FREE TOPOLOGICAL VECTOR SPACES

SAAK S. GABRIELYAN AND SIDNEY A. MORRIS

Abstract. We define and study the free topological vector space $V(X)$ over a Tychonoff space $X$. We prove that $V(X)$ is a $k_{\omega}$-space if and only if $X$ is a $k_{\omega}$-space. If $X$ is infinite, then $V(X)$ contains a closed vector subspace which is topologically isomorphic to $V(\mathbb{N})$. It is proved that if $X$ is a $k$-space, then $V(X)$ is locally convex if and only if $X$ is discrete and countable. If $X$ is a metrizable space it is shown that: (1) $V(X)$ has countable tightness if and only if $X$ is separable, and (2) $V(X)$ is a $k$-space if and only if $X$ is locally compact and separable. It is proved that $V(X)$ is a barrelled topological vector space if and only if $X$ is discrete. This result is applied to free locally convex spaces $L(X)$ over a Tychonoff space $X$ by showing that: (1) $L(X)$ is quasibarrelled if and only if $X$ is discrete, and (2) $L(X)$ is a Baire space if and only if $X$ is finite.

1. Introduction.

Until recently almost all papers in topological vector spaces restricted themselves to locally convex spaces. However in recent years a number of questions about non-locally convex vector spaces have arisen. All topological spaces are assumed here to be Tychonoff and all vector spaces are over the field of real numbers $\mathbb{R}$. The free topological group $F(X)$, the free abelian topological group $A(X)$ and the free locally convex space $L(X)$ over a Tychonoff space $X$ were introduced by Markov [24] and intensively studied over the last half-century, see for example [8, 13, 17, 22, 33, 35]. It has been known for a half a century that the (Freyd) Adjoint Functor Theorem ([24] or Theorem A3.60 of [14]) implies the existence and uniqueness of $F(X), A(X)$ and $L(X)$. This paper focuses on free topological vector spaces. One surprising fact is that free topological vector spaces in some respect behave better than free locally convex spaces.

2. Basic properties of free topological vector spaces

Definition 2.1. The free topological vector space $V(X)$ over a Tychonoff space $X$ is a pair consisting of a topological vector space $V(X)$ and a continuous mapping $i = i_X : X \to V(X)$ such that every continuous mapping $f$ from $X$ to a topological vector space (tvs) $E$ gives rise to a unique continuous linear operator $\bar{f} : V(X) \to E$ with $f = f \circ i$.

In analogy with the Graev free abelian topological group over a Tychonoff space $X$ with a distinguished point $p$, we can define the Graev free topological vector space $V_G(X, p)$ over $(X, p)$.

Definition 2.2. The Graev free topological vector space $V_G(X, e)$ over a Tychonoff space $X$ with a distinguished point $e$ is a pair consisting of a topological vector space $V_G(X, e)$ and a continuous mapping $i = i_X : X \to V_G(X, e)$ such that $i(e) = 0$ and every continuous mapping $f$ from $X$ to a topological vector space $E$ with $f(e) = 0$ gives rise to a unique continuous linear operator $\bar{f} : V_G(X, e) \to E$ with $f = f \circ i$.

2000 Mathematics Subject Classification. Primary 46A03; Secondary 54A25, 54D50.

Key words and phrases. free topological vector space, free locally convex space.
We shall use the notation: for a subset $A$ of a vector space $E$ and a natural number $n \in \mathbb{N}$ we denote by $\text{sp}_n(A)$ the following subset of $E$

$$\text{sp}_n(A) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \in [-n, n], x_i \in A, \forall i = 1, \ldots, n\},$$

and set $\text{sp}(A) := \bigcup_{n \in \mathbb{N}} \text{sp}_n(A)$, the span of $A$ in $E$.

As $X$ is a Tychonoff space, the mapping $i_X$ is an embedding. So we identify the space $X$ with $i(X)$ and regard $X$ as a subspace of $\mathbb{V}(X)$.

**Theorem 2.3.** Let $X$ be a Tychonoff space and $e \in X$ a distinguished subspace. Then

(i) $\mathbb{V}(X)$ and $\mathbb{V}_G(X,e)$ exist (and are Hausdorff);

(ii) $\text{sp}(X) = \mathbb{V}(X)$ and $X$ is a vector space basis for $\mathbb{V}(X)$;

(iii) $\text{sp}(X) = \mathbb{V}_G(X,e)$ and $X \setminus \{e\}$ is a vector space basis for $\mathbb{V}_G(X,e)$;

(iv) $\mathbb{V}(X)$ and $\mathbb{V}_G(X,e)$ are unique up to isomorphism of topological vector spaces;

(v) $X$ is a closed subspace of $\mathbb{V}(X)$ and $\mathbb{V}_G(X,e)$;

(vi) $\text{sp}_n(X)$ is closed in $\mathbb{V}(X)$ and $\mathbb{V}_G(X,e)$, for every $n \in \mathbb{N}$;

(vii) if $q : X \to Y$ is a quotient map of Tychonoff spaces $X$ and $Y$, then $\mathbb{V}(Y)$ is a quotient topological vector space of $\mathbb{V}(X)$;

(viii) if $Y$ is a Tychonoff space with a distinguished point $p$ and $X \setminus Y$ is the wedge sum of $(X,e)$ and $(Y,p)$, then $\mathbb{V}_G(X,e) \times \mathbb{V}_G(Y,p) = \mathbb{V}_G(X \setminus Y, (e,p))$.

**Proof.** (i)-(iv) follow from the Adjoint Functor Theorem.

(v)-(vi) We consider only the case $\mathbb{V}(X)$. Let $\beta X$ be the Stone-Čech compactification of $X$. Then, by the definition of free topological vector space, the natural map $\beta : X \to \beta X \subseteq \mathbb{V}(\beta X)$ can be extended to a continuous injective linear operator $\bar{\beta} : \mathbb{V}(X) \to \mathbb{V}(\beta X)$. Since $\beta X$ and $\text{sp}_n(\beta X)$ are compact subsets of $\mathbb{V}(\beta X)$, $X = \bar{\beta}^{-1}(\beta X)$ and $\text{sp}_n(X) = \bar{\beta}^{-1}(\text{sp}_n(\beta X))$ by the injectivity of $\bar{\beta}$, we obtain that $X$ and $\text{sp}_n(X)$ are closed subsets of $\mathbb{V}(X)$, for every $n \in \mathbb{N}$.

(vii) Let $\bar{q} : \mathbb{V}(X) \to \mathbb{V}(Y)$ be a continuous linear operator extending $q$. Set $H = \ker(\bar{q})$ and let $j : \mathbb{V}(X) \to \mathbb{V}(X)/H$ be the quotient map. Denote by $f : \mathbb{V}(X)/H \to \mathbb{V}(Y)$ the induced continuous linear map. We have to show that the topology of $\mathbb{V}(X)/H$ coincides with the topology of $\mathbb{V}(Y)$.

Let $E$ be an arbitrary tvs and $t : Y \to E$ a continuous map. Then $q \circ t : X \to E$ is continuous. So there is a unique continuous extension $\tilde{q} \circ t : \mathbb{V}(X) \to E$. As $\mathbb{V}(X)/H$ is algebraically $\mathbb{V}(Y)$, we obtain the induced linear map $T : \mathbb{V}(X)/H \to E$. Now if $U$ is open in $E$, then $V := \tilde{q} \circ t^{-1}(U)$ is open in $\mathbb{V}(X)$ and hence $j(V)$ is open in $\mathbb{V}(X)/H$. Since $T(j(V)) = \tilde{q} \circ t(V) = U$, we obtain that $T$ is continuous. Finally the definition of free tvs implies that $\mathbb{V}(X)/H$ is $\mathbb{V}(Y)$.

(viii) follows from the Adjoint Functor Theorem since the left adjoint functor preserves (finite) coproducts. \qed

We shall denote the topology of $\mathbb{V}(X)$ by $\mu_X$. So $\mathbb{V}(X) = (\mathbb{V}_X, \mu_X)$, where $\mathbb{V}_X$ is a vector space with a basis $X$:

$$\mathbb{V}_X := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : n \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i \in X\}.$$  

If $v \in \mathbb{V}(X)$ (or $v \in L(X)$) has a representation

$$v = \lambda_1 x_1 + \cdots + \lambda_n x_n,$$

where $\lambda_i \in \mathbb{R} \setminus \{0\}$ and $x_i \in X$ are distinct, the set $\text{supp}(v) := \{x_1, \ldots, x_n\}$ is called the support of the element $v$.

For a subspace $Z$ of a space $X$, let $\mathbb{V}(Z,X)$ be the vector subspace of $\mathbb{V}(X)$ generated algebraically by $Z$.

**Lemma 2.4.** Let $Z$ be a closed subspace of a space $X$. Then $\mathbb{V}(Z,X)$ is a closed subspace of $\mathbb{V}(X)$.

**Proof.** Assume that $h = \sum_{i=1}^n a_i x_i$ does not belong to $\mathbb{V}(Z,X)$, where $a_i \neq 0$ and $x_i$ are distinct for all $i$. Then there is an index $i$, say $i = 1$, such that $x_1 \notin Z$. Since $X$ is Tychonoff, there is a function $f : X \to \mathbb{R}$ such that $f(x_1) = 1$ and $f(Z \cup \{x_2, \ldots, x_n\}) = 0$. Lift this function to a linear
mapping \( f : \mathbb{V}(X) \to \mathbb{R} \). Now \( \bar{f}(\mathbb{V}(Z, X)) = 0 \) and \( \bar{f}(h) = a_1 f(x_1) = a_1 \neq 0 \). If \( U \) is an open neighborhood of \( a_1 \) not containing zero, then \( \bar{f}^{-1}(U) \) is an open neighborhood of \( h \) which does not intersect \( \mathbb{V}(Z, X) \).

\[ \square \]

**Proposition 2.5.** If \( X \) is a Tychonoff space and \( Z \) is a retract (in particular, a clopen subset) of \( X \), then \( \mathbb{V}(Z) \) embeds onto a closed vector subspace of \( \mathbb{V}(X) \).

**Proof.** Let \( p : X \to Z \) be a retraction. Then \( \mathbb{V}(Z) \) is a quotient topological vector space of \( \mathbb{V}(X) \) under an extension \( \bar{p} \) of \( p \) by Theorem 2.3 (vii). As \( p(z) = z \) we obtain that \( \bar{p} \) is injective on \( \mathbb{V}(Z, X) \) and \( \bar{p}(\mathbb{V}(Z, X)) = \mathbb{V}(Z) \). So the topology \( \tau \) of \( \mathbb{V}(Z, X) \) is finer than the topology \( \mu_Z \) of \( \mathbb{V}(Z) \). Now the definition of \( \mu_Z \) implies that \( \tau = \mu_Z \). Thus \( \mathbb{V}(Z) \) embeds onto \( \mathbb{V}(Z, X) \). Finally \( \mathbb{V}(Z, X) \) is a closed vector subspace of \( \mathbb{V}(X) \) by Lemma 2.4.

\[ \square \]

**Corollary 2.6.** If \( X = Y \cup Z \) is a disjoint union of Tychonoff spaces \( Y \) and \( Z \), then \( \mathbb{V}(X) = \mathbb{V}(Y) \oplus \mathbb{V}(Z) \).

