Studying cosmic reionization with observations of the global 21-cm signal

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ABSTRACT

We explore the ability of observations of the global brightness temperature of the 21-cm signal to constrain the reionization history and the properties of the ionizing sources. In order to describe the reionization signal, we employ either a commonly used toy model or a more realistic structure formation model that parametrizes the properties of the ionizing sources. If the structure formation model captures the actual evolution of the reionization signal, then detecting the signal is somewhat easier than it would be for the toy model; using the toy model in this case also leads to systematic errors in reconstructing the reionization history, though a sufficiently sensitive experiment should be able to distinguish between the two models. We show that under optimistic assumptions regarding systematic noise and foreground removal, 1-year observations of the global 21-cm spectrum should be able to detect a wide range of realistic models and measure the main features of the reionization history while constraining the key properties of the ionizing sources.

Key words: galaxies: formation – galaxies: high-redshift – cosmology: theory.

1 INTRODUCTION

One of the most important frontier fields of cosmology is the evolution of the Universe from the dark ages following hydrogen recombination through to the epoch of reionization. The 21-cm line associated with the hyperfine transition of atomic hydrogen is the most promising signal for detecting and mapping the spatial and redshift distribution of hydrogen in the universe, and for studying the sources responsible for heating and reionizing the intergalactic medium (IGM) at redshifts \( z \gtrsim 6 \). Indeed, an important feature of the 21-cm signal is that the spectral dimension allows in principle 3D tomography of hydrogen as a function of redshift, providing much richer structure than the cosmic microwave background (CMB), which yields just a single sky map; this may help to detect primordial non-Gaussianity and test inflation (Loeb & Zaldarriaga 2004). During reionization, detecting the bubble structure would probe the main sources of ionizing radiation, even if these are otherwise unobservable because, for example, they are too faint to be detected individually. It is also important to characterize when and how long the reionization took place, because of the significant effect of reionization on the subsequent formation of galaxies. In particular, the ultraviolet (UV) radiation heats the gas, raising the Jeans mass and causing a suppression of star-forming galaxies in low-mass haloes (\( \lesssim 10^8 \, M_\odot \)).

A first estimate of the reionization redshift \( z_r \) has been deduced from the CMB polarization, where an additional peak on large angular scales, corresponding to the horizon size at the reionization epoch, is expected due to scattering, with an amplitude related to the total optical depth. The recent analysis based on the 7-year Wilkinson Microwave Anisotropy Probe (WMAP) data (Komatsu et al. 2011) finds a reionization redshift of \( 10.4 \pm 1.2 \) in terms of an equivalent instantaneous reionization, but is still consistent with a wide range of possible reionization histories. Other constraints come from the Ly\( \alpha \) galaxies at redshift 5.7 and 6.5, whose characteristic luminosity function shows a lack of time evolution that is consistent with a fully ionized IGM at \( z \sim 6 \) (Malhotra & Rhoads 2004). Analyses of the spectra of high-\( z \) quasi-stellar objects (Fan et al. 2004; Goto 2006; Willett et al. 2007, 2009; Mortlock et al. 2011) and Gamma Ray Bursts (GRB) (Totani et al. 2006) also suggest that the IGM is still very highly ionized at this redshift.

The expected picture for reionization is thus an inhomogeneous and extended process, for which the nature and the evolution of the ionizing sources are still observationally undetermined. Upcoming and future observational probes should allow us to distinguish among various reionization models and in particular constrain the possible extended or instantaneous nature of this process (Bruscoli, Ferrara & Scannapieco 2002). Low-frequency observations with radio telescope arrays, such as Low-Frequency Array for radio astronomy (LOFAR),\(^1\) Murchison Widefield Array (MWA),\(^2\) Precision Array for Probing the Epoch of Reionization (PAPER)\(^3\) and Square Kilometer Array (SKA),\(^4\) will over the next decade constrain the spatial distribution of the ionization sources, while dipole observations of the spatially integrated 21-cm signal are currently

\(^1\) http://www.lofar.org/ 
\(^2\) http://www.MWATelescope.org/ 
\(^3\) Parsons et al. (2010). 
\(^4\) http://www.skatelescope.org/
underway, although in their infancy, as exemplified by EDGES (Experiment to Detect the Global EoR Signature; Bowman, Rogers & Hewitt 2008; Bowman & Rogers 2010).

Recently Pritchard & Loeb (2010, hereafter PL) made a first theoretical attempt to predict detection limits for future observations of the global 21-cm signal. In this paper, we explore the potential for global 21-cm experiments to constrain reionization starting with the same simple analytical toy model used by PL (based on the tanh function), but then focus on a more realistic, physically based galaxy formation model that parameterizes the properties of the ionizing sources. In the following section, we briefly review the general setup of global 21-cm measurements and the toy model, and then introduce a simple galaxy formation model within cold dark matter (CDM)-dominated hierarchical structure formation. In Section 3 we make predictions for the spatially integrated 21-cm signal that experiments such as EDGES aim to measure. We compare the expected signal from the two models, and show some examples of the expected errors in the model parameters that are reconstructed from observations. In Section 4 we explore the systematic effects of assuming an incorrect model when trying to reconstruct the global 21-cm signal. We finish this part with Section 3.5, which presents our main result, the detection limits of the global 21-cm signal. We summarize and discuss our conclusions in Section 4.

Hereafter, we assume a flat ΛCDM cosmology, with matter density parameter Ωm = 0.272 (dark matter plus baryons), cosmological constant density parameter ΩΛ = 0.728, H0 = 70.4 km s⁻¹ Mpc⁻¹ (Hubble constant), Ωb = 0.045 (baryons), ns = 0.963 (power spectrum index) and σ8 = 0.809 (power spectrum normalization) according to the latest 7-year WMAP results (Jarosik et al. 2011). Unless otherwise stated, we estimate all errors at the 68.3 per cent confidence level.

