Large-dimensional Central Limit Theorem with Fourth-moment Error Bounds on Convex Sets and Balls

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Abstract: We prove the large-dimensional Gaussian approximation of a sum of $n$ independent random vectors in $\mathbb{R}^d$ together with fourth-moment error bounds on convex sets and Euclidean balls. We show that compared with classical third-moment bounds, our bounds can achieve improved and, in the case of balls, optimal dependence $d = o(n)$ on dimension. We discuss an application to the bootstrap. The proof is by recent advances in Stein’s method.

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1 INTRODUCTION

Let $\{\xi_i\}_{i=1}^n$ be a sequence of independent mean-zero random vectors in $\mathbb{R}^d$. Let $W = \sum_{i=1}^n \xi_i$ and $\Sigma = \text{Var}(W)$. It is well known that under finite third-moment conditions and for fixed dimension $d$, the distribution of $W$ can be approximated by a Gaussian distribution with error rate $O(1/\sqrt{n})$.

Motivated by modern statistical applications, we are interested in the large-dimensional setting where $d$ grows with $n$. Numerous studies have provided explicit error bounds on various distributional distances in the Gaussian approximation. See, for example, [Bentkus (2003, 2005)] and [Raic (2019a)] for results for the probabilities of convex sets in $\mathbb{R}^d$; [Chernozhukov, Chetverikov and Kato (2013, 2017), Chernozhukov et al. (2019) and Fang and Koike (2020a)] for results for hyperrectangles; and [Zhai (2018), Eldan, Mikulincer and Zhai (2018), Raić (2019b) and Bonis (2020)] for results for the Wasserstein distance in the approximation. However, the optimal rates, especially in terms of how rapidly $d$ can grow with $n$ while maintaining the validity of the Gaussian approximation, have not been fully addressed and remain a challenging open problem.

In this paper, we consider the approximation of probabilities of convex sets and Euclidean balls. For convex sets, [Bentkus (2005)] proved for the above $W$ that if $\Sigma$ is invertible and $Z \sim N(0, \Sigma)$, then

$$\sup_{A \in \mathcal{A}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq Cd^{1/4} \sum_{i=1}^n \mathbb{E}|\Sigma^{-1/2}\xi_i|^3,$$  (1.1)
where $\mathcal{A}$ is the collection of all measurable convex sets in $\mathbb{R}^d$, $C$ is an absolute constant and $| \cdot |$ denotes the Euclidean norm when applied to a vector. Raič (2019a) obtained an explicit constant in the error bound (1.1). The error bound (1.1) is optimal up to the factor $d^{1/4}$ because, as shown by Nagaev (1976), the bound no longer holds if we replace $d^{1/4}$ by any vanishing quantity. For Euclidean balls, it is known that the factor $d^{1/4}$ can be removed if $\Sigma = I_d$, the $d \times d$ identity matrix. This was proved in Bentkus (2003) for the independent and identically distributed (i.i.d.) case. The general case follows from Raič (2019a, Theorem 1.3 and Example 1.2) and Sazonov (1972, Remark 2.1), for example.

Our first main result (cf. Theorem 2.1) is that up to a logarithmic factor,

$$
\sup_{A \in \mathcal{A}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq \log Cd^{1/4} \left( \sum_{i=1}^{n} \mathbb{E}|\Sigma^{-1/2} \xi_i|^4 \right)^{1/2}.
$$

(1.2)

The bound (1.2) is optimal up to the $d^{1/4}$ and the logarithmic factors (cf. Proposition 2.1). We will argue that (cf. Remark 2.1) under finite fourth-moment conditions, the bound (1.2) has near-optimal dependence on $n$. Moreover, perhaps surprisingly, it can achieve better dependence on dimension compared with (1.1). We note that applying the Cauchy-Schwarz inequality to (1.1) results in a bound as in (1.2), but with an additional factor of $d^{1/2}$. It is the removal of this factor that enables the improvement of the dependence on dimension.

We then consider the Gaussian approximation on the class $\mathcal{B}$ of all Euclidean balls, which is arguably most relevant for statistical applications, e.g., chi-square tests. We show that (cf. Theorem 3.1) the factor $d^{1/4}$ in (1.2) can be removed if we replace $\mathcal{A}$ with $\mathcal{B}$. Furthermore, we obtain an error bound (cf. Theorem 3.2) that typically vanishes as long as $d = o(n)$. Incidentally, the requirement $d = o(n)$ is necessary for the validity of the Gaussian approximation on balls (cf. Proposition 3.1).

We prove our main results using Stein’s method (Stein (1972) and its recent advances. To prove (1.2), we use a Gaussian anti-concentration inequality for convex sets by Ball (1993), the recursive argument of Raič (2019a), a multivariate exchangeable pair coupling (Chatterjee and Meckes (2008) and Reinert and Röllin (2009)) and a symmetry argument in Fang and Koike (2020a,b). To prove the results for balls, we further use a Gaussian anti-concentration inequality for ellipsoids by Giessing and Fan (2020).

The bound (1.1) and its variants have been widely used in the statistics literature, especially in inference for models with large parameter dimensions. See, for example, Spokoiny and Zhilova (2015), Pouzo (2015), Peng and Schick (2018), Shi et al. (2019) and Chen and Zhou (2020). Our new bounds’ improved dependence on dimension may prove useful if we are interested in allowing $d$ to grow as rapidly as possible depending on $n$, which is one of the most important subjects in such literature. We will also discuss an application to the bootstrap that is ubiquitous in this field (see Section 4).

The paper is organized as follows. In Section 2 and 3 we state our main results for the large-dimensional Gaussian approximation of sums of independent random vectors on convex sets and balls, respectively. In Section 4 we discuss an application to the bootstrap. Section 5 contains all of the proofs.

For a matrix $M$, we use $\|M\|_{H.S.}$ to denote its Hilbert-Schmidt norm. We use $C$ to denote positive absolute constants which may differ in different expressions. For a vector
In this section, we consider the Gaussian approximation of sums of independent random vectors on convex sets. Our main result is the following fourth-moment error bound in the approximation.

**Theorem 2.1.** Let \( \xi = \{\xi_i\}_{i=1}^n \) be a sequence of centered independent random vectors in \( \mathbb{R}^d \) with finite fourth moments and set \( W = \sum_{i=1}^n \xi_i \). Assume \( \text{Var}(W) = \Sigma \) and \( \Sigma \) is invertible. Let \( Z \sim \mathcal{N}(0, \Sigma) \) be a centered Gaussian vector in \( \mathbb{R}^d \) with covariance matrix \( \Sigma \). Then,

\[
\sup_{A \in \mathcal{A}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq Cd^{1/4} \left( \sum_{i=1}^n \mathbb{E}|\xi_i|^{-1/2} \right)^{1/2} \left( \log \left( \sum_{i=1}^n \mathbb{E}|\xi_i|^4 \right) \right)^{1/2} \left( \sum_{i=1}^n \mathbb{E}|\xi_i|^4 \right)^{1/2},
\]

where \( \mathcal{A} \) is the collection of all measurable convex sets in \( \mathbb{R}^d \).

The next result shows that the bound (2.1) is optimal up to the \( d^{1/4} \) and the logarithmic factors.

**Proposition 2.1.** There is an absolute constant \( C_0 > 0 \) such that, for sufficiently large \( n \), we can construct centered i.i.d. random vectors \( \xi_1, \ldots, \xi_n \) in \( \mathbb{R}^d \) with finite fourth moments (which may depend on \( n \)) satisfying \( \text{Var}(W) = I_d \) and

\[
\sup_{A \in \mathcal{A}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \geq C_0 \left( \sum_{i=1}^n \mathbb{E}|\xi_i|^4 \right)^{1/2}
\]

as long as \( d \leq \sqrt{n} / \log n \).

We use the next remark to discuss the crucial fact that our bound (2.1) may be preferable to the third-moment bound (1.1) in the large-dimensional setting.

**Remark 2.1.** To understand the typical order of the right-hand side of (2.1), we consider the situation where \( \xi_i = X_i / \sqrt{n} \) and \( \{X_1, X_2, \ldots\} \) is a sequence of i.i.d. mean-zero random vectors in \( \mathbb{R}^d \) with \( \text{Var}(X_i) = I_d \); hence \( \Sigma = I_d \). For the \( d \)-vector \( X_i \), \( \mathbb{E}|X_i|^3 \) and \( \mathbb{E}|X_i|^4 \) are typically proportional to \( d^{3/2} \) and \( d^2 \), respectively. In this case, the right-hand side of (2.1) is of the order \( O(\frac{d^{3/2}}{\sqrt{n}})^{1/2} \) up to a logarithmic factor. In contrast, the right-hand side of (1.1) is of the order \( O(\frac{d^{7/2}}{n})^{1/2} \). Therefore, subject to the requirement of the existence of the fourth moment, (2.1) is preferable to (1.1) in the large-dimensional setting where \( d \to \infty \). We mention in this context that Zhai (2018, Corollary 1.5) obtained a bound typically of the order \( O(\frac{d^{7/2}}{n})^{1/3} \) up to a logarithmic factor under a boundedness condition. He obtained the bound as a by-product of a Wasserstein-2 bound in the Gaussian approximation.
3 APPROXIMATION ON BALLS

In this section, we consider the Gaussian approximation of sums of independent random vectors on Euclidean balls. Our first result shows that the factor $d^{1/4}$ appearing on the right-hand side of (2.1) can be removed if we restrict the approximation to the class of balls. To facilitate the application to the bootstrap in Section 4, here we do not assume $W$ and $Z$ have the same covariance matrix.

**Theorem 3.1.** Let $\xi = \{\xi_i\}_{i=1}^n$ be a sequence of centered independent random vectors in $\mathbb{R}^d$ with finite fourth moments and set $W = \sum_{i=1}^n \xi_i$. Let $Z \sim N(0, \Sigma)$ be a centered Gaussian vector in $\mathbb{R}^d$ with covariance matrix $\Sigma$. Assume $\Sigma$ is invertible. Then

$$\sup_{A \in \mathcal{B}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq C \Psi \left( \delta(W, \Sigma) \right),$$

where $\Psi(x) = x(|\log x| \vee 1)$, $\mathcal{B}$ is the set of all Euclidean balls in $\mathbb{R}^d$ and

$$\delta(W, \Sigma) := \|I_d - \text{Var}(\Sigma^{-1/2}W)\|_{\text{H.S.}} + \left( \sum_{i=1}^n \mathbb{E}|\Sigma^{-1/2}\xi_i|^4 \right)^{1/2}.$$

Following Remark 2.1, we can see that if $\text{Var}(W) = \Sigma$, then the typical order of the right-hand side of (3.1) is $O(d^2 n^{1/2})$ up to a logarithmic factor. It has near-optimal dependence on $n$ and converges to 0 if $d = o(\sqrt{n})$. In the next result, we sacrifice the rate of $n$ to obtain the optimal growth rate of $d = o(n)$ in terms of the dimension (cf. Proposition 3.1 below).

**Theorem 3.2.** Let $\xi, W$ and $Z$ be as in Theorem 3.1. Assume $\text{tr}(\Sigma^2) > 0$. Then

$$\sup_{A \in \mathcal{B}} |\mathbb{P}(W \in A) - \mathbb{P}(Z \in A)| \leq \frac{C}{\text{tr}(\Sigma^2)^{1/4}} \sqrt{\delta(W, \Sigma)},$$

where

$$\delta(W, \Sigma) := \|\Sigma - \text{Var}(W)\|_{\text{H.S.}} + \sum_{j=1}^d |\Sigma_{jj} - \text{Var}(W_{jj})| + \sqrt{\sum_{i=1}^n \mathbb{E}|\xi_i|^4} + \sum_{j=1}^d \sum_{i=1}^n \mathbb{E}|\xi_{ij}|^4.$$

**Remark 3.1.** Since $\mathbb{E}|\xi_i|^4 \leq d \sum_{j=1}^d \mathbb{E}\xi_{ij}^4$, if $\text{Var}(W) = \Sigma = I_d$, the right-hand side of (3.2) is bounded by

$$C \max_{1 \leq j \leq d} \left( d \sum_{i=1}^n \mathbb{E}\xi_{ij}^4 \right)^{1/4}.$$

If $\max_{1 \leq i \leq n} \max_{1 \leq j \leq d} (\mathbb{E}\xi_{ij}^4)^{1/4} = O(1/\sqrt{n})$ as $n \to \infty$ as in the typical case in applications (where $\xi_{ij} = \frac{X_{ij}}{\sqrt{n}}$ for some $X_{ij}$ not depending on $n$), this converges to 0 as long as $d/n \to 0$. It is not difficult to prove this condition is generally necessary for convergence of the quantity on the left-hand side of (3.2):
Proposition 3.1. Let $X_1, \ldots, X_n$ be i.i.d. standard Gaussian vectors in $\mathbb{R}^d$. Let $\{e_i\}_{i=1}^n$ be i.i.d. variables independent of $\{X_i\}_{i=1}^n$ with $\mathbb{E}e_1 = 0$, $\mathbb{E}e_1^2 = 1$, $\mathbb{E}e_1^4 < \infty$ and $\text{Var}(e_1^2) > 0$. Assume the law of $e_1$ does not depend on $n$. Set $W := n^{-1/2} \sum_{i=1}^n e_i X_i$ and let $Z \sim N(0, \mathbb{I}_d)$. If
\[
\sup_{x \geq 0} |\mathbb{P}(|W| \leq x) - \mathbb{P}(|Z| \leq x)| \to 0
\]
as $d, n \to \infty$, we must have $d/n \to 0$.