Below we generalize Corollary 2.6. First we recall some definitions.

For a non-empty family \( \{ E_i \}_{i \in I} \) of vector spaces, the direct sum of \( E_i \) is denoted by

\[ \bigoplus_{i \in I} E_i := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} E_i : x_i = 0, \text{for all but a finite number of } i \right\} , \]

and we denote by \( j_k \) the natural embedding of \( E_k \) into \( \bigoplus_{i \in I} E_i \); that is,

\[ j_k(x) = (x_i)_{i \in I} \in \bigoplus_{i \in I} E_i, \text{ where } x_i = x \text{ if } i = k \text{ and } x_i = 0 \text{ if } i \neq k. \]

If \( \{ E_i \}_{i \in I} \) is a non-empty family of topological vector spaces the final vector space topology \( T_f \) on \( \bigoplus_{i \in I} E_i \) with respect to the family of canonical homomorphisms \( j_k : E_k \to \bigoplus_{i \in I} E_i \) is the finest vector space topology on \( \bigoplus_{i \in I} E_i \) such that all \( j_k \) are continuous.

**Definition 2.7.** Let \( \mathcal{E} = \{ (E_i, T_i) \}_{i \in I} \) be a non-empty family of topological vector spaces. The topological vector space \( (E, T) \) is the coproduct of the family \( \mathcal{E} \) in the category \( \text{TVS} \) of topological vector spaces and continuous linear operators if

(i) for each \( i \in I \) there is an embedding \( j_i : E_i \to E \);

(ii) for any tvs \( V \) and each family \( \{ p_i \}_{i \in I} \) of continuous linear mappings \( p_i : E_i \to V \), there exists a unique continuous linear mapping \( p : E \to V \) such that \( p_i = p \circ j_i \) for every \( i \in I \).

The underlying vector space structure of the coproduct \( (E, T) \) is the direct sum \( \bigoplus_{i \in I} E_i \). The coproduct topology \( T \) on \( E \) coincides with the final vector space topology \( T_f \) with respect to the family of canonical homomorphisms \( j_i : E_i \to E \). Note that a coproduct of a family of tvs is unique up to topological linear isomorphism. If \( I \) is countable, then the coproduct topology \( T \) on \( E \) is the subspace topology on \( E \) induced by the box topology on the product \( \prod_{i \in I} E_i \).

**Proposition 2.8.** Let \( X = \bigcup_{i \in I} X_i \) be a disjoint sum of a nonempty family \( \{ X_i : i \in I \} \) of Tychonoff spaces. Then \( \mathbb{V}(X) \) is topologically isomorphic with the coproduct \( (E, T) \) of the family \( \{ \mathbb{V}(X_i) : i \in I \} \). If the set \( I \) is countable, then the topology \( T \) is the subspace topology on the direct sum induced by the box topology on the product \( \prod_{i \in I} \mathbb{V}(X_i) \).

**Proof.** It is clear that the underlying vector spaces of \( \mathbb{V}(X) \) and \( (E, T) \) is the direct sum \( \mathbb{V}_X = \bigoplus_{i \in I} \mathbb{V}_X \). Let \( id_{\mathbb{V}_X} : (E, T) \to \mathbb{V}(X) \) be the identity map and note that the inclusion \( p_i : \mathbb{V}(X_i) \to \mathbb{V}(X) \) is an embedding by Proposition 2.5 for every \( i \in I \). So, by the definition of coproduct topology, the map \( id_{\mathbb{V}_X} \) is continuous. Now the definition of the free vector space topology shows that \( \mu_X = T \).

\[ \square \]
The next proposition shows that the Graev free topological vector space does not depend on the distinguished point. This we prove analogously to Theorem 2 of [13].

**Proposition 2.9.** Let $X$ be a Tychonoff space and $e, p \in X$ be distinct points. Then $\mathcal{V}_G(X,e)$ and $\mathcal{V}_G(X,p)$ are topologically isomorphic.

**Proof.** Define $\varphi : (X,e) \to \mathcal{V}_G(X,p)$ setting $\varphi(x) := x - e$, where “−” denotes difference in $\mathcal{V}_G(X,p)$. Then $\varphi$ is continuous and $\varphi(e) = 0$. So there is a continuous linear map $\bar{\varphi} : \mathcal{V}_G(X,e) \to \mathcal{V}_G(X,p)$ which extends $\varphi$. Analogously, we define $\psi : (X,p) \to \mathcal{V}_G(X,e)$ setting $\psi(x) := x - p$. Then $\psi$ is continuous and $\psi(p) = 0$. So there is a continuous linear map $\bar{\psi} : \mathcal{V}_G(X,p) \to \mathcal{V}_G(X,e)$ which extends $\psi$. Now, for every $x \in (X,e)$, we have

$$\bar{\psi}\bar{\varphi}(x) = \bar{\psi}(x - e) = \bar{\psi}(x) - \bar{\psi}(e) = \psi(x) - \psi(e) = (x - p) - (e - p) = x - e = x,$$

since $e$ is the identity in the space $\mathcal{V}_G(X,e)$. Since $X$ generates $\mathcal{V}_G(X,e)$, we obtain that $\bar{\psi}\bar{\varphi}$ is the identity map of $\mathcal{V}_G(X,e)$. Analogously, $\bar{\varphi}\bar{\psi}$ is the identity map of $\mathcal{V}_G(X,p)$. Thus $\mathcal{V}_G(X,e)$ and $\mathcal{V}_G(X,p)$ are topologically isomorphic.

So we can write $\mathcal{V}_G(X,e) = \mathcal{V}_G(X)$.

Our next result is analogous to Theorem 3 of [26].

**Proposition 2.10.** Let $X$ be a Tychonoff space and $\{e\}$ be a singleton. Then

$$\mathcal{V}(X) = \mathcal{V}_G(X \sqcup \{e\}) = \mathcal{V}_G(X) \times \mathbb{R}.$$  

**Proof.** Let $\varphi : X \to E$ be a continuous map into a topological vector space $E$. Define $i : X \to X \sqcup \{e\}$ by $i(x) := x$ and $\psi : X \sqcup \{e\} \to E$ by $\psi(x) := x$ and $\psi(e) := 0$. Then we obtain the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & E \\
\downarrow{\bar{i}} & & \\
X \sqcup \{e\} & \xrightarrow{\bar{\psi}} & \mathcal{V}_G(X \sqcup \{e\}, e)
\end{array}
\]

where $\bar{\psi}$ is a linear continuous extension of $\psi$. Therefore the space $H := \mathcal{V}_G(X \sqcup \{e\}, e)$ has the universal property, and hence $\mathcal{V}(X) = H$ by the uniqueness of the free tvs over $X$. This proves the first equality.

Fix arbitrarily a point $p \in X$. The second equality follows from the first equality, Theorem 2.3, Proposition 2.9 and the following chain of equalities

$$\mathcal{V}_G(X,p) \times \mathbb{R} = \mathcal{V}_G(X,p) \times \mathcal{V}_G(\{e,p\}, p) = \mathcal{V}_G((X,p) \sqcup (\{e,p\}, p))$$

$$= \mathcal{V}_G(X \sqcup \{e\}, p) = \mathcal{V}_G(X \sqcup \{e\}, e) = \mathcal{V}(X).$$

\[
\square
\]

Graev [13] used this to show that non-homeomorphic Tychonoff spaces $X$ and $Y$ may have isomorphic the free topological groups $A(X)$ and $A(Y)$. The same holds for free topological vector spaces.

**Example 2.11.** Let $X = [0,1]$, $Y = [1,2]$, $e = 1/2 \in X$ and $p = 1 \in X \cap Y$. Then $(X,p) \sqcup (Y,p) = ([0,2],p)$ is an interval which is not homeomorphic to the space $Z := (X,e) \cap (Y,p)$. However, the Graev free topological vector spaces $\mathcal{V}_G([0,2])$ and $\mathcal{V}_G(Z)$ are topologically isomorphic by Theorem 2.3 and Proposition 2.9 since

$$\mathcal{V}_G([0,2]) = \mathcal{V}_G(X,p) \times \mathcal{V}_G(Y,p) = \mathcal{V}_G(X,e) \times \mathcal{V}_G(Y,p) = \mathcal{V}_G(Z).$$
3. Free topological vector spaces over $k_\omega$-spaces

Let $\{\langle X_n, \tau_n \rangle \}_{n \in \mathbb{N}}$ be a sequence of topological spaces such that $X_n \subseteq X_{n+1}$ and $\tau_{n+1}|_{X_n} = \tau_n$ for all $n \in \mathbb{N}$. The union $X = \bigcup_{n \in \mathbb{N}} X_n$ with the weak topology $\tau$ (i.e., $U \in \tau$ if and only if $U \cap X_n \in \tau_n$ for every $n \in \omega$) is called the inductive limit of the sequence $\{\langle X_n, \tau_n \rangle \}_{n \in \mathbb{N}}$ and it is denoted by $(X, \tau) = \lim_{\longrightarrow} (X_n, \tau_n)$. Recall that a topological space is called a $k_\omega$-space if it is the inductive limit of an increasing sequence of its compact subsets. A topological group $(G, \tau)$ is called a $k_\omega$-group if its underlying topological space is a $k_\omega$-space.

**Theorem 3.1.** If $X$ is a $k_\omega$-space and $X = \bigcup_{n \in \mathbb{N}} C_n$ is a $k_\omega$-decomposition of $X$, then $\mathbb{V}(X)$ is a $k_\omega$-space and $\mathbb{V}(X) = \bigcup_{n \in \mathbb{N}} \text{sp}_n(C_n)$ is a $k_\omega$-decomposition of $\mathbb{V}(X)$.

**Proof.** Let $(X, \tau) = \lim_{\longrightarrow} (C_n, \tau_n)$, where $C_1 \subseteq C_2 \subseteq \ldots$ are compact. For every $n \in \mathbb{N}$ denote by $X_n$ the image of the mapping $T_n : [-n, n]^n \times C_n \to \mathbb{V}(X)$ defined by

$$T_n((a_1, \ldots, a_n), (x_1, \ldots, x_n)) := a_1 x_1 + \cdots + a_n x_n.$$  
Since $\mathbb{V}(X)$ is a tvs, $T_n$ is continuous and hence $X_n = \text{sp}_n(C_n)$ is a compact subspace of $\mathbb{V}(X)$. Clearly, $\mathbb{V}(X) = \bigcup_{n \in \mathbb{N}} X_n$.

