On the continuous dependence of the minimal solution of constrained backward stochastic differential equations

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Abstract

It is well-known that solutions of backward differential equations are continuously dependent on the terminal value. Since the increasing part of the minimal solution of a constrained backward differential equation (shortly CBSDE) varies against terminal value, the continuous dependence property of terminal value is not obvious for it. In this paper, we obtain a result about this problem under some mild assumptions.

The main tool used here is the penalization method to get the minimal solution of a CBSDE and the property of convex functional that it is continuous when it is lower semi-continuous. The comparison theorem of the minimal solution of CBSDE plays a crucial role in our proof.

Keywords: CBSDE, continuous dependence, convex functional, minimal solution

1 Introduction

It is well-known that g-solution and g-supersolution are both continuously dependent on the terminal value, see El Karoui etc.[3] and Pardoux and Peng[5] for references. g-solution and g-supersolution are often used to price or hedge claims in mathematical finance. In practice, some constraints may put on the portfolio and wealth process, and the backward stochastic differential equation subjecting to some constraints comes up to our consideration. For such investigation on constrained backward stochastic equation, we refer to Cvitanic.J, Karatzas.I and Soner.H.M[1]. In their paper, the constraint is \( z(t) \in K \) for some convex set \( K \). Under the assumption that there exist at least one solution to such constrained BSDE, a penalization method can be used to get the minimal solution to this CBSDE. Similarly, Peng[6] pointed out that the smallest g-supersolution, which is denoted by \( \mathcal{E}^{g,t}_g(\xi) \), can be obtained by penalization method, i.e, it can be approximated by an increasing sequence of g-supersolutions suppose that for the square integrable terminal value \( \xi \in L^2_T(\mathbb{R}) \), there exist at least one g-supersolution \((y_t, z_t, C_t)\) satisfying the constraint equation \( \phi(t, y_t, z_t) = 0 \). In these case, it is obvious that the associated increasing part \( C_t \) of the minimal solution of CBSDE varies against terminal value, the usual priori estimation of g-solution or g-supersolution does not work for the continuous dependence property. In this paper, we mainly investigate such a problem in the framework of Peng[6] when both \( g \) and \( \phi \) are convex. The case \( z \in K \) can be concluded in this case taking \( \phi(z) = d(z, K) \), the distance function from \( z \) to the convex set \( K \). It is obvious convex in \( z \).

This paper is organized as follows: In section 2, we state the framework in Peng[6] and some propositions on the smallest g-supersolution, namely the minimal solution of CBSDE with constraint as \( \phi(t, y_t, z_t) = 0 \).

Under the assumption that when the generator \( g \) and constraint function \( \phi \) are both convex, we obtain the continuous dependence property of \( \mathcal{E}^{g,t}_g(\xi) \) \( 0 \leq t \leq T \) in terms of \( \xi \in L^2_T(\mathbb{R}) \) in section 3.

2 BSDE and the minimal solution of CBSDE

Given a probability space \((\Omega, \mathcal{F}, P)\) and \(\mathbb{R}^d\)-valued Brownian motion \( W(t) \), we consider a sequence \( \{(\mathcal{F}_t); t \in [0, T]\} \) of filtrations generated by Brownian motion \( W(t) \) and \( \mathcal{P} \) is the \( \sigma \)-field of predictable sets of \( \Omega \times [0, T] \). We use \( L^2_T(\mathbb{R}^d) \) to denote the space of all \( F_T \)-measurable random variables \( \xi : \Omega \to \mathbb{R}^d \) for which

\[
\| \xi \|^2 = E[|\xi|^2] < +\infty.
\]

and use \( H^2_T(\mathbb{R}^d) \) to denote the space of predictable process \( \varphi : \Omega \times [0, T] \to \mathbb{R}^d \) for which

\[
\| \varphi \|^2 = E[\int_0^T |\varphi|^2] < +\infty.
\]

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The backward stochastic differential equation (shortly BSDE) driven by \( g(t, y, z) \) is given by
\[
-dy_t = g(t, y_t, z_t)dt - z_t^*dW(t)
\]
where \( y_t \in \mathbb{R} \) and \( W(t) \in \mathbb{R}^d \). Suppose that \( \xi \in L^2_t(\mathbb{R}) \), \( g(\cdot, 0, 0) \in H^2_t(\mathbb{R}) \) and \( g \) is uniformly Lipschitz; i.e., there exists \( M > 0 \) such that \( dP \otimes dt \) a.s.
\[
|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq M(|y_1 - y_2| + |z_1 - z_2|), \quad \forall (y_1, z_1), (y_2, z_2)
\]
Pardoux and Peng [5] proved the existence of adapted solution \((y(t), z(t))\) of such BSDE. We call \((g, \xi)\) standard parameters for the BSDE.

In this paper, we assume \( g(\omega, t, y, z) \) are both convex in \((y, z)\).

**Definition 2.1. (super-solution)** A super-solution of a BSDE associated with the standard parameters \((g, \xi)\) is a vector process \((y_t, z_t, C_t)\) satisfying
\[
-dy_t = g(t, y_t, z_t)dt + dC_t - z_t^*dW(t), \quad y_T = \xi,
\]
or being equivalent to
\[
y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s^*dW_s + \int_t^T dC_s,
\]
where \((C_t, t \in [0, T])\) is an increasing, adapted, right-continuous process with \( C_0 = 0 \) and \( z_t^* \) is the transpose of \( z_t \). When \( C_t \equiv 0 \), we call \((y_t, z_t)\) a g-solution.

