PERIODS AND IGUSA ZETA FUNCTIONS

PRAKASH BELKALE AND PATRICK BROSNAN

ABSTRACT. We show that the coefficients in the Laurent series of Igusa Zeta functions $I(s) = \int_C f^s \omega$ are periods. This will be used to show in a subsequent paper (by P. Brosnan) that certain numbers occurring in Feynman amplitudes (upto gamma factors) are periods.

INTRODUCTION

In their paper [8], Kontsevich and Zagier give an elementary definition of a period integral as an absolutely convergent integral of a rational function over a subset of $\mathbb{R}^n$ defined by polynomial inequalities and equalities. They then show that some of the most important quantities in mathematics are periods, and sketch a proof that their notion of a period agrees with the more elaborate notion that algebraic geometers have studied since Riemann and Weierstraß. The last chapter links periods to the “framed motives” studied by A. Goncharov and proposes a structure of a torsor on a certain set of framed motives. The paper is full of interesting examples, however, its main purpose seems to be to justify the following:

Philosophical Principle 0.1. Whenever you meet a new number, and have decided (or convinced yourself) that it is transcendental, try to figure out whether it is a period.

A very interesting class of numbers arises naturally in quantum field theory which we want to prove to be periods. Namely, if $I(D)$ is a Feynman amplitude coming from a scalar field theory corresponding to a Feynman integral with all parameters in $\mathbb{Q}$, then $I(D) = G(D)J(D)$ where $G(D)$ is a relatively simple gamma factor and $J(D)$ is a meromorphic function such that the coefficients in the Laurent series expansion of $J(D)$ at $D = D_0$ are periods for $D_0$ any integer. We remark that this confirms (albeit in a very weak sense) the fact noticed by Kreimer and Broadhurst that the principal parts of the Laurent series for primitive diagrams often have coefficients which are multiple zeta values.

Once dimensional regularization is understood precisely, the proof of the period-icity of these numbers follows from an analogous result for Igusa zeta functions which may be interesting in its own right and which can be explained directly in terms of pure mathematics. We will approach the
purely mathematical side in this paper. A subsequent paper by one of us (Brosnan) will explain the physics of dimensional regularization and on how the theorem on Igusa Zeta functions ties up with regularization.

To explain this result on Igusa Zeta function, let \( \Delta_n \subset \mathbb{R}^{n+1} \) be the standard \( n \)-simplex equipped with the \( n \)-form

\[
\omega = dx_1 \wedge \cdots \wedge dx_n.
\]

Let \( f \in \mathbb{R}[x_0, \ldots, x_n] \) be a polynomial function which is non-negative on \( \Delta \). Then, according to results of Atiyah [1] and Bernstein and Gelfand [2], the function

\[
I(s) = \int_{\Delta_n} f^s \omega
\]

is meromorphic on the complex \( s \)-plane with isolated singularities. These functions are called \textit{Igusa zeta functions}. Our main theorem concerning them is the following:

\textbf{Theorem 0.2.} Suppose that \( f \in \mathbb{Q}[x_0, \ldots, x_n] \) is a polynomial with rational coefficients and let \( s_0 \) be an integer. Let

\[
I(s) = \sum_{i \geq N} a_i(s - s_0)^i
\]

be the Laurent series expansion of \( I(s) \) at \( s_0 \). Then the \( a_i \) are periods.

The above result suffices to show that most of the numbers investigated by Kreimer and Broadhurst are, in fact, periods. However, it will be convenient to prove a version of this result which is more general in the following two senses: (a) the simplex \( \Delta_n \) can be taken to be a general semi-algebraic set defined over \( \mathbb{Q} \), and (b) the function \( f \) can be taken to lie in the function field \( \mathbb{Q}(x_0, \ldots, x_n) \). For (b) we will need to use a more general definition of \( I(s) \) than the one in [1,2]. However, the generalization is necessary to handle many of the Feynman amplitudes with infrared divergences considered by physicists.

We will use the symbol \( P \) to denote the \( \mathbb{Q} \)-algebra of periods and use the definition of a period that appears in [3]. For the convenience of the reader, we also paraphrase this definition in (1.3).

