Undeformed (additive) energy conservation law in Doubly Special Relativity

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Abstract

All the Doubly Special Relativity (DSR) models studied in literature so far involve a deformation of the energy conservation rule that forces us to release the hypothesis of the additivity of the energy for composite systems. In view of the importance of the issue for a consistent formulation of a DSR statistical mechanics and a DSR thermodynamics, we show that DSR models preserving the usual (i.e. additive) energy conservation rule can be found. These models allow the construction of a DSR-covariant extensive energy. The implications of the analysis for the dynamics of DSR-covariant multiparticle systems are also briefly discussed.

1 Introduction

Doubly Special Relativity has been proposed as a possible two-scale extension of Special Relativity [1,2,3,4,5]. The introduction of the second invariant scale \( \lambda \), eventually connected with the Planck length/energy, leads to some departures from Special Relativity already manifesting at the kinematical level. It has been clear from the beginning [1], that the two most immediate consequences of the introduction of this new invariant scale are: for a single particle, the possibility to get a modification of the energy-momentum dispersion relation (and a consequent possible modification of the relation between particle’s velocity and particle’s energy, see also [6,7]); for composite systems, the modification of the energy-momentum conservation rules (see also [8] and references therein).

On a theoretical ground however, whereas the modification of the free propagation does not seem to lead to inconsistencies, the modification of the energy-momentum conservation laws appears to be troublesome. In [9] (see also [10]) a procedure to construct all-order (in the Planck scale) energy-momentum conservation rules has been proposed. When applied to the known DSR models, this procedure produces modifications of both the energy and the spatial momentum conservation rule. Similar modifications are encountered adopting procedures based on non-trivial co-products (see [11,12,13] and references therein).

In these scenarios, in which the energy and the spatial momentum conservation laws are both modified, the deformation of the energy conservation rule appears to be particularly troublesome since it directly leads to difficulties in the construction of DSR-covariant multiparticle systems, statistical mechanics and thermodynamics. Nevertheless DSR-based statistical mechanics has been proposed in [14] and there are a number of studies in the field of cosmology and black-hole physics (see for example [15,16,17,18,19]) that construct thermodynamical quantities from DSR-motivated energy-momentum dispersion relations.

As first noticed in [20], the fact that the total energy of the system cannot be covariantly defined as the sum of the energies of the particles composing the system \( E_{\text{system}} \neq E_1 + E_2 + \ldots + E_n \), has the implication that the energy fails to be a fully DRS-covariant extensive quantity.

Moreover, if we define with \( \bigoplus \) the composition law of the energies and with \( E_1, E_2, E_3 \) the energies of three subsystems which we think the original system composed of, for all the models proposed in literature so far, we get \( E_S = E_1 \bigoplus E_2 \bigoplus E_3 \neq E_1 \bigoplus (E_2 \bigoplus E_3) \neq (E_1 \bigoplus E_2) \bigoplus E_3 \), which means that the energy of the system should depend on the particular way we order the subsystems. The same conclusion holds for the mass of the system (i.e. the zero-momentum energy) that should change depending on the particular choice of the decomposition.

Another problem related to the missing additivity of the energy is the so called “spectator problem”. This problem manifests in the fact that the evolution of a single particle strongly depends on how one

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considers the particle as a part of a larger system. We will discuss in more detail this problem in the next sections.

All the above mentioned problems are automatically absent in every relativity scheme in which the energy of a set of particles is simply the sum of the energies of the single particles composing the system, as it is the case in Galilean Relativity and in Special Relativity. In the next sections we will show that also in DSR there is no need to renounce to the hypothesis of the additivity of the energy.

