GROUP DECISION MAKING APPROACH BASED ON POSSIBILITY DEGREE MEASURE UNDER LINGUISTIC INTERVAL-VALUED INTUITIONISTIC FUZZY SET ENVIRONMENT

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Abstract. In the present article, we extended the idea of the linguistic intuitionistic fuzzy set to linguistic interval-valued intuitionistic fuzzy (LIVIF) set to represent the data by the interval-valued linguistic terms of membership and non-membership degrees. Some of the desirable properties of the proposed set are studied. Also, we propose a new ranking method named as possibility degree measures to compare the two or more different LIVIF numbers. During the aggregation process, some LIVIF weighted and ordered weighted aggregation operators are proposed to aggregate the collections of the LIVIF numbers. Finally, based on these proposed operators and possibility degree measure, a new group decision making approach is presented to rank the different alternatives. A real-life case has been studied to manifest the practicability and feasibility of the proposed group decision making method.

1. Introduction. In real life, multi attributes group decision making (MAGDM) problems are an important part of decision theory in which we choose the best one from the set of finite alternatives to the different attributes. For solving the decision-making (DM) problems, decision-makers provide their judgments or ratings towards the object. But it is not possible that their judgments may be in the form of crisp data due to the fuzziness or vagueness of the data [15, 25]. Instead, it has become popular that these assessments are presented by a fuzzy set or extensions of the fuzzy set. Fuzzy set (FS) [49], proposed by Zadeh, is an effective tool to deal with vagueness and has received much attention. After their successful study, researchers are engaged in their extensions and out of that, intuitionistic fuzzy set (IFS) and interval-valued intuitionistic fuzzy set (IVIFS) theories as proposed by Atanassov[4], Atanassov and Gargov[3] respectively, which takes the degree of membership as well as non-membership simultaneously are widely used in the decision-making process. During the last decades, the researchers were paying more attention to these theories and successfully applied it to the various situations in the decision-making process. The two important aspects of solving the MAGDM problem are (i) to design an
appropriate function which aggregates the different preference of the decision makers into the collective ones, (ii) to design appropriate measures to rank the alternatives. For the former part, an aggregation operator is an important part of the decision making which usually takes the form of mathematical function to aggregate all the individual input data onto a single one. Over the last decade, numerable attempts have been made by different researchers in processing the information values using different aggregation operators under IFS and IVIFS environments [1, 2, 46, 13, 14, 38, 18, 17, 8, 36, 10, 41, 28, 27, 26, 32, 34].

In the aspect of ranking the numbers, Xu and Da [47] defined the possibility degree measure method for ranking the interval numbers. Garg [9] presented a generalized improved score function of ranking the IVIFSs. Zhang et al. [52] presented possibility degree measure method to rank the different interval-valued intuitionistic fuzzy numbers (IVIFNs). Wan and Dong [35] proposes a new ranking method of interval-valued IFNs based on the possibility degree measures from the probability viewpoint and their corresponding decision making method. Garg and Kumar [19] presented an improved possibility degree measure method for ranking the intuitionistic fuzzy numbers. The overview of the different approaches related to possibility degree measures under these existing theories are summarized in [6, 7].

Since all the above existing theories deal with the uncertain information only by quantitative aspects. But in real-life problems, there are many attribute values which are qualitative in nature and cannot be expressed by a numeric value. In such cases, it is easy to describe the preference values as a linguistic variable. For this, Zadeh [50] proposed the concept of a linguistic variable (LV) to represent the rating values towards the object. For instance, in order to measure the performance of a student in an academic year, he may use some of the linguistic terms such as “excellent”, “good”, “average”, “poor”. However, due to the complexity of decision environment and the subjective nature of human thinking, sometimes it is very difficult to express the membership and non-membership degrees by numeric values and is possible to be expressed as LVs. Linguistic approaches [24, 23, 40] provide us more degree of freedom to analyze the imprecise and vague information. In that direction, Xu [43] proposed some linguistic weighted geometric averaging operators while Xu [39] presented a decision-making approach based on possibility degree measures under the uncertain linguistic information. Later on, Zhang [51] introduced a new concept of the linguistic intuitionistic fuzzy set (LIFS) in which the membership degrees are expressed by the LVs. Chen et al. [5] defined averaging and geometric aggregation operators for aggregating LIFNs information and presented a MAGDM approach. Liu and Wang [30] defined some improved operational laws for LIFNs and aggregation operators based on it. Garg and Kumar [20] presented some aggregation operators for LIFSs by using set pair analysis theory. Liu and Qin [29] presented power averaging operator for aggregating the LIFNs and MAGDM method based on it. Xian et al. [37] presented a new hybrid aggregation operator and decision-making approach based on it. Recently, Garg [12], Garg and Nancy [21] presented some new linguistic decision making approaches for solving the decision making problems under different environments.

The above theories have been successfully applied, but sometimes due to the complex fuzzy information, decision-makers cannot provide their opinion by LIFNs in terms of single-valued linguistic terms. Therefore to provide more degree of freedom from decision-makers to express their opinion and better dealing with the complex
fuzzy information, in this article motivated by IVIFS, we have extended the idea of LIFN to the linguistic interval-valued intuitionistic fuzzy number (LIVIFN) by expressing membership and non-membership grades in form of interval-valued linguistic terms. For developing any other theory about the field of the multi-attribute DM, information aggregating techniques and ranking methods have played an important roles. For it, an improved possibility degree measure of comparison between two different LIVIFNs is defined by using the notion of the 2-dimensional random vector, and a new method is then developed to rank the numbers. Afterward, for solving the MAGDM problems, we need some aggregating techniques to collect the information of different attributes. For it, we proposed some geometric and ordered weighted geometric aggregation operators under the LIVIFS environment where preferences are expressed in terms of LIVIFNs. The various desirable properties of the proposed operators are investigated. Therefore under the LIVIFS environment, the objective of this paper is divided into the four parts:

(i) to propose the concept of the LIVIFS, by combining the features of linguistic variables and IVIFS, in which the importance of each object is expressed in the form of the interval-valued linguistic membership and non-membership terms.

(ii) to introduce a new possibility degree measure for ranking the different LIVIFNs and to propose two new weighted geometric aggregation operators namely, LIVIF weighted geometric (LIVIFWG) and LIVIF ordered weighted geometric (LIVIFOWG) of the different LIVIFNs.

(iii) to establish a MAGDM approach based on these proposed operator and possibility degree measure method.

(iv) to illustrate the developed approach with a numerical example and validate it with some validity test criteria.

The remainder of this paper is formulated as follows. In Section 2, we briefly describe the concepts of IVIFS and the linguistic variable. In Section 3, we defined the concept of the linguistic interval-valued intuitionistic fuzzy set and a new possibility degree measure method for ranking different LIVIFSs. In Section 4, we present some new weighted geometric and ordered weighted aggregation operators for aggregating the collections of the LIVIFNs. Various desirable properties of these operators are also investigated. In section 5, a group decision making approach to solve the MAGDM problems is presented based on the proposed operators and possibility degree measure method. Section 6 deals with an illustrative example to show the effectiveness and feasibility of the approach. Finally, section 7 concludes the paper.

2. Preliminaries. In this section, we briefly review some basic concepts of the IVIFSs and linguistic set approaches.

Definition 2.1. [3, 44] Let $X$ be the nonempty finite universal set, an IVIFS $A$ in $X$ is defined as

$$A = \left\{ (x, [u^L_A(x), u^U_A(x)], [v^L_A(x), v^U_A(x)]) \mid x \in X \right\},$$

where, for each $x$, $[u^L_A, u^U_A], [v^L_A, v^U_A] \subseteq [0, 1]$, and represents membership and non-membership grades of $x$ to $A$ respectively, such that $0 \leq u^U_A + v^U_A \leq 1$. This pair $([u^L_A, u^U_A], [v^L_A, v^U_A])$ is called an IVIF number (IVIFN).

Sometimes, a decision maker may give his preferences in terms of a linguistic number than a numerical number. For it, a linguistic term set (LTS) is defined as:
Definition 2.2. [24] Let \( S = \{ s_t \mid t = 0, 1, 2, \ldots, h \} \) be a finite odd cardinality LTS, where \( s_t \) represents a positive integer. Each linguistic term \( s_t \) must have the following characteristics.

(i) The set is ordered: \( s_k \leq s_t \iff k \leq t \)

(ii) There is a negation operator: \( \text{Neg}(s_k) = s_t \) such that \( t = h - k \)

(iii) Max operator: \( \max(s_k, s_t) = s_k \iff s_k \geq s_t \)

(iv) Min operator: \( \min(s_k, s_t) = s_k \iff s_k \leq s_t \)

For instance, consider a linguistic variable “intelligent” and define their corresponding set of seven linguistic terms as \( S = \{ s_0 = \text{N(none)}, s_1 = \text{VL(very low)}, s_2 = \text{L(low)}, s_3 = \text{M(medium)}, s_4 = \text{H(high)}, s_5 = \text{VH(very high)}, s_6 = \text{P(perfect)} \} \).

