Bouncing universe dual to the concordance model: Scalar-tensor theories in the Jordan frame

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Abstract. Scalar-tensor theories of gravity provide mathematically equivalent descriptions of Einstein’s gravity with a scalar field, in a conformally connected spacetime, described in terms of the Jordan frame and the Einstein frame. In this paper, we use the Jordan frame-Einstein frame correspondence to explore dual universes with contrasting cosmological evolutions. We study the mapping between Einstein and Jordan frames where the Einstein frame universe effectively describes the late-time evolution of the physical universe, driven by dark energy and non-relativistic matter. The Brans-Dicke theory of gravity is taken to be the dual scalar-tensor theory in the Jordan frame. We show that the standard Einstein frame universe, with dark energy and non-relativistic matter, always corresponds to a bouncing Jordan frame universe, if it is governed by a Brans-Dicke theory. This essentially leads to an alternative description of the late-time evolution of the physical universe, in terms of a bouncing Brans-Dicke universe in the Jordan frame. Previous studies have shown that for a bouncing Jordan frame, particularly for an early-time accelerating phase of the universe, the map between the Einstein and Jordan frames may become singular in the perturbative regime, causing the conformal correspondence to break down. In order to check whether the present bouncing model for late-time acceleration is free of such instabilities, we study the evolution of scalar metric perturbations. The Jordan frame metric perturbations are numerically solved, first via pulling the perturbations in the Einstein frame using the conformal correspondence, and then directly in the Jordan frame. The evolutions of perturbations obtained in these two cases are in a good agreement. Thus, the duality between the Einstein frame, mimicking the physical universe, and the bouncing Brans-Dicke Jordan frame, is shown to be stable against linear perturbations. An effective bouncing description of the current accelerating universe has interesting implications, for example, one may study the late-time cosmological perturbations as fluctuations in a bouncing scenario.
1 Introduction

It is well-known that some classes of modified theories of gravity, such as scalar-tensor theories \[1–3\] and \(f(R)\) theories \[4–6\], can be recast as Einstein gravity with a minimally coupled scalar-field, in a conformally connected frame. The universe described by the modified gravity action is referred to as the Jordan frame, whereas, the universe described by the Einstein-Hilbert action in the conformally connected frame is called the Einstein frame.

Einstein and Jordan frames are mathematically equivalent, they essentially describe the same theory in terms of different dynamical variables \[7, 8\]. However, the equations of motion in these frames may lead to drastically different evolutions of the corresponding universes. Recent studies exploring conformally connected universes with contrasting cosmological evolutions can be found in \[9–16\]. In \[13\] it is demonstrated that a decelerating Einstein frame universe can be conformally equivalent to accelerating Jordan frame universes, governed by \(f(R)\) and scalar-tensor theories. In \[17\], finite time cosmological singularities are classified into four types, depending on the diverging nature of the scale factor and its derivatives, the effective energy density and the effective pressure of the universe. The conformal correspondence between different types cosmological singularities is explored in \[12\], there it is shown that the type of cosmological singularity may vary depending on the choice of the conformal frame. The duality between an expanding Einstein frame and a collapsing Jordan frame is studied in \[16\]. It is shown that for some viable quintessence models in the Einstein frame, the corresponding Jordan frame, governed by an \(f(R)\) gravity theory, may have collapsing description. A general condition is derived to predict whether a quintessence model, with a given time-dependent equation of state parameter, leads to such an expansion-collapse duality between the conformally connected frames.
In this paper we explore the Einstein frame-Jordan frame correspondence in the cosmological background, where the Jordan frame is governed by a scalar-tensor theory action. We classify scalar-tensor theories based on whether they lead a collapsing Jordan frame, corresponding to an expanding Einstein frame. The general condition for such expansion-collapse duality is then applied to a special case, where the Jordan frame action is taken to be the Brans-Dicke action, and the Einstein frame is modeled to describe the physical universe in the current era.

The late-time acceleration of the physical universe is realized in general relativity by introducing dark energy, an exotic fluid that violates the strong energy condition [18–20]. The $\Lambda CDM$ ($\Lambda$ + Cold Dark Matter) model of the current universe, often referred to as the concordance model of cosmology, interprets dark energy as the cosmological constant ($\Lambda$) in the Einstein field equation [21–23]. We show that an Einstein frame which effectively describes the evolution of the concordance model of cosmology, always corresponds to a bouncing Jordan frame, governed by a Brans-Dicke theory. The transition of the Einstein frame from a dust-dominated phase to an accelerating phase can be associated with the bounce in the Jordan frame. This result is shown to be valid even if we consider dynamical models of dark energy, such as the quintessence model, in the Einstein frame. A conformal map of this kind may lead to an alternative description of the standard late-time cosmological evolution, in terms of the evolution of a bouncing universe.

In general, bouncing models of cosmologies have been explored as a theory of the early universe, alternative to the inflationary scenario [24–27]. For example, bouncing scenarios are realized using scalar-tensor theories in [6, 28, 29]. As for any theory of the early universe, cosmological perturbations play an important role in the bouncing models. The statistical properties of the large-scale structure and CMBR anisotropies as observed today, must be explained from the primordial fluctuation near the bouncing epoch (see, for example, [30–32] and references therein). A viable bouncing scenario in the early universe hence needs to be checked for stability under perturbations.

In order to accommodate for cosmological perturbations in the present bouncing model of the late-time universe, the conformal correspondence between the physical Einstein frame and the bouncing Jordan frame should be stable in the perturbative regime. Previous studies have pointed out that for early universe bouncing scenarios, the linear order conformal correspondence may become singular under certain conditions, breaking the Einstein frame-Jordan frame duality (see, for example, [10, 15]). To check whether such divergences appear in the present case, where the bouncing universe is dual to the standard late-time universe, we study the evolution of scalar perturbations in both Einstein and Jordan frames. The Jordan frame perturbations are numerically solved, first via Einstein frame using the conformal map, then directly in the Jordan frame. The evolution of Jordan frame scalar perturbations obtained in these two ways are in good agreement. This explicitly shows that the map between the linear order perturbations in the conformally connected frames remains valid, even through the bounce in the Jordan frame. The duality between the bouncing universe and the late-time physical universe is hence found to be stable under linear perturbations.

This article is organized in the following way. In section 2 we briefly introduce the Einstein frame-Jordan frame correspondence, where the Jordan frame is governed by a scalar-tensor theory action. The general condition for expansion-collapse duality for scalar-tensor theories is obtained in section 3. In section 4 we show that an Einstein frame, effectively describing the evolution of the concordance model of cosmology, always corresponds to a bouncing Jordan frame governed by a Brans-Dicke theory. We show in section 5 that such an
Einstein frame-Jordan frame duality is stable in the linear perturbation regime. We conclude with summary and discussion in section 6.

Throughout this article Latin indices represent spacetime components, Greek indices represent spatial components. Metric signature is taken to be $(-,+,+,+)$. 

2 Jordan and Einstein frames

We begin with a brief review of Jordan and Einstein frames in the context of scalar-tensor theories, for detailed discussion, see [1–3, 6].

