Induced resonant torques during the descent of a small asymmetric spacecraft in the atmosphere

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Abstract. This paper investigates an uncontrolled descent of an asymmetric spacecraft in the atmosphere. Suppose that the small spacecraft is a rigid body with small mass-inertial and small aerodynamic asymmetries. The aim of the research is to study small mechanical torques caused by the principal resonance at the non-resonant domains. The proposed study solves the following problems: revealing specific features of the mechanical torque, determining the magnitudes of non-resonant domains of activity of the torque and finding methods to measure the torque. One can conclude that the application of the averaging method makes it possible to research the small mechanical torques induced by the principal resonance. This torque leads to a significant angular velocity evolution of a small spacecraft in the non-resonant domains.

1. Introduction

Many recent studies have focused on examining rotational motion observed during re-entry of a spacecraft (SC) into the atmosphere [1-4]. It is known that the presence of a small mass-aerodynamic asymmetry leads to resonance in the atmospheric motion of a SC [5]. It should be noted that prolonged resonance is causing violation of the technological limitations of the angle of attack. In addition, secondary resonance effects can also contribute to the development of an emergency situation during the atmospheric descent of the SC [5]. Indeed, these dynamic phenomena can contribute to a significant increase of the angular velocity of the SC. The secondary resonance effects are evolutionary phenomena induced by the resonance in the non-resonant domains adjacent to the resonance. These resonance effects were first noted in the study of the rapid rotation of a satellite with a magnetic damper [6]. The central issue in the study of resonances is the research of stability of resonance phenomena. Let the internal stability of the resonances take place in a small neighborhood of this resonance, then the external stability of the resonances is the phenomenon leading to the evolution of slow system variables outside the small neighborhood of the resonance.

Many recent studies have focused on the stability of dynamical systems in a small neighborhood of resonances [7]. There has been growing interest in the external stability of resonances [8]. The relationship between the secondary resonance effects and phenomenon of the external stability of resonances has been noted in [8]. In addition, both phenomena are determined by the influence of secular terms with the resonance ratio of frequencies in the denominators. As in the case of the resonance effect, the fulfillment of the conditions for the external stability of the resonance leads to a non-resonant evolution of the values of the system variables to a small resonant domain.
The increasing interest in external stability of resonances has heightened the need to receive small torques induced by the resonance. In addition, the new small mechanical torque caused by the influence of resonance was considered in the problem of ensuring attitude of a microsatellite with three flywheels and strong permanent magnet with hysteresis rods on board [9].

It should be noted that the author does not know the publications about of the mechanical moments induced by resonance in the problem of descent of an asymmetric spacecraft in a viscous medium.

The aim of the paper is to research new small torques induced by the principal resonance during rotational motion of an asymmetric SC in the atmosphere. The secondary purpose is to determine the magnitude of the domain of the resonant torque and to find methods to measure the moments.

The outline of the paper is as follows: In Section 2, the nonlinear equations of motion of the SC are averaged using three approximations of the averaging method. In this case, a non-resonant averaging scheme is applied. In Section 3, the small resonant torques are analysed in greater detail. In this case, specific features of the resonant torques are revealed. The resonant torques of the SC with mass-aerodynamic and inertial-aerodynamic asymmetry are considered separately. Section 4 includes the magnitude of the domain of activity of the resonant torque. Section 5 provides methods capable of measuring induced moments. The work concludes with Section 6, where the numerical results are presented.

2. Mathematical model

Let us write the nonlinear system in a form convenient for the subsequent analysis. To this end, select it in the generalized parameters of asymmetry, as was done in [5]. As a result, the nonlinear system of a relative motion of a SC will take the form

\[
\begin{align*}
\dot{\mathbf{x}}_1 & = -m_x^2 \sin(\theta + \theta_2) + m^2 \omega_1^2 t g^2 \alpha \cos(2\theta + 2\theta_3), \\
F_a & = -e \omega_2 t g \alpha \frac{d\alpha}{dt} - \mu \frac{m^2}{2\omega_a} \cos(\theta + \theta_1) - e \omega_2 t g \alpha \left[10 + \mathbf{I}_x \right] \omega_1 \left[2 - (2 + \mathbf{I}_x) \omega_1^2 \right] m^2 \cos(2\theta + 2\theta_3), \\
d\theta & = \omega_1 - \omega_1^2 , \\
d\omega & = -e \omega \frac{dq}{dt} .
\end{align*}
\]

