A CENTRAL LIMIT THEOREM FOR THE KONTSEVICH-ZORICH COCYCLE

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Abstract. We show that a central limit theorem holds for exterior powers of the Kontsevich-Zorich (KZ) cocycle. In particular, we show that, under the hypothesis that the top Lyapunov exponent on the exterior power is simple, a central limit theorem holds for the lift of the (leafwise) hyperbolic Brownian motion to any strongly irreducible, symplectic, $\text{SL}(2,\mathbb{R})$-invariant subbundle, that is moreover symplectic-orthogonal to the so-called tautological subbundle. We then show that this implies that a central limit theorem holds for the lift of the Teichmüller geodesic flow to the same bundle.

For the random cocycle over the hyperbolic Brownian motion, we prove under the same hypotheses that the variance of the top exponent is strictly positive. For the deterministic cocycle over the Teichmüller geodesic flow we prove that the variance is strictly positive only for the top exponent of the first exterior power (the KZ cocycle itself) under the hypothesis that its Lyapunov spectrum is simple.

1. Introduction

This paper concerns the Kontsevich-Zorich (KZ) cocycle, a much studied dynamical system in the field of Teichmüller dynamics. The KZ cocycle has played a major role in addressing multiple questions of physical interest, including, for example, the computation of diffusion rates on wind-tree models [DHL14], and it itself acts as a renormalizing dynamical system for straight-line flows on translation surfaces. We refer the reader to the surveys [Zor06, FM14, Wri15] for an introduction to this rich area of research.

It is now well-established that Hodge theory, together with classical potential theory, can be brought to bear on this cocycle and its associated Lyapunov exponents, thanks to the pioneering works of M. Kontsevich and A. Zorich [KZ97, Kon97], later developed in [For02]. In these works, the Hodge norm was introduced in Teichmüller dynamics, and in [For02] it was proved that the logarithm of the Hodge norm is a subharmonic function on all exterior powers of the cocycle, hence the KZ cocycle has positive exponents on strata (see also [For06], [For11], [FMZ11], [FMZ12]).

The Hodge norm has since played a crucial role in the developments in Teichmüller dynamics, in particular in the study of the hyperbolicity properties of the KZ cocycle and of the Teichmüller flow, and related questions
in the ergodic theory of translations flows and interval exchange transformations. A very partial list of landmark applications of the Hodge norm includes [ABEM12, EKZ14, EMM15, EM18, Fil16a, Fil16b, Fil17].

The purpose of this paper is to show that probabilistic potential theory (and thus stochastic calculus), can be applied to study the oscillations of the Hodge norm of the KZ cocycle. In fact, we prove a (non-commutative) Central Limit Theorem (CLT) for exterior powers of both the random and the deterministic KZ cocycles, and we moreover prove the non-degeneracy of the CLT for the random cocycles and for the first exterior power of the deterministic KZ cocycle (the KZ cocycle itself) under the natural dynamical assumptions of simplicity of the Lyapunov spectrum. Motivated by computer experiments and the results in [Zor96, Zor97], the simplicity of the KZ spectrum for all strata of the moduli space of Abelian differentials was conjectured by M. Kontsevich and A. Zorich in [KZ97, Kon97]. It was then established by A. Avila and M. Viana in [AV07], and, in genus 2, for all SL(2, R)-invariant orbifolds, by M. Bainbridge in [Bai07].

The problem of finding oscillations of the KZ cocycle has been the subject of recent interest: in [AS21b], a mechanism to produce oscillations of the KZ cocycle was presented, where the basepoint is a fixed surface, and a more refined mechanism was developed by J. Chaika, O. Khalil and J. Smillie in their work on the ergodic measures of the Teichmüller horocycle flow [CKS21]. We expect that the deterministic central limit theorem presented here can be brought to bear on the scope of these results.

The probabilistic ideas that inspired our approach, and their application to geodesic flows, go back to the work of Y. Le Jan [LJ94], and we refer the reader to [FLJ12] for both an introduction to stochastic calculus and to the remarkable ideas that appear in [LJ94].

The approach we follow to prove the CLT for the random cocycle also relies crucially on the analysis of the Brownian semigroup to solve a (leafwise) Poisson equation, by a method reminiscent of [Led95]. In fact we prove exponential mixing for the lift of the Brownian motion to the Hodge bundle, leveraging the exponential mixing of the Teichmüller geodesic flow, due to A. Avila, S. Gouëzel and J.-C. Yoccoz for strata [AGY06] and to A. Avila and S. Gouëzel for all SL(2, R)-invariant orbifolds [AG13].

The CLT for the deterministic cocycle is then derived from the corresponding result for the random cocycle by a stopping time argument based in part on an asymptotic estimate due to A. Ancona [Anc90].

We point out that in the setting of products of independent and identically distributed random matrices, the central limit theorem was established, in varying levels of generality, by Bellman in [Bel54], H. Furstenberg and H. Kesten in [FK60], Tutubalin in [Tut77], Le Page in [LP82], Y. Guivarc’h and A. Raugi in [GR85], I. Ya. Golsheid and G. A. Margulis in [GM89], Hennion in [Hen97], Jan in [Jan00], and more recently, and under an optimal finite second moment condition, by Y. Benoist and J.-F. Quint in [BQ16]. The central limit theorem was also established for
solutions of linear stochastic differential equations with Markovian coefficients by P. Bougerol in [Bou88]. On the other hand, to the best of our knowledge, there are no comparable works in the setting of deterministic cocycles over (non-uniformly) hyperbolic flows, such as the one we treat here. We point to [DKP21], [FK21], [PP22] for some results on the central limit theorem in this direction. However, the cited results are established for cocycles over shifts of finite type (SFT’s), while it is well known that the Teichmüller flow has a symbolic representation as a suspension flow over a Markov shift on a countable alphabet. While not the original aim of this paper, we note that our work addresses, if ever so incrementally, this dearth of central limit theorem results for deterministic cocycles over non-uniformly hyperbolic flows (a related result for the KZ cocycle, based on the study of a transfer operator via anisotropic Banach spaces, has been recently announced by O. Khalil). We finally remark that the positivity of the variance for the particular case of the (deterministic) KZ cocycle, that we prove in this paper under certain hypotheses, would remain a non-trivial question, even if a general theorem were available.

In another direction, the paper of J. Daniels and B. Deroin [DD19] adapted the Teichmüller dynamics methodology to more general compact Kähler manifolds, and one in which the methods in this paper are applicable, provided that we can prove existence of a solution to Poisson’s equation for the corresponding Laplacian.

In [DFV17], D. Dolgopyat, B. Fayad and I. Vinogradov proved a central limit theorem for the Siegel transform of sufficiently regular observables for the diagonal action on the space of lattices. Their methods are in fact much more general and imply in particular a Central Limit Theorem for pushforwards of (unstable) unipotent arcs with respect to the uniform distribution on almost every unipotent orbit [DFV17, Theorem 7.1, Corollary 7.2]. It would be interesting to prove exponential mixing for the action of the Teichmüller flow on the projectivized Hodge bundle (see also Question A.2), with the aim of applying a multiplicative generalization of their results to the KZ cocycle.

After the appearance of a first draft of our paper, F. Arana-Herrera and the second-named author in [AHF24] proved a central limit theorem for sections of the Hodge bundle and a mixing central limit theorem for the Kontsevich-Zorich cocycle, with the results of this paper as a crucial input in a more general approach to mixing limit distributions results.

2. Statement of results

Let \( \pi : \mathbb{P}(\mathbb{H}) \to X \) be the projectivization of a strongly irreducible \( \text{SL}(2, \mathbb{R}) \)-invariant symplectic subbundle of the absolute (real) Hodge bundle over an \( \text{SL}(2, \mathbb{R}) \) orbit closure \( X \), whose fiber over each point in \( X \) is \( H^1(S, \mathbb{R}) \), and with \( \nu \) an ergodic \( \text{SL}(2, \mathbb{R}) \)-invariant probability measure on \( X \). The Kontsevich-Zorich cocycle is the lift of the \( \text{SL}(2, \mathbb{R}) \) action to \( \mathbb{P}(\mathbb{H}) \),
obtained by parallel transport with respect to the Gauss-Manin connection. Furthermore, the cocycle acts symplectically since it preserves the intersection form on $H^1(S, \mathbb{R})$.

For our purposes, we will be concerned with $k$-th exterior powers $H^{(k)}$ of strongly irreducible invariant symplectic components $H$ of the Hodge bundle, which are symplectic orthogonal to the tautological subbundle (spanned for every $\omega \in X$ by $[\text{Re} \, \omega]$ and $[\text{Im} \, \omega]$). We will denote by $\mathbb{P}(H^{(k)})$ the projectivization of the bundle $H^{(k)}$. This bundle supports an SO$(2, \mathbb{R})$-invariant probability measure $\hat{\nu}$ such that, for $\nu$-a.e $\omega$, the conditional measure on $\mathbb{P}(H^{(k)}_{\omega})$ is the Haar measure. An Euclidean structure is in fact given by the Hodge norm (see 3.5 for the definition), and which we use in the sequel.

Therefore, we fix the SO$(2, \mathbb{R})$-invariant Hodge norm $\| \cdot \|^{(k)}_{\pi(\cdot)}$ on $\mathbb{P}(H^{(k)})$. Define $\sigma_k : \text{SL}(2, \mathbb{R}) \times \mathbb{P}(H^{(k)}) \to \mathbb{R}$ by

$$\sigma_k(g, v) = \log \frac{\|g v\|^{(k)}_{\pi(\cdot)}}{\|v\|^{(k)}_{\pi(\cdot)}}.$$

Let $g_t = \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right)$ denote the diagonal subgroup of $\text{SL}(2, \mathbb{R})$, whose action on the orbit closure $X$ yields the Teichmüller flow. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h$ denote the non-negative Lyapunov exponents of the Kontsevich–Zorich cocycle on a symplectic strongly irreducible subbundle $H$ of dimension $2h \in \{2, \ldots , 2g\}$. Since the cocycle is symplectic on $H$, the top $h$ exponents determine the entire Lyapunov spectrum.

For $\omega \in X$, let $v_\omega$ in $\mathbb{P}(H^{(k)}_{\omega})$ be any $k$-dimensional exterior vector (of dimension $k \leq h$) in $H_{\omega}$. For $\nu$-a.e. $v = (\omega, v_\omega)$, it is a consequence of the multiplicative ergodic theorem that

$$\lim_{T \to \infty} T^{-1} \sum_{i=1}^{k} \lambda_i = \frac{1}{\sqrt{2\pi V^{(1)}_{g_{\infty}}}} \int_{a}^{b} \exp(-x^2/V^{(1)}_{g_{\infty}}) dx.$$

Moreover, if the Lyapunov spectrum is simple, then $V^{(1)}_{g_{\infty}} > 0$. 