Remark 3.2. $W$ in Proposition 3.1 can be regarded as a bootstrap approximation of $Z$ (cf. Section 4). Remark 3.1 and Proposition 3.1 suggest that, in general, bootstrapping may not provide a more accurate approximation than the Gaussian approximation in terms of the dependence on dimension.

Remark 3.3. Theorem 3.2 can be used to deduce Central Limit Theorems (CLTs) for $|W|^2$ under suitable conditions. For example, if $a = 0, \Sigma = I_d, \xi_i = X_i/\sqrt{n}$ for an i.i.d. sequence of random vectors $\{X_1, \ldots, X_n\}$ with $\max_{1 \leq j \leq d} \mathbb{E}(X_{ij}^4) \leq C$, then by Theorem 3.2 Remark 3.1 and the CLT for chi-square random variables, we have, for $d \to \infty$ and $d = o(n)$,
\[
\frac{|W|^2 - d}{\sqrt{2d}} \to N(0, 1) \quad \text{in distribution.}
\]
This recovers Corollary 3 of Peng and Schick (2018), who proved the result by regarding $|W|^2$ as a quadratic function of $\{\xi_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq d}$ and using the martingale CLT.

4 APPLICATION TO BOOTSTRAP APPROXIMATION ON BALLS

Let $X = \{X_i\}_{i=1}^n$ be a sequence of centered independent random vectors in $\mathbb{R}^d$ with finite fourth moments and consider the normalized sum $W := n^{-1/2} \sum_{i=1}^n X_i$. Theoretical results developed in the previous section allows us to approximate the probability $\mathbb{P}(W \in A)$ for $A \in \mathcal{B}$ by its Gaussian analog $\mathbb{P}(Z \in A)$, where $Z \sim N(0, \Sigma)$ and $\Sigma := \text{Var}(W)$ even when the dimension $d$ grows with the sample size $n$. Nevertheless, analytical evaluation of $\mathbb{P}(Z \in A)$ could be complicated for a general form of $\Sigma$ and thus we may still need an additional effort to resolve this issue for statistical application. This section develops bootstrap approximation for $\mathbb{P}(W \in A)$, one of the most popular methods to settle this sort of problem. Concrete applications are found in Spokoiny and Zhilova (2015), Pouzo (2015) and Chen and Zhou (2020), for example.

4.1 Empirical bootstrap

First we consider Efron’s empirical bootstrap introduced by Efron (1979). Let $X_1^*, \ldots, X_n^*$ be i.i.d. draws from the empirical distribution of $X$. That is, conditional on $X$, $X_1^*, \ldots, X_n^*$ are independent and each $X_i^*$ is uniformly distributed on $\{X_1, \ldots, X_n\}$. The bootstrap approximation of $W$ is then given by
\[
W^* := \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^* - \bar{X}), \quad \text{where} \quad \bar{X} := \frac{1}{n} \sum_{i=1}^n X_i.
\]
The following theorem provides a bootstrap analog of Theorem 3.2.
Theorem 4.1. If \( \text{tr}(\Sigma^2) > 0 \), we have for any \( K > 0 \),
\[
\mathbb{P} \left( \sup_{A \in B} |\mathbb{P}(W^* \in A|X) - \mathbb{P}(Z \in A)| > K \sqrt{\Delta_n} \right) \leq \frac{C}{K^2},
\] (4.1)
where
\[
\Delta_n := \frac{1}{n \text{tr}(\Sigma^2)^{1/2}} \left( \sqrt{\sum_{i=1}^{n} \mathbb{E}|X_i|^4} + \sum_{j=1}^{d} \sqrt{\sum_{i=1}^{n} \mathbb{E}X_{ij}^4} \right).
\]

Corollary 4.1. If \( \text{tr}(\Sigma^2) > 0 \), we have
\[
\sup_{A \in B} |\mathbb{P}(W \in A) - \mathbb{P}(W^* \in A|X)| = O_p(\sqrt{\Delta_n}) \quad (4.2)
\]
as \( n \to \infty \). Moreover, let \( \alpha \in (0, 1) \) and define
\[
q^*_n(\alpha) := \inf \{ x \in \mathbb{R} : \mathbb{P}(|W^*| > x|X) \leq \alpha \}.
\]
Then we have \( \mathbb{P}(|W| > q^*_n(\alpha)) \to \alpha \) as \( n \to \infty \), provided that \( \Delta_n \to 0 \).

Remark 4.1. If \( \Sigma \) is invertible, it is possible to derive a bootstrap version of Theorem 3.1 yielding a near optimal convergence rate with respect to the sample size \( n \).

Remark 4.2 (Relation to Zhilova (2020)). Theorem 4.1 of Zhilova (2020) gives a non-asymptotic bound for the quantity on the left-hand side of (4.2) under additional distributional assumptions on \( X_i \). While it exhibits better dependence on \( n \) than our result, ours generally provides better dependence on the dimension \( d (d = o(n) \text{ vs. } d = o(n^{1/2})) \), at least when \( \Sigma = I_d \); see Remark 3.1 above and Remark 4.1 of Zhilova (2020). Besides, our result allows \( \Sigma \) to be singular. Also, it is presumably possible to give a non-asymptotic version of our result similarly to (4.1) but an exponential concentration if we additionally assume \( X_i \) are sub-Gaussian as in Zhilova (2020) (see also Remark 4.5 of Zhilova (2020)).

4.2 Wild bootstrap

Next we consider the wild (or multiplier) bootstrap, which was originally suggested in Section 7 of Wu (1986) (see also Liu (1988)). Let \( e_1, \ldots, e_n \) be i.i.d. variables independent of \( X \) with \( \mathbb{E}e_1 = 0, \mathbb{E}e_1^2 = 1 \) and \( \mathbb{E}e_1^4 < \infty \). The wild bootstrap approximation of \( W \) with multiplier variables \( e_1, \ldots, e_n \) is given by
\[
W^\circ := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i X_i.
\]

In this setting, we can establish the following wild bootstrap version of Theorem 4.1.

Theorem 4.2. If \( \text{tr}(\Sigma^2) > 0 \), we have for any \( K > 0 \),
\[
\mathbb{P} \left( \sup_{A \in B} |\mathbb{P}(W^\circ \in A|X) - \mathbb{P}(Z \in A)| > K(\mathbb{E}e_1^4)^{1/4} \sqrt{\Delta_n} \right) \leq \frac{C}{K^2},
\]
where \( \Delta_n \) is defined as in Theorem 4.1.
Corollary 4.2. If $\text{tr}(\Sigma^2) > 0$ and the law of $e_1$ does not depend on $n$, we have
\[
\sup_{A \in \mathcal{B}} |\mathbb{P}(W \in A) - \mathbb{P}(W^o \in A|X)| = O_p(\sqrt{\Delta_n}) \tag{4.3}
\]
as $n \to \infty$. Moreover, let $\alpha \in (0,1)$ and define
\[
q_n^\alpha(\alpha) := \inf \{x \in \mathbb{R} : \mathbb{P}(|W| > x|X) \leq \alpha\}.
\]
Then we have $\mathbb{P}(|W| > q_n^\alpha(\alpha)) \to \alpha$ as $n \to \infty$, provided that $\Delta_n \to 0$.

Remark 4.3. Again, it is possible to derive a wild bootstrap version of Theorem 3.1 when $\Sigma$ is invertible.

Remark 4.4 (Relation to Zhilova (2020)). Theorem 4.3 of Zhilova (2020) establishes a non-asymptotic bound for the quantity on the left-hand side of (4.3) under additional distributional assumptions on $X_i$. Compared to this bound, our result shows better dependence both on $n$ and $d$ ($O(d/n)^{1/4}$ vs. $O(d^2/n)^{1/5}$). In particular, contrary to Remark 4.4 of Zhilova (2020), our result suggests that, in general, there would be no significant difference between the empirical and wild bootstrap approximations in terms of accuracy. It is also worth mentioning that our result does not require the unit skewness assumption $\mathbb{E}e_3^1 = 1$ on the multiplier variables. In addition, similarly to the empirical bootstrap case, one can presumably derive a non-asymptotic version of (4.3) as in Zhilova (2020).

5 PROOFS

We first introduce some notation used throughout the proofs. For two vectors $x, y \in \mathbb{R}^d$, $x \cdot y$ denotes their inner product. For two $d \times d$ matrices $M$ and $N$, we write $\langle M, N \rangle_{H.S.}$ for their Hilbert-Schmidt inner product.

For real-valued functions on $\mathbb{R}^d$ we will write $\partial_i f(x)$ for $\partial f(x)/\partial x_i$, $\partial_{ij} f(x)$ for $\partial^2 f(x)/\partial x_i \partial x_j$ and so forth. We write $\nabla f$ and Hess $f$ for the gradient and Hessian matrix of $f$, respectively. In addition, following Raič (2019a,b), we denote by $\nabla^r f(x)$ the $r$-th derivative of $f$ at $x$ regarded as an $r$-linear form: The value of $\nabla^r f(x)$ evaluated at $u_1, \ldots, u_r \in \mathbb{R}^d$ is given by
\[
\langle \nabla^r f(x), u_1 \otimes \cdots \otimes u_r \rangle = \sum_{j_1, \ldots, j_r=1}^d \partial_{j_1, \ldots, j_r} f(x) u_{1,j_1} \cdots u_{r,j_r}.
\]
When $u_1 = \cdots = u_r =: u$, we write $u_1 \otimes \cdots \otimes u_r = u^\otimes r$ for short.

For any $r$-linear form $T$, its injective norm is defined by
\[
|T|_\vee := \sup_{|u_1|_\vee = \cdots = |u_r|_\vee = 1} |\langle T, u_1 \otimes \cdots \otimes u_r \rangle|.
\]

For an $(r-1)$-times differentiable function $h : \mathbb{R}^d \to \mathbb{R}$, we write
\[
M_r(h) := \sup_{x \neq y} \frac{|\nabla^{r-1} h(x) - \nabla^{r-1} h(y)|_\vee}{|x - y|}.
\]
Note that $M_r(h) = \sup_{x \in \mathbb{R}^d} |\nabla^r h(x)|$ if $h$ is $r$-times differentiable. We refer to the beginning of Raić (2019a, Section 2) and Raić (2019b, Section 5) for more details about these notation.

Finally, we refer to the following bound for derivatives of the $d$-dimensional standard normal density $\phi$, which will be used several times in the following (cf. the inequality after Eq. (4.9) of Raić (2019b)):

$$\int_{\mathbb{R}^d} |\langle \nabla^s \phi(z), u^{\otimes s}\rangle|dz \leq C_s |u|^s$$

for any fixed integer $s$, where $C_s$ is a constant depending only on $s$.

### 5.1 Basic decomposition

The proofs for Theorems 2.1 and 3.1–3.2 start with approximating the indicator function $1_A$ for $A \in \mathcal{A}$ or $A \in \mathcal{B}$ by an appropriate smooth function $h$. Then, the problem amounts to establishing an appropriate bound for $\mathbb{E} h(W) - \mathbb{E} h(Z)$. To accomplish this, we will make use of a decomposition of $\mathbb{E} h(W) - \mathbb{E} h(Z)$ derived from the exchangeable pair approach in Stein’s method for multivariate normal approximation by Chatterjee and Meckes (2008) and Reinert and Röllin (2009) along with a symmetry argument by Fang and Koike (2020a,b) (cf. (5.11)–(5.12) below).

