TWO PROBLEMS ON MATCHINGS IN SET FAMILIES – IN THE FOOTSTEPS OF ERDŐS AND KLEITMAN

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ABSTRACT. The families \( \mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]} \) are called \( q \)-dependent if there are no pairwise disjoint \( F_1 \in \mathcal{F}_1, \ldots, F_s \in \mathcal{F}_s \) satisfying \( |F_1 \cup \ldots \cup F_s| \leq q \). We determine \( \max |\mathcal{F}_1| + \ldots + |\mathcal{F}_s| \) for all values \( n \geq q, s \geq 2 \). The result provides a far-reaching generalization of an important classical result of Kleitman.

The well-known Erdős Matching Conjecture suggests the largest size of a family \( \mathcal{F} \subset \binom{[n]}{k} \) with no \( s \) pairwise disjoint sets. After more than 50 years its full solution is still not in sight. In the present paper we provide a Hilton-Milner-type stability theorem for the Erdős Matching Conjecture in a relatively wide range, in particular, for \( n \geq (2 + o(1))sk \) with \( o(1) \) depending on \( s \) only. This is a considerable improvement of a classical result due to Bollobás, Daykin and Erdős.

We apply our results to advance in the following anti-Ramsey-type problem, proposed by Özkahya and Young. Let \( ar(n,k,s) \) be the minimum number \( x \) of colors such that in any coloring of the \( k \)-element subsets of \([n]\) with \( x \) (non-empty) colors there is a rainbow matching of size \( s \), that is, \( s \) sets of different colors that are pairwise disjoint. We prove a stability result for the problem, which allows to determine \( ar(n,k,s) \) for all \( k \geq 3 \) and \( n \geq sk + (s-1)(k-1) \). Some other consequences of our results are presented as well.

1. Introduction

Let \([n] := \{1,2,\ldots,n\}\) be the standard \( n \)-element set and \( 2^{[n]} \) its power set. A subset \( \mathcal{F} \subset 2^{[n]} \) is called a family. For \( 0 \leq k \leq n \) let \( \binom{[n]}{k} \) denote the family of all \( k \)-subsets of \([n]\).

For a family \( \mathcal{F} \), let \( \nu(\mathcal{F}) \) denote the maximum number of pairwise disjoint members of \( \mathcal{F} \). Note that \( \nu(\mathcal{F}) \leq n \) holds, unless \( \emptyset \in \mathcal{F} \). The fundamental parameter \( \nu(\mathcal{F}) \) is called the independence number or matching number of \( \mathcal{F} \).

Let us introduce an analogous notion for several families.

Definition 1. Suppose that \( \mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]} \), where \( 2 \leq s \leq n \). We say that \( \mathcal{F}_1, \ldots, \mathcal{F}_s \) are cross-dependent if there is no choice of \( F_1 \in \mathcal{F}_1, \ldots, F_s \in \mathcal{F}_s \) such that \( F_1, \ldots, F_s \) are pairwise disjoint.

Note that \( \nu(\mathcal{F}) < s \) is equivalent to saying that \( \mathcal{F}_1, \ldots, \mathcal{F}_s \), where \( \mathcal{F}_i := \mathcal{F} \) for all \( i \in [s] \), are cross-dependent.

Example. Let \( n = sm + s - \ell \) for some \( \ell \in [s] \). Define

\[
\hat{\mathcal{F}}_i := \begin{cases} 
\{F \subset [n] : |F| \geq m\}, & 1 \leq i < \ell, \\
\{F \subset [n] : |F| \geq m + 1\}, & \ell \leq i \leq s.
\end{cases}
\]

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Then \( \tilde{F}_1, \ldots, \tilde{F}_s \) are easily seen to be cross-dependent.

One of the main results of the present paper is as follows.

**Theorem 1.** Choose integers \( s, m, \ell \) satisfying \( s \geq 2, m \geq 0 \) and suppose that \( F_1, \ldots, F_s \subset 2^{[n]} \) are cross-dependent. Then

\[
\sum_{i=1}^{s} |F_i| \leq (\ell - 1) \left( \frac{n}{m} \right) + s \sum_{t \geq m+1} \left( \frac{n}{t} \right) = \sum_{i=1}^{s} |\tilde{F}_i|.
\]

Inequality (1) extends the following important classical result of Kleitman.

**Theorem (Kleitman, [20]).** Let \( s \geq 2 \) be an integer and \( F \subset 2^{[n]} \) a family satisfying \( \nu(F) < s \). Then for \( n = s(m + 1) - \ell \) we have

\[
|F| \leq \frac{\ell - 1}{s} \left( \frac{n}{m} \right) + \sum_{t \geq m+1} \left( \frac{n}{t} \right).
\]

In the case \( n = s(m + 1) - 1 \) the families \( \tilde{F}_i \), from the example above are all the same, and thereby the bound (2) is best possible. It is also best possible for \( \ell = s \), as the following example due to Kleitman shows:

\[
K := \{ K \subset [sm] : |K| \geq m + 1 \} \cup \left[ \left\lfloor \frac{sm - 1}{m} \right\rfloor \right].
\]

(Note that \( \left\lfloor \frac{sm - 1}{m} \right\rfloor = \frac{sm - 1}{s} \left( \frac{sm}{m} \right) \).) In the case \( s = 2 \) the bound (2) reduces to \( |F| \leq 2^{n-1} \). This easy statement was proved already by Erdős, Ko and Rado [6].

Although (2) is a beautiful result, it has no matching lower bound for \( n \not\equiv 0 \) or \(-1\) (mod \( s \)). For \( s = 3 \) the exact answer for the only remaining residue class was given by Quinn [25]. Recently, we made some further progress [11, 12] and in particular completely resolved the case \( n \equiv -2 \) (mod \( s \)).

Let us mention that, except the case \( s = 3 \) and \( \ell = 2 \), an extension of the methods used by Kleitman in [20] is sufficient to prove Theorem 1. However, for that single case it does not seem to work. This forced us to find a very different proof. It is given in Section 3. We proved related results refining the method developed in the present paper, see [10, 12, 13].

Next we discuss a generalization of the notion of cross-dependence.

**Definition 2.** Let \( 2 \leq s \leq n \) and \( q \in [n] \) be fixed integers. The families \( F_1, \ldots, F_s \subset 2^{[n]} \) are called \( q \)-dependent if there are no pairwise disjoint \( F_1 \in F_1, \ldots, F_s \in F_s \) satisfying \( |F_1 \cup \ldots \cup F_s| \leq q \).

For \( q = n \) the notion of \( q \)-dependence reduces to that of cross-dependence. Quite surprisingly, one can determine the exact maximum of \( |F_1| + \ldots + |F_s| \) for \( q \)-dependent families \( F_1, \ldots, F_s \subset 2^{[n]} \) and all values of \( n, q, s \).

Let \( s \geq 2, m \geq 0 \) and \( \ell \in [s] \). If \( n \geq q := sm + s - \ell \), then one can define

\[
\tilde{F}^{n,q}_i := \begin{cases} 
\{ F \subset [n] : |F| \geq m \}, & 1 \leq i < \ell, \\
\{ F \subset [n] : |F| \geq m + 1 \}, & \ell \leq i \leq s.
\end{cases}
\]

(Note that \( \left\lfloor \frac{sm - 1}{m} \right\rfloor = \frac{sm - 1}{s} \left( \frac{sm}{m} \right) \).) In the case \( s = 2 \) the bound (2) reduces to \( |F| \leq 2^{n-1} \). This easy statement was proved already by Erdős, Ko and Rado [6].

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In Section 3 we prove the following generalization of Theorem 1. It follows by induction from Theorem 1, which serves as the base case.

**Theorem 2.** Choose integers $s, m, \ell$ satisfying $s \geq 2$, $m \geq 0$ and $\ell \in [s]$. Put $q = sm + s - \ell$ and suppose that $n \geq q$. If $F_1, \ldots, F_s \subset 2^{[n]}$ are $q$-dependent, then

$$\sum_{i=1}^{s} |F_i| \leq (\ell - 1) \binom{n}{m} + s \sum_{t=m+1}^{n} \binom{n}{t} \left[ = \sum_{i=1}^{s} |\tilde{F}_{i,q}| \right].$$

\[ (4) \]

1.1. **A Hilton-Milner-type result for the Erdős Matching Conjecture.** The Kleitman Theorem was motivated by a conjecture of Erdős (see [20]). Erdős [4] himself studied the uniform case, i.e., the families $F \subset \binom{[n]}{k}$. Let us make a formal definition.

**Definition 3.** For positive integers $n, k, s$ satisfying $s \geq 2$, $n \geq ks$, define

$$e_k(n, s) := \max \{ |F| : F \subset \binom{[n]}{k}, \nu(F) < s \}.$$  

Note that for $s = 2$ the quantity $e_k(n, s)$ was determined by Erdős, Ko and Rado.

**Theorem (Erdős-Ko-Rado [6]).**

$$e_k(n, 2) = \binom{n-1}{k-1} \quad \text{for} \quad n \geq 2k > 0.$$  

(5)

The case $s \geq 3$ is much harder. There are several natural examples of families $A \subset \binom{[n]}{k}$ satisfying $\nu(A) = s$ for $n \geq (s+1)k$. Following [7], for each $i \in [k]$ let us define the families $A_i^{(k)}(n, s)$:

$$A_i^{(k)}(n, s) := \left\{ A \in \binom{[n]}{k} : |A \cap [(s+1)i - 1]| \geq i \right\}. $$

(6)

Note that $|A_i^{(k)}(n, s)| = \binom{n}{k} - \binom{n-s}{k}$.

**Conjecture 1 (Erdős Matching Conjecture [4]).** For $n \geq (s+1)k$ we have

$$e_k(n, s+1) = \max \{|A_1^{(k)}(n, s)|, |A_k^{(k)}(n, s)|\}. $$

(7)

The conjecture (7) is known to be true for $k \leq 3$ (cf. [5], [23] and [9]). Improving earlier results of [1], [2], [17] and [15], the first author [8] proved

$$e_k(n, s+1) = \binom{n}{k} - \binom{n-s}{k} \quad \text{for} \quad n \geq (2s + 1)k - s.$$  

(8)

In the forthcoming paper [14], we improve this bound for large $s$. We note that the conjecture is related to several questions in combinatorics, probability and computer science, cf. [11] [26].

In the case $s = 1$, corresponding to the Erdős-Ko-Rado theorem, one has a very useful stability theorem due to Hilton and Milner [16]. Below we discuss it and its natural generalization to the case $s > 1$. 
Let us define the following families.

\[ \mathcal{H}^{(k)}(n, s) := \left\{ H \in \binom{[n]}{k} : H \cap [s] \neq \emptyset \right\} \cup \left\{ \{s+1, s+k\} \right\} \]

\[ \left\{ H \in \binom{[n]}{k} : H \cap [s] = \{s\}, H \cap [s+1, s+k] = \emptyset \right\}. \]

Note that \( \nu(\mathcal{H}^{(k)}(n, s)) = s \) for \( n \geq sk \) and

\[ |\mathcal{H}^{(k)}(n, s)| = \binom{n}{k} - \binom{n-s}{k} + 1 - \binom{n-s-k}{k-1}. \quad (9) \]

The covering number \( \tau(\mathcal{H}) \) of a family \( \mathcal{H} \) is the minimum of \( |T| \) over all \( T \subset [n] \) satisfying \( T \cap H \neq \emptyset \) for all \( H \in \mathcal{H} \). Recall the definition (6). The equality \( \tau(\mathcal{A}^{(k)}_1(n, s)) = s \) for \( n \geq k + s \) is obvious. At the same time, if \( n \geq k + s \), then \( \tau(\mathcal{H}^{(k)}(n, s)) = s + 1 \) and \( \tau(\mathcal{A}^{(k)}_i(n, s)) > s \) for each \( i \geq 2 \).

Let us make the following conjecture.

