Einstein-aether theory, violation of Lorentz invariance, and metric-affine gravity

Christian Heinicke\textsuperscript{1}, Peter Baekler\textsuperscript{2}, Friedrich W. Hehl\textsuperscript{1,3}

\textsuperscript{1}Institut für Theoretische Physik, Universität zu Köln, 50923 Köln, Germany
\textsuperscript{2}Fachbereich Medien, Fachhochschule Düsseldorf, University of Applied Sciences, Josef-Gockeln-Str. 9, 40474 Düsseldorf, Germany
\textsuperscript{3}Department of Physics and Astronomy, University of Missouri-Columbia, Columbia, MO 65211, USA

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We show that the Einstein-aether theory of Jacobson and Mattingly (J&M) can be understood in the framework of the metric-affine (gauge theory of) gravity (MAG). We achieve this by relating the aether vector field of J&M to certain post-Riemannian nonmetricity pieces contained in an independent linear connection of spacetime. Then, for the aether, a corresponding geometrical curvature-square Lagrangian with a massive piece can be formulated straightforwardly. We find an exact spherically symmetric solution of our model.

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I. INTRODUCTION

In an attempt to violate Lorentz invariance locally, J&M\textsuperscript{37} introduced an extra timelike 4-vector field $u$ of unit length, see also [8, 12, 17, 18, 21, 26, 36, 46] and the earlier work of Gasperini [23, 24] and Kostelecky and Samuel [42]. This “aether field” is governed by a Lagrangian carrying a kinetic term $\sim (\nabla u)^2$ and a massive term $\sim u^2$. The aether field equation is of (massive) Yang-Mills type

$$\nabla H + \text{lower order term} \sim \ell u,$$

(1)
with \( H \sim \nabla u \) as field momentum (excitation) and \( \ell \) as some constant. The aether field \( u \) can be considered to be an analogue of a Yang-Mills potential \( B \).

The purpose of our note is to point out that such an aether theory can be reconstructed quite naturally as a specific example within the framework of MAG \[27, 30\]. The reasons are as follows:

1. Since the aim of the aether theory of J&M is to violate local Lorentz invariance, one should abandon the (flat) Minkowski as well as the (curved) Riemannian spacetime, which are rigidly and locally Lorentz invariant, respectively, and look for a post-Riemannian structure of spacetime appropriate for such an approach. If one introduces in spacetime, besides the metric \( g_{\alpha \beta} \), an independent linear connection 1-form \( \Gamma_{\alpha \beta} \), then it is known from literature, see \[30\], that the nonmetricity

\[
Q_{\alpha \beta} := - \hat{D} g_{\alpha \beta},
\]

where \( \hat{D} \) is the covariant exterior derivative with respect to the connection \( \Gamma_{\alpha \beta} \), is a measure for the violation of local Lorentz invariance. Note that the introduction of a torsion 2-form \( T^\alpha \) is voluntary in this context.

2. In four dimensions the nonmetricity with its 40 components can be split irreducibly into four pieces (see \[30\]):

\[
Q_{\alpha \beta} = {}^{(1)}Q_{\alpha \beta} \oplus {}^{(2)}Q_{\alpha \beta} \oplus {}^{(3)}Q_{\alpha \beta} \oplus {}^{(4)}Q_{\alpha \beta},
\]

\[
40 = 16 \oplus 16 \oplus 4 \oplus 4.
\]

In this article we want to concentrate on reconstructing the J&M theory in the framework of MAG. Therefore we pick the two vector-like quantities \( {}^{(3)}Q_{\alpha \beta} \) and \( {}^{(4)}Q_{\alpha \beta} \), with four independent components each, as relevant for our purpose. The explicit form of these two quantities will be studied below. A result will be that the geometrical interpretation of \( {}^{(4)}Q_{\alpha \beta} \) identifies it as equivalent to the Weyl covector \[84\] of 1918, which is related to scale transformations (and thus extends the Lorentz to the conformal group), whereas \( {}^{(3)}Q_{\alpha \beta} \) is related to shear transformations and, accordingly, is a generic field obstructing local Lorentz invariance. In other words, we identify \( {}^{(3)}Q_{\alpha \beta} \) as analogue of the aether field \( u \) of J&M. Accordingly, by assuming an independent linear connection of spacetime, we arrive straightforwardly at a vector-like quantity of
shear type that dissolves local Lorentz invariance. It is part of the geometric structure of spacetime and represents as such some kind of genuine aether.

3. Identifying \( (3)Q_{\alpha\beta} \) as an analogue of the aether field \( u \) suggests a Yang-Mills type Lagrangian

\[
V_{(3)Q} \sim \Gamma_{\alpha\beta} (3)Q_{\alpha\beta} \wedge \star \Gamma_{\alpha\beta} (3)Q_{\alpha\beta} + \ell (3)Q_{\alpha\beta} \wedge \star (3)Q_{\alpha\beta},
\]

where \( \star \) denotes the Hodge star operator. The first term parallels the kinetic aether term of J&M, see [37], Eq.(2), the second one the massive term of J&M. If a gravitational Hilbert-Einstein Lagrangian is added to (4), we recover the basic structure of the J&M Lagrangian.

However, we stress an important difference. In our approach we proceed from variables of purely geometric origin: metric \( g_{\alpha\beta} \) (defining angles and lengths), coframe \( \vartheta^\alpha \) (representing the local reference frame) and connection \( \Gamma_{\alpha\beta} \) (defining parallel displacement). In contrast, the aether field of J&M is an external quantity with no obvious relation to the geometry of spacetime, it is more like a cosmological fluid. Moreover, in spite of the presumed violation of Lorentz invariance, the J&M aether field \( u \) has a fixed (timelike) magnitude, which doesn’t seem convincing to us.

In actual fact, the geometrical Lagrangian that we are going to study below will be a bit more complicated than (4). This conforms better to the MAG models that we developed earlier. In this context we will also demonstrate that the Lagrangian (4), in spite of the existence of the derivative terms in (4), fits very well into the first order version of MAG.

4. In the J&M model, the coupling of the aether field to matter seems to be an unsolved problem. In our approach, in the sense of Einstein’s equivalence principle, we assume minimal coupling, that is, partial (or exterior) derivatives are substituted by covariant ones: \( d \rightarrow \Gamma D \). Thereby a universal coupling of matter to our aether is guaranteed.

In Sec.II we give a sketch of metric-affine geometry, and in Sec.III we describe the main geometrical properties of the nonmetricity. In Sec.IV we turn to the the different curvature 2-forms, in particular to those that relate to post-Riemannian structures. Subsequently, in Sec.V, we display the general form of the field equations of MAG and compare them with those of J&M. In Sec.VI we formulate the general quadratic Lagrangian of MAG. We
tailor it such that it becomes somewhat analogous to the J&M Lagrangian. We compute
the corresponding excitations (field momenta) explicitly by suitable partial differentiation
of the Lagrangian. Accordingly, the field equations are now known explicitly.

In Sec.VII we present an exact spherically symmetric solution, the mass and the angular
momentum of which are determined in Sec.VIII. The prolongation techniques that we took
for finding exact solutions are explained shortly in Sec.IX. We discuss our results in Sec.X.

Our notation is as follows (see [30, 32]): We use the formalism of exterior differential
forms. We denote the frame by $e_\alpha$, with the anholonomic or frame indices $\alpha, \beta, \ldots = 0, 1, 2, 3$.
Decomposed with respect to a natural frame $\partial_i$, we have $e_\alpha = e^i_\alpha \partial_i$, where $i, j, \ldots = 0, 1, 2, 3$
are holonomic or coordinate indices. The frame $e_\alpha$ is the vector basis of the tangent space
at each point of the 4D spacetime manifold. The symbol $\lbrack \rbrack$ denotes the interior and $\wedge$
the exterior product. The coframe $\vartheta^\beta = e^\beta_j dx^j$ is dual to the frame, i.e., $e_\alpha \lbrack \vartheta^\beta = \delta^\beta_\alpha$. If $*$
denotes the Hodge star operator and if $\vartheta^{\alpha \beta} := \vartheta^\alpha \wedge \vartheta^\beta$, etc., then we can introduce the
eta-basis $\eta := \star 1$, $\eta^\alpha := \star \vartheta^\alpha$, $\eta^{\alpha \beta} := \star \vartheta^{\alpha \beta}$, etc., see also [73]. Parentheses surrounding
indices $(\alpha \beta) := (\alpha \beta + \beta \alpha)/2$ denote symmetrization and brackets $[\alpha \beta] := (\alpha \beta - \beta \alpha)/2$
antisymmetrization.

**II. METRIC-AFFINE GEOMETRY OF SPACETIME**

Spacetime is a 4-dimensional differentiable manifold that is equipped with a metric and
a linear (also known as affine) connection (see [69]). The metric is of Lorentzian signature
$(-+++)$ and is given by

$$g = g_{\alpha \beta} \vartheta^\alpha \otimes \vartheta^\beta. \tag{5}$$

In general, the coframe is left arbitrary, sometimes it is convenient to choose it orthonormal:
$g = o_{\alpha \beta} \vartheta^\alpha \otimes \vartheta^\beta$, with $o_{\alpha \beta} = \text{diag}(-1,+1,+1,+1)$. The linear connection $\Gamma_{\alpha \beta}^i$
governs parallel transfer and allows to define a covariant exterior derivative $D$ (we drop now the $\Gamma$
on top of $D$ for convenience). We can decompose $\Gamma_{\alpha \beta}^i$ with respect to a natural frame:

$$\Gamma_{\alpha \beta}^i = \Gamma_{ia}^\beta dx^i. \tag{6}$$

It has apparently 64 independent components. The notion of a non-trivial linear connection
is decisive in going beyond the (flat) Minkowskian spacetime. To quote Einstein [16]: “... the
essential achievement of general relativity, namely to overcome ‘rigid’ space (i.e., the inertial
frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the ‘displacement field’ ($\Gamma^l_{ik}$), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (i.e., the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of ‘rigid’ space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular $\Gamma$ field can be deduced from a Riemannian metric . . .

The metric $g_{\alpha\beta}$ induces a Riemannian (or Levi-Civita) connection 1-form $\tilde{\Gamma}_\beta^\alpha$. In holonomic coordinates, it reads

$$\tilde{\Gamma}^j_i = \tilde{\Gamma}^j_{ki} dx^k, \quad \tilde{\Gamma}^j_{ki} := \frac{1}{2} g^{jl} (\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}).$$  

(7)

The post-Riemannian part of the connection, the distortion

$$N^\beta_\alpha := \Gamma^\beta_\alpha - \tilde{\Gamma}^\beta_\alpha,$$  

(8)

is a tensor-valued 1-form. Its 64 independent components describes the deviation from Riemannian geometry. In Einstein’s theory of gravity, general relativity (GR), spacetime is Riemannian, that is, the distortion $N_{\alpha\beta}$ vanishes.

| Potential | Field strength | Bianchi identity |
|-----------|----------------|-----------------|
| metric $g_{\alpha\beta}$ | nonmetricity $Q_{\alpha\beta} = -Dg_{\alpha\beta}$ | zeroth $DQ_{\alpha\beta} = 2R(\alpha^\mu g_{\beta\mu})$ |
| coframe $\vartheta^\alpha$ | torsion $T^\alpha = D\vartheta^\alpha$ | first $DT^\alpha = R^\alpha_\mu \wedge \vartheta^\mu$ |
| connection $\Gamma^\beta_\alpha$ | curvature $R^\beta_\alpha = d\Gamma^\beta_\alpha - \Gamma^\beta_\mu \wedge \Gamma^\mu_\beta$ | second $DR^\beta_\alpha = 0$ |

There are two other measures for the deviation of a connection from its Levi-Civita part: The nonmetricity $Q_{\alpha\beta}$ of (2) and the torsion

$$T^\alpha := D\vartheta^\alpha,$$  

(9)

see Table I. If one develops the covariant exterior derivative on the right-hand-side of (2) and substitutes the torsion of (9) suitably, then, after some algebra, one finds for the distortion
explicitly
\[ N_{\alpha\beta} = -e_{[\alpha} T_{\beta]} + \frac{1}{2}(e_{\alpha} e_{\beta} T_{\gamma}) \, \vartheta^\gamma + (e_{[\alpha} Q_{\beta]\gamma}) \, \vartheta^\gamma + \frac{1}{2} Q_{\alpha\beta}. \] (10)
Nonmetricity and torsion can be recovered from \( N_{\alpha\beta} \) straightforwardly:
\[ Q_{\alpha\beta} = 2 N_{(\alpha\beta)}, \quad T^\alpha = N^{\alpha}_{\beta} \wedge \vartheta^\beta. \] (11)
We call the (negative of the) torsion dependent piece of \( N_{\alpha\beta} \) the *contortion* 1-form \( K_{\alpha\beta} \).
Like the torsion \( T^\alpha \), it has 24 independent components, and we have \( T^\alpha = -K^{\alpha}_{\beta} \wedge \vartheta^\beta \).