Here \(e\) is a small parameter, characterizing the smallness of the parameters of mass-inertial and aerodynamic asymmetry, \(\omega\) is the space angle of attack, \(\omega_1\) is the angular velocity of the SC, \(\theta\) is the fast phase, \(\theta = \varphi - \pi / 2\), \(\varphi\) is the aerodynamic roll angle, \(m_x^2\), \(m^2\), \(m^2\), \(\theta_1\), \(\theta_2\), \(\theta_3\) are the functions, characterizing the magnitude and relative position of the mass, aerodynamic and inertial asymmetry of the SC, \(m^2 = \sqrt{(m_1^2)^2 + (m_2^2)^2}\), \(\sin \theta_1 = m_1^2 / m^2\), \(\cos \theta_1 = -m_2^2 / m^2\),

\[
\begin{align*}
m_1^2 & = \frac{\left[1 + \mathbf{I}_x \omega_1 - 3 \omega_1 \omega_2 \omega^2 \right]}{2\omega_a} - C_{xy} \Delta \gamma g \alpha - \omega_2 t g^2 \alpha \left[\left(m^2 f + C_{xy} \Delta \gamma \right) \mu \right] , \\
\mu & = \frac{\omega_2 t g^2 \alpha \left[\omega_1 \left(\omega_1 + \omega_1 t g^2 \alpha \right) - \omega^2 \left(\omega_1 \omega_1 \mu \omega_a \right) \right]}{2\omega_a m_{zn}}, \\
m_2^2 & = \frac{\left[1 + \mathbf{I}_x \omega_1 - 3 \omega_1 \omega_2 \omega^2 \right]}{2\omega_a} - C_{xy} \Delta \gamma g \alpha + \omega_1 t g^2 \alpha \left[\omega_1 \left(\omega_1 + \omega_1 t g^2 \alpha \right) - \omega^2 \left(\omega_1 \omega_1 \mu \omega_a \right) \right] .
\end{align*}
\]
\[ m^A_x = \sqrt{(m^A_{x1})^2 + (m^A_{x2})^2}, \sin \theta_2 = -m^A_{x1} / m^A_x, \cos \theta_2 = m^A_{x2} / m^A_x, \]
\[ m^A_{x1} = -\frac{\omega_y^2}{m_{zn}} (m^f_{x1} + C_{yx} \Delta y) \tan \alpha - l_{xy} \omega_y^2 \tan \alpha, \]
\[ m^A_{x2} = -\frac{\omega_z^2}{m_{zn}} (m^f_{x2} + C_{yz} \Delta z) \tan \alpha - l_{xz} \omega_z^2 \tan \alpha; \]

\[ \omega_a = \sqrt{\frac{2}{l_x} \omega_y^2 + 4 + \omega^2}; \quad m_{zn} \]
is the coefficient of stabilizing moment, \( \omega \) is the frequency of precession with angular velocity \( \omega_x = 0 \), \( m^A = \sqrt{l_{yz}^2 + \Delta^2 / \Delta^2} \), \( \sin 2\theta_3 = -\Delta^2 / m^A \), \( \cos 2\theta_3 = -l_{yz} / m^A \), \( I_x, I_y, I_z, I_{xy}, I_{yz}, I_{xz} \) are the axial and centrifugal moments of inertia of the SC, \( \tilde{I}_x = I_x / I \), \( \tilde{I}_{xy} = I_{xy} / I \), \( \tilde{I}_{xz} = I_{xz} / I \), \( \Delta^2 = \Delta I / I \), \( \omega_{1,2} = \frac{l_x \omega_x}{2} \pm \omega_a \), \( \omega(z) = \omega_x - \omega_{1,2} \) is the resonant ratio; \( C_x, C_{ys} \) are the dimensionless coefficients of aerodynamic forces; \( m^f_y, m^f_z \) are the dimensionless coefficients of small aerodynamic moments caused by asymmetric surface of the SC; \( l_{yz}, \Delta z \) are small dimensionless displacement of the centre of mass of the SC,

\[ F_a = -\frac{\partial m_{zn}}{\partial \alpha} \frac{q S L}{l} \left( \frac{\omega_y^2}{\cos^2 \alpha} + (\tilde{I}_x \omega_x - \omega_{1,2})(\tilde{I}_z \omega_x - 2 \omega_{1,2}) \right), \] \( q \) is the dynamic pressure, \( S \) is the frontal area of the SC; \( L \) is the length of the SC. The equation (2) contains signs \( \pm \) and \( \mu \). The upper sign is selected when fulfilling the condition \( \omega_x > 0 \), and the lower sign is selected when fulfilling the condition \( \omega_x < 0 \).

