BEALE-KATO-MAJDA TYPE CONDITION FOR BURGERS EQUATION

BEN GOLDYS, MISHA NEKLYUDOV

Abstract. We consider a multidimensional Burgers equation on the torus $\mathbb{T}^d$ and the whole space $\mathbb{R}^d$. We show that, in case of the torus, there exists a unique global solution in Lebesgue spaces. For a torus we also provide estimates on the large time behaviour of solutions. In the case of $\mathbb{R}^d$ we establish the existence of a unique global solution if a Beale-Kato-Majda type condition is satisfied. To prove these results we use the probabilistic arguments which seem to be new.

In this paper we are concerned with the following multidimensional Burgers equation:

$$
\frac{\partial u_i}{\partial t} + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i + f_i, \quad t \in [0, T]
$$

$$
u (0) = u_0, i = 1, \ldots, d, x \in \mathcal{O},
$$

$$u_0 \in L^p(\mathcal{O}, \mathbb{R}^d), f \in L^p(0, T; L^p(\mathcal{O}, \mathbb{R}^d)), p \geq d,$$

where $\mathcal{O}$ is either the torus $\mathbb{T}^d$ or the full space $\mathbb{R}^d$. Equations of this type arise in the theory of conservation laws, see for example [17] and are also known as simplified models of turbulence.

If the external force $f$ is of potential type, $f = \nabla U$ and the initial condition $u_0 = \nabla U_0$ is of gradient type as well, the existence and uniqueness of solutions is well known, see for example [18] and references therein. These assumptions however, are too restrictive in many problems. For example the Burgers equation with data of non-potential type arises in some problems of gas dynamics and inelastic granular media (see [2]). It is also important to consider a more general Burgers equation in the analysis of turbulence. The question of the existence and uniqueness of solutions in case of data $f, u_0$ of non-gradient type seems to be completely open. In this paper we will consider a general case, where $f$ and $u_0$ need not be of gradient type. Our main result is that under some, rather mild conditions, the existence of a unique global solution in the whole space is implied by a version of the Beale-Kato-Majda condition, that is well known in the theory of the Navier-Stokes equation. Also we prove, without any additional assumptions, the existence and uniqueness of global solution of Burgers equation on the torus. In the last part of this paper we obtain an upper bound for the growth of solutions for time tending to infinity.

Let us recall some standard notations that will be used throughout the paper. Suppose that $H^{\alpha,p}(\mathcal{O})$ - closure of $C^\infty_0(\mathcal{O})$ w.r. to the norm $\|f\|_{\alpha,p} = \|(I - \Delta)^{\frac{\alpha}{2}} f\|_p$.

This work was supported by an ARC Discovery grant.
\( \alpha \in \mathbb{R}, p \geq 1 \). In what follows we use the notation \( F(u, v) = (u \nabla)v, F(v) = F(u,v), \cdot = \frac{\partial}{\partial t} \).

**Definition 0.1.** We say that \( u \in L^\infty(0,T;L^p(\mathcal{O},\mathbb{R}^d)) \) is a mild solution of Burgers equation with the initial condition \( u_0 \in L^p(\mathcal{O},\mathbb{R}^d) \) and force \( f \in L^1(0,T;L^p(\mathcal{O},\mathbb{R}^d)) \) if \( F(u) \in L^1(0,T;L^p(\mathcal{O},\mathbb{R}^d)) \) and \( u \) satisfies following equality

\[
(0.2) \quad u(t) = S_t u_0 - \int_0^t S_{t-s}(F(u(s)) - f(s))ds, t \in [0,T].
\]

where \( \{S_t = e^{\lambda t \Delta}\}_{t \geq 0} : \mathcal{O} \to \mathbb{R}^d \) is a heat semigroup on \( \mathcal{O} \). We assume that \( S_t \) acts on vector functions componentwise.

1. **Local Existence of Solution**

The local existence of solution to Burgers equation in \( L^p(\mathcal{O},\mathbb{R}^d) \) spaces can be shown in the same way as for the Navier-Stokes equation (see [8],[9],[11],[12],[13] and others). Here we only state main points of the proof following the work of Weissler [12].

We will use following abstract theorem proved in [12] (p.222, theorem 2), see also [9] and [11].

**Theorem 1.1.** Let \( W, X, Y, Z \) be Banach spaces continuously embedded in some topological vector space \( X \). \( R_t = e^{tA}, t \geq 0 \) be \( C_0 \)-semigroup on \( X \), which satisfies the following additional conditions

(a1) For each \( t > 0 \), \( R_t \) extends to a bounded map \( W \to X \). For some \( a > 0 \) there are positive constants \( C \) and \( T \) such that

\[
(1.1) \quad |R_t h|_X \leq Ct^{-a} |h|_W, h \in W, t \in (0,T].
\]

(a2) For each \( t > 0 \), \( R_t \) is a bounded map \( X \to Y \). For some \( b > 0 \) there are positive constants \( C \) and \( T \) such that

\[
(1.2) \quad |R_t h|_Y \leq Ct^{-b} |h|_X, h \in X, t \in (0,T].
\]

Furthermore, function \( |R_t h|_Y \in C((0,T]), h \in X \) and

\[
(1.3) \quad \lim_{t \to 0^+} t^b |R_t h|_Y = 0, \forall h \in X.
\]

(a3) For each \( t > 0 \), \( R_t \) is a bounded map \( X \to Z \). For some \( c > 0 \) there are positive constants \( C \) and \( T \) such that

\[
(1.4) \quad |R_t h|_Z \leq Ct^{-c} |h|_X, h \in X, t \in (0,T].
\]

Furthermore, function \( |R_t h|_Z \in C((0,T]), h \in X \) and

\[
(1.5) \quad \lim_{t \to 0^+} t^c |R_t h|_Z = 0, \forall h \in X.
\]
Let also $G : Y \times Z \to W$ be a bounded bilinear map, and let $G(u) = G(u, u), u \in Y \cap Z, f \in L^\infty(0, T; W)$. Assume also that $a + b + c \leq 1$.

Then for each $u_0 \in X$ there is $T > 0$ and unique function $u : [0, T] \to X$ such that:

(a) $u \in C([0, T], X), u(0) = u_0$.
(b) $u \in C((0, T], Y), \lim_{t \to 0^+} t^b |u(t)|_Y = 0$.
(c) $u \in C((0, T], Z), \lim_{t \to 0^+} t^c |u(t)|_Z = 0$.
(d) $u(t) = R_t u_0 + \int_0^t R_{t-\tau} (G(u(\tau)) + f(\tau)) d\tau, t \in [0, T]$

**Remark 1.2.** Weissler \[12\] considers only the case of $f = 0$. The general case follows similarly (see appendix for the proof).

In the next proposition we will summarize properties of heat semigroup $S_t' = e^{t \triangle}, t \geq 0$ on $\mathcal{O}$.

**Proposition 1.3.**

(i) Let $p \in (1, \infty)$ and $\alpha < \beta$. Then for any $t > 0$ $e^{t \triangle}$ is a bounded map $H^{\alpha, p}(\mathcal{O}, \mathbb{R}^d) \to H^{\beta, p}(\mathcal{O}, \mathbb{R}^d)$. Moreover, for each $T > 0$ there exists $C = C(p, \alpha, \beta)$, such that

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}, 1 < p \leq q < \infty, h \in L^p(\mathcal{O}, \mathbb{R}^d).$$

Furthermore,

$$\lim_{t \to 0^+} t^{\frac{\alpha}{p} + \frac{\beta}{q}} |\nabla^m e^{t \triangle} h|_{L^p(\mathcal{O}, \mathbb{R}^d)} = 0, \quad h \in L^p(\mathcal{O}, \mathbb{R}^d).$$

(ii) Let $p \in (1, \infty)$ and $\alpha < \beta$. Then for any $t > 0$ $e^{t \triangle}$ is a bounded map $H^{\beta, p}(\mathcal{O}, \mathbb{R}^d) \to H^{\beta, p}(\mathcal{O}, \mathbb{R}^d)$. Moreover, for each $T > 0$ there exists $C = C(p, \alpha, \beta)$, such that

$$|e^{t \triangle} h|_{H^{\alpha, p}(\mathcal{O}, \mathbb{R}^d)} \leq Ct^{(\alpha - \beta)/2} |h|_{H^{\alpha, p}(\mathcal{O}, \mathbb{R}^d)}, t \in (0, T], h \in H^{\alpha, p}(\mathcal{O}, \mathbb{R}^d).$$

Furthermore,

$$\lim_{t \to 0^+} t^{(\beta - \alpha)/2} |e^{t \triangle} h|_{H^{\beta, p}} = 0, \quad h \in H^{\alpha, p}(\mathcal{O}, \mathbb{R}^d).$$

(iii) Let $p \in (1, \infty)$. Then for any $t > 0$, $e^{t \triangle} : L^p(\mathcal{O}, \mathbb{R}^d) \to H^{1, p}(\mathcal{O}, \mathbb{R}^d)$ is a bounded map. Moreover, for each $T > 0$ there exists $C = C(p, T)$, such that

$$|e^{t \triangle} h|_{H^{1, p}(\mathcal{O}, \mathbb{R}^d)} \leq Ct^{-\frac{1}{2}} |h|_{L^p(\mathcal{O}, \mathbb{R}^d)}, t \in (0, T], h \in L^p(\mathcal{O}, \mathbb{R}^d).$$

Furthermore,

$$\lim_{t \to 0^+} t^{\frac{1}{2}} |e^{t \triangle} h|_{H^{1, p}(\mathcal{O}, \mathbb{R}^d)} = 0, \quad h \in L^p(\mathcal{O}, \mathbb{R}^d).$$

**Proof:** The results above are well known in case of $\mathcal{O} = \mathbb{R}^d$. If $\mathcal{O} = \mathbb{T}^d$ then the lemma is well known for the Dirichlet boundary conditions, see for example books by Lunardi: Analytic semigroups and optimal regularity in parabolic problems or
by Souplet: Superlinear parabolic problems. Analogous statements for the periodic Laplacian follow easily by the same method.