Later on, Xu [43] extended the discrete LTS to a continuous LTS as

\[
S_{[0,h]} = \left\{ s_z \mid s_0 \leq s_z \leq s_h \right\}
\]

Definition 2.3. [51] Let \( s_\alpha, s_\beta \in S_{[0,h]} \). Addition and multiplication operation laws for linguistic variables based on the t-norm and t-conorm are defined as follows:

\[
s_\alpha + s_\beta = s_\beta + s_\alpha = s_{hE(\frac{\alpha}{\eta}, \frac{\beta}{\eta})},
\]

\[
s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{hG(\frac{\alpha}{h}, \frac{\beta}{h})}
\]

where \( E \left( \frac{\alpha}{\eta}, \frac{\beta}{\eta} \right) \) and \( G \left( \frac{\alpha}{\eta}, \frac{\beta}{\eta} \right) \) are t-conorm and t-norm respectively. Since \( E \left( \frac{\alpha}{\eta}, \frac{\beta}{\eta} \right), G \left( \frac{\alpha}{\eta}, \frac{\beta}{\eta} \right) \in [0, 1] \), therefore \( hE \left( \frac{\alpha}{\eta}, \frac{\beta}{\eta} \right), hE \left( \frac{\alpha}{\eta}, \frac{\beta}{\eta} \right) \in [0, h] \) which implies that \( s_{hE(\frac{\alpha}{\eta}, \frac{\beta}{\eta})}, s_{hE(\frac{\alpha}{\eta}, \frac{\beta}{\eta})} \in S_{[0,h]} \).

Definition 2.4. [51] A linguistic intuitionistic fuzzy set \( A \) in \( X \) is defined as

\[
A = \left\{ (x, s_{\tau(x)}, s_{\theta(x)}) \mid x \in X \right\},
\]

where, \( \forall x, s_{\tau}, s_{\theta} \in S_{[0,h]} \) represent the linguistic membership and nonmembership degrees of \( x \) to \( A \) respectively, such that \( 0 \leq \tau + \theta \leq h \) holds. Usually, the pair \( (s_{\tau}, s_{\theta}) \) is called a linguistic intuitionistic fuzzy number (LIFN).

3. Linguistic interval-valued intuitionistic fuzzy set. In this section, we propose the concept of the linguistic interval-valued intuitionistic fuzzy set (LIVIFS) by extending it from the linguistic intuitionistic fuzzy set. The concept of the LIVIFS is given as follows.

Definition 3.1. Let \( S_{[0,h]} \) be a continuous linguistic term set. A LIVIFS in defined in the finite universe of discourse \( X \) mathematically with the form

\[
A = \left\{ (x, [s_{\tau(x)}, s_{\eta(x)}], [s_{\theta(x)}, s_{\upsilon(x)}]) \mid x \in X \right\},
\]

where \( [s_{\tau}, s_{\eta}] \) and \( [s_{\theta}, s_{\upsilon}] \) are all subsets of \( [s_0, s_h] \) and represent the linguistic membership and nonmembership degrees of \( x \) to \( A \) respectively. For any \( x \in X \), \( s_{\eta(x)} + s_{\upsilon(x)} \leq s_h \) (i.e., \( \eta + \upsilon \leq h \)) is always satisfied, and in turn, the linguistic intuitionistic index of \( x \) to \( A \) is defined as \( s_{\pi_A}(x) = [s_{h-\eta(x)}-\upsilon(x), s_{h-\tau(x)}-\theta(x)] \). Usually, the pair \( ([s_{\tau(x)}, s_{\eta(x)}], [s_{\theta(x)}, s_{\upsilon(x)}]) \) is called linguistic interval-valued intuitionistic fuzzy number (LIVIFN).

For the convenience, we denote the LIVIFN as \( \gamma = ([s_{\tau}, s_{\eta}], [s_{\theta}, s_{\upsilon}]) \), where \( [s_{\tau}, s_{\eta}] \subseteq [s_0, s_h], [s_{\theta}, s_{\upsilon}] \subseteq [s_0, s_h], \eta + \upsilon \leq h \) and also \( s_{\tau}, s_{\eta}, s_{\theta}, s_{\upsilon} \in S_{[0,h]} \) holds.
Definition 3.2. Let $\gamma_1 = ([s_{r1}, s_{n1}], [s_{\theta1}, s_{v1}])$ and $\gamma_2 = ([s_{r2}, s_{n2}], [s_{\theta2}, s_{v2}])$ be two LIVIFNs, then

(i) $\gamma_1 \succeq \gamma_2$ if $\tau_1 \leq \tau_2, \eta_1 \leq \eta_2, \theta_1 \geq \theta_2$, and $v_1 \leq v_2$;
(ii) $\gamma_1 = \gamma_2$ iff $\gamma_1 \succeq \gamma_2$ and $\gamma_2 \succeq \gamma_1$;
(iii) $\gamma_1^c = ([s_{\theta1}, s_{v1}], [s_{r1}, s_{n1}])$ be the complement of $\gamma_1$;
(iv) $\gamma_1 \cup \gamma_2 = \left(\max\{s_{r1}, s_{r2}\}, \min\{s_{\theta1}, s_{\theta2}\}\right)$;
(v) $\gamma_1 \cap \gamma_2 = \left(\min\{s_{r1}, s_{r2}\}, \max\{s_{\theta1}, s_{\theta2}\}\right)$.

Motivated from the idea as presented by Xu and Da [46], we proposed a new possibility degree measure to compare any two LIVIFNs under LIVIFS environment as follows.

Definition 3.3. Let $\gamma_1 = ([s_{r1}, s_{n1}], [s_{\theta1}, s_{v1}])$ and $\gamma_2 = ([s_{r2}, s_{n2}], [s_{\theta2}, s_{v2}])$ be two different LIVIFNs defined over $X$ and $S^{[0,1]}$ be continuous LTS. Then, the possibility degree measure, $p(\gamma_1 \succeq \gamma_2)$, of $\gamma_1 \succeq \gamma_2$ is defined as

$$p(\gamma_1 \succeq \gamma_2) = \frac{1}{2} \left[ \min \left( \max \left( h \left( \frac{\eta_1 - \tau_2}{\eta_1 - \tau_1 + \eta_2 - \tau_2} \right), 0 \right), h \right) + \min \left( \max \left( h \left( \frac{v_2 - \theta_1}{v_1 - \theta_1 + v_2 - \theta_2} \right), 0 \right), h \right) \right]$$

(4)

On the other hand, if $\tau_1 = \tau_2 = \eta_1 = \eta_2$ then the possibility degree measure, $p(\gamma_1 \succeq \gamma_2)$, of $\gamma_1 \succeq \gamma_2$ is defined as

$$p(\gamma_1 \succeq \gamma_2) = \min \left( \max \left( h \left( \frac{v_2 - \theta_1}{v_1 - \theta_1 + v_2 - \theta_2} \right), 0 \right), h \right)$$

(5)

while if $\theta_1 = \theta_2 = v_1 = v_2$ then the possibility degree measure, $p(\gamma_1 \succeq \gamma_2)$, of $\gamma_1 \succeq \gamma_2$ is defined as [42]

$$p(\gamma_1 \succeq \gamma_2) = \min \left( \max \left( h \left( \frac{\eta_1 - \tau_2}{\eta_1 - \tau_1 + \eta_2 - \tau_2} \right), 0 \right), h \right)$$

(6)

Theorem 3.4. Let $\gamma_1$ and $\gamma_2$ be two LIVIFNs and $S^{[0,1]}$ be continuous LTS, then

(i) $0 \leq p(\gamma_1 \succeq \gamma_2) \leq h$;
(ii) $p(\gamma_1 \succeq \gamma_2) = h/2$ if $\gamma_1 = \gamma_2$;
(iii) $p(\gamma_1 \succeq \gamma_2) + p(\gamma_2 \succeq \gamma_1) = h$.

Proof. Let $\gamma_1 = ([s_{r1}, s_{n1}], [s_{\theta1}, s_{v1}])$ and $\gamma_2 = ([s_{r2}, s_{n2}], [s_{\theta2}, s_{v2}])$ be two LIVIFNs.

(i) Since $p(\gamma_1 \succeq \gamma_2) \geq 0$ is obvious, so we need to prove only $p(\gamma_1 \succeq \gamma_2) \leq h$. For it, we take $x = \frac{\eta_1 - \tau_2}{\eta_1 - \eta_1 + \eta_2 - \tau_2}$ and $y = \frac{v_2 - \theta_1}{v_1 - \theta_1 + v_2 - \theta_2}$. Then, the following cases are arising.

(a) If $x \geq 1$ and $y \geq 1$, then

$$p(\gamma_1 \succeq \gamma_2) = \frac{1}{2} \left[ \min \left( \max \left( hx, 0 \right), h \right) + \min \left( \max \left( hy, 0 \right), h \right) \right]$$

$$= \frac{1}{2} (h + h) = h$$
(b) If \(0 \leq x \leq 1\) and \(y \geq 1\), then
\[
p(\gamma_1 \succeq \gamma_2) = \frac{1}{2} \left[ \min(\max(hx, 0), h) + \min(\max(hy, 0), h) \right] \\
= \frac{1}{2} (hx + h) \leq h
\]

(c) If \(0 < x, y < 1\), then
\[
p(\gamma_1 \succeq \gamma_2) = \frac{1}{2} \left[ \min(\max(hx, 0), h) + \min(\max(hy, 0), h) \right] \\
= \frac{1}{2} (hx + hy) \\
< \frac{1}{2} (h + h) = h
\]

Hence, in all cases, we have \(p(\gamma_1 \succeq \gamma_2) \leq h\).

