A STUDY OF QUASI-MODE PARAMETRIC EXCITATIONS IN LOWER-HYBRID HEATING OF TOKAMAK PLASMAS

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ABSTRACT. A detailed linear and non-linear analysis of quasi-mode parametric excitations relevant to experiments in supplementary heating of tokamak plasmas is presented. The linear analysis includes the full ion-cyclotron harmonic quasi-mode spectrum. The non-linear analysis, considering depletion of the pump electric field, is applied to the recent Alcator A heating experiment. Because of the very different characteristics of a tokamak plasma near the wall (in the shadow of the limiter) and inside, the quasi-mode excitations are studied independently for the plasma edge and the main bulk of the plasma, and for two typical regimes in overall density, the low (peak in density, \( n_0 = 1.5 \times 10^{14} \text{cm}^{-3} \)) and high (\( n_0 = 5 \times 10^{14} \text{cm}^{-3} \)) density regimes. At the edge of the plasma and for the low-density regime, it is found that higher \( n_0 \) \((n_0 = c k_x/\omega)\) than those predicted by the linear theory are strongly excited. Inside the plasma, the excitation of higher wave numbers is also significant. These results indicate that a large amount of the RF-power may not penetrate to the plasma centre, but will rather be either Landau-damped on the electrons or mode-converted into thermal modes, close to the plasma edge. Moreover, for sufficiently high peaks in density, it is found that all the RF-power is mode-converted before reaching the plasma centre. Inside the plasma, the power density of the excited sideband fields is shown to be always very small in comparison with their excitation at the plasma edge.

1. INTRODUCTION

In the heating of tokamaks with RF-power near the lower-hybrid frequency, the driven fields can act as a pump for parametric instabilities. The lower-hybrid fields are principally electrostatic, and are characterized by a frequency \( \omega_{\text{LH}} = \omega_{\text{pi}}/(1+\omega_{\text{pe}}/\omega_{\text{ci}})^{1/2} \), and wavevectors \( k_\parallel \) (parallel to the magnetic field \( \mathbf{B} \)) and \( k_\perp \) (perpendicular to \( \mathbf{B} \)) such that \( k_\parallel/k_\perp \ll 1 \). The quasi-mode type of parametric instabilities consists in the excitation of a lower-hybrid sideband wave from the background noise level through low-frequency fluctuations in the plasma. The power levels in current tokamak experiments [1] usually exceed the thresholds for this parametric instability to occur, and these experiments, as well as several other tokamak heating experiments over the past few years [2], all indeed show the presence of such excitations. This has provoked extensive studies of these processes in the last few years. The original, linear analysis of the temporal evolution of this instability [3] has been recently extended and complemented with extensive numerical computations [4]. This instability is, however, a convective one, and its spatial evolution in a pump of finite spatial extent is more relevant. The linear spatial evolution, including group velocities along and transverse to the magnetic field, for a constant pump of a finite spatial extent is discussed in Ref.[5]. The non-linear effects of pump depletion have been formulated and discussed in general terms, in Refs [6, 7]. Other non-linear saturation effects have also been invoked; these relate mainly to models of temporal cascading in which the excited lower-hybrid sidebands are treated in the random-phase approximation [8]. We shall not consider these here any further, especially since the physically important non-linear evolution is a spatial one. In spite of the broad attention that this problem has received, there are still many unresolved questions regarding the relevance and physical consequences of these parametric excitations in today’s tokamak experiments. More specifically, it is of interest to know how physical effects like pump depletion will scale with density and temperature; what the excited spectrum will be in
terms of wave amplitudes and wave numbers, and how this spectrum will change with changes in the density and temperatures. We present specific computational results which should help to answer these questions and give us a better insight into the effects that these parametric processes have in plasma heating.

Our plasma model is a two-dimensional one, the $x$-direction is the direction of the plasma inhomogeneities (i.e. density, temperatures, and toroidal magnetic field inhomogeneities), and the $z$-direction is in the direction of the toroidal magnetic field. We apply the theoretical results to parameters relevant to the recent Alcator A heating experiment [1].

We first start with a detailed analysis of the linear theory (Sections 2 and 3). The low-frequency fluctuations are due to both electrons and ions. In the previous works, it was commonly assumed that the quasi-mode excitation due to scattering off ions is negligible under the approximation $T_e > T_i$. This is never true for tokamak devices, and we shall see that scattering-off ions give important modifications to the excited spectrum. We distinguish between two different spatial regions, the plasma edge (in the shadow of the limiters [9]) and the main bulk of the plasma. We also make a comparative study between the two different regimes, of low and high densities. The low-density regime corresponds to a peak density of about $n_0 = 1.5 \times 10^{14} \text{cm}^{-3}$, and the high-density regime to a peak density around $n_0 = 5 \times 10^{14} \text{cm}^{-3}$. The plasma edge is characterized by a large drop in density and temperatures inside the shadow of the limiter [9]. Parametric excitations are particularly strong here, and the characteristics of the excited spectrum differ importantly from the ones found inside the plasma.

Non-linear calculations are carried out in Sections 4 and 5. The linear analysis presented in the preceding sections then acquires its complete physical meaning. We study how the pump is depleted at the edge and in the plasma. Pump depletion will have important consequences in plasma heating, since it may change completely the wave-number spectrum and the power density inside the plasma. We find that, at the edge of the plasma, strong non-linear excitation of higher $n_z$ may take place that will radically change the power spectrum and the amount of power penetrating into the plasma. Depletion in the plasma depends critically on the density regime we are considering. Depletion increases with increasing density. Inside the plasma the excited sideband field has a field amplitude which, in general, is never comparable to the pump amplitude. The modified wave-number spectrum also plays an important role in determining how much of the total power penetrates up to the centre of the plasma. All the results are summarized in Section 6.

Finally, it should be remarked that there is a variety of non-linear phenomena that can be operative in lower-hybrid heating of a tokamak-type plasma. In this paper, we have focused on the quasi-mode parametric excitations. Other parametric excitations and self-modulation effects, especially at the plasma edge, may also have important consequences in lower-hybrid heating. Their interaction with quasi-mode excitations is beyond the scope of this paper.

2. LINEAR THEORY, SPATIAL GROWTH FACTOR

The linear steady-state evolution of a sideband field $\vec{E}_1$ driven by a pump field $\vec{E}_0$ which decays via non-linear damping on electrons or ions is given by [6],

$$\vec{v}_1 \cdot \nabla E_1^2 = (C(x) E_{0k}^2(x)) E_1^2$$

(1)

where $\vec{v}_1$ is the group velocity of the sideband, $C(x)$ is the coupling coefficient, $E_{0k}^2(x)$ is the magnitude of the pump field as described by the linear equations, and $E_1^2$ is the magnitude of the excited sideband field. The co-ordinate $x$ is in the direction of the spatial inhomogeneities and is normalized to the slab half-width $a$. By integrating Eq.(1) along the group velocity trajectory of the sideband field, one finds the spatial growth factor (Fig.1):

![FIG.1. Linear pump propagation cone and sideband group velocity trajectory. The trajectories of the pump wave lie between the lines $\tau = L/2$ and $\tau = -L/2$. The trajectories of the sideband wave lie along the lines $\theta$ such that $0 \leq \theta \leq L/2$. The $x$- and $z$-directions are in the directions of the plasma inhomogeneities and the toroidal magnetic field, respectively.](image-url)
In previous works [3–7], it was commonly assumed that $T_e \gg T_i$, and thus that $|\chi_{el}| \gg |\chi_{ei}|$. Under these conditions, Eq. (5) becomes equal to

$$x_{el} = \frac{1}{k^2 \lambda^2_{ei}} \sqrt{\frac{\pi}{2}} k_{z} \frac{\omega}{2k_{z} \gamma T_e} \exp \left( - \frac{\omega^2}{2k_{z}^2 \gamma^2 T_e} \right)$$

implying that the scattering is only due to electrons. This, we find, is never true for tokamak-type plasma parameters where, in fact, one finds $T_e \approx T_i$ and ion contributions to the scattering is important.

For lower-hybrid heating of typical tokamak plasmas we find that the low-frequency quasi-mode excitation in the plasma can span several ion cyclotron harmonics. In addition, we find that, inside the plasma, $\Lambda_n(\beta_i) = I_n(\beta_i) \exp(-\beta_i)$ is usually small, since $\beta_i = (k_{i} \sqrt{T_e}/\Omega_i)^2$ is greater than one. This means that the contribution of the corresponding $n$ will not be appreciable unless the $Z$-function contribution is large.

