TETRAVALENT HALF-ARC-TRANSITIVE GRAPHS OF ORDER $12p$

M. GHASEMI¹, N. MEHDIPOOR² AND A.A. TALEBI³

¹ Department of Mathematics, Urmia University, Urmia 57135, Iran. email: m.ghasemi@urmia.ac.ir
² Department of Mathematics, Mazandaran University, email: nargesmehdipoor@yahoo.com
³ Department of Mathematics, Mazandaran University, email: a.talebi@umz.ac.ir

ABSTRACT. A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set, edge set, but not its arc set. In this paper, we study all tetravalent half-arc-transitive graphs of order $12p$.

1. Introduction

In this study, all graphs considered are assumed to be finite, simple and connected. For a graph $X$, $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ denote its vertex set, edge set, arc set, and full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ denotes the edge incident to $u$ and $v$ in $X$, and $N_X(u)$ denotes the neighborhood of $u$ in $X$, that is, the set of vertices adjacent to $u$ in $X$.

A graph $\tilde{X}$ is called a covering of a graph $X$ with projection $p : \tilde{X} \rightarrow X$ if there is a surjection $p : V(\tilde{X}) \rightarrow V(X)$ such that $p|_{N_{\tilde{X}}(\tilde{v})} : N_{\tilde{X}}(\tilde{v}) \rightarrow N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. A permutation group $G$ on a set $\Omega$ is said to be semiregular if the stabilizer $G_v$ of $v$ in $G$ is trivial for each $v \in \Omega$, and is regular if $G$ is transitive, and semiregular. Let $K$ be a subgroup of $\text{Aut}(X)$ such that $K$ is intransitive on $V(X)$. The quotient graph $X/K$ induced by $K$ is defined as the graph such that the set $\Omega$ of $K$-orbits in $V(X)$ is the vertex set of $X/K$ and $B, C \in \Omega$ are adjacent if and only if there exists a $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$. A covering $\tilde{X}$ of $X$ with a projection $p$ is said to be regular (or $K$-covering) if there is a subgroup $K$ of the automorphism group $\text{Aut}({\tilde{X}})$ such that $K$ is semiregular on both $V(\tilde{X})$ and $E(\tilde{X})$ and graph $X$ is isomorphic to the quotient graph $\tilde{X}/K$, say by $h$, and the quotient map $\tilde{X} \rightarrow \tilde{X}/K$ is the composition $ph$ of $p$ and $h$. The group of covering transformations $CT(p)$ of $p : \tilde{X} \rightarrow X$ is the group of all self equivalences of $p$, that is, of all automorphisms $\tilde{\alpha} \in \text{Aut}(\tilde{X})$ such that $p = \tilde{\alpha}p$. If $\tilde{X}$ is connected, $K$ becomes the covering transformation group.

2010 Mathematics Subject Classification. 05C25; 20B25.

Key words and phrases. Half-arc-transitive graph, Tightly attached, Regular covering projection, Solvable groups.
For a graph $X$ and a subgroup $G$ of $\text{Aut}(X)$, $X$ is said to be $G$-vertex-transitive, $G$-edge-transitive or $G$-arc-transitive if $G$ is transitive on $V(X)$, $E(X)$ or $A(X)$, respectively, and $G$-arc-regular if $G$ acts regularly on $A(X)$. A graph $X$ is called vertex-transitive, edge-transitive, arc-transitive, or arc-regular if $X$ is $\text{Aut}(X)$-vertex-transitive, $\text{Aut}(X)$-edge-transitive, $\text{Aut}(X)$-arc-transitive, or $\text{Aut}(X)$-arc-regular, respectively. Let $X$ be a tetravalent $G$-half-arc-transitive graph for a subgroup $G$ of $\text{Aut}(X)$, that is $G$ acts transitively on $V(X)$, $E(X)$, but not $A(X)$. Then under the natural action of $G$ on $V(X) \times V(X)$, $G$ has two orbits on the arc set $A(X)$, say $A_1$ and $A_2$, where $A_2 = \{(v, u) | (u, v) \in A_1\}$. Therefore, one may obtain two oriented graphs with the vertex set $V(X)$ and the arc sets $A_1$ and $A_2$. Assume that $D_G(X)$ be one of the two oriented graphs. Also in the special case, if $G = \text{Aut}(X)$ then $X$ is said to be 1/2-transitive or half-arc-transitive.

By Tutte [29], each connected vertex-transitive and edge-transitive graph of odd valency is arc-transitive. So half-arc-transitive graphs of odd valency do not exist. Bouwer [5] answered Tutte’s question about existence of half-arc-transitive graphs of even valency. A number of authors later studied the construction of these graphs. See, for example [1, 2, 9, 11, 14, 20, 21, 22, 23, 31, 32, 34]. Let $p$ be a prime. There are no half-arc-transitive graphs of order $p$, $p^2$ and $2p$ (see [6, 8]). Feng, Kwak, Wang and Zhou [12] classified the connected tetravalent half-arc-transitive graphs of order $2pq$ for distinct odd primes $p$ and $q$. The tetravalent half-arc-transitive graphs of order $p^5$, $p^4$, $2p^2$, $p^3$ and $2p^3$ are classified in [7, 13, 33, 37, 38] respectively. Wang et al. [35] studied tetravalent half-arc-transitive graphs of order a product of three primes. In [24], Liu studied tetravalent half-arc-transitive graphs of order $p^2q^2$ with $p$, $q$ distinct odd primes. Feng et al. [15] classified the tetravalent half-arc-transitive graphs of order $4p$. In [10] a complete classification of tetravalent half-arc-transitive metacirculants of order 2-powers was given. In [36], a classification of all tetravalent half-arc-transitive graphs of order $8p$ was given. In this paper, we will study tetravalent half-arc-transitive graphs of order $12p$.

