Positive multipeak solutions to a zero mass problem in exterior domains

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Abstract

We establish the existence of positive multipeak solutions to the nonlinear scalar field equation with zero mass

$$-\Delta u = f(u), \quad u \in D^{1,2}_0(\Omega_R),$$

where $\Omega_R := \{ x \in \mathbb{R}^N : |u| > R \}$ with $R > 0$, $N \geq 4$, and the nonlinearity $f$ is subcritical at infinity and supercritical near the origin. We show that the number of positive multipeak solutions becomes arbitrarily large as $R \to \infty$.

Keywords and phrases: Scalar field equation, zero mass, exterior domain, multipeak solutions.

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1 Introduction

This paper is concerned with the existence of multiple positive solutions to the nonlinear scalar field equation

$$-\Delta u = f(u), \quad u \in D^{1,2}_0(\Omega_R),$$

in the exterior domains $\Omega_R := \{ x \in \mathbb{R}^N : |u| > R \}$ with $R > 0$ and $N \geq 4$. The nonlinearity $f$ is assumed to be subcritical at infinity and supercritical

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near the origin. Our precise assumptions on $f$ are given below. They include the model nonlinearity

$$f(s) = \frac{s^{q-1}}{1 + s^{q-p}},$$  \hspace{1cm} (1.2)

with $2 < p < 2^* := \frac{2N}{N-2} < q$.

In their seminal paper [3], Berestycki and Lions considered the zero mass problem

$$-\Delta u = f(u), \quad u \in D^{1,2}(\mathbb{R}^N),$$  \hspace{1cm} (1.3)

in the whole space $\mathbb{R}^N$. They showed that, if $f$ is subcritical at infinity and supercritical near the origin, it has a ground state solution.

The problem

$$-\Delta u = f(u), \quad u \in D^{1,2}_0(\Omega),$$  \hspace{1cm} (1.4)

in an arbitrary exterior domain $\Omega$ (i.e., in a smooth domain whose complement is bounded and nonempty) was studied by Benci and Micheletti for domains whose complement has small enough diameter [2], and by Khatib and Maia in the general case [9]. They showed that (1.4) does not have a least energy solution, but that it does have a positive higher energy solution whenever the limit problem (1.3) has a unique positive solution, up to translations. This is true, e.g., for the model nonlinearity (1.2).

When $\Omega$ is the complement of a ball, the problem (1.1) is known to have a positive radial solution; see [8]. Thus, it is natural to ask whether the solution found in [9] coincides with the radial one or not. We shall see that it does not, if $R$ is sufficiently large. Moreover, we will show that the number of positive nonradial solutions to the problem (1.1) becomes arbitrarily large, as $R \to \infty$.

We assume that $f$ has the following properties:

(f1) $f \in C^1[0, \infty)$, and there are constants $A_1 > 0$ and $2 < p < 2^* < q$ such that, for $m = -1, 0, 1$,

$$|f^{(m)}(s)| \leq \begin{cases} A_1|s|^{p-(m+1)} & \text{if } |s| \geq 1, \\ A_1|s|^{q-(m+1)} & \text{if } |s| \leq 1, \end{cases}$$  \hspace{1cm} (1.5)

where $f^{(-1)} := F$, $f^{(0)} := f$, $f^{(1)} := f'$, and $F(s) := \int_0^s f(t)dt$.

(f2) There is a constant $\theta > 2$ such that $0 \leq \theta F(s) \leq f(s)s < f'(s)s^2$ for all $s > 0$. 

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We write
\[ \|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 \]
for the norm of \( u \) in the space \( D^{1,2}(\mathbb{R}^N) \), and \( D_0^{1,2}(\Omega_R) \) for the closure of \( C^\infty_c(\Omega_R) \) in \( D^{1,2}(\mathbb{R}^N) \). Let
\[ J(u) := \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(u) \]
be the energy functional associated to problem (1.1).

We identify \( \mathbb{R}^N \equiv \mathbb{C} \times \mathbb{R}^{N-2} \) and we write the points in \( \mathbb{R}^N \) as \( x = (z,y) \) with \( z \in \mathbb{C} \) and \( y \in \mathbb{R}^{N-2} \).

We will prove the following result.

**Theorem 1.1.** If \( f \) satisfies (f1) – (f2) then, for any given \( m \in \mathbb{N}, m \geq 2 \), there exists \( R(m) > 0 \) such that, for each \( R > R(m) \), the problem (1.1) has \( m-1 \) positive nonradial solutions \( u_{R,2}, \ldots, u_{R,m} \in D_0^{1,2}(\Omega_R) \) with the following properties: for every \( n = 2, \ldots, m \),

(a) \( u_{R,n}(e^{2\pi ij/n}z, y) = u_{R,n}(z, y) \) for all \( (z, y) \in \Omega_R \) and \( j = 0, \ldots, n-1 \),

(b) \( u_{R,n}(z, y_1) = u_{R,n}(z, y_2) \) if \( |y_1| = |y_2| \),

(c) \( (n-1)c_0 < J(u_{R,n}) < nc_0 \), where \( c_0 \) is the ground state energy of the limit problem (1.3).

Moreover, there are sequences \( R_k > 0, \xi_k = (\zeta_k, 0) \in \Omega_{R_k} \) and a positive least energy radial solution \( \omega \) to the limit problem (1.3) such that
\[
\text{dist}(\xi_k, \partial \Omega_{R_k}) \to \infty \quad \text{as } k \to \infty,
\]
and
\[
\lim_{k \to \infty} \left\| u_{R_k,n} - \sum_{j=0}^{n-1} \omega \left( \cdot - (e^{2\pi ij/n} \zeta_k, 0) \right) \right\| = 0,
\]
and
\[ J(u_{R_k,n}) \to nc_0 \quad \text{as } k \to \infty. \]

The solution \( u_{R,n} \) is obtained by minimizing the energy functional \( J \) on the Nehari manifold of functions which have the symmetries described in (a) and (b). Due to the lack of compactness of the functional \( J \), the existence of these minimizers is not obvious. We prove a splitting lemma for the varying domains \( \Omega_R \) (see Lemma 3.4) which yields a condition for the existence of
Fine estimates allow us to show that this condition is satisfied and, thus, to prove that a minimizer exists for every $R > 0$ and every $2 \leq n \leq \infty$; see Theorem 5.2 below.

The splitting lemma for the varying domains $\Omega_R$ allows us also to show that
$$\lim_{R \to \infty} J(u_{R,n}) = nc_0$$
for each $2 \leq n \leq \infty$, and that the limit profile of $u_{R,n}$, as $R \to \infty$, is the one described in Theorem 1.1.

The symmetries given by (a) and (b) were used by Li in [10] to obtain multiple positive solutions to a subcritical problem in expanding annuli. As pointed out by Byeon in [4], the argument given in [10] does not carry over to dimension 3. The same thing happens in our situation: the energy bounds for the minimizers, when $N = 3$, do not allow us to distinguish them apart; see Theorem 6.1 below. We believe that Theorem 1.1 is true also in dimension 3, but the proof requires a different argument. In fact, Cao and Noussair obtained a similar result in [5] for a semilinear elliptic equation with positive mass and subcritical nonlinearity, which includes dimension $N = 3$.

We wish to stress that, as our solutions are obtained by minimization in a suitable symmetric setting, in contrast to the situation considered in [6,9], we do not need any special properties of the positive solution to the limit problem (1.3), such as uniqueness or nondegeneracy.

This paper is organized as follows: in Section 2 we introduce the variational setting for problem (1.1). Section 3 is devoted to the proof of a splitting lemma for the varying domains $\Omega_R$ in a symmetric setting. In Section 4 we obtain an upper bound for the energy of the symmetric minimizers, that will allow us to derive their existence. Section 5 contains the proof of our main result. Finally, in Section 6 we briefly discuss the 3-dimensional case.

## 2 The limit problem and the variational setting

For $s < 0$ we define $f(s) := -f(-s)$. Then $f \in C^1(\mathbb{R})$. Note that, if $u$ is a positive solution of the problem (1.1) for this new function, it is also a solution of (1.1) for the original function $f$. Hereafter, $f$ will denote this extension. We will assume throughout that $N \geq 4$ and that $f$ satisfies $(f1) - (f2)$.

Let $D^{1,2}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)\}$, with its standard
scalar product and norm

\[ \langle u, v \rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \quad \| u \| := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2}. \]

Since \( f \in C^1(\mathbb{R}) \) and \( f \) satisfies \((f1)\), the limit problem \((1.3)\) has a ground state solution \( \omega \in C^2(\mathbb{R}^N) \), which is positive, radially symmetric and decreasing in the radial direction; see [3, Theorem 4].

