A Finite Landscape?

B.S. Acharya
Abdus Salam
International Centre for Theoretical Physics,
Strada Costiera 11
34014 Trieste. Italy

M.R. Douglas
New High Energy Theory Center,
Rutgers University,
Piscataway, New Jersey
and
I.H.E.S., Bures-sur-Yvette, France

ABSTRACT We present evidence that the number of string/M theory vacua consistent with experiments is finite. We do this both by explicit analysis of infinite sequences of vacua and by applying various mathematical finiteness theorems.
1 Introduction

How many string vacua are there which agree with experimental observations? While obviously a difficult question, perhaps using statistical arguments which take advantage of the large number of vacua, we can estimate this number. If it is small, this will demonstrate that string theory is falsifiable in principle, and help us decide what type of predictions the theory can make.

A first step in this direction would be to establish whether or not this number is finite. One simplification of this is that we might only need a subset of the properties of the Standard Model to establish finiteness. The basic properties we assume are four large dimensions and no massless scalar fields.

In thinking about this question and attempting to construct infinite sequences of vacua we were led to the following conjecture (slightly refined from that made in [1]):

Conjecture 1 The number of 4d vacua with an upper bound on the vacuum energy, an upper bound on the compactification volume, and a lower bound on the mass of the lightest Kaluza-Klein tower, is finite.

We will discuss the precise definition of the quantities which enter this conjecture below. Their simplest motivation is phenomenological and they can all be related to observables in four dimensions.

The only compactifications we really understand well enough at present to use in studying this conjecture are those which are based on large volume limits; i.e. begin with compactification of the $d = 11$ and $d = 10$ supergravity theories, and then add various stringy or $M$ theoretic effects to stabilize the moduli. On general grounds infinite sequences of such vacua might emerge in three different ways:

a) topology: there could exist infinitely many topologies for the extra dimensions, or for the vector bundles used in defining gauge fields in the extra dimensions.

b) fields: there could exist infinitely many distinct solutions of the equations of motion, including Einstein metrics, brane configurations or background gauge fields. A basic example here is to add an arbitrary number $N$ of space-filling brane-antibrane pairs. We suspect that such configurations are always unstable for sufficiently large $N$, and discuss
this in section 3.2. But we haven’t proven it, so our conjecture explicitly postulates an upper bound on the vacuum energy. Such a bound is easy to motivate phenomenologically, as the actual vacuum energy is very small.

Among more subtle results for this question, in [2] it was pointed out that theorems in algebraic geometry bound the allowed values of generation number and the number of branches of bundle moduli space in heterotic string compactification, while in [3] it was recently shown that intersecting brane models on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold are finite in number.

c) *fluxes*: an infinite number of fluxes. This is conceptually not so different from (b), but we separate it out as it can be studied more systematically. By now there are a fair number of results showing that flux vacua are finite in number; let us mention results in [4] and a theorem proven in [5] according to which the integrals of the densities on Calabi-Yau moduli space, which give the asymptotic number of flux vacua in IIb and other compactifications, are finite.

In the next section we will describe sequences of vacua with infinitely many topologies and explain why they do not violate our finiteness conjecture. We go on to describe Cheeger’s finiteness theorem in Riemannian geometry [6], and how it can be applied to demonstrate that certain classes of string vacua (e.g., Calabi-Yau and $G_2$ vacua) contain only finitely many topological choices for the extra dimensions in the supergravity approximation. This establishes that, even if there were infinitely many topologically distinct Calabi-Yau or $G_2$ manifolds, only finitely many of them can be consistent with the Standard Model. In the last section of the paper we address the question of infinitely many field configurations, giving an example of an infinite sequence which does not contradict 1; we finish by relating the finiteness of the number of vacua to Gromov’s compactness result [7] for the space of Riemannian metrics.

The fact that experimental observations put a bound on the allowed topologies of the extra dimensions is, whilst striking from a physical point of view, a consequence of the interplay between ‘dynamics’ and topology in the context of Riemannian geometry.

### 2 Infinite topologies?

Are there infinitely many topologically distinct choices for the extra dimensions in string/M theory? The answer is yes, as we will illustrate by example
shortly. However, we will also demonstrate that in many classes of vacua in the supergravity approximation, *only finitely many topological choices can be consistent with experimental observations*. In particular, it is not known whether or not there are finitely or infinitely many diffeomorphism types of compact Calabi-Yau threefolds or $G_2$-holonomy manifolds. Our results will show that *it does not matter* that the number could be infinite, by proving there can be at most finitely many vacua which are consistent with the conjecture.

Before describing this result, we will begin by discussing explicit examples of infinite sequences of topologies for the extra dimensions. These examples originate in the fact that there exist infinite sequences of topologically distinct Einstein manifolds with positive cosmological constant. For example, consider quotients of round spheres by cyclic groups of arbitrary order. Since in dimension seven such Einstein manifolds form part of the data for Freund-Rubin vacua [8] (see [9] for a review), we have infinite sequences of topologies for the extra dimensions in such vacua. In this section we will explicitly analyse these and more non-trivial infinite sequences of Einstein manifolds and show how they are consistent with the conjecture.

### 2.1 Review of Freund-Rubin vacua

These are near horizon geometries of brane metrics in which there is an $AdS$ factor. We will focus on the $M2$-brane case for which the Freund-Rubin metric takes the form

$$g_{10+1} = g(AdS4_a) + a^2g^{(0)}(X)$$

Here $AdS4_a$ is anti de Sitter space of radius $a$ and $X$ is a compact 7-manifold. We take non-zero electric four-form flux $F \propto \epsilon_{0123}$; then Einstein’s equation

$$R_{ij} = \frac{1}{3}F_{ipqr}F_{j}^{pq} - \frac{g_{ij}}{36}(|F|^2 - m^2)$$

forces $g_{0}(X)$ to be an Einstein metric. We define this to have fixed positive scalar curvature, satisfying

$$R_{ij}(g^{(0)}) = 6g_{ij}^{(0)}.$$  \hspace{1cm} (3)

Note that this condition determines the overall normalization of $g^{(0)}$, and thus we call it a “normalized metric.”

