Jucys-Murphy Elements and Unitary Matrix Integrals

SHO MATSUMOTO
Graduate School of Mathematics, Nagoya University.
Furocho, Chikusa-ku, Nagoya, 464-8602, JAPAN.
sho-matsumoto@math.nagoya-u.ac.jp

and

JONATHAN NOVAK
Department of Mathematics, Massachusetts Institute of Technology
Cambridge MA 02139, USA.
jnovak@math.mit.edu

Abstract

In this paper, we study the relationship between polynomial integrals on the unitary group and the conjugacy class expansion of symmetric functions in Jucys-Murphy elements. Our main result is an explicit formula for the top coefficients in the class expansion of monomial symmetric functions in Jucys-Murphy elements, from which we recover the first order asymptotics of polynomial integrals over $U(N)$ as $N \to \infty$. Our results on class expansion include an analogue of Macdonald’s result for the top connection coefficients of the class algebra, a generalization of Stanley and Olshanski’s result on the polynomiality of content statistics on Plancherel-random partitions, and an exact formula for the multiplicity of the class of full cycles in the expansion of a complete symmetric function in Jucys-Murphy elements. The latter leads to a new combinatorial interpretation of the Carlitz-Riordan central factorial numbers.

1 Introduction

1.1 Polynomial integrals on the unitary group

Let $U(N)$ denote the group of $N \times N$ unitary matrices, equipped with the normalized Haar measure, and let $A \subset L^2(U(N), \text{Borel, Haar})$ denote the algebra of polynomial functions on $U(N)$. Thus $A$ is the set of functions $f : U(N) \to \mathbb{C}$ for which there exists a polynomial $p_f$ in $N^2$ variables with $f(U) = p_f(u_{11}, \ldots, u_{NN})$ for all $U = (u_{ij}) \in U(N)$. How can one compute inner products in $A$?

In addition to being a natural question concerning one of the classical groups, this problem is of importance in the theory of random matrices and various related branches of mathematics and mathematical physics, see e.g. [1, 2, 11 13, 15, 21, 39, 41, 47]. By linearity of the integral, the problem reduces to the computation of monomial inner products.
\[
\langle u_{i(1)}j(1) \cdots u_{i(m)}j(m), u'_{i'(1)}j'(1) \cdots u'_{i'(n)}j'(n) \rangle_N
\]

where \( i, j : [m] \to [N] \) and \( i', j' : [n] \to [N] \) are given functions. Using the invariance of the Haar measure, one may show that this inner product vanishes unless \( m = n \). Thus \( \mathcal{A} \) admits the orthogonal decomposition

\[
\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}^{(n)},
\]

where \( \mathcal{A}^{(n)} \) is the space of homogeneous polynomial functions of degree \( n \). This orthogonality is equivalent to the statement that the the space \( V^\otimes m \otimes V^* \otimes^n \), where \( V \) is the defining representation of \( \text{U}(N) \) and \( V^* \) is the dual representation, admits a non-trivial space of invariants if and only if \( m = n \). When \( m = n \leq N \), the space of \( \text{U}(N) \)-invariants in the representation \( V^\otimes m \otimes V^* \otimes^n \) has a basis indexed by the elements of the symmetric group \( S(n) \) \cite{18}. This fact can be used \cite{14, 15} to derive the formula

\[
(1.1) \quad \langle u_{i(1)}j(1) \cdots u_{i(n)}j(n), u'_{i(1)}j'(1) \cdots u'_{i(n)}j'(n) \rangle_N = \sum_{\pi \in S(n)} \left( \sum_{\sigma \rho^{-1} = \pi} \delta_{i,i'} \delta_{j,j'} \langle u_{11} \cdots u_{nn}, u_{1\pi(1)} \cdots u_{n\pi(n)} \rangle_N \right),
\]

valid for any \( n \leq N \), which further reduces the problem of evaluating inner products in \( \mathcal{A} \) to the evaluation of certain canonical inner products labelled by permutations.

The formula (1.1) is reminiscent of the Wick formula \cite{22}, which reduces the computation of mixed moments of Gaussian random variables to the computation of covariances modulo the application of a combinatorial rule. However, the Wick-type rule (1.1) does not reduce the problem to the computation of covariances, but only to the computation of the basic inner products \( \langle u_{i(1)}j(1) \cdots u_{i(n)}j(n), u_{i(1)}j(1) \cdots u_{i(n)}j(n) \rangle_N \) indexed by permutations. These basic inner products are certain rational functions of \( N \) of which depend on the cycle structure of \( \pi \) in a rather complicated way. Nevertheless, it is known that the behaviour of these averages simplifies dramatically in the limit \( N \to \infty \). Given a permutation \( \pi \in S(n) \), recall that its reduced cycle type is the partition \( \mu \) whose parts \( \mu_1 \geq \mu_2 \geq \ldots \) are the lengths of the cycles of \( \pi \), each reduced by one in order to remove fixed points \cite{23}. For example, the reduced cycle type of the permutation \( \pi = (1 2 3 4)(5 6)(7) \in S(8) \) is \( \mu = (3, 1) \). Let \( C_\mu(n) \) denote the conjugacy class of permutations in \( S(n) \) of reduced cycle type \( \mu \). Then, for any \( \pi \in C_\mu(n) \),

\[
(1.2) \quad (-1)^{|\mu| - |\mu|} \langle u_{11} \cdots u_{nn}, u_{1\pi(1)} \cdots u_{n\pi(n)} \rangle_N = \prod_{i=1}^{\ell(\mu)} \text{Cat}_{\mu_i} + O\left( \frac{1}{N^2} \right)
\]

as \( N \to \infty \), where

\[
\text{Cat}_r = \frac{1}{r + 1} \binom{2r}{r}
\]
are the Catalan numbers. This estimate has been known to physicists for some time, see e.g. [1] for an application to quantum transport in mesoscopic systems.

The first proof of (1.2) is due to Collins [4], and is quite involved. The appearance of the Catalan numbers, which are ubiquitous in enumerative combinatorics [44, Exercise 6.19], motivates the search for combinatorial structure underlying the averages \( \langle u_{11} \ldots u_{nn}, u_{1\pi(1)} \ldots u_{n\pi(n)} \rangle_N \) which might lead to a conceptually simple proof of (1.2). Along these lines, it was observed in [33] that the evaluation of the basic inner products is intimately related to a certain “inverse problem” in the character theory of the symmetric groups. This connection is explored in the present article. Working on the symmetric group side, we obtain a number of results concerning this “inverse problem” which give a new perspective on the structure of polynomial integrals on the unitary group (leading in particular to a new proof of (1.2)) and are moreover of significant interest in their own right.

1.2 The class expansion problem

Consider the group algebra \( \mathbb{C}[S(n)] \) of the symmetric group, and let \( Z(n) \) be its center. Associate to the conjugacy class \( C_\mu(n) \) its indicator function, i.e. the group algebra element

\[
c_\mu(n) := \sum_{\pi \in C_\mu(n)} \pi.
\]

The set \( \{c_\mu(n) : \text{wt}(\mu) \leq n\} \), where \( \text{wt}(\mu) := |\mu| + \ell(\mu) \), is a natural basis of \( Z(n) \), which for this reason is often referred to as the class algebra. The weight statistic \( \text{wt}(\mu) \) is simply the size of the support of any \( \pi \in C_\mu(n) \), i.e. the number of points on which \( \pi \) acts non-trivially, while \( |\mu| \) is the minimal number of transpositions required to generate \( \pi \).

Recall that the equivalence classes of complex, irreducible representations \( V^\lambda \) of \( \mathbb{C}[S(n)] \) are indexed by partitions \( \lambda \) of size \( n \), see e.g. [36]. By Schur’s Lemma, the central element \( c_\mu(n) \) acts as a scalar operator in any irreducible representation \( V^\lambda \),

\[
c_\mu(n) \cdot v = \omega_\mu(\lambda)v
\]

for all \( v \in V^\lambda \). The scalar \( \omega_\mu(\lambda) \) is called the central character of \( c_\mu(n) \) acting in \( V^\lambda \).

A fundamental problem in the representation theory of the symmetric groups is to describe the central character \( \omega_\mu(\lambda) \), viewed as a function of \( \lambda \) with \( \mu \) fixed, in terms of natural statistics on \( \lambda \). For example, a well-known formula going back to Frobenius asserts that the central character of the conjugacy class \( c_{(1)}(n) \) of transpositions acting in \( V^\lambda \) is simply

\[
(1.3) \quad \omega_{(1)}(\lambda) = \sum_{\Box \in \lambda} c(\Box),
\]

the sum of the contents of the Young diagram \( \lambda \). Analogous formulas are known for more complicated partitions \( \mu \), but their structure is very complex, see [26] and references therein.

The “inverse problem” referred to above is the following: construct a group algebra element in \( Z(n) \) whose central character in \( V^\lambda \) admits a simple description. One such construction is the
Okounkov-Pandharipande theory of completed cycles \cite{35}, which exhibits remarkable connections with Gromow-Witten theory. We now describe an older and closely related construction, originally due to Jucys \cite{23}, which turns out to be intimately related to the computation of the basic inner products \(\langle u_{11} \ldots u_{nn}, u_{1\pi(1)} \ldots u_{n\pi(n)} \rangle_N\) in the algebra \(\mathcal{A}\).

The Jucys-Murphy elements \(J_2, \ldots, J_n \in \mathbb{C}[S(n)]\) are the transposition sums

\[
J_2 = (1 \, 2) \\
J_3 = (1 \, 3) + (2 \, 3) \\
J_4 = (1 \, 4) + (2 \, 4) + (3 \, 4) \\
\vdots \\
J_n = (1 \, n) + (2 \, n) + \ldots + (n-1 \, n).
\]

These elements commute, and in fact \(\mathbb{C}[J_2, \ldots, J_n]\) is a maximal commutative subalgebra of \(\mathbb{C}[S(n)]\) known as the Gelfand-Zetlin algebra, see \cite{36}. Although the Jucys-Murphy elements are clearly non-central, Jucys observed that symmetric functions of them are central. More precisely, let \(\Lambda\) denote the \(\mathbb{C}\)-algebra of symmetric functions \cite{28} and consider the multiset (or “alphabet”) \(\Xi_n = \{\{J_2, \ldots, J_n, 0, 0, \ldots\}\}\). Jucys proved that

\[
f(\Xi_n) = f(J_2, \ldots, J_n, 0, 0, \ldots) \in \mathcal{Z}(n)
\]

for any symmetric function \(f\). We thus have a homomorphism of commutative \(\mathbb{C}\)-algebras \(\Lambda \to \mathcal{Z}(n)\) defined by \(f \mapsto f(\Xi_n)\), which will be called the Jucys-Murphy specialization.

Jucys proved the validity of (1.4) by showing directly that the image of the elementary symmetric function \(e_k\) under the JM specialization is

\[
e_k(\Xi_n) = \sum_{|\mu|=k} \epsilon_\mu(n),
\]

which may be visualized as the sum of all permutations on the \(k\)th level of the Cayley graph of \(S(n)\), generated by the set of all transpositions. For a proof of this fact, see \cite{7, 23} or Proposition 2.1 below. Since \(\Lambda = \mathbb{C}[e_1, e_2, \ldots]\) is the algebra of polynomials in the elementary symmetric functions (the "fundamental theorem" of symmetric function theory), (1.5) implies (1.4). Moreover, it is a classical result of Farahat and Higman \cite{11} that the class sums (1.5) generate the algebra \(\mathcal{Z}(n)\), so that the JM specialization is in fact a surjection onto the class algebra.

