ON THE HOMOGENIZATION OF RANDOM STATIONARY ELLIPTIC OPERATORS IN DIVERGENCE FORM

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ABSTRACT. In this note we comment on the homogenization of a random elliptic operator in divergence form \(-\nabla \cdot a \nabla\), where the coefficient field \(a\) is distributed according to a stationary, but not necessarily ergodic, probability measure \(\mathbb{P}\). We generalize the well-known case for \(\mathbb{P}\) stationary and ergodic by showing that the operator \(-\nabla \cdot a(\xi) \nabla\) almost surely homogenizes to a constant-coefficient, random operator \(-\nabla \cdot A_h \nabla\). Furthermore, we use a disintegration formula for \(\mathbb{P}\) with respect to a family of ergodic and stationary probability measures to show that the law of \(A_h\) may be obtained by using the standard homogenization results on each probability measure of the previous family. We finally provide a more explicit formula for \(A_h\) in the case of coefficient fields which are a function of a stationary Gaussian field.

1. Introduction

This note provides a remark on the homogenization of random elliptic operators in divergence form \(-\nabla \cdot a \nabla\), where the coefficient field \(a : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) is symmetric, uniformly elliptic and distributed according to a probability measure \(\mathbb{P}\) which is invariant with respect to the translations \(a(\cdot + x), x \in \mathbb{R}^d\). We do not assume that \(\mathbb{P}\) is ergodic.

In the case of stationary and ergodic measures, it is well-established that for \(\mathbb{P}\)-almost every realization of the coefficient field \(a\), for \(\varepsilon \downarrow 0^+\) the rescaled operator \(-\nabla \cdot a(\varepsilon \xi) \nabla\) homogenizes to \(-\nabla \cdot A_h \nabla\). The homogenized coefficient \(A_h\) is constant, deterministic and satisfies the same ellipticity bounds. Qualitative stochastic homogenization, namely the convergence of solutions \(u_\varepsilon\) associated to \(-\nabla \cdot a(\varepsilon \xi) \nabla\) to the solution \(u_0\) associated to \(-\nabla \cdot A_h \nabla\), has been obtained in [18] and [21]; in the last two decades, a large literature has been developed to upgrade these results into quantitative estimates on the convergence of \(u_\varepsilon\) to \(u_h\) (e.g. \([1, 2, 3, 5, 13, 14]\)). This has led, in addition, to an exhaustive understanding of the fluctuations structure of \(u_\varepsilon\) and of other meaningful quantities related to the random operator \(-\nabla \cdot a \nabla\) \([1, 9, 10, 17]\).

As is well-known in classical stochastic homogenization \([18, 21]\), namely when the measure \(\mathbb{P}\) is stationary and ergodic, the homogenized matrix \(A_h\) may be identified with the large-scale limit of the spacial averages of suitable stationary random fields, namely the flux of the correctors \(a(\varepsilon_i + \nabla \phi_i)\) (see \((2.8)\) and \((2.7)\)). We refer to \([4, 6, 12, 16, 20]\) for an extensive study of the correctors and their properties. This characterization of \(A_h\) allows to appeal to Birkhoff’s ergodic theorem (see e.g. \([19]\)) and infer that \(A_h\) is well-defined for \(\mathbb{P}\)-almost every realization of \(a\) and, by the ergodicity assumption, that \(A_h\) does not depend on \(a\). It is thus intuitive to expect that, if only the ergodicity assumption on \(\mathbb{P}\) fails, one still obtains a homogenization result for \(-\nabla \cdot a(\xi) \nabla\), this time with the homogenized coefficient \(A_h = A_h(a)\) being a random matrix. The first result contained in this note gives a rigorous derivation of this argument (see Theorem 2.1).

If we denote by \(\Omega\) the space of realizations of \(a\) and by \((\Omega, \mathcal{F}, \mathbb{P})\) the associated probability space, Birkhoff’s ergodic theorem also implies that the random matrix \(A_h\) is measurable with respect to the \(\sigma\)-algebra \(\mathcal{I} \subseteq \mathcal{F}\) generated by the subsets of \(\Omega\) which are translation invariant. In other words, in the case of stationary measures, the homogenization process does not remove the randomness from \(A_h\) but leads nonetheless to a reduction in its complexity. We give a further result in this direction by relying on some techniques coming from statistics and dynamical systems that allow to write \(\mathbb{P}\) as a disintegration with respect to a family of stationary and ergodic probability measures. More precisely, if \(\mathbb{P}\) is a stationary probability measure on \((\Omega, \mathcal{F})\), then for all \(B \in \mathcal{F}\)

\[
\mathbb{P}(B) = \int_{\Omega_0} \mathbb{P}_\xi(B) \mathbb{P}(d\xi),
\]  

(1.1)
where \((\Omega_0, \mathcal{I}_0)\) is a measurable space, \(\{P_x\}_{x \in \Omega_0}\) is a family of ergodic and stationary probability measures on \((\Omega, \mathcal{F})\), and \(\tilde{P}\) is a probability measure on \((\Omega_0, \mathcal{I}_0)\) (see [7, 15] and Lemma 2.2). More precisely, the set \(\Omega_0\) is obtained as a quotient \(\Omega/\sim\) with respect to a suitable equivalence relation \(\sim\), and \(\mathcal{I}_0\) is isomorphic to the \(\sigma\)-algebra \(\mathcal{I}\) of the \(\sigma\)-algebra \(\mathcal{F}\) under translations. Using (1.1), we show that \(A_h\) may be identified with a random variable on \((\Omega_0, \mathcal{I}_0, \tilde{P})\) and that we may simply obtain the realization \(A_h(\xi)\) for \(\xi \in \Omega_0\) by appealing to the standard homogenization result for the ergodic and stationary measure \((\Omega, \mathcal{F}, \tilde{P})\) (Corollary 2.3).

As an application of the previous result, we study the case of coefficient fields \(a\) which are Gaussian related, namely when \(a\) is a function of a stationary Gaussian field. The Gaussian setting allows to obtain a more explicit disintegration of \(\mathbb{P}\) and an explicit formula for \(\Omega_0\) and \(\tilde{P}\) which characterize the law of the random matrix \(A_h\) (Corollary 2.6).

We conclude this introduction by mentioning that in [8] a homogenization result for random free-discontinuity functionals has also been obtained in the setting of stationary measures which are not assumed to be ergodic.

