Quantum Lorentz and braided Poincaré groups

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Abstract
Quantum Lorentz groups $H$ admitting quantum Minkowski space $V$ are selected. Natural structure of a quantum space $G = V \times H$ is introduced, defining a quantum group structure on $G$ only for triangular $H (q = 1)$. We show that it defines a braided quantum group structure on $G$ for $|q| = 1$.

0 Introduction
Any example of a quantum Poincaré group [1] is constructed using one of quantum Lorentz groups introduced in [2]. However, only very special cases of the latter (triangular deformations) can be used for this purpose. Cases related to the celebrated $q$-deformation of Drinfeld and Jimbo are, unfortunately, excluded. This is in fact a general feature of inhomogeneous quantum groups [3, 4].

It turned out recently that this obstacle can be circumvented, if one allows the deformed inhomogeneous group to be a braided quantum group rather than an ordinary quantum group. It means that the comultiplication is a morphism into a nontrivial crossed-product algebra rather than the usual product. It turns out that on the level of generators, the only non-trivial cross-relations are those for the translation coordinates. These results have been derived in [5] for the case when the homogeneous part is the standard $q$-deformed (with $|q| = 1$) orthogonal quantum group $SO(p, p)$, $SO(p, p + 1)$ [6] or $SO(p, p + 2)$ [7]. I have learned recently about the paper [8] where results of similar type (without the reality condition) were obtained (cf. also [3, 11]).

In the present paper we study the case when the homogeneous part $H$ is the Lorentz group. This case requires a separate study, because we have the possibility to take into account the complete classification of quantum deformations [2]. Another reason for a separate treatment is that we want to consider the 'more fundamental' simply connected $SL(2, \mathbb{C})$ group instead of $SO(1, 3)$.

The paper is organized as follows. In Section 1 we recall non-triangular, deformation-type cases of quantum Lorentz group $H$. In Section 2 we select those cases which have

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the corresponding quantum Minkowski space $V$ (this happens for $|q| = 1$ or $q^2 \in \mathbb{R}$). In Section 3 we construct a natural crossed ‘cartesian product’ $G$ of $V$ and $H$ (as quantum spaces). In Section 4 we investigate conditions under which the natural formula for the comultiplication on generators, defines a morphism of algebras, the product algebra being understood with suitable crossed (or braided) structure.

The same program on the Poisson level has already been presented in [5].

We conclude in Section 5 with explicit commutation relations for the Minkowski space. Several proofs are shifted to the Appendix.

1 Quantum Lorentz groups

We recall that the $^*$-algebra $A = Poly (H)$ of polynomials on quantum $H = SL(2, \mathbb{C})$ is generated by matrix elements of

$$u = (u_B^A)_{A,B=1,2} = \begin{pmatrix} u_1^2 & u_2^1 \\ u_1^1 & u_2^2 \end{pmatrix}$$

subject to relations

$$u_1 u_2 E = E, \quad E' u_1 u_2 = E', \quad X u_1 \bar{\pi} = \bar{\pi} u_2 X,$$

(1)

where $E$, $E'$ and $X$ are described in Theorem 2.2 of [2]. Here the subscripts 1 and 2 refer to the position of a given object in the tensor product of the underlying ‘arithmetic’ vector space (in this case $\mathbb{C}^2$, with the standard basis $e_1, e_2$). For instance, the first equality means that $u_1^2 u_2^B E = E' u_1^1 u_2^2 E$ (summation convention). We omit the subscripts when the object has only one natural position in a given situation (like $E$ for instance). The complex conjugate $\bar{\pi}$ of $u$ is given by

$$\bar{\pi} = (\bar{u}_B^A)_{A,B=1,2} = \begin{pmatrix} (u_1^1)^* & (u_2^1)^* \\ (u_1^2)^* & (u_2^2)^* \end{pmatrix},$$

i.e. $\bar{\pi} = (u_B^A)^*$.

The ‘barred’ indices refer to the complex conjugated basis $e_\uparrow := \bar{e}_1$, $e_\downarrow := \bar{e}_2$ in $\mathbb{C}^2$.

With the standard comultiplication defined on generators by $\Delta u = uu'$ (primed coordinates refer to the second copy of $H$; in a less compact notation, $\Delta u_B^A = u_B^C \otimes u_B^C \in A \otimes A$), the above $^*$-algebra becomes a Hopf $^*$-algebra. ($\Delta$ preserves the relations, for instance $\Delta u_1 \Delta u_2 E = u_1 u_1' u_2 u_2' E = u_1 u_2 u_1' u_2' E = u_1 u_2 E = E$.)

