Closed Conformal Killing-Yano Initial Data

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Abstract
Through an exhaustive search, we produce a 5-parameter family of propagation identities for the closed conformal Killing-Yano equation on 2-forms, which hold on an Einstein cosmological vacuum spacetime in any dimension \( n > 4 \). It is well-known that spacetimes admitting a non-degenerate 2-form of this type are exhausted by the Kerr-NUT-(A)dS family of exact higher dimensional black hole solutions. As a consequence, we identify a set of necessary and sufficient conditions ensuring that the cosmological vacuum development of an initial data set for Einstein’s field equations admits a closed conformal Killing-Yano 2-form. We refer to these conditions as closed conformal Killing-Yano initial data (cCYKID) equations. The 4-dimensional case is special and is treated separately, where we can also handle the conformal Killing-Yano equation without the closed condition.

1 Introduction
Solutions of the Killing equation are vector fields generating infinitesimal isometries of Lorentzian spacetimes or more generally (pseudo-)Riemannian geometries. Generalizations of the Killing equation to higher rank tensors \cite{9} include the Killing-Stückel \cite{12} equations on symmetric tensors, as well as the Killing-Yano equations on \( p \)-forms \cite{45}. Solutions of these equations, the higher rank Killing tensors, can be associated with so-called hidden symmetries, which are responsible for the integrability of geodesic equation, and the separability of Hamilton-Jacobi or wave/Laplace equations, as well as supersymmetric or spinorial generalizations of any of these equations \cite{9,21,42}. Of particular interest is the closely related equation for closed conformal Killing-Yano 2-form, which is responsible for the complete integrability of Einstein’s equations, resulting in the so-called Kerr-NUT-(A)dS family of higher dimensional rotating
black holes (where this 2-form is called the \textit{principal tensor}). This result has by now a substantial literature, with \cite{25, 24, 23, 27, 41, 32} being some key references and more listed in the extensive review \cite{14}, and it has motivated us to focus on the equation for closed conformal Killing-Yano 2-forms. For our work, the most relevant aspects of this equation are its integrability conditions, which are conveniently summarized in \cite{4}. Later in this work, we make use of some representation theoretic methods and Young diagrams. These tools have also been recently fruitfully used to study the integrability conditions of higher rank Killing-Stäckel and Killing-Yano tensors \cite{26}, a development that goes in a different direction than our work.

In the recent work \cite{19} we returned to the question of how to detect the presence of solutions of a geometric PDE on the bulk of a solution of Einstein’s equations just by looking at the initial data for the metric. This question had been studied and successfully answered, by deriving the corresponding initial data equations, for only for a small number of examples: Killing equation \cite{6, 36, 37, 11, 5}, homothetic Killing equation \cite{7}, and some Killing spinor equations \cite{18, 1, 2, 20} (though, only in 4 dimensions). In \cite{19} we have succeeded in adding the conformal Killing equation to this short list. In this work, motivated by the possibility of characterizing the initial data giving rise to the Kerr-NUT-(A)dS family of spacetimes, we use the methods of \cite{19} to derive the initial data equations for closed conformal Killing-Yano (cCYK) 2-forms. Such an initial data characterization is complementary to the above mentioned local characterization by the existence of a principal tensor. In particular, such an initial data characterization would be independent of the way the corresponding Cauchy surface is embedded in the ambient spacetime (cf. Remark 2.2).

In numerical relativity, a Cauchy surface independent characterization of Kerr initial data, which was constructed by one of us \cite{17} (though using a different approach), has already been fruitfully exploited \cite{8} to quantitatively estimate the convergence of a ringdown simulation to a Kerr background. Our results might lead in the future to similar applications in numerical relativity in higher dimensions, or in the study of non-linear stability of Kerr-NUT-(A)dS black holes in mathematical relativity.

In Section 2 we recall the general strategy from \cite{19}, setup the notation, and recall the simplest example of the Killing initial data (KID). The strategy involves identifying a propagation identity (Proposition 2.1), whose existence is then responsible for the successful identification of the desired initial data conditions. At the moment, such a propagation identity can only be found by trial and error, or by an exhaustive search. In Section 3, we use representation-theoretic ideas to carry out an exhaustive search, at low differential order, for a propagation identity for cCYK 2-forms. Some of the more technical details are relegated to Appendices A and B. The search is successful in all spacetime dimensions higher than $n = 4$ and yields a multi-parameter family of propagation identities (Theorem 3.1). The $n = 4$ case is handled separately (Theorem 3.4), where the needed propagation identity was discovered after some trial and error. The corresponding initial data conditions are derived in Theorems 3.3 and 3.5, respectively. Finally, in Section 4, we adapt our methods to conformal
Killing-Yano 2-forms (without the closed condition) in $n = 4$ dimensions. The equivalent spinorial result was first obtained in [18], but we give a purely tensorial result and derivation, which are in line with our motivation to improve the characterization of the initial data of 4-dimensional rotating black holes [17].

While we concentrate on Lorentzian geometries, all the covariant identities that we present are valid also for pseudo-Riemannian geometries of any signature. All the computations of this paper have been double-checked with the tensor computer algebra system xAct [34, 35].

2 Propagation equations and initial data characterizations

In this section, we give the necessary background information for presenting our new results in Sections 3 and 4. Namely, we describe what we mean by an initial data system or initial data characterization and state the main proposition about propagation identities (Proposition 2.1), which defines the parameters of the exhaustive search we will later perform to find the initial data systems for (closed) conformal Killing-Yano equations. A propagation identity, provided it exists, allows us to conclude that a given auxiliary condition can be propagated to the future or past if it is satisfied on initial data. The initial data characterization for this condition then reduces to the search for a corresponding propagation identity. We have simply formalized a well-known argument: if some dynamical fields obey an evolution (or propagation) equation, then an auxiliary property of the initial data (expressed as a differential operator) is preserved by the evolution if this property itself is propagated by a compatible evolution equation. What we call a propagation identity simply captures this compatibility. A classic non-trivial example is the fact that the harmonic gauge (a.k.a wave gauge) condition on initial data for the Einstein equations is preserved by evolution via the Einstein equations in harmonic gauge, which was used in the proof of local well-posedness of the Cauchy problem in General Relativity [13].

Often, it is the main dynamical evolution equation that is fixed and one searches for convenient auxiliary conditions on the fields that have compatible propagation equations. Instead, we flip the attention to what would be the auxiliary condition (we call it the target geometric PDE and consider it fixed), and then look for compatible evolution equations. Simplifying to the case where all equations are linear, a propagation identity can be illustrated as follows:

\[
\begin{array}{c}
\text{fields} \xrightarrow{\sigma} \text{eqs} \\
\phi \xrightarrow{E} \psi \\
Q \xrightarrow{} P
\end{array}
\]
Here our target geometric PDE is \( E[\phi] = 0 \), where \( \phi \) denotes a set of fields. The operator \( Q \) evolves the fields, the operator \( P \) evolves the target geometric equation components, collectively labelled by \( \psi = E[\phi] \), and the operator \(\sigma \) must exist to close the desired propagation identity: \( P[E[\phi]] = \sigma(Q[\phi]) \). The key implication of this identity is that when \( \phi \) is propagated by \( Q[\phi] = 0 \), then the value of \( \psi = E[\phi] \) is propagated by \( P[\psi] = 0 \). In particular, when \( \phi \) is on-shell (\( Q[\phi] = 0 \)) the vanishing of \( \psi = E[\phi] \) on the initial data hypersurface implies that \( E[\phi] = 0 \) everywhere (in the domain of dependence of the initial data). Since we would like the above propagation identity to be valid for arbitrary solutions of \( E[\phi] = 0 \), the propagation operator for the fields must itself be a consequence of the geometric PDE, that is, we need the extra condition \( Q[\phi] = \rho(E[\phi]) \) for some linear operator \( \rho \). Finally, since we are working on vacuum backgrounds, our identities are allowed error terms that vanish when the Einstein equations \( G[g] = 0 \) are satisfied. This is merely a simplifying assumption and there are examples where vacuum propagation identities can be extended to non-vacuum backgrounds \([39, 40]\). This concludes the explanation of the notation used in the formal statement of Proposition 2.1 below.

From now on, all of our differential operators are presumed to be defined between vector bundles over a manifold \( M \) and have smooth coefficients. What follows is a summary of a more extensive discussion from our previous work \([19]\).

We call a linear partial differential equation (PDE) \( P[\psi] = 0 \) a propagation equation (of order \( k \geq 1 \)) if it has a well-posed initial value problem: given a Cauchy surface \( \Sigma \subset M \) with unit normal \( n^a \), the equation can be put into Cauchy-Kovalevskaya form (solved for the highest time derivative) and for each assignment of arbitrary smooth initial data \( \psi|_\Sigma = \psi_0, \ldots, \nabla_n^{k-1}\psi|_\Sigma = \psi_{k-1} \) (where \( \nabla_n = n^a\nabla_a \)) there exists a unique solution of \( P[\psi] = 0 \) on all of \( M \). In particular, due to the linearity of the propagation equation, if the initial data all vanish, \( \psi_0 = \cdots = \psi_{k-1} = 0 \), then \( \psi = 0 \) is the corresponding unique solution on \( M \).

There are multiple examples of propagation equations: (a) Wave (a.k.a normally-hyperbolic) equations, \( P[\psi] = \square \psi + P'(\nabla \psi, \psi) \) \([3]\), and generalized versions of those \([19, \text{Sec.2}]\). (b) Transport equations, \( P[\psi] = a^a \nabla_a \psi + P'(\psi) \), with \( a^a \) everywhere transverse to \( \Sigma \) \([28]\). (c) Special cases, like \( P_{abcd}[\psi] = \nabla^a \psi_{abcd} \) for \( \psi_{abcd} \) satisfying the symmetry and tracelessness conditions of the Weyl tensor in 4 dimensions (a so-called Weyl candidate \([28]\)).

**Proposition 2.1** \(([19, \text{Lem.1–2}])\). Consider a globally hyperbolic spacetime \((M, g)\), satisfying the Einstein vacuum equations, \( G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0 \). Let \( E[\phi] = 0 \) be a PDE (system) defined on some (possibly multicomponent) field \( \phi \). Suppose that there exist propagation equations \( P[\psi] = 0, Q[\phi] = 0 \) (of respective orders \( k \) and \( l \)), where the differential operators \( P \) and \( Q \) satisfy the identities

\[
P[E[\phi]] = \sigma(Q[\phi]) + \tau_P[G], \quad Q[\phi] = \rho(E[\phi]) + \tau_Q[G] \tag{1}
\]

for some linear differential operators \( \rho, \sigma, \tau_P \) and \( \tau_Q \). Then, given a Cauchy surface \( \Sigma \subset M \) with unit timelike normal \( n^a \), there is a bijection between the
solutions of $E[\phi] = 0$ and the solutions of $Q[\phi] = 0$ whose initial data $\phi|_{\Sigma} = \phi_0, \ldots, \nabla^{l-1}_n \phi|_{\Sigma} = \phi_{l-1}$ satisfies $\psi|_{\Sigma} = 0, \ldots, \nabla^{k-1}_n \psi|_{\Sigma} = 0$, for $\psi = E[\phi]$.

In addition, there exists a purely spatial linear PDE on $\Sigma$, $E^\Sigma[\phi_0, \ldots, \phi_{l-1}] = 0$ such that the conditions $Q[\phi] = 0$ and $E^\Sigma[\phi|_{\Sigma}, \ldots, \nabla^{l-1}_n \phi|_{\Sigma}] = 0$ are equivalent to the vanishing of the initial data $\psi|_{\Sigma} = 0, \ldots, \nabla^{k-1}_n \psi|_{\Sigma} = 0$ for $\psi = E[\phi]$.

We must emphasize that the existence of a propagation identity like (1) for a particular equation $E[\phi] = 0$ is not a given and must be discovered either by trial-and-error or through a systematic search. The bulk of this work, in Sections 3 and 4, is devoted exactly to such a systematic search. When a propagation identity exists, there is no reason for it to be unique. For instance, we will find such identities in continuous families.

**Remark 2.1.** For an operator $E^\Sigma$ satisfying the second part of Proposition 2.1, we call

$$E^\Sigma[\phi_0, \ldots, \phi_{l-1}] = 0$$

(2)

an initial data characterization for the geometric equation $E[\phi] = 0$ or in short a set of $E$-initial data conditions or a $E$-initial data system. Clearly, the operator $E^\Sigma$ is not uniquely fixed. For instance, its components may contain many redundant equations. Thus, in practice, once some $E$-initial data conditions have been obtained, they will be significantly simplified by eliminating as many higher order (in spatial derivatives) terms as possible. Also, when some of the components of $E^\Sigma[\phi_0, \ldots, \phi_{l-1}] = 0$ can be used to directly solve for one of the arguments, say $\phi_{l-1}$, in terms of the remaining ones, we can split the initial data system into (a) $\phi_{l-1} = \cdots$ and (b) a system involving only the remaining arguments, $E^\Sigma[\phi_0, \ldots, \phi_{l-2}] = 0$. When presenting an $E$-initial data system, we will omit from $E^\Sigma$ those components that can be rewritten as type (a) and only write the remaining components of type (b), reduced to the smallest convenient set of arguments. Of course, the derivation of the initial data system will provide the information about how all type (a) components can be recovered.

