Weak phases $\gamma$ and $\alpha$ from $B^+$, or $B^0$ and $B_s$ decays

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Abstract

An improved flavor SU(3) method is presented for determining the weak angle $\gamma$ of the unitarity triangle using decay rates for $B^+ \to K\pi, B^+ \to K^+K^0$ and $B^+ \to \pi^+\eta$ (or $B^0 \to K\pi$ and $B_s \to K\pi$), their CP-conjugate modes and the CP-averaged rate for $B^\pm \to \pi^\pm\pi^0$. Rescattering (color-suppressed) contribution in $B^+(B^0) \to K\pi$, for which an improved bound is obtained, is subtracted away. The only significant SU(3) breaking effects are accounted for in the factorization approximation of tree amplitudes. The weak angle $\alpha$ is obtained as a byproduct.
The determination of the angles of the unitarity triangle is an important goal of physics studies at existing and future $B$ meson facilities. It is expected to provide tests of the CKM mechanism of CP violation in the Standard Model and to shed light on possible new physics. In particular, the determination of the weak phase $\gamma = \text{Arg}(V_{us}^\ast)$ has stimulated a great deal of effort, both on the theoretical and experimental side. A variety of ways have been proposed to extract this angle \cite{11}, ranging from theoretically clean methods \cite{2} applied to $DK$ modes \cite{3} hampered by some very small branching ratios, to approximate methods applied to $K\pi$ modes most of which have already been observed \cite{1}. In the latter case one usually uses approximate flavor SU(3) symmetry of strong interactions \cite{9} to relate $B \to K\pi$ to $B \to \pi\pi$ amplitudes.

In a somewhat simplified version of this idea, Gronau, Rosner and London \cite{6} suggested to determine $\gamma$ through a triangle construction for $B^+ \to K^0\pi^+$ decay amplitudes into $K^0\pi^+, K^+\pi^0$ and $\pi^+\pi^0$, and for the corresponding charge-conjugate decays. It was later noted \cite{7} that higher order electroweak penguin (EWP) contributions upset this triangle construction. Various attempts were made to eliminate the uncertainties due to EWP amplitudes \cite{1}. Recently Neubert and Rosner \cite{8} included the EWP amplitudes in $B^+ \to K\pi$ in a model-independent manner, by relating them to corresponding current-current amplitudes. (One often refers to such amplitudes as “tree” amplitudes, since they are of lowest order in electroweak couplings.) In their revised triangle construction the authors of \cite{8} must rely, however, on the dynamical assumption that the amplitude for $B^+ \to K^0\pi^+$ is dominated completely by a QCD penguin contribution, and involves no term proportional to $e^{i\gamma}$ \cite{9}. This assumption is equivalent to neglecting certain final state rescattering effects \cite{10}. Present experimental limits on such effects from SU(3) related $B \to K\bar{K}$ decays \cite{11,12} are not yet sufficiently strong for ignoring them. Indirect evidence against such effects could also be obtained from future limits on the CP asymmetry in $B^\pm \to K\pi^\pm$.

In view of the possibility that rescattering effects could give rise to a small, however non-negligible, contribution in $B^+ \to K^0\pi^+$ with phase $\gamma$, thus upsetting the construction of Ref. \cite{8}, we propose in the present Letter to combine the processes $B^\pm \to K\pi$ with future information from $B^\pm \to K^\pm K$ and $B^\pm \to \pi^\pm\eta_8$ decays ($\eta_8$ is an SU(3) octet). Alternatively, to avoid the question of $\eta - \eta'$ mixing, the same procedure can be applied by combining $B^0 \to K\pi$ and $B_s \to \bar{K}\pi$ decays. Using a simple SU(3) relation between these pairs of processes, we will show that one can avoid uncertainties due to final state rescattering in $B^+ \to K^0\pi^+$ and due to a color-suppressed amplitude in $B^0 \to K^0\pi^0$. SU(3) breaking effects occurring in these relations will be shown to contribute only a very small uncertainty in $\gamma$. SU(3) breaking, in the relation between tree amplitudes of $B \to K\pi$ and $B \to \pi\pi$, will be accounted for in the factorization approximation. Electroweak penguin effects will be included in a model-independent way \cite{13}.

