The Capture of Particles by Chaotic Resonances During Orbital Migration

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ABSTRACT

Because a chaotic zone can reduce the long timescale capture probabilities and cause catastrophic events such as close encounters with a planet or star during temporary capture, the dynamics of migrating planets is likely to be strongly dependent on the widths of the chaotic zones in their resonances. Previous theoretical work on the resonant capture of particles into mean-motion resonances by orbital migration has been restricted to the study of integrable models. By exploring toy 2 and 4 dimensional drifting Hamiltonian models we illustrate how the structure in phase space of a resonance can be used to generalize this integrable capture theory to include the richer phenomenology of a chaotic resonance. We show that particles are temporarily captured into the chaotic zone of a resonance with fixed shape and width for a time that is approximately given by the width of the chaotic zone divided by the resonance drift rate. Particles can be permanently captured into a drifting chaotic resonance only if they are captured into a growing non-stochastic region. Therefore resonances containing wide chaotic zones have lower permanent capture probabilities than those lacking chaotic zones. We expect large deviations from the predictions of integrable capture theories when the chaotic zone is large and the migration rate is sufficiently long that many Lyapunov times pass while particles are temporarily captured in the resonance.

The 2:1 mean-motion resonance with Neptune in the Kuiper Belt contains a chaotic zone, even when integrated on fairly short timescales such as a million years. Because of the chaotic zone, the capture probability is lower than estimated previously from drifting integrable models. This may offer an explanation for low eccentricity Kuiper Belt objects between 45-47 AU which should have been previously captured in the 2:1 resonance by Neptune’s migration.

1. Introduction

Scenarios incorporating the orbital migration of giant planets have been proposed to explain the orbit of Pluto and the eccentricity distribution of Kuiper Belt Objects (Malhotra 1995), as well as the small orbital semi-major axes of many of the newly discovered extra-solar planets (Murray et al. 1998). A theory of resonant capture exists for well defined adiabatically varying non-chaotic integrable resonant systems (similar to pendulums, Yoder 1979, Henrard 1982, Henrard & Lemaitre 1983). However, it now known that due to multiple resonance overlaps some of the
mean-motion resonances in the solar system are in fact chaotic (Wisdom 1985; Holman & Murray 1996; Murray, Holman & Potter 1998). From their numerical integration, Tittemore & Wisdom (1990) and Dermott, Malhotra & Murray (1988) found that the chaotic nature of the resonances did influence the capture probabilities of resonances driving the evolution of the Uranian satellites. Tittemore & Wisdom (1990) showed that the Uranian satellites were temporarily captured into the chaotic zone associated with the separatrix of a resonance. Because chaotic resonances will always temporarily capture particles into their chaotic zones, the dynamics is fundamentally different than that of integrable resonances where no capture takes place unless the phase space volume of the resonance grows. Previous theoretical work on the capture process during orbital migration has been restricted to integrable models (e.g., Henrard 1982, Malhotra 1988, Borderies & Goldreich 1984). In this paper, by numerically investigating toy Hamiltonian models, we explore the general problem of capture by chaotic resonances.

In our previous work we investigated numerically the affect of an orbiting giant planet on planetesimals interior to the planet (Quillen & Holman 2000). We showed that the strong mean-motion resonances captured particles and caused catastrophic events such as ejection by the planet or an impact with the central star. We proposed that because impacts can enrich the metallicity of a star at a time when the star is no longer fully convective, the migration process offers an explanation for the high metallicities of stars with planets discovered via radial velocity searches. The theory of adiabatically varying integrable systems (e.g., Henrard 1982; Malhotra 1988; Borderies & Goldreich 1984; Murray & Dermott 1999) predicts permanent capture probabilities for integrable mean-motion resonances. However from our simulation we found that particles were often captured only for short periods of time. This was particularly evident in the simulations with slow migration rates. We can try to understand the problem with the understanding that the strong drifting resonances in the problem contain large chaotic zones. As we show below, when there is a chaotic zone, the probability of temporary capture into the resonance is 100%. Because a chaotic zone is incapable of permanently capturing particles in a drifting resonance, we expect that the probability of permanent capture drops as the width of the chaotic zone increases. We expect that the distribution of temporarily capture times and the probability of permanent capture depends on the width and variation of the chaotic zone, the drift rate of the planet, and the distribution of diffusion timescales in the zone.