In the sequel we focus on non-triangular deformations. It means that

$$E = e_1 \otimes e_2 - q e_2 \otimes e_1, \quad E' = e^2 \otimes e^1 - q^{-1} e^1 \otimes e^2 \quad q \in \mathbb{C} \setminus \{0, i, -i\}$$

(2)

(the standard $q$-deformation) and $X$ is given by (13) or (15) of [3], i.e. we have one of the following two cases

1. $X = t \frac{1}{2} (e_T^1 \otimes e_1^\uparrow + e_T^2 \otimes e_2^\uparrow) + t^{-1} \frac{1}{2} (e_T^1 \otimes e_1^\downarrow + e_T^2 \otimes e_2^\downarrow)$ \quad $0 < t \in \mathbb{R}$

2. $X = q \frac{i}{2} (e_T^1 \otimes e_1^\uparrow + e_T^2 \otimes e_2^\uparrow) + q^{-1} \frac{i}{2} (e_T^1 \otimes e_1^\downarrow + e_T^2 \otimes e_2^\downarrow) \pm q \frac{i}{2} e_T^2 \otimes e_1^\uparrow$ \quad (for $0 < q \in \mathbb{R}$)
(with the obvious notation for the matrix units $e_1^\dagger := e_1 \otimes e_1$ etc.).

Any matrix which intertwines $u_1 u_2$ with itself and satisfies the braid equation is proportional to
\[ M := qP' - q^{-1}P \quad \text{or} \quad M^{-1} = q^{-1}P' - qP, \]
where $P := -(q + q^{-1})^{-1}EE'$ (the deformed antisymmetrizer) is the projection on $E$ parallel to $\ker E'$ and $P' := I - P$ (the deformed symmetrizer). Conjugating $M_{\pm 1} u_1 u_2 = u_1 u_2 M_{\pm 1}$ we obtain $K_{\pm 1} u_1 u_2 = u_1 u_2 K_{\pm 1}$, where
\[ K := \tau M \tau = \overline{q} Q' - \overline{q}^{-1} Q, \quad Q := \tau P \tau, \quad Q' := I - Q. \]

Throughout the paper, $\tau$ denotes the permutation in the tensor product.

\section{Quantum Minkowski spaces}

In order to discuss quantum Minkowski spaces, covariant under the quantum Lorentz group, we consider the four-dimensional representation of the latter,
\[ h := u_1 u_2 \quad \text{i.e.} \quad h^{AB}_{CD} := u_1^{AB} u_2^{CD}. \]
Note that $\tau h \tau = h$. It means that in a basis of elements selfadjoint with respect to the natural conjugation $x \mapsto \tau x$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$, such as the basis of Pauli matrices $\sigma_j^{AB}$ ($j = 0, \ldots, 3$), the matrix $h$ has selfadjoint elements. In considerations which refer only to the four-dimensional ('vector') representation, it is often convenient to use exactly the components of $h$ in the basis of Pauli matrices. These components will be denoted by $h_{jk}^k$ ($j, k = 0, \ldots, 3$). In the leg-numbering notation (like in (1)) we shall use bold subscripts for the four-dimensional case. For instance, the tensor square of $h$ will be denoted either by $h_{12} h_{34}$ (referring to the spinor representation) or by $h_{1} h_2$ (referring to the vector representation).

Now we look for appropriate quadratic commutation relations defining the quantum Minkowski space. We use here the standard method of dealing with 'quantum vector spaces'. The algebra of polynomials on quantum Minkowski space should be generated by four generators $x = (x^{AB})_{A, B = 1, 2} = (x^j)_{j = 0, \ldots, 3}$ satisfying the reality condition
\[ \tau x = x \quad \text{i.e.} \quad (x^{AB})^* = x^{BA} \quad \text{or} \quad (x^j)^* = x^j \]
and some quadratic relations $Ax x_2 = 0$, such that $\Delta_V x := hx'$ satisfies the same relations (note, that $\Delta_V x$ satisfies the reality automatically). The last requirement will be satisfied if $A$ is an intertwiner of $h_1 h_2$:
\[ A\Delta_V x_1 x_2 = Ah_1 h_2 x_1 x_2 = h_1 h_2 A x_1 x_2 = 0 \]
(this is the key point of the method of [6]). It remains to choose an appropriate intertwiner: it should be a deformation of the antisymmetrizer (see also Remark 2.2 below).

From $M_{\pm 1}$, $K_{\pm 1}$, $X$ and $X^{-1}$ we can build easily four intertwiners of $h_1 h_2 = h_{12} h_{34}$, namely
\[ \hat{R}_\pm := X_{23}(M_{12} K_{34}^{\pm 1}) X_{23}^{-1} \]
and their inverses. Each of them becomes the permutation in the classical limit.
**Proposition 2.1** Matrices $\hat{R}_\pm$ satisfy the braid equation (with $\mathbb{C}^2 \otimes \mathbb{C}^2$ being the elementary space).

For the proof we refer to Appendix (Section 6.1).

Substituting (3), (4) into (8) we get the spectral decomposition

$$\hat{R}_\pm = X_{23}(q^{±1}P' \otimes Q' + q^{-1}q^{±1}P \otimes Q - q^{-1}q^{±1}P \otimes Q')X_{23}^{-1}. \quad (9)$$

Since the projections $P' \otimes Q, P \otimes Q'$ are 3-dimensional,

$$P^{(-)} := X_{23}(P' \otimes Q + P \otimes Q')X_{23}^{-1}. \quad (10)$$

is a good candidate for the deformed antisymmetrizer. It is in fact easy to see that it becomes the classical antisymmetrizer in the classical limit.