**Remark 2.2.** The geometric meaning of Proposition 2.1 is that the equations that characterize the initial data of the equation $E[\phi] = 0$ given by (2) yields a set of necessary and sufficient conditions for the existence of an isometric embedding between a Riemannian manifold $(\Sigma, h)$, that is isometric to the Cauchy hypersurface $\Sigma$, and a globally hyperbolic $\Lambda$-vacuum solution $(M, g)$ of the Einstein equations possessing a corresponding solution of $E[\phi] = 0$, with $\Sigma$ as its Cauchy hypersurface. Recall that any (vacuum) Cauchy data always has a maximal globally hyperbolic extension [10] that does not necessarily agree with the maximal analytic extension of a vacuum spacetime (the latter might not even be globally hyperbolic). Therefore the globally hyperbolic spacetime $(M, g)$ considered in proposition 2.1 shall be understood as the maximal globally hyperbolic extension of the Cauchy data. In this sense we may generalize the assumptions of Proposition 2.1 and speak of an initial data characterization of the geometric equation $E[\phi] = 0$ in a given ambient vacuum $(M, g)$, by restricting to a globally hyperbolic development of any partial Cauchy surface, even if $(M, g)$ is not globally hyperbolic.
For our purposes, in order to establish that any particular propagation equation has a well-posed initial value problem, it will be sufficient to check that it belongs to the class that we called generalized normally hyperbolic in [19, Sec.2]. We showed how equations from that class inherit the well-posedness properties from the better known normally hyperbolic class [3]. An operator $Q$ (of order $l$) is generalized normally hyperbolic when it is determined (acts between vector bundles of equal rank) and there exists an operator $Q'$ (of order $2m-l$, $m \geq 1$) such that
\[
N[\phi] := Q'[Q[\phi]] = \square^m \phi + \text{l.o.t},
\]
(3)
where l.o.t stands for term of differential order lower than $2m$. That is, the principal symbol of $N[\phi]$ is a power of the wave operator. We call $Q'$ an adjugate operator for $Q$.

One of the consequences [19, Lem.3] of generalized normal hyperbolicity of an operator $Q$ is the non-degeneracy of its principal symbol $\sigma_p(Q)$ as a numerical matrix for any $p \in T^*M$ that is not null (recall that $\sigma_p(Q)$ is a vector bundle morphism obtained by collecting the highest order terms of $Q$ and replacing $\nabla_a$ with multiplication by a covector $p_a$, and hence, for given frames on the source and target vector bundles, the principal symbol is a numerical matrix valued function of $p_a$). Therefore, to show that some $Q$ cannot be generalized normally hyperbolic, it is sufficient to exhibit at least one non-null ($p_a p^a \neq 0$) value of $p \in T^*M$ for which $\sigma_p(Q)$ is singular (equivalently, it possesses at least one left or right null-vector). To save the trouble of explicitly writing down such a covector $p_a$, for instance when there is no single canonical choice, the following is a useful result:

**Lemma 2.2.** Suppose that $Q$ is a determined operator of order $l$ and there exists a non-vanishing differential operator $Q'$ of order $l'$ such that $Q' \circ Q = 0 + \text{l.o.t}$ (or $Q \circ Q' = 0 + \text{l.o.t}$), where $0$ is to be interpreted as a special case of a differential operator of order $l + l'$. Then $Q$ cannot be generalized normally hyperbolic.

**Proof.** The hypotheses basically mean that $\sigma_{p'}(Q')\sigma_p(Q) = 0$ (or $\sigma_p(Q)\sigma_{p'}(Q') = 0$) with $\sigma_{p'}(Q')$ not being identically zero at least for some $x \in M$. Since $\sigma_p(Q')$ depends polynomially on $p \in T^*_x M$, it has rank $\geq 1$ on an open dense subset of $T^*_x M$. Pick any non-null covector $p_a$ from that set, so that the row (column) space of $\sigma_p(Q')$ has at least one non-vanishing element, which is then a left (right) null-vector of $\sigma_p(Q)$. Since this implies that $\sigma_p(Q)$ is singular for a non-null covector $p_a$, the operator $Q$ cannot be generalized normally hyperbolic. \(\square\)

### 2.1 Example: Killing initial data in an Einstein space

To show a practical application of Proposition 2.1 we review here the case of the Killing equation as it was presented in [19, Subsect.2.1],
\[
K_{ab}[v] = \nabla_a v_b + \nabla_b v_a = 0 \quad (E[\phi] = 0),
\]
(4)
whose solution vector fields $v^a$ are infinitesimal isometries of the background metric. If we assume that the Einstein tensor $G_{ab} = 0$ then the corresponding
propagation equations are
\[
\Box v_a + \frac{2\Lambda}{n-2} v_a = 0 \quad (Q[\phi] = 0),
\]
\[
\Box h_{ab} - 2R^c{}_{ab} h_{cd} = 0 \quad (P[\psi] = 0),
\]
where symmetric tensors \( h_{ab} = K_{ab}[v] \) coincide with the target of the Killing equation. The propagation equation \( P[\psi] = 0 \) happens to coincide with the harmonic gauge linearized Einstein evolution equation, with \( h_{ab} \) considered as a linearized perturbation of the background metric. The propagation identities (1) then take the form
\[
\Box K_{ab} - 2R^c{}_{ab} K_{cd} = K_{ab} \left[ \Box v + \frac{2\Lambda}{n-2} v \right]
\]
\[
(P[E[\phi]] = \sigma[Q[\phi]] + \tau[G]),
\]
\[
\Box v_a + \frac{2\Lambda}{n-2} v_a = \nabla^b K_{ab} - \frac{1}{2} \nabla_a K_{bb} + v^b K_{ab} - \frac{1}{2} \nabla_a K_{bb}
\]
\[
(Q[\phi] = \rho[E[\phi]]).
\]

To obtain the K-initial data conditions, or more commonly the Killing initial data (KID) conditions, we must first introduce a space-time split around a Cauchy surface \( \Sigma \subset M \), \( \dim M = n \) and \( \dim \Sigma = n-1 \). Let us use Gaussian normal coordinates to set up a codimension-1 foliation on an open neighborhood \( U \supset \Sigma \) by level sets of a smooth temporal function \( t: U \to \mathbb{R} \), of which \( \Sigma = \{ t = 0 \} \) is the zero level set. Choose \( t \) such that \( n_a = \nabla_a t \) is a unit normal to the level sets of \( t \). Let us identify tensors on \( \Sigma \) by upper case Latin indices \( A, B, C, \ldots \), denote the pullback of the ambient metric to \( \Sigma \) by \( g_{AB} \) and its inverse by \( g^{AB} \), and also denote by \( h^a_A \) the injection \( T\Sigma \rightarrow TM \) induced by the foliation. Raising and lowering the respective indices on \( h^a_A \) with \( g^{ab} \) and \( g_{AB} \), we get the corresponding injections and orthogonal projections between \( T\Sigma, T^*\Sigma, TM \) and \( T^*M \). In our notation, all covariant and contravariant tensors split according to
\[
v_a = v_0 n_a + h^A_a v_A, \quad u^b = -u^0 n^b + h^b_B u^B,
\]
which we also denote by
\[
\begin{pmatrix} v_0 \\ v_A \end{pmatrix}, \quad \begin{pmatrix} u^0 \\ u^B \end{pmatrix}.
\]
\[
\begin{pmatrix} -1 & 0 \\ 0 & g_{AB} \end{pmatrix}.
\]
Thus, in our convention, the ambient metric splits as
\[
\begin{pmatrix} -1 & 0 \\ 0 & g_{AB} \end{pmatrix}.
\]
Let \( D_A \) denote the Levi-Civita connection on \( (\Sigma, g_{AB}) \), depending on the foliation time \( t \) of course, and let \( \partial_t = \mathcal{L}_{-n} \) denote the Lie derivative with respect to the future-pointing normal vector \(-n^a\). The action of \( \partial_t \) extends to \( t \)-dependent
tensors on $\Sigma$ in the natural way. The ($t$-dependent) extrinsic curvature on $\Sigma$ is then defined by
\[ \pi_{AB} = \frac{1}{2} \partial_t g_{AB} \] (12)
and the ambient spacetime connection decomposes as
\[ \nabla_a v_b \to \left[ \nabla_0 v_b \right], \] (13)
where
\[
\begin{align*}
\nabla_0 v_a &\to \begin{bmatrix} \partial_t \\ \partial_0 \delta^B_A - \pi_A B \end{bmatrix} \begin{bmatrix} v_0 \\ v_B \end{bmatrix}, \\
\nabla_A v_b &\to \begin{bmatrix} D_A - \pi_{AB} \\ -\pi_{AB} \\ D_A \delta^C_B \end{bmatrix} \begin{bmatrix} v_0 \\ v_B \\ v_C \end{bmatrix}.
\end{align*}
\] (14)

The ambient $\Lambda$-vacuum Einstein equations $R_{ab} - \frac{2\Lambda}{n-2} g_{ab} = 0$ decompose as
\[
\begin{bmatrix}
-\nabla_0 \pi - \pi \cdot \pi + \frac{2\Lambda}{n-2} \\
D^C \pi_{CB} - D_B \pi
\end{bmatrix}
\begin{bmatrix}
\nabla_0 \pi_{AB} + \pi \pi_{AB} + r_{AB} - \frac{2\Lambda}{n-2} g_{AB}
\end{bmatrix} = 0, \] (16)
where now $r_{AB}$ is the Ricci tensor of $g_{AB}$ on $\Sigma$, $\pi = \pi_C C$, $(\pi \cdot \pi)_{AB} = \pi_A C \pi_{CB}$, and $\pi \cdot \pi = (\pi \cdot \pi)_C C$. Note that we have found it convenient to use the $\nabla_0$ operator instead of $\partial_t$ because of its preservation of both the orthogonal splitting with respect to the foliation and of the spatial metric, $\nabla_0 g_{AB} = \nabla_0 g^{AB} = 0$.

For convenience, we note the commutator
\[
(D_0 D_A - D_A D_0) \begin{bmatrix} v_0 \\ v_B \end{bmatrix} = -\pi_A C D_C \begin{bmatrix} v_0 \\ v_B \end{bmatrix} + \begin{bmatrix} 0 \\ (DC \pi_{AB} - D_B \pi_A C) \end{bmatrix} v_C, \] (17)
and the identity
\[
\nabla_0 r_{ABCD} = -D(A D_C) \pi_{BD} + D(A D_D) \pi_{BC} + D(B D_C) \pi_{AD} - D(B D_D) \pi_{AC}
\] (18)
\[ - (\pi_{[A]} E_{E[B]CD} + \pi_{[C]} E_{E[AB]D]) , \]
which can be obtained by splitting the Bianchi identity $\nabla_a R_{bcde} = 0$. Taking the trace and using the vacuum equations yields
\[

abla_0 r_{AC} = D_A D_C \pi - D_B D_B \pi_{AC} - 2r_{ABCD} \pi_{BD}. \] (19)

According to Proposition 2.1 and the specific identity (7), the Killing equation $K_{ab}[v] = 0$ is satisfied when $v_a$ is any solution of (5) where both $K_{ab}[v]_{\Sigma} = \nabla_0 K_{ab}[v]_{\Sigma} = 0$. Using respectively $K_{00}[v] = 0$ and $K_{0B}[v] = 0$ to eliminate the time derivatives of $v_0$ and $v_B$ from these conditions, while also eliminating the time derivatives of $\pi_{AB}$ using the $\Lambda$-vacuum Einstein equations (16), we obtain the well-known Killing initial data (KID) conditions [5] in the presence of a cosmological constant:
\[
\begin{align*}
D_A v_B + D_B v_A - 2\pi_{AB} v_0 &= 0, \\
D_A D_B v_0 + (2(\pi \cdot \pi)_{AB} - \pi \pi_{AB} - r_{AB}) v_0 \\
- 2\pi_{(B} C) D_{A)} v_C - (D_C \pi_{AB}) v_C + \frac{4\Lambda}{n-2} g_{AB} v_0 &= 0.
\end{align*}
\] (20)
3 Closed Conformal Killing-Yano initial data

Consider an \( n \)-dimensional Lorentzian manifold \((M, g)\) satisfying the vacuum Einstein equations with a cosmological constant \( \Lambda \),

\[
R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0
\]
or equivalently

\[
R_{ab} = \frac{2 \Lambda}{n-2} g_{ab}.
\]

We will restrict ourselves to dimensions \( n > 2 \).

Let \( Y_{ab} = Y\begin{bmatrix} a & b \end{bmatrix} \) be a 2-form. The conformal Killing-Yano (CYK)\(^1\) equation in \( n \) dimensions is

\[
\text{CYK}_{abc} [Y] := 2 \nabla_a Y_{bc} - \nabla_b Y_{ca} + \nabla_c Y_{ba} + 3 \frac{n-1}{n-2} g_{ab} \nabla_d Y_{cd} - 3 \frac{n-1}{n-2} g_{ca} \nabla_d Y_{bd} = 0.
\]

In our index notation \( a:bc \), the : only serves to visually separate groups of indices.

When both \( \text{CYK} [Y] = 0 \) and the exterior derivative \( dY = 0 \), we call \( Y_{ab} \) a closed conformal Killing-Yano (cCYK) 2-form.

