Stability of an electrified liquid jet approaching to critical point

Zixuan Fang¹ and Ping Wang²

Abstract
This article reports the linear stability analysis of a thermodynamic-transcritical jet sprayed to a radial electrical field. An asymptotic approach was used to obtain the stability solution of a supercritical jet subjected to electrical field. In order to obtain the solutions for the electrified supercritical jet, the surface tension was decreased and consequently led the increase in Weber number in the linear governing equation of subcritical charged jet. To investigate the role of surface tension and electric stress playing in the destabilizing process when approaching the critical point, the energy budget is performed by tracing the energy sources. It was found that, when the Weber number is increased to a sufficiently large value, the solution will become an asymptotic value, which can be considered as a solution under the supercritical conditions. The electric stress can increase both the maximum growth rate and the dominant wave number of electrified supercritical jet, that is, higher electrical field intensity would enhance the instability of the electrified supercritical jet and decrease the wavelength of the disturbances.

Keywords
Transcritical jet, electrical field, surface tension, linear stability

Date received: 23 October 2019; accepted: 19 May 2020

Handling Editor: James Baldwin

Introduction
When the fluid jet is subjected to an electrostatic electric field, spray may occur on the surface of the fluid due to an imbalance between the electrostatic stress and the surface tension.¹⁻³ This phenomenon is called electrospray and exhibits a few modes that depend on the ratio of electrostatic to surface tension forces, which have different applications in many fields ranging from aerospace microthrust to electrospray ionization mass spectrometry (ESI-MS) technologies. The instability of the emanated jet has great influences on the performance of electrospray for different applications. For ESI-MS technique, the breakup of the jet is favorable, so we need to enhance the instability. In contrast, for electrospinning technique, the electric field and surface tension forces do not break down the slender jet to produce a fine fiber, thus the instability of jet should be damped. There have been many fundamental studies on the electrified jets’ instability. Parkin⁴ regarded water jet as a perfect conductor and measured the growth rate of perturbations on it. Hartman et al.⁵ proposed an analytical physical model and revealed the relationship between the jet breakup and axisymmetric or varicose instabilities. Turnbull⁶,⁷ studied conductive and dielectric jets’ temporal stability in both radial and axial electric fields. Li et al.⁸ considered the viscous of liquid jet and established a new model under a radial electric field. For the axisymmetric mode, the experimental results coincided well with the theoretically most probable wavelength. Mestel⁹ studied the instability of

¹School of Astronautics, Beihang University, Beijing, China
²Beijing Institute of Control Engineering, Beijing, China

Corresponding author:
Ping Wang, Beijing Institute of Control Engineering, Beijing 100094, China.
Email: wangping_2102@163.com
viscous liquid jets sprayed in the axial electrical field. Ruo et al.\(^\text{10}\) found that the asymmetric mode would be excited by the electrostatic repulsion between the free charges on jet surface and the disturbance wavelength is smaller than that of the noncharged jet. Fu et al.\(^\text{11}\) and Yang et al.\(^\text{12}\) studied the spatial-temporal instability and the breakup of an electrified liquid jet by a linear analysis method. Yang et al.\(^\text{13}\) investigated two-modes disturbance instability based on Fu et al.’s work.

All the above studies on the electrified jets are limited to the condition of subcritical low pressure, in which the mechanism of destabilization for a liquid jet is the competition between the electrical stress and surface tension. A multitude of applications of spray operates at supercritical conditions, ranging from internal-combustion engines, rocket engines to biotechnology. In the biomedical field, injecting a nanofluid jet into a supercritical environment is widely used to manufacture time release and coated nanosized particles.\(^\text{14,15}\) Also, in many power devices, like diesel engines, gas turbines, and rocket engines, the critical pressures of the injected fuel or oxidizer are often exceeded because of the high pressures, which increases public interests in high-pressure combustion to produce high-power energy conversion and thrust.\(^\text{16-19}\) As is known, the surface tension would decrease when the pressure and temperature get close to the critical point and completely disappear when exceeding the critical point.\(^\text{20}\) Hence, it is interesting to see what would happen for the electrified liquid jet when the liquid got close to the critical point and subsequently the surface tension gradually decreased. In these transcritical conditions, the electrical stress would dominate over the surface tension until the surface tension completely disappeared. What the instability characteristics would be when the fluid properties get close to the critical point is of great theoretical interest. However, utilizing electrical force to affect the instability of a supercritical jet is a promising method for flow control in real applications, for example, mixing enhancement in rocket engine\(^\text{21}\) or retarding microjet breakup in biotechnology.\(^\text{22}\) Therefore, it is desirable to better understand the instability mechanism of the transcritical jets under the action of electrical field.