**Remark 2.2** It is not necessary to use the argument of a ‘deformed antisymmetrizer’. In fact, there is a more straightforward (logical) approach. Note that the subspace $V^*$ spanned by $x^j$ is invariant with respect to $H^*$, and we are looking just for a 6-dimensional invariant subspace of $V^* \otimes V^*$. It must be therefore the direct sum of the two 3-dimensional irreducible subrepresentations in $V^* \otimes V^*$.

**Definition 2.3** The *-algebra generated by $(x^j)^* = x^j$ (i.e. $(x^{AB})^* = x^{BA}$) and relations

$$P^{(-)} x_1 x_2 = 0 \quad \text{ (i.e. } P^{(-)} x_{12} x_{34} = 0) \quad \text{(11)}$$

is said to be the *-algebra of polynomials on quantum Minkowski space (and denoted by Poly $(V)$) if it has the classical size (i.e. if the Poincaré-Birkhoff-Witt theorem holds).

**Proposition 2.4** Quantum Minkowski spaces exist only for

$$|q| = 1 \quad \text{ or } \quad \overline{q} = q^2. \quad \text{(12)}$$

For the proof, see Appendix (Section 6.2). Note that this result was conjectured in [11].

In the sequel we assume one of the two possibilities: $q = \overline{q}$ or $|q| = 1$ (we discard $q = -\overline{q}$ as not of deformation type).

Note that

for $q = \overline{q}$, \hspace{1cm} \( \hat{R}_+ = X_{23}(q^2P' \otimes Q' + q^{-2}P \otimes Q - P' \otimes Q - P \otimes Q')X_{23}^{-1} \quad \text{(13)} \)

\( \hat{R}_- = X_{23}(P' \otimes Q + P \otimes Q - q^2P' \otimes Q - q^{-2}P \otimes Q')X_{23}^{-1} \quad \text{(14)} \)

for $|q| = 1$, \hspace{1cm} \( \hat{R}_+ = X_{23}(P' \otimes Q + P \otimes Q - q^2P' \otimes Q - q^{-2}P \otimes Q')X_{23}^{-1} \quad \text{(15)} \)

\( \hat{R}_- = X_{23}(q^2P' \otimes Q' + q^{-2}P \otimes Q - P' \otimes Q - P \otimes Q')X_{23}^{-1} \quad \text{(16)} \)

hence

for $q = \overline{q}$, \hspace{1cm} \( [11] \quad \Longleftrightarrow \quad \hat{R}_- x_1 x_2 = x_1 x_2, \)

for $|q| = 1$, \hspace{1cm} \( [11] \quad \Longleftrightarrow \quad \hat{R}_+ x_1 x_2 = x_1 x_2, \)

which ‘explains’ why for $q = \overline{q}$ or $|q| = 1$ we get the appropriate size of the algebra generated by $x$, namely, different ways of ordering the polynomials of the third degree give the same result, due to the Yang-Baxter property of $\hat{R}_\pm$ (Prop. 2.1):

$$R_{12} R_{13} R_{23} x_1 x_2 x_3 = x_3 x_2 x_1 = R_{23} R_{13} R_{12} x_1 x_2 x_3, \quad (17)$$

where $R = \tau\hat{R}_-$ for $\overline{q} = q$ and $R = \tau\hat{R}_+$ for $|q| = 1$. 
3 Crossed product of Minkowski with Lorentz

In this section we shall introduce a crossed tensor product of Poly (H) and Poly (V) in such a way, that the standard comultiplication
\[ \Delta u = uu', \quad \Delta x = x + hx', \]
preserves ‘as much as possible’ from the algebraic structure (preserves as many relations as possible). Technically (see Theorem 3.1 below for the precise statement), we consider the universal *-algebra \( \mathcal{B} \) generated by \( u_B^A \) and \( x'^j = (x^j)^* \), satisfying \((\ref{1}), (\ref{11})\) and the cross relations
\[ x_{12}u_3 = T u_1 x_{23} \quad \text{(i.e. } x'^A_B u^C_D = T^{A_E}_{E_F} u^E_D x^{K_L} x_{K_L}) \]
for an appropriate matrix \( T \), which we select after some discussion.

Note that the ‘preservation of relations’ by \( \Delta \) means that \( \Delta u \) and \( \Delta x \) do satisfy the same relations as \( u \) and \( x \). Let us check when it happens. Of course, \( \Delta u \) satisfies \((\ref{1})\) as before. Since
\[ \Delta x_1 \Delta u_2 = (x_1 + h_1 x_1') u_3' = T u_1 x_{23} u_1' + h_1 u_3 T u_1' x_{23}, \]
we use here \( x_{12} u_3' = u_3 x_{12}' \) and
\[ T \Delta u_1 \Delta x_2 = T u_1 (x_{23} + h_2 x_{23}') = T u_1 x_{23} u_1' + T u_1 h_2 u_1' x_{23}', \]
\( \Delta x \) and \( \Delta u \) satisfy \((\ref{19})\) if
\[ T u_1 h_3 = h_1 u_3 T, \]
i.e. \( T \in \text{Mor} (u_1 u_2 \overline{u}_3, u_1 u_2 \overline{u}_3) \) \((T \text{ intertwines } u_1 u_2 \overline{u}_3 \text{ with } u_1 u_2 \overline{u}_3)\). It means that
\[ T = X_{23} S_{12} \]
for some \( S \in \text{Mor} (u_1 u_2, u_1 u_2) \), which we assume to be invertible.