The torsion has 3 irreducible pieces (see [30]), its totally antisymmetric piece (computer name \texttt{axitor}, 4 components),
\[ (3)T^\alpha := -\frac{1}{3} e_{\alpha} \left( \vartheta^\beta \wedge T_{\beta} \right), \] (12)
its trace (\texttt{trator}, 4 components)
\[ (2)T^\alpha := \frac{1}{3} \vartheta^\alpha \wedge T \quad \text{with} \quad T := e_{\alpha} [T^\alpha], \] (13)
and its tensor piece (\texttt{tentor}, 16 components)
\[ (1)T^\alpha := T^\alpha - (2)T^\alpha - (3)T^\alpha. \] (14)

For the Riemannian spacetime of GR, we have \( T^\alpha = 0 \) and \( Q_{\alpha\beta} = 0 \). If we relax the former constraint, \( T^\alpha \neq 0 \), we arrive at the Riemann-Cartan (RC) spacetime of the viable Einstein-Cartan theory of gravity, see [9, 27, 52, 75]. Since still \( Q_{\alpha\beta} = 0 \), such a RC-spacetime carries a metric-compatible connection and, accordingly, a length is invariant under parallel displacement. By the same token, a RC-spacetime is locally Lorentz invariant.

Haugan and Lämmerzahl [28], see also [44], argue that the presence of a torsion of spacetime violates local Lorentz invariance. Similarly, Kostelecky [41] and Bluhm and Kostelecky [11] attempt to violate Lorentz invariance already on the level of a RC-spacetime. According to our point of view, it is more natural to have a *non*vanishing nonmetricity under such circumstances. Abandoning Lorentz invariance suggests the presence of nonmetricity, i.e., \( Q_{\alpha\beta} \neq 0 \). This is what we assume for the rest of our article.

A broad overview over the subject of violating Lorentz invariance has been given by Bluhm [10], see also the references given there. In experiment and in observation [45, 51, 55, 74] there is presently no evidence for Lorentz violations. Nevertheless, from a theoretical point of view (string theory, quantum gravity) a violation of Lorentz invariance is expected at some level, for certain models, see [2, 3], e.g..
### III. SOME PROPERTIES OF THE NONMETRICITY

In the presence of nonmetricity $Q_{\alpha\beta} = -Dg_{\alpha\beta} \neq 0$, let us parallelly transport two vectors $u$ and $v$ from a point $P$ along a curve with tangent vector $c$ to a neighboring point $Q$. The scalar product of the two vectors $g(u, v)$ will change according to the Lie derivative $\mathcal{L}_c g(u, v)$. With the gauge covariant Lie derivative of a form $\psi$ (see [30, 37, 73])

$$L_c \psi := c \lrcorner D\psi + D(c \lrcorner \psi),$$

we have

$$L_c g(u, v) = (L_c g(u, v))_{\alpha\beta} u^\alpha v^\beta = (Dg_{\alpha\beta})_{\alpha\beta} u^\alpha v^\beta,$$  \hspace{1cm} (15)

since $L_c u^\alpha = 0$ and $L_c v^\alpha = 0$ because of parallel transfer. Thus,

$$L_c g(u, v) = -(c \lrcorner Q_{\alpha\beta}) u^\alpha v^\beta = -c \left[ Q_{\alpha\beta} u^\alpha v^\beta + Q g(u, v) \right],$$  \hspace{1cm} (16)

where

$$Q := \frac{1}{4} Q^\alpha, \quad Q_{\alpha\beta} := Q_{\alpha\beta} - Q g_{\alpha\beta}$$  \hspace{1cm} (17)

are the trace (Weyl covector) and the traceless (shear) part of the nonmetricity. In the case of vanishing shear, the scalar product just changes by a factor and the light-cone is left intact under parallel transfer. Otherwise, with shear $Q_{\alpha\beta}$, the angle between the vectors $u$ and $v$ does change. Hence, by admitting nonmetricity, we dissolve the local Lorentz invariance of a Riemann-Cartan spacetime. We depicted in Fig.1 how absolute parallelism, length, and angles are successively abandoned.

As we displayed in (3), the nonmetricity can be decomposed into four pieces. We have to recapitulate some of these features. In four dimensions, as symmetric tensor-valued 1-form, the nonmetricity has 40 independent components. Two vector-like pieces can be easily identified. Firstly, the Weyl covector $Q^\alpha$ can be extracted by taking the trace of $Q_{\alpha\beta}$. The remaining tracefree part of the nonmetricity $Q'_{\alpha\beta}$ in (17) contains a second vector-like piece represented by the 1-form $\Lambda$:

$$\Lambda := (\epsilon^\beta \lrcorner Q_{\alpha\beta}) \wedge \vartheta^\alpha.$$  \hspace{1cm} (18)

The 2-form$^1$

$$P_\alpha := Q_{\alpha\beta} \wedge \vartheta^\beta - \frac{1}{3} \vartheta_\alpha \wedge \Lambda,$$  \hspace{1cm} (19)

$^1$ For $n$ dimensions, we have $P_\alpha := Q_{\alpha\beta} \wedge \vartheta^\beta - \frac{1}{n-1} \vartheta_\alpha \wedge \Lambda$. In [30] we introduced a 2-form $\Omega_\alpha$ instead that is related to $P_\alpha$ as follows: $P_\alpha = (-1)^{\text{Ind}(g)} \ast \Omega_\alpha$. 

FIG. 1: Two vectors at a point $P$ span a triangle. If we parallelly transfer both vectors around a closed loop back to $P$, then in the course of the round trip the triangle gets linearly transformed.

\[
\begin{array}{c}
(L_n,g) \xrightarrow{\text{Metric-affine}} \text{rotation} \oplus \text{dilation} \oplus \text{shear} \\
Y_n \xrightarrow{\text{Weyl-Cartan}} \text{rotation} \oplus \text{dilation} \\
U_n \xrightarrow{\text{Riemann-Cartan}} \text{rotation} \\
E_n \xrightarrow{\text{Euclid}} \text{identity}
\end{array}
\]

with the properties

\[
P^\alpha \land \vartheta_\alpha = 0, \quad e_\alpha \rbracket P^\alpha = 0, \quad (20)
\]

turns out to be related to a further irreducible piece of $Q_{\alpha\beta}$. We have (see [30])

\[
\begin{align}
(4)\ Q_{\alpha\beta} &:= Q g_{\alpha\beta}, \\
(3)\ Q_{\alpha\beta} &:= \frac{4}{9} \left( \vartheta_\alpha e_\beta \rbracket - \frac{1}{4} g_{\alpha\beta} \right) \Lambda, \\
(2)\ Q_{\alpha\beta} &:= -\frac{2}{3} e_\alpha \rbracket P_\beta, \\
(1)\ Q_{\alpha\beta} &:= Q_{\alpha\beta} - (2)\ Q_{\alpha\beta} - (3)\ Q_{\alpha\beta} - (4)\ Q_{\alpha\beta}. 
\end{align}
\]

As we want to relate the aether vector field $u$ of J&M to the nonmetricity, it is obvious that $(4)\ Q_{\alpha\beta} \sim Q$ and $(3)\ Q_{\alpha\beta} \sim \Lambda$ are the objects of our main interest.

The nonmetricity $Q_{\alpha\beta} = Q_{\alpha\beta} dx^i$, from a geometrical point of view, can be understood as a strain measure for the different directions specified by the 1-forms $dx^i$. In accordance with what we stated above, $[\underline{12}]$ defines a shear measure $Q_{\alpha\beta}$ since the dilation measure $Q$
is subtracted out. It is then immediately clear that \( Q_{\alpha\beta} \sim Q \) is related to \textit{dilations} and \( Q_{\alpha\beta} \sim \Lambda \) to \textit{shears}. Therefore, generically it is the 1-form \( \Lambda \) that is related to the aether vector \( u \). However, we will keep also the Weyl 1-form \( Q \) since tentatively it seems to be related to the constraint \( u^2 = 1 \) of J&M.

Later, for the prolongation of previously known solutions, we need the expression \( Q_{\alpha\beta} \wedge \vartheta^\beta \). It is useful to express it also in terms of \( Q, \Lambda \), and \( P_\alpha \). If we substitute the irreducible decomposition (3) into \( Q_{\alpha\beta} \wedge \vartheta^\beta \) and remember (21) to (24), we find

\[
Q_{\alpha\beta} \wedge \vartheta^\beta = P_\alpha - \frac{1}{3} (\Lambda - 3Q) \wedge \vartheta_\alpha .
\] (25)

Note that \( Q_{\alpha\beta} \wedge \vartheta^\beta = 0 \), whereas the other irreducible pieces contribute.

As we saw in (4), we need “massive” terms in the nonmetricity for the construction of a J&M type Lagrangian. Since \( *Q_{\alpha\beta} = g_{\alpha\beta} *Q \), we find straightforwardly

\[
(4) Q_{\alpha\beta} \wedge * (4) Q^{\alpha\beta} = 4Q \wedge *Q .
\] (26)

In the case of (3)\( Q_{\alpha\beta} \), things are a bit more complicated. However, if we use the formulas \( \vartheta^\alpha \wedge * \vartheta_\beta = \delta^\alpha_\beta \eta \) and \( \eta = *1 \), then, after some algebra, we arrive at the same type of formula

\[
(3) Q_{\alpha\beta} \wedge * (3) Q^{\alpha\beta} = \frac{4}{9} \Lambda \wedge *\Lambda .
\] (27)

We find the simplicity of (26) and (27) remarkable.

### IV. CURVATURE

The standard definition of the curvature 2-form \( R_{\alpha\beta} \) is given in Table I. The Riemannian curvature (of GR) we denote by \( \widetilde{R}_{\alpha\beta} \); of course, \( \widetilde{R}_{\alpha\beta} = -\widetilde{R}_{\beta\alpha} \). The curvature \( R_{\alpha\beta} \) can be decomposed into its symmetric and antisymmetric part according to

\[
R_{\alpha\beta} = W_{\alpha\beta} + Z_{\alpha\beta} , \quad \text{with} \quad W_{\alpha\beta} := R_{[\alpha\beta]} , \quad Z_{\alpha\beta} := R_{(\alpha\beta)} .
\] (28)

The “rotational” curvature \( W_{\alpha\beta} \) characterizes a RC-space, whereas the “strain” curvature \( Z_{\alpha\beta} \) only emerges if nonmetricity is admitted. Therefore, the investigation of the “Lorentz violating” curvature \( Z_{\alpha\beta} \) is of central importance to our paper.

However, in order to link up our theory to GR and to the Einstein-Cartan theory with its RC-spacetime, we have to take a look at the rotational curvature \( W_{\alpha\beta} \) as well. It can be
decomposed into six irreducible pieces:

\[ W_{\alpha\beta} = (1)W_{\alpha\beta} \oplus (2)W_{\alpha\beta} \oplus (3)W_{\alpha\beta} \oplus (4)W_{\alpha\beta} \oplus (5)W_{\alpha\beta} \oplus (6)W_{\alpha\beta} \]

\[ = \text{weyl} \oplus \text{paircom} \oplus \text{pscalar} \oplus \text{ricsymf} \oplus \text{ricanti} \oplus \text{scalar}, \]

\[ 36 = 10 \oplus 9 \oplus 1 \oplus 9 \oplus 6 \oplus 1. \]

The names are those that we use in our computer algebra programs for the decomposition of \( W_{\alpha\beta} \). In a RC-space, all six pieces in (29) are nonvanishing in general. If torsion \( T^\alpha = 0 \), then, as can be seen from the first Bianchi identity in Table I, paircom, pscalar, and ricanti vanish and we are left with the three pieces known from GR: The Weyl tensor weyl, the symmetric tracefree Ricci tensor ricsymf, and the curvature scalar scalar. Thereby the curvature reduces from the 36 independent components in a RC-space to 20 independent components in a Riemannian space, a result well-known from GR.

