SOME REMARKS ON DP-MINIMAL GROUPS

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Abstract. We prove that \(\omega\)-categorical dp-minimal groups are nilpotent-by-finite, a small step in the general direction of proving the conjecture raised in [Kru12 Conjecture 0.2]. We also show that in dp-minimal definably amenable groups, f-generic global types are strongly f-generic (in the sense of [CS15]).

1. Introduction

A theory \(T\) is \textit{dp-minimal} if the following cannot happen. There are two formulas \(\varphi(x, y)\), \(\psi(x, z)\) with \(x\) a singleton (\(y\) and \(z\) perhaps not), and sequences \(\langle a_i | i < \omega \rangle\) and \(\langle b_j | j < \omega \rangle\) such that \(|a_i| = |y|, |b_j| = |z|\) for all \(i, j < \omega\) and for every \(i, j < \omega\) there is some element \(c_{i,j}\) (all in the monster model \(\mathcal{C} \models T\)) such that for all \(i', j', i, j < \omega\), \(\varphi(c_{i,j}, a_{i'})\) holds iff \(i = i'\) and \(\varphi(c_{i,j}, b_{j'})\) holds iff \(j = j'\).

This notion originated from Shelah’s work on strong dependence [She14] but was first properly defined and studied in [OUT11].

Dp-minimal theories are in some sense the simplest case of NIP theories, but still they include all \(\sigma\)-minimal and \(\epsilon\)-minimal theories and the theory of the \(p\)-adics (see [Sim15 Example 4.28]). This restrictive yet still interesting assumption about \(T\) yields many conclusions, evident by the amount of research done in the area, sometimes with the additional assumption of a group or field structure. See e.g., [KOU13, DGL11, Sim11, KS14, Sim14, Joh15, JSW15] to name a few examples.

This note contains some results (mostly) on dp-minimal groups, contributing to the general research in the area.

In Section 2 we prove that all dp-minimal \(\omega\)-categorical groups are nilpotent-by-finite. In Subsection 2.7 we prove a general result on NIP groups: there is a finite \(A\) with \(C(A)\) abelian.

In Section 3 we prove that in definably amenable dp-minimal groups, being \(f\)-generic is the same as being strongly \(f\)-generic.

All definitions are given in the appropriate sections.

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2. $\omega$-CATEGORICAL DP-MINIMAL GROUPS

2.1. Introduction. It is well known that stable $\omega$-categorical groups are nilpotent-by-finite by [BCM79, Fel78] where in [BCM79] it is proven that $\omega$-categorical $\omega$-stable groups are abelian-by-finite (and it is conjectured that this is true for stable $\omega$-categorical groups as well). In [Mac88] Macpherson proves that $\omega$-categorical NSOP groups (and in particular simple in the model theoretic sense) are also nilpotent-by-finite.

Krupinski generalized the stable case in [Kru12, Theorem 3.4] by proving that that every $\omega$-categorical NIP group that has fsg (finitely satisfiable generics) is nilpotent-by-finite. In [DK13] Krupinski and Dobrowolski extended this result and removed the NIP hypothesis.

In this section we will go in the other direction and remove the assumption of fsg. However, our proof requires the stronger assumption of dp-minimality and not just NIP.

2.2. What we get from $\omega$-categoricity. We will need the following facts about $\omega$-categorical theories.

Suppose that $T$ is $\omega$-categorical.

(1) (Ryll–Nardzewski, see e.g., [TZ12, Theorem 4.3.1]) For all $n < \omega$, there are at most finitely many $\emptyset$-definable sets in $n$ variables.

(2) If $M \models T$ is saturated (in particular, countable) and $X \subseteq M^n$ is invariant under $\text{Aut}(M)$ then $X$ is $\emptyset$-definable.

(3) By (2), an $\omega$-categorical theory $T$ eliminates $\exists^\infty$, which means that for all $\varphi(x, y)$ there is some $n < \omega$ such that for all $a \in M \models T$, $\varphi(M, a)$ is infinite iff $|\varphi(M, a)| \geq n$.

A structure $M$ is $\omega$-categorical if its theory is.

Suppose that $(G, \cdot)$ is an $\omega$-categorical group. Then, it follows easily from (1) that $(G, \cdot)$ is locally finite (every finitely generated subgroup is finite).

We will use the following fact about locally finite groups.

Fact 2.1. [KW73, Corollary 2.5] If $G$ is an infinite locally finite group (every finitely generated subgroup is finite), then $G$ contains an infinite abelian subgroup.