**Conjecture 2.** Suppose that \( \mathcal{F} \subset \binom{[n]}{k} \) satisfies \( \nu(\mathcal{F}) = s, \tau(\mathcal{F}) > s \). Then

\[ |\mathcal{F}| \leq \max\left\{ |\mathcal{A}^{(k)}_i(n, s)| : i = 2, \ldots, k, |\mathcal{H}^{(k)}(n, s)| \right\}. \quad (10) \]

One can verify that \( |\mathcal{A}^{(k)}_i(n, s)| \leq \max\{ |\mathcal{A}^{(k)}_1(n, s)|, |\mathcal{A}^{(k)}_i(n, s)| \} \) for any \( i \in k \). Modulo this verification, Conjecture 2 implies Conjecture 1: indeed, we have \( |\mathcal{H}^{(k)}(n, s)| \leq |\mathcal{A}^{(k)}_1(n, s)| \), and so the maximum on the right hand side of (10) is at most the maximum on the right hand side of (7). The Hilton-Milner theorem shows that (10) is true for \( s = 1 \).

**Theorem (Hilton-Milner [16]).** Suppose that \( n > 2k \) and let \( \mathcal{F} \subset \binom{[n]}{k} \) be a family satisfying \( \nu(\mathcal{F}) = 1 \) and \( \tau(\mathcal{F}) \geq 2 \). Then

\[ |\mathcal{F}| \leq |\mathcal{H}^{(k)}(n, 1)|. \]

We mention that for \( n > 2sk \) the maximum on the RHS of (10) is attained on \( |\mathcal{H}^{(k)}(n, s)| \). For \( n > 2k^3s \), (10) was verified by Bollobás, Daykin and Erdős [2].

Our second main result is the proof of (10) in a much wider range.

**Theorem 3.** Suppose that \( k \geq 3 \) and either \( n \geq (s + \max\{25, 2s+2\})k \) or \( n \geq (2 + o(1))sk \), where \( o(1) \) is with respect to \( s \to \infty \). Then for any \( \mathcal{G} \subset \binom{[n]}{k} \) with \( \nu(\mathcal{G}) = s < \tau(\mathcal{G}) \) we have \( |\mathcal{G}| \leq |\mathcal{H}^{(k)}(n, s)| \).

We prove Theorem 3 in Section 2. We note that stability results for the Erdős Matching Conjecture 3, as well as some important progress for a much more general class of Turan-type problems [19] were obtained recently. However, these results deal with the case \( n > f(s) \cdot k \), where \( f(s) \) is a fast growing function depending on \( s \). Some other Hilton-Milner-related stability results were recently proven in [21] for \( n > n_0(s, k) \).
Let us recall the method of left shifting (or simply shifting). For a given pair of indices \(1 \leq i < j \leq n\) and a set \(A \in 2^n\) we define the \((i, j)\)-shift \(S_{i,j}(A)\) of \(A\) in the following way.

\[
S_{i,j}(A) := \begin{cases} 
A & \text{if } i \in A \text{ or } j \notin A; \\
(A - \{j\}) \cup \{i\} & \text{if } i \notin A \text{ and } j \in A.
\end{cases}
\]

Next, we define the \((i, j)\)-shift \(S_{i,j}(F)\) of a family \(F \subset 2^n\):

\[
S_{i,j}(F) := \{S_{i,j}(A) : A \in F\} \cup \{A : A, S_{i,j}(A) \in F\}.
\]

We call a family \(F\) shifted, if \(S_{i,j}(F) = F\) for all \(1 \leq i < j \leq n\).

2. Proof of Theorem 3

For \(s = 1\) the theorem follows from the Hilton-Milner theorem, therefore we may assume that \(s \geq 2\). Choose an integer \(u\) such that

\[
(u + s)(k - 1) + s + 1 \leq n < (u + s + 1)(k - 1) + s + 1.
\]

Consider a family \(G\) satisfying the requirements of the theorem.

2.1. The case of shifted \(G\). First we prove Theorem 3 in the assumption that \(G\) is shifted. Following [8], we say that the families \(F_1, \ldots, F_{s+1}\) are nested if \(F_1 \supset F_2 \supset \ldots \supset F_{s+1}\). The following lemma is a crucial tool for the proof and may be obtained by a straightforward modification of the proof of [8, Theorem 3.1]:

**Lemma 4** (Frankl [8]). Let \(N \geq (u + s)(k - 1)\) for some \(u \in \mathbb{Z}\), \(u \geq s + 1\), and suppose that \(F_1, \ldots, F_{s+1} \subset \binom{[N]}{k-1}\) are cross-dependent and nested. Then

\[
|F_1| + |F_2| + \ldots + |F_s| + u|F_{s+1}| \leq s \binom{N}{k-1}.
\]

We use the following notation. For a family \(G \subset 2^n\), all \(p \in [n]\) and \(Q \subset [p]\) define

\[
G(Q, p) := \{G \setminus Q : G \in G, G \cap [p] = Q\}.
\]

The first step of the proof of Theorem 3 is the following lemma.

**Lemma 5.** Assume that \(|G| - |G(\emptyset, s)| \leq \binom{n}{k} - \binom{n-s}{k} - C\) for some \(C > 0\) and that \(\nu(G(\emptyset, s)) = x\) for some \(x \in [s]\). Then

\[
|G| \leq \binom{n}{k} - \binom{n-s}{k} - \frac{u-x-1}{u}C.
\]

Moreover, if \(\nu(G(\emptyset, s)) = \tau(G(\emptyset, s)) = 1\), then

\[
|G| \leq \binom{n}{k} - \binom{n-s}{k} - \frac{u-1}{u}C.
\]
Proof. Recall the definition of the immediate shadow of a family $\mathcal{H}$:

$$\partial \mathcal{H} := \{ H : \exists H' \in \mathcal{H}, H \subset H', |H' \setminus H| = 1 \}. \quad \text{(15)}$$

We have $\partial \mathcal{G}(\emptyset, s + 1) \subset \mathcal{G}\{s + 1\}, s + 1\}$ since $\mathcal{G}$ is shifted. In [8, Theorem 1.2] the first author proved the inequality $x|\partial \mathcal{H}| \geq |\mathcal{H}|$, valid for any $\mathcal{H}$ with $\nu(\mathcal{H}) \leq x$. Due to the shiftedness of $\mathcal{G}$, we also have $\mathcal{G}(\emptyset, s + 1) = \emptyset$ if $\tau(\mathcal{G}(\emptyset, s)) = 1$. Combining these three facts, we get that

$$|\mathcal{G}(\emptyset, s + 1)| \leq x'|\mathcal{G}(\{s + 1\}, s + 1)|, \quad \text{(16)}$$

where $x' = x$ if $\tau(\mathcal{G}(\emptyset, s)) > 1$ and $x' = 0$ if $\tau(\mathcal{G}(\emptyset, s)) = 1$. Consequently,

$$|\mathcal{G}(\emptyset, s)| \leq (x' + 1)|\mathcal{G}(\{s + 1\}, s + 1)|. \quad \text{(17)}$$

Using (16) and (12), we have

$$\sum_{i=1}^{s+1} |\mathcal{G}(\{i\}, s + 1)| + |\mathcal{G}(\emptyset, s + 1)| \leq \sum_{i=1}^{s} |\mathcal{G}(\{i\}, s + 1)| + (x' + 1)|\mathcal{G}(\{s + 1\}, s + 1)| \leq$$

$$\leq s \left( \frac{n - s - 1}{k - 1} \right) - (u - x' - 1)|\mathcal{G}(\{s + 1\}, s + 1)|.$$ 

For any $Q \subset [1, s + 1]$, $|Q| \geq 2$, we have $|A_{i}^{(k)}(n, s)(Q, s + 1)| = \binom{s + 2n}{k - |Q|}$, and thus $|\mathcal{G}(Q, s + 1)| \leq |A_{i}^{(k)}(n, s)(Q, s + 1)|$. We also have $A_{i}^{(k)}(n, s)(\emptyset, s + 1) = \emptyset$ and $\sum_{i=1}^{s+1} |A_{i}^{(k)}(n, s)(\{i\}, s + 1)| = s \binom{n-s-1}{k-1}$.

Therefore, $|A_{i}^{(k)}(n, s)| - |\mathcal{G}| \geq (u - x' - 1)|\mathcal{G}(\{s + 1\}, s + 1)| \geq \frac{u-x'-1}{x'+1}|\mathcal{G}(\emptyset, s)|$. On the other hand, the inequality in the assumptions of the lemma may be formulated as $|A_{i}^{(k)}(n, s)| - |\mathcal{G}| \geq C - |\mathcal{G}(\emptyset, s)|$. Adding these two inequalities (the second one multiplied by $\frac{u-x'-1}{x'+1}$), we obtain that $|A_{i}^{(k)}(n, s)| - |\mathcal{G}| \geq \frac{u-x'-1}{u}C$. \hfill $\Box$

Now, to prove Theorem 5 it is sufficient to obtain good bounds on $C$ from the formulation of Lemma 5. We do that in the next two propositions. We use the following simple observation:

**Observation 6.** If for some $C > 0$, $S \subset [s]$ and $B \subset \binom{[s+1,n]}{k-1}$ we have $\sum_{i \in S} |\mathcal{G}(\{i\}, s) \cap B| \leq |S||B| - C$, then both $\sum_{i \in S} |\mathcal{G}(\{i\}, s)| \leq |S|\binom{n-s}{k-1} - C$ and $|\mathcal{G}| - |\mathcal{G}(\emptyset, s)| \leq \binom{n}{k} - \binom{n-s}{k} - C$.

We are going to use the next proposition and lemma for the case $\nu(\mathcal{G}(\emptyset, s)) \geq 2$. Assume that $\mathcal{G}(\emptyset, s)$ contains $x$ pairwise disjoint sets $F_1, \ldots, F_x$ for some $x \in [s]$. Put $B_j := \binom{[s+1,n]}{k-1}\{F_j\}$.

**Proposition 7.** Under the assumption above, choose a positive integer $q$ and integers $0 =: p_0 < p_1 < p_2 < \ldots < p_q := x$. Put $f := \prod_{j=1}^{q} (p_j - p_{j-1})$. Then for $u \geq qf + \frac{q-1}{k-1}$ we have

$$\sum_{i=1}^{s} |\mathcal{G}(\{i\}, s)| \leq s \binom{n-s}{k-1} - q \bigcap_{j=0}^{q-1} \bigcup_{z=p_{j+1}}^{p_{j+1}} B_z. \quad \text{(18)}$$
Proof. For each $i \in [s]$ denote

$$\mathcal{I}(\{i\}, s) := \mathcal{G}(\{i\}, s) \cap \bigcap_{j=0}^{q-1} \left( \bigcup_{z=p_j+1}^{p_{j+1}} \mathcal{B}_z \right).$$

Assume that $|\mathcal{I}(\{s-q+1\}, s)| = y$. Then, since $\mathcal{I}(\{i\}, s) \supset \mathcal{I}(\{i+1\}, s)$ for any $i \in [s-1]$, we have

$$\sum_{i=s-q+1}^{s} |\mathcal{I}(\{i\}, s)| \leq qy.$$  \hspace{1cm} (19)

Applying Observation 6 with $S = [s - q + 1, s]$ and $\mathcal{B} = \bigcap_{j=0}^{q-1} \left( \bigcup_{z=p_j+1}^{p_{j+1}} \mathcal{B}_z \right)$, we get

$$\sum_{i=s-q+1}^{s} |\mathcal{G}(\{i\}, s)| \leq q \binom{n-s}{k-1} - q \left| \bigcap_{j=0}^{q-1} \left( \bigcup_{z=p_j+1}^{p_{j+1}} \mathcal{B}_z \right) \right| + qy.$$  \hspace{1cm} (20)