Let us come back to the strain curvature \( Z_{\alpha\beta} = Z_{\beta\alpha} \). Obviously, it has one distinctive piece, namely its trace \( Z := g^{\alpha\beta} Z_{\alpha\beta} = Z_\gamma^\gamma \). It should be noted that \( Z \) is related to a premetric quantity. In a space in which only a linear connection is specified, the curvature \( R_{\alpha}^\beta \) can be contracted, \( R_\gamma^\gamma \), even if a metric is not present. Thus \( R_\gamma^\gamma \) and, as a consequence, also \( Z \) is rightfully called dilcurv, the part of the curvature related to dil(at)ations. This is an irreducible piece of \( Z_{\alpha\beta} \) and we call it

\[ ^{(4)}Z_{\alpha\beta} := \frac{1}{4} g_{\alpha\beta} Z. \] (30)

Since on the level of the nonmetricity dilations are related to \( ^{(4)}Q_{\alpha\beta} \), we denoted the related curvature piece by the same number. In fact, the zeroth Bianchi identity in Table I, if contracted, yields \( g^{\alpha\beta} DQ_{\alpha\beta} = 2Z_\gamma^\gamma = Z \). By partial integration, we find

\[ Z = 2dQ \quad \text{or} \quad ^{(4)}Z_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} dQ = \frac{1}{2} \left( D^{(4)}Q_{\alpha\beta} + Q_{\alpha\beta} \wedge Q \right). \] (31)

Apparently, to the field strength \( Z \), that is, to dilcurv, there belongs the potential \( Q \), the Weyl covector.\(^2\)

The tracefree part of the strain curvature

\[ Z_{\alpha\beta} := Z_{\alpha\beta} - \frac{1}{4} Z g_{\alpha\beta} \] (32)

---

\( ^2 \) Let us recall, it was dilcurv that Weyl used in his unsuccessful unified field theory of 1918 to describe the electromagnetic field strength \( F \) — and \( Q \) was meant to be the electromagnetic potential \( A \), see Goenner and O’Raifeartaigh.
represents the shear curvature. In terms of $Z_{\alpha\beta}$, a further decomposition of $Z_{\alpha\beta}$ is possible. We have collected the results in Appendix A. We find a decomposition of $Z_{\alpha\beta}$ into five irreducible pieces:

$$Z_{\alpha\beta} = (1)Z_{\alpha\beta} \oplus (2)Z_{\alpha\beta} \oplus (3)Z_{\alpha\beta} \oplus (4)Z_{\alpha\beta} \oplus (5)Z_{\alpha\beta}, \quad (33)$$

$$60 = 30 \oplus 9 \oplus 6 \oplus 6 \oplus 9.$$

Alerted by our results with respect to the nonmetricity, we expect that $(3)Z_{\alpha\beta}$ with its six independent components may be related to $(3)Q_{\alpha\beta}$ in a similar way as $(4)Z_{\alpha\beta}$ is to $(4)Q_{\alpha\beta}$, see (31). Symbolically, we expect

$$\text{Weyl 1-form } Q \sim (4)Q \xrightarrow{d} (4)Z \sim dQ, \quad (34)$$

$$\text{shear 1-form } \Lambda \sim (3)Q \xrightarrow{d} (3)Z \sim d\Lambda. \quad (35)$$

The rigorous form of relation (34) was already presented in (31). What about (35)? Well, life is a bit more complicated than (35) suggests. If we take the definition of $(3)Z_{\alpha\beta}$ from (A3) and the zeroth Bianchi identity $DQ_{\alpha\beta} = 2Z_{\alpha\beta}$ from Table I, then, after some light algebra, we find

$$(3)Z_{\alpha\beta} = \frac{1}{3} (2 \vartheta_{(\alpha} \land e_{\beta)} - g_{\alpha\beta}) \hat{\Delta}, \quad (36)$$

with

$$\hat{\Delta} = \frac{1}{4} \vartheta^\alpha \land e_\beta \left( DQ_{\alpha\beta} - g_{\alpha\beta} DQ \right) = \frac{1}{4} \vartheta^\alpha \land e_\beta \left( DQ_{\alpha\beta} + Q \land Q_{\alpha\beta} \right). \quad (37)$$

Apparently, Eq. (37) turns out to be more complicated than we guessed in (35). We can only hope to find a formula of type (35), if we forbid certain pieces of the connection to occur.

In Appendix 13 we will show that under the conditions

$$(2)Q_{\alpha\beta} = 0, \quad (1)T^\alpha = (3)T^\alpha = 0, \quad (38)$$

we have, see (B15) and (36),

$$\hat{\Delta} = \frac{1}{6} d\Lambda \text{ or } (3)Z_{\alpha\beta} = \frac{1}{18} \left( 2 \vartheta_{(\alpha} \land e_{\beta)} - g_{\alpha\beta} \right) d\Lambda. \quad (39)$$

Accordingly (35) turns out to be correct after all, provided the conditions (38) are met. Thus, modulo the conditions (38), our goal is reached of constructing a gauge Lagrangian à la (4) in terms of $\Lambda$ and $Q$. For $(4)Q_{\alpha\beta}$ we have, see (30) and (31),

$$(4)Z_{\alpha\beta} \land * (4)Z^{\alpha\beta} = \frac{1}{4} Z \land * Z = dQ \land * dQ, \quad (40)$$
and for \((3)Q_{\alpha\beta}\), using \((C1)\) and \((39)\),

\[
(3)Z_{\alpha\beta} \wedge * (3)Z^{\alpha\beta} = \frac{4}{3} \dot{\Delta} \wedge * \dot{\Delta} = \frac{1}{27} d\Delta \wedge * d\Delta.
\] (41)

V. CURRENTS AND FIELD EQUATIONS OF MAG, COMPARISON WITH J&M

Similar as J&M [37], we assume gravity and aether to exist, and then we couple them minimally to certain matter fields \(\Psi\). However, in our case the gauge potentials \((g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha^\beta)\) represent ordinary gravity \((g_{\alpha\beta}, \vartheta^\alpha)\) as well as the image \((\Gamma_\alpha^\beta)\) of the J&M aether, and they are both part of the geometry of spacetime. The total first-order Lagrangian reads

\[
L_{\text{tot}} = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha^\beta) + L(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi).
\] (42)

The independent variables of the action principle are \(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha^\beta\), and \(\Psi\). The variation of the matter Lagrangian

\[
\delta L = \frac{1}{2} \delta g_{\alpha\beta} \sigma^{\alpha\beta} + \delta \vartheta^\alpha \wedge \Sigma_\alpha + \delta \Gamma_\alpha^\beta \wedge \Delta^\alpha_\beta + \delta \Psi \wedge \frac{\delta L}{\delta \Psi}
\] (43)

allows us identify the material currents coupled to the potentials as metric and canonical energy-momentum and as hypermomentum, respectively: \((\sigma_{\alpha\beta}, \Sigma_\alpha, \Delta^\alpha_\beta)\). The energy-momenta \(\sigma_{\alpha\beta}\) and \(\Sigma_\alpha\) are related to each other by a Belinfante-Rosenfeld type of relation. The hypermomentum splits in spin current \(\oplus\) dilation current \(\oplus\) shear current:

\[
\Delta_{\alpha\beta} = \tau_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \Delta^\gamma_\gamma + \hat{\Delta}^\alpha_{\alpha\beta}, \quad \tau_{\alpha\beta} = -\tau_{\beta\alpha}, \quad \hat{\Delta}^\alpha_{\alpha\beta} = \hat{\Delta}^\beta_{\beta\alpha}, \quad \hat{\Delta}^\gamma_\gamma = 0.
\] (44)

The hypothetical shear current \(\hat{\Delta}^\alpha_{\alpha\beta}\) is discussed in [31, 56], see also the literature given there.

Our strategy is to leave open the explicit form of the gauge Lagrangian \(V\) for the time being and to introduce the excitations (or field momenta) of the gauge fields instead:

\[
M^{\alpha\beta} = -2 \frac{\partial V}{\partial Q_{\alpha\beta}}, \quad H_\alpha = -\frac{\partial V}{\partial T^\alpha}, \quad H^{\alpha}_\beta = -\frac{\partial V}{\partial R_\alpha^\beta}.
\] (45)

These three constitutive laws, expressing the excitations in terms of the field strengths, characterize the physical properties of the spacetime continuum under consideration.
The field equations read

\[ DM^{\alpha \beta} - m^{\alpha \beta} = \sigma^{\alpha \beta} \quad \text{(zeroth)}, \quad (46) \]
\[ DH_\alpha - E_\alpha = \Sigma_\alpha \quad \text{(first)}, \quad (47) \]
\[ DH^{\alpha \beta} - E^{\alpha \beta} = \Delta^{\alpha \beta} \quad \text{(second)}, \quad (48) \]
\[ \frac{\delta L}{\delta \Psi} = 0 \quad \text{(matter)}. \quad (49) \]

If the second field equation (48) is fulfilled, then either the zeroth field equation (46) or the first one (47) is redundant due to some Noether identities. Hence we need only to consider (46), (48), (49) or (47), (48), (49).

On the right-hand-side of each of the gauge field equations (46) to (48) we have a material current, on the left-hand-side first the Yang-Mills type term “derivative of excitation” minus, as second term, a gauge current that, together with the material current, features as source of the corresponding gauge field. The gauge currents turn out to be the metrical energy-momentum of the gauge fields

\[ m^{\alpha \beta} := 2 \frac{\partial V}{\partial g_{\alpha \beta}} = \vartheta^{(\alpha} \wedge E^{\beta)} + Q^{(\alpha} \gamma \wedge M^{\beta)\gamma} - T^{(\alpha} \wedge H^{\beta)} - R_{\gamma}^{\alpha} \wedge H^{[\gamma|\beta]} + R^{(\alpha|\gamma) \wedge H^{\beta)\gamma}, \quad (50) \]

the canonical energy-momentum of the gauge fields\(^3\)

\[ E_\alpha := \frac{\partial V}{\partial \vartheta_\alpha} = e_\alpha \wedge V + (e_\alpha \wedge T^{\beta}) \wedge H_\beta + (e_\alpha \wedge R_{\beta\gamma}) \wedge H^{\beta}_{\gamma} + \frac{1}{2} (e_\alpha \wedge Q_{\beta\gamma}) M^{\beta\gamma}, \quad (51) \]

and the hypermomentum of the gauge fields

\[ E^{\alpha \beta} := \frac{\partial V}{\partial \Gamma^{\alpha \beta}} = -\vartheta^{\alpha} \wedge H_{\beta} - g_{\beta\gamma} M^{\alpha\gamma}, \quad (52) \]

respectively.

Like J&M, we will concentrate on the sourcefree region, that is, we assume that the material currents vanish. We discussed them here in order to find the physical interpretations of the gauge currents \(m^{\alpha \beta}, E_\alpha,\) and \(E^{\alpha \beta}\). Since J&M consider a symmetric energy-momentum in their theory, we consider the sourcefree zeroth and second field equations:

\[ DM^{\alpha \beta} = m^{\alpha \beta}, \quad (53) \]
\[ DH^{\alpha \beta} = E^{\alpha \beta}. \quad (54) \]

\(^3\) For the relations between different energy-momentum currents in gravitational theory one should also compare Itin [35].
These are the two field equations that underlie our model of the J&M theory. The rest of the paper will be devoted to making them explicit and for finding exact solutions of them.

