CM GALOIS GROUPS AND MUMFORD-TATE DOMAINS

RYAN KEAST

Abstract. Period domains are the classification spaces of polarized Hodge structures. The given polarization defines a $\mathbb{Q}$-algebraic group $G$ and Lie algebra $\mathfrak{g}$. Through the work of [GGK], to each irreducible CM Hodge structure in a period domain there is an associated CM field. These CM fields give an arithmetically defined basis of $\mathfrak{g}_\mathbb{C}$; their associated Galois groups act on this basis. A rational subalgebra is closed under the action of these groups. Using imprimitive systems, we show that closure under the Galois group has considerable implications. For instance, we generalize Ribet’s nondegeneracy theorem and relate the degree of nilpotent $N$ with the size of Mumford-Tate groups. The most striking observation is this: many irreducible CM Hodge structures are never contained in positive dimension horizontal Mumford-Tate domains nor in rational nilpotent orbits.

1. Introduction

A parameterizing space of polarized Hodge structures is called a period domain. Let $D$ be a period domain with fixed rational vector space $V$ and polarization $Q$. We construct the following algebraic group:

$$G = \{ g \in \text{Aut}(V) | Q(gu, gv) = Q(u, v) \forall u, v \in V \}. $$

We restrict ourselves to the case where the weight is odd so that we may identify $G$ with $\text{Sp}(2n, \mathbb{Q})$. $G_\mathbb{R}$ acts transitively on $D$ and we may identify $D = G_\mathbb{R}/H_\varphi$, where $H_\varphi$ is the subgroup that fixes some $\varphi \in D$. Similarly $G_\mathbb{C}$ acts transitively on the compact dual $\check{D}$. Consider $G(\mathbb{C})'$s associated Lie algebra:

(1.1) $\mathfrak{g}_\mathbb{C} = \{ X \in \text{End}(V_\mathbb{C}) | Q(Xu, v) + Q(u, Xv) = 0 \forall u, v \in V \}.$

Fix a point $\varphi \in D$. $\mathfrak{g}_\mathbb{C}$ inherits a weight 0 Hodge filtration from $\varphi$ and decomposes in the following way: $\mathfrak{g} = \oplus l \mathfrak{g}^{l,-l}$, where

(1.2) $\mathfrak{g}^{l,-l} = \{ X \in \mathfrak{g} | X(V^{p,q}) \subseteq V^{p+l, q-l} \}.$

A subvariety of $D$ will be called horizontal if its tangent bundle lives in the subbundle generated by the orbit of $\mathfrak{g}^{-1,1}$. From Griffiths’s transversality, a subvariety of $D$ that arises from a variation of polarized Hodge structures will always be horizontal.

Regarding a Hodge structure $\varphi \in D$ as a representation of $\mathbb{C}^*$, the first Hodge-Riemann relation indicates $\varphi(S^1) \subset G_\mathbb{R}$. The smallest $\mathbb{Q}$-algebraic group whose real points contain the image of the circle $\varphi(S^1)$ is called the Mumford-Tate group $M_\varphi$. Let $\mathfrak{m}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ be the associated Lie algebra of $M_\varphi(\mathbb{C})$. $\varphi'(1) \in \mathfrak{m}_\mathbb{C}$ is called the grading element. It has the property that given $x \in \mathfrak{g}^{l,-l}$

(1.3) $\frac{1}{2} \varphi'(1), x] = lx$
i.e., the eigenspaces of its adjoint action recovers the Hodge decomposition of $g_C$. The real orbit $M_\varphi(R)\varphi$ is called a Mumford-Tate domain. For a full exposition of the importance of Mumford-Tate domains, we direct the reader to [GGK].

In light of the work of [FL], there is significant interest in the case when the horizontal subvariety is also a Mumford-Tate domain:

**Theorem 1.** [FL] Let $Z$ be a closed horizontal subvariety of a period domain $D$ and $\Gamma \subseteq G(\mathbb{Z})$ be the stabilizer of $Z$. If $\Gamma\backslash Z$ is strongly quasi-projective and $Z$ is semi-algebraic, then $Z$ is a horizontal Mumford-Tate domain.

