A FREE BOUNDARY PROBLEM INSPIRED BY A CONJECTURE OF DE GIORGI

NIKOLA KAMBUROV

ABSTRACT. We study global monotone solutions of the free boundary problem that arises from minimizing the energy functional $I(u) = \int |\nabla u|^2 + V(u)$, where $V(u)$ is the characteristic function of the interval $(-1, 1)$. This functional is a close relative of the scalar Ginzburg-Landau functional $J(u) = \int |\nabla u|^2 + W(u)$, where $W(u) = (1 - u^2)^2/2$ is a standard double-well potential. According to a famous conjecture of De Giorgi, global critical points of $J$ that are bounded and monotone in one direction have level sets that are hyperplanes, at least up to dimension 8. Recently, Del Pino, Kowalczyk and Wei gave an intricate fixed-point-argument construction of a counterexample in dimension 9, whose level sets “follow” the entire minimal non-planar graph, built by Bombieri, De Giorgi and Giusti (BdGG). In this paper we turn to the free boundary variant of the problem and we construct the analogous example; the advantage here is that of geometric transparency as the interphase $\{|u| < 1\}$ will be contained within a unit-width band around the BdGG graph. Furthermore, we avoid the technicalities of Del Pino, Kowalczyk and Wei’s fixed-point argument by using barriers only.

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1. Introduction.

In this paper we study the following free boundary problem:

\[ \Delta u = 0 \quad \text{in} \quad \Omega_{\text{in}} := \{ x \in \Omega : |u(x)| < 1 \} \]
\[ u = \pm 1 \quad \text{in} \quad \Omega \setminus \Omega_{\text{in}} \]
\[ |\nabla u| = 1 \quad \text{on} \quad \partial\Omega_{\text{in}} \cap \Omega = F^+(u) \sqcup F^-(u) \]

where \( \Omega \subseteq \mathbb{R}^n \) is a domain and the free boundary of \( u \) consists of two pieces
\[
F^+(u) := \partial\{ u = 1 \} \cap \Omega \quad F^-(u) := \partial\{ u = -1 \} \cap \Omega.
\]

In particular, we are interested in global solutions (\( \Omega = \mathbb{R}^n \)), which are monotonically increasing in the last coordinate \( x_n \). We pose the following question:

**Problem \(
\star\). Let \( n = 9 \). Does there exist a global solution to (1.1), monotonically increasing in \( x_9 \), such that its level sets are not hyperplanes?**

The question above should be read in the context of the prominent De Giorgi’s conjecture concerning global solutions of the Allen-Cahn equation

\[ \Delta u = (1 - u^2)u \quad \text{in} \quad \mathbb{R}^n, \]

namely:

**Conjecture** (De Giorgi [9]). If \( u \in C^2(\mathbb{R}^n) \) is a global solution of (1.2) such that \( |u| \leq 1 \) and \( \partial_{x_n} u > 0 \), then the level sets \( \{ u = \lambda \} \) are hyperplanes, at least for dimensions \( n \leq 8 \).

The common nature of the PDE’s (1.1) and (1.2) is rooted in the fact that they arise as Euler-Lagrange equations for the closely related energy functionals \( I \) and \( J \), respectively

\[ I(u, \Omega) = \int_{\Omega} |\nabla u|^2 + \mathcal{V}(u) \quad \text{for} \quad u : \Omega \to [-1,1] \]
\[ J(u, \Omega) = \int_{\Omega} |\nabla u|^2 + \mathcal{W}(u), \]

where \( \mathcal{V}(u) := 1_{(-1,1)}(u) \) is a singular version of the standard double-well potential \( \mathcal{W}(u) := \frac{(1-u^2)^2}{2} \).

De Giorgi’s conjecture has been motivated by a fascinating and deep connection between the theory of semilinear elliptic PDE and the theory of minimal surfaces. The connection was first rigorously stated through the notion of \( \Gamma \)-convergence in the work of Modica [16]. Assuming that \( u \) minimizes \( J \) in a large ball \( B_{1/\epsilon} \), \( u_\epsilon(x) = u(x/\epsilon) \) minimizes the rescaled energy

\[ J_\epsilon(v, B_1) = \epsilon \int_{B_1} |\nabla v|^2 + \frac{1}{\epsilon} \int_{B_1} \mathcal{W}(v) \]

in the unit ball. What Modica proved was that as \( \epsilon \to 0 \), a subsequence of minimizers \( u_k \) of \( J_\epsilon(\cdot, B_1) \) of uniformly bounded energy converges to

\[ u_k \rightarrow 1_E - 1_{B_1 \setminus E} \quad \text{in} \quad L^1_{\text{loc}}(B_1), \]

where \( E \) has a perimeter minimizing boundary in \( B_1 \), i.e. \( \partial E \) is a minimal hypersurface. The convergence is in fact stronger: Caffarelli and Cordoba ([7, 8]) later showed that the level sets \( \{ u_k = \lambda \} \) for \(-1 < \lambda < 1\) converge uniformly on compacts to the minimal hypersurface \( \partial E \). Intuitively speaking therefore, the level sets of a global minimizer of \( J \) look like a minimal hypersurface at large scales. The analogous statements can, of course, be made for global minimizers of \( I \).
The monotonicity assumption \( \partial_{x_n} u > 0 \) in De Giorgi’s conjecture implies that \( u \) is not only a stable critical point for \( J \), but that \( u \) is, in fact, a global minimizer of \( J \) in a certain sense (see [1]). Under the natural assumption
\[
\lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1, \tag{1.5}
\]
the level sets of \( u \) are also graphs over \( \mathbb{R}^{n-1} \) in the \( e_n \)-direction. Therefore, the preceding discussion combined with Bernstein’s theorem, which states that an entire minimal graph in \( \mathbb{R}^{n-1} \times \mathbb{R} \) is a hyperplane for \( n \leq 8 \) (cf. Simons [21]), is what gives plausibility to De Giorgi’s conjecture. Moreover, the existence of a non-planar minimal graph in dimension \( n = 9 \), constructed by Bombieri, De Giorgi and Giusti in [5], strongly suggests that the conjecture is likely to be false for \( n \geq 9 \).

There has been a lot of recent work which has almost completely resolved De Giorgi’s conjecture. It was fully established in dimensions \( n = 2 \) by Ghoussoub and Gui [13] and \( n = 3 \) by Ambrosio and Cabré [4], while Savin [18] managed to prove it for dimensions \( 2 \leq n \leq 8 \) under the additional assumption (1.5). Savin’s approach has a broader scope and applies to monotone global minimizers of the functional \( J \), as well (see [19]).

Recently Del Pino, Kowalczyk and Wei [12] successfully constructed a counterexample in dimension \( n = 9 \). Roughly speaking, their strategy is based upon the derivation of a sufficiently good ansatz whose level sets “follow” the Bombieri–De Giorgi–Giusti (BdGG) minimal graph. This allows them to carry out an intricate fixed point argument.

It was a desire to gain a better understanding of precisely what geometric ingredients are responsible for the existence of this important counterexample that led us to formulate and resolve in the affirmative the alternative Problem \( \star \).

**Theorem 1.1.** There exists a solution \( u : \mathbb{R}^9 \to \mathbb{R} \) of (1.1) which is monotonically increasing in \( x_9 \) and whose free boundary \( F(u) = F^+(u) \cup F^-(u) \) consists of two non-planar smooth graphs.

The study of the free boundary problem has an obvious geometric advantage. In this setting, the interphase \( \{|u| < 1\} \) will be contained within a unit-width band around the BdGG minimal graph, so that ones does not have to worry about capturing a non-trivial behaviour of the solution away from the band. We will use the method of barriers which is elementary in nature and allows for a transparent and relatively precise description of the solution (we will be able to trap the solution quite tightly between the two barriers). This way we avoid using fixed point arguments which are arguably the main culprit for the level of technical complexity of the construction by Del Pino, Kowalczyk and Wei.

To construct a solution to (1.1) once we are in possession of a supersolution lying above a subsolution (we will define these notions shortly), we adopt the strategy developed by De Silva [10] in her study of global free boundary graphs that arise from monotone solutions to the classical one-phase free boundary problem:
\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Omega_p(u) := \{x \in \Omega : u(x) > 0\} \\
|\nabla u| &= 1 \quad \text{on} \quad F_p(u) := \Omega \cap \partial \Omega_p(u). \tag{1.6}
\end{align*}
\]
Namely, a global solution to (1.6) is constructed as the limit of a sequence of local minimizers of the one-phase energy functional:
\[
I_0(u, \Omega) = \int_{\Omega} |\nabla u|^2 + 1_{\{u > 0\}} \tag{1.7}
\]
constrained to lie between a fixed strict subsolution and a fixed strict supersolution to (1.6). The strictness condition ensures that the free boundary of each minimizer doesn’t touch the free boundaries of the barriers. Following the classical ideas of Alt, Caffarelli and Friedman (2, 3), De Silva shows that \( u \) is a global energy minimizing viscosity solution, which is locally Lipschitz continuous.
and has non-degenerate growth along its free boundary; moreover, if one assumes that the two barriers are monotonically increasing in $x_n$, the global solution can also be chosen to be monotonically increasing in $x_n$ after a rearrangement. The harder part is the regularity theory: De Silva’s key observation is that the positive phase of the minimizer is locally an NTA (non-tangentially accessible) domain ([14]) which allows the application of the powerful boundary Harnack principle. By comparing the solution with a vertical translate, she rules out the possibility that the free boundary contains any vertical segments, so that it is a graph, and then she shows that the graph is, in fact, continuous. A more sophisticated comparison argument by De Silva and Jerison [11] establishes a Lipschitz bound on the free boundary graph. Hence, by the classical result of Caffarelli [6], the free boundary is locally a $C^{1,\alpha}$ graph, so that the global minimizer is indeed a classical solution to (1.6).

Obviously, the functionals $I$ and $I_0$ are close relatives: if $u$ minimizes $I(\cdot, \Omega)$ and $D \subset \Omega$ is a (nice enough) subdomain, such that $D \cap \{u = 1\} = \emptyset$, then $u + 1$ minimizes $I_0(\cdot, D)$; similarly, if $D \cap \{u = -1\} = \emptyset$, $1 - u$ minimizes $I_0(\cdot, D)$. So, after we construct a global minimizer $u$ to $I$, we will be in a position to directly apply the regularity theory for the one-phase energy minimizers from the discussion above to the free boundary of $u$.

Let us now give a brief outline of the arguments and the structure of our paper.

2. Outline of strategy.

First, let us recall the definition of a classical super/subsolution to the one-phase problem (1.6) (see for example [6]).

**Definition 2.1.** A classical supersolution (resp. subsolution) to (1.6) is a non-negative continuous function $w$ in $\Omega$ such that

- $w \in C^2(\Omega_p(w))$;
- $\Delta w \leq 0$ (resp. $\Delta w \geq 0$) in $\Omega_p(w)$;
- The free boundary $F_p(w)$ is a $C^2$ surface and
  \[ 0 < |\nabla w| \leq 1 \quad (\text{resp. } |\nabla w| \geq 1) \quad \text{on} \quad F_p(w). \]

If the inequality above is strict, we call $w$ a strict super (resp. sub) solution.

The appropriate notion of a classical super/subsolution to our free boundary problem (1.1) is, therefore, the following:

**Definition 2.2.** A classical supersolution (resp. subsolution) to (1.1) is a non-negative continuous function $w$ in $\Omega$ such that

- $w \in C^2(\Omega_{in}(w))$;
- $\Delta w \leq 0$ (resp. $\Delta w \geq 0$) in $\Omega_{in}(w)$;
- The free boundary $F(w) = F^+(w) \sqcup F^-(w)$ consists of two $C^2$ surfaces and
  \[ 0 < |\nabla w| \leq 1 \quad (\text{resp. } |\nabla w| \geq 1) \quad \text{on} \quad F^-(w), \]
  \[ |\nabla w| \geq 1 \quad (\text{resp. } |\nabla w| \leq 1) \quad \text{on} \quad F^+(w). \]

If the inequalities above are strict, we call $w$ a strict super (resp. sub) solution to (1.1).

As mentioned in the introduction, the driving intuition is that the level surfaces of a solution to (1.1) should follow the shape of the BdGG entire minimal graph $\Gamma = \{(x', x_9) \in \mathbb{R}^8 \times \mathbb{R} : x_9 = F(x')\}$. The function $F : \mathbb{R}^8 \to \mathbb{R}$ satisfies the minimal surface equation $H[F] = 0$ where $H[\cdot]$ is the mean curvature operator (MCO)

\[ H[F] := \nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right). \]
Note that there is a whole one-parameter family of such entire minimal non-planar graphs, obtained
by rescaling $\Gamma$:

$$\Gamma_\alpha := \alpha^{-1} \Gamma = \{ x_9 = F_\alpha(x') \}$$

where $\alpha > 0$ and $F_\alpha(x') := \alpha^{-1} F(\alpha x')$. “Following the shape” should be interpreted in the sense
that we would like the solution $u$ and thus the trapping super/subsolution $W$ and $V$ to behave
asymptotically at infinity like the signed distance to $\Gamma_\alpha$ for some $\alpha > 0$:

$$V(x) \leq u(x) \leq W(x) \approx \text{signed dist}(x, \Gamma_\alpha)$$

within their interphases (for the definition of the signed distance: we take the sign to be positive if
the point $x \in \mathbb{R}^9$ lies “above the graph”, i.e. if $x_9 > F_\alpha(x')$, and negative otherwise). This suggests
that the coordinates

$$\mathbb{R}^9 \ni x \rightarrow (y, z) \in \Gamma_\alpha \times \mathbb{R}, \quad x = y + z \nu_\alpha(y),$$

where $\nu_\alpha(y)$ is the unit normal to $\Gamma_\alpha$ at $y$ with $\nu_\alpha(y) \cdot e_9 > 0$, will be particularly well-suited to the
problem. We will later show in Lemma 3.1 that the coordinates

$$\mathbb{R}^9 \ni x \rightarrow (y, z) \in \Gamma_1 \times \mathbb{R}, \quad x = y + z \nu_1(y),$$

are well-defined in a thin band around $\Gamma = \Gamma_1$

$$B_{\Gamma}(d) = \{ x \in \mathbb{R}^9 : \text{dist}(x, \Gamma) < d \}$$

for $d > 0$ small enough. By taking blow-ups of space $x \rightarrow \alpha^{-1} x$ we see the coordinates (2.1) with
respect to the blow-up $\Gamma_\alpha$ will be well defined in the band $B_{\Gamma_\alpha}(\alpha^{-1} d)$. Thus we can ensure that the
coordinate system (2.1) is well defined in a unit-size band $B_{\Gamma_\alpha} = B_{\Gamma_\alpha}(2)$ for all $\alpha > 0$ small enough.
The trick of scaling will prove quite useful in what comes later, as well. Its effect on the geometry in
the unit-width band is described in §3, Lemma 3.4.

Not surprisingly, we look for a supersolution/subsolution that is a polynomial in $z$:

$$w(y, z) = \sum_{k=0}^{m} h_k^\alpha(y) z^k,$$

where $h_k^\alpha(y) \approx 1$ to main order at infinity, whereas all the other coefficient decay appropriately to 0. As
it turns out, $m = 5$ suffices.

The Euclidean Laplacian is given by the following key formula:

$$\Delta = \Delta_{\Gamma_\alpha(z)} + \partial^2 \mu_\alpha(z) - H_{\Gamma_\alpha}(\mu_\alpha(z)) \partial z,$$

where $\Delta_S$ denotes the Laplace-Beltrami operator on a surface $S$,

$$\Gamma_\alpha(z) = \{ y + z \nu_\alpha(y) : y \in \Gamma \}$$

is a level set for the signed distance to $\Gamma_\alpha$ and $H_{\Gamma_\alpha}(\mu_\alpha(z))$ is its mean curvature at $y + z \nu(y)$. Note
that if $k_i^\alpha(y)$ denote the principal curvatures of $\Gamma_\alpha$ at $y$

$$H_{\Gamma_\alpha}(\mu_\alpha(z)) = \sum_{i=1}^{8} \frac{1}{(k_i^\alpha - \frac{1}{z})} = \sum_{i=1}^{8} \frac{k_i^\alpha}{1 - k_i^\alpha z}. (2.3)$$

Expanding (2.3) in $z$, we formally have

$$H_{\Gamma_\alpha}(\mu_\alpha(z)) = (H_{1, \alpha} = 0) + \sum_{l=2}^{\infty} z^{l-1} H_{l, \alpha}, (2.4)$$

where $H_{l, \alpha} = \sum_{i=1}^{8} (k_i^\alpha)^l$ is the sum of the $l$-th powers of the principal curvatures of $\Gamma_\alpha$. It turns out
that the principal curvatures

$$k_i^\alpha(y) = O(\alpha(1 + \alpha |x'|)^{-1}), \quad y = (x', F_\alpha(x')) \in \Gamma_\alpha$$
meaning in particular that the series (2.4) converges rapidly for \((y, z) \in B_{r_\alpha}\) and \(\alpha\) small enough. However, a more refined understanding of the asymptotics of the quantities \(H_k := H_{k,1}\) will be needed.

Del Pino, Kowalczyk and Wei faced the exact same issue in [12]. For the purpose, they introduce the model graph \(\Gamma_\infty\), which has an explicit coordinate description and which matches \(\Gamma\) very well at infinity. This allows one to approximate geometric data and geometric operators on \(\Gamma\) with their counterparts on \(\Gamma_\infty\): for example, the second fundamental form, the quantities \(H_k\), the intrinsic gradient and the Laplace-Beltrami operator. We will briefly present their results concerning the geometry of \(\Gamma\) and the closeness between \(\Gamma\) and \(\Gamma_\infty\), and use their framework to prove additional relevant estimates in Section 3.

Choosing the coefficients \(h_0^\alpha(y)\) so that \(w\) meets the supersolution conditions is the subject of Section 4. In fact, most of the choices will be imposed on us (see Remark 4.1): they will be given in terms of geometric quantities like \(H_{l,\alpha}\) and their covariant derivatives. The upshot is that (see Lemmas 4.1 and 4.3)

\[
\Delta w = J_{\Gamma,\alpha} h_0^\alpha + h_2^\alpha - z^2 H_{3,\alpha} + \text{lower order terms} \quad \text{in } B_{r_\alpha},
\]

\[
|\nabla w| = 1 \pm h_2^\alpha + \text{lower order terms} \quad \text{on } \{w = \pm 1\},
\]

(2.5)

where

\[
J_{\Gamma,\alpha} = \Delta_{\Gamma,\alpha} + |A_{\alpha}|^2
\]

is the Jacobi operator on \(\Gamma_\alpha\) and \(h_2^\alpha = 2 h_0^\alpha - |A_{\alpha}|^2 h_0^\alpha\). Thus, by varying \(h_0^\alpha\) we can satisfy the superharmonicity condition and by varying \(h_2^\alpha\) — the gradient condition on the free boundary. So, \(h_2^\alpha\) needs to be positive and we will also require that \(h_0^\alpha > 0\). That way, we only have to flip the signs of the coefficients \(h_0^\alpha\) and \(h_2^\alpha\) in the ansatz (and leave the remaining ones unchanged) in order to produce a subsolution ansatz that automatically lies underneath the supersolution.

So, we want the function \(h_0^\alpha\) to be a positive supersolution for \(J_{\Gamma,\alpha}\) that satisfies an appropriate differential inequality. It turns out that \(J_{\Gamma}\) admits nonnegative supersolutions \(h\) of the following types:

- **Type 1** is such that \(J_{\Gamma} h\) can absorb terms that decay like \(r^{-k}\) for \(k > 4\). See Proposition 4.1. This is useful when dealing with the lower order terms in (2.5).
- **Type 2** can take care of the \(|H_3|\)-term which is globally \(O(r^{-3})\) but has the important additional property that it vanishes on the Simons cone \(S = \{(\vec{u}, \vec{v}) \in \mathbb{R}^4 \times \mathbb{R}^4 : |\vec{u}| = |\vec{v}|\}\).

The Type 1 supersolution is readily provided by Del Pino, Kowalczyk and Wei’s paper [12, Proposition 4.2(b)]. We build the Type 2 supersolution ourselves in Section 4.2 (asymptotically in Lemma 4.6 and globally in Proposition 4.2) and the construction involves a very delicate patching of two supersolutions in a region of the graph over the Simons cone. The ingredients are contained in the analysis of the linearized mean curvature operator \(H'[F_\infty]\) around \(\Gamma_\infty\) carried out in [12, §7], Lemma 7.2 and 7.3; for the reader’s convenience we state these results in Appendix A.3. The operators \(J_{\Gamma,\infty} := \Delta_{\Gamma,\infty} + |A_\infty|^2\) and \(H'[F_\infty]\) are closely related (see (A.1)) and in turn \(J_{\Gamma,\infty}\) is asymptotically close to \(J_{\Gamma}\) (see (3.24)). This allows one to first build a (weak) supersolution away from the origin, which can then be upgraded to a global smooth supersolution via elliptic theory.

Having these two types of barriers for \(J_{\Gamma}\) we will be able to satisfy the free boundary supersolution conditions far away from the origin. In order to satisfy them globally, we employ the trick of scaling by \(\alpha\). That way we can also ensure that both the supersolution and the subsolution are monotonically increasing in \(x_0\) for all small enough \(\alpha > 0\).

Once we have obtained the monotone sub-supersolution pair \(V \leq W\) we proceed to construct the solution \(u\) to (1.1) as a global minimizer of the functional \(I\), constrained to lie in-between \(V \leq u \leq W\). This is the subject of Section 5.
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3. The Bombieri-De Giorgi-Giusti graph $\Gamma$ and its approximation.