Note that

$$\bigcap_{j=0}^{q-1} \left( \bigcup_{z=p_j+1}^{p_{j+1}} \mathcal{G}(\{s-q+1\}, s) \cap \mathcal{B}_z \right) = \bigcup_{x_0=p_0+1}^{p_1} \bigcup_{z_1=p_1+1}^{p_2} \ldots \bigcup_{z_{q-1}=p_{q-1}+1}^{p_q} \left( \bigcap_{j=0}^{q-1} \mathcal{G}(\{s-q+1\}, s) \cap \mathcal{B}_{z_j} \right).$$

Since $|\mathcal{I}(\{s-q+1\}, s)| = y$, by the pigeon-hole principle from the equality above we infer that there exist a choice of $z'_0 \in [p_0 + 1, p_1], \ldots, z'_{q-1} \in [p_{q-1} + 1, p_q]$, such that

$$\left| \mathcal{G}(\{s-q+1\}, s) \cap \bigcap_{j=0}^{q-1} \mathcal{B}_{z'_j} \right| \geq \frac{y}{f}.$$  \hspace{1cm} (21)

Next, the families $\mathcal{G}_z(\{1\}, s), \ldots, \mathcal{G}_z(\{s-q+1\}, s)$, where $\mathcal{G}_z(\{i\}, s) := \mathcal{G}(\{i\}, s) \cap \bigcap_{j=0}^{q-1} \mathcal{B}_{z'_j}$, are cross-dependent and nested. Indeed, if the families are not cross-dependent, and there exist $G_1 \in \mathcal{G}_z(\{1\}, s), \ldots, G_{s-q+1} \in \mathcal{G}_z(\{s-q+1\}, s)$ that are pairwise disjoint, then $G_1, \ldots, G_{s-q+1}, F_{z'_0}, \ldots, F_{z'_{q-1}}$ form an $(s+1)$-matching in $\mathcal{G}$. Note that $\bigcap_{j=0}^{q-1} \mathcal{B}_{z_j} = (S')$, where $S' := [s+1, n] \setminus (F_{z'_0} \cup \ldots \cup F_{z'_{q-1}})$, and $|S'| = (u+s)(k-1) + 1 - qk$. We have $u' := u - \frac{u-1}{k-1} \geq s-q+1$ since $u \geq s+1$. Moreover, $|S'| = (u' + s - q)(k-1)$. Thus, we may apply (12) with $[u'], [S'], s-q$ playing the roles of $u, N, s$, respectively. From (12) and the inequality $u' = u - \frac{u-1}{k-1} \geq fq$ we get

$$|\mathcal{G}_z(\{1\}, s)| + \ldots + |\mathcal{G}_z(\{s-q\}, s)| + fq|\mathcal{G}_z(\{s-q+1\}, s)| \leq (s-q) \left| \bigcap_{j=0}^{q-1} \mathcal{B}_{z_j} \right|,$$  \hspace{1cm} (22)

which, in view of (21), gives us

$$\sum_{i=1}^{s-q} |\mathcal{G}_z(\{i\}, s)| \leq (s-q) \left| \bigcap_{j=0}^{q-1} \mathcal{B}_{z_j} \right| - qy.$$  \hspace{1cm} (23)
Applying Observation 6 with \( S = [1, s - q] \) and \( B = \bigcap_{j=0}^{q-1} B_{z_j} \), we get
\[
\sum_{i=1}^{s-q} |G(\{i\}, s)| \leq (s - q) \binom{n - s}{k - 1} - qy. \tag{24}
\]
We get the statement of the proposition by summing up (20) and (24). \( \square \)

The main difficulty in using Proposition 7 directly is that it is very difficult to deal with sums/products of binomial coefficients that arise when writing down the subtrahend in (18) explicitly. The proof of the following lemma is a way to work around it. The lemma itself is an important technical ingredient in establishing bounds on \( u \) in Theorem 3.

**Lemma 8.** Assume that \( \nu(G(\emptyset, s)) \geq x \) for \( x \in [s] \). Then we have
\[
\sum_{i=1}^{s} |G(\{i\}, s)| \leq s \binom{n - s}{k - 1} - \gamma \binom{n - k - s}{k - 1}, \tag{25}
\]
where

(i) \( \gamma = \frac{4}{3} \) for \( x = 2 \) and \( u \geq 3 \),

(ii) \( \gamma = \frac{3}{2} \) for \( x = 3 \) and \( u \geq 5 \),

(iii) \( \gamma = \frac{16}{9} \) for \( x = 4, 5 \) and \( u \geq 9 \),

(iv) \( \gamma = 2 \) for \( x \geq 6 \) and \( u \geq 25 \),

(v) \( \gamma = \Omega(x / \log^2 x) \) for \( u \geq 2^x \).

**Proof.** The logic of the proofs of all five statements is similar. We combine the bounds from Proposition 7 with different parameters to get the bound of the form \( \beta \sum_{i=1}^{s} |G(\{i\}, s)| \leq \beta s \binom{n-s}{k-1} - \sum_{z=1}^{x} |B_z| \) for the smallest possible \( \beta \). Then the constant \( \gamma \) from the statement of Lemma 8 is defined as \( \gamma := x / \beta \). Since \( |B_z| = \binom{n-k-s}{k-1} \) for any \( z \), we get the statement, as long as we can guarantee the bounds on \( \beta \) we claim. Therefore, we aim to find a linear combination of equations (18) with coefficients \( \beta_j \), which satisfies the following two conditions:

a) the sum of the subtrahends in the RHS is at least \( \sum_{z=1}^{x} |B_z| \), \( \tag{26} \)
b) \( \beta := \sum \beta_j \) is as small as possible. \( \tag{27} \)

For any \( S \subseteq [x] \), we introduce the following notation:
\[
\bigcap_{j \in S} B_j := \left( \bigcap_{j \in S} B_j \right) \setminus \left( \bigcup_{j \in [x] \setminus S} B_j \right).
\]
(Note that this definition depends on \( x \), but the value of \( x \) will be clear from the context.) Consider the following inclusion-exclusion-type decomposition:
\[
\sum_{z=1}^{x} |B_z| = \left| \bigcup_{z=1}^{x} B_z \right| + \sum_{1 \leq z_1 < z_2 \leq x} |B_{z_1} \cap B_{z_2}| + 2 \sum_{1 \leq z_1 < z_2 < z_3 \leq x} |B_{z_1} \cap B_{z_2} \cap B_{z_3}| + \ldots + (x-1) \left| \bigcap_{j=1}^{x} B_{z_j} \right|. \tag{28}
\]

The cardinalities of intersections in (28) are determined by the number of intersecting families, but do not depend on the actual families that are intersecting. The number of summands of the form \( \left| \bigcap_{j=1}^{q} B_{z_j} \right| \), multiplied by the coefficient \( (\ell - 1) \), is \( (\ell - 1) \binom{\ell}{3} \) for any
2 ≤ ℓ ≤ x. We call each of the families of the form \( \cap_{j=1}^{\ell} B_{z_j} \) an \( \ell \)-intersection. We denote its size by \( \alpha_\ell \). Putting \( \eta_\ell := (\ell - 1)\binom{x}{\ell} \) for each \( 2 ≤ \ell ≤ x \), we can rewrite (28) as

\[
\sum_{z=1}^{x} |B_z| = \left| \bigcup_{z=1}^{x} B_z \right| + \sum_{\ell=2}^{x} \eta_\ell \alpha_\ell.
\]  

(29)

Each subtrahend in (18) also admits a unique decomposition into \( l \)-intersections, analogous to (28) (we will see some examples below). In the proof of each part of Lemma 8 we guarantee (20) by finding the linear combination of (18) with different parameters, such that the resulting subtrahend has the form

\[
\left| \bigcup_{z=1}^{x} B_z \right| + \sum_{\ell=2}^{x} \eta'_\ell \alpha_\ell,
\]

where \( \eta'_\ell ≥ \eta_\ell \) for each \( 2 ≤ \ell ≤ x \). In particular, the term \( \left| \bigcup_{z=1}^{x} B_z \right| \) is always “covered” by the subtrahend in (18) with \( q = 1 \), taken with coefficient 1. For each member of the linear combination we use the following notation: \([\text{parameters substituted in (18); the coefficient}]\).

Since the proofs of (i)–(iv) are almost identical, we present the proofs of (i) and (iv) only. We start with (i). We sum up \([q = 1; \text{coefficient } 1]\) with \([q = 2, p_1 = 1; \text{coefficient } \frac{1}{2}]\). We note that for the latter member of the linear combination (18) gives

\[
\sum_{i=1}^{s} G(\{i\}, s) ≤ s \binom{n-s}{k-1} - 2|B_1 \cap B_2|.
\]

Returning to the linear combination, we get the inequality

\[
\frac{3}{2} \bigcup_{i=1}^{s} G(\{i\}, s) ≤ \frac{3}{2} s \binom{n-s}{k-1} - |B_1 \cup B_2| - |B_1 \cap B_2| = \frac{3}{2} s \binom{n-s}{k-1} - |B_1| - |B_2|.
\]

The condition on \( u \), imposed by the application of (18), is satisfied for \( u ≥ 3 \). It is clear that \( \gamma = \frac{2}{\frac{3}{2}} = \frac{4}{3} \) for this linear combination. This concludes the proof of (i).

The proof of (iv) is more cumbersome. It is sufficient to verify (iv) for \( x = 6 \). Take the following linear combination: \([q = 1; \text{coefficient } 1]\), \([q = 2, p_1 = 3; \text{coefficient } \frac{3}{2}]\), \([q = 3, p_1 = 2, p_2 = 4; \text{coefficient } \frac{1}{3}]\), and \([q = 6, p_i = i \text{ for } i = 0, \ldots, 6; \text{coefficient } \frac{1}{6}]\). First, it is clear that for this combination we have \( \beta = 1 + \frac{3}{2} + \frac{1}{3} + \frac{1}{6} = 3 \), and thus \( \gamma = x/\beta = 2 \). Moreover, it is easy to see that the condition on \( u \), imposed by the applications of (18), is \( u ≥ 25 \) and comes from the third summand. Therefore, we are left to verify that the corresponding \( \eta'_\ell \) satisfy \( \eta'_\ell ≥ \eta_\ell \) for each \( 2 ≤ \ell ≤ 6 \).

For \( x = 6 \), the value of \( \eta_\ell \) is 15, 40, 45, 24, 5 for \( \ell = 2, \ldots, 6 \), respectively. Next, we find the values of \( \eta'_\ell \). As it is not difficult to check, the subtrahends in (18) with \( q = 2, p_1 = 3, q = 3, p_1 = 2, p_2 = 4 \) and \( q = 6, p_i = i \) for \( i = 0, \ldots, 6 \), respectively, have the form

\begin{align*}
(q = 2) & \quad 18\alpha_2 + 36\alpha_3 + 30\alpha_4 + 12\alpha_5 + 2\alpha_6; \\
(q = 3) & \quad 24\alpha_3 + 36\alpha_4 + 18\alpha_5 + 3\alpha_6; \\
(q = 6) & \quad 6\alpha_6.
\end{align*}

(31) | (32) | (33)
Let us elaborate on how we obtain \((31)\). For \(2 \leq \ell \leq 6\), put \(C_{\ell} := \{C \in \binom{[\ell]}{6} : |C \cap [3]| \geq 1, |C \cap [4,6]| \geq 1\}\) and \(C := \bigcup_{\ell=2}^{6} C_{\ell}\). One can easily check that \(|C_{\ell}|\) equals \(9, 18, 15, 6, 1\) for \(\ell = 2, \ldots, 6\), respectively. The subtrahend in \((13)\) has the form
\[
2|B_{1} \cup B_{2} \cup B_{3} \cap (B_{4} \cup B_{5} \cup B_{6})| = 2 \sum_{C \in C} \left| \bigcap_{i \in C} B_{i} \right| = 18\alpha_{2} + 36\alpha_{3} + 30\alpha_{4} + 12\alpha_{5} + 2\alpha_{6}.
\]
Summing up \((31)\), \((32)\), \((33)\) with coefficients \(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\), respectively, we get the expression
\[
27\alpha_{2} + 62\alpha_{3} + 57\alpha_{4} + 24\alpha_{5} + 5\alpha_{6},
\]
which is bigger than \(\sum_{\ell=2}^{6} \eta_{\ell}\alpha_{\ell}\), as claimed. The proof of (iv) is complete. (We note that the choice of the coefficients, and thus the bound on \(\gamma\), is clearly not optimal. It is however sufficient for our purposes.)

