{\sigma}_E|^2). \]
Following the strategy for the estimate of \(B(R, \theta)\) in [23], \(\text{Err}_2\) has estimate
\[ |\text{Err}_2| \leq C\tau h^{2p+1}(|E^n|_{H^{p+1}}^2 + |H^n|_{H^{p+1}}^2). \] (16)

The third error term is the key part which involves the discretization of the nonlinear term. Noting that the fully discrete scheme use an average instead of extrapolation in the discretization of the nonlinear term at the first time step, we shall discuss the estimate of \(\text{Err}_3\) for two different cases. Firstly, we consider the estimate at the first time step, i.e, \(n = 1\). Inserting a intermediate term in \(\text{Err}_3\), we have
\[ |\text{Err}_3| \leq A_1(\tau \tilde{T}_1^E) + |(\sigma(x, |P_h E^1|)P_h E^1, \tau \tilde{T}_1^E) - (\sigma(x, |\tilde{E}_h^1|)\tilde{E}_h^1, \tau \tilde{T}_1^E)|. \]
where \(A_1(\tau \tilde{T}_1^E)\) is defined in (11).
In terms of \(|(\sigma(|P_h E^1|)P_h E^1, \tau \tilde{T}_1^E) - (\sigma(|\tilde{E}_h^1|)\tilde{E}_h^1, \tau \tilde{T}_1^E)|\), we have
\[ |\sigma(|P_h E^1|)P_h E^1, \tau \tilde{T}_1^E) - (\sigma(|\tilde{E}_h^1|)\tilde{E}_h^1, \tau \tilde{T}_1^E)| \leq \xi_1 \leq \|P_h E^1\|_\infty + \|\tilde{E}_h^1\|_\infty \leq C. \]
Hence, the estimate of \(\|P_h E^1 - \tilde{E}_h^1\|_0\) and the bound of \(\tilde{E}_h^1\) are needed. By the definition of \(P_h E^1\) and \(\tilde{E}_h^1\), we know that
\[ P_h E^1 - \tilde{E}_h^1 = \frac{P_h E^0 - \tilde{E}_h^1}{2} + \frac{P_h E^1 - \tilde{E}_h^1}{2}. \]
Hence, they can be obtained by performing error analysis for numerical scheme (6a) and (6b).

Note that (6a) and (6b) are almost the same discretization scheme as (5a) and (5b) except the treatment of nonlinear term. Define
\[ e_1^{E,+} = E^1 - P_h E^1 - (E_h^{1,+} - P_h E^1) := R_{E}^{1,+} - \theta_{E}^{1,+}, \]
\[ e_1^{H,+} = H^1 - P_h H^1 - (H_h^{1,+} - P_h H^1) := R_{H}^{1,+} - \theta_{H}^{1,+}. \]
By the definition of \(E_0^0, H_0^0\), we get \(\theta_{E}^0 = 0, \theta_{H}^0 = 0\). Following the same derivation of the error equation (15), we obtain
\[ \epsilon_0(|\theta_{E}^{1,+}|_{2}^2 - |\theta_{E}^0|_{2}^2) + \mu_0(|\theta_{H}^{1,+}|_{2}^2 - |\theta_{H}^0|_{2}^2) = \sum_{i=1}^{3} \text{Err}_i, \] (18)
where
\[ \text{Err}_1 = -\sum_{i=1}^{3} (S_i^{+}, \tau - \theta_{E}^{1,+} + \theta_{E}^0) - (S_i^+, \tau + \theta_{H}^{1,+}), \]
\[ \text{Err}_2 = B_E(e_1^{E,+} + e_0^0, \tau H_{H}^{1,+}) - B_E(e_1^{H,+} + e_0^0, \tau \theta_{H}^{1,+}), \]
\[ \text{Err}_3 = (\sigma(x, |E_0^0|)E^1 + E_0^0, \tau \theta_{E}^{1,+} + \theta_{E}^0, \tau H_{H}^{1,+}). \] (19)
and
\[ S_1^* = \nabla \times \frac{H^1 + H^0}{2} - \nabla \times H^1, \quad S_2^* = \sigma(x, |E^0|) \frac{E^1 + E^0}{2} - \sigma(x, |E^2|) E^2. \]
\[ S_3^* = c_0(\delta_t E^1 - \partial_t E^2), \quad S_4^* = \mu_0(\delta_t H^1 - \partial_t H^2), \quad S_5^* = \nabla \times \frac{E^1 + E^0}{2} - \nabla \times E^2. \]