(ii) Let \(\gamma_1 = ([s_{\tau_1}, s_{n_1}], [s_{\eta_1}, s_{v_1}]\), \(\gamma_2 = ([s_{\tau_2}, s_{n_2}], [s_{\eta_2}, s_{v_2}]\) be two LIVIFNs. If \(\gamma_1 = \gamma_2\), which implies that \(\tau_1 = \tau_2\), \(\eta_1 = \eta_2\), \(\theta_1 = \theta_2\) and \(v_1 = v_2\). Then, Eq. (4) becomes
\[
p(\gamma_1 \succeq \gamma_2) = \frac{1}{2} \left[ \min \left( \max \left( h \left( \frac{\eta_1 - \tau_2}{\eta_1 - \tau_1 + \eta_2 - \tau_2} \right), 0 \right), h \right) \\
+ \min \left( \max \left( h \left( \frac{v_2 - \theta_1}{v_1 - \theta_1 + v_2 - \theta_2} \right), 0 \right), h \right) \right] \\
= \frac{1}{2} \left[ (h/2) + (h/2) \right] \\
= h/2
\]

(iii) Take \(x = \frac{\eta_1 - \tau_2}{\eta_1 - \tau_1 + \eta_2 - \tau_2}\), \(y = \frac{v_2 - \theta_1}{v_1 - \theta_1 + v_2 - \theta_2}\), \(\rho = \frac{\eta_2 - \tau_1}{\eta_1 - \tau_1 + \eta_2 - \tau_2}\) and \(\varrho = \frac{v_1 - \theta_1 + v_2 - \theta_2}{v_1 - \theta_1 + v_2 - \theta_2}\) such that \(x + \rho = 1\) and \(y + \varrho = 1\). Then, the following cases are arising.

(a) If \(x, y \geq 1\) and \(\rho, \varrho \leq 0\), then
\[
p(\gamma_1 \succeq \gamma_2) + p(\gamma_2 \succeq \gamma_1) \\
= \frac{1}{2} \left[ \min(\max(hx, 0), h) + \min(\max(hy, 0), h) \right] \\
+ \frac{1}{2} \left[ \min(\max(h\rho, 0), h) + \min(\max(h\varrho, 0), h) \right] \\
= \frac{1}{2} (h + h) + \frac{1}{2} (0 + 0) \\
= h
\]

(b) If \(0 < x, y < 1\) and \(0 < \rho, \varrho < 1\), then
\[
p(\gamma_1 \succeq \gamma_2) + p(\gamma_2 \succeq \gamma_1) \\
= \frac{1}{2} \left[ \min(\max(hx, 0), h) + \min(\max(hy, 0), h) \right] \\
+ \frac{1}{2} \left[ \min(\max(h\rho, 0), h) + \min(\max(h\varrho, 0), h) \right] \\
= \frac{1}{2} (hx + hy) + \frac{1}{2} (h\rho + h\varrho) \\
= h
(c) If \( x, y \leq 0 \) and \( \rho, \varrho \geq 1 \), then
\[
p(\gamma_1 \succeq \gamma_2) + p(\gamma_2 \succeq \gamma_1)
= \frac{1}{2} \left[ \min(\max(h \cdot x, 0), h) + \min(\max(h \cdot y, 0), h) \right]
+ \frac{1}{2} \left[ \min(\max(h \cdot \rho, 0), h) + \min(\max(h \cdot \varrho, 0), h) \right]
= \frac{1}{2} (0 + 0) + \frac{1}{2} (h + h)
= h
\]

Hence, in all cases, we have \( p(\gamma_1 \succeq \gamma_2) + p(\gamma_2 \succeq \gamma_1) = h \).

\[\square\]

**Theorem 3.5.** For two LIVIFNs \( \gamma_1 \) and \( \gamma_2 \), possibility degree measure \( p(\gamma_1 \succeq \gamma_2) \) satisfies the following properties :

(i) \( p(\gamma_1 \succeq \gamma_2) = h \) if \( \eta_2 \leq \tau_1 \) and \( \nu_1 \leq \theta_2 \);
(ii) \( p(\gamma_1 \succeq \gamma_2) = 0 \) if \( \eta_1 \leq \tau_2 \) and \( \nu_2 \leq \theta_1 \);
(iii) \( p(\gamma_1 \succeq \gamma_2) = h/2 \) if \( \eta_2 \leq \tau_1 \) and \( \nu_2 \leq \theta_1 \);
(iv) \( p(\gamma_1 \succeq \gamma_2) = h/2 \) if \( \eta_1 \leq \tau_2 \) and \( \nu_1 \leq \theta_2 \).

**Proof.** Here, we shall prove only the parts (i) and (ii), while rest are similar. Let \( \gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{\nu_1}]) \) and \( \gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{\nu_2}]) \) be two LIVIFNs, then we have

(i) If \( \eta_2 \leq \tau_1 \Rightarrow \tau_2 \leq \eta_1 \) which implies that \( \eta_1 - \tau_2 \geq 0 \) and \( \eta_2 - \tau_1 \leq 0 \), then
\[
\min \left( \max \left( h \left( \frac{\eta_1 - \tau_2}{\eta_1 - \tau_1 + \eta_2 - \tau_2} \right), 0 \right), h \right) = h.
\]

Similarly, let \( \nu_1 \leq \theta_2 \Rightarrow \theta_1 \leq \nu_2 \) which implies that \( \nu_2 - \theta_1 \geq 0 \) and \( \nu_1 - \theta_2 \leq 0 \), then
\[
\min \left( \max \left( h \left( \frac{\nu_2 - \theta_1}{\nu_1 - \theta_1 + \nu_2 - \theta_2} \right), 0 \right), h \right) = h.
\]

Hence, \( p(\gamma_1 \succeq \gamma_2) = h \).

(ii) Since \( \eta_1 \leq \tau_2 \Rightarrow \eta_1 - \tau_2 \leq 0 \) and \( \eta_1 - \tau_1 + \eta_2 - \tau_2 \geq 0 \), then
\[
\min \left( \max \left( h \left( \frac{\eta_1 - \tau_2}{\eta_1 - \tau_1 + \eta_2 - \tau_2} \right), 0 \right), h \right) = 0.
\]

Similarly, let \( \nu_2 \leq \theta_1 \Rightarrow \nu_2 - \theta_1 \leq 0 \) and \( \nu_1 - \theta_1 + \nu_2 - \theta_2 \geq 0 \), which implies that
\[
\min \left( \max \left( h \left( \frac{\nu_2 - \theta_1}{\nu_1 - \theta_1 + \nu_2 - \theta_2} \right), 0 \right), h \right) = 0.
\]

Hence, \( p(\gamma_1 \succeq \gamma_2) = 0 \).

\[\square\]

**Example 3.1.** Let \( \gamma_1 = ([s_{\tau_1}, s_{\eta_1}], [s_{\theta_1}, s_{\nu_1}]) \) and \( \gamma_2 = ([s_{\tau_2}, s_{\eta_2}], [s_{\theta_2}, s_{\nu_2}]) \) be two LIVIFNs and \( S_{[0,1]} \) be continuous LTS, then
\[
p(\gamma_1 \succeq \gamma_2) = \frac{1}{2} \left[ \min \left( \max \left( 8 \left( \frac{4 - 3}{4 - 2 + 5 - 3} \right), 0 \right), 8 \right)
+ \min \left( \max \left( 8 \left( \frac{3 - 2}{3 - 2 + 3 - 1} \right), 0 \right), 8 \right) \right]
\]
Then, the ranking order of all alternatives \( \gamma_r \) by using Eq. (7) and get

\[
P = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
& & \ddots & \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{pmatrix}
\]

Based on this matrix, the optimal membership degrees of the numbers are computed

from these values, it is observed that \( \gamma_4 \) of the given LIVIFNs is

\[
(r_k) \text{ for the numbers } \gamma_k(k = 1, 2, \ldots, n) \text{ is given as follows:}
\]

\[
r_k = \frac{1}{n(n-1)} \left( \sum_{t=1}^{n} p_{kt} + \frac{n}{2} - 1 \right)
\]

\[\text{(7)}\]

Then, the ranking order of all alternatives \( \gamma_k, (k = 1, 2, \ldots, n) \) is found according
to decreasing order of the values of \( r_k \)'s and hence choose the best alternative.

**Example 3.2.** Let \( \gamma_1 = ([s_2, s_3], [s_3, s_4]), \gamma_2 = ([s_1, s_2], [s_4, s_3]), \gamma_3 = ([s_3, s_4], [s_2, s_3]), \gamma_4 = ([s_3, s_4], [s_1, s_2]) \) be four LIVIFNs defined over the continuous linguistic term set \( S_{[0,8]} \). In order to rank these given numbers by using our proposed possibility degree measure, we construct the possibility degree matrix \( P = (p_{kt})_{4 \times 4} \) by using Eq. (4) as

\[
P = \begin{pmatrix}
4.0000 & 8.0000 & 0.0000 & 0.0000 \\
0.0000 & 4.0000 & 0.0000 & 0.0000 \\
8.0000 & 8.0000 & 4.0000 & 1.3333 \\
8.0000 & 8.0000 & 6.6667 & 4.0000
\end{pmatrix}
\]

Based on this matrix, the optimal membership degrees of the numbers are computed
by using Eq. (7) and get \( r_1 = 1.0833, r_2 = 0.4167, r_3 = 1.8611 \) and \( r_4 = 2.3056 \).

From these values, it is observed that \( r_4 > r_3 > r_1 > r_2 \) and thus the ranking order of the given LIVIFNs is \( \gamma_4 > \gamma_3 > \gamma_1 > \gamma_2 \), where “\( > \)” refers to “preferred to”.

4. **Aggregating operators for LIVIFNs.** In this section, we define some new geometric aggregation operators for LIVIFNs.