Assumption \((f1)\) implies that \(|f(s)| \leq A_1 |s|^{2^*-1}\) and \(|f'(s)| \leq A_1 |s|^{2^*-2}\), and assumption \((f2)\) yields that \( f(s) > 0 \) if \( s > 0 \). Therefore, every positive solution \( u \) to \((1.3)\) satisfies the decay estimates

\[ A_2 (1 + |x|)^{-(N-2)} \leq u(x) \leq A_3 (1 + |x|)^{-(N-2)}, \]
\[ |\nabla u(x)| \leq A_3 (1 + |x|)^{-(N-1)}, \]

for some positive constants \( A_2 \) and \( A_3 \), and \( u \) is radially symmetric and strictly radially decreasing about some point in \( \mathbb{R}^N \); see Theorem 1.1 and Corollary 1.2 in [12].

Let \( 2 < p < 2^* < q \). The following proposition, combined with assumption \((f1)\), provides the interpolation and boundedness properties that are needed to obtain a good variational setting.

**Proposition 2.1.** Let \( \alpha, \beta > 0 \) and \( h \in C^0(\mathbb{R}) \). Assume that \( \frac{\alpha}{\beta} \leq \frac{p}{q}, \beta \leq q \), and there exists \( M > 0 \) such that

\[ |h(s)| \leq M \min\{|s|^\alpha, |s|^\beta\} \quad \text{for every } s \in \mathbb{R}. \]

Then, for every \( t \in \left[ \frac{q}{\beta}, \frac{p}{\alpha} \right] \), the map \( D^{1,2}(\mathbb{R}^N) \to L^t(\mathbb{R}^N) \) given by \( u \mapsto h(u) \) is well defined, continuous and bounded.

**Proof.** See [1, Proposition 3.5] and [6, Proposition 3.1].

Let \( F(u) := \int_{0}^{u} f(s) \, ds \). As \( |F(s)| \leq A_1 |s|^{2^*} \) and \( |f(s)| \leq A_1 |s|^{2^*-1} \) by assumption \((f1)\), the functionals \( \Phi, \Psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[ \Phi(u) := \int_{\mathbb{R}^N} F(u), \quad \Psi(u) := \int_{\mathbb{R}^N} f(u)u \]

are well defined. Using Proposition 2.1 it is easy to show that \( \Phi \) is of class \( C^2 \) and \( \Psi \) is of class \( C^1 \); see [2, Lemma 2.6] or [1, Proposition 3.8]. Hence, the functional \( J : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[ J(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} F(u), \]

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is of class $C^2$, with derivative
\[
J'(u)v = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \int_{\mathbb{R}^N} f(u)v, \quad u, v \in D^{1,2}(\mathbb{R}^N),
\]
and the functional $D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by
\[
u \mapsto J'(\nu) = \int_{\mathbb{R}^N} |\nabla \nu|^2 - \int_{\mathbb{R}^N} f(\nu)\nu,
\]
is of class $C^1$.

3 Symmetries and concentration

For $R > 0$, let $\Omega_R := \{x \in \mathbb{R}^N : |x| > R\}$. As usual, $D^{1,2}_0(\Omega_R)$ denotes the closure of $C_0^\infty(\Omega_R)$ in $D^{1,2}(\mathbb{R}^N)$.

Let $G$ be a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$. A function $u : \mathbb{R}^N \to \mathbb{R}$ is called $G$-invariant if
\[u(gx) = u(x) \quad \text{for every } g \in G, \ x \in \mathbb{R}.
\]
By the principle of symmetric criticality [11], the $G$-invariant solutions to the problem (1.1) are the critical points of the functional $J$ restricted to the space
\[D^{1,2}_0(\Omega_R)^G := \{u \in D^{1,2}_0(\Omega_R) : u \text{ is } G\text{-invariant}\}.
\]
The nontrivial ones belong to the set
\[N^G_R := \{u \in D^{1,2}_0(\Omega_R)^G : u \neq 0, J'(u)u = 0\}.
\]
It is shown in [8] that, under our assumptions on $f$, the problem (1.1) has a positive radial solution; see also Theorem 5.2 below. Therefore, $N^G_R \neq \emptyset$ and, hence,
\[c^G_R := \inf_{u \in N^G_R} J(u) < \infty.
\]
Our assumptions on $f$ also imply that $N^G_R$ is a $C^1$-submanifold of $D^{1,2}_0(\Omega_R)^G$ and a natural constraint for the functional $J$; see [6, Lemma 3.2].

If $G = \{1\}$ is the trivial group, we write $N_R$ and $c_R$ instead of $N^G_R$ and $c^G_R$. Setting $\Omega_0 := \mathbb{R}^N$ we have that $N_0$ and $c_0$ are the Nehari manifold and the ground state energy of the limit problem (1.3), and there exists $\varrho > 0$ such that
\[
\|u\| \geq \varrho \quad \text{for every } u \in N_0;
\]
\[\text{(3.1)}
\]
see, e.g., [6, Lemma 3.2].

Note that, if $H$ is a closed subgroup of $G$, then
\[ N_R^G \subset N_R^H \subset N_R \subset N_0 \quad \text{and} \quad c_R^G \geq c_R^H \geq c_R \geq c_0 > 0. \] (3.2)

Note also that
\[ N_R^G \subset N_S^G \subset N_0^G \quad \text{and} \quad c_R^G \geq c_S^G \geq c_0^G = c_0 \quad \text{if} \quad S \leq R. \] (3.3)

Next, we introduce the groups that will play a role in the proof of our main result.

Let $S^1 := \{ e^{i\theta} : \theta \in [0, 2\pi) \}$ be the group of unit complex numbers. The proper closed subgroups of $S^1$ are the cyclic groups
\[ \mathbb{Z}_n := \left\{ e^{\frac{2\pi i j}{n}} : j = 0, \ldots, n - 1 \right\}. \]

For $2 \leq n \leq \infty$, we define
\[ \Gamma_\infty := S^1 \times O(N - 2) \quad \text{and} \quad \Gamma_n := \mathbb{Z}_n \times O(N - 2). \]

The group $\Gamma_\infty$ acts coordinatewise on $\mathbb{R}^N \equiv \mathbb{C} \times \mathbb{R}^{N-2}$, i.e., if $(\alpha, \beta) \in S^1 \times O(N - 2)$ and $(z, y) \in \mathbb{C} \times \mathbb{R}^{N-2}$, then
\[ (\alpha, \beta)(z, y) := (\alpha z, \beta y). \]

As usual, we write $Gx := \{ gx : g \in G \}$ for the $G$-orbit of a point $x \in \mathbb{R}^N$.

Our aim is to prove a splitting lemma for the moving domains $\Omega_R$; see Lemma 3.4 below. We start with the following auxiliary lemmas.

**Lemma 3.1.** Let $2 \leq n \leq \infty$ and $(x_k)$ be a sequence in $\mathbb{R}^N$. After passing to a subsequence, there exists a sequence $(\xi_k)$ in $\mathbb{R}^N$ and a constant $C_0 > 0$ such that
\[ \text{dist}(\Gamma_n x_k, \xi_k) \leq C_0 \quad \text{for all} \ k, \]
and one of the following statements holds true:

- either $\xi_k = 0$ for all $k$,
- or $n < \infty$, $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $|\alpha \zeta_k - \tilde{\alpha} \zeta_k| \to \infty$ for every $\alpha, \tilde{\alpha} \in \mathbb{Z}_n$ with $\alpha \neq \tilde{\alpha}$,
- or, for each $m \in \mathbb{N}$, there exist $\gamma_1, \ldots, \gamma_m \in \Gamma_n$ such that $|\gamma_i \xi_k - \gamma_j \xi_k| \to \infty$ if $i \neq j$. 

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Proof. Write \( x_k = (z_k, y_k) \in \mathbb{C} \times \mathbb{R}^{N-2} \). There are four possibilities:

(i) If \( (x_k) \) is bounded, we set \( \xi_k = 0 \) for all \( k \).

(ii) If \( n < \infty \), \( (x_k) \) is unbounded and \( (y_k) \) is bounded, then \( (z_k) \) is unbounded. So, passing to a subsequence, we get that \( z_k \neq 0 \) and, setting \( \xi_k := (z_k, 0) \), we have that \( \text{dist}(\Gamma_n x_k, \xi_k) = |y_k| \) and

\[
|\alpha z_k - \tilde{\alpha} z_k| \geq |e^{2\pi i/n} - 1| |z_k| \to \infty
\]

for every \( \alpha, \tilde{\alpha} \in \mathbb{Z} \) with \( \alpha \neq \tilde{\alpha} \).