2 DSR models preserving the additivity of the energy

2.1 The general procedure

We start our analysis by requiring the covariance of the undeformed energy conservation law in a $n$-to-$m$ particle scattering process:

$$
\sum_{\alpha=1}^{n} E^{(\alpha)} = \sum_{\beta=1}^{m} E^{(\beta)},
$$

under the boost action obtained by commutators of the type $\delta_k E = \xi [N_k, E]$, where $[N_k, E] = \pi_k (\vec{p}, \lambda)$. One gets:

$$
\sum_{\alpha=1}^{n} \pi_k^{(\alpha)} (\vec{p}, \lambda) = \sum_{\beta=1}^{m} \pi_k^{(\beta)} (\vec{p}, \lambda).
$$

The covariance of (2) then implies that

$$
\sum_{\alpha=1}^{n} \sum_{i=1}^{3} [N_j, p_i^{(\alpha)}] \frac{\partial \pi_k^{(\alpha)}}{\partial p_i} = \sum_{\beta=1}^{m} \sum_{i=1}^{3} [N_j, p_i^{(\beta)}] \frac{\partial \pi_k^{(\beta)}}{\partial p_i}.
$$

The above equations are simply solved if the commutators $[N_j, p_i]$ satisfy the relations

$$
\sum_{i=1}^{3} [N_j, p_i] \frac{\partial \pi_k}{\partial p_i} = E \delta_{jk}.
$$

The linear system (4) can be easily solved with respect to $[N_i, p_j]$ providing the action of the boosts on the spatial momenta:

$$
[N_i, p_j] = E \left( \frac{\partial \pi}{\partial p} \right)_{ij}^{-1},
$$

where we have used the shorthand notation $\left( \frac{\partial \pi}{\partial p} \right)_{ij} = \frac{\partial \pi_i}{\partial p_j}$.

From the action of the boosts on the energy and on the spatial momenta one immediately deduces the form of the dispersion relation:

$$
E^2 - \pi^2 (\vec{p}, \lambda) = m^2.
$$

2.2 An explicit realization

To do an explicit example of a DSR model allowing undeformed energy conservation law, let us consider boost actions given by the commutators:

$$
[N_i, E] = \frac{p_i}{(1 - \lambda p)},
$$

$$
[N_i, p_j] = E (1 - \lambda p) \left( \delta_{ij} - \lambda \frac{p_i p_j}{p} \right).
$$

\footnote{We assume, as usual, that the Lorentz group $SO(3, 1)$ is undeformed.}
The energy-momentum dispersion relation then is given by

\[
E = \sqrt{m^2 + \frac{p^2}{(1 - \lambda p)^2}},
\]

(9)

It is straightforward to verify that, when the spatial momentum approaches the (invariant) Planck momentum, \( p \rightarrow \lambda^{-1} \), the energy diverges.

Covariant generalizations of the conservation rules are:

\[
\sum_{\alpha=1}^{n} E^{(\alpha)} = \sum_{\beta=1}^{m} E^{(\beta)},
\]

(10)

\[
\sum_{\alpha=1}^{n} \frac{\vec{p}^{(\alpha)}}{(1 - \lambda p^{(\alpha)})} = \sum_{\beta=1}^{m} \frac{\vec{p}^{(\beta)}}{(1 - \lambda p^{(\beta)})},
\]

(11)

that preserve the additivity of the energy.

3 Implications for the multiparticle dynamics

A “spectator problem” manifests in DSR when one tries to define the dynamics of DSR systems through the usual canonical/Heisenberg evolution scheme. Technically the problem is due to the fact that the time-evolution operator is bilinear in its two arguments, whereas the Hamiltonian of the system, which is the generator of the time evolution, depends nonlinearly on the Hamiltonians of the subsystems.

To be more specific let us consider a \( n \)-particle system whose Hamiltonian is

\[
H_S = H_S (H_1, H_2, ..., H_n),
\]

(12)

and a general observable \( O^{(k)}(p^{(k)}) \) associated to the \( k \)-th particle of the system.

We get for the evolution of the observable

\[
[O^{(k)}(p^{(k)}), H_S] = [O^{(k)}(p^{(k)}), H_k] \frac{\partial H_S(p^{(1)}, p^{(2)}, ..., p^{(n)})}{\partial H_k},
\]

(13)

If the additivity of the energy is missing, the evolution of the observable \( O^{(k)}(p^{(k)}) \) depends, through the term \( \frac{\partial H_S(p^{(1)}, p^{(2)}, ..., p^{(n)})}{\partial H_k} \), on the state of the other particles composing the system. In this case it would be difficult even to define the notion of “free particle” and strong (long-range) interactions would be hard to avoid.