Our main result is the following:

**Theorem 2.1.** Let $H$ be a strongly irreducible, symplectic, $\text{SL}(2, \mathbb{R})$-invariant subbundle, which is symplectic orthogonal to the tautological subbundle. If $\lambda_k > \lambda_{k+1}$, then there exists a real number $V^{(k)}_{g_{\infty}} \geq 0$ such that

$$\lim_{T \to \infty} \nu \left( \left\{ v \in \mathbb{P}(H^{(k)}) : a \leq \frac{1}{\sqrt{T}} \left( \sigma_k(g_T, v) - T \sum_{i=1}^{k} \lambda_i \right) \leq b \right\} \right) = \frac{1}{\sqrt{2\pi V^{(k)}_{g_{\infty}}}} \int_{a}^{b} \exp(-x^2/V^{(k)}_{g_{\infty}}) dx.$$
Remark 2.2. The statement also holds in the event that $V_{g_{\infty}}^{(k)} = 0$, and in that case the resulting distribution would be a delta distribution. The positivity of the variance holds for 2-dimensional subbundles with strictly positive top Lyapunov exponent (for instance on the symplectic orthogonal of the tautological subbundle in genus 2 for any $\text{SL}(2, \mathbb{R})$-invariant measure), as in this case the simplicity condition on the top exponent is trivially satisfied.

Remark 2.3. The simplicity of the Lyapunov spectrum is established for the canonical Masur-Veech measures on strata by A. Avila and M. Viana in [AV07], and we remark that, in genus 2, this is established for all ergodic $\text{SL}(2, \mathbb{R})$-invariant probability measures by M. Bainbridge in [Bai07].

Remark 2.4. The assumption that $H$ is symplectic orthogonal to the tautological subbundle precludes the equality $2h = 2g$, which in turn precludes the equality $k = g$. It also follows by the spectral gap property of the Kontsevich-Zorich cocycle that for any $g_t$-invariant and ergodic probability measure, $\lambda_1 < 1$ [For02] (see also [FMZ12, Corollary 2.2]).

Since in Theorem 2.1 we randomize both the Abelian differential $\omega$ and vector $v_{\omega} \in \mathbb{P}(H_{\omega}^{(k)})$ with respect to the measure $\nu$, it is natural to ask if our results also hold for future-Oseledets-generic sections of $\mathbb{P}(H^{(k)})$ (see Definition 6.1). In particular, for the deterministic cocycle, we derive in Section 6:

Corollary 2.5. Under the hypothesis of Theorem 2.1, there exists a real number $V_{g_{\infty}}^{(1)} \geq 0$ such that for any future-Oseledets-generic section $v = (\omega, v_{\omega})$ of $\mathbb{P}(H^{(k)})$, we have

$$
\lim_{T \to \infty} \nu \left( \left\{ \omega \in X : a \leq \frac{1}{\sqrt{T}} \left( \sigma_k(g_T, v_{\omega}) - T \sum_{i=1}^{k} \lambda_i \right) \leq b \right\} \right) = \frac{1}{\sqrt{2\pi V_{g_{\infty}}^{(k)}}} \int_{a}^{b} \exp(-x^2/V_{g_{\infty}}^{(k)}) dx.
$$

Moreover, if the Lyapunov spectrum is simple, then $V_{g_{\infty}}^{(1)} > 0$.

To prove Theorem 2.1, we will first work with the hyperbolic Brownian motion, which is the diffusion process generated by the foliated hyperbolic Laplacian. Let $\rho$ be a (foliated) hyperbolic Brownian motion trajectory starting at a (generic) basepoint $\omega \in X$, defined almost everywhere with respect to a probability measure $\mathbb{P}_\omega$ on the space of such trajectories $W_\omega$. This process is in fact defined on $X^* = \text{SO}(2, \mathbb{R}) \setminus X$. Moreover, $\rho$ can be lifted to $\text{SL}(2, \mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as $\rho$ by abuse of notation. Additionally, the space $X$ gives rise to a product space $X^W := X \otimes W$ whose fiber over each point $\omega$ in $X$ is $W_\omega$, and which also
supports a measure $\nu_P := \nu \otimes \mathbb{P}$, whose conditional measure over a point $\omega$ is $\mathbb{P}_\omega$. We can thus similarly define the product $W$-Hodge bundle $\mathbb{P}^W(H^{(k)})$, whose fiber over each point $(\omega, \rho)$ in $X^W$ is $H^{(k)}_\omega$. A pair $(\rho, v) \in \mathbb{P}^W(H^{(k)})$ is thus defined to be the lift of the path $\rho$ (starting at $\omega$) to $\mathbb{P}^W(H^{(k)})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_P := \hat{\nu} \otimes \mathbb{P}$ whose conditional measure over a point $v$ is $\mathbb{P}_v$. We therefore also have

**Theorem 2.6.** Let $H$ be a strongly irreducible, symplectic, $SL(2, \mathbb{R})$-invariant subbundle, which is symplectic orthogonal to the tautological subbundle. If $\lambda_k > \lambda_{k+1}$, then there exists a real number $V^{(k)}_{\rho_\infty} > 0$ such that

$$\lim_{T \to \infty} \hat{\nu}_P \left( \left\{ (\rho, v) \in \mathbb{P}^W(H^{(k)}) : a \leq \frac{1}{\sqrt{T}} \left( \sigma_k(\rho_T, v) - T \left( \sum_{i=1}^{k} \lambda_i \right) \right) \leq b \right\} \right) = \frac{1}{\sqrt{2\pi V^{(k)}_{\rho_\infty}}} \int_{a}^{b} \exp \left( -\frac{x^2}{2V^{(k)}_{\rho_\infty}} \right) dx.$$

**Remark 2.7.** Observe that for $g = 2$, the symplectic orthogonal bundle to the tautological bundle has dimension 2, hence it is strongly irreducible. Our two results reduce to ones that concern the second Lyapunov exponent of the Kontsevich-Zorich cocycle on the full Hodge bundle.

In addition to the Hodge theoretic techniques that we employ, some ingredients of our proof include

- Exponential mixing of the Teichmüller geodesic flow (due to Avila-Gouëzel-Yoccoz [AGY06] for strata and Avila-Gouëzel [AG13] for all $SL(2, \mathbb{R})$-invariant orbifolds) to derive the existence of a unique zero-average solution of a Poisson equation (see Appendix A);
- elementary stochastic calculus to extract and control the necessary oscillations;
- and an asymptotic estimate due to Ancona [Anc90] to relate the geodesic flow with the Brownian motion.

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3. Preliminaries

3.1. Translation surfaces. Let $S$ be a Riemann surface of genus $g \geq 2$, and $\omega$ a holomorphic 1-form on $S$. The pair $(S, \omega)$ is called a translation surface, since $\omega$ induces an atlas whose coordinate changes are translations on $C \equiv \mathbb{R}^2$. In other terms, $\omega$ gives a flat metric with finitely many conical singularities and trivial holonomy on $S$, and the zero set of $\omega$ characterizes the singularity set of the conical metric. The area of a translation surface is given by $\int_S \omega \wedge \overline{\omega}$. We will refer to the pair $(S, \omega)$ as just $\omega$.

3.2. Moduli Space. Let $T\mathcal{H}_g$ be the Teichmüller space of unit-area translation surfaces of genus $g \geq 2$, and let $\mathcal{H}_g = T\mathcal{H}_g / \text{Mod}_g$ be the corresponding moduli space, where $\text{Mod}_g$ denotes the mapping class group. The space $\mathcal{H}_g$ is partitioned into strata $\mathcal{H}_\kappa$, which consist of all unit-area translation surfaces whose conical singularities have total angles $2\pi (1 + \kappa_1), \ldots, 2\pi (1 + \kappa_s)$, as $\kappa = (\kappa_1, \ldots, \kappa_s)$ varies over multi-indices with $\sum \kappa_i = 2g - 2$. Local period coordinates on each stratum are defined by the map which takes every holomorphic 1-form $\omega$ to its cohomology class $[\omega]$ in $H^1(S, \Sigma_\omega, \mathbb{C})$, relative to the set $\Sigma_\omega$ of its zeros. The set of all period coordinate maps defines an affine structure on each stratum, since all changes of coordinates are given by affine maps.

3.3. $\text{SL}(2, \mathbb{R})$ action. There is a natural action of $\text{SL}(2, \mathbb{R})$ on the space of all translation surfaces which descends to their Teichmüller and moduli spaces. It is proved in [EM18, EMM15] that, for any $\omega \in H^1(S, \Sigma_\omega, \mathbb{C})$, the closure $X$ of $\text{SL}(2, \mathbb{R}) \cdot \omega$ is an affine invariant suborbifold, and supports a unique ergodic $\text{SL}(2, \mathbb{R})$-invariant probability measure $\nu$ in the Lebesgue measure class, given by the normalized Lebesgue measure in period coordinates.

3.4. Kontsevich-Zorich cocycle. Let $\hat{\mathcal{H}}_{\kappa}(S, \mathbb{R}) = T\mathcal{H}_\kappa \times H^1(S, \mathbb{R})$, and for every $g \in \text{SL}(2, \mathbb{R})$, let $\hat{g} : \hat{\mathcal{H}}_{\kappa}(S, \mathbb{R}) \to \hat{\mathcal{H}}_{\kappa}(S, \mathbb{R})$ be the trivial cocycle map defined as

$$\hat{g}(\omega, c) = (g\omega, c), \quad \text{for } \omega \in T\mathcal{H}_\kappa \text{ and } c \in H^1(S, \mathbb{R}),$$

The absolute (real) Hodge bundle is given by $H_{\kappa}(S, \mathbb{R}) = \hat{\mathcal{H}}_{\kappa}(S, \mathbb{R}) / \text{Mod}_g$ and the Kontsevich-Zorich cocycle $g$ is the projection of $\hat{g}$ to $H_{\kappa}(S, \mathbb{R})$.