In this section we will describe several results concerning the asymptotic geometry of the entire minimal graph in 9 dimensions, constructed by Bombieri, De Giorgi and Giusti in [5]. Some of them have been covered by the analysis of the graph $\Gamma$, carried out by Del Pino, Kowalczyk and Wei in [12] (Lemmas 3.1, 3.2, 3.4 and 3.5 below). We will state those in a form suitable for our later computations. Furthermore, we will establish the important estimates for the covariant derivatives of $H_l$ in Lemma 3.3.

Let us first set notation. Recall, we denote the entire minimal graph by

$$\Gamma = \{(x', x_9) \in \mathbb{R}^8 \times \mathbb{R} : x_9 = F(x')\} \subset \mathbb{R}^8 \times \mathbb{R},$$

where $F : \mathbb{R}^8 \to \mathbb{R}$ is an entire solution to the minimal surface equation (MSE):

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \text{ in } \mathbb{R}^8. \quad (3.1)$$

The graph enjoys certain nice symmetries. Write $x' = (\vec{u}, \vec{v}) \in \mathbb{R}^8$, where $\vec{u}, \vec{v} \in \mathbb{R}^4$ and $u = |\vec{u}|$, $v = |\vec{v}|$. Then $F$ satisfies

$$F \text{ is radially symmetric in both } \vec{u}, \vec{v}, \text{ i.e. } F = F(u, v)$$

$$F(u, v) = -F(v, u), \quad (3.2)$$

so that $F$ vanishes on the Simons cone

$$S = \{u = v\} = \{x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\} \subset \mathbb{R}^8.$$

3.1. Geometry of the unit-width band around $\Gamma$. We will use powers of

$$r(x) = |x'| \text{ for } x = (x', x_9) \in \mathbb{R}^8 \times \mathbb{R} = \mathbb{R}^9,$$

to measure the decay rate of various quantities at infinity. As mentioned in the beginning, we are interested in the domain of definition for the coordinates (2.2):

$$\mathbb{R}^9 \ni x \to (y, z) \in \Gamma \times \mathbb{R} \quad x = y + z\nu(y),$$

where $\nu(y)$ is the unit normal to $\Gamma$ at $y \in \Gamma$ such that $\nu(y) \cdot e_9 > 0$. In particular, we are considering a type of domains which is a thin band around the graph $\Gamma$:

$$\mathcal{B}_\Gamma(d) = \{x \in \mathbb{R}^9 : \text{dist}(x, \Gamma) < d\}.$$

**Lemma 3.1** (cf. Remark 8.1 in [12]). There exists a small enough $d > 0$ such that the coordinates (2.2) are well-defined in $\mathcal{B}_\Gamma(d)$.

The level surfaces for the signed distance to $\Gamma$

$$\Gamma(z) = \{x \in \mathbb{R}^9 : \text{ signed dist}(x, \Gamma) = z\}$$

are prominent in our analysis as it will be necessary to estimate derivative operators on $\Gamma(z)$, for small $z$. According to Lemma 3.1, the coordinates (2.2) are well-defined in a thin-enough band $\mathcal{B}_\Gamma(d)$. Equivalently, the orthogonal projection onto $\Gamma$ is well-defined in $\mathcal{B}_\Gamma(d)$:

$$\pi_\Gamma : \mathcal{B}_\Gamma(d) \to \Gamma \quad \pi_\Gamma(x) = y.$$
Lemma 3.3. The following useful estimates.

We will use the following lemma, quantifying the proximity of the gradient and the Laplace-Beltrami operator on $\Gamma(z)$ acting on $\tilde{f}$ to the gradient and the Laplace-Beltrami operator on $\Gamma$ acting on $f$.

**Lemma 3.2** (cf. §3.1 in [12]). There exists $0 < d < 2$ small enough so that

- The coordinates (2.2) are well-defined;
- If we fix $z$ with $|z| < d$ and assume $\tilde{f} \in C^2(\Gamma(z))$ and $f \in C^2(\Gamma)$ are related via (3.3), we have the following comparison of their gradients, viewed as Euclidean vectors:

$$|\nabla_{\Gamma(z)} \tilde{f} - (\nabla_{\Gamma} f) \circ \pi_{\Gamma}| = O \left( z \frac{|D_{\Gamma} f|}{1 + r} \right) \circ \pi_{\Gamma}, \quad (3.4)$$

while the Laplace-Beltrami operators on $\Gamma(z)$ and $\Gamma$ are related by:

$$\Delta_{\Gamma(z)} \tilde{f} = \left( \Delta_{\Gamma} f + O \left( z \frac{|D_{\Gamma} f|}{1 + r} + z^2 \frac{|D_{\Gamma} f|}{1 + r^2} \right) \right) \circ \pi_{\Gamma}. \quad (3.5)$$

Recall that we are interested in the decay rate of the quantities

$$H_l = \sum_{i=1}^{8} k_i^l,$$

where $\{k_i\}_{i=1}^{8}$ are the principal curvatures of $\Gamma$ – the eigenvalues of the second fundamental form. We prove the following useful estimates.

**Lemma 3.3.** The $k$-th order intrinsic derivatives of the quantity $H_l$ are bounded by

$$|D_{\Gamma}^k H_l(y)| \leq \frac{C_{kl}}{1 + r^{l+k}(y)}, \quad (3.6)$$

for some numerical constants $C_{kl} > 0$.

Finally, we would like to investigate how scaling space $x \to \alpha^{-1} x$ affects the estimates in Lemmas 3.2 and 3.3. For a function $f$ on $\Gamma$, define $f_{\alpha}$ to be the corresponding function on $\Gamma_{\alpha} = \alpha^{-1} \Gamma:

$$f_{\alpha}(y) = f(\alpha y).$$

Also, denote $\{k_i^\alpha\}_{i=1}^{8}$ to be the principal curvatures of $\Gamma_{\alpha}$ and

$$H_{l,\alpha} = \sum_{i=1}^{8} (k_i^\alpha)^l.$$

**Lemma 3.4.** Scaling space $x \to \alpha^{-1} x$ has the following effects:

- The intrinsic $k$-th order derivatives of $f_{\alpha}$, $k = 0, 1, 2, \ldots$, scale like

$$D_{\Gamma_{\alpha}}^k f_{\alpha}(y) = \alpha^k (D_{\Gamma}^k f)(\alpha y) \quad y \in \Gamma_{\alpha}; \quad (3.7)$$

- The quantities

$$D_{\Gamma_{\alpha}}^k H_{l,\alpha}(y) = \alpha^{k+l} (D_{\Gamma}^k H_l)(\alpha y) = O \left( \frac{\alpha^{l+k}}{1 + (\alpha r(y))^{l+k}} \right) \quad y \in \Gamma_{\alpha}; \quad (3.8)$$

- If the coordinates (2.2) are well-defined in a band $B_{\Gamma}(d)$, the coordinates

$$\mathbb{R}^2 \ni x \to (y, z) \in \Gamma_{\alpha} \times \mathbb{R} \quad x = y + z \nu_{\alpha}(y),$$

where $\nu_{\alpha}(y) = \nu(\alpha y)$ is the unit-normal to $\Gamma_{\alpha}$ at $y$, will be well-defined in the band $B_{\Gamma_{\alpha}}(d/\alpha)$. Thus, the orthogonal projection onto $\Gamma_{\alpha}$ is well-defined in $B_{\Gamma_{\alpha}}(d/\alpha)$:

$$\pi_{\Gamma_{\alpha}} : B_{\Gamma_{\alpha}}(d/\alpha) \to \Gamma_{\alpha} \quad \pi_{\Gamma_{\alpha}}(x) = y.$$
• If $|z| < d/\alpha$ is small enough and $\tilde{f}_a \in C^2(\Gamma_\alpha(z))$ and $f_a \in C^2(\Gamma_\alpha)$ are related via $\tilde{f}_a = f_a \circ \pi_{\Gamma_\alpha}$, the estimates corresponding to (3.4) and (3.5) take the form

$$
|\nabla_{\Gamma_\alpha(z)} \tilde{f}_a - (\nabla_{\Gamma_\alpha} f_a) \circ \pi_{\Gamma_\alpha}| = O\left(\frac{\alpha |D_{\Gamma_\alpha} f_a|}{1 + \alpha r}\right) \circ \pi_{\Gamma_\alpha},
$$

(3.9)

$$
\Delta_{\Gamma_\alpha(z)} \tilde{f}_a = \Delta_{\Gamma_\alpha} f_a + O\left(\alpha \frac{|D_{\Gamma_\alpha}^2 f_a|}{1 + \alpha r} + \alpha^2 \frac{|D_{\Gamma_\alpha} f_a|}{1 + (\alpha r)^2}\right) \circ \pi_{\Gamma_\alpha}.
$$

(3.10)

Let us now turn to the proofs of the aforementioned lemmas.

We will take advantage of the following local representation of the minimal graph $\Gamma$. At each $y = (x_0', F(x_0')) \in \Gamma$ denote by $T = T(y)$ the tangent hyperplane to $\Gamma$ at $y$. A simple consequence of the Implicit Function Theorem states that $\Gamma$ can locally be viewed as a smooth (minimal) graph over a neighbourhood in the tangent hyperplane $T$. Concretely, if $\{\tilde{e}_i\}_{i=1}^8$ is an orthonormal basis for $T$ and $\tilde{e}_9 = \nu(y)$ is the unit normal to $T$, there exists a a small enough $a = a(y)$ and a smooth function $G : T \cap B_a(y) \to \mathbb{R}$ so that in a neighbourhood of $x_0'$,

$$(x', F(x')) = (x_0', F(x_0')) + \sum_{i=1}^8 t_i \tilde{e}_i + G(t) \tilde{e}_9 \quad \forall |t| < a.
$$

(3.11)

Moreover, $G(t)$ satisfies the MSE $H[G] = 0$ in $\{|t| < a\}$. In [12] the authors establish the following key estimates for $G$, which provide the basis for the lemmas, stated above.

**Lemma 3.5** (cf. Proposition 3.1 and §8.1 in [12]). Fix $y \in \Gamma$ and let $\rho = 1 + r(y)$. There exists a constant $\beta > 0$, independent of $y$, such that the local representation (3.11) is defined in a neighbourhood $\{|t| < a(y)\} \subset T$ with $a(y) = \beta \rho$. Moreover,

$$
|D_2 G(t)| \leq \frac{c|t|}{\rho}, \quad |D_1^k G(t)| \leq \frac{c_k}{\rho^{k-1}} \quad \text{in} \quad |t| \leq \beta \rho
$$

(3.12)

for $k \in \mathbb{N}$ and some numerical constants $c, c_k > 0$. Also, the unit normal $\nu$ to $\Gamma$ doesn’t tilt significantly over the same neighbourhood:

$$
|\nu(t, G(t)) - \nu(y)| \leq \frac{c|t|}{\rho} \quad |t| \leq \beta \rho.
$$

(3.13)

The proof of Lemma 3.5 is based on Simon’s estimate for the second fundamental form of minimal graphs that admit tangent cylinders at infinity, [20, Thm.4, p.673],

$$
|A|^2(y) \leq \frac{c}{1 + r(y)^2},
$$

and employs standard MSE estimates applied on an appropriate rescale of $G$.

**Lemma 3.1**, the possibility to define the coordinates (2.2) in a thin enough band around $\Gamma$, is a corollary of (3.13).

**Proof of Lemma 3.1.** Assume the contrary: that there doesn’t exist a $d > 0$ for which the coordinates (2.2) are well-defined in $B_\Gamma(d)$. The coordinates will fail to represent a point $x \in B_\Gamma(d)$ uniquely when there exist two points $y_1 \neq y_2 \in \Gamma$ such that

$$
|x - y_1| = |x - y_2| = \text{dist}(x, \Gamma) < d,
$$

i.e. if

$$
x = y_1 + z \nu(y_1) = y_2 + z \nu(y_2).
$$

We have $|y_1 - y_2| = |z| |\nu(y_2) - \nu(y_1)| \leq 2d$. This means that if $d$ is sufficiently small ($d \leq \frac{1}{2} \beta(1 + r(y_1))$, for example), $y_2$ lies in the portion of $\Gamma$ which is a graph over $T(y_1) \cap \{ |t| \leq \beta(1 + r(y_1)) \}$ – a neighbourhood of the tangent hyperplane at $y_1$. But then (3.13) gives us

$$
|y_1 - y_2| \leq d |\nu_1 - \nu_2| \leq \frac{cd}{1 + r(y_1)} |y_1 - y_2| \leq cd |y_1 - y_2|
$$
which is impossible whenever \( d \) is small enough so that \( cd < 1 \).

**Proof of Lemma 3.2.** Let \( \tilde{y} \in \Gamma(z) \), \( y = \pi_\Gamma(\tilde{y}) \) and let \( T = T(y) \) be the tangent hyperplane to \( \Gamma \) at \( y \). Use the Euclidean coordinates \( t \) on \( T \) to parameterize \( \Gamma(z) \) near \( \tilde{y} \):

\[
t \to (t, G(t)) + z\nu(y(t)),
\]

where \( \nu(y(t)) = \frac{(-D_t G, 1)}{\sqrt{1 + |D_t G|^2}} \) is the unit normal to \( \Gamma \) at \( y(t) = (t, G(t)) \). Note that in these coordinates

\[
\hat{f}(t) = f(t).
\]

As before, set \( \rho = 1 + r(y) \) and use Einstein index notation. Because of (3.12), the metric tensor \( g(z) \) on \( \Gamma(z) \) computes to:

\[
g_{ij}(z) = \delta_{ij} + G_i G_j + z(\tilde{e}_i + G_i \nu) \cdot \partial_j \nu + z(\tilde{e}_j + G_j \nu) \cdot \partial_i \nu + z^2 \partial_i \nu \cdot \partial_j \nu
\]

\[
= g_{ij}(0) - 2zG_{ij} + z^2 \partial_i \nu \cdot \partial_j \nu = g_{ij}(0) + O(z \rho^1) + O(z^2 \rho^{-2})
\]

while its inverse

\[
g^{ij}(z) = g^{ij}(0) + O(z \rho^{-1})
\]

for \( |z| \leq d \) small enough. Noting that \( g_{ij} = g_{ij}(0) \) is the metric tensor on \( \Gamma \) in \( t \)-coordinates, we see that the difference of gradients, viewed as vectors in Euclidean space:

\[
|\nabla_{\Gamma(z)} \hat{f}(\tilde{y}) - \nabla_{\Gamma} f(y)| =
\]

\[
= |g^{ij}(z) \partial_j f(0)(\tilde{e}_i + G_i(0) \tilde{e}_9) + z \partial_i \nu(0)) - g^{ij}(0) \partial_j f(0)(\tilde{e}_i + G_i(0) \tilde{e}_9)|
\]

\[
= O(z |D_{\Gamma} f(y)| \rho^{-1}).
\]

The derivatives of the metric tensor satisfy:

\[
\partial_k [g_{ij}(z)] = \partial_k g_{ij} + O(z \rho^{-2})
\]

so that

\[
\partial_k [g^{ij}(z)] = -g^{ij}(z) \partial_k g_{im}(z) g^{mj}(z) = \partial_k g^{ij} + O(z \rho^{-1}).
\]

Thus,

\[
\Delta_{\Gamma(z)} \hat{f}(\tilde{y}) = \frac{1}{\sqrt{|g(z)|}} \partial_1 (g^{ij}(z) \sqrt{|g(z)|} \partial_j f) =
\]

\[
= g^{ij}(z) \partial^2 f + \partial_i g^{ij}(z) \partial^2 f + \frac{g^{ij}(z) \partial_i g(z)}{2 |g(z)|} \partial_j f =
\]

\[
= \Delta_{\Gamma} f(y) + O(z \rho^{-1} |D^2_{\Gamma} f(y)|) + O(z \rho^{-2} |D_{\Gamma} f(y)|).
\]

**Proof of Lemma 3.3.** Set \( \rho = 1 + r(y) \) and write the metric tensor of \( \Gamma \) around \( y \) in the coordinates \( t \), (3.11):

\[
g_{ij} = \delta_{ij} + G_i G_j = \delta_{ij} + O(\beta^2) \quad |t| \leq \beta \rho.
\]

Its inverse takes the form

\[
g^{ij} = \delta_{ij} - \frac{G_i G_j}{1 + |\nabla G|^2} = \delta_{ij} + O(\beta^2) \quad |t| \leq \beta \rho.
\]

Mind that constants in the \( O \)-notation are independent of \( \rho \). Taking into account (3.12) we see that for \( |t| \leq \beta \rho \), \( k = 0, 1, 2 \ldots \)

\[
|D^k_t g_{ij}(t)| \leq \frac{c_k^{t}}{\rho^k} \quad |D^k_{\Gamma} g^{ij}(t)| \leq \frac{c_k^{\rho}}{\rho^k}.
\]
An easy consequence is the fact that the intrinsic $k$-th order derivative of a function $f$ on $\Gamma$ at $y$ will be majorized by the $t$-derivatives of $f$ at $t = 0$ up to order $k$ as follows:

$$|D^k_t f(y)| \leq \sum_{j=1}^k c_j r^{j-k} |D^j_t f(0)|$$  \hspace{1cm} (3.15)$$

for some numerical constants $c_k > 0$. Since $g_{ij} = O(1)$ and $g^{ij} = O(1)$, it suffices to show that

$$\nabla I f = \partial_I f + \sum_{|J| < k} c_J(t) \partial_J f,$$

where

$$|D^m_t c_J(t)| = O(\rho^{-|J|-m}) \quad |t| \leq \beta \rho, \quad m = 0, 1, \ldots$$  \hspace{1cm} (3.16)$$

Here, of course, $I, J$ denote multi-indices (e.g. if $I = (i_1, i_2, \ldots, i_k)$, $|I| = k$), $\nabla$ denotes covariant differentiation and

$$\partial_I f = \partial_{i_1 i_2 \ldots i_k} f$$

$$\nabla I f = (\nabla^k f)(\partial_{i_1}, \partial_{i_2}, \ldots, \partial_{i_k}).$$

We’ll prove (3.16) by induction on $k = |I|$. When $k = 1$, the statement is obviously true and assume it holds up to $k - 1$. For convenience, define the following transformation on multi-indices of length $k - 1$:

$$\sigma^j_l (j_1, j_2, \ldots, j_{k-1}) = (j_1, \ldots, j_l-1, j_l, j_{l+1}, \ldots, j_{k-1}) \quad 1 \leq l \leq k - 1.$$  \hspace{1cm} (3.17)$$

So, if $|I| = k$ and we write $I = (i_1, I')$, $I' = (i_2, \ldots, i_k)$, the covariant differentiation rule gives

$$\nabla I f = \partial_{i_1} (\nabla_{I'} f) - \sum_{1 \leq l = \beta \rho, \quad m = 0, 1, \ldots} |D^m_t c_J(t)| = O(\rho^{-|J|-m}) \quad |t| \leq \beta \rho,$$

where $\Gamma^k_{ij}$ are the Christoffel symbols for the metric tensor $g$ in the coordinates $t$. We only need to check $D^m_t \Gamma^k_{ij} = O(\rho^{-1-m})$, which follows immediately from (3.14). The induction step is complete.

Let us apply (3.15) to the second fundamental form

$$A^j_i = A_{ik} g^{kj} = \frac{G_{ik} g^{kj}}{\sqrt{1 + |\nabla G|^2}} = O(\rho^{-1}) \quad |t| \leq \beta \rho,$$

(we use the Einstein index notation again). Observe that its $t$-derivatives decay like

$$|D^m_t (A^j_i)| = O(\rho^{-1-m}) \quad |t| \leq \beta \rho.$$  \hspace{1cm} (3.18)$$

Since $H_l = \text{Trace}([A^j_i])$, $D^m_t H_l = \text{Trace}(D^m_t [A^j_i]) = O(\rho^{-m-l})$ \hspace{1cm} (3.19)$$

Hence, (3.15) implies the desired

$$|D^k_t H_l(y)| \leq C^l_k r^{l+k}.$$  \hspace{1cm} (3.20)$$

\Box

Proof of Lemma 3.4. The results in the lemma are obtained after simple length-scale considerations. Equation (3.7) is immediate. The principal curvatures $k^\alpha(y)$ scale like distance$^{-1}$, so that

$$k^\alpha_t(y) = \alpha k^\alpha(y) \quad y \in \Gamma_\alpha,$$

and thus

$$H_{l,\alpha}(y) = \alpha^l H_l(\alpha y) = \alpha^l (H_l)_\alpha(y).$$  \hspace{1cm} (3.22)$$

We invoke (3.7) and (3.6) to obtain the full estimate (3.8).
We are left to check (3.9) and (3.10), which follow from (3.4) and (3.5) right after we note that
\[
\begin{align*}
(\nabla_{\Gamma_{\alpha z}} \tilde{f})_\alpha(y + z\nu_\alpha(y)) &= \alpha (\nabla_{\Gamma_{\alpha z}} \tilde{f})(\alpha y + z\nu(y)) \\
(\Delta_{\Gamma_{\alpha z}} \tilde{f}_\alpha)(y + z\nu_\alpha(y)) &= \alpha^2 (\Delta_{\Gamma_{\alpha z}} \tilde{f})(\alpha y + z\nu(y)),
\end{align*}
\]
where \( \tilde{f} = f \circ \pi_\Gamma \) is the lift of \( f \) onto \( \Gamma_{\alpha z} \).

\[\square\]

3.2. Proximity between \( \Gamma \) and the model graph. A more refined knowledge of the asymptotics of \( \Gamma \) is needed in order to carry out the construction of a supersolution to (1.1). To extract better information about geometry of \( \Gamma \) at infinity, Del Pino, Kowalczyk and Wei [12, §2] introduce a model graph \( \Gamma_\infty \), which has an explicit formula and which approximates \( \Gamma \) very well at infinity. Namely, the model graph
\[
\Gamma_\infty = \{(x', x_9) \in \mathbb{R}^8 : x_9 = F_\infty(x')\}
\]
where \( F_\infty : \mathbb{R}^8 \to \mathbb{R} \) solves the “homogenized” MSE:
\[
\nabla \cdot \left( \frac{\nabla F_\infty}{|\nabla F_\infty|} \right) = 0, \tag{3.18}
\]
has the same growth (\( \sim r^3 \)) at infinity as \( F \) and shares the same symmetries (3.2). This determines the function \( F_\infty \) uniquely up to a multiplicative constant: if we use polar coordinates to write
\[
u = r \cos \theta \quad v = r \sin \theta,
\]
the function takes the form \( F_\infty(r, \theta) = r^3 g(\theta) \) where \( g(\theta) \in C^2[0, \pi/2] \) is the unique (up to a scalar multiple) solution to
\[
\frac{21g \sin^2(2\theta)}{\sqrt{9g^2 + g'^2}} + \left( \frac{g' \sin^3(2\theta)}{\sqrt{9g^2 + g'^2}} \right)' = 0, \quad \text{in} \quad \theta \in [0, \pi/2],
\]
g\left(\frac{\pi}{2}\right) = 0.
\]
which is odd with respect to \( \theta = \pi/4 \). For concreteness, we pick the \( g(\theta) \) which satisfies in addition \( g'(\frac{\pi}{4}) = 1 \).

