GENERALIZED CONTACT BUNDLES

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Abstract. In this Note, we propose a line bundle approach to odd-dimensional analogues of generalized complex structures. This new approach has three main advantages: (1) it encompasses all existing ones; (2) it elucidates the geometric meaning of the integrability condition for generalized contact structures; (3) in light of new results on multiplicative forms and Spencer operators [9], it allows a simple interpretation of the defining equations of a generalized contact structure in terms of Lie algebroids and Lie groupoids.

1. Introduction

Generalized complex structures have been introduced by Hitchin in [12] and further investigated by Gualtieri in [11]. They can only be supported by even-dimensional manifolds and encompass symplectic structures and complex structures as extreme cases. Since both of these extreme cases have analogues in odd-dimensional geometry (namely, contact and almost contact structures, respectively), it is natural to ask if there is any natural odd-dimensional analogue of generalized complex structures.

Several approaches to odd-dimensional analogues of generalized complex structures can be found in the literature [13, 22, 18, 19, 1]. They are often named generalized contact structures and all of them include contact structures globally defined by a contact 1-form. However, none of them incorporates non-coorientable contact structures. From a conceptual point of view, contact geometry is the geometry of an hyperplane distribution and the choice of a contact form is just a technical tool making things simpler. Even more, there are interesting contact structures that do not possess any global contact form. Accordingly, it would be nice to define a generalized contact structure “independently of the choice of a contact form”. This Note fills that gap. We call the proposed structure a generalized contact bundle to distinguish it from previously defined generalized contact structures. Generalized contact bundles are just a slight generalization of Iglesias-Wade integrable generalized almost contact structures [13] to the realm of (generically non-trivial) line bundles. Generalized contact bundles encompass (generically non-coorientable) contact structures and complex structures on...
the Atiyah algebroid of a line bundle as extreme cases. This new point of view on
generalized contact geometry could also be useful in studying $T$-duality [1].

In this Note, we interpret the defining equations of a generalized contact structure in
terms of Lie algebroids and Lie groupoids. As a side result we define a novel notion of
multiplicative Atiyah form on a Lie groupoid and identify its infinitesimal counterpart.
This could have an independent interest.

2. The Atiyah algebroid associated to a contact distribution

For a better understanding of the concept of a generalized contact bundle, we briefly
discuss a line bundle approach to contact geometry. By definition, a contact structure on
an odd-dimensional manifold $M$ is a maximally non-integrable hyperplane distribution
$H \subset TM$. In a dual way, any hyperplane distribution $H$ on $M$ can be regarded as
a nowhere vanishing 1-form $\theta : TM \to L$ (its structure form) with values in the line
bundle $L = TM/H$, such that $H = \ker \theta$. Now, consider the so called Atiyah algebroid
$DL \to M$ (also known as gauge algebroid [15], [6]) of the line bundle $L$ [23, Sections 2, 3].
Recall that sections of $DL$ are derivations of $L$, i.e. $R$-linear operators $\Delta : \Gamma(L) \to \Gamma(L)$
such that there exists a, necessarily unique, vector field $\sigma\Delta \in X(M)$, called the
symbol of $\Delta$, such that $\Delta(f\lambda) = (\sigma\Delta)(f)\lambda + f\Delta(\lambda)$ for all
$f \in C^\infty(M)$ and $\lambda \in \Gamma(L)$. This
is a transitive Lie algebroid whose Lie bracket is the commutator, and whose anchor
$DL \to TM$ is the symbol $\sigma$. Additionally, $L$ carries a tautological representation of $DL$
given by the action of an operator on a section. Any $k$-cochain in the de Rham complex
$(\Omega^k_L := \Gamma(\wedge^k(DL)^* \otimes L), d_{DL})$ of $DL$ with coefficients in $L$ will be called an $L$-valued
Atiyah $k$-form. There is a one-to-one correspondence between contact structures $H$
with $TM/H = L$ and non-degenerate, $d_{DL}$-closed, $L$-valued Atiyah 2-forms. Contact
structure $H$, with structure form $\theta : TM \to L$, corresponds to the Atiyah 2-form
$\omega := d_{DL}\sigma^*\theta$, where $\sigma^*\theta(\Delta) := \theta(\sigma\Delta)$.

