Exact interior solutions in $(2 + 1)$ dimensional spacetime.

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Abstract: We provide a new class of exact solutions for the interior in $(2 + 1)$ dimensional spacetime. The solutions obtained for the perfect fluid model both with and without cosmological constant ($\Lambda$) are found to be regular and singularity free. It assume very simple analytical forms that help us to study the various physical properties of the configuration. Solutions without $\Lambda$ are found to be physically acceptable.

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I. INTRODUCTION

Study of exact solutions of Einstein’s Field Equations is an important part of the theory of General Relativity. This importance is not only from more formal mathematical aspects associated with the theory (e.g. the classification of space-times) but also from the growing importance of the application of general relativity to astrophysical phenomena. For example, exact solutions may offer physical insights that numerical solutions cannot. Present time trend of analyzing different aspects of black hole (BH) solutions did lead us to grow our interests in cleaner $(2 + 1)$ dimensional gravity. Discovery of BTZ BH \[1\] ignited the light first. Through this $(2 + 1)$ dimensional model if we need to explore the foundations of classical and quantum gravity we would not find any Newtonian limit and no propagating degrees of freedom will arise. In literature, very easy-to-find works in this aspect comprise the study of quasi normal modes of charged dilaton BHs in $(2 + 1)$ dimensional solutions in low energy string theory with asymptotic anti de-Sitter space times [2]. Hawking radiation from covariant anomalies in $(2 + 1)$ dimensional BHs [3] is another beautiful example. Lastly, we must also name the study of branes with naked singularities analogous to linear or planar defects in crystals and showing that zero-branes in AdS space times are “negative mass BHs!” [4]. Taking charged gravastars as an alternative to charged BHs in $(2 + 1)$ AdS space times is already investigated [5]. Extensions of BTZ BH solutions with charge are also available in the literature. These are obtained by employing non-linear Born Infield electrodynamics to eliminate the inner singularity [6]. The non-static charged BTZ like BHs in $(N + 1)$ dimensions have also been studied [7] which in its static limit, for $N = 2$, reduces to $(2 + 1)$ BTZ BH solutions.

Study of interior solutions in $(2 + 1)$ dimension [8] shows that even the noncommutative-geometry-inspired BTZ BH is not free from any singularity. Study of interior solutions are rarely found in literature. For example, solutions of C. Wolf [9] and S. Yazadjiev [10], solutions in textbook framework of Brans-Dicke theory of gravity by S.M.Kozyrev [11] and new class of solutions corresponding to BTZ exterior spacetime by Sharma et al. [12], which is regular at the centre and it satisfies all the physical requirements except at the boundary where the authors propose a thin ring of matter content with negative energy density so as to prevent collapsing. The discontinuity of the affine connections at the boundary surface provide the above matter confined to the ring. Such a stress-energy tensor is not ruled out from the consideration of Casimir effect for massless fields.

The purpose of the present work is to find exact interior solutions for perfect fluid model both with and without cosmological constant, $\Lambda$. The motivation for doing so is provided by the fact that the assumption of equation of state (EoS), $p = mp$, which seems to be very reasonable for describing the matter distribution in the study of relativistic objects like stars [13, 14], wormholes [15, 16], and gravastars [17, 18].

The structure of our work is as follows: In sec. (II), we derive required Einstein equations. Sec. (III) constitutes of different interior solutions for various cases of EoS. Lastly, in sec. (IV) a brief conclusion is provided.

II. EINSTEIN FIELD EQUATIONS IN $(2 + 1)$ DIMENSION

We take the static metric to describe the interior region of a $(2 + 1)$ dimensional space time as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + r^2d\theta^2,$$

where $\nu(r)$ and $\mu(r)$ are the two unknown metric functions. We take the perfect fluid form of the energy momentum tensor

$$T_{ij} = \text{diag}(-\rho, p, p),$$
where $\rho$ is energy density and $p$ is pressure. Einstein’s field equations with a cosmological constant, $\Lambda$, for the space-time metric (Eq. (1)) together with the energy momentum tensor given in Eq. (2) may be written as

$$2\pi\rho + \Lambda = \mu e^{-2\mu(r)/r},$$
$$2\pi p - \Lambda = \nu'e^{-2\mu(r)/r},$$
$$2\pi p - \Lambda = e^{-2\mu}(\nu'' + \nu'^2 - \nu'\nu').$$

Here superscript $'\nu$' denotes the derivative with respect to $r$. Assuming $G = c = 1$, the generalized Tolman-Oppenheimer-Volkov (TOV) equation may be written as

$$(\rho + p)\nu' + p' = 0,$$

which represents conservation equations in (2+1) dimensions.

We take the EoS of the form

$$p = m\rho,$$

where $m$ is EoS parameter.