Figure 1 shows the frames of SC. In particular, \( OXYZ \) is the frames of references, associated with the SC; \( OXY_n Z_n \) is the frames of references, associated with the space angle of attack; \( OX_k Y_k Z_k \) is the frames of references, associated with the trajectory of SC.

![Figure 1. The frames of the spacecraft.](image)

While obtaining equations (1)-(4), it was assumed that

- the coefficient of aerodynamic moment can be approximated by a Fourier series
  \[ m^f_x \approx m^f_{x1} \sin \varphi + m^f_{x2} \cos \varphi, \]
  \( (5) \)

- the equation (1) - (3) did not account for aerodynamic damping with respect to the angle of attack and angular velocity.

The approximate nonlinear system (1)-(4) can be obtained by the method of integral manifolds [10].
The principal resonance is possible in the system (1)-(4), if the equality \( \frac{d\theta}{dt} = \omega_x - \omega_{1,2} = 0 \) is valid. Since \( \omega_1 > 0 \) and \( \omega_2 < 0 \), the principal resonance \( \omega_x - \omega_{1,2} = 0 \) corresponds to two resonant curves \( \omega_x(t) \). This curves are defined by the equalities \( \omega_1' - \omega_1 = 0 \) (\( \omega_x > 0 \)) and \( \omega_2' - \omega_2 = 0 \) (\( \omega_x < 0 \)). The solution of the equation \( \omega_x - \omega_{1,2} = 0 \) allows us to find the resonant value of the angular velocity

\[
\omega_x' = \pm \frac{\omega}{\sqrt{1 - T_x}}.
\]  

Here \( \omega = \sqrt{-m_{2n} q S L \tan \alpha / I} \), \( q \) is the dynamic pressure, \( S \) is the frontal area of the SC, \( L \) is the length of the SC.

The nonlinear system of equations (1)-(4) is solved together with the system of equations defined motion of the centre of mass of a SC [8]:

\[
\frac{dV}{dt} = -\frac{C_{xy} q S m}{m} - g \sin \theta, \tag{7}
\]

\[
\frac{dH}{dt} = V \sin \theta, \tag{8}
\]

\[
\frac{d\vartheta}{dt} = \frac{C_{yx} S q}{mV} - \frac{g \cos \theta}{V}, \tag{9}
\]

where \( C_{xy}, C_{yx} \) are known aerodynamic coefficients, \( V \) is the air speed of the SC, \( H \) is the altitude of the SC, \( \vartheta \) is the flight-path angle, \( g \) is the gravity acceleration, \( m \) is the mass of the SC.

Suppose that it is required to study the evolution of the rotational motion of SC in the atmosphere. For this, it is necessary to analyse the variation of the slow variables of the system (1)-(4). However, the right-hand sides of equations (1)-(2) contain the fast phase \( \theta \). Hence, similar study directly using the equations (1)-(4) is impossible. It should be noted that the system (1)-(4) is a standard system with one fast phase

\[
\frac{dz}{dt} = \varepsilon Z(z, \theta, \varepsilon), \tag{10}
\]

\[
\frac{d\theta}{dt} = \omega(\theta). \tag{11}
\]

Here \( z = [\omega_x, \varalpha, \omega] \) is the vector of slow variables; \( Z(z, \theta) \) is the vector of the right-hand sides of (1),(2),(4).

Note that resonant zones in such systems have a width of the order of \( \sqrt{\varepsilon} \). In this case, the resonance region of the system (1)-(4) is characterized by the equality \( \frac{d\vartheta}{dt} = o(\sqrt{\varepsilon}) \). It should be noted that equations (7)-(9) do not depend on the fast phase \( \theta \). Therefore, applying the averaging method, it is possible to consider the system (1)-(4) separately from equations (7)-(9).