(iii) If \( n < \infty \) and \( (y_k) \) is unbounded then, after passing to a subsequence, we get that \( y_k \neq 0 \) and

\[
\frac{y_k}{|y_k|} \to y.
\]

As \( y \) lies on the unit sphere \( S^{N-3} \subset \mathbb{R}^{N-2} \) and \( N \geq 4 \), for each \( m \in \mathbb{N} \) there exist \( \beta_1, \ldots, \beta_m \in O(N-2) \) such that \( \beta_i y \neq \beta_j y \) if \( i \neq j \). Then, there exist \( \delta > 0 \) and \( k_0 \in \mathbb{N} \) such that

\[
|\beta_i \frac{y_k}{|y_k|} - \beta_j \frac{y_k}{|y_k|}| \geq \delta \quad \text{if } i \neq j \text{ and } k \geq k_0.
\]

Setting \( \xi_k := x_k \) and \( \gamma_i := (1, \beta_i) \), we have that \( \text{dist}(\Gamma_n x_k, \xi_k) = 0 \) and

\[
|\gamma_i \xi_k - \gamma_j \xi_k| = |\beta_i y_k - \beta_j y_k| \geq \delta |y_k| \to \infty \quad \text{if } i \neq j.
\]

(iv) If \( n = \infty \) and \( (x_k) \) is unbounded, passing to a subsequence, we get that \( x_k \neq 0 \). Since the \( \Gamma_\infty \)-orbit of every \( x \neq 0 \) is homeomorphic either to \( S^1 \) or to \( S^1 \times S^{N-3} \) or to \( S^{N-3} \), as \( N \geq 4 \), it is an infinite set. So, setting \( \xi_k := x_k \) and arguing as in the previous case, we get that \( \text{dist}(\Gamma_\infty x_k, \xi_k) = 0 \) and that, for each \( m \in \mathbb{N} \), there exist \( \gamma_1, \ldots, \gamma_m \in \Gamma_\infty \) such that

\[
|\gamma_i \xi_k - \gamma_j \xi_k| \geq \delta |x_k| \to \infty \quad \text{if } i \neq j.
\]

\[\square\]

Lemma 3.2. If \( u_k \rightharpoonup u \) weakly in \( D^{1,2}(\mathbb{R}^N) \) then, passing to a subsequence,

\[
\begin{align*}
(a) \quad & \int_{\mathbb{R}^N} |f(u_k) - f(u)||\varphi| = o(1) \quad \text{for every } \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N), \\
(b) \quad & \int_{\mathbb{R}^N} F(u_k) - \int_{\mathbb{R}^N} F(u_k - u) = \int_{\mathbb{R}^N} F(u) + o(1), \\
(c) \quad & \int_{\mathbb{R}^N} f(u_k) u_k - \int_{\mathbb{R}^N} f(u_k - u) [u_k - u] = \int_{\mathbb{R}^N} f(u) u + o(1), \\
(d) \quad & |f(u_k) - f(u_k - u) - f(u) - f(u)| \to 0 \text{ in } (D^{1,2}(\mathbb{R}^N))'.
\end{align*}
\]

Proof. The proof of this lemma relies on Proposition 2.1. Statements (a), (b) and (d) are proved in [6, Lemma 3.8]. The proof of (c) uses a similar argument. \[\square\]
The following vanishing lemma is crucial for the proof of Lemma 3.4 below. We write \( B_\varepsilon(y) := \{ x \in \mathbb{R}^N : |x - y| < \varepsilon \} \).

**Lemma 3.3.** If \((u_k)\) is bounded in \(D^{1,2}(\mathbb{R}^N)\) and there exists \( \varepsilon > 0 \) such that
\[
\lim_{k \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_\varepsilon(y)} |u_k|^2 \right) = 0,
\]
then \( \lim_{k \to \infty} \int_{\mathbb{R}^N} f(u_k) u_k = 0. \)

**Proof.** See [6, Lemma 3.5]. \( \square \)

Next, we prove a splitting lemma for moving domains.

**Lemma 3.4.** Fix \( 2 \leq n \leq \infty \), and let \((R_k)\) be a nondecreasing sequence of positive numbers and \( u_k \in D^{1,2}_0(\Omega_{R_k}) \Gamma_n \) be such that

(i) \( u_k \rightharpoonup 0 \) weakly but not strongly in \( D^{1,2}(\mathbb{R}^N) \),

(ii) \( J(u_k) \to c \),

(iii) \( J'(u_k)\varphi_k \to 0 \) for every sequence \( (\varphi_k) \) such that \( \varphi_k \in D^{1,2}_0(\Omega_{R_k}) \) and \( (\varphi_k) \) is bounded in \( D^{1,2}(\mathbb{R}^N) \).

Then, \( n < \infty \) and, after passing to a subsequence, there exist \( \xi_k = (\zeta_k, 0) \in \Omega_{R_k} \) such that \( \text{dist}(\xi_k, \partial \Omega_{R_k}) \to \infty \), and a nontrivial \( O(N - 2) \)-invariant solution \( v \) to the limit problem (1.3) such that \( v \geq 0 \) if \( u_k \geq 0 \) for all \( k \), and

\[
\|u_k\|^2 - \|u_k - w_k\|^2 = n\|v\|^2 + o(1),
\]

\[
\int_{\mathbb{R}^N} F(u_k) - \int_{\mathbb{R}^N} F(u_k - w_k) = n \int_{\mathbb{R}^N} F(v) + o(1),
\]

\[
\int_{\mathbb{R}^N} f(u_k) u_k - \int_{\mathbb{R}^N} f(u_k - w_k)[u_k - w_k] = n \int_{\mathbb{R}^N} f(v) v + o(1),
\]

where
\[
w_k(z, y) := \sum_{j=0}^{n-1} v(e^{-2\pi ij/n} z - \zeta_k, y).
\]

Moreover, \( c \geq nJ(v) \geq nc_0 \).

**Proof.** For simplicity, we write \( \Omega_k := \Omega_{R_k} \).
By property \((i)\), after passing to a subsequence, we may assume that 
\((u_k)\) is bounded and bounded away from 0 in \(D^{1,2}_0(\mathbb{R}^N)\). Then, property \((iii)\) implies that

\[
\|u_k\|^2 - \int_{\mathbb{R}^N} f(u_k)u_k = J'(u_k)u_k = o(1),
\]

and using assumption \((f1)\) we obtain that

\[
0 < a_1 < \int_{\mathbb{R}^N} f(u_k)u_k \leq A_0|u_k|_{2^*}^{2^*},
\]

where \(|\cdot|_{2^*}\) is the norm in \(L^{2^*}(\mathbb{R}^N)\). By the vanishing lemma (Lemma 3.3), there exist \(a_2 > 0\) and a sequence \((x_k)\) in \(\mathbb{R}^N\) such that

\[
\int_{B_1(x_k)} |u_k|^2 = \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |u_k|^2 \geq a_2 > 0 \quad \text{for all } k.
\]

For the sequence \((x_k)\) we choose \((\xi_k)\) as in Lemma 3.1. Then, \(|\gamma_kx_k - \xi_k| \leq C_0\) for some \(\gamma_k \in \Gamma_n\) and, as \(u_k\) is \(\Gamma_n\)-invariant, we obtain that

\[
\int_{B_{C_0+1}(\xi_k)} |u_k|^2 \geq \int_{B_1(\gamma_kx_k)} |u_k|^2 \geq a_2 \quad \text{for all } k. \tag{3.4}
\]

Since \(\text{supp}(u_k) \subset \Omega_k\), this implies that

\[
\text{dist}(\xi_k, \Omega_k) \leq C_0 + 1. \tag{3.5}
\]

Set \(v_k(x) := u_k(x + \xi_k)\). Passing to a subsequence, we have that \(v_k \rightharpoonup v\) weakly in \(D^{1,2}(\mathbb{R}^N)\), \(v_k \to v\) a.e. in \(\mathbb{R}^N\) and \(v_k \to v\) strongly in \(L^2_{\text{loc}}(\mathbb{R}^N)\). Then, \(v \geq 0\) if \(u_k \geq 0\) for all \(k\). Inequality (3.4) implies that \(v \neq 0\) and, as \(u_k \to 0\) weakly in \(D^{1,2}(\mathbb{R}^N)\), we deduce that \(|\xi_k| \to \infty\).

