The position of the quasielastic peak and electron Coulomb distortion in $(e, e')$ scattering

Andreas Aste
Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland

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Abstract
The position of the quasielastic peak for $(e, e')$ scattering off $^{208}$Pb extracted from a selected data set measured at Saclay is related to a heuristic theoretical description. An analysis of the data shows that the peak position can be described very accurately by a simple equation in the relevant kinematic region where a pronounced peak is observable. The simple findings result in a concluding comment related to recent calculations concerning the Coulomb distortion in $(e, e')$ scattering for heavy nuclei.

Keywords: Quasielastic electron scattering, Coulomb corrections

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1 Introduction

Inclusive \((e,e')\) scattering has several features which make it a very useful tool for investigating the properties of nuclei, since the interaction of the electron with nuclei is well understood in terms of the electromagnetic and the electroweak interaction. Inclusive scattering, where only the scattered electron is observed, provides information on a number of interesting nuclear properties like, e.g., the Fermi momentum [1], high-momentum components in nuclear wave functions [2], modifications of nuclear form factors in the nuclear medium [3], the scaling properties of the quasielastic response allow to study the reaction mechanism [4], and extrapolation of the quasielastic response to infinite nucleon number \(A = \infty\) provides us with a very valuable observable of infinite nuclear matter [5]. In this paper we focus on the position the quasielastic peak, which is clearly visible when the \((e,e')\) cross section is plotted versus the energy loss of the scattered electron for an energy range of the electrons in the region of some few hundred MeV. For the present analysis, data taken at Saclay [6] for \(^{208}\)Pb are studied. The presented study also sheds some light on the problem of Coulomb distortions in \((e,e')\) scattering off heavy nuclei, which has regained some recent interest.

2 Position of the quasielastic peak

From a simplified point of view, quasielastic \((e,e')\) scattering may be described as the scattering process of an electron off all the individual nucleons constituting the nucleus. Many theoretical calculations for inclusive scattering in connection with the problem of Coulomb distortions, which are presently available, rely on this simplified picture, and are correspondingly based on single particle shell model descriptions. The width of the peak is mainly due to the Fermi motion of the nucleons inside the nucleus, whereas the position of the peak can be approximately inferred from a simple classical consideration. One first assumes that the electron with initial and final momentum \(\vec{k}_i\) and \(\vec{k}_f\) knocks a nucleon at rest inside the fixed nucleon, transferring thereby an energy \(\omega\) to the nucleon, which is given by the difference of the initial and final electron energy \(\epsilon_i - \epsilon_f\), with \(\epsilon_{i,f} = |\vec{k}_{i,f}|\) for highly relativistic electrons in the present case. The heavy nucleus acts solely as a spectator within this simplified picture. The three-momentum transferred to the nucleon is \(\vec{q} = \vec{k}_i - \vec{k}_f\), such that from energy conservation follows

\[
\omega = (\vec{p}_f^2 + m_n^2)^{1/2} - m_n = (\vec{q}^2 + m_n^2)^{1/2} - m_n, \quad (1)
\]

where \(m_n\) is the (in-medium) mass of the nucleon and \(\vec{p}_f\) the momentum of the knocked nucleon. Squaring Eq. (1) one obtains

\[
Q^2 = \vec{q}^2 - \omega^2 = 2m_n \omega, \quad \text{or} \quad \omega = \frac{Q^2}{2m_n}. \quad (2)
\]

The four-momentum transfer squared \(Q^2\) itself is \(\omega\)-dependent. Therefore, if the electron scattering angle \(\vartheta\) is given, one may write in a more explicit form

\[
\omega = \frac{\epsilon_i^2(1 - \cos \vartheta)}{m_n + \epsilon_i(1 - \cos \vartheta)}. \quad (3)
\]

However, the energy of the nucleons that leave the nucleus is not given by the energy loss of the electron, but is reduced by a removal energy of the nucleon. An more ambitious expression for the position of the quasielastic peak, which can be often found in the literature, is

\[
\omega = \frac{Q^2}{2m_n} + \bar{E}_{\text{rem}}, \quad (4)
\]

where \(\bar{E}_{\text{rem}}\) is an average nucleon removal energy. A slightly modified expression for the position of the quasielastic peak follows

\[
\omega = \frac{\epsilon_i^2(1 - \cos \vartheta) + m_n \bar{E}_{\text{rem}}}{m_n + \epsilon_i(1 - \cos \vartheta)}. \quad (5)
\]