2. The Forced Pendulum Analogy

These phenomena can be illustrated simply with a mathematical model for the forced pendulum. This is a simple two dimensional Hamiltonian system which exhibits the chaos caused by resonant overlap (e.g., Holman & Murray 1996). The Hamiltonian is

\[
H(P, \psi) = \frac{1}{2} P^2 + K \left[ 1 + a \cos(\nu t) \right] \cos(\psi)
\] (1)
\begin{equation}
H(P, \psi) = \frac{1}{2} P^2 + K \cos(\psi) + \frac{K a}{2} \left[ \cos(\psi + \nu t) + \cos(\psi - \nu t) \right]
\end{equation}

and has resonances at \( p = 0, \pm \nu \) with \( \psi = \pi \). To drift the resonance we modify the Hamiltonian

\begin{equation}
H(P, \psi) = \frac{1}{2} P^2 + K \left[ 1 + a \cos(\nu t) \right] \cos(\psi) + bP.
\end{equation}

For the resonance to migrate or drift, \( b \) must be a function of \( t \). We can allow the resonance to grow without changing its shape if we set \( \dot{a} = 0 \) and \( \dot{K} > 0 \).

In Figure 1 we show the result of drifting two different systems, one with a large chaotic zone and the other without. We numerically integrated the above Hamiltonian using a conventional Burlisch-Stoer numerical scheme. Both numerical integrations pertain to systems with constant resonant widths and shapes \((\dot{K}, \dot{a} = 0)\). Since the size and shape of the resonance does not change with time, the capture probability predicted for a non-chaotic resonance is zero (e.g., Henrard 1982). However when the chaotic zone is large, particles can spend a significant amount of time in the chaotic zone associated with the separatrix before passing to the other side of the resonance. We can define an effective width for the chaotic zone as \( \Delta P_z = \frac{V_z}{2\pi} \) where \( V_z \) is the volume in phase space of the chaotic zone connected to the separatrix. Particles spend different time periods temporarily captured in the chaotic zone. The distribution of temporary capture times has a lower edge at nearly zero time in the resonance. The mean length of time captured into the resonance is

\begin{equation}
\Delta T_c = \Delta P_z / |\dot{b}|
\end{equation}

when the shape of the resonance is held fixed.

Because the chaotic zone may contain regions with different diffusion timescales, we expect that the final particle distribution will have a dependence on drift rate. To explore this we performed integrations for 3 different drift rates for the system shown on the right hand side of Figure 1. The final particle probability distributions are shown in Figure 2. For the faster drift rates, the final particle distribution is nearly flat, however for the slower drift rates, the distribution is more triangular and has a longer tail. In the longer integrations we expect that particles can be trapped in regions with longer diffusion timescales.

2.1. The Probability of Capture

As shown by Yoder 1979 and Henrard 1982, we can estimate the capture probability for an adiabatically drifting resonance by considering the volume of phase space per unit time which is passed by the separatrices of the resonance. We are referring to a system with 2 separatrices (the non-chaotic example shown in Figure 1). For \( P_+(\psi, t) \) and \( P_- (\psi, t) \) the momenta of the upper and lower separatrices as a function of angle and time, the rate of volume swept by the upper separatrix is \( B_+ \) where \( B_+ \equiv \int_0^{2\pi} P_+(\psi, t) d\psi \). The growth rate of phase space volume in the resonance is \( B_+ - B_- \), where \( B_- \) is the corresponding expression for the lower separatrix. Particles
swept up by the resonance must either be captured or ejected. The capture rate depends on the ratio of volume increase in the resonance to that swept up by the resonance and so is given by

\[ P_c = \frac{(B_+ - B_-)}{B_+}. \] (4)

