UNIVERSAL VALUED ABELIAN GROUPS

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Abstract. The counterparts of the Urysohn universal space in category of metric spaces and the Gurari˘ı space in category of Banach spaces are constructed for separable valued Abelian groups of fixed (finite) exponents (and for valued groups of similar type) and their uniqueness is established. Geometry of these groups, denoted by $G_r(N)$, is investigated and it is shown that each of $G_r(N)$’s is homeomorphic to the Hilbert space $l^2$. Those of $G_r(N)$’s which are Urysohn as metric spaces are recognized. ‘Linear-like’ structures on $G_r(N)$ are studied and it is proved that every separable metrizable topological vector space may be enlarged to $G_r(0)$ with a ‘linear-like’ structure which extends the linear structure of the given space.

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1. Introduction

In several branches of mathematics, such as metric space theory or functional analysis, there are known examples of the so-called universal spaces. Probably the most famous example in this subject is the Banach space $C([0,1])$ of all real-valued continuous functions on $[0,1]$ which is universal for separable normed vector spaces (over $\mathbb{R}$) as well as for separable metric spaces. Here universality means that every separable normed vector space (respectively separable metric space) is isometrically-linearly isomorphic (respectively isometric) to a linear subspace (to a subset) of $C([0,1])$. This is known as the Banach-Mazur theorem. Undoubtedly, the ‘concreteness’ of the universal space in this theorem merits its fame. However, it is no so easy to characterize $C([0,1])$ among Banach (or metric) spaces. There is also a general idea, in the spirit of Fraissé limits, of constructing universal spaces which are simultaneously uniquely determined by certain conditions related to the so-called universal disposition property (see below; with this terminology we follow e.g. Wieslaw Kubis). Surely, the first example of a universal space of this kind was given by Urysohn [24, 25]. The Urysohn universal metric space, usually denoted by $U$, is a unique (up to isometry) separable complete metric space which has universal disposition property for finite metric spaces. That is:
Every isometric map of a subset of a finite metric space $X$ into $U$ is extendable to an isometric map of the whole space $X$ into $U$.

There is also a bounded version of $U$. It may be defined in the following way. A metric space $X$ is said to be *Urysohn* iff

(U0) $X$ is separable and complete,
(U1) (universality) every separable metric space of diameter no greater than the diameter of $X$ is isometrically embeddable into $X$,
(U2) ($\omega$-homogeneity) every isometry between two arbitrary finite subsets of $X$ is extendable to an isometry of $X$ onto $X$.

A fundamental result on Urysohn spaces states that for every $r \in [0, \infty]$ there is a unique (up to isometry) Urysohn metric space (denoted by $U_r$) of diameter $r$. It is also easily shown that $U_r$ is uniquely determined (among separable complete metric spaces of diameter no greater than $r$) by universal disposition property for finite metric spaces of diameter no greater than $r$.

If we pass from the category of metric spaces to Banach spaces, universal disposition property may be defined as follows (cf. [8]). A Banach space $E$ is said to have universal disposition property (for finite dimensional Banach spaces) iff every isometric linear map of a linear subspace of a finite dimensional Banach space $F$ into $E$ is extendable to an isometric linear map of $F$ into $E$. Gurarii [8] has proved that there is no separable Banach space with universal disposition property. In the same paper he has constructed a separable Banach space $G$, which is nowadays called the Gurarii space, with almost universal disposition property defined in the following way (the same space was also built by Lazar and Lindenstrauss [11] in a different context):

Every isometric linear map $\psi: E \to G$ of a linear subspace $E$ of a finite dimensional Banach space $F$ admits a linear extension $\hat{\psi}: F \to G$ such that $(1 - \varepsilon)\|x\| \leq \|\hat{\psi}(x)\| \leq (1 + \varepsilon)\|x\|$ for all $x \in F$ where $\varepsilon$ is an arbitrarily given real number from $(0, 1)$.

Gurarii has also shown that $G$ is universal for separable Banach spaces (i.e. that every separable Banach space admits an isometric linear embedding into $G$). It was Lusky [12] who first proved the uniqueness of $G$ up to isometric linear isomorphism (other proof of the uniqueness is contained in [13]; the proof which involves the back-and-forth technique was given by Solecki [20]). Thus, the Gurarii space is a unique separable Banach space with almost universal disposition property.

There is a striking resemblance between (almost) universal disposition properties of $U$ and $G$. In a sense, both these spaces correspond to each other in categories of metric spaces and Banach spaces. The aim of this paper is to prove the existence (together with uniqueness)
of a universal (in the sense of embedding by isometric group homomorphisms) group with universal disposition property (for finite groups) in the class (and some of their subclasses) of all separable valued Abelian groups of class $O_0$, defined by the following condition:

A valued Abelian group $(G, +, p)$ is said to be of class $O_0$ iff $\lim_{n \to \infty} \frac{p(x^n)}{n} = 0$ for every $a \in G$.

(In particular, if $p$ is a value on an Abelian group $(G, +)$, then each of the valued groups $(G, +, \min(p, 1))$, $(G, +, \frac{p}{p+1})$ and $(G, +, p^\alpha)$ for $0 < \alpha < 1$ is of class $O_0$.) What we exactly mean is explained below.

Denote by $\mathfrak{G}$ the class of all separable valued Abelian groups. Let $\mathfrak{G}_1(0)$ and $\mathfrak{G}_1(1)$ stand for the classes of all groups $(G, +, +, p) \in \mathfrak{G}$ of class $O_0$ and, respectively, for which $p \leq 1$. Note that $\mathfrak{G}_1(0) \subset \mathfrak{G}_1(1)$. Additionally, for natural $N > 1$ let $\mathfrak{G}_N(1) \subset \mathfrak{G}_N(0)$.

Finally, put $\mathfrak{G}_1(N) = \mathfrak{G}_1(0) \cap \mathfrak{G}_N(1)$.

We say a function $\omega: [0, \infty) \to [0, \infty)$ is a modulus of continuity iff

$(\omega 1)$ $\omega$ is monotone increasing, that is, $\omega(x) \leq \omega(y)$ provided $0 \leq x \leq y$,

$(\omega 2)$ $\omega(x + y) \leq \omega(x) + \omega(y)$ for any $x, y \geq 0$,

$(\omega 3)$ $\lim_{t \to 0^+} \omega(t) = \omega(0) = 0$.

(Observe that we allow the zero function to be a modulus of continuity.)

The main two results of the paper are:

1.1. Theorem. Let $r \in \{1, \infty\}$ and $N \in \{0, 2, 3, 4, \ldots\}$. There is a unique (up to isometric group isomorphism) valued Abelian group, denoted by $\mathbb{G}_r(N)$, with the following three properties:

(G1) $\mathbb{G}_r(N)$ is complete and $\mathbb{G}_r(N) \in \mathfrak{G}_r(N)$,

(G2) whenever $(H, +, q)$ is a finite Abelian group (of exponent $N$, provided $N \neq 0$) with $q \leq r$, $K$ is a subgroup of $H$ and $\varphi: K \to \mathbb{G}_r(N)$ is an isometric group homomorphism, then for every $\varepsilon \in (0, 1)$ there is a group homomorphism $\varphi_\varepsilon: H \to \mathbb{G}_r(N)$ such that

$$\max_{x \in K} p(\varphi(x) - \varphi_\varepsilon(x)) \leq \varepsilon$$

and

$$(1-\varepsilon)q(y) \leq p(\varphi_\varepsilon(y)) \leq (1+\varepsilon)q(y)$$

for $y \in H$, where $p$ is the value of $\mathbb{G}_r(N)$,

(G3) if $N = 0$, finite rank elements of $\mathbb{G}_r(N)$ form a dense subset of $\mathbb{G}_r(N)$.

1.2. Theorem. Let $r \in \{1, \infty\}$ and $N \in \{0, 2, 3, 4, \ldots\}$ and let $p$ stand for the value of $\mathbb{G}_r(N)$.

(A) Let $(H, +, q) \in \mathfrak{G}_r(N)$, $K$ be a closed subgroup of $H$ and $\varphi: K \to \mathbb{G}_r(N)$ be a continuous group homomorphism whose range has compact closure in $\mathbb{G}_r(N)$. 
There are moduli of continuity $\omega \not\equiv 0$, $\varrho \not\equiv 0$ and $\tau$ such that for each $x \in K$,

\begin{equation}
 p(\varphi(x)) \leq (\omega \circ q)(x),
\end{equation}

\begin{equation}
 \tau(\text{dist}_q(x, \ker \varphi)) \leq (\varrho \circ p)(\varphi(x))
\end{equation}

(\text{where } \text{dist}_q(x, \ker \varphi) = \inf\{q(x - y) : y \in \ker \varphi\}) \text{ and }

\begin{equation}
 (\omega - \varrho - \tau)
\end{equation}

\begin{align*}
 \tau(t) + \varrho(s) & \leq \varrho(\omega(t) + s) \text{ for any } t, s \geq 0, \\
 \tau(1) & \leq \varrho(1) \text{ provided } r = 1.
\end{align*}

If $\varphi$ is open as a map of $K$ onto $\varphi(K)$, the above $\tau$ may be chosen nonzero.

For every modulus of continuity $\omega \not\equiv 0$ such that (1-2) is fulfilled for all $x \in K$ there is a continuous group homomorphism $\varphi_\omega : H \to \mathbb{G}_r(N)$ extending $\varphi$ and satisfying (1-2) for each $x \in H$ with $\varphi$ replaced by $\varphi_\omega$, and such that $\ker \varphi_\omega = \ker \varphi$ and whenever (1-3) and (\omega-\varrho-\tau) are fulfilled for $\tau$ and $\varrho \not\equiv 0$ (and $x \in K$), then (1-3) is satisfied for any $x \in H$ with $\varphi$ replaced by $\varphi_\omega$.

In particular, if $\varphi$ is isometric, it admits an extension being an isometric group homomorphism.

Every (topological) isomorphism between two compact subgroups of $\mathbb{G}_r(N)$ is extendable to an automorphism of the topological group $\mathbb{G}_r(N)$. What is more, if $K$ is a compact subgroup of $\mathbb{G}_r(N)$, $\varphi : K \to \mathbb{G}_r(N)$ is a group homomorphism, $\omega \not\equiv 0$ and $\tau \not\equiv 0$ are moduli of continuity such that $(\omega - \text{id})$ and $(\tau - \omega - \text{id})$ are fulfilled (where id denotes the identity map on $[0, \infty)$) and for each $x \in K$,

\begin{equation}
 p(\varphi(x)) \leq (\omega \circ p)(x) \text{ and } p(x) \leq (\tau \circ p)(\varphi(x)),
\end{equation}

then there is a group automorphism $\psi : \mathbb{G}_r(N) \to \mathbb{G}_r(N)$ extending $\varphi$ such that (1-4) is satisfied for each $x \in \mathbb{G}_r(N)$ with $\varphi$ replaced by $\psi$.

In particular, if $\varphi$ is isometric, it admits an extension being an isometric automorphism.

Note that the point (G2) (in the statement of Theorem 1.1) is a counterpart of the condition, proposed by Solecki [20], which characterizes the above mentioned Gurarii space up to isometric linear isomorphism. Observe also that (G3) is quite natural, since (G2) refers only to finite rank elements of the group.

Theorem 1.2 implies that every member of $\mathbb{G}_r(N)$ admits an embedding to $\mathbb{G}_r(N)$ by an isometric group homomorphism and that in condition (G2) one may substitute $\varepsilon = 0$, that is, $\mathbb{G}_r(N)$ has universal disposition property for finite groups of class $\mathbb{G}_r(N)$.

We shall also show that the group $\mathbb{G}_r(N)$ as a metric space is universal for separable metric spaces of diameter no greater than $r$, that
is, that every such space is isometrically embeddable into the metric space \( \mathbb{G}_r(N) \). It turns out that \( \mathbb{G}_r(N) \) (as a metric space) is Urysohn iff \( N \in \{0,2\} \). In particular, the groups \( \mathbb{G}_r(2) \) are Boolean Urysohn groups introduced by us in [17]. What is more, for different pairs \((N,r)\) and \((M,s)\) the metric spaces \( \mathbb{G}_r(N) \) and \( \mathbb{G}_s(M) \) are isometric iff \( r = s \) and \( \{N,M\} = \{0,2\} \). Also all the groups \( \mathbb{G}_r(N) \) are pairwise nonisomorphic as topological groups.

In Section 8 we prove that each of the topological spaces \( \mathbb{G}_r(N) \) is homeomorphic to the Hilbert space \( l^2 \). To establish this result we develop our earlier study of the so-called topological pseudovector groups, introduced in [18]. Namely, a pseudovector Abelian group is a triple \((G,+,\ast)\) such that \((G,+)\) is an Abelian group, \(\ast\) is an action of \([0,\infty)\) on \(G\) satisfying the following axioms: \(0 \ast x = 0_G, 1 \ast x = x, (st) \ast x = s \ast (t \ast x)\) for all \(x \in G\) and \(s,t \geq 0\), and for every \(t \geq 0\) the function \(G \ni x \mapsto t \ast x \in G\) is a group homomorphism. If, in addition, \(G\) is a topological group, it is said to be a topological pseudovector group provided the action \(\ast\), as a function of \([0,\infty) \times G\) into \(G\), is continuous. A value \(p\) on the pseudovector Abelian group \(G\) is called a norm if \(p(t \ast x) = tp(x)\) for any \(t \geq 0\) and \(x \in G\), and it is called a subnorm iff \(p(x) \leq p(s \ast x) \leq sp(x)\) for each \(s \geq 1\) and \(x \in G\). If a (sub)norm \(p\) induces the topology on \(G\) with respect to which the action \(\ast\) is continuous, then the quadruple \((G,+,\ast,p)\) is called a (sub)normed topological pseudovector group. In Section 7 we show that every metrizable topological pseudovector group admits a subnorm inducing its topology. The main result on pseudovector structures of \(\mathbb{G}_r(N)\) is the following

1.3. **Theorem.** Let \(r \in \{1,\infty\} \) and \(N \in \{0,2,3,4,\ldots\}\). Every nontrivial (sub)normed topological pseudovector group \((G,+,\ast,\|\cdot\|_G)\) such that \((G,+,\|\cdot\|_G) \in \mathbb{G}_r(N)\) may be enlarged to a (sub)normed pseudovector topological group \((\tilde{G},+,\ast,\|\cdot\|)\) such that the valued groups \(G\) and \(\mathbb{G}_r(N)\) are isometrically group isomorphic.

In the above result one may erase the word ‘nontrivial’ provided that \(r = \infty\) or the final value \(\|\cdot\|\) has to be a subnorm rather than a norm. As a main application of the above result we obtain the above mentioned theorem which says that \(\mathbb{G}_r(N)\) is homeomorphic to \(l^2\). (This result may be immediately deduced for \(N = 0,2\) from the fact that the metric spaces \(\mathbb{G}_r(N)\) with \(N = 0,2\) are Urysohn and from Uspenskij’s theorem [28] on the topology of the Urysohn space.)

The proof of the existence of the groups \(\mathbb{G}_r(N)\) is based on the general technique of Fraïssé limits. First we construct a countable valued Abelian group \(\mathbb{Q}\mathbb{G}_r(N)\), a counterpart of the so-called rational Urysohn metric space, and then we prove that its completion is the group we search for. Although this approach is just an adaptation of the original construction of the Urysohn space \(\mathbb{U}\) described in [24, 25], the details
are more complicated, mainly because finite groups admit no one-point extensions, in the opposite to finite metric spaces. The unusual property of compact subgroups of $\mathbb{G}_r(N)$, formulated in Theorem 1.2, corresponds to Huhumävili’s theorem [9] for compact subsets of $\mathbb{U}$ (which says that every isometry between two compact subsets of $\mathbb{U}$ is extendable to an isometry of $\mathbb{U}$ onto itself). It is not so strong for $N \neq 0$, since every compact metric Abelian group of finite exponent is totally disconnected. In contrast, $\mathbb{G}_r(0)$ contains a copy of every metrizable compact Abelian group, among which one may find groups which are universal (in the sense of topological embedding), as topological spaces, for separable metrizable topological spaces, such as the countable infinite Cartesian power of $\mathbb{R}/\mathbb{Z}$.

There are known examples of the so-called universal Polish groups, that is, of completely metrizable separable (non-Abelian) topological groups $G$ such that every Polish group is isomorphic (as a topological group) to a closed subgroup of $G$. For example, Uspenskij have shown that the homeomorphism group of the Hilbert cube [26] as well as the isometry group of the Urysohn universal metric space [27] are (nonisomorphic) universal Polish groups (cf. Remark just after Theorem 5.2 of [15]). In the opposite to this, the author knows no example of a universal Polish Abelian group (i.e. a Polish Abelian group which is universal for Polish Abelian groups). In this paper we give two such examples: $\mathbb{G}_1(0)$ and $\mathbb{G}_\infty(0)$. (Both these valued groups are of course ‘metrically’ universal for separable valued Abelian groups with values bounded by 1.)

The part on pseudovector structures extends our earlier (introductory) work [18] in this subject. The proofs presented here are quite new and much more general. Theorem 1.3 generalizes and strengthens Theorem 4.3 of [18]. Since every norm on a nontrivial pseudovector group is unbounded, to equip the groups $\mathbb{G}_1(N)$ with ‘normed-like’ pseudovector structures we have to extend the notion of a norm to a subnorm, which is done in the recent paper. It seems to be interesting whether one may distinguish a special subnormed topological pseudovector structure on $\mathbb{G}_r(N)$ which will make this group universal for subnormed topological pseudovector groups of ‘$\mathfrak{G}_r(N)$-like’ class (this cannot be inferred from Theorem 1.3). The one idea is to define a counterpart of the Gurarii space for pseudovector groups of suitable class (this is discussed in details in Section 9). We do not know yet whether such a pseudovector group, denoted by $\mathbb{PVG}_r(N)$, exists and we leave this as an open problem. However, we prove that if it only exists, it has to be isometrically group isomorphic to $\mathbb{G}_r(N)$ and that $\mathbb{PVG}_r(N)$ as a PV group is unique up to isometric linear isomorphism. What is more, every subnormed topological PV group of suitable class (to which $\mathbb{PVG}_r(N)$ belongs) is embeddable into $\mathbb{PVG}_r(N)$ by means of an isometric linear homomorphism.
For well understanding of this exposition it is enough to know basic facts on metrizable and valued Abelian groups, e.g. the material of Chapter 1 of [2]. The reader interested in Urysohn universal space is referred to a survey article on the subject [15] or to [5], [10], [24, 25].

**Notation and terminology.** In this paper $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ stand for the sets of all real, all rational and, respectively, all integer numbers. The symbol ‘$\text{id}$’ is reserved to denote the identity map. All groups are Abelian and we use the additive notation. For simplicity, the action of any group is always denoted by (the same sign) ‘$+$’ and its neutral element is denoted by $0$. If $G$ is a group, $a \in G$ and $k \in \mathbb{Z}$, the value at $k$ of a unique group homomorphism of $\mathbb{Z}$ into $G$ which sends $1$ to $a$ is denoted by $ka$ or $k \cdot a$. The group $G$ is said to be of exponent $N$ (where $N \in \mathbb{Z}$, $N \geq 2$) iff $N \cdot x = 0$ for any $x \in G$. The subgroup of $G$ consisting of all finite rank elements of $G$ is denoted by $G_{\text{fin}}$.

For $s, t \in [0, \infty]$ let $s \wedge t$ and $s \vee t$ stand for the minimum and the maximum of $s$ and $t$ (respectively). Similarly, if $f$ is a real-valued function and $t \in [0, \infty]$, $f \wedge t$ is a function with the same domain as $f$ and $(f \wedge t)(x) = f(x) \wedge t$. In a similar manner we define $f \vee t$, and $f \wedge g$ and $f \vee g$ for two real-valued functions $f$ and $g$ with common domain.

A value on a group $G$ is a function $p: G \to [0, \infty)$ such that for any $x, y \in G$,

(V1) $p(x) = 0 \iff x = 0$,
(V2) $p(-x) = p(x)$,
(V3) $p(x + y) \leq p(x) + p(y)$.

If in the above the condition (V1) is replaced by ‘$p(0) = 0$’, $p$ is called a *semivalue*. Every value $p$ on $G$ induces an invariant metric $G \times G \ni (x, y) \mapsto p(x - y) \in [0, \infty)$. The topology induced by the value $p$ is the topology induced by the latter metric. So, we may speak of a separable, complete, compact (etc.) valued group. Two values on a group are equivalent iff they induce the same topology. A value on a topological group is said to be compatible if it induces the given topology of the group. By $\delta_G$ we denote the discrete value on $G$ defined by $\delta_G(q) = 1$ for $q \in G \setminus \{0\}$. For two valued groups $(G, +, p)$ and $(G', +, p')$ we shall write $(G', +, p') \supset (G, +, p)$ iff $G \subset G'$, $p'$ extends $p$ and the addition of $G'$ extends the addition of $G$. If this happens, we say that $(G, +, p)$ is enlarged to $(G', +, p')$.

Subgroups need not be closed. A subgroup generated by a subset $A$ of a group $G$ is denoted by $\langle A \rangle$. We write $\langle a_1, \ldots, a_n \rangle$ instead of $\langle \{a_1, \ldots, a_n\} \rangle$. If $\psi: G \to H$ is a homomorphism between valued groups $(G, +, p)$ and $(H, +, q)$, we use the term a group homomorphism to underline that we make no additional assumptions on the topological behavior of $\psi$. The group homomorphism is isometric if $q(\psi(x)) = p(x)$ for each $x \in G$. Adapting terminology of functional analysis, we say $\psi$ is $\varepsilon$-almost isometric with $\varepsilon \in (0, 1)$ provided
\[(1 - \varepsilon)p(x) \leq q(\psi(x)) \leq (1 + \varepsilon)p(x)\] for every \(x \in G\) (compare with (1-1)).

The classes \(G\) and \(G_r(N)\) (with \(r \in \{1, \infty\}\) and \(N \in \mathbb{Z}_+ \setminus \{1\}\)) have the same meaning as in Introduction. A metric space \((X,d)\) is said to be topologically (respectively metrically) universal for a class of metric spaces provided every member of the class is homeomorphic (respectively isometric) to a subset of \(X\).

Whenever \(f\) and \(g\) are two real-valued functions defined on a common nonempty domain, the supremum distance \(\|f - g\|_\infty\) of \(f\) and \(g\) is denoted by \(\|f - g\|_\infty\). Similarly, if \(f\) and \(g\) take values in a valued group \((G, +, p)\) and there is no danger of confusion, we shall also write \(\|f - g\|_\infty\) for the supremum distance of \(f\) and \(g\) induced by \(p\). The diameter of a metric space \((X,d)\) is denoted by \(\text{diam}(X,d)\).

Whenever \(\psi\) is a group homomorphism \(\psi: (G, +) \rightarrow (G', +)\) we may associate a semivalue \(p_\psi\) on \(G\): \(p_\psi = p' \circ \psi\). It is easily seen that \(\psi\) is continuous iff \(p_\psi\) is continuous (with respect to \(p\)). We say that \(\psi\) is bounded iff \(p_\psi\) is \(p\)-bounded. That is:

\[
\psi \text{ is bounded } \iff \exists \omega \in \Omega: p' \circ \psi \leq \omega \circ p.
\]
The following are left as exercises. Some of them are quite easy, other are well-known results of real analysis:

(MC1) For every $\omega \in \Omega$ there is a finite limit $\lim_{x \to \infty} \frac{\omega(x)}{x}$ and it is equal to $\inf_{x > 0} \omega(x)$.

(MC2) If $\omega, \tau \in \Omega$, then $\omega \circ \tau \in \Omega$.

(MC3) Every $\omega \in \Omega$ is uniformly continuous, and $\omega^{-1}([0]) = \{0\}$ provided $\omega \neq 0$.

(MC4) If $\omega \in \Omega^*$ and $(x_n)_{n=1}^\infty \subset \mathbb{R}^+$, then $\lim_{n \to \infty} \omega(x_n) = 0 \iff \lim_{n \to \infty} x_n = 0$.

For each $r \in [0, \infty)$ denote by $\Omega_r$ the set of all $\omega \in \Omega$ such that $\lim_{x \to \infty} \frac{\omega(x)}{x} \leq r$ (cf. (MC1)). For us, the set $\Omega_0$ is of great importance.

2.1. Example. Every bounded modulus of continuity belongs to $\Omega_0$.

If $\omega \in \Omega$ and $\tau \in \Omega_0$, then $\omega \circ \tau, \tau \circ \omega \in \Omega_0$. The following functions are members of $\Omega_0$:

$x \mapsto x \wedge 1, \ x \mapsto \frac{x}{x + 1}$ and $x \mapsto x^\alpha \ (0 < \alpha < 1)$.

The next result is well-known. It is a variation of [1, §1, Theorem 1]. We shall use it to characterize bounded group homomorphisms.

2.2. Lemma. For a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ the following conditions are equivalent:

(i) there is $\omega \in \Omega$ such that $f \leq \omega$,
(ii) $\limsup_{x \to 0^+} f(x) = f(0) = 0$ and 

$$
 r := \limsup_{n \to \infty} \sup_{f([0, 2^n])} \frac{2^n}{2^n} < \infty.
$$

What is more, if (ii) is fulfilled [and $f$ is bounded by $M$], there is $\omega \in \Omega_{2r}$ [bounded by $M$] such that $f \leq \omega$.

As an immediate consequence of Lemma 2.2 we obtain

2.3. Proposition. Let $\psi : (G, +, p) \to (G', +, p')$ be a group homomorphism and let

$$
 f : \mathbb{R}^+ \ni x \mapsto \sup \{p'(\psi(x)) : x \in G, \ p(x) \leq \xi \} \in [0, \infty].
$$

Then:

(a) $\psi$ is continuous iff $\lim_{t \to 0^+} f(t) = 0$,
(b) $\psi$ is bounded iff $\lim_{t \to 0^+} f(t) = 0$ and $\limsup_{t \to \infty} \frac{f(t)}{t} < \infty$.

2.4. Corollary. If $\psi : (G, +, p) \to (G', +, p')$ is a continuous group homomorphism such that the set $\psi(G)$ is bounded in the metric space $(G', p')$, then $\psi$ is bounded.

2.5. Corollary. A group homomorphism $\psi : (G, +, p) \to (G', +, p')$ is continuous iff there are $\omega, \tau \in \Omega^*$ such that $\tau \circ p' \circ \psi \leq \omega \circ p$.

Proof. Sufficiency is clear (thanks to (MC4)). To prove the necessity, put $\tau(t) = t \wedge 1$ and apply Lemma 2.2 for suitable $f : \mathbb{R}^+ \to [0, 1]$. □
2B. Extending a value.

2.6. Lemma. Let $Q$ be as in (2-1). Let $(D, +, \lambda)$ be a finite valued group and $D_0$ its subgroup such that $\lambda(D_0) \subseteq Q$. Then for every $\varepsilon > 0$ there is a value $\lambda$ on $D$ extending $\lambda|_{D_0}$ such that $\lambda(D) \subseteq Q$ and $\|\lambda - \lambda\|_\infty \leq \varepsilon$. What is more, if $\lambda \leq 1$, then $\hat{\lambda} \leq 1$.