Using the notations of \cite{13}, we write the neutral and charged $B$ decay amplitudes into $K\pi$ states in terms of graphical SU(3) amplitudes

\[ A(B^0 \to K^+\pi^-) = |\lambda_u^{(s)}|e^{i\gamma}(-T - P_{uc}) - |\lambda_t^{(s)}|(-P_{ct} + P_{1\text{EW}}^\text{EW}) , \]
\[ \sqrt{2}A(B^0 \to K^0\pi^0) = |\lambda_u^{(s)}|e^{i\gamma}(-C + P_{uc}) - |\lambda_t^{(s)}|(P_{ct} + \sqrt{2}P_{2\text{EW}}^\text{EW}) , \]
\[ A(B^+ \to K^0\pi^+) = |\lambda_u^{(s)}|e^{i\gamma}(A + P_{uc}) - |\lambda_t^{(s)}|(P_{ct} + P_{3\text{EW}}^\text{EW}) , \]
\[ \sqrt{2}A(B^+ \to K^+\pi^0) = |\lambda_u^{(s)}|e^{i\gamma}(-T - C - A - P_{uc}) - |\lambda_t^{(s)}|(-P_{ct} + \sqrt{2}P_{4\text{EW}}^\text{EW}) , \]
where $\lambda_q^{(i)} = V_{qs}^* V_{q'd}$, the amplitudes $T, C, A, P$ include unknown strong phases, and $P_{EW}^{1-4}$ are the respective EWP contributions to these decays. The amplitudes (1)-(4) satisfy the two triangle relations [3,4]

$$\sqrt{2}A(B^+ \rightarrow K^+\pi^0) + A(B^+ \rightarrow K^0\pi^+) = \sqrt{2}A(B^0 \rightarrow K^0\pi^0) + A(B^0 \rightarrow K^+\pi^-) = \sqrt{2}\lambda|A(B^+ \rightarrow \pi^+\pi^0)|e^{i(\gamma+\phi)}\rho \left(1 - \delta_{EW} e^{-i\gamma}\right).$$

Here we denote $\lambda = V_{us}/V_{ud}$, $\delta_{EW} = -(3/2)|\lambda_1^{(s)}/\lambda_1^{(d)}|\kappa \simeq 0.66$ ($\kappa \equiv (c_9 + c_{10})/(c_1 + c_2) = -8.8 \cdot 10^{-3}$), while $\phi$ is an unknown strong phase. The second term in the brackets represents the sum of EWP contributions to the amplitudes on the left-hand-sides [3,3]. The correction factor $\rho = (f_K/f_\pi)|1 + (3/2)\kappa|\lambda_1^{(d)}/\lambda_1^{(d)}|\exp(i\alpha)|^{-1} \approx 1.22$ accounts for factorizable SU(3) breaking effects and EWP contributions to the amplitude $A(B^+ \rightarrow \pi^+\pi^0)$ respectively. Numerically this factor is dominated by the former contribution.

Each of the two amplitude triangles [3] cannot be used by itself, together with the corresponding relation for the CP-conjugate amplitudes $\tilde{A}(\tilde{B} \rightarrow \tilde{f}) \equiv e^{2i\gamma}A(\tilde{B} \rightarrow \tilde{f})$, to allow a determination of $\gamma$. The reason is that in general all four amplitudes in (1)-(4) involve two terms with different weak phases. $\gamma$ can be determined only when one of these terms can be neglected in one of the amplitudes as assumed in [3,3]. Although the first terms in (2) and (3) are likely to be significantly smaller than the second terms, we will not neglect them in the forthcoming discussion. For definiteness, we present in detail the version of our method applied to neutral $B$ decays. A similar brief treatment of $B^+$ decays precedes the conclusion.

Normalizing amplitudes by $A(B^+ \rightarrow \pi^+\pi^0)$, we define reduced amplitudes

$$x_{+-} = \frac{1}{\sqrt{2}\rho} \frac{|A(B^0 \rightarrow K^+\pi^-)|}{|A(B^0 \rightarrow \pi^+\pi^0)|}, \quad x_{00} = \frac{1}{\lambda\rho} \frac{|A(B^0 \rightarrow K^0\pi^0)|}{|A(B^0 \rightarrow \pi^+\pi^0)|},$$

$$\bar{x}_{-+} = \frac{1}{\sqrt{2}\rho} \frac{|A(\bar{B}^0 \rightarrow K^-\pi^+)|}{|A(\bar{B}^0 \rightarrow \pi^+\pi^0)|}, \quad \bar{x}_{00} = \frac{1}{\lambda\rho} \frac{|A(\bar{B}^0 \rightarrow \bar{K}^0\pi^0)|}{|A(\bar{B}^0 \rightarrow \pi^+\pi^0)|}. \quad (6)$$

