Klein-Fock equation, proper-time formalism and symmetry of Hydrogen atom

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Abstract
We present main points of some of Fock papers in Quantum Theory, which were not properly followed when published.

Introduction
Fock papers on Quantum Theory have quite different lifes in Physics. Some of them were immediately recognized with Fock name attached to them. Hartree-Fock approach, Fock space and Fock representation are very well known examples. Another ones were not properly used when published. To this class belong papers where Fock was ahead of time, and his contemporaries were not able to really understand their importance. Klein-Fock equation, proper-time formalism and symmetry of Hydrogen atom are best examples of Fock papers within such a class.

In this lecture we explain main points of these Fock papers and remind Fock interpretation of energy-time uncertainty relation, as well as his ideas on generalization of the concept of physical space published in his last paper.

*To be published in the Proceedings of the UNESCO International School of Physics "Quantum Theory" part 2, commemorating the 100th anniversary of the birth of V.A.Fock, St.Peterburg, SPbU, 1999.
This work was supported in part by RFBR (Grant 97-01-01186) and by GRACENAS (Grant 97-0-14.1-61).
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Generalization of Schrödinger equation and Klein-Fock equation

V.A. Fock sent his first paper on Quantum Mechanics for publication in Zs.f. Physik one week after Schrödinger’s first paper had reached Leningrad. Fock generalized Schrödinger equation for the case of magnetic field and obtained an expression for the (normal) level splitting in a magnetic field as well as level splitting of Hydrogen atom in an electric field [1].

In his second paper on Quantum Mechanics [2] V.A. Fock presented relativistic generalization of Schrödinger equation for a particle in electromagnetic field on curved space. O. Klein [3] has also derived a relativistic generalization of Schrödinger equation and published it in Zs.f. Physik, vol.37 before Fock. But the paper by Fock in vol.39 was received by this journal before publication of Klein’s paper. The third paper on the same subject for the simplest case of a free particle on a flat space was written by W. Gordon [4]; it was received by Zs.F. Physik after paper by Klein was published and appeared in vol.40. Nevertheless, in many textbooks the equation for spin 0 particle is called as the Klein-Gordon equation.

One might see the reason for such discrimination of Fock in a character of the paper [2]. This paper contains much more material than a simple relativistic generalization of Schrödinger Eq. to Eq. for a free spin 0 particle. At that time, this material could not be appreciated by most physicists.

Let us consider briefly main points of Fock’s paper [2]. Fock introduces five dimensional space with the metric depending on electromagnetic potential $A_\mu$

$$d\sigma^2 = g_{\mu\nu}dx_\mu dx_\nu - \frac{e^2}{m^2c^4}(A_\nu dx_\nu + du)^2$$

where $u$ is an additional coordinate. In classical physics null geodesic line $d\sigma = 0$ describes a trajectory of charged particle in this space. Corresponding action $S$ will have five dimensional gradient squared equal to zero. Four dimensional action $W$ is related to $S$

$$S = \frac{e}{c}u + W$$
Five dimensional equations are invariant under transformation

\[ A_\nu = A'_\nu + \partial_\nu f \]
\[ u = u' - f \]

which later [5] were called as **gradient transformation**.

Both classical and quantum equations act in the same space. Therefore, the corresponding quantum wave equation for the wave function \( \Psi \) is the d’Alambert equation in five dimensional space

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial \Psi}{\partial x_\nu} \right) - 2A_\nu \frac{\partial^2 \Psi}{\partial u \partial x_\nu} + \left( A_\nu A^\nu - \frac{m^2 c^4}{e^2} \right) \frac{\partial^2 \Psi}{\partial u^2} = 0
\]

Four dimensional wave function \( \psi \) can be found from \( \Psi \) by a phase transformation

\[ \Psi = e^{i\frac{\pi}{2}\mu} \psi \]

and five dimensional eq. for \( \Psi \) produces immediately the Klein-Fock equation for \( \psi \) in electromagnetic field and on the curved space.

Gradient transformation of \( A_\nu \) and \( u \) induces transformation of the wave function

\[ \psi' = e^{-i\frac{\pi}{2}\mu} \psi \]

This transformation rule of wave function under gauge tranformation was first written in Fock’s paper on the Klein-Fock equation.

These two papers of young Fock made his name known to theoreticians. He got Rockefeller fellowship and went to study in Goettingen and Paris.

