On SUSY Breaking And $\chi$SB
From String Duals

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Abstract

We find regular string duals of three dimensional $\mathcal{N} = 1$ SYM with a Chern-Simons interaction at level $k$ for $SO$ and $Sp$ gauge groups. Using the string dual we exactly reproduce the conjectured pattern of supersymmetry breaking proposed by Witten by showing that there is dynamical supersymmetry breaking for $k < h/2$ while supersymmetry remains unbroken for $k \geq h/2$, where $h$ is the dual Coxeter number of the gauge group. We also find regular string duals of four dimensional $\mathcal{N} = 1$ SYM for $SO$ and $Sp$ gauge groups and exactly reproduce the expected pattern of chiral symmetry breaking $Z_{2h} \to Z_2$ by analyzing the symmetries of the string solution.
1. Introduction

The description of asymptotically free, confining gauge theories via gravity has shed new light on familiar questions regarding the non-trivial infrared dynamics of gauge theory. Despite the lack of proper string techniques to study the strong coupling dynamics of an asymptotically free dual gauge theory, many expectations from gauge theory have been realized in gravity duals. Understanding how to deal with string theory in highly curved backgrounds in the presence of Ramond-Ramond fields will eventually yield a calculationally reliable solution of the large N limit of gauge theory.

In this paper we find the string duals of minimally supersymmetric gauge theories in three and four dimensions for $SO$ and $Sp$ gauge groups and exactly reproduce some of the expected non-trivial infrared dynamics of these gauge theories.

In the three dimensional case we generalize the construction of Maldacena and Nastase [1] and construct regular string duals of $\mathcal{N} = 1$ SYM with a Chern-Simons interaction at level $k$ for $SO$ and $Sp$ gauge groups. Witten has conjectured that supersymmetry is spontaneously broken in this theory when $k < \frac{h}{2}$, where $h$ is the dual Coxeter number of the gauge group, and has shown that supersymmetry is unbroken for $k \geq \frac{h}{2}$. Generalizing the results of [1] we show that the string dual we describe exactly reproduces the conjectured pattern of supersymmetry breaking.

We also construct a regular supergravity solution describing four dimensional $\mathcal{N} = 1$ SYM for $SO$ and $Sp$ gauge groups. Here we generalize the solution of Maldacena and Nuñez [2] to find solutions in which the expected pattern of chiral symmetry breaking $Z_{2h} \to Z_2$ in gauge theory is realized in the string construction by showing that an asymptotic symmetry of the solution that exists in the UV region is broken in the IR region.

An important tool in finding the string duals of these gauge theories is the proper treatment of orientifold projections when branes wrap supersymmetric cycles and understanding how to extend the orientifold action in the gravity dual. Precise agreement with field theory expectations is obtained by taking into account an interesting subtlety on the proper description of the Wess-Zumino coupling on a D-brane in models with unoriented open strings which we describe in section 3.

The plan of the rest of the paper is as follows. In section 2 we briefly summarize the field theory results in [4] and the solution found in [1] which is dual to $D = 3$ SYM for

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1 In a similar context to the one studied here, reference [3] considered orientifolds of wrapped branes.
SU(N) gauge group. In section 4 we make a suitable orientifold construction which generalizes the gauge groups to SO and Sp and identify the effect of the orientifold construction in the gravity dual. Following [1] we realize Witten’s conjecture about supersymmetry breaking. In section 4 we recall the field theory results on chiral symmetry breaking in $D = 4 \, \mathcal{N} = 1$ SYM and the dual supergravity solution in [2] when the gauge group is SU(N). In section 5 we find the string dual for SO and Sp gauge groups and reproduce using the string dual the expected pattern of chiral symmetry breaking. Finally, the appendix rederives the BPS equations in [1] from an effective superpotential method.

2. SUGRA dual to $D = 3 \, \mathcal{N} = 1$ SYM with a Chern-Simons Interaction

A supersymmetric Chern-Simons interaction at level $k$ can be added to $\mathcal{N} = 1$ SYM with gauge group $G$

$$L_{CS} = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A + \bar{\lambda} \lambda \right).$$  \hspace{1cm} (2.1)

Classically, the supersymmetric gauge theory has a mass gap when $k \neq 0$ [3]. The mass of the gluon and gluino fields is of order $g^2 k$ and for $k >> 1$ one can integrate out the massive gluinos while retaining a local purely bosonic low energy effective action. Integrating out a massive fermion at one loop in a three dimensional gauge theory induces a Chern-Simons interaction whose level is determined by the sign of the fermion mass and the representation of $G$ under which the fermion transforms [3][5]. Therefore, in the supersymmetric example we are considering, integrating out the massive gluino shifts the effective value of $k$ appearing in the bosonic Chern-Simons term to $k_{eff} = k - h/2$ [3], where $h$ is the dual Coxeter number of G. At very low energies, where the Yang-Mills term can be neglected in comparison with the Chern-Simons term, the effective description of the model is – for $k >> 1$ – given by bosonic Chern-Simons theory with gauge group $G$ at level $k_{eff}$. This result was extended to all $k$ in [4] where it was shown that the low energy effective description of $\mathcal{N} = 1$ SYM with gauge group $G$ with a Chern-Simons interaction at level $k$ is given by bosonic Chern-Simons theory with gauge group $G$ at level $k_{eff} = k - h/2$.

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2 In this paper we will take $k > 0$.

3 The induced level is given by $\delta k = -\frac{1}{2} C_R \text{sign}(m)$, where $C_R$ is the index of the representation $R$ under which the fermion transforms and $m$ is the fermion mass, i.e. $\text{tr}(t^a_R t^b_R) = 2C_R \delta^{ab}$. For the adjoint representation $C_R = h$. 

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Using this effective description, Witten \cite{4} computed the supersymmetry index \cite{9} 
\[ \text{Tr}(-1)^F \] of three dimensional $\mathcal{N} = 1$ SYM with gauge group $G$ with a Chern-Simons interaction of level $k$. The number of supersymmetric vacua equal the number of zero energy states of the effective bosonic Chern-Simons theory at level $k_{\text{eff}}$, which in turn equal the number of conformal blocks of the corresponding WZW model. The result is

\[ \text{Tr}(-1)^F = \begin{cases} 0 & k \geq \frac{h}{2} \\ \neq 0 & k < \frac{h}{2} \end{cases} \]

(2.2)

which shows that supersymmetry is unbroken for $k \geq \frac{h}{2}$. The supersymmetry index vanishes for $k < \frac{h}{2}$, which is insufficient to decide whether supersymmetry is broken in the infinite volume limit. However, in \cite{4} Witten conjectured that dynamical supersymmetry breaking does occur for $k < \frac{h}{2}$.

In section 3 we generalize the work of Maldacena and Nastase \cite{1} and exactly reproduce the conjectured pattern of supersymmetry breaking for SO and Sp gauge groups using a string dual description. Here we summarize the value of the dual Coxeter number of the gauge groups for which we find a supergravity dual solution.

|        | $SU(N)$ | $SO(2N)$ | $Sp(N)$ | $SO(2N + 1)$ |
|--------|---------|----------|---------|--------------|
| $h$    | $N$     | $2N - 2$ | $N + 1$ | $2N - 1$     |

Table 1: Value of dual Coxeter number for several gauge groups.