3.5. Hodge inner product and the second fundamental form. Given two holomorphic 1-forms $\omega_1, \omega_2$ in $\Omega(S)$, where $\Omega(S)$ is the vector space of holomorphic 1-forms on $S$, the Hodge inner product is given by the formula

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_S \omega_1 \wedge \overline{\omega_2}$$

Moreover, the Hodge representation theorem implies that for any given cohomology class $c \in H^1(S, \mathbb{R})$, there is a unique holomorphic 1-form $h(c) \in H^1(S, \mathbb{R})$. The Hodge inner product is an $\text{SL}(2, \mathbb{R})$-invariant bilinear form on $H_{\kappa}(S, \mathbb{R})$. The second fundamental form $\mathcal{F}_{\kappa}(g)$ is given by

$$\mathcal{F}_{\kappa}(g) = \frac{1}{2} \int_S \langle \omega_1, \omega_2 \rangle$$

for any $\omega_1, \omega_2 \in H^1(S, \mathbb{R})$. The second fundamental form is a bilinear form on the tangent space $T_{\omega} H_{\kappa}(S, \mathbb{R})$ at each point $\omega$. The absolute (real) Hodge bundle is given by $H_{\kappa}(S, \mathbb{R}) = \hat{\mathcal{H}}_{\kappa}(S, \mathbb{R}) / \text{Mod}_g$ and the Kontsevich-Zorich cocycle $g$ is the projection of $\hat{g}$ to $H_{\kappa}(S, \mathbb{R})$.
The Hodge inner product for two real cohomology classes $c_1, c_2 \in H^1(S, \mathbb{R})$ is defined as

$$A_\omega(c_1, c_2) := \langle h(c_1), h(c_2) \rangle.$$ 

The second fundamental form $B_\omega$ (of the Gauss-Manin connection with respect to the Chern connection for the holomorphic structure of the Hodge filtration) is defined as

$$B_\omega(c_1, c_2) := \frac{i}{2} \int_S \frac{h(c_1)h(c_2)}{\omega^2} \omega \wedge \overline{\omega}.$$ 

Let $H_\omega$ denote the curvature operator of the second fundamental form.

**Remark 3.1.** It is known that $B_\omega$ vanishes identically in the symplectic orthogonal of the tautological subbundle on only two orbit closures, namely the *Eierlegende Wollmilchsau* and *Ornithorynque*, and this follows from the works [Aul16, EKZ14, Möl11, AN20]. By a result of S. Filip [Fil17], the rank of the second fundamental form $B$ equals the number of strictly positive Lyapunov exponents of the Kontsevich–Zorich cocycle.

In the following $\mathcal{H}$ will denote a strongly irreducible, symplectic, SL$(2, \mathbb{R})$-invariant subbundle, which is symplectic orthogonal to the tautological bundle. For any isotropic $k$-dimensional exterior vector $c_\omega$ in $\mathcal{H}^{(k)}$, it also follows by [For02] (see also [FMZ12, Corollary 2.2]) that

$$\left| \frac{d}{dt} \sigma_k(g_t, c_\omega) \right| < k$$

(3.5.1)

For $h \in \{1, \ldots, g-1\}$, let $\{c_1, c_2, \ldots, c_h\}$ be a Hodge-orthonormal basis of $\mathcal{H} \subset H^1(S, \mathbb{R})$, and let $A_\omega^{(h)}$ (resp., $B_\omega^{(h)}$) be the corresponding representation matrix of the Hodge inner product $A_\omega$ (resp., of the second fundamental form $B_\omega$). Let $H_\omega^{(h)} = B_\omega^{(h)} \overline{B_\omega^{(h)}}$ be the matrix of the curvature operator, which is Hermitian non-negative, since $B_\omega^{(h)}$ is symmetric. The eigenvalues of $B_\omega^{(h)}$ are denoted by $\Lambda_i(\omega)$, where $|\Lambda_1| > |\Lambda_2| \geq \cdots \geq |\Lambda_h| \geq 0$. Moreover, the norm squared of these eigenvalues, $|\Lambda_i(\omega)|^2$, are the eigenvalues of the curvature matrix $H_\omega^{(h)}$, which are continuous, bounded functions on $\mathcal{H}_g$ (cf. [FMZ12], Lemma 2.3).

For any $k$-dimensional exterior vector $v \in \mathbb{P}(\mathcal{H}^{(k)})$, let

$$\{c_1, c_2, \ldots, c_k, c_{k+1}, \ldots, c_h\} \subset \mathcal{H}$$

be an ordered orthonormal basis such that $\{c_1, c_2, \ldots, c_k\}$ is a basis of $v$. We let $A_\omega^{(k)}(v)$ (resp., $B_\omega^{(k)}(v)$) be the corresponding representation matrix of the Hodge inner product $A_\omega$ (resp., of the second fundamental form $B_\omega$) restricted to $v$ with respect to the basis $\{c_1, c_2, \ldots, c_k\}$. We let $H_\omega^{(k)}(v)$ be the representation matrix of the restriction of the curvature operator $H_\omega$ to $v$ with respect to the basis $\{c_1, c_2, \ldots, c_k\}$.
3.6. Foliated Hyperbolic Laplacian. The space $\mathcal{H}_g$, is foliated by the orbits of the $\SL(2,\mathbb{R})$-action, whose leaves are isometric to the unit cotangent bundle of the Poincaré disk $\mathbb{D}$. For $\omega \in \mathcal{H}_g$, the Teichmüller disk $L_\omega := \SL(2,\mathbb{R})/\SO(2,\mathbb{R}) \cdot \omega$ is isometric to $\mathbb{D}$, and so is endowed with the (foliated) hyperbolic gradient $\nabla_{L_\omega}$ and hyperbolic Laplacian $\Delta_{L_\omega}$. Let 

$$r_\theta = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

**Remark 3.2.** Observe that for $\omega \in X$, the Teichmüller disk $L_\omega$ is identified with $\mathbb{D}$ via the map $(t, \theta) \mapsto \SO(2,\mathbb{R}) \cdot g_\Omega r_\theta \omega$.

Now suppose that $f : X \to \mathbb{R}$ is an $\SO(2,\mathbb{R})$-invariant $C^\infty$-function in the direction of the leaf. For $\omega \in X$ and for $L_\omega$ the Teichmüller disk passing through $\omega$, we define $\Delta f(\omega) := \Delta_{L_\omega} f|_{L_\omega}(\omega)$, where $f|_{L_\omega}$ is the restriction of $f$ to $L_\omega$. We also define the leafwise gradient similarly.

Observe that the Hodge inner product $A_{\omega}(\cdot, \cdot)$ is invariant under the action of $\SO(2,\mathbb{R})$, and so defines a real-analytic function on the Teichmüller disk. In the sequel, we will only work in a given Teichmüller disk, so the norm will read $(\cdot, \cdot)_\omega$ for a complex parameter $z \in \mathbb{D}$. For any $k$-dimensional exterior vector $v = (\omega, v_\omega)$ in the symplectic orthogonal of the tautological subbundle (with the origin $z = 0$ corresponding to $\omega$ as in 3.2), define

$$\sigma_k(z, v) := \log |\det A_z^{(k)}(v)|^{1/2},$$

where $A_z^{(k)}(v) = A_z(v_i, v_j)$ and $\{v_i\}$ is an ordered basis of $v$.

**Remark 3.3.** In fact, this is an abuse of notation since we originally lifted elements of $\SL(2,\mathbb{R})$ to the Hodge bundle. This is not an issue since the Hodge norm is $\SO(2,\mathbb{R})$-invariant.

We recall the following fundamental fact

**Theorem 3.4. [For02, FMZ12]** For every $1 \leq k \leq h$ there exist smooth functions $\Phi_k : \mathbb{P}(H^{(k)}) \to [0, k]$ and $\Psi_k : \mathbb{P}(H^{(k)}) \to D(0, k) \subset \mathbb{C}$ such that the following holds. For any $k$-dimensional exterior vector $v \in \mathbb{P}(H^{(k)})$, we have the following identities:

$$\Delta_{L_\omega} \sigma_k(z, v) = 2 \Phi_k(z, v) \quad \text{and} \quad \nabla_{L_\omega} \sigma_k(z, v) = \Psi_k(z, v). \quad (3.6.1)$$

In the particular case that $k = h$, there exist functions $\Lambda_i : X \to D(0, 1)$ for all $i \in \{1, \ldots, h\}$ such that

$$\Delta_{L_\omega} \sigma_h(z, v) = 2 \sum_{i=1}^h |\Lambda_i(z)|^2 \quad \text{and} \quad \nabla_{L_\omega} \sigma_h(z, v) = \sum_{i=1}^h \Lambda_i(z).$$

In particular, in this case, the Laplacian and the gradient are independent of the choice of a maximal isotropic (Lagrangian) subspace $v \in \mathbb{P}(H^{(h)})$. Moreover, for all $k \in \{1, \ldots, h\}$, under the condition that $\lambda_k > \lambda_{k+1}$, which implies that the unstable Oseledets isotropic $k$-dimensional distribution $E_k^+$
is well-defined, for \( \nu \). In particular, in this case, the Laplacian and the gradient are independent of the choice of a maximal isotropic (Lagrangian) subspace \( \mathbf{v} \in \mathbb{P}(H^k) \). Moreover, for all \( k \in \{1, \ldots, h\} \), under the condition that \( \lambda_k > \lambda_{k+1} \), which implies that the unstable Oseledets isotropic \( k \)-dimensional distribution \( E^+_k \) is well-defined, for \( \nu \) almost all \( \mathbf{v} \in \mathbb{P}(H^k) \) we have that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \Delta_L \sigma_k (g_t, \mathbf{v}) \, dt = \int_X 2 \Phi_k (\omega, E^+_k (\omega)) \, d\nu = 2 \sum_{i=1}^k \lambda_i.
\]

**Remark 3.5.** The functions \( \Phi_k \) and \( \Psi_k \) can be written as follows. Let \( B_z^{(k)}(\mathbf{v}) \) and \( H_z^{(k)}(\mathbf{v}) \) denote the restrictions of the second fundamental form and of the curvature to the \( k \)-dimensional exterior vector \( \mathbf{v} \in \mathbb{P}(H^k) \). By definition \( B^{(k)} \) and \( H^{(k)} \) are functions on \( \mathbb{P}(H^k) \) with values in the subspace of complex symmetric \( k \times k \) matrices and non-negative Hermitian \( k \times k \) matrices. The following formulas hold, for all \( (\omega, \mathbf{v}) \in \mathbb{P}(H^k) \):

\[
\Phi_k (\omega, \mathbf{v}) = 2 \text{tr}(H^{(k)}(\mathbf{v})) - \text{tr} \left( B^{(k)}(\mathbf{v}) \bar{B}^{(k)}(\mathbf{v}) \right); \quad \Psi_k (\omega, \mathbf{v}) = \text{tr}(\bar{B}^{(k)}(\mathbf{v})).
\] (3.6.2)

An important property of the function \( \Psi_k \), which is relevant in the proof of positivity of the variance (for \( k = 1 \)) in Section 5.2, is that for all \( \omega \in X \) the set of its critical point is a subset of the level set

\[
\{ \mathbf{v} \in \mathbb{P}_{\omega}(H^k) | \Psi_k (\omega, \mathbf{v}) = 0 \}.
\]