In this paper, we consider g-supersolutions \((y_t, z_t, C_t)\) satisfying the constraint
\[
\phi(t, y_t, z_t) = 0,
\]
where \( \phi(t, y, z) : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \). In such case, we give the following definition,

**Definition 2.2. (The smallest g-supersolution or the minimal solution)** A g-supersolution \((y_t, z_t, C_t)\) is said to be the smallest g-supersolution or the minimal solution, given \( y_T = \xi \), subjecting to the constraint (2.3) if for any other g-supersolution \((y'_t, z'_t, C'_t)\) satisfies (2.3) with \( y'_T = \xi \), we have \( y_t \leq y'_t \) a.e., a.s., the smallest g-supersolution is denoted by \( E_t^{g, \phi}(\xi) \).

For any \( \xi \in L^2_t(\mathbb{R}) \), we denote \( H(\phi)(\xi) \) as the set of g-supersolutions \((y_t, z_t, C_t)\) subjecting to (2.3) with \( y_T = \xi \). When \( H(\phi)(\xi) \) is not empty, Peng[6] proved that the smallest g-supersolution exists for \( \xi \in L^2_t(\mathbb{R}) \). In this paper, for simplicity, we first consider the continuous dependence property of \( E_t^{g, \phi}(\xi) \) at \( t = 0 \).

The convexity of \( E_t^{g, \phi}(\xi) \) can be easily deduced from the same propositions of g-solutions or g-supersolutions when both \( g \) and \( \phi \) are convex.

**Proposition 2.1.** Let \( \phi(t, y, z) \) be a function: \([0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \). If \( \phi(t, y, z) \) is uniformly Lipschitz and convex in \((y, z)\), then under the assumption that \( g(t, y, z) \) is also uniformly Lipschitz and convex in \((y, z)\) and \( g(\cdot, 0, 0), \phi(\cdot, 0, 0) \in H^2_t(\mathbb{R}) \), we have
\[
E_t^{g, \phi}(a\xi + (1 - a)\eta) \leq aE_t^{g, \phi}(\xi) + (1 - a)E_t^{g, \phi}(\eta) \quad \forall t \in [0, T]
\]
for any \( \xi, \eta \in L^2_t(\mathbb{R}) \) and \( a \in [0, 1] \).

**Proof** According to Peng[6], the solutions \( y_t^m(\xi) \) of
\[
y_t^m(\xi) = \xi + \int_t^T g(y_s^m(\xi), z_s^m, s)ds + A_t^m - A_t - \int_t^T z_s^m dW_s.
\]
is an increasing sequence and converges to \( E_t^{g, \phi}(\xi) \), where
\[
A_t^m := m \int_0^t \phi(y_s^m, z_s^m, s)ds.
\]

For any fixed \( m \), by the convexity of \( g \) and \( \phi \), \( y_t^m(\xi) \) is a convex in \( \xi \), that is
\[
y_t^m(a\xi + (1 - a)\eta) \leq a y_t^m(\xi) + (1 - a)y_t^m(\eta),
\]
taking limit as \( m \to \infty \), we get the required result.

By the same method of penalization, we can get the comparison theorem of \( \mathcal{E}_{t}^{\phi}(\xi) \).

**Proposition 2.2.** Let \( \phi(t, y, z) \) be a function: \([0, T] \times R \times R^{d} \to R^{+}\), then under the same assumptions as above proposition, we have

\[
\mathcal{E}_{t}^{\phi}(\xi) \leq \mathcal{E}_{t}^{\phi}(\eta)
\]

for any \( \xi, \eta \in L_{T}^{2}(R) \) when \( P(\eta \geq \xi) = 1 \).

### 3 Continuous dependence of the minimal solution with terminal value

Under the same assumptions as last section, for any \( k \in R \), we define the \( k \)-level set of \( \mathcal{E}_{t}^{\phi}(\xi) \) as

\[
A_{k} \triangleq \{ \xi \in L_{T}^{2}(R) | \mathcal{E}_{t}^{\phi}(\xi) \leq k \}.
\]

For such sets, we have the following result.

**Lemma 3.1.** For any \( k \in R \), the set \( A_{k} \) is closed in \( L_{T}^{2}(R) \)-norm.

**Proof** Suppose a sequence \( \{ \xi_{n}, n = 1, 2 \cdots \} \subset A_{k} \) converges under norm to some \( \xi \in L_{T}^{2}(R) \). For any \( \xi_{n} \), we take \( y_{0}^{m}(\xi_{n}) \) as in proposition 2.1. Since \( y_{0}^{m}(\xi_{n}) \) converges increasingly to \( \mathcal{E}_{t}^{\phi}(\xi_{n}) \leq k \) as \( m \to \infty \), \( y_{0}^{m}(\xi_{n}) \leq k \) holds for any \( n \) and \( m \).

For any fixed \( m \), take \( g_{m} = g + m \phi \), by the continuous dependence property of \( g_{m}\)-solution, we have \( y_{0}^{m}(\xi_{n}) \to y_{0}^{m}(\xi) \) as \( n \to \infty \) and \( y_{0}^{m}(\xi) \leq k \) is obtained for any \( m \). Again, for the fixed \( \xi \in L_{T}^{2}(R) \), \( y_{0}^{m}(\xi) \to \mathcal{E}_{t}^{\phi}(\xi) \) as \( m \to \infty \). Thus one has \( \mathcal{E}_{t}^{\phi}(\xi) \leq k \), this means \( A_{k} \) is closed under norm in \( L_{T}^{2}(R) \).

By lemma 3.1, \( \varphi(\xi) = \mathcal{E}_{t}^{\phi}(\xi) \) is lower semi-continuous on its domain of definition, but it is well-known that a convex functional is continuous if and only if it is lower semi-continuous, then we have the following theorem.