Let $\tau_n$ be the topology of the compact space $X_n$. Clearly, $\tau_{n+1}|_{X_n} = \tau_n$. So we can define the inductive limit $\mathbb{V}(X, \tau) = \lim_{\longrightarrow} (X_n, \tau_n)$. Then $(\mathbb{V}(X, \tau))$ is a $k_\omega$-space and every compact subset of $(\mathbb{V}(X, \tau))$ is contained in some $X_n$ by [14, Lemma 9.3]. It is clear that $\mu_X \leq \tau$. So to prove the proposition it is enough to show that $\tau$ is a vector space topology on $\mathbb{V}(X)$.

To this end we must prove that the map $T : \mathbb{R} \times (\mathbb{V}(X, \tau)) \to (\mathbb{V}(X, \tau))$, $T(\lambda, x, y) := \lambda x + y$, is continuous. Since $(\mathbb{V}(X, \tau))$ is a $k_\omega$-space, the space $Z := \mathbb{R} \times (\mathbb{V}(X, \tau)) \times (\mathbb{V}(X, \tau))$ is also a $k_\omega$-space. Thus, to show that $T$ is continuous we only have to show that $T$ is continuous on all compact subsets of $Z$.

Let $K$ be a compact subset of $Z$. Then $K \subseteq [-n, n] \times X_n \times X_n$ for some $n \in \mathbb{N}$. Thus

$$T(K) \subseteq T([-n, n] \times X_n \times X_n) \subseteq X_n^{2+n}.$$  
Noting that $K$ is compact and $\mu_X \leq \tau$, we see that $K$ has the same induced topology as a subset of $Z$ as it has as a subset of $\mathbb{R} \times \mathbb{V}(X) \times \mathbb{V}(X)$. Since $\tau_{|X_n^{2+n}} = \mu_X|X_n^{2+n}$ and $\mu_X$ is a vector space topology, $T : K \to X_n^{2+n}$ is continuous. So $T$ is continuous and hence $(\mathbb{V}(X, \tau))$ is a topological vector space. Thus $\tau$ is $\mu_X$ by the definition of $\mu_X$. \hfill $\Box$

**Remark 3.2.** Banakh proved Theorem 3.1 for the special case of $X$ being a submetrizable $k_\omega$-space. The research in [14] and the research in this paper were done independently.

The (Weil) completeness is one of the most important properties of topological groups. Recall that any $k_\omega$ topological group is complete by [15, Theorem 2].

**Corollary 3.3.** If $X$ is a $k_\omega$-space, then $\mathbb{V}(X)$ is complete.

**Corollary 3.4.** Let $X$ be a Tychonoff space. Then for every compact subset $K$ of $\mathbb{V}(X)$ there is an $n \in \mathbb{N}$ such that $K \subseteq \text{sp}_n(X)$.

**Proof.** Denote by $\beta X$ the Stone–Čech compactification of $X$, and let $\overline{\beta} : \mathbb{V}(X) \to \mathbb{V}(\beta X)$ be an extension of $\beta$. Consider the following sequence

$$K \xrightarrow{id_K} \mathbb{V}(X) \xrightarrow{\overline{\beta}} \mathbb{V}(\beta X).$$  
Then, by Theorem 3.1 there is an $n \in \mathbb{N}$ such that $\overline{\beta}(K) \subseteq \text{sp}_n(\beta X)$. Since $\overline{\beta}$ is injective we obtain $K \subseteq \overline{\beta}^{-1}(\mathbb{V}(X) \cap \text{sp}_n(\beta X)) = \text{sp}_n(X)$. \hfill $\Box$
Proposition 3.9. Proof. Let $X$ be a Tychonoff space. But $L(X)$ is the free topological vector space generated algebraically by $X$. Topology of $L(X)$ is the finest vector topology which induces the given topology on $X$. By the proof of Theorem 3.1, the space $L(X)$ is a $k$-space if and only if $X$ is a countable discrete space.

We shall denote the topology of $L(X)$ by $\nu_X$, so $L(X) = (\forall X, \nu_X)$.

Fact 3.6. For a Tychonoff space $X$, the space $L(X)$ is a $k$-space if and only if $X$ is a countable discrete space.

Let us recall also the definition of free abelian topological groups.

Definition 3.7. Let $X$ be a Tychonoff space. An abelian topological group $A(X)$ is called the free abelian topological group over $X$ if $A(X)$ satisfies the following conditions:

(i) there is a continuous mapping $i : X \to A(X)$ such that $i(X)$ algebraically generates $A(X)$;

(ii) if $f : X \to G$ is a continuous mapping to an abelian topological group $G$, then there exists a continuous homomorphism $\bar{f} : A(X) \to G$ such that $f = \bar{f} \circ i$.

The fact that $\forall(X)$ is complete when $X$ is a $k_\omega$-space contradicts the known result (Fact 3.8) that the free locally convex space $L(X)$ is not complete if $X$ has any infinite compact subspace.

Fact 3.8. Let $X$ be a Tychonoff space. Then

(i) $A(X)$ is complete if and only if $X$ is Dieudonné complete;

(ii) $L(X)$ is complete if and only if $X$ is Dieudonné complete and does not have infinite compact subsets.

Proposition 3.9. Let $X = \bigcup_{n \in \mathbb{N}} C_n$ be a $k_\omega$-decomposition of a $k_\omega$-space $X$ into compact sets and let $E$ be a topological vector space generated algebraically by $X$. Further, let $E$ have the property that a subset $A$ of $E$ is closed in $E$ if and only if $A \cap \text{sp}_n(C_n)$ is compact for all $n \in \mathbb{N}$. Then the topology of $E$ is the finest vector topology which induces the given topology on $X$, and so $E$ is the free topological vector space over $X$.

Proof. Let $\tau$ be the given topology on $E$ and $\tau' \supseteq \tau$ the finest vector topology inducing the given topology on $X$. By the proof of Theorem 3.1, $A \subseteq E$ is closed in $\tau'$ if and only if each $A \cap \text{sp}_n(C_n)$ is compact. But $\tau$ and $\tau'$ induce the same topology on $X$, hence on $C_n$ and hence also on $\text{sp}_n(C_n)$. Thus $\tau' = \tau$ as desired.

Proposition 3.10. Let $X = \bigcup_{n \in \mathbb{N}} C_n$ be a $k_\omega$-space and let $Y$ be a subset of $\forall(X)$ such that $Y$ is a free vector space basis for the subspace, $\text{sp}(Y)$, that it generates. Assume that $K_1, K_2, \ldots$ is a sequence of compact subsets of $Y$ such that $Y = \bigcup_{n \in \mathbb{N}} K_n$ is a $k_\omega$-decomposition of $Y$ inducing the same topology on $Y$ that $Y$ inherits as a subset of $\forall(X)$. If for every $n \in \mathbb{N}$ there is a natural number $m$ such that $\text{sp}(Y) \cap \text{sp}_n(C_n) \subseteq \text{sp}_m(K_m)$, then $\text{sp}(Y)$ is $\forall(Y)$, and both $\text{sp}(Y)$ and $Y$ are closed subsets of $\forall(X)$.

Proof. It follows from the proof of Theorem 3.1 that, to prove $\text{sp}(Y)$ is closed in $\forall(X)$, we only have to show that $\text{sp}(Y) \cap \text{sp}_n(C_n)$ is compact for each $n \in \mathbb{N}$. Now

$$\text{sp}(Y) \cap \text{sp}_n(C_n) = \text{sp}(Y) \cap \text{sp}_n(C_n) \cap \text{sp}_m(K_m) = \text{sp}_n(C_n) \cap \text{sp}_m(K_m),$$

and hence is compact. Thus $\text{sp}(Y)$ is closed in $\forall(X)$. Analogously, $Y$ is closed in $\forall(X)$.

Using Proposition 3.9 to prove that $\text{sp}(Y)$ is the free topological vector space on $Y$, it suffices to show that a subset $A$ of $\text{sp}(Y)$ is closed if $A \cap \text{sp}_n(K_n)$ is compact for all $n \in \mathbb{N}$. Consider $A \cap \text{sp}_n(C_n)$, for any $n$. Then there is an $m \in \mathbb{N}$ such that $\text{sp}(Y) \cap \text{sp}_n(C_n) \subseteq \text{sp}_m(K_m)$ and hence $A \cap \text{sp}_n(C_n) = A \cap \text{sp}_n(C_n) \cap \text{sp}(Y) = A \cap \text{sp}_n(C_n) \cap \text{sp}_m(K_m) = (A \cap \text{sp}_m(K_m)) \cap \text{sp}_n(C_n)$. 

Theorem 3.1 is surprising since it contrasts with what is known about free locally convex spaces, see Fact 3.6.
Since both $A \cap \text{sp}_n(K_m)$ and $\text{sp}_n(C_n)$ are compact, $A \cap \text{sp}_n(C_n)$ is compact in $V(X)$. Thus $A$ is a closed subset of $V(X)$ and the proof is complete. \hfill \Box

Since a closed subspace of a $k_\omega$-space is also a $k_\omega$-space, Lemma 2.4 and Proposition 3.10 implies

**Corollary 3.11.** If $Y$ is a closed subspace of a $k_\omega$-space $X$, then the closed subspace $V(Y, X)$ of $V(X)$ is $V(Y)$.

**Proposition 3.12.** If $K$ is a compact subspace of a Tychonoff space $X$, then $V(K, X)$ is $V(K)$.

**Proof.** Denote by $\beta X$ the Stone–Čech compactification of $X$. So we obtain the following commutative diagram

$$
\begin{array}{c}
K \xrightarrow{i} X \xrightarrow{\beta} \beta X \\
\downarrow \quad \downarrow \quad \downarrow \\
V(K) \xrightarrow{\bar{i}} V(X) \xrightarrow{\bar{\beta}} V(\beta X)
\end{array}
$$

Then $V(K, \beta X) = V(K)$ by Corollary 3.11. So we obtain

$$V(K) \xrightarrow{\bar{i}} \bar{i}(V(K)) = V(K, X) \xrightarrow{\bar{\beta}} V(K),$$

and so $V(K) = V(K, X)$. \hfill \Box

**Proposition 3.13.** If $X$ is a $k_\omega$-space, then every metrizable (in particular, Banach) vector subspace $E$ of $V(X)$ is finite-dimensional.