2. Preliminaries

Let $X$ be a graph and $K$ be a finite group. By $a^{-1}$ we mean the reverse arc to an arc $a$. A voltage assignment (or $K$-voltage assignment) of $X$ is a function $\xi : A(X) \to K$ with the property that $\xi(a^{-1}) = \xi(a)^{-1}$ for each arc $a \in A(X)$. The values of $\xi$ are called voltages, and $K$ is the voltage group. The graph $X \times_\xi K$ derived from a voltage assignment $\xi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge $(e, g)$ of $X \times K$ joins a vertex $(u, g)$ to $(v, \xi((a)g))$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = \{u, v\}$. Clearly, the derived graph $X \times_\xi K$ is a covering of $X$ with the first coordinate projection $p : X \times_\xi K \to X$, which is called the natural projection. By defining $(u, g)^2 = (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\xi K)$, $K$ becomes a subgroup of $\text{Aut}(X \times_\xi K)$ which acts semiregularly on $V(X \times_\xi K)$. Therefore, $X \times_\xi K$ can be viewed as a $K$-covering. For each $u \in V(X)$ and $\{u, v\} \in E(X)$, the vertex set
\{(u, g) | g \in K\} \) is the fibre of \( u \) and the edge set \( \{(u, g)(v, \xi(a)g) | g \in K\} \) is the fibre of \( \{u, v\} \), where \( a = (u, v) \). The group \( K \) of automorphisms of \( X \) fixing every fibre setwise is called the covering transformation group. Conversely, each regular covering \( \tilde{X} \) of \( X \) with a covering transformation group \( \tilde{K} \) can be derived from a \( K \)-voltage assignment. Given a spanning tree \( T \) of the graph \( X \), a voltage assignment \( \xi \) is said to be \( T \)-reduced if the voltages on the tree arcs are the identity. Gross and Tucker in [18] showed that every regular covering \( \tilde{X} \) of a graph \( X \) can be derived from a \( T \)-reduced voltage assignment \( \tilde{X} \) with respect to an arbitrary fixed spanning tree \( T \) of \( X \).

Let \( \tilde{X} \) be a \( K \)-covering of \( X \) with a projection \( p \). If \( \alpha \in \text{Aut}(X) \) and \( \tilde{\alpha} \in \text{Aut}(\tilde{X}) \) satisfy \( \tilde{\alpha}p = p\alpha \), we call \( \tilde{\alpha} \) a lift of \( \alpha \), and \( \alpha \) the projection of \( \tilde{\alpha} \). The lifts and projections of such subgroups are of course subgroups in \( \text{Aut}(\tilde{X}) \) and \( \text{Aut}(X) \), respectively.

Let \( G \) be a group, and let \( S \subseteq G \) be a set of group elements such that the identity element \( 1 \) not in \( S \). The Cayley graph associated with \( (G, S) \) is defined as the graph having one vertex associated with each group element, edges \((g, h)\) whenever \( hg^{-1} \in S \). The Cayley graph \( X \) is denoted by \( \text{Cay}(G, S) \). In graph theory, the lexicographic product or (graph) composition \( G[H] \) of graphs \( G \) and \( H \) is a graph such that the vertex set of \( G[H] \) is the cartesian product \( V(G) \times V(H) \); and any two vertices \((x, y)\) and \((v, w)\) are adjacent in \( G[H] \) if and only if either \( x \) is adjacent with \( v \) in \( G \) or \( v = x \) and \( w \) is adjacent with \( y \) in \( H \). Clearly, if \( G \) and \( H \) are arc-transitive then \( G[H] \) is arc-transitive.

Let \( X \) be a tetravalent \( G \)-half-arc-transitive graph for some \( G \leq \text{Aut}(X) \). Then no element of \( G \) can interchange a pair of adjacent vertices in \( X \). By [19], there is no half-arc-transitive graph with less than 27 vertices. Half-arc-transitive graphs have even valencies. An even length cycle \( C \) in \( X \) is a \( G \)-alternating cycle if every other vertex of \( C \) is the head and every other vertex of \( C \) is the tail of its two incident edges in \( D_G(X) \). All \( G \)-alternating cycles in \( X \) have the same length. The radius of graph is half of the length of an alternating cycle. Any two adjacent \( G \)-alternating cycles in \( X \) intersect in the same number of vertices, called the \( G \)-attachment number of \( X \). The intersection of two adjacent \( G \)-alternating cycles is called a \( G \)-attachment set. We say that \( X \) is tightly attached if the attachment number of \( X \) equal with its radius.

Now we introduce graph \( X(r; m, n) \) and a result due to Marušič.

Suppose that \( m \geq 3 \) be an integer, \( n \geq 3 \) an odd integer and let \( r \in \mathbb{Z}_n^\ast \) satisfy \( r^m = \pm 1 \). The graph \( X(r; m, n) \) is defined to have vertex set \( V = \{u_i^j | i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\} \) and edge set \( E = \{u_i^j, u_i^{j \pm r_i} | i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\} \).

**Proposition 2.1.** [25, Theorem 3.4] A connected tetravalent graph \( X \) is a tightly attached half-arc-transitive graph of odd radius \( n \) if and only if \( X \cong X(r; m, n) \), where \( m \geq 3 \), and \( r \in \mathbb{Z}_n^\ast \) satisfying \( r^m = \pm 1 \), and moreover none of the following conditions is fulfilled:

1. \( r^2 = \pm 1 \);
2. \((r; m, n) = (2; 3, 7)\);
(3) \((r; m, n) = (r; 6, 7k), \) where \(k \geq 1\) is odd, \((7, k) = 1, r^6 = 1,\) and there exists a unique solution \(q \in \{r, -r, r^{-1}, -r^{-1}\}\) of the equation \(x^2 + x - 2 = 0\) such that \(7(q - 1) = 0\) and \(q \equiv 5 \pmod{7}\).