The discussion of when \( \Delta \) preserves \((\ref{11})\) will be postponed till the next section.

By taking the star operation of \((\ref{19})\), we obtain
\[ \overline{u}_{12} x_{23} = X_{12}(S \tau)^{-1}_{23}x_{12} u_3 \]
we have used the property \( \tau X = X \), hence
\[ x_{12} \overline{u}_3 = (S \tau^{-1})_{23} X_{12}^{-1} u_1 x_{23}. \]
It follows that
\[ x_{12} h_{34} = X_{23} S_{12} (S \tau^{-1})_{34} X_{23}^{-1} h_{12} x_{34} \]
and the matrix governing the commutation of \( x \) and \( h \) has similar structure to \( \hat{R}_- \) in \((\ref{8})\). It suggests that \( S \) should be proportional to \( M \) or \( M^{-1} \). We shall show that it is indeed the case, if we require \( \mathcal{B} \) to have a correct size.

Recall that a crossed tensor product of two algebras, \( C \) and \( D \), is the tensor product of vector spaces \( C \otimes D \) equipped with the multiplication
\[ m = (m_C \otimes m_D)(\text{id} \otimes s \otimes \text{id}), \]
$m_C$ and $m_D$ being the multiplication maps in $C$ and $D$, where $s: D \otimes C \to C \otimes D$ is a linear map satisfying

$$(\id \otimes s)(m_D \otimes \id ) = (\id \otimes m_D)(s \otimes \id ) (\id \otimes s), \quad (s \otimes \id )(\id \otimes m_C) = (m_C \otimes \id )(\id \otimes s)(s \otimes \id )$$

(26)

(this condition is equivalent to the associativity of $m$). For unital algebras we require additionally $s(I \otimes c) = c \otimes I$, $s(d \otimes I) = (I \otimes d)$ and for $^*$-algebras, we require that $^*_{12}^*_{12} = \id$, where

$${^*}_{12} = s(^* \otimes ^*) \tau.$$  

(27)

Under these conditions $C \otimes D$ becomes a unital $^*$-algebra (called the crossed tensor product of $C$ and $D$) and the inclusions $c \mapsto c \otimes I$, $d \mapsto I \otimes d$ are unital $^*$-homomorphisms (cf. for instance [12]).

**Theorem 3.1** If there exists a crossed tensor product of $^*$-algebras $\text{Poly}(H)$ and $\text{Poly}(V)$, compatible with (19), i.e. such that

$$s(x^{AB} \otimes u^C_D) = T_{EKL}^{ABC}u^E_D \otimes x^{KL},$$

(28)

then it is unique. It exists if and only if

$$S = q^{-\frac{1}{2}} M \quad \text{or} \quad S = q^\frac{1}{2} M^{-1}$$

(square roots defined up to sign).

The proof is given in the Appendix (Section 6.3).

From now on we shall consider the case when $S = q^{-\frac{1}{2}} M$ (the second case in (29) is completely analogous). We can write (24) as

$$x_1 h_2 = \hat{W} h_1 x_2,$$

(30)

where

$$\hat{W} = \hat{R}_- \quad \text{for} \quad \bar{q} = q, \quad \hat{W} = q^{-1} \hat{R}_- \quad \text{for} \quad |q| = 1.$$  

(31)

4 Poincaré group with braided translations — only for $|q| = 1$

Now we can return to the problem when $\Delta$ preserves (11), i.e. when $P^{-} x_1 x_2 = 0$ implies $P^{-} \Delta x_1 \Delta x_2 = 0$. Assuming (11), first two terms in

$$\Delta x_1 \Delta x_2 = (x_1 + h_1 x'_1)(x_2 + h_2 x'_2) = x_1 x_2 + h_1 x'_1 h_2 x'_2 + x_1 h_2 x'_2 + h_1 x'_1 x_2$$

(32)

are obviously annihilated by $P^{-}$ (second, because $P^{-} h_1 h_2 x'_1 x'_2 = h_1 h_2 P^{-} x'_1 x'_2 = 0$). In the last term we shall need to commute $x'_1$ with $x_2$. Normally they just commute, but, it will be convenient to consider here the following more general situation

$$x'_1 x_2 = \hat{B} x_1 x'_2 \quad \text{or} \quad x'_2 x_1 = B x_1 x'_2 \quad (\hat{B} = \tau B)$$

(33)
(for some matrix $B$). In particular, if $B = \text{id}$, $x_1'$ and $x_2$ commute. Note that this more general assumption does not affect previous results on the preservation of (1) and (13).

