Mean Curvature Flow with Boundary

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Abstract. We develop a theory of surfaces with boundary moving by mean curvature flow. In particular, we prove a general existence theorem using elliptic regularization, and we prove boundary regularity at all positive times under very mild hypotheses.

Keywords. Mean curvature flow, boundary, regularity

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1. Introduction

In this paper, we study mean curvature flow for surfaces with boundary: each point moves so that the normal component of its velocity is equal to the mean curvature, and the boundary remains fixed. (More generally, the boundary can be time-dependent, but prescribed.) In particular,

1. We define integral Brakke flows with boundary and prove the basic properties. This is a rather general class that includes network flows. See §5.

2. We define the subclass of standard Brakke flows with boundary. These are flows as in (1) with additional nice properties. In particular, for almost all times, the moving surface has the prescribed boundary in the sense of mod 2 homology. This condition excludes, for example, surfaces with triple junctions (or, more generally, with odd-order junctions). See §13.

3. Following Ilmanen [Ilm94], we use elliptic regularization to prove existence of standard Brakke flows with boundary for any prescribed initial surface. See §14.

4. We prove a strong boundary regularity theorem for standard Brakke flows with boundary. See §17.

As a special case of some of the results (Theorems 14.1 and 17.1), we have

**Theorem 1.1.** Let $N$ be a smooth, compact, $(m + 1)$-dimensional Riemannian manifold with smooth, strictly mean-convex boundary. Let $M_0$ be a smoothly embedded $m$-dimensional submanifold of $N$ whose boundary is a smooth submanifold $\Gamma$ of $\partial N$. (More generally, $M_0$ can be any $m$-rectifiable set of finite $m$-dimensional measure whose boundary, in the sense of mod 2 flat chains, is a smoothly embedded submanifold $\Gamma$ of $\partial N$.) Then there is a standard Brakke flow

$$t \in [0, \infty) \mapsto M(t)$$

with boundary $\Gamma$ such that $M(0) = M_0$. Furthermore, if $M(\cdot)$ is any standard Brakke flow with boundary $\Gamma$, then the flow is smooth (with multiplicity one) in a spacetime neighborhood of each point $(p, t)$ with $p \in \Gamma$ and $t > 0$.

Thus (under the hypotheses of the theorem) we have boundary regularity at all positive times, even after interior singularities may have occurred.
The regularity in Theorem 1.1 is uniform as $t \to \infty$. For given any sequence of times $t_i \to \infty$, there is a subsequence $t_{i(j)}$ such that the time-translated flows

$$t \in [-t_{i(j)}, \infty) \mapsto M_{t_{i(j)}}(t) := M(t_{i(j)} + t)$$

converge to a standard eternal limit flow $M'(\cdot)$ by §10 and Theorem 13.1. Since the area of $M(t)$ is a decreasing function of $t$, the area of $M'(t)$ is constant (it is equal to the limit as $t \to \infty$ of the area of $M(t)$). It follows that the $M'(t)$ are stationary integral varifolds and therefore non-moving (i.e., independent of $t$). The limit flow is regular at the boundary by Theorem 17.1, and thus the convergence $M_{t_{i(j)}}(\cdot) \to M'(\cdot)$ is smooth near the boundary by the local regularity theory in [Whi05].

The notion of standard Brakke flow with boundary is crucial in Theorem 1.1: the regularity assertion of Theorem 1.1 is false for general integral Brakke flows with boundary, because interior singularities can move into the boundary. Consider, for example, three points $A$, $B$, and $C$ on the unit circle in $\mathbb{R}^2$ and consider a configuration consisting of three curves in the interior of the triangle $ABC$ such that the three curves meet at equal angles at a point $P$ in the interior of the disk and such the other endpoints of the curves are the three points $A$, $B$, and $C$. The configuration evolves so that the three points on the unit circle are fixed, and so that interior points move with normal velocity equal to the curvature. This implies that the triple junction $P(t)$ moves in such a way that the curves continue to meet at equal angles at the junction. If each interior angle of the triangle $ABC$ is less than $120^\circ$, then the triple junction remains in the interior, and we have boundary regularity at all times. However, if one of the angles is greater than $120^\circ$, then $P(t)$ will bump into the corresponding vertex in finite time, thus creating a boundary singularity.

The flow described in the previous paragraph is an integral Brakke flow with boundary $\{A, B, C\}$. However, it is not a standard Brakke flow with boundary $\{A, B, C\}$, because if we think of the network as a mod 2 chain, then the boundary contains $P(t)$ in addition to $A$, $B$, and $C$.

It is natural to wonder whether such a boundary singularity could occur if the original surface is smooth and embedded. In the case of curves, the answer is “no”: the flow would remain smooth everywhere for all time by the analog of Grayson’s Theorem. However, although I do not yet have a proof, I believe that there is an integral Brakke flow

$$t \in [0, \infty) \mapsto M(t)$$

with boundary $\Gamma$, where $\Gamma$ consists of smooth embedded curves in the unit sphere in $\mathbb{R}^3$, such that $M(0)$ is a smoothly embedded surface in the unit ball and such that later the moving surface develops a triple junction curve that eventually bumps into the boundary. Note that this could only happen if we had non-uniqueness, since by Theorem 1.1 there is a standard Brakke flow $M'(\cdot)$ with the same initial surface and the same boundary, and that flow never develops boundary singularities. Of course the two flows are equal at least until singularities occur, but they must differ as soon as $M(\cdot)$ has a triple junction curve.

In Theorem 1.1, the condition that $\Gamma$ lie in the boundary of $N$ is also crucial. In another paper [Whi21], we show that there is a standard mean curvature flow with boundary that starts with a smoothly embedded Möbius strip in $\mathbb{R}^3$ and that develops a boundary...
singularity at which the tangent flow is given by a smoothly embedded, non-orientable shrinker with straight line boundary. For oriented surfaces, the situation is very different: we prove [Whi21, Theorem 1] that if \( M \) is an \( m \)-dimensional, smoothly embedded shrinker in \( \mathbb{R}^{m+1} \) with an \((m - 1)\)-dimensional linear subspace as boundary, then \( M \) is a flat halfspace.

The regularity part of Theorem 1.1 is a consequence of the following general theorem (see Theorem 16.2):

**Theorem 1.2.** Suppose that \( t \in [0, T] \mapsto M(t) \) is an \( m \)-dimensional standard mean curvature flow with boundary \( \Gamma \) in a smooth, \((m + 1)\)-dimensional Riemannian manifold. If a tangent flow at \((p, t)\) is contained in a wedge, where \( p \in \Gamma \) and \( t > 0 \), then \((p, t)\) is a regular point of the flow \( M(t) \).

Two features of this paper seem to be new even for Brakke flows without boundary. First, when taking limits of Brakke flows, we get improved subsequential convergence of the mean curvature for almost all times; see Remark 10.3. Second, to prove Huisken's monotonicity formula, one needs to know that the mean curvature vector is orthogonal to the variety almost everywhere. Brakke [Bra78, §5] proved such orthogonality for arbitrary integral varifolds of bounded first variation. However, the proof is rather long (40 pages). In this paper, we give a much easier proof that such orthogonality is preserved when taking weak limits of mean curvature flows. Thus, in particular, orthogonality holds in flows coming from elliptic regularization. For this reason, we have chosen to include orthogonality of mean curvature as part of the definition of Brakke flow.

For simplicity, in most of the paper we consider flows in which the boundary is fixed. In §18, we indicate how to modify the theory for moving boundaries.

In §19 we show that a certain weak notion of orientability is preserved when taking weak limits of flows.

Although mean curvature flow has been extensively studied, there have been only a few investigations of mean curvature flow of surfaces with boundary. The papers [Whi95] and [Whi05] dealt with mean curvature flow of surfaces both with and without boundary. In [Sto96], Stone proved a theorem analogous to the boundary regularity part of Theorem 1.1, but only at the first singular time and under additional, rather restrictive hypotheses. In particular, the moving surface was assumed to be mean convex and to satisfy a Type I estimate. In [IW15], mean curvature flow with boundary was used to prove sharp lower density bounds for area-minimizing hypercones.

### 2. Notation

In this paper, \( U \) is a smooth Riemannian manifold (possibly with smooth boundary). We do not assume that \( U \) is complete: it may be an open subset of a larger Riemannian manifold. We let \( G_m(U) \) denote the Grassman bundle of pairs \((x, P)\) where \( x \in U \) and \( P \) is an \( m \)-dimensional linear subspace of \( \text{Tan}(U, x) \). We let \( \mathcal{X}(U) \) denote the space of continuous, compactly supported vectorfields on \( U \). We let \( \mathcal{X}_m(U) \) denote the space of continuous, compactly supported functions on \( G_m(U) \) that assign to each \((x, P)\) in \( G_m(U) \) a vector in \( \text{Tan}(U, x) \).
If $M$ is a Radon Measure on $U$ and if $f$ is a function on $U$, we let

$$Mf = \int f \, dM.$$ 

If $\Gamma$ is a $k$-dimensional submanifold of $U$ (or, more generally, a $k$-rectifiable set of locally finite $k$-dimensional measure), then (by slight abuse of notation) we will also use $\Gamma$ to denote the associated Radon measure. Thus

$$\int f \, d\Gamma = \int f \, \mathcal{H}^k$$

and

$$\Gamma(K) = \mathcal{H}^k(\Gamma \cap K).$$

3. $L^p$ vectorfields

In the following theorem, $1_K$ denotes the characteristic function of the set $K$. Thus if $M$ is a Radon measure on $U$ and if $p < \infty$, then

$$\|Y1_K\|_{L^p(M)} = \left( \int_K |Y|^p \, dM \right)^{1/p}.$$ 

Similarly,

$$\|Y1_K\|_{L^\infty(M)}$$

is the essential supremum of $|Y|$ on the set $K$ with respect to the measure $M$.

**Theorem 3.1.** Let $V_i$ and $V$ be rectifiable $m$-varifolds in $U$ such that $V_i \rightharpoonup V$. Let $M_i$ and $M$ be the associated Radon measures on $U$. Suppose that $p \in (1, \infty]$ and that $Y_i$ is a Borel vectorfield on $U$ such that

$$c_K := \sup_i \|Y_i1_K\|_{L^p(M_i)} < \infty$$

for every $K \subset U$. Then, after passing to a subsequence, there is a Borel vectorfield $Y$ on $U$ in $L^p_{\text{loc}}(M)$ such that

$$\int X(x, \text{Tan}(M_i(x))) \cdot Y_i(x) \, dM_i(x) \rightharpoonup \int X(x, \text{Tan}(M(x))) \cdot Y(x) \, dM(x)$$

for every $X \in \mathcal{X}^m(U)$.

**Proof.** Define

$$L_i : \mathcal{X}^m(U) \to \mathbb{R},$$

$$L_i(X) = \int X(x, \text{Tan}(M_i(x))) \cdot Y_i(x) \, dM_i(x).$$

If $K \subset U$ and if $X \in \mathcal{X}^m(U)$ is supported in $\{(x, P) : x \in K\}$, then by Hölder's Inequality,

$$|L_i(X)| \leq c_K \left( \int |X(x, \text{Tan}(M_i, x))|^q \, dM_i(x) \right)^{1/q},$$

(3.1)
where \( q = p/(p-1) \), or, equivalently,
\[
|L_i(X)| \leq c_K \left( \int |X(x, P)|^q \, dV_i(x, P) \right)^{1/q}.
\]
From (3.1), we see that
\[
|L_i(X)| \leq c_K d_K^{1/q} \sup |X|
\]
where \( d_K := \sup_i M_i(K) \). Note that \( d_K < \infty \) by weak convergence of \( M_i \) to \( M \).

By (3.3) and Banach-Alaoglu, we can assume, after passing to a subsequence, that there is an \( L : X_\mu(U) \to \mathbb{R} \) such that
\[
L_i(X) \to L(X) \quad \text{for every } X \in X_\mu(U).
\]
Letting \( i \to \infty \) in (3.2) gives
\[
|L(X)| \leq c_K \left( \int |X(x, P)|^q \, dV(x, P) \right)^{1/q}.
\]

By the Riesz Representation Theorem, there is an \( M \)-measurable vectorfield \( \tilde{Y}(x, P) \) such that
\[
L(X) = \int X(x, P) \cdot \tilde{Y}(x, P) \, dV(x, P)
\]
for every \( X \in X_\mu(U) \). Since \( V \) is rectifiable and since \( M \) is the associated Radon measure on \( U \), we can rewrite (3.5) as
\[
L(X) = \int X(x, \text{Tan}(M, x)) \cdot \tilde{Y}(x, \text{Tan}(M, x)) \, dM(x).
\]
Thus if we set \( Y(x) = \tilde{Y}(x, \text{Tan}(M, x)) \), then we have
\[
L(X) = \int X(x, \text{Tan}(M, x)) \cdot Y(x) \, dM(x),
\]
as desired. \( \Box \)

**Corollary 3.2.** *In Theorem 3.1, if \( Y_i(x) \) is perpendicular to \( \text{Tan}(M_i, x) \) for \( M_i \)-almost every \( x \), then \( Y(x) \) is perpendicular to \( \text{Tan}(M, x) \) for \( M \)-almost every \( x \).*

**Proof.** Suppose \( X \in X_\mu(U) \). Let
\[
\tilde{X}(x, P) = \Pi_P X(x, P).
\]
Then \( \tilde{X} \) is also in \( X_\mu(U) \). Hence
\[
\int \tilde{X}(x, \text{Tan}(M_i, x)) \cdot Y_i(x) \, dM_i(x) \to \int \tilde{X}(x, \text{Tan}(M, x)) \cdot Y(x) \, dM(x),
\]
i.e.,
\[
\int \Pi_{\text{Tan}(M_i, x)} X(x) \cdot Y(x) \, dM_i(x) \to \int \Pi_{\text{Tan}(M, x)} X(x) \cdot Y(x) \, dM(x)
\]
The left hand side is 0, so
\[
\int \Pi_{\text{Tan}(M, x)} X(x) \cdot Y(x) \, dM(x) = 0
\]
or, equivalently,
\[ \int X(x) \cdot \Pi_{\operatorname{tan}(M,x)} Y(x) \, dM(x) = 0 \]
for all \( X \in \mathcal{X}(U) \). Since \( \mathcal{X}(U) \) is dense in \( L^q_{\text{loc}}(M) \) (cf. \cite{Ilm94}, §7.4), it follows that
\[ \Pi_{\operatorname{tan}(M,x)} Y(x) = 0 \]
for \( M \)-almost every \( x \).

\[ \square \]

**Theorem 3.3.** Suppose \( \beta_i (i = 1, 2, \ldots) \) and \( \beta \) are Radon measures on \( U \) such that \( \beta_i \) converges to \( \beta \). Suppose that \( p \in (1, \infty) \) and that \( Y_i \) is a Borel vectorfield on \( U \) such that
\[ \sup_i \|1_K Y_i\|_{L^p(\beta_i)} < \infty \]
for every \( K \subset U \). Then (after passing to a subsequence) there is a Borel vectorfield \( Y \) on \( U \) in \( L^p_{\text{loc}}(M) \) such that
\[ \hat{\int} X \cdot Y_i \, d\beta_i \rightarrow \hat{\int} X \cdot Y \, d\beta \]
for all \( X \) in \( \mathcal{X}(U) \).

The proof is essentially the same as the proof of Theorem 3.1, except that we work in \( U \) rather than in \( G_m(U) \).

4. A Varifold Closure Theorem

Let \( U \) be an open subset of a smooth Riemannian manifold, let \( \mathcal{M}(U) \) be the set of all Radon measures on \( U \), and let \( \mathcal{M}_k(U) \) be the set of Radon measures associated to \( k \)-dimensional rectifiable varifolds in \( U \). Equivalently, \( \mathcal{M}_k(U) \) is the set of Radon measures \( M \) such that

(i) \( M(U \setminus S) = 0 \) for some countable union \( S \) of \( k \)-dimensional \( C^1 \) submanifolds of \( U \), and

(ii) \( M \) is absolutely continuous with respect to \( \mathcal{H}^k \).

Let \( I.\mathcal{M}_k(U) \) be the set of \( M \in \mathcal{M}_k(U) \) such that \( \Theta(M, x) \) is an integer for \( M \)-almost every \( x \).