Proof. We may assume that $\varepsilon \in (0, 1)$. For $h \in D_0$ let $\lambda'(h) = 0$ and for $h \in D \setminus D_0$ let $\lambda'(h) \in [0, \varepsilon]$ be such that $\lambda'(-h) = \lambda'(h)$ and $\lambda(h) + \lambda'(h) \in Q$. For $x \in D$ put

$$\tilde{\lambda}(x) = \inf \{\sum_{j=1}^{n} (\lambda(h_j) + \lambda'(h_j)) : n \geq 1, h_1, \ldots, h_n \in D, x = \sum_{j=1}^{n} h_j\}.$$

It is easily seen that $\tilde{\lambda}$ is a value on $D$ such that $\lambda \leq \tilde{\lambda} \leq \lambda + \lambda'$. This yields that $\|\lambda - \lambda\|_\infty \leq \varepsilon$ and $\tilde{\lambda} = \lambda$ on $D_0$. Further, since $D$ is finite, the infimum in the formula for $\tilde{\lambda}(x)$ is reached and therefore $\tilde{\lambda}(D) \subseteq Q$. Finally, if $\lambda \leq 1$, take $M \in Q \cap [1 - \varepsilon, 1]$ such that $\lambda(D_0) \subseteq [0, M]$ and replace $\lambda$ by $\lambda \wedge M$. $\square$

2.7. Lemma. Let $r \in \{1, \infty\}$. Let $(D, +, \lambda)$ be a valued group with $\lambda \leq r$, $D_0$ its closed subgroup, $\lambda_0$ a semivalue on $D_0$ and let $\omega \in \Omega^*$ be such that $\lambda_0 \leq (\omega \circ \lambda)|_{D_0}$. Then there is a semivalue $\tilde{\lambda}$ on $D$ which extends $\lambda_0$ and satisfies the following conditions:

(a) $\tilde{\lambda} \leq \omega \circ \lambda$,
(b) $\tilde{\lambda}^{-1}(\{0\}) = \lambda_0^{-1}(\{0\})$; in particular: $\tilde{\lambda}$ is a value provided so is $\lambda_0$,
(c) if $\lambda_0 \leq r$, then $\tilde{\lambda} \leq r$,
(d) if $\lambda_0$ is a value equivalent to $\lambda|_{D_0}$, then $\tilde{\lambda}$ is equivalent to $\lambda$,
(e) if for some $\tau, \varrho \in \Omega$ with $\varrho \neq 0$ and $(\omega \circ \tau)$ one has

$$\tau(\text{dist}_\lambda(h, \lambda_0^{-1}(\{0\}))) \leq (\varrho \circ \lambda_0)(h)$$

for every $h \in D_0$, then $\tau(\text{dist}_\lambda(x, \lambda_0^{-1}(\{0\}))) \leq (\varrho \circ \tilde{\lambda})(x)$ for $x \in D$.

Proof. For $x \in D$ put

$$\tilde{\lambda}(x) = \inf \{(\omega \circ \lambda)(x - h) + \lambda_0(h) : h \in D_0\}.$$

Further, if $\lambda_0 \leq r$, replace $\tilde{\lambda}$ by $\tilde{\lambda} \wedge r$. One easily verifies that $\tilde{\lambda}$ is a semivalue extending $\lambda_0$ which satisfies (a) and (d). The point (b) follows from the closedness of $D_0$. We shall only show (e). For simplicity, put $q(x) = \text{dist}_\lambda(x, \lambda_0^{-1}(\{0\}))$ ($x \in D$). Thanks to the continuity of $\varrho$, it suffices to check that $(\tau \circ q)(x) \leq \varrho((\omega \circ \lambda)(x - h) + \lambda_0(h))$ for $x \in D$ and $h \in D_0$ (the second condition in $(\omega \circ \tau)$ allows us to replace $\tilde{\lambda}$ by $\lambda \wedge r$). But $(\tau \circ q)(x) \leq (\tau \circ \lambda)(x - h) + (\varrho \circ \lambda_0)(h)$ and hence it is enough to show that

$$(\tau \circ \lambda)(x - h) + (\varrho \circ \lambda_0)(h) \leq \varrho((\omega \circ \lambda)(x - h) + \lambda_0(h)),$$

which is the first condition in $(\omega \circ \tau)$ with $t = \lambda(x - h)$ and $s = \lambda_0(h)$. $\square$
2.8. Example. Let \( \omega_0, \varrho_0 \in \Omega^*, \tau_0 \in \Omega \) and \( r \in \{1, \infty\} \). Put \( \omega = \omega_0 \lor \tau_0, \varrho = \varrho_0 + \text{id} \) and \( \tau = \tau_0 \land \varrho(r) \) \( (\varrho(\infty) := \lim_{t \to \infty} \varrho(t)) \). Then \( \omega, \varrho \) and \( \tau \) are moduli of continuity such that \( \tau \leq \tau_0, \varrho \leq \varrho, \omega_0 \leq \omega \) and \( (\omega, \varrho, \tau) \) is fulfilled. The example shows that we may always replace moduli of continuity appearing in the inequalities in points (a) and (e) of Lemma 2.7 by other moduli in such a way that these inequalities are still satisfied and the strange condition \( (\omega, \varrho, \tau) \) is fulfilled.

2C. The class \( O_0 \). We call a group of class \( O_{\text{fin}} \) iff every its element has finite rank. That is, \( G \) is of class \( O_{\text{fin}} \) if \( G = G_{\text{fin}} \).

Let \( (G, +, p) \) be an arbitrary valued group and \( x \) its element. Since the sequence \( a_n := p(nx) \) \( (n \geq 1) \) satisfies the condition
\[
a_{n+m} \leq a_n + a_m \quad (n, m \geq 1),
\]
the sequence \( \left( \frac{a_n}{n} \right)_{n=1}^\infty \) has finite limit and \( \lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n} \). Put \( p_0^*: G \to \mathbb{R}_+ \)
\[
p_0^*(x) = \lim_{n \to \infty} \frac{p(nx)}{n}.
\]
Observe that \( p_0^* \) is a semivalue on \( G \) such that \( p_0^* \leq p \). Thus \( p_0^* \) is \( p \)-bounded and hence the set \( G_0^* = p_0^{-1}(\{0\}) \) is a closed subgroup of \( (G, +, p) \). Let \( G' \) be the quotient group \( G / G_0^* \) equipped with the value \( p' \) (naturally) induced by \( p_0^* \). Observe that
\[
(2-3) \quad p'(kx) = |k|p'(x)
\]
for every \( x \in G' \) and \( k \in \mathbb{Z} \) and hence \( p_0^{*'} = p' \).

We say the valued group \( (G, +, p) \) is of class \( O_0 \) iff \( G_0^* = G \) or, equivalently, if \( p_0^* \equiv 0 \). It is of class \( O_\infty \) iff \( p_0^* = p \) (that is, if (2-3) is fulfilled with \( p' \) replaced by \( p \)). Finally, \( (G, +, p) \) is of class \( O_1 \) iff \( p_0^* \) is a value on \( G \) or, equivalently, if \( G_0^* \) is trivial. Note that \( O_\infty \subset O_1 \).

We begin with an interesting characterization of members of the above mentioned classes.

Making use of (2-3) and repeating the proof of the Hahn-Banach theorem one easily gets

2.9. Lemma. Let \( (G, +, p) \) be a valued group.

(A) If \( \psi: G \to \mathbb{R} \) is a group homomorphism such that for each \( x \in G \),
\[
(2-4) \quad |\psi(x)| \leq p(x),
\]
then \( |\psi(x)| \leq p_0^*(x) \) for every \( x \in G \).

(B) For each \( a \in G \) there is a group homomorphism \( \psi: G \to \mathbb{R} \) satisfying (2-4) for all \( x \in G \) such that \( \psi(a) = p_0^*(a) \).

Let us call a group homomorphism \( \psi: G \to \mathbb{R} \) satisfying (2-4) non-expansive. The above result yields

2.10. Theorem. Let \( (G, +, p) \) be a valued group.
I) $G$ is of class $O_0$ iff $G$ admits no nonzero bounded group homomorphisms into Banach spaces.

II) $G$ is of class $O_\infty$ iff $G$ admits an isometric group homomorphism into a Banach space.

III) $G$ is of class $O_1$ iff $G$ admits a nonexpansive group homomorphism with trivial kernel into a Banach space, iff $G$ admits a bounded group homomorphism with trivial kernel into a Banach space.

Proof. (I): Suppose $(G, +, p)$ is of class $O_0$, $(E, \| \cdot \|)$ is a Banach space, $\omega \in \Omega$ and $\psi: G \to E$ is a group homomorphism such that $\|\psi(x)\| \leq (\omega \circ p)(x)$ for $x \in G$. Then we have

\begin{equation}
\|\psi(x)\| \leq \frac{(\omega \circ p)(nx)}{n} = \frac{\omega\left(n \cdot \frac{p(nx)}{n}\right)}{n} \leq \frac{p(nx)}{n} \to 0.
\end{equation}

The inverse implication follows from the point (B) of Lemma 2.9.

(II): Assume $(G, +, p)$ is of class $O_\infty$. Let $Z$ be the set of all nonexpansive group homomorphisms of $G$ into $\mathbb{R}$ and let $E$ be the Banach space of all real-valued bounded functions on $Z$, equipped with the supremum norm. For $g \in G$ let $e_g \in E$ be given by $e_g(\xi) = \xi(g)$. Then the function $G \ni g \mapsto e_g \in E$ is an isometric group homomorphism (again by Lemma 2.9). The inverse implication is immediate.

(III): If $(G, +, p)$ is of class $O_1$, then $(G, +, p_0^*)$ is a valued group of class $O_\infty$ and thus the assertion follows from (II). Conversely, if $\psi: G \to E$ is a bounded group homomorphism of $G$ into a Banach space $E$, (2-5) shows that $p_0^*(x) \geq \|\psi(x)\|$. \hfill \Box

2.11. Remark. Theorem 2.10 expresses how far is boundedness from continuity: only valued groups of class $O_0$ admit bounded group homomorphisms with trivial kernels into Banach spaces. In contrast, there are valued groups of class $O_0$ which are group isomorphic to Banach spaces: if $(E, \| \cdot \|)$ is a Banach space, then $(E, \| \cdot \| \wedge 1)$ is of class $O_0$. By the way, the latter example shows that membership to any of the classes $O_0$, $O_1$, $O_\infty$ is not a topological group invariant.

Theorem 2.10 also implies that every valued group $G$ has a unique closed subgroup $H$ (namely, $G_0^*$) such that $H$ is of class $O_0$ and $G/H$ admits a nonexpansive group homomorphism with trivial kernel into a Banach space.

It also follows from Theorem 2.10 and the well-known theorem of Banach and Mazur that the Banach space $C([0,1])$ of all continuous real-valued functions on $[0,1]$, as a valued group, is universal for separable valued Abelian groups of class $O_\infty$.

From now on, we are only interested in valued groups of class $O_0$. The reader will easily check that every group with bounded value is of this class and that $O_{fin}^* \subset O_0$. It turns out that, in a sense, $O_0$ is the ‘closure’ of $O_{fin}$. Formally this is formulated in the following
2.12. **Theorem.** A valued group \((G, +, p)\) is of class \(O_0\) iff it may be enlarged to a valued group \((\bar{G}, +, \bar{p})\) such that \(G_{fin}\) is dense in \(\bar{G}\).

**Proof.** The sufficiency is clear, since \(G_{0*}\) is closed and contains \(G_{fin}\).

To prove the necessity, thanks to transfinite induction, it is enough to show that if \(p_0(a) = 0\), then for every \(\varepsilon > 0\) there is a valued group \((\bar{G}, +, \bar{p}) \supset (G, +, p)\) and \(b \in \bar{G}_{fin}\) such that \(\bar{p}(a - b) \leq \varepsilon\).

Let \(N \geq 2\) be such that

\[ (2-6) \quad p(na) \leq \varepsilon n \quad \text{for } n \geq N. \]

Take \(M > 0\) such that \(M \geq p(ja)\) for \(j = 1, \ldots, N\). Let \(H = \langle b \rangle\) be a cyclic group of rank \(N\). We identify \(g \in G\) with \((g, 0) \in G \times H =: \bar{G}\).

Define \(\bar{p}: \bar{G} \to \mathbb{R}_+\) by

\[ \bar{p}(g, h) = \inf \{ p(g - ka) + |k|\varepsilon + M\delta_H(kb + h) : k \in \mathbb{Z} \}. \]

It is clear that \(\bar{p}\) is a value on \(\bar{G}\) such that \(\bar{p}(g, 0) \leq p(g)\) for \(g \in G\) and \(\bar{p}(0, b) \leq \varepsilon).\) So, it suffices to show that \(p(g) \leq \bar{p}(g, 0)\), which is equivalent to

\[ (2-7) \quad p(ka) \leq |k|\varepsilon + M\delta_H(kb) \]

for \(k \in \mathbb{Z}\). We may assume \(k \neq 0\). If \(|k| \geq N\), \((2-7)\) is covered by \((2-6)\). Finally, if \(0 < |k| < N\), then \(p(ka) \leq M = M \cdot \delta_H(kb)\) and we are done.

Note that if in the above theorem \(G\) is separable, \(\bar{G}\) may be constructed to be separable as well and that in that case the proof is constructive. It is also easily seen that we may force \(\bar{G}\) to have the same diameter as \(G\).

2.13. **Example.** Let \((G, +, p)\) be a valued group and \(\omega \in \Omega^*\). If \(G\) is of class \(O_0\) or \(\omega \in \Omega_0\), then \((G, +, \omega \circ p)\) is of class \(O_0\). In particular, for every separable valued group \((G, +, p)\), \((G, +, p^\alpha) \in \mathfrak{G}_\infty(0)\) for \(0 < \alpha < 1\) and \((G, +, p \wedge 1), (G, +, p_{p+1}) \in \mathfrak{G}_1(0)\). The value \(p\) may be reconstructed from each of the values \(p^\alpha\) \((0 < \alpha < 1)\) and \(p_{p+1}\), and since \(p(x) = \lim_{\alpha \to 1^-} p^\alpha(x)\) for every \(x \in G\), one may say that each valued group can be ‘approximated’ by valued groups of class \(O_0\). It is also clear that every metrizable topological group admits a compatible value under which it becomes a valued group of class \(O_0\).

Taking into account Remark 2.11, we see that the image of a valued group of class \(O_0\) under a continuous group homomorphism need not be of class \(O_0\). However, the reader will easily check that

2.14. **Proposition.** A subgroup (equipped with the inherited value) and the completion of a valued group of class \(O_0\) is of class \(O_0\) as well. The image of a valued group of class \(O_0\) under a bounded group homomorphism is of class \(O_0\).
2D. Enlarging a finite valued group. A very useful method of constructing the Urysohn universal space, introduced by Katětov [10], involves the technique of the so-called Katětov maps which correspond to one-point extensions of metric spaces. In this part we shall describe how to enlarge valued groups by means of these maps. As a corollary of the results presented below we shall obtain in the sequel a rather surprising result that \( \mathcal{G}(N) \) is Urysohn iff \( N \in \{0,2\} \).

Recall that a Katětov map on a metric space \((X,d)\) is a function \( f : X \to \mathbb{R}_+ \) such that \( |f(x) - f(y)| \leq d(x,y) \leq f(x) + f(y) \) for any \( x,y \in X \). The set of all Katětov maps on \( X \) is denoted by \( E(X) \). Additionally, for \( r \in [0,\infty] \) let \( E_r(X) \) be the set of all \( f \in E(X) \) such that \( f(x) \leq r \) for each \( x \in X \), and let \( E'(X) = E_{\text{diam}X}(X) \). Katětov maps belonging to \( E'(X) \) are called inner. If \( A \subset B \subset X \), a Katětov map \( f : B \to \mathbb{R}_+ \) is said to be trivial on \( A \) in \( X \) iff there is \( b \in X \) such that \( f(a) = d(a,b) \) for each \( a \in A \). The map \( f \) is trivial in \( X \) if \( f \) is trivial on its domain in \( X \).

A fundamental result on the Urysohn space says that a nonempty separable complete metric space \( X \) is Urysohn iff every inner Katětov map is trivial in \( X \) on every finite subset of the space (see e.g. [15]).

We begin with

\[ \text{2.15. Proposition.} \] Let \((G,+,p)\) be a valued group of exponent \( N \geq 3 \). If \( A \subset G \) and \( f \in E(A) \) is trivial in \( G \), then for any \( a_1, \ldots, a_N \in A \),

\[ (2-8) \quad |p(\sum_{k=1}^N a_k) - f(a_N)| \leq \sum_{k=1}^{N-1} f(a_k). \]

\[ \text{Proof.} \] Let \( x \in G \) be such that \( f(a) = p(x - a) \) for \( a \in A \). Then, since \( N \cdot x = 0 \),

\[ |p(\sum_{k=1}^N a_k) - f(a_N)| \leq p(\sum_{k=1}^N a_k + (x - a_N)) = p(\sum_{k=1}^{N-1} (a_k - x)) \leq \sum_{k=1}^{N-1} p(a_k - x) = \sum_{k=1}^{N-1} f(a_k). \]

\[ \square \]

\[ \text{2.16. Remark.} \] If \((G,+,p)\) is a valued group of exponent \( N \geq 3 \) and \( f \) is a Katětov map whose domain \( H \) is a subgroup of \( G \), then \((2-8)\) is equivalent to

\[ (2-9) \quad f(- \sum_{k=1}^{N-1} a_k) \leq \sum_{k=1}^{N-1} f(a_k) \quad \text{for any } a_1, \ldots, a_{N-1} \in H. \]

Indeed, if \( a_1, \ldots, a_{N-1} \in H \) are fixed, then, since \(- \sum_{k=1}^{N-1} a_k \in H\),

\[ \sup_{a_N \in H} |p(\sum_{k=1}^N a_k) - f(a_N)| = f(- \sum_{k=1}^{N-1} a_k). \]
Our aim is to prove, in a sense, the converse of Proposition 2.15.

2.17. Theorem. Let \( r \in \{0, \infty\} \), \( N \in \mathbb{Z}_+ \setminus \{1\} \) and let \((G, +, p) \in \mathfrak{G}_r(N)\) be a bounded valued group. Let \( A \) be a nonempty subset of \( G \) and \( f \in E_r(A) \). If \( N > 2 \), we assume that \( f \) satisfies (2-8) for all \( a_1, \ldots, a_N \in A \). Then there is a bounded valued group \((\hat{G}, +, \hat{p}) \in \mathfrak{G}_r(N)\) such that \((\hat{G}, +, \hat{p}) \supset (G, +, p)\) and \( f \) is trivial in \( \hat{G} \).

If, in addition, \( Q \) is as in (2-1), \( G \) is finite and \( p(G) \cup f(A) \subset Q \), the group \( \hat{G} \) may be taken so that it is finite and \( \hat{p}(\hat{G}) \subset Q \).

Proof. First of all observe that (2-8) is fulfilled for any \( a_1, \ldots, a_N \in A \) also when \( N = 2 \), because then \( a_1 + a_2 = a_1 - a_2 \). Note that if \( \inf f(A) = 0 \), then \( f \) is trivial in the completion of \( G \), so we may and do assume that \( c := \inf f(A) > 0 \). Further, let \( M = \sup p(G) \vee \sup f(A) \) (\( M \in Q \) provided all additional conditions of the theorem are satisfied; \( f \) is bounded because \( G \) is so). If \( N \neq 0 \), let \( m = N \). Otherwise, take \( m \geq 2 \) such that \( m - 1 \geq \frac{M}{c} \). Let \( H = \langle b \rangle \) be a cyclic group of rank \( m \). Put \( \tilde{G} = G \times H \). We identify each \( g \in G \) with \((g, 0) \in \tilde{G} \). Let us agree that \( \sum_{j=1}^b = 0 \) provided \( a > b \). Define \( \tilde{p} \colon G \to \mathbb{R}_+ \) by the formula

\[
\tilde{p}(g, h) = \inf \{ p(g - \sum_{j=1}^n \varepsilon_j a_j) + \sum_{j=1}^n f(a_j) + M \delta_H(\sum_{j=1}^n \varepsilon_j b - h) : n \geq 0, a_1, \ldots, a_n \in A, \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\} \}.
\]

It is easy to verify that \( \tilde{p} \) is a value. What is more, if all additional conditions of the theorem are satisfied, the infimum in the formula for \( \tilde{p}(g, h) \) is reached and thus \( \tilde{p}(\hat{G}) \subset Q \). Observe that \( \tilde{p}(g, 0) \leq p(g) \) for each \( g \in G \) and \( \tilde{p}(a, b) \leq f(a) \) for \( a \in A \). If we show that in both these inequalities one may put the equality sign, the proof will be completed. (Indeed, to make then the value \( \tilde{p} \) bounded by \( r \), it suffices to replace \( \tilde{p} \) by \( \tilde{p} \wedge M \).)

First we will check that \( \tilde{p}(g, 0) = p(g) \) for \( g \in G \). Equivalently, we have to show that

\[
(2-10) \quad \sum_{j=1}^n f(a_j) + M \delta_H(\sum_{j=1}^n \varepsilon_j b) \geq p(\sum_{j=1}^n \varepsilon_j a_j)
\]

for any \( n \geq 1 \) (for \( n = 0 \) it is clear), \( a_1, \ldots, a_n \in A \) and \( \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\} \). If \( \sum_{j=1}^n \varepsilon_j b \neq 0 \), then \( \delta_H(\sum_{j=1}^n \varepsilon_j b) = 1 \) and hence (2-10) is fulfilled, by the definition of \( M \). Thus we may assume that \( \sum_{j=1}^n \varepsilon_j b = 0 \), that is, \( m \mid \sum_{j=1}^n \varepsilon_j \).

If \( \sum_{j=1}^n \varepsilon_j \neq 0 \) and \( N = 0 \), then \( n \geq |\sum_{j=1}^n \varepsilon_j| \geq m \), so \( \sum_{j=1}^n f(a_j) \geq nc \geq mc \geq M \geq p(\sum_{j=1}^n \varepsilon_j a_j) \) and consequently (2-10) is fulfilled.

If \( \sum_{j=1}^n \varepsilon_j \neq 0 \) and \( N \neq 0 \), then \( m = N \) and \( \sum_{j=1}^n \varepsilon_j = lN \) for some \( l \in \mathbb{Z} \setminus \{0\} \). This means that, after renumeration of \( \varepsilon_j \)’s (and
$a_j$’s), we may assume $\varepsilon_j = l/|l|$ for $j = 1, \ldots, |l|N$ and $\sum_{j=|l|N+1}^{n} \varepsilon_j = 0$. Now making use of (2-8) we infer that $\sum_{k=1}^{|l|N} p(\sum_{j=(k-1)N+1}^{N} a_j) \leq \sum_{j=1}^{|l|N} f(a_j)$. So, if only $p(\sum_{j=|l|N+1}^{n} \varepsilon_j a_j) \leq \sum_{j=|l|N+1}^{n} f(a_j)$, (2-10) will be satisfied. This reduces the problem to the case when $\sum_{j=1}^{n} \varepsilon_j = 0$ (and $N$ is arbitrary).

Finally, if $\sum_{j=1}^{n} \varepsilon_j = 0$, then $n$ is even, say $n = 2k$, and—after renumeration—we may assume $\varepsilon_j = (-1)^{j-1}$ for $j = 1, \ldots, n$. But then

$$p(\sum_{j=1}^{n} \varepsilon_j a_j) = p(\sum_{j=1}^{k} (a_{2j-1} - a_{2j})) \leq \sum_{j=1}^{k} (f(a_{2j-1}) + f(a_{2j})) = \sum_{j=1}^{n} f(a_j)$$

which finishes the proof of (2-10). Now we pass to proving that $\tilde{p}(a, b) = f(a)$ for $a \in A$. This is equivalent to

$$(2-11) \quad p(a - \sum_{j=1}^{n} \varepsilon_j a_j) + \sum_{j=1}^{n} f(a_j) + M \delta_H(\sum_{j=1}^{n} \varepsilon_j b - b) \geq f(a)$$

for each $n \geq 0$ and arbitrary $a_1, \ldots, a_n \in A$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. As before, if $\sum_{j=1}^{n} \varepsilon_j b \neq b$, then, thanks to the definition of $M$, (2-11) is fulfilled. Thus we may assume $m!|\sum_{j=1}^{n} \varepsilon_j - 1$ (so, $n > 0$).

If $\sum_{j=1}^{n} \varepsilon_j \neq 1$ and $N = 0$, $n \geq |\sum_{j=1}^{n} \varepsilon_j| \geq m - 1 \geq \frac{M}{c}$ and hence $\sum_{j=1}^{n} f(a_j) \geq nc \geq M \geq f(a)$.

If $\sum_{j=1}^{n} \varepsilon_j \neq 1$ and $N \neq 0$, then $\sum_{j=1}^{n} \varepsilon_j = lN+1$ for some $l \in \mathbb{Z}\setminus\{0\}$. Then either $\varepsilon_1 = \ldots = \varepsilon_n = -1$ or at least $|l|N$ of $\varepsilon_j$’s are equal to $l/|l|$. In the first case we get $n = |l|N - 1$ and, $|l|$ times making use of (2-8):

$$p(a + \sum_{j=1}^{n} a_j) + \sum_{j=1}^{n} f(a_j) \geq p(a + \sum_{j=1}^{N-1} a_{lN-j}) + \sum_{j=1}^{N-1} f(a_{lN-j})$$

$$+ \sum_{k=1}^{l-1N-1} f(a_{kN-j}) - p(\sum_{j=0}^{n} a_{kN-j}) \geq f(a),$$

which gives (2-11). In the second case we may assume, after renumeration, that $\varepsilon_j = l/|l|$ for $j = 1, \ldots, |l|N$ and $\sum_{j=|l|N+1}^{n} \varepsilon_j = 1$. Then, again by (2-8), $\sum_{k=1}^{l-1N-1} f(a_{kN-j}) - p(\sum_{j=0}^{n-1} a_{kN-j}) \geq 0$. So, if only $p(a - \sum_{j=|l|N+1}^{n} \varepsilon_j a_j) + \sum_{j=|l|N+1}^{n} f(a_j) \geq f(a)$, (2-11) will be satisfied. This reduces the problem to the case when $\sum_{j=1}^{n} \varepsilon_j = 1$ (and $N$ is arbitrary).
Finally, when $\sum_{j=1}^{n} \varepsilon_j = 1$, then $n$ is odd, say $n = 2k + 1$, and (after renumeration) $\varepsilon_j = (-1)^{j-1}$. But then

\[
p(a - \sum_{j=1}^{n} \varepsilon_j a_j) + \sum_{j=1}^{n} f(a_j) \geq p(a - a_n) + f(a_n) + \sum_{j=1}^{k} [f(a_{2j-1}) + f(a_{2j}) - p(a_{2j} - a_{2j-1})] \geq f(a).
\]

□

In [17] we have shown that the Urysohn universal metric space $U$ admits a (unique) structure of a metric group of exponent 2 (which, in fact, will turn out to be isometrically group isomorphic to $\mathbb{G}_\infty(2)$). At the end of [17] we asked (following the referee) whether $U$ admits group structures of other exponents. Applying Remark 2.16, below we give a partial (negative) answer to this problem.

2.18. **Proposition.** There is no nontrivial valued Abelian group of exponent 3 which is Urysohn as a metric space.

**Proof.** Let $(H, +, q)$ be a nontrivial valued Abelian group. Take $h \in H$ such that $0 < q(h) < \frac{1}{2} \sup q(H)$ (if there is no such $h$, then $H$ is automatically non-Urysohn). Notice that $q(h) = q(-h) = q(h - (-h))$. Let $f: \{0, h, -h\} \to \mathbb{R}_+$ be given by $f(h) = f(-h) = \frac{1}{2}q(h)$ and $f(0) = \frac{3}{2}q(h)$. It is easily seen that $f$ is a Katětov map bounded by the diameter of $H$. If $f$ was trivial in $H$, we would have (by (2-9)) $f(0) \leq f(h) + f(-h)$, which fails to be true. □

We do not know whether it is possible to show, using only (2-9), the counterpart of Proposition 2.18 for exponents greater than 3.