**Proof.** Let $\bar{E}$ be the closure of $E$ in $V(X)$. Then $\bar{E}$ is a metrizable closed subspace of $V(X)$. As $V(X)$ is a $k_\omega$-space by Theorem 3.4, $\bar{E}$ is also a $k_\omega$-space. Therefore $\bar{E}$ is a locally compact by [7, 3.4.E], and hence it is finite-dimensional, see [20, §15.7]. Thus $E$ is finite-dimensional as well. \hfill \Box

On the other hand, we now see that every infinite-dimensional space $V(X)$ contains the space $\varphi = V(\mathbb{N})$. But $\varphi$ is the inductive limit of $\mathbb{R}^n$, and so $\varphi$ is a locally convex space and therefore $\varphi = L(\mathbb{N})$.

For every $n \in \mathbb{N}$, set

$$T_n := 1 + \cdots + n \quad \text{and} \quad S_n := T_1 + \cdots + T_n.$$  

(3.1)

**Theorem 3.14.** If $X$ is an infinite Tychonoff space, then $V(X)$ contains a closed vector subspace which is topologically isomorphic to $\varphi = V(\mathbb{N}) = L(\mathbb{N})$.

**Proof.** First we assume that $X$ is an infinite compact space. Take arbitrarily a sequence $\{z_n\}_{n \in \mathbb{N}}$ of distinct elements of $X$. For every $n \in \mathbb{N}$ and $S_n$ defined in (3.1), set

$$y_1 := z_1, \quad y_2 := 2z_1 + z_2 + z_3, \quad y_n := nz_1 + z_{S_{n-1}+1} + z_{S_{n-1}+2} + \cdots + z_{S_n}, \quad n > 2,$$

where the $S_n$ are as in (3.1) and “+” denotes the vector space addition in $V(X)$. Since the sequence $\{nz_1\}_{n \in \mathbb{N}}$ is discrete and closed in $\mathbb{R}z_1 \subset V(X)$, the sequence $Y := \{y_n\}_{n \in \mathbb{N}}$ is discrete and closed in $V(X)$. So $\text{sp}(Y) = V(Y, X)$ is a closed vector subspace in $V(X)$ by Lemma 2.4. Let us show that $\text{sp}(Y)$ is topologically isomorphic to $\varphi$. For every $n \in \mathbb{N}$, set $K_n := \{y_1, \ldots, y_n\}$.

We claim that $\text{sp}(Y) \cap \text{sp}_n(X) \subseteq \text{sp}_n(K_n)$. Indeed, fix $t \in \text{sp}(Y) \cap \text{sp}_n(X)$. So there are distinct $x_1, \ldots, x_n \in X$, $i_1 \cdot \ldots \cdot i_m$, nonzero real numbers $a_1, \ldots, a_m$ and nonzero numbers
\( \lambda_1, \ldots, \lambda_n \in [-n,n] \) such that
\[
  t = a_1y_1 + \cdots + a_my_m
  = (a_1n_1 + \cdots + a_mn_m)z_1 + \sum_{i=1}^{m} \sum_{j=1}^{i} a_i z_{S_{i-1}+j}
  = \lambda_1x_1 + \cdots + \lambda_nx_n.
\]
But since all the elements \( z_{S_{i-1}+j} \) are distinct elements of the canonical basis of \( V(X) \), this equality implies that \( i_m \leq n \) and \( \{a_1, \ldots, a_m\} \subseteq \{\lambda_1, \ldots, \lambda_n\} \). Thus \( t \in sp_n(K_n) \).

The topology \( \tau \) of \( sp(Y) \) induced from \( V(X) \) is defined by the sequence \( \{sp(Y) \cap sp_n(X)\}_{n \in \mathbb{N}} \) of compact sets by Theorem \ref{thm:compactness} and closedness of \( sp(Y) \). Note that for every \( n \in \mathbb{N} \) there is an \( m \in \mathbb{N} \) such that \( sp_n(K_n) \subseteq sp(Y) \cap sp_n(X) \). This inclusion and the claim imply that \( \tau \) coincides with the free topology \( \mu_Y \) on \( sp(Y) \) defined by the sequence \( \{sp_n(K_n)\}_{n \in \mathbb{N}} \). Thus \( sp(Y) \) is topologically isomorphic to \( \varphi \).

Now let the space \( X \) be arbitrary and let \( \beta X \) be the Stone-Čech compactification of \( X \). So there is a continuous linear monomorphism \( \iota : V(X) \to \mathbb{V}(\beta X) \). As \( X \) is infinite, \( \beta X \) contains a sequence \( \{z_n\}_{n \in \mathbb{N}} \) of distinct elements of \( X \). So \( \mathbb{V}(\beta X) \) contains a closed and discrete subset \( Y \) such that \( sp(Y) \) is a closed vector subspace of \( \mathbb{V}(\beta X) \) which is topologically isomorphic to \( \varphi \) by the first step. Clearly, \( \iota^{-1}(Y) \) is a closed and discrete subset of \( \mathbb{V}(X) \) and \( E := \iota^{-1}(sp(Y)) \) is a closed vector subspace of \( \mathbb{V}(X) \). So the topology \( \tau' \) on \( E \) induced from \( \mathbb{V}(X) \) is finer than the free topology \( \mu_Y \) on \( E \). Therefore \( \tau' = \mu_Y \). Thus \( \varphi \) is topologically isomorphic to the closed vector subspace \( E \) of \( \mathbb{V}(X) \).

Theorem \ref{thm:compactness} is of interest also because such a result does not hold for free locally convex spaces. Indeed, Theorem 4.3 of \cite{GABRIELYAN} states that if the free locally convex space \( L(X) \) over a Tychonoff space \( X \) embeds into \( L(I) \), where \( I = [0,1] \), then \( X \) is a metrizable countable-dimensional compactum. In particular, the space \( \varphi = L(\mathbb{N}) \) does not embed into \( L(I) \).

\begin{thm}
Let \( \{K_n\}_{n \in \mathbb{N}} \) be a sequence of disjoint compact subsets of \( \mathbb{R} \). Then \( \mathbb{V}(\bigcup_{n \in \mathbb{N}} K_n) \) embeds onto a closed vector subspace of \( \mathbb{V}(I) \).
\end{thm}

\textbf{Proof.} Take two sequences \( \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \subseteq I \) such that \( a_1 = 0 \) and \( a_n < b_n < a_{n+1} < 1 \) for every \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \), set \( I_n := [a_n, b_n] \) and \( X_n := \bigcup_{i \leq n} I_i \). Since each \( K_n \) is homeomorphic to a closed subset of \( I_n \), by Corollary \ref{cor:homeomorphism}, we can assume that \( K_n = I_n \) for every \( n \in \mathbb{N} \). Set \( X := \bigcup_{n \in \mathbb{N}} I_n \).

For every \( n \in \mathbb{N} \) and \( 1 \leq i \leq A_n := \lfloor S_n/2 \rfloor \), the integer part of \( S_n/2 \) as defined in \( \ref{def:integer-part} \), we define the closed interval
\[
  I_{i,n} := \left[ a_n + \frac{b_n - a_n}{S_n}(2i - 1), a_n + \frac{b_n - a_n}{S_n}(2i) \right],
\]
and define the homeomorphism \( g_{i,n} : I_n \to I_{i,n} \) by
\[
  g_{i,n}(x) := \frac{1}{S_n}x + a_n \left( 1 - \frac{1}{S_n} \right) + \frac{b_n - a_n}{S_n}(2i - 1).
\]
For every \( n \in \mathbb{N} \), define the map \( h_n : I_n \to \mathbb{V}(I) \) by
\[
  h_n(x) := g_{1,n}(x) + g_{2,n}(x) + \cdots + g_{A_n,n}(x),
\]
where “+” denotes the vector space addition in \( \mathbb{V}(I) \). Now we define the map \( \chi : X \to \mathbb{V}(I) \) setting
\[
  \chi(x) := h_n(x), \quad \text{if } x \in I_n.
\]
Clearly, \( \chi \) is continuous and one-to-one. Set \( Y := \chi(X) \) and \( Y_n := \chi(X_n) \) for every \( n \in \mathbb{N} \).
We claim that \( \text{sp}(Y) \cap \text{sp}_n(\mathbb{I}) \subseteq \text{sp}_n(Y_n) \) for every \( n \in \mathbb{N} \). Indeed, fix \( t \in \text{sp}(Y) \cap \text{sp}_n(\mathbb{I}) \). So there are distinct \( x_1, \ldots, x_n \in \mathbb{I} \), distinct \( y_1, \ldots, y_m \in Y \), nonzero real numbers \( a_1, \ldots, a_m \) and nonzero numbers \( \lambda_1, \ldots, \lambda_n \in [-n, n] \) such that
\[
(3.2) \quad t = a_1y_1 + \cdots + a_my_m = \lambda_1x_1 + \cdots + \lambda_n x_n.
\]
For every \( 1 \leq i \leq m \) take \( x_i \in X \) and \( n_i \in \mathbb{N} \) such that
\[
y_i = \chi(x_i) \text{ and } x_i \in I_{n_i}, \text{ so } y_i = \sum_{j \leq A_{n_i}} g_{j,n_i}(x_i).
\]
Taking into account that all \( g_{j,n_i}(x_i) \) are distinct elements of the basis \( \mathbb{I} \) of \( \mathbb{V}(\mathbb{I}) \), equality 3.2 implies that \( m \leq n \) and \( \{a_1, \ldots, a_m\} \subseteq \{\lambda_1, \ldots, \lambda_n\} \). Thus \( t \in \text{sp}_n(Y_n) \).

For every \( n \in \mathbb{N} \), take the maximal \( l(n) \in \mathbb{N} \) such that \( S_{l(n)} \subseteq n \) (note that \( S_1 = 1 \)). Then the same argument as in the claim shows that \( Y \cap \text{sp}_n(\mathbb{I}) = Y_{S_{l(n)}} \), for every \( n \in \mathbb{N} \). Fix a closed subset \( F \) of \( X \). Then for every \( n \in \mathbb{N} \) we have
\[
\chi(F) \cap \text{sp}_n(\mathbb{I}) = \chi(F) \cap Y \cap \text{sp}_n(\mathbb{I}) = \chi(F) \cap Y_{S_{l(n)}} = \chi(F) \cap \chi(X_{S_{l(n)}}) = \chi(F \cap X_{S_{l(n)}})
\]
is a closed subset of \( \text{sp}_n(\mathbb{I}) \). Since \( \mathbb{V}(\mathbb{I}) = \bigcup_n \text{sp}_n(\mathbb{I}) \) is a \( k \)-space by Theorem 3.1 we obtain that \( \chi(F) \) is closed in \( \mathbb{V}(\mathbb{I}) \). Therefore \( \chi \) is a closed map. Thus \( \chi \) is a homeomorphism of \( X \) onto \( Y \). Finally, Proposition 3.10 implies that \( \text{sp}(Y) \) is a closed subspace of \( \mathbb{V}(\mathbb{I}) \) and is topologically isomorphic to \( \mathbb{V}(X) \).