Del Pino, Kowalczyk and Wei then prove the following result quantifying the asymptotic proximity between the BdGG graph and the model graph.

**Theorem 3.1** (cf. Theorem 2 in [12]). *There exists a function \( F = F(u, v) \), an entire solution to the minimal surface equation (3.1) which has the symmetries (3.2) and satisfies
\[
F_\infty \leq F \leq F_\infty + \frac{C}{r^\sigma} \min\{F_\infty, 1\} \quad \text{in} \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \quad r > R_0 \tag{3.19}
\]
for some constants \( C, R_0 > 0 \) and \( 0 < \sigma < 1 \).*

The proximity of \( \Gamma \) and \( \Gamma_\infty \) at infinity allows one to approximate geometric data and geometric operators defined on the non-explicit \( \Gamma \) with their counterparts on the explicit \( \Gamma_\infty \). To put it more concretely: one can use the orthogonal projection \( \pi_\Gamma \) onto \( \Gamma \) to identify functions \( f \) defined on \( \Gamma \) with functions \( f_\infty \) defined on \( \Gamma_\infty \) far away from the origin:
\[
f_\infty = f \circ \pi_\Gamma \quad \Gamma_\infty \cap \{r > R\} \tag{3.20}
\]
for \( R > 0 \) large enough. Then one can compare \( (\nabla_{\Gamma_{\alpha z}} f) \circ \pi_\Gamma \) to \( \nabla_{\Gamma_\infty} f_\infty \) and \( (\Delta_{\Gamma_{\alpha z}} f) \circ \pi_\Gamma \) to \( \Delta_{\Gamma_\infty} f_\infty \). Also, if we denote \( \{(k_{\infty,i})_i\}_{i=1}^8 \) to be the principal curvatures of \( \Gamma_\infty \), and
\[
H_{\infty, i} = \sum_{i=1}^8 (k_{\infty,i})^i,
\]
for \( i = 1, \ldots, 8 \).
one can use the explicit $H_{\infty}t$ to approximate the curvature quantities $H_t$ associated with $\Gamma$. The ultimate goal is to approximate the Jacobi operator on $\Gamma$,

$$J_\Gamma f = (\Delta_\Gamma + H_2)f$$

asymptotically with the Jacobi operator on $\Gamma_\infty$,

$$J_{\Gamma_\infty} f_\infty = (\Delta_{\Gamma_\infty} + H_{\infty,2})f_\infty.$$

On the basis of Theorem 3.1, Del Pino, Kowalczyk and Wei establish the following list of results.

**Lemma 3.6** (cf. §8.2, 8.3 in [12]). For enough away from the origin, $r > R$, for some constant $0 < \sigma < 1$,

- One can express $\Gamma_\infty$ locally as a graph of a function $G_\infty(t)$ over a neighbourhood of the tangent hyperplane $T = T(y)$ to $y \in \Gamma$ with $r(y) > R$. Moreover, for some constants $C_k > 0$, $k = 0, 1, \ldots$

$$|D_t^k(G - G_\infty)|_{t=0} \leq \frac{C_k}{r(y)^{k+1+\sigma}},$$

where $(t, G(t))$ is the local parametrization (3.11) of $\Gamma$.

- The Laplace-Beltrami operator on $\Gamma$ can be approximated with the Laplace-Beltrami operator on $\Gamma_\infty$ as follows:

$$\boxed{(\Delta_\Gamma f) \circ \pi_\Gamma = \Delta_{\Gamma_\infty} f_\infty + O(r^{-2-\sigma}|D_t^2 f_\infty| + r^{-3-\sigma}|D_t f_\infty|),}$$

where $f$ and $f_\infty$ are related via (3.20).

- The quantities $H_2 = |A|^2$ and $H_3$ of $\Gamma$ are approximated by $H_{\infty,2}$ and $H_{\infty,3}$, respectively, as follows:

$$H_2 \circ \pi_\Gamma = H_{\infty,2} + O(r^{-4-\sigma})$$

$$H_3 \circ \pi_\Gamma = H_{\infty,3} + O(r^{-5-\sigma}).$$

- Therefore, the Jacobi operators on $\Gamma$ and $\Gamma_\infty$ are related by

$$\boxed{(J_\Gamma f) \circ \pi_\Gamma = J_{\Gamma_\infty} f_\infty + O(r^{-2-\sigma}|D_t^2 f_\infty| + r^{-3-\sigma}|D_t f_\infty| + r^{-4-\sigma}|f_\infty|.}$$

4. Construction of the super and subsolution.

4.1. The ansatz. Define the $L^\infty$ weighted norms

$$\|f\|_{k,L^\infty(\Omega)} = \|(1 + r(y)^k)f(y)\|_{L^\infty(\Omega)}$$

for regions $\Omega \subseteq \Gamma$. Use the short-hand $\| \cdot \|_{k,\infty}$ when $\Omega = \Gamma$.

Recall that for $\alpha > 0$ small enough the coordinates (2.1) are well-defined in the band $B_{\Gamma_\alpha} = B_{\Gamma_\alpha,2}$. We will work with the following ansatz $w : B_{\Gamma_\alpha} \to \mathbb{R}$:

$$w(y, z) = h_0^\alpha(y) + z h_1^\alpha(y) + z^2 h_2^\alpha(y) + z^3 h_3^\alpha(y) + z^4 h_4^\alpha(y) + z^5 h_5^\alpha(y),$$

where $h_i^\alpha = h_i^\alpha(y)$ are functions on $B_{\Gamma_\alpha}$, independent of the $z$-variable. The coefficients $h_1^\alpha, h_2^\alpha, h_3^\alpha, h_4^\alpha, h_5^\alpha$ are explicitly specified in terms of geometric quantities associated with $\Gamma_\alpha$:

$$h_1^\alpha = 1 - \frac{|A_\alpha|^2}{2} + h_1^\alpha \quad h_1^\alpha = -\frac{5}{24} (\Delta_{\Gamma_\alpha} + |A_\alpha|^2) |A_\alpha|^2 - \frac{H_{4,\alpha}}{4}$$

$$h_2^\alpha = \frac{1}{6} (|A_\alpha|^2 + h_2^\alpha) \quad h_3^\alpha = \frac{1}{2} (\Delta_{\Gamma_\alpha} |A_\alpha|^2)$$

$$h_4^\alpha = \frac{1}{20} (\frac{1}{2} + H_{4,\alpha} - \frac{\Delta_{\Gamma_\alpha} |A_\alpha|^2}{6}).$$

$$\frac{1}{2} + H_{4,\alpha} - \frac{\Delta_{\Gamma_\alpha} |A_\alpha|^2}{6}).$$
According to (3.8) of Lemma 3.4, the size of $h = (h_1^2 - 1), h_2^2$ and their covariant derivatives up to second order on $\Gamma,$

$$h = O\left(\frac{\alpha^2}{1 + (\alpha r)^2}\right), \quad |D_{\Gamma, a} h| = O\left(\frac{\alpha^3}{1 + (\alpha r)^3}\right), \quad |D_{\Gamma, a}^2 h| = O\left(\frac{\alpha^4}{1 + (\alpha r)^4}\right).$$  \hfill (4.3)

while for $h = h_1^2, h_2^2, h_3^2$

$$h = O\left(\frac{\alpha^4}{1 + (\alpha r)^4}\right), \quad |D_{\Gamma, a} h| = O\left(\frac{\alpha^5}{1 + (\alpha r)^5}\right), \quad |D_{\Gamma, a}^2 h| = O\left(\frac{\alpha^6}{1 + (\alpha r)^6}\right).$$  \hfill (4.4)

We set $h_1^2 = 0.$ The coefficients $h_0^2$ and $h_2^2$ will be specified later so that the ansatz meets the supersolution conditions in Definition 2.2, but from the very start we will require that they satisfy the following properties:

- $h_0^2 > 0$ is strictly positive and scales like
  $$h_0^2(y) = \alpha^p h_0(\alpha y),$$
  where $0 < p < 1$ and $h_0 \in C^2(\Gamma)$ is positive with
  $$\|D_\Gamma^2 h_0\|_{1, \infty} + \|D_{\Gamma} h_0\|_{2, \infty} + \|h_0\|_{1, \infty} \leq C_1.$$  \hfill (4.5)

for some positive constant $C_1.$ So,

$$h_0^2 = O\left(\frac{\alpha^p}{1 + (\alpha r)^2}\right), \quad |D_{\Gamma, a} h_0^2| = O\left(\frac{\alpha^{1+p}}{1 + (\alpha r)^3}\right), \quad |D_{\Gamma, a}^2 h_0^2| = O\left(\frac{\alpha^{2+p}}{1 + (\alpha r)^4}\right).$$

- $h_2^2$ equals
  $$h_2^2 = \frac{1}{2}(|A_\alpha|^2 h_0^2 + h_2^2),$$
  where the correction $h_2^\alpha$ scales like
  $$h_2^\alpha(y) = \alpha^{2+p} h_2^\alpha(\alpha y)$$
  and $h_2^\alpha \in C^2(\Gamma)$ is positive with
  $$\|D_\Gamma^2 h_2^\alpha\|_{5, \infty} + \|D_{\Gamma} h_2^\alpha\|_{4, \infty} + \|h_2^\alpha\|_{3, \infty} \leq C_2.$$  \hfill (4.6)

for some positive constant $C_2.$ Thus,

$$h_2^2 = O\left(\frac{\alpha^{2+p}}{1 + (\alpha r)^3}\right), \quad |D_{\Gamma, a} h_2^2| = O\left(\frac{\alpha^{3+p}}{1 + (\alpha r)^4}\right), \quad |D_{\Gamma, a}^2 h_2^2| = O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right).$$

**Remark 4.1.** At first look, the choices for $h_0^\alpha$ above may seem somewhat arbitrary but they are prompted by the supersolution conditions. The fact that we expect the solution to behave asymptotically like $z$ suggests that $h_0^\alpha \approx 1$ to main order. Thus, the main order term in $H_{\Gamma_a(z)} \partial_z w$ is $z |A_\alpha|^2$ which has to be cancelled by the $z^1$-term in $\partial_z^2 w$; thus, $h_3 \approx \frac{|A_\alpha|^2}{2}.$ Now $\partial_z w \approx h_1^2 + z^2 |A_\alpha|^2$ and since $w$ achieves values $\pm 1$ at $z \approx \pm 1$ and $|\nabla w| \approx \partial_z w,$ the supersolution gradient condition demands that we refine $h_1^2$ to equal $h_1^2 \approx 1 - \frac{|A_\alpha|^2}{2}.$ The form of $h_2^2$ is contingent upon the fact that $w(y, \cdot)$ attains the values $\pm 1$ asymptotically at $z_\pm \approx \pm 1 - h_0^2,$ so that

$$\partial_z w(y, z_\pm) \approx 1 \pm (2h_2^2 - |A_\alpha|^2 h_0^2),$$

requiring the positivity of $h_2^\alpha = 2h_0^2 - |A_\alpha|^2 h_0^2.$ The remaining choices (and further refinements) are made so that no terms that decay at a rate $r^{-1}$ (and no better) at infinity are present in the expansions of $\Delta w$ or $\partial_z w(y, z_\pm)$.

All this will become transparent once we carry out the computations of the Laplacian of $w$ in Lemma 4.4 and of the gradient of $w$ on $\{w = \pm 1\}$ in Lemmas 4.2 and 4.3 below.
NB. In what follows the constants in the \(O\)-notation depend solely on \(p\), \(C_1\), \(C_2\) and the minimal graph \(\Gamma\), but not on the scaling parameter \(\alpha\).

**Lemma 4.1.** The Laplacian of \(w\) in \(B_{\Gamma,\alpha}\) can be estimated by

\[
\Delta w(y, z) = (\Delta_{\Gamma,\alpha}(z) + |A_\alpha|^2)h_0^\alpha + h_2^\alpha - z^2 H_{3,\alpha} + O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right). \tag{4.7}
\]

**Proof.** Compute in succession:

\[
\partial_z w = 1 - \frac{|A_\alpha|^2}{2} + h_1^\alpha + z(|A_\alpha|^2 h_0^\alpha + h_2^\alpha) + z^2 |A_\alpha|^2 \frac{h_0^\alpha + h_3^\alpha}{2} + 5z^4 h_5^\alpha \tag{4.8}
\]

\[
\partial_{y, z}^2 w = |A_\alpha|^2 h_0^\alpha + h_2^\alpha + z(|A_\alpha|^2 + h_3^\alpha) + 20z^3 h_5^\alpha \tag{4.9}
\]

\[
H_{\Gamma,\alpha}(z)\partial_z w = \left(z |A_\alpha|^2 + z^2 H_{3,\alpha} + z^3 H_{4,\alpha} + O\left(z^4 \frac{\alpha^5}{1 + (\alpha r)^5}\right)\right)\partial_z w = \\
= z \left(|A_\alpha|^2 - \frac{|A_\alpha|^4}{2} + O\left(\frac{\alpha^6}{1 + (\alpha r)^6}\right)\right) + z^2 \left(H_{3,\alpha} + O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right)\right) \tag{4.10}
\]

\[
+ z^3 \left(\frac{|A_\alpha|^4}{2} + H_{4,\alpha} + O\left(\frac{\alpha^{5+p}}{1 + (\alpha r)^6}\right)\right) + O\left(z^4 \frac{\alpha^5}{1 + (\alpha r)^5}\right)
\]

Because of (3.10) and (4.4),

\[
|\Delta_{\Gamma,\alpha}(z) h_1^\alpha| + |\Delta_{\Gamma,\alpha}(z) h_3^\alpha| + |\Delta_{\Gamma,\alpha}(z) h_5^\alpha| = O\left(\frac{\alpha^6}{1 + (\alpha r)^6}\right)
\]

\[
\Delta_{\Gamma,\alpha}(z) h_2^\alpha = O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right)
\]

\[
\Delta_{\Gamma,\alpha}(z) |A_\alpha|^2 = \Delta_{\Gamma,\alpha} |A_\alpha|^2 + O\left(z^4 \frac{\alpha^5}{1 + (\alpha r)^5}\right)
\]

so that

\[
\Delta_{\Gamma,\alpha}(z) w = \Delta_{\Gamma,\alpha}(z) h_0^\alpha - z \frac{\Delta_{\Gamma,\alpha} |A_\alpha|^2}{2} + z^3 \frac{\Delta_{\Gamma,\alpha} |A_\alpha|^2}{6} + O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right). \tag{4.11}
\]

Combining (4.9), (4.10) and (4.11) we derive that in \(B_{\Gamma,\alpha}\)

\[
\Delta w = (\Delta_{\Gamma,\alpha}(z) + |A_\alpha|^2)h_0^\alpha + h_2^\alpha - z^2 H_{3,\alpha}
\]

\[
+ z \left(- \frac{\Delta_{\Gamma,\alpha} |A_\alpha|^2}{2} + |A_\alpha|^2 + h_3^\alpha - |A_\alpha|^2 + \frac{|A_\alpha|^4}{2}\right) + \\
+ z^3 \left(\frac{\Delta_{\Gamma,\alpha} |A_\alpha|^2}{6} + 20h_5^\alpha - \frac{|A_\alpha|^4}{2} - H_{4,\alpha}\right) = \\
= (\Delta_{\Gamma,\alpha}(z) + |A_\alpha|^2)h_0^\alpha + h_2^\alpha - z^2 H_{3,\alpha} + O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right).
\]

\(\square\)

Now we would like to determine how far the level surfaces \(\{w = \pm 1\}\) stand from the graph \(\Gamma\).

Note that for \(|z| \leq 2\) and uniformly in \(y \in \Gamma\), we have \(w = z + O(\alpha^p)\) and \(\partial_z w = 1 + O(\alpha^2)\). Thus for all small enough \(\alpha > 0\), \(w(y, \cdot)\) is strictly increasing and attains the values \(\pm 1\) for unique \(z_{\pm}(y)\) with \(|z_{\pm}(y)| \leq 2\).

**Lemma 4.2.** For all small enough \(\alpha > 0\) (so that \(z_{\pm}\) is well-defined),

\[
\partial_z w(y, z_{\pm}(y)) = 1 \pm h_2^\alpha - \frac{|A_\alpha|^2(h_0^\alpha)^2}{2} - h_2^\alpha h_0^\alpha + O\left(\frac{\alpha^{2+3p}}{1 + (\alpha r)^5}\right).
\]
Proof. Fix $y \in \Gamma_\alpha$ and let us estimate $z_\pm(y)$. Write $z_\pm = \pm 1 + \delta_\pm$, where $\delta_\pm = o(1)$ as $\alpha \to 0$. We compute

$$w(y, z_\pm) = h_0^\alpha + (1 - \frac{|A_\alpha|}{2})^2 (\pm 1 + \delta_\pm) + h^2_0 (\pm 1 + \delta_\pm)^2 +$$

$$+ \frac{|A_\alpha|^2}{6} (\pm 1 + \delta_\pm)^3 + O\left(\frac{\alpha^4}{1 + (\alpha r)^4}\right)$$

$$= h_0^\alpha (1 - \frac{|A_\alpha|^2}{2}) + h^2_0 \pm \frac{|A_\alpha|^2}{6} +$$

$$+ \delta_\pm (1 + 2h^2_0) + \delta^2_\pm (\frac{|A_\alpha|^2}{2} + h^2_0) + \delta^3_\pm \frac{|A_\alpha|^2}{6} + O\left(\frac{\alpha^4}{1 + (\alpha r)^4}\right).$$

Thus,

$$h_0^\alpha = \frac{|A_\alpha|^2}{3} + h^2_0 + \delta_\pm (1 + 2h^2_0) + \delta^2_\pm (\frac{|A_\alpha|^2}{2} + h^2_0) + \delta^3_\pm \frac{|A_\alpha|^2}{6} = O\left(\frac{\alpha^4}{1 + (\alpha r)^4}\right),$$

so that

$$\delta_\pm = O\left(h_0^\alpha + h^2_0 + \frac{|A_\alpha|^2}{3}\right) = O\left(\frac{\alpha^p}{1 + \alpha r}\right),$$

which in turn implies

$$\delta_\pm = -h_0^\alpha \pm \frac{|A_\alpha|^2}{3} - h^2_0 + O\left(\frac{\alpha^{p+2}}{1 + (\alpha r)^4}\right),$$

(4.12)

We can now estimate $\partial_z w(y, z_\pm(y))$:

$$\partial_z w(y, z_\pm(y)) = 1 - \frac{|A_\alpha|^2}{2} + h_1^\alpha + 2h^2_0 (\pm 1 - h_0^\alpha) +$$

$$+ \frac{|A_\alpha|^2 + h_1^\alpha}{2} (\pm 1 - h_0^\alpha) \pm \frac{|A_\alpha|^2}{3} + 5h_0^\alpha + O\left(\frac{\alpha^{p+3}}{1 + (\alpha r)^5}\right) =$$

$$= 1 + (\pm 2h^2_0 \mp |A_\alpha|^2 h_0^\alpha) + (-2h^2_0 h_0^\alpha + \frac{|A_\alpha|^2 h_0^\alpha}{2}) +$$

$$+ (h_1^\alpha + \frac{h_3^\alpha}{3}) + \frac{|A_\alpha|^4}{3} + 5h_0^\alpha + O\left(\frac{\alpha^{p+3}}{1 + (\alpha r)^5}\right)$$

$$= 1 \pm h_2^\alpha - \frac{|A_\alpha|^2(h_0^\alpha)^2}{2} - h_2^\alpha h_0^\alpha + O\left(\frac{\alpha^{p+3}}{1 + (\alpha r)^5}\right).$$

Straightforward derivative estimates using (3.9) yield

**Lemma 4.3.** We have

$$|\nabla_{\Gamma_\alpha(z_\pm)} w|^2 = |\nabla_{\Gamma_\alpha(z_\pm)} h_0^\alpha|^2 + O\left(\frac{\alpha^{4+p}}{1 + (\alpha r)^5}\right)$$

and thus

$$|\nabla w|^2(y, z_\pm(y)) = (\partial_z w)^2 + |\nabla_{\Gamma_\alpha(z_\pm)} w|^2 =$$

$$= 1 \pm 2h_2^\alpha (1 + O(\alpha^p)) - |A_\alpha|^2(h_0^\alpha)^2 + |\nabla_{\Gamma_\alpha(z_\pm)} h_0^\alpha|^2 + O\left(\frac{\alpha^{p+3}}{1 + (\alpha r)^5}\right).$$

Our ansatz has the very nice, extra feature that it is strictly increasing in $\mathcal{B}_{\Gamma_\alpha}$ in the direction of $c_9$.