2 MODELLING THE 21-CM SIGNAL

2.1 The 21-cm foreground and signal model

In general, a global 21-cm measurement yields the antenna temperature \( T_{\text{sky}}(\nu) = T_{\text{bg}}(\nu) + T_{\text{b}}(\nu) \), where \( T_{\text{bg}}(\nu) \) and \( T_{\text{b}}(\nu) \) are the foreground and cosmological 21-cm brightness temperatures, respectively. For the cosmic signal, we assume that the dipole antenna temperature essentially measures a sky average, since fluctuations are expected to be present only on angular scales that correspond to small fractions of the sky. The foregrounds (i.e. our Galactic emission and radio emission from other galaxies) have large-scale angular structure, but even if they are convolved with an angular dipole response, this does not affect our analysis, which only assumes that they are smooth as a function of frequency.

For the foreground brightness temperature \( T_{\text{bg}} \), we assume a polynomial fit of the form

\[
\log T_{\text{bg}} = \sum_{i=0}^{N_{\text{pol}}} a_i \log(\nu/v_0) \tag{1}
\]

In particular, we use the third-order polynomial fit from PL, who fitted the model of the sky put together by de Oliveira-Costa et al. (2008) using all existing observations, by averaging the foregrounds over the dipole’s angular response:

\[
\log T_{\text{bg}} = \log T_0 + a_1 \log(\nu/v_0) + a_2 \log(\nu/v_0)^2
+ a_3 \log(\nu/v_0)^3, \tag{2}
\]

with parameter values \( v_0 = 150 \text{ MHz}, T_0 = 320 \text{ K}, a_1 = -2.54, a_2 = -0.074 \) and \( a_3 = 0.013 \) chosen from fitting to the band \( \nu = 100-200 \text{ MHz} \). Note that at these frequencies \( T_{\text{bg}} \) is dominated by diffuse synchrotron radiation from the Galaxy. The residuals related to such a parameterization of the foreground are dominated by limitations of the adapted sky model (de Oliveira-Costa et al. 2008) and they are \(<1 \text{ mK} \) averaged over the band. While in principle higher-order polynomials may be needed to reduce such residuals in the future, given the smoothness of the spectrum of the foreground, low-order polynomials are the key to avoid throwing the signal away with the foreground and to reduce the statistical errors (Section 3.5).

For the cosmic 21-cm signal, the brightness temperature through the IGM is \( T_{\text{CMB}} = T_0 \left( \frac{\Omega_b h^2}{0.03} \right) \left( \frac{\Omega_m}{0.3} \right)^{-1/2} (1+z)^{-1/2} \), \( z \ll 1 \) the optical depth at \( 21(1+z) \) cm produced by a patch of neutral hydrogen at the mean density and with a uniform 21-cm spin temperature \( T_s \),

\[
\tau(z) = 9.0 \times 10^{-3} \left( \frac{T_{\text{CMB}}}{T_s} \right) \left( \frac{\Omega_b h^2}{0.03} \right) \left( \frac{\Omega_m}{0.3} \right)^{-1/2} (1+z)^{-1/2} \tag{3}
\]

During the epoch of reionization the Lyman α and X-ray radiation backgrounds are expected to be strong enough to bring the spin temperature \( T_s \) to the gas temperature and heat the cosmic gas well above the CMB temperature (Madau, Meiksin & Rees 1997). Under these conditions, the observed 21-cm brightness temperature \( T_b \) relative to the CMB temperature \( T_{\text{CMB}} \) is independent of \( T_s \). Therefore, \( T_b \) (hereafter measured relative to \( T_{\text{CMB}} \)) is given by

\[
T_b(z) = (T_s - T_{\text{CMB}})(1-e^{-z})Q_{\text{HI}}
= T_{21} \left( \frac{1+z}{10} \right)^{1/2} Q_{\text{HI}}, \tag{4}
\]

where \( T_{21} = 9.0 \times 10^{-3} (\Omega_b h^2/0.03)(\Omega_m/0.3)^{-1/2} T_{\text{CMB}} \lesssim 27.2 \text{ mK} \), and \( Q_{\text{HI}} = N_{\text{HI}}/(N_{\text{HI}} + N_{\text{HII}}) \) is the neutral hydrogen fraction. Note that the ionized fraction is \( Q_{\text{HII}} = 1 - Q_{\text{HI}} \). Throughout this paper, given that we are interested in the spatially integrated 21-cm signal, we consider only the cosmic mean neutral or ionized fraction, and neglect spatial fluctuations in the 21-cm signal from density and peculiar velocity fluctuations.

We consider an experiment covering the frequency range 100–250 MHz in 50 bins of bandwidth \( B = 3 \text{ MHz} \) for each of the receiver frequency channels, and integrating time \( t_{\text{int}} = 500 \text{ h} \) (these parameters mimic EDGES with an order of magnitude longer integration time). Under these assumptions, the thermal noise in the \( i \)th receiver frequency channel is given by the radiometer equation:

\[
\sigma_i^2 = \frac{T_{21}^2}{B_{\text{int}}}, \tag{5}
\]

We note that the frequency range we consider corresponds to the redshift range 4.7–13.2.

Our model thus consists of the foreground brightness temperature \( T_{\text{bg}}(\nu) \) and a suitable model for the cosmological 21-cm signal \( T_{\text{b}}(\nu) \). To derive the parameter errors, we directly calculate the Fisher matrix of the foreground plus 21-cm signal parameters \( \nu \) expected with the above thermal noise \( \sigma_i \),

\[
F_{ij} = \sum_{n=1}^{N_{\text{channel}}} \frac{1}{\sigma_n^2} \left( \frac{\partial T_{\text{sky}}(\nu_n; \nu; p)}{\partial p_i} \frac{\partial T_{\text{sky}}(\nu_n; \nu; p)}{\partial p_j} \right). \tag{6}
\]

This equation provides an estimate of the covariance matrix \( C = F^{-1} \), and therefore of the parameter uncertainty in dipole observations. Note that this is equivalent to finding the covariance matrix near the minimum \( \chi^2 \). These errors should be accurate as long as they are small. However, in many cases we consider regions of parameter space where the errors are large, e.g. when we calculate the detection limit of an experiment, or more generally due to parameter degeneracies. Thus, we often use a more generally valid
and computationally intensive Monte Carlo (MC) error analysis. We generate a large number of MC simulations of the measurement noise, finding the best-fitting parameters in each case by minimizing the $\chi^2$:

$$
\chi^2 = \sum_{n=1}^{N_{\text{ trials}}} \frac{1}{\sigma_n^2} \left[ \Delta T_{\text{sky}}(v_n; \hat{p}) \right]^2 ,
$$

where $\Delta T_{\text{sky}}$ is the difference between the measured and predicted total 21-cm sky temperature in channel $n$ (centred at the frequency $v_n$). The distribution of best-fitting parameters in the MC trials yields parameter errors and their correlations.