2. Notation and abstract result

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space equipped with a group of transformations \(\{\tau_x : \Omega \to \Omega\}_{x \in \mathbb{R}^d}\), \(d \geq 2\), with respect to which the measure \(\mathbb{P}\) is stationary, i.e.

\[
\mathbb{P} \circ \tau_x = \mathbb{P} \quad \forall x \in \mathbb{R}^d.
\]

We assume that \(\mathcal{F}\) is countably generated and that for all \(B \in \mathcal{F}\)

\[
\lim_{|x| \to 0} \int |1_B(\tau_x \omega) - 1_B(\omega)| \mathbb{P}(d\omega) = 0.
\]

Form this it follows that the joint map \(\tau : \Omega \times \mathbb{R}^d \to \Omega, \tau(x, \omega) = \tau_x \omega\) is measurable with respect to the tensor \(\sigma\)-algebra of \(\mathcal{F}\) and of the Lebesgue measurable sets of \(\mathbb{R}^d\).

In addition, the assumptions on \(\mathcal{F}\) also imply that the spaces \(L^p(\Omega, \mathcal{F}, \mathbb{P})\) are separable for all \(1 \leq p < +\infty\) and that the maps \(T_x : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to L^p(\Omega, \mathcal{F}, \mathbb{P}), T_x f := f \circ \tau_x\) are strongly continuous for all \(x \in \mathbb{R}^d\) and \(1 \leq p < +\infty\). For \(F \in L^1(\Omega, \mathcal{F}, \mathbb{P})\), we define

\[
\langle F \rangle := \int \Omega F(\omega) \mathbb{P}(d\omega).
\]

Let \(\mathcal{M}_{d, \text{sym}}\) denote the space of symmetric \(d \times d\) real matrices and let \((\Omega, \mathcal{F}, \mathbb{P})\) be as above. We define the random coefficient field \(a\) as follows: Let \(A : \Omega \to \mathcal{M}_{d, \text{sym}}\) be a \((\text{measurable, matrix-valued})\) random variable satisfying for \(0 < \lambda \leq \Lambda\) and \(\mathbb{P}\)-almost every \(\omega \in \mathbb{R}^d\)

\[
\lambda |\xi|^2 \leq \xi \cdot A(\omega) \xi \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d.
\]

We set

\[
a : \mathbb{R}^d \times \Omega \to \mathcal{M}_{d, \text{sym}}, \quad a(\omega, x) = T_x A(\omega) = A(\tau_x \omega).
\]

Thus, for \(\mathbb{P}\)-almost every \(\omega \in \Omega\) the operator \(-\nabla \cdot a \nabla\) is bounded and uniformly elliptic.

We emphasize that we do not require that \(\mathbb{P}\) is ergodic. By Birkhoff’s ergodic theorem [19] we have that for any \(F \in L^1(\Omega, \mathcal{F}, \mathbb{P})\) and \(\mathbb{P}\)-almost every \(\omega \in \Omega\)

\[
\lim_{R \to +\infty} \int_{|x| < R} F(\tau_x \omega) \, dx = \langle F \rangle |\mathcal{I}|,
\]

where the right-hand side is the conditional expectation of \(F\) with respect to the \(\sigma\)-algebra \(\mathcal{I} \subseteq \mathcal{F}\) generated by the sets

\[
A \in \mathcal{F}, \quad \tau_x A = A \quad \forall x \in \mathbb{R}^d.
\]

We recall that in the ergodic case, i.e. when \(\mathcal{I}\) is trivial and the right-hand side of (2.6) is given by \(\langle F \rangle\), it is well-known [18, 21] that the operator \(-\nabla \cdot a(\omega, \cdot) \nabla\) homogenizes \(\mathbb{P}\)-almost surely to the operator \(-\nabla \cdot A_h \nabla\). The matrix \(A_h \in \mathcal{M}_{d, \text{sym}}\) is constant and deterministic and is given by the formula

\[
e_i \cdot A_h e_j = \langle (e_i + \nabla \phi_i(0)) \cdot A(e_j + \nabla \phi_j(0)) \rangle, \quad i, j = 1, \cdots, d.
\]
Here, for each \( i = 1, \cdots, d \) the random fields \( \phi_i(\omega, \cdot) \in H^1_{\text{loc}}(\mathbb{R}^d) \) are the first-order correctors [14, 18, 21] satisfying for almost every \( \omega \in \Omega \)

\[
-\nabla \cdot a(\omega, x) \nabla (\phi_i(\omega, x) + x_i) = 0 \quad \text{in } \mathbb{R}^d,
\]

\[
\lim_{R \to \infty} R^{-2} \int_{|x| < R} |\phi_i(\omega, x) - \int_{|y| < R} \phi_i(\omega, y) dy|^2 dx = 0. \tag{2.8}
\]

Note that the functions \( \phi_i(\omega, \cdot) \) are uniquely defined up to a random variable.

2.1. Abstract results.

**Theorem 2.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be as above and let \( a \) be as in (2.5). Then, there exists a random variable \( A_h : \Omega \to \mathcal{M}_{d, \text{sym}} \) such that for any bounded open set \( D \subseteq \mathbb{R}^d \), \( f \in H^{-1}(D) \) and almost every \( \omega \in \Omega \), the solutions to the Dirichlet problem

\[
\begin{cases}
-\nabla \cdot \nabla u_e(x) = f(x) & \text{in } D \\
u_e = 0 & \text{on } \partial D,
\end{cases}
\]

converge weakly in \( H^1_0(D) \) to the (random) solution of

\[
\begin{cases}
-\nabla \cdot A_h(\omega) \nabla u_h(\omega, x) = f(x) & \text{in } D \\
u_h(\omega, x) = 0 & \text{on } \partial D.
\end{cases}
\]

Moreover,

\[
e_i \cdot A_h(\omega) e_j = \langle (e_i + \nabla \phi_i(0)) \cdot A(e_i + \nabla \phi_j(0)) \rangle_I,
\]

where each \( \phi_i(\omega, \cdot), i = 1, \cdots, d \) satisfies (2.8) with respect to the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

The term on the right-hand side of (2.9) admits a further reformulation in terms of the ergodic decomposition for the measure \( \mathbb{P} \). This is a standard result in the theory of asymptotically mean stationary processes (see, e.g., [15][Chapter 7, Theorem 7.4.1]):

**Lemma 2.2 (Ergodic decomposition).** There exist a family \( \{ \mathbb{P}_\xi \}_{\xi \in \Omega_0} \) of ergodic and stationary probability measures on \((\Omega, \mathcal{F})\) and a probability space \((\Omega_0, \mathcal{I}_0, \mathbb{P})\) such that the measure \( \mathbb{P} \) admits the disintegration

\[
\mathbb{P}(F) = \int_{\Omega_0} \left( \int_{\Omega} F(\omega) \mathbb{P}_\xi(d\omega) \right) \mathbb{P}(d\xi) \quad \forall F \in L^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.10}
\]