**Lemma 4.4.** For all $\alpha$ small enough

$$\partial_{x_9} w > 0 \quad \text{in} \quad \mathcal{B}_{\Gamma_\alpha}.$$
Proof. Computing in the coordinates (2.1)
\[
\partial_{x_9} w(y, z) = \nabla w \cdot e_9 = (\nabla_{\Gamma_\alpha} w + (\partial_z w) \nu(y)) \cdot e_9
\]
\[
\geq \frac{\partial_z w}{\sqrt{1 + |\nabla_{\Gamma_\alpha}|^2}} - |\nabla_{\Gamma_\alpha} w| = \frac{1 + O(\alpha^2)}{\sqrt{1 + |\nabla_{\Gamma_\alpha}|^2}} + O\left(\frac{\alpha^{1+p}}{1 + (\alpha r)^2}\right). \tag{4.13}
\]
Since
\[
\frac{1}{\sqrt{1 + |\nabla_{\Gamma_\alpha}|^2}} \geq c \frac{1}{1 + (\alpha r)^2}
\]
for some positive constant \(c > 0\) (see Remark 8.2 in [12]), (4.13) yields
\[
\partial_{x_9} w > 0
\]
for all small enough \(\alpha > 0\). \(\square\)

4.2. Supersolutions for the Jacobi operator. As we have noticed from Lemma 4.1, the sign of \(\Delta w\) depends crucially on whether the Jacobi operator \(J_{\Gamma, \alpha}\) admits positive supersolutions that satisfy appropriate differential inequalities. We will show that the Jacobi operator \(J_{\Gamma}\) on the (non-rescaled) minimal graph \(\Gamma\) admits the following two types of smooth supersolutions:
- **Type 1** is a positive supersolution \(h \in C^2(\Gamma)\) such that for some \(0 < \epsilon < 1\)
  \[
  J_{\Gamma} h(y) \leq -\frac{1}{1 + r^{4+\epsilon}(y)}.
  \]
- **Type 2** is a positive supersolution \(h \in C^2(\Gamma)\) such that
  \[
  J_{\Gamma} h(y) \leq -\frac{\theta(y) - \pi/4}{1 + r^{3}(y)}.
  \]

The Type 1 supersolution is readily provided by [12, Proposition 4.2(b)] (our Proposition 4.1 below is a straightforward modification). We construct the Type 2 supersolution in Proposition 4.2 and the supporting Lemma 4.6.

**Proposition 4.1.** Let \(0 < \epsilon < 1\). There exists a positive function \(h \in C^2(\Gamma)\) such that
\[
\|D_\Gamma^2 h\|_{3+\epsilon, \infty} + \|D_\Gamma h\|_{2+\epsilon, \infty} + \|h\|_{2+\epsilon, \infty} < \infty
\]
and
\[
J_{\Gamma} h \leq -\frac{1}{1 + r^{4+\epsilon}}.
\]

**Proposition 4.2.** There exists a non-negative function \(h \in C^2(\Gamma)\) such that
\[
\|D_\Gamma^2 h\|_{3, \infty} + \|D_\Gamma h\|_{2, \infty} + \|h\|_{1, \infty} < \infty
\]
and
\[
J_{\Gamma} h \leq -\frac{\theta - \pi/4}{1 + r^{3}}.
\]
Moreover, there is a \(\frac{1}{2} < \tau < \frac{2}{3}\) (e.g. \(\tau = \frac{5}{8}\)) such that for every \(0 \leq \delta < \delta' \leq \frac{3}{2}\)
\[
\|h\|_{1+\delta \tau, L^{\infty}(S(-\delta'))} + \|D_\Gamma h\|_{2+\delta \tau, L^{\infty}(S(-\delta'))} + \|D_\Gamma^2 h\|_{3+\delta \tau, L^{\infty}(S(-\delta'))} < \infty \tag{4.14}
\]
where \(S(-\delta') = \{|\theta - \frac{\pi}{4}| \leq (1 + r)^{-\delta'}\} \subset \Gamma\).
Before we venture into proving these two propositions, recall that
\[ J_Γ h \leq f \] in the weak sense if
\[ (J_Γ h - f)[φ] \leq 0 \] for all non-negative \( φ \in C^1_c(Γ) \).

Above we have used the notation
\[ f[φ] = \int_Γ fφ \quad \text{for} \quad f \in L^1_{loc}(Γ) \]
\[ (J_Γ h)[φ] = \int_Γ -∇_Γ h \cdot ∇_Γ φ + |A|^2 φ \]
where test functions \( φ \in C^1_c(Γ) \).

Let us make the important remark that the operator \( J_Γ \) satisfies the maximum principle.

**Remark 4.2** (Maximum principle for \( J_Γ \)). Since \( h_0 := \frac{1}{\sqrt{1 + |∇F|^2}} > 0 \) solves \( J_Γ h_0 = 0 \) (see (A.2)), the elliptic operator
\[ L := h_0 \Delta_Γ + 2∇_Γ h_0 \cdot ∇_Γ \]
satisfies
\[ J_Γ h = L(h/h_0). \]
Thus, if \( h \) is a supersolution for \( J_Γ \) (in the weak sense) in a bounded domain \( U \subset Γ \) and \( h \in C(\overline{U}) \),
\[ 0 \geq J_Γ h = L(h/h_0) \quad \text{in} \quad U, \]
so that the quotient \( h/h_0 \) doesn’t achieve its minimum at an interior point of \( U \) unless \( h/h_0 \) is constant in \( U \).

In fact, we will construct the supersolutions in Propositions 4.1 and 4.2 as solutions to appropriate elliptic differential equations rather than inequalities. This approach will pay off, because in the end we will automatically possess global smooth supersolutions, whose first and second derivatives will have the appropriate decay rates at infinity. Specifically, we will investigate the linear problem
\[ J_Γ h = f \quad \text{in} \quad Γ, \quad (4.15) \]
where \( h \) and \( f \) are in appropriately weighted Hölder-type spaces. As usual, we first study the problem (4.15) in bounded domains \( Γ_R := Γ \cap \{ r < R \} \)
\[ J_Γ h_R = f \quad \text{in} \quad Γ_R \]
\[ h_R = 0 \quad \text{on} \quad ∂Γ_R. \quad (4.16) \]

Because \( J_Γ \) satisfies the maximum principle, the problem (4.16) is uniquely solvable for all \( R \). In order then to run a compactness argument which takes a sequence \( h_{R_n}, R_n \nearrow \infty \) and produces a globally-defined \( h : Γ \to ℝ \) that solves (4.15), we need two important ingredients – the existence of suitable global barrier functions and an a priori estimate (Lemma 4.7) for the solution to (4.16).

We first exhibit functions that are (weak) supersolutions for \( J_Γ \) far away from the origin. Later, we will be able to modify and extend them to barrier functions on the whole of \( Γ \).

**Lemma 4.5** (cf. Lemma 7.2 in [12]). Let \( 0 < \epsilon < 1 \). There exists a positive function \( h \), such that for some \( R > 0 \) and constants \( c, C > 0 \)
\[ J_Γ h(y) \leq -\frac{1}{1 + r^{4+\epsilon}} \quad \text{in} \quad \{ r(y) > R \} \quad (4.17) \]
\[ \frac{c}{1 + r^{2+\epsilon}} \leq h(y) \leq \frac{C}{1 + r^{2+\epsilon}} \quad \text{in} \quad \{ r(y) > R \}. \quad (4.18) \]
Proof. It follows from the existence of the Type 1–supersolution $h_{1,\infty} \in C^2(\Gamma_\infty)$ for $J_{\Gamma_\infty}$ far away from the origin \{r > R\} (See (A.7) of Appendix A):

$$J_{\Gamma_\infty} h_{1,\infty}(\tilde{y}) \leq -\frac{1}{1 + r(\tilde{y})^{4+\epsilon}} \quad \tilde{y} \in \Gamma_\infty \cap \{r > R\}.$$  

Use the orthogonal projection $\pi_\Gamma$ to lift $h_{1,\infty}$ to a function $h_1$ on $\Gamma \cap \{r > R\}$

$$h_1 \circ \pi_\Gamma = h_{1,\infty} \quad \text{on} \quad \Gamma \cap \{r > R\}.$$  

Then according to (3.24) and the gradient and hessian estimates in Lemma B.1 (see Appendix B),

$$J_{\Gamma} h(\pi_\Gamma(\tilde{y})) = J_{\Gamma_\infty} h_{1,\infty}(\tilde{y}) + O\left(r^{-2-\sigma}|D^2_{\Gamma_\infty} h_{1,\infty}| + r^{-3-\sigma}|D^2_{\Gamma_\infty} h_{1,\infty}| + r^{-4-\sigma}|h_{1,\infty}|\right)(\tilde{y})$$

$$\leq -\frac{1}{2(1 + r^{4+\epsilon}(\tilde{y}))}$$

for $r(\tilde{y}) > R$ large enough. Equations (4.17) and (4.18) are obtained once we note that, according to Lemma 3.6, \(y := \pi_\Gamma(\tilde{y})\) is very close to \(\tilde{y}\) :

$$|y - \tilde{y}| = O(r^{-1-\sigma}(y))$$

in \{r > R\} for a large enough $R$.

□

Lemma 4.6. There exists a locally Lipschitz, non-negative function $h$, which is a weak supersolution for $J_{\Gamma}$ away from the origin and which satisfies

$$J_{\Gamma} h(y) \leq -\frac{\theta - \pi/4}{1 + r^3} \quad \text{on} \quad \{r(y) > r_0\} \quad (4.19)$$

for some large enough $r_0 > 0$. Moreover,

$$h = O\left(\frac{\theta - \pi/4}{1 + r^3} + \frac{1}{1 + r^{2+\epsilon}}\right) \quad (4.20)$$

for some $\tau \in (\frac{1}{2}, \frac{2}{3})$ and some $\epsilon \in (0, 1)$ (e.g. $\tau = \frac{5}{8}$ and $\epsilon = \frac{1}{8}$ do the job).

Proof. The construction of the weak supersolution in this case is achieved by patching up two smooth supersolutions, defined on overlapping regions of $\Gamma$, via the min operation. The resultant function is obviously locally Lipschitz.

One of the building blocks is the Type 2 supersolution for $J_{\Gamma_\infty}$ at infinity (A.8)– call it $\tilde{h}_{\text{ext}} \in C^2(\Gamma_\infty \cap \{\pi/4 < \theta \leq \pi/2\})$ here:

$$\tilde{h}_{\text{ext}}(\tilde{y}) = \frac{r q_2(\theta(\tilde{y}))}{\sqrt{1 + |\nabla F_\infty|^2}}$$

where $q_2(\theta)$ has the following expansion near $\theta = \frac{\pi}{4}$:

$$q_2(\theta) = (\theta - \frac{\pi}{4})^\tau a_0 + a_2(\theta - \frac{\pi}{4})^2 + \cdots \quad a_0 > 0,$$

and

$$J_{\Gamma_\infty} \tilde{h}_{\text{ext}} \leq -\frac{(\theta - \pi/4)^\tau}{1 + r^3} \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \quad \text{and} \quad r > r_0,$$

for some large $r_0 > 0$. Define for

$$-2 \leq \alpha_2 < \alpha_1 < 0 \quad (4.21)$$

the following subregions of the model graph $\Gamma_\infty$

$$\Gamma_{\infty,\text{int}} = \{\theta - \frac{\pi}{4} \mid \theta > r^{\alpha_1}\} \cap \{r > r_0\} \subset \Gamma_\infty$$

$$\Gamma_{\infty,\text{ext}} = \{\theta - \frac{\pi}{4} \mid \theta > r^{\alpha_2}\} \cap \{r > r_0\} \subset \Gamma_\infty.$$
Note that $\Gamma_{\infty,\text{int}}$ and $\Gamma_{\infty,\text{ext}}$ have a non-empty overlap and that they cover all of $\Gamma_{\infty} \cap \{r > r_0\}$. Define $\tilde{h}_{\text{int}} \in C^2(\Gamma_{\infty})$ by

$$\tilde{h}_{\text{int}} = \frac{(\theta - \pi/4)^2 r^{3+\delta}}{\sqrt{1 + |\nabla F_{\infty}|^2}}, \tag{4.22}$$

for some $0 < \delta < 1$, which will be specified later, and extend $\tilde{h}_{\text{ext}}$ on the whole of $\Gamma_{\infty} \cap \{r > r_0\}$ so that it is even about $\theta = \frac{\pi}{4}$:

$$\tilde{h}_{\text{ext}}(r, \theta) = \tilde{h}_{\text{ext}}(r, \frac{\pi}{2} - \theta).$$

The goal is that the lifts of $\tilde{h}_{\text{int}}, \tilde{h}_{\text{ext}}$ onto $\Gamma$:

$$h_{\text{int}}(\pi_\Gamma(\tilde{y})) = h_{\text{int}}(\tilde{y}) \quad h_{\text{ext}}(\pi_\Gamma(\tilde{y})) = h_{\text{ext}}(\tilde{y}) \quad \tilde{y} \in \Gamma_{\infty}$$

corrected by an appropriate asymptotic supersolution of Type 1 (given by the previous Lemma 4.5), will satisfy the desired differential inequality (4.19) in the respective regions

$$\Gamma_{\text{int}} = \pi_\Gamma(\Gamma_{\infty,\text{int}}) \subset \Gamma \quad \Gamma_{\text{ext}} = \pi_\Gamma(\Gamma_{\infty,\text{ext}}) \subset \Gamma.$$

Applying the gradient and hessian estimates of Lemma B.1 from Appendix B and the fact that $-2 \leq \alpha_2 < \alpha_1$, we derive that in $\Gamma_{\infty,\text{int}}$

$$|\tilde{h}_{\text{int}}| = O(r^{1+\delta + 2\alpha_1})$$

$$|D_{\Gamma_{\infty}} \tilde{h}_{\text{int}}| = O\left((\theta - \frac{\pi}{4})^2 r^{\delta + 2\alpha_1 + r^{2+\delta}}|\theta - \frac{\pi}{4}| = O(r^{\delta + 2\alpha_1} + r^{\alpha_1 + \delta - 2}) = O(r^{\delta + 2\alpha_1}) \right)$$

$$|D_{\Gamma_{\infty}}^2 \tilde{h}_{\text{int}}| = O\left((\theta - \frac{\pi}{4})^2 r^{1+\delta + 3+\delta} |\theta - \frac{\pi}{4}| + r^{-5+\delta}\right) = O(r^{-1 + \delta + 2\alpha_1}).$$

On the other hand, $\tilde{h}_{\text{ext}}$ satisfies in $\Gamma_{\infty,\text{ext}}$

$$|\tilde{h}_{\text{ext}}| = O(|\theta - \frac{\pi}{4}| r^{-1})$$

$$|D_{\Gamma_{\infty}} \tilde{h}_{\text{ext}}| = O\left(|\theta - \frac{\pi}{4}| r^{-2} + r^{-4}|\theta - \frac{\pi}{4}| r^{-1}\right) = O(|\theta - \frac{\pi}{4}| r^{-2})$$

$$|D_{\Gamma_{\infty}}^2 \tilde{h}_{\text{ext}}| = O\left(|\theta - \frac{\pi}{4}| r^{-3} + r^{-5}|\theta - \frac{\pi}{4}| r^{-2}\right) = O(|\theta - \frac{\pi}{4}| r^{-3}).$$

Thus, the proximity (3.24) between $J_T$ and $J_{\Gamma_{\infty}}$ implies

$$J_T h_{\text{int}}(\pi_\Gamma(\tilde{y})) = O(r^{1+\delta + 2\alpha_1} \tilde{y})) \quad \tilde{y} \in \Gamma_{\infty,\text{int}}$$

$$J_T h_{\text{ext}}(\pi_\Gamma(\tilde{y})) \leq \frac{|\theta(\tilde{y}) - \pi/4|}{1 + r^3(\tilde{y})} + O\left(|\theta(\tilde{y}) - \pi/4| r^{-5-\sigma}(\tilde{y})\right)$$

$$\leq -\frac{1}{2} \frac{|\theta(\tilde{y}) - \pi/4|}{1 + r^3(\tilde{y})} \leq -\frac{C|\theta(\tilde{y}) - \pi/4|}{1 + r^3(\tilde{y})} \quad \tilde{y} \in \Gamma_{\infty,\text{ext}}$$

for large enough $r(\tilde{y}) > r_0$. According to Lemma 3.6, if $y = \pi_\Gamma(\tilde{y})$,

$$|\tilde{y} - y| = O(r(y)^{-1-\sigma})$$

for some $0 < \sigma < 1$ and $r(y) > r_0$ large enough. Therefore, the pair $(r(\tilde{y}), \theta(\tilde{y}))$ is asymptotically equal to $(r(y), \theta(y))$:

$$|r(y) - r(\tilde{y})| = O(r^{-1-\sigma}(y)) \quad |\theta(y) - \theta(\tilde{y})| = O(r^{-2-\sigma}(y)).$$

Thus,

$$\frac{|\theta(\tilde{y}) - \pi/4|}{1 + r^3(\tilde{y})} = \frac{|\theta(y) - \pi/4|}{1 + r^3(y)} + O(r(y)^{-5-\sigma})$$
so that

\[ J_{\Gamma}h_{\text{int}}(y) + \frac{C}{2} \frac{|\theta(y) - \pi/4|}{1 + r^3(y)} = O(r^{1+\delta+2\alpha_1}(y) + r^{\alpha_1-3}(y)) = \]

\[ = O(r^{1+\delta+2\alpha_1}(y)) \quad y \in \Gamma_{\text{int}} \]  

(4.23)

\[ J_{\Gamma}h_{\text{ext}}(y) + \frac{C}{2} \frac{|\theta(y) - \pi/4|}{1 + r^3(y)} \leq -C \frac{|\theta(y) - \pi/4|}{2} \frac{1 + r^3(y)}{1 + r^3(y)} + O(r^{-5-\sigma}(y)) \quad y \in \Gamma_{\text{ext}}. \]  

(4.24)

Let \( h' \) be the supersolution for \( J_{\Gamma} \), provided by Lemma 4.5:

\[ J_{\Gamma}h' \leq -\frac{1}{1 + r^4 + \tau} \quad r > r_0 \]  

(4.25)

for some \( 0 < \epsilon < 1 \) which we’ll pick shortly. Below we will define the functions \( h_1 \) and \( h_2 \), which will be supersolutions for \( J_{\Gamma} \) on \( \Gamma_{\text{int}} \) and \( \Gamma_{\text{ext}} \), respectively, and patch them into a (weak) supersolution \( h \), defined on \( \Gamma \cap \{ r > r_0 \} \), via the min-operation:

\[ h = \min(h_1, h_2). \]

In order for the operation to succeed, we have to verify the following:

- \( h_1 := h_{\text{int}} + h' \) satisfies the differential inequality (4.19) in \( \Gamma_{\text{int}} \) for large \( r_0 \):

\[ J_{\Gamma}(h_{\text{int}} + h') + \frac{C}{2} \frac{|\theta - \pi/4|}{1 + r^3} \leq 0 \quad \text{in} \quad \Gamma_{\text{int}}. \]

Because of (4.23) and (4.25), it suffices

\[-1 + \delta + 2\alpha_1 < -4 - \epsilon \quad \Rightarrow \quad \alpha_1 < \frac{-3 - \epsilon - \delta}{2} \]

- \( h_2 := h_{\text{ext}} + h' \) satisfies (4.19) in \( \Gamma_{\text{ext}} \) for large \( r_0 \):

\[ J_{\Gamma}(h_{\text{ext}} + h') + \frac{C}{2} \frac{|\theta - \pi/4|}{1 + r^3} \leq 0 \quad \text{in} \quad y \in \Gamma_{\text{ext}}. \]  

(4.26)

By (4.24) and (4.25) this holds for a sufficiently large \( r_0 \).

- \( h_1 < h_2 \) on \( \Gamma_{\text{int}} \setminus \Gamma_{\text{ext}} \) and \( h_1 > h_2 \) in \( \Gamma_{\text{ext}} \setminus \Gamma_{\text{int}} \), i.e. we would like to have \( \tilde{h}_{\text{int}} < \tilde{h}_{\text{ext}} \) on \( \Gamma_{\infty,\text{int}} \) and \( \tilde{h}_{\text{int}} > \tilde{h}_{\text{ext}} \) in \( \Gamma_{\infty,\text{ext}} \). This will be the case for large enough \( r_0 \) if

\[ \alpha_2 < \frac{-2 + \delta}{2 - \tau} < \alpha_1 \]  

(4.27)

Collect conditions (4.21), (4.26) and (4.27) in

\[-2 \leq \alpha_2 < \frac{-2 + \delta}{2 - \tau} < \alpha_1 < \frac{-3 - \epsilon - \delta}{2}. \]  

(4.28)

Moreover, (4.28) needs to be compatible with

\[ \epsilon, \delta \in (0, 1), \quad \frac{1}{3} < \tau < \frac{2}{3}. \]

Condition (4.28) is fairly tight, but not void: indeed, for \( \delta = \frac{1}{2} \), \( \epsilon = \frac{1}{8} \in (0, 1) \) and \( \tau = \frac{5}{8} \in (\frac{1}{3}, \frac{2}{3}) \), we have

\[-2 \leq \alpha_2 < \frac{-20}{11} < \alpha_1 < \frac{-29}{16}. \]

So setting the parameters appropriately, we can conclude that

\[ h = \begin{cases}  
  h_1 & \text{in} \quad \Gamma_{\text{int}} \setminus \Gamma_{\text{ext}} \\
  \min(h_1, h_2) & \text{in} \quad \Gamma_{\text{int}} \cap \Gamma_{\text{ext}} \\
  h_2 & \text{in} \quad \Gamma_{\text{ext}} \setminus \Gamma_{\text{int}} 
\end{cases} \]  

(4.29)

is a weak, locally Lipschitz, supersolution for \( J_{\Gamma} \) in \( r > r_0 \) for a large enough \( r_0 > 0 \) that satisfies (4.19) and (4.34). \qed
The second ingredient is an a priori estimate for the solution \( h \) to (4.16). Introduce the Hölder-type norms:

\[
|f|_{C^\gamma(\Omega)} = \sup_{y_1 \neq y_2 \in \Omega} \frac{|f(y_1) - f(y_2)|}{\text{dist}_F(y_1, y_2)\gamma}
\]

\[
\|f\|_{k, C^\gamma(\Omega)} = \|f\|_{k, L^\infty(\Omega)} + \|(1 + |\tau|^{k+\gamma})|f|_{C^\gamma(C_{\beta}(1+\tau))}\|_{L^\infty(\Omega)}
\]

where \( \Omega \subseteq \Gamma \), \( k \geq 0 \), \( 0 < \gamma < 1 \), \( \text{dist}_F(y_1, y_2) \) is the intrinsic distance on \( \Gamma \) and

\[
C_r(y) = \{ \sum_{i=1}^8 t_i \mathcal{E}_i + lv(y) : |t| < r, l \in \mathbb{R} \}
\]
is the infinite right cylinder with a base \( B_r \subseteq \mathbb{R}^2 \) and its derivatives, and the estimates (3.6) on the second fundamental form (3.11) to recall notation.

