On Scott power spaces*

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Abstract

In this paper, we mainly discuss some basic properties of Scott power spaces. For a \( T_0 \) space \( X \), let \( K(X) \) be the poset of all nonempty compact saturated subsets of \( X \) endowed with the Smyth order. It is proved that the Scott power space \( \Sigma K(X) \) of a well-filtered space \( X \) is still well-filtered, and a \( T_0 \) space \( Y \) is well-filtered if \( \Sigma K(Y) \) is well-filtered and the upper Vietoris topology is coarser than the Scott topology on \( K(Y) \). A sober space is constructed for which its Scott power space is not sober. A few sufficient conditions are given under which a Scott power space is sober. Some other properties, such as local compactness, first-countability, Rudin property and well-filtered determinedness, of Smyth power spaces and Scott power spaces are also investigated. 

Keywords: Scott power space; Smyth power space; Sobriety; Well-filteredness; Local compactness; First-countability

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1. Introduction

An important problem in domain theory is the modelling of non-deterministic features of programming languages and of parallel features treated in a non-deterministic way. If a non-deterministic program runs several times with the same input, it may produce different outputs. To describe this behaviour, powerdomains were introduced by Plotkin [20, 21] and Smyth [25] to give denotational semantics to non-deterministic choice in higher-order programming languages. The three main such powerdomains are the Smyth powerdomain for demonic non-determinism, the Hoare powerdomain for angelic non-determinism, and the Plotkin powerdomain for erratic non-determinism. This viewpoint traditionally stays with the category of dcpos, but is easily and profitably extended to general topological spaces (see, for example, [1, Sections 6.2.3 and 6.2.4] and [23]).

A subset \( A \) of a \( T_0 \) space \( X \) is called saturated if \( A \) equals the intersection of all open sets containing it (or equivalently, \( A \) is an upper set in the specialization order). We shall use \( K(X) \) to denote the set of all nonempty compact saturated subsets of \( X \) and endow it with the Smyth preorder, that is, for \( K_1, K_2 \in K(X), K_1 \subseteq K_2 \) iff \( K_2 \subseteq K_1 \). The upper Vietoris topology on \( K(X) \) is the topology that has \( \{ \square U : U \in \mathcal{O}(X) \} \) as a base, where \( \square U = \{ K \in K(X) : K \subseteq U \} \), and the resulting space is called the Smyth power space or upper space of \( X \) and is denoted by \( P^s(X) \).

There is another prominent topology one can put on \( K(X) \), namely, the Scott topology. We call the space \( \Sigma K(X) = (K(X), \sigma(K(X))) \) the Scott power space of \( X \). It is well-known that when \( X \) is well-filtered,
K(X) is a dcpo, with least upper bounds of directed families computed as filtered intersections, and the Scott topology is finer than the upper Vietoris topology; when X is locally compact and well-filtered (equivalently, locally compact and well-filtered), the two topologies coincide.

In this paper, we mainly discuss some basic properties of Scott power spaces. The paper is organized as follows:

In Section 2, some standard definitions and notations are introduced which will be used in the whole paper. A few basic properties of irreducible sets and compact saturated sets are listed.

In Section 3, we briefly recall the concepts of Scott topology and continuous domains and some fundamental results about them. A countable algebraic lattice L is given for which the poset \( \text{Fin}L \) of all upper finitely generated sets of L (with the reverse inclusion order) is not a dcpo.

In Section 4, we list a few important results of \( d \)-spaces, well-filtered spaces and sober spaces that will be used in other sections.

In Section 5, we recall some concepts and results about the topological Rudin Lemma, Rudin spaces and well-filtered determinedness of Smyth power spaces and Scott power spaces.

In Section 6, we mainly investigate the well-filteredness of Scott power spaces. It is proved that the Scott power space of a well-filtered space is well-filtered, and a \( T_0 \) space \( X \) is well-filtered iff the upper Vietoris topology is coarser than the Scott topology on \( K(X) \) and \( \Sigma K(X) \) is well-filtered.

In Section 7, a sober space is constructed for which its Scott power space is not sober.

In Section 8, we study the question under what conditions the Scott power space \( \Sigma K(X) \) of a sober space \( X \) is sober. This question is related to the investigation of conditions under which the upper Vietoris topology coincides with the Scott topology on \( K(X) \), and further it is closely related to the local compactness and first-countability of \( X \).

In section 9, the Rudin property and well-filtered determinedness of Smyth power spaces and Scott power spaces are discussed.

2. Preliminaries

In this section, we briefly recall some standard definitions and notations that will be used in the paper. Some basic properties of irreducible sets and compact saturated sets are presented.

For a set \( X \), \( |X| \) will denote the cardinality of \( X \). Let \( \mathbb{N} \) denote the set of all natural numbers with the usual order if no other explanation and \( \omega = |\mathbb{N}| \). The set of all subsets of \( X \) is denoted by \( 2^X \). Let \( X^{(\leq \omega)} = \{ F \subseteq X : F \) is a finite set \} and \( X^{(\leq \omega)} = \{ F \subseteq X : F \) is a countable set \}.

For a poset \( P \) and \( A \subseteq P \), let \( \downarrow A = \{ x \in P : x \leq a \) for some \( a \in A \} \) and \( \uparrow A = \{ x \in P : x \geq a \) for some \( a \in A \} \). For \( x \in P \), we write \( \downarrow x \) for \( \downarrow \{ x \} \) and \( \uparrow x \) for \( \uparrow \{ x \} \). A subset \( A \) is called a lower set (resp., an upper set) if \( A = \downarrow A \) (resp., \( A = \uparrow A \)). Let \( \text{Fin}P = \{ \uparrow F : F \in P^{(\leq \omega)} \} \). For a nonempty subset \( A \) of \( P \), define \( \text{min}(A) = \{ u \in A : u \) is a minimal element of \( A \} \) and \( \text{max}(A) = \{ v \in A : v \) is a maximal element of \( A \} \).

A nonempty subset \( D \) of a poset \( P \) is directed if every two elements in \( D \) have an upper bound in \( D \). The set of all directed sets of \( P \) is denoted by \( \mathcal{D}(P) \). \( I \subseteq P \) is called an ideal of \( P \) if \( I \) is a directed lower subset of \( P \). Let \( \text{Id}(P) \) be the poset (with the order of set inclusion) of all ideals of \( P \). Dually, we define the notion of filters and denote the poset of all filters of \( P \) by \( \text{Filt}(P) \). The poset \( P \) is called a directed complete poset, or dcpo for short, if for any \( D \in \mathcal{D}(P) \), \( \vee D \) exists in \( P \).

The poset \( P \) is said to be Noetherian if it satisfies the ascending chain condition (ACC for short): every ascending chain has a greatest member. Clearly, \( P \) is Noetherian iff every directed set of \( P \) has a largest element (or equivalently, every ideal of \( P \) is principal). A topological space \( X \) is said to be a Noetherian space if every open subset is compact (see \[3\] Definition 9.7.1).

As in \[3\], a topological space \( X \) is locally hypercompact if for each \( x \in X \) and each open neighborhood \( U \) of \( x \), there is \( \uparrow F \in \text{Fin}X \) such that \( x \in \text{int} \uparrow F \subseteq \uparrow F \subseteq U \). The space \( X \) is called a c-space if for each \( x \in X \) and each open neighborhood \( U \) of \( x \), there is \( \uparrow x \subseteq X \) such that \( x \in \text{int} \uparrow u \subseteq \uparrow u \subseteq U \). A set \( K \) of \( X \) is called supercompact if for any family \( \{ U_i : i \in I \} \subseteq \mathcal{O}(X) \), \( K \subseteq \bigcup_{i \in I} U_i \) implies \( K \subseteq U \) for some \( i \in I \). It is easy to verify that the supercompact saturated sets of \( X \) are exactly the sets \( \uparrow x \) with \( x \in X \) (see \[10\] Fact 2.2). It is well-known that \( X \) is a c-space iff \( \mathcal{O}(X) \) is a completely distributive lattice (cf. \[4\]).
The category of all $T_0$ spaces is denoted by $\textbf{Top}_0$. For $X \in \text{ob}(\textbf{Top}_0)$, we use $\leq_X$ to denote the specialization order of $X$: $x \leq_X y$ iff $x \in \{y\}$. In the following, when a $T_0$ space $X$ is considered as a poset, the order always refers to the specialization order if no other explanation. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of $X$, and let $\mathcal{S}^u(X) = \{\{x\} : x \in X\}$ and $\mathcal{D}_r(X) = \{\mathcal{D} : D \in \mathcal{D}(X)\}$.

It is straightforward to verify the following.

**Remark 2.1.** Let $X$ be a topological space and $A, B \subseteq X$. Then

(1) $\overline{A} = \overline{B}$ if and only if for any $U \in \mathcal{O}(X)$, $A \cap U \neq \emptyset$ iff $B \cap U \neq \emptyset$.

(2) If $\tau_1, \tau_2$ are two topologies on the set $X$ and $\tau_1 \subseteq \tau_2$, then $\text{cl}_{\tau_2} A = \text{cl}_{\tau_1} A$ implies $\text{cl}_{\tau_2} A = \text{cl}_{\tau_1} B$.

For a $T_0$ space $X$ and a nonempty subset $A$ of $X$, $A$ is irreducible if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\text{irr}(X)$ (resp., $\text{irr}_r(X)$) the set of all irreducible (resp., irreducible) closed subsets of $X$. Clearly, every subset of $X$ that is directed under $\leq_X$ is irreducible.

The following lemma is well-known and can be easily verified.

**Lemma 2.2.** If $f : X \rightarrow Y$ is continuous and $A \in \text{irr}(X)$, then $f(A) \in \text{irr}(Y)$.

For any $T_0$ space $X$, the lower Vietoris topology on $\text{irr}_r(X)$ is the topology $\{\Diamond U : U \in \mathcal{O}(X)\}$, where $\Diamond U = \{A \in \text{irr}_r(X) : A \cap U \neq \emptyset\}$. The resulting space, denoted by $X^s$, with the canonical mapping $\eta_X : X \rightarrow X^s, x \mapsto \{x\}$, is the sobrification of $X$ (cf. [6, 8]). Clearly, $\eta_X : X \rightarrow X^s$ is an order and topological embedding (cf. [6, 8, 23]).

**Remark 2.3.** For a $T_0$ space $X$, $\eta_X : X \rightarrow X^s$ is a dense topological embedding (cf. [6, 8, 23]).

A subset $A$ of a space $X$ is called saturated if $A$ equals the intersection of all open sets containing it (or equivalently, $A$ is an upper set in the specialization order). We shall use $K(X)$ to denote the set of all nonempty compact saturated subsets of $X$ and endow it with the Smyth preorder, that is, for $K_1, K_2 \in K(X), K_1 \subseteq K_2$ iff $K_2 \subseteq K_1$. The upper Vietoris topology on $K(X)$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a base, where $\Diamond U = \{K \in K(X) : K \subseteq U\}$, and the resulting space is called the Smyth power space or upper space of $X$ and is denoted by $P_S(X)$ (cf. [9, 23]).

**Remark 2.4.** Let $X$ be a $T_0$ space.

(1) The specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \subseteq$).

(2) The canonical mapping $\xi_X : X \rightarrow P_S(X), x \mapsto \uparrow x$, is an order and topological embedding (cf. [9, 10, 23]).

(3) $X$ is homeomorphic to the subspace $S^u(X)$ of $P_S(X)$ by means of $\xi_X$.

**Lemma 2.5.** For a $T_0$ space $X$ and $A \subseteq X$, $\text{cl}_{\text{cl}(P_S(X))}\xi_X(A) = \Diamond \overline{A}$.

**Proof.** Clearly, $\Diamond \overline{A} = K(X) \setminus \Diamond (X \setminus \overline{A})$ is closed in $P_S(X)$ and hence $\text{cl}_{\text{cl}(P_S(X))}\xi_X(A) \subseteq \Diamond \overline{A}$. Since $\Diamond C : C \in \mathcal{C}(X)$ is a (closed) base of $P_S(X)$, there is a family $\{C_i : i \in I\} \subseteq \mathcal{C}(X)$ such that $\text{cl}_{\text{cl}(P_S(X))}\xi_X(A) = \bigcap_{i \in I} \Diamond C_i$. Then for each $i \in I$, $\xi_X(A) \subseteq \Diamond C_i$, and consequently, $\uparrow a \cap C_i \neq \emptyset$ for each $a \in A$; whence, for each $a \in A$, $a \in C_i$ as $C_i = \downarrow \xi_X(A)$. It follow that $\overline{A} \subseteq C_i$ for each $i \in I$ and hence $\Diamond \overline{A} \subseteq \bigcap_{i \in I} \Diamond C_i = \text{cl}_{\text{cl}(P_S(X))}\xi_X(A) = \Diamond \overline{A}$.

**Proposition 2.6.** (25, Lemma 2.19) $P_S : \textbf{Top}_0 \rightarrow \textbf{Top}_0$ is a covariant functor, where for any $f : X \rightarrow Y$ in $\textbf{Top}_0$, $P_S(f) : P_S(X) \rightarrow P_S(Y)$ is defined by $P_S(f)(K) = \uparrow f(K)$ for all $K \in K(X)$.

**Corollary 2.7.** Let $X$ and $Y$ be two $T_0$ spaces. If $Y$ is a retract of $X$, then $P_S(Y)$ is a retract of $P_S(X)$.

For a nonempty subset $C$ of a $T_0$ space, it is easy to see that $C$ is compact iff $\uparrow C \in K(X)$ and endow it with the Smyth power space of $X$ and is denoted by $P_S(X)$. Furthermore, we have the following useful result (see, e.g., [4, pp.2068]).
Lemma 2.8. Let \( X \) be a \( T_0 \) space and \( C \in K(X) \). Then \( C = \uparrow \min(C) \) and \( \min(C) \) is compact.

Lemma 2.9. Let \( X \) be a \( T_0 \) space. For any nonempty family \( \{ K_i : i \in I \} \subseteq K(X) \), \( \bigvee_{i \in I} K_i \) exists in \( K(X) \) iff \( \bigcap_{i \in I} K_i = \bigcap_{i \in I} K_i \). In this case \( \bigvee_{i \in I} K_i = \bigcap_{i \in I} K_i \).

Proof. Suppose that \( \{ K_i : i \in I \} \subseteq K(X) \) is a nonempty family and \( \bigvee_{i \in I} K_i \) exists in \( K(X) \). Let \( K = \bigvee_{i \in I} K_i \). Then \( K \subseteq K_i \) for all \( i \in I \), and hence \( K \subseteq \bigcap_{i \in I} K_i \). For any \( x \in \bigcap_{i \in I} K_i \), \( \uparrow x \) is a upper bound of \( \{ K_i : i \in I \} \) in \( K(X) \), whence \( K \subseteq \uparrow x \) or, equivalently, \( \uparrow x \subseteq K \). Therefore, \( \bigcap_{i \in I} K_i \subseteq K \). Thus \( \bigcap_{i \in I} K_i = K \in K(X) \).

Conversely, if \( \bigcap_{i \in I} K_i = K \in K(X) \), then \( \bigcap_{i \in I} K_i \) is an upper bound of \( \{ K_i : i \in I \} \) in \( K(X) \). Let \( G \in K(X) \) be another upper bound of \( \{ K_i : i \in I \} \) in \( K(X) \). Let \( G \in K(X) \) be another upper bound of \( \{ K_i : i \in I \} \) in \( K(X) \). Let \( \bigcap_{i \in I} K_i \subseteq G \), proving that \( \bigvee_{i \in I} K_i = \bigcap_{i \in I} K_i \).

Similarly, we have the following.

Lemma 2.10. Let \( P \) be a poset. For any nonempty family \( \{ \uparrow F_i : i \in I \} \subseteq \text{Fin} P \), \( \bigvee_{i \in I} \uparrow F_i \) exists in \( \text{Fin} P \) iff \( \bigcap_{i \in I} \uparrow F_i = \bigcap_{i \in I} \uparrow F_i \).

Lemma 2.11. \((\text{[28] Proposition 7.21})\) Let \( X \) be a \( T_0 \) space.

1. If \( K \in K(P_S(X)) \), then \( \bigcup K \in K(X) \).

2. The mapping \( \bigcup : P_S(P_S(X)) \rightarrow P_S(X) \), \( K \mapsto \bigcup K \), is continuous.

For a metric space \( (X,d) \), \( x \in X \) and a positive number \( r \), let \( B(x,r) = \{ y \in Y : d(x,y) < r \} \) be the \( r \)-ball about \( x \). For a set \( A \subseteq X \) and a positive number \( r \), by the \( r \)-ball about \( A \) we mean the set \( B(A,r) = \bigcup_{a \in A} B(a,r) \).

The following result is well-known (cf. [3].)

Lemma 2.12. Let \( (X,d) \) be a metric space and \( K \) a compact set of \( X \). Then for any open set \( U \) containing \( K \), there is an \( r > 0 \) such that \( K \subseteq B(K,r) \subseteq U \).

3. Scott topology and continuous domains

For a poset \( P \), a subset \( U \) of \( P \) is Scott open if (i) \( U = \uparrow U \) and (ii) for any directed subset \( D \) with \( \forall D \in U \) implies \( D \cap U \neq \emptyset \). All Scott open subsets of \( P \) form a topology, called the Scott topology on \( P \) and denoted by \( \sigma(P) \). The space \( \Sigma P = (P, \sigma(P)) \) is called the Scott space of \( P \). For the chain \( 2 = \{ 0,1 \} \) (with the order \( 1 < 2 \)), we have \( \sigma(2) = \{ \emptyset, \{ 1 \}, \{ 0,1 \} \} \). The space \( \Sigma 2 \) is well-known under the name of Sierpiński space. The upper topology on \( P \), generated by the complements of the principal ideals of \( P \), is denoted by \( v(P) \). The upper sets of \( P \) form the (upper) Alexandroff topology \( \alpha(P) \).