Since the actual metric on the seven extra dimensions is rescaled by $a^2$, the scalar curvature of $X$ is of order $a^{-2}$, as is that of the four dimensional AdS universe. Because of this, if the volume of $X$ as measured by $g_{0}(X)$ is order
one, then the masses of Kaluza-Klein modes are of order the gravitational mass in AdS. In this case, there would be no meaningful four dimensional limit in which one can ignore the dynamics of the Kaluza-Klein particles. While this is true for $S^7$, if we could find an example in which the radii of $X$ in the $g^{(0)}$ metric were much smaller than one, there would be a mass gap between the gravitational fluctuations and Kaluza-Klein modes, and there would be a four dimensional limit [10].

2.1.1 Quantization of $a$

If we think of the metric as the near horizon geometry of a brane metric, then the parameter $a$ is related to the integer number $N$ of branes, as

$$a^6 = \frac{NL_p^6}{V_0(X)},$$

where $L_p = 1/M_p$ is the eleven dimensional Planck length and $V_0$ is the volume of $X$ as measured by $g_0(X)$.

To show that Eq. (4) is correct, we need the formulae for $g(AdS4)$ and the $G$-flux in the vacuum. These are respectively

$$g(AdS4_a) = \frac{a^2}{r^2} dr^2 + \frac{r^4}{a^4} g_{2+1} \tag{5}$$

where $g_{2+1}$ is the $2 + 1$-dimensional Minkowski metric, and

$$G \sim a dVol(AdS4_a) = \frac{r^5}{a^4} dV_{2+1} \wedge dr \tag{6}$$

where $dV_{2+1}$ is the volume form on the Minkowski space.

Because this is the background solution for $N M2$-branes, the $M2$-brane charge, which is the integral of $\ast G$ over $X$ in the metric Eq. (1), must be $N$. Hence

$$N = \int_X \ast G = a^6 V_0 \tag{7}$$

We prefer to write physical quantities in terms of $N$ rather than $a$.

The four dimensional Planck scale $M_{pl4}$ is related to the volume of $X$ in 11d Planck units as

$$\frac{M_{pl4}^2}{M_p^2} = V(X) \sim \frac{N^7}{V_0^4} \tag{8}$$

Notice that as $V_0$ decreases, $V$ actually increases. This may at first seem counterintuitive, but actually it is a simple consequence of the quantization
of flux. In order to maintain the integrality of Eq. (7), a decrease in \( V_0 \) must be accompanied by an increase in \( a \) and thus \( V \). This fact will play an important role in what follows.

The (negative) cosmological constant is of order

\[
|\Lambda| \sim M_{\text{pl}}^4 \frac{\frac{1}{2} V_0^2}{N^2}
\]  

(9)

The above two formulae can also be rewritten as:

\[
V^3|\Lambda| = N^2
\]  

(10)

and

\[
V^9|\Lambda|^7 = V_0^2
\]  

(11)

2.2 Finite number of solutions

Let us begin by considering \( X = S^7 \). From the relations above, we see that as \( N \) tends to infinity, the physical volume goes to infinity. Thus, Freund-Rubin vacua are simple examples of infinite sequences of vacua which do not violate the finiteness conjecture; rather they violate one of its assumptions.

What if we consider more general compactification manifolds? Since the relation Eq. (8) between \( V_0 \) and the four dimensional Planck scale followed directly from Eq. (7), which is true regardless of the topology of the extra dimensions, we can make the same argument for any infinite sequence of vacua in which the normalized volumes approach zero, to show that the actual volume of the extra dimensions will again go to infinity.

A simple example is the infinite sequence of Einstein 7-manifolds constructed by taking a \( Z_k \subset U(1) \) quotient of an Einstein manifold with \( U(1) \)-symmetry. Here \( V_0 = (V_0(k = 1))/k \) by symmetry.

What about the opposite possibility, a sequence \( X_i \) in which the normalized volumes go to infinity? This is not possible, because of

**Theorem 2.1** (Bishop, 1963 [11]) A \( d \)-dimensional manifold \( X \) with a lower bound on the Ricci tensor,

\[
R_{ij} \geq (d - 1)k g_{ij}
\]  

(12)

with \( k > 0 \), has volume less than or equal to that of the round \( (SO(d + 1)\)-symmetric) sphere of curvature \( k \),

\[
\text{vol } M \leq \text{vol } S^d(k),
\]

with equality only for \( X \cong S^d \) (Cheng 1975).
Our normalized metrics satisfy Eq. (12) with $k = 1$, so at fixed flux $N$, all of
the others lead to larger physical volumes $V$ than the round sphere.

Thus, to invalidate the conjecture with generalized Freund-Rubin vacua,
we must find an infinite sequence of Einstein metrics with finite normalized
volume. Now there do exist much more non-trivial infinite sequences
of distinct Einstein metrics. For instance, [12] construct several infinite se-
quencies of Einstein 7-manifolds by constructing sequences of conical eight
dimensional hyperkahler quotients. The bases of these cones are Einstein
7-manifolds. Unfortunately, to our knowledge the volumes have not been
computed in these examples. We will argue later that they cannot provide
counterexamples, but this will follow from more abstract considerations; here
let us turn to a more explicit case.

