LONG-TIME BEHAVIOR OF SOLUTIONS OF DISSIPATIVE SUPERLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

LUAN HOANG

ABSTRACT. This paper establishes the precise asymptotic behavior, as time $t$ tends to infinity, for nontrivial, decaying solutions of genuinely nonlinear dissipative systems of ordinary differential equations. The lowest order term in these systems, when the spatial variables are small, is not linear, but rather positively homogeneous of a degree greater than one. We prove that the solution behaves like $\xi t^{-p}$, as $t \to \infty$, for a nonzero vector $\xi$ and an explicit number $p > 0$.

CONTENTS

1. Introduction 1
2. A basic case 13
3. Background 17
4. The case of symmetric matrix 20
5. The general case 29
References 34

1. INTRODUCTION

This paper answers a fundamental question about the precise asymptotic behavior of decaying solutions, as time tends to infinity, of a genuine nonlinear dissipative system of ordinary differential equations. The word “genuine” is used roughly here to refer to the case when the main dissipative term in the system is nonlinear. This situation is different from many previously studied dissipative systems of ordinary differential equations (ODE) and partial differential equations (PDE). Our starting point is the following result by Foias and Saut [13] for solutions of the Navier–Stokes equations (NSE) in bounded or periodic domains. They consider the NSE written in the functional form in an appropriate functional space as

$$u' + Au + B(u, u) = 0,$$  \hfill (1.1)

where $A$ is a linear operator with positive eigenvalues and $B(\cdot, \cdot)$ is a bilinear form.

For any nontrivial solution $u(t)$ of (1.1), there exist an eigenvalue $\Lambda$ of $A$ and an eigenfunction $\xi$ of $A$ associated with $\Lambda$ such that

$$e^{\Lambda t} u(t) \to \xi \text{ as } t \to \infty.$$  \hfill (1.2)
Above, the limit holds in any $C^m$-norms.

This result is extended to more general differential inequalities in [15]. As a consequence of [15], if $u(t)$ is a nontrivial, decaying solution of

$$u_t + Au = F(u),$$  \hspace{1cm} (1.3)

where $F(u)$ is a higher order term, as $u \to 0$, then the asymptotic approximation (1.2) holds. The proof in [15] follows that of Foias and Saut in [13]. For systems of ODE, this result is re-established in [12] using a different method.

Note that Foias and Saut also obtain the asymptotic expansions for the solutions of the NSE (1.1) in [14]. However, these asymptotic expansions can be obtained independently from the first approximation in (1.2). The interested reader can find other results on the asymptotic expansions for different ODE and PDE without forcing functions in [12,18,20,23,24], and with forcing functions in [9–11,16,17,19]. In particular, [12] obtains the asymptotic expansions of nontrivial, decaying solutions of (1.3) in the case $F(u)$ is not smooth in any neighborhood of the origin. Roughly speaking, $F(u)$ can be approximated, near the origin, by a finite sum or a series of positively homogeneous terms of possibly nonintegral degrees, see Definition 3.2 below. A crucial step in [12] is the very first asymptotic approximation (1.2).

All of the mentioned papers deal with the equations when the main dissipation term is linear. Having motivated by the general class of positively homogeneous functions in [12], we study a quite different class of equations when the lowest order term is nonlinear. More specifically, we consider the following ODE system in $\mathbb{R}^n$

$$y' = -H(y)Ay + G(t, y),$$  \hspace{1cm} (1.4)

where $A$ is a constant $n \times n$ matrix with positive eigenvalues, $H$ is a positively homogeneous function of degree $\alpha > 0$, and $G$ is a higher order term. More precise conditions will be stated in Sections 3, 4, 5 below.

We study the nontrivial, decaying solutions $y(t)$ of (1.4) as $t \to \infty$. Because of the lack of the linear term, $y(t)$ will not decay exponentially. Rather, it will decay as a power function, see Theorem 3.5 below. We present here a heuristic argument in order to find out certain information about a possible asymptotic approximation for $y(t)$.

We assume, as $t \to \infty$, that

$$y(t) \sim \xi t^{-p}$$ for some nonzero vector $\xi \in \mathbb{R}^n$ and real number $p > 0$.  \hspace{1cm} (1.5)

Using this to approximate both sides of equation (1.4) and ignoring the higher order term $G(t, y)$, we obtain

$$-pt^{-p-1}\xi \sim -t^{-p(\alpha+1)}H(\xi)A\xi. $$

Matching the power of $t$ and the coefficients from both sides gives

$$p = 1/\alpha \text{ and } A\xi = \frac{1}{\alpha H(\xi)}\xi. $$  \hspace{1cm} (1.6)

Thus, $\Lambda \triangleq 1/(\alpha H(\xi))$ is an eigenvalue of $A$, and $\xi$ is an eigenvector of $A$ associated with $\Lambda$. It turns out that the asymptotic approximation (1.5) with (1.6) is exactly what we will obtain rigorously under some appropriate conditions on $A$, $H$ and $G$. This is the goal of the current paper.

The paper is organized as follows. In Section 2 we establish in Theorem 2.1 the asymptotic behavior of the solutions to a simple system which comes up in many applications. In addition
to its own merit, the result serves as a key step in the proof of the main results for more general cases in Sections 4 and 5. In Section 3, we specify the conditions for $T$ and $G$. The basic issues of global existence, uniqueness and asymptotic estimates are established in Theorem 3.5. The estimates in (3.6) actually justify the correct decaying mode that we attempted in the heuristic argument (1.3). Section 4 treats the case when the matrix $A$ is symmetric. The main result in this section is Theorem 4.9 which basically proves (1.5) and (1.6). Preparations for it consist of many steps from Lemma 4.2 to Lemma 4.6. The proof of Theorem 4.9 inherits some original ideas from the work of Foias and Saut [13]. They include studying the asymptotic behavior of the “Dirichlet” quotient $\lambda(t)$ and the function $v(t)$ - a normalization of the solution $y(t)$, see (4.3). However, to treat the genuine nonlinearity in the problem, new ideas and techniques are needed. Notable among them are a new perturbation argument and the newly realized property (HC) for the function $H$.

Notation. For any vector $x \in \mathbb{R}^n$, we denote by $|x|$ its Euclidean norm.

For an $n \times n$ real matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, its Euclidean norm is $\|A\| = (\sum_{i,j=1}^m a_{ij}^2)^{1/2}$, while its norm as a bounded linear operator is $\|A\|_{op} = \max\{|Ax| : |x| = 1\}$.

For two functions $f, g : [T, \infty) \to [0, \infty)$, for some number $T \in \mathbb{R}$, we write

$$f(t) = \mathcal{O}(g(t)) \text{ as } t \to \infty,$$

if there are $T' \geq T$ and $C > 0$ such that $f(t) \leq C g(t)$ for all $t \geq T'$. Very often, “as $t \to \infty$” is implicitly understood and, hence, omitted.

Hereafter, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ is the spatial dimension and is fixed.

2. A Basic Case

Let $a > 0$, $\alpha > 0$ and $t_\ast \geq 0$ be given numbers.

Assume $y \in C^1([t_\ast, \infty), \mathbb{R}^n)$ satisfies $y(t) \neq 0$ for all $t \geq t_\ast$,

$$\lim_{t \to \infty} y(t) = 0,$$

and

$$y' = -a|y|^{\alpha} y + f(t) \text{ for } t > t_\ast,$$

where $f$ is a continuous function from $[t_\ast, \infty)$ to $\mathbb{R}^n$ such that

$$|f(t)| \leq M|y(t)|^{\alpha+1+\delta}, \text{ for all } t \geq t_\ast \text{ and some constants } M, \delta > 0.$$

The asymptotic behavior of $y(t)$, as $t \to \infty$, is the following.
Theorem 2.1. There exists a nonzero vector $\xi_1 \in \mathbb{R}^n$ such that, as $t \to \infty,$
\[
|y(t) - \xi_1 t^{-1/\alpha}| = \mathcal{O}(t^{-1/\alpha-\varepsilon}) \text{ for some } \varepsilon > 0.
\] (2.4)

Moreover,
\[
\alpha a |\xi_1|^{\alpha} = 1.
\] (2.5)

Proof. By increasing the initial time $t_*$ to be sufficiently large, we can assume, without loss of generality, that $|y(t)| \leq 1$ for all $t \geq t_*$, and $\delta < \alpha.$

For $t > t_*$, we calculate
\[
\frac{d}{dt}(|y|^{-\alpha}) = -\frac{\alpha}{|y|^{\alpha+2}} f(t) \cdot y = \alpha a - \frac{\alpha f(t) \cdot y}{|y|^{\alpha+2}}.
\] (2.6)

By Cauchy-Schwarz’s inequality and condition (2.3), we have
\[
\frac{|f(t) \cdot y(t)|}{|y(t)|^{\alpha+2}} \leq M |y(t)|^\delta \text{ for } t > t_*.
\] (2.7)

We fix two positive numbers $a_1$ and $a_2$ such that $a_1 < a < a_2$. Thanks to (2.4) and (2.7), there is a positive number $t_0 \geq t_*$ such that one has
\[
\alpha a_1 \leq \frac{d}{dt} |y|^{-\alpha} \leq \alpha a_2 \text{ for all } t > t_0.
\]

Hence, for $t \geq t_0 > 0$,
\[
|y(t)|^\alpha \geq \frac{1}{|y(t_0)|^{-\alpha} + \alpha a_2 (t - t_0)} \geq C_1^\alpha t^{-1},
\]
\[
|y(t)|^\alpha \leq \frac{1}{|y(t_0)|^{-\alpha} + \alpha a_1 (t - t_0)} \leq C_2^\alpha t^{-1},
\]

where $C_1$ and $C_2$ are some positive numbers. We obtain
\[
C_1 t^{-1/\alpha} \leq |y(t)| \leq C_2 t^{-1/\alpha} \text{ for all } t \geq t_0.
\] (2.8)

As a consequence of (2.3) and (2.8), we have
\[
|f(t)| \leq M_1 t^{-1-1/\alpha-\delta/\alpha} \text{ for all } t \geq t_0, \text{ where } M_1 = MC_2^{1+\alpha+\delta} > 0.
\] (2.9)

Integrating equation (2.6) gives
\[
|y(t)|^{-\alpha} - |y(t_0)|^{-\alpha} = a a_1 (t - t_0) + g(t), \text{ where } g(t) = -\frac{\lambda}{|y(t)|^{-\alpha} + \alpha a_1 (t - t_0) + g(t)}.
\]

Hence, for all $t \geq t_0$, one has $|y(t_0)|^{-\alpha} + \alpha a_1 (t - t_0) + g(t) > 0$ and
\[
|y(t)|^\alpha = \frac{1}{|y(t_0)|^{-\alpha} + a a_1 (t - t_0) + g(t)}.
\] (2.10)

Using (2.7) and the upper bound of $|y(t)|$ in (2.8), we estimate
\[
|g(t)| \leq \frac{\alpha}{1 - \delta/\alpha} \int_{t_0}^t \frac{|f(\tau)| \cdot y(\tau)}{|y(\tau)|^{\alpha+2}} d\tau \leq \alpha M \int_{t_0}^t |y(\tau)|^\delta d\tau
\]
\[
\leq \alpha MC_2^\delta \int_{t_0}^t \tau^{-\delta/\alpha} d\tau = \frac{\alpha MC_2^\delta}{1 - \delta/\alpha} (t^{1-\delta/\alpha} - t_0^{1-\delta/\alpha}).
\]

Setting $C_3 = \alpha^2 MC_2^\delta / (\alpha - \delta) > 0$, we obtain
\[
|g(t)| \leq C_3 t_0^{1-\delta/\alpha} \text{ for all } t \geq t_0.
\] (2.11)
We consider equation (2.2) as a linear equation of $y$ with time-dependent coefficient $-a|y(t)|^\alpha$ and forcing function $f(t)$. By the variation of constants formula, we solve for $y(t)$ explicitly as

$$y(t) = e^{-J(t)} \left( y(t_0) + \int_{t_0}^{t} e^{J(\tau)} f(\tau) d\tau \right) \text{ for } t \geq t_0,$$

where

$$J(t) = a \int_{t_0}^{t} |y(\tau)|^\alpha d\tau.$$  \hspace{1cm} (2.12)