4. Topological properties of free topological vector spaces

For a subset \( A \) of a tvs \( E \), we denote the convex hull of \( A \) by \( \text{conv}(A) \), so
\[
\text{conv}(A) := \left\{ \lambda_1a_1 + \cdots + \lambda_na_n : \lambda_1, \ldots, \lambda_n \geq 0, \sum_{i=1}^n \lambda_i = 1, a_1, \ldots, a_n \in A, n \in \mathbb{N} \right\}.
\]
Denote by \( \text{LCS} \) the category of all locally convex spaces and continuous linear operators. Let \( (E, \tau) \) be a topological vector space and let \( \mathcal{N}(E) \) be a base of neighborhoods at zero of \( E \). Then the family \( \hat{\mathcal{N}}(E) := \{ \text{conv}(U) : U \in \mathcal{N}(E) \} \) forms a locally convex vector topology \( \hat{\tau} \) on \( E \). So \( \hat{\tau} \) is the strongest locally convex vector topology on \( E \) which is coarser than the origin topology \( \tau \). The lcs \( (E, \hat{\tau}) \) is called the locally convex modification of \( E \). Let \( E \) and \( H \) be topological vector spaces and \( T : E \to H \) a continuous linear operator. Define the functor \( \mathcal{L} : \text{TVS} \to \text{LCS} \) by the assignment
\[
(E, \tau) \to \mathcal{L}(E) := (E, \hat{\tau}), \quad \mathcal{L}(T) := T.
\]
For a Tychonoff space \( X \), the underlying group of \( A(X) \) we denote by \( A_a(X) \), and the underlying vector spaces of \( L(X) \) and \( \mathbb{V}(X) \) are denoted by \( L_a(X) \) and \( \mathbb{V}_a(X) \), respectively. Below we obtain some relations between \( \mathbb{V}(X) \), \( L(X) \) and \( A(X) \).

Proposition 4.1. Let \( X \) be a Tychonoff space. Then
(i) \( \mathcal{L}(\mathbb{V}(X)) = L(X) \);
(ii) the identity map \( \text{id}_X : X \to X \) extends to a canonical homomorphisms \( \text{id}_{A(X)} : A(X) \to \mathbb{V}(X) \), which is an embedding of topological groups;
(iii) \( \text{id}_{A(X)}(A(X)) \) is closed in \( \mathbb{V}(X) \).

Proof. (i) follows from the definitions of free lcs and free tvs.

(ii) As \( \nu_X \leq \mu_X \), we obtain \( \nu_X |_{A_a(X)} \leq \mu_X |_{A_a(X)} \). On the other hand, by the definition of \( A(X) \), there is a continuous homomorphism from \( A(X) \) into \( \mathbb{V}(X) \), and since the topology \( \tau_{A(X)} \) of \( A(X) \) is \( \nu_X |_{A_a(X)} \) by \( \text{SN} \), we obtain that \( \nu_X |_{A_a(X)} \geq \mu_X |_{A_a(X)} \). Therefore \( \nu_X |_{A_a(X)} = \mu_X |_{A_a(X)} = \tau_{A(X)} \). Thus \( \text{id}_{A(X)} \) is an embedding.

(iii) By \( \text{SN} \), \( A(X) \) is closed in the topology \( \nu_X \). Thus \( A(X) \) is also closed in \( \mu_X \).
Proposition 4.2 allows us to reduce easily the study of some topological properties of \( L(X) \) and \( \mathbb{V}(X) \) to the study of the corresponding properties for \( A(X) \). We demonstrate below such a reduction.

It is known that \( A(X) \) is Lindelöf if and only if \( X^n \) is Lindelöf for every \( n \in \mathbb{N} \), see Corollary 7.1.18 in [3]. An analogous result holds for \( L(X) \) and \( \mathbb{V}(X) \).

**Proposition 4.2.** Let \( X \) be a Tychonoff space. Then

(i) \( L(X) \) is Lindelöf if and only if \( X^n \) is Lindelöf for every \( n \in \mathbb{N} \);

(ii) \( \mathbb{V}(X) \) is Lindelöf if and only if \( X^n \) is Lindelöf for every \( n \in \mathbb{N} \).

**Proof.** We prove only (ii) as (i) is proved in an analogous manner. Assume that \( \mathbb{V}(X) \) is a Lindelöf space. By Proposition 4.1, \( A(X) \) is a closed subspace of \( L(X) \). Hence \( A(X) \) is also Lindelöf. Therefore \( X^n \) is Lindelöf for every \( n \in \mathbb{N} \) by Corollary 7.1.18 of [3].

Conversely, let \( X^n \) be a Lindelöf space for every \( n \in \mathbb{N} \). Then the disjoint sum

\[
Y := \bigcup_{n \in \mathbb{N}} Y_n, \quad \text{where} \quad Y_n := [-n, n] \times X^n,
\]

is also a Lindelöf space, see Corollary 3.8.10 in [4]. Consider the map \( T : Y \rightarrow \mathbb{V}(X) \) defined by

\[
T(y) := T_n(y) \text{ if } y = (a_1, \ldots, a_n) \times (x_1, \ldots, x_n) \in Y_n \text{ and } T_n(y) := a_1x_1 + \cdots + a_nx_n.
\]

Clearly, the map \( T \) is continuous. Thus \( \mathbb{V}(X) \) is a Lindelöf space. \( \square \)

Bel’nov [6] proved that if a topological group \( G \) is algebraically generated by a Lindelöf subspace, then \( G \) is topologically isomorphic to a subgroup of the product of some family of second-countable groups. As \( \mathbb{V}(X) \) and \( L(X) \) are algebraically generated by the continuous image of \([-1, 1] \times X \) and a product of a compact space and a Lindelöf space is also Lindelöf, we obtain

**Proposition 4.3.** Let \( X \) be a Lindelöf space. Then \( \mathbb{V}(X) \) and \( L(X) \) are topologically isomorphic to a subgroup of the product of some family of second-countable groups.

Recall that a subset \( A \) of a topological space \( X \) is called functionally bounded if every continuous real-valued function \( f \in C(X) \) is bounded on \( A \). We shall use the following result, see Lemma 10.11.3 in [3].

**Fact 4.4** ([4]). Let \( X \) be a Tychonoff space and \( A \) be a functionally bounded subset of \( L(X) \). Then the set \( \bigcup_{v \in A} \text{supp}(v) \) is functionally bounded in \( X \) and there is an \( n \in \mathbb{N} \) such that \( A \subseteq \text{sp}_n(X) \).

An analogous result holds also for \( \mathbb{V}(X) \).

**Proposition 4.5.** Let \( X \) be a Tychonoff space and \( A \) be a functionally bounded subset of \( \mathbb{V}(X) \). Then the set \( \bigcup_{v \in A} \text{supp}(v) \) is functionally bounded in \( X \) and there is an \( n \in \mathbb{N} \) such that \( A \subseteq \text{sp}_n(X) \).

**Proof.** Proposition 4.1 implies that the identity map \( id : \mathbb{V}(X) \rightarrow L(X) \) is continuous. So \( A \) is also a functionally bounded subset of \( L(X) \) and Fact 4.4 applies. \( \square \)

Two Tychonoff spaces \( X \) and \( Y \) are called \( \mathbb{V} \)-equivalent if the free topological vector spaces \( \mathbb{V}(X) \) and \( \mathbb{V}(Y) \) are isomorphic as topological vector spaces. Analogously, \( X \) and \( Y \) are said to be \( L \)-equivalent if the free locally convex spaces \( L(X) \) and \( L(Y) \) are isomorphic as topological vector spaces. A topological property \( P \) is called \( \mathbb{V} \)-invariant (\( L \)-invariant) if every space \( Y \) which is \( \mathbb{V} \)-equivalent (respectively, \( L \)-equivalent) to a Tychonoff space \( X \) with \( P \) also has the property \( P \).

A subspace \( Y \) of \( \mathbb{V}(X) \) (\( L(X) \)) is called a topological basis of \( \mathbb{V}(X) \) (respectively, \( L(X) \)) if \( Y \) is a vector basis of \( \mathbb{V}(X) \) (\( L(X) \)) and the maximal vector space topology (maximal locally convex space topology) on the abstract vector space \( \mathbb{V}_X \) which induces on \( Y \) its original topology coincides with the topology of \( \mathbb{V}(X) \) (respectively, \( L(X) \)). The next theorem has a similar proof to the proof of Theorem 7.10.10 of [3].
Theorem 4.6. Pseudocompactness is a $\mathbb{V}$-invariant property and an $L$-invariant property.

Proof. We prove the theorem only for the case $\mathbb{V}(X)$. Let $X$ and $Y$ be $\mathbb{V}$-equivalent spaces. The theorem is clear if $X$ or $Y$ is finite, so we assume that $X$ and $Y$ are infinite. Assume that $Y$ is pseudocompact. So we can assume that $Y$ is a topological basis of the space $\mathbb{V}(X)$. Note that $Y$ is a closed subspace of $\mathbb{V}(X)$ since $Y$ is closed in $\mathbb{V}(Y)$. Suppose for a contradiction that $X$ is not pseudocompact. Then $X$ contains a discrete family $\mathcal{U} = \{U_n : n \geq 0\}$ of non-empty open sets. For every $n \in \mathbb{N}$, choose a point $x_n \in U_n$.

Define the function $d(x, v) : X \times \mathbb{V}(X) \to \mathbb{R}$ as follows: if $v = \lambda_1 x_1 + \cdots + \lambda_k x_k \in \mathbb{V}(X)$, where all $x_i$ are distinct and $\lambda_i \neq 0$ for every $i = 1, \ldots, k$, then $d(x, v) = \lambda_i$ if $x = x_i$ for some $1 \leq i \leq k$, and $d(x, v) = 0$ otherwise.

Since $Y$ is a vector basis of $\mathbb{V}(X)$, for every $n \in \mathbb{N}$ there exists $y_n \in Y$ such that $d(x_n, y_n) \neq 0$. So, for every $n \in \mathbb{N}$, we have

$$y_n = \lambda_{0,n} x_n + \sum_{j=1}^{k_n} \lambda_{j,n} t_{j,n}, \quad \text{where } \lambda_{j,n} \neq 0 \text{ for every } 0 \leq j \leq k_n,$$

and all letters $t_{1,n}, \ldots, t_{k_n,n} \in X \setminus \{x_n\}$ are distinct (possibly, $k_n = 0$). Passing to a subsequence of $\{y_n : n \in \mathbb{N}\}$ if it is needed, we can assume that $d(x_j, y_n) = 0$ whenever $n < j$.