We will now establish the following regularity estimate.

**Lemma 4.7** (compare to Lemma 7.5 in [12]). Let \( R > 0 \) be finite or infinite and assume \( h \in C^{2,\gamma}(\Gamma_R) \) is a solution to (4.16) with \( f \in C(\Gamma_R) \). Then

\[
\|D^2_\tau f\|_{k+2, C^\gamma(\Gamma_{R/2})} + \|D_\tau f\|_{k+1, L^\infty(\Gamma_{R/2})} \leq C(\|h\|_{k, L^\infty(\Gamma_R)} + \|f\|_{k+2, C^\gamma(\Gamma_R)})
\]

(4.30)

with a constant \( C > 0 \), independent of \( R \).

**Proof.** The proof is based on a rescaling technique. We may assume

\[
\|h\|_{k, L^\infty(\Gamma_R)} + \|f\|_{k+2, C^\gamma(\Gamma_R)} \leq 1.
\]

Pick \( y \in \Gamma_{R/2} \), set \( \rho = 1 + r(y) \) and express the operator \( J_\Gamma \) in \( C_{\beta \rho}(y) \cap \Gamma_R \) using the coordinates \( t \) (3.11):

\[
J_\Gamma h = \frac{1}{\sqrt{|g|}} \partial_i (g^{ij} \sqrt{|g|} \partial_j h) + |A|^2 h = g^{ij} \partial_i \partial_j h + \frac{g^{ij} |\partial_j| |g|}{2} \partial_j h + |A|^2 h = a^{ij} \partial_i \partial_j h(t) + b^i \partial_i h(t) + |A|^2 h(t) = f(t) \quad t \in B'_{\beta \rho}.
\]

Rescaling to size one,

\[
\tilde{h}(t) = \rho^k h(\rho y), \quad \tilde{f}(t) = \rho^{k+2} g(\rho t), \quad \tilde{a}^{ij}(t) = a^{ij}(\rho t), \quad \tilde{b}^i = \rho b^i(\rho t),
\]

we get

\[
\tilde{a}^{ij} \partial_i \partial_j \tilde{h}(t) + \tilde{b}^i \partial_i \tilde{h}(t) + |A_{\rho}|^2(t) \tilde{h}(t) = \tilde{f}(t) \quad \text{in} \quad B'_{\beta \rho}.
\]

Recall the standard Hölder norm of a function \( q \) defined on a domain \( U \subseteq \mathbb{R}^8 \):

\[
\|q\|_{C^\gamma(U)} := \|q\|_{L^\infty(U)} + \sup_{t \neq s \in U} \frac{|q(t) - q(s)|}{|t - s|\gamma}.
\]

Because of the estimates (3.14) on the metric tensor \( g \) and its derivatives, and the estimates (3.6) on the second fundamental form \( |A|^2 \) and its derivatives, we can bound

\[
\|\tilde{a}^{ij}\|_{C^\gamma(B'_{\beta \rho})}, \|\tilde{b}^i\|_{C^\gamma(B'_{\beta \rho})}, \|A_{\rho}|^2\|_{C^\gamma(B'_{\beta \rho})} \leq K
\]

by a universal constant \( K \). Thus, by interior Schauder estimates,

\[
\|D^2_\tau \tilde{h}\|_{C^\gamma(B'_{\beta \rho/2})} + \|D_\tau \tilde{h}\|_{L^\infty(B'_{\beta \rho/2})} \leq C(\|\tilde{h}\|_{L^\infty(B'_{\beta \rho/2})} + \|\tilde{f}\|_{C^\gamma(B'_{\beta \rho/2})}) \leq C'
\]

so that

\[
\rho^{k+2+\gamma} |D^2_\tau \tilde{h}(y)|_{C^\gamma(\Gamma_{R/2} \cap C_{\beta\rho/2}(y))} + \rho^{k+2} |D_\tau \tilde{h}(y)| + \rho^{k+1} |D_\Gamma \tilde{h}(y)| \leq C'',
\]

(4.31)

for each \( y \in \Gamma_{R/2} \). \( \square \)
Let us now show that
\[ J_{f} h = f \quad \text{in} \quad \Gamma, \]
where the right-hand side \( \| f \|_{k+2, C^\gamma(\Gamma)} < \infty \) for some \( k > 2 \), is uniquely solvable when \( \| h \|_{k, \infty} < \infty \).

**Proposition 4.3.** Let \( k > 2, 0 < \gamma < 1 \) and \( \| f \|_{k+2, C^\gamma(\Gamma)} < \infty \). There exists a unique solution \( h \in C^2(\Gamma) \) to (4.15) such that \( \| h \|_{k, L^\infty(\Gamma)} < \infty \). Moreover,
\[
\| D^2_{f} h \|_{k+2, L^\infty(\Gamma)} + \| D_{f} h \|_{k+1, L^\infty(\Gamma)} + \| h \|_{k, L^\infty(\Gamma)} \leq C \| f \|_{k+2, C^\gamma(\Gamma)}. \tag{4.32}
\]

**Proof.** Uniqueness follows from the maximum principle (Remark 4.2) and the fact that \( |h/h_0| \leq C r^{2-k} \to 0 \) as \( r \to \infty \).

To establish existence, consider the Dirichlet problem in expanding bounded domains:
\[
J_{\Gamma} h_n = f \quad \text{in} \quad \Gamma_{R_n},
\]
\[
h_n = 0 \quad \text{on} \quad \partial \Gamma_{R_n},
\]
where \( R_n \not\to \infty \). First claim that
\[
\| h_n \|_{k, L^\infty(\Gamma_{R_n})} \leq C \| f \|_{k+2, C^\gamma(\Gamma_{R_n})}. \tag{4.33}
\]
for some constant independent of \( n \). Assume not; then there is a subsequence (call it \( R_n \) again) such that
\[
\| h_n \|_{k, L^\infty(\Gamma_{R_n})} \geq n \| f \|_{k+2, C^\gamma(\Gamma_{R_n})}.
\]
If we set \( \bar{f}_n = f/\| h_n \|_{k, L^\infty(\Gamma_{R_n})} \), \( \bar{h}_n = h_n/\| h_n \|_{k, L^\infty(\Gamma_{R_n})} \), we see that
\[
J_{\Gamma} \bar{h}_n = \bar{f}_n
\]
with \( \| \bar{h}_n \|_{k, L^\infty(\Gamma_{R_n})} = 1 \) and \( \| \bar{f}_n \|_{k+2, C^\gamma(\Gamma_{R_n})} \leq 1/n \). The a priori estimate (4.30) implies, after possibly passing to a subsequence, that \( h_n \) converge uniformly on compact sets to a \( C^2(\Gamma) \)-function \( \bar{h} \) with \( \| \bar{h} \|_{k, \infty} < \infty \) which solves
\[
J_{\Gamma} \bar{h} = 0 \quad \text{in} \quad \Gamma.
\]
Uniqueness requires that \( h = 0 \). Let \( h'_{\infty} \) be the supersolution for \( J_{\Gamma} \) provided by Lemma 4.5 with some \( 0 < \varepsilon < k - 2 \) and \( r_0 \) large enough:
\[
J_{\Gamma} h'_{\infty} \leq -\frac{1}{1 + r^{4+\varepsilon}}, \quad h'_{\infty} \geq \frac{c}{1 + r^{2+\varepsilon}} \quad r > r_0.
\]
Since \( h_n \to 0 \) uniformly on compact sets, \( s_n := \sup_{r_0} h_n \to 0 \). Therefore,
\[
\pm h_n + \mu_n h'_{\infty} \geq 0 \quad \text{on} \quad r = r_0 \quad \text{and} \quad r = R_n
\]
\[
J_{\Gamma}(\pm h_n + \mu_n h'_{\infty}) \leq 0 \quad \text{in} \quad r_0 < r < R_n
\]
for \( \mu_n = \max \{ s_n c^{-1}, \frac{1}{n} \} \to 0 \). An application of the maximum principle yields
\[
|h_n| \leq \mu_n h'_{\infty} \quad \text{in} \quad r_0 < r < R_n.
\]
Combine this with the fact that \( \| h_n \|_{k, L^\infty(\Gamma_{r_0})} \leq s_n r_0^k \) to conclude
\[
\| h_n \|_{k, L^\infty(\Gamma_{R_n})} \to 0 \quad \text{as} \quad n \to \infty
\]
which is a contradiction. Hence, (4.33) holds and the a priori estimate (4.30) becomes
\[
\| D^2_{f} h_n \|_{k+2, C^\gamma(\Gamma_{R_n/2})} + \| D_{f} h_n \|_{k+1, L^\infty(\Gamma_{R_n/2})} + \| h_n \|_{k, L^\infty(\Gamma_{R_n/2})} \leq C \| f \|_{k+2, C^\gamma(\Gamma_{R_n})}
\]
for some constant \( C \), independent of \( R_n \). Now a standard compactness argument produces a \( C^2(\Gamma) \)-function \( h \) which solves (4.15) and satisfies the estimate (4.32).

Proposition 4.1 is an immediate corollary.
Proof of Proposition 4.1. Let $h$ solve (4.15) with a right-hand side $f = -\frac{1}{1 + r \theta}$. We are only left with checking that $h$ is strictly positive. This is a consequence of the strong maximum principle (Remark 4.2) and the fact that $|h/h_0(y)| = O(r^{-\epsilon}(y)) \to 0$, as $r(y) \to \infty$. \qed

Now we would like to construct a global barrier function (not-necessarily smooth) for (4.15) with a right-hand-side

$$f = -\frac{\theta - \pi/4}{1 + r^3}.$$  

Lemma 4.8. There exists a globally defined, locally Lipschitz function $h \geq 0$ which is a weak supersolution for $J_\Gamma$ and which satisfies

$$J_\Gamma h \leq -\frac{|\theta - \pi/4|}{1 + r^3} \text{ in } \Gamma.$$  

Moreover,

$$h = O\left(\frac{|\theta - \pi/4|^\tau}{1 + r} + \frac{1}{1 + r^{2+\epsilon}}\right)$$

for some $\tau \in (\frac{1}{2}, \frac{3}{4})$ and some $\epsilon \in (0, 1)$ (e.g. $\tau = \frac{5}{8}$ and $\epsilon = \frac{1}{8}$).

Proof. Let $h''_\infty$ be the weak supersolution for $J_\Gamma$ in $\Gamma_{r_0}^c$, provided by Lemma 4.6:

$$(J_\Gamma h''_\infty - f)[\phi] \leq 0$$

for every non-negative $\phi \in C^1_c(\Gamma_{r_0}^c)$. Now let $\psi \in C^\infty(\Gamma)$ be a non-negative cutoff function such that $\psi(y) = 0$ for $r(y) \leq r_0$ and $\psi(y) = 1$ for $r(y) \geq r_0 + 1$.

Define a function $h''$ on the whole of $\Gamma$ by

$$h''(y) = \begin{cases} 0 & \text{in } r < r_0 \\ \psi(y)h''_\infty(y) & \text{in } r \geq r_0 \end{cases}$$

Finally set

$$h = Ch' + h'',$$

where $h'$ is the supersolution provided by Proposition 4.1 and $C > 0$ is some large constant, to be fixed shortly. Now for any nonnegative $\phi \in C^1_c(\Gamma)$, the fact that $(J_\Gamma h''_\infty - f)[\phi] \leq 0$ implies

$$(J_\Gamma h - f)[\phi] = C(J_\Gamma h')[\phi] - f[\phi] + \int -h''_\infty \nabla_\Gamma \psi \cdot \nabla_\Gamma \phi - \nabla_\Gamma h''_\infty \cdot (\psi \nabla_\Gamma \phi) - |A|^2 h''_\infty \psi \phi$$

$$= C(J_\Gamma h')[\phi] - (1 - \psi)f[\phi] + (J_\Gamma h''_\infty)[\psi \phi] - f[\psi \phi] + \int (2\nabla_\Gamma \psi \cdot \nabla_\Gamma h''_\infty + h''_\infty \Delta_\Gamma \psi)\phi \leq -C(J_\Gamma h')[\phi] + k[\phi]$$

where $k$ is a bounded function, compactly supported in $\Gamma_{r_0+1}$. We were able to carry out the integration by parts, since $h''_\infty$ is locally Lipschitz. Taking $C > 0$ large enough we conclude that $J_\Gamma h \leq f$ globally, in the weak sense. \qed

We now possess all the means to prove Proposition 4.2.

Proof of Proposition 4.2. Pick $f \in C^{0, \gamma}(\Gamma)$ such that $f \leq 0$ and

$$f \circ \pi_\Gamma(\hat{y}) = -\frac{|\theta(\hat{y}) - \pi/4|}{1 + r^3(\hat{y})} \quad \hat{y} \in \Gamma_\infty \cap \{r > r_0\}$$

for a large enough $r_0$. It is not hard to verify that $\|f\|_{3, C^{\gamma}(\Gamma)} < \infty$ by transferring the computation onto $\Gamma_\infty$ via (3.4) and employing the gradient estimate in Lemma B.1.
Let $h_n$ solve the Dirichlet problem (4.16) in the expanding bounded domains $\Gamma_{R_n}$, $R_n \to \infty$

\begin{align*}
J_\Gamma h_n = f & \quad \text{in } \Gamma_{R_n} \\
h_n = 0 & \quad \text{on } \partial \Gamma_{R_n}.
\end{align*}

Since $f$ is non-positive, the weak maximum principle implies $h_n \geq 0$. Let $h'$ be a Type 1 supersolution, provided by Proposition 4.1, and let $h''$ be the weak Type 2 supersolution which we constructed in Lemma 4.8. Noting again that

\[ \frac{|\theta(y) - \pi/4|}{1 + r^3(y)} = O(r(y)^{-5-\sigma}) \quad y = \pi\tau(y) \]

for some $\sigma > 0$, we obtain

\[ J_\Gamma(-h_n + h'' + Ch') \leq 0 \]

for a large enough $C$. Moreover, since $-h_n + Ch' + h'' \geq 0$ on $\partial \Gamma_{R_n}$, the maximum principle implies

\[ 0 \leq h_n \leq h'' + Ch' \quad \text{in } \Gamma_{R_n}. \]

Thus, \[ \|h_n\|_{1,\infty(\Gamma_{R_n})} \leq C' \] for an absolute constant $C'$ independent of $n$. We can now employ the a priori estimate (4.30) into a standard compactness argument that yields a non-negative $C^2$–function $\tilde{h}$ solving

\[ J_\Gamma \tilde{h} = f \quad \text{in } \Gamma \]

with

\[ 0 \leq \tilde{h} \leq h'' + O(h') \]

\[ \|D^2_{\Gamma}\tilde{h}\|_{3,\infty} + \|D_{\Gamma}\tilde{h}\|_{2,\infty} + \|\tilde{h}\|_{1,\infty} < \infty. \] (4.35)

After possibly correcting $\tilde{h}$ by a supersolution of Type 1, $h = \tilde{h} + ch'$, we can conclude that

\[ J_\Gamma h(y) = f + cJ_\Gamma h' \leq -\frac{|\theta(y) - \pi/4|}{1 + r^3(y)}. \]

To establish the second statement in the proposition, namely the refinement of decay of $h$ near $\theta = \pi/4$, we notice that on $S(-\delta)$ with $0 < \delta < \frac{3}{2}$

\[ \tilde{h} = O\left(\frac{|\theta - \pi/4|}{1 + r} + \frac{1}{1 + r^3}\right) = O(r^{-1-\delta\tau}), \]

as $\delta\tau + 1 < 2$. Also, \[ \|f\|_{3+\delta\tau, C^\gamma(S(-\delta))} < \infty. \]

Then an argument, based on rescaling and interior elliptic estimates – absolutely analogous to the one for the a priori estimate (Lemma 4.7) – gives us the interior estimate (4.14) (for $\tilde{h}$ and thus for $h$ itself) on $S(-\delta') \subseteq S(-\delta)$. There is a caveat: the same argument will carry through to the present situation, once we ascertain that $S(-\delta)$ contains “balls” of size $\sim r$, centered on points in $S(-\delta')$ far away from the origin. More precisely, we want for some $r_0$ large enough,

\[ C_{\beta r(y)}(y) \cap \Gamma \subseteq S(-\delta) \quad \text{for every } y \in S(-\delta') \cap \{r(y) > r_0\} \] (4.36)

Note that according to Lemma 3.5, the fact that $\Gamma$ is a graph $\{(t, G(t)) \}$ over $B'_{\beta r(y)}(y)$ with

\[ |G(t)| \leq Cr(y) \]

implies

\[ C_{\beta r(y)}(y) \cap \Gamma \subseteq B_{c_0r(y)}(y) \cap \Gamma \]

for a large enough numerical constant $c_0 > 0$. Suppose that (4.36) is not true: then there exist $y' \in \partial S(-\delta')$ and $y \in \partial S(-\delta)$ with $r' = r(y')$, $r = r(y)$ arbitrarily large such that $|y - y'| < c_0r'$.
Denote the projections of $y'$ and $y$ onto $\mathbb{R}^8$ by $(\overrightarrow{w}, \overrightarrow{v})$ and $(\overrightarrow{u}, \overrightarrow{v})$, respectively. Obviously, $r/r' \sim 1$ and for $0 \leq \delta < \delta' < 2$

$$|y' - y|^2 = |\overrightarrow{u} - \overrightarrow{w}|^2 + |\overrightarrow{v} - \overrightarrow{v}'|^2 + |F(r', (1 + r')^{-\delta}) - F(r, (1 + r)^{-\delta})|^2$$

$$\geq (u - u')^2 + (v - v')^2 + |F(r', (1 + r')^{-\delta}) - F(r, (1 + r)^{-\delta})|^2$$

$$\geq (r' - r)^2 + |F_\infty(r', (1 + r')^{-\delta}) - F_\infty(r, (1 + r)^{-\delta})|^2 - C'r^{-2\sigma}$$

$$\geq -C'r^{-2\sigma} + (r' - r)^2 + c^2 \bigg| r^{-\delta} \left( \frac{r}{r'} \right)^{3-\delta} \bigg| - 1 \bigg| (r')^{2(3-\delta)} \bigg| \gg (r')^2$$

which is a contradiction. □

4.3. The free boundary super and subsolution. In correspondence with the form of the supersolution ansatz (4.1), define the subsolution ansatz $v : B_{\Gamma_\alpha} \to \mathbb{R}$ by

$$v(y, z) = -h_0^\alpha(y) + z h_1^\alpha(y) + z^2(-h_2^\alpha(y)) + z^3 h_3^\alpha(y) + z^5 h_5^\alpha(y). \tag{4.37}$$

Since we require $h_0^\alpha > 0$ and $h_2^\alpha > 0$, we will automatically have $v < w$ in $B_{\Gamma_\alpha}$. Also,

$$0 < w - v = 2(h_0^\alpha + z^2 h_2^\alpha) = O \left( \frac{\alpha^p}{1 + \alpha r} \right).$$

**Proposition 4.4.** Fix $0 < p < 1$. There exist $C_1, C_2 > 0$ and $\alpha_0 > 0$ such that for all small enough $\alpha \leq \alpha_0$, $w$ given by (4.1) satisfies

$$\Delta w < 0 \quad \text{in} \quad B_{\Gamma_\alpha}$$

$$|\nabla w|^2 > 1 \quad \text{on} \quad \{w = 1\} \quad \text{and} \quad (4.38)$$

$$|\nabla w|^2 < 1 \quad \text{on} \quad \{w = -1\},$$

while

$$\Delta v > 0 \quad \text{in} \quad B_{\Gamma_\alpha}$$

$$|\nabla v|^2 < 1 \quad \text{on} \quad \{v = 1\} \quad \text{and} \quad (4.39)$$

$$|\nabla v|^2 > 1 \quad \text{on} \quad \{v = -1\}.$$ 

Moreover, $0 < w - v \leq \frac{1}{2}$, $\partial_{x_9} v > 0$ and $\partial_{x_9} w > 0$ in $B_{\Gamma_\alpha}$.