The application of the well-known averaging method makes it possible to simplify the equations (10)-(11) in the non-resonant case. The substitution of the variables of the averaging method gives [11]:

\[
z = z^0 + \varepsilon z_1(z^0, \theta^0) + \varepsilon^2 z_2(z^0, \theta^0) + ..., \tag{12}
\]

\[
\theta = \theta^0 + \varepsilon \theta_1(z^0, \theta^0) + \varepsilon^2 \theta_2(z^0, \theta^0) + ..., \tag{13}
\]

where the functions \( z_i(z^0, \theta^0), \theta_i(z^0, \theta^0) \) are periodic with respect to the \( \theta^0 \) and they have a zero mean.

It is required that variables \( z^0 \) and \( \theta^0 \) should satisfy equations do not include the fast phase \( \theta^0 \):
The functions \( A_i, B_i, z_i, \theta_i \), where \( i=1,2, \ldots \), are defined according to the method [11].

The equations (1)-(2) contain the fast variable \( \theta \), which complicates the use of these equations for analysis of non-resonant evolutions of the system. Averaging the equations (1)-(2), taking into account the first three approximations and using the non-resonant scheme of the method [11], the author obtains:

\[
\left\{ \frac{d\omega_x}{dt} \right\} = \varepsilon^2 A_2(\omega_x) + \varepsilon^3 A_3(\omega_x).
\]

Here \( A_1(\omega_x), A_2(\omega_x), A_3(\omega_x) \) are functions of slow variables. This paragraph follows a section title so it should not be indented.

3. Analysis of resonant torques

Without loss of generality, two cases are considered separately.

3.1. Resonant torque in the second approximation

Let the equality \( m = 0 \) is valid. In this case, the non-resonant averaging method is applied to the system (1)-(4). The equation for the angular velocity \( \omega_x \) taking into account the first three approximations \( A_1(\omega_x), A_2(\omega_x), A_3(\omega_x) \) is represented in the form:

\[
\frac{d\omega_x}{dt} = \varepsilon^2 A_2(\omega_x).
\]

Here \( A_1(\omega_x) = 0 \), \( A_3(\omega_x) = 0 \), \( A_2(\omega_x) = \pm \frac{\dot{A}}{m_x} \cos(\theta_1 - \theta_2) \pm \frac{m_x}{m} \frac{\dot{A}}{m} \cos(\theta_1 - \theta_2) \pm \frac{\dot{m}}{m} \frac{A}{\Delta} \frac{1}{\Delta} \cos(\theta_1 - \theta_2) \pm \frac{\ddot{m}}{\Delta} \frac{A}{\Delta} \frac{1}{\Delta} \cos(\theta_1 - \theta_2).

Here \( m_x, m^2 \) are dimensionless parameters of magnitudes of asymmetries.

Consequently, the resonant mechanical torque of the second order of smallness is equal to:

\[
M_2^{(\omega_x)} = I_x A_2^{(\omega_x)}.
\]

The torque (18) is represented in the following form:

\[
M_2^{(\omega_x)} = M_{21}^{(\omega_x)}(\Delta) + M_{22}^{(\omega_x)}(\Delta^2).
\]

Here \( M_{21}^{(\omega_x)} = \pm I_x m \frac{\dot{A}}{m_x} \cos(\theta_1 - \theta_2) \pm I_x \frac{\dot{m}}{m} \frac{\dot{A}}{m} \cos(\theta_1 - \theta_2) \pm \frac{\ddot{m}}{\Delta} \frac{A}{\Delta} \frac{1}{\Delta} \cos(\theta_1 - \theta_2) \pm \frac{\ddot{m}}{\Delta} \frac{A}{\Delta} \frac{1}{\Delta} \cos(\theta_1 - \theta_2).

Here \( m_x, m^2 \) are dimensionless parameters of magnitudes of asymmetries.
As follows from equation (17), the first and third approximations become equal to zero in the process of averaging equation (1). Hence, the evolution of the averaged velocity $\omega_x$ is determined by the terms of the second approximation $A_2^{(\omega_x)} \neq 0$. At the same time, the small resonant torque (18) determines the terms obtained by combinations of small parameters from the action of mass-inertial asymmetry ($\frac{-m^A}{m_x} \frac{\partial m^A}{\partial \alpha}, \theta_2$) and aerodynamic asymmetry ($\frac{-m^A}{m^g} \frac{\partial m^g}{\partial \alpha} , \theta_1$).