2E. **Amalgamation lemmas.** In constructions of Fraïssé limits, possibility of amalgamating (i.e. ‘gluing’) spaces is the crucial axiom. However, only the first of the results stated below will be used for this purpose. From now to the end of the section, $r \in \{1, \infty\}$ and $N \in \mathbb{Z}_+ \setminus \{1\}$ are fixed. To avoid repetitions, let us make the following preliminary annotation (which will be used in the proofs of the next three lemmas):

\[
(2-12) \quad \begin{cases}
\hat{D} = (D_1 \times D_2)/\hat{D}_0 \\
p: D_1 \times D_2 \to \hat{D} \text{ the quotient group homomorphism} \\
\hat{\psi}_1: D_1 \ni x \mapsto \pi(x, 0) \in \hat{D} \\
\hat{\psi}_2: D_2 \ni y \mapsto \pi(0, y) \in \hat{D}
\end{cases}
\]

Note that below Proposition 2.14 is applied several times without referring to it.
2.19. Lemma. Let \((D_j, +, \lambda_j) \in \mathfrak{G}_r(N)\) \((j = 0, 1, 2)\) and for \(j = 1, 2\) let 
\(\varphi_j: D_0 \to D_j\) be an isometric group homomorphism. Then there exist a finite valued group \((D, +, \lambda) \in \mathfrak{G}_r(N)\) and isometric group homomorphisms 
\(\psi_j: D_j \to D\) \((j = 1, 2)\) such that \(\psi_2 \circ \varphi_2 = \psi_1 \circ \varphi_1\). If, in addition, \(Q\) 
is as in (2-1), \(D_1\) and \(D_2\) are finite and \(\lambda_j(D_j) \subset Q\) \(j = 1, 2\), then 
\(D\) is finite as well and \(\lambda(D) \subset Q\).

Proof. Let \(\tilde{D}_0 = \{(\varphi_1(x), -\varphi_2(x))\}: x \in D_0\) \(\subset D_1 \times D_2\) and \(\tilde{D}, \pi, \tilde{\psi}_1\) 
and \(\tilde{\psi}_2\) be as in (2-12). Then \(\tilde{\psi}_2 \circ \varphi_2 = \tilde{\psi}_1 \circ \varphi_1\). Define \(\lambda: \tilde{D} \to \mathbb{R}_+\) by 
the formula

\[\tilde{\lambda}(z) = \inf\{\lambda_1(x_1) + \lambda_2(x_2)\}: (x_1, x_2) \in \pi^{-1}(\{z\})\}.
\]

It is clear that \(\tilde{\lambda}\) is a well defined semivalue on \(\tilde{D}\). Now put \(D = \tilde{D}/\tilde{\lambda}^{-1}(\{0\})\) and define \(\lambda, \psi_1\) and \(\psi_2\) by the rules \(\lambda \circ \tilde{\pi} = \tilde{\lambda}\) and \(\psi_j = \tilde{\psi}_j \circ \varphi_j\) \((j = 1, 2)\) where \(\tilde{\pi}: \tilde{D} \to D\) is the quotient group homomorphism. Finally, replace \(\lambda\) by \(\lambda \wedge 1\) if \(r = 1\). We leave this as a simple exercise that all assertions are satisfied. \(\Box\)

2.20. Lemma. Let \((D_1, +, \lambda_1), (D_2, +, \lambda_2) \in \mathfrak{G}_r(N)\) be finite valued groups and \(D_0\) be a subgroup of \(D_1\). Let \(u: D_0 \to D_2\) and \(v: D_1 \to D_2\) 
be an isometric and, respectively, an \(\varepsilon\)-almost isometric group homomorphism (where \(\varepsilon \in (0, 1)\)) such that

\[(2-13) \quad \|u - v\|_{D_0} \leq \varepsilon.\]

Then there are a finite valued group \((D, +, \lambda) \in \mathfrak{G}_r(N)\) and isometric group homomorphisms \(w_j: D_j \to D\) \((j = 1, 2)\) such that \(w_1|_{D_0} = w_2 \circ u\) and

\[(2-14) \quad \|w_1 - w_2 \circ v\|_{\infty} \leq A\varepsilon\]

where \(A = 1 + \text{diam}(D_1, \lambda_1)\).

Proof. Let \(\tilde{D}_0 = \{(x, -u(x))\}: x \in D_0\) \(\subset D_1 \times D_2\) and \(\tilde{D}, \pi, \tilde{\psi}_1\) and \(\tilde{\psi}_2\) 
be as in (2-12). Put \(D = \tilde{D}\) and \(w_j = \tilde{\psi}_j\) \((j = 1, 2)\). It is easily checked that 
\(w_2 \circ u = w_1|_{D_0}\) and \(D = w_1(D_1) + w_2(D_2)\). Define \(\lambda: D \to \mathbb{R}_+\) by

\[\lambda(z) = \inf\{\lambda_1(x_1 - x_0) + A\varepsilon \delta_D(w_1(x_0) - (w_2 \circ v)(x_0)) + \lambda_2(x_2 + v(x_0))\}: x_0, x_1 \in D_1, x_2 \in D_2, z = w_1(x_1) + w_2(x_2)\}.
\]

We see that \(\lambda\) is a semivalue on \(D\) and, since \(D_1\) and \(D_2\) are finite, the infimum in the formula for \(\lambda(z)\) is reached. Therefore, if \(\lambda(z) = 0\), then for some \(x_0, x_1 \in D_1\) and \(x_2 \in D_2\) one has \(z = w_1(x_1) + w_2(x_2)\) and 
\(x_1 - x_0 = 0\), \(w_1(x_0) - (w_2 \circ v)(x_0) = 0\) and \(x_2 + v(x_0) = 0\). These yield 
\(x_1 = x_0\), \(x_2 = -v(x_1)\) and \(w_1(x_1) = -w_2(x_2)\) and consequently \(z = 0\). So, \(\lambda\) is a value. Moreover, it follows from the definition of \(\lambda\) that 
(2-14) is satisfied and that \(\lambda(w_j(x)) \leq \lambda_j(x)\) for \(x \in D_j\) \((j = 1, 2)\). We
shall now prove that in the latter inequalities one may put the equality sign. Fix \( j \in \{1, 2\} \). For \( x \in D_j \) we have to show that

\[
\lambda_1(x_1 - x_0) + A\varepsilon\delta_D(w_1(x_0) - (w_2 \circ v)(x_0)) + \lambda_2(x_2 + v(x_0)) \geq \lambda_j(x)
\]

provided \( x_0, x_1 \in D_1, x_2 \in D_2 \) and

\[
w_1(x_1) + w_2(x_2) = w_j(x).
\]

First assume \( j = 1 \). In that case (2-16) means that \( (x_1 - x, x_2) \in D_0 \) and thus \( h := x - x_1 \in D_0 \) and \( x_2 = u(h) \). So, (2-15) has the form

\[
\lambda_1(x_1 - x_0) + A\varepsilon\delta_D(w_1(x_0) - (w_2 \circ v)(x_0)) + \lambda_2(u(h) + v(x_0)) \geq \lambda_1(x_1 + h)
\]

which, after substitution \( x_1 := x_0 \), is equivalent to

\[
A\varepsilon\delta_D(w_1(x_0) - (w_2 \circ v)(x_0)) + \lambda_2(u(h) + v(x_0)) \geq \lambda_1(x_0 + h).
\]

If \( w_1(x_0) = w_2(v(x_0)) \), then \( x_0 \in D_0 \) and \( v(x_0) = u(x_0) \) and consequently (2-17) changes into \( \lambda_2(u(h) + u(x_0)) \geq \lambda_1(x_0 + h) \) which is fulfilled because \( u \) is isometric. So, we may assume that \( w_1(x_0) \neq w_2(v(x_0)) \) and then we have to show that \( A\varepsilon + \lambda_2(u(h) + v(x_0)) \geq \lambda_1(x_0 + h) \). But \( v \) is \( \varepsilon \)-almost isometric, hence \( \lambda_2(v(x_0 + h)) \geq (1 - \varepsilon)\lambda_1(x_0 + h) \). So, thanks to (2-13), we obtain

\[
A\varepsilon + \lambda_2(u(h) + v(x_0)) \geq \varepsilon \lambda_1(x_0 + h) + \varepsilon + \lambda_2(v(x_0) + v(h)) - \lambda_2(u(h) - v(h)) \geq \lambda_1(x_0 + h).
\]

When \( j = 2 \), we argue in a similar way. In that case (2-16) gives \( h := x_1 \in D_0 \) and \( x - x_2 = u(h) \). After substitution \( x_2 := -v(x_0) \) the inequality (2-15) changes into \( \lambda_1(h - x_0) + A\varepsilon\delta_D(w_1(x_0) - (w_2 \circ v)(x_0)) \geq \lambda_2(u(h) - v(x_0)) \). As before, the situation when \( w_1(x_0) = w_2(v(x_0)) \) is simple. Otherwise we have to prove that \( A\varepsilon + \lambda_1(h - x_0) \geq \lambda_2(u(h) - v(x_0)) \). We have

\[
\lambda_2(u(h) - v(x_0)) \leq \lambda_2(u(h) - v(h)) + \lambda_2(v(h) - v(x_0)) \leq \varepsilon + (1 + \varepsilon)\lambda_1(h - x_0) \leq A\varepsilon + \lambda_1(h - x_0).
\]

To end the proof, replace \( \lambda \) by \( \lambda \wedge r \) to guarantee that \( (D, +, \lambda) \in \mathfrak{G}_r(N) \). \( \square \)

2.21. Lemma. Let \( (D_j, +, \lambda_j) \in \mathfrak{G}_r(N) \) \((j = 1, 2)\), \( E_1 \) and \( E_2 \) be subgroups of \( D_1 \) and \( \varphi_j : E_j \to D_2 \) \((j = 1, 2)\) be isometric group homomorphisms such that for all \( (x_1, x_2) \in E_1 \times E_2 \),

\[
|\lambda_2(\varphi_1(x_1) - \varphi_2(x_2)) - \lambda_1(x_1 - x_2)| \leq \varepsilon
\]

(where \( \varepsilon > 0 \)). Then there exist a valued group \( (D, +, \lambda) \in \mathfrak{G}_r(N) \) and isometric group homomorphisms \( \psi_j : D_j \to D \) \((j = 1, 2)\) such that \( \psi_2 \circ \varphi_1 = \psi_1 |_{E_1} \) and

\[
\|\psi_1 |_{E_2} - \psi_2 \circ \varphi_2\|_\infty \leq \varepsilon.
\]
Proof. Let $\tilde{D}_0 = \{(x, -\varphi_1(x)) : x \in E_1\} \subset D_1 \times D_2$ and $\tilde{D}$, $\pi$, $\tilde{\psi}_1$ and $\tilde{\psi}_2$ be as in (2-12). Define $\tilde{\lambda} : \tilde{D} \to \mathbb{R}_+$ by

$$
\tilde{\lambda}(z) = \inf \{\lambda_1(x_1 - x_2) + \varepsilon \delta_\tilde{D}(\tilde{\psi}_1(x_2) - (\tilde{\psi}_2 \circ \varphi_2)(x_2)) + \lambda_2(y + \varphi_2(x_2)) : x_1 \in D_1, x_2 \in E_2, y \in D_2, z = \tilde{\psi}_1(x_1) + \tilde{\psi}_2(y)\}.
$$

As usual, $\tilde{\lambda}$ is a semivalue on $\tilde{D}$. We see that $\tilde{\lambda}(\tilde{\psi}_1(x_2) - (\tilde{\psi}_2 \circ \varphi_2)(x_2)) \leq \varepsilon$ for $x_2 \in E_2$ (this corresponds to (2-19)) and $\tilde{\lambda}(\tilde{\psi}_j(x)) \leq \lambda_j(x)$ for $x \in D_j$ ($j = 1, 2$). We want to show that in fact $\tilde{\lambda}(\tilde{\psi}_j(z)) = \lambda_j(z)$ for $z \in D_j$, which is equivalent to

$$(2-20) \quad \lambda_1(x - h) + \varepsilon \delta_\tilde{D}(\tilde{\psi}_1(h) - (\tilde{\psi}_2 \circ \varphi_2)(h)) + \lambda_2(y + \varphi_2(h)) \geq \lambda_j(z)$$

provided $x \in D_1$, $h \in E_2$, $y \in D_2$ and $z = \tilde{\psi}_1(x) + \tilde{\psi}_2(y)$. First assume $j = 1$. We infer from (2-21) that $k := x - z \in E_1$ and $y = -\varphi_1(k)$. Then (2-20) is equivalent to

$$(2-22) \quad \varepsilon \delta_\tilde{D}(\tilde{\psi}_1(h) - (\tilde{\psi}_2 \circ \varphi_2)(h)) + \lambda_2(h - \varphi_2(h)) \geq \lambda_1(h - k).$$

When $\tilde{\psi}_1(h) = (\tilde{\psi}_2 \circ \varphi_2)(h)$, $\varphi_2(h) = \varphi_1(h)$ and (2-22) is satisfied, thanks to the isometricity of $\varphi_1$. Otherwise, (2-22) follows from (2-18).

The case of $j = 2$ is similar and is left for the reader.

To finish the proof, define $D$, $\lambda$, $\psi_1$ and $\psi_2$ in exactly the same way as at the end of the proof of Lemma 2.19. \qed

3. Proof of Theorem 1.1

For the duration of this section $r \in \{1, \infty\}$, $N \in \mathbb{Z}_+ \setminus \{1\}$ and a set $Q \subset \mathbb{R}$ such as in (2-1) are fixed. Note that it is not assumed that $Q$ is countable. However, in some results its countability is necessary and then we add this assumption to their statements. For simplicity, we put the following

3.1. Definition. Suppose $Q$ is countable. A valued group $(G, +, p)$ is said to be a $Q$-group iff $G$ is finite or a countable group of class $O_{\text{fin}}$ and $p(G) \subset Q$.

As a special case of the general technique of Fraïssé limits, we get

3.2. Theorem. Suppose $Q$ is countable. There is a unique (up to isometric group isomorphism) $Q$-group $QG_r(N) \in G_r(N)$ with the following property. Whenever $(H, +, q) \in G_r(N)$ is a finite $Q$-group and $K$ is a subgroup of $H$, every isometric group homomorphism of $K$ into $QG_r(N)$ is extendable to an isometric group homomorphism of $H$ into $QG_r(N)$. 

Proof. The uniqueness follows from the back-and-forth method. The existence may be provide in a standard way. There are only countably many (up to isometric group isomorphism) finite $Q$-groups belonging to $\mathfrak{G}_r(N)$ and thus there is a sequence $(H_n, +, q_n)_{n=1}^\infty \subset \mathfrak{G}_r(N)$ of finite $Q$-groups in which every such group appears infinitely many times. Using repeatedly Lemma 2.19, inductively define a sequence of finite $Q$-groups $(G_n, +, p_n) \in \mathfrak{G}_r(N)$, starting with $G_0 = \{0\}$, such that $(G_n, +, p_n) \subset (G_{n+1}, +, p_{n+1})$ and every isometric group homomorphism of a subgroup of $H_n$ is extendable to an isometric group homomorphism of $H_n$ into $G_{n+1}$. Finally, put $(Q\mathfrak{G}_r(N), +, p) = \bigcup_{n=1}^\infty (G_n, +, p_n)$. The details are left for the reader. \[\Box\]

It is clear that the completion of a member of $\mathfrak{G}_r(N)$ is of the same class. In what follows, we fix the following situation. We assume that $(G, +, p) \in \mathfrak{G}_r(N)$ is complete and for some dense subgroup $G_0$ of $G$ the following conditions are fulfilled:

(QG1) whenever $(H, +, q) \in \mathfrak{G}_r(N)$ is a finite group with $Q$-valued value, $K$ is its subgroup and $\varphi: K \to G_0$ is an isometric group homomorphism, then for every $\varepsilon \in (0, 1)$ there is an $\varepsilon$-almost isometric group homomorphism $\psi: H \to G_0$ such that $\|\psi|_K - \varphi\|_{\infty} \leq \varepsilon$.

(QG2) $p(G_0) \subset Q$ and the set of finite rank elements of $G_0$ is dense in $G_0$.

(We underline that it is not assumed here that $Q$ is countable.) Our aim is to show that

(UEP) whenever $(H, +, q) \in \mathfrak{G}_r(N)$ is a finite valued group, $K$ is its subgroup and $\varphi: K \to G$ is an isometric group homomorphism, there is an isometric group homomorphism $\psi: H \to G$ which extends $\varphi$.

The proof of (UEP) is preceded by a few lemmas.

3.3. Lemma. Let $a \in G$ be of finite rank $k \geq 2$. For every $\varepsilon > 0$ there is $b \in G_0$ such that $\text{rank}(b) = k$ and $p(ja - jb) \leq \varepsilon$ for each $j \in \mathbb{Z}$.

Proof. Let $\delta \in (0, 1)$ be such that $\delta \leq \frac{\varepsilon}{4k}$. By (QG2), there is a finite rank element $c \in G_0$ for which $p(a - c) \leq \delta$. Let $H = \langle a, c \rangle$ and $K = \langle c \rangle \subset H$. Notice that $p(K) \subset Q$. We conclude from Lemma 2.6 that there is a $Q$-valued value $\varphi$ on $H$ which extends $p|_K$, is bounded by $r$ and satisfies $\|p|_H - q\|_{\infty} \leq \delta$. We see that $(H, +, q) \in \mathfrak{G}_r(N)$. So, by (QG1) applied to $\text{id}: K \to G_0$, there is a $\delta$-almost isometric (with respect to $q$) group homomorphism $\varphi: H \to G_0$ such that $p(x - \varphi(x)) \leq \delta$ for $x \in K$. Put $b = \varphi(a)$. Then $\text{rank}(b) = \text{rank}(a)$ and

\[p(a - b) \leq p(a - c) + p(c - \varphi(c)) + p(\varphi(c) - \varphi(a)) \leq 2\delta + 2\|q(c - a)\|_{\infty} \leq 2\delta + 2(p(c - a) + \delta) \leq 6\delta \leq \frac{\varepsilon}{k},\]
so \( p(ja – jb) \leq \varepsilon \) for \( j = 0, 1, \ldots, k – 1 \) and we are done. \( \square \)

3.4. Lemma. Let \( H \) be a finite subgroup of \( G \). For each \( \varepsilon > 0 \) there is a group homomorphism \( \varphi: H \to G_0 \) with trivial kernel such that \( p(\varphi(x) - x) \leq \varepsilon \) for each \( x \in H \).

Proof. We assume \( H \) is isomorphic to a direct product of cyclic groups, there is \( p \) By Lemma 3.4, there is a group homomorphism \( \sum \phi \) is a group homomorphism (thanks to (3-1) and (3-2)).

\[
\Lambda = \Lambda_1 \times \ldots \times \Lambda_s: \langle a_1 \rangle \times \ldots \times \langle a_s \rangle \to \langle b_1 \rangle \times \ldots \times \langle b_s \rangle
\]

Further, if \( l_1, \ldots, l_s \in \mathbb{Z} \), then \( p(\sum_{j=1}^s l_j a_j - \sum_{j=1}^s l_j b_j) \leq \sum_{j=1}^s p(l_j a_j - l_j b_j) \leq \varepsilon < \mu \) and thus \( \sum_{j=1}^s l_j a_j = 0 \) iff \( \sum_{j=1}^s l_j b_j = 0 \). This yields that the group homomorphism

\[
\Psi: \langle b_1 \rangle \times \ldots \times \langle b_s \rangle \ni \langle y_1, \ldots, y_s \rangle \mapsto \sum_{j=1}^s y_j \in G_0
\]

has trivial kernel. Now it suffices to put \( \varphi = \Psi \circ \Lambda \circ \Phi^{-1} \). \( \square \)

3.5. Lemma. Let \( (H, +, q) \in \mathcal{G}_r(N) \) be a finite group, \( K \) its subgroup and let \( \varphi: K \to G \) be an isometric group homomorphism. Then for every \( \varepsilon \in (0, 1) \) there is an \( \varepsilon \)-almost isometric group homomorphism \( \psi: H \to G \) such that \( \|\psi|_K - \varphi\|_{\infty} \leq \varepsilon \).

Proof. Again, we assume \( H \) is nontrivial. Let \( \mu = \min\{q(h): h \in H \setminus \{0\}\} \). Take \( \delta \) such that

\[
\delta \in (0, \frac{1}{2}), \quad \delta < c, \quad (1 + 2\delta)^2 \leq 1 + \varepsilon, \quad (1 - 2\delta)^2 \geq 1 - \varepsilon.
\]

By Lemma 3.4, there is a group homomorphism \( \kappa: \varphi(K) \to G_0 \) with trivial kernel such that \( p(\kappa(x) - x) \leq \delta^2 \). Put \( \psi_0 = \kappa \circ \varphi: K \to G_0 \). Then \( \psi_0 \) has trivial kernel and

\[
\|\psi_0 - \varphi\|_{\infty} \leq \delta^2.
\]

Let \( \lambda_0: K \ni x \mapsto p(\psi_0(x)) \in \mathbb{R}_+ \). Observe that \( \lambda_0 \) is a value bounded by \( r \) and \( \lambda_0(K) \subset Q \) (see (QG2)). Moreover, for \( x \in K \setminus \{0\} \) we have (thanks to (3-1) and (3-2)):

\[
q(x) = p(\varphi(x)) \leq p(\psi_0(x)) + \delta^2 \leq \lambda_0(x) + \delta q(x)
\]

and

\[
\lambda_0(x) \leq p(\varphi(x)) + \delta^2 \leq q(x) + \delta q(x) = (1 + \delta)q(x).
\]
The above estimations gives \( q \big|_K \leq \frac{1}{1 + \delta} \lambda_0 \) and \( \lambda_0 \leq (1 + \delta)q \big|_K \). Now by Lemma 2.7 (with \( \omega(t) = (1 + \delta)t, \varphi(t) = \frac{t}{1 + \delta} \)) and \( \tau(t) = t \), there is a value \( \lambda_1 \) on \( H \) which extends \( \lambda_0 \), is bounded by \( r \) and satisfies

\[(3-3) \quad (1 - \delta)q \leq \lambda_1 \leq (1 + \delta)q.\]

Further, since \( \lambda_1(K) = \lambda_0(K) \subset Q \), we infer from Lemma 2.6 that there is a value \( \lambda \) on \( H \) bounded by \( r \), extending \( \lambda_1 \big|_K \) and satisfying \( \lambda(H) \subset Q \) and

\[(3-4) \quad \|\lambda_1 - \lambda\|_\infty \leq \delta^2.\]

Note that \( \psi_0: K \rightarrow G_0 \) is isometric with respect to \( \lambda \) and therefore, by (QG1), there is a \( \delta \)-almost isometric group homomorphism \( \psi: (H, +, \lambda) \rightarrow (G, +, p) \) such that \( \|\psi \big|_K - \psi_0\|_\infty \leq \delta \). The latter inequality combined with (3-1) and (3-2) gives \( \|\psi \big|_K - \varphi\|_\infty \leq \varepsilon \). So, we only need to check that \( \psi \), as a group homomorphism of \( (H, +, q) \) into \( (G, +, p) \), is \( \varepsilon \)-almost isometric. For \( h \in H \setminus \{0\} \) we have, thanks to (3-1), (3-3) and (3-4):

\[p(\psi(h)) \leq (1 + \delta)\lambda(h) \leq (1 + \delta)(\lambda_1(h) + \delta^2)\]
\[\leq (1 + \delta)[(1 + \delta)q(h) + \delta q(h)] \leq (1 + 2\delta)^2q(h) \leq (1 + \varepsilon)q(h)\]

and

\[p(\psi(h)) \geq (1 - \delta)\lambda(h) \geq (1 - \delta)(\lambda_1(h) - \delta^2)\]
\[\geq (1 - \delta)[(1 - \delta)q(h) - \delta q(h)] \geq (1 - 2\delta)^2q(h) \geq (1 - \varepsilon)q(h)\]

which finishes the proof. \( \Box \)

3.6. Remark. From the beginning of the section to this moment we have never used the assumption that \( G \) is complete. So, the assertion of Lemma 3.5 is fulfilled for every group \( G \) which is ‘between’ \( G_0 \) and its completion. In particular, Lemma 3.5 holds true for \( G = G_0 \).

3.7. Lemma. Let \( (H, +, q) \in \mathfrak{G}_r(N) \) be a finite group and \( K \) its subgroup. Further, let \( \varphi: K \rightarrow G \) and \( \psi: H \rightarrow G \) be, respectively, an isometric and an \( \varepsilon \)-almost isometric group homomorphism (where \( \varepsilon \in (0, 1) \)) such that \( \|\psi \big|_K - \varphi\|_\infty \leq \varepsilon \). For every \( \delta \in (0, \varepsilon) \) there is a \( \delta \)-almost isometric group homomorphism \( \psi': H \rightarrow G \) such that \( \|\psi' \big|_K - \varphi\|_\infty \leq \delta \) and \( \|\psi - \psi'\|_\infty \leq C\varepsilon \) where \( C = 3 + 2 \text{diam}(H, q) \).

Proof. Let \( L = \varphi(K) + \psi(H) \) and \( p' = p \big|_L \). Then \( (L, +, p') \in \mathfrak{G}_r(N) \) and \( L \) is finite. By Lemma 2.20, there are a finite valued group \( (D, +, \lambda) \in \mathfrak{G}_r(N) \) and isometric group homomorphisms \( w_H: H \rightarrow D \) and \( w_L: L \rightarrow D \) such that

\[(3-5) \quad w_L \circ \varphi = w_H \big|_K \quad \text{and} \quad \|w_H - w_L \circ \psi\|_\infty \leq A\varepsilon\]

and
where \( A = 1 + \text{diam}(H, q) \). Put \( D_0 = w_L(L) \). Observe that \( w_L^{-1} : D_0 \to L \subset G \) is isometric. Hence, by Lemma 3.5, there is a \( \delta \)-almost isometric group homomorphism \( v : D \to G \) such that

\[
(3-6) \quad \|v|_{D_0} - w_L^{-1}\|_\infty \leq \delta.
\]

Then \( \psi' = v \circ w_H : H \to G \) is \( \delta \)-almost isometric. Now if \( x \in K \), then by (3-5), \( w_H(x) = (w_L \circ \varphi)(x) \) and thus

\[
p(\varphi(x) - \psi'(x)) = p(w_L^{-1}((w_L \circ \varphi)(x)) - v((w_L \circ \varphi)(x))) \leq \delta,
\]

thanks to (3-6). Finally, if \( h \in H \), then \( \psi(h) \in L \) and (by (3-5) and (3-6)):

\[
p(\psi(h) - \psi'(h)) \leq p(w_L^{-1}((w_L \circ \psi)(h)) - v((w_L \circ \psi)(h)))
\]

\[
+ p(v((w_L \circ \psi)(h)) - v(w_H(h))) \leq \|w_L^{-1} - v|_{D_0}\|_\infty
\]

\[
+ (1 + \delta)p((w_L \circ \psi)(h) - w_H(h)) \leq \delta + 2\|w_L \circ \psi - w_H\|_\infty
\]

\[
\leq \varepsilon + 2A\varepsilon = C\varepsilon.
\]

\( \square \)

Now we are ready to prove

3.8. Theorem. The group \( G \) satisfies (UEP).

Proof. Let \((H, +, q), K\) and \( \varphi \) be as in (UEP). Put \( C = 3 + 2\text{diam}(H, q) \) and \( \varepsilon_n = 2^{-n} \) \((n \geq 1)\). By Lemma 3.5, there is an \( \varepsilon_1 \)-almost isometric group homomorphism \( \psi_1 : H \to G \) such that \( \|\psi_1|_K - \varphi\|_\infty \leq \varepsilon_1 \). Suppose we have \( \psi_n\) for some \( n \geq 2 \). Applying Lemma 3.7, we obtain \( \psi_n : H \to G \) such that

\[
(1_n) \quad \psi_n \text{ is an } \varepsilon_n\text{-almost isometric group homomorphism},
\]

\[
(2_n) \|\psi_n|_K - \varphi\|_\infty \leq \varepsilon_n,
\]

\[
(3_n) \|\psi_n - \psi_{n-1}\|_\infty \leq C\varepsilon_{n-1}.
\]

The condition (3_n) ensures that the sequence \( (\psi_n(h))_{n=1}^\infty \) converges for every \( h \in H \) and thus we may define a group homomorphism \( \psi : H \to G \) by \( \psi(h) = \lim_{n \to \infty} \psi_n(h) \). Then (1_n) yields that \( \psi \) is isometric and (2_n) gives \( \psi|_K = \varphi \).

