QUASI-PARABOLIC COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. In this work we study the essential spectra of composition operators on weighted Bergman spaces of analytic functions which might be termed as “quasi-parabolic.” This is the class of composition operators on $A^2_\alpha$ with symbols whose conjugate with the Cayley transform on the upper half-plane are of the form $\varphi(z) = z + \psi(z)$, where $\psi \in H^\infty(\mathbb{H})$ and $\Im(\psi(z)) > \epsilon > 0$. We especially examine the case where $\psi$ is discontinuous at infinity. A new method is devised to show that this type of composition operators fall in a C*-algebra of Toeplitz operators and Fourier multipliers. This method enables us to provide new examples of essentially normal composition operators and to calculate their essential spectra.

1. Introduction

This paper is a continuation of our work [6] on quasi-parabolic composition operators on the Hardy space $H^2$. In this work we investigate the same class of operators on weighted Bergman spaces $A^2_\alpha(\mathbb{D})$.

Quasi-parabolic composition operators is a generalization of the composition operators induced by parabolic linear fractional non-automorphisms of the unit disc that fix a point $\xi$ on the boundary. These linear fractional transformations for $\xi = 1$ take the form

$$\varphi_a(z) = \frac{2iz + a(1 - z)}{2i + a(1 - z)}$$

with $\Im(a) > 0$. Quasi-parabolic composition operators on $H^2(\mathbb{D})$ are composition operators induced by the symbols where ‘$a$’ is replaced by a bounded analytic function ‘$\psi$’ for which $\Im(\psi(z)) > \delta > 0 \ \forall z \in \mathbb{D}$. We recall that the local essential range $\mathcal{R}_\xi(\psi^*)$ of $\psi \in H^\infty(\mathbb{D})$ at $\xi \in \mathbb{T}$ is defined to be the set of points $\zeta \in \mathbb{C}$ for which the set $\{z \in \mathbb{T} : |\psi^*(z) - \zeta| < \varepsilon\} \cap S_{\xi,r}$ has positive Lebesgue measure $\forall \varepsilon > 0$ and $\forall r > 0$ where $S_{\xi,r} = \{z \in \mathbb{T} : |z - \xi| < r\}$ and $\psi^* \in L^\infty(\mathbb{T})$ is the boundary value function of $\psi$. In [6] we showed that if $\psi \in QC(\mathbb{T}) \cap H^\infty(\mathbb{D})$ then these composition operators are essentially normal and their essential spectra are given as

$$\sigma_e(C_\varphi) = \{e^{izt} : t \in [0, \infty], z \in \mathcal{R}_1(\psi^*)\} \cup \{0\}$$

where $\mathcal{R}_1(\psi^*)$ is the local essential range of $\psi$ at 1.

In the weighted Bergman space setting $QC$ is replaced by $VMO_\partial$. The class of “Vanishing Mean Oscillation near the Boundary” functions is defined as the set of

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functions \( f \in L^1(\mathbb{D}, dA) \) satisfying
\[
\lim_{|z| \to 1} \frac{1}{|Q_z|} \int_{Q_z} |f(w) - 1| \, dA(u) \, |dA(w) = 0
\]
where \( Q_z = \{w \in \mathbb{D} : |w| \geq |z|, |\arg w - \arg z| \leq 1 - |z|\} \) and \( |Q_z| = (1 + |z|)(1 - |z|)^2 \) is the \( dA \) measure of \( Q_z \). We have the following very similar result in the weighted Bergman space setting:

**Main Theorem 1.** Let \( \varphi : \mathbb{D} \to \mathbb{D} \) be an analytic self-map of \( \mathbb{D} \) such that
\[
\varphi(z) = \frac{2iz + \eta(z)(1-z)}{2i + \eta(z)(1-z)}
\]
where \( \eta \in H^\infty(\mathbb{D}) \) with \( \Im(\varphi(z)) > \epsilon > 0 \) for all \( z \in \mathbb{D} \). If \( \eta \in \text{VMO}_0(\mathbb{D}) \cap H^\infty \) then we have

- (i) \( C_\varphi : A^2_\mathbb{D}(\mathbb{D}) \to A^2_\mathbb{D}(\mathbb{D}) \) is essentially normal
- (ii) \( \sigma_\epsilon(C_\varphi) = \{e^{izt} : t \in [0, \infty], z \in \mathcal{R}_1(\eta^*)\} \cup \{0\} \)

where \( \mathcal{R}_1(\eta^*) \) is the local essential range of \( \eta^* \in L^\infty(\mathbb{T}) \) at 1 and \( \eta^* \) is the boundary limit value function of \( \eta \).

In the upper half-plane for \( \psi \in \text{VMO}_0(\mathbb{H}) \cap H^\infty(\mathbb{H}) \) the local essential range \( \mathcal{R}_\infty(\psi) \) of \( \psi \) at \( \infty \) is defined to be the set of points \( z \in \mathbb{C} \) so that, for all \( \epsilon > 0 \) and \( n > 0 \), we have
\[
\lambda((\psi^*)^{-1}(B(z, \epsilon)) \cap (\mathbb{R} - [-n, n])) > 0,
\]
where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \) and \( \psi^* \) is the boundary value function of \( \psi \). We have the following result for the upper half-plane case:

**Main Theorem 2.** Let \( \psi \in \text{VMO}_0(\mathbb{H}) \cap H^\infty(\mathbb{H}) \) such that \( \Im(\psi(z)) > \epsilon > 0 \) for all \( z \in \mathbb{H} \) then for \( \varphi(z) = z + \psi(z) \) and \( \alpha > -1 \) we have

- (i) \( C_\varphi : A^2_\mathbb{H}(\mathbb{H}) \to A^2_\mathbb{H}(\mathbb{H}) \) is essentially normal
- (ii) \( \sigma_\epsilon(C_\varphi) = \{e^{izt} : t \in [0, \infty], z \in \mathcal{R}_\infty(\psi^*)\} \cup \{0\} \)

where \( \mathcal{R}_\infty(\psi^*) \) is the local essential range of \( \psi^* \in L^\infty(\mathbb{R}) \) at \( \infty \) and \( \psi^* \) is the boundary limit value of \( \psi \).

2. **Notation and Preliminaries**

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let \( S \) be a compact Hausdorff topological space. The space of all complex valued continuous functions on \( S \) will be denoted by \( C(S) \). For any \( f \in C(S) \), \( \| f \|_\infty \) will denote the sup-norm of \( f \), i.e.
\[
\| f \|_\infty = \sup\{| f(s) | : s \in S\}.
\]

For a Banach space \( X \), \( K(X) \) will denote the space of all compact operators on \( X \) and \( B(X) \) will denote the space of all bounded linear operators on \( X \). The open unit disc will be denoted by \( \mathbb{D} \), the open upper half-plane will be denoted by \( \mathbb{H} \), the real line will be denoted by \( \mathbb{R} \) and the complex plane will be denoted by \( \mathbb{C} \). The one point compactification of \( \mathbb{R} \) will be denoted by \( \hat{\mathbb{R}} \) which is homeomorphic to \( \mathbb{T} \). For any \( z \in \mathbb{C} \), \( \Re(z) \) will denote the real part, and \( \Im(z) \) will denote the imaginary part of \( z \), respectively. For any subset \( S \subset B(H) \), where \( H \) is a Hilbert space, the
C*-algebra generated by $S$ will be denoted by $C^*(S)$. The Cayley transform $\mathcal{C}$ will be defined by

$$\mathcal{C}(z) = \frac{z - i}{z + i}.$$  

For any $a \in L^\infty(\mathbb{H})$ (or $a \in L^\infty(\mathbb{D})$), $M_a$ will be the multiplication operator on $L^2(\mathbb{H})$ (or $L^2(\mathbb{D})$) defined as

$$M_a(f)(x) = a(x)f(x).$$

For convenience, we remind the reader of the rudiments of Gelfand theory of commutative Banach algebras and Toeplitz operators.

Let $A$ be a commutative Banach algebra. Then its maximal ideal space $M(A)$ is defined as

$$M(A) = \{ x \in A^* : x(ab) = x(a)x(b) \quad \forall a,b \in A \}$$

where $A^*$ is the dual space of $A$. If $A$ has identity then $M(A)$ is a compact Hausdorff topological space with the weak* topology. The Gelfand transform $\Gamma : A \rightarrow C(M(A))$ is defined as

$$\Gamma(a)(x) = x(a).$$

If $A$ is a commutative C*-algebra with identity, then $\Gamma$ is an isometric *-isomorphism between $A$ and $C(M(A))$. If $A$ is a C*-algebra and $I$ is a two-sided closed ideal of $A$, then the quotient algebra $A/I$ is also a C*-algebra (see [1] and [11]). For $a \in A$ the spectrum $\sigma_A(a)$ of $a$ on $A$ is defined as

$$\sigma_A(a) = \{ \lambda \in \mathbb{C} : \lambda e - a \text{ is not invertible in } A \},$$

where $e$ is the identity of $A$. We will use the spectral permanency property of C*-algebras (see [11], pp. 283); i.e. if $A$ is a C*-algebra with identity and $B$ is a closed *-subalgebra of $A$, then for any $b \in B$ we have

$$\sigma_B(b) = \sigma_A(b).$$

To compute essential spectra we employ the following important fact (see [11], pp. 268): If $A$ is a commutative Banach algebra with identity then for any $a \in A$ we have

$$\sigma_A(a) = \{ \Gamma(a)(x) = x(a) : x \in M(A) \}. \quad (2)$$

In general (for $A$ not necessarily commutative), we have

$$\sigma_A(a) \supseteq \{ x(a) : x \in M(A) \}. \quad (3)$$

For a Banach algebra $A$, we denote by $com(A)$ the closed ideal in $A$ generated by the commutators $\{ a_1a_2 - a_2a_1 : a_1, a_2 \in A \}$. It is an algebraic fact that the quotient algebra $A/com(A)$ is a commutative Banach algebra. The reader can find detailed information about Banach and C*-algebras in [11] related to what we have reviewed so far.