The triangle relation (3) for $B^0$ decays and its CP-conjugate are given by

$$x_{00}e^{i\phi_1} + x_{+-}e^{i\phi'_1} = 1 - \delta_{EW} e^{-i\gamma}, \quad (8)$$

$$\bar{x}_{00}e^{i\phi_1} + \bar{x}_{-+}e^{i\phi'_1} = 1 - \delta_{EW} e^{i\gamma}, \quad (9)$$

where $\phi_1, \phi'_1, \tilde{\phi}_1, \tilde{\phi}'_1$ contain both strong and weak phases. These triangles are represented in Fig. 1. The angles $\phi_1$ and $\tilde{\phi}_1$ are functions of $\cos \gamma$ defined by second order equations

$$\frac{1}{2} - \delta_{EW} \cos \gamma \cos \phi_1 + \delta_{EW} \sin \gamma \sin \phi_1 = \frac{x_{00}^2 - x_{+-}^2 + (1 + \delta_{EW}^2 - 2\delta_{EW} \cos \gamma)}{2x_{00}}, \quad (10)$$

$$\frac{1}{2} - \delta_{EW} \cos \gamma \cos \tilde{\phi}_1 - \delta_{EW} \sin \gamma \sin \tilde{\phi}_1 = \frac{\bar{x}_{00}^2 - \bar{x}_{-+}^2 + (1 + \delta_{EW}^2 - 2\delta_{EW} \cos \gamma)}{2\bar{x}_{00}}. \quad (11)$$

As mentioned, a major simplification occurs when the color-suppressed amplitude $-C + P_{uc}$ in $A(B^0 \rightarrow K^0\pi^0)$ (4) is neglected relative to the dominant penguin contribution. In this limit, the angle between the amplitudes $\sqrt{2}A(B^0 \rightarrow K^0\pi^0)$ and $\sqrt{2}\tilde{A}(B^0 \rightarrow \bar{K}^0\pi^0)$ in Fig. 1 is $2\gamma$. This implies $\cos 2\gamma = \cos(\phi_1 - \tilde{\phi}_1)$ which determines $\gamma$. A similar argument
applies in charged $B$ decays \cite{8} when the annihilation amplitude $A + P_{uc}$ is neglected in $A(B^+ \to K^0\pi^+)$ \cite{3}. In general, without neglecting these terms, the two triangles in Fig. 1 involve an arbitrary relative angle which prohibits a determination of $\gamma$.

In order to avoid these dynamical assumptions and to establish another constraint on the relative angles between the above two triangles, let us consider together with $B^0 \to K\pi$ also the following $B_s$ decay amplitudes \cite{13}

$$
A(B_s \to K^+\pi^-) = |\lambda_u^{(d)}|e^{i\gamma}(-T' - P_{uc}') + |\lambda_t^{(d)}|e^{-i\beta}(-P_{ct}' + P_1'^{EW}) ,
$$

$$
\sqrt{2}A(B_s \to K^0\pi^0) = |\lambda_u^{(d)}|e^{i\gamma}(-C' + P_{uc}') + |\lambda_t^{(d)}|e^{-i\beta}(P_{ct'} + \sqrt{2}P_2'^{EW}) .
$$

In the SU(3) symmetric limit the reduced amplitudes appearing in these expressions are equal to those appearing in Eqs. \cite{4} and \cite{3}.

In the first case this follows simply from U-spin. The amplitudes \cite{12} and \cite{13} satisfy a triangle relation similar to \cite{5}.

In the SU(3) symmetric limit this relation is exact, even accounting for EWP contributions. The factor $\rho' = (\mathcal{F}(B_s - M_{K_s}^2)F_{B_sK}(M_{K_s}^2))/(\mathcal{F}(B - M_{\pi}^2)F_{B\pi}(M_{\pi}^2))$ parametrizes the leading factorizable SU(3) breaking effects.