4.1. **Some operational laws.** Motivated by t-norm and t-conorm, we propose the following operations laws for LIVIFNs.

**Definition 4.1.** Let \( \gamma_1 = ([s_{\tau_1}, s_{\theta_1}, s_{v_1}]), \gamma_2 = ([s_{\tau_2}, s_{\theta_2}, s_{v_2}]) \) be two LIVIFNs and \( \lambda > 0 \), then, based on Definition 2.3, some operational laws defined as follows:

(i) \( \gamma_1 \oplus \gamma_2 = \left([s_{\tau_1} + \tau_2 - \tau_{12}, s_{\theta_1} + \theta_2 - \theta_{12}, s_{v_1} + v_2 - v_{12}] \right); \)

(ii) \( \gamma_1 \odot \gamma_2 = \left([s_{\frac{\tau_1 + \tau_2}{2}}, s_{\frac{\theta_1 + \theta_2}{2}}, s_{\frac{v_1 + v_2}{2}}] \right); \)

(iii) \( \lambda \gamma_1 = \left([s_{\frac{\tau_1}{\lambda}}, s_{\frac{\theta_1}{\lambda}}, s_{\frac{v_1}{\lambda}}] \right); \)
Let $H$. Hence, on the other hand, we have similarly, we have.

**Proof.** Here, we shall prove only $\gamma_1 \oplus \gamma_2$ is LIVIFN, while other can be proven similar. For two LIVIFNs $\gamma_1 = ([s_{r_1}, s_{\eta_1}], [s_{\theta_1}, s_{v_1}])$ and $\gamma_2 = ([s_{r_2}, s_{\eta_2}], [s_{\theta_2}, s_{v_2}]),$ we have $0 \leq \eta_1, \eta_2, \upsilon_1, \upsilon_2 \leq h, \eta_1 + \upsilon_1 \leq h$ and $\eta_2 + \upsilon_2 \leq h$, therefore we get

\[
0 \leq \left(1 - \frac{\eta_1}{h}\right) \left(1 - \frac{\eta_2}{h}\right) \leq 1
\]

\[
\Rightarrow 0 \leq 1 - \left(1 - \frac{\eta_1}{h}\right) \left(1 - \frac{\eta_2}{h}\right) \leq 1
\]

\[
\Rightarrow 0 \leq h \left[1 - \left(1 - \frac{\eta_1}{h}\right) \left(1 - \frac{\eta_2}{h}\right)\right] \leq h.
\]

Similarly, we have

\[
0 \leq \frac{\upsilon_1 \upsilon_2}{h} \leq h.
\]

On the other hand, we have

\[
\eta_1 + \eta_2 - \frac{\eta_1 \eta_2}{h} + \frac{\upsilon_1 \upsilon_2}{h} \leq \eta_1 + \eta_2 - \frac{\eta_1 \eta_2}{h} + \frac{(h - \eta_1)(h - \eta_2)}{h} = h.
\]

Hence, $\gamma_1 \oplus \gamma_2$ is a LIVIFN.

**Theorem 4.3.** Let $\gamma = ([s_r, s_\eta], [s_\theta, s_v]), \gamma_1 = ([s_{r_1}, s_{\eta_1}], [s_{\theta_1}, s_{v_1}]), \gamma_2 = ([s_{r_2}, s_{\eta_2}], [s_{\theta_2}, s_{v_2}])$ be three LIVIFNs, and $\lambda, \lambda_1, \lambda_2 > 0$ be three real numbers, then

(i) $\gamma_1 \oplus \gamma_2 = \gamma_2 \oplus \gamma_1$;
(ii) $\lambda(\gamma_1 \oplus \gamma_2) = \lambda \gamma_1 \oplus \lambda \gamma_2$;
(iii) $\lambda_1 \gamma_1 \oplus \lambda_2 \gamma = (\lambda_1 + \lambda_2) \gamma$;
(iv) $\gamma_1 \otimes \gamma_2 = \gamma_2 \otimes \gamma_1$;
(v) $\gamma_1 \otimes \gamma_2 = \gamma_1 \otimes \gamma_2$;
(vi) $\gamma_1 \gamma_2 = (\gamma_1 \otimes \gamma_2)^\lambda$.

**Proof.** Here, we prove the parts (i)-(iii), while others can be proven similarly.

(i) According to Definition 4.1, we have

\[
\gamma_1 \oplus \gamma_2 = \left([s_{r_1} \tau_{r_2} - \frac{\tau_1}{h}, s_{\eta_1} \eta_2 - \frac{\eta_1}{h}, \frac{s_{\theta_1} \theta_2}{h}, \frac{s_{\theta_1} \eta_2}{h} + \frac{s_{\theta_1} \eta_2}{h} + \frac{s_{\theta_1} \eta_2}{h}], \frac{s_{\theta_1} \theta_2}{h}, \frac{s_{\theta_1} \eta_2}{h} + \frac{s_{\theta_1} \eta_2}{h} + \frac{s_{\theta_1} \eta_2}{h} \right)
\]

\[
= \left([s_{r_1} \tau_{r_2} - \frac{\tau_1}{h}, s_{\eta_1} \eta_2 - \frac{\eta_1}{h}, \frac{s_{\theta_1} \theta_2}{h}, \frac{s_{\theta_1} \eta_2}{h} + \frac{s_{\theta_1} \eta_2}{h} + \frac{s_{\theta_1} \eta_2}{h} \right)
\]

(ii) For a real number $\lambda > 0$, we have

\[
\lambda \gamma_1 = \left([s_{h(1 - (1 - \frac{\lambda}{h}) \lambda)}, s_{h(1 - (1 - \frac{\lambda}{h}) \lambda)}], \frac{s_{h(\frac{\lambda}{h})}}, \frac{s_{h(\frac{\lambda}{h})}} \right)
\]

and

\[
\lambda \gamma_2 = \left([s_{h(1 - (1 - \frac{\lambda}{h}) \lambda)}, s_{h(1 - (1 - \frac{\lambda}{h}) \lambda)}], \frac{s_{h(\frac{\lambda}{h})}}, \frac{s_{h(\frac{\lambda}{h})}} \right)
\]

...
Theorem 4.5. Let Definition 4.4. geometric aggregation operator for a collection of LIVIFNs denoted by \( \Omega \). We define LIVIF weighted geometric operator, which is the weighted geometric operator as presented by Xu and Yager [48], we define LIVIF weighted geometric (LIVIFG) operator, which is defined as
\[
\text{LIVIFG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma_1^{\omega_1} \otimes \gamma_2^{\omega_2} \otimes \cdots \otimes \gamma_n^{\omega_n},
\]
where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the weight vector of \( \gamma_t \) such that \( \omega_t > 0, \sum_{t=1}^{n} \omega_t = 1 \). Then LIVIFG is called as linguistic interval-valued intuitionistic fuzzy weighted geometric operator.

(ii) For positive real numbers \( \lambda_1, \lambda_2 \) and LIVIFN \( \gamma \), we get
\[
\lambda_1 \gamma + \lambda_2 \gamma = \begin{pmatrix}
\left[ s_h\left(1 - \left(\frac{1}{1+\lambda_1}\right)^\lambda \right), s_h\left(1 - \left(\frac{1}{1+\lambda_1}\right)^\lambda \right) \right],
\left[ s_h\left(\frac{\theta_1}{\tau_1}\right)^{\lambda_1}, s_h\left(\frac{\tau_1}{\theta_1}\right)^{\lambda_1} \right],
\left[ s_h\left(\frac{\theta_2}{\tau_2}\right)^{\lambda_1}, s_h\left(\frac{\tau_2}{\theta_2}\right)^{\lambda_1} \right],
\left[ s_h\left(\frac{\theta_n}{\tau_n}\right)^{\lambda_1}, s_h\left(\frac{\tau_n}{\theta_n}\right)^{\lambda_1} \right],
\left[ s_h\left(\frac{\theta_1}{\tau_1}\right)^{\lambda_2}, s_h\left(\frac{\tau_1}{\theta_1}\right)^{\lambda_2} \right],
\left[ s_h\left(\frac{\theta_2}{\tau_2}\right)^{\lambda_2}, s_h\left(\frac{\tau_2}{\theta_2}\right)^{\lambda_2} \right],
\left[ s_h\left(\frac{\theta_n}{\tau_n}\right)^{\lambda_2}, s_h\left(\frac{\tau_n}{\theta_n}\right)^{\lambda_2} \right]
\end{pmatrix}
\]
\[
= \left[ s_h\left(\frac{\theta_1}{\tau_1}\right)^{\lambda_1+\lambda_2}, s_h\left(\frac{\tau_1}{\theta_1}\right)^{\lambda_1+\lambda_2} \right],
\left[ s_h\left(\frac{\theta_2}{\tau_2}\right)^{\lambda_1+\lambda_2}, s_h\left(\frac{\tau_2}{\theta_2}\right)^{\lambda_1+\lambda_2} \right],
\left[ s_h\left(\frac{\theta_n}{\tau_n}\right)^{\lambda_1+\lambda_2}, s_h\left(\frac{\tau_n}{\theta_n}\right)^{\lambda_1+\lambda_2} \right]
\end{pmatrix}
\]
\[
= \lambda(\lambda_1 + \lambda_2)\gamma
\]
\[
= \lambda(\gamma_1 + \gamma_2)
\]
\[
\square
\]

4.2. Weighted geometric operator. In this section, motivated from the idea of geometric operators as presented by Xu and Yager [48], we define LIVIF weighted geometric aggregation operator for a collection of LIVIFNs denoted by \( \Omega \).