In scalar-tensor theories, the gravity sector is governed by a scalar field ($\lambda$) along with the metric tensor field ($g_{ab}$). The action of a general scalar-tensor theory can be written as [1, 3]

$$S_J = \int d^4x \sqrt{-g} \left( f(\lambda) R - \frac{1}{2} h(\lambda) g^{ab} \partial_a \lambda \partial_b \lambda - U(\lambda) \right), \quad (2.1)$$

where $f(\lambda), h(\lambda), U(\lambda)$ are arbitrary functions of the scalar field $\lambda$. The universe described by this action is referred to as the Jordan frame universe. Scalar-tensor theories belong to a class of extended theories of gravity which can be recast as general relativity, with a canonical scalar field, in a conformally connected frame. With the following conformal transformation [1, 3]

$$\tilde{g}_{ab} = \Omega^2(x) g_{ab}, \quad (2.2a)$$
$$\Omega^2 = \frac{16}{\pi G f(\lambda)}, \quad (2.2b)$$

the action (2.1) can be written as

$$S_J = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{16\pi G} \tilde{R} - \frac{1}{2} K[\lambda] \tilde{g}^{ab} \partial_a \lambda \partial_b \lambda - \frac{U(\lambda)}{(16\pi G f(\lambda))^2} \right], \quad (2.3)$$

where,

$$K[\lambda] = \frac{1}{16\pi G f^2(\lambda)} \left( h(\lambda) f(\lambda) + 3f^2_{\lambda} \right). \quad (2.4)$$

The first term in eq. (2.3) is the Einstein-Hilbert action with respect to the metric $\tilde{g}_{ab}$. The remaining terms describe a minimally coupled scalar field, with non-canonical kinetic term. One may define a new scalar field $\varphi$ by

$$\frac{d\varphi}{d\lambda} = \sqrt{K[\lambda]}, \quad K[\lambda] > 0, \quad (2.5)$$

such that in terms of $\varphi$, the action (2.3) becomes [1, 3]

$$S_E = \int d^4x \sqrt{-\bar{g}} \left[ \frac{1}{16\pi G} \bar{R} - \frac{1}{2} \bar{g}^{ab} \partial_a \varphi \partial_b \varphi - V(\varphi) \right], \quad (2.6)$$

where,

$$V(\varphi) = \frac{U(\lambda(\varphi))}{(16\pi G f(\lambda(\varphi)))^2}. \quad (2.7)$$

Note that $K[\lambda] > 0$ is necessary for the field $\varphi$ to be real. The action (2.6) describes a minimally coupled canonical scalar field $\varphi$, with potential $V(\varphi)$, in Einstein’s gravity. The universe governed by this action is referred to as the Einstein frame universe.

We now move on to the expansion-collapse duality between the conformally connected frames.
3 Expansion-collapse duality between Einstein and Jordan frames

As discussed above, a minimally-coupled scalar field in general relativity can have an alternative description given by a scalar-tensor theory in the Jordan frame. In this paper we are interested in a scenario where the Einstein frame is the physical universe, undergoing the dark energy-dominated late time accelerating phase, with non-negligible subdominant presence of non-relativistic matter. Then corresponding to this Einstein frame, we seek a class of scalar-tensor theories leading to a collapsing Jordan frame universe. Such a class of scalar-tensor theories, if exists, can provide an effective description of the expanding physical universe in the Einstein frame, in terms of a collapsing one, in the Jordan frame. In this section we find a general condition for such an expansion-collapse duality between the conformal frames.

Let us consider that both Jordan and Einstein frame spacetimes are described by spatially flat FRW metrics, i.e.,

\[ g_{ab} = \text{diag} \left[ -1, a^2(t), a^2(t), a^2(t) \right] \quad \text{and} \quad g_{ab} = \text{diag} \left[ -1, \tilde{a}^2(\tilde{t}), \tilde{a}^2(\tilde{t}), \tilde{a}^2(\tilde{t}) \right], \]

respectively. Then, according to the conformal transformation (2.2), Einstein and Jordan frame scale factors and coordinate times are related by [1]

\[ \tilde{a} = \Omega a = \sqrt{16\pi G f(\lambda)} a \] (3.2a)
\[ d\tilde{t} = \Omega dt = \sqrt{16\pi G f(\lambda)} dt. \] (3.2b)

The Einstein frame action (2.6) leads to the usual Friedmann equations

\[ \tilde{H}^2 = \left( \frac{1}{a} \frac{da}{dt} \right)^2 = \frac{\kappa^2}{3} \rho_\varphi(\tilde{a}) \] (3.3a)
\[ \frac{d\tilde{H}}{d\tilde{t}} = -\frac{\kappa^2}{2} (\rho_\varphi + P_\varphi) = -\frac{\kappa^2}{2} \rho_\varphi(1 + w_\varphi), \] (3.3b)

where \( \kappa^2 = 8\pi G \), \( \rho_\varphi \) and \( P_\varphi \) are the energy density and pressure associated with the scalar field \( \varphi \),

\[ \rho_\varphi = \frac{1}{2} \left( \frac{d\varphi}{d\tilde{t}} \right)^2 + V(\varphi), \] (3.4a)
\[ P_\varphi = \frac{1}{2} \left( \frac{d\varphi}{d\tilde{t}} \right)^2 - V(\varphi), \] (3.4b)

\( \tilde{H} \) is the Einstein frame Hubble parameter, and \( w_\varphi = P_\varphi/\rho_\varphi \) is the equation of state parameter corresponding to the scalar field \( \varphi \). From the above relations, the time derivative of the field can be written as

\[ \frac{d\varphi}{d\tilde{t}} = \sqrt{\rho_\varphi(1 + w_\varphi)}. \] (3.5)

Starting with the relation between the scale factors in the two conformal frames (3.2a) and using eqs. (2.5) and (3.5), one can find

\[ \frac{da}{d\tilde{a}} = \frac{1}{\sqrt{16\pi G}} f^{-\frac{1}{2}} \left( 1 - \frac{1}{2} \frac{f}{\tilde{f}} K^{-\frac{1}{2}} |\lambda| \sqrt{\frac{(1 + w_\varphi)\rho_\varphi}{H^2}} \right), \] (3.6)
where the subscript \((\cdot, \lambda)\) represents derivative with respect to the Jordan frame scalar field \(\lambda\). The condition for the expansion-collapse duality between Jordan and Einstein frames is then obtained by setting

\[
\frac{da}{d\tilde{a}} < 0 \tag{3.7}
\]

leading to

\[
\frac{1}{\sqrt{16\pi G}} f^{-\frac{1}{2}} \left( 1 - \frac{1}{2} \frac{f_{\lambda}}{f} K^{-\frac{1}{2}}[\lambda] \sqrt{\frac{(1 + w_\varphi)\rho_\varphi}{H^2}} \right) < 0, \tag{3.8}
\]

Note that \(f(\lambda) > 0\) is required to ensure that the conformal factor is real, i.e. \(\Omega^2 > 0\). Let us further consider \(f_{\lambda} > 0\), then using the Friedmann equation (3.3a) the expansion-collapse condition can be put in the form

\[
1 + w_\varphi > \frac{2}{3} \left( \frac{fh}{f_{\lambda}^2} + 3 \right). \tag{3.9}
\]

In general, both sides of the above inequality may evolve in time. For a scalar field in the Einstein frame, with arbitrary time-dependent \(w_\varphi(\tilde{a})\), and a scalar-tensor theory in the Jordan frame, specified by the functions \(f(\lambda), h(\lambda) (f_{\lambda} > 0)\), there may exist periods of evolution when \(w_\varphi(\tilde{a})\) satisfies the above inequality. During such a period, for an expanding (collapsing) Einstein frame universe, the corresponding Jordan frame collapses (expands).

Here we would like to mention that in a previous study [16], we obtained a similar expansion-collapse duality condition between Einstein and Jordan frames, where the Jordan frame was governed by an \(f(R)\) theory. The expansion-collapse condition in the case of \(f(R)\) theories is solely determined by the Einstein frame quantities \((w_\varphi, \rho_\varphi, \ddot{H})\), the Jordan frame function \(f(R)\) does not appear explicitly in the condition. However, in the present case, one requires the expressions for the Jordan frame functions \(f(\lambda)\) and \(h(\lambda)\) in order to check the validity of the condition (3.9). This additional requirement in the case of scalar-tensor theories is expected, simply because the Jordan frame action has three unspecified functions \((f, h, U)\), whereas, the Jordan frame action for \(f(R)\) theories has a single unspecified function, \(f(R)\).

We now explore the expansion-collapse duality between an Einstein frame which describes the late-time evolution of the physical universe, and a suitable scalar-tensor theory in the Jordan frame.