**Theorem 3.1.** Suppose the generator function \( g(t, y, z) \) and constraint function \( \phi(t, y, z) \) are both convex and uniformly Lipschitz in \((y, z)\), \( g(\cdot, 0, 0)\phi(\cdot, 0, 0) \in H_{T}^{2}(R) \), then \( \mathcal{E}_{t}^{\phi}(\xi) \) is a continuous functional in its domain of definition.

Next we want to investigate the continuous dependence property of \( \mathcal{E}_{t}^{\phi}(\xi) \) when \( t \neq 0 \). For this aim, we suppose \( \{ \xi_{n}, n = 1, 2 \cdots \} \) is a sequence in the domain of definition of \( \mathcal{E}_{t}^{\phi}(\xi) \).

**Theorem 3.2.** Suppose the generator function \( g(t, y, z) \) and constraint function \( \phi(t, y, z) \) are both convex and uniformly Lipschitz in \((y, z)\), \( g(\cdot, 0, 0)\phi(\cdot, 0, 0) \in H_{T}^{2}(R) \). Then for random variables \( \xi_{n}, \xi, \xi \in L_{T}^{2}(R) \), we have

\[
E[|\mathcal{E}_{t}^{\phi}(\xi_{n}) - \mathcal{E}_{t}^{\phi}(\xi)|^{2}] \to 0
\]

if \( E[|\xi_{n} - \xi|^{2}] \to 0 \) as \( n \to \infty \).

**Proof** Let \( \xi_{n} = \xi_{n} \vee \xi, \xi_{n} = \xi_{n} \wedge \xi \), by assumption, \( \xi_{n}, \xi_{n} \in L_{T}^{2}(R) \) and they both converge to \( \xi \) in the space \( L_{T}^{2}(R) \) with norm.

First, we consider \( \alpha_{n} \triangleq E[|\mathcal{E}_{t}^{\phi}(\xi_{n}) - \mathcal{E}_{t}^{\phi}(\xi)|^{2}] \) and show that \( \alpha_{n} \to 0 \) as \( n \to \infty \).

In fact, for any fixed \( \xi \in L_{T}^{2}(R) \), by proposition 2.1, the functional \( \varphi(\eta) \triangleq E[|\mathcal{E}_{t}^{\phi}(\eta) - \mathcal{E}_{t}^{\phi}(\xi)|^{2}] \) is convex on the set \( K = \{ \eta \in L_{T}^{2}(R) | \eta \geq \xi \text{ a.s.} \} \). It is also easy to show by the same technique used in lemma 3.1 that \( \varphi(\eta) \) is lower semi-continuous on \( K \), thus \( \alpha_{n} \to 0 \) comes true.

Secondly, let \( \beta_{n} \triangleq E[|\mathcal{E}_{t}^{\phi}(\xi_{n}) - \mathcal{E}_{t}^{\phi}(\xi)|^{2}] \) and we can prove \( \beta_{n} \to 0 \) as \( n \to \infty \).

This time let us consider the functional \( \varphi(\eta) \triangleq E[|\mathcal{E}_{t}^{\phi}(\eta) - \mathcal{E}_{t}^{\phi}(\xi)|^{2}] \) on the convex set \( \hat{K} = \{ \eta \in L_{T}^{2}(R) | \eta \leq \xi \text{ a.s.} \} \). For any sequence \( \{ \eta_{n} \in \hat{K}, n = 1, 2 \cdots \} \) which converges to \( \eta \in \hat{K} \), we can show that \( \varphi(\eta_{n}) \geq c \) if \( \varphi(\eta_{n}) \geq c \) for all \( n \). In fact, let \( y_{0}^{m}(\eta_{n}) \) be the approximating sequence of \( \mathcal{E}_{t}^{\phi}(\eta_{n}) \) as in proposition 2.1, that is \( y_{t}^{m}(\eta_{n}) \leq \mathcal{E}_{t}^{\phi}(\eta_{n}) \) and converges increasingly to \( \mathcal{E}_{t}^{\phi}(\eta_{n}) \) as \( m \to \infty \). Since \( \eta_{n} \leq \xi \text{ a.s.} \), thus
$E[|y_m^n(\eta_n) - \mathcal{E}_i^{\phi}(\xi)|^2 \geq E[|\mathcal{E}_i^{\phi}(\eta_n) - \mathcal{E}_i^{\phi}(\xi)|^2 \geq c$ for any $n$ and $m$. For any fixed $m$, by continuous dependence property of unconstrained BSDE, $E[|y_m^n(\eta_n) - y_m^n(\eta)|^2 \to 0$ as $n \to \infty$, this gives $E[|y_m^n(\eta) - \mathcal{E}_i^{\phi}(\xi)|^2 \geq c$ for any $m$, it is then easy to conclude that $E[|\mathcal{E}_i^{\phi}(\eta) - \mathcal{E}_i^{\phi}(\xi)|^2 \geq c$ by monotone convergence theorem.

Now suppose on the contrary, if $\beta_n \to 0$ as $n \to \infty$, then there is some subsequence of $\{\xi_n, n = 1, 2, \cdots\}$ (for convenience, we still denote it as $\{\xi_n, n = 1, 2, \cdots\}$), such that $\beta_n \triangleq E[|\mathcal{E}_i^{\phi}(\xi_n) - \mathcal{E}_i^{\phi}(\xi)|^2 \geq \delta$ for some $\delta > 0$. But if we take $\xi_n$ as $\eta_n$ in the above argument, it will be a contradiction noting that $\xi_n \to \xi \in \bar{K}$.