The essential spectrum $\sigma_e(T)$ of an operator $T$ acting on a Banach space $X$ is the spectrum of the coset of $T$ in the Calkin algebra $B(X)/K(X)$, the algebra of bounded linear operators modulo compact operators. The well known Atkinson’s theorem identifies the essential spectrum of $T$ as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not a Fredholm operator. The essential norm of $T$ will be denoted by $\| T \|_e$ which is defined as

$$\| T \|_e = \inf \{ \| T + K \| : K \in K(X) \}$$
The bracket $[\cdot]$ will denote the equivalence class modulo $K(X)$. An operator $T \in B(H)$ is called essentially normal if $T^*T - TT^* \in K(H)$ where $H$ is a Hilbert space and $T^*$ denotes the Hilbert space adjoint of $T$.

For $\alpha > -1$ the weighted Bergman space $A^2_\alpha(\mathbb{H})$ of the upper half-plane is defined as

$$A^2_\alpha(\mathbb{H}) = \{ f : \mathbb{H} \to \mathbb{C} : f \text{ is analytic and } \int_{\mathbb{H}} |f(x + iy)|^2 y^\alpha dx dy < \infty \}$$

The weighted Bergman spaces $A^2_\alpha$ are reproducing kernel Hilbert spaces with kernel functions

$$k_w(z) = \frac{1}{(\bar{w} - z)^{\alpha + 2}}$$

(see [3]). For $H^2(\mathbb{D})$, the Hardy space of the unit disc it is quite an obvious fact that if $f \in L^2(\mathbb{T})$, $f(z) = \sum_{n} \hat{f}(n) z^n$ then

$$f \in H^2 \iff \hat{f}(n) = 0 \quad \forall n < 0$$

i.e. $f \in H^2$ if and only if its negative Fourier coefficients are zero.

A similar fact arises for $H^2(\mathbb{H})$ as the Paley-Wiener theorem:

**Paley-Wiener Theorem.** T.F.A.E

a. $F \in H^2(\mathbb{H})$

b. $\exists f \in L^2(\mathbb{R}^+) \text{ s.t. } F(z) = \int_0^\infty f(t)e^{2\pi itz} dt \quad z \in \mathbb{H}$

Moreover the correspondence $F \to f$ is an isometric isomorphism of $H^2(\mathbb{H})$ onto $L^2(\mathbb{R}^+)$. For weighted Bergman spaces $A^2_\alpha(\mathbb{H})$ the Paley Wiener theorem as proved by P. Duren, E. Gallardo-Gutierrez and A. Montes-Rodriguez(see [3]) takes the following form:

**Paley-Wiener Theorem for Weighted Bergman Spaces.** T.F.A.E

(a). $F \in A^2_\alpha(\mathbb{H})$

(b). $\exists f \in L^2_{\alpha + 1}(\mathbb{R}^+) \text{ s.t. } F(z) = \int_0^\infty f(t)e^{2\pi itz} dt \quad z \in \mathbb{H}$

where

$$L^2_\beta(\mathbb{R}^+) = \{ f : \mathbb{R}^+ \to \mathbb{C} : \int_0^\infty |f(t)|^2 t^{-\beta} dt < \infty \}$$

The weighted space $L^2_\beta(\mathbb{R}^+)$ is equipped with the norm

$$(\| f \|_{L^2_\beta})^2 = \frac{\Gamma(\beta)}{2\beta} \int_0^\infty |f(t)|^2 t^{-\beta} dt$$

The correspondence $F \to f$ is again an isometric isomorphism of $A^2_\alpha(\mathbb{H})$ onto $L^2_\beta(\mathbb{R}^+)$. It is easily seen that this correspondence is the well-known Fourier transform.

Since the map $F^{-1} : L^2_{\alpha + 1}(\mathbb{R}^+) \to A^2_\alpha(\mathbb{H})$,

$$F^{-1}(f)(z) = \int_0^\infty f(t)e^{2\pi itz} dt$$

is an isometric isomorphism of Hilbert spaces, it is unitary and hence we have

$$\langle F^{-1}(f), g \rangle_{A^2_\alpha(\mathbb{H})} = \int_{\mathbb{H}} F^{-1}(f)(z)g(z)dA_\alpha(z) = \int_{\mathbb{H}} ( \int_0^\infty f(t)e^{2\pi itz} dt )g(z)dA_\alpha(z) = \langle f, F(g) \rangle_{L^2_{\alpha + 1}(\mathbb{R}^+)}.$$
∀his proofs work for the general weighted case i.e. for $\alpha$ satisfying $f$ that for
\[(1+|z|)^{\alpha}\]is defined as $f$ operator and $P$ $\langle f, F(g) \rangle_{L^2_{\alpha+1}(\mathbb{R}^+)} = \frac{\Gamma(\alpha + 1)}{2\alpha + 1} \int_0^\infty f(t) F(g)(t) \frac{dt}{t^{1+\alpha}}$

Hence we have the representation of $F : A^2_\alpha(\mathbb{H}) \rightarrow L^2_{\alpha+1}(\mathbb{R}^+)$ as follows:

$$F(g)(t) = \frac{\alpha + 1}{\Gamma(\alpha + 1)} \int_\mathbb{H} e^{-2\pi itz} g(z) dA_\alpha(z)$$

By the help of this fact one can distinguish a class of $C^*$ algebras of operators acting on $A^2_\alpha(\mathbb{H})$. For $X$ being a $C^*$ algebra of functions of $\mathbb{R}^+$ s.t. $X \subseteq L^\infty(\mathbb{R}^+)$ the Fourier multiplier algebra on $A^2_\alpha$ associated to $X$ is defined to be

$$D_\phi : A^2_\alpha(\mathbb{H}) \rightarrow A^2_\alpha(\mathbb{H}) \text{ will denote the Fourier multiplier defined as}$$

$$D_\phi = F^{-1} M_\phi F$$

For any $f \in L^\infty(\mathbb{D})$, the Toeplitz operator $T_f : A^2_\alpha(\mathbb{D}) \rightarrow A^2_\alpha(\mathbb{D})$ with symbol $f$ is defined as

$$T_f = PM_f$$

where $P : L^2(\mathbb{D}, dA_\alpha) \rightarrow A^2_\alpha(\mathbb{D})$ is the orthogonal projection and $M_f : L^2(\mathbb{D}, dA_\alpha) \rightarrow L^2(\mathbb{D}, dA_\alpha)$ is the multiplication operator. Similarly the Toeplitz operator $T_f : A^2_\alpha(\mathbb{H}) \rightarrow A^2_\alpha(\mathbb{H})$ on $A^2_\alpha(\mathbb{H})$ is defined as $T_f = PM_f$ where $P : L^2(\mathbb{H}, dA_\alpha) \rightarrow A^2_\alpha(\mathbb{H})$ is the orthogonal projection, $M_f : L^2(\mathbb{H}, dA_\alpha) \rightarrow L^2(\mathbb{H}, dA_\alpha)$ is the multiplication operator and $f \in L^\infty(\mathbb{H})$. In [18] Zhu introduced the space of functions $VMO_\alpha(\mathbb{D})$ of “vanishing mean oscillation near the boundary” which is defined as the set of functions $f \in L^3(\mathbb{D}, dA)$ satisfying

$$\lim_{|z| \rightarrow 1} \frac{1}{|Q_z|} \int_{Q_z} |f(w) - \frac{1}{|Q_z|} \int_{Q_z} f(u) dA(u)| \, dA(w) = 0$$

where $Q_z = \{w \in \mathbb{D} : |w| > 1-|z|\}$ and $Q_z := (1+|z|)|1-z|^2$ is the $dA$ measure of $Q_z$. Zhu showed for the case $\alpha = 0$ that for $f \in L^\infty(\mathbb{D})$, the semi-commutator $T_\alpha f - T_\alpha T_f$ is compact on $A^2 \forall g \in L^\infty(\mathbb{D})$ if and only if $f \in VMO_\alpha(\mathbb{D})$. Although Zhu proved this fact for $\alpha = 0$, his proofs work for the general weighted case i.e. for $\alpha \neq 0$. Zhu also introduced the space of functions $ESV(\mathbb{D})$ of “eventually slowly varying” which is defined as the set of functions $f \in L^\infty(\mathbb{D})$ satisfying for any $\varepsilon > 0$ and $\kappa \in (0, 1)$ there is $\delta_0 > 0$ such that

$$|f(z) - f(w)| < \varepsilon$$
have induced by the maps of the form $\phi$, bounded (see [2]), this implies that $\Phi$ becomes under intertwining with this isomorphism, i.e. for $\varphi$ is an isometric isomorphism. It is of interest to us what the composition operators on $A^2_\alpha$ will be helpful in our task: reproducing kernel Hilbert spaces. We begin with a simple geometric lemma that will be helpful in our task:

$$\Phi(f)(z) = \frac{2^{\alpha+1}}{(z+i)^{\alpha+2}}f\left(\frac{z-i}{z+i}\right)$$

is an isomorphism. It is of interest to us what the composition operators become under intertwining with this isomorphism, i.e. for $\varphi : \mathbb{D} \to \mathbb{D}$ what is $\Phi C_\varphi \Phi^{-1}$? We have the following answer to this question:

$$\Phi C_\varphi \Phi^{-1} = M_{\tau \varphi} C_{\hat{\varphi}}$$

where $M_\tau f(z) = \tau(z)^2 f(z)$ is the multiplication operator, $\tau(z) = \frac{\bar{z}(z+i)}{z+i}$, $\hat{\varphi} = C^{-1} \circ \varphi \circ C$ and $C(z) = \frac{z+i}{\bar{z}+i}$ is the Cayley transform. This gives us the boundedness of $C_\varphi : A^2_\alpha(\mathbb{H}) \to A^2_\alpha(\mathbb{H})$ for $\varphi(z) = pz + \psi(z)$ where $p > 0$, $\psi \in H^\infty$ and the closure of the image $\overline{\psi(\mathbb{H})} \subset \mathbb{H}$ is compact in $\mathbb{H}$.