\( \square \)

Proof of Theorem 1.1. Let \( G \) be the completion of \( \mathbb{Q}G_r(N) \) (cf. Theorem 3.2). By Theorem 3.8, \( G \) satisfies (UEP). This gives the existence of \( \mathbb{G}_r(N) \). The uniqueness follows again from Theorem 3.8, applied to the situation when \( Q = \mathbb{R} \), (G3) and the back-and-forth method.

\( \square \)

Note that we have proved that \( \mathbb{G}_r(N) \) fulfills (UEP). This fact will be used in the next section. The results of this section gives more than just the assertion of Theorem 1.1. Namely,

3.9. Proposition. In each of the following cases the completion of a valued group \((G, +, p)\) is isometrically group isomorphic to \( \mathbb{G}_r(N) \).
(A) \( G = Q\mathbb{G}_r(N) \) where \( Q \) is a countable set as in (2-1) (cf. Theorem 3.2).

(B) \((G,+,p) \in \mathbb{G}_r(N) \) and \( G \) satisfies conditions (G2) and (G3) of Theorem 1.1 (with \( \mathbb{G}_r(N) \) replaced by \( G \)).

(C) \((G,+,p) \) satisfies conditions (QG1) and (QG2) (with \( G_0 \) replaced by \( G \)) for some set \( Q \) as in (2-1).

4. Proof of Theorem 1.2

As in the previous section, we fix \( r \in \{1, \infty\} \) and \( N \in \mathbb{Z}_+ \setminus \{1\} \). Our first aim of this part is to show that

(CEP) Whenever \((L,+,q) \in \mathbb{G}_r(N), K \) and \( H \) are, respectively, a compact and a finite subgroup of \( L \), and \( \varphi: K \to \mathbb{G}_r(N) \) is an isometric group homomorphism, then there is an isometric group homomorphism \( \psi: K + H \to \mathbb{G}_r(N) \) which extends \( \varphi \).

Similarly as in the previous section, the proof of (CEP) is preceded by a few auxiliary lemmas. In some of them we use the Hausdorff distance, which is denoted by us by \( \text{dist}_q(A,B) \) if only \( A \) and \( B \) are two compact nonempty subsets of a valued group with value \( q \).

4.1. Lemma. Let \( a \in \mathbb{G}_r(0) \) be such that the closure \( K \) of \( \langle a \rangle \) is compact. Then for every \( \varepsilon > 0 \) there is a finite rank element \( b \) of \( \mathbb{G}_r(0) \) such that \( \text{dist}_p([b], K) \leq \varepsilon \) where \( p \) is the value of \( \mathbb{G}_r(0) \).

Proof. We assume \( \text{rank}(a) = \infty \). Since the elements of the sequence \( (na)_{n=1}^{\infty} \) form a dense subset of \( K \), there is \( m \geq 2 \) such that the set \( \{0,a,\ldots,(m-1)a\} \) is an \((\varepsilon/4)\)-net for \( K \) and

\[(4-1) \quad p(ma) \leq \frac{\varepsilon}{8}.
\]

By (G3), there is a finite rank element \( c \in \mathbb{G}_r(0) \) for which

\[(4-2) \quad p(a - c) \leq \frac{\varepsilon}{4m}.
\]

Put \( H = \langle c \rangle \) and let \( H' = \langle c' \rangle \) be a cyclic group of rank \( m \). For each \( k \in \mathbb{Z} \) let \( s(k) \in \{0,1,\ldots,m-1\} \) be such that \( m|k - s(k)| \). Define \( q_0: H \times H' \to \mathbb{R}_+ \) by \( q_0(x,lc') = p(x + s(l)a) + \delta_{H'}(lc')\varepsilon/8 \) \((l \in \mathbb{Z})\). Observe that \( q_0 \) is well defined; \( q_0(x,y) = 0 \) iff \( x = 0 \) and \( y = 0 \); and \( q_0(x,0) = p(x) \). We claim that \( q_0 \) satisfies the triangle inequality, that is, \( q_0(x + x', (l + l')c') \leq q_0(x,lc') + q_0(x',l'c') \). Indeed, if \( s(l) + s(l') < m \), then \( s(l + l') = s(l) + s(l') \) and then the latter inequality is immediate. And when \( s(l) + s(l') \geq m \), we have \( s(l) \geq 0 \), \( s(l') \geq 0 \) and \( s(l + l') = s(l) + s(l') - m \) and hence, by (4-1),

\[
q_0(x,lc') + q_0(x',l'c') \geq p(x + x' + (s(l) + s(l'))a) + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \geq p(x + x' + s(l + l')a - ma) + p(ma) + \frac{\varepsilon}{8} \delta_{H'}((l + l')c') \geq q_0(x + x', (l + l')c').
\]
Now let \( q: H \times H' \to \mathbb{R}_+ \) be given by

\[
q(x, y) = \left[ \frac{1}{2} (q_0(x, y) + q_0(-x, -y)) \right] \wedge r.
\]

It follows from the properties of \( q_0 \) that \( q \) is a value and \( q(x, 0) = p(x) \) for \( x \in H \). Moreover, for \( j \in \{0,1,\ldots,m-1\} \) one has

\[
q(jc, -jc') \leq \frac{\varepsilon}{2}.
\]

Indeed, we may assume that \( j \neq 0 \) and then \( s(j) = j \) and \( s(-j) = m-j \). So, by (4-1) and (4-2),

\[
q_0(jc, -jc') \leq p(j(c-a)) + p(ma) + \frac{\varepsilon}{8} \leq \frac{\varepsilon}{2}
\]

and \( q_0(-jc, jc') = p(j(c-a)) + \frac{\varepsilon}{8} \leq \frac{\varepsilon}{2} \). Further, since the function \( H \times \{0\} \ni (h, 0) \mapsto h \in G_r(0) \), we conclude from (UEP) that there exists an isometric group homomorphism \( \psi: H \times H' \to G_r(0) \) with \( \psi(h, 0) = h \) for any \( h \in H \). Put \( b = \psi(0, c') \). Then \( \langle b \rangle = \{0, b, \ldots, (m-1)b\} \). Now if \( j \in \{0,1,\ldots,m-1\} \), we get (see (4-2) and (4-3)):

\[
p(jb - ja) \leq p(jc - jb) + p(jc - ja) \leq p(\psi(jc, 0) - j\psi(0, c')) + jp(c-a)
\]

\[
\leq p(\psi(jc, -jc')) + \frac{\varepsilon}{4} = q(jc, -jc') + \frac{\varepsilon}{4} \leq \frac{3}{4}\varepsilon.
\]

Conversely, if \( x \in K \), there is \( j \in \{0,1,\ldots,m-1\} \) such that \( p(x - ja) \leq \varepsilon/4 \) and then \( p(x - jb) \leq p(x - ja) + p(ja - jb) \leq \varepsilon \) which finally gives

\[
\text{dist}_p(\langle b \rangle, K) \leq \varepsilon.
\]

From now to the end of the next section \( p \) denotes the value of \( G_r(N) \).

4.2. Lemma. Let \( K \) be a compact subgroup of \( G_r(N) \). For each \( \varepsilon > 0 \) there is a finite subgroup \( H \) of \( G_r(N) \) such that \( \text{dist}_p(H, K) \leq \varepsilon \).

Proof. There is \( s \geq 2 \) and \( a_1, \ldots, a_s \in K \) such that \( \{a_1, \ldots, a_s\} \) is an \((\varepsilon/2)\)-net for \( K \). If \( N \neq 0 \), put \( H = \langle a_1, \ldots, a_s \rangle \) and notice that \( \text{dist}_p(H, K) \leq \varepsilon/2 \) since \( \{a_1, \ldots, a_s\} \subset H \subset K \). We now assume \( N = 0 \). By Lemma 4.1, there are finite rank elements \( b_1, \ldots, b_s \in G_r(0) \) with \( \text{dist}_p(\langle b_j \rangle, K_j) \leq \varepsilon/s \) where \( K_j \) is the closure of \( \langle a_j \rangle \) \( (j = 1, \ldots, s) \). Let \( H = \langle b_1, \ldots, b_s \rangle \). For every \( x \in H \) there are \( x_j \in \langle b_j \rangle \) \( (j = 1, \ldots, s) \) for which \( x = \sum_{j=1}^s x_j \). Then for every \( j \) one can find \( y_j \in K_j \) such that \( p(x_j - y_j) \leq \varepsilon/s \) and hence \( p(x - y) \leq \varepsilon \) for \( y = \sum_{j=1}^s y_j \in K \). Conversely, if \( y \in K \), there is \( j \) such that \( p(x - a_j) \leq \varepsilon/2 \) and there is \( w \in \langle b_j \rangle \subset H \) with \( p(a_j - w) \leq \varepsilon/s \leq \varepsilon/2 \) which yields \( p(x - w) \leq \varepsilon \) and we are done.

For need of the nearest three results let \((L,+,q)\), \( K \), \( H \) and \( \varphi \) be as in (CEP).

4.3. Lemma. For every \( \varepsilon > 0 \) there is an isometric group homomorphism \( \psi: H \to G_r(N) \) such that for any \( h \in H \) and \( k \in K \),

\[
|p(\psi(h) - \varphi(k)) - q(h - k)| \leq \varepsilon.
\]
Proof. Let $\tilde{K} = \varphi(K)$. By Lemma 4.2, there is a finite subgroup $F$ of $\mathbb{G}_r(N)$ such that
\begin{equation}
\text{dist}_p(F, \tilde{K}) \leq \frac{\varepsilon}{2}.
\end{equation}

It follows from Lemma 2.19, applied to $\varphi_1 = \varphi: K \to \tilde{K} + F$ and $\varphi_2 = \text{id}: K \to L$, that there exist a valued group $(D, +, \lambda) \in \mathcal{G}_r(N)$ and isometric group homomorphisms $\Phi: L \to D$ and $\Psi: \tilde{K} + F \to D$ for which $\Psi \circ \varphi = \Phi|_K$. Further, the group $\Phi(H) + \Psi(F)$ is a finite subgroup of $D$ and the group homomorphism $\Psi^{-1}|_{\Psi(F)}: \Psi(F) \to \mathbb{G}_r(N)$ is isometric. So, thanks to (UEP), there is an isometric group homomorphism $\tau: \Phi(H) + \Psi(F) \to \mathbb{G}_r(N)$ which extends $\Psi^{-1}|_{\Psi(F)}$.

Put $\psi = \tau \circ \Phi|_H$. Fix $h \in H$ and $k \in K$. Take $f \in F$ such that $p(f - \varphi(k)) \leq \varepsilon/2$ (see (4-5)). Then $p(\psi(h) - \varphi(k)) \leq p(\tau(\Phi(h)) - f) + \varepsilon/2$ and
\begin{align*}
p(\tau(\Phi(h)) - f) &= p(\tau(\Phi(h)) - \tau(\Phi(f))) = \lambda(\Phi(h) - \Phi(f)) + \lambda(\Phi(k) - \Phi(f)) = q(h - k) \\
&\quad + \lambda(\Phi(k) - \Phi(f)) = q(h - k) + p(\varphi(k) - f) \leq q(h - k) + \frac{\varepsilon}{2}
\end{align*}
which shows that $p(\psi(h) - \varphi(k)) \leq q(h - k) + \varepsilon$. Conversely,
\begin{align*}
q(h - k) &= \lambda(\Phi(h) - \Phi(k)) \leq \lambda(\Phi(h) - \Phi(f)) + \lambda(\Phi(f) - \Phi(k)) \\
&= p(\tau(\Phi(h)) - \tau(\Phi(f))) + \lambda(\Phi(f) - \Phi(k)) \\
&= p(\psi(h) - f) + p(f - \varphi(k)) \leq p(\psi(h) - \varphi(k)) + 2p(f - \varphi(k))
\end{align*}
which finishes the proof of (4-4), because $2p(f - \varphi(k)) \leq \varepsilon$. \hfill \Box

4.4. Lemma. Let $\psi: H \to \mathbb{G}_r(N)$ be an isometric group homomorphism such that (4-4) is fulfilled for any $h \in H$ and $k \in K$. For each $\delta > 0$ there exists an isometric group homomorphism $\psi': H \to \mathbb{G}_r(N)$ for which $\|\psi - \psi'\|_\infty \leq \varepsilon + \delta$ and for all $h \in H$ and $k \in K$,
\begin{equation}
|p(\psi'(h) - \varphi(k)) - q(h - k)| \leq \delta.
\end{equation}

Proof. By Lemma 2.21, there are a valued group $(D, +, \lambda) \in \mathcal{G}_r(N)$ and isometric group homomorphisms $w_L: L \to D$ and $w_G: \varphi(K) + \psi(H) \to D$ such that $w_L|_K = w_G \circ \varphi$ and
\begin{equation}
\|w_L|_H - w_G \circ \psi\|_\infty \leq \varepsilon.
\end{equation}
Then $Z := w_G(\varphi(K) + \psi(H))$ and $F := w_L(H) + w_G(\psi(H))$ are, respectively, a compact and a finite subgroup of $D$, and $w_G^{-1}: Z \to \mathbb{G}_r(N)$ is isometric. So, thanks to Lemma 4.3, there is an isometric group homomorphism $\xi: F \to \mathbb{G}_r(N)$ such that for any $z \in Z$ and $f \in F$,
\begin{equation}
|p(w_G^{-1}(z) - \xi(f)) - \lambda(z - f)| \leq \delta.
\end{equation}
Hence (4-6) follows from (4-8). It suffices to show that 

$$z \phi \epsilon$$

Put

$$\psi = \xi \circ w_L|_H : H \to G_r(N)$$

Now fix \( h \in H \) and \( k \in K \) and put \( z = w_G(\varphi(k)) \in Z \) and \( f = w_L(h) \in F \). Then we have \( p(\psi'(h) - \varphi(k)) = p(\xi(f) - w_L^{-1}(z)) \) and (since \( w_G \circ \varphi = w_L\) )

\[
q(h - k) = \lambda(w_L(h) - w_L(k)) = \lambda(w_L(h) - w_G(\varphi(k))) = \lambda(f - z).
\]

Hence (4-6) follows from (4-8). It suffices to show that \( ||\psi - \psi'||_\infty \leq \varepsilon + \delta \). For \( h \in H \) we get

\[
p(\psi(h) - \psi'(h)) \leq p(\psi(h) - \xi(\{w_G \circ \psi\}(h))) + p(\xi(\{w_G \circ \psi\}(h)) - \xi(w_L(h)))
\]

and, by (4-7), \( p(\xi(\{w_G \circ \psi\}(h)) - \xi(w_L(h))) = \lambda(\{w_G \circ \psi\}(h) - w_L(h)) \leq \varepsilon \). So, it remains to check that \( p(\psi(h) - \xi(\{w_G \circ \psi\}(h))) \leq \delta \). But this follows from (4-8) for \( z = f = (w_G \circ \psi)(h) \in Z \cap F \).

Finally, we have

4.5. Theorem. The assertion of (CEP) is satisfied.

Proof. Put \( \varepsilon_n = \frac{1}{2^n} \). By Lemma 4.3, there is an isometric group homomorphism \( \psi_1 : H \to G_r(N) \) such that (4-4) (with \( \psi \) replaced by \( \psi_1 \)) is fulfilled for any \( h \in H \) and \( k \in K \). Now suppose that \( \psi_{n-1} : H \to G_r(N) \) is constructed. We infer from Lemma 4.4 that there is a group homomorphism \( \psi_n : H \to G_r(N) \) such that

\[
(1_n) \psi_n \text{ is isometric},
\]

\[
(2_n) ||\psi_{n-1} - \psi_n||_\infty \leq \varepsilon_{n-1} + \varepsilon_n \leq 2\varepsilon_{n-1},
\]

\[
(3_n) |p(\psi_n(h) - \varphi(k)) - q(h - k)| \leq \varepsilon_n \text{ for all } h \in H \text{ and } k \in K.
\]

We conclude from (1_n) and (2_n) that a group homomorphism \( \varphi_0 : H \to G_r(N) \) given by \( \varphi_0(h) = \lim_{n \to \infty} \psi_n(h) \) is well defined and isometric. What is more, (3_n) yields

\[
(4-9) \quad p(\varphi_0(h) - \varphi(k)) = q(h - k)
\]

for any \( h \in H \) and \( k \in K \). We easily deduce from (4-9) that the formula \( \psi(h + k) = \varphi_0(h) + \psi(k) \) (where \( h \in H \) and \( k \in K \)) well defines an isometric group homomorphism \( \psi : K + H \to G_r(N) \) which extends \( \varphi \). □

Proof of Theorem 1.2. We begin with the note that the assertion of part (B) follows from (A), (G3) and the back-and-forth method. Thus, it suffices to prove (A). First we shall show the easier point, (i). When \( \varphi \) is not open, it suffices to put \( \tau \equiv 0 \), \( q = \text{id} \) and to apply Corollary 2.4 in order to find \( \omega \in \Omega^* \). Similarly, if \( \varphi \) is open as a map of \( K \) onto \( \varphi(K) \), the semivalue \( q_0 : K \ni x \mapsto \text{dist}_q(x, \ker \varphi) \in \mathbb{R}_+ \) is continuous with respect to \( p \circ \varphi \) and hence for \( \tau_0 = \frac{\text{id}}{\text{id} + 1} \) one may find (thanks to Lemma 2.2) \( q_0 \in \Omega^* \) such that \( \tau_0 \circ q_0 \leq q_0 \circ p \circ \varphi \). Further, take suitable \( \omega_0 \in \Omega^* \) and apply the idea of Example 2.8 to guarantee \( (\omega \circ q) \). We now pass to the main part of the theorem—to point (ii).

Put \( \lambda_0 : K \ni x \mapsto p(\varphi(x)) \in \mathbb{R}_+. \) Then \( \lambda_0 \) is a semivalue on \( K \) such that \( \lambda_0 \leq (\omega \circ q)|_K \). Lemma 2.7 ensures us the existence of a
semivalue \( \lambda \) on \( H \) bounded by \( r \) which extends \( \lambda_0 \) and satisfies suitable conditions. Put \( H' = H/\ker \varphi \). Let \( \pi : H \to H' \) be the quotient group homomorphism, \( \lambda' \) the value induced by \( \lambda \) (recall that \( \lambda^{-1}\{0\} = \lambda_0^{-1}\{0\} = \ker \varphi \)), \( K' = \pi(K) \) and let \( \varphi' : K' \to \mathcal{G}_r(N) \) be a group homomorphism such that \( \varphi' \circ \pi|_K = \varphi \). Notice that \( (H', +, \lambda') \in \mathcal{G}_r(N) \) (thanks to Proposition 2.14) and
\[
(4-10) \quad p(\varphi'(x)) = \lambda'(x)
\]
for every \( x \in K' \) (because \( \lambda \) extends \( \lambda_0 \)). Now let \( (\tilde{H}, +, \tilde{\lambda}) \in \mathcal{G}_r(N) \) be the completion of \( (H', +, \lambda') \). The relation (4-10) ensures us that \( \varphi' \) extends to an isometric group homomorphism \( \tilde{\varphi} : \tilde{K} \to \mathcal{G}_r(N) \) defined on a closed subgroup of \( \tilde{H} \). Since the closure of \( \varphi(K) \) is compact in \( \mathcal{G}_r(N) \), \( \tilde{K} \) is compact. Enlarging, if needed, the group \( \tilde{H} \) (making use of Theorem 2.12) we may assume that \( \tilde{H}_{fin} \) is dense in \( \tilde{H} \). Now thanks to (CEP) and the induction argument we see that there is an isometric group homomorphism \( \tilde{\psi} : \tilde{H} \to \mathcal{G}_r(N) \) which extends \( \tilde{\varphi} \). Define \( \varphi_\omega : H \to \mathcal{G}_r(N) \) by \( \varphi_\omega = \tilde{\psi} \circ \pi \) and observe that \( \ker \varphi = \ker \varphi_\omega \) and \( p(\varphi_\omega(h)) = \lambda(h) \) for each \( h \in H \). The verification that all other assertions are fulfilled is left for the reader.

In the next section we shall show that \( \mathcal{G}_r(2) \) is Urysohn as a metric space. Thus, Theorem 1.2 extends and strengthens the results of [17].

4.6. Corollary. (A) The group \( \mathcal{G}_r(N) \) is universal for the class \( \mathcal{G}_r(N) \); that is, every member of \( \mathcal{G}_r(N) \) admits an isometric group homomorphism into \( \mathcal{G}_r(N) \).
(B) The groups \( \mathcal{G}_1(0) \) and \( \mathcal{G}_\infty(0) \) are topologically universal for the class of separable metrizable topological Abelian groups. What is more, for every \( (G_+, +, q) \in \mathcal{G} \) the valued group \( (G_+, +, q \wedge 1) \) (respectively \( (G_+, +, q^\alpha) \) with \( 0 < \alpha < 1 \)) admits an isometric group homomorphism into \( \mathcal{G}_1(0) \) (respectively into \( \mathcal{G}_\infty(0) \)).

For two pairs \( (r, N), (s, M) \in \{1, \infty\} \times (\mathbb{Z}_+ \setminus \{1\}) \) let us write \( (r, N) \preceq (s, M) \) iff \( r \leq s \) and \( N|M \). The reader will easily check that

4.7. Proposition. Let \( (r, N) \in \{1, \infty\} \times (\mathbb{Z}_+ \setminus \{1\}) \).
(A) \( \mathcal{G}_s(M) \) is embeddable in \( \mathcal{G}_r(N) \) by means of an isometric group homomorphism iff \( (s, M) \preceq (r, N) \).
(B) Let \( M|N \) (and \( M \neq 1 \)). Then the group \( \mathcal{G}_r(N, M) := \{x \in \mathcal{G}_r(N) : M \cdot x = 0\} \) is isometrically group isomorphic to \( \mathcal{G}_r(M) \).

The above simple result has two interesting consequence, formulated in the next two results.

4.8. Proposition. There is a family
\[\{\Phi_{s,M}^{r,N} : \mathcal{G}_s(M) \to \mathcal{G}_r(N)\}_{(s,M) \preceq (r,N)}\]
of isometric group homomorphisms such that for each \( (r, N), (s, M), (t, L) \) with \( (r, N) \preceq (s, M) \preceq (t, L) \):
Proof. Let $\Psi_{r,N} : G_1(0) \to G_\infty(0)$ be an isometric group homomorphism. Further, let $\Psi : G \to G$ be group isomorphisms (where $G$ is as in the statement of Proposition 4.7). Now it suffices to put $\Phi_{r,N} = \Phi_{r,N}^{-1} \circ \Psi_{r,N}$ provided $(s, M) \preceq (r, N)$. □

4.9 Proposition. Suppose $N \neq 0$. If $N_1, \ldots, N_s \geq 2$ are mutually coprime and $N = N_1 \cdot \ldots \cdot N_s$, then $G_r(N)$ and $G_r(N_1) \times \ldots \times G_r(N_s)$ are isomorphic as topological groups.

Proof. The assertion follows from Proposition 4.7 and the fact that the function

$$G_r(N_1) \times \ldots \times G_r(N_s) \ni (x_1, \ldots, x_s) \mapsto \sum_{j=1}^{s} x_j \in G_r(N)$$

is an isomorphism of topological groups. □

In the next section we shall show that $G_r(N)$’s are pairwise nonisomorphic as topological groups.

The main assumption of Theorem 1.2 is that the closure of the image of a group homomorphism is compact. One may ask whether one may weaken this condition. As the next result shows, nothing else may be done in this direction for group homomorphisms with (metrically) bounded images. (This result has its natural well-known counterpart for the Urysohn metric space.)

4.10 Proposition. Let $K$ be a closed bounded subgroup of $G_r(N)$ such that for every finite group $H$ and every value $q$ on $K \times H$ such that $(K \times H, +, q) \in G_r(N)$ and $q(x, 0) = p(x)$ for $x \in K$ there is an isometric group homomorphism $\psi : K \times H \to G_r(N)$ with $\psi(x, 0) = x$ for $x \in K$. Then $K$ is compact.

Proof. Suppose $K$ is noncompact. We infer from the completeness of $K$ that then there exist $\varepsilon > 0$ and a sequence $(x_n)_{n=1}^\infty \subset K$ such that $p(x_n - x_m) \geq \varepsilon$ for distinct $n$ and $m$. Put $A = \{ x_n : n \geq 1 \}$. For every $x \in G_r(N)$ let $e_x : A \to \mathbb{R}_+$ be given by $e_x(a) = p(x - a)$. It is easily seen that the set $E = \{ e_x : x \in G_r(N) \} \subset E_r(A)$ is separable (with respect to the supremum metric), since the map $G_r(N) \ni x \mapsto e_x \in E$ is a nonexpansive surjection. However, we shall show that $E$ contains an uncountable discrete subset (which will finish the proof).

Suppose $f \in E_r(A)$ is such that $f$ satisfies (2-8) for any $a_1, \ldots, a_N \in A$ provided $N > 2$. Then, by Theorem 2.17, $f$ is trivial in some valued group belonging to $G_r(N)$ and of the form $K \times H$ with $H$ finite. So,
thanks to our assumption on $K$, $f$ is trivial in $\mathbb{G}_r(N)$ which means that $f \in E$.

Let $M = \text{diam}(K) > 0$. Take $\delta \in (0, \varepsilon)$ such that $\delta < M/2$. For a subset $J$ of $A$ let $f_J: A \to \mathbb{R}_+$ be given by: $f_J(a) = M$ if $a \in J$ and $f_J(a) = M - \delta$ otherwise. A direct calculation shows that $f_J \in E$ and $f_J$ satisfies (2-8) provided $N \neq 0$. So, according to the previous paragraph, $f_J \in E$. But $\|f_J - f_{J'}\|_{\infty} = \delta$ whenever $J$ and $J'$ are different subsets of $A$ and hence $E$ cannot be separable. □

4.11. Remark. Melleray [14, 15] has shown that if every isometry between to subsets of the Urysohn metric space $U$ which are isometric to a given separable complete metric space $X$ is extendable to an isometry of $U$ onto itself, then $X$ is compact. A counterpart of this result in category of valued groups reads as follows:

If every isometric group isomorphism between two subgroups of $\mathbb{G}_r(N)$ which are isometrically group isomorphic to a complete group $(H, +, q) \in \mathbb{G}_r(N)$ is extendable to an isometric group automorphism of $\mathbb{G}_r(N)$, then $H$ is compact.

It follows from Proposition 4.10 that every group $H$ having the above property and bounded value has to be compact. However, the problem whether the above stated result is true in $\mathbb{G}_\infty(N)$ we leave open.

5. Geometry of $\mathbb{G}_r(N)$’s

The part is mainly devoted to investigations of the groups $\mathbb{G}_r(N)$’s as metric spaces. We begin with

5.1. Theorem. The metric spaces $\mathbb{G}_r(N)$ with $N \in \{0, 2\}$ are Urysohn. In particular, $\mathbb{G}_r(2)$ is a Boolean Urysohn metric group introduced in [17].