In fact, the following linear algebra result holds:

**Lemma 3.6.** For every \( \omega \in X \), the set of critical points of the function \( \Psi_k (\omega, \cdot) \) equals the set

\[
\{ \mathbf{v} \in \mathbb{P}(H^k) | B_z^{(k)}(\mathbf{v}) = 0 \} \subset \{ \mathbf{v} \in \mathbb{P}(H^k) | \Psi_k (\omega, \mathbf{v}) = 0 \}.
\]

In addition, for all \( \omega \in X \) and \( \mathbf{v} \in \mathbb{P}_{\omega}(H^k) \), we have

\[
\| D_\mathbf{v} \Psi_k (\omega, \mathbf{v}) \| \geq \| \Psi_k (\omega, \mathbf{v}) \| / k.
\]

**Proof.** By its definition, the function \( \Psi_k \) is computed as follows.

Let \( \{v_1, \ldots, v_k\} \) denote any Hodge orthonormal basis of the \( k \)-dimensional isotropic subspace \( \mathbf{v} \in \mathbb{P}_{\omega}(H^k) \), then

\[
\Psi_k (\omega, \mathbf{v}) = \sum_{i=1}^k B_\omega (v_i, v_i).
\]

It can be seen that the above expression depends only on the isotropic subspace \( \mathbf{v} \) and not on its orthonormal basis. By fixing an orthonormal basis \( \{w_1, \ldots, w_h\} \) of a maximal isotropic (Lagrangian) subspace, the space of all orthonormal bases is in bijective correspondence with the group of complex unitary matrices.
Let then $U = (U_{ij})$ denote a complex unitary matrix (such that $UU^* = I$) and for all $i \in \{1, \ldots, h\}$ let

$$w_i = \sum_{a=1}^{h} U_{ia} v_a.$$ 

We then have

$$\Psi_k(\omega, w) = \sum_{i=1}^{k} B_{\omega}(w_i, w_i) = \sum_{i=1}^{k} \sum_{a,b=1}^{h} U_{ia} U_{ib} B(v_a, v_b).$$

The tangent space at the identity of the unitary group is the vector space of anti-Hermitian matrices, that is, the matrices $T$ such that $T^* = -T$. By differentiating the above formula with respect to $U$ along $T$ we have

$$(DU\Psi_k)(\omega, v, T) = 2 \sum_{a,b=1}^{k} T_{ab} B(v_a, v_b).$$ (3.6.3)

It follows that, if $v$ is a critical point, for $T$ such that $T_{\alpha\beta} = 0$ for all $(\alpha, \beta) \notin \{(a, b), (b, a)\}$ we have

$$(T_{ab} + T_{ba}) B(v_a, v_b) = 0$$

which if $T_{ab} + T_{ba} = T_{ab} - \overline{T_{ab}} \neq 0$, implies $B(v_a, v_b) = 0$.

Finally, the stated lower bound follows from formula (3.6.3) since

$$\|Dv\Psi_k(\omega, v)\| \geq \max_{i \in \{1, \ldots, k\}} |B_{\omega}(v_i, v_i)| \geq |\Psi_k(\omega, v)| / k.$$ 

The proof is therefore complete. \qed

3.7. Harmonic measures. A probability measure $\nu$ on $SO(2, \mathbb{R}) \setminus X$ is called harmonic if for all bounded functions $f : SO(2, \mathbb{R}) \setminus X \to \mathbb{R}$ of class $C^\infty$ in the leaf direction,

$$\int_{SO(2,\mathbb{R}) \setminus X} \Delta f(\omega) \, d\nu = \int_{SO(2,\mathbb{R}) \setminus X} \Delta_{\omega} f|_{\omega}(\omega) \, d\nu = 0.$$ 

Such a measure is also ergodic if $SO(2, \mathbb{R}) \setminus X$ cannot be partitioned into two union of leaves, each of which having positive $\nu$ measure. We refer the reader to the interesting paper of Lucy Garnett [Gar83] for details and for an ergodic theorem for such measures. It is also a fact, due to Bakhtin-Martinez [BM08], that harmonic measures on $SO(2, \mathbb{R}) \setminus X$ are in one-to-one correspondence with $P$-invariant measures on $X$. This is closely related to a classical fact due to Furstenberg [Fur63a, Fur63b] that $P$-invariant measures are in one-to-one correspondence with (admissible) stationary measures, and that harmonic measures are stationary. In the case of $SL(2, \mathbb{R})$, these three notions are therefore closely related.
3.8. Hyperbolic Brownian Motion. Following the normalization used in [For02] (which is a standard normalization, see also [Hel00]), for \( z = re^{i\theta} \) with \( \theta \in [0, 2\pi] \), write

\[
t := \frac{1}{2} \log \frac{1 + r}{1 - r}.
\]

(3.8.1)

Since the Hodge norm is \( \text{SO}(2, \mathbb{R}) \)-invariant, it suffices to study the diffusion process generated by \( \frac{1}{2} \Delta_{L_\omega} \), where the leafwise hyperbolic Laplacian in geodesic polar coordinates is given by

\[
\Delta_{L_\omega} = \frac{\partial^2}{\partial t^2} + 2 \coth(2t) \frac{\partial}{\partial t} + \frac{4}{\sinh^2(2t)} \frac{\partial^2}{\partial \theta^2}.
\]

(3.8.2)

Moreover, let \((W^{(i)}_\omega, \mathbb{P}^{(i)}_\omega), i = 1, 2\), be two copies of the space of Brownian trajectories \( C(\mathbb{R}^+, \mathbb{R}) \) starting at the origin (with the origin corresponding to a random point \( \omega \)), together with the standard Wiener measure, and such that \( W^{(1)}_\omega \) and \( W^{(2)}_\omega \) are independent. Set \( W_\omega = W^{(1)}_\omega \times W^{(2)}_\omega \) and \( \mathbb{P}_\omega = \mathbb{P}^{(1)}_\omega \times \mathbb{P}^{(2)}_\omega \). The hyperbolic Brownian motion is the diffusion process \( \rho_s = (t(s), \theta(s)) \) generated by the (leafwise) hyperbolic Laplacian. It follows by Ito’s formula [FLJ12, Theorem VI.5.6] that the generator determines the trajectories of the diffusion process \( \rho_s \) which are solutions of the following stochastic differential equations

\[
dt(s) = dW^{(1)}_s + \coth(2t(s)) ds
\]

(3.8.3)

\[
d\theta(s) = \frac{2}{\sinh(2t(s))} dW^{(2)}_s
\]

(3.8.4)

with \( t(0) = 0 \) and \( \theta(0) \) being uniformly distributed on \( S^1 \).

In addition, for an \( \text{SO}(2, \mathbb{R}) \)-invariant function \( f : \mathbb{P}(\mathbb{H}) \rightarrow \mathbb{R} \), where \( f \) is of class \( W^{2,2} \) along \( \text{SL}(2, \mathbb{R}) \) orbits, Ito’s formula gives

\[
f(\rho_T, \nu) - f(\rho_0, \nu) = \int_0^T \left( \frac{\partial}{\partial t} f(\rho_s, \nu) \cdot \frac{2}{\sinh(2t(s))} \frac{\partial}{\partial \theta} f(\rho_s, \nu) \right) \cdot (dW^{(1)}_s, dW^{(2)}_s) + \int_0^T \frac{1}{2} \frac{\partial^2}{\partial t^2} f(\rho_s, \nu) + \frac{1}{2} 2 \coth(2t(s)) \frac{\partial}{\partial t} f(\rho_s, \nu) + \frac{4}{\sinh^2(2t(s))} \frac{\partial^2}{\partial \theta^2} f(\rho_s, \nu) \right) ds
\]

\[
= \int_0^T \nabla_{L_\omega} f(\rho_s, \nu) \cdot (dW^{(1)}_s, dW^{(2)}_s) + \frac{1}{2} \int_0^T \Delta_{L_\omega} f(\rho_s, \nu) ds.
\]

Finally, we note that the foliated heat semigroup \( D_t \) is given as follows

\[
D_s f(x, \nu) := \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty f(z, \nu) p_\omega(t, s) \sinh(t) dt d\theta
\]

where \( p_\omega(t, s) \) is the (foliated) hyperbolic heat kernel at time \( s \); in other words, for \( x, y \in L_\omega \), this is the transition probability kernel \( p_\omega(x, y; s) \), with \( d_D(x, y) = t \).
4. Proofs of Main Theorems

4.1. Distributional Convergence in Theorem 2.6. Recall that $\rho_s$ is the diffusion process generated by the foliated hyperbolic Laplacian. We are interested in studying the term

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_T, v) - T\sum_{i=1}^{k} \lambda_i). \quad (4.1.1)$$

Set $\lambda(k) = \sum_{i=1}^{k} \lambda_i$. By applying Ito’s formula, we obtain,

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_T, v) - T\lambda(k)) = \frac{\sigma_k(\rho_0, v)}{\sqrt{T}} + \frac{1}{\sqrt{T}} \int_0^T \nabla_{L,\omega} \sigma_k(\rho_s, v) \cdot (dW_s^{(1)}, dW_s^{(2)}) \quad (4.1.2)$$

$$+ \frac{1}{2\sqrt{T}} \int_0^T (\Delta_{L,\omega} \sigma_k(\rho_s, v) - 2\lambda(k)) ds \quad (4.1.3)$$

Let $W^{2,2}(\mathbb{P}(H), \nu)$ denote the (foliated) Sobolev space of functions which belong to $L^2(\mathbb{P}(H), \nu)$ together with all their derivatives up to second order, in all directions tangent to $\text{SL}(2, \mathbb{R})$ orbits:

$$f \in W^{2,2}(\mathbb{P}(H), \nu) \iff f, Vf, VWf \in L^2(\mathbb{P}(H), \nu), \quad \text{for all } V, W \in \mathfrak{sl}(2, \mathbb{R}).$$

It follows by Lemma A.3 that the equation

$$\Delta_{L,\omega} u(\omega, v) = \Delta_{L,\omega} \sigma(\omega, v) - 2\lambda(k), \quad \text{for } (\omega, v, \omega) \in \mathbb{P}(H), \quad (4.1.4)$$

has an $\text{SO}(2, \mathbb{R})$-invariant solution $u^{(k)} \in W^{2,2}(\mathbb{P}(H), \nu)$, the space of functions with all $\mathfrak{sl}(2, \mathbb{R})$-derivatives up to second order in $L^2(\mathbb{P}(H), \nu)$.