Since Mumford-Tate groups are $\mathbb{Q}$-algebraic groups, their Lie algebra is defined over $\mathbb{Q}$. If a Mumford-Tate group is to give rise to a horizontal sub-domain, the Lie algebra must live in $F^{-1}(g_C)$. To explore the existence of such horizontal Mumford-Tate domains passing through a given $\varphi \in D$, we define the following index:

**Definition 2.** Let $\varphi \in D$ be a polarized Hodge structure with the associated Hodge filtration on $g_C$. We define $r(\varphi)$ to be the smallest positive $k$ such that there exists a vector $v \in g_Q$ with $v \in F^{-k}(g_C)$ and $v \notin F^0(g_C)$.

Let $N \in g_Q$ be a nilpotent element coming from Schmid’s nilpotent orbit theorem[?]. The requirement that the orbit be horizontal indicates $r(\varphi) > 1$ would prohibit a rational nilpotent orbit from passing $\varphi$.

For example, Cattani and Kaplan [CK] classify all the real $SL_2$ of $D$ orbits by constructing a set of $\text{Ad}(H_\varphi)$ conjugacy classes of homorphisms from $sl_2$ to $g$. If $r(\varphi) > 1$, then these conjugacy classes will never permit $\mathbb{Q}$-representations.

Every Mumford-Tate domain passes through a CM point, a Hodge structure whose Mumford-Tate group is abelian. From the work of [GGK], to each CM point we assign a compisitium of CM fields. The CM points gives rise to an arithmetically defined basis of $g_C$, of which the Galois groups of these CM fields acts on. Using these Galois groups, [GGK] developed an algorithm for classifying all Mumford-Tate groups from a given period domain.

In [D], the Galois groups of CM fields are characterized as imprimitive permutation groups. The following observation can be viewed as a synthesis of the work in [GGK] and [D]: For an irreducible CM Hodge structure $\varphi \in D$ and its associated CM field $L$, each $v \in g_Q$ induces an imprimitive system on the set of embeddings of $L$. Using relatively basic algebraic techniques, we can reach some rather striking results concerning Mumford-Tate groups whose domains pass through irreducible CM points.

The first statement generalizes a known result concerning the dimension of the Mumford-Tate groups of certain CM points. While the original result is concerned with weight one Hodge structures, we extend it to all odd weights:

**Theorem 3.** Let $\varphi \in D$ be an irreducible polarized odd weight CM Hodge structure. If $\text{dim}(V) = 2p$ where $p > 2$ is prime, then $\text{dim}(M_\varphi) = p$.

The second main result concerns the degree of nilpotent $N \in m_C$ and the size of $m_C$:

**Theorem 4.** Let $\varphi \in D$ be CM and nondegenerate [nondegenerate implies irreducible]. Assume $M$ is a Mumford-Tate group whose Mumford-Tate domain passes through $M$. Let $N \in m_Q(\subseteq \text{End}(V))$ be nilpotent. If $N^{l-1} \neq 0$ with $l > \text{dim}(V)/2$ then $m_C = g_C$. 
The final main result concerns irreducible CM Hodge structures that lie somewhere between CY [first Hodge number is one] and principal [all Hodge numbers are one] Hodge structures.

**Theorem 5.** Let $D$ be a period domain of polarized odd weight $n$ Hodge structures on the rational vector space $V$ with $h^{n,0} = h^{n-1,1} = 1$ and $4 \nmid \dim(V)$. If $\varphi$ is irreducible and CM, then $r(\varphi) > 1$.

Since CM Hodge structures are always motivic, this means that there is a large class of motivic Hodge structures which are never contained in positive dimension horizontal Mumford-Tate domains or nilpotent orbits. This means that non-trivial variations of Hodge structure passing through such points can never satisfy the conditions given by the theorem of [FL].

### 2. Oriented CM Fields and CM Hodge Structures

**Definition 6.** A complex multiplication (CM) field $L$ is a totally imaginary number field that is a quadratic extension of a totally real field.

Let $L^c$ be the Galois closure of a CM field. A critical feature of CM field is that $Gal(L^c/\mathbb{Q})$ has an involution $\rho$ in its center that corresponds to complex conjugation. Since $L$ is totally imaginary, its embeddings will come in complex conjugate pairs: If $[L : \mathbb{Q}] = 2n$, $L$ will have $2n$ embeddings

$$\text{Hom}(L, \mathbb{C}) = \{\theta_1, ... \theta_n, \theta_{-n}, ... \theta_{-1}\} \theta_{-k} = \bar{\theta}_k.$$

Since $L^c$ is Galois, every embedding into $\mathbb{C}$ has the same image, so we fix an embedding $\Theta$ and we regard $L^c$ as a sub-field of $\mathbb{C}$. Via the action on the image of $\Theta(L^c)$, $Gal(\Theta(L^c)/\mathbb{Q})$ acts on the set $\{\theta_1, ... \theta_n, \theta_{-n}, ... \theta_{-1}\}$.