Finally, one may try to improve expression Eq. (5) further by replacing the initial electron energy by an effective value \(\epsilon'_i = \epsilon_i + 20\,\text{MeV}\), since the average enhancement of the kinetic energy of the electron inside the nuclear region due to the strong attractive Coulomb potential is given by \(\sim 20\,\text{MeV}\) in the case of lead, such that we arrive at a simple heuristic equation for the theoretical position of the quasielastic peak

\[
\omega^{th} = \frac{\epsilon'_i^2(1 - \cos \vartheta) + m_n \bar{E}_{\text{rem}}}{m_n + \epsilon'_i(1 - \cos \vartheta)}. \quad (6)
\]

A typical example for a quasielastic peak is shown in Fig. 1, where Saclay data for \(^{208}\)Pb [6] are displayed for initial electron energy \(\epsilon_i = 485\,\text{MeV}\) and
scattering angle \( \theta = 60^\circ \). For this scattering angle, a quasielastic peak is visible for initial electron energy \( \epsilon_i = 262, 310, 354, 420, 485, 550, 600, \) and 645 MeV, whereas for a scattering angle of 143\(^\circ\), the experimental data display a peak for \( \epsilon_i = 140, 206, 262, 310, 354, \) and 420 MeV. The position of each peak and the corresponding inaccuracy of the peak position was extracted from the data by the help of polynomial least squares fits with varying order. However, if one plots the theoretical position Eq. (6) of the corresponding peaks against the measured positions with a typical nucleon mass of \( m_n = 939 \) MeV, one finds by a least squares fit that the position of the peaks is described much more accurately by

\[
\omega^{th} = \frac{\sigma \epsilon_i^2 (1 - \cos \theta) + \tilde{m}_n \tilde{E}}{m_n + \epsilon_i^2 (1 - \cos \theta)}
\]

with \( \tilde{m}_n \) and \( \tilde{E} \) as fitting parameters is shown in Figs. 4 and 5. The experimental data are reproduced in a very satisfactory manner by Eq. (8) for \( \theta = 60^\circ \) with \( \tilde{m}_n = 721 \) MeV and \( \tilde{E} = 12.8 \) MeV and for \( \theta = 143^\circ \) with \( \tilde{m}_n = 662 \) MeV and \( \tilde{E} = 10.0 \) MeV.

There is no need to interpret the effective mass parameter \( \tilde{m}_n \) within a physical picture. It simply incorporates in an efficient way the complex interaction processes that are taking place inside the nucleus, i.e. the interaction of the knocked effective nucleon with the other nucleons inside the nucleus or even effects like pion production which becomes important at higher energy transfer. In this sense, the mass parameter \( \tilde{m}_n \) should not be related directly to the effective (momentum transfer dependent) nucleon mass as it is investigated in [6], although it is clear that the size of \( \tilde{m}_n \sim 0.7 \ldots 0.8 m_n \) is quite typical. Naturally, the above description breaks down at high energy transfer, when the quasielastic peak is no longer visible in the \((e, e')\) cross section data. One may mention [7] as another early work which
addressed the position of the quasielastic peak in a more theoretical framework.

3 Coulomb corrections

This section comments on calculations of the Ohio group [9] and the Basel group [10], which dealt with the problem of Coulomb distortion in the case of the highly charged 208Pb nucleus.