This is shown with more rigor by Henrard 1982 for the drifting pendulum (Hamiltonian in equation (2) restricted to \( a = 0 \)). The probability of capture

\[ P_c = \begin{cases} f & \text{if } 0 < f < 1; \\ 1 & \text{if } f \geq 1 \end{cases} \] (5)

where

\[ f = \frac{2}{1 - \frac{b}{K} \sqrt{K}}. \] (6)

We have assumed that particles start at large \( P \) and the growing resonance \( \dot{K} > 0 \) drifts upwards \( \dot{b} < 0 \). If the resonance shrinks, \( \dot{K} \leq 0 \), then the permanent capture probability is zero. We have computed permanent capture probabilities for a series of systems with different sized chaotic zones by choosing different values for \( a \) in each system. Since we do not allow \( a \) to vary in each individual simulation, the shape (not size) of the resonance remains the same while the resonance drifts. For these simulations we set \( \dot{K} \) such that when \( a = 0 \) the probability of capture is \( P_c = 1 \). We see in Figure 3 that the permanent capture probability of the resonance drops when it has an decreasing volume fraction covered by integrable motion or stable islands. The fraction of the resonance covered by islands drops exponentially as \( a \to 0 \) so the capture probability exponentially approaches 1 for small \( a \).

When the resonance contains a chaotic separatrix, we expect its permanent capture probability to differ from that of a resonance of similar shape lacking a chaotic separatrix. In the limit of an entirely chaotic resonance, the resonance cannot permanently capture particles unless the resonance width grows faster than the drift rate and the resonance is effectively stationary. A drifting resonance requires a stable, integrable, non-chaotic, growing island to capture particles. We expect the capture probability to be given by

\[ P_c = \frac{\dot{V}_i}{B_+}. \] (7)

where \( \dot{V}_i \) is the growth rate of the island or islands of non-chaotic phase space volume, and \( B_+ \) is the rate that particles are swept into the chaotic zone of the resonance. When the volume of the chaotic zone shrinks to zero we recover the formalism of the integrable model.

2.2. Escape from the 2:1 resonance in the Kuiper Belt

Malhotra 1995 proposed that Neptune’s outwards migration was responsible for the capture of Pluto and other Kuiper belt objects into the 3:2 and 2:1 mean-motion resonances with Neptune.
Because the capture probability predicted with the integrable formalism is 1 for low eccentricity objects, particles are not expected to escape the 2:1 resonance. The existence of low eccentricity objects between 45-47 AU, just within the 2:1 resonance, has posed a challenge to explain. Nevertheless, the numerical simulations of the migrating major planets by [Malhotra 1995] and [Hahn & Malhotra 1999] showed that particles with initially low eccentricity could pass through this resonance. [Hahn & Malhotra 1999] suggested that the stochastic migration rate of Neptune seen in their simulations might result in particles leaving the 2:1 resonance, however the simulation with a smooth migration by [Malhotra 1995] still showed that low eccentricity particles could avoid capture. Numerical integrations have shown that on long timescales (4 Gyrs) the 2:1 is possibly entirely chaotic (Malhotra 2000). Based on our understanding of our toy problems we may be able to offer an alternative explanation for passage of particles through the 2:1 resonance even when Neptune’s migration is smooth.

The numerical simulations of the 2 dimensional toy model we discussed above were in the regime \( \nu \sim \omega_0 \) where \( \omega_0 = \sqrt{K} \) (see equation (2)). This regime is appropriate for the asteroid belt ([Holman & Murray 1996]) and for the simulation of inwards orbital migration of a Jovian sized giant planet (as by [Quillen & Holman 2000]). However in the Kuiper Belt secular oscillation frequencies are slow compared to the mean-motion resonance oscillation frequencies. If the simple model described by equation(2) were appropriate we would expect \( \nu \ll \omega_0 \) and that the resonances are highly overlapped. In this regime on long timescales the resonance is entirely chaotic. However, on short timescales we can consider the resonance to be a slowly varying integrable system. In this case we could use the formalism for calculating capture for slowly varying separatrices (e.g., [Haberman & Ho 1993; Neishtadt, Sidorenko, & Treschev 1997]).