2.3. Equivalent conditions for being nilpotent-by-finite.

Remark 2.2. If $G$ is nilpotent-by-finite, and $H \equiv G$ then $H$ is nilpotent-by-finite. Why? Suppose that $\lambda^+ = 2^\lambda > |G|, |H|$ and let $G^*$ be a saturated extension of $G$ of size $\lambda^+$, which, we may assume, also contains $H$. Then it is enough to show that $G^*$ is nilpotent-by-finite (being nilpotent-by-finite transfers to subgroups). Suppose that $G_0 \leq G$ is nilpotent of finite index. Let $(S^*, S^*_0)$

\footnote{On the face of it, they asked that the group is generically stable. However, by [Kru12, Remark 1.8], under NIP and $\omega$-categoricity, a definable generically stable group has fsg iff it is generically stable.}
be a saturated extension of \((G, G_0)\) of size \(\lambda\). Then \(S_0^* \leq S^*\) is nilpotent of finite index in \(S^*\) and \(G^* \cong S^*\).

Recall that in any NIP group \(G\), the connected component \(G^{00}\) (the smallest type-definable subgroup of bounded index of \(\mathcal{C}\)) exists (see [She08]), so when \(G\) is \(\omega\)-categorical, it must be \(\emptyset\)-definable of finite index (so we can talk about it in \(G\) without going to a saturated extension).

**Proposition 2.3.** Suppose that \(C\) is a class of countable \(\omega\)-categorical NIP groups (in the pure group language) satisfying: if \(G \in C\), \(H \trianglelefteq G\) definable (over \(\emptyset\)) then \(G/H \in C\) and \(H \in C\). Then the following statements are equivalent:

1. Every \(G \in C\) is nilpotent-by-finite.
2. Every infinite characteristically simple \(G \in C\) is abelian.
3. Every infinite \(G \in C\) contains an infinite \(\emptyset\)-definable abelian subgroup.

**Proof.** (2) implies (1) is essentially Krupinski’s argument from [Kru12]. Suppose that \(G \in C\) and we wish to show that it is nilpotent-by-finite. We may of course assume that \(G\) is infinite. We may assume that \(G_0^0 = G\) so that \(G\) has no definable subgroups of finite index.

By \(\omega\)-categoricity, we can write \(\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G\) where the groups \(G_i\) are \(0\)-definable, and this is a maximal (length-wise) such chain. The groups \(G_i\) are invariant under \(\text{Aut}(G)\) so normal, and by assumption \(G_i \in C\). The proof is now by induction on \(n \geq 1\). For \(n = 1\), this follows immediately from (2) (i.e., \(G_1\) will be abelian).

Now note that by the induction hypothesis if \(H \trianglelefteq G\) is \(0\)-definable and non-trivial then \(G/H\) is nilpotent: \(G/H\) is in \(C\) (as \(G = G^{00}, G/H\) is infinite). Also \((G/H)^{00} = (G/H)\). But the maximal length of a chain as above which suits \(G/H\) must be shorter than \(n\). Hence \(G/H\) is nilpotent-by-finite. It follows by Fitting’s theorem (see [Rob93] Theorem 5.2.8) that the Fitting subgroup of \(G/H\) — the product of all normal nilpotent subgroups of finite index — is itself a nilpotent group which is also \(0\)-definable (by \(\omega\)-categoricity). Together \(G/H\) is itself nilpotent.

Hence we may assume that \(Z(G)\) is trivial (otherwise \(G/Z(G)\) is nilpotent and so \(G\) is too). This in turn implies that \(G_1\) is infinite (if not, then \(C_G(G_1)\) is of finite index in \(G\), and hence equals \(G\), but then \(G_1 \subseteq Z(G)\)). Now we use (2) on \(G_1\) to finish: \(G_1\) is abelian and \(G/G_1\) is nilpotent, so both are solvable, and hence \(G\) is solvable. However, by [AM97] Theorem 1.2, if \(G\) is not nilpotent-by-finite (equivalently, nilpotent, since we already assumed \(G = G^{00}\), using the Fitting subgroup as above), it interprets the infinite atomless boolean algebra, and has IP.

(3) implies (2) is obvious.