**Remark 4.1.** Recall that the trajectory $\rho$ was lifted to $\text{SL}(2, \mathbb{R})$, and was moreover defined in the introduction by taking the outward radial unit tangent vector at all points. We denote the lifted path by $\hat{\rho}$. As consequence the unstable subspace $E^+_k(\hat{\rho}_s)$ is defined at almost all $s \in \mathbb{R}^+$ with probability one as the unstable Oseledets subspace at the radial outward unit tangent vector $\hat{\rho}_s \in X$ at the point $\rho_s \in D = \text{SO}(2, \mathbb{R}) \setminus \text{SL}(2, \mathbb{R}) \omega$.

Let $W^{2,\infty}(\mathbb{D})$ denote the Sobolev space of essentially bounded functions on the unit disc $\mathbb{D}$ with essentially bounded weak derivatives up to order 2. The function $u(z) := u^{(k)}(z)$ belongs to the space $W^{2,\infty}(\mathbb{D})$, and it is not necessarily $C^2$ along $\text{SL}(2, \mathbb{R})$ orbits. However, a version of Ito’s formula for weakly differentiable functions, known as Ito-Krylov’s formula (see for instance [Aeb96, FP00, Kry10]) applies.
For all $s > 0$, let $\rho_s = (t(s), \theta(s))$ in geodesic polar coordinates. By Ito-Krylov’s formula we get,

$$\frac{1}{\sqrt{T}}(u(\rho_T, \mathbf{v}) - u(\rho_0, \mathbf{v})) = \frac{1}{\sqrt{T}} \int_0^T \nabla_{\rho_s} u(\rho_s, \mathbf{v}) \cdot (dW^{(1)}_s, dW^{(2)}_s)$$

$$+ \frac{1}{2\sqrt{T}} \int_0^T \Delta_{\rho_s} u(\rho_s) ds$$

So we have that

$$\frac{1}{2\sqrt{T}} \left( \int_0^T (\Delta_{\rho_s} \sigma_k(\rho_s, \mathbf{v}) - 2\lambda_k) ds \right)$$

$$= \frac{1}{\sqrt{T}}(u(\rho_T) - u(\rho_0))$$

$$- \frac{1}{\sqrt{T}} \int_0^T \nabla_{\rho_s} u(\rho_s) \cdot (dW^{(1)}_s, dW^{(2)}_s).$$

Define

$$M_T = \int_0^T \nabla_{\rho_s} (\sigma_k(\rho_s, \mathbf{v}) - u(\rho_s, \mathbf{v})) \cdot (dW^{(1)}_s, dW^{(2)}_s)$$

(4.1.5)

We then have

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_T, \mathbf{v}) - T \sum_{i=1}^k \lambda_i) = \frac{1}{\sqrt{T}}(u(\rho_T, \mathbf{v}) - u(\rho_0, \mathbf{v}) + \sigma_k(\rho_0, \mathbf{v}))$$

$$+ \frac{1}{\sqrt{T}} M_T.$$

Next, we study the quadratic variation $\langle M_T, M_T \rangle_{\nu_p}$. Recalling that the co-variance of two Ito integrals with respect to independent Brownian motions is zero, we have:

$$\langle M_T, M_T \rangle_{\nu_p} = \mathbb{E}_{\nu_p} \left[ \left( \int_0^T (\nabla_{\rho_s} \sigma_k(\rho_s, \mathbf{v}) - \nabla_{\rho_s} u(\rho_s, \mathbf{v})) \cdot (dW^{(1)}_s, dW^{(2)}_s) \right)^2 \right]$$

$$= \mathbb{E}_{\nu_p} \left[ \left( \int_0^T \frac{\partial \sigma_k}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s, \mathbf{v}) \right) dW^{(1)}_s \right]^2$$

$$+ \mathbb{E}_{\nu_p} \left[ \left( \int_0^T \frac{2}{\sinh(2t(s))} \frac{\partial \sigma_k}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s, \mathbf{v}) \right) dW^{(2)}_s \right]^2$$
Applying Ito’s isometry [FLJ12, Lemma VI.4.3] on the expectation of the square of the Ito integrals on the RHS yields

\[
\langle M_T, M_T \rangle_{\hat{\nu}} = \mathbb{E}_{\hat{\nu}} \left[ \int_0^T \left( \frac{\partial \sigma_k}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s, \mathbf{v}) \right)^2 ds \right] + \mathbb{E}_{\hat{\nu}} \left[ \int_0^T \left( \frac{2}{\sinh(2t(s))} \left( \frac{\partial \sigma_k}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s, \mathbf{v}) \right) \right)^2 ds \right]
\]

\[
= \mathbb{E}_{\hat{\nu}} \left[ \int_0^T |\nabla_L \sigma_k(\rho_s, \mathbf{v}) - \nabla_L u(\rho_s, \mathbf{v})|^2 ds \right]
\]

Observe that \(|\nabla u|^2 \in L^2(\mathbb{P}(H), \nu)| by Lemma A.3. Therefore, by Oseledets’ theorem, Fubini’s theorem, and the dominated convergence theorem, we have the convergence with respect to the measure \(\hat{\nu}\) on \(\mathbb{P}(H(k))\):

\[
V_{\rho^\infty}^{(k)} := \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_{\hat{\nu}} \left[ \int_0^T |\nabla_L \sigma_k(\rho_s, \mathbf{v}) - \nabla_L u(\rho_s, \mathbf{v})|^2 ds \right] = \int_{\mathbb{P}(H(k))} |\Psi_k(\mathbf{v}) - \nabla_L u(\omega, \mathbf{v})|^2 d\hat{\nu}
\]

\[
= \int_X |\Psi_k(E_k^+(\omega)) - \nabla_L u(\omega, E_k^+(\omega))|^2 d\nu.
\]

See also [For02, Corollary 5.5]. The above formula, together with [FLJ12, Lemma VIII.7.4], implies that the random variables \(M_T/\sqrt{T}\), hence the random variables \((\sigma(\rho_T, \mathbf{v}) - \lambda_k T)/\sqrt{T}\), converge in distribution to a centered Gaussian distribution of variance \(V_{\rho^\infty}^{(k)}\).

4.2. Distributional Convergence in Theorem 2.1. Observe that \(t(s) = d_D(0, \rho_s)\), and that it is rotationally invariant. We will need the following useful lemma:

**Lemma 4.2.** [FLJ12, Lemma VII.7.2.1] For all \(\omega \in X\), there exists an \(\mathbb{P}_\omega\)-almost everywhere converging process \(\eta_s\) such that \(t(s) = W_s^{(1)} + s + \eta_s\).

**Proof.** It is a classical fact that \(t(s) \to \infty \mathbb{P}_\omega\)-almost everywhere. This implies that \(\lim_{s \to \infty} \coth(2t(s)) = 1\) almost everywhere. Setting \(\eta_s := t(s) - W_s^{(1)} - s\), so that, together with 3.8.3, we get

\[
\eta_s = \int_0^s \left( \frac{1}{\coth(2t(s))} - 1 \right) ds = \int_0^s \frac{2ds}{e^{4t(s)} - 1},
\]

which converges almost everywhere, as desired. \(\square\)

Next, it will be crucial to stop the radial process before it exits the region bounded by a circle of geodesic radius \(T\), and so for each \(T\), we define the
stopping time $\tau_T$ as follows
\[
\tau_T := \inf \{ s > 0 : T = d_D(0, \rho_s) \}
= \inf \{ s > 0 : T = W_s^{(1)} + s + \eta_s \}
\]
where the second equality follows by Lemma 4.2. Next, we will need the following lemma:

**Lemma 4.3.** For all $\omega \in X$, we have $\lim_{T \to \infty} \tau_T / T = 1$ $\mathbb{P}_\omega$-almost everywhere. Moreover, we have that as $T \to \infty$, $\tau_T \to \infty$ $\mathbb{P}_\omega$-almost everywhere.

**Proof.** Observe that we have $\tau_T = T - W_s^{(1)} - \eta_s$. The lemma then follows immediately from the definition of the stopping time and the law of the iterated logarithm. $\square$

See also [EFLJ01, Lemma 4.2] for related and interesting results on this stopping time.

Recall that $\mathbb{P}_\omega$ is the Wiener measure on the space of all Brownian trajectories $W_\omega$ starting at the origin (corresponding to the random point $\omega$). Let $\mathbb{P}^\theta_\omega$ be the Wiener measure on the space $W^\theta_\omega$ corresponding to all paths starting at the origin and conditioned to exit at the point $e^{i\theta}$ in $\partial D^2$. To relate the conditioned process $\rho^\theta_s$ to the unconditioned process $\rho_s$, we will need the following lemma:

**Lemma 4.4.**
\[
\mathbb{P}_\omega = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{P}^\theta_\omega d\theta \quad (4.2.1)
\]

**Proof.** Recall that $W_\omega$ is the space of all hyperbolic Brownian motion trajectories starting at the origin, with $\mathbb{P}_\omega$ the corresponding Wiener measure. There exists a map $\Theta : W_\omega \to \partial D^2$, defined $\mathbb{P}_\omega$-almost everywhere, such that $\Theta(\rho) = \rho_\infty$, where $\rho_\infty$ is the limit point of $\rho$ on $\partial D^2$. It is a classical fact that the pushforward measure $\Theta_* (\mathbb{P}_\omega)$ equals $\text{Leb}$, where $\text{Leb}$ is the normalized Lebesgue measure on $[0, 2\pi]$. We also recall that the foliated process is in fact defined on $SO(2, \mathbb{R}) \setminus X$ and that $\check{\nu}$ is $SO(2, \mathbb{R})$-invariant, and so our disintegration claim follows. $\square$

**Remark 4.5.** See also [Fra05, Lemma 8] for a short potential theoretic proof (using Doob’s $h$-process) of this fact. The approach to proving the central limit theorem in [Fra05], with the aid of a stopping time, is what we will essentially follow in the sequel, though in our case the proof here is simpler, in view of the Lipschitz property of the Kontsevich-Zorich cocycle and Ancona’s estimate.

**Remark 4.6.** It is worth repeating and adapting what is written in the introduction in view of the application of the conditioned process in the sequel. The conditioned process is in fact defined on $X^* = SO(2, \mathbb{R}) \setminus X$. Moreover, $\rho^\theta$ can be lifted to $SL(2, \mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to
the lifted path as \( \rho^\theta \) by abuse of notation. Additionally, the space \( X \) gives rise to a product space \( X^\rho^\theta := X \otimes W^\rho \) whose fiber over each point \( \omega \) in \( X \) is \( W^\rho_{\omega} \), and which also supports a measure \( \nu_{\rho^\theta} := \nu \otimes \mathbb{P}^\rho \), whose conditional measure over a point \( \omega \) is \( \mathbb{P}^\rho_{\omega} \). We can thus similarly define the product \( W^\rho\)-Hodge bundle \( \mathbb{P}^W_{\rho^\theta}(\mathbb{H}^{(k)}) \), whose fiber over each point \( (\omega, \rho^\theta) \) in \( X^\rho^\theta \) is \( \mathbb{H}^{(k)}_{\omega} \). A pair \((\rho^\theta, \mathbf{v}) \in \mathbb{P}^W_{\rho^\theta}(\mathbb{H}^{(k)})\) is thus defined to be the lift of the path \( \rho^\theta \) (starting at \( \omega \)) to \( \mathbb{P}^W_{\rho^\theta}(\mathbb{H}^{(k)}) \), obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure \( \hat{\nu}_{\rho^\theta} := \hat{\nu} \otimes \mathbb{P}^\rho \) whose conditional measure over a point \( \mathbf{v} \) is \( \mathbb{P}^\rho_{\omega} \).