The proof of (v) is the most technical. Since we can always replace \(x\) by any smaller positive integer, we assume for simplicity that \(x = 2^{r}\) for some positive integer \(r\), at the expense of factor 2 that goes into the \(\Omega\)-notation. Consider the following linear combination:
\[
M_{j} := [q(j) = 2^{j}, \ p_{i}(j) = ix/q(j) \text{ for } i = 0, \ldots, q(j); \text{ coefficient } 4(j+2)], \text{ where } j = 0, \ldots, r.
\]
First we verify that the linear combination above has enough \(\ell\)-intersections for each \(\ell\). The union of all \(B_{z}\) is given by \(M_{0}\). For larger \(\ell\) we need to do an auxiliary calculation.

Let us determine, how many different \(\ell\)-intersections are contained in the family
\[
\bigcap_{i=0}^{q(j)-1} \left( \bigcup_{z=p_{i}(j)+1}^{p_{i+1}(j)} B_{z} \right). \tag{34}
\]
For \(\ell \geq q(j)\), the \(\ell\)-subsets of \([x]\) corresponding to the \(\ell\)-intersections contained in the family above form the family \(C_{\ell}(j) := \{C \subset \binom{[x]}{\ell} : C \cap [p_{i}(j)+1, p_{i+1}(j)] \neq \emptyset \text{ for all } i = 0, \ldots, q(j)-1\}\) (cf. the considerations after \((33)\)). We can bound the number of such \(\ell\)-intersections from below by \(\binom{\ell}{r}(1-q(j)(\frac{2^{q(j)-1}}{q(j)})^{\ell}) \geq \binom{\ell}{r}(1-q(j)e^{-\ell/q(j)})\). For \(\ell \geq q(j)\log_{2}(2q(j))\) this is at more than \(\frac{1}{2}\binom{n}{\ell}\). We conclude that the family \((34)\) contains more than \(\frac{1}{2}\binom{n}{\ell}\) different \(\ell\)-intersections for \(\ell \geq q(j)\log_{2}(2q(j)) = 2^{j+1}\).

Since the subtrahend in \(M_{j}\) is the size of the family \((34)\) multiplied by the factor \(2^{j+2}(j+2)\), we conclude that \(M_{j}\) contributes more than \(2^{j+1}(j+2)^{\binom{n}{\ell}}\) to \(\eta'_{\ell}\) for \(\ell \geq 2^{j}(j+1)\) (see \((30)\) for the definition of \(\eta'_{\ell}\)). Next, for each \(\ell \geq 2\), find the largest integer \(j\) such that \(2^{j}(j+1) \leq \ell\). It is clear that \(2^{j+1}(j+2) > \ell\). As we have shown above, \(M_{j}\) contributes at least \(2^{j+1}(j+2)^{\binom{n}{\ell}} > \ell^{\binom{n}{\ell}} > \eta_{\ell}\) to the coefficient in front of \(\alpha_{\ell}\). Thus, \(\gamma'_{\ell} \geq \gamma_{\ell}\). One may also note that for \(\ell = 2, 3\) the largest \(j\) as defined above is 0, and then the situation is slightly different since we have used the term \(M_{0}\) to “cover” the union in \((30)\). However, the “unused” coefficient for \(M_{0}\) is 7, and it is straightforward to see that \(M_{0}\) contributes \(7^{\binom{n}{\ell}}\) to \(\eta'_{\ell}\), which is clearly sufficient for \(\ell = 2, 3\).

Second, we calculate the sum \(\beta\) of the coefficients of \(M_{j}\). We have
\[
\beta = \sum_{i=0}^{r} 4(i+2) \leq 4 \binom{r+3}{2} = O(\log^{2}x).
\]
Thus, \(\gamma = x/\beta = \Omega(x / \log^{2}x)\).
Finally, we verify that the condition imposed on $u$ is sufficient for the applications of (18) we used. For $M_j$ the restriction is satisfied for $u \geq 2^j (2^{r-j})^{2^j} + 2^j = 2^{j+(r-j)2^j} + 2^j$. This expression is clearly maximized when $j = r - 1$, and in that case the inequality is $u \geq 2^{r-1} + 2^{r-1} + 2^{r-1}$. The latter expression is smaller than $2^{2^r}$ for any $r \geq 1$. Thus, the condition $u \geq 2^x \geq 2^{2^r}$ is sufficient.

In the case $\nu(\mathcal{G}(\emptyset, s)) = 1$ we need a proposition which is more fine-grained than Proposition 7. For each $j \in [k+1]$ put $D_j := [s+1, s+k+1] \setminus \{s+j\}$ and define the families $C_j := (E_j)$, where $E_j := [s+1, \ldots, n] \setminus D_j$ and $|E_j| \geq (u+s)(k-1)+1-k = (u+s-1)(k-1)$.

**Proposition 9.** Assume that $\nu(\mathcal{G}(\emptyset, s)) = 1$ and put $v := \max\{1, k+2-u\}$.

(i) If $\tau(\mathcal{G}(\emptyset, s)) > 1$, then

$$\sum_{i=1}^{s} |\mathcal{G}(\{i\}, s)| \leq s \left( \frac{n-s}{k-1} \right) - \left| \bigcup_{j=v}^{k+1} C_j \right|. \tag{35}$$

(ii) If $\tau(\mathcal{G}(\emptyset, s)) = 1$ and for some integer $t \in [v, k]$ we have $|\mathcal{G}(\emptyset, s)| > \left( \frac{n-s-t}{k-t} \right)$, then

$$\sum_{i=1}^{s} |\mathcal{G}(\{i\}, s)| \leq s \left( \frac{n-s}{k-1} \right) - \left| \bigcup_{j=t}^{k+1} C_j \right|. \tag{36}$$

**Proof.** (i) Since $\mathcal{G}(\emptyset, s)$ is shifted and $\tau(\mathcal{G}(\emptyset, s)) > 1$, the set $D_1$ is contained in $\mathcal{G}(\emptyset, s)$. Then, by shiftedness, for each $j \in [k+1]$ $D_j$ is contained in $\mathcal{G}(\emptyset, s)$. Arguing as in the proof of Proposition 7 let $\left| \left( \bigcup_{j=v}^{k+1} C_j \right) \cap \mathcal{G}(\{s\}) \right| = y$. Then there is an index $j \in [v, k+1]$ such that $\left| C_j \cap \mathcal{G}(\{s\}) \right| \geq \left( \frac{y}{k+2-v} \right) \geq \frac{u}{u}$. The rest of the argument is very similar to the argument in Proposition 7 which we reproduce for completeness.

Put $\mathcal{G}_j(\{i\}, s) := \mathcal{G}(\{i\}, s) \cap C_j$. The families $\mathcal{G}_j(\{1\}, s), \ldots, \mathcal{G}_j(\{s\}, s) \subset (E_j)$ are cross-dependent and nested. Recall that $|E_j| \geq (u+s-1)(k-1)$. From (12) we get

$$|\mathcal{G}_j(\{1\}, s)| + \ldots + |\mathcal{G}_j(\{s-1\}, s)| + u|\mathcal{G}_j(\{s\}, s)| \leq (s-1) \left( \frac{|E_j|}{k-1} \right),$$

which, in view of $|\mathcal{G}_j(\{s\}, s)| \geq \frac{u}{u}$, gives us

$$\left| \left( \bigcup_{j=v}^{k+1} C_j \right) \cap \mathcal{G}(\{s\}) \right| + \sum_{i=1}^{s-1} |\mathcal{G}_j(\{i\}, s)| \leq (s-1) \left( \frac{|E_j|}{k-1} \right).$$

Applying Observation 5 with $S = [1, s-1]$ and $B = \bigcup_{j=v}^{k+1} (E_j)$, we get (i).

(ii) Similarly, since $\mathcal{G}(\emptyset, s)$ is shifted and $|\mathcal{G}(\emptyset, s)| > \left( \frac{n-s-t}{k-t} \right)$, the set $D_t$ must be contained in $\mathcal{G}(\emptyset, s)$. Therefore, each $D_j$ for $j \in [t, s+1]$ is contained in $\mathcal{G}(\emptyset, s)$, and we conclude as before.

We go on to the proof of Theorem 3. In the case $|\mathcal{G}(\emptyset, s)| = 1$ we get exactly the bound stated in the theorem since $|\mathcal{G}| = |\mathcal{G}| - |\mathcal{G}(\emptyset, s)| + 1 \leq \left( \binom{n}{k} \right) - \left( \binom{n-s}{k} \right) - |\mathcal{B}_1| + 1$. Thus, for the rest of the proof we assume that $|\mathcal{G}(\emptyset, s)| > 1$. 

Case 1. \( \nu(\mathcal{G}(\emptyset, s)) = \tau(\mathcal{G}(\emptyset, s)) = 1 \). If \( 1 < |\mathcal{G}(\emptyset, s)| \leq \binom{n-s-k+1}{1} = n-s-k+1 \) then by Proposition \( \mathbb{P} \) (ii) (with \( t = k \)) we have \( C \geq |\mathcal{C}_k \cup \mathcal{C}_{k+1}| = \binom{n-k-s}{k-1} + \binom{n-k-s-1}{k-2} \) for \( C \) from Lemma \( \mathbb{S} \). For \( k \geq 4 \) and \( s \geq 2 \) we have
\[
\binom{n-s-k-1}{k-2} \geq n-s-k \geq |\mathcal{G}(\emptyset, s)| - 1,
\]
thus the theorem holds in this case. For \( k = 3 \) one can see that \( C \geq \binom{n-s-2}{2} \) if \( |\mathcal{G}(\emptyset, s)| \geq 4 \), and thus we have \( C - \binom{n-s-3}{2} = n-s-3 \geq |\mathcal{G}(\emptyset, s)| - 1 \).

If \( |\mathcal{G}(\emptyset, s)| > n-s-k+1 \) then we use the following bound:
\[
C \geq \left| \bigcup_{j=k-1}^{k+1} \mathcal{C}_j \right| = \binom{n-k-s}{k-1} + 2 \binom{n-k-s-1}{k-2} =
\]
\[
= \left( 1 + \frac{2(k-1)}{n-k-s} \right) \binom{n-k-s}{k-1} \geq \left( 1 + \frac{2}{u+s} \right) \binom{n-k-s}{k-1} \text{(37)}
\]
The last expression is at least \( \frac{n}{u} \binom{n-k-s}{k-1} \) if \( \frac{u+s+2}{u-1} \geq \frac{n}{u} \), which holds for \( u \geq s+2 \). Since \( u \geq s+2 \), we can apply (14) and conclude that Theorem 3 holds in this case.

Case 2. \( \nu(\mathcal{G}(\emptyset, s)) = 1 < \tau(\mathcal{G}(\emptyset, s)) \). Analogously to (37), Proposition \( \mathbb{P} \) implies that
\[
C \geq \left| \bigcup_{j=z}^{k+1} \mathcal{C}_j \right| = \left( 1 + \frac{\min\{k+1,u\}}{u+s-1} \right) \binom{n-k-s}{k-1}.
\]
The inequality \( \frac{u+s+\min\{k+1,u\}}{u+s} \geq \frac{u}{u-2} \) holds for \( u \geq s+4 \) and \( k \geq 3 \), and we can apply (13).