Using (2.10) in (2.12), we rewrite $J(t)$ as

$$J(t) = \int_{t_0}^{t} \frac{a}{|y(t_0)|^{-\alpha} + a\alpha(\tau - t_0) + g(\tau)} d\tau = J_1(t) - J_2(t),$$

where

$$J_1(t) = \int_{t_0}^{t} \frac{a}{|y(t_0)|^{-\alpha} + a\alpha(\tau - t_0)} d\tau \text{ and } J_2(t) = \int_{t_0}^{t} h(\tau) d\tau,$$  \hspace{1cm} (2.13)

with

$$h(\tau) = \frac{ag(\tau)}{(|y(t_0)|^{-\alpha} + a\alpha(\tau - t_0))(|y(t_0)|^{-\alpha} + a\alpha(\tau - t_0) + g(\tau))}.$$  \hspace{1cm} (2.14)

Clearly,

$$J_1(t) = \frac{1}{\alpha} \ln(1 + |y(t_0)|^\alpha a\alpha(t - t_0)).$$  \hspace{1cm} (2.15)

Therefore,

$$y(t) = \frac{e^{J_2(t)} \left( y(t_0) + \int_{t_0}^{t} e^{J(\tau)} f(\tau) d\tau \right)}{(1 + |y(t_0)|^\alpha a\alpha(t - t_0))^{1/\alpha}}.$$  \hspace{1cm} (2.16)

Consider the integrand $h(\tau)$ of $J_2(t)$. Taking into account the estimate of $|g(\tau)|$ in (2.11), we assert that there is $t_1 \geq t_0$ such that, for $\tau \geq t_1$,

$$|y(t_0)|^{-\alpha} + a\alpha(\tau - t_0) \geq a\alpha \tau/2,$$

$$|y(t_0)|^{-\alpha} + a\alpha(\tau - t_0) + g(\tau) \geq a\alpha(\tau - t_0) - C_3 \tau^{1-\delta/\alpha} \geq a\alpha \tau/2.$$

Combining these estimates with (2.11) yields, as $\tau \to \infty$,

$$|h(\tau)| = O(|g(\tau)|^{\tau^{-2}}) = O(\tau^{1-\delta/\alpha} \tau^{-2}) = O(\tau^{-1-\delta/\alpha}).$$  \hspace{1cm} (2.17)

Therefore,

$$\lim_{t \to \infty} J_2(t) = \int_{t_0}^{\infty} h(\tau) d\tau = J_* \in \mathbb{R},$$  \hspace{1cm} (2.18)

and

$$J_2(t) = J_* + h_1(t), \text{ where } h_1(t) = -\int_{t}^{\infty} h(\tau) d\tau.$$  \hspace{1cm} (2.19)

Estimate (2.15) implies

$$|h_1(t)| = O(t^{-\delta/\alpha}).$$  \hspace{1cm} (2.20)

Consequently,

$$e^{J_2(t)} = e^{J_*} e^{h_1(t)} = e^{J_*} + h_2(t), \text{ where } h_2(t) = e^{J_*} (e^{h_1(t)}) - 1.$$  \hspace{1cm} (2.21)
Since \( h_1(t) \to 0 \) as \( t \to \infty \), we have
\[
|h_2(t)| = O(|h_1(t)|) = O(t^{-\delta/\alpha}).
\] (2.18)

Regarding the integral in formula (2.14), we use (2.9), (2.13) and (2.16) to have
\[
e^{J(t)}|f(t)| = (1 + |y(t_0)|^\alpha a\alpha(t - t_0))^{1/\alpha} e^{-J_2(t)}|f(t)|
= O(t^{1/\alpha} \cdot 1 \cdot t^{-1-1/\alpha-\delta/\alpha}) = O(t^{-1-\delta/\alpha}).
\] (2.19)

Therefore,
\[
\lim_{t \to \infty} \int_t^{t_0} e^{J(\tau)} f(\tau) d\tau = \int_{t_0}^{\infty} e^{J(\tau)} f(\tau) d\tau = \eta_\ast \in \mathbb{R}^n,
\]
and
\[
\int_{t_0}^{t} e^{J(\tau)} f(\tau) d\tau = \eta + \eta(t), \text{ where } \eta(t) = -\int_t^{\infty} e^{J(\tau)} f(\tau) d\tau.
\] (2.20)

It follows (2.19) that
\[
|\eta(t)| = O(t^{-\delta/\alpha}).
\] (2.21)

Combining (2.14), (2.17) and (2.20) gives
\[
y(t) = \frac{e^{J_\ast} + h_2(t)}{(1 + |y(t_0)|^\alpha a\alpha(t - t_0))^{1/\alpha}} (y(t_0) + \eta_\ast + \eta(t)) \text{ for } t \geq t_0.
\]

This expression and properties (2.18), (2.21) imply
\[
\left| y(t) - \frac{e^{J_\ast}(y(t_0) + \eta_\ast)}{(1 + |y(t_0)|^\alpha a\alpha(t - t_0))^{1/\alpha}} \right| = O(t^{-1/\alpha - \delta/\alpha}).
\] (2.22)

We write
\[
\frac{1}{(1 + |y(t_0)|^\alpha a\alpha(t - t_0))^{1/\alpha}} = \frac{1}{|y(t_0)|^\alpha (a\alpha t_0) \cdot \left(1 + \frac{1 - |y(t_0)|^\alpha a\alpha t_0}{|y(t_0)|^\alpha a\alpha t_0} \right)^{-1/\alpha}}.
\]

We use the approximation \(|(1 + x)^{-1/\alpha} - 1| = O(|x|)\) as \( x \in \mathbb{R}, x \to 0 \), and then substitute \( x = x(t) := \frac{1 - \lim_{t_0}^t a\alpha t_0}{|y(t_0)|^\alpha a\alpha t_0} \) when \( t \) is large. We obtain
\[
\left| \frac{1}{(1 + |y(t_0)|^\alpha a\alpha(t - t_0))^{1/\alpha}} - \frac{1}{|y(t_0)|^\alpha (a\alpha t_0) \cdot \left(1 + \frac{1 - |y(t_0)|^\alpha a\alpha t_0}{|y(t_0)|^\alpha a\alpha t_0} \right)^{-1/\alpha}} \right| = O\left(\frac{|x(t)|}{|y(t_0)|^\alpha (a\alpha t_0) \cdot \left(1 + \frac{1 - |y(t_0)|^\alpha a\alpha t_0}{|y(t_0)|^\alpha a\alpha t_0} \right)^{-1/\alpha}}\right) = O(t^{-1/\alpha - 1}).
\]

Combining this fact with (2.22), we derive
\[
|y(t) - \xi_\ast t^{-\alpha}| = O(t^{-1/\alpha - \delta/\alpha} + t^{-1/\alpha - 1}) = O(t^{-1/\alpha - \delta/\alpha}),
\]
with
\[
\xi_\ast = \frac{e^{J_\ast}}{|y(t_0)|^\alpha (a\alpha t_0) \cdot \left(1 + \frac{1 - |y(t_0)|^\alpha a\alpha t_0}{|y(t_0)|^\alpha a\alpha t_0} \right)^{-1/\alpha}} (y(t_0) + \eta_\ast) \in \mathbb{R}^n.
\]

Therefore, we obtain the desired estimate (2.4).

Because of the lower bound of \(|y(t)|\) in (2.8), the vector \( \xi_\ast \) in (2.4) must be non-zero.

It remains to prove identity (2.5). By the triangle inequality and (2.4), one has
\[
||y(t)| - |\xi_\ast| t^{-\alpha}| = O(t^{-1/\alpha - \varepsilon}).
\] (2.23)

From (2.10),
\[
|y(t)| = \frac{1}{(a\alpha t_0)^{1/\alpha}} \cdot \left(1 + \frac{|y(t_0)|^{-\alpha} - a\alpha t_0 + g(t)}{a\alpha} \right)^{-1/\alpha}.
\]
Taking into account estimate (2.11) of \(|g(t)|\), we have, as \(t \to \infty\),
\[
\left| y(t) \right| - \frac{t^{-1/\alpha}}{(a\alpha)^{1/\alpha}} = O \left( \frac{1}{t^{1/\alpha}} \cdot \frac{|y(t_0)|^{1-\alpha} - a\alpha t_0 + g(t)}{a\alpha t} \right) = O \left( \frac{1}{t^{1/\alpha}} \cdot \frac{|g(t)|}{t} \right),
\]
which yields
\[
\left| y(t) \right| - \frac{t^{-1/\alpha}}{(a\alpha)^{1/\alpha}} = O(t^{-1/\alpha - \delta/\alpha}).
\] (2.24)

Comparing two asymptotic approximations of the norm \(|y(t)|\), as \(t \to \infty\), in (2.23) and (2.24), one must have \(|\xi_*| = 1/(a\alpha)^{1/\alpha}\), which proves (2.25). The proof is complete. \(\square\)

We note from (2.23) and (2.25) that the norm \(|y(t)|\) can be asymptotically approximated, as \(t \to \infty\), by \(|\xi_*|t^{-1/\alpha}\) with \(|\xi_*| = (a\alpha)^{-1/\alpha}\) independent of the solution \(y(t)\). This, in fact, agrees with formula (2.11).

3. Background

Consider the ODE system (1.4) in \(\mathbb{R}^n\). We specify the conditions for \(A, H\) and \(G\) in the following.

**Assumption 3.1.** Hereafter, \(A\) is a (real) diagonalizable \(n \times n\) matrix with positive eigenvalues.

Thanks to Assumption 3.1 the spectrum \(\sigma(A)\) of matrix \(A\) consists of eigenvalues \(\Lambda_k\)'s, for \(1 \leq k \leq n\), which are positive and increasing in \(k\). Then there exists an invertible \(n \times n\) (real) matrix \(S\) such that
\[
A = S^{-1}A_0S, \quad \text{where} \quad A_0 = \text{diag}[\Lambda_1, \Lambda_2, \ldots, \Lambda_n].
\] (3.1)

Denote the distinct eigenvalues of \(A\) by \(\lambda_j\)'s which are strictly increasing in \(j\), i.e.,
\[
0 < \lambda_1 = \Lambda_1 < \lambda_2 < \ldots < \lambda_d = \Lambda_n \quad \text{for some integer} \ d \in [1, n].
\]

In the case \(A\) is symmetric, the matrix \(S\) is orthogonal, i.e., \(S^{-1} = S^T\), and
\[
\Lambda_1|x|^2 \leq x \cdot Ax \leq \Lambda_n|x|^2 \quad \text{for all} \ x \in \mathbb{R}^n.
\] (3.2)

Regarding the first nonlinearity in (1.4), the function \(H\) will be in the following class.

**Definition 3.2.** Let \(X\) and \(Y\) be two (real) linear spaces.

A function \(F : X \to Y\) is positively homogeneous of degree \(\beta > 0\) if
\[
F(tx) = t^\beta F(x) \quad \text{for any} \ x \in X \ \text{and} \ t > 0.
\] (3.3)

Define \(\mathcal{H}_\beta(X, Y)\) to be the set of positively homogeneous functions of degree \(\beta\) from \(X\) to \(Y\).

In Definition 3.2 by taking \(x = 0\) and \(t = 2\) in (3.3), one has \(F(0) = 0\). Consequently, (3.3) holds for all \(t \geq 0\).

Clearly, each \(\mathcal{H}_\beta(X, Y)\) is a linear space, and the zero function belongs to \(\mathcal{H}_\beta(X, Y)\) for all \(\beta > 0\). If \(F : X \to Y\) is a homogeneous polynomial of degree \(m \in \mathbb{N}\), then \(F \in \mathcal{H}_m(X, Y)\). The spaces \(\mathcal{H}_\beta(X, Y)\)'s can contain much more complicated functions than polynomials and finite sums of power monomials, see [12] for many examples.