If \( M \in \mathcal{M}_k(U) \), we let \( \operatorname{Var}(M) \) be the associated \( k \)-dimensional varifold in \( U \). Thus \( M \in I.\mathcal{M}_k(U) \) if and only if \( \operatorname{Var}(M) \) is an integral varifold.

Now suppose that \( M \in \mathcal{M}_k(U) \) and that \( \operatorname{Var}(M) \) has bounded first variation. Then there exist an \( M \)-locally integrable vectorfield \( H(\cdot) = H(M, \cdot) \), a Radon measure \( \beta(M) \) that is singular with respect to \( M \), and a \( \beta(M) \)-locally integrable unit vectorfield \( \eta(\cdot) = \eta(M, \cdot) \) with the following property: if \( X \) is any compactly supported, \( C^1 \) vectorfield on \( U \), then
\[ \int_M \operatorname{Div}_M X \, dM = - \int H \cdot X \, dM + \int X \cdot \eta \, d\beta. \]

**Definition 4.1.** If \( \Gamma \) is a properly embedded \((m - 1)\)-dimensional submanifold of \( U \), then \( \mathcal{V}_m(U, \Gamma) \) is the space of \( M \in I.\mathcal{M}_m(U) \) such that \( M \) has bounded first variation and such that

(1) \( \beta(M) \leqslant \mathcal{H}^{m-1} \cap \Gamma \).

(2) \( H(\cdot) \) and \( \operatorname{Tan}(M, \cdot) \) are perpendicular \( M \)-almost everywhere.
(As mentioned in the introduction, the orthogonality condition (2) in Definition [4.1] is superfluous according to a theorem of Brakke [Bra78, §5], but the proof of that theorem is rather difficult. Including Condition (2) in the definition makes that theorem unnecessary for us.)

Let
\[
\nu(M, x) = \lim_{r \to 0} \frac{1}{\omega_{m-1} r^{m-1}} \int_{B(x, r)} \eta(\cdot) d\beta
\]

where the limit exists, and let \( \nu(M, x) = 0 \) where the limit does not exist. Note that the limit exists \( H^{m-1} \) almost everywhere. Note also that we can rewrite (4.1) as
\[
(\hat{\text{Div}}_M X dM = - H \cdot X dM + \int_{\Gamma} \nu \cdot X dH^{m-1}
\]
or (using the notational conventions described in Section 2) as
\[
\int \hat{\text{Div}}_M X dM = - \int H \cdot X dM + \int \nu \cdot X d\Gamma.
\]

**Remark 4.2.** The condition that \( \beta \leq H^{m-1} \) on \( \Gamma \) is equivalent to the condition that \( |\nu(x)| \leq 1 \) for \( H^{m-1} \) almost every \( x \in \Gamma \).

**Remark 4.3.** If \( M \in \mathcal{V}_m(U, \Gamma) \), then \( \nu(M, \cdot) \) is perpendicular to \( \Gamma \) at almost every point of \( \Gamma \) by [All75, §3.1].

In the following theorem, we write \( H_i(\cdot) \) and \( H(\cdot) \) for \( H(M_i, \cdot) \) and \( H(M, \cdot) \), and \( \nu_i(\cdot) \) and \( \nu(\cdot) \) for \( \nu(M_i, \cdot) \) and \( \nu(M, \cdot) \).

**Theorem 4.4** (Varifold Closure Theorem). **Suppose for** \( i = 1, 2, \ldots \) **that** \( M_i \in \mathcal{V}_m(U, \Gamma_i) \), **where the** \( \Gamma_i \) **are smooth** \( (m - 1) \)-dimensional submanifolds of \( U \) **that converge in** \( C^1 \) **to a smooth manifold** \( \Gamma \). **Suppose that the** \( M_i \) **converge to a Radon measure** \( M \) **and that**
\[
d_K := \sup_i \int_K |H_i|^2 dM_i < \infty
\]
for every \( K \Subset U \). **Then**

1. **For every** \( K \Subset U \),
   \[
   \sup_i \left( \int_K |H_i| dM_i + \beta(M_i)(K) \right) < \infty.
   \]

2. **If** \( M \in \mathcal{I}(U) \) **and** \( \text{Var}(M_i) \) **converges to** \( \text{Var}(M) \). **Thus if** \( f : \mathcal{G}_m(U) \to \mathbb{R} \) **is continuous and compactly supported, then**
   \[
   \int f(x, \text{Tan}(M_i, x)) dM_i(x) \to \int f(x, \text{Tan}(M, x)) dM(x)
   \]

3. **If** \( X \in \mathcal{X}_m(U) \), **then**
   \[
   \int X(x, \text{Tan}(M_i, x)) \cdot H_i(x) dM_i \to \int X(x, \text{Tan}(M, x)) \cdot H(x) dM(x).
   \]

4. **If** \( Z \in \mathcal{X}_{m-1}(U) \), **then**
   \[
   \int Z(x, \text{Tan}(\Gamma_i, x)) \cdot \nu_i(x) d\Gamma_i(x) \to \int Z(x, \text{Tan}(\Gamma, x)) \cdot \nu(x) d\Gamma(x).
   \]
(5) \( M \in \mathcal{V}_m(U, \Gamma) \).

Proof. Since the \( M_i \) converge to \( M \),
\[
(4.4) \quad c_K := \sup_i M_i(K) < \infty
\]
for every \( K \subset U \). Thus
\[
\int_K |H_i| dM_i \leq (M_i(K))^{1/2} \left( \int_K |H_i|^2 dM_i \right)^{1/2} \leq (c_K d_K)^{1/2}
\]
Also,
\[
\sup_i \beta(M_i)(K) \leq \sup_i \mathcal{H}^{m-1}(\Gamma_i \cap K) < \infty
\]
since \( \Gamma_i \) converges in \( C^1 \) to \( \Gamma \). This proves Assertion (1).

By Assertion (1) and by Allard’s Closure Theorem for Integral Varifolds ([All72, Theorem 6.4] or [Sim83, §42.8] or [Sim18, chapter 8, §5.9]), the varifolds \( \text{Var}(M_i) \) converge (after passing to a subsequence) to an integral varifold \( V \) of bounded first variation. Note that \( \mu_V = M \), so the limit \( V = \text{Var}(M) \) does not depend on the choice of subsequence. Thus the original sequence \( \text{Var}(M_i) \) converges to \( \text{Var}(M) \). Thus we have proved Assertion (2) of the theorem.

By Theorem 3.1 every sequence of \( i \) tending to infinity has a subsequence \( i(j) \) for which there exist an \( M \)-measurable vectorfield \( \tilde{H} \) and a \( \Gamma \)-measurable vectorfield \( \tilde{\nu} \) such that
\[
(4.5) \quad \int X(x, \text{Tan}(M_i(j), x)) \cdot H_i(j)(x) dM_i(j)(x) \rightarrow \int X(x, \text{Tan}(M, x)) \cdot \tilde{H}(x) dM(x)
\]
for every \( X \in \mathcal{X}_m(U) \) and
\[
(4.6) \quad \int Z(x, \text{Tan}(\Gamma_i(j), x)) \cdot \nu_i(j)(x) d\Gamma_i(j)(x) \rightarrow \int Z(x, \text{Tan}(\Gamma, x)) \cdot \tilde{\nu}(x) d\Gamma(x)
\]
for every \( Z \in \mathcal{X}_{m-1}(U) \). Furthermore, by Corollary 3.2, the perpendicularly almost everywhere of \( H(M_i, \cdot) \) and \( \text{Tan}(M_i, \cdot) \) implies the perpendicularly almost everywhere of \( \tilde{H}(\cdot) \) and \( \text{Tan}(M, \cdot) \). Also, from (4.6) (and Remark 4.2) we see that \( |\tilde{\nu}(\cdot)| \leq 1 \) almost everywhere with respect to \( \Gamma \).

For every \( C^1 \), compactly supported vectorfield \( X \) on \( U \), we have
\[
\int \text{Div}_{M_i(j)} X dM_i(j) = -\int H_i(j) \cdot X dM_i(j)(\cdot) + \int \nu_i(j) \cdot X d\Gamma_i(j)(\cdot).
\]
By the convergence \( \text{Var}(M_i(j)) \) to \( \text{Var}(M) \) and by (4.5) and (4.6), it follows that
\[
\int \text{Div}_M X dM = -\int \tilde{H} \cdot X dM_i + \int \tilde{\nu} \cdot X d\Gamma.
\]
Consequently, \( M \in \mathcal{V}_m(U, \Gamma) \), \( \tilde{H}(\cdot) = H(M, \cdot) \) and \( \tilde{\nu}(\cdot) = \nu(M, \cdot) \). We passed to a subsequence \( i(j) \), but since the limits \( H(M, \cdot) \) and \( \nu(M, \cdot) \) are independent of the choice of subsequence, in fact (4.5) and (4.6) hold for the original sequence.

\[ \square \]

5. **Brakke Flows with Boundary**

**Definition 5.1.** An \( m \)-dimensional integral Brakke flow with boundary in \( U \) is a pair \((M(\cdot), \Gamma)\) where \( \Gamma \) is a smooth, properly embedded \((m - 1)\)-dimensional submanifold of \( U \) and where

\[ t \in I \mapsto M(t) \]

is a Borel map from an interval \( I \) to the space \( \mathcal{M}(U) \) of Radon measures in \( U \) such that

1. For almost every \( t \in I \), \( M(t) \) is in \( \mathcal{V}_m(U, \Gamma) \) (see Definition 4.1).
2. If \([a, b] \subset I \) and if \( K \subset U \) is compact, then

\[ \int_a^b \int_K (1 + |H|^2) \, dM(t) \, dt < \infty. \]

3. If \([a, b] \subset I \) and if \( u \) is a nonnegative, compactly supported, \( C^2 \) function on \( U \times [a, b] \), then

\[ (Mu)(a) - (Mu)(b) \geq \int_a^b \int \left( |H|^2 - H \cdot \nabla u - \frac{\partial u}{\partial t} \right) \, dM(t) \, dt. \]

We also say that “\( M(\cdot) \) is an integral Brakke flow with boundary \( \Gamma \)”.

By (2), the integral in (3) is finite.

(The condition that \( t \mapsto M(\cdot) \) is a Borel map is equivalent to the condition that \( t \mapsto M(t) \) is a Borel map for every continuous, compactly supported function \( f \) on \( U \).

**Proposition 5.2.** If \( t \mapsto M(t) \) is a Brakke flow with boundary \( \Gamma \), then the defining inequality (3) in Definition 5.1 holds for every nonnegative, compactly supported, Lipschitz function \( u \) on \( U \) that is \( C^1 \) on \( \{ u > 0 \} \).

**Proof.** Approximate \( u \) by \( C^2 \) functions \( u_n \) and use the Dominated Convergence Theorem. \( \square \)

**Lemma 5.3.** Suppose that \( M \) is a rectifiable varifold of bounded first variation and that \( u \) is a nonnegative, compactly supported, Lipschitz function such that \( u|_{\{ u > 0 \}} \) is \( C^2 \) and such that

\[ \sup_{u(x) > 0} |\nabla^2 u(x)| < \infty. \]

Then

\[ \int \text{Div}_M \nabla u \, dM \leq - \int H(M, \cdot) \cdot \nabla u \, dM + \int \eta(M, \cdot) \cdot \nabla u \, d\beta(M). \]

**Proof.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a smooth increasing function such that \( \phi(x) = 0 \) for \( x < 1 \), \( \phi(x) = x - 1 \) for \( x \geq 3 \), and such that \( \phi'' \geq 0 \) everywhere. Let \( \kappa > 0 \), and apply the Divergence Theorem to \( \kappa^{-1} \phi(\kappa u) \):

\[ \int \text{Div}_M \nabla (\kappa^{-1} \phi(\kappa u)) \, dM = - \int H \cdot \nabla (\kappa^{-1} \phi(\kappa u)) \, dM + \int \eta \cdot \nabla (\kappa^{-1} \phi(\kappa u)) \, d\beta \]
Now $\nabla (\kappa^{-1} \phi(\kappa u)) = \phi'(\kappa u) \nabla u$, so

$$\text{Div}_M \nabla (\kappa^{-1} \phi(\kappa u)) = \text{Div}_M (\phi'(\kappa u) \nabla u)$$

$$= \phi''(\kappa u) |\nabla_M u|^2 \kappa + \phi'(\kappa u) \text{Div}_M \nabla u$$

$$\geq \phi'(\kappa u) \text{Div}_M \nabla u.$$ 

Thus

$$\int \phi'(\kappa u) \text{Div}_M \nabla u \, dM \leq - \int \phi'(\kappa u) H \cdot \nabla u \, dM + \int \phi'(\kappa u) \eta \cdot \nabla u \, d\beta$$

Now let $\kappa \to \infty$ and use the Dominated Convergence Theorem. $\Box$

If $S$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, let $\text{Trace}_k(S) = \sum_{i=1}^k \lambda_i$. Thus if $M$ is an $m$-dimensional submanifold (or, more generally, if $M$ is in $\mathcal{M}_m(U)$), then

$$\text{Div}_M \nabla u \geq \text{Trace}_m(\nabla^2 u).$$

**Corollary 5.4.** If $t \in [a, b] \mapsto M(t)$ is a Brakke Flow with boundary $\Gamma$, if

$$f : U \times [a, b] \to \mathbb{R}$$

is a nonnegative, $C^2$ function with $\{f > 0\}$ compact, and if $u := 1_{f > 0} f$, then

$$(Mu)(a) - (Mu)(b)$$

$$\geq \int_a^b \int \left( |u|^2 + \text{Div}_M \nabla u - \frac{\partial u}{\partial t} \right) \, dM(t) \, dt - \int_a^b \int \nabla \cdot \nabla u \, d\Gamma \, dt$$

$$\geq \int_a^b \int \left( |u|^2 + \text{Trace}_m(\nabla^2 u) - \frac{\partial u}{\partial t} \right) \, dM(t) \, dt - \int_a^b \int \nabla \cdot \nabla u \, d\Gamma \, dt.$$

**Proof.** This follows immediately from Proposition 5.2 and Lemma 5.3. $\Box$

As a special case of Corollary 5.4, we have

**Theorem 5.5.** Let $t \in [0, T] \mapsto M(t)$ be a Brakke Flow with boundary $\Gamma$. Suppose that

$\overline{B(x, R)}$ is a compact subset of $U$,

$\text{dist}(\cdot, x)^2$ is smooth on $\overline{B(x, R)}$, and

$\text{Trace}_m(-\nabla^2(\text{dist}(\cdot, x)^2)) \geq -4m$ on $\overline{B(x, R)}$.

Let $u = (R^2 - \text{dist}(\cdot, x)^2 - 4mt)^+$. Then for $t \in [0, T]$,

$$(Mu)(t) \leq (Mu)(0) + 2tR \mathcal{H}^{m-1}(\Gamma \cap B(x, R)).$$

Note that $\text{Trace}_m(-\nabla^2(\text{dist}(\cdot, x)^2)) = -2m$ at $x$, so the hypotheses in Theorem 5.5 are satisfied if $R$ is sufficiently small.
Theorem 5.6. Let \( t \in I \mapsto M(t) \) be a Brakke Flow with boundary \( \Gamma \). Let \( u \) be a nonnegative, compactly supported, \( C^2 \) function on \( U \). Then for \( [a, b] \subset I \),
\[
\frac{1}{2} \int_a^b \int u |H|^2 \, dM(t) \, dt \leq M(a)u - M(b)u + (b - a)(\max |\nabla^2 u|) K_{[a,b]},
\]
where
\[
K_{[a,b]} := \sup_{t \in [a, b]} M(t)(\text{spt } u).
\]

Proof. By [Ilm94, Lemma 6.6],
\[
|\nabla u|^2 \leq 2 |u| \max |\nabla^2 u|.
\]
Thus wherever \( u > 0 \),
\[
u |H|^2 - H \cdot \nabla u = \frac{1}{2} |H|^2 + \frac{1}{2} |u|^{1/2} H - u^{-1/2}(\nabla u)^2 - \frac{1}{2} \frac{|\nabla u|^2}{|u|}
\geq \frac{1}{2} |H|^2 - \max |\nabla^2 u|
\]
Consequently,
\[
M(a)u - M(b)u \geq \frac{1}{2} \int_a^b \int u |H|^2 - H \cdot \nabla u \, dM_i(t) \, dt
\geq \frac{1}{2} \int_a^b \int u |H|^2 \, dM_i(t) \, dt - \max |\nabla^2 u| \int_a^b M(\text{spt } u) \, dt
\geq \frac{1}{2} \int_a^b \int u |H|^2 \, dM(t) \, dt - (b - a) \max |\nabla^2 u| K_{[a,b]}.
\]

Corollary 5.7. If \( K_I < \infty \), then
\[
t \in I \mapsto M(t)u - (\max |\nabla^2 u|) K_I t
\]
is a non-increasing function of \( t \).