If instead we adopt additive energy composition law, being

\[
H_S = H_1 + H_2 + ... + H_n,
\]

(14)

and

\[
\frac{\partial H_S(p^{(1)}, p^{(2)}, ..., p^{(n)})}{\partial H_k} = 1,
\]

(15)

we obtain

\[
[O^{(k)}(p^{(k)}), H_S] = [O^{(k)}(p^{(k)}), H_k],
\]

(16)

which is free from the unlikely non-local effects.
4 Comparison with a previous analysis

In Ref. [21] a model has been studied in which both the energy and the spatial momentum compose linearly. The energy-momentum dispersion relation analyzed in Ref. [21] is the one originally proposed in [10]:

\[ E^2 = p^2 + m^2 \left( 1 + \frac{2k}{m} E \right). \]  \hspace{1cm} (17)

The authors of Ref. [21] find that the energy-momentum dispersion relation (17) is covariant under the linear, but inhomogenous, boost action given by

\[ p'_{\mu} = \Lambda_{\mu}^{\nu} p_{\nu} + (\delta_{\mu}^{\nu} - \Lambda_{\mu}^{\nu})mk, \]  \hspace{1cm} (18)

where the Einstein notation has been used for the energy-momentum vector \( p_{\mu} \equiv (E, \vec{p}) \) and for the Lorentz transformation tensor \( \Lambda_{\mu}^{\nu} \). \( k \) is a deformation parameter.

The linearity of the composition of both the energy and the spatial momentum follows directly from the linearity of the boost action (18).

However, from the point of view of a scattering process, as analyzed in the previous sections, important differences emerge with respect to the special relativistic conservation laws. In fact, if we consider an \( n \)-to-\( m \) particle scattering process, the request of covariance of the conservation laws:

\[ \sum_{\alpha=1}^{n} E^{(\alpha)} = \sum_{\beta=1}^{m} E^{(\beta)}, \]  \hspace{1cm} (19)

\[ \sum_{\alpha=1}^{n} \vec{p}^{(\alpha)} = \sum_{\beta=1}^{m} \vec{p}^{(\beta)}, \]  \hspace{1cm} (20)

under the action of (18), also implies that

\[ \sum_{\alpha=1}^{n} km^{(\alpha)} = \sum_{\beta=1}^{m} km^{(\beta)}. \]  \hspace{1cm} (21)

Equation (21) results as a constraint on the allowed particle scattering. To better understand the meaning of this constraint we have to make explicit the relation among the parameter \( k \), the particle mass \( m \), and the Planck parameter \( \lambda \) we used in this paper. Various possibilities have been discussed in the same Ref. [21]. The cases there taken in to the account are: \( k \propto \lambda \), \( k \propto m\lambda \) and \( k \propto \lambda/m \).

In the first case \( (k \propto \lambda) \) we would get

\[ \sum_{\alpha=1}^{n} m^{(\alpha)} = \sum_{\beta=1}^{m} m^{(\beta)}, \]  \hspace{1cm} (22)

in the second case \( (k \propto m\lambda) \) we would get

\[ \sum_{\alpha=1}^{n} m^2^{(\alpha)} = \sum_{\beta=1}^{m} m^2^{(\beta)}, \]  \hspace{1cm} (23)

whereas, in the third case \( (k \propto \lambda/m) \), we would obtain \( n = m \).

In the first two cases \( (k \propto \lambda, k \propto m\lambda) \) the constraints regard the relations between the masses of the incoming particles and that of the outcoming particles. In the third case \( (k \propto \lambda/m) \), the constraint only allows scattering processes preserving the particle number \( (n = m) \). All the constraints we have found are violated in the observed scattering processes\(^{4}\). Thus, we have to conclude that the model presented in Ref. [21] is not viable from the perspective adopted in this paper.

\(^{4}\)Notice that for massless particles the constraints are trivially satisfied but, in that case, the model simply reduces to Special Relativity.
5 Conclusions

We have shown that the non-additivity of the energy is not a common feature of DSR theories. In the framework of DSR theories with additive-energy composition law one avoids many problems that mine the consistency of DSR statistical mechanics and DSR thermodynamics. In particular: i) the energy (and the mass) of a composite system is a extensive quantity; ii) the energy (and the mass) of a system does not depend on how one thinks a system decomposed in subsystems; iii) the time evolution of multiparticle systems, obtainable by means of the usual canonical/Heisenberg formalism, is free from long-range nonlocal effects.

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