(1) implies (3). Without loss of generality, \(G = G^{00}\). If \(Z(G)\) is infinite, we are done. Otherwise, \(G/Z(G)\) is centerless: if \(x \in G\) with \(yx^{-1}xy \in xZ(G)\) for all \(y \in G\), then \(C(x)\) has a finite index in \(G\). Hence \(C(x) = G\) so \(x \in Z(G)\). But by (1) and as \(G = G^{00}\) (and by taking the Fitting subgroup as above), \(G\) is nilpotent so we have a contradiction. \(\square\)
Remark 2.4. Note that the class of all (countable) NIP ω-categorical groups satisfy the conditions in Proposition 2.3. So does the class of ω-categorical dp-minimal groups (taking quotients of the universe $M$, as opposed to e.g., $M^2$, preserves dp-minimality).

Remark 2.5. In [Kru12], Krupinski proved that (2) in Proposition 2.3 holds for the class of NIP ω-categorical groups with fsg. His proof uses a classification theorem on ω-stable characteristically simple groups due to Wilson and Apps [Wil82, App83] (see remarks after Problem 2.16). By that theorem and [Kru12, Proposition 3.2], it follows that ω-categorical characteristically simple groups with NIP are $p$-groups for some $p$. By the argument in the proof of [Kru12, Proposition 3.1], it follows that for such groups $G$, if $a_1, \ldots, a_n \in G$ then $C(a_1) \cap \cdots \cap C(a_n)$ is infinite.

However, we will avoid using the classification theorem, and prove (3) directly for dp-minimal ω-categorical groups.

Remark 2.6. Note also that Fact 2.1 alone is not enough, even though it may seem so in light of the fact that if $G$ is both NIP and contains an infinite abelian subgroup, then $G$ contains an infinite definable abelian subgroup by [She09, Claim 4.3]. However this subgroup is not necessarily $\emptyset$-definable.

2.4. What we get from dp-minimality and NIP. The only use of dp-minimality in the proof is the following basic observation.

Fact 2.7. [Sim15, Claim in proof of Proposition 4.31] If $(G, \cdot)$ is a dp-minimal group then for every definable subgroups $H_1, H_2 \leq G$ either $[H_1 : H_1 \cap H_2] < \infty$ or $[H_2 : H_1 \cap H_2] < \infty$.

We will also use the Baldwin-Saxl lemma, which is true for all NIP groups.

Fact 2.8. [BS76] Let $(G, \cdot)$ be NIP. Suppose that $\varphi(x, y)$ is a formula and that $\{ \varphi(x, c) \mid c \in C\}$ defines a family of subgroups of $G$. Then there is a number $n < \omega$ (depending only on $\varphi$) such that any finite intersection of groups from this family is already an intersection of $n$ of them.

2.5. Proof of the main result.

Theorem 2.9. If $(G, \cdot)$ is an infinite dp-minimal ω-categorical group then $G$ contains an infinite $\emptyset$-definable abelian subgroup.

By Proposition 2.3 we get the following.

Corollary 2.10. If $(G, \cdot)$ is a dp-minimal ω-categorical group then $G$ is nilpotent-by-finite.

For the proof we work in a countable (so ω-saturated) model. So fix such a group $G$. By ω-categoricity, there is a minimal infinite $\emptyset$-definable subgroup $G_0 \leq G$ (i.e., $G_0$ contains no $\emptyset$-definable infinite subgroups), so we may assume that $G = G_0$. 

Lemma 2.11. For every \( a, b \in G \) either \( [C(a) : C(b) \cap C(a)] < \infty \) or \( [C(b) : C(b) \cap C(a)] < \infty \).

Proof. This follows directly from Fact 2.7. \( \square \)

Let \( X = \{ a \in G \mid |C(a)| = \infty \} \). By elimination of \( \exists^\infty \), \( X \) is definable. By Fact 2.11 \( X \) is infinite. For \( a, b \in X \), by Lemma 2.11 it follows that either \( C(a) \cap C(b) \) has finite index in \( C(a) \) or in \( C(b) \). In either case, \( C(a) \cap C(b) \) is infinite. Since \( C(a) \cap C(b) \subseteq C(ab) \), it follows that \( X \) is a group. By our assumption on \( G \) (it contains no infinite \( 0 \)-definable subgroups), \( G = X \).

Compare the following corollary with Remark 2.5.

Corollary 2.12. For every \( a_0, \ldots, a_{n-1} \in G \), \( \bigcap \{ C(a_i) \mid i < n \} \) is infinite.