Let $\theta : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that $\hat{\theta} = C^{-1} \circ \theta \circ C = \varphi$ then we have $\Phi C_\varphi \Phi^{-1} = M_{\tau \varphi} C_{\hat{\varphi}}$ where $\tau(z) = \frac{\bar{z}(z+i)}{z+i}$. If $\varphi(z) = pz + \psi(z)$ with $p > 0$, $\psi \in H^\infty$ and $\Re(\psi(z)) > \delta > 0$ then $M_\tau$ is a bounded operator. Since $C_\varphi$ is always bounded (see [2]), this implies that $\Phi C_\varphi \Phi^{-1}$ is also bounded and we conclude that $C_\varphi$ is bounded on $A^2_\alpha(\mathbb{H})$ (see also [8]). We also observe that for any $f \in L^\infty(\mathbb{D})$ we have

$$\Phi \circ T_f \circ \Phi^{-1} = T_{f \circ \varphi}$$

3. Approximation Scheme for Composition Operators on Weighted Bergman Spaces of the Upper Half-Plane

In this section we develop an approximation scheme for composition operators induced by the maps of the form $\phi(z) = pz + \psi(z)$ where $p > 0$ and $\psi \in H^\infty$ such that the closure of the image $\overline{\psi(\mathbb{H})} \subset \mathbb{H}$ is compact in $\mathbb{H}$, by linear combinations of $T_\psi$ and Fourier multipliers where $T_\psi$ is the Toeplitz operator with symbol $\psi$. By the preceding section we know that these maps induce bounded composition operators on $A^2_\alpha(\mathbb{H})$. In establishing this approximation scheme our main tools are the integral representation formulas which come from the fact that these spaces are reproducing kernel Hilbert spaces. We begin with a simple geometric lemma that will be helpful in our task:

**Lemma 1.** Let $K \subset \mathbb{H}$ be a compact subset of $\mathbb{H}$. Then $\exists \beta \in \mathbb{R}^+$ such that $\sup\{|\beta \frac{z-i}{p}| : z \in K\} < \delta < 1$ for some $\delta \in (0,1)$

**Proof.** See [6] \hfill \Box

We also need the following lemma which characterizes certain integral operators as Fourier multipliers:
Lemma 2. Let $\alpha \in \mathbb{R}$ s.t. $\alpha > -1$ and $\beta \in \mathbb{H}$. Then the operator $M_n : A^2_n(\mathbb{H}) \rightarrow A^2_n(\mathbb{H})$ defined as

$$(M_n f)(z) = \frac{-1}{\pi} \int_{\mathbb{H}} \frac{f(w) dA_{n}(w)}{(\bar{w} - z - \beta)^{\alpha + 2}}$$

where $n \in \mathbb{N} \setminus \{0\}$, is the Fourier multiplier $D_{\phi_n}$ where

$$\phi_n(t) = \frac{(2\pi i t)^n e^{2\pi i \beta t}}{(\alpha + 2)(\alpha + 3)...(\alpha + n + 1)}$$

i.e. $M_n = D_{\phi_n}$.

Proof. Let $f \in L^2_{\alpha+1}(\mathbb{H}^+)$ then for

$$g(z) = F^{-1}(f)(z) = \int_{0}^{\infty} f(t) e^{2\pi i t z} dt$$

we have $g \in A^2_n(\mathbb{H})$. Since $A^2_n(\mathbb{H})$ is a reproducing kernel Hilbert space we have

$$g(z_0) = \frac{-1}{\pi} \int_{\mathbb{H}} \frac{f(w) dA_{n}(w)}{(\bar{w} - z_0)^{\alpha + 2}}$$

Hence differentiating under the integral sign $n$ times and substituting $z_0 = z + \beta$ we have

$$M_n g(z) = \frac{1}{(\alpha + 2)(\alpha + 3)...(\alpha + n + 1)} g^{(n)}(z + \beta)$$

Combining this with equation (4) we have

$$M_n g(z) = \frac{1}{(\alpha + 2)(\alpha + 3)...(\alpha + n + 1)} \int_{0}^{\infty} f(t) e^{2\pi it(z+\beta)} dt$$

Interchanging the differentiation and integration in the above equation we have

$$M_n g(z) = \frac{1}{(\alpha + 2)(\alpha + 3)...(\alpha + n + 1)} \int_{0}^{\infty} (2\pi it)^n e^{2\pi i \beta t} f(t) e^{2\pi itz} dt = F^{-1}(\phi_n f)(z)$$

where

$$\phi_n(t) = \frac{(2\pi it)^n e^{2\pi i \beta t}}{(\alpha + 2)(\alpha + 3)...(\alpha + n + 1)}$$

This means that

$$M_n (F^{-1}(f))(z) = F^{-1}(\phi_n f)(z)$$

which implies that

$$M_n = F^{-1} M_{\phi_n} F = D_{\phi_n}$$

Proposition 3. Let $\varphi : \mathbb{H} \rightarrow \mathbb{H}$ be an analytic self-map of the form $\varphi(z) = pz + \psi(z)$ where $p > 0$ and $\psi \in H^\infty$ with the closure of the image $\overline{\psi(\mathbb{H})} \subset \mathbb{H}$ is compact in $\mathbb{H}$. Then $\exists \beta > 0$ such that for $C_{\varphi} : A^2_{\alpha} \rightarrow A^2_{\alpha}$ we have

$$C_{\varphi} = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n! \Gamma(\alpha + 2)} T^n D_{\phi_n} V_p$$

where the convergence of the series is in operator norm, $T^n f(z) = \tau^n(z) f(z)$, $\tau(z) = \overline{\psi(z) - i\beta}$, $\overline{\psi(z) = \psi(\bar{z})}$, $V_p f(z) = f(pz)$ is the dilation by $p$ and $\phi_n(t) = \frac{(2\pi it)^n e^{-2\pi i \beta t}}{(\alpha + \beta)(\alpha + 3)...(\alpha + n + 1)}$ for $n \geq 1$ and $\phi_0(t) = e^{-2\pi i \beta t}$. 
Proof. The integral representation formula is as follows: For \( f \in A^2_\alpha(\mathbb{H}) \) we have
\[
f(z) = -\frac{1}{\pi} \int_{\mathbb{H}} \frac{f(w)dA_\alpha(w)}{(\bar{w} - z)^{\alpha+2}}
\]
where \( dA_\alpha(w) = (\Im(w))^\alpha dA(w) \) is a translation invariant measure on \( \mathbb{H} \).

For \( \varphi : \mathbb{H} \to \mathbb{H} \) an analytic self-map of \( \mathbb{H} \), we can insert the substitution \( z \to \varphi(z) \) in order to get an integral representation of the composition operator \( C_\varphi : A^2_\alpha \to A^2_\alpha \):
\[
C_\varphi(f)(z) = -\frac{1}{\pi} \int_{\mathbb{H}} \frac{f(w)dA_\alpha(w)}{(\bar{w} - \varphi(z))^{\alpha+2}}
\]
Let \( \varphi(z) = p\bar{z} + \psi(z) \) where \( \psi \in H^\infty \) with \( \Im(\psi(z)) > \delta > 0 \) \( \forall z \in \mathbb{H} \) and \( p > 0 \). Then for \( C_\varphi : A^2_\alpha \to A^2_\alpha \) we have
\[
(C_\varphi V_\frac{\alpha}{p})f(z) = -\frac{1}{\pi} \int_{\mathbb{H}} \frac{f(w)dA_\alpha(w)}{(\bar{w} - z - \psi(z))^{\alpha+2}}
\]
where \( \tilde{\psi}(z) = \psi(\frac{z}{p}) \). We look at
\[
\frac{1}{(\bar{w} - z - \psi(z))^{\alpha+2}} = \frac{1}{(\bar{w} - z - i\beta - (\psi(z) - i\beta))^{\alpha+2}}
\]
\[
= \frac{1}{(\bar{w} - z - i\beta)^{\alpha+2}} \left( 1 - \frac{\tilde{\psi}(z) - i\beta}{\bar{w} - z - i\beta} \right)^{\alpha+2}
\]
We apply Lemma 1 to have \( \beta > 0 \) such that
\[
|\frac{\tilde{\psi}(z) - i\beta}{\bar{w} - z - i\beta}| < \delta < 1
\]
Here we have the geometric series formula as
\[
\frac{1}{1 - \frac{\tilde{\psi}(z) - i\beta}{\bar{w} - z - i\beta}}^{\alpha+2} = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n!\Gamma(\alpha + 2)} \left( \frac{\tilde{\psi}(z) - i\beta}{\bar{w} - z - i\beta} \right)^n + Q_{M+1}(w, z)
\]
Inserting this into (5) we have
\[
C_\varphi f(z) = \sum_{n=0}^{M} \frac{\Gamma(n + 2 + \alpha)}{n!\Gamma(\alpha + 2)} T_n D_{\phi_n} f(z) + \int_{\mathbb{H}} \frac{Q_{M+1}(w, z)f(w)dA_\alpha(w)}{(\bar{w} - z - i\beta)^{\alpha+2}}
\]
where \( T_n f(z) = \tau^n(z)f(z) \), \( \tau(z) = \tilde{\psi}(z) - i\beta \) and \( D_{\phi_n} \) is the Fourier multiplier.