By induction on $n \geq 0$, we define continuous real-valued functions $f_n$ on $X$ as follows. Set $f_0 \equiv 0$. Assume that we have defined $f_0, \ldots, f_{n-1}$. Put $g_n := \sum_{i=0}^{n-1} f_i$. Then there exists a continuous real-valued function $f_n$ on $X$ such that $f_n(X \setminus U_n) = \{0\}$ and $f_n(t_{j,i}) = 0$ at each point $t_{j,i}$ with $i \leq n$ that belongs to $U_n$ and also satisfies

$$f_n(x_n) = \frac{n}{\lambda_{0,n}} + \frac{1}{\lambda_{0,n}} \sum_{j \in A_n} |\lambda_{j,n} g_n(t_{j,n})|,$$

where $A_n$ is the set of all $1 \leq j \leq k_n$ such that $t_{j,n} \in U_0 \cup \cdots \cup U_{n-1}$.

Since the family $\mathcal{U}$ is discrete, the function $f := \sum_{n \in \mathbb{N}} f_n$ is continuous on $X$. Moreover,

$$f = f_n \text{ on } U_n, \text{ and } f = g_n \text{ on } U_0 \cup \cdots \cup U_{n-1}.$$

In addition, the definition of $f$ implies that for all $n \in \mathbb{N}$ and all $1 \leq j \leq k_n$,

$$f(t_{j,n}) = 0 \quad \text{whenever } j \notin A_n.$$

Extend $f$ to a continuous functional $\tilde{f} : \mathbb{V}(X) \to \mathbb{R}$. From (4.1) it follows that

$$\tilde{f}(y_n) = \lambda_{0,n} f(x_n) + \sum_{j=1}^{k_n} \lambda_{j,n} f(t_{j,n}), \quad \text{by (4.1)}$$

$$= \lambda_{0,n} f(x_n) + \sum_{j \in A_n} \lambda_{j,n} f(t_{j,n}), \quad \text{by (4.2) and (4.3)}$$

$$= n + \sum_{j \in A_n} |\lambda_{j,n} g_n(t_{j,n})| + \sum_{j \in A_n} \lambda_{j,n} g_n(t_{j,n}) \geq n$$

for every $n \in \mathbb{N}$. As $\{y_n : n \in \mathbb{N}\} \subset Y$, we conclude that $Y$ is not pseudocompact. This contradiction completes the proof. 

Graev proved in [13] that compactness is an $A$-invariant property. Below we prove an analogous result with a similar proof.

Theorem 4.7. Let $X$ and $Y$ be any $\mathbb{V}$-equivalent spaces or $L$-equivalent spaces. Then

(i) if $X$ is compact, then so is $Y$;

(ii) if $X$ is compact and metrizable, then so is $Y$. 

Proof. We prove the theorem only for $\nu$-equivalent spaces.

(i) Since $Y$ is closed in $\nu(Y)$ it is also closed in $\nu(X)$. Theorem 4.6 implies that $Y$ is pseudo-compact. So, for some $n \in \mathbb{N}$, $Y$ is a closed subset of the compact set $sp_n(X)$ by Proposition 4.5. Thus $Y$ is compact.

(ii) By the proof of (i), $Y$ is a closed subset of $sp_n(X)$ for some $n \in \mathbb{N}$. Since $sp_n(X)$ is a continuous image of the compact metrizable space $[-n,n]^n \times X^n$ we obtain that $sp_n(X)$ is a compact metrizable space. Thus $Y$ is compact and metrizable. \(\square\)

Proposition 4.8. If $X$ is a Tychonoff non-normal space, then $L(X)$ and $\nu(X)$ are also not normal spaces.

Proof. The proposition follows from the fact that $X$ is a closed subspace of $L(X)$ and $\nu(X)$ and the fact that a closed subspace of a normal space is also normal. \(\square\)

Remark 4.9. We note that even if $X$ is a normal space, $\nu(X)$ and $L(X)$ need not be normal spaces. This follows from the following three facts: (1) $A(X)$ is a closed subgroup of $\nu(X)$ and $L(X)$, (2) $A(X)$ contains a closed homeomorphic copy of $X^n$ for every $n \in \mathbb{N}$, see Corollary 7.1.16 of [3], and (3) the square of a normal space can be not normal, see Example 2.3.12 of [7].

For a tvs $E$, we denote by $E'$ the topological dual space of $E$.

Proposition 4.10. For every Tychonoff space $X$, the topologies $\nu_X$ and $\mu_X$ are compatible, i.e. $\nu(X)$ and $L(X)$ have the same continuous functionals.

Proof. The topology $\tau$ on $\nu_X$ defined by $\nu(X)'$ is locally convex. So, by the definition of $\nu_X$, we have $\tau \leq \nu_X$. Thus any $\chi \in \nu(X)'$ is also continuous in $\nu_X$, and hence $\nu(X)' \subseteq L(X)'$. The converse assertion is trivial. \(\square\)

Using Fact 3.8 we obtain

Proposition 4.11. If $X$ is a Tychonoff space such that $\nu(X)$ is complete, then $X$ is Dieudonné complete.

Proof. Note that $A(X)$ is a closed subspace of $\nu(X)$ by Theorem 2.8. So, if $\nu(X)$ is complete, then $A(X)$ is also complete. Thus $X$ is Dieudonné complete by Fact 3.8(i). \(\square\)

We do not know whether the converse is true

Question 4.12. Let $X$ be a Dieudonné complete Tychonoff space. Is $\nu(X)$ complete?

Note that Protasov [31] proved the completeness of $\nu(\kappa)$ for any cardinal $\kappa$.

A natural question arises: For which Tychonoff spaces is $X$ the space $\nu(X)$ locally convex? Note that $\nu(X)$ is locally convex if and only if $\nu(X) = L(X)$. Below we obtain a complete answer to this question in the case that $X$ is a $k$-space. We start with a necessary condition for the equality $\mu_X = \nu_X$.

Proposition 4.13. If $\nu(X)$ is locally convex, then $X$ does not contain infinite compact subsets.

Proof. Suppose that $X$ contains an infinite compact subset $K$. By Proposition 3.12, $\nu(K)$ is also locally convex. So $\nu(K) = L(K)$ and hence $L(K)$ is complete by Theorem 3.1. Since $K$ is also Dieudonné complete, we obtain a contradiction with Fact 3.8(ii). \(\square\)

Remark 4.14. It is known that the family $\mathcal{S}$ of all seminorms on $\nu_a(X)$ which are continuous on $X$ defines a free locally convex vector topology $\nu_X$ on $\nu_a(X)$. So the family $\mathcal{S}$ defines the topology $\mu_X$ of $\nu(X)$ if and only if $\nu(X) = L(X)$.

Any topological vector space which is a reflexive abelian group is locally convex, see [5]. This remark and Propositions 4.10 and 4.13 imply
Corollary 4.15. If $V(X)$ is a reflexive group, then $V(X) = L(X)$. In particular, $X$ does not contain infinite compact sets.

We do not know whether the converse is true:

Question 4.16. If $X$ does not contain infinite compact subsets, is $V(X)$ locally convex?

Theorem 4.17. If $X$ is a $k$-space, then $V(X)$ is locally convex if and only if $X$ is a discrete countable space, that is, $V(X)$ equals $\varphi$ or $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Proof. Let $X$ be a $k$-space such that $V(X)$ is locally convex. By Proposition 4.13, $X$ does not contain infinite compact subsets. So being a $k$-space, the space $X$ is discrete. I. Protasov [31] (see also [11]) proved that $V(D)$ is not locally convex for every uncountable discrete space $D$. Thus $X$ must be countable. Conversely, if $X$ is a discrete countable space, then $V(X) = L(X)$ by Proposition 4.14 of [16].

Proposition 4.13 and Theorem 4.17 motivate the following question: When does $V(X)$ contain an infinite-dimensional locally convex subspaces (for example $L(Y)$ for some infinite $Y$)?

It is well known that the free groups $F(X)$ and $A(X)$ are Fréchet–Urysohn spaces and only if $X$ is a discrete space (see [30]), and the space $L(X)$ is a $k$-space if and only if $X$ is a discrete countable space by Fact 3.6. In Theorem 4.20 below we prove an analogous result for $V(X)$. We need the following result.

Fact 4.18 ([11]). If $X$ is an uncountable Tychonoff space, then $V(X)$ has uncountable tightness and is not a $k$-space.

Proposition 4.19. Let $V(X)$ be a sequential space.

(i) If $X$ is non-discrete, then $V(X)$ has sequential order $\omega_1$.

(ii) If $X$ is discrete, then $X$ is countable; so $V(X)$ equals $\varphi$ or $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

Proof. (i) Since $X$ is a closed subspace of the sequential space $V(X)$ by Theorem 2.3, $X$ is also sequential. Being non-discrete $X$ contains a non-trivial convergent sequence $\mathfrak{s}$. Proposition 3.12 implies that $V(\mathfrak{s})$ is isomorphic to a closed subspace of $V(X)$. So $V(\mathfrak{s})$ is a sequential space. As $A(\mathfrak{s})$ has sequential order $\omega_1$ by [30] Theorem 3.9] (see also [32] Theorem 2.3.10), we obtain that also $V(X)$ has sequential order $\omega_1$.

(ii) follows from Fact 4.18. $\square$

It is well known that $\varphi$ is a sequential non-Fréchet–Urysohn space.

Theorem 4.20. For a Tychonoff space $X$, the space $V(X)$ is Fréchet–Urysohn if and only if $X$ is finite. In particular, $V(X)$ is metrizable if and only if $X$ is finite.

Proof. Assume that $V(X)$ is a Fréchet–Urysohn space. Then $X$ is discrete by Proposition 4.19. If $X$ is infinite, then $V(X)$ contains $\varphi$ as a direct summand. So $V(X)$ is not Fréchet–Urysohn, a contradiction. Thus $X$ must be finite. Conversely, if $X$ is finite, then $V(X) = \mathbb{R}^{|X|}$ is a locally compact metrizable space. $\square$

Remark 4.21. Note that $V(X)$ is a locally compact space if and only if $X$ is finite. This follows from the Principal Structure Theorem for locally compact abelian groups (Theorem 25 of [27]) observing that $V(X)$ is a torsion-free divisible abelian group and so $V(X) = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

Recall (see [25]) that a topological space $X$ is called cosmic, if $X$ is a regular space with a countable network (a family $\mathcal{N}$ of subsets of $X$ is called a network in $X$ if, whenever $x \in U$ with $U$ open in $X$, then $x \in N \subseteq U$ for some $N \in \mathcal{N}$). Michael proved in [25] that a regular space $X$ is cosmic if and only if $X$ is a continuous image of a separable metric space.
Proposition 4.22. Let $X$ be a Tychonoff space. Then $\mathbb{V}(X)$ is a cosmic space if and only if $X$ is cosmic.

Proof. If $\mathbb{V}(X)$ is cosmic, then $X$ is also cosmic as a subspace of a cosmic space.