Now we can formulate the theorems:

**Theorem 1.4.** For all \( u_0 \in L^p(\mathcal{O}, \mathbb{R}^d), f \in L^\infty([0,T], L^\frac{2p}{p}(\mathcal{O}, \mathbb{R}^d)), p \geq d \) there exists \( T_0 = T_0(\nu, |u_0|_{L^p(\mathcal{O}, \mathbb{R}^d)}, |f|_{L^\frac{2p}{p}(\mathcal{O}, \mathbb{R}^d)}) > 0 \) such that there exists unique mild solution \( u \in L^\infty(0,T_0; L^p(\mathcal{O}, \mathbb{R}^d)) \) of Burgers equation. Furthermore

(a) \( u : [0,T_0] \to L^p(\mathcal{O}, \mathbb{R}^d) \) is continuous and \( u(0) = u_0 \).

(b) \( u : (0,T_0) \to L^{2p}(\mathcal{O}, \mathbb{R}^d) \) is continuous and \( \lim_{t \to 0} t^{\frac{d}{2p}}|u(t)|_{L^{2p}(\mathcal{O}, \mathbb{R}^d)} = 0 \).

(c) \( u : (0,T_0) \to H^{1,p}(\mathcal{O}, \mathbb{R}^d) \) is continuous and \( \lim_{t \to 0} t^{\frac{d}{p}}|u(t)|_{H^{1,p}(\mathcal{O}, \mathbb{R}^d)} = 0 \).

**Proof of Theorem 1.4.** We use theorem (1.1) with following data \( X = L^p(\mathcal{O}, \mathbb{R}^d), Y = L^{2p}(\mathcal{O}, \mathbb{R}^d), Z = H^{1,p}(\mathcal{O}, \mathbb{R}^d), W = L^{\frac{2p}{d}}(\mathcal{O}, \mathbb{R}^d) \). Then it follows from Hölder inequality that \( F : L^{2p}(\mathcal{O}, \mathbb{R}^d) \times H^{1,p}(\mathcal{O}, \mathbb{R}^d) \to L^{\frac{2p}{d}}(\mathcal{O}, \mathbb{R}^d) \) is a bounded bilinear map. Conditions (1.1) is satisfied with \( a = \frac{d}{2p} \) by estimate (1.6). Conditions (1.2),(1.3) are satisfied with \( b = \frac{d}{4p} \) by (1.6) and (1.7). Conditions (1.4),(1.5) are satisfied with \( c = \frac{1}{2} \) by (1.10) and (1.11).

**Corollary 1.5.** Let \( p \geq d, \theta \in (0,1), u_0 \in L^p(\mathcal{O}, \mathbb{R}^d), f \in L^\infty([0,T], L^{\frac{2p}{d}}(\mathcal{O}, \mathbb{R}^d) \cap L^p(\mathcal{O}, \mathbb{R}^d)), f \in C^d(\varepsilon,T], L^p(\mathcal{O}, \mathbb{R}^d)), \forall \varepsilon > 0. \) Then there exist \( T_2 > 0 \) such that \( u \in C^1((0,T_2]; L^p(\mathcal{O}, \mathbb{R}^d) \cap C((0,T_2]; H^{1,2p}(\mathcal{O}, \mathbb{R}^d)) \cap C^d(\varepsilon,T_2], H^{2,2p}(\mathcal{O}, \mathbb{R}^d)) \cap C^{1+\theta}(\varepsilon,T_2], L^p(\mathcal{O}, \mathbb{R}^d)) \), \( \forall \varepsilon > 0 \) and \( u \) satisfies Burgers equation

\[
(1.12) \quad u' = \nu \triangle u - F(u(t)) + f(t),
\]

**Proof.** By theorem [1.4] we have that \( u(t) \in L^{2p}(\mathcal{O}, \mathbb{R}^d), t \in (0,T_0) \). Let us show that there exist \( T_1 \) such that \( u \in C((0,T_1], H^{1,2p}(\mathcal{O}, \mathbb{R}^d)) \) and \( \lim_{t \to 0} t^{\frac{d}{2p}}|u(t)|_{H^{1,2p}(\mathcal{O}, \mathbb{R}^d)} = 0 \). We apply Theorem (1.1) with following data \( X = Y = L^p(\mathcal{O}, \mathbb{R}^d), Z = H^{1,2p}(\mathcal{O}, \mathbb{R}^d), W = L^{\frac{2p}{d}}(\mathcal{O}, \mathbb{R}^d) \). Then it follows from Hölder inequality that \( F : L^{2p}(\mathcal{O}, \mathbb{R}^d) \times H^{1,2p}(\mathcal{O}, \mathbb{R}^d) \to L^{\frac{2p}{d}}(\mathcal{O}, \mathbb{R}^d) \) is a bounded bilinear map. Conditions (1.1) is satisfied with \( a = \frac{d}{2p} \) by estimate (1.6). Conditions (1.2),(1.3) are satisfied with arbitrary \( b > 0 \) because heat semigroup is analytic on \( L^p(\mathcal{O}, \mathbb{R}^d) \). Conditions (1.4),(1.5) are satisfied with \( c = \frac{1}{2} \) by (1.10) and (1.11).

As the result by part c of the Theorem 1.1 we get existence of \( T_1 \) such that \( u \in C((0,T_1], H^{1,2p}(\mathcal{O}, \mathbb{R}^d)) \) and \( \lim_{t \to 0} t^{\frac{d}{2p}}|u(t)|_{H^{1,2p}(\mathcal{O}, \mathbb{R}^d)} = 0 \). Put \( T_2 = \min\{T,T_0,T_1\} \).
Therefore, we have
\[
|F(u)|_{L^1(0,T_2;L^p(O,R^d))} \leq \int_0^{T_2} |u(s)|_{L^2p(O,R^d)} |\nabla u|_{L^{2p}(O,R^d)} ds
\]
(1.14)
\[
\leq \int_0^{T_2} \frac{1}{s^{\frac{d}{2p} + \frac{1}{2}}} \sup_s (s^{\frac{d}{2p}} |u(s)|_{L^{2p}(O,R^d)}) \sup_s (s^{\frac{1}{2p}} |u(s)|_{H^{1,2p}(O,R^d)}) ds
\]
(1.13)
\[
sup_s (s^{\frac{d}{2p}} |u(s)|_{L^{2p}(O,R^d)}) \sup_s (s^{\frac{1}{2p}} |u(s)|_{H^{1,2p}(O,R^d)}) T_2^{\frac{1}{2} - \frac{d}{2p}} < \infty
\]

Let us show that \(F(u(\cdot)) : [\varepsilon, T_2] \to L^p(O, R^d)\) is a H"older continuous for any \(\varepsilon > 0\). Then the result will follow from theorem 4.3.4, p.137 in [16], (1.13) and assumption \(f \in L^1([0, T]; L^p(O, R^d)) \cap C^\theta([\varepsilon, T], L^p(O, R^d)), \forall \varepsilon > 0\). Since \(F : H^{1,2p}(O, R^d) \to L^p(O, R^d)\) is locally Lipschitz it is easy to notice that it is enough to prove that \(u : [\varepsilon, T_2] \to H^{1,2p}(O, R^d)\) is a H"older continuous for any \(\varepsilon > 0\). Since we have representation
\[
(1.14) \quad u(t) = S^\nu_{t-\varepsilon} u(\varepsilon) - \int_\varepsilon^t S^\nu_{t-s} (F(u(s)) - f(s)) ds, t \in [\varepsilon, T_2],
\]
for \(u\) it is enough to show that each term of this representation is H"older continuous. Similarly to (1.13) we have
\[
(1.15) \quad \sup_{t \in [0, T_2]} t^{\frac{1}{2} + \frac{d}{2p}} |F(u(t))|_{L^p(O, R^d)} \leq \sup_s s^{\frac{d}{2p}} |u(s)|_{L^{2p}(O,R^d)} \sup_s s^{\frac{1}{2p}} |u(s)|_{H^{1,2p}(O,R^d)} < \infty
\]
and it follows by proposition 4.2.3 part (i), p.130 of [16] that \(\int_0^t S^\nu_{t-s} F(u(s)) ds \in C^{\theta}(0, T_2; L^p(O, R^d))\). Similarly, we have that \(\int_\varepsilon^t S^\nu_{t-s} f(s) ds \in C^{\theta}(0, T_2; L^p(O, R^d))\) and the result follows.

