Quantum corrections to leptogenesis from the gradient expansion

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Using the closed-time-path formalism we quantify gradient corrections to the kinetic equations for leptogenesis, that are neglected in the standard Boltzmann approach. In particular we show that an additional CP-violating source term arises, which is non-zero even when all species are in local thermal equilibrium. In the early universe it is proportional to the expansion rate and would vanish for static equilibrium configurations, in accordance with the Sakharov conditions. We find that for thermal leptogenesis in a standard cosmological background the additional source term is small. However, it can become the dominant source in the limit of ultra-strong washout.

Keywords: Kadanoff–Baym equations, Boltzmann equation, CP-violation, expanding universe, leptogenesis

I. INTRODUCTION

Today the observed universe almost entirely consists of matter, i.e. is baryonically asymmetric. An attractive explanation of the observed asymmetry is provided by the baryogenesis via leptogenesis scenario \cite{Fukugita:1986hr}. In this scenario the Standard Model is supplemented by heavy Majorana neutrinos. The CP- and lepton-number violating decay of the heavy neutrinos produces a net lepton asymmetry.

The rapid expansion of the universe ensures that this asymmetry is not washed out by the inverse decay and scattering processes. Finally, anomalous electroweak processes convert the generated lepton asymmetry to the observed baryon asymmetry \cite{Fukugita:1986hr}.

The computation of the asymmetry in terms of the neutrino masses and mixing parameters requires a microscopic description of the out-of-equilibrium decay process. The canonical approach is based on Boltzmann equations, furnished with decay and scattering rates computed from the vacuum S-matrix elements. The size of various corrections to this approximation has been investigated by including e.g. deviations from kinetic equilibrium, quantum statistical terms, thermal masses and matrix elements \cite{Fukugita:1986hr}.

Beyond this it is important to check the validity of the semi-classical treatment embodied by the Boltzmann approach. This is particularly relevant for leptogenesis where loop effects and unstable particles are essential for the generation of the asymmetry. Within the standard Boltzmann (bottom-up) approach, these typically give rise to double counting problems as well as ambiguities related to the application of equilibrium quantum field theory for the analysis of out-of-equilibrium processes.

These ambiguities can be resolved within the nonequilibrium closed-time-path or Schwinger-Keldysh formalism. In contrast to the bottom-up approach this may be viewed as a top-down approach. Here, the underlying microscopic description is based on the full quantum-mechanical evolution equation for the expectation value of the lepton current. Starting from the latter, quantum-corrected Boltzmann-like equations can be derived, which are inherently free of the double-counting problem, and include medium corrections to the CP-violating parameters in a consistent way \cite{Garny:2009uj}. Furthermore, a possible influence of off-shell and memory effects has also been studied in this approach \cite{Garny:2009uj}.

The reduction of the full quantum equations of motion to Boltzmann-like equation requires a so-called gradient expansion in powers of space-time gradients $\partial_X$. Physically, it amounts to an expansion in powers of the ratio of the microscopic time-scale $t_{\text{mic}} \sim 1/M_i$ and the macroscopic scales $t_{\text{mac}} \sim 1/\Gamma_i, 1/H$. This means, it requires that the decay rates $\Gamma_i$ and the cosmic expansion rate $H$ are much smaller than the corresponding right-handed neutrino masses $M_i$.

The standard Boltzmann treatment of leptogenesis relies on the zeroth order in the gradient expansion, i.e. neglects all gradient terms. Although the condition $t_{\text{mic}} \ll t_{\text{mac}}$ is typically well fulfilled in thermal leptogenesis, higher gradient terms could still be important for the calculation of the asymmetry, since the latter depends on the tiny difference between particle and antiparticle interactions and densities. At zeroth order the deviation from equilibrium, which is crucial according to the third Sakharov condition, is described by the nonequilibrium distribution function of the heavy Majorana neutrinos: $\Delta f_{\psi}(p) = f_{\psi}(p) - f_{\psi}^0(p)$. Consequently, the CP-violating source term is proportional to $\Delta f_{\psi}$.

The gradient terms capture another potential source for a deviation from thermal equilibrium, which is due to the time-dependence of the effective temperature $T(t)$ of the thermal bath of leptons, quarks, gauge- and Higgs bosons. Consequently, we expect that the gradient CP-violating source term is proportional to $\dot{T} \equiv dT/dt$. In contrast to the standard source, it remains non-zero.
in the limit $\Delta f_{\psi_i} \to 0$, and therefore could be important even though it is suppressed by the Hubble scale, $T \approx -HT$. Furthermore, it is conceivable that leptogenesis or baryogenesis occurs in the early Universe simultaneously with other non-equilibrium phenomena such as (p)reheating or phase transitions in which case gradient contributions could be strongly enhanced. Note that such gradient terms are crucial for electroweak baryogenesis, see e.g. [21, 22]. Finally, on the formal level, it is well-known that including first-order gradient terms ensures the validity of exact conservation laws of the kinetic equations [23].

The aim of this work is to quantify the leading CP-violating source term proportional to $T$ arising from gradient corrections to the kinetic equations. In section II we briefly review the Boltzmann and the closed-time-path approaches and set up our notation. In the same section we discuss the gradient expansion and the Boltzmann limit. Next we analyze the new source term and discuss the results.

(i) As we demonstrate in section III the additional source term has a qualitatively new structure and does not vanish even if all species are in local thermal equilibrium. Furthermore, we demonstrate that it becomes important for very heavy Majorana neutrinos and ultra-strong washout.

(ii) In section IV we argue that the gradient corrections also modify the well-known standard source term.

Finally we summarize our results and conclude in section V.

II. GENERATION OF AN ASYMMETRY IN THE CTP APPROACH

In this section, we first review the standard Boltzmann approach, and then the closed-time-path (CTP) approach for the generation of a $B-L$ asymmetry. In order to illustrate the effect of the gradient corrections, we consider a toy model which has also been used in [11, 12, 29]. However, the derivation and the structure of the results are generic, and similar gradient corrections will also be present in phenomenological scenarios such as thermal leptogenesis. The Lagrangian of the model reads

$$\mathcal{L} = \frac{1}{2} \partial^\mu \psi_i \partial^\mu \psi_i - \frac{1}{2} M_i^2 \psi_i \psi_i + \partial^\mu \bar{b} \partial^\mu b - m^2 \bar{b} b - \frac{\lambda}{2^{2/3}} \bar{b} \psi_i b \psi_i - \frac{\lambda}{2^{2/3}} \bar{b} \psi_i b \psi_i, \quad i = 1, 2. \quad (1)$$

It can be considered as toy model for a generic baryogenesis scenario in which the asymmetry is produced by the out-of-equilibrium decay of some heavy species and where the CP-asymmetry in the decay is induced by the one-loop contributions depicted in Fig 1. For example, in thermal leptogenesis the real scalar fields $\psi_i$ model the heavy Majorana (s)neutrinos, whereas the complex field $b$ represents the (s)leptons. In GUT baryogenesis the real scalar fields model the heavy bosons and the complex field $b$ the baryons. In the following, we shall simply refer to $b$ as toy-baryons and to $\psi_i$ as toy-neutrinos.

A. Boltzmann approach

The standard approach is based on generalized Boltzmann equations for the distribution functions $f_a(X, p)$ of on-shell particle species $a$ [30, 31],

$$p^a \partial_a f_a(p) = C_a = C_a^{\text{gain}} [1 + f_a(p)] - C_a^{\text{loss}} f_a(p), \quad (2)$$

where $D_a$ is the covariant derivative, $C_a$ are collision integrals comprising gain and loss terms, and we suppress the space-time coordinate $X$ for brevity. The latter take decays, inverse decays and scatterings into account, with rates inferred from the $S$-matrix (in-out formalism). If one considers only the decay and inverse decay processes, see Fig 1 then for the toy-baryons

$$C_b^{\text{gain}}(p) = \int d\Pi_3^b d\Pi_3^\psi \delta(k - p - q)|M|_{\psi_i \to b}^2 \times f_{\psi_i}(k[1 + f_b(q)]), \quad (3a)$$

$$C_b^{\text{loss}}(p) = \int d\Pi_3^b d\Pi_3^\psi \delta(k - p - q)|M|_{b \to \psi_i}^2 \times [1 + f_\psi(k)] f_b(q), \quad (3b)$$

where $d\Pi_p^3 \equiv d^3 p/[(2\pi)^3 2E_p]$. For the antibaryons $f_b$ is replaced by $f_{\bar{b}}$. CPT invariance implies for the in-out matrix elements: $|M|_{\psi_i \to b b}^\pm = |M|_{\bar{b} \to \bar{b} b}^{\pm \psi_i}$ and $|M|_{\bar{b} \to \psi_i} = |M|_{b \to \baru} = \frac{1}{2} |g_i|^2 (1 + \epsilon_{ \psi_i}^\pm)$ where $\epsilon_{ \psi_i}^\pm$ is the usual CP-violating parameter. As is well-known, certain scattering contributions $bb \leftrightarrow b\bar{b}$ also have to be taken into account for consistency within the standard Boltzmann approach, see below.

