A FAITHFUL 2-DIMENSIONAL TQFT

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Abstract. It has been shown in this paper that the commutative Frobenius algebra \( \mathbb{QZ}_5 \otimes \mathbb{Z} \mathbb{QS}_3 \) provides a complete invariant for two-dimensional cobordisms, i.e., that the corresponding two-dimensional quantum field theory is faithful. The essential role in the proof of this result plays Zsigmondy’s Theorem.

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1. Introduction

It is evident that one aspect of topological quantum field theories (TQFTs) concerns with the corresponding invariants of manifolds. However, the completeness of these invariants is seldom investigated in the literature. There is a result in [5, Section 14] that claims faithfulness of a 1-dimensional TQFT, which is inspired by [4]. The second author, in her recent work [15], has shown that every 1-dimensional TQFT, with respect to the field of characteristic zero, is faithful. This means that every such 1-dimensional TQFT provides a complete invariant for 1-cobordisms.

In the current article we show that there is a faithful 2-dimensional TQFT. At first sight, this result has stronger algebraic than topological impact. It says that there is a commutative Frobenius algebra, which satisfies only the equalities in the language: multiplication, unit, comultiplication and counit that are necessary to define this notion.

Since this structure is free of additional equations, one could be tempted to call it “free commutative Frobenius algebra”. However, since the category of commutative Frobenius algebras is a groupoid—every homomorphism is an isomorphism (cf. [9, Lemma 2.4.5]—there are no freely generated objects in this category. We find that the existence of such a commutative Frobenius algebra justifies the whole notion.

On the other hand, we do not know whether there exists a faithful \( n \)-dimensional TQFT for \( n \geq 3 \). An important step towards a solution of this problem was given in [8], where the author presents the cobordism category in arbitrary dimension \( n \) with generators and relations. Our proof for 2-dimensional case suggests that for \( n \geq 3 \), particular difficulties could be caused by closed manifolds with many connected components. As it was shown in [6], neither Turaev-Viro, [17], nor Reshetikhin-Turaev, [16], 3-dimensional TQFTs are faithful. We are aware of the fact that even a negative answer to this question might be conclusive—it suggests that TQFTs should search for a more “expressive” targets than the category of vector spaces.

In order to keep this paper as short as possible, we rely on [9] for basic definitions, and suggest the reader to be acquainted with now classical works [2], [12], [1] and more recent [10].

2. The category \( 2Cob \) and 2TQFTs

Let \( 2Cob \) be the category whose objects are \( 0, 1, 2, \ldots \), where \( n \) is the sequence of \( n \) circles and whose arrows are the equivalence classes of 2-cobordisms defined
as in [9, Section 1.2]. We denote cobordisms by $K, L, \ldots$, and $K = L$ means that $K$ and $L$ belong to the same equivalence class.

Let $K : n \rightarrow m$ be a 2-cobordism whose ingoing and outgoing boundaries are respectively the sequences of circles $(\Sigma_0, \ldots, \Sigma_{n-1})$ and $(\Sigma_0', \ldots, \Sigma_{m-1}')$. We define an equivalence relation $\rho_K$ on the set 
\[
\{(0, \ldots, n-1) \times \{0\} \cup \{(0, \ldots, m-1) \times \{1\}\}
\]
such that $(i, k) \rho_K (j, l)$ when $\Sigma_k^i$ and $\Sigma_l^j$ belong to the same connected component in $K$ (cf. [3, Section 8]). For example, if $K : 3 \rightarrow 4$ is

\begin{center}
\includegraphics[width=0.3\textwidth]{cobordism_example.png}
\end{center}

then the equivalence classes of $\rho_K$ are
\[
\{(0, 0), (2, 0), (0, 1), (2, 1)\} \quad \text{and} \quad \{(1, 0), (1, 1), (3, 1)\}.
\]

Also, we denote by $(g_k^i)_K$ the genus of the connected component of $K$ containing $\Sigma_k^i$.

The category $2\text{Cob}$ is a symmetric monoidal with the tensor product $\otimes$ given by “putting side by side” and symmetry generated by the transpositions:

\begin{center}
\includegraphics[width=0.3\textwidth]{tensor_product.png}
\end{center}

Let $\text{Vect}$ be the category of vector spaces over a fixed field whose symmetric monoidal structure is given by the tensor product and the usual symmetry. According to Atiyah’s axioms (see [2, Section 2]), a 2-dimensional quantum field theory (2TQFT) is a symmetric, strong monoidal functor (cf. [11, Section XI.2]) from $2\text{Cob}$ to $\text{Vect}$.