2.1. Supergravity Dual For $G = SU(N)$

A simple way\footnote{To properly define the index one considers the theory on $\mathbb{R} \times T^2$ and the number vacua equals the number of conformal blocks of the WZW model on a Riemann surface with $T^2$ topology. On $T^2$, the conformal blocks are in one to one correspondence with the integrable representations of the affine Lie algebra $\hat{G}$ at level $k_{\text{eff}}$. Moreover, all vacua are bosonic or fermionic so that $\text{Tr}(-1)^F = \pm$ number of conformal blocks.} of realizing $D = 3$, $\mathcal{N} = 1$ SU(N) SYM is to study Type II string theory on $\mathbb{R}^{1,2} \times X$, where $X$ is a seven dimensional manifold of $G_2$ holonomy and to consider the low energy effective field theory living on $N$ suitably wrapped branes.\footnote{For an alternative realization of $SU(N)$ SYM with a Chern-Simons term see for example \cite{10,11}, where a description of supersymmetry breaking for $SU(N)$ was given.} There are two
choices that lead to the desired gauge theory. One can either wrap $N$ Type IIB five-branes on the supersymmetric $S^3$ cycle of $X = S(S^3)$ – the spin bundle over $S^3$ – or wrap $N$ Type IIA D6-branes on the supersymmetric four cycle $Z$ of $X = \Sigma(Z)$ – the bundle of anti-self-dual two-forms over $Z$ – where $Z$ is $S^4$ or $\mathbb{P}^2$. Since the cycle which the branes wrap is rigid, the familiar massless scalars describing the motion of the branes in the transverse directions are absent in the low energy field theory and one just gets pure $SU(N)$ gauge theory in three dimensions. In this paper we consider the Type IIB realization with wrapped five-branes.

In the Type IIB realization a Chern-Simons interaction at level $k$ can be incorporated by turning on three-form flux through $S^3 \in S(S^3)$ which induces the following bosonic low energy coupling on the five-branes

$$L_{WZ} = \frac{1}{8\pi^2} \int_{R^{1,2} \times S^3} B \wedge \text{Tr}(F \wedge F) = -\frac{1}{8\pi^2} \int_{R^{1,2} \times S^3} H \wedge \text{Tr}(A \wedge dA + \frac{2}{3} A^3) = -\frac{k_0}{4\pi} \int_{R^{1,2}} \text{Tr}(A \wedge dA + \frac{2}{3} A^3)$$

(2.3)

which together with its superpartner generates the $\mathcal{N} = 1$ supersymmetric Chern-Simons interaction (2.1).

We have been careful in differentiating between the levels $k_0$ and $k$. The reason for this is that the low energy effective field theory on the branes is $SU(N)$ SYM with a Chern-Simons interaction with bare level $k_0$. However, massive $\mathcal{N} = 1$ Kaluza-Klein multiplets arise by reducing the six dimensional massless vector multiplet on the five-branes over $S^3$ and integrating out the massive Kaluza-Klein adjoint fermions may shift the bare Chern-Simons level $k_0$ if there is a net number of multiplets with a given sign for their mass.

It has been shown in [1] that there is a single Kaluza-Klein adjoint massive chiral multiplet

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7 Since $\mathbb{P}^2$ is not a Spin manifold, one must turn on non-zero magnetic flux through $\mathbb{P}^1 \in \mathbb{P}^2$.

8 Since the shift of the level depends only on the sign of the fermion mass term, fermions with opposite mass leave $k_0$ invariant and the net shift of the level depends only on the net number of fermions with a specific mass term sign.

9 One can start with the topological twist corresponding to $D = 3$ $\mathcal{N} = 2$ SYM, so that the Kaluza-Klein modes come in $\mathcal{N} = 2$ chiral multiplets which have two fermions one with each sign for the mass and therefore produce no shift of the level. One can continuously twist the theory to $\mathcal{N} = 1$ by smoothly changing the gauge field on the normal bundle. To determine the number of unpaired fermions one computes the number of eigenvalues of the Dirac operator which change sign under a gauge transformation of the normal bundle, which is just given by the winding number of the gauge transformation, which is 1.
which does not have a partner with opposite mass so that the effective theory on the branes after integrating out all Kaluza-Klein modes is three dimensional $\mathcal{N} = 1 \ SU(N) \ SYM$ with a Chern-Simons interaction at level $k = k_0 - h/2 = k_0 - N/2$.

Recently, Maldacena and Nastase [1] have interpreted a solution found by Chamseddine and Volkov [17] as describing a collection of $N$ five-branes wrapping $S^3 \in S(S^3)$ when $k_0 = N$. This regular supersymmetric solution is therefore dual to $D = 3$, $\mathcal{N} = 1 \ SU(N) \ SYM$ with a Chern-Simons term at level $k = N/2$.

The Type IIB supergravity solution [1] obtained by lifting the gauged supergravity solution in [17] to ten dimensions is in the class of supersymmetric supergravity solutions considered in [19]. It is a compactification of Type IIB string theory on $\mathbb{R}^{1,2} \times M_7$ with a nontrivial three-form field strength $H$ and dilaton profile over $M_7$, where $M_7$ denotes the internal space. The metric of the solution is given by

$$\begin{align*}
    ds^2 &= ds^2(\mathbb{R}^{1,2}) + N[dr^2 + \frac{1}{4}R^2(r)w_a w_a] + N\frac{1}{4}(\tilde{w}_a - A_a)^2 \\
    A_a &= \frac{1 + b(r)}{2} w_a,
\end{align*}
$$

where $w_a, a = 1, 2, 3$ parametrize $S^3$ and $\tilde{w}_a, a = 1, 2, 3$ are another set of $SU(2)$ left invariant one forms parametrizing $\tilde{S}^3$. The radial function $b(r)$ is monotonically increasing and goes from 0 at $r \to \infty$ to 1 at $r = 0$. On the other hand, $R^2(r)$ behaves linearly in $r$ near $r \to \infty$ while $R^2(r) \simeq r^2$ near $r = 0$, so that the metric inside the square brackets in (2.4) is the flat $\mathbb{R}^4$ near the origin. Topologically, $M_7$ is asymptotically conical with base $S^3 \times \tilde{S}^3$ and in the interior the $\tilde{S}^3$ stays of finite size while $S^3$ becomes contractible. The supersymmetry left unbroken by the solution, which is $\mathcal{N} = 1$ in three dimensions, guarantees that $M_7$ is a $G_2$ holonomy manifold with respect to the connection with torsion $\nabla = \nabla + \pi H$, where $\nabla$ is the usual Levi-Civita connection and $H$ is the three-form piercing $M_7$.