### III. INTERIOR SOLUTIONS

We first obtain interior solutions without any cosmological constant, thereby taking $\Lambda = 0$. Latter on, we generalize our study to non-zero value of $\Lambda$. We choose various cases of EoS parameter for both the choices of $\Lambda$.

**A. With no cosmological constant ($\Lambda = 0$)**

1. $0 < m < 1$

For $\Lambda = 0$, the field equations (3)-(6) become

$$2\pi\rho = \mu'e^{-2\mu(r)/r},$$
$$2\pi m\rho = \nu'e^{-2\mu(r)/r},$$
$$2\pi m\rho = e^{-2\mu}(\nu'^2 + \nu'' - \nu'\nu').$$

The TOV equation (11) takes the form,

$$(\rho + m\rho)\nu' + m\rho' = 0,$$

Equation (11) yields

$$\rho^{m}e^{(1+m)\nu} = C.$$

Solving equations (8) and (9), we get

$$\nu = m\mu + A.$$  

Equating Eq.(9) with Eq.(10), we get

$$e^\mu = e^\nu\nu'/r.$$  

Now, solving equations (13) and (14), we obtain

$$\nu = \frac{A}{1 - m} - \frac{m}{1 - m}\ln\left\{\frac{1 - m}{m}\left(B - \frac{r^2}{2}\right)^{\frac{2}{1 - m}}\right\},$$

$$\mu = \frac{1}{1 - m}\left[A - \ln\left\{\frac{1 - m}{m}\left(B - \frac{r^2}{2}\right)^{\frac{2}{1 - m}}\right\}\right],$$

$$\rho = \frac{1}{2\pi m}e^{-\frac{2\mu}{1 - m}}\left[\frac{1 - m}{m}\left(B - \frac{r^2}{2}\right)^{\frac{2}{1 - m}}\right].$$

Here $C$, $A$ and $B$ are integration constants.

For the consistency of solutions, the constants should follow the constraint equation,

$$A = m\ln(2\pi m) + \ln C.$$  

These solutions are regular at the center. The central density is given by

$$\rho_c = \frac{1}{(2\pi m)}e^{-\frac{2\mu}{1 - m}}[B(1 - m)/m]^{\frac{1}{1 - m}}.$$  

The interior solution is valid up to the radius $r < \sqrt{2B}$. For a physically meaningful solution the radial and tangential pressure should be decreasing function of $r$. From equation (17), we find

$$\frac{dp}{dr} = \frac{1}{m}\frac{dp}{dr} < 0,$$

which gives density and pressure as decreasing functions of $r$. At $r = 0$, one can get

$$\frac{dp}{dr} = 0, \frac{dp}{dr} = 0 \text{ and } \frac{d^2p}{dr^2} = \left[\frac{(1 - m)B}{m}\right]^{\frac{2m}{1 - m}} < 0,$$

which support maximality of central density and radial central pressure. Here, density and pressure decrease radially outward as shown in FIG. 1.
The above TOV (Eq. 11) may be re-written as

\[ \frac{M_G (\rho + p)}{r} e^{\frac{\nu}{r}} + \frac{dp}{dr} = 0, \]  

(22)

where \( M_G = M_G(r) \) is the gravitational mass inside a sphere of radius \( r \) and is given by Tolman-Whittaker formula, which may be derived from field equations,

\[ M_G(r) = re^{\frac{\nu}{r}} \nu'. \]  

(23)

This modified form of TOV equation indicates the equilibrium condition for the fluid sphere subject to the gravitational and hydrostatic forces,

\[ F_g + F_h = 0, \]  

(24)

where

\[ F_g = \nu' (\rho + p) = r + m \frac{e^{-\frac{2A}{r^2}}}{m} \left[ 1 - m \left( B - \frac{r^2}{2} \right) \right] e^{\frac{2m}{r}}, \]  

(25)

\[ F_h = \frac{dp}{dr} = -F_g. \]  

(26)

The profiles of \( F_g \) and \( F_h \) for the specific values of the parameters are shown in FIG. 2 which provides the information about the static equilibrium due to the combined effect of gravitational and hydrostatic forces.

Mass, \( M(r) \), within a radius \( r \), is calculated as

\[ M(r) = \int_0^r 2\pi \rho \delta d\nu = \frac{1}{2} e^{-\frac{2A}{r^2}} \left[ B \left( 1 - \frac{m}{m} \right) \right] e^{\frac{2m}{r}} \right] \frac{2m}{r} \right. \]  

(27)

\[ + \frac{1}{2} e^{-\frac{2A}{r^2}} \left[ \left( B - \frac{r^2}{2} \right) \left( 1 - \frac{m}{m} \right) \right] \frac{2m}{r} \right. \]  

FIG. 2: Variation of compactness (\( u = \frac{M(r)}{r} \)) in the interior region. Description of curves is the same as in FIG. 1.

