WEAK HARNACK INEQUALITY AND HÖLDER REGULARITY FOR
SYMMETRIC STABLE LÉVY PROCESSES

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Abstract. In this paper we consider weak Harnack inequality and Hölder regularity estimates for symmetric \( \alpha \)-stable Lévy process in \( \mathbb{R}^d \), \( \alpha \in (0, 2), d \geq 2 \). We consider a symmetric \( \alpha \)-stable Lévy process \( X \) for which a spherical part \( \mu \) of the Lévy measure is a spectral measure. In addition, we assume that \( \mu \) is absolutely continuous with respect to the uniform measure \( \sigma \) on the sphere and impose certain bounds on the corresponding density. Eventually, we show that the weak Harnack inequality holds, which we apply to prove Hölder regularity results.

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1. Introduction

In the paper, we use the notation
\[
B(x_0, r) = \{ x : |x - x_0| \leq r \}, r \geq 0.
\]

We consider a symmetric \( \alpha \)-stable Lévy process \( X \), which has the characteristic function of the form
\[
\mathbb{E}[e^{iu \cdot X_t}] = e^{-\Phi(u)}, u \in \mathbb{R}^d, t \geq 0,
\]
where the characteristic exponent \( \Phi \) is given by
\[
\Phi(u) = \int_{S^{d-1}} |u \cdot \xi|^\alpha \mu(d\xi).
\]
The measure \( \mu \) is symmetric, finite and non-zero on \( S^{d-1} \) (see [6], Theorem 14.13). Let the measure \( \mu \) be absolutely continuous with respect to the uniform surface measure on \( S^{d-1} \) and denote its density by \( f_\mu \).

The potential density \( p(t, x, y) = p(t, y - x) \) is determined by the Fourier transform
\[
\int_{\mathbb{R}^d} e^{i\xi \cdot x} \, p(t, x) \, dx = e^{-\Phi(\xi)}, \xi \in \mathbb{R}^d, t \geq 0.
\]

Definition 1.1. The Green function is defined by
\[
G(x, y) = \int_0^\infty p(t, x, y) \, dt, \ x, y \in \mathbb{R}^d.
\]

Definition 1.2. Let \( D \) be an open set, \( D \subset \mathbb{R}^d \). The Green function of \( X^D \) is defined by
\[
G_D(x, y) = G(x, y) - \mathbb{E}[G(X_{t_0}, y)], \ x, y \in D.
\]

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Definition 1.3. A measure $\lambda$ on $\mathbb{R}^d$ is called degenerate if there is a proper linear subspace $M$ of $\mathbb{R}^d$ such that $Spt(\lambda) \subset M$, where $Spt(\lambda)$ denotes the support of the measure $\lambda$.

A measure $\lambda$ is called non-degenerate if it is not degenerate.

Definition 1.4. A measure $\mu$ on $\mathbb{S}^{d-1}$ is called a spectral measure if it is positive, finite, non-degenerate and symmetric.

2. Weak Harnack Inequality

In comparison to [7], here we show the weak Harnack inequality for $X$, where a function $u$ is bounded, but may be non-negative. The definition follows.

Definition 2.1. The weak Harnack inequality for a symmetric $\alpha$-stable Lévy processes $X$ holds if there is a constant $c = c(\alpha, d)$ such that for every bounded function $u$ on $\mathbb{R}^d$, which is harmonic in $B_r(x_0)$ with respect to $X$, $x_0 \in \mathbb{R}^d$, $0 < r \leq r_0$, and non-negative on $B_r(x_0)$, the inequality

$$\int_{B_r(x_0)} u(x) \, dx \leq c \inf_{B_{\theta r}(x_0)} u + c \sup_{z \in B_{\theta r}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\mu(y - z) \, dy$$

holds, where $\theta > \lambda > 1, 2\theta > \sigma > 1$.

We prove the following result.