Jucys’ second remarkable discovery concerning the eponymous elements \(\Xi_n\) is that the central character of \(f(\Xi_n)\) acting in \(V^\lambda\) is given by the simple substitution rule \(f(\Xi_n) \mapsto f(A_\lambda)\), where \(A_\lambda = \{\epsilon(\square) : \square \in \lambda\}\) is the content alphabet of \(\lambda\). That is, we have the action

\[
f(\Xi_n) \cdot v = f(A_\lambda)v
\]
for all \( f \in \Lambda, v \in V^\lambda \). A proof of this property may be found in [34]. Thus the JM specialization furnishes group algebra elements whose central character in a given representation takes an especially simple form, namely a symmetric polynomial function of the contents of the partition indexing the representation. As an example, one can conclude that the central character of the Farahat-Higman generator (1.5) acting in \( V^\lambda \) is simply

\[
e_k(A_\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} c(\Box_{i_1}) c(\Box_{i_2}) \cdots c(\Box_{i_k}),
\]

where \( \Box_1, \Box_2, \ldots, \Box_n \) is an arbitrary enumeration of the cells of the Young diagram \( \lambda \). When \( k = 1 \), this reduces to Frobenius’ formula (1.3). More generally, given a symmetric function \( f \in \Lambda \), if one can explicitly resolve the central element \( f(\Xi_n) \) into a linear sum of conjugacy classes \( c_\mu(n) \), then one will have explicitly constructed an element of the group algebra whose central character in the irreducible representation \( V^\lambda \) is \( f(A_\lambda) \). This is the class expansion problem: given \( f \in \Lambda \), explicitly determine its coordinates

\[
f(\Xi_n) = \sum_\mu G_\mu(f, n) c_\mu(n)
\]

relative to the conjugacy class basis.

The class expansion problem has been completely solved for two polynomial bases of \( \Lambda \), namely the elementary symmetric functions \( \{e_k\} \) and the power-sum symmetric functions \( \{p_k\} \) [25]. Ideally, one would like to obtain the solution for a linear basis of \( \Lambda \). In this article, we focus on the class expansion of the monomial symmetric functions \( \{m_\lambda\} \), which constitute the most natural linear basis of \( \Lambda \). This choice of basis is further motivated by the surprising fact that its solution would enable one to integrate arbitrary polynomial functions on \( U(N) \); as observed in [33], we have

\[
\langle u_{11} \ldots u_{nn}, u_{11} \pi(1) \ldots u_{nn} \pi(n) \rangle_N = \frac{1}{N^n} \sum_{k=0}^{\infty} (-1)^k G_\mu(h_k, n) N^k
\]

for any \( \pi \in C_\mu(n) \) and \( N \geq n \), where \( h_k \) is the complete homogeneous symmetric function of degree \( k \). Heuristically, this formula can be thought of as follows. Since the columns of a unitary matrix are unit vectors, one expects that the entries of a Haar-distributed random unitary matrix should be of order \( 1/N \), and indeed it is true that \( \langle u_{ij}, u_{ij} \rangle_N = 1/N \). Furthermore, for any fixed \( n \), any \( n \times n \) submatrix of an \( N \times N \) random Haar-distributed unitary matrix converges to a matrix of independent random variables as \( N \to \infty \) (see [32] [38] for a precise statement), so the inner product \( \langle u_{11} \ldots u_{nn}, u_{11} \ldots u_{nn} \rangle \) should be of order \( N^{-n} \). The (convergent) power series in (1.7) gives the corrections to this estimate for \( N \geq n \) finite and \( \pi \in S(n) \) different from the identity.

1.3 Main results and organization

In Section 2 we present our main result: an explicit formula for the class coefficients \( G_\mu(m_\lambda, n) \) in the “top” case \( |\mu| = |\lambda| \). This formula involves certain refinements of the Catalan numbers
originally considered by Haiman [17]. Via the fundamental relation (1.7), the main formula yields a conceptually simple derivation of the first-order asymptotics (1.2) as a corollary.

In Section 3, we obtain general properties of the class coefficients $G_\mu(f,n)$ analogous to the properties of the connection coefficients governing multiplication of conjugacy classes in the algebra $\mathcal{Z}(n)$. We prove that the class coefficients $G_\mu(f,n)$ are polynomial functions of $n$, and as an application of this result we generalize a theorem of Stanley [45] and Olshanski [37] on the polynomiality of content statistics on Plancherel-random partitions. We furthermore use Lagrange inversion to give an analogue of Macdonald’s construction [15, 28] for top connection coefficients in the setting of top class coefficients.

In Section 4, we use the character theory of the symmetric groups to obtain an exact formula for the multiplicity of the class of full cycles in the conjugacy class expansion of a complete homogeneous symmetric function in Jucys-Murphy elements. Quite surprisingly, the formula obtained involves the central factorial numbers introduced by Carlitz and Riordan [3] in their study of the divided difference operator. One consequence of this result is an elegant new combinatorial interpretation of the central factorial numbers as counting “primitive” factorizations of a full cycle into transpositions. Another consequence is an exact formula for cyclic inner products in $\mathcal{A}$ previously stated by Collins [4].

Finally, in the Appendix we give explicit examples of the class expansion of monomial symmetric functions as well as tables of top class coefficients.

1.4 Acknowledgements

Both authors would like to thank Benoît Collins for helpful and inspiring conversations. Parts of this paper were written while the second author was visiting Ecole Normale Supérieure de Lyon, and J. N. would like to thank Alice Guionnet for the opportunity to visit.

An earlier version of this paper led to further recent research into the class expansion problem and its relationship with group integrals, see [10, 27, 29, 48]. We would like to acknowledge helpful conversations and correspondence with Valentin Féray and Michel Lassalle regarding their works [10, 27].

2 Top class coefficients of $m_\lambda(\Xi_n)$

The monomial symmetric functions are perhaps the most natural class of symmetric functions; they are constructed by taking an arbitrary monomial as a seed, and growing a power series from this monomial by symmetrizing it. More precisely, let $x_1, x_2, x_3, \ldots$ be formal variables, and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots)$ be a partition. By definition, the monomial symmetric function of type $\lambda$ in the variables $x_i$ is the formal power series

$$m_\lambda(x_1, x_2, x_3, \ldots) = \sum x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots,$$

where the summation is over all distinct permutations $(a_1, a_2, a_3, \ldots)$ of the vector $(\lambda_1, \lambda_2, \lambda_3, \ldots)$. Since the latter vector has only $\ell(\lambda)$ non-zero entries, this definition yields a well-defined formal power series each of whose terms is a monomial of degree $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \ldots$. If we set
The monomial symmetric function specializes to a symmetric polynomial $m_\lambda(x_1, x_2, \ldots, x_n)$ which is homogenous of degree $n$ provided $\ell(\lambda) \leq n$, but is identically zero if $\ell(\lambda) > n$. Many other basic families of symmetric functions may be obtained from the monomial symmetric functions, for example the elementary symmetric functions $e_k = m_{(1^k)}$, the Newton power-sums $p_k = m_{(k)}$, and the complete symmetric functions $h_k = \sum_{\lambda \vdash k} m_\lambda$. General references on symmetric function theory are [28] and [44, Chapter 7].

In this section, we study the image of the monomial symmetric functions in the JM specialization. Let $L_\mu^\lambda(n) := G_\mu(m_\lambda, n)$, so that

$$m_\lambda(\Xi_n) = \sum_\mu L_\mu^\lambda(n) c_\mu(n).$$

Since the alphabet $\Xi_n$ contains $n - 1$ non-zero elements, we may restrict our study to partitions $\lambda$ with $\ell(\lambda) \leq n - 1$. From the definition of $m_\lambda$ and the JM elements, one finds that the class coefficient $L_\mu^\lambda(n)$ has the following combinatorial interpretation: it is equal to the number of factorizations of a fixed representative $\pi$ of the conjugacy class $C_\mu(n)$ into $|\lambda|$ transpositions of the form

$$\pi = (\ast 2) \cdots (\ast 2) (\ast 3) \cdots (\ast 3) \cdots (\ast n) \cdots (\ast n),$$

where $(a_1, a_2, \ldots, a_{n-1})$ is a permutation of $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$. Since the minimal number of transpositions required to generate a permutation of reduced cycle type $\mu$ is $|\mu|$, it is clear from this combinatorial interpretation that $L_\mu^\lambda(n)$ vanishes unless $|\mu| \leq |\lambda|$. Furthermore, in order for $L_\mu^\lambda(n)$ to be non-zero, we must have $|\mu| + 2g = |\lambda|$ for some integer $g \geq 0$; this is simply the fact that any permutation is either even or odd (Corollary 2.8 below). In this section we explicitly evaluate the class coefficient $L_\mu^\lambda(n)$ in the “top” case $|\mu| = |\lambda|$ by counting all factorizations (2.1) with the minimal number of factors.

2.1 Jucys’ theorem

Before treating the class expansion problem for arbitrary monomial symmetric functions, let us revisit the case $m_{(1^k)} = e_k$ considered by Jucys.

Proposition 2.1 ([23]). For any $k \geq 0$,

$$e_k(\Xi_n) = \sum_{|\mu| = k} c_\mu(n).$$

Proof. Consider first the case $k \geq n$. Then it is clear that $e_k(\Xi_n) = 0$, since each term in the sum

$$e_k(\Xi_n) = \sum_{2 \leq t_1 \leq \cdots \leq t_k} J_{t_1} \cdots J_{t_k}$$

is identical to the empty product 1.
defining \( e_k(\Xi_n) \) is a product of \( k \) distinct factors and the alphabet \( \Xi_n \) only contains \( n-1 \) non-zero elements. Moreover, if \(|\mu| \geq n\) then \( \text{wt}(\mu) \geq n+1 \), and hence the conjugacy class \( C_\mu(n) \) is empty and \( e_\mu(n) = 0 \), whence the sum on the right hand side of the desired equality is 0. Thus the claim holds for \( k \geq n \).

Now suppose \( 0 \leq k \leq n-1 \). Then

\[
\sum_{|\mu|=k} e_\mu(n) = \sum_{\sigma \in S(n)} \sigma, \\
\]

where \( \#(\sigma) \) denotes the number of cycles of \( \sigma \in S(n) \). To prove that the claim holds in the range \( 0 \leq k \leq n-1 \) we proceed by induction on \( n \).

If \( n = 2 \), then \( e_1(\Xi_2) = J_2 = (1, 2) \) and the claim is trivial.

Let \( n > 2 \) and suppose the claim holds true for \( e_k(\Xi_{n-1}) \) with any \( k \). We define the projection \( P_n \) from \( S(n) \) to \( S(n-1) \) by

\[
P_n(\sigma)(i) = \begin{cases} 
\sigma(i) & \text{if } \sigma(i) \neq n \\
\sigma(n) & \text{if } \sigma(i) = n,
\end{cases}
\]

for \( \sigma \in S(n) \) and \( 1 \leq i \leq n-1 \). In other words, \( P_n(\sigma) \) is defined to be the permutation whose cycle decomposition is obtained by erasing the letter \( n \) in the cycle decomposition of \( \sigma \). For each \( \tau \in S(n-1) \), we have \( P_n^{-1}(\tau) = \{\tau(s, n) \mid 1 \leq s \leq n-1\} \cup \{\tau \cdot (n)\} \). Here \( \tau \cdot (n) \) is an image under the natural injection \( S(n-1) \hookrightarrow S(n) \). Observe \( \#(\tau(s, n)) = \#(\tau) \) and \( \#(\tau \cdot (n)) = \#(\tau) + 1 \). Thus, the sum over permutations with exactly \( n-k \) cycles equals

\[
\sum_{\tau \in S(n-1)} \sum_{\sigma \in P_n^{-1}(\tau)} \sigma = \sum_{\tau \in S(n-1)} \tau \cdot (n) + \sum_{\tau \in S(n-1)} \sum_{s=1}^{n-1} \tau(s, n).
\]

By the induction hypothesis, the first sum on the right hand side equals \( e_k(J_1, \ldots, J_{n-1}) \) Since \( e_k(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_{n-1}) + e_{k-1}(x_1, \ldots, x_{n-1})x_n \), we obtain the desired equality for \( n \).

\[\square\]

### 2.2 Explicit formula for top class coefficients

Let \( \text{Cat}_r = \frac{1}{r+1} \binom{2r}{r} \) be the \( r \)th Catalan number:

| \( r \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| \( \text{Cat}_r \) | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 |

It is well known that Catalan numbers satisfy the recurrence

\[
\text{Cat}_r = \sum_{q=0}^{r-1} \text{Cat}_q \text{Cat}_{r-1-q}.
\]

(2.2)
The Catalan numbers admit a vast array of combinatorial interpretations, see [44, Exercise 6.19]. We will use the following interpretation of the Catalan numbers. For a positive integer \( k \), let \( E(k) \) be the set of all weakly increasing sequences \((i_1, \ldots, i_k)\) of \( k \) positive integers satisfying \( i_p \geq p \) for \( 1 \leq p \leq k - 1 \) and \( i_k = k \). For example,

\[
E(3) = \{(123), (133), (223), (233), (333)\}.
\]

Then, as proved below, the cardinality of \( E(k) \) is \( \text{Cat}_k \).