Given a twice differentiable function $h : \mathbb{R}^d \to \mathbb{R}$ with bounded partial derivatives, we consider the Stein equation

$$\langle \text{Hess } f(w), \Sigma \rangle_{H.S.} - w \cdot \nabla f(w) = h(w) - \mathbb{E} h(Z), \quad w \in \mathbb{R}^d.$$  

(5.2)

It can be verified directly that

$$f(w) = \int_0^1 \frac{1}{2(1-s)} \int_{\mathbb{R}^d} [h(\sqrt{1-sw} + \sqrt{s} \Sigma^{1/2}z) - \mathbb{E} h(Z)] \phi(z)dzds$$

(5.3)

is a solution to (5.2) (cf. Goetze (1991) and Meckes (2009)). In the following we assume that $f$ is thrice differentiable with bounded partial derivatives. This is true if $\Sigma$ is invertible or $h$ is thrice differentiable with bounded partial derivatives.

Let $\{\xi_1, \ldots, \xi_n\}$ be an independent copy of $\{\xi_1, \ldots, \xi_n\}$, and let $I$ be a random index uniformly chosen from $\{1, \ldots, n\}$ and independent of $\{\xi_1, \ldots, \xi_n, \xi_1', \ldots, \xi_n'\}$. Define $W' = W - \xi_I + \xi_I'$. It is easy to verify that $(W, W')$ has the same distribution as $(W', W)$ (exchangeability) and

$$\mathbb{E}(W' - W|W) = -\frac{W}{n}.$$  

(5.4)

From exchangeability and (5.4), we have, with $D = W' - W$,

$$0 = \frac{n}{2} \mathbb{E}[D \cdot (\nabla f(W') + \nabla f(W))]$$

$$= \mathbb{E} \left[ \frac{n}{2} D \cdot (\nabla f(W') - \nabla f(W)) + nD \cdot \nabla f(W) \right]$$

$$= \mathbb{E} \left[ \frac{n}{2} \sum_{j,k=1}^d D_j D_k \partial_{jk} f(W) + R_2 + nD \cdot \nabla f(W) \right]$$

(5.5)
\[ \mathbb{E} \left[ \langle \text{Hess} f(W), \Sigma \rangle_{H.S.} - R_1 + R_2 - W \cdot \nabla f(W) \right], \]

where

\[ R_1 = \sum_{j,k=1}^{d} \mathbb{E} \{ (\Sigma_{jk} - \frac{n}{2} D_j D_k) \partial_{jk} f(W) \}, \tag{5.6} \]

\[ R_2 = \frac{n}{2} \sum_{j,k,l=1}^{d} \mathbb{E} D_j D_k D_l U \partial_{jkl} f(W + (1 - U)D) \tag{5.7} \]

and \( U \) is a uniform random variable on \([0,1] \) independent of everything else. From (5.2) and (5.5), we have

\[ \mathbb{E} h(W) - \mathbb{E} h(Z) = R_1 - R_2. \tag{5.8} \]

We further rewrite \( R_1 \) and \( R_2 \) respectively as follows. First, set

\[ V = (V_{jk})_{1 \leq j,k \leq d} := \left( \mathbb{E} \left[ \Sigma_{jk} - \frac{n}{2} D_j D_k |\xi| \right] \right)_{1 \leq j,k \leq d}. \]

Then we evidently have

\[ R_1 = \sum_{j,k=1}^{d} \mathbb{E} V_{jk} \partial_{jk} f(W) = \mathbb{E} \langle V, \text{Hess} f(W) \rangle_{H.S.}. \tag{5.9} \]

Also, one can easily verify that (cf. Eq.(22) of Chernozhukov, Chetverikov and Kato (2014))

\[ V = \Sigma - \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} [\xi_i \xi_i^\top] - \frac{1}{2} \sum_{i=1}^{n} \xi_i \xi_i^\top = (\Sigma - \text{Var}(W)) - \frac{1}{2} \sum_{i=1}^{n} (\xi_i \xi_i^\top - \mathbb{E} [\xi_i \xi_i^\top]). \tag{5.10} \]

Next, by exchangeability we have

\[ \mathbb{E} [D_j D_k D_l U \partial_{jkl} f(W + (1 - U)D)] \]
\[ = -\mathbb{E} [D_j D_k D_l U \partial_{jkl} f(W' - (1 - U)D)] \tag{5.11} \]
\[ = -\mathbb{E} [D_j D_k D_l U \partial_{jkl} f(W + UD)]. \]

Hence we obtain

\[ R_2 = \frac{n}{4} \sum_{j,k,l=1}^{d} \mathbb{E} [D_j D_k D_l U \{ \partial_{jkl} f(W + (1 - U)D) - \partial_{jkl} f(W + UD) \}]. \tag{5.12} \]

**5.2 Proof of Theorem 2.1**

Since \( \Sigma^{-1/2} W = \sum_{i=1}^{n} \Sigma^{-1/2} \xi_i \) and \( \{ \Sigma^{-1/2} x : x \in A \} \in \mathcal{A} \) for all \( A \in \mathcal{A} \), it suffices to consider the case \( \Sigma = I_d \). The proof is a combination of Bentkus (2003)'s smoothing, the decomposition (5.8), and a recursive argument by Raid (2019b).
Fix $\beta_0 > 0$. Define

$$K(\beta_0) = \sup_W \max\left\{ \beta_0, \left( \sum_{i \in I} \mathbb{E} \left| \xi_i \right|^4 \right)^{1/2} \left( \left| \log \left( \sum_{i \in I} \mathbb{E} \left| \xi_i \right|^4 \right) \right| \lor 1 \right) \right\},$$

(5.13)

where the first supremum is taken over the family of all sums $W = \sum_{i \in I} \xi_i$ of finite number of independent mean-zero random vectors with $\mathbb{E} \left| \xi_i \right|^4 < \infty$ and $\text{Var}(W) = I_d$. We will obtain a recursive inequality for $K(\beta_0)$ and prove that

$$K(\beta_0) \leq C d^{1/4}$$

(5.14)

for an absolute constant $C$ that does not depend on $\beta_0$. Eq. (2.1) then follows by sending $\beta_0 \to 0$.

Now we fix a $W = \sum_{i=1}^n \xi_i$, $n \geq 1$, in the aforementioned family (will take sup in (5.3)). Let

$$\bar{\beta} = \max\left\{ \beta_0, \left( \sum_{i=1}^n \mathbb{E} \left| \xi_i \right|^4 \right)^{1/2} \left( \left| \log \left( \sum_{i=1}^n \mathbb{E} \left| \xi_i \right|^4 \right) \right| \lor 1 \right) \right\}.$$  

(5.15)

Next, recall that $A$ is the collection of all convex sets in $\mathbb{R}^d$. For $A \in \mathcal{A}$, $\varepsilon > 0$, define

$$A^\varepsilon = \{ x \in \mathbb{R}^d : \text{dist}(x, A) \leq \varepsilon \},$$

where $\text{dist}(x, A) = \inf_{y \in A} |x - y|$.

**Lemma 5.1** (Lemma 2.3 of Bentkus (2003)). For any $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists a function $h_{A,\varepsilon}$ (which depends only on $A$ and $\varepsilon$) such that

$$h_{A,\varepsilon}(x) = 1 \text{ for } x \in A, \quad h_{A,\varepsilon}(x) = 0 \text{ for } x \in \mathbb{R}^d \setminus A^\varepsilon, \quad 0 \leq h_{A,\varepsilon}(x) \leq 1,$$

and

$$M_1(h_{A,\varepsilon}) \leq \frac{C}{\varepsilon}, \quad M_2(h_{A,\varepsilon}) \leq \frac{C}{\varepsilon^2},$$

(5.16)

where $C$ is an absolute constant that does not depend on $A$ and $\varepsilon$.

**Lemma 5.2** (Theorem 4 of Ball (1993)). Let $\phi$ be the standard Gaussian density on $\mathbb{R}^d$, $d \geq 2$, and let $A$ be a convex set in $\mathbb{R}^d$. Then

$$\int_{\partial A} \phi \leq 4d^{1/4}.$$  

(5.17)

From the Gaussian anti-concentration inequality (5.17), it is not difficult to obtain the following smoothing lemma.

**Lemma 5.3** (Lemma 4.2 of Fang and R"ollin (2015)). For any $d$-dimensional random vector $W$ and any $\varepsilon > 0$,

$$\sup_{A \in \mathcal{A}} |P(W \in A) - P(Z \in A)| \leq 4d^{1/4} \varepsilon + \sup_{A \in \mathcal{A}} |\mathbb{E} h_{A,\varepsilon}(W) - \mathbb{E} h_{A,\varepsilon}(Z)|,$$

(5.18)

where $h_{A,\varepsilon}$ is as in Lemma 5.1.
We now fix $A \in \mathcal{A}$ (will take sup in (5.32)), $0 < \varepsilon \leq 1$, write $h := h_{A, \varepsilon}$ and proceed to bound $|\mathbb{E} h(W) - \mathbb{E} h(Z)|$ by the decomposition (5.8). Consider the solution $f$ to the Stein equation (5.2) with $\Sigma = I_d$, which is given by (5.3). Since $h$ has bounded partial derivatives up to the second order and $\Sigma = I_d$ is invertible, $f$ is thrice differentiable with bounded partial derivatives. Using the integration by parts formula, we have for $1 \leq j, k, l \leq d$ and any constant $0 \leq c_0 \leq 1$ that

$$\partial_{jk} f(w) = \int_{0}^{c_0} \frac{1}{2\sqrt{s}} \int_{\mathbb{R}^d} \partial_j h(\sqrt{1 - sw} + \sqrt{s}z) \partial_k \phi(z) dz ds$$

$$+ \int_{c_0}^{1} \frac{1}{2s} \int_{\mathbb{R}^d} h(\sqrt{1 - sw} + \sqrt{s}z) \partial_{jk} \phi(z) dz ds$$

(5.19)

and

$$\partial_{jkl} f(w) = \int_{0}^{c_0} \frac{\sqrt{1 - s}}{2\sqrt{s}} \int_{\mathbb{R}^d} \partial_{jk} h(\sqrt{1 - sw} + \sqrt{s}z) \partial_l \phi(z) dz ds$$

$$+ \int_{c_0}^{1} \frac{\sqrt{1 - s}}{2s} \int_{\mathbb{R}^d} \partial_j h(\sqrt{1 - sw} + \sqrt{s}z) \partial_{kl} \phi(z) dz ds.$$  

(5.20)

We first bound $R_1$ in (5.9). We will utilize the following lemma.

