BRACKET WIDTH OF THE LIE ALGEBRA OF VECTOR FIELDS ON A SMOOTH AFFINE CURVE

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Abstract. We prove that the bracket width of the simple Lie algebra of vector fields $\text{Vec}(C)$ of a smooth irreducible affine curve $C$ with a trivial tangent sheaf is at most three. In addition, if $C$ is a plane curve, the bracket width of $\text{Vec}(C)$ is at most two and if moreover $C$ has a unique place at infinity, the bracket width of $\text{Vec}(C)$ is exactly two. We also show that in case $C$ is rational, the width of $\text{Vec}(C)$ equals one.

1. Introduction

Given a Lie algebra $L$ over an infinite field $\mathbb{k}$, we define its bracket width as the supremum of lengths $\ell(x)$, where $x$ runs over the derived algebra $[L, L]$ and $\ell(x)$ is defined as the smallest number $n$ of Lie brackets $[y_i, z_i]$ needed to represent $x$ in the form

$$\sum_{i=1}^{n} [y_i, z_i].$$

The bracket width applies in studying different aspects of Lie algebras, see [Rom16]. In particular, in [Rom16] the author provides many examples of Lie algebras with the bracket width strictly bigger than one. However, the first example of a simple Lie algebra with the bracket width strictly bigger than one was found only very recently in [DKR21, Theorem A] among Lie algebras of vector fields of smooth affine curves which are simple by [Jor86] and [Sie96, Proposition 1]. In the current note we provide an upper bound on the bracket width of a Lie algebra of vector fields on an irreducible smooth affine curve $C$ with certain properties. Our first main result is the following statement which partially answers [DKR21, Question 2].

Theorem A. Let $C$ be an irreducible smooth affine curve with trivial tangent sheaf. Then the bracket width of the Lie algebra $\text{Vec}(C)$ is smaller than or equal to three. In addition, if $C$ is a plane curve, the bracket width of $\text{Vec}(C)$ is smaller than or equal to two.

The upper bound for the bracket width given in Theorem A is, in particular, of interest since it allows us to compute the bracket width of $\text{Vec}(C)$ for a certain family of smooth plane affine curves.

Corollary 1. Let $C$ be an irreducible non-rational smooth plane affine curve with a unique place at infinity. Then the width of the simple Lie algebra $\text{Vec}(C)$ equals two.

There are many examples of affine curves with only one place at infinity (see Definition 1), and they were studied in many different contexts, see, e.g., a paper of Kollár [Kol20] and the references therein. A simple class of examples is given by affine hyperelliptic plane curves.

2020 Mathematics Subject Classification. 14H52, 17B66.
$C \subset \mathbb{A}^2$ defined by equations $y^2 = h(x)$, where $h(x)$ is a monic polynomial of odd degree strictly greater than one which has only simple roots ([DKR21, Example 2]).

We believe that the assumption on the curve $C$ in Corollary 1 can be lightened and we have the following conjecture.

**Conjecture 1.** Assume $C$ is a non-rational affine smooth plane curve. Then the bracket width of $\text{Vec}(C)$ is exactly two.

Assume $f$ is a regular function on $C$. We define a principal open subset $C_f \subset C$ as

$$\{ x \in C \mid f(x) \neq 0 \} \subset C.$$ 

Note that $C_f$ is a smooth affine curve itself.

**Theorem B.** Let $C$ be an irreducible smooth affine curve and $C_f$ be its principal open subset. Then the bracket width of $\text{Vec}(C_f)$ is smaller than or equal to the bracket width of $\text{Vec}(C)$.

We do not know an example of a smooth affine curve $C$ with a principal open subset $U \subset C$ such that the width of $\text{Vec}(U)$ is strictly smaller than $\text{Vec}(C)$.

As a consequence of Theorem B we have the following statement that disproves [DKR21, Conjecture 1].

**Corollary 2.** If $\mathbb{k}$ is algebraically closed, the bracket width of the Lie algebra $\text{Vec}(C)$ of a rational smooth affine curve $C$ is one.