For $m, k, n \geq 0$, let $E_{m,k,n}$ denote the connected 2-cobordism with $n$ ingoing boundaries, $m$ outgoing boundaries and genus $k$.

\begin{center}
\includegraphics[width=0.3\textwidth]{2cobordism.png}
\end{center}

As a part of a relation between 2TQFTs and commutative Frobenius algebras, which is thoroughly explained in [9, Section 3.3], we have that if $F$ is a 2TQFT, then for
\[
\mu = F(E_{1,0,2}), \quad \eta = F(E_{1,0,0}), \quad \delta = F(E_{2,0,1}), \quad \text{and} \quad \varepsilon = F(E_{0,0,1}),
\]
(F1, μ, η, δ, ε), is a commutative Frobenius algebra. Conversely, if (A, μ, η, δ, ε) is a commutative Frobenius algebra, then there is a 2TQFT, which we denote by F_A, mapping 1 into A, and E_{1,0,0}, E_{1,0,1}, E_{2,0,1} and \( E_{0,0,1} \) into \( μ, η, δ \) and \( ε \), respectively.

For such an \( F_A \), we denote \( F_A K \) by \( (K)_A \), and abbreviate \( F_A K = F_A L \) by \( K = A L \).

The following three lemmata hold since 2TQFT is a monoidal functor.

**Lemma 2.1 (FILLING HOLES).** If \( K = A L \) for \( K, L : n \rightarrow m \), then for every \( 0 < i \leq n \) and \( 0 < j \leq m - 1 \), we have that

\[
K \circ (\text{id}_i \otimes E_{1,0,0} \otimes \text{id}_{n-i-1}) = A L \circ (\text{id}_i \otimes E_{1,0,0} \otimes \text{id}_{n-i-1})
\]

and

\[
(\text{id}_j \otimes E_{0,0,1} \otimes \text{id}_{m-j-1}) \circ K = A (\text{id}_j \otimes E_{0,0,1} \otimes \text{id}_{m-j-1}) \circ L.
\]

**Lemma 2.2 (STRETCHING 1).** If \( K = A L \) for \( K, L : 1 \rightarrow 0 \), then we have that

\[
(K \otimes \text{id}_1) \circ E_{2,0,1} = A (L \otimes \text{id}_1) \circ E_{2,0,1},
\]

and if \( K = A L \) for \( K, L : 0 \rightarrow 1 \), then we have that

\[
E_{1,0,2} \circ (K \otimes \text{id}_1) = A E_{1,0,2} \circ (L \otimes \text{id}_1).
\]

**Lemma 2.3 (STRETCHING 2).** If \( K = A L \) for \( K, L : 2 \rightarrow 0 \), then we have that

\[
(K \otimes \text{id}_1) \circ (\text{id}_1 \otimes E_{2,0,0}) = A (L \otimes \text{id}_1) \circ (\text{id}_1 \otimes E_{2,0,0}),
\]

and if \( K = A L \) for \( K, L : 0 \rightarrow 2 \), then we have that

\[
(\text{id}_1 \otimes E_{0,0,2}) \circ (K \otimes \text{id}_1) = A (\text{id}_1 \otimes E_{0,0,2}) \circ (L \otimes \text{id}_1).
\]

**Proposition 2.4 (MAXIMALITY).** If for \( K \neq L \), we have that \( K = A L \), where \( \text{dim}(A) > 1 \), then for some \( k_1 \geq \ldots \geq k_n \geq 0 \) and \( l_1 \geq \ldots \geq l_m \geq 0 \) such that \( (k_1, \ldots, k_n) \neq (l_1, \ldots, l_m) \), we have that

\[
\bigotimes_{i=1}^{n} E_{0,k_i,0} = A \bigotimes_{j=1}^{m} E_{0,l_j,0}.
\]

**Proof.** Since \( \text{dim}(A) > 1 \), the cobordisms \( K \) and \( L \) must have the same source and target. Also, \( K \neq L \) entails that either \( \rho K \neq \rho L \), or \( \rho K = \rho L \) and there is \( (i,k) \) such that \( (g^k_k) \neq (g^k_k)_L \), or \( \rho K = \rho L \) and for every \( (i,k) \), \( (g^k_k) = (g^k_k)_L \) and \( K \) and \( L \) differ in their closed components.
We start with the last and simplest case. If \( \rho_K = \rho_L \) and for every \((i, k)\) we have that \((g_k^i)_K = (g_k^i)_L \), then by applying Lemma 2.1 for all the boundary components, we arrive at the equality of the form (2.1).