For the operator
\[
R_{M+1} f(z) = \int_{\mathbb{H}} \frac{Q_{M+1}(w, z)f(w)dA_\alpha(w)}{(\bar{w} - z - i\beta)^{\alpha+2}}
\]
we have \( \|R_{M+1}\| \leq \|Q_{M+1}\| \|S_{i\beta}\| \) where \( S_{i\beta} f(z) = f(z + i\beta) \) and \( \|Q_{M+1}\| \to \infty = \sup_{(z, w) \in \mathbb{H}^2} |Q_{M+1}(w, z)| \). Since \( \|Q_{M+1}\| \to \infty \) as \( M \to \infty \) we have \( \|R_{M+1}\| \to 0 \) as \( M \to \infty \). □
4. A $C^*$-algebra of Operators on $A^2_\alpha(\mathbb{H})$

In the preceding section we have shown that “quasi-parabolic” composition operators on the upper half-plane lie in the $C^*$-algebra generated by certain Toeplitz operators and Fourier multipliers. In this section we will identify the character space of the $C^*$-algebra generated by Toeplitz operators with a class of symbols and Fourier multipliers.

In the Hardy space case we observed that if $\varphi \in QC$ and $\theta \in C([0, \infty])$ then the commutator $T_\varphi D_\theta - D_\theta T_\varphi \in K(H^2)$ is compact on the Hardy space $H^2$. In the weighted Bergman space case which we are considering in this paper, QC will be replaced by VMO. The upper half-plane versions of VMO and ESV are defined as follows:

$$VMO_\theta(\mathbb{H}) = \{f \circ \mathcal{C} : f \in VMO_\theta(\mathbb{D})\}$$

and

$$ESV(\mathbb{H}) = \{f \circ \mathcal{C} : f \in ESV(\mathbb{D})\}.$$ 

It is not difficult to see that by definition and Zhu’s result\(^{[13]}\) we have

$$VMO_\theta(\mathbb{H}) \cap H^\infty(\mathbb{H}) = ESV(\mathbb{H}) \cap H^\infty(\mathbb{H}),$$

Lemma 4. Let $\psi \in ESV(\mathbb{H}) \cap H^\infty(\mathbb{H}) = VMO_\theta(\mathbb{H}) \cap H^\infty(\mathbb{H})$, $C_{\psi,a} : A^2_\alpha(\mathbb{H}) \to A^2_\alpha(\mathbb{H})$ is compact where

$$C_{\psi,a}(f) = (\psi(z + a) - \psi(z))f(z + a)$$

where $a \in \mathbb{H}$.

Proof. Since $\psi \in ESV$, $\forall \varepsilon > 0$ there exists $m \in \mathbb{N}$ and a compact subset $K \subset \mathbb{H}$ so that $|\psi(z + \frac{ka}{m}) - \psi(z)| < \frac{\varepsilon}{m}$ for all $z \notin K$. So for all $k \in \{1, 2, ..., m\}$ there is a compact subset $K_k \subset \mathbb{H}$ so that $\forall z \notin K_k$ we have

$$|\psi(z + \frac{ka}{m}) - \psi(z + \frac{(k-1)a}{m})| < \frac{\varepsilon}{m}.$$ 

Hence there is a compact set $K = \bigcup_{k=1}^{m} K_k$ so that $\forall z \notin K$ we have

$$|\psi(z + a) - \psi(z)| \leq \sum_{k=0}^{m-1} |\psi(z + \frac{(k+1)a}{m}) - \psi(z + \frac{ka}{m})| < m \sum_{k=0}^{m-1} \frac{\varepsilon}{m} = \varepsilon.$$ 

So we have $\forall \varepsilon > 0$, there is a compact subset $K \subset \mathbb{H}$ so that

$$|\psi(z + a) - \psi(z)| < \varepsilon \quad \forall z \notin K.$$ 

Let $\{f_n\}_{n=1}^{\infty} \subset A^2_\alpha(\mathbb{H})$ so that $\|f_n\|_{A^2_\alpha} \leq 1 \forall n \in \mathbb{N}$ and let $g_n = C_{\psi,a}(f_n)$. Let $K_m \subset \mathbb{H}$ be a sequence of compact subsets of $\mathbb{H}$ so that $K_m \subset (K_{m+1})^c$ and $\bigcup_{m \in \mathbb{N}} K_m = \mathbb{H}$. Since $\{g_n\}$ is equi-bounded on $K_1$ by Montel’s theorem it has a subsequence $\{g_{n_k}\}$ so that it converges uniformly on $K_1$. Since $\{g_{n_k}\}$ is equi-bounded on $K_2$ it has a further subsequence $\{g_{n_{k_l}}\}$ so that it converges uniformly on $K_2$. Proceeding in this way using Cantor’s diagonal argument one can extract a subsequence $\{g_k\}$ so that $\{g_k\}$ is uniformly convergent on $K_m \forall m \in \mathbb{N}$.

Let $\varepsilon > 0$ be given then $\exists m \in \mathbb{N}$ so that

$$|\psi(z + a) - \psi(z)| < \varepsilon \quad \forall z \notin K_m.$$
Consider
\[ \| g_k - g_l \|_{A^2_\alpha}^2 = \int_{\mathbb{H}} |\psi(z + a) - \psi(z)|^2 |(f_k - f_l)(z + a)|^2 \, dA_\alpha(z) \]
\[ = \int_{K_m} |\psi(z + a) - \psi(z)|^2 |(f_k - f_l)(z + a)|^2 \, dA_\alpha(z) \]
\[ + \int_{\mathbb{H} \setminus K_m} |\psi(z + a) - \psi(z)|^2 |(f_k - f_l)(z + a)|^2 \, dA_\alpha(z) \]
\[ = \int_{K_m} |g_k(z) - g_l(z)|^2 \, dA_\alpha(z) \]
\[ + \int_{\mathbb{H} \setminus K_m} |\psi(z + a) - \psi(z)|^2 |(f_k - f_l)(z + a)|^2 \, dA_\alpha(z) \]

Since \( \{g_k\} \) is uniformly convergent on \( K_m \), \( \exists n_0 \in \mathbb{N} \) so that \( \forall k, l > n_0 \)
\[ \| g_k - g_l \|_{K_m} < \varepsilon \]
where \( \lambda_\alpha(K_m) = \int_{K_m} dA_\alpha(z) \) is the \( dA_\alpha \) measure of \( K_m \). Hence
\[ \int_{K_m} |g_k(z) - g_l(z)|^2 \, dA_\alpha(z) < \varepsilon \quad \forall k, l > n_0 \]
and we have
\[ \int_{\mathbb{H} \setminus K_m} |\psi(z + a) - \psi(z)|^2 |(f_k - f_l)(z + a)|^2 \, dA_\alpha(z) \leq 2 \| f_k \|_{A^2_\alpha} \leq 2\varepsilon \]
since \( \| f_k \|_{A^2_\alpha} \leq 1 \ \forall k \in \mathbb{N} \). Hence \( \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \) so that \( \forall k, l > n_0 \)
\[ \| g_k - g_l \|_{A^2_\alpha} < 3\varepsilon \]
This implies that \( \{g_k\} \) is a Cauchy sequence in \( A^2_\alpha \). So for any sequence \( \{f_n\} \subset A^2_\alpha \)
satisfying \( \| f_n \|_{A^2_\alpha} \leq 1 \), \( g_n = C_{\psi, \alpha}(f_n) \) has a convergent subsequence in \( A^2_\alpha \). This implies that \( C_{\psi, \alpha} \) is compact on \( A^2_\alpha \). \( \square \)

**Corollary 5.** Let \( \psi \in VMO_\alpha \cap L^\infty(\mathbb{H}) \) and \( \theta \in C([0, \infty]) \) then the commutator \( T_\psi D_\theta - D_\theta T_\psi \in K(A^2_\alpha(\mathbb{H})) \) is compact on \( A^2_\alpha \).