Definition 4.4. Let \( \gamma_t (t = 1, 2, \ldots, n) \) be a collection of LIVIFNs. A LIVIFWG operator is a mapping \( \text{LIVIFWG} : \Omega^n \to \Omega \) defined as
\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma_1^{\omega_1} \otimes \gamma_2^{\omega_2} \otimes \cdots \otimes \gamma_n^{\omega_n},
\]
where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the weight vector of \( \gamma_t \) such that \( \omega_t > 0, \sum_{t=1}^{n} \omega_t = 1 \), then LIVIFWG is called as linguistic interval-valued intuitionistic fuzzy weighted geometric operator.

Especially, if \( \omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T \), then the LIVIFWG operator is reduced to linguistic interval-valued intuitionistic fuzzy geometric (LIVIFG) operator, which is defined as
\[
\text{LIVIFG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = (\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_n)^{\frac{1}{n}}.
\]

Theorem 4.5. Let \( \gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{\epsilon_t}]), t = 1, 2, \ldots, n; \) be a collection of LIVIFNs, and \( \omega_t \) be the weight vector of \( \gamma_t \) such that \( \omega_t > 0, \sum_{t=1}^{n} \omega_t = 1 \), then the
aggregated value by using LIVIFWG operator is also a LIVIFN, and given by
\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \left( \begin{array}{c}
\sum_{t=1}^{S} h \left( \prod_{i=1}^{n} \left( \frac{T_i}{h} \right)^{\omega t} \right), \\
\sum_{t=1}^{S} h \left( \prod_{i=1}^{n} \left( 1 - \frac{T_i}{h} \right)^{\omega t} \right), \\
\sum_{t=1}^{S} h \left( \prod_{i=1}^{n} \left( 1 - \frac{V_i}{h} \right)^{\omega t} \right) \end{array} \right)
\]
(9)

Proof. The first result holds immediately from Theorem 4.2. Now to prove Eq. (9), we use the principle of mathematical induction on \( n \).

For \( n = 2 \) and by Definition 4.1, we have
\[
\gamma_{1}^{\omega 1} = \left( \begin{array}{c}
S h \left( \frac{T_1}{h} \right)^{-1}, S h \left( \frac{\eta_1}{h} \right)^{-1} \\
S h \left( 1 - \left( 1 - \frac{T_1}{h} \right)^{-1} \right) \end{array} \right), \quad \gamma_{2}^{\omega 2} = \left( \begin{array}{c}
S h \left( \frac{T_2}{h} \right)^{-1}, S h \left( \frac{\eta_2}{h} \right)^{-1} \\
S h \left( 1 - \left( 1 - \frac{T_2}{h} \right)^{-1} \right) \end{array} \right).
\]

Therefore,
\[
\text{LIVIFWG}(\gamma_1, \gamma_2) = \gamma_{1}^{\omega 1} \otimes \gamma_{2}^{\omega 2}
\]
\[
= \left( \begin{array}{c}
S h \left( \frac{T_1}{h} \right)^{-2} \left( \frac{T_2}{h} \right)^{-1}, S h \left( 1 - \left( 1 - \frac{T_1}{h} \right)^{-1} \right) \left( 1 - \frac{T_2}{h} \right)^{-1} \\
S h \left( \frac{T_2}{h} \right)^{-2} \left( \frac{T_1}{h} \right)^{-1}, S h \left( 1 - \left( 1 - \frac{T_2}{h} \right)^{-1} \right) \left( 1 - \frac{T_1}{h} \right)^{-1} \end{array} \right).
\]

Hence, Eq. (9) hold for \( n = 2 \).

Secondly, assume that the Eq. (9) hold for \( n = k \), that is
\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_k) = \gamma_{1}^{\omega 1} \otimes \gamma_{2}^{\omega 2} \otimes \cdots \otimes \gamma_{k}^{\omega k}
\]
\[
= \left( \begin{array}{c}
S h \left( \prod_{i=1}^{k} \left( \frac{T_i}{h} \right)^{-1} \right), \\
S h \left( \prod_{i=1}^{k} \left( 1 - \frac{T_i}{h} \right)^{-1} \right), \\
S h \left( \prod_{i=1}^{k} \left( 1 - \frac{V_i}{h} \right)^{-1} \right) \end{array} \right).
\]

Now, for \( n = k + 1 \), by Definition 4.1, we have
\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_{k+1}) = \text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_k) \otimes \gamma_{k+1}^{\omega_{k+1}}
\]
\[
= \left( \begin{array}{c}
S h \left( \prod_{i=1}^{k+1} \left( \frac{T_i}{h} \right)^{-1} \right), \\
S h \left( \prod_{i=1}^{k+1} \left( 1 - \frac{T_i}{h} \right)^{-1} \right), \\
S h \left( \prod_{i=1}^{k+1} \left( 1 - \frac{V_i}{h} \right)^{-1} \right) \end{array} \right).
\]

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which is true for \( n = k + 1 \). Hence, by the principle of mathematical induction, result given in Eq. (9) hold for positive integer \( n \), which completes the proof of the theorem. \( \square \)

The following example illustrate the working of the proposed geometric aggregation operator.

**Example 4.1.** Let \( S_{[0,8]} = \{ s_z \mid s_0 \leq s_z \leq s_8 \} \) be a continuous LTS and let \( \gamma_1 = ([s_1, s_2], [s_2, s_4]), \gamma_2 = ([s_1, s_4], [s_2, s_3]) \), and \( \gamma_3 = ([s_2, s_3], [s_3, s_5]) \) be three LIVIFNs and \( \omega = (0.3, 0.4, 0.3)^T \) be the weight vector of \( \gamma(t = 1, 2, 3) \), then

\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \gamma_3) = \left( \begin{array}{c}
\sum_{\tau_1} \left( \prod_{i=1}^{3} \left( \frac{z}{z_i} \right)^{\omega_i} \right) \\
\sum_{\tau_1} \left( \prod_{i=1}^{3} \left( \frac{z_{1-i}}{z_i} \right)^{\omega_i} \right) \\
\sum_{\tau_1} \left( \prod_{i=1}^{3} \left( \frac{z_{1-i}}{z_i} \right)^{\omega_i} \right)
\end{array} \right) = \left( \begin{array}{c}
\sum_{\tau_1} \left( \prod_{i=1}^{3} \left( \frac{z}{z_i} \right)^{0.3} \right) \\
\sum_{\tau_1} \left( \prod_{i=1}^{3} \left( \frac{z_{1-i}}{z_i} \right)^{0.3} \right) \\
\sum_{\tau_1} \left( \prod_{i=1}^{3} \left( \frac{z_{1-i}}{z_i} \right)^{0.3} \right)
\end{array} \right)
\]

**Theorem 4.6.** Let \( \gamma(t = 1, 2, \ldots, n) \) be a collection of LIVIFNs, and \( \omega_i \) be the normalized weight vector of \( \gamma_t \), then LIVIFWG operator satisfies the following properties:

(P1) **(Idempotency)** If \( \gamma_t = \gamma \) for all \( t \), then

\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma.
\]

(P2) **(Boundedness)** Let \( \gamma^- = ([s_{\tau^-}, s_{\eta^-}], [s_{\theta^-}, s_{\nu^-}]) \) and \( \gamma^+ = ([s_{\tau^+}, s_{\eta^+}], [s_{\theta^+}, s_{\nu^+}]) \), where \( \tau^- = \min\{\tau_1\}, \eta^- = \min\{\eta_1\}, \theta^- = \max\{\theta_1\}, \nu^- = \max\{\nu_1\}, \tau^+ = \max\{\tau_1\}, \eta^+ = \max\{\eta_1\}, \theta^+ = \min\{\theta_1\}, \nu^+ = \min\{\nu_1\} \), then we have

\[
\gamma^- \leq \text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) \leq \gamma^+.
\]

(P3) **(Monotonicity)** If \( \tau_t \leq \tau_t^*, \eta_t \leq \eta_t^*, \theta_t \geq \theta_t^* \) and \( \nu_t \geq \nu_t^* \) for all \( t \) where \( \gamma_t^* = ([s_{\tau_t^*}, s_{\eta_t^*}], [s_{\theta_t^*}, s_{\nu_t^*}]) \) is another LIVIFN, then

\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) \leq \text{LIVIFWG}(\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*).
\]

**Proof.** For a collection of LIVIFNs \( \gamma(t = 1, 2, \ldots, n) \), we have

(P1) Since \( \gamma_t = \gamma, \forall t \), then

\[
\text{LIVIFWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma_1^{\omega_1} \otimes \gamma_2^{\omega_2} \otimes \ldots \otimes \gamma_n^{\omega_n}
\]

\[
= \gamma_1^{\omega_1} \otimes \gamma_2^{\omega_2} \otimes \ldots \otimes \gamma_n^{\omega_n}
\]

\[
= \sum_{t=1}^{\omega} \gamma_t^n = \gamma.
\]
(P2) For LIVIFNs $\gamma_t = ([s_{\tau_t}, s_{\eta_t}], [s_{\theta_t}, s_{\upsilon_t}])$, $t = 1, 2, \ldots, n$, we have
\[
\tau^- = \min_t \{\tau_t\} \leq \tau_t \leq \max_t \{\tau_t\} = \tau^+
\]
\[
\Rightarrow \frac{\tau^-}{h} \leq \frac{\tau_t}{h} \leq \frac{\tau^+}{h}
\]
\[
\Rightarrow \prod_{t=1}^{n} \left( \frac{\tau^-}{h} \right)^{\omega_t} \leq \prod_{t=1}^{n} \left( \frac{\tau_t}{h} \right)^{\omega_t} \leq \prod_{t=1}^{n} \left( \frac{\tau^+}{h} \right)^{\omega_t}
\]
\[
\Rightarrow \left( \frac{\tau^-}{h} \right)^{\sum_{t=1}^{n} \omega_t} \leq \prod_{t=1}^{n} \left( \frac{\tau_t}{h} \right)^{\omega_t} \leq \left( \frac{\tau^+}{h} \right)^{\sum_{t=1}^{n} \omega_t}
\]
\[
\Rightarrow \tau^- \leq h \prod_{t=1}^{n} \left( \frac{\tau_t}{h} \right)^{\omega_t} \leq \tau^+.
\]