The field equation (53) is of an Einsteinian type. If the gauge Lagrangian $V$ depends on a Hilbert-Einstein term $R_{sc}$, inter alia (curvature scalar), then, within $m^{\alpha\beta}$, the Einstein 3-form emerges. The left hand side of (53) should depend on the shear 1-form $\Lambda$. This can be achieved by putting $M^{\alpha\beta} = -2\partial V/\partial Q_{\alpha\beta}$ proportional to $(3)Q^{\alpha\beta}$, see (22). As a consequence, the Lagrangian $V$ carries a quadratic $(3)Q$ piece and we expect $V \sim R_{sc} + (3)Q^2$. Under these circumstances, Eq.(53) is the analogue of (37), Eq.(5). Both equations have 10 independent components.4

Our second field equation (54), with the gauge hypermomentum as source, has 64 independent components. For this reason, as we argued above, we have to kill all components apart from those 4 components related to the shear 1-form $\Lambda \sim (3)Q_{\alpha\beta}$. At a first glance, this seems to be an impossible task. A little reflection shows that the situation is not hopeless at all. Substitute (52) into (54):

$$DH^{\alpha\beta} = -\partial^\alpha \wedge H_\beta - M^{\alpha\beta}.$$ \hspace{1cm} (55)

According to the the last paragraph, we have $M^{\alpha\beta} \sim (3)Q^{\alpha\beta}$, that is, $M^{\alpha\beta}$ on the right hand side of (55) depends on 4 independent components. The term with $H^\alpha = -\partial V/\partial T^\alpha$ can be chosen to vanish by forbidding explicit torsion dependent terms in the Lagrangian.

Left over for discussion is $H^{\alpha\beta} = -\partial V/\partial R_{\alpha\beta}$ on the left hand side of (55). Clearly we want this term to depend in an essential way on the shear 1-form $\Lambda$. A look at (41) convinces us to take $H^{\alpha\beta} \sim (3)Z^{\alpha\beta} \sim d\Lambda$. Then the left hand side of (55) becomes a wave type expression $\sim \Box \Lambda$ with four essential components. Hence our equation reduces to just four components, like the corresponding J&M equation. Thus, our second field equation (54) is undoubtedly the analogue of (37), Eq.(4). At the same time it is also clear that we could introduce a richer aether structure than the one J&M studied by means of their vector field $u$.

4 In our approach we don’t find the analogue of the $\nabla(Ju)$ terms in the first line of the aether stress tensor of J&M (37), Eq.(9). The reason is clear. In the J&M aether these terms arise, see (17), p.2, “from varying the metric dependence of the connection.” However, in our case the connection is an independent variable. Incidentally, an aether stress tensor that depends, as in the J&M theory, on second derivatives of the field variable is not particularly plausible anyway.
If we collect our heuristic arguments for constructing the MAG analogue of the J&M aether, then we arrive at

\[ V_{\text{J&M}} \sim \frac{1}{\kappa} (R_{\text{sc}} + (3)Q^2) + \frac{1}{\rho} (3)Z^2, \]  

(56)

with \( \kappa \) as gravitational constant and \( \rho \) as a dimensionless coupling constant. This first toy Lagrangian should be compared with [37], Eq.(1). Our massive term \( (3)Q^2 \) resembles the constraint piece in the J&M Lagrangian, whereas

\[ (3)Z^2 \sim d\Lambda \wedge *d\Lambda \sim g^{\gamma[a} g^{\beta]\delta} (\partial_{a}\Lambda_{\beta}) (\partial_{\gamma}\Lambda_{\delta}) \eta \]  

(57)

is the analogue of \( K^{ab}_{\text{mm}} \nabla_{a}u^{m}\nabla_{b}u^{n} \). However, J&M have, in \( K^{ab}_{\text{mm}} \), four open constants \( c_1, c_2, c_3, c_4 \). In the specific model investigated below, we concentrate on the simple Maxwell-type kinetic term (57). In doing so, we seem closer to Kostelecky and Samuel [42] than to J&M. However, the general MAG Lagrangian, see Appendix D, Eq.(D1), encompasses kinetic terms of all 11 irreducible pieces of the curvature and thereby generalizes the 4 parameters of J&M considerably.

Let us look back to our first ansatz for an aether Lagrangian in Eq.(4). There we had derivative terms of the nonmetricity. However, such terms are not allowed in first order MAG, see the gauge Lagrangian \( V \) in (42). Only an algebraic dependency of the field strengths \( Q_{\alpha\beta}, T^\alpha, R_{\alpha\beta} \), mostly quadratic for dimensional reasons, is allowed. Nevertheless, by the zeroth Bianchi identity, see Table I, the derivatives can be removed and transformed to terms algebraic in the curvature; this is at least possible for \( (3)Q_{\alpha\beta} \) and \( (4)Q_{\alpha\beta} \). Thus also the Lagrangian (4) falls into the category of allowed Lagrangians within MAG. At the same time we see again how closely nonmetricity and shear curvature are interwoven.

We would like to stress that the formalism of MAG that we developed in this paper up to now is exact and free of any hand waving arguments. It is a first order Lagrange-Noether gauge formalism of non-Abelian nature and our gauge field equations (53) and (54) are coupled nonlinear partial differential equations of second order in the gauge potentials \( g_{\alpha\beta}, \Gamma_{\alpha\beta} \). The only hand waving is involved in the explicit choice of the gauge Lagrangian \( V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_{\alpha\beta}) \) in (42). Since we want to develop a model that is an image of the J&M theory, we have to hand pick a suitable \( V \). In (56) we made a first attempt.
VI. A LAGRANGIAN FOR GRAVITY AND AETHER

In gauge theories the Lagrangian is assumed to be quadratic in the field strengths, in our case in $R_{\alpha\beta}$, $T^\alpha$, and $Q_{\alpha\beta}$. The most general parity conserving Lagrangian of such a type has been displayed in Appendix D. For modeling the J&M aether theory, we don’t need this very complicated expression in its full generality. Nevertheless, let us look at its basic structure:

$$V_{\text{MAG}} \sim \frac{1}{\kappa} (R_{\text{sc}} + \lambda_0 + T^2 + TQ + Q^2) + \frac{1}{\rho} (W^2 + Z^2).$$

(58)

All indices are suppressed. The expression in the first parentheses describes $(g_{\alpha\beta}, \vartheta^\alpha)$-gravity of the Newton-Einstein type, including a cosmological term with $\lambda_0$. This “weak” gravity is governed by the conventional gravitational constant $\kappa$. If only these terms are present, the propagation of $\Gamma_{\alpha\beta}$ is inhibited. Then, in addition to conventional gravity, only new contact interactions emerge that are glued to matter. The viable Einstein-Cartan theory with its spin-spin contact interaction is an example.

If one desires to make the connection $\Gamma_{\alpha\beta}$ propagating, see [43, 70] for vanishing and [4, 14, 20, 80, 81, 82, 83] for nonvanishing nonmetricity, then one has to allow for curvature-square pieces $W^2$ and $Z^2$, as shown in (58). This “strong gravity”, the potential of which is $\Gamma_{\alpha\beta}$, is governed by a new dimensionless coupling constant $\rho$. Our hypothesis is that such a universal strong gravitational interaction is present in nature.

The J&M aether, in our interpretation, allows at least the $(3)Q_{\alpha\beta}$ piece of the connection to propagate, see the $(3)Z^2$ piece in the ansatz (56). We would like to stick to (56) as closely as possible. Since we search for an exact spherically symmetric solution of our model to be defined, we need to be flexible in the exact choice of the Lagrangian. After some computer algebra experiments, we came up with the following toy Lagrangian that we are going to investigate in the context of the J&M aether theory:

$$V = \frac{1}{2\kappa} \left[ -a_0 \left( R^{\alpha\beta} \wedge \eta_{\alpha\beta} + 2\lambda_0 \eta \right) + Q_{\alpha\beta} \wedge^* \left( b_1 (1) Q^{\alpha\beta} + b_3 (3) Q^{\alpha\beta} \right) \right] - \frac{z_3}{2\rho} R^{\alpha\beta} \wedge^* (3) Z_{\alpha\beta}.$$  

(59)

We have the following constants of order unity: $a_0, b_1, b_3, z_3$. As compared to (56), we have no torsion piece. However, we added in a $(1)Q^{\alpha\beta}$ piece. The most general quadratic Lagrangian (D1) is appreciably more complicated than (59). Nevertheless, without our computer algebra programs (see [71], [33], [32]) we would not have been able to handle the messy expressions.
Once the Lagrangian is specified, it is simple to calculate the gauge excitations by partial differentiation of (59) with respect to $Q_{\alpha\beta}$, $T^\alpha$, and $R_{\alpha}^\beta$:

$$M^\alpha_{\beta} = -\frac{2}{\kappa} \star (b_1 Q_{\alpha\beta} + b_3 Q_{\alpha\beta}) ,$$

$$H_\alpha = 0 ,$$

$$H^\alpha_{\beta} = \frac{a_0}{2\kappa} \eta^\alpha_{\beta} + \frac{z_3}{\rho} \star (3) Z^\alpha_{\beta} .$$

These excitations have to be substituted into the field equations (53), (54) and into the gauge currents (50), (51), (52), respectively.

**VII. A SIMPLE SPHERICALLY SYMMETRIC SOLUTION OF MAG**

We look for exact spherically symmetric solutions of the field equations belonging to the Lagrangian (59). For this purpose, the coframe $\vartheta^\alpha$ is assumed to be of Schwarzschild-deSitter (or Kottler) form,

$$\vartheta^0 = e^{\mu(r)} dt , \quad \vartheta^1 = e^{-\mu(r)} dr , \quad \vartheta^2 = r d\theta , \quad \vartheta^3 = r \sin \theta d\phi ,$$

with the function

$$e^{2\mu(r)} = 1 - \frac{2m}{r} - \frac{\lambda_0}{3} r^2 .$$

We use Schwarzschild coordinates $x^i = (t, r, \theta, \phi)$. Since the coframe is assumed to be orthonormal, the metric reads

$$g = -\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3 .$$

The nonmetricity $Q_{\alpha\beta}$ is given by

$$Q_{\alpha\beta} = \frac{\ell_0 e^{-\mu(r)}}{2r^2} \left( \begin{array}{cccc} \vartheta^1 & 0 & 0 & 0 \\ 0 & \vartheta^2 & \vartheta^3 \\ 0 & \vartheta^2 & 0 & 0 \\ 0 & \vartheta^3 & 0 & 0 \end{array} \right) + \frac{\ell_1 e^{-\mu(r)}}{2r^2} \left( \begin{array}{cccc} 3\vartheta^0 & 0 & -\vartheta^2 & -\vartheta^3 \\ 0 & -\vartheta^0 - 3\vartheta^1 & 0 & 0 \\ -\vartheta^2 & 0 & 0 & 0 \\ -\vartheta^3 & 0 & 0 & 0 \end{array} \right)$$

$$= \frac{e^{-\mu(r)}}{2r^2} \left( \begin{array}{cccc} 3\ell_1 \vartheta^0 + \ell_0 \vartheta^1 & 0 & -\ell_1 \vartheta^2 & -\ell_1 \vartheta^3 \\ 0 & -\ell_1 (\vartheta^0 - 3\vartheta^1) & \ell_0 \vartheta^2 & \ell_0 \vartheta^3 \\ -\ell_1 \vartheta^2 & \ell_0 \vartheta^2 & 0 & 0 \\ -\ell_1 \vartheta^3 & \ell_0 \vartheta^3 & 0 & 0 \end{array} \right) .$$

(66)
The integration constants \( \ell_0 \) and \( \ell_1 \) can be interpreted as a measure for the violation of Lorentz invariance. According to (66), we have \((2)\) \( Q^{\alpha\beta} = 0 \) or \( P_\alpha = 0 \). All other irreducible pieces of \( Q_{\alpha\beta} \) are nonvanishing: \((1)\) \( Q^{\alpha\beta} \neq 0 \), \((3)\) \( Q^{\alpha\beta} \neq 0 \), \((4)\) \( Q^{\alpha\beta} \neq 0 \). In particular, we find for the shear and the Weyl 1-forms

\[
\Lambda = \frac{9e^{-\mu(r)}}{8r^2} (\ell_0 + \ell_1) \phi^1 ,
\]

\[
Q = \frac{e^{-\mu(r)}}{8r^2} \left[ -4\ell_1 \phi^0 + (3\ell_1 - \ell_0) \phi^1 \right].
\]

The torsion 2-form turns out to be

\[
T^\alpha = \ell_0 \frac{e^{-\mu(r)}}{4r^2} \begin{pmatrix}
\phi^{01} \\
0 \\
-\phi^{12} \\
-\phi^{13}
\end{pmatrix} - \ell_1 \frac{e^{-\mu(r)}}{4r^2} \begin{pmatrix}
0 \\
\phi^{01} \\
\phi^{02} \\
\phi^{03}
\end{pmatrix} \frac{e^{-\mu(r)}}{4r^2} \begin{pmatrix}
\ell_0 \phi^{01} \\
-\ell_1 \phi^{01} \\
-\ell_1 \phi^{02} - \ell_0 \phi^{12} \\
-\ell_1 \phi^{03} - \ell_0 \phi^{13}
\end{pmatrix}.
\]

As a consequence, \((1)\) \( T^\alpha = (3)\) \( T^\alpha = 0 \) and only \((2)\) \( T^\alpha \neq 0 \). By contraction of (69) with \( e^\alpha \rfloor \) we find (recall \( T = e^\alpha \rfloor T^\alpha \))

\[
T = \frac{3e^{-\mu(r)}}{4r^2} \left( \ell_1 \phi^0 + \ell_0 \phi^1 \right).
\]

The requirement that the torsion (69) and the nonmetricity (66) together with the orthonormal coframe field (63) be a solution of the field equations of the Lagrangian (59) implies some constraints on the coupling constants. If we take care of these constraints in (59), we find the truncated Lagrangian

\[
V = \frac{1}{2\kappa} \left[ -R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_0 \eta + Q_{\alpha\beta} \wedge (\frac{1}{4} (1) Q^{\alpha\beta} - \frac{1}{2} (3) Q^{\alpha\beta}) \right]
- \frac{z_3}{2\rho} \frac{(3) Z^{\alpha\beta} \wedge (3) Z_{\alpha\beta}}{2}.
\]

In other words, our exact solution (63)-(66),(69) solves the field equations of the Lagrangian (71). Note that \( z_3 \) is left arbitrary.