**Remark 3.1.** The CYK\(_{abc}\) operator takes values in 3-tensors that transform pointwise irreducibly under \( SO(1, n-1) \) (the group of orientation preserving linear transformations respecting the Lorentzian metric \( g_{ab} \)), which is traditionally labelled by the Young tableau \( \mathcal{Y}^\Gamma \). Given any such tableau (consisting of left-aligned rows of boxes of non-increasing length) filled with tensor indices, the corresponding subspace carrying the irreducible representation is obtained by first symmetrizing over the rows, then antisymmetrizing over the columns and finally subtracting all the traces. The resulting representation is always irreducible, with the possible exception of tableaux with columns of length exactly \( n/2 \) (due to the possibility of decomposing \((n/2)\)-forms into self-dual and anti-self-dual subspaces). But such exceptions only occur for some dimensions and signatures and we will not encounter them below (since we use Lorentzian signature and real representation), with the exception of 3-forms \( \mathcal{Y}^\Gamma \) in dimension \( n = 6 \). If we do not subtract the traces, then we obtain a subspace transforming irreducibly under \( GL(n) \) (the group of general linear transformations), but which may be reducible with respect to \( SO(1, n-1) \). Below, we will freely use Young diagrams (unfilled Young tableaux) to label other irreducible tensor representations. Although we will not need more of them, basic facts about Young diagrams and their relation to \( GL(n) \) and \( SO(p,q) \) representation theory can be found in \([16, 15, 22]\).

The representation type of the conformal Killing-Yano operator implies that it must satisfy the following identities

\[
\text{CYK}_{a:bc} [Y] = \text{CYK}_{[abc]} [Y] = 0, \quad (22)
\]

which can be straightforwardly verified from its definition in \((21)\). Our use of : is to separate out the antisymmetric index group \( bc \).
The covariant derivative of $Y_{ab}$ decomposes as

$$\text{(a}\nabla_a)(\mathbf{b}Y_{bc}) = \frac{1}{3} \mathbf{P} \text{CYK}_{abc}[Y] + \frac{1}{6} \mathbf{b} dY_{abc} - \frac{2}{n-1} g_{a[b}(\mathbf{a}\delta Y)_{c]}$$ \hfill (23)

with $$(dY)_{abc} := 2(\nabla_a Y_{bc} + \nabla_b Y_{ca} + \nabla_c Y_{ab})$$ \hfill (24)

and $$(\delta Y)_b := \nabla^a Y_{ab}$$ \hfill (25)

where we have prefixed $Y_{ab}$ and various operators with Young diagrams indicating that they take values in the corresponding irreducible $SO(1,n-1)$ representation. The most important information contained in (23) can be summarized representation-theoretically by the decomposition of the following tensor product of representations into irreducible ones: $\mathbf{a}\mathbf{b} = \mathbf{a} + \mathbf{P} + \mathbf{b}$. Since each irreducible representation on the right-hand side appears with multiplicity one, Schur’s lemma guarantees that the projection of $\nabla_a Y_{bc}$ onto the corresponding representation is uniquely fixed up to a scalar multiple, which we explicitly fix in the definitions (21), (24) and (25). In the case of multiplicity greater than one, there will exist multiple independent projectors onto the same representation, with the number of independent ones equal to the multiplicity. Of course, the choice of basis in this space of projectors is not unique and has to be made by hand.

The above basic ideas from representation theory will help us carry out an exhaustive search for a covariant, second order propagation identity of the form (1) for the cCYK equations. A priori, the order of the differential operators in (1) is not bounded, nor do they have to be covariant, but for practical reasons, we have restricted our search to operators $P$ and $Q$ that are of second order and covariantly constructed from the Levi-Civita connection $\nabla_a$, the metric $g_{ab}$ and the Riemann tensor\(^2\) $R_{abcd}$. Expanding the search to higher orders would be prohibitively expensive, at least without significant automation of our methods. In any case, our search will succeed (Theorem 3.1) in all dimensions ($n > 2$) except $n = 4$. The 4-dimensional case will be handled separately in Section 4.

Let us also briefly remark that the two separate CYK$\mathbb{Y} = 0$ and $dY = 0$ equations can be combined into the single equivalent equation

$$\nabla_a Y_{bc} - \frac{2}{n-1} g_{a[b}\nabla^d Y_{c]d} = 0,$$ \hfill (26)

where the left-hand side is just the traceless part of $\nabla_a Y_{bc}$. The tensor type of this equation is not irreducible in the sense of $SO(1,n-1)$ representations. So it would not be as helpful in the representation-theoretic exhaustive search described above. Projecting this equation onto the $\mathbf{P}$ and $\mathbf{b}$ tensor types recovers the original separate equations, also demonstrating the complete equivalence of the two formulations.

### 3.1 Dimensions $n < 4$

The lowest dimension in which the CYK operator makes sense is $n = 2$. However, in that case, the $SO(1,1)$ representations $\mathbf{P}$ and $\mathbf{b}$ are both 0-dimensional,\(^2\) Our conventions are $2\nabla_{[a} \nabla_{b]} v_c = R_{abc}^d v_d$ and $R_{ab} = R_{ach}^c$.\(^2\)
meaning that the equations CYK[Y] = 0 and dY = 0 are both trivial conditions of the form 0 = 0.

In dimension n = 3, we can represent any 2-form as Y_{ab} = \eta_{abc} Y_c, where Y_c is a 1-form and \eta_{abc} is the Levi-Civita tensor. We then have the identities

\[
\frac{1}{3} \eta_{(a}^{cd} \text{CYK}_{b)cd}[Y] = \nabla_a Y_b + \nabla_b Y_a - \frac{2}{3} g_{ab} \nabla^c Y_c, \quad (27)
\]

\[
- \frac{1}{2} \eta^{abc} (dY)_{abc} = \nabla^c Y_c. \quad (28)
\]

This means that CYK[Y] = 0 is equivalent to the conformal Killing equation on Y_c,

\[
\text{CK}_{ab}[Y] := \nabla_a Y_b + \nabla_b Y_a - \frac{2}{n} g_{ab} \nabla^c Y_c = 0, \quad (29)
\]

while imposing the additional condition (dY)_{abc} = 0, or the equivalent \nabla^c Y_c = 0, turns it into the Killing equation. The propagation identities and initial data for the conformal Killing equation were found in our previous work [19], where we also reviewed the analogous well-known results for the Killing equation as well as their history.

Thus, in the rest of this work we concentrate on dimension \( n \geq 4 \).

### 3.2 Propagation identity in dimension \( n > 4 \)

Our search strategy has the following steps: (a) identify a basis for the potential \( P, Q, \rho, \) and \( \sigma \) operators in the propagation identity (1), (b) find the most general solution for these operators and identify the free parameters that it depends on, (c) check for which values of the free parameters the operators \( P \) and \( Q \) are generalized normally hyperbolic. The result of this search, recorded in Theorem 3.1, is that there exists a 5-parameter family of identities satisfying all of our search criteria.

(a) We start by listing the basis elements of for all the operators we want to parametrize. The following schematic identity illustrates the number of basis elements and their labels:

\[
\begin{align*}
\mathbb{P} \left[ \rho_{1}^{1,2,3,4,5,6} \right] & \mathbb{P} \left[ \rho_{7,8}^{1,2,3,4} \right] \mathbb{P} \left[ \text{CYK} \right] \mathbb{P} \left[ \sigma_{1}^{1,2,3,4} \right] \mathbb{P} \left[ \rho_{1}^{1} \rho_{2}^{1} \right] \mathbb{P} \left[ \text{CYK} \right] \mathbb{B} \\
\mathbb{P} \left[ \rho_{7,8}^{1,2,3,4} \right] & \mathbb{P} \left[ \text{CYK} \right] \mathbb{P} \left[ \sigma_{1}^{1,2,3,4} \right] \mathbb{P} \left[ \rho_{1}^{1} \rho_{2}^{1} \right] \mathbb{P} \left[ \text{CYK} \right] \mathbb{B} \\
\mathbb{P} \left[ \rho_{1}^{1} \rho_{2}^{1} \right] \mathbb{P} \left[ \text{CYK} \right] \mathbb{P} \left[ \sigma_{1}^{1,2,3,4} \right] & \mathbb{B} = \mathbb{B} Q_{1,2,3,4}^{1,2,3,4} \mathbb{B}.
\end{align*}
\]

(30a)

(30b)

To visually help the reader, we have inserted Young diagram labels to illustrate the irreducible tensor representations that each operator acts between.

To explain the size of each basis, we first need to define precisely what we mean by a second order operator. Obviously, it cannot contain terms with more
than two iterated \( \nabla_a \) derivatives. But each subleading term should also be of total order two. Being covariant, each such subleading term consists of a number of iterated \( \nabla_a \) derivatives multiplied by copies of the Riemann tensor \( R \) and its covariant derivatives. We count the total order as follows: it is additive for products, \( \nabla^k \) has total order \( k \), \( \nabla^l R \) has total \( 2 + l \), the cosmological constant \( \Lambda \) has total order 2, while other constants, the metric \( g_{ab} \) and tensor contractions have total order zero. It is easy to see that this total order is preserved by the Leibniz rule, exchange of covariant derivatives and substitution of Einstein’s equations, while it is additive under operator composition.

Thus, in the identities \((30)\), CYK, \( d, \rho^1, \rho^2, \sigma^1, \sigma^1 \) are all of total order one, \( P^i, \tilde{P}^i, Q^i \) are all of total order two, and \( T^i, \tilde{T}^i \) are all of total order three. To see how many independent ways there are to combine two \( (\nabla_a) \) derivatives with a tensor like \( C_{a:bc} \): consider the tensor product decomposition

\[
(\otimes C) \otimes C = (\mathbb{R} + \mathbb{I}) \otimes C = (\mathbb{R}^3) + (\mathbb{R}^3 + \mathbb{R}^3 + \mathbb{R}^3 + \mathbb{R}^3 + \mathbb{R}^3 + \mathbb{R}^3 + 2\mathbb{R}^3 + \mathbb{R}^3 + \mathbb{R}^3),
\]

where in the first product, the trivial representation \( \mathbb{R} \) corresponds to \( \Box = g^{ab} \nabla_a \nabla_b \), while \( \mathbb{I} \) corresponds to the traceless symmetrized projection of \( \nabla_a \nabla_b \). We need only consider symmetrized derivatives \( (\otimes) \nabla_c \nabla_d \), since the antisymmetric part is equivalent to contractions with the Riemann tensor. The total multiplicity of \( \mathbb{I} \) appearing on the right-hand side is \( 3 = 1 + 2 \), thus there are three independent ways of applying two derivatives to \( C_{a:bc} \). This number automatically counts all possible index permutations, index contractions and products with metric \( g_{ab} \) or Levi-Civita \( \eta_{a_1 \cdots a_n} \) tensors, as all such operations are \( SO(1, n - 1) \) equivariant. Similarly, we can work out the number of independent ways to combine the Riemann tensor \( R_{abcd} \) with a tensor argument by recalling \[38, p.193\] that the \( \Lambda \)-vacuum Weyl tensor \( W_{abcd} = R_{abcd} - \frac{4\Lambda g_{[a|c} g_{d]b}}{(n-1)(n-2)} \) (due to the algebraic Bianchi identity and being fully traceless) belongs to a representation of type \( \mathbb{H} \) while the (appropriately symmetrized traceless) derivatives \( \nabla W, \nabla^2 W, \ldots \) (due to the differential Bianchi identity and its contractions) have independent projections only onto the respective representations \( \mathbb{H}, \mathbb{H}^2, \ldots \), with other projections being expressible in terms of lower order derivatives.

For sufficiently large \( n \) (cf. Remark 3.2), the following tensor decomposition shows how all the operator basis elements from \((30)\) fit into the above scheme, where to save space we have dropped all summands that are irrelevant for identity \((30)\), with multiplicity indicated by listing multiple basis element labels. For economy of notation, we refer directly to the Riemann tensor and its derivatives \( R \) and \( \nabla R \), rather than the Weyl tensor expressions \( W \) and \( \nabla W \), since in explicit computations \( W_{abcd} \) must anyway be expressed in terms of \( R_{abcd} \) and \( \Lambda \) (cf. Remark A.1). For the second order operators:
And the same for the third order operators:

|   | $\nabla Y_{ab}$ | $\nabla C_{a:bc}$ | $\nabla \Xi_{abc}$ |
|---|-----------------|-------------------|-----------------|
| $\nabla$ | $\nabla Y_{ab}$ | $\nabla C_{a:bc}$ | $\nabla \Xi_{abc}$ |
| $\nabla \nabla$ | $\nabla \nabla Y_{ab}$ | $\nabla \nabla C_{a:bc}$ | $\nabla \nabla \Xi_{abc}$ |
| $R$ | $\nabla R$ | $\nabla R$ | $\nabla R$ |
| $\Lambda$ | $\nabla \Lambda$ | $\nabla \Lambda$ | $\nabla \Lambda$ |

Remark 3.2. For sufficiently large $n$ (larger than the total number of boxes involved in the product, for instance) these product tables can be checked using Littlewood’s rule [33, Thm.I] (cf. [30, 31] for complete proofs and modern generalizations). In our case, we need to take $n \geq 8$ for Littlewood’s rule to apply for all products that we are interested in. For small values of $n$, namely $4 < n < 8$, the above multiplication tables can be checked using computer algebra [43, 44]. We have found exceptions only in dimension $n = 6$. Basically, the 3-form representation becomes reducible for $n = 6$ in Lorentzian signature, and splits into the eigen-subspaces of the Hodge $\ast$ operator. Thus the tensor product tables need to be rewritten taking that into account. In addition, the multiplicity of $\nabla$ in the product $\nabla \nabla \nabla$ is 3 instead of 2. Thus, we cannot claim that our lists of operators give complete bases in dimension $n = 6$.