Theorem 2.2 (Weak Harnack Inequality). Let $X$ be a symmetric $\alpha$-stable Lévy process in $\mathbb{R}^d$, $d \geq 2$, with index of stability $\alpha \in (0, 2)$ and the characteristic function of the form

$$\mathbb{E}_0 e^{iu \cdot X_t} = e^{-t \Phi(u)}, \quad u \in \mathbb{R}^d, \quad t \geq 0,$$

where the characteristic exponent is given by

$$\Phi(u) = \int_{\mathbb{S}^{d-1}} |\mu \cdot \xi|^\alpha \mu(d\xi),$$

and $\mu$ is a spectral measure. Furthermore, let $\mu$ be absolutely continuous with respect to the uniform measure $\sigma$ on the sphere $\mathbb{S}^{d-1}$ and denote by $f_\mu$ its density. Assume that there is a positive constant $m$ such that

$$0 \leq f_\mu(\xi) \leq m, \quad \xi \in \mathbb{S}^{d-1}.$$

Then the weak Harnack inequality for $X$ holds.

In order to show the theorem, we use the results that follow.

Lemma 2.3. Let $\alpha \in (0, 2), d \geq 2$. There is a constant $c_1$ such that for every $z \in B_{\alpha r}(x_0)$, the inequality

$$\int_{B_{\alpha r}(x_0)} G_{B_{\alpha r}(x_0)}(x, z) \, dx \leq c_1$$

holds, where $1/\lambda < \alpha < 1$. 
Proof. By the inequality $p_D \leq p$ and the estimate of the transition density $p$ for small times (see e.g. [8], Theorem 1), it follows:

\[
\begin{align*}
&\int_{B_r(x_0)} G_{B_t(x_0)}(x, z) \, dx \\
&= \int_{B_r(x_0)} \left[ \int_{0}^{\infty} p_{B_t(x_0)}(t, x, z) \, dt \right] \, dx \\
&\leq \int_{B_r(x_0)} \left[ \int_{0}^{\infty} p(t, x, z) \, dt \right] \, dx \\
&\leq c(\alpha, d) \int_{B_r(x_0)} \left[ \int_{0}^{\infty} \left( t^{-d/\alpha} \wedge \frac{t}{|x - z|^{d + d}} \right) \, dt \right] \, dx \\
&\quad + \int_{B_r(x_0)} \left[ \int_{1}^{\infty} t^{-d/\alpha} \cdot p(1, t^{-1/\alpha}x, t^{-1/\alpha}z) \, dt \right] \, dx \\
&\leq c(\alpha, d) \int_{B_r(x_0)} \left[ \int_{0}^{\infty} t^{-d/\alpha} \, dt \right] \, dx \\
&\quad + \int_{B_r(x_0)} \left[ \int_{1}^{\infty} t^{-d/\alpha} \, dt \right] \, dx \\
&\leq c(\alpha, d) \int_{B_r(x_0)} |x - z|^{\alpha - d} \, dx \\
&\quad + \frac{\alpha}{d - \alpha} \int_{B_r(x_0)} \left[ (|x - z|^{\alpha} \wedge 1)^{\frac{d}{d - \alpha}} - 1 \right] \, dx \\
&\quad + c(\alpha, d) \frac{\alpha}{d - \alpha} |B_{r/\lambda}(x_0)| \\
&\leq c(\alpha, d, r, \lambda).
\end{align*}
\]

\[\Box\]

Lemma 2.4. Let $\alpha \in (0, 2), d \geq 2$. There exist $\delta_1 = \delta_1(\alpha, d) > 0$ and $c_2 = c_2(\alpha, d, \delta_1)$ such that for every $\tilde{x} \in B_{r/\theta}(x_0)$ and every $z \in B(\tilde{x}, \delta_1)$ the inequality

\[G_{B_{\theta}(x_0)}(\tilde{x}, z) \geq c_2\]

holds.

Proof. Using

\[G_{B_{\theta}(x_0)}(\tilde{x}, z) = G(\tilde{x}, z) - \mathbb{E}^{\tilde{x}}[G(X_{r/\theta}(x_0), z)],\]

in order to prove the lemma, we compute the estimates for $G(\tilde{x}, z)$ from below and the estimates for $\mathbb{E}^{\tilde{x}}[G(X_{r/\theta}(x_0), z)]$ from above. Using the heat kernel estimates for small times
We obtain