Case 3. \( \nu(\mathcal{G}(\emptyset, s)) = x \geq 2 \). We make use of Lemma \( \mathbb{S} \). We are done in this case as long as, in terms of Lemma \( \mathbb{S} \)
\[
\gamma \cdot \frac{u-x-1}{u} \geq 1 \text{ (38)}
\]
Using Lemma \( \mathbb{S} \) (i)–(iv), one can see that (38) holds provided \( u \geq \max\{25,2s+2\} \). Indeed, let us verify this technical claim. It is clearly sufficient to verify it for \( u = \max\{25,2s+2\} \).
\begin{itemize}
  \item If \( x = 2 \), then \( \gamma = \frac{4}{3} \) and the left hand side of (38) is at least \( \frac{4}{3} \cdot \frac{22}{25} > 1 \).
  \item If \( x = 3 \), then the LHS is at least \( \frac{3}{2} \cdot \frac{24}{25} > 1 \).
  \item If \( x = 4 \), then the LHS is at least \( \frac{16}{9} \cdot \frac{20}{25} > 1 \).
  \item If \( x = 5 \), then the LHS is at least \( \frac{16}{9} \cdot \frac{19}{25} > 1 \).
  \item If \( x \geq 6 \), then, using \( s \geq x \), the LHS is at least \( 2 \cdot \frac{x+1}{2x+2} = 1 \).
\end{itemize}
To conclude this case, we remark that the inequalities \( u \geq \max\{25,2s+2\} \), \( n \geq (s+u)(k-1) + s+1 \), and \( k \geq 3 \) are sufficient for all the considerations above to work.

Next, using the fifth statement from Lemma \( \mathbb{S} \) we obtain that (38) is satisfied for \( u = s + o(s) \). Indeed, take sufficiently large \( s \) and put \( u = s + \delta \frac{s(\log \log s)^2}{\log s} \) with some \( \delta > 0 \) that will be determined later. If \( 2x+25 \leq s \), then the argument given in the previous paragraph shows that the condition \( u \geq s \) is sufficient. Thus, we may assume that \( x \geq (s-25)/2 \geq \log_2 s \),
where the second inequality holds for all sufficiently large \( s \). Then, applying the fifth point of Lemma 8 with \( x = \log_2 s \) (note that \( u > s = 2^x \), and thus the condition of Lemma 8 are satisfied), we get

\[
\gamma \cdot \frac{u - x - 1}{u} \geq \Omega \left( \frac{\log s}{(\log \log s)^2} \right) \cdot \frac{u - s - 1}{u} = \Omega \left( \frac{\log s}{(\log \log s)^2} \right) \cdot \frac{\delta s (\log \log s)^2}{\log s} > 1,
\]

if \( \delta \) is sufficiently large. The proof of Theorem 3 for shifted families is complete.

2.2. The case of not shifted \( \mathcal{G} \). Consider a family \( \mathcal{G} \) satisfying the requirements of the theorem. Since the property \( \tau(\mathcal{G}) > s \) is not necessarily maintained by \((i, j)\)-shifts, we cannot assume that the family \( \mathcal{G} \) is shifted right away. However, each \((i, j)\)-shift for \( 1 \leq i < j \leq n \), decreases \( \tau(\mathcal{G}) \) by at most 1. Perform the \((i, j)\)-shifts \((1 \leq i < j \leq n)\) one by one until either \( \mathcal{G} \) becomes shifted or \( \tau(\mathcal{G}) = s + 1 \). In the former case we fall into the situation of the previous subsection.

Now suppose that \( \tau(\mathcal{G}) = s + 1 \) and that each set from \( \mathcal{G} \) intersects \([s + 1]\). Then each family \( \mathcal{G}(\{i\}, s + 1) \) for \( i \in [s + 1] \) is nonempty. Make the family \( \mathcal{G} \) shifted in coordinates \( s + 2, \ldots, n \) by performing all possible \((i, j)\)-shifts with \( s + 2 \leq i < j \leq n \). Denote the new family by \( \mathcal{G} \) again. Since shifts do not increase the matching number, we have \( \nu(\mathcal{G}) \leq s \) and \( \tau(\mathcal{G}) \leq s + 1 \). Each family \( \mathcal{G}(\{i\}, s + 1) \) contains the set \([s + 2, s + k]\).

Next, perform all possible shifts on coordinates \( 1, \ldots, s + 1 \) and denote the resulting family by \( \mathcal{G}' \). We have \( |\mathcal{G}'| = |\mathcal{G}|, \nu(\mathcal{G}') \leq s \), and, most importantly,

\[
\mathcal{G}'(\{1\}, s + 1), \ldots, \mathcal{G}'(\{s + 1\}, s + 1) \text{ are nested and non-empty.} \quad (39)
\]

These families are non-empty due to the fact that each of them contained the set \([s + 2, s + k]\) before the shifts on \([s + 1]\).

We can actually apply the proof from the previous subsection to \( \mathcal{G}' \). Indeed, the main consequence of the shiftedness we were using is (39). The other consequence of the shiftedness was the bound (16), which automatically holds in this case since every set from \( \mathcal{G}' \) intersects \([s + 1]\) and thus \( \mathcal{G}'(\emptyset, s + 1) \) is empty. The proof of Theorem 3 is complete.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Take \( s \) cross-dependent families \( \mathcal{F}_1, \ldots, \mathcal{F}_s \). For \( s = 2 \) the bound (1) states that \( |\mathcal{F}_1| + |\mathcal{F}_2| \leq 2^{[n]} \), which follows from the following trivial observation: if \( A \in \mathcal{F}_1 \) then \([n] \setminus A \notin \mathcal{F}_2 \). Thus, we may assume that \( s \geq 3 \). Also, the case of \( m = 0 \) is very easy to verify for any \( s \), and so we assume that \( m \geq 1 \).

Put \( n := s(m + 1) - \ell \) for the rest of this section. Recall that \( \mathcal{F} \) is called an up-set if for any \( F \in \mathcal{F} \) all sets that contain \( F \) are also in \( \mathcal{F} \). When dealing with cross-dependent and \( q \)-dependent families, we may restrict our attention to the families that are up-sets and shifted (e.g. see [11, Claim 17]), which we assume for the rest of the section.

Let us first treat the case \( \ell = 1 \). This is the easiest case, and it will provide the reader with a good overview of the technique we use. Take \( s \) pairwise disjoint sets \( H_1, \ldots, H_s \) of size \( m \) at random. To simplify notation, assume that the \( s - 1 \) elements of \([n] \setminus \bigcup_{i=1}^{s} H_i \) form the set \([s - 1]\).
For each \( i \in [s] \) let \( \emptyset =: H_i^{(0)} \subset \ldots \subset H_i^{(m)} := H_i \) be a randomly chosen full chain in \( H_i \). For all \( i \in [s] \), \( 0 \leq j \leq m \) and \( S \subset [s-1] \) define the random variables \( \beta^{(j)}_i \) and \( \beta(S) \):

\[
\beta^{(j)}_i = \begin{cases} 1 & \text{if } H_i^{(j)} \in \mathcal{F}_i, \\ 0 & \text{if } H_i^{(j)} \notin \mathcal{F}_i; \end{cases} \quad \beta(S) = \begin{cases} 1 & \text{if } H_i \cup S \in \mathcal{F}_i, \\ 0 & \text{if } H_i \cup S \notin \mathcal{F}_i. \end{cases} \tag{40}
\]

Note that \( S \) may be the empty set and that \( \beta^{(m)}_i = \beta(\emptyset) \). The cross-dependence of \( \mathcal{F}_1, \ldots, \mathcal{F}_s \) implies

\[
\beta(S_1) \beta(S_2) \cdots \beta(S_s) = 0 \quad \text{whenever } S_1, \ldots, S_s \subset [s-1] \text{ are pairwise disjoint}. \tag{41}
\]

For all \( 0 \leq j \leq m \) and \( S \subset [s-1] \) the expectations \( E[\beta^{(j)}_i] \) and \( E[\beta(S)] \) satisfy

\[
E[\beta^{(j)}_i] = \left| \frac{|\mathcal{F}_i \cap \binom{[n]}{j}|}{\binom{n}{j}} \right|, \quad E[\beta(S)] = \left| \frac{|\mathcal{F}_i \cap \binom{[m+|S|]}{m+|S|}|}{\binom{n}{m+|S|}} \right|. \tag{42}
\]

Our aim is to prove the following statement.

**Lemma 10.** For every choice of \( H_1, \ldots, H_s \) and the full chains one has

\[
\sum_{i=1}^{s} \sum_{j=0}^{m} \binom{n}{j} \beta^{(j)}_i + \sum_{S \in ([s-1])} \frac{\binom{n}{m+1}}{\binom{n}{s-1}} \beta_i(S) \leq s \left( \frac{n}{m+1} + \frac{n}{m+2} \right), \tag{43}
\]

where \( \beta^{(j)}_i \) and \( \beta(S) \) are as defined in (40).

Replacing the left hand side of (43) with its expected value, it is straightforward to see that (42) implies

\[
\sum_{i=1}^{s} \sum_{j=0}^{m+2} |\mathcal{F}_i \cap \binom{[n]}{j}| \leq s \left( \frac{n}{m+1} + \frac{n}{m+2} \right),
\]

from which (41) follows.

**Proof of Lemma 10.** If for all \( i \in [s] \) we have \( \beta_i^{(m)} = 0 \) then we are done. Assume that \( \sum_{i=1}^{s} \beta_i^{(m)} = p \) for some \( p \geq 1 \). Clearly, \( p \leq s-1 \) since \( \mathcal{F}_1, \ldots, \mathcal{F}_s \) are cross-dependent. W.l.o.g., we assume that \( \beta_i^{(m)} = 1 \) if and only if \( i \in [p] \).

If \( p \leq s-2 \), then we proceed as follows. We have \( \sum_{i=p+1}^{s} \beta_i(\{k_i\}) \leq s-p-1 \) for any distinct \( k_{p+1}, \ldots, k_s \in [s-1] \). Averaging over the choice of \( k_{p+1}, \ldots, k_s \), it is easy to see that at least a \( 1/(s-p) \)-fraction of the pairs \((i, k)\), where \( p+1 \leq i \leq s \) and \( k \in [s-1] \), satisfies \( \beta_i(\{k\}) = 0 \). In other words, there are at least \( s-1 \) such pairs. Consequently, we get that the sum on the right hand side of (43) is at most \( p \sum_{j=1}^{m} \binom{n}{j} + (s-1) \binom{n}{m+1} + s \binom{n}{m+2} \) and the difference between the RHS and the LHS of (43) is at least \( D := \binom{n}{m+1} - p \sum_{j=1}^{m} \binom{n}{j} \). It is easy to check that for \( n = sm + s - 1 \) we have \( \binom{n}{m+1} = (s-1) \binom{n}{m} \), and thus \( D \geq \binom{n}{m} - p \sum_{j=1}^{m-1} \binom{n}{j} \).

For \( n = sm + s - \ell \) with \( \ell \in [s] \) and \( m \geq 1 \) the following holds.

\[
\frac{\binom{m-j-1}{n-j}}{\binom{m-j}{n-j}} = \frac{m-j}{n-m+j+1} \leq \frac{m-1}{(s-1)m} \quad \text{for } j \geq 1. \tag{44}
\]
Using (44) we may obtain that
\[ \sum_{j=1}^{m} (s-2) \left( \begin{array}{c} n \\ m-j \end{array} \right) \leq (s-2) \left( 1 + \sum_{i=1}^{\infty} \frac{m-1}{m(s-1)^i} \right) \left( \begin{array}{c} n \\ m-1 \end{array} \right) = \]
\[ (s-2) \left( 1 + \frac{m-1}{m(s-2)} \right) \left( \begin{array}{c} n \\ m-1 \end{array} \right) = \frac{(s-1-\frac{1}{m})m}{(s-1)m+s-\ell+1} \left( \begin{array}{c} n \\ m \end{array} \right). \] (45)

Using (45), it is easy to conclude that \( D > 0 \) in this case.