If \( \rho_K \neq \rho_L \) and there is \((i, k)\) such that \((g_k^i)_K \neq (g_k^i)_L \), then by applying Lemma 2.1 for all the boundary components except the one corresponding to \((i, k)\), and then by applying Lemma 2.2 we arrive at the equality of the form

\[
E_{1, p, 1} \otimes (\bigotimes_{i=1}^{n} E_{0, k_i, 0}) = A \ E_{1, q, 0} \otimes (\bigotimes_{j=1}^{m} E_{0, l_j, 0}),
\]

for some \( n, m, p, q \geq 0 \) such that \( p \neq q \), and \( k_1 \geq \ldots \geq k_n \geq 0, l_1 \geq \ldots \geq l_m \geq 0 \).

If \( \rho_K \neq \rho_L \) and \((i, k) \rho_K(j, l) \), while not \((i, k) \rho_L(j, l) \), then by applying Lemma 2.1 for all the boundary components except those corresponding to \((i, k)\) and \((j, l)\) we arrive either directly at the equality of the form

\[
E_{1, p, 1} \otimes (\bigotimes_{i=1}^{n} E_{0, k_i, 0}) = A \ E_{1, q, 0} \otimes (\bigotimes_{j=1}^{m} E_{0, l_j, 0}),
\]

for some \( n, m, p, q, r \geq 0 \) and \( k_1 \geq \ldots \geq k_n \geq 0, l_1 \geq \ldots \geq l_m \geq 0, \) or this equality is obtained by a further application of Lemma 2.2.

For \( a \geq \max\{k_1, l_1\} \), put the both sides of the equalities (2.2) and (2.3) in the context \( E_{0, a, 1} \otimes \_ \otimes E_{1, a, 0} \) in order to obtain the equality of the form (2.1). \( \square \)

3. Frobenius Algebras \( \mathbb{QZ}_5 \) and \( Z(\mathbb{QS}_3) \)

For all the examples below, when we fix a basis \( \langle \beta_1, \ldots, \beta_n \rangle \) of a vector space \( V \), then we assume that the tensor product \( V \otimes V \) has the fixed basis

\[ \langle \beta_1 \otimes \beta_1, \beta_1 \otimes \beta_2, \ldots, \beta_2 \otimes \beta_1, \ldots, \beta_n \otimes \beta_n \rangle, \]

and we represent the linear transformations by matrices with respect to these bases.

For \( \mathbb{Z}_5 \) being the cyclic group of order 5, with the generator \( a \), let \( \mathbb{QZ}_5 \) be the group algebra and let \( \langle e, a, a^2, a^3, a^4 \rangle \) be its basis. The multiplication \( \mu : \mathbb{QZ}_5 \otimes \mathbb{QZ}_5 \to \mathbb{QZ}_5 \) is represented by the \( 5 \times 25 \) matrix \( M \)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

while the unit \( \eta : \mathbb{Q} \to \mathbb{QZ}_5 \) is represented by the \( 5 \times 1 \) matrix:

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

The comultiplication \( \delta : \mathbb{QZ}_5 \to \mathbb{QZ}_5 \otimes \mathbb{QZ}_5 \) is represented by the \( 25 \times 5 \) matrix \( \frac{1}{5} M^T \), and the counit \( \varepsilon : \mathbb{QZ}_5 \to \mathbb{Q} \) is represented by the \( 1 \times 5 \) matrix

\[
\begin{bmatrix}
5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The structure \( (\mathbb{QZ}_5, \mu, \eta, \delta, \varepsilon) \) is a commutative Frobenius algebra and it is special in the sense that for every \( k \)

\[ E_{1, k, 1} = q_{\mathbb{Z}_5} E_{1, 0, 1}. \]

Note that \( (E_{0, k, 0})_{\mathbb{QZ}_5} \) is represented by the \( 1 \times 1 \) matrix, i.e. the rational number 5.
For \( S_3 \) being the symmetric group of order 6, let \( Z(\mathbb{Q}S_3) \) be the center of the group algebra \( \mathbb{Q}S_3 \). Denote the three conjugacy classes of \( S_3 \) by \( C_1 = \{ e \} \), \( C_2 = \{(12),(13),(23)\} \) and \( C_3 = \{(123),(132)\} \). By [7, Proposition 12.22] one can fix
\[
\langle e, (12) + (13) + (23), (123) + (132) \rangle
\]
as the basis of \( Z(\mathbb{Q}S_3) \).