In the following sections we will focus on the modelling of the 21-cm signal, in particular of the neutral fraction $Q_{\text{HI}}$ in equation (4). Given the great uncertainty associated with the evolution of the neutral fraction $Q_{\text{HI}}$ due to the uncertain astrophysics of the ionizing sources, we will begin with a toy model, namely the tanh-based parametrization used by previous authors, which simply characterizes when and for how long the reionization occurs; then we will consider a more complex and physically motivated structure formation model, in order to better describe the reionization process and extract interesting astrophysical information, such as the mass of the smallest galaxies that can form and contribute to the redshift evolution of the ionizing sources.

2.2 The tanh-based model of reionization

The tanh-based parametrization is characterized by two parameters describing the two main features of reionization: its mid-point $z_i$ and duration $\Delta z$. This approach was used by PL for the 21-cm signal [note that Bowman et al. (2008) used a somewhat different parametrization], and a similar tanh-based fitting function is the default parametrization of reionization in the Code for Anisotropies in the Microwave Background (CAMB) (although there it is based on the optical depth for CMB scattering) (Lewis 2008). Under the assumptions outlined above for the gas state during reionization, the 21-cm signal is given by

$$
T_b(z) = T_2 \left(1 + \frac{z - z_i}{10} \right)^{1/2} \frac{1}{2} \left[ \tanh \left( \frac{z - z_i}{\Delta z} \right) + 1 \right].
$$

(8)

Note that $z_i$ is the redshift at which the ionized fraction $Q_{\text{HI}} = 50$ per cent, while $z_i + \Delta z$ and $z_i - \Delta z$ are the redshifts at which $Q_{\text{HI}} = 11.9$ and 88.1 per cent, respectively. This parametrization is a convenient mathematical toy model but it does not have any particular physical motivation. We consider both the case where we fix the amplitude of the signal $T_{21}$ to its known value (equation 4), and the case where we leave it as a free parameter (following PL).

2.3 A simple CDM-dominated galaxy formation model

In the previous section, we considered a toy model that has been used in previous observational and theoretical papers. While a toy model can be justified as an unbiased analysis tool, especially given the large current uncertainty in the astrophysics of high-redshift galaxies, such an approach is also problematic. The particular model assumed (with a fixed, arbitrarily chosen shape) may lead to systematically biased results if it cannot reasonably approximate the real reionization (we consider this issue further below.). In addition, it can be hard to interpret any results of a toy model in terms of the underlying parameters of interest. In particular, the redshift evolution of reionization is closely related to structure formation. Indeed, reionization is driven by the intergalactic ionizing radiation field, which (we expect) is the result of the ionizing radiation escaping from stars and quasars within galaxies. While astrophysical aspects (such as star formation and feedback) play a significant role, the evolution of galaxies is driven by the properties of the host dark matter haloes. A major reason for studying reionization is to learn more about both galaxy formation and the astrophysical properties of galaxies in the reionization era. Thus, a more realistic and useful approach is to use models based on our understanding of CDM-driven galaxy and structure formation, a model with many successes at lower redshifts, and to include some flexibility in order to account for the uncertain astrophysical parameters. Here we take the first step in this process by using a simple model that is based on the standard theory of galaxy formation.

We begin with the equation from Barkana & Loeb (2001), based on Shapiro & Giroux (1987), that statistically describes the transition from a neutral universe to a fully ionized one; in particular this equation describes the evolution of the H II filling factor $Q_{\text{HI}}$, i.e. the fraction of the volume of the universe which is filled by H II regions.

$$
\frac{dQ_{\text{HI}}}{dr} = \frac{N_{\text{ion}}}{0.76} \frac{dF_{\text{cool}}}{dt} - a_B \frac{C}{a^3} \frac{\bar{n}_H}{Q_{\text{HI}}},
$$

(9)

assuming a primordial mass fraction of hydrogen of 0.76. In this equation $N_{\text{ion}} = N_f f_{\text{star}} f_{\text{esc}}$ is an efficiency parameter that gives the overall number of ionizing photons per baryon; for instance, if we assume that baryons are incorporated into stars with an efficiency of $f_{\text{star}} = 10^{-2}$ per cent, the escape fraction for the resulting ionizing radiation is $f_{\text{esc}} = 5$ per cent and $N_f \approx 4000$ ionizing photons are produced per baryon in stars (for a stellar initial mass function similar to the one measured locally but with a metallicity equal to 1/20 of the solar value), we infer that for every baryon in galaxies $\sim 20$ escaping ionizing photons are produced by stars. We obtain a similar result if we consider mini-quasars rather than stars (Barkana & Loeb 2001). It is possible to get a substantially higher $N_{\text{ion}}$ using Population III stars or by assuming a high escape fraction. $N_{\text{ion}}$ also determines the maximum comoving radius of the region that a halo of mass $M$ can ionize on its own (neglecting recombinations),

$$
r_{\text{max}} = 675 \text{kpc} \left( \frac{N_{\text{ion}}}{40} \frac{M}{10^9 M_\odot} \right)^{1/3},
$$

(10)

a radius that is larger than the halo virial radius by a factor of $\sim 20$ (essentially independent of redshift and halo mass).

Also in equation (9), $a = 1/(1 + z)$ is the scale factor, $\bar{n}_H$ is the present number density of hydrogen, $a_B$ is the case B recombination coefficient of hydrogen and $C$ represents the volume-averaged clumping factor (in general time-dependent).

$$
C = \langle \bar{n}_H^3 \rangle / \bar{n}_H^2.
$$

(11)

This factor crudely accounts for a non-uniform IGM that includes high-density clumps. Since each ionized bubble is far larger than the typical scale of clumping, so that many clumps are averaged over, $C$ can be assumed to be approximately spatially uniform.