The explicit expressions for the strain and the rotational curvature can be found in Appendix F. The Weyl and the scalar parts of the rotational curvature \( W_{\alpha\beta} \), see (F1) and (F6), are composed of a Riemannian and a post-Riemannian piece. The other parts of \( W_{\alpha\beta} \) as well as of \( Z_{\alpha\beta} \) are purely post-Riemannian. We were surprised that \((3) Z^{\alpha\beta} = 0 \); particularly simple is \( \text{dilcurv} \sim \ell_1/(2r^3) \).

\[\text{5} \text{ According to the classification scheme of Baekler et al. [7], this solution is a special subcase of class Va.}\]
The set of the three one-forms \( \{Q, T, \Lambda\} \) are related by
\[
3Q + 2T - \Lambda = 0,
\]
as can be checked easily. In this way, the torsion 1-form is closely related to the shear and the Weyl 1-forms. This is a relation which follows from Baekler’s general prolongation ansatz [5] to solve the field equations of MAG.

Our solution looks like a superposition of two elementary solutions. For \( \ell_1 = 0 \) we find one elementary solution, a second one for \( \ell_0 = 0 \), both for the same set of coupling constants. Also this fact can be understood from the point of view of the prolongation technique.

\textbf{VIII. KILLING VECTORS AND QUASILOCAL CHARGES}

Let us determine the mass and the angular momentum of our exact solution. In a Riemannian space we call \( \xi = \xi^a e_\alpha \) a Killing vector if the latter is the generator of a symmetry transformation of the metric, i.e.,
\[
\mathcal{L}_\xi g = 0.
\]
In metric-affine space, coframe and connection are independent. Hence, Eq. (73) has to be supplemented by a corresponding requirement for the connection [30, p.83],
\[
\mathcal{L}_\xi \Gamma_\alpha^\beta = 0.
\]
These two relations can be recast into a more convenient form,
\[
e_\alpha \left[ \tilde{D} \xi_\beta \right] = 0, \quad (75)
\]
\[
D \left( e_\alpha \left[ \tilde{D} \xi_\beta \right] \right) + \xi_\alpha R_\alpha^\beta = 0, \quad (76)
\]
where \( \tilde{D} \) refers to the Riemannian part of the connection (Levi-Civita connection) and \( \tilde{D} \) to the transposed connection: \( \tilde{D} := d + \tilde{\Gamma}_\alpha^\beta := d + \Gamma_\alpha^\beta + e_\alpha \] \( T^\beta \).

For our solution the Killing vectors are the same as in case of the Schwarzschild-de Sitter metric in Riemannian spacetime,
\[
\begin{align*}
(0) & \quad \xi = \partial_t, \\
(1) & \quad \xi = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \\
(2) & \quad \xi = -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi, \\
(3) & \quad \xi = \partial_\phi.
\end{align*}
\]
Subsequently we can compute the quasilocal charges by using formulas of Nester, Chen, Tung, and Wu [13, 34, 85, 86, 87], for related work see [6, 53], e.g. The barred quantities refer to a background solution, the symbol \( \Delta \) denotes the difference between a solution and the background, \( \Delta \alpha = \alpha - \alpha \).

\[
\mathfrak{B}(N) := - \begin{cases} 
\frac{1}{2} \Delta g_{\alpha\beta} \left( N \right) \tilde{M}^{\alpha\beta} 
\end{cases}
- \begin{cases} 
\frac{1}{2} \Delta g_{\alpha\beta} \left( N \right) M^{\alpha\beta} 
\end{cases}
- \begin{cases} 
(N) \partial^\alpha \Delta H_\alpha + \Delta \partial^\alpha \wedge (N) P_\alpha 
\end{cases}
- \begin{cases} 
(N) \bar{\partial}^\alpha \Delta H_\alpha + \Delta \bar{\partial}^\alpha \wedge (N) H_\alpha 
\end{cases}
- \begin{cases} 
\bar{D}_\alpha N^\beta \Delta H^\alpha_\beta + \Delta \Gamma^\alpha_\beta \wedge (N) \bar{H}^\alpha_\beta 
\end{cases}
- \begin{cases} 
D_\alpha N^\beta \Delta H^\alpha_\beta + \Delta \Gamma^\alpha_\beta \wedge (N) H^\alpha_\beta 
\end{cases}
\] (81)

The upper (lower) line in the braces is chosen if the field strengths (momenta) are prescribed on the boundary. By taking \( N = \partial_t \) and integrating \( \mathfrak{B} \) over a 2-sphere and performing the limit \( r \to \infty \), we get the total energy. As background solution we assume our solution with \( m = 0 \) and \( \ell_0 = \ell_1 = 0 \). Similarly, by taking \( N = \partial_\phi \) we obtain the total angular momentum,

\[
E_{\infty} = \lim_{r \to \infty} \int_{S^2} \mathfrak{B}(\partial_t) = - \frac{8\pi m}{\kappa}, \tag{82}
\]
\[
L_{\infty} = \lim_{r \to \infty} \int_{S^2} \mathfrak{B}(\partial_\phi) = 0. \tag{83}
\]

**IX. REMARKS ON THE PROLONGATION TECHNIQUE**

Previously numerous exact solutions of MAG have already been found. Let us mention, as examples, the papers [29, 34, 39, 54, 59, 65, 68, 76, 77, 79, 82, 83, 85, 86] and references given there. Even possible links to observation were discussed in [1, 63, 64, 66, 67, 72]. The solution in Sec. VII was found by using prolongation methods. Such a method for MAG was proposed by Baekler et al. [5, 7]. In the sequel, we will explain how we applied the prolongation method to our case in question.

For this purpose, we start, in the framework of the Poincaré gauge theory (see [2, 27]), from a known exact solution with Schwarzschild metric and \( 1/r^2 \)-torsion in a RC-spacetime, i.e., the nonmetricity vanishes. Then, for generating nonmetricity, we make the ansatz (Weyl 1-form \( Q = Q_\alpha^\alpha/4 \))

\[
T^\alpha = \xi_0 Q^\alpha_\beta \wedge \partial^\beta + \xi_1 Q \wedge \partial^\alpha + (3)T^\alpha \\
= \xi_0 Q^\alpha_\beta \wedge \partial^\beta + (\xi_0 + \xi_1) Q \wedge \partial^\alpha + (3)T^\alpha, \tag{84}
\]

\[
24 = 16 + 4 + 4.
\]
with arbitrary constants \( \xi_0 \) and \( \xi_1 \) to be determined by the field equations. The 24 components of \( T^\alpha \) are related to the 40 components of \( Q_{\alpha\beta} \). Because of the property \( (1) Q_{\alpha\beta} \land \vartheta^\beta = 0 \), the irreducible part \( (1) Q_{\alpha\beta} \) (16 independent components) does not contribute to the torsion. Furthermore \( (Q_{\alpha\beta} \land \vartheta^\beta) \land \vartheta_\alpha = 0 \) (4 independent components). Hence, our ansatz relates the 24 components of the torsion to \( 16 + 4 + 4 = 40 \) independent components of \( Q_{\alpha\beta} \land \vartheta^\beta \), \( Q \land \vartheta^\alpha \), and \( (3) T^\alpha \). In the end, the second order partial differential equations (53), (54) become nonlinear algebraic equations. We solve them by requiring certain constraints on the coupling constants. In this way we find exact solutions with \( 1/r^\nu \)-behavior of \( Q_{\alpha\beta} \), here \( \nu = 1, 2, 3 \).

If we substitute the decomposition formula (25) into (84), we find

\[
T^\alpha = \xi_0 P^\alpha - \frac{1}{3} [\xi_0 \Lambda - 3(\xi_0 + \xi_1) Q] \land \vartheta^\alpha + (3) T^\alpha .
\] (85)

Contraction yields

\[
\xi_0 \Lambda - 3(\xi_0 + \xi_1) Q - T = 0 .
\] (86)

Here it is useful to take recourse to the first Bianchi identity. Provided that the conditions

\[
T^\alpha = (2) T = \frac{1}{3} \vartheta^\alpha \land T, \quad (2) Q_{\alpha\beta} = 0 ,
\] (87)

are fulfilled, we derive in Appendix B in (E13) that \([R_{\text{ica}} := \vartheta^\alpha \land (e_\beta \, (5) W_{\alpha\beta})] \nabla^\alpha - \frac{1}{3} d (2T + 3Q - \Lambda) = 0 \]. (88)

This equation can be understood as an integrability condition for (86). Substitution of (86) into (88) yields

\[
0 = R_{\text{ica}} + \frac{1}{3} d \left\{ (2\xi_0 - 1) \Lambda + 3 [1 - 2(\xi_0 + \xi_1)] Q \right\} .
\] (89)

We have two free parameters. Thus, on this level, we can always find solutions with \( R_{\text{ica}} = 0 \). However, Eq. (89) also implies the non-trivial result

\[
d R_{\text{ica}} = 0 .
\] (90)

The second Bianchi identity may yield more conditions.

We now turn to the spherically symmetric solution of Sec VII. In this case, we have the prolongation ansatz (84) with

\[
\xi_0 = \frac{1}{2}, \quad \xi_1 = 0 , \quad (3) T^\alpha = 0 .
\] (91)
Then (86) becomes
\[ \Lambda - 3Q - 2T = 0. \] (92)

This is consistent with the first Bianchi identity, see (E14). Since only \((2)T^\alpha \neq 0\), we have
\[ \Lambda - 3Q - 2T = \text{exact form}. \] (93)

The distortion 1-form (10) can be taken from (66) and (69) or from \[7\], Eq.(35):
\[ N_{\alpha\beta} = \frac{1}{2} Q_{\alpha\beta}. \] (94)

Note that \(N_{[\alpha\beta]} = 0\). In this special case, the curvature can be easily decomposed in Riemannian and post-Riemannian pieces,
\[ R_{\alpha\beta} = \tilde{R}_{\alpha\beta} + \frac{1}{2} \tilde{D}Q_{\alpha\beta} - \frac{1}{4} \partial_{\alpha} \gamma \wedge \partial_{\gamma} \beta, \] (95)
where \(\tilde{R}_{\alpha\beta}\) denotes the purely Riemannian part of the curvature and \(\tilde{D}\) the exterior covariant derivative with respect to the Riemannian connection.

Possibly, for a real “liberated” aether dynamics, one is forced to allow for \(Rica \neq 0\). Apparently \(Rica\) is the non-exact piece of \(\Lambda, Q, T\) and as such contributes generically to \((3)Z_{\alpha\beta}\) and \((4)Z_{\alpha\beta}\).