Define \(\Omega_k := \{x \in \mathbb{R}^N : x + \xi_k \in \Omega_k\}\). Note that, if \(\varphi \in C_c^\infty(\mathbb{R}^N)\) and \(\text{supp}(\varphi) \subset \Omega_k\) for \(k\) large enough, then \(\varphi_k(x) := \varphi(x - \xi_k)\) satisfies \(\varphi_k \in C_c^\infty(\Omega_k)\) for \(k\) large enough, and property \((iii)\) yields

\[
J'(v_k)\varphi = \int_{\mathbb{R}^N} (\nabla v_k \cdot \nabla \varphi - f(v_k)\varphi)
= \int_{\Omega_k} (\nabla u_k \cdot \nabla \varphi_k - f(u_k)\varphi_k) = o(1). \tag{3.6}
\]

Set

\[
d_k := \text{dist}(\xi_k, \partial \Omega_k),
\]
and consider the interior unit normal \( \eta_k := \frac{\xi_k}{|\xi_k|} \) to \( \partial \Omega_k \) at the point \( \frac{R_k \xi_k}{|R_k \xi_k|} \).

Passing to a subsequence, we have that \( \eta_k \to \eta \). We claim that the sequence \( (d_k) \) is unbounded.

If the sequence \( (R_k) \) is bounded then, as \( |\xi_k| \to \infty \), this is immediately true.

If \( R_k \to \infty \), arguing by contradiction, we assume that \( (d_k) \) is bounded. Then, after passing to a subsequence, we have that \( d_k \to d \in [0, \infty) \). We consider two cases. If a subsequence of \( (\xi_k) \) satisfies that \( \xi_k \in \bar{\Omega}_k \), we set
\[
H := \{ x \in \mathbb{R}_N : \eta \cdot x > -d \}.
\]

Since \( R_k \to \infty \), every compact subset in \( \mathbb{R}_N \setminus H \) is contained in \( \mathbb{R}_N \setminus \bar{\Omega}_k \) for \( k \) large enough and, as \( v_k \equiv 0 \) in \( \mathbb{R}_N \setminus \bar{\Omega}_k \), we have that \( v \in D^{1,2}_0(H) \).

Moreover, every compact subset of \( H \) is contained in \( \bar{\Omega}_k \) for \( k \) large enough. So, if \( \varphi \in C_c^\infty(H) \), then \( \text{supp}(\varphi) \subset \bar{\Omega}_k \) for \( k \) large enough. Passing to the limit in equation (3.6), using Lemma 3.2(a), we conclude that \( J'(v) \varphi = 0 \) for every \( \varphi \in C_c^\infty(H) \). It follows that \( v \) is a nontrivial solution of
\[
-\Delta v = f(v), \quad v \in D^{1,2}_0(H),
\]
contradicting the fact that this problem has only the trivial solution; see [8,12]. Likewise, if a subsequence of \( (\xi_k) \) satisfies that \( \xi_k \in \mathbb{R}_N \setminus \Omega_k \), we set
\[
H := \{ x \in \mathbb{R}_N : \eta \cdot x > d \},
\]
and a similar argument yields a contradiction.

This proves that \( (d_k) \) is unbounded, and the inequality (3.5) implies that \( \xi_k \in \Omega_k \) and that every compact subset of \( \mathbb{R}_N \) is contained in \( \Omega_k \) for \( k \) large enough. So, passing to the limit in equation (3.6), we conclude that \( v \) is a nontrivial solution to the limit problem (1.3).

Let \( \gamma_1, \ldots, \gamma_m \in \Gamma_n \) be such that \( |\gamma_i \xi_k - \gamma_j \xi_k| \to \infty \) if \( i \neq j \). Then, for each \( j \in \{1, \ldots, m\} \),
\[
v_k \circ \gamma_j^{-1} - \sum_{i \neq j} v \circ \gamma_i^{-1}(\cdot - \gamma_i \xi_k + \gamma_j \xi_k) \to v \circ \gamma_j^{-1}
\]
weakly in \( D^{1,2}(\mathbb{R}_N) \), where the sum is defined to be 0 if \( j = m \). Hence,
\[
\left\| v_k \circ \gamma_j^{-1} - \sum_{i \neq j} v \circ \gamma_i^{-1}(\cdot - \gamma_i \xi_k + \gamma_j \xi_k) \right\|^2 = \left\| v_k \circ \gamma_j^{-1} - \sum_{i \neq j} v \circ \gamma_i^{-1}(\cdot - \gamma_i \xi_k + \gamma_j \xi_k) \right\|^2 + \left\| v \circ \gamma_j^{-1} \right\|^2 + o(1).
\]
Performing the change of variable $x + \gamma_j \xi_k = \tilde{x}$, recalling that $v_k(x) = u_k(x + \xi_k)$ and taking into account that $u_k$ is $\Gamma_n$-invariant we get
\[
\left\| u_k - \sum_{i=j+1}^{m} v \circ \gamma_i^{-1}(\cdot - \gamma_i \xi_k) \right\|^2 = \left\| u_k - \sum_{i=j}^{m} v \circ \gamma_i^{-1}(\cdot - \gamma_i \xi_k) \right\|^2 + \|v\|^2 + o(1),
\]
and iterating this identity for $j = 1, \ldots, m$ we obtain
\[
\|u_k\|^2 = \|u_k - w_k\|^2 + m \|v\|^2 + o(1),
\] (3.7)
where
\[
w_k := \sum_{i=1}^{m} v \circ \gamma_i^{-1}(\cdot - \gamma_i \xi_k).
\]
Similarly, using statements (b) and (c) of Lemma 3.2 we get that
\[
\int_{\mathbb{R}^N} F(u_k) = \int_{\mathbb{R}^N} F(u_k - w_k) + m \int_{\mathbb{R}^N} F(v) + o(1)
\] (3.8)
and
\[
\int_{\mathbb{R}^N} f(u_k)u_k - \int_{\mathbb{R}^N} f(u_k - w_k)[u_k - w_k] = m \int_{\mathbb{R}^N} f(v)v + o(1).
\] (3.9)
As $v$ solves the problem (1.3), property (iii) and equations (3.7) and (3.9) yield
\[
\|u_k - w_k\|^2 = \int_{\mathbb{R}^N} f(u_k - w_k)[u_k - w_k] + o(1).
\]
Note that assumption (f2) implies that $f(s)s - 2F(s) \geq 0$ for every $s \in \mathbb{R}$. Therefore,
\[
\frac{1}{2} \|u_k - w_k\|^2 - \int_{\mathbb{R}^N} F(u_k - w_k) + o(1)
= \int_{\mathbb{R}^N} \left( \frac{1}{2} f(u_k - w_k)[u_k - w_k] - F(u_k - w_k) \right) \geq 0,
\]
and from property (ii) and equations (3.7) and (3.8) we get that
\[
c = \lim_{k \to \infty} J(u_k) \geq mJ(v) \geq mc_0.
\] (3.10)
This says that $m$ cannot be arbitrarily large. So, as $|\xi_k| \to \infty$, the only possibility left in Lemma 3.1 is that $n < \infty$, $\xi_k = (\zeta_k, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$ and $|e^{2\pi i/n} \zeta_k - e^{2\pi i/j} \zeta_k| \to \infty$ for $i, j \in \{0, \ldots, n - 1\}$ with $i \neq j$.

Then, $v_k(z, y) = u_k(z + \zeta_k, y)$. As $u_k$ is $O(N-2)$-invariant, we get that $v_k$ is $O(N-2)$-invariant and, since $v_k \to v$ a.e. in $\mathbb{R}^N$, we have that $v$ is also $O(N-2)$-invariant.

Finally, if we take $m := n$ and $\gamma_i := e^{2\pi i(i-1)/n}$, $i = 1, \ldots, n$, the statements (3.7), (3.8), (3.9) and (3.10) complete the proof of the lemma.

4 An upper bound for the energy of symmetric minimizers

Fix $2 \leq n < \infty$ and a positive radial ground state solution $\omega$ to the limit problem (1.3). Then, $\omega$ satisfies the decay estimates (2.1).

Set $\xi_j := (e^{2\pi i/j/n}, 0)$ and, for each $\rho > 0$, let

$$\omega_{j, \rho}(x) := \omega(x - \rho \xi_j) \quad \text{and} \quad \sigma_{\rho} := \sum_{j=0}^{n-1} \omega_{j, \rho}.$$  

Fix $R > 0$ and a radial cut-off function $\psi = \psi_R \in C_\infty^\infty(\mathbb{R}^N)$ such that

$$0 \leq \psi(x) \leq 1, \quad \psi(x) = 0 \text{ if } |x| \leq R, \quad \psi(x) = 1 \text{ if } |x| \geq 2R. \quad (4.1)$$

Then, $\psi \sigma_{\rho} \in D_0^{1,2}(\Omega_R)^{\Gamma_R}$. The aim of this section is to prove the following result.