We immediately derive as a corollary:

**Corollary 4.1.** Let $v$, $w$, $0 < \alpha \leq \alpha_0$ be as in Proposition 4.4 above. Then the function $W : \mathbb{R}^9 \to \mathbb{R}$, given by

$$W(x) = \begin{cases} 
  w(x) & \text{for} \ x \in B_{\Gamma_\alpha} \cap \{|w| \leq 1\} \\
  1 & \text{for} \ x \in (B_{\Gamma_\alpha} \cap \{|w| \leq 1\})^c \cap \{x_9 > F(x')\} \\
  -1 & \text{for} \ x \in (B_{\Gamma_\alpha} \cap \{|w| \leq 1\})^c \cap \{x_9 < F(x')\} 
\end{cases}$$

is a classical strict supersolution to (1.1), while the function $V : \mathbb{R}^9 \to \mathbb{R}$, given by

$$V(x) = \begin{cases} 
  v(x) & \text{for} \ x \in B_{\Gamma_\alpha} \cap \{|v| \leq 1\} \\
  1 & \text{for} \ x \in (B_{\Gamma_\alpha} \cap \{|v| \leq 1\})^c \cap \{x_9 > F(x')\} \\
  -1 & \text{for} \ x \in (B_{\Gamma_\alpha} \cap \{|v| \leq 1\})^c \cap \{x_9 < F(x')\} 
\end{cases}$$

is a classical strict subsolution. Moreover, $0 \leq W - V \leq \frac{1}{2}$, both $V$ and $W$ are monotonically increasing in $x_9$ and strictly increasing in $x_9$ inside $\Omega_m(V)$, $\Omega_m(W)$, respectively.
Proof of Proposition 4.4. Fix some $0 < \delta < \frac{1}{2}$ and $0 < \epsilon < \delta \tau$, where $\frac{1}{2} < \tau < \frac{2}{3}$ is provided by Proposition 4.2 and let $h' > 0$, $h'' \geq 0$ be the $J_\tau$-supersolutions given by Lemma 4.1 and 4.2, respectively. Remember that we only need to set the values of $h_0$ and $h'_2$ in order to determine the ansatz (4.1) completely. So, let

$$h_0 = h' + h'',$$

and

$$h'_2 = \frac{1}{2} \left( \frac{1}{1 + r^{1+\epsilon}} + \frac{c \cos^2(2\theta)}{1 + r^3} \right),$$

where $c > 0$ is such that $2c \cos^2(2\theta) \leq |\theta - \pi/4|$. Set

$$C_1 = \|D^2_r h_0\|_{3,\infty} + \|D_r h_0\|_{2,\infty} + \|h_0\|_{1,\infty}$$

and

$$C_2 = \|D^2_r h'_2\|_{5,\infty} + \|D_r h'_2\|_{4,\infty} + \|h'_2\|_{3,\infty}.$$  

Claim that for all small enough $\alpha > 0$, $\Delta w < 0$ in $B_{\Gamma_n}$. This is a consequence of the following computations.

- For $\alpha > 0$ small enough,

$$J_{\Gamma_n} h_0^\alpha - z^2 H_{3,\alpha} \leq -\frac{\alpha^{2+p}}{1 + (\alpha r)^{4+\epsilon}} - \frac{\alpha^{2+p} |\theta - \pi/4|}{1 + (\alpha r)^3} + O\left( \frac{\alpha^3 |\theta - \pi/4|}{1 + (\alpha r)^3} \right).$$

so that

$$J_{\Gamma_n} h_0^\alpha - z^2 H_{3,\alpha} + h_2^\alpha \leq -\frac{\alpha^{2+p}}{4} \left( \frac{2}{1 + (\alpha r)^{4+\epsilon}} + \frac{\alpha^3 |\theta - \pi/4|}{1 + (\alpha r)^3} \right). \tag{4.40}$$

- In $S_n(-1) = \{|\theta - \frac{\pi}{4}| \leq (1 + \alpha r)^{-1}\}$, Proposition 4.2 and (3.10) imply

$$(\Delta_{\Gamma_n(z)} + |A_\alpha|^2) h_0^\alpha = J_{\Gamma_n} h_0^\alpha + O\left( \frac{\alpha^{3+p}}{1 + (\alpha r)^{4+\delta \tau}} \right). \tag{4.41}$$

Then (4.7), (4.40) and (4.41) yield the desired

$$\Delta w(y, z) < 0 \quad \text{for } y \in S_n \text{ and } (y, z) \in B_{\Gamma_n},$$

and all small enough $\alpha > 0$.

- In $S_n^c(-1) = \{|\theta - \frac{\pi}{4}| > (1 + \alpha r)^{-1}\}$, (4.40) can be estimated further by

$$J_{\Gamma_n} h_0^\alpha - z^2 H_{3,\alpha} + h_2^\alpha \leq -\frac{\alpha^{2+p}}{4} \left( \frac{2}{1 + (\alpha r)^{4+\epsilon}} + \frac{c'}{1 + (\alpha r)^4} \right). \tag{4.42}$$

Because of (3.10) we have

$$(\Delta_{\Gamma_n(z)} + |A_\alpha|^2) h_0^\alpha = J_{\Gamma_n} h_0^\alpha + O\left( \frac{\alpha^{3+p}}{1 + (\alpha r)^{4+\delta \tau}} \right). \tag{4.43}$$

Thus, (4.7), (4.42) and (4.43) yield

$$\Delta w(y, z) < 0 \quad \text{for } y \in S_n^c \text{ and } (y, z) \in B_{\Gamma_n},$$

and all small enough $\alpha > 0$.

To verify that necessary gradient conditions (4.39) are also met, we need to check that for small enough $\alpha > 0$, $h_0^\alpha$ majorizes both $|A_\alpha|^2(h_0^\alpha)^2$ and $|\nabla_{\Gamma_n(z)} h_0^\alpha|^2$ (see Lemma 4.3). Indeed,

- in $S_n(-\frac{1}{2}) = \{|\theta - \frac{\pi}{4}| \leq (1 + \alpha r)^{-\frac{1}{2}}\}$

$$|A_\alpha|^2(h_0^\alpha)^2 + |\nabla_{\Gamma_n(z)} h_0^\alpha|^2 = O\left( \frac{\alpha^{2+2p}}{1 + (\alpha r)^{4+2\delta \tau}} \right)$$

is dominated by $h_2^\alpha \geq \frac{1}{2} \frac{\alpha^{2+p}}{1 + (\alpha r)^{4+\tau}}$. 
• in $S_0^\alpha(-\frac{1}{2}) = \{ |\theta - \frac{\pi}{4}| > (1 + \alpha r)^{-\frac{1}{2}} \}$

$$|A_\alpha|^2(h_0^{\alpha})^2 + |\nabla_{\Gamma_\alpha(x)}h_0^\alpha|^2 = O\left(\frac{\alpha^{2+2p}}{1+(\alpha r)^4}\right)$$

is dominated by $h_2^{\alpha} \geq \frac{1}{2} \frac{\alpha^{2+2p}\cos^2(2\theta)}{1+(\alpha r)^4} \geq c'' \alpha^{2+2p}$.

Checking that $v$ meets the conditions for a subsolution is absolutely analogous. In view of Lemma 4.4, $\partial_n w > 0$ and similarly $\partial_n v > 0$.

5. The solution. Existence and regularity.

We have at our disposal a globally defined classical strict subsolution $V$ to (1.1) lying below a classical strict supersolution $W$ both of which are monotonically increasing in $x_n$ (in fact, strictly increasing in their interphases $\mathbb{R}_{[0]}^n$). In this section we will explain why this engenders the existence of a classical solution $u$ to (1.1), trapped in-between. Moreover, the solution will inherit some of the nice properties of the barriers $V, W$, such as monotonicity in $x_n$ and graph free boundaries $F^+(u)$ and $F^-(u)$.

We will construct $u$ as a global minimizer of $I$, constrained to lie between $V$ and $W$.

**Definition 5.1.** A function $u \in H^1_{loc}(\mathbb{R}^n)$ is a global minimizer of $I$, constrained between $V \leq W$ if for any bounded right cylinder $\Omega \subset \mathbb{R}^n$

$$I(u, \Omega) \leq I(v, \Omega) \quad \text{for all } v \in H^1(\Omega) \text{ such that } V \leq v \leq W \text{ and } u - v \in H^1_0(\Omega).$$

As usual, we obtain a global (constrained) minimizer $u$ as a sequence of local (constrained) minimizers on expanding bounded domains. For the purpose, we will verify that local minimizers are Lipschitz continuous with a universal bound on the local Lipschitz constant. This is done in the spirit of [10].

Afterwards, we will show that a global minimizer $u$ which, in addition, meets certain simple geometric constraints, is actually a classical solution to our free boundary problem. This is achieved almost for free – by applying the regularity theory of minimizers to the energy functional $I_0$, developed in [10] and [11], to the functions $1 \pm u$.

5.1. Existence of a local minimizer. Let $\Omega \subset \mathbb{R}^n$ be a cylinder

$$C_{R,h} = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < R, |x_n| < h \}$$

and consider the minimization problem for the functional

$$I(v, \Omega) = \int_{\Omega} |\nabla v|^2 + 1_{|v|<1},$$

where $v$ ranges over the following closed convex subset of $H^1(\Omega)$:

$$S(\Omega) = \{ v \in H^1(\Omega) : V \leq v \leq W \text{ a.e.} \}.$$

Let us show that there exists $u \in S(\Omega)$ for which the infimum of $I(\cdot, \Omega)$ over $S(\Omega)$ is attained.

**Proposition 5.1** (Existence of monotone local minimizers). There exists $u \in S(\Omega)$ such that

$$I(u, \Omega) = m := \inf_{v \in S(\Omega)} I(v, \Omega).$$

Moreover, given that $V$ and $W$ are monotonically increasing in the $x_n$-variable, $u$ can also be taken to be monotonically increasing in the $x_n$-variable.
Proof. For convenience use the simplified notation $I(v) = I(v, \Omega)$. Obviously, the infimum $m$ is non-negative and finite:

$$0 \leq m \leq C_0 := \min(I(V), I(W)).$$

Take a sequence $u_k \in S(\Omega)$ such that $C_0 \geq I(u_k) \downarrow m$. Then

$$\|u_k\|^2_{H^1} = \|u_k\|^2_{L^2} + \|\nabla u_k\|^2_{L^2} \leq |\Omega| + C_0$$

is uniformly bounded, so by compactness we can extract a subsequence (call it $u_k$ again) such that

$$u_k \to u \text{ in } L^2 \text{ and a.e. and } \nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^2$$

for some $u \in S(\Omega)$. Claim that $I(u) = m$. It suffices to show that $I$ is lower semicontinuous with respect to the weak-$H^1$ topology, i.e.

$$I(u) \leq \liminf_{k \to \infty} I(u_k) \tag{5.1}$$

which is done analogously as in [2].

We can produce a minimizer, which is monotonically increasing in the $x_n$-variable by applying a rearrangement. A monotone-increasing rearrangement in the $x_n$-variable, $f \to f^*$ satisfies the following properties (cf. [15]):

1. If $f$ is monotonically increasing in the $x_n$-variable, $f^* = f$.
2. The functions $f$ and $f^*$ are equimeasurable, i.e. $|f^{-1}(O)| = |(f^*)^{-1}(O)|$ for any open interval $O \subseteq \mathbb{R}$.
3. The mapping $f \to f^*$ is order-preserving, i.e. if $f \leq g$ then $f^* \leq g^*$.
4. If $f \in H^1(C_{R,h})$, then $f^* \in H^1(C_{R,h})$ and

$$\|\nabla f^*\|^2_{L^2} \leq \|\nabla f\|^2_{L^2}.$$

Since $V, W$ are monotonically increasing in the $x_n$-variable, $V^* = V$ and $W^* = W$; thus $V \leq u^* \leq W$ by order preservation under rearrangements. Moreover, $u^* \in H^1(\Omega)$, so that $u^* \in S(\Omega)$ and because of properties 2 and 4 above,

$$m \leq I(u^*) \leq I(u) = m.$$ 

Thus, $u^*$ is a minimizer to $I$ over $S(\Omega)$, monotonically increasing in the $x_n$-variable. \(\square\)

5.2. Lipschitz continuity of local minimizers. Employing standard arguments, we first establish continuity of local minimizers before we prove Lipschitz continuity with a universal bound on the local Lipschitz constant.

We adapt the technique of harmonic replacements used by [2].

**Definition 5.2.** The harmonic replacement of $u$ in the ball $B \subseteq \Omega$ is the unique function $v \in H^1(\Omega)$ that is harmonic in $B$ and agrees with $u$ on $\Omega \setminus B$.

Let

$$B_{V,W} = \Omega_{in}(V) \cup \Omega_{in}(W)$$

and note that $V$ is subharmonic in $B_{V,W}$ whereas $W$ is superharmonic in $B_{V,W}$.

Below we show that the function $u$, constructed in Proposition 5.1, is continuous in $B_{V,W}$.

**Proposition 5.2** (Continuity). Let $D \subseteq B_{V,W} \subseteq \Omega$. Then the minimizer $u$, constructed in Proposition 5.1, is in a Hölder class $C^\alpha(D)$ for some $\alpha > 0$, depending on $D$. In particular, $u$ is continuous in $B_{V,W}$.
Proof. Let $B_\rho \subseteq D$ denote a ball of radius $\rho$, centered at some fixed point in $D$. Let $v_r$ be the harmonic replacement of $u$ in the concentric $B_r \subseteq B_\rho$. Since $B_r \subset B_{V,W}$, where $V,W$ are subharmonic and superharmonic, respectively, the weak maximum principle implies that $V \leq v_r \leq W$ a.e. in $B_r$. Thus $v_r \in S(\Omega)$ and $I(u) \leq I(v_r)$. Therefore,

$$\int_{B_r} |\nabla (u - v_r)|^2 = \int_{B_r} ((\nabla u)^2 - |\nabla v_r|^2) \leq 2|B_r| = c_0 r^n \quad \forall 0 < r \leq \rho,$$

for some dimensional constant $c_0$. Whence a standard dyadic argument in the spirit of [17, Theorem 5.3.6] yields

$$\int_{B_{r/4}} |\nabla u|^2 \leq C(1 + \rho^{-1})(1 + \log^2(\rho/r)) \quad \forall 0 < r \leq \rho,$$

from which the statement of the proposition follows as in [17, Theorem 3.5.2].

$\square$

**Corollary 5.1.** The function $u$ is harmonic in $\Omega_{in}(u) = \{|u| < 1\}$, subharmonic in $\{u < 1\}$ and superharmonic in $\{u > -1\}$.

Proof. From the previous proposition we know that $u$ is continuous in $B_{V,W}$, therefore $\Omega_{in}(u) \subseteq B_{V,W}$ is an open set. Thus for any $x \in \Omega_{in}(u)$ we can find a small enough closed ball $\overline{B} = B_r(x) \subseteq \Omega_{in}$. Let $v$ be the harmonic replacement of $u$ in $B$. Since, $|u| < 1$ on $\partial B$, the maximum principle implies that $|v| < 1$. Combining the latter with the fact that harmonic extensions minimize the Dirichlet energy, we get that $I(u, B) \geq I(v, B)$. However, by minimality, $I(u, B) \leq I(v, B)$. Hence, $I(u, B) = I(v, B)$, which in turn implies that

$$\|\nabla u\|_{L^2(B)}^2 = \|\nabla v\|_{L^2(B)}^2.$$

So, $u$ is itself the minimizer of the Dirichlet energy, meaning that $u$ is harmonic in $B$. Since $x \in \Omega_{in}$ is arbitrary, we conclude that $u$ is harmonic in $\Omega_{in}$.

The fact that $u$ is subharmonic in $\{u < 1\}$ and superharmonic in $\{u > -1\}$ now follows from the mean-value characterization of sub/super-harmonic functions.

$\square$

Before we proceed to establish Lipschitz continuity, let us state the following definition related to the geometry of the pair of barriers $V, W$:

**Definition 5.3.** We call the subsolution-supersolution pair $(V,W)$ **nicely intertwined in the bounded domain $\Omega$** if

$$F^+(W) \cap \Omega \subseteq \Omega_{in}(V) \quad \text{and} \quad F^-(V) \cap \Omega \subseteq \Omega_{in}(W).$$

so that $F^+(W) \cap \Omega$ stays a positive distance away from $\{V = \pm 1\} \cap \Omega$ and $F^-(V) \cap \Omega$ stays a positive distance away from $\{W = \pm 1\} \cap \Omega$. We say that $V, W : \mathbb{R}^n \to \mathbb{R}$ are **nicely intertwined globally** if for every $R > 0$, there exists an $h_0 = h_0(R)$ large enough, such that $(V,W)$ is nicely intertwined in all cylinders $\Omega = C_{R,h}$ for $h \geq h_0(R)$.

**Proposition 5.3.** Let $D \subseteq D' \subseteq \Omega$ be compactly contained cylinders and suppose $(V,W)$ is nicely intertwined in $\Omega$. Then there exists a constant $K$, depending on $n$, $d(\partial D, \partial D')$, $d(F^+(W) \cap D', F^-(V) \cap D')$ and the Lipschitz constant of $V, W$ in $D'$, such that $|\nabla u| \leq K$ in $D$. That is, $u$ is Lipschitz-continuous in $D$.

Proof. Since $u \in H^1(\Omega)$ and $|u| \leq 1$,

$$\nabla u = \nabla u 1_{|u|<1} \quad \text{a.e.}$$
Obviously, in the situation when \( r < \frac{d(\partial D, \partial D')}{2} \); for otherwise, using the gradient estimate for harmonic functions,

\[
|\nabla u(x_0)| \leq C \frac{r}{\rho} \int_{B_r} |u| \leq \frac{C}{d(\partial D, \partial D')}.
\]

Figure 1. The free boundaries of a nicely intertwined pair \((V, W)\) in a cylinder \(\Omega\).

Thus, it suffices to bound the gradient at points \(x_0 \in \Omega_{\text{in}}(u) \cap D\). Let \(B_r = B_r(x_0)\) be the largest ball contained in \(\Omega_{\text{in}} \cap D'\), centered at \(x_0\). We may restrict our attention to the situation when \(r < \frac{d(\partial D, \partial D')}{2}\); for otherwise, using the gradient estimate for harmonic functions,

\[
|\nabla u(x_0)| \leq \frac{C}{r} \int_{B_r} |u| \leq \frac{C}{d(\partial D, \partial D')}.
\]

Obviously, in the situation when \(r < \frac{d(\partial D, \partial D')}{2}\), \(B_r\) must touch the free boundary \(F(u)\) and not the fixed boundary \(\partial D'\).

Assume \(B_r(x_0)\) touches \(F^+(u)\) at a point \(x_1\). We consider two cases determined by how close \(x_1\) is to \(F^+(V)\).

- Assume \(x_1\) is relatively close to \(F^+(V)\):

\[
d(x_1, F^+(V)) = |x_2 - x_1| \leq r/2 \text{ for some } x_2 \in F^+(V).
\]

Note that \(|x_2 - x_0| \leq 3r/2 < d(D, D')\), thus the segment between \(x_0\) and \(x_2\) is contained in \(D'\). The Lipschitz continuity of \(V\) in \(D'\) yields

\[
1 - u(x_0) \leq 1 - V(x_0) \leq \|\nabla V\|_{L^\infty(D')} |x_2 - x_0| \leq \|\nabla V\|_{L^\infty(D')} 3r/2.
\]

Because \(1 - u \geq 0\) is harmonic in \(B_r(x_0)\), Harnack’s inequality implies that

\[
1 - u \leq c(1 - u)(x_0) \leq c' \|\nabla V\|_{L^\infty(D')} r \text{ in } B_{r/2}.
\]

Hence, by the gradient estimate for harmonic functions we get the desired

\[
|\nabla u|(x_0) = |\nabla(1 - u)|(x_0) \leq C \int_{B_{r/2}(x_0)} (1 - u) \leq C \|\nabla V\|_{L^\infty(D')}.
\]

- Assume that \(d(x_1, F^+(V)) > r/2\). Certainly, \(B_{r/2}(x_1) \subseteq D'\), as \(r \leq d(\partial D, \partial D')/2\). We may also assume that \(r \leq L = d(F^+(W) \cap D', F^-(V) \cap D')\), for otherwise the gradient estimate for harmonic functions will immediately give us

\[
|\nabla u|(x_0) \leq \frac{C}{L}.
\]

With these assumptions in mind we see that \(B_{r/2}(x_1) \subseteq \Omega_{\text{in}}(V) \cap D'\), so that \(u > -1\) on \(B_{r/2}(x_1)\).
Let \( v \) be the harmonic replacement of \( u \) in \( B_{r/2}(x_1) \). By the strong maximum principle \(|v| < 1\), so by minimality,
\[
\int_{B_{r/2}(x_1)} |\nabla (u-v)|^2 \leq \int_{B_{r/2}} (1_{|v|<1} - 1_{|u|<1}) = |B_{r/2}(x_1) \cap \{u = 1\}|. \tag{5.2}
\]

Now the argument for Lipschitz continuity of \([2]\) goes through. It is based on the following bound for the measure of \( B_{r/2}(x_1) \cap \{u = 1\} \):
\[
|B_{r/2}(x_1) \cap \{u = 1\}| \left( \int_{\partial B_{r/2}(x_1)} (1 - v) \right)^2 \leq Cr^2 \int_{B_{r/2}(x_1)} |\nabla (u-v)|^2. \tag{5.3}
\]

Hence, (5.2) and (5.3) imply
\[
\int_{\partial B_{r/2}(x_1)} (1 - v) \leq Cr.
\]

Let \( x_3 \) be a point on the segment between \( x_0 \) and \( x_1 \), which is at a distance \( r/4 \) from \( x_1 \). According to Corollary 5.1, \( u \) is superharmonic in \( B_{r/2}(x_1) \), so \( 1 - u(x_3) \leq 1 - v(x_3) \). On the other hand, using a Poisson kernel estimate
\[
1 - v(x_3) \leq C \int_{\partial B_{r/2}(x_1)} (1 - v) \leq Cr
\]
for some dimensional constant \( C \). Then by Harnack inequality,
\[
\sup_{B_{r/4}(x_0)} (1 - u) \leq C'(1 - u(x_3)) \leq C' r.
\]

Applying a gradient estimate, we can conclude \(|\nabla u|(x_0) \leq C\).