Characteristic feature of the resonant torque (18) is the presence of the resonant ratios $\Delta, \Delta^2$ in the denominators. The magnitudes of these ratios determine influence of the principal resonance into the evolution of the torque (18) in the non-resonant domains adjacent to the resonance. In addition, it follows from (19) that the terms containing $\Delta, \Delta^2$ can have different signs. Consequently, a special case is possible, when these terms completely compensate each other. In this case, the resonant moment (19) takes the zero value. In fact, in this case the first three approximations of the averaging method are zero.

3.2. Resonant torque in third approximation

Let the equality $m_x^A = 0$ is valid. Similarly, the averaging method is applied to the system (1)-(4). In addition, the effect of asymmetry on the non-resonant evolution of the angular velocity $\omega_x$ is determined by terms of the third approximation:

$$A_3^{(\omega_x)} = e^3 \left( \frac{g_1 g_2}{\Delta^3} \frac{\partial (m^A g_1 \partial \Delta)}{\partial \alpha} - \frac{g_1}{\Delta^2} \frac{\partial (m^A g_1 \partial g_2)}{\partial \alpha} + \frac{3m^A g_1^2}{\Delta^4} (\Delta \frac{\partial \Delta}{\partial \alpha} \frac{\partial g_2}{\partial \alpha} - g_2 \left( \frac{\partial \Delta}{\partial \alpha} \right)^2 \right) \right).$$

Here, the torque components are

$$M_3^{(\omega_x)} = M_{31}^{(\omega_x)} (\Delta^2) + M_{32}^{(\omega_x)} (\Delta^3) + M_{33}^{(\omega_x)} (\Delta^4).$$

As follows from equation (20), the first and second approximations become equal to zero in the process of averaging equation (1). Indeed, the evolution of the averaged velocity $\omega_x$ is determined by
the terms of the third approximation \( A_3(\omega_3) \neq 0 \). Besides, the small resonant torque (21) is defined by the terms obtained by combinations of small parameters from the action of mass-aerodynamic asymmetry \((\bar{m}^A, \theta_1)\) and inertial asymmetry \((m^A, \theta_3)\). It should be noted that characteristic feature for the resonant torque (21) is the presence of resonant ratios \( \Delta^2, \Delta^3, \Delta^4 \) in the denominators. Similarly, the magnitudes of these ratios determine influence of the principal resonance into the evolution of the torque (21) in the non-resonant domains adjacent to the resonance. In addition, it follows from (22) that the terms containing \( \Delta^3 \) and \( \Delta^2, \Delta^4 \) can have different signs. Hence, a case is possible, when these terms completely compensate each other. In this case, the resonant moment (22) takes the zero value.

4. The magnitudes of non-resonant domains of activity of the torques
This study is designed to evaluate of the domains of activity of the induced torques. The boundaries of the domains of activity are defined as follows. Note that in the resonant domains \( \Delta = o(\sqrt{\varepsilon}) \) it is not possible to use the torques (19), (22).

Indeed, the non-resonance scheme of averaging assumes a finding the system (1)-(4) beyond of the resonant domains. Thus, the first boundary of the domain is \( \varepsilon_0 = \Delta < 1/\sqrt{\varepsilon} \). The order of smallness of the torque increases by one order, if the torque is removed from the boundary. Hence, the averaged value of angular velocity \( \omega_3 \) is stabilized. Thus, the second boundary of the domain is determined.