**Corollary 1.6.** Suppose that assumptions of the corollary (1.5) are satisfied. Assume also that \(f \in C^\theta([\varepsilon, T], H^{k+2p}(O, R^d)), \forall \varepsilon > 0\) for some \(k \in \mathbb{N}\). Then \(u \in C^\theta([\varepsilon, T], H^{k+2p}(O, R^d)) \cap C^{1+\theta}([\varepsilon, T], H^{k+2p}(O, R^d)), \forall \varepsilon > 0\).

**Proof:** We will show the result for \(k = 1\). General case follows similarly. We have that \(u(t) \in L^2p(O, R^d), t > 0\). As a result, following the proof of the previous corollary we can get that
\[
(1.16) \quad u \in C^\theta([\varepsilon, T], H^{2,2p}(O, R^d)) \cap C^{1+\theta}([\varepsilon, T], L^2p(O, R^d)), \forall \varepsilon > 0.
\]
Therefore, we have following estimates for nonlinearity
\[
|F(u)|_{C^\theta([\varepsilon, T], L^p(O, R^d))} \leq |u|_{L^\infty(\varepsilon, T; L^2p(O, R^d))} |\nabla u|_{C^\theta([\varepsilon, T], L^2p(O, R^d))}
+ |\nabla u|_{L^\infty(\varepsilon, T; L^2p(O, R^d))} |u|_{C^\theta([\varepsilon, T], L^2p(O, R^d))} < \infty
(1.17)
\]
where we have used (1.16). Furthermore,
\[
\begin{align*}
|\nabla F(u)|_{C^0([\varepsilon,T],L^p(\mathbb{R}^d))} & \leq C|\nabla u|_{L^\infty(\varepsilon,T;L^2p(\mathbb{R}^d))} |\nabla u|_{C^0([\varepsilon,T],L^2p(\mathbb{R}^d))} \\
& \quad + |u|_{L^\infty(\varepsilon,T;L^2p(\mathbb{R}^d))} |\Delta u|_{C^0([\varepsilon,T],L^2p(\mathbb{R}^d))} \\
& \quad + |\Delta u|_{L^\infty(\varepsilon,T;L^2p(\mathbb{R}^d))} |u|_{C^0([\varepsilon,T],L^2p(\mathbb{R}^d))} < \infty,
\end{align*}
\] (1.18)
where we have used (1.16). Thus, combining (1.17) and (1.18) we get $F(u) \in C^0([\varepsilon,T],H^{1-p})$, $\forall \varepsilon > 0$. In the same time, by assumption we have that $f \in C^0([\varepsilon,T],H^{1-p}(\mathbb{R}^d))$, $\forall \varepsilon > 0$. Therefore by maximal regularity result, theorem 4.3.1, p.134 of [16], it follows that $u \in C^0([\varepsilon,T],H^{3-p}(\mathbb{R}^d)) \cap C^{1+\theta}([\varepsilon,T],H^{1-p}(\mathbb{R}^d))$.

In the next lemma we will show that either local solution defined in previous theorems is global or it blows up. Let us denote $T_{max}$ maximal existence time of solution.

**Lemma 1.7.** Assume that $u_0 \in L^p(\mathbb{R}^d)$, $f \in L^\infty([0,T],L^{2p}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d))$, $p > d$ and $T_{max} < \infty$. Let $u \in L^\infty([0,T_{max});L^p(\mathbb{R}^d))$ be maximal local mild solution of Burgers equation (0.2). Then
\[
\lim\sup_{t/T_{max}} |u(t)|_{L^p(\mathbb{R}^d)}^2 = \infty.
\] (1.19)

**Proof of Lemma 1.7** We will argue by contradiction. Assume that
\[
\lim\sup_{t/T_{max}} |u(t)|_{L^p(\mathbb{R}^d)}^2 < \infty.
\] (1.20)
Then there exist $T_1$ such that
\[
K_1 = \sup_{t \in [T_1,T_{max}]} |u(t)|_{L^p(\mathbb{R}^d)} < \infty.
\] (1.21)
We will show that there exist constant $C, \alpha > 0$ such that
\[
|u(t) - u(\tau)|_{L^p(\mathbb{R}^d)} \leq C|t - \tau|^\alpha, \; t, \tau \in [T_1,T_{max}), T_1 \leq T_2 < T_{max}.\] (1.22)
Then it follows from (1.20) and (1.22) that there exist $y \in L^p$ such that
\[
\lim_{t/T_{max}} |u(t) - y|_{L^p(\mathbb{R}^d)} = 0,
\] (1.23)
and we have a contradiction with definition of $T_{max}$. Thus, we need to show (1.22). Let us show first that there exist $T_3 < T_{max}$ such that
\[
K_2 = \sup_{t \in [T_3,T_{max}]} |u(t)|_{H^{1-p}(\mathbb{R}^d)} < \infty.
\] (1.24)
It is enough to show
\[
\sup_{t \in [T_3,T_{max}]} |\nabla u(t)|_{L^p(\mathbb{R}^d)} < \infty,
\] (1.25)
for some $T_1 \leq T_3 < T_{max}$. Indeed, (1.24) immediately follows from (1.21) and (1.25). We have
\[
\nabla u(t) = \nabla \int_0^t S_{t-s}^\nu (F(u(s)) - f(s)) ds.
\] (1.26)
Hence,

\[
\begin{align*}
|\nabla u(t)|_{L^p(\mathbb{O}, \mathbb{R}^d)} & \leq |\nabla S^\nu_t u_0|_{L^p(\mathbb{O}, \mathbb{R}^d)} \\
& + \int_0^t |\nabla S^\nu_{t-s} f(s)|_{L^p(\mathbb{O}, \mathbb{R}^d)} ds + \int_0^t |\nabla S^\nu_{t-s} F(u(s))|_{L^p(\mathbb{O}, \mathbb{R}^d)} ds \\
& \leq \frac{C|u_0|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{t^{1/2}} + \int_0^t \frac{|f(s)|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{|t-s|^{1/2}} ds \\
& + C \int_0^t \frac{|S^\nu_{(t-s)/2} F(u(s))|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{|t-s|^{1/2}} ds \\
& \leq \frac{C|u_0|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{t^{1/2}} + 2\sqrt{t} \sup_{s \in [0,t]} |f(s)|_{L^p(\mathbb{O}, \mathbb{R}^d)} \\
& + C \int_0^t \frac{|F(u(s))|_{L^{p/2}(\mathbb{O}, \mathbb{R}^d)}}{|t-s|^{1/2 + \frac{d}{2p}}} ds \\
& \leq \frac{C|u_0|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{t^{1/2}} + 2\sqrt{t} \sup_{s \in [0,t]} |f(s)|_{L^p(\mathbb{O}, \mathbb{R}^d)} \\
& + C \int_0^t \frac{|u(t)|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{|t-s|^{1/2 + \frac{d}{2p}}} |\nabla u(t)|_{L^p(\mathbb{O}, \mathbb{R}^d)} ds \\
& \leq \frac{C|u_0|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{t^{1/2}} + 2\sqrt{t} \sup_{s \in [0,t]} |f(s)|_{L^p(\mathbb{O}, \mathbb{R}^d)} \\
& + CK \int_0^t \frac{|\nabla u(t)|_{L^p(\mathbb{O}, \mathbb{R}^d)}}{|t-s|^{1/2 + \frac{d}{2p}}} ds,
\end{align*}
\]

(1.27)

where second and third inequalities follow from (1.6), forth inequality follows from Hölder inequality and assumption (1.21) is used in the fifth one. Now if \( \frac{1}{2} + \frac{d}{2p} < 1 \) (i.e. if \( p > d \)) we can use Gronwall inequality ([10], Lemma 7.1.1, p. 188) to conclude that the estimate (1.25) holds. Thus we get an estimate (1.24).