The case when the ball \( B_4(x_0) \) touches \( F^-(u) \) is treated analogously. \( \square \)

5.3. **Construction of a global minimizer.** Take an increasing sequence of cylinders \( \Omega_k = C_{R_k,h_k} \) with \( R_k, h_k \nearrow \infty \) and let \( u_k \) be the minimizers to \( I(\cdot, \Omega_k) \) over \( S(\Omega_k) \) constructed in Proposition 5.1. If \((V,W)\) is a nicely intertwined pair, Proposition 5.3 implies that (for all large enough \( k \)) \( u_k \) are uniformly Lipschitz-continuous on compact subsets of \( \mathbb{R}^n \). Therefore, one can extract a subsequence (call it again \( \{u_k\} \)) which converges uniformly to a globally defined, locally Lipschitz continuous function \( u : \mathbb{R}^n \to \mathbb{R} \), so that in addition \( \nabla u_k \rightharpoonup \nabla u \) weakly in \( L^\infty_{\text{loc}}(\mathbb{R}^n) \).

**Proposition 5.4.** Assume that \( V,W \) is a pair of a globally defined subsolution and supersolution to (1.1), which are monotonically increasing in \( x_n \) and nicely intertwined with \( V \leq W \). Then the locally Lipschitz-continuous function \( u : \mathbb{R}^n \to \mathbb{R} \), produced above, is monotonically increasing in the \( x_n \)-variable, satisfies \( V \leq u \leq W \) and is harmonic in \( \{u > 0\} \). Moreover, for any cylinder \( \Omega \subset \mathbb{R}^n \) \( u \) minimizes \( I(\cdot, \Omega) \) among all competitors \( v \in S(\Omega) \) such that \( v - u \in H^1_0(\Omega) \).

**Proof.** The first three properties follow from the uniform convergence \( u_k \to u \) on compact sets. Let us concentrate on the minimization property: assume that there exists a cylinder \( \Omega \) and a competitor \( v \in S(\Omega) \), \( v - u \in H^1_0(\Omega) \) such that
\[
I(v, \Omega) \leq I(u, \Omega) - \delta
\]
for some \( \delta > 0 \). Denote by \( \mathcal{N}_t(\Omega) \) the \( t \)-thickening of \( \Omega \):
\[
\mathcal{N}_t(\Omega) = \{ x \in \mathbb{R}^n : \text{dist}(x, \Omega) < t \}.
\]
For $k$ large enough so that $\Omega \in \Omega_k$ construct the following competitor $v_k : \Omega_k \to \mathbb{R}$ for $u_k$:

$$v_k(x) = \begin{cases} v(x) & \text{in } \Omega \\ (1 - \frac{d(x, \Omega)}{t}) u(x) + \frac{d(x, \Omega)}{t} u_k(x) & \text{in } A_k := \mathcal{N}_t(\Omega) \setminus \Omega \\ u_k(x) & \text{in } \Omega_k \setminus \mathcal{N}_t(\Omega) \end{cases}$$

for some small enough $t > 0$ which will be chosen later. It is easy to check that $v_k \in S(\Omega_k)$, therefore

$$0 \leq I(v_k, \Omega_k) - I(u_k, \Omega_k) \leq$$

$$\leq (I(v, \Omega) - I(u, \Omega)) + (I(u, \Omega) - I(u_k, \Omega)) + (I(v_k, A_k) - I(u_k, A_k))$$

$$\leq -\delta + (I(u, \Omega) - I(u_k, \Omega)) + (I(v_k, A_k) - I(u_k, A_k)).$$

By the lower semicontinuity of $I(\cdot, \Omega)$ there exists a subsequence $u_{k_l}$ such that

$$I(u, \Omega) - I(u_{k_l}, \Omega) < \delta/2.$$

Now we claim that we can choose $t$ so small that for all $l$ large enough

$$I(v_{k_l}, A_k) - I(u_{k_l}, A_k) < \delta/2$$

which will lead to a contradiction in (5.4). Indeed, $|\nabla u_k| \leq K$ is uniformly bounded on some fixed large cylinder $\Omega' \supseteq \mathcal{N}_t(\Omega)$, so

$$I(u_{k_l}, A_k) \leq (K + 1)|A_k| \leq Ct.$$

Also, $|\nabla u| \leq K$ on $\Omega'$

$$|\nabla v_k| = \left| \nabla \left( u + \frac{d(x, \Omega)}{t} (u_{k_l} - u) \right) \right| \leq |\nabla u| + \frac{1}{t} |u_{k_l} - u| + |\nabla u_{k_l} - \nabla u|$$

$$\leq 3K + \frac{1}{t} |u_{k_l} - u|.$$  

Thus, if $\epsilon_k = \sup_{\Omega'} |u_k - u|$ 

$$I(v_{k_l}, A_k) \leq C'|A_k|(1 + \epsilon_{k_l}/t + (\epsilon_{k_l}/t)^2) \leq Ct.$$  

for all $l$ large enough, so that $\epsilon_{k_l} \leq t$. Thus, if we choose $t < \delta/(2C)$, the estimate (5.5) will be satisfied for all $l$ large enough. \qed

5.4. Regularity of global minimizers. As mentioned in the introduction, there is an intimate connection between the energy functional $I$ and the standard one-phase energy functional

$$I_0(u, \Omega) = \int_{\Omega} |\nabla u|^2 + 1_{\{u > 0\}} \quad u \in H^1(\Omega).$$

as well as between the notions of viscosity (sub-/super-) solutions to (1.1) and (1.6). Recall,

**Definition 5.4.** A viscosity solution to (1.6) is a non-negative continuous function $u$ in $\Omega$ such that

- $\Delta u = 0$ in $\Omega_p(u)$;
- If there is a tangent ball $B$ to $F_p(u)$ at some $x_0 \in F_p(u)$ from either the positive or zero side, then

$$u(x) = \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \quad \text{as } x \to x_0,$$

where $\nu$ is the unit normal to $\partial B$ at $x_0$ directed into $\Omega_p(u)$.

Equivalently, a viscosity solution cannot be touched from above by a strict classical supersolution or from below by strict classical subsolution at a free boundary point.

**Definition 5.5.** A viscosity subsolution (resp. supersolution) to (1.6) is a non-negative continuous function $v$ in $\Omega$ such that

- $\Delta u \geq 0$ in $\Omega_p(u)$;
If there is a tangent ball from the positive side $B \subset \Omega_p(v)$ (resp. zero side $\Omega_n(v)$) to $F_0(v)$ at some $x_0 \in F_0(v)$ and $\nu$ denotes the unit inner (resp. outer) normal to $\partial B$ at $x_0$, then

$$v(x) = \alpha (x - x_0, \nu)^+ + \alpha (|x - x_0|) \quad \text{as} \quad x \to x_0,$$

for some $\alpha \geq 1$ (resp. $\alpha \leq 1$).

If $\alpha$ is strictly greater (resp. smaller) than 1, then $v$ is called a strict viscosity subsolution (resp. supersolution).

**Remark 5.1.** Indeed, suppose $u$ minimizes $I(\Omega)$ among all $v \in H^1(\Omega)$, such that $V \leq v \leq W$, where $V$ is a subsolution and $W$ is a supersolution to (1.1) and let $D \subseteq \Omega$ be a (regular enough) domain such that either

$$D \cap \{V = -1\} = \emptyset \quad \text{or} \quad D \cap \{W = 1\} = \emptyset.$$

In the first case, we readily see that $1 - u$ minimizes $I_0(v_0, D)$ among all admissible $1 - W \leq v_0 \leq 1 - V$, with $1 - W$ being a subsolution and $1 - V$ a supersolution to (1.6) in $D$. Similarly, in the second case, $u_0 + 1$ minimizes $I_0(v_0, D)$ among all admissible $V + 1 \leq v_0 \leq W + 1$, with $V + 1$ being a subsolution and $W + 1$ a supersolution to (1.6) in $D$.

Below we will collect the regularity results concerning constrained minimizers of $I_0$, developed by [10]. For completeness, we will lay out the natural sequence of establishing regularity: starting from weaker notions and ending with the optimum, classical regularity.

When a viscosity solution $u$ to (1.6) arises from a minimization problem, $u$ exhibits a non-degenerate behaviour at the free boundary in the sense that $u$ grows linearly away from its free boundary in the positive phase. To make that statement precise we need the following definition.

**Definition 5.6.** A continuous nonnegative function $u$ is non-degenerate along its free boundary $F_p(u)$ in $\Omega$ if for every $G \Subset \Omega$, there exists a constant $K = K(G) > 0$ such that for every $x_0 \in F_p(u) \cap G$ and every ball $B_r(x_0) \subseteq G$,

$$\sup_{B_r(x_0)} u \geq Kr.$$

On the way to establishing strong regularity properties for free boundary $F_p(u)$ one needs certain weaker, measure-theoretic notions of regularity.

**Definition 5.7.** The free boundary $F_p(u)$ satisfies the density property (D) if for every $G \Subset \Omega$ there exists a constant $c = c(G) > 0$ such that for every ball $B_r \subseteq G$, centered at a free boundary point,

$$c \leq \frac{|B_r \cap \Omega_p(u)|}{|B_r|} \leq 1 - c.$$

Here we should also recall the notion of nontangentially accessible (NTA) domains [14], which admit the application of the powerful boundary Harnack principles.

**Definition 5.8.** A bounded domain $D \subset \mathbb{R}^n$ is NTA if for some constants $M > 0$ and $r_0 > 0$ it satisfies the following three conditions:

- **(Corkscrew condition).** For any $x \in \partial D$, $r < r_0$, there exists $y = y_r(x) \in D$ such that $M^{-1}r < |y - x| < r$ and $\text{dist}(y, \partial D) > M^{-1}r$;

- **(Density condition).** The Lebesgue density of $D^c = \mathbb{R}^n \setminus D$ at every point $x \in D^c$ is uniformly bounded from below by some positive $c > 0$

$$\frac{|B_r(x) \cap D^c|}{|B_r(x)|} \geq c \quad \forall \quad x \in D^c \quad 0 < r < r_0;$$

- **(Harnack chain condition)** If $x_1, x_2 \in D$, $\text{dist}(x_i, \partial D) > \epsilon$, $i = 1, 2$ and $|x_1 - x_2| < m$ there exists a sequence of $N = N(m)$ balls $\{B_{r_j}\}^N_{j=1}$ in $D$, such that $x_1 \in B_{r_1}$, $x_2 \in B_{r_N}$, successive balls intersect and $M^{-1}r_j < \text{dist}(B_{r_j}, \partial D) < Mr_j$, $j = 1, \ldots, N$. 

We can now state the regularity results proved in [10] and [11] concerning constrained minimizers of \( I_0 \). We will say that the triple of functions \((V_0, u_0, W_0)\) defined on a vertical right cylinder \( \Omega = C_{R,h} \) satisfies the hypotheses \( H(\Omega) \) if

- \( V_0 \) is a strict classical subsolution and \( W_0 \) is a strict classical supersolution to \((1.6)\) in \( \Omega \) such that \( V_0 \leq W_0, \partial_{x_n} V_0 > 0 \) in \( \{ V_0 > 0 \} \) and \( \partial_{x_n} W_0 > 0 \) in \( \{ W_0 > 0 \} \). Moreover, \( F_0(V_0), F_0(W_0) \) are a positive distance away from the top and bottom sections of \( \Omega \).
- The function \( u_0 \) is monotonically increasing in \( x_n \) and minimizes \( I_0(\cdot, \Omega) \) among all competitors \( v \in H^1(\Omega) \) such that \( v - u \in H^1_0(\Omega) \) and \( V_0 \leq v \leq W_0 \) in \( \Omega \).

**Theorem 5.1** ([10], [11]). If \((V_0, u_0, W_0)\) satisfies the hypotheses \( H(\Omega) \), then:

- [10], \( u \) is Lipschitz-continuous and non-degenerate along its free boundary \( F_p(u) \);
- [10], \( F_p(u) \) touches neither \( F_p(V_0) \) nor \( F_p(W_0) \);
- [10], \( u \) is a viscosity solution to \((1.6)\);
- [10], For any vertical cylinder \( D \subseteq \Omega \) the positive phase \( D \cap \Omega_p(u) \) is an NTA domain;
- [10], The free boundary \( F_p(u) \cap C_{R,h} \) is given by the graph of a continuous function \( \phi \), \( F_p(u) = \{(x', x_n) : |x'| < 3R/4, x_n = \phi(x')\} \).
- [11], If \( \max_{|x'| < 3R/4} |\phi(x)| \leq h - \epsilon \), and \( \epsilon \ll R < h \) then
  \[
  \sup_{|x'| < \epsilon/2} |\nabla \phi| \leq C,
  \]
  where \( C \) depends on the dimension \( n \), the Lipschitz constant of \( u \), on \( h, \epsilon \) and the NTA constants of \( \Omega_p(u) \cap C_{3R/4,h} \). By the work of Caffarelli [6], this implies \( \phi(x') \) is smooth in \( \{|x'| \leq \epsilon/4\} \).

Let us revert our attention to the original problem. According to Proposition 4.4, we are in possession of a pair of a strict classical supersolution \( W : \mathbb{R}^9 \to \mathbb{R} \) and a strict classical subsolution \( V : \mathbb{R}^9 \to \mathbb{R} \), such that \( V \leq W \), both are monotonically increasing in the \( x_9 \)-variable (strictly increasing in that direction when away from their \( \pm 1 \) phases), and are, in addition, nicely intertwined (see Definition 5.3). By Proposition 5.4, we can then construct a globally defined, monotonically increasing in \( x_9 \) function \( u : \mathbb{R}^9 \to \mathbb{R} \), such that \( u \) minimizes \( I(\cdot, \Omega) \) among \( v \in S(\Omega) \) for any vertical cylinder \( \Omega \). Taking into account the observations we made in Remark 5.1, we can utilize the regularity results (Theorem 5.1) once we simply show that around every free boundary point \( x_+ \in F^+(u) \) there exists a vertical cylinder \( \Omega_+ \ni x_+ \), such that \( \Omega_+ \cap \{ V = -1 \} = \emptyset \) and around every free boundary point \( x_- \in F^-(u) \) there exists a cylinder \( \Omega_- \ni x_- \), such that \( \Omega_- \cap \{ W = 1 \} = \emptyset \). Furthermore, we’ll need to show \( F^+(u) \cap \Omega_+ \) and \( F^-(u) \cap \Omega_- \) stay a positive distance away from the top and bottom of \( \Omega_+ \), respectively \( \Omega_- \). This is the content of the next lemma.

**Lemma 5.1.** Let \( V, W \) be the strict subsolution/supersolution provided by Corollary 4.1 and \( u \) – the function constructed in §5.3. For every \( y' \in \mathbb{R}^8 \) there exist an \( y_n \in \mathbb{R} \) and an \( R > 0 \) small enough such that

\[
-\frac{1}{2} \leq V(x', y_n) \leq W(x', y_n) \leq \frac{1}{2}
\]

for all \( x' \in \mathbb{R}^8 \) with \( |x' - y'| \leq R \). Thus, if

\[
\Omega_+(y') = \{(x', x_n) : |x' - y'| < R, 0 < x_n - y_n < h\}
\]

\[
\Omega_-(y') = \{(x', x_n) : |x' - y'| < R, 0 < y_n - x_n < h\}
\]

and \( h > 0 \) is large enough, the monotonicity of \( V \) and \( W \) in the \( x_n \)-direction guarantees that

- \( \Omega_+(y') \cap \{ V = -1 \} = \emptyset \) and \( F^+(V), F^+(W) \) exit from the side of \( \Omega_+ \);
- \( \Omega_-(y') \cap \{ W = 1 \} = \emptyset \) and \( F^-(V), F^-(W) \) exit from the side of \( \Omega_- \).
In particular, \((1 - W, 1 - u, 1 - V)(x', - x_n)\) satisfy hypotheses \(H(\Omega_+(y'))\), while \((V + 1, u + 1, W + 1)(x', x_n)\) satisfy hypotheses \(H(\Omega_-(y'))\).

**Proof.** Fix \(y' \in \mathbb{R}^8\). Since \(W(y', x_n) = \pm 1\) for all large positive (negative) \(x_n\), there certainly exists a \(y_n\) such that \(W(y', y_n) = \frac{1}{4}\). For a small enough \(R > 0\), we can ensure \(0 \leq W(x', y_n) \leq \frac{1}{2}\) whenever \(|x' - y'| < R\). Since, \(W\) and \(V\) were constructed so that

\[
0 \leq W(x) - V(x) \leq \frac{1}{2} \quad \forall x \in \mathbb{R}^9 \quad \text{(Corollary 4.1)}
\]

we see that

\[
-\frac{1}{2} \leq V(x', y_n) \leq W(x', x_n) \leq \frac{1}{2} \quad |x' - y'| < R.
\]

\(\square\)

Localizing at any free boundary point, we immediately invoke the lemma above and the regularity results (Theorem 5.1), establishing the desired Theorem 1.1.

**Appendix A. Supersolutions for \(J_{\Gamma_{\infty}}\)**

The two objectives of this appendix are

- To state the results of Del Pino, Kowalczyk and Wei [12] concerning supersolutions for the linearized mean curvature operator on \(F_\infty\):

\[
H'[F_\infty](\phi) = \frac{d}{dt} \bigg|_{t=0} H[F_\infty + t\phi] = \text{div} \left( \frac{\nabla \phi}{\sqrt{1 + |\nabla F_\infty|^2}} - \frac{(\nabla F_\infty \cdot \nabla \phi) \nabla F_\infty}{(1 + |\nabla F_\infty|^2)^{3/2}} \right)
\]

where \(H[\cdot]\) is the mean curvature operator (MCO), and to describe the relation between \(H'[F_{\infty}]\) and the Jacobi operator \(J_{\Gamma_{\infty}}\) on \(\Gamma_{\infty}\).

- To obtain the refined estimate \(H_{\infty, 3} = O \left( \frac{|\theta - \pi/4|}{1 + |\theta|} \right)\).

We recall that in polar coordinates \((r, \theta)\), \(F_\infty(r, \theta) = r^3 g(\theta)\) and the function \(g(\theta)\) is smooth and satisfies:

1. \(g(\theta) = -g(\frac{\pi}{2} - \theta)\)
2. \(g(\theta) = (\theta - \frac{\pi}{4})(1 + c_3(\theta - \frac{\pi}{4})^2 + \cdots)\) near \(\theta = \frac{\pi}{4}\)
3. \(g'(\theta) > 0\) for \(\theta \in (0, \frac{\pi}{4})\) and \(g'(0) = g'(\pi/2) = 0\).

**A.1. The relation between the Jacobi operator and the linearized MCO.** For a domain \(U \subset \mathbb{R}^8\), let \(S : U \to \mathbb{R}\) be an arbitrary \(C^2(U)\) function and denote by \(\Sigma = \{(x', x_9) \in U \times \mathbb{R} : x_9 = S(x')\}\) its graph. Denote the standard projection onto \(\mathbb{R}^9\) by

\[
\pi : \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}^8.
\]

We can identify functions \(\phi\) defined on \(U\) with functions \(\phi_\Sigma\) on \(\Sigma\) in the usual way:

\[
\phi_\Sigma = \phi \circ \pi.
\]

We’ll abuse notation and use the same symbol \(\phi\) to denote both. A long, but straightforward computation yields the following interesting formula relating the linearized MCO associated with \(S\) to the Jacobi operator \(J_{\Sigma} := \Delta_{\Sigma} + |A_\Sigma|^2\) on the graph \(\Sigma:\)

\[
J_{\Sigma} \left( \frac{\phi}{\sqrt{1 + |\nabla S|^2}} \right) = H'[S](\phi) - \frac{\nabla (H[S]) \cdot \nabla S}{\sqrt{1 + |\nabla S|^2}} \frac{\phi}{\sqrt{1 + |\nabla S|^2}}.
\]

(\text{A.1})

Note that if \(\Sigma\) is a minimal graph, i.e. \(H[S] = 0\), we recover the well-known relation

\[
J_{\Sigma} \left( \frac{\phi}{\sqrt{1 + |\nabla S|^2}} \right) = H'[S](\phi).
\]

(\text{A.2})
For our purposes, we would like to estimate the size of the error term in (A.1) when $S = F_{\infty}$.

