Vacuum Einstein equations in terms of curvature forms

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Abstract

A closed explicit representation of the vacuum Einstein equations in terms of components of curvature 2-forms is given. The discussion is restricted to the case of non-vanishing cubic invariant of conformal curvature spinor. The complete set of algebraic and differential identities connecting particular equations is presented and their consistency conditions are analyzed.

Short title: Einstein equations in $S$-forms

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1. INTRODUCTION

As it is well known, in the standard formulation of general relativity theory the spacetime metric is considered as a fundamental gravitational field variable while the connection and curvature are in a sense secondary and are derived from it. All the three are physically significant objects: the metric defines causal space-time structure and determines time and length scales, the curvature distinguishes true gravitational effects from purely inertial ones, while the connection encompasses the possibility of treating gravity as a gauge theory which is important, in particular, in attempts of finding unification with the other types of interaction. Quite remarkably, the beautiful and clear mathematical formalism of classical general relativity allows for several alternative dynamical descriptions of the same physical system.

Historically, the components of the metric tensor (in a specific gauge) were first used as phase space variables (“gravitational potentials”) of the gravitational field (e.g., ADM approach, [1]). Closely related are the approaches using the (pseudo-) orthonormal tetrads or similar algebraic structures.

A more recent Ashtekar’s approach [2] exploits the connection as a source of phase space variables, and a number of concrete realizations of such an idea have been developed (see [3,4] and references therein).

There are, however, some other choices of a basic structure describing a state of the gravitational field. Recently a considerable attention has been attracted to the formulations in which the self-dual two-forms play a central role [5–10] (see a review and some further references in [11]). It is certainly worthwhile to mention an important contribution to this subject by Plebański [12,13] (cf. also [14]) who investigated relations between the metric and special bases of self- and anti-self-dual complex 2-forms and proved a number of statements which underlie many of the modern developments [5–10]. In particular, the first order chiral gravitational action for vacuum general relativity was introduced in [12].

Additional interest to the alternative gravitational field variables formulations is at-
tracted by recent studies of formal similarities and exact mappings between Yang-Mills gauge theories of internal symmetry groups and gauge gravity models \cite{15-19}. These approaches proved to be useful, in particular, for the problems of constructing new exact solutions in gravity and Yang-Mills gauge theories. The relations between gravitational instantons and Yang-Mills gauge fields are discussed in \cite{20}, while new singular $SU(2)$ spherically and cylindrically symmetric solutions with confining properties are reported in \cite{21}.

In the present work, we investigate the case when the space-time curvature is considered as a primary characteristic of the gravitational field. Previously, there were attempts (using different, mainly purely algebraic methods) to derive space-time metric in terms of curvature \cite{22-24}, or in terms of connection \cite{25}. In the so-called purely connection formulation of general relativity \cite{3-4} the metric is effectively eliminated in favour of connection. Our aim is to go one step further and to eliminate both the metric and connection, leaving only the curvature components as fundamental variables. We do not address at the moment such problems as a construction of an action, Hamiltonian, momenta etc, leaving this for further study. As a first necessary step, the very possibility of deriving a closed expression for the vacuum Einstein field equations directly in terms of the curvature alone is demonstrated in an explicit form. This is the main new result reported in our paper.

The structure of the paper is as follows. After explaining some basic notations and conventions in section 2, in section 3 we introduce the notion of $S$-forms and discuss the properties of these objects which play a central role in our approach. Our new observations are formulated in the proposition 6 and corollaries 7,8. These technical results essentially underlie all further derivations in the paper. In section 4 we show how one can construct $S$-forms from the curvature $\Omega$-forms. The crucial importance of $S$-forms in the description of a four-dimensional Lorentzian geometry is outlined in section 5. Section 6 contains our main result, the theorem 11, which gives a self-consistent formulation of the vacuum Einstein equations in terms of the curvature two-forms. The mutual relations between different subsets of the resulting algebraic-differential system of field equations are found in section 7. Our conclusions are summarised in section 8.
2. PRELIMINARIES AND NOTATIONS

We shall consider a complexified tangent space over some point of a 4-dimensional space-time $M$. The Greek indices $\alpha, \beta, \ldots$ are running over the set $\{1,2,3,4\}$ enumerating the local space-time coordinates. The capital Latin (spinor) indices, undotted $A, B, \ldots$ and dotted $\dot{A}, \dot{B}, \ldots$, run over the 2-element sets $\{0, 1\}$ and $\{0, 1\}$, respectively.

Further, $\varepsilon^{\alpha\beta\mu\nu}, \varepsilon_{\alpha\beta\mu\nu}$ and $\epsilon^{AB}, \epsilon_{AB}, \epsilon^{\dot{A}\dot{B}}, \epsilon_{\dot{A}\dot{B}}$ are the standard Levi-Civita symbols, 4-dimensional and 2-dimensional, respectively. Spinorial $\epsilon$’s are used for lowering and raising of spinor indices in accordance with the rules $t^B = \iota_A \epsilon^{AB}, t_A = \epsilon_{AB} t^B, \dot{t}^B = \iota_{\dot{A}} \epsilon^{\dot{A}\dot{B}}, \dot{t}_{\dot{A}} = \epsilon_{\dot{A}\dot{B}} \dot{t}^B$ (i.e. $\delta^B_A = \epsilon^B_A, \delta^B_{\dot{A}} = \epsilon^B_{\dot{A}}$).

Finally, we recall the general definition of Hodge dual and of the so-called $\eta$-basis of the exterior algebra over $M$. Let us denote a coframe 1–form $\vartheta^a$ (which forms an orthonormal basis of the (complexified) cotangent space $\Lambda^1$) and its dual frame $e_a$, such that $e_a \downarrow \vartheta^b = \delta^b_a$. Let the volume 4–form be $\eta = \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \wedge \vartheta^4$. Then the $\eta$-basis and the Hodge duals are defined as follows:

$$\eta_a = e_a \downarrow \eta = *\vartheta_a,$$

(2.1)

$$\eta_{ab} = e_b \downarrow \eta_a = *(\vartheta_a \wedge \vartheta_b),$$

(2.2)

$$\eta_{abc} = e_c \downarrow \eta_{ab} = *(\vartheta_a \wedge \vartheta_b \wedge \vartheta_c),$$

(2.3)

$$\eta_{abcd} = e_d \downarrow \eta_{abc} = *(\vartheta_a \wedge \vartheta_b \wedge \vartheta_c \wedge \vartheta_d).$$

(2.4)

3. ALGEBRA OF S-FORMS

Many of the results presented in this section seem to be well known, however we find it useful to collect all the facts together and summarize them in a lucid way using the
convenient formalism of spinor–valued (complex) exterior forms on \(M\). The main result of this section is formulated in the proposition 6.

Let us start our discussion with introducing a fundamental object – a triad of complex 2-forms \(S_{AB} = S_{AB\mu\nu} dx^\mu \wedge dx^\nu\) which are labeled by a pair of spinor indices \(A, B\). This object is assumed to be symmetric, \(S_{AB} = S_{BA}\), thus indeed representing three forms denoted by \(S_{00}, S_{01} = S_{10}, S_{11}\).

Their properties are as follows. These forms are assumed to be non-degenerate in the sense that

\[
S^{KL} \wedge S_{KL} \neq 0, \tag{3.1}
\]

and form a complete set satisfying

\[
S^{AB} \wedge S_{CD} = \frac{1}{3} \delta^A_C \delta^B_D S^{KL} \wedge S_{KL}. \tag{3.2}
\]

In fact, the non-degeneracy condition (3.1) introduces on the space-time manifold a non-trivial volume 4-form which we denote

\[
\eta := -\frac{1}{3} S^{KL} \wedge S_{KL}. \tag{3.3}
\]

The sign is related to the orientation chosen on \(M\) and in fact one can introduce a different orientation. The numerical factor is included for convenience of simplifying various relations below. Such a formally defined volume is complex, in general (for the Lorentzian geometry it is purely imaginary, see below).

Both relations (3.1) and (3.2) are explicitly covariant with respect to the \(SL(2, C)\) transformations defined on the objects with the “undotted spinor indices” in accordance with

\[
t \rightarrow \ell \cdot t : \quad (\ell \cdot t)^A = \ell^A_B t_B, \quad (\ell \cdot t)_B = \ell^A_B t_A, \tag{3.4}
\]

where the complex transformation matrix \(|\ell^A_B|\) obeys the only constraint

\[
\epsilon^{CD} \epsilon_{AB} \ell^A_C \ell^B_D = 2,
\]

i.e. is unimodular, \(|\ell^A_B| \in SL(2, C)\).
We shall name the 2-forms $S_{AB}$ whose components obey the above conditions $S$-forms for the sake of brevity.