Proof. This is an almost immediate consequence of (UEP) and Theorem 2.17. Let $G_0 = \mathbb{G}_r(N)_{\text{fin}}$. Let $f \in E(G_0)$ and $B$ be a finite nonempty subset of $G_0$. Put $H = \langle B \rangle$. By Theorem 2.17, $H$ may be enlarged to a finite group belonging to $\mathbb{G}_r(N)$ in which $f|_B$ is trivial. Now we infer from (UEP) that $f|_B$ is indeed trivial in $G_0$. So, it follows from the well-known result on the Urysohn space (see e.g. [24, 25], [10], [15]) that $\mathbb{G}_r(N)$, as the completion of $G_0$, is Urysohn. □

5.2. Remark. The same argument as in the proof of Theorem 5.1 shows that the metric space $Q\mathbb{G}_r(N)$ for $N \in \{0, 2\}$ is the so-called rational Urysohn space.

Theorem 5.1 tells us ‘everything’ about the metric spaces $\mathbb{G}_r(N)$ with $N \in \{0, 2\}$. Therefore from now on, we assume $N > 2$. In Theorem 5.5 we shall show that in that case $\mathbb{G}_r(N)$ is not Urysohn.
In the same way as in the proof of Theorem 5.1 one shows the next result the proof of which is left as an exercise (use Theorem 2.17 and Remark 2.16; recall that all elements are of finite rank).

5.3. **Proposition.** (A) Let $B$ be a finite nonempty subset of $\mathcal{G}_r(N)$ and $f \in E_r(B)$. Then $f$ is trivial in $\mathcal{G}_r(N)$ iff $f$ fulfills (2.8) for any $a_1, \ldots, a_N \in B$. In particular, $f$ is trivial in $\mathcal{G}_r(N)$ iff $f|_A$ is trivial in $\mathcal{G}_r(N)$ for every subset $A$ of $B$ such that $0 < \text{card}(A) \leq N$.

(B) Let $H$ be a finite subgroup of $\mathcal{G}_r(N)$ and $f \in E_r(H)$. The map $f$ is trivial in $\mathcal{G}_r(N)$ iff $f$ fulfills (2-9).

It turns out that the exponent $N$ is determined by the metric of $\mathcal{G}_r(N)$. This is a consequence of the above result and the following

5.4. **Example.** Let $Z = \langle b \rangle$ be a cyclic group of rank $N$ and let $H = \mathcal{G}_r(N)$. For $j = 1, \ldots, N$ let $e_j = (e_j, \ldots, e_j) \in H$ with $e_{jj} = b$ and $e_{jk} = 0$ for $k \neq j$. Further, let $F = \{\pm e_j : j = 1, \ldots, e_N\} \cup \{e_j - e_k : j, k = 1, \ldots, N\} \subset H$. Of course, $F$ generates $H$. Let $\|\cdot\|_F$ be the ‘norm’ on $H$ generated by $F$ (in the terminology of finitely generated groups, cf. [7]), i.e. $\|\cdot\|_F$ is a value such that for nonzero $h \in H$,

\[
\|h\|_F = \min\{n \geq 1 | \exists f_1, \ldots, f_n \in F : h = \sum_{s=1}^n f_s\}
\]

(note that $F = -F$). We see that

\[
(5-1) \quad \|e_j - e_k\|_F = 1 \quad \text{for} \quad j \neq k.
\]

We claim that if $n_1, \ldots, n_N \in \mathbb{Z}_+$,

\[
(5-2) \quad \sum_{j=1}^N n_je_j\|_F \leq N - \max(n_1, \ldots, n_N) \quad \text{provided} \quad \sum_{j=1}^N n_j = N.
\]

Indeed, suppose that e.g. $\max(n_1, \ldots, n_N) = n_N$ and observe that $\sum_{j=1}^N n_je_j = \sum_{j=1}^{N-1} n_j(e_j - e_N)$, thanks to the assumption in (5-2). The latter equality gives (5-2). Further, we have

\[
(5-3) \quad \|\sum_{j=1}^N e_j\|_F = N - 1.
\]

To see this, suppose (for the contrary) that $\|\sum_{j=1}^N e_j\|_F \leq N - 2$. This means (since $0 \in F$) that there are $f_1, \ldots, f_{N-2} \in F$ which sum up to $\sum_{j=1}^N e_j$. Write $f_s = \sum_{j=1}^{N-s} \varepsilon_{js} e_j$ where $\varepsilon_{js} \in \{0, 1, -1\}$. Since the rank of $b$ is $N$, we conclude from this that $\sum_{s=1}^{N-2} \varepsilon_{js} = 1$ for $j = 1, \ldots, N$ and hence $\sum_{j=1}^N \sum_{s=1}^{N-2} \varepsilon_{js} = N$. However, $\sum_{j=1}^N \sum_{s=1}^{N-2} \varepsilon_{js} \in \{-1, 0, 1\}$ for $s = 1, \ldots, N - 2$ (because $f_s \in F$) and therefore $\sum_{s=1}^{N-2} |\sum_{j=1}^N \varepsilon_{js}| < N - 1$ which denies earlier conclusion. So, (5-3) is fulfilled (by (5-2)).
Now put \( c = \max(1/2, 1 - 2/N) \) and let \( f: \{e_1, \ldots, e_N\} \to \mathbb{R}_+ \) be constantly equal to \( c \). By (5-1), \( f \) is a Katětov map. Observe that (thanks to (5-3))

\[
(5-4) \quad \| \sum_{j=1}^N e_j \|_F > \sum_{j=1}^N f(e_j)
\]

(here is the only moment where we need to have \( N \geq 3 \)). So, \( f \) fails to satisfy (2-8) and thus \( H \) cannot be enlarged to a valued group of exponent \( N \) in which \( f \) is trivial (by Proposition 2.15). However, if \( A \) is a proper nonempty subset of \( \{e_1, \ldots, e_N\} \) and \( a_1, \ldots, a_N \in A \), then the inequality (2-8) is fulfilled, by (5-2). Consequently, \( H \) may be enlarged to a finite group belonging to \( \mathfrak{G}_r(N) \) in which \( f \big|_A \) is trivial.

If we now replace \( \| : \|_F \) by \( \alpha \| : \|_F \) with small enough \( \alpha > 0 \), we shall obtain an analogous example in the class \( \mathfrak{G}_1(N) \).

As a corollary of Theorem 5.1, Proposition 5.3 and the above example we obtain

5.5. Theorem. When \( N > 2 \), \( N \) is the least natural number \( k \) with the following property. For every finite nonempty subset \( A \) of \( \mathfrak{G}_r(N) \) and each \( f \in E_r(A) \), \( f \) is trivial in \( \mathfrak{G}_r(N) \) iff \( f \big|_B \) is trivial in \( \mathfrak{G}_r(N) \) for any nonempty \( B \subset A \) with \( \operatorname{card}(B) \leq k \).

In particular, for two distinct pairs \((r, N), (s, M) \in \{1, \infty \} \times (\mathbb{Z}_+ \setminus \{1\})\), the metric spaces \( \mathfrak{G}_r(N) \) and \( \mathfrak{G}_s(M) \) are isometric iff \( r = s \) and \( \{N, M\} = \{0, 2\} \). Hence, \( \mathfrak{G}_r(N) \) is non-Urysohn for \( N > 2 \).

Further geometric properties of \( \mathfrak{G}_r(N) \)'s are stated below.

5.6. Proposition. Let \( f \in E^r(\mathfrak{G}_r(N)) \).

\( \text{(A)} \) For every two-point subset \( B \) of \( \mathfrak{G}_r(N) \) the map \( f \big|_B \) is trivial in \( \mathfrak{G}_r(N) \).

\( \text{(B)} \) If \( N = 4 \), \( f \) is trivial (in \( \mathfrak{G}_r(N) \)) on every three-point subset of the space.

\( \text{(C)} \) If \( N \neq 4 \), there is a three-point set \( C \subset \mathfrak{G}_r(N) \) and a map \( g \in E_r(C) \) which is nontrivial in \( \mathfrak{G}_r(N) \).

Proof. The points (b) and (c) are left as exercises. (To prove (c) for \( N > 4 \), take \( Z \) as in Example 5.4, \( H = Z^3 \), define \( e_1, e_2, e_3, F \) and \( \| : \|_F \) in a similar manner and consider \( f: \{e_1, e_2, e_3\} \to \mathbb{R}_+ \) constantly equal to \( 1/2 \). Show that \( \|(k+r_1)e_1+(k+r_2)e_2+(k+r_3)e_3\|_F \geq \frac{2}{5}(N-1) > \frac{N}{2} \) for \( r_1, r_2, r_3 \in \{0, 1, 2\} \) which sum up to \( r \in \{0, 1, 2\} \) such that \( N = 3k + r \).

To show (a), it remains to check that (2-8) is fulfilled whenever \( a_1, \ldots, a_N \in \{a, b\} \subset \mathfrak{G}_r(N) \). In that case (2-8) reduces to \( |p(j(a-b)) - f(a)| \leq (j-1)f(a) + (N-j)f(b) \) (since \( (N-j)b = -jb \)) with \( j = 2, \ldots, N-1 \) (for \( j = 1 \) this inequality follows from the definition of a Katětov map). Of course \( f(a) - p(j(a-b)) \leq (j-1)f(a) \), so we
only need to check that \( p(j(a - b)) \leq jf(a) + (N - j)f(b) \). By the symmetry \( j(a - b) = -(N - j)(a - b) \), we may assume that \( j \leq N/2 \). But then \( p(j(a - b)) \leq jf(a) + f(b) \leq jf(a) + (N - j)f(b) \). □

5.7. Corollary. For any two distinct points \( x \) and \( y \) of \( \mathbb{G}_r(N) \) there is an isometric arc of \([0, c]\) to \( \mathbb{G}_r(N) \) joining \( x \) and \( y \), where \( c = p(x - y) \).

\[
\frac{1}{2} \leq \frac{p(x - z) - p(y - z)}{p(x - y)} \leq 1.
\]

Proof. By Proposition 5.6, for any \( x, y \in \mathbb{G}_r(N) \) there is \( z \in \mathbb{G}_r(N) \) such that \( p(x - z) = p(y - z) = p(x - y)/2 \). Since \( \mathbb{G}_r(N) \) is complete, the assertion follows. □

The above result implies that every metric space which is the image of \( \mathbb{G}_1(N) \) under a uniformly continuous function has bounded metric. Since continuous group homomorphisms are uniformly continuous, as a consequence of this we obtain the result announced in the previous section.

5.8. Corollary. The groups \( \mathbb{G}_r(N) \)'s are pairwise nonisomorphic as topological groups.

The inequality (5.4) in Example 5.4 implies that there are points \( x_1, \ldots, x_N \) in \( \mathbb{G}_r(N) \) and radii \( r_1, \ldots, r_N > 0 \) such that \( p(x_j - x_k) < r_j + r_k \) for any \( j, k \), but the closed balls \( \overline{B}(x_j, r_j) \) \((j = 1, \ldots, N)\) have empty intersection. We conclude from this that the metric space \( \mathbb{G}_r(N) \) fails to have the property of extending isometric maps between finite subsets of \( \mathbb{G}_r(N) \) to Lipschitz maps with Lipschitz constants arbitrarily close to 1. This is why it is not so easy (as in case of \( \mathbb{G}_r(0) \) and \( \mathbb{G}_r(2) \)) which are Urysohn spaces; compare with [28] or [19] where it is shown that the Urysohn space is homeomorphic to \( 
\mathbb{I}^2 \)
) to prove that \( \mathbb{G}_r(N) \) is an absolute retract (for metric spaces). We shall do this in Section 8.

Our last aim of this part is to show that each of the metric spaces \( \mathbb{G}_r(N) \)'s is metrically universal for separable metric spaces of diameter no greater than \( r \).

In what follows, \( N \geq 2 \) and \( \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \) represents a cyclic group of rank \( N \). Moreover, let \( \epsilon \) stand for the generator of \( \mathbb{Z}_N \). Adapting the idea of Lipschitz-free Banach spaces generated by metric spaces (see e.g. [29], [6]), we introduce

5.9. Definition. Let \( X \) be a nonempty set. For \( x \in X \) let \( H_x = \mathbb{Z}_N \) and let \( \mathbb{Z}_N[[X]] = \bigoplus_{x \in X} H_x \) be the direct product of groups \( H_x \)'s. That is, \( \mathbb{Z}_N[[X]] \) consists of all functions \( f : X \rightarrow \mathbb{Z}_N \) for which the set \( \text{supp} f := \{ x \in X : f(x) \neq 0 \} \) is finite and \( \mathbb{Z}_N[[X]] \) is equipped with the pointwise addition. Further, let \( \mathbb{Z}_N[X] = \{ f \in \mathbb{Z}_N[[X]] : \sum_{x \in X} f(x) = 0 \} \). It is clear that \( \mathbb{Z}_N[X] \) is a subgroup of \( \mathbb{Z}_N[[X]] \).

For every \( x \in X \) let \( \hat{x} \in \mathbb{Z}_N[[X]] \) be such that \( \hat{x}(x) = \epsilon \) and \( \hat{x}(y) = 0 \) for \( y \neq x \). The set \( \{ \hat{x} - \hat{y} : x, y \in X \} \) generates the group \( \mathbb{Z}_N[X] \).
Whenever \( d \) is a metric on \( X \), define \( p_d : Z_N[X] \to \mathbb{R}_+ \) by

\[
(5-5) \quad p_d(f) = \inf \left\{ \sum_{j=1}^{n} d(x_j, y_j) : n \geq 1, \ x_1, y_1, \ldots, x_n, y_n \in X, \ f = \sum_{j=1}^{n} (\hat{x}_j - \hat{y}_j) \right\}.
\]

The triple \((Z_N[X], +, p_d)\) is called the valued Abelian group of exponent \( N \) generated by the metric space \((X, d)\).

As we will see in the next result, \( p_d \) is indeed a value. For need of this, let us introduce the following notation. Let \((X, d)\) be a finite metric space. If \( \text{card}(X) < 2 \), let \( \mu(X) := 0 \). Otherwise let

\[
\mu(X) := \max \{ \min \{d(x, y) : y \in X, y \neq x\} : x \in X, y \in X, y \neq x \} > 0.
\]

Then we have

5.10. Proposition. Let \((X, d)\) be a nonempty metric space. For every \( f \in Z_N[X] \):

(A) \( p_d(f) \geq \mu(\text{supp } f) \); in particular, \( p_d \) is a value,

(B) if \( N = 2 \) and \( f \neq 0 \),

\[
(5-6) \quad p_d(f) = \min \left\{ \sum_{j=1}^{k} d(x_j, y_j) : x_1, y_1, \ldots, x_k, y_k \text{ are all different} \right\}
\]

and \( \{x_1, y_1, \ldots, x_k, y_k\} = \text{supp}(f)\).

Proof. First observe that—thanks to the triangle inequality—in the formula \((5-5)\) we may consider only such systems \( x_1, y_1, \ldots, x_n, y_n \in X \) that

\[
(5-7) \quad \{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\} = \emptyset
\]

and—for the same reason—if \( N = 2 \), we may also restrict to systems \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) in which elements are different. This proves (B). Now assume that \( N \geq 3 \), that

\[
(5-8) \quad f = \sum_{j=1}^{n} (\hat{x}_j - \hat{y}_j)
\]

and \((5-7)\) is fulfilled. Let \( F = \{x_1, y_1, \ldots, x_n, y_n\} \). For \( a, b \in F \) we write \( a \sim b \) provided there is \( j \in \{1, \ldots, n\} \) with \( \{a, b\} = \{x_j, y_j\} \); and \( a \equiv b \) iff either \( a = b \) or there are \( c_0, c_1, \ldots, c_k \in F \) for which \( c_0 = a, c_k = b \) and \( c_j \sim c_{j-1} \) for \( j = 1, \ldots, k \). It is clear that ‘\( \equiv \)’ is an equivalence on \( F \) such that

\[
(5-9) \quad \{j : x_j \equiv a\} = \{j : y_j \equiv a\} \quad \text{for each } a \in A.
\]
We infer from (5-7) and (5-8) that \( \text{supp}(f) \subset F \) and for each \( a \in A \), \( f(a) = \text{card}\{j \in \{1, \ldots, n\} : x_j = a\}e \) or \( f(a) = -\text{card}\{j \in \{1, \ldots, n\} : y_j = a\}e \), and hence

\[
\text{(5-10)} \quad \begin{cases} 
\text{card}\{j : a \in \{x_j, y_j\}\} \neq 0 \mod N & \text{if } a \in \text{supp}(f), \\
\text{card}\{j : a \in \{x_j, y_j\}\} \equiv 0 \mod N & \text{otherwise}.
\end{cases}
\]

Fix \( a \in A \). It follows from (5-9) and (5-10) that there is \( b \in \text{supp}(g) \setminus \{a\} \) such that \( b \equiv a \). So, there are \( c_0, \ldots, c_k \in A \) for which \( c_0 = a \), \( c_k = b \) and \( c_j \sim c_{j-1} \) for \( j = 1, \ldots, k \). Passing into a suitable subset of \( \{0, \ldots, k\} \) we may assume that all \( c_j \)'s are different. This means that there are distinct indices \( \nu_1, \ldots, \nu_k \) such that \( \{c_j, c_{j-1}\} = \{x_{\nu_1}, y_{\nu_2}\} \).

But then \( \sum_{j=1}^n d(x_j, y_j) \geq \sum_{s=1}^k d(c_s, c_{s-1}) \geq d(a, b) \). So, the assertion follows from the arbitrariness of \( a \in \text{supp}(f) \). □

5.11. **Corollary.** Let \( (X, d) \) be a nonempty metric space and \( a \in X \).

The function

\[
(X, d) \ni x \mapsto \hat{x} - \hat{a} \in (\mathbb{Z}_N[X], p_d)
\]

is isometric.

If we replace in the above result \( p_d \) by \( p_d \wedge r \), the assertion of Corollary 5.11 well remain true. It is also clear that \( \mathbb{Z}_N[X] \) is separable provided so is \( X \). So, an application of Corollary 4.6 yields

5.12. **Theorem.** Every separable metric space of diameter no greater than \( r \) admits an isometric embedding into the metric space \( \mathbb{G}_r(N) \).

5.13. **Example.** If \( X \) is a nonempty set, \( H \) is a group of exponent \( N \), every function \( u: X \to H \) induces a unique (well defined) group homomorphism \( \hat{u}: \mathbb{Z}_N[X] \to H \) such that \( \hat{u}(\hat{x} - \hat{y}) = u(x) - u(y) \) for any \( x, y \in X \). What is more, if \( d \) is a metric on \( X \), \( q \) is a value on \( H \) and \( u: (X, d) \to (H, q) \) is Lipschitz, it easily follows from the formula for \( p_d \) that \( \hat{u}: (\mathbb{Z}_N[X], p_d) \to (H, q) \) satisfies the Lipschitz condition with the same constant as \( u \). Similarly, every function \( v: X \to Y \) between two sets induces a group homomorphism \( \hat{v}: \mathbb{Z}_N[X] \to \mathbb{Z}_N[Y] \) and if \( v: (X, d) \to (Y, g) \) is Lipschitz, then \( \hat{v}: (\mathbb{Z}_N[X], p_d) \to (\mathbb{Z}_N[Y], p_g) \) satisfies the Lipschitz condition with the same constant as \( v \).

In particular, if \( A \) is a subset of \( (X, d) \), the inclusion map of \( A \) into \( X \) induces a nonexpansive group homomorphism of \( \mathbb{Z}_N[A] \) into \( \mathbb{Z}_N[X] \). In case of \( N = 2 \), the latter group homomorphism is isometric, which easily follows from Proposition 5.10 (similar result holds true for the Lipschitz-free Banach spaces generated by metric spaces—see [29]). It turns out that for \( N > 2 \) this homomorphism may not be isometric, which is rather strange. Let us give an example based on Example 5.4.

Let \( X = \{0, 1, 2, \ldots, N\} \) and \( A = X \setminus \{0\} \). We equip \( X \) with a metric \( d \) such that for distinct \( j, k \in A \), \( d(j, k) = 1 \) and \( d(0, j) = \max(1/2, 1 - 2/N) \). For clarity, let \( q = d_{A \times A} \). Put \( f = \sum_{a \in A} \hat{a} \in \mathbb{Z}_N[A] \). It follows
from the argument in Example 5.4 that $p_{\mathbb{V}}(f) = N - 1$. However, in $\mathbb{Z}_N[X]$, $f = \sum_{a \in A} (\hat{a} - \hat{0})$ and thus $p_d(f) \leq N \cdot \max(1/2, 1 - 2/N) < N - 1$.

Taking into account Theorem 2.17, Theorem 5.1, Theorem 5.5 and the above example, we see that the case of $N > 2$ is ‘singular’.

6. Pseudovector groups

Pseudovector groups were introduced in [18]. Here we shall generalize the results of [18] on pseudovector structures on the (unbounded) Boolean Urysohn group, mainly in order to prove that all the groups $G_r(N)$’s are homeomorphic to the Hilbert space. However, some of results of this part may be seen interesting and can play important role in theory of topological pseudovector groups. All terms, beside the notion of a $\nabla$-norm, are repeated from the introductory work [18]. As before, we restrict our considerations only to Abelian groups.

We begin with

6.1. Definition. Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. A triple $(G, +, \ast)$ is said to be a pseudovector (Abelian) group over $\mathbb{F}$ (briefly, an $\mathbb{F}$-PV group, or a PV group provided $\mathbb{F} = \mathbb{R}$) iff $(G, +)$ is an Abelian group and $\ast : \mathbb{F}^+ \times G \to G$ is an action such that for any $s, t \in \mathbb{F}^+$ and $x \in G$,

$0 \ast x = 0, 1 \ast x = x, (st) \ast x = s \ast (t \ast x)$ and the function $G \ni y \mapsto t \ast y \in G$ is a group homomorphism.

A topological pseudovector group over $\mathbb{F}$ is an $\mathbb{F}$-PV group $(G, +, \ast)$ equipped with a topology $\tau$ such that $(G, +, \tau)$ is a topological group and the action ‘$\ast$’ is continuous.

A norm on an $\mathbb{F}$-PV group $(G, +, \ast)$ is a value $\| \cdot \| : G \to \mathbb{R}_+$ such that $\| t \ast x \| = t \| x \|$ for each $t \in \mathbb{F}^+$ and $x \in G$. The norm $\| \cdot \|$ is topological iff the action ‘$\ast$’ is continuous with respect to $\| \cdot \|$. In that case we speak of a normed topological pseudovector group over $\mathbb{F}$ (for short, a NTPV group over $\mathbb{F}$).

If $G$ is an $\mathbb{F}$-PV group, $\mathbb{K}$ is a subfield of $\mathbb{F}$ and $A$ any subset of $G$, by $\text{lin}_\mathbb{K} A$ we denote the $\mathbb{K}$-PV subgroup of $G$ generated by $A$, that is, $\text{lin}_\mathbb{K} A = \{ \sum_{j=1}^n t_j \ast a_j : n \geq 1, t_1, \ldots, t_n \in \mathbb{K}^+, a_1, \ldots, a_n \in A \cup (-A) \cup \{0\} \}$. If $\mathbb{K} = \mathbb{R}$, we write $\text{lin} A$ instead of $\text{lin}_\mathbb{R} A$.

A group homomorphism $u : G \to H$ between two $\mathbb{F}$-PV groups is linear if $u(t \ast x) = t \ast u(x)$ for every $t \in \mathbb{F}^+$ and $x \in G$.

Our aim is to show that each of the groups $G_r(N)$’s may be endowed with a ‘normed-like’ topological pseudovector structure. Since every norm on a nontrivial group is unbounded, none of the groups $G_1(N)$ is isometrically group isomorphic to a NTPV group. Thus, we have to generalize the notion of a norm.

6.2. Definition. A function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be norming iff $\kappa$ satisfies the following conditions:
\( \kappa(1) = 1 \) and \( \kappa(x) \geq x \) for each \( x \geq 0 \),
\[(\text{NF1}) \quad \kappa(xy) \leq \kappa(x)\kappa(y) \quad \text{for any } x, y \in \mathbb{R}_+, \]
\[(\text{NF2}) \quad \kappa(x) \leq x \quad \text{for each } x \geq 0, \]
\[(\text{NF3}) \quad \kappa \text{ is monotone increasing, i.e. } \kappa(x) \leq \kappa(y) \text{ provided } 0 \leq x \leq y, \]
\[(\text{NF4}) \quad \kappa \text{ is Lipschitz, that is, there is a constant } L' > 0 \text{ such that} \]
\[|\kappa(x) - \kappa(y)| \leq L'|x - y| \quad \text{for } x, y \geq 0. \]

(Notice that it is not assumed that \( \kappa(0) = 0 \).)

A (semi)value \( \| \cdot \| \) on an \( \mathbb{F}-PV \) group \((G, +, \ast)\) is said to be a \( \kappa \)-\textit{(semi)norm} iff \( \|t \ast x\| \leq \kappa(t)\|x\| \) for each \( t \in \mathbb{F}_+ \) and \( x \in G \). When \( \| \cdot \| \) is a \( \kappa \)-norm on an \( \mathbb{F}-PV \) group \((G, +, \ast)\), the quadruple \((G, +, \ast, \| \cdot \|)\) is said to be a \( \kappa \)-\textit{normed} \( \mathbb{F}-PV \) group.

6.3. \textbf{Remark.} In the whole paper, we use (NF3) only two times: in (P6) (which finds no application in this paper) and in the proof of Lemma 6.13 below, which may be improved so that (NF3) will not be applied. The reason for adding (NF3) to the axioms of a norming function is just a matter of our personal ‘taste’.

The most important examples on norming functions are \( \text{id} : \mathbb{R}_+ \to \mathbb{R}_+ \) and \( \nabla = \text{id} \lor 1 \). We leave these as simple exercises that \( \nabla \) is indeed norming and that a value is an id-norm iff it is a norm. We call a \( \nabla \)-norm also a \textit{subnorm}, and \textit{subnormed} means \( \nabla \)-normed. In fact we need only these two functions. However, no additional work is needed for arbitrary norming functions and it seems to us instructive to point out which properties of \( \text{id} \) and \( \nabla \) are relevant in our considerations.

The reader should check with no difficulties the following properties of every norming function \( \kappa \). Below we assume that \( L' > 0 \) as in (NF4).

(P1) If \( \kappa(0) = 0 \), then \( \text{id} \leq \kappa \leq L' \text{id} \).

(P2) If \( \kappa(0) \neq 0 \), then \( \nabla \leq \kappa \leq (L' + 1)\nabla \).

(P3) Every norm is a \( \kappa \)-norm.

(P4) If \( \kappa(0) \neq 0 \), every \( \nabla \)-norm is a \( \kappa \)-norm.

(P5) if \( \kappa(0) \neq 0 \) and \( \| \cdot \| \) is a \( \kappa \)-norm, then \( \| \cdot \| \land 1, \frac{\| \cdot \|}{1 + \| \cdot \|} \) and \( \| \cdot \|^\alpha \) with \( 0 < \alpha < 1 \) are \( \kappa \)-norms as well.

(P6) Composition of two norming functions is a norming function.

As the following result shows, \( \nabla \)-norms appear much more often than norms.

6.4. \textbf{Proposition.} Every topological pseudovector group metrizable as a topological space admits a \( \nabla \)-norm inducing its topology.