**Theorem 7.** Let $L$ be a CM field with the corresponding Galois closure $L^c$. The action of $Gal(L^c/\mathbb{Q})$ acting on $\{\theta_1, ... \theta_n, \theta_{-n}, ... \theta_{-1}\}$ gives it the structure of an imprimitive permutation group with $n$ imprimitive domains of size 2.

The imprimitive domains are of the form $\{\theta_k, \theta_{-k}\}$.

**Definition 8.** A CM Hodge structure is a Hodge structure $\varphi$ such that $M_{\varphi} \subseteq H_{\varphi}$.

**Proposition 9.** Equivalently a CM Hodge structure is a Hodge structure where $M_{\varphi}$ is abelian.

**Definition 10.** A CM field with a CM type is a choice from each conjugate pair, a single embedding.

As one might guess, CM types are intimately connected to weight one Hodge structure. To generalize to higher weight we need to add extra structure:

**Definition 11.** A weight $n$ oriented CM field is a pair

$$\left(L, \{\prod_{p+q=n} \theta\}_{p+q=n}\right)$$

where $L$ is a CM field and $\{\prod_{p+q=n} \theta\}_{p+q=n}$ is a partition of the set of embeddings of $L$ into $\mathbb{C}$ with the following property:

$$\theta \in \prod_{p+q=n} \Leftrightarrow \bar{\theta} \in \prod_{q-p}.$$
To obtain the Hodge structure, we view the CM field $L$ itself as a $\mathbb{Q}$ vector space. An element $l \in L$ acts on $L$ by left multiplication, the action decomposes $L \otimes \mathbb{C}$ into one dimensional $\theta_k(l)$ eigenspaces we label $E_{\theta_k}$. We gain a Hodge decomposition by setting $V^{p,q} = \oplus_{\theta_k \in \prod_{p,q}} E_{\theta_k}$.

**Theorem 12.** [GGK] Every Hodge structure constructed in the previous manner is a polarizable CM Hodge structure [not necessarily irreducible] and every odd weight irreducible polarized CM Hodge structure comes from such a construction.

An irreducible CM Hodge structure $\varphi \in D$ will then give an arithmetically defined basis of $V_{\mathbb{C}}$ that respects the Hodge structure.

**Definition 13.** The Hodge-Galois basis of $L \otimes \mathbb{C}$ is the basis $\{w_k\}$ with $w_k \in E_{\theta_k}$, scaled so that for $l \in L$, $l = \sum_{\theta_k(l)} w_k$ when considered as an element in $L \otimes \mathbb{C}$.

Via the action on the embeddings, $Gal(L^c/\mathbb{Q})$ acts on the Hodge-Galois basis. Since $L^c$ is Galois, every embedding has the same image. We fix some embedding $\Theta : L^c \to \mathbb{C}$. For $a \in \Theta(L^c)$, $a = \Theta(l)$ for some $l \in L^c$. We define for $g \in Gal(L^c/\mathbb{Q})$, $g(a) = \Theta(gl)$. For $a_k \in \Theta(L^c)$, we have

$$g \left( \sum a_k w \right) = \sum g(a_k)g(w_k).$$

By the construction of the Hodge-Galois basis, $\sum a_k w$ is in the original rational vector space $L$ if and only if it is fixed by $Gal(L^c/\mathbb{Q})$.

A polarization of a CM Hodge structure will be called a Hodge-Galois polarization if the following condition is met:

$$Q(w_l, \bar{w}_k) = 0 \text{ unless } l = k.$$

This condition makes it much easier to explicitly handle $g_{\mathbb{C}}$. Luckily, for irreducible CM Hodge structures this condition is trivial.

**Proposition 14.** [GGK] Every polarization of an irreducible CM Hodge structure is a Hodge-Galois polarization.

For odd weight period domains with polarization $Q$, $g_{\mathbb{C}} \cong \mathfrak{sp}(2n, \mathbb{Q})$. The Hodge-Galois polarization allows us to explicitly construct an arithmetically defined Hodge basis of $g_{\mathbb{C}}$.