$$\mathbb{E}[G(X_{t,B_\alpha(x)})]$$

$$= \int_{B_\alpha(x)} G(u, z) P_{B_\alpha(x)}(\bar{x}, u) \, du$$

$$= \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_0^\infty p(t, u, z) \, dt \right] \, du$$

$$= \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_0^1 p(t, u, z) \, dt \right] \, du$$

$$+ \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_1^\infty p(t, u, z) \, dt \right] \, du$$

$$\leq c(\alpha, d) \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_0^1 \left( t^{-d/\alpha} \wedge \frac{t}{|u - z|^\alpha + d} \right) \, dt \right] \, du$$

$$+ c(\alpha, d) \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_1^\infty r^{-d/\alpha} \cdot p(1, r^{-1/\alpha} u, r^{-1/\alpha} z) \, dr \right] \, du$$

$$\leq c(\alpha, d) \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_0^1 \left( t^{-d/\alpha} \wedge \frac{t}{|u - z|^\alpha + d} \right) \, dt \right] \, du$$

$$+ c(\alpha, d) \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_1^\infty r^{-d/\alpha} \, dr \right] \, du$$

$$\leq c(\alpha, d) \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_0^1 \left( t^{-d/\alpha} \wedge \frac{t}{|u - z|^\alpha + d} \right) \, dt \right] \, du$$

(2.3)

Examining the integral in (2.3) more closely, we obtain:

$$\int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \left[ \int_0^1 \left( t^{-d/\alpha} \wedge \frac{t}{|u - z|^\alpha + d} \right) \, dt \right] \, du$$

$$= \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \mathbf{1}_{|u - z| < 1}(u) \left[ \int_0^1 \left( t^{-d/\alpha} \wedge \frac{t}{|u - z|^\alpha + d} \right) \, dt \right] \, du$$

$$+ \int_{B_\alpha(x)} P_{B_\alpha(x)}(\bar{x}, u) \mathbf{1}_{|u - z| > 1}(u) \left[ \int_0^1 \left( t^{-d/\alpha} \wedge \frac{t}{|u - z|^\alpha + d} \right) \, dt \right] \, du$$

(2.4)

$$= I_1 + I_2.$$
where in the last inequality we used (2.5) and (2.6), we obtain

\[ I_1 = \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|<1}(u) \]

\[ = \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|<1}(u) \left[ \int_0^1 \left( r^{-d/\alpha} \wedge \frac{r}{|z-u|^{\alpha d}} \right) dr \right] du \]

\[ = c_1 \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|<1}(u) |z-u|^{-\alpha d} du \]

\[ + c_2 \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|<1}(u) \frac{\alpha}{d-\alpha} (|z-u|^{-\alpha d} - 1) du \]

(2.5)

\[ \leq c_1(\alpha, d, \tilde{\delta}_1, a, r), \]

where in the last inequality we used \( z \in B(\tilde{x}, \tilde{\delta}_1) \), for \( \tilde{\delta}_1 > 0 \) small enough.

\[ I_2 = \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|>1}(u) \]

\[ = \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|>1}(u) \left[ \int_0^1 \left( r^{-d/\alpha} \wedge \frac{r}{|z-u|^{\alpha d}} \right) dr \right] du \]

\[ = c \int_{B_1(x_0)} P_{B_{s_0}}(x, u) 1_{|z-u|>1}(u) |z-u|^{-\alpha d} du \]

(2.6)

\[ \leq c_2. \]

In conclusion, by (2.3), (2.4), (2.5) and (2.6), we obtain

\[ \mathbb{E}^z[G(X_{t_{B_{s_0}, 0}}, z)] \leq \tilde{c}_1(\alpha, d, \tilde{\delta}_1, a, r), \]

for all \( z \in B(\tilde{x}, \tilde{\delta}_1) \) and \( \tilde{\delta}_1 > 0 \) small enough.

To estimate \( G(\tilde{x}, z) \) from below, we use the continuity of the potential density (cf. [9]).