For more details on Atiyah forms as well as their functorial properties, see [23, Section 3].

3. Generalized contact bundles and contact-Hitchin pairs

Recall that a generalized almost complex structure on a manifold $M$ is an endomorphism
$J : TM \to TM$ of the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$ such that
(1) $J^2 = - \text{id}$, and (2) $J$ is skew-symmetric with respect to the natural pairing on
$\mathbb{T}M$. If, additionally, (3) the $\sqrt{-1}$-eigenbundle of $J$ in the complexification $\mathbb{T}M \otimes \mathbb{C}$
in involutive relative to the Dorfman (equivalently, the Courant) bracket, then $J$ is said to
be integrable, and $(M, J)$ is called a generalized complex manifold (see [11] for more
details).

Replacing the tangent algebroid with the Atiyah algebroid of a line bundle in the
definition of a generalized complex manifold, we obtain the notion of generalized contact
bundle. More precisely, let $L \to M$ be a line bundle. For a vector bundle $V \to M$, there is an obvious $L$-valued (duality) pairing $\langle -, - \rangle_L : V \otimes (V^* \otimes L) \to L$, and for
every vector bundle morphism \( F : V \to W \) (covering the identity) there is an adjoint morphism \( F^\dagger : W^* \otimes L \to V^* \otimes L \) uniquely determined by \( \langle F^\dagger(\phi), v \rangle_L = \langle \phi, F(v) \rangle_L \), \( \phi \in W^* \otimes L, v \in V \). Clearly, \((DL)^* \otimes L = J^1L\), the first jet bundle of \( L \). The direct sum \( DL := DL \oplus J^1L \) is a contact-Courant algebroid (in the sense of Grabowski [10]), and is called an omni-Lie algebroid in [6]. We denote by \([-,-] : \Gamma(DL) \times \Gamma(DL) \to \Gamma(DL)\) and \(\langle-, -\rangle : DL \otimes DL \to L\), the Dorfman-Jacobi bracket and the \(L\)-valued symmetric pairing, respectively. Namely, for all \( \Delta, \nabla \in \Gamma(DL)\), \( \phi, \psi \in \Gamma(J^1L)\),

\[
\langle (\Delta, \phi), (\nabla, \psi) \rangle := \langle \Delta, \psi \rangle_L + \langle \nabla; \phi \rangle_L,
\]

and

\[
\langle (\Delta, \phi), (\nabla, \psi) \rangle := \langle \Delta, \nabla \rangle, L_\Delta \psi - i_\nabla d_{DL} \phi.
\]

See e.g. [23] for the main properties of these structures.

**Definition 3.1.** A generalized almost contact bundle is a line bundle \( L \to M \) equipped with a generalized almost contact structure, i.e. an endomorphism \( I : DL \to DL \) such that \( I^2 = -id \), and \( I^\dagger = -I \).

**Remark 3.2.** Similarly as for generalized almost complex structures, it is easy to see that, in the case \( L = M \times \mathbb{R} \), one recovers [13] Definition 4.1 which is equivalent to Sekiya’s generalized \( f \)-almost contact structures [19]. In particular, Poon-Wade’s generalized almost contact pairs are special cases of Definition 3.1.

Using direct sum decomposition \( DL = DL \oplus J^1L \), one sees that every generalized almost contact structure on \( L \) is of the form

\[
I = \begin{pmatrix}
\varphi & J^2 \\
\omega^3 & -\varphi^\dagger
\end{pmatrix}
\] (3.1)

where

(i) \( \varphi : DL \to DL \) is a vector bundle endomorphism,

(ii) \( J : \wedge^2 J^1L \to L \) is a 2-form with associated morphism \( J^2 : J^1L \to DL \), and

(iii) \( \omega : \wedge^2 DL \to L \) is a 2-form with associated morphism \( \omega^3 : DL \to J^1L \),

satisfying the relations:

\[
\varphi J^2 = J^2 \varphi^\dagger; \quad \varphi^2 = -id - J^2 \omega^3; \quad \text{and} \quad \omega^3 \varphi = \varphi^\dagger \omega^3.
\] (3.2)

