Conformal Perturbations of modified Novikov Operators and the
Kastler-Kalau-Walze type theorem

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Abstract

In this paper, we obtain two Kastler-Kalau-Walze type theorems for conformal perturbations of modified
Novikov Operators on four-dimensional and six-dimensional compact manifolds with (respectively without)
boundary.

Keywords: Conformal perturbations of modified Novikov Operators; noncommutative residue;
Kastler-Kalau-Walze type theorem.

1. Introduction

The noncommutative residue found in \cite{1, 2} plays a prominent role in noncommutative geometry. For
one-dimensional manifolds, the noncommutative residue was discovered by Adler \cite{3} in connection with
geometric aspects of nonlinear partial differential equations. For arbitrary closed compact \(n\)-dimensional
manifolds, the noncommutative residue was introduced by Wodzicki in \cite{2} using the theory of zeta functions
of elliptic pseudodifferential operators. In \cite{4}, Connes used the noncommutative residue to derive a conformal
4-dimensional Polyakov action analogy. Furthermore, Connes made a challenging observation that the
noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-
Hilbert action in \cite{5}. In \cite{6}, Kastler gave a brute-force proof of this theorem. In \cite{7}, Kalau and Walze
proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the
Wodzicki residue of the square of the inverse of the Dirac operator \(\text{Wres}(D^{-2})\) in turn is essentially the
second coefficient of the heat kernel expansion of \(D^2\) in \cite{8}.

In \cite{9}, Ponge defined lower dimensional volumes of Riemannian manifolds by the Wodzicki residue.
Fedosov et al. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a
unique continuous trace in \cite{10}. In \cite{11}, Schrohe gave the relation between the Dixmier trace and the
noncommutative residue for manifolds with boundary. In \cite{12}, Wang generalized the Kastler-Kalau-Walze
type theorem to the cases of 3, 4-dimensional spin manifolds with boundary and proved a Kastler-Kalau-
Walze type theorem. In \cite{12, 13, 14, 15, 16, 17}, Y. Wang and his coauthors computed the lower dimensional
volumes for 5, 6, 7-dimensional spin manifolds with boundary and also got some Kastler-Kalau-Walze type
theorems. In \cite{17}, authors computed \(\text{Wres}[(\pi^{+}D^{-2}) \circ (\pi^{+}D^{-n+2})]\) for any-dimensional manifolds with
boundary, and proved a general Kastler-Kalau-Walze type theorem.

In \cite{19}, Y. Wang proves a Kastler-Kalau-Walze type theorem for perturbations of Dirac operators on
compact manifolds with or without boundary. In \cite{20}, J. Wang and Y. Wang proved two kinds of Kastler-
Kalau-Walze type theorems for conformal perturbations of twisted Dirac operators and conformal perturba-
tions of signature operators by a vector bundle with a non-unitary connection on four-dimensional manifolds

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with (respectively without) boundary. In [21], López and his collaborators introduced an elliptic differential operator which is called Novikov operator. In [22], we prove Kastler-Kalau-Walze-type theorems for modified Novikov operators on compact manifolds with (respectively without) a boundary.

The motivation of this paper is to establish two Kastler-Kalau-Walze type theorems for conformal perturbations of modified Novikov operators on manifolds with boundary. We know that the leading symbol of conformal perturbations of modified Novikov operators is not \(ic(\xi)\). This is the reason that we study the residue of conformal perturbations of modified Novikov operators.

This paper is organized as follows: In Section 2, we recall some basic facts and formulas about Boutet de Monvel’s calculus and modified Novikov Operators. In Section 3, we give a Kastler-Kalau-Walze type theorem for conformal perturbations of modified Novikov Operators on four-dimensional manifolds with boundary. In Section 4, we give a Kastler-Kalau-Walze type theorem for conformal perturbations of modified Novikov Operators on six-dimensional manifolds with boundary. The main results are Theorem 3.4 and Theorem 4.3 in this paper.

2. Boutet de Monvel’s calculus and modified Novikov Operators

In this section, we shall recall some basic facts and formulas about Boutet de Monvel’s calculus. Let

\[F: L^2(R) \to L^2(R); \quad F(u)(v) = \int_R e^{-i\nu t}u(t)dt\]

denote the Fourier transformation and \(\varphi(R^+)\) (similarly define \(\varphi(R^-)\)), where \(\varphi(R)\) denotes the Schwartz space and

\[r^+: C^\infty(R) \to C^\infty(R^+); \quad f \mapsto f|_{R^+}; \quad \mathbb{R}^+ = \{x \geq 0; x \in \mathbb{R}\}. \tag{2.1}\]

We define \(H^+ = F(\varphi(R^+))\); \(H_0^- = F(\varphi(R^-))\) which are orthogonal to each other. We have the following property: \(h \in H^+\) (resp. \(H^-\)) if and only if \(h \in C^\infty(R)\) which has an analytic extension to the lower (resp. upper) complex half-plane \(\{\text{Im} \xi < 0\}\) (resp. \(\{\text{Im} \xi > 0\}\)) such that for all nonnegative integer \(l,

\[\frac{d^l}{d\xi^l}(\xi) = \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l}(\frac{c_k}{\xi^{k+l}}) \tag{2.2}\]

as \(|\xi| \to +\infty, \text{Im} \xi \leq 0\) (resp. \(\text{Im} \xi \geq 0\)).

Let \(H'\) be the space of all polynomials and \(H^- = H_0^- \bigoplus H'\); \(H = H^+ \bigoplus H^-\). Denote by \(\pi^+\) (resp. \(\pi^-\)) respectively the projection on \(H^+\) (resp. \(H^-\)). For calculations, we take \(H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}\) (\(\tilde{H}\) is a dense set in the topology of \(H\)). Then on \(\tilde{H}\),

\[\pi^+h(\xi_0) = \frac{1}{2\pi i} \lim_{\nu \to 0^+} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + \nu \xi - \xi}d\xi, \tag{2.3}\]

where \(\Gamma^+\) is a Jordan close curve included \(\text{Im} \xi > 0\), surrounding all the singularities of \(h\) in the upper half-plane and \(\xi_0 \in \mathbb{R}\). Similarly, define \(\pi^-\) on \(\tilde{H}\),

\[\pi^-h = \frac{1}{2\pi} \int_{\Gamma^-} h(\xi)d\xi. \tag{2.4}\]

So, \(\pi^-(H^-) = 0\). For \(h \in H \bigcap L^1(R)\), \(\pi^+h = \frac{1}{2\pi} \int_R h(v)dv\) and for \(h \in H^+ \bigcap L^1(R)\), \(\pi^-h = 0\).

Denote by \(\mathcal{B}\) Boutet de Monvel’s algebra. For a detailed introduction to Boutet de Monvel’s algebra see Boutet de Monvel [24], Grubb [25], Rempel-Schulze [26] or Schrohe-Schulze [27]. In the following we will give a review of some basic facts we need.

An operator of order \(m \in \mathbb{Z}\) and type \(d\) is a matrix

\[A = \begin{pmatrix} \pi^+P + G & K \\ T & S \end{pmatrix}: C^\infty(X, E_1) \bigoplus C^\infty(\partial X, F_1) \to C^\infty(X, E_2) \bigoplus C^\infty(\partial X, F_2), \]
where $X$ is a manifold with boundary $\partial X$ and $E_1, E_2$ (resp. $F_1, F_2$) are vector bundles over $X$ (resp. $\partial X$). Here, $P : C^\infty(\Omega, E_1) \rightarrow C^\infty(\Omega, E_2)$ is a classical pseudodifferential operator of order $m$ on $\Omega$, where $\Omega$ is an open neighborhood of $X$ and $E_i[X = E_i] (i = 1, 2)$. Then $P$ has an extension: $\mathcal{E}(\Omega, E_1) \rightarrow \mathcal{D}^r(\Omega, E_2)$, where $\mathcal{E}(\Omega, E_1)$ and $\mathcal{D}^r(\Omega, E_2)$ are the dual space of $C^\infty(\Omega, E_1)$ and $C^\infty(\Omega, E_2)$. Let $e^+ : C^\infty(X, E_1) \rightarrow \mathcal{E}(\Omega, E_1)$ denote extension by zero from $X$ to $\Omega$ and $r^+ : \mathcal{D}^r(\Omega, E_2) \rightarrow \mathcal{D}^r(\Omega, E_2)$ denote the restriction from $\Omega$ to $X$, then define

$$\pi^+ P = r^+ P e^+ : C^\infty(X, E_1) \rightarrow \mathcal{D}^r(\Omega, E_2).$$

In addition, $P$ is supposed to have the transmission property; this means that, for all $j, k, \alpha$, the homogeneous component $p_j$ of order $j$ in the asymptotic expansion of the symbol $p$ of $P$ in local coordinates near the boundary satisfies:

$$\partial_x^j \partial_\xi^k p_j(x', 0, 0, +1) = (-1)^{j - |\alpha|} \partial_{x'}^j \partial_\xi^k p_j(x', 0, 0, -1),$$

then $\pi^+ P$ maps $C^\infty(X, E_1)$ into $C^\infty(X, E_2)$ by Section 2.1 of [14].

Let $G, T$ be respectively the singular Green operator and the trace operator of order $m$ and type $d$. $K$ is a potential operator and $S$ is a classical pseudodifferential operator of order $m$ along the boundary (for detailed definition, see [11]). Denote by $B^{m,d}$ the collection of all operators of order $m$ and type $d$, and $\mathcal{S}$ is the union over all $m$ and $d$.

Recall $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^{m,d} \times B^{m',d'} \rightarrow B^{m+m',\max\{m'+d,d\}}$. Write

$$\tilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in B^{m,d}, \tilde{A}' = \begin{pmatrix} \pi^+ P' + G' & K' \\ T' & S' \end{pmatrix} \in B^{m',d'}.$$

The composition $\tilde{A} \tilde{A}'$ is obtained by multiplication of the matrices (for more details see [14]). For example $\pi^+ P \circ G'$ and $G \circ G'$ are singular Green operators of type $d'$ and

$$\pi^+ P \circ \pi^+ P' = \pi^+ (P P') + L(P, P').$$

Here $P P'$ is the usual composition of pseudodifferential operators and $L(P, P')$ called leftover term is a singular Green operator of type $m' + d$. For our case, $P, P'$ are classical pseudo differential operators, in other words $\pi^+ P \in \mathcal{B}^\infty$ and $\pi^+ P' \in \mathcal{B}^\infty$.

In the following, write $\pi^+ D^{-1} = \begin{pmatrix} \pi^+ D^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. Let $M$ be a compact manifold with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,$$  \quad (2.5)

where $g^{\partial M}$ is the metric on $\partial M$. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic $\partial M \times [0, 1)$. By the definition of $h(x_n) \in C^\infty([0, 1])$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{(0,1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\tilde{g}$ on $\tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\tilde{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2,$$  \quad (2.6)

such that $\tilde{g}|M = g$. We fix a metric $\tilde{g}$ on the $\tilde{M}$ such that $\tilde{g}|M = g$.

Consider the $n - 1$-form

$$\sigma(\xi) = \sum_{j=1}^{n} (-1)^{j+1} \xi_j dx_1 \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge dx_n,$$

where the hat indicates that the corresponding factor has been omitted.
Restricted $\sigma(\xi)$ to the $n - 1$-dimensional unit sphere $|\xi| = 1$, $\sigma(\xi)$ gives the volume form on $|\xi| = 1$. Denoting by $|\xi'| = 1$ and $\sigma(\xi')$ the $n - 2$-dimensional unit sphere and the corresponding $n - 2$-form.

Denote by $B^\infty$ the algebra of all operators in Boutet de Monvel’s calculus (with integral order) and by $B^{-\infty}$ the ideal of all smoothing operators in $B^\infty$. Now we recall the main theorem in [10].

**Theorem 2.1. (Fedosov-Golse-Leichtnam-Schrohe)** Let $X$ and $\partial X$ be connected, $\dim X = n \geq 3$, $A = \left( \begin{array}{ccc} \pi^+P + G & K \\ T & S \end{array} \right) \in B$, and denote by $p, b$ and $s$ the local symbols of $P, G$ and $S$ respectively.

Define:

$$\hat{\text{Wres}}(A) = \int_X \int_{|\xi|=1} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx$$

$$+ 2\pi \int_X \int_{|\xi'|=1} \left\{ \text{tr}_E [(trb_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \right\} \sigma(\xi') dx' ,$$

then

a) $\hat{\text{Wres}}([A, B]) = 0$, for any $A, B \in B$;

b) It is a unique continuous trace on $B/B^{-\infty}$.

Secondly, we recall the definition of Novikov Operator (see details in [21]). Let $M$ be a $n$-dimensional $(n \geq 3)$ oriented compact Riemannian manifold with a Riemannian metric $g$. The de Rham derivative $d$ is a differential operator on $C^\infty(M; \wedge^*TM)$. Then we have the de Rham coderivative $\delta = d^*$, the symmetric operators $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ (the Laplacian).

With more generality, we take any closed $\theta \in C^\infty(M; T^*M)$. For the sake of simplicity, we assume that $\theta$ is real. Then we have the Novikov operators defined by $\theta$, depending on $z \in \mathbb{C}$ in [21],

$$d_z = d + z(\theta \wedge), \quad \bar{d}_z = d_z^* = \delta + \bar{\pi}(\theta \wedge)^*$$

$$D_z = d_z + \bar{d}_z = (d + \delta) + z(\theta \wedge) + \bar{\pi}(\theta \wedge)^*$$

$$= (d + \delta) + [\text{Re}z(\theta \wedge) + \text{Re}z(\theta \wedge)^*] + i[\text{Im}z(\theta \wedge) - \text{Im}z(\theta \wedge)^*]$$

$$= (d + \delta) + \text{Re}z(\theta \wedge - (\theta \wedge)^*) + i\text{Im}z[\theta \wedge - (\theta \wedge)^*]$$

$$= (d + \delta) + \text{Re}z(\theta) + i\text{Im}z\bar{c}(\theta),$$

where $\text{Re}z$ is the real part of $z$, $\text{Im}z$ is the imaginary part of $z$, $\bar{c}(\theta) = (\theta)^* \wedge + (\theta \wedge)^*$ and $c(\theta) = (\theta)^* \wedge - (\theta \wedge)^*$.