Let $\tilde{w}_{nm}$ indicate the endomorphism that takes $w_m$ to $w_n$ and kill everything else. Denoting $Q_i = Q(w_i, \tilde{w}_i)$, we can form a Hodge basis of $g_{\mathbb{C}}$

$$\left\{ \tilde{w}_{i,j} + \frac{Q_i}{-Q_j} \tilde{w}_{-j,-i} \right\}.$$

If the $\alpha_{i,j}$’s are in $\Theta(L^c)$, $Gal(L/\mathbb{Q})$ also has natural action given by

$$g \left( \sum \alpha_{i,j} \left( \tilde{w}_{i,j} + \frac{Q_i}{-Q_j} \tilde{w}_{-j,-i} \right) \right) = \sum \left( g(\alpha_{i,j}) \tilde{w}_{g(i),g(j)} + \frac{Q_{g(i)}}{-Q_{g(j)}} \tilde{w}_{g(-j),g(-i)} \right).$$

Again we have $\sum \alpha_{i,j} \left( w_{i,j} + \frac{Q_i}{-Q_j} w_{-j,-i} \right)$ is in the original rational vector space if and only if it is fixed by $Gal(L^c/\mathbb{Q})$. 

3. Classification Theorem

**Proposition 15.** [GGK] Let $D$ be a period domain and $\varphi \in D$ a CM Hodge structure. If the Mumford-Tate domain of a Mumford-Tate group $M$ passes through $\varphi$, then the Lie algebra $m$ is a sub-Hodge structure of $(g_Q, \text{ad}\varphi)$ in which $m_C$ contains $\varphi'(1)$. The requirement that $m$ is a sub-Hodge structure means that it $m_C = \oplus \Lambda^i$ where $\Lambda^i$ is a subspace of $g_i - i$ defined over $L'$. Since $M$ is defined over $\mathbb{Q}$, if

$$\sum \alpha_{i,j} \left( \hat{w}_{i,j} + \frac{Q_i}{-Q_j} \hat{w}_{j,-i} \right) \in m_C$$

and the $\alpha_{ij}$ are in $\Theta(L')$, then for all $g \in \text{Gal}(L'/\mathbb{Q})$

$$g \left( \sum \alpha_{i,j} \left( \hat{w}_{i,j} + \frac{Q_i}{-Q_j} \hat{w}_{j,-i} \right) \right) \in m_C.$$

4. $g_Q$ and $F^{-1}(g_Q)$: Examples

We recall the index discussed in the introduction:

**Definition 16.** Let $\varphi \in D$ be a polarized Hodge structure with the associated Hodge filtration on $g_C$. We define $r(\varphi)$ to be the smallest positive $k$ such that there exists a vector $v \in g_Q$ with $v \in F^{-k}(g_C)$ and $v \notin F^0(g_C)$.

If $\varphi$ is irreducible and CM, the existence of $v \in g_Q$ as in the definition will depend on how the associated Galois group interacts with the Hodge filtration. We provide the following examples for two reasons. The first is to give concrete constructions of some Hodge bases discussed above. The other is to show some cases where the Galois group forces $r(\varphi) > 1$.

**Example 17.** Consider the oriented CM field

$$(\mathbb{Q}(\zeta_5)|\theta_1 \in \Pi^{3,0}, \theta_2 \in \Pi^{2,1}, \theta_{-2} \in \Pi^{1,2}, \theta_{-1} \in \Pi^{0,3})$$

where $\zeta_5$ is a primitive $5^{th}$ root of unity. We construct a polarized CM Hodge structure and fix a Hodge basis: $w_1 \in E_{\theta_1}$, $w_2 \in E_{\theta_2}$, $w_{-2} \in E_{\theta_{-2}}$, $w_{-1} \in E_{\theta_{-1}}$. Let $\hat{w}_{nm}$ indicate the endomorphism that takes $w_m$ to $w_n$ and kills everything else. Denoting $Q_i = Q(w_i, w_{-i})$ we construct the following Hodge basis of $g_C = \{X \in \text{End}(V_C)|Q(Xu, v) + Q(u, Xv) = 0 \forall u, v \in V\}$

$$g^{0,0} \left\{ \begin{array}{c} \hat{w}_{11} - \hat{w}_{-1-1} \\ \hat{w}_{22} - \hat{w}_{-2-2} \end{array} \right\}$$

$$g^{1,-1} \left\{ \begin{array}{c} \hat{w}_{12} + \frac{Q_1}{Q_2} \hat{w}_{-2-1} \\ \hat{w}_{-2} \end{array} \right\}$$