**Assumption 3.3.** The function \(H\) belongs to \(\mathcal{H}_\alpha(\mathbb{R}^n, \mathbb{R})\) for some \(\alpha > 0\), is continuous on \(\mathbb{R}^n\), and \(H(x) > 0\) for all \(x \in \mathbb{R}^n \setminus \{0\}\).
One has from Assumption 3.3 that
\[ 0 < c_1 = \min_{|x|=1} H(x) \leq \max_{|x|=1} H(x) = c_2 < \infty. \]

For \( x \in \mathbb{R}^n \setminus \{0\} \), we have \( H(x) = |x|^\alpha H(x/|x|) \in [c_1|x|^\alpha, c_2|x|^\alpha] \). Together with the fact \( H(0) = 0 \), this yields
\[ c_1|x|^\alpha \leq H(x) \leq c_2|x|^\alpha \text{ for all } x \in \mathbb{R}^n. \quad (3.4) \]

**Assumption 3.4.** We assume the followings.
(i) The function \( x \in \mathbb{R}^n \rightarrow H(x)Ax \) is locally Lipschitz.
(ii) There is \( t_0 \geq 0 \) such that the function \( G(t, x) \) is continuous on \([t_0, \infty) \times \mathbb{R}^n\), Lipschitz with respect to \( x \) on any compact subsets of \([t_0, \infty) \times \mathbb{R}^n\).
(iii) There exist positive numbers \( c_*, r_*, \delta \) such that
\[ |G(t, x)| \leq c_*|x|^{1+\alpha+\delta} \text{ for all } t \geq t_*, \text{ and all } x \in \mathbb{R}^n \text{ with } |x| \leq r_. \]

It follows (3.5) that \( G(t, 0) = 0 \) for all \( t \geq t_* \). Hence, \( y \equiv 0 \) on \([t_*, \infty)\) is the trivial solution of (1.4).

We first obtain the global existence and uniqueness of solutions of (1.4) with small initial data, and find their lower and upper bounds for all time.

**Theorem 3.5.** There is \( r_0 > 0 \) such that if \( y_0 \in \mathbb{R}^n \) satisfies \( 0 < |y_0| < r_0 \), then there exists a unique solution \( y(t) \in C^1([t_*, \infty), \mathbb{R}^n) \) of (1.4) on \((t_*, \infty)\) and \( y(t_*) = y_0 \).

Moreover, there are two positive constants \( C_1 \) and \( C_2 \) such that
\[ C_1(1+t)^{-1/\alpha} \leq |y(t)| \leq C_2(1+t)^{-1/\alpha} \text{ for all } t \geq t_. \quad (3.6) \]

**Proof.** Part 1. Consider \( A \) is symmetric first. Select \( r_0 > 0 \) such that
\[ 2r_0 \leq r_* \text{ and } c_*(2r_0)^\delta \leq c_1\Lambda_1. \quad (3.7) \]

Let \( y_0 \) be any vector in \( \mathbb{R}^n \) with \( 0 < |y_0| < r_0 \). There exists a maximal \( T_{\text{max}} \in (t_*, \infty) \) and a unique solution \( y(t) \in C^1([t_*, T_{\text{max}}), \mathbb{R}^n) \) of (1.4) on \((t_*, T_{\text{max}})\) with \( y(t_*) = y_0 \) such that
\[ |y(t)| < 2r_0 \text{ for all } t \in [t_*, T_{\text{max}}). \quad (3.8) \]

We claim that \( T_{\text{max}} = \infty \). Suppose this is not true, i.e., \( T_{\text{max}} < \infty \). Then
\[ \lim_{t \to T_{\text{max}}} |y(t)| = 2r_0. \quad (3.9) \]

Taking the dot product of the equation with \( y \) gives
\[ \frac{1}{2} \frac{d}{dt} |y|^2 = -H(y)(Ay) \cdot y + G(t, y) \cdot y. \]

By (3.8) and (3.7), \( |y(t)| \leq r_* \text{ on } [t_*, T_{\text{max}}] \). Together with (3.5), this implies
\[ |G(t, y(t)) \cdot y(t)| \leq c_*|y(t)|^{2+\alpha+\delta} \text{ for all } t \in [t_*, T_{\text{max}}). \quad (3.10) \]

By (3.4), (3.2) and (3.10), one has
\[ \frac{1}{2} \frac{d}{dt} |y|^2 \leq -(c_1\Lambda_1 - c_*|y|^\delta)|y|^{2+\alpha} \text{ for all } t \in (t_*, T_{\text{max}}). \]

Because of (3.8) and (3.7), we have
\[ c_*|y(t)|^\delta \leq c_*(2r_0)^\delta \leq c_1\Lambda_1 \text{ for all } t \in [t_*, T_{\text{max}}). \]
Thus, \( \frac{d}{dt}|y(t)|^2 \leq 0 \) on \((t_*, T_{\text{max}})\), which implies
\[
|y(t)| \leq |y_0| < r_0 \text{ for all } t \in [t_*, T_{\text{max}}).
\] (3.11)
This yields a contradiction to (3.9). Therefore, \( T_{\text{max}} = \infty \).

We prove (3.6) now. By the uniqueness/backward uniqueness of the solutions of (1.4), one has \( y(t) \neq 0 \) for all \( t \in [t_*, \infty) \). We calculate, for \( t > t_* \),
\[
\frac{d}{dt}(|y|^{-\alpha}) = \alpha \left( \frac{H(y)(Ay) \cdot y}{|y|^{\alpha+2}} - \frac{G(t, y) \cdot y}{|y|^{\alpha+2}} \right). \tag{3.12}
\]
Utilizing (3.4), (3.2) and (3.10) again, one has
\[
\alpha(c_1 \Lambda_1 - c_* |y|^{\delta}) \leq \frac{d}{dt}(|y|^{-\alpha}) \leq \alpha(c_2 \Lambda_n + c_* |y|^{\delta}).
\]
Let \( a_1, a_2 \) be any positive numbers such that \( a_1 < c_1 \Lambda_1 \) and \( a_2 > c_2 \Lambda_n \). We select \( r_0 \) that satisfies the following additional condition
\[
c_* r_0^{\delta} \leq \min\{c_1 \Lambda_1 - a_1, a_2 - c_2 \Lambda_n\}. \tag{3.13}
\]
Together with (3.11), we obtain \( \alpha a_1 \leq \frac{d}{dt}(|y|^{-\alpha}) \leq \alpha a_2 \) for all \( t > t_* \). Hence,
\[
\frac{1}{|y_0|^{-\alpha} + \alpha a_2 (t - t_*)} \leq |y(t)|^{\alpha} \leq \frac{1}{|y_0|^{-\alpha} + \alpha a_1 (t - t_*)} \tag{3.14}
\]
for all \( t \geq t_* \). Then the desired estimates in (3.6) follow.

**Part 2.** Now, consider the general case. Let \( A = S^{-1} A_0 S \) as in (3.11). We make the change of variables \( z = S y \). Note that
\[
\|S^{-1}\|_{\text{op}}^{-1} |y| \leq |z| \leq \|S\|_{\text{op}} |y|. \tag{3.15}
\]
Then equation (1.4) with initial condition \( y(t_*) = y_0 \) is equivalent to
\[
z' = -\tilde{H}(z) A_0 z + \tilde{G}(t, z), \quad z(t_*) = z_0 \overset{\text{def}}{=} S y_0, \tag{3.16}
\]
where
\[
\tilde{H}(z) = H(S^{-1} z) \text{ and } \tilde{G}(t, z) = S G(t, S^{-1} z) \text{ for } z \in \mathbb{R}^n. \tag{3.17}
\]
Using the relations in (3.15), we can apply the result in Part 1 to equation (3.16). If \( |z_0| \neq 0 \) is sufficiently small, the unique solution \( z(t) \) exists for all \( t \geq t_* \), and satisfies
\[
C_1'(1 + t)^{-1/\alpha} \leq |z(t)| \leq C_2'(1 + t)^{-1/\alpha} \text{ for all } t \geq t_*, \tag{3.18}
\]
where \( C_1', C_2' \) are two positive constants. Consequently, when \( |y_0| \neq 0 \) is sufficiently small, the unique solution \( y(t) \) of (1.4) on \((t_*, \infty)\) with \( y(t_*) = y_0 \) is \( S^{-1} z(t) \), for \( t \geq t_* \), and, thanks to (3.15) and (3.18), the estimates in (3.6) hold true. We omit the details. \( \square \)

As a consequence of (3.6), the solution \( y(t) \) in Theorem 3.5 goes to zero as \( t \to \infty \). This type of solutions will be the subject of our investigations in the next two sections.
4. The Case of Symmetric Matrix

Assume, throughout this section, that the matrix $A$ is symmetric.

Together with Assumption 3.1, this implies that the matrix $A$ is positive definite. Denote by $A^{1/2}$ the square root matrix of $A$, which is symmetric, positive definite and $(A^{1/2})^2 = A$.

For $j = 1, 2, \ldots, d$, denote by $R_{\lambda_j}$ the orthogonal projection from $\mathbb{R}^n$ to the eigenspace of $A$ associated with the eigenvalue $\lambda_j$. Then

$$|R_{\lambda_j} x| \leq |x| \text{ for } j = 1, 2, \ldots, d \text{ and all } x \in \mathbb{R}^n. \quad (4.1)$$

Let the function $H$ satisfy Assumption 3.3. Assume the following conditions on $G(t, x)$.

**Assumption 4.1.** The function $G(t, x)$ is continuous on $[t_*, \infty) \times \mathbb{R}^n$ for some $t_*, \delta > 0$, and there exist numbers $T_*, T > T_*$ and $c_*, r_*, \delta > 0$ such that

$$|G(t, x)| \leq c_*|x|^{1+\alpha+\delta} \text{ for all } t \geq T_*, \text{ and all } x \in \mathbb{R}^n \text{ with } |x| \leq r_* \quad (4.2)$$

Assume $y(t)$ is a function in $C^1([t_*, \infty), \mathbb{R}^n)$ that solves equation (1.4) on $(t_*, \infty)$, $y(t) \neq 0$ for all $t \geq t_*$, and

$$\lim_{t \to \infty} y(t) = 0. \quad (4.3)$$

Recall that such a solution $y(t)$ exists under appropriate conditions as shown in Theorem 3.5. Moreover, this theorem essentially gives the following preliminary asymptotic estimates for $|y(t)|$.

**Lemma 4.2.** There exist numbers $\bar{T} > T_*$ and $C_1, C_2 > 0$ such that

$$C_1 t^{-1/\alpha} \leq |y(t)| \leq C_2 t^{-1/\alpha} \text{ for all } t \geq \bar{T}. \quad (4.4)$$

Moreover, one has

$$\frac{1}{(\alpha c_2 \Lambda_n)^{1/\alpha}} \leq \lim inf_{t \to \infty} t^{1/\alpha} |y(t)| \leq \lim sup_{t \to \infty} t^{1/\alpha} |y(t)| \leq \frac{1}{(\alpha c_1 \Lambda_1)^{1/\alpha}}. \quad (4.5)$$

**Proof.** Suppose $a_1, a_2 > 0$ are two numbers such that $a_1 < c_1 \Lambda_1 \leq c_2 \Lambda_n < a_2$.

Take $r_0 > 0$ that satisfies $r_0 \leq r_*$ and (3.13). Because of the limit (4.3), there is a number $\bar{T} > T_*$ such that

$$|y(t)| < r_0 \text{ for all } t \geq \bar{T}. \quad (4.6)$$

Performing the same calculations as in the proof of Theorem 3.5 from (3.12) to the end of Part 1 with $\bar{T}$ replacing $t_*$, we obtain, similar to (3.14),

$$\frac{1}{(|y(\bar{T})|^{-\alpha} + \alpha a_2(t - \bar{T}))^{1/\alpha}} \leq |y(t)| \leq \frac{1}{(|y(\bar{T})|^{-\alpha} + \alpha a_1(t - \bar{T}))^{1/\alpha}}. \quad (4.7)$$

for all $t \geq \bar{T}$. Because $T > 0$ and $|y(T)| > 0$, we derive from (4.7) the estimates in (4.4) for some positive constants $C_1, C_2$ depending on $a_1, a_2$. By the particular choice $a_1 = c_1 \Lambda_1/2$ and $a_2 = c_2 \Lambda_n + 1$, the numbers $\bar{T}$ and $C_1, C_2$ are now fixed in (4.4).