The collapsed fraction $f_{\text{cool}}$ is the fraction of all the baryons in the universe that is in galaxies, i.e. the fraction of gas which settles into haloes and cools efficiently inside them. A simple estimate of the collapse fraction at high redshift is the halo mass fraction above some cooling threshold. More generally, we include haloes above some minimum circular velocity $V_c$. We use the Sheth–Tormen halo mass function, which accurately fits the mean halo abundance in simulations (Sheth & Tormen 2002). We calculate the power spectrum transfer function using the CAMB code (Lewis, Challinor & Lasenby 2000).
The solution of equation (9) is (Barkana & Loeb 2001)

\[ Q_{\text{HII}}(t) = \int_0^t N_{\text{ion}} \frac{dF_{\text{col}}}{dt'} e^{f(t', t)} \, dt', \]  

(12)

where (if \( C \) is time-independent)

\[ F(t', t) = -\frac{2}{3} \frac{\alphaHI0}{\sqrt{\Omega_m H_0}} C [f(t') - f(t)], \]  

(13)

and where (in flat ΛCDM)

\[ f(t) = \left[ \frac{1}{a^3} + \frac{1 - \Omega_n}{\Omega_m} \right]^{1/2}. \]  

(14)

Once \( Q_{\text{HII}}(t) \) reaches unity, the universe becomes fully reionized and remains so within our model.

Equation (12) allows us to quickly calculate the time evolution of the ionized fraction of the universe once we fix the IGM clumping factor and the parameters related to the ionizing sources. Also, in equation (6) we calculate accurate derivatives for the 21-cm signal as numerical integrals of partial derivatives of the integrand in equation (12). Hereafter, we refer to this CDM-dominated galaxy formation model as the CDM model.

Within the CDM model, the parameters that determine the redshift evolution of \( Q_{\text{HII}} \) and \( T_b \) are \( N_{\text{ion}} \) (which we assume is a constant in this first investigation of fitting global 21-cm signals from a galaxy formation model), \( C \) (likewise assumed constant) and the minimum halo circular velocity \( V_c \) (equivalent to a minimum mass) required for haloes that host galaxies. We allow \( V_c \) to vary, since while cooling sets a minimum value for it, feedback (radiative or from supernovae) may in reality set a higher threshold for effective star formation. We set \( C = 1 \) as our standard value (i.e. corresponding to a uniform IGM), and discuss in several cases the effect of allowing \( C \) to vary.

3 RESULTS

3.1 Global properties of the two models

We begin by visually comparing our two models, the \( \tanh \)-based toy model and the more realistic CDM-based model. In Fig. 1 we plot a few examples of the global (volume-averaged) 21-cm signal for each model over the assumed experimental frequency range (100–250 MHz). For the CDM model, we consider a minimum halo circular velocity \( V_c \) \{4.5, 16.5, 36, 64\} km s\(^{-1}\) from left to right). We fix \( C = 1 \) and \( N_{\text{ion}} = 20 \). We also show (dashed curves) the global 21-cm signal for a \( \tanh \) model of reionization with \( \Delta z = 2 \) and \( z_r = \{6, 8, 10\} \) (from right to left). The signal for a fully neutral universe is shown for comparison (dotted curve). Note some redshift values (at the top) that fall within the experimental frequency range (\( z = 4.7-13.2 \)).

Figure 1. Global 21-cm signal predicted by the CDM model (solid curves) for various values of the minimum circular velocity of galactic haloes \( V_c = \{4.5, 16.5, 36, 64\} \) from left to right). We fix \( C = 1 \) and \( N_{\text{ion}} = 20 \). We also show (dashed curves) the global 21-cm signal for a \( \tanh \) model of reionization with \( \Delta z = 2 \) and \( z_r = \{6, 8, 10\} \) (from right to left). The signal for a fully neutral universe is shown for comparison (dotted curve). Note some redshift values (at the top) that fall within the experimental frequency range (\( z = 4.7-13.2 \)).

The main qualitative difference between the two models is that the CDM model shows a steady rise of \( Q_{\text{HII}} \), while the toy model is much more round in shape, in particular showing a slowdown of reionization during its last quarter or so. The toy model is explicitly symmetric in redshift about the mid-point of reionization, while in the CDM model reionization starts slowly but ends quickly. The steady acceleration of reionization in the CDM model is driven by the exponential rise of the ionizing sources, which correspond to rare haloes at these redshifts. However, our simple model is by no means fully general, so we treat our conclusions with caution, as discussed further below. We consider the CDM model to be an example of a realistic model, which may be quantitatively plausible if some of the missing complications turn out to have a relatively minor effect on the global 21-cm signal. We note, however, that some complications may tend to make the CDM model more similar in shape to the \( \tanh \) model (see Section 4).

The \( \tanh \)-based model is explicitly expressed in terms of the mid-point \( z_r \) and duration \( \Delta z \) of reionization, while in the CDM model these are derived parameters. While the mid-point \( z_r \) is naturally defined as \( Q_{\text{HII}} = 0.50 \) per cent, there is some ambiguity in \( \Delta z \). For the toy model, we have chosen to follow previous PL in defining \( \Delta z \) as above, a definition that is natural for the \( \tanh \) function, and implies that \( z_r + \Delta z = z_{\text{ion}} \). The overall spread of \( 2\Delta z \) delineates the central 76.2 per cent of reionization. However, for the CDM model we use a definition that should be the natural one more generally: \( \Delta z = (z_{\text{ion}} - z_{\text{1σ}})/2 \), with \( z_{\text{1σ}} \) being the redshifts corresponding to \( Q_{\text{HII}} = 0.16 \) and \( Q_{\text{HII}} = 0.84 \), respectively. Thus, in the CDM model a spread of \( 2\Delta z \) marks the central 68 per cent. In the context of the \( \tanh \) model, this definition...
would give a value of $\Delta z$ smaller by a factor of 1.2 than the definition that we have followed.