**Proposition 4.1.** For every $R > 0$ and $2 \leq n < \infty$ there exists $\rho_0 > 0$ such that, for each $\rho > \rho_0$,

(a) there is a unique $t_{\rho} = t_{R,n,\rho} \in (0, \infty)$ such that $t_{\rho} \psi \sigma_{\rho} \in \mathcal{N}_R^{\Gamma_R},$

(b) $c_{\rho_R}^{\Gamma_R} \leq J(t_{\rho} \psi \sigma_{\rho}) < n c_0.$

We start with some lemmas.

**Lemma 4.2.** (a) If $y_0, y \in \mathbb{R}^N$, $y_0 \neq y$, and $\alpha$ and $\beta$ are positive constants such that $\alpha + \beta > N$, then there exists $C_1 = C_1(\alpha, \beta, |y - y_0|) > 0$ such that

$$\int_{\mathbb{R}^N} \frac{d\mathbf{x}}{(1 + |x - \rho y_0|)^\alpha (1 + |x - \rho y|)^\beta} \leq C_1 \rho^{-\mu}$$

for all $R \geq 1$, where $\mu := \min\{\alpha, \beta, \alpha + \beta - N\}$. 13
(b) If \( y_0, y \in \mathbb{R}^N \setminus \{0\} \), and \( \kappa \) and \( \vartheta \) are positive constants such that 
\( \kappa + 2\vartheta > N \), then there exists \( C_2 = C_2(\kappa, \vartheta, |y_0|, |y|) > 0 \) such that 
\[
\int_{\mathbb{R}^N} \frac{\mathrm{d}x}{(1 + |x|)^{\kappa}(1 + |x - \rho y_0|)^\vartheta(1 + |x - \rho y|)^\vartheta} \leq C_2 \rho^{-\tau},
\]
for all \( R \geq 1 \), where \( \tau := \min\{\kappa, 2\vartheta, \kappa + 2\vartheta - N\} \).

**Proof.** See [6, Lemma 4.1]. \( \square \)

For \( i, j \in \{0, \ldots, n - 1\} \), \( \rho > 0 \), define
\[
\varepsilon_{i,j,\rho} := \int_{\mathbb{R}^N} \nabla \omega_{i,\rho} \cdot \nabla \omega_{j,\rho} = \int_{\mathbb{R}^N} f(\omega_{i,\rho}) \omega_{j,\rho},
\]
\[
\varepsilon_\rho := \sum_{i \neq j} \varepsilon_{i,j,\rho}.
\]

**Lemma 4.3.** (a) There are positive constants \( C_1 \) and \( C_2 \) such that, for every \( i \neq j \) and \( \rho \) large enough,
\[
C_1 \rho^{-(N-2)} \leq \varepsilon_{i,j,\rho} \leq C_2 \rho^{-(N-2)}.
\]
Hence, \( \varepsilon_\rho \to 0 \) as \( \rho \to \infty \).

(b) There exists \( C_0 > 0 \) such that
\[
\int_{\mathbb{R}^N} f(\omega_{i,\rho}) \omega_{j,\rho} \geq C_0 \rho^{-(N-2)}
\]
for every \( s, t \geq \frac{1}{2} \) and \( \rho \) large enough.

(c) If \( \nu > 0 \) and \( i \neq j \) then, as \( \rho \to \infty \),
\[
\int_{\mathbb{R}^N} (\omega_{i,\rho} \omega_{j,\rho})^{1+\frac{\nu}{2}} = o(\varepsilon_\rho).
\]

(d) For every \( r > 1 \) and every compact subset \( K \) of \( \mathbb{R}^N \), we have that
\[
\int_K |\sigma_\rho|^r = o(\varepsilon_\rho) \quad \text{and} \quad \int_K |\nabla \sigma_\rho|^r = o(\varepsilon_\rho).
\]

**Proof.** The first inequality in statement (a) is a special case of (b). The second one is proved in [6, Lemma 4.2]. Statement (b) is proved in [6, Lemma 4.3].
(c) Lemma 4.2 with \(\alpha = \beta = (1 + 2\nu)(N - 2)\) and (2.1) imply that, for some \(\mu > N - 2\),
\[
\int_{\mathbb{R}^N} (\omega_{i,\rho} \omega_{j,\rho})^{1 + \frac{\mu}{2}} \leq C \rho^{-\mu} = o(\varepsilon_{\rho}).
\]

(d) Since \(K\) is compact, there exists \(C_K > 0\) such that \(|x - \rho \xi| \leq \rho + C_K\) for all \(x \in K\). So from the decay estimates (2.1) we obtain that
\[
\int_{K} \omega_{i,\rho}^r \leq C \int_{K} |x - \rho \xi|^{-r(N-2)} dx \leq C \rho^{-r(N-2)},
\]
for \(\rho\) large enough. As \(r > 1\), statement (a) yields
\[
\frac{1}{\varepsilon_{\rho}} \int_{K} \omega_{i,\rho}^r \leq C \rho^{-(r-1)(N-2)} \to 0 \quad \text{as} \quad \rho \to \infty.
\]
Therefore,
\[
\int_{K} \sigma_{\rho}^r \leq C \sum_{i=0}^{n-1} \int_{K} \omega_{i,\rho}^r = o(\varepsilon_{\rho}),
\]
as claimed. The other estimate is obtained similarly. \qed

Lemma 4.4. For every \(t \in (0, \infty)\) we have that
\[
\|\psi \sigma_{\rho}\|^2 = \|\sigma_{\rho}\|^2 + o(\varepsilon_{\rho}), \tag{4.2}
\]
\[
\int_{\mathbb{R}^N} F(t \psi \sigma_{\rho}) = \int_{\mathbb{R}^N} F(t \sigma_{\rho}) + t^2 o(\varepsilon_{\rho}), \tag{4.3}
\]
\[
\int_{\mathbb{R}^N} f(t \psi \sigma_{\rho}) |\psi \sigma_{\rho}| = \int_{\mathbb{R}^N} f(t \sigma_{\rho}) \sigma_{\rho} + t^2 o(\varepsilon_{\rho}). \tag{4.4}
\]

Proof. Let \(u \in D^{1,2}(\mathbb{R}^N)\). An easy computation shows that
\[
\|\psi u\|^2 = \|u\|^2 + \int_{\mathbb{R}^N} (\psi^2 - 1)|\nabla u|^2 - \int_{\mathbb{R}^N} (\psi \Delta \psi) u^2.
\]
Setting \(u = \sigma_{\rho}\) and applying statement (d) of Lemma 4.3 we obtain (4.2).
Next, note that
\[
\int_{\mathbb{R}^N} (F(t \psi u) - F(tu)) = \int_{B_{2R}(0)} (F(t \psi u) - F(tu)).
\]
By the mean value theorem and assumption (f1) there exists \(s = s(x) \in (0, 1)\) such that
\[
|F(t \psi u) - F(tu)| = |f((1 - s)\psi + s|tu||(|1 - \psi|tu| \leq A_0|tu|^{2^*}.
\]
pointwise. Hence
\[
\int_{B_{2R}(0)} |F(t\psi u) - F(tu)| \leq A_0 t^{2^*} \int_{B_{2R}(0)} |u|^{2^*}.
\]
Setting \(u = \sigma_\rho\) and applying statement \((d)\) of Lemma 4.3 we obtain (4.3). The proof of (4.4) is obtained in a similar way.

**Lemma 4.5.** For every \(\tau > 1\) there exists \(C_\tau > 0\) such that
\[
\left| \int_{\mathbb{R}^N} (tf(\omega_{i,\rho}) - f(t\omega_{i,\rho})) \omega_{j,\rho} \right| \leq C_\tau |t - 1|\varepsilon_{i,j,\rho},
\]
for all \(t \in [0, \tau]\).