We thank A. Goncharov, D. Kreimer, M. V. Nori and H. Rossi for useful communication. The idea of using Picard-Fuchs equations in Theorem 1.8 comes from discussions with Madhav Nori. This idea is ‘standard’ when studying periods of powers of functions, but it came somewhat of a surprise that there were no ‘Gamma factors’ at the end. Also historical precedents to the ‘functional equation’ in theorem 1.8 should be noted. They appear in Bernstein’s paper.
1. IGUSA ZETA FUNCTIONS

1.1. Atiyah’s Theorem. Let $X$ be a smooth complex algebraic variety defined over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the real points of $X$ and let $G$ be a semi-algebraic subset of $X(\mathbb{R})$ defined by inequalities

$$G = \{ x \in X(\mathbb{R}) \mid g_i(x) \geq 0 \text{ for all } i \}$$

where here the $g_i$ are real-analytic functions on $X(\mathbb{R})$. Let $f$ be a real-analytic function on $X(\mathbb{R})$ which is non-negative and not identically zero on $G$. Let $\Gamma$ denote the characteristic function of $G$. In this notation Atiyah’s theorem [1] can be stated as follows.

Theorem 1.1 (Atiyah). The function $f^s\Gamma$, which is locally integrable for $\Re(s) > 0$, extends analytically to a distribution on $X$ which is a meromorphic function of $s$ in the whole complex plane. Over any relatively compact open set $U$ in $X$ the poles of $f^s\Gamma$ occur at the points of the form $-r/N, r = 1, 2, \ldots$, where $N$ is a fixed integer (depending on $f$ and $U$) and the order of any pole does not exceed the dimension of $X$. Moreover, $f^0\Gamma = \Gamma$.

1.2. Sem-algebraic Sets. The following definition is given in [4].

Definition 1.2. A region $C \subset \mathbb{R}^n$ is semi-algebraic if it is a union of intersections of sets of the form $\{ x \in X(\mathbb{R}) \mid f(x) > 0 \}$ or $\{ x \in X(\mathbb{R}) \mid f(x) = 0 \}$ with $f \in \mathbb{R}[x_1, \ldots, x_n]$.

We will say that $C \subset \mathbb{R}^n$ is semi-arithmetic if the functions $f$ appearing in the definition are in $\mathbb{R}_{\text{alg}}[x_1, \ldots, x_n]$ with $\mathbb{R}_{\text{alg}} = \mathbb{R} \cap \mathbb{Q}$.

Definition 1.3. A period is a number whose real and imaginary parts are given by absolutely convergent integrals of the form $\int_C f \, d\mu$ where $C \subset \mathbb{R}^n$ is a semi-arithmetic set, $f \in \mathbb{R}_{\text{alg}}(x_1, \ldots, x_n)$ and $\mu$ is Lebesgue measure on $\mathbb{R}^n$.

Assume that $X$ is a variety defined over $\mathbb{R}$. For $f \in \mathbb{R}[X]$, let $X_{f \geq 0}$ denote the set

$$\{ x \in X(\mathbb{R}) \mid f(x) \geq 0 \}.$$
subset of $X(\mathbb{R})$. It is known that the interior of $C$ contains a semi-algebraic (resp. semi-arithmetic) dense open subset $U \subset C$ which is smooth and orientable. (This follows from Proposition 2.9.10 of [4].) By a pre-orientation of $C$, we mean a choice of such a subset $U$ along with an orientation of $U$. If $\omega \in \Omega^n(X)$ is a differential form and $C$ is pre-oriented, then we make the definition

$$\int_C \omega \overset{\text{def}}{=} \int_U \omega. \quad (2)$$

If $C \subset \mathbb{R}^n$ then the interior of $C$ is smooth and comes with a canonical pre-orientation inherited from the standard orientation on $\mathbb{R}^n$. For $C$ compact, the orientation gives a class in $\sigma \in H_n(C, \partial C)$ where $\partial C$ is the topological boundary of $C$. To use Atiyah’s theorem in the context of semi-algebraic sets, we need to be able to convert an integral $\int_C \omega$ over an arbitrary semi-algebraic set into a sum of integrals over sets of the form of the set $G$ in (1). The following lemma is needed to this end.