Proof. By induction on \( n \). For \( n = 1 \) and \( n = 2 \) we just gave the argument. For larger \( n \) it is exactly the same: \( \bigcap \{ C(a_i) \mid i < n \} \) has infinite index in one of \( \bigcap \{ C(a_i) \mid i < n-1 \} \) or \( \bigcap \{ C(a_i) \mid 1 \leq i < n \} \), both infinite. \( \square \)

For every \( a \in G \) let \( H_a = \{ b \in G \mid [C(a) : C(b) \cap C(a)] < \infty \} \), and define

\[ C^0(a) = \bigcap \{ C(b) \mid b \in H_a \} . \]

Observe that \( C^0(a) \) is definable over \( a \) since \( H_a \) is definable.

By \( \omega \)-categoricity there exists some \( n_* \) such that for all \( a, b \in G \), \( [C(a) : C(b) \cap C(a)] < n_* \) iff \( [C(a) : C(b) \cap C(a)] < \infty \).

Main Lemma 2.13. For every \( a, b \in G \) either \( C^0(a) \subseteq C^0(b) \) or \( C^0(b) \subseteq C^0(a) \).

Proof. By Fact 2.7 either \( [C(a) : C(b) \cap C(a)] < n_* \) or \( [C(b) : C(b) \cap C(a)] < n_* \). Suppose that the former happens. Then \( C^0(a) \subseteq C^0(b) \): if \( d \in H_b \), \( [C(b) : C(b) \cap C(d)] < n_* \), so

\[
[C(a) : C(a) \cap C(d)] \leq [C(a) : C(a) \cap C(b) \cap C(d)] \\
\leq [C(a) : C(a) \cap C(b)] \cdot [C(b) : C(b) \cap C(d)] < n_*^2 .
\]

Hence \( d \in H_a \). \( \square \)

Lemma 2.14. For every \( a \in G \) the group \( C^0(a) \) is infinite. Moreover, \( [C(a) : C^0(a)] < \infty \).

Proof. By Fact 2.8 there is some \( N \) such that for every \( k < \omega \) and every \( a_i \in G \) for \( i < k \), \( \bigcap \{ C(a_i) \mid i < k \} = \bigcap \{ C(a_i) \mid i \in I_0 \} \) where \( I_0 \subseteq k \) is of size \( \leq N \). Find \( a_1, \ldots, a_N \in H_a \) with \( \bigcap \{ C(a_i) \mid i < N \} \cap C(a) \) of maximal index in \( C(a) \) (this index is bounded by \( n_*^N \)). Let \( D = \bigcap \{ C(a_i) \mid i < N \} \cap C(a) \). Then, for every \( b \in H_a \), \( C(b) \cap D \) equals to some sub-intersection \( D' \) of size \( N \), but then \( [C(a) : D'] = [C(a) : D] \) so \( D' = D \) and hence \( \bigcap \{ C(b) \mid b \in H_a \} = D \) and in particular it is infinite and of finite index in \( C(a) \). \( \square \)

Proof of Theorem 2.3. Split into two cases.
Case 1. The set \( Y = \{ a \in G \mid a \in C^0(a) \} \) is infinite.

In this case, note that if \( a, b \in Y \) then by Main Lemma 2.13, we may assume that \( C^0(a) \subseteq C^0(b) \subseteq C(b) \) (because \( b \in H_b \)). But then \( a \in C(b) \), so \( Y \) is an infinite commutative \( \emptyset \)-definable set. Hence the group generated by \( Y \) must be abelian, and it must be \( G \) by our choice of \( G \), so we are done.

Case 2. The set \( Y \) is finite.

Pick some \( a_0 \not\in Y \). By induction on \( n < \omega \), choose \( a_n \in C^0(a_{n-1}) \setminus Y \). We can find such elements by Lemma 2.14. For \( n < \omega \), if \( C^0(a_n) \subseteq C^0(a_{n+1}) \) then \( a_{n+1} \in C^0(a_{n+1}) \) which cannot be, so by Main Lemma 2.13 \( C^0(a_{n+1}) \subseteq C^0(a_n) \).

Let \( K > [C(a_n) : C^0(a_n)] \) for all \( n < \omega \). As \( a_K \in C^0(a_i) \) for all \( i < K, a_i \in C(a_K) \). Hence for some \( i < j < K, a_i^{-1}a_j \in C^0(a_K) \subseteq C^0(a_i) \). But \( a_j \in C^0(a_i) \) as well, so \( a_i \in C^0(a_i) \) — contradiction.

\[ \square \]

2.6. Concluding remarks.

**Problem 2.15.** Can we generalize this result to work under weaker assumptions than \( \omega \)-categoricity, such as elimination of \( \exists^\infty \)? (i.e., assume that \( G \) is a dp-minimal group eliminating \( \exists^\infty \), also for imaginaries, with an infinite abelian subgroup, then does it contain an infinite \( \emptyset \)-definable subgroup?)