Now we can turn to the proof of (1.22). We have

(1.28) \[ u(t) - u(\tau) = S^\nu_{t-\tau} u(\tau) - u(\tau) + \int_\tau^t S^\nu_{t-s} (f(s) - F(u(s))) ds. \]
Then

\[
|u(t) - u(\tau)|_{L^p(\mathcal{O}, \mathbb{R}^d)} \leq |S'_\tau u(\tau) - u(\tau)|_{L^p(\mathcal{O}, \mathbb{R}^d)} + \int_\tau^t |S'_{t-s} F(u(s))|_{L^p(\mathcal{O}, \mathbb{R}^d)} ds
\]

(1.29) 

\[
(I) = \int_\tau^t \nu \triangle S'_s u(s) ds \leq \nu \int_\tau^t |\nabla S'_s (\nabla u(s))|_{L^p(\mathcal{O}, \mathbb{R}^d)} ds 
\]

\[
\leq \nu \int_\tau^t \frac{|\nabla u(s)|_{L^p(\mathcal{O}, \mathbb{R}^d)}}{s^{1/2}} ds \leq K_2 t^{1/2} |t - \tau|.
\]

(1.30)

For the second term we have

\[
(II) \leq \sup_{s \in [\tau,t]} |f(s)|_{L^p(\mathcal{O}, \mathbb{R}^d)} |t - \tau|.
\]

(1.31)

Third term is estimated as follows

\[
(III) \leq \int_\tau^t \frac{|F(u(s))|_{L^{p/2}(\mathcal{O}, \mathbb{R}^d)}}{|t - s|^{\frac{p}{2p}}} ds \leq \int_\tau^t \frac{|u(t)|_{L^p(\mathcal{O}, \mathbb{R}^d)} |\nabla u(t)|_{L^p(\mathcal{O}, \mathbb{R}^d)}}{|t - s|^{\frac{p}{2p}}} ds 
\]

\[
\leq CK_2^2 |t - \tau|^{1-\frac{d}{p}},
\]

(1.32)

where first inequality follows from (1.6), second one follows from Hölder inequality and the last inequality follows from estimate (1.24).

Combining (1.30), (1.31) and (1.32) we get (1.22).

\[\Box\]

**Remark 1.8.** Authors believe that the Lemma 1.7 holds also for the critical case of \(p = d\). It would be interesting to prove this fact.

## 2. Global existence of solution on the torus \(\mathbb{T}^d\)

In this section we establish main results of the article. First, we will show that there exist global solution of Burgers equation on torus.

**Theorem 2.1.** Fix \(p > d\). Let \(\theta \in (0, 1)\), \(u_0 \in L^p(\mathbb{T}^d, \mathbb{R}^d)\), \(f \in L^\infty([0, T], L^\frac{p}{2}(\mathbb{T}^d, \mathbb{R}^d) \cap L^p(\mathbb{T}^d, \mathbb{R}^d))\), \(f \in L^1([0, T]; L^\infty(\mathbb{T}^d, \mathbb{R}^d))\), \(f \in C^\theta([\varepsilon, T], L^p(\mathbb{T}^d, \mathbb{R}^d))\), \(\forall \varepsilon > 0\). Then there exist global solution \(u \in C([0, T], L^p(\mathbb{T}^d, \mathbb{R}^d)) \cap C^1([0, T]; L^p(\mathbb{T}^d, \mathbb{R}^d)) \) and \(C((0, T]; H^{2p}(\mathbb{T}^d, \mathbb{R}^d)) \cap C^\theta([\varepsilon, T], H^{2p}(\mathbb{T}^d, \mathbb{R}^d)) \cap C^1+\theta([\varepsilon, T], L^p(\mathbb{T}^d, \mathbb{R}^d))\), \(\forall \varepsilon > 0\) which satisfies Burgers equation (1.12).
Proof of Proposition 2.7. We have according to the Corollary 1.5 that there exist local solution on interval \([0, T_{\text{max}}]\). Furthermore, we have by Lemma 1.7 blow-up of the solution when \(t \to T_{\text{max}}\). Thus it is enough to prove \(L^p\) estimate uniform on semiinterval \([T_0, T_{\text{max}}]\) for some \(T_0 < T_{\text{max}}\). Fix \(0 < \delta < T < T_{\text{max}}\). By Corollary 1.5 we can assume that \(u \in C([\varepsilon, T], H^{2,2p}(\mathbb{T}^d, \mathbb{R}^d)) \cap C^1([\varepsilon, T], L^{2p}(\mathbb{T}^d, \mathbb{R}^d)) \forall \varepsilon > 0\). Define flow
\[
  dX_t(x) = -u(T-t, X_t(x))dt + \sqrt{2\nu}dW_t \\
  X_0(x) = x, x \in \mathbb{T}^d, 0 \leq t \leq T - \delta
\]
(2.1)
Notice that \(u \in C([\delta, T], H^{2-p}(\mathbb{T}^d, \mathbb{R}^d)) \subset C([\delta, T], C^{2-d/p}(\mathbb{T}^d, \mathbb{R}^d))\) and, therefore, the flow is correctly defined and does not blow up. Now we will deduce Feynman-Kac type representation for solution of Burgers equation. Let \(\{u_\varepsilon\}_{\varepsilon > 0} \in C^1([\delta, T], L^2(\mathbb{T}^d, \mathbb{R}^d))\) be a sequence of functions converging to \(u\) in \(C^1([\delta, T], L^{2p}(\mathbb{T}^d, \mathbb{R}^d)) \cap C([\delta, T], H^{2,2p}(\mathbb{T}^d, \mathbb{R}^d))\). Such sequence can be constructed, for example, by mollifying of \(u\). Then we have by Ito formula that
\[
  u_\varepsilon(T-t, X_t(x)) = u_\varepsilon(T, x) + \int_0^t (\nu \Delta u_\varepsilon(T-s, X_s) - (u \nabla)u_\varepsilon(T-s, X_s) - \frac{\partial u_\varepsilon}{\partial t}(T-s, X_s))ds \\
  + \sqrt{2\nu} \int_0^t \frac{\partial u_\varepsilon}{\partial x_j}(T-s, X_s)dW_s, t \in [0, T - \delta].
\]
(2.2)
The last term is a martingale because \(\nabla u_\varepsilon \in C([\delta, T], H^{1-p}(\mathbb{T}^d, \mathbb{R}^d)) \subset C([\delta, T] \times \mathbb{T}^d, \mathbb{R}^d), p > d\) by Sobolev embedding theorem. Hence applying mathematical expectation to equality (2.2) we get
\[
  u_\varepsilon(T, x) = \mathbb{E}u_\varepsilon(T-t, X_t(x)) + \int_0^t \mathbb{E}((u \nabla)u_\varepsilon + \frac{\partial u_\varepsilon}{\partial t} - \nu \Delta u_\varepsilon(T-s, X_s))ds, t \in [0, T - \delta]
\]
(2.3)
Now let us show convergence (w.r.t. norm of \(L^\infty(\mathbb{T}^d, \mathbb{R}^d)\)) of all terms in (2.3) when we tend \(\varepsilon\) to 0. We have
\[
  \sup_{\mathbb{T}^d} |u_\varepsilon(T, x) - u(T, x)| \leq |u_\varepsilon(T) - u(T)|_{H^{1,2p}(\mathbb{T}^d, \mathbb{R}^d)} \to 0, \varepsilon \to 0,
\]
by definition of \(u_\varepsilon\). Fix \(t \in (0, T - \delta]\). Similarly,
\[
  \mathbb{E}|u_\varepsilon(T-t, X_t(\cdot)) - \mathbb{E}u(T-t, X_t(\cdot))|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} \leq
\]
(2.4)
\[
  |u_\varepsilon(T-t) - u(T-t)|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} \varepsilon \to 0.
\]
(2.5)
result follows from Theorem 5.3, p.142 in [15]. We can notice that

\[ \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \]

Notice that by Girsanov type Theorem (see [15], p. 180-181) we have that

\[ \int_0^T \mathbb{E}(u \nabla)[u_\epsilon - u](T-s, X_s)ds )_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \leq \]

\[ \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \int_0^T \mathbb{E}[|\nabla(u_\epsilon - u)(T-s, X_s)|]ds )_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} = (I) \]

Denote

\[ M_t^\nu = e^{-\int_0^t u(T-s, X_s)ds - \nu \int_0^t |u|^2 (T-s, X_s)ds}, t \in [0, T - \delta], \nu > 0 \]

\[ M_t^\nu \] is a continuous martingale. Indeed, \( u \) is bounded continuous function and the result follows from Theorem 5.3, p.142 in [15]. We can notice that

\[ (2.6) \quad \mathbb{E}(M_t^\nu)^2 \leq e^{\nu(T-\delta)} \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} = K < \infty \]

Notice that by Girsanov type Theorem (see [15], p. 180-181) we have that

\[ Eg(X_t(x)) = EM_t^\nu g(x + \sqrt{2\nu} W_t), g \in L^p(\mathbb{T}^d, \mathbb{R}). \]