There is a further property of our specific exact solution which is of interest. The ansatz (84), together with the first Bianchi identity, yields
\[ (Q_{\alpha\mu} \wedge Q_{\beta}^\mu + 4R_{[\alpha\beta]}) \wedge \vartheta^\beta = \] (96)
\[ (Q_{\alpha\mu} \wedge Q_{\beta}^\mu + 4R_{[\alpha\beta]}) \wedge \vartheta^\beta = 0. \]

Compare now (95) with (96) and find
\[ R_{\alpha\beta} \wedge \vartheta^\beta = \tilde{R}_{\alpha\beta} \wedge \vartheta^\beta + W_{\alpha\beta} \wedge \vartheta^\beta + \frac{1}{2} (\tilde{D}Q_{\alpha\beta}) \wedge \vartheta^\beta, \] (97)
i.e., we have \(\xi_0 = 1/2\) and \(\xi_1 = 0\). This implies for the symmetric (strain) curvature
\[ Z_{\alpha\beta} \wedge \vartheta^\beta = \tilde{R}_{\alpha\beta} \wedge \vartheta^\beta + \frac{1}{2} (\tilde{D}Q_{\alpha\beta}) \wedge \vartheta^\beta. \] (98)

We decompose the curvature into symmetric and antisymmetric pieces. This yields
\[ Z_{\alpha\beta} = \frac{1}{2} (\tilde{D}Q_{\alpha\beta}), \] (99)
\[ W_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \frac{1}{4} \partial_{[\alpha} \gamma \wedge \partial_{\beta]} \gamma. \] (100)
Thus an extra field equation for $Q_{\alpha\beta}$ is implied,

$$(\tilde{D}Q_{\alpha\beta}) \wedge \vartheta^\beta = 0.$$  \hfill (101)

Generally, this equation implies further integrability conditions. With

$$DDQ_{\alpha\beta} = -2R_{(\alpha} \wedge Q_{\beta)\gamma}$$  \hfill (102)

and the ansatz (84), we find the algebraic constraint

$$\frac{1}{\xi_0} \tilde{R}_{\alpha\beta} \wedge T^\beta = 0,$$  \hfill (103)

which has to be fulfilled by this solution.

**X. DISCUSSION**

We constructed a model within MAG which exhibits vector-like Lorentz violating fields. It may be seen as analogue to the Einstein-aether theory of Jacobson et al. We were able to find a simple exact spherically symmetric solution for the field equations of a truncated Lagrangian. Our solution distorts the conventional spherically symmetric Schwarzschild-de Sitter spacetime by nonmetricity and torsion. The presence of nonmetricity will obstruct the local Lorentz invariance of the Riemannian spacetime. Further investigations should include the search for wave-like aether solution which, most likely, will require a more complicated “background” than simple Schwarzschild spacetime, cf. plane-wave solutions in MAG [22, 50, 58].

It should be understood that MAG is a comprehensive framework for classical gravitational field theories. Different authors started with MAG and, by using the nonlinear realization technique, tried to “freeze out” certain degrees of freedom like, e.g., the nonmetricity. Percacci [62], Tresguerres and Mielke [79] (see also [38, 47]), and, most recently, Kirsch [40] developed models of such a kind, for earlier work one should compare Lord and Goswami [48, 49], see also Tresguerres [78]. There is not much doubt that gravity has a metric-affine structure. Therefore, MAG seems an appropriate framework for classical gravity. But there are different ways to realize this structure. Still, if Lorentz invariance turns out to be violated, then the nonmetricity of spacetime should play a leading role.
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APPENDIX A: IRREDUCIBLE DECOMPOSITION OF THE STRAIN CURVATURE $Z_{\alpha\beta}$ IN $n$ DIMENSIONS

The 2-form $Z_{\alpha\beta}$ in (32) — for $n$ dimensions we have $1/n$ instead of $1/4$ — can be cut into different pieces by contraction with $e_\alpha$, transvecting with $\vartheta^\alpha$, and by “hodge”-ing the corresponding expressions:

$$\hat{\Delta} := \frac{1}{n-2} \vartheta^{\alpha} \wedge Z_{\alpha}, \quad Y_\alpha := \ast(Z_{\alpha\beta} \wedge \vartheta^\beta).$$

(A1)

Subsequently we can subtract out traces:

$$\Xi_\alpha := Z_\alpha - \frac{1}{2} e_\alpha \ast(\vartheta^\gamma \wedge Z_\gamma), \quad Y_\alpha := Y_\alpha - \frac{1}{n-2} e_\alpha \ast(\vartheta^\gamma \wedge Y_\gamma).$$

(A2)

The irreducible pieces may then be written as (see [30])

$$\begin{align*}
(2)Z_{\alpha\beta} &:= -\frac{1}{2} \ast(\vartheta^{(\alpha} \wedge Y^{\beta)}), \\
(3)Z_{\alpha\beta} &:= \frac{1}{n+2} \left( n \vartheta^{(\alpha} \wedge e^{\beta)} - 2 g_{\alpha\beta} \right) \hat{\Delta}, \\
(4)Z_{\alpha\beta} &:= \frac{1}{n} g_{\alpha\beta} Z, \\
(5)Z_{\alpha\beta} &:= \frac{2}{n} \vartheta^{(\alpha} \wedge \Xi^{\beta)}, \\
(1)Z_{\alpha\beta} &:= Z_{\alpha\beta} - (2)Z_{\alpha\beta} - (3)Z_{\alpha\beta} - (4)Z_{\alpha\beta} - (5)Z_{\alpha\beta}.
\end{align*}$$

(A3-A7)

APPENDIX B: EXPRESSING THE CURVATURE $^{(3)}Z_{\alpha\beta}$ IN TERMS OF NONMETRICITY AND TORSION

According to (A4), the curvature $^{(3)}Z_{\alpha\beta}$ can be expressed in terms of $\hat{\Delta}$. Thus, we start from the definition of $\hat{\Delta}$ in (A1) and move the interior product to the left:

$$\hat{\Delta} = \frac{1}{n-2} \vartheta^{\alpha} \wedge (e^{\beta} \wedge Z_{\alpha\beta}) = \frac{1}{n-2} [-e^{\beta} \ast(\vartheta^{\alpha} \wedge Z_{\alpha\beta}) + Z_{\alpha}^\alpha] = -\frac{1}{n-2} e^{\alpha} \ast(Z_{\alpha\beta} \wedge \vartheta^\beta).$$

(B1)
Obviously, we have to express $Z_{\alpha\beta} \wedge \vartheta^\beta$ in terms of nonmetricity and torsion. This should be possible by means of the zeroth Bianchi identity

$$DQ_{\alpha\beta} = -D Dg_{\alpha\beta} = R_\alpha^\gamma g_{\gamma\beta} + R_{\beta\gamma} g_{\alpha\gamma} = 2R_{(\alpha\beta)} = 2Z_{\alpha\beta}.$$  \hfill (B2)

We wedge with $\vartheta^\beta$ from the right and obtain

$$D (Q_{\alpha\beta} \wedge \vartheta^\beta) = 2Z_{\alpha\beta} \wedge \vartheta^\beta - Q_{\alpha\beta} \wedge T^\beta.$$  \hfill (B3)

On the other hand, by making use of (25) (suitably generalized for $n$ dimensions) and of

$$D\vartheta^\alpha = (Dg_{\alpha\beta}) \wedge \vartheta^\beta + g_{\alpha\beta} D\vartheta^\beta = -Q_{\alpha\beta} \wedge \vartheta^\beta + T^\alpha,$$  \hfill (B4)

we can calculate

$$D (Q_{\alpha\beta} \wedge \vartheta^\beta) \overset{\text{(25)}}{=} D \left( Q \wedge \vartheta^\alpha + \frac{1}{n-1} \vartheta^\alpha \wedge \Lambda + P_{\alpha} \right)$$

$$= dQ \wedge \vartheta^\alpha - Q \wedge D\vartheta^\alpha + \frac{1}{n-1} D\vartheta^\alpha \wedge \Lambda - \frac{1}{n-1} \vartheta^\alpha \wedge d\Lambda + DP_{\alpha}$$

$$\overset{\text{(B4)}}{=} dQ \wedge \vartheta^\alpha - Q \wedge (-Q_{\alpha\beta} \wedge \vartheta^\beta + T^\alpha)$$

$$+ \frac{1}{n-1} (-Q_{\alpha\beta} \wedge \vartheta^\beta + T^\alpha) \wedge \Lambda - \frac{1}{n-1} \vartheta^\alpha \wedge d\Lambda + DP_{\alpha}$$

$$\overset{\text{(25)}}{=} dQ \wedge \vartheta^\alpha + Q \wedge \left( Q \wedge \vartheta^\alpha + \frac{1}{n-1} \vartheta^\alpha \wedge \Lambda + P_{\beta} \right) - Q \wedge T^\alpha$$

$$- \frac{1}{n-1} \left( Q \wedge \vartheta^\alpha + \frac{1}{n-1} \vartheta^\alpha \wedge \Lambda + P_{\alpha} \right) \wedge \Lambda + \frac{1}{n-1} T^\alpha \wedge \Lambda$$

$$- \frac{1}{n-1} \vartheta^\alpha \wedge d\Lambda + DP_{\alpha}$$

$$= dQ \wedge \vartheta^\alpha + Q \wedge \vartheta^\alpha + \frac{1}{n-1} Q \wedge \vartheta^\alpha \wedge \Lambda + Q \wedge P_{\alpha} - Q \wedge T^\alpha$$

$$- \frac{1}{n-1} Q \wedge \vartheta^\alpha \wedge \Lambda - \frac{1}{(n-1)^2} \vartheta^\alpha \wedge \Lambda \wedge \Lambda - \frac{1}{n-1} P_{\alpha} \wedge \Lambda + \frac{1}{n-1} T^\alpha \wedge \Lambda$$

$$- \frac{1}{n-1} \vartheta^\alpha \wedge d\Lambda + DP_{\alpha}$$

$$= dQ \wedge \vartheta^\alpha + Q \wedge P_{\alpha} - Q \wedge T^\alpha - \frac{1}{n-1} P_{\alpha} \wedge \Lambda + \frac{1}{n-1} T^\alpha \wedge \Lambda$$

$$- \frac{1}{n-1} \vartheta^\alpha \wedge d\Lambda + DP_{\alpha}.$$  \hfill (B5)

Now we can compare (B3) and (B5). We find

$$2Z_{\alpha\beta} \wedge \vartheta^\beta = 2Z_{\alpha\beta} \wedge \vartheta^\beta - 2^{(4)}Z_{\alpha\beta} \wedge \vartheta^\beta$$

$$= Q \wedge P_{\alpha} - Q \wedge T^\alpha - \frac{1}{n-1} P_{\alpha} \wedge \Lambda$$

$$+ \frac{1}{n-1} T^\alpha \wedge \Lambda - \frac{1}{n-1} \vartheta^\alpha \wedge d\Lambda + DP_{\alpha} + Q_{\alpha\beta} \wedge T^\beta.$$  \hfill (B6)
We expand the last term by means of the irreducible decomposition of torsion and non-metricity:

\[ Q_{\alpha\beta} \wedge T^\beta = \mathcal{Q}_{\alpha\beta} \wedge T^\beta + Q \wedge T_\alpha \]
\[ = \mathcal{Q}_{\alpha\beta} \wedge \left( (1)^{T^\beta} + (3)^{T^\beta} \right) + \mathcal{Q}_{\alpha\beta} \wedge \left( \frac{1}{n-1} \vartheta^\beta \wedge T \right) + Q \wedge T_\alpha \]
\[ = \mathcal{Q}_{\alpha\beta} \wedge \left( (1)^{T^\beta} + (3)^{T^\beta} \right) + \frac{1}{(n-1)^2} \vartheta_\alpha \wedge \Lambda \wedge T + \frac{1}{n-1} P_\alpha \wedge T + Q \wedge T_\alpha . \quad \text{(B7)} \]

By the same token,
\[ \frac{1}{n-1} T_\alpha \wedge \Lambda = \frac{1}{n-1} \left( (1)^{T_\alpha} + (3)^{T_\alpha} \right) \wedge \Lambda + \frac{1}{(n-1)^2} \vartheta_\alpha \wedge T \wedge \Lambda . \quad \text{(B8)} \]

Substituting (B7, B8) into (B6) yields
\[ 2 \not{Z} \wedge \vartheta^\beta = DP_\alpha - \frac{1}{n-1} \vartheta_\alpha \wedge d\Lambda + \mathcal{Q}_{\alpha\beta} \wedge \left( (1)^{T^\beta} + (3)^{T^\beta} \right) \]
\[ + \frac{1}{n-1} \left( (1)^{T_\alpha} + (3)^{T_\alpha} \right) \wedge \Lambda + P_\alpha \wedge \left[ Q - \frac{1}{n-1} (\Lambda - T) \right] . \quad \text{(B9)} \]