**Proposition 1** For each set of $S$-forms which satisfy the completeness (3.2) and the non-degeneracy (3.1) conditions, there exists a basis $\theta_{AB}$ of the complexified cotangent bundle $\Lambda^1$ such that

$$S_{AB} = \frac{1}{2} \theta_A^K \wedge \theta_{BK}. \quad (3.5)$$

**Proof:** Eqs. (3.2) include in particular the equations $S_{00} \wedge S_{00} = S_{11} \wedge S_{11} = 0$, $S_{00} \wedge S_{11} = -\eta$, which imply an existence of the linearly independent 1-forms $\theta^j$, $j = 1, 2, 3, 4$, such that $S_{00} = \theta^3 \wedge \theta^1$, $S_{11} = \theta^4 \wedge \theta^2$, $\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = \eta \neq 0$. The latter relations are not changed under the transformations of the 1-forms,

$$\left( \begin{array}{c} \theta^3 \\ \theta^1 \end{array} \right) \rightarrow \ell_{(1)} \cdot \left( \begin{array}{c} \theta^3 \\ \theta^1 \end{array} \right), \quad \left( \begin{array}{c} \theta^4 \\ \theta^2 \end{array} \right) \rightarrow \ell_{(2)} \cdot \left( \begin{array}{c} \theta^4 \\ \theta^2 \end{array} \right), \quad (3.6)$$

where $\ell_{(1)}, \ell_{(2)}$ are arbitrary complex $2 \times 2$ unimodular matrices. On the account of the further equations (3.2), $S_{01} \wedge S_{00} = S_{01} \wedge S_{11} = 0$, $S_{01} \wedge S_{01} = \frac{1}{2} \eta$, one notices that the 2-form $S_{01} = S_{10}$ admits the expansion $S_{01} = a\theta^3 \wedge \theta^2 + b\theta^4 \wedge \theta^1 + p\theta^1 \wedge \theta^2 + q\theta^3 \wedge \theta^4$, where the coefficients $a, b, p, q$ satisfy $ab + pq = \frac{1}{4}$. Applying the $SL(2,C)$ transformations (3.6) one can reduce this to $-2S_{01} = \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4$. Finally, introducing the spinor indexing of $\theta$-tetrad by means of the correspondence

$$\theta^3 \mapsto \theta_{00}, \quad \theta^1 \mapsto \theta_{01}, \quad \theta^2 \mapsto \theta_{10}, \quad \theta^4 \mapsto -\theta_{11}, \quad (3.7)$$

Eq. (3.5) immediately follows. □

**Corollary 2** Given the 1-forms $\theta_{AB}$ from the proposition 1, a family of 2-forms $S_{\dot{A}\dot{B}} = S_{\dot{B}\dot{A}}$, defined by

$$S_{\dot{A}\dot{B}} = \frac{1}{2} \theta_{\dot{A}}^K \wedge \theta_{K\dot{B}} \quad (3.8)$$

and consisting of three elements $S_{00}, S_{01} = S_{10}, S_{11}$, obeys the “dotted” version of eqs. (3.2),
\[ S^\dot{A}\dot{B} \land S_{\dot{C}\dot{D}} = \frac{1}{3} \delta^\dot{A}_{\langle B} \delta^\dot{C}_{D \rangle} S^{\dot{K}\dot{L}} \land S_{\dot{K}\dot{L}}, \] (3.9)

and the identities

\[ S_{AB} \land S_{\dot{C}\dot{D}} = 0, \quad S^{\dot{K}\dot{L}} \land S_{KL} + S^{\dot{K}\dot{L}} \land S_{\dot{K}\dot{L}} = 0. \] (3.10)

**Proof:** The most straightforward way to prove eqs. (3.9)-(3.10) is to notice that \( S_{AB} \) and \( S_{\dot{A}\dot{B}} \) together form the basis \( \theta_{\dot{A}\dot{B}} \land \theta_{\dot{C}\dot{D}} \) of the complexified 2-forms \( \Lambda^2 \) bundle as follows:

\[ \theta_{\dot{A}\dot{B}} \land \theta_{\dot{C}\dot{D}} = \epsilon_{AC} S_{BD} + \epsilon_{BD} S_{AC}. \] (3.11)

The equation (3.11) is directly proved with the help of the standard spinor algebra methods. □

Notice that all the above equations are invariant with respect to another independent copy of the group \( SL(2, C) \) besides (3.4). The second \( SL(2, C) \) transformation group acts on “dotted spinor indices” by the similar rules \( \iota^A \rightarrow \iota^B \iota^A, \theta^A \rightarrow \iota^B \theta^A \), where \( |\iota^A| \in SL(2, C) \).

In general, these two transformation groups are unrelated to each other.

An immediate consequence of the proposition 1 is the existence of metric on \( M \) which can be defined in an \( SL(2, C) \) invariant manner as \( g = \theta^\dot{A}\dot{B} \otimes \theta^A_B \). In the local coordinates \( x^\alpha \) the complex frame has the components \( \theta^{\dot{A}\dot{B}}_\alpha = \partial_{\alpha} \theta^{\dot{A}\dot{B}} \) and hence the components of the metric are

\[ g_{\alpha\beta} = \theta^\dot{A}\dot{B} \theta^A_B. \] (3.12)

From the proposition 1 it follows that this symmetric tensor is non-degenerate.

Given the volume 4–form (3.3) and the corresponding coframe 1-form \( \theta^A_B \) (metric \( g_{\alpha\beta} \)), one can define the \( \eta \)-basis of the exterior algebra and compute the Hodge duals. Let us denote \( e^\dot{A}\dot{B} \) the complex frame (basis of the complexified tangent space) dual to the fundamental coframe 1-form, i.e.

\[ e^\dot{A}\dot{B} \theta^C_D = \delta^C_A \delta^D_B. \] (3.13)
Proposition 3 The Hodge duals and the $\eta$-basis related to the coframe $\theta^A_B$ are as follows:

\[
\eta^A_K = e_A^\dot{B} \eta = *\theta^A_K \\
= \frac{2}{3} S_{AC} \wedge \theta^{CK},
\]

(3.14)

\[
\eta^{AB}\dot{K}\dot{L} = e_B^\dot{L} \eta_A^\dot{K} = *(\theta^A_K \wedge \theta_B^L)
\]

\[
= \epsilon_{AB}S^{\dot{K}\dot{L}} - \epsilon^{\dot{K}\dot{L}}S_{AB},
\]

(3.15)

\[
\eta^{ABC}\dot{K}\dot{L}\dot{M} = e_C^\dot{M} \eta_A^\dot{K} = *(\theta^A_K \wedge \theta_B^L \wedge \theta_C^M)
\]

\[
= \epsilon_{ABC}S^{\dot{K}\dot{L}\dot{M}} - \epsilon^{\dot{K}\dot{L}\dot{M}}S_{ABC},
\]

(3.16)

\[
\eta^{ABCD}\dot{K}\dot{L}\dot{M}\dot{N} = e_D^\dot{N} \eta_{ABC}^\dot{K}\dot{L}\dot{M} = *(\theta^A_K \wedge \theta_B^L \wedge \theta_C^M \wedge \theta_D^N)
\]

\[
= -\epsilon_{AB} \epsilon_{CD} \epsilon^{\dot{K}\dot{L}\dot{M}\dot{N}} + \epsilon_{D(A} \epsilon_{B)C} \epsilon^{\dot{K}\dot{L}\dot{M}\dot{N}}.
\]

(3.17)

Proof: One should straightforwardly calculate subsequent interior products. For example, starting from (3.3) and using (3.5) and (3.13), one finds

\[
e_A^\dot{B} \eta = \frac{1}{3} S_{KL} \wedge e_A^\dot{B}((\theta^K_M \wedge \theta^{LM}) = \frac{2}{3} S_{AL} \wedge \theta^{L\dot{B}}.
\]

This proves (3.14), the rest relations (3.15)-(3.17) are demonstrated analogously. □

