Solution of inverse fractional Fisher’s equation by differential quadrature method

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Abstract. This work is an attempt to solve the inverse problem on fractional Fisher’s equation. A method comprising of Lubich’s approach to discretize the time fractional derivative and differential quadrature method with modified B-spline basis function to approximate the space derivatives is proposed to find the numerical solution of the equation. A stable numerical solution is obtained for this problem and then a comparison is made with the existing results. The obtained results are presented in form of tables and figures. The proposed method can be applied to similar fractional equations.

1. Introduction
For a completely known physical system, the mathematical description of its uniqueness, stability, existence of a solution etc. and all the parameters must be known. But if one of the unknown parameters describing this system is to be found from some extra information, then that kind of problem is called an inverse problem. In literature, there are mainly three categories of inverse problem viz. coefficient inverse problems, boundary value inverse problems and evolutionary inverse problems. In this paper we intend to discuss the boundary value inverse problem on fractional Fisher equation.

The Fisher’s equation is a reaction diffusion equation which represents the problem of biological invasion. It finds its place in ecology, physiology, and in general phase transition problems etc. and is considered a population growth model. Its equation is given as

\[
\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + \alpha u(x,t)(1 - \frac{u(x,t)}{u_\infty})
\]  

(1)

where \( u \) is the population, \( x \) is the space coordinate, \( t \) is time coordinate and \( D, \alpha, u_\infty \) are positive constant parameters that represent diffusivity, growth rate and number of individuals respectively.

Using the following transformations:

\[
\tau = \alpha t
\]

\[
z = x \sqrt{\frac{\alpha}{D}}
\]

and

\[
v = \frac{u(x,t)}{u_\infty}
\]
its much popular dimensionless form is given as:

$$\frac{\partial v(z, \tau)}{\partial \tau} = \frac{\partial^2 v(z, \tau)}{\partial z^2} + v(1-v)$$

Writing the time derivative as Caputo’s fractional derivative, one easily gets the fractional form of Fisher’s equation:

$$\frac{\partial^\alpha v(z, \tau)}{\partial \tau^\alpha} = \frac{\partial^2 v(z, \tau)}{\partial z^2} + v(1-v)$$

In the recent past the fractional differential equations have proved to be better models [3]. Fractional partial differential equations (FPDEs) have generated quite interest in mathematics fraternity. Besides modeling the physical, biological and chemical processes [3,11], they have applications in sampling, signal processing etc. There are various analytic and numerical methods [2,4,5,6] for solving nonlinear FPDEs. In this manuscript, the focus is to solve the inverse problem of fractional Fisher’s equation using additional information. In the next section of the manuscript, the inverse problem has been formulated followed by the various discretizations used. Fourth and fifth section is about the algorithm of direct and inverse problem respectively. The subsequent section contains three numerical examples to check the methodology.

2. Inverse problem

Inverse problems came into picture around twentieth century. It is a mathematical problem in which some unknown fact can be found out with the help of information in hand. This unknown element can be a coefficient, a boundary condition or an initial condition primarily. In this work, the inverse problem is to find the unknown boundary condition. The problem is solved in two parts, wherein the first part is a direct problem defined on one part say \(P\) of the domain and the inverse problem is then defined on the second say \(\bar{P}\) as mentioned below. In this work, we consider time fractional inverse Fisher’s equation of order \(\alpha\), \(0 \leq \alpha \leq 1\) for the function \(u(x,t)\) as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t)] + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T$$

with initial condition

$$u(x,0) = a(x), \quad 0 \leq x \leq 1, \quad (3)$$

and boundary conditions

$$u(x^@,t) = b(t), \quad u(1,t) = c(t), \quad 0 < x^@ \leq 1, \quad 0 \leq t \leq T$$

(4)

where \(a(x), b(t), c(t)\) are known functions and \(a(x)\) is continuous while \(b(t)\) and \(c(t)\) are infinitely differentiable and \(T\) is the known time level. The function \(u(x,t)\) and boundary value \(u(0,t)\) is to be determined. This problem can be bifurcated on two sub-parts of the domain i.e. \(P = \{x: 0 \leq x \leq x^@\}\) and on \(\bar{P} = \{x: x^@ \leq x \leq 1\}\).