The following is the main result of the paper tetravalent half-transitive graphs of order \(4p\).

Proposition 2.2. [[15], Theorem 3.3] Let \(p\) be a prime and \(X\) a tetravalent graph of order \(4p\). Then, \(X\) is half-transitive if and only if \(p \equiv 1 \pmod{8}\) and \(X \cong X(r; 4, p)\) (denote by \(X(4, p)\) the graph \(X(r; 4, p)\)).

Now we express an observations about tetravalent half-arc-transitive graphs.

Proposition 2.3. [[26], Lemma 3.5] Let \(X\) be a connected tetravalent \(G\)-half-arc-transitive graph for some \(G \leq \text{Aut}(X)\), and let \(\Delta\) be a \(G\)-attachment set of \(X\). If \(|\Delta| \geq 3\), then the vertex-stabilizer of \(v \in V(X)\) in \(G\) is of order \(2\).

Proposition 2.4. [17] A non-abelian simple group whose order has at most three prime divisors is isomorphic to one of the following groups:

\[
\text{A}_5, \text{A}_6, \text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSL}(2, 17), \text{PSL}(3, 3), \text{PSU}(3, 3), \text{PSU}(4, 2),
\]

whose orders are \(2^2 \cdot 3 \cdot 5, 2^3 \cdot 3^2 \cdot 5, 2^3 \cdot 3 \cdot 7, 2^4 \cdot 3^2 \cdot 7, 2^4 \cdot 3^3 \cdot 13, 2^5 \cdot 3^3 \cdot 7, 2^6 \cdot 3^4 \cdot 5\), respectively.

The following result is extracted from [[4], Theorem 1].

Proposition 2.5. Let \(X\) be a tetravalent arc-transitive graph of order \(2pq\) where \(p\) and \(q\) are odd and distinct primes. Then one of the following holds:

(1) \(X\) is arc-regular and appears in [39];
(2) \(X\) is isomorphic to the lexicographic product \(C_{pq}[2K_1]\) of the cycle \(C_{pq}\) and the edgeless graph on two vertices \(2K_1\).

In the following, we describe the structure of the graphs required in this paper [[27], [28], [40]].

The Rose Window graph \(R_6(5, 4)\) is a tetravalent graph with 12 vertices. Its vertex set is \(\{S_i, Q_i | i \in Z_6\}\). The graph has four kinds of edges: kind of edges: \(S_iS_{i+1}\) (rim edges), \(S_iQ_i\) (inspoke edges), \(S_{i+5}Q_i\) (outspoke edges) and \(Q_iQ_{i+4}\) (hub edges). \(|\text{Aut}(R_6(5, 4))| = 48\). Fig 1 shows \(R_6(5, 4)\).

A general Wreath graph \(W(6, 2)\) has 12 vertices and it is regular of valency 4. Its vertex set is \(\{E_i, F_i | i \in Z_6\}\), where \(E_i = (i, 0)\) and \(F_i = (i, 1)\). Its edges are \(\{E_i, E_{i+1}\}, \{E_i, F_{i+1}\}, \{F_i, E_{i+1}\}\) and \(\{F_i, F_{i+1}\}\). \(|\text{Aut}(W(6, 2))| = 768\). See Fig 2.
The graph $C(2; p, 2)$ was first defined by Praeger and Xu [28, Definition 2.1 (b)]. Let $p$ be an odd prime. The graph $C(2; p, 2)$ has vertex set $\mathbb{Z}_p \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and its edges are defined by \{$(i, (x, y)), (i + 1, (y, z))$\} $\in E(C(2; p, 2))$ for all $i \in \mathbb{Z}_p$ and $x, y, z \in \mathbb{Z}_2$. Aut($C(2; p, 2)$) $\cong D_{2p} \rtimes \mathbb{Z}_2$.

Let $p \equiv 1 \pmod{4}$, where $p$ is a prime and $w$ is an element of order 4 in $\mathbb{Z}_p^*$. The graph $CA^0_{4p}$ is Cay($G, \{a, a^{-1}, a^w b, a^{-w} b\}$) and the graph $CA^1_{4p}$ is Cay($G, \{a, a^{-1}, a^w b, a^{-w} b\}$), where $G = \langle a > \times < b > \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$.

3. Main Results

In this section, we study all tetravalent half-arc-transitive graphs of order $12p$ where $p$ is a prime. To do this, we prove the following results.

**Lemma 3.1.** Let $X$ be a graph, $G \leq \text{Aut}(X)$, $N \leq G$ and $X$ be $N$-regular covering of $X_N$. Then $X$ is $G$-half-arc-transitive if and only if $X_N$ is $G/N$-half-arc-transitive.

**Proof.** Suppose that $N \leq G$ and $X$ is $G$-half-arc-transitive. Since $X$ is $N$-regular covering of $X_N$, it follows that $K = N$ and $G/N \leq \text{Aut}(X_N)$, where $K$ is the kernel of $G$ acting on orbits of $N$. Let $x^N, y^N$ be two arbitrary vertices of graph $X_N$. By our assumption there exits $g \in G$ such that $x^g = y$. Now $(x^N)^N_g = (x^g)^N = y^N$. It implies that $X_N$ is $G/N$-vertex-transitive. Now, suppose that $\{x^N, y^N\}$ and $\{u^N, v^N\}$ are two arbitrary edges of $X_N$. Without loss of generality, we may suppose that $\{x, y\}$ and $\{u, v\}$ are two edges of $X$. By our assumption there exits $g \in G$ such that $\{x, y\}^g = \{u, v\}$. Then we may assume that $x^g = u$ and $y^g = v$. Hence $(x^N)^N_g = x^N g = x^{g^N} = u^N$ and $(y^N)^N_g = y^N g = y^{g^N} = v^N$. Then $X_N$ is $G/N$-edge-transitive. Suppose to contrary that $X_N$ is $G/N$-arc-transitive. Let $(x, y)$ and $(u, v)$ are two arcs of graph $X$. Now $(x^N, y^N)$ and $(u^N, v^N)$ are two arcs of graph $X_N$. By our assumption, there exits $Ng \in G/N$ such that $(x^N, y^N)^N_{g} = (u^N, v^N)$. Therefore $(x^N)^{Ng} = u^N$ and $(y^N)^{N_{g}} = v^N$. Thus $x^{N_{g}} = u^{N}$ and $y^{N_{g}} = v^{N}$. Then $x^g = u$ and $y^g = v^{n'}$ for $n, n' \in N$ and so $(x, y)^g = (u^n, v^{n'})$. 