Assume now that \( p = s-1 \). Then \( \beta_s(\{k\}) = 0 \) for any \( k \in [s-1] \). Then the LHS of (43) is at most \((s-1) \sum_{j=1}^{m+2} \binom{n}{j}\), and the difference \( D \) between the RHS and the LHS of (43) satisfies \( D \geq \binom{n}{m+2} + \binom{n}{m+1} - (s-1) \sum_{j=0}^{m} \binom{n}{j} \). Using the calculations from the previous case, we get \( D \geq \binom{n}{m+2} - \sum_{j=0}^{m} \binom{n}{j} \). On the other hand, we have \( \binom{n}{m+2} \geq \binom{n}{m+1} \) since \( n = s(m+1) - 1 \geq 2m + 3 \) for \( m \geq 1 \) and \( s \geq 3 \). Thus, \( D \geq \binom{n}{m+1} - \sum_{j=0}^{m} \binom{n}{j} > 0 \). □

From now on, we assume that \( \ell \geq 2 \). We use the same idea, however, we will average over a slightly different collection of sets. Take \( s \) pairwise disjoint sets \( H_1, \ldots, H_s \) of size \( m-1 \) at random. W.l.o.g. assume that the \( 2s-\ell \) elements of \([n] \setminus \bigcup_{i=1}^{s} H_i\) form the set \([2s-\ell]\).

For all \( i \in [s], 0 \leq j \leq m-1 \) and \( S \subseteq [2s-\ell] \), define the random variables \( \beta_{ij}^{(s)} \) and \( \beta_{i}(S) \) analogously to how it is done in (40). Note that the analogues of (41) and (42) hold in this case as well.

The analogue of Lemma 10 in this case is the following statement. Once verified, the rest of the argument is the same.

**Lemma 11.** For every choice of \( H_1, \ldots, H_s \) and the full chains one has
\[ \sum_{i=1}^{s} \left[ \sum_{j=1}^{m-2} \binom{n}{j} \beta_{ij}^{(s)} + \sum_{S \subseteq [2s-\ell], |S| \leq 3} \frac{\binom{n}{m-1+|S|}}{\binom{2s-\ell}{|S|}} \beta_{i}(S) \right] \leq s \sum_{j=m+1}^{m+2} \binom{n}{j} + (\ell-1) \binom{n}{m}. \] (46)

The proof of Lemma 11 uses the following proposition.

**Proposition 12.** For \( n' := 2s' - \ell' \) with \( \ell' \in [s'] \) and for \( s' \) cross-dependent families \( F_1', \ldots, F_{s'}' \subseteq \binom{[n']}{1} \cup \binom{[n']}{2} \) we have
\[ \sum_{i=1}^{s'} |F_i'| \leq (\ell'-1)n' + s' \binom{n'}{2}. \] (47)

**Proof of Proposition 12.** Fix a random ordering on \([n']\). For \( S \subseteq [n'] \), put \( \vartheta_i(S) = 1 \) if \( S \in F_i' \) and \( \vartheta_i(S) = 0 \) otherwise. Similarly to (43), (46), it is sufficient to prove
\[ \sum_{i=1}^{s'} \left[ \binom{n'}{1} \vartheta_i(\{i\}) + \sum_{x=s'+1}^{n'} \frac{\binom{n'}{2}}{n'-s'} \vartheta_i(\{i,x\}) \right] \leq (\ell'-1)n' + s' \binom{n'}{2}. \] (48)

W.l.o.g., suppose that \( \vartheta_i(\{i\}) = 1 \) if and only if \( i \in [p] \). If \( p \leq \ell'-1 \), then we are done. If \( \ell' = s' \) then the statement of the proposition is obvious since at least one of \( \vartheta_i(\{i\}) \) is equal to 0. Thus, we assume that \( \ell' \leq p \leq s'-1 \).
Recall that $n' = s' + (s' - \ell')$. For any collection of distinct elements $x_{p+1}, \ldots, x_{s'} \in [s'+1, n']$ we have $\sum_{i=p+1}^{s'} \vartheta_i(\{i, x_i\}) \leq s' - p - 1$. By a simple averaging argument we get that at least a $1/(s' - p)$-proportion of pairs $(i, x)$, where $i = p + 1, \ldots, s'$ and $x = s' + 1, \ldots, n'$, satisfies $\vartheta_i(\{i, x\}) = 0$. This accounts for at least $n' - s' = s' - \ell'$ such pairs. Therefore, the left hand side of (48) in this case does not exceed $(s' - 1)\left(\begin{array}{c} n' \\ 2 \end{array}\right)$, which is at most the right hand side of (48) since $(\begin{array}{c} n' \\ 2 \end{array}) = n'(s' - \ell' + 1)/2 > (s' - \ell')n'$.

\[\square\]

Proof of Lemma 11. We have $\sum_{i=1}^{s} \beta_i(\emptyset) = p$ for some $0 \leq p \leq s - 1$. W.l.o.g. assume that $\beta_i(\emptyset) = 1$ if and only if $i \in [p]$.

Assume that $\ell/2 \leq p \leq s - 2$. In this case we have $s - p \leq s - \ell/2 = \frac{1}{2}(2s - \ell)$. Then $\sum_{i=p+1}^{s} \beta_i(S_i) \leq s - p - 1$ for any pairwise disjoint $S_{p+1}, \ldots, S_s$ of cardinality two. By a simple averaging argument we immediately get that at least a $1/(s - p)$-proportion of all pairs $(i, S)$ satisfy $\beta_i(S) = 0$, where $i = p + 1, \ldots, s$ and $S \in \binom{[2s-\ell]}{2}$. In other words,

$$\beta_i(S) = 0 \text{ for at least } \left(\frac{2s - \ell}{2}\right) \text{ pairs } (i, S), \quad \text{where } |S| \in \binom{2s - \ell}{2} \text{ and } i \in [p+1, s].$$

(49)

Since the families $F_1, \ldots, F_s$ are cross-dependent, we similarly get that

$$\beta_i(S) = 0 \text{ for at least } 2s - \ell \text{ pairs } (i, S), \quad \text{where } |S| \in \binom{2s - \ell}{1} \text{ and } i \in [s].$$

(50)

Therefore, the left hand side of (46) is at most $p \sum_{j=0}^{m-1} \binom{n}{j} + (s - 1) \sum_{j=m}^{m+1} \binom{n}{j} + s \binom{n}{m+2}$. Consider the difference $D$ between the RHS and the LHS of (46). Then $D$ is at least

$$\left(\binom{n}{m+1} - (s - \ell)\binom{n}{m} - (s - 2)\sum_{j=0}^{m-1} \binom{n}{j}\right).$$

(51)

Next we show that this expression is always nonnegative. We have

$$\left(\binom{n}{m+1} - (s - \ell)\binom{n}{m}\right) - (s - \ell)\left(\binom{n}{m+1} - (s - \ell)\binom{n}{m}\right) = \frac{m(s - 1) + \ell - s}{m + 1} - (s - \ell)\left(\binom{n}{m+1} - (s - \ell)\binom{n}{m}\right).$$

(52)

It is easy to see that the right hand side of (52) is bigger than the right hand side of (45) for both $\ell = 2$ and $\ell \geq 3$, which proves (46) in this case.

If $p = s - 1$, then $\beta_s(S) = 0$ for all $S \subset [2s - \ell], |S| \leq 3$. Therefore, the LHS (46) is at most $(s - 1) \sum_{j=0}^{m+2} \binom{n}{j}$, and $D$ is at least

$$\left(\binom{n}{m+2} + \binom{n}{m+1} - (s - \ell)\binom{n}{m} - (s - 1)\sum_{j=0}^{m-1} \binom{n}{j}\right).$$

(53)

Recall that $m \geq 1$. If $\ell \geq 3$, then one can see from (52) and (45) that

$$\left(\binom{n}{m+1} - (s - \ell)\binom{n}{m} - (s - 1)\sum_{j=0}^{m-1} \binom{n}{j}\right) \geq \frac{m(\ell - 1) - s - 1}{m + 1} - \frac{(s - 1)m - 1}{s - 2} \frac{(s - 1)m + s - \ell + 1}{m},$$

which is always nonnegative. Indeed, it is easy to see for $\ell \geq 4$ (and thus $s \geq 4$), and for $\ell = 3$ it can be verified separately.
If \( \ell = 2 \) then due to (52) and (45) the expression (53) is at least \((n \choose m + 2) - \sum_{j=0}^{m-1} (n \choose j)\). We have \((n \choose m + 2) \geq (n \choose m)\) since \(n = s(m + 1) - \ell \geq sm + 1 \geq 2m + 2\) for any \(s \geq 3\) and \(m \geq 1\). Finally, \((n \choose m) > \sum_{j=0}^{m-1} (n \choose j)\) by (45).

Suppose that \(p < \ell/2\). Then again \(\prod_{i=p+1}^{s} \beta_i(S_i) = 0\) for any \(s - p\) pairwise disjoint \(S_i\). For each \(i \in [p + 1, s]\), consider the family \(F_i' := \{S \subset [2s - \ell] : \beta_i(S) = 1, |S| \leq 2\}\). These families are cross-dependent. Applying (47) to \(F_i'\) with \(2(s - p) - (\ell - 2p), s - p\) and \(\ell - 2p\) playing the roles of \(n', s'\) and \(\ell'\), respectively, we get that

\[
\sum_{i=p+1}^{s} |F_i'| \leq (\ell - 2p - 1)(2s - \ell) + (s - p)\left(\frac{2s - \ell}{2}\right).
\]

(Note that we tacitly use that none of the \(F_{p+1}', \ldots, F_s'\) contain the empty set by the assumption.)

We conclude that among the coefficients \(\beta_i(S)\), where \(i \in [s]\) and \(|S| \in [2]\), there are at least \((2s - \ell)(s - \ell + p + 1)\) that are equal to zero. The following observation is verified by a simple calculation.

**Observation 13.** Suppose \(s \geq 3\) and \(m \geq 1\). In the summation over \(S\) in (46), the coefficient in front of \(\beta_i(S_1)\) for \(|S_1| = 1\) is not bigger than the coefficient in front of \(\beta_i(S_2)\) for \(|S_2| = 2\). That is,

\[
\frac{(n \choose m+1)}{(2s-\ell+1)} \geq \frac{(n \choose m)}{(2s-\ell+2)}.
\]

Using the observation and the conclusion of the paragraph above it, we get that the left hand side of (46) is at most

\[
p \sum_{j=0}^{m-1} (n \choose j) \geq (\ell - p - 1)(n \choose m) + s\left(n \choose m + 1\right) + s\left(n \choose m + 2\right).
\]

Since \((n \choose m) > \sum_{j=0}^{m-1} (n \choose j)\), the last expression is smaller than \((\ell - 1)(n \choose m) + s(n \choose m+1) + s(n \choose m+2)\). \( \square \)

### 3.1. Proof of Theorem 2.2
We prove the theorem by double induction. We apply induction on \(m\), and for fixed \(m\) we apply induction on \(n\). The case \(m = 0\) of (11) is very easy to verify. The case \(n = q\) is the bound (11).