### 2. Proof of Theorem A

**Proof of Theorem A.** Denote by $\mathcal{O}(C)$ the ring of regular functions on $C$. Since by hypothesis the tangent sheaf of $C$ is trivial, we have $\text{Vec}(C) = \mathcal{O}(C) \cdot \tau$ for a certain nowhere vanishing global vector field $\tau \in \text{Vec}(C)$, unique up to multiplication by a nonzero constant. It is well-known that every smooth affine variety of dimension $d$ can be embedded into $\mathbb{A}^{2d+1}$ and that the bound $2d + 1$ is optimal ([Sr91, Corollary 1]). In particular, a smooth affine curve can be embedded into $\mathbb{A}^3$ and this bound is sharp. Hence, $\mathcal{O}(C) \cong \mathbb{k}[x,y,z]/I$, where $I \subset \mathbb{k}[x,y,z]$ is some ideal and we have the natural surjections

$$\pi : \mathbb{k}[x,y,z] \twoheadrightarrow \mathbb{k}[x,y,z]/I$$

and

$$\pi_* : \{ \nu \in \text{Der} \mathbb{k}[x,y,z] = \text{Vec}(\mathbb{A}^3) \mid \nu(I) \subset I \} \twoheadrightarrow \text{Vec}(C).$$

Note that $\{ \nu \in \text{Der} \mathbb{k}[x,y,z] \mid \nu(I) \subset I \} \subset \text{Der} \mathbb{k}[x,y,z]$ is a Lie subalgebra and $\pi_*$ is a homomorphism of Lie algebras. Then $\tau$ is the image of a derivation $\tilde{\tau} = \tilde{P} \frac{\partial}{\partial x} + \tilde{Q} \frac{\partial}{\partial y} + \tilde{R} \frac{\partial}{\partial z}$ that preserves the ideal $I$. Further, define $P, Q, R \in \mathcal{O}(C)$, $P = \pi(\tilde{P})$, $Q = \pi(\tilde{Q})$, $R = \pi(\tilde{R})$.

**Claim 1.**

$$\left\{ [\tilde{f}, \tilde{g}], [y\tilde{f}, \tilde{g}], [z\tilde{f}, \tilde{g}], [\tilde{f}, \tilde{g}] , \tilde{h} \in \mathbb{k}[x,y,z] \right\} = (\tilde{P}, \tilde{Q}, \tilde{R})\tau,$$

where $(\tilde{P}, \tilde{Q}, \tilde{R})$ denotes the ideal of $\mathbb{k}[x,y,z]$ generated by $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$. 

[2]
Indeed,

\[
[\bar{\tau}, \bar{f} \bar{\tau}] + [y \bar{\tau}, \bar{g} \bar{\tau}] + [z \bar{\tau}, \bar{h} \bar{\tau}] = (\bar{P}\bar{f}_y' + \bar{Q}\bar{g}_y' + \bar{R}\bar{h}_y') + y(\bar{P}\bar{g}_z + \bar{Q}\bar{g}_y + \bar{R}\bar{g}_z') - \bar{g}\bar{Q} + z(\bar{P}\bar{h}_x' + \bar{Q}\bar{h}_y' + \bar{R}\bar{h}_z') - \bar{h}\bar{R})\bar{\tau}
\]

which equals

\[
(\bar{P}(\bar{f}_x' + y\bar{g}_x' + z\bar{h}_x') + \bar{Q}(\bar{f}_y' + y\bar{g}_y' + z\bar{h}_y') + \bar{R}(\bar{f}_z' + y\bar{g}_z' + z\bar{h}_z') - \bar{g})\bar{\tau}
\]

(2) \( (\bar{P}(\bar{f}_x' + y\bar{g}_x' + z\bar{h}_x') + \bar{Q}(\bar{f}_y' + y\bar{g}_y' + z\bar{h}_y') + \bar{R}(\bar{f}_z' + y\bar{g}_z' + z\bar{h}_z') - \bar{g})\bar{\tau} = (\bar{P}(\bar{f} + y\bar{g} + z\bar{h})_x' + \bar{Q}((\bar{f} + y\bar{g} + z\bar{h})_y - 2\bar{g}) + \bar{R}((\bar{f} + y\bar{g} + z\bar{h})_z - 2\bar{h}))\bar{\tau} \).