We first comment on Fig. 4 in [9]. The left part of the figure displays theoretical calculations for (e, e’) scattering cross sections with \( \epsilon_i = 310 \text{ MeV} \) and \( \vartheta = 143^\circ \). The peak of the plane wave Born approximation (PWBA) curve is located at \( \omega = 140 \text{ MeV} \), whereas the peak of the effective momentum approximation (EMA) is located at \( \omega = 163 \text{ MeV} \). However, the EMA as it is applied in [9] is a plane wave calculation with the initial electron energy of 310 MeV replaced by an effective electron energy of 310 MeV – \( V(0) = -25 \text{ MeV} \), combined with a renormalization of the cross section by a constant focusing factor depending on the initial electron energy only, such that the position of the peak is not influenced by this factor. Furthermore, the energy transfer \( \omega \) is left unchanged in the formal calculations, such that, e.g., \( Q^2 = q^2 - \omega^2 = 2\epsilon_i \epsilon_f (1 - \cos \vartheta) \) is replaced by \( Q_{eff}^2 = 2(\epsilon_i + 25 \text{ MeV})(\epsilon_i - \omega + 25 \text{ MeV})(1 - \cos \vartheta) \).

The focusing factor for the final state electron wave function is not neglected in this approach, but already absorbed in the enhanced phase space factor of the final state electron. \( V(0) = -25 \text{ MeV} \) is the electrostatic potential energy of the electron in the center of the nucleus. There exist different descriptions of how to apply the EMA, which are all, of course, equivalent (see, e.g., also the introduction in [11] for an alternative description). We will argue below that the EMA is in fact a valuable tool to calculate Coulomb effects in (e, e’) scattering, however, an average potential value \( \bar{V} \simeq -19 \text{ MeV} \) should be used instead of the \textit{ad hoc} value \( V(0) \).

The essence of the EMA is to account for two effects of the attractive Coulomb field of the nucleus. First, the electron wave function is focused towards the nuclear region, and second, the electron momentum is enhanced due to the attractive Coulomb force. Both the focusing and the momentum of the electron vary inside the nucleus, but the average effect can be described by the average (or effective) Coulomb potential of the nucleus which is given by \( \bar{V} \simeq -19 \text{ MeV} \) for 208Pb. It has already been observed by Rosenfelder [12], that for high-energy electrons, the distorted electron wave can be approximated by (\( \psi_0 \) is the constant spin-dependent Dirac spinor)

\[
\psi_{\vec{k}}(\vec{r}) = \frac{|\vec{k}_{eff}|}{|k|} \psi_0 e^{i\vec{k}_{eff}\vec{r}},
\]
with $|\vec{k}_{eff}| = |\vec{k}| - \bar{V}$ for both the initial and final electron momentum. It should be noted here that Rosenfelder explicitly mentioned already in [12] that $\bar{V}$ is close to the mean value of the electrostatic potential of the nucleus. Unfortunately, he then wrote down the explicit expression for the central value

$$V(0) = \frac{3Z\alpha}{2R}$$

of the potential of a homogeneously charged sphere with radius $R$, which is related to the corresponding mean value by $\bar{V} = 4V(0)/5$. This minor, but not irrelevant misapprehension for the case of the EMA, has propagated in the literature since then (see, e.g., [7]).

With a little abuse of notation, we will disregard the negative sign of $V(0)$ and $\bar{V}$ in the following. According to the considerations above in connection with Eq. (8), the PWBA peak should be located approximately at $(\bar{m}_n = 662\text{ MeV}, \bar{E} = 10\text{ MeV})$

$$\omega_{PWBA} \simeq \frac{310\text{ MeV}^2(1 - \cos 143^\circ) + \bar{m}_n \bar{E}}{\bar{m}_n + 310\text{ MeV}(1 - \cos 143^\circ)}$$

$$\simeq 147.2\text{ MeV},$$

and the EMA peak at

$$\omega_{EMA} \simeq \frac{335\text{ MeV}^2(1 - \cos 143^\circ) + \bar{m}_n \bar{E}}{\bar{m}_n + 335\text{ MeV}(1 - \cos 143^\circ)}$$