Corollary 4 Under the conditions of the proposition 1 the 2-forms $S_{AB}$ and $S_{\dot{A}\dot{B}}$ are, respectively, anti-self-dual and self-dual with respect to the metric (3.12) they define,

\[
* S_{AB} = -S_{AB}, \quad * S_{\dot{A}\dot{B}} = S_{\dot{A}\dot{B}}.
\]

(3.18)

Two contractions of (3.13) yield (3.18). □

Corollary 5 Under the conditions of the proposition 1 the components of the metric (3.12) and its determinant are expressed in terms of the rational functions of the components of 2-forms $S_{AB}$ via the Urbantke formula,

\[
g_{\alpha\beta} = \frac{2}{3} *(S_B^A \wedge S_B^C \wedge S_C^A) ,
\]

(3.19)
where $S_{AB\alpha} = \partial_{\alpha} \cdot S_{AB}$,

$$3 = - *(S^{KL} \wedge S_{KL}). \tag{3.20}$$

**Proof:** Noticing that

$$S_{AB\alpha} = \theta_{\alpha} (A \dot{K} |_{(\alpha)} \theta_{B}) K \tag{3.21}$$

and using (3.5), one should apply (3.17) to the r.h.s. of (3.19). Analogously, making complete pairwise contraction of all indices in (3.17), one finds (3.20). \(\Box\)

It seems worthwhile to write (3.19) and (3.20) explicitly in terms of the $S$–forms components. Recalling that $S_{AB} = S_{AB\mu\nu} dx^\mu \wedge dx^\nu$, and hence $S_{AB\alpha} = 2 S_{AB\alpha\beta} dx^\beta$, one finds for (3.19) and (3.20), respectively,

$$\sqrt{\text{det } g_{\alpha\beta}} = - \frac{8}{3} \epsilon_{\mu\nu\rho\sigma} S^{B}_{A \alpha\mu} S_{B \rho\sigma} S^{C}_{C \nu\beta}, \tag{3.22}$$

$$3 \sqrt{\text{det } g} = - \epsilon^{\mu\nu\rho\sigma} S^{KL}_{\mu\nu} S_{KL\rho\sigma}. \tag{3.23}$$

Hence we see that indeed the metric components and its determinant are the rational functions of components of $S$–forms. One should keep in mind that the signs on the r.h.s.’s of (3.22)-(3.23) depend on the orientation chosen on $M$. With another (opposite) choice both signs would be pluses. A different, purely algebraic mechanism which changes these signs is provided by what can be called an “anti-dotting” operation. In simple terms, this means that the dotted $S$-forms $S_{\dot{A}\dot{B}}$ replace the original $S_{AB}$ in all formulas, and vice versa. This reflects the completely equivalent role of dotted and undotted $S$-forms in determining the metric structure.

The 3-forms $\eta_{\dot{A}\dot{B}}$ indeed form the basis of the space of all complex 3-forms over $M$, and this is expressed in the identity

$$\theta^{A}_{K} \wedge \theta^{B}_{L} \wedge \theta^{C}_{M} = \epsilon^{A(B \eta^{C})}_{K} \epsilon_{L|M} - \epsilon_{K(L \eta^{A})_{M}} \epsilon^{BC}. \tag{3.24}$$

The latter can be proven with the standard spinor indices manipulation tricks after noticing that the l.h.s. is antisymmetric under the permutation of any pair of indices $\{A_{K}\}, \{B_{L}\}, \{C_{M}\}$. 

9
The forms $\eta_{A\dot{B}}$ are of particular importance, because their components actually determine the inverse of the metric tensor (3.12). Indeed, let us first notice that

$$\theta^{A}_{B} \wedge \eta_{C\dot{D}} = \delta^{A}_{C} \delta_{B}^{D} \eta, \quad (3.25)$$

which follows when computing the interior product $e_{C\dot{D}}(\theta^{A}_{B} \wedge \eta)$ for the zero 5-form inside the parentheses. On the other hand, when taking explicitly the coordinate basis $e_{\alpha} = \partial_{\alpha}$ and the corresponding coordinate coframe $dx^{\alpha}$ one finds analogously $dx^{\alpha} \wedge \eta_{\beta} = \delta_{\beta}^{\alpha} \eta$ with

$$\eta_{\alpha} = \partial_{\alpha} \eta \quad (3.26)$$
as the coordinate 3-form of $\eta$-basis. Expanding $\eta_{A\dot{B}}$ with respect to the coordinate basis (3.26),

$$\eta_{A\dot{B}} = e_{A}^{\dot{B} \alpha} \eta_{\alpha}, \quad (3.27)$$

we immediately see that (3.25) yields

$$e_{A}^{\dot{B} \alpha} \theta^{C}_{\dot{D} \alpha} = \delta^{C}_{A} \delta_{\dot{D}}^{\dot{B}}. \quad (3.28)$$

From this one evidently has

$$e_{A}^{\dot{B} \alpha} \theta^{A}_{\dot{B} \beta} = \delta_{\beta}^{\alpha}, \quad (3.29)$$

and consequently, the components of the inverse metric tensor are given by

$$g^{\alpha\beta} = e_{A}^{\dot{B} \alpha} e_{B}^{\dot{A} \beta}. \quad (3.30)$$

It is straightforward to prove the identity $g^{\alpha\beta} g_{\beta\gamma} = \delta_{\gamma}^{\alpha}$ substituting the definitions (3.12) and (3.30) and (3.28)-(3.29). Contracting (3.30) with the components of the coframe, and (3.12) with that of the dual frame, one finds the useful relations,

$$g^{\alpha\beta} \theta^{A}_{\dot{B} \beta} = e_{A}^{\dot{B} \alpha}, \quad g_{\alpha\beta} e_{A}^{\dot{B} \beta} = \theta^{A}_{\dot{B} \alpha}. \quad (3.31)$$

It is clear that the components of $\eta_{A\dot{B}}$ describe the expansion of the coordinate coframe with respect to the fundamental $\theta^{A}_{B}$ coframe, namely
\[ dx^\alpha = e_A^{\dot{B} \alpha} \theta^A_{\dot{B}}. \] 

(3.32)

[Let us mention also that the frame of tangent space evidently reads \( e_A^{\dot{B} \alpha} = e_A^{\dot{B} \alpha} \partial_\alpha \), this explains the notation for the components \( e_A^{\dot{B} \alpha} \).]

It is straightforward to see that the components of the inverse metric (3.30) are, like (3.22)-(3.23), the rational functions of the \( S \)-forms components, namely,

\[ \frac{3}{4} (\text{det } g) \ g^{\alpha \beta} = -\varepsilon^{\alpha \mu \nu \gamma} \varepsilon^{\beta \rho \sigma \delta} S_A^{\ B \mu \nu} S_B^{\ C \gamma \delta} S_C^{\ A \rho \sigma}. \] 

(3.33)

It seems worthwhile to mention that unlike the relatively well known so-called Urbantke formula (3.22), the inverse metric expression (3.33) was not reported in the literature, at least to our knowledge. Both formulas play an important role in the subsequent reformulation of the vacuum Einstein equations in terms of the curvature components.

It is clear that \( g_{\mu \nu}, g^{\mu \nu} \) may be regarded as metric tensors. In general, they are complex. However, if the equation

\[ S_{AB} \wedge S_{CD} = 0 \] 

(3.34)

is fulfilled (where the overbar denotes complex conjugation), i.e. the subspaces of \( \Lambda^2 \) spanned by the triads \( S_{AB} \) and \( S_{\overline{AB}} \), respectively, are “wedge orthogonal”, then the metric is conformal to real one possessing the Lorentz signature. Moreover, if \( \Re(S^{KL} \wedge S_{KL}) = 0 \) the conformal factor reduces to one of the fourth roots of 1, i.e. \( 1, -1, i \) or \( -i \). Then the obvious constant rescaling of \( S \)-forms leads to the real Lorentzian metric.

We shall not give a proof of that claim here but mention that in the latter case \( (\text{det } g) \) is real negative. Bearing this in mind, we remark that perhaps it would be better to rewrite all the component formulas, namely, (3.22), (3.23) and (3.33), with the metric determinant in the form \( \sqrt{\text{det } g} = i \sqrt{-\text{det } g} \). The definition of the volume 4-form should be then \( \eta = \pm i^{1/3} S^{KL} \wedge S_{KL} \) instead of (3.3), with the Hodge duals in the proposition 3 modified correspondingly. Then the imaginary unit would take its usual place in the duality equations (3.18). However, the use of the complex formally Euclidean definitions simplifies all the
calculations below, and we thus will not change the original notation as soon as the reality conditions are not imposed until the very end.