Furthermore, there exists a measurable map

\[
\Pi : (\Omega, \mathcal{I}) \to (\Omega_0, \mathcal{I}_0)
\]

such that for every \( F \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) the conditional expectation \( \mathbb{E}(F | \mathcal{I}) \) may be identified with a random variable in \( L^1(\Omega_0, \mathcal{I}_0, \mathbb{P}_\xi) \) via the relation

\[
\mathbb{E}(F | \mathcal{I})(\omega) = \int_{\Omega_0} F(\tilde{\omega}) P_{\Pi(\omega)}(d\tilde{\omega}) \quad \text{for } \mathbb{P}-\text{almost every } \omega \in \Omega. \tag{2.11}
\]

The next corollary relies on the previous decomposition of \( \mathbb{P} \) to show that the random homogenized matrix of Theorem 2.1 may be obtained by fixing the element \( \xi \in \Omega_0 \) and applying on the probability space \((\Omega, \mathcal{F}, \mathbb{P}_\xi)\) the standard homogenization results [14, 18, 21] for stationary and ergodic measures. For \( \xi \in \Omega_0 \) fixed, let indeed \( a_{h, \xi} \) be the deterministic homogenized matrix obtained by means of classical homogenization and defined as

\[
e_i \cdot a_{h, \xi} e_j = \int_{\Omega} (e_i + \nabla \phi_{\xi,i}(0, \omega)) \cdot a(\omega)(e_j + \nabla \phi_{\xi,j}(0, \omega)) \mathbb{P}_\xi(d\omega),
\]

with \( \phi_{\xi,i} \) the correctors solving (2.8) with respect to \((\Omega, \mathcal{F}, \mathbb{P}_\xi)\). Then:

**Corollary 2.3.** Let \( A_h \) be the homogenized matrix introduced in Theorem 2.1 and let \( \Pi \) be the projection map of Lemma 2.10. Then, for \( \mathbb{P}-\text{almost every } \xi \in \Omega_0 \) and all \( \omega \in \Pi^{-1}(\xi) \) we have

\[
A_h(\omega) = a_{h, \xi}. \tag{2.12}
\]

Therefore, for all \( B \in \mathbb{B}(\mathcal{M}_{d, \text{sym}}) \) we have

\[
\mathbb{P}(\{ \omega : A_h(\omega) \in B \}) = \mathbb{P}(\{ \xi : a_{h, \xi} \in B \}).
\]
As we show in the next section, this abstract result admits a more explicit formulation in the case of coefficients being generated by a stationary Gaussian field.

Remark 2.4. Convex combination of stationary measures. Lemma 2.2 is a generalization of the fact that stationarity is closed under convex combination and that ergodic measures are extremal points of any convex set. More precisely, let \( \{P_i\}_{i=1}^N \) be \( N < +\infty \) be distinct stationary and ergodic probability measures on \((\Omega, \mathcal{F})\), with \( \mathcal{F} \) countably generated. Let \( \mathbb{P} \) be the measure obtained as the convex combination:

\[
\mathbb{P} = \sum_{k=1}^N \alpha_k P_k, \quad \sum_{k=1}^N \alpha_k = 1, \quad 0 \leq \alpha_k \leq 1.
\]

(2.13)

It is easy to check that \( \mathbb{P} \) is a stationary probability measure. Moreover, \( \mathbb{P} \) is ergodic if and only if it is an extremal point, i.e. there exists \( \alpha_k = 1 \) for some \( k \in \{1, \cdots, N\} \).

We argue the only non trivial implication of the previous statement: Let us assume that \( \mathbb{P} \) is ergodic. We show that if there exists \( i \in \{1, \cdots, N\} \) such that \( \alpha_i \in (0,1) \), then the ergodicity property is contradicted. Indeed, by the last two conditions in (2.13), the previous assumption implies that there exists another \( j \in \{1, \cdots, N\} \), \( j \neq i \), such that \( \alpha_j \in (0,1) \). Moreover, since all the measures are distinct, we may find a set \( B \in \mathcal{F} \) such that \( \mathbb{P}_i(B) > 0 \) and \( \mathbb{P}_j(B) \neq \mathbb{P}_j(B) \). By Birkhoff’s theorem and the assumption on the ergodicity of each measure \( P_k \), it follows that the set

\[
A = \{\omega \in \Omega : \lim_{R \uparrow +\infty} \int_{|x| < R} T_x 1_B(\omega) = \mathbb{P}_i(B)\}
\]

satisfies \( \mathbb{P}_i(A) = 1, \mathbb{P}_j(A) = 0 \). Note that the set \( A \in \mathcal{I} \). Hence, we have by (2.13) that

\[
\mathbb{P}(A) = \alpha_i + \sum_{k=1, \, k \neq i, j}^N \alpha_k P_k(A).
\]

Since we assumed that \( \alpha_i, \alpha_j \in (0,1) \), \( P_k \) are probability measures and \( \sum_{k=1}^N \alpha_k = 1 \), we infer that \( \mathbb{P}(A) \in (0,1) \). This yields a contradiction.

Similarly, we note that since \( \mathcal{F} \) is countably generated, i.e. \( \mathcal{F} = \sigma(\{B_n\}_{n \in \mathbb{N}}) \), also the sets

\[
C_i = \bigcap_{n \in \mathbb{N}} \{\omega \in \Omega : \lim_{R \uparrow +\infty} \int_{|x| < R} T_x 1_{B_n}(\omega) = \mathbb{P}_i(B_n)\}
\]

satisfy \( \mathbb{P}_i(C_j) = \delta_{ij} \) for all \( i, j = 1, \cdots, N \). In particular, \( \{C_i\}_{i=1}^N \) provide a \( \mathbb{P} \)-essential partition for \( \Omega \) in each one of which the limit of the spatial averages are given by integration in \( \mathbb{P}_i \). Hence, in this easy case the set \( \Omega_0 \) of Lemma 2.2 is just \( \Omega_0 = \{1, \cdots, N\} \), the family of ergodic probabilities is \( \{P_i\}_{i=1}^N \) and \( \tilde{\mathbb{P}} \) is the measure on \( \mathcal{I} = \sigma(\{1, \cdots, N\}) \) (uniquely) defined by \( \tilde{\mathbb{P}}(i) = \alpha_i \) for \( i = 1, \cdots, N \).