We may assume that all the \(F_i\) are shifted. The following two families on \([n - 1]\) are typically defined for a family \(\mathcal{F} \subset 2^{[n]}:\)

\[
\mathcal{F}(n) := \{A - \{n\} : n \in A, A \in \mathcal{F}\},
\]

\[
\mathcal{F}(\bar{n}) := \{A : n \notin A, A \in \mathcal{F}\}.
\]

It is clear that \(\mathcal{F}(\bar{n}), \ldots, \mathcal{F}_s(\bar{n})\) are \(q\)-dependent. Next we show that \(\mathcal{F}_1(n), \ldots, \mathcal{F}_s(n)\) are \((q - s)\)-dependent. Assume for contradiction that \(F_1, \ldots, F_s\), where \(F_i \in \mathcal{F}_i(n)\), are pairwise disjoint and that \(H := \bigcup F_i\) has size at most \(q - s\). Since \(n \geq q\), we have \(n - (q - s) \geq s\). Therefore, we can find distinct elements \(x_1, \ldots, x_s \in [n] - H\). Since \(\mathcal{F}_i\) are shifted, we have \(F_i \cup \{x_i\} \in \mathcal{F}_i\) for \(i \in [s]\), and the sets \(F_i \cup \{x_i\}\) are pairwise disjoint. Their union \(H \cup \{x_1, \ldots, x_s\}\) has size \(|H| + s \leq q\), contradicting the assumptions of the theorem.
Recall the definition (3). The induction hypothesis for $F_i(n)$ gives
\[
\sum_{i=1}^{s} |F_i(n)| \leq \sum_{i=1}^{s} |F_i^{n-1,q}|.
\] (54)

The induction hypothesis applied to $F_i(n)$ with $n - 1, q - s, m - 1$ playing the roles of $n, q, m$ gives
\[
\sum_{i=1}^{s} |F_i(n)| \leq \sum_{i=1}^{s} |F_i^{n-1,q-s}|.
\] (55)

Adding up (54) and (55), we get
\[
\sum_{i=1}^{s} |F_i| = \sum_{i=1}^{s} (|F_i(n)| + |F_i(n)|) \leq \sum_{i=1}^{s} (|F_i^{n-1,q}| + |F_i^{n-1,q-s}|).
\]

We have $F_i^{n-1,q} = F_i^{n,q}(n)$ and $F_i^{n-1,q-s} = F_i^{n,q}(n)$. Thus, for any $i \in [s]
\[
|F_i^{n-1,q}| + |F_i^{n-1,q-s}| = |F_i^{n,q}(n)| + |F_i^{n,q}(n)| = |F_i^{n,q}|
\]

4. APPLICATION OF THEOREM 3 TO AN ANTI-RAMSEY PROBLEM

We call a partition $F_1 \cup \ldots \cup F_M$ of $\binom{[n]}{k}$ into non-empty families $F_i$ an $M$-coloring. Let
the anti-Ramsey number $ar(n, k, s)$ be the minimum $M$ such that in any $M$-coloring there is a rainbow $s$-matching, that is, a set of $s$ pairwise disjoint $k$-sets from pairwise distinct $F_i$.

This quantity was studied by Özkahya and Young [24], who have made the following conjecture.

**Conjecture 3** ([24]). One has $ar(n, k, s) = e_k(n, s - 1) + 2$ for all $n > sk$.

It is not difficult to see that $ar(n, k, s) \geq e_k(n, s - 1) + 2$ for any $n, k, s$. Indeed, consider a maximum size family of $k$-sets with no $(s - 1)$-matching and assign a different color to each of these sets. Next, assign one new color to all the remaining sets. This is a coloring of $\binom{[n]}{k}$ without a rainbow $s$-matching. In [24] the authors proved Conjecture 3 for $s = 3$ and for $n \geq 2k^3s$. They also obtained the bound $ar(n, k, s) \leq e_k(n, s - 1) + s$ for $n \geq sk + (s - 1)(k - 1)$.

In this section we first state and prove a result for $n \geq sk + (s - 1)(k - 1)$, which is much stronger than Conjecture 3 in that range. We say that the $M$-coloring of $\binom{[n]}{k}$ is $s$-star-like if there exists a set $Y \in \binom{[n]}{s-2}$ and a number $i \in [M]$ such that for every $j \in [M] - \{i\}$ and $F \in F_j$, $F$ intersects $Y$. Clearly, each $s$-star-like coloring has at most $e_k(n, s - 1) + 1$ colors. For convenience, we define the quantity
\[
h(n, k, s) := \max\{|F| : F \subset \binom{[n]}{k}, \nu(F) < s, \tau(F) \geq s\},
\]
which was determined for a certain range in Theorem 3.

**Theorem 14.** Let $s \geq 3, k \geq 2$ and $n \geq sk + (s - 1)(k - 1)$ be some integers. Consider an $M$-coloring of $\binom{[n]}{k}$ without a rainbow $s$-matching. Then either this coloring is $s$-star-like, or $M \leq h(n, k, s - 1) + s$. 
Conjecture 3 follows from Theorem 14 once we can apply Theorem 3 or an analogous statement. Indeed, it is shown in Theorem 3 that \( \varphi(n, k, s-1) \) is much smaller than \( e(n, s-1) \), moreover, it implies that in most cases \( h(n, k, s-1) + s \) is smaller than \( e_k(n, s-1) + 2 \). We do not need an exact Hilton–Milner-type result to deduce Conjecture 3. We may use a weaker form of Theorem 3 that was proven in [11] and is valid for a slightly wider range of parameters than Theorem 3. In particular, [11] Theorem 5 implies that for \( n \geq sk + (s-1)(k-1) \) we have \( e_k(n, s-1) - h(n, k, s-1) \geq \frac{1}{s} (n-s-k+2)^2 \). For any \( n \geq sk + (s-1)(k-1) \) and \( k \geq 3 \) we have \( \frac{1}{s} (n-s-k+2)^2 > s - 2 \). Thus, Theorem 14 implies Conjecture 3 for this range.

**Corollary 15.** We have \( ar(n, k, s) = e_k(n, s-1) + 2 \) for \( n \geq sk + (s-1)(k-1) \) and \( k \geq 3 \).

Note that this is the same range in which Özkahya and Young got a weaker bound \( ar(n, k, s) \leq e_k(n, s-1) + s \). We remark that the case \( k = 2 \) has already been settled for all values of parameters (see [24] for the history of the problem).

**Proof of Theorem 14.** Arguing indirectly, fix an \( M \)-coloring with \( M \geq h(n, k, s-1) + s \) and with no rainbow \( s \)-matching. We may assume that there is a rainbow \((s-1)\)-matching \( F_1, \ldots, F_{s-1} \) with, say, \( F_i \in \mathcal{F}_{M-i+1} \) for each \( i \in [s-1] \). Otherwise, recolor some of the sets in new colors so that the number of colors increases and a rainbow \((s-1)\)-matching (but no rainbow \( s \)-matching) appears.

For each \( i \in [M-s+1] \), choose \( G_i \in \mathcal{F}_i \). Note that \( M - s + 1 > h(n, k, s-1) \). Thus, either \( G := \{G_1, \ldots, G_{M-s+1}\} \subset \binom{[n]}{k} - \binom{[n]}{k} \) for a suitable \( U \subset \binom{[n]}{n-s+2} \) and for any choice of \( G_i \), or for some choice of \( G_i \) there is an \((s-1)\)-matching, say \( G_1, \ldots, G_{s-1}, \) in \( G \).

In the latter case we can apply the argument used by Özkahya and Young: since the colors of \( F_1, \ldots, F_{s-1}, G_1, \ldots, G_{s-1} \) are all distinct, any \( k \)-set from \([n] \setminus \bigcup_{i=1}^{s-1} G_i \cup F_i \) forms a rainbow \( s \)-matching with one of these two \((s-1)\)-matchings. Moreover, \( G_i \cap (F_1 \cup \ldots \cup F_{s-1}) \neq \emptyset \) holds for each \( i \in [s-1] \). Thus, \( |\bigcup_{i=1}^{s-1} G_i \cup F_i| \leq (2k-1)(s-1) \), and we are done provided \( |[n] \setminus \bigcup_{i=1}^{s-1} G_i \cup F_i| \geq k \), which holds for \( n \geq k + (2k-1)(s-1) = sk + (s-1)(k-1) \).

In the former case the family \( G \) must satisfy \( \tau(G) \leq s - 2 \) for all choices of \( G_i \in \mathcal{F}_1, \ldots, G_{M-s+1} \in \mathcal{F}_{M-s+1} \).

**Claim 16.** Fix \( N > h(n, k, s-1) \) and pairwise disjoint families \( \mathcal{H}_1, \ldots, \mathcal{H}_N \) of \( k \)-subsets of \([n]\). If for any set of representatives \( \mathcal{H} := \{H_1, \ldots, H_N\} \) with \( H_i \in \mathcal{H}_i \) we have \( \tau(\mathcal{H}) \leq s-2 \), then \( \tau(\bigcup_{i=1}^{N} H_i) \leq s-2 \).

**Proof.** Fix one choice of \( \mathcal{H} \) and let \( T \) be a hitting set of size \( s-2 \) for \( \mathcal{H} \). Arguing indirectly, assume that there is a set \( H' \in \mathcal{H}_1 \) such that \( H' \cap T = \emptyset \). The family \( \mathcal{H}' := \{H', H_2, \ldots, H_N\} \) also satisfies \( \tau(\mathcal{H}') = s-2 \), and so there is a set \( T' \neq T \), \( |T'| \leq s-2 \), such that \( H_2, \ldots, H_N \) all intersect \( T' \). Define \( m(T, T') := |\{F \subset \binom{[n]}{k} : F \cap T \neq \emptyset \neq F \cap T'\}| \). We want to show that for any distinct \( T, T' \) of size \( s-2 \) the quantity \( m(T, T') \) is strictly smaller than \( h(n, k, s-1) \). This contradicts the fact that \( H_2, \ldots, H_N \) all intersect \( T \) and \( T' \).

Let us show that \( m(T, T') \) is maximal when \( |T \cap T'| = |T| - 1 = s - 3 \). Indeed if \( |T \cap T'| < |T| - 1 \), then we may choose \( x \in T \setminus T' \) and define \( T'' := (T' \setminus \{y\}) \cup \{x\} \). Let \( F \) be an arbitrary set satisfying \( F \cap T \neq \emptyset \), \( F \cap T' \neq \emptyset \) and \( F \cap T'' = \emptyset \). This means that \( F \cap (T' \cup T'') = \{y\} \), \( F \cap (T - \{x\}) \setminus T' \neq \emptyset \). Setting \( |T \cap T'| = t \), the number of such sets \( F \in \binom{[n]}{k} \) is \( \binom{n-t}{k-1} - \binom{n-2e-2t}{k-1} \).
On the other hand, the sets \( F \) satisfying \( F \cap T \neq \emptyset, F \cap T^c \neq \emptyset, \) and \( F \cap T' = \emptyset \) are those with \( F \cap (T' \cup T^c) = \{ x \} \). Their number is \( \binom{n-(s-2)-1}{k-1} \), which is clearly bigger.

Suppose now that \( |T \cap T'| = s - 3 \). Then \( m(T, T') = \binom{n}{k} - \binom{n-s+3}{k} + \binom{n-s+1}{k-2} \). Since \( h(n, k, s - 1) \geq \binom{n}{k} - \binom{n-s+2}{k} - \binom{n-s+2-k}{k-1} + 1 \), we have \( h(n, k, s - 1) - m(T, T') > \binom{n-s+1}{k-1} - \binom{n-s+2-k}{k-1} > 0 \). This completes the proof of the claim.

Applying Claim \ref{claim:16} to the first \( M - s + 1 \) color classes, we get that they all intersect a set \( T \) of size \( s - 2 \). To complete the proof, we need to show that the same holds for some \( M - 1 \) colors.

Note that since \( \sum_{i=1}^{M-s+1} |F_i| \leq \binom{n}{k} - \binom{n-s+2}{k} \), one of the last \( s - 1 \) color classes, say \( F_M \), has size at least \( \frac{1}{s-1} \binom{n-s+2}{k} \).

\textbf{Claim 17.} In every rainbow \((s - 1)\)-matching, one of the \( k \)-sets belongs to \( F_M \).