In this study, a linear stability analysis was performed to understand the breakup mechanism of an electrified liquid jet in a transcritical environment. We emphasized on the instability of the electrified jet in the transition process from subcritical to supercritical condition. During this process, the surface tension does exist but decreases when the condition approaches the critical point. The stability analysis has been done on an electrified cylindrical liquid jet. To obtain the final solution, the surface tension was gradually decreased and hence the Weber number was increased in the dispersion relation. The energy budget was used to analyze the role of surface tension and electric stress in the transition process. The asymptotic analysis was performed to forecast the maximum growth rate at extremely high Weber numbers, considered to be an analogy to the supercritical condition.

**Methodology**

We consider a viscoelastic liquid jet of radius \(R\), sprayed into a quiescent, inviscid, and incompressible gas medium, for axisymmetric disturbances (as shown in Figure 1). At the beginning of the process, the liquid jet has velocity \(U = (U_z, U_r)\), where \(U_z\) is the unperturbed axial velocity, \(U_r\) is the unperturbed radial velocity and was assumed to be zero in the study. The densities of liquid and gas are defined as \(\rho_l\) and \(\rho_g\), respectively, and \(\sigma\) is the liquid’s surface tension. The liquid jet was electrified by being positioned in an earthed coaxial annular electrode of radius \(R_0\). The voltage \(V_0\) is on the jet surface. The liquid is regarded as a perfect conductor, while the ambient gas as a perfect dielectric. Therefore, in the ambient gas, a basic radial electric field of magnitude \(-\frac{V_0}{\sigma \ln (R/R_0)}\) is formed. The density of free charge on the initial interface is \(-\varepsilon_0 \frac{V_0}{R \ln (R/R_0)}\), where \(\varepsilon_0\) is the dielectric constant of ambient gas.

Because the fluid behaves as non-Newtonian fluids in some cases for biomedical manufacturing, the viscoelastic constitutive model was used without the loss of generality. In this study, we focused on Newtonian fluid. The dispersion relation of the electrified Newtonian liquid jet can be obtained by degenerating that of the electrified viscoelastic liquid jet.
A corotational model with the constitutive equation used to describe the viscoelastic properties of the fluid, in which the stress tensor $\tau$ could be expressed as (equations (3)–(5) in Fu et al.\textsuperscript{23})

$$
\tau + \lambda_1 \frac{D\tau}{Dt} + \frac{1}{2} \mu_0 (\sigma \tau) \dot{\gamma} - \frac{1}{2} \mu_1 (\sigma \cdot \dot{\gamma} + \dot{\gamma} \cdot \tau)
+ \frac{1}{2} \nu_1 (\sigma : \ddot{\gamma})
+ \frac{1}{2} \nu_2 (\dot{\gamma} : \ddot{\gamma})
= - \mu_0 \left[ \dot{\gamma} + \lambda_2 \frac{D\gamma}{Dt} - \mu_2 (\sigma \cdot \dot{\gamma} + \dot{\gamma} \cdot \sigma) \right]
$$

(1)

where $\mu_0$ represents the initial shear viscosity equaling to zero, $\lambda_1$ represents the relaxation time stress, $\lambda_2$ represents the deformation retardation time of the liquid, and $\delta$ represents the unit tensor. The quantities $\mu_0$, $\mu_1$, $\mu_2$, $\nu_1$, and $\nu_2$ are all time constants. The rate of strain tensor $\dot{\gamma}$ and vorticity tensor $\Omega$ are defined by

$$
\dot{\gamma} = \nabla U + (\nabla U)^T
$$

(2)

$$
\Omega = \nabla U - (\nabla U)^T
$$

(3)