Define \( \bar{\tau} = \bar{f} + y\bar{g} + z\bar{h} \in \mathcal{O}(C) \). Now, the expression (2) can be written as

\[
(\bar{P}\bar{f}_x' + \bar{Q}(\bar{f}_y' - 2\bar{g}) + \bar{R}(\bar{f}_z' - 2\bar{h}))\bar{\tau}.
\]

For any \( F, G, H \in \mathbb{k}[x, y, z] \) there exist \( \bar{\tau}, \bar{g}, \bar{h} \) such that \( F = \bar{f}_x', G = \bar{f}_y' - 2\bar{g} \) and \( H = \bar{f}_z' - 2\bar{h} \). Hence,

\[
\left\{ (\bar{P}\bar{f}_x' + \bar{Q}(\bar{f}_y' - 2\bar{g}) + \bar{R}(\bar{f}_z' - 2\bar{h}))\bar{\tau} \mid \bar{\tau}, \bar{g}, \bar{h} \in \mathbb{k}[x, y, z] \right\} = \left\{ (\bar{P}F + \bar{Q}G + \bar{R}H)\bar{\tau} \mid F, G, H \in \mathbb{k}[x, y, z] \right\} = (\bar{P}, \bar{Q}, \bar{R})\bar{\tau}.
\]

This proves Claim 1.

We claim now that any vector field from \( \text{Vec}(C) \) can be written as the sum

\[
[\tau, f\tau] + [\pi(y)\tau, g\tau] + [\pi(z)\tau, h\tau]
\]

for some regular functions \( f, g, h \in \mathcal{O}(C) \). Indeed, \( (\bar{P}, \bar{Q}, \bar{R}) \) is preserved by \( \bar{\tau} \), hence \( \pi((\bar{P}, \bar{Q}, \bar{R})) \) is preserved by \( \tau \). Whence \( \pi((\bar{P}, \bar{Q}, \bar{R}))\tau \) is the ideal in \( \text{Vec}(C) \). Therefore, because of simplicity of \( \text{Vec}(C) \) we have

\[
\pi_*((\bar{P}, \bar{Q}, \bar{R})\tau) = \text{Vec}(C) \text{ or equivalently } \pi((\bar{P}, \bar{Q}, \bar{R})) = \mathcal{O}(C).
\]

Finally, by Claim 1 we have

\[
\left\{ \pi_*([\tau, \bar{f} \bar{\tau}]) + \pi_*([y \bar{\tau}, \bar{g} \bar{\tau}]) + \pi_*([z \bar{\tau}, \bar{h} \bar{\tau}]) \mid \bar{\tau}, \bar{g}, \bar{h} \in \mathbb{k}[x, y, z] \right\} = \pi_*((\bar{P}, \bar{Q}, \bar{R})\tau) = \text{Vec}(C).
\]

Thus, the first statement of the theorem follows as \( \pi_* \) is a homomorphism of Lie algebras.

If \( C \) is a plane curve, then, similarly as above, \( \text{Vec}(C) = \mathcal{O}(C) \cdot \tau, \mathcal{O}(C) \simeq \mathbb{k}[x, y]/I \), where \( I \subset \mathbb{k}[x, y] \) is an ideal and we have the natural surjections

\[
\pi: \mathbb{k}[x, y] \twoheadrightarrow \mathbb{k}[x, y]/I
\]

and

\[
\pi_*: \left\{ \nu \in \text{Der}\mathbb{k}[x, y] = \text{Vec}(\mathbb{A}^2) \mid \nu(I) \subset I \right\} \twoheadrightarrow \text{Vec}(C).
\]