---

**Fig 1.** The Rose Window graph $R_6(5, 4)$

**Fig 2.** The Wreath graph $W(6, 2)$
There exits \( n'' \in N \) such that \( (u^n, v^{n''})^{n''} = (u, v) \). Then \( (x, y)^{g^{n''}} = (u^n, v^{n''})^{n''} = (u, v) \). Therefore \( X \) is \( G \)-arc-transitive, a contradiction. Then \( X_N \) is \( G/N \)-half-arc-transitive.

Now suppose that \( X_N \) is \( G/N \)-half-arc-transitive. Thus \( G/N \) acts transitively on \( V(X_N) \). Let \( u, v \in V(X) \) and \( u^N, v^N \in V(X_N) \). Then there is \( Ng \in G/N \) such that \( (u^N)^Ng = v^N \) and hence, there is \( n' \in N \) such that \( u^g = v^{n'} \) and \( u^{g(n')} = v \). Then since \( g(n')^{-1} \in G \), it implies that \( X \) is vertex-transitive. For any \( \{u, v\}, \{x, y\} \in E(X) \), we have \( \{u^N, v^N\}, \{x^N, y^N\} \in E(X_N) \). Since \( X_N \) is \( G/N \)-edge-transitive, we have \( Ng \in G/N \) such that \( \{u^N, v^N\}^Ng = \{x^N, y^N\} \) and \( \{(u^N)^Ng, (v^N)^Ng\} = \{x^N, y^N\} \). Without loss of generality, we may suppose that \( (u^N)^Ng = (u)^Ng = x^N \) and \( (v^N)^Ng = (v)^Ng = y^N \).

There exits \( n', n'' \in N \) such that \( (u, v)^g = (x^{n'}, y^{n''}) \). Also there exits \( n \in N \) such that \( (x^{n'}, y^{n''})^n = \{x, y\} \). Thus we may assume that \( (u, v)^g = \{x, y\} \) and so \( X \) is \( G \)-edge-transitive. Similar to the previous, it can be shown that if \( X_N \) is not \( G/N \)-arc-transitive then \( X \) is not \( G \)-arc-transitive. Therefore \( X \) is \( G \)-half-arc-transitive.

The following lemma is basic for the main result.

**Lemma 3.2.** Let \( X \) be a half-arc-transitive graph, \( p \) is a prime and \( N \trianglelefteq \text{Aut}(X) \), where \( N \cong \mathbb{Z}_p \). If the quotient graph \( X_N \) is a Cayley graph and has the same valency with \( X \) then \( X \) is a \( N \)-regular covering of \( X_N \) and \( X \) is a Cayley graph.

**Proof.** Let \( N \) be a normal subgroup of \( A := \text{Aut}(X) \) and \( X_N \) be the quotient graph of \( X \) with respect to the orbits of \( N \) on \( V(X) \). Assume that \( K \) is the kernel of \( A \) acting on \( V(X_N) \). The stabilizer \( K_v \) of \( v \in V(X) \) in \( K \) fixes the neighborhood of \( v \) in \( X \). The connectivity of \( X \) implies \( K_v = 1 \) for any \( v \in V(X) \) and hence \( N_v = 1 \). If \( N_{\{\alpha, \beta\}} \neq 1 \) then \( N_{\{\alpha, \beta\}} \neq N \). Since \( X \) is connected, there is a \( \{\beta, \gamma\} \in E(X) \) where \( \beta, \gamma \in V(X) \). Then we have \( g \in A \) such that \( \{\alpha, \beta\} = \{\beta, \gamma\}^g \) because \( X \) is an edge-transitive graph. Hence \( N_{\{\alpha, \beta\}} = N_{\{\beta, \gamma\}} = g^{-1}N_{\{\beta, \gamma\}}g = N_{\{\beta, \gamma\}} \). It is a contradiction and so \( N_{\{\alpha, \beta\}} = 1 \). Therefore \( X \) is a \( \mathbb{Z}_p \)-regular covering of \( X_N \). Now we prove that \( X \) is a Cayley graph. Let \( X_N \cong \text{Cay}(G, S) \), \( X \cong X_N \times \mathbb{Z}_p \) where \( \xi \) is the \( T \)-reduced voltage assignment and \( \tilde{G} \) is a lift of \( G \) such that \( \tilde{\alpha}p = p\alpha \) where \( p : X \to X_N \) is regular covering projection, \( \alpha \in \text{Aut}(X_N) \) and \( \tilde{\alpha} \in A \). For any \( (x, k), (y, k') \in V(X) \) where \( k, k' \in \mathbb{Z}_p \) and \( x, y \in V(X_N) \), we have \( \alpha \in \text{Aut}(X_N) \) such that \( x^\alpha = y \). For \( k'' \in \mathbb{Z}_p \), \( (x, k)^{\tilde{\alpha}p} = (z, k'')^p = z \) where \( (x, k)^\tilde{\alpha} = (z, k'') \). Also \( (x, k)^p\alpha = x^\alpha = y \). Then \( y = z \) and hence \( (y, k), (y, k'') \in p^{-1}(y) \). Therefore \( \tilde{G} \) is transitive on \( V(X) \). Now, we prove that \( \tilde{G} \) is semiregular. Suppose that \( (x, k)^\tilde{\alpha} = (x, k) \). Now, since \( G \) is semiregular and \( \tilde{\alpha}p = p\alpha \), it implies that \( x = (x, k)^{\tilde{\alpha}p} = (x, k)^p\alpha = x^\alpha \). Then \( \alpha = 1 \) and hence \( \tilde{\alpha}p = p \). Therefore \( \tilde{\alpha} \in \text{CT}(p) = \mathbb{Z}_p \) and since \( \text{CT}(p) \) is semiregular, it follows that \( \tilde{\alpha} = 1 \). \( \Box \)