**Proof.** Fix \(u \in \mathbb{R}\) and consider the function \(h(t) := tf(u) - f(tu)\). Assumption \((f1)\) implies that \(|h'(t)| \leq C|u|^{2^*-1}\) for all \(t \in [0, \tau]\). So, by the mean value theorem,
\[
|tf(u) - f(tu)| = |h(t) - h(1)| \leq C|u|^{2^*-1}|t - 1|
\]
for all \(t \in [0, \tau]\). By assumption \((f1)\), we have that \(|f(s)| \leq A_1 |s|^{2^*-1}\). On the other hand, from the estimates (2.1) and Lemma 4.2(a) we obtain
\[
\int_{\mathbb{R}^N} \omega_{i,\rho}^{2^*-1} \omega_{j,\rho} \leq C \int_{\mathbb{R}^N} \frac{dx}{(1 + |x - \rho\xi_i|)^{N+2}(1 + |x - \rho\xi_j|)^{N-2}} \leq C \rho^{-(N-2)}.
\]
Hence, Lemma 4.3 yields
\[
\left| \int_{\mathbb{R}^N} (tf(\omega_{i,\rho}) - f(t\omega_{i,\rho})) \omega_{j,\rho} \right| \leq C |t - 1| \int_{\mathbb{R}^N} \omega_{i,\rho}^{2^*-1} \omega_{j,\rho} \leq C_\tau |t - 1|\varepsilon_{i,j,\rho},
\]
as claimed.

**Lemma 4.6.** There exists \(\rho_0 > 0\) such that, for each \(\rho > \rho_0\), there is a unique \(t_\rho \in (0, \infty)\) which satisfies that \(t_\rho \psi_\sigma_\rho \in N^T_{R}\), and there exists \(t_0 > 1\) such that \(t_\rho \in (0, t_0)\) for every \(\rho \geq \rho_0\).

**Proof.** Assumption \((f2)\) implies that the function \(\frac{f(t)}{t}\) is strictly increasing in \((0, \infty)\). Hence, for each positive function \(u \in D^{1,2}_{\mathbb{R}^N}\), the function
\[
\frac{J'(tu)}{t} = \|u\|^2 - \int_{\mathbb{R}^N} \frac{f(tu)}{tu} u^2
\]
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is strictly decreasing in $t \in (0, \infty)$. Therefore, if there exists $t_u \in (0, \infty)$ such that $J'(t_u)u = 0$, this number will be unique. Observe that $J'(tu)u > 0$ for $t$ small enough.

We claim that there exist $\rho_1 > 0$, $t_0 > 1$ and $M_0 > 0$ such that

$$\frac{J'(t_0\rho)}{t_0} < -M_0 \quad \text{for all } \rho \geq \rho_1.$$  \hspace{1cm} (4.5)

Indeed, as $\omega \in \mathcal{N}_0$, there exist $M > 0$ and $t_0 > 1$ such that $\frac{J'(t_0\omega)}{t_0} \leq -M$. Assumption $(f2)$ implies that $f$ is increasing. Hence,

$$\frac{J'(t_0\rho)}{t_0} = \|\rho\|^2 - \frac{1}{t_0} \int f(t_0\rho) - \sum_{i=0}^{n-1} \|\omega_{i,\rho}\|^2 + \epsilon - \sum_{i=0}^{n-1} \frac{1}{t_0} \int f(t_0\rho) - \sum_{i=0}^{n-1} \frac{1}{t_0} \int (f(t_0\rho) - f(t_0\omega_{i,\rho})) \omega_{i,\rho}
\leq -Mn + o_{\rho}(1),$$

which immediately yields (4.5).

Lemma 4.4 implies that there exists $\rho_0 \geq \rho_1$ such that

$$\frac{J'(t_0\psi)\psi}{t_0} = \frac{J'(t_0\rho)}{t_0} + o(\epsilon) < -\frac{M_0}{2} \quad \text{if } \rho > \rho_0.$$

Hence, there is a unique $t_\rho \in (0, t_0)$ such that $J'(t_\rho\psi)\psi = 0$, as claimed.$\square$

**Lemma 4.7.** $t_\rho \to 1$ as $\rho \to \infty$.

**Proof.** From Lemma 4.4 we get that

$$J'(\psi)\psi = \|\psi\|^2 - \int f(\sigma) \sigma + o(\epsilon)$$

$$= \sum_{i=0}^{n-1} J'(\omega_{i,\rho}) \omega_{i,\rho} + \sum_{i=0}^{n-1} \int f(\sigma) - f(\omega_{i,\rho}) \omega_{i,\rho}$$

$$= \|\psi\|^2 - \sum_{i=0}^{n-1} \int f(\sigma) - f(\omega_{i,\rho}) \omega_{i,\rho}. $$

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By the mean value theorem and assumption \((f1)\) there exists \(s = s(x) \in (0, 1)\) such that

\[
|f(\sigma) - f(\omega_i, \rho)| = |f'((1-s)\sigma + s\omega_i, \rho)||\sigma - \omega_i, \rho|
\leq C|\sigma|^{2^* - 2} \sum_{j \neq i} \omega_{i, \rho}
\]

pointwise. Hence, using Hölder’s inequality and the fact that \(|\rho(\xi_j - \xi_i)| \to \infty\) as \(\rho \to \infty\), we get that

\[
\int_{\mathbb{R}^N} |f(\sigma) - f(\omega_i, \rho)| \omega_{i, \rho} \leq \sum_{j \neq i} C|\sigma|^{2^* - 2} \omega_{i, \rho} \omega_{j, \rho}
\leq C \sum_{j \neq i} |\sigma|^{2^* - 2} \left( \int_{\mathbb{R}^N} (\omega_{i, \rho} \omega_{j, \rho})^{2^* / 2} \right)^{2 / 2^*}
\leq C \sum_{j \neq i} \left( \int_{\mathbb{R}^N} \omega_{i, \rho}^{2^* / 2} (\cdot - \rho(\xi_j - \xi_i)) \omega_{j, \rho}^{2^* / 2} \right)^{2 / 2^*} = o_\rho(1).
\]

We conclude that \(J'(\psi(\sigma))\omega_{\sigma, \rho} = o_\rho(1)\). This implies that \(t_\rho \to 1\) as \(\rho \to \infty\), as claimed.

**Lemma 4.8.** Given \(r > 0, m \in \mathbb{N}\) and \(\nu \in (0, q - 2)\), there exists a constant \(C_{r, m, \nu} > 0\) such that, for any finite set of numbers \(a_1, \ldots, a_m \in (0, r]\),

\[
F\left( \sum_{i=1}^m a_i \right) - \sum_{i=1}^m F(a_i) - \sum_{i,j=1 \atop i \neq j}^m f(a_i)a_j \geq -C_{r, m, \nu} \sum_{i,j=1 \atop i \neq j}^m (a_ia_j)^{1+\frac{\nu}{2}}. \quad (4.6)
\]

**Proof.** First, we claim that, for any \(a, b > 0\),

\[
f(a + b) \geq f(a) + f(b). \quad (4.7)
\]

To prove this inequality, observe that we may assume that \(a \geq b\). Note that assumption \((f2)\) implies that \(\frac{f(t)}{t}\) is increasing in \((0, \infty)\). Then, using \((f2)\) we obtain that

\[
f(a + b) - f(a) - f(b) = \int_a^{a+b} f'(t) \, dt - f(b) \geq \int_a^{a+b} \frac{f(t)}{t} \, dt - f(b)
\geq b \left( \frac{f(a)}{a} - \frac{f(b)}{b} \right) \geq 0,
\]

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as claimed.

Now we prove inequality (4.6) by induction on \( m \). Hereafter, \( C \) will denote some positive constant, not necessarily the same one, which depends only on \( r, m, \nu \).

By [6, Lemma 4.8] we have that, for any \( a_1, a_2 \in (0, r] \),

\[
F(a_1 + a_2) - F(a_1) - F(a_2) - f(a_1)a_2 - f(a_1)a_2 \geq -C|a_1a_2|^{1+\frac{\nu}{2}}. \tag{4.8}
\]

Let \( m \geq 3 \) and assume that the inequality (4.6) is true for \( m-1 \). Then, using the inequalities (4.7) and (4.8) we obtain

\[
F \left( \sum_{i=1}^{m} a_i \right) \geq F \left( \sum_{i=1}^{m-1} a_i \right) + F(a_m) + f \left( \sum_{i=1}^{m-1} a_i \right) a_m
\]

\[
+ f(a_m) \left( \sum_{i=1}^{m-1} a_i \right) - C \left( \sum_{i=1}^{m-1} a_i a_m \right)^{1+\frac{\nu}{2}}
\]

\[
\geq \sum_{i=1}^{m} F(a_i) + \sum_{i,j=1, i \neq j} f(a_i)a_j + f \left( \sum_{i=1}^{m-1} a_i \right) a_m + f(a_m) \left( \sum_{i=1}^{m-1} a_i \right)
\]

\[
- C \sum_{i,j=1, i \neq j} (a_i a_j)^{1+\frac{\nu}{2}} - C \left( \sum_{i=1}^{m-1} a_i a_m \right)^{1+\frac{\nu}{2}}
\]

\[
\geq \sum_{i=1}^{m} F(a_i) + \sum_{i,j=1, i \neq j} f(a_i)a_j - C \sum_{i,j=1, i \neq j} (a_i a_j)^{1+\frac{\nu}{2}},
\]

as claimed. \( \square \)

**Proof of Proposition 4.1.** Statement (a) was proved in Lemma 4.6. The first inequality in statement (b) follows from (a). Next, we prove the second inequality.