Let \((i_1, \ldots, i_k)\) be a weakly increasing sequence of \( k \) positive integers. We say that \((i_1, \ldots, i_k)\) is of type \( \lambda \vdash k \) if \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a permutation of \((b_1, b_2, \ldots)\), where, for each \( p \geq 1 \), \( b_p \) is the multiplicity of \( p \) in \((i_1, \ldots, i_k)\).

Example 2.1. The sequences \((1233)\), \((1334)\), and \((1134)\) are of type \((2, 1, 1)\), while the sequences \((4447799)\), \((555669999)\) are of type \((4, 3, 2)\).

Definition 2.1. Given a partition \( \lambda \vdash k \), the refined Catalan number \( \text{RC}(\lambda) \) counts sequences \((i_1, \ldots, i_k)\) in \( E(k) \) of type \( \lambda \). If \( \lambda \) is the empty partition, set \( \text{RC}(\lambda) = 1 \).

Example 2.2. The four sequences \((1444)\), \((2444)\), \((3444)\), \((3334)\) in \( E(4) \) are all of type \((3, 1)\), and indeed \( \text{RC}(3, 1) = 4 \). We have \( \text{RC}(k) = \text{RC}(1^k) = 1 \).

Proposition 2.2. The sum of \( \text{RC}(\lambda) \) over \( \lambda \vdash k \) equals \( \text{Cat}_k \):

\[
\sum_{\lambda \vdash k} \text{RC}(\lambda) = \text{Cat}_k.
\]

Proof. This is a direct consequence of the fact that \( |E(k)| = \text{Cat}_k \).

Example 2.3. We give some examples of \( \text{RC}(\lambda) \) for small \( |\lambda| \). “SUM” stands for the sum \( \sum_{\lambda \vdash k} \text{RC}(\lambda) = \text{Cat}_k \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\lambda & 1 & 2^1 & \text{SUM} & 3 \ 2^1 \ i^1 \ \text{SUM} \\
\hline
\text{RC}(\lambda) & 1 & 1 & 2 & 1 \ 1 \ 1 \ 5 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\lambda & 4 & 31 & 2^2 \ 1^4 & \text{SUM} \\
\hline
\text{RC}(\lambda) & 1 \ 4 \ 2 \ 6 \ 1 \ 14 \\
\hline
\end{array}
\]

An explicit formula for \( \text{RC}(\lambda) \) is known and given as follows. See [43] and also [17].

Proposition 2.3 ([43]). For any partition \( \lambda \),

\[
\text{RC}(\lambda) = \frac{|\lambda|!}{(|\lambda| - \ell(\lambda) + 1)! \prod_{i \geq 1} m_i(\lambda)!} = \frac{1}{|\lambda| + 1} m_\lambda(1^{|\lambda|+1}).
\]

Here \( m_i(\lambda) \) is the multiplicity of \( i \) in \( \lambda = (\lambda_1, \lambda_2, \ldots) \).
Note that $RC(a^m) = \frac{1}{(a-1)m+1} \binom{am}{m}$ is often called a higher Catalan number or sometimes a Fuss-Catalan number. In particular, $RC(2^m) = \text{Cat}_m$.

**Definition 2.2.** Given two partitions $\lambda, \mu \vdash k$, let $\mathbb{R}(\lambda, \mu)$ denote the set of sequences of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell(\mu)))}$ such that $\lambda^{(i)} \vdash \mu_i$ for $1 \leq i \leq \ell(\mu)$ and $\lambda^{(1)} \cup \lambda^{(2)} \cup \cdots \cup \lambda^{(\ell(\mu)))} = \lambda$. Here $\lambda^{(1)} \cup \lambda^{(2)} \cup \cdots \cup \lambda^{(\ell(\mu)))}$ is the partition obtained by rearranging the juxtaposed sequence of parts of the partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell(\mu)))}$ in weakly decreasing order. If $\mathbb{R}(\lambda, \mu) \neq \emptyset$, we say that $\lambda$ is a refinement of $\mu$.

Informally, $\lambda$ is a refinement of $\mu$ if it is obtained by “splitting” parts of $\mu$ into smaller pieces. The following assertions are immediate:

- $\mathbb{R}(\lambda, (k))$ consists of one element $(\lambda)$;
- $\mathbb{R}(\lambda, (1^k))$ consists of one element $((1), (1), \ldots, (1))$ if $\lambda = (1^k)$, or is empty otherwise;
- $\mathbb{R}((k), \mu)$ consists of one element $((k))$ if $\mu = (k)$, and is empty otherwise;
- $\mathbb{R}((1^k), \mu)$ consists of one element $((1^\mu_1), (1^\mu_2), \ldots, (1^\ell(\mu)))$;
- Suppose $\ell(\lambda) = \ell(\mu)$. Then $\mathbb{R}(\lambda, \mu)$ consists of one element $((\lambda_1), (\lambda_2), \ldots, (\lambda_{\ell(\lambda)}))$ if $\lambda = \mu$, and is empty otherwise.
- $\mathbb{R}(\lambda, \mu) = \emptyset$ unless $\lambda \leq \mu$. Here $\leq$ stands for the dominance partial ordering: $\lambda \leq \mu \iff \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$. (This is may be found in [28, I (6.10)].)

**Example 2.4.** The set $\mathbb{R}((3, 2, 2, 1), (5, 3))$ consists of two elements given by $((3, 2), (2, 1))$ and $((2, 2, 1), (3))$.

We are now ready to state our formula for top class coefficients.

**Theorem 2.4.** Let $\mu, \lambda$ be partitions, $|\mu| = |\lambda|$. Then the top class coefficient $L_\mu^\lambda(n) = G_\mu(m\lambda, n)$ is given by

\[
L_\mu^\lambda(n) = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell(\mu)))}) \in \mathbb{R}(\lambda, \mu)} \text{RC}(\lambda^{(1)}) \text{RC}(\lambda^{(2)}) \cdots \text{RC}(\lambda^{(\ell(\mu)))}.
\]

In particular, $L_\mu^\lambda = \delta_{\lambda, \mu}$ is zero unless $\lambda$ is a refinement of $\mu$.

Observe that for $\lambda, \mu \vdash k$,

\[
L_{\lambda(k)} = \text{RC}(\lambda), \quad L_{(k)}^{(\lambda)} = \delta_{\lambda, (k)}, \quad L_\mu^{(k)} = \delta_{\mu, (k)}, \quad L_\mu^{(1^k)} = 1, \quad L_\lambda = 1.
\]

The equality $L_\mu^{(1^k)} = 1$ is compatible with Proposition 2.1.

Since $L_\mu^{\lambda} = 0$ unless $\lambda$ is a refinement of $\mu$, the matrix $(L_\mu^{\lambda})_{\lambda, \mu \vdash k}$ is strictly lower unitriangular in the sense of [28, I-6].
The proof of Theorem 2.4 in the next subsections. The numbers \( L^\lambda \) for \( \lambda, \mu \vdash k \) for \( k \leq 7 \) are tabulated in the Appendix. Before moving on to the proof, let us state two consequences of this theorem.

Define \( F^k_k(\mu)(n) := G_\mu(h_k, n) \), where \( h_k = \sum_{\lambda \vdash k} m_\lambda \) is the complete symmetric function of degree \( k \). Thus

\[
h_k(\Xi_n) = \sum_\mu F^k_\mu(n)c_\mu(n).
\]

**Corollary 2.5.** We have

\[
F^{\vert \mu \vert}_\mu = \prod_{i \geq 1} \text{Cat}_{\mu_i}.
\]

**Proof.** For \( k = \vert \mu \vert \), we have

\[
F^k_\mu = \sum_{\lambda \vdash k} L^\lambda = \sum_{\lambda \vdash k} \sum_{(\lambda^{(1)}, \lambda^{(2)}, \ldots) \in \mathbb{R}(\lambda, \mu)} \text{RC}(\lambda^{(1)}) \text{RC}(\lambda^{(2)}) \cdots \text{RC}(\lambda^{(\ell(\mu))})
\]

by Theorem 2.4. By the definition of \( \mathbb{R}(\lambda, \mu) \), we see that

\[
\bigcup_{\lambda \vdash k} \mathbb{R}(\lambda, \mu) = \{(\lambda^{(1)}, \lambda^{(2)}, \ldots) \mid \lambda^{(i)} \vdash \mu_i (i \geq 1)\},
\]

so that, by Proposition 2.2,

\[
F^k_\mu = \prod_{i \geq 1} \sum_{\lambda^{(i)} \vdash \mu_i} \text{RC}(\lambda^{(i)}) = \prod_{i \geq 1} \text{Cat}_{\mu_i}.
\]

**Remark:** For the double covering \( \tilde{S}(n) \) of the symmetric group, a result similar to Theorem 2.5 was obtained by Tysse and Wang [46]. They deal with \( e_k(M_1^2, \ldots, M_n^2) \), where the \( M_i \) are elements of the spin group algebra of \( \tilde{S}(n) \) called odd Jucys-Murphy elements.

Corollary 2.5 was first obtained by Murray [31, Corollary 6.4] in the framework of the Farahat-Higman algebra, and independently rediscovered by the second author [33] via Collins’ work [4] on unitary matrix integrals. The proof given here is different from either of these, and is completely combinatorial. In fact, when combined with the \( 1/N \) expansion (1.7) obtained in [33], it yields an elementary and transparent proof of the first order asymptotics (1.2) of the basic inner products \( \langle u_{11}, \ldots, u_{nn}, u_{\pi(1)} \ldots u_{\pi(n)} \rangle_N \).

**Corollary 2.6.** Let \( N \geq n \) be positive integers. For any \( \pi \in C_\mu(n) \), we have

\[
(-1)^{|\mu|} N^{n+|\mu|} \langle u_{11}, \ldots, u_{nn}, u_{\pi(1)} \ldots u_{\pi(n)} \rangle = \prod_{i=1}^{\ell(\mu)} \text{Cat}_{\mu_i} + O\left(\frac{1}{N^2}\right)
\]

as \( N \to \infty \).
Proof. The fundamental relation (1.7) between the basic inner products and the class expansion problem reads

\[ \langle u_{11} \ldots u_{nn}, u_{1\pi(1)} \ldots u_{n\pi(n)} \rangle_N = \frac{1}{N^n} \sum_{k=0}^{\infty} (-1)^k \frac{F_k^\mu(N)}{N^k}. \]

Since \( F^\mu_k(n) \) is non-zero only if \( k = |\mu| + 2g \) for some integer \( g \geq 0 \), this becomes

\[
(-1)^{|\mu|} N^{|\mu|} + |\mu| \langle u_{11} \ldots u_{nn}, u_{1\pi(1)} \ldots u_{n\pi(n)} \rangle_N = \sum_{g=0}^{\infty} F^{|\mu|+2g}_k(n) \frac{N^{2g}}{N^{2g}}
= F^{|\mu|}_k + \sum_{g=1}^{\infty} F^{|\mu|+2g}_k(n) \frac{N^{2g}}{N^{2g}}
= F^{|\mu|}_k + \text{const}(n, \mu) \frac{N^{2g}}{N^2},
\]

where \( \text{const}(n, \mu) \) is a number depending only on \( n \) and \( \mu \). The result now follows from Corollary 2.5.

\[ \square \]

2.3 Proof of Theorem 2.4

In this subsection we give the proof of Theorem 2.4.

2.3.1 Basic lemmas

Define the support of a permutation \( \sigma \) to be the number of points on which it acts non-trivially:

\[ \text{supp}(\sigma) = \{ i \mid \sigma(i) \neq i \}. \]

If the reduced cycle-type of \( \sigma \) is \( \mu \), then \( |\text{supp}(\sigma)| = \text{wt}(\mu) \).