At last, by the comparison theorem of proposition 2.2, we have $E[\mathcal{E}_i^{\phi}(\xi_n) \leq \mathcal{E}_i^{\phi}(\xi_n) \leq \mathcal{E}_i^{\phi}(\xi_n)$ and

$$E[\mathcal{E}_i^{\phi}(\xi_n) - \mathcal{E}_i^{\phi}(\xi)|^2 \leq \max\{E[\mathcal{E}_i^{\phi}(\xi_n) - \mathcal{E}_i^{\phi}(\xi)|^2, E[\mathcal{E}_i^{\phi}(\xi_n) - \mathcal{E}_i^{\phi}(\xi)|^2]\},$$

thus complete our proof. \qed

From the proof of theorem 3.2, we have following corollary for general generator function and constraint function,

**Corollary 3.1.** Suppose the generator function $g(t, y, z)$ and constraint function $\phi(t, y, z)$ are both uniformly Lipschitz in $(y, z)$ and $g(\cdot, 0, 0, \phi(\cdot, 0, 0) \in H^2_t(R)$. If $\{\xi_n \in L_T^2(R), n = 1, 2, \cdots\}$ is a sequence which converges to $\xi \in L_T^2(R)$ with $\xi_n \leq \xi$ a.s for any $n$, then we have

$$E[\mathcal{E}_i^{\phi}(\xi_n) - \mathcal{E}_i^{\phi}(\xi)|^2 \to 0 \ \forall t \in [0, T].$$

The proof is just a restatement of the second part of the proof of theorem 3.2.

**Remark 3.1.** El Karoui etc.[2] proved the existence of solution of reflected backward stochastic differential equation (shortly RBSDE) reflected by $S_t(0 \leq t \leq T)$, it is just the smallest $g$-supersolution or minimal solution with the constraint $\phi(t, y, z) = (y_t - S_t)^{-} = 0$. The domain of definition of the RBSDE is $\{\xi \in L_T^2(R)|\xi \geq S_{T_t}, a.s\}$, it is a convex closed set in $L_T^2(R)$. By priori estimation, the minimal solution is uniformly continuous with terminal value.

**Remark 3.2.** Note that in El Karoui etc.[2], the continuity of $\mathcal{E}_i^{\phi}(\xi)$ for any $t \in [0, T]$ is uniform and the generator function need not to be convex. Furthermore, under the assumption of $\mathcal{E}_0^{\phi}(\xi) < \infty$ for all $\xi \in L_T^2(R)$, with the constraint function $\phi(t, y, z)$ being the distance function of $z/\sigma y$ from a convex closed set in $R^n$, Karatzas and Shreve[4] proved that $\mathcal{E}_0^{\phi}(\xi)$ can be represent as a supremum of a family of linear functional on $L_T^2(R)$, and then by the resonance theorem, $\mathcal{E}_0^{\phi}(\xi)$ is also uniformly continuous with $\xi$, although the constraint function $\phi(t, y, z)$ is not convex. Inspired by these facts, we conjecture more general continuous dependence theorem may exist.

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Continuous property and Fatou property of $g\Gamma$-solution

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Abstract

It is well-known that solutions of backward differential equations are continuously dependent on the terminal value. Since the increasing part of the $g\Gamma$-solution of a constrained backward differential equation (shortly CBSDE) varies against terminal value, the continuous property is not obvious for it. Although the definition domain of $g\Gamma$-solution is not very clear, we can still make it well defined on some appropriate space such as all (P)-essentially bounded functions on some probability space $(\Omega, \mathcal{F}, P)$. In general case, we first prove the continuous property from below. Interestingly, when both $g$ and $\Gamma$ are convex, this means that the risk measure induced by $g\Gamma$-solution satisfies the important Fatou property. Furthermore, with the help of existing wonderful result in convex analysis, we prove the continuous property of $g\Gamma$-solution in strong sense in suitable Banach space. The comparison theorem of $g\Gamma$-solution plays a crucial role in our proof.

1 Introduction

Suppose $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(W_t)_{0 \leq t \leq T}$ is a Brownian motion based on it. Let $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$ be the filtration generated by Brownian motion and augmented by $P$-null sets. Denote $\mathcal{F}_T^2(R)$ as the space of $\mathcal{F}_T$ measurable square integrable random variables.

For any given $\xi \in \mathcal{F}_T^2(R)$, which always be viewed as a contingent claim in mathematical finance, a BSDE with terminal value $\xi$ evolves as

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s^*dW_s$$

(1.1)

where $g(t, y, z) : [0, T] \times R \times R^d \to R$ is a function satisfying uniformly Lipschitz condition

$$|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq M(|y_1 - y_2| + |z_1 - z_2|), \quad \forall (y_1, z_1), (y_2, z_2)$$

(A1)

for some $M > 0$.

If we assume further that

$$g(\cdot, 0, 0) \in H_T^2(R)$$

(A2)

holds, Paradoux and Peng [7] proved the existence and uniqueness of the adapted solution $(y(t), z(t))$ of (1.1) by a useful priori estimation which also ensures the continuous property of $y(t)$ with terminal value, see also El Karoui etc. [4] and Pardoux and Peng [7] for references.

We call $(y(t), z(t))$ as $g$-solution and $g$ as the generator function.

BSDE is always used to price or hedge claims in financial market. Usually, $y(t)$ represents the wealth process and $z(t)$ corresponds to the portfolio process in investment. But in true financial environment, some constraints are inevitable to put on $(y(t), z(t))$. For example, in Civitanic Karatzas [2], the author want $z(t) \in K$ for some convex set. More generally, Peng and Xu [9] ask

$$(y(t), z(t)) \in \Gamma$$

(1.2)

where $\Gamma = \{(y, z)|\phi(y, z) = 0\} \subset R \times R^d$ and $\phi(y, z) : R \times R^d \to R^+$. 
When some constraints are considered, one always need another increasing, right continuous process $Y(t)$ to join the solution to make solution still be adapted, that is, one always consider $g$-supersolution $(y(t), z(t), C(t))$ satisfying

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s^*dW_s + \int_t^T dC_s, \quad (1.3)$$

and $y(t), z(t)$ satisfying (1.2).