Let us fix $\omega$ and $x$ and define $r(\omega, x)$ as the closest integer to $\omega/2\pi|x|$, we thus approximate $\chi_{IR}$ by

$$\chi_{IR} = \frac{1}{k^2 \lambda^2_{ei}} \left[ 1 + \frac{\omega}{\sqrt{2k_{z} \gamma T_i}} \right]$$

where we have reduced the infinite sum in all the cyclotron harmonics ($n = -\infty, \ldots, 0, \ldots, \infty$) to only three terms, and $Z_R$ denotes the real part of the $Z$-function.

For the imaginary part of $\chi_i$, we have

$$\chi_i = \frac{1}{k^2 \lambda^2_{ei}} \sqrt{\frac{\pi}{2}} k_{z} \frac{\omega}{2k_{z} \gamma T_i} \exp \left( - \frac{(\omega - r\Omega_i)^2}{2k_{z}^2 \gamma^2 T_i} \right)$$

The behaviour of the real and imaginary parts of $Z$ with respect to its argument and the fact that $|k_{z} \sqrt{T_e}/\Omega_i| \ll 1$ lead to only large values of $Z$ in a narrow interval of the frequency spectrum around the point $\omega = n\Omega_i$, this interval being even narrower for the imaginary part of $Z$, for which one can say that it is only different from zero when $\omega$ is very close to $r\Omega_i$. Thus, for fixed $x$ and $\omega$, we need only to take account of the closest harmonics as expressed in
Eqs (8) and (9). When $\omega < \Omega_i$, the main contributing term is for $n = 0$ which turns out to be always inappreciable for $|x| < 1$, since it requires $\omega/\omega_0 \approx k_x |v_{Te}|/\omega_0 \ll 1$ and thus that $\omega_1$ be very close to $\omega_0$, where the growth factor is always small. For this situation, we can approximate $\chi_1$ by the real quantity $\chi_1 \approx 1/k^2 \lambda^2_{De}$.

We note that the growth factor of the interaction is only appreciable when electron Landau damping for the quasi-mode is important, i.e. $\omega/(k_x |v_{Te}|) < 3.5$. The ion-cyclotron harmonic dependence of the quasi-mode gives important modifications to the growth factor as a function of frequency. This can be appreciated from the following: let us assume that $\chi_{el} \equiv 0$ (i.e. $\omega/(k_x |v_{Te}|) > 3.5$). Equation (5) now becomes

$$2k^2 \lambda^2_{De} \text{Im}\left(\frac{(1+x_1) x_e}{1+x_1+2x_e} / \frac{k^2 \lambda^2_{De} x_e x_i}{1+x_1+x_e} \right)^2$$

where $x_e = (-1/k^2 \lambda^2_{De}) (k_x v_{Te})^2 / (1+x_1)$ and $k^2 \lambda^2_{De} < 1$. Equation (10) is always very small, even if $\omega = n \Omega_i$, since $k^2 \lambda^2_{De} |x_e| \approx 1$. The only exception is when, for certain $\omega \approx n \Omega_i$, $(1+x_1) \approx 0$, then reducing to $k^2 \lambda^2_{De} x_e x_i / x_i$ which can give large growth factors if $x_i \approx (1+x_1)^2$. However, for $1+x_1+x_e \approx 0$, the low-frequency mode is no longer a quasi-mode but a true mode; thus the interaction becomes a resonant three-wave interaction and is no longer described by the quasi-mode equation, Eq.(1). Here, we shall not pursue such resonant interactions.

To calculate $\Gamma$ as a function of $\omega_1$ for a fixed $x$ and a given pump frequency $\omega_p$ (e.g. $\omega_p = 2.45$ GHz for Alcator A), we shall proceed by first choosing some values for $n_{0z} = c k_{0z}/\omega_p$ and $n_{1z} = c k_{1z}/\omega_1$. The parameter $n_{0z}$ is really fixed by the RF-power spectrum distribution which we shall specify in the next section. Besides, there are two physical conditions limiting the range of variation for $n_{0z}$ and $n_{1z}$: a) Pump and sideband are fluid-like waves, that is, they cannot be electron-Landau-damped ($\omega_e/(k_{0z} v_{Te}) > 3.5$ and $\omega_1/(k_{1z} v_{Te}) > 3.5$); this gives the upper bound

$$n_{0z} < \frac{1}{3.5} \frac{c}{v_{Te}} \quad (11)$$

b) The lower bound is given by the requirement $k_{1z} > 0$ in the region where the interaction takes place.

The excited frequency spectrum is found by using the condition of strong electron Landau damping of the quasi-mode $\omega/(k_x |v_{Te}|) < 3.5$,

$$\frac{\omega_z + 3.5}{\beta_z} < \frac{\omega_1}{\omega_0} < 1$$

where $\alpha_z = \omega_0 / k_{0z} v_{Te}$, $\beta_z = \omega_1 / k_{1z} v_{Te}$, and we assume that $k_{1z} > 0$. The minus sign is taken when $\alpha_z < \beta_z$ and the plus when $\alpha_z > \beta_z$. Once we choose $n_{0z}$, $n_{1z}$ and $\omega_1$, we can solve for $k_{0z} = k_{ox}$ and $k_{1z}$ using the lower-hybrid dispersion relation corrected for finite-temperature effects,

$$k_4^2 + (k^2 k_x^2 K_2) k_2^2 + c k_2^2 - k_2^2 K_1 = 0 \quad (13)$$

where $a$, $b$ and $c$ are given elsewhere.

Let us now determine the lower limit of the integral $x_1$, in Eq.(2), for a given $x$. Denoting by $v_0$ the group velocity of the pump field, we find

$$\Gamma(x) = \frac{\gamma(x)}{v_{1z}} \left( \frac{L}{\frac{v_{1z}}{v_{1x}} - \frac{v_{0z}}{v_{0x}}} \right) \quad (15)$$

This growth factor can now be maximized with respect to the angle $\phi$. $\Gamma$ dependences on $\phi$ come through $\gamma(x)$ which behaves like $\sin^2(\phi)$ (see Eq.(3)), and through

$$v_{1z} = - \frac{\omega_1}{k_{1z}} \frac{\cos(\phi)}{1 + \frac{\omega^2}{\omega_{pe}^2}} \sqrt{\frac{K_1^2}{-K_1^2}} \quad (16)$$

$$\frac{v_{1z}}{v_{1x}} - \frac{v_{0z}}{v_{0x}} = - \left( \frac{1}{\cos \phi} \sqrt{\frac{-K_1}{K_1}} - \sqrt{\frac{K_0}{K_0}} \right) \quad (17)$$

$K_1$ and $K_0$ are elements of the cold-plasma dielectric tensor for the frequency $\omega_1$. Putting these together.
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in Eq. (15) leads to the total angular dependences of the growth factor \( \Gamma \), which can be easily maximized to obtain

\[
\cos(\phi) = \sqrt{\frac{(-K_{t1}) K_{u1}}{K_{ii}} \cdot \frac{(-K_{t1}) K_{u1}}{K_{ii} (-K_{01})}} - 1 \quad (18)
\]

In the computations we shall only consider the maximum growth factor with respect to \( \phi \). We note that, in this section and for the rest of the paper, we limit ourselves to the case \( k_{t1} > 0 \) which is maximized according with Eq. (18) for \( v_{t1} < 0 \). This case represents waves travelling toward the centre of the plasma.

When \( k_{t1} < 0 \), Eq. (15) is maximized with respect to \( \cos(\phi) \), for \( v_{t1} > 0 \) representing waves travelling toward the plasma edge. We are not considering this last possibility since the results will be similar in both cases, and only the direction of propagation is reversed. However, one has always to remember that when parametric excitations become strong and the RF-power is transferred to the excited waves, a large part of this power will travel down to the plasma edge. In general, we find that \( \cos(\phi) > 0(k_{t1} > 0) \) and as \( \omega_1 \to \omega_0 \), \( \cos(\phi) \to 1 \) (i.e. \( k_{ii} \to k_{t1} \) and \( k_{1y} \to 0 \)). It should be noted that in the temporal evolution of the instability, the temporal growth factor, given by Eq. (3), maximizes with respect to \( \phi \) when \( \cos(\phi) \) is zero, i.e. when \( k_{ii} = k_{1y} \) and \( k_{1x} = 0 \) representing waves which will never get into the centre of the plasma.