Then \( \tau \) is the image of a derivation \( \bar{\tau} = \bar{P}\frac{\partial}{\partial x} + \bar{Q}\frac{\partial}{\partial y} \) that preserves the ideal \( I \). Further, define \( P, Q \in \mathcal{O}(C), P = \pi(\bar{P}), Q = \pi(\bar{Q}) \). The second statement of the theorem follows if we prove that any vector field of \( \text{Vec}(C) \) can be written as the sum

\[
[\tau, f\tau] + [\pi(y)\tau, g\tau]
\]

for some regular functions \( f, g \in \mathcal{O}(C) \). Similarly as above (4) follows from the next equality

\[
\left\{ [\bar{\tau}, \bar{f} \bar{\tau}] + [y \bar{\tau}, \bar{g} \bar{\tau}] \mid \bar{\tau}, \bar{g} \in \mathbb{k}[x, y] \right\} = (\bar{P}, \bar{Q})\bar{\tau}
\]

which is proved analogously as Claim 1. \( \square \)
Definition 1. An irreducible smooth affine curve $C$ is said to have a unique place at infinity if it is equal to the complement of a single closed point in a smooth projective curve $\bar{C}$.

Proof of Corollary 1. By [DKR21, Theorem A] the bracket width of Vec($C$) is strictly greater than one. Since $C$ is a smooth irreducible plane affine curve, its tangent sheaf is trivial. Now by Theorem A the width of Vec($C$) is less or equal than two which proves the claim. □

3. Proof of Theorem B

Proof of Theorem B. Denote by $K(C)$ the field of fractions of $O(C)$. Then

(5) $K(C) \otimes_{O(C)} \text{Der}(O(C)) \simeq \text{Der}(K(C))$

and the latter is known to be a free $K(C)$-module of rank equal to dim $C = 1$. By (5) Vec($C$) = $\text{Der}(O(C)) \subset \text{Vec}(C) = K(C)\tau$, where $\tau \in \text{Vec}(C)$ is a global vector field. Hence, Vec($C_f$) = $R(C)[f^{-1}]\tau$. Now, for $a, b \in R(C)$, $\frac{a}{f^k}\tau, \frac{b}{f^k}\tau \in \text{Vec}(C_f) = R(C)[f^{-1}]\tau$ and we have:

(6) $\left[ \frac{a}{f^k}\tau, \frac{b}{f^k}\tau \right] = \left( \frac{a}{f^k}\tau \left( \frac{b}{f^k}\tau \right) - \frac{b}{f^k}\tau \left( \frac{a}{f^k}\tau \right) \right) \tau = \frac{1}{f^{2k}} [a, b] \tau.$

Further, for any $h \in R(C)[f^{-1}]$ there exists $g \in R(C)$ such that

$h = \frac{g}{f^{2k}}$

for some $k \in \mathbb{N}$. Assume that

$g\tau = [a_1\tau, b_1\tau] + \cdots + [a_n\tau, b_n\tau].$

Then using (6) we have

$h\tau = \frac{g}{f^{2k}}\tau = \left[ \frac{a_1}{f^k}\tau, \frac{b_1}{f^k}\tau \right] + \cdots + \left[ \frac{a_n}{f^k}\tau, \frac{b_n}{f^k}\tau \right].$

This completes the proof. □

Proof of Corollary 2. Let us recall that every rational smooth affine curve $C$ is isomorphic to $A^1 \setminus \Lambda$, where $\Lambda$ is a finite set of $r \geq 0$ points. In particular, it admits a closed embedding into $A^2$. Indeed, $C$ can also be seen as the complement in $A^1$ of a finite number (possibly zero) of points and we can consider the closed embedding $\mu: C \rightarrow A^2$ given by $x \mapsto (x, \frac{1}{f(x)})$, where $f \in \mathbb{k}[x]$ is a polynomial whose roots are exactly the removed points. Note that the image of $\mu$ is the curve of $A^2$ defined by the equation $f(x)y = 1$.

By [DKR21, Proposition 1] the bracket width of Vec($A^1$) is one. By Theorem B the bracket width of Vec($A^1 \setminus \Lambda$) is smaller than or equal to the bracket width of Vec($A^1$) which is one. The proof follows. □

Acknowledgements. We thank Adrien Dubouloz and Boris Kunyavskii for useful discussions.
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