### 4 Concordance model: Bounce in Jordan frame

The current accelerating expansion of the physical universe is considered to be driven by a strong energy condition-violating exotic dark energy [18–20]. The equation of state parameter of dark energy \(w_{\text{de}}\) must be smaller than \(-1/3\) and the value \(w_{\text{de}} \approx -1\) is favoured by observations [33]. The cosmological constant \(\Lambda\) is the simplest implementation of dark energy within the framework of general relativity. Although being consistent with observations, the cosmological constant model suffers from several fine-tuning problems [21, 22]. A wide class of dynamical models of dark energy has been explored as an alternative for the \(\Lambda\) model (see, for example, [18–20] and references therein). Quintessence, or canonical scalar field
minimally coupled to gravity, with suitable equation of state parameter, is the simplest of such dynamical models of dark energy [20, 34, 35]. In the following discussion, we will mainly consider the cosmological constant model of dark energy as an example. However, we will see that the results obtained for the $\Lambda CDM$ model may be generalized for dynamical models of dark energy as well.

4.1 Einstein frame: Concordance model

In this section we set up the Einstein frame for it to describe the current accelerating epoch of the universe, consisting of dark energy and non-relativistic matter or dust. Since the Einstein frame already comes with a canonical scalar field, it would therefore be an obvious choice to identify the Einstein frame scalar field as the quintessence field. Although dark energy is the dominant component in the current epoch, the subdominant presence of non-relativistic matter can not be ignored. In order to implement a realistic physical scenario in the Einstein frame, one must take into account the presence of dust in the Einstein frame as well. However, due to the nature of the conformal transformation, the minimally coupled matter component in the Einstein frame gets non-minimally coupled to gravity in the Jordan frame, leading to non-trivial equations of motion in the Jordan frame.

Instead of identifying the Einstein frame scalar field ($\varphi$) as a quintessence model, we consider that the scalar field effectively describes both dark energy and non-relativistic matter in the field equation level. The energy density of the scalar field, $\rho_\varphi$, is taken to be

$$\rho_\varphi(\tilde{a}) = \rho_{de}(\tilde{a}) + \rho_m(\tilde{a}) = \rho_{de}(\tilde{a}) + \rho_{m0}\tilde{a}^{-3},$$

where $\rho_{m0}$ is the energy density of dust at current epoch ($\tilde{a} = 1$). $\rho_{de}$ is the energy density of the dark energy component, which is a constant for the cosmological constant model, or can be time-dependent for a dynamical model for dark energy. With this role of the Einstein frame scalar field, there is no need to add extra matter component in the Einstein frame, as the single scalar field takes into account both dark energy and matter. Since the scalar field is the sole component in the Einstein frame, the Jordan frame action remains a pure gravity action, governed by only the metric $g_{ab}$ and the Jordan frame scalar field $\lambda$.

For simplicity, let us consider a constant $\rho_{de}$, i.e., the Einstein frame effectively describes the $\Lambda CDM$ or the concordance model (see, for example, [23]). The energy density of $\varphi$ becomes,

$$\rho_\varphi(\tilde{a}) = \rho_\Lambda + \rho_m = \rho_\Lambda + \rho_{m0}\tilde{a}^{-3}$$

$$= \rho_c \left( \Omega_\Lambda + \Omega_{m0}\tilde{a}^{-3} \right),$$

where $\rho_c = 3\bar{H}_0^2/\kappa^2$ is the critical density of the universe, $\bar{H}_0 = \bar{H}(\tilde{a} = 1)$, $\Omega_\Lambda = \rho_\Lambda/\rho_c$ and $\Omega_{m0} = \rho_{m0}/\rho_c$. The Einstein frame scalar field $\varphi$ is hereafter referred to as the concordance field, in order to distinguish it from a quintessence model.

Starting with the above $\rho_\varphi(\tilde{a})$, one can reconstruct the action of the concordance field as follows. Using the Friedmann equations in the Einstein frame (3.3), the equation of state $w_\varphi$, which is the effective equation of state of the $\Lambda CDM$ model, can be written as a function of the scale factor as (see fig. 1),

$$w_\varphi = -1 - 2 \frac{1}{3} \frac{d\bar{H}}{d\tilde{t}} = -\frac{\Omega_\Lambda \tilde{a}^3}{1 - \Omega_\Lambda + \Omega_\Lambda \tilde{a}^3}. $$


Figure 1: Equation of state parameter of the concordance field is plotted with respect to the Einstein frame scale factor. At the early times $\tilde{a} \to 0$, $w_\phi \to 0$, corresponds to the dust-dominated phase of the universe. In the late times $w_\phi$ approaches $-1$, describing a $\Lambda$-dominated universe. The vertical line represents the period of dust-dark energy equivalence.

This, along with eq. (3.5), leads to the concordance field as a function of the scale factor,

$$\kappa \varphi(\tilde{a}) = \frac{1}{\sqrt{3}} \left[ \ln \left( \sqrt{\frac{\Omega_\Lambda}{1 - \Omega_\Lambda}} \tilde{a}^3 + 1 - 1 \right) - \ln \left( \sqrt{\frac{\Omega_\Lambda}{1 - \Omega_\Lambda}} \tilde{a}^3 + 1 + 1 \right) \right]. \quad (4.4)$$

Finally, from eq. (3.4), the concordance potential

$$V = \frac{1}{2} (1 - w_\phi(\tilde{a})) \rho_\phi(\tilde{a}) \quad (4.5)$$

can be written as a function $\varphi$ as

$$V(\varphi) = \frac{3 H_0^2}{8 \kappa^2 \Omega_\Lambda} \left[ 6 + \exp \left( -\sqrt{3} \kappa \varphi \right) + \exp \left( \sqrt{3} \kappa \varphi \right) \right]. \quad (4.6)$$

where $\tilde{a}$ is replaced with $\varphi$ according to eq. (4.4). In other words, a canonical scalar field with the above potential in the Einstein frame, leads to equations of motion which are exactly the same as those of the $\Lambda CDM$ or the concordance cosmology. One can solve the Friedmann equation for the concordance model to find the Einstein frame scale factor as

$$\tilde{a}(\tilde{t}) = \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \right)^{\frac{1}{3}} \sinh^2 \left( \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 (\tilde{t} - \tilde{t}_i) \right), \quad (4.7)$$

where,

$$\tilde{t}_i = \frac{2}{3H_0 \sqrt{\Omega_\Lambda}} \arcsinh \left( -\sqrt{\frac{\Omega_\Lambda}{1 - \Omega_\Lambda}} \right), \quad (4.8)$$

which is, as expected, the scale factor of the $\Lambda CDM$ model. Note that the origin of the Einstein frame coordinate time is chosen to be the current epoch $\tilde{t}_0$, i.e. $\tilde{t}_0 = 0$ and $\tilde{a}(\tilde{t} = \tilde{t}_0 = 0) = 1$, whereas, at $\tilde{t} = \tilde{t}_i$ the scale factor $\tilde{a}(\tilde{t} = \tilde{t}_i) = 0$.

Having set up the Einstein frame, we will now move to a suitable Jordan frame description of the universe.
4.2 Jordan frame: Brans-Dicke theory

The concordance scalar field in the Einstein frame, with potential (4.6), can be mapped to a wide class of scalar-tensor theories, depending on choices of the functions \( f(\lambda), h(\lambda) \) in the Jordan frame action (2.1). This correspondence is given by the relations (2.5) and (2.7), as discussed before. We are interested in an example of the scalar-tensor theories, dual to the concordance model, such that the conformally connected frames possess the expansion-collapse duality feature as discussed in section 3.

The widely-studied Brans-Dicke theory of gravity is the prototype of scalar-tensor theories [1, 3, 36–40]. A Brans-Dicke theory dual to the concordance model may lead to the expansion-collapse duality under certain conditions, as we will see. Apart from this, the Brans-Dicke action is simple enough for the equations of motion in the Jordan frame to be solved analytically. For this reason, we will consider the Brans-Dicke theory as an example of scalar-tensor theories in the Jordan frame for the rest of the discussion.