\textbf{Proof.} Let \((G, +, \ast)\) be a topological pseudovector group. By a well-known theorem (see e.g. [2, Theorem 8.8]), there is a value \( p \) on \( G \), bounded by 1, which induces the topology of \( G \). Now define \( \| \cdot \| : G \to \mathbb{R}_+ \) by

\[\|x\| = \sup\left\{ \frac{p(\alpha \ast x)}{\alpha \lor 1} : \alpha \geq 0 \right\}.\]

We leave this as an exercise that \( \| \cdot \| \) is a subnorm equivalent to \( p \) (use the boundedness of \( p \) and the continuity of ‘\( \ast \)’). \( \square \)
Note that the above result may be applied to any metrizable topological vector space.

From now to the end of the section $\kappa$ is a fixed norming function and $L \geq 1$ is such a constant that for all $x, y \geq 0$,

$$
\begin{align*}
|\kappa(x) - \kappa(y)| & \leq L|x - y|, \\
\kappa(x) & \leq L\nabla(x)
\end{align*}
$$

(cf. (NF4), (P1) and (P2)). Note also that it follows from (P1) that $\kappa(x) \leq Lx$ for $x \geq 0$ provided $\kappa(0) = 0$.

Everywhere below $\mathbb{F}$ is a subfield of $\mathbb{R}$ and $(G, +, *, \parallel \cdot \parallel_G)$ is a $\kappa$-normed pseudovector group. At this moment, we do not assume that the action ‘*’ is continuous.

As in [18], we call an element $a \in G$ continuous (respectively Lipschitz) provided the function $\mathbb{F}_+ \ni t \mapsto t*a \in G$ is continuous (Lipschitz). The set of all continuous (Lipschitz) elements of $G$ is denoted by $C_\mathbb{F}(G)$ ($\mathcal{L}_\mathbb{F}(G)$) and in case $\mathbb{F} = \mathbb{R}$ we write $C(G)$ ($\mathcal{L}(G)$). $G$ is said to be Lipschitz if $\mathcal{L}_\mathbb{F}(G) = G$. Similarly, a $\kappa$-seminorm $\parallel \cdot \parallel_G$ on $G$ is Lipschitz iff for every $a \in G$ there is $C_a \in \mathbb{R}_+$ such that $\parallel t*a - s*a \parallel_G \leq C_a|t-s|$ for any $s, t \in \mathbb{R}_+$.

The proofs of the next two results are omitted.

6.5. Proposition. (A) For $a \in G, a \in C_\mathbb{F}(G)$ iff

$$
\lim_{t \to 0} \|t*a\|_G = 0 \quad \text{and} \quad \lim_{t \to 1} \|a - t*a\|_G = 0.
$$

(B) For $a \in G, a \in \mathcal{L}_\mathbb{F}(G)$ iff the function $\mathbb{F} \cap [0, 1] \ni t \mapsto t*a \in G$ is Lipschitz.

(C) The set $C_\mathbb{F}(G)$ is a closed pseudovector subgroup (over $\mathbb{F}$) of $G$ and $C_\mathbb{F}(G)$ is a topological pseudovector group. The set $\mathcal{L}_\mathbb{F}(G)$ is a pseudovector subgroup (over $\mathbb{F}$) of $C_\mathbb{F}(G)$.

(D) For every $a \in C_\mathbb{F}(G)$ the map $\mathbb{F}_+ \ni t \mapsto t*a \in G$ is uniformly continuous (as a map between metric spaces) on every bounded subset of $\mathbb{F}_+$.

6.6. Proposition. Suppose $C_\mathbb{F}(G) = G$. Let $(\bar{G}, +, \parallel \cdot \parallel)$ be the completion of $(G, +, \parallel \cdot \parallel)$. There is an action $\bar{*} : \mathbb{R}_+ \times \bar{G} \to \bar{G}$ such that $(\bar{G}, +, \bar{*}, \parallel \parallel)$ is a $\kappa$-normed topological PV group and $t*x = t\bar{*}x$ for $t \in \mathbb{F}_+$ and $x \in G$. Moreover, $\mathcal{L}_\mathbb{F}(G) = \mathcal{L}(G) \cap G$.

The next result will be used several times by us.

6.7. Lemma. If there is $a \in G \setminus \{0\}$ such that $M := \sup\{\|t*a\|_G : t \in \mathbb{F}_+\} < \infty$, then $\kappa(0) \neq 0$.

Proof. For positive $t, s \in \mathbb{F}$ we have $\|t*x\|_G \leq \kappa(s)\|\frac{t}{s}x\|_G \leq M\kappa(s)$ and thus $M \leq M\kappa(s)$. So, $\kappa(s) \geq 1$ and the assertion follows. $\square$
6.8. **Lemma.** For every \( a \in \mathcal{C}_F(G) \) and \( \delta > 0 \) there exists a constant \( M = M(a, \delta) \in \mathbb{R}_+ \) such that for any \( s, t \in \mathbb{R}_+ \),
\[
\|s \ast a - t \ast a\|_G \leq M|s - t| + \delta \kappa(s \vee t).
\]

**Proof.** Let
\[
M = L \sup \left\{ \frac{\|s \ast a - t \ast a\|_G - \delta}{|s - t|} : s, t \in \mathbb{F} \cap [0, 1], \ s \neq t \right\}.
\]
We infer from point (D) of Proposition 6.5 that \( M < \infty \). We shall show that \( M \) satisfies (6-2). First assume that \( \kappa(0) = 0 \). We know that then \( \kappa(t) \leq Lt \) for every \( t \in \mathbb{R}_+ \). So, for \( t > s > 0 \) we have
\[
\|s \ast a - t \ast a\|_G \leq \kappa(t)(s/t) \ast a - a\|_G \leq \kappa(t)((M/L)(1 - s/t) + \delta) \leq M(t - s) + \delta \kappa(t).
\]
When \( \kappa(0) \neq 0 \), then \( \kappa(t) \geq 1 \) for every \( t \in \mathbb{R}_+ \) and hence (6-2) is fulfilled when \( t \vee s \leq 1 \), by the definition of \( M \). Finally, for \( t > s \geq 0 \) and \( t > 1 \), \( \kappa(t) \leq Lt \) (see (6-1)) and therefore we may repeat the estimations from the previous paragraph to get the assertion. \( \square \)

6.9. **Lemma.** Let \( (D_j, +, *, ||\cdot||_j) \in \mathfrak{G}_r(N) \) be Lipschitz \( \kappa \)-normed PV groups over \( \mathbb{F} \) \((j = 0, 1, 2)\) and \( \varphi_j : D_0 \to D_j \ (j = 1, 2) \) be isometric linear homomorphisms. Then there is a Lipschitz PV group \( (D, +, *, ||\cdot||_\ast) \in \mathfrak{G}_r(N) \) over \( \mathbb{F} \) and isometric linear homomorphisms \( \psi_j : D_j \to D \ (j = 1, 2) \) such that \( \psi_0 \circ \varphi_2 = \psi_1 \circ \varphi_1 \).

**Proof.** If both \( D_1 \) and \( D_2 \) are trivial, it suffices to take \( D \) trivial as well. If \( D_1 \) or \( D_2 \) is nontrivial, we may apply Lemma 6.7 from which it follows that \( \kappa(0) \neq 0 \) if \( r < \infty \). Thus, in the latter case we may involve (P5). These comments ensure us that we may repeat the proof of Lemma 2.19. The details are skipped. \( \square \)

For every valued group \((H, +, q)\) let \( L_0[H] \) consist of all functions \( u : \mathbb{R}_+ \to H \) for which there are numbers \( 0 = t_0 < t_1 < \ldots < t_n < \infty \) such that \( u \) is constant on each of the intervals \([t_{j-1}, t_j) \) \((j = 1, \ldots, n)\), say \( u([t_{j-1}, t_j)) = \{u_j\} \), and \( u(t) = 0 \) for \( t \geq t_n \). We shall abbreviate this by writing
\[
u = \sum_{j=1}^{n} u_j \chi_{[t_{j-1}, t_j)}.
\]

\( L_0[H] \) is a **Lipschitz** normed PV group when it is equipped with the pointwise addition, the action given by \((t * u)(s) = u(s/t)\) for \( t > 0 \), \( s \geq 0 \) and \( u \in L_0[H] \) (of course, \( 0 * u \equiv 0 \)) and the norm \( ||u||_q = \int_0^\infty q(u(t)) \, dt \) (cf. [18]). What is more, for every nonzero \( u \in L_0[H] \) there are unique \( n \geq 1 \), real numbers \( 0 < t_1 < \ldots < t_n \) and nonzero elements \( h_1, \ldots, h_n \in H \) such that \( u = \sum_{j=1}^{n} t_{j} * \hat{h}_j \) where for \( h \in H \), \( \hat{h}(s) = h \) for \( s \in [0, 1) \) and \( \hat{h}(s) = 0 \) for \( s \geq 1 \). Note that the function \( H \ni h \mapsto \hat{h} \in L_0[H] \) is an isometric group homomorphism and every group homomorphism \( \psi : H \to K \) of \( H \) into a group \( K \) induces a unique linear homomorphism \( \hat{\psi} : L_0[H] \to K \) determined by condition
\[ \hat{\psi}(\hat{h}) = \psi(h) \] for \( h \in H \). Thus, \( (L_0[H], +, *) \) may be called the free PV group generated by \( H \). A straightforward verification shows that \( (L_0[H], +, \| \cdot \|_q) \) is of class \( O_0 \) or of exponent \( N \) provided so is \( (H, +, q) \).

From now on, \( (G, +, *, \| \cdot \|_G) \) is a \( \kappa \)-normed PV group. As in the previous sections, we fix \( r \in \{1, \infty\} \) and \( N \in \mathbb{Z}_+ \setminus \{1\} \). Additionally, we assume the following elementary property:

\[ (AX) \quad G \neq \{0\} \quad \text{or} \quad \kappa(0) \neq 0 \quad \text{or} \quad r = \infty. \]

(This is because of Lemma 6.7. Precisely, we want to enlarge \( \kappa \)-normed PV groups of class \( \mathfrak{G}_r(N) \). When \( G \) is trivial and \( r \) is finite, this is impossible unless \( \kappa(0) \neq 0 \).)

Our aim is to prove counterparts of several results from Section 1 on enlarging groups. As we will see, in case of PV groups this is much more complicated.

6.10. **Theorem.** Let \( (H, +, q) \) be a finite valued group, \( K \) its subgroup and \( \| \cdot \|_0 \) a Lipschitz \( \kappa \)-seminorm on \( L_0[K] \) such that \( \| \hat{h} \|_0 = q(h) \) for \( h \in K \). Then there is a Lipschitz \( \kappa \)-seminorm on \( L_0[H] \) such that \( \| f \| = \| f \|_0 \) for \( f \in L_0[K] \) and \( \| \hat{h} \| = q(h) \) for \( h \in H \).

**Proof.** We assume that \( H \neq K \). Since \( K \) is finite and \( \| \cdot \|_0 \) is Lipschitz, there is a constant \( \lambda > 1 \) such that \( \| s \hat{h} - t \hat{h} \|_0 \leq \lambda |t - s| \) for every \( h \in H \) and \( s, t \in \mathbb{R}_+ \). Let \( M = \max q(H) \), \( \mu = \min \{ q(h) : h \in H \setminus \{0\} \} \) and \( A = (\lambda + ML)/\mu \). For \( f \in L_0[H] \) put

\[
\| f \| = \inf \{ \| A \| f - g - \sum_{j=1}^n t_j \hat{h}_j \|_q + \sum_{j=1}^n \kappa(t_j)q(h_j) + \| g \|_0 : \\
\quad n \geq 1, \; t_1, \ldots, t_n \geq 0, \; h_1, \ldots, h_n \in H, \; g \in L_0[K] \}. 
\]

It is clear that \( \| \cdot \| \) is a \( \kappa \)-seminorm on \( L_0[H] \) (by (NF2) and (P3)). What is more, \( \| \cdot \| \leq A \| \cdot \|_q \) which yields that \( \| \cdot \| \) is Lipschitz. Thus, we only need to show that \( \| f \| = \| f \|_0 \) and \( \| \hat{h} \| = q(h) \) for \( f \in L_0[K] \) and \( h \in H \). The inequalities \( \leq \) are immediate (since \( \kappa(1) = 1 \)). To prove the inverse inequalities, first of all note that

\[
\| f \| = \inf \left\{ \| \sum_{k=1}^m s_k \hat{g}_k \|_0 + A \sum_{j=1}^m (s_j - s_{j-1})q(\sum_{k=j}^m \varepsilon_k) + \sum_{k=1}^m \kappa(s_k)q(f_k - f_{k+1} + g_k + \varepsilon_k) : \\
\quad m \geq 1, \; 0 = s_0 < \ldots < s_m, \; \varepsilon_1, \ldots, \varepsilon_m, f_1, \ldots, f_m \in H, \; f_{m+1} = 0, \; g_1, \ldots, g_m \in K, \; \\
\quad f = \sum_{k=1}^m s_k (\hat{f}_k - \hat{f}_{k+1}) \right\}. 
\]
Indeed, under the notation of (6-4), it suffices to substitute

\[ t_k = s_k, \quad g = - \sum_{k=1}^{m} s_k \ast \hat{g}_k \quad \text{and} \quad h_k = f_k - f_{k+1} + g_k + \varepsilon_k \]

to obtain the expression as is (6-3) (observe that \( f = \sum_{k=1}^{m} s_k \ast (\hat{f}_k - \hat{f}_{k+1}) \) iff \( f = \sum_{k=1}^{m} f_k \chi_{[s_{k-1}, s_k)} \); and \( \| f - g - \sum_{j=1}^{m} s_j \ast \hat{h}_j \|_q = \sum_{j=1}^{m} (s_j - s_{j-1})q(\sum_{k=j}^{m} \varepsilon_k) \)). Conversely, in (6-3) we may assume that \( t_1, \ldots, t_n \) are positive and different and \( h_j \neq 0 \). Consequently, we may assume that \( 0 < t_1 < \ldots < t_n < \infty \). Then take \( 0 = s_0 < \ldots < s_m < \infty \) such that \( f = \sum_{k=1}^{m} f_k \chi_{[s_{k-1}, s_k)} \), \( g = \sum_{k=1}^{m} u_k \chi_{[s_{k-1}, s_k)} \) \((v_k \in K)\) and \( \sum_{j=1}^{m} t_j \ast \hat{h}_j = \sum_{k=1}^{m} u_k \chi_{[s_{k-1}, s_k)} \). Put \( f_{m+1} = g_{m+1} = u_{m+1} = 0 \), \( g_k = v_{k+1} - v_k \in K \) and \( \varepsilon_k = u_k - u_{k+1} - (f_k - f_{k+1}) - g_k \) and check that \( \sum_{j=1}^{m} \kappa(t_j)q(h_j) = \sum_{k=1}^{m} \kappa(s_k)q(u_k - u_{k+1}) \); \( \| f - g - \sum_{j=1}^{m} t_j \ast \hat{h}_j \|_q = \sum_{j=1}^{m} (s_j - s_{j-1})q(f_j - v_j - u_j) \) and \( f_j - v_j - u_j = - \sum_{k=j}^{m} \varepsilon_k \). This proves the equivalence of (6-3) and (6-4) (the details are left for the reader).

So, if \( f \in L_0[K] \), the inequality \( \| f \| \geq A\| f \|_0 \) is equivalent to

\[
\sum_{k=1}^{m} \kappa(s_k)q(f_k - f_{k+1} + g_k + \varepsilon_k) + A \sum_{j=1}^{m} (s_j - s_{j-1})q(\sum_{k=j}^{m} \varepsilon_k) \geq \| \sum_{k=1}^{m} s_k \ast (\hat{f}_k - \hat{f}_{k+1}) \|_0
\]

(under the notation of (6-4); in that case \( f_1, \ldots, f_m \in K \)). So, substituting \( h_k = f_k - f_{k+1} + g_k \in K \) and \( \Delta s_j = s_j - s_{j-1} \), the latter inequality follows from

\[
(6-5) \quad A \sum_{j=1}^{m} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k=1}^{m} \kappa(s_k)q(h_k + \varepsilon_k) \geq \| \sum_{k=1}^{m} s_k \ast \hat{h}_k \|_0.
\]

We shall prove (6-5) by induction on \( m \).

When \( m = 1 \), we have to show that \( Asq(\varepsilon) + \kappa(s)q(h + \varepsilon) \geq \| s \ast \hat{h} \|_0 \). If \( \varepsilon = 0 \), this follows from the fact that \( \| \cdot \|_0 \) is a \( \kappa \)-seminorm. And if \( \varepsilon \neq 0 \), we have

\[
Asq(\varepsilon) + \kappa(s)q(h + \varepsilon) \geq A\mu s \geq \lambda |s - 0| \geq \| s \ast \hat{h} \|_0.
\]

Now assume that (6-5) is satisfied for \( m - 1 \). We distinguish between three cases.

Firstly, assume that \( \sum_{k=j}^{m} \varepsilon_k \neq 0 \) for each \( j \in \{1, \ldots, m\} \). In that case the left-hand side expression of (6-5) is no less than \( A\mu \sum_{j=1}^{m} \Delta s_j \),
and

\[ A\mu \sum_{j=1}^{m} \Delta s_j \geq \lambda \sum_{j=1}^{m} |s_j - s_{j-1}| \geq \] 

\[ \geq \left\| \sum_{j=1}^{m} [s_j * (\sum_{k=j}^{m} \hat{h}_k) - s_{j-1} * (\sum_{k=j}^{m} \hat{h}_k)] \right\|_0 = \left\| \sum_{k=1}^{m} s_k * \hat{h}_k \right\|_0, \]

which gives (6-5). Secondly, if \( \varepsilon_m = 0 \), the assertion follows from (6-5) for \( 'm - 1' \).

Thirdly, assume that \( \varepsilon_m \neq 0 \) and there is \( z \in \{1, \ldots, m\} \) such that \( \sum_{k=z}^{m} \varepsilon_k = 0 \). We may assume that \( z \) is the largest natural number with this property. This yields that \( z < m \) and

\[ \sum_{k=z}^{m} \varepsilon_k = 0 \neq \sum_{k=z+1}^{m} \varepsilon_k. \]

Let us define

\[ s'_0, \ldots, s'_{m-1} \equiv s_0, \ldots, s_{z-1}, s_z+1, \ldots, s_m, \]

\[ \varepsilon'_1, \ldots, \varepsilon'_{m-1} \equiv \varepsilon_1, \ldots, \varepsilon_z, \varepsilon_z+1, \varepsilon_{z+2}, \ldots, \varepsilon_m, \]

\[ h'_1, \ldots, h'_{m-1} \equiv h_1, \ldots, h_{z-1}, h_z + h_{z+1}, h_{z+2}, \ldots, h_m. \]

We infer from induction hypothesis that

\[ A \sum_{j=1}^{m-1} (\Delta s'_j) q(\sum_{k=j}^{m-1} \varepsilon'_k) + \sum_{k=1}^{m-1} \kappa(s'_k) q(h'_k + \varepsilon'_k) \geq \left\| \sum_{k=1}^{m-1} s'_k * \hat{h}'_k \right\|_0. \]

It is easily verified, thanks to (6-6) and (6-1), that

\[ A \sum_{j=1}^{m} (\Delta s_j) q(\sum_{k=j}^{m} \varepsilon_k) = \]

\[ = A \sum_{j=1}^{m-1} (\Delta s'_j) q(\sum_{k=j}^{m-1} \varepsilon'_k) + A (\Delta s_{z+1}) q(\sum_{k=z+1}^{m} \varepsilon_k) \]

and

\[ A(\Delta s_{z+1}) q(\sum_{k=z+1}^{m} \varepsilon_k) \geq (\lambda + ML)(\Delta s_{z+1}) \geq \]

\[ \geq \| s_{z+1} * \hat{h}_z - s_z * \hat{h}_z \|_0 + q(h_z + \varepsilon_z)[\kappa(s_{z+1}) - \kappa(s_z)]. \]

Further

\[ \sum_{k=1}^{m} \kappa(s_k) q(h_k + \varepsilon_k) = \sum_{k=1}^{m-1} \kappa(s'_k) q(h'_k + \varepsilon'_k) \]

\[ + [\kappa(s_z) q(h_z + \varepsilon_z) + \kappa(s_{z+1}) q(h_{z+1} + \varepsilon_{z+1}) - \kappa(s'_z) q(h'_z + \varepsilon'_z)]. \]
and

\[(6-12) \quad [q(h_z + \varepsilon_z) + q(h_{z+1} + \varepsilon_{z+1})]\kappa(s_{z+1}) - \kappa(s'_z q(h'_z + \varepsilon'_z) \geq 0.
\]

So, (6-9), (6-10), (6-11) and (6-12) yield

\[
A \sum_{j=1}^{m} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k=1}^{m} \kappa(s_k)q(h_k + \varepsilon_k) \geq A \sum_{j=1}^{m-1} (\Delta s'_j)q(\sum_{k=j}^{m-1} \varepsilon'_k)
\]

\[
+ \sum_{k=1}^{m-1} \kappa(s'_k)q(h'_k + \varepsilon'_k) + \|s_{z+1} \ast \hat{h}_z - s_z \ast \hat{h}_z\|_0
\]

and hence (6-5) follows from (6-8).

We now pass to the proof that \(|\hat{x}| \geq q(x)| for \(x \in H \setminus \{0\}|. Under the notation of (6-4), when \(f = \hat{x}, there is \(l \in \{1, \ldots, m\}| for which \(s_l = 1| and then \(f_j = x| for \(j = 1, \ldots, l| and \(f_j = 0\) otherwise. Hence, (6-4) has the form (recall that \(\kappa(1) = 1):

\[
A \sum_{j=1}^{m} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k \neq l}^{m} \kappa(s_k)q(g_k + \varepsilon_k) + \|\sum_{k=1}^{m} s_k \ast \hat{g}_k\|_0
\]

\[
+ q(x + g_l + \varepsilon_l) \geq q(x),
\]

which is equivalent, after replacing \(g_j\) by \(h_j\), to

\[(6-13) \quad A \sum_{j=1}^{m} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k \neq l}^{m} \kappa(s_k)q(h_k + \varepsilon_k)
\]

\[
+ \|\sum_{k=1}^{m} s_k \ast \hat{h}_k\|_0 \geq q(h_l + \varepsilon_l)
\]

(where \(h_j \in K\). As before, we shall prove (6-13) by induction on \(m\).

When \(m = 1\), also \(l = 1\) and hence we need to show that

\[
Ag(\varepsilon) + \|\hat{h}\|_0 \geq q(h + \varepsilon)
\]

which easily follows since \(A \geq 1\) and \(\|\hat{h}\|_0 = q(h)| for \(h \in K\).

Suppose (6-13) is fulfilled for \(m - 1\). We distinguish between several cases. When \(\sum_{k=j}^{m} \varepsilon_k \neq 0\) for each \(j\), the left-hand side expression of (6-13) is no less than \(A \mu s_m \geq M s_l \geq q(h_l + \varepsilon_l)\).

If \(\varepsilon_m = 0\) and \(m \neq l\), (6-13) follows from induction hypothesis. Now assume that \(\varepsilon_l = 0\). Let \(s'_j, h'_j, \varepsilon'_j\ (j = 1, \ldots, m - 1)\) be systems obtained from \(s_j, h_j, \varepsilon_j\) by erasing the \(l\)-th members. Making use of
(6-5), we see that

\[
A \sum_{j=1}^{m} (\Delta s_j) q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k \neq l} \kappa(s_k) q(h_k + \varepsilon_k) + \| \sum_{k=1}^{m} s_k \ast \hat{h}_k \|_0 = \\
A \sum_{j=1}^{m-1} (\Delta s'_j) q(\sum_{k=j}^{m-1} \varepsilon'_k) + \sum_{k \neq l} \kappa(s'_k) q(h'_k + \varepsilon'_k) + \| \sum_{k=1}^{m-1} s'_k \ast \hat{h}'_k + s_l \ast \hat{h}_l \|_0 \\
\geq \| \sum_{k=1}^{m-1} s'_k \ast \hat{h}'_k \|_0 + \| \sum_{k=1}^{m-1} s'_k \ast \hat{h}'_k + \hat{h}_l \|_0 \geq \| \hat{h}_l \|_0 = q(h_l + \varepsilon_l).
\]

So, we may now assume that \( \varepsilon_m \neq 0 \) and there is \( z \) such that \( \sum_{k=z}^{m} \varepsilon_k = 0 \). Again, let \( z \) be the largest natural number with this property. We conclude from this that \( z < m \) and (6-6). We shall separately consider the cases when \( z \not\in \{l - 1, l\} \), \( z = l - 1 \) or \( z = l \).

When \( z \neq l - 1, l \), define systems \( s'_j, \varepsilon'_j \) and \( h'_j \) as in (6-7), and let \( l' \in \{1, \ldots, m - 1\} \) corresponds to \( l \). Note that (6-9), (6-10) and (6-12) are fulfilled. Moreover, instead of (6-11) we have

\[
\sum_{k \neq l} \kappa(s_k) q(h_k + \varepsilon_k) = \sum_{k \neq z, z+1, l} \kappa(s_k) q(h_k + \varepsilon_k) \\
+ [\kappa(s_z) q(h_z + \varepsilon_z) + \kappa(s_{z+1}) q(h_{z+1} + \varepsilon_{z+1})] = \sum_{k < m, k \neq l'} \kappa(s'_k) q(h'_k + \varepsilon'_k) \\
+ [\kappa(s_z) q(h_z + \varepsilon_z) + \kappa(s_{z+1}) q(h_{z+1} + \varepsilon_{z+1}) - \kappa(s'_z) q(h'_z + \varepsilon'_z)].
\]

The reader will now easily check that (6-13) is satisfied, thanks to the induction hypothesis (since \( h'_l + \varepsilon'_l = h_l + \varepsilon_l \)).

If \( z = l - 1 \ (l > 1) \), proceed in the same way. Here \( l' = l - 1 \) and instead of (6-11) one has

\[
\sum_{k \neq l} \kappa(s_k) q(h_k + \varepsilon_k) = \sum_{k \neq l-1, l} \kappa(s_k) q(h_k + \varepsilon_k) + \kappa(s_{l-1}) q(h_{l-1} + \varepsilon_{l-1}) \\
= \sum_{k < m, k \neq l'} \kappa(s'_k) q(h'_k + \varepsilon'_k) + \kappa(s_{l-1}) q(h_{l-1} + \varepsilon_{l-1}).
\]

Combination of induction hypothesis, (6-9), (6-10) and the above inequality reduces (6-13) to

\[
q(h'_l + \varepsilon'_l) + q(h_{l-1} + \varepsilon_{l-1}) \geq q(h_l + \varepsilon_l)
\]

which is fulfilled because \( h'_l = h_{l-1} + h_l \) and \( \varepsilon'_l = \varepsilon_{l-1} + \varepsilon_l \).
It remains to check what happens if $z = l$. From the maximality of $z$ we infer that $\sum_{k=j}^{m} \varepsilon_k \neq 0$ for $j > l$ and thus

\[(6-14)\quad A \sum_{j=l+1}^{m} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) \geq A \mu \sum_{j=l+1}^{m} (s_j - s_{j-1}) \geq \lambda \sum_{j=l+1}^{m} |s_j - s_{j-1}| \geq \sum_{j=l+1}^{m} [s_j * (\sum_{k=j}^{m} \hat{h}_k) - s_{j-1} * (\sum_{k=j}^{m} \hat{h}_k)]_0 = \sum_{k=l+1}^{m} [s_k * \hat{h}_k * s_l * \hat{h}_k]_0 = \sum_{k=l+1}^{m} s_k * \hat{h}_k - \sum_{k=l+1}^{m} \hat{h}_k].

Further, it follows from (6-5) that

\[A \sum_{j=1}^{l-1} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k=1}^{l-1} \kappa(s_k)q(h_k + \varepsilon_k) \geq \sum_{k=1}^{l-1} s_k * \hat{h}_k].