The simplest possibilities are for the $\sigma$ and $\rho$ operators:

\[
\sigma_{a:bc}^{1}[Y] = \text{CYK}_{a:bc}[Y], \tag{32}
\]
\[
\sigma_{a:bc}^{1}[Y] = (dY)_{abc}; \tag{33}
\]
\[
\rho_{ab}^{1}[C] = \nabla c C_{c:ab}[Y], \tag{34}
\]
\[
\rho_{ab}^{1}[\Xi] = (\delta \Xi)_{ab} := \nabla c \Xi_{c:ab}. \tag{35}
\]

The definitions of the operators $P^i, \hat{P}^i, T^i, \hat{T}^i$ and $Q^i$ are somewhat lengthy and their precise form can be found in Appendix A. What is salient about these operators are their composition rules, which are reported in the next paragraph.

(b) The left-hand side in the schematic identity (30a) is parametrized by the coefficients in front of the $P^i, \hat{P}^i$, and $\rho^i$ terms (up to rescaling, there is a
unique possibility for each of the $\sigma^1$ and $\hat{\sigma}^1$ operators, so their coefficients can be absorbed into those of the $P^i$ and $\hat{P}^i$, respectively), while the right-hand side is parametrized by the coefficients of $T^i$ and $\hat{T}^i$. By explicitly computing all the relevant operator compositions, we end up with the following matrix identities:

\[
\begin{bmatrix}
P^1 \circ \text{CYK} \\
P^2 \circ \text{CYK} \\
P^3 \circ \text{CYK} \\
P^4 \circ \text{CYK} \\
P^5 \circ \text{CYK} \\
\sigma^1 \circ \rho^1 \circ \text{CYK} \\
\sigma^1 \circ \rho^2 \circ \text{d}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & -2 & 0 \\
0 & -2 & 3 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 3 & 0 & -3 & 0 & 6 & -12 \\
2 & -6 & 1 & -2 & 0 & -3 & 0 \\
\end{bmatrix},
\]

(36a)

\[
\begin{bmatrix}
\hat{P}^1 \circ \text{d} \\
\hat{P}^2 \circ \text{d} \\
\hat{P}^3 \circ \text{d} \\
\hat{P}^4 \circ \text{d} \\
\hat{P}^5 \circ \text{CYK} \\
\hat{P}^6 \circ \text{CYK} \\
\hat{\sigma}^1 \circ \rho^1 \circ \text{CYK} \\
\hat{\sigma}^1 \circ \rho^2 \circ \text{d}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
2 & -2 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & -\frac{12}{n-2} & 0 & 0 & 0 \\
\end{bmatrix},
\]

(36b)

The goal now is to find a set of coefficients for the left-hand side of the propagation identity (30a) such that the right-hand side is identically zero. It is easy to see that such coefficients correspond exactly to left null-vectors of the matrix $C$ defined by the identity

\[
\begin{bmatrix}
P^{1,2,3,4,5,6} \circ \text{CYK} \\
P^{7,8} \circ \text{d} \\
P^{1,2,3,4} \circ \text{d} \\
P^{5,6} \circ \text{CYK} \\
\sigma^1 \circ \rho^1 \circ \text{CYK} \\
\sigma^1 \circ \rho^2 \circ \text{d} \\
\sigma^1 \circ \rho^1 \circ \text{CYK} \\
\sigma^1 \circ \rho^2 \circ \text{d}
\end{bmatrix}
= \mathcal{C}
\begin{bmatrix}
T^{1,2,3,4,5,6,7,8} \\
\hat{T}^{1,2,3,4}
\end{bmatrix}
\]

(37)

The matrix $C$ can be easily constructed from the blocks of the coefficient matrices in (36a) and (36b). It is now a matter of basic linear algebra to check that (for $n > 2$) $\mathcal{C}$ has a 5-dimensional left null-space. The general 5-parameter solution for a left-null vector recorded in Theorem 3.1.

(c) It remains to check that the possible $P$ and $Q$ operators in the propagation identity (30) are generalized normally hyperbolic (according to the definition given in Section 2). To that end, only the coefficients of the operators that contribute to the principal symbol are important (those actually of second
differential order). For $P$, these are $P^{1,2,3,7,5}$, $\hat{P}^{1,2}$ and $\hat{P}^5$, while for $Q$ these are $Q^{1,2}$. Note that, explicitly working out the operator compositions in (30b) gives the following relation between the coefficients of $\rho^{1,2}$ and those of $Q^{1,2,3,4}$:

\[
\begin{bmatrix}
\rho^1 \circ \text{CYK} \\
\rho^2 \circ d
\end{bmatrix} = \begin{bmatrix}
2 & \frac{(n-4)}{(n-2)} & -1 & \frac{1}{n-2} \\
2 & -\frac{8}{n-2} & 2 & -\frac{1}{n-2}
\end{bmatrix}
\begin{bmatrix}
Q^1 \\
Q^2 \\
Q^3 \\
Q^4
\end{bmatrix}.
\] (38)

Checking generalized normal-hyperbolicity of an operator comes down to parametrizing an ansatz for the adjugate operator and checking whether the key identity (3) can be satisfied for some values of the parameters. In Appendix B, we have recorded the necessary and sufficient conditions for generalized normal hyperbolicity of $Q$ in Lemma B.1 and of $P$ in Lemma B.2. Applied to the family of identities obtained in step (b), we find that for $n > 2$ (with the exception of $n = 4$) a generic element of the family both $P$ and $Q$ are generalized normally hyperbolic, with the exceptional values of the parameters consisting of the union of certain hyperplanes. The full result is recorded in Theorem 3.1.

We are now ready to state the main result of this section in

**Theorem 3.1.** For $n > 2$, there exists the following identity of the form (30) (with vanishing right-hand side):

\[
P = \sum_{i=1}^{6} p_i P_i \quad \rho_7 P^7 + p_8 P^8, \quad \hat{P}_5 \hat{P}^5 + \hat{P}_6 \hat{P}^6 \sum_{i=1}^{4} \hat{p}_i \hat{P}^i, \quad Q = r_1 \rho^1 \circ \text{CYK} + r_2 \rho^2 \circ d,
\] (39)

with

\[
p_1 = x, \quad \hat{p}_1 = y, \quad p_2 = -\frac{x}{n-2}, \quad \hat{p}_2 = \frac{x_1 + y + z}{2},
\]
\[
p_3 = \frac{x}{3} - y_1, \quad \hat{p}_3 = \frac{2y + z}{2} - y_1, \quad p_4 = \frac{x}{6}, \quad \hat{p}_4 = -\frac{6(y + z)}{n-2} + \frac{12}{n-2} y_1,
\]
\[
p_5 = -x, \quad \hat{p}_5 = z, \quad p_6 = -\frac{2x}{n-2}, \quad \hat{p}_6 = z - 2y_1,
\]
\[
p_7 = -\frac{x}{6} - \frac{x_1}{2} - y_1, \quad r_1 = y_1, \quad p_8 = -x, \quad r_2 = -\frac{x_1}{2} - y_1.
\] (40)

where $x$, $y$, $z$, $x_1$ and $y_1$ are free parameters. Moreover, for $n > 2$, necessary and sufficient conditions on these free parameters for the generalized normal
hyperbolicity of $Q$ consist of
\[ x_1 \neq 0, \quad y_1 \neq 0, \quad (41a) \]
and for $P$ they consist of
\[ x \neq 0, \quad (n-4)x \neq 0, \quad y \neq 0, \quad x_1(3z - x - 6y_1) \neq 0, \quad y_1 \neq 0. \quad (41b) \]
For $n > 4$ and $n \neq 6$, the family of operators $P$ and $Q$ in (39) is the most general one of its kind.

Proof. The theorem follows from the calculations discussed in steps (a), (b) and (c) above. As explained in Remark 3.2, the structure of the tensor product decomposition tables from step (a) allows us to claim that we have carried out an exhaustive search in all dimensions $n > 4$, with the exception of $n = 6$. \[ \square \]

Remark 3.3. The propagation identity from the above theorem can now be used to construct cCYKID conditions, for generic values of the free parameters. The free parameters can also be chosen to simplify $P$ in some ways. For instance, we can make its principal symbol block diagonal ($p_7 = p_5 = 0$) by setting $z = 0$ and $x_1 = -(x + 6y_1)/3$. But we cannot make $P$ either lower ($p_7 = p_8 = 0$) or upper ($p_5 = p_6 = 0$) block triangular without violating at least one of the hyperbolicity inequalities. As a consequence, it is also impossible to decouple the CYK[Y] = 0 equation from the dY = 0 equation ($p_7 = p_8 = r_2 = 0$).

The exhaustive nature of the search which produced Theorem 3.1 then leads to the following

Corollary 3.2. In dimensions $n > 4$, $n \neq 6$, there does not exist a propagation identity for the equation $CYK[Y] = 0$ with operators $P$ and $Q$ of total order 2.

Likely, dimension $n = 6$ is not an exception to the corollary, but our analysis would have to be extended to arrive at that conclusion rigorously (cf. Remark 3.2).

3.3 Construction of cCYKID in dimension $n > 4$

Let us denote by $C_{a;bc}$ the left-hand side of (26), the combined form of the cCYK operator. Then we have the following integrability condition
\[
\nabla_{[d}C_{a];bc} - \frac{g_{c[d}\nabla^eC_{a];be} - g_{b[d}\nabla^eC_{a];ce}}{(n-2)} \\
= R_{dace}[bY_c] + 2\Lambda g_{c[d}Y_{a]b} - g_{b[d}Y_{a]c} + \frac{(g_{b[d}R_{a]ce} + - g_{c[d}R_{a]be}Y^{ce})}{2(n-2)} \\
=: I_{da;bc}[Y]. \quad (42)
\]

The zeroth order operator $I_{da;bc}[Y]$ acting on $Y_{ab}$ is traceless and antisymmetric in both groups of : separated indices, but has no other symmetries. It will be useful in giving the precise form of the cCYKID conditions below.

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Theorem 3.3. Consider a globally hyperbolic Einstein $\Lambda$-vacuum Lorentzian manifold, $(M,g)$ of dimension $n > 4$ with $R_{ab} = \frac{2\Lambda}{n-2}g_{ab}$, and a Cauchy surface $\Sigma \subset M$. The necessary and sufficient conditions yielding a set of closed conformal Killing-Yano initial data (cCYKID) for $Y_{ab}$ on $\Sigma$ are given by the following equations, where we also indicate the provenance of each equation, and each equality holds modulo the preceding ones.

\begin{align*}
\frac{1}{3} (CYK_{AB0}[Y] + \frac{1}{2} (dY)_{AB0}) \big|_{\Sigma} &= 0: \\
&= D_A Y_{B0} - \frac{1}{n-1} g_{AB} D^C Y_{C0} - \pi_A^C Y_{BC} = 0 , \tag{43a} \\
\frac{1}{3} (CYK_{ABC}[Y] + \frac{1}{2} (dY)_{ABC}) \big|_{\Sigma} &= 0: \\
&= D_A Y_{BC} - \frac{2}{n-2} g_{A[B} D^D Y_{D]C} \\
&+ 2 \pi_{A[B} Y_{C]0} - \frac{2}{n-2} g_{A[B} (\pi g_{C]D} - \pi_{C]D}) Y_{D0} = 0 , \tag{43b} \\
\frac{(n-3)}{3(n-2)} \nabla_0 CYK_{(A;B)0}[Y] \big|_{\Sigma} &= 0: \\
&= 2 I_{0(A;B)0}[Y] = (D_C \pi_{AB} - D_{(A} \pi_{B)C}) Y_{C0} \\
&+ \pi_{(A}^C Y_{B)C} + \pi_{(A|C} \pi^D Y_{D]B]} + r_{(A}^C Y_{B)C} = 0 , \tag{43c} \\
- \frac{1}{6} \nabla_0 CYK_{A;BC}[Y] \big|_{\Sigma} &= 0: \\
&= -I_{0A;BC}[Y] = (D_{[C} \pi_B^E Y_{AE} + (D_{[C} \pi_A^E Y_{E]B}) \\
&+ (D^E \pi_{A[B} Y_{C]E} + \frac{1}{2} (r_{BCA}^E + 2 \pi_{A[B} \pi_{C]E}) Y_{E0} \\
&+ (r_{A[B} - \pi_{AE} \pi^E_{[B} + \pi_{A|B} - \frac{2\Lambda}{n-2} g_{A[B} ) Y_{C]0} = 0 , \tag{43d} \\
\frac{(n-3)}{6} \nabla_0 (dY)_{AB0} \big|_{\Sigma} &= 0: \\
&= 2(n-2) I_{0(A;B)0}[Y] = \frac{1}{2} (2 \pi_A^C \pi_B^D Y_{CD} \\
&- (n-2)(\pi^C_{AB} Y_{B[D} - \pi_{A|C} Y_{B]C} - r_{[A}^C Y_{B]C}) \\
&+ (n-4) D_{[A} \pi_{B]} Y_{C0} + \frac{2(n-3)}{(n-2)} \Delta Y_{AB} = 0 . \tag{43e}
\end{align*}