The total baryon density is given by the time component of the baryon current:

$$n_B(t) \equiv V^{-1} \int dV \dot{j}_B(t, x) = n_\psi - n_\bar{b} = \int \frac{d^3 p}{2\pi^3} [f_b - f_{\bar{b}}]. \quad (4)$$

To stress the analogy with phenomenological models, we consider the difference $n_B \to n_{B-L} \equiv n_B - n_L$, where $n_L$ (or $n_{\bar{b}}$, depending on the interpretation of $b$ as baryons or leptons, respectively) vanishes in the toy model. Within
the standard Boltzmann approach, an evolution equation for \( n_{B-L} \) can be obtained by subtracting the Boltzmann equations \( \text{(2a)} \) for \( f_b \) and \( f_\bar{b} \). Dividing the left- and right-hand sides of Eq. \( \text{(2b)} \) by \( \rho^2 \equiv E_F \) and integrating over the momentum space one finds \( \text{(32)} \):

\[
\frac{1}{a^3} \frac{d}{dt} (a^3 n_{B-L}) = \int d\Omega y_f |c_b^\text{gain} (1 + f_b) - c_\bar{b}^\text{loss} f_\bar{b}|
\]

\[-c_b^\text{gain} (1 + f_b) + c_\bar{b}^\text{loss} f_\bar{b} \]  

where \( a \) is the cosmic scale factor and \( t \) the proper time. Let us stress that, when inserting Eq. \( \text{(3)} \), the structure of the in-out matrix elements would lead to a generation of an asymmetry even in equilibrium. This inconsistency originates from a double counting of decay followed by inverse decay and scattering with a \( \psi_i \) in the intermediate state. If the quantum statistical terms are neglected it can be removed by explicitly subtracting the on-shell part of the \( s \)-channel scattering, a procedure known as real intermediate state subtraction \( \text{(33, 34)} \). This is an example for double-counting mentioned in the introduction, and can be completely resolved in the CTP approach \( \text{(11, 12, 13, 14)} \).

The processes which contribute to the right-hand side of Eq. \( \text{(5)} \) can be classified as source terms \( S_0 \), which account for the generation of an asymmetry, or washout terms \( W_0 \), which tend to deplete the asymmetry. The subscript should remind the reader that, in the standard approach, only zero-order gradient contributions are included. In the hierarchical limit, \( M_2 \gg M_1 \equiv M \), the integrated Boltzmann equation for the \( B - L \) number density in the comoving volume can be cast into the form \( \text{(35–37)} \)

\[
\frac{dY_{B-L}}{dz} = S_0(z) - W_0(z) Y_{B-L},
\]

where \( Y = n/s \) is the yield and \( z \equiv M/T \) is the inverse temperature normalized by the mass of the lightest toy-neutrino\(^1\). Performing the usual approximations, one finds from Eq. \( \text{(3)} \) that the source term is given by

\[
S_0(z) = \epsilon_{\text{vac}} \kappa z \gamma_D (Y_{\psi_i} - Y^\text{eq}_{\psi_i}),
\]

where \( \kappa \equiv \Gamma/H_{\text{T}=M} \) is the so-called washout parameter and the thermally averaged dilution factor is given by the ratio of two modified Bessel functions, \( \gamma_D \equiv K_1(z)/K_2(z) \). Note that the structure is completely analogous to the phenomenological case.

Equations \( \text{(5)} \) also imply that the washout term is given by

\[
W_0(z) = \kappa z \gamma_{B-L},
\]

where \( \gamma_{B-L} \equiv z^2 K_1(z) \). Note that in a symmetric configuration, i.e. for \( Y_{B-L} = 0 \), the contribution of the washout term in Eq. \( \text{(6)} \) vanishes. We will use this property to calculate the source term in the closed-time-path formalism later on. The yield \( Y_{\psi_i}(z) \) obeys an equation similar to Eq. \( \text{(4)} \),

\[
dY_{\psi_i}/dz = -\kappa z \gamma_D (Y_{\psi_i} - Y^\text{eq}_{\psi_i}).
\]

In the strong washout regime, i.e. for large \( \kappa \), there is a well known asymptotic solution of this system. Up to an overall numerical factor \( \mathcal{O}(1) \) it reads

\[
\eta_0 \equiv Y_{B-L} (t \to \infty) \propto \frac{\epsilon_{\text{vac}}}{z_f} \frac{1}{\kappa},
\]

where \( z_f \) is the so-called freeze-out inverse temperature which is determined by the solution of \( \kappa z_f \gamma_{B-L} = 1 \).

**B. Closed-time-path approach**

Quantum corrections to the semi-classical Boltzmann approach can be studied using nonequilibrium quantum field theory techniques which rely on the closed time path formalism. In the remainder of this section we briefly review the closed time path approach, its relation to the Boltzmann approach, and discuss the gradient expansion.

In general the \( B - L \) asymmetry \( \text{(1)} \) is given by the zero-component of the expectation value of the corresponding quantum mechanical current operator:

\[
J_{\mu}(x) = 2i \{ [D_{\mu} b(x)] \bar{b}(x) - b(x) D_{\mu} \bar{b}(x) \}. \tag{10}
\]

Note that \( \langle \cdot \rangle \equiv \text{Tr}(\cdot) \), where the density matrix \( \rho \) characterizes the system at some initial time \( t_{\text{init}} \). The time-evolution of such \( in-in \) expectation values can be described within the closed-time-path or Schwinger-Keldysh approach. A useful quantity are the Wightman propagators,

\[
D_{\gamma}(x, y) \equiv \bar{D}_{\gamma}(y, x) \equiv \langle b(y) \bar{b}(x) \rangle, \quad D_{\gamma}(x, y) \equiv \bar{D}_{\gamma}(y, x) \equiv \langle b(y) \bar{b}(x) \rangle. \tag{11a}
\]

\[
D_{\gamma}(x, y) \equiv \bar{D}_{\gamma}(y, x) \equiv \langle b(y) \bar{b}(x) \rangle. \tag{11b}
\]

In terms of these, the \( B - L \) current can be expressed as

\[
J_{\mu}(x) = 2i D_{\mu,0} \big[ D_{\gamma}(x, y) - D_{\gamma}(y, x) \big] |_{y=x}. \tag{12}
\]

The time-evolution of the Wightmann propagators is described by so-called Kadanoff-Baym equations, which are self-consistent Schwinger-Dyson equations formulated on the closed time path. In the limit of coinciding arguments they read \( \text{(11, 12)} \)

\[
[\Box_x + m^2] D_{\gamma}(x, y)|_{y=x} = i \int_{t_{\text{init}}}^{t} dz \{ \Sigma_{\gamma}(x, z) D_{\gamma}(z, x) - \Sigma_{\gamma}(x, z) D_{\gamma}(z, x) \}. \tag{13}
\]

The self-energies \( \Sigma_{\gamma} \) and \( \Sigma_{\gamma} \) in Eq. \( \text{(13)} \) can be interpreted as generalizations of gain and loss terms, respectively \( \text{(11, 12)} \). At one-loop level, see Fig. \( \text{2} \) they read

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\(^1\) We have replaced the derivative with respect to the proper time \( t \) in Eq. \( \text{(9)} \) by the derivative with respect to the dimensionless inverse temperature using the relation \( \frac{1}{a^3} \frac{d}{dt} = \frac{\dot{a}}{a} \frac{dt}{d\tau} = \frac{dt}{d\tau} \) valid in the FRW universe.
Formally, this equation describes the full quantum time-evolution of the expectation value of the Hamiltonian. Note that in the comoving frame in a holographically dual setup, it is possible to determine the Schwinger-Dyson equations for the toy-baryons and toy-neutrinos, respectively.