The SU(3) relations between the terms of definite CKM factors in \cite{4} and \cite{3} respectively, allow a simple geometrical interpretation. Drawing the amplitudes \cite{4}, \cite{13}-scaled by $\lambda$ and their CP-conjugates ($A(B \to f) \equiv e^{i\gamma}A(B \to \bar{f})$), such that all amplitudes originate in a common point, the other ends of the four amplitudes form a quadrangle as shown in Fig. 2. (The point of origin is not shown in this figure. In Fig. 1 it is chosen as the point $O$.) This quadrangle is not determined by rate measurements alone, since it involves the unknown relative angle between the triangles \cite{3}, \cite{14} and their charge-conjugates, which depends on $\gamma$ through $\phi_1, \phi_1$. We will show now that the quadrangle provides another condition on $\gamma$ which fixes this phase.

Consider the four sides of the quadrangle in Fig. 2 given in the SU(3) limit by (with $p \equiv P_{ct} + \sqrt{2}P_2^{EW}$)

$$
v = |\lambda_t^{(s)}|(1 - e^{2i\gamma})p ,
$$

$$
x = (\lambda|\lambda_t^{(d)}|e^{i(\beta + 2\gamma)} + |\lambda_t^{(s)}|) p ,
$$

$$
z = (\lambda|\lambda_t^{(d)}|e^{-i\beta} + |\lambda_t^{(s)}|e^{2i\gamma}) p ,
$$

$$
y = \lambda|\lambda_t^{(d)}|(e^{i(\beta + 2\gamma)} - e^{-i\beta}) p .
$$

Since all four sides of the quadrangle are proportional to a single hadronic amplitude $p \equiv P_{ct} + \sqrt{2}P_2^{EW}$, its shape is determined exclusively by CKM parameters. In fact this quadrangle is an isosceles trapezoid, $|x| = |z|$, whose sides $v$ and $y$ are parallel. We will select a point $X$ on the median of the trapezoid (the line bisecting the sides $v$ and $y$ perpendicularly) with the property that its distances to the vertices of the trapezoid are in the following ratio

$$
r = \frac{AX}{CX} = \frac{BX}{DX} = \frac{|\lambda_t^{(s)}|}{\lambda|\lambda_t^{(d)}|} = \frac{1}{\lambda|V_{td}|} = 22 \pm 4 .
$$
The value of $|V_{ts}/V_{td}|$ is taken from a recent global analysis of the unitarity triangle \cite{15}. It is easy to see that the angles through which the sides $\nu$ and $y$ are seen from the point $X$ are $2\gamma$ and $2\alpha$, respectively. (See Fig. 2). The new condition on $\gamma$, together with \cite{10}, \cite{11} illustrated in Fig. 1, are sufficient for determining this phase up to discrete ambiguities (to be discussed below). Fig. 2 can also be used to measure $\alpha$.

The conditions \cite{10} can be applied to determine the point $X$ in Fig. 1 in the following way. First, we note that the points $C$ and $D$ are fixed by Eq. \cite{14} and its charge-conjugate. Then, recall that the set of points $X$, for which the ratio of the distances to two given points $A$ and $C$ takes a fixed value $r$, is a circle given by

$$|X - \frac{Cr^2 - A}{r^2 - 1}| = \frac{r|A - C|}{r^2 - 1}.$$ \hspace{1cm} (17)

Using $r^2 \gg 1$, $|A| \sim 20|C|$ and $|C|r^2 \gg |A|$ (see discussion below), where $A$ and $C$ are the coordinates of these points with respect to the origin $O$ shown in Fig. 1, the circle is approximated by $|X - C| = |A - C|/r \simeq |A|/r$. The second condition \cite{14}, applied to $B$ and $D$, has a similar form, $|X - D| = |B - D|/r \simeq |B|/r$. The two circles of equal radii, $|A|/r \approx |B|/r$, with centers at $C$ and $D$, intersect at $X$ and determine this point up to a possible two-fold ambiguity. $\gamma$ is generally determined up to an eight-fold ambiguity due to an additional up-down ambiguity of the two $B_s$ triangles. In practice, four of these possibilities might be eliminated if the two circles do not intersect.