However, this time, the triple $(y(t), z(t), C(t))$ which satisfying (1.3) and (1.2) may be not unique and the minimal one becomes something. The definition of the minimal solution which was called $g_\Gamma$-solution and denoted by $\mathcal{E}^t_{\Gamma, \phi}(\xi)$ in Peng and Xu [9] will be given in section 2.

Under the assumption that there exist at least one solution $(y(t), z(t), C(t))$ of (1.3) satisfying (1.2), a penalization method can be used to get the minimal solution to this CBSDE, i.e, it can be approximated by an increasing sequence of $g$-supersolutions. In such case, it is obvious that the associated increasing part $C_t$ of $g_\Gamma$-solution varies against terminal value, the usual priori estimation of $g$-solution does not work for the continuous property.

In this paper, We first prove the continuous property from below of $g\_\Gamma$-solution in general case. When both $g$ and $\phi$ are convex, with the help of wonderful result in convex analysis, the continuous property comes true in the interior of the definition domain of $g\_\Gamma$-solution.

This paper is organized as follows: In section 2, We state the framework in Peng[8] and some propositions about $g\_\Gamma$-solution. The main results are obtained in section 3.

## 2 BSDE and $g\_\Gamma$-solution of CBSDE

Given a probability space $(\Omega, \mathcal{F}, P)$ and $R^d$-valued Brownian motion $W(t)$, we consider a sequence $\{(\mathcal{F}_t): t \in [0, T]\}$ of filtrations generated by Brownian motion $W(t)$ and augmented by $P$-null sets. $\mathcal{P}$ is the $\sigma$-field of predictable sets of $\Omega \times [0, T]$. We use $L^2_\mathcal{F}(R^d)$ to denote the space of all $\mathcal{F}_t$-measurable random variables $\xi : \Omega \rightarrow R^d$ for which

$$\| \xi \|^2 = E[|\xi|^2] < +\infty.$$  

and use $H^2_\mathcal{F}(R^d)$ to denote the space of predictable process $\varphi : \Omega \times [0, T] \rightarrow R^d$ for which

$$\| \varphi \|^2 = E[\int_0^T |\varphi|^2] < +\infty.$$  

The backward stochastic differential equation (shortly BSDE ) driven by $g(t, y, z)$ is given by

$$-dy_t = g(t, y_t, z_t)dt - z_t^*dW(t) \quad (2.1)$$

where $y_t \in R$ and $W(t) \in R^d$. Suppose that $\xi \in L^2_\mathcal{F}(R)$ and $g$ satisfies (A1) and (A2), Pardoux and Peng [7] proved the existence of adapted solution $(y(t), z(t))$ of such BSDE. We call $(g, \xi)$ standard parameters for the BSDE.

The following definitions is necessary to help us go on with our study.

**Definition 2.1.** *(super-solution)* A super-solution of a BSDE associated with the standard parameters $(g, \xi)$ is a vector process $(y_t, z_t, C_t)$ satisfying

$$-dy_t = g(t, y_t, z_t)dt + dC_t - z_t^*dW(t), \quad y_T = \xi, \quad (2.2)$$
or being equivalent to
\[ y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s, \]
where \((C_t, t \in [0, T])\) is an increasing, adapted, right-continuous process with \(C_0 = 0\) and \(z_t^*\) is the transpose of \(z_t\). When \(C_t \equiv 0\), we call \((y_t, z_t)\) a \(g\)-solution.

Constraints like (1.2) is always considered in this paper. In such case, we give the following definition.

**Definition 2.2.** (\(g_T\)-solution or the minimal solution) A \(g\)-supersolution \((y_t, z_t, C_t)\) is said to be the the minimal solution, given \(y_T = \xi\), subjected to the constraint (1.2) if for any other \(g\)-supersolution \((y'_t, z'_t, C'_t)\) satisfying (1.2) with \(y'_T = \xi\), we have \(y_t \leq y'_t\) a.e., a.s.. The minimal solution is denoted by \(E^{\phi, \xi}_t\) and for convenience called as \(g_T\)-solution.

For any \(\xi \in L^2_T(R)\), we denote \(H^\phi(\xi)\) as the set of \(g\)-supersolutions \((y_t, z_t, C_t)\) subjecting to (1.2) with \(y_T = \xi\). When \(H^\phi(\xi)\) is not empty, Peng [8] proved that \(g_T\)-solution exists.

In general case, unlike \(g\)-solution, the increasing part of \(g_T\)-solution is different with different terminal value and it is impossible to get a similar priori estimation. The continuous property seems hard to hold, however, we can prove it is still continuous from below. In order to obtain a whole continuity, we consider the convex case, that is we assume both \(g\) and \(\phi\) are convex.

The convexity of \(E^{\phi, \xi}_t\) can be easily deduced from the same proposition of solution of BSDE with convex generator function.

**Proposition 2.1.** Let \(\phi(t, y, z)\) be a function: \([0, T] \times R \times R^d \rightarrow R^+\) and \(g(t, y, z)\) is a function: \([0, T] \times R \times R^d \rightarrow R\). Suppose \(\phi(t, y, z)\) and \(g(t, y, z)\) are both convex in \((y, z)\) and satisfy (A1) and (A2), then
\[ E^{\phi, \xi}_t(a\xi + (1 - a)\eta) \leq aE^{\phi, \xi}_t(\xi) + (1 - a)E^{\phi, \eta}_t(\eta) \quad \forall t \in [0, T] \]
holds for any \(\xi, \eta \in L^2_T(R)\) and \(a \in [0, 1]\).