The direct problem is stated as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t)] + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T$$

with initial condition

$$u(x,0) = a(x), \quad x^@ \leq x \leq 1,$$

and boundary conditions

$$u(x^@,t) = b(t), \quad u(1,t) = c(t), \quad 0 < x^@ \leq 1, \quad 0 \leq t \leq T$$

(5)
Using the discretizations as in Table 1, the equation 2 can be written in the following form:

\[
\frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{1}{\tau^\alpha} \sum_{k=1}^{N} \int_{t_{n-k+1}}^{t_{n}} u(x, t_k) - \frac{1}{\tau^\alpha} \sum_{k=1}^{n} \int_{t_{n-k+1}}^{t_{n}} u(x, t_k) \, dt_k
\]

with initial condition

\[
u(x, 0) = a(x), \quad 0 \leq x \leq x^* \quad 0 \leq t \leq T
\]

and boundary conditions

\[
u(x^*, t) = b(t), \quad 0 \leq t \leq T
\]

In this manuscript, the direct problem is solved with the help of differential quadrature method (DQM) using B-spline [29] and comparison has been made with the solutions obtained by DQM using Chebyshev-Gauss-Lobatto (CGL) points [25] and the inverse problem is solved using fourth order accurate compact approximation finite difference method.

3. The discretizations

Consider a uniform mesh of size \( h \) and \( \tau \) on \( x \) and \( t \) axes respectively such that \( h = \frac{1}{N} \) and \( \tau = \frac{T}{N} \). The grid points are defined by

\[
x_i = ih_i, \quad i = 1, 2, ..., N
\]

\[
t_k = k\tau, \quad k = 1, 2, ..., N
\]

Let \( u_i^k \approx u(x_i, t_k) \) be the numerical approximation and \( x^* \) be arbitrarily any point in the grid, according to which we bifurcate the \( x \)-axis into two parts. As \( x^* \) is an interior point, let \( x^* = mh = x_m \) for some integer \( 2 \leq m \leq N - 1 \). The direct problem and then the inverse problem are solved on the suitable parts. The solution of inverse problem [27, 28] depends upon the solution of direct problem. For the direct problem, the second order derivative is discretized using the DQM [29] and for inverse problem, fourth order accurate compact approximation finite difference method [30] is used to discretize the second order derivative. In both cases, Caputo’s definition of fractional derivative has been used and is discretized using Lubich’s approach as in [10, 19, 29].

The symbols have usual meanings as described in the reference [29].

4. Solution of the direct problem

Using the discretizations as in Table 1, the equation 2 can be written in the following form:

\[
\frac{1}{\tau^\alpha} \sum_{k=1}^{N} I^{\alpha N-k+1} u_i^k - \frac{1}{\tau^\alpha} \sum_{k=1}^{N} I^{\alpha N} u_i^k = \sum_{j=1}^{N} b_{ij} u_j^N + u_i^N (1 - u_i^N) + f_i^N
\]

### Table 1. Discretization of derivatives

| Direct / Inverse problem | Discretization |
|--------------------------|----------------|
| Direct                   | \( \frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{1}{\tau^\alpha} \sum_{k=1}^{N} \int_{t_{n-k+1}}^{t_{n}} u(x, t_k) - \frac{1}{\tau^\alpha} \sum_{k=1}^{n} \int_{t_{n-k+1}}^{t_{n}} u(x, t_k) \, dt_k \) |
| Inverse                  | \( \frac{\partial^\alpha u}{\partial t^\alpha} \approx \frac{1}{\tau^\alpha} \sum_{k=1}^{n} \int_{t_{n-k+1}}^{t_{n}} u(x, t_k) - \frac{1}{\tau^\alpha} \sum_{k=1}^{N} \int_{t_{n-k+1}}^{t_{n}} u(x, t_k) \, dt_k \) |

The inverse problem is then defined as

\[
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t)[1 - u(x, t)] + f(x, t), \quad 0 \leq x \leq x^*, \quad 0 \leq t \leq T
\]

with initial condition

\[
u(x, 0) = a(x), \quad 0 \leq x \leq x^*\)

and boundary conditions

\[
u(x^*, t) = b(t), \quad 0 \leq t \leq T
\]
RAFAS compared the results with the DQM with CGL points. The method presented above can be used to solve inverse problems and some examples are given to illustrate its application.