Lemma 2.7. Given a permutation \( \pi \) and a transposition \( (s t) \), let \( \Pi = \pi(s t) \). Suppose that \( \Lambda = (\Lambda_1, \Lambda_2, \ldots) \) and \( \lambda = (\lambda_1, \lambda_2, \ldots) \) are the reduced cycle-types of \( \Pi \) and \( \pi \), respectively. Then we have \(|\Lambda| = |\lambda| \pm 1\). Furthermore, if \(|\Lambda| = |\lambda| + 1\), then \( \text{supp}(\Pi) = \text{supp}(\pi) \cup \{s, t\} \), and \( s, t \) belong to the same cycle of \( \Pi \).

Proof. Given a permutation \( \pi \) and a transposition \( (s t) \), the following four cases may occur: (i) \(|\text{supp}(\pi) \cap \{s, t\}| = 0\); (ii) \(|\text{supp}(\pi) \cap \{s, t\}| = 1\); (iii) \( s, t \in \text{supp}(\pi) \), and \( s, t \) belong to different cycles of \( \pi \); (iv) \( s, t \in \text{supp}(\pi) \), and \( s, t \) belong to the same cycle of \( \pi \).

For the case (i), we obtain \( \Lambda = \lambda \cup (1) \) immediately. In the case (ii), we may suppose \( \text{supp}(\pi) \cap \{s, t\} = \{s\} \). Then \( \pi \) has a cycle \((\ldots, s, \pi(s), \ldots)\), and \( \Pi \) has the cycle \((\ldots, s, t, \pi(s), \ldots)\). Therefore \( \Lambda \) has a part equal to \( \lambda_j + 1 \). In the case (iii), \( \pi \) has two cycles of the forms \((\ldots, \pi^{-1}(s), s, \pi(s), \ldots)\) and \((\ldots, \pi^{-1}(t), t, \pi(t), \ldots)\). Therefore \( \Pi \) has the
combined cycle \((\ldots, \pi^{-1}(s), s, \pi(t), \ldots, \pi^{-1}(t), t, \pi(s), \ldots)\). Thus, a certain part \(\Lambda_k\) of \(\Lambda\) equals \(\lambda_i + \lambda_j + 1\) for some \(1 \leq i < j \leq \ell(\lambda)\). In the case (iv), \(\pi\) has a cycle of the form 
\[(\ldots, \pi^{-1}(s), s, \pi(s), \ldots, \pi^{-1}(t), t, \pi(t), \ldots),\]
and so \(\Pi\) has divided cycles \((\ldots, \pi^{-1}(s), s, \pi(t), \ldots)\) and \((\pi(s) \ldots \pi^{-1}(t) \; t)\). Thus, there are \(\Lambda_j\) and \(\Lambda_k\) equal to \(r - 1\) and \(\lambda_i - r\) for some \(\lambda_i\) and \(r \geq 1\).

For the case (iv), \(\Lambda\) and \(\lambda\) satisfy the identity \(|\Lambda| = |\lambda| - 1\). For other cases (i),(ii), and (iii), we have \(|\Lambda| = |\lambda| + 1\). The rest of the claims are seen above.

**Corollary 2.8.** Let \(\sigma\) be a permutation of reduced cycle-type \(\lambda\). Suppose that \(\sigma\) factors as \((s_1, t_1) \cdots (s_p, t_p)\), where \(s_i < t_i\) \((1 \leq i \leq p)\). Then \(|\lambda| \leq p\) and \(|\lambda| \equiv p \pmod{2}\).

If \(\sigma\) is a permutation of reduced cycle-type \(\lambda \vdash r\), and if \(\sigma\) may be factored into \(r\) transpositions
\[(2.3) \quad \sigma = (s_1, t_1) \cdots (s_r, t_r),\]
then we say that \((2.3)\) is a minimal factorization of \(\sigma\).

**Lemma 2.9.** Let \(\lambda \vdash r\) and let \(\sigma\) be a permutation of reduced cycle-type \(\lambda\). Suppose that \(\sigma\) factors as \((s_1, t_1)(s_2, t_2) \cdots (s_r, t_r)\), where \(s_i < t_i\) \((1 \leq i \leq r)\) and \(2 \leq t_1 \leq \cdots \leq t_r\). Then \(\supp(\sigma) = \{s_1, t_1, s_2, t_2, \ldots, s_r, t_r\}\). Furthermore, for each \(i\), the letters \(s_i, t_i\) belong to the same cycle of \(\sigma\).

**Proof.** For each \(1 \leq i \leq r\), define \(\sigma_i = (s_1, t_1) \cdots (s_i, t_i)\). It follows by Lemma 2.7 that the size of the reduced cycle-type of \(\sigma_i\) must be \(i\), and that \(\supp(\sigma_i) = \supp(\sigma_{i-1}) \cup \{s_i, t_i\}\). In addition, \(s_i, t_i\) belong to the same cycle of \(\sigma_i\), and therefore to the one of \(\sigma\).

**Lemma 2.10.** Let \(\tau^{(1)}\) and \(\tau^{(2)}\) be permutations such that \(i < j\) for all \(i \in \supp(\tau^{(1)})\) and \(j \in \supp(\tau^{(2)})\). Suppose that the reduced cycle-types of \(\tau^{(1)}\) and \(\tau^{(2)}\) have weights \(r_1\) and \(r_2\), respectively. Also, suppose that \(\sigma := \tau^{(1)} \tau^{(2)}\) may be expressed as \(\sigma = (s_1, t_1) \cdots (s_r, t_r)\), where \(r = r_1 + r_2\), \(s_i < t_i\) \((1 \leq i \leq r)\), and \(2 \leq t_1 \leq \cdots \leq t_r\).

Then,
\[
\tau^{(1)} = (s_1, t_1) \cdots (s_r, t_r), \quad \tau^{(2)} = (s_{r_1+1}, t_{r_1+1}) \cdots (s_r, t_r).
\]

**Proof.** By Lemma 2.9 we see \(\supp(\tau^{(1)}) \cup \supp(\tau^{(2)}) = \supp(\sigma) = \{s_1, t_1, \ldots, s_r, t_r\}\). Since \(t_i\) are not decreasing, there exists an integer \(p\) such that \(t_1, \ldots, t_p \in \supp(\tau^{(1)})\) and \(t_{p+1}, \ldots, t_r \in \supp(\tau^{(2)})\). Furthermore, applying Lemma 2.9 again, we see that \(s_i, t_i\) belong to the same cycle of \(\sigma\), and so that \(\supp(\tau^{(1)}) = \{s_1, t_1, \ldots, s_p, t_p\}\) and \(\supp(\tau^{(2)}) = \{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\}\). In particular, for any \(i \in \{s_1, t_1, \ldots, s_p, t_p\}\) and \(j \in \{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\}\), we have \(\tau^{(1)}(i) = \sigma(i)\) and \(\tau^{(2)}(j) = \sigma(j)\).

Let \(\rho^{(1)} = (s_1, t_1) \cdots (s_p, t_p)\) and \(\rho^{(2)} = (s_{p+1}, t_{p+1}) \cdots (s_r, t_r)\). Since \(\sigma = \rho^{(1)} \rho^{(2)}\) we have \(\{s_1, t_1, \ldots, s_p, t_p\} = \supp(\rho^{(1)})\) and \(\{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\} = \supp(\rho^{(2)})\). Therefore for any \(i \in \{s_1, t_1, \ldots, s_p, t_p\}\) and \(j \in \{s_{p+1}, t_{p+1}, \ldots, s_r, t_r\}\), we have \(\rho^{(1)}(i) = \sigma(i)\) and \(\rho^{(2)}(j) = \sigma(j)\). This means \(\tau^{(1)} = \rho^{(1)}\) and \(\tau^{(2)} = \rho^{(2)}\). In particular, the sizes of the reduced cycle-type of \(\rho^{(1)}\) and \(\rho^{(2)}\) are \(r_1\) and \(r_2\), respectively. By definition of \(\rho^{(i)}\) and Corollary 2.8 we have \(r_1 \leq p\) and \(r_2 \leq r - p\). But \(r = r_1 + r_2\) so that \(p = r_1\). Therefore \(\tau^{(1)} = \rho^{(1)} = (s_1, t_1) \cdots (s_{r_1}, t_{r_1})\). The desired expression for \(\tau^{(2)}\) also follows.
2.3.2 Expression for cycles

Let \( a, r \) be non-negative integers. Define the set \( \mathbb{E}(a; r) \) by
\[
\mathbb{E}(a; r) = \{(i_1, \ldots, i_r) \in \mathbb{Z}^r \mid i_1 \leq \cdots \leq i_r, \quad i_p \geq a + p (1 \leq p \leq r - 1), \quad i_r = a + r\}
\]
for \( r \geq 1 \) and let \( \mathbb{E}(a; 0) = \emptyset \). This extends the above definition of \( \mathbb{E}(r) = \mathbb{E}(0; r) \), and the mapping \((i_1, \ldots, i_r) \mapsto (a + i_1, \ldots, a + i_r)\) gives a bijection from \( \mathbb{E}(r) \) to \( \mathbb{E}(a; r) \). Put
\[
\mathbb{E}_0(a; r) = \{(i_1, \ldots, i_r) \in \mathbb{E}(a; r) \mid i_p > a + p (1 \leq p \leq r - 1)\},
\]
\[
\mathbb{E}_1(a; r) = \{(i_1, \ldots, i_r) \in \mathbb{E}(a; r) \mid i_1 = a + 1, \quad i_p > a + p (2 \leq p \leq r - 1)\},
\]
\[
\vdots
\]
\[
\mathbb{E}_{q-1}(a; r) = \{(i_1, \ldots, i_r) \in \mathbb{E}(a; r) \mid i_{q-1} = a + r - 1\}.
\]

Then we obtain the decomposition \( \mathbb{E}(a; r) = \bigcup_{q=0}^{r-1} \mathbb{E}_q(a; r) \). For each \((i_1, \ldots, i_r) \in \mathbb{E}_q(a; r)\) with \(0 \leq q \leq r - 2\), we have \(i_{r-1} = a + r\). Therefore, for each \(0 \leq q \leq r - 1\), the mapping
\[
(i_1, \ldots, i_q, i_{q+1}, \ldots, i_r) \mapsto ((i_1, \ldots, i_q), (i_{q+1}, \ldots, i_{r-1}))
\]
gives a bijection from \( \mathbb{E}_q(a; r) \) to \( \mathbb{E}(a; q) \times \mathbb{E}(a + q + 1; r - 1 - q) \). Here when either \( q = 0 \) or \( q = r - 1 \), we regard the set \( \mathbb{E}(a; q) \times \mathbb{E}(a + q + 1; r - 1 - q) \) as \( \mathbb{E}(a + 1; r - 1) \) or \( \mathbb{E}(a; r - 1) \), respectively. Thus, we obtain a natural identification
\[
\mathbb{E}(a; r) = \mathbb{E}_0(a; r) \sqcup \left( \bigcup_{q=1}^{r-2} \mathbb{E}_q(a; r) \right) \sqcup \mathbb{E}_{r-1}(a; r)
\]
\[
\cong \mathbb{E}(a + 1; r - 1) \sqcup \left( \bigcup_{q=1}^{r-2} (\mathbb{E}(a; q) \times \mathbb{E}(a + q + 1; r - 1 - q)) \right) \sqcup \mathbb{E}(a; r - 1).
\]

In particular, \(| \mathbb{E}(r) | = | \mathbb{E}(r - 1) | + \sum_{q=1}^{r-2} | \mathbb{E}(q) || \mathbb{E}(r - 1 - q) | + | \mathbb{E}(r - 1) | \) for \( r \geq 2 \). Comparing this equation with the Catalan recurrence, we have \(| \mathbb{E}(a; r) | = | \mathbb{E}(r) | = \text{Cat}_r \) for all \( r \geq 1 \).

For two positive integers \( a, r \), we define the cycle \( \xi(a; r) \) of length \( r + 1 \) by
\[
\xi(a; r) = (a, a + 1, \ldots, a + r).
\]

For convenience, we take \( \xi(a; 0) \) to be the identity permutation. The following proposition is the key to our proof of Theorem 2.3.