Lemma 3.1. \((\text{[3] Proposition II-2.1})\) For posets \( P, Q \) and \( f : P \rightarrow Q \), the following two conditions are equivalent:

1. \( f : \Sigma P \rightarrow \Sigma Q \) is continuous.

2. For each \( D \in \mathcal{D}(P) \) for which \( \forall D \exists \text{ in } P \), \( \forall f(D) \exists \text{ in } Q \) and \( f(\forall D) = \forall f(D) \).

For a dcpo \( P \) and \( A, B \subseteq P \), we say \( A \) is way below \( B \), written \( A \ll B \), if for each \( D \in \mathcal{D}(P) \), \( \forall D \in \uparrow B \) implies \( D \cap \uparrow A \neq \emptyset \). For \( B = \{ x \} \), a singleton, \( A \ll B \) is written \( A \ll x \) for short. For \( x \in P \), let \( w(x) = \{ F \in P(\leq x) : F \preceq x \} \), \( \down x = \{ u \in P : u \ll x \} \) and \( K(P) = \{ k \in P : k \ll k \} \). Points in \( K(P) \) are called compact elements of \( P \).

For the following definition and related conceptions, please refer to [3].

Definition 3.2. Let \( P \) be a dcpo and \( X \) a \( T_0 \) space.

1. \( P \) is called a continuous domain, if for each \( x \in P \), \( \down x \) is directed and \( x = \bigvee \down x \). When a complete lattice \( L \) is continuous, we call \( L \) a continuous lattice.
(2) $P$ is called an algebraic domain, if for each $x \in P$, $\{ k \in K(P) : k \leq x \}$ is directed and $x = \vee \{ k \in K(P) : k \leq x \}$. When a complete lattice $L$ is algebraic, we call $L$ an algebraic lattice.

(3) $P$ is called a quasicontinuous domain, if for each $x \in P$, $\{ \uparrow F : F \in w(x) \}$ is filtered and $\uparrow x = \bigcap \{ \uparrow F : F \in w(x) \}$.

(4) $X$ is called core-compact if $\mathcal{O}(X)$ is a continuous lattice.

**Remark 3.3.** It is well-known that if a topological space $X$ is locally compact, then it is core-compact (see, e.g., [6, Examples I-1.7]). In [11, Section 7] (see also [6, Exercise V-5.25]) Hofmann and Lawson gave a second-countable core-compact $T_0$ space $X$ in which every compact subset of $X$ has empty interior and hence it is not locally compact.

The following result is well-known (see [6]).

**Theorem 3.4.** Let $P$ be a dcpo.

1. If $P$ is algebraic, then it is continuous.
2. If $P$ is continuous, then it is quasicontinuous.
3. $P$ is continuous iff $\Sigma P$ is a $c$-space.
4. $P$ is quasicontinuous iff $\Sigma P$ is locally hypercompact.

The following example show that there is even a countable algebraic lattice $L$ such that $\text{Fin} L$ (note that the order on $\text{Fin} L$ is the reverse inclusion order $\supseteq$) is not a dcpo.

**Example 3.5.** Let $L = \mathbb{N} \cup \{ \omega_0, \omega_1, \cdots, \omega_n, \cdots \} \cup \{ \infty \}$. Define an order on $L$ as follows (see Figure 1):

(i) $1 < 2 < 3 < \cdots < n < n + 1 < \cdots < \omega_0$,
(ii) $n < \omega_n$ for each $n \in \mathbb{N}$, and
(iii) $\infty$ is a largest element of $L$.

It is easy to see that $L$ is a complete lattice. Moreover, for each $x \in L \setminus \{ \omega_0 \}$, $x \ll x$, whence $L$ is a countable algebraic lattice.

Now we show that $\text{Fin} L$ is not a dcpo. Let

\[
\uparrow F_n = \uparrow n \cup \{ \omega_1, \cdots, \omega_{n-1} \} = \uparrow n \cup \{ \omega_1, \cdots, \omega_{n-1} \} \cup \{ \infty \} = \{ n, n + 1, \cdots \} \cup \{ \omega_0, \omega_1, \cdots, \omega_n, \cdots \} \cup \{ \infty \}.
\]

![Figure 1: An algebraic lattice $L$ with non-dcpo $\text{Fin} L$](image)

It is clear that $\{ \uparrow F_n : n \in \mathbb{N} \} \subseteq \mathcal{D}(\text{Fin} L)$ and

\[
\bigcap_{n \in \mathbb{N}} \uparrow F_n = \{ \omega_0, \omega_1, \cdots, \omega_n, \cdots \} \cup \{ \infty \} \not\in \text{Fin} L.
\]

By Lemma 2.10 $\{ \uparrow F_n : n \in \mathbb{N} \}$ has no join in $\text{Fin} L$. Thus $\text{Fin} L$ is not a dcpo.
As \( \text{Fin} L \) is not a dcpo, it cannot be used as a mathematical model for denotational semantics of non-deterministic programs. So one should look for other novel mathematical structures, such as the poset of all nonempty compact saturated sets of a suitable \( T_0 \) space \( X \) and the poset of all strongly compact saturated sets of \( X \) (\( S \subseteq X \) is strongly compact if for all open sets \( U \) with \( S \subseteq U \), there is a finite set \( F \) with \( S \subseteq F \subseteq U \)) (cf. [9]).

4. \( d \)-spaces, well-filtered spaces and sober spaces

A \( T_0 \) space \( X \) is called a \( d \)-space (or monotone convergence space) if \( X \) (with the specialization order) is a dcpo and \( \mathcal{O}(X) \subseteq \sigma(X) \) (cf. [6, 26]).

It is easy to verify the following result (cf. [6, 30]).

**Proposition 4.1.** For a \( T_0 \) space \( X \), the following conditions are equivalent:

(1) \( X \) is a \( d \)-space.

(2) \( \mathcal{D}_c(X) = \mathcal{S}_c(X) \).

(3) \( X \) is a dcpo, and \( \mathcal{D} = \{ \vee D \} \) for any \( D \in \mathcal{D}(X) \).

(4) For any \( D \in \mathcal{D}(X) \) and \( U \in \mathcal{O}(X) \), \( \bigcap_{d \in D} \uparrow d \subseteq U \) implies \( \uparrow d \subseteq U \) (i.e., \( d \in U \)) for some \( d \in D \).

**Lemma 4.2.** ([34, Lemma 2.1] ) Let \( X \) be a \( d \)-space. Then for any nonempty closed subset \( A \) of \( X \), \( A = \downarrow \max(A) \), and hence \( \max(A) \neq \emptyset \).

A topological space \( X \) is called sober, if for any \( F \in \text{irr}(X) \), there is a unique point \( a \in X \) such that \( F = \{ a \} \). Hausdorff spaces are always sober (see, e.g., [5, Proposition 8.2.12]) and sober spaces are always \( T_0 \) since \( \{ x \} = \{ y \} \) always implies \( x = y \). The Sierpinski space \( \Sigma 2 \) is sober but not \( T_1 \) and an infinite set with the co-finite topology is \( T_1 \) but not sober (see Example 6.4).

The following conclusion is well-known (see, e.g., [6, 7, 9]).

**Proposition 4.3.** For a quasicontinuous domain \( P \), \( \Sigma P \) is sober.

For the sobriety of the Smyth power spaces, we have the following well-known result.

**Theorem 4.4.** (Heckmann-Keimel-Schalk Theorem) ([14, Theorem 3.13], [23, Lemma 7.20]) For a \( T_0 \) space \( X \), the following conditions are equivalent:

(1) \( X \) is sober.

(2) For any \( A \in \text{irr}(P_S(X)) \) and \( U \in \mathcal{O}(X) \), \( \bigcap K \subseteq U \) implies \( K \subseteq U \) for some \( K \in A \).

(3) \( P_S(X) \) is sober.

A \( T_0 \) space \( X \) is called well-filtered if for any filtered family \( K \subseteq K(X) \) and open set \( U \), \( \bigcap K \subseteq U \) implies \( K \subseteq U \) for some \( K \in K \).

**Remark 4.5.** The following implications are well-known (which are irreversible) (cf. [9]):

sobriety \( \Rightarrow \) well-filteredness \( \Rightarrow \) \( d \)-space.

In [27] and [11, 15], the following two useful results were given.

**Proposition 4.6.** ([27, Corollary 3.2]) If a dcpo \( P \) endowed with the Lawson topology is compact (in particular, \( P \) is a complete lattice), then \( \Sigma P \) is well-filtered.

**Theorem 4.7.** ([14, Corollary 4.6], [15, Theorem 2.3]) For a \( T_0 \) space \( X \), the following conditions are equivalent:

(1) \( X \) locally compact and sober.

(2) \( X \) is locally compact and well-filtered.
(3) $X$ is core-compact and sober.

For the well-filteredness of topological spaces, a similar result to Theorem 4.4 was proved in [32] (see also [30]).

**Theorem 4.8.** ([32, Theorem 5.3], [32, Theorem 4]) For a $T_0$ space, the following conditions are equivalent:

1. $X$ is well-filtered.
2. $P_S(X)$ is a $d$-space.
3. $P_S(X)$ is well-filtered.

**Corollary 4.9.** For a well-filtered space (especially, a sober space) $X$, $K(X)$ (with the Smyth order) is a dcpo and the upper Vietoris topology is coarser than the Scott topology on $K(X)$.

By Theorem 4.8 and Corollary 4.9 we know that for a $T_0$ space $X$, if $P_S(X)$ is a $d$-space (equivalently, $X$ is a well-filtered space), then $ΣK(X)$ is a $d$-space. Example 7.2 below shows that $ΣK(X)$ is a sober space does not imply that $X$ is well-filtered (i.e., $P_S(X)$ is a $d$-space) in general.

**Example 4.10.** (Johnstone’s dcpo adding a top element) Let $J = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with ordering defined by $(j, k) \leq (m, n)$ if $j = m$ and $k \leq n$, or $n = \infty$ and $k \leq m$. $J$ is a well-known dcpo constructed by Johnstone in [14] (see Figure 2).

The set $J_{\text{max}} = \{(n, \infty) : n \in \mathbb{N}\}$ is the set of all maximal elements of $J$. Adding top $\top$ to $J$ yields a dcpo $J_\top = J \cup \{\top\}$ ($x \leq \top$ for any $x \in J$). Then $\top$ is the largest element of $J_\top$ and $\{\top\} \in \sigma(J_\top)$. The following three conclusions about $\Sigma J$ are known (see, for example, [17, Example 3.1] and [18, Lemma 3.1]):

(i) $\text{Irrc}(\Sigma J) = \{\{x\} = \downarrow_J x : x \in J\} \cup \{\emptyset\}$.
(ii) $K(\Sigma J) = (2^{J_{\text{max}}} \setminus \{\emptyset\}) \cup \text{Fin}\ J$.
(iii) $\Sigma J$ is not well-filtered.

Hence we have

(a) $\text{Irrc}(\Sigma J_\top) = \{\{x\} = \downarrow_{J_\top} x : x \in J_\top\} \cup \{\emptyset\}$ by (i).
(b) $K(\Sigma J_\top) = \{\uparrow_{J_\top} G : G \text{ is nonempty and } G \subseteq J_{\text{max}} \cup \{\top\}\} \cup \text{Fin}\ J_\top$ by (ii).
(c) $\Sigma J$ is not a dcpo.

$\mathcal{G} = \{J \setminus F : F \in (J_{\text{max}})^\omega\}$. Then by (ii), $\mathcal{G} \subseteq K(\Sigma J_\top)$ is a filtered family and $\bigcap \mathcal{G} = \bigcap_{F \in (J_{\text{max}})^\omega} (J \setminus F) = J_{\text{max}} \setminus (\bigcup (J_{\text{max}})^\omega) = \emptyset$, whence by Lemma 2.9 $\mathcal{G}$ has no least upper bound in $K(X)$. Thus $K(\Sigma J)$ is not a dcpo.
(d) $K(\Sigma^\gamma_T)$ is a dcpo.

Suppose that $\{K_d : d \in D\}$ is directed in $K(\Sigma^\gamma_T)$ (with the Smyth order). Then $\top \in \bigcap_{d \in D} K_d$ and hence $\bigcap_{d \in D} K_d \neq \emptyset$. Now we show that $\bigcap_{d \in D} K_d \in K(\Sigma^\gamma_T)$. If $\bigcap_{d \in D} K_d = \{\top\}$, then obviously $\bigcap_{d \in D} K_d \in K(\Sigma^\gamma_T)$. Now we assume $\bigcap_{d \in D} K_d \in K(\Sigma^\gamma_T) \neq \{\top\}$ and $\{V_i : i \in I\} \subseteq \sigma(\Sigma^\gamma_T)$ is an open cover of $\bigcap_{d \in D} K_d$. For each $d \in D$ and $i \in I$, let $H_d = K_d \setminus \{\top\}$ and $U_i = V_i \setminus \{\top\}$. Then $H_d \in K(\Sigma^\gamma_T) (d \in D)$, $U_i \in \sigma(\Sigma^\gamma_T) (i \in I)$ and $\emptyset \neq \bigcap_{d \in D} H_d = \bigcap_{d \in D} K_d \setminus \{\top\} \subseteq \bigcup_{i \in I} V_i \setminus \{\top\} = \bigcup_{i \in I} U_i$. By Example 3.1, there is $d_0 \in D$ such that $H_{d_0} \subseteq \bigcup_{i \in I} U_i$, and consequently, there is $J \in I(<\omega)$ such that $H_d \subseteq \bigcup_{i \in J} U_i$. It follows that $\bigcap_{d \in D} K_d \subseteq K_{d_0} \subseteq \bigcup_{i \in J} V_i$. Thus $\bigcap_{d \in D} K_d \in K(\Sigma^\gamma_T)$. By Lemma 2.9 $K(\Sigma^\gamma_T)$ is a dcpo.

(e) $\Sigma^\gamma_T$ is not well-filtered.

Indeed, let $\mathcal{K} = \{\top_{\gamma_{J_0}(\max \setminus F) : F \in (\max)^{(<\omega)})\}$. Then by (b), $\mathcal{K} \subseteq K(\Sigma^\gamma_T)$ is a filtered family and $\bigcap \mathcal{K} = \bigcap_{F \in (\max)^{(<\omega)}} \top_{\gamma_{J_0}(\max \setminus F) \cap \top_{\gamma_{J_0}(\max \setminus F)} = \bigcup_{F \in (\max)^{(<\omega)}} \top_{J_0}(\max \setminus F)} \top_{\gamma_{J_0}(\max \setminus F) \cup \top_{\gamma_{J_0}(\max \setminus F)} = \top} \cup \bigcup_{F \in (\max)^{(<\omega)}} \top_{\gamma_{J_0}(\max \setminus F)} = \top$. Therefore, $\Sigma^\gamma_T$ is not well-filtered.

5. Topological Rudin Lemma, Rudin spaces and well-filtered determined spaces

In Section 5, we recall some concepts and results about the topological Rudin Lemma, Rudin spaces, $\omega$-Rudin spaces, well-filtered determined spaces and $\omega$-well-filtered determined spaces in [16, 24, 29, 30, 31] that will be used in the next four sections.

Rudin’s Lemma is a useful tool in non-Hausdorff topology and plays a crucial role in domain theory (see [4, 7, 9]). Rudin [22] proved her lemma by transfinite methods, using the Axiom of Choice. Heckmann and Keimel [10] presented the following topological variant of Rudin’s Lemma.

Lemma 5.1. (Topological Rudin Lemma) Let $X$ be a topological space and $A$ an irreducible subset of the Smyth power space $P_\Sigma(X)$. Then every closed set $C \subseteq X$ that meets all members of $A$ contains a minimal irreducible closed subset $A$ that still meets all members of $A$.

Applying Lemma 5.1, to the Alexandroff topology on a poset $P$, one obtains the original Rudin’s Lemma.

Corollary 5.2. (Rudin’s Lemma) Let $P$ be a poset, $C$ a nonempty lower subset of $P$ and $F \in \text{Fin}P$ a filtered family with $F \subseteq \bigcup C$. Then there exists a directed subset $D$ of $C$ such that $F \subseteq \bigcup_{d \in D} F$.

For a $T_0$ space $X$ and $\mathcal{K} \subseteq K(X)$, let $M(\mathcal{K}) = \{A \in C(X) : K \cap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \bigcap A$) and $m(\mathcal{K}) = \{A \in C(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

In [24, 30], based on topological Rudin’s Lemma, Rudin spaces and well-filtered spaces (WD spaces for short) were introduced and investigated. These two spaces are closely related to sober spaces and well-filtered spaces (see [24, 30]).

Definition 5.3. (24, 30) Let $X$ be a $T_0$ space.

1. A nonempty subset $A$ of $X$ is said to have the Rudin property, if there exists a filtered family $\mathcal{K} \subseteq K(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\text{RD}(X) = \{A \in C(X) : A \text{ has Rudin property}\}$. The sets in $\text{RD}(X)$ will also be called Rudin sets.

2. $X$ is called a Rudin space, RD space for short, if $\text{Irr}_c(X) = \text{RD}(X)$, that is, all irreducible closed sets of $X$ are Rudin sets.

The Rudin property is called the compactly filtered property in [24]. In order to emphasize its origin from (topological) Rudin’s Lemma, such a property was called the Rudin property in [30]. Clearly, $A$ has Rudin property iff $\overline{A}$ has Rudin property (that is, $\overline{A}$ is a Rudin set).

Proposition 5.4. (24, 30) Let $X$ be a $T_0$ space and $Y$ a well-filtered space. If $f : X \rightarrow Y$ is continuous and $A \subseteq X$ has Rudin property, then there exists a unique $y_A \in X$ such that $f(A) = \{y_A\}$.

Motivated by Proposition 5.4 the following concept was introduced in [30].
Definition 5.5. ([30]) Let $X$ be a $T_0$ space.