$$\simeq 164.9\text{ MeV}.\quad (12)$$

The theoretical (model dependent) and experimental peak positions do not have to coincide extremely well, still one finds that they agree in a satisfactory way. On the other hand, the distance $\omega_{EMA} - \omega_{PWBA}$ between the peaks is a robust quantity, since it depends only on a comparably small change of the momentum transfer $Q^2$ into $Q_{eff}^2$, and the behavior of the peak position as a function of the momentum transfer shows a universal behavior and is obviously well under control. Therefore, one should expect a peak shift of $\Delta \omega = 164.9 - 147.2 = 17.7\text{ MeV}$. However, the peak shift of 23 MeV displayed in [9] would rather correspond to an effective potential of 32.5 MeV. If a free nucleon mass of 939 MeV was used in the theoretical calculations in conjunction with adapted removal energies or nuclear potentials in order to obtain the correct position of the peaks, the situation is worse, since the expected peak shift $\simeq Q_{eff}^2/2m_n - Q^2/2m_n$ would then correspond to $V(0) \simeq 37\text{ MeV}$. Interestingly, the distance between the peaks of the Coulomb corrected DWBA curve, where exact electron wave functions have been used for the calculation, and the PWBA curve, is given by $16.5\text{ MeV}$, corresponding to an effective energy shift of $\sim 23\text{ MeV}$, or, if vacuum nucleon masses were used, to $V(0) \sim 18\text{ MeV}$. The definition of the nucleon current in [9] indeed suggests that the free nucleon mass has been used. The figure displays an EMA curve with a peak shift against the PWBA curve which is obviously too large, but the shift of the DWBA curve is compatible with the EMA. But there remains a problem with the normalization of the theoretical data.

The situation is harder to interpret for the left plot with theoretical results for $\epsilon_r = 485\text{ MeV}$ and $\vartheta = 60^\circ$. Magnifying the plot one finds that the PWBA and the DWBA peaks are separated only by approximately 5 MeV, compatible with an EMA calculation using an electrostatic potential value of $V(0) \simeq 15\text{ MeV}$, if a nucleon mass of 939 MeV is used (otherwise, the situation is worse). The separation of the PWBA and the EMA peak is again too large and corresponds to an EMA calculation with $V(0) \simeq 30\text{ MeV}$ (for $\bar{m}_n = 721\text{ MeV}$) or even $V(0) \simeq 39\text{ MeV}$ (for $m_n = 939\text{ MeV}$).

Again, the figure supports rather an EMA type behavior of the DWBA results contrary to the initial intention of [9], but there is again a problem with
the normalization. The observations above have lead to the attempt to describe the theoretical DWBA results presented in [9] by an improved EMA, where the normalization of EMA results was modified in a suitable manner [13].

We finally comment on the eikonal calculations (EDWBA) presented in [10], which originally seemed to be compatible with the results in [9]. In this paper, Coulomb corrected cross sections were calculated based on the eikonal approximation, where the distortion of the electron wave functions is described by the help of an eikonal integral, which is easily accessible by numerical calculations. It should be pointed out that the paper contains a pedagogical introduction, which may mislead to the assumption that the electron current and matrix elements (like Eq. (40) in [10]) were calculated within a simplified Klein-Gordon model, however, the full spinor formalism was applied to electrons within the eikonal framework [14]. The PWBA and EDWBA curves in Fig. 3 of [10] show a indeed similar behavior as the PWBA and DWBA results in [9]. As in [9],

calculations in [10], a focusing value for the electron cross section was used which corresponds basically to the central value of the electrostatic potential, i.e. \( V(0) = 25 \text{MeV} \). But exact solutions of the Dirac equation for electrons in the electrostatic potential of a \(^{208}\text{Pb}\) nucleus reveal that the average focusing inside the nucleus is lower than the central value, and can be calculated reliably from an effective potential value of approximately \( 19 - 20 \text{MeV} \) [15]. This lead to an overestimation of the Coulomb corrected cross sections. Note that the average momentum of the electrons is well described by the eikonal integral and corresponds also to an average value of approximately 19 MeV.