Even though the secular oscillations frequencies are slow in the Kuiper belt, the edges of the resonances are still found numerically to exhibit chaotic motion even on the short timescales of millions of years (e.g., [Morbidelli 1997; Malhotra 2000]). So the theory of capture in the regime of slowly varying separatrices, is still not applicable to the theory of capture by Kuiper Belt resonances. The numerical integrations (e.g. [Morbidelli 1997] on the 3:2 and [Malhotra 2000] on the 2:1 resonances) show that the problem is not as simple as a simple model of multiple resonances restricted to a few terms and that even on short timescales, the resonances contain fairly wide chaotic zones.

When a resonance contains regions with widely different diffusion timescales, the problem is more complicated than illustrated with our simple 2D model explored above. However if the migration rate is fast compared to the diffusion timescale in a particular region of phase space, we can consider that region to be integrable. This is equivalent to modeling the resonance as a series of overlapped resonances with different frequencies, and ignoring the terms with the slowest frequencies.

If a particle can remain in a region of libration for \( 10^6 - 10^7 \) years then for orbital migration rates such that the resonance is passed on this timescale (\( \Delta a/\dot{a} \sim 10^6 - 10^7 \) years, for \( \Delta a \) the
width of the resonance and \( \dot{a} \) the resonance migration rate) we can consider that region to be integrable, and so capable of capturing particles. For low initial particle eccentricities, \( e_{\text{init}} < 0.1 \), Malhotra 2000 found that about a fifth of phase space was likely to be chaotic in the 2:1 resonance on fairly short timescales. In other words, Malhotra 2000 found no regions of stable libration for about 1/5 of possible values for the resonant angle. Even though the integrable model predicts a 100% capture probability for \( e_{\text{init}} < 0.06 \), if 1/5 of the resonance is chaotic, then the capture probability is likely to be only \( \sim 4/5 \) for migration rates \( \sim 10^6 - 10^7 \) years. This is one way that low eccentricity objects could pass through the 2:1 resonance, and a possible explanation for low eccentricity Kuiper Belt objects between 45-47 AU which should have been previously captured in the 2:1 resonance by Neptune’s migration.

Following a period of fairly swift migration, it is likely that Neptune underwent migration at a slower rate. For slower migration rates we can consider a larger fraction of the 2:1 resonance to be chaotic and the capture probability will be even smaller. Particles pumped to high eccentricity while caught in the resonance, can subsequently escape. In our previous simulations (Quillen & Holman 2000), we saw that particles could be temporarily captured, have their eccentricities increase while in the resonance and then could escape with eccentricities that were not necessarily well above their initial value. Any objects released at high eccentricity from the 2:1 resonance in the Kuiper Belt would not be observed today. Because of their high eccentricities, following escape from the resonance they would be likely to suffer encounters with Neptune. Low eccentricity particles which escaped the 2:1 resonance are likely to remain in stable orbits for the age of the solar system. The remaining objects we expect to find today should either reside in stable high eccentricity regions of the 2:1 resonance or at low eccentricity inside the semi-major axis of this resonance. As the observational constraints on the orbital elements of Kuiper Belt objects improve, we expect it will be possible to numerically explore the validity of this kind of scenario. Because of the different different diffusion timescales in the resonance, such a study may provide constraints on Neptune’s migration rate as a function of time.

3. The Four Dimensional Analogy

The celestial dynamics problem restricted to the plane containing the planet and sun is a 4 not 2 dimensional problem. Oscillations in semi-major axis are coupled to those in eccentricity. While a particle is temporarily captured into a resonance, large excursions in eccentricity can also take place (as discussed by Tittemore & Wisdom 1990 and Dermott, Malhotra & Murray 1988).