1. A subset $A$ of $X$ is called a well-filtered determined set, WD set for short, if for any continuous mapping $f : X \to Y$ to a well-filtered space $Y$, there exists a unique $y_A \in Y$ such that $f(A) = \{y_A\}$. Denote by $\text{WD}(X)$ the set of all closed well-filtered determined subsets of $X$.

2. $X$ is called a well-filtered determined space, WD space for short, if all irreducible closed subsets of $X$ are well-filtered determined, that is, $\text{Irr}(X) = \text{WD}(X)$.

Obviously, a subset $A$ of a space $X$ is well-filtered determined iff $A$ is well-filtered determined.

Proposition 5.6. ([24, 30]) Let $X$ be a $T_0$ space. Then $\mathcal{S}_c(X) \subseteq \mathcal{D}_c(X) \subseteq \mathcal{RD}(X) \subseteq \text{WD}(X) \subseteq \text{Irr}(X)$.

Definition 5.7. ([30]) A $T_0$ space $X$ is called a directed closure space, DC space for short, if $\text{Irr}(X) = \mathcal{D}_c(X)$, that is, for each $A \in \text{Irr}(X)$, there exists a directed subset of $X$ such that $A = \downarrow A$.

Corollary 5.8. ([30, Corollary 6.3]) Sober $\Rightarrow$ DC $\Rightarrow$ RD $\Rightarrow$ WD.

Proposition 5.9. ([30, Corollary 7.11]) For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is well-filtered.
2. $\text{RD}(X) = \mathcal{S}_c(X)$.
3. $\text{WD}(X) = \mathcal{S}_c(X)$.

Theorem 5.10. ([30, Theorem 6.6]) For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is sober.
2. $X$ is a DC d-space.
3. $X$ is a well-filtered DC space.
4. $X$ is a well-filtered Rudin space.
5. $X$ is a well-filtered WD space.

Proposition 5.11. ([5, Proposition 3.2]) Let $X$ be a locally hypercompact $T_0$ space and $A \subseteq \text{Irr}(X)$. Then there exists a directed subset $D \subseteq \downarrow A$ such that $A = \downarrow D$. Therefore, $X$ is a DC space.

Proposition 5.12. ([30, Theorem 6.10 and Theorem 6.15]) Let $X$ be a $T_0$ space.

1. If $X$ is locally compact, then $X$ is a Rudin space.
2. If $X$ is core-compact, then $X$ is a WD space.

It is still not known whether every core-compact $T_0$ space is a Rudin space (see [34, Question 5.14]).

Question 5.13. For a core-compact $T_0$ space $X$, is the Smyth power space $P_S(X)$ a WD space? Is the Scott power space $\Sigma_K(X)$ a WD space?

From Theorem 5.10 and Proposition 5.12 one can immediately get the following result, which was first proved by Lawson, Wu and Xi [16] using a different method.

Corollary 5.14. ([16, 30]) Every core-compact well-filtered space is sober.

By Corollary 5.14, Theorem 4.7 can be strengthened into the following one.

Theorem 5.15. For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ locally compact and sober.
2. $X$ locally compact and well-filtered.
3. $X$ core-compact and sober.
4. $X$ core-compact and well-filtered.
Figure 3: Certain relations among some kinds of spaces

In order to emphasize the Scott topology, we introduce the following notions.

**Definition 5.16.** A poset $P$ is called a **sober dcpo** (resp., a **well-filtered dcpo**) if $\Sigma P$ is a sober space (resp., well-filtered space).

Clearly, a sober dcpo is a well-filtered dcpo. For Isbile’s lattice $L$ constructed in [12], $\Sigma L$ is non-sober, namely, $L$ is not a sober dcpo, and by Corollary 4.6, $L$ is well-filtered. The Johnstone’s dcpo $\mathbb{J}$ (see Example 4.10) is not well-filtered.

**Definition 5.17.** Let $P$ be a poset.

1. $P$ is said to be a **DC poset** if $\Sigma P$ is a DC space.
2. $P$ is said to be a **Rudin poset** if $\Sigma P$ is a Rudin space.
3. $P$ is said to be a **well-filtered determined poset**, a **WD-poset** for short, if $\Sigma P$ is a well-filtered determined space.
4. When a dcpo $P$ is a Rudin poset (resp., a well-filtered determined poset), we will call $P$ a **Rudin dcpo** (resp., a **well-filtered determined dcpo**).

The following corollary follows directly from Theorem 5.10.

**Corollary 5.18.** For a poset $P$, the following conditions are equivalent:

1. $P$ is a sober dcpo.
2. $P$ is a DC dcpo.
3. $P$ is a DC well-filtered dcpo.
4. $P$ is a Rudin well-filtered dcpo.
5. $P$ is a WD well-filtered dcpo.

In [29], the following countable versions of Rudin spaces and WD spaces were introduced and studied.

**Definition 5.19.** ([29, Definition 5.1]) Let $X$ be a $T_0$ space and $A$ a nonempty subset of $X$.

(a) The set $A$ is said to be an **$\omega$-Rudin set**, if there exists a countable filtered family $\mathcal{K} \subseteq \mathcal{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\text{RD}_\omega(X)$ denote the set of all closed $\omega$-Rudin sets of $X$.

(b) The space $X$ is called **$\omega$-Rudin space**, if $\text{Irr}_c(X) = \text{RD}_\omega(X)$ or, equivalently, all irreducible (closed) subsets of $X$ are $\omega$-Rudin sets.

**Definition 5.20.** ([29, Definition 3.9]) A $T_0$ space $X$ is called **$\omega$-well-filtered**, if for any countable filtered family $\{K_n : n < \omega\} \subseteq \mathcal{K}(X)$ and $U \in \mathcal{O}(X)$, it holds that

$$\bigcap_{n < \omega} K_n \subseteq U \Rightarrow \exists n_0 < \omega, K_{n_0} \subseteq U.$$
Definition 5.21. ([29] Definition 5.4]) Let $X$ be a $T_0$ space and $A$ a nonempty subset of $X$.

(a) The set $A$ is called an $\omega$-well-filtered determined set, $\omega$-WD set for short, if for any continuous mapping $f : X \to Y$ to an $\omega$-well-filtered space $Y$, there exists a (unique) $y_A \in Y$ such that $f(A) = \{y_A\}$. Denote by $\text{WD}_\omega(X)$ the set of all closed $\omega$-well-filtered determined subsets of $X$.

(b) The space $X$ is called $\omega$-well-filtered determined, $\omega$-WD space for short, if $\text{irr}_c(X) = \text{WD}_\omega(X)$ or, equivalently, all irreducible (closed) subsets of $X$ are $\omega$-well-filtered determined.

For a $T_0$ space $X$, it was proved in [29] Proposition 5.5] that $\mathcal{S}_c(X) \subseteq \text{RD}_\omega(X) \subseteq \text{WD}_\omega(X) \subseteq \text{irr}_c(X)$. Therefore, every $\omega$-Rudin space is $\omega$-well-filtered determined.

By [29] Theorem 5.11, we have the following similar result to Theorem 5.10.

Proposition 5.22. For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is sober.
2. $X$ is an $\omega$-Rudin and $\omega$-well-filtered space.
3. $X$ is an $\omega$-well-filtered determined and $\omega$-well-filtered space.

Theorem 5.23. ([31] Theorem 5.6 and Theorem 6.12]) Let $X$ be a $T_0$ space.

1. If the sobrification $X^s$ of $X$ is first-countable, then $X$ is an $\omega$-Rudin space.
2. If $X$ is first-countable, then $X$ is a WD space.

From Theorem 5.10 and Theorem 5.23 we immediately deduce the following result.

Corollary 5.24. ([29] Theorem 4.2]) Every first-countable well-filtered $T_0$ space is sober.

It is still not known whether a first-countable $T_0$ space is a Rudin space (see [31] Problem 6.15]). Since the first-countability is a hereditary property, from Remark 2.4 and Theorem 5.23 we know that if the Smyth power space $P_\omega(X)$ of a $T_0$ space $X$ is first-countable, then $X$ is a WD space.

So naturally we ask the following question.

Question 5.25. Is a $T_0$ space with a first-countable Smyth power space a Rudin space?

In Example 7.1 a $T_0$ space $X$ is given for which the Scott power space $\Sigma \mathcal{K}(X)$ is a first-countable sober $c$-space but $X$ is not a WD space (and hence not a Rudin space).

By Proposition 5.22 and Theorem 5.23 we have the following result.

Corollary 5.26. ([31] Theorem 5.9]) Every $\omega$-well-filtered space with a first-countable sobrification is sober.

In Theorem 5.23 and Corollary 5.26 the first-countability of $X^s$ can not be weakened to that of $X$ as shown in the following example. It is also shows that the first-countability of a $T_0$ space $X$ does not imply the first-countability of $X^s$ in general.

Example 5.27. Let $\omega_1$ be the first uncountable ordinal number and $P = [0, \omega_1)$. Then

(a) $\mathcal{C}(\Sigma P) = \{\uparrow t : t \in P\} \cup \{\emptyset, P\}$.

(b) $\Sigma P$ is first-countable and compact (since $P$ has a least element 0).

(c) $(\Sigma P)^s$ is not first-countable.

In fact, it is easy to verify that $(\Sigma P)^s$ is homeomorphic to $\Sigma[0, \omega_1]$. Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, $\Sigma[0, \omega_1]$ has no countable base at the point $\omega_1$.

(d) $K(\Sigma P) = \{\uparrow x : x \in P\}$ and $\Sigma P$ is not an $\omega$-Rudin space.

For $K \in K(\Sigma P)$, we have $\inf K \in K$, and hence $K = \uparrow \inf K$. So $K(\Sigma P) = \{\uparrow x : x \in P\}$. Now we show that the irreducible closed set $P$ is not an $\omega$-Rudin set. For any countable filtered family $\{\uparrow \alpha_n : n \in \mathbb{N}\} \subseteq K(\Sigma P)$, let $\beta = \sup \{\alpha_n : n \in \mathbb{N}\}$. If $\beta$ is still a countable ordinal number, clearly, $\downarrow \beta \in M(\{\uparrow \alpha_n : n \in \mathbb{N}\})$ and $P \neq \downarrow \beta$. Therefore, $P \notin m(\{\uparrow \alpha_n : n \in \mathbb{N}\})$. Thus $P$ is not an $\omega$-Rudin set, and hence $\Sigma P$ is not an $\omega$-Rudin space.
(e) \( \Sigma P \) is a Rudin space.

It is easy to check that \( \text{Ir}(\Sigma P) = \{ \downarrow x : x \in P \} \cup \{ P \} \). Clearly, \( \downarrow x \) is a Rudin set for each \( x \in P \).

Now we show that \( P \) is a Rudin set. First, \( \{ \uparrow s : s \in P \} \) is filtered. Second, \( P \in M(\{ \uparrow s : s \in P \}) \). For a closed subset \( B \) of \( \Sigma P \), if \( B \neq \emptyset \), then \( B = \emptyset \) for some \( t \in P \), and hence \( \uparrow (t + 1) \cap \downarrow t = \emptyset \). Thus \( B \notin M(\{ \uparrow s : s \in P \}) \), proving that \( P \) is a Rudin set.

(f) \( P \) is not a dcpo (note that \( P \) is directed and \( \lor P \) does not exist). So \( \Sigma P \) is not a d-space, and hence \( \Sigma P \) is neither well-filtered nor sober.

(g) \( \Sigma P \) is \( \omega \)-well-filtered.

If \( \{ \uparrow x_n : n \in \mathbb{N} \} \subseteq K(\Sigma P) \) is countable filtered family and \( U \in \sigma(P) \) with \( \bigcap_{n \in \mathbb{N}} \uparrow x_n \subseteq U \), then \( \{ x_n : i \in \mathbb{N} \} \) is a countable subset of \( P = [0, \omega_1) \). Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, we have \( \beta = \sup\{ x_n : n \in \mathbb{N} \} \in P \), and hence \( \uparrow \beta = \bigcap_{n \in \mathbb{N}} \uparrow x_n \subseteq U \). Therefore, \( \beta \in U \), and consequently, \( x_n \in U \) for some \( n \in \mathbb{N} \) or, equivalently, \( \uparrow x_n \subseteq U \), proving that \( \Sigma P \) is \( \omega \)-well-filtered.

6. Well-filteredness of Scott power spaces

In this section, we mainly discuss the following two questions:

**Question 1.** Is the Scott power space \( \Sigma K(X) \) of a d-space \( X \) a d-space?

**Question 2.** Is the Scott power space \( \Sigma K(X) \) of a well-filtered space \( X \) well-filtered?

First, Example 6.4 below shows that there is a second-countable Noetherian d-space \( X \) for which \( K(X) \) is not a dcpo and hence neither the Smyth power space \( P_S(X) \) nor the Scott power space \( \Sigma K(X) \) is a d-space, which gives a negative answer to Question 1.

In order to present the example, we need the following lemma.

**Lemma 6.1.** (23, Lemma 7.26) For a locally compact \( T_0 \) space \( X \), the Scott topology is coarser than the upper Vietoris topology on \( K(X) \), that is, \( \sigma(K(X)) \subseteq \mathcal{O}(P_S(X)) \).

**Proof.** It was proved by Schalk in 23 (see 23, the proof of Lemma 7.26). We present a more direct proof here.

Suppose that \( U \in \sigma(K(X)) \) and \( K \in U \). Let \( \mathcal{K} = \{ G \in K(X) : K \subseteq \text{int } G \} \). Now we show that \( \mathcal{K} \) is filtered and \( G = \bigcap \mathcal{K} \).

1° For each \( U \in \mathcal{O}(X) \) with \( K \subseteq U \), there is \( G_U \in \mathcal{K} \) with \( G_U \subseteq U \).

If \( U \in \mathcal{O}(X) \) for which \( K \subseteq U \), then for each \( x \in K \), there is \( K_x \in K(X) \) such that \( x \in \text{int } K_x \subseteq K_x \subseteq U \) since \( X \) is locally compact. By the compactness of \( K \), there is \( \{ x_1, x_2, \ldots, x_n \} \subseteq K \) such that \( K \subseteq \bigcup_{i=1}^{n} \text{int } K_{x_i} \).

Let \( G_U = \bigcup_{i=1}^{n} K_{x_i} \). Then \( K \subseteq \text{int } G_U \subseteq G_U \subseteq U \), whence \( G_U \in \mathcal{K} \) and \( G_U \subseteq U \).

2° \( \mathcal{K} \) is filtered.

Suppose that \( G_1, G_2 \in \mathcal{K} \). Then \( K \subseteq \text{int } G_1 \cap \text{int } G_2 \). Hence, by what was shown above, there is \( G_3 \in \mathcal{K} \) with \( G_3 \subseteq \text{int } G_1 \cap \text{int } G_2 \), proving the filteredness of \( \mathcal{K} \).

By 1° and 2°, \( K \subseteq \bigcap \mathcal{K} \subseteq \{ U \in \mathcal{O}(X) : K \subseteq U \} = K \), whence \( K = \bigcap \mathcal{K} = \bigvee_{K(X)} K \) by Lemma 2.9.

Since \( K \in U \in \sigma(K(X)) \), \( G \in U \) for some \( G \in \mathcal{K} \). Hence \( K \in \sigma(K(X)) \), proving the filteredness of \( \mathcal{K} \).

By Corollary 4.9 and Lemma 6.1, we get the following corollary.

**Corollary 6.2.** (23, Lemma 7.26) If \( X \) is a locally compact sober space (equivalently, a locally compact well-filtered space or a core-compact well-filtered space), then the upper Vietoris topology and the Scott topology on \( K(X) \) coincide.

Considering Remark 3.3 and Lemma 6.1 we have the following question.
Question 6.3. For a core-compact $T_0$ space $X$, is the Scott topology coarser than the upper Vietoris topology on $K(X)$?

Example 6.4. Let $X$ be a countably infinite set (for example, $X = \mathbb{N}$) and $X_{cof}$ the space equipped with the co-finite topology (the empty set and the complements of finite subsets of $X$ are open). Then

(a) $C(X_{cof}) = \{\emptyset, X\} \cup X^{<\omega}$, $X_{cof}$ is $T_1$ and hence a $d$-space.
(b) $irr(X_{cof}) = \{\{x\} : x \in X\} \cup \{X\}$.
(c) $K(X_{cof}) = 2^X \setminus \{\emptyset\}$.
(d) $X_{cof}$ is second-countable.

Clearly, $\mathcal{O}(X_{cof})$ is countable, and hence $X_{cof}$ is second-countable.

(e) $X_{cof}$ is Noetherian and hence locally compact.

Since every subset of $X$ is compact in $X_{cof}$, the space $X_{cof}$ is a Noetherian space and hence a locally compact space.

(f) $X_{cof}$ is a Rudin space.

By (e) and Proposition 5.12 (or by (d) and Corollary 8.14 below), $X_{cof}$ is a Rudin space.

(g) $K(X_{cof})$ is not a dcpo and hence $X_{cof}$ is neither well-filtered nor sober.

$k = \{X \setminus F : F \in X^{<\omega}\} \subseteq K(X_{cof})$ is countable filtered and $\bigcap K_X = X \setminus \bigcup X^{<\omega} = X \setminus \emptyset$, whence $\bigvee K \not\in K(X_{cof})$ by Lemma 2.9. Thus $K(X_{cof})$ is not a dcpo, whence by Remark 4.9 and Theorem 4.8 $X_{cof}$ is neither well-filtered nor sober.

(h) The upper Vietoris topology and the Scott topology on $K(X_{cof})$ agree.