2.2. Application to stationary Gaussian fields. The results of this section rely on [24][Theorem 5 and Theorem 6]. For \( d \geq 2, \, n \geq 1 \), let \( X \) be a stationary \( \mathbb{R}^n \)-valued Gaussian field on \( \mathbb{R}^d \) having continuous trajectories, i.e. the space of trajectories is given by \( \Omega = C^0(\mathbb{R}^d, \mathbb{R}^n) \) with \( \mathcal{F} \) the \( \sigma \)-algebra of the cylindrical sets. We assume that the group of transformations \( \{\tau_x\}_{x \in \mathbb{R}^d} \) acts on each trajectory in \( \Omega \) as \( \tau_x X(\cdot) = X(\cdot + x) \). With this choice of \( \mathcal{F} \) and \( \Omega \) condition (2.3) is satisfied.

Let \( X \) be centered. We recall that for a given Gaussian field \( X \), the autocorrelation matrix is given by

\[
C(x) := \langle X(x) \otimes X(0) \rangle \in \mathbb{R}^{n \times n}, \quad x \in \mathbb{R}^d.
\]

Note that by stationarity we have that for all \( x, y \in \mathbb{R}^d \) we have

\[
\langle X(x) \otimes X(y) \rangle = C(x - y), \quad C(x) = C^t(-x).
\]

(2.14)

For \( C \in C^0(\mathbb{R}^d) \), by Bochner’s theorem [26][Chapter XI, Section 14] we may write

\[
C(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \hat{C}(\xi) \, d\xi,
\]
where \( \hat{C}(\xi) \), usually known as spectral measure, is a positive definite \( \mathbb{C}^{n \times n} \)-valued measure on \( \mathbb{R}^d \). We remark that by (2.14) it is easy to check that \( \hat{C} \) satisfies

\[
\hat{C}(\xi) = \hat{C}(\xi)^*, \quad \hat{C}(\xi) = \hat{C}(-\xi)^t.
\]  

(2.15)

In the case of stationary Gaussian field, the ergodicity of the process \( X \) under the translation group \( \{\tau_x\}_{x \in \mathbb{R}^d} \) is equivalent to requiring that spectral measure \( \hat{C} \) does not have an atomic part [7, 11]. This and (2.15) yield that for any stationary Gaussian field, the non-ergodic behaviour is related to the presence in \( \hat{C} \) of linear combinations of the form

\[
\alpha_0 \delta_0 + \sum_{i=1}^N \alpha_i \delta_{-\omega_i} + \alpha_i^t \delta_{\omega_i},
\]

(2.16)

for a positive-definite \( \alpha_0 \in \mathbb{R}^{n \times n} \), hermitian matrices \( \{\alpha_i\}_{i=1}^N \subseteq \mathbb{C}^{n \times n} \) and \( \{\omega_i\}_{i=1}^N \subseteq \mathbb{R}^d \). We remark that the terms in the sum above may also be infinite, i.e. \( N = +\infty \), but from now on we restrict ourselves to the case \( N \in \mathbb{N} \).

The special structure of Gaussian fields allows to extract a more explicit formulation for the \( \{\omega\} \) values \( \Omega = \omega \) where \( \hat{C} \). Corollary 2.6. Let

\[
\{\omega_i\}_{i=1}^N \subseteq \mathbb{C}^{n \times n}, \quad \{\omega_i\}_{i=1}^N \subseteq \mathbb{R}^d.
\]

We remark that by (2.14) it is easy to check that \( \hat{C} \) does not have an atomic part [7, 11]. This and (2.15) yield that for any stationary Gaussian field, the non-ergodic behaviour is related to the presence in \( \hat{C} \) of linear combinations of the form

\[
\alpha_0 \delta_0 + \sum_{i=1}^N \alpha_i \delta_{-\omega_i} + \alpha_i^t \delta_{\omega_i},
\]

(2.16)

for a positive-definite \( \alpha_0 \in \mathbb{R}^{n \times n} \), hermitian matrices \( \{\alpha_i\}_{i=1}^N \subseteq \mathbb{C}^{n \times n} \) and \( \{\omega_i\}_{i=1}^N \subseteq \mathbb{R}^d \). We remark that the terms in the sum above may also be infinite, i.e. \( N = +\infty \), but from now on we restrict ourselves to the case \( N \in \mathbb{N} \).

The special structure of Gaussian fields allows to extract a more explicit formulation for the \( \sigma \)-algebra \( I \) of the invariant sets. As we show in the proof of the next statement, the presence of the non-zero atoms in the spectral measure \( \hat{C} \) corresponds to cosine terms in the process \( X \). This yields that the large-scale behaviour of \( X \) and the \( \sigma \)-algebra \( I \) of the invariant sets crucially depend on possible resonances between the frequencies of oscillations. To this purpose, for any collection of values \( \Omega = \{\omega_i\}_{i=1}^N \subseteq \mathbb{R}^d \), with \( 1 \leq N < +\infty \), in (2.16), we introduce the subset of \( \mathbb{Z}^N \) defined by

\[
\mathcal{R}_\Omega := \{k \in \mathbb{Z}^N : \sum_{i=1}^N k_i \omega_i = 0\}.
\]

Remark 2.5. If \( \mathcal{R}_\Omega \) is non-trivial, then we may always write

\[
\mathcal{R}_\Omega = \text{Span}_\mathbb{Z}(v^1, \cdots, v^r),
\]

(2.17)

for \( 1 \leq r \leq N - 1 \) and with \( \{v^1, \cdots, v^r\} \subseteq \mathbb{Z}^N \setminus \{0\} \) satisfying the condition of linear integer-independence

\[
\sum_{j=1}^r m_j v^j = 0 \iff m_j = 0, \quad \text{for all } j = 1, \cdots, r.
\]

This results follows from the classical theory of linear diophantine equations: In fact, up to a permutation of the elements in \( \Omega \), we may always assume that there exists an index \( 1 \leq M \leq N \) such that the values \( \omega_1, \cdots, \omega_M \) are all rationally incommensurable and, if \( M < N \), that for all \( j = M + 1, \cdots, N \) we have \( \omega_j = \sum_{i=1}^M q_i^j \omega_i \) for a unique \( M \)-tuple \( \{q_1^j, \cdots, q_M^j\} \subseteq \mathbb{Q} \). By using this decomposition, solving \( \sum_{i=1}^N k_i \omega_i = 0 \) for \( k \in \mathbb{Z}^N \) reduces to solving the system of \( M \) equations with rational coefficients

\[
k_i - \sum_{j=M+1}^N q_j^i k_j = 0, \quad i = 1, \cdots, M,
\]

for the \( N \) integer variables \( k_1, \cdots, k_N \). This system has at most \( N - M \) linearly integer-independent solutions \( v^1, \cdots, v^{N-M} \in \mathbb{Z}^N \) [23][Chapter 4, Corollary 4.1c and formula (6)]. Since \( N - M \leq N - 1 \), identity (2.17) is obtained.