**Proof.** Let \( S_a : A^2_\alpha \to A^2_\alpha \) be the translation operator \( S_a f(z) = f(z + a) \) where \( a \in \mathbb{H} \). Then we observe that \( S_a = D_{\phi_a} \) where \( \phi_a(t) = e^{2\pi iat} \). Since for \( \psi \in VMO_\alpha \cap H^\infty(\mathbb{H}) \) \( C_{\psi, \alpha} = S_a T_\psi - T_\psi S_a \) we have \( T_\psi D_{\phi_a} - D_{\phi_a} T_\psi \in K(A^2_\alpha(\mathbb{H})) \) by lemma 4. We also observe that \( S_{-\bar{a}} = D_{\phi_{-\bar{a}}} \) and hence \( S_{-\bar{a}} T_\psi - T_\psi S_{-\bar{a}} \in K(A^2_\alpha) \) since \( -\bar{a} \in \mathbb{H} \) for \( a \in \mathbb{H} \). By Stone-Weierstrass theorem the set of functions of the form \( p(\phi_a, \phi_{-\bar{a}}) \) where \( p(z, w) = \sum_{j=0}^{\infty} c_j z^j w^j \) is dense in \( C([0, \infty]) \), hence \( T_\psi D_\theta - D_\theta T_\psi \in K(A^2_\alpha) \) \( \forall \theta \in C([0, \infty]) \). Since \( VMO_\alpha \cap L^\infty(\mathbb{H}) \) is generated by functions \( \psi \) and \( \psi \) where \( \psi \in VMO_\alpha \cap H^\infty(\mathbb{H}) \) we have \( T_\psi D_\theta - D_\theta T_\psi \in K(A^2_\alpha) \) \( \forall \psi \in VMO_\alpha \cap L^\infty(\mathbb{H}) \) and \( \forall \theta \in C([0, \infty]) \). \( \square \)

Now we are ready to construct our C*-algebra of operators. Let
\[ T(VMO_\alpha) = C^*(\{T_f : f \in VMO_\alpha(\mathbb{H}) \cap L^\infty(\mathbb{H})\}) \subset B(A^2_\alpha(\mathbb{H})) \]
be the C*-algebra of Toeplitz operators on \( A^2_\alpha(\mathbb{H}) \) with symbols in \( VMO_\alpha \). By Zhu’s result([13]), since the commutators \( T_f T_g - T_g T_f \in K(A^2_\alpha(\mathbb{H})) \) are compact for all \( f, g \in VMO_\alpha \cap L^\infty(\mathbb{H}) \), it is easy to see that the quotient C*-algebra
$\mathcal{T}(VMO_0)/K(A_n^2(\mathbb{H}))$ is a unital commutative $C^*$-algebra. Let $\mathcal{M}$ be the maximal ideal space of $\mathcal{T}(VMO_0)/K(A_n^2(\mathbb{H}))$. It is a well known fact in the theory of Toeplitz operators on Bergman spaces that $\mathcal{T}(C(\mathbb{D}))/K(A_n^2(\mathbb{D}))$ is isometrically isomorphic to $C(\mathbb{T})$ and the isometric isomorphism is given by the correspondence $[T_f] \mapsto f|_{\mathbb{T}}$ (see [12]). This fact can be carried over to the upper half-plane by using the Cayley transform: Let $\mathbb{H} = \{ z \in \mathbb{C} : \Im(z) \geq 0 \}$ and $\mathbb{H}$ be the one point compactification of $\mathbb{H}$, then the correspondence $[T_f] \mapsto f|_{\mathbb{H}}$ is an isometric isomorphism between $\mathcal{T}(C(\mathbb{H}))/K(A_n^2(\mathbb{H}))$ and $C(\mathbb{R})$. Since $\mathcal{T}(C(\mathbb{H}))/K(A_n^2(\mathbb{H}))$ is a subalgebra of $\mathcal{T}(VMO_0)/K(A_n^2(\mathbb{H}))$ and $\mathcal{T}(C(\mathbb{H}))/K(A_n^2(\mathbb{H}))$ is isometrically isomorphic to $C(\mathbb{R})$, the maximal ideal space $\mathcal{M}$ can be thought as fibered over $\mathbb{R}$. For $x \in \mathbb{R}$, let

$$\mathcal{M}_x = \{ \phi \in \mathcal{M} : \phi|_{\mathcal{T}(C(\mathbb{H}))/K(A_n^2(\mathbb{H}))} = \delta_x \}$$

where $\delta_x([T_f]) = f(x)$. Then we have

$$\mathcal{M} = \bigcup_{x \in \mathbb{R}} \mathcal{M}_x$$

and for $x_1 \neq x_2$ we have $\mathcal{M}_{x_1} \cap \mathcal{M}_{x_2} = \emptyset$. Now let $F_{C([0, \infty])}^{A_n^2}$ be the $C^*$-algebra generated by Fourier multipliers with symbols in $C([0, \infty])$, our $C^*$-algebra is the following

$$\Psi(VMO_0, C([0, \infty])) = C^*(\mathcal{T}(VMO_0) \cup F_{C([0, \infty])}^{A_n^2}) \subset B(A_n^2(\mathbb{H}))$$

By corollary 5, $\Psi(VMO_0, C([0, \infty]))$ is a unital commutative $C^*$-algebra. It is of interest to ask for its maximal ideal space.

We will use the following theorem of Power (see [9] and [10]) to characterize its maximal ideal space:

**Power’s Theorem.** Let $C_1$, $C_2$ be two $C^*$-subalgebras of $B(H)$ with identity, where $H$ is a separable Hilbert space, such that $M(C_i) \neq \emptyset$, where $M(C_i)$ is the space of multiplicative linear functionals of $C_i$, $i = 1, 2$ and let $C$ be the $C^*$-algebra they generate. Then for the commutative $C^*$-algebra $\check{C} = C/com(C)$ we have $M(\check{C}) = P(C_1, C_2) \subset M(C_1) \times M(C_2)$, where $P(C_1, C_2)$ is defined to be the set of points $(x_1, x_2) \in M(C_1) \times M(C_2)$ satisfying the condition:

Given $0 \leq a_1 \leq 1$, $0 \leq a_2 \leq 1$, $a_1 \in C_1$, $a_2 \in C_2$

$$x_i(a_i) = 1 \quad \text{with} \quad i = 1, 2 \quad \Rightarrow \quad \|a_1 a_2\| = 1.$$ 

**Proof.** See [10].

**Theorem 6.** Let

$$\Psi(VMO_0, C([0, \infty])) = C^*(\mathcal{T}(VMO_0) \cup F_{C([0, \infty])}^{A_n^2}) \subset B(A_n^2(\mathbb{H}))$$

then $\Psi(VMO_0, C([0, \infty]))/K(A_n^2(\mathbb{H}))$ is a unital commutative $C^*$-algebra. For its maximal ideal space $M(\Psi)$ we have

$$M(\Psi) \cong (\mathcal{M} \times \{ \infty \}) \cup (\mathcal{M}_\infty \times [0, \infty])$$

where

$$\mathcal{M}_\infty = \{ \phi \in \mathcal{M} : \phi|_{\mathcal{T}(C(\mathbb{H}))/K(A_n^2(\mathbb{H}))} = \delta_\infty \}$$

is the fiber of $\mathcal{M}$ at $\infty$ with $\delta_\infty([T_f]) = f(\infty)$. 