Due to

\[ p(1, 0) \geq c, \]

by continuity of \( p(1, \cdot) \) in \( x = 0 \), there is \( R > 0 \) such that \( p(1, x) > \frac{1}{2} \cdot p(1, 0) \), for all \( |x| < R. \)

Furthermore, for \( |\xi| = 1 \), since:

\[ G(0, \xi) \geq \int_{R^+} p(t, \xi) dt \]

\[ = \int_{R^+} r^{-d/\alpha} \cdot p(1, \frac{\xi}{r^{\alpha d}}) dr \geq \frac{1}{2} \cdot \int_{R^+} r^{-d/\alpha} \cdot p(1, 0) dr \]

\[ = c_1(\alpha, d) > 0, \]

we obtain

(2.8) \[ G(0, \xi) \geq c_1(\alpha, d). \]
For $|x| \neq 0$, by scaling and (2.8)

$$G(0, x) = |x|^{d-d} \cdot G\left(0, \frac{x}{|x|}\right) \geq c_1 \cdot |x|^{d-d}.$$ 

Therefore,

(2.9) \hspace{1cm} G(\bar{x}, z) \geq \bar{c}_2 \cdot |\bar{x} - z|^{d-d}.

Now, choose $\delta_1$ such that $\delta_1 < (\bar{c}_2/(\bar{c}_1 + c))^{\frac{1}{d-d}} \cdot \delta_1$, where $c > 0$. Then for every $\bar{x} \in B_{r/\theta}(x_0)$ and $z \in B(\bar{x}, \delta_1)$, combining (2.2), (2.7) and (2.9), we obtain

$$G_{B_{r}(x_0)}(\bar{x}, z) \geq \bar{c}_2 \cdot |\bar{x} - z|^{d-d} - \bar{c}_1 \geq c > 0.$$ 

Define $c_2 = c$ and now the statement follows. \hfill \Box

Remark 2.5. Notice that, according to the Lemma 2.3 and Lemma 2.4, there are $\bar{c} = \bar{c}(\alpha, d)$ and $\delta_1 = \delta_1(\alpha, d)$ such that for every $\bar{x} \in B_{r/\theta}(x_0)$ and for every $z \in B(\bar{x}, \delta_1)$ the inequality

$$\int_{B_{r}(x_0)} G_{B_{r}(x_0)}(x, z) \, dx \leq \bar{c} \cdot G_{B_{r}(x_0)}(\bar{x}, z)$$

holds.

Lemma 2.6. For $\alpha \in (0, 2)$ and $d \geq 2$, let $\delta_1 > 0$ be as in Lemma 2.4. There is a constant $c_3 = c_3(\alpha, d)$ such that for every $\bar{x} \in B_{r/\theta}(x_0)$ and every $\bar{u} \in B_{r}(x_0) \setminus B(\bar{x}, \delta_1)$ the inequality

$$\int_{B_{r}(x_0)} G_{B_{r}(x_0)}(x, \bar{u}) \, dx \leq c_3 \cdot G_{B_{r}(x_0)}(\bar{x}, \bar{u})$$

holds.

Proof. The proof relies on the maximum principle (cf. [5]). We use the fact that $G_D(\bar{x}, \cdot)$ is regular harmonic in $D \setminus B(\bar{x}, \varepsilon)$ with respect to $X$ for every $\varepsilon > 0$ (cf. [1]).
Define $c_3 = \tilde{c}$ and the lemma follows. \hfill \Box

**Proof of Theorem 2.2.** Let $u \geq 0$ and $1/\lambda < a < 1$.

(2.10) \quad \int_{B_{\epsilon / 2}(x_0)} u(x) \, dx
\begin{align*}
&= \int_{B_{\epsilon / 2}(x_0)} \mathbb{E}[u(X_{\tau_{B_{\epsilon / 2}(x_0)}})] \, dx \\
&= \int_{B_{\epsilon / 2}(x_0)} \left[ \int_{(B_{\epsilon / 2}(x_0))^c} u(y) P_{B_{\epsilon / 2}(x_0)}(x, y) \, dy \right] \, dx \\
&= \int_{B_{\epsilon / 2}(x_0)} \left[ \int_{(B_{\epsilon / 2}(x_0))^c} u(y) \left( \int_{B_{\epsilon / 2}(x_0)} f_y(y - z) G_{B_{\epsilon / 2}(x_0)}(x, z) \, dz \right) \, dy \right] \, dx
\end{align*}