Therefore, the results presented in [10] have to be corrected in two steps, with the result illustrated below for Fig. 4 in [10], which is displayed again as Fig. 7 in this paper. The figure shows the ratio of the cross sections, calculated in PWBA, with the Coulomb corrected cross sections according to the EDWBA and EMA. First, the EMA curve has to be recalculated for an effective potential of 19 MeV instead of 25 MeV. This slightly reduces the ratio \( \sigma_{\text{PWBA}}/\sigma_{\text{EMA}} \) and moves the corresponding dotted curve closer to one (the horizontal line). Second, the focusing factors of the EDWBA calculations must be corrected. As an example, for \( \epsilon_i = 485 \text{MeV} \) and \( \epsilon_f = 385 \text{MeV} \), the original focusing factor was given by

\[
f = \frac{(485 + 25)^2 \times (385 + 25)^2}{485^2 \times 385^2} \approx 1.254. \tag{13}
\]

The correct focusing factor should rather be

\[
f_{\text{corr}} = \frac{(485 + 19)^2 \times (385 + 19)^2}{485^2 \times 385^2} \approx 1.189. \tag{14}
\]

Accordingly, the EDWBA cross section has to be reduced by 5.2% at \( \omega = 100 \text{MeV} \) and the corresponding solid curve moves upwards in the plot. The result is shown in Fig. 8. Note that for Fig. 8 also locally varying focusing factors obtained from exact solutions of the Dirac equation were used in conjunction with the eikonal approximation for the phase of the electron wave functions. An attempt to calculate corrections to the focusing near the nuclear center has already been presented in [16], which, however, does not lead to reliable predictions in the important surface region of the nucleus. The results for a

![Figure 6: Cross sections for \((e,e')\) scattering with \(\epsilon_i = 310 \text{MeV} \) and \(\vartheta = 143^\circ\) obtained in [10]. The plots displays a similar behavior as the results presented in [9], however, the Coulomb corrected eikonal cross sections are too large due an overestimation of the electron wave function focusing.](image-url)
calculation using a locally varying focusing or a calculation using a corresponding, but constant mean focusing, do not differ significantly. Therefore, the calculational examples Eqs. (13) and (14) contain the main essence of the present consideration, i.e., that the focusing used in [10] corresponds to an effective potential of 25 MeV which is too large. For the EMA calculation presented in Fig. 8, a slightly smaller value $\bar{V} = 18.7\,\text{MeV}$ than in the calculational example above was used, since this value has been determined experimentally to be $\bar{V} = 18.7 \pm 1.5\,\text{MeV}$ for $^{208}\text{Pb}$ [11].

One observes that the EMA and the EDWBA agree very well, if the correct effective potential and focusing factors are used. This result was also obtained in [15], where exact solutions of the Dirac equation were used in conjunction with a simplified model for the nuclear current. Detailed calculations with realistic nuclear current will be presented in a forthcoming paper in the near future. Note that a small discrepancy of the order of $1-2\%$ as in Fig. 8 between the EMA and the EDWBA still leaves the possibility for an improved EMA as proposed in [13], however, such an attempt to make the EMA perfect must be based on full DWBA calculations. Detailed DWBA calculations will also be necessary in order to find optimal EMA values for $\bar{V}$ and the related effective kinematic variables for different kinematic settings.

One has to conclude that using exact Dirac wave functions or an eikonal approximation with correct normalization of the wave function in the nuclear region confirm the usefulness of the EMA, if an appropriate effective potential is used in the kinematic regions considered in this work. Additionally, the energy of the final state electron should be larger than 200 MeV and the momentum transfer $Q^2$ larger than $(300\,\text{MeV})^2$ [15], such that the length scale of the electron wave functions and of the exchanged virtual photon is sufficiently small compared to the size of the nucleus and a semiclassical behavior sets in. It is a pleasant fact that the focusing factors of the electron wave functions and the modification of the momentum transfer due to the attractive Coulomb potential can be calculated from the same effective potential $\bar{V}$. It must be pointed out that this fact is to some extent accidental, since it is only typical for charge distributions which are close to the charge distribution of a homogeneously charged sphere. In a more ambitious approach to the EMA, one would use two different $k_{\text{eff}}$’s in Eq. (11).

![Figure 7: Comparison of Coulomb corrections ($\epsilon_i = 485\,\text{MeV}, \vartheta = 60^\circ$) for different approaches in [10]. For the EMA, an effective potential value of 25 MeV was used.](image)

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Figure 8: The same as in Fig. 7 but with corrected focusing in the eikonal calculation and an EMA curve obtained by using an average potential $\bar{V} = 18.7$ MeV.

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