For $\theta, \theta' \in \Gamma(TM)$, we consider the modified Novikov operators. We define that

$$\tilde{D}_N = d + \delta + \bar{c}(\theta) + c(\theta'), \quad \tilde{D}_N^* = d + \delta + \bar{c}(\theta) - c(\theta'),$$

where $\bar{c}(\theta) = (\theta)^* \wedge + (\theta \wedge)^*$ and $c(\theta') = (\theta')^* \wedge - (\theta' \wedge)^*$.

Let $\nabla^L$ be the Levi-Civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, the connection matrix $(\omega_{a,i})$ is defined by

$$\nabla^L(\tilde{e}_1, \cdots, \tilde{e}_n) = (\tilde{e}_1, \cdots, \tilde{e}_n)(\omega_{a,i}).$$

(2.8)

Let $\epsilon(\tilde{e}_j^*)$, $\iota(\tilde{e}_j^*)$ be the exterior and interior multiplications respectively and $c(\tilde{e}_j)$ be the Clifford action. Suppose that $\partial_\theta$ is a natural local frame on $TM$ and $(g^{ij})_{1 \leq i, j \leq n}$ is the inverse matrix associated to the metric matrix $(g_{ij})_{1 \leq i, j \leq n}$ on $M$. Write

$$c(\tilde{e}_j) = \epsilon(\tilde{e}_j^*) - \iota(\tilde{e}_j^*) \quad \text{and} \quad \bar{c}(\tilde{e}_j) = \epsilon(\tilde{e}_j^*) + \iota(\tilde{e}_j^*).$$

(2.9)

The modified Novikov Operators $\tilde{D}_N$ and $\tilde{D}_N^*$ are defined by

$$\tilde{D}_N = d + \delta + \bar{c}(\theta) + c(\theta') = \sum_{i=1}^n c(\tilde{e}_i) \left[ \tilde{e}_i + \frac{1}{4} \sum_{s, t} \omega_{s,t}(\tilde{e}_i) [\bar{c}(\tilde{e}_s)\bar{c}(\tilde{e}_t) - c(\tilde{e}_s)c(\tilde{e}_t)] \right] + \bar{c}(\theta) + c(\theta');$$

(2.10)
\[ \hat{D}_N = d + \delta + \bar{c}(\theta) - c(\theta') = \sum_{i=1}^{n} c(\tilde{e}_i) \left[ \tilde{e}_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i)(\bar{c}(\tilde{e}_s)\bar{c}(\tilde{e}_t) - c(\tilde{e}_s)c(\tilde{e}_t)) \right] + \bar{c}(\theta) - c(\theta'). \] (2.11)

Let \( g^{ij} = g(dx_i, dx_j) \), \( \xi = \sum_k \xi_k dx_k \) and \( \nabla_{\partial_k} = \sum_k \Gamma_{ij}^k \partial_k \), we denote that

\[
\begin{align*}
\sigma_i &= -\frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t); \\
\xi^j &= g^{ij}\xi_i; \\
\Gamma^{k} &= g^{ij}\Gamma_{ij}^k; \\
\sigma^j &= g^{ij}\sigma_i; \\
a^j &= g^{ij}a_i.
\end{align*}
\] (2.12)

Then the modified Novikov Operators \( \hat{D}_N \) and \( \hat{D}_N^* \) can be written as

\[
\begin{align*}
\hat{D}_N &= \sum_{i=1}^{n} c(\tilde{e}_i)[\tilde{e}_i + a_i + \sigma_i] + \bar{c}(\theta) + c(\theta'); \\
\hat{D}_N^* &= \sum_{i=1}^{n} c(\tilde{e}_i)[\tilde{e}_i + a_i + \sigma_i] + \bar{c}(\theta) - c(\theta').
\end{align*}
\] (2.13)

By Lemma 1 in [13] and Lemma 2.1 in [12], for any fixed point \( x_0 \in \partial M \), choosing the normal coordinates \( U \) of \( x_0 \) in \( \partial M \) (not in \( M \)). Denote by \( \sigma_l(P) \) the \( l \)-order symbol of an operator \( P \). By the composition formula and (2.2.11) in [12], we obtain in [19, Lemma 2.6],

**Lemma 2.2.** The following identities hold:

\[
\begin{align*}
\sigma_1(\hat{D}_N) &= \sigma_1(\hat{D}_N^*) = ic(\xi); \\
\sigma_0(\hat{D}_N) &= \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t)c(\tilde{e}_i) + \bar{c}(\theta) + c(\theta'); \\
\sigma_0(\hat{D}_N^*) &= \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) - \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t)c(\tilde{e}_i) + \bar{c}(\theta) - c(\theta').
\end{align*}
\] (2.15)

Write

\[
D_x^\alpha = (-i)^{\alpha} \partial_{x_1}^\alpha; \quad \sigma(D) = p_1 + p_0; \quad \sigma(D^{-1}) = \sum_{j=1}^{\infty} q_{-j}. \] (2.16)

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(D \circ D^{-1}) = \sum_{\alpha} \frac{1}{\alpha!} \partial_{x_1}^\alpha[\sigma(D)]D_x^\alpha[\sigma(D^{-1})]
\]

\[
= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \cdots) \\
+ \sum_j (\partial_{x_j}p_1 + \partial_{x_j}p_0)(D_{x_j}q_{-1} + D_{x_j}q_{-2} + D_{x_j}q_{-3} + \cdots)
\]

\[
= p_1q_{-1} + (p_1q_{-2} + p_0q_{-1} + \sum_j \partial_{x_j}p_1 D_{x_j}q_{-1}) + \cdots, \] (2.17)

so

\[
q_{-1} = p_{1}^{-1}; \quad q_{-2} = -p_{1}^{-1}[p_0p_{1}^{-1} + \sum_j \partial_{x_j}p_1 D_{x_j}(p_{1}^{-1})]. \] (2.18)

By Lemma 2.2, we have some symbols of operators.
Lemma 2.3. The following identities hold:

\begin{align*}
\sigma_{-1}(\tilde{D}^{-1}_N) &= \sigma_{-1}(\tilde{D}^{-1}_N) = \frac{\iota c(\xi)}{\xi^2}; \\
\sigma_{-2}(\tilde{D}^{-1}_N) &= \frac{c(\xi)\sigma_0(\tilde{D}^{-1}_N) c(\xi)}{\xi^4} + \frac{c(\xi)}{\xi^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))\xi^2 - c(\xi)\partial_{x_j}(\xi^2) \right]; \\
\sigma_{-2}((\tilde{D}^*_N)^{-1}) &= \frac{c(\xi)\sigma_0((\tilde{D}^*_N)^{-1}) c(\xi)}{\xi^4} + \frac{c(\xi)}{\xi^6} \sum_j c(dx_j) \left[ \partial_{x_j}(c(\xi))\xi^2 - c(\xi)\partial_{x_j}(\xi^2) \right].
\end{align*}

(2.19)

3. A Kastler-Kalau-Walze type theorem for four-dimensional manifolds with boundary

Let \( M \) be 4-dimensional compact manifolds with the boundary \( \partial M \). In the following, we will compute the more general case \( \overline{Wres}[\pi^+(f\tilde{D}^{-1}_N) \circ \pi^+(f^{-1}(\tilde{D}^{-1}_N)^{-1})] \) for nonzero smooth functions \( f, f^{-1} \). An application of (3.5) and (3.6) in [14] shows that

\[ \overline{Wres}[\pi^+(f\tilde{D}^{-1}_N) \circ \pi^+(f^{-1}(\tilde{D}^*_N)^{-1})] = Wres[f\tilde{D}^{-1}_N \circ f^{-1}(\tilde{D}^*_N)^{-1}] + \int_{\partial M} \Phi, \]

where

\[ \Phi = \int_{|\xi|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum_{\alpha} \frac{(-i)^{\alpha+j+k+2}}{\alpha!(j+k+1)!} \text{trace}_{\gamma,T,M} \left[ \partial_{x_{\alpha}} \partial_{\xi}^j \partial_{\xi}^k \sigma_j(f\tilde{D}^{-1}_N)(x',0,\xi',\xi) \right] \times \partial_{x_{\alpha}}^j \partial_{\xi}^k \sigma_j \left( (f^{-1}(\tilde{D}^*_N)^{-1})(x',0,\xi',\xi) \right) d\xi d\xi' d\xi', \]

(3.2)

and the sum is taken over \( r - k + |\alpha| + \ell - j - 1 = -n = -4 \), \( r \leq -1, \ell \leq -1 \).

Note that

\[ f\tilde{D}^{-1}_N \circ f^{-1}(\tilde{D}^*_N)^{-1} = (\tilde{D}^*_N \tilde{D}_N - \tilde{D}^*_N c(df)f^{-1})^{-1}. \]

(3.3)

We first establish the main theorem in this section. One has the following Lichnerowicz formula.

Theorem 3.1. The following equalities hold:

\[ \tilde{D}^*_N \tilde{D}_N - \tilde{D}^*_N c(df)f^{-1} \]

\[ = -g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_i} \nabla_{\partial_j}) - \frac{1}{8} \sum_{ijkl} R_{ijkl} c(\xi_i) c(\xi_j) c(\xi_k) c(\xi_l) + \sum_i c(\xi_i) c(\nabla_{\xi_i} T M \theta) + \frac{1}{8} c(\theta') \]

\[ \times c(\theta) + c(\theta') c(\theta'') + |\theta'|^2 + |\theta''|^2 + \frac{1}{4} \sum_i c(\xi_i) c(\theta') c(\theta''), \]

\[ \times \frac{\partial_i (c(df)f^{-1})}{\partial x_j} + \left( c(\theta') - c(\theta) \right) c(df)f^{-1} - \partial_i \left( \frac{1}{2} c(\partial_i c(df)f^{-1}) \right) - \frac{1}{4} \sum_i \left[ c(\xi_i) c(\theta') c(\theta'') \right], \]

(3.4)

where \( s \) is the scalar curvature.

In order to prove Theorem 3.1, we recall the basic notions of Laplace type operators. Let \( M \) be smooth compact oriented Riemannian \( n \)-dimensional manifolds without boundary and \( V' \) be a vector bundle on \( M \). Any differential operator \( P \) of Laplace type has locally the form

\[ P = -\left( g^{ij} \partial_i \partial_j + A_i \partial_i + B \right), \]

(3.5)
where $\partial_i$ is a natural local frame on $TM$ and $(g^{ij})_{1 \leq i, j \leq n}$ is the inverse matrix associated to the metric matrix $(g_{ij})_{1 \leq i, j \leq n}$ on $M$, and $A'$ and $B$ are smooth sections of $\text{End}(V')$ on $M$ (endomorphism). If $P$ is a Laplace type operator with the form (3.5), then there is a unique connection $\nabla$ on $V'$ and a unique endomorphism $E$ such that

$$P = - \left[ g^{ij}(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\nabla^{\nabla}_{\partial_i}} \partial_j) + E \right].$$

(3.6)

where $\nabla^{\nabla}$ is the Levi-Civita connection on $M$. Moreover (with local frames of $T^*M$ and $V'$), $\nabla_{\partial_i} = \partial_i + \omega_i$ and $E$ are related to $g^{ij}$, $A'$ and $B$ through

$$\omega_i = \frac{1}{2} g_{ij} \{ A^i + g^{kl} \Gamma^j_{kl} \text{id} \},$$

(3.7)

and

$$E = B - g^{ij} (\partial_i (\omega_j) + \omega_i \omega_j - \omega_k \Gamma^k_{ij}),$$

(3.8)

where $\Gamma^j_{kl}$ is the Christoffel coefficient of $\nabla^{\nabla}$.

By Proposition 4.6 of [28], we have

$$(d + \delta + c(\theta))^2 = (d + \delta)^2 + \sum_i c(\xi_i) c(\nabla^{\nabla}_{\xi_i} \theta) + |\theta|^2.$$  

(3.9)

By [28], the local expression of $(d + \delta)^2$ is

$$(d + \delta)^2 = -\Delta_0 - \frac{1}{8} \sum_{ijkl} R_{ijkl} c(\xi_i) c(\xi_j) c(\xi_k) c(\xi_l) + \frac{1}{4} g.$$  

(3.10)

By [28] and [8], we have

$$-\Delta_0 = \Delta = -g^{ij} \nabla^L_i \nabla^L_j (\nabla^L_{ij} - \Gamma^j_{ij} \nabla^L_k).$$  

(3.11)

We note that

$$\hat{D}_N \hat{D}_N = (d + \delta + c(\theta))^2 + (d + \delta)c(\theta') + c(\theta)c(\theta') - c(\theta')(d + \delta) - c(\theta')c(\theta) + |\theta'|^2,$$  

(3.12)

$$-c(\theta')(d + \delta) + (d + \delta)c(\theta') = \sum_{i,j} g^{ij} \left[ c(\partial_i) c(\theta') - c(\theta)c(\partial_i) \right] \partial_j - \sum_{i,j} g^{ij} \left[ c(\theta') c(\partial_i) \sigma_i + c(\theta') c(\partial_i) a_i \right] + c(\partial_i) \partial_j (c(\theta')) + c(\partial_i) \sigma_i c(\theta') + c(\partial_i) a_i c(\theta'),$$  

(3.13)

then we obtain

$$\hat{D}_N \hat{D}_N - \hat{D}_N c(df) f^{-1}$$

$$= - \sum_{i,j} g^{ij} \left[ \partial_i \partial_j + 2a_i \partial_j + 2a_j \partial_i - \Gamma^k_{ij} \partial_k + (\partial_i \sigma_j) + (\partial_j \sigma_i) + \sigma_i a_j + a_i \sigma_j + a_i a_j + a_j a_j - \Gamma^k_{ij} \sigma_k \right]$$

$$+ g^{ij} \left[ \sigma_i a_j c(\theta') - c(\partial_i) a_i c(\theta') \right] \partial_j - \sum_{i,j} g^{ij} \left[ c(\theta') c(\partial_i) \sigma_i + c(\theta') c(\partial_i) a_i - c(\partial_i) \partial_i c(\theta') \right]$$

$$-c(\partial_i) \sigma_i c(\theta') - c(\partial_i) a_i c(\theta') \frac{1}{8} \sum_{ijkl} R_{ijkl} c(\xi_i) c(\xi_j) c(\xi_k) c(\xi_l) + \frac{1}{4} g + \sum_i c(\xi_i) c(\nabla^{\nabla}_{\xi_i} \theta) + |\theta|^2$$

$$+|\theta'|^2 + c(\bar{\theta}) c(\theta') - c(\theta') c(\bar{\theta})$$

$$= - \sum_{i,j} g^{ij} (\partial_i) c(\partial_j) - \sum_{i,j} g^{ij} (\partial_i) \frac{\partial_i (\partial_j) c(df) f^{-1}}{\partial x_j} \right)$$

$$+ \sum_{i,j} g^{ij} (a_i) c(df) f^{-1} - \left( c(\bar{\theta}) - c(\theta') \right) c(df) f^{-1}.$$  