3. LINEAR THEORY, CALCULATIONS

A. Inside the plasma

In our computer calculations we consider a D-\( \rho \)-plasma and take the density, temperatures and toroidal magnetic field to be spatially inhomogeneous obeying the following profiles, consistent with measurements on Alcator A [12]:

\[
n = n_0 \left( 1 - \frac{x^2}{1.47} \right)^{3/2}
\]

\[
T = T_0 e^{-2x^2}
\]

\[
B = B_0 \left( 1 - \frac{x}{5.68} \right)
\]

where \( n_0, T_0 \) and \( B_0 \) are the density temperatures and magnetic field values at the centre of the plasma, \( x = 0 \). We shall take \( B_0 \) fixed at 60 kG, and consider two typical values for the peak density, \( n_0 = 1.5 \times 10^{14} \), and \( 5 \times 10^{14} \text{cm}^{-3} \). The corresponding peaks in temperatures are, for the low-density regime: \( T_{e0} = 1.3 \text{ keV}, T_{i0} = 0.75 \text{ keV} \), and for the high-density regime: \( T_{e0} = 1.15 \text{ keV}, T_{i0} = 0.68 \text{ keV} \). The accessible \( n_x \) are \( n_{x1} = 1.6 \) and 2, respectively.

As an example we apply the theory previously outlined to the experimental characteristics of Alcator A. We use the maximum applied RF-power as 80 kW, driven at the plasma wall through a two-waveguide array, each of width \( w/2 = 1.25 \text{ cm} \), and height \( h = 8 \text{ cm} \). Reflection back into the waveguides, reduces the available power by 20\%. According to linear theory \[13, 14\], from the remaining 64 kW, about 35 to 40\% is below accessibility. This power is fed to surface magnetic field layer near the plasma wall and never penetrates in the main bulk of the plasma. Let us restrict ourselves to the propagation cone, \( k_{oz} > 0 \). Following linear theory, the accessible power is distributed within the wave numbers \( n_x = 2 \) to 5 as follows: between \( n_x = 2 \) and 3, there is an average of approximately 7.5 kW, between \( n_x = 3 \) and 4, it is 5.5 kW, and between \( n_x = 4 \) to 5, it is 3 kW; in addition, there are about 2 kW of power beyond \( n_x = 5 \). The total power in a resonance cone will be taken as approximately 17.5 kW. We are modelling this power distribution with the function

\[
\frac{2}{\pi} \frac{c}{\omega_0} \frac{P_0(x)}{L} \frac{\sin^2((n_x - n_{oz}) A)}{(n_x - n_{oz})^2}
\]

where \( A = (\omega_0/2c)L \), and \( L(x) \) is the width of the pump cone, which depends on the amount of power, \( P_0(x) \), penetrating to a certain point in the plasma.

In addition to accessibility which limits the power penetration into the plasma, mode conversion of the pump electric field \[11\] will also restrict the amount of power getting to the plasma centre. In Fig. 2, we show how the different wave numbers \( n_{oz} \) are mode-converted as they propagate into the plasma, for the two regimes of low (Fig. 2A) and high (Fig. 2B) densities. We see that, for the low-density regime, the whole linear power spectrum reaches the plasma centre. In the high-density regime, and for \( x < 0.5 \), only the part of the spectrum between the wave numbers, \( n_x = 2 \) to 3, penetrates. These considerations, and the \( n_x \) power spectrum distribution predicted by the linear theory, allow us to make the following estimation: Inside the
FIG. 2. Mode conversion of the pump electric field at different x for different $n_z$. A) $n_0 = 1.5 \times 10^{14} \text{cm}^{-3}$, $T_{e0} = 1.3 \text{ keV}$, $T_{i0} = 0.75 \text{ keV}$; B) $n_0 = 5 \times 10^{14} \text{cm}^{-3}$, $T_{e0} = 1.15 \text{ keV}$, $T_{i0} = 0.68 \text{ keV}$.

FIG. 3. Inside the plasma. Spatial growth factor versus sideband frequency spectrum. A) $n_0 = 1.5 \times 10^{14} \text{cm}^{-3}$, $T_{e0} = 1.3 \text{ keV}$, $T_{i0} = 0.75 \text{ keV}$, $n_{z0} = 3.5$, $n_{z} = 2$, $P_0 = 17.5 \text{ kW}$; B) $n_0 = 5 \times 10^{14} \text{cm}^{-3}$, $T_{e0} = 1.15 \text{ keV}$, $T_{i0} = 0.68 \text{ keV}$, $n_{z0} = 2$. We take at $x = 0.75$, $n_0 = 3.5$ with $P_0 = 17.5 \text{ kW}$, and at $x = 0.25$, $n_0 = 2.5$ with $P_0 = 7.5 \text{ kW}$; C) $n_0 = 1.5 \times 10^{14} \text{cm}^{-3}$, $T_{e0} = 1.3 \text{ keV}$, $T_{i0} = 0.75 \text{ keV}$, $n_{z0} = 3.5$, $n_{z} = 5$, $P_0 = 17.5 \text{ kW}$; D) $n_0 = 5 \times 10^{14} \text{cm}^{-3}$, $T_{e0} = 1.15 \text{ keV}$, $T_{i0} = 0.68 \text{ keV}$, $n_{z0} = 3.5$, $n_{z} = 5$, $P_0 = 17.5 \text{ kW}$.
plasma, $x < 1$ for the low-density regime and $0.5 < x < 1$ for the high-density regime, we shall assume, for linear calculations, that $P_0 \sim 17.5$ kW, and the pump spectrum is centred at $n_{oz} = 3.5$, with $L \sim 1.3$ $\omega$. For the high-density regime, and for $x < 0.5$, we have, because of mode conversion, $P_0 \sim 7.5$ kW, the spectrum being centred at $n_{oz} = 2.5$, with $L \sim 4$ $\omega$, i.e. the propagation cone is four times larger than the width of the waveguides. In any of the former cases, the power density is obtained by dividing the RF-power available by the area of the corresponding pump cone. We note that $\Gamma$ is independent of the pump width as can easily be seen from Eqs (3), (4) and (15).

In Fig.3, we represent $\Gamma$ as a function of the frequency spectrum for the two different peaks in density and for two different positions in the plasma, near the edge, $x = 0.75$, and near the centre, $x = 0.25$. The power distribution fixes $n_{oz} = 3.5$ except for $n_o = 5 \times 10^{14}$ cm$^{-3}$ and $x = 0.25$, where, as explained before, $n_{oz} = 2.5$. The power available is 17.5 kW except for the high-density regime and $x = 0.25$, where, owing to mode conversion, the power available is 7.5 kW. We have chosen $n_{lz} = 2$ in Figs 3A and B, and $n_{lz} = 5$ in Figs 3C and D. Comparing Figs 3A and B, we see that the spatial growth factor increases very rapidly with density because of the WKB enhancement and cylindrical focusing factor of the pump field; we ignore the focusing factor at about half-way to the centre of the plasma. We also note that $v_{1x}$ gives an additional WKB-type of enhancement for $\Gamma$ (see Eqs (15) and (16)). Moreover, we find that both $v_{ox}$ and $v_{1x}$ are small, and that pump and sideband move parallel to the magnetic field: $|v_{1x}/v_{1x}| > |v_{ox}/v_{ox}| > 30$, and they reach values of the order of 100 at the highest densities. The number of resonant peaks observed in the calculations varies with the width of the excited spectrum, which depends on $n_{oz}$, $n_{lz}$ and $x$ (see Eq.(12)). When $n_{oz}$ and $n_{lz}$ are close in value, the number of resonant harmonics, and the width of the excited spectrum, are smaller than when they are far from each other. This is illustrated in Figs 3A and C, where the density is the same but $n_{lz}$ is different. We also note that $\Gamma$ increases with increasing $n_{lz}$, this being due to the $v_{1x}$-dependence upon $n_{lz}$, as shown in Eq.(16). We shall later on discuss the important consequences of this fact. Figures 3B and D also differ in the value of $n_{lz}$. As the density is already too high, sideband fields with high $n_{lz}$ can only be excited close to the edge of the plasma, $x \sim 0.75$, as is shown in Fig.3D. This is due to mode conversion of the excited fields. Mode conversion for the pump electric field was shown to occur at different $n_{lz}$ for different peaks in density (see Fig.2). Similarly, the values that $n_{lz}$ and $\omega_1$ can take for a given density are also restricted by mode conversion. As the density increases, $\omega_1$ has to be closer to $\omega_0$ and $n_{lz}$ closer to accessibility in order for Eq.(13) to give a real $k_{1z}$. For example, in Fig. 3D and for $x < 0.5$, no sideband fields can be excited since $n_{lz}$ is too large ($n_{lz} = 5$) for such densities. Similar restrictions due to mode conversion of the sideband are found in Fig.3B, for $x = 0.25$ and $\omega_1/\omega_0 < 0.96$. In Fig.4, we are representing $\Gamma$ for the low-density case at $x = 0.75$, and taking $n_{lz} = 12$ and $n_{oz} = 6.5$. The reasons why we have shifted the pump spectrum toward higher $n_{lz}$ will be explained in the next sections. What we want to notice here, is that $\Gamma$ also increases linearly with $n_{oz}$ (see Eq.(4)), and that parametric excitations for such high values of $n_{lz}$ can only take place close to the edge ($x \sim 0.75$).