In order to demonstrate the algebraic solution for the two weak phases, let us introduce also reduced amplitudes for $B_s$ decays

$$y_{-+} = \frac{1}{\sqrt{2\rho'}} \frac{|A(B_s \to K^-\pi^+)|}{|A(B^+ \to \pi^+\pi^0)|}, \quad y_{00} = \frac{1}{\rho'} \frac{|A(B_s \to \bar{K}^0\pi^0)|}{|A(B^+ \to \pi^+\pi^0)|},$$ \hspace{1cm} (18)

$$\bar{y}_{-+} = \frac{1}{\sqrt{2\rho'}} \frac{|A(B_s \to K^+\pi^-)|}{|A(B^+ \to \pi^+\pi^0)|}, \quad \bar{y}_{00} = \frac{1}{\rho'} \frac{|A(B_s \to \bar{K}^0\pi^0)|}{|A(B^+ \to \pi^+\pi^0)|}.$$ \hspace{1cm} (19)

These amplitudes satisfy the triangle relations

$$y_{00}e^{i\phi_2} + y_{-+}e^{i\phi_2'} = 1, \quad \bar{y}_{00}e^{i\hat{\phi}_2} + \bar{y}_{-+}e^{i\hat{\phi}_2'} = 1,$$ \hspace{1cm} (20)

where tiny EWP contributions to the amplitude $A(B^+ \to \pi^+\pi^0)$ are neglected. (Their effects will be estimated below). The phases $\phi_2$ and $\hat{\phi}_2$ are determined from rate measurements through \cite{20}

$$\cos \phi_2 = \frac{1 + y_{00}^2 - y_{-+}^2}{2y_{00}}, \quad \cos \hat{\phi}_2 = \frac{1 + \bar{y}_{00}^2 - \bar{y}_{-+}^2}{2\bar{y}_{00}}.$$ \hspace{1cm} (21)

The angle $\gamma$ is extracted as the root of the equation $\cos(BXA) = \cos 2\gamma$. Denoting the position of the point $X$ by $pe^{i\phi}$, determined as explained above, an explicit form for this equation is

$$2x_{00}\bar{x}_{00}\sin(\phi_1 - \phi_1) \sin 2\gamma - 2[\rho^2 - x_{00}\bar{x}_{00}\cos(\phi_1 - \phi_1)] \cos 2\gamma = x_{00}^2 + \bar{x}_{00}^2 - 2\rho^2.$$ \hspace{1cm} (22)

This determines $\gamma$ when combined with Eqs. \cite{10} \cite{11}. The angle $\alpha$ is given directly by the angle $CXD$. 

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In order to evaluate the precision of this method, let us first consider the magnitudes of the amplitude ratios appearing in the triangle relations (3), (4) and (21). Using the measured decay rates of $B \to K\pi$ and $B \to \pi\pi$ [4], one estimates from the dominant terms $\bar{\epsilon}_{00} \simeq x_{00} \simeq \bar{x}_{-+} \simeq \pm |\lambda_0^{(s)}| P_{ct}/\lambda_0^{(s)}(T+C)| \simeq 4$. Similarly, $y_{00} \simeq y_{00} \simeq |C/T| \simeq 0.2$. We also note that since $B_X/CX = |\lambda_0^{(s)}|/|\lambda_0^{(d)}| = r$, the isosceles triangle in Fig. 2 with angle $2\gamma$ is about 20 times larger than the one with angle $2\alpha$. These estimates justify the approximations made below Eq. (17). The errors in the distances of the center points of the two circles from $O$, and the errors in the radii of these circles are of the order of $x_{00}/r^2 \simeq 0.01$ and can be neglected.

We now discuss the theoretical errors in the determination of $\gamma$ and $\alpha$. An intrinsic source of uncertainty is the parameter $\delta_{EW} \simeq 0.63 \pm 0.11$ [3], where the 5% shift from 0.66 accounts for factorizable SU(3)-breaking corrections, and the error is dominated by the present poorly known ratio of CKM matrix elements $|V_{ub}/V_{cb}|$. Its effect on the extraction of $\gamma$ was examined in detail in [8], and we have nothing new to add to that discussion.