**Proof** According to Peng [8], the solutions \(y^m_t(\xi)\) of
\[ y^m_t(\xi) = \xi + \int_t^T g(y^m_s(\xi), z^m_s, s)ds + A^m_t - A^m_t - \int_t^T z^m_s dW_s, \]
is an increasing sequence and converges to \(E^{\phi, \xi}_t\), where
\[ A^m_t := m \int_0^t \phi(y^m_s, z^m_s, s)ds. \]
For any fixed \(m\), by the convexity of \(g\) and \(\phi\), \(y^m_t(\xi)\) is a convex in \(\xi\), that is
\[ y^m_t(a\xi + (1 - a)\eta) \leq ay^m_t(\xi) + (1 - a)y^m_t(\eta), \]
taking limit as \(m \to \infty\), we get the required result.

By the same method of penalization, we can get the comparison theorem of \(E^{\phi, \xi}_t\).

**Proposition 2.2.** Under the same assumptions as above proposition, we have
\[ E^{\phi, \xi}_t(\xi) \leq E^{\phi, \xi}_t(\eta) \]
for any \(\xi, \eta \in L^2_T(R)\) when \(P(\eta \geq \xi) = 1\).
3 Continuous property about \( g_t \)-solution

In this paper, we first study a semi continuous property of \( \mathcal{E}_t^{g,\phi}(\xi) \) when \( t = 0 \) in general case.

The first result is similar to the Fatou property of convex risk measure studied in Föllmer and Schied [6].

**Theorem 3.1.** Suppose the generator function \( g(t, y, z) \) and the constraint function \( \phi(t, y, z) \) both satisfy conditions (A1) and (A2), \( \{\xi_n \in L^2_t(P), n = 1, 2, \ldots\} \) is an bounded increasing sequence and converges almost surely to \( \xi \). If \( \mathcal{E}_t^{g,\phi}(\xi) \) exists for \( \zeta = \xi, \xi_n, n = 1, 2, \ldots \),

\[
\lim_{n \to \infty} \mathcal{E}_t^{g,\phi}(\xi_n) = \mathcal{E}_t^{g,\phi}(\xi).
\]

**Proof** Taking \( y_n^m(\xi) \) as in proposition 2.1. By proposition 2.2, \( \{\mathcal{E}_t^{g,\phi}(\xi_n), n = 1, 2, \ldots\} \) is an increasing sequence. We denote its limit at \( t = 0 \) as \( a \), then \( a \leq \mathcal{E}_0^{g,\phi}(\xi) \). Since \( \xi_n \) converges almost surely increasingly to \( \xi \in L^2_t(R) \), by dominated convergence theorem, it also converges strongly in \( L^2_t(R) \), then by the continuous dependence property of \( g \)-supersolution, the limit of \( \{y_n^m(\xi_n)\}_{n=1}^{\infty} \) is \( y_0^m(\xi) \) for any fixed \( m \).

We want to show that \( a = \mathcal{E}_0^{g,\phi}(\xi) \). If on the contrary on has \( a < \mathcal{E}_0^{g,\phi}(\xi) \), then there is some \( \delta > 0 \) such that \( \mathcal{E}_0^{g,\phi}(\xi) - \mathcal{E}_0^{g,\phi}(\xi_n) > \delta \) for any \( n \). On the other hand, for any \( \epsilon > 0 \), \( 0 \leq \mathcal{E}_0^{g,\phi}(\xi) - y_0^m(\xi) \leq \epsilon \) holds for some larger \( m_0 \). Fixing \( m_0, \epsilon \), there is some \( n_0 \) which depends on \( m_0, \epsilon \) such that \( 0 \leq y_{m_0}^m(\xi) - y_0^m(\xi_{n_0}) \leq \epsilon \), so \( \mathcal{E}_0^{g,\phi}(\xi) - y_{m_0}^m(\xi_{n_0}) \leq 2\epsilon \), but we have \( \mathcal{E}_0^{g,\phi}(\xi) - y_{m_0}^m(\xi_{n_0}) \geq \mathcal{E}_0^{g,\phi}(\xi) - \mathcal{E}_0^{g,\phi}(\xi_{n_0}) > \delta \), this is impossible for \( \epsilon < \frac{\delta}{2} \).

Suppose \( g(t, y, z) \) and \( \phi(t, y, z) \) are both independent of \( y \) and convex in \( z \) satisfying conditions (A1), (A2) and

\[
g(\cdot, \cdot, 0) = 0. \tag{A3}
\]

then according to Peng and Xu [9], Emanuela Rosazza Gianin [5], \( \mathcal{E}_t^{g,\phi}(\xi) \) defines well a convex risk measure on \( L^2_t(P) \), the Banach space of all \( P \)-essentially bounded real functions on a probability space \( (\Omega, \mathcal{F}_T, P) \). The above theorem tell us that such risk measure satisfies the important Fatou property.

In the case \( t \neq 0 \), we can still get some kind of semi continuous property in some sense. We will show this result as a corollary later.

We then go on to investigate the whole continuity of \( g_t \)-solution.