The solution has non-trivial H-flux (see eqns. A.5 and A.15 in the Appendix). The number of five-branes wrapping $S^3 \in S(S^3)$ is encoded as magnetic flux of $H$ through $\tilde{S}^3$. The solution also has flux through $S^3$, and its asymptotic value at the boundary $r \to \infty$,
as explained previously, determines the Chern-Simons level of the holographic field theory dual. The fluxes of the solution are

$$\int_{\tilde{S}^3} \frac{H}{2\pi} = \int_{S^3} \frac{H_\infty}{2\pi} = N.$$  \hfill (2.5)

Since \(k_0 = N\), this supergravity solution is dual to \(SU(N)\) SYM with \(k = N/2\). An external quark is represented by a string stretching from the boundary and since the solution is completely nonsingular and without a horizon it exhibits confinement as well as a mass gap as expected from the field theory analysis \[4\][21].

We note that this duality realizes the general philosophy advocated in \[22\] in which the infrared description of the gauge theory on a collection of wrapped branes is given by a topologically distinct background in which branes have been replaced by fluxes. In the original description the branes wrap \(S^3 \in S(S^3)\) which is asymptotically a cone over \(S^3 \times \tilde{S}^3\), where \(\tilde{S}^3\) is contractible in the full geometry while \(S^3\) is topologically non-trivial. The dual description of the gauge theory is given by a geometry \(M_7\) in which the roles of \(S^3\) and \(\tilde{S}^3\) are reversed, so that \(\tilde{S}^3\) is now topologically non-trivial and supports the flux left behind by the branes once they disappear. In effect the infrared dynamics of the gauge theory is captured by the “flopped” geometry which the branes wrapped \[23\].

Since the supergravity solution is supersymmetric, this shows that \(SU(N)\) SYM with a Chern-Simons term at level \(k = N/2\) has unbroken supersymmetry. Maldacena and Nastase showed that one can use this solution to explore what happens when \(k \neq N/2\). Changing to \(k = N/2 + n\) – or equivalently changing to \(k_0 = N + n\) – is accomplished by wrapping \(n = k - N/2\) five-branes\[14\] on \(\tilde{S}^3 \in M_7\), which adds \(n\) units of H-flux through \(S^3 \in M_7\) so that the level is \(k = N/2 + n\). This can be done reliably for \(n \ll N\), when the backreaction of the \(n\) wrapped branes can be safely ignored. Adding \(n > 0\) five-branes does not break supersymmetry\[15\], since the wrapped branes preserve the same supersymmetry as the background \[1\]. On the other hand, adding \(n < 0\) five-branes – that is \(|n|\) anti-branes – breaks supersymmetry completely. Therefore, using supergravity one

\[13\] The flux of \(H\) through \(S^3\) vanishes at \(r = 0\) as required for a smooth solution since \(S^3\) vanishes in the interior.

\[14\] Since the external quark is a string which can now end on the five-branes, this shows that for \(k \neq N/2\) that there is no confinement, in agreement with the field theory analysis of \[4\].

\[15\] The \(G_2\) structure of \(M_7\), which is a three-form, is such that when pulled back to \(\tilde{S}^3\) is proportional to the volume form on \(\tilde{S}^3\).
realizes the expected pattern of supersymmetry breaking for \( G = SU(N) \). For \( k \geq N/2 \) supersymmetry is unbroken and for \( k < N/2 \) supersymmetry is broken.

For \( k > N/2 \) the low energy effective description of the physics is given by the field theory on the \( n \) five-branes wrapped over \( S^3 \). At very low energies, where the Yang-Mills term can be neglected, one obtains \( N = 1 \) \( SU(n) \) Chern-Simons theory with level \( k = N \). Integrating out the \( SU(n) \) adjoint gluinos and Kaluza-Klein modes result in a negligible shift of the Chern-Simons level (of \( \mathcal{O}(n) \)) in the regime where backreaction can be ignored so that the effective description is given by \( SU(n) \) Chern-Simons theory with level \( k = N \) which is precisely the level-rank \([24]\) dual of what was found in \([4]\) which is \( SU(N) \) Chern-Simons theory with \( k_{\text{eff}} = n \). In the next section we generalize this duality to SO and Sp gauge groups.

3. Supergravity Dual For \( G = SO \) and \( G = Sp \)

The \( SU(N) \) theory is realized by wrapping \( N \) D5-branes\footnote{In the previous section we wrote the solution corresponding to wrapped NS5-branes. One can trivially act with S-duality and write the corresponding solution for wrapped D5-branes, which is given by \( ds^2(D5) = e^{-\phi_{NS}} ds^2(NS5), C_2^R = -B_2^R \) and \( \phi_{D5} = -\phi_{NS} \).} on \( S^3 \in S(S^3) \). A simple way to generalize the construction to other gauge groups is to superimpose an orientifold five-plane – an \( O5 \)-plane – on top of the five-branes. Since the branes are curved this requires performing an orientifold projection which leaves an \( O5 \)-plane along \( R^{1,2} \times S^3 \), which is the worldvolume of the five-branes. This can be accomplished by considering the following projection

\[
\text{Type IIB on } (R^{1,2} \times S(S^3)) / \{1, \Omega \cdot \sigma \}, \tag{3.1}\]

where \( \Omega \) is the usual worldsheet parity operator and \( \sigma \) is the generator of a \( Z_2 \) symmetry of \( S(S^3) \) which we will now describe.

The topology of \( S(S^3) \) where the five-branes are supersymmetrically embedded can be characterized by the following hypersurface in \( R^8 \)

\[
\sum_{i=1}^{4} a_i^2 - b_i^2 = V, \tag{3.2} \]

with \( a_i, b_i \in R \). The space is asymptotically conical and the base of the cone is \( S^3 \times \tilde{S}^3 \). These spheres are parametrized by the coordinates \( a_i \) and \( b_i \) respectively. For \( V > 0 \),
$S^3$ is topologically non-trivial while $\tilde{S}^3$ shrinks smoothly in the interior and is therefore topologically trivial. For $V < 0$ the roles of $S^3$ and $\tilde{S}^3$ are reversed and now it is $\tilde{S}^3$ which is topologically non-trivial. The transformation $V \to -V$ can then be interpreted as a flop transition [23].

In this description of $S(S^3)$ the relevant $\sigma$ acts by

$$\sigma : \begin{cases} a_i \to a_i \\ b_i \to -b_i \end{cases} \quad (3.3)$$

If one considers the geometry near infinity after quotienting by $\sigma$ one gets a cone with base $S^3 \times \tilde{S}^3/Z_2$. Since $\tilde{S}^3$ is contractible in the full geometry the $Z_2$ action has fixed points. Therefore, an $O5$-plane sits at the fixed locus of $\sigma$ – that is $b_i = 0$ – which is $R^{1,2} \times S^3$, which is what we were after. Therefore, we have identified a suitable orientifold projection in string theory that results, at energy scales smaller than the scale of the $S^3$ to a three-dimensional $\mathcal{N} = 1$ gauge theory with $SO$ or $Sp$ gauge groups. For future reference we note that when $V < 0$, that is in the flopped geometry, that $\sigma$ is a freely acting involution since the fixed point set $b_i = 0$ is not part of the manifold and thus there is no orientifold plane.