Fu et al.\textsuperscript{11} focused on the electrified viscoelastic liquid jet and the derivation of dispersion relation for axisymmetric mode can be written as

$$
(\omega + ik)^2 - QC_0 \omega^2 \frac{I_1(k)}{I_0(k)} + \frac{2k^2}{Re} (\omega + ik) \left( 2 - \frac{I_1(k)}{kI_0(k)} \right)
+ 4k^3 \left( k - \frac{I_1(k)}{I_0(k)} \right)
- \frac{k}{Re} (1 - k^2) \frac{I_1(k)}{I_0(k)} + Eu(1 + k^2)k \frac{I_1(k)}{I_0(k)} = 0
$$

(4)

where $k$ is the wave number of the perturbations in the $x$ direction, $\omega$ is the complex frequency, of which the real part, $\omega_r$, is the growth rate of the disturbance. $I_n(x)$ and $K_n(x)$ are the modified Bessel functions. The density and radial parameters are nondimensionalized in the dispersion relation, written as $Q = \rho_g / \rho_t$ and $b = R_0 / R$. Besides, the equivalent Reynolds number $Re_l = \rho_t U_2 R / \mu_t$, the Weber number $We = \rho_t U_2^2 R / \sigma$, the electrical Euler number $Eu = \epsilon_0 \sigma_0^2 / \rho_t U_2^2 R^2 \ln b$, $l = (k^2 + Re(\omega + ik))^{1/2}$, $C_0 = I_0(k) K_1(k) + K_0(k) I_1(k)$, $I_1(k) K_1(k) - K_1(k) I_1(k)$, $\zeta_1 = I_1(k) K_0(k) + K_1(k) I_0(k)$, $I_0(k) K_0(k) - K_0(k) I_0(k)$, $Re = \rho_t U_2 / \mu_0$, and $\mu_t = \mu_0 (1 + \omega_0 \lambda_2) / (1 + \omega_0 \lambda_1)$ are defined. For Newtonian fluids, $\lambda_1 = \lambda_2 = 0$.

By solving the dispersion equation numerically, it can be found that the liquid jet tends to become unstable when the minimum value of disturbance growth rate is positive.\textsuperscript{23–29} In supercritical cases, the linear stability analysis method fails to accurately predict the stability of electrified jets. Due to the disappearance of surface tension, which forces the Weber number tends to infinity, the supercritical regime is exceedingly complicated. This made the solution to diverge indefinitely, and no valid solution can be found. Nowadays, the solution to the dispersion equation in the supercritical case is given, the purpose of which is to obtain a solution for exceedingly large Weber numbers. The dispersion equation (4) was solved numerically with commercial calculation software Maple.

To investigate the role of surface tension and electric stress playing in the destabilizing process when approaching critical point, we use energy analysis, which solves the problem by tracing the sources of energy. Lin\textsuperscript{30} raised a model to describe the energy budget. A control volume of the liquid is established, which is based on one wavelength $\lambda_d = 2\pi / k$ of the disturbance. After forming the dot product of conservation of the momentum equation with the perturbation velocity vector $\mathbf{v}$, Gauss theorem was used to transform some of the volume integrals to surface integrals over the control volume. The results were averaged over one wavelength $\lambda_d$ and the energy equation in integral form can be derived as\textsuperscript{30}

$$
\frac{1}{\lambda_d} \int_V \mathbf{v} \cdot \left( \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial z} \right) dV = - \frac{1}{\lambda_d} \int_A \mathbf{p} \mathbf{v} \cdot dS
+ \frac{1}{\lambda_d} \int_V \mathbf{v} \cdot (\nabla \cdot \mathbf{T}) dV
$$

(5)

where $V$ and $A$ stand for the control volume and surface area, respectively.

At the interface, the normal stress is continuous; thus, the dynamic boundary conditions in the direction normal to the interface can be written as (equation (11) in Fu et al.\textsuperscript{23})

$$
\rho_l \frac{1}{2} \varepsilon_0 E^2 - 2\mu \frac{\partial U}{\partial r} + \sigma \left( \frac{\eta}{R^2} + \frac{\partial^2 \eta}{\partial z^2} \right) = p_g \quad \text{at} \quad r = R
$$

(6)

where $p_g$ is the gas pressure and $\mu$ is the apparent viscosity of the fluid.