By [27], all tetravalent half-arc-transitive graphs of order \( 12p \) where \( p \leq 53 \) is a prime, are classified. Then in the following, we may assume that \( p > 53 \).
Lemma 3.3. Let $X$ be a tetravalent half-arc-transitive graph of order $12p$, where $p$ is a prime. Then $\text{Aut}(X)$ has a normal Sylow $p$-subgroup or $X$ is $\mathbb{Z}_3$-regular covering of $C(2; p, 2)$ or $C_{2p}[2K_1]$.

Proof. Let $X$ be a tetravalent half-arc-transitive graph of order $12p$ where $p$ is a prime. Let $A := \text{Aut}(X)$. Since the stabilizer $A_v$ of $v \in V(X)$ is a 2-group, we have $|A| = 2^{m+2}.3.p$, for some nonnegative integer $m$. Suppose to the contrary that $A$ has no normal Sylow $p$-subgroups. Let $N$ be a minimal normal subgroup of $A$. We claim that $N$ is solvable. Otherwise, by Proposition 2.4 and since $p > 53$, we get a contradiction. Then $N$ is solvable and hence it is an elementary abelian 2-, 3- or $p$-group.

Case I: $N$ is a 2-group.

Let $X_N$ be the quotient graph of $X$ corresponding to the orbits of $N$ on $V(X)$. Then $|V(X_N)| = 6p$ or $3p$.

Subcase 1: $|V(X_N)| = 6p$.

Since $X$ is edge-transitive, $X_N$ has valency 2 or 4. Suppose that $X_N$ has valency 2. Then $X \cong C_{6p}[2K_1]$, which is arc-transitive. It is a contradiction. Assume now that $X_N$ has valency 4. If $X_N$ is half-arc-transitive then by [[12], Theorem 4.1], $|\text{Aut}(X_N)| = 2^2.3.p$. Let $K$ be the kernel of $A$ acting on $V(X_N)$. Since $K$ fixes each orbit of $N$, the stabilizer $K_v = 1$ for any $v \in V(X)$. Then $|N| = |K|$. On the other hand $A/K \leq \text{Aut}(X_N)$. Since $A/K$ acts transitively on $V(X_N)$ and $E(X_N)$, $|A| = 24p$. Then $1 + np | 24$. Since $p > 53$ then $P \leq A$, a contradiction. Now, suppose that $X_N$ is arc-transitive. Let $X_N$ has valency 4. By Proposition 2.5, if $X_N$ is arc-regular then $|\text{Aut}(X_N)| = 24p$. By lemma 3.1, $A/K$ is half-arc-transitive and hence $|A| = 24p$. Then $P \leq A$ because $p > 53$. It is a contradiction. If $X_N$ not be arc-regular then by Proposition 2.5, $Y = X_N \cong C_{3p}[2K_1]$ and $B = \text{Aut}(Y)$, $|B| = 2^{3p+1}.3.p$. Assume that $M$ is a minimal normal subgroup of $B$. By the same argument as in the first paragraph, $M$ is solvable and hence it is an elementary abelian 2-, 3- or $p$-group. First, assume that $M$ is a 2-group and $Y_M$ is the quotient graph of $Y$ corresponding to the orbits of $M$ on $V(Y)$. The quotient graph $Y_M$ has order $3p$ and valency 2 or 4. If $Y_M$ has valency 4 then $M_v = 1$ for $v \in V(Y)$. Assume that $K_1$ be the kernel of $B$ acting on $V(Y_M)$. Hence $|K_1| = |M|$. Thus $B/K_1 \leq \text{Aut}(Y_M)$. It is a contradiction because $|\text{Aut}(Y_M)| = 12p$ by [[30], Theorem 5]. If $Y_M$ has valency 2 then $Y_M \cong C_{3p}$ and $\text{Aut}(Y_M) \cong D_{6p}$. Since $|K_1| \leq 2$, we have $|B| \leq 12p$. We get a contradiction because $p > 53$. Now, suppose that $M$ be a 3-group. Then $|V(Y_M)| = 2p$. Since $M_v = 1$ for $v \in V(Y)$ by using [[16], Theorem 1.1(4)], $Y_M$ has valency 4. By [[6], Table 1], $Y_M \cong G(2, p, r)$ or $G(2p, r)$. Then $|K_1| = |M|$ and hence $B/K_1 \leq \text{Aut}(Y_M)$. It is a contradiction because $|\text{Aut}(Y_M)| = 2^{p+1}.p$ or $8p$ and $p > 53$. Let $M$ be a $p$-group. Then $|Y_M| = 6$. Since $M_v \leq M$ we have $|M_v| = 1$. By [[16], Theorem 1.1(4)], $Y_M$ has valency 4. By [[27], $|\text{Aut}(Y_M)| = 48$. Hence $B/K_1 \leq \text{Aut}(Y_M)$. It is a contradiction.

Subcase 2: $|V(X_N)| = 3p$.