This, combined with (6-14), gives

\[A \sum_{j=1}^{m} (\Delta s_j)q(\sum_{k=j}^{m} \varepsilon_k) + \sum_{k=1}^{l-1} \kappa(s_k)q(h_k + \varepsilon_k) + \sum_{k=l+1}^{m} s_k * \hat{h}_k = \sum_{k=l+1}^{m} s_k * \hat{h}_k + (l-1) \sum_{k=l+1}^{m} \hat{h}_k]_0 = \sum_{k=l+1}^{m} s_k * \hat{h}_k - \sum_{k=l+1}^{m} \hat{h}_k].

but $-\sum_{k=l+1}^{m} \varepsilon_k = \varepsilon_l$ (by (6-6)), which finally finishes the proof. □

As a corollary of the above result, we obtain

6.11. Theorem. Let $(H, +, \varepsilon) \in \mathfrak{G}_r(N)$ be a finite valued group, $K$ be a subgroup of $H$, $(G, +, *, \| \cdot \|_G)$ be a Lipschitz $\kappa$-normed pseudovector group such that $(G, +, \| \cdot \|_G) \in \mathfrak{G}_r(N)$ and $(AX)$ is fulfilled. And let $\varphi: K \to G$ be an isometric group homomorphism. There is a Lipschitz $\kappa$-normed PV group $(\hat{G}, +, *, \| \cdot \|_G)$ such that $(\hat{G}, +, *, \| \cdot \|_G) \in \mathfrak{G}_r(N)$ and an isometric group homomorphism $\psi: H \to \hat{G}$ such that $(\hat{G}, +, *, \| \cdot \|_G) \in \mathfrak{G}_r(N)$ and $\psi|_K = \varphi$. 
Proof. Let $\tilde{\varphi}: L_0[K] \to G$ be the linear homomorphism induced by $\varphi$. Define $\| \cdot \|_0: L_0[K] \to \mathbb{R}_+$ by $\|f\|_0 = \|\tilde{\varphi}(f)\|_G$. Then $\| \cdot \|_0$ is a Lipschitz $\kappa$-seminorm on $L_0[K]$ such that $\|h\|_0 = q(h)$ for $h \in K$ (since $\varphi$ is isometric). So, according to Theorem 6.10, there is a Lipschitz $\kappa$-seminorm $\| \cdot \|_H$ on $L_0[H]$ which extends $\| \cdot \|_0$ and satisfies $\|\tilde{h}\|_H = q(h)$ for $h \in H$. Now let $\tilde{H} = L_0[H]/\{f \in L_0[H]: \|f\|_H = 0\}$ be the PV group equipped with the Lipschitz $\kappa$-norm induced by $\| \cdot \|_H$, let $\pi: L_0[H] \to \tilde{H}$ be the quotient linear homomorphism and $\tilde{K} = \pi(L_0[K])$. Observe that there is an isometric linear homomorphism $\tilde{\varphi}: \tilde{K} \to G$ such that $\tilde{\varphi} \circ \pi|_{L_0[K]} = \tilde{\varphi}$. To this end, apply Lemma 6.9 to $\tilde{\varphi}$ and the inclusion map of $K$ into $\tilde{H}$ in order to obtain $\tilde{G}$. Finally, involve Lemma 6.7 and (P5) if needed. \hfill \Box

6.12. Lemma. Suppose $a \in C(G) \setminus \{0\}$ is of finite rank. For each $\varepsilon > 0$ there is a PV group $(\tilde{G}, +, \ast, \| \cdot \|) \supset (G, +, \ast, \| \cdot \|_G)$ and $b \in \mathcal{L}(\tilde{G})$ such that $\text{rank}(b) = \text{rank}(a)$ and $\|a - b\| \leq \varepsilon$.

Proof. Let $H = \langle a \rangle$. The homomorphism $Z \ni k \mapsto ka \in H \subset G$ induces linear homomorphisms $\psi_G: L_0[Z] \to G$ and $\psi_H: L_0[Z] \to L_0[H]$ such that $\psi_G(1) = a$ and $\psi_H(1) = \tilde{a}$. Let $\delta$ be the discrete value on $H$ and $N \geq 2$ be the rank of $a$. We equip $L_0[Z]$ with the $\kappa$-norm $\| \cdot \|_\kappa$ defined as follows. When $f \in L_0[Z] \setminus \{0\}$, there are unique $n \geq 1$, $0 < t_1 < \ldots < t_n < \infty$ and $k_1, \ldots, k_n \in Z \setminus \{0\}$ such that $f = \sum_{j=1}^n t_j \ast k_j$. We put $\|f\|_\kappa = \sum_{j=1}^n \kappa(t_j)$. Further, let $M = M(a, \varepsilon/N)$ be as in Lemma 6.8. Put $A = NM$, $\tilde{G} = G \times L_0[H]$ and define a $\kappa$-seminorm $\| \cdot \|$ on $\tilde{G}$ by

$$
\|(g, f)\| = \inf \{\|g - \psi_G(u)\|_G + \varepsilon\|u\|_\kappa + A\|\psi_H(u) - f\|_\delta: u \in L_0[Z]\}.
$$

It is easily seen that $\| \cdot \|$ is a $\kappa$-seminorm on $\tilde{G}$ such that $\| (g, f) \| > 0$ for $f \neq 0$ and $\|(g, 0)\| \leq \|g\|_G$ for every $g \in G$; $\|(a, \tilde{a})\| \leq \varepsilon$ and $\|(0, f)\| \leq A\|f\|_\delta$ for $f \in L_0[H]$. The latter will imply that $(0, \tilde{a}) \in \mathcal{L}(\tilde{G})$. So, if we show that $\|(g, 0)\| \geq \|g\|_G$, $\| \cdot \|$ will be a $\kappa$-norm and the proof will be completed. The latter is equivalent to

$$(6-15) \quad \varepsilon\|u\|_\kappa + A\|\psi_H(u)\|_\delta \geq \|\psi_G(u)\|_G$$

where $u \in L_0[Z]$. We may assume that $u \neq 0$. Writing $u = \sum_{k=1}^m s_k \ast \tilde{l}_k$ with $0 = s_0 < \ldots < s_m$ and $l_1, \ldots, l_m \in Z \setminus \{0\}$, and putting $\nu_j = \sum_{k=j}^m l_k$ we obtain

$$
\|u\|_\kappa = \sum_{k=1}^m \kappa(s_k), u = \sum_{k=1}^m (s_k \ast \tilde{\nu}_k - s_k - (\nu_k a)), \psi_H(u) = \sum_{k=1}^m (s_k \ast \tilde{\nu}_k a - s_k \ast \tilde{\nu}_k a) \text{ and } \psi_G(u) = \sum_{k=1}^m [s_k \ast (\nu_k a) - s_k - (\nu_k a)].
$$

Consequently, $\|\psi_H(u)\|_\delta = \sum_{k=1}^m (s_k - s_{k-1})\delta(\nu_k a)$ and

$$
\varepsilon\|u\|_\kappa + A\|\psi_H(u)\|_\delta - \|\psi_G(u)\|_G \geq \sum_{k=1}^m [\varepsilon\kappa(s_k) + A(s_k - s_{k-1})\delta(\nu_k a)] - \|s_k \ast (\nu_k a) - s_k \ast (\nu_k a)\|_G.
$$
Let $H \in (6-17)$ for every $\kappa$ value on may assume that (6-18) changes into $s$ may have $0 = s$ continuous with respect to both the left-hand and the right-hand side expressions of (6-14) are $m$ and an element $6.13$. Lemma. and we are done. □

Proof. Take $F$ as in the previous proof, observe that $\parallel \cdot \parallel$ for every $s,t$ $(6-16)$ $< m < N$.

But then, by the definition of $M$, $\parallel s* (ma) - t* (ma) \parallel_G \leq m \parallel s*a - t*a \parallel_G \leq N [M \parallel s-t \parallel (\varepsilon/N) \kappa (s \lor t)] = A \parallel s-t \parallel \delta (ma) + \varepsilon \kappa (s \lor t)$ and we are done.

6.13. Lemma. Suppose $a \in C(G) \setminus \{0\} \text{ is such that } \lim_{n \to \infty} \parallel na \parallel_G/n = 0$. For every $\varepsilon > 0$ there is a PV group $(G,+,\star, \parallel \cdot \parallel) \ni (G,+,\star, \parallel \cdot \parallel_G)$ and an element $b \in L(G)_{fin}$ such that $\parallel a - b \parallel \leq \varepsilon$.

Proof. Take $N \geq 2$ such that for every $n \geq N$,

(6-16) \[
\frac{\parallel na \parallel_G}{n} \leq \frac{\varepsilon}{2}.
\]

Further, let $M = M(a, \varepsilon/(2N))$ be as in Lemma 6.8. Put $A = NM$. Let $H = (b)$ be a cyclic group of rank $N$ and $\delta$ denote the discrete value on $H$. Now put $\tilde{G} = G \times L_0[H]$ and define $\parallel \cdot \parallel : \tilde{G} \to \mathbb{R}_+$ by

\[
\parallel (g,f) \parallel = \inf \{\parallel g - \sum_{j=1}^{m} s_j (l_j a) \parallel_G + \varepsilon \sum_{j=1}^{m} \kappa (s_j) + A \parallel \sum_{j=1}^{m} s_j (l_j \tilde{b}) - f \parallel_\delta : m \geq 0, \ s_1, \ldots, s_m \geq 0, \ l_1, \ldots, l_m \in \{-1,1\} \}.
\]

As in the previous proof, observe that $\parallel \cdot \parallel$ is a $\kappa$-seminorm such that $\parallel (g,f) \parallel \geq 0$ provided $f \neq 0$, $\parallel (g,0) \parallel \leq \parallel g \parallel_G$, $\parallel (0,f) \parallel \leq A \parallel f \parallel_\delta$ ($g \in G$, $f \in L_0[H]$) and $\parallel (a,\tilde{b}) \parallel \leq \varepsilon$. So, if only we show that $\parallel (g,0) \parallel \geq \parallel g \parallel_G$ for $g \in G$, the proof will be completed (because then $\parallel \cdot \parallel$ will be a $\kappa$-norm and $\tilde{b} \in L(\tilde{G})$). The latter is equivalent to

(6-17) \[
\frac{\varepsilon}{2} \sum_{k=1}^{m} \kappa (s_k) + A \parallel \sum_{k=1}^{m} s_k (l_k \tilde{b}) \parallel_\delta \geq \parallel \sum_{k=1}^{m} s_k (l_k a) \parallel_G
\]

for every $m \geq 0, \ s_1, \ldots, s_m \in \mathbb{R}_+$ and $l_1, \ldots, l_m \in \{-1,1\}$. We may assume that $m \geq 1$ and then, when $m$ and $l_1, \ldots, l_m$ are fixed, both the left-hand and the right-hand side expressions of (6-14) are continuous with respect to $s_1, \ldots, s_m$. Hence we may assume that $s_1, \ldots, s_m$ are positive and different. Finally, after renumberation, we may have $0 = s_0 < s_1 < \ldots < s_m$. But then $\parallel \sum_{k=1}^{m} s_k (l_k \tilde{b}) \parallel_\delta = \sum_{j=1}^{m} (\Delta s_j) \delta (\sum_{k=j}^{m} l_k b)$ (where, as usually, $\Delta s_j = s_j - s_{j-1}$). So, (6-17) changes into

(6-18) \[
\frac{\varepsilon}{2} \sum_{j=1}^{m} \kappa (s_j) + \sum_{j=1}^{m} [A (\Delta s_j) \delta (\sum_{k=j}^{m} l_k b) + \frac{\varepsilon}{2} \kappa (s_j)] \geq \parallel \sum_{j=1}^{m} s_j (l_j a) \parallel_G.
\]
Now we shall construct a system of natural numbers \( \{\nu_0, \ldots, \nu_z\} \) (for some \( z \geq 1 \)) for which

\[
\text{(E0)} \quad 0 = \nu_z < \ldots < \nu_0 = m + 1,
\]

\[
\text{(E1)} \quad \text{for every } j, k \text{ with } 1 \leq k \leq z \text{ and } \nu_k < j < \nu_{k-1}, \quad \quad |\sum_{s=j}^{\nu_{k-1}-1} l_s| < N,
\]

\[
\text{(E2)} \quad \text{for every } k \text{ with } 1 \leq k < z, \quad |\sum_{s=\nu_k}^{\nu_{k-1}-1} l_s| = N.
\]

Put \( \nu_0 = m + 1 \) and suppose \( \nu_p \) is defined for some \( p \geq 0 \). If either \( \nu_p = 1 \) or \( |\sum_{s=\nu_p+1}^{\nu_p-1} l_s| < N \) for every \( j \in \{1, \ldots, \nu_p - 1\} \), put \( \nu_{p+1} = 0 \) and \( z = p + 1 \) and finish the construction. Otherwise, take the greatest natural number \( \nu_{p+1} \in \{1, \ldots, \nu_p - 1\} \) such that \( |\sum_{s=\nu_p+1}^{\nu_p-1} l_s| \geq N \).

Since \( |l_s| = 1 \) for every \( s \), one has \( |\sum_{s=\nu_p+1}^{\nu_p-1} l_s| = N \). The verification that \( \text{(E0)}-\text{(E2)} \) are fulfilled is left as an exercise.

For simplicity, let \( l_0 = 0 \). Now it follows from \( \text{(E2)} \) that

\[
\text{(E3)} \quad \text{for every } j, k \text{ with } 0 \leq k < z \text{ and } \nu_{k+1} \leq j < \nu_k, \quad \quad \sum_{s=j}^{\nu_k-1} l_s b = \sum_{s=\nu_k}^{\nu_{k+1}-1} l_s b
\]

(recall that \( \text{rank}(b) = N \)). Thanks to \( \text{(E3)} \), we have

\[
(6-19) \quad \sum_{j=1}^{m} [A(\Delta s_j)\delta(\sum_{k=j}^{m} l_k b) + \frac{\varepsilon}{2} \kappa(s_j)] \geq \sum_{k=0}^{z-1} \sum_{j=\nu_{k+1}+1}^{\nu_k-1} [A(\Delta s_j)\delta(\sum_{q=j}^{\nu_k-1} l_q b) + \frac{\varepsilon}{2} \kappa(s_j)].
\]

Further, when \( 0 \leq k < z \),

\[
\sum_{j=\nu_{k+1}}^{\nu_k-1} s_j * (l_j a) = \sum_{j=\nu_{k+1}}^{\nu_k-1} s_j * (\sum_{q=j}^{\nu_k-1} l_q a) - \sum_{q=j}^{\nu_k-1} l_q a = \sum_{j=\nu_{k+1}}^{\nu_k-1} s_j * (\sum_{q=j}^{\nu_k-1} l_q a) - \sum_{j=\nu_{k+1}+1}^{\nu_k} s_{j-1} * (\sum_{q=j}^{\nu_k-1} l_q a) = \sum_{j=\nu_{k+1}+1}^{\nu_k-1} (\sum_{q=j}^{\nu_k-1} l_q)(s_j * a - s_{j-1} * a) + s_{\nu_{k+1}} * (\sum_{q=\nu_{k+1}}^{\nu_k-1} l_q a)
\]

and therefore (by (E2), (6-16) and the fact that \( s_{v_k} = 0 \)):

\[
\| \sum_{j=v_{k+1}}^{v_k-1} s_j \ast (l_j a) \|_G = \| \sum_{k=0}^{v_k-1} \sum_{j=v_{k+1}}^{v_k-1} s_j \ast (l_j a) \|_G \leq \sum_{k=0}^{v_k-1} \sum_{j=v_{k+1}}^{v_k-1} |l_q| \cdot \| s_j \ast a - s_{j-1} \ast a \|_G + \sum_{k=1}^{z-1} \kappa(s_{v_k}) \| N a \|_G \leq \sum_{k=0}^{v_k-1} \sum_{j=v_{k+1}}^{v_k-1} |l_q| \cdot \| s_j \ast a - s_{j-1} \ast a \|_G + \frac{\varepsilon}{2} N \sum_{k=1}^{z-1} \kappa(s_{v_k}).
\]

So, taking into account (6-19), we see that (6-18) will be satisfied provided

\[
(6-20) \quad |l_q| \cdot \| s_j \ast a - s_{j-1} \ast a \|_G \leq A(\Delta s_j) \delta(\sum_{q=j}^{v_k-1} l_q) + \frac{\varepsilon}{2} \kappa(s_j)
\]

whenever \( v_{k+1} < j < v_k \) \((k = 0, \ldots, z - 1)\) and

\[
(6-21) \quad N \sum_{k=1}^{z-1} \kappa(s_{v_k}) \leq \sum_{j=1}^{m} \kappa(s_j).
\]

To prove (6-20), put \( \lambda = |\sum_{q=j}^{v_k-1} l_q|, \; t = s_j \) and \( s = s_{j-1} \). By (E1), \( 0 \leq \lambda < N \). When \( \lambda = 0 \), (6-20) is clear. And if \( \lambda \neq 0 \), \( \delta(\lambda b) = 1 \) and then, by the definition of \( M \),

\[
\lambda \| t \ast a - s \ast a \|_G \leq N(\| M |t - s| + \frac{\varepsilon}{2N} \kappa(t \lor s)) = A|t - s| \delta(\lambda b) + \frac{\varepsilon}{2} \kappa(t)
\]

which gives (6-20).

Now we pass to (6-21). By (E2), \( N \leq v_{k+1} - v_k \) for \( k = 1, \ldots, z - 1 \). So, by (NF3) we obtain

\[
N \sum_{k=1}^{z-1} \kappa(s_{v_k}) \leq \sum_{k=1}^{z-1} \sum_{j=v_{k+1}}^{v_k-1} \kappa(s_{v_k}) \leq \sum_{k=1}^{z-1} \sum_{j=v_k}^{v_{k-1}} \kappa(s_j) \leq \sum_{j=1}^{m} \kappa(s_j)
\]

which finishes the proof. \( \square \)

As an immediate consequence of (P5) and Lemmas 6.7, 6.12 and 6.13 we obtain

6.14. **Theorem.** If \((G, +, \ast, \| \cdot \|_G)\) is a \( \kappa \)-normed topological pseudovector group such that \((G, +, \| \cdot \|_G) \in \mathfrak{G}_r(N)\), there is a \( \kappa \)-normed pseudovector group \((\bar{G}, +, \ast, \| \cdot \|_G) \supset (G, +, \ast, \| \cdot \|_G)\) such that \((\bar{G}, +, \| \cdot \|) \in \mathfrak{G}_r(N)\) and the set \( L(G)_{fin} \) is dense in \( G \).
7. Proof of Theorem 1.3

Let \( r \in \{1, \infty\} \), \( N \in \mathbb{Z}_+ \setminus \{1\} \) and \( \kappa \) be a norming function. As it is easily seen, Theorem 1.3 immediately follows from

7.1. Theorem. Let \( (G, +, *, \| \cdot \|_G) \) be a \( \kappa \)-normed topological pseudovector group such that \( (G, +, \| \cdot \|_G) \in \mathfrak{S}_r(N) \) and \( (AX) \) is fulfilled. There is a \( \kappa \)-normed pseudovector group \( (G, +, *, \| \cdot \|) \supset (G, +, *, \| \cdot \|_G) \) such that the set \( \mathcal{L}(G)_{\text{fin}} \) is dense in \( G \) and the valued groups \( G \) and \( \mathbb{G}_r(N) \) are isometrically group isomorphic.

The above result strengthens and generalizes [18, Theorem 4.3]. The proof of Theorem 7.1 will be preceded by

7.2. Lemma. Let \( (G', +, *, \| \cdot \|') \) be a Lipschitz \( \kappa \)-normed pseudovector group such that \( (G', +, \| \cdot \|') \in \mathfrak{S}_r(N) \) and \( (AX) \) is satisfied. Let \( D \) and \( \mathbb{F} \) be, respectively, a countable subset of \( G' \) and a countable subfield of \( \mathbb{R} \). For every countable family \( \mathfrak{S} \) of finite valued groups of class \( \mathfrak{S}_r(N) \) there is a Lipschitz \( \kappa \)-normed pseudovector group \( (G''', +, *, \| \cdot \|''') \supset (G', +, *, \| \cdot \|') \) such that \( (G''', +, \| \cdot \|''') \in \mathfrak{S}_r(N) \) and with the following property. Whenever \( H \in \mathfrak{S}, K \) is a subgroup of \( H \) and \( \varphi: K \to \text{lin}_\mathbb{F} D \subset G' \) is an isometric group homomorphism, there is an isometric group homomorphism \( \psi: H \to G''' \) which extends \( \varphi \).

Proof. Let \( D' = \text{lin}_\mathbb{F} D \). Since every member of \( \mathfrak{S} \) is a finite group and \( D' \) is countable, the family \( \{\varphi: K \to D'\} \), where \( K \) runs over all subgroups of members of \( \mathfrak{S} \) and \( \varphi \) is an isometric group homomorphism, is countable. So, the assertion follows from induction and Theorem 6.11.

Proof of Theorem 7.1. Thanks to Theorem 6.14, we may and do assume that \( A = \mathcal{L}(G)_{\text{fin}} \) is dense in \( G \). Let \( A_0 \) be a countable dense subset of \( A \). Note that \( \text{lin}_{A_0} \) is a dense Lipschitz PV subgroup of \( G \) and, as a group, of class \( \mathcal{O}_{\text{fin}} \). For purpose of this proof, let us call a \( \kappa \)-normed PV group \( (E, +, *, \| \cdot \|_E) \) of class \( \mathcal{L}_r(N)_{\text{fin}} \) provided \( (E, +, \| \cdot \|_E) \in \mathfrak{S}_r(N) \) and \( E = \mathcal{L}(E)_{\text{fin}} \).

We shall construct, making use of induction, sequences \( (\mathbb{F}_n)_{n=0}^\infty \), \( (G_n, +, *, \| \cdot \|_n) \) and \( (D_n)_{n=0}^\infty \) such that for every \( n \geq 0 \):

1. \( \mathbb{F}_n \) is a countable subfield of \( \mathbb{R} \),
2. \( (G_n, +, *, \| \cdot \|_n) \in \mathfrak{S}_r(N)_{\text{fin}} \),
3. \( D_n \) is a countable subset of \( G_n \) and \( G_n = \text{lin} D_n \),
4. \( 0 = \mathbb{Q} \) and for \( n > 0 \),
   \[ \mathbb{F}_n \supset \mathbb{F}_{n-1} \cup \{\|g\|_{n-1}: g \in \text{lin}_{\mathbb{F}_{n-1}} D_{n-1}\}, \]
5. \( G_0 = \text{lin} A_0 \) and \( \| \cdot \|_0 \) is inherited from \( \| \cdot \|_G \); and for \( n > 0 \),
   \[ (G_n, +, *, \| \cdot \|_n) \supset (G_{n-1}, +, *, \| \cdot \|_{n-1}) \] and \( D_n \supset D_{n-1} \),
6. for \( n > 0 \): whenever \( (H, +, q) \in \mathfrak{S}_r(N) \) is a finite valued group with \( q(H) \subset \mathbb{F}_{n-1} \) and \( \varphi: K \to \text{lin}_{\mathbb{F}_{n-1}} D_{n-1} \) is an isometric group homomorphism.
homomorphism of a subgroup $K$ of $H$, there is an isometric group homomorphism $\psi: H \to \operatorname{lin}_{\mathbb{F}} n$, which extends $\varphi$.

Define $\mathbb{F}_0$ and $G_0$ as in (4n) and (5n) and put $D_0 = A_0$. Suppose that for some $n > 0$ we have defined $\mathbb{F}_{n-1}$, $G_{n-1}$ and $D_{n-1}$ with suitable properties. Let $\mathcal{F}$ be a countable family of all (up to isometric group isomorphism) finite $\mathbb{F}_{n-1}$-groups of class $\mathcal{G}_r(N)$ (cf. Definition 3.1).

Take a Lipschitz $\kappa$-normed PV group $(G'', +, *, || \cdot ||'')$ which witnesses the assertion of Lemma 7.2 for $\mathcal{F}$, $G' = G_{n-1}$, $\mathbb{F} = \mathbb{F}_{n-1}$ and $D = D_{n-1}$. Further, let $\mathcal{F}$ be the family of all pairs $(\varphi, H)$ with $H \in \mathcal{F}$ and $\varphi: K \to \operatorname{lin}_{\mathbb{F}_{n-1}} D_{n-1}$ an isometric group homomorphisms where $K$ is a subgroup of $H$. The collection $\mathcal{F}$ is countable. For each $(\varphi, H) \in \mathcal{F}$ let $\hat{\varphi}: H \to G''$ be an isometric group homomorphism extending $\varphi$. Now put $D_n = D_{n-1} \cup \bigcup_{(\varphi, H) \in \mathcal{F}} \hat{\varphi}(H)$ and $G_n = \operatorname{lin} D_n \subset G''$. Let $|| \cdot ||_n$ be the $\kappa$-norm on $G_n$, inherited from $|| \cdot ||''$ and $\mathbb{F}$ be the subfield of $\mathbb{R}$ generated by $\mathbb{F}_{n-1} \cup \{||g||_n: g \in D_n\}$. It is easy to verify that conditions (1n)–(6n) are fulfilled.

Having the sequences $(\mathbb{F}_n)_{n=0}^\infty$: $(D_n)_{n=0}^\infty$ and $(G_n)_{n=0}^\infty$, define the PV group $(G, +, *, || \cdot ||)$ as the completion of $\bigcup_{n=0}^\infty (G_n, +, *, || \cdot ||_n)$ (cf. Proposition 6.6), $\mathbb{F} = \bigcup_{n=0}^\infty \mathbb{F}_n$ and $D = \bigcup_{n=0}^\infty D_n$. Note that

$$(G, +, *, || \cdot ||) \supset (G_0, +, *, || \cdot ||_0)$$

(since $G_0$ is dense in $G$) and $\hat{D} = \operatorname{lin}_{\mathbb{F}} D$ is dense in $\hat{G}$. Further, $\hat{D}$ is isometrically group isomorphic to $\mathbb{F}\mathcal{G}_r(N)$, which follows from the construction. Thus, to this end it remains to apply Proposition 3.9. □

As consequences of Theorem 7.1 we obtain (see Proposition 6.4)

7.3. **Theorem.** Let $r \in \{1, \infty\}$. Every separable metrizable topological pseudovector (Abelian) group is isomorphic (as a topological PV group) to a pseudovector subgroup of a subnormed topological PV group which is isometrically group isomorphic to $\mathcal{G}_r(0)$.

8. **Topology of $\mathcal{G}_r(N)$’s**

This part is devoted to the proof of the following

8.1. **Theorem.** Each of the topological spaces $\mathcal{G}_r(N)$’s is homeomorphic to the Hilbert space $l_2$.

Note that Theorem 8.1 for $N \in \{0, 2\}$ follows from Theorem 5.1 and the result of Uspenskij [28]. However, the proof presented here needs no additional work for such $N$ and thus we give the proof of Theorem 8.1 in its full generality. We shall do this making use of Theorem 1.3.

Before we pass to the proof of Theorem 8.1, we have to recall some notions and results of infinite-dimensional topology. First of all, recall that the space $l_2$ is the Banach space consisting of all square summable real sequences equipped with the norm $||(a_n)_{n=1}^\infty||_2 = (\sum_{n=1}^\infty a_n^2)^{1/2}$. 

A metrizable space $X$ is said to be an **absolute retract** iff it is a retract of any metrizable space in which $X$ is embedded as a closed set. The space $X$ is **homotopically trivial** provided every map of the boundary of $[0, 1]^n$ (for arbitrary $n \geq 1$) into $X$ is extendable to a map of the whole cube $[0, 1]^n$ into $X$. Observe that under such a definition the empty space is homotopically trivial. A subset $Y$ of the space $X$ is said to be **contractible in** $X$ if there is a map $H : Y \times [0, 1] \to X$ such that $H(y, 1) = y$ for each $y \in Y$ and the map $H(\cdot, 0)$ is constant. An elementary result concerning contractibility and homotopical triviality says that if every compact nonempty subset of a metrizable space $M$ is contractible in $M$, then $M$ is homotopically trivial.