By the local compactness of $X_{cof}$ and Lemma 6.1, we have $\sigma(K(X_{cof})) \subseteq \mathcal{O}(P_S(X_{cof}))$. Now we show that $\square U \subseteq \sigma(K(X_{cof}))$ for each $U \in \mathcal{O}(X_{cof}) \setminus \{\emptyset\}$. Clearly, $\square U = \bigcup_{K \in K(X_{cof})} U$. Suppose that $K_D = \{K_d : d \in D\} \in \mathcal{D}(K(X_{cof}))$ and $\bigvee_{K \in K(X_{cof})} K_d \in \square U$. Then by Lemma 2.9, $\bigcup_{d \in D} K_d = \bigvee_{K \in K(X_{cof})} K_d \subseteq \square U$ or, equivalently, $X \setminus U \subseteq \bigcup_{d \in D} (X \setminus K_d)$. Since $X \setminus U$ is finite and $\{X \setminus K_d : d \in D\}$ is directed, there is $d_0 \in D$ with $X \setminus U \subseteq X \setminus K_{d_0}$, whence $K_{d_0} \setminus U$. Thus $\square U \in \sigma(K(X_{cof}))$. Therefore, $\mathcal{O}(P_S(X_{cof})) \subseteq \sigma(K(X_{cof}))$ and hence $\sigma(K(X_{cof})) = \mathcal{O}(P_S(X_{cof}))$.

(i) $\Sigma K(X_{cof})$ is not a $d$-space and hence it is neither a well-filtered space nor a sober space.

Since $K(X_{cof})$ is not a dcpo, $\Sigma K(X_{cof})$ is not a $d$-space. By Remark 4.5, $\Sigma K(X_{cof})$ is neither a well-filtered space nor a sober space.

Now we investigate Question 2. First, as one of the main results of this paper, we have the following conclusion.

**Theorem 6.5.** For a well-filtered space $X$, $\Sigma K(X)$ is well-filtered.

**Proof.** By Corollary 4.9, $K(X)$ is a dcpo and $\mathcal{O}(P_S(X)) \subseteq \sigma(K(X))$ (i.e., $\square U \subseteq \sigma(K(X))$ for all $U \in \mathcal{O}(X)$). Suppose that $\{K_d : d \in D\} \subseteq \Sigma K(X)$ is filtered, $U \in \sigma(K(X))$ and $\bigcap_{d \in D} K_d \subseteq U$. If $K_d \subseteq U$ for each $d \in D$, that is, $K_d \cap (K(X) \setminus U) \neq \emptyset$, then $\{K_d : d \in D\} \in \mathcal{I}(P_S(\Sigma K(X)))$ and hence by Lemma 5.1 $K(X) \setminus U$ contains a minimal irreducible closed subset $A$ that still meets all members $K_d$. For each $d \in D$, let $K_d = \bigcup_{K \in K(X)} (K_d \cap A)$.

**Claim 1:** For each $d \in D$, $K_d \in K(X)$ and $K_d \in A$.

By $K_d \in \Sigma K(X)$ and $A \in \mathcal{C}(\Sigma K(X))$, we have that $\bigcup_{K \in K(X)} (K_d \cap A) \in \Sigma K(X)$, and hence $\bigcup_{K \in K(X)} (K_d \cap A) \in K(P_S(X))$ by $\mathcal{O}(P_S(X)) \subseteq \mathcal{C}(K(X))$. By Lemma 2.11, $K_d = \bigcup_{K \in K(X)} (K_d \cap A) = \bigcup(K_d \cap A) \in K(X)$. Since $A = \bigcup_{K \in K(X)} A$ and $K_d \cap A \neq \emptyset$, we have $K_d \in A$.

**Claim 2:** $\{K_d : d \in D\} \subseteq K(X)$ is filtered (by Claim 1 and the filteredness of $\{K_d : d \in D\}$).

**Claim 3:** $K = \bigcap_{d \in D} K_d \in K(X)$. By Claim 1, Claim 2 and Lemma 2.9, $K = \bigcap_{d \in D} (K_d : d \in D) \in A$ since $A \in \mathcal{C}(\sigma(K(X)))$.

**Claim 4:** For each $k \in K$, $A \subseteq \bigcap_{K \in K(X)} \{K\}$. For each $d \in D$, we have $k \in K \subseteq K_d = \bigcup(K_d \cap A)$, whence there is $G_d \in K_d \cap A$ such that $k \in G_d$, and consequently, $G_d \in K_d \cap A \cap \bigcap_{K \in K(X)} \{K\}$. Therefore, $A \cap \bigcap_{K \in K(X)} \{K\} \in M(\{K_d : d \in D\})$. By the minimality of $A$ and $\bigcap_{K \in K(X)} \{K\} \in \mathcal{C}(P_S(X)) \subseteq \mathcal{C}(\Sigma K(X))$, we have $A = \bigcap_{K \in K(X)} \{K\} \cap A$, that is, $A \subseteq \bigcap_{K \in K(X)} \{K\}$. 

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Claim 5: \( A = \downarrow_{K(X)} K \).

By Claim 3 and Claim 4, \( \downarrow_{K(X)} K \subseteq A \subseteq \bigcap_{k \in K} \Diamond_{K(k)} \{ k \} \). Clearly,

\[
G \in \bigcap_{k \in K} \Diamond_{K(k)} \{ k \} \iff \forall k \in K, G \in \Diamond_{K(k)} \{ k \} \iff \forall k \in K, G \cap \{ k \} \neq \emptyset \iff \forall k \in K, k \in G \iff K \subseteq G.
\]

This implies that \( \bigcap_{k \in K} \Diamond_{K(k)} \{ k \} = \downarrow_{K(X)} K \), and hence \( A = \downarrow_{K(X)} K \).

Claim 5: \( K \in \bigcap_{d \in D} \mathcal{K}_d \).

For each \( d \in D \), by \( \mathcal{K}_d \bigcap \mathcal{A} = \emptyset, \mathcal{K}_d = \uparrow_{K(X)} \mathcal{K}_d \) and \( A = \downarrow_{K(X)} K \), we have \( K \in \mathcal{K}_d \), whence \( K \in \bigcap_{d \in D} \mathcal{K}_d \subseteq \mathcal{U} \), being a contraction with \( K \in \mathcal{A} \subseteq K(X) \setminus \mathcal{U} \).

Therefore, there is \( d_0 \in D \) such that \( \mathcal{K}_{d_0} \subseteq \mathcal{U} \), proving that \( \Sigma K(X) \) is well-filtered.

Example 7.2 shows that unlike Smyth power spaces (see Theorem 4.8), the converse of Theorem 6.5 does not hold.

From Theorem 5.15 and Theorem 6.5 we deduce the following result.

Corollary 6.6. For a well-filtered space \( X \), the following two conditions are equivalent:

1. \( \Sigma K(X) \) is core-compact.
2. \( \Sigma K(X) \) is locally compact.

By Theorem 5.10, Theorem 5.15 and Theorem 6.5, we have the following two corollaries.

Corollary 6.7. For a well-filtered space \( X \), the following three conditions are equivalent:

1. \( \Sigma K(X) \) is sober.
2. \( \Sigma K(X) \) is Rudin.
3. \( \Sigma K(X) \) is well-filtered determined.

Corollary 6.8. Let \( X \) be a well-filtered space.

1. If \( K \in K(\Sigma K(X)) \), then \( \bigcup K \in K(X) \).
2. The mapping \( \bigcup : \Sigma K(\Sigma K(X)) \to \Sigma K(X) \), \( K \to \bigcup K \), is continuous.

Proof. (1): By Corollary 4.9, \( \mathcal{O}(P_S(X)) \subseteq \sigma(\mathcal{K}(X)) \). For \( K \in K(\Sigma K(X)) \), we have \( K \in K(\mathcal{P}_S(X)) \) since \( \mathcal{O}(P_S(X)) \subseteq \sigma(K(X)) \). By Lemma 2.11, \( \bigcup \mathcal{K} \in K(X) \).

(2): Suppose that \( \{ \mathcal{K}_d : d \in D \} \subseteq K(\Sigma K(X)) \) is directed (with the Smyth order) for which \( \bigcup_{K(\Sigma K(X))} \{ \mathcal{K}_d : d \in D \} = \bigcap_{d \in D} \mathcal{K}_d \). It follows that \( \bigcup_{K(\Sigma K(X))} \{ \mathcal{K}_d : d \in D \} = \bigcup_{d \in D} \mathcal{K}_d \). For each \( K \in \bigcap_{d \in D} \mathcal{K}_d \), define \( \varphi(K) = \bigcap_{d \in D} \mathcal{K}_d \). Hence \( \bigcup_{d \in D} \varphi(K) = K \). Therefore, \( \bigcup_{d \in D} \varphi(K) \subseteq \bigcup_{d \in D} \mathcal{K}_d \).

Conversely, for each \( \varphi \in \bigcup_{d \in D} \mathcal{K}_d \), \( x \in \bigcap_{d \in D} \varphi(d) \) and \( d \in D \), we have that \( \uparrow x \subseteq \varphi(d') \in \mathcal{K}_d \), and consequently, \( \uparrow x \subseteq \bigcup_{d \in D} \mathcal{K}_d \). It follows that \( \bigcap_{d \in D} \varphi(d) \subseteq \bigcup_{d \in D} \mathcal{K}_d \). Therefore, \( \bigcup_{d \in D} \varphi(d) \subseteq \bigcup_{d \in D} \mathcal{K}_d \).

Thus \( \bigcup_{K(\Sigma K(X))} \{ \mathcal{K}_d : d \in D \} = \bigcup_{d \in D} \mathcal{K}_d \). By Lemma 3.1, \( \bigcup : \Sigma K(\Sigma K(X)) \to \Sigma K(X) \) is continuous.

Proposition 6.9. Let \( X \) be a \( T_0 \) space. If the upper Vietoris topology is coarser than the Scott topology on \( K(X) \) (that is, \( \mathcal{O}(P_S(X)) \subseteq \sigma(K(X)) \), and \( \Sigma K(X) \) is well-filtered, then \( X \) is well-filtered.
Proof. Suppose that \( \{K_d : d \in D\} \subseteq K(X) \) is filtered and \( U \in \mathcal{O}(X) \) with \( \bigcap_{d \in D} K_d \subseteq U \). Then \( \{\uparrow_{K(X)} K_d : d \in D\} \subseteq K(\Sigma K(X)) \) is filtered, \( \cap U \in \mathcal{O}(P_S(X)) \subseteq \sigma(K(X)) \) and \( \bigcap_{d \in D} \uparrow_{K(X)} K_d \subseteq \square U \). By the well-filteredness of \( \Sigma K(X) \), there is \( d \in D \) such that \( \uparrow_{K(X)} K_d \subseteq \square U \), and hence \( K_d \subseteq U \). Thus \( X \) is well-filtered.

Example 7.2 below shows that when \( X \) lacks the condition of \( \mathcal{O}(P_S(X)) \subseteq \sigma(K(X)) \), Proposition 6.9 may not hold.

Corollary 6.10. For a \( T_0 \) space \( X \), the following conditions are equivalent:

1. \( X \) is well-filtered.
2. The upper Vietoris topology is coarser than the Scott topology on \( K(X) \), and \( \Sigma K(X) \) is well-filtered.
3. The upper Vietoris topology is coarser than the Scott topology on \( K(X) \), and \( \Sigma K(X) \) is a \( d \)-space.
4. \( K(X) \) is a dcpo, and the upper Vietoris topology is coarser than the Scott topology on \( K(X) \).

Proof. (1) \( \Rightarrow \) (2): By Corollary 4.9 and Theorem 6.5.

(2) \( \Rightarrow \) (3): By Remark 4.5.

(3) \( \Rightarrow \) (4): Trivial.

(4) \( \Rightarrow \) (1): By (4), \( P_S(X) \) is a \( d \)-space, whence \( X \) is well-filtered by Theorem 4.8.

7. Non-sobriety of Scott power space of a sober space

In this section, we investigate the following question:

Question 3. Is the Scott power space \( \Sigma K(X) \) of a sober space \( X \) sober?

First, the following example shows that there is a well-filtered space \( X \) for which its Scott power space \( \Sigma K(X) \) is a first-countable sober \( c \)-space, but \( X \) is not sober although its Scott power space \( \Sigma K(X) \) is sober by Corollary 8.1. Hence, by Corollary 5.14 and Corollary 5.24 \( X \) is neither core-compact nor first-countable. So the sobriety of the Scott power space of a \( T_0 \) space \( X \) does not imply the sobriety of \( X \) in general.

Example 7.1. Let \( X \) be an uncountably infinite set and \( X_{\text{coc}} \) the space equipped with the \( \text{co-countable topology} \) (the empty set and the complements of countable subsets of \( X \) are open). Then

(a) \( C(X_{\text{coc}}) = \emptyset \cup X^{(\leq \omega)} \), \( X_{\text{coc}} \) is \( T_1 \) and hence a \( d \)-space, and the specialization order on \( X_{\text{coc}} \) is the discrete order.

(b) Neither \( X_{\text{coc}} \) nor \( P_S(X_{\text{coc}}) \) is first-countable.

For a point \( x \in X \), suppose that there is a countable base \( \{X \setminus C_n : n \in \mathbb{N}, C_n \in X^{(\leq \omega)}\} \) at \( x \) in \( X_{\text{coc}} \). Let \( C = \bigcup_{n \in \mathbb{N}} C_n \). Then \( C \in X^{(\leq \omega)} \). Select \( t \in X \setminus (C \cup \{x\}) \) and let \( U = X \setminus \{t\} \). Then \( x \in U \in \mathcal{O}(X_{\text{coc}}) \). But \( X \setminus C_n \not\subseteq U \) for every \( n \in \mathbb{N} \), a contradiction. Thus \( X_{\text{coc}} \) is not first-countable.

Since the first-countability is a hereditary property and \( X_{\text{coc}} \) is homeomorphic to the subspace \( S^+ (X_{\text{coc}}) \) of \( P_S(X_{\text{coc}}) \) (see Remark 2.4 or Proposition 8.9 below), \( P_S(X_{\text{coc}}) \) is not first-countable.

(c) \( \text{irr}_{\text{coc}} (X_{\text{coc}}) = \{\{x\} : x \in X\} \cup \{X\} \cup \{\{x\} : x \in X\} \cup \{X\} \). Therefore, \( X_{\text{coc}} \) is not sober.

(d) \( K(X_{\text{coc}}) = X^{(\leq \omega)} \setminus \{\emptyset\} \) and \( \text{int} K = \emptyset \) for all \( K \in K(X_{\text{coc}}) \), and hence \( X_{\text{coc}} \) is not locally compact.

Clearly, every finite subset is compact. Conversely, if \( C \subseteq X \) is infinite, then \( C \) has an infinite countable subset \( \{c_n : n \in \mathbb{N}\} \). Let \( C_0 = \{c_n : n \in \mathbb{N}\} \) and \( U_m = (X \setminus C_0) \cup \{c_m\} \) for each \( m \in \mathbb{N} \). Then \( \{U_n : n \in \mathbb{N}\} \) is an open cover of \( C \), but has no finite subcover. Hence \( C \) is not compact. Thus \( K(X_{\text{coc}}) = X^{(\leq \omega)} \setminus \{\emptyset\} \). Clearly, \( \text{int} K = \emptyset \) for all \( K \in K(X_{\text{coc}}) \). Hence \( X_{\text{coc}} \) is not locally compact.

(e) \( X_{\text{coc}} \) is well-filtered and well-compact.

Suppose that \( \{F_d : d \in D\} \subseteq K(X_{\text{coc}}) \) is a filtered family and \( U \in \mathcal{O}(X_{\text{coc}}) \) with \( \bigcap_{d \in D} F_d \subseteq U \). As \( \{F_d : d \in D\} \) is filtered and all \( F_d \) are finite, \( \{F_d : d \in D\} \) has a least element \( F_{d_0} \), and hence \( F_{d_0} = \bigcap_{d \in D} F_d \subseteq U \), proving that \( X_{\text{coc}} \) is well-filtered. By (d) and Theorem 5.13, \( X_{\text{coc}} \) is not core-compact.
(f) $K(X_{coc})$ is a Noetherian dcpo and hence $\Sigma K(X_{coc}) = (K(X_{coc}), \alpha(K(X_{coc}))$ is first-countable.
Clearly, $K(X_{coc}) = X^{(\omega)} \but \{\emptyset\}$ (with the Smyth order) is a Noetherian dcpo and $\sigma(K(X_{coc})) = \alpha(K(X_{coc}))$. For any $F \in K(X_{coc}) = X^{(\omega)} \but \{\emptyset\}$, $\uparrow_{K(X_{coc})} F$ is a base at $F$ in $\Sigma K(X_{coc})$. Hence $\Sigma K(X_{coc})$ is first-countable.

(g) The upper Vietoris topology and the Scott topology on $K(X_{coc})$ do not agree.
By (e) and Corollary 4.9, $\mathcal{O}(P_S(X_{coc})) \subseteq \sigma(K(X_{coc}))$. For $F \in X^{(\omega)} \but \{\emptyset\}$, $\uparrow_{K(X_{coc})} F \notin \sigma(P_S(X_{coc}))$ since there is no $G \in X^{(\omega)}$ with $F \in \Box(X \but G) = (X \but G)^{(\omega)} \but \{\emptyset\} \subseteq \uparrow_{K(X_{coc})} F$. Thus $\sigma(P_S(X_{coc})) \notin \mathcal{O}(P_S(X_{coc}))$.

(h) The Scott power space $\Sigma K(X_{coc})$ is a sober c-space. So it is Rudin and well-filtered determined.
$K(X_{coc}) = X^{(\omega)} \but \{\emptyset\}$ (with the Smyth order) is a Noetherian dcpo and hence it is an algebraic domain.
By Theorem 3.4 and Proposition 4.3, $K(X_{coc})$ is a sober c-space. Hence by Theorem 5.10 $\Sigma K(X)$ is Rudin and well-filtered determined.

(i) $X_{coc}$ is neither a Rudin space nor a WD space.
By (e) and Theorem 5.10 $X_{coc}$ is neither a Rudin space nor a WD space.