Moreover, direct calculation leads to
\[ E \epsilon_\sigma(x, |E^0|) (P_h \frac{E^1 + E^0}{2} - \frac{E_{h^1}^1 + P_h E^0}{2}, \frac{E_{h^2}^1 + P_h E^0}{2}) \leq C \tau \| \theta_{E^1}^{1,*} \|_0^2. \]

Similar as (16), we have estimates
\[ |\tilde{E}_{r2}| \leq C \tau h^{2p+1} (||E^1||_p + ||H^1||_p), \]
\[ |\tilde{E}_{r1}| \leq C \tau^3 + \tau (||\theta_{E^1}^{1,*}||_0^2 + ||\theta_{H}^{1,*}||_0^2). \]

Moreover, direct calculation leads to
\[ |\tilde{E}_{r3}| \leq \tilde{h}_1 (\tau \| \theta_{E^1}^{1,*} \|_0^2) + \left( \sigma(x, |P_h E^0|) (P_h \frac{E^1 + E^0}{2} - \frac{E_{h^1}^1 + P_h E^0}{2}, \frac{E_{h^2}^1 + P_h E^0}{2}) \right)_{T_h}, \]
where \( E_0^0 = P_h E^0 \) is used. Then the following estimate is obtained
\[ \left( \sigma(x, |P_h E^0|) (P_h \frac{E^1 + E^0}{2} - \frac{E_{h^1}^1 + P_h E^0}{2}, \frac{E_{h^2}^1 + P_h E^0}{2}) \right)_{T_h} \leq C \tau \| \theta_{E^1}^{1,*} \|_0^2. \]

Together with Lemma 3.4, we obtain
\[ \left| \tilde{E}_{r3} \right| \leq C \tau^3 + \tau h^{2p+2} + C \tau (||\theta_{E^1}^{1,*}||_0^2). \]

By applying (20)-(21) in (18), we get
\[ c_0(\theta_{E^1}^{1,*})_0^2 + \mu_0(\theta_{H}^{1,*})_0^2 \leq C \tau^3 + h^{2p+1}. \]

Recalling the definition of \( \bar{E}_h^1 \), we have
\[ ||\bar{E}_h^1||_\infty \leq \frac{1}{2} ||E_h^{1,*}||_\infty + ||E_h^0||_\infty \leq \frac{1}{2} ||\theta_{E^1}^{1,*}||_\infty + \frac{1}{2} ||P_h E^0 + P_h E^1||_\infty. \]

Together with error estimate (22) and time step size assumption \( \tau \leq \frac{1}{2} \), we obtain
\[ ||\bar{E}_h^1||_\infty \leq C \tau^{-\frac{1}{2}} (\tau^{\frac{1}{2}} + h^{p+\frac{1}{2}}) + ||P_h E^1||_\infty \leq C, \]
by using inverse inequality [1]
\[ ||v||_{\infty, K} \leq C h^{-\frac{1}{2}} ||v||_{0, K}, \quad \forall v \in P^k(K), \quad d = 2, 3. \]

Substituting error estimate (22) and infinity bound (23) into (17) leads to
\[ \left( \sigma(x, |P_h E^1|) P_h E^1 - \sigma(x, |\bar{E}_h^1|) \bar{E}_h^1, \frac{P_h E^0}{2} \right)_{T_h} \leq C (\tau^4 + h^{2p+1}). \]

Then, the combination of (15), (16) and (24) leads
\[ c_0(\theta_{E^1}^{1,*})_0^2 + \mu_0(\theta_{H}^{1,*})_0^2 \leq C (\tau^4 + h^{2p+1}). \]

Applying triangle inequality and approximatin results in Lemma 3.1 gives
\[ \frac{1}{2} (||E^1||_0 + ||H^1||_0) \leq C (\tau^2 + h^{p+\frac{1}{2}}), \]
which completes the proof of Theorem 2.1 for \( n = 1 \).