The multiplication \( \mu : Z(\mathbb{Q}S_3) \otimes Z(\mathbb{Q}S_3) \to Z(\mathbb{Q}S_3) \) is represented by the \( 3 \times 9 \) matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 1
\end{pmatrix},
\]
while the unit \( \eta : \mathbb{Q} \to Z(\mathbb{Q}S_3) \) is represented by the \( 3 \times 1 \) matrix
\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}.
\]

Let the comultiplication \( \delta : Z(\mathbb{Q}S_3) \to Z(\mathbb{Q}S_3) \otimes Z(\mathbb{Q}S_3) \) be represented by the \( 9 \times 3 \) matrix, which is the transpose of
\[
\begin{pmatrix}
1 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & \frac{1}{3} & 0 & 1 & 0 & \frac{1}{3}
\end{pmatrix},
\]
and let the counit \( \varepsilon : Z(\mathbb{Q}S_3) \to \mathbb{Q} \) be represented by the \( 1 \times 3 \) matrix
\[
\begin{pmatrix}
1 & 0 & 0
\end{pmatrix}.
\]

It is easy to check that \((Z(\mathbb{Q}S_3), \mu, \eta, \delta, \varepsilon)\) is a commutative Frobenius algebra and note that \((E_{0,k,0})_{Z(\mathbb{Q}S_3)}\) is represented by the rational number
\[
\left( \frac{3}{2} \right)^{k-1} \left( 2^{2k-1} + 1 \right).
\]

4. Faithfulness

In this section we denote the tensor product \( \mathbb{Q}Z_\delta \otimes Z(\mathbb{Q}S_3) \) by \( A \). The algebra \( A \) is equipped with the commutative Frobenius structure as the tensor product of two such algebras (cf. [9, Section 2.4]). Note that \((E_{0,k,0})_{A}\) is represented by the rational number
\[
5 \cdot \left( \frac{3}{2} \right)^{k-1} \left( 2^{2k-1} + 1 \right).
\]

The following lemma is crucial for the proof of the faithfulness of the 2TQFT corresponding to \( A \).

**Lemma 4.1.** If for \( k_1 \geq \ldots \geq k_n \geq 0 \) and \( l_1 \geq \ldots \geq l_m \geq 0 \)
\[
\prod_{i=1}^{n} \left( 5 \cdot \left( \frac{3}{2} \right)^{k_i-1} \left( 2^{2k_i-1} + 1 \right) \right) = \prod_{j=1}^{m} \left( 5 \cdot \left( \frac{3}{2} \right)^{l_j-1} \left( 2^{2l_j-1} + 1 \right) \right),
\]
then \( n = m \) and \((k_1, \ldots, k_n) = (l_1, \ldots, l_m)\).

**Proof.** Let \( p \) and \( q \) be such that \( k_p, l_q > 0 \) and \( k_{p+1} = 0 = l_{q+1} \) (if there are any). Then the above equality reads
\[
5^n \cdot \frac{3^{\sum_{i=1}^{p} k_i} - p}{2^{\sum_{i=1}^{p} k_i}} \prod_{i=p+1}^{n} \left( 2^{2k_i-1} + 1 \right) = 5^m \cdot \frac{3^{\sum_{j=1}^{q} l_j} - q}{2^{\sum_{j=1}^{q} l_j}} \prod_{j=q+1}^{m} \left( 2^{2l_j-1} + 1 \right).
\]
Since the last digit in $2^{2k-1} + 1$ is either 3 or 9, such a factor is not divisible by 5, and we may conclude that $n = m$. Since all the factors but $2^{(\sum_{i=1}^p k_i) - p}$ and $2^{(\sum_{j=1}^q l_j) - q}$ are odd, we may conclude that $(\sum_{i=1}^p k_i) - p = (\sum_{j=1}^q l_j) - q$, and that
\[
\prod_{i=1}^p (2^{2k_i-1} + 1) = \prod_{j=1}^q (2^{2j-1} + 1).
\]

If $(k_1, \ldots, k_p) \neq (l_1, \ldots, l_q)$, then, after cancelation, we may assume that every $k_i$ is different from every $l_j$. Assume also that $k_1 > l_1$. It is not possible that $k_1 = 2$, since then $l_1 = \ldots = l_q = 1$ and $(\sum_{j=1}^q l_j) - q = 0 < (\sum_{i=1}^p k_i) - p$. Hence, $k_1 \geq 3$ and $2k_1 - 1 \geq 5$. By applying Zsigmondy’s Theorem for sums [13] (see also [18], P1.7 and [14]), there would be a prime that divides $2^{2k_1-1} + 1$ and for every $1 \leq j \leq q$ it does not divide $2^{2j-1} + 1$, which contradicts the above equality. \hfill \Box

**Theorem 4.2.** The 2TQFT, $F_k$ is faithful and injective on objects.

**Proof.** Since $\dim(A) > 1$, the functor $F_k$ is injective on objects. By Lemma 4.1 we have that for every $k_1 \geq \ldots \geq k_n \geq 0$ and $l_1 \geq \ldots \geq l_m \geq 0$ such that $(k_1, \ldots, k_n) \neq (l_1, \ldots, l_m)$
\[
\bigotimes_{i=1}^n E_{0,k_i,0} \neq \bigotimes_{j=1}^m E_{0,l_j,0},
\]
and it remains for Proposition 2.4 to be applied. \hfill \Box

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