With the following choice of the functions [1, 3],

\[
\begin{align*}
 f(\lambda) &= \frac{\lambda}{16\pi}, \\
 h(\lambda) &= \frac{w_{BD}}{8\pi\lambda},
\end{align*}
\]

(4.9a)

(4.9b)

the Jordan frame action (2.1) becomes the Brans-Dicke action

\[
S_{BD}^J = \int d^4x \sqrt{-g} \left( \frac{\lambda}{16\pi} R - \frac{w_{BD}}{16\pi\lambda} g^{ab} \partial_b \lambda \partial_a \lambda - U(\lambda) \right),
\]

(4.10)

where \( w_{BD} \), the constant Brans-Dicke parameter, must satisfy \( w_{BD} > -3/2 \) in order for the concordance field \( \varphi \) to be real (see eq. (2.5)). Note that \( f_\lambda = \frac{1}{16\pi}, f_\lambda > 0 \) is ensured for all \( \lambda \). One can then apply eq. (3.9) to determine the condition for a collapsing Jordan frame,

\[
w_\varphi(\bar{a}) > \frac{4}{3} w_{BD} + 1.
\]

(4.11)

For the Brans-Dicke model, the expansion-collapse duality condition becomes significantly simple, as it only requires the knowledge of the equation of state \( w_\varphi(\bar{a}) \). Given the evolution of \( w_\varphi(\bar{a}) \), the expanding and collapsing phases of the Jordan frame, determined by the condition (4.11), can be visualized in fig. 2. The shaded region in the figure depicts the domain in the \((w_\varphi, w_{BD})\) space, where the condition (4.11) is satisfied. That is, for a Brans-Dicke Jordan frame specified by \( w_{BD} \), at a given value of the Einstein frame scale factor \( \bar{a} \), if the pair \((w_\varphi(\bar{a}), w_{BD})\) lies within the shaded region, then we can conclude that the Jordan frame is collapsing at \( \bar{a} = \bar{a}_* \).

The Einstein frame is matter dominated at early-times, i.e., for \( \bar{a} \to 0 \), the equation of state parameter of the concordance field \( w_\varphi \approx 0 \) (see fig. 1). Eventually, as the scale factor increases, \( w_\varphi \) decreases. In the late-time of the Einstein frame, i.e., as \( \bar{a} \to \infty, w_\varphi \to -1 \), depicting the dark-energy dominated era. We see from fig. 2 that for \( w_{BD} > -3/4 \), the Jordan frame is never collapsing. For \(-3/2 < w_{BD} < -3/4\), the points with \( w_\varphi = 0 \) always lie in the shaded region. This implies that at the beginning of the matter-dominated era of the Einstein frame \((w_\varphi \approx 0)\), the Jordan frame is collapsing. For the same value of \( w_{BD} \), \( w_\varphi \) will then monotonically decrease towards \(-1\) with increasing \( \bar{a} \). Thus, at some time, the trajectory of the Einstein frame universe in the \((w_\varphi, w_{BD})\) space will inevitably come out of the shaded region and enter the white region, where the Jordan frame is expanding. The
Figure 2: Condition for a collapsing Brans-Dicke Jordan frame, dual to the concordance model, determined by \( w_{BD} \) and \( w_{\varphi} \); where \( w_{BD} \) is the constant Brans-Dicke parameter specifying the Jordan frame, \( w_{\varphi} \) is the dynamic equation of state of the concordance field. At a given instant, if a pair \((w_{\varphi}, w_{BD})\) lies in the shaded region, then the corresponding Brans-Dicke universe is collapsing at that instant.

transition of the trajectory from the shaded region (collapsing Jordan frame) to the white region (expanding Jordan frame), represents a bounce in the Jordan frame. The Einstein frame scale factor at the point of the bounce, \( \tilde{a}_b \), satisfies

\[
 w_{\varphi}(\tilde{a}_b) = \frac{4}{3} w_{BD} + 1. \tag{4.12}
\]

Note that, the argument made for the existence of bouncing Brans-Dicke models dual to the physical universe, is not sensitive to the exact nature of \( w_{\varphi}(\tilde{a}) \). We will show that this is a generic feature of any model depicting the late-time standard cosmological evolution.

For the concordance scalar field in the Einstein frame, the equations of motion of the corresponding Jordan frame can be solved analytically. Starting with eq. (2.5) and using eq. (4.9), one can write the Brans-Dicke field \( \lambda \) as a function of the concordance field as,

\[
 \lambda(\varphi) = \lambda_0 \exp \left( \sqrt{\frac{2}{\varpi}} \kappa \varphi \right), \tag{4.13}
\]

where we define

\[
 \varpi = 2w_{BD} + 3, \tag{4.14}
\]

and \( \lambda_0 \) is an integration constant such that \( \lambda(\varphi = 0) = \lambda_0 \). Given the concordance potential (4.6), one can obtain the corresponding Brans-Dicke potential using eqs. (2.7) and (4.13) as

\[
 U(\lambda) = \frac{3G}{64\pi} H_0^2 \Omega_\Lambda \lambda^2 \left( 6 + \left( \frac{\lambda}{\lambda_0} \right) \sqrt{\frac{2}{\varpi}} + \left( \frac{\lambda}{\lambda_0} \right)^{-\sqrt{\frac{2}{\varpi}}} \right). \tag{4.15}
\]
Figure 3: Bouncing behaviour of the Brans-Dicke Jordan frame, dual to the concordance model. Scale factors of different Jordan frames, specified by different $w_{BD}$ values, are plotted with respect to the Einstein frame scale factor. The gray plot below is the equation of state of the concordance field $w_\phi$. The Vertical gray line represents the epoch of dust-$\Lambda$ equivalence in the Einstein frame. The occurrence of the Jordan frame bounce shifts towards future with decreasing $w_{BD}$.

Now, from eq. (3.2a), the Brans-Dicke frame scale factor $a$ is related to the Einstein frame scale factor as

$$a = \frac{\tilde{a}}{\sqrt{G\lambda}}. \quad (4.16)$$

Using eqs. (4.4) and (4.13), we replace $\lambda$ with $\tilde{a}$ in the above expression to write the Jordan frame scale factor as a function of the Einstein frame scale factor,

$$a(\tilde{a}) = \sqrt{8\pi \kappa^2 \Lambda_0} \left( \frac{\Omega_\Lambda}{\Omega_\Lambda - \Omega_M} \frac{\tilde{a}^3 + 1 + 1}{\tilde{a}^3 + 1 - 1} \right)^{\frac{1}{6\omega}}. \quad (4.17)$$

Evolution of the Jordan frame scale factor $a(\tilde{a})$ for different values of $w_{BD}$ is shown in fig. 3. For all the plots, the Jordan frame universe is seen to be collapsing at the early matter-dominated phase in the Einstein frame. Eventually, the Jordan frame goes through a non-singular bounce and starts expanding with the Einstein frame. One can find the Einstein frame scale factor at the point of bounce in the Jordan frame to be

$$\tilde{a}_b = \left( \frac{1 - \Omega_\Lambda}{2\Omega_\Lambda} \right)^{\frac{1}{3}} \left( \frac{-3 + 4w_{BD}}{3 + 2w_{BD}} \right)^{\frac{1}{3}}, \quad (4.18)$$

for $-\frac{3}{2} < w_{BD} < -\frac{3}{4}$ (see fig. 2). We see, depending on $w_{BD}$, the Jordan frame bounce can occur corresponding to a value of the Einstein frame scale factor anywhere within $\tilde{a}_b \to 0$ (for $w_{BD} \to -\frac{3}{2}$) and $\tilde{a}_b \to \infty$ (for $w_{BD} \to -\frac{3}{4}$), i.e., anywhere within the entire concordance era. The time of the Jordan frame bounce shifts to the future with increasing value of $w_{BD}$ parameter, as it can be seen from fig. 4.
Figure 4: Einstein frame scale factor at the point of the Jordan frame bounce $\tilde{a}_b$ is plotted with respect to the Brans-Dicke parameter $w_{BD}$. Different plots are for $\Omega_{\Lambda} = 0.6, 0.7, 0.8$. $\tilde{a}_b$ can take any possible value depending on $-\frac{3}{2} < w_{BD} < -\frac{3}{4}$. For smaller values of $w_{BD}$, the Jordan frame bounce occurs at later times in the Einstein frame.