Three different gauge groups can be realized by placing D5-branes on top of an $O5$-plane. The three types of orientifold planes$^{17}$ differ in the choice of discrete fluxes in the geometry transverse to the orientifold plane, which in this case is $\tilde{RP}^3 = \tilde{S}^3/Z_2$.

| $O5^-$ | $O5^+$ | $\tilde{O5}^-$ |
|--------|--------|--------------|
| $G$    | $SO(2N)$ | $Sp(N)$ | $SO(2N + 1)$ |

Table 2: Gauge group on $2N$ five-branes coincident with an $O5$-plane.

We can now construct SUGRA duals to these gauge theories by implementing the orientifold action we have just described on the SUGRA solution presented in the previous section. That background was given by Type IIB string theory on $R^{1,2} \times M_7$ where $M_7$ is a $G_2$ holonomy manifold with respect to a connection with torsion. As discussed in the previous section, topologically $M_7$ is asymptotic to a cone over $S^3 \times \tilde{S}^3$ just like $S(S^3)$. However, $M_7$ is topologically the flopped version of $S(S^3)$. That is, in $M_7$ $S^3$

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$^{17}$ There is actually a fourth type [25] of $O5$-plane, which yields $Sp(N)$ gauge symmetry. It differs from the familiar $O5$-plane yielding the same gauge group by a discrete Ramond-Ramond 0-form $\theta$-angle. These two theories differ only in their nonperturbative spectrum.
is contractible while $\tilde{S}^3$ is topologically non-trivial. Therefore, $\sigma$ acts freely on $M_7$ since $\tilde{\mathbb{R}P}^3$ never shrinks. Thus, the string dual to $D = 3 \, \mathcal{N} = 1$ SYM theories with $SO$ and $Sp$ gauge groups is described by unoriented closed string theory on the smooth background $\mathbb{R}^{1,2} \times M_7 / \mathbb{Z}_2$, without any open strings nor orientifold planes.

In the SUGRA dual solution the branes wrapped on $S^3$ disappear and are replaced by flux over $\tilde{S}^3$, which is now non-contractible. This still occurs when an $O5$-plane is added in the gauge theory realization. The only two differences are that one must take into account the total five-brane charge – that of the five-branes plus that of the $O5$-plane – which is then converted into flux in the dual theory and replace $\tilde{S}^3$ by $\tilde{\mathbb{R}P}^3$ due to the action of $\sigma$. The total five-brane charge can be easily computed by remembering that the gauge groups that we are considering appear by projecting the massless open strings on $N$ branes together with their images – so that there are $2N$ branes in total – approaching an orientifold plane and that the charge of an $O_p^\pm$ plane is $\pm 32 \times 2^{p-9}$. Finally, one must recall that an $\tilde{O_p}^-$ plane has an extra Dp-brane which can’t be moved from the orientifold plane since it has no image under the orientifold action. Therefore, taking into account the charge of the orientifold we see that the total five-brane charge is given by

|     | $O5^-$ | $O5^+$ | $\tilde{O5}^-$ |
|-----|--------|--------|----------------|
| $G$ | $SO(2N)$ | $Sp(N)$ | $SO(2N + 1)$ |
| $Q$ | $2N - 2$ | $2N + 2$ | $2N - 1$ |

Table 3: Five-brane charge of a configuration of $2N$ branes on top of an $O5$-plane.

As described above the corresponding SUGRA solution is found from that in [1] by replacing everywhere in the solution $N \rightarrow Q$ and $\tilde{S}^3 \rightarrow \tilde{\mathbb{R}P}^3$. Therefore, the fluxes of the supergravity solution are given by

$$\int_{\tilde{S}^3} \frac{H}{2\pi} = \int_{S^3} \frac{H}{2\pi} = Q.$$  \hspace{1cm} (3.4)

Since there is non-zero flux of $H$ through $S^3$ this solution describes $D = 3 \, \mathcal{N} = 1$ SYM with a non-vanishing Chern-Simons interaction. In order to identify the correct value of the level of the Chern-Simons interaction, one must take into account the following subtlety.\hspace{1cm} \textsuperscript{18}

\hspace{1cm} \textsuperscript{18} A similar discussion appeared recently in [26].
The Wess-Zumino term

\[ L_{WZ} = \frac{1}{8\pi^2} \int B \wedge \text{Tr} F \wedge F \]  

(3.5)

is the familiar coupling in the worldvolume theory of $N$ D5-branes with gauge group $SU(N)$. This coupling implies that a D1-brane dissolved inside N D5-branes is equivalent to a single instanton in the $SU(N)$ gauge theory on the five-branes, so that the induced D1-string charge is the same as the instanton number in the $SU(N)$ gauge theory. Is this coupling modified when one considers the worldvolume theory of D-branes with unoriented open strings? First let’s consider the case where the gauge theory on the brane is $SO(2N)$\(^{19}\). When the five-branes approach the O5\(^{-}\)-plane the gauge symmetry is enhanced from $SU(N)$ to $SO(2N)$. Since a single instanton in $SU(N)$ corresponds to a unit of D1-string charge, in order to determine the correct normalization of the Wess-Zumino coupling for an $SO(2N)$ gauge theory, one must determine the $SO(2N)$ instanton number of a $SU(N)$ gauge field configuration when it is embedded in $SO(2N)$. It is straightforward\(^{20}\) to show that an instanton number one gauge field in $SU(N)$ has instanton number one when embedded in $SO(2N)$, which in turn corresponds a single unit of induced D1-string charge. Therefore the correct coupling is given by

\[ L_{WZ}^{SO} = L_{WZ}. \]  

(3.6)

On the other hand, when the five-branes approach the O5\(^{+}\)-plane the gauge symmetry is enhanced from $SU(N)$ to $Sp(N)$. Here an instanton number one gauge field in $SU(N)$ has instanton number two when embedded in $Sp(N)$. This follows by embedding an $SU(2)$ instanton inside $SU(N)$ into $Sp(N)$ via the splitting $SU(N - 2) \times SU(2) \subset Sp(N - 2) \times Sp(2)$. If we denote by $t$ an $SU(2)$ gauge field configuration, then the minimal embedding of $SU(2)$ inside $Sp(2)$ is given by the following $Sp(2)$ Lie algebra valued matrix

\[ T_{Sp} = \begin{pmatrix} t & 0 \\ 0 & -t^T \end{pmatrix}. \]  

(3.7)

\(^{19}\) The discussion that follows trivially extends when the gauge group is $SO(2N + 1)$.

\(^{20}\) One can embed a $SU(N)$ instanton in $SO(2N)$ by turning on a $SU(2)$ instanton in $SU(N)$ via the embedding given by $SU(N - 2) \times SU(2) \subset SO(2N - 4) \times SO(4)$. The minimal $SO(2N)$ instanton is then obtained by embedding the $SU(2) \subset SU(N)$ instanton in an $SU(2)$ subgroup of $SO(4)$, which clearly has instanton number one in $SO(2N)$. 