Proof. First, we compute the split form of the independent and non-trivial components of these CYK$[Y]$ and $dY$ operators:
CYK_{0;BC}[Y] = \frac{3}{n-1} \left( 2\pi F_{F} Y_{C}^{0} - D_{F} Y_{C}^{F} - (n-2)\nabla_{0} Y_{0} \right), \quad (44a)

CYK_{0;BC}[Y] = 2\text{CYK}_{[B;C]0}[Y]
= 2(\pi_{[B}^{A} Y_{C]A} - D_{[B} Y_{C]0} + \nabla_{0} Y_{BC}), \quad (44b)

CYK_{(A;B)0}[Y] = \frac{3}{2} \left( \text{CYK}_{AB}[Y_{0}] - 2\pi_{(A}^{D} Y_{B)D} \right), \quad (44c)

\text{CYK}_{AB;BC}[Y] = \text{MAC}_{AB;BC}[Y]
+ 6\pi_{A[B} Y_{C]0} - \frac{6}{n-2} g_{A[B}(\pi_{gC]D} - \pi_{C[D})Y^{D} \right) \quad (44d)

(dY)_{AB0} = 2(2D_{[A} Y_{B]0} - 2\pi_{[A}^{C} Y_{B]C} + \nabla_{0} Y_{AB}), \quad (44e)

(dY)_{ABC} = (dY)_{ABC}, \quad (44f)

where we have used \text{CK}, \text{CYK}, \text{d} to denote the \((n-1)\)-dimensional versions of the operators defined in (29), (21) and (24) respectively. We will use CYK_{0;BC}[Y] = 0 and CYK_{0;BC}[Y] = 0 to systematically eliminate \(\nabla_{0}\) derivatives of \(Y_{0}\) and \(Y_{AB}\) from the rest of the calculations. Specifically, this results in the substitutions

\nabla_{0} Y_{0} = \frac{1}{n-2} \left( D^{C} Y_{CA} - \pi_{A}^{C} Y_{0C} + \pi Y_{0A} \right), \quad (45a)

\nabla_{0} Y_{AB} = D_{[A} Y_{B]0} - \pi_{[B}^{C} Y_{A]C}. \quad (45b)

Similarly, we will use the Einstein equations (16) to systematically eliminate \(\nabla_{0} \pi_{AB}, \nabla^{B} \pi_{AB}\) and the spatial Ricci scalar \(r\) throughout our calculations.

Further, the conditions \(\nabla_{0} \text{CYK}_{0;OC}[Y] = \nabla_{0} \text{CYK}_{0;BC}[Y] = 0\) will appear as part of setting to zero the \(\nabla_{0}\) derivatives of all the components in (44). However, they can always be satisfied by solving for \(\nabla_{0}^{2} Y_{0}\) and \(\nabla_{0}^{2} Y_{AB}\), in analogy with (45). Strictly speaking, these second order derivatives are constrained by the propagation equation \(Q_{ab}[Y] = 0\). But one of the requirements on the propagation identities, imposed by Proposition 2.1 and verified by Theorem 3.1, is that \(Q[Y]\) factors through the \(\text{CYK}[Y]\) and \(dY\), which means that we can solve for \(\nabla_{0}^{2} Y_{0}\) and \(\nabla_{0}^{2} Y_{AB}\) just by differentiating (45) and the conditions \(\nabla_{0} \text{CYK}_{0;OC}[Y] = \nabla_{0} \text{CYK}_{0;BC}[Y] = 0\) do not impose any independent purely spatial constraints on the initial data for \(Y_{ab}\).

Now, in the same way that we obtained the combined form (26) of the spacetime cCYK operator, combining CYK_{(A;B)0}[Y] = 0 and (dY)_{AB0} = 0 immediately gives (43a), while combining CYK_{A;BC}[Y] = 0 and (dY)_{ABC} = 0 immediately gives (43b), the first two cCYKID conditions.

It now remains to take the \(\nabla_{0}\) derivatives of the already obtained (43a) and (43b), systematically eliminate \(\nabla_{0} Y_{0}\) and \(\nabla_{0} Y_{AB}\) as above, and to simplify the results (meaning trying to eliminate as many high order spatial derivatives of \(Y\) as possible) using purely spatial integrability conditions of the same equations. We can shortcut this process by taking advantage of the spacetime
integrability condition (42). Splitting that identity results in the following relevant components:

\[(n-3) \frac{6}{n-2} \nabla_0 \text{CYK}_{(A:B)0}[Y] + O(\text{CYK}[Y]) + O(dY) = I_{0(A:B)0}[Y], \quad (46a)\]

\[\frac{1}{6} \nabla_0 \text{CYK}_{A:BC}[Y] + O(\text{CYK}[Y]) + O(dY) = I_{0A:BC}[Y], \quad (46b)\]

\[(n-3) \frac{12}{n-2} \nabla_0 (dY)_{AB0}[Y] + O(\text{CYK}[Y]) + O(dY) = I_{0[A:B]0}[Y], \quad (46c)\]

\[\frac{1}{12} \nabla_0 (dY)_{ABC}[Y] + O(\text{CYK}[Y]) + O(dY) = I_{0[A:BC]0}[Y], \quad (46d)\]

where \(O(\cdot)\) denotes linear dependence on the argument and any of its spatial derivatives. Each right-hand side is already a zeroth order operator acting on \(Y\). Computing the components of \(I_{0(A:B)0}[Y]\) and \(I_{0[A:B]0}[Y]\) directly gives us the desired cCYKID conditions (43c) and (43e). Next, we find

\[I_{0[A:BC]0}[Y] = -Y_{|A} D^D B^e C^e |D, \quad (47)\]

which is of spatial tensor type \(\mathbb{F}\). It happens to be proportional to an integrability condition obtained by applying \(D_C\) to (43a) and projecting onto \(\mathbb{F}\). Another integrability condition that we can get is the projection of the derivative of (43a) onto spatial tensors of type \(\mathbb{F}\) (in one of the two possible ways), which helps us simplify the explicit expression for \(I_{0A:BC}[Y]\). The resulting simplified expression is our remaining cCYKID condition (43d).

Finally, having established that the cCYKID conditions (43) are equivalent to \(\text{CYK}[Y] = 0, (dY) = 0, \) and \(\nabla_0 \text{CYK}[Y] = 0, \nabla_0 (dY) = 0 \) on \(\Sigma\), a joint application of Theorem 3.1 and Proposition 2.1 completes the proof.

Remark 3.4. It is well-known that the Killing equation and its generalizations, including Killing-Yano, Killing-Stäckel equations and their conformal versions, tend to have at most a finite number of linearly independent solutions, with the maximal number achieved on maximally symmetric backgrounds [27]. This behavior is characteristic of PDEs of so-called finite type [29, Apx.A], which are defined by the property that the Taylor expansion of a general solution at any point admits only finitely many independent coefficients. The finite type property depends only on the symbol of the equaiton (the coefficients of the highest derivative terms) and it may be enough to check a subsystem. In fact, considering more equations only increases the constraints on the number of linearly independent solutions, while adding subleading terms may only add integrability conditions, which do the same. So, given that the cCYK equation is of finite type and that our Theorem 3.3 establishes a bijection between solutions to the cCYKID conditions (43) on in initial data surface \(\Sigma\) and solutions to the cCYK equations on the domain of dependence of \(\Sigma\), the cCYKID equations should themselves be of finite type. Indeed, this can be checked explicitly by noting that the symbol of (43b) coincides with the symbol of the form (26) of the cCYK equation for \(Y_{BC}\) on \(\Sigma\), while the symbol of the \((AB)\) symmetrization
of (43a) coincides with the symbol of the CK equation (29) for $Y_{b0}$ on $\Sigma$, which is also well-known to be of finite type.

### 3.4 Propagation identity in dimension $n = 4$

The 5-parameter propagation identity from Theorem 3.1 can be specialized to dimension $n = 4$, but it fails one of the seven inequalities needed to establish hyperbolicity of the $P$ and $Q$ operators, for any value of the parameters. More specifically, it fails the inequality associated with the coefficient of the operator $P^2$. Fortunately, we can use the same trick that was used for the conformal Killing operator in [19]. The idea is to reduce $P^2$ from a second order to a first order operator by decoupling a differential consequence of the cCYK system and propagating it independently. Ultimately, instead of a second order, we will find a fourth order propagation identity for the cCYK system in dimension $n = 4$.

Note the following identity (valid in general dimension and without restriction on the Ricci tensor $R$):

\[ K_{ab}[\delta Y] = \frac{(n-1)}{3(n-2)} S_{ab}[\text{CYK}[Y]] + \frac{(n-1)}{(n-2)} 2 R_{(a}^c Y_{b)c}, \]

with $S_{ab}[C] := 2 \nabla^c C_{(a;b)c}$. (48)

The Ricci-dependent term vanishes for $\Lambda$-vacua, when $R_{ab} = \frac{2\Lambda}{n-2} g_{ab}$. This identity can be used to factor

\[ P^2_{a;bc}[\text{CYK}[Y]] = -\frac{3(n-2)}{(n-1)} \tilde{P}_{a;bc}[K[\delta Y]], \]

with $\tilde{P}_{a;bc}[h] := 2 \nabla [b h]_c a - \frac{2}{n-1} g_{a[b} \nabla^d h_{c]d} + \frac{2}{n-1} g_{a[b} \nabla^d h_{c]d}$. (50)

But the Killing operator $K_{ab}[\nu]$ satisfies its own propagation identity (7), which is compatible with that from Theorem 3.1 in the sense that

\[ (\Box + \frac{2\Lambda}{n-2}) (\delta Y)_a + \frac{(n-1)}{3(n-2)} R^b_{abcd} \text{CYK}_{c;cd}[Y] = \frac{1}{y_1} \frac{(n-1)}{3(n-2)} (\delta Q[Y])_a, \]

with $Q[Y]$ defined by Theorem 3.1. Writing the propagation identity (7) in terms of $Y$, we get

\[ \Box K_{ab}[\delta Y] - \frac{(n-1)}{(n-2)} 2 R^c_{ab} S_{cd}[\text{CYK}[Y]] + \frac{(n-1)}{3(n-2)} K_{ab}[R \cdot \text{CYK}[Y]] = \frac{1}{y_1} \frac{(n-1)}{3(n-2)} K_{ab}[\delta Q[Y]], \]

where $(R \cdot \text{CYK}[Y])_a = R_{a;bc;cd}$.

We can now use the same strategy as was used for the conformal Killing equation in [19, Sec.4] to prove
Theorem 3.4. Under the same hypotheses as Theorem 3.1, but for \( n = 4 \), there exists a 6-parameter family of 4th order propagation identities of the form

\[
\square \left[ p_1 P^1 + (p_2 - y_2) P^2 + p_3 P^3 \right] + \hat{p}_5 P^5 + \hat{p}_1 P^3 + \hat{p}_2 P^2 \]

\[
\square \left[ \sigma^1 - \frac{3}{n} \bar{P} \circ K \circ \delta \right] \left[ r_1 \rho^1 \quad r_2 \rho^2 \right] \left[ \begin{array}{c} \text{CYK} \\ \text{d} \end{array} \right] + l.o.t
\]

where \( l.o.t \) stands for operators of differential order three or lower acting on the cCYK system, while the \( p_i \), \( \hat{p}_j \) and \( r_k \) coefficients depend on the free parameters \( x, y, z, x_1, y_1 \) in the same way as in Theorem 3.1 and \( y_2 \) is an additional free parameter. The necessary and sufficient conditions for the generalized normal hyperbolicity of the corresponding \( Q \) operator are still

\[
x_1 \neq 0, \quad y_1 \neq 0,
\]

and for the corresponding \( P \) operator they are now

\[
x \neq 0, \quad y_2 \neq 0, \quad y \neq 0, \quad x_1 (3z - x - 6y_1) \neq 0, \quad \text{and} \quad y_1 \neq 0.
\]

Proof. The first step is to apply the wave operator \( \square \) to both sides of the propagation identity from Theorem 3.1 restricted to \( n = 4 \) dimensions. Then, note that we are completely free to do the following rewriting:

\[
\square p_2 P^2 \circ \text{CYK} = \square (p_2 - y_2) P^2 \circ \text{CYK} - y_2 \frac{3(n-2)}{(n-1)} \square P \circ K \circ \delta
\]

\[
= \square (p_2 - y_2) P^2 \circ \text{CYK} - y_2 \frac{3(n-2)}{(n-1)} \hat{P} \circ \square K \circ \delta + l.o.t.
\]

Finally, using (53) to eliminate \( \square K \circ \delta \) from the above formula, we arrive directly at the desired propagation identity (54). Recalling the relevant hyperbolicity conditions from Lemmas B.1 and B.2, which are unchanged when the operators contributing to the principal symbol are multiplied by a power of \( \square \), we get the corresponding inequalities (55).