Toruńczyk in his famous works [22, 23] has given a characterization of metric spaces which are homeomorphic to $l^2$. Based on his results, Dobrowolski and Toruńczyk [4] has proved the following theorem, which will be one of tools of this section.

**8.2. Theorem.** A separable completely metrizable topological group is homeomorphic to $l^2$ iff it is a non-locally compact absolute retract.

Thus, according to Theorem 8.2, in order to prove Theorem 8.1, we only need to show that each of the spaces $G_r(N)'s$ is an absolute retract (that $G_r(N)$ is non-locally compact it easily follows from Theorem 5.12). We shall prove this involving a very convenient criterion due to Toruńczyk [21] (cf. [16, Theorem ??]) a special case of which is formulated below (compare with the proof in [28]).

**8.3. Theorem.** If the intersection of every finite nonempty collection of open balls in a metric space $(X, d)$ with centres in a given dense subset of $X$ is homotopically trivial, then $X$ is an absolute retract.

Now we are ready to prove Theorem 8.1. It turns out that the argument is slightly more complicated in case of bounded groups than in the unbounded case (the main reason for this is that there are no nontrivial bounded norms on PV groups). Thus, we divide the proof into these two cases.

**Proof of Theorem 8.1 for $r = \infty$.** Let $D = G_r(N)_{fin}$ and $p$ be the value of $G_\infty(N)$. By (G3), $D$ is dense in $G_\infty(N)$. Take arbitrary points $x_1, \ldots, x_n \in D$ and radii $r_1, \ldots, r_n > 0$. Let $B = \bigcap_{j=1}^n B_p(x_j, r_j)$ where $B_p(x_j, r_j) = \{z \in G_\infty(N) : p(z - x_j) < r_j\}$. We may assume that $B$ is nonempty. We shall show that $B$ is contractible (in itself). Since translations $x \mapsto x + a$ ($a \in G_\infty(N)$) are (bijective) isometries on $G_\infty(N)$, we may also assume that $0 \in B$. This means that

$$p(x_j) < r_j \quad (j = 1, \ldots, n).$$

Let $H = \langle x_1, \ldots, x_n \rangle$. Observe that $H$ is a finite group. Denote by $q$ the restriction of $p$ to $H$. Then $(H, +, q) \in \mathcal{G}_\infty(N)$ and consequently $(L_0[H], +, \| \cdot \|_q) \in \mathcal{G}_\infty(N)$. Notice that $\varphi : H \ni h \mapsto \hat{h} \in L_0[H]$ is an
isometric group homomorphism. By Theorem 1.3, there is a normed topological PV group \((G, +, *, \| \cdot \|) \supseteq (L_0[H], +, *, \| \cdot \|_q)\) such that \(G\) and \(\mathbb{G}_\infty(N)\) are isometrically group isomorphic. So, thanks to (UEP), there is an isometric group isomorphism \(\psi: \mathbb{G}_\infty(N) \to G\) which extends \(\varphi\). Define \(H: B \times [0, 1] \to \mathbb{G}_\infty(N)\) by \(H(y, t) = \psi^{-1}(t \ast \psi(y))\). We see that \(H\) is continuous and \(H(y, 0) = 0\) and \(H(y, 1) = y\) for \(y \in B\). So, it suffices to check that \(H(B \times [0, 1]) \subset B\).

For \(y \in B, t \in [0, 1]\) and \(j \in \{1, \ldots, n\}\) we have, by (8-1),

\[
p(H(y, t) - x_j) = \|t \ast \psi(y) - \psi(x_j)\| \\
\leq \|t \ast \psi(y) - t \ast \psi(x_j)\| + \|t \ast \psi(x_j) - \psi(x_j)\| = \\
t\|\psi(y - x_j)\| + \|t \ast \hat{x}_j - \hat{x}_j\|_q = tp(y - x_j) + (1 - t)p(x_j) < r_j
\]

and we are done. \(\Box\)

Proof of Theorem 8.1 (for arbitrary \(r\)). We shall improve the previous argument so that it will work also in bounded case. Let \(D; p; x_1, \ldots, x_n \in D; r_1, \ldots, r_n > 0\) and \(B\) be as in the previous proof. As there, we may and do assume that (8-1) is fulfilled. Let \(K\) be a compact subset of \(B\). We shall show that \(K\) is contractible in \(B\).

For \(j \in \{1, \ldots, n\}\), let \(g_j = \max\{p(z - x_j); z \in K \cup \{0\}\} < r_j\). Let \(\epsilon > 0\) be such that

\[
g_j + 2\epsilon < r_j \quad (j = 1, \ldots, n). \tag{8-2}
\]

Further, take \(w_1, \ldots, w_l \in D\) for which

\[
B_p(w_k, \epsilon) \cap K \neq \emptyset \quad (k = 1, \ldots, l) \quad \text{and} \quad K \subset \bigcup_{k=1}^l B_p(w_k, \epsilon). \tag{8-3}
\]

The above implies that

\[
p(w_k - x_j) \leq g_j + \epsilon \quad (j \in \{1, \ldots, n\}, \ k \in \{1, \ldots, l\}). \tag{8-4}
\]

Let \(H = \langle x_1, \ldots, x_n; w_1, \ldots, w_l\rangle, q\) denote the restriction of \(p\) to \(H\) and let \(\| \cdot \|_H = \| \cdot \|_q \wedge r\). Then \((L_0[H], +, *, \| \cdot \|_H)\) is a Lipschitz subnormed PV group such that \((L_0[H], +, *, \| \cdot \|_H)\) is a Lipschitz subnormed PV group such that \((L_0[H], +, *, \| \cdot \|)\) in \(G_r(N)\) and the function \(\varphi: H \ni h \mapsto \hat{h} \in L_0[H]\) is an isometric group homomorphism. Again, we conclude from Theorem 1.3 that there is a subnormed topological PV group \((G, +, *, \| \cdot \|) \supseteq (L_0[H], +, *, \| \cdot \|_H)\) such that \(G\) and \(G_r(N)\) are isometrically group isomorphic. Therefore, there is an isometric group isomorphism \(\psi: G_r(N) \to G\) which extends \(\varphi\). Define \(H: K \times [0, 1] \to G_r(N)\) as before: \(H(y, t) = \psi^{-1}(t \ast \psi(y))\). We only need to show that \(H\) takes values in \(B\). To this end, let \(y \in K, \ t \in [0, 1]\) and \(j \in \{1, \ldots, n\}\). Take \(k \in \{1, \ldots, l\}\) such that \(p(y - w_k) < \epsilon\) and, using (8-2) and (8-4),
observe that
\[ p(H(y, t) - x_j) = \|t \ast \psi(y) - \psi(x_j)\| \leq \]
\[ \leq \|t \ast \psi(y - w_k)\| + \|t \ast \psi(w_k - \psi(x_j))\| \leq \]
\[ \leq \nabla(t)\|\psi(y - w_k)\| + \|t \ast \hat{w}_k - \hat{x}_j\| = \]
\[ = p(y - w_k) + tp(w_k - x_j) + (1 - t)p(x_j) \leq \]
\[ \leq \varepsilon + t(\varrho_j + \varepsilon) + (1 - t)\varrho_j \leq \varrho_j + 2\varepsilon < r_j \]
which finishes the proof. □

9. Subnormed topological PV groups of class \( O_{00} \)

In this section all groups are subnormed topological pseudovector. Observe that if \( \| \cdot \| \) is a subnorm on \( G \), then for every \( g \in G \) the function \( (0, \infty) \ni t \mapsto \|t \ast g\|/t \in \mathbb{R}_+ \) is monotone decreasing. Consequently, there is a finite limit
\[ \|g\|_{*0} = \lim_{t \to \infty} \frac{\|t \ast g\|}{t}. \]
The function \( \| \cdot \|_{*0} : G \to \mathbb{R}_+ \) is a seminorm on \( G \) such that \( \| \cdot \|_{*0} \leq \| \cdot \| \). In particular, \( G_{*0} = \{g \in G : \|g\|_{*0} = 0\} \) is a closed PV subgroup of \( G \). We call \( G \) of class \( O_{*0} \) iff \( \| \cdot \|_{*0} \equiv 0 \) or, equivalently, if \( G = G_{*0} \).

9.1. Definition. A subnormed PV group \( G \) is of class \( O_{00} \) if \( G \) is of both classes \( O_0 \) and \( O_{*0} \). That is, \( G \) is of class \( O_{00} \) if for every \( g \in G \),
\[ \lim_{t \to \infty} \frac{\|t \ast g\|}{t} = \lim_{n \to \infty} \frac{\|ng\|}{n} = 0. \]
We call an element \( a \) of \( G \) bounded provided \( \text{lin}\{a\} \) is a (metrically) bounded subset of \( G \). Let us denote by \( G_{bd} \) the set of all bounded elements of \( G \). Let \( E(G) = L(G) \cap G_{fin} \cap G_{bd} \). So, \( E(G) \) is a PV subgroup of \( G \) consisting of all finite rank elements which are both Lipschitz and bounded. It is easily seen that \( G \) is of class \( O_{00} \) provided \( E(G) \) is dense. We want to prove the ‘converse’ of this statement, namely

9.2. Theorem. A subnormed topological PV group \( G \) is of class \( O_{00} \) iff it may be enlarged to a subnormed PV group \( \widetilde{G} \) such that \( E(\widetilde{G}) \) is dense in \( \widetilde{G} \). Moreover, if \( G \in \mathfrak{G}_r(N) \cap O_{00} \), the above \( \widetilde{G} \) may be chosen so that \( \widetilde{G} \in \mathfrak{G}_r(N) \).

Theorem 9.2 is an almost immediate consequence of

9.3. Lemma. Let \( a \in E(G) \setminus \{0\} \) be such that \( a \in G_{0*} \cap G_{*0} \). For every \( \varepsilon > 0 \) there is a subnormed PV group \( (\widetilde{G}, +, \ast, \| \cdot \|) \) enlarging \( G \) and \( b \in E(\widetilde{G}) \) such that \( \|a - b\| \leq \varepsilon \). What is more, if \( \text{rank}(a) < \infty \), then \( \text{rank}(b) = \text{rank}(a) \); and \( \| \cdot \| \) is bounded by \( 1 \) provided so is the subnorm of \( G \).
Proof. We mimic the proofs of Lemma 6.12 and Lemma 6.13. Let \( \| \cdot \|_G \) denote the subnorm of \( G \). If \( a \) has finite rank, let \( N = \text{rank}(a) \). Otherwise let \( N \geq 2 \) be as in (6-16). Put \( \kappa = \nabla \). Take \( T > 1 \) such that for each \( t \geq T \),

(9-1) \[ N\|t \ast a\|_G \leq \varepsilon t. \]

If \( \text{rank}(a) < \infty \), repeat the proof of Lemma 6.12 (below we use the same notation as there) with \( \| \cdot \| \) replaced by

\[
\|(g,f)\| = \inf \{\|g - \psi_G(u)\|_G + \varepsilon\|u\|_\kappa + A(\|\psi_H(u) - f\|_\delta \wedge T) : u \in L_0[\mathbb{Z}] \}.
\]

If \( \text{rank}(a) = \infty \), repeat the proof of Lemma 6.13 with

\[
\|(g,f)\| = \inf \{\|g - \sum_{j=1}^m s_j \ast (l_j a)\|_G + \varepsilon \sum_{j=1}^m \kappa(s_j)
+ A\left[\|\sum_{j=1}^m s_j \ast (l_j b) - f\|_\delta \wedge T\right] : m \geq 0, s_1, \ldots, s_m \geq 0, l_1, \ldots, l_m \in \{-1,1\}\}.
\]

In order to show that \( \|(g,0)\| \geq \|g\|_G \) for \( g \in G \), distinguish between two cases: when \( \|\psi_H(u)\|_\delta \leq T \) (respectively \( \|\sum_{j=1}^m s_j \ast (l_j b)\|_\delta \leq T \)) and when the latter is false. In the first case just copy the original proof of suitable lemma. Here we only show how to derive the second case.

First assume \( \text{rank}(a) < \infty \). Write \( u = \sum_{k=1}^m s_k \ast \hat{l}_k \) with \( 0 = s_0 < \ldots < s_m \) and \( l_1, \ldots, l_m \in \mathbb{Z} \setminus \{0\} \). Let \( z \in \{0, \ldots, m\} \) be such that \( s_z \leq T \) and \( s_{z+1} > T \). Put \( u' = \sum_{k=0}^z s_k \ast \hat{l}_k \in L_0[\mathbb{Z}] \). Notice that

\[
\|\psi_H(u')\|_\delta \leq T, \quad \|\psi_G(u')\|_G \leq \|\psi_G(u')\|_G + \sum_{k=z+1}^m \|s_k \ast (l_k a)\|_G \quad \text{and, by (9-1), } \|s_k \ast (l_k a)\|_G \leq \varepsilon s_k = \varepsilon \kappa(s_k) \text{ for } k > z.
\]

So,

\[
\|\psi_G(u)\|_G \leq \|\psi_G(u')\|_G + \varepsilon \sum_{k=z+1}^m \kappa(s_k) \leq \varepsilon \sum_{k=1}^z \kappa(s_k) + A\|\psi_H(u')\|_\delta + \varepsilon \sum_{k=z+1}^m \kappa(s_k) \leq \varepsilon \sum_{k=1}^m \kappa(s_k) + AT = \varepsilon\|u\|_\kappa + A(\|\psi_H(u)\|_\delta \wedge T)
\]

which gives the assertion.

When \( \text{rank}(a) = \infty \), argue in a similar way. Assuming that \( 0 = s_0 < \ldots < s_m \) and taking \( z \in \{0, \ldots, m\} \) for which \( s_z \leq T \) and \( s_{z+1} > T \), observe that \( \| \sum_{k=1}^z s_k \ast (l_k \hat{b})\|_\delta \leq T \) and (by Lemma 6.13)

\[
\| \sum_{k=1}^z s_k \ast (l_k a)\|_G \leq \varepsilon \sum_{k=1}^z \kappa(s_k) + A\| \sum_{k=1}^z s_k \ast (l_k \hat{b})\|_\delta.
\]
Further, it follows from (9-1) that \( \|s_k \ast (l_k a)\|_G \leq \varepsilon \kappa(s_k) \) for \( k > z \) (recall that \( l_k \in \{-1, 1\} \)). Hence

\[
\| \sum_{k=1}^{m} s_k \ast (l_k a) \|_G \leq \| \sum_{k=1}^{z} s_k \ast (l_k a) \|_G + \varepsilon \sum_{k=z+1}^{m} \kappa(s_k) \leq \\
\leq \varepsilon \sum_{k=1}^{m} \kappa(s_k) + AT \leq \varepsilon \sum_{k=1}^{m} \kappa(s_k) + A \left[ \| \sum_{k=1}^{m} s_k \ast (l_k b) \|_G \right] \\
\]

which is equivalent to \( \|(g, 0)\| \geq \|g\|_G \).

Finally, replace \( \| \cdot \| \) by \( \| \cdot \| \wedge 1 \) if \( \| \cdot \|_G \leq 1 \). The details are left for the reader. □

Theorem 1.1 and the existence of the Gurariǐ Banach space suggests to adapt these ideas in the realm of subnormed topological pseudovector (Abelian) groups. It may be done in a few ways. One of them is proposed below.

We fix \( r \in \{1, \infty\}, N \in \mathbb{Z}_+ \setminus \{0\} \). For simplicity, denote by \( \mathcal{PV}\mathcal{G}_r(N) \) the class of all subnormed topological PV groups which belong, as valued groups, to \( \mathcal{G}_r(N) \) and are of class \( \mathcal{O}_{00} \). In particular, \( \mathcal{PV}\mathcal{G}_\infty(0) \) consists of all separable subnormed topological PV groups of class \( \mathcal{O}_{00} \).

Let us call a subnormed PV group \( H \in \mathcal{PV}\mathcal{G}_r(N) \) of class \( \mathcal{E}_r(N)_{\text{fin}} \) finitely generated PV subgroup of \( H \) and \( H \) is finitely generated, that is, \( H = \text{lin} F \) for some finite subset of \( H \). Observe that the subnorm of a PV group of class \( \mathcal{E}_r(N)_{\text{fin}} \) is bounded.

‘Gurariǐ-like’ space in category of subnormed topological PV groups of class \( \mathcal{PV}\mathcal{G}_r(N) \) may be defined as follows.

9.4. Definition. A subnormed PV group \((G, +, \ast, \| \cdot \|_G)\) is said to be \( \mathcal{E}_r(N)\)-Gurariǐ iff the following two conditions are satisfied:

(GPV1) \((G, +, \| \cdot \|_G) \in \mathcal{G}_r(N), G \) is complete and the set \( \mathcal{E}(G) \) is dense in \( G \),

(GPV2) whenever \((H, +, \ast, \| \cdot \|_H) \in \mathcal{E}_r(N)_{\text{fin}}, K \) is a finitely generated PV subgroup of \( H \) and \( \varphi: K \to G \) is an isometric linear homomorphism, for every \( \varepsilon \in (0, 1) \) there exists an \( \varepsilon \)-almost isometric linear homomorphism \( \psi: H \to G \) such that \( \| \varphi(x) - \psi(x) \|_G \leq \varepsilon \) for every \( x \in K \).

Notice that, thanks to (GPV1), every \( \mathcal{E}_r(N)\)-Gurariǐ PV group is of class \( \mathcal{PV}\mathcal{G}_r(N) \) (by Proposition 6.5 and Theorem 9.2). Below we list other properties of them.

9.5. Proposition. Every \( \mathcal{E}_r(N)\)-Gurariǐ pseudovector group is isometrically group isomorphic to \( \mathcal{G}_r(N) \).

Proof. Let \( G \) be a subnormed \( \mathcal{E}_r(N)\)-Gurariǐ PV group. Thanks to Proposition 3.9, it suffices to check that \( G_0 = \mathcal{E}(G) \) satisfies conditions
(QG1) and (QG2) for $Q = \mathbb{R}_+$. But this easily follows from Theorem 6.11 and (GPV2). (Indeed, note that the PV group $G$ obtained from Theorem 6.11 may be constructed so that $G \in \mathcal{E}_r(N)_{\text{fin}}$.)

We shall now show that every $\mathcal{E}_r(N)$-Gurariǐ PV group satisfies the counterpart of (UEP). Precisely, that in (GPV2) one may put $\varepsilon = 0$. We can provide this by repeating the arguments of Sections 2 and 3. The crucial point here is that the PV groups of class $\mathcal{E}_r(N)_{\text{fin}}$ have bounded subnorms. We start with

9.6. Lemma. Let $(D_j, +, *, \| \cdot \|_j) \in \mathcal{E}_r(N)_{\text{fin}}$ ($j = 1, 2$) and $D_0$ be a finitely generated PV subgroup of $D_1$. Let $u: D_0 \to D_2$ and $v: D_1 \to D_2$ be an isometric and, respectively, an $\varepsilon$-almost isometric linear homomorphism (where $\varepsilon \in (0, 1)$) such that $\|u - v\|_{D_0} \leq \varepsilon$. Then there are a subnormed PV group $(G, +, +, \| \cdot \|)$ in $\mathcal{E}_r(N)_{\text{fin}}$ and isometric linear homomorphisms $\psi_j: D_j \to G$ ($j = 1, 2$) such that $\psi_1|_{D_0} = \psi_2 \circ u$ and $\|\psi_1 - \psi_2 \circ v\|_\infty \leq A\varepsilon$ where $A = 1 + \text{diam}(D_1, \| \cdot \|_1)$.

Proof. Define $D$, $w_1$, $w_2$ and a semivalue $\lambda$ on $D$ in exactly the same way as in the proof of Lemma 2.20. Observe that $D$ is a PV group, $\lambda$ is a seminorm and $w_1$ and $w_2$ are linear homomorphisms. To this end, let $G$ be the quotient subnormed PV group $D/\lambda^{-1}(\{0\})$ and $\psi_1$ and $\psi_2$ be the homomorphisms naturally induced by $w_1$ and $w_2$. \qed

Now repeating the proofs of Lemma 3.7 and Theorem 3.8 we obtain the next two results.

9.7. Lemma. Let $G$ be an $\mathcal{E}_r(N)$-Gurariǐ PV group, $(H, +, *, \| \cdot \|_H) \in \mathcal{E}_r(N)_{\text{fin}}$ be a subnormed PV group and $K$ its finitely generated PV subgroup. Further, let $\varphi: K \to G$ and $\psi: H \to G$ be, respectively, an isometric and an $\varepsilon$-almost isometric linear homomorphism (where $\varepsilon \in (0, 1)$) such that $\|\psi\|_K - \varphi\|_\infty \leq \varepsilon$. For every $\delta \in (0, \varepsilon)$ there is a $\delta$-almost isometric linear homomorphism $\psi': H \to G$ such that $\|\psi'\|_K - \varphi\|_\infty \leq \delta$ and $\|\psi - \psi'\|_\infty \leq C\varepsilon$ where $C = 3 + 2\text{diam}(H, \| \cdot \|_H)$.

9.8. Theorem. Let $G$ be an $\mathcal{E}_r(N)$-Gurariǐ PV group, let $H \in \mathcal{E}_r(N)_{\text{fin}}$ and $K$ be a finitely generated PV subgroup of $H$. Every isometric linear homomorphism of $K$ into $G$ is extendable to an isometric linear homomorphism of $H$ into $G$.

9.9. Corollary. Each two subnormed $\mathcal{E}_r(N)$-Gurariǐ PV groups are isometrically isomorphic as subnormed PV groups.

Thanks to Corollary 9.9, we may reserve a special symbol for a $\mathcal{E}_r(N)$-Gurariǐ PV group. We shall denote it (if it only exists) by $\mathbb{PVG}_{r}(N)$.

It follows from Theorem 9.2 and Theorem 9.8 that

9.10. Proposition. Every subnormed PV group of class $\mathbb{PVG}_{r}(N)$ is embeddable into $\mathbb{PVG}_{r}(N)$ by means of an isometric linear homomorphism.
We believe $\mathbb{PVG}_r(N)$ exists for every $r$ and $N$. The existence of it for $N = 0$ would have an interesting application in functional analysis (cf. Proposition 6.4).

9.11. Proposition. If there exists $\mathbb{PVG}_r(0)$, there is a separable complete subnormed topological vector space $V$ of class $\mathbb{PVG}_r(0)$ which is universal for all subnormed topological vector spaces of this class. Precisely, every subnormed topological vector space of class $O_0$ whose subnorm is bounded by $r$ admits an isometric linear homomorphism into $V$. Moreover, every separable metrizable topological vector space admits a homeomorphic linear embedding into $V$.

Proof. Put

$$V = \{x \in \mathbb{PVG}_r(0): (t + s)x = t \cdot x + s \cdot y \text{ for every } t, s \in \mathbb{R}_+\}.$$ 

Equip $V$ with the subnorm inherited from the one of $\mathbb{PVG}_r(0)$. We leave this as a simple exercise that the multiplication by reals is defined by $tv = t \cdot v$ and $(-t)v = -(t \cdot v)$ for $t \in \mathbb{R}_+$ and $v \in V$; and that the assertion of the proposition is fulfilled (use Proposition 6.4 and Proposition 9.10). □

Then next result corresponds to Proposition 4.7.

9.12. Proposition. If $M \mid N$, then

$$\mathbb{PVG}_r(N, M) := \{x \in \mathbb{PVG}_r(N): M \cdot x = 0\}$$

is isometrically isomorphic as a subnormed PV group to $\mathbb{PVG}_r(M)$.

Proof. For simplicity put $G = \mathbb{PVG}_r(N, M)$. We only need to check that the set $E(G)$ is dense if $G$. Let $a \in G$, $\varepsilon > 0$ and let $\| \cdot \|$ stand for the subnorm of $\mathbb{PVG}_r(N)$. By (GPV1), there is $b \in E(\mathbb{PVG}_r(N))$ such that $\|a - b\| \leq \varepsilon$. At the same time, according to Theorem 9.2, there is a subnormed PV group $(G', +, *, \| \cdot \|')$ of class $\mathbb{PVG}_r(M)$ which enlarges $\mathbb{lin}\{a\}$ and such that $E(G')$ is dense in $G'$. In particular, there is $a' \in E(G')$ with $\|a - a'\|' \leq \varepsilon$. Now using a standard argument of amalgamation, we see that there is a subnormed PV group $(D, +, *, \| \cdot \|_D)$ and isometric linear homomorphisms $\psi: \mathbb{lin}\{a, b\} \rightarrow D$ and $\psi': G' \rightarrow D$ which coincide on $\mathbb{lin}\{a\}$ (we omit the details). Put $H = \mathbb{lin}\{\psi(b), \psi'(a')\} \subset D$ and $K = \mathbb{lin}\{\psi'(b)\} \subset H$. Notice that $H \in E_r(N)_f$ and apply (GPV2) to get an $\varepsilon$-almost isometric linear homomorphism $\varphi: H \rightarrow \mathbb{PVG}_r(N)$ for which $\|b - \varphi(a')\| \leq \varepsilon$. To this end, observe that $\varphi(a') \in E(G)$ and $\|a - \varphi(a')\| \leq 2\varepsilon$. □

9.13. Corollary. $\mathbb{PVG}_r(N)$ exists for each $N \geq 2$ provided $\mathbb{PVG}_r(0)$ exists.

9.14. Example. Let $(H, +, *, \| \cdot \|)$ be a separable subnormed topological PV group. Then $(H, +, *, \| \cdot \|') \in \mathbb{PVG}_\infty(0)$ for every $\alpha \in (0, 1)$. So, as in case of valued groups, every separable subnormed topological PV group may be ‘approximated’ by members of class $\mathbb{PVG}_\infty(0)$.
10. Concluding remarks

In Proposition 2.18 we have proved that the Urysohn universal space $U$ admits no structure of an Abelian valued group of exponent 3. At the same time, as it was shown in [17], $U$ admits a unique structure of a valued group of exponent 2. Taking into account these two remarks, we pose the following

**Conjecture.** For $r \in \{1, \infty\}$, $G_r(2)$ is a unique (up to isometric group isomorphism) valued Abelian group of finite exponent and of diameter $r$ which is Urysohn as a metric space.

If the above conjecture was false, it would be interesting to know for which exponents suitable groups exist and for which exponents they are unique.

Cameron and Vershik [3] have shown that the Urysohn universal metric space admits a structure of a monothetic valued group. Taking this into account, the following may also be interesting.

**Question 1.** Are the topological groups $G_1(0)$ and $G_\infty(0)$ monothetic?

There is a fascinating phenomenon related to universal disposition property. Namely, the Urysohn universal space $U$ is a unique (separable complete) metric space with universal disposition property for finite metric spaces; $G_\infty(0)$ is a unique (separable complete with dense set of finite rank elements) valued Abelian group with universal disposition property for finite valued Abelian groups and $\text{PVG}_\infty(0)$ (if only exists) is a unique (separable complete with dense set of finite rank bounded Lipschitz elements) subnormed topological PV group with universal disposition property for finitely generated bounded Lipschitz subnormed PV groups. It follows from Theorem 5.1 and Proposition 9.5 that all these three metric spaces are isometric and both the above valued groups are isometrically group isomorphic. So, in fact we just enrich the structure of a single space. It seems to us important to answer the following question with which we end the paper.

**Question 2.** Do there exist the PV groups $\text{PVG}_1(0)$ and $\text{PVG}_\infty(0)$?

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