Using Eq. (13) we can next derive an equation for the divergence of the baryon current. It reads
\[
D^\mu j_\mu(x) = 2i (\square_y + m^2) [D_>(x,y) - D_<(x,y)] |_{y=x} = - \int \mathcal{D}^4 z \left\{ \Sigma_<(x,z) D_>(z,x) - \Sigma_>(x,z) D_<(z,x) \right\}.
\]

Formally, this equation describes the full quantum time-evolution of the expectation value of the B - L current starting from arbitrary (Gaussian) initial states at \( t = t_{init} \). Therefore, it is a suitable starting point to derive non-equilibrium quantum corrections to the standard results. Note that in the comoving frame in a homogeneous FRW space-time \( j_\mu = (n_{B-L},0) \), i.e.
\[
D^\mu j_\mu(x) = \frac{1}{a^3} \frac{d}{dt} \left( a^4 n_{B-L} \right) = \frac{1}{K} \frac{dY_{B-L}}{dz},
\]

where \( K^{-1} \equiv s dz/dt = \frac{s}{sH} = \frac{sH}{|\mathcal{T}|} \). Thus, Eq. (15) indeed constitutes a quantum generalization of Eqs. (1) and (2).

Typically, there exists a separation of fast microscopic time-scales, e.g. \( t_{mic} \sim 1/M \), and the macroscopic evolution characterized by decay and expansion rates, \( t_{mac} \sim 1/\Gamma, 1/H \). The crucial observation is that the former sets the scale for the variation of the two-point functions \( \Sigma(x,y) \) with respect to the central coordinate, given by \( s = x - y \), while the latter set the scale for the variation with respect to the central coordinate, given by \( X = (x + y)/2 \). If there is a strong hierarchy between microscopic and macroscopic time-scales, which is typically the case for thermal leptogenesis, it is possible to expand Eq. (15) in gradients with respect to \( X \), and keep only terms up to a certain order (see e.g. [21]). For that purpose, it is natural to express the correlation functions in the Wigner representation, which describes the “fast” variations along the relative coordinate \( s \) in momentum space,
\[
D_\Sigma(X,p) = \int d^4 s e^{ips} D_\Sigma(X + s/2, X - s/2).
\]

Then, the gradient expansion of Eq. (15) in the limit \( t_{init} \rightarrow -\infty \) can be obtained using the general relation [22]
\[
\mathcal{D}^4 z A(x,z) B(z,x) = \int d\Pi_p e^{-i\phi} \{ A(X,p) \} \{ B(X,p) \},
\]

where \( d\Pi_p \equiv d^4 p/(2\pi)^4 \). The derivative operator
\[
\mathcal{D}^\mu j_\mu \rightarrow \mathcal{D}^\mu j_\mu |_0 + \mathcal{D}^\mu j_\mu |_1 + \ldots.
\]

As mentioned above, the standard Boltzmann limit is based on the zeroth order in the gradient expansion, which reads (see Appendix A)
\[
\mathcal{D}^\mu j_\mu |_0 = - \int d\Pi_p \Theta(p_0) \left\{ \Sigma_0^<(1 + f_k) - \Sigma_0^< f_k - \Sigma_0^< f_k + \Sigma_0^< f_k \right\} D_\rho.
\]

Note the close similarity of Eqs. (21) and (5). The first two terms on the right-hand side can be interpreted as gain- and loss terms for particles, respectively, while the last two terms represent gain and loss terms for anti-particles. Note that anti-baryons have negative baryon-number, hence the relative minus sign of the second compared to the first line. Within the CTP approach, the rates of gain and loss processes are described self-consistently by the self-energies \( \Sigma_0^< \) and \( \Sigma_0^> \). Their structure is fixed by the CTP formalism, which resolves the ambiguities of the standard Boltzmann approach mentioned before. Note that in Eq. (21), we already introduced the distribution functions for baryons, \( f_k \), according to the so-called Kadanoff-Baym ansatz,
\[
D_\Sigma = f_k D_\rho, \quad D_\Sigma = (1 + f_k) D_\rho,
\]

together with an analogous relation for anti-baryons involving \( f_\bar{k} \). Here \( D_\rho = D_\Sigma - D_\tilde{\Sigma} \) is the spectral function, which has a Breit-Wigner-like shape. In the quasiparticle limit it reads
\[
D_\rho(X,p) = 2\pi \text{sign}(p_0) \delta \left( g_{\mu\nu} p_\mu p_\nu - m^2 \right).
\]

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2 See [21, 38] for a proper generalization to curved space-time.
The corresponding self-energies in Wigner representation, see Eq. (17), read [11]

\[ \Sigma_{\xi}^{(1)}(X,p) = -\int d\Pi_k d\Pi_q (2\pi)^4 \delta(k-q-p) \times g_i^* g_j G_{\xi}^{ij}(X,k) D_\xi(X,q), \]

\[ \Sigma_{\xi}^{(2)}(X,p) = -\int d\Pi_k d\Pi_q (2\pi)^4 \delta(k-q-p) \times g_i g_j^* G_{\xi}^{ij}(X,k) D_\xi(X,q), \]

(24a)

(24b)

where \( d\Pi_k \equiv d^3 p/(2\pi)^3 \), and \( G_{\xi}^{ij} \) and \( D_\xi \) denote the Wightman propagators of the toy-neutrinos and the baryons, respectively (see Appendix [B] for more details).

The Wightman propagator of the toy-neutrinos is a two-by-two matrix with non-zero off-diagonal elements. The latter describe mixing of the heavy fields and, in particular, carry information on the CP-violation in the system. For a hierarchical mass spectrum of the toy-neutrinos one has [12]:

\[ G_{\xi}^{ij} = \varepsilon_i G_{\xi}^{ii} + \varepsilon_j^* G_{\xi}^{jj}, \]

(25)

so that Eq. (24) can be recast in terms of the CP-violating parameters,

\[ \varepsilon_i = -2 \text{Im} (g_i^*/g_i) \text{Im} \varepsilon, \]

(26)

and the diagonal components \( G_{\xi}^{ii} \) of the Wightmann function.

Using the quasiparticle approximation and integrating Eq. (24) over the time-component of the four-momenta we obtain gain and loss terms similar to those in Eq. (16), see below.

Let us now discuss the CP-violating source term within the CTP approach. As has been mentioned above the contribution of the wout neutron term vanishes in the symmetric configuration, i.e. for \( f_1 = f_2 \), and only the source term contributes to the baryon current.

Comparing the integrated Boltzmann equation (6) for the yield and Eq. (16) suggests to define the CP-violating source term as

\[ S(z) \equiv K \cdot \mathcal{D}^\mu j_\mu|_{\text{sym}}, \]

(27)

where the subscript means that one evaluates the right-hand side of Eq. (16) for a symmetric configuration. Using the zero-order gradient contribution [24] and inserting the explicit expressions for the self-energies [24], we find for the source term in the CTP approach, to zeroth order in the gradients:

\[ S_0 \equiv K \cdot \mathcal{D}^\mu j_\mu|_{0, \text{sym}} = 2K |g_1|^2 d\Pi_k d\Pi_q \Theta(p_0)(2\pi)^4 \delta(k-p-q) \times G_{\xi}^{11}(k) D_\rho(p) D_\rho(q) \cdot \epsilon(k,T) \left[ f_{\psi_1}(k) - f_{\psi_1}^\text{eq}(k) \right] \times \left[ (1 + f_{\psi_1}^\text{eq}(p))(1 + f_{\psi_1}^\text{eq}(q)) - f_{\psi_1}^\text{eq}(p) f_{\psi_1}^\text{eq}(q) \right], \]

(28)

where \( f_{\psi_1} \) and \( G_{\xi}^{11} \) are the distribution- and spectral function of the lightest toy-neutrino, respectively, and we have assumed that the baryons are in equilibrium. The effective CP-violating parameter \( \epsilon(k,T) = \epsilon_{\text{vac}} + \epsilon_{\text{med}}(k,T) \) agrees with the one obtained from the quantum-corrected Boltzmann equations derived in [11] and incorporates medium corrections [13]. In the limit of an hierarchical mass spectrum [11, 12]:

\[ \epsilon(k,T) = \epsilon_{\text{vac}} \times \left( 1 + \frac{3}{2} \frac{d^2 \alpha}{dT^2} [f_1(E_1) + f_2(E_2)] \right), \]

(29)

where the second factor accounts for the medium effects and \( E_1(2) \equiv \frac{1}{2} \left[ (M^2 + |k|^2)^2 \pm (1 - 4m^2/M^2)^2 |k| \cos \theta \right] \).