(j) The Smyth power space $P_S(X_{coc})$ is well-filtered but non-sober. Hence it is neither a Rudin space nor a WD space.
By (e) and Theorem 4.4 and Theorem 4.8 $P_S(X_{coc})$ is well-filtered and non-sober. Hence $P_S(X_{coc})$ is neither a Rudin space nor a WD space by Theorem 5.10.

(k) $P_S(X_{coc})$ is not core-compact.
By (j) and Theorem 5.10 it needs only to show that $P_S(X_{coc})$ is not locally compact. Assume, on the contrary, that $P_S(X_{coc})$ is not locally compact. For $x \in X$ and $U \in \mathcal{O}(X_{coc})$ with $x \in U$, then by the local compactness of $P_S(X_{coc})$, there is $V \in \mathcal{O}(X_{coc})$ and $K \in K(X_{coc})$ such that $x \in V \subseteq K \subseteq U$.
Let $K = \bigcup K$. Then by Lemma 2.11 $K \in K(X)$ and $x \in V = \bigcup V \subseteq K \subseteq \bigcup U = U$. It follows that $X_{coc}$ is locally compact, which is in contradiction with (e). Thus $P_S(X_{coc})$ is not core-compact.

The following example shows there is even a second-countable Noetherian $T_0$ space $X$ such that the Scott power space $\Sigma K(X)$ is a second-countable sober space but $X$ is not well-filtered (and hence not sober).

**Example 7.2.** Let $P = \mathbb{N} \cup \{\infty\}$ and define an order on $P$ by $x \leq_P y$ iff $x = y$ or $x \in \mathbb{N}$ and $y = \infty$ (see Figure 4).

Let $\tau = \{(\mathbb{N} \but F) \cup \{\infty\} : F \in \mathbb{N}^{(\omega)} \but \{\emptyset, P\} \cup \{\infty\}\}$. It is straightforward to verify that $\tau$ is a $T_0$ topology on $P$ and the specialization order of $(P, \tau)$ agrees with the original order on $P$. Now we have

(a) $\mathcal{C}((P, \tau)) = \mathbb{N}^{(\omega)} \but \{\emptyset, P\} \cup \{\infty\}$.
(b) $\text{Irr}_c((P, \tau)) = \{\overline{n} : n \in \mathbb{N}\} \cup \{\overline{\infty} = P\} \cup \{\infty\}$ and hence $(P, \tau)$ is not sober.
(c) $K((P, \tau)) = \{A \cup \{\infty\} : A \subseteq \mathbb{N}\}$.
(d) $(P, \tau)$ is not well-filtered.

Let $\mathcal{K} = \{(\mathbb{N} \but F) \cup \{\infty\} : F \in \mathbb{N}^{(\omega)}\}$. Then $\mathcal{K} \subseteq K((P, \tau))$ is a filtered family and $\bigcap \mathcal{K} = \{\infty\} \in \tau$.
But there is no $F \in \mathbb{N}^{(\omega)}$ with $(\mathbb{N} \but F) \cup \{\infty\} = \{\infty\}$. Thus $(P, \tau)$ is not well-filtered. In fact, $(P, \tau)$ is not weak well-filtered in the sense of [17].
(e) \((P, \tau)\) is Noetherian and second-countable and hence it is a Rudin space. Since \(|\tau| = \omega\), \((P, \tau)\) is second-countable. As every subset of \(P\) is compact in \((P, \tau)\), the space \((P, \tau)\) is a Noetherian space (and hence a locally compact space). Hence by Proposition 5.12, \((P, \tau)\) is a Rudin space.

(f) \(\Sigma K((P, \tau))\) is a second-countable sober space. Clearly, \(K((P, \tau))\) is isomorphic with the algebraic lattice \(2^\mathbb{N}\) (with the order of set inclusion) via the poset isomorphism \(\varphi : K((P, \tau)) \to 2^\mathbb{N}\) defined by \(\varphi(A \cup \{\infty\}) = \mathbb{N} \setminus A\) for each \(A \in 2^\mathbb{N}\) (note that the order on \(K((P, \tau))\) is the Szymk order). Hence \(\Sigma K((P, \tau)) \cong \Sigma 2^\mathbb{N}\). Clearly, \(2^\mathbb{N}\) is an algebraic lattice, whence by Theorem 3.4 and Proposition 13, \(\Sigma 2^\mathbb{N}\) is sober and hence \(\Sigma K((P, \tau))\) is sober. Clearly, \(\Sigma 2^\mathbb{N}\) is second-countable since \(\{\uparrow F : F \in (2^\mathbb{N})^{(<\omega)}\}\) is a countable base of \(\Sigma 2^\mathbb{N}\). So \(\Sigma K((P, \tau))\) is second-countable.

(g) \(P_2((P, \tau))\) is second-countable by Proposition 8.8.

(h) \(\sigma(K((P, \tau))) \subseteq O(P_3((P, \tau)))\) but \(O(P_3((P, \tau))) \not\subseteq \sigma(K((P, \tau)))\).

Since \((P, \tau)\) is locally compact, \(\sigma(K((P, \tau))) \subseteq O(P_3((P, \tau)))\) by Lemma 6.1. Clearly, \(\bigsquare\{\infty\} = \{\{\infty\}\} \in O(P_3((P, \tau)))\). Now we show that \(\bigsquare\{\infty\} \not\in \sigma(K((P, \tau)))\). By Lemma 2.9, \(\bigvee\{F \cup \{\infty\} : F \in (\mathbb{N})^{(<\omega)} \setminus \{\emptyset\}\} = \bigcap\{F \cup \{\infty\} : F \in (\mathbb{N})^{(<\omega)} \setminus \{\emptyset\}\} = \{\infty\} \in \bigsquare\{\infty\}\), but there is no \(F \in (\mathbb{N})^{(<\omega)} \setminus \{\emptyset\}\) with \(F \cup \{\infty\} \in \bigsquare\{\infty\}\). Thus \(\bigsquare\{\infty\} \not\in \sigma(K((P, \tau)))\).

In the following we will construct a sober space \(X\) for which its Scott power space is non-sober (see Theorem 7.17 below). Let \(\mathcal{L} = \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\infty\})\), where \(\mathbb{N}\) is the set of natural numbers with the usual order. Define an order \(\leq\) on \(\mathcal{L}\) as follows:

- \((i_1, j_1, k_1) \leq (i_2, j_2, k_2)\) if and only if:
  1. \(i_1 = i_2, j_1 = j_2, k_1 \leq k_2 \leq \infty;\) or
  2. \(i_2 = i_1 + 1, k_1 \leq j_2, k_2 = \infty\).

\(\mathcal{L}\) is a known dcpo constructed by Jia in [13, Example 2.6.1]. It can be easily depicted as in Figure 5 taken from [13].

![Figure 5: A non-sober well-filtered dcpo \(\mathcal{L}\)](image)

For each \((n, m)\) \(\in \mathbb{N} \times \mathbb{N}\), let

\[
\mathcal{L}_n = \{(n, j, l) : j \in \mathbb{N}, l \in \mathbb{N} \cup \{\infty\}\},
\]

\[
\mathcal{L}_n^- = \{(n, j, \infty) : j \in \mathbb{N}\},
\]

\[
\mathcal{L}^\infty = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n^- = \{(i, j, \infty) : (i, j) \in \mathbb{N} \times \mathbb{N}\} \text{ (the set of all maximal elements of } \mathcal{L})\),
\]

\[
\mathcal{L}^{\leq \infty} = \bigcap_{i=1}^{\infty} \mathcal{L}_i = \{\text{the set of all elements of finite height}\},
\]

\[
\mathcal{L}_{\leq n}^\infty = \bigcup_{i=1}^{n} \mathcal{L}_i = \{(i, j, \infty) : i \leq n, j \in \mathbb{N}, l \in \mathbb{N} \cup \{\infty\}\},
\]

\[
\mathcal{L}_{\leq (n+1)}^\infty = \bigcup_{i=1}^{n} \mathcal{L}_i^\infty = \{\{i, j, \infty) : i \leq n, j \in \mathbb{N}\},
\]

\[
\mathcal{L}_{\leq (n, m)}^\infty = \{(i, j, \infty) : j \geq m\}.
\]

**Lemma 7.3.** Suppose that \(D\) is an infinite directed subset of \(\mathcal{L}\). Then there is a unique \((i, j, \infty) \in \mathcal{L}^\infty\) such that \((i, j, \infty)\) is a largest element of \(D\) or the following two conditions are satisfied:
(i) \((i, j, \infty) \notin D\), and
(ii) for each \(d = (i_d, j_d, l_d) \in D\), \(i_d = i, j_d = j\) and \(l_d < \infty\) (i.e., \(l_d \in \mathbb{N}\)).

**Proof.** If there is \(d_0 = (i, j, \infty) \in D \cap \mathcal{L}^\infty\), then for each \(d = (i_d, j_d, l_d) \in D\), there is \(d^* = (i_d^*, j_d^*, l_d^*) \in D\) such that \(d_0 = (i, j, \infty) \leq d^* = (i_d^*, j_d^*, l_d^*)\) and \(d = (i_d, j_d, l_d) \leq d^* = (i_d^*, j_d^*, l_d^*)\). Hence \(l_d^* = \infty, i_d^* = i, j_d^* = j\) (i.e., \(d^* = d_0\)) and \(l_d \leq d^* = d_0\). Hence \(d_0 = (i, j, \infty)\) is the (unique) largest element of \(D\).

Now suppose that \(D \cap \mathcal{L}^\infty = \emptyset\), that is, \(D \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}\). Select a \(d_1 = (i_{d_1}, j_{d_1}, l_{d_1}) \in D\). Then for each \(d = (i_d, j_d, l_d) \in D\), by the directedness of \(D\), there is \(d^* = (i_d^*, j_d^*, l_d^*) \in D\) such that \(d_1 = (i_{d_1}, j_{d_1}, l_{d_1}) \leq d^* = (i_d^*, j_d^*, l_d^*)\) and \(d = (i_d, j_d, l_d) \leq d^* = (i_d^*, j_d^*, l_d^*)\). Hence \(i_d = i_{d_1}, j_d = j_{d_1}\) and \(l_d \leq l_d^* \leq l_d^*\). Let \(i = i_d\) and \(j = j_d\). Then \(D \subseteq \{(i, j, l) : l \in \mathbb{N}\}\). Clearly, \((i, j, \infty)\) is the unique element of \(\mathcal{L}^\infty\) satisfying conditions (i) and (ii), and \((i, j, \infty) = \bigvee_{\mathcal{L}} D\).

For any infinite directed subset \(D\) of \(\mathcal{L}\), by Lemma 7.3 \(D\) is contained in \(\downarrow(i, j, \infty)\) for some \(i, j \in \mathbb{N}\) with its supremum being \((i, j, \infty)\). Hence \(\mathcal{L}\) is a dcop.

**Corollary 7.4.** Let \(A\) be a nonempty subset of \(\mathcal{L}\) for which \(A = \downarrow_{\text{max}}(A)\). For an infinite directed subset \(D\) of \(A\), if \(D\) has no largest element, then there is a unique \((i, j, \infty) \in \mathcal{L}^\infty\) such that

1. \(D \subseteq \{(i, j, l) : l \in \mathbb{N}\}\),
2. \((i, j, \infty) = \bigvee_{\mathcal{L}} D\), and
3. \(\text{max}(A) \cap \mathcal{L}^\infty_{i+1} = \text{is infinite}.

**Proof.** Since \(D\) has no largest element, by Lemma 7.3 there is a unique \((i, j, \infty) \in \mathcal{L}^\infty\) such that the following two conditions are satisfied:

(i) \((i, j, \infty) \notin D\), and
(ii) for each \(d = (i_d, j_d, l_d) \in D\), \(i_d = i, j_d = j\) and \(l_d < \infty\) (i.e., \(l_d \in \mathbb{N}\)).

So \(D \subseteq \{(i, j, l) : (i, j, l) \in \mathbb{N}\}\) and \((i, j, \infty) = \bigvee_{\mathcal{L}} D\). Now we show that \(\text{max}(A) \cap \mathcal{L}^\infty_{i+1}\) is infinite. For each \(d = (i, j, l) \in D\), by \(A = \downarrow_{\text{max}}(A)\), there is \((i(d), j(d), l(d)) \in \text{max}(A)\) with \(d = (i(d), j(d), l(d))\).

If \(i(d) = i\), then \(j(d) = j\) and \(l(d) \leq l(d)\). Since \(D \subseteq A\), \(\{d : d^* = (i, j, l) \in D\} \subseteq \text{max}(A)\). We have that \(l(d) = \infty\), which is in contradiction with condition (i).

Therefore, \(i(d) = i + 1\) and hence \(l(d) = \infty\) and \(l(d) \leq j(d)\). Hence \((i, j, l) \leq (i(d), j(d), l(d)) = (i + 1, j, l)\).

Since \(\{d : d = (i, j, l) \in D\} \subseteq \mathbb{N}\) is infinite, \(\{(i(d), j(d), l(d)) = (i + 1, j, l, \infty) : d \in D\} \subseteq \text{max}(A) \cap \mathcal{L}^\infty\) is infinite (note that \(l(d) \leq j(d)\) for each \(d \in D\)). Thus \(\text{max}(A) \cap \mathcal{L}^\infty_{i+1}\) is infinite.

**Lemma 7.5.** Let \(A \subseteq \mathcal{L}\) be a nonempty set and \(A \neq \mathcal{L}\). Then \(A\) is Scott closed if and only if \(A = \downarrow_{\text{max}}(A)\) and one of the following three conditions are satisfied:

1. \(A \subseteq \mathcal{L}^\infty\) (or equivalently, \(\text{max}(A) \subseteq \mathcal{L}^\infty\)),
2. \(A \cap \mathcal{L}^\infty \neq \emptyset\) and \(|A \cap \mathcal{L}^\infty| < \omega\) for each \(i \in \mathbb{N}\),
3. \(i(A) = \max\{i \in \mathbb{N} : |A \cap \mathcal{L}^\infty| = \omega\}\) exists and \(A_i \subseteq A\) for each \(n \leq i(A) - 1\).

**Proof.** Suppose that \(A\) is Scott closed. Then \(A = \downarrow_{\text{max}}(A)\) by Lemma 4.2. Now we show that \(A\) satisfies one of conditions (1)-(3). If neither condition (1) nor condition (2) holds. Then there is some \(i_0 \in \mathbb{N}\) such that \(A \cap \mathcal{L}^\infty_{i_0}\) is infinite. We first show that \(A\) satisfies the following property Q:

(Q) For \(n \in \mathbb{N}\), if \(A \cap \mathcal{L}^\infty_n\) is infinite, then \(A_i \subseteq A\) for each \(i \leq n - 1\).

If \(n = 1\), then \(A_{-1} = A_0 = \emptyset \subseteq A\). Now we assume \(2 \leq n\). For each \((j, l) \in \mathbb{N} \times \mathbb{N}\), since \(A \cap \mathcal{L}^\infty_n\) is infinite (i.e., \(\{j^n \cap \mathbb{N} : (j^n, \infty) \in \mathcal{A}\}\)) is infinite), there is \(j^n \in \mathbb{N}\) such that \(j^n, \infty) \in \mathcal{A}\) and \(l \leq j^n\), whence \((n - 1, j, l) \in \mathbb{N} \times \mathbb{N}\). Hence \(\{(n - 1, j, l) : (j, l) \in \mathbb{N} \times \mathbb{N}\} \subseteq \bigvee_{\mathcal{L}} \mathcal{L}^\infty_n\) \subseteq \mathcal{A} = A\). For each \(j \in \mathbb{N}\), since \((n - 1, j, \infty) = \bigvee_{\mathcal{L}^\infty_n} (n - 1, j, l)\) and \(A\) is Scott closed, we have \((n - 1, j, \infty) = A\). Hence \(A_{-1} \subseteq A\). In particular, \(A_{-1} \subseteq A\). Then by induction we get that \(A_i \subseteq A\) for any \(1 \leq i \leq n - 1\).

By property Q, if \(\{i \in \mathbb{N} : A \cap \mathcal{L}^\infty_i = \omega\}\) is infinite, then for each \(n \in \mathbb{N}\), \(A_{-n - 1} \subseteq A\). Hence \(A = \bigcup_{n \in \mathbb{N}} A_{|n|} \subseteq A\), which contradicts \(A \neq \mathcal{L}\). Hence \(\{i \in \mathbb{N} : |A \cap \mathcal{L}^\infty_i = \omega\}\) is a nonempty finite subset of \(\mathbb{N}\) and hence \(i(A) = \max\{i \in \mathbb{N} : |A \cap \mathcal{L}^\infty_i = \omega\}\) exists. By property Q we have \(A_i \subseteq A\) for each \(i \leq i(A) - 1\), proving that condition (3) holds.
Conversely, assume that $A = \downarrow \max(A)$ and one of conditions (1)-(3) is satisfied. We will show that $A$ is Scott closed.

**Case 1.** $A \subseteq L^{<\omega}$ (resp., $A \cap L^{<\omega} \neq \emptyset$ and $|A \cap L^{<\omega}| \prec \omega$ for each $i \in \mathbb{N}$).

Suppose that $D$ is a directed subset of $A$. If $D$ has no largest element, then $D$ is infinite, whence by Corollary 7.4, there is a unique $(i,j,l) \in L^{<\omega}$ such that $D \subseteq \{(i,j,l) : l \in \mathbb{N}\}$, $(i,j,\infty) = \bigvee_{\mathcal{L}} D$ and $\max(A) \cap L^{<\omega}$ is infinite, being a contradiction with $A \subseteq L^{<\omega}$ (resp., $A \cap L^{<\omega} \neq \emptyset$ and $|A \cap L^{<\omega}| \prec \omega$ for each $i \in \mathbb{N}$). Therefore, $D$ has a largest element $d_0$ and hence $\bigvee_{\mathcal{L}} D = d_0 \in A$. Thus $A$ is Scott closed.