It follows (4.6) that

$$\frac{1}{(\alpha a_2)^{1/\alpha}} \leq \lim inf_{t \to \infty} t^{1/\alpha} |y(t)| \leq \lim sup_{t \to \infty} t^{1/\alpha} |y(t)| \leq \frac{1}{(\alpha a_1)^{1/\alpha}}$$

for any numbers $a_1 \in (0, c_1 \Lambda_1)$ and $a_2 \in (c_2 \Lambda_n, \infty)$. Thus, we obtain (4.5). \qed
As a consequence of (4.4) and (3.4), one has, for all \( t \geq T \),
\[
C_3 t^{-1} \leq H(y(t)) \leq C_4 t^{-1},
\]
where \( C_3 = c_1 C_1^\alpha \) and \( C_4 = c_2 C_2^\alpha \).

For \( t \geq t_* \), define
\[
\lambda(t) = \frac{|A^{1/2} y(t)|^2}{|y(t)|^2} = \frac{y(t) \cdot Ay(t)}{|y(t)|^2} \quad \text{and} \quad v(t) = \frac{y(t)}{|y(t)|}.
\]
Then \( \lambda \in C^1([t_*, \infty)) \) and, thanks to (3.2),
\[
\Lambda_1 \leq \lambda(t) \leq \Lambda_n \leq \|A\| \text{ for all } t \geq t_*,
\]
Hence,
\[
\Lambda_1 \leq \liminf_{t \to \infty} \lambda(t) \leq \limsup_{t \to \infty} \lambda(t) \leq \Lambda_n.
\]
Also, \( v \in C^1([t_*, \infty), \mathbb{R}^n) \) and
\[
|v(t)| = 1 \text{ for all } t \geq t_*. \tag{4.11}
\]

**Lemma 4.3.** The quotient \( \lambda(t) \) converges, as \( t \to \infty \), to an eigenvalue \( \Lambda \) of \( A \).

**Proof.** We find a differential equation for \( \lambda(t) \). Using the fact that \( A \) is symmetric, we have, for \( t > t_* \),
\[
\lambda'(t) = \frac{1}{|y|^2} \frac{d}{dt} |A^{1/2} y|^2 - \frac{|A^{1/2} y|^2}{|y|^4} \frac{d}{dt} |y|^2 = \frac{2}{|y|^2} y \cdot Ay - \frac{2|A^{1/2} y|^2}{|y|^4} y \cdot y.
\]
Hence,
\[
\lambda'(t) = \frac{2}{|y|^2} y \cdot (Ay - \lambda y). \tag{4.12}
\]
We rewrite equation (1.3) as
\[
y' = -H(y)(Ay - \lambda y) - \lambda H(y)y + G(t,y).
\]
Using this expression of \( y' \) in (4.12) yields
\[
\lambda'(t) = -\frac{2 H(y)}{|y|^2} Ay - \lambda y \frac{2 \lambda H(y)}{|y|^2} y \cdot (Ay - \lambda y) + h(t), \tag{4.13}
\]
where
\[
h(t) = \frac{2}{|y(t)|^2} G(t, y(t)) \cdot (Ay(t) - \lambda(t)y(t)).
\]
Note that the second term on the right-hand side of (4.13) vanishes thanks to the fact \( y \cdot (Ay - \lambda y) = 0 \). Then equation (4.13) reduces to
\[
\lambda'(t) = -2 H(y)|Av - \lambda v|^2 + h(t). \tag{4.14}
\]
Using (4.2), (4.9), (4.11) and, then, (4.1), we estimate
\[
|h(t)| \leq 2c_*|y(t)|^{\alpha + \delta}(2\|A\|)^2 \leq C_5 t^{-1 - \delta/\alpha}, \tag{4.15}
\]
for all \( t \geq \bar{T} \), where \( C_5 = 8c_*\|A\|^2 C_4^{\alpha + \delta} \).

For \( t' > t \geq \bar{T} \), integrating equation (4.14) from \( t \) to \( t' \) gives
\[
\lambda(t') - \lambda(t) + 2 \int_t^{t'} H(y(\tau))|Av(\tau) - \lambda(\tau)v(\tau)|^2 d\tau = \int_t^{t'} h(\tau) d\tau. \tag{4.16}
\]
Note from (4.15) that
\[ \left| \int_t^{t'} h(\tau) \, d\tau \right| \leq \frac{\alpha C_5}{\delta} t^{-\delta/\alpha}. \]
Thus, taking the limit superior of (4.16), as \( t' \to \infty \), yields
\[ \lim_{t' \to \infty} \sup \lambda(t') \leq \lambda(t) + \frac{\alpha C_5}{\delta} t^{-\delta/\alpha} < \infty. \] (4.17)

Then taking the limit inferior of (4.17), as \( t \to \infty \), gives
\[ \lim_{t' \to \infty} \sup \lambda(t') \leq \lim_{t \to \infty} \inf \lambda(t). \]
This and (4.10) imply \( \lambda(t) \) converges as \( t \to \infty \), and
\[ \lim_{t \to \infty} \lambda(t) = \Lambda \in [\Lambda_1, \Lambda_n]. \] (4.18)

Next, we prove that \( \Lambda \) is an eigenvalue of \( A \). Using property (4.18) in (4.16) and the Cauchy criterion, as \( t, t' \to \infty \), we obtain
\[ \int_{\tau}^{\infty} H(y(\tau)) |Av(\tau) - \lambda(\tau)v(\tau)|^2 \, d\tau < \infty. \] (4.19)

**Claim.** \( \forall \varepsilon > 0, \exists T \geq \max\{ \frac{\delta}{\alpha} \} : |Av(t) - \lambda(t)v(t)| < \varepsilon. \)

We accept this claim momentarily. Then there exists a sequence \( t_j \to \infty \), as \( j \to \infty \), such that
\[ \lim_{j \to \infty} |Av(t_j) - \lambda(t_j)v(t_j)| = 0. \] (4.20)

Because of (4.11) and by taking a subsequence of \( (t_j)_{j=1}^{\infty} \), we can assume \( v(t_j) \to \bar{v} \in \mathbb{R}^n \), as \( j \to \infty \), with \( |\bar{v}| = 1 \). We already have from (4.18) that \( \lambda(t_j) \to \Lambda \). Then passing \( j \to \infty \) in (4.20) gives \( A\bar{v} = \Lambda \bar{v} \). Thus, \( \Lambda \) is an eigenvalue of \( A \).

Finally, we prove the Claim. Suppose the Claim is not true. Then
\[ \exists \varepsilon_0 > 0, \forall t \geq \max\{ \frac{\delta}{\alpha}, 1/\varepsilon_0 \} : |Av(t) - \lambda(t)v(t)| \geq \varepsilon_0. \] (4.21)

Let \( T = \max\{ \frac{\delta}{\alpha}, 1/\varepsilon_0 \} \). Combining (4.21) with property (4.17), we have
\[ \int_{T}^{\infty} H(y(\tau)) |Av(\tau) - \lambda(\tau)v(\tau)|^2 \, d\tau \geq \int_{T}^{\infty} C_3 \tau^{-1} \varepsilon_0^2 \, d\tau = \infty, \]
which contradicts (4.19). Hence, the Claim is true and the proof of Lemma 4.3 is complete.

For the remainder of this section, \( \Lambda \) is the eigenvalue in Lemma 4.3.

**Lemma 4.4.** There is \( \varepsilon > 0 \) such that
\[ |(\text{Id} - R_{\Lambda})v(t)| = O(t^{-\varepsilon}) \text{ as } t \to \infty. \] (4.22)

**Proof.** We find a differential equation for \( v(t) \). We compute
\[ v' = \frac{1}{|y|} y' - \frac{1}{|y|^3} (y' \cdot y) y \]
\[ = \frac{1}{|y|} (-H(y)Ay + G(t,y)) - \frac{1}{|y|^3} ((-H(y)Ay + G(t,y)) \cdot y) y \]
\[ = -H(y)Av + H(y)|A^{1/2}v|^2 v + g(t), \]
where \( g : [t_*, \infty) \to \mathbb{R}^n \) is defined by
\[
g(t) = \frac{1}{|y(t)|} G(t, y(t)) - \frac{G(t, y(t)) \cdot y(t)}{|y(t)|^3} y(t).
\]
Thanks to the fact \(|A^{1/2}v|^2 = \lambda(t)\), we have
\[
v' = -H(y)(Av - \lambda v) + g(t) \quad \text{for all } t > t_*. \tag{4.23}
\]
Using property (4.2) of \( G(t, y) \), one can estimate
\[
|g(t)| \leq 2c_\ast |y(t)|^{\alpha + \delta} \quad \text{for all } t \geq T_\ast. \tag{4.24}
\]
We write
\[
(Id - R_\lambda)v = \sum_{1 \leq j \leq d, \lambda_j \neq \lambda} R_{\lambda_j}v. \tag{4.25}
\]
We will estimate each \(|R_{\lambda_j}v(t)|\) on the right-hand side of (4.25).
Define \( \mu = \min\{|\lambda_j - \Lambda| : 1 \leq j \leq d, \lambda_j \neq \Lambda\} > 0. \)
Let \( \lambda_j \in \sigma(A) \setminus \{\Lambda\}. \) Applying \( R_{\lambda_j} \) to equation (4.23) and taking the dot product with \( R_{\lambda_j}v \)
\[
\frac{1}{2} \frac{d}{dt}|R_{\lambda_j}v|^2 = -H(y)(\lambda_j - \lambda)|R_{\lambda_j}v|^2 + R_{\lambda_j}g(t) \cdot R_{\lambda_j}v. \tag{4.26}
\]
By Cauchy–Schwarz’s inequality, then Cauchy’s inequality, and property (3.4), we have
\[
|R_{\lambda_j}g(t) \cdot R_{\lambda_j}v| \leq 2c_\ast |y|^{\alpha + \delta}|R_{\lambda_j}v| \leq \frac{\mu}{4} H(y)|R_{\lambda_j}v|^2 + \frac{4c_\ast^2 |y|^{2\alpha + 2\delta}}{\mu H(y)}
\]
\[
\leq \frac{\mu}{4} H(y)|R_{\lambda_j}v|^2 + \frac{4c_\ast^2}{\mu c_1} |y|^{\alpha + 2\delta}.
\]
Together with the use of (4.4) to estimate the last norm \(|y(t)|\), this implies, for \( t \geq \bar{T} \),
\[
|R_{\lambda_j}g(t) \cdot R_{\lambda_j}v| \leq \frac{\mu}{4} H(y)|R_{\lambda_j}v|^2 + \frac{C_6}{2} t^{1 - 2\delta/\alpha}, \quad \text{with } C_6 = \frac{8c_\ast^2 C_2^{\alpha + 2\delta}}{\mu c_1}. \tag{4.27}
\]
**Case \( \lambda_j > \Lambda \).** In this case, combining (4.26) and (4.27) yields, for \( t \geq \bar{T} \),
\[
\frac{1}{2} \frac{d}{dt}|R_{\lambda_j}v|^2 \leq -(\lambda_j - \lambda - \frac{\mu}{4}) H(y)|R_{\lambda_j}v|^2 + \frac{C_6}{2} t^{1 - 2\delta/\alpha}.
\]
Note, as \( t \to \infty \), that
\[
\lambda_j - \lambda(t) - \frac{\mu}{4} \to \lambda_j - \Lambda - \frac{\mu}{4} \geq \mu - \frac{\mu}{4} = \frac{3\mu}{4}.
\]
One has \( \lambda_j - \lambda(t) - \frac{\mu}{4} \geq \frac{\mu}{2} \) for \( t \) sufficiently large. Thus, for sufficiently large \( t \),
\[
\frac{d}{dt}|R_{\lambda_j}v|^2 \leq -\mu H(y)|R_{\lambda_j}v|^2 + C_6 t^{1 - 2\delta/\alpha}. \tag{4.28}
\]
Below, \( T \in [T, \infty) \) is fixed and can be taken sufficiently large.
Let \( t \) and \( t_0 \) be any numbers in \([T, \infty)\) with \( t > t_0 \). It follows (4.28) that
\[
|R_{\lambda_j}v(t)|^2 \leq e^{-\mu \int_{t_0}^t H(y(\tau))d\tau} |R_{\lambda_j}v(t_0)|^2 + C_6 \int_{t_0}^t e^{-\mu \int_{t_0}^s H(y(\tau))d\tau} t^{1 - 2\delta/\alpha} d\tau. \tag{4.29}
\]
Regarding the first inequality in (4.27), we fix a number \( \theta > 0 \) such that
\[
\theta \leq C_3 \text{ and } \theta \mu < 2\delta/\alpha.
\]
Then
\[ H(y(t)) \geq \theta t^{-1} \text{ for all } t \geq T. \] (4.30)