(2.11) \quad \int_{(B_{\epsilon / 2}(x_0))^c} u(y) \left( \int_{B_{\epsilon / 2}(x_0)} f_y(y - z) G_{B_{\epsilon / 2}(x_0)}(x, z) \, dz \right) \, dy
By Remark 2.5 and Lemma 2.6 there is a constant $c$ such that for every $\bar{x} \in B_{r/\theta}(x_0)$

\begin{equation}
(2.11) \leq c \cdot \int_{B_{r}(x_0)} u(y) \left[ \int_{B_{r}(x_0)} 1_{B(\bar{x}, \delta)}(z) \right. \\
\hspace{2cm} f_r(y-z) \cdot G_{B_{r}(x_0)}(\bar{x}, z) \, dz \left. \right] dy \\
+ c \cdot \int_{(B_{r}(x_0))^c} u(y) \left[ \int_{B_{r}(x_0)} 1_{B(\bar{x}, \delta)}(z) \right. \\
\hspace{2cm} f_r(y-z) \cdot G_{B_{r}(x_0)}(\bar{x}, z) \, dz \left. \right] dy \\
= c \cdot \int_{(B_{r}(x_0))^c} u(y) \left[ \int_{B_{r}(x_0)} f_r(y-z) \cdot G_{B_{r}(x_0)}(\bar{x}, z) \, dz \right] dy \\
= c \cdot \int_{(B_{r}(x_0))^c} u(y) P_{B_{r}(x_0)}(\bar{x}, y) \, dy \\
= c \cdot E^\delta [u(X_{r\theta}(x_0))]
\end{equation}

Since the inequality

$$
\int_{B_{r/\theta}(x_0)} u(x) \, dx \leq c \cdot u(\bar{x}),
$$

holds for any $\bar{x} \in B_{r/\theta}(x_0)$, we obtain

$$
\int_{B_{r/\theta}(x_0)} u(x) \, dx \leq c \cdot \inf_{B_{r/\theta}(x_0)} u.
$$

Therefore, the theorem is proved for non-negative functions $u$.

Let $u = u^+ - u^-$, where

$$
u^+ = \max\{u, 0\}, \quad u^- = -\min\{u, 0\}.
$$

By the first part of the proof,

\begin{equation}
(2.12) \leq c \cdot \int_{B_{r/\theta}(x_0)} u^+ \, dx \\
\hspace{2cm} \leq c \cdot u^+(\bar{x}) \\
= c \cdot (u(\bar{x}) + u^-(\bar{x}))
\end{equation}

for any $\bar{x} \in B_{r/\theta}(x_0)$. Set $c_0 = r/2\theta$. Since $u^-$ is harmonic in $B_{r/\theta}(x_0)$, we obtain

$$
u^-(\bar{x}) = E^\delta [u^-(X_{r\theta}(x_0))]
$$

\begin{align}
&= \int_{(B_{r}(x_0))^c} u^{-}(y) P_{B_{r}(x_0)}(\bar{x}, y) \, dy \\
&= \int_{(B_{r}(x_0))^c} u^{-}(y) \left[ \int_{B_{r}(x_0)} f_r(y-z) \cdot G_{B_{r}(x_0)}(\bar{x}, z) \, dz \right] dy \\
&= \int_{B_{r}(x_0)} G_{B_{r}(x_0)}(\bar{x}, y) \left[ \int_{(B_{r}(x_0))^c} u^- (y) f_r(y-z) \, dz \right] dy \\
&= \int_{B_{r}(x_0)} G_{B_{r}(x_0)}(\bar{x}, y) f_r(y-v) \, dy.
\end{align}
Examining the inner integral in (2.13) more closely, it follows:

\[
\int_{(B_{\delta}(x_0))^c} u^-(y) f_\alpha(y - v) \, dy \leq \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - v) \, dy \\
\leq \sup_{z \in B_{\delta}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - z) \, dy.
\]

Therefore,

\[
(2.13) \leq \int_{B_{\delta}(x_0)} G_{B_{\delta}(x_0)}(\tilde{x}, v) \sup_{z \in B_{\delta}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - z) \, dy \, dv
\]

\[
= \int_{B_{\delta}(x_0)} G_{B_{\delta}(x_0)}(\tilde{x}, v) \, dv \cdot \sup_{z \in B_{\delta}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - z) \, dy
\]

\[
\leq c \cdot \sup_{z \in B_{\delta}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - z) \, dy.
\]