Let $|V(X_N)| = 3p$ and $X_N$ has valency 2. Then $X \cong C_{3p}[2K_1]$. This leads to a contradiction. If $X_N$ has valency 4 and it is half-arc-transitive then by [[2], Theorem 2.5],
Case II: $N$ is 3-group.

If $|V(X_N)| = 4p$ and $X_N$ has valency 2, then $X_N \cong C_{4p}$ and hence $\text{Aut}(X_N) \cong D_{8p}$. Since $K = K_vN$ for any $v \in V(X)$ and $K$ acts faithfully on $V(X)$, we have $K \leq S_3$ and hence $K_v \leq 2$. Then $|A| \leq 48p$. Therefore $P \leq A$ because according to assumption $p > 53$. This leads to a contradiction. Now let $|V(X_N)| = 4p$ and $X_N$ has valency 4. Then $X_N$ is arc-transitive or half-arc-transitive. By [[40], Table 1] and Proposition 2.2, $X_N \cong C(2; p, 2)$, $C_{2p}[2K_1]$, $CA_{4p}^0$, $CA_{4p}^1$ or $X(4, p)$. Let $X_N \cong C(2; p, 2)$ or $C_{2p}[2K_1]$. Since $X_N$ has valency 4, $N$ acts semiregularly on $V(X)$ and so $X$ is a $Z_3$-regular covering of $C(2; p, 2)$ or $C_{2p}[2K_1]$. Assume that $Y = X_N \cong CA_{4p}^0$ or $CA_{4p}^1$ and $B = \text{Aut}(Y)$. Since $|K| = |N|$, we have $A/K \leq B$ and hence $|A| \leq 48p$. Then $P \leq A$. Suppose that $Y = X_N \cong X(4, p)$ and $B = \text{Aut}(Y)$. Since $Y$ is half-arc-transitive, we have $|B| = 2^{m+2}.p$, for some nonnegative integer $m$. Let $M$ be a minimal normal subgroup of $B$. Thus $M$ is an elementary abelian 2- or p-group. First, assume that $M$ is a p-group and $Y_M$ be the quotient graph of $Y$ corresponding to the orbits of $M$ on $V(Y)$. Then $|V(Y_M)| = 4$. Since $Y$ is an edge-transitive graph and $M_v = 1$ for $v \in V(Y)$, $Y_M$ has valency 4, a contradiction. Suppose that $M$ is a 2-group. Therefore $|V(Y_M)| = 2p$ or $p$ and $Y_M$ has valency 2 or 4.

Subcase 1: $|V(Y_M)| = 2p$.

If $Y_M$ has valency 2 then $Y \cong C_{2p}[2K_1]$, which is arc-transitive. Since $Y$ is half-arc-transitive, we get a contradiction. Suppose now that $Y_M$ has valency 4. By [[8], Table 1], $Y_M \cong G(2p, 4)$ or $G(2, p, 2)$. Assume that $Y_M \cong G(2p, 4)$. Since $(K_1)_v = 1$, $|B/K_1| \leq 8p$ and hence $|A| \leq 48p$. It is a contradiction because $p > 53$. Suppose that $Y_M \cong G(2, p, 2)$. Let $Z = Y_M \cong G(2, p, 2)$ and $C = \text{Aut}(Z)$. Let $H$ be a minimal normal subgroup of $C$ and let $Z_H$ be the quotient graph of $Z$ with respect to the orbits of $H$. Since $|C| = 2^{p+1}.p$, $H$ is 2- or p-group. Assume that $H$ is a 2-group. Thus $|Z_H| = p$ and $Z_H$ has valency 2 or 4. By [[6], Theorem 3], $|\text{Aut}(Z_H)| = 2p$ or $4p$. Assume that $K_1$ be the kernel of $C$ acting on $V(Z_H)$. If $Z_H$ has valency 4 then $|K_1| = |H| = 2$ because $|(K_1)_v| = 1$. Then $C/K_1 \leq 16p$ and hence $2^{p+1} \leq 8p$. We get a contradiction because $p > 53$. If $Z_H$ has valency 2 then $|K_1| \leq 8$ because $|(K_1)_v| \leq 2$. Thus $C/K_1 \leq 16p$ and hence $2^{p+1} \leq 8p$, a contradiction can be obtained. Now, suppose that $H$ is a p-group. Then $|Z_H| = 2$ with valency 2, 4, a contradiction.

Subcase 2: $|V(Y_M)| = p$.

If $Y_M$ has valency 4 then by lemma 3.2, $Y$ is $Z_2$-regular cover of $Y_M$ and $Y$ is a Cayley graph. But by [[15], $X(4, p)$ is not a Cayley graph, a contradiction. Suppose that $Y_M$ has valency 2 and hence $Y_M \cong C_p$. Assume that $K_1$ is the kernel of $B$ acting on $V(Y_M)$ and $(K_1)_v = 1$. Then $B/K_1 \leq \text{Aut}(Y_M)$ and so $|B| \leq 8p$. Therefore $|A| \leq 24p$ and hence
TETRAVALENT HALF-ARC-TRANSITIVE GRAPHS OF ORDER 12p