Thus we have

\[ \mathbb{E}[|\nabla(u_\epsilon - u)(T-s, X_s)|] = \mathbb{E}M_t^\nu |\nabla(u_\epsilon - u)(T-s, x + \sqrt{2\nu} W_s)| \]

and

\[ (I) \leq \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{T}\left( \int_0^T (\mathbb{E}|\nabla(u_\epsilon - u)(T-s, X_s)|)^2 ds \right)^{1/2}\]

\[ \leq \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{T}\left( \int_0^T \mathbb{E}(M_s^\nu)^2 \mathbb{E}|\nabla(u_\epsilon - u)(T-s, x + \sqrt{2\nu} W_s)|^2 ds \right)^{1/2}\]

\[ \leq \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{T}\left( \int_0^T |S_s^\nu[|\nabla(u_\epsilon - u)(T-s, x)|^2]|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} ds \right)^{1/2}\]

\[ \leq \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{T}\left( \int_0^T |S_s^\nu[|\nabla(u_\epsilon - u)(T-s, x)|^2]|_{H^1,p(\mathbb{T}^d,\mathbb{R}^d)} ds \right)^{1/2}\]

\[ \leq \sup_{t \in [δ,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{T}\left( \int_0^T \frac{1}{s^{1/2}} |\nabla(u_\epsilon - u)(T-s, x)|^2 |_{L^p(\mathbb{T}^d,\mathbb{R}^d)} ds \right)^{1/2} \]
\[
\begin{align*}
&\leq \sup_{t \in [\delta,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{KT} \left( \int_0^t \frac{|u_\varepsilon - u(T - s, x)|_{H^1,2p(\mathbb{T}^d,\mathbb{R}^d)}^2}{s^{1/2}} \, ds \right)^{1/2} \\
&\leq \sup_{t \in [\delta,T]} |u(t)|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \sqrt{KT} \frac{3}{4} \sup_{s \in [\delta,T]} |u_\varepsilon - u(s, x)|_{H^1,2p(\mathbb{T}^d,\mathbb{R}^d)} \rightarrow 0, t \in (0, T - \delta].
\end{align*}
\]

Thus we have shown convergence of \( \int_0^t \mathbb{E}(u\nabla)u_\varepsilon(T - s, X_s)ds \) to \( \int_0^t \mathbb{E}(u\nabla)u(T - s, X_s)ds \) in \( L^\infty(\mathbb{T}^d,\mathbb{R}^d) \)-norm. Similarly, we have

\[
\begin{align*}
&\left| \int_0^t \mathbb{E}(u_\varepsilon')(T - s, X_s)ds \right|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \\
&= \left| \int_0^t \mathbb{E}M_s\nu(u_\varepsilon' - u')(T - s, x + \sqrt{2\nu W_s})ds \right|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \\
&\leq \sqrt{T} \left( \int_0^t \mathbb{E}[M_s^\nu(u_\varepsilon' - u')(T - s, x + \sqrt{2\nu W_s})]^2 \, ds \right)^{1/2} \\
&\leq \sqrt{T} \int_0^t \mathbb{E}(M_s^\nu)^{\frac{3}{4}} \mathbb{E}[(u_\varepsilon' - u')(T - s, x + \sqrt{2\nu W_s})]^2 \, ds^{1/2} \\
&\leq \sqrt{TK} \left( \int_0^t |S_s^\nu[(u_\varepsilon' - u')(T - s, x)]^2|_{H^1,2p(\mathbb{T}^d,\mathbb{R}^d)} \, ds \right)^{1/2} \\
&\leq \sqrt{TK} \left( \int_0^t |\mathbb{E}S_s^\nu[(u_\varepsilon' - u')(T - s, x)]_{H^1,2p(\mathbb{T}^d,\mathbb{R}^d)} \, ds \right)^{1/2} \\
&\leq \sqrt{TK} \left( \int_0^t \frac{|u_\varepsilon' - u'(T - s, x)|_{L^p(\mathbb{T}^d,\mathbb{R}^d)}^2}{s^{1/2}} \, ds \right)^{1/2} \\
&\leq \sqrt{KT^{3/4}} \sup_{s \in [\delta,T]} |u_\varepsilon'(s) - u'(s)|_{L^p(\mathbb{T}^d,\mathbb{R}^d)} \rightarrow 0, \varepsilon \rightarrow 0, t \in (0, T - \delta].
\end{align*}
\]
For the last term we have an estimate

\[
| \int_0^t \mathbb{E}(\Delta u_\varepsilon - \Delta u)(T-s, X_s) ds |_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)}
\]

\[
= | \int_0^t \mathbb{E} M'_s(\Delta u_\varepsilon - \Delta u)(T-s, x + \sqrt{2\nu} W_s) ds |_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)}
\]

\[
\leq | \sqrt{T} \int_0^t (\mathbb{E}[M'_s(\Delta u_\varepsilon - \Delta u)(T-s, x + \sqrt{2\nu} W_s)])^2 ds |_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)}^{1/2}
\]

\[
\leq \sqrt{T} \int_0^t \mathbb{E}(M'_s)^2 |\Delta(u_\varepsilon - u)(T-s, x)|^2 ds |_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)}^{1/2}
\]

\[
\leq \sqrt{TK} \int_0^t S'_s |\Delta(u_\varepsilon - u)(T-s, x)|^2 ds |_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)}^{1/2}
\]

\[
\leq \sqrt{TK} (\int_0^t |S'_s |\Delta(u_\varepsilon - u)(T-s, x)|^2 |_{H^1(\mathbb{R}^d, \mathbb{R}^d)} ds )^{1/2}
\]

\[
\leq \sqrt{TK} (\int_0^t |\Delta(u_\varepsilon - u)(T-s, x)|^2 |_{L^p(\mathbb{R}^d, \mathbb{R}^d)} ds )^{1/2}
\]

\[
\leq \sqrt{KT^{3/4}} \sup_{s \in [\delta, T]} |u_\varepsilon(s) - u(s)|_{H^2(\mathbb{R}^d, \mathbb{R}^d)} \rightarrow 0, \varepsilon \rightarrow 0, t \in (0, T - \delta].
\]

Thus, we have shown that we can tend \( \varepsilon \rightarrow 0 \) in equality (2.3). As a result we get

\[
uu(T, x) = \mathbb{E} u(T - t, X_t(x)) +
\]

(2.7) \[
\int_0^t \mathbb{E}(\nu \Delta u + \frac{\partial u}{\partial t} - \nu \Delta u(T-s, X_s)) ds, t \in [0, T - \delta].
\]

Put \( t = T - \delta \) in equality (2.7). We have

\[
uu(T, x) = \mathbb{E} u(\delta, X_t(x)) +
\]

(2.8) \[
\int_0^{T-\delta} \mathbb{E} f(T-s, X_s) ds.
\]

As a consequence we immediately get

(2.9) \[
|u(T)|_{L^\infty} \leq |u(\delta)|_{L^\infty} + \int_0^T |f(s)|_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)} ds.
\]
Therefore, because torus is compact we have $L^\infty(\mathbb{T}^d, \mathbb{R}^d) \subset L^p(\mathbb{T}^d, \mathbb{R}^d)$ and
\begin{equation}
|u(T)|_{L^p(\mathbb{T}^d, \mathbb{R}^d)} \leq C(|u(\delta)|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} + \int_0^T |f(s)|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} ds).
\end{equation}

Since $u \in C((0, T], H^{1,p}(\mathbb{T}^d, \mathbb{R}^d))$ and $\delta > 0$ is arbitrary small we have $|u(\delta)|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} \leq |u(\delta)|_{H^{1,p}(\mathbb{T}^d, \mathbb{R}^d)} < \infty$. Tending $T \to T_{\max}$ in (2.10) we get our estimate.

The case of Burgers equation in Euclidean space is much more difficult because $L^\infty$ estimate does not allow us to deduce estimate in $L^p$. In this case we have only following "conditional" Theorem.

**Theorem 2.2.** Fix $p \in (d, \infty)$. Assume that $u \in L^\infty(0, T; L^p(\mathbb{R}^d, \mathbb{R}^d))$, $\forall T < T_0$ ($T_0$ is such that $\limsup_{t \to T_0} |u(t)|^2_{L^p} = \infty$) local solution of Burgers equation such that
\begin{equation}
0 \leq |f| \in C^1((0, T] \times \mathbb{R}^d), u(0) = u_0 \in L^p(\mathbb{R}^d, \mathbb{R}^d), f \in L^p(0, T; L^p(\mathbb{R}^d, \mathbb{R}^d)) \cap C^{0,1}((0, T] \times \mathbb{R}^d), \text{ div } f \in L^\infty(0, T; L^\infty(\mathbb{R}^d)).
\end{equation}
Assume also that
\begin{equation}
\omega = \text{curl } u \in L^\infty(0, T_0; L^\infty(\mathbb{R}^d)),
\end{equation}
and for any $\delta > 0$ there exists $0 \leq t_\delta < \delta$ such that $\text{div } u$ satisfies following growth condition:
\begin{equation}
\exists C > 0 \liminf_{R \to \infty} e^{-cR^2} \max_{|x| = R, t \in [t_\delta, T]} \text{div } u(x, t) \leq 0, \forall T < T_0,
\end{equation}
\begin{equation}
\exists 0 < t_0 < T \sup_{x} \text{div } u(t_0, x) \leq M < \infty.
\end{equation}
Furthermore, we assume that $u$ has no more than linear growth at infinity:
\begin{equation}
\limsup_{R \to \infty} \max_{|x| = R, t \in [t_\delta, T]} |u(x, t)| \leq \infty, \forall T < T_0.
\end{equation}

Let $K = p + M + |\omega|_{L^\infty(0, T_0; L^\infty(\mathbb{R}^d))} + |\text{div } f|_{L^\infty(0, T_0; L^\infty(\mathbb{R}^d))}. Then $T_0 = \infty$. Moreover, if $K \geq 0$ we have
\begin{equation}
|u(t)|_{L^p(\mathbb{R}^d, \mathbb{R}^d)}^p + \nu p(p - 1) \int_0^t \int_{\mathbb{R}^d} \sum_i |u^i|^{p-2}(s, x)|\nabla u^i(s, x)|^2 dxds \leq |u_0|_{L^p}^p e^{Kt} + \int_0^t |f(s)|_{L^p}^p e^{K(t-s)} ds, t \in (0, \infty).
\end{equation}
Furthermore, if $K < 0$ we have
\begin{equation}
|u(t)|_{L^p(\mathbb{R}^d, \mathbb{R}^d)}^p \leq |u_0|_{L^p}^p e^{Kt} + \int_0^t |f(s)|_{L^p}^p e^{K(t-s)} ds, t \in (0, \infty).
\end{equation}

**Remark 2.3.** Similar condition for Navier-Stokes equation is called Beale-Kato-Majda condition (see [14]).
Remark 2.4. In the case when compatibility conditions are satisfied and we have that \( \text{div} \ u \in C([0, T] \times \mathbb{R}^d) \) we can put \( t_0 = 0 \) in the condition (2.13).