The 1–forms $S_{AB\alpha}$ prove to be an extremely useful objects in the subsequent discussion of relations between $S$-forms and connection and curvature forms. The following statement summarizes some of the key properties of these forms.

**Proposition 6** Under the conditions of the proposition 1, the 1-forms $S_{AB\alpha}$ satisfy the following identities

$$S_{AB\alpha} \wedge S_{CD} = \delta_{(C}^{(A} \ast S_{D)}^{B)}\eta_{\alpha} - \frac{1}{2} \delta_{(C}^{A} \delta_{D}^{B)} \eta_{\alpha},$$  \hspace{1cm} (3.35)

$$S_{CD\alpha} \ast (dx^{\alpha} \wedge dx^{\beta} \wedge S^{AB}) = - \ast (dx^{\beta} \wedge S^{(A}(c\delta_{D}^{B)}) - \frac{1}{2} \delta_{(C}^{A} \delta_{D}^{B)} dx^{\beta},$$  \hspace{1cm} (3.36)

where, as usually in this paper, the Hodge star dual is determined by the metric defined by the $S$–forms (as summarized in (3.14)-(3.17)).

**Proof:** We start directly from the completeness condition and apply the interior product with the coordinate basis $\partial_{\alpha}$ to (3.2). This yields

$$S_{AB\alpha} \wedge S_{CD} + S_{AB\alpha} \wedge S_{CD\alpha} = - \delta_{(C}^{A} \delta_{D}^{B)} \eta_{\alpha},$$  \hspace{1cm} (3.37)

where we have used (3.3) and (3.26). Now from (3.24) we find

$$\theta_{A}^{\dot{D}} \wedge S_{BC} = \delta_{(B}^{A} \eta_{C)\dot{D}}.$$  \hspace{1cm} (3.38)

and using (3.21), one straightforwardly computes

$$S_{AB\alpha} \wedge S_{CD} - S_{AB\alpha} \wedge S_{CD\alpha} = \theta_{(A}^{(K} \delta_{\alpha}\eta_{B)}^{\dot{K}} + \theta_{(C}^{\dot{K} \delta_{D}^{A)} \wedge \eta_{\beta)^{\dot{K}}}.$$  \hspace{1cm} (3.39)

Applying Hodge operator to (3.21), one gets

$$* S_{AB\alpha} = \theta_{(A}^{K} \delta_{(\alpha}^{B)} \eta_{B)^{K}},$$  \hspace{1cm} (3.40)

and it remains only to rearrange the r.h.s. of (3.39) with the help of (3.40), with the final formula.
Combining (3.37) and (3.41), one proves the first identity (3.35).

The second identity (3.36) is simply the Hodge dual of the first one, although some efforts are required to demonstrate this explicitly. To begin with, one easily notices that since \( \eta_\alpha = *dx_\alpha \) (hereafter lower coordinate index is understood to be moved with the help of the metric \( g_{\alpha\beta} \)), the last terms on the r.h.sides of (3.35) and (3.36) are dual. As for the first terms on the r.h.sides, one derives

\[
dx_\beta \wedge S_{BC} = g_{\alpha\beta} dx^\alpha \wedge S_{BC} = g_{\alpha\beta} e_A B^A \theta^A D \wedge S_{BC} = \theta(B_{\beta}) |C_D | \eta_C D,
\]

(3.42)

where we used (3.32), (3.31), and (3.38). Thus in view of (3.40),

\[
* (dx_\beta \wedge S_{BC}) = -S_{BC\beta},
\]

(3.43)

and we have proven that the r.h.s. of (3.35) is a Hodge dual of the r.h.s. of (3.36). Now let us prove the duality of the left hand sides. We will several times make use of the well known identity

\[
\Phi \wedge * \Psi = \Psi \wedge * \Phi,
\]

(3.44)

which holds for any forms \( \Phi, \Psi \) of equal degree. As a preliminary step we notice that

\[
dx_\gamma \wedge S^{AB\beta} \wedge S_{CD} = dx_\beta \wedge S_{CD\gamma} \wedge S^{AB}.
\]

(3.45)

This is directly seen when multiplying (3.35) by the coordinate coframe 1-form from the left, and then recalling that \( dx^\gamma \wedge \eta_\beta = \delta_\beta^\gamma \eta = dx_\beta \wedge \eta_\gamma \), while using (3.44),

\[
dx_\gamma \wedge * S_{BD\beta} = S_{BD\beta} \wedge \eta_\gamma = 2S_{BD\beta_\gamma} dx^\alpha \wedge \eta_\gamma = 2S_{BD\beta_\gamma} \eta
\]

(3.46)

\[
= -dx_\beta \wedge * S_{BD\gamma}.
\]

And now we can complete the demonstration. By using repeatedly (3.44) and (3.46), we find
\[ dx_\gamma \wedge \ast [S_{CD\alpha}(dx^\alpha \wedge dx^\beta \wedge S^{AB})] = [S_{CD\alpha}(dx^\alpha \wedge dx^\beta \wedge S^{AB})] \wedge \eta_\gamma \]

\[ = 2S_{CD\alpha\gamma}(dx^\alpha \wedge dx^\beta \wedge S^{AB})\eta \]

\[ = 2S_{CD\alpha\gamma}dx^\alpha \wedge dx^\beta \wedge S^{AB} \]

\[ = dx_\beta \wedge S_{CD\gamma} \wedge S^{AB} \]

\[ = dx_\gamma \wedge [S^{AB\beta} \wedge S_{CD}], \quad (3.47) \]

where we have used (3.45) at the last line. Thus by Cartan’s lemma we see that

\[ S^{AB\beta} \wedge S_{CD} = \ast [S_{CD\alpha}(dx^\alpha \wedge dx^\beta \wedge S^{AB})], \]

and this ends the proof of the second identity (3.36). \(\Box\)

**Corollary 7** An arbitrary spinor-valued symmetric 1–form \(T_{AB} = T_{BA}\) and any 1-form \(\psi\) satisfy the identities

\[ \ast (\psi \wedge S_{AB}) \wedge S^{CD} = \psi \wedge S^{(C}_{(A} \delta^{D)}_{B)} + \frac{1}{2} \ast \psi \delta^{C}_{(A} \delta^{D)}_{B)}, \quad (3.48) \]

\[ T_{AB} = -\left( \ast (dx^\alpha \wedge U_{AC})S^{C}_{B\alpha} + \ast (dx^\alpha \wedge U_{BC})S^{C}_{A\alpha} \right) + \ast U_{AB}, \quad (3.49) \]

\[ T_{AB} \wedge S^{CD} = -U^{(C}_{(A} \delta^{D)}_{B)} + \ast (U_{AB}) \wedge S_{CD} + \ast (U^{(C}_{(A} \wedge S^{D)}_{B)} + \ast (U^{K}_{(A}) \delta^{(C)}_{B} \wedge S^{D)}_{K}), \quad (3.50) \]

with

\[ U_{AB} = T^{D}_{(A} \wedge S_{B)D}, \]

where the \(S\)–forms satisfy the conditions of the proposition 1, and the Hodge dual is defined by the \(S\)-corresponding metric.

**Proof:** Take the coefficients of the \(T\)-form in coordinate basis, \(T_{AB\beta}dx^\beta\) and compute the contraction of the identity (3.36) with \(2\delta^{D}_{(M} \epsilon^{N)}_{(A} \delta^{C}_{B)K}T^{K}_{B\beta}\). This proves (3.49), while (3.50) follows from the latter with the help of (3.35). The identity (3.48) is in fact a different form of (3.35) with the relation (3.43) inserted. \(\Box\)

Immediate consequences of (3.48)-(3.50) are
\[(\psi \wedge S_{AB}) \wedge S^{AB} = \frac{3}{2} * \psi, \quad (3.51)\]

\[T_{AB} \wedge S^{AB} = - * (T^D (A \wedge S_{BD}) \wedge S^{AB}), \quad (3.52)\]

which are identically satisfied for any symmetric 1-form \(T_{AB}\) and any 1-form \(\psi\).