**Case 2.** $i(A) = \max\{i \in \mathbb{N} : |A \cap L^{<\omega}| = \omega\}$ exists and $\mathcal{L}_0 \subseteq A$ for each $n \leq i(A) - 1$.

For an infinite directed subset $D$ of $A$, if $D$ has a largest element, then clearly $\bigvee_{\mathcal{L}} D \in A$. If $D$ has no largest element, then by Corollary 7.4, there is a unique $(i,j,\infty) \in L^{<\omega}$ such that $D \subseteq \{(i,j,l) : l \in \mathbb{N}\}$, $(i,j,\infty) = \bigvee_{\mathcal{L}} D$ and $\max(A) \cap L^{<\omega}$ is infinite. Since $i(A) = \max\{i \in \mathbb{N} : |A \cap L^{<\omega}| = \omega\}$ exists and $\mathcal{L}_0 \subseteq A$ for each $n \leq i(A) - 1$, we have $i + 1 \leq i(A)$ and hence $\bigvee_{\mathcal{L}} D = (i,j,\infty) \in L_i \subseteq A$. So $A$ is Scott closed.

**Lemma 7.6.** $\text{Irr}_c(\Sigma \mathcal{L}) = \{[x] = \downarrow x : x \in \mathcal{L}\}$.

**Proof.** Clearly, $\{[x] = \downarrow x : x \in \mathcal{L}\} \subseteq \text{Irr}_c(\Sigma \mathcal{L})$. It was proved in [13] Example 2.6.1] that $\mathcal{L} \in \text{Irr}_c(\Sigma \mathcal{L})$. Suppose that $A \in \text{Irr}_c(\Sigma \mathcal{L})$ and $A \neq \mathcal{L}$. Then by Lemma 7.5, $A = \downarrow \max(A) = \downarrow (\max(A) \cap \mathcal{L}^{<\omega}) (\max(A) \cap \mathcal{L}^{\infty} \text{ or } \max(A) \cap \mathcal{L}^{<\omega} \text{ may be the empty set})$ and one of the following three conditions are satisfied:

(i) $A \subseteq L^{<\omega}$ (or equivalently, $\max(A) \subseteq L^{<\omega}$).

(ii) $\max(A) \neq \emptyset$ and $|A \cap L^{<\omega}| \prec \omega$ for each $i \in \mathbb{N}$.

(iii) $i(A) = \max\{i \in \mathbb{N} : |A \cap L^{<\omega}| = \omega\}$ exists and $\mathcal{L}_n \subseteq A$ for each $n \leq i(A) - 1$.

Let $B = \downarrow (\max(A) \cap \mathcal{L}^{<\omega})$ and $C = \downarrow (\max(A) \cap \mathcal{L}^{\infty})$. Then $B = \downarrow \max(B)$ and $C = \downarrow \max(C)$. Clearly, $C \subseteq L^{<\omega}$. Hence $C$ is Scott closed by Lemma 7.5. If $B \neq \emptyset$ (i.e., $\max(A) \cap \mathcal{L}^{<\omega} \neq \emptyset$), then $A$ satisfies one of conditions (2) and (3) of Lemma 7.5. If $B = \emptyset$, then $A$ satisfies one of conditions (2) and (3) of Lemma 7.5. and hence by Lemma 7.5, again, $B$ is Scott closed. By the irreducibility of $A$, we have $A = B$ or $A = C$.

**Case 1.** $\max(A) \cap L^{<\omega} \neq \emptyset$ and $A = \downarrow (\max(A) \cap L^{<\omega})$.

If $i(A) = \max\{i \in \mathbb{N} : |A \cap L^{<\omega}| = \omega\}$ exists and $\mathcal{L}_n \subseteq A$ for each $n \leq i(A) - 1$, then there is $i(A), j, \infty) \in A \cap L^{<\omega}(i(A))$. By Lemma 7.3, we can easily verify that $\bigcup_{n=1}^{i(A)-1} \downarrow (A \cap L^{<\omega}_{i+1} \setminus \{(i(A), j, \infty)\})$ is Scott closed. Clearly, $A = \bigcup_{n=1}^{i(A)-1} \downarrow (A \cap L^{<\omega}_{i+1} \setminus \{(i(A), j, \infty)\}) \cup \downarrow (i(A), j, \infty)$. By the irreducibility of $A$ and $A \neq \downarrow (i(A), j, \infty)$ (since $A \cap L^{<\omega}(i(A))$ is infinite), we have $A = \bigcup_{n=1}^{i(A)-1} \downarrow (A \cap L^{<\omega}_{i+1} \setminus \{(i(A), j, \infty)\})$, which contradicts $i(A), j, \infty) \notin \bigcup_{n=1}^{i(A)-1} \downarrow (A \cap L^{<\omega}_{i+1} \setminus \{(i(A), j, \infty)\})$.

So $|A \cap L^{<\omega}| \prec \omega$ for each $i \in \mathbb{N}$. Choose a point $(i,j,\infty) \in \max(A) \cap L^{<\omega}$. By Lemma 7.3, we can easily check that $\downarrow (\max(A) \cap L^{<\omega} \setminus \{(i,j,\infty)\})$ is Scott closed and $A = \downarrow (\max(A) \cap L^{<\omega} \setminus \{(i,j,\infty)\}) \cup \downarrow (i,j,\infty)$. By the irreducibility of $A$, we have $A = \downarrow (i,j,\infty)$ or $A = \downarrow (\max(A) \cap L^{<\omega} \setminus \{(i,j,\infty)\})$ (which contradicts $(i,j,\infty) \notin \downarrow (\max(A) \cap L^{<\omega} \setminus \{(i,j,\infty)\})$). Thus $A = \downarrow (i,j,\infty)$ and $\max(A) \cap L^{<\omega} = \{(i,j,\infty)\}$.

**Case 2.** $A = \downarrow (\max(A) \cap L^{<\omega})$.

Choose a point $(i,j,l) \in \max(A)$. Then by Lemma 7.5, $\downarrow (\max(A) \setminus \{(i,j,l)\})$ is Scott closed and $A = \downarrow (\max(A) \setminus \{(i,j,l)\}) \cup \downarrow (i,j,l)$. It follows that $A = \downarrow (i,j,l)$ or $A = \downarrow (\max(A) \setminus \{(i,j,l)\})$, which contradicts $(i,j,l) \notin \downarrow (\max(A) \setminus \{(i,j,l)\})$. So $A = \downarrow (i,j,l)$ and $\max(A) = \{(i,j,l)\}$.

It follows from the above that $\text{Irr}_c(\Sigma \mathcal{L}) = \{[x] = \downarrow x : x \in \mathcal{L}\}$.

**Lemma 7.7.** [13] Example 2.6.1] For a nonempty saturated subset $K \subseteq \mathcal{L}$, $K$ is compact in $\Sigma \mathcal{L}$ if and only if the following three conditions are satisfied:

(1) $\text{min}(K) \cap L^{<\omega}$ is finite,
(2) there exists \( i_0, j_0 \in \mathbb{N} \) such that \( (L^\infty_{<i_0} \cup L^\infty_{i_0 \geq j_0}) \cap K = \{(i_0, j_0, \infty)\} \), and

(3) the set \( \{n \in \mathbb{N} : L^\infty_n \cap K \neq \emptyset\} \) is finite.

**Corollary 7.8.** For any filtered family \( \{K_d : d \in D\} \subseteq K(\Sigma L) \), \( \bigcap_{d \in D} K_d \neq \emptyset \).

**Proof.** We can assume that \( D \) is directed and \( K_{d_0} \subseteq K_{d_1} \) if \( d_1 \leq d_2 \) (indeed, \( D \) can be defined an order by \( d_1 \leq d_2 \) iff \( K_{d_1} \subseteq K_{d_2} \)). For each \( d \in D \), by Lemma 7.7 there exist \( i_d, j_d \in \mathbb{N} \) such that \( (L^\infty_{i_d} \cup L^\infty_{i_d \geq j_d}) \cap K_d = \{(i_d, j_d, \infty)\} \) and the set \( \mathbb{N}_d = \{n \in \mathbb{N} : L^\infty_n \cap K_d \neq \emptyset\} \) is finite. Select a \( d_0 \in D \). Then for each \( d \in D \) with \( d_0 \leq d \) (whence \( K_{d_0} \subseteq K_d \)), we have that \( i_d \in \mathbb{N}_d \subseteq \mathbb{N}_{d_0} \) and \( i_{d_0} \leq i_d \) (otherwise, \( i_{d_0} > i_d \) would imply that \( (i_d, j_d, \infty) \in (L^\infty_{i_d} \cup L^\infty_{i_d \geq j_d}) \cap K_d \subseteq L^\infty_{i_d} \cap K_d \), which contradicts \( L^\infty_{i_d} \cap K_{d_0} = \emptyset \)). Let \( D_{d_0} = \{d \in D : d_0 \leq d\} \) and \( D_i = \{d \in D : i_d = i\} \) for each \( i \in \mathbb{N}_{d_0} \). Since \( \mathbb{N}_{d_0} \) is finite, \( D_{d_0} \) is directed and \( D_{d_0} = \bigcup_{i \in \mathbb{N}_{d_0}} D_i \), there is \( i_0 \in \mathbb{N}_{d_0} \) such that \( D_{i_0} \) is a cofinal subset of \( D_{d_0} \) and hence a cofinal subset of \( D \), more precisely, for each \( d \in D \), there is \( d^* \in D \) such that \( d^* \in \bigcup_{i \in \mathbb{N}_{d_0}} d_i \) and \( i_{d^*} = i_0 \).

Clearly, \( D_{i_0} \) is also directed. Select a \( d_1 \in D_{i_0} \). Then \( D_{d_1} = \{d \in D_{i_0} : d_1 \leq d\} \) is a directed and cofinal subset of \( D_{i_0} \) and hence a directed and cofinal subset of \( D \). For each \( d \in D_{d_1} \) (note that \( K_{d_0} \subseteq K_{d_1} \)), we have that \( i_{d_1} = i_1 = 0, \{i_1 = i_0, i_1 < j_1, \infty\} \in K_{d_1} \subseteq K_{d_0} \) and \( L^\infty_{0 \geq j_1} \cap K_{d_1} = \{(i_0, j_1, \infty)\} \). It follows that \( j_1 \leq j_{d_1} \). For each \( 1 \leq j \leq j_{d_1} \), let \( \tilde{D}_j = \{d \in D_{d_1} : j_d = j\} \). Since \( \{1, 2, \ldots, j_{d_1}\} \) is finite, \( D_{d_1} \) is directed and \( D_{d_1} = \bigcup_{j \in \mathbb{N}_{d_0}} \tilde{D}_j \), there is \( 1 \leq j_0 \leq j_{d_1} \) such that \( \tilde{D}_{j_0} \) is a cofinal subset of \( D_{d_1} \), and hence a cofinal subset of \( D \), more precisely, for each \( d \in D \), there is \( d' \in D \) such that \( d' \in \bigcup_{i \in \mathbb{N}_{d_0}} d_i \cap \bigcap d_i \), \( i_{d'} = i_0 \) and \( j_{d'} = j_0 \). It follows that \( (i_0, j_0, \infty) \in \bigcap_{d \in D} K_d \).

**Proposition 7.9.** (\cite{13} Example 2.6.1) \( \Sigma L \) is well-filtered but non-sober.

Indeed, \( L \) is an irreducible closed subset of \( \Sigma L \) but has no largest element, so \( \Sigma L \) is non-sober.

Using Topological Rudin Lemma, Lemma 7.6 and Corollary 7.8, we can give a short proof of the well-filteredness of \( \Sigma L \). Suppose that \( \{K_d : d \in D\} \subseteq K(\Sigma L) \) is a filtered family and \( U \in \sigma(\mathcal{L}) \) with \( \bigcap_{d \in D} K_d \subseteq U \). Assume, on the contrary, that \( K_d \not\subseteq U \) for each \( d \in D \) (whence \( U \neq \emptyset \)). Then by Lemma 5.4 \( \mathcal{L} \cap U \) contains a minimal irreducible closed subset \( A \) that still meets all members \( K_d \). By Corollary 7.8 \( \bigcap_{d \in D} K_d \neq \emptyset \), whence \( U \neq \emptyset \) and \( A \neq L \). It follows from Lemma 7.6 that \( A = \{x\} \) for some \( x \in L \). Then \( x \in \bigcap_{d \in D} K_d \subseteq U \), which contradicts \( x \in A \subseteq L \setminus U \). Thus \( \Sigma L \) is well-filtered.

**Definition 7.10.** Let \( X \) be a \( T_0 \) space for which \( X \) is irreducible (i.e., \( X \in \text{Ir}(\mathcal{X}) \)). Choose a point \( \top \) such that \( \top \notin X \). Then \( \mathcal{C}(X) \setminus \{X\} \cup \{X \cup \{\top\}\} \) (as the set of all closed sets) is a topology on \( X \cup \{\top\} \). The resulting space is denoted by \( X_\top \). Define a mapping \( \xi_X : X \to X_\top \) by \( \xi_X(x) = x \) for each \( x \in X \). Clearly, \( \eta_X \) is a topological embedding.

As \( X \) is \( T_0 \), \( X_\top \) is also \( T_0 \) and \( \overline{\{\top\}} = X \cup \{\top\} \) in \( X_\top \). Hence \( \top \) is a largest element of \( X_\top \) and for \( x, y \in X, x \leq y \) iff \( x \leq y \). It is worthy noting that the set \( \{\top\} \) is not open in \( X_\top \).

**Remark 7.11.** If \( X \) is not irreducible, then there exist \( A, B \in \mathcal{C}(X) \setminus \{X\} \) such that \( X = A \cup B \), whence \( \mathcal{C}(X) \setminus \{X\} \cup \{X \cup \{\top\}\} \) is not a topology on \( X \cup \{\top\} \).

**Lemma 7.12.** Let \( X \) be a \( T_0 \) space for which \( X \) is irreducible. Then \( K(X_\top) = \{G \cup \{\top\} : G \in K(X)\} \cup \{\{\top\}\} \).

**Proof.** Clearly, \( \mathcal{O}(X_\top) = \{U \cup \{\top\} : U \in \sigma(\mathcal{L}) \cup \{\emptyset\}\} \cup \{\emptyset\} \).

First, if \( K \in K(X_\top) \setminus \{\{\top\}\} \), then \( G = K \setminus \{\top\} \) is a nonempty saturated subset of \( X \). Now we verify that \( G \) is a compact subset of \( X \). Suppose that \( \{U_i : i \in I\} \subseteq \mathcal{O}(X) \setminus \{\emptyset\} \) is an open cover of \( G \). Then \( \{U_i \cup \{\top\} : i \in I\} \subseteq \mathcal{O}(X_\top) \) is an open cover of \( K \subseteq G \cup \{\top\} \). By the compactness of \( K \), there is \( I_0 \in I^{<\omega} \) such that \( K \subseteq \bigcup_{i \in I_0} U_i \cup \{\top\} \), whence \( G = K \setminus \{\top\} \subseteq \bigcup_{i \in I_0} U_i \), proving that \( G \in K(\Sigma L) \).

Conversely, assume that \( G \in K(X) \) and \( \{W_j : j \in J\} \subseteq \mathcal{O}(\Sigma L) \cap \{\emptyset\} \) is an open cover of \( K = G \cup \{\top\} \). Then \( K \) is saturated and for each \( j \in J \), there is \( V_j \in \mathcal{O}(X) \) such that \( W_j = V_j \cup \{\top\} \). Hence \( \{V_j : j \in J\} \subseteq \mathcal{O}(X) \setminus \{\emptyset\} \). Thus \( \mathcal{O}(X_\top) \).

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Lemma 7.13. Suppose that $X$ is a non-sober $T_0$ space for which $\text{irr}_c(X) = \{\overline{x} : x \in X\} \cup \{X\}$. Then $\langle X_T, \zeta_X \rangle$ is a sobrification of $X$.

Proof. Since $X$ is non-sober and $\text{irr}_c(X) = \{\overline{x} : x \in X\} \cup \{X\}, X \neq \overline{x}$ for every $x \in X$. It is well-known that the space $X^*$ with the canonical mapping $\eta_X : X \to X^*$, $\eta_X(x) = x$, is a sobrification of $X$ (see, for example, [6, Exercise V-4.9]). For $C \in \mathcal{C}(X)$, we have

$$\Box_{\text{irr}_c(X)} C = \{A \in \text{irr}_c(X) : A \subseteq C\} = \left\{ \begin{array}{ll} \overline{\{c\}} & c \in C, \\ \overline{\{x\}} : x \in X \cup \{X\} & C = X. \end{array} \right.$$ 

Define a mapping $f : X^* \to X_T$ by

$$f(A) = \begin{cases} x & A = \overline{x}, x \in X, \\ \top & A = X. \end{cases}$$

For each $C \in (\mathcal{C}(X) \setminus \{X\}) \cup \{X_T\}$ and $B \in \mathcal{C}(X)$, we have $f^{-1}(C) = \Box_{\text{irr}_c(X)} C$ and

$$f(\Box_{\text{irr}_c(X)} B) = \begin{cases} B & B \neq X, \\ X \cup \{\top\} & B = X. \end{cases}$$

Thus $f$ is a homeomorphism.

So $\langle X_T, \zeta_X = f \circ \eta_X \rangle$ is a sobrification of $X$. □

The following corollary is straightforward from Lemma 7.6, Proposition 7.9 and Lemma 7.13.

Corollary 7.14. $\langle (\Sigma \mathcal{L})_T, \zeta_L \rangle$ is a sobrification of $\Sigma \mathcal{L}$, where $\zeta_L : \Sigma \mathcal{L} \to (\Sigma \mathcal{L})_T$ is defined by $\zeta_L(x) = x$ for each $x \in \mathcal{L}$.