We recall the following fundamental result due to Ancona [Anc90] (see also [Gru98, Lemma 4.1]):

**Theorem 4.7.** [Anc90, Théorème 7.3] For all \( \omega \in X \), and \( \mathbb{P}^\rho_{\omega} \)-almost all paths \( \rho \) starting at \( \omega \), we have that \( d_{\mathbb{P}}(\rho_0\rho_\infty, \rho_T) = O(\log T) \) as \( T \to \infty \), where \( \rho_0\rho_\infty \) is the geodesic ray with \( \rho_0 \in \mathbb{D} \) and \( \rho_\infty \in \partial \mathbb{D} \).

Now observe that our aim is to study

\[
\Sigma^\rho(T, [a, b]) := \hat{\nu} \left( \left\{ \mathbf{v} \in \mathbb{P}(\mathbb{H}^{(k)}) : a \leq \frac{1}{\sqrt{T}}(\sigma_k(g_T, \mathbf{v}) - T\lambda_{(k)}) \leq b \right\} \right)
\]

as \( T \to \infty \).

Let

\[
\Sigma^\rho(T, [a, b]) := \hat{\nu}_{\mathbb{P}} \left( \left\{ (\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbb{H}^{(k)}) : a \leq \frac{1}{\sqrt{T}}(\sigma_k(\rho_T, \mathbf{v}) - T\lambda_{(k)}) \leq b \right\} \right)
\]

**Lemma 4.8.** The quantity

\[
|\Sigma^\rho(T, [a, b]) - \Sigma^\rho(T, [a, b])| \to 0 \tag{4.2.2}
\]

as \( T \to \infty \), \( \mathbb{P}^\rho_{\omega} \)-almost everywhere and for all \( \omega \in X \).

**Proof.** By applying the disintegration in Lemma 4.4, (4.2.2) is also equal to

\[
\Sigma^\rho(T, [a, b]) = \text{Leb} \otimes \hat{\nu}_{\mathbb{P}} \left( \left\{ (\theta, \rho^\theta, \mathbf{v}) \in [0, 2\pi] \otimes \mathbb{P}^W_{\rho^\theta}(\mathbb{H}) : a \leq \frac{1}{\sqrt{T}}(\sigma_k(\rho_T^\rho, \mathbf{v}) - T\lambda_{(k)}) \leq b \right\} \right)
\]

\[
= \text{Leb} \otimes \hat{\nu}_{\mathbb{P}} \left( \left\{ (\theta, \rho^\theta, \mathbf{v}) \in [0, 2\pi] \otimes \mathbb{P}^W_{\rho^\theta}(\mathbb{H}) : a \leq \frac{1}{\sqrt{T}}(\sigma_k(g_T\tau, \mathbf{v}) - T\lambda_{(k)} + \sigma_k(\rho_T^\rho, \mathbf{v}) - \sigma_k(g_T\tau, \mathbf{v})) \leq b \right\} \right).
\]

Theorem 4.7 applied to \( \tau_T \) gives that, for all \( \omega \in X \), \( d_{\mathbb{P}}(g_T\tau, 0, \rho_T^\rho) = O(\log \tau_T) \) \( \mathbb{P}^\rho_{\omega} \)-almost everywhere as \( T \to \infty \). Together with Lemma 4.3, the lemma now follows by the Lipschitz property of the Kontsevich-Zorich cocycle (by the derivative bound in 3.5.1). \( \square \)
Therefore, it suffices to study the limiting distribution of the quantity

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau T}, v) - T\lambda(k)).$$

Observe that we have that for all $\omega \in X$, and $\mathbb{P}_\omega$-almost everywhere, $\tau_t \to \infty$ as $T \to \infty$. By applying the stopping time identity $T = \tau_T + W_{\tau_T}^{(1)} + \eta_{\tau_T}$, a straightforward calculation shows the following equality:

$$\frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau T}, v) - T\lambda(k)) = -\frac{1}{\sqrt{T}}\eta_{\tau T}\lambda(k)$$

(4.2.3)

$$- \frac{1}{\sqrt{T}}W_{\tau T}^{(1)}\lambda(k)$$

(4.2.4)

$$+ \frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau T}, v) - \tau_T\lambda(k))$$

(4.2.5)

So this reduces the proof of the theorem to controlling three terms on the RHS of the previous equality. First, we can observe that 4.2.3 clearly converges to zero $\mathbb{P}_\omega$-almost everywhere by Lemma 4.2. Next, it follows by Lemma 4.3 that

$$\lim_{T \to \infty} \frac{\lambda(k)}{\sqrt{T}} W_{\tau T}^{(1)} \xrightarrow{d} \lambda^2(k)$$

and in particular the variance of 4.2.4 is $\lambda^2(k)$. The variance of 4.2.5 converges to $\nu_{\rho_{\infty}}^{(k)}$ by Theorem 2.6, together with the simple observation that, since by Lemma 4.3 $\tau_T/T \to 1$ and $M_T$ is a (deterministic) Ito process (integral) we have

$$\lim_{T \to +\infty} \frac{1}{\sqrt{T}}(\sigma_k(\rho_{\tau T}, v) - \tau_T\lambda(k))$$

$$= \lim_{T \to +\infty} \frac{1}{\sqrt{T}}M_{\tau_T} = \lim_{T \to +\infty} M_{\tau_T}/T = M_1,$$

and similarly

$$\lim_{T \to +\infty} \frac{1}{\sqrt{T}}(\sigma_k(\rho_{T}, v) - T\lambda(k))$$

$$= \lim_{T \to +\infty} \frac{1}{\sqrt{T}}M_T = M_1.$$

Remark 4.9. In fact, by the above argument it follows also that $\frac{1}{\sqrt{T}}M_{\tau_T}$ converges in distribution to a centered Gaussian random variable with variance $\nu_{\rho_{\infty}}^{(k)}$.

The following lemma concerns the covariance of the terms 4.2.4 and 4.2.5, and shows that it converges almost everywhere:

Lemma 4.10. $\text{Cov}_{\nu_{\rho}} \left( \frac{1}{\sqrt{T}}M_{\tau_T}, -\frac{\lambda(k)}{\sqrt{T}}W_{\tau_T}^{(1)} \right) \to -\lambda^2(k)$.
Proof. It follows by Eqs 3.6.1 and 4.1.4 that we have

$$\Delta_{L_\omega} \sigma_k(z, v) - \Delta_{L_\omega} u(z, v) = 2\lambda_{(k)}.$$ 

Hence, by [For02, Lemma 3.1], we conclude that

$$\frac{\partial}{\partial t} \frac{1}{2\pi} \int_0^{2\pi} (\sigma_k(z, v) - u(z, v)) d\theta$$

$$= \frac{1}{\sinh(2t)} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} (\Delta_{L_\omega} \sigma_k - \Delta_{L_\omega} u) d\theta \sinh(2r) dr$$

$$= \lambda_{(k)} \frac{\cosh(2t) - 1}{\sinh(2t)} + o(1) = \lambda_{(k)} \tanh(t) + o(1).$$

We are now ready to calculate the covariance. We have

$$\text{Cov}_{\bar{\nu}} \left( \frac{1}{\sqrt{T}} M_{\bar{\nu}}, \frac{-\lambda_{(k)}}{\sqrt{T}} W_{(1)}^1 \right)$$

$$= -E_{\bar{\nu}} \left[ \frac{\lambda_{(k)}}{T} \int_0^T (\nabla_{L_\omega} \sigma(\rho_s, v) - \nabla_{L_\omega} u(\rho_s, v)) \cdot (dW_s^{(1)}, dW_s^{(2)}) \int_0^T dW_s^{(1)} \right]$$

$$= -E_{\bar{\nu}} \left[ \frac{\lambda_{(k)}}{T} \int_0^T (\frac{\partial \sigma}{\partial t}(\rho_s, v) - \frac{\partial u}{\partial t}(\rho_s, v)) dW_s^{(1)} \int_0^T dW_s^{(1)} \right]$$

$$= -E_{\bar{\nu}} \left[ \frac{\lambda_{(k)}}{T} \int_0^T (\frac{\partial \sigma}{\partial t}(\rho_s, v) - \frac{\partial u}{\partial t}(\rho_s, v)) ds \right]$$

$$= -E_{\bar{\nu}} \left[ \frac{\lambda_{(k)}}{T} \int_0^T (\frac{\partial \sigma}{\partial t}(\rho_s, v) - \frac{\partial u}{\partial t}(\rho_s, v)) ds \right] + o(1)$$

$$= -\frac{\lambda_{(k)}^2}{T} E_{\bar{\nu}} \left[ \int_0^T \tanh(t(s)) ds \right] + o(1)$$

$$\rightarrow -\lambda_{(k)}^2,$$

where 4.2.8 follows by the independence of $W_s^{(1)}$ and $W_s^{(2)}$, and where 4.2.9 follows by an application of Ito’s inner product (a more general case of Ito’s isometry, which follows by applying the polarization identity), which also holds for our stopping time – in fact, Ito’s isometry holds for stochastic integrals with infinite time horizon, and so it also follows for our defined stopping time (see also [FLJ12, Lemma VI.4.3]). We also note that 4.2.10 holds thanks to Lemma 4.3 and the identity within its proof. Finally, 4.2.11 holds thanks to 4.2.6, together with the rotational invariance of the hyperbolic heat kernel. □
To conclude the proof of Theorem 2.1, and by writing $W^{(1)}_{\tau_T} = \int_0^{\tau_T} dW_s^{(1)}$, we have

\[
\frac{1}{\sqrt{T}} \left( \sigma_k(\rho_{\tau_T}, \mathbf{v}) - T\lambda(k) \right) = - \frac{1}{\sqrt{T}} W^{(1)}_{\tau_T} \lambda(k) + \frac{1}{\sqrt{T}} M_{\tau_T} + o(1)
\]

\[
= \frac{1}{\sqrt{T}} \int_0^{\tau_T} (-\lambda(k) + \frac{\partial \sigma}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s, \mathbf{v})) dW_s^{(1)}
\]

\[
+ \frac{1}{\sqrt{T}} \int_0^{\tau_T} \frac{2}{\sinh(2t(s))} \left( \frac{\partial \sigma}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s, \mathbf{v}) \right) dW_s^{(2)} + o(1)
\]