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It then follows that the instanton number of $T_{Sp}$ is twice that of $t$. Therefore, the induced D1-string charge on the D5-brane worldvolume equals twice the instanton number of $Sp(N)$ so that the correct Wess-Zumino coupling is given by

$$L_{WZ}^{Sp} = \frac{1}{2} L_{WZ}.$$ (3.8)

Taking into account this subtlety for $Sp(N)$ one finds that the flux of $H$ over $S^3$ in Table 3 induces a bare Chern-Simons interaction at level $k_0 = h$ (see Table 1) on the worldvolume of the wrapped branes for all gauge groups. As explained in [1], $k_0$ is shifted by the presence of an unpaired massive Kaluza-Klein fermion, which is not projected out by the presence of the $O_5$-plane but transforms now in the adjoint representation of $G$. Integrating this fermion out shifts $k_0$ to $k = k_0 - h/2 = h/2$. Therefore, we have generalized the results of of [1] to find the string dual of three dimensional $\mathcal{N} = 1$ SYM with gauge group $G$ and a Chern-Simons coupling at level $k = h/2$ and have seen that supersymmetry is unbroken. Regularity of the solution and the lack of a horizon show that the dual gauge theory confines and has a mass gap as expected from the field theory analysis of [4][21].

One can change the value of the level to $k = h/2 + n$ by wrapping $n$ five-branes over $\tilde{\mathbb{RP}}^3 \in M_7$, which in turn adds $n$ units of flux through $S^3$. When $n > 0$, adding five-branes does not break any further supersymmetry and therefore supersymmetry is unbroken. On the other hand, when $n < 0$, anti-five-branes must be added to the background, which break supersymmetry completely. Therefore, using SUGRA we have exactly reproduced the conjecture by Witten regarding supersymmetry breaking as a function of $k$ for these theories, that is for $k \geq h/2$ supersymmetry is unbroken while supersymmetry is spontaneously broken for $k < h/2$.

The low energy effective theory when $n > 0$ or $k > h/2$ is properly described by the field theory on the $n$ wrapped branes over $\tilde{\mathbb{RP}}^3$. The statement that the five-branes wrap $\tilde{\mathbb{RP}}^3$ can be formalized by saying that there is a map $\Phi$ between $Y$ and $\tilde{\mathbb{RP}}^3$, where $Y$ is a closed three-manifold. Since the background we are considering is orientifolded, this means that in going around a noncontractible loop in the target space, that the orientation of the worldsheet must be reversed, which means that $\Phi$ must satisfy [23] $\Phi^t(x) = w_1(Y)$, where $x \in H^1(\tilde{\mathbb{RP}}^3, \mathbb{Z}_2)$ and $w_1(Y)$ is the obstruction to the orientability of $Y$. Following [23], one can show that $\Phi$ must a map of even degree so we take, for example, $Y = \tilde{S}^3$. By integrating the appropriate Wess-Zumino coupling on the worldvolume of the $n$ wrapped

\footnote{Just as in the case discussed in [1], this analysis is self-consistent when $n << h$.}
five-branes, we find that the flux over $Y$ is twice as large as that over $\mathbb{RP}^3$. Therefore, the effective field theory on the branes in the limit where $n \ll h$ in which backreaction can be ignored is given by

- $G = SO(N) : SO(n)$ Chern-Simons at level $k = N$
- $G = Sp(N) : Sp(n)$ Chern-Simons at level $k = N$

This is to be compared with the expected low energy description found by Witten and described in section 2

- $G = SO(N) : SO(N)$ Chern-Simons at level $k_{eff} = n$
- $G = Sp(N) : Sp(N)$ Chern-Simons at level $k_{eff} = n$

Remarkably, the effective description predicted by the string dual is precisely the level-rank dual [24] of the expected field theory result, which has the same number of ground states.

4. SUGRA dual to $D = 4 \mathcal{N} = 1$ SYM

Four dimensional $\mathcal{N} = 1$ SYM with gauge group $G$ has at the classical level a chiral $U(1)_R$ symmetry that acts on the gluino fields by

$$\lambda \rightarrow e^{-i\alpha} \lambda, \quad \bar{\lambda} \rightarrow e^{i\alpha} \bar{\lambda}. \quad (4.1)$$

Likewise, $U(1)_R$ acts with the same charges on the supersymmetry generators of positive and negative chirality $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$ respectively. Quantum mechanically, the presence of zero modes of $\lambda$ in an instanton background induces a 't Hooft vertex which explicitly breaks the $U(1)_R$ symmetry down to an anomaly free chiral $Z_{2h}$ subgroup, where $h$ is the dual Coxeter number of $G$. The $Z_{2h}$ symmetry is then spontaneously broken to $Z_2$, which acts by $-1$ on $\lambda$ and $\bar{\lambda}$, by the presence of a gluino condensate

$$< \lambda \lambda >^h = \Lambda^{3h}_{QCD}, \quad (4.2)$$

which gives rise to $h$ supersymmetric vacua with a mass gap and confinement.

4.1. Supergravity Dual For $G = SU(N)$

A simple way of realizing $D = 4 \mathcal{N} = 1 SU(N)$ SYM is to consider the low energy effective field theory on $N$ five-branes wrapping the $S^2$ of the resolved conifold geometry,
which is topologically the $X = O(-1) \oplus O(-1)$ bundle over $S^2$ and admits a metric with $SU(3)$ holonomy\textsuperscript{22}.

Recently, Maldacena and Nuñez \textsuperscript{2} have interpreted a gauged supergravity solution found by Chamseddine and Volkov \textsuperscript{28,29} as the supergravity description of the above wrapped five-brane configuration. This smooth supergravity solution is thus dual to $D = 4 \mathcal{N} = 1 SU(N)$ SYM.

The solution they found is a compactification\textsuperscript{23} of Type IIB string theory on $R^{1,3} \times M_6$ with a non-trivial three-form field strength $H$ and dilaton over $M_6$. The metric of the solution is given by

$$\begin{align*}
    ds^2 &= ds^2(R^{1,3}) + N[dr^2 + e^{2h(r)}(d\theta^2 + \sin^2 \theta \, d\phi^2)] + N \frac{1}{4}(\tilde{w}_a - A_a)^2, \\
    A_1 &= a(r)d\theta, \quad A_2 = -a(r)\sin \theta d\phi, \quad A_3 = \cos \theta d\phi
\end{align*}$$

(4.3)

where $\theta$ and $\phi$ parametrize the $S^2$ which the branes wrapped and $\tilde{w}_a$ are $SU(2)$ left invariant one-forms parametrizing the transverse $\tilde{S}^3$ to the five-branes

$$\begin{align*}
    w_1 &= \cos \tilde{\psi} \, d\tilde{\theta} + \sin \tilde{\psi} \sin \tilde{\theta} \, d\tilde{\phi} \\
    w_2 &= -\sin \tilde{\psi} \, d\tilde{\theta} + \cos \tilde{\psi} \sin \tilde{\theta} \, d\tilde{\phi} \\
    w_3 &= d\tilde{\psi} + \cos \tilde{\theta} \, d\tilde{\phi}.
\end{align*}$$