**Lemma 5.4** (Lemma 4.3 of Fang and Röllin (2015)). For $k \geq 1$ and each map $a : \{1, \ldots, d\}^k \to \mathbb{R}$, we have

$$\int_{\mathbb{R}^d} \left( \sum_{i_1, \ldots, i_k=1}^{d} a(i_1, \ldots, i_k) \frac{\partial_{i_1 \ldots i_k} \phi(z)}{\phi(z)} \right)^2 \phi(z) dz \leq k! \sum_{i_1, \ldots, i_k=1}^{d} (a(i_1, \ldots, i_k))^2. \quad (5.21)$$

Now, using the expression of $\partial_{jk} f$ in (5.19) with $c_0 = \varepsilon^2$, we have

$$R_1 = R_{11} + R_{12},$$

where

$$R_{11} = \sum_{j, k=1}^{d} \mathbb{E} \left[ V_{jk} \int_{0}^{\varepsilon^2} \frac{1}{2\sqrt{s}} \int_{\mathbb{R}^d} \partial_j h(\sqrt{1 - sw} + \sqrt{s}z) \partial_k \phi(z) dz ds \right]$$

and

$$R_{12} = \sum_{j, k=1}^{d} \mathbb{E} \left[ V_{jk} \int_{\varepsilon^2}^{1} \frac{1}{2s} \int_{\mathbb{R}^d} h(\sqrt{1 - sw} + \sqrt{s}z) \partial_{jk} \phi(z) dz ds \right].$$

For $R_{11}$, we use the Cauchy-Schwarz inequality and the bounds (5.16) and (5.21), and
Applying similar arguments, we have, for $R$

\[ |R| = \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E \left( \sum_{j=1}^d \partial_j h(\sqrt{1 - sW + \sqrt{s}z}) \sum_{k=1}^d V_{jk} \frac{\partial_k \phi(z)}{\phi(z)} \phi(z) dz ds \right) \]

\[ \leq C \varepsilon \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E \left( \sum_{j=1}^d \left( \sum_{k=1}^d V_{jk} \frac{\partial_k \phi(z)}{\phi(z)} \right)^2 \phi(z) dz ds \right)^{1/2} \]

\[ \leq C \varepsilon \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E \left( \sum_{j=1}^d \left( \sum_{k=1}^d V_{jk} \frac{\partial_k \phi(z)}{\phi(z)} \right)^2 \phi(z) dz \right) ds \]

\[ \leq C \varepsilon \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E \left( \sum_{j=1}^d \sum_{k=1}^d V_{jk}^2 \right)^{1/2} ds \leq C \left( \sum_{j,k=1}^d EV_{jk}^2 \right)^{1/2}. \]  

(5.22)

Since for $V$ in (5.10) with $\text{Var}(W) = \Sigma$,

\[ \text{EV}_{jk}^2 = \frac{1}{4} E \text{Var} \left[ \sum_{i=1}^n \xi_{ij} \xi_{ik} \right] = \frac{1}{4} \sum_{i=1}^n \text{Var} \left[ \xi_{ij} \xi_{ik} \right] \leq \frac{1}{4} \sum_{i=1}^n E \left[ \xi_{ij}^2 \xi_{ik}^2 \right], \]

we obtain

\[ |R| \leq C \left( \sum_{j,k=1}^d \sum_{i=1}^n E \left[ \xi_{ij}^2 \xi_{ik}^2 \right] \right)^{1/2} = C \left( \sum_{i=1}^n E \left[ \sum_{j=1}^d \xi_{ij}^2 \right]^2 \right)^{1/2} = C \left( \sum_{i=1}^n E |\xi_i|^4 \right)^{1/2}. \]

Applying similar arguments, we have, for $R_{12}$,

\[ |R_{12}| = \left| \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E h(\sqrt{1 - sW + \sqrt{s}z}) \sum_{j,k=1}^d V_{jk} \frac{\partial_{jk} \phi(z)}{\phi(z)} \phi(z) dz ds \right| \]

\[ \leq \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E \left( \sum_{j,k=1}^d V_{jk} \frac{\partial_{jk} \phi(z)}{\phi(z)} \right) \phi(z) dz ds \]

\[ \leq \int_0^{\varepsilon^2} \frac{1}{2} \int_{\mathbb{R}^d} E \left( \sum_{j,k=1}^d V_{jk} \frac{\partial_{jk} \phi(z)}{\phi(z)} \right)^2 \phi(z) dz ds \]

\[ \leq C \left( \sum_{j,k=1}^d \sum_{i=1}^d V_{jk}^2 \right)^{1/2} \leq C |\varepsilon| \left( \sum_{i=1}^n E |\xi_i|^4 \right)^{1/2}. \]  

(5.23)

Therefore,

\[ |R_1| \leq C(|\log \varepsilon| + 1) \left( \sum_{i=1}^n E |\xi_i|^4 \right)^{1/2}. \]  

(5.24)
Next, we bound $R_2$. Take $0 < \eta \leq 1$ arbitrarily. Using the expression of $\partial_{jkl}f$ in (5.20) with $c_0 = \eta^2$ and the two equivalent expressions (5.7) and (5.12) for $R_2$, we have

$$R_2 = R_{21} + R_{22},$$

where

$$R_{21} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j,k,l=1}^{d} \mathbb{E}U(\xi_{ij} - \xi_{ij})(\xi_{ik} - \xi_{ik})(\xi_{il} - \xi_{il}) \int_{\eta^2}^{1} \frac{\sqrt{1-s}}{2\sqrt{s}} d s \times \int_{\mathbb{R}^d} \partial_{jkl}h(\sqrt{1-s}(W + (1-U)(\xi'_i - \xi_i)) + \sqrt{s}z) \partial_k \phi(z) d z d s$$

and

$$R_{22} = \frac{1}{4} \sum_{i=1}^{n} \sum_{j,k,l,m=1}^{d} \mathbb{E}U(1 - 2U)(\xi_{ij} - \xi_{ij})(\xi_{ik} - \xi_{ik})(\xi_{il} - \xi_{il})(\xi_{im} - \xi_{im}) \int_{\eta^2}^{1} \frac{1-s}{2s} d s \times \int_{\mathbb{R}^d} \partial_{jlm}h(\sqrt{1-s}(W + (U + (1-2U)U')(\xi'_i - \xi_i)) + \sqrt{s}z) \partial_{kl} \phi(z) d z d s,$$

(5.25)

where $U'$ is a uniform random variable on $[0, 1]$ independent of everything else and we used the mean value theorem in the last equality. Let $W^{(i)} = W - \xi_i$ for $i \in \{1, \ldots, n\}$. We will use the fact that $\nabla h$ is non-zero only in $A^c \setminus A$ and bound

$$\mathbb{P}(\sqrt{1-s}W^{(i)} \in A^c \setminus A | U, U', \xi_i, \xi'_i),$$

where $0 < s < 1$ and $A_i$ is a convex set which may depend on $U, U', \xi_i, \xi'_i, s$ and $z$. Let $\Sigma_i$ be the covariance matrix of $W^{(i)}$ and let $\sigma_i$ be its smallest eigenvalue, which will be assumed to be positive in Case 1 below. We have

$$\mathbb{P}(\sqrt{1-s}W^{(i)} \in A^c \setminus A | U, U', \xi_i, \xi'_i) = \mathbb{P}(\Sigma_i^{-1/2}W^{(i)} \in \frac{1}{\sqrt{1-s}} \Sigma_i^{-1/2}(A^c \setminus A) | U, U', \xi_i, \xi'_i) \leq 4d^{1/4} \frac{\varepsilon}{\sigma_i \sqrt{1-s}} + 2 \sup_{A \in A} | \mathbb{P}(\Sigma_i^{-1/2}W^{(i)} \in A) - \mathbb{P}(Z \in A) |,$$

(5.26)

where we used the $4d^{1/4}$ upper bound for the Gaussian surface area of any convex set in Lemma 5.2. From (5.13), we have

$$\sup_{A \in A} | \mathbb{P}(\Sigma_i^{-1/2}W^{(i)} \in A) - \mathbb{P}(Z \in A) | \leq K(\beta_0) \max \left \{ \beta_0 \left( \sum_{j=1}^{n} \mathbb{E}|\Sigma_i^{-1/2} \xi_j|^4 \right)^{1/2} \left( \log \left( \sum_{j=1}^{n} \mathbb{E}|\Sigma_i^{-1/2} \xi_j|^4 \right) \vee 1 \right) \right \},$$

(5.27)
Let \( \beta_s = 0.19, \quad \sigma_s = (1 - \beta_s)^{1/2} = 0.9. \)

We first consider **Case 1**: \( \bar{\beta} \leq \beta_s/d^{1/4} \) (cf. (5.15)). In this case, because \( \mathbb{E}|\xi_i|^2 \leq \sqrt{\mathbb{E}|\xi_i|^4} \leq \bar{\beta} \leq \beta_s \) and for each unit vector \( u \in \mathbb{R}^d, \)

\[
\langle \Sigma_i u, u \rangle = u^\top \Sigma_i u = u^\top (I_d - \mathbb{E}\xi_i \xi_i^\top) u = 1 - \mathbb{E}(\xi_i \cdot u)^2 \geq 1 - \mathbb{E}|\xi_i|^2 \geq 1 - \beta_s, \]

we have \( \sigma_i \geq \sigma_s. \) Note that \( x(|\log x|/1) \) is an increasing function. Therefore, from (5.27), we have, by increasing \( \sum_{i=1}^n \mathbb{E}\Sigma_i^{-1/2} |\xi_j|^4 \) to \( \frac{1}{\sigma^2} \sum_{j=1}^n \mathbb{E}|\xi_j|^4, \)

\[
\sup_{A \in \mathcal{A}} |\mathbb{P}(\Sigma_i^{-1/2} W(i) \in A) - P(Z \in A)| \leq \beta_0 \max \left\{ \beta_0, \frac{2\beta}{\sigma^2} \right\} \leq CK(\beta_0)\bar{\beta}. \quad (5.28) \]

Applying (5.16), (5.26), (5.28), and (5.1), we have

\[
|R_{21}| \leq C \sum_{i=1}^n \mathbb{E}|\xi_i|^3 \left( d^{1/4} \varepsilon + K(\beta_0)\bar{\beta} \right) \eta \quad (5.29) \]

and

\[
|R_{22}| \leq C \sum_{i=1}^n \mathbb{E}|\xi_i|^4 \left( d^{1/4} \varepsilon + K(\beta_0)\bar{\beta} \right) |\log \eta|. \quad (5.30) \]

Now, if \( \sum_{i=1}^n \mathbb{E}|\xi_i|^4 < \sum_{i=1}^n \mathbb{E}|\xi_i|^3, \) choose \( \eta = \sum_{i=1}^n \mathbb{E}|\xi_i|^4 / \sum_{i=1}^n \mathbb{E}|\xi_i|^3 < 1. \) Note that we have by the Cauchy-Schwarz inequality

\[
\sum_{i=1}^n \mathbb{E}|\xi_i|^3 \leq \sqrt{\sum_{i=1}^n \mathbb{E}|\xi_i|^2 \sum_{i=1}^n \mathbb{E}|\xi_i|^4} = \sqrt{d \sum_{i=1}^n \mathbb{E}|\xi_i|^4}. \]

Thus we obtain

\[
|\log \eta| \leq \frac{1}{2} \log d - \frac{1}{2} \log \left( \sum_{i=1}^n \mathbb{E}|\xi_i|^4 \right). \]

Since \( (\sum_{i=1}^n \mathbb{E}|\xi_i|^4)^{1/2} \leq \bar{\beta} \) and in the case under consideration, \( \bar{\beta} \leq \beta_s/d^{1/4}, \) we have

\[
|\log \eta| \leq C |\log (\sum_{i=1}^n \mathbb{E}|\xi_i|^4)|. \]

Therefore, (5.29)–(5.30) yield

\[
|R_{21}| + |R_{22}| \leq C \sum_{i=1}^n \mathbb{E}|\xi_i|^4 \left( d^{1/4} \varepsilon + K(\beta_0)\bar{\beta} \right) \left( |\log (\sum_{i=1}^n \mathbb{E}|\xi_i|^4)| \lor 1 \right) \quad (5.31) \]
This inequality also holds true if $\sum_{i=1}^{n} E|\xi_i|^4 \geq \sum_{i=1}^{n} E|\xi_i|^3$ by taking $\eta = 1$ in (5.29)–(5.30). From (5.18), (5.8), (5.24), (5.31), we have

$$\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq 4d^{1/4} \varepsilon + C(|\log | \bigvee 1) \left( \sum_{i=1}^{n} E|\xi_i|^4 \right)^{1/2}$$

(5.32)

Choose $\varepsilon = \min\{ 2C \sum_{i=1}^{n} E|\xi_i|^4 (|\log (\sum_{i=1}^{n} E|\xi_i|^4) \bigvee 1) \}^{1/2}, 1 \}$ with the same absolute constant $C$ as in the third term on the right-hand side of (5.32). If $\varepsilon < 1$, then from (5.32),

$$\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq \left( Cd^{1/4} + \frac{K(\beta_0)}{2} \right) \beta;$$

hence

$$\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq Cd^{1/4} + \frac{K(\beta_0)}{2}.$$  (5.33)

If $\varepsilon = 1$, then $\sum_{i=1}^{n} E|\xi_i|^4$ and $\beta$ are bounded away from 0 by an absolute constant; hence

$$\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq \frac{1}{\beta} \leq C.$$  (5.34)

We now consider **Case 2**: $\beta > \beta_*/d^{1/4}$. We trivially estimate

$$\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq \frac{1}{\beta} \leq \frac{d^{1/4}}{\beta_*} \leq Cd^{1/4}.$$  (5.35)

Combining (5.33), (5.34) and (5.35), we obtain

$$\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq Cd^{1/4} + \frac{K(\beta_0)}{2}.$$  (5.36)

Note that the right-hand side of the above bound does not depend on $W$. Taking supremum over $W$, we obtain

$$K(\beta_0) \leq Cd^{1/4} + \frac{K(\beta_0)}{2}.$$  (5.37)

This implies (5.14), hence (2.7).