### 5. Solution of inverse problem

The inverse problem of the fractional Fisher’s equation is defined in equation (2) and on applying the discretizations as mentioned in Table 1, the following formulation is obtained:

\[
\frac{1}{\tau^\alpha} \sum_{k=1}^{N} l_k^\alpha u_l^{N-k+1} - \frac{1}{\tau^\alpha} \sum_{k=1}^{N} l_k^\alpha u^l_1 = \frac{6}{5} u_{l+1}^{k+1} - \frac{2}{h^2} u_l^{k+1} + u_l^{k-1} + f_i^{k+1}
\]  

(8)

Varying \(i\) from 2 to \(m-1\) equation (2) results in the matrix system \(AU = B\) where

\[
A = \begin{bmatrix}
\tau^\alpha & -b_{(m+1)3} \tau^\alpha & \ldots & b_{(m+1)(N-1)} \tau^\alpha \\
-b_{(m+2)2} \tau^\alpha & \tau^\alpha & \ldots & b_{(m+2)(N-1)} \tau^\alpha \\
\vdots & \ddots & \ddots & \vdots \\
-b_{(N-1)2} \tau^\alpha & \ldots & -b_{(N-1)3} \tau^\alpha & \tau^\alpha \\
\end{bmatrix}
\]

\[
U = [u_{m+1}^N \ u_{m+2}^N \ \ldots \ u_{N-1}^N]'
\]

and

\[
B = \begin{bmatrix}
\tau^\alpha (f_{m+1}^N + b_{(m+1)} u_m^N + b_{(m+1)N} u_{N}^N) + \sum_{k=1}^{N} l_k^\alpha u_{m+1}^l - \sum_{k=2}^{N-1} l_k^\alpha u_{N-k+1}^{N-k+1} \\
\tau^\alpha (f_{m+2}^N + b_{(m+2)} u_m^N + b_{(m+2)N} u_{N}^N) + \sum_{k=1}^{N} l_k^\alpha u_{m+2}^l - \sum_{k=2}^{N-1} l_k^\alpha u_{N-k+1}^{N-k+1} \\
\vdots \\
\tau^\alpha (f_{N}^N + b_{(N)} u_m^N + b_{(N)N} u_{N}^N) + \sum_{k=1}^{N} l_k^\alpha u_{N-1}^l - \sum_{k=2}^{N} l_k^\alpha u_{N-k+1}^{N-k+1}
\end{bmatrix}
\]

The matrix system can be solved for \(u(x,t)\) in part \(P\) of the domain.

### 6. Examples

The method presented above can be used to solve inverse problems and some examples are given to illustrate its application. We have used the DQM with hybrid B spline in solving the direct problem and compared the results with the DQM with CGL points.
Table 2. Table of errors in DQM with B-spline and DQM with CGL points for different values of N and α = 0.2 for Example1

| N | Direct/Inverse problem | DQM with B-spline | DQM with CGL points |
|---|------------------------|-------------------|---------------------|
| 7 | Direct                 | 8.1778e-06        | 3.7370e-06          |
|   | Inverse                | 0.0070            | 0.0515              |
| 9 | Direct                 | 5.6622e-06        | 3.2963e-06          |
|   | Inverse                | 0.0062            | 0.0374              |
| 11| Direct                 | 4.5481e-06        | 3.0955e-06          |
|   | Inverse                | 0.0058            | 0.0290              |
| 21| Direct                 | 3.1091e-06        | 2.7561e-06          |
|   | Inverse                | 0.0049            | 0.0125              |
| 41| Direct                 | 2.7580e-06        | 2.6711e-06          |
|   | Inverse                | 0.0045            | 0.0045              |
| 45| Direct                 | 2.7162e-06        | 2.6665e-06          |
|   | Inverse                | 0.0044            | 0.0037              |
| 51| Direct                 | 2.7162e-06        | 2.6620e-06          |
|   | Inverse                | 0.0044            | 0.0029              |