\[ \Box \]
Proof. We will use Power’s theorem. In our case,
\[ H = A^2_\alpha(H), C_1 = T(VMO_\theta), C_2 = F^A_\alpha(C([0, \infty])), \text{ and } \hat{C} = \Psi(VMO_\theta, C([0, \infty]))/K(A^2_\alpha(H)). \]
We have
\[ M(C_1) = \mathcal{M} \text{ and } M(C_2) = [0, \infty]. \]
So we need to determine \((x, y) \in \mathcal{M} \times [0, \infty]\) so that for all \(f \in VMO_\theta(H)\) and \(\vartheta \in C([0, \infty])\) with \(0 < f, \vartheta \leq 1\), we have
\[
\hat{T}_f(x) = \vartheta(y) = 1 \Rightarrow \| T_f D_\theta \| = 1 \quad \text{or} \quad \| D_\theta T_f \| = 1.
\]
Let \(x \in \mathcal{M}\) such that \(x \in \mathcal{M}_t\) with \(t \neq \infty\) and \(y \in [0, \infty)\). Choose \(f \in C(H)\) and \(\vartheta \in C([0, \infty])\) such that
\[
\hat{T}_f(x) = f(t) = \vartheta(y) = 1, \quad 0 \leq f \leq 1, \quad 0 \leq \vartheta \leq 1, \quad f(z) < 1
\]
for all \(z \in \mathbb{H}\setminus\{t\}\) and \(\vartheta(w) < 1\) for all \(w \in [0, \infty]\setminus\{y\}\), where both \(f\) and \(\vartheta\) have compact supports. Consider \(D_\theta M_f : L^2_\alpha(H) \to L^2_\alpha(H)\): we have
\[
(D_\theta M_f)(g)(z) = \frac{2^{\alpha+1}}{\Gamma(\alpha + 1)} \int_0^\infty \hat{T}_f(z) e^{2\pi i t} \int_{\mathbb{H}} t^{1+\alpha} e^{-2\pi i \hat{z}} f(\zeta) g(\zeta) dA_\alpha(\zeta) dt
\]
\[
= \frac{2^{\alpha+1}}{\Gamma(\alpha + 1)} \int_0^\infty (f(\zeta)) \int_0^\infty \vartheta(t) t^{\alpha+1} e^{2\pi i t (z-\hat{z})} dt g(\zeta) dA_\alpha(\zeta)
\]
\[
= \int_{\mathbb{H}} g(z) k(z, \zeta) dA_\alpha(z)
\]
where \(k(z, \zeta) = \frac{2^{\alpha+1}}{\Gamma(\alpha + 1)} f(\zeta) f_0^\infty \vartheta(t) t^{\alpha+1} e^{2\pi i t (z-\hat{z})} dt\). Since \(f\) and \(\vartheta\) have compact supports we have
\[
\int_{\mathbb{H}} \int_{\mathbb{H}} |k(z, \zeta)|^2 dA_\alpha(z) dA_\alpha(\zeta) \leq \frac{2^{\alpha+1}}{\Gamma(\alpha + 1)} \| f \|_V^2 \| A_\alpha(K_1) \| \| \vartheta \|_{L^2(R)}^2 \| A_\alpha(K_2)
\]
where \(K_1, K_2 \subset \mathbb{H}\) are supports of \(f\) and \(\vartheta\) respectively, and
\[
A_\alpha(K) = \int_K dA_\alpha(z)
\]
is the \(dA_\alpha\) measure of the compact subset \(K \subset \mathbb{H}\). This implies that \(D_\theta M_f\) is Hilbert-Schmidt on \(L^2_\alpha(H)\) and hence is compact. Since
\[
\| T_f D_\theta \|_{L^2_\alpha} \leq \| M_f D_\theta \|_{L^2_\alpha}
\]
and
\[
\| M_f D_\theta \|_{L^2_\alpha} = \| (M_f D_\theta)^* \|_{L^2_\alpha} = \| D_\theta M_f \|_{L^2_\alpha}
\]
we have by C*-equality
\[
\| T_f D_\theta \|_{L^2_\alpha} \leq \| M_f D_\theta \|_{L^2_\alpha} = \| D_\theta M_f \|_{L^2_\alpha}
\]
\[
= \| (D_\theta M_f)^* (D_\theta M_f) \|_{L^2_\alpha} = \| M_f D_\theta^2 M_f \|_{L^2_\alpha}
\]
Since \(M_f D_\theta^2 M_f\) is a compact self-adjoint operator \(\| M_f D_\theta^2 M_f \|_{L^2_\alpha} = \lambda\) where \(\lambda\) is the largest eigenvalue of \(M_f D_\theta^2 M_f\). Let \(g \in L^2_\alpha\) be the corresponding eigenvector with \(\| g \|_{L^2_\alpha} = 1\) i.e. \((M_f D_\theta^2 M_f)g) = \lambda g\). Since \(f(z) < 1 \forall z \in \mathbb{H}\setminus\{t\}\) we have
\[
\| (M_f) h \|_{L^2_\alpha} < \| h \|_{L^2_\alpha} \quad \forall h \in L^2_\alpha. \quad \text{Hence we have}
\]
\[
\lambda = \| \lambda g \|_{L^2_\alpha} = \| (M_f D_\theta^2 M_f)(g) \|_{L^2_\alpha} < \| (D_\theta^2 M_f)(g) \|_{L^2_\alpha} \leq 1.
\]
And this implies that
\[ \| T_f D_\theta \|_{A^2_0}^2 \leq \lambda < 1. \]
Hence under these conditions we have
\[ \| D_\theta M_f \|_{L^2_0(H)} < 1 \Rightarrow \| D_\theta T_f \|_{A^2_0(H)} < 1 \Rightarrow (x, y) \notin M(\hat{C}), \]
so if \((x, y) \in M(\hat{C})\), then either \(y = \infty\) or \(x \in M_\infty\).
Let \(y = \infty\) and \(x \in M\). Let \(f \in \text{VMO}_0(\mathbb{H})\) and \(\vartheta \in C([0, \infty))\) such that
\[ 0 \leq f, \vartheta \leq 1 \quad \text{and} \quad \hat{T}_f(x) = \vartheta(y) = 1. \]
Let \(\varepsilon > 0\) be given and let \(t_0 \in (0, \infty)\) so that
\[ 1 - \varepsilon \leq \vartheta(t) < 1 \quad \forall t \in [t_0, \infty). \]
Let \(S_{t_0} : L^2_{0+1}(\mathbb{R}^+) \to L^2_{0+1}(\mathbb{R}^+)\) be defined as
\[ S_{t_0}f(t) = \begin{cases} f(t - t_0) & \text{if } t \geq t_0 \\ 0 & \text{otherwise} \end{cases} \]
Since \(\sup_{z \in \mathbb{H}} \{ |e^{it_0z}| \} = 1 \ \forall z \in \mathbb{H}\) we have
\[ \| T_{e^{it_0z}} f \|_{A^2_0} = 1 \]
Hence there is \(g \in A^2_0(\mathbb{H})\) so that \(\| g \|_{A^2_0} = 1\) and \(\| T_{e^{it_0z}} g \|_{A^2_0} > 1 - \varepsilon\). Since \(e^{it_0z} \in H^\infty(\mathbb{H})\) we have
\[ T_{e^{it_0z}} = M_{e^{it_0z}} T_f. \]
We observe that
\[ \mathcal{F}^{-1} S_{t_0} = M_{e^{it_0z}} \mathcal{F}^{-1} \]
and this implies that
\[ M_{e^{it_0z}} = \mathcal{F} S_{t_0} \mathcal{F}^{-1}. \]
So we have
\[ \mathcal{F} M_{e^{it_0z}} T_f = \mathcal{F}(\mathcal{F}^{-1} S_{t_0} \mathcal{F}) T_f = S_{t_0} \mathcal{F} T_f. \]
And this implies that
\[ \| D_\theta T_f(e^{it_0z}g) \|_{A^2_0} = \| \mathcal{F}^{-1} M_\theta \mathcal{F} T_{e^{it_0z}} f(g) \|_{A^2_0} = \| \mathcal{F}^{-1} M_\theta \mathcal{F} M_{e^{it_0z}} T_f(g) \|_{A^2_0} \]
\[ = \| \mathcal{F}^{-1} M_\theta S_{t_0} \mathcal{F} T_f(g) \|_{A^2_0} = \| M_\theta S_{t_0} \mathcal{F} T_f(g) \|_{L^2_{0+1}} \]
\[ S_{t_0} \mathcal{F} T_f(g) \] is supported on \([t_0, \infty)\), since \(T_{e^{it_0z}} f(g) = M_{e^{it_0z}} T_f(g) = \mathcal{F}^{-1} S_{t_0} \mathcal{F} T_f \) and \(\mathcal{F}^{-1}\) is an isometry we have
\[ \| S_{t_0} \mathcal{F} T_f(g) \|_{L^2_{0+1}} \geq 1 - \varepsilon. \]
This implies that
\[ \| D_\theta T_f(e^{it_0z}g) \|_{A^2_0} = \| M_\theta S_{t_0} \mathcal{F} T_f(g) \|_{L^2_{0+1}} \]
\[ \geq \inf \{ \vartheta(t) : t > t_0 \} \| S_{t_0} \mathcal{F} T_f(g) \|_{L^2_{0+1}} \geq (1 - \varepsilon)^2 \]
And since \(\| e^{it_0z} g \|_{A^2_0} \leq 1\) we have
\[ \| D_\theta T_f \|_{A^2_0(H)} \geq (1 - \varepsilon)^2 \]
for any \(\varepsilon > 0\). Hence we conclude that
\[ \| D_\theta T_f \|_{A^2_0(H)} = 1 \Rightarrow (x, \infty) \in M(\hat{C}) \quad \forall x \in M(C_1). \]
Now let \( x \in M_\infty \) and \( y \in [0, \infty) \). Let \( f \in VMO_\partial(H) \cap L^\infty(H) \) and \( \vartheta \in C([0, \infty]) \) such that
\[
\tilde{T}_f(x) = \partial(y) = 1 \quad \text{and} \quad 0 \leq f, \vartheta \leq 1.
\]
Since \( VMO_\partial(H) \cap L^\infty(H) \) is generated by linear algebraic combinations of functions \( f \) and \( \tilde{f} \) where \( f \in VMO_\partial(H) \cap H^\infty(H) \) w.l.o.g one may assume that \( f = f_0 \tilde{f}_0 \) where \( f_0 \in VMO_\partial(H) \cap H^\infty(H) \). By [13] we have \( H_{f_0} : A^2_\alpha \to (A^2_\alpha)'^* \) is compact where
\[
H_{f_0} = (I - P)M_{f_0}
\]
is the Hankel operator with symbol \( f_0 \). Since \( H_f = H_{f_0}M_{f_0} \), \( H_f \) is also compact.

Since \( f \in VMO_\partial(H) \cap L^\infty(H) \), for a given \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that
\[
\| \tilde{T}_f(x) - \frac{1}{Q_z} \int_{Q_z} f \circ \mathcal{C}^{-1}(w) dA(w) \| \leq \varepsilon.
\]
for all \( z \in \mathbb{D} \) with \( |1 - z| < \delta \) where \( Q_z = \{ w \in \mathbb{D} : |w| > |z|, \arg z - \arg w| < 1 - |z| \} \) and \( |Q_z| = (1 + |z|)(1 - |z|)^2 \) is the \( dA \) measure of \( Q_z \). Since \( \tilde{T}_f(x) = 1 \) and \( 0 \leq f \leq 1 \), this implies that for all \( \varepsilon > 0 \) there exists \( w_0 > 0 \) such that \( 1 - \varepsilon \leq f(w) \leq 1 \) for all \( w \in H \) with \( |w| > w_0 \). We have \( M_f = T_f + H_f \) which is valid in much more general contexts. Since \( H_f \) is compact for any \( g \in A^2_\alpha(H) \), we have
\[
\lim_{w \to \infty} \| H_f(S_w g(\cdot))\|_{L^2_\alpha(H)} = 0
\]
where \( S_w g(z) = g(z - w) \) \( (w \in \mathbb{R}) \) is the translation operator, since \( S_w g \) converges to \( 0 \) weakly as \( w \) tends to infinity and \( H_f \) is compact. Let \( g \in A^2_\alpha(H) \) so that \( \|g\|_{A^2_\alpha} = 1 \) and \( \|D_{\partial} g\|_{A^2_\alpha} > 1 - \frac{\varepsilon}{2} \). Let \( K \subset H \) be such that \( K \subset C \) is compact and
\[
\left( \int_K |D_{\partial} g(z)|^2 dA_\alpha(z) \right)^{1/2} > 1 - \varepsilon.
\]
Let \( w > w_0 \) so that
\[
\| H_f(S_w D_{\partial} g) \|_{L^2_\alpha(H)} \leq \varepsilon.
\]
Then we have
\[
\| T_f(S_w D_{\partial} g) \|_{A^2_\alpha(H)} = \| (M_f - H_f)(S_w D_{\partial} g) \|_{A^2_\alpha(H)} \geq \| M_f(S_w D_{\partial} g) \|_{L^2_\alpha(H)} - \| H_f(S_w D_{\partial} g) \|_{L^2_\alpha(H)} \geq \| M_f(S_w D_{\partial} g) \|_{L^2_\alpha(H)} - \varepsilon
\]
Since \( f(z) > 1 - \varepsilon \ \forall \ z > w_0 \) we have
\[
\| M_f S_w D_{\partial} g \|_{L^2_\alpha(H)} \geq \left( \int_{w+K} |f(z) D_{\partial}(g)(z - w)|^2 dA_\alpha(z) \right)^{1/2} = \left( \int_K |f(u + w) D_{\partial}(g)(u)|^2 dA_\alpha(u) \right)^{1/2} \geq \inf \{ f(z) : z \in w + K \} \left( \int_K |D_{\partial}(g)(z)|^2 dA_\alpha(z) \right)^{1/2} \geq (1 - \varepsilon)^2
\]
So we have
\[
\| T_f S_w D_{\partial} g \|_{A^2_\alpha} = \| T_f D_{\partial} S_w g \|_{A^2_\alpha} \geq (1 - \varepsilon)^2 - \varepsilon
\]
since \( D_{\partial} S_w = S_w D_{\partial} \forall w \in \mathbb{R} (S_w = D_{\epsilon^{2n} \alpha w}) \). Since \( S_w \) is unitary for all \( w \in \mathbb{R} \), we have \( \| S_w g \|_{A^2_\alpha} = 1 \) and this implies that
\[
\| T_f D_{\partial} \|_{A^2_\alpha} \geq (1 - \varepsilon)^2 - \varepsilon
\]
Proof. Let $A$ be the local essential range of $\psi$. Since $A$ is also a C*-subalgebra of $\Psi$, we have