**Problem 2.16.** What about inp-minimal groups? Inp-minimality is the analogous notion to dp-minimality for NTP\(_2\) [Che14], so it makes sense that this result still holds there, as it does in both the simple (by [Mac88]) and NIP case.

Using the same notation as in [Kru12], we let \( B(F) \) be the group of all continuous functions from the cantor space \( 2^\omega \) into a finite simple non-abelian group \( F \). We also let \( B^-(F) \) be the group of all such functions sending a fixed point \( x_0 \in 2^\omega \) to \( e \in F \). By [Wil82, App83, Theorem 2.3] they are characteristically simple and \( \omega \)-categorical, and in fact by a theorem of Wilson [Wil82], a countably infinite \( \omega \)-categorical characteristically simple group is either isomorphic to one of them, is an abelian \( p \)-group or is a perfect \( p \)-group. Neither groups is nilpotent-by-finite. If they were nilpotent-by-finite, then there would be a normal nilpotent subgroup of finite index, so they would be nilpotent (by Fitting’s theorem, see the proof of Proposition 2.3). But then they must be abelian, which they are not.

It is worthwhile to note the following.

**Proposition 2.17.** For a finite simple non-abelian group \( F \), both \( B(F) \) and \( B^-(F) \) have TP\(_2\) and in particular are not inp-minimal.

For the proof we will need the following simple criterion for having TP\(_2\).
Lemma 2.18. Suppose that $A$ is some infinite set in $\mathfrak{C}$ and $\varphi (x,y)$ is a formula such that for some $k < \omega$, for every sequence $\langle A_i \mid i < \omega \rangle$ of pairwise disjoint subsets of $A$, there are $\langle b_i \mid i < \omega \rangle$ such that $A_i \subseteq \varphi (\mathfrak{C}, b_i)$ and $\varphi (x,b_i) \cup \{\varphi (x,b_i) \mid i < \omega \}$ is $k$-inconsistent. Then $T$ has $\text{TP}_2$.

Proof. We may enumerate $A$ as $\langle a_s \mid s \in \omega^\omega \land |\text{supp} (s)| < \omega \rangle$, where $\text{supp} (s) = \{i \in \omega \mid s(i) \neq 0\}$. Let $A_{i,j} = \{a_s \mid s(i) = j\}$. Then for each $i < \omega$, $\{A_{i,j} \mid j < \omega\}$ are mutually disjoint. By assumption we can find $b_{i,j}$ for $i, j < \omega$ such that $\{\varphi (x,b_{i,j}) \mid j < \omega\}$ are $k$-inconsistent and $A_{i,j} \subseteq \varphi (\mathfrak{C}, b_{i,j})$. Then $\{\varphi (x,b_{i,j}) \mid i,j < \omega\}$ witness the tree property of the second kind.

2.7. A theorem on NIP groups. We end this section with a general remark on NIP groups (without any other assumptions).

Theorem 2.19. Suppose that $(G, \cdot)$ is an NIP group (or more generally, a type-definable group in an NIP theory). Then there is some finite set $A$ such that $C (A)$ is abelian.

Proof. Assume that $G$ is a type-definable group, defined by the type $\pi(x)$ (and multiplication $\cdot_G$ and the unit $e_G$ are definable).

If there is some finite $B \subseteq \mathfrak{C}$, $C (B)$ is finite, then take $A = C (B) \cup B$. So assume not: for all $B$, $C (B)$ is infinite.

Let $M$ be any $|\pi|^+$-saturated model, so that $G (M) \prec G (\mathfrak{C})$ by the Tarski-Vaught test.

Let $p_0$ be a partial type containing the formulas $x \in C (A)$ for all finite sets $A$ of $G (M)$. By assumption, $p_0$ is finitely satisfiable in $G (M)$. Let $S$ be the set of all global types in $S_G (\mathfrak{C})$ containing $p_0$ and f.s. in $G (M)$. All of these types are in particular invariant over $M$, so their product is well defined. (For the precise definition of a product of global invariant types, see [Sim15, 2.2.1], but one can understand it from the proof.)
Claim. For \( p, q \in S, p(x) \otimes q(y) \models x \cdot y = y \cdot x \).