The sum of the last two terms in (32) is equal

$$(\hat{W} h_1 x_2 x_2' + h_1 x_1' x_2)_{jk} = \hat{W}_{jk} h_1 x_2 x_2' + h_1 B_{bc} x_2 x_2' = (\hat{W}_{jk} + \delta_{a} B_{bc}) h_1 x_2 x_2',$$

hence, finally, $\Delta$ preserves (11) when

$$P_{12}(-\hat{W}_{12} + B_{23}) = 0.$$  \hspace{1cm} (34)

Now, if $B = I$, then using (31) we see that the above equality is possible only for $q^2 = 1$.

This is one more manifestation of the fact that the standard $q$-deformation is not compatible with inhomogeneous groups.

On the other hand, if we could manage that

$$P(-\hat{W} + \sigma I) = 0 \quad \text{for some } \sigma,$$  \hspace{1cm} (35)

then $B = \sigma I$ satisfies (31). In this case $\Delta$ preserves (11) provided we consider the ‘braiding’

$$x'^j x^k = \sigma x^k x'^j.$$  \hspace{1cm} (36)

Taking into account that $P(-)$ is a projection and a function of $\hat{W}$, condition (31) means that $P(-)$ is a spectral projection of $\hat{W}$ corresponding to a single eigenvalue (equal to $-\sigma$). From (31), (13) and (16) it is clear that this is possible only for $|q| = 1$ and in this case $\sigma = q^{-1}$.

It is easy to check that (36) define consistently a crossed tensor product of $B$ with itself. Concluding, we have a family of braided Poincaré groups, labelled by two parameters: $|q| = 1$ and $t > 0$.

5 Minkowski space

We present here explicitly the defining relations (11) for the quantum Minkowski space corresponding to $|q| = 1$ and $t > 0$:

$$\alpha\beta = t q \beta \alpha$$
$$\alpha\gamma = t^{-1} q \gamma \alpha$$
$$\beta\delta = t q \delta \beta$$
$$\gamma\delta = t^{-1} q \delta \gamma$$
$$\beta\gamma = \gamma \beta$$
$$[\alpha, \delta] = t^{-1} (q - q^{-1}) \beta \gamma$$

and

$$\alpha^* = \alpha, \quad \delta^* = \delta, \quad \beta^* = \gamma$$  \hspace{1cm} (37)

(cf. (13)–(16) and (11)). We have denoted the elements $x^{AB}$ as follows

$$x = \begin{pmatrix} x^1 & x^2 \\ x^2 & x^3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$  \hspace{1cm} (38)
We may introduce a complex parameter $z := q/t \neq 0$. The corresponding quantum Minkowski space $\mathcal{M}_z$ is described by the $\ast$-algebra $\text{Poly}(\mathcal{M}_z)$ generated by elements $\alpha, \beta, \gamma$ satisfying

$$\begin{align*}
\alpha^* &= \alpha, \\
\delta^* &= \delta, \\
\gamma^* \gamma &= \gamma \gamma^*,
\end{align*}$$

such that

\begin{align*}
\alpha \gamma &= z \gamma \alpha \\
\gamma \delta &= z \delta \gamma \\
[\alpha, \delta] &= (z - \frac{1}{z}) \gamma \gamma^*.
\end{align*}

The invariant *Minkowski length* (obtained as $E_{12}'(\tau E')^{34} X_{23}^{-1} x_{12} x_{34}$) is a central element of $\text{Poly}(V)$, equal

$$\frac{\alpha \delta}{2z} + \frac{\delta \alpha}{2z} - \gamma^* \gamma.$$
6.2 Selection of parameters for Minkowski

We shall write relations (11) in explicit form. The two cases of \( X \) may be written in one formula

\[
X = e_1^T \otimes e_1^2 + e_2^T \otimes e_2^2 + t^{-1}(e_1^2 \otimes e_1^2 + e_2^2 \otimes e_2^2) + \varepsilon e_1^2 \otimes e_1^2,
\]

where \( \varepsilon = 0 \) in case 1 and \( \varepsilon = \pm 1 \), \( t = q \) in case 2 (we have rescaled \( X \) for convenience).

We have

\[
X^{-1} = e_1^T \otimes e_1^2 + e_2^T \otimes e_2^2 + t(e_1^2 \otimes e_1^2 + e_2^2 \otimes e_2^2) - \varepsilon e_1^2 \otimes e_1^2.
\]

Of course, (11) is equivalent to

\[
(P' \otimes Q)X_{23}^{-1}x_{12}x_{34} = 0 \quad \text{and} \quad (\mathcal{P} \otimes Q')X_{23}^{-1}x_{12}x_{34} = 0.
\]

Using

\[
\ker \mathcal{P}' = \langle E \rangle = \ker \langle E \rangle^0 = \ker \langle e^{11}, e^{22}, e^{21} + qe^{12} \rangle,
\]

\[
\ker \mathcal{Q} = \ker \tau E^T = \ker(\varepsilon e^{11} - e^{21}), \quad \ker \mathcal{P} = \ker \mathcal{P}' = \ker(qe^{21} - e^{12}),
\]

\[
\ker \mathcal{Q}' = \langle \tau E \rangle = \ker \langle \tau E \rangle^0 = \ker \langle e^{11}, e^{22}, \varepsilon e^{11} + e^{22} \rangle,
\]

we see that \( \mathcal{P}' \otimes \mathcal{Q} \) is composed of vectors which are annihilated by the following three functionals

\[
\{e^{11}, e^{22}, e^{21} + qe^{12}\} \otimes (\varepsilon e^{11} - e^{21}) =
\]

\[
= \{\varepsilon e^{11} - e^{11}, \varepsilon e^{22} - e^{22}, |q|^2 e^{21} + \varepsilon e^{11} - qe^{12} - e^{21}\}
\]