In order to gain intuition on how the characteristics of reionization are set in the CDM model, in Fig. 2 we show the dependence of $z_r$ and $\Delta z$ on $N_{\text{ion}}$ for one value of $V_c$ and several values of $C$. Larger values of $N_{\text{ion}}$ lead to earlier reionization (i.e. higher $z_r$) at a time when the ionizing sources are brighter and rarer, so their rarity leads to a shorter span $\Delta z$ for reionization. We compare $C = 0$ (fast recombinations) and $C = 10$. At least during most of the reionization $C$ is likely to be of the order of unity, since the low-density IGM gets reionized first, and the denser gas is left for the final stages of reionization. We find that a high clumping factor can be essentially counterbalanced by a higher value of $N_{\text{ion}}$, at least during the central portion of reionization that defines $z_r$ and $\Delta z$. So that including $C$ as a free parameter mostly includes the degeneracy of the parameters but does not significantly change the allowed parameter space of $z_r$ and $\Delta z$.

A more complete picture of the allowed parameter space is shown in Fig. 3, where we present the isocontours of $z_r$ and $\Delta z$ in the $N_{\text{ion}}-V_c$ plane. For reasonable values of $N_{\text{ion}}$ and $V_c$, the reionization mid-point $z_r$ varies widely (from below 6 to above 18), while the span of reionization $\Delta z$ covers roughly the range 1–3. As noted above, these values of $\Delta z$ should be multiplied by a factor of 1.2 for a fair comparison with $\Delta z$ in the $tanh$ model.

### 3.2 Expected parameter errors: the $tanh$ model

We now derive parameter errors for some specific instances of potential global 21-cm observations. We begin with the $tanh$ model, and test the Fisher matrix formalism against the MC error analysis. Taking fiducial values of $z_r = 8$, $\Delta z = 1$, and assuming the model parameters for the foreground as in Section 2.1, we generate $10^5$ MC simulations of the noise, finding the best-fitting parameters in each case. The resulting parameter contours are shown in Fig. 4 along with the corresponding Fisher matrix constraints. We find good agreement between the two methods in this example.

We consider both the case where $T_{21}$ is a free parameter (as assumed by PL) and where it is fixed at its known value. The error ellipses show that there is a strong positive correlation between $T_{21}$ and $\Delta z$, i.e. there is an uncertainty in distinguishing between a higher amplitude extended scenario and a lower amplitude quicker scenario, since both produce a similar slope with frequency in the 21-cm signal, and it is this sharp slope that can be distinguished.

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**Figure 2.** Reionization characteristics $z_r$ and $\Delta z$ as a function of $N_{\text{ion}}$ for the CDM model. We fix $V_c = 16.5$ km s$^{-1}$ and consider $C = \{0, 1, 10\}$ (dotted, solid and dashed curves, respectively).

**Figure 3.** Isocontours of $z_r$ (solid lines) and $\Delta z$ (dashed lines) derived from the CDM model in the $N_{\text{ion}}-V_c$ plane (for $C = 1$). The parameters ($N_{\text{ion}}$, $V_c$) are allowed to vary over $N_{\text{ion}} \in \{2, 1000\}$ and $V_c \in \{4.5, 100\}$ km s$^{-1}$.

**Figure 4.** The 68 and 95 per cent confidence regions of various parameter pairs for the $tanh$ model of reionization, comparing the MC likelihood (green/bright and red/dark shaded regions, respectively) to the Fisher matrix (solid ellipses) calculations. For comparison with PL, we set $z_r = 8$ and $\Delta z = 1$, $T_{21}$ free and fit four polynomial (foreground) parameters (i.e. $N_{\text{poly}} = 3$). In the panels on the right, we also plot the Fisher matrix (dashed ellipses) results for the same model with $T_{21}$ fixed at its known value. We assume an integration time $t_{\text{int}} = 500$ h.
from the foregrounds (which are smooth and thus can be modelled by a low-order polynomial). There are also significant correlations among the other parameters. Comparing with PL, we note that the amplitude of the errors that we find is smaller in all four panels, and also the sign of the correlation is different in the $z_1 = T_{21}$ and $z_2 = \Delta z$ relations. Note that PL used a different frequency interval, i.e. 100–200 MHz (Pritchard & Loeb, private communication).

The results thus show significant correlations among the parameters when $T_{21}$ is free, but substantially reduced errors and correlations when $T_{21}$ is fixed. We conclude that it is possible to obtain a direct observational estimate for $T_{21}$ from these type of data, in order to check consistency with the theoretically expected value, but in order to constrain reionization it is very helpful to use our independent knowledge of $T_{21}$. In this example we have assumed an integration time $t_{int} = 500$ and fitted a foreground polynomial of degree $N_{poly} = 3$ over the entire frequency range, assuming no remaining foreground or systematic residuals. This represents a quite optimistic assumption regarding the level of systematic noise and the ease of foreground removal, far beyond the current EDGES experiment, as discussed further below.

Since we have just considered a rather optimistic experimental scenario, it is interesting to consider more realistic possibilities. One way to do this is to vary the integration time, thus increasing the errors. We can take this also as a rough indication of the effect of increasing the foreground or systematic residuals to various levels (still with $N_{poly} = 3$). In the case considered in Fig. 4, the errors per frequency bin range from 0.4 mK in the lowest-frequency bin to an order of magnitude lower at the highest-frequency bin. More generally, the noise varies with the integration time $t_{int}$ of the bolometer as $t_{int}^{-0.5}$ (equation 5). In Fig. 5 we consider the fractional error on $z_1$ as a function of $t_{int}$ for $\Delta z = \{1, 2, 3\}$ and $z_2 = 8$. Note that the fractional error in $z_1$ varies approximately as $t_{int}^{-0.5}$ since the errors are relatively small over most of the plotted range (which makes the model behave approximately like a linear model). More extended reionization scenarios increase the errors significantly. Fixing $T_{21}$ at its known value reduces the errors by 15–50 per cent.