Utilizing this estimate in (4.29) gives, for \( t \geq T \),
\[
|R_{\lambda_j} v(t)|^2 \leq e^{-\theta \mu \int_{0}^{t} \tau^{-1} d\tau} |R_{\lambda_j} v(t_0)|^2 + C_6 \int_{t_0}^{t} e^{-\theta \mu \int_{\tau}^{t} s^{-1} ds} \tau^{-1-2\delta/\alpha} d\tau
\]
\[
= \frac{t_0}{t^{\theta \mu}} |R_{\lambda_j} v(t_0)|^2 + C_6 \int_{t_0}^{t} \tau^{-1-2\delta/\alpha} d\tau
\]
\[
= \frac{t_0}{t^{\theta \mu}} |R_{\lambda_j} v(t_0)|^2 + \frac{C_6}{t^{\theta \mu}(2\delta/\alpha - \theta \mu)} \left( t_0^{\theta \mu - 2\delta/\alpha} - t^{\theta \mu - 2\delta/\alpha} \right).
\]

Thus,
\[
|R_{\lambda_j} v(t)|^2 \leq \frac{t_0}{t^{\theta \mu}} |R_{\lambda_j} v(t_0)|^2 + \frac{C_6}{t^{\theta \mu}(2\delta/\alpha - \theta \mu)}.
\]
(4.31)

With \( t_0 \) fixed in (4.31), we obtain
\[
|R_{\lambda_j} v(t)| = \mathcal{O}(t^{-\theta \mu/2}) \text{ as } t \to \infty.
\] (4.32)

\underline{Case \( \lambda_j < \Lambda \).} Using (4.27) to have a lower bound for the last term in (4.26), we have
\[
\frac{1}{2} \frac{d}{dt} |R_{\lambda_j} v|^2 \geq (\lambda - \lambda_j - \frac{\mu}{4}) H(y) |R_{\lambda_j} v|^2 - \frac{C_6 t^{-1-2\delta/\alpha}}{2}.
\]

As \( t \to \infty \),
\[
\lambda(t) - \lambda_j - \frac{\mu}{4} \to \Lambda - \lambda_j - \frac{\mu}{4} \geq \mu - \frac{\mu}{4} = \frac{3}{4} \mu.
\]

Thus, for sufficiently large \( t \), one has \( \lambda(t) - \lambda_j - \frac{\mu}{4} \geq \frac{\mu}{2} \), and hence,
\[
\frac{d}{dt} |R_{\lambda_j} v|^2 \geq \mu H(y) |R_{\lambda_j} v|^2 - C_6 t^{-1-2\delta/\alpha}.
\]

Again, we can take a number \( T \in [\bar{T}, \infty) \) sufficiently large in the calculations below. Then, for any \( t, t_0 \) such that \( t > t_0 \geq T \), one has
\[
e^{-\mu \int_{t_0}^{t} H(y(\tau)) d\tau} |R_{\lambda_j} v(t)|^2 - |R_{\lambda_j} v(t_0)|^2 \geq -C_6 \int_{t_0}^{t} e^{-\mu \int_{\tau}^{t} H(y(s)) ds} \tau^{-1-2\delta/\alpha} d\tau. \] (4.33)

Note that \( \int_{t_0}^{\infty} H(y(\tau)) d\tau = \infty \) and \( |R_{\lambda_j} v(t)| \leq |v(t)| = 1 \). Then
\[
\lim_{t \to \infty} e^{-\mu \int_{t_0}^{t} H(y(\tau)) d\tau} |R_{\lambda_j} v(t)|^2 = 0.
\]

Letting \( t \to \infty \) in (4.33) and using (4.30) yield
\[
|R_{\lambda_j} v(t_0)|^2 \leq C_6 \int_{t_0}^{\infty} e^{-\mu \int_{t_0}^{\tau} H(y(s)) ds} \tau^{-1-2\delta/\alpha} d\tau \leq C_6 \int_{t_0}^{\infty} \tau^{-1-2\delta/\alpha} d\tau = \frac{C_6}{\theta \mu + 2\delta/\alpha} t_0^{-2\delta/\alpha}.
\]

Therefore, we obtain
\[
|R_{\lambda_j} v(t_0)| = \mathcal{O}(t_0^{-\delta/\alpha}) \text{ as } t_0 \to \infty.
\] (4.34)

By the expression (4.25) of \( (\text{Id} - R_\Lambda)v(t) \), estimate (4.32) of \( |R_{\lambda_j} v(t)| \) for all \( \lambda_j > \Lambda \), and estimate (4.34) of \( |R_{\lambda_j} v(t)| \) for all \( \lambda_j < \Lambda \), we obtain (4.22) with \( \varepsilon = \min\{\theta \mu/2, \delta/\alpha\} = \theta \mu/2 \). □
Lemma 4.4 results in the following estimates for $y(t)$ which refine (4.4).

**Corollary 4.5.** Let $\varepsilon > 0$ be as in Lemma 4.4. Then

$$|(\text{Id} - R_A)y(t)| = O(t^{-1/\alpha - \varepsilon}) \text{ as } t \to \infty,$$

(4.35)

and there exist numbers $T_0 \geq \bar{T}$ and $C_7, C_8 > 0$ such that

$$C_7 t^{-1/\alpha} \leq |R_A y(t)| \leq C_8 t^{-1/\alpha} \text{ for all } t \geq T_0.$$

(4.36)

**Proof.** On the one hand, we have from (4.4) and (4.22) that

$$|(\text{Id} - R_A)y(t)| = |y(t)| \cdot |(\text{Id} - R_A)v(t)| = O(t^{-1/\alpha} \cdot t^{-\varepsilon}),$$

which proves (4.35). On the other hand, by the triangle inequality and (4.4), one has

$$|R_A y(t)| \leq |y(t)| + |(\text{Id} - R_A)y(t)| \leq C_2 t^{-1/\alpha} + |(\text{Id} - R_A)y(t)|,$$

$$|R_A y(t)| \geq |y(t)| - |(\text{Id} - R_A)y(t)| \geq C_1 t^{-1/\alpha} - |(\text{Id} - R_A)y(t)|.$$

These inequalities and estimate (4.35) for $|(\text{Id} - R_A)y(t)|$ imply the desired lower and upper bounds for $|R_A y(t)|$ in (4.36) when $t$ is sufficiently large. \hfill \Box

**Lemma 4.6.** There exists a unit vector $v_* \in \mathbb{R}^n$ such that

$$|R_A v(t) - v_*| = O(t^{-\varepsilon}), \text{ as } t \to \infty, \text{ for some } \varepsilon > 0,$$

(4.37)

$$|v(t) - v_*| = O(t^{-\varepsilon}), \text{ as } t \to \infty, \text{ for some } \varepsilon > 0.$$

(4.38)

Consequently, one has

$$\lim_{t \to \infty} R_A v(t) = \lim_{t \to \infty} v(t) = v_*.$$

(4.39)

**Proof.** We prove (4.37) first. Let $\varepsilon_0 > 0$ be such that (4.22) holds for $\varepsilon = \varepsilon_0$. Then one has

$$|1 - |R_A v(t)|| = ||v(t)| - |R_A v(t)|| \leq |v(t) - R_A v(t)| = O(t^{-\varepsilon_0}).$$

(4.40)

As a consequence of (4.40), $|R_A v(t)| \to 1$ as $t \to \infty$. Hence, $R_A v(t) \neq 0$ for large $t$. Applying $R_A$ to equation (4.23) yields, for $t > t_*$,

$$\frac{d}{dt} R_A v = -H(y)(\Lambda - \lambda) R_A v + R_A g(t).$$

(4.41)

Then, for large $t$,

$$\frac{d}{dt} |R_A v| = \frac{1}{|R_A v|} \left( \frac{d}{dt} R_A v \cdot R_A v = -H(y)(\Lambda - \lambda) |R_A v| \right) + g_1(t),$$

(4.42)

where

$$g_1(t) = \frac{1}{|R_A v(t)|} R_A g(t) \cdot R_A v(t).$$

Consider $T \in [\bar{T}, \infty)$ sufficiently large. Solving for solution $|R_A v(t)|$ by the variation of constants formula from the differential equation (4.42) gives, for $t > t_0 \geq T$,

$$|R_A v(t)| = e^{-\int_t^{t_0} H(y(\tau))(\Lambda - \lambda(\tau))d\tau} \left( |R_A v(t_0)| + \int_t^{t_0} e^{\int_0^\tau H(y(s))(\Lambda - \lambda(s))ds} g_1(\tau) d\tau \right).$$
It yields
\[
\int_{t_0}^{t} H(y(\tau))(\Lambda - \lambda(\tau))d\tau
= \ln \left( \left| R_\Lambda v(t_0) \right| + \int_{t_0}^{t} e^{\int_{t_0}^{\tau} H(y(s))(\Lambda - \lambda(s))ds} g_1(\tau)d\tau \right) - \ln |R_\Lambda v(t)|.
\] (4.43)

We have from (4.1), (4.24) and (4.4) that, for \( t \geq T \),
\[
|g_1(t)| \leq |R_\Lambda g(t)| \leq |g(t)| \leq C_9 t^{-(1+\delta/\alpha)}, \quad \text{where} \quad C_9 = 2c_s C_2^{\alpha+\delta}.
\] (4.44)

Recall that \( C_4 \) is the positive constant in (4.7). We take \( \varepsilon_1 > 0 \) small such that
\[
C_4 \varepsilon_1 < \delta/\alpha.
\] (4.45)

Thanks to Lemma 4.3, we can assume \( T \) is sufficiently large so that
\[
|\Lambda - \lambda(s)| < \varepsilon_1 \text{ for all } s \geq T.
\] Together with (4.44), we have, for \( \tau \in [t_0, t] \),
\[
e^{\int_{t_0}^{\tau} H(y(s))(\Lambda - \lambda(s))ds} g_1(\tau) \leq e^{\int_{t_0}^{\tau} C_4 \varepsilon_1 s^{-1}ds} C_9 \tau^{-(1+\delta/\alpha)} = \frac{\tau^{C_4 \varepsilon_1}}{t_0^{C_4 \varepsilon_1}} C_9 \tau^{-(1+\delta/\alpha)}
\]
\[
= \frac{C_9}{t_0^{C_4 \varepsilon_1}} \tau^{1-\delta/\alpha+C_4 \varepsilon_1}.
\]

Thanks to this and (4.45),
\[
\lim_{t \to \infty} \int_{t_0}^{t} e^{\int_{t_0}^{\tau} H(y(s))(\Lambda - \lambda(s))ds} g_1(\tau)d\tau = \int_{t_0}^{\infty} e^{\int_{t_0}^{\tau} H(y(s))(\Lambda - \lambda(s))ds} g_1(\tau)d\tau = \eta(t_0) \in \mathbb{R}.
\] (4.46)

Note that
\[
|\eta(t_0)| \leq \frac{C_9}{t_0^{C_4 \varepsilon_1}} \int_{t_0}^{\infty} \tau^{1-\delta/\alpha+C_4 \varepsilon_1}d\tau = \frac{C_9}{\delta/\alpha - C_4 \varepsilon_1 t_0^{-\delta/\alpha}}.
\] (4.47)

Passing to the limit as \( t \to \infty \) in (4.43), we have
\[
\int_{t_0}^{\infty} H(y(\tau))(\Lambda - \lambda(\tau))d\tau = \ln(|R_\Lambda v(t_0)| + \eta(t_0)) - \ln 1 \in \mathbb{R}.
\] (4.48)

By (4.48), we can define, for \( t_0 \geq T \),
\[
h(t_0) = \int_{t_0}^{\infty} H(y(\tau))(\Lambda - \lambda(\tau))d\tau \in \mathbb{R}.
\]

We rewrite (4.48) as
\[
h(t_0) = \ln(|R_\Lambda v(t_0)| + \eta(t_0)) = \ln(1 + (|R_\Lambda v(t_0)| - 1) + \eta(t_0)).
\]

With this expression and properties (4.40) and (4.47), we have, as \( t_0 \to \infty \),
\[
|h(t_0)| = \mathcal{O} (|R_\Lambda v(t_0)| - 1 + |\eta(t_0)|) = \mathcal{O}(t_0^{-\varepsilon_0} + t_0^{-\delta/\alpha}) = \mathcal{O}(t_0^{-\varepsilon_2}),
\] (4.49)

where \( \varepsilon_2 = \min\{\varepsilon_0, \delta/\alpha\} \).