3.5 Construction of cCYKID in dimension \( n = 4 \)

In contrast to the case of \( n > 4 \) dimensions (Section 3.3), the fact that in \( n = 4 \) dimensions we must use the fourth order propagation identity from Theorem 3.4 to apply Proposition 2.1 means that the corresponding cCYKID conditions must be obtained by evaluating \( \nabla_0^k \text{CYK} |_{\Sigma} = 0 \) and \( \nabla_0^k \text{d}Y |_{\Sigma} = 0 \) for \( k = 0, 1, 2, 3 \). But, our task is simplified by the observation, already exploited in the proof of Theorem 3.4, that the fourth order identity (54) follows from the coupled set of second order propagation identities (53) for \( K[\delta Y] = 0 \) and (39) for \( \text{CYK}[Y] = 0, \text{dY} = 0 \). We have previously encountered an analogous situation...
in the construction of the conformal Killing initial data [19]. The same argument as in the proof of Theorem 3 of [19], which we do not reproduce here, shows that it is in fact sufficient to evaluate the initial data conditions \( \nabla^k \text{CYK}[Y] \big|_{\Sigma} = 0 \), \( \nabla^k dY \big|_{\Sigma} = 0 \) and \( \nabla^k K[\delta Y] \big|_{\Sigma} = 0 \) only for \( k = 0, 1 \).

**Theorem 3.5.** Consider a globally hyperbolic Einstein \( \Lambda \)-vacuum Lorentzian manifold, \((M,g)\) of dimension \( n = 4 \) with \( R_{ab} = \Lambda g_{ab} \), and a Cauchy surface \( \Sigma \subset M \). The necessary and sufficient conditions yielding a set of closed conformal Killing-Yano initial data \( c\text{CYKID} \) for \( Y_{ab} \) on \( \Sigma \) are the initial data conditions of Theorem 3.3 (specialized to \( n = 4 \)) together with the KID conditions (20) applied to \( v = \delta Y \), which can be rewritten in two equivalent ways, modulo the conditions already included in Theorem 3.3. The first consists of only the (20b) condition

\[
D_AD_Bv_0 + (2(\pi \cdot \pi)_{AB} - \pi \pi_{AB} - r_{AB})v_0
- 2\pi_{(B,C}D_{A)C}v_C - (D^C\pi_{AB})v_C + \frac{4\Lambda}{n-2}g_{AB}v_0 = 0, \quad (57)
\]

with

\[
v_0 = DBY_{B0}, \quad v_A = \frac{3}{2}(DBY_{BA} - \pi_A BY_{B0} + \pi Y_{A0}). \quad (58)
\]

The second equivalent condition is

\[
12\pi_{(A|B)}^{BD} Y_{D|C} + 12\pi_{(A|B)}^{BD} r_{(A|B)Y_{D|C}} - 12\pi r_{(A|B)Y_{B|C}}
+ 24\pi_{(A|B)Y_{D|C}} - 12\left(\Lambda + \pi DE^D \pi + \pi^2\right) \pi_{(A|B)Y_{B|C}}
- 12Y_{(A|B)D|C} DB \pi + 12Y_{B}^{(A|B)D|C}B
+ 6[5\pi_{(A|B)DB} + \pi_{AB}^{AB} + 2\pi_{AC} DB \pi]
- 3\pi_{(A|B)D} + 4\pi_{D(A|B)C}DB - \pi D_{(A|B)C}DB - 2\pi_{B(A|B)C}DB
- 4\pi_{(A|B)DB} DB \pi_{(C)C}B - 2\pi_{(A|B)D} DB \pi_{(C)C}DB Y_{B} = 0. \quad (59)
\]

**Proof.** As summarized before the statement of the theorem, in imitation of the proof of [19, Thm.3], an application of Proposition 2.1 implies that the conditions necessary and sufficient to identify the initial data of a cCYK 2-form \( Y_{ab} \) are equivalent to

\[
\nabla^k \text{CYK}[Y] \big|_{\Sigma} = 0, \quad (60a)
\]

\[
\nabla^k dY \big|_{\Sigma} = 0, \quad (60b)
\]

\[
\nabla^k K[\delta Y] \big|_{\Sigma} = 0, \quad (60c)
\]

for \( k = 0, 1 \). In Theorem 3.3, we have already given a set of initial data conditions that are intrinsic to \( \Sigma \) and are equivalent to (60a) and (60b). Though these results were stated for \( n > 4 \), all the same calculations remain valid in dimension \( n = 4 \).

On the other hand, when \( n = 4 \), the propagation identity (39) fails to be generalized normally hyperbolic and so (60a) and (60b) cannot be used to solve for...
the $\nabla^2_0 \text{CYK}[Y]$ and $\nabla^2_0 dY$. So these conditions may no longer be sufficient. Sufficiency is restored by adding the conditions (60c), which are of course equivalent to the well-known KID conditions (20) applied to $v_a = (\delta Y)_a$, whose components specialize to (58) after eliminating $\nabla_0 Y_{A0}$ and $\nabla_0 Y_{AB}$ using $\text{CYK}_{0;0C} = 0$ and $\text{CYK}_{0,BC} = 0$.

However, these additional KID conditions are not all independent. Namely, splitting the identity (48) gives us the schematic identities

$$
\text{K}_{00}[\delta Y] = O(\text{CYK}[Y]), \quad \text{K}_{A0}[\delta Y] = O(\text{CYK}[Y]), \quad \text{K}_{AB}[\delta Y] = -\nabla_0 \text{CYK}_{(A;B)0}[Y] + O(\text{CYK}[Y]),
$$

(61)

where $O(-)$ denotes linear dependence on the argument and any of its spatial derivatives. Hence, the only independent initial data conditions will come from $\nabla_0 \text{K}_{AB}[\delta Y]|_{\Sigma} = 0$ or only the (20b) part of the KID conditions, which we have copied to (57) in the statement of the theorem. Equivalently, as can be seen from the preceding identities, this remaining independent condition can be replaced by $\nabla_0^2 \text{CYK}_{(A;B)0}[Y]|_{\Sigma} = 0$. In the proof of Theorem 3.3, we have already shown that the condition $\nabla_0 \text{CYK}_{(A:B)0}[Y]|_{\Sigma} = 0$ is equivalent to (43c), which no longer contains any spatial derivatives of $Y_{ab}$. Thus, to obtain the new independent condition on $Y_{ab}$, it is sufficient to apply $\nabla_0$ to (43c) and once again eliminate all $\nabla_0 Y_{ab}$. In this way, while also eliminating $\nabla_0 \pi_{AB}$ using the Einstein equations (16) and $\nabla_0 r_{ABCD}$ using (18), direct calculation gives us the desired initial data condition (59).

$$
\square
$$

4 Conformal Killing Yano initial data

In a general dimension $n$ it is not yet known how to construct a propagation identity for the conformal Killing-Yano (CYK) system, without the closed condition that was used in the successful construction in Section 3 on an Einstein (Λ-vacuum) background. But, as we will analyze in this section, the problem can be solved if $n = 4$. In principle, the solution can be extracted from the previously studied case of the Killing $(2, 0)$-spinor [18, 1, 2], which is the spinorial version of a self-dual conformal Killing-Yano 2-form. Instead, we give a purely tensorial derivation, taking advantage of the explicit calculations from Section 3 and the conceptually clear approach to the problem that we have described in Section 2 and our previous work [19]. Below, we will freely use the notation and results introduced in Sections 2 and 3.

Recall formula (48), which factors $K_{ab}[\delta Y]$ through $\text{CYK}_{a;bc}[Y]$ in general dimension. And also note the following formula, which is valid only in 4 dimensions (but without restriction on the Ricci tensor $R_{ab}$):

$$
K_{ab}[\ast dY] = 3S_{ab}[\text{CYK}[Y]] - 18R^e_{(a} \eta_{b)c} dY_{de},
$$

(62)

with $S_{ab}[C] = 2\eta_{[a} c_{(d} \nabla_{e} C_{b);cd}$. (63)
The Ricci-dependent term vanishes for \( \Lambda \)-vacua, when \( R_{ab} = \frac{2\Lambda}{n+2} \). For reference, our 4-dimensional conventions for the Hodge \( * \) operation are

\[
(*v)_{abc} = \eta_{abc} v_d, \quad (*Y)_{ab} = \frac{1}{2} \eta_{ab} Y_{cd}, \quad (*w)_a = \frac{1}{6} \eta_{abc} w_{bcd},
\]

with \( \eta_{abcd} \) being the Levi-Civita tensor.

Recall also that the general 5-parameter propagation identity can be specialized to both \( n = 4 \) dimensions and also to the case which decouples the CYK\([Y]\) operator from the exterior derivative \( dY \) (setting \( p_7 = p_8 = r_2 = 0 \)). But in both cases at least one of the inequalities from the hyperbolicity conditions (55) fails. In Section 3.4, this failure when \( n = 4 \) was fixed by decoupling \( K_{ab}[\delta Y] \) operator and propagating it separately. By analogy, in this section, we will fix the failure of hyperbolicity for the \( n = 4 \) propagation identity decoupled from \( dY \), by also decoupling \( K_{ab}[*dY] \) and propagating it separately. In general dimension, restoring the hyperbolicity for the decoupled case remains an open problem.

By decoupling the general 5-parameter propagation cCYK identity (Theorem 3.1) from \( dY \) (setting \( p_7 = p_8 = r_2 = 0 \)), what remains is the following 1-parameter identity

\[
-P_3 = \sigma^1 \circ \rho^1 \circ \text{CYK},
\]

where the single parameter is just an overall multiplicative constant. To apply the decoupling strategy in Section 3.4, we used the fact that \( P^2_{abc}[\text{CYK}[Y]] \) factors through \( K_{ab}[\delta Y] \), according to (50). Unfortunately, \( P^3_{abc}[\text{CYK}[Y]] \) does not directly factor through \( K_{ab}[*dY] \). Thus, it is convenient to introduce the alternative operator \( \tilde{P}^3 \), which does factor:

\[
\tilde{P}^3_{a:bc}[\text{CYK}[Y]] := \frac{3}{2}(\star \tilde{P}[K[*dY]])_{a:bc},
\]

where the operator \( \tilde{P} \) was defined in (51) and we have extended the Hodge \( * \) operator to

\[
(*C)_{a:bc}[Y] := \frac{2}{3}(\eta_{bc} p^q C_{a:pq} + \eta_{a[p} b^q C_{c]:pq}).
\]

With this choice, identity (65) gets rewritten as

\[
3P^1 - \frac{3}{2}P^2 - \frac{1}{2}P^3 + \frac{1}{2}P^4 - 3P^5 - 5P^6 = \sigma^1 \circ \rho^1 \circ \text{CYK}.
\]

To clarify the structure of this identity, let us rewrite it more explicitly as

\[
3\Box \text{CYK}_{a:bc}[Y] + 3\tilde{P}_{a:bc}[K[\delta Y]] - \frac{3}{4}(\star \tilde{P}[K[*dY]])_{a:bc} + \text{l.o.t}
\]

\[= \text{CYK}_{a:bc}[Q[Y]], \]

where both \( P^1_{a:bc}[C] = \Box C_{a:bc} \) and \( Q[Y] = \rho^1 \circ \text{CYK}[Y] \) are generalized normally hyperbolic in the required way (Theorem 3.4), while the \( P^2 \) and \( \tilde{P}^3 \) terms have been rewritten as first order operators on \( K[\delta] \) and \( K[*dY] \).
Before proceeding, let us specialize identity (52) to our choice of dimension and parameters \((n = 4, y_1 = 1)\), which shows that the \(K_{ab}[\delta Y]\) propagates in a way compatible with our choice of \(Q[Y]\):

\[
(\Box + \Lambda)(\delta Y)_a + \frac{1}{2} R_{a \ b c d} CYK_{b c d}[Y] = \frac{1}{2} (\delta Q[Y])_a,
\]

where \((R \cdot CYK[Y])_a = R_{a \ b c d} CYK_{b c d}\). Similarly, we must verify that \(K_{ab}[*dY]\) also propagates in a way that is compatible with \(Q[Y]\). A direct calculation shows that

\[
(\Box + \Lambda)(*dY)_a + R_{a \ b c d} CYK_{b c d}[Y] = \frac{1}{2} (*dQ[Y])_a.
\]

Alternatively, substituting \(Y \mapsto \ast Y\) into (70), we immediately get (71), after using the following helpful identities:

\[
(\delta \ast Y)_a = \frac{1}{2} (*dY)_a; \\
CYK_{a \ b c}[\ast Y] = \ast CYK_{a \ b c}[Y], \\
Q_{a b}[\ast Y] = (\ast Q[Y])_{a b}.
\]

Thus, rewriting the Killing equation propagation identity (7) adapted to our situation, we get

\[
\Box K_{a b}[\delta Y] - R_{c \ a b}^e dy^d S_{c d}[CYK[Y]] + \frac{1}{2} K_{a b}[R \cdot CYK[Y]] = \frac{1}{2} K_{a b}[\delta Q[Y]],
\]

\[
\Box K_{a b}[\ast dY] - 2R_{c \ a b}^e dy^d \ast S_{c d}[\ast CYK[Y]] + K_{a b}[R \cdot \ast CYK[Y]] = \frac{1}{2} K_{a b}[\ast dQ[Y]]
\]

where again \((R \cdot C)_a = R_{a \ b c d} C_{b c d}\).