Remark 2.5. If \( K < 0 \) and \( \int_0^t |f(s)|_p^p e^{K(t-s)} ds \to 0, t \to \infty \) than we immediately get that \( u(t) \to 0, t \to \infty \) in \( L^p \) norm.

Proof of Theorem 2.2. Assume \( T_0 < \infty \). Fix \( t_0 > 0 \) such that
\[
\sup_x \text{div} \ u(t_0, x) \leq M + 1
\]
and
\[
\liminf_{R \to \infty} e^{-cR^2} \left[ \max_{|x|=R, t \in [t_0, T]} \text{div} \ u(x, t) \right] \leq 0, \forall T < T_0.
\]
Existence of such \( t_0 \) follows from (2.13) and (2.12). Let us multiply \( i \)-th equation of system (0.1) on \( \text{sgn}(u^i) |u^i|^{p-1} \), \( i = 1, \ldots, d \), take a sum w.r.t. \( i \) and integrate w.r.t. to space variable. We get
\[
\frac{d}{dt} |u(t)|_{L^p}^p + \nu p(p-1) \int \sum_i |u^i|^{p-2} |\nabla u^i(s, x)|^2 dx
\]
(2.19)
\[
= \int \sum_i |u_i(t, x)|^p \text{div} u dx + p \int \sum_i f^i \text{sgn}(u^i)|u^i|^{p-1} dx
\]
Fix \( t_1 \geq t_0 \). Integrating w.r.t. to time from \( t_1 \) to \( t \) and applying Young inequality we get
\[
|u(t)|_{L^p}^p + \nu p(p-1) \int_{t_1}^t \int \sum_i |u^i|^{p-2} |\nabla u^i(s, x)|^2 dx ds
\]
\[
\leq |u(t_1)|_{L^p}^p + p \int_{t_1}^t \int \sum_i |u^i|^{p-1} |f_i| dx ds +
\]
\[
\int_{t_1}^t \int \sum_i |u_i(s, x)|^p \text{div} u(s, x) dx ds
\]
\[
\leq |u(t_1)|_{L^p}^p + \int_{t_1}^t |f(s)|_{L_p}^p ds + (p-1) \int_{t_1}^t |u(s)|_{L_p}^p ds +
\]
(2.20)
\[
\int_{t_1}^t \int \sum_i |u_i(s, x)|^p \text{div} u(s, x) dx ds.
\]
Now let us denote \( r = \text{div} \ u \). Taking \( \text{div} \) of equation (0.1) we get
\[
\frac{\partial r}{\partial t} + (u \nabla) r - \nu \Delta r + |\nabla u|^2 - |\text{curl} \ u|^2 - \text{div} f = 0
\]
(2.21)
Indeed

\[
\text{div}(u \nabla) u = (u \nabla) \text{div} u + \sum_{i,j} \frac{\partial u_i}{\partial x^j} \frac{\partial u_j}{\partial x^i}
\]

\[
= (u \nabla) \text{div} u + \sum_{i,j} \left( \frac{\partial u_i}{\partial x^j} \right)^2 + \frac{\partial u_i}{\partial x^j} \left( \frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j} \right)
\]

(2.22)

\[
= (u \nabla) \text{div} u + \sum_{i<j} \left( \frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right)^2 + \frac{\partial u_i}{\partial x^j} \left( \frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j} \right)
\]

Let us denote

\[
D = \{(t, x) \in [t_0, T] \times \mathbb{R}^d | r(t, x) \geq 0\},
\]

\[
D^+ = \{(t, x) \in [t_0, T] \times \mathbb{R}^d | r(t, x) \geq 0, \left| \nabla u \right|^2(t, x) - | \text{curl} u|^2(t, x) - \text{div} f \geq 0\},
\]

\[
D^- = \{(t, x) \in [t_0, T] \times \mathbb{R}^d | r(t, x) \geq 0, \left| \nabla u \right|^2(t, x) - | \text{curl} u|^2(t, x) - \text{div} f < 0\}.
\]

Then \( D = D^+ \cup D^- \) and we have that

(2.23)

\[
r(t, x) = \text{div} u(t, x) \leq \left| \nabla u \right|(t, x) < | \text{curl} u|(t, x) + | \text{div} f |, (t, x) \in D^-.
\]

Furthermore, for all \((t, x) \in D^+\) we have that

(2.24)

\[
u \Delta r - (u \nabla) r - \frac{\partial r}{\partial t} = |\nabla u|^2 - | \text{curl} u|^2 - \text{div} f \geq 0,
\]

\( u \) has no more than linear growth on the set \([t_0, T] \times \mathbb{R}^d \) because \( u \in C^{1,2}([t_0, T] \times \mathbb{R}^d) \) and condition (2.14) is satisfied. Moreover, condition (2.18) is also satisfied.

Therefore, by Phragmen-Lindelof principle (see [6], chapter 3, section 6, theorem 10 and remark (i) after the proof of thm. 10) we have that

\[
r(t, x) \leq \max \left( \sup_{y \in \mathbb{R}^d} \text{div} u(t_0, y), \sup_{s \in (t_0, T), y \in \partial D^+} r(s, y) \right) \leq \sup_{y \in \mathbb{R}^d} \text{div} u(t_0, y) + \sup_{s \in (t_0, T), y \in \partial D^+} | \text{curl} u(s, y) | + | \text{div} f | \leq M + 1
\]

(2.25)

\[
+ | \text{curl} u |_{L^\infty(0, T_0; L^\infty(\mathbb{R}^d))} + | \text{div} f |_{L^\infty(0, T_0; L^\infty(\mathbb{R}^d))},
\]

\((t, x) \in D^+ \cap \{ t_0 < t < T, x \in \mathbb{R}^d \}.\)

Combining estimates (2.23) and (2.25) we get

(2.26)

\[
r(t, x) = \text{div} u(t, x) \leq M + 1 + | \text{curl} u |_{L^\infty(0, T_0; L^\infty(\mathbb{R}^d))} + | \text{div} f |_{L^\infty(0, T_0; L^\infty(\mathbb{R}^d))}, (t, x) \in D \cap \{ t_0 < t < T, x \in \mathbb{R}^d \}.\)
Thus, combining estimate (2.26) and inequality (2.20) we get

\[ |u(t)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} + \nu p(p - 1) \int_{t_1}^{t} \int_{\mathbb{R}^d} \sum_i |u^i|^{p-2}(s, x)|\nabla u^i(s, x)|^2 \, dx \, ds \leq |u(t_1)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} + \int_{t_1}^{t} |f(s)|^p_{L^p} \, ds \]

(2.27)

\[ + \int_{t_1}^{t} (p + M + |\text{curl } u|_{L^\infty(0, T; L^\infty(\mathbb{R}^d)))} + |\text{div } f|_{L^\infty(0, T; L^\infty(\mathbb{R}^d)))}) |u(s)|_{L^p} \, ds \]

Denote \( K = p + M + |\omega|_{L^\infty(0, T; L^\infty(\mathbb{R}^d)))} + |\text{div } f|_{L^\infty(0, T; L^\infty(\mathbb{R}^d)))}. \) Then we can rewrite (2.27) as follows

\[ |u(t)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} - |u(t_1)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} + \nu p(p - 1) \int_{t_1}^{t} \int_{\mathbb{R}^d} \sum_i |u^i|^{p-2}(s, x)|\nabla u^i(s, x)|^2 \, dx \, ds \]

(2.28)

\[ \leq \int_{t_1}^{t} |f(s)|^p_{L^p} \, ds + \int_{t_1}^{t} K |u(s)|_{L^p} \, ds \]

Dividing (2.28) on \((t - t_1)\) and tending \( t_1 \) to \( t \) we get

\[ \frac{d}{dt} |u(t)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} + \nu p(p - 1) \int_{\mathbb{R}^d} \sum_i |u^i|^{p-2}(t, x)|\nabla u^i(t, x)|^2 \, dx \leq |f(t)|^p_{L^p} + K |u(t)|^p_{L^p}, t \in (t_0, T). \]