After applying equation (6) to the interface (similar to equations (3.7)–(3.15) in Xie et al.\textsuperscript{31}) and in the axial direction taking spatial periodicity into account, the energy budget equation can be written as

$$
\epsilon_{ce} = w_{Re} + w_{El} + w_g + w_{We} + w_{Eu}
$$

(7)

where
\[ e_{ke} = \frac{1}{\lambda_d} \int_0^{2\pi \lambda_d} \int_0^1 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left( u_r^2 + u_\theta^2 + u_z^2 \right) r dr dz \]  

\[ w_{ve} = -\frac{1}{Re\lambda_d} \int_0^{2\pi \lambda_d} \int_0^1 \left[ 2 \left( \frac{\partial u_r}{\partial r} \right)^2 + \left( \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right)^2 \right. \]
\[ + 2 \left( \frac{\partial u_\theta}{\partial r} + \frac{u_r}{r} \right)^2 + \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_\theta}{\partial z} \right)^2 + \left( \frac{\partial u_z}{\partial z} \right)^2 \]  

\[ r dr dz \]  

\[ w_{Re} = -\frac{1}{Re\lambda_d} \int_0^{2\pi \lambda_d} \int_0^1 \left[ 2 \left( \frac{\partial u_r}{\partial r} \right)^2 + \left( \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right)^2 \right. \]
\[ + 2 \left( \frac{\partial u_\theta}{\partial r} + \frac{u_r}{r} \right)^2 + \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_\theta}{\partial z} \right)^2 + \left( \frac{\partial u_z}{\partial z} \right)^2 \]  

\[ r dr dz \]  

\[ w_{El} = w_{ve} - w_{Re} \]

\[ w_g = -\frac{1}{\lambda_d} \int_0^{2\pi \lambda_d} \int_0^1 \left( p_g u_r \right) r dr dz \]

\[ w_{We} = \frac{1}{We\lambda_d} \int_0^{2\pi \lambda_d} \int_0^1 \left[ 1 + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] \xi_0 u_r r dr dz \]

\[ w_{Eu} = -\frac{1}{\lambda_d} \int_0^{2\pi \lambda_d} \int_0^1 \left[ Eu(1 + k\xi_0) u_r \right] r dr dz \]

**Results and discussion**

With the critical point gradually being approached, the value of the surface tension decreases; at the critical point, surface tension completely disappears, which indicates that the Weber number tends to infinity at the critical point. As the jet gradually approaches the critical point, the value of the surface tension decreases. The surface tension completely disappears when it reaches the critical point, which indicates that the Weber number tends to infinity at this point. Hence, to characterize the stability behavior of the electrified jet at supercritical conditions as accurately as possible, the supercritical point was approached here asymptotically. Gradually increasing the Weber numbers and plotting the trends of the growth rate as well as wave number, the stability curves can be obtained, which equals to the process of reaching the critical conditions asymptotically. The peak of the curve stands for the growth rate for the most amplified disturbance.

Figure 2 shows the relationship between the nondimensional temporal growth rate \( \omega_t \) and the nondimensional wave number \( k \) for the electrified jet injected at different Weber numbers. It can be found that when the Weber number is smaller than 6500 approximately, the curves are habitual for the growth rate curves. Before the maximum growth rate is reached, the curves increase with wave number monotonically. The growth rate increases as a whole substantially as the Weber number increases. It can be explained as follows, both decrease in the surface tension and the increase in the liquid sheet’s velocity might cause the growth of liquid Weber number. Aerodynamic effect plays an essential role in the instability-enhancement mechanism for liquid sheet velocity. The effect of surface tension is taken into consideration because surface tension tends to smooth out the disturbances on the interface between the liquid and the gas. As the surface tension decreases, the flow becomes more unstable. Besides, the dominant wave number, which corresponds to the maximum
growth rate, increases greatly when the liquid Weber number increases, that is, the disturbance wavelength can be decreased by decreasing the surface tension.

When Weber number is sufficiently large (\(>6500\)), the range of instability also becomes very large. In this study, considering the large time-consuming and the actual purpose, the abscissa-crossing points for these cases were not calculated. As Weber number becomes more than 325,000, the dispersion relation curves corresponding to different Weber number almost overlap. Hence, it can be concluded that the dispersion curve when Weber number approaches infinity almost coincides with one certain curve when the Weber number is large enough (in the order of \(10^5\) in this case), that is, when the Weber number is quite large, the dispersion relation curves corresponding to the supercritical condition can be approached.