**Corollary 8** Under the conditions of the proposition 1, the following cubic identity is fulfilled (cf. (3.19)),

\[S_{AB}^\alpha \wedge S_{CD} \wedge S_{EF}^\beta = \eta \left( \frac{1}{2} g_{\alpha \beta} \delta_{CD}(E) \delta_{EF} - 2 \delta_{(A} g_{\beta)(B} \delta_{CD}) \right). \quad (3.53)\]

**Proof:** Multiply the identity (3.35) by \(S_{EF}^\beta\) and then use (3.44), (3.46) and (3.35). \(\Box\)

### 4. Algebraic Relations of S- and \(\Omega\)-Forms

Let us introduce some convenient notations. We define the quadratic \((I)\) and cubic \((J)\) “totally contracted” operators over the objects \(Z^{ABCD}\) with four spinor indices which are symmetric with respect to the first and last pairs, \(Z^{ABCD} = Z^{(AB)(CD)} = Z^{(CD)(AB)}\).

Specifically, let

\[I[Z] = Z_{KL}^{MN} Z_{MN}^{KL}, \quad J[Z] = Z_{KL}^{MN} Z_{MN}^{PQ} Z_{PQ}^{KL}, \quad J_k[Z] = J[Z] - k (\text{tr}[Z])^3, \quad \text{where} \quad \text{tr}[Z] = Z_{KL}^{KL}. \quad (4.1)\]

Let us consider, preliminarily, the problem of the inverting of a \(3 \times 3\) matrix formed by the components of a totally symmetric 4-index spinor \(C^{ABCD} = C^{(ABCD)}\). (A particular example of such an object is provided by the Weyl conformal curvature spinor, see also section 5). This problem arises, in particular, in the pure connection formulation of general relativity [9] but it was only briefly discussed there.

**Proposition 9** Let the 4-index spinor \(Z_{ABCD}\) be symmetric in the first and last index pairs, \(Z_{ABCD} = Z_{(AB)(CD)} = Z_{(CD)(AB)}\), and obey the algebraic constraints.
\[ Z^K_{(AB)K} = 0, \quad I[Z] = (\text{tr}[Z])^2. \] (4.2)

Then

\[ \tilde{C}_{ABCD} = Z^K_{AB} Z^C_{KL} Z^D_{KL} - Z_{ABCD} Z^K_{KL} \] (4.3)

is totally symmetric and obeys the equations

\[ Z^K_{AB} \tilde{C}^K_{KL} = \frac{1}{3} J_1[Z] \delta^C_{(A} \delta^D_{B)}, \] (4.4)
\[ \frac{1}{3} J_1[Z] Z^K_{AB} = \tilde{C}^K_{AB} \tilde{C}^K_{KL} \delta^C_{(A} \delta^D_{B)} \] (4.5)

**Proof:** Regarding the index pairs \(AB\) and \(CD\) as multi-indices, \(Z^K_{ABCD}\) may be interpreted as a \(3 \times 3\) matrix \([Z]\). As such, it annihilates its own characteristic polynomial which equals

\[ [Z]^3 - (\text{tr}[Z])[Z]^2 - \frac{1}{2} \{\text{tr}([Z]^2) - (\text{tr}[Z])^2\} [Z] \]
\[ - \frac{1}{6} \{(\text{tr}[Z])^3 - 3 \text{tr}[Z] \text{tr}([Z]^2) + 2 \text{tr}([Z]^3)\} [I]. \]

In our case \(\text{tr}([Z]^2) = I[Z], \text{tr}([Z]^3) = J[Z],\) the unit matrix \([I] = |\delta^A_B\delta^B_D|\). Using the second of eqs. (4.2), one finds

\[ [Z] \left\{ [Z]^2 - [Z] \text{tr}[Z] \right\} = \frac{1}{3} [I] \left\{ J[Z] - (\text{tr}[Z])^3 \right\}, \]

and (4.4) immediately follows.

Next, eqs. (4.3), (4.4) yield

\[ \tilde{C}^K_{AB} \tilde{C}^K_{KL} = \frac{1}{3} J_1[Z] \left( Z^K_{AB} - Z^K_{KL} \delta^C_{(A} \delta^D_{B)} \right) \]

and hence \(I[\tilde{C}] = -\frac{2}{3} \text{tr}[Z] J_1[Z],\) that proves (4.3).

Finally, the first of eqs. (4.2) entails the total symmetry of \(\tilde{C}_{ABCD}\): it is easy to show that any contraction of the r.h.s. of (4.3) vanishes. \(\square\)

Now let us assume \(Z_{ABCD}\) to be not an arbitrary but constructed from the components of a set of 2-forms \(\Omega_{AB} = \Omega_{AB\mu\nu} dx^\mu \wedge dx^\nu\) obeying the index symmetries of \(S\)-forms: \(\Omega_{AB\mu\nu} = \Omega_{(AB)[\mu\nu]}\). Specifically, let
\[ Z_{AB}^{CD} = \Omega_{AB}^{CD} - \frac{1}{2} \Omega_{KL}^{KL} \delta_{(A}^{C} \delta_{B)}^{D}, \]  

where we denoted
\[ \Omega_{AB}^{CD} = \varepsilon^{\alpha\beta\mu\nu} \Omega_{AB\alpha\beta} \Omega_{CD}^{\mu\nu}. \]

Notice that, by construction, \( \Omega_{(AB)K} = 0 \).

Further, we introduce another set of two-forms,
\[ \tilde{S}_{AB} = Z_{AB}^{KL} \Omega_{KL} = \Omega_{AB}^{KL} \Omega_{KL} - \frac{1}{2} \Omega_{KL}^{KL} \Omega_{AB} \]  

which possess the same index symmetries as \( \Omega \)- and \( S \)-forms and are the homogeneous cubic polynomials in the components \( \Omega_{AB\mu\nu} \).

**Proposition 10** If the components \( \Omega_{AB\mu\nu} \) obey the constraints
\[ 2\Omega_{AB}^{CD} \Omega_{CD}^{AB} = (\Omega_{KL}^{KL})^2, \]  

\[ J_1[\Omega] = \Omega_{AB}^{CD} \Omega_{CD}^{PQ} \Omega_{PQ}^{AB} - \frac{1}{4}(\Omega_{KL}^{KL})^3 \neq 0, \]  

then the 2-forms \( \tilde{S}_{AB} \) satisfy the conditions of the proposition 1, namely
\[ \tilde{S}_{AB} \wedge \tilde{S}_{CD} = \frac{1}{3} \delta^A_{(C} \delta^B_{D)} \tilde{S}_{KL} \wedge \tilde{S}_{KL}, \quad \tilde{S}_{KL} \wedge \tilde{S}_{KL} \neq 0. \]

**Proof:** Equation \( (4.9) \) guarantees the satisfaction of the conditions \( (4.2) \) of the proposition 9 for \( (4.6) \). The corresponding totally symmetric spinor reads
\[ \tilde{C}_{CD}^{KL} = \Omega_{CD}^{MN} \Omega_{MN}^{KL} - \frac{1}{2} \Omega_{CD}^{KL} \Omega_{PQ}^{PQ} \]  

\[ = Z_{CD}^{MN} \Omega_{MN}^{KL} \]  

In view of \( (4.4) \) and \( (4.12) \) we find
\[ \tilde{S}_{AB} \wedge \tilde{S}_{CD} = \frac{1}{3} \delta^A_{(C} \delta^B_{D)} \ J_1[Z] dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \]  

It is easy to see that \( tr[Z] = -\frac{1}{2} \Omega_{KL}^{KL}, \ I[Z] = \frac{1}{4}(\Omega_{KL}^{KL})^2, \) and \( J_1[Z] = J_1[\Omega], \) hence...
\[ S^{KL} \wedge \bar{S}_{KL} = J_\frac{1}{4}[\Omega]dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \neq 0. \] (4.14)

This ends the proof. □

These results solve the problem of inverting the relation (4.8). The inverse reads

\[ \frac{1}{3} J_\frac{1}{4}[\Omega] \Omega_{AB} = \bar{C}_{AB}^{KL} \bar{S}_{KL}. \] (4.15)

For further applications we now introduce new objects

\[ S_{AB} = e^\phi \bar{S}_{AB}, \quad C_{ABCD} = 3e^{-\phi}(J_\frac{1}{4}[\Omega])^{-1}\bar{C}_{ABCD} \] (4.16)

with some yet unspecified scalar \( \phi \). Obviously,

\[ \Omega_{AB} = C_{AB}^{KL} S_{KL}. \] (4.17)

Notice that the 2-forms \( S_{AB} \) evidently also satisfy the conditions of the proposition 1 with

\[ S^{KL} \wedge S_{KL} = e^{2\phi} J_\frac{1}{4}[\Omega] dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \] (4.18)