We will focus instead on the SU(3) breaking effects introduced by the additional amplitudes considered in this method. They show up as differences between the amplitudes contributing to $B^0 \to K^0\pi^0$ [2] and $B_s \to \bar{K}^0\pi^0$ [13], $|c'| \neq |c|$ (with $c \equiv C - P_{uc}$) and analogous inequalities holding for the corresponding penguin amplitudes, $|p'| \neq |p|$ (with $p \equiv P_{ct} + \sqrt{2} P_{EW}$). One expects these amplitudes to differ by at most 30%. A smaller uncertainty exists in the factor $\rho'$, for which the deviation from unity can be taken from quark models. Fixing $|p|$ and $|c|$ and allowing $|p'/\rho'|$ and $|c'/\rho'|$ to vary within 40%, the points $C$ and $D$ in Fig. 2 can vary within small circles of radius $\Delta y_{00} \simeq 0.4 y_{00} \simeq 0.08$. Therefore, the corrections to $\gamma$ due to SU(3) breaking are expected to be small. (This is due to our judicious choice of origin about which the triangles in Fig. 1 are rotated, this point being adjacent to the color-suppressed amplitude for $B_s$ decay). To estimate the absolute value of the error in $\gamma$ arising from SU(3) breaking, we consider the most unfavorable case of a simultaneous shift of $y_{00}$ and $\bar{y}_{00}$ by $\Delta y_{00} = 0.08$. This translates into an error in $\gamma$ of $\Delta \gamma \simeq \Delta y_{00}/x_{00} \simeq 0.02$ which is about 1°.

The ratio $r$ introduced in (13) is known with an error of about 20%. This affects the determined position of the point $X$ through the radii of the circles centered at $C$ and $D$. The radii of these circles are of the order of $x_{00}/r \simeq 0.2$, implying an error in the position of the point $X$ of the order of 0.04, which is half of the uncertainty arising from SU(3) breaking. Combining these two errors in quadrature, one obtains a total error in $\gamma$ of about 1.3°.

Another source of theoretical uncertainty is connected with the neglect of EWP contributions in $A(B^+ \to \pi^+\pi^0)$. We have recently shown that when these effects are included, the relation between this decay amplitude and its CP conjugate is [13,16]

$$A(B^+ \to \pi^+\pi^0) = e^{2i\xi} \tilde{A}(B^- \to \pi^-\pi^0), \quad \tan \xi = \frac{x \sin \alpha}{1 + x \cos \alpha},$$

where $x = -(3/2)\kappa \sin \alpha / \sin(\alpha + \gamma)$. Numerically the angle $2\xi$ is seen to be very small, under 2°. This uncertainty will affect only the relative orientation of the two $B_s$ triangles, shifting the angles $\phi_2$ and $\tilde{\phi}_2$ by an amount $\Delta \phi_2 = -\Delta \tilde{\phi}_2 = \xi$. (The effect of these EWP on
the \(B \rightarrow K \pi\) triangles (3) enters only through the factor \(\rho\), to which they contribute at the level of 1%). The corresponding error in the positions of \(C\) and \(D\) is of order 0.2\(\zeta \approx 0.003\) which is well under the uncertainty arising from the other sources discussed above.

On the other hand, the smallness of the \(CXD\) triangle implies that \(SU(3)\) breaking effects will have a larger impact on the extraction of \(\alpha\) from this method. The estimates given above indicate that the error in such a determination is at the level of 30%.

A similar method can be applied to the determination of \(\gamma\) from \(B^+ \rightarrow K^0 \pi^+\) and \(B^+ \rightarrow K^+ \pi^0\) decays. In this case uncertainties due to rescattering in \(B^+ \rightarrow K^0 \pi^+\) can be eliminated by considering in addition the decays \(B^+ \rightarrow K^+ \bar{K}^0\) and \(B^+ \rightarrow \pi^+ \eta_8\), where \(\eta_8\) is an \(SU(3)\) octet. Their amplitudes are given by 5,13

\[
A(B^+ \rightarrow K^+ \bar{K}^0) = |\lambda_u^{(d)}| e^{i\gamma} (A + P_{uc}) + |\lambda_t^{(d)}| e^{-i\beta} (P_{ct} + P_3^{EW}),
\]

\[
\sqrt{6} A(B^+ \rightarrow \pi^+ \eta_8) = |\lambda_u^{(d)}| e^{i\gamma} (-C - 2A - 2P_{uc}) + |\lambda_t^{(d)}| e^{-i\beta} (-2P_{ct} + P_5^{EW}) ,
\]

and are closely related to (3) and (4). Their relative orientation with respect to (3) and (4) can be fixed as in the \(B^0\) case with the help of the (exact) triangle relation

\[
A(B^+ \rightarrow K^+ \bar{K}^0) + \frac{3}{\sqrt{2}} A(B^+ \rightarrow \pi^+ \eta_8) = \frac{1}{\sqrt{2}} A(B^+ \rightarrow \pi^+ \pi^0) .
\]