(\( \varepsilon e^{11} := e^{11} \otimes e^{22} \) etc.) and \( \mathcal{P} \otimes \mathcal{Q}' \) is composed of vectors which are annihilated by the following three functionals

\[
(qe^{21} - e^{12}) \otimes \{e^{11}, e^{22}, \varepsilon e^{11} + e^{22}\} =
\]

\[
= \{qe^{21} - e^{12}, qe^{22} - e^{21}, |q|^2 e^{21} + qe^{11} - \varepsilon e^{12} - e^{21}\}
\]

Composing all the six functionals with \( X_{23}^{-1} \), we get the following functionals

\[
\{\varepsilon e^{11} - \varepsilon e^{22}\} - te^{12} + \varepsilon e^{21} - e^{22}, |q|^2 te^{12} + \varepsilon e^{21} - e^{22} - qe^{12} - te^{21},
\]

\[
q(e^{21} - e^{22}) - te^{12}, \varepsilon e^{21} - e^{22}, |q|^2 te^{12} + q(e^{21} - e^{22}) - e^{12} - e^{21},
\]

\[
q(e^{21} - e^{22}) - te^{12}, \varepsilon e^{21} - e^{22}, |q|^2 te^{12} + q(e^{21} - e^{22}) - e^{12} - e^{21},
\]

and (11) is equivalent to vanishing of these functionals on \( x_{12}x_{34} \). This way we obtain six following relations:

\[
|q|^2 e^{12} - e^{12} - q x_{12} x_{21} = 0
\]

\[
|q|^2 e^{22} - e^{22} - t x_{12} x_{21} = 0
\]

\[
|q|^2 e^{21} - e^{21} - t x_{12} x_{21} = 0
\]

\[
|q|^2 e^{22} - e^{22} - t x_{12} x_{21} = 0
\]

\[
|q|^2 e^{12} - e^{12} - q x_{12} x_{21} = 0
\]

\[
|q|^2 e^{21} - e^{21} - t x_{12} x_{21} = 0.
\]
Substituting
\[
\begin{pmatrix}
  x_1^\top & x_2^\top \\
  x_2 & x_2^2
\end{pmatrix} = 
\begin{pmatrix}
  \alpha & \beta \\
  \gamma & \delta
\end{pmatrix},
\]
we can write these relations as follows
\[
\begin{align*}
\overline{q}(\alpha \beta - \varepsilon \beta \delta) - t \beta \alpha &= 0 \quad (47) \\
\overline{q} t \gamma \delta - \delta \gamma &= 0 \quad (48) \\
|q|^2 t \alpha \delta + \overline{q}(\gamma \beta - \varepsilon \delta \delta) - q \beta \gamma - t \delta \alpha &= 0 \quad (49) \\
q(\gamma \alpha - \varepsilon \delta \gamma) - t \alpha \gamma &= 0 \quad (50) \\
q t \delta \beta - \beta \delta &= 0 \quad (51) \\
|q|^2 t \delta \alpha + q(\gamma \beta - \varepsilon \delta \delta) - \overline{q} \beta \gamma - t \delta \alpha &= 0. \quad (52)
\end{align*}
\]
Now we shall show that for the PBW theorem, condition (12) is necessary. We thus consider the case 1 i.e. \( \varepsilon = 0 \). In this case, the commutation relations take the form
\[
\begin{align*}
\beta \alpha &= \overline{q} t^{-1} \alpha \beta \quad (53) \\
\gamma \alpha &= q^{-1} t \alpha \gamma \quad (54) \\
\delta \gamma &= \overline{q} t \gamma \delta \quad (55) \\
\delta \beta &= q^{-1} t^{-1} \beta \delta \quad (56) \\
|q|^2 t \delta \alpha + q \gamma \beta &= t \alpha \delta + \overline{q} \beta \gamma \quad (57) \\
-t \delta \alpha + \overline{q} \gamma \beta &= -|q|^2 t \alpha \delta + q \beta \gamma. \quad (58)
\end{align*}
\]
Taking \( \overline{q}(57) - q(58) \) and \( \overline{q} + |q|^2 (58) \) instead of \( \overline{q} \) and \( 58 \), we obtain:
\[
\begin{align*}
q(\overline{q}^2 + 1) t \delta \alpha &= q(\overline{q}^2 + 1) t \alpha \delta + (\overline{q}^2 - q^2) \beta \gamma \quad (59) \\
q(\overline{q}^2 + 1) \gamma \beta &= t(1 - |q|^4) \alpha \delta + \overline{q}(\overline{q}^2 + 1) \beta \gamma. \quad (60)
\end{align*}
\]
Using \( 53-56 \) and \( 59-60 \) it is easy to see that each element of the algebra can be written as a sum of (alphabetically) ordered monomials in \( \alpha, \beta, \gamma, \delta \). Now, if we perform the two independent ways of ordering of \( q(\overline{q}^2 + 1) \gamma \beta \alpha \), we obtain
\[
q(\overline{q}^2 + 1) \gamma (\beta \alpha) = q(\overline{q}^2 + 1) \overline{q} t^{-1} \gamma \alpha \beta = \overline{q}(\overline{q}^2 + 1) \alpha \gamma \beta = \frac{\overline{q}}{q} t(1 - |q|^4) \alpha \delta + \overline{q}(\overline{q}^2 + 1) \beta \gamma
\]
on one hand, and
\[
q(\overline{q}^2 + 1)(\gamma \beta) \alpha = [t(1 - |q|^4) \alpha \delta + \overline{q}(\overline{q}^2 + 1) \beta \gamma] \alpha = \frac{1 - |q|^4}{q(\overline{q}^2 + 1)} \alpha t(1 - |q|^4) \alpha \delta + \overline{q}(\overline{q}^2 + 1) \beta \gamma + \overline{q} \frac{1 - |q|^4}{q(\overline{q}^2 + 1)} \alpha \gamma \beta + \overline{q} \frac{\overline{q}^2}{q(\overline{q}^2 + 1) \alpha \beta \gamma}
\]
on the other. Comparing the coefficients at \( \alpha \alpha \delta \) we get (12). Comparing at \( \alpha \beta \gamma \) gives exactly the same. (We assume of course that \( \alpha \alpha \delta \) and \( \alpha \beta \gamma \) are linearly independent.)
For \( |q| = 1 \), relations (59) and (60) are equivalent to
\[
\gamma \beta = \beta \gamma, \quad [\alpha, \delta] = \frac{1}{t} (q - q^{-1}) \beta \gamma,
\]
and the algebra \( \text{Poly} \ (V) \) resembles the usual \( GL_q(2) \) algebra (it is the same, if \( t = 1 \)). One can easily check that \( \text{Poly} \ (V) \) is a \( q \)-enveloping algebra in the sense of [13], if we order the generators as follows:
\[
e_1 := \alpha, \ e_2 := \beta, \ e_3 := \gamma, \ e_4 := \delta,
\]
hence the PBW theorem holds in this case (see Theorem 2.8.1 of [13]).