6. Monotonicity with Boundary in a Manifold

Now consider mean curvature flow in a smooth Riemannian manifold \( N \). We embed \( N \) isometrically in a Euclidean space \( \mathbb{R}^d \). By spacetime translation, it suffices to consider monotonicity about the origin in spacetime. By parabolic scaling, we can assume that \( N \) is properly embedded in an open subset of \( \mathbb{R}^d \) that contains \( B^d(0, 1) \). If \( M \) is an \( m \)-dimensional submanifold of \( N \), we let \( H \) be the mean curvature as a submanifold of \( \mathbb{R}^d \), and we let \( H_N \) and \( H_N^\perp \) be the projections of \( H \) to \( \text{Tan}(N, \cdot) \) and to \( \text{Tan}(N^\perp, \cdot) \). Thus \( H_N \) is the mean curvature of \( M \) as a submanifold of \( N \).

For \( x \in \mathbb{R}^d \) and \( t < 0 \), let
\[
\rho(x, t) = \frac{1}{(4\pi |t|)^{m/2}} \exp \left( -\frac{|x|^2}{4|t|} \right)
\]
and
\[ \hat{\rho}(x, t) = \phi(|x|) \rho(x, t), \]
where \( \phi \) is a smooth function compactly supported in \([0, 1)\) such that \( \phi = 1 \) on \([0, 1/2]\), and \( \phi' \leq 0 \).

A straightforward calculation (see [KT14 §6.1]) shows that
\[ K := \sup \left| \frac{\partial \hat{\rho}}{\partial t} + \text{Div}_M \nabla \hat{\rho} \right| < \infty \]
and that if \( \Gamma \) is a smooth, properly embedded, \((m - 1)\)-dimensional submanifold of \( \overline{B(0, 1)} \), then
\[ C_\Gamma := \sup_{t < 0} \int |(\nabla \hat{\rho}(\cdot, t))_{\Gamma^\perp}| d\Gamma < \infty. \]

**Theorem 6.1** (Huisken Monotonicity). *Let \( U \) be an open subset of \( \mathbb{R}^d \) that contains \( \overline{B^d(0, 1)} \) and \( N \) be a smooth, properly embedded submanifold of \( U \). Let \( \Gamma \) be a smooth, properly embedded \((m - 1)\)-dimensional submanifold of \( U \). Let \( t \in I \mapsto M(t) \) be a Brakke flow in \( N \) with boundary \( \Gamma \). Suppose that
\[ M(t) \mathbb{B}(0, 1) \leq C \]
for \( t \in I \) and that the norm of the second fundamental form of \( N \) is bounded by \( A \). Then for \( a, b \in I \) with \( a \leq b < 0 \),
\[ (M\hat{\rho})(b) - (M\hat{\rho})(a) \geq \int_a^b \int \left( H_N - \frac{(\nabla^\perp \hat{\rho})_N}{\hat{\rho}} \right)^2 \hat{\rho} dM(t) \, d t \]
\[ + \int_a^b \int v_M \cdot \nabla \hat{\rho} d\Gamma d t \]
\[ - m A^2 \int_a^b \hat{\rho} \, dM(t) \, d t \]
\[ - CK(b - a) \]
\[ (6.4) \]
where \( C \) and \( K \) are as in \([6.1] \) and \([6.3] \). Furthermore,
\[ e^{-m A^2 t} \left( (M\hat{\rho})(t) + \int_{\tau = T_0}^t \int v_M \cdot \nabla \hat{\rho} \, d\Gamma \, d\tau - CK t \right) \]
is a decreasing function of \( t \) for \( t < 0 \) in \( I \).
Proof. 

\[(M\dot{\rho})(a) - (M\dot{\rho})(b)\]

\[\geq \int_a^b \int \left( \dot{\rho}|H_N|^2 - H_N \cdot \nabla \dot{\rho} - \frac{\partial \dot{\rho}}{\partial t} \right) dM(t) dt\]

\[\geq \int_a^b \int \left( \dot{\rho}|H_N|^2 - H_N \cdot \nabla \dot{\rho} + \text{Div}_M \nabla \dot{\rho} + \frac{|\nabla \perp \dot{\rho}|^2}{\dot{\rho}} - K1_B \right) dM(t) dt\]

\[= \int_a^b \int \left( \dot{\rho}|H_N|^2 - H_N \cdot \nabla \dot{\rho} - H \cdot \nabla \dot{\rho} + \frac{|\nabla \perp \dot{\rho}|^2}{\dot{\rho}} - K1_B \right) dM(t) dt\]

\[\geq \int_a^b \int \nu_M \cdot \nabla \dot{\rho} d\Gamma dt\]

\[= \int_a^b \int \left( \dot{\rho}|H_N|^2 - H_N \cdot \nabla \dot{\rho} - H \cdot \nabla \dot{\rho} + \frac{|\nabla \perp \dot{\rho}|^2}{\dot{\rho}} - K1_B \right) dM(t) dt\]

\[+ \int_a^b \int \nu_M \cdot \nabla \dot{\rho} d\Gamma dt - KC(b - a).\]

We rewrite the penultimate the integrand in (6.5) as follows, using the orthogonality of the mean curvature:

\[\dot{\rho}|H_N|^2 - H_N \cdot \nabla \dot{\rho} - H \cdot \nabla \dot{\rho} + \frac{|\nabla \perp \dot{\rho}|^2}{\dot{\rho}}\]

\[= \dot{\rho}|H_N|^2 - 2H_N \cdot \nabla \dot{\rho} - H_{N\perp} \cdot \nabla \dot{\rho} + \frac{|\nabla \perp \dot{\rho}|^2}{\dot{\rho}}\]

\[= \dot{\rho}|H_N|^2 - 2H_N \cdot \nabla \perp \dot{\rho} - H_{N\perp} \cdot \nabla \dot{\rho} + \frac{|\nabla \perp \dot{\rho}|^2}{\dot{\rho}}\]

\[= \dot{\rho}|H_N|^2 - 2H_N \cdot (\nabla \perp \dot{\rho})_N + \frac{|(\nabla \perp \dot{\rho})_N|^2}{\dot{\rho}} - H_{N\perp} \cdot (\nabla \dot{\rho})_{N\perp} = \frac{|(\nabla \perp \dot{\rho})_N|^2}{\dot{\rho}}\]

\[= \left| H_N - \frac{(\nabla \perp \dot{\rho})_N}{\dot{\rho}} \right|^2 \dot{\rho} + \frac{1}{2} H_{N\perp} - \frac{(\nabla \dot{\rho})_N}{\dot{\rho}} \right|^2 \dot{\rho} - \frac{1}{4} |H_{N\perp}|^2 \dot{\rho}\]

\[\geq \left| H_N - \frac{(\nabla \perp \dot{\rho})_N}{\dot{\rho}} \right|^2 \dot{\rho} - mA^2\dot{\rho}\]

since \(|H_{N\perp}|^2 \leq mA^2\). Substituting this into (6.5) gives (6.4).

Now let

(6.6) \[f(t) = (M\dot{\rho})(t) + \int_{\tau=0}^{t} \nu_M \cdot \nabla \dot{\rho} d\Gamma d\tau - CKt.\]

By (6.4), we have \[f'(t) \leq mA^2 f(t)\]

in the distributional sense, which immediately implies that \[\frac{d}{dt} \left( e^{-mA^2 t} f(t) \right) \leq 0.\]
Corollary 6.2. The quantity $(M\hat{\rho})(t)$ has a finite limit as $t \to 0$.

Proof. Since $e^{-mA^2 t} f(t)$ is in a non-increasing function of $t$ (where $f$ is given by (6.6)), $\lim_{t \uparrow} f(t)$ exists and is in $[-\infty, \infty)$. By (6.2),

$$\lim_{t \uparrow 0} \int_{\tau = T_0}^t \nu_M \cdot \nabla \hat{\rho} \ d\Gamma \ d\tau$$

exists and is finite. Thus $\lim_{t \to 0} (M\hat{\rho})(t)$ exists and is $< \infty$. Since $(M\hat{\rho})(t) \geq 0$, the limit is $\geq 0$, and thus is a finite, nonnegative number. □

Definition 6.3. The Gauss density of $M(\cdot)$ at $(0,0)$ is

$$\Theta(M(\cdot), (0,0)) = \lim_{t \uparrow 10} (M\hat{\rho})(t)$$

The extended Gauss density of $M(\cdot)$ at $(0,0)$ is

$$\Theta_e(M(\cdot), (0,0)) = \begin{cases} \Theta(M(\cdot), (0,0)) & \text{if } 0 \text{ is not in } \Gamma, \\
\Theta(M(\cdot), (0,0)) + \frac{1}{2} & \text{if } 0 \in \Gamma. \end{cases}$$

It is straightforward to prove that the Gauss density does not depend on the isometric embedding of $N$ into $\mathbb{R}^d$ or on the choice of the cutoff function $\phi$.

7. Monotonicity with Boundary in Euclidean Space

The monotonicity inequality becomes simpler for mean curvature flow with fixed boundary in a Euclidean space. (The material in this section and in Sections 8 and 9 is not used in the rest of the paper.)

Let $\Gamma$ be a smooth, properly embedded $(m - 1)$-dimensional manifold in $\mathbb{R}^n$. For $v \in \mathbb{R}^n$, the exterior cone over $\Gamma$ with vertex $v$ is

$$\{v + s(x - v) : x \in \Gamma, s \geq 1\}.$$

The multiplicity $\theta(p) = \theta_{\Gamma,v}(p)$ of the exterior cone at a point $p \in \mathbb{R}^n$ is the number of points $(x,s) \in \Gamma \times [1,\infty)$ such that

$$v + s(x - v) = p.$$

The exterior cone (counting multiplicity) determines a Radon measure $E = E_{\Gamma,v}$ on $\mathbb{R}^n$:

$$dE_{\Gamma,v} = \theta_{\Gamma,v} d\mathcal{H}^m.$$

Theorem 7.1. Suppose $t \in [a, b] \mapsto M(t)$ is a $m$-dimensional Brakke flow in $\mathbb{R}^n$ with boundary $\Gamma$. Let $v \in \mathbb{R}^n$, $t_0 \geq b$, and

$$\psi(x, t) = \psi_{v, t_0}(x, t) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{|x - v|^2}{4(t_0 - t)}\right).$$

Then

$$(M(t) + E_{\Gamma,v})\psi$$
is a decreasing function of \( t \) for \( t \in [a, b] \). Indeed,

\[
(M(a) + E_{\Gamma,t})\psi(\cdot, a) - (M(b) + E_{\Gamma,t})\psi(\cdot, b)
\]

\[= \int_a^b \int H - \frac{\nabla^\perp \psi}{\psi} \frac{\psi}{\rho} \rho \, dM(t) \, dt + \int_a^b \int |(\nu_M + \nu_E) \cdot \nabla \psi| \, d\Gamma \, dt.
\]

Furthermore, if

\[
(M(a) + E_{\Gamma,t})\psi(\cdot, a) = (M(b) + E_{\Gamma,t})\psi(\cdot, b),
\]

then for almost every \( t \in [a, b] \),

\[
(7.2)
\]

\[H = \nabla^\perp \psi / \psi \]

holds \( M(t) \)-almost everywhere, and

\[
(7.3)
\]

\[\nu_{M(t)}(x) = \frac{(x - v)^\perp}{|(x - v)^\perp|} \]

holds for almost every \( x \in \Gamma \) for which \((x - v)^\perp\) is nonzero, where \((x - v)^\perp\) is the projection of \((x - v)\) to \(\text{Tan}(\Gamma, x)^\perp\).

The theorem says that, although the integral of \( \psi(\cdot, t) \) over \( M(t) \) need not be decreasing (as a function of \( t \)), if we extend \( M(t) \) by attaching the exterior cone, i.e., if we replace \( M(t) \) by \( M(t) + E_{\Gamma,t} \), then the integral of \( \psi(\cdot, t) \) over the extended surface is decreasing.

(This is very analogous to the extended monotonicity formula for minimal surfaces in \[EWW02\].)

**Proof.** By translating and parabolically dilating, it suffices to consider the case when \((v, t_0) = (0, 0)\) and \( a < b < 0 \). Thus \( \psi = \rho \), where \( \rho \) is as in Section 6. For simplicity, we give the proof in the case that \( \Gamma \) and the supports of the \( M(t) \) lie in a compact set. (Otherwise, one uses cutoff functions and then lets the cutoff functions tend to the constant function 1.)

In this case, the inequality \(6.4\) in the Monotonicity Theorem \(6.1\) becomes equality with \( \rho \) in place of \( \hat{\rho} \) and with \( A = K = 0 \):

\[
(7.4)
\]

\[
(M\rho)(a) - (M\rho)(b) = \int_a^b \int H - \frac{(\nabla^\perp \rho)}{\rho} \rho \, dM(t) \, dt
+ \int_a^b \int \nu_M \cdot \nabla \rho \, d\Gamma \, dt.
\]

For notational simplicity, let us assume the exterior cone over \( \Gamma \) (with vertex 0) is embedded, i.e., that the multiplicity \( \theta = \theta_{\Gamma,0} \) is 1 at all points on the cone. We will use \( E = E_{\Gamma,0} \) to denote both the exterior cone and the associated Radon measure.

A straightforward calculation shows that on \( E \),

\[
\frac{\partial \rho}{\partial t} = -\text{Div}_E \nabla \rho.
\]
Thus
\[
\frac{d}{dt} E\rho(\cdot, t) = \frac{d}{dt} \int_E \rho \, d\mathcal{H}^m \\
= \int_E \frac{\partial \rho}{\partial t} \, d\mathcal{H}^m \\
= -\int_E \text{Div}_E \nabla \rho \, d\mathcal{H}^m \\
= \int_E H_E \cdot \nabla \rho \, d\mathcal{H}^m - \int_\Gamma v_E \cdot \nabla \rho \, d\mathcal{H}^{m-1} \\
= -\int_\Gamma v_E \cdot \nabla \rho \, d\mathcal{H}^{m-1}.
\]

(Note that \(H_E \cdot \nabla \rho \equiv 0\) since \(H_E\) is perpendicular to \(E\) and \(\nabla \rho\) is tangent to \(E\) because \(E\) is a portion of a cone with vertex at the origin.) Thus
\[
E\rho(\cdot, a) - E\rho(\cdot, b) = -\int_a^b \frac{d}{dt} E\rho(\cdot, t) \, dt \\
= \int_a^b \int_\Gamma v_E \cdot \nabla \rho \, d\Gamma.
\]

Adding (7.4) and (7.5) gives
\[
(M(a) + E_{\Gamma, v})\rho(\cdot, a) - (M(b) + E_{\Gamma, v})\rho(\cdot, b) \\
= \int_a^b \int_\Gamma \left| H - \frac{\nabla \rho}{\rho} \right|^2 \rho \, dM(t) \, dt + \int_a^b \int_\Gamma (v_M + v_E) \cdot \nabla \rho \, d\Gamma \, dt.
\]

Claim 7.2. If \(x^\perp = 0\) (i.e., if \(x \in \text{Tan}(\Gamma, x)\)), then
\[
(u + v_E) \cdot \nabla \rho = 0
\]
for all \(u \in \text{Tan}(\Gamma, x)^\perp\). If \(x^\perp \neq 0\), then
\[
v_E = \frac{x^\perp}{|x^\perp|}
\]
and if \(u\) is a vector in \(\text{Tan}(\Gamma, x)^\perp\) with \(|u| \leq 1\), then
\[
(u + v_E) \cdot \nabla \rho \geq 0,
\]
with equality if and only if \(u = -v_E\).

The proof of the claim is straightforward vector geometry.