Note also that the structure of the CTP source term automatically guarantees that no asymmetry is produced in equilibrium. This means that it is inherently free of the double-counting problem and no explicit real intermediate state subtraction is required. Finally, when neglecting quantum statistical terms and medium corrections, assuming kinetic equilibrium for \( \psi_1 \), and inserting the quasi-particle approximation [26], one recovers the standard source term \( S_0 \) given in Eq. (7).

III. ADDITIONAL SOURCE TERM FROM HUBBLE EXPANSION

In this section, we derive and discuss the additional CP-violating source term for the \( B - L \) asymmetry, which is generated due to the time-dependence of the temperature of the thermal bath, \( T = -H T \).

An important feature of the CTP approach is that the CP-violating source term [27] automatically vanishes in complete thermal equilibrium [11, 12]. Thus, it must be possible to express it in terms of quantities which are nonzero only out of equilibrium. Obviously, the deviation of the toy-neutrino distribution from the equilibrium one, \( \Delta f_{\psi_1} = f_{\psi_1} - f_{\psi_1}^\text{eq} \), is such a quantity. As discussed above, a second one is given by \( \dot{T} \equiv dT/dt \). Thus, one may decompose the source term [27] according to

\[ S(z) \equiv K \cdot \mathcal{D}^\mu j_\mu|_{\text{asy}} = S(z)|_{\Delta f} + S(z)|_{\dot{T}} + S(z)|_{\Delta f \times \dot{T}}. \]

(30)

The first contribution is given by Eq. (28), of zeroth order in the gradients,

\[ S(z)|_{\Delta f} = S(z)|_0 = K \cdot \mathcal{D}^\mu j_\mu|_{0, \text{sym}}. \]

(31)

The second contribution appears only at first order in the gradients. It can be computed by setting all species into local thermal equilibrium (LTE) (i.e. setting \( \Delta f_{\psi_1} = 0 \) with time-dependent temperature \( T(t) \)),

\[ S(z)|_{\dot{T}} = S(z)|_{\text{LTE}} = K \cdot \mathcal{D}^\mu j_\mu|_{\text{LTE}}. \]

(32)

Finally, the third contribution requires both \( \Delta f_{\psi_1} \neq 0 \) and \( \dot{T} \neq 0 \). In the remaining part of this section, we will show that a source term proportional to the expansion rate, \( S \propto H \), indeed follows from \( S(z)|_{\dot{T}} \). For a discussion of \( S|_{\Delta f \times \dot{T}} \) we refer to section [IV].
The first-order gradient contribution to the full expression for the $N - L$ current, Eq. (15), can be obtained straightforwardly using Eq. (18), and keeping linear terms in the derivative operator ("Poisson bracket") defined in Eq. (19) (see Appendix A),

$$
\mathcal{D}^\mu j_\mu|_1 = -\int d^4\pi \Theta(p_0) \left[ \Sigma^\nu_1(1 + f_0) - \Sigma^\nu_0 f_b \right]
$$

$$
- \Sigma^\nu_0 (1 + f_b) - \Sigma^\nu_1 f_b D_\rho
$$

$$
+ 2\delta(\Sigma^\nu_1) \{ D_h \} + 2\delta(\Sigma^\nu_0) \{ D_F \}
$$

$$
- 2\delta(\Sigma^\nu_1) \{ D_h \} - 2\delta(\Sigma^\nu_0) \{ D_F \}.
$$

The self-energies $\Sigma^\nu_i$ are given by Eq. (22) with the first-order gradient solutions of the Kadanoff-Baym equations for the toy-neutrino propagator,

$$
\tilde{G}^1_{\nu \nu} = -i\delta(\tilde{G}_{R}, \tilde{G}_{A}) + \delta(\tilde{G}_{R}, \tilde{G}_{A}, \tilde{G}_{A}) + \delta(\tilde{G}_{R}, \tilde{G}_{A}, \tilde{G}_{A}) \right),
$$

where we have introduced $\delta(\tilde{G}_R, \tilde{G}_A) \equiv \tilde{G}_R \tilde{G}_A$ and $\delta(\tilde{G}_R, \tilde{G}_A, \tilde{G}_A) \equiv \tilde{G}_R \tilde{G}_A \tilde{G}_A$ (see Appendix B). Furthermore, $D_F = (D_\nu + D_\rho)/2$, with analogous relations for self-energies, and $D_h = (D_\nu + D_\rho)/2$, involving retarded and advanced functions (see Appendix A for more details).

As discussed above, the source term $S(z)$ can be obtained by evaluating Eq. (33) for a symmetric system in local thermal equilibrium. After a somewhat tedious calculation, for which we refer to Appendix C one finds that leading order in the toy-neutrino coupling, and in the hierarchical limit, it is given by

$$
S(z) \equiv K \cdot \mathcal{D}^\mu j_\mu|_1, \text{sym}
$$

$$
= 2K |g_0|^2 \int d^4 \pi d^4 \pi \Theta(p_0)(2\pi)^4 \delta(k - p - q)
$$

$$
\times G^1_\nu(k) D_\rho(p) D_\rho(q)
$$

$$
\times \int \mathcal{E}^T(k, T) \left[ f^{eq}_\psi(k)[1 + f^{eq}_b(p)][1 + f^{eq}_b(q)]
$$

$$
+ [1 + f^{eq}_\psi(k)][1 + f^{eq}_\psi(p)][1 + f^{eq}_\psi(q)] \right].
$$

The self-energies $\Sigma^\nu_i$ are defined in Eq. (19) (see Appendix A),

$$
\bar{\Sigma}^\nu = -\frac{\pi T}{3} \times \left( \frac{1}{2} + 3 f^{eq}_\psi(\ell) \right)
$$

$$
\times \left[ \frac{\mathcal{P}}{(k + \ell)^2 - m^2} + \frac{\mathcal{P}}{(k - \ell)^2 - m^2} \right].
$$

This additional source term is the main result of this work. Note that it is proportional to the expansion rate, since $T = -\frac{\partial}{\partial T}$, as expected from a gradient contribution. Its structure is qualitatively different from the zero-order source term $\bar{\Sigma}^\nu$, since it can be non-zero even when the heavy toy-neutrinos are in equilibrium with the thermal bath. We stress that the structure of this source term is, nevertheless, in agreement with the Sakharov conditions: it is proportional to $\epsilon^{vac}$, i.e. it requires CP violation, and it requires a deviation from equilibrium, which is, however, described by $T$ here.

Let us now discuss the implications of this source term. An explicit expression for $\epsilon_T$ can be found in Eq. (22). In the strong washout regime most of the asymmetry is generated at temperatures $T \lesssim M$. In this limit we obtain for the new CP-violating parameter

$$
\epsilon_T(k, T) \approx -\epsilon^{vac} \times \frac{H}{2\pi T}.
$$

Inserting $H = 1.66 \sqrt{g_0} T^2/M_{Pl}$ with $g_0 = 106.75$, we find

$$
\epsilon_T(k, T) \approx -2.7 \epsilon^{vac} \times \frac{T}{M_{Pl}}.
$$

This is clearly much smaller than the CP-violating parameter $\epsilon^{vac}$ itself for typical temperatures $T \sim 10^9$ GeV. However, one should keep in mind that the structure of the source term (28) differs from the usual one. In particular, if the neutrino is very close to equilibrium, the zero-order source term (28) is suppressed, which could partly compensate the smallness of $\epsilon_T$. This is typically relevant in the strong washout regime. Therefore, we expect that it is legitimate to neglect the gradient correction (28), unless for extremely strong washout. In order to estimate its impact, we apply the approximations common for the standard approach to Eq. (25), and use Eq. (45). In this limit Eq. (45) can be re-written as an additional source term to Eq. (25),

$$
\frac{dY_B - L}{dz} = \epsilon_T(z) = -\kappa \gamma_B \frac{H M_{Pl}}{2\pi^2} \epsilon^{vac}.
$$

Note that we neglected gradient corrections to the washout term here. This can be justified by the observation that the latter have the same qualitative structure than the zero-order washout terms, unlike the source terms.