Assume that $X$ is a cosmic space. So there is a separable metric space $M$ and a continuous surjective map $f : M \to X$, see [25]. For every $n \in \mathbb{N}$ set
\[ Y_n := [-n, n]^n \times M^n, \]
and define the map $T_n : Y_n \to \text{sp}_n(X)$ by
\[ T_n(a_1, \ldots, a_n, y_1, \ldots, y_n) := a_1 f(y_1) + \cdots + a_n f(y_n), \quad a_i \in [-n, n], y_i \in M, i = 1, \ldots, n. \]
Clearly, $Y_n$ is a separable metric space and $T_n$ is continuous and onto.

Set $Y := \bigsqcup_{n \in \mathbb{N}} Y_n$ and define the map $T : Y \to \mathbb{V}(X)$ by
\[ T(y) := T_n(y), \quad \text{if } y \in Y_n. \]
Clearly, $Y$ is a separable metric space and $T$ is continuous and onto. Thus $\mathbb{V}(X)$ is a cosmic space by [25].

Corollary 4.23. For a Tychonoff space $X$ the following assertions are equivalent:

(i) $X$ is a cosmic space;
(ii) $A(X)$ is a cosmic space;
(iii) $L(X)$ is a cosmic space.

Proof. (i)⇒(iii) follows from Proposition 4.22 since the topology $\nu_X$ of $L(X)$ is weaker than the topology $\mu_X$ of $\mathbb{V}(X)$: if $\mathbb{V}(X)$ is the image of a separable metric space under a continuous map, then so is $L(X)$.

(iii)⇒(ii) and (ii)⇒(i) follow from the facts that $A(X)$ is a subspace of $L(X)$ and $X$ is a subspace of $A(X)$.

Recall that a space $X$ has countable tightness if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$. We use the following remarkable result of Arhangel’skii, Okunev and Pestov which shows that the topology of $A(X)$ is rather complicated and unpleasant even for the simplest case of a metrizable space $X$. Denote by $X'$ the set of all non-isolated points in a space $X$.

Fact 4.24 ([2]). Let $X$ be a metrizable space. Then

(i) the tightness of $A(X)$ is countable if and only if the set $X'$ is separable;
(ii) $A(X)$ is a k-space if and only if $X$ is locally compact and the set $X'$ is separable.

For a metric space $X$, the space $L(X)$ has countable tightness if and only if $X$ is separable, see [12]. The same holds also for $\mathbb{V}(X)$.

Theorem 4.25. Let $X$ be a metrizable space. Then $\mathbb{V}(X)$ has countable tightness if and only if $X$ is separable.

Proof. Assume that $\mathbb{V}(X)$ has countable tightness. Then $A(X)$ has countable tightness as a subspace of $\mathbb{V}(X)$, see Proposition 4.1. So $X'$ is separable by Fact 4.24. To prove that $X$ is separable we have to show that the set $D := X \setminus X'$ is countable.

Suppose for a contradiction that $D$ is uncountable. Then there is a positive number $c$ and an uncountable subset $D_0$ of $D$ such that $B_c(d) = \{d\}$ for every $d \in D_0$, where $B_c(d)$ is the $c$-ball centered at $d$. It is easy to see that $D_0$ is a clopen subset of $X$. So $X = X_0 \cup D_0$, where $X_0 := X \setminus D_0$. Now Corollary 4.6 implies that $\mathbb{V}(X) = \mathbb{V}(X_0) \sqcup \mathbb{V}(D_0)$. Hence $\mathbb{V}(D_0)$ also has countable tightness, but this contradicts Fact 4.18. Thus $D$ is countable, and hence $X$ is separable.

Conversely, if $X$ is separable it is a cosmic space. So $\mathbb{V}(X)$ is a cosmic space by Proposition 4.22. Thus $\mathbb{V}(X)$ has countable tightness, see [25].
By Fact 3.6, \( L(X) \) is a \( k \)-space if and only if \( X \) is a countable discrete space. For the case \( V(X) \) the situation is more complicated.

**Theorem 4.26.** For an infinite metrizable space \( X \) the following assertions are equivalent:

(i) \( X \) is a locally compact separable (metric) space;
(ii) \( V(X) \) is a non-Fréchet–Urysohn sequential space;
(iii) \( V(X) \) is a \( k \)-space;
(iv) \( V(X) \) is a \( k_\omega \)-space.

**Proof.** (i)\( \Rightarrow \)(vi). Since \( X \) is locally compact metrizable and separable, \( X \) is a \( k \)-space. So \( V(X) \) is a \( k_\omega \)-space by Theorem 3.1.

(iv)\( \Rightarrow \)(iii) is trivial.

(iii)\( \Rightarrow \)(i). If \( V(X) \) is a \( k \)-space, then \( A(X) \) is a \( k \)-space as a closed subspace of \( V(X) \). Then \( X \) is locally compact and the set \( X' \) is separable by Fact 4.24. To prove that \( X \) is separable we have to show that the set \( D := X \setminus X' \) is countable. We repeat our argument from the proof of Theorem 4.25.

Suppose for a contradiction that \( D \) is uncountable. Then there is a positive number \( c \) and an uncountable subset \( D_0 \) of \( D \) such that \( B_c(d) = \{d\} \) for every \( d \in D_0 \), where \( B_c(d) \) is the \( c \)-ball centered at \( d \). It is easy to see that \( D_0 \) is a clopen subset of \( X \). So \( X = X_0 \cup D_0 \), where \( X_0 := X \setminus D_0 \).

Now Corollary 2.6 implies that \( V(X) = V(X_0) \oplus V(D_0) \). Hence \( V(D_0) \) also is a \( k \)-space, but this contradicts Fact 4.18. Thus \( D \) is countable, and hence \( X \) is separable.

(iii)\( \Rightarrow \)(ii) is trivial.

Let us prove (iv)\( \Rightarrow \)(i). Since \( V(X) \) is a \( k_\omega \)-space, the closed subset \( X \) is also a \( k_\omega \)-space. Let \( X = \bigcup_{n \in \mathbb{N}} C_n \) be a \( k_\omega \)-decomposition of \( X \). Then \( V(X) = \bigcup_{n \in \mathbb{N}} \text{sp}_n(C_n) \) is a \( k_\omega \)-decomposition of \( V(X) \), see Theorem 3.1. As each \( \text{sp}_n(C_n) \) is a metrizable compactum, the space \( V(X) \) is sequential. Since \( X \) is infinite, Theorem 4.20 implies that \( V(X) \) is a non-Fréchet–Urysohn space.

\[ \square \]

5. Vector space properties of free topological vector spaces

First we note that any topological vector space is a quotient space of a free topological vector space.

**Proposition 5.1.** Let \( E \) be any topological vector space and \( V(E) \) the free topological vector space on \( E \). Then the canonical continuous linear map of \( V(E) \) onto \( E \) is a quotient map.

**Proof.** Denote by \( \tau \) the topology of \( E \). The identity map \( \varphi : E \to E \) can be extended to a continuous linear map \( \Phi : V(E) \to E \). If \( \Phi \) is not a quotient map, then the quotient vector space topology \( \tau_1 \) on the underlying vector space \( E_a \) of \( E \) is strictly finer than \( \tau \). Therefore \( \varphi \) is not continuous, a contradiction. Thus \( \tau_1 = \tau \).

\[ \square \]

Recall that a topological vector space \( E \) is called *barrelled* if every barrel in \( E \) is a neighborhood of zero. Following Saxon [34], a topological vector space \( E \) is called *Baire-like* if every increasing sequence \( \{A_n\}_{n \in \mathbb{N}} \) of absolutely convex closed subsets covering \( E \) contains a member which is a neighborhood of zero. Clearly, Baire lcs \( \Rightarrow \) Baire-like.

**Theorem 5.2.** Let \( X \) be a Tychonoff space. Then

(i) \( V(X) \) is barrelled if and only if \( X \) is discrete.
(ii) Let \( X \) be discrete. Then \( V(X) \) is a Baire-like space if and only if \( X \) is finite.
(iii) Let \( X \) be discrete. Then \( V(X) \) is a Baire space if and only if \( X \) is finite.

**Proof.** (i) Assume that \( V(X) \) is a barrelled tvs. Suppose for a contradiction that \( X \) is not discrete. Let \( x_0 \in X \) be a non-isolated point and take a net \( N = \{x_i\}_{i \in I} \) in \( X \) converging to \( x_0 \) (we assume
that $x_0 \not\in N)$. Set

$$A := \bigcup \left\{ \left[\frac{-1}{2}, \frac{1}{2}\right] x_i : i \in I \right\} \cup \bigcup \left\{ [-1,1] x : x \in X \setminus N \right\} \subset \mathbb{V}_X,$$

and let $B$ be the absolute convex hull of $A$. Then $B$ is a barrel in $\mathbb{V}(X)$ and $B = \text{conv}(A)$. Note that, for every $i \in I$,

$$\lambda x_i + \mu x_0 \in B \text{ if and only if } 2\lambda, \mu \in [-1,1].$$

We show that $B$ is not a neighborhood of zero in $\mathbb{V}(X)$. Indeed, otherwise we can find a neighborhood $U$ of $x_0$ such that $x - x_0 \in B$ for every $x \in U$. So, for every $j \in I$ such that $x_j \in U$, we obtain $x_j - x_0 \in B$ that contradicts (5.1). Thus $X$ is discrete.

Conversely, let $X$ be a discrete space. We shall use the following simple description of the topology $\mu_X$ of $\mathbb{V}(X)$ given in the proof of Theorem 1 in [34]. For each $x \in X$, choose some $\lambda_x > 0$, and denote by $S_X$ the family of all subsets of $\mathbb{V}_X$ of the form

$$\bigcup \{ [-\lambda_x, \lambda_x] x : x \in X \}.$$ 

For every sequence $\{S_k\}_{k \geq 0}$ in $S_X$, we put

$$\sum_{k \geq 0} S_k := \bigcup_{k \geq 0} (S_0 + S_1 + \cdots + S_k),$$

and denote by $\mathcal{N}_X$ the family of all subsets of $\mathbb{V}_X$ of the form $\sum_{k \geq 0} S_k$. Then $\mathcal{N}_X$ is a basis at zero, 0, for $\mu_X$.

Now let $B$ be a barrel in $\mathbb{V}(X)$. For every $x \in X$ choose $\lambda_x > 0$ such that $[-\lambda_x, \lambda_x] x \subseteq B$ and set

$$B_0 := \text{conv} \left\{ \bigcup \{ [-\lambda_x, \lambda_x] x : x \in X \} \right\}.$$ 

Then $B_0$ is a barrel in $\mathbb{V}(X)$ and $B_0 \subseteq B$. For every integer $k \geq 0$, set

$$S_k := \bigcup \left\{ \left[ \frac{-\lambda_x}{2k+1}, \frac{\lambda_x}{2k+1} \right] x : x \in X \right\}.$$ 

Then the neighborhood $\sum_{k \geq 0} S_k$ of zero in $\mathbb{V}(X)$ is a subset of $B_0$. Therefore $B$ is also a neighborhood of zero in $\mathbb{V}(X)$. Thus $\mathbb{V}(X)$ is a barrelled space.