P ⊆ A because p > 53. Then (K1)v ̸= 1. Let V(YM) = {Ω0, Ω1, Ω2, ..., Ωp−1}. The subgraph induced by any two adjacent orbits is either a cycle of length 8 or a union of two cycles of length 4. Suppose that ⟨Ωi ∪ Ωi+1⟩ is an 8-cycle. Thus K1 acts faithfully on each Ωi and hence (K1)v ∼= Z2. It implies that |K1| = 8. Since M is transitive on each Ωi and (K1)v > 1, all edges in the induced subgraph ⟨Ωi ∪ Ωi+1⟩ have the same direction either from Ωi to Ωi+1 or from Ωi+1 to Ωi in the oriented graph DB(Y). It follows that B/K1 ∼= Zp and |B| ≤ 8p. Since M is transitive on each Ωi and (K1)v > 1, all edges in the induced subgraph ⟨Ωi ∪ Ωi+1⟩ have the same direction either from Ωi to Ωi+1 or from Ωi+1 to Ωi in the oriented graph DB(Y). It follows that B/K1 ∼= Zp and |B| ≤ 8p. Therefore |A| ≤ 24p and hence P ⊆ A because p > 53. Assume that ⟨Ωi ∪ Ωi+1⟩ is a union of two 4-cycles. Let Ωi = {u0, u1, u2, u3} for any i in Zp. Then B has an automorphism α of order p such that for any i in Zp, Ωαi = Ωi+1. Let (ui)ε = uεi+1 for i in Zp and j in Z4. Consider a 4-cycle C in the induced subgraph ⟨Ω0 ∪ Ω1⟩ and let n be the number of edges of C which are in some orbit of α. Then n = 0, 1, or 2. Consequently, the induced subgraph ⟨Ω0 ∪ Ω1⟩ is one of the of the following three cases.

In the Case 1, Y is disconnected, a contradiction. In the Case 2, Y ∼= C2p[2K1]. We get a contradiction because Y ∼= X(4, p). In the Case 3, Y ∼= C(2; p, 2) that is arc-transitive. It is a contradiction because X(4, p) is a half-arc-transitive graph.

![Case 1](image1)

![Case 2](image2)

![Case 3](image3)

**Fig 3.** The induced subgraph ⟨Ω0 ∪ Ω1⟩.

**Case III:** N is p-group.

If |N| = p then N is a normal Sylow p-subgroup of A as claimed. □

**Theorem 3.4.** Let X be a connected tetravalent vertex-transitive and edge-transitive graph of order 12p, where p > 53 is a prime. Then one of the following statements holds:

1. X is half-arc-transitive if and only if X ∼= X(r; 12, p) such that r ∈ Zp satisfying r12 = ±1.
2. X is half-arc-transitive Cayley graph if and only if X ∼= Y ×ξ Zp, where Y = W(6, 2) or R0(5, 4) and ξ : A(Y) → Zp.
3. If X is half-arc-transitive then X ∼= C(2; p, 2) ×ξ Z3, where ξ : A(C(2; p, 2)) → Z3.
(4) if $X$ is half-arc-transitive then $X$ is a Cayley graph and $X \cong C_{2p}[2K_1] \times _\xi \mathbb{Z}_3$, where $\xi : A(C_{2p}[2K_1]) \rightarrow \mathbb{Z}_3$.

**Proof.** Let $X$ be a tetravalent half-arc-transitive graph of order $12p$ and hence $|A| = 2^{m+2}.3.p$ for some integer $m \geq 0$. By Lemma 3.3, either $P \trianglelefteq A$ or $X$ is a $\mathbb{Z}_3$-regular covering of $C(2; p, 2)$ or $C_{2p}[2K_1]$. First, suppose that $P \trianglelefteq A$. Now, let $X_P$ be the quotient graph of $X$ corresponding to the orbits of $P$. Assume that $K$ is the kernel of $A$ acting on $V(X_P)$. Then $V(X_P) = 12$ and $X_P$ has valency 2 or 4. If $X_P$ has valency 2 then $X_P \cong C_{12}$ and hence $\text{Aut}(X_P) \cong D_{24}$. By Proposition 2.3, $A_v \cong \mathbb{Z}_2$ and hence $|A| = 24p$. The attachment number of $X$ is equal to its radius. So $X$ is a tetravalent tightly attached half-arc-transitive graph of odd radius $p$. By Proposition 2.1, $X \cong X(r; 12, p)$ and $|A| = 24p$. Also, by Proposition 2.1, it is trivial that $X(r; 12, p)$ is tetravalent half-arc-transitive graph of order $12p$. Assume that $X_P$ has valency 4 and $X_P$ is arc-transitive or half-arc-transitive. There is no half-arc-transitive graph of order 12. Suppose that $X_P$ is an arc-transitive graph. By [27], $W(6, 2)$ and $R_6(5, 4)$ are the only two arc-transitive graphs of order 12. These graphs are Cayley graphs by [3]. Since $P$ acts semiregular on $V(X)$ and $E(X)$, by Lemma 3.2, $X$ is a $\mathbb{Z}_p$-regular covering of $X_P$ and $X$ is a Cayley graph. For convenience, consider the graphs $W(6, 2)$ and $R_6(5, 4)$. By [3], these graphs have half-arc-transitive subgroups. By Lemma 3.1, since $X_P$ is $A/P$-half-arc-transitive then $X$ is half-arc-transitive. Now suppose that $X$ is half-arc-transitive which is $\mathbb{Z}_3$-regular covering of $X_P$. By Lemma 3.3 and Lemma 3.2, the cases (3) and (4) holds.

$\square$

**References**

[1] I. Antončič and P. Šparl, *Classification of quartic half-arc-transitive weak metacirculants of girth at most 4*. Discrete Math. 339 (2016), 931-945.

[2] B. Alspach and M.Y. Xu, *1/2-transitive graphs of order $3p$*. J. Algebraic Combin. 3 (1994), 347-355.

[3] R.A. Beezer, *Sage for Linear Algebra A Supplement to a First course in Linear Algebra*. Sage web site http://www.sagemath.org. 2011.

[4] K. Berčič and M. Ghasemi, *Tetravalent arc-transitive graphs of order twice a product of two primes*. Discrete Math. 312 (2012), 3643-3648.

[5] I.Z. Bouwer, *Vertex and edge transitive, but not 1-transitive graphs*. Can. Math. Bull. 13 (1970), 231-237.

[6] C.Y. Chao, *On the classification of symmetric graphs with a prime number of vertices*. Trans. Amer. Math. Soc. 158 (1971), 247-256.