Let us assume now, that $\kappa$ is very large and the system is very close to thermal equilibrium. In this limit $Y_{\psi} \approx Y^{vac}_{\psi}$ and the standard source term can be neglected. Thus, the rate equation for the baryon number density in the comoving volume takes the form:

$$
\frac{dY_B - L}{dz} \approx \epsilon_T(z) = -\frac{W_0(z)Y_B - L}{M_{Pl}}.
$$

Using the method of steepest descent we obtain an approximate solution for the asymptotic value of the generated asymmetry:

$$
\eta_{grad} \equiv Y_B - L(t \to \infty) \approx \frac{\epsilon^{vac} M_{Pl}}{z_f M_{Pl}},
$$

where $z_f$ denotes the inhomogeneous diffusion length, and $\epsilon^{vac}$ is the initial CP-violating parameter.
where the freeze-out temperature $z_f$ is determined by the same equation as in the zero-order calculation. Comparing this result with Eq. (14) we see that the asymmetry is not suppressed by the washout factor $\kappa$ but on the other hand it is strongly suppressed by the ratio of the right-handed toy-neutrino mass to the Planck scale. The numerical analysis confirms that for large values of the washout parameter Eq. (13) indeed well approximates the exact result.

In Fig. 3 we present the ratio of numerical solutions of Eq. (40) to those of Eq. (6). For a wide range of $\kappa$ and $M$ the gradient terms are subdominant and can safely be neglected. However, in the ultra-strong washout regime and for very heavy Majorana neutrinos they become the dominant source of the asymmetry.

Important is not only the relative size of the two contributions but also the absolute value of the generated asymmetry. As is evident from Fig. 4 although for large $\kappa$ the contribution of the gradient source term dominates, the efficiency of leptogenesis in the standard cosmological setting is too small to reproduce the observed value of the asymmetry. However, the relative importance of the gradient terms could be strongly enhanced if other non-equilibrium phenomena would occur simultaneously with leptogenesis. Essentially, the size of gradient terms is determined by the scale of temporal or spatial inhomogeneities. Since leptogenesis is often regarded to occur shortly after reheating, it is possible that non-equilibrium fluctuations occurring for example during (p)reheating are still present. Another possibility would be that a phase transition occurs at the temperature relevant for leptogenesis, possibly related to the breaking of $B-L$ symmetry. If the phase transition is of first order gradient terms could have a great impact, similarly as for electroweak baryogenesis. Finally, we note that the analysis presented here also applies to alternative mechanisms such as GUT-scale baryogenesis. Due to the much higher mass scale of $\sim 10^{16}$ GeV, the gradient corrections are also larger in this case.

IV. GRADIENT CORRECTION TO CP-VIOLATING PARAMETER

In this section we discuss the last term in the expansion (39) of the source term. As we will see, it can be interpreted as a correction to the CP-violating parameters $\epsilon_i$. As has been argued above, $\epsilon_i$ are generated by the off-diagonal components of the Wightman propagators of the toy-neutrinos provided that the latter are complex-valued. The first-order solution (34) does have an imaginary component. Thus, it also contributes to the CP-violation in the system. Particularly interesting are the following two terms of Eq. (34):

$$-i[\hat{G}_L\{\hat{H}_{A}\}^\dagger\{\hat{g}_A\} + \{\hat{g}_R\}^\dagger\{\hat{H}_{L}\}^\dagger\hat{G}_L],$$

(44)

because they satisfy the condition (28). To evaluate them we need explicit expressions for the self-energies and the propagators. In a state with zero (or small) asymmetry the self-energies corresponding to Fig. 2a read

$$\Pi^j_{R(A)} = \Pi^j_{R^j} = \frac{1}{2} \Pi^j_{R^j} = \frac{1}{1024} (g_i^j g_j^i + g_j^j g_i^i) [L_h + \frac{1}{2} L_p],$$

(45)
where the functions $L_h$ and $L_ρ$ are defined in Appendix B. It has also been shown in [12] that the diagonal retarded and advanced propagators may be split into two real-valued diagonal matrices:

$$\mathcal{G}_{R(A)}^{ij} = \frac{1}{2} \mathcal{G}_h^{ij} \pm i \mathcal{G}_ρ^{ij},$$

where the off-shell diagonal propagator and the spectral function are given respectively by [12, 25]

$$\begin{align*}
\mathcal{G}_h^{ij} &= -\frac{k^2 - M_i^2 - \Pi_h^{ij}}{(k^2 - M_j^2 - \Pi_h^{ij})^2 + \frac{1}{4}(\Pi_h^{ij})^2}, \\
\mathcal{G}_ρ^{ij} &= -\frac{\Pi_h^{ij}}{(k^2 - M_j^2 - \Pi_h^{ij})^2 + \frac{1}{4}(\Pi_h^{ij})^2}.
\end{align*}$$

Substituting Eqs. (45) and (46) into Eq. (44) we find

$$\text{Im} \, \epsilon_i = -\frac{1}{T} \{\Pi_h^{ij}\} \{\mathcal{G}_h^{ij}\} + \frac{1}{2} \{\Pi_h^{in}\} \{\mathcal{G}_ρ^{in}\}.$$  (48a)

In the hierarchical case to leading order:

$$\begin{align*}
\{\Pi_h^{ij}\} \{\mathcal{G}_h^{ij}\} &\approx \{k^{a}D_{a} \Pi_h^{ij}\} \mathcal{G}_h^{ij}, \\
\{\Pi_h^{in}\} \{\mathcal{G}_ρ^{in}\} &\approx 2 \{k^{a}D_{a} \Pi_h^{in}\} \mathcal{G}_ρ^{in}.
\end{align*}$$

The decomposition coefficient $\epsilon_i$ must be evaluated on the mass shell of the corresponding quasiparticle. Since we assume strong hierarchy of the masses here, $\mathcal{G}_ρ^{ij}$ evaluated on the mass shell of the $i$th quasiparticle is negligibly small for $i \neq j$. Thus, the contribution of Eq. (48a) can be neglected. Evaluating the contribution of Eq. (48b) we obtain for the gradient correction to the $CP$-violating parameter:

$$\epsilon_{g \times T} = -\epsilon^{vac}_{\alpha c} \cdot \frac{2 [k^{a}D_{a}L_h]}{M_j^2 - M_i^2}.$$  (50)

In the strong washout regime the toy-baryons are very close to equilibrium so that $L_h(X, p)$ depends on time only through the dependence of the temperature $T$ on time. Using once again the relation $T = -HT$ we find

$$k^{a}D_{a}L_h = \left( \frac{\partial}{\partial T} - HT \frac{\partial}{\partial E} \right) L_h = HT \left( \left( \frac{\partial}{\partial T} - \frac{\partial}{\partial E} \right) \frac{\partial}{\partial (T/E)} \right) L_h \equiv HT \cdot F,$$  (51)

where $E$ and $k$ are the energy and momentum of the decaying toy-neutrino and $F$ is a dimensionless function of $T$, $k$, and $M_i$. In agreement with our expectations the correction to the $CP$-violating parameter is proportional to the Hubble parameter $H$. Evaluating Eq. (50) in the radiation dominated universe we obtain in the hierarchical case:

$$\epsilon_{g \times T} = -\epsilon^{vac}_{\alpha c} \cdot \frac{M_1}{M_π} \cdot \frac{M_i^2}{M_π^2} \cdot \frac{F}{z^3},$$  (52)

where we have introduced $F \equiv 2 \cdot 1.66 \cdot F$ for convenience. Let us analyze Eq. (52) term by term. Since most of the asymmetry is generated at $T \sim M_1$, at the epoch of leptogenesis $z \sim 1$. The dependence of the function $F$ on the dimensionless inverse temperature for the momenta of the decaying particle in the range $0.25 \leq |k|/T \leq 4$.

\begin{figure}[ht]
\centering
\includegraphics[width=0.8\textwidth]{f.png}
\caption{ Dependence of the function $F$ on the dimensionless inverse temperature for the momenta of the decaying particle in the range $0.25 \leq |k|/T \leq 4$.}
\end{figure}

Roughly speaking $F \sim O(10)$, i.e. it is of the order of $2 \cdot 1.66 \sqrt{G}$. For hierarchical mass spectrum, $M_1^2/M_i^2 \lesssim 0.1$, the requirement of successful leptogenesis implies that the mass of the lightest right-handed neutrino should lie in the range $M_1 \approx 10^{9} - 10^{11}$ GeV. Even if we take the larger value, $M_1 \approx 10^{11}$ GeV, its relative size compared to the Planck mass, $M_π = 1.2 \cdot 10^{19}$ GeV, is still very small, of order of $10^{-8}$. Consequently, the new contribution to the $CP$-violating parameter is strongly suppressed, primarily due to the smallness of the mass of the lightest right-handed neutrino as compared to the Planck mass. Let us note again that in GUT baryogenesis scenarios where the generation of the asymmetry takes place at higher temperatures, $T \approx M_{GUT}$, the relative suppression of the new contributions would be less pronounced.