(3.14)

By (3.7), (3.8) and (3.14), we have

$$(\omega_i) \hat{D}_N - \hat{D}_N c(\theta) f^{-1} = \sigma_i + a_i - \frac{1}{2} \left[ c(\partial_i) c(\theta') - c(\theta') c(\partial_i) \right] + \frac{1}{2} c(\partial_i) c(df) f^{-1}.$$  

(3.15)
By (3.8), we have

\[
E_{\tilde{D}_N} - \tilde{\Delta}_N c(df)f^{-1}
\]

\[
= -\sum_{i,j} g^{ij} c(\partial_i) \frac{\partial (c(df)f^{-1})}{\partial x_j} - \sum_{i,j} g^{ij} (a_i + \sigma_i) c(df)f^{-1} - \left( \tilde{c}(\theta) - c(\theta') \right) c(df)f^{-1} - \partial_i \left( \frac{1}{2} c(\partial_i) \right)
\]

\[
\times c(df)f^{-1} - \left( \sigma^i + a^i \right) \frac{1}{2} c(\partial_i) c(df)f^{-1} + \frac{g^{ij}}{4} \left[ c(\partial_i) c(\theta') - c(\theta') c(\partial_i) \right] c(\partial_j) c(df)f^{-1}
\]

\[
- \frac{1}{2} c(\partial_i) c(df)f^{-1} \left( \sigma^i + a^i \right) + \frac{g^{ij}}{4} c(\partial_i) c(df)f^{-1} \left[ c(\partial_j) c(\theta') - c(\theta') c(\partial_j) \right] + \frac{1}{2} c(\partial_i) c(df)f^{-1} \Gamma^k
\]

\[-c(\partial_i) \sigma^i c(\theta') - c(\partial_i) a^i c(\theta') + \frac{1}{8} \sum_{ijkl} R_{ijkl} c(\tilde{e}_i) c(\tilde{e}_j) c(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i} c(\tilde{e}_i) c(\nabla^T \nabla \theta) - |\theta|^2 - |\theta'|^2
\]

\[
- \frac{1}{4} \left[ c(\partial_i) c(\theta') - c(\theta') c(\partial_i) \right] \left[ |\sigma^i + a^i| - \frac{g^{ij}}{4} c(\partial_i) c(\theta') - c(\theta') c(\partial_i) \right] c(\partial_j) c(\theta') - c(\theta') c(\partial_j)
\]

\[
- \frac{1}{2} \left[ c(\partial_i) c(\theta') - c(\theta') c(\partial_i) \right] c(\partial_j) c(\theta') - c(\theta') c(\partial_j) - \frac{1}{2} \left[ \sigma^j + a^j \right] c(\partial_j) c(\theta') - c(\theta') c(\partial_j).
\]

(3.16)

For a smooth vector field \( Y \) on \( M \), let \( c(Y) \) denote the Clifford action. Since \( E \) is globally defined on \( M \), taking normal coordinates at \( x_0 \), we have \( \sigma^i(x_0) = 0, a^i(x_0) = 0, \partial^i c(\partial_j)(x_0) = 0, \Gamma^k(x_0) = 0, g^{ij}(x_0) = \delta^i_j \), so that

\[
E_{\tilde{D}_N} - \tilde{\Delta}_N c(df)f^{-1}(x_0)
\]

\[
= -\sum_{i} c(e_i) \frac{\partial (c(df)f^{-1})}{\partial x_j} - \left( \tilde{c}(\theta) - c(\theta') \right) c(df)f^{-1} + \partial_i \left( \frac{1}{2} c(\partial_i) c(df)f^{-1} \right) + \frac{1}{4} \sum_{i} \left[ c(e_i) c(\theta') \right]
\]

\[-c(\theta') c(e_i) \right] c(e_i) c(df)f^{-1} + \frac{1}{4} \sum_{i} c(e_i) c(df)f^{-1} \left[ c(e_i) c(\theta') - c(\theta') c(e_i) \right] + \frac{1}{4} \sum_{ijkl} R_{ijkl} c(\tilde{e}_i) c(\tilde{e}_j)
\]

\[
\times c(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i} c(\tilde{e}_i) c(\nabla^T \nabla \theta) - \frac{1}{4} \left[ c(\theta') c(\tilde{e}_i) c(\tilde{e}_j) \right] + \tilde{\Delta}_N c(df)f^{-1}(x_0)
\]

\[
= -\sum_{i} c(e_i) \frac{\partial (c(df)f^{-1})}{\partial x_j} - \left( \tilde{c}(\theta) - c(\theta') \right) c(df)f^{-1} + \partial_i \left( \frac{1}{2} c(e_i) c(df)f^{-1} \right) + \frac{1}{4} \sum_{i} \left[ c(e_i) c(\theta') \right]
\]

\[-c(\theta') c(e_i) \right] c(e_i) c(df)f^{-1} + \frac{1}{4} \sum_{i} c(e_i) c(df)f^{-1} \left[ c(e_i) c(\theta') - c(\theta') c(e_i) \right] + \frac{1}{4} \sum_{ijkl} R_{ijkl} c(\tilde{e}_i) c(\tilde{e}_j)
\]

\[
\times c(\tilde{e}_k) c(\tilde{e}_l) - \sum_{i} c(\tilde{e}_i) c(\nabla^T \nabla \theta) - \frac{1}{4} \left[ c(\theta') c(\tilde{e}_i) c(\tilde{e}_j) \right] + \tilde{\Delta}_N c(df)f^{-1}(x_0)
\]

\[
- c(\theta') c(\tilde{e}_i) \right] c(\theta') c(\tilde{e}_j) - \frac{1}{4} \sum_{i} c(\tilde{e}_i) c(\theta') - c(\theta') c(\tilde{e}_i) - |\theta|^2 - |\theta'|^2 - \frac{1}{4} \sum_{i} c(\tilde{e}_i) c(\theta')
\]

\[
- c(\theta') c(\tilde{e}_i) \right] c(\theta') c(\tilde{e}_j) - \frac{1}{4} \left[ c(\theta') c(\tilde{e}_i) c(\tilde{e}_j) \right] + \tilde{\Delta}_N c(df)f^{-1}(x_0),
\]

(3.17)

which, together with (3.6), yields Theorem 3.1.

The non-commutative residue of a generalized laplacian \( \tilde{\Delta} \) is expressed as by

\[
(n - 2) \Phi_2(\tilde{\Delta}) = (4\pi)^{-\frac{n}{2}} \Gamma \left( \frac{n}{2} \right) r e s (\tilde{\Delta}^{n + 1}),
\]

(3.18)

where \( \Phi_2(\tilde{\Delta}) \) denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of \( \tilde{\Delta} \). Now let \( \tilde{\Delta} = \tilde{D}_N \tilde{D}_N - \tilde{\Delta}_N c(df)f^{-1} \). Since \( \tilde{D}_N \tilde{D}_N - \tilde{\Delta}_N c(df)f^{-1} \) is a generalized laplacian \( \tilde{\Delta} \), we can
suppose $\widehat{D}_N^*\widehat{D}_N - \widehat{D}_N^*c(df) f^{-1} = \Delta - E$, then we have

$$\text{Wres}(\widehat{D}_N^*\widehat{D}_N - \widehat{D}_N^*c(df) f^{-1}) = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M \text{tr} \left( \frac{1}{6} s + E \widehat{D}_N^*\widehat{D}_N - \widehat{D}_N^*c(df) f^{-1} \right) d\text{Vol}_M, \quad (3.19)$$

where $\text{Wres}$ denote the noncommutative residue.

We use tr as shorthand of trace. One has the following Lemma.

**Lemma 3.2.** The following identity holds

$$\text{tr} \left[ \frac{\partial_i (c(df)^{-1})}{\partial x_i} \right] (x_0) = \left[ - f^{-1} \Delta(f) - \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right] (x_0) \text{tr}[\text{id}];$$

$$\text{tr} \left[ \frac{\partial_i (c\theta c(df)^{-1})}{\partial x_i} \right] (x_0) = \left[ - f^{-1} \Delta(f)(x_0) - \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right] (x_0) \text{tr}[\text{id}]. \quad (3.20)$$

Combining (3.19) with (3.20), we have

**Theorem 3.3.** For even $n$-dimensional compact oriented manifolds without boundary, the following equalities holds:

$$\text{Wres}(\widehat{D}_N^*\widehat{D}_N - \widehat{D}_N^*c(df) f^{-1}) = \frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!} \int_M 2^n \left\{ \left( - \frac{1}{12} s - |\theta|^2 + (n-2)|\theta'|^2 + g(e_j, \nabla_{\theta'} \theta') \right) + \left[ f^{-1} \Delta(f) + \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right] \left[ f^{-1} \Delta(f) + \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right] \right\} d\text{Vol}_M, \quad (3.21)$$

where $s$ is the scalar curvature.

Locally we can use Theorem 3.3 to compute the interior of $\text{Wres}[\pi^+(f\widehat{D}_N^{-1}) \circ \pi^+(f^{-1}(\widehat{D}_N^{-1}))]$, we have

$$\int_M \int_{\xi=1} \text{tr}_{\Lambda^* T^* M} \left[ \sigma_{-4}((\widehat{D}_N^*\widehat{D}_N - \widehat{D}_N^*c(df) f^{-1})) \right] \sigma(\xi) d\xi = 32\pi^2 \int_M \left\{ 16g(e_j, \nabla_{\theta'} \theta') - \frac{4}{3} - 16|\theta'|^2 + 32|\theta|^2 \right\} + 16 \left[ f^{-1} \Delta(f) + \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right] \left[ f^{-1} \Delta(f) + \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right] + \sum_i \left[ e_i \theta c(\theta') c(\theta) - c(\theta') c(e_i) \right] d\text{Vol}_M. \quad (3.22)$$

So we only need to compute $\int_{\partial M} \Phi$. From the remark above, now we can compute $\Phi$ (see formula (3.2) for the definition of $\Phi$). Since $n = 4$, then $\text{tr}_{\Lambda^* T^* M}[\text{id}] = \text{dim}(\Lambda^*(4)) = 16$, since the sum is taken over $r + l - k - j - |\alpha| = -3$, $r \leq -1, l \leq -1$, then we have the following five cases:

**case (a) (1)** $r = -1, l = -1, k = j = 0, |\alpha| = 1$
By (2.3), we get

\[
\text{case (a) (I)} = - \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial^2_{\xi} \pi^+_n \sigma_{-1}(\hat{D}^{-1}_N) \times \partial^2_{\xi} \sigma_{-1}((\hat{D}^*_N)^{-1})](x_0) d\xi_n \sigma(\xi') dx' - f \sum_{j<n} \partial_j(f^{-1}) \\
\times \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr}[\partial^2_{\xi} \pi^+_n \sigma_{-1}(\hat{D}^{-1}_N) \times \partial^2_{\xi} \sigma_{-1}(\hat{D}^*_N)^{-1}](x_0) d\xi_n \sigma(\xi') dx' \\
= 0,
\]

(3.23)

so case (a) (I) vanishes.

\textbf{case (a) (II)} \quad r = -1, \quad l = -1, \quad k = |\alpha| = 0, \quad j = 1

By (3.2), we get

\[
\text{case (a) (II)} = -\frac{1}{2} \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi^+_n \sigma_{-1}(\hat{D}^{-1}_N) \times \partial^2_{\xi} \sigma_{-1}((\hat{D}^*_N)^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\
-\frac{1}{2} f^{-1} \partial_{x_n}(f) \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \sigma_{-1}(\hat{D}^{-1}_N) \times \partial^2_{\xi} \sigma_{-1}(\hat{D}^*_N)^{-1}](x_0) d\xi_n \sigma(\xi') dx'.
\]

(3.24)

By Lemma 2.3, we have

\[
\partial^2_{\xi} \sigma_{-1}((\hat{D}^*_N)^{-1})(x_0) = i \left( -\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n c(\xi)}{|\xi|^6} \right); \tag{3.25}
\]

\[
\partial_{x_n} \sigma_{-1}(\hat{D}^{-1}_N)(x_0) = \frac{i\partial_{x_n} c(\xi')(x_0)}{|\xi|^2} - \frac{i c(\xi)|\xi'|^2 h'(0)}{|\xi|^4}. \tag{3.26}
\]

By (2.3), (2.4) and the Cauchy integral formula we have

\[
\pi^+_n \left[ \frac{c(\xi)}{|\xi|^4} \right](x_0)|_{|\xi|=1} = \pi^+_n \left[ \frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right] = \frac{1}{2\pi i} \lim_{\nu_0 \to 0} \int_{\Gamma^+} \frac{c(\xi') + \nu_0 c(dx_n)}{(\xi_n + \nu_0 - \eta_0)^2} d\eta_0 = -\frac{(i\xi_n + 2)c(\xi') + ic(dx_n)}{4(\xi_n - i)^2}. \tag{3.27}
\]

Similarly we have,

\[
\pi^+_n \left[ \frac{i\partial_{x_n} c(\xi')}{|\xi|^2} \right](x_0)|_{|\xi|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)}. \tag{3.28}
\]

By (3.26), then

\[
\pi^+_n \partial_{x_n} \sigma_{-1}(\hat{D}^{-1}_N)|_{|\xi|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)} + i h'(0) \left[ \frac{(i\xi_n + 2)c(\xi') + ic(dx_n)}{4(\xi_n - i)^2} \right]. \tag{3.29}
\]
By the relation of the Clifford action and $\text{tr} AB = \text{tr} BA$, we have the equalities:

$$
\text{tr}[c(\xi')c(dx_n)] = 0; \quad \text{tr}[c(dx_n)^3] = -16; \quad \text{tr}[c(\xi')^2(x_0)]|_{\xi'|=1} = -16;
$$

$$
\text{tr}[\partial_{x_n} c(\xi')c(dx_n)] = 0; \quad \text{tr}[\partial_{x_n} c(\xi')c(\xi')](x_n)|_{\xi'|=1} = -8h'(0); \quad \text{tr}[\partial(\xi_n)c(\xi_n)\partial(\overline{\xi}_n)\partial(\xi_n)] = 0 (i \neq j). \quad (3.30)
$$