We will directly consider detection limits in a later section, but one way to define a successful detection of reionization is when observations yield a meaningful constraint on the most interesting single number associated with reionization, namely $z_1$. Within the $tanh$ model, rough (10 per cent) constraints on $z_1$ are expected for $t_{int} = 26$ h (if $\Delta z = 3$) or 1.9 h (if $\Delta z = 2$), while tight (1 per cent) constraints require $t_{int} = 848$ h (if $\Delta z = 3$), 51 h (if $\Delta z = 2$) or 3.1 h (for sharper reionization, with $\Delta z = 1$).

In Fig. 6 we show a different range of the parameter space, considering three possible values of $z_1$, while varying $\Delta z$ over a wide range, all for $t_{int} = 500$ h. Here we show the relative errors on both $\Delta z$ and $z_1$, finding that $z_1$ is generally better constrained, by up to an order of magnitude. The errors increase with $\Delta z$, roughly saturating at 30 per cent for $z_1$ and 50 per cent for $\Delta z$ (i.e. the errors only increase slowly beyond these values as $\Delta z$ is further increased beyond $\sim 5$). As before, fixing $T_{21}$ at its known value can make a big difference (compared to allowing it to be a free parameter), especially in constraining $\Delta z$ (except when all the errors are large, for high $\Delta z$). The relative errors vary weakly with $z_1$ over the range of 6–10 (a range which is all well within our assumed experimental frequency window).

### 3.3 Expected parameter errors: the CDM model

We begin our exploration of global 21-cm measurements in the context of the CDM model with Fig. 7, where we show parameter errors and correlations for the fiducial values of $(20, 1, 16.5)$ for $(V_c, C, V_e)$, respectively. The error ellipses show that there is a strong positive correlation between $V_c$ and $N_{ion}$; indeed, this is a partial degeneracy, since while the error ellipse covers a small total area, each of these parameters is uncertain at a relatively high ($\sim 10$ per cent) level. From Fig. 3 it is apparent that this degeneracy with a positive correlation is driven by the value of $z_1$, which is the main constraint from these observations (at least in the example we are considering a high-precision experiment.

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**Figure 5.** Relative error on $z_1$ as a function of $t_{int}$ for $\Delta z = \{1, 2, 3\}$ (from bottom to top in each case) and $z_2 = 8$ for the $tanh$ model of reionization. We consider $T_{21}$ fixed or free in the analysis (solid and dashed curves, respectively). The errors have been calculated via MC analysis.

**Figure 6.** Relative error on $\Delta z$ and $z_1$ as a function of $\Delta z$ for the $tanh$ model of reionization, for $\Delta z = \{6, 8, 10\}$ (dotted, dashed and solid curves, respectively). In each case we consider $T_{21}$ to be fixed or free (where fixed corresponds to the lower curve at the left end of the plot). The errors have been calculated via MC analysis.
with low noise). There is also a strong anti-correlation between \( V_c \) and \( T_0 - \langle T_o \rangle \), demonstrating how the foreground fitting removes power from the total signal \( T_{sky}(v) \), making it more difficult to determine the parameters of reionization. The Fisher matrix and MC formalisms yield a reasonable agreement, but there are bigger differences compared to the \textit{tanh} model, likely because the partial degeneracy gives larger errors in some directions in the CDM model.

Fortunately, the partial degeneracy in the parameters of the CDM model is relatively harmless in terms of measuring the characteristics of reionization. For the case considered in Fig. 7, we measure \( z_r = 8.74 \pm 0.02 \) and \( \Delta z = 1.83 \pm 0.02 \). The two-parameter contour is shown in Fig. 8. The relative errors in \( z_r \) and \( \Delta z \) are much smaller than in \( V_c \) and \( N_{\text{ion}} \), showing that the global 21-cm measurements constrain these quantities rather directly, somewhat independently of the underlying galaxy and halo parameters. Also, as noted above, the fractional error on \( z_r \) is significantly smaller than in \( \Delta z \). The plotted results assume \( C = 1 \), but we find that if we allow the clumping factor to be a free parameter in the fit, this increases the CDM model parameter degeneracies but it does not significantly affect the errors on \( z_r \) and \( \Delta z \).

As we did for the \textit{tanh} model, we vary the integration time and consider the expected experimental accuracy in measuring the most important quantity, \( z_r \). Fig. 9 shows the fractional error on \( z_r \) for \( \Delta z = \{1.5, 2, 2.5\} \). As in Fig. 5, the fractional error varies approximately as \( t_{\text{int}}^{-0.5} \), and increases for larger \( \Delta z \). For \( C = 1 \), tight (1 per cent) constraints on \( z_r \) require \( t_{\text{int}} = 68 \) h (if \( \Delta z = 2.5 \)), 29 h (if \( \Delta z = 2 \)) or 13 h (if \( \Delta z = 1.5 \)). The CDM model gives somewhat better accuracy than the \textit{tanh} model, for similar values of \( z_r \) and \( \Delta z \), though the numbers are comparable.

### 3.4 Systematic effect of the choice of reionization model

Our use of two different models allows us to explore the systematic effects of assuming an incorrect model when trying to reconstruct the global 21-cm signal from observations. We assume our more realistic CDM model as the input model, and try to fit the resulting 21-cm signal with the \textit{tanh} model. In Fig. 10 we plot the 21-cm signal as inferred from the fit of the \textit{tanh} model + foreground to the 21-cm data generated from the CDM model + foreground with \((N_{\text{ion}}, C, V_c) = (20, 1, 16.5)\), corresponding to \( z_r = 8.74 \) and \( \Delta z = 1.83 \). The fit of the \textit{tanh}-based model + foregrounds, after the subtraction of the best-fitting foreground polynomial (which takes...
out part of the signal together with the original input foreground), leads to a quite different output profile of $T_\nu(v)$ compared to the input one, and to biased values of the mid-point and duration of reionization. The best-fitting parameters are $z_i = 8.19 \pm 0.01$ and $\Delta z = 1.28 \pm 0.01$; the latter is even more discrepant than may appear, because the input CDM value of $\Delta z$ should be multiplied by 1.2 for a fair comparison with the $\tanh$ model. While the statistical errors of the fit are tiny (for $t_{\text{int}} = 500$ h and $N_{\text{poly}} = 3$), the systematic errors are quite large. The systematic errors are related to the inadequacy of the $\tanh$ model in representing the reionization signal and to the presence of the foreground which must be fitted with a polynomial.