V. SUMMARY AND OUTLOOK

In this paper we have derived gradient corrections to the kinetic equations for leptogenesis, that are neglected in the standard Boltzmann approach.

We have found that there is an additional $CP$-violating source term $S_F$ with a qualitatively new structure, which arises due to the time-dependence of the effective temperature of the thermal bath. It does not vanish even if all particle species are in local thermal equilibrium.

\footnote{This effect exists also for the vertex contribution to the $CP$-violating parameter. However, there it is of a higher order in the coupling constants and for this reason we do not consider it here.}
For a standard cosmological background it is comparable to the conventional one only if the washout parameter $\kappa \sim M_{\text{Pl}}/M_1$ and can therefore be safely neglected in standard thermal leptogenesis. However, it becomes dominant in the limit of ultra-strong washout and for very heavy Majorana neutrinos, and can play an important role for alternative baryogenesis mechanisms operating at very high scales, e.g. at the GUT scale.

We have also analyzed a contribution to the effective in-medium CP-violating parameter which is induced by the gradient terms. Just like the $S_T$ term, the new contribution to the CP-violating parameter is suppressed by the small ratio of the heavy neutrino mass to the Planck scale.

The gradient terms could be greatly enhanced in a non-thermal environment, where large temporal or spatial gradients can occur. This may be relevant when the reheating temperature is very close to the right-handed neutrino mass, as is often required to avoid the overproduction of gravitinos in supersymmetric scenarios. Another conceivable situation is that the seesaw scale is associated to the breaking of a symmetry, possibly $\tilde{B} - L$, in which case a phase transition could occur at temperatures relevant for leptogenesis. If it is of first order, gradient terms can play a major role, similar to electroweak baryogenesis. The additional gradient source term could then even allow to lower the scale of leptogenesis without having to rely on resonance effects. This is left for future work.

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Appendix A: Gradient Expansion

In this appendix we derive the gradient expansion of the CTP evolution equation Eq. (15) for the $B - L$ current, following the lines of [22, 39, 41]. For that purpose, it is convenient to switch from the Wightman functions to an equivalent representation in terms of the so-called statistical propagator $D_F$ and the spectral function $D_\rho$,

\[ D_F(x,y) \equiv \frac{i}{2} [D_>(x,y) + D_<(x,y)], \quad (A1a) \]

\[ D_\rho(x,y) \equiv i [D_>(x,y) - D_<(x,y)]. \quad (A1b) \]

Using analogous definitions for $\bar{D}$ and the self-energies, Eq. (15) can equivalently be written as

\[ D^\mu j_\mu(x) = i \int d^4z \Theta(x^0 - z^0) \Theta(z^0 - t_{\text{init}}) (A2) \]

\[ \times \left[ \Sigma_F(x,z) D_\rho(z,x) - \Sigma_\rho(x,z) D_F(z,x) - \Sigma_F(z,x) D_\rho(z,x) + \Sigma_\rho(z,x) D_F(z,x) \right]. \]

Note that we have expressed the integration limits in terms of the usual $\Theta$-function. It is helpful to absorb $\Theta(x^0 - z^0)$ into retarded and advanced propagators,

\[ D_R(x,y) \equiv \Theta(x^0 - y^0) D_\rho(x,y), \quad (A3) \]

\[ D_A(x,y) \equiv -\Theta(y^0 - x^0) D_\rho(x,y), \quad (A4) \]

again with analogous definitions for $\bar{D}$ and the self-energies. The Wigner representation of the various two-point functions reads

\[ A(X,p) = (-i)^p \int d^4s e^{is_p} A(X + s/2, X - s/2), \quad (A5) \]

where $p = 1$ for $A = D_\rho, \bar{D}_\rho, \Sigma_\rho, \Sigma_F$, and zero otherwise. Using Eq. (13) and assuming $t_{\text{init}} \to -\infty$, the Wigner representation of Eq. (A2) is

\[ D^\mu j_\mu(x) = -i \int d\Pi_p \left[ \Sigma_\rho D_A + \Sigma_F D_\rho \right. \]

\[ - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right. \right. \]

\[ \left. \left. - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right\} \right\} \right\} \right]. \]

All two-point functions are evaluated at the point $(X,p)$ in phase-space. The gradient expansion formally follows by expanding the exponentials in powers of the derivative operator $\partial \propto \partial_X$ defined in Eq. (19).

1. Zeroth order

Evaluating Eq. (A5) at zeroth order in the gradient expansion yields

\[ D^\mu j_\mu(x)|_0 = -i \int d\Pi_p \left[ \Sigma_0^F D_A + \Sigma_R^0 D_F \right. \]

\[ - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right. \right. \]

\[ \left. \left. - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right\} \right\} \right\} \right]. \quad (A7) \]

By substituting $p \to -p$ in the second line and using

\[ D_{(F,A,R;\rho)}(X,p) = \bar{D}_{(F,A,R;\rho)}(X,-p), \quad (A8) \]

\[ D_\rho(X,p) = -D_\rho(X,-p) \]

\[ = -i[D_R(X,p) - D_A(X,p)], \]

this equation can be simplified to

\[ D^\mu j_\mu(x)|_0 = -\int d\Pi_p \left[ \Sigma_0^F D_A - \Sigma_0^R D_F \right. \]

\[ - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right. \right. \]

\[ \left. \left. - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right\} \right\} \right\} \right]. \quad (A9) \]

Using Eq. (A1a) to re-express this result in terms of Wightman functions (note factors $i$ in Eq. (A5) yields

\[ D^\mu j_\mu(x)|_0 = -\int d\Pi_p \left[ \Sigma_0^F D_A - \Sigma_0^R D_F \right. \]

\[ - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right. \right. \]

\[ \left. \left. - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right\} \right\} \right\} \right]. \quad (A10) \]

Finally, inserting the identity $1 = \Theta(p_0) + \Theta(-p_0)$ and using Eq. (A8) yields,

\[ D^\mu j_\mu(x)|_0 = -\int d\Pi_p \Theta(p_0) \left[ \Sigma_0^F D_A - \Sigma_0^R D_F \right. \]

\[ - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right. \right. \]

\[ \left. \left. - e^{-i\phi} \left\{ \Sigma_F D_A - e^{-i\phi} \left\{ \Sigma_R D_F \right\} \right\} \right\} \right]. \quad (A11) \]

which corresponds to Eq. (21).

---

4 For simplicity, we will work in flat space-time here. The relevant equations can be easily generalized to FRW space-time using the results of ref. [24].
2. First order

The first-order gradient contribution to Eq. (A6) consists of two parts. One obviously involves the linear term in the expansion of $e^{-iQ}$, in addition, it is import-tant to realize that the self-energies, see Eq. (13), contain an internal toy-neutrino line described by the out-of-equilibrium propagator $G^{ij}$. In order to obtain a consistent gradient expansion, it is important to perform also a gradient expansion of the equation of motion for $G^{ij}$, which is discussed in Appendix B. This means we also have to expand the self-energies,\[\Sigma = \Sigma^0 + \Sigma^1 + \ldots \quad \text{(A12)}\]
doing similar manipulations as for the zeroth-order equation yields\[D^\mu f_{\mu s}^{(1)} = \int d\Pi_p \left[ \Sigma^0_s D_s^\mu - \Sigma^1_s D_s \right] + 2\diamond \{ \Sigma^0_s \} \{ D_s \} + 2\diamond \{ \Sigma^0_s \} \{ D_s \},\quad \text{(A13)}\]
where $D_s(X, p) \equiv [D_R(X, p) + D_A(X, p)]/2$. Inserting again $1 = \Theta(p_0) + \Theta(-p_0)$ and using Eq. (A5) yields Eq. (33).