To conclude,

\[
(2.14) \quad u^-(\tilde{x}) \leq c \cdot \sup_{z \in B_{\delta}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - z) \, dy
\]

holds, for any \( \tilde{x} \in B_{\delta}(x_0) \). Therefore, (2.12) yields

\[
\int_{B_{\delta}(x_0)} u(x) \, dx - c u^-(\tilde{x}) \leq c \cdot \inf_{B_{\delta}(x_0)} u,
\]

from where we obtain

\[
\int_{B_{\delta}(x_0)} u(x) \, dx \leq c \cdot \inf_{B_{\delta}(x_0)} u + c u^-(\tilde{x}),
\]

which, together with (2.14), implies

\[
\int_{B_{\delta}(x_0)} u(x) \, dx \leq c \cdot \inf_{B_{\delta}(x_0)} u + c \cdot \sup_{z \in B_{\delta}(x_0)} \int_{\mathbb{R}^d} u^-(y) f_\alpha(y - z) \, dy,
\]

and hence the theorem. \( \square \)

3. Hölder Regularity Estimates

**Theorem 3.1.** Let \( x_0 \in \mathbb{R}^d, d \geq 2, r_0 > 0 \) and \( \theta > \lambda > 1, 2 \theta > \sigma > 1 \). Let \( X \) be a symmetric \( \alpha \)-stable Lévy process in \( \mathbb{R}^d \), with index of stability \( \alpha \in (0, 2) \) and the characteristic function of the form

\[
\mathbb{E}^0 e^{i\varphi X_t} = e^{-\Phi(u)}, \quad u \in \mathbb{R}^d, \ t \geq 0,
\]

where the characteristic exponent is given by

\[
\Phi(u) = \int_{S^{d-1}} |u \cdot \xi|^\alpha \mu(d\xi),
\]

and \( \mu \) is a spectral measure. Furthermore, let \( \mu \) be absolutely continuous with respect to the uniform measure \( \sigma \) on the sphere \( S^{d-1} \) and denote by \( f_\mu \) its density. Assume that there is a positive constant \( m \) such that

\[
0 \leq f_\mu(\xi) \leq m, \ \xi \in S^{d-1}.
\]
Then there exists $\beta \in (0, 1)$ such that for every bounded function $u$ on $\mathbb{R}^d$, which is harmonic in $B_r(x_0)$ with respect to $X$, $0 < r \leq r_0$, and non-negative on $B_r(x_0)$

$$|u(x) - u(y)| \leq c \|u\|_\infty \left(\frac{|x - y|}{\rho}\right)^{\beta},$$

for almost all $x, y \in B_\rho(x_0), \rho \in (0, r/2)$.

**Proof.** For $r > 0$, denote $B_r := B_r(x_0), x_0 \in \mathbb{R}^d$. Let $c_1$ be the constant from the Theorem 2.2, $r \in (0, r_0)$. Set

$$k := (4c_1)^{-1},$$

$$\beta := \log \left(\frac{2 - \kappa}{\log(\theta)}\right).$$

Furthermore, for $n \in \mathbb{N}$ set $M_0 := \|u\|_\infty, M_{-n} := M_0, m_0 := \inf_{\mathbb{R}^d} u, m_{-n} := m_0$. We construct an increasing sequence $(m_n)$ and a decreasing sequence $(M_n)$ such that for $n \in \mathbb{Z}$ the following holds:

$$m_n \leq u(z) \leq M_n, \quad \text{for almost all } z \in B_{r^\theta^{-n}},$$

$$M_n - m_n = K\theta^{-n\beta},$$

where $K = M_0 - m_0 \in [0, 2\|u\|_\infty]$.

In the case $|x - y| \geq \rho$, the theorem holds true. Let us examine the case

$$|x - y| < \rho.$$

Choose $j \in \mathbb{N}_0$ such that

$$\rho \cdot \theta^{-j-1} \leq |x - y| \leq \rho \theta^{-j}.$$

From (3.1) we obtain that for almost all $x, y \in B_\rho(x_0)$

$$\frac{|u(x) - u(y)|}{|x - y|^{\beta}} \leq \frac{\text{osc}_{B_{\rho \theta^{-j}(x)}}u}{|x - y|^{\beta}} \leq K \left(\frac{\theta^{-j}}{|x - y|}\right)^{\beta} \leq K \theta^{\beta} \rho^{-\beta},$$

which proves the theorem.