To illustrate what happens in the 4 dimensional system we explore a toy model with Hamiltonian similar to that given in equation (27) of Murray & Holman 1997

\[
H(P, \psi; I, \phi) = \frac{1}{2} P^2 + KI^q [1 + a \cos(\phi)] \cos(\psi) + bP + \nu I
\]

\[
= \frac{1}{2} P^2 + KI^q \left[ \cos(\psi) + \frac{a}{2} \cos(\psi + \phi) + \frac{a}{2} \cos(\psi - \phi) \right] + bP + \nu I
\]
In the celestial dynamics problem, $I$ is primarily related to the eccentricity of the particle, $P$ to the semi-major axis, $\psi$ corresponds to resonant angle and $\phi$ to the longitude of perihelion. $\nu$ corresponds to the precession frequency and is given by secular theory, and $q$ depends on the order of the resonance. In the celestial dynamics problem, $K$ and $a$ instead of being constants would be functions of the semi-major axis or $P$. If we do a canonical transformation to variables $\Gamma = kI; \theta = (\psi - \phi)/k$ for $k = 2q$ then we recover a Hamiltonian in the form $H \propto \frac{1}{2} \Gamma^2 + b/\Gamma + K' \Gamma^{k/2} \cos(k\theta) + \ldots$. This Hamiltonian is the integral model often used to estimate capture probabilities into $k$’th order mean-motion resonances (e.g., Henrard 1982; Malhotra 1988; Borderies & Goldreich 1984; Murray & Dermott 1999).

An integration of the 4 dimensional Hamiltonian given above with initial conditions $I = 1$ so that it is close analogy with the two dimensional analog discussed above, yields phenomenology (see Figure 4) remarkably similar to that we observed in our orbital migration numerical integrations (Quillen & Holman 2000). We see in Figure 4 that when temporary capture takes places there tends to be an increase in the mean value of $I$. After capture $I_{\text{mean}} \sim (H_0 - 1/2P_0^2)/\nu$ where $H_0$ is the value of the Hamiltonian and $P_0$ is the centre of the resonance when capture takes place. Since excursions in $I$ can take place, resulting in a wider resonance, particles can remain in the resonance for longer times than possible in the two dimensional analog. The resulting momentum probability distributions are broader (shown in Figure 5), particularly when the migration rates are slow.

Because $I$ is related to the particle eccentricity, excursions in $I$ represent excursions in eccentricity which could result in catastrophic events such as encounters with a planet or star. If the eccentricity undergoes a random walk once a particle is captured into the resonance (as discussed in the diffusion model by Murray & Holman 1997), then when the capture time is long compared to the diffusion timescale, catastrophic events would be more likely to occur. In this situation the dynamics would be strongly dependent upon the planet’s orbital migration rate. Murray & Holman 1997 discusses the diffusion timescale in terms of the Lyapunov time which for the simulations shown in Figures 4 and 5 is $T_L \sim 2\pi$. The slower drift rate integrations shown in Figure 5 represent cases where particles are trapped for many Lyapunov times. This may explain why their final momentum probability distributions exhibit larger tails.

We now discuss the phenomenology seen in our previous numerical simulations (Quillen & Holman 2000). Though initial particle eccentricities were fairly low, $e_{\text{init}} \sim 0.1$, because we chose planet eccentricities of $e_p = 0.3$ for many of the simulations, the forced eccentricities were high and the mean initial eccentricities tended to be $\sim 0.05 - 0.3$. The integrable model (Malhotra 1988) predicts a 100% capture probability for a Jupiter mass planet into the 3:1 resonance for eccentricities less than $\epsilon < 0.07$. Therefore we expect fairly low permanent capture probabilities of $0.2 - 0.5$ into this resonance (Borderies & Goldreich 1984).

For the 3:1 mean-motion resonance with a Jovian sized planet, the Lyapunov timescale (which is related to the secular oscillation frequency, Holman & Murray 1996) is about $10^3$ times
the planet period. For a resonance width that is about 0.02 the radius of the planet, we expect
temporary capture to last a time of \(2 \times 10^4\) planet periods which is only 20 Lyapunov times for a
migration rates \(D_a \equiv \frac{P \dot{a}}{a} = 10^{-6}\) given in units of the initial planet orbital period. For most of
the simulations, temporary capture times were fairly short and deviations in eccentricity caused
by temporary capture into the chaotic zone were limited. We also did 2 simulations with slower
migration rates of \(D_a = 3 \times 10^{-7}\). In these simulations particles are expected to remain in the
chaotic zone for longer, \(~ 60\) Lyapunov times. In the slower migration simulations we saw more
examples of resonances temporarily capturing particles. The probability of capture into the 3:1
(including temporary captures) was larger than that predicted from the integrable theory (using
rough numbers from Table 2 of Quillen & Holman 2000). The simulations showed that some of
the temporary captures resulted in catastrophic events such as encounters with the planet or star.
As expected from the final momentum distributions shown in Figure 5, in the slower simulations,
the passage of the resonances heated the eccentricities and semi-major axis distribution to a larger
extent than in the faster migration rate simulations.