**Theorem 4.1.** Under the same hypotheses as Theorem 3.1, but for \(n = 4\), there exists the following 2-parameter family of 4th order propagation identities of the form

\[
\Box (3P^1 + (y_2 - 3)\frac{1}{2} P^2 + (y_3 - 1)\frac{1}{2} \bar{P}^3) \circ CYK + l.o.t \\
= (\Box \sigma^1 + \frac{1}{2} y_2 \bar{P} \circ K \circ \delta + \frac{1}{2} y_3 P \circ K \circ \ast d) \circ \rho^1 \circ CYK,
\]

where \(l.o.t\) stands for operators of differential order three or lower acting on the \(CYK\) operator and \(y_2, y_3\) are free parameters. The necessary and sufficient conditions for the generalized normal hyperbolicity of the corresponding \(P\) operator are

\[y_3 \neq 0, \quad y_2 \neq 0, \quad \text{and} \quad y_3 \neq -2.\]

and the corresponding \(Q = \rho^1 \circ CYK\) operator is always normally-hyperbolic (being independent of the free parameters).

The proof is directly analogous to that of Theorem 3.4.
Proof. The first step is to apply the wave operator $\Box$ to both sides of the propagation identity (68). Then, note that we are completely free to do the following rewriting:

$$\Box(\Box^{-3/2})P^2 \circ \text{CYK} = \Box(y_2 - 3)\frac{1}{2}P^2 \circ \text{CYK} - y_2 \Box P \circ K \circ \delta \tag{79}$$

$$= \Box(y_2 - 3)\frac{1}{2}P^2 \circ \text{CYK} - y_2 \Box P \circ \Box K \circ \delta + \text{l.o.t.}$$

$$\Box(\Box^{-1/2})P^3 \circ \text{CYK} = \Box(y_3 - 1)\frac{1}{2}P^3 \circ \text{CYK} - \frac{3}{4}y_3 \Box \circ K \circ \circ \delta$$

$$= \Box(y_3 - 1)\frac{1}{2}P^3 \circ \text{CYK} - \frac{3}{4}y_3 \Box P \circ \Box K \circ \circ \circ \delta + \text{l.o.t.} \tag{80}$$

Finally, using (75) and (76) to eliminate $\Box K \circ \circ \circ \delta$ and $\Box K \circ \circ \circ \delta$ from the above formulas, we arrive directly at the desired propagation identity (77). Recalling the relevant hyperbolicity conditions from Lemmas B.1 and B.2 (the latter lemma is adapted by setting $x = y = 0$ and $w = 1$ to decouple the $d$ and CYK operators, while the translation from the $\Box^3$ to the $\Box^3$ operator is done by comparing (65) and (68)), which are unchanged when the operators contributing to the principal symbol are multiplied by a power of $\Box$, we get the corresponding inequalities (78).

Next we use the previous propagation equations to construct conformal Killing-Yano initial data (CYKID) in dimension four.

**Theorem 4.2.** Consider a globally hyperbolic Einstein $\Lambda$-vacuum Lorentzian manifold, $(M, g)$ of dimension $n = 4$, with $R_{ab} = \Lambda g_{ab}$, and a Cauchy hypersurface $\Sigma \subset M$. The necessary and sufficient conditions yielding a set of conformal Killing-Yano initial data (CYKID) for $Y_{ab}$ on $\Sigma$ are the following equations, where we indicate the provenance of each of them. Each equality holds modulo the ones preceding it.

$$\frac{3}{4} \text{CYK}_{(A:B)}[Y]_\Sigma = 0: \; \Box \; \text{part of \; (43a) \; for \; } n = 4, \; \text{or}$$

$$\Box \Box K[Y]_{AB} - 2\pi_{(A} C Y_{B)C} = 0, \tag{81a}$$

$$\text{CYK}_{A:BC}[Y]_\Sigma = 0: \; \Box \; \text{part of \; (43b) \; for \; } n = 4, \; \text{or}$$

$$\Box \text{CYK}[Y]_{A:BC} - 6\pi_{A[B} Y_{0C]} - 3g_{A[B} \pi_{C]} E Y_{6E} + 3\pi g_{A[B} Y_{0C]} = 0. \tag{81b}$$

$$\frac{1}{6} \nabla_0 \text{CYK}_{(A:B)}[Y]_\Sigma = 0: \; \text{(43c) \; for \; } n = 4, \; \text{or}$$

$$(D_{C} \pi_{AB} - D_{(A \pi_{B)C})} Y^{C}_{0}) + \pi_{(A} C Y_{B)C} + \pi_{(A} C \pi_{C} D Y_{D]} + r_{(A} C Y_{B)C} = 0, \tag{81c}$$

$$-\frac{1}{6} \nabla_0 \text{CYK}_{A:BC}[Y]_\Sigma = 0: \; \text{(43d) \; for \; } n = 4, \; \text{or}$$

$$(D_{C} \pi_{B]} E) Y_{AE} + (D_{C} \pi_{A} E) Y_{B]E}$$

$$+ (D_{E} \pi_{A[B} Y_{C]E} + \frac{1}{2} (r_{BCA} E + 2\pi_{A[B} \pi_{C]} E) Y_{E0}$$

$$+ \left(r_{A[B - \pi_{AE} E} B + \pi_{A[B - \frac{2\Lambda}{n - 2} g_{A[B}} \right) Y_{C]0} = 0. \tag{81d}$$
\[ \nabla_0 K_{AB}[\delta Y]_{|\Sigma} = 0: \]

\[
D_A D_B v_0 + (2(\pi \cdot \pi)_{AB} - \pi \pi_{AB} - \tau_{AB}) v_0 \\
- 2\pi_{(B} C_{D_A)} v_C - (D^C \pi_{AB}) v_C + 2\Lambda g_{AB} v_0 = 0,
\]

\( (81e) \)

\[ \nabla_0 K_{AB}[*dY]_{|\Sigma} = 0: \]

\[
D_A D_B(*w)_0 + (2(\pi \cdot \pi)_{AB} - \pi \pi_{AB} - \tau_{AB})(*w)_0 \\
- 2\pi_{(B} C_{D_A)}(*w)_C - (D^C \pi_{AB})(*w)_C + 2\Lambda g_{AB}(*w)_0 = 0,
\]

\( (81f) \)

where we have used the same spatial differential operators \( \nabla K \) and \( \nabla Y \) as in the proof of Theorem 3.3, and the components \( v_0, v_A \) of \( v_a = (\delta Y)_a \) are the same as in (58), while the \( (*w)_0, (*w)_A \) components of \( (w)_a = (*dY)_a \) are

\[ (*w)_0 := \varepsilon_{ABC} D^C Y^{AB}, \quad (*w)_A := \varepsilon_{ABC} (\pi^{DB} Y_D^C + 3D^B Y^C_0), \]

\( (82) \)

with \( \varepsilon_{ABC} := \eta_{0ABC}. \)

The proof is directly analogous to that of Theorem 3.5.

**Proof.** To construct the CYKID conditions, it is sufficient to, once again, apply Proposition 2.1 to the 4th order propagation identity (77) for the \( K[Y] \) operator obtained in Theorem 4.1. But, more practically, following the logic explained in the proof of Theorem 3.5, since the 4th order identity was obtained by combining compatible second order propagation identities for the \( K[Y] \), \( K[\delta Y] \) and \( K[*dY] \) operators, the following initial data conditions are sufficient:

\[ \nabla_0^k CYK[Y]_{|\Sigma} = 0, \]

\( (83a) \)

\[ \nabla_0^k K[\delta Y]_{|\Sigma} = 0, \]

\( (83b) \)

\[ \nabla_0^k K[*dY]_{|\Sigma} = 0, \]

\( (83c) \)

for \( k = 0, 1. \)

We can get (81a) and (81b) by computing \( CYK_{(A:B)|0}[Y] \) and \( CYK_{A:BC}[Y] \) from the formulas (44), or just take the appropriately symmetrized projections of (83a) and (83b), specialized to \( n = 4 \), as indicated in the theorem. The remaining components allow us to systematically eliminate any \( \nabla_0 \) derivatives of \( Y_A0 \) and \( Y_{AB} \) in the remaining calculations. The computation of \( \nabla_0 CYK_{(A:B)|0}[Y] \) and \( \nabla_0 CYK_{A:BC}[Y] \) follows the same logic as in the proof of Theorem 3.3, producing (81c) and (81d) as indicated in the theorem.

Again following the logic of the proof of Theorem 3.5, splitting the identities (48) and (62) and systematically eliminating all \( \nabla_0 \) derivatives of \( Y_A0 \) and \( Y_{AB} \), allows us to write

\[ K_{00}[\delta Y] = O(CYK[Y]), \quad K_{A0}[\delta Y] = O(CYK[Y]), \]

\[ K_{AB}[\delta Y] = -\nabla_0 CYK_{(A:B)|0}[Y] + O(CYK[Y]), \]

\( (84) \)
\[
K_{00}[\ast dY] = O(CYK[Y]), \quad K_{A0}[\ast dY] = O(CYK[Y]),
K_{AB}[\ast dY] = -6\varepsilon^{CDE}g_{C(A}\nabla_0 CYK[Y]_{B);DE} + O(CYK[Y]),
\]

where, as usual, \(O(\cdot)\) denotes linear dependence on the argument and any of its spatial derivatives. Hence, setting any of the above expressions to zero does not add any new independent initial data conditions. Obviously, the same will be true of \(\nabla_0 K_{00}[\delta Y], \nabla_0 K_{A0}[\delta Y]\), and \(\nabla_0 K_{00}[\ast dY]\) and \(\nabla_0 K_{A0}[\ast dY]\).

It remains only to compute the initial data conditions from \(\nabla_0 K_{AB}[\delta Y]\) and \(\nabla_0 K_{AB}[\ast dY]\). Again, as in the proof of Theorem 3.5, we know that it would be sufficient to plug into the second KID condition (20b) the vectors \(v_a = (\delta Y)_a\) and \((\ast w)_a = (\ast dY)_a\), whose split components, with \(\nabla_0\) derivatives eliminated, are by direct computation given by (58) and (82) respectively. The result gives us the remaining CYKID conditions (81e) and (81f). The resulting expressions are third order spatial differential operators on \(Y_{A0}\) and \(Y_{AB}\). We can reduce them to first order differential operators by simply applying \(\nabla_0\) to (84) and (85), respectively substituting (81c) and (81d) for \(\nabla_0 CYK_{(A:B)0}[Y]\) and \(\nabla_0 CYK_{A:BC}[Y]\), and systematically eliminating \(\nabla_0 Y_{A0}\) and \(\nabla_0 Y_{AB}\). However the resulting expressions become rather long and unenlightening, so we omit them.

Remark 3.4 applies equally well to check the finite type property of the 4-dimensional CYKID conditions (81); it is sufficient to look at the symbols of (81a) and (81b).

5 Discussion

We derived a set of necessary and sufficient conditions (the cCYKID equations) ensuring that a \(\Lambda\)-vacuum initial data set for the Einstein equations admits a closed conformal Killing-Yano 2-form (Theorem 3.3 in dimensions \(n > 4\), and Theorem 3.5 for \(n = 4\)) or, in the special dimension \(n = 4\), just a conformal Killing-Yano 2-form (Theorem 4.2). These initial data equations include both differential conditions on the spatial components \(Y_{A0}\) and \(Y_{AB}\) of the spacetime 2-form \(Y_{ab}\), as well as purely algebraic conditions that involve the intrinsic and extrinsic geometry of the initial data surface. While these results are special to Lorentzian signature, the propagation identities (Theorems 3.1, 3.4 and 4.1) that we have used to derive the initial data conditions are fully covariant and hence remain valid in any pseudo-Riemannian signature. The method that we have used to arrive at Theorem 3.1 is a representation-theoretic exhaustive search based on covariance and fixed total degree of various differential operators. In fact, we have shown that, in dimensions \(n > 4\) (excluding \(n = 6\)), the result is guaranteed to be the most general one. As a result, we have also concluded (Corollary 3.2) that there does not exist a second order covariant propagation identity for non-closed CYK 2-forms.

As we have indicated in the Introduction, the ability to describe \(\Lambda\)-vacuum Einstein initial data giving rise to a cCYK (or, for \(n = 4\), also CYK) 2-form can improve the initial data characterization given in [17] of the Kerr rotating black
hole solution. In higher dimensions, the same idea could be used to give the first initial data characterizations of members of the Kerr-NUT-(A)dS rotating black holes. It would be interesting to explore these possibilities in future work. As discussed in Remark 2.2 these prospective results admit the interpretation of necessary and sufficient conditions for the existence of isometric embeddings of Riemannian manifolds in the corresponding ambient spacetimes. Moreover, a general formulation of the non-linear stability problem for the Kerr-NUT-(A)dS rotating black holes must include the idea of general vacuum initial data close in some topology to general Kerr-NUT-(A)dS data, constructed from the cCYKID initial data characterization of Theorem 4.2.

At the moment, no propagation identity is known for the higher dimensional CYK 2-forms ($n > 4$) or for higher rank cCYK $p$-forms ($p > 2$). It would be interesting to study these equations using the approaches used in this work: representation-theoretic exhaustive search and clever decoupling of independently propagated integrability conditions.

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A  Tensor bases

In this appendix, we list several sequences of tensor-valued covariant differential operators which, according to the representation-theoretic discussion in Section 3 span the space of operators of a certain total order and tensor type. The notation for these operators follows tables preceding Remark 3.2. We presume throughout that the cosmological vacuum Einstein equations hold, $R_{ab} = \frac{2\Lambda}{n} g_{ab}$, so that the Ricci tensor never appears in the formulas below.