The nodes of the growth factor occur when $\omega$ is near $n \Omega_1$, $\chi_{il} = 0$ and $(1 + \chi_{ilR}) = 0$. As pointed out, $\chi_{il}$ is only different from zero in a tiny region around $\omega = n \Omega_1$, thus for $\omega \neq n \Omega_1$ we can take

$$\text{Im} \frac{(1 + \chi_{il}) x_e}{1 + x_i + x_e} = x_{el} \frac{(1 + \chi_{ilR})^2}{(1 + x_{ilR} + x_{elR}^2 + x_{el}^2)}$$

(22)

which becomes zero for $\chi_{ilR} = -1$. As we are considering an inhomogeneous magnetic field, the position of the resonant $\omega$ will change in space and so will also the positions of the growth factor nodes. This means that, for a fixed $\omega_1$, there will be regions in $x$ where
the pump and sideband are coupled, and others where
the coupling is very small. The maximum values for $\Gamma$
between nodes are approximately given by the condition:
$1 + x_{IR} + x_{R} \sim 0$ (except when $x_{IR}$ is very
small and then $1 + x_{IR}$ becomes zero simultaneously).
We note that the function $1 + x_{IR} + x_{R}$ has a zero
between each two harmonics, these zeros are very close
to those of $1 + x_{IR}$. At these zeros, Eq.(22) behaves
like $(1 + x_{IR})/x_{IR}$. When $x_{IR}$ is small, that is when $\omega$
is far away from the main bulk of the electron damping
(e.g. $\omega[(k_x l)_{VT_e}] > 3$ or $\omega[(k_x l)_{VT_i}] < 0.5$), the zeros
of $1 + x_{IR} + x_{R}$ can lead to singularly high values
for $\Gamma$, as can be observed in the different figures
presented. These singular values correspond to fre-
quencies fulfilling, $1 + x_{I} + x_{e} \sim 0$, which, as remarked
before, is the transition condition between a well
defined low-frequency mode and a quasi-mode (i.e. 
one for which $1 + x_{I} + x_{e} \neq 0$). Our formulation is no
longer valid here, and such values of $\Gamma$ should not be
considered as reliable; they would have to be calculated
from a resonant three-wave interaction
formulation.

B. The plasma edge

Let us now study in detail the quasi-mode para-
metric excitations in a special region of the plasma
situated in the shadow of the limiters of the tokamak.
This region is characterized by a large drop in density
and temperatures [9]. The drop in density can still be
approximately described by Eq.(19), taking $|x| \sim 1$.

The corresponding drop in temperatures is about a
factor 10 larger than the one prescribed by Eq.(20),
taking $|x| \sim 1$. This leads to very high values of $\mu^2$,
given by Eq.(4), and thus to very large values for the
spatial growth factor. We are going to explore the
regimes of low ($n_0 = 1.5 \times 10^{14} \text{cm}^{-3}$) and high
($n_0 = 5 \times 10^{14} \text{cm}^{-3}$) central densities. According to
the experimental results in Ref.[9], for the high-
density regime we find at the edge: $n \sim 6.7 \times 10^{13} \text{cm}^{-3}$
and $T_e \sim 10 \text{eV}$; for the low-density regime we have:
$n \sim 2 \times 10^{13} \text{cm}^{-3}$, and $T_e \sim 10 \text{eV}$. The toroidal
magnetic field is about 50 kG and the ratio between
electron and ion temperatures will be assumed to be
the same as in the centre of the plasma, so that $T_i \sim T_e$.
It should be remarked that the ion temperature in the
shadow of the limiter was not measured in Ref.[9].
The possibility exists that $T_e > T_i$ and hence the para-
metric excitation could be a resonant one involving
ion-cyclotron and/or ion acoustic waves. Here, we
assume $T_e \sim T_i$ for which case the quasi-mode excitation
is dominant. At the plasma edge, $x \sim 1$, the available
power in a cone will be assumed to be $32 \text{ kW}$, centred
around $n_{oz} = 3$, and $L$ taken to be just the width of the
waveguides, $w = 2.5 \text{ cm}$. We remark that by taking
into account that part of the power which is not
accessible to the plasma, we consider the possibility
that by decaying into higher $n_z$ it might become
accessible and then penetrate into the plasma.

We have already commented that $\Gamma$ increases
linearly with $n_{12}$; this will have important con-
sequences in this region of the plasma. The parameter

![FIG.5. Plasma edge. Spatial growth factor versus sideband frequency spectrum. A) $n_0 = 1.5 \times 10^{14} \text{cm}^{-3}$ (local density,
$2 \times 10^{13} \text{cm}^{-3}$), $n_{oz} = 3$, $P_0 = 32 \text{ kW}$; B) $n_0 = 5 \times 10^{14} \text{cm}^{-3}$ (local density, $6.7 \times 10^{13} \text{cm}^{-3}$), $n_{oz} = 3$, $P_0 = 32 \text{ kW}$. For both cases the
local temperature is $T_e \approx T_i \approx 10 \text{ eV}$.)](https://example.com/figure5.png)
n_{lz} is restricted by the condition of Eq. (11) and by mode conversion. At the edge, because of the large drop in temperatures, n_{lz} can vary from 1 to 50. Hence, the growth factors will increase to enormous values for the higher n_{lz}. In Fig. 5, we represent $\Gamma$ as a function of $\omega_z$ for the two regimes and for different values of n_{lz}; for the low-density regime (Fig. 5A), we take n_{lz} = 6 and 8, while for the high-density regime (Fig. 5B), we take n_{lz} = 1.5 and 4. We see that similar values for $\Gamma$ are obtained for different values of n_{lz} as one considers the regimes of low or high densities. As the density decreases, n_{lz} has to become larger in order to keep $\Gamma$ as large as when the density is high and n_{lz} small. We also note that the frequency spectrum is much narrower than the one usually found inside the plasma; this is also due to T_e being small: The pump spectrum is centred at $n_{oz} = 3$, implying $\omega_{oz} = \omega_0/(k_{oz}v_{Te}) \approx 70$, and the lower limit of Eq.(12) becomes now approximately equal to $\beta_z/(\beta_z \pm 3.5)$. Thus, if n_{lz} < 10, that is if $\beta_z = \omega_1/(k_{lz}v_{Te}) \gg 3.5$, one finds that $\omega_1/\omega_0$ is very close to 1. Given these large growth factors, one can conclude, and we shall show it in the non-linear calculations, that the pump will be completely depleted in the limiter region. The excited sideband fields have a frequency close to the pump frequency, but the wave-number spectrum may be very different from the initial linear spectrum. It should also be noted that in Fig. 5 the narrow regions of growth factor very close to the pump frequency $\omega_0$ can be attributed to ion-acoustic quasi-mode interactions, i.e. those driven by ion-Landau damping of the quasi-mode, these giving rise to the peak very near $\omega_0$. However, at such low frequencies, our quasi-mode formulation may not be the most appropriate one since $\chi_{el} \sim 0$, and, again, a resonant three-wave interaction with $\omega$ very small should be investigated.

Regarding the computational methods used in this region, we have to remark that as the plasma parameters are quite different from the ones inside the plasma $|k_x| < 1$, the discussion presented in Section 1 to derive Eqs (8) and (9) is no longer valid here. In fact, now $\beta_z = (k_xv_{Ti}/\Omega_i)^2 \gg 1$, and, for a given $\omega$, we cannot be sure that the main contributing harmonic is the closest to $\omega = n_{lz}\Omega_i$, since the Bessel functions are not as small as they were before. In all the calculations we have used a sufficient number of harmonics to obtain accurate results.

4. NON-LINEAR DEPLETION

The depletion of the pump field and the non-linear evolution of the excited sideband field are described by the following non-linear equations [6]:

$$v_{ox}(x) \frac{\partial}{\partial x} + v_{oz}(x) \frac{\partial}{\partial z} E_0^2 = -C(x) E_0^2 E_1^2$$  \hspace{1cm} (23)

$$v_{1x}(x) \frac{\partial}{\partial x} + v_{1z}(x) \frac{\partial}{\partial z} E_1^2 = C(x) E_0^2 E_1$$  \hspace{1cm} (24)

Equations (23) and (24) have been obtained through a WKB mode analysis assuming the plasma to be weakly inhomogeneous.