Note that although the set $\{\top\}$ is open in $\Sigma(\mathcal{L})_T$ (or equivalently, $\top$ is a compact element of the dcpo $\mathcal{L} \cup \{\top\}$), it is not open in $(\Sigma \mathcal{L})_T$.

By Lemma 7.6 and Lemma 7.12 we get the following.

Corollary 7.15. $K((\Sigma \mathcal{L})_T) = \{G \cup \{\top\} : G \in K(\Sigma \mathcal{L})) \cup \{\{\top\}\}.$

Lemma 7.16. $\{\top\}$ is a compact element in the dcpo $K((\Sigma \mathcal{L})_T)$. Hence $\{\top\}$ is open in the Scott space $\Sigma K((\Sigma \mathcal{L})_T)$.

Proof. By Proposition 7.9 $\Sigma \mathcal{L}$ is well-filtered and hence $K((\Sigma \mathcal{L})_T)$ (with the Smyth order) is a dcpo. Hence by Corollary 7.15 $K((\Sigma \mathcal{L})_T)$ is a dcpo. Now we show that $\{\top\} \in K((\Sigma \mathcal{L})_T)$. Suppose that $\{K_d : d \in D\}$ is a directed subset of $K((\Sigma \mathcal{L})_T)$ and $\{\top\} \subseteq \bigvee_{d \in D} K_d$. Then by Lemma 2.9 $\bigvee_{d \in D} K_d = \bigcap_{d \in D} K_d$ and hence $\bigcap_{d \in D} K_d \subseteq \{\top\}$. It follows that $\bigcap_{d \in D} (K_d \setminus \{\top\}) = \emptyset$. By Corollary 7.8 and Corollary 7.15 there is $d \in D$ such that $K_d \setminus \{\top\} = \emptyset$, that is, $K_d = \{\top\}$. Thus $\{\top\} \in K((\Sigma \mathcal{L})_T)$.

Theorem 7.17. The Scott power space $\Sigma K((\Sigma \mathcal{L})_T)$ of the sober space $(\Sigma \mathcal{L})_T$ is non-sober.
Corollary 8.1. \(\Sigma X\)

In the next section we will show that Scott power spaces of locally compact (especially, compact) \(T\) and \(G\)

Proof. For simplicity, let \(A = \{G \cup \{\top\} : G \in K(\Sigma L)\}\).

Claim 1: \(A\) is a closed subset of \(\Sigma K((\Sigma L)_{\tau})\).

By Corollary 7.15 and Lemma 7.16, \(A\) is Scott closed.

Claim 2: \(A\) is irreducible.

Suppose that \(U, V \in \sigma(K((\Sigma L)_{\tau}))\) and \(A \cap U \neq \emptyset \neq A \cap V\). Then by Corollary 7.15 there are some \(G_1, G_2 \in K(\Sigma L)\) such that \(G_1 \cup \{\top\} \in A \cap U\) and \(G_2 \cup \{\top\} \in A \cap V\), and hence \((i_1, j_1, \infty) \in G_1\) and \((i_2, j_2, \infty) \in G_2\) for some \((i_1, j_1), (i_2, j_2) \in \mathbb{N} \times \mathbb{N}\). Hence by Corollary 7.15 \(\uparrow\{i_1, j_1, \infty\} \cup \{\top\} \in A \cap U\) and \(\uparrow\{i_2, j_2, \infty\} \cup \{\top\} \in A \cap V\) (note that \(U, V\) are upper sets and \(G_1 \cup \{\top\} \subseteq \uparrow\{i_1, j_1, \infty\} \cup \{\top\}, G_2 \cup \{\top\} \subseteq \uparrow\{i_2, j_2, \infty\} \cup \{\top\}\)). Without loss of generality, we assume \(i_1 \leq i_2\). Since \(\bigvee_{i \in \mathbb{N}}(\uparrow\{i_1, j_1, \infty\} \cup \{\top\}) = \bigcap_{i \in \mathbb{N}}(\uparrow\{i_1, j_1, \infty\} \cup \{\top\}) = \uparrow\{i_1, j_1, \infty\} \cup \{\top\} \in U \in \sigma(K((\Sigma L)_{\tau}))\), we have some \(l_1 \in \mathbb{N}\) such that \(\uparrow\{i_1, j_1, l_1\} \in U\). Then by Corollary 7.15 \(\uparrow\{i_1 + 1, l_1, \infty\} \cup \{\top\} \subseteq U \in \sigma(K((\Sigma L)_{\tau}))\) since \((i_1, j_1, l_1) \leq (i_1 + 1, l_1, \infty)\) and \(U\) is an upper set. Then by induction we have \(\uparrow\{i_2, j', \infty\} \cup \{\top\} \subseteq U\) for some \(j' \in \mathbb{N}\). Again, since \(\bigvee_{i \in \mathbb{N}}(\uparrow\{i_2, j', \infty\} \cup \{\top\}) = \bigcap_{i \in \mathbb{N}}(\uparrow\{i_2, j', \infty\} \cup \{\top\}) = \uparrow\{i_2, j', \infty\} \cup \{\top\} \in U \in \sigma(K((\Sigma L)_{\tau}))\), we have some \(k_1, k_2 \in \mathbb{N}\) such that \(\uparrow\{i_2, j', k_1\} \subseteq U\) and \(\uparrow\{i_2, j', k_2\} \in V\). Take \(m = \max\{k_1, k_2\}\). Then \(\uparrow\{i_2, m, \infty\} \cup \{\top\} \in A \cap U \cup \cap V\). Thus \(A \in \mathcal{I}rr_2(\Sigma K((\Sigma L)_{\tau}))\).

Claim 1: \(A\) has no largest element.

Clearly, \(\{\uparrow\{i, j, \infty\} \cup \{\top\} : i, j \in \mathbb{N}\}\) is the set of all maximal elements of \(A\) and hence \(A\) has no largest element.

By Claims 1-3, the Scott power space \(\Sigma K((\Sigma L)_{\tau})\) is non-sober.

We know that every \(T_2\) space is sober and hence its Scott power space is well-filtered by Theorem 6.5.

In the next section we will show that Scott power spaces of locally compact (especially, compact) \(T_2\) spaces are sober (see Corollary 8.6 below).

By Theorem 4.4 Corollary 3.9 Theorem 7.17 and Corollary 8.6 below, we naturally pose the following question.

Question 7.18. For a \(T_2\) space \(X\), is the Scott power space \(\Sigma K(X)\) sober?

8. Local compactness, first-countability and sobriety of Scott power spaces

In this section, we investigate the conditions under which the Scott power space of a sober space is still sober. We will see that Question 3 is related to the investigation of conditions under which the upper Vietoris topology coincides with the Scott topology on \(K(X)\), and further it is closely related to the local compactness and first-countability of \(X\).

First, by Corollary 5.14 Corollary 5.24 and Theorem 6.5 we get the following.

Corollary 8.1. If \(X\) is a well-filtered space for which the Scott power space \(\Sigma K(X)\) is first-countable or core-compact (especially, locally compact), then \(\Sigma K(X)\) is sober.

For the local compactness of Smyth power spaces, we have the following.

Lemma 8.2. (Theorem 3.1) For a \(T_0\) space, the following conditions are equivalent:

1. \(X\) is locally compact.
2. \(P_2(X)\) is core-compact.
3. \(P_2(X)\) is locally compact.
4. \(P_5(X)\) is locally hypercompact.
5. \(P_5(X)\) is a e-space.

The following corollary follows directly from Proposition 6.11 and Lemma 8.2.

Corollary 8.3. For a locally compact \(T_0\) space \(X\), the Smyth power space \(P_5(X)\) is a DC space.
Question 8.4. For a locally compact $T_0$ space $X$, is the Scott power space $\Sigma K(X)$ a Rudin space or a WD space?

Proposition 8.5. Let $X$ be a locally compact sober space. Then

(1) the Scott power space of $X$ and the Synth power space of $X$ coincide, that is, $\Sigma K(X) = P_S(X)$.
(2) $K(X)$ is a continuous domain.
(3) $\Sigma K(X)$ is a sober c-space.

Proof. By Corollary 4.9 and Lemma 6.1 $\Sigma K(X) = P_S(X)$. By Proposition 1-1.24.2, $K(X)$ is a continuous semilattice, and hence by Theorem 3.4 and Proposition 4.3 $\Sigma K(X)$ is a sober c-space. \hfill $\square$

Corollary 8.6. If $X$ is a locally compact $T_2$ (especially, a compact $T_2$) space, then

(1) $\Sigma K(X) = P_S(X)$.
(2) $K(X)$ is a continuous domain.
(3) $\Sigma K(X)$ is a sober c-space.

By Theorem 3.4, Proposition 4.3 and Proposition 8.5, we have the following corollary.

Corollary 8.7. Let $P$ be a quasicontinuous domain. Then

(1) the upper Vietoris topology agrees with the Scott topology on $K(\Sigma P)$.
(2) $K(\Sigma P)$ is a continuous semilattice.
(3) the Scott power space $\Sigma K(\Sigma P)$ is a sober c-space.

Now we discuss the first-countability of the Scott power spaces. First, for the Smyth power spaces and sobrifications of $T_0$ spaces, we have the following conclusion.

Proposition 8.8. (3, 37, 33) For a $T_0$ space, the following conditions are equivalent:

(1) $X$ is second-countable.
(2) $P_S(X)$ is second-countable.
(3) $X^*$ is second-countable.

Since first-countability is a hereditary property, by Remark 2.3 and Remark 2.4, we get the following result.

Proposition 8.9. Let $X$ be a $T_0$ space. If $X^*$ is first-countable or $P_S(X)$ is first-countable, then $X$ is first-countable.

Example 7.1 shows that unlike the Smyth power space, the first-countability of the Scott power space of a $T_0$ space $X$ does not imply the first-countability of $X$ in general.

The converse of Proposition 8.9 does not hold in general, as shown in Example 8.27 and the following example. It also shows that even for a compact Hausdorff first-countable space $X$, the Scott power space of $X$ and the Smyth power space of $X$ may not be first-countable.

There is even a $T_0$ space $X$ for which the Scott power space $\Sigma K(X)$ is second-countable but $X$ is not first-countable (see Example 8.26 below). So for Scott power spaces, the analogous results to Proposition 8.8 and Proposition 8.9 do not hold.

Example 8.10. Consider in the plane $\mathbb{R}^2$ two concentric circles $C_i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = i\}$, where $i = 1, 2$, and their union $X = C_1 \cup C_2$; the projection of $C_1$ onto $C_2$ from the point $(0, 0)$ is denoted by $p$. On the set $X$ we generate a topology by defining a neighbourhood system $\{B(z) : z \in X\}$ as follows: $B(z) = \{z\}$ for $z \in C_2$ and $B(z) = \{U_j(z) : j \in \mathbb{N}\}$ for $z \in C_1$, where $U_j = V_j \cup \{p(V_j \setminus \{z\})\}$ and $V_j$ is the arc of $C_1$ with center at $z$ and of length $1/j$. The space $X$ is called the Alexandroff double circle (cf. 3). The following conclusions about $X$ are known (see, for example, 3, Example 3.1.26).

(a) $X$ is Hausdorff and first-countable.
(b) $X$ is compact and locally compact.
(c) $X$ is not separable, and hence not second-countable.
(d) $C_1$ is a compact subspace of $X$.
(e) $C_2$ is a discrete subspace of $X$.

There is no countable base at $C_1$ in $P_S(X)$. Thus $P_S(X)$ is not first-countable. For details, see [33, Example 4.4]. By Corollary 8.6 $\Sigma K(X) = P_S(X)$, whence the Scott power space of $X$ is not first-countable.

Proposition 8.11. ([33 Proposition 4.5]) Let $X$ be a first-countable $T_0$ space. If $\min(K)$ is countable for any $K \in K(X)$, then $P_S(X)$ is first-countable.

Proposition 8.12. For a metric space $(X,d)$, $P_S((X,d))$ is first-countable.

Proof. For $K \in K((X,d))$, let $B_K = \{B(K,1/n) : n \in \mathbb{N}\}$. Then by Proposition 2.12 $B_K = \{B(K,1/n) : n \in \mathbb{N}\}$ is a countable base at $K$ in $P_S((X,d))$. Thus $P_S((X,d))$ is first-countable.

For a countable $T_0$ space $X$, it is easy to see that $X$ is second-countable if $X$ is first-countable. Indeed, let $X = \{x_n : n \in \mathbb{N}\}$. If $X$ is first-countable, then for each $n \in \mathbb{N}$, there is a countable base $B_n$ at $x_n$. Let $B = \bigcup_{n \in \mathbb{N}} B_n$. Then $B$ is a countable base of $X$. Thus $X$ is second-countable. Therefore, by Proposition 8.8 and Proposition 8.9, we have the following.

Corollary 8.13. For a countable $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is first-countable.
2. $X$ is second-countable.
3. $X^*$ is first-countable.
4. $X^*$ is second-countable.
5. $P_S(X)$ is first-countable.
6. $P_S(X)$ is second-countable.

It is worth noting that the Scott topology on a countable complete lattice may not be first-countable, see [29, Example 4.8].

By Proposition 5.22, Theorem 5.23, Proposition 8.8 and Corollary 8.13 we deduce the following two results.

Corollary 8.14. ([31 Corollary 5.7 and Corollary 5.8]) Every second-countable (especially, countable first-countable) $T_0$ space is an $\omega$-Rudin space.

Corollary 8.15. Every second-countable (especially, countable first-countable) $\omega$-well-filtered space is sober.

For a $T_0$ space $X$ with a first-countable Smyth power space, we have a similar result to Lemma 6.1.

Lemma 8.16. Let $X$ be a $T_0$ space for which the Smyth power space $P_S(X)$ is first-countable. Then the Scott topology is coarser than the upper Vietoris topology on $K(X)$.

Proof. See the proof of [33 Theorem 5.7].

The following conclusion is straightforward from Theorem 4.8 Corollary 4.9 Corollary 5.24 and Lemma 8.16

Corollary 8.17. ([33 Theorem 5.7]) Let $X$ be a well-filtered space for which the Smyth power space $P_S(X)$ is first-countable. Then

1. the upper Vietoris topology agrees with the Scott topology on $K(X)$.
2. the Scott power space $\Sigma K(X)$ is a first-countable sober space.

By Proposition 8.11 and Corollary 8.17 we obtain the following.
Corollary 8.18. ([33, Corollary 5.10]) Let $X$ be a first-countable well-filtered space $X$ in which all compact subsets are countable (especially, $|X| \leq \omega$). Then

1. the upper Vietoris topology agrees with the Scott topology on $K(X)$.
2. the Scott power space $\Sigma K(X)$ is a first-countable sober space.

Let $X_{\text{coc}}$ be the space in Example 7.1. Then $X_{\text{coc}}$ is well-filtered and not first-countable, and the Scott power space $\Sigma K(X)$ is a first-countable sober c-space, but $\sigma(K(X_{\text{coc}})) \not\subseteq O(P_S(X_{\text{coc}}))$.

By Example 7.1 Lemma 8.16 and Corollary 8.17, we naturally pose the following four questions.

Question 8.19. For a first-countable $T_0$ space $X$, is the Scott topology coarser than the upper Vietoris topology on $K(X)$?

Question 8.20. For a first-countable well-filtered (or equivalently, a first-countable sober) space $X$, does the upper Vietoris topology and the Scott topology on $K(X)$ coincide?

Question 8.21. For a first-countable $T_2$ space $X$, does the upper Vietoris topology and the Scott topology on $K(X)$ coincide?

Question 8.22. Is the Scott power space of a first-countable well-filtered (or equivalently, a first-countable sober) space sober?

Since every metric space is $T_2$ (and hence sober), by Proposition 8.5, Proposition 8.12 and Corollary 8.17, we get the following conclusion.

Corollary 8.23. Let $(X, d)$ be a metric space. Then

1. the upper Vietoris topology agrees with the Scott topology on $K((X, d))$.
2. the Scott power space $\Sigma K((X, d))$ is a first-countable sober space.

If, in addition, $(X, d)$ is locally compact (especially, compact), then

3. $K((X, d))$ is a continuous semilattice.
4. the Scott power space $\Sigma K((X, d))$ is a c-space.

The following two conclusions follow directly from Proposition 8.8, Lemma 8.16 and Corollary 8.17.

Corollary 8.24. Let $X$ be a second-countable $T_0$ space. Then the Scott topology is coarser than the upper Vietoris topology on $K(X)$.

Corollary 8.25. Let $X$ be a second-countable well-filtered space (or equivalently, a second-countable sober space). Then

1. the Scott topology agrees with the upper Vietoris topology on $K(X)$.
2. the Scott power space of $X$ is a second-countable sober space.

The following example shows that there is a countable Hausdorff space $X$ for which the Scott power space $\Sigma K(X)$ is second-countable but $X$ is not first-countable (and hence $P_S(X)$ is not first-countable).

Example 8.26. Let $p$ be a point in $\beta(\mathbb{N}) \setminus \mathbb{N}$, where $\beta(\mathbb{N})$ is the Stone-Cech compactification of the discrete space of natural numbers, and consider on $X = \mathbb{N} \cup \{p\}$ the induced topology (cf. [9, Example II-1.25]). Then

(a) $|X| = \omega$ and $X$ is a non-discrete Hausdorff space and hence a sober space.
(b) $K(X) = X^{(<\omega)} \setminus \{\emptyset\}$ and $\text{int} K = \emptyset$ for each $K \in K(X)$. So $X$ is not locally compact.
(c) $K(X)$ is a Noetherian poset and $|K(X)| = \omega$. Hence the Scott power space $\Sigma K(X)$ is a second-countable sober c-space.