Let us give also the following useful higher order identities:

\[ \Omega_{AB}^{KL} \bar{C}_{KL}^{CD} = \frac{1}{2} \text{tr}[\Omega] \bar{C}_{AB}^{CD} + \frac{1}{3} J_\frac{1}{4}[\Omega] \delta_C^{(A} \delta_D^{B)}, \] (4.19)

\[ \Omega_A^{KL} B \bar{C}_{KL}^{CD} = \frac{1}{3} J_\frac{1}{4}[\Omega] \delta_C^{(A} \delta_D^{B)}, \] (4.20)

\[ \bar{C}_{AB}^{KL} \bar{C}_{KL}^{CD} = \frac{1}{3} J_\frac{1}{4}[\Omega] \Omega_{AB}^{CD}, \] (4.21)

\[ \bar{C}_{AB}^{KL} \bar{C}_{KL}^{MN} \bar{C}_{MN}^{CD} = \frac{1}{3} J_\frac{1}{4}[\Omega] \left( \frac{1}{2} \text{tr}[\Omega] \bar{C}_{AB}^{CD} + \frac{1}{3} J_\frac{1}{4}[\Omega] \delta_C^{(A} \delta_D^{B)} \right). \] (4.22)

From these one finds

\[ I[\bar{C}] = \frac{1}{3} J_\frac{1}{4}[\Omega] \text{tr}[\Omega], \quad J[\bar{C}] = \frac{1}{3} (J_\frac{1}{4}[\Omega])^2, \quad J[C] = 9e^{-3\phi}(J_\frac{1}{4}[\Omega])^{-1}. \] (4.23)
5. LORENTZIAN GEOMETRY IN TERMS OF S-FORMS: BASIC EQUATIONS

Here we briefly outline the method of describing the 4-dimensional Lorentzian geometry in terms of special sets of $S$-forms. Our consideration is everywhere local.

As it was shown in the previous sections, the metrical properties of the Lorentzian 4-dimensional space-time can be exhaustively described by the triad of complex spinor-valued 2-forms $S_{AB} = S_{BA} = S_{AB}^\mu dx^\mu \wedge dx^\nu$ which obey

(i) the non-degeneracy condition (3.1)

(ii) the completeness condition (3.2) and

(iii) the reality conditions

$$S_{AB} \wedge \overline{S_{CD}} = 0, \quad \Re(S^{KL} \wedge S_{KL}) = 0. \quad (5.1)$$

The very metric can be restored from the $S$-forms by means of the Urbantke equations (3.22), (3.23), (3.33). However such basic characteristics of the geometry as the connection and the curvature do not require the immediate use of the metric tensors and can be completely described in terms of the $S$-forms alone. In particular, the symmetric (torsion-free) metric-compatible connection is described by the complex-valued 1-forms $\Gamma_{AB} = \Gamma_{BA}$ obeying the first structure equations

$$dS_{AB} + 2\Gamma^{K}_{(A} \wedge S_{B)K} = 0. \quad (5.2)$$

The complex-valued 2-forms $\Omega_{AB} = \Omega_{BA}$ of the the curvature associated with connection $\Gamma_{AB}$ are expressed in terms of the latter as follows (the second structure equations):

$$\Omega^{A}_{B} = d\Gamma^{A}_{B} + \Gamma^{K}_{B} \wedge \Gamma^{A}_{K}. \quad (5.3)$$

Curvature and connection forms satisfy also the Bianchi identities

$$d\Omega_{AB} + 2\Gamma^{K}_{(A} \wedge \Omega_{B)K} = 0. \quad (5.4)$$
As it is well known, the *vacuum Einstein equations* are equivalent to the existence of the expansion (cf., e.g., [12,5,9])

$$\Omega_{AB} = Y_{AB}{}^{KL} S_{KL}$$

(5.5)

for some coefficients $Y_{ABKL} = Y_{(AB)(KL)}$. The latter can be written

$$Y_{AB}{}^{CD} = \frac{1}{2} \Psi_{AB}{}^{CD} - \frac{1}{12} \delta^C_D \delta^D_A R$$

(5.6)

in terms of the totally symmetric Weyl spinor (spinor of conformal curvature) $\Psi_{ABCD} = \Psi_{(ABCD)} (= 2C_{ABCD})$ and the scalar curvature $R$ which is constant by virtue of the vacuum version of Bianchi identities. It follows that $Y^K_{(AB)K} = 0$. We shall assume $R = 0$ below, therefore restricting our consideration to the case of the vanishing cosmological term. Then the relation (4.17), more restrictive than (5.5), takes place.

Assuming that $\Gamma_{AB}$ and $\Omega_{AB}$ are arbitrary symmetric 1- and 2-forms (i.e., they do not necessarily satisfy the eqs. (5.3) and (5.4)), let us introduce the following differential operators

\[
\mathcal{I}^A_B = \mathcal{I}^A_B \alpha \beta \omega d\omega^\alpha \wedge d\omega^\beta = d \Gamma^A_B + \Gamma^K_B \wedge \Gamma^A_K - \Omega^A_B, \tag{5.7}
\]

\[
\mathcal{B}_{AB} = \mathcal{B}_{AB \alpha \beta \gamma} d\omega^\alpha \wedge d\omega^\beta \wedge d\omega^\gamma = d \Omega_{AB} + 2 \Gamma^K_{(A} \wedge \Omega^K_{B)K}. \tag{5.8}
\]

As it is easily seen, the following identities

\[
d \mathcal{B}_{AB} + 2 \Gamma^K_{(A} \wedge \mathcal{B}_{B)K} \equiv 2 \mathcal{I}^K_{(A} \wedge \Omega_{B)K}, \tag{5.9}
\]

\[
d \mathcal{I}^A_B + \Gamma^K_B \wedge \mathcal{I}^A_K - \Gamma^K_L \wedge \mathcal{I}^L_B \equiv - \mathcal{B}^A_B \tag{5.10}
\]

hold true for any $\Gamma^A_B$ and $\Omega_{AB}$.

Further, assuming the fulfillment of the first structure equation (5.2), one obtains

$$\mathcal{I}^K_{(A} \wedge S_{B)K} \equiv 0,$$

(5.11)

*provided* the equation $\Omega^K_{(A} \wedge S_{B)K} = 0$ is fulfilled (that holds true in particular in the case of the relation (5.5)).
6. CLOSED FORM OF VACUUM EINSTEIN EQUATIONS

At first, it would be convenient to define two cubic $G$ operators in a class of the symmetric spinor-valued 2-forms which include, in particular, the objects described above as the $S$-forms (cf. eqs. (3.22), (3.33)):

$$G[S]_{\alpha\beta} = \varepsilon^{\mu\rho\sigma} S^B_{\alpha\mu} S^C_{\beta\rho} S^A_{\mu\sigma}, \quad (6.1)$$

$$G[S]^{\alpha\beta} = \varepsilon_{\alpha\mu\gamma} \varepsilon^{\beta\rho\delta} S^B_{\mu\nu} S^C_{\gamma\delta} S^A_{\rho\sigma}. \quad (6.2)$$

Notice that the eqs. (6.1)-(6.2) specify the tensor densities with respect to coordinate transformations.

In accordance with the eqs. (3.19), (3.20), components of the metric tensor can be directly expressed via the corresponding $S$-forms. On the other hand if the vacuum Einstein equations are fulfilled then the $S$-forms and the curvature $\Omega$-forms are closely connected by means of the simple equation (4.17). It may be supposed therefore that the metric can be expressed algebraically in terms of the curvature forms as a rational function of components of latter. Such an algebraic problem was investigated in [22–24], proving this conjecture to be true in a wide class of curvature structures. In our approach this is true provided the generic condition $J_4[\Omega] \neq 0$ is fulfilled, which is equivalent to the non-vanishing of the cubic invariant $J[\Psi]$ of the undotted Weyl spinor. However, the algebraic relations alone do not completely suffice to determine the metric from the curvature, and a differential equation is to be solved to obtain a scalar scaling factor.