4.3 Bouncing Jordan frame dual to dynamical dark energy models

So far in the discussion we have considered that the Einstein frame effectively mimics the evolution of the $\Lambda CDM$ universe. However, even if we consider models of dark energy other than the cosmological constant, it is possible to find a corresponding Brans-Dicke Jordan frame with bouncing behaviour. This can simply be understood again from the condition (4.11) and fig. 2 as follows.

Let us now consider that instead of the cosmological constant $\Lambda$, dark energy is now implemented by a dynamical model in the Einstein frame (reviews of viable dynamical models of dark energy can be found, for example, in [18–20]). Then, as discussed in section 4.1, the energy contribution of dark energy $\rho_{de}(\tilde{a})$ in eq. (4.1) must be set according to the dynamical model of dark energy. The Einstein frame scalar field $\varphi$ now gives an effective combined description of the dynamical model of dark energy and the non-relativistic matter component. The equation of state of the scalar field $w_{\varphi}$, which is essentially the effective equation of state in the Einstein frame, is set according to

$$w_{\varphi}(\tilde{a}) = -1 - \frac{2}{3} \frac{1}{H^2} \frac{d\dot{H}}{dt}$$

as before (see eq. (4.3)). The condition for the Jordan frame collapse, inequality (4.11), remains the same, as it is applicable to an arbitrary equations of state. The exact evolution of $w_{\varphi}(\tilde{a})$ depends on the choice of the dark energy model, however, we are only interested in the following generic feature of $w_{\varphi}(\tilde{a})$. Since the Einstein frame universe transitions from a matter-dominated phase to an accelerating phase, $w_{\varphi}$ must also go through a transition from a value $w_{\varphi} \approx 0$ to $w_{\varphi} < -\frac{1}{3}$. Now corresponding to this Einstein frame, we want to find a Brans-Dicke Jordan frame which is collapsing in the matter-dominated phase and expanding in the accelerating phase. This can be achieved by selecting the $w_{BD}$ parameter within the range

$$-\frac{3}{4} > w_{BD} > -1,$$

or

$$0 > 1 + \frac{4}{3} w_{BD} > -\frac{1}{3}.$$
To see this, let $\tilde{a}_m$ and $\tilde{a}_a$ be two values of the Einstein frame scale factor somewhere in the matter-dominated phase and in the accelerating phase, respectively, i.e., $w_\varphi(\tilde{a}_m) \approx 0$ and $w_\varphi(\tilde{a}_a) < -\frac{1}{3}$. The inequality (4.20b) then leads to
\[
 w_\varphi(\tilde{a}_m) > \frac{4}{3} w_{BD} + 1 > w_\varphi(\tilde{a}_a),
\]
(4.21)

According to the condition for collapsing Jordan frame (4.11), the first part of the above inequality, $w_\varphi(\tilde{a}_m) > \frac{4}{3} w_{BD} + 1$, shows that the Jordan frame is collapsing in the matter-dominated phase, when the Einstein frame scale factor is $\tilde{a}_m$. The second part, $\frac{4}{3} w_{BD} + 1 > w_\varphi(\tilde{a}_a)$, shows that in the accelerating phase, when the Einstein frame scale factor is $\tilde{a}_a$, the Jordan frame is expanding. This implies that the Jordan frame must go through a bounce at an instant, somewhere between the matter-dominated and accelerating phase of the Einstein frame. Corresponding to the Einstein frame, it is hence always possible to find a class of Brans-Dicke theories, specified by $w_{BD}$ from the range (4.20a), producing bouncing Jordan frames. Therefore, we show that the duality between the late-time physical universe, effectively implemented in the Einstein frame, and a bouncing Brans-Dicke Jordan frame, not only exits for the concordance ($\Lambda CDM$) model, it is rather a generic feature of any late-time model of the universe with dark-energy and non-relativistic matter.

In the next section we explore the map between Einstein and Jordan frame in the perturbative regime. We verify whether the conformal correspondence discussed in this section remains stable under linear perturbations.

5 Einstein frame-Jordan frame correspondence: Effects of linear perturbations

The Einstein and Jordan frame universes are connected via conformal transformation of the metric. We have so far considered that both the Einstein and Jordan frame metrics take the form of the spatially flat FRW spacetime, as given in eq. (2.6). However, in general, both the conformally connected universes can possess small fluctuations on the background of FRW spacetime. A robust conformal correspondence should provide a regular map between the perturbations in the Einstein and Jordan frames as well.

The study of cosmological perturbations in several classes of modified theories of gravity, particularly exploring the Einstein frame-Jordan frame mapping, can be found in [41–47]. In [41–43] perturbations in generalized $f(\phi, R)$ theories are studied using the Einstein frame description. Perturbation in the early universe $f(R)$ models is explored, for example, in [44].

In this section we introduce linear scalar perturbations in the background metrics of both the conformally connected frames. Following the treatment in [41–44], we briefly review the relation between metric perturbations in Einstein and Jordan frames.

5.1 Metric potentials in Einstein and Jordan frames

Let us consider scalar perturbations in the Jordan and Einstein frame metrics. The line elements in the Jordan and Einstein frames, written in the Newtonian gauge, are
\[
d s^2 = a^2(\eta) \left[ -(1 + 2\Phi) d\eta^2 + (1 - 2\Psi) \delta_{\alpha\beta} dx^\alpha dx^\beta \right],
\]
(5.1a)
\[
d \tilde{s}^2 = \tilde{a}^2(\eta) \left[ -(1 + 2\tilde{\Phi}) d\eta^2 + (1 - 2\tilde{\Psi}) \delta_{\alpha\beta} dx^\alpha dx^\beta \right],
\]
(5.1b)
where \((\Phi, \Psi)\) and \((\tilde{\Phi}, \tilde{\Psi})\) are the metric potentials in the Jordan and Einstein frames, respectively. Note that we will be using the conformal time \((\eta)\) in this section, which is same for both the frames \(^1\). The above mentioned perturbed line elements are related via the conformal transformation (2.2a) as
\[
\text{d}\tilde{s}^2 = \Omega^2 \text{d}s^2, \tag{5.2}
\]
where the conformal parameter is perturbed as well
\[
\Omega(\eta, x^\alpha) = \tilde{\Omega}(\eta) + \delta\Omega(\eta, x^\alpha). \tag{5.3}
\]
Comparing the first order terms in eq. (5.2), one can relate the metric potentials in the two frames as [41–44]
\[
\Phi = \tilde{\Phi} - \frac{\delta\Omega}{\tilde{\Omega}}, \tag{5.4a}
\]
\[
\Psi = \tilde{\Psi} + \frac{\delta\Omega}{\tilde{\Omega}}. \tag{5.4b}
\]
Let us now come to the example where the Einstein and Jordan frames are governed by the concordance scalar field \((\varphi)\) and the Brans-Dicke field \((\lambda)\), where the fields are now perturbed,
\[
\lambda(\eta, x^\alpha) = \tilde{\lambda}(\eta) + \delta\lambda(\eta, x^\alpha), \tag{5.5a}
\]
\[
\varphi(\eta, x^\alpha) = \tilde{\varphi}(\eta) + \delta\varphi(\eta, x^\alpha). \tag{5.5b}
\]
Since the Einstein frame is free from anisotropic stress, we further take \(\tilde{\Phi} = \tilde{\Psi} \) [44]. The relations between the metric potentials in the two frames then reduce to
\[
\Phi = \tilde{\Phi} - \frac{\delta\lambda}{2\lambda}, \tag{5.6a}
\]
\[
\Psi = \tilde{\Psi} + \frac{\delta\lambda}{2\lambda}, \tag{5.6b}
\]
whereas, the two potentials in the Jordan frame are related by
\[
\Psi = \Phi + \frac{\delta\lambda}{\lambda}. \tag{5.7}
\]
This shows that unlike the Einstein frame, the two metric potentials in the Jordan frame are not equal, which is a well-known characteristic of modified theories of gravity [41, 44].