A.2. Computation of $H[F_{\infty}]$ and $|\nabla(H[F_{\infty}])|$. First, we compute $H[F_{\infty}]$. Since $\text{div} \left( \frac{\nabla F_{\infty}}{|\nabla F_{\infty}|^2} \right) = 0$,

$$H[F_{\infty}] = \text{div} \left( \frac{\nabla F_{\infty}}{\sqrt{1 + |\nabla F_{\infty}|^2}} \right) = -\text{div} \left( \frac{\nabla F_{\infty}}{|\nabla F_{\infty}|^2 (|\nabla F_{\infty}| + \sqrt{1 + |\nabla F_{\infty}|^2})} \right) = -\frac{\nabla F_{\infty}}{|\nabla F_{\infty}|} \cdot \nabla \left( \frac{1}{|\nabla F_{\infty}|^2 (|\nabla F_{\infty}| + \sqrt{1 + |\nabla F_{\infty}|^2})} \right) = \frac{\nabla F_{\infty}}{|\nabla F_{\infty}|} \cdot \nabla Q,$$

where $Q(x') := \sqrt{1 + |\nabla F_{\infty}|^2 (|\nabla F_{\infty}| + \sqrt{1 + |\nabla F_{\infty}|^2})}$. Note that $Q$ is bounded from below by

$$Q(x') \geq 2|\nabla F_{\infty}|^2 = 2r^4(9g(\theta)^2 + g'(\theta)^2) \geq 2mr^4,$$

where $m = \min_{\theta \in [0, \pi/2]} (9g(\theta)^2 + g'(\theta)^2) > 0$. Also,

$$|\nabla |\nabla F_{\infty}|^2| = (2r \sqrt{9g^2 + g'^2})^2 + \left( r g' - \frac{9g + g''}{\sqrt{9g^2 + g'^2}} \right)^2 = O(r^2)$$

and

$$|\nabla \sqrt{1 + |\nabla F_{\infty}|^2}|^2 = \left( |\nabla F_{\infty}| \frac{1}{\sqrt{1 + |\nabla F_{\infty}|^2}} \right)^2 = |\nabla F_{\infty}|^2 \frac{|\nabla F_{\infty}|^2}{1 + |\nabla F_{\infty}|^2} = O(r^2).$$

Thus,

$$|\nabla Q| = O(r^3) \quad (A.3)$$

and

$$|H[F_{\infty}]| \leq \frac{|\nabla Q|}{Q^2} = O(r^{-5}). \quad (A.4)$$

To compute $|\nabla H[F_{\infty}]|$, observe that

$$\nabla (H[F_{\infty}]) = \frac{\nabla (\nabla F_{\infty} \cdot \nabla Q)}{|\nabla F_{\infty}| Q^2} - H[F_{\infty}] \frac{|\nabla F_{\infty}|}{Q^2} - 2H[F_{\infty}] \frac{\nabla Q}{Q}.$$

The last two summands are obviously $O(r^{-6})$. Let us bound

$$|\nabla (\nabla F_{\infty} \cdot \nabla Q)|^2 = \left( \partial_r ((F_{\infty})_r Q_r + r^{-2}(F_{\infty})_r Q_r) \right)^2 + r^{-2} \left( \partial_\theta ((F_{\infty})_r Q_r + r^{-2}(F_{\infty})_{r \theta}) \right)^2.$$

Because of (A.3), $Q_r, r^{-1}Q_\theta = O(r^3)$. Furthermore,

$$F_{\infty,rr}, r^{-1}F_{\infty,rr}, r^{-1}F_{\infty,\theta} = O(r),$$

and

$$Q_{rr}, r^{-1}Q_{r\theta}, r^{-2}Q_{\theta} = O(r^2).$$

Thus, $|\nabla (\nabla F_{\infty} \cdot \nabla Q)|^2 = O(r^8)$ and the first summand in (A.4) is then

$$\frac{\nabla (\nabla F_{\infty} \cdot \nabla Q)}{|\nabla F_{\infty}| Q^2} = O \left( \frac{r^4}{r^{16}} \right) = O(r^{-6}),$$

as well. We conclude

$$|\nabla (H[F_{\infty}])| = O(r^{-6}). \quad (A.5)$$
A.3. Supersolutions for $J_{\Gamma,\infty}$. In [12, §7.2] the authors study the linearized MCO $H'[F_{\infty}]$ and show that it admits two types of supersolutions away from the origin. We will call those Type 1 and Type 2 in parallel with the labels we used in Section §4.2. Because of (A.5), formula (A.1) becomes

$$J_{\Gamma,\infty}\left(\frac{\phi}{\sqrt{1+|\nabla F_{\infty}|^2}}\right) = H'[F_{\infty}](\phi) + O\left(\frac{r^{-6}|\phi|}{\sqrt{1+|\nabla F_{\infty}|^2}}\right)$$

(A.6)

so that one can then cook up supersolutions for the Jacobi operator $J_{\Gamma,\infty}$.

- Type 1 supersolution for the linearized MCO (cf. the proof of Lemma 7.2 in [12]):
  There exists a smooth function $\phi_1 = \phi_1(r,\theta) = r^{-\tau}q_1(\theta)$ with $q_1(\theta) > 0$ and even about $\theta = \pi/4$ that satisfies the differential inequality

$$H'[F_{\infty}](\phi_1) \leq -\frac{1}{r^{4+\epsilon}} \quad r > r_0$$

for sufficiently large $r_0$. Thus, (A.6) implies that $h_1 = \frac{\phi_1}{\sqrt{1+|\nabla F_{\infty}|^2}} \in C^\infty(\Gamma_{\infty})$ satisfies

$$J_{\Gamma,\infty} h_1 \leq -\frac{1}{r^{4+\epsilon}} + O(r^{-8-\epsilon}) \leq -\frac{1}{1+\epsilon}$$

(A.7)

for sufficiently large $r > r_0$.

- Type 2 supersolution for the linearized MCO (cf. the proof of Lemma 7.3 in [12]):
  For every $\frac{1}{3} < \tau < \frac{2}{3}$ there exists a function $\phi_2 = \phi_2(r,\theta) = r q_2(\theta)$, defined for $\theta \in \{\pi/4 < \theta \leq \pi/2\}$, such that

$$H'[F_{\infty}](\phi_2) \leq -\frac{g(\theta)^{\tau}}{r^3}, \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right], \quad r > r_0$$

for sufficiently large $r_0$. Moreover, $q_2(\theta)$ is smooth in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ and has the following expansion near $\theta = \frac{\pi}{4}$:

$$q_2(\theta) = (\theta - \frac{\pi}{4})^{\tau} \left(a_0 + a_2(\theta - \frac{\pi}{4})^2 + \cdots\right) \quad \text{where} \quad a_0 > 0.$$

Therefore, $h_2 = \frac{\phi_2}{\sqrt{1+|\nabla F_{\infty}|^2}}$ satisfies the differential inequality

$$J_{\Gamma,\infty} h_2 \leq -\frac{g(\theta)^{\tau}}{r^3} + O(r^{-7}g(\theta)^{\tau}) \leq -\frac{g(\theta)^{\tau}}{1+r^3}$$

(A.8)

in $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ for sufficiently large $r > r_0$.

A.4. Computation of $H_{\infty,3}$. We will compute the second fundamental form $A$ of the graph $\Gamma_{\infty}$ and then estimate the sizes of the principal curvatures.

Let $y = (\tilde{u}r \cos \theta, \tilde{u}r \sin \theta, r^3g(\theta)) \in \Gamma_{\infty}$, with $\tilde{u}, \tilde{v} \in S^3 \subset \mathbb{R}^4$ and consider local parametrizations $\tilde{u}(t_1, t_2, t_3)$, and $\tilde{v}(s_1, s_2, s_3)$ of $S^3$ around $\tilde{u}$ and $\tilde{v}$, respectively, such that

$$\tilde{u}(0) = \tilde{u} \quad \partial_{t_i} \tilde{u}(0) = t_i, \quad i = 1, 2, 3$$

$$\tilde{v}(0) = \tilde{v} \quad \partial_{s_i} \tilde{v}(0) = \sigma_i, \quad i = 1, 2, 3$$

where $\{t_i\}, \{\sigma_i\}$ are orthonormal bases for $T_{\tilde{u}}S^3$ and $T_{\tilde{v}}S^3$, respectively. Then

$$P(r, \theta, t_1, s_1) = (\tilde{u}r \cos \theta, \tilde{u}r \sin \theta, r^3g(\theta))$$

(A.9)

defines a local parametrization of $\Gamma_{\infty}$ near $y$.

In the system of coordinates $\{r, \theta, t_1, s_1\}$ the metric tensor near $y$ takes the form $g = g_2 + g_{\text{sym}}$, where

$$g_2 = \begin{pmatrix}
1 + 9r^4g^2 & 3r^5gg' & \frac{3r^5gg'}{r^2(1+g^2r^4)} \\
3r^5gg' & r^2(1+g^2r^4) & (r^2\cos^2\theta)U_3 \\
\frac{3r^5gg'}{r^2(1+g^2r^4)} & (r^2\cos^2\theta)U_3 & (r^2\sin^2\theta)V_3
\end{pmatrix}
\quad (A.10)$$

$$g_{\text{sym}} = \begin{pmatrix}
r^2(1+g^2r^4) & 0 & 0 \\
0 & (r^2\cos^2\theta)U_3 & 0 \\
0 & 0 & (r^2\sin^2\theta)V_3
\end{pmatrix}$$
In the expression for $g_{\text{sym}}$ above, $U_3, V_3$ are $3 \times 3$ matrices that depend only on $\{t_k, s_k\}$ with $U_3(0) = V_3(0) = I_3$, the identity $3 \times 3$ matrix. We will also need the inverse of $g$, $g^{-1} = g_{2}^{-1} \oplus g_{\text{sym}}^{-1}$, where

\[
g_2^{-1}(y) = \frac{1}{\sigma} \left( \begin{array}{cc} 1 + r^4 g'^2 & -r^3 g' g' \\ -r^3 g' g' & -r^2 + 9g'^2 r^2 \end{array} \right) \sigma := 1 + r^4(9g^2 + g'^2) \tag{A.11}
g_{\text{sym}}^{-1}(y) = r^{-2} \left( \begin{array}{cc} (\cos \theta)^{-2} & I_3 \\ (\sin \theta)^{-2} & I_3 \end{array} \right).
\]

The unit normal $\nu(y)$ is given by

\[
\nu(y) = \frac{1}{\sqrt{\sigma}} \left( (F_\infty)_v \hat{u}, (F_\infty)_v \hat{v}, -1 \right) = -\frac{1}{\sqrt{\sigma}} (r^2(3g \cos \theta - g' \sin \theta) \hat{u}, r^2(3g \cos \theta + g' \sin \theta) \hat{v}, -1).
\]

We calculate the second fundamental form $(A_\infty)_{ij} = -\partial_i P \cdot \partial_j \nu = \partial_i P \cdot \nu$ at $y$ to be $A_\infty = (A_\infty)_2 \oplus (A_\infty)_{\text{sym}}$, where

\[
(A_\infty)_2(y) = \frac{1}{\sqrt{\sigma}} \left( \begin{array}{cc} 6rg & 2r^2 g' \\ 2r^2 g' & r^3(3g + g'') \end{array} \right)
\]

\[
(A_\infty)_{\text{sym}}(y) = \frac{1}{\sqrt{\sigma}} \left( \begin{array}{cc} r \cos \theta(F_\infty)_u I_3 \\ r \sin \theta(F_\infty)_v I_3 \end{array} \right).
\]

The principal curvatures of $\Gamma_\infty$ at $y$ are the eigenvalues of the matrix $A_\infty g_{\text{sym}}^{-1}$, i.e. the eigenvalues $\mu_1, \mu_2$ (each of multiplicity 3) of $(A_\infty)_{\text{sym}}(g_{\text{sym}})_{\text{sym}}^{-1}$:

\[
\mu_1 = \frac{1}{\sqrt{\sigma}} (F_\infty)_u \frac{r}{\sqrt{\sigma}}(3g - g' \tan \theta)
\]

\[
\mu_2 = \frac{1}{\sqrt{\sigma}} (F_\infty)_v \frac{r}{\sqrt{\sigma}}(3g + g' \cot \theta)
\]

and the eigenvalues $\lambda_1, \lambda_2$ of

\[
(A_\infty)_2(g_2)^{-1} = \frac{1}{\sigma^{3/2}} \left( \begin{array}{cc} O(r(\theta - \frac{\pi}{4})) & * \\ * & O(r^5(\theta - \frac{\pi}{4})) \end{array} \right).
\]

Since $g' \cot \theta = O(1)$ and $g' \tan \theta = O(1)$, we see that $\mu_1$ and $\mu_2$ are $O((1 + r)^{-1})$. Note further that

\[
\mu_1 + \mu_2 = \frac{r}{\sqrt{\sigma}}(6g + 2g' \cot 2\theta) = O\left(\frac{\theta - \pi/4}{1 + r}\right). \tag{A.12}
\]

On the other hand, $\lambda_{1,2} = O((1 + r)^{-1})$ as well, since

\[
\lambda_1 + \lambda_2 = \text{Trace}((A_\infty)_2(g_2)^{-1}) = O\left(\frac{\theta - \pi/4}{1 + r}\right) \tag{A.13}
\]

\[
\lambda_1 \lambda_2 = \text{det}(A_\infty)_2 \text{det}(g_2)^{-1} = O(1) \frac{1}{r^2 \sigma} = O(r^{-6}).
\]

Now (A.12) and (A.13), combined with the fact that the principal curvatures are all of order $O((1 + r)^{-1})$ imply that

\[
H_{\infty,3} = 3(\mu_1 + \mu_2)(\mu_1^2 - \mu_1 \mu_2 + \mu_2^2) + (\lambda_1 + \lambda_2)(\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2) = O\left(\frac{\theta - \pi/4}{1 + r^3}\right).
\]
Appendix B. Bounds on the gradient and hessian of $h(r, \theta)$ on $\Gamma_\infty$.

Once again we will make use of the \{r, \theta, \tau, s_i\} system of coordinates (A.9) in order to estimate the first and second covariant derivatives of a function $h = h(r, \theta) \in C^2(\Gamma_\infty)$ which depends only on $r$ and $\theta$.

**Lemma B.1.** The gradient and hessian of $h = h(r, \theta) \in C^2(\Gamma_\infty)$ satisfy:

$$|D_{r \infty} h| = O\left( |\partial_r h| + (r^{-1} \vartheta + r^{-3}) |\partial_\theta h| \right) \tag{B.1}$$

$$|D^2_{r \infty} h| = O\left( |\partial^2_r h| + (r^{-1} \vartheta + r^{-3}) |\partial^2_\theta h| + (r^{-1} \vartheta + r^{-3})^2 |\partial^3_\theta h| \right) + r^{-1} O\left( |\partial_r h| + (r^{-1} \vartheta + r^{-3}) |\partial_\theta h| \right) + O\left( (r^{-6} + r^{-2} \vartheta^2) \left| \frac{\partial_\theta h}{g'} \right| \right) \tag{B.2}$$

where $\vartheta := |\theta - \pi/4|$ and the constants in the $O$-notation depend on $g$.

**Proof.** In order to carry out the computations, we adopt the standard Einstein index notation. That way, we write

$$|D_{r \infty} h|^2 = h^i h_i \quad \text{and} \quad |D^2_{r \infty} h|^2 = h^i_j h^j_i,$$

where $i, j$ range over the list of coordinates \{r, \theta, \{t_k\}, \{s_k\}\}

$$h_i = \partial_i h, \quad h^i = g^{ik} h_k \quad \text{with} \quad g^{ij} = (g^{-1})_{ij}$$

and

$$h^i_j = \partial_i h^j + \Gamma^i_{jk} h^k.$$

In the expression above, $\Gamma^i_{jk}$ are, of course, the Christoffel symbols:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}). \tag{B.3}$$

Since $h = h(r, \theta)$, we have $|D_{r \infty} h|^2 = h^r h_r + h^\theta h_\theta$. Using (A.11) we calculate

$$h^r = \frac{1 + r^4 g^2}{\sigma} h_r - \frac{3 r^3 g g'}{\sigma} h_\theta$$

$$h^\theta = - \frac{3 r^3 g g'}{\sigma} h_r + \frac{r^{-2} + 9 g^2 r^2}{\sigma} h_\theta$$

and taking into account that $g(\theta) = O(\vartheta)$ and $9g^2 + g'g'$ is uniformly bounded from above and from below by positive constants, we conclude

$$|D_{r \infty} h|^2 = O\left( |h_r|^2 + r^{-1} \vartheta |h_r| |h_\theta| + (r^{-3} + r^{-1} \vartheta)^2 |h_\theta|^2 \right)$$

so that (B.1) is verified.

The computation of the hessian is slightly more involved. For convenience we will denote by Greek letters $\alpha, \beta, \gamma, \ldots$ indices that correspond to coordinates $r, \theta$, and by Latin $l, m, n, \ldots$ indices that correspond to coordinates \{t_i\}, \{s_i\}. First, note that the “cross term” contribution $h_\alpha^i h^i_\alpha = 0$, because

$$h^i_\alpha = \partial_i h^\alpha + \Gamma^i_{\alpha j} h^j = 0,$$

as $\partial_i h^\alpha = 0$ and

$$\Gamma^\alpha_{i\beta} = \frac{1}{2} g^{\alpha \gamma} (\partial_i g_{\gamma \beta} + \partial_\beta g_{i \gamma} - \partial_\gamma g_{i \beta}) = 0.$$

Thus, $|D^2_{r \infty} h|^2 = S_2 + S_{\text{sym}}$, where

$$S_2 := h_\alpha^\beta h^\beta_\alpha, \quad S_{\text{sym}} := h^m_i h^j_m.$$

Let us first deal with $S_{\text{sym}}$. We see that

$$S_{\text{sym}} = \Gamma^m_{i\alpha} \Gamma^l_{m \beta} h^\alpha h^\beta,$$
where
\[ \Gamma^m_{lm} = \frac{1}{2} g^{mm} \left( \partial_m g_{\alpha \alpha} + \partial_\alpha g_{lm} - \partial_m g_{\alpha l} \right) = \frac{1}{2} g^{mm} \partial_\alpha g_{lm}. \]

Noting that
\[ \partial_\alpha g_{\text{sym}}(y) = \begin{pmatrix} \partial_\alpha (r^2 \cos^2 \theta) I_3 \\ \partial_\alpha (r^2 \sin^2 \theta) I_3 \end{pmatrix}, \]
we obtain
\[ S_{\text{sym}} = \left( h^r \right)^2 \frac{6}{r^2} + 2 h^r h^\theta \frac{3}{r} \left( \cot \theta - \tan \theta \right) + \left( h^\theta \right)^2 3 \left( \cot^2 \theta + \tan^2 \theta \right). \]

A more refined estimation of \( h^\theta \) gives
\[ h^\theta = O \left( \left| g' \right| \left( \left| h_r \right| / r + (r^{-6} + r^{-2} \theta^2) |h_\theta / g'| \right) \right) \]
and since
\[ \cot \theta - \tan \theta = O(1/g') \quad \cot^2 \theta + \tan^2 \theta = O(1/g'^2) \]
we conclude
\[ S_{\text{sym}} = O(\left| h_r \right|^2 r^{-2} + r^{-2} \theta^2 |h_\theta|^2 + (r^{-6} + r^{-2} \theta^2)^2 |h_\theta / g'|^2). \]  

Now we proceed with the computation of \( S_2 \). For the purpose we need to calculate the Christoffel symbols \( \Gamma^\gamma_{\alpha \beta} \). The derivatives of \( g_2 \) are
\[ \partial_r g_2 = \begin{pmatrix} 36 r^3 g^2 \\ 15 r^4 g' \\ 2 + 6 g'^2 r^5 \end{pmatrix}, \quad \partial_\theta g_2 = \begin{pmatrix} 18 r^4 g' \\ 3 r^2 (g'^2 + g'') \\ 2 r^6 g' \end{pmatrix} \]
which we then plug in (B.3) to obtain:
\[
\begin{pmatrix}
\Gamma^\gamma_{rr} \\
\Gamma^\gamma_{r\theta} \\
\Gamma^\gamma_{\theta \theta}
\end{pmatrix} = \begin{pmatrix}
g_2^{-1} \frac{1}{2} \begin{pmatrix}
\partial_r g_{rr} & \partial_r g_{r\theta} & \partial_r g_{\theta \theta} \\
\partial_\theta g_{rr} & \partial_\theta g_{r\theta} & \partial_\theta g_{\theta \theta}
\end{pmatrix} \\
\partial_\theta g_{\gamma r} & \partial_\theta g_{\gamma \theta} & \partial_\theta g_{\gamma \theta}
\end{pmatrix} = \begin{pmatrix}
\frac{6}{\sigma} \begin{pmatrix} 3 r^3 g^2 & r^2 g' \\
1 & 6 r^4 g' & 3 \sigma \left( g'^2 + g'^2 \right)
\end{pmatrix} \\
0 & 0 & \frac{1}{\sigma} \begin{pmatrix} r^5 & 3 r^4 g' & r^{-1} (3 g g' - g'^2) - r \\
0 & r^4 g' & g' - 3 g
\end{pmatrix}
\end{pmatrix}
\]

For convenience define the following expressions that measure the magnitude of the first and second derivatives of \( h \):
\[
\mathcal{F}(h_\alpha) := |h_r| + (\partial r^{-1} + r^{-3}) |h_\theta| \\
S(\partial^2 h_{\alpha \beta}, h, h_\alpha) := |\partial^2 r h| + (\partial r^{-1} + r^{-3}) |\partial^2_\theta h| + (\partial r^{-1} + r^{-3})^2 |\partial^2_\theta h| + r^{-1} \mathcal{F}(h_\alpha)
\]
A straightforward computation yields:
\[
h^r_r = \partial_r h^r + \Gamma^r_{rr} h^r + \Gamma^r_{r\theta} h^\theta = O(S(\partial^2 h_{\alpha \beta}, h_\alpha))
\]
\[
h^\theta_\theta = \partial_\theta h^\theta + \Gamma^\theta_{\theta r} h^r + \Gamma^\theta_{\theta \theta} h^\theta = O(S(\partial^2 h_{\alpha \beta}, h_\alpha))
\]
\[
h^r_\theta = \partial_r h^\theta + \Gamma^r_{r \theta} h^r + \Gamma^r_{r \theta} h^\theta =
\]
\[= O(r^{-7} h_\theta + r^{-6} \partial^2_\theta h + \partial r^{-1} S(\partial^2 h_{\alpha \beta}, h_\alpha)) + O(\partial r^{-2} \mathcal{F}(h_\alpha)) =
\]
\[= O((r^{-3} + \partial r^{-1}) S(\partial^2 h_{\alpha \beta}, h_\alpha))
\]
\[
h^\theta_\theta = \partial_\theta h^r + \Gamma^\theta_{\theta r} h^r + \Gamma^\theta_{\theta \theta} h^\theta =
\]
\[= O(\partial^2_\theta h + r^{-1} \partial^2_\theta h + \partial h_r + r^{-1} h_\theta) + O(\partial h_r + (r^{-5} + \partial^2 r^{-1}) h_\theta) =
\]
\[= O((r^{-3} + \partial r^{-1})^{-1} S(\partial^2 h_{\alpha \beta}, h_\alpha)),
\]
whence we conclude
\[ S_2 = O(S^2(\partial^2_{\alpha\beta} h, h)). \] (B.5)

Equations (B.4) and (B.5) yield the estimate (B.2) for the hessian. □

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