**Lemma 1.4.** Let $f_i (1 \leq i \leq n)$ and $g_j (1 \leq j \leq m)$ be two sets of functions in $\mathbb{R}[X]$. Let $U = U_1 \cup U_2$ be an oriented open set with

$$U_1 = \{ x \in X | f_i > 0, 1 \leq i \leq n \}, \quad (3)$$
$$U_2 = \{ x \in X | g_i > 0, 1 \leq j \leq m \}. \quad (4)$$

Consider strings of the form

$$e = (a_1, \ldots, a_n, b_1, \ldots, b_m)$$

where the $a_i$, $b_j$ are in $\{+1, -1\}$ and either all the $a$’s are $+1$ or all the $b$’s are $+1$.

Consider $U_e = \{ x \in X | a_i f_i > 0, b_j g_j > 0, 1 \leq i \leq n; 1 \leq j \leq m \}$

Then, for a form $\omega \in \Omega^n(X)$ (with $n = \dim X$),

$$\int_U \omega = \sum_e \int_{U_e} \omega \quad (5)$$

where the $e$ are subject to the above constraints.

Since our domains of integration are going to be semi-algebraic sets, we need a more flexible definition of Periods. This is equivalent to the definition of periods above (1.3), a proof is sketched in [8], page 3,31. The definition below is the definition of periods for the purposes of this paper.
**Definition 1.5.** Let $X$ be a smooth algebraic variety of dimension $d$ defined over $\mathbb{Q}$, $D \subset X$ a divisor with normal crossings, $\omega \in \Omega^d(X)$ an algebraic differential form on $X$ of the top degree, and $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$ a (homology class of) a singular $d-$ chain on the complex manifold $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$. Periods are the ring (over $\mathbb{Q}$) generated by numbers of the form $\int_{\gamma} \omega$.

We could have replaced $\mathbb{Q}$ by $\overline{\mathbb{Q}}$ above, and obtained the same ring (as Kontsevich and Zagier remark). This is easy because a variety defined over $\overline{\mathbb{Q}}$ can be viewed as defined over $\mathbb{Q}$, but we get several copies over the algebraic closure. But there is one more modification that one can make which is a bit more subtle. This is to allow for absolutely convergent integrals. Most examples (e.g., multiple zeta values) are not directly periods in the above sense, the integrals defining them can have singularities on the boundary. To take care of this we note the following theorem which will be proved in Section 2.

**Theorem 1.6.** Let $X$ be a smooth $n$-dimensional algebraic variety defined over a field $k \subset \mathbb{R}_{\text{alg}}$. Let $F$ be a reduced effective divisor and let $\omega \in \Omega^n(X - F)$ be an $n$-form. Let $C \subset X(\mathbb{R})$ be a pre-oriented semi-arithmetic set with non-empty interior $C^0$. Then the integral $\int_C \omega \in P$ provided that it is absolutely convergent.

**Remark 1.7.** Already known to Kontsevich and Zagier, as in page 31 of [8]. We wanted to elaborate on their comment that this follows from resolution of singularities in characteristic 0.

We now turn to the theorem on Igusa Zeta functions.

**Theorem 1.8.** Let $X$ be a smooth variety defined over $k \subset \mathbb{R}_{\text{alg}}$ and let $f \in \mathcal{O}(X)$ be a function. Let $C$ be a compact pre-oriented semi-arithmetic subset of $X_{f \geq 0}(\mathbb{R})$ defined over $k$. Then, if $\omega \in \Omega^n(X)$ is a differential form, the function

$$I(s) = \int_C f^s \omega$$

extends meromorphically to all of $\mathbb{C}$ with poles occurring only at negative integers. Moreover, for any $s_0 \in \mathbb{Z}$, the coefficients $a_i$ in the Laurent expansion