Claim 7.2 implies that in (7.6), we can replace \((v_M + v_E) \cdot \nabla \rho\) by its absolute value, which gives (7.1). The “Furthermore” assertion follows immediately from (7.1) and Claim 7.2.

For the next corollary, we consider the full cone \(C_{\Gamma, v}\) over \(\Gamma\) with vertex \(v\), namely
\[
\{ v + s(x - v) : x \in \Gamma, s \geq 0 \}.\]
As above, we make $C_{\Gamma,v}$ into a measure (counting multiplicity) by setting:

$$dC_{\Gamma,v} = \theta \, d\mathcal{H}^m,$$

where $\theta(p)$ is the number of points $(x,s) \in \Gamma \times [0,\infty)$ such that

$$v + s(x-v) = p.$$

Since $C_{\Gamma,v}$ is a cone with vertex $v$, the ratio $\theta(C_{\Gamma,v}) = C_{\Gamma,v}(B(v,r)) / (\omega_m r^m)$ is independent of $r$.

**Corollary 7.3.**

\begin{equation}
(M(b) + E_{\Gamma,v})\psi(\cdot,a) \leq \frac{M(a)(\mathbb{R}^n)}{(4\pi(t_0-a))^{m/2}} + E_{\Gamma,v}\psi(\cdot,a)
\end{equation}

\begin{equation}
\leq \frac{M(a)(\mathbb{R}^n)}{(4\pi(t_0-a))^{m/2}} + \theta_{\Gamma,v}.
\end{equation}

Furthermore, if the flow is ancient and if $\sup_t M(t) \mathbb{R}^n < \infty$, then

\begin{equation}
(M(b) + E_{\Gamma,v})\psi(\cdot,b) \leq \theta_{\Gamma,v}.
\end{equation}

**Proof.** The inequality (7.7) follows from (7.1) since $\psi(\cdot,a)$ attains its maximum value $(4\pi(t_0-a))^{-m/2}$ at the point $v$. The inequality (7.8) follows since $E_{\Gamma,v} \leq C_{\Gamma,v}$ and since a straightforward calculation shows that

\begin{equation}
C_{\Gamma,v}\psi(\cdot,a) = \theta_{\Gamma,v}.
\end{equation}

(Alternatively, (7.10) follows immediately from Remark 9.2 below.) The inequality (7.9) follows by letting $a \to -\infty$. \qed

8. Entropy

In this section, we adapt the concept of entropy in mean curvature flow to mean curvature flow with boundary.

Suppose that $M$ is a properly embedded $m$-dimensional manifold (without boundary) in $\mathbb{R}^n$. Recall that the entropy of $M$ is

\begin{equation}
e(M) = \sup_{a \in \mathbb{R}^n, \lambda > 0} (4\pi \lambda)^{-m/2} \int_{x \in M} e^{-|x-a|^2/(4\lambda)} \, dx
\end{equation}

\begin{equation}
= \sup_{\tilde{M}} (4\pi)^{-m/2} \int_{x \in \tilde{M}} e^{-|x|^2/4} \, d\tilde{x},
\end{equation}

where the second supremum is over all surfaces $\tilde{M}$ obtained from $M$ by translating and dilating.

More generally, suppose $M$ is a Radon measure on $\mathbb{R}^n$. The $m$-dimensional entropy of $M$ is

\begin{equation}
e(M) = \sup_{a \in \mathbb{R}^n, \lambda > 0} (4\pi \lambda)^{-m/2} \int_{x \in M} e^{-|x-a|^2/(4\lambda)} \, dM
\end{equation}

\begin{equation}
= \sup_{\tilde{M}} (4\pi)^{-m/2} \int_{x \in \tilde{M}} e^{-|x|^2/4} \, d\tilde{M}.
\end{equation}
where the supremum is over all Radon measures \( \hat{M} \) obtained from \( M \) by translation and \( m \)-dimensional scaling. (If \( M \) is a Radon measure and \( \lambda > 0 \), the \( m \)-dimensional rescaling of \( M \) by \( \lambda \) is the measure \( \lambda \# M \) obtained by pushing \( M \) forward by \( x \mapsto \lambda x \) and then multiplying by \( \lambda^m \).)

If \( t \in I \mapsto M(t) \) is a smooth mean curvature flow of properly immersed \( m \)-manifolds in \( \mathbb{R}^n \) or, more generally, if it is an \( m \)-dimensional Brakke flow in \( \mathbb{R}^n \), then, by Huisken’s monotonicity formula, the entropy \( e(M(t)) \) is a decreasing function of \( t \in I \).

Now suppose that \( \Gamma \) is a smooth, properly embedded \((m - 1)\)-dimensional manifold in \( \mathbb{R}^n \), and suppose that \( M \) is an \( m \)-manifold with boundary \( \Gamma \). If \( v \in \mathbb{R}^n \), we let \([M, \Gamma, v]\) be the piecewise-smooth manifold obtained by attaching the exterior cone \( \{v + s(x - v) : v \geq 1\} \) to \( M \). More generally, if \( M \) is any Radon measure on \( \mathbb{R}^n \), we let \([M, \Gamma, v]\) be the Radon measure \( M + E_{\Gamma, v} \).

We define

\[
(8.3) \quad \Theta_{\text{gauss}}(M, \Gamma, v, r) = \int \frac{\exp(-|x - v|^2/(4r^2))}{(4\pi r^2)^{m/2}} d[M, \Gamma, v] x + \frac{1}{2} 1_{\Gamma}(x),
\]

where

\[
1_{\Gamma}(x) = \begin{cases} 
1 & \text{if } x \in \Gamma, \\
0 & \text{if } x \notin \Gamma.
\end{cases}
\]

The term \( \frac{1}{2} 1_{\Gamma}(x) \) is necessary to make \( \Theta_{\text{gauss}}(M, \Gamma, v, r) \) continuous as a function of \( v \). (To see that \( \Theta_{\text{gauss}}(M, \Gamma, v, r) \) depends continuously on \( v \), note that if \( v_i \notin \Gamma \) converges to \( v \in \Gamma \), then, after passing to a subsequence, \( E_{\Gamma, v_i} \) converges to \( E_{\Gamma, v} \) together with a halfplane bounded by \( \text{Tan}(\Gamma, v) \).)

We define the \( m \)-dimensional entropy \( e(M; \Gamma) \) of the pair \((M, \Gamma)\) to be

\[
(8.4) \quad e(M; \Gamma) = \sup_{a \in \mathbb{R}^n, r > 0} \Theta_{\text{gauss}}(M, \Gamma, a, r) = \sup_{(M, \Gamma)} \Theta_{\text{gauss}}(\hat{M}, \hat{\Gamma}, 0, 1),
\]

where the second supremum is over all pairs \((\hat{M}, \hat{\Gamma})\) obtained from \((M, \Gamma)\) by translation and dilation. Note that \( e(M; \emptyset) = e(M) \).

**Theorem 8.1.** Let \( t \in I \mapsto M(t) \) be an \( m \)-dimensional Brakke flow in \( \mathbb{R}^n \) with boundary \( \Gamma \). Then

\[
t \in I \mapsto e(M(t); \Gamma)
\]

is a decreasing function of \( t \).

**Proof.** This is an immediate consequence of the Monotonicity Theorem 7.1. \( \Box \)

**Theorem 8.2.** Suppose that \( M \) is a shrinker, i.e., that

\[
t \in (-\infty, 0) \mapsto |t|^{1/2} M
\]
is a mean curvature flow. Then, in the definition (8.1) of entropy, the supremum is attained for \( a = 0 \) and \( \lambda = 1 \). Furthermore, the density of \( M \) at every point \( p \in M \) is \( \leq e(M) \), and if equality holds at any point, then \( M \) is a cone about that point.

See for example [CM12, lemma 7.01], or, for the first assertion, the proof of Theorem 8.3 below. (In [CM12, lemma 7.01], the shrinker is assumed to have polynomial area growth, but that assumption is not necessary since every shrinker has polynomial area growth. See for example [Bre16, Proposition 10].)

**Theorem 8.3.** Suppose that \( M \) is an \( m \)-dimensional shrinker in \( \mathbb{R}^n \) whose boundary is an \((m - 1)\)-dimensional linear subspace \( L \). Then in the definition (8.4) of \( e(M; L) \), the supremum is attained for \( a = 0 \) and \( r = 1 \). Furthermore, the extended density of \( \Sigma \) at each point is \( \leq e(M, L) \), and if equality holds at any point, then \( \Sigma \) is a cone.

**Proof.** The proof is essentially the same as in the boundaryless case. For the reader’s convenience, we give the proof of the first assertion: that the supremum in (8.4) is attained for \( a = 0 \) and \( r = 1 \). Let

\[
M(t) = \begin{cases} 
|t|_{\#}^{1/2} M & (t \leq 0), \\
0 & (t > 0).
\end{cases}
\]

Thus \( M(\cdot) \) is a mean curvature flow with boundary \( L \). Let

\[
\Theta = \Theta_{\text{gauss}}(M, \Gamma, 0, 1) = \Theta_{\text{gauss}}(M(-1), \Gamma, 0, 1)
\]

Trivially \( e(M; L) \geq \Theta \). We must show that \( e(M, L) \leq \Theta \). By self-similarity,

\[
\Theta = \Theta_{\text{gauss}}(M(-r^2), \Gamma, 0, r)
\]

for all \( r > 0 \). By monotonicity (Theorem 7.1),

\[
\Theta_{\text{gauss}}(M(t - r^2), \Gamma, a, r)
\]

is an increasing function of \( r \) for every spacetime point \((a, t)\). From polynomial area growth of \( M \) [Bre16, Proposition 10], one easily checks that

\[
\lim_{r \to \infty} \Theta_{\text{gauss}}(M(t - r^2, \Gamma, a, r) = \lim_{r \to \infty} \Theta_{\text{gauss}}(M(-r^2), \Gamma, 0, r) = \Theta.
\]

Thus

\[
\Theta \geq \Theta_{\text{gauss}}(M(t - r^2, \Gamma, a, r)
\]

for all \( r > 0 \). Since \( t \) is arbitrary, we see that for all \( \rho, a, \) and \( r \),

\[
\Theta \geq \Theta_{\text{Gauss}}(M(-\rho^2), \Gamma, a, r)
= \Theta(\rho \# M, \Gamma, a, r)
= \Theta(M, \Gamma, a/\rho, r/\rho).
\]

Since \( a \) and \( \rho \) are arbitrary, we see that

\[
\Theta \geq \sup_{p,R} \Theta(M, \Gamma, p, R) = e(M; L).
\]

\( \Box \)
9. Entropy and Maximal Density Ratio

Entropy is closely related to a more geometrically intuitive notion, namely the maximal density ratio. In particular, for any surface (or Radon measure) the ratio of the two quantities is bounded above and below.

The \( (m\text{-dimensional}) \) maximal density ratio \( \text{mdr}(M) \) of a surface \( M \) in \( \mathbb{R}^n \) is defined to be

\[
\text{mdr}(M) := \sup_{x \in \mathbb{R}^n, r > 0} \frac{\mathcal{H}^m(M \cap B(x, r))}{\omega_m r^m},
\]

where \( \omega_m \) is the volume of the unit ball in \( \mathbb{R}^m \). Of course \( \text{mdr}(M) \) is invariant under rigid motions and scaling. More generally, if \( M \) is a Radon measure on \( \mathbb{R}^n \), the the \( m \)-dimensional maximal density ratio of \( M \) is

\[
(9.1) \quad \text{mdr}(M) := \sup_{a \in \mathbb{R}^n, r > 0} \frac{M B(x, r)}{\omega_m r^m}.
\]

**Theorem 9.1.** \( \text{mdr}(M) \geq e(M) \geq c_m \text{mdr}(M) \) for some \( c_m > 0 \).

**Proof.** Let us assume that \( M \) is an \( m \)-manifold; except for notation, the same proof works for Radon measures. Let \( \tilde{M} \) be a surface obtained from \( M \) by translating and scaling. Let

\[
V(r) = \mathcal{H}^m(\tilde{M} \cap B(0, r))
\]

and \( \theta = \text{mdr}(M) \). Thus \( V(r) \leq \omega_m \theta r^m \), so

\[
\int_{\tilde{M} \cap \{|x| \leq R\}} e^{-|x|^2/4} \, dx = \int_{r=0}^{R} e^{-r^2/4} V'(r) \, dr
\]

\[
= e^{-R^2/4} V(R) - \int_{r=0}^{R} \frac{d}{dr} \left(e^{-r^2/4}\right) V(r) \, dr
\]

\[
= e^{-R^2/4} V(R) + \int_{r=0}^{R} \frac{1}{2} r e^{-r^2/4} V(r) \, dr
\]

\[
\geq e^{-R^2/4} V(R).
\]

To prove the lower bound for \( e(M) \), let \( 0 < \eta < 1 \). We may assume by translating and scaling that \( V(1) \geq \eta \omega_m \text{mdr}(M) \). Thus by (9.2) with \( R = 1 \),

\[
(4\pi)^{m/2} e(M) \geq e^{-1/4} V(1) \geq e^{-1/4} (\eta \omega_m) \text{mdr}(M).
\]

Since this holds for all \( \eta \in (0, 1) \) it also holds for \( \eta = 1 \). Thus

\[
e(M) \geq (4\pi)^{-m/2} e^{-1/4} \omega_m \text{mdr}(M).
\]

To prove the upper bound for \( e(M) \), we may assume that \( \theta := \text{mdr}(M) \) is finite. Now \( V(r) \leq \omega_m \theta r^m \), so from (9.2) we see that

\[
\int_{\tilde{M} \cap \{|x| \leq R\}} e^{-|x|^2/4} \, dx = e^{-R^2/4} V(R) + \int_{r=0}^{R} \frac{1}{2} r e^{-r^2/4} V(r) \, dr
\]

\[
\leq e^{-R^2/4} \omega_m \theta R^m + \int_{r=0}^{R} \frac{1}{2} r e^{-r^2/4} \omega_m \theta r^m \, dr.
\]
Letting $R \to \infty$ and multiplying by $(4\pi)^{-m/2}$ gives

\[(4\pi)^{-m/2} \int_{\mathcal{M}} e^{-|x|^2/4} \, dx \leq \theta \int_0^R (4\pi)^{-m/2} (\omega_m/2) r^{1+m} e^{-r^2/4} \, dr \]

\[= \theta.\]

Taking the supremum over all $\mathcal{M}$ gives $e(M) \leq \theta = \text{mdr}(M)$.

(Here is one way to see that the definite integral on the right side of (9.4) is 1. Note that if $\mathcal{M}$ is an $m$-plane through the origin, then $V(r) \equiv \omega_m r^m$, so the inequalities in (9.3) and in (9.4) become equalities. Furthermore, in this case, the left side of (9.4) is 1 and $\theta = 1$, so the definite integral on the right side of (9.4) must be equal to 1.)

**Remark 9.4.** The proof shows that if $M$ is a Radon measure on $\mathbb{R}^n$, then $M\rho$ is a weighted average of the density ratios $(M\mathcal{B}(0, r))/\omega_m r^m$ over $r \in (0, \infty)$. More generally, $M\psi_{v,a}$ is a weighted average of $(M\mathcal{B}(v, r))/\omega_m r^m$ over $r \in (0, \infty)$.

Now we turn to manifolds (or varieties) with boundary. Suppose $\Gamma$ is a smooth $(m - 1)$-dimensional manifold without boundary in $\mathbb{R}^n$. If $M$ is an $m$-dimensional manifold with boundary $\Gamma$, we let

\[\Theta(M, \Gamma, a, r) = \frac{\text{area}(M \cup E_{\Gamma, a} \cap \mathcal{B}(a, r))}{\omega_m r^m} + \frac{1}{2} \mathcal{I}_\Gamma(x),\]

where $E_{\Gamma, \nu}$ is the exterior cone over $\Gamma$ with vertex $\nu$ (as in Section 7).

More generally, if $M$ is a Radon measure on $\mathbb{R}^n$, we let

\[\Theta(M, \Gamma, a, r) = \frac{(M + E_{\Gamma, a})\mathcal{B}(a, r)}{\omega_m r^m} + \frac{1}{2} \mathcal{I}_\Gamma(x),\]

where $E_{\Gamma, a}$ is the Radon measure associated to the exterior cone.