Assume that there is $k \in \mathbb{N}$ and that there are $m_n, M_n$ such that (3.1) holds for $n \leq k - 1$. We construct $m_k$ and $M_k$ such that (3.1) holds for $n = k$.

Define

$$v(z) = \left(u(z) - \frac{m_{k-1} + M_{k-1}}{2}\right) \frac{2\theta^{(k-1)\beta}}{K}, \quad z \in \mathbb{R}^d.$$

From here we have

$$|v(z)| \leq 1, \quad \text{for almost all } z \in B_{\rho \theta^{-k}}.$$

We show that Theorem 2.2 implies either

$$v(z) \leq 1 - \kappa, \quad z \in B_{\rho \theta^{-k}}$$

or

$$v(z) \geq -1 + \kappa, \quad z \in B_{\rho \theta^{-k}}.$$

Let us consider two cases:

$$|\{x \in B_{\rho \theta^{-k-1/4}} : v(x) \leq 0\}| \geq 1/2|B_{\rho \theta^{-k-1/4}}|,$$

or

$$|\{x \in B_{\rho \theta^{-k-1/4}} : v(x) > 0\}| \geq 1/2|B_{\rho \theta^{-k-1/4}}|.$$
Let $z \in \mathbb{R}^d$ be such that $|z - x_0| \geq r\theta^{-(k-1)}$. There exists $j \in \mathbb{N}$ with the property

$$r^{-(k-j)} \leq |z - x_0| \leq r^{-(k-j-1)}.$$  

Consequently,

$$\frac{K}{2\theta^{k-1} \beta} v(z) = u(z) - \frac{m_{k-1} + M_{k-1}}{2}$$

$$\leq M_{k-(j+1)} - m_{k-(j+1)} + m_{k-(j+1)} - \frac{m_{k-1} + M_{k-1}}{2}$$

$$\leq M_{k-(j+1)} - m_{k-(j+1)} + M_{k-1} - \frac{m_{k-1} + M_{k-1}}{2}$$

$$= M_{k-(j+1)} - m_{k-(j+1)} - \frac{M_{k-1} - m_{k-1}}{2}$$

$$\leq K\theta^{-(k-j-1)\beta} \leq \frac{K}{2}\theta^{-(k-1)\beta}$$

$$= K\theta^{-(k-1)\beta}(\theta^{\beta} - 1/2)$$

$$= \frac{K}{2}\theta^{(k-1)\beta}(2\theta^{\beta} - 1),$$

from where we obtain

$$v(z) \leq 2\theta^{\beta} - 1$$

$$\leq 2\left(\theta \cdot \frac{|z - x_0|}{r\theta^{-(k-1)}}\right)^\beta - 1.$$

Notice that the second inequality follows from (3.5). Similarly,

$$\frac{K}{2\theta^{k-1} \beta} v(z) = u(z) - \frac{m_{k-1} + M_{k-1}}{2}$$

$$\geq m_{k-(j+1)} - M_{k-(j+1)} + M_{k-(j+1)} - \frac{m_{k-1} + M_{k-1}}{2}$$

$$\geq m_{k-(j+1)} - M_{k-(j+1)} + M_{k-1} - \frac{m_{k-1} + M_{k-1}}{2}$$

$$= -(M_{k-(j+1)} - m_{k-(j+1)}) + \frac{M_{k-1} - m_{k-1}}{2}$$

$$\geq -K\theta^{-(k-j-1)\beta} + \frac{K}{2}\theta^{-(k-1)\beta}$$

$$= K\theta^{-(k-1)\beta}(1/2 - \theta^{\beta})$$

$$= \frac{K}{2}\theta^{(k-1)\beta}(1 - 2\theta^{\beta}),$$

from where we obtain

$$v(z) \geq 1 - 2\theta^{\beta}$$

$$\geq 1 - 2\left(\theta \cdot \frac{|z - x_0|}{r\theta^{-(k-1)}}\right)^\beta.$$