Clearly, $K(X) = X^{(<\omega)} \setminus \{\emptyset\}$ (with the Smyth order) is Noetherian (and hence algebraic) and $|K(X)| = \omega$ since $|X| = \omega$. Therefore, $\sigma(K(X)) = \alpha(K(X))$ and $\{\uparrow_{K(X)} F : F \in X^{(<\omega)} \setminus \{\emptyset\}\} \subseteq \Sigma K(X)$ is a countable base of $\Sigma K(X)$. By Theorem 5.4 and Proposition 4.3, $\Sigma K(X)$ is a sober c-space.
(d) the upper Vietoris topology and the Scott topology on \(K(X)\) do not coincide, or more precisely, \(\sigma(K(X)) \not\subseteq \mathcal{O}(P_S(X))\).

By Corollary 1.9, \(\mathcal{O}(P_S(X)) \subseteq \sigma(K(X))\). Clearly, for any \(F \in X^{(<\omega)} \setminus \{\emptyset\}\), \(\uparrow_{K(X)} F \in \sigma(K(X))\) but \(\uparrow_{K(X)} F \notin \mathcal{O}(P_S(X))\).

(e) Neither \(X\) nor \(P_S(X)\) is first-countable.

By (d), Proposition 8.11 and Lemma 8.16, neither \(P_S(X)\) nor \(X\) is first-countable (cf. [3] Corollary 3.6.17).

The above example also shows that if the Smyth power space is replaced with the Scott power space in the conditions of Lemma 8.16 and Corollary 8.17, the analogous results to Lemma 8.16 and Corollary 8.17 do not hold.

By Proposition 8.8, Lemma 8.16, Corollary 8.17 and Corollary 8.25, we raise the following question.

**Question 8.27.** For a second-countable \(T_0\) space \(X\), is the Scott power space of \(X\) second-countable?

**9. Rudin property and well-filtered determinedness of Smyth power spaces and Scott power spaces**

Firstly, we discuss the Rudin property and well-filtered determinedness of Smyth power spaces. The following result was proved in [30].

**Proposition 9.1.** ([30, Theorem 7.21]) Let \(X\) be a \(T_0\) space. If \(P_S(X)\) is well-filtered determined, then \(X\) is well-filtered determined.

By Theorem 4.8 and Theorem 5.10, we have the following.

**Proposition 9.2.** Let \(X\) be a well-filtered space. Then the following conditions are equivalent:

1. \(X\) is a Rudin space.
2. \(X\) is a WD space.
3. \(P_S(X)\) is a Rudin space.
4. \(P_S(X)\) is a WD space.

It is still not known whether the converse of Proposition 9.1 holds (that is, whether the Smyth power space \(P_S(X)\) of a well-filtered determined \(T_0\) space \(X\) is well-filtered determined) (see [30, Question 8.6]).

**Theorem 9.3.** Let \(X\) be a \(T_0\) space. If \(P_S(X)\) is a Rudin space, then \(X\) is a Rudin space.

**Proof.** Let \(A \in \text{Ir}_{\tau}(X)\). Then by Lemma 2.5, \(\xi_X(A) = \triangle A \in \text{Ir}_{\tau}(P_S(X))\), where \(\xi_X : X \to P_S(X)\) is the canonical embedding (see Remark 2.4). Since \(P_S(X)\) is a Rudin space, there is a filtered family \(\{K_d : d \in D\} \subseteq K(P_S(X))\) such that \(\diamond A \in m(\{K_d : d \in D\})\). For each \(d \in D\), let \(K_d = \bigcup K_d\). Then by Lemma 2.11, \(\{K_d : d \in D\} \subseteq K(X)\) is filtered. Clearly, \(A \in m(\{K_d : d \in D\})\). For any proper closed subset \(B\) of \(A\), we have that \(\diamond B \in \mathcal{C}(P_S(X))\) and \(\diamond B\) is a proper closed subset of \(\diamond U\) (for any \(a \in A \setminus B\), \(\uparrow a \in \diamond A \setminus \diamond B\)). By the minimality of \(\diamond U\), there is a \(d \in D\) such that \(\diamond B \cap K_d = \emptyset\), and consequently, \(B \cap K_d = \emptyset\). Thus \(B \notin m(\{K_d : d \in D\})\), and hence \(A \in m(\{K_d : d \in D\})\).

**Question 9.4.** Is the Smyth power space \(P_S(X)\) of a Rudin space \(X\) still a Rudin space?

Now we discuss the Rudin property and well-filtered determinedness of Scott power spaces.

First, even for a sober space \(X\) (whence it is both a Rudin space and a WD space by Theorem 5.10), its Scott power space may not be a WD space (and hence not a WD space). Indeed, let \((\Sigma L)\tau\) be as in Theorem 7.17. Then \((\Sigma L)\tau\) is a sober space. By Theorem 6.5 and Theorem 7.17, the Scott power space \(\Sigma K((\Sigma L)\tau)\) is well-filtered but non-sober. Hence by Theorem 5.10, \(\Sigma K((\Sigma L)\tau)\) is neither a Rudin space nor a WD space.

Conversely, Example 7.1 shows that there is a well-filtered space \(X\) such that
(a) the Scott power space $\Sigma K(X)$ is a first-countable sober $c$-space and hence $\Sigma K(X)$ is both Rudin and WD.
(b) $X$ is neither a Rudin space nor a WD space.
(c) the Smyth power space $P_S(X)$ is neither a Rudin space nor a WD space.

Then we investigate some sufficient conditions under which the well-filtered determinedness (resp. the Rudin property) of Scott power space of a $T_0$ space $X$ implies that of $X$.

**Definition 9.5.** A $T_0$ space $X$ is said to have property S if for each $A \in \text{Irr}_c(X)$, $\{\uparrow a : a \in A\} \in \text{Irr}(\Sigma K(X))$ or $\Diamond A \in \text{Irr}_c(\Sigma K(X))$. A poset $P$ is said to have property S if $\Sigma P$ has property S.

**Remark 9.6.** Let $X$ be a $T_0$ space and $A \in \text{Irr}_c(X)$.

1. Since $\xi_X : X \to P_S(K(X)), x \mapsto \uparrow x$, is continuous, $\{\uparrow a : a \in A\} \in \text{Irr}(P_S(X))$ and $\text{cl}_{O(P_S(X))}\{\uparrow a : a \in A\} = \Diamond A \in \text{Irr}_c(P_S(X))$ by Lemma 2.2 and Lemma 2.5.

2. If $\sigma(K(X)) \subseteq O(P_S(X))$, then $\xi_X : X \to \Sigma K(X)$ is continuous by Remark 2.4 and hence $X$ has property S.

3. For a poset $P$, by Lemma 3.1 and Lemma 2.9, the mapping $\xi_P : \Sigma P \to \Sigma K(\Sigma P), x \mapsto \uparrow x$, is continuous. Therefore, $P$ has property S.

**Proposition 9.7.** Suppose that a $T_0$ space $X$ has property S and $O(P_S(X)) \subseteq \sigma(K(X))$. If $\Sigma K(X)$ is a Rudin space, then $X$ is a Rudin space.

**Proof.** Let $A \in \text{Irr}_c(X)$. Then by the property S of X, $\{\uparrow a : a \in A\} \in \text{Irr}(\Sigma K(X))$ or $\Diamond A \in \text{Irr}_c(\Sigma K(X))$.

**Case 1:** $\{\uparrow a : a \in A\} \in \text{Irr}(\Sigma K(X))$.

Since $\Sigma K(X)$ is a Rudin space, there is a filtered family $\{K_d : d \in D\} \subseteq K(\Sigma K(X))$ such that $\text{cl}_{\sigma(K(X))}\{\uparrow a : a \in A\} \in m(\{K_d : d \in D\})$. As $O(P_S(X)) \subseteq \sigma(K(X))$, we have that $\text{cl}_{\sigma(K(X))}\{\uparrow a : a \in A\} \subseteq \Diamond A \in C(P_S(X)) \subseteq C(\Sigma K(X))$ and $\{K_d : d \in D\} \subseteq K(P_S(X))$. Therefore, $\Diamond A \in M(\{K_d : d \in D\})$. For each $d \in D$, let $K_d = \bigcup K_d$. Then by Lemma 2.11, $\{K_d : d \in D\} \subseteq K(X)$ is filtered. Since $\Diamond A \in M(\{K_d : d \in D\})$, $A \in M(\{K_d : d \in D\})$. Now we show that $A \in m(\{K_d : d \in D\})$.

Suppose that $B$ is a proper closed subset of $A$. Then there is $a \in A \cap (X \setminus B)$, and hence $\uparrow a \in \Box (X \setminus B) \in O(P_S(X)) \subseteq \sigma(K(X))$. Clearly, $\{\uparrow b : b \in B\} \cap \Box (X \setminus B) = \emptyset$, and consequently, $\uparrow a \notin \text{cl}_{\sigma(K(X))}\{\uparrow b : b \in B\}$. Therefore, $\text{cl}_{\sigma(K(X))}\{\uparrow b : b \in B\}$ is a proper subset of $\text{cl}_{\sigma(K(X))}\{\uparrow a : a \in A\}$. By $\text{cl}_{\sigma(K(X))}\{\uparrow a : a \in A\} \in m(\{K_d : d \in D\})$, there is $d_0 \in D$ such that $\text{cl}_{\sigma(K(X))}\{\uparrow b : b \in B\} \cap K_{d_0} = \emptyset$, and hence $\{\uparrow b : b \in B\} \cap K_{d_0} = \emptyset$. Since $K_{d_0} = \uparrow K(X)K_{d_0}$, we have that $B \cap K_{d_0} = B \cap (\bigcup K_{d_0}) = \emptyset$. Thus $A \in m(\{K_d : d \in D\})$.

**Case 2:** $\Diamond A \in \text{Irr}_c(\Sigma K(X))$.

Since $\Sigma K(X)$ is a Rudin space, there is a filtered family $\{K_d : d \in D\} \subseteq K(\Sigma K(X))$ such that $\Diamond A \in m(\{K_d : d \in D\})$.

As carried out in the proof of Case 1, $A$ is a Rudin set of $X$.

Thus $X$ is a Rudin space.

**Corollary 9.8.** Suppose that $X$ is a well-filtered space with property S. If $\Sigma K(X)$ is a WD space (especially, a Rudin space), then both $\Sigma K(X)$ and $X$ are sober.

**Proof.** By Theorem 6.5, $\Sigma K(X)$ is well-filtered. As $\Sigma K(X)$ is WD (if $\Sigma K(X)$ is Rudin, then by Proposition 5.8 it is WD), by Theorem 5.10, $\Sigma K(X)$ is sober. Hence, by Theorem 5.10 and Corollary 4.9, $\Sigma K(X)$ is Rudin and $O(P_S(X)) \subseteq \sigma(K(X))$, and consequently, $X$ is Rudin by Proposition 9.7. It follows from Theorem 5.10 that $X$ is sober.

By Remark 9.6 and Corollary 9.8 we have the following corollary.

**Corollary 9.9.** Let $X$ be a well-filtered space. If $\xi_X : X \to \Sigma K(X)$ is continuous and $\Sigma K(X)$ is a WD space (especially, a Rudin space), then both $\Sigma K(X)$ and $X$ are sober.

As an immediate corollary of Corollary 9.9 we get the following result.
Corollary 9.10. ([32] Theorem 2]) Suppose that $X$ is a well-filtered space and $\xi^*_X : X \to \Sigma K(X)$ is continuous. If $\Sigma K(X)$ is sober, then $X$ is sober. Therefore, if $X$ is non-sober, then its Scott power space $\Sigma K(X)$ is non-sober.

Example 7.1 shows that when $X$ lacks the property S or the continuity of $\xi^*_X : X \to \Sigma K(X)$, Proposition 9.15. Corollary 9.8, Corollary 9.9 and Corollary 9.10 may not hold.

By Remark 9.6 Proposition 9.7 Corollary 9.8 and Corollary 9.10 we deduce the following three corollaries.

Corollary 9.11. Let $P$ be a poset. If $O(P_S(\Sigma P)) \subseteq \sigma(\Sigma P)$ and $K(\Sigma P)$ is a Rudin poset, then $P$ is a Rudin dcpo.

Corollary 9.12. Let $P$ be a well-filtered dcpo. If $K(\Sigma P)$ is a WD dcpo (especially, a Rudin dcpo), then both $K(\Sigma P)$ and $P$ are sober dcpos.

Corollary 9.13. Let $P$ be a well-filtered dcpo. If $K(\Sigma P)$ is a sober dcpo, then $P$ is a sober dcpo. Therefore, if $P$ is not a sober dcpo, then $K(\Sigma P)$ is not a sober dcpo.

Example 9.14. Let $L$ be the Isbell’s lattice constructed in [12]. Then
(a) $L$ is not a sober dcpo (see [12]).
(b) $L$ is a well-filtered dcpo by Proposition 4.6.
(c) $L$ is neither a Rudin dcpo nor a WD dcpo by (a)(b) and Corollary 9.18.
(d) $K(\Sigma L)$ is a well-filtered dcpo by Theorem 6.5.
(e) $K(\Sigma L)$ is a spatial frame (see [32] Lemma 1).
(f) $K(\Sigma L)$ is not a sober dcpo by (a) and Corollary 9.13.
(g) $K(\Sigma L)$ is neither a Rudin dcpo nor a WD dcpo by (a)(b) and Corollary 6.12.

Proposition 9.15. Suppose that $X$ is a $T_0$ space for which $\sigma(K(X)) \subseteq O(P_S(X))$. If $\Sigma K(X)$ is well-filtered determined, then $X$ is well-filtered determined.

Proof. By Remark 9.6 $X$ has property S. Let $A \in Irr_c(X)$, $Y$ a well-filtered space and $f : X \to Y$ a continuous mapping. Then by $\sigma(K(X)) \subseteq O(P_S(X))$, Lemma 2.6 and Theorem 6.5 $\Sigma K(Y)$ is well-filtered and $P_S^c(f) : P_S(X) \to \Sigma K(Y)$ is continuous, where $P_S^c(f)(K) = \uparrow f(K)$ for all $K \in K(X)$. By assumption, \{a : a \in A\} $\in Irr(K(X))$ or $A \in Irr_c(\Sigma K(X))$, and hence by the well-filtered determinedness of $\Sigma K(Y)$ and the continuity of $P_S^c(f)$, there exists a unique $Q \in \Sigma K(Y)$ such that \{a \in A\} = $P^c_S(f)(\{\uparrow a : a \in A\}) = \{Q\}$. By Remark 9.6 $\Sigma K(Y)$ is sober dcpo. Therefore, $\uparrow f(a) : a \in A\} = \{Q\} \subseteq \Sigma K(Y)$. By assumption, \{a \in A\} = $P^c_S(f)(\{\uparrow a : a \in A\}) = \{Q\}$ in $\Sigma K(Y)$, since $P^c_S(f)(\{\uparrow a : a \in A\}) = \{Q\} \subseteq \Sigma K(Y)$. Since $Y$ is well-filtered, by Corollary 4.9 $O(P_S(Y)) \subseteq \sigma(K(X))$. Hence by Remark 9.6 $\Sigma K(Y)$ is well-filtered determined.

Claim 1: $Q$ is supercompact.

Let $\{V_j : j \in J\} \subseteq O(Y)$ with $Q \subseteq \bigcup_{j \in J} V_j$, i.e., $Q \subseteq \bigcup_{j \in J} V_j$. Since $\bigcap_{j \in J} \{\uparrow a : a \in A\} \cap \bigcup_{j \in J} V_j \neq \emptyset$. Then there exists $a_0 \in A$ such that $\uparrow f(a_0) \subseteq V_j$, and consequently, $\{a_0 : a \in A\} \cap \bigcup_{j \in J} V_j \neq \emptyset$. By Corollary 9.6 $\Sigma K(X)$ is well-filtered determined again, we have $Q \subseteq \bigcup_{j \in J} V_j$, that is, $Q \subseteq U_j$.

Hence, by [10] Fact 2.2, there exists $y_Q \in Y$ such that $Q = \uparrow f(y_Q).

Claim 2: $f(A) = \{y_Q\}$ in $Y$.

For each $y \in f(A)$, by Corollary 9.6 $\{\uparrow a : a \in A\} = \bigcap_{j \in J} V_j$, we have that $\uparrow y \in \bigcap_{j \in J} V_j$. This implies that $f(A) \subseteq \{y_Q\}$. In addition, since $\{y_Q\} \subseteq \bigcap_{j \in J} V_j$, we have that $y_Q \in \bigcap_{j \in J} V_j$. This implies that $f(A) \subseteq \{y_Q\}$. In addition, since $\{y_Q\} \subseteq \bigcap_{j \in J} V_j$, we have that $y_Q \in \bigcap_{j \in J} V_j$. This implies that $f(A) \subseteq \{y_Q\}$.

By Claim 2, $A \in WD(X)$, proving that $X$ is well-filtered determined.

Corollary 9.16. For a poset $P$, if $\sigma(K(\Sigma P)) \subseteq O(P_S(\Sigma P))$ and $K(\Sigma P)$ is a WD poset, then $P$ is a WD poset.
From Corollary 5.18 and Corollary 9.16 we deduce the following result.

**Corollary 9.17.** If $P$ is a well-filtered dcpo, $\sigma(K(\Sigma P)) \subseteq \mathcal{O}(P(\Sigma P))$ and $K(\Sigma P)$ is a WD dcpo (especially, a Rudin dcpo), then both $K(\Sigma P)$ and $P$ are sober dcpos.

Finally, by Lemma 6.1, Lemma 8.16 and Proposition 9.15, we get the following two corollaries.

**Corollary 9.18.** If $X$ is a locally compact $T_0$ space and $\Sigma K(X)$ is well-filtered determined, then $X$ is well-filtered determined.

**Corollary 9.19.** Suppose that $X$ is a $T_0$ space for which the Smyth power space $P_0(X)$ is first-countable. If $\Sigma K(X)$ is well-filtered determined, then $X$ is well-filtered determined.

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