### 5.3 Proof of Proposition 2.1

It is not difficult to see that Nagaev (1976)'s example indeed satisfies the conditions stated in the proposition. We briefly summarize the construction for the sake of completeness.

First, given an integer $n \geq 3$, let $\{\eta_i\}_{i=1}^{n}$ be i.i.d. variables such that

$$\mathbb{P}(\eta_1 < y) = \Phi \left( \frac{y + a_n}{\sigma_n} \right) (1 - p_n) + p_n 1_{(x_n, \infty)}(y), \quad y \in \mathbb{R},$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution, $\sigma_n = \sqrt{n}$, and $a_n = \log \sigma_n - \log \log \sigma_n$. The constants $p_n$ and $x_n$ are chosen such that $p_n \to 1$ and $x_n \to \infty$ as $n \to \infty$. The parameter $\beta_*$ is defined similarly to $\beta$.

This completes the proof of Proposition 2.1.
where $\Phi$ is the standard normal distribution function and $x_n, p_n, a_n, \sigma_n$ are positive constants satisfying the following conditions:

\[ x_n = \frac{\sqrt{n}}{\log n}, \quad p_n x_n = a_n (1 - p_n), \quad p_n x_n^2 = \frac{1}{2}, \quad \frac{1}{2} + (\sigma_n^2 + a_n^2)(1 - p_n) = 1. \]

By construction, we have

\[ \mathbb{E} \eta_1 = 0, \quad \mathbb{E} \eta_1^2 = 1, \quad \mathbb{E} \eta_1^3 = x_n - (3a_n \sigma_n^2 + a_n^3)(1 - p_n) \]

and

\[ \frac{x_n^2}{2} \leq \mathbb{E} \eta_1^4 \leq \frac{x_n^2}{2} + 3(a_n + \sigma_n)^4. \]

Moreover, Nagaev (1976) has shown that, for sufficiently large $n$,

\[ \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_i < 0 \right) - \frac{1}{2} > \frac{\mathbb{E} \eta_1^3}{7 \sqrt{2} n}. \]

Since we have

\[ \frac{x_n}{\sqrt{2}} \leq \sqrt{\mathbb{E} \eta_1^4} \leq \frac{x_n}{\sqrt{2}} + \sqrt{3(a_n + \sigma_n)^2} \leq \frac{\mathbb{E} \eta_1^3}{\sqrt{2}} + \frac{3a_n \sigma_n^2 + a_n^3}{\sqrt{2}} + \sqrt{3(a_n + \sigma_n)^2}, \]

and $\sigma_n^2 + a_n^2 = \frac{1}{2(1 - p_n)} \leq 1$ for sufficiently large $n$, we conclude

\[ \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_i < 0 \right) - \frac{1}{2} > \frac{\sqrt{\mathbb{E} \eta_1^3}}{8 \sqrt{n}} \]

for sufficiently large $n$.

Now let $\zeta_{i,j} (i, j = 1, 2, \ldots)$ be independent standard normal variables independent of $\{\xi_i\}_{i=1}^n$. Then we define the independent random vectors $\{\xi_i\}_{i=1}^n$ in $\mathbb{R}^d$ by

\[ \xi_i := \frac{1}{\sqrt{n}} (\eta_i, \zeta_{i,1}, \ldots, \zeta_{i,d-1})^\top, \quad i = 1, \ldots, n. \]

We have

\[ \sum_{i=1}^n \mathbb{E} |\xi_i|^4 = n \mathbb{E} |\xi_1|^4 \leq \frac{1}{n} \left\{ 2\mathbb{E} \eta_1^4 + 2\mathbb{E} \left( \sum_{j=1}^{d-1} \zeta_{i,j}^2 \right)^2 \right\} \leq \frac{2\mathbb{E} \eta_1^4 + 6d^2}{n}. \]

Therefore, if $d \leq \sqrt{n} / \log n = x_n$, we obtain

\[ \sum_{i=1}^n \mathbb{E} |\xi_i|^4 \leq \frac{14 \mathbb{E} \eta_1^4}{n}. \]

Thus, we conclude

\[ \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_i < 0 \right) - \frac{1}{2} > C_0 \sqrt{\sum_{i=1}^n \mathbb{E} |\xi_i|^4}, \]

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where \( C_0 := (8\sqrt{14\pi})^{-1} \). Hence, for \( A = \{ x \in \mathbb{R}^d : x_1 = 0 \} \), we have

\[
| \mathbb{P}(W \in A) - \mathbb{P}(Z \in A) | > C_0 \sqrt{\sum_{i=1}^n \mathbb{E} |\xi_i|^4}.
\]

This completes the proof.

### 5.4 Proof of Theorem 3.1

We first note that, for any \( d \times d \) orthogonal matrix \( U \), we have

\[
UW = \sum_{i=1}^n U \xi_i, \quad UZ \sim N(0, U \Sigma U^\top), \quad \delta(UW, U \Sigma U^\top) = \delta(W, \Sigma) \text{ and } UB \in \mathcal{B} \text{ for all } B \in \mathcal{B}.
\]

Therefore, it is enough to prove (3.1) when \( \Sigma \) is diagonal with positive entries. The proof is a combination of Zhilova (2020)’s smoothing, a Gaussian anti-concentration inequality for ellipsoids by Giessing and Fan (2020), the decomposition (5.8), and a recursive argument by Räic (2019a).

Fix \( \beta_0 > 0 \). Define

\[
K'(\beta_0) = \sup_{W, \Sigma} \frac{\mathbb{P}(W \in A) - \mathbb{P}(\Sigma^{1/2} Z_0 \in A)}{\max \{ \beta_0, \Psi (\delta(W, \Sigma)) \}},
\]

where \( Z_0 \sim N(0, I_d) \) and the first supremum is taken over the family of all sums \( W = \sum_{i \in I} \xi_i \) of finite number of independent centered random vectors with \( \mathbb{E} |\xi_i|^4 < \infty \), and diagonal matrices \( \Sigma \) with positive entries. We will obtain a recursive inequality for \( K'(\beta_0) \) and prove that

\[
K'(\beta_0) \leq C
\]

for an absolute constant \( C \) that does not depend on \( \beta_0 \). Eq.(3.1) then follows by sending \( \beta_0 \to 0 \).

Now we fix a \( W = \sum_{i=1}^n \xi_i, \ n \geq 1, \) and \( \Sigma \) in the aforementioned family (will take sup in (5.53)). Let

\[
\tilde{\beta} = \max \{ \beta_0, \Psi (\delta(W, \Sigma)) \}.
\]

We write \( \sigma_j \) for the \( j \)-th diagonal entry of \( \Sigma^{1/2} \).

**Lemma 5.5.** For any \( A \in \mathcal{B} \) and \( \varepsilon > 0 \), there exists a \( C^\infty \) function \( \tilde{h}_{A, \varepsilon} \) (which depends only on \( A \) and \( \varepsilon \)) such that

\[
\tilde{h}_{A, \varepsilon}(x) = 1 \text{ for } x \in A, \quad \tilde{h}_{A, \varepsilon}(x) = 0 \text{ for } x \in \mathbb{R}^d \setminus A, \quad 0 \leq \tilde{h}_{A, \varepsilon}(x) \leq 1,
\]

and

\[
M_r(\tilde{h}_{A, \varepsilon}) \leq \frac{C}{\varepsilon^r} \quad \text{for } r = 1, 2, 3, 4,
\]

and

\[
\sup_{x \in \mathbb{R}^d} \left| (M, \text{Hess } \tilde{h}_{A, \varepsilon}(x))_{\text{H.S.}} \right| \leq \frac{C}{\varepsilon^2} \left( \| M \|_{\text{H.S.}} + \sum_{j=1}^d |M_{jj}| \right)
\]

for any \( d \times d \) matrix \( M = (M_{jk})_{1 \leq j, k \leq d} \), where \( C \) is an absolute constant that does not depend on \( A, \varepsilon \) or \( M \).
**Proof.** We follow the construction of [Zhilova (2020, Lemma A.3)](https://www.example.com). Take a $C^\infty$ function $g : \mathbb{R} \to [0, 1]$ satisfying $g(x) = 1$ for $x \leq 0$ and $g(x) = 0$ for $x \geq 1$. Let the ball $A$ have center $a$ and radius $r$, i.e. $A = \{ x \in \mathbb{R}^d : |x - a| \leq r \}$. Define the function $\tilde{\rho} : \mathbb{R}^d \to \mathbb{R}$ by

$$
\tilde{\rho}(x) = \begin{cases} 
|x - a|^2 - r^2 & \text{if } x \notin A, \\
0 & \text{if } x \in A.
\end{cases}
$$

Also, set $\hat{\varepsilon} := \varepsilon^2 + 2r\varepsilon$. Then, we define the function $\tilde{h}_{A, \varepsilon} : \mathbb{R}^d \to [0, 1]$ by $\tilde{h}_{A, \varepsilon}(x) = g(\tilde{\rho}(x)/\hat{\varepsilon})$. From the proof of [Zhilova (2020, Lemma A.3)](https://www.example.com), this $\tilde{h}_{A, \varepsilon}$ satisfies \(5.40\)–\(5.41\), so it remains to check \(5.42\). We have

$$
\partial_{jk} \tilde{h}_{A, \varepsilon}(x) = 4 \frac{\varepsilon^2}{\hat{\varepsilon}^2} g''(\tilde{\rho}(x)/\hat{\varepsilon})(x_j - a_j)(x_k - a_k) + 2 \frac{2}{\hat{\varepsilon}} g'(\tilde{\rho}(x)/\hat{\varepsilon}) \delta_{jk}.
$$

Therefore, noting that $g''(x) = 0$ if $x \notin (0, 1)$, we obtain

$$
\langle M, \text{Hess} \tilde{h}_{A, \varepsilon}(x) \rangle_{H.S.} \leq 4 \frac{\varepsilon^2}{\hat{\varepsilon}^2} g''(\tilde{\rho}(x)/\hat{\varepsilon}) ||M||_{H.S.} |x - a|^2 + 2 \frac{2}{\hat{\varepsilon}} g'(\tilde{\rho}(x)/\hat{\varepsilon}) \sum_{j=1}^{d} \left|M_{jj}\right|.
$$

This completes the proof. \(\square\)

**Lemma 5.6** (Lemma A.4 of [Zhilova (2020)](https://www.example.com)). For any $d$-dimensional random vector $W$ and any $\varepsilon > 0$,

$$
\sup_{A \in B} |P(W \in A) - P(Z \in A)| \leq \sup_{A \in B} \mathbb{P}(Z \in A^c \setminus A) + \sup_{A \in B} \left|\mathbb{E}\tilde{h}_{A, \varepsilon}(W) - \mathbb{E}\tilde{h}_{A, \varepsilon}(Z)\right|,
$$

(5.43)

where $\tilde{h}_{A, \varepsilon}$ is as in Lemma 5.5.

Set $\tilde{\sigma} := \text{tr}(\Sigma^2)^{1/4}$. The following anti-concentration inequality is an immediate consequence of [Gieissing and Fan (2020, Corollary 5)](https://www.example.com):

**Lemma 5.7.** Assume $\tilde{\sigma} > 0$. For any $\varepsilon > 0$,\n
$$
\sup_{A \in B} \mathbb{P}(Z \in A^c \setminus A) \leq C\tilde{\sigma}^{-1}\varepsilon.
$$

**Proof.** Take $A = \{ x \in \mathbb{R}^d : |x - a| \leq r \} \in B$ arbitrarily. We have by Corollary 5 of [Gieissing and Fan (2020)](https://www.example.com)

$$
\mathbb{P}(Z \in A^c \setminus A) = \mathbb{P}(r < |Z - a| \leq r + \varepsilon) \leq \frac{C\varepsilon}{(\text{tr}(\Sigma^2) + a^\top \Sigma a)^{1/4}}.
$$

Since $a^\top \Sigma a \geq 0$, we obtain the desired result. \(\square\)

The following lemma can be shown by elementary calculation, so we omit its proof.
Lemma 5.8. \( \Psi \) is an increasing function on \((0, \infty)\). Moreover, \( \Psi(cx) \leq (c + \Psi(c))\Psi(x) \) for all \( x > 0 \) and \( c \geq 1 \).