Given a point $a \in \mathbb{R}^n$, note that $(M + E_{\Gamma, a})\partial \mathcal{B}(a, r) = 0$ for almost all $r$, and that for such $r$, the function $\Theta(M, \Gamma, \cdot, r)$ is continuous at $a$.

We also let

\[\text{mdr}(M; \Gamma) = \sup_{a \in \mathbb{R}^n, r > 0} \Theta(M, \Gamma, a, r).\]

**Theorem 9.3.** Suppose that $\Gamma$ is a smooth, property embedded $(m - 1)$-dimensional submanifold of $\mathbb{R}^n$ and that $M$ is a Radon measure on $\mathbb{R}^n$. Then

\[\text{mdr}(M; \Gamma) \geq e(M; \Gamma) \geq c_m \text{mdr}(M; \Gamma).\]

The proof is essentially identical to the proof of Theorem 9.1.

**Remark 9.4.** In the definition (9.4) of $e(M; \Gamma)$, if we fix the point $\nu$ and take the supremum over $r > 0$, we get an entropy-like quantity $e(M, \Gamma)(\nu)$ that depends on $\nu$:

\[e(M, \Gamma)(\nu) := \sup_{r > 0} \Theta_{\text{gauss}}(M, \Gamma, \nu, r).\]

Likewise, we can define a fixed-center-point version of the maximal density ratio:

\[\text{mdr}(M; \Gamma)(a) := \sup_{r > 0} \Theta(M, \Gamma, a, r).\]
Of course \( e(M; \Gamma) = \sup_v e(M; \Gamma)(v) \) and \( \text{mdr}(M; \Gamma) = \sup_a \text{mdr}(M; \Gamma)(a) \). Furthermore, for each \( v \in \mathbb{R}^n \),

\[
\text{mdr}(M; \Gamma)(v) \geq e(M; \Gamma)(v) \geq c_m \text{mdr}(M; \Gamma)(v),
\]

and if \( t \in I \mapsto M(t) \) is an \( m \)-dimensional Brakke flow with boundary \( \Gamma \), then \( e(M(t); \Gamma)(v) \) is a decreasing function of \( t \). The proofs are identical to the proofs of Theorems 9.3 and 8.1.

10. COMPACTNESS THEOREMS

**Theorem 10.1.** Suppose for \( i = 1, 2, \ldots \) that

\[
t \in [0, T] \mapsto M_i(t)
\]

is an integral Brakke flow in \( U \) with boundary \( \Gamma_i \). Suppose also that the \( \Gamma_i \) converge in \( C^1 \) to a smooth, properly embedded \((m - 1)\)-dimensional submanifold \( \Gamma \) of \( U \), and that

\[
c_K := \sup_i \sup_{t \in [0, T]} M_i(t) K < \infty
\]

for each compact subset \( K \) of \( U \). Then there is a subsequence \( i(j) \) such that for each \( t \in [0, T] \), \( M_{i(j)}(t) \) converges to a Radon measure \( M(t) \).

**Proof.** Let \( \mathcal{F} \) be a countable collection of \( C^2 \), nonnegative, compactly supported functions on \( U \) such that the linear span of \( \mathcal{F} \) is dense in the space of all continuous, compactly supported functions. By Corollary 5.7, for each \( u \in \mathcal{F} \), the function

\[
t \in [0, T] \mapsto M(t) u - c_{\text{spt}(u)}(\max|\nabla^2 u|) t
\]

is non-increasing. Each such function is also bounded. Hence by passing to a subsequence, we can assume that each of the functions (10.1) converges to a limit function. Theorem 10.1 follows immediately from the Riesz Representation Theorem. \( \square \)

In the following theorem, we write \( H_i(x, t) \) for \( H(M_i(t), x) \) and \( v_i(x, t) \) for \( v(M_i(t), x) \).

**Theorem 10.2.** Suppose

(1) \( \Gamma_i \) \((i \in \mathbb{N})\) and \( \Gamma \) are smooth, \((m - 1)\)-dimensional submanifolds of \( U \), and the \( \Gamma_i \) converge smoothly to \( \Gamma \).

(2) For \( i \in \mathbb{N} \), \( t \in [0, T] \mapsto M_i(t) \) is an \( m \)-dimensional integral Brakke flow with boundary \( \Gamma_i \).

(3) For each \( t \), \( M_i(t) \) converges to a Radon measure \( M(t) \).

Then \( t \mapsto M(t) \) is an integral Brakke Flow with boundary \( \Gamma \), and

\[
c_K := \sup_i \sup_{t \in [0, T]} M_i(t) K < \infty.
\]

Furthermore, there is a continuous, everywhere positive function \( \phi : U \to \mathbb{R} \) with the following property. For almost every \( t \), there is a subsequence \( i(j) \) such that:

\[
\sup_j \int |H_i(j)(t, \cdot)|^2 \, dM_i(j)(t) < \infty,
\]

\[
\sup_j \left( \int_K |H_i(j)(t, \cdot)| \, dM_i(j)(t) + \beta(M_i(j)(t))K \right) < \infty \quad \text{for all } K \subset U,
\]

where \( \beta \) is a constant related to the Brakke flow.
\( (10.5) \quad \text{Var}(M_{i(j)}(t)) \rightarrow \text{Var}(M(t)), \)

\[ \int H_{i(j)}(x) \cdot X(x, \text{Tan}(M_{i(j)}(x))) \, dM_i(t) \]

\( (10.6) \quad \rightarrow \int H(x) \cdot X(x, \text{Tan}(M(x))) \, dM(t), \)

\[ \int \nu_{i(j)}(x) \cdot Y(x, \text{Tan}(\Gamma_{i(j)}(x))) \, d\Gamma_{i(j)}(x) \]

\( (10.7) \quad \rightarrow \int \nu(x) \cdot Y(x, \text{Tan}(\Gamma(x))) \, d\Gamma(x) \)

for every \( X \in \mathcal{X}_m(U) \) and for every \( Y \in \mathcal{X}_{m-1}(U) \).

**Remark 10.3.** Even for Brakke flows without boundary, the fact that \( X \) in (10.6) can depend on \( x \) and \( \text{Tan}(M_{i(j)}(x)) \) (rather than just on \( x \)) seems to be new.

**Proof.** The finiteness of \( c_K \) follows immediately from Theorem 5.5. By Theorem 5.6,

\[ d_K := \sup_i \int_0^T \int_K |H_i|^2 \, dM_i(t) \, dt < \infty. \]

It follows that there is a continuous, everywhere positive function \( \phi : U \rightarrow \mathbb{R} \) such that

\( (10.8) \quad C := \sup_i \int_0^T \int_\Omega |H_i|^2 \, dM_i(t) \, dt < \infty. \)

By Fatou’s Lemma,

\[ \int_0^T \left( \liminf_i \int \phi |H_i|^2 \, dM_i(t) \right) \, dt < \infty, \]

so for almost every \( t \),

\[ \liminf_i \int \phi |H_i|^2 \, dM_i(t) < \infty. \]

For every such \( t \), there is a subsequence \( i(j) \) such that

\[ \sup_j \int \phi |H_{i(j)}|^2 \, dM_{i(j)}(t) < \infty. \]

By the Varifold Closure Theorem 4.4, \( M(t) \in \mathcal{V}_m(U, \Gamma) \), and (10.4), (10.5), (10.6), and (10.7) hold.

It remains only to show that \( t \rightarrow M(t) \) is an integral Brakke flow with boundary \( \Gamma \). Let \( u : U \times [0, T] \rightarrow \mathbb{R} \) be a nonnegative, compactly supported, \( C^2 \) function. For each \( i \),

\[ (M_i u)(a) - (M_i u)(b) \]

\[ \geq \int_a^b \int \left( u |H_i|^2 - H_i \cdot \nabla u - \frac{\partial u}{\partial t} \right) \, dM_i(t) \, dt \]

\[ = \int_a^b \int \left( u^{1/2} H_i - \frac{1}{2} u^{-1/2} \nabla u \right)^2 \left( u^{-1} \frac{\partial |u|^2}{\partial t} + \frac{1}{4} \frac{\partial u}{u} \right) \, dM_i(t) \, dt \]
Therefore by (10.8),
\[(M_i u)(a) - (M_i u)(b) + \epsilon C \geq \int_a^b \int \left( \left| u^{1/2} H_i - \frac{1}{2} u^{-1/2} \nabla u \right|^2 + \epsilon \phi |H_i|^2 - \frac{1}{4} \frac{|\nabla u|^2}{u} - \frac{\partial u}{\partial t} \right) dM_i(t) \, dt.\]

Letting \( i \to \infty \) gives, by Fatou’s Lemma,
\[(M u)(a) - (M u)(b) + \epsilon C \geq \int_a^b \lambda_\epsilon(t) \, dt - \int_a^b \int \left( \frac{1}{4} \frac{|\nabla u|^2}{u} + \frac{\partial u}{\partial t} \right) dM(t) \, dt,\]

where
\[\lambda_\epsilon(t) = \liminf_i \int \left( \left| u^{1/2} H_i - \frac{1}{2} u^{-1/2} \nabla u \right|^2 + \epsilon \phi |H_i|^2 \right) dM_i(t).\]

For each \( t \) with \( \lambda_\epsilon(t) < \infty \), there is a subsequence \( i(j) \) such that
\[\int \left( \left| u^{1/2} H_{i(j)} - \frac{1}{2} u^{-1/2} \nabla u \right|^2 + \epsilon \phi |H_{i(j)}|^2 \right) dM_i(j)(t) \to \lambda_\epsilon(t).\]

For such \( t \), we have (as above), \( \text{Var}(M_{i(j)}(t)) \to \text{Var}(M(t)) \) and
\[\int H_{i(j)} \cdot X \, dM_{i(j)}(t) \to \int H \cdot X \, dM(t),\]

for all \( X \in \mathcal{X}(U) \). Consequently,
\[\int \left( u^{1/2} H_{i(j)} - \frac{1}{2} u^{-1/2} \nabla u \right) \cdot X \, dM_{i(j)}(t) \to \int \left( u^{1/2} H - \frac{1}{2} u^{-1/2} \nabla u \right) \cdot X \, dM(t).\]

Since this holds for all \( X \in \mathcal{X}(U) \),
\[\int \left| u^{1/2} H - \frac{1}{2} u^{-1/2} \nabla u \right|^2 dM(t) \leq \liminf \int \left| u^{1/2} H_{i(j)} - \frac{1}{2} u^{-1/2} \nabla u \right|^2 dM_{i(j)}(t) \leq \lambda_\epsilon(t)\]

Substituting this into (10.9) and letting \( \epsilon \to 0 \) gives
\[(M u)(a) - (M u)(b) \geq \int_a^b \int \left( \left| u^{1/2} H - \frac{1}{2} u^{-1/2} \nabla u \right|^2 + \frac{1}{4} \frac{|\nabla u|^2}{u} + \frac{\partial u}{\partial t} \right) dM(t) \, dt \]

\[= \int_a^b \int \left( |uH|^2 - H \cdot \nabla u - \frac{\partial u}{\partial t} \right) dM(t) \, dt \]

\( \square \)
11. Tangent Flows

Consider an integral Brakke flow \( t \in I \mapsto M(t) \) in \( N \) with boundary \( \Gamma \). As in \S 6, we isometrically embed \( N \) in a Euclidean space \( \mathbb{R}^d \). We now discuss tangent flows at a space-time point \((p_0, t_0)\). By making a spacetime translation, it suffices to consider the case \((p_0, t_0) = (0,0)\).

For \( \lambda > 0 \), let \( M^\lambda \) be the result of applying the parabolic dilation \( D_\lambda : (x,t) \mapsto (\lambda x, \lambda^2 t) \) to the flow \( t \in I \cap (-\infty,0) \mapsto M(t) \).

Just as for mean curvature flow without boundary, monotonicity together with the compactness and closure theorems in \S 10 implies existence of tangent flows: for every sequence \( \lambda(i) \to \infty \), there is a subsequence \( \lambda(i(j)) \) such that the flows \( M^{\lambda(i(j))}(\cdot) \) converge to a flow \( M'(\cdot) : t \in (-\infty,0] \mapsto M'(t) \). If \( 0 \notin \Gamma \), it is an integral Brakke flow in the Euclidean space \( \text{Tan}(N,0) \). If \( 0 \in \Gamma \), it is an integral Brakke flow in \( \text{Tan}(N,0) \) with boundary \( \text{Tan}(\Gamma,0) \).

In either case, the tangent flow \( M'(\cdot) \) is self-similar: it is invariant under parabolic dilations \( D_\lambda \) with \( \lambda > 0 \).

**Definition 11.1.** We say that an integral Brakke flow \( t \in I \mapsto M(t) \) with boundary \( \Gamma \) is **unit-regular** provided the following holds:

For each \( p \in N \) and \( t \in I \), if one of the tangent flows at \((p,t)\) is a multiplicity-1 plane or halfplane, then the flow \( M(\cdot) \) is fully smooth in a spacetime neighborhood of \((p,t)\).

Equivalently,

For each \( p \in N \) and \( t \in I \), if the extended Gauss density \( \Theta_e(M(\cdot),(p,t)) \) is 1, then the flow \( M(\cdot) \) is fully smooth in a spacetime neighborhood of \((p,t)\).

(See Definition 6.3 for extended Gauss density.)

Here “fully smooth” means “smooth and with no sudden vanishing”.

Unit-regularity does not prevent sudden vanishing; it just implies that such vanishing can only occur where the multiplicity is greater than one. For example, consider the Brakke flow that consists of a nonmoving plane with multiplicity \( k \) for \( t \leq 0 \) and that vanishes at time 0. The flow is not unit-regular if \( k = 1 \), but it is (vacuously) unit-regular if \( k > 1 \).

**Remark 11.2.** If \( p \notin \Gamma \) and if \( \Theta_e(M,(p,t)) = 1 \), then \((p,t)\) is a backwardly \( C^{1,\alpha} \)-regular point of the flow by Brakke’s Regularity Theorem [Bra78] if the ambient space is Euclidean or by the Kasai-Tonegawa [KT14] generalization of that theorem for general ambient manifolds, and consequently is a backwardly \( C^\infty \)-regular point by [Jon14]. Presumably the analogous theorems are true for \((p,t)\) with \( p \in \Gamma \) and with \( \Theta_e(M(\cdot),(p,t)) = 1 \). If so, then every integral Brakke flow with boundary would have the backward smoothness (but not necessarily the full smoothness) described in the definition of unit-regularity. However, none of those
facts are required for this paper; here, the simpler local regularity theorems in \cite{Whi05} suffice.

12. Mod 2 Flat Chains

Let $L_m^\text{rec}(U, \mathbb{Z}^+)$ denote the space of functions on $U$ that take values in the nonnegative integers, that are locally $L^1$ with respect to Hausdorff $m$-dimensional measure on $U$, and that vanish except on a countable union of $m$-dimensional $C^1$-submanifolds of $U$. We identify functions that agree except on a set of Hausdorff $m$-dimensional measure zero. Let $L_m^\text{rec}(U, \mathbb{Z}_2)$ be the corresponding space with the nonnegative integers $\mathbb{Z}^+$ replaced by $\mathbb{Z}_2$, the integers mod 2. The space $IM_m(U)$ (defined in §2) is naturally isomorphic to $L_m^\text{rec}(U, \mathbb{Z}^+)$: given any $M \in IM_m(U)$, the corresponding function in $L_m^\text{rec}(U, \mathbb{Z}^+)$ is the density function $\Theta(M, \cdot)$ given by

$$\Theta(M, x) = \lim_{r \to 0} \frac{MB(x, r)}{\omega_m r^m},$$

where $\omega_m$ is the volume of the unit ball in $\mathbb{R}^m$. In particular, this limit exists and is a non-negative integer for $H^m$-almost every $x \in U$. Similarly, the space of $m$-dimensional rectifiable mod 2 flat chains$^1$ in $U$ is naturally isomorphic to $L_m^\text{rec}(U, \mathbb{Z}_2)$: given any such flat chain $A$, the corresponding function is the density function $\Theta(A, \cdot)$ given by

$$\Theta(A, x) := \lim_{r \to 0} \frac{\mu_A B(x, r)}{\omega_m r^m},$$

where $\mu_A$ is the Radon measure on $U$ determined by $A$. In particular, this limit exists and is 0 or 1 for $H^m$-almost every $x \in U$.