We conclude that the chaotic zone in is likely to have the strongest affect for slow migration
rates. We expect large deviations from the predictions of integrable capture theories when the
chaotic zone is large and the migration rates is sufficiently long that many Lyapunov times pass
while particles are temporarily captured in the resonance.

In the simulations shown in Figures 4 and 5, because \(K\) and \(a\) are constants, the size of the
resonance does not change as \(P\) varies. If we allowed \(K\) and \(a\) to be functions of \(P\) we could allow
the size of the resonant islands to grow as the resonance drifts. This would then would allow the
integrable islands in the resonance to capture particles for longer periods of time. The separation
between overlapping resonances (here described by \(\nu\)) is given by secular theory (e.g., Holman
& Murray 1996) and so will not vary quickly once a particle is trapped in one of the integrable
islands. However, as the eccentricity grows, the widths of the individual sub-resonances will also
grow and the resonances will overlap to a larger degree. We therefore expect the size of the chaotic
region to grow after a particle is trapped in the resonance. Because the volume in the integrable
islands will at some time shrink instead of growing, particles will eventually escape the resonance.
This provides a complete analogy for the process of resonant capture and escape that we saw in
our previous simulations (Quillen & Holman 2000).

4. Summary and Discussion

By exploring toy Hamiltonian systems we have shown how the capture process is fundamentally
different for drifting chaotic resonances than for drifting integrable systems. Previous theoretical
work on resonant capture has been limited to integrable models. In this paper, we have illustrated
how an understanding of the structure in phase space of a resonance can be used to generalize this
integrable theory to include the richer phenomenology of a chaotic resonance.
1) We have shown that particles are temporarily captured into the chaotic zone of a drifting resonance. For a resonance of fixed shape, the capture time depends on the effective width of the zone and the drift rate. In fact, temporary capture will take place even when the resonant width is shrinking. This is not true in the case of an integrable resonance.

2) The permanent capture probability of a resonance depends on the ratio of phase space volume growth rate in its integrable islands compared to that swept up by the resonance. This implies that permanent capture probabilities are lower for resonances containing larger chaotic zones than those estimated from drifting integrable models. This offers a possible explanation for the passage of particles through the 2:1 resonance in the Kuiper Belt following migration by Neptune, and for the temporary captures seen in our previous simulations (Quillen & Holman 2000).

The continued migration of a planet via ejection of planetesimals depends on the fraction of particles remaining after a strong resonance has swept through the disk. Since catastrophic events such as close encounters with a planet or star can take place during temporary resonant capture, and because particles are more likely to escape drifting chaotic resonances, the dynamics of migrating systems may be strongly influenced by this process. We expect the largest deviations from the predictions of integrable capture theories when the chaotic zones are large and migration rates are sufficiently long that many Lyapunov times pass while particles are temporarily captured in the resonance.

3) The passage of chaotic resonances results in a heating of the particle momentum distribution. The momentum distribution width is increased by the effective momentum width of the chaotic zone. The final momentum distribution shape is dependent on the drift rate.

In this work we have numerically explored some simple drifting Hamiltonian systems. We suspect that the final momentum distributions of a drifting chaotic resonance is sensitive to the the distribution of diffusion times in the resonance. One way to explore this possibility would be to numerically investigate toy models with different structure in their chaotic zones.