Table 3. Table of errors in DQM with B-spline and DQM with CGL points for different values of α for Example1

| α | Direct/Inverse problem | DQM with B-spline | DQM with CGL points |
|---|------------------------|-------------------|---------------------|
| 0.1| Direct                 | 2.8166e-06        | 2.7282e-06          |
|   | Inverse                | 0.0027            | 0.0027              |
| 0.25| Direct                | 2.7173e-06        | 2.6313e-06          |
|    | Inverse                | 0.0057            | 0.0057              |
| 0.5 | Direct                 | 2.3503e-06        | 2.2748e-06          |
|    | Inverse                | 0.0177            | 0.0177              |
| 0.75| Direct                 | 1.5337e-06        | 1.4804e-06          |
|    | Inverse                | 0.0317            | 0.0317              |
| 1.0 | Direct                 | 1.1091e-06        | 1.1862e-06          |
|    | Inverse                | 0.0471            | 0.0471              |

6.1. Example 1
Consider equation (2) - (4) with the following conditions:

\[ u(x,0) = 0, \quad 0 \leq x \leq 1, \]
\[ u(x^@,t) = e^{\alpha} t^2, \quad 0 \leq t \leq T, \]
\[ u(1,t) = e^{1/2}, \quad 0 \leq t \leq T \quad \text{where} \ x^@ = 0.5. \]

The exact solution of the problem is \( u(x,t) = e^{x^2} t^2 \) and

\[ f(x,t) = \frac{2e^{\alpha} t^{2-\alpha}}{\Gamma(3-\alpha)} - 2e^{x^2} t^2 + e^{2x^4} \]

For fixed value of \( k = 0.01, T = 1, \alpha = 0.2 \) the absolute errors were found for various number of mesh points and recorded in the Table 2.

From the Table 2 it can be concluded that \( N = 41 \) is the best choice for the number of grid points. For \( N = 41, k = 0.01, T = 0.1, \) the change in the behaviour of the solution can be observed with different values of \( \alpha. \)
Figure 1. Example 1 using DQM with B-spline

Figure 2. Example 1 using DQM with CGL points
Table 4. Table of errors in DQM with B-spline and DQM with CGL points for different values of $\alpha$ in Example 2

| $\alpha$ | Direct/Inverse problem | DQM with B-spline | DQM with CGL points |
|----------|------------------------|-------------------|---------------------|
| 0.1      | Direct                 | 1.9667e-05        | 2.0304e-05          |
| 0.1      | Inverse                | 0.0054            | 0.0690              |
| 0.25     | Direct                 | 1.9801e-05        | 2.0471e-05          |
| 0.25     | Inverse                | 0.0045            | 0.0754              |
| 0.5      | Direct                 | 2.0688e-05        | 2.1297e-05          |
| 0.5      | Inverse                | 0.0487            | 0.1002              |
| 0.75     | Direct                 | 2.8089e-05        | 2.7660e-05          |
| 0.75     | Inverse                | 0.1429            | 0.1453              |
| 1.0      | Direct                 | 1.0753e-04        | 1.0434e-04          |
| 1.0      | Inverse                | 0.1404            | 0.1553              |

Figure 3. Example 2 using DQM with B-spline

6.2. Example 2
Consider equation (2) - (4) with the following conditions:

$u(x,0) = 0, \quad 0 \leq x \leq 1,$
$u(x@, t) = 0, \quad 0 \leq t \leq T,$
$u(1, t) = -t, \quad 0 \leq t \leq T \quad \text{where } x@ = 0.5.$