Remark A.1. In the representation-theoretic discussion of Section 3, we noted that, being traceless, the Weyl tensor $W_{abcd}$ is precisely of representation type $\mathbf{H}$. The Riemann tensor $R_{abcd}$ has all the same symmetries, but is not traceless, due to a possibly non-vanishing cosmological constant $\Lambda$. Thus, following strict representation-theoretic logic, we should write all terms involving $W_{abcd}$ and $\Lambda$ separately, like so:

$$c_1 O(W_{abcd}) + c_2 O(\Lambda) + \cdots. \quad (86)$$

However, expressing $W_{abcd}$ in terms $R_{abcd}$, $g_{ab}$ and $\Lambda$, any such expression becomes

$$c_1 O(R_{abcd}) + c_2 O(\Lambda) + \cdots, \quad (87)$$

where $c_1$ stays the same but $c_2$ may now be different. Since for the purposes of computer algebra it is more economical to work directly with $R_{abcd}$, rather than $W_{abcd}$, we choose to work with the coefficients $c_1$ and $c_2$ in the second formulation directly.
The following is a basis of the possible second total order covariant differential operators of type \( B Y \rightarrow B Q \):

\[
Q_{ab}^1[Y] = \Box Y_{ab},
\]

\[
Q_{ab}^2[Y] = 2 \nabla_a \nabla^d Y_{bd} = -(d \delta Y)_{ab},
\]

\[
Q_{ab}^3[Y] = R_{ab}^{\;\;de} Y_{de},
\]

\[
Q_{ab}^4[Y] = \Lambda Y_{ab}.
\]

The following are bases of the possible second total order covariant differential operators of type \( B P \rightarrow B P \) and \( B \Xi \rightarrow B P \):

\[
P_{abc;[C]} = \Box C_{abc},
\]

\[
P_{abc;[C]} = \nabla_b \nabla^d (C_{acde} + C_{cda}) - \nabla_c \nabla^d (C_{abde} + C_{bdae}) - \frac{2}{n-1} g_{ab[bc} \nabla^e \nabla^f C_{(ef)c]d} \right) + \frac{3}{2(n-1)} g_{ab[bc} R_{cd]ef} C_{de],}
\]

\[
P_{abc;[C]} = 2 \nabla_a \nabla^d C_{d xcb} - \nabla_b \nabla^d C_{d ac} + \nabla_c \nabla^d C_{d ab} + \frac{6}{n-1} g_{ab[bc} \nabla^e \nabla^f C_{(ef)c]d} \right) + \frac{3}{2(n-1)} g_{ab[bc} R_{cd]ef} C_{de],}
\]

\[
P_{abc;[C]} = 2 R_{bc}^{\;\;ef} C_{a;ef} - R_{ca}^{\;\;ef} C_{b;ef} + R_{ba}^{\;\;ef} C_{c;ef} + \frac{6}{(n-1)} g_{ab[bc} R_{de]}^{\;\;ef} C_{d ef],}
\]

\[
P_{abc;[C]} = R_a^{\;\;e f} (C_{c;ef} + C_{f;ac}) - R_a^{\;\;e f} (C_{c;bf} + C_{f;bc}) + \frac{3}{(n-1)} g_{ab[bc} R_{de]}^{\;\;ef} C_{d ef],}
\]

\[
P_{abc;[C]} = \Lambda C_{abc},
\]

\[
P_{abc;[\Xi]} = CYK_{abc;[\delta \Xi]},
\]

\[
P_{abc;[\Xi]} = \frac{1}{6} (2 R_{bc}^{\;\;de} \Xi_{ade} - R_{ab}^{\;\;de} \Xi_{cde} + R_{ac}^{\;\;de} \Xi_{bde}).
\]

The following are bases of the possible second total order covariant differential operators of type \( B \Xi \rightarrow B \Pi \) and \( B P \rightarrow B \Pi \):

\[
\hat{P}_{abc;[\Xi]} = \Box \Xi_{abc},
\]

\[
\hat{P}_{abc;[\Xi]} = -(d \delta \Xi)_{abc},
\]

\[
\hat{P}_{abc;[\Xi]} = R_{ab}^{\;\;de} \Xi_{cde} + R_{bc}^{\;\;de} \Xi_{ade} + R_{ca}^{\;\;de} \Xi_{bde},
\]

\[
\hat{P}_{abc;[\Xi]} = \Lambda \Xi_{abc},
\]

\[
\hat{P}_{abc;[C]} = \nabla_a \nabla^d C_{d xcb} + \nabla_b \nabla^d C_{d ac} + \nabla_c \nabla^d C_{d ab},
\]

\[
\hat{P}_{abc;[C]} = R_{ab}^{\;\;de} C_{cde} + R_{bc}^{\;\;de} C_{ade} + R_{ca}^{\;\;de} C_{bde}.
\]

The following is a basis of the possible third total order covariant differential
operators of type $\mathcal{B}Y \to \mathcal{B}T$:
\[
T^1_{abc}[Y] = \Box CYK_{abc}[Y],
\]
\[
T^2_{abc}[Y] = \nabla_{(a} \nabla_{b)}(\delta Y)_{c} - \nabla_{(a} \nabla_{c)}(\delta Y)_{b} - \frac{2\Lambda}{(n-1)(n-2)}g_{[a\beta}[\Box(\delta Y)_{c]},
\]
\[
T^3_{abc}[Y] = 2\Lambda_{\beta\gamma}^{\alpha\delta\epsilon\zeta}CYK_{\alpha\delta\epsilon\beta\gamma}[Y] - R_{\gamma a}^{\epsilon\delta\epsilon\zeta}CYK_{\beta\epsilon\beta\gamma}[Y] + R_{\beta a}^{\epsilon\gamma\epsilon\zeta}CYK_{\gamma\beta\gamma}[Y]
+ \frac{6}{(n-1)}g_{[a\beta}[R_{\gamma\delta\epsilon\zeta}]^{\alpha\delta\epsilon\gamma}[Y],
\]
\[
T^4_{abc}[Y] = 2\Lambda_{\beta\gamma}^{\alpha\delta\epsilon\zeta}CYK_{\alpha\delta\epsilon\beta\gamma}[Y] - 2\Lambda_{\beta\gamma}^{\alpha\delta\epsilon\zeta}CYK_{\gamma\beta\gamma}[Y]
+ \frac{3}{(n-1)}g_{[a\beta}[R_{\gamma\delta\epsilon\zeta}]^{\alpha\delta\epsilon\gamma}[Y],
\]
\[
T^5_{abc}[Y] = \frac{2}{(n-1)}R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{ade} - \frac{1}{6}R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{cde} + \frac{1}{6}R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{bde},
\]
\[
T^6_{abc}[Y] = -\frac{1}{(n-1)}R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{d} + \frac{4\Lambda}{(n-1)^2(n-2)}g_{[a\beta}[\Box(\delta Y)_{c}],
\]
\[
T^7_{abc}[Y] = \nabla_{\gamma}R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{d},
\]
\[
T^8_{abc}[Y] = \Lambda CYK_{\alpha\beta\gamma}[Y].
\]

The following is a basis of the possible third total order covariant differential operators of type $\mathcal{B}Y \to \mathcal{B}T'$:
\[
\hat{T}^1_{abc}[Y] = \Box(dY)_{abc},
\]
\[
\hat{T}^2_{abc}[Y] = R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{cde} + R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{ade} + R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{bde},
\]
\[
\hat{T}^3_{abc}[Y] = R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{d} + R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{cde} + R_{\beta\gamma}^{\alpha\delta\epsilon\zeta}C_{\delta\epsilon\gamma\beta}[Y]_{ade},
\]
\[
\hat{T}^4_{abc}[Y] = \Lambda(dY)_{abc}.
\]

### B Generalized normal-hyperbolicity

In this appendix, we find the necessary and sufficient conditions for which the generic $P$ and $Q$ operators from identity (30) are generalized normally hyperbolic. Our strategy is to first pick a differential order, say $k$, and to parametrize the most general covariant ansatz for the principal symbols of the potential adjugate operators $P'$ and $Q'$ at that order. Then one can check whether the adjugate identity (3) could be satisfied at that order.

Before proceeding, in addition to the second order $P^i$, $\hat{P}^i$ and $Q^i$ operators introduced in Section 3 and explicitly defined in Appendix A, we also need to define a fourth order operator
\[
P^9_{abc}[C] = \nabla^{(e\gamma\delta\epsilon\zeta)}(\nabla_{(a} \nabla_{b)}C_{\epsilon\gamma\delta\epsilon\zeta} - \nabla_{(a} \nabla_{c)}C_{\epsilon\gamma\delta\epsilon\zeta})
+ \frac{1}{n-1}\Box(d\gamma\delta\epsilon\zeta)(g_{ab}C_{d\epsilon\zeta} - g_{ac}C_{d\epsilon\zeta}).
\]
Since \( P \) and \( Q \) are of second order, they act (by pre-composition and up to lower-order terms) as a linear map between the spaces of principal symbols of order \( k \) and \( k+2 \). As can be seen from the following tensor product decomposition table (cf. Remark 3.2 and the explanations of the tables in Section 3)

|     | \( \Box \) | \( \nabla^2 \) | \( \nabla^4 \) | \( \nabla^6 \) | \( \nabla^8 \) | \( \ldots \) |
|-----|-------------|-------------|-------------|-------------|-------------|-----------|
| \( Y_{ab} \) | \( \Box Q^1 \) | \( \Box Q^2 \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( C_{abc} \) | \( \Box P^1 \) | \( \Box P^2 + \Box P^3 \) | \( \Box P^5 \) | \( \Box P^9 \) | \( \ldots \) | \( \ldots \) |
| \( \Xi_{abc} \) | \( \Box P^1 \) | \( \Box P^2 + \Box P^7 \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |

there is an order \( k \) (\( k = 2 \) for \( Q \), and \( k = 4 \) for \( P \)) after which there are essentially no new principal symbols, meaning that for \( k' \geq k \) all operators with independent principal symbols of order \( k' + 2 \) can be obtained by acting with \( \Box \) on the operators of order \( k' \). Therefore, starting at order \( k \), the pre-composition action of say \( P \) on the space of potential adjugate operators \( P' \) can be represented by a square matrix. If this matrix is invertible, then the adjugate identity (3) can be satisfied, which shows generalized normal hyperbolicity of \( P \). If this matrix is singular, then it has a right null-vector, which parametrizes an operator \( P' \) such that \( P' \circ P = 0 + \text{l.o.t.} \). But then, by Lemma 2.2, \( P \) cannot be generalized normally hyperbolic. The same argument works for \( Q \).

In the next two Lemmas, we record the results of these calculations for \( P \) and \( Q \) from (30).

**Lemma B.1.** An operator of the form

\[
Q_{ab}[Y] = sQ_{ab}[Y] + (s - t)Q^2[Y]_{ab} + \text{l.o.t},
\]

is generalized normally hyperbolic iff

\[
s \neq 0, \quad t \neq 0,
\]

due to the adjugate identity

\[
\left[ \frac{1}{s} Q^1 + \left( \frac{1}{s} - \frac{1}{t} \right) Q^2 \right] \circ Q = \Box^2 + \text{l.o.t.}
\]

**Lemma B.2.** An operator of the form

\[
P = \begin{bmatrix}
u P^1 + (v - u)\frac{1}{2} P^2 + u \frac{1}{2} P^3 & 0 & 0 \\
0 & q \hat{P}^1 + q \frac{1}{2} \hat{P}^2 & x P^7 \\
u Y P^5 & y \hat{P}^5 & \frac{1}{2} P^2
\end{bmatrix}
\]

is generalized normally hyperbolic iff

\[
u \neq 0, \quad v \neq 0, \quad q \neq 0,
\]

\[
xy - wz \neq 0, \quad \text{and} \quad (n - 2)(6w - v + 2u) - 2u \neq 0,
\]

\[32\]
due to the adjugate identity
\[ P' \circ P = \begin{bmatrix} \Box^3 & 0 \\ 0 & \Box^3 \end{bmatrix} + l.o.t \] (123)
for
\[ P' = \begin{bmatrix} \Box (u'P^1 + (v' - u')^2 P^2 + u' \frac{1}{3} P^3) + (p' - \frac{2}{3} w') P^9 \\ 0 \end{bmatrix} \begin{bmatrix} \Box^3 \hat{P}^1 + \hat{q} \frac{1}{2} \hat{P}^2 \\ \Box \end{bmatrix} \]
\[ + \Box \begin{bmatrix} w' \frac{1}{3} P^3 \\ y' \frac{2}{3} \hat{P}^5 \\ z' \frac{1}{2} \hat{P}^2 \end{bmatrix}, \] (124)
with
\[ u' = \frac{1}{u}, \quad v' = \frac{1}{v}, \quad q' = \frac{1}{q}, \] (125)
\[ \begin{bmatrix} w' \\ x' \\ y' \\ z' \end{bmatrix} = \frac{1}{xy - wz} \begin{bmatrix} -z \\ x \\ y \\ -w \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}^{-1}, \] (126)
\[ p' = \frac{(n - 3)(2(u - v)^2 + 3(uw + 2uw - 4vw)) + 3u(5v + 2w)}{2uv[(n - 2)(6w - v + 2u) - 2u]} \] (127)

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