$$I(s) = \sum_{i \geq N} a_i (s - s_0)^i$$

are periods.
Our first step is to prove the theorem for $s_0 > 0$. In this case, Atiyah’s theorem shows that the integral for $I(s)$ converges and is analytic in a neighborhood of $s_0$. Thus, assuming $f \neq 0$, we can differentiate under the integral sign to obtain

$$I^{(l)}(s_0) = \int_C f^{s_0} \log^l(f) \omega. \quad (8)$$

Now

$$\log f(x) = \int_0^1 \frac{f(x) - 1}{(f(x) - 1)t + 1} dt. \quad (9)$$

Thus we can write the log factors in (8) as period integrals.

To do this explicitly, set $Y = X \times \mathbb{A}^1$, $D = C \times [0, 1]$ and

$$\eta = \omega \wedge \frac{f(x) - 1}{(f(x) - 1)t_1 + 1} dt_1 \wedge \cdots \wedge \frac{f(x) - 1}{(f(x) - 1)t_l + 1} dt_l.$$ 

We then have

$$\int_D f^{s_0} \eta = \int_C f^{s_0} \log^l(f) \omega. \quad (10)$$

The left hand side is absolutely convergent (in fact bounded on the domain of integration\textsuperscript{1}). Thus, $I^{(l)}(s_0)$ is a period for all $l$ as long as $s_0 > 0$ (theorem 1.6), and the theorem is verified for $s_0 > 0$.

To verify the theorem for $s_0 \leq 0$, we use an auxiliary function and the Picard-Fuchs equation. Set

$$J(t) = \int_C \frac{\omega}{1 - tf} \quad (11)$$

viewing the integrand as an $n$-form on $X \times \mathbb{A}^1$. Then $J(t) = \sum_{l \geq 0} I(l)t^l$ for all $t$ such that the sum converges. Since $C$ is compact, $f$ is bounded on $C$ by some constant $R$. Thus, for $t < 1/R$, $C$ does not intersect the divisor $Z = V(1 - tf)$ where the integrand may have a pole, and the integral (11) converges.

Using the triangulation theorem for semi-algebraic sets (\textsuperscript{4} Theorem 9.2.1), we can assume that $C$ is homeomorphic analytically to an $n$-simplicial complex with one $n$-cell and that $\partial C$ is contained in a divisor $D \subset X$ (defined over $k$) Let $\sigma \in H_n(X(\mathbb{R}) - Z(\mathbb{R}), D(\mathbb{R}) - Z(\mathbb{R}); Z)$ be the class

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\textsuperscript{1}Consider the integral $\int_{x \geq y \geq 0} \frac{\omega}{y} dxdy$, the integrand is bounded, yet the differential form $\frac{\omega}{y}dxdy$ has a pole at $x = 0$. A blow up at $(0, 0)$ resolves this problem and converts this to $\int_{0 \leq x \leq 1, 0 \leq y \leq 1} uxdxdy.$
represented by integration over the points of $C$ that are smooth in $X$. Then, for each $t$ with $|t| < 1/R$,

$$J(t) = \int_\sigma \frac{\omega}{1 - tf}.$$  

There is an algebraic vector bundle $V = H^m_{DR}(X - Z, D - Z)$ over $\mathbb{A}^1 - S$ where $S$ is a finite subset of $\mathbb{A}^1$ which can include 0 (but defined over $k$). The stalks of $V$ are equal to the de Rham cohomology of $H^n(X_y - Z_y, D_y - Z_y)$ over the field $k(y)$ for all $y \in \mathbb{A}^1 - S$. The integrand $s = \frac{\omega}{1 - tf}$ can be thought of as a global section of $V$ (because it is an algebraic differential form of the top degree it is closed and vanishes when restricted to $D - Z$).

This bundle $V$ carries an algebraic connection $\nabla$, an isomorphism over $\mathbb{A}^1 - S$ (of analytic vector bundles)

$$V_{\mathbb{A}^1} \to \mathcal{L} \otimes_{\mathbb{Z}} O_{Y_C}$$

where $\mathcal{L}$ is the local system whose fiber at $y \in Y_C$ is the singular cohomology of the pair $(X_y - Z_y, D_y - Z_y)$. The connection is integrable, has regular singular points and the sheaf of flat sections is the sheaf $\mathcal{L}$.