For \( F : \mathbb{R}^n \rightarrow M_{d,\text{sym}} \) continuous and pointwise elliptic in the sense of (2.4), we define

\[
A(X) := F \circ X(0), \quad a(X, x) := F \circ X(x).
\]

(2.18)

In the sake of a leaner notation, we state the following corollary in the special case \( n = 1 \), i.e. when the Gaussian field \( X \) is real-valued, and comment afterwards on the generalization of this result to \( n \geq 1 \).

Corollary 2.6. Let \( X \) be a stationary, centered, Gaussian field having continuous correlation function \( C \) and spectral measure \( \hat{C} \) with an atomic part given by (2.16) for \( N < +\infty \). Let \( a \) be defined as in (2.18). Then

(a) If \( \mathcal{R}_\Omega = \{0\} \), i.e. the values \( \{\omega_i\}_{i=1}^N \) are rationally incommensurable, then for every \( B \in \mathbb{B}(M_{d,\text{sym}}) \) we have

\[
\mathbb{P}(\{\omega : A_\mathbf{h}(\omega) \in B\}) = \mathbb{P}(\{(x, r) \in \mathbb{R} \times (\mathbb{R}_+)^N : a_{\mathbf{h}}(x, r) \in B\}),
\]

where \( C(\xi) \), usually known as spectral measure, is a positive definite \( \mathbb{C}^{n \times n} \)-valued measure on \( \mathbb{R}^d \). We remark that by (2.14) it is easy to check that \( C \) satisfies

\[
C(\xi) = C(\xi)^*, \quad C(\xi) = C(-\xi)^t.
\]

(2.15)
with
\[
\hat{P}(dx,dr) = \frac{e^{-\frac{|x|^2}{2\alpha^2}}}{(2\pi\alpha^2)^{\frac{d}{2}}} \prod_{i=1}^{N} \frac{e^{-\frac{r_i^2}{\alpha_i^2}}}{\alpha_i^2} \, dx \, dr_1 \cdots dr_N,
\]
where \(\{\alpha_i\}_{i=0}^{N}\) are the amplitudes in (2.16). In other words, \(\hat{P}\) is the probability measure associated to an independent Gaussian random variable and \(N\) independent Rayleigh random variables.

(b) If, otherwise, \(r \geq 1\) in (2.17), then for every \(B \in \mathcal{B}(\mathcal{M}_{d,sym})\) we have as well
\[
\mathbb{P}(\{\omega : A_h(\omega) \in B\}) = \hat{P}(\{(x,r,\eta) \in \mathbb{R} \times (\mathbb{R}_+)^N \times \mathbb{R}^r : a_n(x,r,\eta) \in B\}).
\]
Here,
\[
\hat{P}(dx,dr,d\eta) = \hat{P}_1(dx,dr)\hat{P}_2(d\eta)
\]
with \(\hat{P}_1\) as in case (a) and \(\hat{P}_2\) the probability associated to the vector \(\eta = \{\eta_1, \cdots, \eta_r\}\) obtained for each \(j = 1, \cdots, r\) as
\[
\eta_j = \sum_{i=1}^{N} v^j_i \phi_i \mod(2\pi),
\]
for \(v^j\) as in (2.17) and \(\{\phi_i\}_{i=1}^{N}\) independent random variables which are uniformly distributed on \([0, 2\pi)\).

The analogue of the previous result holds also in the higher-dimensional case \(n \geq 1\), provided the random variables \(x, r_1, \cdots, r_N\) are \(\mathbb{R}^{n \times n}\)-valued with each component independent and distributed as in the case \(n = 1\) above.

3. Proofs

Proof of Theorem 2.1. We resort to the proof of Theorem 2.1 in the ergodic case \([13, 14, 18]\) and show that only few modifications are needed in order to adapt it to our setting.

Also in this case we rely on the construction of the sub-linear corrector \(\phi = \{\phi_i\}_{i=1}^{d}\) satisfying (2.8). More precisely, for every \(i = 1, \cdots, d\), we construct a random variable \(\chi_i \in [L^2(\Omega, \mathcal{F}, \mathbb{P})]^d\) which satisfies
\[
\langle \chi_i | \mathcal{I} \rangle = \langle \chi_i \rangle = 0 \tag{3.19}
\]
and such that
\[
\nabla \phi_i(\omega, x) = \chi_i(\tau_x \omega), \tag{3.20}
\]
with \(\phi_i\) solving (2.8) for \(\mathbb{P}\) almost every \(\omega \in \Omega\).

To prove the existence of \(\chi\) as above, we modify the argument of [13] and enumerate below only the (few) steps which require a non-trivial adaptation to our setting. Let \(i = 1, \cdots, d\) be fixed and let us write \(\phi\) instead of \(\phi_i\). Moreover, since no ambiguity on the measure \(\mathbb{P}\) considered occurs, we write \(L^p(\Omega)\) instead of \(L^p(\Omega, \mathcal{F}, \mathbb{P})\). For any \(F \in L^2(\Omega)\) and \(x \in \mathbb{R}^d\) let \(T_x F := F \circ \tau_x\). Thanks to (2.2) and (2.3), the group of transformations \(\{T_x\}_{x \in \mathbb{R}^d}\) provides a unitary and strongly continuous group of operators on \(L^2(\Omega)\). We may thus denote by \(D_j, j = 1, \cdots, d\), the infinitesimal generators of \(T_{x-e_j}\) [22][Subsection VIII.4], namely
\[
\lim_{h \to 0^+} \frac{T_{he_j} - I}{h} = D_j \ \text{in} \ L^2(\Omega).
\]
We denote by \(D := \bigcap_{j=1}^{d} D(D_j) \subseteq L^2(\Omega)\) the domain of the operator \(D := (D_1, \cdots, D_d)\) and note that, again by (2.3), this set is dense in \(L^2(\Omega)\).