**Proposition 2.11.** Let \( t_1, \ldots, t_r \) be positive integers satisfying \( 2 \leq t_1 \leq \cdots \leq t_r \). The cycle \( \xi(a; r) \) may be expressed as a product of \( r \) transpositions
\[
\xi(a; r) = (s_1, t_1)(s_2, t_2) \cdots (s_r, t_r), \quad s_i < t_i \ (1 \leq i \leq r)
\]
if and only if
\[
(t_1, \ldots, t_r) \in \mathbb{E}(a; r).
\]

Furthermore, for each \((t_1, \ldots, t_r) \in \mathbb{E}(a; r)\), the expression \((2.5)\) of \( \xi(a; r) \) is unique.
Proof of Proposition 2.11. We proceed by induction on \( r \). When \( r = 1 \), since \( \xi(a; r) = (a, a + 1) \), and since \( \mathbb{E}(a; 1) \) consists of a sequence \((a + 1)\) of length 1, our claims are trivial. Let \( r > 1 \) and suppose that for cycles of length \( r - 1 \), all claims in the theorem hold true.

(i) First, we suppose that the cycle \( \xi(a; r) \) is given by the form \((2.7)\). Then we have \( t_r = a + r \) because \( t_r \) is the maximum among \( \text{supp}(\xi(a; r)) \), where \( \text{supp}(\xi(a; r)) = \{s_1, t_1, \ldots, s_r, t_r\} \) by Lemma 2.10. If we write as \( s_r = a + q \) with \( 0 \leq q \leq r - 1 \), we have

\[
(s_1, t_1) \cdot (s_{r-1}, t_{r-1}) = (a, a + 1, \ldots, a + q)(a + q + 1, a + q + 2, \ldots, a + r).
\]

By Lemma 2.10, we see that

\[
(2.7) \quad (s_1, t_1) \cdot (s_q, t_q) = (a, a + 1, \ldots, a + q), \\
(s_{q+1}, t_{q+1}) \cdot (s_{r-1}, t_{r-1}) = (a + q + 1, a + q + 2, \ldots, a + r).
\]

By the induction hypothesis for cycles of length \( q + 1 \) and of length \( r - q \), we have \((t_1, \ldots, t_q) \in \mathbb{E}(a; q) \) and \((t_{q+1}, \ldots, t_{r-1}) \in \mathbb{E}(a + q + 1; r - 1 - q) \). This fact and Equation \((2.4)\) imply \((t_1, \ldots, t_q, t_{q+1}, \ldots, t_{r-1}, t_r) \in \mathbb{E}(a; r) \subset \mathbb{E}(a; r) \).

(ii) Next, we suppose \((t_1, \ldots, t_r) \in \mathbb{E}(a; r) \). According to the decomposition \( \mathbb{E}(a; r) = \bigcup_{q=0}^{r-1} \mathbb{E}_q(a; r) \), there exists a unique number \( q \) such that \( 0 \leq q \leq r - 1 \) and \((t_1, \ldots, t_r) \in \mathbb{E}_q(a; r) \), and then \((t_1, \ldots, t_q) \in \mathbb{E}(a; q) \) and \((t_{q+1}, \ldots, t_{r-1}) \in \mathbb{E}(a + q + 1; r - 1 - q) \). By the induction assumption, there exist sequences \((s_1, s_2, \ldots, s_q) \) and \((s_{q+1}, \ldots, s_{r-1}) \) satisfying \((2.7)\). Therefore we obtain the expression

\[
\xi(a; r) = (s_1, t_1) \cdot (s_q, t_q)(s_{q+1}, t_{q+1}) \cdots (s_{r-1}, t_{r-1})(a + q, a + r),
\]
as required.

(iii) It remains to prove the uniqueness of the expression \((2.5)\). Assume that the cycle \( \xi(a; r) \) has two expressions

\[
(s_1, t_1) (s_2, t_2) \cdots (s_r, t_r) \quad \text{and} \quad (s_1', t_1) (s_2', t_2) \cdots (s_r', t_r),
\]

where \( s_i, s_i' < t_i \) (1 \( \leq i \leq r \)). Write as \( s_r = a + q \) and \( s_r' = a + q' \). As we saw in the part (i), the sequence \((t_1, \ldots, t_r) \) belongs to \( \mathbb{E}_q(a; r) \cap \mathbb{E}_{q'}(a; r) \). But, since \( \mathbb{E}_q(a; r) \cap \mathbb{E}_{q'}(a; r) = \emptyset \) if \( q \neq q' \), we have \( q = q' \) so that \( s_r = s_r' \). Now, as like \((2.7)\), we have \((t_1, \ldots, t_q) \in \mathbb{E}(a; q) \) and \((t_{q+1}, \ldots, t_{r-1}) \in \mathbb{E}(a + q + 1; r - 1 - q) \), and

\[
(s_1, t_1) \cdots (s_q, t_q) = (s_1', t_1) \cdots (s_q', t_q) = (a, a + 1, \ldots, a + q), \\
(s_{q+1}, t_{q+1}) \cdots (s_{r-1}, t_{r-1}) = (s_{q+1}', t_{q+1}) \cdots (s_{r-1}', t_{r-1}) = (a + q + 1, a + q + 2, \ldots, a + r).
\]

By the induction assumption, we obtain \( s_1 = s_1', \ldots, s_q = s_q', s_{q+1} = s_{q+1}', \ldots, s_{r-1} = s_{r-1}' \).
2.3.3 Proof of Theorem 2.4

Recall the definition of the Jucys-Murphy elements: $J_k = \sum_{1 \leq s < t} (s, t)$. For a permutation $\sigma \in S(n)$ and a polynomial $f$ in $n$ variables, denote by $[\sigma]f(\Xi_n)$ the multiplicity of $\sigma$ in $f(J_1, \ldots, J_n)$:

$$f(\Xi_n) = \sum_{\sigma \in S(n)} ([\sigma]f(\Xi_n)) \sigma \in \mathbb{C}[S_n].$$

For a partition $\mu$ with size $k$ and length $l$, we define the canonical permutation $\sigma_{\mu}$ of reduced cycle-type $\mu$ by

$$\sigma_{\mu} = (1, 2, \ldots, n)(1 + \mu_1, 2 + \mu_1, \ldots, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_l, \ldots, n + \mu_1).$$

Proposition 2.12. Let $\mu$ be a partition of $k$ and let $(t_1, \ldots, t_k)$ be a sequence of positive integers such that $2 \leq t_i \leq \cdots \leq t_k$. Then $[\sigma_{\mu}]J_{t_1} \cdot \cdot \cdot J_{t_k} = 1$ if $(t_1, \ldots, t_k)$ satisfies

$$(2.8) \quad (t_{\mu_1 + \cdots + \mu_{i-1} + 1}, \ldots, t_{\mu_1 + \cdots + \mu_{i-1} + \mu_i}) \in \mathbb{E}(\mu + \cdots + \mu_{i-1} + i; \mu_i)$$

for all $1 \leq i \leq \ell(\mu)$, and $[\sigma_{\mu}]J_{t_1} \cdot \cdot \cdot J_{t_k} = 0$ otherwise.

Proof. The value $[\sigma_{\mu}]J_{t_1} \cdot \cdot \cdot J_{t_k}$ is the number of sequences $(s_1, \ldots, s_k)$ satisfying

$$\sigma_{\mu} = \prod_{i=1}^{\ell(\mu)} \xi(\mu_1 + \cdots + \mu_{i-1} + i; \mu_i) = (s_{\mu_1 + \cdots + \mu_{i-1} + 1}, t_{\mu_1 + \cdots + \mu_{i-1} + 1}) \cdots (s_{\mu_1 + \cdots + \mu_{i-1} + \mu_i}, t_{\mu_1 + \cdots + \mu_{i-1} + \mu_i})$$

for all $1 \leq i \leq \ell(\mu)$. It follows by Proposition 2.11 that $[\sigma_{\mu}]J_{t_1} \cdot \cdot \cdot J_{t_k}$ equals to 1 if $(2.8)$ holds true for all $i$, and to 0 otherwise.

Example 2.6. Let $2 \leq t_1 \leq \cdots \leq t_6$ and consider $\sigma_{(3,2,1)} = (1, 2, 3, 4)(5, 6, 7)(8, 9)$. Suppose $[\sigma_{(3,2,1)}]J_{t_1} \cdot \cdot \cdot J_{t_6} = 1$. Then, Proposition 2.12 claims

$$(t_1, t_2, t_3) \in \mathbb{E}(1; 3), \quad (t_4, t_5) \in \mathbb{E}(5; 2), \quad (t_6) \in \mathbb{E}(8; 1).$$

Therefore, $(t_1, t_2) \in \{(2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$, $t_3 = 4$, $t_4 \in \{6, 7\}$, $t_5 = 7$, and $t_6 = 9$.

As defined above, a weakly increasing sequence $(t_1, \ldots, t_r)$ is of type $\lambda \vdash r$ with $\ell(\lambda) = l$ if there exists a permutation $(\alpha_1, \ldots, \alpha_l)$ of $(\lambda_1, \ldots, \lambda_l)$ such that

$$t_1 = t_2 = \cdots = t_{\alpha_1} < t_{\alpha_1 + 1} = t_{\alpha_1 + 2} = \cdots = t_{\alpha_1 + \alpha_2} < t_{\alpha_1 + \alpha_2 + 1} = \cdots$$

The monomial symmetric polynomial $m_{\lambda}(\Xi_n)$, $\lambda \vdash k$, is written as

$$m_{\lambda}(\Xi_n) = \sum_{2 \leq t_1 \leq \cdots \leq t_k \leq n \atop (t_1, \ldots, t_k): \text{type } \lambda} J_{t_1}J_{t_2} \cdot \cdot \cdot J_{t_k} = \sum_{2 \leq t_1 \leq \cdots \leq t_k \leq n \atop (t_1, \ldots, t_k): \text{type } \lambda} \sum_{s_1 = 1}^{t_1-1} \cdots \sum_{s_k = 1}^{t_k-1} (s_1, t_1) \cdots (s_k, t_k).$$
Let $\mu$ be a partition of $k$. We now evaluate the coefficient $L_\mu^\lambda(n)$ of $c_\mu(n)$ in $m_\lambda(J_1, \ldots, J_n)$, which equals $L_\mu^\lambda(n) = [\sigma_\mu]m_\lambda(J_1, \ldots, J_n)$. By the assumption $n \geq k + \ell(\mu)$, the permutation $\sigma_\mu$ lives in $S(n)$. It follows by Proposition 2.12 that $L_\mu^\lambda(n)$ is the number of weakly increasing sequences $(t_1, \ldots, t_k)$ of type $\lambda$, satisfying (2.8) for all $1 \leq i \leq \ell(\mu)$. If $(t_1, \ldots, t_k)$ is such a sequence and if we let $\lambda(1) \cup \lambda(2) \cup \cdots$ so that $(\lambda(1), \lambda(2), \ldots) \in R(\lambda, \mu)$. Thus, $L_\mu^\lambda(n)$ coincides with

$$\sum_{(\lambda^{(1)}, \lambda^{(2)}, \ldots) \in R(\lambda, \mu)} \prod_{i=1}^{\ell(\mu)} RC(\lambda^{(i)}).$$

This completes the proof of Theorem 2.4.

3 Class coefficients and connection coefficients

In this section, we pursue a certain analogy between the class coefficients $G_\mu(f, n)$ arising in the expansion of symmetric functions in Jucys-Murphy elements, and the connection coefficients $A_{\alpha\beta}^\mu(n)$ of the class algebra $Z(n)$. The latter are by definition the structure constants of $Z(n)$, i.e.

$$c_{\alpha}(n)c_{\beta}(n) = \sum_{\mu} A_{\alpha\beta}^\mu(n)c_{\mu}(n).$$

Like the class coefficient $L_\mu^\lambda(n)$, the connection coefficient $A_{\mu}^{\alpha\beta}(n)$ has an immediate combinatorial interpretation: it is equal to the number of factorizations

$$\pi = \sigma \rho$$

of a fixed representative $\pi$ of $C_\mu(n)$ into a permutation $\sigma$ of reduced cycle type $\alpha$ and a permutation $\rho$ of reduced cycle type $\beta$. The following properties of connection coefficients are well known:

1. $A_{\mu}^{\alpha\beta}(n)$ is a polynomial function of $n$;
2. $A_{\mu}^{\alpha\beta}(n)$ vanishes unless $|\mu| + 2g = |\alpha| + |\beta|$ for some integer $g \geq 0$;
3. In the “top” case $|\mu| = |\alpha| + |\beta|$, the connection coefficient $A_{\mu}^{\alpha\beta} = A_{\mu}^{\alpha\beta}(n)$ is independent of $n$, and vanishes unless $\alpha \cup \beta$ is a refinement of $\mu$.