By (3.25), (3.29) and a direct computation, we have

$$
\text{By the relation of the Clifford action and tr} \ AB = \text{tr} BA, we have
$$

Then

$$
\text{Similarly, we have}
$$

Then

$$
\text{Similarly, we get}
$$

Then, we obtain

**case (a) (II)**

$$
-2\pi i \Omega_3 f^{-1} \partial_{x_n}(f) dx'.
$$

**case (a) (III)**

$$
-1, \quad l = -1, \quad j = |\alpha| = 0, \quad k = 1
$$

By (3.2), we get

$$
\text{case (a) (III)}
$$

$$
-\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\hat{D}_N^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\hat{D}_N^{-1})] (x_0) d\xi_n d\sigma(\xi') dx'.
$$

$$
= -2\pi i \Omega_3 f^{-1} \partial_{x_n}(f) dx'.
$$

(3.33)

(3.34)

(3.35)
By Lemma 2.3, we have
\[ \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\hat{D}_N^*)^{-1})(x_0)|_{[\xi]=1} \]
\[ = -ih'(0) \left[ \frac{c(dx_n)}{[\xi]^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{[\xi]^0} \right] - \frac{2\xi_n i\partial_{x_n} c(\xi')(x_0)}{[\xi]^4}; \quad (3.36) \]
\[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})(x_0)|_{[\xi]=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \quad (3.37) \]

Similar to case (a) (II), we have
\[ \text{tr} \left\{ \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \times i h'(0) \left[ \frac{c(dx_n)}{[\xi]^4} - 4\xi_n \frac{c(\xi') + \xi_n c(dx_n)}{[\xi]^0} \right] \right\} = 8h'(0) \frac{i - 3\xi_n}{(\xi_n - i)^4(\xi_n + i)^2}. \quad (3.38) \]
and
\[ \text{tr} \left[ \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2} \times \frac{2\xi_n i\partial_{x_n} c(\xi')(x_0)}{[\xi]^4} \right] = -8ih'(0)\xi_n \frac{i - 3\xi_n}{(\xi_n - i)^4(\xi_n + i)^2}. \quad (3.39) \]

So we have
\[ -\frac{1}{2} \int_{[\xi]=1}^{\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})(x_0)] d\xi_n \sigma(\xi') dx' \]
\[ = -\int_{[\xi]=1}^{\infty} \int_{-\infty}^{+\infty} h'(0) 4i(\xi + 3\xi_n) \frac{1}{(\xi_n + i)^4(\xi_n + i)^2} d\xi_n \sigma(\xi') dx' \]
\[ = -h'(0)\Omega_3 \frac{2\pi i}{3} \frac{4i(\xi + 3\xi_n)}{(\xi_n + i)^4(\xi_n + i)^2} |_{\xi_n = -i} dx' + h'(0)\Omega_3 \frac{2\pi i}{3} \frac{4i\xi}{(\xi_n + i)^2} |_{\xi_n = -i} dx' \]
\[ = \frac{3}{2} \pi h'(0)\Omega_3 dx'. \quad (3.40) \]

Similarly, we get
\[ -\frac{1}{2} f \partial_{x_n}(f^{-1}) \int_{[\xi]=1}^{\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})(x_0)] d\xi_n \sigma(\xi') dx' \]
\[ = 2\pi i \Omega_3 f \partial_{x_n}(f^{-1}) dx'. \quad (3.41) \]

Then, we obtain
\[ \text{case (a) (III)} = \frac{3}{2} \pi h'(0)\Omega_3 dx' + 2\pi i \Omega_3 f \partial_{x_n}(f^{-1}) dx'. \quad (3.42) \]

**case (b)** \( r = -2, \ l = -1, \ k = j = |a| = 0 \)

By (3.2), we get
\[ \text{case (b)} = -i \int_{[\xi]=1}^{\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-2}(f \hat{D}_N^{-1})(x_0)] d\xi_n \sigma(\xi') dx' \]
\[ = -i \int_{[\xi]=1}^{\infty} \int_{-\infty}^{+\infty} \text{tr}[\pi^+_{\xi_n} \sigma_{-2}(\hat{D}_N^{-1})(x_0)] d\xi_n \sigma(\xi') dx'. \quad (3.43) \]

By Lemma 2.3, we have
\[ \sigma_{-2}(\hat{D}_N^{-1})(x_0) = \frac{c(\xi_0) c(\hat{D}_N)(x_0)c(\xi)}{[\xi]^4} + \frac{c(\xi)}{[\xi]^0} c(dx_n) \left[ \partial_{x_n} [c(\xi')(x_0)] [\xi]^2 - c(\xi) h'(0) [\xi]^2 \right]. \quad (3.44) \]
\( \sigma_0(\hat{D}_N) = \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) - \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i)) + \hat{c}(\theta) + \hat{c}(\theta'). \) (3.45)

We denote \( b_0^b(x_0) = \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(\hat{e}_i)(x_0) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i); \) \( b_0^b(x_0) = -\frac{1}{4} \sum_{s,t,i} \omega_{s,t}(\hat{e}_i)(x_0) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i)). \)

Then
\[
\begin{align*}
\pi^+_{\xi_n} \sigma_{-2}(\hat{D}_N^{-1}(x_0))|_{|\xi'|=1} &= \pi^+_{\xi_n} \left[ \frac{c(\xi) b_0^b(x_0) c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi^+_{\xi_n} \left[ \frac{c(\xi) \hat{c}(\theta) + c(\theta')(x_0) c(\xi)}{(1 + \xi_n^2)^2} \right] \\
&\quad + \pi^+_c \left[ \frac{c(\xi) b_0^b(x_0) c(\xi) + c(\xi)c(dx_n) \partial_{x_n}[c(\xi')](x_0)}{1 + \xi_n^2} \right] + \hat{h}(0) \frac{c(\xi)c(dx_n) c(\xi)}{(1 + \xi_n^2)^3}.
\end{align*}
\]

By direct calculation, we have
\[
\begin{align*}
\pi^+_{\xi_n} \left[ \frac{c(\xi) b_0^b(x_0) c(\xi)}{(1 + \xi_n^2)^2} \right] &= \pi^+_{\xi_n} \left[ \frac{c(\xi') b_0^b(x_0) c(\xi')}{(1 + \xi_n^2)^2} \right] + \pi^+_{\xi_n} \left[ \frac{\xi_n c(\xi) b_0^b(x_0) c(dx_n)}{(1 + \xi_n^2)^2} \right] \\
&\quad + \pi^+_{\xi_n} \left[ \frac{\xi_n c(dx_n) b_0^b(x_0) c(\xi')}{(1 + \xi_n^2)^2} \right] + \pi^+_{\xi_n} \left[ \frac{\xi_n^2 c(dx_n) b_0^b(x_0) c(dx_n)}{(1 + \xi_n^2)^2} \right] \\
&= \frac{c(\xi') b_0^b(x_0) c(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} + \frac{ic(\xi') b_0^b(x_0) c(dx_n)}{4(\xi_n - i)^2} \\
&\quad + \frac{ic(dx_n) b_0^b(x_0) c(\xi')}{4(\xi_n - i)^2} + \frac{-i\xi_n c(dx_n) b_0^b(x_0) c(dx_n)}{4(\xi_n - i)^2}.
\end{align*}
\]

Since
\[
c(dx_n) b_0^b(x_0) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} c(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i),
\]
then by the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA, \) we have the equalities:
\[
\begin{align*}
\text{tr}[c(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i) \tilde{c}(\hat{e}_i)] &= 0 \ (i < n); \quad \text{tr}[b_0^b c(dx_n)] = 0; \quad \text{tr}[c(\theta) c(dx_n)] = 0; \\
\text{tr}[c(\theta') c(dx_n)] &= -16g(\theta', dx_n); \quad \text{tr}[c(\xi') \tilde{c}(dx_n)] = 0.
\end{align*}
\]

Since
\[
\partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1}) = \partial_{\xi_n} q^{-1}(x_0)|_{|\xi'|=1} = 1 \left[ \frac{c(dx_n)}{1 + \xi_n} - \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1 + \xi_n^2)^2} \right],
\]

By (3.47) and (3.50), we have
\[
\begin{align*}
\text{tr}[\pi^+_{\xi_n} \left[ \frac{c(\xi) b_0^b(x_0) c(\xi)}{(1 + \xi_n^2)^2} \right] \times \partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})(x_0)|_{|\xi'|=1} &= \frac{1}{2(1 + \xi_n^2)^2} \text{tr}[c(\xi') b_0^b(x_0)] + \frac{i}{2(1 + \xi_n^2)^2} \text{tr}[c(dx_n) b_0^b(x_0)] \\
&= \frac{1}{2(1 + \xi_n^2)^2} \text{tr}[c(\xi') b_0^b(x_0)].
\end{align*}
\]

We note that \( i < n, \) \( \int_{|\xi'|=1} \{\xi_1, \xi_2, \ldots, \xi_{2d+1}\} c(\xi') = 0, \) so \( \text{tr}[c(\xi') b_0^b(x_0)] \) has no contribution for computing case (b).
By direct calculation we have

\[
\pi^+_{\xi_n}\left[\frac{c(\xi)b_0^2(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{\xi_n}[c(\xi')](x_0)}{(1 + \xi_n^2)}\right] - h'(0)\pi^+_{\xi_n}\left[\frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n)^2}\right] = B_1 - B_2, \tag{3.52}
\]

where

\[
B_1 = \frac{-1}{4(\xi_n - i)^2}(2 + i\xi_n)c(\xi')b_0^2(x_0)c(\xi') + i\xi_n(c(dx_n)b_0^2(x_0)c(dx_n) + (2 + i\xi_n)c(dx_n)\partial_{\xi_n}c(\xi') + ic(dx_n)b_0^2(x_0)c(\xi') + \partial_{\xi_n}c(\xi')) \tag{3.53}
\]

and

\[
B_2 = \frac{h'(0)}{2}\left[\frac{c(dx_n) - ic(\xi')}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2}[ic(\xi') - c(dx_n)]\right]. \tag{3.54}
\]

By (3.50) and (3.54), we have

\[
\text{tr}[B_2 \times \partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})]\|_{\xi'\|=1} = \frac{i}{2}h'(0)\frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2}\text{tr}[\text{id}]
\]

\[
= 8ih'(0)\frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2}. \tag{3.55}
\]

By (3.50) and (3.53), we have

\[
\text{tr}[B_1 \times \partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})]\|_{\xi'\|=1} = \frac{-8i\epsilon_0}{(1 + \xi_n^2)^2} + 2h'(0)\frac{\xi_n^2 - \xi_n - 2}{(\xi_n - i)(1 + \xi_n^2)^2}, \tag{3.56}
\]

where \(b_0^2 = c_0(c(dx_n))\) and \(c_0 = -\frac{1}{2}h'(0)\).

By (3.55) and (3.56), we have

\[
-\frac{i}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[(B_1 - B_2) \times \partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})](x_0)d\xi_n \sigma_{\xi'}(x_0)dx'
\]

\[
= -\Omega_3 \int_{\pi^+}^{\pi^-} \frac{8\epsilon_0(\xi_n - i) + ih'(0)}{(\xi_n - i)^2(\xi_n + i)^2} d\xi_n dx'
\]

\[
= \frac{9}{2} \pi h'(0)\Omega_3 dx'. \tag{3.57}
\]

Similar to (3.51), we have

\[
\text{tr}[\pi^+_{\xi_n}\left[\frac{c(\xi)c(\theta)(x_0)c(\xi)}{(1 + \xi_n^2)^2}\right] \times \partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})(x_0)]\|_{\xi'\|=1}
\]

\[
= \frac{1}{2(1 + \xi_n^2)^2}\text{tr}[c(\xi)c(\theta)(x_0)] + \frac{i}{2(1 + \xi_n^2)^2}\text{tr}[c(dx_n)c(\theta)(x_0)]
\]

\[
= \frac{1}{2(1 + \xi_n^2)^2}\text{tr}[c(\xi)c(\theta)(x_0)]. \tag{3.58}
\]

Similar to (3.51), we have

\[
\text{tr}[\pi^+_{\xi_n}\left[\frac{c(\xi)c(\theta')(x_0)c(\xi)}{(1 + \xi_n^2)^2}\right] \times \partial_{\xi_n} \sigma_{-1}(\hat{D}_N^{-1})(x_0)]\|_{\xi'\|=1} = \frac{i}{2(1 + \xi_n^2)^2}\text{tr}[c(dx_n)c(\theta')(x_0)]. \tag{3.59}
\]
By (3.58) and (3.59), we have
\[
-\frac{\pi}{4} \text{tr}[\sigma_0 c(dx_n) c(\theta')]\Omega_3 dx'
= -4\pi g(\theta', dx_n)\Omega_3 dx'.
\] (3.60)

By (3.51), (3.57) and (3.60), we have
\[
\text{case (c)} = \frac{9}{2} \pi h'(0)\Omega_3 dx' - 4\pi g(\theta', dx_n)\Omega_3 dx'.
\] (3.61)

By (3.64), Lemma 2.3, we have
\[
\sigma_0(\bar{D}_N^*)^1(x_0) = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.
\] (3.63)

Since
\[
\sigma_{-2}(\bar{D}_N^*)^1(x_0) = \frac{c(\xi')\sigma_0(\bar{D}_N^*)^1(x_0)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \left[ \partial_{x_n}[c(\xi')(x_0)]|\xi|^2 - c(\xi)h'(0)\right]_{x_n=0},
\] (3.64)

then
\[
\sigma_0(\bar{D}_N^*)^1(x_0) = \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(\bar{e}_i)(x_0) c(\bar{e}_i)c(\bar{e}_i)c(\bar{e}_i)c(\bar{e}_i) - \frac{1}{4} \sum_{s,t,i} \omega_{s,t}(\bar{e}_i)(x_0) c(\bar{e}_i)c(\bar{e}_i)c(\bar{e}_i) + (\theta - c(\theta'))(x_0)
= b_0^1(x_0) + b_0^2(x_0) + (\theta - c(\theta'))(x_0),
\] (3.65)