Solving for \( R_\Lambda v(t) \) from (4.41) by the variation of constants formula, one has
\[
R_\Lambda v(t) = e^{\int_{t_0}^{t} H(y(\tau))(\Lambda - \lambda(\tau))d\tau} \left( R_\Lambda v(t_0) + \int_{t_0}^{t} e^{\int_{t_0}^{\tau} H(y(s))(\Lambda - \lambda(s))ds} R_\Lambda g(\tau)d\tau \right).
\] (4.50)
Recall $|R_A g(\tau)| \leq |g(\tau)|$. Same arguments as those from \((4.44)\) to \((4.47)\) with $R_A g(\tau)$ replacing $g_1(\tau)$, we obtain, similar to \((4.46)\) and \((4.47)\), that

$$
\lim_{t \to \infty} \int_{t_0}^{t} e^{\int_{s}^{t} H(y(s))(\Lambda - \lambda(s))ds} R_A g(\tau) = \int_{t_0}^{\infty} e^{\int_{s}^{t} H(y(s))(\Lambda - \lambda(s))ds} R_A g(\tau) d\tau = X(t_0) \in \mathbb{R}^n
$$

for $t_0 \geq T$, and

$$|X(t_0)| = \mathcal{O}(t_0^{-\varepsilon_2}) \text{ as } t_0 \to \infty. \quad (4.51)$$

Taking $t \to \infty$ in \((4.50)\) gives

$$
\lim_{t \to \infty} R_A v(t) = v_* \overset{\text{def}}{=} e^{-h(t_0)}(R_A v(t_0) + X(t_0)) \in \mathbb{R}^n.
$$

We rewrite \((4.50)\) as

$$
R_A v(t) = e^{h(t)-h(t_0)}(R_A v(t_0) + X(t_0) - X(t)) = e^{h(t)}v_* - e^{h(t)-h(t_0)}X(t).
$$

Thus,

$$
|R_A v(t) - v_*| \leq |e^{h(t)} - 1| \cdot |v_*| + e^{h(t)-h(t_0)}|X(t)|.
$$

Having $t_0$ fixed in the above formula and using \((4.49)\) and \((4.51)\), we deduce, as $t \to \infty$,

$$
|R_A v(t) - v_*| = \mathcal{O}(|e^{h(t)} - 1| + |X(t)|) = \mathcal{O}(|h(t)| + |X(t)|) = \mathcal{O}(t^{-\varepsilon_2}).
$$

Therefore, we obtain the desired estimate \((4.37)\).

Next, by the triangle inequality,

$$|v(t) - v_*| \leq |v(t) - R_A v(t)| + |R_A v(t) - v_*|.
$$

This inequality and estimates \((4.22), (4.37)\) imply \((4.38)\).

Finally, \((4.39)\) is a direct consequence of \((4.37)\) and \((4.38)\). The proof is complete. \(\square\)

To derive further properties of $y(t)$, we require more conditions on the function $H$.

**Definition 4.7.** Let $E$ be a nonempty subset of $\mathbb{R}^n$ and $F$ be a function from $E$ to $\mathbb{R}$. We say $F$ has property (HC) on $E$ if, for any $x_0 \in E$, there exist numbers $r, C, \gamma > 0$ such that

$$
|F(x) - F(x_0)| \leq C|x - x_0|^{\gamma}
$$

for any $x \in E$ with $|x - x_0| < r$.

Because the power $\gamma$ is allowed to depend on each $x_0$, the property (HC) is weaker than a Hölder continuity requirement such as $F \in C^{0, \gamma}(E)$ for some $\gamma \in (0, 1)$.

The next condition imposed on $H$ is the following.

**Assumption 4.8.** The function $H$ has property (HC) on the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$.

The asymptotic behavior of $y(t)$ can be described in the theorem below.

**Theorem 4.9.** There exists a nonzero vector $\xi_* \in \mathbb{R}^n$ such that, as $t \to \infty$,

$$
|y(t) - \xi_* t^{-1/\alpha}| = \mathcal{O}(t^{-1/\alpha - \varepsilon}) \text{ for some } \varepsilon > 0. \quad (4.53)
$$

In fact, $\xi_*$ is an eigenvector of $A$ associated with the eigenvalue $\Lambda$, and one has

$$
|R_A y(t) - \xi_* t^{-1/\alpha}| = \mathcal{O}(t^{-1/\alpha - \varepsilon}) \text{ for some } \varepsilon > 0. \quad (4.54)
$$

Moreover,

$$
\alpha \Lambda H(\xi_*) = 1. \quad (4.55)
$$
Proof. We prove \((4.54)\) first. We look for a differential equation for \(R_\Lambda y(t)\). Applying \(R_\Lambda\) to equation \((1.21)\) and rewriting \(H(y) = |y|^\alpha H(v)\), we have
\[
(R_\Lambda y)' = -\Lambda |y|^\alpha H(v) R_\Lambda y + R_\Lambda(G(t,y)).
\] (4.56)

Let \(v_*\) be the unit vector in Lemma \(4.6\) and \(\varepsilon_0 > 0\) be such that \((4.22), (4.35), (4.37),\) and \((4.38)\) hold for \(\varepsilon = \varepsilon_0\).

We approximate \(|y|^\alpha H(v)\) on the right-hand side of \((4.56)\) in the following way
\[
|y|^\alpha H(v) = |R_\Lambda y|^\alpha H(v_*) + g_1(t),
\]
where
\[
g_1(t) = |y(t)|^\alpha (H(v(t)) - H(v_*)) + (|y(t)|^\alpha - |R_\Lambda y(t)|^\alpha)H(v_*)
\]
\[
= |y(t)|^\alpha \{H(v(t)) - H(v_*) + (1 - |R_\Lambda v(t)|^\alpha)H(v_*)\}.
\]

Then
\[
(R_\Lambda y)' = -\Lambda H(v_*)|R_\Lambda y|^\alpha R_\Lambda y + g(t),
\] (4.57)

where \(g(t) = -\Lambda g_1(t)R_\Lambda y + R_\Lambda(G(t,y))\).

We estimate \(|g_1(t)|\) first and then \(|g(t)|\). Take \(F = H, E = \)the unit sphere, the unit vector \(x_0 = v_*\) in Definition \(4.7\). Then there exists a number \(\gamma > 0\) depending on \(v_*\) such that, according to \((4.52)\) with the unit vector \(x = v(t)\) for sufficiently large \(t\),
\[
|H(v(t)) - H(v_*)| = O(|v(t) - v_*|^\gamma) = O(t^{-\gamma \varepsilon_0}).
\] (4.58)

Recall that \(\lim_{t \to \infty} |R_\Lambda v(t)| = 1\). We use the approximation \(|s^\alpha - 1| = O(|s - 1|)\) when the real number \(s \to 1\). As \(t \to \infty\), by taking \(s = |R_\Lambda v(t)|\) and using estimate \((4.40)\), we derive
\[
|1 - |R_\Lambda v(t)||^\alpha = O(|1 - |R_\Lambda v(t)||^\alpha) = O(t^{-\varepsilon_1}).
\] (4.59)

Combining \((4.58), (4.59)\) with \((4.4)\), we obtain
\[
|g_1(t)| = O(|y(t)|^\alpha (t^{-\gamma \varepsilon_0} + t^{-\varepsilon_0})) = O(t^{-1-\varepsilon_1}),
\] (4.60)
where \(\varepsilon_1 = \varepsilon_0 \min\{1, \gamma\}\).

For the last term in \(|g(t)|\), we have from \((4.2)\) and \((4.4)\) that
\[
|R_\Lambda(G(t,y(t)))| = O(|y(t)|^{-\alpha - 1 - \delta}) = O(t^{-1-\alpha-\delta/\alpha}).
\] (4.61)

Combining \((4.61)\) and \((4.61)\) gives
\[
|g(t)| = O(t^{-1-\varepsilon_1}|R_\Lambda y(t)| + t^{-1-\alpha-\delta/\alpha}) = O(t^{-1-1/\alpha-\varepsilon_2/\alpha}),
\]
where \(\varepsilon_2 = \min\{\varepsilon_1, \alpha, \delta\}\).

By the lower bound of \(|R_\Lambda y(t)|\) in \((4.36)\), we have
\[
|g(t)| = O(|R_\Lambda y(t)|^{1+\alpha+\varepsilon_2}).
\]

We apply Theorem \((2.1)\) to solution \(R_\Lambda y(t)\) of equation \((4.57)\), for \(t > T\), with a sufficiently large \(T > 0\), constant \(a = \Lambda H(v_*)\) and \(f = g\) in \((2.2)\). It results in the existence of a nonzero vector \(\xi_* \in \mathbb{R}^n\) such that
\[
|R_\Lambda y(t) - \xi_* t^{-1/\alpha}| = O(t^{-1/\alpha-\varepsilon_3}) \text{ for some } \varepsilon_3 > 0,
\] (4.62)

and
\[
a \Lambda H(v_*)|\xi_*|^\alpha = 1.
\] (4.63)

The desired statement \((4.54)\) immediately follows \((4.62)\).
Because $\xi_* = \lim_{t\to \infty} t^{1/\alpha} R_{\Lambda} y(t)$, by (4.54), and the fact $\xi_* \neq 0$, we have $\xi_* \in R_{\Lambda}(\mathbb{R}^n) \setminus \{0\}$. Hence, $\xi_*$ is an eigenvector of $A$ associated with $\Lambda$.

Next, we prove (4.53). Writing $y(t) - \xi_* t^{-1/\alpha} = (I - R_{\Lambda}) y(t) + (R_{\Lambda} y(t) - \xi_* t^{-1/\alpha})$, and using the estimate (4.35), with $\varepsilon = \varepsilon_0$, and estimate (1.62) yield
\[
|y(t) - \xi_* t^{-1/\alpha}| = \mathcal{O}(t^{-1/\alpha - \varepsilon_0} + t^{-1/\alpha - \varepsilon_3}).
\]
This implies (4.53) with $\varepsilon = \min\{\varepsilon_0, \varepsilon_3\}$.

Finally, we prove (4.55). Let $w(t) = t^{1/\alpha} y(t)$. As $t \to \infty$, we have $v(t) \to v_*$ and $w(t) \to \xi_*$, thanks to (4.38) and (4.53). By writing $v(t) = w(t)/|w(t)|$ and passing $t \to \infty$, we obtain
\[
v_* = \xi_* / |\xi_*| \quad (4.64)
\]
Then (4.55) follows (4.63) and (4.64). The proof is complete.

\[\square\]

5. The general case

In this section, we again study equation (1.4), where the matrix $A$ is as in Section 3 and the function $G(t, x)$ is as in Assumption 4.1. Let $y(t)$ be a solution as in Section 4.

For $1 \leq k, \ell \leq n$, let $E_{k\ell}$ be the elementary $n \times n$ matrix $(\delta_{ki} \delta_{\ell j})_{1 \leq i, j \leq n}$, where $\delta_{ki}$ and $\delta_{\ell j}$ are the Kronecker delta symbols. For $j = 1, 2, \ldots, d$, define
\[
\hat{R}_{\lambda_j} = \sum_{1 \leq i \leq n, \lambda_i = \lambda_j} E_{ii} \text{ and } R_{\lambda_j} = S^{-1} \hat{R}_{\lambda_j} S. \quad (5.1)
\]
Then one immediately has
\[
I_n = \sum_{j=1}^d R_{\lambda_j}, \quad R_{\lambda_j} R_{\lambda_j} = \delta_{ij} R_{\lambda_j}, \quad AR_{\lambda_j} = R_{\lambda_j} A = \lambda_j R_{\lambda_j}. \quad (5.2)
\]

Thanks to (5.2), each $R_{\lambda_j}$ is a projection, and $R_{\lambda_j}(\mathbb{R}^n)$ is the eigenspace of $A$ associated with the eigenvalue $\lambda_j$.