The news, though, is not all bad, since an experiment with such low noise levels would result in high, strongly discrepant, $\chi^2$ values for such a poor fit, giving a clear indication that the template being used must, indeed, be modified. In particular, we find $\chi^2 = 1044$ for 44 degrees of freedom. This means that in this example, only a much reduced experimental sensitivity corresponding to $t_{\text{int}} \sim 20$ h would give a reduced $\chi^2$ of the order of unity for the $\tanh$ model fit. Any experiment above this sensitivity would be able to discriminate between the CDM and $\tanh$ models.

In Fig. 11 we explore these kinds of systematic errors over a wider range of the parameter space. We compare the best-fitting parameters $z_i$ and $\Delta z$ from fitting the $\tanh$ model + foreground with the true values for an input CDM model of reionization. We fix the input $\Delta z = 1.5$ (equivalent to $\sim 1.8$ in the $\tanh$ model), and vary $z_i$ over the range 6–10. This wider range shows similar results to the example given above, where the best-fitting $z_i$ in the $\tanh$ model is underestimated typically by 5–10 per cent, while $\Delta z$ is underestimated by much more. We consider several different values of the assumed clumping factor in the CDM model, and find that the $C = 0$ and $C = 1$ curves lie on top of each other, while $C = 10$ is only slightly different (where the comparison is made with fixed $z_i$ and $\Delta z$ values in the input CDM model).

### 3.5 Detection limits of the global 21-cm signal

In this section we present our main result, i.e. the experimental sensitivity that is required to detect the global 21-cm signal, as predicted by each of the reionization models. A range of different results is summarized in Fig. 12. First we display the full range of allowed values of the mid-point and span of reionization within the CDM model (grey shaded region), where the parameters ($N_{\text{ion}}, V_c$) are allowed to vary over $N_{\text{ion}} \in \{2, 1000\}$ and $V_c \in \{4.5, 100\}$ km s$^{-1}$, fixing $C = 1$. This region reflects Fig. 3, showing that a wide range of $z_i$ is plausible, while the most relevant range of $z_i = 6–12$ includes some models with $\Delta z$ as low as $\sim 1$.

The figure also shows curves which delineate the 95 per cent detection region for the different reionization models, for various polynomial orders of the fit ($N_{\text{poly}} = \{3, 6, 9, 12\}$ in equation 1) and several possible values of the integration time $t_{\text{int}}$. We define a model as detected if it is inconsistent with a fit that does not include the reionization signal. Thus, we fit each model signal with a foreground polynomial of a particular $N_{\text{poly}}$, and if the resulting minimum $\chi^2$ is inconsistent with zero at greater than 95 per cent confidence, then that model is included within the detection region. For the CDM model, the unrealistic sudden end to reionization raises the $\chi^2$ values somewhat, so we reduce our sensitivity to this by removing from the $\chi^2$ value the contribution of the 12 MHz band centred on the end of reionization for each model. For both models there are some oscillations in the detection limit curves due to the degeneracy between the 21-cm signal and the polynomial fitting of the foreground.

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Figure 12. The 95% detection region for global 21-cm experiments, in terms of the mid-point $z_r$ and span $\Delta z$ of reionization. We consider polynomial order in the fit $N_{\text{poly}} = \{3, 6, 9, 12\}$ in various panels. We consider both the CDM model (solid curves) and the tanh model (dashed curves), in each case for an observational integration time $t_{\text{int}} = 500, 50$ or 10 h (red, blue and green, respectively, also from top to bottom). We also show the full range of allowed values of $z_r$ and $\Delta z$ of reionization (grey shaded area) assuming the parameter space $N_{\text{ion}} \in \{2, 1000\}$, $V_c \in \{4.5, 100\}$ km s$^{-1}$, $C = 1$.

For the tanh model, for the same case as PL ($t_{\text{int}} = 500$ h) we find significantly better prospects for detectability, with our limits on $\Delta z$ for a given $z_r$ typically higher by a factor of $\sim 1.5$ than their result. It is possible to constrain some models with $\Delta z \sim 1$ even with $t_{\text{int}} = 10$ h, if $N_{\text{poly}} = 3$ suffices for removing the foreground and other systematics, or with $t_{\text{int}} = 500$ h if $N_{\text{poly}} = 9$. The worst case of $N_{\text{poly}} = 12$ only allows the detection (or ruling out) of very sharp reionization models that are probably unrealistic.

Our CDM model gives comparable constraints to the tanh model for low values of $N_{\text{poly}}$, but it is significantly more detectable with higher-order polynomials. The rapid rise of reionization up until its sharp end is easier to distinguish from a high-order polynomial compared with the smooth tanh model, even after the removal (in the CDM case) of the frequency interval right near the end of reionization. We thus find that an interesting parameter space of CDM models can be detected even with $N_{\text{poly}} = 12$, for integration times of at least $\sim 50$ h.

We have not tried to indicate current constraints on reionization in Fig. 12 to avoid overcrowding the figure, especially since these constraints are not directly expressed in terms of $z_r$ and $\Delta z$, and the conversion to these variables would differ somewhat between the tanh and CDM models. Roughly, for these models, the 7-year WMAP data imply a 95% confidence limit of $z_r \gtrsim 8$, while the absorption constraints that show a high ionization fraction at $z \sim 6.5$ imply a...
minimum $z$, that increases beyond 8 if $\Delta z \gtrsim 1$ (see also the Introduction and the discussion in PL). The relation between these constraints on reionization and those from global 21-cm measurements would change for more complex models of reionization.

4 CONCLUSIONS

The aim of this paper was to investigate the possibility that global 21-cm observations during the epoch of reionization can probe the evolution of the IGM and the physical properties of the ionizing sources. Detecting the 21-cm signal in the presence of the large foregrounds is challenging and it is important to explore all avenues. While interferometric radio arrays are gearing up to measure 21-cm fluctuations, global measurements with a single-dipole experiment can provide an independent and complementary method for detecting and/or constraining reionization.