The exact solution of the problem is $u(x, t) = t \cos \pi x$ and

$$f(x, t) = \frac{t^{1-\alpha} \cos \pi x}{\Gamma(2-\alpha)} + t \cos \pi x \left( \pi^2 - 1 + t \cos \pi x \right)$$

6.3. Example 3
Consider equation (2) - (4) with the following conditions:

$u(x, 0) = 0, \quad 0 \leq x \leq 1,$
Solution of direct problem in Example2 using DQM using CGL points

\[ u(x,t) = t^2, \quad 0 \leq t \leq T, \]

\[ u(1,t) = -t^2, \quad 0 \leq t \leq T \quad \text{where} \quad x^@ = 0.5. \]

The exact solution of the problem is \( u(x,t) = t^2 (\sin \pi x + \cos \pi x) \) and

\[ f(x,t) = (\sin \pi x + \cos \pi x) \left[ \frac{2 t^2 - \alpha}{\Gamma(3 - \alpha)} + t^2 (\pi^2 - 1 + \sin \pi x + \cos \pi x) \right] \]

**Table 5.** Table of errors in DQM with B-spline and DQM with CGL points for different values of \( \alpha \) in Example3

| \( \alpha \) | Direct/Inverse problem | DQM with B-spline | DQM with CGL points |
|--------------|------------------------|------------------|---------------------|
| 0.1          | Direct                 | 5.3382e-07       | 2.3550e-06          |
|              | Inverse                | 4.3953e-04       | 0.0081              |
| 0.25         | Direct                 | 5.3334e-07       | 2.3465e-06          |
|              | Inverse                | 0.0023           | 0.0106              |
| 0.5          | Direct                 | 5.3338e-07       | 2.3260e-06          |
|              | Inverse                | 0.0138           | 0.0206              |
| 0.75         | Direct                 | 5.3158e-07       | 2.2852e-06          |
|              | Inverse                | 0.0311           | 0.0318              |
| 1.0          | Direct                 | 5.1385e-07       | 2.1846e-06          |
|              | Inverse                | 0.0226           | 0.0236              |

**Figure 4.** Example 2 using DQM with CGL points

7. Conclusion

As established in the popular book by G.D. Smith [23], the scheme

\[ TU^{k+1} = Q^{-k+1} + R^{k+1} + S^k \]
Figure 5. Example 3 using DQM with B-spline

Figure 6. Example 3 using DQM with CGL points
is stable if the absolute value of eigen values of the inverse of the matrix $P$ are less than or equal to one. In the numerical scheme given by equation (8) the eigen values of inverse of the matrix $T$ satisfy

$$\left| \frac{k^\alpha h^2}{h^2} \right| \geq 1$$

$$h \leq k^\alpha/2$$

Thus for all suitable values of $h$, $k$ and $\alpha$ the system is stable. And in all the three examples the absolute values of the eigen values of the inverse of matrix involved for direct as well as inverse problem are less than or equal to one, hence the scheme is stable. Therefore, the methodology presented here can be used to solve inverse problems of similar nature.