If \( q = \bar{q} \), relations (59) and (60) are equivalent to
\[
\delta \alpha = \alpha \delta, \quad [\beta, \gamma] = t(q - q^{-1}) \alpha \delta.
\]
Replacing \( \alpha \leftrightarrow \beta, \ \gamma \leftrightarrow \delta, \ t \leftrightarrow t^{-1} \), we get the same relations as in the previous case, hence the PBW theorem holds also in this case. Similarly, it holds for \( \bar{q} = -q \).

The case when \( \varepsilon = \pm 1 \) and \( q = t > 0 \) corresponds to the standard quantum deformation of the Lorentz group, containing as a subgroup \( SU_q(2) \) or \( SU_q(1,1) \) (depending on the sign of \( \varepsilon (q - 1) \)). Relations (17)–(22) are then equivalent to those considered by many authors [14, 15, 16]. It can be easily shown that the PBW theorem holds in this case, using the Diamond Lemma [17] (choose \( \beta < \alpha < \delta < \gamma \) as the total ordering).

### 6.3 The algebra \( B \)

We set \( \mathcal{A} := \text{Poly} \ (H), \ \mathcal{C} := \text{Poly} \ (V) \).

The uniqueness of \( S \) is obvious, since its value on any monomial can be reduced by (26) to the case (19).

Writing (19) as
\[
x_{12} u_3 = T u_1 x_{23}
\]
(in case the crossed product exists), we get
\[
x_{12} E_{34} = x_{12} u_3 u_4 E_{34} = T_{123} u_1 x_{23} u_4 E_{14} = T_{123} T_{234} u_1 u_2 x_{34} E_{12} = T_{123} T_{234} E_{12} x_{34},
\]
hence \( T \) must satisfy
\[
T_{123} T_{234} E_{12} = E_{34}.
\]
Taking into account that \( X_{23} X_{12} E_{23} = E_{12} \), it means that
\[
S_{12} S_{23} E_{12} = E_{23}.
\]
It is easy to see that the only solutions of (54) which are intertwiners of \( u_1 u_2 \) (hence of the form \( a I + b E E' \)) are (23).