We now fix \( A \in \mathcal{B} \) (will take sup in (5.51)), \( 0 < \varepsilon \leq \tilde{\sigma} \), write \( h := \tilde{h}_{A, \varepsilon} \) and proceed to bound \( |Eh(W) - Eh(Z)| \) by the decomposition (5.8). Consider the solution \( f \) to the Stein equation (5.2), which is given by (5.3). Since \( h \) has bounded partial derivatives up to the third order, \( f \) is thrice differentiable with bounded partial derivatives. Using the integration by parts formula, we have for \( 1 \leq j, k, l \leq d \) and any \( 0 \leq c_0 \leq 1 \) that

\[
\partial_{jkl}f(w) = \int_0^{c_0} \frac{1}{2\sqrt{s}} \int_{\mathbb{R}^d} \partial_j h(\sqrt{1 - sw + \sqrt{s}\Sigma^{1/2}z})\sigma_k^{-1}\partial_k \phi(z)dz ds \\
+ \int_0^{c_0} \frac{1}{2s} \int_{\mathbb{R}^d} h(\sqrt{1 - sw + \sqrt{s}\Sigma^{1/2}z})\sigma_j^{-1}\sigma_k^{-1}\partial_{jkl} \phi(z)dz ds
\]

and

\[
\partial_{jk}f(w) = \int_0^{c_0} \frac{1}{2\sqrt{s}} \int_{\mathbb{R}^d} \partial_j h(\sqrt{1 - sw + \sqrt{s}\Sigma^{1/2}z})\sigma_k^{-1}\partial_k \phi(z)dz ds \\
+ \int_0^{c_0} \frac{1}{2s^{3/2}} \int_{\mathbb{R}^d} h(\sqrt{1 - sw + \sqrt{s}\Sigma^{1/2}z})\sigma_j^{-1}\sigma_k^{-1}\partial_{jk} \phi(z)dz ds.
\]

We first bound \( R_1 \) in (5.9). Using the expression of \( \partial_{jk}f \) in (5.44) with \( c_0 = (\varepsilon/\tilde{\sigma})^2 \), we have

\[
R_1 = R_{11} + R_{12},
\]

where

\[
R_{11} = \sum_{j,k=1}^d \mathbb{E} \left[ V_{jk} \int_0^{(\varepsilon/\tilde{\sigma})^2} \frac{1}{2\sqrt{s}} \int_{\mathbb{R}^d} \partial_j h(\sqrt{1 - sw + \sqrt{s}\Sigma^{1/2}z})\sigma_k^{-1}\partial_k \phi(z)dz ds \right]
\]

and

\[
R_{12} = \sum_{j,k=1}^d \mathbb{E} \left[ V_{jk} \int_0^{(\varepsilon/\tilde{\sigma})^2} \frac{1}{2s} \int_{\mathbb{R}^d} h(\sqrt{1 - sw + \sqrt{s}\Sigma^{1/2}z})\sigma_j^{-1}\sigma_k^{-1}\partial_{jk} \phi(z)dz ds \right].
\]

For \( R_{11} \), applying analogous arguments to (5.22), we obtain

\[
|R_{11}| \leq C\tilde{\sigma}^{-1} \left\{ \sum_{j,k=1}^d \sigma_k^{-2}EY_{jk}^2 \right\}^{1/2} \leq C \left\{ \sum_{j,k=1}^d (\sigma_j\sigma_k)^{-2}EY_{jk}^2 \right\}^{1/2},
\]

where we used the inequality \( \tilde{\sigma} \geq \sigma_j \) to derive the last inequality. The triangle inequality yields, for \( V \) in (5.10),

\[
\left\{ \sum_{j,k=1}^d (\sigma_j\sigma_k)^{-2}EY_{jk}^2 \right\}^{1/2} \leq \|I_d - \text{Var}(\Sigma^{-1/2}W)\|_{H.S.}
\]

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Besides, we have

$$\left\{ \sum_{j,k=1}^{d} (\sigma_j \sigma_k)^{-2} \text{Var} \left[ \sum_{i=1}^{n} \xi_{ij} \xi_{ik} \right] \right\}^{1/2} = \left\{ \sum_{i=1}^{n} \sum_{j,k=1}^{d} (\sigma_j \sigma_k)^{-2} \text{Var}[\xi_{ij} \xi_{ik}] \right\}^{1/2}$$

and

$$\leq \left\{ \sum_{i=1}^{n} \left( \sum_{j,k=1}^{d} (\sigma_j \sigma_k)^{-2} \mathbb{E} \xi_{ij}^2 \xi_{ik}^2 \right) \right\}^{1/2} = \left\{ \sum_{i=1}^{n} \mathbb{E} \left[ \left( \sum_{j=1}^{d} \sigma_j^{-2} \xi_{ij}^2 \right) \right] \right\}^{1/2} = \left( \sum_{i=1}^{n} \mathbb{E} [\Sigma^{-1/2} \xi_i^4] \right)^{1/2}.$$  

Consequently, we obtain

$$|R_{11}| \leq C \delta(W, \Sigma).$$

For $R_{12}$, we apply analogous arguments to (5.23) and obtain

$$|R_{12}| \leq C |\log(\varepsilon/\sigma)| \left\{ \sum_{j,k=1}^{d} (\sigma_j \sigma_k)^{-2} \mathbb{E} V_{jk}^2 \right\}^{1/2} \leq C |\log(\varepsilon/\sigma)| \delta(W, \Sigma).$$

Therefore,

$$|R_1| \leq C (|\log(\varepsilon/\sigma)| \lor 1) \delta(W, \Sigma). \quad (5.46)$$

Next, we bound $R_2$ in (5.12). Using the expression of $\partial_{jkl} f$ in (5.45) with $\epsilon_0 = (\varepsilon/\sigma)^2$, we have

$$R_2 = R_{21} + R_{22},$$

where

$$R_{21} = \frac{1}{4} \sum_{i=1}^{n} \sum_{j,k,l=1}^{d} \mathbb{E} U(\xi_{ij}' - \xi_{ij})(\xi_{ik}' - \xi_{ik})(\xi_{il}' - \xi_{il}) \int_{(\varepsilon/\sigma)^2}^{\frac{\sqrt{1-s}}{2\sqrt{s}}} \int_{\mathbb{R}^d} [\partial_{jkl} h(\sqrt{1-s}(W + (1-U)(\xi_i' - \xi_i)) + \sqrt{s} \Sigma^{1/2} z) \right] \times \sigma_i^{-1} \partial_i \phi(z) dz ds$$

and

$$R_{22} = \frac{1}{4} \sum_{i=1}^{n} \sum_{j,k,l=1}^{d} \mathbb{E} U(\xi_{ij}' - \xi_{ij})(\xi_{ik}' - \xi_{ik})(\xi_{il}' - \xi_{il}) \int_{(\varepsilon/\sigma)^2}^{\frac{\sqrt{1-s}}{2s^{3/2}}} \int_{\mathbb{R}^d} [h(\sqrt{1-s}(W + (1-U)(\xi_i' - \xi_i)) + \sqrt{s} \Sigma^{1/2} z) - h(\sqrt{1-s}(W + U(\xi_i' - \xi_i)) + \sqrt{s} \Sigma^{1/2} z)] \times \sigma_i^{-1} \sigma_j^{-1} \partial_{jkl} \phi(z) dz ds.$$
Let $W^{(i)} = W - \xi_i$ for $i \in \{1, \ldots, n\}$. We will use the mean value theorem for the differences involving $h$ in the above two expressions as in (5.25), the fact that $\nabla h$ is non-zero only in $A^c \setminus A$ and bound
\[
P(\sqrt{1-s}W^{(i)} \in A_i^c \setminus A_i|U, U', \xi_i, \xi'_i),
\]
where $0 < s < 1$, $U'$ is a uniform random variable on $[0, 1]$ independent of everything else, and $A_i$ is a Euclidean ball which may depend on $U$, $U'$, $\xi_i$, $\xi'_i$, $s$ and $\Sigma^{1/2}z$. We have by Lemma 5.7
\[
P(\sqrt{1-s}W^{(i)} \in A_i^c \setminus A_i|U, U', \xi_i, \xi'_i) \leq C \frac{\varepsilon}{\delta \sqrt{1-s}} + 2 \sup_{A \in B} \left| P(W^{(i)} \in A) - P(Z \in A) \right|.
\]
From (5.37), we have
\[
\sup_{A \in B} \left| P(W^{(i)} \in A) - P(Z \in A) \right| \leq K'(\beta_0) \max \left\{ \beta_0, \Psi(\delta(W^{(i)}, \Sigma)) \right\}.
\]
(5.48)
Since
\[
\| \text{Var}(\Sigma^{-1/2}W) - \text{Var}(\Sigma^{-1/2}W^{(i)}) \|_{H.S.} = \sqrt{\sum_{j,k=1}^{d} (\mathbb{E}(\Sigma^{-1/2}x_j)(\Sigma^{-1/2}x_k))^2} \leq \sqrt{\mathbb{E}|\Sigma^{-1/2}x|^4}
\]
and $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$ for any $x, y \geq 0$, we have
\[
\delta(W^{(i)}, \Sigma) \leq \| I_d - \text{Var}(\Sigma^{-1/2}W) \|_{H.S.} + \sqrt{\mathbb{E}|\Sigma^{-1/2}x|^4} + \sqrt{\sum_{j=1}^{n} \mathbb{E}|\Sigma^{-1/2}x_j|^4} \leq \sqrt{2}\delta(W, \Sigma).
\]
Hence, we obtain by Lemma 5.8
\[
\Psi(\delta(W^{(i)}, \Sigma)) \leq 2\sqrt{2}\Psi(\delta(W, \Sigma)) \leq 2\sqrt{2}\beta.
\]
(5.49)
Thus we conclude
\[
\sup_{A \in B} \left| P(W^{(i)} \in A) - P(Z \in A) \right| \leq K'(\beta_0) \max \left\{ \beta_0, 2\sqrt{2}\beta \right\} = 2\sqrt{2}K'(\beta_0)\beta.
\]
(5.49)
Using the mean value theorem for $R_{21}$, $R_{22}$ and applying (5.41), (5.47), (5.49) and (5.41), we have
\[
|R_{21}| + |R_{22}| \leq \frac{C\sigma^2}{\varepsilon^2} \sum_{i=1}^{n} \mathbb{E}|\Sigma^{-1/2}x_i|^4 \left( \frac{\varepsilon}{\sigma} + 2\sqrt{2}K'(\beta_0)\beta \right),
\]
(5.50)
where we also used the inequality \( \max_{1 \leq j \leq d} \sigma_j \leq \tilde{\sigma} \). From Lemmas 5.6–5.7, (5.8), (5.46), (5.50), we have

\[
\sup_{A \in B} |P(W \in A) - P(Z \in A)| \leq C\tilde{\sigma}^{-1} \varepsilon + C(\log(\varepsilon/\tilde{\sigma}) \vee 1)\delta(W, \Sigma) + \frac{C\tilde{\sigma}^2}{\varepsilon^2} \sum_{i=1}^n \mathbb{E} |\Sigma^{-1/2} \xi_i|^4 \left( \frac{\varepsilon}{\tilde{\sigma}} + K'(\beta_0) \tilde{\beta} \right). \tag{5.51}
\]

Choose \( \varepsilon = \min\{\tilde{\sigma} \left[ 2C \sum_{i=1}^n \mathbb{E} |\Sigma^{-1/2} \xi_i|^4 \right]^{1/2}, \tilde{\sigma} \} \) for the same absolute constant \( C \) as in the third term on the right-hand side of (5.51). If \( \varepsilon < \tilde{\sigma} \), then from (5.51),

\[
\sup_{A \in B} |P(W \in A) - P(Z \in A)| \leq \left( C + \frac{K'(\beta_0)}{2} \right) \tilde{\beta};
\]

hence

\[
\sup_{A \in B} |P(W \in A) - P(Z \in A)| \leq C + \frac{K'(\beta_0)}{2}. \tag{5.52}
\]

If \( \varepsilon = \tilde{\sigma} \), then \( \sum_{i=1}^n \mathbb{E} |\Sigma^{-1/2} \xi_i|^4 \) and \( \tilde{\beta} \) are bounded away from 0 by an absolute constant; hence

\[
\sup_{A \in A} |P(W \in A) - P(Z \in A)| \leq \frac{1}{\tilde{\beta}} \leq C.
\]

Note that the right-hand sides of the above two bounds do not depend on \( W \) or \( \Sigma \). Taking supremum over \( W \) and \( \Sigma \), we obtain

\[
K'(\beta_0) \leq C + \frac{K'(\beta_0)}{2}. \tag{5.53}
\]

This implies (5.38), hence (3.1).