References

[1] Isakov V 2006 Inverse Problems for Partial Differential Equations Applied Mathematical Sciences 127(2)
[2] Kheiri H, Mojaver A and Shahi S 2015 Analytical solutions for the fractional Fisher’s equation Sahand Communications in Mathematical Analysis 2(1) 27–49
[3] Podlubny I 1999 Fractional Differential Equations (Academic Press, San Diego, CA)
[4] Sabatier J, Agrawal O P and Machado J A T 2007 Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering (Springer, Dordrecht, The Netherlands)
[5] Khader M M and Saad K M 2018 A numerical approach for solving the problem of biological invasion (fractional Fisher equation) using Chebyshev spectral collocation method Chaos Solitons Fractals 110 169-77
[6] Khader M M 2011 On the numerical solutions for the fractional diffusion equation Communications in Nonlinear Science and Numerical Simulation 16, 2535-42
[7] Tadjeran C, Meerschaert M M and Scheffler H 2006 A second-order accurate numerical approximation for the fractional diffusion equation Journal of Computational Physics 213(1) 205–13
[8] Meerschaert M M, Scheffler H and Tadjeran C 2006 Finite difference methods for two-dimensional fractional dispersion equation Journal of Computational Physics 211(1) 249–61
[9] Zheng G H and Wei T 2011 A new regularization method for solving a time-fractional inverse diffusion problem Journal of Mathematical Analysis and Applications 378(2) 418–31
[10] Chen M and Deng W 2013 WSLD operators II: the new fourth order difference approximations for space Riemann-Liouville derivative (ArXiv e-prints, arXiv:1306.5900 (math.NA))
[11] Miller K S and Ross B 1993 An Introduction to the Fractional Calculus and Fractional Differential Equations (Wiley, New York)
[12] Bellman R, Kashef B G and Casti J 1972 Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations Journal of Computational Physics 10(1) 40–52
[13] Korkmaz A and Dag I 2011 Polynomial based differential quadrature method for numerical solution of nonlinear Burgers equations Journal of the Franklin Institute 348 2863–75
[14] Bashan A 2018 An effective application of differential quadrature method based on modified cubic B-splines to numerical solutions of the KdV equation Turkish Journal of Mathematics 42 373–94
[15] Bashan A, Yagmurlu N M, Ucar Y and Esen A 2017 An effective approach to numerical soliton solutions for the Schrödinger equation via modified cubic B-spline differential quadrature method Chaos, Solitons and Fractals 100 45–56
[16] Arora G and Joshi V 2017 A computational approach using modified trigonometric cubic B-spline for numerical solution of Burgers equation in one and two dimensions Alexandria Engineering Journal 57(2) 1087–98
[17] Bashan A, Karakoc S B G and Geyikli T 2015 B-spline Differential Quadrature Method for the Modified Burgers Equation Cankaya University Science and Engineering 12(1) 001–013
[18] Schoenberg I J 1964 On trigonometric spline interpolation Journal of Mathematics and Mechanics 13 795–825
[19] Zhu X G, Nie Y F, Zhang W W 2017 An efficient differential quadrature method for fractional advection-diffusion equation Nonlinear Dynamics 90 1807–27
[20] Mittal R C and Jain R K 2012 Numerical solutions of non linear Burgers’ equation with modified cubic B-splines collocation method Applied Mathematics and Computation 218(15) 7839–55
[21] Butt C W and Malik M 1996 Differential quadrature method in computational mechanics Applied Mechanics Review 49(1) 1–28
[22] Tomasiello S 2003 Stability and accuracy of the iterative differential quadrature method International Journal for Numerical Methods in Engineering 58(9) 1277–96
[23] Smith G D 1985 Numerical solution of partial differential equations (Oxford University Press)
[24] Ghasemi M 2017 High order approximations using spline-based differential quadrature method: Implementation to the multi-dimensional PDEs Applied Mathematical Modelling 46 63–80
[25] Korkmaz A and Dag I 2009 Crank-Nicolson Differential quadrature algorithms for the Kawahara equation Chaos, Solitons and Fractals 42 65-73
[26] Maini P K, Mcelwain D L S and Leavesley D 2004 Travelling Waves in a Wound Healing Assay Applied Mathematics Letters 17 575–80
[27] Zeidabadi H, Pourgholi R and Tabasi S H 2018 A hybrid scheme for time fractional inverse parabolic problem Waves in Random and Complex Media https://doi.org/10.1080/17455030.2018.1511073
[28] Pourgholi R, Tabasi S H and Zeidabadi H 2017 Numerical techniques for solving system of nonlinear inverse problem Engineering with Computers https://doi.org/10.1007/s00366-017-0554-6.
[29] Arora G, Pratiksha 2019 Solution of fractional Burgers equation using advanced differential quadrature method Nonlinear Studies 26(3) 1–16
[30] Mehra M and Patel K S 2017 Algorithm 986: A Suite of Compact Finite Difference Schemes ACM Trans. Math. Softw. 44(2) Article 23