Since $\tau_T = T - W^{(1)}_{\tau_T} - \eta_{\tau_T}$, we have

\[
\frac{1}{\sqrt{T}} \left( \sigma_k(\rho_{\tau_T}, \mathbf{v}) - T\lambda(k) \right) + o(1) = \frac{1}{\sqrt{T}} \int_0^{\tau_T} (-\lambda(k) + \frac{\partial \sigma}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s, \mathbf{v})) dW_s^{(1)}
\]

\[
+ \frac{1}{\sqrt{T}} \int_0^{T} \frac{2}{\sinh(2t(s))} \left( \frac{\partial \sigma}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s, \mathbf{v}) \right) dW_s^{(2)}
\]

\[
- \frac{1}{\sqrt{T}} \int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} (-\lambda(k) + \frac{\partial \sigma}{\partial t}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial t}(\rho_s, \mathbf{v})) dW_s^{(1)}
\]

\[
- \frac{1}{\sqrt{T}} \int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} \frac{2}{\sinh(2t(s))} \left( \frac{\partial \sigma}{\partial \theta}(\rho_s, \mathbf{v}) - \frac{\partial u}{\partial \theta}(\rho_s, \mathbf{v}) \right) dW_s^{(2)}
\]

We remark that the stochastic integrals 4.2.14 and 4.2.15 converge to 0 in probability. Indeed, since the Brownian motion $W^{(1)}$ has mean zero and $\eta_t$ is convergent, we have that $(W^{(1)}_{\tau_T} + \eta_{\tau_T})/T \to 0$, hence for any square integrable function $f$ it follows that

\[
\frac{1}{\sqrt{T}} \int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} f(\rho_s, \mathbf{v}) dW_s^{(1)}
\]

\[
= \int_{T-W^{(1)}_{\tau_T}-\eta_{\tau_T}}^{T} f(\rho_s, \mathbf{v}) dW_s^{(1)}
\]

\[
= \int_{1-(W^{(1)}_{\tau_T}+\eta_{\tau_T})/T}^{1} f(\rho_T s, \mathbf{v}) dW_s^{(1)} \to 0.
\]

In addition, it follows by [FLJ12, Lemma VIII.7.4] that the sum of the normalized stochastic integrals 4.2.14 and 4.2.15 converges to a centered Gaussian distribution. This, together with convergence of the asymptotic covariance in Lemma 4.10, completes the proof, and in particular we have
that the asymptotic variance $V_{g\infty}^{(k)}$ is

$$V_{g\infty}^{(k)} = V(k)_{g\infty} + \lambda^2(k) + 2 \lim_{T \to \infty} \text{Cov} \left( \frac{2}{\sqrt{T}} M_{TT}, -\frac{\lambda(k)}{\sqrt{T}} W_{TT}^{(1)} \right)$$  \hspace{1cm} (4.2.18)

$$= V(k)_{\rho\infty} + \lambda^2(k) - 2\lambda^2(k) = V(k)_{\rho\infty} - \lambda^2(k).$$  \hspace{1cm} (4.2.19)

\[\square\]

5. Positivity of the variance

5.1. Random cocycle. Recall that 4.2.19 says that $V_{g\infty}^{(k)} = V(k)_{\rho\infty} - \lambda^2(k)$, and so we also have the following important corollary:

**Corollary 5.1.** If $\lambda_k > \lambda_{k+1}$, then $V_{\rho\infty}^{(k)} > 0$.

**Proof.** Since, by construction, $V_{g\infty}^{(k)} \geq 0$, and we have that $V_{\rho\infty}^{(k)} \geq 0$, and it is clear that, since $\lambda(k) = \sum_{i=1}^{k} \lambda_i$, we have $\lambda^2(k) \geq \lambda^2_1 > 0$. \[\square\]

5.2. Deterministic cocycle. While 4.2.19 ensures convergence of the asymptotic variance for the deterministic cocycle, it is not clear to us how it can be leveraged to deduce its positivity. Instead, we approach the positivity of the variance for the deterministic cocycle directly, in the spirit of the potential theoretic approach in [For02]. We first observe that a direct expression of the converging asymptotic variance for the deterministic cocycle is

$$V_{g\infty}^{(k)} = \lim_{T \to \infty} \frac{1}{T} \int_{P(H)} \left[ \sigma_k(g_T, v) - \lambda(k)T \right]^2 d\nu$$  \hspace{1cm} (5.2.1)

The existence and the regularity of the solution $U$ of the Poisson equation 4.1.4 will be again crucial for our approach towards the positivity of $V_{g\infty}^{(k)}$ (see also the proof of Lemma 4.10).

Let $F_k(g_T\theta, v) := \sigma_k(g_T\theta, v) - \lambda(k)T$. In fact, we will study an auxiliary random variable $F_k - u$, and use it at the end to deduce the positivity of the asymptotic variance $V_{g\infty}^{(k)}$.

Let $\Psi_k$ the vector valued function defined in formula (3.6.2):

$$\Psi_k(\omega, v) = \text{tr} \left( B_{\omega}^{(k)}(v) \right), \quad \text{for all } (\omega, v) \in P(H^{(k)}),$$

We prove below the following condition for the vanishing of the deterministic variance.

**Lemma 5.2.** The variance $V_{g\infty}^{(k)}$ of the deterministic cocycle (see formula (5.2.1)) vanishes if and only if

$$\Psi_k \circ E_k^+ - (\lambda(k), 0) - \nabla_L u \circ E_k^+ = 0 \quad \nu\text{-almost everywhere}.$$
Proof. We first prove that the normalized asymptotic variance of the random variable $F_k$ coincides with that of $F_k - U$.

It follows by an immediate application of [For02, Lemma 3.1] that, for any smooth function $F$ and any function $u \in W^{2,\infty}$ on the Poincaré disk, with respect to hyperbolic geodesic polar coordinates $z = (t, \theta)$, we have the formula

$$
\frac{1}{2\pi} \frac{\partial}{\partial t} \int_0^{2\pi} (F - u)^2(t, \theta) d\theta = \frac{1}{2} \tan(t) \frac{1}{|D_t|} \int_{D_t} \Delta_{L_\omega}((F - u)^2) \omega_P
$$

$$
= \tan(t) \frac{1}{|D_t|} \int_{D_t} (F - u) \Delta_{L_\omega}(F - u) \omega_P
$$

$$
+ \tan(t) \frac{1}{|D_t|} \int_{D_t} |\nabla_{L_\omega}(F - u)|^2 \omega_P
$$

where $|D_t|$ is the hyperbolic area element of the disk $D_t$ of geodesic radius $t > 0$ that is centered at the origin, and $\omega_P$ is the hyperbolic area on the Poincaré disk.

By applying the above formula to the function $F(t, \theta) = F_k(g_t r \theta, v)$, for every $v \in \mathbb{P}(H(k))$ (we recall that $\mathbb{P}(H(k))$ denotes the projective Hodge bundle over an $\text{SL}(2, \mathbb{R})$-invariant sub-orbifold $X$ of the moduli space of Abelian differentials) and $u(t, \theta) = U(g_t r \theta(\omega))$ and by integrating over $\mathbb{P}(H(k))$ with respect to the measure $\hat{\nu}$ (defined as the $\text{SO}(2, \mathbb{R})$-invariant Haar measures on the fibers), we have

$$
\int_{\mathbb{P}(H(k))} \frac{\partial}{\partial t} (F_k - u)^2(t, \theta) d\hat{\nu}
$$

$$
= \int_{\mathbb{P}(H(k))} \frac{\tan(t)}{\sinh^2(t)} \int_0^t (F_k - u) \Delta_{L_\omega}(F_k - u)(\tau, \theta) d(\sinh^2 \tau) d\hat{\nu}
$$

$$
+ \int_{\mathbb{P}(H(k))} \frac{\tan(t)}{\sinh^2(t)} \int_0^t |\nabla_{L_\omega}(F_k - u)|^2(\tau, \theta) d(\sinh^2 \tau) d\hat{\nu}
$$

By integrating over $[0, T]$ with respect to $dt$, we have

$$
\frac{1}{T} \left[ \int_{\mathbb{P}(H(k))} [(F_k - u)^2(g_T, v) - (F_k - u)^2(g_0, v)] d\hat{\nu} \right]
$$

$$
= \frac{1}{T} \int_0^T \int_{\mathbb{P}(H(k))} \frac{\tan(t)}{\sinh^2(t)} \int_0^t (F_k - u) \Delta_{L_\omega}(F_k - u) d(\sinh^2 \tau) d\hat{\nu} dt
$$

$$
+ \frac{1}{T} \int_0^T \int_{\mathbb{P}(H(k))} \frac{\tan(t)}{\sinh^2(t)} \int_0^t |\nabla_{L_\omega}(F_k - u)|^2 d(\sinh^2 \tau) d\hat{\nu} dt
$$
By Eq. 4.1.4, we observe that
\[ \Delta L_\omega(F_k - u)(t, \theta) = \Delta L_\omega(\sigma_k(g_t g^\omega, \nu)) - \Delta L_\omega(\lambda_k t) = 2\lambda_k(1 - \coth(2t)) \to 0 \]
exponentially as \( t \to \infty \). It follows therefore that 5.2.3 converges to 0 as \( t \to \infty \), and we thus have that
\[ \lim_{T \to \infty} \frac{1}{T} \int_{T_{(k)}} (F_k - u)(t, \theta)(g_t, v) d\nu \]
\[ = \int_{T_{(k)}} \left| \Psi_k(E^+_k(\omega)) - (\lambda_k, 0) - \nabla L_\omega u(\omega, E^+_k(\omega)) \right|^2 d\nu. \]

**Remark 5.3.** The above steps follow closely the outline of the proof of [FMZ12, Theorem 1], which we refer to for more details.