### 5.5 Proof of Theorem 3.2

Without loss of generality, we may assume \( Z \sim \mathcal{N}(0, \Sigma) \) to be independent of everything else. The proof is a combination of Zhilova (2020)’s smoothing, a Gaussian anti-concentration inequality for ellipsoids by Giessing and Fan (2020), the decomposition (5.8), and a concentration inequality type argument by Chernozhukov et al. (2019).

Fix \( A \in B \) (will take sup in (5.53), \( \varepsilon > 0 \) (to be chosen above (5.59)), write \( h := \tilde{h}_{A,\varepsilon} \) as in Lemma 5.5 and proceed to bound \( |\mathbb{E}h(W) - \mathbb{E}h(Z)| \) by the decomposition (5.8).

Consider the solution \( f \) to the Stein equation (5.2), which is given by (5.3). Note that we can rewrite \( f \) as

\[
f(w) = \int_0^1 -\frac{1}{2(1-s)} \mathbb{E} [h(\sqrt{1-s}w + Z) - \mathbb{E}h(Z)] ds.
\]

Since \( h \) has bounded partial derivatives up to the fourth order, \( f \) is four times differentiable and

\[
\nabla^r f(w) = \int_0^1 -\frac{(1-s)^{r/2-1}}{2} \mathbb{E} [\nabla^r h(\sqrt{1-s}w + \sqrt{s}Z)] ds \quad \text{for any } r = 1, 2, 3, 4. \tag{5.54}
\]
We first bound \( R_1 \) in (5.9). Using (5.54), we obtain

\[
R_1 = -\frac{1}{2} \int_0^1 \mathbb{E}\left[\langle V, \text{Hess} h(W^s) \rangle_{H.S.}\right] ds,
\]

where \( W^s := \sqrt{1 - sW} + \sqrt{sZ} \). Since \( \nabla h \) is non-zero only in \( A^\varepsilon \setminus A \), we have

\[
R_1 = -\frac{1}{2} \int_0^1 \mathbb{E}\left[\langle V, \text{Hess} h(W^s) \rangle_{H.S.} 1_{\{W^s \in A^\varepsilon \setminus A\}}\right] ds.
\]

Therefore, using (5.42), we obtain

\[
|R_1| \leq \frac{1}{2} \int_0^1 \mathbb{E}\left[\langle V, \text{Hess} h(W^s) \rangle_{H.S.} 1_{\{W^s \in A^\varepsilon \setminus A\}}\right] ds
\]

\[
\leq \frac{C}{\varepsilon^2} \int_0^1 \mathbb{E}\left(\|V\|_{H.S.} + \sum_{j=1}^d |V_{jj}| \right) 1_{\{W^s \in A^\varepsilon \setminus A\}} ds
\]

\[
= \frac{C}{\varepsilon^2} \int_0^1 \mathbb{E}\left(\|V\|_{H.S.} + \sum_{j=1}^d |V_{jj}| \right) \mathbb{P}(W^s \in A^\varepsilon \setminus A|\xi) ds.
\]

Since \( Z \) is independent of \( \xi \), Lemma 5.7 yields

\[
\mathbb{P}(W^s \in A^\varepsilon \setminus A|\xi) \leq \frac{C\varepsilon}{\sigma\sqrt{s}}.
\]

Thus we deduce

\[
|R_1| \leq \frac{C}{\sigma\varepsilon} \mathbb{E}\left(\|V\|_{H.S.} + \sum_{j=1}^d |V_{jj}| \right) \int_0^1 \frac{1}{\sqrt{s}} ds \leq \frac{C}{\sigma\varepsilon} \mathbb{E}\left(\|V\|_{H.S.} + \sum_{j=1}^d |V_{jj}| \right).
\]

Using (5.10) and the triangle inequality, we obtain

\[
\mathbb{E}\left(\|V\|_{H.S.} + \sum_{j=1}^d |V_{jj}| \right)
\]

\[
\leq \|\Sigma - \text{Var}(W)\|_{H.S.} + \sum_{j=1}^d |\Sigma_{jj} - \text{Var}(W_j)| + \frac{1}{2} \sqrt{\sum_{j,k=1}^d \sum_{i=1}^n \text{Var}(\xi_{ij}\xi_{ik})} + \frac{1}{2} \sum_{j=1}^d \sqrt{\sum_{i=1}^n \text{Var}(\xi_{ij}^2)}
\]

\[
\leq \|\Sigma - \text{Var}(W)\|_{H.S.} + \sum_{j=1}^d |\Sigma_{jj} - \text{Var}(W_j)| + \frac{1}{2} \sqrt{\sum_{j,k=1}^d \sum_{i=1}^n \mathbb{E}[\xi_{ij}^2\xi_{ik}^2]} + \frac{1}{2} \sum_{j=1}^d \sqrt{\sum_{i=1}^n \mathbb{E}[\xi_{ij}^4]}
\]

\[
\leq \tilde{\delta}(W, \Sigma).
\]

Therefore, we conclude

\[
|R_1| \leq \frac{C}{\sigma\varepsilon} \tilde{\delta}(W, \Sigma).
\]
Next we bound $R_2$ in (5.12). We rewrite it as

$$R_2 = \frac{n}{4} \sum_{j,k,l,m=1}^{d} \mathbb{E}[D_j D_k D_l D_m U (1 - 2U) \partial_{jklm} f (W + \tilde{D})],$$

(5.56)

where $\tilde{D} := UD + U'(1 - 2U)D$ and $U'$ is a uniform random variable on $[0, 1]$ independent of everything else. Now we set $\tilde{W}^s := \sqrt{1 - s(W + \tilde{D})} + \sqrt{s}Z$. Then, using (5.54), we can rewrite $R_2$ as

$$R_2 = n \int_0^1 -\frac{1-s}{8} \mathbb{E}U(1 - 2U)\langle \nabla^4 h(\tilde{W}^s), D^{\otimes 4} \rangle ds.$$

Since $\nabla h$ is non-zero only in $A^c \setminus A$, we can further rewrite it as

$$R_2 = n \int_0^1 -\frac{1-s}{8} \mathbb{E}U(1 - 2U)\langle \nabla^4 h(\tilde{W}^s), D^{\otimes 4} \rangle 1_{\{\tilde{W}^s \in A^c \setminus A\}} ds.$$

Therefore, using (5.41), we obtain

$$|R_2| \leq \frac{n}{\varepsilon^4} \int_0^1 \mathbb{E}|\nabla^4 h(\tilde{W}^s), D^{\otimes 4}| 1_{\{\tilde{W}^s \in A^c \setminus A\}} ds \leq \frac{Cn}{\varepsilon^4} \int_0^1 \mathbb{E}|D|^{4} 1_{\{\tilde{W}^s \in A^c \setminus A\}} ds$$

$$= \frac{Cn}{\varepsilon^4} \int_0^1 \mathbb{E}|D|^{4} P(\tilde{W}^s \in A^c \setminus A| D, U, U') ds.$$

Since $Z$ is independent of $D, U$ and $U'$, Lemma 5.7 yields

$$P(\tilde{W}^s \in A^c \setminus A| D, U, U') \leq \frac{C\varepsilon}{\delta \sqrt{s}}.$$

Thus we conclude

$$|R_2| \leq \frac{Cn}{\varepsilon^4} \mathbb{E}|D|^{4} \int_0^1 \frac{1}{\sqrt{s}} ds \leq \frac{C}{\sigma \varepsilon^3} \sum_{i=1}^{n} \mathbb{E} |\xi_i|^{4}. \quad (5.57)$$

From Lemmas 5.6, 5.7, 5.8, 5.55, 5.57, we have

$$\sup_{A \in \mathcal{B}} |P(W \in A) - P(Z \in A)| \leq C\sigma^{-1} \varepsilon + \frac{C}{\sigma \varepsilon} \delta(W, \Sigma) + \frac{C}{\sigma \varepsilon^3} \sum_{i=1}^{n} \mathbb{E} |\xi_i|^{4}, \quad (5.58)$$

and by choosing $\varepsilon = \sqrt{\delta(W, \Sigma)}$, we obtain

$$\sup_{A \in \mathcal{B}} |P(W \in A) - P(Z \in A)| \leq \frac{C}{\sigma} \sqrt{\delta(W, \Sigma)}. \quad (5.59)$$

This completes the proof.
5.6 Proof of Proposition 3.1

Since \(|Z|^2 - d)/\sqrt{2d}\) converges in law to \(N(0, 1)\) as \(d \to \infty\), \(|W|^2 - d)/\sqrt{2d}\) also converges in law to \(N(0, 1)\). Since \(W\) has the same law as \(\sqrt{V}Z'\) by assumption, where \(V := n^{-1} \sum_{i=1}^n e_i^2\) and \(Z' \sim N(0, I_d)\) is independent of \(\{e_i\}_{i=1}^\infty\), \((V|Z|^2 - d)/\sqrt{2d}\) should also converge in law to \(N(0, 1)\). Since

\[
\frac{V|Z|^2 - d}{\sqrt{2d}} = V \frac{|Z|^2 - d}{\sqrt{2d}} + \sqrt{\frac{d}{2}}(V - 1) = (V - 1) \frac{|Z'|^2 - d}{\sqrt{2d}} + \frac{|Z'|^2 - d}{\sqrt{2d}} + \sqrt{\frac{d}{2}}(V - 1)
\]

and the first term converges to 0 in probability,

\[
\frac{|Z'|^2 - d}{\sqrt{2d}} + \sqrt{\frac{d}{2}}(V - 1)
\]

must converge in law to \(N(0, 1)\). In the above expression, the first term converges in law to \(N(0, 1)\) and the first and second terms are independent, so this implies \(\sqrt{d}(V - 1) = o_p(1)\) as \(n \to \infty\). Since \(\sqrt{n}(V - 1)\) converges in law to \(N(0, \text{Var}(e_1^2))\), we must have \(d/n \to 0\).

5.7 Proof of Theorem 4.1 and Corollary 4.1

First we prove Theorem 4.1. Conditional on \(X, X_1^* - \bar{X}, \ldots, X_n^* - \bar{X}\) are i.i.d. with mean 0 and covariance matrix \(\hat{\Sigma} := n^{-1} \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^\top\). Therefore, applying Theorem 3.2 conditional on \(X\), we obtain

\[
\sup_{A \in B} |\mathbb{P}(W^* \in A|\xi) - \mathbb{P}(Z \in A)| \leq \frac{C}{\text{tr}(\Sigma^2)^{1/4}} \sqrt{\delta^*}, \tag{5.60}
\]

where

\[
\delta^* := ||\Sigma - \hat{\Sigma}||_{H.S.} + \frac{d}{n} \sum_{j=1}^d |\Sigma_{jj} - \hat{\Sigma}_{jj}| + \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i^* - \bar{X}|^4|X| + \frac{1}{n} \sum_{j=1}^d \sum_{i=1}^n \mathbb{E}|X_{ij}^* - \bar{X}_j|^4|X|
\]

\[
= ||\Sigma - \hat{\Sigma}||_{H.S.} + \frac{d}{n} \sum_{j=1}^d |\Sigma_{jj} - \hat{\Sigma}_{jj}| + \frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}|^4 + \frac{1}{n} \sum_{j=1}^d \sum_{i=1}^n |X_{ij} - \bar{X}_j|^4.
\]

Noting \(\hat{\Sigma} = n^{-1} \sum_{i=1}^n X_i X_i^\top - \bar{X} \bar{X}^\top\), we obtain

\[
\mathbb{E}|\Sigma_{jk} - \hat{\Sigma}_{jk}|^2 \leq 2 \text{Var} \left[ \frac{1}{n} \sum_{i=1}^n X_{ij} X_{ik} \right] + 2 \mathbb{E}[\bar{X}_j^2 \bar{X}_k^2] \leq 2 \frac{2}{n^2} \sum_{i=1}^n \mathbb{E}[X_{ij}^2 X_{ik}^2] + 2 \sqrt{\mathbb{E}[\bar{X}_j]^4 \mathbb{E}[\bar{X}_k]^4}.
\]