If $\sigma$ is a flat section of the dual local system $\mathcal{L}^*$ (which is the local system of the homology of pairs $H^n(X_y - Z_y, D_y - Z_y)$) over an open set $U \subset \mathbb{A}^1 - S$ in the analytic topology, then we can form a function on $U$: $g(y) = \int_\sigma s_y$.

If $T$ is a tangent vector field on $U$, we have the formula

$$T(g) = \int_\sigma \nabla_T(s)_y.$$  

Now, $V$ is a vector bundle of finite rank so given any section $s$ over $\mathbb{A}^1 - S$, there is a relation of the form

$$\sum_{i=0}^r q_i(t) \nabla^i_T(s)_y.$$  

where the $q_i$ are rational functions in $t$ with coefficients in $k$. We can assume that they are polynomials by multiplying the equation by a polynomial $\in k[t]$.

Integrating this against the $\sigma$ obtained from $C$ and $s = \frac{\omega}{1 - tf}$ we obtain a nontrivial linear relation of the form

$$\sum_{i=0}^r q_i(t) J^{(i)}(t) = 0$$  

where the $q_i(t) \in k(t)$.

For a complete reference to the Picard-Fuchs theory see [6].
Clearing denominators in (13), we can assume that the $q_i(t) \in k[t]$. Expanding out $q_i(t) = \sum_{j=0}^{d_i} a_{i,j} t^j$ (for some $a_{i,j} \in k$) and $J^{(i)}(t) = \sum_{j \geq 0} \frac{j!}{(j-i)!} t^{j-i} I(j)$ and equating terms with the same power of $t$, we obtain a set of relations between the $I(j)$'s. Explicitly, we obtain the relation

$$
\sum_{s \geq 0} \sum_{i=1}^{d} \sum_{j=0}^{d} a_{i,j} \frac{(s + i - j)!}{(s-j)!} I(s + i - j) t^d = 0
$$

where $d = \max d_i$.

Noting that, for each pair $(i, j)$, the coefficient $a_{i,j} \frac{(s + i - j)!}{(s-j)!}$ is a polynomial of degree $i$ in $s$, we see that we have a relation of the form

$$
\sum_{i=0}^{d+r} c_i(s) I(s + i) = 0
$$

with the $c_i$ polynomials in $k[s]$. Note that, as long as $f$ and $\omega$ are nonzero the relation (15) is nontrivial.

We wish to show that (15) holds for all complex values of $s$. By the uniqueness of analytic continuation, it is enough to show that this is so for $\Re(s) > d + r$. We then use the following corollary of a result from [5] (p. 953).

**Theorem 1.9** (Carleson). Let $h(z)$ be holomorphic for $\Re(z) > 0$ and assume $h(n) = 0$ for $n \in \mathbb{N}$. Then $h(z) = 0$ if $h(z) \leq K e^{m\Re(z)}$ for a constants $m$ and $K$.

To use Carleson’s theorem, let $Q(z)$ be the left hand side of (15) viewed as a function of a complex variable $z = s - d - r$. Then $Q(z)$ is holomorphic for $\Re(z) > 0$. Moreover, since $f$ is bounded on the semi-algebraic set $C$ by a number $R$, $|I(s)|$ is bounded by $A R^{\Re(s)}$ for some constant $A$. Thus $Q(z)$ is bounded by $K e^{m\Re(s)}$ for some constants $m$ and $K$. It follows from Carleson’s theorem that $Q(z) = 0$ for $\Re(s) > 0$. Thus, by uniqueness of analytic continuation, it follows that $Q(z) = 0$ for all $z$.