For a homogeneous medium, Eqs (23) and (24) admit the exact solutions [15]:

$$\psi_0 = -\frac{\partial}{\partial t} \ln \left( S(t) - T(r) \right)$$  \hspace{1cm} (25)

$$\psi_1 = +\frac{\partial}{\partial \theta} \ln \left( S(t) - T(r) \right)$$  \hspace{1cm} (26)

where $\psi_0 = CE_0^2/\nu_1$, $\psi_1 = CE_1^2/\nu_0$, $v = v_{oz}v_0 - v_{iz}/v_{1x}$.

$$\theta = z - a \int_{1}^{x} \frac{v_{iz}}{v_{1x}} \, dx'$$

and

$$\tau = - \left( z - a \int_{1}^{x} \frac{v_{oz}}{v_{ox}} \, dx' \right)$$

We shall assume a finite uniform extended pump at $x = 1$, $A_0 = \psi_0(x = 1, |z| < w/2)$, and denote by $A_1 = \psi_1(x = 1)$ the noise level. The functions $S(\theta)$ and $T(\tau)$ are uniquely specified in terms of $A_0$ and $A_1$ for the two different regions: $|r| < L/2$, $|\theta| < L/2$ (region I) and $|r| < L/2$, $\theta < -L/2$ (region II), see Fig.1, and their analytic expressions can be found in Ref.[15].

For an inhomogeneous plasma we shall also describe the dynamics of the interaction through Eqs (25) and (26), but now $A_0$ and $A_1$ become slowly-varying functions of the inhomogeneity co-ordinate $x$:

$$A_0(x) = \frac{C(x)}{v(x)} \frac{E_0^2(x)}{v_{ox}(x)}$$  \hspace{1cm} (27)

$$A_1(x) = \frac{C(x)}{v(x)} \frac{E_1^2(x)}{v_{ox}(x)}$$  \hspace{1cm} (28)
where $E_{0}(x)$ and $E_{1}(x)$ are the pump and sideband electric fields as described by the linear equations,

$$E_{x}^{2} = \frac{2}{\epsilon_{0}} \frac{p}{|x|} \frac{k_{x}}{\omega} \sqrt{-\frac{k_{x}^{2}}{k_{l}^{2}}}$$

(29)

$p$ is the power density which, for the pump, has already been specified in the former sections, and for the sideband we assume $p_{s} = p_{1}/p_{0} = 5 \times 10^{-4}$. Equations (25), (26) and (27), (28) describe the evolution of the interaction in a WKB sense, that is, if this evolution is faster than the variation of plasma parameters with $x$.

As a measure of power depletion we compute the average of the square pump electric field along the width of the resonance cone, and the sideband field coming out from the resonance cone (i.e. at $r = -L/2$). For $(1-x) > \Delta x_{w}$, where $\Delta x_{w} = (w/a)/|v_{lz}/v_{lx} - v_{o}/v_{ox}|$, one finds approximately (see Appendix):

$$\frac{1}{L} \int_{-\infty}^{+\infty} \frac{E_{0}^{2}(x,z)}{E_{0}(1)} \ dx = \frac{E_{0}^{2}(x)}{E_{0}(1)} \left(\frac{1}{2\Gamma} \ln \left(e^{\frac{A_{l}(\theta + \tau)}{2\Gamma}} - 1 - e^{-2\Gamma}\right)\right)$$

(30)

$$\frac{E_{1}^{2}(x,\tau = -L/2)}{E_{0}(1)} = \frac{E_{1g}(x)}{E_{0}(1)} \frac{1}{1 + e^{-A_{l}(\theta + \tau)} e^{-A_{l}L(-1 + e^{-2\Gamma})}}$$

(31)

where

$$\theta + \tau = a \int_{1}^{x} \left(\frac{v_{ox}}{v_{ox}} - \frac{v_{lz}}{v_{lx}}\right) \ dx', \text{ and } \Gamma = -(A_{o} + A_{l}) L/2$$

is the spatial growth factor. It should be noted that Eq.(30) is independent of the width of the propagation cone, and so is also, approximately, Eq.(31) since $(\theta + \tau) > L$.

In region I the electric fields are independent of $z$ and there solutions are as found from a one-dimensional analysis [15]:

$$\frac{E_{x}^{2}(x)}{E_{0}^{2}(1)} = \frac{E_{ox}^{2}(x)}{E_{0}^{2}(1)} \frac{1}{1 + p_{r}(A_{o} + A_{l})(\theta + \tau)}$$

(32)

$$\frac{E_{z}^{2}(x)}{E_{0}^{2}(1)} = \frac{E_{1g}(x)}{E_{0}(1)} \frac{1}{p_{r} e^{-\frac{A_{l}(\theta + \tau)}{2\Gamma}}}$$

(33)

and satisfy $E_{x}^{2} + (v_{lx}/v_{ox}) E_{ox}^{2} = E_{1g}^{2} + (v_{ox}/v_{hx}) E_{hx}^{2}$. Then, if $E_{0}^{2} \to 0$ one has $E_{1}^{2} \to (v_{ox}/v_{hx}) E_{hx}(x)$, and the sideband will come out with all the pump power. On the other hand, if the depletion of the pump occurs in region II, then the power density of the sideband can be much less than that of the pump. Complete depletion of the pump in region II occurs when $\exp(-A_{l}(\theta + \tau)) = 0$. At that $x$, (30) and (31) give $E_{0}^{2} = 0$ and $E_{1}^{2} = E_{1g}^{2}$.

Let us now carefully analyse Eq.(31). The sideband electric field $E_{1}^{2}$ gets larger when the denominator of Eq.(31) gets smaller, which happens when $\exp(-2\Gamma) < 1$ and $\exp(-A_{l}(\theta + \tau)) \sim 1$, (we note that $|A_{l}L| \ll A_{l}(\theta + \tau)$ and that $\exp(-A_{l}L) \sim 1$). In order for $\exp(-2\Gamma) < 1$, we need large growth factors (e.g. $\Gamma > 1$), that is either high densities, or low densities and large $n_{z}$. At high densities or sufficiently high $n_{z}$, we have $\exp(-2\Gamma) \sim 0$, but we still need $\exp(-A_{l}(\theta + \tau)) \sim 1$, i.e. $A_{l}(\theta + \tau) \sim 0$. Combining Eq.(28) with the definition of $(\theta + \tau)$, we obtain $A_{l}(\theta + \tau) \approx C_{g} r_{1} a(x-1)/v_{ox}$, and thus, inside the plasma, $|x| < 1$, $A_{l}(\theta + \tau)$ will be small only if $v_{ox}$ is sufficiently large. However, as has already been pointed out, $v_{ox}$ is always small and gets smaller as one increases the density. Hence, no strong sideband fields are expected to be created in the main bulk of the plasma. The total RF-power is, of course, conserved, but it is now spread out in a larger region of space following the group velocity trajectory of the pump field (i.e. along $\tau = -L/2$). This power is introduced in the plasma through a two-waveguide array, which for Alcator A has an area $h_{w} = 20 \text{ cm}^{2}$; as the pump gets in the plasma the sideband it generates will be spread out in an area:

$$h(2L + a) \int_{1}^{x} v_{ox} \ dx'$$

For $h_{w} = 76 \text{ cm}^{2}$, and $|v_{oz}/v_{ox}| > 30$, we find

$$h \int_{1}^{x} \frac{v_{oz}}{v_{ox}} \ dx' > 2.3 \times 10^{3}(1-x) \text{ cm}^{2}$$

Thus, a strongly depleted pump does not necessarily lead to large-amplitude sideband fields. The excitation of strong fields requires a strong interaction between the pump and the noise (say, $\Gamma > 1$), and that this interaction is to occur near the mouth of the waveguides ($x \sim 1$). At the edge, $x$ very close to 1, $\Gamma$ is very large, larger than anywhere else; besides, $\Delta x_{w}$ is also large and increases with decreasing density. Thus, at
QUASI-MODE EXCITATIONS

5. NON-LINEAR THEORY, CALCULATIONS

A. The plasma edge

Let us now apply the non-linear description of the previous section of the Alcator A experiment, and let us start with the plasma edge. In Fig. 6, we are representing the averaged square pump electric field, as given by Eq.(30), for the minimum depletion distance out of the resonance cone, $\Delta x_w = (w/a)/[v_{12}/v_{1x} - v_{0x}/v_{0z}]$, as a function of the sideband frequency spectrum. We take the RF-power to be $32\text{ kW}$ and, according to the linear theory, $n_{0z} = 3$. In the case of low density, Fig. 6A, we see that the wave numbers $n_{1z} = 6$ and 8, deplete most of the pump power. For the high-density case, Fig. 6B, the wave numbers, $n_{1z} = 1.5$ and 4 are already able to essentially deplete the pump. Let us call $n_{zth}$ the minimum $n_z$ for which the pump suffers an appreciable depletion in the distance $\Delta x_w$. This threshold will become smaller, and eventually equal to 1, as the density increases. When the density is very small one has $n_{zth} \gg 1$. In our two examples, one can approximately fix $n_{zth} \sim 7$ for the low-density regime and $n_{zth} \sim 3$ for the high-density regime.