When $A$ is symmetric, the $R_{\lambda_j}$’s defined in (5.1) are the orthogonal projections defined in Section 4.

Regarding the function $H$, we examine Assumption 4.8 further.

Lemma 5.1. Let $F$ be a function in $\mathcal{H}_\alpha(\mathbb{R}^n, \mathbb{R})$ for some $\alpha > 0$. Then $F$ has property (HC) on the unit sphere if and only if $F$ has property (HC) on $\mathbb{R}^n \setminus \{0\}$.

Proof. It is clear that if $F$ has property (HC) on $\mathbb{R}^n \setminus \{0\}$ then it has property (HC) on the unit sphere.

Now, suppose $F$ has property (HC) on the unit sphere. Then $F$ is continuous on the unit sphere. Consequently,
\[
\max_{|x| = 1} |F(x)| = M_0 \in [0, \infty). \quad (5.3)
\]

Let $\xi \neq 0$. In Definition 4.7 for the set $E$ being the unit sphere, we take $x_0$ to be the unit vector $\xi/|\xi|$. Then there are $r_1, C_0, \gamma > 0$ such that if $x \in \mathbb{R}^n \setminus \{0\}$ and $|x/|x| - \xi/|\xi|| < r_1$, then
\[
|F(x/|x|) - F(\xi/|\xi|)| \leq C_0 |x/|x| - \xi/|\xi||^\gamma. \quad (5.4)
\]
Note that the functions $x \in \mathbb{R}^n \setminus \{0\} \mapsto |x|^\alpha$ and $x \in \mathbb{R}^n \setminus \{0\} \mapsto x/|x|$ are $C^1$-functions. Let $r_2 = |\xi|/2$. Then there is a constant $M_1 > 0$ such that if $|x - \xi| < r_2$ then
\[
\left| |x|^\alpha - |\xi|^\alpha \right| \leq M_1 |x - \xi| \text{ and } |x/|x| - \xi/|\xi|| \leq M_1 |x - \xi| \quad (5.5)
\]
Set \( r_0 = \min\{1, r_1/M_1, r_2\} \). Let \( x \in \mathbb{R}^n \) and \( |x - \xi| < r_0 \). We have \( x \neq 0 \) and, from the second inequality in (5.5),
\[
|x/| - \xi/|| \leq M_1|x - \xi| < r_1.
\] (5.6)

Combining (5.4), (5.5), (5.6) and (5.3) yields
\[
|F(x) - F(\xi)| = \left| |x|^\alpha F(x/|x|) - |\xi|^\alpha F(\xi/|\xi|) \right|
\leq \left| |x|^\alpha - |\xi|^\alpha \right| \cdot |F(x/|x|)| + |\xi|^\alpha |F(x/|x|) - F(\xi/|\xi|)|
\leq M_1|x - \xi||M_0 + |\xi|^\alpha C_0 x/|x| - \xi/||\xi||^\gamma \leq M_1M_0|x - \xi| + |\xi|^\alpha C_0 M_1^\gamma |x - \xi|^\gamma.
\]

Thus,
\[
|F(x) - F(\xi)| \leq (M_1M_0 + |\xi|^\alpha C_0 M_1^\gamma) |x - \xi|^{\min\{1, \gamma\}}.
\]

Therefore, \( F \) has property (HC) on \( \mathbb{R}^n \setminus \{0\} \). \( \square \)

Thanks to Lemma 5.1, Assumptions 3.3 and 4.8 can be combined into a simple form as the following.

**Assumption 5.2.** The function \( H \) belongs to \( \mathcal{H}_\alpha(\mathbb{R}^n, \mathbb{R}) \) for some \( \alpha > 0 \), has property (HC) on the unit sphere, and \( H > 0 \) on the unit sphere.

Indeed, by the virtue of Lemma 5.1, such a function \( H \) in Assumption 5.2 has property (HC) on \( \mathbb{R}^n \setminus \{0\} \). Consequently, it is continuous on \( \mathbb{R}^n \setminus \{0\} \). By the fact (5.3) for \( H \) in place of \( F \), one has \( |H(x)| \leq M_0|x|^\alpha \) for all \( x \neq 0 \). Together with the fact \( H(0) = 0 \), we have \( H \) is also continuous at the origin, and, hence, on \( \mathbb{R}^n \). Finally, for \( x \neq 0 \), \( H(x) = |x|^\alpha H(x/|x|) > 0 \). Thus, \( H \) satisfies both Assumptions 3.3 and 4.8.

We are ready to present our main result.

**Theorem 5.3.** Under Assumption 5.2, there exist an eigenvalue \( \Lambda \) of \( A \) and an eigenvector \( \xi_+ \) of \( A \) associated with \( \Lambda \) such that, as \( t \to \infty \),
\[
|y(t) - \xi_+ t^{-1/\alpha}| = \mathcal{O}(t^{-1/\alpha-\varepsilon}) \text{ for some } \varepsilon > 0.
\] (5.7)

More specifically,
\[
|(I_n - R_\Lambda)y(t)|, |R_\Lambda y(t) - \xi_+ t^{-1/\alpha}| = \mathcal{O}(t^{-1/\alpha-\varepsilon}) \text{ for some } \varepsilon > 0,
\] (5.8)

and relation (4.55) holds true.

**Proof.** Setting \( z(t) = Sy(t) \), we have \( z(t) \) satisfies equation (3.16) with \( \tilde{H} \) and \( \tilde{G} \) defined in (3.17). We will apply Theorem 4.9 to equation (3.16).

**Verification of Assumption 5.2** for \( \tilde{H} \). Clearly, \( H \in \mathcal{H}_\alpha(\mathbb{R}^n) \) and \( \tilde{H}(z) > 0 \) for \( |z| = 1 \).

Let \( \xi \) be a unit vector in \( \mathbb{R}^n \). By Lemma 5.1, the function \( H \) has property (HC) on \( \mathbb{R}^n \setminus \{0\} \). Let \( x_0 = S^{-1}\xi \neq 0 \) in Definition 4.7 for \( E = \mathbb{R}^n \setminus \{0\} \). Then there are numbers \( r \in (0, |x_0|) \) and \( C, \gamma > 0 \) such that
\[
|H(x) - H(S^{-1}\xi)| \leq C|x - S^{-1}\xi|^\gamma,
\] (5.9)
for any \( x \in \mathbb{R}^n \) with \( |x - S^{-1}\xi| < r \). (Note that such \( x \) is already a nonzero vector.)

Take \( r' = r/\|S^{-1}\| > 0 \). Let \( z \) be any unit vector in \( \mathbb{R}^n \) and \( |z - \xi| < r' \). Set \( x = S^{-1}z \). Then
\[
|x - S^{-1}\xi| = \|S^{-1}(z - \xi)| \leq \|S^{-1}\| \cdot |z - \xi| < \|S^{-1}\| r' = r.
\] (5.10)

It follows (5.9) and (5.10) that
\[
|\tilde{H}(z) - \tilde{H}(\xi)| \leq C|x - S^{-1}\xi|^\gamma \leq C\|S^{-1}\|\gamma |z - \xi|^\gamma.
\]
Therefore, the function $\tilde{H}$ has property (HC) on the unit sphere.

Thanks to \((3.15)\), one can also verify that the function $\tilde{G}$ satisfies Assumption \((4.1)\) with the same $\alpha, \delta, t_*, T_*$, and a new constant $c_*$.

We apply Theorem \((4.9)\) to equation \((3.16)\) and solution $z(t)$, with $A_0$ replacing $A$ and $\hat{R}_{\lambda_j}$ replacing $R_{\lambda_j}$. Then there exist an eigenvalue $\Lambda$ of $A_0$ and an eigenvector $\xi_0$ of $A_0$ associated with $\Lambda$ such that

\[
|z(t) - \xi_0 t^{-1/\alpha}| = O(t^{-1/\alpha-\varepsilon}),
\]

\[
|(I_n - \hat{R}_\Lambda)z(t)|, |\hat{R}_\Lambda z(t) - \xi_0 t^{-1/\alpha}| = O(t^{-1/\alpha-\varepsilon}),
\]

for some number $\varepsilon > 0$, and

\[
\alpha \Lambda \tilde{H}(\xi_0) = 1.
\]

Let $\xi_* = S^{-1} \xi_0$ which is a nonzero vector. It is clear that $\Lambda$ is an eigenvalue of $A$ and $\xi_*$ is an eigenvector of $A$ associated with $\Lambda$.

It follows \((5.11)\) that $|S(y(t) - \xi_* t^{-1/\alpha})| = O(t^{-1/\alpha-\varepsilon})$, which implies \((5.7)\).

Similarly, we convert \((5.12)\) to

\[
|S(I_n - R_\Lambda)y(t)|, |S(R_\Lambda y(t) - \xi_* t^{-1/\alpha})| = O(t^{-1/\alpha-\varepsilon}),
\]

which yields \((5.8)\).

Finally, relation \((5.13)\) and the fact $\tilde{H}(\xi_0) = H(\xi_*)$ imply \((4.55)\). The proof is complete.

\[\square\]

The following remarks on Theorem \((5.3)\) are in order.

(a) As a consequence of \((4.55)\) and \((3.4)\), one can estimate $|\xi_*|$ from above and below by

\[
\frac{1}{(\alpha c_2 \Lambda_n)^{1/\alpha}} \leq |\xi_*| \leq \frac{1}{(\alpha c_1 \Lambda_1)^{1/\alpha}}.
\]

These estimates agree with \((4.5)\) derived previously for the symmetric matrix case.

(b) One observes that the bounds in \((5.14)\) are independent of the solution $y(t)$. This is different from the case of the previously studied ODE systems with the lowest order term being linear such as \((1.3)\).

(c) The actual value of $|\xi_*|$ may still depend on the solution $y(t)$, while, for the basic case in Section \((2)\), it does not.

In the above proof of Theorem \((5.3)\) the properties of $\xi_*$ are obtained by using the approximate equation \((1.57)\) and the result proved for it in Theorem \((2.1)\). However, as we will see below, the asymptotic approximation \((5.7)\), once established, already determines those properties of $\xi_*$. It is proved independently without using equation \((1.57)\).

**Theorem 5.4.** Under Assumption \((5.2)\) let $y \in C^1([T, \infty), \mathbb{R}^n)$ be a solution of \((1.4)\) on $(T, \infty)$ for some $T \geq t_*$. Suppose there exists a vector $\xi \in \mathbb{R}^n \setminus \{0\}$ such that

\[
|y(t) - \xi t^{-1/\alpha}| = O(t^{-1/\alpha-\varepsilon}) \text{ for some } \varepsilon > 0.
\]

Then $\xi$ satisfies

\[
H(\xi) A \xi = \frac{1}{\alpha} \xi.
\]

Consequently, $\xi$ is an eigenvector of $A$ associated with the eigenvalue $\Lambda \overset{\text{def}}{=} 1/(\alpha H(\xi))$. 
Proof. First of all, one has from (5.13) that
\[ |y(t)| = \mathcal{O}(t^{-1/\alpha}). \]  
(5.17)

Set \( w(t) = t^{1/\alpha}y(t). \) Then (5.13) implies
\[ |w(t) - \xi| = \mathcal{O}(t^{-\varepsilon}) \] and, consequently, \( \lim_{t \to \infty} w(t) = \xi. \)
(5.18)

We find an asymptotic approximation for \( w'(t). \) For sufficiently large \( t, \) we calculate
\[ w' = \frac{1}{\alpha t} w + t^{1/\alpha}(-H(y)Ay + G(t, y)) = \frac{1}{\alpha t} w - \frac{1}{t}H(w)Aw + t^{1/\alpha}G(t, y). \]

Then
\[ w' = \frac{1}{t} \left( \frac{1}{\alpha} \xi - H(\xi)A\xi \right) + h(t), \]  
(5.19)

where
\[ h(t) = \frac{1}{\alpha t} (w(t) - \xi) - \frac{1}{t} (H(w(t))Aw(t) - H(\xi)A\xi) + t^{1/\alpha}G(t, y(t)). \]