From Lemmas 4.6 and 4.7 we have that \( t_\rho \in \left[ \frac{1}{2}, t_0 \right] \) for large enough \( \rho \).
So, from Lemmas 4.4, 4.8, 4.3 and 4.5 we get that
\[ J(t_\rho \psi \sigma_\rho) = J(t_\rho \sigma_\rho) + o(\varepsilon_\rho) \]
\[ = \sum_{j=0}^{n-1} J(t_\rho \omega_{j,\rho}) + \frac{t_\rho^2}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} \nabla \omega_{i,\rho} \cdot \nabla \omega_{j,\rho} \]
\[ + \sum_{j=0}^{n-1} \int_{\mathbb{R}^N} F(t_\rho \omega_{j,\rho}) - \int_{\mathbb{R}^N} F(t_\rho \sigma_\rho) + o(\varepsilon_\rho) \]
\[ \leq nc_0 + \frac{t_\rho^2}{2} \sum_{i \neq j} \int_{\mathbb{R}^N} f(\omega_{i,\rho}) \omega_{j,\rho} - \frac{t_\rho^2}{2} \varepsilon_\rho \]
\[ - \sum_{i \neq j} \int_{\mathbb{R}^N} f(t_\rho \omega_{i,\rho}) t_\rho \omega_{j,\rho} + C \sum_{i \neq j} \int_{\mathbb{R}^N} \left( \frac{\varepsilon_\rho}{\rho} \right)^{2+\nu} + o(\varepsilon_\rho) \]
\[ \leq nc_0 + \sum_{i \neq j} \int_{\mathbb{R}^N} (t_\rho f(\omega_{i,\rho}) - f(t_\rho \omega_{i,\rho})) t_\rho \omega_{j,\rho} - \frac{t_\rho^2}{2} \varepsilon_\rho + o(\varepsilon_\rho) \]
\[ \leq nc_0 + C|t_\rho - 1|\varepsilon_\rho - \frac{t_\rho^2}{2} \varepsilon_\rho + o(\varepsilon_\rho). \]

Lemma 4.7 says that \( t_\rho \to 1 \) as \( \rho \to \infty \). This yields the conclusion.

**5 Multiple positive solutions**

This section is devoted to the proof of the Theorem 1.1.

Fix \( R > 0 \). We write \( \nabla J(u) \) for the gradient of the functional \( J : D_0^{1,2}(\Omega_R) \to \mathbb{R} \) at \( u \) and we write \( \nabla_{N^G_R} J(u) \) for the orthogonal projection of \( \nabla J(u) \) onto the tangent space to the Nehari manifold \( N^G_R \) at the point \( u \).

A sequence \( (u_k) \) will be called a \((PS)_c\)-sequence for \( J \) on \( N^G_R \) if
\[ u_k \in N^G_R, \quad J(u_k) \to c \quad \text{and} \quad \| \nabla_{N^G_R} J(u_k) \| \to 0. \]

**Lemma 5.1.** Every \((PS)_c\)-sequence \( (u_k) \) for \( J \) on \( N^G_R \) contains a subsequence which is bounded in \( D^{1,2}(\mathbb{R}^N) \) and satisfies \( \| \nabla J(u_k) \| \to 0. \)

**Proof.** By [6, Lemma 3.6], \( (u_k) \) contains a bounded subsequence. The same argument given to prove [6, Lemma 3.7] shows that \( (u_k) \) contains a subsequence such that \( \| \nabla G_J(u_k) \| \to 0 \), where \( \nabla G_J \) is the gradient of the functional \( J : D_0^{1,2}(\Omega_R)^G \to \mathbb{R} \) at \( u_k \).

Observe that, if \( u \in D_0^{1,2}(\Omega_R)^G \), then \( \nabla J(u) \in D_0^{1,2}(\Omega_R)^G \). Therefore, \( \nabla J(u) = \nabla G J(u) \), and the proof is complete. \( \square \)
Theorem 5.2. Fix $R > 0$ and let $G$ be either the group $O(N)$ or one of the groups $\Gamma_n$ with $2 \leq n \leq \infty$. Then the problem (1.1) has a positive $G$-invariant solution $u$ such that

$$c_0 < J(u) = c^G_R < nc_0.$$  

Proof. Let $u_k \in \mathcal{N}_R^G$ be such that $J(u_k) \to c^G_R$. Ekeland’s variational principle for $C^1$-manifolds [7, Theorem 2.1], together with Lemma 5.1, allows us to assume that $(u_k)$ is bounded in $D^{1,2}_0(\Omega_R)$ and that $\|\nabla J(u_k)\| \to 0$.

Passing to a subsequence, we have that $u_k \rightharpoonup u$ weakly in $D^{1,2}_0(\Omega_R)$. Set $v_k := u_k - u \in D^{1,2}_0(\Omega_R)^G$. Using Lemma 3.2 we obtain that $u$ is a solution to the problem (1.1), and that

$$J(u_k) = J(v_k) + J(u) + o(1) \quad \text{and} \quad \nabla J(u_k) = \nabla J(v_k) + o(1).$$  

We claim that $v_k \to 0$ strongly in $D^{1,2}_0(\Omega_R)$.

Arguing by contradiction, assume it does not. Set $R_k := R$. Note that $D^{1,2}_0(\Omega_R)^O(N) \subset D^{1,2}_0(\Omega_R)^{\Gamma_\infty}$. Then, up to a subsequence, $(v_k)$ satisfies the assumptions (i), (ii) and (iii) of Lemma 3.4 for some $c \leq c^G_R$, as $(f_2)$ yields $J(u) \geq 0$. Consequently, $G \neq O(N)$ and $G \neq \Gamma_\infty$. Moreover, if $G = \Gamma_n$ with $n < \infty$ then

$$c^G_R \geq \lim_{k \to \infty} J(v_k) \geq nc_0,$$

contradicting Proposition 4.1.

This proves that $u_k \to u$ strongly in $D^{1,2}_0(\Omega_R)$. Hence, $u \in \mathcal{N}_R^G$ and $J(u) = c^G_R$. It is shown in [9, Lemma 2.9] that $c_0$ is the ground state energy of the problem without symmetries in an exterior domain, and that it is not attained. From this fact, and Proposition 4.1, we derive that $c_0 < J(u) = c^G_R < nc_0$.

Finally, since $|u|$ is also a minimizer of $J$ on $\mathcal{N}_R^G$, the problem (1.1) has a positive least energy $G$-invariant solution, as claimed.

As we mentioned earlier, the existence of a positive least energy radial solution to the problem (1.1) was proved in [8]. Theorem 1.1 asserts that this is not the only positive solution if $R$ is large enough. The main step in its proof is the following lemma.

Lemma 5.3. For each $2 \leq n < \infty$,

$$\sup_{R > 0} c^\Gamma_R = nc_0.$$
Moreover, if \( u_{R,n} \in \mathcal{N}_{R}^{\Gamma_n} \) is such that \( u_{R,n} > 0 \) and \( J(u_{R,n}) = c_{R}^{\Gamma_n} \), then there exist \( R_k \to \infty, \xi_k = (\zeta_k,0) \in \Omega_{R_k} \) and a positive least energy radial solution \( \omega \) to the limit problem (1.3) such that

\[
\text{dist}(\xi_k, \partial \Omega_{R_k}) \to \infty
\]

and

\[
\lim_{k \to \infty} \left\| u_{R_k,n} - \sum_{j=0}^{n-1} \omega \left( \cdot - e^{2\pi i j/n} \xi_k \right) \right\| = 0,
\]

where \( e^{2\pi i j/n} \xi_k \) := \((e^{2\pi i j/n} \zeta_k,0)\).