By direct calculation, we have
\[
\partial_{x_n} \left\{ \frac{c(\xi')(x_0) + c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) \right\}
= \frac{c(dx_n) b_0^1(x_0) c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6} c(dx_n) - \frac{4|\xi|^6 c(\xi) b_0^1(x_0) c(\xi)}{|\xi|^4};
\] (3.67)
\[
\partial_{\xi_n} \left\{ \frac{c(\xi)(\xi(\theta) - c(\theta'))(x_0)c(\xi)}{\|\xi\|^4} \right\} = \frac{c(dx_n)(\xi(\theta) - c(\theta'))(x_0)c(\xi)}{\|\xi\|^4} + \frac{c(\xi)(\xi(\theta) - c(\theta'))(x_0)c(dx_n)}{\|\xi\|^4} - 4\xi_n c(\xi)(\xi(\theta) - c(\theta'))(x_0)c(\xi),
\]

and

\[
\partial_{\xi_n} \left\{ \frac{c(\xi)c(dx_n)[\partial_{x_n}[c(\xi')](x_0)]\|\xi\|^2 - c(\xi)h'(0)]}{\|\xi\|^4} \right\} + \partial_{\xi_n} \left\{ \frac{c(\xi)b_0^2(x_0)c(\xi)}{\|\xi\|^4} \right\} = \frac{1}{(1 + \xi_n^2)^3} \left[ (2\xi_n - 2\xi_n^3) c(dx_n) b_0^2 c(dx_n) + (1 - 3\xi_n^2) c(dx_n) b_0^2 c(\xi') + (1 - 3\xi_n^2) c(\xi') b_0^2 c(dx_n) - 4\xi_n c(\xi') b_0^2 c(\xi') + (3\xi_n^2 - 1) \partial_{x_n} c(\xi') - 4\xi_n c(\xi') c(dx_n) \partial_{x_n} c(\xi') + 2h'(0) c(\xi') + 2h'(0) \xi_n c(dx_n) \right] + 6\xi_n h'(0) \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^4}.
\]

By (3.63) and (3.67), we have

\[
\text{tr} \left[ \frac{\pi_n^+ \sigma_- (\hat{D}_{N-1}^{-1}) \times \partial_{\xi_n} c(\xi) b_0^2 c(\xi)}{\|\xi\|^4} \right](x_0)|_{\|\xi\|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi') b_0^2(x_0)].
\]

By (3.48) and (3.49), we have

\[
\text{tr} \left[ \frac{\pi_n^+ \sigma_- (\hat{D}_{N-1}^{-1}) \times \partial_{\xi_n} c(\xi) b_0^2 c(\xi)}{\|\xi\|^4} \right](x_0)|_{\|\xi\|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi') b_0^2(x_0)].
\]

We note that \(i < n\), \(\int_{\|\xi\|=1} \{\xi_1, \xi_2, \ldots, \xi_{2d+1}\} \sigma(\xi') = 0\), so \(\text{tr}[c(\xi') b_0^2(x_0)]\) has no contribution for computing case (c).

By (3.63) and (3.69), we have

\[
\text{tr} \left\{ \frac{\pi_n^+ \sigma_- (\hat{D}_{N-1}^{-1}) \times \partial_{\xi_n} c(\xi) c(dx_n) [\partial_{x_n}[c(\xi')](x_0)]\|\xi\|^2 - c(\xi)h'(0)]}{\|\xi\|^4} + \frac{c(\xi) b_0^2(x_0)c(\xi)}{\|\xi\|^4} \right\}(x_0)|_{\|\xi\|=1} = \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^4(\xi + i)^4} + \frac{48h'(0)i\xi_n}{(\xi - i)^4(\xi + i)^4},
\]

then

\[
-i\Omega_3 \int_{\Gamma_{\xi}} \left[ \frac{12h'(0)(i\xi_n^2 + \xi_n - 2i)}{(\xi - i)^4(\xi + i)^3} + \frac{48h'(0)i\xi_n}{(\xi - i)^4(\xi + i)^4} \right] d\xi_n d\xi' = -\frac{9}{2} h'(0) \Omega_3 d\xi'.
\]

By (3.63) and (3.68), we have

\[
\text{tr} \left[ \frac{\pi_n^+ \sigma_- (\hat{D}_{N-1}^{-1}) \times \partial_{\xi_n} c(\xi)(\xi(\theta) - c(\theta'))c(\xi)}{\|\xi\|^4} \right](x_0)|_{\|\xi\|=1} = \frac{-1}{(\xi - i)(\xi + i)^3} \text{tr}[c(\xi')(\xi(\theta) - c(\theta'))(x_0)] + \frac{i}{(\xi - i)(\xi + i)^3} \text{tr}[c(dx_n)(\xi(\theta) - c(\theta'))(x_0)].
\]

\[\]
By \( \int_{|\xi|=1} \sigma_x \sigma_y \) for six-dimensional manifolds with boundary \( M \) and the metric \( g^M \) as above, \( \Delta_N \) and \( \hat{\Delta}_N \) be modified Novikov operators on \( M \), then

\[
\widetilde{W}_{res}[\pi^+(f \hat{\Delta}_N^{-1}) \circ \pi^+(f^{-1}(\hat{\Delta}_N^{-1})^{-1})] = \sum_{l=1}^{\infty} \int_{|\xi|=1} \left\{ \sum_{j,k=0}^{\infty} \frac{(-i)^{a+j+k+l}}{a! (j+k+1)!} \frac{1}{\alpha^a} \frac{1}{\alpha^b} \frac{1}{\alpha^c} \frac{1}{\alpha^d} \right\} \sigma_x \sigma_y \sigma_x \sigma_y \sigma_x \sigma_y dx \]

where \( s \) is the scalar curvature.

### 4. A Kastler-Kalau-Walze type theorem for six-dimensional manifolds with boundary

Let \( M \) be 6-dimensional compact manifolds with the boundary \( \partial M \). In the following, we will compute the more general case \( \widetilde{W}_{res}[\pi^+(f \hat{\Delta}_N^{-1}) \circ \pi^+(f^{-1}(\hat{\Delta}_N^{-1})^{-1} \cdot f \hat{\Delta}_N^{-1} \cdot f^{-1}(\hat{\Delta}_N^{-1})^{-1})] \) for nonzero smooth functions \( f, f^{-1} \). An application of (3.5) and (3.6) in [14] shows that

\[
\widetilde{W}_{res}[\pi^+(f \hat{\Delta}_N^{-1}) \circ \pi^+(f^{-1}(\hat{\Delta}_N^{-1})^{-1} \cdot f \hat{\Delta}_N^{-1} \cdot f^{-1}(\hat{\Delta}_N^{-1})^{-1})] = \int_M \int_{|\xi|=1} \left\{ \sum_{j,k=0}^{\infty} \frac{(-i)^{a+j+k+l}}{a! (j+k+1)!} \frac{1}{\alpha^a} \frac{1}{\alpha^b} \frac{1}{\alpha^c} \frac{1}{\alpha^d} \right\} \sigma_x \sigma_y \sigma_x \sigma_y \sigma_x \sigma_y dx + \int_{\partial M} \Psi,
\]

where

\[
\Psi = \int_{|\xi|=1}^{\infty} \sum_{j,k=0}^{\infty} \frac{(-i)^{a+j+k+l}}{a! (j+k+1)!} \frac{1}{\alpha^a} \frac{1}{\alpha^b} \frac{1}{\alpha^c} \frac{1}{\alpha^d} \right\} \sigma_x \sigma_y \sigma_x \sigma_y \sigma_x \sigma_y dx.
\]
and the sum is taken over \( r - k + |\alpha| + \ell - j - 1 = -n = -6, r \leq -1, \ell \leq -3. \)

Note that
\[
\begin{align*}
&f^{-1}((\tilde{D}_N^* N)^{-1} \cdot f \tilde{D}_N^{-1} \cdot f^{-1}(\tilde{D}_N^* N)^{-1} \\
= & (\tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot \tilde{D}_N^* f)^{-1} \\
= & \left( \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot [\tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot f] \right)^{-1} \\
= & \left( \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot f \cdot \tilde{D}_N^* \tilde{D}_N^{-1} \cdot \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot f \right)^{-1} \\
= & \left( f \cdot \tilde{D}_N^* \tilde{D}_N^{-1} \tilde{D}_N^* f + [\tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \tilde{D}_N^* f - \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} f] \right)^{-1} \\
= & \left( f \cdot \tilde{D}_N^* \tilde{D}_N^{-1} \tilde{D}_N^* f + [\tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \tilde{D}_N^* f - \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} f] \right)^{-1} \\
= & \left( f \cdot \tilde{D}_N^* \tilde{D}_N^{-1} \tilde{D}_N^* f + \tilde{D}_N^* \tilde{D}_N^{-1} \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} f \right)^{-1}. \quad (4.3)
\end{align*}
\]

In order to get the symbol of operators \( \tilde{D}_N^* f \cdot \tilde{D}_N^{-1} \cdot \tilde{D}_N f \). We first give the specification of \( \tilde{D}_N^* \tilde{D}_N \) and \( \tilde{D}_N^* \tilde{D}_N \). By (2.10) and (2.11), we have

\[
\begin{align*}
\tilde{D}_N^* \tilde{D}_N &= - \sum_{i,j} g^{ij} \left[ \partial_i \partial_j + 2 \sigma_i \partial_j + 2 a_i \partial_j - \Gamma^k_{ij} \partial_k + (\partial_i \sigma_j) + (\partial_j \sigma_i) + a_i \sigma_j + a_j \sigma_i + a_i a_j - \Gamma^{ik}_{ij} \sigma_k \\
&\quad - \Gamma^k_{ij} a_k \right] - \frac{1}{8} \sum_{ijkl} R_{ijkl} c(\partial_i c(\partial_j c(\partial_k c(\partial_l)))) + \frac{1}{4} \sum_i (\partial_i \theta)(\partial_i \theta) + |\theta|^2 \\
&\quad + \frac{1}{2} \sum_{ijkl} R_{ijkl} c(\partial_i c(\partial_j c(\partial_k c(\partial_l)))) + \frac{1}{4} \sum_i (\partial_i \theta)(\partial_i \theta) + |\theta|^2
\end{align*}
\]

and

\[
\begin{align*}
\tilde{D}_N \tilde{D}_N &= - \sum_{i,j} g^{ij} \left[ \partial_i \partial_j + 2 \sigma_i \partial_j + 2 a_i \partial_j - \Gamma^k_{ij} \partial_k + (\partial_i \sigma_j) + (\partial_j \sigma_i) + a_i \sigma_j + a_j \sigma_i + a_i a_j - \Gamma^{ik}_{ij} \sigma_k \\
&\quad - \Gamma^k_{ij} a_k \right] + \frac{1}{8} \sum_{ijkl} R_{ijkl} c(\partial_i c(\partial_j c(\partial_k c(\partial_l)))) + \frac{1}{4} \sum_i (\partial_i \theta)(\partial_i \theta) + |\theta|^2 \\
&\quad + \frac{1}{2} \sum_{ijkl} R_{ijkl} c(\partial_i c(\partial_j c(\partial_k c(\partial_l)))) + \frac{1}{4} \sum_i (\partial_i \theta)(\partial_i \theta) + |\theta|^2
\end{align*}
\]
Combining (2.11) and (4.4), we obtain
\[
\tilde{D}_N^N \tilde{D}_N \tilde{D}_N^N = \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) (-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) \left\{ - (\partial_i g^{ij}) \partial_i \partial_j - g^{ij} \left( 4(\sigma_i + a_i) \partial_j - 2 \Gamma_{ij}^k \partial_k \right) \partial_j \right\} \\
+ \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) \left\{ - 2 (\partial_i g^{ij})(\sigma_i + a_i) \partial_j + g^{ij} (\partial_i \Gamma_{ij}^k) \partial_k - 2 g^{ij} [(\partial_i \sigma_i) + (\partial_i a_i)] \partial_j + (\partial_i g^{ij}) \Gamma_{ij}^k \partial_k \right\} \\
+ \sum_{j,k} \left[ \partial_i \left( c(\theta')c(\tilde{e}_j) - c(\tilde{e}_j)c(\theta') \right) \right](\tilde{e}_j, dx^k) \partial_k + \sum_{j,k} \left( c(\theta')c(\tilde{e}_j) - c(\tilde{e}_j)c(\theta') \right) \left[ \partial_i (\tilde{e}_j, dx^k) \partial_k \right] \\
+ \sum_{i,j} g^{ij} \left[ c(\theta')c(\partial_i \sigma_i + c(\theta') c(\partial_i a_i) - c(\partial_i) \partial_i (c(\theta')) - c(\partial_i) \sigma_i c(\theta') - c(\partial_i) a_i c(\theta') \right] \right\} \\
+ \sum_{i,j} g^{ij} \left[ c(\theta')c(\partial_i \sigma_i + c(\theta') c(\partial_i a_i) - c(\partial_i) \partial_i (c(\theta')) - c(\partial_i) \sigma_i c(\theta') - c(\partial_i) a_i c(\theta') \right] \left( \frac{1}{4} \mathbf{s} \right) ^2 \\
- \frac{1}{8} \sum_{i,j,k} R_{ijkl} c(\tilde{e}_i)(\tilde{e}_j)(\tilde{e}_k) c(\tilde{e}_l) + \sum_{i} c(\tilde{e}_i)(\tilde{e}_i, dx_i) \left\{ 2 \sum_{j,k} \left[ c(\theta')c(\tilde{e}_j) - c(\tilde{e}_j)c(\theta') \right] \right\} \\
\times (\tilde{e}_i, dx_k) \} \partial_i \partial_k + \sum_{i,j} g^{ij} \left[ 2 \sigma_i \partial_j + 2 a_i \partial_j - \Gamma_{ij}^k \partial_k + (\partial_i \sigma_j) \right] \\
+ (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \left] - \sum_{i,j} g^{ij} \left[ c(\partial_i) c(\theta') - c(\theta') c(\partial_i) \right] \partial_j \\
+ \sum_{i,j} g^{ij} \left[ c(\theta') c(\partial_i) \sigma_i + c(\theta') c(\partial_i) a_i - c(\partial_i) \partial_i (c(\theta')) - c(\partial_i) \sigma_i c(\theta') - c(\partial_i) a_i c(\theta') \right] \left( \frac{1}{4} \mathbf{s} \right) ^2 \\
- \frac{1}{8} \sum_{i,j,k} R_{ijkl} c(\tilde{e}_i)(\tilde{e}_j)(\tilde{e}_k) c(\tilde{e}_l) + \sum_{i} c(\tilde{e}_i)(\tilde{e}_i, dx_i) \left\{ 2 \sum_{j,k} \left[ c(\theta')c(\tilde{e}_j) - c(\tilde{e}_j)c(\theta') \right] \right\} \left( \frac{1}{4} \mathbf{s} \right) ^2 \right\}.
\]