Property (1) above is a classical result due to Farahat and Higman [11], see also [20]. In the previous section, we proved analogues of properties (2) and (3) for the class coefficients $L_\mu^\lambda(n)$. In this section, we will prove the analogue of (1), namely that for any fixed symmetric function
and partition \( \mu \), the class coefficient \( G_\mu(f, n) \) is a polynomial function of \( n \). By combining this fact with the spectral properties of Jucys-Murphy elements in irreducible symmetric group representations, we recover and generalize a recent result of Stanley [45] and Olshanski [37] on the polynomiality of certain statistics on Plancherel-random partitions.

Macdonald [28], see also [15], used Lagrange inversion to construct a basis \( \{ g_\mu \} \) of the algebra of symmetric functions which encodes the top class coefficients \( A^\alpha_\beta \):

\[
g_\alpha g_\beta = \sum_{|\mu|=|\alpha|+|\beta|} A^\alpha_\beta g_\mu.
\]

We conclude this section by obtaining an analogue of Macdonald’s result, which realizes the top class coefficients \( L^\lambda_\mu \) intrinsically as part of the algebraic structure of \( \Lambda \), without any reference to Jucys-Murphy elements. This computation constitutes a new change of basis formula in the algebra of symmetric functions.

### 3.1 Polynomaility

**Theorem 3.1.** Fix a symmetric function \( f \in \Lambda \) and a partition \( \mu \). The class coefficient \( G_\mu(f, n) \) is a polynomial function of \( n \).

**Proof.** Since \( \Lambda = \mathbb{C}[e_1, e_2, \ldots] \) there exists a polynomial, say \( p_f \), such that

\[
f = p_f(e_{i_1}, \ldots, e_{i_k})
\]

for some elementary symmetric functions \( e_{i_1}, \ldots, e_{i_k} \). By Proposition 2.1 we have

\[
f(\Xi_n) = p_f \left( \sum_{|\mu|=i_1} e_\mu(n), \ldots, \sum_{|\mu|=i_k} e_\mu(n) \right),
\]

and the result now follows from the polynomiality of the connection coefficients \( A^\alpha_\beta(g_\mu) \).

From the isotypic decomposition

\[
\mathbb{C}[S(n)] = \bigoplus_{\lambda \vdash n} \dim \lambda \mathcal{V}^\lambda,
\]

where \( \dim \lambda \) denotes the dimension of the irreducible representation \( \mathcal{V}^\lambda \), one obtains Burnside’s identity:

\[
\sum_{\lambda \vdash n} (\dim \lambda)^2 = n!.
\]

Burnside’s identity implies that the function

\[
\lambda \mapsto \frac{(\dim \lambda)^2}{n!}
\]
defines a probability measure, known as the Plancherel measure, on the sample space \( \mathcal{Y}_n = \{ \lambda \vdash n \} \). Given a function \( f: \mathcal{Y} \to \mathbb{C} \) defined on the set of partitions, let

\[
\langle f \rangle_n = \sum_{\lambda \vdash n} f(\lambda) \frac{\dim \lambda^2}{n!}
\]

denote its expected value with respect to the Plancherel measure on \( \mathcal{Y}_n \). The following polynomiality property of Plancherel averages was proved by Stanley [45] and Olshanski [37], by different methods. As an application of Theorem 3.1, we give a third proof.

**Theorem 3.2.** Let \( f \in \Lambda \) be a symmetric function. The Plancherel expectation \( \langle f(A_\lambda) \rangle_n \), where \( A_\lambda \) is the content alphabet of \( \lambda \), is a polynomial function of \( n \).

**Proof.** We will prove the following more general fact. Let \( \chi^\lambda_\mu \) denote the trace of any representative of the conjugacy class \( C_\mu(n) \) in the irreducible representation \( V^\lambda \). Then, the sum

\[
\sum_{\lambda \vdash n} f(A_\lambda) \chi^\lambda_\mu \frac{\dim \lambda}{n!}
\]

is a polynomial function of \( n \). When \( \mu \) is the empty partition, this reduces to the statement of the theorem.

We prove the more general assertion as follows. Put

\[
\chi^\lambda = \sum_\mu \chi^\lambda_\mu c_\mu(n).
\]

Then it is a basic fact that \( \{ \chi^\lambda : \lambda \vdash n \} \) constitutes a basis of the class algebra \( \mathcal{Z}(n) \), and the coordinates of the identity are given in this basis by

\[
c_{(0)}(n) = \sum_{\lambda \vdash n} \frac{\dim \lambda}{n!} \chi^\lambda.
\]

Since the central character of \( f(\Xi_n) \) in \( V^\lambda \) is \( f(A_\lambda) \), we have \( f(\Xi_n)\chi^\lambda = f(A_\lambda)\chi^\lambda \) in \( \mathcal{Z}(n) \). Thus the coordinates of \( f(\Xi_n) \) relative to the character basis are

\[
f(\Xi_n) = f(\Xi_n)c_{(0)}(n) = \sum_{\lambda \vdash n} f(A_\lambda) \frac{\dim \lambda}{n!} \chi^\lambda,
\]

so that

\[
G_\mu(f, n) = \sum_{\lambda \vdash n} f(A_\lambda) \frac{\dim \lambda}{n!} \chi^\lambda_\mu.
\]

The result now follows from Theorem 3.1. \( \square \)
3.2 An analogue of Macdonald’s result for top connection coefficients

Macdonald [28, Chapter I.7, Example 25], see also [15], used Lagrange inversion to construct a basis \( \{ g_\mu \} \) of the algebra of symmetric functions whose connection coefficients coincide with the top connection coefficients \( A^\mu_\nu \) of the class algebra, as in equation (3.1).

We now give an analogue of Macdonald’s result for the top class coefficients \( L_\mu^\lambda \): for each \( k \geq 1 \) we realize the matrix \( (L_\mu^\lambda)_{|\mu|=|\lambda|=k} \) as the transition matrix between two bases of the degree \( k \) component of the graded algebra \( \Lambda \).

Since the elementary symmetric functions \( e_k \) are algebraically independent and generate \( \Lambda \), we may define an endomorphism \( \psi : \Lambda \to \Lambda \) by \( \psi(e_k) = h_k \). This endomorphism is in fact involutive: \( \psi(h_k) = e_k \). The image \( f_\lambda := \psi(m_\lambda) \) of the monomial symmetric function of type \( \lambda \) under \( \psi \) is known as the forgotten symmetric function of type \( \lambda \), see [44, Exercise 7.9].

**Theorem 3.3.** Let \( |\lambda| = k \). Then

\[
(-1)^k f_\lambda = \sum_{|\mu|=k} L_\mu^\lambda g_\mu.
\]

**Proof.** Let

\[
u = t + \sum_{r=1}^{\infty} h_r t^{r+1}.
\]

Then \( t \) can be expressed as a power series in \( u \). Define symmetric functions \( h_r^* \), \( r = 1, 2, \ldots \), via

\[
u = u + \sum_{r=1}^{\infty} h_r^* u^{r+1}.
\]

From the Lagrange inversion formula, the symmetric functions \( h_r^* \) are explicitly given by

\[
 h_r^* = (-1)^r \sum_{\lambda \vdash r} \text{RC}(\lambda) e_\lambda,
\]

where \( e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \), see [31 (3.6)] and also [17], [28, Ch. I, Example 2.24], [43].

Let \( h_\lambda^* = h_\lambda^* h_\lambda^* \cdots \). Then \( \{h_\lambda^*\} \) is a basis of \( \Lambda \).

Let \( \mu \) be a partition of \( k \). We will now prove that

\[
(-1)^{E} h_\mu^* = \sum_{\lambda \vdash k} L_\mu^\lambda e_\lambda,
\]

and thus the matrix \( (L_\mu^\lambda)_{|\mu|=|\lambda|=k} \) is the transition matrix from the basis \( \{(-1)^{|\lambda|} h_\lambda^*\} \) to the basis \( \{e_\lambda\} \) in the \( k \)th component of \( \Lambda \). The proof goes as follows: let \( l = \ell(\mu) \). It follows from Theorem...
2.4 that

\[ h_{\mu} = h_{\mu_1}^* h_{\mu_2}^* \cdots h_{\mu_l}^* \]

\[ = (-1)^k \sum_{\lambda(1)\vdash \mu_1} \sum_{\lambda(2)\vdash \mu_2} \cdots \sum_{\lambda(l)\vdash \mu_l} RC(\lambda(1)) RC(\lambda(2)) \cdots RC(\lambda(l)) e_{\lambda(1)\cup \lambda(2)\cup \cdots \cup \lambda(l)} \]

\[ = (-1)^k \sum_{\lambda \vdash k} \sum_{(\lambda(1),\lambda(2),\ldots,\lambda(l)) \in R(\lambda, \mu)} RC(\lambda(1)) RC(\lambda(2)) \cdots RC(\lambda(l)) e_{\lambda} \]

Let \( \langle \cdot, \cdot \rangle \) be the scalar product on \( \Lambda \) defined by \( \langle h_\lambda, m_\mu \rangle = \delta_{\lambda \mu} \). With respect to this scalar product, the dual bases of \( \{ h_\lambda^* \} \) and \( \{ e_\lambda \} \) are, respectively, Macdonald’s symmetric functions \( \{ g_\lambda \} \) and the forgotten symmetric functions \( \{ f_\lambda \} \), see [28, Ch. I.2], [28, Ch. I, Example 7.25], [31]. Thus (3.2) is equivalent to

\[ (-1)^k f_\lambda = \sum_{\mu \vdash k} L_\mu^\lambda g_\mu, \quad \lambda \vdash k. \]

4 Coefficient of a full cycle and central factorial numbers

Given a permutation \( \pi \in S(n) \) and an integer \( k \geq 0 \), how many factorizations

\[ \pi = (s_1 \ t_1)(s_2 \ t_2) \cdots (s_k \ t_k) \]

of \( \pi \) into \( k \) transpositions are there? This is a very natural question. Geometrically, the problem is to count the number of \( k \)-step walks from the identity to \( \pi \) on the Cayley graph of \( S(n) \). Algebraically, this is a special case of the connection coefficient problem, since if \( \pi \in C_\mu(n) \) then the desired number, call it \( A_\mu^n(n) \), is the coefficient of \( c_\mu(n) \) in the product \( c_1(n) c_1(n) \cdots c_1(n) \) of \( k \) copies of the class of transpositions. In the special case where \( \pi \in C_{(n-1)}(n) \) is a full cycle, the number of factorizations of \( \pi \) into \( n-1 \) transpositions (i.e. the minimal number required) is the Cayley number,

\[ A_{(n-1)}^{n-1}(n) = \text{Cay}_{n-1} = n^{n-2}, \]

which the reader will recognize as the number of trees on the vertex set \([n-1]\). This formula was first obtained by Hurwitz [19] as a corollary of his exact enumeration of holomorphic maps \( \mathbb{P}^1 \to \mathbb{P}^1 \) from the Riemann sphere to itself with one degenerate branch point. Various bijective proofs of this result have since been found, see [13] and references therein. It follows that the number of factorizations of a fixed representative of \( C_\mu(n) \) into \(|\mu|\) transpositions, the minimal number required, is
The number $A_{(n-1)}^k(n)$ of factorizations of a fixed representative of the class $C_{(n-1)}(n)$ of full cycles into any number $k$ of transpositions was determined by Jackson [21] using a character-theoretic argument. The number of factorizations is non-zero if and only if $k = n - 1 + 2g$ for some integer $g \geq 0$, and Jackson’s formula is

\[(n-1+2g)\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left( \binom{n}{2} - kn \right)^{n-1+2g}.\]

Jackson’s formula was rediscovered in the context of singularity theory by Shapiro, Shapiro and Vainshtein [42], and a bijective proof was found by Goulden [14].