The compactness of the fluid sphere, \( u(r) \), is thus defined as be found as

\[ u(r) = M(r)/r. \]  

(28)

This is an increasing function of the radial parameter (see figure 3). Correspondingly, the surface redshift (\( Z_s \)) is given by

\[ Z_s(r) = (1 - 2u(r))^{-\frac{1}{2}} - 1. \]  

(29)

FIG. 3: Variation of the forces in the interior region. Description of curves is the same as in FIG. 1.

FIG. 4: Variation of redshift in the interior region. Description of curves is same as in FIG. 1.

2. \( m = 1 \)

For stiff fluid model, \( p = \rho \), and with \( \Lambda = 0 \), the field equations (3)-(6) yield following solutions

\[ \nu = e^{-D} \left( \frac{r^2}{2} / 2 \right) + E, \]  

(30)

\[ \mu = -D + e^{-D} \left( \frac{r^2}{2} / 2 \right) + E, \]  

(31)

\[ \rho = F e^{-2E - r^2 e^{-2D}}. \]  

(32)

Here \( D, E \) and \( F \) are integration constants.

For the consistency of the solutions, the constants should follow the following constraint equation.

\[ D = \ln(2\pi F). \]  

(33)

This ensures that \( F > 0 \).

The solutions are regular at the center and are valid for
infinite large sphere. The central density is $\rho_c = F e^{-2E}$.

The TOV equation yields

$$F_g + F_h = 0,$$

where

$$F_g = -F_h = (r/\pi)e^{-2E - r^2 e^{-D}}.$$  \hfill (37)

From Eq. (32), we find at $r = 0$,

$$\frac{dp}{dr} = 0, \quad \frac{dp}{dr} = 0 \quad \text{and} \quad \frac{d^2 p}{dr^2} = \frac{1}{\pi} \left[-e^{-2E}\right] < 0$$  \hfill (34)

Thus central density is maximum.

The mass, $M(r)$, within a radial distance $r$ is given by

$$M(r) = e^D \left[e^{D - 2E} - e^{D - 2E - r^2 e^{-D}}\right]/2.$$  \hfill (35)

The compactness of the fluid sphere is thus, $u(r) = M(r)/r$. Having $u$, the $Z_s$ is determined using Eq. (29). The important physical characteristics such as density, compactness and redshift are shown in FIGs. 5-7.

The equation of state of the kind, $p = -\rho$ is related to the $\Lambda$ – dark energy, an agent responsible for the second phase of the inflation of Hot Big Bang theory. Using the equation of state of the kind, $p = -\rho$, and with $\Lambda = 0$, the field equations (3)-(6) yield following solutions
\[
\begin{align*}
\rho &= -p = J, \quad (38) \\
\nu &= \frac{[H + \ln(r^2 + K)]}{2}, \quad (39) \\
\mu &= \frac{[H - \ln(r^2 + K)]}{2}. \quad (40)
\end{align*}
\]

Here, \(J\), \(K\) and \(H\) are integration constants. Solutions hold good for the following constraint equation
\[
2\pi J + e^{-H} = 0. \quad (41)
\]

These are regular at the center if \(K\) is positive and the solution is valid for the infinite large fluid sphere. However, for \(K < 0\), solution is valid for \(r > \sqrt{-K}\) up to infinite large radius.

4. \(m = 0\)

For the dust case i.e. when \(p = 0\) and \(\rho \neq 0\), the field equations (3)-(6) reduce to
\[
\nu = \nu_0 \quad (42)
\]
and
\[
e^{-2\mu} = \mu_0 - \int 4\pi r \rho dr. \quad (43)
\]

Here, \(\nu_0\) and \(\mu_0\) are integration constants.

Unless specifying the energy density, one can not get exact analytical solution of the field equations. Thus dust model in \((2+1)\) dimensional space time is possible for known energy density.

**B. With cosmological constant \((\Lambda \neq 0)\)**

1. \(m = -1\)

As before for the equation of state of the kind \(p = -\rho\) with non zero \(\Lambda\), the field equations (3)-(6) yield \(\rho = c_4\). The metric coefficients may be obtained as
\[
\nu = \ln(r^2 + B_5)/2 + D_5 \quad (44)
\]
and
\[
\mu = -\ln(r^2 + B_5)/2 - D_5 + A_4. \quad (45)
\]

Here, \(c_4\), \(A_4\), \(B_5\) and \(D_5\) are integration constants.

These solutions are consistent if
\[
2\pi c_4 + \Lambda + e^{2D_5 - 2A_4} = 0. \quad (46)
\]

These solutions are regular at the center if \(B_5\) is positive and the solution is valid for the infinite large fluid sphere. However, for \(B_5 < 0\), the solution is valid for \(r > \sqrt{-B_5}\) up to infinite large radius. The nature of the solutions of the metric potentials is independent of the sign of \(\Lambda\).