$$g^{-1,1} \left\{ \begin{array}{c} \hat{w}_{-1,-2} + \frac{Q_{-1}}{Q_2} \hat{w}_{2,1} \\ \hat{w}_{-2} \end{array} \right\}$$

$$g^{2,-2} \left\{ \begin{array}{c} \hat{w}_{12} + \frac{Q_1}{Q_2} \hat{w}_{-1} \\ \hat{w}_{-2} \end{array} \right\}$$

$$g^{-2,2} \left\{ \begin{array}{c} \hat{w}_{-1,2} + \frac{Q_{-1}}{Q_2} \hat{w}_{1} \\ \hat{w}_{-2} \end{array} \right\}$$

$$g^{3,-3} \left\{ \begin{array}{c} \hat{w}_{1,-1} \end{array} \right\}$$
\[ \mathfrak{g}^{-3,3} \left\{ \begin{array}{l}
\hat{w}_{-1,1} \\
\hat{w}_{-2,2} \\
\hat{w}_{11} - \hat{w}_{-1,1} \\
\hat{w}_{22} - \hat{w}_{-2,2} \\
\hat{w}_{12} + \frac{Q_1}{Q_2} \hat{w}_{-2,1} \\
\hat{w}_{1,2} + \frac{Q_1}{Q_2} \hat{w}_{-2,1} \\
\hat{w}_{-1,2} + \frac{Q_1}{Q_2} \hat{w}_{2,1} \\
\hat{w}_{1,1} - \hat{w}_{-2,2} \\
\hat{w}_{2,2} - \hat{w}_{-2,2} \\
\hat{w}_{3,3} - \hat{w}_{-3,3} \\
\hat{w}_{-1,2} + \frac{Q_5}{Q_3} \hat{w}_{21} \\
\hat{w}_{3,2} + \frac{Q_5}{Q_3} \hat{w}_{2,3} \\
\hat{w}_{3,3} - \hat{w}_{-3,3} \\
\hat{w}_{-1,3} + \frac{Q_5}{Q_3} \hat{w}_{31} \\
\hat{w}_{2,3} + \frac{Q_5}{Q_3} \hat{w}_{3,2} \\
\hat{w}_{2,2} - \hat{w}_{-2,2} \\
\hat{w}_{1,3} + \frac{Q_5}{Q_3} \hat{w}_{3,-1} \\
\hat{w}_{-2,3} + \frac{Q_5}{Q_3} \hat{w}_{3,-2} \\
\hat{w}_{1,3} + \frac{Q_5}{Q_3} \hat{w}_{3,-1} \\
\hat{w}_{1,3} + \frac{Q_5}{Q_3} \hat{w}_{3,-1} \\
\end{array} \right. \]

\[ \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) = \mathbb{Z}_4 \] and we write the generator as \((1, 2, -1, -2)\). We group the orbits together:

\[ \hat{w}_{11} - \hat{w}_{-1,1} \]
\[ \hat{w}_{22} - \hat{w}_{-2,2} \]
\[ \hat{w}_{12} + \frac{Q_1}{Q_2} \hat{w}_{-2,1} \]
\[ \hat{w}_{1,2} + \frac{Q_1}{Q_2} \hat{w}_{-2,1} \]
\[ \hat{w}_{-1,2} + \frac{Q_1}{Q_2} \hat{w}_{2,1} \]
\[ \hat{w}_{1,1} - \hat{w}_{-2,2} \]
\[ \hat{w}_{2,2} - \hat{w}_{-2,2} \]
\[ \hat{w}_{3,3} - \hat{w}_{-3,3} \]
\[ \hat{w}_{-1,2} + \frac{Q_5}{Q_3} \hat{w}_{21} \]
\[ \hat{w}_{3,2} + \frac{Q_5}{Q_3} \hat{w}_{2,3} \]
\[ \hat{w}_{3,3} - \hat{w}_{-3,3} \]
\[ \hat{w}_{-1,3} + \frac{Q_5}{Q_3} \hat{w}_{31} \]
\[ \hat{w}_{2,3} + \frac{Q_5}{Q_3} \hat{w}_{3,2} \]
\[ \hat{w}_{2,2} - \hat{w}_{-2,2} \]
\[ \hat{w}_{1,3} + \frac{Q_5}{Q_3} \hat{w}_{3,-1} \]

The thing to note is that each vector that lives in \(\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}\) gets taken outside of \(\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{-1,1}\) by the Galois group action. It follows that the \(r\) index of this CM Hodge structure is 2.