$$\| T_f D_\theta \|_{A^2_\infty} = 1$$

and $(x, y) \in M(\tilde{C})$ for all $x \in M_\infty$. □

5. MAIN RESULTS

In this section we characterize the essential spectra of quasi-parabolic composition operators with translation functions in $VMO_0$ class which is the main aim of the paper. In doing this we will heavily use Banach algebraic methods.

We will need the following result whose proof uses a theorem of [7](p. 171) and which gives the values $\hat{T}_f(x)$ of $T_f$ for $x \in M_\infty$ on the fibers of $M$ the maximal ideal space of $T(VMO_0)/K(A_\infty^0(\mathbb{H}))$ at infinity:

**Proposition 7.** Let $\psi \in VMO_0 \cap H^\infty(\mathbb{H})$ and $M$ be the maximal ideal space of $T(VMO_0)/K(A_\infty^0(\mathbb{H}))$. Let $M_\infty$ be the fiber of $M$ at infinity which is defined as

$$M_\infty = \{x \in M : x|_{T(C(\mathbb{H}))} = \delta_\infty\}$$

where $\delta_\infty([T_f]) = f(\infty)$. Then we have

$$\{[\hat{T}_\psi](x) : x \in M_\infty\} = R_\infty(\psi^*)$$

where $\psi^* \in L^\infty(\mathbb{R})$ is the boundary value function of $\psi$ i.e. $\psi^*(x) = \lim_{y \to 0} \psi(x + iy)$ and

$$R_\infty(\psi^*) = \{\zeta \in \mathbb{C} : |x : \psi^*(x) - \zeta| \leq \varepsilon\} \cap (\mathbb{R} \setminus [-n, n]) \geq 0, \forall \varepsilon > 0, \forall n \in \mathbb{N}\}

is the local essential range of $\psi^*$ at infinity.

**Proof.** Let $A = C^*(\{T_\psi\} \cup \{T_f : f \in C(\mathbb{H})\})/K(A_\infty^0(\mathbb{H})) \subset B(A_\infty^0(\mathbb{H}))/K(A_\infty^0(\mathbb{H}))$. $A$ is also a C*-subalgebra of $\Psi(VMO_0, C([0, \infty]))/K(A_\infty^0(\mathbb{H}))$. We observe that $A$ is isometrically isomorphic to $\tilde{A}$ where $\tilde{A} = C^*(\{\psi^*\} \cup C(\mathbb{R})) \subset L^\infty(\mathbb{R})$, via the correspondence $f \to [T_f]$. Since the ideal generated by a maximal ideal $I \subset A$ in $\Psi = \Psi(VMO_0, C([0, \infty]))/K(A_\infty^0(\mathbb{H}))$ is contained in a maximal ideal of $\Psi$, we have

$$\{[\hat{T}_\psi](x) : x \in M_\infty\} = \{[\hat{T}_\psi](x) : x \in M(A)_\infty\}$$

where $M(A)_\infty = \{x \in M(A) : x|_{T(C(\mathbb{H}))}/K(A_\infty^0(\mathbb{H})) = \delta_\infty\}$ with $\delta_\infty([T_f]) = f(\infty)$. Since $A$ is isometrically isomorphic to $\tilde{A}$ we have

$$\{[\hat{T}_\psi](x) : x \in M(A)_\infty\} = \{\hat{\psi^*}(x) : x \in M(\tilde{A})\}$$

Similarly since the ideal generated by a maximal ideal $I \subset \tilde{A}$ in $L^\infty(\mathbb{R})$ is contained in a maximal ideal in $L^\infty(\mathbb{R})$, we have

$$\{\hat{\psi^*}(x) : x \in M(\tilde{A})_\infty\} = \{\hat{\psi^*}(x) : x \in M(L^\infty(\mathbb{R}))_\infty\}$$

By the theorem of [7](p. 171) we have

$$\{\hat{\psi^*}(x) : x \in M(L^\infty(\mathbb{R}))_\infty\} = R_\infty(\psi^*)$$

Therefore we have

$$\{[\hat{T}_\psi](x) : x \in M_\infty\} = R_\infty(\psi^*)$$

Firstly we have the following result on the upper half-plane:
Theorem A. Let $\psi \in VMO_\partial(\mathbb{H}) \cap H^\infty(\mathbb{H})$ such that $\Im(\psi(z)) > \epsilon > 0$ for all $z \in \mathbb{H}$ then for $\varphi(z) = z + \psi(z)$ and $\alpha > -1$ we have

- (i) $C_\varphi : A^2_\alpha(\mathbb{H}) \to A^2_\alpha(\mathbb{H})$ is essentially normal
- (ii) $\sigma_e(C_\varphi) = \{e^{izt} : t \in [0, \infty], z \in \mathbb{R}\}$ where $\mathbb{R}^\infty(\psi^*)$ is the local essential range of $\psi^* \in L^\infty(\mathbb{R})$ at $\infty$ and $\psi^*$ is the boundary limit value function of $\psi$.

Proof. By Proposition 3 we have the following series expansion for $C_\varphi$:

$$C_\varphi = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{n!\Gamma(\alpha + 2)} T_{\tau_n} D_{\phi_n}$$

where $\tau(z) = \psi(z) - i\beta$ and $\phi_n(t) = \frac{(2\pi it)^n e^{-2\pi \beta t}}{(\alpha+2)(\alpha+3)\ldots(\alpha+n+1)}$. So we conclude that if $\psi \in VMO_\partial(\mathbb{H}) \cap H^\infty(\mathbb{H})$ with $\Im(\psi(z)) > \epsilon > 0$ then

$$C_\varphi \in \Psi(VMO_\partial, C([0, \infty]))$$

where $\varphi(z) = z + \psi(z)$. Since $\Psi(VMO_\partial, C([0, \infty])) / K(A^2_\alpha(\mathbb{H}))$ is commutative, for any $T \in \Psi(VMO_\partial, C([0, \infty]))$ and $\tau \in \Psi(VMO_\partial, C([0, \infty]))$ we have

$$[TT^*] = [T][T^*] = [T^*[T] = [T^*T].$$

This implies that $(TT^* - T^*T) \in K(A^2_\alpha(\mathbb{H}))$. Since $C_\varphi \in \Psi(VMO_\partial, C([0, \infty]))$ we also have

$$(C_\varphi C_\varphi - C_\varphi C_\varphi^*) \in K(A^2_\alpha(\mathbb{H})).$$

This proves (i).

For (ii) we look at the values of $\Gamma[C_\varphi]$ at $M(\Psi) = M(\Psi(VMO_\partial, C([0, \infty])) / K(A^2_\alpha(\mathbb{H})))$ where $\Gamma$ is the Gelfand transform of $\Psi(VMO_\partial, C([0, \infty])) / K(A^2_\alpha(\mathbb{H}))$. By Theorem 6 we have

$$M(\Psi) = (M \times \{\infty\}) \cup (M \times [0, \infty])$$

where $M = M(T(VMO_\partial) / K(A^2_\alpha(\mathbb{H})))$. By equation (6) we have the Gelfand transform $\Gamma[C_\varphi]$ of $C_\varphi$ at $t = \infty$ as

$$\Gamma[C_\varphi](x, \infty) = \sum_{j=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{j!\Gamma(\alpha + 2)} (\tau_j(x))^j \phi_j(\infty) = 0 \quad \forall x \in M$$

since $\phi_j(\infty) = 0$ for all $j \in \mathbb{N}$ where $\phi_j(t) = \frac{(2\pi it)^j e^{-2\pi \beta t}}{(\alpha+2)(\alpha+3)\ldots(\alpha+n+1)}$. We calculate $\Gamma[C_\varphi]$ of $C_\varphi$ for $x \in M_\infty$ as

$$\Gamma[C_\varphi](x, t) = \left( \sum_{j=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{j!\Gamma(\alpha + 2)} (\tau_j(x))^j D_{\phi_j}(\frac{2\pi it)^j e^{-2\pi \beta t}}{(\alpha+2)(\alpha+3)\ldots(\alpha+n+1)}) \right)(x, t) = \sum_{j=0}^{\infty} \frac{\Gamma(n + 2 + \alpha)}{j!\Gamma(\alpha + 2)} (\tau_j(x))^j e^{2\pi i T_{\tau_j}(x)},$$

where

$$\tau_j(x) = \frac{(2\pi it)^j e^{-2\pi \beta t}}{(\alpha+2)(\alpha+3)\ldots(\alpha+n+1)}$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!}(\tau_j(x))^j (2\pi it)^j e^{-2\pi \beta t} = e^{2\pi i T_{\tau_j}(x)}t.$$
for all $x \in \mathcal{M}_\infty$ and $t \in [0, \infty)$ since

$$(\alpha + 2)(\alpha + 3)\ldots(\alpha + n + 1) = \frac{\Gamma(n + 2 + \alpha)}{\Gamma(\alpha + 2)}.$$ 