Recently in the context of supergravity solutions and the AdS/CFT corre-
spondence, Gauntlett et al [13] have constructed infinite sequences of Einstein
7-metrics, again with positive scalar curvature. The volumes of these ex-
amples have been computed in unpublished work of Martelli and Sparks [16].
These Einstein 7-manifolds $Y^{p,k}(M)$ are labelled by two integers $(p,k)$ and
are $S^3/Z_p$ bundles over Kahler Einstein 4-manifolds $M$. The two classes of
explicit examples are when $M$ is either $\mathbb{C}P^2$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$. More recently,
[14, 15] have shown that these $Y^{p,k}$ metrics are special cases of more general
families of Einstein metrics which depend upon three integers.

Let us discuss the manifolds $Y^{p,k}(\mathbb{C}P^2)$; the others are very similar. The
integer $k$ is bounded by the values of $p$ as

$$3p/2 \leq k \leq 3p.$$  \hspace{1cm} (13)

Thus, to obtain an infinite sequence we must take $p$ to infinity. The most
important result of [13] for our purposes is the nature of the manifold for the
‘boundary values’ of $k$. One can show that

$$Y^{p,3p}(\mathbb{C}P^2) = S^7/Z_{3p}$$  \hspace{1cm} (14)

and that

$$Y^{p,3p/2}(\mathbb{C}P^2) = M^{3,2}/Z_{p/2}$$  \hspace{1cm} (15)

where $M^{3,2}$ is the homogeneous Stiefel manifold.

Thus the two limiting values of $k$ are cyclic quotients of homogeneous
Einstein manifolds. To proceed further we need some information about the
volumes of $Y^{p,k}(\mathbb{C}P^2)$. One can show (cf [16]) that

$$V_0(M^{3,2}/Z_{p/2}) > V_0(Y^{p,k}(\mathbb{C}P^2)) > V_0(S^7/Z_{3p})$$  \hspace{1cm} (16)

From these bounds it follows that as $p$ tends to infinity $V_0(Y^{p,k}(\mathbb{C}P^2))$
goes to zero. Hence by our previous discussion the actual volume of the extra
dimensions diverges owing to the flux quantization condition.
To summarise: we have shown in several explicit infinite sequences of four dimensional Freund-Rubin vacua that only finitely many can satisfy an upper bound for the volume of the extra dimensions.

3 Cheeger’s Finiteness Theorem

A remarkable theorem of Cheeger can help explain such results.

**Theorem 3.1 (Cheeger, 1970 [6]):**

In a (potentially infinite) sequence of Riemannian manifolds $M_i$ with metrics such that

1. the sectional curvatures $K$ are all bounded, say $|K| \leq 1$
2. the volumes are bounded below; $V_i \geq V_{\text{min}}$
3. the diameters are bounded above, $D_i \leq D_{\text{max}}$

there can only be a finite number of diffeomorphism types.

In less formal language, this theorem states that given that all components of the Riemannian curvature tensors (in an orthonormal basis) of all the manifolds in the sequence remain bounded, and that the manifolds do not become too elongated in some direction, and that their volumes do not go to zero, then there can be only finitely many topological types in the sequence.

Let us first explain how we can use this theorem to establish our conjecture, in the supergravity approximation, in a wide variety of cases. We will then explain the mathematics behind it in an intuitive way. Note that the conditions of the theorem involve dimensionful quantities. We will take these to be measured in units of the largest fundamental length scale (the string scale or the 10 or 11-dimensional Planck scale).

We first want to argue that the hypotheses of Conjecture 1 imply the hypotheses of Cheeger’s theorem. First, conditions 1 and 2 are trivial in the following sense. In general, string and $M$ theory physics can be described using supergravity and Riemannian geometry in the “supergravity regime,” in which curvatures and other field strengths are small compared to the string or higher dimensional Planck scales, all closed geodesics are longer than the string scale, volumes of minimal cycles are larger than these scales, and so on. Thus, we must impose conditions 1 and 2 with $V_{\text{min}} = 1$, just to remain in this regime.

Of course, one knows that string/$M$ theory is sensible beyond this regime, and thus our finiteness arguments cannot cover all possible vacua. The point however is that we want to exclude violations of our conjecture in this regime, a problem which can be addressed using present-day mathematics. If it holds there, we can go on to think about other, more stringy regimes later.
This leaves condition 3. This condition bounds the diameter $D$, defined as the supremum of the distance between any pair of points. Intuitively, one can argue that as the diameter becomes large, the manifold elongates and the smallest eigenvalues of the Laplacian will go to zero, so that a Kaluza-Klein mode becomes light. Thus, a lower bound on the KK scale, should imply an upper bound on the diameter. Phenomenological bounds on the KK scale will be around 1TeV (for particle with Standard Model quantum numbers), or $(10 \mu m)^{-1} \sim 10^{-3}$eV for graviton KK modes, which would lead to corrections to the inverse square law for gravity.

To make this precise, we begin by considering the smallest non-zero eigenvalue $\lambda_1$ of the scalar Laplacian on the extra dimensions,

$$\Delta \phi = \lambda_1 \phi. \tag{17}$$

Writing

$$g_{\mu\nu}^{(d+4)} = g_{\mu\nu}^{(4)} \cdot \phi + \ldots, \tag{18}$$

we see that this will be the mass squared of a graviton KK mode in four dimensions.

Next, by taking powers $\phi^n$ of the wave function, we might expect to get approximate eigenfunctions with eigenvalues $n\lambda_1$, and an entire Kaluza-Klein tower. However this is not obvious; a single field might be becoming light, which could happen in various ways unrelated to the diameter condition. Nevertheless, let us provisionally take $\lambda_1$ as the definition of the Kaluza-Klein mass squared, and return to this point below.