We set
\[
U := \{\xi \in L^2(\Omega) : T_{x} \xi = \xi \ \forall x \in \mathbb{R}^d\},
\]
\[
V(\Omega) := \{(D \xi : \xi \in D)\}^{L^2(\Omega)}.
\]
Then, since
\[ U = \{ \xi \in L^2(\Omega) : D\xi = 0 \text{ in } L^2(\Omega) \}, \]
it follows that
\[ V(\Omega) \subseteq (U^\perp)^d. \]
and that any element \( \Psi \in V(\Omega) \) satisfies
\[ \langle \Psi \rangle = \langle \Psi | I \rangle = 0. \]
Therefore, we define \( \Psi \in V(\Omega) \) as the Lax-Milgram solution of
\[ \langle \Psi \cdot A\chi \rangle = \langle \Psi \cdot A\epsilon_i \rangle, \quad \forall \Psi \in V(\Omega), \quad (3.21) \]
with \( A \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \) as in (2.5).
With this definition of \( \chi \in V(\Omega) \) as the solution of (3.21), the same arguments used in [14] yield (3.20) and the first line of (2.8). To conclude the proof of (2.8), we first observe that by \( \chi \in L^2(\Omega) \) and the first identity in (3.20), Neumann’s ergodic theorem [19][Theorem 1.4] yields also that for almost every \( \omega \in \Omega \)
\[ \lim \limits_{R \uparrow +\infty} \langle | \int_{|x|<R} \nabla \phi_i(\omega, x)^2 | \rangle = 0. \]
From this identity we may argue exactly as in [13][Proof of Corollary 1] and obtain also the last sub-linearity property in (2.8).
Equipped with the correctors \( \{ \phi_i \}_{i=1}^d \) as above, we argue as in the ergodic case to show Theorem 2.1:
By (3.20) and Birkhoff’s ergodic theorem we have indeed that for almost every \( \omega \in \Omega \) and every \( R > 0 \)
\[ \lim \limits_{\varepsilon \downarrow 0^+} \int_{|x|<R} | \nabla \phi_i(\omega, x, \varepsilon) |^2 = \langle | |^2 | I \rangle. \]
Furthermore, another application of Birkhoff’s ergodic theorem together with a standard separability argument implies that for almost every \( \omega \in \Omega \) and every \( \rho \in C_0^\infty(\mathbb{R}^d) \)
\[ \lim \limits_{\varepsilon \downarrow 0^+} \int \rho(x) \nabla \phi_i(\omega, x, \varepsilon) = \langle \chi_i | I \rangle \int \rho(x) dx \quad (3.19). \]
These two limits yield for the whole family \( \varepsilon \downarrow 0^+ \)
\[ \nabla \phi_i(\omega, x, \varepsilon) \rightarrow 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^d). \]
Hence, for any bounded domain \( D \subseteq B_R \), for some \( R > 0 \), the functions
\[ w^\varepsilon_i(\omega, x) := x_i + \varepsilon (\phi_i(\omega, x, \varepsilon) - \int_{|x|<R} \phi_i(\omega, y, \varepsilon) dy) \]
satisfy for almost every \( \omega \in \Omega \)
\[ w^\varepsilon_i(\omega, \cdot) \rightharpoonup x_i \quad \text{in } H^1(D). \quad (3.22) \]
By (3.20) and the stationarity of \( a \), we may argue similarly to obtain that for \( \mathbb{P} \)-almost every \( \omega \in \Omega \)
\[ e_j \cdot a(\omega, x, \varepsilon) \nabla w^\varepsilon_i(\omega) \rightharpoonup A_{h,i,j}(\omega) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^d). \quad (3.23) \]
We remark that the identification of the above limit with \( A_{h,i,j} \) as in (2.9) follows by
\[ \langle \chi_j \cdot A(\epsilon_i + \chi_i) | I \rangle = 0. \]
This identity is implied in turn by Birkhoff’s ergodic theorem, (2.8) and the bounds (2.4) for \( a \) after taking the limit \( R \uparrow +\infty \) in the estimate
\[ \left| \int_{|x|<R} \nabla \phi_j \cdot a(\epsilon_i + \phi_i) \right| \leq CR^{-1} \left( \int_{|x|<2R} | \phi_j - \int_{|x|<2R} \phi_j |^2 \right)^{\frac{1}{2}} \left( 1 + \int |\chi_i|^2 \right)^{\frac{1}{2}}. \]
Here, the constant \( C = C(d) < +\infty \). This estimate in turn easily follows by testing equation (2.8) for \( \phi_i \) with \( \eta_R(\phi_j - \int_{|x|<2R} \phi_j) \), where \( \eta_R \) is a cut-off function for \( \{|x|<R\} \) in \( \{|x|<2R\} \).
Convergences (3.22) and (3.23) allow us to apply Tartar’s Div-Curl lemma [25][Chapter 7, Lemma 7.2] as in the ergodic case and conclude the proof of Theorem 2.1. □
Proof of Lemma 2.2. We begin by constructing the family \( \{ \mathbb{P}_x \}_{x \in \mathbb{R}^d} \) of stationary and ergodic measures on \( (\Omega, \mathcal{F}) \): Let \( \mathcal{S} \) be a countable collection of sets generating \( \mathcal{F} \). By Birkhoff’s ergodic theorem, for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), we may define the probability measure \( \mathbb{P}_\omega \) on \( (\Omega, \mathcal{F}) \) as

\[
\mathbb{P}_\omega(B) := \lim_{R \to +\infty} \frac{1}{2R} \int_{|x| < R} 1_B(\tau_x \omega) \, dx, \quad B \in \mathcal{S}. \tag{3.24}
\]

Since each probability measure is uniquely defined by its value on the generating set \( \mathcal{S} \), it is immediate to check that \( \mathbb{P}_\omega \) is stationary. In addition, since if \( I \in \mathcal{I} \) then the limit above exists for each \( \omega \in \Omega \) and coincides with \( 1_I(\omega) \), from definition (3.24) it follows that

\[
\mathbb{P}_\omega(I) = 1_I(\omega) \in \{0, 1\}. \tag{3.25}
\]

Equivalently, \( \mathbb{P}_\omega \) is ergodic with respect to \( \{ \tau_x \}_{x \in \mathbb{R}^d} \).