In order to derive quantitative predictions, we have implemented both a previously used toy model and a more realistic and physically motivated model for reionization. The first one, the tanh model, is expressed in terms of two parameters, namely the two main characteristics of the overall reionization process, its mid-point $z_*$ and span $\Delta z$; the particular form of the model is merely mathematically convenient, with no real physical significance, and it restricts reionization to be smooth and symmetric about its mid-point. The second model, the CDM model, is based on the standard understanding of galaxy formation within CDM-dominated haloes. It assumes a fixed overall ionizing efficiency $N_{\text{ion}}$ (number of ionizing photons per baryon), a density clumping factor $C$ and a minimum halo circular velocity $V_c$ for galactic haloes, and it yields reionization models with up to three parameters ($C$ is relatively minor and we typically held it fixed in the fitting). Unlike the tanh model, the CDM model is asymmetric, with the exponentially increasing halo abundance leading to an acceleration of reionization in its later stages.

Despite the fact that the tanh model is a simple parametrization that has often been used in the literature, we have shown that it leads to substantial systematic errors if it is assumed when fitting a 21-cm signal that is described by the CDM model. In particular, the best-fitting $z_*$ in the tanh model is underestimated typically by 5–10 per cent, while $\Delta z$ is underestimated by tens of per cent. However, a sufficiently sensitive experiment (e.g. with an integration time $t_{\text{int}} > 20$ h for the case of a foreground polynomial of degree $N_{\text{poly}} = 3$) would be able to discriminate between the CDM and tanh models based on the $\chi^2$ value of the best-fitting model.

Our main result is a detailed plot of the detection limits of global 21-cm experiments (Fig. 12). We find that the CDM model can produce quick reionization scenarios (with a redshift span $\Delta z \sim 1$) if feedback makes large haloes dominate, which then requires a high ionizing efficiency in these haloes (see also Fig. 3). Some of these realistically possible models can be ruled out with 50-h global 21-cm experiments even in the pessimistic case where a polynomial of degree $N_{\text{poly}} = 12$ is required for removing the foregrounds (or other systematic effects). If somewhat more ambitious experiments are achievable, then a broad range of scenarios up to $z_* \sim 12$ can be probed within our CDM model. The smooth and symmetric tanh model is more difficult to differentiate from the foreground polynomial, and it requires greater integration times and lower $N_{\text{poly}}$ in order to rule out for similar reionization characteristics.

Our conclusions are generally optimistic in terms of the possibility for global 21-cm experiments to reconstructing the reionization history and constrain the properties of the ionizing sources. In particular, 1-yr EDGES observations may allow a remarkably precise reconstruction. However, the polynomial degree $N_{\text{poly}}$ that is required for removing the foreground and systematic effects plays an important role. In the most optimistic case, where $N_{\text{poly}} = 3$ suffices over the entire frequency range of 100–250 MHz, 1 per cent errors on $z_*$ are achievable; they require $t_{\text{int}} = 51$ h with the tanh model or 29 h with the CDM model (in each case with $\Delta z = 2$ as defined in that model). These, of course, are only statistical errors, while we have shown that there can be much larger systematic errors if the assumed reionization model cannot reproduce the real 21-cm signal from reionization.

Indeed, our results merit some caution, since our investigation indicates that a broader range of flexible and realistic models of reionization should be studied before we can be confident that the results are robust. For instance, the parameters of the model ($N_{\text{ion}}$, $V_c$ and $C$) could change with redshift due to evolving feedbacks such as metal enrichment or the effect of photoheating on suppressing gas accretion on to galaxies in the reionized regions (this effect is large if reionization is initially dominated by relatively small haloes). Such an evolution could, e.g., be parametrized as in Barkana (2009), while such a complication of the model would no doubt lead to serious partial degeneracies among the parameters, hopefully the main characteristics of reionization would remain measurable at high accuracy. Another possible complication that could be added is the increasing effect of recombinations near the end of reionization due to the optical depth of dense clumps within the then-large HII bubbles (Furlanetto & Oh 2005). We note that some of these effects, such as the feedback and the increasing recombinations, should slow the progress of reionization as it nears its end, thus rounding the shape of the CDM model and perhaps reaching a result that is more similar to the tanh model. We plan to study these more realistic models.

We note that in the radiometer equation we neglected the effects of the instrumental response (or bandpass) on both the foreground and the cosmological signal (i.e. we assumed a flat bandpass filter). Indeed, in a real observation this would raise three issues, but it should be possible to deal with them (Judd Bowman, private communication). First, our theoretical model must be convolved with the instrumental response in order to compare to the observations. This can be done if the bandpass is known with sufficient accuracy ($\sim 1$ per cent in current experiments and expected to improve). The foregrounds must also be convolved with the response, but as long as the instrumental response is smooth, i.e. does not introduce sharp frequency features, the foregrounds can still be removed by low-order polynomial fitting. Finally, another effect of a less than perfect response is a decrease in the sensitivity of the observation with respect to a flat receiver response. We can simply compensate for this effect by slightly increasing the observing time (by a few tens of per cent for current experiments).

Recently, Bowman & Rogers (2010) reported substantial new results from their upgraded EDGES experiment. Taking only the cleanest data out of their observational run, they had a thermal noise level equivalent to $t_{\text{int}} = 0.8$ h (with our idealized assumptions) within their 100–200 MHz band. They also found it necessary to use $N_{\text{poly}} = 5$ in order to remove the foreground and systematics, and reach the thermal noise level, within 20 MHz sub-bands in their spectrum. This is still somewhat worse than even our conservative $N_{\text{poly}} = 12$ case over the full 100–250 MHz range. Even with these limitations, Bowman & Rogers (2010) reached an observational milestone, namely the first direct observational limit on the rapidity of cosmic reionization. In particular, using the tanh model, they set a 95 per cent confidence lower limit of $\Delta z > 0.06$.
for the duration of the reionization epoch. Global 21-cm experiments are still in their infancy and clearly have a quite promising future.

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