We want to argue that, if we put a phenomenological lower bound on $\lambda_1$,

$$M_{KK}^2 \leq \lambda_1, \tag{19}$$

this will enforce an upper bound on the diameter, justifying condition 3. This will be true if we can find a mathematical upper bound on $\lambda_1$ in terms of the diameter, say of the general form

$$\lambda_1 \leq \frac{C}{D^2} \tag{20}$$

suggested by dimensional analysis, where $C$ might depend on other data in some bounded way. Combining this with the phenomenological bound Eq. (19), we would have the desired upper bound,

$$D^2 \leq \frac{C}{M_{KK}^2}.$$ 

In fact a bound Eq. (20) can be proven by variational arguments [18]. We will explain the details of this shortly, but first let us discuss what this will
imply. We have now argued that, if we restrict attention to compactifications in the supergravity regime, a sequence of vacua satisfying the hypotheses of conjecture 1 will come from a sequence of compactification manifolds satisfying the hypotheses of Cheeger’s theorem. Therefore, the sequence can only contain a finite number of distinct topologies. In particular the sequence could be “all” vacua satisfying our conditions, so to get all vacua we would only need to consider a finite list of possible topologies, even if there turn out to be infinitely many topologies of (say) Calabi-Yau manifolds. All but a finite number of these would lead to vacua with unobserved light KK modes.

Why should Cheeger’s theorem be true? The basic intuition can perhaps be seen by thinking of a Riemann surface of constant negative curvature. To increase the genus, one must attach handles; this always increases the volume (due to the simplicity of two dimensions) but does not “obviously” increase the diameter. However, if one thinks of the Riemann surface as some piece of the Poincaré disk with identifications along its boundaries, it is clear that increasing the volume must increase the diameter, albeit at a very slow (logarithmic) rate.

The actual proof is rather intricate but breaks down into several components. The basic idea is to use the hypotheses to show that any manifold $M_i$ in the sequence can be covered by a definite finite number $N$ of convex balls of a fixed radius $r$, and use this to reduce the problem to combinatorics. To show this, one combines various theorems in “comparison geometry,” according to which the local geometry of any manifold satisfying curvature bounds has to be “similar” (in some precise sense) to a constant curvature space. This allows bounding the minimal volume of a ball, which is clearly necessary. Subtle arguments are needed to show that one can actually use balls of a definite radius; in particular one might worry that the manifolds might contain very short noncontractable loops, which would be a problem. A lower bound on the length of such loops is the injectivity radius $r_i$, defined as the minimum over all points $p$ of the radius at which Riemann normal coordinates around $p$ break down (are no longer one-to-one). This will be the shorter of half the length of the shortest periodic geodesic, or else the shortest distance between conjugate points. It turns out that the injectivity radius can be bounded below, and that doing this requires imposing an upper bound on the diameter, rather than the volume or something else.

Once we have shown that $M_i$ can be covered by a finite number of balls, we can imagine describing the topology of $M_i$ as a simplicial complex, in which balls become vertices, a pairwise overlap becomes a link, a triple overlap becomes a face, and so on. Because the balls are convex, the overlap regions are all contractible, so this complex will have the same homotopy type as the original manifold. In addition, the curvature conditions can be used to
show that the transition functions between the balls are diffeomorphisms, so the complex actually determines the diffeomorphism type. Then, the evident fact that given a finite number of vertices, one can make only a finite (though very large) number of different simplicial complexes, implies the theorem.

Now let us come back to Eq. (20). The basic intuition here is simple; one chooses two points \( p \) and \( q \) whose separation \( d(p, q) \) is the diameter \( D \), and considers a wavefunction with derivatives of order \( \pi/D \),

\[
\phi(x) = \sin \frac{\pi d(p, x)}{D} - \text{const.}
\]

where the constant is chosen to make it orthogonal to the ground state, \( 0 = \int \phi \). A variational argument should then tell us that

\[
\lambda_1 \leq \frac{\int \phi \Delta \phi}{\int \phi^2} \sim \frac{\pi^2}{D^2}.
\]  

This is basically right but there is an important caveat which emerges from making a proof along these lines. The standard proof [18] is made by dividing the manifold into two pieces with boundary, \( M_p \) within distance \( D/2 \) of \( p \) and \( M_q \) within distance \( D/2 \) of \( q \). Assume we have a lower bound on the Ricci curvature:

\[
R_{ij} \geq (d - 1)kg_{ij}
\]

for some real constant \( k \). One can then show using a variational argument that the first nontrivial eigenvalues on each of the two pieces with Dirichlet boundary conditions are bounded above,

\[
\lambda_1 \leq \lambda_1|_{M_p}, \lambda_1|_{M_q} \leq \lambda_1(k, D/2),
\]  

by the Dirichlet eigenvalue on a ball of radius \( D/2 \) and constant curvature \( (d - 1)k \). The smaller of these is then an upper bound for \( \lambda_1 \) on \( M \).

A similar argument can be used to bound the \( m \)th eigenvalue, by dividing \( M \) into \( m + 1 \) regions. One finds

\[
\lambda_m \leq \lambda_1(k, \frac{D}{2m})
\]

and thus \( \lambda_m \sim m^2/D^2 \) as is appropriate for a KK tower. This justifies the definition we made, in the sense that while the true \( \lambda_1 \) might not be the first mode of a KK tower, the \( \lambda_1 \) which is bounded by Eq. (22) will be the first of a tower.

The eigenvalue on the ball \( \lambda_1(k, D/2) \) can be written explicitly in terms of Bessel functions. For non-negative Ricci curvature, it indeed falls off as
On the other hand, large curvature can also make the lowest non-zero eigenvalue large, as one can see from the ball; for negative $k << -1/D^2$ we have $\lambda(k, D/2) \sim (d - 1)^2|k|/4$.