We estimate \( |h(t)|. \) For the first term of \( h(t), \) we have
\[ \frac{1}{\alpha t} |w(t) - \xi| = \mathcal{O}(t^{-1-\varepsilon}). \]

For the middle term of \( h(t), \) we have
\[ H(w)Aw - H(\xi)A\xi = (H(w) - H(\xi))Aw + H(\xi)A(w - \xi). \]

By Lemma 5.1, the function \( H \) has property (HC) on the set \( E = \mathbb{R}^n \setminus \{0\}. \) By Definition 4.7 with \( F = H \) and \( x_0 = \xi, \) and the facts in (5.18), there exists \( \gamma > 0 \) such that
\[ |H(w(t)) - H(\xi)| \cdot |Aw(t)| = \mathcal{O}(|w(t) - \xi|^\gamma \cdot 1) = \mathcal{O}(t^{-\gamma\varepsilon}). \]

Clearly, \( H(\xi)|A(w(t) - \xi)| = \mathcal{O}(|w(t) - \xi|) = \mathcal{O}(t^{-\varepsilon}). \) Thus,
\[ t^{-1}|H(w(t))Aw(t) - H(\xi)A\xi| = \mathcal{O}(t^{-1-\varepsilon} + t^{-1-\gamma\varepsilon}). \]

For the last term of \( h(t), \) it follows (4.2) and (5.17) that
\[ t^{1/\alpha}|G(t, y(t))| = \mathcal{O}(t^{1/\alpha} \cdot t^{-(1+\alpha+\delta)/\alpha}) = \mathcal{O}(t^{-1-\delta/\alpha}). \]

Therefore,
\[ |h(t)| = \mathcal{O}(t^{-1-\varepsilon} + t^{-1-\gamma\varepsilon} + t^{-1-\delta/\alpha}) = \mathcal{O}(t^{-1-\delta_1}), \]  
(5.20)

where \( \delta_1 = \min\{\varepsilon, \gamma\varepsilon, \delta/\alpha\}. \)

Let \( t > s \) be sufficiently large numbers. Integrating equation (5.19) from \( s \) to \( t, \) and taking into account (5.20) give
\[ \left| w(t) - w(s) - \left( \frac{1}{\alpha} \xi - H(\xi)A\xi \right) \ln(t/s) \right| \leq C(s^{-\delta_1} - t^{-\delta_1}), \]

for some constant \( C > 0. \) Letting \( t = 2s \) and taking \( s \to \infty \) yield
\[ \frac{1}{\alpha} \xi - H(\xi)A\xi = 0, \]
which proves (5.16). The last statement in Theorem 5.4 is an obvious consequence of (5.16). \( \square \)

Below, we provide explicit examples for \( H \) satisfying Assumption 5.2. Firstly, here are some elementary facts.
Lemma 5.5. Let $E$ be a nonempty subset of $\mathbb{R}^n$. Suppose two functions $F_1, F_2 : E \to \mathbb{R}$ have property (HC) on $E$. Then so do the functions $aF_1 + bF_2$ and $F_1F_2$ for any $a, b \in \mathbb{R}$.

Proof. Suppose $F_i$, for $i = 1, 2$, satisfies \((1.52)\) with $r_i, C_i, \gamma_i > 0$. Let $r = \min\{1, r_1, r_2\}$, $C = \max\{C_1, C_2\}$ and $\gamma = \min\{\gamma_1, \gamma_2\}$. Then both $F_1$ and $F_2$ satisfy \((1.52)\) with the same numbers $r, C, \gamma$. The statement for $aF_1 + bF_2$ is now obviously true. For the product $F_1F_2$, one observes that $F_1$ and $F_2$ are bounded on the set $\{x \in E : |x - x_0| < r\}$. The rest of the proof is standard. We omit the details.

For $p \geq 1$, the $l^p$-norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is $\|x\|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$.

Lemma 5.6. Let $E = \mathbb{R}^n \setminus \{0\}$ and $p \geq 1$, $\alpha > 0$. Then the function $H(x) = \|x\|_p^\alpha$, for $x \in \mathbb{R}^n$, has property (HC) on $E$ with the same power $\gamma = 1$ in \((1.52)\).

Proof. When $p > 1$, one has $H \in C^1(E)$, hence the statement is true. Consider the case $p = 1$. Let $\xi \in E$ and set $r = \|\xi\|_1 > 0$. Note that there is $C > 0$ such that $|t^\alpha - r^\alpha| \leq C|t - r|$ for all $t \in I \equiv [r/2, 3r/2]$.

Let $x \in E$ with $|x - \xi| < r/(2\sqrt{n})$. One has $\|\|x\|_1 - \|\xi\|_1\| \leq \|x - \xi\|_1 \leq \sqrt{n}|x - \xi| < r/2,$

which implies $t \equiv \|x\|_1 \in I$. Thus,

$$|H(x) - H(\xi)| = \|x\|_1^\alpha - \|\xi\|_1^\alpha \leq C\|x\|_1 - \|\xi\|_1 \leq C\sqrt{n}|x - \xi|.$$ 

Therefore, $H$ has property (HC) on $E$ with the same power $\gamma = 1$.

Example 5.7. The requirement of $H$ having property (HC) on the unit sphere is not strict. In many cases, $H$, in fact, is a $C^1$-function on $\mathbb{R}^n \setminus \{0\}$, hence, it meets this condition. For example, as in Section 2, $H(x) = |x|^\alpha$ for $\alpha > 0$. A generalization is

$$H(x) = \|K_1x\|_{p_1}^{\alpha_1}\|K_2x\|_{p_2}^{\alpha_2} \cdots \|K_mx\|_{p_m}^{\alpha_m},$$

where, for $j = 1, 2, \ldots, m$, $K_j$ is an invertible $n \times n$ matrix, $\alpha_j > 0$, and $p_j > 1$. In this case, $H$ satisfies Assumption \(5.21\) with

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_m.$$  \(5.22\)

In fact, thanks to Lemmas 5.6 and 5.5, we allow $p_j \geq 1$ in \((5.21)\).

Of course, $H$ can also be a linear combination, with positive coefficients, of the functions of the form \((5.21)\) having possibly different $m$'s, but resulting in the same $\alpha$ in \((5.22)\).

The function $H$ can be even more complicated. Here are some examples when $n = 2$. For $x = (x_1, x_2) \in \mathbb{R}^2$,

$$H(x) = (x_1^2 + 3x_2^2)^{3/4}, \quad H(x) = (x_1^4 - x_1^2x_2^2 + x_2^4)^{5/3},$$

$$H(x) = (|x_1^6 + 5x_2^6|^{1/3} + |2x_1^6 - x_2^6|^{4/3})^{1/7}, \quad H(x) = (\|x\|_{5/3}^6 + \|x\|_{7/4}^6)^{11/8}.$$ 

More examples of $H$ can be constructed by the similar investigation in section 6 of [12].

Remark 5.8. The study of the solutions of ODE systems near an equilibrium has a long history. One of the long-standing methods for detailed descriptions of their asymptotic behavior is the Poincaré–Dulac normal form \([1][2][22]\). This has been developed much further by many researchers over the years, see the books \([7][8]\), monograph \([21]\), and, for example, papers \([3][6]\) and references therein. However, the techniques from this approach, such as the
generalized normal forms and power geometry in [7,8], are not applicable to the equations of our current interest. In fact, our class of equations, problems, techniques and those in [7,8,21] are quite different, and can be considered as complementary to each other. For instance, the main task in [3–6,21] is to find a solution with a certain type of expansions. On contrary, our result establishes the exact asymptotic approximation for any (nontrivial, decaying) solution.

References

[1] V. I. Arnol’d. Geometrical methods in the theory of ordinary differential equations, volume 250 of A Series of Comprehensive Studies in Mathematics. Springer-Verlag, New York, second edition, 1988.
[2] Yuri N. Bibikov. Local theory of nonlinear analytic ordinary differential equations, volume 702 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1979.
[3] A. D. Bruno. Asymptotic behavior and expansions of solutions of an ordinary differential equation. Uspekhi Mat. Nauk, 59(3(357)):31–80, 2004.
[4] A. D. Bruno. Power-logarithmic expansions of solutions of a system of ordinary differential equations. Dokl. Akad. Nauk, 419(3):298–302, 2008.
[5] A. D. Bruno. Power-exponential expansions of solutions of an ordinary differential equation. Dokl. Akad. Nauk, 444(2):137–142, 2012.
[6] A. D. Bruno. On complicated expansions of solutions to ODES. Comput. Math. Math. Phys., 58(3):328–347, 2018.
[7] Alexander D. Bruno. Local methods in nonlinear differential equations. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1989.
[8] Alexander D. Bruno. Power geometry in algebraic and differential equations, volume 57 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 2000.
[9] Dat Cao and Luan Hoang. Asymptotic expansions in a general system of decaying functions for solutions of the Navier-Stokes equations. Ann. Mat. Pura Appl. (4), 199(3):1023–1072, 2020.
[10] Dat Cao and Luan Hoang. Long-time asymptotic expansions for Navier-Stokes equations with power-decaying forces. Proc. Roy. Soc. Edinburgh Sect. A, 150(2):569–606, 2020.
[11] Dat Cao and Luan Hoang. Asymptotic expansions with exponential, power, and logarithmic functions for non-autonomous nonlinear differential equations. J. Evol. Equ., 21(2):1179–1225, 2021.
[12] Dat Cao, Luan Hoang, and Thinh Kieu. Infinite series asymptotic expansions for decaying solutions of dissipative differential equations with non-smooth nonlinearity. Qual. Theory Dyn. Syst., 20(3):Paper No. 62, 38 pp, 2021.
[13] C. Foias and J.-C. Saut. Asymptotic behavior, as \( t \to +\infty \), of solutions of Navier-Stokes equations and nonlinear spectral manifolds. Indiana Univ. Math. J., 33(3):459–477, 1984.
[14] C. Foias and J.-C. Saut. Asymptotic integration of Navier-Stokes equations with potential forces. I. Indiana Univ. Math. J., 40(1):305–320, 1991.
[15] Jean-Michel Ghidaglia. Long time behaviour of solutions of abstract inequalities: applications to thermohydraulic and magnetohydrodynamic equations. J. Differential Equations, 61(2):268–294, 1986.
[16] Luan Hoang. Asymptotic expansions about infinity for solutions of nonlinear differential equations with coherently decaying forcing functions. The Annali della Scuola Normale di Pisa - Classe di Scienze, pages 1–48, 2022. accepted. Preprint DOI:10.48550/arXiv.2108.03724.
[17] Luan Hoang. The Navier–Stokes equations with body forces decaying coherently in time. pages 1–36, 2022. submitted. Preprint DOI:10.48550/arXiv.2204.05247.
[18] Luan T. Hoang and Vincent R. Martinez. Asymptotic expansion in Gevrey spaces for solutions of Navier-Stokes equations. Asymptot. Anal., 104(3–4):167–190, 2017.
[19] Luan T. Hoang and Vincent R. Martinez. Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces. J. Math. Anal. Appl., 462(1):84–113, 2018.
[20] Luan T. Hoang and Edriss S. Titi. Asymptotic expansions in time for rotating incompressible viscous fluids. Ann. Inst. H. Poincaré Anal. Non Linéaire, 38(1):109–137, 2021.
[21] Valery V. Kozlov and Stanislav D. Furta. *Asymptotic solutions of strongly nonlinear systems of differential equations*. Springer Monographs in Mathematics. Springer, Heidelberg, 2013. Translated from the 2009 Russian second edition by Lester J. Senechal.

[22] Solomon Lefschetz. *Differential equations: geometric theory*. Dover Publications, Inc., New York, second edition, 1977.

[23] Gheorghe Minea. Investigation of the Foias-Saut normalization in the finite-dimensional case. *J. Dynam. Differential Equations*, 10(1):189–207, 1998.

[24] Y. Shi. A Foias-Saut type of expansion for dissipative wave equations. *Comm. Partial Differential Equations*, 25(11-12):2287–2331, 2000.

Department of Mathematics and Statistics, Texas Tech University, 1108 Memorial Circle, Lubbock, TX 79409–1042, U. S. A.

Email address: luan.hoang@ttu.edu