We use the following properties of the irreducible pieces:
\[ e^\alpha \left| P_\alpha = e^\alpha \left| (1)^{T_\alpha} = e^\alpha \left| (3)^{T_\alpha} = 0, \quad e^\alpha \left| Q_{\alpha\beta} = \Lambda_\beta . \quad \text{(B10)} \right. \right. \]

Then we find
\[ 2e^\alpha \left| (Z_{\alpha\beta} \wedge \vartheta^\beta) = P^\alpha e_\alpha \left[ Q - \frac{1}{n-1} (\Lambda - T) \right] + \frac{n}{n-1} \left( (1)^{T_\alpha} + (3)^{T_\alpha} \right) \Lambda^\alpha \]
\[ - \frac{n-2}{n-1} d\Lambda + e^\alpha \left| DP_\alpha - \mathcal{Q}_{\alpha\beta} \wedge e^\alpha \left| (1)^{T^\beta} + (3)^{T^\beta} \right) \right. \quad \text{(B11)} \]

We can further simplify the last term. First we note that
\[ e^\alpha \left| (3)^{T^\beta} = (-1)^s e^\alpha \left[ \frac{1}{3} \ast \left( \vartheta^\beta \wedge \ast (T^\gamma \wedge \vartheta_\gamma) \right) \right] = (1)^s e^\alpha \left[ \frac{1}{3} \ast \left( \vartheta^\beta \wedge \ast (T^\gamma \wedge \vartheta_\gamma) \wedge \vartheta^\alpha \right) \right] = -e^\beta \left| (3)^{T^\alpha} . \quad \text{(B12)} \right. \]

Hence,
\[ \mathcal{Q}_{\alpha\beta} \wedge e^\alpha \left| ((1)^{T^\beta} + (3)^{T^\beta}) = \mathcal{Q}_{\alpha\beta} \wedge e^\alpha \left| (1)^{T^\beta} = (1)^{Q_{\alpha\beta} \wedge e^\alpha \left| (1)^{T^\beta} \right. \right. \right. \]
\[ + (2)^{Q_{\alpha\beta} \wedge e^\alpha \left| (1)^{T^\beta} + (3)^{Q_{\alpha\beta} \wedge e^\alpha \left| (1)^{T^\beta . \quad \text{(B13)} \right. \right. \right. \]

The last term can be further rewritten as
\[ (3)^{Q_{\alpha\beta} \wedge e^\alpha \left| (1)^{T^\beta} = \frac{2n}{(n-1)(n+2)} \left( \vartheta_\alpha \Lambda_\beta - \frac{1}{n} g_{\alpha\beta} \Lambda \right) \wedge e^\alpha \left| (1)^{T^\beta} \right. \right. \]
\[
\begin{align*}
&= \frac{n}{(n-1)(n+2)} \left( \partial^\beta \Lambda_\alpha \wedge e^{\alpha} [(1) T^\beta + \Lambda_\beta \partial^\alpha e^{\alpha} [(1) T^\beta] \\
&= \frac{n}{(n-1)(n+2)} \left[ \Lambda_\alpha \left( -e^{\alpha} [(\partial^\beta \wedge (1) T^\beta)] + (1) T^\alpha \right) + 2 \Lambda_\beta (1) T^\beta \right] \\
&= \frac{3n}{(n-1)(n+2)} \Lambda_\alpha (1) T^\alpha ,
\end{align*}
\]
where we used \( \partial^\beta \wedge (1) T^\beta = 0 \). Finally we arrive at
\[
\hat{\Delta} = \frac{1}{2(n-1)} d\Lambda - \frac{1}{2(n-2)} e^\alpha | DP_\alpha \\
- \frac{1}{2(n-2)} \left\{ \frac{1}{n-1} P_\alpha e^\alpha | [(n-1) Q + \Lambda - T] + \left( \frac{n+1}{n+2} (1) T_\alpha + \frac{n}{n-1} (3) T_\alpha \right) \Lambda^\alpha \\
- (1) Q_{\alpha \beta} + (2) Q_{\alpha \beta} \wedge e^\alpha [(1) T^\beta] \right\} .
\]
\[
(B15)
\]
Note that in the last line we could substitute \( (2) Q_{\alpha \beta} = -2 e_{(\alpha} [P_{\beta)}]/3 \).

**APPENDIX C: THE \((3) Z^{\alpha \beta} \wedge \ast (3) Z_{\alpha \beta}\) PIECE OF THE LAGRANGIAN**

We prove that the following relation holds for arbitrary spacetimes:
\[
(3) Z^{\alpha \beta} \wedge \ast (3) Z_{\alpha \beta} = \frac{n(n-2)}{n+2} \hat{\Delta} \wedge \ast \hat{\Delta} .
\]
\[
(C1)
\]
This comes about since \((3) Z^{\alpha \beta}\) corresponds to a scalar-valued degree of freedom, namely to the two-form \( \hat{\Delta} \), see (A4). For a p-form \( \phi \), we have the rules for the Hodge dual \( \ast \phi = (-1)^{p(n-p)} \ast \phi \) in the case of Lorentz signature, furthermore, \( \partial^\alpha \wedge (e_\alpha [\phi] = p \phi \) and \( \ast (\phi \wedge \partial_\alpha) = e_\alpha [\ast \phi] \). Thus, the terms quadratic in contractions of \( \hat{\Delta} \) in the end evaluate to \( \hat{\Delta} \wedge \ast \hat{\Delta} \),
\[
(e^\alpha [\hat{\Delta}] \wedge \ast (e_\alpha [\hat{\Delta}]) = -(e^\alpha [\hat{\Delta}] \wedge (e_\alpha [\ast \hat{\Delta}) = -(e^\alpha [\hat{\Delta}] \wedge \partial_\alpha \wedge \ast \hat{\Delta} = 2 \hat{\Delta} \wedge \ast \hat{\Delta} .
\]
\[
(C2)
\]
We recall the definition (A4)
\[
(3) Z_{\alpha \beta} = \frac{1}{n+2} \left[ n \partial_\alpha \wedge (e_\beta [\hat{\Delta} - 2 g_{\alpha \beta} \hat{\Delta} \right] .
\]
\[
(C3)
\]
It is symmetric \((3) Z^{[\alpha \beta] = 0\) and tracefree \((3) Z^{\gamma \gamma} = 0\). Thus,
\[
(3) Z_{\alpha \beta} \wedge \ast (3) Z^{\alpha \beta} = \frac{1}{n+2} \left[ n \partial_\alpha \wedge (e_\beta [\hat{\Delta} \wedge \ast (3) Z^{\alpha \beta} \right]
= \frac{1}{(n+2)^2} \left[ n^2 \partial_\alpha \wedge (e_\beta [\hat{\Delta}] \wedge ( (\partial^\alpha \wedge (e^{\beta} [\hat{\Delta}]) - 2n \partial_\alpha \wedge (e_\beta [\hat{\Delta}] \wedge g^{\alpha \beta} \ast \hat{\Delta} \right] .
\]
\[
(C4)
\]
In order to calculate the first term, we apply the rules for commuting the Hodge star with the exterior/interior product.

\[
\star \left( \vartheta^{(\alpha \wedge (e^{\beta} \hat{\Delta})} \right) = - \star \left[ (e^{(\beta \hat{\Delta}) \wedge \vartheta^{(\alpha)}} \right] = - e^{(\alpha \wedge (e^{\beta} \hat{\Delta})
\]
\[= -(-1)^{n-1} e^{(\alpha \wedge \vartheta^{(\beta)}} \right] = e^{(\alpha \wedge \vartheta)} \wedge \hat{\Delta})
\[
= g^{\alpha\beta} \Delta - \vartheta^{(\alpha \wedge (e^{\beta} \hat{\Delta})}. \quad (C5)
\]

The second term in the brackets of \([C4]\) simply evaluates to

\[-2n \vartheta_{\alpha \wedge (e_{\beta} \hat{\Delta})} \wedge g^{\alpha\beta} \Delta = -2n \vartheta^{\alpha \wedge (e_{\alpha} \hat{\Delta})} \wedge \hat{\Delta} = -4n \hat{\Delta} \wedge \hat{\Delta}. \quad (C6)
\]

Substituting \([C5]\) and \([C6]\) into \([C4]\), we find

\[\begin{align*}
(3) Z^{\alpha \beta} \wedge \star (3) Z^{\alpha \beta} &= \frac{1}{(n+2)^2} \left[ 2n^2 \hat{\Delta} \wedge \star \hat{\Delta} - n^2 \vartheta_{\alpha \wedge (e_{\beta} \hat{\Delta})} \wedge \vartheta^{(\alpha \wedge (e^{\beta} \hat{\Delta})} - 4n \hat{\Delta} \wedge \star \hat{\Delta} \right]. \quad (C7)
\end{align*}\]

The term in the middle yields \((\vartheta^{\alpha \wedge \vartheta_{\alpha}} = 0)\)

\[\begin{align*}
(\vartheta_{\alpha \wedge e_{\beta} \hat{\Delta}}) \wedge (\vartheta^{(\alpha \wedge (e^{\beta} \hat{\Delta})} &= \frac{1}{2} \left[ \vartheta_{\alpha \wedge (e_{\beta} \hat{\Delta})} \wedge \vartheta^{\alpha \wedge (e^{\beta} \hat{\Delta})} + \vartheta_{\alpha \wedge (e_{\beta} \hat{\Delta})} \wedge \vartheta^{\beta \wedge (e^{\alpha} \hat{\Delta})} \right]
\end{align*}\]
\[= \frac{1}{2} \left[ -\vartheta^{\beta \wedge (e_{\beta} \hat{\Delta})} \wedge \vartheta_{\alpha \wedge (e^{\alpha} \hat{\Delta})} \right] = -(n-2) \hat{\Delta} \wedge \star \hat{\Delta}. \quad (C8)
\]

Eventually,

\[\begin{align*}
(3) Z^{\alpha \beta} \wedge \star (3) Z^{\alpha \beta} &= \frac{1}{(n+2)^2} \left[ 2n^2 + n^2(n-2) - 4n \right] \hat{\Delta} \wedge \star \hat{\Delta} = \frac{n(n-2)}{n+2} \hat{\Delta} \wedge \star \hat{\Delta}. \quad (C9)
\end{align*}\]

**APPENDIX D: THE MAG LAGRANGIAN QUADRATIC IN CURVATURE, TORSION, AND NONMETRICITY**

The most general parity conserving quadratic Lagrangian, which is expressed in terms of the \(4 + 3 + 6 + 5\) irreducible pieces of \(Q_{\alpha\beta}, T^{\alpha}, W^{\alpha}_{\beta},\) and \(Z^{\alpha\beta},\) respectively, reads (see [19], [59], [20], and references given):

\[
V_{\text{MAG}} = \frac{1}{2\kappa} \left[ -a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda_0 \eta + T^{\alpha} \wedge \star \left( \sum_{I=1}^{3} a_I^{(I)} T_{\alpha} \right) \right.
\]
\[+ 2 \left( \sum_{I=2}^{4} c_I^{(I)} Q_{\alpha\beta} \right) \wedge \vartheta^{\alpha} \wedge \star T^{\beta} + Q_{\alpha\beta} \wedge \star \left( \sum_{I=1}^{4} b_I^{(I)} Q^{\alpha\beta} \right) \]
\[+ b_5 \left( Q_{\alpha\gamma} \wedge \vartheta^{\alpha} \right) \wedge \star \left( Q^{\beta\gamma} \wedge \vartheta_{\beta} \right) \]
- $\frac{1}{2\rho} R^{\alpha\beta} \wedge \star \left( \sum_{I=1}^{6} w_I W_{\alpha\beta} + w_7 \vartheta_\alpha \wedge (e_\gamma)^{(5)} W^\gamma_{\beta} \right) \\
+ \sum_{I=1}^{5} z_I^{(I)} Z_{\alpha\beta} + z_6 \vartheta_\gamma \wedge (e_\alpha)^{(2)} Z^\gamma_{\beta} + \sum_{I=7}^{9} z_I \vartheta_\alpha \wedge (e_\gamma)^{(I-4)} Z^\gamma_{\beta} \right) \right). \quad \text{(D1)}$

Here $\kappa$ is the dimensionful (weak) gravitational constant, $\lambda_0$ the “bare” cosmological constant, and the dimensionless $\rho$ is the strong gravity coupling constant. The constants $a_0, a_1, a_3, b_1, b_5, c_2, c_3, c_4, w_1, w_7, z_1, z_9$ are dimensionless and should be of order unity. Note the nontrivial formula for the Hilbert-Einstein type of Lagrangian

$$R_{\alpha\beta} \wedge \eta^{\alpha\beta} = (6) W_{\alpha\beta} \wedge \eta^{\alpha\beta}. \quad \text{(D2)}$$

Because of the irreducibility, we have

$$T^2 \sim T^\alpha \wedge \star \left( a_1^{(1)} T_\alpha + a_2^{(2)} T_\alpha + a_3^{(3)} T_\alpha \right)$$

$$= a_1 \star^{(1)} T_\alpha \wedge (1) T_\alpha + a_2 \star^{(2)} T_\alpha \wedge (2) T_\alpha + a_3 \star^{(3)} T_\alpha \wedge (3) T_\alpha, \quad \text{(D3)}$$

and similar formulas for the pure square pieces of $Q_{\alpha\beta}, W_{\alpha\beta},$ and $Z_{\alpha\beta}$. In the curvature square terms in (D1) we introduced the irreducible pieces of the antisymmetric part $W_{\alpha\beta} := R_{[\alpha\beta]}$ (rotation part) and the symmetric part $Z_{\alpha\beta} := R_{(\alpha\beta)}$ (strain part) of the curvature 2-form $R_{\alpha\beta}$. In $Z_{\alpha\beta}$, we have the purely post-Riemannian part of the curvature. Note the peculiar cross terms with $c_I$ and $b_5$, and the exotic terms with $w_7$ and $z_6, z_7, z_8, z_9$.