This caveat is not important for the Freund-Rubin case (actually Bishop’s theorem forces $k \sim 1/D^2$ for positive curvature anyways), so we have now justified the claims made in section 2, and can assert that the number of topologies which can lead to quasi-realistic vacua is finite. This is not the end of the story as we can imagine infinitely many solutions on a single topology, but we discuss this in section 4.

The caveat is also unimportant for Ricci flat compactification manifolds such as Calabi-Yau and $G_2$-holonomy, so we can make the same statement there: even if there were an infinite number of Calabi-Yaus or $G_2$-holonomy manifolds, only finitely many topologies can be used as models for the extra dimensions in the supergravity approximation.

What about the case of negative Ricci curvature? One might at first think that this would be ruled out by some sort of positive energy theorem, but this is not so obvious; for example the trace term in Einstein’s equations Eq. (2) can be negative in the presence of magnetic flux.

We should distinguish two cases. On the one hand, more realistic solutions will have branes, flux, quantum effects and so on, potentially leading to corrections to the metric and small amounts of negative Ricci curvature. This will not significantly affect the spectrum of the Laplacian and the bound Eq. (22), so is not a problem for the argument.

On the other hand, if we could find supergravity solutions with compactification manifolds with large negative Ricci curvature compared to $1/D^2$, we might well find that the Laplacian has a gap independent of the diameter. While Cheeger’s theorem would still tell us that any infinite series of compactifications must run off to infinite diameter, if the volume and the KK scale stay bounded, there might be no sign of this from four dimensional physics.

This seems on the face of it unlikely, and indeed the flux contributions to the stress-energy will fall off with compactification volume. Actually, we do not know of static supergravity solutions with negative curvature, so perhaps there is a general argument against this possibility.

### 3.1 Singularities

Cheeger’s theorem applies to smooth manifolds. In string theory however, special kinds of singularities are physically acceptable. For example orbifold singularities and conical singularities in space are often physically sensible.
Is there a natural extension of Cheeger’s theorem which includes singularities of certain kinds?

In fact for 4-dimensional spaces there is such a generalisation, due to Anderson [19]. Anderson proves that the set of compact four dimensional orbifolds in which the magnitude of the Ricci tensor is bounded, the volume is bounded below and the diameter above contains finitely many topological types. In other words, if one replaces condition 1 of Cheeger’s theorem with a bound on the Ricci tensor and allows orbifold singularities then there are again finitely many topological types. From a physical point of view the bound on the Ricci tensor is much more natural since this is equivalent to a bound on the energy-momentum tensor which is completely reasonable. Unfortunately, extensions of Anderson’s result to higher dimensions have only been obtained with an additional bound on the $L^{n/2}$ norm of the Riemann tensor, which does not appear to have a clear physical interpretation. This latter condition implies that the curvature of the extra dimensions cannot diverge too quickly at the singularities. It would be interesting to see, if by including more general conical singularities, a more “physically” reasonable version of Anderson’s theorem applies.

3.2 Infinite sequences of non-supersymmetric vacua

One can also motivate other ways to set the bounds in the theorem.

For non-supersymmetric vacua, we could try to argue for condition 1, the curvature bound, on grounds of stability. Let us consider the fluctuations of the metric $g_{ij}(X)$ in the extra dimensions. The modes $\delta g_{ij}$ corresponding to scalar fields in four dimensions have a contribution to their mass-squared given by eigenvalues of the Lichnerowicz operator on $X$,

$$-\nabla_i^2 \delta g_{ij} + 2R^k_i \delta g_{jk} - 2R^m_{\ j} \ ^n \delta g_{mn} \equiv \Delta_L \delta g_{ij} \propto m^2 \delta g_{ij} \quad (23)$$

Unlike the Laplacian, $\Delta_L$ can have negative eigenvalues on a compact manifold, leading to possible tachyons and instability. This is of course classical and small tachyonic masses might be compensated for by quantum corrections, but large Riemann curvatures will lead to large tree level masses which cannot be compensated. Thus, stability suggests the imposition of the first condition.

How generic a problem is this? For supersymmetric compactifications, a scalar mass squared will be related to a fermion mass $m_F$ as

$$m^2 = m_F(m_F - \sqrt{3|\Lambda|/M_{pl}^4}). \quad (24)$$

For Minkowski compactifications, this is manifestly non-negative, while more generally minimizing with respect to $m_F$ leads to the Breitenlohner-Freedman
bound. For phenomenology, we are most interested in either the physical case \( \Lambda \sim 0 \), or the case of \( \Lambda = -3|W|^2 \ll M_{\text{pl}}^4 \) so that adding small supersymmetry breaking effects will bring \( \Lambda \) to zero. Either way, there is a small window \( 0 < m_F < \sqrt{3|\Lambda|/M_{\text{pl}}^4} \) for instability, while large variations of the Riemann tensor in Eq. (23) would presumably lead to large variations of \( m_F \) and push \( m^2 \) positive again. Thus this general type of instability due to KK modes would be controlled, as long as the supersymmetry breaking scale is below the Planck scale.

More generally, this type of consideration motivates placing an upper bound on the supersymmetry breaking scale. By the standard supergravity formula

\[
V = |F|^2 + |D|^2 - 3|W|^2,
\]

this will imply an upper bound on the vacuum energy.