Similarly, we have
\[
\eta^- \leq h \prod_{t=1}^{n} \left( \frac{\eta_t}{h} \right)^{\omega_t} \leq \eta^+.
\]

On the other hand, for non-membership part, we have
\[
\theta^+ = \min_t \{\theta_t\} \leq \theta_t \leq \max_t \{\theta_t\} = \theta^-
\]
\[
\Rightarrow \frac{\theta^+}{h} \leq \frac{\theta_t}{h} \leq \frac{\theta^-}{h}
\]
\[
\Rightarrow 1 - \frac{\theta^+}{h} \geq 1 - \frac{\theta_t}{h} \geq 1 - \frac{\theta^-}{h}
\]
\[
\Rightarrow \prod_{t=1}^{n} \left( 1 - \frac{\theta^+}{h} \right)^{\omega_t} \geq \prod_{t=1}^{n} \left( 1 - \frac{\theta_t}{h} \right)^{\omega_t} \geq \prod_{t=1}^{n} \left( 1 - \frac{\theta^-}{h} \right)^{\omega_t}
\]
\[
\Rightarrow h \left( 1 - \left( 1 - \frac{\theta^+}{h} \right)^{\sum_{t=1}^{n} \omega_t} \right) \leq h \left( 1 - \prod_{t=1}^{n} \left( 1 - \frac{\theta_t}{h} \right)^{\omega_t} \right)
\]
\[
\leq h \left( 1 - \left( 1 - \frac{\theta^-}{h} \right)^{\sum_{t=1}^{n} \omega_t} \right)
\]
\[
\Rightarrow \theta^+ \leq h \left( 1 - \prod_{t=1}^{n} \left( 1 - \frac{\theta_t}{h} \right)^{\omega_t} \right) \leq \theta^-.
\]

Similarly,
\[
\upsilon^+ \leq h \left( 1 - \prod_{t=1}^{n} \left( 1 - \frac{\upsilon_t}{h} \right)^{\omega_t} \right) \leq \upsilon^-.
\]

Hence, according to Definition 3.2, we obtain
\[
\gamma^- \leq LIVIFWG(\gamma_1, \gamma_2, \ldots, \gamma_n) \leq \gamma^+.
\]

(P3) The monotonicity of the LIVIFWG operator can be obtained by similar proving method.
4.3. Ordered weighted geometric operator. In this section, we define ordered weighted LIVIF geometric aggregation operator for a collection of LIVIFNs denoted by Ω.

**Definition 4.7.** Let \( \gamma_t (t = 1, 2, \ldots, n) \) be a collection of LIVIFNs, then a linguistic interval-valued intuitionistic fuzzy ordered weighted geometric operator is a mapping \( \text{LIVIFOWG} : \Omega^n \rightarrow \Omega \) defined as

\[
\text{LIVIFOWG} (\alpha_1, \gamma_2, \ldots, \gamma_n) = \gamma_{\sigma(1)}^{w_1} \otimes \gamma_{\sigma(2)}^{w_2} \otimes \ldots \otimes \gamma_{\sigma(n)}^{w_n}
\]

(10)

where \( \gamma_{\sigma(t)} = ([s_{r_{\sigma(t)}}, s_{\theta_{\sigma(t)}}], [s_{g_{\sigma(t)}}, s_{\upsilon_{\sigma(t)}}]) \) is the \( t \)th largest value of the \( \gamma_t \) and \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector of LIVIFOWG operator with \( w_t > 0, \sum_{t=1}^{n} w_t = 1 \).

**Theorem 4.8.** The aggregated value for a collection of LIVIFNs \( \gamma_t = ([s_{r_t}, s_{\theta_t}], [s_{g_t}, s_{\upsilon_t}]) (t = 1, 2, \ldots, n) \) by using LIVIFOWG operator is also a LIVIFN, and is given by

\[
\text{LIVIFOWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \left( \begin{array}{c}
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right), \\
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right), \\
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right)
\end{array} \right)
\]

(11)

**Proof.** The proof of this theorem is similar to Theorem 4.5, so we omit here.  

**Example 4.2.** Let \( S_{[0, 8]} \) be a continuous LTS and let \( \gamma_1 = ([s_3, s_4], [s_2, s_3]), \gamma_2 = ([s_4, s_5], [s_1, s_2]), \) and \( \gamma_3 = ([s_2, s_4], [s_2, s_3]) \) be three LIVIFNs and \( w = (0.3, 0.4, 0.3)^T \) be the weight vector of \( \gamma_t (t = 1, 2, 3) \). To aggregate these numbers by LIVIFOWG operator, we firstly compute the ordering of these numbers by using the proposed possibility degree measure. For it, we construct the possibility degree matrix \( P = (p_{kt})_{3 \times 3} \) by using Eq. (4) as

\[
P = \begin{bmatrix}
4.0000 & 0.0000 & 4.6667 \\
8.0000 & 4.0000 & 8.0000 \\
3.3333 & 0.0000 & 4.0000
\end{bmatrix}
\]

Based on this matrix, the optimal membership degrees of the numbers \( \gamma_t (t = 1, 2, 3) \) are computed by using Eq. (10) and get \( r_1 = 1.5278, r_2 = 3.4167 \) and \( r_3 = 1.3056 \). From these values, it is seen that \( r_2 > r_1 > r_3 \) and thus ordering of the given numbers is \( \gamma_2 \succ \gamma_1 \succ \gamma_3 \). Therefore, \( \gamma_{\sigma(1)} = \gamma_2, \gamma_{\sigma(2)} = \gamma_1 \) and \( \gamma_{\sigma(3)} = \gamma_3 \). Now, based on these ordering numbers and by using the proposed LIVIFOWG operator we get

\[
\text{LIVIFOWG}(\gamma_1, \gamma_2, \gamma_3) = \left( \begin{array}{c}
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right), \\
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right), \\
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right)
\end{array} \right)
\]

(11)

\[
= \left( \begin{array}{c}
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right), \\
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right), \\
S \left( \frac{3}{\Pi_{t=1}^{n} (\gamma_{\sigma(t)})^{w_t}} \right)
\end{array} \right)
\]

= \([s_{8.9558}, s_{8.2769}], [s_{8.1760}, s_{8.7189}])
\]
Similar to Theorem 4.6, LIVIFOWG operator also satisfies the following properties, which are stated as follows:

**Theorem 4.9.** Let \( \gamma_t (t = 1, 2, \ldots, n) \) be a collection of LIVIFNs, then LIVIFOWG operator satisfies the following properties:

1. **(Idempotency)** If \( \gamma_t = \gamma \) for all \( t \), then
   \[
   \text{LIVIFOWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma.
   \]

2. **(Boundedness)** Let \( \gamma^- = ([s_{\tau^-}, s_{\eta^-}], [s_{\theta^-}, s_{v^-}]) \) and \( \gamma^+ = ([s_{\tau^+}, s_{\eta^+}], [s_{\theta^+}, s_{v^+}]) \), where \( \tau^- = \min_t \{\tau_t\}, \eta^- = \min_t \{\eta_t\}, \theta^- = \max_t \{\theta_t\}, v^- = \max_t \{v_t\} \), then we have
   \[
   \gamma^- \leq \text{LIVIFOWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) \leq \gamma^+.
   \]

3. **(Monotonicity)** If \( \tau_t \leq \tau_t^*, \eta_t \leq \eta_t^*, \theta_t \geq \theta_t^* \) and \( v_t \geq v_t^* \) for all \( t \) where \( \gamma_t^* = ([s_{\tau_t^*}, s_{\eta_t^*}], [s_{\theta_t^*}, s_{v_t^*}]) \) is another LIVIFN, then
   \[
   \text{LIVIFOWG}(\gamma_1, \gamma_2, \ldots, \gamma_n) \leq \text{LIVIFOWG}(\gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*).
   \]

**Proof.** Similar to above properties. \( \square \)

5. **Proposed MAGDM approach problem under LIVIF information.** In this section, we present a decision making method based on the proposed geometric aggregation operators and the possibility degree measure for solving the MAGDM problems under the LIVIFS environment.