Consider now the following restricted variant of the above question: given a permutation $\pi$ and an integer $k \geq 0$, how many factorizations $\pi = (s_1 t_1)(s_2 t_2) \ldots (s_k t_k)$ of $\pi$ into $k$ transpositions are there satisfying the constraint $t_1 \leq t_2 \leq \cdots \leq t_k$? Factorizations of this form were called \textit{primitive} in [12]. Algebraically, this is a special case of the class expansion problem: if $\pi \in C_{\mu}(n)$, then the required number is the coefficient $F_{\mu}^k(n)$ of $c_{\mu}(n)$ in $h_k(\Xi_n)$, where $h_k$ is the complete homogeneous symmetric function of degree $k$. For the class of full cycles, the analogue of (4.1) was first obtained by Gewurz and Merola [12]:

$$F_{(n-1)}^{(n-1)} = \text{Cat}_{n-1}.$$ 

The general solution to this problem in the minimal case $k = |\mu|$ is given by Corollary 2.5 which gives the analogue of (4.2):

$$F_{\mu}^{\ell(\mu)} = \prod_{i=1}^{\ell(\mu)} \text{Cat}_{\mu_i}.$$ 

Note that the order constraint has the effect of desymmetrizing the Cayley numbers to the Catalan numbers and removing the shuffle factor. In this language, Theorem 2.4 is an exact enumeration of minimal primitive factorizations by type.

In this final section, we obtain the analogue of Jackson’s formula (4.3). That is, we give an exact enumeration of the primitive factorizations of any full cycle, of any length. Like Jackson’s argument, our method relies on techniques from the character theory of the symmetric groups. We prove that
\( F_{(n-1)}^{n-1+2g} = \text{Cat}_{n-1} \cdot T(n-1+g, n-1) \)

for any integer \( g \geq 0 \), where

\[
T(a, b) = 2 \sum_{j=1}^{b} (-1)^{b-j} \frac{j^{2a}}{(b-j)!(b+j)!}
\]

is the Carlitz-Riordan central factorial number of the second kind. \( T(a, b) \) is equal to the number of partitions of the set \( \{1, 1', 2, 2', \ldots, a, a'\} \) into \( b \) disjoint non-empty subsets \( V_1, \ldots, V_b \) such that, for each \( 1 \leq k \leq b \), if \( i \) is the least integer such that either \( i \) or \( i' \) belongs to \( V_k \) then \( \{i, i'\} \subseteq V_k \).

Thus \( T(a, b) \) may be thought of as a two-coloured Stirling number of the second kind. Indeed, central factorial numbers first appeared in Carlitz and Riordan’s investigation of the central difference operator \([3]\), where they play the same role as the Stirling numbers for the usual difference operator. For more information and references regarding central factorial numbers, see \([40\text{, Section 6.5]}\) and \([44\text{, Exercise 5.8]}\). The result presented in this section gives a new and rather surprising combinatorial interpretation of the central factorial numbers as counting primitive factorizations of a full cycle. It would be very interesting to obtain a bijective proof.

4.1 Coefficient of \( c_{(n-1)}(n) \) in \( h_k(\Xi_n) \)

Let \( q \) be a formal variable, and introduce the generating function

\[
\Phi(q, n) = \sum_{k=0}^{\infty} h_k(\Xi_n)q^k.
\]

The generating series \( \Phi(q, n) \) is an element of the ring \( \mathcal{Z}(n)[[q]] \) of formal power series in \( q \) with coefficients in \( \mathcal{Z}(n) \). Via the hook-length formula, Jucys’ result \((1.6)\) may be re-stated as follows: for any symmetric function \( f \in \Lambda \), the central element \( f(\Xi_n) \in \mathcal{Z}(n) \) has coordinates

\[
f(\Xi_n) = \sum_{\lambda \vdash n} \frac{f(A_\lambda)}{H_\lambda} \chi^\lambda
\]

with respect to the character basis of \( \mathcal{Z}(n) \), where \( H_\lambda = \prod_{\Box \in \lambda} h(\Box) \) is the product of the hook-lengths over the cells of \( \lambda \). Substituting the character expansion of \( h_k(\Xi_n) \) into the generating function \((4.8)\), we obtain

\[
\Phi(q, n) = \sum_{k=0}^{\infty} \left( \sum_{\lambda \vdash n} \frac{f(A_\lambda)}{H_\lambda} \chi^\lambda \right) q^k.
\]

Changing order of summation and appealing to the generating function

\[
\sum_{k=0}^{\infty} h_k(x_1, x_2, x_3, \ldots)q^k = \prod_{i=1}^{\infty} \frac{1}{1-x_iq}
\]
of the complete symmetric functions, this becomes

\begin{equation} \tag{4.9} \Phi(q, n) = \sum_{\lambda \vdash n} \frac{\chi^\lambda}{H_{\lambda} \prod_{\Box \in \lambda} (1 - c(\Box)q)}. \end{equation}

Now let \( \mu \) be a partition, and set

\[ \Phi_\mu(q, n) = \sum_{k=0}^{\infty} F^k_{\mu}(n)q^k. \]

Then from (4.9) we obtain

\[ \Phi_\mu(q, n) = \sum_{\lambda \vdash n} \frac{\chi^\lambda_{\mu}}{H_{\lambda} \prod_{\Box \in \lambda} (1 - c(\Box)q)}, \]

from which it is clear that \( \Phi_\mu(q, n) \) is a rational function of \( q \), with coefficients in \( \mathbb{Q} \). Now, the trace of a full cycle \( \pi \in C_{(n-1)}(n) \) in an irreducible representation \( V^\lambda \) of \( S(n) \) is non-zero if and only if \( \lambda \) is a “hook” partition:

\[ \chi^\lambda_{(n-1)} = \begin{cases} (-1)^r, & \text{if } \lambda = (n-k, 1^k) \\ 0, & \text{otherwise} \end{cases}, \]

see e.g. [7] for a proof of this classical fact. When \( \lambda = (n-k, 1^k) \) is a hook, the content alphabet of \( \lambda \) is simply

\[ A_\lambda = \{1, 2, \ldots, n-k-1\} \sqcup \{-1, -2, \ldots, -(k-1)\}. \]

Thus, summing over all hook representations, we have

\[ \Phi_{(n-1)}(q, n) = \sum_{k=0}^{n-1} \frac{(-1)^k}{H_{(n-k,1^k)} \prod_{j=1}^{n-k-1} (1 - jq) \prod_{j=1}^{k-1} (1 + jq)}, \]

which is a rational function of the form

\[ \Phi_{(n-1)}(q, n) = \frac{a_0 + a_1 q + \cdots + a_{n-1} q^{n-1}}{\prod_{j=1}^{n-1} (1 - j^2 q^2)}. \]

On the other hand, we know from Corollary 2.5 that

\[ \Phi_{(n-1)}(q, n) = \sum_{g=0}^{\infty} F^{n-1+2g}_{(n-1)}(n)q^{n-1+2g}, \quad F^{n-1}_{(n-1)} = \text{Cat}_{n-1}, \]

so that \( a_0 = a_1 = \cdots = a_{n-2} = 0 \) and \( a_{n-1} = \text{Cat}_{n-1} \). We have thus proved the following.

**Theorem 4.1.** We have

\[ \Phi_{(n-1)}(q, n) = \frac{\text{Cat}_{n-1} q^{n-1}}{(1 - q^2)(1 - 4q^2)(1 - 9q^2) \cdots (1 - (n-1)^2 q^2)}. \]
Up to the factor $\text{Cat}_{n-1}$, the rational function appearing in Theorem 4.1 is an ordinary generating function for the central factorial numbers:

$$
\frac{q^{n-1}}{(1-q^2)(1-4q^2)(1-9q^2)\ldots(1-(n-1)^2q^2)} = \sum_{g \geq 0} T(n-1+g,n-1)q^{n-1+g},
$$

see [44, Exercise 5.8]. Thus Theorem 4.1 is equivalent to the identity (4.6) stated above. Finally, making the substitution $q = -1/N$ and appealing to the fundamental identity (1.7), we see that Theorem 4.1 implies the following exact formula [4] for cyclic inner products in the algebra $A$:

$$
\langle u_{11}u_{22}u_{33}\ldots u_{nn},u_{12}u_{23}u_{34}\ldots u_{nn} \rangle_N = \frac{(-1)^{n-1}\text{Cat}_{n-1}}{N(N^2-1)(N^2-4)(N^2-9)\ldots(N^2-(n-1)^2)}.
$$

5 Appendix A: Examples

5.1 A.1: Class expansion of $m_{\lambda}(\Xi_n)$ for $|\lambda| \leq 4$.

$|\lambda| = 1$

$$
m_{(1)}(\Xi_n) = c_{(1)}(n).
$$

$|\lambda| = 2$

$$
m_{(2)}(\Xi_n) = c_{(2)}(n) + \frac{1}{2}n(n-1)c_{(0)}(n),
m_{(1,1)}(\Xi_n) = c_{(2)}(n) + c_{(1,1)}(n),
h_2(\Xi_n) = 2c_{(2)}(n) + c_{(1,1)}(n) + \frac{1}{2}n(n-1)c_{(0)}(n).
$$

$|\lambda| = 3$

$$
m_{(3)}(\Xi_n) = c_{(3)}(n) + (2n-3)c_{(1)}(n),
m_{(2,1)}(\Xi_n) = 3c_{(3)}(n) + c_{(2,1)}(n) + \frac{1}{2}(n-2)(n+1)c_{(1)}(n),
m_{(1,3)}(\Xi_n) = c_{(3)}(n) + c_{(2,1)}(n) + c_{(1,3)}(n),
h_3(\Xi_n) = 5c_{(3)}(n) + 2c_{(2,1)}(n) + c_{(1,3)}(n) + \frac{1}{2}(n^2 + 3n - 8)c_{(1)}(n).
$$
\(|\lambda| = 4\)

\[
m_{(4)}(\Xi_n) = c_4(n) + (3n - 4)c_{(2)}(n) + 4c_{(12)}(n) + \frac{1}{6}n(n - 1)(4n - 5)c_{(0)}(n).
\]

\[
m_{(3,1)}(\Xi_n) = 4c_4(n) + c_{(3,1)}(n) + 2(3n - 7)c_{(2)}(n) + 2(2n - 3)c_{(12)}(n)
+ \frac{1}{3}n(n - 1)(n - 2)c_{(0)}(n).
\]

\[
m_{(2)}(\Xi_n) = 2c_4(n) + c_{(22)}(n) + \frac{1}{2}(n^2 - n - 4)c_{(2)}(n) + 2c_{(12)}(n)
+ \frac{1}{24}n(n - 1)(n - 2)(3n - 1)c_{(0)}(n).
\]

\[
m_{(2,1^2)}(\Xi_n) = 6c_4(n) + 3c_{(3,1)}(n) + 2c_{(22)}(n) + c_{(2,12)}(n) + \frac{1}{2}(n - 3)(n + 2)c_{(2)}(n)
+ \frac{1}{2}(n^2 - n - 4)c_{(12)}(n).
\]

\[
m_{(1^4)}(\Xi_n) = c_4(n) + c_{(3,1)}(n) + c_{(22)}(n) + c_{(2,12)}(n) + c_{(1^4)}(n).
\]

\[
h_4(\Xi_n) = 14c_4(n) + 5c_{(3,1)}(n) + 4c_{(22)}(n) + 2c_{(2,12)}(n) + c_{(1^4)}(n)
+ (n^2 + 8n - 23)c_{(2)}(n) + \frac{1}{2}(n^2 + 7n - 4)c_{(12)}(n)
+ \frac{1}{24}n(n - 1)(3n^2 + 17n - 34)c_{(0)}(n).
\]

5.2 A.2: Tables of \(L^\lambda_\mu\)

We now give tables of \(L^\lambda_\mu\), which can be compared with the class expansions in Appendix A.1. The row labelled “SUM” stands for \(\sum_{\lambda \vdash k} L^\lambda_\mu\) for each column associated with \(\mu\), which equals \(\prod \text{Cat}_\mu\).