4 Infinite classes of solutions

One must next ask whether there can exist infinitely many distinct vacuum metrics for a fixed topology.\(^1\) For instance, infinitely many Einstein metrics on a fixed topology. In the Calabi-Yau and \( G_2 \)-holonomy cases the answer is of course yes, there is a continuous moduli space of metrics. In these cases however, this degeneracy is removed by fluxes and/or other contributions to the moduli potential, and finiteness in the physically relevant case of vacua without massless fields is addressed by studying flux vacua.

In the Freund-Rubin case, for instance, the situation is different. Einstein metrics of positive scalar curvature tend to be rigid. However, for a fixed topology there can exist infinitely many distinct Einstein metrics! To study this concretely, we need an explicit infinite sequence for which the volumes are known. In dimension 7 we do not know of such examples, however there does exist an infinite series of Einstein metrics on \( S^2 \times S^3 \), which were also discovered in [13] and depend upon two integers \((p, q)\). Analogously to equation Eq. (16), one can show that the volumes of these manifolds are bounded above and below by the volumes of \( \mathbb{Z}_p \) quotients of homogeneous spaces [17]. This shows that only finitely many of these manifolds have a finite volume. It would be interesting to find out whether this type of result generalizes to any infinite sequence of Einstein metrics with positive cosmological constant on a space with fixed topology.

\(^1\)We particularly thank Is Singer for emphasizing this issue.
4.1 Finiteness, barriers and distances between vacua

The examples of continuous Calabi-Yau and $G_2$ moduli spaces show that we will in any case need to bring in more than Riemannian geometry and Einstein’s equations to proceed. Since the full problem of moduli stabilization is very complicated, we again ask what simplifications could be made which would still suffice to get a convincing argument for finiteness, and an estimate of the number of solutions.

One observation of this type [1] is that, if we know in some example that a potential is generated (say by nonperturbative effects), we can estimate the number of vacua by counting the number of minima of a “generic” potential. For the example of supergravity, this suggests that the number of vacua is the integral of a natural characteristic class for the bundle in which the covariant gradient of the superpotential $D_i W$ takes values, and this argument leads to the Ashok-Douglas density [22].

Let us make a simpler observation of this type. Suppose we can formulate a problem of finding stabilized vacua in terms of a potential $V(\phi)$ which is a function of some scalar fields $\phi^a$. These fields might have been moduli in a related theory with more supersymmetry, or not; the main point however is that a four-dimensional effective action containing these fields,

$$\int d^4 x G_{ab}(\phi) \partial \phi^a \partial \phi^b - V(\phi) + \ldots$$

will define a metric $G_{ab}$ on the configuration space parameterized by $\phi$. Now computing this metric exactly is probably even more difficult than computing the potential. But qualitative properties of the metric which imply finiteness might not be so hard to get.

A reasonable conjecture, made in various talks of the second author around 2004–2005, is that the finite number of minima of $V(\phi)$ in some region of configuration space is tied to the finiteness of the volume of that region, in the volume form $\sqrt{\det G}$. This suggests the general conjecture [24, 26, 27] that the volumes of all of the moduli spaces which arise naturally in string compactification (Calabi-Yau moduli space, moduli spaces of conformal field theories, and so on) are finite. Of course just saying this is not in itself a physical argument for why such volumes should be finite; other arguments which have been proposed have been based on the renormalization group in CFT [26], or consistency after incorporating gravity [27].

However, one can easily imagine loopholes to a connection between numbers of vacua, and the volume of configuration space. Mathematically, the basic example is the function

$$V(\phi) = \sin \frac{1}{\phi}$$
which has an infinite sequence of minima accumulating at zero. If we take
the canonical kinetic term for $\phi$, there are infinitely many minima in a region
of finite volume.

Of course this is just postulating a function and there is no reason to
think that such a potential can arise physically. On the other hand we would
like some criterion which distinguishes it from the potentials we think can
arise physically, to see why we should not worry about this possibility.\(^2\)

One such is to ask that the distance between vacua in field space, be
comparable to the energy scale $\left(\Delta V\right)^{1/4}$ set by the height of the potential
barrier between the vacua. This is certainly a natural condition if the physics
does not depend on additional scales; whether or not it is universally true,
we do not know. In any case there are toy potentials satisfying this condition
with an infinite number of vacua, for example

$$V(\phi) = \phi^8 \sin \frac{1}{\phi}. \quad (26)$$

However such potentials will (by definition) lead to sequences of vacua for
which the successive barrier heights go to zero. Thus, if we were working at
any fixed energy scale $E$, as we approached $\phi \to 0$ we would eventually decide
that the potential was becoming unimportant. This suggests a definition in
which, if two vacua are only separated by a potential barrier less than some
prespecified minimal height $\epsilon$, we should consider them as the same physical
vacuum.

A related but slightly different criterion would be to insist that physically
distinct vacua have some minimal separation in configuration space. One
reason to prefer this definition is that vacua with different vacuum expecta-
tion values for fields will in general make different predictions even if not
separated by a potential barrier, as is familiar for compactifications with ex-
tended supersymmetry (consider masses of BPS states). Conversely, if two
minima of $V(\phi)$ are separated by a distance less than some prescribed mini-
mal separation $\epsilon$ in configuration space, we might consider them as the same
physical vacuum. One might argue that in this case quantum fluctuations
will produce transitions between the different vacua.

Another argument for this definition would be to think about the physical
observables as functions of the field values in configuration space: say masses,
couplings, etc. are all written as

$$m(\phi), g(\phi), \ldots$$

\(^2\)A somewhat more physical model of this type was proposed in \([28]\), but so far as we
know it does not come out of string theory compactification.
To the extent that these are continuous functions, there will be some \( \Delta \phi \) below which we cannot distinguish the physical predictions. This argument also suggests its own limitations – if there are phase transitions, i.e. other parameters in the configuration which we neglected, which after a small shift in \( \phi \) can vary by large amounts, this need not be true. So we get an idea of a “sufficient” specification of the configuration, i.e. one in which we included all the fields which could vary significantly and which affect the observables.