**Example 18.** We proceed in a nearly identical fashion except we have the CM field \(\mathbb{Q}(\zeta_7)\) and Hodge numbers \((1, 1, 1, 1, 1)\)

\[ (\mathbb{Q}(\zeta_7)|\theta_1 \in \Pi^{5,0} \theta_2 \in \Pi^{4,1} \theta_3 \in \Pi^{2,3} \theta_{-3} \in \Pi^{3,2} \theta_{-2} \in \Pi^{1,4} \theta_{-1} \in \Pi^{0,5} \) \]
We write the generator as \((1, 2, 3, -1, -2, -3)\). We group the orbits and label them by the smallest \(k\) such that \(F^{-k}\) that contains them:

\[
\begin{align*}
\mathcal{g}^{-3,3} & \left\{ \begin{array}{l}
\hat{w}_{-2,2} \\
\hat{w}_{-1,3} + \frac{Q_5}{Q_6} \hat{w}_{-3,1}
\end{array} \right. \\
\mathcal{g}^{4,-4} & \left\{ \begin{array}{l}
\hat{w}_{1,-2} + \frac{Q_1}{Q_2} \hat{w}_{2,-1}
\end{array} \right. \\
\mathcal{g}^{-4,4} & \left\{ \begin{array}{l}
\hat{w}_{-1,2} + \frac{Q_1}{Q_2} \hat{w}_{2,1}
\end{array} \right. \\
\mathcal{g}^{5,-5} & \left\{ \begin{array}{l}
\hat{w}_{1,-1}
\end{array} \right. \\
\mathcal{g}^{-5,5} & \left\{ \begin{array}{l}
\hat{w}_{1,-1}
\end{array} \right.
\end{align*}
\]

We conclude that \(r\) index is 3.

Using similar ideas as the examples above, we now prove results in the more general setting. To avoid some rather unenlightening complications and to make use of the previous examples, we shall restrict to the case where the CM Hodge structures in question are odd weight and irreducible.
5. NONDEGENERACY

Definition 19. Let \( \varphi \in D \) be a polarized CM Hodge structure. We say \( \varphi \) is non-degenerate if \( \dim(M_\varphi) = \dim_Q(V)/2 \)

The Ribet’s nondegeneracy theorem (c.f.\([D]\) ) says that if the weight is one and \( \dim(V) = 2p \) with \( p > 2 \) and prime, then being irreducible implies imprimitivity.

Theorem 20. [The Generalized Ribet Non-degeneration Theorem] Let \((V, Q, \varphi)\) be an irreducible polarized odd weight CM Hodge structure. If \( \dim(V) = 2p \) where \( p > 2 \) is prime, then \( \dim(M_\varphi) = p \).

Proof. Let \( F \) be the associated CM field, and let \( F^c \) be its Galois closure. Since \( p \) is prime and divides \( |Gal(F^c)| \), basic group theory informs us that \( Gal(F^c/Q) \) contains the cyclic subgroup \( \mathbb{Z}_p \). We write the generator of \( \mathbb{Z}_p \) as \((0, 1, 2, \ldots, p-1)\).

We express the grading element as a sum of the Hodge-Galois basis of \( \mathbb{Z} \) numbers, \( f \) where \( \varphi(1) \) must satisfy the condition in \([3]\) \( [\frac{1}{2} \varphi(1), w_{m,-m}] = A_m w_{m,-m} \). With the weight being odd, we are guaranteed that all the \( A_m \) are odd integers.

Since the \( m \) must be defined over \( Q \), it must contain \( g(\varphi(1)) \) for all \( g \in Gal(F^c) \). Since it contains the cyclic subgroup \( \mathbb{Z}_p \), the dimension of \( m \) must be at least the size of the rank of the following circulant matrix:

\[
\begin{pmatrix}
A_0 & \cdots & \cdots & 0 \\
A_1 & A_2 & \cdots & A_{p-1} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & A_{p-1}
\end{pmatrix}
\]