Let \( \Sigma_0 \in \mathcal{F} \) be the (\( \mathbb{P} \)-zero measure set) of elements \( \omega \in \Omega \) for which \( \mathbb{P}_\omega \) defined in (3.24) does not exist. We introduce the equivalence relation on \( \Omega \)

\[
\omega \sim \tilde{\omega} \iff \mathbb{P}_\omega = \mathbb{P}_{\tilde{\omega}} \quad \text{or} \quad \omega, \tilde{\omega} \in \Sigma_0, \tag{3.26}
\]

and define the quotient space \( \Omega_0 := \Omega / \sim \) and the projection operator

\[
\Pi : \Omega \to \Omega_0, \quad \omega \mapsto \xi = \{ \tilde{\omega} \in \Omega : \tilde{\omega} \sim \omega \}. \tag{3.27}
\]

Hence, thanks to (3.24), \( \{ \mathbb{P}_\omega \}_{\omega \in \Omega_0} = \{ \mathbb{P}_x \}_{x \in \mathbb{R}^d} \) is a family of ergodic and stationary probability measures on \( (\Omega, \mathcal{F}) \). Rigorously, the probability \( P_x \) corresponding to \( \xi = \Pi(\Sigma_0) \in \Omega_0 \) is not well-defined. However, since we take as measure \( \mathbb{P} \) the push-forward \( \mathbb{P} \circ \Pi^{-1} \), it follows that \( \mathbb{P}(\Pi(\Sigma_0)) = 0 \) and thus that in the decomposition (2.10) the measure \( \mathbb{P}(\Pi(\Sigma_0)) \) is negligible.

We now define the \( \sigma \)-algebra \( \mathcal{I}_0 \) as the image of \( \mathcal{I} \) under \( \Pi \), i.e.

\[
\mathcal{I}_0 := \{ \Pi(I) : I \in \mathcal{I} \}, \quad \Pi(I) := \{ \Pi(\omega) : \omega \in I \} \subseteq \Omega_0,
\]

and argue that the above definition is well-posed and that \( \mathcal{I}_0 \) is a \( \sigma \)-algebra isomorphic to \( \mathcal{I} \) in the sense that

\[
I = \Pi^{-1} \circ \Pi(I),
\]

for every \( I \in \mathcal{I} \). To do so, it suffices to observe that for every \( I \in \mathcal{I} \) and \( \xi \in \Omega_0 \)

\[
\Pi^{-1}(\xi) \subseteq I \iff \Pi^{-1}(\xi) \cap I = \emptyset.
\]

The \( \Rightarrow \) implication is trivial. For the \( \Leftarrow \) implication we observe that whenever \( \omega \in \Pi^{-1}(\xi) \cap I \), then by (3.26) and (3.25) for every \( \tilde{\omega} \in \Pi^{-1}(\xi) \) we have that \( 1_I(\tilde{\omega}) = 1_I(\omega) = 1 \).

From the previous argument and the fact that \( \mathcal{I} \subseteq \mathcal{F} \), it follows that the map \( \Pi \) is measurable from \( (\Omega, \mathcal{I}) \) to \( (\Omega_0, \mathcal{I}_0) \), as well as from \( (\Omega, \mathcal{F}) \) to \( (\Omega_0, \mathcal{I}_0) \). We define the probability measure \( \mathbb{P} \) on \( (\Omega_0, \mathcal{I}_0) \) as the push-forward of \( \mathbb{P} \) under \( \Pi \), i.e.

\[
\mathbb{P} = \mathbb{P} \circ \Pi^{-1}. \tag{3.28}
\]

With these definitions of \( \{ \mathbb{P}_x \}_{x \in \mathbb{R}^d} \) and \( (\Omega_0, \mathcal{I}_0, \mathbb{P}) \), it remains to establish (2.10), (2.11). We begin with (2.11) and use a standard approximation argument: Let \( \mathcal{F} = \sigma(\mathcal{S}) \). For any \( A \in \mathcal{F} \), by Birkhoff’s ergodic theorem we may construct for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) a probability measure \( \mathbb{P}_\omega \) on \( (\Omega, \mathcal{F}) \) such that for all \( B \in \mathcal{S} \cup \{ A \} \) it holds

\[
\mathbb{P}_\omega(B) = \lim_{R \to +\infty} \frac{1}{2R} \int_{|x| < R} T_x 1_B(\omega) \, dx = \langle 1_B \mid I \rangle.
\]

Since \( \mathbb{P}_\omega \) and \( \mathbb{P}_\omega \) coincide on the set of generators \( \mathcal{S} \), it follows by uniqueness that \( \mathbb{P}_\omega(A) = \langle A \mid I \rangle \) for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). Therefore,

\[
\langle 1_A \rangle = \int_{\Omega} \langle A \mid I \rangle \mathbb{P}(d\omega) = \int_{\Omega} \mathbb{P}_\omega(A) \mathbb{P}(d\omega).
\]

By arguing similarly and using (2.3), for every \( F \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) we have

\[
\langle F \rangle = \int_{\Omega} \int_{\Omega} F(\tilde{\omega}) \mathbb{P}(d\omega) \mathbb{P}(d\tilde{\omega}).
\]
We now appeal to the definitions (3.27) and (3.28) to conclude that

\[ \langle F \rangle = \int_{\Omega} \int_{\Omega} F(\omega) \mathbb{P}_\xi(d\omega) \tilde{\mathbb{P}}(d\xi), \]

i.e. formula (2.10). The proof of this lemma is complete. \qed

**Proof of Corollary 2.3.** By (2.9) of Theorem 2.1 and (2.11) of Lemma 2.2, we may rewrite for \( \mathbb{P} \)-almost every \( \xi \in \Omega_0 \) and \( \omega \in \Omega \) with \( \Pi(\omega) = \xi \)

\[ e_i \cdot A_h(\omega)e_j = e_i \cdot A_h(\xi)e_j = \int_{\Omega_0} (e_i + \nabla \phi_i(\omega,0)) \cdot a(0)(e_j + \nabla \phi_j(\omega,0)) \mathbb{P}_\xi(d\omega). \]  

(3.29)

It thus remains to show that in the right-hand side above we may substitute the random variables \( \nabla \phi_i, \nabla \phi_j \) with \( \nabla \phi \xi;i, \nabla \phi \xi;j \). To do so, we resort to the construction of \( \phi \) obtained in the proof of Theorem 2.1 via the random variable \( \chi \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \) (see (3.20)). We also remark that the same holds for \( \phi \xi \), where \( \nabla \phi \xi = \chi \xi \) with \( \chi \xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}_\xi) \). This either follows directly from the homogenization results for ergodic measures [14][Chapter 6, Section 6.1], or by the exact same argument used in the proof of Theorem 2.1 for \( \phi \).