[7] H. Cheng and L. Cui, *Tetravalent half-arc-transitive graphs of order $p^5$*. Appl. Math. Comput. 332 (2018), 506-518.

[8] Y. Cheng and J. Oxley, *On weakly symmetric graphs of order twice a prime*. J. Combin. Theory Ser. B 42 (2018), 196-211.

[9] M.D.E. Conder and A. Žitnik, *Half-arc-transitive graphs of arbitrary even valency greater than 2*. European J. Combin. 54 (2016), 177-186.
[10] L. Cui and J.X. Zhou, *A classification of tetravalent half-arc-transitive metacirculants of 2-power orders*. J. Applied Mathematics and Computation. 392 (2021), 1-14.

[11] S.F. Du and M.Y. Xu, *Vertex-primitive 1/2-arc-transitive graphs of smallest order*. Commun. Algebra. 27 (1999), 163-171.

[12] Y.Q. Feng, J.H. Kwak, X. Wang and J.X. Zhou, *Tetravalent half-arc-transitive graphs of order 2pq*. J. Algebraic Combin. 33 (2011), 543-553.

[13] Y.Q. Feng, J.H. Kwak, M. Y. Xu and J.X. Zhou, *Tetravalent half-arc-transitive graphs of order p^4*. European J. Combin. 29 (2008), 555-567.

[14] Y.Q. Feng, J.H. Kwak, and C. Zhou, *Constructing even radius tightly attached half-arc-transitive graphs of valency four*. J. Algebraic Combin. 26 (2007), 431-451.

[15] Y.Q. Feng, K.S. Wang, and C. Zhou, *Tetravalent half-transitive graphs of order 4p*. European J. Combin. 28 (2007), 726-733.

[16] A. Gardiner and C.E. Praeger, *On 4-valent symmetric graphs*. European J. Combin. 15 (1994), 375-381.

[17] D. Gorenstein, *Finite Simple Groups*. Plenum, New York (1982).

[18] J.L. Gross and T.W. Tucker, *Generating all graph coverings by permutation voltage assignments*, Discrete Math. 18 (1977), 273-283.

[19] D.F. Holt, *A graph which is edge transitive but not arc transitive*. J. Graph Theory 5 (1981), 201-204.

[20] A. Hujdurović, K. Kutnar, D. Marušić, *Half-arc-transitive group actions with a small number of alternets*. J. Combin. Theory, Ser. A 124 (2014), 114-129.

[21] K. Kutnar, D. Marušić, P. Šparl, *An infinite family of half-arc-transitive graphs with universal reachability relation*. European J. Combin. 31 (2010), 1725-1734.

[22] K. Kutnar, D. Marušić, P. Šparl, R. J. Wang and M. Y. Xu, *Classification of half-arctransitive graphs of order 4p*. European J. Combin. 34 (2013), 1158-1176.

[23] C.H. Li and H.S. Sim, *On half-transitive metacirculant graphs of prime-power order*. J. Combin. Theory Ser. B 81 (2001), 45-57.

[24] H. Liu, B. lou and B. ling, *Tetravalent half-arc-transitive graphs of order p^2q^2*. J. Czechoslovak Mathematical, 69 (144) (2019), 391-401.

[25] D. Marušić, *Half-transitive groups actions on finite graphs of valency 4*. J. Combin. Theory Ser. B 73 (1998), 41-76.

[26] D. Marušić and C.E. Praeger, *Tetravalent graphs admitting half-transitive group action: alternating cycles*. J. Combin. Theory Ser. B 75 (1999), 188-205.

[27] P. Potočnik and S. Wilson, *A Census of edge-transitive tetravalent graphs*, http://jimmie.tcu.edu/~swilson/C4Site/index.html.

[28] C.E. Praeger and M.Y. Xu, *A characterization of a class of symmetric graphs of twice prime valency*. European J. Combin. 10 (1989), 91-102.

[29] W.T. Tutte, *Connectivity in Graphs*. Mathematical Expositions 15, University of Toronto Press, Toronto; Oxford University Press, London, (1966).

[30] R.J. Wang and M.Y. Xu, *A classification of symmetric graphs of order 3p*. J. Combin Theory 58 (1989), 197-216.

[31] R.J. Wang, *Half-transitive graphs of order a product of two distinct primes*. J. Commun. Algebra 22 (1994), 915-927.

[32] X. Wang and Y.Q. Feng, *Half-arc-transitive graphs of order 4p of valency twice a prime*. Ars Math. Contemp. 3 (2010), 151-163.

[33] X. Wang and Y.Q. Feng, *There exists no tetravalent half-arc-transitive graph of order 2p^2*. Discrete Math. 310 (2010), 1721-1724.
34] Y. Wang and Y.Q. Feng, *Half-arc-transitive graphs of prime-cube order of small valencies*. Ars Math. Contemp. 13 (2017), 343-353.

35] X. Wang, Y. Feng, J.X. Zhou, J. Wang and Q. Ma, *Tetravalent half-arc-transitive graphs of order a product of three primes*. Discrete Math. 339 (2016), 1566-1573.

36] X. Wang, J. Wang and Y. Liu, *Tetravalent half-arc-transitive graphs of order 8p*. J. Algebraic Combin. 51 (2019), 237-246.

37] M.Y. Xu, *Half-transitive graphs of prime-cube order*. J. Algebraic Combin. 1 (1992), 275-282.

38] M.M. Zhang and J.X. Zhou, *Tetravalent half-arc-transitive bi-p-metacirculants*. J. Graph Theory 92 (2019), 19-38.

39] J.X. Zhou and Y.Q. Feng, *Tetravalent one-regular graphs of order 2pq*. J. Algebraic Combin. 29 (2009), 457-471.

40] J.X. Zhou, *Tetravalent s-transitive graphs of order 4p*. Discrete Math. 309 (2009), 6081-6086.