\textit{Proof.} Arguing indirectly, assume that there is a rainbow \((s - 1)\)-matching in colors \( F_{M-s+1}, \ldots, F_{M-1} \). Applying Claim \ref{claim:16} to \( F_1, \ldots, F_{M-s}, F_M \), we find a hitting set \( T \) of size \( s - 2 \) for \( F_1 \cup \ldots \cup F_{M-s} \cup F_M \). We infer

\[ M - s + \frac{1}{s-1} \binom{n-s+2}{k} \leq |F_1 \cup \ldots \cup F_{M-s} \cup F_M| \leq \binom{n}{k} - \binom{n-s+2}{k}. \quad (56) \]

We have \( M - s \geq h(n, k, s - 1) > \binom{n}{k} - \binom{n-s+3}{k} \). Also, we have

\[ \binom{n-s+3}{k} = \frac{n-s+3}{n-s+3-k} \binom{n-s+2}{k} \leq \frac{s+1}{s} \binom{n-s+2}{k}, \]

provided \( n - s + 3 - k \geq sk \). The inequalities above contradict (56), and so the claim follows.

We conclude that there is no rainbow \((s - 1)\)-matching in \( F_1 \cup \ldots \cup F_{M-1} \), which implies that there is a cover of size \( s - 2 \) for any set of distinct representatives of the color classes. In turn, Claim \ref{claim:16} implies that \( F_1 \cup \ldots \cup F_{M-1} \) can be covered by a set \( T \) of size \( s - 2 \), i.e., the coloring is \( s \)-star-like. Theorem \ref{thm:14} is proved.

Next, we prove a weaker bound on \( ar(n, k, s) \), which is valid for all \( n > sk \):

\textbf{Theorem 18.} We have \( ar(n, k, s) \leq e_k(n, s - 1) + \frac{(s-1)\binom{n}{k}}{(n-(s-1)k)} + 1 \) for any \( n > ks \).

\textit{Proof.} Fix an \( M \)-coloring of \( \binom{[n]}{k} \) with no rainbow \( s \)-matching and let \( G_1, \ldots, G_M \) be a set of representatives of the color classes. Reordering \( G_i \) if necessary, we may assume that for some integer \( T \) the collections \( \{G_1, \ldots, G_{s-1}\}, \{G_s, \ldots, G_{2(s-1)}\}, \ldots, \{G_{(T-1)(s-1)+1}, \ldots, G_{T(s-1)}\} \) are rainbow \((s - 1)\)-matchings, while the remaining collection \( \{G_{T(s-1)+1}, \ldots, G_M\} \) does not contain a rainbow \((s - 1)\)-matching. We have

\[ M - T(s-1) \leq e_k(n, s - 1). \quad (57) \]

Set \( H_i := [n] - (G_{i(s-1)+1} \cup \ldots \cup G_{(i+1)(s-1)}) \) for \( i = 0, \ldots, T - 1 \). The non-existence of rainbow \( s \)-matchings in the coloring implies that in the coloring of \( \binom{H_i}{k} \) we only used the colors of the \( s - 1 \) sets \( G_{i(s-1)+1}, \ldots, G_{(i+1)(s-1)} \). Consequently, \( \binom{H_i}{k} \) and \( \binom{H_i}{k} \) must be
disjoint for \(i \neq j\). From this \(\binom{n}{k} \geq T^{(n-(s-1)k)}\) follows. Combining with (57), we obtain 
\[
M \leq e_k(n, s - 1) + (s - 1)\binom{n}{k} / (n - (s-1)k),
\]
as claimed. \(\square\)

Finally, we prove a bound that is valid for \(n > (s + \sqrt{s})k\) and is stronger than the previous one in many cases.

**Theorem 19.** We have \(ar(n, k, s) < e_k(n, s - 1) + \frac{(s-1)n-(sk)}{(n-sk)^2-k((2s-1)k-n)} + 1\) for any \(n > (s + \sqrt{s})k\). In particular, if \(n = csk\) for some fixed \(c > 1\) and \(s \to \infty\), then \(ar(n, k, s) \leq e_k(n, s - 1) + (1 + o(1))s^{-c}\).

**Proof.** Following the notations of the previous proof, put \(F_i := G_{i(s-1)+1} \cup \ldots \cup G_{i+1(s-1)}\) for \(i = 0, \ldots, T - 1\). Note that \(|F_i| = (s-1)k\). Then for each \(i \neq j\) we have \(|F_i \cup F_j| > n - k\) since otherwise any \(k\)-set in \([n] \setminus (F_i \cup F_j)\) will form a rainbow \(s\)-matching with either the sets forming \(F_i\) or the sets forming \(F_j\). Thus, we have \(|F_i \cap F_j| = 2|F_i| - |F_i \cup F_j| < (2s-1)k - n\) and

\[
\sum_{0 \leq i < j \leq T-1} |F_i \cap F_j| < \binom{T}{2}(2s-1)k - n.
\]

On the other hand, if an element \(x\) is contained in \(d_x\) sets \(F_i\), then it contributes \(\frac{d_x}{2}\) to the sum \(\sum_{0 \leq i < j \leq T-1} |F_i \cap F_j|\). The average number \(d\) of sets containing an element of \([n]\) satisfies \(d = \frac{T(s-1)k}{n}\). Therefore, we get

\[
\sum_{0 \leq i < j \leq T-1} |F_i \cap F_j| = \sum_{x \in [n]} \binom{d_x}{2} \geq n \binom{d}{2}.
\]

Combining the two displayed formulas, we get

\[
T(T-1)((2s-1)k - n)) > n \frac{T(s-1)k}{n} \left( \frac{T(s-1)k}{n} - 1 \right).
\]

After rearranging and simplifying the expression above we get

\[
\left( \frac{(s-1)k^2}{n} - (2s-1)k + n \right) T < (s-1)k - (2s-1)k + n,
\]

which is equivalent to

\[
((n - sk)^2 - (2s-1)k^2 + nk) T < n(n - sk).
\] (58)

The left hand side is positive provided \(n = sk + x\), where \(x^2 \geq k((s-1)k - x)\). This holds for \(x = k\sqrt{s}\). In this assumption it is clear that (58) combined with (57) implies the first statement of the theorem. If \(s \to \infty\) and \(n = csk\) with \(c > 1\), then (58) transforms into \((1 + o(1))(c - 1)^2 T \leq c(c - 1)\), which, together with (57), implies the second statement of the theorem. \(\square\)

We remark that a similar proof, applied in a more general scenario, appeared in [22].
5. Almost matchings

Let us say that the sets $F_1, \ldots, F_s$ form an almost matching if the family $\mathcal{F} := \{F_1, \ldots, F_s\}$ has at most one vertex of degree greater than one and that vertex has degree at most two.

Define

$$a(n, s) := \max \{ |\mathcal{F}| : \mathcal{F} \subset 2^n, \mathcal{F} \text{ contains no almost-matching of size } s \}.$$ 

**Theorem 20.** The equality $a(sm - 2, s) = \sum_{t=m}^{sm-2} \binom{sm-2}{t}$ holds for all $s \geq 2, m \geq 1$. Moreover, the equality is achieved only by the family $\{F \subset [sm-2] : |F| \geq m\}$.

**Proof.** Suppose $\mathcal{F} \subset 2^n$ contains no almost-matching of size $s$, where $n = sm - 2$. We may suppose that $\mathcal{F}$ is an up-set. Consider the families $\mathcal{F}_1, \ldots, \mathcal{F}_s$, where $\mathcal{F}_i := \mathcal{F}$ for $i = 1, \ldots, s - 1$, and $\mathcal{F}_s := \partial \mathcal{F} \cup \{[n]\}$ (see (15)).

**Claim 21.** The families $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are cross-dependent.

**Proof.** Indeed, if $F_1, \ldots, F_s$, where $F_i \in \mathcal{F}_i$, are pairwise disjoint then, replacing $F_s$ by some $F \in \mathcal{F}$, $F_s \subset F$, $|F \setminus F_s| = 1$, we obtain $s$ members $F_1, \ldots, F_{s-1}, F$ of $\mathcal{F}$ that form an almost-matching.

Applying Theorem 20 yields

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leq \left( \frac{n}{m-1} \right) + s \sum_{t=m}^{n} \binom{n}{t}. \tag{59}$$

**Lemma 22.** Let $\mathcal{F} \subset 2^n$ be an up-set and suppose that for some $1 \leq k \leq n$, $|\mathcal{F}| = \sum_{i=k}^{n} \binom{n}{i}$. Then $\partial \mathcal{F}$ satisfies

$$\partial(\mathcal{F}) \geq \sum_{i=k-1}^{n-1} \binom{n}{i} \tag{60}$$

with equality if and only if $\mathcal{F} = \binom{[n]}{\geq k}$.

Let us first deduce Theorem 20 from Lemma 22. By the lemma, if $|\mathcal{F}| > \sum_{i=m}^{n} \binom{n}{i}$ or $|\mathcal{F}| = \sum_{i=m}^{n} \binom{n}{i}$ and $\mathcal{F} \neq \binom{[n]}{\geq m}$ then $|\mathcal{F}_s| > \sum_{i=m-1}^{n} \binom{n}{i}$, which contradicts (59). This yields the statement of the theorem.

**Proof of Lemma 22.** We apply induction on $n$. The cases $n = 1, 2$ are trivial, since $\binom{[n]}{\geq k}$ is the only up-set of the given size. For $n = 3, k = 1$ we have one more shifted up-set, namely $\{P : \{1\} \subset P \subset [3]\}$. For this family one has strict inequality in (60).

Suppose now that (60) is true for $n$ and all $k$ and let us prove it for $n + 1$ and all $k$. Note that the cases $k = 1, k = n + 1$ are obvious and suppose that $1 < k \leq n$. First suppose that $\mathcal{F}$ is shifted.

**Claim 23.** $|\mathcal{F}(1)| \geq \sum_{i=k}^{n+1} \binom{n}{i-1}$.

**Proof.** The opposite would imply $|\mathcal{F}(\bar{1})| > \sum_{i=k}^{n+1} \binom{n}{i+1} - \binom{n}{i-1} = \sum_{i=k}^{n} \binom{n}{i}$. By the induction hypothesis, $\partial \mathcal{F}(\bar{1}) \geq \sum_{i=k-1}^{n-1} \binom{n}{i} = \sum_{j=k}^{n+1} \binom{n}{j-1} - 1$. By shiftedness, if $G \in \partial \mathcal{F}(\bar{1})$ then $G \cup \{1\} \in \mathcal{F}$, i.e., $G \in \mathcal{F}(1)$. Noting that $[n + 1] \in \mathcal{F}$ implies that $[2, n + 1] \in \mathcal{F}(1)$ as well, the claim follows. \qed
Applying the induction hypothesis for \( F(1) \) yields

\[
|\partial F(1)| \geq \sum_{i=k}^{n+1} \binom{n}{i-2}.
\]

Together with the claim and the fact that \( \partial F \supset F(1) \) we infer

\[
|\partial F| \geq \sum_{i=k}^{n+1} \binom{n}{i-1} + \binom{n}{i-2} = \sum_{i=k}^{n+1} \binom{n+1}{i-1},
\]

equivalent to the bound in Lemma 22.

In case of equality the induction hypothesis implies \( F(1) = \binom{\lfloor n \rfloor}{\geq k-1} \). By shiftedness, \( |F| \geq k \) for all \( F \in F \). This in turn implies \( F \subset \binom{\lfloor n+1 \rfloor}{\geq k} \).

Finally, we note that shifting does not change the size of the sets. Thus, \( S_{ij}(F) = \binom{n+1}{\geq k} \) can only occur if already \( F = \binom{\lfloor n+1 \rfloor}{\geq k} \). Therefore, the lemma is true for not necessarily shifted families as well. \( \square \)

6. Conclusion

In this paper we have obtained several results related to families of sets with no \( s \) pairwise disjoint sets. The stability results in the spirit of Theorem 3 have proven to be useful. We have found two applications so far: to Conjecture 3 and to Erdős and Kleitman’s problem on families with no matchings of size \( s \), studied in [11].

The method we developed for the proof of Theorem 1 may be applied in different scenarios. Apart from the result on almost matchings, we have applied a modification of this method in [12] to determine the size of the largest families with no matchings of size 3 and 4, as well as to an old problem concerning families with no partitions [13]. Similar ideas also appeared in [10].

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