(4.4)

The explicit form of the radial functions $h(r), a(r)$ can be found in \textsuperscript{2}. Here we will only need its asymptotics which near $r \to \infty$ are $e^{2h} \simeq r$ and $a(r) \simeq 0$ while near $r \simeq 0$ they behave as $e^{2h} \simeq r^2$ and $a(r) \simeq 1$ so that the metric in the square brackets in (4.3) is the flat $\mathbb{R}^3$ metric near the origin. Topologically, $M_6$ is asymptotically conical with base $S^2 \times \tilde{S}^3$ and in the interior $S^3$ stays of finite size while $S^2$ becomes contractible. The solution has $\mathcal{N} = 1$ supersymmetry in four dimensions so that $M_6$ is a $SU(3)$ holonomy manifold with respect to the connection with torsion $\nabla = \nabla + \pi H$, where $\nabla$ is the usual Levi-Civita connection\textsuperscript{24} while $H$ is the three-form flux through $M_6$.

\textsuperscript{22} There is a “mirror” realization, in which pure SYM appears by wrapping $N$ D6-branes on the deformed conifold. It is in this Type IIA picture where one can make a direct connection with M-theory on $G_2$ holonomy spaces \textsuperscript{27,23}.

\textsuperscript{23} The internal geometry is nevertheless non-compact.

\textsuperscript{24} The complex geometry of $M_6$ and the derivation of the BPS equations via an effective superpotential for the ansatz can be found in \textsuperscript{30}.
The solution also has the following non-trivial three-form field strength

\[ H = \frac{N}{2\pi} \left[ -\frac{1}{4}(w_1 - A_1) \wedge (w_2 - A_2) \wedge (w_3 - A_3) + \frac{1}{4} F_a \wedge (\tilde{w}_a - A_a) \right] \] (4.5)

whose flux over \( \tilde{S}^3 \) encodes number of five-branes wrapping \( S^2 \) in the \( X = O(-1) \oplus O(-1) \) bundle over \( S^2 \)

\[ \int_{\tilde{S}^3} \frac{H}{2\pi} = N. \] (4.6)

We note that this solution also realizes\(^\text{25}\) the general philosophy of \[22\]. Here the original geometry of \( X \) is such that it is asymptotically a cone over \( S^2 \times \tilde{S}^3 \) and five-branes wrap the topologically non-trivial \( S^2 \) while \( \tilde{S}^3 \) is contractible in the full geometry. In the dual supergravity solution, the roles of \( S^2 \) and \( \tilde{S}^3 \) are reversed since in \( M_6 \) \( S^2 \) is contractible while \( \tilde{S}^3 \) supports the flux left behind by the branes once they disappear. The topology of \( M_6 \) is that of the deformed conifold.

One can use this supergravity solution, which manifestly exhibits a mass gap and confinement, to explore the gravity realization\(^2\) of the expected chiral symmetry breaking \( Z_{2N} \rightarrow Z_2 \). From the implementation of the twist required to obtain a supersymmetric gauge theory on the wrapped brane one can identify the classical \( U(1)_R \) symmetry of the gauge theory with shift symmetry of \( \tilde{\psi} \), which acts by \( \tilde{\psi} \rightarrow \tilde{\psi} + c \). Clearly, from the form of metric (4.3) in the interior (IR) of \( M_6 \) we see that the metric is only invariant under shifts of \( \tilde{\psi} \) by \( 2\pi \tilde{\psi} \rightarrow \tilde{\psi} + 2\pi \), which due to the \( \tilde{\psi} \) periodicity \( \tilde{\psi} \simeq \tilde{\psi} + 4\pi \) generates a \( Z_2 \) symmetry, which is identified with the \( Z_2 \) symmetry of \( \mathcal{N} = 1 \) SYM preserved by the gluino condensate.

In the UV region of the geometry the full \( U(1)_R \) symmetry acting by arbitrary shifts on \( \tilde{\psi} \) is restored. On the other hand the field theory expectation is that this symmetry is broken down to \( Z_{2N} \). In field theory, this explicit breaking is due to Yang-Mills instantons and in \[2\] this expectation was realized by taking into account the effect of a Yang-Mills instanton in the brane realization. By taking into account the Wess-Zumino term (3.5) one can indentify a SYM instanton with a string worldsheet wrapping an \( S^2 \) in \( M_6 \). As explained in \[2\] the appropriate description of a SYM instanton is given by a string worldsheet wrapping an \( S^2 \) in \( M_6 \) parametrized by \( \theta \) and \( \phi \) and with \( \tilde{\theta} = \theta \) and \( \tilde{\phi} = \phi \). Then the amplitude for a string instanton wrapping this cycle is given by

\[ v = \exp \left( i \int_{S^2} B \right). \] (4.7)

\(^{25}\) We note that a very interesting description of a confining gauge theory in which branes are replaced by fluxes also appeared in \[31\] which is dual to a cascading gauge theory.
By using the asymptotic expression for $H$ in (4.5) one finds that

$$\int_{S^2} B = -2N\tilde{\psi}.$$ (4.8)

Therefore, the instanton corrected UV string solution is only invariant under $\tilde{\psi} \rightarrow \tilde{\psi} + 2\pi k/N$, where $n = 1, \ldots, 2N$ and therefore reproduce the expected anomaly free $Z_{2N}$ R-symmetry.

In this way the expected chiral symmetry breaking pattern $Z_{2N} \rightarrow Z_2$ is realized in the string dual as the symmetry group of the string solution being broken from the UV to the IR. The $N$ vacua can then be easily constructed \cite{2} by performing a discrete set of $N$ rotations of the $SU(2)$ gauge fields $A_\alpha$ in (4.3) described in \cite{4} such that the UV solution is unchanged while it is rotated but still smooth in the interior. We now extend this realization to SO and Sp gauge groups.

5. Supergravity Dual For $G = SO$ and $G = Sp$

One can generalize the construction in the previous section to SO and Sp gauge groups by introducing an O5-plane on top of the five-branes.\cite{26} In this case the O5-plane must along $R^{1,3} \times S^2$, where $S^2$ is the supersymmetric cycle of the resolved conifold geometry $X = O(-1) \oplus O(-1)$ bundle over $S^2$. This can be accomplished by considering

$$\text{Type IIB on } (R^{1,2} \times X) / \{1, \Omega \cdot \sigma\}, \quad (5.1)$$

where $\Omega$ is the usual worldsheet parity operator and $\sigma$ is $Z_2$ symmetry of $X$ whose fixed point set is $R^{1,3} \times S^2$.