Proof. We need to show that if \( N \supseteq M, a \models q|_N, b \models p|_N \), then \( a \cdot b = b \cdot a \). If not, then \( b \notin C(a) \), so for some \( b_0 \in G(M), b_0 \notin C(a) \), so \( a \notin C(b_0) \) — contradiction.

By \cite{Sim15} Lemma 2.26, it follows that for any \( a \models p, b \models q, a \cdot b = b \cdot a \) (the proof there works just fine for type-definable groups, because it only uses that the formula for multiplication is NIP, but multiplication is definable).

By compactness for every \( p(x), q(y) \in S \) there are formulas \( \psi_{p,q}(x) \in p, \varphi_{p,q}(y) \in q \) such that for every \( a \models \psi_{p,q}, b \models \varphi_{p,q} \) in \( G, a \cdot b = b \cdot a \). Fix \( p \). By compactness (as \( S \) is closed), there is a finite set of types \( q_i \in S \) for \( i < n \) such that \( \varphi_p = \bigvee_{i < n} \varphi_{p,q_i} \) contains \( S \). Let \( \psi_p = \bigwedge_{i < n} \psi_{p,q_i} \). Again by compactness there are \( p_i \) for \( i < m \) such that \( \bigvee_{i < m} \psi_{p_i} \) contains \( S \). Let \( \chi = (\bigwedge_{i < m} \varphi_{p_i}) \land (\bigvee_{i < m} \psi_{p_i}) \), then \( \chi \) contains \( S \) and for every \( a, b \models \chi \) in \( G(C) \), \( a \cdot b = b \cdot a \).

It cannot be that for all finite \( A \subseteq G(M), \neg \chi(M) \cap C(A) \neq \emptyset \) (otherwise we can define a type, f.s. in \( M \), containing \( p_0 \), so in \( S \), but not satisfying \( \chi \)). Hence there is some finite \( A \subseteq G(M) \) such that \( C(A)(M) \models \chi \). Hence \( C(A)(M) \) is abelian, but as \( G(M) \prec G(C) \), so is \( C(A) \).

Remark 2.20. When the group \( G \) is an \( \omega \)-categorical characteristically simple NIP group, then by Proposition 2.17 and the remark before it (or just \cite{Kru12} Fact 0.1 and Proposition 3.2), Krupinski’s proof of \cite{Kru12} Proposition 3.1] gives us that for any finite set \( A, C(A) \) is infinite. Together with 2.19 we know that we can find some \( A \) such that \( C(A) \) is abelian and infinite.

3. \( f \)-generic is the same as strongly \( f \)-generic

Assume that \( G \) is definably-amenable and dp-minimal. The main theorem here says that any definable \( X \subseteq G \) which divides over a small model also \( G \)-divides. This means that that are at most boundedly many global \( f \)-generic types and a global type is \( f \)-generic if it is strongly \( f \)-generic (see Corollaries 3.8 and 3.10).

Let us first recall the definitions. Throughout we assume \( T \) is NIP.

Definition 3.1. A definable group \( G \) is definably amenable if it admits a \( G \)-invariant Keisler measure on its definable subsets.

A Keisler measure is a finitely additive probability measure on definable subsets of \( G \). We will not use this definition, so there is no need for us to get too deeply into Keisler measures. Instead we will use the following characterization from \cite{CST15} given in terms of \( G \)-dividing.

Definition 3.2. For \( X \subseteq G \) definable, we say that \( G \)-divides if there is an indiscernible sequence \( \langle g_i \mid i < \omega \rangle \) of elements from \( G \) over the parameters defining \( X \) such that \( \{g_iX \mid i < \omega \} \) is inconsistent (equivalently, remove the indiscernibility assumption and replace it with \( k \)-inconsistency).
Similarly, we say that $X$ right-$G$-divides if there is a sequence as above such that $\langle Xg_i \mid i < \omega \rangle$ is inconsistent.

**Fact 3.3.** [CS15 Corollary 3.5] Let $G$ be a group definable in an NIP theory. Then if $G$ is definably amenable then the family of $G$-dividing subsets of $G$ forms an ideal. Hence in this case any non-$G$-dividing partial type can be extended to a global one.

As an example which relates to the previous section, we note that any countable $\omega$-categorical group is locally finite and hence it is amenable by [Run02 Example 1.2.13] and so any group elementarily equivalent to it is definably amenable [Sim15 Example 8.13].