We will now replace the Brans-Dicke field terms \((\tilde{\lambda}, \delta\lambda)\) in eq. (5.6) with those of the concordance field. Noting that \(\varphi\) and \(\lambda\) are related as (see eq. (2.5))
\[
\frac{\text{d}\varphi}{\text{d}\lambda} = \sqrt{K[\lambda]} = \frac{1}{\sqrt{2\kappa^2}} \frac{\sqrt{\varpi}}{\lambda}, \tag{5.8}
\]
we get
\[
\frac{\delta\lambda}{\lambda} = \frac{\sqrt{2\kappa}}{\sqrt{\varpi}} \frac{\delta\varphi}{\varphi} \tag{5.9a}
\]
\[
= \frac{\sqrt{2}}{2\kappa \varpi} \left( \dot{\Phi}' + \ddot{\Phi} \right), \tag{5.9b}
\]
\(^1\)as it can be seen from eq. (3.2), \(\text{d}\eta = (1/\dot{a}(t)) \text{d}\tilde{t} = (1/a(t)) \text{d}t\)

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where the prime denotes derivative with respect to conformal time \( \eta \), \( \ddot{H} = \dot{a}'/\dot{a} \). The last line is derived using the space-time component of the perturbed Einstein equation in the Einstein frame [44],

\[
\ddot{\Phi} + \dot{H}\ddot{\Phi} = \frac{k^2}{2} \varphi' \delta \varphi. \tag{5.10}
\]

Using eq. (5.9b) in eq. (5.6) we can write

\[
\Phi = \ddot{\Phi} - \frac{\sqrt{2}}{\sqrt{\omega}} \frac{1}{\kappa \dot{\varphi}^2} \left( \dot{\Phi}' + \dot{H} \ddot{\Phi} \right) \tag{5.11a}
\]

and

\[
\Psi = \ddot{\Phi} + \frac{\sqrt{2}}{\sqrt{\omega}} \frac{1}{\kappa \dot{\varphi}^2} \left( \dot{\Phi}' + \dot{H} \ddot{\Phi} \right). \tag{5.11b}
\]

These are the relations between the metric potentials in the two conformal frames. Thus, once the evolution of the metric potential in the Einstein frame is known, along with the background quantities \( \dot{a}(\eta) \) and \( \ddot{\varphi}(\eta) \), one can obtain the solutions for the metric potentials in the Jordan frame.

It is, however, possible that under certain circumstances the above relations become singular, breaking the perturbative map between the two conformal frames. This issue is addressed, for example, in [10, 15], in the context of early universe bouncing \( f(R) \) theories. It is shown that the relation between metric potentials can diverge when the background scalar field in the Einstein frame goes through an extremum, i.e., \( \varphi' \to 0 \). The same issue can potentially arise in the case of Brans-Dicke theory governed Jordan frame. From the relations (5.11) we see that if the background concordance field goes through an extremum at an instant, i.e. \( \varphi' \to 0 \), the Jordan frame potentials \( \Phi, \Psi \) can diverge, even if the Einstein frame potential and its derivative (\( \ddot{\Phi}, \dot{\Phi}' \)) remain finite. This can cause the Einstein frame-Jordan frame correspondence to break down in the perturbative regime. We now check whether such a divergence occurs in the present model of the bouncing universe dual to the physical universe.

From eq. (3.5), the conformal time derivative of the background field can be written as

\[
\varphi' = \ddot{\varphi} = \ddot{\varphi} \sqrt{(1 + w_{\varphi}(\ddot{\varphi}))\rho_{\varphi}(\ddot{\varphi})} \tag{5.12a}
\]

and

\[
\varphi' = \ddot{\varphi} = \sqrt{\frac{\rho_{\varphi} - \Omega_{\Lambda}}{\ddot{\varphi}}}, \tag{5.12b}
\]

where we have used eqs. (4.2b) and (4.3) to obtain eq. (5.12b). This shows that \( \varphi' \) is non-zero for any finite value of the Einstein frame scale factor \( \ddot{\varphi} \). Also from eq. (4.7), \( \ddot{\varphi} \) is finite-valued as long as the coordinate time \( \ddot{\varphi} \) is finite, hence \( \varphi' \) never becomes 0 at an instant. Unlike the models explored in [10, 15], the issue of diverging \( \Phi, \Psi \) caused by \( \varphi' \to 0 \) is not present in the current model.

Therefore, we expect the Jordan frame-Einstein frame map to behave regularly in the perturbative regime, even when the Jordan frame goes through the bounce. In the following discussion we explicitly verify the stability of the perturbative conformal map. To do this, we first numerically solve the Einstein frame metric perturbation, the result is then transported to the Jordan frame using the conformal correspondence. For the second case, we solve the perturbation in the Jordan directly, the result is then compared with those obtained via the Einstein frame. If the conformal map is robust under perturbation, then the Jordan frame metric potentials obtained in these two cases should be in agreement.

\[2\text{see [44] for similar relations between metric potentials in the context of } f(R) \text{ theories.}\]
Figure 5: Numerical solution for Einstein frame metric perturbation $\tilde{\Phi}$ is plotted with respect to the conformal time $\eta$, for Fourier mode $k = 15$. $\eta$ is given in the unit of 13.5 Gy, $\eta = 0$ corresponds to the current Einstein frame epoch, $\tilde{a} = 1$.

5.1.1 Numerical results in the Einstein frame

Let us first consider the evolution of perturbations in the Einstein frame. Cosmological perturbation in the presence of a canonical scalar field has been broadly studied, both in the context of inflationary models and dark energy models (see, for example, [44, 48] and references therein). For the perturbed Einstein frame metric (5.1b) and the perturbed concordance field (5.5b), the Einstein field equation leads to [44]

$$
\ddot{\Phi} + 2 \frac{d}{d\eta} \left( \frac{\dot{a}}{\dot{\varphi}} \right) \left( \frac{\ddot{\varphi}}{\dot{\varphi}} \right)^{-1} \dot{\Phi} - \nabla^2 \Phi + 2 \varphi \frac{d}{d\eta} \left( \frac{\dot{H}}{\varphi} \right) \Phi = 0,
$$

(5.13)

where $\nabla^2$ is the spatial Laplacian. Following the treatment of [44, 49], the above equation for a Fourier mode $k$ can be put in a relatively simple form,

$$
u''(\eta) + \left( k^2 - \frac{\theta''(\eta)}{\theta(\eta)} \right) u(\eta) = 0,
$$

(5.14)

where the first order quantity $u$ and background quantity $\theta$ are defined as

$$u(\eta) = \tilde{a}(\eta) \frac{\tilde{\Phi}(\eta)}{\varphi'(\eta)},
$$

(5.15a)

$$\theta(\eta) = \frac{\dot{H}}{\dot{\varphi}^2}.
$$

(5.15b)

For small scale modes, or for large $k$ ($k^2 \gg \theta''/\theta$), the contribution from the $\theta''/\theta$ term in eq. (5.14) becomes minimal, leading to an oscillatory solution for $u$. The Einstein frame metric potential is then obtained using $\tilde{\Phi} = (\tilde{\varphi}'/\tilde{a})u$, where, from eq. (5.12b), the factor $(\tilde{\varphi}'/\tilde{a})$ can be shown to decay as $1/\tilde{a}^2$. Hence, for $k^2 \gg \theta''/\theta$, we expect $\tilde{\Phi}$ to have the profile of a damped oscillator.
The background quantity \( \theta(\eta) \) can be obtained analytically, from eqs. (4.7) and (5.12b). Using this we numerically solve the perturbation equation (5.14) for such a sufficiently large \( k = 15 \), with initial conditions

\[
\tilde{\Phi}(\eta = 0) = 10^{-3}, \quad \tilde{\Phi}'(\eta = 0) = 10^{-2}.
\]  

Other background parameters are taken to be \( \Omega_\Lambda = 0.7, \) \( H_0 = 70 \text{ km sec}^{-1} \text{ Mpc}^{-1} \). The origin of conformal time \( \eta = \int \frac{dt}{\tilde{a}} \) is chosen such that \( \eta = 0 \) coincides with the current epoch, i.e., \( \tilde{a}(\tilde{t} = \tilde{t}_0 = 0) = \tilde{a}(\eta = 0) = 1 \) in the Einstein frame. Figure 5 shows the numerical evolution of the Einstein frame perturbation with conformal time. As discussed above, \( \tilde{\Phi} \) exhibits oscillation with decreasing amplitude.