Conversely, every triple \( (\varphi, J, \omega) \) as above determines a generalized almost contact structure via [3,11]. From the third equation in (3.2), putting \( \omega_\varphi(\Delta, \nabla) := \omega(\varphi \Delta, \nabla) \), we get a well defined Atiyah 2-form \( \omega_\varphi \). Following [13] we introduce the:

**Definition 3.3.** A generalized almost contact structure \( I \) on \( L \) is integrable if its Nijenhuis torsion \( N_I : \Gamma(DL) \times \Gamma(DL) \to \Gamma(DL) \), defined by \( N_I(\alpha, \beta) := [I\alpha, I\beta] - [\alpha, \beta] - I[I\alpha, \beta] - I[I\alpha, I\beta] \), vanishes identically. A generalized contact structure is an integrable generalized almost contact structure. A generalized contact bundle is a line bundle equipped with a generalized contact structure.
Now, a section $J \in \Gamma(\wedge^2(J^1L)^* \otimes L)$ defines both a skew-symmetric bracket $\{-, -\}_J$ on $\Gamma(L)$ and a skew-symmetric bracket $[-, -]_J$ on $\Gamma(J^1L)$ via
\[
\{\lambda, \mu\}_J := J(j^1\lambda, j^1\mu) \quad \text{and} \quad [\phi, \psi]_J := \mathcal{L}_{J\phi}\psi - \mathcal{L}_{J\psi}\phi - d_{DL}J(\phi, \psi).
\]
It is easy to see that $(L, \{\cdot, \cdot\}_J)$ is a Jacobi bundle (see, e.g., [17]) if and only if $(J^1L, [-, -]_J, \sigma J^2)$ is a Lie algebroid (see, e.g., [8, 16]), in this case we say that $J$ is a Jacobi structure on $L$.

**Proposition 3.4.** Let $\mathcal{I}$ be a generalized almost contact structure on $L$. It is integrable if and only if, for all $\sigma, \tau \in \Gamma(J^1L)$, $\Delta, \nabla, \Box \in \Gamma(DL)$,
\[
J^2[\phi, \psi]_J = [J^2\phi, J^2\psi]; \quad (3.3)
\]
\[
\varphi^\dagger[\phi, \psi]_J = \mathcal{L}_{J\phi}\varphi^\dagger\psi - \mathcal{L}_{J\psi}\varphi^\dagger\phi - d_{DL}J(\varphi\phi, \psi); \quad (3.4)
\]
\[
\mathcal{N}_{\varphi}(\Delta, \nabla) = J^2(i_\Delta i_\nabla d_{DL}\omega); \quad (3.5)
\]
and
\[
(d_{DL}\omega_{\varphi})(\Delta, \nabla, \Box) = (d_{DL}\omega)(\varphi\Delta, \nabla, \Box) + (d_{DL}\omega)(\Delta, \varphi\nabla, \Box) + (d_{DL}\omega)(\Delta, \nabla, \varphi\Box); \quad (3.6)
\]
where $\mathcal{N}_{\varphi}(\Delta, \nabla) := [\varphi\Delta, \varphi\nabla] + \varphi^2[\Delta, \nabla] - \varphi[\varphi\Delta, \nabla] - \varphi[\Delta, \varphi\nabla]$.

Equations (3.2) and (3.3)-(3.6) should be seen as structure equations of a generalized contact structure.

**Example 3.5.** Let $\mathcal{I}$ be a generalized almost contact structure on $L \to M$ given by (3.1). As for generalized complex structures there are two extreme cases. The first one is when $\varphi = 0$, hence $J^\dagger = (\omega^\dagger)^{-1}$, and $\mathcal{I}$ is completely determined by $\omega$ which is a non-degenerate Atiyah 2-form. Now, $\mathcal{I}$ is integrable if and only if $d_{DL}\omega = 0$, hence $\omega$ corresponds to a contact structure $H$ on $M$ such that $TM/H = L$. The second extreme case is when $J = \omega = 0$, hence $\varphi^2 = -\text{id}$, i.e., $\varphi$ is an almost complex structure on the Atiyah algebroid $DL$, and $\mathcal{I}$ is integrable if and only if $\varphi$ is a complex structure [3].

Equation (3.3) says that $J$ is a Jacobi structure. So every generalized contact bundle has an underlying Jacobi structure. Equation (3.4) describes a compatibility condition between $J$ and $\varphi$. Equation (3.5) measures the non-integrability of $\varphi$ while (3.6) is a compatibility condition between $\varphi$ and $\omega$.