Energy budget is used to demonstrate the effect of surface tension and electric stress. The rate of change in kinetic energy and various works as a function of wave number are given in Figure 3. It is shown in Figure 4 that the factors playing major roles are the work made by surface tension and electric stress. The work made by surface tension, denoted by \(w_{WT}\), is negative; thus, the surface tension has a stabilizing effect, which verifies the analysis above. The work done by electric stress \(w_{Eu}\) is positive, attributing the most to the destabilizing of jets. The fluid in this case is assumed to be Newtonian fluid, so \(w_{ve}\) equals to \(w_{Re}\), which means there are no contribution of elasticity. The work done by viscous dissipation, \(w_{ve}\) is negative and small, implying that fluid viscosity inhibits instability but its effect is weak. The effect of surrounding gas is small, which can be seen from the value of \(w_{g}\). When Weber number or electrical Euler number increases (compared to the condition in Figure 3), the effects of work done by different factors are generally the same, except that the instability range increases, as shown in Figures 4 and 5.

Figure 6 shows the work done by surface tension when Weber number varies. From the picture, it can be seen from the curves that, although the range of wave number attributing to the stabilization of jet extends as the Weber number increases, the curve generally bends toward the horizontal axis, which means the work done by surface tension decreases. It also can be deduced through the expression of work done by surface tension equation (13) that, when Weber number approach infinity, the work done by surface tension would decrease to nearly zero. That means the attribution of surface tension would disappear as the Weber number increase to infinity, and the dispersion curves would approximate to one limited curve, which is the supercritical jet condition.
In Figure 7, the relationship between the maximum growth rate and different electrical Euler number is shown. The maximum growth rate changes with Weber number. By varying the surface tension, the Weber number is increased up to $10^8$, which is sufficiently large to be regarded as the supercritical conditions. Four different electrical Euler numbers have been examined, and all of them show an obviously character: as Weber number increases up to the value of $10^3$, the maximum growth rate increase steeply, and then gradually reach an asymptote after $We = 10^4$. When Weber number becomes larger than about $10^5$ level, the maximum growth rate does not vary with Weber number. This “platform” value of growth rate represents the growth rate under the supercritical condition for corresponding electrical Euler number. It is not difficult to conclude that the Weber number at which the maximum growth rate reaches the asymptote value is nearly the same for different Euler numbers. At the same time, the asymptote value increases as the Euler number increases, that is, with the increase in Euler number, the maximum growth rate of the liquid jet under supercritical condition increases. Hence, it can be concluded that the applied electrical field would enhance the instability of a jet in supercritical environment, and the maximum growth rate can be obtained by this asymptotic method.

Figure 8 shows the relationship between the dominant wave number and varying Weber numbers under different electrical Euler numbers. The trend of dominant wave number is somewhat different with that of maximum growth rate. For low Weber numbers, the dominant wave number increases sharply with the growing Weber number. As Weber number increases, the increasing rate of dominant wave number with Weber number reduces. The dominant wave number does not reach an asymptotic value as the maximum growth rate does. And we cannot obtain the dominant wave number at even higher Weber number because the solution would be diverging when increasing Weber number further. However, we can conclude from Figure 2 that the dominant wave number would also approximate to a limited value at infinity Weber number, and this asymptotic behavior in some way represents the behavior of the wavelength of disturbance under supercritical condition. Also, it can be deduced from Figure 8 that the dominant disturbance wave number under a supercritical condition increases as the
Euler number becomes larger, that is, the disturbance wavelength in this study becomes shorter as the applied electrical field intensifies under supercritical conditions.

Conclusion

A linear stability analysis method was carried out in this article to study an electrified supercritical jet. The stability solution was found here using an asymptotic approach. Unlike other studies, we obtained the final solutions for electrified supercritical jet in an approximate way. The surface tension was decreased conceptually to approach supercritical conditions, while mathematically the Weber number increased in our governing equations of an electrified subcritical liquid jet. According to our method, as Weber number increases to a sufficiently large value, the solution will converge toward an asymptotic value, which can be deemed to be equivalent solution under supercritical conditions.