For strong excitation of large-amplitude sideband fields, we require that pump depletion is to occur mainly in region I, that is in a distance from the edge approximately equal to $\Delta x_w$. This characteristic distance, $\Delta x_w$ is a function of density and sideband frequency $\omega_1$. At the edge, $\omega_1$ is so close to $\omega_0$ and the density is so small that $\Delta x_w$ is usually large. For example, in Fig. 6, and for those $\omega_1$ that give the strongest depletion, we find, for the low-density regime $\Delta x_w \sim 0.07$, and for the high-density regime, $\Delta x_w \sim 0.04$, (to recover $\Delta x_w$ in real magnitude one has to multiply the above dimensionless numbers by the minor radius of the tokamak, $a = 10\text{ cm}$). The large-amplitude excited fields, $n_{1z} \gg n_{zth}$, will propagate in a cone of width equal to the pump width, $L \sim 2.5\text{ cm}$. In terms of wave numbers, this means that the width of the excited spectrum is of the order of the width of the pump spectrum, $\Delta n_{1z} \sim 4$ (in general, one has $\Delta n_{1z} \leq \Delta n_{0z} \sim 4$). The question that arises now is for $n_{1z} \gg n_{zth}$, which are the selected $n_z$ that deplete the pump. We are not going to carry out, in the present paper, the calculations that lead to the excited spectrum. As a general criterion, we take those $n_{1z} \gg n_{zth}$ that 'deplete first' (i.e. those with the larger group velocities) as the ones that deplete the pump. To support this criterion, let us present some physical arguments on what would happen if the excited fields had the largest possible $n_{1z}$. Initially, the pump spectrum is centred around the value $n_{0z} \sim 3$; after decaying, the excited spectrum will be centred at, say, $n_{1z} \sim 45$. Let us consider the excited spectrum as a

FIG. 6. Plasma edge. Non-linear depletion of the square of the pump electric field versus sideband frequency spectrum at the fixed position from the edge. $\Delta x_w = (w/a)/[v_{12}/v_{1x} - v_{0x}/v_{0z}]$. A) $n_0 = 1.5 \times 10^{14}\text{ cm}^{-3}$, (local density, $2 \times 10^{13}\text{ cm}^{-3}$), $n_{0z} = 3$, $P_0 = 32\text{ kW}$; B) $n_0 = 5 \times 10^{14}\text{ cm}^{-3}$, (local density, $6.7 \times 10^{13}\text{ cm}^{-3}$), $n_{0z} = 3$, $P_0 = 32\text{ kW}$. For both cases the local temperature is $T_e = T_i \approx 10\text{ eV}$. 
new pump which essentially differs from the initial pump in the wave-number spectrum. This new pump, centred at \( n_{0z} \approx 45 \), may decay again. The spatial growth factor increases linearly with \( n_{0z} \), see Eq.(4); this means that \( \Gamma \) is multiplied by a factor of 15 with respect to the values it took when \( n_{0z} = 3 \), which makes this new pump highly unstable. This would initiate a quick cascade evolution toward smaller \( n_z \) until the pump reaches the most stable parametric decay, i.e. near the lowest \( n_z \) for which the pump completely depletes, which will be what we call the threshold \( n_{2th} \). We note that if the highest possible \( n_{lz} \) are excited, they could damp a certain amount of the RF-power on the electrons before the new pump is able to penetrate inside the plasma. Nevertheless, to achieve a very high excited new pump (i.e. \( n_{lz} > 45 \)) one needs to decrease drastically the density of the plasma, since otherwise the rate of decay toward smaller \( n_z \) will be always larger than the linear Landau-damping rate, and thus the cascading processes toward smaller \( n_z \) will dominate.

It should be noted that \( n_{2th} \) and the pump spectrum penetrating the plasma depend on the assumed noise level. The thermal noise level is too low, and non-linear processes always enhance it. The experimental data show that for \( n_0 = 1.5 \times 10^{14} \text{cm}^{-3} \), the electrons are heated close to the centre of the plasma. For this to occur, we require the excitation of the wave numbers, \( n_{lz} = 6 \) and 7, which in turn leads to the estimation \( A_t/A_0 = 5 \times 10^{-4} \) so that appreciable pump depletion should occur into these wave numbers. We want to remark that it is beyond the scope of this paper to present quantitative results on the stability of the excited spectrum; with arguments as the ones presented in this section, we only attempt to present a qualitative picture of the evolution for these quasi-mode excitations and mainly wish to point out how they scale in the different regimes of plasma parameters.

Now, we can summarize the situation at the plasma edge as follows: The pump is completely depleted in this region, and the excited spectrum has a frequency very close to the initial pump frequency and propagates in a cone of width approximately equal to the waveguide width. However, the wave-number spectrum may be very different from the initial linear spectrum. For \( n_0 = 1.5 \times 10^{14} \text{cm}^{-3} \), \( 5 < n_{lz} < 10 \), and for \( n_0 = 5 \times 10^{14} \text{cm}^{-3} \), \( 1 < n_{lz} < 6 \). The excited spectrum will act as a new pump inside the plasma, which is now centred at about \( n_{0z} \approx 7 \), for the low-density regime, and around \( n_{0z} \approx 3 \), for the high-density regime. We note that by decreasing the density, \( n_{0z} \) will become larger, and all the RF-power may be mode-converted or damped on the electrons, before getting into the main bulk of the plasma.

**B. Inside the plasma**

The excited spectrum at the plasma edge will act as a new pump inside the plasma, centred at \( n_{0z} = 7 \) and \( n_{0z} = 3 \), for the low- and high-density regimes, respectively. We assume that 40% of the RF-power will remain close to the edge and never be able to act in the plasma. This may occur because the pump can excite, close to the plasma edge, new waves of higher \( n_z \) (e.g. \( n_{lz} > 12 \), for the low-density regime, and \( n_{lz} > 5 \) for the high-density regime). The part of the power which goes into these high \( n_{lz} \) will be either mode-converted or transferred to the electrons via Landau damping, before the new pump penetrates well inside the plasma. Under this assumption, the amount of power penetrating into the plasma in a resonance cone is, for both regimes, about 17.5 kW.

The essential difference between the main bulk of the plasma and the plasma edge is the rapid rise in the electron temperature, which is faster than the corresponding rise in density. We have already seen that the growth factors are here smaller than at the plasma edge. Consequently, pump depletion is expected to be weaker, and occur in a distance, \( \Delta x_d \), much larger than \( \Delta x_w \). The excitation of sideband fields will occur mostly in region II, and their amplitudes will be much smaller than that of the launched pump electric field. The way in which the excited spectrum will be selected is also different from the plasma edge. First of all, the range of variation for \( n_{lz} \) is now more restricted by the increasing electron temperature and by the increasing plasma density. The largest possible \( n_{lz} \) (e.g. \( n_{lz} \sim 12 \) for the low-density regime and \( n_{lz} \sim 5 \) for the high-density regime) will interact with the new pump first because they have the largest growth factors, and will deplete a large amount of the RF-power before the new pump is able to get well inside the plasma. However, as the magnitude of the excited sideband electric fields is very small, the growth rates that they generate, acting as a pump, are now only moderate and because of that we assume that cascading to lower \( n_z \) does not take place. Thus, the part of the power that is transferred to the largest possible \( n_{lz} \), will be either mode-converted into thermal modes or Landau-damped on the electrons. The amount of RF-power transferred to the highest \( n_{lz} \) increases by increasing the central peak density. For the high-density regime, we find that the new pump is completely depleted close to the plasma limiter. In the low-density
FIG. 7. Inside the plasma. Non-linear evolution of the square of the pump electric field (full line) and of the pump power density (broken line) versus plasma position, \( n_0 = 1.5 \times 10^{14} \text{cm}^{-3} \), \( T_{e0} = 1.3 \text{ keV} \), \( P_0 = 0.75 \text{ keV} \), \( n_{oz} = 6.5 \), \( P_0 = 17.5 \text{ kW} \). A) \( n_{lz} = 2 \), and 5, \( \omega_1/\omega_0 = 0.86 \); B) \( n_{lz} = 12 \), \( \omega_1/\omega_0 = 0.9 \).