Thus, using (4.3)-(4.6), we get the specification of \( \tilde{D}_N f \cdot \tilde{D}_N f^{-1} \cdot \tilde{D}_N f \).
\[
\tilde{D}_N f \cdot \tilde{D}_N f^{-1} \cdot \tilde{D}_N f \]
\[
= f \cdot \tilde{D}_N^N \tilde{D}_N \tilde{D}_N f + c(df) \tilde{D}_N \tilde{D}_N f - \tilde{D}_N \tilde{D}_N f \cdot c(df^{-1}) \cdot f + \tilde{D}_N f \cdot c(df^{-1}) \cdot f
\]
\[
= f \cdot \left\{ \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) (-g^{ij} \partial_i \partial_j) + \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) \left\{ - (\partial_i g^{ij}) \partial_i \partial_j - g^{ij} \left( 4(\sigma_i + a_i) \partial_j - 2 \Gamma_{ij}^k \partial_k \right) \partial_j \right\} \\
+ \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) \left\{ - 2 (\partial_i g^{ij})(\sigma_i + a_i) \partial_j + g^{ij} (\partial_i \Gamma_{ij}^k) \partial_k - 2 g^{ij} [(\partial_i \sigma_i) + (\partial_i a_i)] \partial_j + (\partial_i g^{ij}) \Gamma_{ij}^k \partial_k \right\} \\
+ \sum_{j,k} \left[ \partial_i \left( c(\theta')c(\tilde{e}_j) - c(\tilde{e}_j)c(\theta') \right) \right](\tilde{e}_j, dx^k) \partial_k + \sum_{j,k} \left( c(\theta')c(\tilde{e}_j) - c(\tilde{e}_j)c(\theta') \right) \left[ \partial_i (\tilde{e}_j, dx^k) \partial_k \right] \\
+ \sum_{i=1}^n c(\tilde{e}_i)(\tilde{e}_i, dx_i) \partial_i \left\{ - g^{ij} \left[ (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma_{ij}^k \sigma_k - \Gamma_{ij}^k a_k \right] \right\} \right\}.
\]
+ \sum_{i,j} g^{i,j} \left[ c(\theta')c(\partial_i)\sigma_i + c(\theta')c(\partial_i)a_i - c(\partial_i)\partial_i(c(\theta')) - c(\partial_i)\sigma_i c(\theta') - c(\partial_i)a_i c(\theta') \right] + \frac{1}{4} s \\
- \frac{1}{8} \sum_{ijkl} R_{ijkl} \tilde{e}(\tilde{e}_i)(\tilde{e}_j)(\tilde{e}_k) + \frac{1}{2} \left( c(\tilde{e}_i) c(\tilde{e}_j)(\tilde{e}_k) \right) + \frac{1}{2} \left( c(\tilde{e}_i) c(\tilde{e}_j)(\tilde{e}_k) \right) \left\{ 2 \sum_{j,k} c(\theta')c(\tilde{e}_i) - c(\tilde{e}_j) c(\theta') \right\} \\
+ \left[ (\sigma_i + a_i) + (\tilde{e}(\theta) - c(\theta')) \right] \left\{ - g^{i,j} \partial_i \partial_j + \frac{1}{2} \sum_{i=1}^n \tilde{c}(\tilde{e}_i, dx_i) \right\} \left\{ 2 \sum_{j,k} c(\theta')c(\tilde{e}_i) - c(\tilde{e}_j) c(\theta') \right\} \\
\times (\tilde{e}_i, dx_i) \partial_i \partial_k + \left[ (\sigma_i + a_i) + (\tilde{e}(\theta) - c(\theta')) \right] \left\{ - \sum_{i,j} g^{i,j} \left[ 2 \sigma_i \partial_j + 2 a_i \partial_j - \Gamma^k_{i,j} \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma^k_{i,j} \sigma_k - \Gamma^k_{i,j} a_k \right] \right\} \\
+ \left[ g^{i,j} \left[ c(\tilde{e}_i) c(\tilde{e}_j) \sigma_i + c(\tilde{e}_i) c(\tilde{e}_j) a_i - c(\partial_i) \partial_i(c(\theta')) - c(\tilde{e}_i) \sigma_i c(\theta') - c(\tilde{e}_i) a_i c(\theta') \right] + \frac{1}{4} s + |\theta'|^2 \right\} \\
- \frac{1}{8} \sum_{ijkl} R_{ijkl} \tilde{e}(\tilde{e}_i)(\tilde{e}_j)(\tilde{e}_k) + \frac{1}{2} \left( c(\tilde{e}_i) c(\tilde{e}_j)(\tilde{e}_k) \right) + \frac{1}{2} \left( c(\tilde{e}_i) c(\tilde{e}_j)(\tilde{e}_k) \right) \left\{ 2 \sum_{j,k} c(\theta')c(\tilde{e}_i) - c(\tilde{e}_j) c(\theta') \right\} \\
\times \left\{ - \sum_{i,j} g^{i,j} \left[ \partial_i \partial_j + 2 a_i \partial_j - \Gamma^k_{i,j} \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma^k_{i,j} \sigma_k - \Gamma^k_{i,j} a_k \right] \right\} \\
- \left[ g^{i,j} \left[ c(\tilde{e}_i) c(\tilde{e}_j) \sigma_i + c(\tilde{e}_i) c(\tilde{e}_j) a_i - c(\partial_i) \partial_i(c(\theta')) - c(\tilde{e}_i) \sigma_i c(\theta') - c(\tilde{e}_i) a_i c(\theta') \right] + \frac{1}{4} s + |\theta'|^2 \right\} \\
+ |\theta'|^2 - c(\theta)c(\theta') + c(\theta')c(\theta') \right\} - \left\{ - \sum_{i,j} g^{i,j} \left[ \partial_i \partial_j + 2 a_i \partial_j - \Gamma^k_{i,j} \partial_k + (\partial_i \sigma_j) + (\partial_i a_j) + \sigma_i \sigma_j + \sigma_i a_j + a_i \sigma_j + a_i a_j - \Gamma^k_{i,j} \sigma_k - \Gamma^k_{i,j} a_k \right] \right\} \\
- \left[ g^{i,j} \left[ c(\theta')c(\partial_i)\sigma_i + c(\theta')c(\partial_i)a_i - c(\partial_i)\partial_i(c(\theta')) - c(\partial_i)\sigma_i c(\theta') - c(\partial_i)a_i c(\theta') \right] + \frac{1}{4} s + \sum_{i=1}^n c(\tilde{e}_i) c(\tilde{e}_j) + |\theta'|^2 + |\theta'|^2 \right\} \\
- c(\theta')c(\theta') \right\} f \cdot c(df^{-1}) \cdot f + \left\{ \sum_{i,j=1}^n g^{i,j} \left[ \partial_i \partial_j + a_i a_i \right] + c(\theta) - c(\theta') \right\} \cdot c(df)c(df^{-1})f. \quad (4.7)
In order to get the symbol of operators \( \hat{D}_N f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N f \). We first give the following formulas:

\[
D_\sigma^\omega = (-\sqrt{-1})^{\alpha} \partial_\sigma^\omega; \quad \sigma(\hat{D}_N f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N f) = p_1 + p_2 + p_1 + p_0; \\
\sigma((\hat{D}_N f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N f)^{-1}) = \sum_{j=3}^\infty q_{-j}. \tag{4.10}
\]

By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma([\hat{D}_N f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N f] \circ (\hat{D}_N f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N f)^{-1}) \\
= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots) \\
+ \sum_j (\partial_\xi_j p_3 + \partial_\xi_j p_2 + \partial_\xi_j p_1 + \partial_\xi_j p_0) (D_{x_j} q_{-3} + D_{x_j} q_{-4} + D_{x_j} q_{-5} + \cdots) \\
= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial_\xi_j p_3 D_{x_j} q_{-3} + \cdots). \tag{4.11}
\]

Then

\[
q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1} [p_2 p_3^{-1} + \sum_j \partial_\xi_j p_3 D_{x_j} (p_3^{-1})]. \tag{4.12}
\]

By Lemma 2.1 in [12] and Lemma 4.1, we obtain

**Lemma 4.2.** Let \( \hat{D}_N \) and \( \hat{D}_N^* \) be modified Novikov operators on \( \hat{M} \), then

\[
\sigma_{-3}(\hat{D}_N^* f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N^* f)^{-1} = f^{-1} \sigma_{-3}(\hat{D}_N^* \hat{D}_N \hat{D}_N^*)^{-1} = \frac{\sqrt{1+\nu} \xi}{f |\xi|^4}; \tag{4.13}
\]

\[
\sigma_{-4}(\hat{D}_N^* f \cdot \hat{D}_N f^{-1} \cdot \hat{D}_N^* f)^{-1} = f^{-1} \sigma_{-4}(\hat{D}_N^* \hat{D}_N \hat{D}_N^*)^{-1} + \frac{2 \nu \xi |f| c(f) c(\xi)}{f^2 |\xi|^6} \\
+ \frac{ic(\xi) \sum_j (c(d x_j)|\xi|^2 + 2 \xi c(\xi)) D_{x_j} (f^{-1}) c(\xi)}{|\xi|^8}. \tag{4.14}
\]

where

\[
\sigma_{-3}(\hat{D}_N^* \hat{D}_N \hat{D}_N^*)^{-1} = \frac{ic(\xi)}{|\xi|^4}; \\
\sigma_{-4}(\hat{D}_N^* \hat{D}_N \hat{D}_N^*)^{-1} = \frac{\xi |\xi|^2 c(\xi) c(\xi) \xi + \frac{ic(\xi)}{|\xi|^8} \left( |\xi|^4 c(d x_n) \partial_{z_n} c(\xi) - 2 h(0) c(x_n) c(\xi) + 2 \xi_n c(\xi) \partial_{z_n} c(\xi) + 4 \xi_n h(0) \right). \tag{4.15}
\]

Locally we can use Theorem 3.3 to compute the interior term of (4.1), then

\[
\int_M \int_{|\xi|=1} tr_{\lambda \cdot T \cdot M} [\sigma_{-n}(\hat{D}_N^* \hat{D}_N f^{-1})^{-1}] \sigma(\xi) d\lambda \\
= 128 \pi^3 \int_M 2^n \left\{ \left( - \frac{1}{12} |\theta|^2 + (n - 2) |\theta|^2 + g(\varepsilon_j, \nabla_{T M} \varepsilon_j) \right) + \left[ f^{-1} \Delta(f) + \langle \text{grad}_M f, \text{grad}_M f' \rangle^{-1} \right] \\
- (e(\theta) - e(\theta')) c(df) f^{-1} - \frac{1}{2} \left[ f^{-1} \Delta(f)(x) + \langle \text{grad}_M f, \text{grad}_M f' \rangle^{-1} \right] + \frac{1}{4} \sum_i \left[ c(e_i) c(\theta') - c(\theta') c(e_i) \right] \right\} dVol_M. \tag{4.16}
\]
So we only need to compute $\int_{\partial M} \Psi$.

From the formula (4.2) for the definition of $\Psi$, now we can compute $\Psi$. Since the sum is taken over $r + \ell - k - j - |\alpha| - 1 = -6$, $r \leq -1, \ell \leq -3$, then we have the $\int_{\partial M} \Psi$ is the sum of the following five cases:

case (a) (I) $r = -1, \ell = -3, j = k = 0, |\alpha| = 1$.

By (4.2), we get

\[
\int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum |\alpha|=1 \text{tr} \left[ \partial^\alpha \pi^{\alpha}_x \sigma_{-1}(f \tilde{D}_N^{-1}) \times \partial^\alpha_x \partial_\xi \sigma_{-3}(f^{-1} \tilde{D}_N^{-1}) \cdot f \tilde{D}_N^{-1} \right] (x_0)d\xi_n \sigma(\xi')dx' \]

Then for $i < n$, we have

\[
\partial_\xi \sigma_{-3}(\hat{\tilde{D}}_N^{-1}\tilde{D}_N\hat{\tilde{D}}_N^{-1})(x_0) = \partial_\xi \left[ \frac{\sqrt{-1}c(\xi)}{|\xi|^4} \right] (x_0) = \sqrt{-1}\partial_\xi \left[ c(\xi) \right] |\xi|^{-4}(x_0) - 2\sqrt{-1}c(\xi) \partial_\xi \left[ |\xi|^2 \right] |\xi|^{-6}(x_0) = 0. \tag{4.18}
\]

Thus we have

\[
- \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum |\alpha|=1 \text{tr} \left[ \partial^\alpha \pi^{\alpha}_x \sigma_{-1}(\hat{\tilde{D}}_N^{-1}) \times \partial^\alpha_x \partial_\xi \sigma_{-3}(\hat{\tilde{D}}_N^{-1}\tilde{D}_N\hat{\tilde{D}}_N^{-1}) \right] (x_0)d\xi_n \sigma(\xi')dx' = 0. \tag{4.19}
\]

By Lemma 2.2 in [12] and (4.15), for $i < n$, we obtain

\[
\partial^\alpha_x \partial_\xi \sigma_{-3}(\hat{\tilde{D}}_N^{-1}\tilde{D}_N\hat{\tilde{D}}_N^{-1})(x_0) = \partial^\alpha_x \partial_\xi \left[ \frac{\sqrt{-1}c(\xi)}{|\xi|^4} \right] (x_0) = -\xi \partial_\xi \left[ c(\xi) \frac{\sqrt{-1}c(\xi) + \xi c(\xi)}{|\xi|^4} \right]. \tag{4.20}
\]

and we get

\[
\partial_\xi \sigma_{-3}(\hat{\tilde{D}}_N^{-1}\tilde{D}_N\hat{\tilde{D}}_N^{-1})^{-1} = \frac{\sqrt{-1}c(\xi)}{|\xi|^4} + \frac{4\sqrt{-1}|\xi|^2 c(\xi) |\xi|^6}{|\xi|^6} \tag{4.21}
\]

Then for $i < n$, we have

\[
\text{tr} \left[ \partial^\alpha_x \pi^{\alpha}_x \sigma_{-1}(\hat{\tilde{D}}_N^{-1}) \times \partial_\xi \sigma_{-3}(\hat{\tilde{D}}_N^{-1}\tilde{D}_N\hat{\tilde{D}}_N^{-1}) \right] (x_0) = -\xi \text{tr} \left[ \frac{\sqrt{-1}c(\xi)}{|\xi|^4} + \frac{4\sqrt{-1}|\xi|^2 c(\xi) |\xi|^6}{|\xi|^6} \right]. \tag{4.22}
\]
We note that $i < n$, $\int_{|\xi'|=1} \xi_i \sigma(\xi') = 0$, so

$$-\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} \left[ \partial_i \pi_\xi^+ \sigma_{-1}((\tilde{D}_N^* \tilde{D}_N \tilde{D}_N)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' = 0. \tag{4.23}$$

Then we have case (a) (I) = 0.