To demonstrate how the curvature is connected with the metric in the vacuum case let us calculate the cubic densities $G[\Omega]_{\mu\nu}$ and $G[\Omega]^{\mu\nu}$. They can be easily found using eqs. (3.53) and (4.22). In the case of totally symmetric $Y^{ABCD} = C^{ABCD}$, eq. (3.53) implies

$$G[\Omega]_{\mu\nu} = \frac{1}{3} J[C] G[S]_{\mu\nu}, \quad G[\Omega]^{\mu\nu} = \frac{1}{3} J[C] G[S]^{\mu\nu}. \quad (6.3)$$

At the same time, proposition 10 tells us that the 2-forms $\tilde{S}_{AB}$, which components are directly constructed from the curvature (4.8), also define a metric on $M$. We will denote
this auxiliary metric $\tilde{g}_{\alpha\beta}$, and the corresponding Hodge operator will be also denoted by the tilde, $\tilde{\ast}$.

In accordance with eqs. (4.14), (4.23), (4.18), one finds

\begin{equation}
  g_{\mu\nu} = e^{\phi} \tilde{g}_{\mu\nu}, \quad g^{\mu\nu} = e^{-\phi} \tilde{g}^{\mu\nu},
\end{equation}

\begin{equation}
  \tilde{g}_{\mu\nu} = \frac{8}{3} G_{[\Omega]_{\mu\nu}}, \quad \tilde{g}^{\mu\nu} = -4(J_{\frac{1}{4}[\Omega]})^{-1} G_{[\Omega]^{\mu\nu}}.
\end{equation}

As we see, the $S$-forms (and the space-time metric $g$) cannot be completely determined in a purely algebraic way in terms of the curvature forms from eq. (4.17) because both $C_{ABKL}$ and $S_{KL}$ are unknown. The degree of the corresponding ambiguity is however expressed by a single scalar function (generally complex) $\phi$, which is reflected in the fact that the eqs. (6.4) involve the yet unspecified factors $e^{\pm \phi}$. We shall see that $\phi$ can be fixed but differential equations rather than algebraic ones have to be drawn here. Specifically, these additional equations are the consequence of the of the Bianchi equations.

A possible way of solving the problem of finding the local geometry of a generic vacuum space-time from its curvature is described as follows.

Assuming the fulfillment of conditions (i)-(iii) of the section 5 we are precisely in a position of the proposition 1 (see section 3) and may exploit eqs. (3.35), (3.36) (as well as further algebraic relations given in sections 3,4). In particular, the straightforward application of the identity (3.49), with an arbitrary 1-form $T_{AB}$ replaced by the connection, allows to show that the only solution of the eq. (5.2) with respect to the connection forms $\Gamma_{AB}$ is described by the formula (see earlier discussion in [26]):

\begin{equation}
  \Gamma_{AB} = -\ast(d\alpha^\alpha \wedge dS^K (A)S_B)_{K\alpha} - \frac{1}{2} \ast dS_{AB}.
\end{equation}

Using (3.49)-(3.48) and (4.19)-(4.23), (after some lengthy algebra) one finds

\begin{equation}
  2C^{ABCD} \ast (\Gamma^K (A \wedge \Omega_B)_{K}) \wedge \Omega_{CD} = \frac{1}{3} J[C] \Gamma_{AB} \wedge S^{AB},
\end{equation}

and hence from (3.52) and the structure equations (5.2) we get an immediate consequence of the Bianchi identities (5.4),

\begin{equation}
  2C^{ABCD} \ast (d\Omega_{AB}) \wedge \Omega_{CD} + \frac{1}{3} J[C] \ast (dS_{AB}) \wedge S^{AB} = 0.
\end{equation}
Substituting $C_{ABCD}$ and $S_{AB}$ expressions provided by the eqs. (4.16), and using (4.23), one reduces the last equation to

$$d\phi = \Phi[\Omega],$$

(6.7)

where

$$\Phi[\Omega] := \frac{2}{3} \tilde{*} \left( 2 \tilde{C}^{ABCD} \ast (d\Omega_{AB}) \wedge \Omega_{CD} + \tilde{*} (d\tilde{S}_{AB}) \wedge \tilde{S}^{AB} \right).$$

(6.8)

Here one should keep in mind that $\tilde{C}_{ABCD}$ and $\tilde{S}_{AB}$ are determined by the curvature according to the eqs. (4.12), (4.8), and all the Hodge duals $\tilde{*}$ are also defined by the curvature with the help of the auxiliary metric $\tilde{g}$ which is explicitly constructed as the rational function of the curvature components in accordance with (6.5).

A remarkable feature of eq. (6.7) is thus that besides $d\phi$ the only functions involved in it are the $\Omega$-forms components. This differential equation fixes the conformal factor $\phi$.

We can now finalize the work, rewriting explicitly all the objects and relations in terms of the curvature. Using the equations (4.16), (6.7) and (3.51), one finds the closed expression of the connection forms in terms of the curvature components and their derivatives,

$$\Gamma_{AB} = \frac{1}{2} \ast \left( \Phi[\Omega] \wedge \tilde{S}_{AB} \right) - \ast (d\alpha \wedge \tilde{S}^{K}_{A} (A) \tilde{S}_{B}) - \frac{1}{2} \ast (d\tilde{S}_{AB}).$$

(6.9)

A similar transformation (in which (3.51) plays a central role) re-casts vacuum Bianchi equations (5.4) to the form expressed in terms of the curvature components alone,

$$\nabla [\Omega]_{AB} \equiv d\Omega_{AB} + \frac{1}{2} \Phi[\Omega] \wedge \Omega_{AB} - \ast (d\tilde{S}^{K}_{A} (A) \wedge \Omega_{BK})
+ \frac{3}{2} \tilde{J}_{4}[\Omega] \left( \tilde{C}_{ABKL} d\tilde{S}_{KL} - 2 \ast (d\tilde{S}^{KL}) \wedge \tilde{S}^{M}_{A} (A) \tilde{C}_{B} KLM \right) = 0.$$

(6.10)

The above facts are summarized in the form of a theorem.

**Theorem 11** In case of non-zero cubic invariant of the conformal curvature $J[\Psi] = \Psi^{CD}_{AB} \Psi^{EF}_{CD} \Psi^{EF}_{AB}$ the vacuum Einstein equations (with zero cosmological term) can be presented in the closed form in terms of the components $\Omega_{AB\mu}$ of the curvature 2-forms $\Omega_{AB} = \Omega_{AB\mu} d\mu \wedge d\nu$. The complete set of equations is separated into the following families:
(A) *algebraic constraints*

\[ 2\Omega^{AB}_{\quad CD} \Omega^{CD}_{\quad AB} = (\Omega^{AB}_{\quad AB})^2, \quad J_{\frac{1}{4}}[\Omega] \neq 0, \]

where \( \Omega^{AB}_{\quad CD} = \varepsilon^{\alpha\beta\mu\nu} \Omega_{A\alpha\beta} \Omega_{C\mu\nu}, \) and \( J_{\frac{1}{4}}[\ ] \) is defined by (4.1);

(B) *second order scaling equation*

\[ d(\Phi[\Omega]) = 0, \]

where \( \Phi[\Omega] \) is defined by (6.8) with \( \tilde{C}_{ABCD} \) defined by (4.12), and \( \tilde{S}_{AB} \) defined by (4.8);

(C) *first order Bianchi equations* (6.10) with the Hodge duals \( \tilde{*} \) defined by the auxiliary metric (6.3) where \( G[\ ]_{\mu\nu}, G'[\ ]^{\mu\nu} \) are defined by (6.2), (6.1);

(D) *second order structure equations*

\[ II[\Omega]_{\quad AB} \equiv d\Gamma[\Omega]_{\quad AB} + \Gamma[\Omega]_{\quad K B} \wedge \Gamma[\Omega]_{\quad A K} - \Omega_{\quad AB} = 0, \]

where \( \Gamma[\Omega]_{\quad AB} \) is defined by (6.9);

(E) *non-holomorphic reality conditions*

\[ \Omega_{\quad AB} \wedge \overline{\Omega}_{\quad CD} = 0, \quad 3\left(2\Phi[\Omega] + (J_{\frac{1}{4}}[\Omega])^{-1}dJ_{\frac{1}{4}}[\Omega]\right) = 0. \]

With the item (E) dropped out, the conditions (A)–(D) discriminate a complex vacuum solution of Einstein equations.

It can be seen that all the equations mentioned in theorem are invariant with respect to the two transformation groups: the group of general coordinate transformations and the group \( SL(2, C) \) of spinorial transformations (essentially, its 2-fold covering group \( SO(3, C) \) which is isomorphic to the special orthochronous Lorentz group).