Denoting \( D = \{\xi \in L^2_t(R) | -\infty < \mathcal{E}_0^{g,\phi}(\xi) < \infty\} \) as the definition domain of \( \mathcal{E}_0^{g,\phi}(\xi) \), we give below assumption

\[
D \quad \text{is norm closed in} \quad L^2(R) \tag{A4}
\]

For any \( k \in R \), we define the \( k \)-level set of \( \mathcal{E}_0^{g,\phi}(\xi) \) as

\[
A_k \triangleq \{\xi \in D | \mathcal{E}_0^{g,\phi}(\xi) \leq k\}.
\]

For such sets, we have the following result.

**Lemma 3.1.** Let \( \phi(t, y, z) : [0, T] \times R \times R^d \to R^+ \) and \( g(t, y, z) : [0, T] \times R \times R^d \to R \). Suppose \( \phi(t, y, z) \) and \( g(t, y, z) \) both satisfy the assumptions (A1), (A2) and (A4), then for any \( k \in R \), the set \( A_k \) is closed in \( L^2_t(R)-\text{norm}\).

**Proof** Suppose a sequence \( \{\xi_n, n = 1, 2, \cdots\} \subset A_k \) converges under norm to some \( \xi \in L^2_t(R) \). By the closedness of \( D, \xi \in D \). For any \( \xi_n \), we take \( y_{m_n}^n(\xi_n) \) as in proposition 2.1. Since \( y_{m_n}^n(\xi_n) \) converges increasingly to \( \mathcal{E}_0^{g,\phi}(\xi_n) \leq k \) as \( m \to \infty \), \( y_n^m(\xi_n) \leq k \) holds for any \( n \) and \( m \).
For any fixed \( m \), take \( g_m = g + m \phi \), by the continuous dependence property of \( g_m \)-solution, we have \( y_0^n(\xi_n) \to y_0^n(\xi) \) as \( n \to \infty \) and \( y_0^n(\xi) \leq k \) is obtained for any \( m \). Again, for the fixed \( \xi \in L^2_T(R) \), \( y_0^n(\xi) \to \mathcal{E}_0^0(\xi) \) as \( m \to \infty \). Thus one has \( \mathcal{E}_0^0(\xi) \leq k \), this means \( A_k \) is closed under norm in \( L^2_T(R) \).

To our aim, we consider the convex case, namely we ask \( g \) and \( \phi \) to be convex.

**Lemma 3.2.** Let \( \phi(t, y, z) : [0, T] \times R \times R^d \to R^+ \) and \( g(t, y, z) : [0, T] \times R \times R^d \to R \) be convex functions. Suppose \( \phi(t, y, z) \) and \( g(t, y, z) \) both satisfy the assumptions (A1), (A2), then the definition domain \( D \) of \( \mathcal{E}_s^g,\phi(\xi) \) is convex in \( L^2_T(R) \).

**Proof** In fact, suppose \( \xi_i \in D \), \( i = 1, 2 \) and \( (y_i(t) = \mathcal{E}_s^g,\phi(\xi_i), z_i(t), C_i(t)) \) respective is the minimal solution with terminal value \( \xi_i \). That is

\[
y_i(t) = \xi_i + \int_t^T g(s, y_i(s), z_i(s))ds - \int_t^T z_i^*(s)dW_s + \int_t^T dC_i(s), i = 1, 2. \tag{3.1}
\]

and

\[
\phi(y_i(t), z_i(t)) = 0, i = 1, 2. \tag{3.2}
\]

Take \( 0 \leq a \leq 1 \), let \( \bar{y}(t) := ay_1(t) + (1-a)y_2(t) \), \( \bar{z}(t) := az_1(t) + (1-a)z_2(t) \) and \( \bar{C}(t) := aC_1(t) + (1-a)C_2(t), \xi := a\xi_1 + (1-a)\xi_2 \), then we have

\[
\bar{y}(t) = \bar{\xi} + \int_t^T (ag(s, y_1(s), z_1(s)) + (1-a)g(s, y_2(s), z_2(s)))ds - \int_t^T \bar{z}(s)^*dW_s + \int_t^T d\bar{C}(s).
\]

Thanks to the convexity of \( g \) and \( \phi \), we have

\[
ag(s, y_1(s), z_1(s)) + (1-a)g(s, y_2(s), z_2(s)) \geq g(s, \bar{y}(s), \bar{z}(s))
\]

and

\[
0 \leq \phi(\bar{y}(s), \bar{z}(s)) \leq a\phi(y_1(t), z_1(t)) + (1-a)\phi(y_2(t), z_2(t)) = 0.
\]

Thus \((\bar{y}(s), \bar{z}(s), C(t))\) is a supersolution of \((2.2')\) satisfying constraint \((1.2)\), where

\[
C(t) = ag(s, y_1(t), z_1(t)) + (1-a)g(t, y_2(t), z_2(t)) - g(t, \bar{y}(t), \bar{z}(t)) + \bar{C}(t).
\]

According to Peng [8], the minimal solution exists with terminal value \( \bar{\xi} \), that is \( D \) is convex.

Furthermore, if both \( g \) and \( \phi \) are convex, proposition 2.1 tells us \( \mathcal{E}_0^g,\phi(\xi) \) is a convex functional on \( D \). This make it possible for us to use powerful tool in convex analysis.

By lemma 3.1, \( \mathcal{E}_0^g,\phi(\xi) \) is closed, namely it is semi lower-continuous in norm, the following theorem comes out, see Aubin [1] corollary 2.3 for reference.

**Theorem 3.2.** Suppose \( g(t, y, z) \) and \( \phi(t, y, z) \) are both convex satisfying \((Ai)\), \( i = 1, 2, 3, 4 \), then \( \mathcal{E}_s^g,\phi(\xi) \) is continuous in \( \bar{D} \) under norm.

Next we turn to investigate the continuous property of \( \mathcal{E}_s^g,\phi(\xi) \) when \( t \neq 0 \).