**Remark 3.6.** Consider the manifold $\widetilde{M} := L^* \setminus 0$ ($0$ being the image of the zero section). Recall that $\widetilde{M}$ is a principal $\mathbb{R}^\times$-bundle (and every principal $\mathbb{R}^\times$-bundle arise in this way). In particular $\widetilde{M}$ comes equipped with an homogeneity structure $h : [0, \infty) \times M \to M$ in the sense of Grabowski (see, e.g., [2]). The fundamental vector field corresponding to the canonical generator $1$ in the Lie algebra $\mathbb{R}$ of $\mathbb{R}^\times$ will be denoted by $\mathcal{E}$.

**Proposition 3.7.** Generalized contact structures on $L$ are in one-to-one correspondence with homogeneous generalized complex structures on $\widetilde{M}$.
Let $\mathcal{J} : \mathbb{T}\tilde{M} \to \mathbb{T}\tilde{M}$ be a generalized complex structures. Using decomposition $\mathbb{T}\tilde{M} = \mathbb{T}\tilde{M} \oplus T^*\tilde{M}$ we see that $\mathcal{J}$ is the same as a triple $(a, \pi, \sigma)$ where $a$ is an endomorphism of $\mathbb{T}\tilde{M}$, $\pi$ is a bi-vector field, and $\sigma$ is a 2-form on $\tilde{M}$ satisfying suitable identities [7]. We say that $\mathcal{J}$ is homogeneous if 1) $a$ is homogeneous of degree 0, i.e. $L_Ea = 0$, 2) $\pi$ is homogeneous of degree $-1$, i.e. $L_E\pi = -\pi$, and 3) $\omega$ is homogeneous of degree 1, i.e. $L_E\sigma = \sigma$. Now, Proposition 3.7 is a straightforward consequence of [23, Theorem A.4]. There is an alternative elegant way of explaining the homogeneity of $\mathcal{J}$. Namely, homogeneity structure $h$ lifts to an homogeneity structure $h_T$ on the generalized tangent bundle, namely the direct sum of the tangent and the phase lifts of $h$ [2, Section 2]. It is easy to check that $\mathcal{J}$ is homogeneous in the above sense iff it is equivariant with respect to $h_T$ (see also [2, Theorem 2.3]).

It is useful to characterize those generalized contact structures such that $\mathcal{J}$ is non-degenerate. In this case there is a unique non-degenerate Atiyah 2-form $\omega_J$, also denoted $J^{-1}$, such that $J'\omega_J = \text{id}$ and (3.3) says that $\omega_J$ is $d_{DL}$-closed. Hence it comes from a contact structure $H_J \subset TM$ such that $TM/H_J = L$. Following Crainic [7] we introduce the following notion:

**Definition 3.8.** A contact-Hitchin pair on a line bundle $L \to M$ is a pair $(H, \Phi)$ consisting of a contact structure $H \subset TM$ with $TM/H = L$, and an endomorphism $\Phi : DL \to DL$ such that (i) $\Omega^\Phi = \Phi^\dagger \Omega^\phi$ (so that the Atiyah 2-form $\Omega_\Phi$ is well-defined), and (ii) $d_{DL} \Omega_\Phi = 0$, where $\Omega$ is the Atiyah 2-form corresponding to $H$, i.e. $\Omega := d_{DL}^* \theta^\phi$, and $\theta : TM \to L$ is the structure form of $H$, i.e. $H = \ker \theta$.

**Proposition 3.9.** There is a one-to-one correspondence between generalized contact structures on $L$ given by (3.7), with $J$ non-degenerate, and contact Hitchin pairs $(H, \Phi)$ on $L$. In this correspondence $H$ is the contact structure corresponding to $\omega_J = J^{-1}$, and moreover:

$\Phi = \varphi$ and $\omega = - (\omega_J + \varphi^* \omega_J)$, where $(\varphi^* \omega_J)(\Delta, \nabla) := \omega_J(\varphi \Delta, \varphi \nabla)$.

The proof of Proposition 3.9 is similar to that of Proposition 2.6 in [7].