The surjective homomorphism

$$[] : \mathbb{Z}^+ \to \mathbb{Z}_2,$$

$$k \mapsto [k]$$

determines a homomorphism from $L_m^\text{rec}(U, \mathbb{Z}^+)$ to $L_m^\text{rec}(U, \mathbb{Z}_2)$ and thus also a homomorphism from the additive semigroup $IM_m(U)$ to the additive group of $m$-dimensional rectifiable mod 2 flat chains in $U$. If $M \in IM_m(U)$, we let $[M]$ denote the corresponding rectifiable mod 2 flat chain. Thus $[M]$ is the unique rectifiable mod 2 flat chain in $U$ such that

$$\Theta([M], x) = [\Theta(M, x)]$$

for $H^m$-almost every $x \in U$.

The following is Theorem 3.3 in \cite{Whi09}:

**Theorem 12.1.** Suppose that $M_i$ ($i = 1, 2, \ldots$) and $M$ are Radon measures in $IM_m(U)$ with the following properties:

1. $\text{Var}(M_i) \to \text{Var}(M)$.

$^1$As in \cite{Sim83} and in \cite{Whi09}, we do not require flat chains to have compact support. In Federer’s terminology \cite{Fed69}, they would be called “locally flat chains”. See the discussion in \cite{Whi09} §2.1.
Each $M_i$ has bounded first variation, and
$$q_K := \sup_i \left( \int_K |H(M_i, \cdot)| dM_i + \beta(M_i)(K) \right) < \infty.$$  
(3) The $\partial [M_i]$ converge (in the flat topology) to a mod 2 flat chain $C$.

Then the $[M_i]$ converge (in the flat topology) to $[M]$. In particular, $\partial [M_i] = C$.

13. STANDARD BRAKKE FLOWS AND THE CLOSURE THEOREM

Suppose that $(M_i(\cdot), \Gamma_i)$ ($i \in \mathbb{N}$) and $(M(\cdot), \Gamma)$ are integral Brakke flows with boundary in $U$ defined on a time interval $I$. We say that the $(M_i(\cdot), \Gamma_i)$ converge to $(M(\cdot), \Gamma)$ if $M_i(t) \to M(t)$ for each $t \in I$ and the $\Gamma_i$ converge smoothly to $\Gamma$.

**Theorem 13.1** (Closure Theorem). Suppose $(M_i(\cdot), \Gamma_i)$ converges to $(M(\cdot), \Gamma)$, where each $M_i : t \in [0, T] \to M_i(t)$ is an integral Brakke flow with boundary $\Gamma_i$.

(1) If the flows $M_i(\cdot)$ are unit-regular, then so is the flow $M(\cdot)$.

(2) If $\partial [M_i(t)] = [\Gamma_i(t)]$ for almost every $t$, then $\partial [M(t)] = [\Gamma(t)]$ for almost every $t$.

**Proof.** See [SW20] Theorem 4.2 for the proof of Assertion [1]. (The proof in [SW20] is for Brakke flows without boundary, but the same proof works for flows with boundary: the proof is based on the local regularity theorems in [Whi05], which are stated and proved both with and without boundary.)

We now prove Assertion [2]. By Theorem [10.2] for almost every $t$, there is a subsequence $M_{i(j)}(t)$ such that

\[(13.1) \quad \text{Var}(M_{i(j)}(t)) \to \text{Var}(M(t))\]

and

\[(13.2) \quad \sup_j \left( \int_K |H_{i(j)}(t, \cdot)| dM_{i(j)}(t) + \beta(M_{i(j)}(t))K \right) < \infty \quad \text{for all } K \subset U.\]

Since the $\Gamma_i$ converge smoothly to $\Gamma$, it follows that the $[\Gamma_i](= \partial [M_i])$ converge to $[\Gamma]$ as mod 2 flat chains. By Theorem [12.1] $[M_{i(j)}(t)] \to [M(t)]$ and $\partial [M(t)] = [\Gamma]$. \qed

**Definition 13.2.** An integral Brakke flow $M(\cdot)$ with boundary $\Gamma$ is called **standard** if it has properties described in Theorem [13.1].

(1) the flow is unit-regular, and

(2) $\partial [M(t)] = [\Gamma]$ for almost every $t$.

14. EXISTENCE

**Theorem 14.1.** Let $N$ be a smooth Riemannian manifold. If $N$ has nonempty boundary, we assume that the boundary is smooth and $m$-convex. Let $\Gamma$ be smooth, properly embedded $(m-1)$-dimensional manifold in $N$. Let $M_0$ be a smoothly embedded $m$-dimensional manifold in $N$ with boundary $\Gamma$ and with finite area, or, more generally, let $M_0$ be an $m$-rectifiable set of finite $m$-dimensional measure such that $\partial [M_0] = [\Gamma]$. Then there exists a standard Brakke flow

$$t \in [0, \infty) \mapsto M(t)$$
with boundary $\Gamma$ such that

$$M(t) \sim \mathcal{H}^m \cap M_0$$

as $t \to 0$.

If $p \notin \Gamma$ and if $M_0$ is smooth in a neighborhood of $p$, then the flow is smooth in a spacetime neighborhood of $(p,0)$. If $p \in \Gamma$ and if $M_0$ is $C^{1,\alpha}$ in a neighborhood of $p$, then the flow is parabolically $C^1$ in a spacetime neighborhood of $(p,0)$.

Recall that $m$-convexity of $\partial N$ at a point $p$ is the condition that the sum of the smallest $m$ principal curvatures of $\partial N$ at $p$ with respect to the inward unit normal is nonnegative. (Thus $1$-convexity is convexity, and if $\dim N = m+1$, then $m$-convexity is mean-convexity.)

We say that $\partial N$ is strictly $m$-convex at $p \in \partial N$ if the sum of the smallest $m$-principal curvatures at $p$ is greater than $0$.

**Proof.** Let us first prove the theorem assuming that $\partial N$ is strictly $m$-convex.

Let $\Gamma^* = (M_0 \times \{0\}) \cup (\Gamma \times [0,\infty))$. Let $\lambda > 0$. Consider a mod 2 flat chain $\Sigma^\lambda$ in $N \times \mathbb{R}$ that minimizes

$$\int_{(x,z) \in N \times \mathbb{R}} e^{-\lambda z} d\mu^{\lambda}(x,z)$$

subject to $\partial \Sigma^\lambda = [\Gamma^*]$. Let $M^\lambda$ be the associated Radon measure; thus $\Sigma^\lambda = [M^\lambda]$.

Note that $\Sigma^\lambda$ is mass-minimizing with respect to the Ilmanen metric $e^{-2\lambda z/m}(g + dz^2)$ (where $g$ is the metric on $N$.) Thus $\text{spt}(M^\lambda) \setminus \Gamma^*$ is smooth (with multiplicity $1$) away from a closed set of Hausdorff dimension $\leq m-1$ [Fed70].

(Here is where strict $m$-convexity of $\partial N$ is used. Note that strict $m$-convexity of $\partial N$ implies strict $(m+1)$-convexity of $(\partial N) \times \mathbb{R}$ with respect to the Ilmanen metric. By the maximum principle in [Whi10], $\Sigma^\lambda$ cannot touch $(\partial N) \times \mathbb{R}$ at any interior point of $\Sigma^\lambda$.)

For $t \geq 0$, let $M^\lambda(t)$ be the portion of $M^\lambda - \lambda t(0,1)$ in $\tilde{N} = N \times (0,\infty)$. Then

$$t \in [0,\infty) \mapsto M^\lambda(t)$$

is an integral Brakke flow with boundary $\tilde{\Gamma} := \Gamma \times (0,\infty)$ in $\tilde{N}$. In fact, it is standard:

1. Because $M^\lambda$ is smooth almost everywhere, the mean curvature vector is orthogonal to the surface almost everywhere.
2. Unit regularity follows from Allard’s Regularity Theorem [All72, §8] and Boundary Regularity Theorem [All75, §4] applied to $M^\lambda$.
3. The mod 2 boundary condition holds by construction.

By [Ilm94, 5.1, 3.2(ii)], the areas of the $\mathcal{M}^\lambda(t)$ have a uniform upper bound on area as $\lambda \to \infty$:

$$\text{area}(\mathcal{M}^\lambda_0 \cap \{a < z < a + b\}) \leq (b + \lambda^{-1}) \text{area}(M_0)$$

for any $a, b > 0$, and thus

$$\text{area}(\mathcal{M}^\lambda(t) \cap \{0 < z < b\}) \leq (b + \lambda^{-1}) \text{area}(M_0).$$

Consequently (by Theorems [10.1] [10.2] and [13.1]), the flows $t \in [0,\infty) \mapsto \mathcal{M}^\lambda(t)$ converge as $\lambda \to \infty$ (after passing to a subsequence) to a standard Brakke flow $\mathcal{M}(\cdot)$ in $\tilde{N}$ with boundary $\tilde{\Gamma}$. 
Furthermore, as in [Ilm94],
\[ \mathcal{M}(0) = M_0 \times (0, \infty) \]
and
\[ \mathcal{M}(t) = M(t) \times (0, \infty) \]
(except possibly for countably many \( t \)), where \( M(t) \) is a Radon measure in \( N \).

Since \( \mathcal{M}(\cdot) \) is a standard Brakke flow with boundary \( \Gamma^* \) in \( N^* \), it follows that \( t \in [0, \infty) \mapsto M(t) \) is a standard Brakke flow in \( N \) with boundary \( \Gamma \) and with
\[ M(0) = M_0. \]

(See [Ilm94, 8.9].)

If \( M_0 \) is \( C^1,\alpha \) in a neighborhood of a point \( p \), then the flow is parabolically \( C^1 \) in a space-time neighborhood of \((p,0)\) by [Whi05]. If \( M_0 \) is smooth in a neighborhood of a point \( p \in N \setminus \Gamma \), then the flow is smooth in a spacetime neighborhood of \((p,0)\) by [SW20, Corollary A.3].

This completes the proof assuming strict \( m \)-convexity. For the general case, let \( g_i \) be a sequence of smooth metrics on \( N \) such that the \( g_i \) converge smoothly to the original metric and such that \( \partial N \) is strictly \( m \)-convex with respect to each \( g_i \). Then we get a suitable Brakke flow for each metric \( g_i \). By Theorems 10.1, 10.2, and 13.1 a subsequence of those Brakke flows will converge to a suitable Brakke flow for the metric \( g \).

(Theorems 10.1, 10.2, and 13.1 are stated for a fixed metric, but they hold, with the same proofs, for sequences of metrics.)

\[ \square \]

15. Self-Similar Flows

**Theorem 15.1** (Shrinker Theorem). Let \( \Gamma \) be an \( (m-1) \)-dimensional linear subspace of \( \mathbb{R}^{m+1} \). Suppose that
\[ t \in (-\infty,0) \mapsto S(t) \]
is an \( m \)-dimensional integral Brakke flow in \( \mathbb{R}^{m+1} \setminus \Gamma \) that is self-similar (i.e, invariant under parabolic dilations \( \mathcal{D}_{\lambda} : (x,t) \mapsto (\lambda x, \lambda^2 t) \) with \( \lambda > 0 \)). Let \( M := S(-1) \), and suppose that \( \text{spt}(M) \setminus \Gamma \) is disjoint from some \( m \)-dimensional halfplane \( P \) with boundary \( \Gamma \). Then \( M \) is a sum of half-planes (each with boundary \( \Gamma \)) with multiplicities.

**Proof.** Let \( \mathcal{D} \) be the set of halfplanes with boundary \( \Gamma \) that are disjoint from \( (\text{spt} M) \setminus \Gamma \). We claim that \( \mathcal{D} \) is open. To see this, suppose \( P \in \mathcal{D} \). By rotating, we can assume that \( P \) is the halfplane \( \{ x_2 = 0, x_1 \geq 0 \} \). Let \( M^+ \) and \( S^+(t) \) be the portions of \( M \) and \( S(t) \) in the region \( \{ x_2 \geq 0, x_1 > 0 \} \).

Let \( f : \mathbb{B}^m(0,1) \cap \{ x_1 > 0 \} \to \mathbb{R} \) be a smooth, compactly supported, nonnegative function that is \( > 0 \) at some points. By multiplying by a small positive constant, we can assume that the graph of \( f \) lies below \( M^+ \). Extend \( f \) to \( \mathbb{B}^m(0,1) \) so that it is odd in \( x_1 \):
\[ f(-x_1, x_2, \ldots, x_m) = -f(x_1, x_2, \ldots, x_m). \]

Now let
\[ u : \mathbb{B}^m(0,1) \times (-1, \infty) \to \mathbb{R} \]
be the solution of the nonparametric MCF equation with
\[ u(\cdot, -1) = f, \]
\[ u(\cdot, t)|_{\partial B^m} \equiv 0. \]

By the boundary maximum principle, \( c := \frac{\partial}{\partial x_1} u(0, 0) > 0 \). Let \( G(t, \epsilon) \) be the graph of \( u(\cdot, t) - \epsilon \). Let \( \epsilon > 0 \). By the maximum principle (Theorem 20.1), \( \text{spt}(S^+(t)) \) lies above \( G(t, \epsilon) \) for all \( t \in [-1, 0] \). Hence \( \text{spt}(S^+(t)) \) lies in the closed region above \( G(t) \) for \( t \in [-1, 0] \). Equivalently, \( \text{spt}(M) \) lies in the closed region above \( G^*(t) := |t|^{-1/2} G(t) \) for all \( t \in [-1, 0] \). At \( t \to 0 \), \( G^*(t) \) converges to the plane \( \{ x : x_2 = cx_1 \} \). Thus we see that the halfplane \( P_\lambda := \{ x : x_2 = \lambda x_1, x_1 > 0 \} \) is in \( \mathcal{P} \) for all \( \lambda \in (c', 0) \). Likewise there is a \( c'' < 0 \) such that the halfplane \( P_\lambda \) is disjoint from \( M \) for all \( \lambda \in (c'', 0) \). This completes the proof of openness of \( \mathcal{P} \).

Now let \( P \) be a halfplane in the boundary of \( \mathcal{P} \). Then \( \text{spt}(M) \) touches \( P \), so, by the strong maximum principle, \( M \) contains \( P \). (Note that \( P \) and the varifold associated to \( M \) are both stationary with respect to the shrinker metric, so \( M \) contains \( P \) by the strong maximum principle in [SW89].) Now repeat the process with \( S(t) \) replaced by \[ t \in (-\infty, 0) \mapsto S(t) - (\mathcal{H}^m \cap P). \]

The process must stop in finitely many steps, since otherwise \( M \) would contain infinitely many halfplanes and thus would not have locally finite area in \( \mathbb{R}^{m+1} \setminus \Gamma \).

**Definition 15.2.** Consider two distinct \( m \)-dimensional linear subspaces \( P \) and \( P' \) of \( \mathbb{R}^{m+1} \). The closure of a component of \( \mathbb{R}^{m+1} \setminus (P \cup P') \) is called a **wedge**, and \( P \cap P' \) is the **edge** of the wedge.

**Corollary 15.3.** Suppose \( t \in (-\infty, 0) \mapsto M(t) \) is a nontrivial \( m \)-dimensional self-similar, integral Brakke flow (without boundary) in \( \mathbb{R}^{m+1} \). Then \( \text{spt}(M(-1)) \) is not contained in any wedge.

**Proof.** If \( \text{spt}(M(-1)) \) were contained in such a wedge \( W \), then by Theorem 15.1 it would be a union of half-planes in \( W \), which is impossible. \[ \square \]

**Corollary 15.4.** Let \( V \) be a stationary integral \( m \)-varifold in \( \mathbb{R}^{m+1} \setminus \Gamma \) that is invariant under positive dilations about 0. Suppose there is an open halfplane with boundary \( \Gamma \) that is disjoint from the support of \( V \). Then \( V \) is a sum of halfplanes with multiplicities.

This is the special case of Theorem 15.1 when the shrinker is a minimal cone.