Notice that the second inequality follows from (3.5).
In the case (3.3), we wish to show $v(z) \leq 1 - \kappa$, for almost all $z \in B_{r^k}$ and some $\kappa \in (0, 1)$, since then

$$u(z) \leq \frac{K(1 - \kappa)}{2} \theta^{-(k-1)\beta} + \frac{m_{k-1} + M_{k-1}}{2} \leq \frac{K(1 - \kappa)}{2} \theta^{-(k-1)\beta} + \frac{m_{k-1} - m_{k-1}}{2} + m_{k-1} \leq m_{k-1} + \frac{K(1 - \kappa)}{2} \theta^{-(k-1)\beta} + \frac{K}{2} \theta^{-(k-1)\beta} \leq m_{k-1} + K \theta^{-k\beta}.$$ 

Then we set

$$m_k = m_{k-1}, \quad M_k = m_k + K \theta^{-k\beta},$$

and obtain that for almost all $z \in B_{r^k}$

$$m_k \leq u(z) \leq M_k.$$ 

Define $w = 1 - v$. The Theorem 2.2 yields

$$\int_{B_{r^k \setminus B_{r^k}}} w(x) \, dx \leq c_1 \inf_{B_{r^k}} w + c_1 \sup_{x \in B_{r^k \setminus B_{r^{k-j}}} \setminus B_{r^k \setminus B_{r^{k-j}}}} \int_{\mathbb{R}^d} w^{-}(z) f_s(x - z) \, dz \quad (3.6)$$

In the case (3.3), the left-hand side of (3.6) is bounded below by $1/2$.

Denote by $A_{rR} = B_R(x_0) \setminus B_r(x_0)$, $r \leq R$. Then,

$$\inf_{B_{r^k}} w \geq \frac{1}{2c_1} - \frac{1}{\lambda} \sup_{x \in B_{r^{k-j}}} \int_{\mathbb{R}^d} w^{-}(z) f_s(x - z) \, dz \geq \frac{1}{2c_1} - \sum_{j=1}^{\infty} \sup_{x \in B_{r^{k-j}}} \int_{\mathbb{R}^d} 1_{A_{r^{k-j}}} w^{-}(z) f_s(x - z) \, dz \geq \frac{1}{2c_1} - \sum_{j=1}^{\infty} \frac{2 \theta^{\beta j} - 2 \eta}{\lambda}.$$ 

where

$$\eta = \sup_{x \in B_{r^{k-j}}} \int_{A_{r^{k-j}}} f_s(x - z) \, dz.$$ 

Since

$$\eta \leq c \cdot \zeta^{-j-1}, \quad \zeta > 1, \quad c > 0,$$

it follows

$$\inf_{B_{r^k}} w \geq \frac{1}{2c_1} - \frac{c}{\lambda} \sum_{j=1}^{\infty} (\theta^{\beta j} - 1) \zeta^{-j-1}.$$ 

Note that for $\beta > 0$ small enough

$$\sum_{j=1}^{\infty} \theta^{\beta j} \zeta^{-j-1} < \infty,$$
which implies
\[
\sum_{j=l+1}^{\infty} (\theta^\beta - 1) \zeta^{-j-1} \leq \sum_{j=l+1}^{\infty} \theta^\beta \zeta^{-j-1} \leq 1/(16c_1),
\]
for some \( l \in \mathbb{N} \). Given \( l \in \mathbb{N} \), if needed, choose smaller \( \beta > 0 \) such that
\[
\sum_{j=1}^{l} (\theta^\beta - 1) \zeta^{-j-1} \leq 1/(16c_1).
\]
Therefore, \( w \geq \kappa \) on \( B_{r_0^\beta} \), which implies \( v \leq 1 - \kappa \) on \( B_{r_0^\beta} \). In the case (3.4), we aim to show \( v(x) \geq -1 + \kappa \). To this end, set \( w = 1 + v \). Following the previous strategy, one sets
\[
M_k = M_{k-1},
\]
\[
m_k = M_k - K \cdot \theta^{-k\beta},
\]
and the result follows. \( \square \)

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