(2.29)

Denote

\[ v(t) = |u(t_0)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} e^{K(t-t_0)} + \int_{t_0}^{t} |f(s)|^p_{L^p} e^{K(t-s)} \, ds, t \in [t_0, T]. \]

Then \( v(t_0) = |u(t_0)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} \) and

\[ \frac{d}{dt} v(t) = |f(t)|^p_{L^p} + K v(t), t \in [t_0, T]. \]

(2.30)

Consequently, \( |u(t)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} - v(t_0) = 0 \) and

\[ \frac{d}{dt} (|u(t)|^p_{L^p(\mathbb{R}^d, \mathbb{R}^d)} - v(t)) \leq K \left( |u(t)|^{p}_{L^p(\mathbb{R}^d, \mathbb{R}^d)} - v(t) \right), t \in (t_0, T), \]

(2.31)
Therefore, we can deduce that

\[(2.32)\]

\[|u(t)|_{L^p(\mathbb{R}^d, \mathbb{R}^d)}^p \leq |u(t_0)|_{L^p(\mathbb{R}^d, \mathbb{R}^d)}^p e^{K(t-t_0)} + \int_{t_0}^{t} |f(s)|_{L^p}^p e^{K(t-s)} ds, t \in [t_0, T].\]

Tending \(t_0\) to 0 we get inequality \(2.16\). Furthermore, in the case of \(K \geq 0\), inserting inequality \(2.32\) in the right part of inequality \(2.27\) we get

\[(2.33)\]

\[|u(t)|_{L^p(\mathbb{R}^d, \mathbb{R}^d)}^p + \nu p (p - 1) \int_{t_0}^{t} \int_{\mathbb{R}^d} |u^i|^{|p-2}(s, x)|\nabla u^i(s, x)|^2 \, dx \, ds \leq |u(t_0)|_{L^p(\mathbb{R}^d, \mathbb{R}^d)}^p e^{K(t-t_0)} + \int_{t_0}^{t} |f(s)|_{L^p}^p e^{K(t-s)} ds, t \in [t_0, T].\]

Tending \(t_0\) to 0 we get inequality \(2.15\). Tending \(t\) to \(T_0\) we get contradiction. \(\square\)

**Corollary 2.6.** Fix \(p > d\). Assume that \(u_0 \in L^p(\mathbb{R}^d, \mathbb{R}^d)\), \(f \in L^\infty([0, T], L^\frac{2}{d}(\mathbb{R}^d, \mathbb{R}^d) \cap L^p(\mathbb{R}^d, \mathbb{R}^d))\), \(f \in C^{\theta}(\mathbb{E}, T), H^4_p(\mathbb{R}^d, \mathbb{R}^d)\), \(\forall \mathbb{E} > 0\), \(\text{div} \, f \in L^\infty(0, T; L^\infty(\mathbb{R}^d))\). Then there exists unique local solution \(u \in L^\infty(0, T_0; L^p(\mathbb{R}^d, \mathbb{R}^d)) \cap C^{1,2}((0, T_0] \times \mathbb{R}^d)\) for some \(T_0 < T\). Furthermore, if this local solution satisfies conditions \(2.11\), \(2.12\), \(2.13\) on interval \([0, T_0]\) than it is global solution i.e. \(T_0 = T\) and energy type inequality \(2.15\) is satisfied (with corresponding \(p\)).

**Proof of Corollary 2.6.** Existence of local solution follows from Corollary 1.6. Local solution satisfies condition \(2.14\) by Sobolev Embedding Theorem (Proposition 2.4, p.5 of [18]). Now Proof immediately follows from Lemma 1.7 and Theorem 2.2. \(\square\)

**Remark 2.7.** It is possible to prove in the same way similar theorem and corollary for torus. In this case, conditions \(2.12\), \(2.13\) and \(2.14\) will disappear.

**Remark 2.8.** If initial condition \(u_0\) and force \(f\) are irrotational (i.e. \(\text{curl} \, u_0 = \text{curl} \, f = 0\) than \(\text{curl} \, u(t) = 0\) and condition \(2.11\) is satisfied.

**Remark 2.9.** Let us consider case \(d = 2\) and assume for simplicity that \(\text{div} \, f = 0\). Then on the boundary of \(D^+\) we will have that

\[|\nabla u|^2(t, x) = |\text{curl} \, u|^2(t, x), (t, x) \in \partial D^+\]

Therefore, we can deduce that

\[2 \det \nabla u(t, x) = (\text{div} \, u)^2(t, x), (t, x) \in \partial D^+.\]

Similarly, we would get

\[2 \det \nabla u(t, x) > (\text{div} \, u)^2(t, x), (t, x) \in D^-\]
As a result, one can consider instead of the assumption that vorticity is bounded, assumption that jacobian is bounded. It would be interesting to understand physical meaning of such assumption. It would also be interesting to acquire better understanding of the structure of the boundary $\partial D^+$.

**Remark 2.10.** If we consider 2D Navier-Stokes equation without force then equation for pressure has form

$$\Delta p = -2 \det \nabla v,$$

where $p$ is a pressure, $v$ is a velocity. As a result, we have that $p$ is a subharmonic (resp. superharmonic) function if $v$ conserves (resp. changes) orientation. It would be of interest to understand physical consequences of this fact.

In the next theorem we show the application of the Corollary 2.6 to the Kardar-Zhang-Parisi (KZP) equation. We formulate it for torus to get rid of the assumptions on behavior of the solution when $|x| \to \infty$.

**Theorem 2.11.** Fix $p > d$. Let $\psi_0 \in H^{1,p}(\mathbb{T}^d)$, $h \in L^\infty([0,T], H^{1,p}(\mathbb{T}^d) \cap H^{1,p}(\mathbb{T}^d))$, $h \in C^\theta([\varepsilon,T], H^{1,p}(\mathbb{T}^d))$, $\forall \varepsilon > 0$, $\Delta h \in L^\infty(0,T; L^\infty(\mathbb{T}^d))$. Then there exists unique solution $\psi^\nu \in C(0,T; H^{1,p}(\mathbb{T}^d)) \cap C^1((0,T] \times \mathbb{T}^d)$ of equation

$$\frac{\partial \psi^\nu}{\partial t} + |\nabla \psi^\nu|^2 = \nu \Delta \psi^\nu + h$$

(2.34)

$$\psi^\nu(0) = \psi_0, t \in [0,T], \nu > 0.$$  

Furthermore,

$$|\psi^\nu(t)|^p_{H^{1,p}} \leq |\psi_0|^p_{H^{1,p}} e^{Kt} + \int_0^t |h(s)|^p_{H^{1,p}} e^{K(t-s)} ds, t > 0,$$

(2.36)

where $K = K(h, p, \psi_0)$.

**Proof of Theorem 2.11** Proof immediately follows from Corollary 2.6 and the fact that gradient of solution of KZP equation is a solution of Burgers equation.

We can notice that estimate (2.36) is uniform w.r.t. $\nu$. This leads us to the following Corollary.

**Corollary 2.12.** Fix $p > d$. Let $\psi_0 \in H^{1,p}(\mathbb{T}^d)$, $\nabla h \in L^1(0,T; L^\infty(\mathbb{T}^d))$, $\Delta h \in L^\infty(0,T; L^\infty(\mathbb{T}^d))$. Then there exists unique viscosity solution $\psi \in C(0,T; H^{1,p}(\mathbb{T}^d))$ of equation

$$\frac{\partial \psi}{\partial t} + |\nabla \psi|^2 = h$$

(2.37)

$$\psi(0) = \psi_0, t \in [0,T].$$

Furthermore,

$$|\psi(t)|^p_{H^{1,p}} \leq |\psi_0|^p_{H^{1,p}} e^{Kt} + \int_0^t |h(s)|^p_{H^{1,p}} e^{K(t-s)} ds, t > 0,$$

(2.39)
where $K = K(h, p, d, \psi_0)$.

**Remark 2.13.** The main point of this corollary is an estimate (2.39). Existence and uniqueness of viscosity solutions has been shown in many works (see survey [4], books [7],[1] and references therein).