Consider a group DM problem in which there are ‘\( m \)’ alternatives \( A_1, A_2, \ldots, A_m \) which are evaluated under the set of ‘\( n \)’ different attributes \( G_1, G_2, \ldots, G_n \). For it, consider a set of ‘\( l \)’ decision makers \( D^{(1)}, D^{(2)}, \ldots, D^{(l)} \) whose weight vectors is \( w = (w^{(1)}, w^{(2)}, \ldots, w^{(l)})^T \) such that \( w^{(q)} > 0 \) and \( \sum_{q=1}^{n} w^{(q)} = 1 \) which are going to evaluate the given alternatives. These decision makers give their preferences, towards evaluating the alternatives \( A_k (k = 1, 2, \ldots, m) \) under the attributes \( G_t (t = 1, 2, \ldots, n) \), in terms of LIVIFNs \( \tilde{\gamma}_{kt}^{(q)} = ([\tilde{s}_{\tau_{kt}}^{(q)}, \tilde{s}_{\eta_{kt}}^{(q)}], [\tilde{s}_{\theta_{kt}}^{(q)}, \tilde{s}_{v_{kt}}^{(q)}]) \), where \( [\tilde{s}_{\tau_{kt}}^{(q)}, \tilde{s}_{\eta_{kt}}^{(q)}], [\tilde{s}_{\theta_{kt}}^{(q)}, \tilde{s}_{v_{kt}}^{(q)}] \subseteq [0, h] \) and \( \eta_{kt}^{(q)} + v_{kt}^{(q)} \leq h \). Further, considering the importance of the given attributes corresponding to each decision maker \( D^{(q)} \) with weight vector \( \omega^{(q)} = (\omega_1^{(q)}, \omega_2^{(q)}, \ldots, \omega_n^{(q)})^T \) such that \( \sum_{t=1}^{n} \omega_t^{(q)} = 1 \).

Then, in the following, we develop a method based on the proposed operators and the possibility degree measure to solve the decision making problems with LIVIFS information, which involves the following steps.

**Step 1.** Arrange the collective information of the alternatives given by the decision makers in the form of the decision matrices \( \tilde{R}^{(q)} = \left( \tilde{\gamma}_{kt}^{(q)} \right)_{m \times n} \) as

\[
\tilde{R}^{(q)} = \begin{pmatrix}
G_1 & G_2 & \cdots & G_n \\
A_1 & \tilde{\gamma}_{11}^{(q)} & \tilde{\gamma}_{12}^{(q)} & \cdots & \tilde{\gamma}_{1n}^{(q)} \\
A_2 & \tilde{\gamma}_{21}^{(q)} & \tilde{\gamma}_{22}^{(q)} & \cdots & \tilde{\gamma}_{2n}^{(q)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_m & \tilde{\gamma}_{m1}^{(q)} & \tilde{\gamma}_{m2}^{(q)} & \cdots & \tilde{\gamma}_{mn}^{(q)}
\end{pmatrix}
\]
Step 2. Normalize the collective information, if required, by converting the cost type attributes into the benefit type by using Eq. (12) to balance the physical dimensions of the rating values as

\[
\gamma_{k}^{(q)} = \left[ \left( \frac{1}{h} \prod_{t=1}^{m} \left( \frac{\gamma_{k}^{(q)}}{h} \right)^{w(\gamma)} \right) \right]
\]

Step 3. For each decision maker \( D(q), q = 1, 2, \ldots, l \), individual overall performances \( \gamma_{k}^{(q)} \) corresponding to each alternative \( A_k, k = 1, 2, \ldots, m \), are calculated by utilizing the LIVIFWG operator as follows:

\[
\gamma_{k}^{(q)} = \text{LIVIFWG}(\gamma_{k1}^{(q)}, \gamma_{k2}^{(q)}, \ldots, \gamma_{kn}^{(q)})
\]

Step 4. Utilizing the LIVIFOWG operator to aggregate individual overall performances \( \gamma_{k}^{(q)} \) into collective overall performances \( \gamma_{k} \) corresponding to each alternative \( A_k, k = 1, 2, \ldots, m \), by assigning the priority value, \( w = (w^{(1)}, w^{(2)}, \ldots, w^{(l)})^T \) such that \( w^{(q)} > 0 \) and \( \sum_{q=1}^{l} w^{(q)} = 1 \), to each expert as follows:

\[
\gamma_k = \text{LIVIFOWG}(\gamma_k^{(1)}, \gamma_k^{(2)}, \ldots, \gamma_k^{(l)})
\]

Step 5. Compute the possibility degree matrix \( P = (p_{kj})_{m \times m} \) as

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1m} \\
p_{21} & p_{22} & \cdots & p_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
p_{m1} & p_{m2} & \cdots & p_{mm}
\end{bmatrix}
\]
where \( p_{kt} = [p(\gamma_k \succeq \gamma_j)]_{m \times m} \) is defined either as

\[
p(\gamma_k \succeq \gamma_j) = \frac{1}{2} \left[ \min \left( \max \left( h \left( \frac{\eta_k - \tau_j}{\eta_k - \tau_k + \eta_j - \tau_j} \right), 0 \right), h \right) + \min \left( \max \left( h \left( \frac{\upsilon_k - \theta_k}{\upsilon_k - \theta_k + \upsilon_j - \theta_j} \right), 0 \right), h \right) \right]
\]

or by Eq. (5) or Eq. (6).

Further, it is clearly observed that \( 0 \leq p_{kj} \leq h \) and \( p_{kj} + p_{jk} = h \); \((k, j = 1, 2, \ldots, m)\) which implies that \( P \) is the fuzzy-complementary-judgement matrix.

Step 6. The ranking value which represents the optimal degree of the membership for alternative \( A_k (k = 1, 2, \ldots, m) \) is computed by using

\[
r_k = \frac{1}{m(m-1)} \left( \sum_{j=1}^{m} p_{kj} + \frac{m}{2} - 1 \right)
\]

Thus, based on these membership values, the ranking order of all alternatives is found according to decreasing order of the values of \( r_k (k = 1, 2, \ldots, m) \)'s and hence choose the best alternative.

6. Illustrative example. The above mentioned approach is illustrated with a numerical example under the LIVIFNs environment which are stated below.

Consider a problem of a pharmaceutical company which wants to select a lab technician for the micro-bio laboratory. For this, company published notification in the newspaper and consider the four attributes required for technician selection, namely, academic record (\( G_1 \)), personal interview evaluation (\( G_2 \)), experience (\( G_3 \)) and technical capability (\( G_4 \)). The relative importance of these attributes is taken in the form of weight \( \omega = (\omega_1, \omega_2, \omega_3, \omega_4) \). On the basis of the notification conditions, the four candidates \( A_1, A_2, A_3 \) and \( A_4 \) are shortlisted for the interview and considered as alternatives. Then, the main object of the company is to choose the best candidate among them for the task. For it, a panel of three experts \( D^{(1)} \) ("Director"), \( D^{(2)} \) ("Head of the Department"), \( D^{(3)} \) ("Human resources manager") are invited to evaluate the given alternatives under each attribute according to linguistic variables defined as \( s_0 = "extremely poor", s_1 = "very poor", s_2 = "poor", s_3 = "slightly poor", s_4 = "fair", s_5 = "slightly good", s_6 = "good", s_7 = "very good", s_8 = "extremely good" \). In order to fulfill it, they have evaluated these and give their preferences in the term of LIVIFNs.

6.1. By proposed approach. The following steps of the proposed approach are executed to find the best alternative(s) for the required post.

Step 1. The rating values of each expert on the alternatives under set of the attributes are summarized in Tables 1, 2 and 3 in the form of LIVIFNs.

Step 2. Since all attributes are of the same types, so there is no need of the normalization process.

Step 3. Aggregate the preference of each alternative \( A_k (k = 1, 2, 3, 4) \) by using LIVIFWG operator defined in Eq. (12). The result corresponding to it is summarized in Table 4.

Step 4. If we assign the weights vector \( w = (0.25, 0.55, 0.20)^T \) of decision maker's then collective overall performance \( \gamma_k \) corresponding to each alternative \( A_k \)
TABLE 1. LIVIFNs decision matrix $\tilde{R}^{(1)}$ of decision maker $D^{(1)}$

|   | $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|-------|-------|-------|-------|
| $A_1$ | $[83, 85, 82, 83]$ | $[84, 85, 81, 82]$ | $[84, 85, 82, 83]$ | $[84, 85, 81, 82]$ |
| $A_2$ | $[83, 85, 82, 83]$ | $[82, 83, 81, 82]$ | $[82, 83, 82, 83]$ | $[83, 85, 82, 83]$ |
| $A_3$ | $[84, 86, 81, 82]$ | $[85, 86, 81, 81]$ | $[83, 84, 82, 83]$ | $[84, 85, 81, 84]$ |
| $A_4$ | $[84, 85, 82, 83]$ | $[81, 83, 83, 84]$ | $[83, 85, 81, 83]$ | $[86, 87, 81, 81]$ |

Weights

0.40 0.25 0.20 0.15

TABLE 2. LIVIFNs decision matrix $\tilde{R}^{(2)}$ of decision maker $D^{(2)}$

|   | $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|-------|-------|-------|-------|
| $A_1$ | $[82, 84, 81, 83]$ | $[84, 85, 81, 82]$ | $[84, 85, 81, 83]$ | $[83, 86, 81, 82]$ |
| $A_2$ | $[83, 85, 81, 83]$ | $[81, 82, 81, 84]$ | $[82, 83, 83, 84]$ | $[83, 85, 81, 84]$ |
| $A_3$ | $[83, 84, 82, 83]$ | $[85, 86, 81, 82]$ | $[83, 85, 82, 83]$ | $[84, 85, 81, 84]$ |
| $A_4$ | $[84, 85, 81, 82]$ | $[81, 83, 83, 85]$ | $[83, 84, 82, 83]$ | $[82, 83, 81, 82]$ |