Conversely, we shall show that if \( S \) is given by (23) then there exists \( s: \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{C} \) with required properties. Let \( \tilde{\mathcal{A}} \) (\( \tilde{\mathcal{C}} \)) be the free \(*\)-algebra generated by \( u_B^A \) (\( x^A \)). We have
\[
\mathcal{A} = \tilde{\mathcal{A}} / \mathcal{J}_A, \quad \mathcal{C} = \tilde{\mathcal{C}} / \mathcal{J}_C,
\]
where $\mathcal{J}_A = \langle \mathcal{J}_A^0 \rangle$ is the ideal generated by $\mathcal{J}_A^0 := \{ u_1 u_2 E - E, E' u_1 u_2 - E', X u_1 \overline{u}_2 - \overline{u}_1 u_2 X \}$ in $\overline{A}$ and $\mathcal{J}_C = \langle \mathcal{J}_C^0 \rangle$ is the ideal generated by $\mathcal{J}_C^0 := \{ \hat{R} x_1 x_2 - x_1 x_2, (x A^T)^* = x B^T \}$ in $\overline{C}$.

Here $\overline{R} = \overline{R}_-$ for $\overline{q} = q$ and $\overline{R} = \overline{R}_+$ for $|q| = 1$ (cf. the discussion near (17)). It is easy to see that there exists a (unique) map $\tilde{s}: \overline{C} \otimes \overline{A} \to \overline{A} \otimes \overline{C}$ satisfying (19) and (26), with $\overline{A}, \overline{C}, s$ replaced by $\hat{A}, \hat{C}, \hat{s}$.

The proof will be finished if we show that

$$\tilde{s}(\overline{C} \otimes \mathcal{J}_A) \subset \mathcal{J}_A \otimes \overline{C}, \quad \tilde{s}(\mathcal{J}_C \otimes \overline{A}) \subset \overline{A} \otimes \mathcal{J}_C.$$  

Since $\{ a \in \overline{A} : \tilde{s}(\overline{C} \otimes a) \subset \mathcal{J}_A \otimes \overline{C} \}$ is an ideal in $\overline{A}$, it is sufficient to show that

$$\tilde{s}(\overline{C} \otimes \mathcal{J}_A^0) \subset \mathcal{J}_A \otimes \overline{C}, \quad \tilde{s}(\mathcal{J}_C^0 \otimes \overline{A}) \subset \overline{A} \otimes \mathcal{J}_C.$$  

and, similarly,

$$\tilde{s}(\mathcal{J}_C^0 \otimes \overline{A}) \subset \overline{A} \otimes \mathcal{J}_C.$$  

We shall show that

$$\tilde{s}(\overline{C}^{(1)} \otimes \mathcal{J}_A^0) \subset \mathcal{J}_A \otimes \overline{C}^{(1)}, \quad \tilde{s}(\mathcal{J}_C^0 \otimes \overline{A}^{(1)}) \subset \overline{A}^{(1)} \otimes \mathcal{J}_C,$$  

where $\overline{A}^{(1)}$ and $\overline{C}^{(1)}$ denote the linear subspaces spanned by the corresponding generators. It is sufficient, because then from (26) it follows that

$$\tilde{s}(\overline{C}^{(n)} \otimes \mathcal{J}_A^0) \subset \mathcal{J}_A \otimes \overline{C}^{(n)}, \quad \tilde{s}(\mathcal{J}_C^0 \otimes \overline{A}^{(n)}) \subset \overline{A}^{(n)} \otimes \mathcal{J}_C,$$  

where $\overline{A}^{(n)}$ and $\overline{C}^{(n)}$ denote the subspaces spanned by monomials of order $n$.

To show (68), note that

$$x_{12}(u_3 u_4 E_{34} - E_{34}) = T_{123}T_{234}u_1 u_2 x_{34} E_{12} - x_{12} E_{34} = x_{12}(u_3 u_4 E - E) x_{34} + T_{123}T_{234}u_1 u_2 E_{12} - x_{12} E_{34} = T_{123}T_{234}(u_1 u_2 E_{12} - E_{12}) x_{34}$$

belongs to $\mathcal{J}_A \otimes \overline{C}$. Similarly, $x_{12}(E'_{34} u_3 u_4 - E'_{34}) \in \mathcal{J}_A \otimes \overline{C}$ and

$$x_{12}(X_{34} u_3 \overline{u}_4 - \overline{u}_3 u_4) = X_{34} T_{123}T_{234}u_1 \overline{u}_2 x_{34} - T_{123}T_{234}u_1 x_{34} X_{12} = 0,$$

where $T' = (\tau S^{-1} \tau)_{23} X_{12}^{-1}$ is the matrix appearing in (23). The equality $X_{34} T_{123}T'_{234} = T'_{123}T_{234} X_{12}$ is proved using formulas of type (10)-(15). Furthermore we have

$$(\hat{R}_{1234} x_{12} x_{34} - x_{12} x_{34}) u_5 = \hat{R}_{1234} x_{12} T_{345} u_3 x_{45} - x_{12} T_{345} u_3 x_{45} = \hat{R}_{1234} T_{345} T_{123} u_1 x_{23} x_{45} - T_{345} T_{123} u_1 x_{23} x_{45} = T_{345} T_{123} u_1 (\hat{R}_{2345} x_{23} x_{45} - x_{23} x_{45}) \in \hat{A} \otimes \mathcal{J}_C^0,$$

since $\hat{R}_{1234} T_{345} T_{123} = T_{345} T_{123} \hat{R}_{2345}$ (it also follows from (10)-(13)).

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