\[
\begin{array}{c|c|c}
\lambda \setminus \mu & 2 & 1^2 \\
\hline
2 & 1 & \\
1^2 & 1 & 1 \\
SUM & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\lambda \setminus \mu & 3 & 21 & 1^3 \\
\hline
3 & 1 & \\
21 & 3 & 1 \\
1^3 & 1 & 1 & 1 \\
SUM & 5 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c}
\lambda \setminus \mu & 4 & 31 & 2^2 & 21^2 & 1^4 \\
\hline
4 & 1 & \\
31 & 4 & 1 \\
2^2 & 2 & 0 & 1 \\
21^2 & 6 & 3 & 2 & 1 \\
1^4 & 1 & 1 & 1 & 1 & 1 \\
SUM & 14 & 5 & 4 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\lambda \setminus \mu & 5 & 41 & 32 & 31^2 & 2^21 & 21^3 & 1^5 \\
\hline
5 & 1 & \\
41 & 5 & 1 \\
32 & 5 & 0 & 1 \\
31^2 & 10 & 4 & 1 & 1 \\
2^21 & 10 & 2 & 3 & 0 & 1 \\
21^3 & 10 & 6 & 4 & 3 & 2 & 1 \\
1^5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
SUM & 42 & 14 & 10 & 5 & 4 & 2 & 1 \\
\end{array}
\]
\[
\begin{array}{c|cccccccccccc}
\lambda \setminus \mu & 6 & 51 & 42 & 41^2 & 3^2 & 321 & 31^3 & 2^3 & 2^21^2 & 21^4 & 1^6 \\
\hline
6 & 1 & & & & & & & & & & \\
51 & 6 & 1 & & & & & & & & & \\
42 & 6 & 0 & 1 & & & & & & & & \\
41^2 & 15 & 5 & 1 & 1 & & & & & & & \\
3^2 & 3 & 0 & 0 & 0 & 1 & & & & & & \\
321 & 30 & 5 & 4 & 0 & 6 & 1 & & & & & \\
31^3 & 20 & 10 & 4 & 4 & 2 & 1 & 1 & & & & \\
2^3 & 5 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & & & \\
2^21^2 & 30 & 10 & 8 & 2 & 9 & 3 & 0 & 3 & 1 & & \\
21^4 & 15 & 10 & 7 & 6 & 6 & 4 & 3 & 3 & 2 & 1 & \\
1^6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\text{SUM} & 132 & 42 & 28 & 14 & 25 & 10 & 5 & 8 & 4 & 2 & 1
\end{array}
\]

\[
\begin{array}{c|cccccccccccc}
\lambda \setminus \mu & 7 & 61 & 52 & 51^2 & 43 & 421 & 41^4 & 3^21 & 32^2 & 321^2 & 31^4 & 2^31 & 2^21^3 & 21^5 & 1^7 \\
\hline
7 & 1 & & & & & & & & & & & & & & \\
61 & 7 & 1 & & & & & & & & & & & & & & \\
52 & 7 & 0 & 1 & & & & & & & & & & & & & & \\
51^2 & 21 & 6 & 1 & 1 & & & & & & & & & & & & & & \\
43 & 7 & 0 & 0 & 0 & 1 & & & & & & & & & & & & & & \\
421 & 42 & 6 & 5 & 0 & 3 & 1 & & & & & & & & & & & & & & \\
41^4 & 35 & 15 & 5 & 1 & 1 & 1 & & & & & & & & & & & & & & \\
3^21 & 21 & 3 & 0 & 0 & 4 & 0 & 0 & 1 & & & & & & & & & & & & \\
32^2 & 21 & 0 & 5 & 0 & 2 & 0 & 0 & 0 & 1 & & & & & & & & & & & & \\
321^2 & 105 & 30 & 15 & 5 & 18 & 4 & 0 & 6 & 2 & 1 & & & & & & & & & & & & \\
31^4 & 35 & 20 & 10 & 10 & 5 & 4 & 4 & 2 & 1 & 1 & 1 & & & & & & & & & & & & \\
2^31 & 35 & 5 & 10 & 0 & 6 & 2 & 0 & 0 & 3 & 0 & 0 & 1 & & & & & & & & & & & & \\
2^21^2 & 70 & 30 & 20 & 10 & 20 & 8 & 2 & 9 & 7 & 3 & 0 & 3 & 1 & & & & & & & & & & & & \\
21^5 & 21 & 15 & 11 & 10 & 9 & 7 & 6 & 6 & 5 & 4 & 3 & 3 & 2 & 1 & & & & & & & & & & & & \\
1^7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & & & & & \\
\hline
\text{SUM} & 429 & 132 & 84 & 42 & 70 & 28 & 14 & 25 & 20 & 10 & 5 & 8 & 4 & 2 & 1 & & & & & & & & & & & & \\
\end{array}
\]

References

[1] Beenaker, C. W. J., and P. W. Brouwer. “Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems.” Journal of Mathematical Physics 37, no. 10 (1996): 4904–4934.

[2] Berkolaiko, G. “Diagonal approximation of the form factor of the unitary group.” Journal of Physics A: Mathematical and General 39, (2006): L77 – L84.

[3] Carlitz, L., and J. Riordan. “The divided central differences of zero.” Canadian Journal of Mathematics 15, no. 1 (1963): 94–101.
[4] Collins, B. “Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability.” *International Mathematics Research Notices*, no. 17 (2003): 953–982.

[5] Collins, B., and P. Śniady. “Integration with respect to the Haar measure on unitary, orthogonal and symplectic group.” *Communications in Mathematical Physics* 264, (2006): 773-795.

[6] Diaconis, P., and A. Gamburd. “Random matrices, magic squares and matching polynomials. *The Electronic Journal of Combinatorics* 11, no. 2 (2004): R2.

[7] Diaconis, P., and C. Greene. “Applications of Murphy’s elements.” *Stanford University Technical Reports* 335, (1989): 1–22.

[8] De Wit, B., and G. ’t Hooft. “Nonconvergence of the 1/N expansion for SU(N) gauge fields on a lattice.” *Physics Letters* 69B, no. 1 (1977): 61-64.

[9] Féray, V. “Partial Jucys-Murphy elements and star factorizations.” *arXiv:0904.4854v1* (2009).

[10] Féray, V. “On complete functions in Jucys-Murphy elements.” *Annals of Combinatorics*, to appear. *arXiv:1009.0144v3* (2011).

[11] Farahat, H. K., and G. Higman. “The centres of symmetric group rings.” *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 250, no. 1261 (1959): 212-221.

[12] Gewurz, D. A., and F. Merola. “Some factorisations counted by Catalan numbers.” *European Journal of Combinatorics* 27, (2006): 990-994.

[13] Goulden, I. P., and S. Pepper. “Labelled trees and factorizations of a cycle into transpositions.” *Discrete Mathematics* 113, (1993): 263–268.

[14] Goulden, I. P. “A differential operator for symmetric functions and the combinatorics of multiplying transpositions.” *Transactions of the American Mathematical Society* 344, no. 1 (1994): 421–440.

[15] Goulden, I. P., and D. M. Jackson. “Symmetric functions and Macdonald’s result for top connexion coefficients in the symmetric group.” *Journal of Algebra* 166, (1994): 364–378.

[16] Goulden, I. P., and D. M. Jackson, “Transitive powers of Young-Jucys-Murphy elements are central.” *Journal of Algebra* 321, (2009): 1826–1835.

[17] Haiman, M. “Conjectures on the quotient ring by diagonal invariants.” *Journal of Algebraic Combinatorics* 3, (1994): 17–76.

[18] Howe, R. “Remarks on classical invariant theory.” *Transactions of the American Mathematical Society* 313, no. 2 (1989): 539–570.

[19] Hurwitz, A. “Über die Anzahl der Riemann’schen Flächen mit gegebenen Verzweigungs punkten.” *Mathematische Annalen* 55, (1902): 53–66.
[20] Ivanov, V., and S. Kerov. “The algebra of conjugacy classes in symmetric groups and partial permutations.” *Journal of Mathematical Sciences* 107, no. 5 (2001): 4212–4230.

[21] Jackson, D. M. “Some combinatorial problems associated with products of conjugacy classes of the symmetric group.” *Journal of Combinatorial Theory, Series A* 49, (1988): 363–369.

[22] Janson, S. *Gaussian Hilbert Spaces*. Cambridge Tracts in Mathematics 129, Cambridge University Press, 1997.

[23] Jucys, A.-A. A. “Symmetric polynomials and the center of the symmetric group ring.” *Reports on Mathematical Physics* 5, no. 1 (1974): 107–112.

[24] Keating, J. P., Mezzadri, F., and B. Singphu. “Rate of convergence of linear functions on the unitary group.” *Journal of Physics A: Mathematical and Theoretical* 44, (2011): 1–27.

[25] Lascoux, A., and J.-Y. Thibon. “Vertex operators and the class algebras of symmetric groups.” *Journal of Mathematical Sciences* 121, no. 3 (2004): 2380–2392.

[26] Lassalle, M. “An explicit formula for the characters of the symmetric group.” *Mathematische Annalen* 340, (2008): 383–405.

[27] Lassalle, M. “Class expansion of some symmetric functions in Jucys-Murphy elements.” [arXiv:1005.2346v1](https://arxiv.org/abs/1005.2346) (2010).

[28] Macdonald, I. G. *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford University Press, reprinted 2008.

[29] Matsumoto, S. “Jucys-Murphy elements, orthogonal matrix integral, and Jack measures.” *Ramanujan Journal*, to appear. [arXiv:1001.2345v2](https://arxiv.org/abs/1001.2345) (2011).

[30] Morozov, A. “Unitary integrals and related matrix models.” *Teoreticheskaya i Matematicheskaya Fizika* 162, no. 1 (2010): 3–40.

[31] Murray, J. “Generators for the center of a symmetric group algebra.” *Journal of Algebra* 271, (2004): 725–748.

[32] Novak, J. “Vicious walkers and random contraction matrices.” *International Mathematics Research Notices*, no. 17 (2009): 3310–3327.

[33] Novak, J. “Jucys-Murphy elements and the unitary Weingarten function.” *Banach Center Publications* 89, (2010): 231-235.

[34] Okounkov, A. “Young basis, Wick formula, and higher Capelli identities.” *International Mathematics Research Notices*, no. 17 (1996): 817–839.

[35] Okounkov, A., and R. Pandharipande. “Gromov-Witten theory, Hurwitz theory, and completed cycles.” *Annals of Mathematics* 163, (2006): 517–560.

[36] Okounkov, A., and A. Vershik. “A new approach to the representation theory of the symmetric groups.” *Selecta Mathematica, New Series* 2, no. 4 (1996): 581–605.
[37] Olshanski, G. “Plancherel averages: remarks on a paper by Stanley.” *The Electronic Journal of Combinatorics* 17, (2010): #R43.

[38] Petz, D., and J. Reffy. “On asymptotics of large Haar distributed unitary matrices.” *Periodica Mathematica Hungarica* 49, no. 1 (2004): 103–117.

[39] Rains, E. “Increasing subsequences and the classical groups.” *The Electronic Journal of Combinatorics* 5, (1998): #R12.

[40] Riordan, J. *Combinatorial Identities*. Robert E. Krieger Publishing Company, Huntington, New York, 1979.

[41] Samuel, S. “$U(N)$ integrals, $1/N$, and the De Wit-'t Hooft anomalies.” *Journal of Mathematical Physics* 21, no. 12 (1980): 2696–2703.

[42] Shapiro, B., Shapiro, M., and A. Vainshtein. “Ramified coverings of $S^2$ with one degenerate branching point and enumeration of edge-ordered graphs.” *Advances in the Mathematical Sciences* (AMS Translations) 34, (1997): 219–228.

[43] Stanley, R. P. “Parking functions and noncrossing partitions.” *The Electronic Journal of Combinatorics* 4, no. 2 (1997): #R20.

[44] Stanley, R. P. *Enumerative Combinatorics, Volume 2*. Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, 1999.

[45] Stanley, R. P. “Some combinatorial properties of hook lengths, contents, and parts of partitions.” *The Ramanujan Journal* 23, (2010): 91–105.

[46] Tysse, J., and W. Wang. “The centers of spin symmetric group algebras and Catalan numbers.” *Journal of Algebraic Combinatorics* 29, (2009): 175–193.

[47] Xu, F. “A random matrix model from two-dimensional Yang-Mills theory.” *Communications in Mathematical Physics* 190, (1997): 287–307.

[48] Zinn-Justin, P. “Jucys-Murphy elements and Weingarten matrices.” *Letters in Mathematical Physics* 91, (2010): 119–127.