**case (a) (II)** $r = -1, l = -3, |\alpha| = k = 0, j = 1$.

By (4.2), we have

$$\text{case (a) (II)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} \left[ \partial_i \pi_\xi^+ \sigma_{-1}(f \tilde{D}_N^* \dot{\xi}_n) \times \partial_i \pi_\xi \sigma_{-3}(f^{-1}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \tag{4.24}$$

By (2.3), (2.4) and (3.26), we have

$$\pi_\xi^+ \partial_i \pi_\xi \sigma_{-1}(\tilde{D}_N^{*}) (x_0)_{|\xi'|=1} = \frac{\partial_i \pi_\xi^+ \sigma_{-1}(f \tilde{D}_N^*) (x_0)_{|\xi'|=1}}{2(\xi_n - \sqrt{-1})} + \sqrt{-1} h'(0) \left[ \frac{\sqrt{-1} c(\xi')}{4(\xi_n - \sqrt{-1})} + \frac{c(\xi') + \sqrt{-1} c(dx_n)}{4(\xi_n - \sqrt{-1})^2} \right]. \tag{4.25}$$

By (4.25)-(4.28), we get

$$\text{tr} \left[ \partial_i \pi_\xi^+ \sigma_{-1}(\tilde{D}_N^{*}) \times \partial_i \pi_\xi \sigma_{-3}(\tilde{D}_N^{*} \tilde{D}_N \tilde{D}_N)^{-1} \right] (x_0) = \frac{64 h'(0)(-1 - 3\xi_n \sqrt{-1} + 5\xi_n^2 + 3\sqrt{-1} \xi_n^3)}{(\xi_n - \sqrt{-1})^2(\xi_n + \sqrt{-1})^2}. \tag{4.29}$$

Then we obtain

$$-\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} \left[ \partial_i \pi_\xi^+ \sigma_{-1}(\tilde{D}_N^{*}) \times \partial_i \pi_\xi \sigma_{-3}(\tilde{D}_N^{*} \tilde{D}_N \tilde{D}_N)^{-1} \right] (x_0) d\xi_n \sigma(\xi') dx' \tag{4.30}$$
On the other hand, by calculations, we have
\[ \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_N^{-1}) (x_0) \big|_{\xi^*} = 1 = \frac{e(\xi') + \sqrt{-1}\xi(dx_n)}{2(\xi_n - \sqrt{-1})}, \] (4.31)

By (4.25), (4.27) and (4.31), we get
\[ \text{tr} \left[ \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_N^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N^*)^{-1} \right] (x_0) = \frac{-128(5\xi_n^2\sqrt{-1} - \sqrt{-1} - 3\xi_n^3 + 3\xi_n)}{(\xi_n - \sqrt{-1})^5(\xi_n + \sqrt{-1})^4}. \] (4.32)

Then we obtain
\[ -\frac{1}{2} f^{-1} \partial_{x_n} (f) \int_{\xi^* = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_N^{-1}) \times \partial^2_{\xi_n} \sigma_{-3}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N^*)^{-1} \right] (x_0) d\xi_n \sigma(\xi') dx' = \frac{5\sqrt{-1} + 44\pi f^{-1} \partial_{x_n} (f) \cdot 8\Omega_4}{4} dx', \] (4.33)

where \( \Omega_4 \) is the canonical volume of \( S_4 \).

Combining (4.24), (4.30) and (4.33), we obtain

\[ \text{case (a) (II)} = \frac{15}{2} \pi h'(0) \Omega_4 dx' + (10\sqrt{-1} + 88) \pi f^{-1} \partial_{x_n} (f) \cdot 8\Omega_4 dx'. \] (4.34)

\[ \text{case (a) (III)} = -1, l = -3, |\alpha| = j = 0, k = 1. \] (4.35)

By (4.2), we have

\[ \text{case (a) (III)} = -\frac{1}{2} \int_{\xi^* = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(f \tilde{D}_N^{-1}) \times \partial_{\xi_n} \sigma_{-3}(f^{-1}(\tilde{D}_N)^{-1}) \cdot f \tilde{D}_N^{-1} \right] (x_0) d\xi_n \sigma(\xi') dx' = -\frac{1}{2} \int_{\xi^* = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_N^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N)^{-1} \right] (x_0) d\xi_n \sigma(\xi') dx'. \] (4.36)

By (4.6) and direct calculations, we have
\[ \partial_{\xi_n} \sigma_{-3}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N^{-1}) = \frac{-4\sqrt{-1}h(0)\xi_n e(\xi')}{(1 + \xi_n^2)^3} + \frac{12\sqrt{-1}h(0)\xi_n e(\xi')}{(1 + \xi_n^2)^4} - \frac{\sqrt{-1}(2 - 10\xi_n^2)h(0)e(dx_n)}{(1 + \xi_n^2)^4}. \] (4.37)

Combining (3.37) and (4.6), we have
\[ \text{tr} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_N^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N^{-1}) \right] (x_0) \big|_{\xi^*} = 1 = 64h'(0)(\sqrt{-1} - 4\xi_n - \sqrt{-1}\xi_n^3) \frac{2h(0)(\xi_n - \sqrt{-1})^5(\xi_n + i)^4}{(\xi_n - \sqrt{-1})^5(\xi_n + \sqrt{-1})^4}, \] (4.38)

and
\[ \text{tr} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(\tilde{D}_N^{-1}) \times \partial_{\xi_n} \sigma_{-3}(\tilde{D}_N^* \tilde{D}_N \tilde{D}_N^{-1}) \right] (x_0) \big|_{\xi^*} = 1 = -32 \frac{4\sqrt{-1}\xi_n + 1 - 3\xi_n^2}{(\xi_n - \sqrt{-1})^5(\xi_n + \sqrt{-1})^3}. \] (4.39)
Then
\[-\frac{1}{2} \int_{\xi' = 1}^{\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_+^{\prime} (\sigma_{-1}(\hat{D}_N^{-1})) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_{-3}(\hat{D}_N^{-1} \hat{D}_N \hat{D}_N^{-1})^{-1} \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= \frac{25}{2} \pi h'(0) \Omega_4 dx', \tag{4.39} \]
and
\[-\frac{1}{2} f \partial_{x_n} (f^{-1}) \int_{\xi' = 1}^{\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_+^{\prime} (\sigma_{-1}(\hat{D}_N^{-1})) \times \partial_{\xi_n} \sigma_{-3}(\hat{D}_N^{-1} \hat{D}_N \hat{D}_N^{-1})^{-1} \right] (x_0) d\xi_n \sigma(\xi') dx' \]
\[= \frac{\pi \sqrt{-1}}{2} f \cdot \partial_{x_n} (f^{-1}) \Omega_4 dx', \tag{4.40} \]
where \( \Omega_4 \) is the canonical volume of \( S_4 \).

Then
\[
\text{case (a) (III) } = \left[ \frac{25}{2} \pi h'(0) + \frac{\pi \sqrt{-1}}{2} f \cdot \partial_{x_n} (f^{-1}) \right] \Omega_4 dx'. \tag{4.41} \]

\textbf{case (b) } r = -1, l = -4, |\alpha| = j = k = 0.

By (4.2), we have
\[
\text{case (b) } = -i \int_{\xi' = 1}^{\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \pi_+^{\prime} \sigma_{-1}(f \hat{D}_N^{-1}) \times \partial_{\xi_n} \sigma_{-4}(f^{-1}(\hat{D}_N)^{-1} \cdot f \hat{D}_N^{-1} \cdot f^{-1}(\hat{D}_N)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
= \left[ \frac{25}{2} \pi \sqrt{-1} \cdot f \cdot \partial_{x_n} (f^{-1}) \right] \Omega_4 dx'. \tag{4.42} \]

In the normal coordinate, \( g^{ij}(x_0) = \delta^i_j \) and \( \partial_{x_j} (g^{\alpha \beta})(x_0) = 0 \), if \( j < n \); \( \partial_{x_j} (g^{\alpha \beta})(x_0) = h'(0) \delta^j_\beta \), if \( j = n \). So by Lemma A.2 in [12], we have \( \Gamma^n(x_0) = \frac{h'}{2} h'(0) \) and \( \Gamma^k(x_0) = 0 \) for \( k < n \). By the definition of \( \delta^k \) and Lemma 2.3 in [12], we have \( \delta^\alpha(x_0) = 0 \) and \( \delta^\beta = \frac{1}{4} h'(0) c(\xi_j) c(\xi_n) \) for \( k < n \). By Lemma 4.2, we obtain
\[
\begin{align*}
  \sigma_{-4}(\hat{D}_N^* \hat{D}_m^* \hat{D}_N^*)^{-1}(x_0)|_{\xi'|=1} &= \frac{c(\xi)\sigma_2((\hat{D}_N^* \hat{D}_m^* \hat{D}_N^*)^{-1}(x_0))}{|\xi'|^4} - \frac{c(\xi)}{|\xi'|^4} \sum_j \partial_{\xi_j} (c(\xi)|\xi|^2) D_{x_j}(\frac{i\xi}{|\xi'|^4}) \\
  &= \frac{1}{|\xi'|^4} c(\xi) \left( \frac{1}{2} h'(0)c(\xi) \sum_{k<n} \xi_k e(\xi_k) c(\xi_k) - \frac{1}{2} h'(0)c(\xi) \sum_{k<n} \xi_k e(\xi_k) c(\xi_k) - \frac{5}{2} h'(0)\xi_n c(\xi) \\
  &\quad - \frac{1}{4} h'(0)|\xi|^2 c(dx_n) - 2[c(\xi)e(\theta')c(\xi) + |\xi|^2 c(\theta') + |\xi|^2 (e(\theta) - c(\theta'))] c(\xi) \\
  &\quad + \frac{i c(\xi)}{|\xi'|^4} \left( \xi'|c(dx_n)\partial_{x_n} c(\xi') - 2h'(0)c(dx_n)c(\xi) + 2\xi_n c(\xi)\partial_{x_n} c(\xi') + 4\xi_n h'(0) \right). \tag{4.43}
\end{align*}
\]

By (3.37) and (4.43), we have
\[
\begin{align*}
  \text{tr}[\partial_{\xi_n} \pi_{\xi_n}' \sigma_{-1}(\hat{D}_N^*)] \times \sigma_{-4}(\hat{D}_N^* \hat{D}_m^* \hat{D}_N^*)^{-1}|_{\xi'|=1} &= \frac{1}{2(\xi_n - i)^2(1+\xi_n^2)^2} \left[ \frac{3}{4} + 2 + (3 + 4i)\xi_n + (-6 + 2i)\xi_n^2 + 3\xi_n^3 + \frac{9i}{4} \xi_n^4 \right] h'(0) \text{tr}[\text{id}] \\
  &\quad + \frac{1}{2(\xi_n - i)^2(1+\xi_n^2)^2} \left( -1 - 3i\xi_n - 2\xi_n^2 - 4i\xi_n^3 - \xi_n^4 \right) \text{tr}[c(\xi)\partial_{\xi_n} c(\xi)] \\
  &\quad - \frac{1}{2(\xi_n - i)^2(1+\xi_n^2)^2} \left( \frac{1}{2} i + \frac{1}{2} \xi_n + \frac{1}{2} \xi_n^2 + \frac{1}{2} \xi_n^3 \right) \text{tr}[c(\xi)' c(\xi) c(dx_n) c(dx_n)] \\
  &\quad + \frac{-\xi_n^4 + 3}{2(\xi_n - i)^4(i + \xi_n)^3} \text{tr}[c(\theta') c(dx_n)] - \frac{3\xi_n + i}{2(\xi_n - i)^4(i + \xi_n)^3} \text{tr}[c(\theta') c(\xi')]. \tag{4.44}
\end{align*}
\]

By direct calculation and the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA \), then we have equalities:
\[
\begin{align*}
  \text{tr}[c(\theta')(x_0)c(dx_n)] &= -64g(\theta', dx_n); \quad \text{tr}[c(\theta')(x_0)c(\xi')] = -64g(\theta', \xi'); \\
  \text{tr}[c(e_i)\xi_j c(e_j) c(\xi_n) c(dx_n)] &= 0 \quad (i < n). \tag{4.45}
\end{align*}
\]

Then
\[
\begin{align*}
  \text{tr}[c(\xi') c(\xi') c(dx_n) c(dx_n)] &= \sum_{i<n, j<n} \text{tr}[\xi_i \xi_j c(\xi_i) c(\xi_j) c(dx_n) c(dx_n)] = 0. \tag{4.46}
\end{align*}
\]