So we have $\Gamma[C_\varphi]$ as the following:

$$\Gamma([C_\varphi])(x, t) = \begin{cases} e^{2\pi i x t} & \text{if } x \in \mathcal{M}_\infty \\ 0 & \text{if } t = \infty \end{cases} \quad (10)$$

Since $\Psi = \Psi(VMO_0, C([0, \infty]))/K(A^2_{\alpha}(\mathbb{H}))$ is a commutative Banach algebra with identity, by equations (2) and (10) we have

$$\sigma_{\Psi}([C_\varphi]) = \{\Gamma([C_\varphi])(x, t) : (x, t) \in M(\Psi(VMO_0, C([0, \infty]))/K(A^2_{\alpha}(\mathbb{H}))) \} = \{e^{2\pi i x t} : x \in \mathcal{M}_\infty, t \in [0, \infty) \} \cup \{0\} \quad (11)$$

Since $\Psi$ is a closed *-subalgebra of the Calkin algebra $\mathcal{B}(A^2_{\alpha}(\mathbb{H}))/K(A^2_{\alpha}(\mathbb{H}))$ which is also a C*-algebra, by equation (1) we have

$$\sigma_{\Psi}([C_\varphi]) = \sigma_{\mathcal{B}(A^2_{\alpha})/K(A^2_{\alpha})}([C_\varphi]). \quad (12)$$

But by definition $\sigma_{\mathcal{B}(A^2_{\alpha})/K(A^2_{\alpha})}([C_\varphi])$ is the essential spectrum of $C_\varphi$. Hence we have

$$\sigma_e(C_\varphi) = \{e^{ix t} : x \in \mathcal{M}_\infty, t \in [0, \infty) \} \cup \{0\}. \quad (13)$$

Now it only remains for us to understand what the set $\{[T_\psi](x) = x([T_\psi]) : x \in \mathcal{M}_\infty \}$ looks like, where $\mathcal{M}_\infty$ is as defined in Theorem 6. By Proposition 7 we have

$$\{[T_\psi](x) : x \in \mathcal{M}_\infty \} = \mathcal{R}_\infty(\psi^*).$$

By Proposition 7 and equation (13) we have

$$\sigma_e(C_\varphi) = \{\{\Gamma[C_\varphi]\}(x, t) : (x, t) \in M(\Psi(VMO_0, C([0, \infty]))/K(A^2_{\alpha}(\mathbb{H}))) \} = \{e^{iz t} : t \in [0, \infty), z \in \mathcal{R}_\infty(\psi^*) \} \cup \{0\}$$

The local essential range $\mathcal{R}_1(f)$ of $f \in L^\infty(\mathbb{T})$ at 1 is defined to be the set of points $\zeta \in \mathbb{C}$ for which the set $\{z \in \mathbb{T} : |f(z) - \zeta| < \varepsilon \} \cap S_{1, r}$ has positive Lebesgue measure $\forall \varepsilon > 0$ and $\forall r > 0$ where $S_{1, r} = \{z \in \mathbb{T} : |z - 1| < r \}$. Conjugating by Cayley transform we observe that

$$\mathcal{R}_1(f) = \mathcal{R}_\infty(f \circ \mathcal{C}).$$

Using this observation we prove the analogous result for the unit disc case:

**Theorem B.** Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that

$$\varphi(z) = \frac{2iz + \eta(z)(1 - z)}{2\eta(z) + (1 - z)}$$

where $\eta \in H^\infty(\mathbb{D})$ with $\Re(\eta(z)) > \varepsilon > 0$ for all $z \in \mathbb{D}$. If $\eta \in VMO_0(\mathbb{D}) \cap H^\infty$ then we have

- (i) $C_\varphi : A^2_{\alpha}(\mathbb{D}) \to A^2_{\alpha}(\mathbb{D})$ is essentially normal
- (ii) $\sigma_e(C_\varphi) = \{e^{iz t} : t \in [0, \infty), z \in \mathcal{R}_1(\eta^*) \} \cup \{0\}$

where $\mathcal{R}_1(\eta^*)$ is the local essential range of $\eta^* \in L^\infty(\mathbb{T})$ at 1 and $\eta^*$ is the boundary limit value function of $\eta$. 
Proof. Using the isometric isomorphism $\Phi : A^2(\mathbb{D}) \to A^2(\mathbb{H})$ introduced in section 2, if $\varphi : \mathbb{D} \to \mathbb{D}$ is of the form

$$\varphi(z) = \frac{2iz + \eta(z)(1 - z)}{2t + \eta(z)(1 - z)}$$

where $\eta \in H^\infty(\mathbb{D})$ satisfies $\Im(\eta(z)) > \delta > 0$ then, by equation (9), for $\tilde{\varphi} = \mathcal{C}^{-1} \circ \varphi \circ \mathcal{C}$ we have $\tilde{\varphi}(z) = z + \eta \circ \mathcal{C}(z)$ and

$$\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1} = T_{\frac{2i\alpha + \eta \circ \mathcal{C}}{2t + \eta \circ \mathcal{C}}}(z) C_{\tilde{\varphi}}.$$  \hfill (14)

For $\eta \in VMO_\partial \cap H^\infty(\mathbb{D})$ we have both

$$C_{\tilde{\varphi}} \in \Psi(VMO_\partial, C([0, \infty]))$$

and hence

$$\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1} \in \Psi(VMO_\partial, C([0, \infty])).$$

Since $\Psi(VMO_\partial, C([0, \infty]))/K(A^2_\partial(\mathbb{H}))$ is commutative and $\Phi$ is an isometric isomorphism, (i) follows from the argument $C_{\tilde{\varphi}}$ is essentially normal if and only if $\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}$ is essentially normal and by equation (14).

For (ii) we look at the values of $\Gamma[\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}]$ at $M(\Psi(VMO_\partial, C([0, \infty]))/K(A^2_\partial))$ where $\Gamma$ is the Gelfand transform of $\Psi(VMO_\partial, C([0, \infty]))/K(A^2_\partial)$. Again applying the Gelfand transform for

$$(x, \infty) \in M(VMO_\partial, C([0, \infty]))/K(A^2_\partial) \subset \mathcal{M} \times [0, \infty]$$

we have

$$\Gamma[\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}](x, \infty) = ((\Gamma[T_{\frac{2i\alpha + \eta \circ \mathcal{C}}{2t + \eta \circ \mathcal{C}}]}(x, \infty))(\Gamma[C_{\tilde{\varphi}}])(x, \infty))$$

Appealing to equation (8) we have $\Gamma[C_{\tilde{\varphi}}](x, \infty) = 0$ for all $x \in \mathcal{M}$ hence we have

$$\Gamma[\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}])(x, \infty) = 0$$

for all $x \in \mathcal{M}$. Applying the Gelfand transform for

$$(x, t) \in \mathcal{M}_\infty \times [0, \infty] \subset M(\Psi(VMO_\partial, C([0, \infty]))/K(A^2_\partial))$$

we have

$$\Gamma[\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}](x, t) = ((\Gamma[T_{\frac{2i\alpha + \eta \circ \mathcal{C}}{2t + \eta \circ \mathcal{C}}]}(x, t))(\Gamma[C_{\tilde{\varphi}}](x, t))$$

$$= ((1 + \Gamma[T_{\eta \circ \mathcal{C}}](x, t))((\Gamma[C_{\tilde{\varphi}}])(x, t))$$

Since $x \in \mathcal{M}_\infty$ we have

$$\Gamma[T_{\frac{2i\alpha + \eta \circ \mathcal{C}}{2t + \eta \circ \mathcal{C}}]}(x, t) = T_{\frac{2i\alpha + \eta \circ \mathcal{C}}{2t + \eta \circ \mathcal{C}}}(x) = x(T_{\frac{2i\alpha + \eta \circ \mathcal{C}}{2t + \eta \circ \mathcal{C}}} = 0.$$}

Hence we have

$$\Gamma[\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}](x, t) = (\Gamma[C_{\tilde{\varphi}}])(x, t)$$

for all $(x, t) \in \mathcal{M}_\infty \times [0, \infty]$. Moreover we have

$$\Gamma[\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}](x, t) = (\Gamma[C_{\tilde{\varphi}}])(x, t)$$

for all $(x, t) \in M(\Psi(VMO_\partial, C([0, \infty]))/K(A^2_\partial))$. Therefore by similar arguments in Theorem A (equations (11) and (12)) we have

$$\sigma_e(\Phi \circ C_{\tilde{\varphi}} \circ \Phi^{-1}) = \sigma_e(C_{\tilde{\varphi}}).$$

By Theorem A (together with equation (14)) we have

$$\sigma_e(C_{\varphi}) = \sigma_e(C_{\tilde{\varphi}}) = \{e^{izt} : z \in \mathcal{R}_\infty(\eta \circ \mathcal{C}^*) = \mathcal{R}_\infty(\eta^*) \cap [0, \infty\right).$$
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