Hence we have

\[
\mathbb{E}|\Sigma - \hat{\Sigma}|_{H.S.} \leq \sqrt{\frac{2}{n^2} \sum_{i=1}^n \mathbb{E}[X_i]^4 + 2 \left( \sum_{j=1}^d \sqrt{\mathbb{E}[\bar{X}_j]^4} \right)^2}
\]
\[
\mathbb{E} \left[ \sum_{j=1}^{d} |\Sigma_{jj} - \widehat{\Sigma}_{jj}| \right] \leq \sum_{j=1}^{d} \sqrt{\frac{2}{n^2} \sum_{i=1}^{n} \mathbb{E} [X^4_{ij}]} + 2 \mathbb{E} \bar{X}^4.
\]

The Marcinkiewicz–Zygmund inequality and the Jensen inequality yield
\[
\mathbb{E} \bar{X}^4_j \leq C \mathbb{E} \left( \frac{1}{n^2} \sum_{i=1}^{n} X^2_{ij} \right)^2 \leq C \frac{n}{n^3} \sum_{i=1}^{n} \mathbb{E} X^4_{ij}.
\]

so we obtain
\[
\mathbb{E} \|\Sigma - \widehat{\Sigma}\|_{H.S.} + \mathbb{E} \left[ \sum_{j=1}^{d} |\Sigma_{jj} - \widehat{\Sigma}_{jj}| \right] \leq C \frac{1}{n} \left( \sqrt{\sum_{i=1}^{n} \mathbb{E} |X_i|^4} + \sum_{j=1}^{d} \sqrt{\sum_{i=1}^{n} \mathbb{E} X^4_{ij}} \right). \tag{5.62}
\]

We also have
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} |X_i - \bar{X}|^4 \right] \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} |X_i - \bar{X}|^4 \leq \frac{2\sqrt{2}}{n} \sqrt{\sum_{i=1}^{n} \mathbb{E} |X_i|^4 + n \mathbb{E} \bar{X}^4}.
\]

Since the Jensen inequality yields
\[
|\bar{X}|^4 = \left\{ \sum_{j=1}^{d} \left( \frac{1}{n} \sum_{i=1}^{n} X^2_{ij} \right) \right\}^2 \leq \left\{ \sum_{j=1}^{d} \frac{1}{n} \sum_{i=1}^{n} X^2_{ij} \right\}^2 \leq \frac{1}{n} \sum_{i=1}^{n} |X_i|^4,
\]

we obtain
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} |X_i - \bar{X}|^4 \right] \leq \frac{4}{n} \sqrt{\mathbb{E} \sum_{i=1}^{n} |X_i|^4}. \tag{5.63}
\]

Besides, we have
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{d} \sum_{i=1}^{n} |X_{ij} - \bar{X}_j|^4 \right] \leq \frac{1}{n} \sum_{j=1}^{d} \sqrt{\sum_{i=1}^{n} \mathbb{E} |X_{ij} - \bar{X}_j|^4} \leq \frac{2\sqrt{2}}{n} \sum_{j=1}^{d} \sqrt{\sum_{i=1}^{n} \mathbb{E} X^4_{ij} + n \mathbb{E} \bar{X}^4_{ij}}.
\]

So \[5.61\] yields
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{d} \sum_{i=1}^{n} |X_{ij} - \bar{X}_j|^4 \right] \leq C \frac{1}{n} \sum_{j=1}^{d} \sum_{i=1}^{n} \mathbb{E} X^4_{ij}. \tag{5.64}
\]
and (5.62)–(5.64) imply that
\[ E \sup_{A \in B} |P(W^* \in A | X) - P(Z \in A)|^2 \leq C \Delta_n. \]

Hence Theorem 4.1 follows from Markov’s inequality.

Next we prove Corollary 4.1. The first claim immediately follows from Theorems 3.2 and 4.1. Besides, Since
\[ P(|W^*| > x | X) = 1 - P(|W^*| \leq x | X) \text{ and } P(|W| \leq q_n^*(\alpha)) = 1 - P(|W| \leq q_n^*(\alpha)), \]
the second claim follows from Theorems 3.2 and 4.1 along with Proposition 3.2 of Koike (2019).

5.8 Proof of Theorem 4.2 and Corollary 4.2

Conditional on \( X \), we apply Theorem 3.2 to \( W^\circ \) and obtain
\[ \sup_{A \in B} |P(W^\circ \in A | X) - P(Z \in A)|^2 \leq \frac{C}{\text{tr}(\Sigma^2)^{1/4}} \sqrt{\delta_1^2 + \delta_2^2}, \]  
(5.65)

where
\[ \delta_1^0 := \| \Sigma - \text{Var}(W^\circ | X) \|_{H.S.} + \sum_{j=1}^{d} |\Sigma_{jj} - \text{Var}(W^\circ_j | X)| \]

and
\[ \delta_2^2 := \frac{1}{n} \left[ \sum_{i=1}^{n} \mathbb{E}[|e_i X_i|^4 | X] + \frac{1}{n} \sum_{j=1}^{d} \left[ \sum_{i=1}^{n} \mathbb{E}[|e_i X_{ij}|^4 | X] \right] \right] \]
\[ = \frac{\sqrt{\mathbb{E}e_1^4}}{n} \left[ \sum_{i=1}^{n} |X_i|^4 \right] + \frac{\sqrt{\mathbb{E}e_1^4}}{n} \sum_{j=1}^{d} \left[ \sum_{i=1}^{n} X_{ij}^4 \right]. \]

Since we have
\[ \text{Var}(W^\circ | X) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[e_i X_i (e_i X_i)^\top | X] = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^\top, \]
we obtain
\[ \mathbb{E} |\Sigma_{jk} - \text{Cov}(W^\circ_j, W^\circ_k | X)|^2 = \text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} X_{ij} X_{ik} \right] \leq \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}[X_{ij}^2 X_{ik}^2]. \]

Hence we deduce
\[ \mathbb{E} \delta_1^0 \leq \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}[|X_i|^4] + \sum_{j=1}^{d} \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}[X_{ij}^4]. \]
(5.66)

From (5.65)–(5.66) and \( \sqrt{\mathbb{E}e_1^4} \geq \mathbb{E}e_1^2 = 1 \), we obtain
\[ \mathbb{E} \sup_{A \in B} |P(W^\circ \in A | X) - P(Z \in A)|^2 \leq C \sqrt{\mathbb{E}e_1^4 \Delta_n}. \]

Hence Theorem 4.2 follows from Markov’s inequality. We can prove Corollary 4.2 analogously to Corollary 4.1.
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REFERENCES

K. Ball (1993). The reverse isoperimetric problem for Gaussian measure. *Discrete Comput. Geom.* **10**, 411–420.

V. Bentkus (2003). On the dependence of the Berry–Esseen bound on dimension. *J. Statist. Plann. Inference* **113**, 385–402.

V. Bentkus (2005). A Lyapunov type bound in $\mathbb{R}^d$. *Theory Probab. Appl.* **49**, 311-323.

T. Bonis (2020). Stein’s method for normal approximation in Wasserstein distances with application to the multivariate central limit theorem. *Probab. Theory Related Fields*. [https://doi.org/10.1007/s00440-020-00989-4](https://doi.org/10.1007/s00440-020-00989-4)

X. Chen and W. X. Zhou (2020). Robust inference via multiplier bootstrap. *Ann. Statist.* **48**, 1665–1691.

S. Chatterjee and E. Meckes (2008). Multivariate normal approximation using exchangeable pairs. *Alea* **4**, 257–283.

V. Chernozhukov, D. Chetverikov and K. Kato (2013). Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *Ann. Statist.* **41**, 2786–2819.

V. Chernozhukov, D. Chetverikov and K. Kato (2014). Gaussian approximation of suprema of empirical processes. *Ann. Statist.* **42**, 1564–1597.

V. Chernozhukov, D. Chetverikov and K. Kato (2017). Central limit theorems and bootstrap in high dimensions. *Ann. Probab.* **45**, 2309–2352.

V. Chernozhukov, D. Chetverikov, K. Kato and Y. Koike (2019). Improved central limit theorem and bootstrap approximation in high dimensions. *Preprint*. Available at [https://arxiv.org/abs/1912.10529](https://arxiv.org/abs/1912.10529)

B. Efron (1979). Bootstrap methods: Another look at the jackknife. *Ann. Statist.* **7**, 1–26.

R. Eldan, D. Mikulincer and A. Zhai (2018). The CLT in high dimensions: quantitative bounds via martingale embedding. To appear in *Ann. Probab*. Preprint available at [https://arxiv.org/abs/1806.09087](https://arxiv.org/abs/1806.09087)

X. Fang and Y. Koike (2020a). High-dimensional central limit theorems by Stein’s method. *Preprint*. Available at [https://arxiv.org/abs/2001.10917](https://arxiv.org/abs/2001.10917)
X. Fang and Y. Koike (2020b). New error bounds in multivariate normal approximations via exchangeable pairs with applications to Wishart matrices and fourth moment theorems. Preprint. Available at https://arxiv.org/abs/2004.02101

X. Fang and A. Röllin (2015). Rates of convergence for multivariate normal approximation with applications to dense graphs and doubly indexed permutation statistics. Bernoulli 21, 2157–2189.

A. Giessing and J. Fan (2020). Bootstrapping ℓp-statistics in high dimensions. Preprint. Available at https://arxiv.org/abs/2006.13099

F. Götze (1991). On the rate of convergence in the multivariate CLT. Ann. Probab. 19, 724–739.

Y. Koike (2019). Mixed-normal limit theorems for multiple Skorohod integrals in high-dimensions, with application to realized covariance. Electron. J. Stat. 13, 1443–1522.

R. Y. Liu (1988). Bootstrap procedures under some non-i.i.d. models. Ann. Statist. 16, 1696–1708.

E. Meckes (2009). On Stein’s method for multivariate normal approximation. In: C. Houdré, V. Koltchinskii, D. M. Mason and M. Peligrad (eds.) High Dimensional Probability V: The Luminy Volume. 5, 153–178. Institute of Mathematical Statistics, Beachwood, Ohio, USA.

S. V. Nagaev (1976). An estimate of the remainder term in the multidimensional central limit theorem. In Proceedings of the Third Japan-USSR Symposium on Probability Theory, pages 419–438. Springer.

H. Peng and A. Schick (2018). Asymptotic normality of quadratic forms with random vectors of increasing dimension. J. Multivariate Anal. 164, 22–39.

D. Pouzo (2015). Bootstrap consistency for quadratic forms of sample averages with increasing dimension. Electron. J. Stat. 9, 3046–3097.

M. Raić (2019a). A multivariate Berry-Esseen theorem with explicit constants. Bernoulli 25, 2824–2853.

M. Raić (2019b). A multivariate central limit theorem for Lipschitz and smooth test functions. Preprint. Available at https://arxiv.org/abs/1812.08268

G. Reinert and A. Röllin (2009). Multivariate normal approximation with Stein’s method of exchangeable pairs under a general linearity condition. Ann. Probab. 37, 2150–2173.

V. V. Sazonov (1972). On a bound for the rate of convergence in the multidimensional central limit theorem. Proc. Sixth Berkeley Symp. on Math. Statist. and Prob. 2, 563–581.

C. Shi, R. Song, Z. Chen and R. Li (2019). Linear hypothesis testing for high dimensional generalized linear models. Ann. Statist. 47, 2671–2703.
V. Spokoiny and M. Zhilova (2015). Bootstrap confidence sets under model misspecification. *Ann. Statist.* **43**, 2653–2675.

C. Stein (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pp. 583–602. Univ. California Press, Berkeley, Calif.

C. F. J. Wu (1986). Jackknife, bootstrap and other resampling methods in regression analysis. *Ann. Statist.* **14**, 1261–1295.

A. Zhai. (2018). A high-dimensional CLT in $W_2$ distance with near optimal convergence rate. *Probab. Theory Related Fields* **170**, no. 3-4, 821–845.

M. Zhilova (2020). Non-classical Berry-Esseen inequalities and accuracy of the bootstrap. *Ann. Statist.* **48**, 1922–1939.