The results showed that in the subcritical condition, both the dominant wave number and maximum growth rate increase as Weber number becomes larger.

1. When Weber number slowly increases and achieves a certain value (up to $10^5$ order in the case here), an asymptote for maximum growth rate appears which is considered as the solution of the supercritical condition.

2. For dominant, there is no asymptote in the calculated wave number range, but only a decrease in the varying rate of dominant wave number with Weber number is observed. It is expected that the dominant wave number would also approximate to a limited value when Weber number become even larger.

3. For all electrical Euler numbers, the solution of maximum growth rate is approached asymptotically at around the same Weber number; however, the Weber number at which the varying rate changes for dominant wave number varies for different Euler numbers. The dominant wave number and maximum growth rate for electrified supercritical jet increase with Euler number, that is, higher electrical field intensity would enhance the instability of the electrified supercritical jet and decrease the wavelength of the disturbances.

Author contributions

P.W. was involved in conceptualization, resources, writing—review and editing, supervision, project administration, and funding acquisition. Z.F. was involved in methodology, software, formal analysis, investigation, data curation, writing—original draft preparation, and visualization.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

References

1. Smith KL, Alexander MS and Stark JP. The sensitivity of volumetric flow rate to applied voltage in cone-jet mode electrospray and the influence of solution properties and emitter geometry. Phys Fluids 2006; 18: 092104.
2. Kourmatzis A and Shrimpton JS. Electrical and transient atomization characteristics of a pulsed charge injection atomizer using electrically insulating liquids. J Electrostat 2011; 69: 157–167.
3. Shrimpton JS and Yule AJ. Electrohydrodynamics of charge injection atomization: regimes and fundamental limits. Atom Sprays 2003; 13: 173–190.
4. Parkin CS. The production of droplets from liquid jets by capillary and electrohydrodynamic instabilities. Doctoral dissertation, Loughborough University of Technology, Loughborough, 1973.
5. Hartman RPA, Brunner DJ, Camelot DMA, et al. Jet break-up in electrohydrodynamic atomization in the cone-jet mode. J Aerosol Sci 2000; 31: 65–95.
6. Turnbull RJ. On the instability of an electrostatically sprayed liquid jet. IEEE Trans Ind Appl 1992; 28: 1432–1438.
7. Turnbull RJ. Finite conductivity effects on electrostatically sprayed liquid jets. IEEE Trans Ind Appl 1996; 32: 837–843.
8. Li F, Liu ZY, Yin XZ, et al. Theoretical and experimental investigation on instability of a conducting liquid jet under a radial electric field. Chin Quart Mech 2007; 28: 517–520.
9. Mestel AJ. Electrohydrodynamic stability of a highly viscous jet. J Fluid Mech 1996; 312: 311–326.
10. Fu QF, Yang LJ, Chen PM, et al. Spatial-temporal stability of an electrified viscoelastic liquid jet. J Fluid Eng 2013; 135: 094501.
11. Yang LJ, Liu YX and Fu QF. Linear stability analysis of an electrified viscoelastic liquid jet. J Fluid Eng Trans ASME 2012; 134: 071303.
12. Yang LJ, Liu YX and Fu QF. Linear stability analysis of electrified viscoelastic liquid sheets. Atom Sprays 2012; 22: 951–982.
13. Gokhale A, Khusid B, Dave RN, et al. Effect of solvent strength and operating pressure on the formation of submicrometer polymer particles in supercritical microjets. J Supercrit Fluid 2007; 43: 341–356.
15. Sekhon BS. Supercritical fluid technology: an overview of pharmaceutical applications. *Int J Pharmtech Res* 2010; 2: 810–826.

16. Branam R and Mayer W. Characterization of cryogenic injection at supercritical pressure. *J Propuls Pow* 2003; 19: 342–355.

17. Mayer W, Schik A, Schäffler M, et al. Injection and mixing processes in high-pressure liquid oxygen/gaseous hydrogen rocket combustors. *J Propuls Pow* 2000; 16: 823–828.

18. Chehroudi B, Talley D and Coy E. Visual characteristics and initial growth rates of round cryogenic jets at subcritical and supercritical pressures. *Phys Fluid* 2002; 14: 850–861.