**Proper time formalism and Fock gauge condition (1937)**

In this paper V.A.Fock [6] considers Dirac equation with an external field and develops a relativistically and gauge invariant quasiclassical method of integration. At that time, such a method was needed to deal with vacuum polarization in the Dirac positron theory. It was shown by V.Weisskopf [7]
that logarithmic divergences appear in calculation of vacuum polarization. The problem was how to separate finite parts from divergent expressions in a unique relativistically and gauge invariant way.

Fock proposed to ensure invariance and uniqueness of calculations by using invariant quantities only. He introduces an invariant parameter to consider evolution in a symmetric manner with respect to all four relativistic coordinates and shows how to relate new five dimensional space to the four dimensional one in a unique way so that an additional parameter acquires a meaning of proper time. Fock notes that working in five dimensional space is necessary, because by this trick one gets uniqueness of divergent expressions due to the fact that the Riemann fundamental solution can be defined uniquely only in spaces of odd dimensions. Fock introduced also a new gauge condition for electromagnetic potential without singularities which enables to express potentials in terms of field strengths uniquely.

Fock considers second order Dirac equation in the four dimensional space

\[
\{P_\mu P^\mu + m^2 c^2 + \frac{e}{c} (\vec{\sigma} \cdot \vec{H}) - \frac{i e}{c} (\vec{\alpha} \cdot \vec{E})\} \Psi = 0
\]

or symbolically

\[
h^2 \Lambda \Psi = 0
\]

where \( \Lambda \) is a second order operator. One can present a solution as an integral

\[
\Psi = \int_C F d\tau
\]

over a function \( F \) of five variables \( x_\mu, \tau \) with a given integration path \( C \) in variable \( \tau \). Then \( F \) should be a solution of Dirac equation with proper time \( \tau \)

\[
\frac{h^2}{2m} \Lambda F = i h \frac{\partial F}{\partial \tau}
\]

subject to condition

\[
\int_C \frac{\partial F}{\partial \tau} d\tau = F_C = 0
\]

where \( F_C \) is function \( F \) taken at the path \( C \). Fock shows how to choose the path \( C \) to give \( \tau \) a meaning of proper time.
All calculations should be done in five dimensional space where separation of divergences is unique. Integration over proper time completes calculations. Proper time integrals does not depend on gauge and are relativistically invariant by their definition.

Fock proper time formalism was developed by J.Schwinger [8] in 1951 and by B.S.De Witt [9] in 1965. It is usually referred to as Schwinger-De Witt proper time method, although it was formulated by Fock. Fock paper was too much ahead of time.

Fock gauge condition for electromagnetic potential reads

\[(x_\nu - x_\nu^0)A^\nu = 0\]

assuming that potential \(A^\nu\) is non-singular. The potential can be expressed in terms of field strengths \(A_{\mu\nu}\) by averaging between points \(x\) and \(x^0\) according to

\[
\mathcal{F} = 2 \int_0^1 f \left[ x^0 + (x - x^0) u \right] u du
\]

Then

\[A^\nu = \frac{1}{2} (x - x^0)_\mu \mathcal{A}^{\mu\nu}\]

Fock gauge condition is especially useful when dealing with selfdual fields. Fock condition has another advantage: as it was found later [10], in quantum field theory the Faddeev-Popov ghosts decouple.

**Symmetry of Hydrogen atom and dynamical groups**

According to Schroedinger equation, energy levels of a charge in a spherically symmetric field are characterized by two quantum numbers: principal quantum number and eigenvalue of angular momentum. However, in Hydrogen atom energy levels depend on principal quantum number only. The origin of degeneracy was guessed to be in additional symmetry of Hydrogen atom. This problem was known long before Fock, but only Fock was able to solve the problem. He did it in paper "Hydrogen atom and non-Euclidean geometry" [11] in a simple and elegant way.
Fock starts by writing down the Schroedinger equation with Coulomb potential in momentum space as an integral equation

\[
\frac{1}{2m} p^2 \psi(p) = -\frac{Ze^2}{2\pi^2\hbar} \int \frac{\psi(p')}{|p - p'|} dp'
\]

For discrete spectrum an average square momentum is

\[ p_0 = \sqrt{-2mE} \]

Fock introduces coordinates of a stereographic projection of a unit sphere in four-dimensional Euclidean space

\[
\begin{align*}
\xi &= \frac{2p_0 p_x}{p_0^2 + p^2} = \sin \alpha \sin \vartheta \cos \varphi \\
\eta &= \frac{2p_0 p_y}{p_0^2 + p^2} = \sin \alpha \sin \vartheta \sin \varphi \\
\zeta &= \frac{2p_0 p_z}{p_0^2 + p^2} = \sin \alpha \cos \vartheta \\
\chi &= \frac{p_0^2 - p^2}{p_0^2 + p^2} = \cos \alpha
\end{align*}
\]

so that

\[ \xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1 \]