As a general fact of circulant matrices\([4]\), the rank is given \( p - \deg[gcd(f, x^p - 1)] \) where \( f = A_0 + A_1 x + \cdots + A_{p-1} x^{p-1} \). Since all of the coefficients of \( f \) are odd numbers, 1 is not a root of \( f \). \( x^p - 1 \) is the minimal polynomial of the \( p^{th} \) roots of unity. If it shared a root with \( f \) it would have to be a rational multiple of \( x^{p-1} + x^{p-2} + \ldots + 1 \). So either the rank is \( p \) or \( A_0 = A_1 = \ldots = A_{p-1} \). The latter case is then covered by the original Ribet Nondegeneracy theorem (c.f.\([D]\) ). \( \square \)

6. IMPRIMITIVE DOMAINS

Let \( \varphi \in D \) be irreducible and CM with associated CM field \( L \). Let \( v \in g_Q \). Consider \( v \) as a matrix defined by the ordered Hodge-Galois basis \((w_1, \ldots, w_n, w_{-1}, \ldots, w_{-n})\) of \( V_C \). We construct a graph in the following manner: the vertices are defined to be the elements of the Hodge-Galois basis. Two vertices are connected by an edge if either entry \( a_{i,j} \) or \( a_{j,i} \) in \( M \) are non-zero. [Note: Because it is fixed under complex conjugation and must respect the Hodge-Galois polarization, \( a_{i,j} \) and \( a_{j,i} \) are either both non-zero or both zero.] The connected components of this graph gives us a partition of the Hodge-Galois basis.

Definition 21. For \( v \in g_Q \) \( \Gamma_v \) and \( \pi_v \) are respectively the graph and partition constructed above. Using the same set of vertices, for a Mumford-Tate group \( M \), we define the following graph \( \Gamma_M \): Two vertices in \( \Gamma_M \) are connected if and only if there is a \( v \in m_Q \) where the vertices are connected in \( \Gamma_v \). In an identical manner we derive a partition \( \pi_M \).
Theorem 22. The partitions $\pi_v$ and $\pi_M$ both give an imprimitive system for the $\text{Gal}(L^c/Q)$ action on the Hodge basis $(w_1, \ldots, w_n, w_{-1}, \ldots, w_{-n})$.

Proof. An alteration of the partition would contradict the fact that for all $v \in g\mathbb{Q}$, $v$ is fixed by $g \in \text{Gal}(L^c/Q)$. \hfill \square

Corollary 23. Let $N \in m_{\mathbb{Q}}$ be nilpotent, if $N^{l-1} \neq 0$ and $l > [L:Q]/2$ then the partition $\pi_N$ is trivial.

Proof. The degree of $N$ is bounded by the largest size of a connected component $\Gamma_N$. Since the group action is transitive, each primitive domain must have the same number of elements. If a domain of imprimitivity contains more than half of the elements, it must be the whole set. \hfill \square

Lemma 24. Assume $\varphi \in D$ is CM and nondegenerate and odd weight. Let $M$ be a Mumford-Tate group whose Mumford-Tate domain passes through $\varphi$. Let $v \in \mathfrak{m}_C$. We express $v$ using the Hodge-Galois basis of $g\mathbb{C}$: $v = \sum \alpha_{i,j} \left( \hat{w}_{i,j} + \frac{Q_i}{Q_j} \hat{w}_{-j,-i} \right)$. For each $\alpha_{i,j} \neq 0$, $\left( \hat{w}_{i,j} + \frac{Q_i}{Q_j} \hat{w}_{-j,-i} \right) \in \mathfrak{m}_C$.

Proof. Since $\varphi$ is CM, $M_{\mathbb{Z}}$ is abelian. Because of nondegeneracy, $\mathfrak{m}_{-\mathbb{C}}$ is maximal and abelian, hence is a Cartan subalgebra of $\mathfrak{sp}(2n,Q)$. From [CGK] we know that for $i \neq j \left( \hat{w}_{i,j} + \frac{Q_i}{Q_j} \hat{w}_{-j,-i} \right)$ is a root vector. For $i = j$, $\left( \hat{w}_{i,i} - \hat{w}_{-i,-i} \right)$ is in the Cartan. Under closure of the Lie bracket, the adjoint action of the Cartan means that for each $\alpha_{i,j} \neq 0$, $\left( \hat{w}_{i,j} + \frac{Q_i}{Q_j} \hat{w}_{-j,-i} \right) \in \mathfrak{m}_C$. \hfill \square

Lemma 25. Assume $\varphi \in D$ is CM, nondegenerate, and odd weight. Let $M$ be a Mumford-Tate group whose Mumford-Tate domain passes through $\varphi$. Let $v \in \mathfrak{m}_{\mathbb{Q}}$. If $\pi_v$ is trivial, then $\Gamma_M$ contains the complete graph and $\mathfrak{m} = \mathfrak{g}$.