Once the evolution of \( \tilde{\Phi} \) is known, one can obtain the Jordan frame metric perturbations \( \Phi, \Psi \), from the relations (5.11). For the numerical results, we choose the Jordan frame with Brans-Dicke parameter \( w_{\text{BD}} = -\frac{3}{4} (1 + \Omega_\Lambda) \). For this choice of \( w_{\text{BD}} \), the Jordan frame bounce occurs exactly at the current epoch of the Einstein frame, i.e., \( \tilde{a} = 1 \). The numerical evolutions of \( \Phi \) and \( \Psi \) are plotted in figs. 6a and 6b.

### 5.1.2 Numerical results in the Jordan frame

Having solved the Jordan frame perturbations using the conformal correspondence, in this section we study the evolution of perturbations directly in the Jordan frame. We then compare these results with those obtained via the Einstein frame in the previous section. If the Einstein frame-Jordan frame correspondence remains valid in the linear perturbation regime, then the solutions obtained for the two cases are expected to be in agreement.

Let us first consider the background evolution of the Brans-Dicke Jordan frame. Starting with the action (4.10) and using the background metric (3.1b), one can obtain the background equations of motion for the Brans-Dicke field and the Jordan frame scale factor as
Figure 7: Jordan frame scale factor $a$ is plotted with respect to the Jordan frame coordinate time $t$, where $w_{BD} = -\frac{3}{4} (1 + \Omega_A)$, $t$ is given in the unit of 13.5 Gy. The Jordan frame bounce occurs at $t = 0$. The blue plot is the numerical solution obtained directly in the Jordan frame, whereas, the black dashed plot is the analytical results obtained via the Einstein frame. The Einstein and Jordan frame results are shown to be in agreement.

(see appendix A)

$$\dot{H} = -\frac{w_{BD}}{2} \left(\frac{\dot{\lambda}}{\lambda}\right)^2 + 2H\frac{\dot{\lambda}}{\lambda} + \frac{1}{2\omega\lambda} \frac{16\pi}{\omega} \left[\lambda U,\lambda(\hat{\lambda}) - 2U(\hat{\lambda})\right] \quad (5.17a)$$

$$\ddot{\lambda} + 3H\dot{\lambda} = -\frac{16\pi}{\omega} \left[\lambda U,\lambda(\hat{\lambda}) - 2U(\hat{\lambda})\right], \quad (5.17b)$$

where overdots represent derivatives with respect to the Jordan frame coordinate time $t$, $H = \dot{a}/a$ is Jordan frame Hubble parameter. The Brans-Dicke potential $U(\lambda)$, dual to the concordance potential, is given in eq. (4.15). Using this we numerically solve the coupled eqs. (5.17a) and (5.17b) simultaneously to obtained $a(t)$ and $\hat{\lambda}(t)$. For this and all following numerical calculations, the origin of Jordan frame coordinate time ($t$) is taken such that it coincides with the Einstein frame coordinate time at the current epoch, i.e., $\tilde{t}_0 = 0$, $t(\tilde{t} = \tilde{t}_0) = 0$, $\eta(\tilde{t} = \tilde{t}_0) = 0$. The initial conditions are fixed it $t = 0$ such that they coincide with the Einstein frame solutions. Figure 7 shows the evolution of the Jordan frame scale factor with respect to the Jordan frame coordinate time $t$. For the chosen Brans-Dicke theory, $w_{BD} = -\frac{3}{4} (1 + \Omega_A)$, the Jordan frame bounce occurs at $t = 0$, which corresponds to the current epoch in the physical universe. The same plot also shows the analytical solution for $a(t)$ obtained using the Einstein frame (from eq. (4.17), using eqs. (3.2b) and (4.7)). We see that the numerical solution obtained directly in the Jordan frame is in agreement with its counterpart, analytically obtained in the Einstein frame.

Having obtained the background evolution in the Jordan frame, we now move on to the perturbations. For the perturbed Brans-Dicke field

$$\lambda = \tilde{\lambda}(t) + \delta\lambda(t, x^\alpha), \quad (5.18)$$
and the perturbed Jordan frame metric in the Newtonian gauge (written in terms of the coordinate time \( t \)),
\[
\text{d}s^2 = -\text{d}t^2(1 + 2\Phi) + a^2(t)(1 - 2\Psi)\delta_{\alpha\beta}\text{d}x^\alpha\text{d}x^\beta,
\] (5.19)
the linear order equations of motions, governing \( \delta\lambda \) and \( \Phi \), can be written for Fourier mode \( k \) as (see appendix A)
\[
\ddot{\delta}\lambda + \dot{\delta}\lambda \left( 3H + \frac{\dot{\lambda}}{\lambda} \right) + \delta\lambda \left( \frac{k^2}{a^2} + \frac{6H\dot{\lambda}}{\lambda} - \frac{2\dot{\lambda}^2}{\lambda} + \frac{16\pi\lambda U_{,\lambda\lambda}}{\psi} - \frac{16\pi U_{,\lambda}}{\psi} \right) - 6H\dot{\lambda}\dot{\Psi} - 2\dot{\Psi}\ddot{\lambda} - 4\dddot{\lambda}\dot{\Psi} = 0,
\] (5.20)
and
\[
-6\dot{\lambda}\dot{\omega}\dot{\Psi} - 30H\dot{\lambda}\dot{\omega}\Psi + \Psi \left( -\frac{2k^2\dot{\lambda}\omega}{a^2} - 24H^2\dot{\lambda}\omega - 12H\dot{\lambda}\omega - \frac{2\dot{\lambda}^2w\omega}{\lambda} \right) + \delta\lambda \left( -\frac{2k^2}{a^2} + 36H^2\dot{\omega} - \frac{6H\dot{\lambda}\omega}{\lambda} + 18\dot{H}\dot{\omega} - 48\pi\dot{\lambda}U_{,\lambda\omega} + 48\pi\dot{U}_{,\lambda\omega} + \frac{5\dot{\lambda}^2w\omega}{\lambda^2} \right) + \dot{\delta}\lambda \left( 6H\dot{\omega} + \frac{2\dot{\lambda}w\omega}{\lambda} \right) = 0
\] (5.21)
For the Fourier mode \( k = 15 \), the coupled eqs. (5.20) and (5.21) are numerically solved together to obtain \( \Psi(t) \) and \( \delta\lambda(t) \). We choose the following initial conditions at \( t = 0 \),
\[
\Psi(0) = 0.0263, \quad \dot{\Psi}(0) = -0.1623, \quad \delta\lambda(0) = 0.1130, \quad \dot{\delta}\lambda(0) = -0.6745,
\] (5.22)
such that they coincide with the Einstein frame initial conditions (5.16) at \( \eta = t = 0 \). Using the solutions of \( \Psi(t) \) and \( \delta\lambda(t) \), the other Jordan frame metric potential \( \Phi(t) \) can be obtained from eq. (5.7). Figures 6a and 6b show the evolution of the Jordan frame metric perturbations \( \Phi, \Psi \), together with their counterparts obtained from the Einstein frame. Note that the Jordan frame solutions \( \Psi(t), \Phi(t) \) are converted to functions of the conformal time \( \eta \), in order to compare them with the Einstein frame solutions. We see that for both the metric potentials, the Jordan and Einstein frame solutions are in good agreement.