We recall that the conifold singularity

$$z_1z_2 - z_3z_4 = 0 \quad (5.2)$$

can be resolved by replacing (5.2) by the following pair of equations

$$\begin{pmatrix} z_1 & z_3 \\ z_4 & z_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \quad (5.3)$$

\footnote{In the previous section we described the solution with NS5-branes but can obtain the corresponding one for D5-branes by performing an S-duality.}
where \((\lambda_1, \lambda_2) \simeq \lambda (\lambda_1, \lambda_2)\) don’t vanish simultaneously and therefore parametrize an \(S^2\). The system (5.3) is the familiar small resolution of the conifold singularity and is topologically \(X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)\) bundle over \(S^2\). When \(z_i = 0 \forall i\) (5.3) defines an entire \(S^2\) and therefore resolves the conical singularity at the apex while when any \(z_i \neq 0\) (5.3) determines a unique point in \(S^2\). \(X\) is asymptotically conical with base \(S^2 \times \tilde{S}^3\) but \(S^3\) is contractible. With this description of \(X\) it is easy to identify the \(Z_2\) involution we were after, it is given by

\[
\sigma : z_i \to -z_i \quad i = 1, \ldots, 4. \tag{5.4}
\]

so that the fixed point set is \(z_i = 0\) where the \(S^2\) sits. Therefore, the orientifold (5.1) results in an \(O_5\)-plane along \(R^{1,3} \times S^2\).

We note for future reference that the singularity (5.2) has an alternative desingularization, in which the singularity is deformed as

\[
z_1 z_2 - z_3 z_4 = V. \tag{5.5}
\]

The action of \(\sigma\) on this geometry is now free since the fixed point set \(z_i = 0\) is not part of the geometry, so no orientifold plane appears in this case.

The Types of gauge groups that one can obtain by performing this orientifold are given in Table 2. Likewise, the total five-brane charge – that of the five-branes plus that of the \(O_5\)-plane – is given in Table 3.

Just like in the discussion in section 3 the corresponding string duals for these other gauge groups are obtained by implementing the orientifold action above on the supergravity solution described in the last section. Therefore, the string dual is given by an orientifold by \(\sigma\) of Type IIB on \(R^{1,3} \times M_6\). \(M_6\) is topologically like the deformed conifold geometry of (5.2) so that in this case the action of \(\sigma\) is free and one obtains a dual given by unoriented closed string theory on the regular geometry \(R^{1,3} \times M_6/Z_2\), without any open strings nor orientifold planes. In this case the action of \(\sigma\) is such that the non-contractible \(\tilde{S}^3\) inside \(M_7\) is replaced by \(\tilde{\mathbb{RP}}^3\).

Apart from replacing \(\tilde{S}^3 \to \tilde{\mathbb{RP}}^3\) in \(M_6\) one must take into account the total five-brane charge. Just like in section 3 the total charge due to the branes and the \(O_5\)-plane that we have on \(X\) is replaced by flux over the non-contractible \(\tilde{\mathbb{RP}}^3\) of \(M_6\) so that we can

\footnote{This follows from the following observation. The real section of equation (5.3) describes \(\tilde{S}^3\) so that the effect of the \(Z_2\) transformation \(z_i \to -z_i\) is to replace it by \(\tilde{\mathbb{RP}}^3\). Topologically, the space is \(T^*\tilde{\mathbb{RP}}^3\).}
replace in the solution found in [2] $N \to Q$, where $Q$ is the total five-brane charge given in Table 3.

We can understand the expected pattern of chiral symmetry breaking for these gauge theories by studying the symmetries of the string solution in the UV and in the IR. In the IR, the metric looks the same modulo the replacement $S^3 \to \tilde{RP}^3$ which makes the $\tilde{\psi}$ coordinate $2\pi$ periodic, that is $\tilde{\psi} \simeq \tilde{\psi} + 2\pi$. Just as in the previous section, the symmetry in the IR is just that given by shifts by $2\pi \tilde{\psi} \simeq \tilde{\psi} + 2\pi$. This shift acts as a trivial rotation in space-time but since the Killing spinors of the supergravity solution are unmodified by the orientifold projection, a rotation by $2\pi$ on $\tilde{\psi}$ generates a $Z_2$ transformation on the Killins spinors which is the expected unbroken symmetry group after gluino condensation.

What is the symmetry in the UV? The symmetry of the metric is again $U(1)_R$ acting by arbitrary shifts on $\tilde{\psi}$. We must analyze the effect on the $U(1)_R$ symmetry of a SYM instanton in the string theory realization. This can be accomplished by taking into account the way a SYM instanton is represented in terms of branes in string theory. Care must be taken in performing this identification since as we discussed in section 3 there are subtleties in the relation between SYM instantons and D-branes for unoriented open strings. We showed that for $SO(N)$ gauge groups, via Wess-Zumino coupling (3.5), that a D1-string on the D5-brane worldvolume corresponds to a single $SO(N)$ SYM instanton. Therefore, the SYM instanton is represented by a euclidean D1-string wrapping the $S^2 \in M_6/Z_2$ that we described in the previous section. The amplitude for such a wrapped D1-string is given by

$$v = \exp \left( i \int_{S^2} B \right). \quad (5.6)$$

By using the asymptotic expression for $H$ in (4.5) and the required replacement of $N \to Q$ one finds that

$$\int_{S^2} B = -2Q\tilde{\psi}. \quad (5.7)$$

Therefore, the discrete rotations on $\tilde{\psi}$ that act non-trivially on the Killing spinors are given by $\tilde{\psi} \to \tilde{\psi} + 2\pi k/Q$ with $k = 1, \ldots, 2Q$. By using Tables 3 and 1 we see that from the string dual that the UV symmetry of the solution is precisely $Z_{2h}$ as required by the field theory analysis.

For the $Sp(N)$ case there is an interesting subtlety. As we discussed in section 3 the relation between brane charge and instanton number is modified for $Sp(N)$ gauge theories. For an $Sp(N)$ gauge theory on a collection of D5-branes we established that a
D1-string dissolved on the D5-branes actually corresponds to two instantons in the $Sp(N)$ gauge theory. Therefore, the amplitude of a single SYM instanton is given by $v^{1/2}$ where $v$ is given by (5.4). The discrete rotations on $\tilde{\psi}$ that leave $v^{1/2}$ invariant and act non-trivially on the Killing spinors are given by $\tilde{\psi} \to \tilde{\psi} + 2\pi k/Q$ with $k = 1, \ldots, Q$. By taking the expression for $Q$ in Table 3 corresponding to the $Sp(N)$ case we see that we exactly reproduce the expected $Z_{2h}$ symmetry as can be seen from Table 1.

Therefore, we have realized the expected pattern of chiral symmetry breaking for $SO$ gauge groups $Z_{2h} \to Z_2$ as the breaking of the UV symmetry of the string solution to that of the IR. The $h$ vacua, just like in the $SU(N)$ case, are obtained by performing the $h$ allowed discrete rotations on the $SU(2)$ gauge fields $A_a$ of the solution such that the new solutions are nonsingular in the interior but are equivalent in the UV region.