**Definition 3.4.** A global type is called $f$-generic if it contains no $G$-dividing formula.

**Remark 3.5.** [CS15 Proposition 3.4] If $G$ is definably amenable, then $p$ is $f$-generic iff all its formula are $f$-generic, which means that for every $\varphi \in p$, no translate of $\varphi$ forks over $M$ where $M$ is some small model containing the parameters of $\varphi$. It is also proved there that a formula is $f$-generic iff it does not $G$-divide, so we will use these terms interchangeably. Similarly, we will write right-$f$-generic for non-right-$G$-dividing.

**Fact 3.6.** [CS15 Proposition 3.9] When $G$ is definably amenable then a global type is $f$-generic type iff it is $G^{00}$-invariant.

**Theorem 3.7.** Suppose that $G$ is dp-minimal and definably amenable. Then if $\varphi(x,c)$ forks over a small model $M$, then it $G$-divides.

**Proof.** Suppose not.

By assumption (and as forking equals dividing over models, see [CMR12]), there is an $M$-indiscernible sequence $\langle c_j \mid j < \omega \rangle$ such that $\langle \varphi(x,c_j) \mid j < \omega \rangle$ is inconsistent.

However, $\varphi(x,c_j)$ is still $f$-generic.

We now divide into two cases: either there is a formula $\psi_0(x,b)$ which is right-$f$-generic but not $f$-generic (call this case 0), or not (case 1). In case 0, let $\xi_0(x,y,b) = \psi_0(y^{-1}x,b)$.

Note that if case 0 does not occur, then every $G$-dividing formula also right-$G$-divides. As $X$ is $f$-generic iff $X^{-1}$ is right-$f$-generic, this means that every right-$G$-dividing formula also $G$-divides.

In case 1, choose a formula $\psi_1(x,b)$ for which, for every formula $\chi(x) \supseteq G^{00}$ (with no parameters) both $\psi_1(x,b) \land \chi(x)$ and $\neg\psi_1(x,b) \land \chi(x)$ are $f$-generic (such a formula exists, as otherwise there is a unique non-$G$-dividing type concentrating on $G^{00}$, so the number of $f$-generic types is bounded, but by assumption there are unboundedly many). Let $\zeta_1(x,y,z,b) = \psi_1(y^{-1}x,b) \land \neg\psi_1(z^{-1}x,b)$. We may assume that $\langle c_j \mid i < \omega \rangle$ is indiscernible over $Mb$.

Depending on the case, let $\zeta(x,y,z,b)$ be either $\zeta_0$ or $\zeta_1$ (so $z$ might be redundant). Construct a sequence $\langle I_i, g_i, h_i \mid i < \omega \rangle$ such that:
In case 0, $g, h_i \in G$. In case 1, $g, h_i \in G^{00}$.

- $I_i$ is indiscernible, $I_i = \langle e_{i,j} \mid j < \omega \rangle$, $e_{i,j} \models \varphi(x, c_j)$ for all $i, j < \omega$.
- $e_{i,j} \models \zeta(x, g, h_i, b)$ for all $i, j < \omega$.
- $e_{i,j} \not\models \zeta(x, g, h_i, b)$ for all $i', i, j < \omega$ whenever $i' > i$.

(In case 0, we only need $g_i$.) How?

Note that for any $g, h \in G^{00}$, $\zeta(x, g, h, b)$ does $G$-divide by Fact 8.3. Hence $\varphi(x, c_j) \setminus \zeta(x, g, h, b)$ is not empty for all $j$, and hence we may find a sequence $e_{n,j} \models \varphi(x, c_j) \setminus \zeta(x)$. Consider the sequence $I = \langle (e_{0,j}, \ldots, e_{n,j}, c_j) \mid j < \omega \rangle$. By Ramsey and compactness there is an $Mh_{<n}g_{<0}b$-indiscernible sequence $I'$ with the same EM-type as $I$ over $Mh_{<n}g_{<0}b$. There is an automorphism taking $\langle e_j' \mid j < \omega \rangle$ to $\langle e_j \mid j < \omega \rangle$ over $Mh$, and applying it we are in the same situation as before (changing $h_{<n}g_{<0}$ and $e_{i,j}$) but now $I_n = \langle e_{n,j} \mid j < \omega \rangle$ is indiscernible. This takes cares of all the bullets except the third one, for which needs to find $g_n, h_n$.

In case 0, the set $\{ \psi_0(x, b) \cdot e_{n,j}^{-1} \mid j < \omega \}$ is consistent (as $\psi_0(x, b)$ does not right-$G$-divide), so contains some $g \in G$, hence $g \models e_{n,j} = \psi_0(x, b)$, i.e., $e_{n,j} \models g^{-1} \cdot \psi_0(x, b) = \zeta_0(x, g^{-1}, b)$ so let $g_n = g^{-1}$.