FIG. 8. Inside the plasma. Non-linear evolution of the square of the excited sideband electric field (full line) and linear evolution of the electric noise (broken line) versus plasma position, \( n_0 = 1.5 \times 10^{14} \text{cm}^{-3} \), \( T_{e0} = 1.3 \text{ keV} \), \( T_{i0} = 0.75 \text{ keV} \), \( n_{oz} = 6.5 \), \( P_0 = 17.5 \text{ kW} \). A) \( n_{lz} = 5 \), \( \omega_1/\omega_0 = 0.86 \); B) \( n_{lz} = 12 \), \( \omega_1/\omega_0 = 0.9 \).

regime, the depletion is not so complete and there may be still an appreciable amount of power getting into the plasma.

To illustrate the above comments, let us present some calculations, first for the low-density regime. The non-linear power spectrum is assumed centred at \( n_{oz} \sim 7 \); we remind that \( \Gamma \) increases with increasing \( n_{oz} \), so the depletion is stronger for the non-linearly generated spectrum than it would be for the original linear one (i.e. \( n_{oz} \sim 3.5 \)). In Fig. 7, we are representing with continuous lines the square pump field averaged along the resonance cone, as given in Eq. (30), and with dotted lines the evolution of the RF-power, averaged also in the pump cone and normalized to the initial RF-power of 17.5 kW. The pump electric field suffers simultaneously the effect of the depletion due to the parametric processes, and a spatial enhancement due to the inhomogeneities of the plasma. When the depletion is small the pump electric field will grow with \( x \) until the depletion overcomes this growth, and makes \( E_0^2 \) decrease down to zero. We take \( n_{lz} = 2 \) and 5 in Fig. 7A, and \( n_{lz} = 12 \) in Fig. 7B. We also take one
FIG. 9. Inside the plasma. Non-linear evolution of the square of the pump electric field (full line) and of the pump power density (broken line) versus plasma position, \( n_0 = 5 \times 10^{14} \text{ cm}^{-3} \), \( T_{e0} = 1.15 \text{ keV} \), \( T_{\theta 0} = 0.68 \text{ keV} \), \( n_{Oz} = 3.5 \), \( P_0 = 17.5 \text{ kW} \); A) \( n_{lz} = 2 \), \( \omega_1/\omega_0 = 0.95 \); B) \( n_{lz} = 5 \), \( \omega_1/\omega_0 = 0.95 \).

FIG. 10. Inside the plasma. Non-linear evolution of the square of the excited sideband electric field (full line) and linear evolution of the electric noise (broken line) versus plasma position, \( n_0 = 5 \times 10^{14} \text{ cm}^{-3} \), \( T_{e0} = 1.15 \text{ keV} \), \( T_{\theta 0} = 0.68 \text{ keV} \), \( n_{Oz} = 3.5 \), \( P_0 = 17.5 \text{ kW} \); A) \( n_{lz} = 2 \), \( \omega_1/\omega_0 = 0.95 \); B) \( n_{lz} = 5 \), \( \omega_1/\omega_0 = 0.95 \).

of the maximum growing frequencies \( \omega_1/\omega_0 = 0.86 \) (Fig. 7A) and \( \omega_1/\omega_0 = 0.9 \) (Fig. 7B) and for these \( \omega_1 \), we find \( \Delta x_w = 0.01 \) and 0.015, respectively. For \( n_{lz} = 2 \), the depletion is negligible; for \( n_{lz} = 5 \) pump depletion starts at \( x \sim 0.5 \) and the overall depletion is around 35%. In Fig. 7B, \( x \) goes only up to 0.5, and this is due to mode conversion of the excited sideband electric field of wave number \( n_{lz} = 12 \) and frequency \( \omega_1/\omega_0 = 0.9 \). However, this excited field starts transferring energy to the electrons at about \( x = 0.75 \), before mode-conversion into thermal modes takes place. Thus, an appreciable amount of the available RF-power, say 40%, may be damped on the electrons and heats the plasma close to the plasma limiter. Besides, this will reduce the amount of power penetrating into the plasma centre. The part of the power that penetrates may excite lower wave-numbers, such as \( n_{lz} = 5 \), close to the plasma centre. However, to properly calculate how much power goes into the smallest \( n_{lz} \) one should make a more careful analysis.
than the one presented here, and subtract from the total power that part that has already gone into the excitation of the highest \( n_{\text{LZ}} \).

In Fig. 8 we study, for the low-density regime, how the excited sideband electric field evolves in \( x \) (continuous lines) compared to the liner evolution of the noise electric field (dotted lines), as given in Eqs (29) and (31), for \( n_{\text{LZ}} = 5 \), \( \omega_1/\omega_0 = 0.86 \) (Fig.8A), and \( n_{\text{LZ}} = 12 \), \( \omega_1/\omega_0 = 0.9 \) (Fig.8B). We see that the amplitudes of the excited fields are never comparable to the amplitude of the launched pump field. This, as explained in the last section, is due to the fact that the pump is moving parallel to the magnetic field. Thus, a pump depleted in a distance, \( \Delta x_d \), in \( x \) of the order of 2 or 3 cm has travelled a much larger distance in the \( z \)-direction, along which the power is continuously spread out. It should be noted that the sideband fields created close to the plasma edge, as in Fig.8B, are always larger than the ones created well inside the plasma, as in Fig.8A. This is due to the fact that \( \omega_0 x \) is larger when the density decreases, and about the same amount of power will be put in a smaller distance following the pump trajectory. The ripple in Fig.8A at \( x \sim 0.7 \), is due to the fact that pump and sideband are occasionally uncoupled because of the spatial inhomogeneities of the toroidal magnetic field.

Next, we shall present some non-linear calculations for the high-density regime. In Fig.9 we study how the pump is depleted in \( x \), for two different values of \( n_{\text{LZ}} \), \( n_{\text{LZ}} = 2 \) (Fig.9A) and \( n_{\text{LZ}} = 5 \) (Fig.9B). The pump power spectrum is centred at \( n_{\text{OZ}} \sim 3 \), the available power is in both examples, 17.5 kW, and the frequency of the excited fields is the same in both cases, \( \omega_1/\omega_0 = 0.95 \). For this \( \omega_1 \), we find that \( \Delta x_{\text{D}} \sim 0.015 \), and the distance in which the pump is depleted is of the order of 2 or 3 cm. We see that for \( n_{\text{LZ}} = 2 \) (Fig.9A), pump depletion starts at \( x \sim 0.7 \). For \( n_{\text{LZ}} = 5 \) (Fig.9B), the depletion starts at \( x \sim 0.9 \), and the pump is effectively depleted (there is only a 10% left) at \( x \sim 0.6 \). Thus, the smallest \( n_{\text{LZ}} \) will not deplete the pump since the highest \( n_{\text{LZ}} \) will interact sooner and do it first. In Fig.9A, the narrow range of \( x \) (from 1 to 0.3) is due to mode conversion of the pump electric field centred at \( n_{\text{OZ}} = 3.5 \). From \( x = 0.3 \) to 0, one should take into account only that part of the spectrum, if any, in between \( n_{\text{OZ}} = 2 \) and 3 (i.e. a centred spectrum at \( n_{\text{OZ}} = 2.5 \)). In Fig.9B, \( x \) varies from 1 to 0.5, but the reason is now mode conversion of the excited sideband electric field whose frequency and wave number are, respectively, \( \omega_1 \sim 0.95 \omega_0 \) and \( n_{\text{LZ}} = 5 \). This effect of mode conversion of the excited spectrum is having crucial consequences, since all the RF-power will be transferred to the highest \( n_{\text{LZ}} \) and mode converted into thermal modes, before reaching the plasma centre. A cut-off in the amount of RF-power getting into the plasma centre is thus predicted for sufficiently high densities. In Fig.10, we show the excited fields for \( n_{\text{LZ}} = 2 \) (Fig.10A), \( n_{\text{LZ}} = 5 \) (Fig.10B), and \( \omega_1/\omega_0 = 0.95 \). We observe again that their amplitudes (continuous lines) are very small compared to the initial pump amplitude, but considerably much larger than the amplitude of the electric noise (dotted lines). We also note that fields excited closer to the plasma edge (Fig.10B) have larger amplitudes than those excited near the plasma centre (Fig.10A).