The upshot of this rather general discussion is the following

**Hypothesis 1** There exists a minimal distance \( \epsilon \) in configuration space between physically distinct vacua.

Its justification comes from two alternatives. The simpler is that this is true of the actual potentials which arise from string theory. But, although we do not know of examples, one can imagine potentials such as Eq. (26) which describe infinite series of vacua with accumulation points. If these lead to distinct observable physics, then Conjecture 1 will be wrong. On the other hand, if the physical observables also converge in the limit, we should count vacua which are separated by distance less than \( \epsilon \) as physically “the same,” also justifying the hypothesis.

### 4.2 Convergence and precompactness

One reason we bring this topic up is that in the mathematics literature we have been citing, a rather central idea is that of a topology on the space of Riemannian manifolds. In other words, one wants to be able to say when a sequence of manifolds \( M_i \) converges, and if so, to what.

This can be related to the idea of distance on configuration space we were just discussing. Suppose we have a family of metrics for the compactification manifold \( X \) with parameters \( \phi^i \); then the parameters can be interpreted as fields in four dimensions, the Einstein (or supergravity) action specialized to the family can be interpreted as a potential

\[
V(\phi) = \int_X \sqrt{g} R[g(\phi)],
\]

and the metric on the space of metrics will give the kinetic term,

\[
G_{ab}(\phi) = \int_X \sqrt{g} g^{ij} \partial g_{ij} \partial g_{kl} \frac{\partial \phi^a}{\partial \phi^b} \frac{\partial \phi^b}{\partial \phi^b}.
\]

In particular, the metric on configuration space will imply a topology on the configuration space.
Of course topology is a weaker concept than metric; if we modify our
definition of distance in any continuous way, we will get the same topology.
In fact it is really too weak to make any statement analogous to “finiteness
of volume.” For example, the interval \( \phi \in (0, 1) \) which contained an infinite
number of vacua for the potential Eq. (26), is topologically equivalent, say
under the map \( \phi \to \phi/(1 - \phi) \), to the entire positive real axis \( \phi \in (0, \infty) \). We
would not be surprised to learn that the latter configuration space contained
an infinite number of vacua.

However the mathematicians work with something in between topology
and metric, which we now explain. Suppose we have a distance function
\( d(x, y) \) between pairs of points (say points in moduli space). Such a function
(satisfying simple axioms such as positivity and the triangle inequality) is
the general notion of metric, as opposed to a Riemannian metric. We then
define convergence in terms of this as

\[
\lim_{i \to \infty} x_i = x \iff \lim_{i \to \infty} d(x_i, x) = 0.
\]

Now suppose we had another metric \( d'(x, y) \), with

\[
d'(x, y) = \lambda(x, y)d(x, y) \quad \forall x, y.
\]

If the \( \lambda \)'s are bounded away from zero and infinity, we will get the same idea
of convergence. Thus such an equivalence class of distance functions incorpo-
rates the topology of the underlying space. And it carries more information,
for example \( d(x, y) = |x - y| \) on the interval \( (0, 1) \) is not equivalent in this
sense to \( d(x, y) = |1/x - 1/y| \) on \( (0, 1) \).

Now, given this structure, we can talk about a set \( X \) having “finite diame-
ter” or “infinite diameter,” according to whether the set of pairwise distances

\[
\{d(x, y) : x, y \in X\}
\]

is bounded or not. Rather than finite diameter, one usually speaks of \( X \)
being totally bounded. This is true if \( X \) can be covered by a finite number
of sets, each of finite diameter. This includes the case in which \( X \) has more
than one connected component.

One can prove that this is equivalent to the condition that \( X \) is precom-
pact. As the name suggests, this means that \( X \) can be Cauchy completed
(as in Cauchy’s construction of the real numbers) and the resulting space is
compact (every sequence contains a convergent subsequence).

It is tempting to add to the earlier conjecture, that a finite volume config-
uration space can only contain a finite number of vacua, the conjecture that
a precompact configuration space can only contain a finite number of vacua.
Now if we accept the hypothesis of the last subsection, this is not a conjecture but in fact follows from the definitions, since regions of configuration space of diameter less than $\epsilon$ can contain at most one vacuum.

We now quote

**Theorem 4.1 (Gromov 1981 [7])**

Let $\mathcal{M}_{d,D,k}$ be the space of $d$-dimensional Riemannian manifolds with diameter not larger than $D$, and Ricci $\geq (d - 1)k$.

This space is precompact in the Gromov-Hausdorff metric.

This is a configuration space which describes all possible metrics (satisfying the conditions), including the Einstein metrics and all other metrics. If we believe the notion of distance given by the Gromov-Hausdorff metric has any relation to physical distance, and accept the arguments of the previous subsection, this shows that there can only be a finite number of vacua including not just all topologies, but all solutions to the Einstein equations as well.

The Gromov-Hausdorff metric is not very physical. A better model for the dependence of physical observables on the underlying manifold is the spectral geometry approach of Bérard, Besson and Gallot [29]. As briefly summarized in [25], BBG define a metric between manifolds in terms of the distances between the entire set of eigenfunctions of the Laplacian. So, convergence in this metric corresponds much more directly to convergence of observables, the spectrum and wave function overlaps. One could also use this to show that this form of convergence agrees with the one defined using distances in the moduli space metric when that makes sense (this one is more general).

BBG then show that $\mathcal{M}_{d,D,k}$ is precompact using this notion of distance. So, there are at most a finite number of compactifications in this sense. Admittedly, it is hard to imagine that such a general result (which does not even impose the equations of motion) will lead to a useful bound on the number, but this illustrates the idea.