Let the inequality \( \left| M_{21}'(\omega_3) \right| > \left| M_{22}'(\omega_3) \right| \) be true. In this case, from expression (19) implies that the domain of the resonant torque is given by \( \sqrt{\varepsilon} < |\Delta| < 1/\varepsilon \). On the contrary, if the inequality \( \left| M_{21}'(\omega_3) \right| < \left| M_{22}'(\omega_3) \right| \) is true, then the domain of the resonant torque satisfies the inequality \( \sqrt{\varepsilon} < \Delta < 1/\sqrt{\varepsilon} \). Let the inequality \( \left| M_{31}'(\omega_3) \right| > \left| M_{32}'(\omega_3) + M_{33}'(\omega_3) \right| \) be true. In this case, from expression (22) implies that the domain of the resonant torque is given by \( \sqrt{\varepsilon} < |\Delta| < 1/\sqrt{\varepsilon} \). Suppose that the inequality \( \left| M_{32}'(\omega_3) \right| > \left| M_{31}'(\omega_3) + M_{33}'(\omega_3) \right| \) is be true. Then the domain of the resonant torque is equal to \( \sqrt{\varepsilon} < |\Delta| < 1/3\sqrt{\varepsilon} \). Let the condition \( \left| M_{33}'(\omega_3) \right| > \left| M_{31}'(\omega_3) + M_{32}'(\omega_3) \right| \) is true. In this case, the domain of activity of the resonant torque is equal to \( \sqrt{\varepsilon} < |\Delta| < 1/4\sqrt{\varepsilon} \).

Table 1 shows magnitudes of domains of existence of the mechanical torques induced by the principal resonance. As shown in Table 1 the magnitude of the domain decreases with increases of the degree of the ratio \( \Delta \).

| Torque | Magnitude of a domain | Rating of the values of the domains |
|--------|----------------------|-----------------------------------|
| \( M_{21}'(\omega_3) = \frac{\varepsilon^2 f_1}{\Delta} \) | \( \sqrt{\varepsilon} < |\Delta| < 1/\varepsilon \) | 1 |
| \( M_{22}'(\omega_3) = \frac{\varepsilon^2 f_2}{\Delta} \) | \( \sqrt{\varepsilon} < |\Delta| < 1/\sqrt{\varepsilon} \) | 2 |
| \( M_{31}'(\omega_3) = \frac{\varepsilon^2 f_3}{\Delta} \) | \( \sqrt{\varepsilon} < |\Delta| < 1/\sqrt{\varepsilon} \) | 2 |
| \( M_{32}'(\omega_3) = \frac{\varepsilon^2 f_4}{\Delta} \) | \( \sqrt{\varepsilon} < |\Delta| < 1/3\sqrt{\varepsilon} \) | 3 |
| \( M_{33}'(\omega_3) = \frac{\varepsilon^2 f_5}{\Delta} \) | \( \sqrt{\varepsilon} < |\Delta| < 1/4\sqrt{\varepsilon} \) | 4 |
5. Measurement of the torque
The author proposes to measure a small mechanical torque induced by the principal resonance, in one of three following ways:

- measurement induced torque by a sensor of mechanical torque;
- measurement of the angular acceleration by accelerometer and then calculation of induced torque on the basis of the known dynamics equation;
- measurement of the spacecraft angular velocity by an angular velocity sensor, with subsequent calculation of the spacecraft angular acceleration and of induced mechanical torque.

It should be noted that the apparatus measurement of small induced moments is a separate scientific and technical problem. When solving this problem, it is necessary to solve a number of problems. In particular, it is required to perform the separation of slow and fast components of the movement of the SC. Thus, the solution of this problem of measuring small induced moments requires a separate consideration in subsequent publications.

6. Numerical results
Suppose that the small parameter is \( \varepsilon = 0.3 \). Note that for a given maximum value of a small parameter, the induced torque can assume the largest values. Consequently, the calculation of the magnitude of the torque for a small parameter \( \varepsilon = 0.3 \) has the greatest scientific and practical interest.

Figures 2 and 3 summarize the main numerical findings of the study. Figure 2 shows the magnitude of the induced torque \( M_2^{(\omega_x)}(\Delta) \) obtained if the condition \( |M_{21}^{(\omega_x)}| > |M_{22}^{(\omega_x)}| \) is valid.

Two domains of the induced torque are presented in Figure 2 as grey rectangles. As shown in Figure 2, it is observed an increase of the absolute magnitude of the induced torque \( M_2^{(\omega_x)}(\Delta) \) when approaching of the system to the resonance domain of the order of \( \Delta \in [-\varepsilon, \varepsilon] \). Consequently, Figure 2 shows that in the case of passage through resonance domain is implemented by changing the sign of the torque \( M_2^{(\omega_x)}(\Delta) \). As can be seen in Figure 2, the induced torque \( M_2^{(\omega_x)}(\Delta) \) (when \( |M_{21}^{(\omega_x)}| > |M_{22}^{(\omega_x)}| \)) can provide the evolution of the averaged value \( \omega_x(t) \) to the of resonant domain, as for \( \Delta<0 \), as for \( \Delta>0 \).