**Remark 15.5.** Theorem 15.1 and Corollary 15.3 remain true (with the same proofs) if the hypothesis that \( M(\cdot) \) is an integral Brakke flow is replaced by the hypothesis that \( M(\cdot) \) is a Brakke flow such that the density of \( M(-1) \) is \( \geq a > 0 \) almost everywhere (with respect to \( M(-1) \)). Likewise Corollary 15.4 remains true for any stationary varifold \( V \) such that \( \Theta(V, \cdot) \geq a > 0 \) holds \( \mu_V \) almost everywhere.
16. THE WEDGE THEOREM AND BOUNDARY REGULARITY

Theorem 16.1 (Wedge Theorem). Suppose $W$ is a wedge (see Definition 15.2) in $\mathbb{R}^{m+1}$ with edge $\Gamma$. Suppose

$$t \in (-\infty, 0) \mapsto S(t)$$

is a self-similar, standard Brakke flow in $W$ with boundary $\Gamma$. Then $S(\cdot)$ is a non-moving halfplane with multiplicity 1.

Proof. Let $M = S(-1)$. By Theorem 15.1 $M = \sum_{i=1}^k P_i$ where each $P_i$ is a multiplicity-one halfplane with boundary $\Gamma$. (The $P_i$ need not be distinct, since a halfplane in the support of $M$ is allowed to have any positive integer multiplicity.) Since $\partial [M] = [\Gamma]$, $k$ is odd.

For each $i$, let $v_i$ be the unit vector in the plane of $P_i$ that is normal to $\Gamma$ and that points out from $P_i$. For any smooth, compactly supported vectorfield $X$,

$$\int \text{Div}_{P_i} X \, dP_i = \int_{\Gamma} X \cdot v_i \, d\mathcal{H}^{m-1},$$

so

$$\int \text{Div}_M X \, dM = \int_{\Gamma} \left( \sum_{i=1}^k v_i \right) \cdot X \, d\mathcal{H}^{m-1}.$$

Thus

$$v(M, \cdot) = \sum_{i=1}^k v_k.$$

By definition of mean curvature flow with boundary, $|v(M, \cdot)| \leq 1$. Now we use the following elementary fact: if $v' = \sum_{i=1}^k v_i$ where the $v_i = (\cos \theta_i, \sin \theta_i)$ are unit vectors with $|\theta_i| \leq \theta < \pi/2$, then

$$(16.1) \quad |v'| \geq \begin{cases} k \cos \theta & \text{if } k \text{ is even,} \\ \sqrt{1 + (k^2 - 1) \cos^2 \theta} & \text{if } k \text{ is odd.} \end{cases}$$

(The inequalities (16.1) can be proved as follows. Given $k$ and $\theta$, it is easy to show that at the minimum of $|v'|$, each $\theta_i$ is $\pm \theta$. If $k$ is even, the minimum is attained by having half of the $\theta_i$ equal to $\theta$ and the other half equal to $-\theta$. If $k = 2j + 1$ is odd, the minimum is attained when $j$ of the $\theta_i$ are equal to $\theta$ and $j + 1$ are equal to $-\theta$.)

In our case, the number of planes (counting multiplicity) in $M'$ is odd, so

$$1 \geq |v'| \geq \sqrt{1 + (k^2 - 1) \cos^2 \theta}.$$ 

Therefore $k = 1$. □

As an immediate consequence of Theorem 16.1 we have

Theorem 16.2. Suppose $t \in [0, T] \mapsto M(t)$ is an $m$-dimensional standard mean curvature flow with boundary $\Gamma$ in a smooth, $(m+1)$-dimensional Riemannian manifold. If a tangent flow at $(p, t)$ is contained in a wedge, where $p \in \Gamma$ and $t > 0$, then $(p, t)$ is a regular point of the flow.
17. A Boundary Regularity Theorem

**Theorem 17.1.** Suppose $N$ is a smooth, $(m+1)$-dimensional Riemannian manifold with smooth, weakly mean-convex boundary. Suppose $\Gamma$ is a smooth, properly embedded $(m-1)$-dimensional submanifold of $N$. Suppose $t \in [0, T] \mapsto M(t)$ is a standard Brakke flow in $N$ with boundary $\Gamma$. If

1. $\partial N$ is strictly mean convex, or if
2. $\text{spt}(M(0)) \cap \partial N = \Gamma \cap \partial N$,

then for every $p$ in $\Gamma \cap \partial N$ and for every $t \in (0, T]$ the spacetime point $(p, t)$ is a regular point of the flow.

**Proof.** Note that if $\partial N$ is strictly mean convex, then by the maximum principle ([Ilm94, 10.5] or [HW18, theorem 27]),

$$\text{spt}(M(t)) \cap \partial N = \Gamma \cap \partial N$$

for all $t > 0$. In other words, as soon as $t > 0$, Hypothesis (2) holds. Thus it suffices to prove Theorem 17.1 under Hypothesis (2).

Since the result is local, it suffices to work in a small neighborhood of the point $p$. Such a neighborhood is diffeomorphic to a halfspace, so we may assume that

$$N = \{x \in \mathbb{R}^{m+1} : x_{m+1} \geq 0\}$$

with some smooth Riemannian metric $g$. We may also assume that $p$ is the origin and that the metric is Euclidean at the origin (i.e., that $g_{ij}(0) = \delta_{ij}$). In the rest of the proof, dist$(\cdot, \cdot)$ refers to $g$-distance, but $B^{m+1}(a, r)$ is the Euclidean ball of radius $r$ about $a$, and if $f$ is a function from a domain in $\mathbb{R}^m$ to $[0, \infty)$ (so that the graph lies in $N$), then expressions such as $\nabla f$ and $\|f\|_{C^2}$ are with respect to the Euclidean metric.

**Lemma 17.2.** For every $\epsilon > 0$, there is a $\delta > 0$ with the following property. If $a \in \partial N \cong \mathbb{R}^m$, $|a| \leq \delta$, $0 < r \leq \delta$, and

$$f : B^m(a, r) \rightarrow [0, \infty)$$

is a function with $\|f\|_{C^3} \leq \epsilon$, then there is a solution $F : B^m(a, r) \times [0, \infty) \rightarrow \mathbb{R}$ of the nonparametric mean curvature flow equation (with respect to the metric on $N$) such that

$$F(\cdot, 0) = f(\cdot),$$

$$F(\cdot, t)|\partial B^m(a, r) = f|\partial B^m(a, r),$$

and such that $|\nabla F| < \epsilon$ at all points.
**Proof of lemma.** Let \( \epsilon > 0 \). If there were no suitable \( \delta > 0 \), there would be a sequence of solutions

\[
F_i : B^m(a_i, r_i) \times [0, T_i] \to [0, \infty)
\]

of the nonparametric \( g \)-mean-curvature flow equation with \(|a_i| \to 0\) and \( r_i \to 0 \) such that

\[
(17.1) \quad \|F_i(\cdot, 0)\|_{C^3} \to 0
\]

and such that

\[
\max|\nabla F_i| = |\nabla F_i(x_i, T_i)| = \epsilon.
\]

for some \( x_i \). Note that

\[T := \liminf \frac{T_i}{r_i^2} > 0\]

by, for example, the local regularity theory in [Whi05]. By passing to a subsequence, we can assume that \( T_i/r_i^2 \to T \).

Let \( D_i(t) \) be the graph of \( F_i(\cdot, t) \). By (17.1),

\[
\frac{\text{area}_g(D_i(0))}{\omega_m r_i^m} \to 1
\]

and thus

\[
(17.2) \quad \sup_{t \in [0, T_i]} \frac{\text{area}_g(D_i(t))}{\omega_m r_i^m} = \frac{\text{area}_g(D_i(0))}{\omega_m r_i^m} \to 1
\]

as \( i \to \infty \). Let \( c_i \) be the average of \( f \) over \( \partial B(a_i, r_i) \). Translating in space by \(- (a_i, c_i)\) and in time by \(- T_i\), dilating parabolically by \(1/r_i\), and passing to a subsequential limit gives a smooth solution

\[
F : B^m(0, 1) \times (-T, 0] \to \mathbb{R}
\]

of the Euclidean nonparametric mean curvature flow equation such that

\[
F(\cdot, t)|\partial B^m(0, 1) \equiv 0, \quad \text{and}
\]

\[
\sup|\nabla F| = \sup|\nabla F(\cdot, 0)| = \epsilon.
\]

But by (17.2), the area of the graph of \( F(\cdot, t) \) is less than or equal to area of \( B^m(0, 1) \), and thus \( F \equiv 0 \), a contradiction. This proves the lemma. \( \square \)

We now complete the proof of Theorem 17.1. Choose \( a \in \text{Tan}(\Gamma, 0) \perp \partial N \) with \( 0 < |a| < \delta \) (where \( \delta \) is as in Lemma 17.2 for \( \epsilon = 1 \)). We choose \( a \) sufficiently close to 0 that the Euclidean ball \( B^{m+1}(a, |a|) \) intersects \( \Gamma \) in the single point 0. Let \( 2R = |a| \).

Let \( f : \mathbb{R}^m \to [0, \infty) \) be a smooth function such that: \( f(a) > 0 \), \( f \) is supported in the interior of \( B^m(a, R) \), \( \|f\|_{C^3} < \delta \), and

\[
(17.3) \quad \text{spt} M(0) \setminus \Gamma \text{ lies in the set } \{x : x_{m+1} > f(x_1, \ldots, x_m)\}.
\]

For \( R \leq r \leq 2R \), let

\[
F_r : B^m(a, r) \times [0, T] \to \mathbb{R}
\]
be the solution of the nonparametric mean curvature flow equation (with respect to the metric $g$) such that

$$F_r(\cdot) = 0 \text{ on } \partial B^m(a, r), \text{ and}$$

$$F_r(\cdot, 0) = f(\cdot) \text{ on } B^m(a, r).$$

By choice of $a$,

$$|\nabla F_r(x, t)| \leq 1$$

for all $x \in B^m(a, r)$ and $t \in [0, T]$. It follows that the graph $D_r(t)$ of $F_r(\cdot, t)$ is contained in the ball $B^{m+1}(a, r)$, and thus

$$D_r(t) \cap \Gamma = \emptyset \quad \text{for all } r \in [R, 2R] \text{ and } t \in [0, T]. \quad (17.4)$$

By the strong maximum principle,

$$F_r(x, t) > 0 \text{ for all } x \text{ in the interior of } B(a, r) \text{ and all } t \in [0, T]$$

and

$$|\nabla F_r(x, t)| \neq 0 \text{ for all } x \in \partial B^m(a, r) \text{ and } t \in (0, T]. \quad (17.5)$$

Let

$$Q_r(t) = \{x : 0 < x^{m+1} < F_r(x_1, \ldots, x_m, t)\}.$$

We claim that

$$Q_r(t) \text{ is disjoint from spt } M(t) \text{ for } t \in [0, T]. \quad (17.6)$$

For suppose not. At the first time $t$ of contact, let $q$ be a point in spt $M(t) \cap Q_r(t)$. Then $q$ is in the graph $D_r(t)$ of $F_r(\cdot, t)$. If $q$ were in $\partial D_r(t)$, then the tangent flow to $M(\cdot)$ at $(q, t)$ would be contained in a wedge (by (17.5)), which is impossible (see Corollary 15.3). Thus

$$\partial D(t) \cap M(t) = \emptyset \quad \text{for all } t \in [0, T], \quad (17.7)$$

and $q \in D(t) \setminus \partial D(t)$. But this (together with (17.3) and (17.4)) violates the maximum principle (Theorem 20.1). This proves (17.6).

From (17.6), we see that

$$\text{spt } M(t) \cap Q_{2R}(t) = \emptyset \quad \text{since } \bigcup_{r < 2R} Q_r(t) = Q_{2R}(t). \quad (17.8)$$

Now let $t \in (0, T]$. By (17.8) and (17.5), any tangent flow to $M(\cdot)$ at $(p, t)$ must be contained in a wedge. Thus $(p, t)$ is a regular point of the flow $M(\cdot)$ by Theorem 16.2. \hfill \Box

18. MOVING BOUNDARIES

Let $I$ be an interval in $\mathbb{R}$. We say that $\Gamma$ is a **moving** $(m - 1)$-**dimensional boundary** in $U \times I$ if $\Gamma$ is a smooth, properly embedded, $m$-dimensional manifold-with-boundary in $U \times I$ such that the boundary of $\Gamma$ is

$$\Gamma \cap (U \times \partial I)$$

and such that the time function $(x, t) \in \Gamma \mapsto t$ has no critical points on $\Gamma$. For $t \in I$, we let $\Gamma(t) = \{x : (x, t) \in \Gamma\}$. For $(x, t) \in \Gamma$, we let $\hat{\Gamma}(x, t)$ be the normal velocity of $\Gamma(t)$ at $x$: it is the
unique vector \( v \in \Tan(\Gamma(t), x)^\perp \) such that \((v, 1)\) is tangent to \(\Gamma\) at \((x, t)\). Note that each \(\Gamma(t)\) is a smooth, properly embedded \(m\)-dimensional submanifold of \(U\).

**Definition 18.1.** Let \(\Gamma \subset U \times I\) be a moving \((m - 1)\)-dimensional boundary. A **Brakke flow with (moving) boundary** \(\Gamma\) is a Borel map

\[
t \in I \mapsto M(t) \in \mathcal{M}(U)
\]

such that

1. For almost every \(t \in I\), \(M(t)\) is in \(\mathcal{V}_m(U, \Gamma(t))\).
2. If \([a, b] \subset I\) and \(K \subset U\) is compact, then

\[
\int_a^b \int_K (1 + |H|^2) dM(t) dt < \infty.
\]
3. If \([a, b] \subset I\) and if \(u\) is a nonnegative, compactly supported \(C^2\) function on \(U \times [a, b]\), then

\[
(Mu)(a) - (Mu)(b) \geq \int_a^b \mathcal{H}^{m-1}(\{u > 0\}) dt
\]

**Theorem 18.2.** Suppose \(t \in I \mapsto M(t)\) is a Brakke flow with moving boundary \(\Gamma\).

1. The defining inequality [3] in Definition [18.1] holds for every nonnegative, compactly supported, Lipschitz function \(u\) on \(U\) that is \(C^1\) on \(\{u > 0\}\).

2. If

\[
f : U \times [a, b] \to \mathbb{R}
\]

is a nonnegative, \(C^2\) function with \(\{f > 0\}\) compact, and if \(u := 1_{f > 0} f\), then

\[
(Mu)(a) - (Mu)(b) \geq \int_a^b \left( u |H|^2 - H \cdot \nabla u - \frac{\partial u}{\partial t} \right) dM(t) dt + \int_a^b \int_\Gamma u \cdot \dot{\Gamma} d\Gamma dt.
\]

3. Suppose that

\[
\overline{B(x, R)} \text{ is a compact subset of } U,
\]
\[
\text{dist}(\cdot, x)^2 \text{ is smooth on } \overline{B(x, R)},
\]
\[
\text{Trace}_m(-\nabla^2(\text{dist}(\cdot, x)^2)) \geq -4m \text{ on } \overline{B(x, R)}, \text{ and}
\]
\[
|\dot{\Gamma}| \leq \delta \text{ on } \Gamma \cap (\overline{B(x, R)} \times [0, T]).
\]

Let \(u = (R^2 - \text{dist}(\cdot, x)^2 - 4mt)^+\). Then for \(t \in [0, T]\),

\[
(Mu)(t) \leq (Mu)(0) + t(R + \delta R^2).\mathcal{H}^{m-1}(\Gamma \cap \overline{B(x, R)}).
\]
(4) Let \( u \) be a \( C^2 \), nonnegative, compactly supported function on \( U \).

\[
\frac{1}{2} \int_a^b \int u|H|^2 \, dM(t) \, dt \leq M(a)u - M(b)u \\
+ (b - a)\max|\nabla^2 u|K_{[a,b]} \\
+ (b - a)L_{[a,b]}
\]

for \( [a, b] \subset I \), where

\[
K_{[a,b]} := \sup_{t \in [a,b]} M(t)(spt u),
\]

\[
L_{[a,b]} := \sup_{t \in [a,b]} \int u|\dot{\Gamma}| \, d\Gamma(t).
\]

The proofs are almost identical to the proofs of Proposition 5.2, Corollary 5.4, Theorem 5.5, and Theorem 5.6.