Appendix B: Gradient expansion for real scalar fields

The dynamics of the system of real scalar fields is described by the non-equilibrium generalization of the Schwinger–Dyson equation 12\[(G^{-1})^{ij}(x, y) = \langle \phi^{ij}(x, y) - \Pi^{ij}(x, y) \rangle, \quad \text{(B1)}\]
where $G^{ij}$ is the full dressed propagator of the “heavy neutrinos”, $\phi^{ij}$ is the diagonal propagator of the free fields and $\Pi^{ij}$ is the self-energy. Let us now split the self-energy matrix $\Pi$ into the diagonal, $\Pi^D$, and off-diagonal, $\Pi^O$, components and introduce a diagonal propagator $\hat{G}$ defined by the equation \[\hat{G}^{-1}(x, y) = \hat{G}^{-1}(x, y) - \hat{\Pi}(x, y). \quad \text{(B2)}\]
Subtracting Eq. (B2) from Eq. (B1) we find:
\[\hat{G}^{-1}(x, y) = \hat{G}^{-1}(x, y) - \hat{\Pi}^D(x, y). \quad \text{(B3)}\]
Multiplying Eq. (B3) by $\hat{G}$ from the left, by $\hat{G}$ from the right and integrating over the closed-time-path contour 42 43 we obtain a formal solution for the full non-equilibrium propagator 12:\[\hat{G}(x, y) = \hat{G}(x, y) - \int d^4u d^4v \theta(u^0) \theta(v^0) \times \left[ \hat{G}(x, u) \hat{\Pi}_D(u, v) \hat{\Pi}_A(v, y) + \hat{G}(x, u) \hat{\Pi}_A(u, v) \hat{\Pi}_D(v, y) + \hat{G}(x, u) \hat{\Pi}_D(u, v) \hat{\Pi}_A(v, y) \right]. \quad \text{(B4)}\]
If the mass spectrum of the heavy scalars is strongly hierarchical, i.e. $M_1^2 \ll M_2^2$, one can approximate the full propagators $\hat{G}$ on the right-hand side of Eq. (B4) by the corresponding diagonal propagators $\hat{G}^D$. That is, in this approximation the dynamics of the diagonal and off-diagonal components of $\hat{G}$ is completely determined by the dynamics of $\hat{G}^D$. Wigner-transforming the resulting expression we obtain to leading order in the gradients $\hat{G}^D = \hat{G}_0^D + \hat{G}_1^D$, where \[\hat{G}_0^D \approx \hat{G}_0 - \left[ \hat{G}_R \hat{\Pi}_D \hat{G}_A^* + \hat{G}_R \hat{\Pi}_D \hat{G}_A + \hat{G}_R \hat{\Pi}_D \hat{G}_A^* \right] \quad \text{(B5a)}\]
\[\hat{G}_1^D \approx -i \left[ \diamond \{ \hat{G}_R, \hat{\Pi}_D, \hat{G}_A \} + \diamond \{ \hat{G}_R, \hat{\Pi}_D, \hat{G}_A^* \} + \diamond \{ \hat{G}_R, \hat{\Pi}_D, \hat{G}_A \} + \diamond \{ \hat{G}_R, \hat{\Pi}_D, \hat{G}_A^* \} \right]. \quad \text{(B5b)}\]
The generalized derivative operators in Eq. (B5b) are defined by \[\diamond \{ A, B, C \} \equiv \{ A \} \{ BC \} + A \{ B \} \{ C \}. \quad \text{(B6)}\]

A system in thermal equilibrium is stationary. Therefore, in equilibrium the right-hand side of Eq. (B5b) vanishes. One can expect that in the early universe this contribution is proportional to the expansion rate $H$ of the universe. A substitution of Eq. (B5b) into the expressions for the Wigner-transforms of the self-energy, Eq. (24), gives us the first-order corrections $\Sigma^1$ to the self-energies.

Appendix C: Derivation of the additional source term

In this appendix we derive the additional CP-violating source term $S_F(z).$ Our starting point is the CPT evolution equation for the $B - L$ current 15. The additional source term arises at first order in the gradient expansion, see Eq. (A13). Inserting baryon-symmetric propagators, $\hat{D} \rightarrow \hat{D}_{\text{sym}}$, as well as the self-energy given in Eq. (24) into Eq. (A13) yields \[S(z)|_{\text{sym}} \equiv K \cdot \hat{D}^{\mu} f_{\mu}(x)|_{\text{sym}} \simeq 2K \int d\Pi_p \int d\Pi_q \int d\Pi \Theta(p_0)(2\pi)^4 \delta(k - p - q) \times \left[ \text{Im}(G_{\text{sym}}^{ij}) D_{\text{sym}}^{ij} - \text{Im}(G^{ij}_{\text{sym}}) \hat{D}_{\text{sym}}^{ij} \hat{D}_{\text{sym}}^{ij} \right]. \quad \text{(C1)}\]
Here we have neglected the contributions in the second line of Eq. (A13), similar as in 24, and the first-order toy-neutrino propagators $G^1$ are given by Eq. (B5b). For brevity, we suppress the superscript of $D^{sym}$ in the following.

Next, we set all particle species into local thermal equilibrium in order to obtain $S_T = S_{1\text{LTE}}^T$. This means that the propagators fulfill the Kubo-Martin-Schwinger (KMS) relation \[D_{\text{LTE}}^{ij}(X, k) = e^{\beta(ku)} D_{\text{LTE}}^{ij}(X, k). \quad \text{(C2)}\]
where \( u = (1, 0, 0, 0) \) in the co-moving frame of the thermal bath, and \( \beta = \beta(t) \equiv 1/T(t) \). Note that this condition implies that \( f_u = 1/[e^{\beta(k \cdot u) - 1}] \), see Eq. (22). For the toy-neutrino, the condition of local equilibrium implies that the zero-order propagators \( G^{(0)} \) fulfill KMS. In particular, this implies that

\[
G^{LTE}_>(X,k) = e^{\beta(k \cdot u)} G^{LTE}_<(X,k) .
\]  

(C3)

Again, in the following we suppress the superscript. These relations greatly simplify Eq. (C1) after inserting Eq. (B5). As an example, we consider the following contribution:

\[
\hat{\Delta} \{ \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \} D_\perp D_- = (\hat{\Delta}_\perp \hat{G}^i_{<\beta}) D_\perp D_- = (\hat{\Delta}_\perp \hat{G}^i_{<\beta}) D_\perp D_- = e^{\beta(k \cdot u)} \hat{G}^i_{<\beta} .
\]  

(C5)

Inserting these relations in the first and second line of the above equation, we find that there are again some cancellations,

\[
\hat{\Delta} \{ \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \} D_\perp D_- = 2 T \hat{G}^i_{<\beta} D_\perp D_- = 2 T \hat{G}^i_{<\beta} .
\]  

(C7)

Similar simplifications are obtained for the two first-order gradient terms of \( G^{(1)} \), see Eq. (B5). To shorten the notation we introduce the differential operator

\[
\hat{\Delta} \alpha \equiv \hat{\Delta} \alpha - \frac{1}{\beta} k^\alpha (\hat{\Delta} \alpha \beta) \partial \beta .
\]  

(C8)

Then, putting everything together, yields

\[
S(z) | T = S(z) | T^{LTE} = -K \text{Im}(g^*_i g_j) \int d\Pi d\Pi d\Pi d\Pi \int_0^{(2\pi)^4} \delta(k - p - q) \Theta(p_0) \times \beta u_a \left[ \text{Im}(\hat{\Delta} \alpha \{ \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \} D_\perp D_- \right] - \text{Im}(\hat{\Delta} \alpha \{ \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \} D_\perp D_-) = \hat{\Delta} \alpha \{ \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \} D_\perp D_- \right].
\]  