However, sign of \(\Lambda\) plays a crucial role to get positive energy density. For positivity of energy density, one should take negative \(\Lambda\).

Note that without any loss of generality, we can take \(D_5 = 0\) as it can be absorbed by re-scaling the coordinates. We match the interior solution to the exterior BTZ black hole metric
\[
ds^2 = -(M_0 - \Lambda r^2)dt^2 + (M_0 - \Lambda r^2)^{-1}dr^2 + r^2 d\theta^2,
\]
at the boundary \(r = R\), which yield
\[
(M_0 - \Lambda R^2) = R^2 + B_5, \quad (47)
\]
\[
(M_0 - \Lambda R^2)^{-1} = e^{2A_4}(R^2 + B_5)^{-1}. \quad (48)
\]

Solving these two equations, we get
\[
B_5 = M_0 - (\Lambda + 1)R^2 \quad \text{and} \quad A_4 = 0. \quad (49)
\]

The consistency relation assumes the form
\[
2\pi c_4 + \Lambda + 1 = 0. \quad (50)
\]

2. \(0 < m \leq 1\)

From the field equations (3)-(6) after some manipulation, we arrive at
\[
2\pi (1 + m)A_1 r^{m-1}/\nu' = 2r\nu' + r\nu'' - \nu'. \quad (51)
\]

One can observe that \(\nu' = 0\) will be a particular solution of this equation. This yields \(\nu = constant\). Equation (4) implies \(p = \Lambda/2\pi\). Finally, we get the following solution for \(\mu\) as
\[
\mu = -\ln [A_3 - N r^2] / 2. \quad (52)
\]

Here, \(N = \Lambda(1 + m)/m\) and \(A_3\) is integration constant.

For positivity of energy density, one should take positive \(\Lambda\). The solutions are regular at the center if \(A_3 > 0\) and valid up to \(r < \sqrt{A_3/N}\). In this case, we can not match the interior solution to the exterior BTZ black hole metric which is vacuum solution with negative \(\Lambda\).

3. \(m = 0\)

For the dust case i.e. when \(p = 0\) and \(\rho \neq 0\), one can not obtain the exact analytical solution of the field equations. Thus dust model in \((2+1)\) dimensional spacetime with non zero \(\Lambda\) is not possible.

**IV. CONCLUSION**

In this paper we have obtained a new class of exact interior solution of Einstein field equation in \((2+1)\) dimensional space time assuming the equation of state \(p = m\rho\)
(where \(m\) is the equation of state parameter). The interior solutions obtained without cosmological, \(\Lambda\), are physically acceptable for the following reasons:
(i) the solutions are regular at the origin,
(ii) both the pressure (\(p\)) and energy density (\(\rho\)) are positive definite at the origin,
(iii) the pressure reduce to zero at some finite boundary radius \(r_0 > 0\),
(iv) both the pressure and energy density are monotonically decreasing to the boundary,
(v) the subluminal sound speed (\(v_s^2 = \frac{d p}{d \rho} = m \leq 1\))
(vi) and Ricci scalar is non zero i.e. spacetime is non flat.

It is to be noted that at very high densities the adiabatic sound speed may not equal the actual propagation speed of the signal. By studying TOV equation, we have shown that equilibrium stage of the interior region without \(\Lambda\) can be achieved due to the combined effect of gravitational and hydrostatic forces. We know BTZ exterior vacuum solution in (2+1) dimension is valid only for non zero \(\Lambda\). Therefore, it is not possible to match our interior solution (without \(\Lambda\)) with BTZ spacetime at some boundary. We emphasis the following fact that any interior solution in four dimensional space made with a perfect fluid must be glued with an exterior vacuum solution only at a regular surface \(p = 0\) (this is consequence of the well-known Israel matching conditions for the related problem). For barotropic equation of state the configurations present \(p = 0\) surfaces at the same location where \(\rho = 0\). For the solutions (15)-(17) with \(0 < m < 1\) this occurs at \(r = \sqrt{2B}\). However, the metric coefficients are singular at the same locus, in fact, there is a curvature singularity at \(r = \sqrt{2B}\). Hence, this is not a regular region where spacetime can be continuously glued with other spacetime. Hence, one should take \(B\) as large as possible so that the solution is valid for the infinite large fluid sphere and we don’t have the vacuum region left. For \(m=1\) case, there does not exist any radius for which \(p = \rho = 0\), hence, Israel matching condition does not occur.

While finding interior solution with non zero \(\Lambda\), we note that density and pressure remain constant. Interestingly, we observe that it is not possible to get dust model in (2 + 1) dimensional spacetime with non zero \(\Lambda\). Investigation on full collapsing model of a (2 + 1) dimensional configuration will be a future project.

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