The Einstein frame solutions behave regularly throughout the evolution, including the point of the Jordan frame bounce at \( \eta = 0 \). The conformal map between the Einstein and Jordan frames is thus able to accommodate for cosmological perturbations present in the two frames. Therefore, the bouncing universe description of the concordance model of cosmology is shown to be stable against linear perturbations.

6 Summary and discussion

The scalar-tensor theories belong to a class of modified theories of gravity, which can be mapped to Einstein’s gravity using the Jordan frame-Einstein frame duality. The Einstein and Jordan frame descriptions are mathematically equivalent, it is, however, possible that the Jordan frame universe goes through a quite distinct cosmological evolution, in comparison
with its Einstein frame counterpart [9–16]. In this paper we study a class of dual universes
with contrasting nature of evolutions.

We consider an Einstein frame where the universe goes through the standard late-time
cosmological evolution, driven by dark energy and non-relativistic matter. Dual to this, we
find a class of scalar-tensor theories which may lead to a collapsing Jordan frame. A general
condition for such an expansion-collapse duality is obtained to predict whether, for a given
scalar-tensor theory, the Jordan frame is collapsing during a period of evolution.

The Einstein frame scalar field is set up to provide an effective description of the Λ
\textit{CDM} or the concordance model of cosmology. As an example of the scalar-tensor theories, we con-
sider the Brans-Dicke model in the Jordan frame. Using the general condition of expansion-
collapse duality, we show that concordance model of cosmology can always be mapped to a
bouncing Brans-Dicke Jordan frame. This implies that the Λ\textit{CDM} model of the late-time
universe can have an effective description in terms of the dynamics of a bouncing universe.
This result is shown to be valid for dynamical models of dark energy as well. The conformal
duality with a bouncing universe is, in fact, a generic feature of any late-time model of the
universe, describing the standard cosmological evolution driven by dark energy and matter.
The bouncing behaviour in the Jordan frame can be associated with the transition of the
physical universe from a matter-dominated phase to an accelerating phase.

A robust conformal correspondence between the Einstein and Jordan frames should
provide a regular map between cosmological perturbations therein. We investigate whether
the dual bouncing universe description of the standard cosmology is robust against linear
perturbations. To see this, we numerically solve scalar metric perturbations in both the con-
formal frames. The Einstein frame solutions are then transported to the Jordan frame, using
the linear order conformal map. These are compared with the Jordan frame perturbations,
directly obtained in the Jordan frame. The solutions of the Jordan frame perturbations in
these two cases are in agreement. Therefore, the conformal duality between the late-time
physical universe and the bouncing Brans-Dicke Jordan frame universe is shown to be stable
under linear perturbations.

Bouncing scenarios, in general, have been explored in the literature as a candidate for
the early universe model, alternative to the inflationary theory. In this paper we show that
a perturbatively stable, effective bouncing description is also possible for the standard late-
time evolution of the universe. This may have interesting implications. As we have seen, the
nature of the cosmological evolution depends on the conformal frame, however, it is shown
in the literature that physical observables, such as the redshift, galaxy number count [14],
Sachs-Wolfe effect, curvature perturbation (see [50, 51] and references therein) are indepen-
dent of the choice of the conformal frame. One can use these frame invariant observables
in the current model to find a map between the late-time cosmological perturbations and
fluctuations in a bouncing model. Also, by arranging the Jordan frame bounce somewhat
earlier in the Einstein frame, we can study the growth of perturbation as of today. This can
lead to interesting insights about perturbations both in the late-time universe and in the
bouncing universe.

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A Field equations in the Jordan frame

Here we summarize the derivations of the Brans-Dicke theory equations of motion in the Jordan frame, eqs. (5.17), (5.20) and (5.21). For detailed derivation see, for example [1].

The variation of the Brans-Dicke action (4.10) with respect to the metric $g_{ab}$ leads to the field equation

$$\lambda G_{ab} = \frac{w_{BD}}{\lambda} \left( \nabla_a \lambda \nabla_b \lambda - \frac{1}{2} g_{ab} \nabla^c \lambda \nabla_c \lambda \right) + \left( \nabla_a \nabla_b \lambda - g_{ab} \Box \lambda \right) - 8 \pi U g_{ab}, \quad \text{(A.1)}$$

the trace of which is

$$\lambda R = \frac{w_{BD}}{\lambda} \nabla^c \lambda \nabla_c \lambda + 3 \Box \lambda + 32 \pi U. \quad \text{(A.2)}$$

The equation of motion of the Brans-Dicke field $\lambda$ can be derived as

$$2 w_{BD} \Box \lambda + \lambda R - w_{BD} \lambda \nabla_c \lambda \nabla^c \lambda - 16 \pi \lambda U,_{,\lambda} = 0. \quad \text{(A.3)}$$

Using eqs. (A.2) and (A.3), one can find the following dynamical equations for the metric and for $\lambda$,

$$\varpi \lambda R = \varpi \frac{w_{BD}}{\lambda} \nabla^c \lambda \nabla_c \lambda + 16 \pi \left( 3 \lambda U,_{,\lambda} + 4 w_{BD} U \right). \quad \text{(A.4a)}$$

$$\varpi \Box \lambda = 16 \pi \left( \lambda U,_{,\lambda} - 2U \right). \quad \text{(A.4b)}$$

For the homogeneous background field $\lambda(t)$ and spatially flat FRW line element (written in the coordinate time) in the Jordan frame,

$$ds^2 = -dt^2 + a^2(t) \delta_{\alpha \beta} dx^\alpha dx^\beta, \quad \text{(A.5)}$$

the above equations lead to

$$\dot{H} = -\frac{w_{BD}}{2} \left( \frac{\dot{\lambda}}{\lambda} \right)^2 + 2 \frac{\dot{\lambda}}{\lambda} + \frac{1}{2 \varpi \lambda} 16 \pi \left[ \lambda U,_{,\lambda} - 2U \right] \quad \text{(A.6a)}$$

$$\ddot{\lambda} + 3H \dot{\lambda} = -\frac{16 \pi}{\varpi} \left[ \lambda U,_{,\lambda} - 2U \right]. \quad \text{(A.6b)}$$

Similarly, using the perturbed metric (5.19) and perturbed field (5.18) in eqs. (A.4a) and (A.4b), one can find the linear order equation

$$-6 \lambda \varpi \ddot{\Psi} - 30 H \dot{\lambda} \Psi + \Psi \left( -\frac{2 k^2 \dot{\lambda} \varpi}{a^2} - 24 H^2 \dot{\lambda} \varpi - 12 \dot{H} \lambda \varpi - \frac{2 \dot{\lambda}^2 w \varpi}{\lambda} \right) + \delta \lambda \left( -\frac{2 k^2 \varpi}{a^2} + 36 H^2 \varpi - \frac{6 H \dot{\lambda} \varpi}{\lambda} + 18 \dot{H} \varpi - 48 \pi \lambda U,_{,\lambda \lambda} - 48 \pi U,_{\lambda} - 64 \pi w U,_{\lambda} + \frac{\dot{\lambda}^2 w \varpi}{\lambda^2} \right) + \dot{\delta} \lambda \left( 6 H \varpi + \frac{2 \dot{\lambda} w \varpi}{\lambda} \right) = 0 \quad \text{(A.7)}$$

and

$$\delta \ddot{\lambda} + \delta \dot{\lambda} \left( 3H + \frac{\dot{\lambda}}{\lambda} \right) + \delta \lambda \left( \frac{k^2}{a^2} + \frac{6 H}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} + \frac{2 \dot{\lambda}}{\lambda} + \frac{16 \pi \lambda U,_{,\lambda \lambda}}{\varpi} - \frac{16 \pi U,_{\lambda \lambda}}{\varpi} \right) - 6 \dot{H} \dot{\Psi} - 2 \Psi \ddot{\lambda} - 4 \dot{\lambda} \ddot{\Psi} = 0, \quad \text{(A.8)}$$

where all the terms involving $\Phi$ are replaced with $\Psi$ using eq. (5.7).
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