Next, we will prove error estimates for \( n \geq 2 \) by induction. Suppose Theorem 2.1 holds for all time steps \( l \) before \( n \) (i.e., \( l < n \)), then
\[ ||E_h^l||_\infty \leq ||\theta_{E^1}^{1,*}||_\infty + ||P_h E^l||_\infty \leq C h^{-\frac{1}{2}} (\tau^2 + h^{p+\frac{1}{2}}) + ||P_h E^l||_\infty, \]
for all $l = 1, 2, \cdots, n - 1$ by using inverse inequality (24) again. Hence, we have $\|E_h^l\|_\infty \leq C$ for all $l = 1, 2, \cdots, n - 1$, if the time step size $\tau$ satisfies $\tau \leq h^4$.

Recalling (15)–(16), we only need to give estimate for $Err_3$. By inserting an intermediate term $\sigma(u, [P_h E^n]\|P_h E^n)$ we have

$$\|Err_3\| \leq A_n(\tau \|E^n\|_{\bar{E}}, \|E^n\|_{\bar{E}}) \leq A_n(\tau \|E^n\|_{\bar{E}}, \|E^n\|_{\bar{E}}) \tau_h.$$ 

Estimate for $A_n(\tau \|E^n\|_{\bar{E}})$ can be obtained by Lemma 3.4, that is

$$A_n(\tau \|E^n\|_{\bar{E}}) \leq C \tau^4 + C \tau h^{-2p+2} + \tau \|\|E^n\|_{\bar{E}}\|^2.$$ 

The properties of $\sigma$ and $E_h^n$ lead to

$$(\sigma([P_h E^n])\|P_h E^n\|_{\bar{E}, \tau_h} - (\|E^n\|_{\bar{E}, \tau_h}) \leq C \|\|E^n\|_{\bar{E}}\|_{p+1} + \|\|E^n\|_{\bar{E}}\|_{2p} \leq C \tau^4 + h^{-2p+1}.$$ 

Then, the proof is finished by using triangle inequality. \hfill $\square$

4. **Numerical results.** Error estimate for 2-D nonlinear Maxwell’s equations can also be obtained in the same way while introducing the scalar curl operator $\text{curl} u = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$ for vector function $u = (u_1, u_2)$ and the vector curl operator $\text{curl} v = (\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x})^T$ for scalar function $v$. Here, we give both 2D and 3D numerical examples to validate our theoretical prediction.

**2D Example.** Let $\Omega = [0, 1]^2$, $\epsilon_0 = \mu_0 = 1$, $\sigma(|E|) = |E|^2$. We choose right hand side such that the exact solution is

$$(E_x, H_x) = \begin{pmatrix} \sin(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{pmatrix} te^{-t}.$$ 

We choose the time step size $\tau$ equal to the spatial mesh size $h$ when $p = 1$ and $\tau$ equal to $h^2$ when $p = 2$. The $L^2$ error and corresponding convergence order at the final time $T = 1$ are shown in Fig. 1. It observed that the convergence rate of both $E_h$ and $H_h$ in $L^2$ norm is $O(\tau^2 + h^{k+1})$, which is better than our theoretical prediction.

**3D Example.** Let $\Omega = [0, 1]^3$, $\epsilon_0 = \mu_0 = 1$, $\sigma(|E|) = |E|^3$. We choose right hand side such that the exact solution is

$$E(x, t) = \begin{pmatrix} \cos(\pi x)\sin(\pi y)\sin(\pi z) \\ 2\sin(\pi x)\cos(\pi y)\sin(\pi z) \\ 3\sin(\pi x)\sin(\pi y)\cos(\pi z) \end{pmatrix} te^{-t},$$

$$H(x, t) = \begin{pmatrix} \pi\sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -2\pi\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ -\pi\cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix} te^{-t}.$$
Again, we choose the time step size $\tau$ equal to the spatial mesh size $h$ when $p = 1$ and $\tau$ equal to $h^2$ when $p = 2$. The $L^2$ error and corresponding convergence order at the final time $T = 1$ are shown in Fig. 2. It is observed that the convergence rate of both $E_h$ and $H_h$ in $L^2$ norm is $O(\tau^2 + h^{k+1})$, which is also better than our theoretical prediction.

5. **Conclusion.** In this paper, we analyze an efficient discontinuous Galerkin method for nonlinear Maxwell’s equations. A semi-explicit extrapolation technique is adopted to linearize the nonlinear term. The fully discrete scheme results in a linear system at each time step. Hence, it enjoys high numerical efficiency. Meanwhile, optimal $L^2$ error estimate is derived under weak time step size condition $\tau < h^{\frac{4}{3}}$.

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