Without loss of generality, we can assume that $c_0(s)$ in (15) is nonzero. Then we have a relation

$$
I(s) = \sum_{i=1}^{d+r} l_i(s) I(s + i).
$$

where $l_i = \frac{-c_i(s)}{c_0(s)}$. Using (16), we can complete the proof of Theorem 1.8 by descending induction on $s_0$. For $s_0 > 0$, the theorem is established.
Suppose then that the theorem is established for $s_0 > M$. We can then use the Laurent expansions for the terms on right hand side of (16) to write out the Laurent expansion for the left hand side. Using the fact that the $l_i$ are rational function in $k(t)$ and using the Laurent expansions of $I(s)$ at $s_0 > M$, it is easy to see that the theorem holds for $s = s_0$.

Remark 1.10. Historical precedents to the ‘functional equation’ in theorem 1.8 should be noted. They first appear in Bernstein’s paper [3]. Using the theory of $D$-modules, he shows that if the domain of of integration was all of $\mathbb{R}^n$ and the polynomial function $f$, satisfied a growth rate of the form

$$|f(X)| \geq C||X||^A$$

for $A > 0$ and $||(x_1, \ldots, x_n)|| = \sum x_i^2$, then functions of the type

$$H(s) = \int_{\mathbb{R}^n} f^{-s} dx_1, \ldots, dx_n$$

satisfied functional equations. This was achieved beautifully using the theory of $D$–modules. But this approach fails (or atleast we could not make it work) when the domain of integration is an arbitrary semi-algebraic set.

2. Periods and semi-arithmetic sets

In this section we prove a theorem tacitly used in [8] relating integrals over semi-arithmetic sets to the integrals over cohomology classes which are more widely thought of as period integrals. The main tool is the same corollary of resolution of singularities used by Atiyah to prove theorem [1.1].

We state it here in the form that we will use.

**Theorem 2.1 (Resolution Theorem).** Let $F \in O(X)$ be a nonzero function on a smooth, complex $n$-dimensional algebraic variety. Let $\omega \in \Omega^n(X - E)$ be a differential $n$-form where $E$ is a divisor. Let $Z(\omega)$ denote the zero set of $\omega$. Then there is a proper morphism $\varphi : \bar{X} \to X$ from a smooth variety $\bar{X}$ such that

(i) $\varphi : \bar{X} - \bar{A} \to X - A$ is an isomorphism, where $A = F^{-1}(0) \cup E \cup Z(\omega)$ and $\bar{A} = \varphi^{-1}(A)$.

(ii) for each $P \in \bar{X}$ there are local coordinates $(y_1, \ldots, y_n)$ centered at $P$ so that, locally near $P$,

$$F \circ \varphi = \epsilon \prod_{j=1}^n y_j^{k_j}$$

$$\omega = \delta \prod_{j=1}^n y_j^{l_j} dy_1 \wedge \cdots \wedge dy_n$$
where \( \epsilon, \delta \) are units in \( \mathcal{O}_{X,P} \), the \( k_j \) are non-negative integers and the \( l_j \) are arbitrary integers.

The theorem, the statement of which is very close to the statement of Atiyah’s resolution theorem on p. 147 of [11], is proved by applying Main Theorem II in [7] to the ideals \( F\mathcal{O}_X, E \) and \( Z(\omega) \).

**Proposition 2.2.** Let \( X \) be a smooth \( n \)-dimensional algebraic variety defined over \( \mathbb{R}_{\text{alg}} \). Let \( F \) be a reduced effective divisor and let \( \omega \in \Omega^n(X - E) \) be an \( n \)-form. Let

\[
G = \{ x \in X(\mathbb{R}) | g_i(x) \geq 0 \}
\]

for some set \( \{ g_i \}_{i=1}^m \) of functions in \( \mathcal{O}(X) \). be a compact, pre-oriented semi-algebraic set with non-empty interior \( G^0 \). Then \( \int_G \omega \) converges absolutely only if there is a smooth \( n \)-dimensional algebraic variety \( \tilde{X} \) with proper, birational morphism \( \varphi : \tilde{X} \to X \) and a compact semi-algebraic set \( \tilde{G} \) such that

(i) \( \int_{\tilde{G}} \varphi^* \omega = \int_G \omega. \)

(ii) \( \varphi^* \omega \) is holomorphic on \( \tilde{G} \).