**Theorem 18.3.** Let \( U \) be an open subset of \( \mathbb{R}^d \) containing \( B^d(0, 1) \). Let \( N \) be a smooth, properly embedded submanifold of \( U \). Let \( \Gamma \) be an \((m - 1)\)-dimensional moving boundary in \( N \times [T_0, 0] \). Let \( t \in [T_0, 0] \mapsto M(t) \) be an integral Brakke flow in \( N \) with moving boundary \( \Gamma \). Suppose that

\[
M(t)B(0, 1) \leq C
\]

for \( t \in I \) and that the norm of the second fundamental form of \( N \) is bounded by \( A \). Then for \( a, b \in [T_0, 0] \) with \( a < b < 0 \), then for \( a, b \in I \) with \( a < b < 0 \),

\[
(M\hat{\rho})(a) - (M\hat{\rho})(b) \geq \int_a^b \int \left| H_N - \frac{(\nabla^\perp \hat{\rho})_N}{\hat{\rho}} \right|^2 \hat{\rho} \, dM(t) \, dt \\
+ \int_a^b \int v_M \cdot (\nabla \hat{\rho} - \hat{\rho} \dot{\Gamma}) \, d\Gamma \, dt \\
- mA^2 \int_a^b \int \hat{\rho} \, dM(t) \, dt \\
- CK(b - a)
\]

(18.1)

where \( C \) and \( K \) are as in (6.1) and (6.3). Furthermore,

\[
e^{-mA^2 t} \left( (M\hat{\rho})(t) + \int_{\tau = T_0}^t \int v_M \cdot (\nabla \hat{\rho} - \hat{\rho} \dot{\Gamma}) \, d\Gamma \, d\tau - CKt \right)
\]

is a decreasing function of \( t \) for \( t < 0 \) in \( I \).

**Proof.** The proof is exactly like the proof of the proof of the Monotonicity Theorem 6.1, except that, starting with the right hand side of the first inequality in (6.5), there is one additional extra term:

\[
- \int_a^b \int \hat{\rho} \, \cdot \, v \, d\Gamma \, dt.
\]

\[\square\]

**Corollary 18.4.** As \( t \uparrow 0 \), \((M\hat{\rho})(t)\) converges to a finite limit.
Proof. A straightforward computation shows that

\[
\int_0^1 \int_{\Gamma_t} |(\nabla \hat{\rho} - \hat{\rho} \hat{\Gamma})_{\hat{\Gamma}^\perp} | \, d\Gamma \, dt < \infty.
\]

The corollary now follows immediately from the monotonicity of (18.2). \qed

The compactness and closure theorems, existence of tangent flows, and the definition of standard flows are the exact analogs of the corresponding theorems and definition for fixed boundaries (§10, §11, §13) so we will not state them. For example, in the statement of the Compactness Theorem [10.1] one simply replaces “Brakke flow with boundary \( \Gamma \)” by “Brakke flow with moving boundary \( \Gamma \)” and “smooth, properly embedded \((m - 1)\)-dimensional submanifold \( \Gamma \) of \( U \)” by “moving \((m - 1)\)-dimensional boundary \( \Gamma \) in \( U \times [0, T] \)”. Likewise, in the statement of Theorem [10.2] one simply replaces “smooth, \((m - 1)\)-dimensional submanifolds of \( U \)” by “moving \((m - 1)\)-dimensional boundaries in \( U \times [0, T] \)” and “Brakke flow with boundary” by “Brakke flow with moving boundary”.

For those various theorems, the extra term arising from the motion of boundary is easy to control, so only trivial modifications of the proofs are required.

Just as for fixed boundaries, we have (as an immediate consequence of the Wedge Theorem [16.1]),

**Theorem 18.5.** Suppose that \( M : [0, T] \mapsto M(t) \) is an \( m \)-dimensional standard Brakke flow with moving boundary \( \Gamma \) in an \((m + 1)\)-dimensional Riemannian manifold. If \( t > 0 \), if \( p \in \Gamma(t) \), and if a tangent flow at \((p, t)\) is contained in a wedge, then \((p, t)\) is a regular point of the flow \( M(\cdot) \).

Furthermore, the main boundary regularity theorem, Theorem [17.1] continues to hold for moving boundaries. The statement and the proof are almost exactly as in the non-moving versions, so we omit them.

Finally, the Existence Theorem [14.1] in Section [14] has an analog for moving boundaries:

**Theorem 18.6.** Let \( N \) be a smooth Riemannian manifold. If \( N \) has nonempty boundary, we assume that the boundary is smooth and \( m \)-convex. Suppose that \( \Gamma \) is a moving \((m - 1)\)-dimensional boundary in \( N \times [0, \infty) \) such that

\[
\bigcup_{0 \leq t \leq T} \Gamma(t) \text{ is compact for every } T < \infty.
\]

Let \( M_0 \) be a smoothly embedded \( m \)-dimensional manifold in \( N \) with moving boundary \( \Gamma(0) \) and with finite area, or, more generally, let \( M_0 \) be an \( m \)-rectifiable set of finite \( m \)-dimensional measure such that \( \partial[M_0] = [\Gamma] \). Then there exists a standard Brakke flow

\[
t \in [0, \infty) \mapsto M(t)
\]

with moving boundary \( \Gamma \) such that

\[
M(t) \to \mathcal{H}^m \setminus M_0
\]

as \( t \to 0 \).
If \( p \notin \Gamma(0) \) and if \( M_0 \) is smooth in a neighborhood of \( p \), then the flow is smooth in a spacetime neighborhood of \((p,0)\). If \( p \in \Gamma(0) \) and if \( M_0 \) is \( C^1, a \)-in a neighborhood of \( p \), then the flow is parabolically \( C^1 \) in a spacetime neighborhood of \((p,0)\).

**Sketch of proof.** It is convenient to first prove the Theorem under the additional assumption that there is a \( T < \infty \) such that

\[
\Gamma(t) = \Gamma(T) \quad \text{for all } t \geq T.
\]

With this assumption, the proof is almost identical to the proof of Theorem 14.1. (The compactness hypothesis (18.4) together with the assumption (18.5) guarantee that the analog of \( \Sigma^A \) in the proof of Theorem 14.1 has finite area with respect to the Ilmanen metric \( e^{-2hz/m} (g + dz^2) \).

For the general case, one takes a sequence \( \Gamma^n \) of moving boundaries that \( \Gamma^n(t) = \Gamma(t) \) for \( 0 \leq t \leq n \),

\[
\Gamma^n(t) = \Gamma^n(n + 1) \quad \text{for } t \geq n + 1,
\]

and such that the compactness hypothesis (18.4) holds for each \( \Gamma^n \). By the special case of the Theorem, there exists a standard Brakke flow \( \mathcal{M}^n \) with moving boundary \( \Gamma^n \) satisfying the conclusions of the Theorem. By the moving boundary analogs of the Compactness Theorem 10.1 and the Closure Theorem 10.2, a subsequence of the flows \( \mathcal{M}^n \) will converge to a standard Brakke flow \( \mathcal{M} \) satisfying the conclusions of the Theorem.

---

19. Orientation

In this section, we show that a certain weak notion of orientability is preserved when taking weak limits of flows.

Suppose \( \Gamma \) is an \((m - 1)\)-dimensional submanifold of \( U \). An **orientation** of \( \Gamma \) is a continuous function \( \gamma \) that assigns to each \( x \in \Gamma \) a unit \((m - 1)\)-vector in \( \text{Tan}(\Gamma, x) \). The pair \((\Gamma, \gamma)\) is called an **oriented submanifold** of \( U \).

If \( \Gamma \) is a moving \((m - 1)\)-dimensional boundary in \( U \times I \), an orientation on \( \Gamma \) is a continuous function that assigns to each \((x, t) \in \Gamma \) a unit \((m - 1)\)-vector in \( \text{Tan}(\Gamma(t), x) \).

Now suppose that \((\Gamma, \gamma)\) is an \((m - 1)\)-dimensional, smooth, oriented submanifold of \( U \) and that \( M \in \mathcal{I}M_{m}(U) \). We say that \( M \) is **boundary-compatible** with \((\Gamma, \gamma)\) provided there is an \( m \)-dimensional, integer-multiplicity rectifiable current \( A \) such that

1. \( \partial A \) is the current given by \( \Gamma \) with orientation \( \gamma \) and multiplicity 1.
2. \( M = \mu_A + 2Z \) for some \( Z \in \mathcal{I}M_{m}(U) \).

Condition (2) is equivalent to

\[
\Theta(M, x) - \Theta(\mu_A, x) \text{ is a nonnegative even integer for } \mathcal{H}^m \text{ almost every } x.
\]

It is also equivalent to:

\[ \mu_A \leq M, \text{ and } A \text{ and } M \text{ determine the same flat chain mod 2.} \]

Note that if \( M \) is boundary compatible with \((\Gamma, \gamma)\), then \( \partial[M] = [\Gamma] \).

The following theorem is an immediate consequence of Theorem 1.2 in [Whi09]:

---
Theorem 19.1. Suppose that \((\Gamma_i, \gamma_i)\) \((i = 1, 2, \ldots)\) and \((\Gamma, \gamma)\) are smooth oriented \((m - 1)\)-dimensional submanifolds of \(U\) such that the \((\Gamma_i, \gamma_i)\) converge weakly (i.e., as rectifiable currents) to \((\Gamma, \gamma)\). Suppose that \(M_i\) \((i = 1, 2, \ldots)\) and \(M\) are Radon measures in \(\mathcal{M}_m(U)\) with the following properties:

1. \(\text{Var}(M_i) \to \text{Var}(M)\).
2. Each \(M_i\) has bounded first variation, and
   \[q_K := \sup_i \left( \int_K |H(M_i, \cdot)|\ dM_i + \beta(M_i)(K) \right) < \infty.\]
3. Each \(M_i\) is boundary-compatible with \((\Gamma_i, \gamma_i)\).

Then \(M\) is boundary-compatible with \((\Gamma, \gamma)\).

Definition 19.2. Let \((\Gamma, \gamma)\) be an oriented \((m - 1)\)-dimensional moving boundary in \(U \times I\). An oriented Brakke flow \((\Gamma, \gamma)\) is a standard mean curvature flow
\[t \in I \mapsto M(t)\]
with boundary \(\Gamma\) such that for almost every \(t \in I\), \(M(t)\) is boundary-compatible with \((\Gamma(t), \gamma(t))\).

Theorem 19.3. Suppose that

1. \((\Gamma_i, \gamma_i)\) \((i = 1, 2, \ldots)\) and \((\Gamma, \gamma)\) are moving \((m - 1)\)-dimensional boundaries in \(U \times I\).
2. The \((\Gamma_i, \gamma_i)\) converge smoothly to \((\Gamma, \gamma)\).
3. For \(i = 1, 2, \ldots\), the map \(t \in I \mapsto M_i(t)\) and is an oriented Brakke flow with moving boundary \((\Gamma_i, \gamma_i)\).
4. \(M_i(t)\) converges to \(M(t)\) for all \(t\).

Then \(t \mapsto M(t)\) is an oriented Brakke flow with boundary \((\Gamma, \gamma)\).

Roughly speaking, Theorem 19.3 says that the class of oriented Brakke flows is closed under taking weak limits.

Proof. By the moving boundary version (see §19) of Theorem 13.1, \(t \mapsto M(t)\) is a standard Brakke flow with boundary \(\Gamma\). By the moving boundary version of Theorem 10.2, for almost every \(t\), there is a subsequence \(M_{i(j)}(t)\) such that

\[\text{Var}(M_{i(j)}(t)) \to \text{Var}(M(t))\]

and

\[\sup_j \left( \int_K |H_{i(j)}(t, \cdot)|\ dM_{i(j)}(t) + \beta(M_{i(j)})(K) \right) < \infty \quad \text{for all } K \subset U.\]

Thus Hypotheses 1 and 2 of Theorem 19.1 hold for the sequence \(M_{i(j)}(t)\). Since \(M_i(t)\) is boundary-compatible with \((\Gamma_i(t), \gamma_i(t))\) for almost every \(t\), Hypothesis 3 of Theorem 19.1 holds. Thus the conclusion holds: \(M(t)\) is boundary-compatible with \((\Gamma(t), \gamma(t))\) for almost every \(t\). \(\square\)

Theorem 19.4. Let \(N\) be a smooth Riemannian manifold. If \(N\) has nonempty boundary, we assume that the boundary is smooth and \(m\)-convex. Let \((\Gamma, \gamma)\) be smooth, properly embedded, oriented \((m - 1)\)-dimensional manifold in \(N\). Let \(M_0\) be an oriented, smoothly embedded \(m\)-dimensional manifold in \(N\) with oriented boundary \((\Gamma, \gamma)\). More generally, \(M_0\) can
be a m-rectifiable set of finite m-dimensional Hausdorff measure for which there is a Borel-measurable orientation of the tangent planes to $M_0$ such that the resulting multiplicity-one current has boundary given by $(\Gamma, \gamma)$. Then there exists an oriented Brakke flow

$$t \in [0, \infty) \mapsto M(t)$$

with boundary $(\Gamma, \gamma)$ such that

$$M(t) \to \mathcal{H}^m \downarrow M_0$$

as $t \to 0$, and such that $M(t)$ is boundary-compatible with $(\Gamma, \gamma)$ for almost every $t$.

If $p \notin \Gamma$ and if $M_0$ is smooth in a neighborhood of $p$, then the flow is smooth in a spacetime neighborhood of $(p, 0)$. If $p \in \Gamma$ and if $M_0$ is $C^{1,\alpha}$ in a neighborhood of $p$, then the flow is parabolically $C^1$ in a spacetime neighborhood of $(p, 0)$.

The proof is exactly like the proof of Theorem 14.1, except that one uses integral currents instead flat chains mod 2 in the elliptic regularization construction, and one uses Theorem 19.3 when passing to the limit.

Theorem 19.4 generalizes in a straightforward way to moving boundaries. As in Theorem 18.6, one adds the extra compactness hypothesis 18.4 for the reason explained in the proof of that theorem.

20. Appendix: A Strong Maximum Principle

**Theorem 20.1.** Suppose that $t \in [0, T] \mapsto M(t)$ is an integral Brakke flow with moving boundary $\Gamma$ in a smooth Riemannian manifold $N$. Suppose that $t \in [0, T] \mapsto D(t)$ is a smooth, one-parameter family of compact, smoothly embedded, m-dimensional manifolds with boundary in $N$ that are moving by mean curvature. (In other words, the normal velocity at each point is equal to the mean curvature vector at that point.) If

- $D(0)$ is disjoint from $\text{spt}(M(0))$,
- $\partial D(t)$ is disjoint from $\text{spt} M(t)$ for all $t \in [0, T]$, and
- $\Gamma(t)$ is disjoint from $D(t)$ for all $t \in [0, T]$,

then $D(t)$ is disjoint from $\text{spt}(M(t))$ for all $t \in [0, T]$.

**Proof.** Choose $\delta > 0$ small enough so that

$$Q := \bigcup_{t \in [0, T]} \{ x \in N : \text{dist}(x, D(t)) \leq \delta \}$$

is compact, and so that for $t \in [0, T]$,

$$\text{dist}(p, q) > \delta \quad (20.1)$$

for all $p \in \Gamma(t)$ and $q \in D(t)$, and for all $p \in \text{spt} M(t)$ and $q \in \partial D(t)$.

Let $\lambda < 0$ be a strict lower bound for the Ricci curvature of $N$ in the set $Q$. We claim that

$$e^{-\lambda t} \text{dist}(D(t), \text{spt} M(t)) > \delta \quad (20.2)$$

for all $t \in [0, T]$. For if not, there would be first time $t > 0$ such that

$$e^{-\lambda t} \text{dist}(D(t), \text{spt} M(t)) = \delta.$$
At that time, there would be an arc-length parametrized geodesic
\[ \gamma : [0, L] \to N \]
such that
\[ \gamma(0) \in D(t), \quad \gamma(L) \in \text{spt} M(t), \quad \text{and} \]
\[ L = \text{dist}(D(t), \text{spt} M(t)) \]
Note that \( \gamma(0) \notin \partial D(t) \) and \( \gamma(L) \notin \Gamma(t) \) by (20.1) and (20.2). (Recall that \( \lambda < 0 \)). By [HW18, Lemma 11] and [HW18, Theorem 28], at the point \( \gamma(0) \in D(t) \), the surface \( D(\cdot) \) moves in the direction \( \gamma'(0) \) faster than the mean curvature in that direction, a contradiction. \( \square \)

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