(C9)

The parts where the space-time derivative acts on a retarded or advanced propagator \( \hat{G}_{R(A)} \) are suppressed, since they only depend on temperature via the thermal mass, which is of higher order in the coupling within the toy-model. Therefore, we neglect the third line. In addition, using relations between retarded and advanced quantities and interchanging \( i \leftrightarrow j \) in the second line gives

\[
S(z) | T = 2K | g_i|^2 \int d\Pi d\Pi d\Pi d\Pi \int_0^{(2\pi)^4} \delta(k - p - q) \Theta(p_0) \times c^T \{ k, T \} \left[ G^i_{<\beta} D_\perp D_- + G^i_{<\beta} D_\perp D_- \right] ,
\]  

(C10)

where

\[
c^T \{ k, T \} = \frac{1}{2} \text{Im}(g_i/g_j) \beta u_a \hat{\Delta} \alpha \text{Im}[i \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \{ k, T \}].
\]  

(C11)

In the hierarchical limit, one obtains for the lightest toy-neutrino (\( i = 1 \)):

\[
e_{\tau}(k, T) \equiv e^\tau_{\text{vac}}(k, T) = -e^\text{vac} \frac{\beta}{2} u_a \hat{\Delta} \alpha \text{Im}[i \hat{G}^i_{<\beta} \Pi^{ij}_A \hat{G}^{ij}_A \{ k, T \}] .
\]  

(C12)

This result can easily be generalized to a frame that is boosted with respect to the co-moving frame:

\[
u_a \hat{\Delta} \alpha = \hat{\Delta} \alpha - \frac{\beta}{2} k^\alpha \partial \beta = T \partial T + \frac{T}{k} \partial \beta \partial \beta .
\]  

(C14)

Thus:

\[
e_{\tau}(k, T) = e^\text{vac} \times \frac{\beta H}{2} T \partial T + (u \cdot k)(u \cdot \partial k) \times L_n(k, T).
\]  

(C16)

In the quasi-particle limit, the loop integral is given by

\[
L_n(k, T) = 16\pi \int d\Pi d\Pi d\Pi d\Pi \int_0^{(2\pi)^4} \left[ \frac{1}{2} + f_b(q \cdot u; T) \right] \frac{-p}{(k - q)^2 - m^2} + \frac{1}{2} + f_b(q \cdot u; T) \left( \frac{-p}{(k + q)^2 - m^2} \right),
\]  

(C17)

where \( q^2 = m^2, q^0 = \sqrt{m^2 + q^2} \) and \( f_b \approx f_b = f^{BE} \) for an approximately symmetric medium. The temperature-derivative can be evaluated using the result in [12]:

\[
T \partial T L_n(k, T) = \frac{1}{\pi |k|} \int_0^{\infty} dE \left[ T \partial T f^{BE}(E; T) \right] \times \text{ln} \left[ \frac{(2E + |k|^2)^2 - k^2}{(2E - |k|^2)^2 - k^2} \right],
\]  

(C18)
where $k^2 = M^2$ and $m \approx 0$. The derivative with respect to the momentum is given by

$$\frac{\partial}{\partial k} L_h(k; T) = 16\pi \int d\Omega^4 \left[ \frac{1}{2} + f_b^{BE}(q \cdot u; T) \right] \times P\left[ \frac{2(k-q)}{[(k-q)^2 - m^2]^2} + \frac{2(k+q)}{[(k+q)^2 - m^2]^2} \right]. \quad (C18)$$

This expression is covariant, i.e. valid in any frame. In the comoving frame, where $u = (1, 0, 0, 0)$, we need to compute only the derivative with respect to $k^0$. In a general frame, this corresponds to $u \cdot \partial_k$. Thus,

$$k^0 \partial_k L_h(k; T) \bigg|_{\text{comoving}} = (u \cdot k) \times u \cdot \frac{\partial}{\partial k} L_h(k; T) = 16\pi \int d\Omega^4 \left[ \frac{1}{2} + f_b^{BE}(q \cdot u; T) \right] \times P\left[ \frac{2u \cdot (k-q)}{[(k-q)^2 - m^2]^2} + \frac{2u \cdot (k+q)}{[(k+q)^2 - m^2]^2} \right]. \quad (C19)$$

The above integral contains a “vacuum” and a “medium” part, where the latter is proportional to $f_b^{BE}(q \cdot u; T)$. An explicit calculation shows that both integrals are well-defined (no UV, IR or on-shell-pole divergences), and yields (for $m = 0$)

$$(u \cdot k)(u \cdot \partial_k) L_h^{vac}(k; T) = \frac{E_k}{2\pi|k|} \ln \frac{|E_k - |k||}{|E_k + |k||}, \quad (C20)$$

$$(u \cdot k)(u \cdot \partial_k) L_h^{med}(k; T) = -\frac{E_k}{2\pi|k|} \times \int_0^\infty \frac{dE f_b^{BE}}{E} \ln \left| \frac{M^4 - 4E^2(E_k - |k|)^2}{M^4 - 4E^2(E_k + |k|)^2} \right| \quad (C21)$$

Here $|k|$ is the momentum of the decaying particle in the comoving frame (rest-frame of the medium), and $E_k = \sqrt{M^2 + k^2}$ is its energy. For the part containing the temperature-derivative of $L_h$, we use

$$T \frac{\partial}{\partial T} f_b^{BE}(E; T) = -E \frac{\partial}{\partial E} f_b^{BE}(E; T). \quad (C22)$$

Then one obtains

$$\left[ T \frac{\partial}{\partial T} + (u \cdot k)(u \cdot \partial_k) \right] L_h(k; T) = \frac{E_k}{2\pi|k|} \ln \frac{|E_k - |k||}{|E_k + |k||} - \int_0^\infty \frac{dE f_b^{BE}}{E} \ln \left| \frac{M^4 - 4E^2(E_k - |k|)^2}{M^4 - 4E^2(E_k + |k|)^2} \right| + \frac{2E}{E_k} \ln \left| \frac{(2E + |k|)^2 - M^2}{(2E - |k|)^2 - M^2} \right| \right]. \quad (C23)$$

It is easy to convince oneself that the remaining integral over the energy $E$ is free of UV or IR divergences, and that the integral exists in the vicinity of the zeros of the arguments of the logarithm occurring inside the integration region. Thus, we finally arrive at the result

$$\epsilon_T(k, T) = \epsilon^{vac} \times \frac{\beta H}{2} \times \frac{E_k}{2\pi|k|} \ln \frac{|E_k - |k||}{|E_k + |k||} - \int_0^\infty dE \frac{\partial f_b^{BE}}{\partial E} \ln \left| \frac{M^4 - 4E^2(E_k - |k|)^2}{M^4 - 4E^2(E_k + |k|)^2} \right| + \frac{2E}{E_k} \ln \left| \frac{(2E + |k|)^2 - M^2}{(2E - |k|)^2 - M^2} \right| \right). \quad (C24)$$

In the non-relativistic limit $|k| \sim T \ll M$, the main contribution to the integration over $E$ comes from the region $E \ll M$, due to the exponential suppression in the Bose-Einstein function. Therefore, we may expand the logarithms in the above expression for $|k|, E \ll M$:

$$\ln \frac{|E_k - |k||}{|E_k + |k||} \rightarrow -2|k|/E_k, \quad (C25)$$

$$\{ \ldots \} \rightarrow 16E^2|k|^3/(M^4E_k). \quad (C26)$$

Then it is easy to perform the energy integral:

$$\int_0^\infty dE \frac{\partial f_b^{BE}}{\partial E} E^2 = -\int_0^\infty dE f_b^{BE} 2E = -\frac{T^2k^2}{3}. \quad (C27)$$

Using this, we obtain the leading contributions in the non-relativistic limit,

$$\epsilon_T(k, T) \rightarrow \epsilon^{vac} \times \frac{\beta H}{2} \left( -\frac{1}{\pi} + \frac{8\pi T^2k^2}{3M^4} \right). \quad (C28)$$

The second contribution in the brackets is suppressed for $T \ll M$, so that we finally obtain

$$\epsilon_T(k, T) \approx -\epsilon^{vac} \times \frac{H}{2\pi T}. \quad (C29)$$

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