In the component formalism, Esser [19] has carefully enumerated all different pieces of the quadratic MAG Lagrangian; for the corresponding nonmetricity and torsion pieces, see also Duan et al. [15]. Accordingly, Eq.(D1) represents the most general quadratic parity-conserving MAG-Lagrangian. All previously published quadratic parity-conserving Lagrangians are subcases of (D1). Hence (D1) is a safe starting point for our future considerations.

**APPENDIX E: THE 2-FORM $R_{\alpha\beta}$ AND THE FIRST BIANCHI IDENTITY**

The first Bianchi identity $B^\alpha \equiv 0$, with the 3-form

$$B^\alpha := DT^\alpha - R_\beta^\alpha \wedge \vartheta^\beta, \quad \text{(E1)}$$
interrelates torsion and curvature. The irreducible decomposition of the first Bianchi identity reads (see \[30\])

\[(2)\ B_\alpha = \frac{1}{n-2} (e_\beta B_\beta) \wedge \vartheta^\alpha, \quad \text{(E2)}\]
\[(3)\ B_\alpha = \frac{1}{4} e_\alpha (\vartheta^\beta \wedge B_\beta), \quad \text{(E3)}\]
\[(1)\ B_\alpha = B_\alpha - (2)B_\alpha - (3)B_\alpha. \quad \text{(E4)}\]

In the last term of (E1), because of the contraction with the coframe, some of the irreducible components of the curvature drop out, see \[30\], Eqs.(B.4.15) and (B.4.29):

\[R_\beta^\alpha \wedge \vartheta^\beta = (2)W_\beta^\alpha + (3)W_\beta^\alpha + (5)W_\beta^\alpha + (2)Z_\beta^\alpha + (3)Z_\beta^\alpha + (4)Z_\beta^\alpha \wedge \vartheta^\beta. \quad \text{(E5)}\]

We contract this piece by the frame:

\[e_\alpha (R_\beta^\alpha \wedge \vartheta^\beta) = e_\alpha ((5)W_\beta^\alpha \wedge \vartheta^\beta) + e_\alpha ((3)Z_\beta^\alpha \wedge \vartheta^\beta) + e_\alpha ((5)Z_\beta^\alpha \wedge \vartheta^\beta)\]
\[= -R_{\text{ica}} - (n - 2) \hat{\Delta} + \frac{n - 2}{2} dQ. \quad \text{(E6)}\]

We introduced as abbreviation the antisymmetric rotational Ricci 2-form for $W_{\alpha\beta}$ by

\[R_{\text{ica}} := \vartheta^\alpha \wedge (e_\beta W_\alpha^\beta) = \vartheta^\alpha \wedge (e_\beta W_\alpha^\beta), \quad \text{(E7)}\]

see \[30\], Eqs.(B.4.8) and (B.4.11). $R_{\text{ica}}$ has $n(n-1)/2$ independent components. Note that there is a subtlety involved here. The (premetric) Ricci 1-form is usually defined in terms of the total curvature according to $\text{Ric}_\alpha := e_\beta (R_{\alpha\beta}$. Here, however, our definition refers only to the rotational curvature $W_{\alpha\beta}$, see \[29\]. Similarly we obtain

\[\vartheta^\alpha \wedge (R_{\beta\alpha} \wedge \vartheta^\beta) = \vartheta^\alpha \wedge ((3)W_{\beta\alpha} \wedge \vartheta^\beta) =: \hat{X}. \quad \text{(E8)}\]

Before we substitute (E6) and (E8) into the first Bianchi identity, we should first compute the corresponding torsion pieces:

\[e_\alpha DT^\alpha = e_\alpha \left[ D((1)T^\alpha + (3)T^\alpha) \right] + \frac{1}{n-1} (1)T^\alpha + (3)T^\alpha \] \[= -\frac{n - 2}{n - 1} dT, \quad \text{(E9)}\]
\[\vartheta^\alpha \wedge DT_\alpha = -D(\vartheta^\alpha \wedge (3)T_\alpha) + T^\alpha \wedge T_\alpha. \quad \text{(E10)}\]

In this way, we find for the 3-form $B^\alpha$ eventually,

\[e_\beta B^\beta = e_\alpha \left[ D((1)T^\alpha + (3)T^\alpha) \right] + \frac{1}{n-1} (1)T^\alpha + (3)T^\alpha \] \[= -\frac{n - 2}{n - 1} dT + R_{\text{ica}} + (n - 2) \hat{\Delta} - \frac{n - 2}{2} dQ = 0, \quad \text{(E11)}\]
\[\vartheta^\beta \wedge B_\beta = -D(\vartheta^\alpha \wedge (3)T_\alpha) + T^\alpha \wedge T_\alpha - \hat{X} = 0. \quad \text{(E12)}\]
Provided the torsion possesses only its trace piece, that is \( (1) T^\alpha = (3) T^\alpha = 0 \), and furthermore \( (2) Q_{\alpha \beta} = 0 \), see the constraints (38), we have \( \hat{\Delta} = d\Lambda/\left[2(n-1)\right] \) and

\[
\mathcal{R}ica - \frac{n-2}{2(n-1)} d \left[ 2T + (n-1) Q - \Lambda \right] = 0.
\]

Thus, in \( n = 4 \), with the constraints (38) fulfilled and for \( \mathcal{R}ica = 0 \), the first Bianchi identity yields

\[
d \left( 3Q + 2T - \Lambda \right) = 0.
\]

Accordingly, under these conditions, the Weyl, the torsion, and the shear 1-forms are algebraically related to each other.

**APPENDIX F: CURVATURE OF THE SPHERICALLY SYMMETRIC AETHER SOLUTION OF SEC.VII**

With the help of our Reduce-Excalc computer algebra programs, we calculate the rotational and the strain curvature, respectively (the diamonds \( \diamond \) and the bullets \( \bullet \) denote those matrix elements that are already known because of the antisymmetry or the symmetry of the matrix involved):

\[
(1) W_{\alpha \beta} = \left( \frac{m}{r^3} - \frac{(\ell_0 + \ell_1)(4\ell_1 - \ell_0)}{96r^4 e^{2\mu(r)}} \right) \left( \begin{array}{ccc} 0 & 2\vartheta^{01} & -\vartheta^{02} & -\vartheta^{03} \\ \vartheta & 0 & \vartheta^{12} & \vartheta^{13} \\ \vartheta & \vartheta & 0 & -2\vartheta^{23} \\ \vartheta & \vartheta & \vartheta & 0 \end{array} \right) \quad \text{(weyl)}, \quad (F1)
\]

\[
(2) W_{\alpha \beta} = 0 \quad \text{(paircom = 0)}, \quad (F2)
\]

\[
(3) W_{\alpha \beta} = 0 \quad \text{(pscalar = 0)}, \quad (F3)
\]

\[
(4) W_{\alpha \beta} = -\vartheta_{[\alpha} \wedge \phi_{\beta]} \quad \text{(ricsymf)}, \quad (F4)
\]

\[
\phi_0 = \frac{(\ell_0 + 7\ell_1)(\ell_0 - \ell_1)}{32r^4 e^{2\mu(r)}} \vartheta^0 - 4\ell_0 \ell_1 \vartheta^1,
\]

\[
\phi_1 = -\frac{4\ell_0 \ell_1 \vartheta^0 + (\ell_0 - 5\ell_1)(\ell_0 - \ell_1)}{32r^4 e^{2\mu(r)}} \vartheta^1,
\]

\[
\phi_2 = \frac{(\ell_0 + \ell_1)(\ell_0 - \ell_1)}{32r^4 e^{2\mu(r)}} \vartheta^2,
\]

\[
\phi_3 = \frac{(\ell_0 + \ell_1)(\ell_0 - \ell_1)}{32r^4 e^{2\mu(r)}} \vartheta^3,
\]

\[
(5) W_{\alpha \beta} = 0 \quad \text{(ricanti = 0)}, \quad (F5)
\]
\[ W_{\alpha\beta} = -\frac{1}{12} W \vartheta_{\alpha\beta}, \quad W = 4\lambda_0 + \frac{(\ell_0 + 5\ell_1)(\ell_0 + \ell_1)}{8r^4 e^{2\mu(r)}} \quad \text{(scalar).} \]  

(6) \[ A_1 = \ell_1(4\lambda_0 r^3 + 15m - 9r), \quad A_2 = \ell_0(2\lambda_0 r^3 + 3m - 3r) + 9\ell_1(3m - r), \quad A_3 = 3\ell_1(3m - r), \]  

(1) \[ Z_{\alpha\beta} = \frac{1}{48r^4 e^{2\mu(r)}} \begin{pmatrix} -4A_1 \vartheta^{01} & -2A_2 \vartheta^{01} & A_2 \vartheta^{03} + 2A_1 \vartheta^{13} \\ \bullet & 4A_3 \vartheta^{01} & -2A_3 \vartheta^{02} + 2A_2 \vartheta^{12} -2A_3 \vartheta^{03} + A_2 \vartheta^{13} \\ \bullet & \bullet & 24\ell_1 r e^{2\mu(r)} \vartheta^{01} \end{pmatrix}, \quad \text{(F7)} \]  

(2) \[ Z_{\alpha\beta} = 0, \quad \text{(F8)} \]  

(3) \[ Z_{\alpha\beta} = 0, \quad \text{(F9)} \]  

(4) \[ Z_{\alpha\beta} = -\frac{\ell_1}{2r^3} g_{\alpha\beta} \vartheta^{01} \quad \text{(dilcurv)}, \quad \text{(F10)} \]  

(5) \[ Z_{\alpha\beta} = \frac{1}{2} \partial_{(\alpha} \vartheta_{\beta)} \vartheta^{01} \quad \text{(F11)} \]  

\[ \Xi_0 = -\frac{\ell_0 + \ell_1}{4r^4 e^{2\mu(r)}} \left[ (\lambda_0 r^3 - 3m)(\ell_0 + \ell_1) \vartheta^0 - 2(3m - r)\ell_1 \vartheta^1 \right], \]  

\[ \Xi_1 = -\frac{[3(\ell_0 + 9\ell_1)m - 6(\ell_0 + 3\ell_1)r + (5\ell_0 + 9\ell_1)\lambda_0 r^3]}{12r^4 e^{2\mu(r)}} \vartheta^1 - 6(3m - r)\ell_1 \vartheta^0, \]  

\[ \Xi_2 = -\frac{\ell_0 + 3\ell_1}{4r^3} \vartheta^2, \quad \Xi_2 = -\frac{\ell_0 + 3\ell_1}{4r^3} \vartheta^3. \]  

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