In case 1, the set $\{ (\chi(x) \land \psi_1(x, b)) \cdot e_{n,j}^{-1} \mid G^{00} \subseteq \chi(x), j < \omega \}$ is consistent (as $G$-dividing $= \chi$-$G$-dividing in this case) so again we can find $g \in G^{00}$ realizing it, so in particular $e_{n,j} \models g^{-1} \cdot \psi_1(x, b)$. Similarly, there is some $h \in G^{00}$ such that $e_{n,j} \models h^{-1} \cdot \neg \psi_1(x, b)$. Finally, choose $g_n = g^{-1}$ and $h_n = h^{-1}$.

This finishes the construction.

Now by Ramsey and compactness we may assume that $(I_i, g_i, h_i \mid i < \omega)$ is indiscernible over $Mg$ and that $(\langle e_{i,j} \mid i < \omega \rangle \setminus \psi_0(x, b)) \models G^{00}$.

- The $\zeta(x, g, h_i, b)$-type is called strongly $f$-generic if it is $f$-generic and does not fork over some small model (this is not the original definition, but see [CS15] Proposition 3.10].

**Corollary 3.8.** If $G$ is a dp-minimal definably amenable group, then any global $f$-generic 1-type $p$ is strongly $f$-generic.

**Proof.** Take any small model $M$. Then $p$ cannot divide over $M$. □
Remark 3.9. Theorem 3.7 does not hold for a group definable in a dp-minimal theory. Consider $T = RCF$, and let $G = R^2$ where $R$ is a saturated model of $T$. Example 3.11 in [CS15] gives a $G$-invariant type $r(x, y)$ (so does not $G$-divide) which is not invariant over any small model $M$.

**Corollary 3.10.** Suppose that $G$ is as above. Then there are boundedly many global $f$-generic types.

**Proof.** Fix some small model $M$. This follows by NIP and (the proof of) Corollary 3.8, as there are boundedly many global types non-forking over $M$ (by NIP they must be invariant over $M$).

**Corollary 3.11.** Suppose $G$ is as above. Then there are boundedly many $G$-invariant Keisler measures.

**Proof.** Suppose that there are unboundedly many such measures $\mu_i$. Fix some small model $M$. By Erdős-Rado, we may find a formula $\varphi(x, y)$, a type $p \in S_y(M)$ some numbers $\alpha \neq \beta \in [0, 1]$ and a sequence of $G$-invariant Keisler measures $(\mu_i | i < \omega)$ such that for all $i < j < \omega$, $\mu_i(\varphi(x, a)) = \alpha$ and $\mu_j(\varphi(x, a)) = \beta$ for some $a = p$ in $C$.

Then there are $a, b \models p$ such that $\alpha = \mu_0(\varphi(x, a)) \neq \mu_1(\varphi(x, a)) = \beta$ and $\alpha = \mu_1(\varphi(x, b)) \neq \mu_2(\varphi(x, b)) = \beta$. In particular $\mu_1(\varphi(x, a) \triangledown \varphi(x, b)) \neq 0$. As $\mu_1$ is $G$-invariant, $\varphi(x, a) \triangledown \varphi(x, b)$ does not $G$-divide (see [CS15] Theorem 3.38], but this follows easily from the definitions), but it forks by NIP.

**Corollary 3.12.** Suppose that $(F, +, \cdot)$ is a dp-minimal field. Then every additive $f$-generic set (i.e., with respect to $(F, +, 0)$) is also multiplicatively $f$-generic.

**Proof.** Note that any abelian group is definably amenable (see [Sim15] Example 8.13]).

Suppose that $X$ is additively $f$-generic. For any $a \in F^\times$, $a \cdot X$ is also additively $f$-generic. Hence, if $X$ is not multiplicatively $f$-generic, then there is an indiscernible sequence $\langle a_i | i < \omega \rangle$ over the parameters defining $X$ such that $\{a_iX | i < \omega \}$ is inconsistent (so $k$-inconsistent for some $k < \omega$). Increasing the sequence to any length, by Fact 3.8 we get unboundedly many additively $f$-generic global types. Contradicting Corollary 3.10.

**Problem 3.13.** Is there a dp-minimal group which is not definably amenable?

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SOME REMARKS ON DP-MINIMAL GROUPS

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