Acknowledgments

We would like to thank A. Brandhuber, E. Diaconescu, A. Kapustin, J. Maldacena and E. Witten for discussions. The work of J.G. is supported in part by the National Science Foundation under grant No. PHY99-07949 and by the DOE under grant No. DE-FG03-92ER40701.
Appendix A. NS5 branes with $SU(2)_L$ gauge fields for 2+1 SYM

Here we rederive the solution found by Maldacena and Nastase by obtaining the first order BPS equations from an effective quantum mechanical Lagrangian derived from gauged supergravity. The relevant gauged supergravity is seven dimensional $SO(4)$ gauged supergravity, which appears by reducing Type IIB supergravity on the near horizon geometry of a NS5-brane. Since we are interested in finding a supergravity dual to $D = 3 \mathcal{N} = 1$ SYM, the supersymmetry of the field theory on the wrapped branes requires considering an $SU(2)_L$ truncation of $SO(4)$ gauged supergravity\(^{28}\).

The ansatz for the solution is determined by the symmetries of the background, which is given by five-branes wrapping $S^3 \in S(S^3)$. This requires considering the most general ansatz invariant under the $SO(4)$ isometries of $S^3$. In the string frame, the ansatz is therefore

$$ds^2 = ds^2(\mathbb{R}^{1,2}) + dr^2 + \varepsilon^{2h(r)} w_aw_a$$

$$A_a = \frac{1 + b(r)}{2} w_a,$$  \hspace{1cm} (A.1)

where $w_a, a = 1, 2, 3$ are a set of $SU(2)$ left invariant one-forms satisfying

$$dw_a = -\frac{1}{2} \epsilon_{abc} w_b \wedge w_c.$$  \hspace{1cm} (A.2)

The three-form field strength $h$ of $SU(2)_L$ gauged supergravity satisfies the following Bianchi identity\(^{29}\)

$$dh = \frac{1}{8\pi} F_a \wedge F_a.$$  \hspace{1cm} (A.3)

The $SO(4)$ symmetry together with

$$F_a = \frac{\dot{b}}{2} dr \wedge w_a + \frac{1}{8}(b^2 - 1) \epsilon_{abc} w_b \wedge w_c$$  \hspace{1cm} (A.4)

determine the three-form from (A.3)\(^{30}\)

$$h = \frac{1}{32\pi}(b^3 - 3b + 2) w_1 \wedge w_2 \wedge w_3.$$  \hspace{1cm} (A.5)

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\(^{28}\) One can also have an $SU(2)_D = (SU(2)_L \times SU(2)_R)_{diag}$ truncation. Then , the solution has double the amount of supersymmetry. This truncation was used in \(^{32}\)\(^{20}\) to find the supergravity dual to $D=3 \mathcal{N} = 2$ SYM.

\(^{29}\) We are writing the action such that the flux of $h/2\pi$ is quantized in integral units.

\(^{30}\) The constant piece is only fixed by requiring the solution we will find to be regular.
Plugging the ansatz into the bosonic $SU(2)_L$ gauged supergravity Lagrangian
\[ \mathcal{L} = \sqrt{g} e^{-2\phi} \left( R + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{8} F^a_{\mu \nu} F^a_{\mu \nu} - \frac{\pi^2}{3} h_{\mu \nu \rho} h^{\mu \nu \rho} + 4 \right) \] (A.6)
one obtains the following effective quantum mechanical system
\[ \mathcal{L}_{\text{eff}} = e^{2\alpha} (T - V), \] (A.7)
where
\[ T = 4 \dot{\alpha}^2 - 3 \dot{h}^2 - \frac{3}{4} e^{-2h} b^2 \] (A.8)
\[ V = \frac{1}{8} e^{-6h} (b^3 - 3b + 2)^2 + \frac{3}{4} e^{-4h} (b^2 - 1)^2 - 6e^{-2h} - 4, \]
with
\[ \alpha = \frac{3}{2} h - \phi \quad \dot{\alpha} = \frac{d\alpha}{dr}, \ldots \] (A.9)
The $SU(2)_L$ gauged supergravity equations of motion follow from (A.7) together with the Hamiltonian constraint which follows from having fixed the radial reparametrization invariance of the ansatz (A.1)
\[ H = T + V = 0. \] (A.10)

One can find a set of first order BPS equations whose solutions also solve the equations of motion if the potential of the effective quantum mechanics can be written in terms of a superpotential as follows
\[ V = \frac{1}{12} (\partial_h W)^2 + \frac{1}{3} e^{2h} (\partial_b W)^2 - \frac{1}{4} W^2. \] (A.11)
The required superpotential is given by
\[ W = 6e^{-h} \sqrt{M}, \] (A.12)
where
\[ M = \left( \frac{1}{12} e^{-2h} (b^3 - 3b + 2) - b \right)^2 + \frac{1}{4} e^{-2h} (b^2 - 1)^2 - \frac{2}{3} (b^2 - 1) + \frac{4}{9} e^{2h}. \] (A.13)
The BPS equations following from this superpotential are given by
\[ \dot{\alpha} = \frac{1}{4} e^{-h} \sqrt{M} \]
\[ \dot{b} = -\frac{2}{3} e^{2h} \partial_b W = \frac{2}{3 \sqrt{M}} e^h \left[ \frac{1}{8} e^{-4h} (b^3 - 3b + 2)(1 - b^2) + (1 - b^3)e^{-2h} - 2b \right] \]
\[ \dot{h} = -\frac{1}{6} e^{-h} \partial_h W = \frac{e^{-h} (b^3 - 3b + 2)^2}{16} + \frac{e^{-2h}}{2} (3(b^2 - 1)^2 - 2(b^4 - 3b^2 + 2b) + b^2 + 2). \] (A.14)

There is a typo in the Appendix of [1], the second equation in (4.4) should an extra multiplicative factor of 1/2 in the second term.
The asymptotics of the solution to these equations can be found in \[1\].

This gauged supergravity solution can then be lifted to a solution of Type IIB supergravity using the formulae in \[33\]. In the string frame the Type IIB solution is

\[
ds^2 = ds^2(\mathbb{R}^{1,2}) + N[dr^2 + \frac{1}{4}e^{2h}w_aw_a] + N\frac{1}{4}(\tilde{w}_a - A_a)^2
\]

\[
H = h + \frac{N}{2\pi} \left[ -\frac{1}{4}(\tilde{w}_1 - A_1) \wedge (\tilde{w}_2 - A_2) \wedge (\tilde{w}_3 - A_3) + \frac{1}{4}F_a \wedge (\tilde{w}_a - A_a) \right]
\]

(A.15)

where $\tilde{w}_a, a = 1, 2, 3$ are another set of SU(2) left invariant one forms parameterizing $\tilde{S}^3$.

This metric is defined on $\mathbb{R}^{1,2} \times M_7$ where $M_7$ is parametrized by $r, w_a$ and $\tilde{w}_a$. $w_a$ and $\tilde{w}_a$ parameterize respectively $S^3$ and $\tilde{S}^3$. As described in the text $M_7$ is asymptotically topologically a cone over $S^3 \times \tilde{S}^3$ but $S^3$ is contractible in the full geometry while $\tilde{S}^3$ is topologically non-trivial.
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