6. SUMMARY AND DISCUSSION

Parametric excitations of the quasi-mode type present different characteristics in two distinct regions of a tokamak plasma, the plasma edge and the main bulk of the plasma. They also lead to different results when considering the two regimes of low and high densities, which are represented, respectively, by the peak in densities: \( n_0 = 1.5 \times 10^{14} \text{cm}^{-3} \) and \( 5 \times 10^{14} \text{cm}^{-3} \).

At the edge of the plasma, we have found that parametric excitations are very strong and that, in fact, the pump is depleted in this region. The excited spectrum has a frequency very close to the initial pump frequency, while the wave-number spectrum may be very different from the initial linear spectrum (i.e. \( 1 < n_{\text{OZ}} < 5 \)). As the temperature is very low here, the excited spectrum can, in principle, run from \( n_{\text{LZ}} = 1 \) to, say, \( n_{\text{LZ}} = 50 \). The spatial growth factor increases linearly with \( n_{\text{LZ}} \). In addition, using the non-linear theory, we have argued that there exists a certain threshold, that we call \( n_{\text{LZ},\text{th}} \), such that for \( n_{\text{LZ}} > n_{\text{LZ},\text{th}} \), the pump will be essentially depleted. We have not calculated the detailed excited spectrum and, as a criterion, have taken those \( n_{\text{LZ}} > n_{\text{LZ},\text{th}} \) which 'deplete first' (i.e. those which have the larger group velocities) as the ones that deplete the pump. By passing from the high-density regime to the low one, \( n_{\text{LZ},\text{th}} \) becomes larger and, eventually, by decreasing sufficiently the density, \( n_{\text{LZ},\text{th}} \) can be large enough so that the excited spectrum is mode-converted into thermal modes or Landau-damped on the electrons, as soon as it gets into the plasma. For sufficiently high densities, and following the above criterion, the excited spectrum will be similar in wave numbers to the initial linear spectrum. To support this criterion, we have presented...
the following argument: Let us assume that the pump decays into the highest possible \( n_{lz} \), and let us consider the excited spectrum as a new pump, centred at, say, \( n_{oz} \sim 45 \), which may decay again. The spatial growth factor \( \Gamma \) increases linearly with \( n_{oz} \) and \( n_{lz} \). This means that \( \Gamma \) is multiplied by a factor of 15 with respect to the values it took when \( n_{oz} = 3 \). This new pump is now highly unstable to quasi-mode parametric decay, and it will quickly cascade toward smaller \( n_{lz} \) to become more stable. This process will be stopped when the new \( n_{oz} \) and \( \Gamma \) are sufficiently small, so that the excited sideband spectrum cannot further decay. Thus, the most stable parametric decay is, in general, well represented by the value \( n_{zth} \). The highest-\( n_{lz} \) excited spectra can be reached just by sufficiently decreasing the density of the plasma. It should be noted that if the excited fields have large \( n_{oz} \) (e.g. \( n_{oz} \sim 50 \)) they can transfer part of their energy to the electrons via Landau damping. However, this can only happen for very low densities, since for sufficiently high densities the rate of decaying into smaller \( n_{oz} \) is always very large and, in fact, it is larger than the linear damping rates. Then a cascading process towards smaller \( n_{oz} \) would dominate, and the energy will not be transferred to the particles at the edge.

As one gets inside the plasma, the plasma parameters will change: The density will increase and so will the temperatures; the increase in the temperatures being much faster than the corresponding one in density. This will also change the characteristics of the excited spectrum. Thus, the most likely parametric excitations will be different, in terms of wave-numbers, frequency, and power density, from the ones found at the edge. We start with the non-linear spectrum excited at the plasma edge. The frequency is just the same as the initial pump frequency. The power density, or otherwise the width of the excited sideband cone, is not different from the initial one, at the mouth of the waveguides. But the wave-number spectrum differs from the linear spectrum: for the low-density regime, it is centred at \( n_{oz} \sim 7 \), and, for the high-density regime, it is centred at about \( n_{oz} \sim 3 \). Note that we are already considering the excited spectrum as a new pump, which may decay into new \( n_{oz} \) as it propagates in the plasma. The possible \( n_{oz} \) that can be excited in the plasma are not as large as they were at the plasma edge (say, \( n_{oz} \leq 12 \), for the low-density case, and \( n_{oz} \leq 6 \), for the high-density case). The spatial growth factors are usually smaller than the ones found at the plasma edge, because of rapid rise in the electron temperature.

In the low-density regime, a large amount of the available RF-power may decay into the highest \( n_{lz} \), as, say, \( n_{lz} = 12 \), before reaching the plasma centre. This part of the power will be transferred to the electrons, via Landau damping, and never reach the plasma centre. In the high-density regime further decays into new wave numbers will also occur as soon as the new pump penetrates into the plasma. The RF-power will be completely transferred to the larger \( n_{lz} \), such as \( n_{lz} = 5 \), and mode-converted into thermal modes before it is able to reach the plasma centre. The lowest wave numbers, such as \( n_{lz} = 2 \), may be excited close to the plasma centre, but by then only a fraction of the power, or very little, may be left.

The excited spectrum presents inside the plasma the following characteristics: The frequency is usually lower than the pump frequency, and becomes closer to it with increasing density. The power density is much lower than the original one, at the waveguide mouth, and has spread to many other wave numbers. Thus, the RF-power will no longer be confined to propagate in a well defined resonance cone, but will be dispersed across the plasma cross-section.

**Appendix**

The function \( S(\theta) - T(\tau) \) is given in terms of \( A_1 \) and \( A_0(z) \) (\( A_0(z) = A_0 \) if \( |z| < L/2 \) and \( A_0(z) = 0 \) if \( |z| > L/2 \)), by means of the following equation:

\[
S(\theta) - T(\tau) = 1 + \int_0^\theta dy A_1 \exp \left( \int_y^\infty (A_0 + A_1) \, dz \right) \nonumber
\]

\[ + \int_0^-\tau dy A_0 \exp \left( \int_y^\infty (A_0 + A_1) \, dz \right) \quad (A.1) \]

In region II \( |\tau| \ll L/2, \ \theta \ll -L/2 \), we find:

\[
S(\theta) - T(\tau) = \frac{1}{(A_0 + A_1)} \left[-A_0 e^{-A_0 + A_1} \frac{L}{2} \right. 
\]

\[+ (A_0 + A_1) e^{-A_0 L/2} A_0 \theta \nonumber \]

\[+ A_0 e^{-A_0 L/2} (A_0 + A_1) \tau \left. e^{-A_0 (\theta + \tau)} \right] \quad (A.2) \]

and through Eqs (25) and (26):

\[
\psi_0 = \frac{A_0 (A_0 + A_1)}{(A_0 + A_1) e^{A_1 (\theta + \tau)} e^{-A_0 (\tau - L/2) - A_0 e^{A_0 (\theta + A_1) (\tau - L/2)}} + A_0} \quad (A.3)
\]
\[ \psi_1 = \frac{A_1 \left( A_0 + A_1 \right)}{(A_0 + A_1) - A_0 e^{A_1 \left( \theta + L/2 \right)} - A_0 e^{A_1 \left( \theta + r \right)} e^{A_0 \left( \tau - L/2 \right)}} \quad (A.4) \]

From Eq. (A.4), taking \( r = -L/2 \) and contemplating the definitions, Eqs (27), (28) and (29), it is easy to recover the form of Eq.(31). To derive Eq.(30), let us consider that \( A_1 \left( \tau - L/2 \right) < A_0 \left( \tau - L/2 \right) \) since \( p_2 = A_1/A_0 = 5 \times 10^{-4} \), and that \( A_1 \left( \tau - L/2 \right) < A_1 \left( \theta + r \right) \) since \( \theta < -L/2 \). These estimations allow us to write

\[ \psi_0 = \frac{A_0 + A_1}{e^{A_0 \left( \tau - L/2 \right)} \left[ -1 + (1 + p_r) e^{A_1 \left( \theta + r \right)} \right] + 1} \quad (A.5) \]

where now the only \( z \)-dependencies come from

\[ z_t \equiv -\left( \tau - L/2 \right) = z - a \int_1^x \frac{v_0}{v_{ox}} \, dx' + L/2 \]

Eq.(A.5) can be integrated exactly with respect to \( z_t \) from \( z_t = 0 \) to \( z_t = L \), and the result is just Eq.(30).

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