Kontsevich and Soibelman have suggested that this type of definition can be made for conformal field theories as well, and have conjectured

**Conjecture 2 (Kontsevich and Soibelman 2000 [30])**

Let $\mathcal{M}_{c,\Delta}$ be the space of two-dimensional conformal field theories with central charge $c$, and with no operators of dimension less than $\Delta$ (except the identity). This space is precompact in a “natural” metric.
The diameter condition is replaced by a lower bound on operator dimensions using the sort of relation we gave in section 3; Kontsevich has argued that CFT’s always satisfy a version of “Ricci ≥ 0” avoiding the subtlety discussed there. Note also that there is no volume condition; T-duality suggests that there is no limit in which the volume can go to zero. Clearly it would be very interesting to make a definition of the space of CFT’s and develop this; see [31] for work in this direction.

Another corollary of these theorems is that any infinite distance limit in a moduli space of metrics is associated to a violation of the hypotheses; restricting attention to the supergravity regime, it must be associated to the diameter of the manifold going to infinity. By our discussion in section 3, this implies that a tower of KK modes is becoming light. This is “Conjecture 2” in the recent [32]; we see that in the case of sigma models in the large volume limit it follows from known facts in geometry, while for CFT’s it would follow from Kontsevich and Soibelman’s conjecture.

Acknowledgements. An early version of this work was presented by the second author at Strings 2005 in Toronto, and both authors thank the organizers of that meeting for hospitality there. MRD particularly thanks Maxim Kontsevich and Alex Nabutovsky for inspiring discussions; BSA is especially grateful to Dario Martelli and James Sparks for explaining [16]. We also thank Frederik Denef, Greg Moore, Sameer Murthy, Eva Silverstein, Is Singer and Cumrun Vafa for discussions. The work of MRD is supported in part by DOE grant DE-FG02-96ER40959.

References

[1] M. R. Douglas, JHEP 0305 (2003) 046 [arXiv:hep-th/0303194].

[2] M. R. Douglas and C. G. Zhou, JHEP 0406, 014 (2004) [arXiv:hep-th/0403018].

[3] M. R. Douglas and W. Taylor, arXiv:hep-th/0606109.

[4] T. Eguchi and Y. Tachikawa, JHEP 0601, 100 (2006) [arXiv:hep-th/0510061].

[5] M. R. Douglas and Z. Lu, “On the Geometry of Moduli Space of Polarized Calabi-Yau manifolds,” math.DG/0603414.

[6] J. Cheeger, “Finiteness theorems for Riemannian Manifolds” Amer. J. Math. 92 (1970) 61.
[7] M. Gromov, “Structures metriques pour les varietes Riemanniennes,” CEDIC, Paris 1981, Eds. J. Lafontaine and P. Pansu.

[8] P. G. O. Freund and M. A. Rubin, Phys. Lett. B 97 (1980) 233.

[9] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Phys. Rept. 130, 1 (1986).

[10] B. S. Acharya, F. Denef, C. Hofman and N. Lambert, arXiv:hep-th/0308046.

[11] M. Berger, “A Panoramic View of Riemannian Geometry,” Springer 2003.

[12] C. Boyer, K. Galicki, B. Mann, E. Rees, “Compact 3-Sasakian 7-manifolds with arbitrary second Betti number.” Invent. Math. 131 (1998), no. 2, 321–344.

[13] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, Adv. Theor. Math. Phys. 8, 987 (2006) [arXiv:hep-th/0403038].

[14] M. Cvetic, H. Lu, D. N. Page and C. N. Pope, Phys. Rev. Lett. 95, 071101 (2005) [arXiv:hep-th/0504225].

[15] D. Martelli and J. Sparks, Phys. Lett. B 621, 208 (2005) [arXiv:hep-th/0505027].

[16] D. Martelli and J. Sparks, unpublished notes.

[17] D. Martelli and J. Sparks, Commun. Math. Phys. 262, 51 (2006) [arXiv:hep-th/0411238].

[18] I. Chavel, “Eigenvalues in Riemannian Geometry,” Academic Press 1984.

[19] M. Anderson, “Convergence and Rigidity Under Ricci Curvature Bounds,” Invent. Math. 102 (1990) 429.

[20] B. S. Acharya, arXiv:hep-th/0212294.

[21] B. S. Acharya, F. Denef and R. Valandro, JHEP 0506, 056 (2005) [arXiv:hep-th/0502060].

[22] S. Ashok and M. R. Douglas, JHEP 0401 (2004) 060 [arXiv:hep-th/0307049].
[23] O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, JHEP 0507, 066 (2005) [arXiv:hep-th/0505160].

[24] J. H. Horne and G. W. Moore, Nucl. Phys. B 432, 109 (1994) [arXiv:hep-th/9403058].

[25] M. R. Douglas, arXiv:hep-th/0602266.

[26] M. R. Douglas and Z. Lu, arXiv:hep-th/0509224.

[27] C. Vafa, arXiv:hep-th/0509212.

[28] G. Dvali and A. Vilenkin, Phys. Rev. D 70, 063501 (2004) [arXiv:hep-th/0304043].

[29] P. Berard, G. Besson, S. Gallot, “Embedding Manifolds by Their Heat Kernel,” Geom. Funct. Anal. 4 (1994) 373.

[30] M. Kontsevich and Y. Soibelman, [arXiv:math.SG/0011041]

[31] D. Roggenkamp and K. Wendland, Commun. Math. Phys. 251, 589 (2004) [arXiv:hep-th/0308143].

[32] H. Ooguri and C. Vafa, [arXiv:hep-th/0605264].