In future, by exploring in detail the timescales and structure of solar systems resonances, and comparing the results of numerical simulations with the observed distribution of objects, we suspect that the form of orbital migration of the planets may be constrained. The study of drifting chaotic resonances will also be applied to other fields. Resonances play an important role in the stellar theory of dynamical friction. The possibility that additional heating can be caused by chaotic resonances has not yet been explored. The brightness of some zodiacal and Kuiper Belt dust belts depends on the lifetime for dust particles to remain in resonances. We expect that studies which take into account the phase space structure of solar system resonances may be used to derive better estimates for the lifetime of dust particles in these belts.

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Fig. 1.— Forced pendulum numerical integrations. Integrations have $K = 0.17$ and $\nu = 1$ (see Hamiltonian given in equation 2). The horizontal axes are the angle $\psi$ and the vertical axes are the momentum $P$. Points are plotted every $t = 0$ modulo $2\pi/\nu$. a) The structure of the fixed stationary resonance ($\dot{b} = 0$) with $a = 0.001$. b) The structure of the fixed stationary resonance with $a = 0.2$. c) The resonance with $a = 0.001$ was begun centered at the location shown in the upper figures and then allowed to drift upwards at a rate of $dP/dt = -\dot{b} \sim 10^{-3}$. 20 particles were integrated with initial conditions $P \approx 2.0, \psi \approx \pi$. The final orbits of the 20 particles are shown. d) Same as c) but for the system with $a = 0.2$. Since the system on the left (shown in a) and c)) is not chaotic, the final particle distribution is identical to that of the initial one only shifted by the effective width of the resonance (the volume of the resonance divided by $2\pi$). Since the resonance width is held fixed, particles are not captured into the resonance but simply jump from one side of the resonance to the other. Because the system on the right (shown in b) and d) contains a large chaotic zone, particles are temporarily captured into the chaotic zone. The final particle momentum distribution has been heated and has width approximately equal to effective width of the chaotic zone. The length of time captured is directly related to the final momentum reached.
Fig. 2.— Momentum probability distributions following the passage of a chaotic resonance. For the 2 dimensional system with $K = 0.17$, $\nu = 1$, $a = 0.2$ shown in Figure 1b) and 1d), the final momentum distributions for 100 particles with initial conditions $P = 2$ and $\psi \sim \pi$ are shown for the different drift rates $-\dot{b} = 10^{-2}$, $10^{-3}$ and $10^{-4}$. The Lyapunov time for this resonance is $\sim 2\pi$. The dispersion of the distribution is nearly constant among the three integrations. However the distribution is more nearly flat for the faster drift rates and more triangular for the slower drift rates. Following passage of a resonance lacking a stochastic zone, the momentum probability is a delta function (as shown in Figure 1c).
The capture probability drops as the width of the chaos zone increases. We plot the capture probability computed from simulations each with 100 particles and $K = 0.17$, $\hat{K} = 1.5 \times 10^{-3}$, $\hat{b} = 10^{-3}$, $\nu = 1$ and $\hat{a} = 0$. The value of $a$ is given as the horizontal axis and the capture probability on the vertical axis. The form of the function is expected if the fraction of the resonance filled with stable islands drops exponentially as $a \to 0$. 
Fig. 4.— Numerical integrations of the four dimensional system (given by equation 8) are shown for 10 particles. a) The $I$ momenta as a function of time in units of $10^4$. b) The $P$ momenta as a function of time in units of $10^4$. This system has $K = 0.17$, $\nu = 1$, $a = 0.20$ (similar to the 2 dimensional system shown in Figure 1, a drift rate of $-\dot{b} = 10^{-4}$ and $q = \frac{1}{2}$). Particles were initially set with $P = 2$, $I = 1$ and randomly distributed angles. The phenomenology of this system is remarkably similar to that seen in the orbital migration numerical simulations of Quillen & Holman (2000).
Fig. 5.— Momentum probability distributions following passage of the resonance for the 4 dimensional system described in Figure 4 but for three different drift rates $-\dot{b} = 10^{-2}, 10^{-3}$ and $10^{-4}$. 100 particles were integrated for each system. Drift rates are shown in the upper right of the $I$ plots. The more slowly drifting system shows larger tails in both momentum distributions.
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