4. **Multiplicative Atiyah forms on Lie groupoids and generalized contact structures**

As we have seen above, every generalized contact bundle $(L, \mathcal{I})$ has an underlying Jacobi structure $J$. Jacobi structures are the infinitesimal counterparts of multiplicative contact structures on Lie groupoids [8] (see also [14] for the case $L = M \times \mathbb{R}$). So, it is natural to ask: are the remaining components $\varphi, \omega$ of $\mathcal{I}$ also infinitesimal counterparts of suitable (multiplicative) structures on $\mathcal{G}$? Theorem 4.7 below answers this question (see [7] Theorems 3.2, 3.3, 3.4] for the generalized complex case). In order to state it, we need to introduce a novel notion of multiplicative Atiyah form.

Multiplicative forms and their infinitesimal counterparts are extensively studied in [4]. Vector-bundle valued differential forms and their infinitesimal counterparts, Spencer
operators, are studied in [9]. In what follows, we outline a similar theory for Atiyah forms.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we will denote the source by $s$, the target by $t$ and the unit by $u$. Moreover, we identify $M$ with its image under $u$. Denote by $\mathcal{G}_x = \{(g_1, g_2) \in \mathcal{G} \times \mathcal{G} : s(g_1) = t(g_2)\}$ the manifold of composable arrows and let $m : \mathcal{G}_2 \to \mathcal{G}$, $(g_1, g_2) \mapsto g_1g_2$ be the multiplication. We denote by $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}_2 \to \mathcal{G}$ the projections onto the first and second factor respectively.

Recall that the Lie algebroid $A$ of $\mathcal{G}$ consists of tangent vectors to the source fibers at points of $M$. Every section $\alpha$ of $A$ corresponds to a unique right invariant, $s$-vertical vector field $\alpha^r$ on $\mathcal{G}$ such that $\alpha = \alpha^r|_M$. Now let $E \to M$ be a vector bundle carrying a representation of $\mathcal{G}$. Thus, there is a flat $A$-connection $\nabla$ in $E$. As shown in [23, Proposition 10.1], there is a canonical flat $\ker ds$-connection $\nabla^s : \ker ds \to D(t^*E)$ in the pull-back bundle $t^*E$ such that $\nabla_\alpha = t_*(\nabla^{\alpha^r}|_M)$ for all $\alpha \in \Gamma(A)$. Additionally, there is a natural vector bundle isomorphism (covering the identity) $i : (t \circ \mathrm{pr}_1)^*E \to (t \circ \mathrm{pr}_2)^*E$ defined as follows. For $((g_1, g_2), e) \in (t \circ \mathrm{pr}_1)^*E$, $e \in E_{t(g_1)}$ put $i((g_1, g_2), e) := ((g_1, g_2), g_1^{-1} \cdot e) \in (t \circ \mathrm{pr}_2)^*E$. In the following $E = L$ is a line bundle. In particular, $(t \circ \mathrm{pr}_2)^*L$-valued Atiyah forms can be pulled-back to $(t \circ \mathrm{pr}_1)^*L$-valued Atiyah forms along $i$.

**Definition 4.1.** An Atiyah form $\omega \in \Omega^{\bullet}_L$ is **multiplicative** if $m^*\omega = \mathrm{pr}_1^*\omega + i^*\mathrm{pr}_2^*\omega$.

We also need the following:

**Definition 4.2.** An endomorphism $\Phi : D(t^*L) \to D(t^*L)$ is **multiplicative** if, for every $\square \in D(m^*t^*L)$, there is a, necessarily unique, $\Phi \square \in D(m^*t^*L)$ such that (1) $\mathrm{pr}_1^*\Phi \square = \Phi \mathrm{pr}_1^*\square$, (2) $\mathrm{pr}_2^*\Phi \square = \Phi \mathrm{pr}_2^*\square$, and (3) $m_*\Phi \square = \Phi m_*\square$.

The following definition and Theorem [4.7] provide the infinitesimal counterpart of multiplicative Atiyah forms. Let $L \to M$ be a line bundle carrying a representation of a Lie algebroid $A \to M$.