Schroedinger equation in new coordinates takes the following form

\[
\Psi(\alpha, \vartheta, \varphi) = \frac{\lambda}{2\pi^2} \int \frac{\Psi(\alpha', \vartheta', \varphi')}{4 \sin^2 \frac{\omega}{2}} d\Omega'
\]

where

\[
\lambda = \frac{Ze^2}{\hbar \sqrt{-2mE}}
\]

and \(2 \sin \frac{\omega}{2}\) is distance between points \(\alpha, \vartheta, \varphi\) and \(\alpha', \vartheta', \varphi'\) on four-dimensional unit sphere
\[ 4 \sin^2 \frac{\omega}{2} = (\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2 + (\chi - \chi')^2 \]

The equation for \( \Psi (\alpha, \vartheta, \varphi) \) is nothing but an integral equation for four-dimensional spherical functions. Function \( \Psi (\alpha, \vartheta, \varphi) \) is related to the wave function in momentum space \( \psi (p) \) as follows

\[ \Psi (\alpha, \vartheta, \varphi) = \frac{\pi}{\sqrt{8}} p_0^{-3/2} \left( p_0^2 + p^2 \right)^2 \psi (p) \]

Integral equation for \( \Psi (\alpha, \vartheta, \varphi) \) contains important information:
1. An invariance group of the Hydrogen-like atom is a group of four dimensional rotations. This explains why energy levels are independent of azimuthal quantum number and introduces hyperspherical functions in calculations. The invariance group discovered by Fock is especially useful in averaging or summation within the layer with given principal quantum number.
2. In the case of continuous spectrum of Hydrogen-like atoms analogous approach leads to the Lorentz group symmetry on four-dimensional hyper-paraboloid.
3. The invariance group of the Hydrogen-like atom discovered by Fock is not a kinematical group which transforms only coordinates or momenta. \( \alpha \)-rotations connect functions with different energy levels \( E' \). Such symmetry was named by Fock as dynamical. Search for new dynamical groups became quite popular after 1960, i.e. 25 years after this paper by Fock.
4. Fock has also shown that momentum space of Hydrogen-like atoms is non-Euclidean. It is endowed by the Riemannian geometry of constant positive curvature in the case of discrete spectra, and by Lobachevskian geometry with constant negative curvature in the case of continuous spectra.

**Energy-time uncertainty relation**

Uncertainty relation \( \Delta E \Delta t \geq h \) was originally considered as a relativization of Heisenberg’s relations \( \Delta x \Delta p_x \geq h \). Analysis of Fock and Krylov [12] has shown that energy-time uncertainty relation should be written as

\[ \Delta (E' - E) \Delta t \geq h \]

where \( \Delta (E' - E) \) is uncertainty in energy change in transition from one state to another, and \( \Delta t \) is time for which transition probability is close to
unity. This relation is applicable to any chosen experiment and results from "non-absolute description" of microsysteme by the wave function.

For quasistationary state $E'$ in transition to exactly stationary state $E$ uncertainty relation defines level width $\Delta E'$. If this width is small so that $\Delta t \approx \tau$ is great, decay probability is constant in time, and exponential decay follows with lifetime $\tau$. One should remind that behaviour of state during experiment (i.e.measurement) cannot be described by Schroedinger equation. Uncertainty in energy change in a system means that uncertainty of interaction energy of a system in transition grows when time of change shortens.

A possible generalization of the concept of physical space

Fock writes [13]: "The concept of physical space is closely related with that of the motion of a physical body. This connection is quite natural since we learn the properties of space by studying the motion of physical bodies. .. And the physical space (as distinguished from the configuration space) was always thought of as a manifold connected with the degrees of freedom of a single mass point. In quantum physics the simplest kind of physical object (a material point) may be supposed to have the properties of an electron." Fock describes spin degree of freedom of electron and the Pauli principle and concludes: "Applying to this case the (tacitly assumed) classical presumption that the physical space is defined by totality of variables describing the degrees of freedom of the simplest physical body (conventionally called a mass point), ...we arrive at the generalization of the concept of physical space. The generalized physical space defined above can be called spinor space." The Pauli principle corresponds to the impossibility for two electrons to occupy one and the same point in spinor space.

This Fock paper may be considered as a physical motivation for Supersymmetry [14].

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