Weights

0.30 0.35 0.25 0.10

TABLE 3. LIVIFNs decision matrix $\tilde{R}^{(3)}$ of decision maker $D^{(3)}$

|   | $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|-------|-------|-------|-------|
| $A_1$ | $[82, 84, 81, 82]$ | $[82, 83, 82, 84]$ | $[83, 85, 82, 83]$ | $[85, 86, 81, 82]$ |
| $A_2$ | $[81, 83, 82, 83]$ | $[84, 85, 81, 82]$ | $[82, 84, 81, 83]$ | $[83, 84, 82, 84]$ |
| $A_3$ | $[82, 83, 81, 83]$ | $[83, 85, 82, 83]$ | $[83, 85, 81, 83]$ | $[83, 85, 82, 83]$ |
| $A_4$ | $[83, 84, 82, 83]$ | $[81, 82, 83, 84]$ | $[83, 85, 81, 82]$ | $[85, 86, 81, 82]$ |

Weights

0.35 0.40 0.15 0.10

TABLE 4. Collective individual performance of each decision maker

|   | $D^{(1)}$ | $D^{(2)}$ | $D^{(3)}$ |
|---|-----------|-----------|-----------|
| $A_1$ | $[83.4146, 84.8354, 81.6194, 82.6217]$ | $[81.1569, 84.7623, 81.0568, 82.5725]$ | $[82.3294, 84.6821, 81.4609, 84.0159]$ |
| $A_2$ | $[82.1199, 84.1887, 81.0975, 82.9952]$ | $[81.0485, 83.1932, 81.5467, 83.6266]$ | $[82.1562, 84.3734, 81.4651, 84.7464]$ |
| $A_3$ | $[83.9090, 85.3834, 81.2215, 82.1497]$ | $[83.5873, 84.8744, 81.6074, 82.6705]$ | $[82.6931, 84.1814, 81.5193, 83.0000]$ |
| $A_4$ | $[82.3737, 84.6284, 81.9466, 83.0265]$ | $[82.1380, 83.0342, 82.0129, 83.5622]$ | $[82.0345, 83.2643, 82.2028, 81.2197]$ |

$(k = 1, 2, 3, 4)$ is calculated by aggregating the different values using Eq. (13). The results corresponding to LIVIFOWG operator are given as

- $\gamma_1 = ([83.1017, 84.5998], [81.4590, 82.6952])$,
- $\gamma_2 = ([82.0707, 84.0105], [81.7779, 83.0673])$,
- $\gamma_3 = ([83.4558, 84.8460], [81.5267, 82.6140])$,
- $\gamma_4 = ([82.2330, 83.5104], [82.1026, 83.2273])$.

Step 5. Utilize Eq. (15) to construct the possibility degree measure matrix as

$$P = \begin{bmatrix}
4.0000 & 5.4900 & 3.5728 & 6.4071 \\
2.5100 & 4.0000 & 2.0734 & 4.6113 \\
4.4272 & 5.9266 & 4.0000 & 6.9933 \\
1.5929 & 3.3887 & 1.0067 & 4.0000
\end{bmatrix}$$

Step 6. The optimal value $r_k$ of the membership degree of the alternatives $A_k (k = 1, 2, 3, 4)$ can be computed by using Eq. (16) and get $r_1 = 1.7058$, $r_2 =$...
1.1829, \( r_3 = 1.8623 \) and \( r_4 = 0.9157 \). Since \( r_3 > r_1 > r_2 > r_4 \), therefore ranking order of the alternatives is \( A_3 \succ A_1 \succ A_2 \succ A_4 \). Thus, the best alternative for the required task is \( A_3 \).

6.2. Validity test. Wang and Triantaphyllou [36] established the following testing criterions to evaluate the validity of MAGDM methods.

Test criterion 1: “An effective MAGDM method does not change the index of the best alternative by replacing a non optimal alternative with a worse alternative without shifting the corresponding importance of every decision attribute”.

Test criterion 2: “To an effective MAGDM method must satisfy transitive property”.

Test criterion 3: “If we decomposed a MAGDM problem into the smaller DM problems by deleting some of the alternatives and same MAGDM method is utilized on these problems to rank alternatives, collective ranking of alternatives must be identical to ranking of un-decomposed DM problem”.

Validity of the proposed approach is tested by using these criteria as follows:

6.2.1. Validity test by criterion 1. For testing the validity of proposed MAGDM approach under the test criterion 1, we replace the non optimal alternative \( A_4 \) with the worse alternative \( A'_4 \) in original decision matrices for each decision maker \( D(q) \), \( q = 1, 2, 3 \). The rating values of \( A'_4 \) is chosen as an arbitrary and summarized in Table 5. Then, by applying the proposed MAGDM approach to transform data we get the optimal degrees of membership for candidates \( A_k \) (\( k = 1, 2, 3, 4 \)) as 1.8498, 1.4720, 1.9282 and 0.4167. Thus, ranking order of the candidates is \( A_3 \succ A_1 \succ A_2 \succ A'_4 \) which is similar to the ranking order of original problem and best candidate remains same i.e., \( A_3 \) and it validates the test criterion 1.

6.2.2. Validity test by criteria 2 and 3. For evaluating the proposed MAGDM approach under the test criteria 2 and 3, we have decomposed original DM problem into three smaller MAGDM problems which contains the alternatives \( \{A_1, A_2, A_4\} \), \( \{A_1, A_3, A_4\} \) and \( \{A_2, A_3, A_4\} \). If we apply the proposed MAGDM approach on these problems then we get ranking order of alternatives as \( A_1 \succ A_2 \succ A_4 \), \( A_3 \succ A_1 \succ A_4 \) and \( A_3 \succ A_2 \succ A_4 \) respectively. After combining together the ranking of the alternatives of these smaller problem, we get the final ranking order as \( A_3 \succ A_1 \succ A_2 \succ A_4 \) which is same as un-decomposed DM problem and shows transitive property. Hence, the proposed MAGDM approach is valid under the test criteria 2 and 3.

6.3. Further discussion. In the following we give some characteristics comparison of our proposed method and the aforementioned methods, which are listed in Table 6. The method proposed by Xu and Yager [48] adopts IFNs to aggregate the uncertain information using geometric operators only by quantitative aspects. On the other hand, the method described by the author in Xu [44] represent the

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### Table 5. Worse alternative \( A'_4 \) for each decision maker

|          | \( G_1 \) | \( G_2 \) | \( G_3 \) | \( G_4 \) |
|----------|-----------|-----------|-----------|-----------|
| \( D^{(1)} \) | \( [s_2, s_3], [s_3, s_4] \) | \( [s_0, s_3], [s_4, s_5] \) | \( [s_2, s_4], [s_2, s_4] \) | \( [s_3, s_4], [s_2, s_3] \) |
| \( D^{(2)} \) | \( [s_2, s_3], [s_3, s_4] \) | \( [s_0, s_1], [s_4, s_6] \) | \( [s_1, s_2], [s_3, s_4] \) | \( [s_1, s_2], [s_3, s_4] \) |
| \( D^{(3)} \) | \( [s_1, s_2], [s_3, s_5] \) | \( [s_1, s_1], [s_4, s_5] \) | \( [s_2, s_4], [s_3, s_4] \) | \( [s_2, s_3], [s_2, s_3] \) |
Table 6. The characteristic comparisons of different methods

| Methods                  | Whether express a wider range of information | Whether describe information using linguistic features | Whether describe information by interval-valued numbers | Whether have the characteristic of generalization |
|--------------------------|---------------------------------------------|--------------------------------------------------------|--------------------------------------------------------|-----------------------------------------------|
| Xu and Yager [48]        | no                                          | no                                                     | no                                                     | no                                            |
| Xu [44]                  | yes                                         | no                                                     | yes                                                    | no                                            |
| Zhang [51]               | no                                          | yes                                                    | no                                                     | yes                                           |
| The proposed method      | yes                                         | yes                                                    | yes                                                    | yes                                           |

wider range of the information in terms of the interval-valued membership degrees. But their approach is also limited to only quantitative aspects and does not apply the linguistic information. Apart from these, the method proposed by Zhang [51] adopts LIFNs to describe the uncertainties in the data as a crisp number. However, in the present study, we proposed the LIVIFNs to describe the uncertainties in terms of linguistic interval pairs of the membership degrees which can easily express the information in a more semantics and concise way and hence can reduce the information loss.

In addition, LIVIFNs used in the new method can model the uncertain and fuzzy information more flexible by its linguistic interval-valued intuitionistic fuzzy numbers during the evaluation process, which can reflect the inherent thoughts of decision makers more accurately. Further, it has been analyzed that the operators defined by the author in [51] can be considered as a special case of the proposed operator by setting the lower and upper bound of membership degrees are equal. Thus, the proposed aggregation operators are more generalized and capture the more information during the analysis.

7. Conclusion. LIVIFS is the generalization of LIFS in which membership and nonmembership degree represented by the interval-valued linguistic terms for better dealing with fuzzy information under the qualitative aspect. In this paper, we have presented a MAGDM approach under the LIVIFS environment. For it, firstly, we defined possibility degree measure method to compare the LIVIFNs along with some properties of it. Afterward, we proposed some weighted and ordered weighted geometric aggregation operators for the collection of the different LIVIF information. Thee desirable properties, namely, idempotency, monotonicity, and boundedness are investigated in details. Finally, a real-life case has been discussed to describe the decision step and illustrate the feasibility of the proposed approach. From the results of the validity test and numerical example, we have concluded that the proposed approach solve the real-life problem effectively. The special advantage of the proposed approach is that possibility degree ranking method reflects the uncertainty of the information and provide the more reasonable results. In the future, the result of this paper can be extended to some other uncertain and fuzzy environments [11, 16, 22, 33].

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