Proof. Fix \( 2 \leq n < \infty \). From the inequalities (3.3) and Proposition 4.1 we get that

\[
0 < c_0 \leq c := \sup_{R>0} c_{R}^{\Gamma_n} \leq nc_0.
\]

Fix a sequence \( R_k < R_{k+1} \) with \( R_k \to \infty \), and let \( u_k := u_{R_k,n} \in \mathcal{N}_{R_k}^{\Gamma_n} \) be such that \( u_k > 0 \) and \( J(u_k) = c_{R_k}^{\Gamma_n} \). Then, as stated in (3.1), \( \|u_k\| \geq \bar{q} > 0 \) for all \( k \). So \((u_k)\) satisfies the assumptions (i), (ii) and (iii) of Lemma 3.4 and, consequently, there exist \( \xi_k = (\zeta_k,0) \in \Omega_{R_k} \) such that \( \text{dist}(\xi_k, \partial \Omega_{R_k}) \to \infty \), and a positive \( O(N-2) \)-invariant solution \( v \) to the limit problem (1.3) such that

\[
nc_0 \leq nJ(v) \leq c \leq nc_0.
\]

Therefore, \( c = nc_0 \) and \( J(v) = c_0 \). By [12, Corollary 1.2], \( v \) is radially symmetric about some point and, as \( v \) is \( O(N-2) \)-invariant, this point is of the form \((\zeta_*,0) \in \mathbb{C} \times \mathbb{R}^{N-2} \). We set \( \omega(z,y) := v(z + \zeta_*, y) \) and \( \zeta_k = \tilde{\zeta}_k + \zeta_* \). Then, from Lemma 3.4 we get that

\[
J(u_k - w_k) = \frac{1}{2} \|u_k - w_k\|^2 - \int_{\mathbb{R}^N} F(u_k - w_k) = o(1), \quad (5.1)
\]

\[
\|u_k - w_k\|^2 = \int_{\mathbb{R}^N} f(u_k - w_k)|u_k - w_k| + o(1), \quad (5.2)
\]

where

\[
w_k(z,y) := \sum_{j=0}^{n-1} v(e^{-2\pi i j/n} z - \tilde{\zeta}_k, y) = \sum_{j=0}^{n-1} \omega(z - e^{2\pi i j/n} \tilde{\zeta}_k, y).
\]

Next we show that \( \|u_k - w_k\|^2 \to 0 \).
Set \( v_k := u_k - w_k \). Arguing by contradiction, assume that a subsequence satisfies that \( \|v_k\|^2 \geq 2a_1 > 0 \). Then, from equation (5.2) and assumption (f1) we obtain that
\[
0 < a_1 < \int_{\mathbb{R}^N} f(v_k)v_k \leq A_0\|v_k\|_{2^*}^{2^*}.
\]
By Lemma 3.3, there exist \( a_2 > 0 \) and a sequence \( (y_k) \) in \( \mathbb{R}^N \) such that
\[
\int_{B_1(y_k)} |v_k|^2 = \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |v_k|^2 \geq a_2 > 0 \quad \text{for all } k.
\]
Define \( \tilde{v}_k(x) := v_k(x + y_k) \). As \( \tilde{v}_k \) is bounded in \( D^{1,2}(\mathbb{R}^N) \), after passing to a subsequence, \( \tilde{v}_k \rightharpoonup \tilde{v} \) weakly in \( D^{1,2}(\mathbb{R}^N) \), \( \tilde{v}_k \rightarrow \tilde{v} \) a.e. in \( \mathbb{R}^N \) and \( \tilde{v}_k \rightarrow \tilde{v} \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \). Hence, \( \tilde{v} \neq 0 \). Assumption (f2) implies that \( f(s)s - 2F(s) > 0 \) if \( s \neq 0 \). So, from equations (5.1) and (5.2) and Fatou’s lemma, we obtain that
\[
0 = \lim_{k \to \infty} \int_{\mathbb{R}^N} \left( \frac{1}{2} f(v_k)v_k - F(v_k) \right) \geq \int_{\mathbb{R}^N} \left( \frac{1}{2} f(\tilde{v})\tilde{v} - F(\tilde{v}) \right) > 0,
\]
which is a contradiction. This proves that \( \|u_k - w_k\|^2 \to 0 \), and finishes the proof of the lemma. \( \square \)

**Proof of Theorem 1.1.** Fix \( m \in \mathbb{N}, m \geq 2 \). For each \( 2 \leq n \leq m \) and \( R > 0 \), Theorem 5.2 yields a positive least energy \( \Gamma_n \)-invariant solution \( u_{R,n} \) to the problem (1.1). By Proposition 4.1 and Lemma 5.3, there exists \( \tilde{R}_n > 0 \) such that
\[
(n - 1)c_0 < J(u_{R,n}) < nc_0 \quad \text{for all } R > \tilde{R}_n.
\]
Setting \( R(m) := \max\{\tilde{R}_1, \ldots, \tilde{R}_m\} \) we obtain the first part of the statement. The second part is given by Lemma 5.3. \( \square \)

Lemma 5.3 yields also the following result.

**Corollary 5.4.** The ground state energies of the radial and the \( \Gamma_\infty \)-invariant solutions satisfy
\[
\sup_{R > 0} c_R^{\Gamma_\infty} = \sup_{R > 0} c_R^{O(N)} = \infty.
\]

**Proof.** Indeed, by Lemma 5.3 and the inequalities (3.2),
\[
nc_0 = \sup_{R > 0} c_R^n \leq \sup_{R > 0} c_R^{\Gamma_\infty} \leq \sup_{R > 0} c_R^{O(N)}
\]
for every \( 2 \leq n < \infty \). \( \square \)
Remark 5.5. Since the energy of the solution obtained in [9] is less than $2c_0$, Corollary 5.4 implies that, for $R$ sufficiently large, that solution is different from the radial one if $N \neq 3$.

6 Some remarks on the 3-dimensional case

If $N = 3$ the situation is quite different. For every $2 \leq n \leq \infty$ there are $\Gamma_n$-orbits in $\Omega_R$ which consist of only two points, namely those of the form $\{(0, y), (0, -y)\}$. Therefore, compactness of $\Gamma_n$-invariant $(PS)_{c_0}$-sequences is lost already at the level $2c_0$. Lemma 5.3 is no longer true in dimension 3. In fact, one has the following result.

Theorem 6.1. Let $N = 3$. Fix $R > 0$ and $2 \leq n \leq \infty$. Then, the problem (1.1) has a positive $\Gamma_n$-invariant solution $u_{R,n}$ which satisfies

$$c_0 < J(u_{R,n}) = c_{R,n}^\Gamma < 2c_0.$$ 

Proof. We give a sketch of the proof. For each $\rho > 0$, let

$$\tilde{\sigma}_p := \sigma_\rho + \omega_{-\rho},$$

where $\omega_r(z, y) := \omega(z, y-r)$ for every $(z, y) \in \mathbb{C} \times \mathbb{R}$ and $r \in \mathbb{R}$. Let $\psi = \psi_R$ be a radial cut-off function as in (4.1). Then, $\psi \tilde{\sigma}_p$ is $[O(2) \times O(1)]$-invariant and, hence, $\psi \tilde{\sigma}_p \in D_{\Gamma_0}(\Omega_R)^\Gamma_n$ for every $2 \leq n \leq \infty$.

As in Lemma 4.6 one proves that, for $\rho$ large enough, there exists $t_0 \in (0, \infty)$ such that $t_0 \psi \tilde{\sigma}_p \in \mathcal{N}_{R,n}^\Gamma_R$. Therefore, $J(t_0 \psi \tilde{\sigma}_p) \geq c_{R,n}^\Gamma$. Moreover, the argument given to prove statement (b) of Proposition 4.1 can be easily adapted to show that $J(t_0 \psi \tilde{\sigma}_p) < 2c_\infty$.

One can also adapt the proof of Lemma 3.4 to show that $J : \mathcal{N}_{R,n}^\Gamma_R \to \mathbb{R}$ satisfies the Palais-Smale condition at every $c < 2c_0$. Then, using Ekeland’s variational principle, we obtain that $c_{R,n}^\Gamma$ is attained by $J$ at some positive function $u_{R,n} \in \mathcal{N}_{R,n}^\Gamma_R$ which satisfies

$$c_0 < J(u_{R,n}) = c_{R,n}^\Gamma < 2c_0,$$

as claimed. \qed

Remark 6.2. Theorem 6.1 implies that

$$\sup_{R > 0} c_{R,n}^\Gamma \leq 2c_0 \quad \text{for every } 2 \leq n < \infty, \quad \text{if } N = 3.$$
This stands in contrast with the statement of Lemma 5.3, which says that
\[
\sup_{R>0} c^R_n = n c_0 \quad \text{for each } 2 \leq n < \infty, \quad \text{if } N \geq 4.
\]

So, if \( N = 3 \), the energy bounds for the minimizers do not allow us to
distinguish them apart, as they do when \( N \neq 3 \).

Moreover, higher energy solutions are not easy to get because compactness is lost at many energy levels, for instance, at the levels \( 2 j c_0 \) and \( c^R_n + 2 j c_0 \) for each \( j \in \mathbb{N} \), \( j \geq 1 \).

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