**Definition 4.3.** An $L$-valued infinitesimal multiplicative (IM) Atiyah k-form on $A$ is a pair $(\mathcal{D}, l)$, where $\mathcal{D} : \Gamma(A) \to \Omega^k_L$ is a first order differential operator, and $l : A \to \wedge^{k-1}(DL)^* \otimes L$ is a vector bundle morphism such that, for all $\alpha, \beta \in \Gamma(A)$, and $f \in C^\infty(M)$,

$$\mathcal{D}(f\alpha) = f\mathcal{D}(\alpha) + d_{DL}f \wedge l(\alpha),$$

and

$$\mathcal{L}_{\nabla_\alpha} \mathcal{D}(\beta) - \mathcal{L}_{\nabla_\beta} \mathcal{D}(\alpha) = \mathcal{D}([\alpha, \beta]);$$

$$\mathcal{L}_{\nabla_\alpha} l(\beta) - i_{\nabla_\beta} \mathcal{D}(\alpha) = l([\alpha, \beta]);$$

$$i_{\nabla_\alpha} l(\beta) + i_{\nabla_\beta} l(\alpha) = 0.$$

Now on, $L \to M$ is a line bundle carrying a representation of a source simply connected Lie groupoid $\mathcal{G} \rightrightarrows M$, and $A$ is the Lie algebroid of $\mathcal{G}$.
Theorem 4.4. There is a one-to-one correspondence between $t^*L$-valued multiplicative Atiyah $k$-forms $\omega$ and $L$-valued IM Atiyah $k$-forms $(\mathcal{D}, l)$ on $A$. In this correspondence

$$\mathcal{D}(\alpha) = u^*(\mathcal{L}_{\nabla^G} \omega) \quad \text{and} \quad l(\alpha) = u^*(i_{\nabla^G} \omega).$$

Proof. There is a direct sum decomposition $\Omega^*_{t^*L} = \Omega^*(\mathcal{G}, t^*L) \oplus \Omega^*(\mathcal{G}, t^*L)[1]$ given by $\omega \equiv (\omega_0, \omega_1)$, with $\omega = \sigma^* \omega_0 + d_{DL} \sigma^* \omega_1$, and $\omega$ is multiplicative if and only if $(\omega_0, \omega_1)$ are so. Using [9, Theorem 1], we see that $(\omega_0, \omega_1)$ correspond to Spencer operators on $\Lambda[9, \text{Definition 2.6}]$. Finally check that, similarly as for Atiyah forms, IM Atiyah forms decompose canonically into a direct sum of Spencer operators. □

Remark 4.5. Let $H \subset T\mathcal{G}$ be a multiplicative contact structure on $\mathcal{G}$, with $T\mathcal{G}/H = t^*L$ and let $\Omega$ be the corresponding Atiyah 2-form. When specialized to $\Omega$, Theorem 4.4 gives an isomorphism $l : A \to J^1L$ of $A$ with the Lie algebroid $(J^1L, [-, -], \sigma J^1\mathcal{L})$ corresponding to a unique Jacobi structure $J$ on $L$.

Definition 4.6. A contact-Hitchin groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with

1. a line bundle $L \to M$ carrying a representation of $\mathcal{G}$,
2. a multiplicative contact-Hitchin pair $(H, \Phi)$ on $t^*L$, i.e. both $H$ and $\Phi$ are multiplicative, and
3. an $L$-valued Atiyah 2-form $\omega$ on $M$ such that $\Omega + \Phi^* \Omega = s^* \omega - t^* \omega$,

where $\Omega$ is the Atiyah 2-form corresponding to $H$.

Theorem 4.7. There is a one-to-one correspondence between contact-Hitchin groupoid structures $(H, \Phi, \Omega)$ on $\mathcal{G}$ and triples $(J, \varphi, \omega)$ satisfying Equations (3.3)-(3.5), and the first two equations in (3.2). In this correspondence, $J$ is the Jacobi structure corresponding to $H$ (Remark 4.5), and $\varphi : DL \to DL$ is the (well-defined) restriction of $\Phi$ to $DL$.

Theorem 4.7 can be proved using arguments similar to those in Theorems 3.3 and 3.4 in [7]. Alternatively, one could use a conceptual approach similar to that of [21], exploiting the notion of Jacobi quasi-Nijenhuis structure [20, 5]. Finally, we observe that the last equation of (3.2) and Equation (3.6) do not have a Lie groupoid/Lie algebroid interpretation.

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