We fix an index \( i = 1, \ldots, d \) and drop it in the notation for \( \phi_i \). On the one hand, by (2.8), for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) we have that \( \phi(\omega, x) \) solves (2.8). We use (2.10) of Lemma 2.2 to infer that also for \( \hat{\mathbb{P}} \)-almost every \( \xi \in \Omega_0 \) and \( \mathbb{P}_\xi \)-almost every \( \omega \in \Omega \) the functions \( \phi(\omega, \cdot) \) satisfy (2.8). On the other hand, by Lemma 2.2 for \( \hat{\mathbb{P}} \)-almost every \( \xi \in \Omega_0 \) the probability measure \( \mathbb{P}_\xi \) in \( (\Omega, \mathcal{F}) \) is stationary and ergodic with respect to the translations \( \{\tau_x\}_{x \in \mathbb{R}^d} \). We thus appeal to the standard results in homogenization [13, 14, 18], to infer that there exists a random field \( \phi \xi \), having stationary gradient, solving (2.8) for \( \mathbb{P}_\xi \)-almost every \( \omega \in \Omega \). Therefore, for \( \mathbb{P} \)-almost every \( \xi \in \Omega_0 \) and \( \mathbb{P}_\xi \)-almost every \( \omega \in \Omega \) we have that the difference \( \phi(\omega, \cdot) - \phi \xi(\omega, \cdot) \) satisfies

\[ -\nabla \cdot a(\omega, x) \nabla (\phi(\omega, x) - \phi \xi(\omega, x)) = 0 \quad \text{in} \; \mathbb{R}^d. \]

This, together with the sub-linearity condition of (2.8) for both \( \phi(\omega, \cdot) \) and \( \phi \xi(\omega, \cdot) \) implies

\[ \nabla \phi(\omega, \cdot) = \nabla \phi \xi(\omega, \cdot) \quad \text{in} \; L^2(\mathbb{R}^d, \mathbb{R}^d). \]

(3.30)

We now appeal to (3.20) for both the gradients \( \nabla \phi, \nabla \phi \xi \) to write

\[ \nabla \phi(\omega, x) = \chi(\tau_x \omega), \quad \nabla \phi \xi(\omega, x) = \chi \xi(\tau_x \omega) \]

for \( \chi \in [L^2(\Omega, \mathcal{F}, \mathbb{P})]^d \) and \( \chi \xi \in [L^2(\Omega, \mathcal{F}, \mathbb{P}_\xi)]^d \). Note that again by (2.10), we have that for \( \hat{\mathbb{P}} \)-almost every \( \xi \in \Omega_0 \) the random variable \( \chi \in [L^2(\Omega, \mathcal{F}, \mathbb{P}_\xi)]^d \). This, the above identities and (3.30) imply that for all \( \rho \in C_0^\infty(\mathbb{R}^d) \) and \( \psi \in [L^2(\Omega, \mathcal{F}, \mathbb{P}_\xi)]^d \)

\[ \int_{\Omega} \psi(\omega) \cdot (\int_{\mathbb{R}^d} \rho(x) \chi(\tau_x \omega) \, dx) \mathbb{P}_\xi(d\omega) = \int_{\Omega} \psi(\omega) \cdot (\int_{\mathbb{R}^d} \rho(x) \chi \xi(\tau_x \omega) \, dx) \mathbb{P}_\xi(d\omega). \]

By stationarity of the measure \( \mathbb{P}_\xi \) this may be rewritten as

\[ \int_{\Omega} (\int_{\mathbb{R}^d} \rho(x) \psi(\tau_{-x} \omega) \, dx) \cdot \chi(\omega) \mathbb{P}_\xi(d\omega) = \int_{\Omega} (\int_{\mathbb{R}^d} \rho(x) \psi(\tau_{-x} \omega) \, dx) \cdot \chi \xi(\omega) \mathbb{P}_\xi(d\omega). \]

We now choose a sequence \( \hat{\phi}_\varepsilon = \varepsilon^{-d} \hat{\phi}(\hat{\tau}_\varepsilon) \) with \( \hat{\phi} \in C_0^\infty(B_1) \) a mollifier and appeal to (2.3) to conclude that for every \( \psi \in L^2(\Omega, \mathcal{F}, \mathbb{P}_\xi) \)

\[ \int_{\Omega} \psi(\omega) \cdot \chi(\omega) \mathbb{P}_\xi(d\omega) = \int_{\Omega} \psi(\omega) \cdot \chi \xi(\omega) \mathbb{P}_\xi(d\omega). \]

In particular, by applying this identity twice, first with \( \Psi = a\chi \xi \) and secondly with \( \Psi = a\chi \), we get that the right-hand side of (3.29) equals to the right-hand side of (2.12) in Corollary 2.3. \qed

**Proof of Corollary 2.6.** Let us split the spectral measure into the two components

\[ \hat{C}(\xi) = \hat{C}_c(\xi) + \hat{C}_a(\xi), \]

with the (positive) measure \( \hat{C}_a \) being the purely atomic part (2.16). From this decomposition it follows that also \( X \) may be decomposed into the two independent processes \( X = X_c + X_a \), having spectral
measure $\hat{C}_c$ and $\hat{C}_a$, respectively. Moreover, the process $X_c$ is ergodic since $\hat{C}_c$ does not contain atoms [7, 11]. By (2.16), the correlation function of the process $X_a$ may be written as

$$C(x) = C_0 + \sum_{j=1}^N \alpha_j \cos(\omega_j \cdot x),$$

which corresponds to the centred stationary Gaussian process

$$X_a(x) = x_0 + \sum_{j=1}^N (\zeta_j \cos(\omega_j \cdot x) + \zeta_j' \sin(\omega_j \cdot x)) = x_0 + \sum_{j=1}^N \Re((\zeta_j + i\zeta_j')e^{i\omega_j \cdot x}),$$

for the independent random variables $x_0 \sim N(0, \alpha_0)$, $\zeta_j, \zeta_j' \sim N(0, \alpha_j)$ for all $j = 1, \cdots, N$. In particular, we remark that if we set $\zeta_j + i\zeta_j' = R_j e^{i\phi_j}$, then the above process may be also rewritten as

$$X_a(x) = x_0 + \sum_{j=1}^N R_j \cos(\omega_j \cdot x + \phi_j)$$

(3.31)

where all $R_j, \phi_j$ are independent and $R_j \sim Ray(\alpha_j)$, $\phi_j \in U([0, 2\pi))$.

By relying on (3.31) and the decomposition for $X$, we appeal to [24] [Theorem 5 and Theorem 6] to identify the $\sigma$-algebra of the invariant sets $I$ in terms of the random variables in (3.31). This, together with Corollary 2.3, concludes the proof. \square

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