So, we have
\[
\begin{align*}
  -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \pi_{\xi_n}' \sigma_{-1}(\hat{D}_N^*) \times \partial_{\xi_n} \left( \sigma_{-4}(\hat{D}_N^* \hat{D}_m^* \hat{D}_N^*)^{-1} \right) \right] (x_0)|\xi_n(\xi') dx' \\
  = i h'(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{\frac{1}{2} i + 3 + 4i\xi_n + (-6 + 2i)\xi_n^2 + 3\xi_n^3 + \frac{9i}{4} \xi_n^4}{2(\xi_n - i)^3(1+\xi_n^2)^3} dx' \\
  + i h'(0) \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{32 \xi_n + i - 2\xi_n^i + 1}{2(\xi_n - i)^3(i + \xi_n)^3} dx' \\
  + i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{\xi_n - i - 2\xi_n^i + 1}{2(\xi_n - i)^3(i + \xi_n)^3} dx' \\
  - i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{3\xi_n + i}{2(\xi_n - i)^3(i + \xi_n)^3} dx' \\
  = (-\frac{19}{4} - 15)i h'(0)\Omega_3 dx' + (-\frac{3}{8} - \frac{75}{8})i h'(0)\Omega_4 dx' + 120i\pi g(dx_n, \theta') \Omega_4 dx' \\
  = \left( -\frac{41}{8} - \frac{195}{8} \right) i h'(0)\Omega_4 dx' + 120i\pi g(dx_n, \theta') \Omega_4 dx'. \tag{4.47}
\end{align*}
\]
Since
\[
\partial_{\xi_n} \left( \frac{c(\xi)c(df)c(\xi)}{|\xi|^6} \right) = \frac{c(dx_n)c(df)c(\xi) + c(\xi')c(df)c(dx_n) + 2\xi_n c(dx_n)c(df)c(dx_n)}{(1 + \xi_n^2)^3}
- \frac{6\xi_n c(\xi)c(df)c(\xi)}{(1 + \xi_n^2)^4}
\]
and
\[
\partial_{\xi_n} \left( \frac{ic(\xi)\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) c(\xi)}{|\xi|^8} \right) = \left\{ \frac{c(dx_n)\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) c(\xi) + c(\xi')\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) c(dx_n)}{(1 + \xi_n^2)^4} \right\} (1 + \xi_n^2)^{-4}
- \frac{8\xi_n c(\xi)\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) c(\xi)}{(1 + \xi_n^2)^5},
\]
(4.48)

then we have
\[
\text{tr} \left[ \pi_+^+_n \sigma_{-1} (\hat{D}_N^{-1}) \right] \times \partial_{\xi_n} \left( \frac{c(\xi)c(df)c(\xi)}{|\xi|^6} \right) (x_0)
= \frac{(4\xi_n i + 2) i}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{tr}[c(\xi')c(df)] + \frac{4\xi_n i + 2}{2(\xi_n + i)(1 + \xi_n^2)^3} \text{tr}[c(dx_n)c(df)]
\]
and
\[
\text{tr} \left[ \pi_+^+_n \sigma_{-1} (\hat{D}_N^{-1}) \times \partial_{\xi_n} \left( \frac{ic(\xi)\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) c(\xi)}{|\xi|^8} \right) \right] (x_0)
= \frac{(3\xi_n - i) i}{(\xi_n + i)(1 + \xi_n^2)^3} \text{tr} \left[ c(\xi')\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right]
+ \frac{3\xi_n - i}{(\xi_n + i)(1 + \xi_n^2)^3} \text{tr} \left[ c(dx_n)\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right].
\]
(4.49)

By the relation of the Clifford action and \(\text{tr}QP = \text{tr}PQ\), then we have the following equalities
\[
\text{tr}[c(dx_n)c(df)] = -g(dx_n, df)
\]
and
\[
\text{tr} \left[ c(dx_n)\sum_j [c(dx_j)|\xi|^2 + 2\xi_j c(\xi)] D_{x_j}(f^{-1}) \right]
= \text{tr} (-id)|\xi|^2 \left( -i\partial_{x_n}(f) f^{-1} \right) + 2 \sum_j \xi_j \xi_n \text{tr} (-id) \left( -i\partial_{x_j}(f) f^{-1} \right)
= -64|\xi|^2 \left( -i\partial_{x_n}(f) f^{-1} \right) + 2 \sum_j \xi_j \xi_n \text{tr} (-id) \left( -i\partial_{x_j}(f) f^{-1} \right).
\]
27
We note that \( i < n \), \( \int_{|\xi|=1} \xi_\alpha \sigma(\xi') = 0 \), so \( \text{tr}[c(\xi')c(df)] \), \( \text{tr}[c(\xi') \sum_j c(dx_j)|\xi'|^2 + 2\xi_\xi c(\xi)]D_{x_j}(f^{-1}) \) and \( 2i \sum_j \xi_j \partial x_j (f) f^{-1} \text{tr}[-id] \) have no contribution for computing \textbf{case (b)}. Then we obtain

\[
-2i f^{-1} \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \pi^+_{\xi_n} \sigma_{-1}(\hat{D}^{-1}_N) \times \partial_{\xi_n} \left( \frac{c(\xi)c(df)c(\xi)}{|\xi|^8} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' = \frac{3}{8f} \pi g(dx_n, df) \Omega_4 dx'.
\]

(4.50)

and

\[
-f i \int_{|\xi|=1}^{+\infty} \text{tr} \left[ \pi^+_{\xi_n} \sigma_{-1}(\hat{D}^{-1}_N) \times \partial_{\xi_n} \left( \frac{ic(\xi) \sum_j c(dx_j)|\xi|^2 + 2\xi_\xi c(\xi)}{|\xi|^8} \right) \right] \times (x_0) d\xi_n \sigma(\xi') dx' = -\frac{15i}{2} \partial_{x_n}(f) \pi 8\Omega_4 dx'.
\]

(4.51)

Thus we have

\[
\textbf{case (b)} = (-\frac{41}{8} i - \frac{195}{8}) \pi h'(0) \Omega_4 dx' + 120i \pi g(dx_n, \theta') \Omega_4 dx' + \frac{3}{8f} \pi g(dx_n, df) \Omega_4 dx' - \frac{15i}{2} \partial_{x_n}(f) \pi 8\Omega_4 dx'.
\]

(4.52)

\textbf{case (c)} \( r = -2, l = -3, |a| = j = k = 0 \).

By (4.2), we have

\[
\textbf{case (c)} = -i \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \pi^+_{\xi_n} \sigma_{-2}(\hat{D}^{-1}_N) \times \partial_{\xi_n} \sigma_{-3}(f^{-1}(\hat{D}^{-1}_N) \cdot f \hat{D}^{-1}_N \cdot f^{-1}(\hat{D}^{-1}_N) -(x_0)\right] d\xi_n \sigma(\xi') dx' = -i \int_{|\xi|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \pi^+_{\xi_n} \sigma_{-2}(\hat{D}^{-1}_N) \times \partial_{\xi_n} \sigma_{-3}(f \hat{D}^{-1}_N \hat{D}^{-1}_N -(x_0)\right] d\xi_n \sigma(\xi') dx'.
\]

(4.53)

By (4.26), we have

\[
\partial_{\xi_n} \sigma_{-3}(f \hat{D}^{-1}_N \hat{D}^{-1}_N -(x_0) = \frac{-4i\xi_\xi c(\xi')}{(1 + \xi_n^2)^3} + \frac{i(1 - 3\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3}.
\]

(4.54)

By (3.46), we obtain

\[
\pi^+_{\xi_n} (\sigma_{-2}(\hat{D}^{-1}_N) (x_0) )|_{|\xi|=1} = \pi^+_{\xi_n} \left[ \frac{c(\xi)h_0''(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi)](x_0)}{(1 + \xi_n^2)^2} \right] + \pi^+_{\xi_n} \left[ \frac{c(\xi)[b_1'(x_0)]c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi^+_{\xi_n} \left[ \frac{c(\xi)[c(\theta) + c(\theta')]c(\xi)(x_0)}{(1 + \xi_n^2)^2} \right].
\]

(4.55)
Furthermore,

\[
\begin{align*}
\pi_{\xi_n}^+ \left[ \frac{c(\xi)[\bar{c}(\theta) + c(\theta')](x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] \\
= \pi_{\xi_n}^+ \left[ \frac{c(\xi)[\bar{c}(\theta) + c(\theta')](x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] + \pi_{\xi_n}^+ \left[ \frac{\xi_n c(\xi)[\bar{c}(\theta) + c(\theta')](x_0)c(dx_n)}{(1 + \xi_n^2)^2} \right] \\
= -\frac{c(\xi')[\bar{c}(\theta) + c(\theta')](x_0)c(\xi')(2 + i\xi_n)}{4(\xi_n - i)^2} + \frac{ic(\xi')[\bar{c}(\theta) + c(\theta')](x_0)c(\xi')}{4(\xi_n - i)^2} \\
+ \frac{ic(dx_n)[\bar{c}(\theta) + c(\theta')](x_0)c(\xi')}{4(\xi_n - i)^2} + \frac{-i\xi_n c(dx_n)[\bar{c}(\theta) + c(\theta')](x_0)c(dx_n)}{4(\xi_n - i)^2}.
\end{align*}
\]

By (3.47)-(3.49) and (4.54), we have

\[
\text{tr} \left[ \pi_{\xi_n}^+ \left( \frac{c(\xi)[\bar{c}(\theta) + c(\theta')](x_0)c(\xi)}{(1 + \xi_n^2)^2} \right) \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_N^* \hat{D}_N)^{-1})(x_0) \right] \bigg|_{\xi^2 = 1} = \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr} [b_0^*(x_0)c(\xi')],
\]

By (3.52)-(3.54), we have

\[
\pi_{\xi_n}^+ \left[ \frac{c(\xi)b_0^2(x_0)c(\xi)}{(1 + \xi_n^2)^2} \right] - \frac{h'(0)}{2} \pi_{\xi_n}^+ \left[ \frac{c(\xi)c(dx_n)c(\xi)}{(1 + \xi_n^2)^3} \right] = B_1 - B_2,
\]

where

\[
B_1 = \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi')b_0^2c(\xi') + i\xi_n c(dx_n)b_0^2c(dx_n) \right] \\
+ \left[ (2 + i\xi_n)c(dx_n)[\bar{c}(\theta) + c(\theta')](x_0)c(\xi') + ic(dx_n)b_0^2c(\xi') + i\xi_n c(dx_n) - i\xi_n c(\xi') \right] \\
= \frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') \right] - \frac{5i}{2} h'(0)c(dx_n) \partial_{\xi_n} c(\xi') + i\xi_n c(\xi'),
\]

\[
B_2 = \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} \left( ic(\xi') - c(dx_n) \right) \right],
\]

By (4.54) and (6.60), we have

\[
\text{tr} [B_2 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_N^* \hat{D}_N)^{-1})(x_0)]]_{\xi^2 = 1} = \text{tr} \left\{ \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} \left( ic(\xi') - c(dx_n) \right) \right] \right\} \\
\times \left[ \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{8(\xi_n - i)^3} \right] \\
= \frac{8h'(0)}{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^3(\xi_n + i)}.
\]
Similarly, we have

\[
\text{tr}[B_1 \times \partial_{\xi_n} \sigma_{-3}((\hat{D}_N \hat{D}_N \hat{D}_N^{-1})(x_0))]|\xi'|=1
\]

\[
= \text{tr}\left\{ \frac{1}{4(\xi_n - i)^2} \left[ \frac{5}{2} h'(0)c(dx_n) - \frac{5i}{2} h'(0)c(\xi') - (2 + i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}c(\xi') + i\partial_{\xi_n}c(\xi') \right] \right. \\
\times \left. -4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n) \right\} \\
= 8h'(0) \left\{ \frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^2} \right\} \left[ \frac{\sigma_{-3}((\hat{D}_N \hat{D}_N \hat{D}_N^{-1})(x_0))}{(1 + \xi_n^2)^4} \right] \\
= \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[\hat{\sigma}(\theta) + c(\theta')c(dx_n)\partial_{\xi_n}c(\xi')] \\
= \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \text{tr}[c(\theta')(x_0)c(\xi')] \\
= \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} \left[ -g(\theta', \xi') \right] \text{tr}[\text{id}].
\]  

(4.62)

By \( \int_{|\xi'|=1} \xi_1 \cdots \xi_{2p+1} \sigma(\xi') = 0 \), we have

\[
\text{case (c)} = -ih'(0) \left\{ \int_{|\xi'|=1} \int_{-\infty}^{\infty} 8 \times \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n - i)^3(\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \right. \\
- \left. i \int_{|\xi'|=1} \int_{-\infty}^{\infty} \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1 + \xi_n^2)^3} [-g(\theta', \xi') \text{tr}[\text{id}]] dx' \right\} \\
= -8ih'(0) \times \frac{2\pi i}{4!} \left[ \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n + i)^5} \right]_{\xi_n=\Omega_4} dx' \\
= \frac{55}{2} \pi h'(0) \Omega_4 dx'.
\]  

(4.63)

(4.64)

Now \( \Psi \) is the sum of the case (a), case (b) and case (c), then

\[
\Psi = (10i + 88)\pi f^{-1} \partial_{x_n} (f) \cdot \Omega_4 dx' + \frac{\pi i}{2} f \cdot \partial_{x_n} (f^{-1}) \Omega_4 dx' - \frac{41i}{8} \pi h'(0) \Omega_4 dx' + 120i \pi g(dx_n, \theta') \Omega_4 dx' + 3 \frac{3}{8} f \pi g(dx_n, df) \Omega_4 dx' - 60i \partial_{x_n} (f) \pi \Omega_4 dx'.
\]  

(4.65)

**Theorem 4.3.** Let \( M \) be 6-dimensional oriented compact manifolds with the boundary \( \partial M \) and the metric
$g^M$ as above, $\hat{D}_N$ and $\hat{D}_N^*$ be modified Novikov operators on $\hat{M}$, then

\[
\text{Wres}\left[\pi^+(f\hat{D}_N) \circ \pi^+(f^{-1}(\hat{D}_N^*)^{-1} \cdot f\hat{D}_N^* \cdot f^{-1}(\hat{D}_N)^{-1})\right] = 128\pi^3 \int_M 2\theta \left\{ \left( -\frac{1}{12}s - |\theta|^2 + (n-2)|\theta|^2 + g(\hat{\epsilon}_j, \nabla_{\hat{\epsilon}_j}^T \theta) \right) + \left( f^{-1}\Delta(f) + \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right) \right.
\]

\[
- \left( c(\theta) - c(\theta') \right) (\text{grad}_M f) f^{-1} - \frac{1}{2} \left( f^{-1}\Delta(f)(x_0) + \langle \text{grad}_M f, \text{grad}_M f^{-1} \rangle \right) + \frac{1}{4} \sum_i \left[ c(\epsilon_i) c(\theta') - c(\theta') c(\epsilon_i) \right]
\]

\[
\times c(\epsilon_i) c(df) f^{-1} + \frac{1}{4} \sum_i c(\epsilon_i) c(df) f^{-1} \left[ c(\epsilon_i) c(\theta') - c(\theta') c(\epsilon_i) \right] \} d\text{Vol}_M + \int_{\partial M} \left\{ 10i + 88\pi f^{-1}\partial_{x_n}(f) \right.
\]

\[
\times \Omega_4 dx' + \frac{\pi i}{2} f \cdot \partial_{x_n}(f^{-1})\Omega_4 dx' \left( -\frac{41}{8} + \frac{65}{8} \right) \pi h'(0) \Omega_4 dx' + 120i\pi g(dx_n, \theta') \Omega_4 dx' + \frac{3}{8} \pi g(dx_n, df) \Omega_4 dx'
\]

\[
- 60i\partial_{x_n}(f) \pi \Omega_4 dx' \} d\text{vol}_M,
\]

(4.66)

where $s$ is the scalar curvature.

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