*Proof of theorem:* The scaling equation (B) implies a local existence of the scalar \( \phi \) such that \( d\phi = \Phi[\Omega] \), determining it up to an arbitrary complex constant. By virtue of the algebraic constraints, the eqs. (4.6), (4.8), (4.16) determine a family of 2-forms \( S_{AB} = S_{AB\mu\nu} \, dx^\mu \wedge dx^\nu \) which in accordance with the proposition 10 satisfy the conditions of
proposition 1, i.e. are the $S$-forms. Moreover it can be shown that equations (C),(D) of theorem are nothing else but the eqs. (5.4), (5.3), respectively, for the connection form specified by (5.6), thus automatically obeying (5.2).

Concerning the basic field equations listed in the section 3 it remains to discuss the reality condition $S_{AB} \wedge S_{CD} = 0$, $\Re(S_{KL} \wedge S^{KL}) = 0$. The first of them is equivalent to the first equation of reality condition (E) of theorem since $S_{AB}$ and $\Omega_{AB}$ span the same subspaces of the complex valued 2-forms space. Further, the second of equations (E) is integrated by virtue of the scaling equation and then exponentiated to

$$\Im\left(e^{i(\text{real constant})} \cdot e^{2\phi} J_{\frac{1}{4}}[\Omega]\right) = 0.$$  

Notice now that all the equations of theorem are invariant with respect to the shift $\phi \rightarrow (\phi + \text{complex constant})$ that allows to re-cast the above equation to $\Im(e^{2\phi} J_{\frac{1}{4}}[\Omega]) \equiv \Re(e^{2\phi} J_{\frac{1}{4}}[\Omega]) = 0$ which in its turn coincides with the second reality condition (5.1). Theorem is therefore proven. \square

It is worthwhile to note that the theorem does not claim a solution of the equations listed above immediately yields a real metric (by means of the eqs. (5.5)). Indeed, it can be seen from the proof that such a metric may be only conformal to a real Lorentzian one but the conformal factor is necessarily a constant (all the equations of the theorem are invariant with respect to the constant conformal rescaling). Then a certain complex shift of the scalar $\phi$ has to be applied to provide a real metric. This ambiguity seems however to be unessential and so the equations are equivalent to the original vacuum Einstein equations.

7. CONSISTENCY CONDITIONS

The equations listed in theorem 11 are not totally independent but manifest some differential and algebraic relations. The latter represent the adapted form of the general identities (5.9)-(5.11) in fact. It is useful to give an explicit form of such a consistency conditions of the vacuum Einstein equations which are important in the construction of the evolution system and for the counting of the number of degrees of freedom of the field.
It has been mentioned that (5.9) and (5.10) hold true for arbitrary $\Gamma_{AB}$ and $\Omega_{AB}$. In particular, the connection in terms of the curvature $\Gamma[\Omega]_{AB}$, given by (5.9), may be applied. Then the identity (5.10) reads

\[ dII[\Omega]_{AB} + 2\Gamma[\Omega]^K_{(A \wedge II)B}K + B[\Omega]_{AB} = 0. \] (7.1)

We see that the fulfillment of (D) yields (C) of the theorem 11. It is worth noting that the algebraic constraints (A) are always assumed to be true, which is necessary for the derivation of (7.1) from (5.10).

Similarly, it follows from (5.9)

\[ dB[\Omega]_{AB} + 2\Gamma[\Omega]^K_{(A \wedge B)[\Omega]B}K = 2II[\Omega]^K_{(A \wedge \Omega)B}K, \] (7.2)

In contrast to (7.1), the above equation does not imply $II[\Omega]_{AB} = 0$ whenever the Bianchi equations $B[\Omega]_{AB} = 0$ are satisfied, but rather a less restrictive constraint $II[\Omega]^K_{(A \wedge \Omega)B}K = 0$ is entailed. Nevertheless some equations from the closed system become linear dependent.

Further, the way of introduction of the operator $\Phi[\Omega]$ implies the additional identity

\[ \tilde{C}^{ABCD}\tilde{\ast}(B[\Omega]_{AB}) \wedge \Omega_{CD} = 0, \] (7.3)

which in its turn entails, by virtue of (7.1), the closed linear homogeneous equation restricting possible values of $II[\Omega]_{AB}$ with $\Omega$’s obeying algebraic constraints (A):

\[ \tilde{C}^{ABCD}\tilde{\ast}(dII[\Omega]_{AB} + 2\Gamma[\Omega]^K_{(A \wedge II)B}K) \wedge \Omega_{CD} = 0. \] (7.4)

A consequence of the last identity (5.11) is a little subtle. In contrast to (5.9), (5.10) which are true for arbitrary $\Gamma, \Omega$, the eq. (5.11) is valid only if the first structure equations (5.2) are fulfilled. In the framework of the current section, the connection (6.6) obeys the structure equations (5.2) automatically but the expression (5.9) does not, in general. However it is easy to see that $\Gamma[\Omega]_{AB}$ still obeys (5.2) provided $\Phi[\Omega] = d\phi$ for some (arbitrary) function $\phi$ and $S_{AB} = e^\phi \tilde{S}_{AB}$, the latter relation being regarded here as the definition of its l.h.s. In such a case therefore $II[\Omega]^K_{(A \wedge \tilde{S})BK} = 0$ for arbitrary $\phi$. On the other hand, from
the formal point of view this equation could fail in general case, that is after the replacement
\( d\phi \rightarrow \Phi[\Omega] \), only because then \( d\Phi[\Omega] \), unlike \( dd\phi = 0 \), does not vanish identically. Thus the
restoring of \( \Phi[\Omega] \) in place of \( d\phi \) in the adapted version of the identity (5.11) introduces the
additional term proportional to \( d\Phi[\Omega] \). The latter can easily be calculated yielding the final
identity

\[
- 3d\Phi[\Omega] \wedge \tilde{S}_{AB} + 2\mathbb{II}[\Omega]^{K}(A \wedge \tilde{S}_{B})K = 0.
\]  

(7.5)

Note that if the structure equations \( \mathbb{II}[\Omega]_{AB} = 0 \) are fulfilled then \( \Phi[\Omega] \) obeys the complexified
source-free Maxwell-type equations.

We have proven

**Proposition 12** If the algebraic constraints (A) of theorem 11 are fulfilled then the
l.h.s.’s of the equations (B)-(D) obey a series of identities presented by (7.1)–(7.5).

8. DISCUSSION

The physical importance of space-time curvature which distinguishes the gravitational
and purely inertial effects suggests that the components of the curvature, rather than that
of the metric or connection, should be interpreted as the mathematical representative of
the “true” gravitational field. Then the problem arises to describe the main physical and
geometrical structures in terms of this fundamental object. The first tractable case of a
significant physical interest is the one of the absence of extended sources of gravity, i.e. the
case of a vacuum space-time which was considered above.

In the case of a non-zero cubic conformal curvature invariant, a generic vacuum Einstein
space-time curvature is endowed with an extremely simple algebraic structure. The crucial
point is the use of a special family of 2-forms, named above \( S \)-forms, which span the subspace
of anti-self-dual elements of the complexified 2-forms space and are fixed up to \( SO(3, C) \)
group transformations. A remarkably simple quartic constraint imposed on the components
of the (anti-self-dual complex of) curvature 2-forms (see item (A) of the theorem 11) is
necessary for the latter to be associated with a vacuum geometry. If it is fulfilled, the metric can be restored from the curvature components up to a conformal factor (see the eq. (6.3)), with the conformal metric components being homogeneous rational functions of the components of curvature 2-forms.

Further, the conformal factor $\phi$, which is necessary for the complete determination of the geometry, is calculated from the equation $d\phi = \Phi[\Omega]$ formulated in terms of curvature components. The latter equation is automatically consistent due to the Bianchi identities.

Keeping these basic relations in mind, it is straightforward to deduce a complete set of the vacuum gravitational field equations (including the Bianchi equations) in a closed form in terms of the curvature components alone. These are listed in theorem 11. An important intermediate step, which seems to be worthwhile mentioning, is the derivation of an explicit closed representation for the connection 1-forms in terms of the curvature components (eq. (6.9)).

The auxiliary metric $\tilde{g}$ with the relevant Hodge operator $\tilde{*}$ turn out to be a convenient technical tool which enables the complete reformulation of the vacuum Einstein theory in terms of the curvature. Of more fundamental importance is the general formalism of $S$-forms which is very helpful, in particular, in discussing exact solutions of the gravitational field equations. The present paper contains the formal general framework. Its applications to the study of exact solutions will be considered elsewhere.

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