**Theorem 3.3.** Suppose \( g(t, y, z) \) and \( \phi(t, y, z) \) are both convex functions satisfying \((Ai)\), \( i = 1, 2, 3, 4 \), then for random variables \( \xi_n, \xi \in \bar{D} \), we have

\[
E|\mathcal{E}_s^g,\phi(\xi_n) - \mathcal{E}_s^g,\phi(\xi)|^2 \to 0
\]

if \( E|\xi_n - \xi|^2 \to 0 \) as \( n \to \infty \).
Proof Let $\xi_n = \xi_n \lor \xi$, $\xi_n = \xi_n \land \xi$. Since they both converge to $\xi$ with norm and by assumption, $\xi_n, \xi_n \in D$ for $n$ large enough, without loss of generality, we assume this holds for each $n$.

First, we consider $\alpha_n \triangleq E|E_{t_1}^{\xi_n}(\xi_n) - E_t^{\phi}(\xi)|^2$ and show that $\alpha_n \to 0$ as $n \to \infty$.

In fact, for any fixed $\xi \in L^2(\mathbb{R})$, by proposition 2.1, $\varphi(\xi) \triangleq E|E_t^{\phi}(\xi)|^2$ is convex on the set $K = \{\eta \in D| \eta \geq \xi \ a.s\}$. It is also easy to show by the same technique used in lemma 3.1 that $\varphi(\xi)$ is lower semi-continuous on $K$, thus $\alpha_n \to 0$ comes true.

Secondly, let $\beta_n \triangleq E|E_{t_1}^{\xi_n}(\xi_n) - E_t^{\phi}(\xi)|^2$ and we can prove $\beta_n \to 0$ as $n \to \infty$.

This time let us consider the functional $\varphi(\xi) \triangleq E|E_{t_1}^{\phi}(\xi)|^2$ on the convex set $K = \{\eta \in D| \eta \leq \xi \ a.s\}$. For any sequence $\{\eta_n \in K, n = 1, 2, \cdots\}$ which converges to $\eta \in K$, we can show that $\varphi(\eta) \geq c$ if $\varphi(\eta_n) \geq c$ for all $n$. In fact, let $y_m(\eta_n)$ be the approximating sequence of $E_{t_1}^{\phi}(\eta_n)$ as in proposition 2.1, that is $y_m(\eta_n) \leq E_{t_1}^{\phi}(\eta_n)$ and converges increasingly to $E_{t_1}^{\phi}(\eta_n)$ as $m \to \infty$. Since $\eta_n \leq \xi$ a.s, thus $E|y_m(\eta_n) - E_{t_1}^{\phi}(\xi)|^2 \geq E|E_{t_1}^{\phi}(\eta_n) - E_{t_1}^{\phi}(\xi)|^2 \geq c$ for any $n$ and $m$. For any fixed $m$, by continuous property of unconstrained BSDE, $E|y_m(\eta_n) - y_m^{\eta}(\eta)|^2 \to 0$ as $n \to \infty$, this gives $E|y_m(\eta_n) - E_{t_1}^{\phi}(\xi)|^2 \geq c$ for any $m$, it is then easy to conclude that $E|E_{t_1}^{\phi}(\eta_n) - E_{t_1}^{\phi}(\xi)|^2 \geq c$ by monotone convergence theorem.

Now suppose on the contrary, if $\beta_n \to 0$ as $n \to \infty$, then there is some subsequence of $\{\xi_n, n = 1, 2, \cdots\}$ (for convenience, we still denote it as $\{\xi_n, n = 1, 2, \cdots\}$), such that $\beta_n \triangleq E|E_{t_1}^{\phi}(\eta_n) - E_{t_1}^{\phi}(\xi)|^2 \geq \delta$ for some $\delta > 0$. But if we take $\xi_n$ as $\eta_n$ in the above argument, it will be a contradiction since $\xi_n \to \xi \in K$.

At last, by the comparison property of proposition 2.2, we have $E_{t_1}^{\phi}(\xi_n) \leq E_{t_1}^{\phi}(\xi_n) \leq E_{t_1}^{\phi}(\xi_n)$ and

$$E|E_{t_1}^{\phi}(\xi_n) - E_{t_1}^{\phi}(\xi_n)|^2 \leq E|E_{t_1}^{\phi}(\xi_n) - E_{t_1}^{\phi}(\xi_n)|^2,$$

thus complete our proof.

From the proof of theorem 3.3, we have following corollary for general generator function and constraint function,

**Corollary 3.1.** Suppose $g(t, y, z)$ and $\phi(t, y, z)$ both satisfy assumptions (A1) and (A2). If $\{\xi_n \in D, n = 1, 2, \cdots\}$ is a sequence which converges to $\xi \in D$ with $\xi_n \leq \xi$ a.s for any $n$, then we have

$$E|E_{t_1}^{\phi}(\xi_n) - E_{t_1}^{\phi}(\xi_n)|^2 \to 0 \quad \forall t \in [0, T].$$

The proof is just a restatement of the second part of the proof of theorem 3.3.

We give an perfect example at last.

**Example 3.1.** A kind of reflected BSDE which was considered in El Karoui etc. [3] can be viewed as CBSDE with constraint being $\phi(t, y_t, z_t) = (y_t - S_t)^- = 0$, where $S_t(0 \leq t \leq T)$ is the lower reflection barrier. Just as Peng and Xu [10] pointed out, the solution of such reflected BSDE is the same with minimal solution of corresponding CBSDE. In this special case, by a priori estimation, the author proved the fact that the minimal solution is uniformly continuous with terminal value, even without the convex assumption about the generator function $g$.

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