Proof. $\Gamma_v$ has a single connected component. By construction, $\Gamma_M$ has a single connected component. By Lemma 24 if the vertices $w_i$ and $w_j$ are connected in $\Gamma_v$, then

$$\hat{w}_{i,j} + \frac{Q_i}{Q_j} \hat{w}_{-j,-i} \in \mathfrak{m}_C.$$

Assume vertices $l$ and $m$ are both connected to vertex $k$. By closure under Lie bracket we have

$$\left( \hat{w}_{l,k} + \frac{Q_i}{Q_k} \hat{w}_{-k,-i} \right) \left( \hat{w}_{k,m} + \frac{Q_i}{Q_j} \hat{w}_{-m,-i} \right) = \left( \hat{w}_{l,m} + \frac{Q_i}{Q_m} \hat{w}_{-m,-i} \right) \in \mathfrak{m}_C.$$

It follows that there exists $v' \in \mathfrak{m}_{\mathbb{Q}}$ where $w_l$ and $w_m$ are connected in $\Gamma_{\varphi}$ and hence $\Gamma_M$. Proceeding inductively, we conclude from $\Gamma_M$ having a single connected component that it contains the complete graph. Invoking 24 again, we have that the entire Hodge basis of $g\mathbb{C}$ is contained in $\mathfrak{m}_C$. \hfill \square

Theorem 26. Assume $\varphi \in D$ be CM and nondegenerate (nondegenerate implies irreducible). Assume $M$ is a Mumford-Tate group whose Mumford-Tate domain passes through $\varphi$. Let $N \in m_{\mathbb{Q}}$ be nilpotent. If $N^{l-1} \neq 0$ with $l > \dim(V)/2$ then $\mathfrak{m}_C = \mathfrak{g}_C$.

Proof. Follows from Lemma 25 and Corollary 23. \hfill \square

Theorem 27. Let $D$ be a period domain with $h^{n,0} = h^{n-1,1} = 1$ and $4 \nmid (\dim_{\mathbb{Q}} V)$. If $\varphi$ is irreducible and CM then $r(\varphi) > 1$. 

Proof. Assume \( r(\varphi) = 1 \). We set the Hodge-Galois basis as \( \{ w_1, \ldots, w_n, w_{-1}, \ldots, w_{-n} \} \) with \( w_1 \in H^{n,0} \) and \( w_2 \in H^{n-1,1} \). Since \( r(\varphi) = 1 \), there exists a \( v \in (\mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1}) \cap \mathfrak{g}_Q \) and \( v \notin F^0(\mathfrak{g}_C) \). We write \( v \) in the Hodge-Galois basis of \( \mathfrak{g}_C \): 

\[
    v = \sum \alpha_{i,j} \left( \hat{w}_{i,j} + \frac{Q_{i,j}}{Q_{-j,-1}} \hat{w}_{-j,-1} \right)
\]

Since \( h^{n,0} = h^{n-1,1} = 1 \) we have the following fact:

\[
    (6.1) \quad \hat{w}_{1,m} + \frac{Q_{1}}{-Q_{m}} \hat{w}_{-m,-1} \in \mathfrak{g}^{1,-1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{-1,1} \implies m = 2 \text{ or } m = 1.
\]

This implies that for \( \Gamma_v \), vertex \( w_1 \) can only be connected to \( w_2 \) or itself. Since \( v \notin F^0(\mathfrak{g}_C) \), \( w_1 \) and \( w_2 \) must be connected.

Assume \( w_2 \) is connected to \( w_n \) with \( n \neq 1 \) nor \( 2 \). Let \( g \in \text{Gal}(L^c/Q) \) be the element that takes \( w_n \) to \( w_1 \). By \( 6.1 \) it follows that \( g \) takes \( w_2 \) to \( w_2 \). The same reasoning leads us to the contradiction that \( g \) takes \( w_1 \) to \( w_1 \). It follows that that \( w_1 \) and \( w_2 \) are not connected to any other vertices.

Since the two vertices form a single connected component, \( \{ w_1, w_2 \} \) is an imprimitive domain. By Dodson's imprimitivity theorem, \( \{ w_1, w_2, \bar{w}_1, \bar{w}_2 \} \) also generates an imprimitive partition, implying that \( 4 \mid (\dim_Q L) \), which is a contradiction. \( \square \)

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