![Figure 2](image)

**Figure 2.** The resonant torque and its domain under the condition \( |M_{21}^{(\omega_x)}| > |M_{22}^{(\omega_x)}| \).

As shown in Figure 3, an increase of value of the induced torque \( M_3^{(\omega_x)}(\Delta) \) is observed when approaching to the resonance domain of the order of \( \Delta \in [-\varepsilon, \varepsilon] \). However, the sign of the induced torque \( M_3^{(\omega_x)} \) does not change in the case of passing through the resonance domain, if the condition \( |M_{31}^{(\omega_x)}| > |M_{32}^{(\omega_x)} + M_{33}^{(\omega_x)}| \) is valid. In addition, if \( \Delta<0 \), then the induced torque \( M_3^{(\omega_x)} \) can provide
the evolution of the averaged value $\omega_x(t)$ closer to the resonant domain. Thus, for $\Delta>0$ the induced torque $M_{3}^{(\omega_x)}$ contributes to the evolution of the angular velocity $\omega_x(t)$ from resonant domain.

\[
M_{3}^{(\omega_x)} (Nm)
\]

**Figure 3.** The resonant torque and its domain under the condition $M_{31}^{(\omega_x)} > M_{32}^{(\omega_x)} + M_{33}^{(\omega_x)}$.

Figure 4 shows the effect of the induced torque on the magnitude of the angular velocity $\omega_x(t)$. In Figure 4, curve 1 describes the change of the resonant angular velocity $\omega_x(t)$. Curve 2 characterizes the change of the angular velocity $\omega_x(t)$ calculated from the averaged equations (17). Curve 3 characterizes the change of the angular velocity $\omega_x(t)$ calculated from the nonlinear system (1)-(4). As shown in Fig. 4, the induced torque leads to a change in the angular velocity from small to resonant values. Moreover, that curve 2 describes only the non-resonant evolution of the averaged angular velocity. On the contrary, curve 3 shows the change in the angular velocity both in the non-resonant and in the resonant case. It should be noted that there is a qualitative coincidence of the results of calculating the angular velocity $\omega_x(t)$ obtained from the averaged equation (17) and the nonlinear system (1)-(4). However, as shown in curves 2 and 3, the quantitative coincidence of the results is violated at the approach to the resonant angular velocity $\omega_x(t)$. Figure 5 describes the evolution of the angle of attack before and after resonance. As shown in Fig. 5, in the resonant case, a significant increase in the angle of attack occurs.

**Figure 4.** The evolution of the angular speed from small to resonance magnitudes.

**Figure 5.** The evolution of the angle of attack before and after resonance.
In the construction of Figures 4 and 5, the following initial conditions and mass-inertial parameters of the spacecraft were used: \( H(0) = 110 \text{ km}, V(0) = 7800 \text{ m/s}^{-1}, \theta(0) = -0.18 \text{ rad}, \omega_z(0) = 0.01 \text{ s}^{-1}, \alpha(0) = 0.1 \text{ rad}, \theta(0)=0, m/S = 230 \text{ kg m}^{-2}, L = 0.6 \text{ m}, I = 1 \text{ kg m}^2, I_x = 0.3, \theta_1 - \theta_2 = \pi, m_A^A m^A = 10^{-6}, m^A = 0. \) When obtaining the numerical results shown in Figures 4-5, it was assumed that the equality \( m^A = 0 \) is valid. However, similar numerical results can be obtained in the case when the equality \( m^A = 0 \) is invalid.

7. Conclusion

Application of averaging in the non-resonant case, it is possible to determine the approximate analytical expressions for the new mechanical torque induced by the principal resonance. The induced resonant torques (19) represents the expressions of second order of smallness. However, the induced resonance torques (21) represents the expressions of third order of smallness. It should be noted that this small mechanical torque can lead to significant evolution the angle of attack in the resonant case.

Note that the use of modern measuring devices makes it possible to measure small resonance moments. From a practical point of view, it would be interesting to elaborate a method taking into account the induced resonant torques in the attitude control of the spacecraft.

8. References

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10