**Proof.** We can find $h^\nu \in L^\infty([0,T],H^{1,p}(\mathbb{T}^d) \cap H^{1,1}(\mathbb{T}^d))$, $h^\nu \in C^\theta(\bar{\mathbb{T}^d})$, $\forall \nu > 0$, $\triangle h^\nu \in L^\infty(0,T;L^\infty(\mathbb{T}^d))$ such that

$$\int_0^T |\nabla h^\nu(s) - \nabla h(s)|_{L^\infty(\mathbb{T}^d)} ds \to 0, \nu \to 0.$$

Let $\{\psi^\nu\}_{\nu > 0} \in C(0,T;H^{1,p}(\mathbb{T}^d)) \cap C^{1,2}((0,T) \times \mathbb{T}^d)$ be sequence of solutions of the system (2.34)-(2.35) where we use $h^\nu$ instead of $h$ in equality (2.34). Since $H^{1,p}(\mathbb{T}^d) \subset C(\mathbb{T}^d)$, $p > d$ and estimate (2.36) we have uniform w.r.t. $\nu$ estimate

$$|\psi^\nu|_{C(0,T;C(\mathbb{T}^d))} \leq K(T,\psi_0,h,d), T > 0, p > d.$$ (2.40)

Then according to Theorem 1.1, p. 175 in [3] we have that there exist uniformly bounded upper continuous subsolution $\psi^* = \lim sup_{\nu \to 0} \psi^\nu$ and uniformly bounded lower continuous supersolution $\psi_* = \lim inf_{\nu \to 0} \psi^\nu$ of system (2.37)-(2.38). Therefore, by comparison principle for viscosity solutions of Hamilton-Jacobi equations (see Theorem 2, p.585 and Remark 3, p. 593 of [5]), $\psi^* \leq \psi_*$ and $\psi = \psi^* = \psi_*$. Thus, $\psi^\nu$ locally uniformly converges to unique viscosity solution $\psi$ of equation (2.37)-(2.38). Estimate (2.36) implies that $\psi$ satisfies (2.39). \qed

### 3. APPENDIX

Let $S$ be an interval of the real line of the form $[a,b]$ or $[a,\infty)$ with $a < b$. Denote $\hat{S}$ interior part of $S$.

**Lemma 3.1 (Gronwall lemma in differential form).** Let $u, \beta \in C(S)$, $u$ is differentiable in $\hat{S}$ and

$$u'(t) \leq \beta(t)u(t), t \in \hat{S}.$$

Then

$$u(t) \leq u(a)e^{\int_a^t \beta(s)ds}.$$

**Remark 3.2.** Notice that there is no assumption that $\beta$ is nonnegative.

**Proof of Lemma 3.1** Let $v(t) = e^{\int_a^t \beta(s)ds}, t \in S$. Then

$$v'(t) = \beta(t)v(t), t \in S.$$

Notice that $v(t) > 0$, $t \in S$ and, therefore,

$$\frac{d}{dt} \frac{u(t)}{v(t)} = \frac{u'v - uv'}{v^2} \leq \frac{\beta uv - \beta vu}{v^2} = 0, t \in \hat{S},$$
i.e.
\[
\frac{u(t)}{v(t)} \leq \frac{u(a)}{v(a)} = u(a),
\]
and the result follows. \(\square\)

**Proof of Theorem 1.1.** Denote
\[
Q_T = \{ u \in C([0, T]; X) \cap C((0, T], Y) \cap C((0, T]; Z) ||u||_{Q_T} < \infty \}
\]
where
\[
|| \cdot ||_{Q_T} = || \cdot ||_{C([0, T]; X)} + \sup_{t \in (0, T]} t^b|u(t)|_Y + \sup_{t \in (0, T]} t^c|u(t)|_Z.
\]
Then \(Q_T\) is a complete metric space.

Fix \(u_0 \in X, g \in L^\infty(0, T; W)\) and let \(\alpha, \beta \) and \(T_1 > 0\) be such that
\[
|R_tu_0 + \int_0^t R_{t-s}gds|_X \leq \alpha, t \in (0, T].
\]
(3.1)

\[
t^b|R_tu_0 + \int_0^t R_{t-s}gds|_Y \leq \beta,
\]
(3.2)

\[
t^c|R_tu_0 + \int_0^t R_{t-s}gds|_Z \leq \beta, t \in (0, T_1].
\]
(3.3)

Existence of \(\alpha\) satisfying (3.1) follows from the fact that \(\{R_t\}_{t \geq 0}\) is \(C_0\)-semigroup in \(X\) and following estimate
\[
|\int_0^t R_{t-s}gds|_X \leq C \int_0^t \frac{|g(s)|_W}{|t-s|^a}ds \leq C|g|_{L^\infty(0, T; W)}t^{1-a}.
\]
(3.4)

Estimate (3.4) and assumptions b) and c) of the theorem imply that for any \(\beta > 0\) inequalities (3.2), (3.3) are true for all sufficiently small \(T_1 > 0\).

Let
\[
M(\alpha, \beta, T) = \left\{ u \in Q_T ||u||_{C([0, T]; X)} \leq 2\alpha, \sup_{t \in (0, T]} t^b|u(t)|_Y \leq 2\beta, \right. \\
\left. \sup_{t \in (0, T]} t^c|u(t)|_Z \leq 2\beta \right\}.
\]

Then \(M(\alpha, \beta, T)\) endowed with norm \(Q_T\) is also complete metric space and we will show that if \(\beta, T = T_1 > 0\) are small enough than the map \(F : u \mapsto R_tu_0 + \int_0^t R_{t-s}gds + \int_0^t R_{t-s}G(u(s))ds\) is a contraction on \(M(\alpha, \beta, T)\).
Similarly, we get that if \( Q_t \leq C \beta < 1 \) then \( F \) is a contraction on \( Q_t \). Furthermore, it follows from inequalities \( (3.4), (3.5), (3.6) \) and \( (3.7) \) that \( F \) is a map from \( M(\alpha, \beta, T) \) to \( M(\alpha, \beta, T) \). Thus there exists a unique fixed point \( u \) of the map \( F : M(\alpha, \beta, T) \rightarrow M(\alpha, \beta, T) \). It remains to show that \( u \) has designated asymptotic behavior when \( t \rightarrow 0 \). It can be done in the same way as in \([12]\), p.223-224.

\[ \square \]

**Acknowledgement.** The authors are indebted to the anonymous referee for helpful comments.
REFERENCES

[1] Bardi, M.; Crandall, M. G.; Evans, L. C.; Soner, H. M.; Souganidis, P. E. VISCOSITY SOLUTIONS AND APPLICATIONS, Lectures given at the 2nd C.I.M.E. Session held in Monteagutini Terme, June 12–20, 1995. Edited by I. Capuzzo Dolcetta and P. L. Lions. Lecture Notes in Mathematics, 1600. Fondazione C.I.M.E., [C.I.M.E. Foundation] Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 1997. x+259 pp.

[2] Ben-Naim E., Chen S.Y., Doolen G.D. and Redner S. Shocklike dynamics of inelastic gases, Phys. Rev. Lett. E 54 (1996), 2564-2572

[3] Barles, G. A new stability result for viscosity solutions of nonlinear parabolic equations with weak convergence in time. C. R. Math. Acad. Sci. Paris 334 (2006), no. 3, 173–178.

[4] Crandall, M. G.; Ishii, H; Lions, P.-L. User’s guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67.

[5] Crandall, M. G.; Ishii, H.; Lions, P.-L. Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited. J. Math. Soc. Japan 39 (1987), no. 4, 581–596.

[6] M. H. Protter, H. F. Weinberger, MAXIMUM PRINCIPLES IN DIFFERENTIAL EQUATIONS, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1967 x+261 pp.

[7] Fleming, W. H.; Soner, H. M. CONTROLLED MARKOV PROCESSES AND VISCOSITY SOLUTIONS, Applications of Mathematics (New York), 25. Springer-Verlag, New York, 1993. xvi+428 pp.

[8] Fujita H.; Kato T. On the Navier-Stokes initial value problem. I. Arch. Rational Mech. Anal. 16 1964 269–315.

[9] Kato, T.; Fujita, H. On the nonstationary Navier-Stokes system. Rend. Sem. Mat. Univ. Padova 32 1962 243–260.

[10] Henry, D. GEOMETRIC THEORY OF SEMILINEAR PARABOLIC EQUATIONS, Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981. iv+348 pp.

[11] Kato T., Strong $L^p$-solutions of the Navier-Stokes equation in $R^m$, with applications to weak solutions. Math. Z. 187 (1984), no. 4, 471–480.

[12] Weissler, Fred B. The Navier-Stokes initial value problem in $L^p$. Arch. Rational Mech. Anal. 74 (1980), no. 3, 219–230.

[13] Fabes, E. B.; Jones, B. F.; Riviere, N. M. The initial value problem for the Navier-Stokes equations with data in $L^p$. Arch. Rational Mech. Anal. 45 (1972), 222–240.

[14] Beale, J. T.; Kato, T.; Majda, A. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Comm. Math. Phys. 94 (1984), no. 1, 61–66.

[15] Ikeda, N., Watanabe, S. STOCHASTIC DIFFERENTIAL EQUATIONS AND DIFFUSION PROCESSES. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981. xiv+464 pp.

[16] Lunardi, A. ANALYTIC SEMIGROUPS AND OPTIMAL REGULARITY IN PARABOLIC PROBLEMS, Progress in Nonlinear Differential Equations and their Applications, 16. Birkhauser Verlag, Basel, 1995. xviii+424 pp.

[17] Serre D. SYSTEMS OF CONSERVATION LAWS CUP, 1999

[18] Taylor, M. E. PARTIAL DIFFERENTIAL EQUATIONS. III. NONLINEAR EQUATIONS, Applied Mathematical Sciences, 117. Springer-Verlag, New York, 1997. xxii+608 pp.

School of Mathematics and Statistics, University of NSW, Sydney, Australia