This triangle relation replaces Eq. (14) in the case of neutral \(B\) decays. Instead of the quadrangle of Eqs. (13), one now constructs a quadrangle from \(A(B^+ \rightarrow K^0 \pi^+\), \(\lambda A(B^+ \rightarrow K^+ \bar{K}^0)\) and their charge-conjugates, the four sides of which are all proportional to \(P_{ct} + P_3^{EW}\). The extraction of \(\gamma\) and \(\alpha\) follows in a similar way. This set of processes is experimentally more accessible than \(B_s \rightarrow \bar{K}^0 \pi^0\), however \(B^+ \rightarrow \pi^+ \eta_8\) involves a certain amount of model-dependence related to \(\eta - \eta'\) mixing 17.

One can use the arguments presented here, with figures similar to the above drawn for \(B^+\) decay amplitudes, to obtain an upper bound on rescattering effects in \(B^+ \rightarrow K^0 \pi^+\) in terms of the charge-averaged \(B^\pm \rightarrow K^\pm \bar{K}^0\) rate. The amplitudes of interest in \(B^+\) decays are given by line segments analogous to those in Fig. 1. (no normalization by \(\sqrt{2} A(B^+ \rightarrow \pi^+ \pi^0)\) is used.)

\[
|OX| = |\lambda_u^{(s)}(A + P_{uc})|, \quad |XA| = |\lambda_t^{(s)}(P_{ct} + P_3^{EW})| .
\]

Simple geometry implies

\[
\epsilon_A \equiv \frac{|\lambda_u^{(s)}(A + P_{uc})|}{|\lambda_t^{(s)}(P_{ct} + P_3^{EW})|} = \frac{|OX|}{|XA|} \leq \frac{\min(|OC|, |OD|) + \frac{1}{r}|OA|}{|OA|} \leq \frac{1}{r} + \lambda \sqrt{\frac{B(B^+ \rightarrow K^\pm \bar{K}^0)}{B(B^\pm \rightarrow K^0 \pi^\pm)}} .
\]

The ratio \(\epsilon_A\), describing rescattering in \(B^+ \rightarrow K^0 \pi^+\), takes its maximum value when \(|OC| = |OD|\) (for fixed \(|OC|^2 + |OD|^2\)). The expression on the right-hand side is accurate up to corrections of order \(|OX|/|OA| \approx 0.05\) of its magnitude (due to the approximation \(|XA| \approx |OA|\) used in the second step). A previous bound 11, based on the assumption of constructive interference between the two terms in Eq. (23), omitted the \(1/r\) term.

In conclusion, we have presented a new method for extracting the weak angle \(\gamma\) using combined \(B^0\) and \(B_s\) decays, or combining \(B^+ \rightarrow K \pi\) with \(B^+ \rightarrow K^+ \bar{K}^0\) and \(B^+ \rightarrow \pi^+ \eta\). This
method represents an improvement of the method suggested in [8] in that color-suppressed contributions in $B^0$ decay, or rescattering effects in case of $B^+$ decay, are eliminated with the help of SU(3) flavor symmetry. The additional SU(3) breaking corrections were shown to be negligible. Under ideal experimental conditions, this method would allow a substantial improvement in the precision of determining $\gamma$. In reality, $B_s$ decay modes involving neutral pions pose a particularly difficult experimental challenge. Alternatively, the use of charged $B$ decays involves a slight theoretical complication due to $\eta - \eta'$ mixing which must be resolved.

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FIG. 1. Relative orientation of $B^0$ amplitude triangles. $C$ and $D$ are the tips of the $B_s$ triangles (not shown for clarity), which determine the point $X$ as explained in the text below Eq. (17) (see also Fig. 2).

FIG. 2. Quadrangle formed by the tips of the triangles for $B^0$ and $B_s$ decays

$A = \sqrt{2}A(B^0 \rightarrow K^0\pi^0)$, $B = \sqrt{2}\lambda\overline{A}(\overline{B}^0 \rightarrow \overline{K^0}\pi^0)$, $C = \lambda\sqrt{2}A(B_s \rightarrow K^0\pi^0)$ and $D = \lambda\sqrt{2}\overline{A}(\overline{B}_s \rightarrow \overline{K}^0\pi^0)$. The point $X$ is determined by the intersection of the two circles of radius $x_{00}/r \approx \tilde{x}_{00}/r$ centered at $C$ and $D$ respectively.