(ii) If $X$ is infinite and $S = \{x_n\}_{n \in \mathbb{N}}$ is a sequence in $X$ consisting of distinct elements, then

$$\mathbb{V}(X) = \bigcup_{n \in \mathbb{N}} A_n,$$

where $A_n := [-n, n] x_1 + \cdots + [-n, n] x_n + \mathbb{V}_X \setminus S$.

Since $X$ is discrete, $A_n$ is closed, and hence $A_n$ is a meager closed subset of $\mathbb{V}(X)$ for every $n \in \mathbb{N}$. Therefore $\mathbb{V}(X)$ is not Baire. Thus $X$ is finite.

If $X$ is finite, then $\mathbb{V}(X) = \mathbb{R}^{\lvert X \rvert}$ is a Baire space.

(iii) follows from (ii). \hfill $\square$

We shall identify elements $\delta(x) \in L(X)$ with the Dirac measure $\delta_x$ on $X$. So for every element $\mu = a_1 x_1 + \cdots + a_n x_n$ of $L(X)$ with distinct $x_1, \ldots, x_n$ we can define the norm of $\mu$ setting

$$\| \mu \| := |a_1| + \cdots + |a_n|.$$ 

We need the following lemma whose proof actually can be extracted from the proof of Lemma 10.11.3 of [4]. Recall that a subset $M$ of a topological vector space $E$ is called bounded if for every neighborhood $U$ of zero there is $\lambda > 0$ such that $M \subseteq \lambda U$.

**Lemma 5.3.** Let $X$ be a Tychonoff space and $M$ a bounded subset of $L(X)$. Then the set $\{ \| \mu \| : \mu \in M \}$ is bounded in $\mathbb{R}$.
Proof. Let \( i : X \rightarrow \beta X \) be the identity inclusion of \( X \) into the Stone–Čech compactification \( \beta X \) of \( X \), and let \( \bar{i} : L(X) \rightarrow L(\beta X) \) be an injective linear continuous extension of \( i \). Since \( L(\beta X) \) is a subspace of \( C_k(C_k(\beta X)) \) (see [10, 40]) and the map \( \bar{i} \) preserves the norm of measures \( \mu \in L(X) \), it follows that \( \bar{i}(M) \) is a bounded subset of the dual Banach space \( C_k(\beta X)' \) endowed with the weak\(^*\) topology. Now the Banach–Steinhaus theorem [8, 3.88] implies that \( M \) is bounded. \( \square \)

Theorem 5.4. For a Tychonoff space \( X \), the following assertions are equivalent:

(i) \( X \) is discrete;
(ii) \( L(X) \) is barrelled;
(iii) \( L(X) \) is quasibarrelled.

Proof. (i) \( \Rightarrow \) (ii) Let \( B \) be a barrel in \( L(X) \). Then \( B \) is a neighborhood of zero in \( \mathbb{V}(X) \) by Theorem 5.2. Since \( \text{conv}(B) = B \), Proposition 4.1 implies that \( B \) is a neighborhood of zero in \( L(X) \). Thus \( L(X) \) is barrelled. (ii) \( \Rightarrow \) (iii) is clear. Let us prove (iii) \( \Rightarrow \) (i).

Suppose for a contradiction that \( X \) is not discrete and \( x_0 \in X \) is a non-isolated point of \( X \). As in the proof of (i) of Theorem 5.2 take a net \( N = \{x_i\}_{i \in I} \) in \( X \) converging to \( x_0 \) (we assume that \( x_0 \not\in N \)) and set

\[
A := \bigcup \left\{ \left[ \frac{1}{2}, \frac{1}{2} \right] x_i : i \in I \right\} \cup \bigcup \left\{ [-1,1] x : x \in X \setminus N \right\} \subset \mathbb{V}_X,
\]

and let \( B \) be the absolute convex hull of \( A \). Then \( B \) is a barrel in \( L(X) \) which is not a neighbourhood of zero even in \( \mathbb{V}(X) \) by the proof of (i) of Theorem 5.2. The construction of \( B \) and Lemma 5.3 imply that \( B \) is bornivorous. Thus \( L(X) \) is not quasibarrelled. This contradiction shows that \( X \) is discrete. \( \square \)

Corollary 5.5. Let \( X \) be a Tychonoff space. Then \( L(X) \) is a Baire space if and only if \( X \) is finite.

Proof. Assume that \( L(X) \) is Baire. Then \( L(X) \) is a barrelled space by [23, Theorem 11.8.6]. Therefore \( X \) is discrete by Theorem 5.4. Thus \( X \) is finite by (the proof of) Theorem 5.2(ii). The converse assertion is trivial. \( \square \)

6. Acknowledgments

The second author gratefully acknowledges the financial support of the research Center for Advanced Studies in Mathematics of the Ben-Gurion University of the Negev.

References

1. A. V. Arhangelskii, *Topological function spaces*, Math. Appl. **78**, Kluwer Academic Publishers, Dordrecht, 1992.
2. A. V. Arhangelskii, O. G. Okunev, V. G. Pestov, Free topological groups over metrizable spaces, Topology Appl. **33** (1989), 63–76.
3. A. V. Arhangelskii, M. G. Tkachenko, *Topological groups and related structures*, Atlantis Press/World Scientific, Amsterdam-Raris, 2008.
4. T. Banakh, Fans and their applications in General Topology, Functional Analysis and Topological Algebra, available in arXiv:1602.04857
5. W. Banaszczyk, *Additive subgroups of topological vector spaces*, LNM 1466, Berlin-Heidelberg-New York 1991.
6. V. K. Bel’nov, On dimension of free topological groups, Proc. IVth Tiraspol Symposium on General Topology, (1979), 14–15 (in Russian).
7. R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
8. M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, Springer, New York, 2010.
9. J. Flood, Free topological vector spaces, Ph. D. thesis, Australian National University, Canberra, 109 pp., 1975.
10. J. Flood, Free locally convex spaces, Dissertationes Math. CCXXI, PWN, Warczawa, 1984.
11. S. Gabriyelyan, The \( k \)-space property for free locally convex spaces, Canadian Math. Bull. **57** (2014), 803–809.
12. S. Gabriyelyan, A characterization of free locally convex spaces over metrizable spaces which have countable tightness, Scientiae Mathematicae Japonicae **78** (2015), 201–205.
13. M. Graev, Free topological groups, Izv. Akad. Nauk SSSR Ser. Mat. 12 (1948), 278–324 (In Russian). Topology and Topological Algebra. Translation Series 1, 8 (1962), 305–364.
14. K. H. Hofmann, S. A. Morris, The Structure of Compact Groups: A Primer for the Student – A Handbook for the Expert, 3rd edition, de Gruyter, Studies in Mathematics 25, Berlin, 2013.
15. D. C. Hunt, S. A. Morris, Free subgroups of free topological groups, Proc. Second Internat. Conf. Theory of Groups, Canberra, Lecture Notes in Mathematics 372 (Sptinger, Berlin, 1974), pp. 377–387.
16. H. Jarchow, Locally Convex Spaces, B.G. Teubner, Stuttgart, 1981.
17. S. Kakutani, Free topological groups and infinite direct product of topological groups, Proc. Imp. Acad. Tokyo 20 (1944), 595–598.
18. E. Katz, S. A. Morris, P. Nickolas, A free subgroups of the free abelian topological groups on the unit interval, Bull. London Math. Soc. 14 (1982), 399–402.
19. J. L. Kelley, General topology, D. van Nostrand, New York, 1955.
20. G. Köthe, Topological vector spaces, Vol. I, Springer-Verlag, Berlin, 1969.
21. A. Leiderman, S. A. Morris, V. Pestov, The free abelian topological group and the free locally convex space on the unit interval, J. London Math. Soc. 56 (1997), 529–538.
22. J. Mack, S. A. Morris, E. T. Ordman, Free topological groups and the projective dimension of a locally compact abelian groups, Proc. Amer. Math. Soc. 40 (1973), 303–308.
23. S. MacLane, Categories for the Working Mathematician, Springer-Verlag, New York, 1971.
24. A. A. Markov, On free topological groups, Dokl. Akad. Nauk SSSR 31 (1941), 299–301.
25. E. Michael, ℵ₀-spaces, J. Math. Mech. 15 (1966), 983–1002.
26. S.A. Morris, Varieties of topological groups and left adjoint functors, J. Austral. Math. Soc. 16 (1973), 220–227.
27. S.A. Morris, Pontryagin duality and the structure of locally compact abelian groups, Cambridge Univ. Press, Cambridge, 1977.
28. L. Narici, E. Beckenstein, Topological vector spaces, Second Edition, CRC Press, New York, 2011.
29. P. Nickolas, A Kurosh subgroup theorem for topological groups, Proc. London Math. Soc. 42 (1981), 461–477.
30. E. T. Ordman, B. V. Smith-Thomas, Sequential conditions and free topological groups, Proc. Amer. Math. Soc. 79 (1980), 319–326.
31. I. Protasov, Maximal vector topologies, Topology Appl. 159 (2012), 2510–2512.
32. I. V. Protasov, E. G. Zelenyuk, Topologies on groups determined by sequences, Monograph Series, Math. Studies VNTL, Lviv, 1999.
33. D. A. Raïkov, Free locally convex spaces for uniform spaces, Math. Sb. 63 (1964), 582–590.
34. S. A. Saxon, Nuclear and product spaces, Baire-like spaces, and the strongest locally convex topology, Math. Ann. 197 (1972), 87–106.
35. O. V. Sipacheva, The topology of a free topological group, (Russian) Fundam. Prikl. Mat. 9 (2003), no. 2, 99–204; translation in J. Math. Sci. (N. Y.) 131 (2005), no. 4, 5765–5838.
36. B. V. Smith-Thomas, Free topological groups, General Topology Appl. 4 (1974), 51–72.
37. N. E. Steenrod, A convenient category of topological spaces, Michigan Math. J. 14 (1967), 133–152.
38. M. G. Tkachenko, On completeness of free abelian topological groups, Soviet Math. Dokl. 27 (1983), 341–345.
39. V. V. Uspenskiĭ, On the topology of free locally convex spaces, Soviet Math. Dokl. 27 (1983), 781–785.
40. V. V. Uspenskiĭ, Free topological groups of metrizable spaces, Math. USSR-Izv. 37 (1991), 657–680.

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, P.O. 653, Israel
E-mail address: saak@math.bgu.ac.il

Faculty of Science and Technology, Federation University Australia, PO Box 663, Ballarat, Victoria, 3353, Australia & Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria, 3086, Australia
E-mail address: morris.sidney@gmail.com