**Proof.** Using the resolution theorem with \( F = \prod_{i=1}^m g_i \), we can find a smooth variety \( \tilde{X} \) with a proper, birational morphism to \( X \) such that for every point \( P \in \tilde{X} \) we have local parameters \( (y_1, \ldots, y_n) \) defined in a neighborhood of \( P \) with

\[
g_i \circ \varphi = \epsilon_i \prod_{j=1}^n y_j^{k_{ij}}, \quad \varphi^* \omega = \delta \prod_{j=1}^n y_j^{l_j} dy_1 \wedge \cdots \wedge dy_n.
\]

Here the \( \epsilon_i \) and \( \delta \) are invertible near \( P \). Set \( \tilde{G} \) equal to the analytic closure of \( \varphi^{-1}(G - A) \) with \( A \) as in the resolution theorem. Then \( \int_{\tilde{G}} \varphi^* \omega = \int_G \omega \) because \( \tilde{G} \) and \( G \) differ only by measure 0 sets. Moreover, since \( \varphi \) is proper and \( G \) is a closed subset of \( \varphi^{-1}G \), \( \tilde{G} \) is compact.

To see that \( \varphi^* \omega \) is holomorphic on \( \tilde{G} \), let \( P \in \tilde{G} \) be a point and let \( \tilde{U} \) be a neighborhood of \( P \) with a local coordinate system \( (y_1, \ldots, y_n) \) as in the resolution theorem. Since \( P \) is in the closure of \( \varphi^{-1}(G - A) \), \( g_i(P) \geq 0 \) for all \( i \). Let \( s_i \) be the sign (±1) of \( \epsilon_i(P) \). Then, since \( \int_G \omega \) is absolutely convergent, it follows that

\[
\int_{0 < s_i y_i(p) < r} \varphi^* \omega = \int_{0 < s_i y_i(p) < r} \prod_{j=1}^n y_j^{l_j} dy_1 \wedge \cdots \wedge dy_n.
\]
is absolutely convergent for a sufficiently small $r$. It is easy to see that this is not possible unless $l_j \geq 0$ for all $j$. Thus $\varphi^* \omega$ is holomorphic at $P$. \hfill \Box

**Proposition 2.3.** Let $X$ be a smooth algebraic variety over $\mathbb{R}_{\text{alg}}$ and let $G = \{x \in X(\mathbb{R}) | g_i(x) \geq 0\}$ be a compact pre-oriented set. Let $\omega \in \mathcal{O}_X(X)$ be a differential $n$-form. Then there is a divisor $D \subset X$ and a chain $\sigma \in H_n(X, D)$ such that $\int_G \omega = \int_\sigma \omega$.

**Proof.** The pre-orientation on $G$ gives us a dense, smooth, open semi-algebraic subset $U$ in $G$ with an orientation on $U$. We, therefore, obtain a chain $\sigma \in H_n(X, D)$ where $D$ is the set of zeroes of the functions $g_i$ defining $G$. This $\sigma$ corresponds to the orientation on the open subset $U$ so we have $\int_\sigma \omega = \int_G \omega$. \hfill \Box

Therefore we conclude:

**Theorem 2.4.** Let $X$ be a smooth $n$-dimensional algebraic variety defined over a field $k \subset \mathbb{R}_{\text{alg}}$. Let $F$ be a reduced effective divisor and let $\omega \in \Omega^n(X - F)$ be an $n$-form. Let $C \subset X(\mathbb{R})$ be a pre-oriented semi-arithmetic set with non-empty interior $C^0$. Then the integral $\int_C \omega \in \mathbb{P}$ provided that it is absolutely convergent.

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MATHEMATICS DEPARTMENT, CB #3250, PHILLIPS HALL, UNC-CHAPEL HILL, CHAPEL HILL, NC 27599

E-mail address: beikale@email.unc.edu

UCLA MATHEMATICS DEPARTMENT, BOX 951555, LOS ANGELES, CA 90095-1555

E-mail address: pbrosnan@math.ucla.edu