19. Bellan J. Supercritical (and subcritical) fluid behavior and modeling: drops, streams, shear and mixing layers, jets and sprays. *Prog Energy Combust Sci* 2000; 26: 329–366.

20. Mayer WOH, Schik AHA, Vielle B, et al. Atomization and breakup of cryogenic propellants under high-pressure subcritical and supercritical conditions. *J Propuls Pow* 1998; 14: 835–842.

21. Zong N and Yang V. Cryogenic fluid jets and mixing layers in transcritical and supercritical environments. *Combust Sci Technol* 2006; 178: 193–227.

22. Duarte ARC, Simplicio AL, Vega-González A, et al. Supercritical fluid impregnation of a biocompatible polymer for ophthalmic drug delivery. *J Supercrit Fluid* 2007; 42: 373–377.

23. Yang LJ, Xu BR and Fu QF. Linear instability analysis of planar non-Newtonian liquid sheets in two gas streams of unequal velocities. *J Non Newton Fluid Mech* 2012; 167: 50–58.

24. Yang LJ, Tong MX and Fu QF. Linear stability analysis of a three-dimensional viscoelastic liquid jet surrounded by a swirling air stream. *J Non Newton Fluid Mech* 2013; 191: 1–13.

25. Yang LJ, Wang C and Fu QF. Weakly nonlinear instability of planar viscous sheets. *J Fluid Mech* 2013; 735: 249–287.

26. Tong MX, Yang LJ and Fu QF. Thermocapillary instability of a two-dimensional viscoelastic planar liquid sheet in surrounding gas. *Phys Fluid* 2014; 26: 033105.

27. Yang LJ and Fu QF. Stability of confined gas-liquid shear flows in recessed shear coaxial injectors. *J Propuls Power* 2012; 28: 1413–1424.

28. Fu QF, Deng XD, Jia BQ, et al. Temporal instability of a confined liquid film with heat and mass transfer. *AIAA J* 2018; 56: 1–8.

29. Fu QF, Deng XD and Yang LJ. Kelvin-Helmholtz instability analysis of confined Oldroyd-B liquid film with heat and mass transfer. *J Non Newton Fluid Mech* 2019; 267: 28–34.

30. Lin SP. *Breakup of liquid sheets and jets*. Cambridge: Cambridge University Press, 2003.

31. Xie L, Jia BQ, Cui X, et al. Linear analysis and energy budget of viscous liquid jets in both axial and radial electric fields. *Appl Math Modell* 2020; 83: 400–418.

**Appendix**

**Notation**

- $A$: surface area
- $b$: dimensionless radius
- $\delta$: rate change in kinetic energy disturbance
- $E_u$: electrical Euler number
- $I_n(x)$, $K_n(x)$: modified Bessel functions
- $p_g$, $p_l$: gas pressure, liquid pressure
- $Q$: dimensionless density
- $R$: radius of liquid jet
- $R_{el}$: equivalent Reynolds number
- $R_0$: position of the annular electrode
- $U$: liquid jet’s velocity
- $U_r$: unperturbed radial velocity
- $U_z$: unperturbed axial velocity
- $V$: control volume
- $V_0$: voltage in the liquid’s surface
- $w_{El}$: contribution of elastic force
- $w_{Eu}$: contribution of electric stress
- $w_g$: contribution of gas pressure
- $w_{Re}$: contribution of viscous dissipation
- $w_{We}$: contribution of surface tension
- $w_{ve}$: total contribution of viscoelasticity
- $W_c$: Weber number
- $x$: strain tensor rate
- $\gamma$: unit tensor
- $\delta$: dielectric constant of ambient gas
- $\lambda_d$: wavelength of disturbance
- $\lambda_1$, $\lambda_2$: relaxation time of stress, deformation retardation time of the liquid
- $\mu$: apparent viscosity of the fluid
- $\mu_0$: initial shear viscosity
- $\mu_1$, $\mu_2$, $\nu_1$, $\nu_2$: time constants
- $\rho_g$, $\rho_l$: the density of gas, the density of liquid
- $\sigma$: liquid’s surface tension
- $\tau$: stress tensor
- $\Omega$: vorticity tensor
- $\omega$: complex frequency
- $\omega_r$: growth rate of the disturbance