We have therefore shown that the normalized asymptotic variance of \( F_k - U \) is strictly positive if the function \( \Psi_k \circ E^+_k - \lambda_k - DU \) is not identically zero. The final claim in the argument is that the asymptotic variance of \( F_k \) is no smaller than that of \( F_k - U \), and this follows by an immediate application of the triangle inequality, as follows
\[ \left[ \frac{1}{T} \int_{T_{(k)}} (F_k - u)^2(g_T, v) d\nu \right]^{1/2} \leq \left[ \frac{1}{T} \int_{T_{(k)}} F_k^2(g_T, v) d\nu \right]^{1/2} \]
\[ + \left[ \frac{1}{T} \int_X u^2(g_T, \theta) d\nu \right]^{1/2} \]
\[ = \left[ \frac{1}{T} \int_{T_{(k)}} F_k^2(g_T, v) d\nu \right]^{1/2} \]
\[ + \frac{1}{\sqrt{T}} \| u \|_{L^2(\nu)} \]

\[ = \int_{T_{(k)}} \left| \Psi_k(E^+_k(\omega)) - (\lambda_k, 0) - \nabla L_\omega u(\omega, E^+_k(\omega)) \right|^2 d\nu. \]

\[ \lim_{T \to \infty} \frac{1}{T} \int_{T_{(k)}} (F_k - u)^2(g_T, v) d\nu \]

\[ \text{together with the square integrability of } U. \]

We have therefore shown that if the asymptotic variance for the deterministic cocycle \( V_{g^\infty} \) is equal to zero, then
\[ \Psi_k \circ E_k^+(\omega) - (\lambda_k, 0) - \nabla L_\omega u \circ E_k^+(\omega) = 0 \quad \nu\text{-almost everywhere.} \]

(5.2.5)

\[ \square \]

Now let \( \{X, Y, \Theta\} \) be the standard generators of the Lie algebra of \( SL(2, \mathbb{R}) \) corresponding to the geodesic flow, the orthogonal geodesic flow and the maximal compact subgroup \( SO(2, \mathbb{R}) \), given by the formulas:
\[ X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}. \]
In order to prove a crucial regularity result for the unstable Oseledets subspace under the assumption of zero variance, it will be useful to derive the following lemma for the lift of our \( \text{SO}(2,\mathbb{R}) \)-invariant function \( u \) from \( \text{SO}(2,\mathbb{R}) \setminus \mathbb{P}(H) \) to \( \mathbb{P}(H) \) (which we continue to call \( u \) by abuse of notation):

**Lemma 5.4.** The hyperbolic gradient (in the radial and tangential directions) and Laplacian is given (in the weak sense) by the following formulas:

\[
\nabla L_\omega u(g_t r_\theta, v) = 2 (X u, Y u)(g_t r_\theta, v),
\]

\[
\Delta L_\omega u(g_t r_\theta, v) = 4(X^2 + Y^2)u(g_t r_\theta, v).
\]

**Proof.** By definition we have

\[
\nabla L_\omega u(g_t r_\theta, v) = 2(X u)(g_t r_\theta, v).
\]

The computation of the angular derivative is based on the formula

\[
\exp(\theta \Theta) \exp(2tX) = \exp(2tX) \exp(-2tX) \exp(\theta \Theta) \exp(2tX)
\]

\[
= \exp(2tX) \text{Ad}_{\exp(-2tX)}(\exp(\theta \Theta))
\]

\[
= \exp(2tX) \exp(e^{ad_{-2tX}}(\theta \Theta))
\]

\[
= \exp(2tX) \exp(\theta(cosh(2t)\Theta + sinh(2t)Y)).
\]

The above formula is computed with respect to standard generators \( \{X, Y, \Theta\} \) of the Lie algebra \( \mathfrak{sl}(2,\mathbb{R}) \) which satisfy the commutation relations

\[
[\Theta, X] = Y, \quad [\Theta, Y] = -X, \quad [X, Y] = -\Theta.
\]

Under the convention in [For02], the curvature of the Poincaré plane is taken to be \(-4\), which corresponds to the choice of generators \( \{2X, 2Y, \Theta\} \).

It follows that

\[
\frac{\partial}{\partial \theta} u(g_t r_\theta, v) = \sinh(2t)Y u(g_t r_\theta, v).
\]

We can now compute the hyperbolic gradient and Laplacian. We have

\[
\nabla L_\omega u(g_t r_\theta, v) = \left( \frac{\partial}{\partial t} u(g_t r_\theta, v), \frac{2}{\sinh(2t)} \frac{\partial}{\partial \theta} u(g_t r_\theta, v) \right)
\]

\[
= 2(X u, Y u)(g_t r_\theta, v).
\]

By the commutation relation \([\Theta, Y] = -X\), we also have

\[
\Delta L_\omega u(g_t r_\theta, v) = \left( \frac{\partial^2}{\partial t^2} + 2\coth(2t) \frac{\partial}{\partial t} + \frac{4}{\sinh^2(2t)} \frac{\partial^2}{\partial \theta^2} \right) u(g_t r_\theta, v)
\]

\[
= (4X^2 + 4\coth(2t)X + \frac{4}{\sinh^2(2t)}(\cosh(2t)\Theta + \sinh(2t)Y)^2)u(g_t r_\theta, v)
\]

\[
= (4(X^2 + Y^2 + \coth^2(2t)\Theta^2 + 2\coth(2t)Y^2))u(g_t r_\theta, v)
\]

\[
= 4(X^2 + Y^2)u(g_t r_\theta, v).
\]

The computation is completed. \(\square\)
Let \( h_t^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \) denote the stable horocyclic (unipotent) subgroup of the group \( \text{SL}(2, \mathbb{R}) \). In our notation, the generator of the flow \( h_t^- \) is the vector field \( H := Y + \Theta \). In fact, we have

\[
[X, Y + \Theta] = -\Theta - Y = -(Y + \Theta)
\]

and we follow the convention that \( \text{SL}(2, \mathbb{R}) \) acts on itself and its quotients on the right by multiplication on the left.

**Lemma 5.5.** Assume the variance \( V_{g_\infty}^{(k)} = 0 \). Then for \( \nu \)-almost all \( \omega \in X \) the function

\[
\Psi_k(E_k^+(h_s^- \omega)), \quad s \in \mathbb{R},
\]

is a Lipschitz function.

**Proof.** Since by assumption the variance \( V_{g_\infty}^{(k)} = 0 \) the identity (5.2.5) holds, hence in particular we have

\[
X u \circ E_k^+(\omega) = \frac{1}{2} \left( \text{Re} \left( \Psi_k \circ E_k^+ \right) - \lambda_{(k)} \right).
\]

Since the unstable space \( E_k^+ \) of the cocycle is invariant under the Teichmüller flow, it follows that the function \( \Psi_k \circ E_k^+ \) is for almost all \( \omega \in X \) smooth along the Teichmüller orbit, and by a similar argument along the orbit of the unstable Teichmüller horocycle flow. It follows then that the function \( u \) is infinitely differentiable, with derivatives uniformly bounded almost everywhere, along the geodesic flow orbit for almost all \( \omega \in X \).

By the construction of the function \( u \) as a solution of the equation

\[
( X^2 + Y^2 ) u \circ E_k^+(\omega) = \frac{1}{4} \Delta_{L \omega} u \circ E_k^+(\omega) = \frac{1}{2} \left( \Phi_k \circ E_k^+ - \lambda_{(k)} \right),
\]

since \( X u \circ E_k^+ \), \( X^2 u \circ E_k^+ \) and \( \Delta_{L \omega} u \circ E_k^+ \) are bounded and, by definition \( H = Y + \Theta \), it follows that, for \( \nu \)-almost all \( \omega \in X \), we have

\[
\frac{d^2}{ds^2} u \circ E_k^+(h_s^- \omega) = H^2 u \circ E_k^+(h_s^- \omega) = Y H u \circ E_k^+(h_s^- \omega) = Y u \circ E_k^+(h_s^- \omega)
\]

is a bounded function, which in turn implies that

\[
\frac{1}{2} \text{Im}(\Psi_k \circ E_k^+)(h_s^- \omega) = H u \circ E_k^+(h_s^- \omega)
\]

has a uniformly bounded derivative, hence it is a Lipschitz function. For the real part of the function, we argue that

\[
H \text{Re}(\Psi_k \circ E_k^+) = H X u \circ E_k^+ = [H, X] u \circ E_k^+ + X H u \circ E_k^+ = H u \circ E_k^+ + X H u \circ E_k^+ = \frac{1}{2} (I + X) \text{Im}(\Psi_k \circ E_k^+)
\]

which is again a function uniformly bounded almost everywhere, hence the function \( \text{Re}(\Psi_k \circ E_k^+) \) is also Lipschitz along almost all horocycle orbits. \( \square \)
Let $H$ be an $\text{SL}(2, \mathbb{R})$-invariant, symplectic subbundle of the Hodge bundle, symplectically orthogonal to the tautological subbundle.

**Corollary 5.6.** If the Lyapunov spectrum of the Kontsevich–Zorich cocycle on $H$ is simple, that is, if $\lambda_1 > \cdots > \lambda_h$, then the deterministic Kontsevich–Zorich cocycle on $H^{(1)} = H$ has strictly positive variance, that is, $V_{g_{\infty}}^{(1)} > 0$.

**Proof.** The proof is by contradiction. Let us assume that $V_{g_{\infty}}^{(1)} = 0$. By Lemma 5.5 for $\nu$-almost all $\omega \in X$ the function

$$\Psi_1(E_{1}^+(h_s^+ \omega)), \quad s \in \mathbb{R},$$

is a Lipschitz function.

The strategy of the argument, based on the so-called freezing argument from [CF20], consists in deriving from the above Lipschitz property the existence of a proper $\text{SL}(2, \mathbb{R})$-invariant subbundle of $P(H)$, thereby contradicting the strong irreducibility assumption.

Recall that the function $\Psi_1$ is given by the formula (see formula (3.6.2))

$$\Psi_1(\omega, v) = \text{tr}(B^{(1)}_{\omega}(v)) = B_{\omega}(v), \quad \text{for } (\omega, v) \in H.$$ 

Since for every $\omega \in X$ the matrix $B_{\omega}$ is a complex symmetric matrix, with entries given by a complex quadratic form, it follows that the function $\Psi_1$ is a quadratic polynomial function (with respect to projective coordinates) on every fiber $\tilde{H}_{\omega}$. In addition, the function $\Psi_1$ is non-constant along circle orbits, since

$$\Psi_1(r^\theta \omega, v) = e^{-2\theta} \Psi_1(\omega, v), \quad \text{for all } v \in H_{\omega}.$$ 

We define a measurable subbundle of the bundle $H$ as follows. Since by assumption the Lyapunov exponent $\lambda_1 > 0$, it follows that

$$|\Psi_1(E_{1}^+(\omega))| > 0 \quad \text{almost everywhere.}$$

In fact, otherwise the second identity in formula (3.6.1) (for $k = 1$), since the bundle $E_{1}^+$ is invariant under the Teichmüller flow, would imply that $\lambda_1 = 0$. It then follows that there exists $c > 0$ and a compact set $\mathcal{K} \subset X$ such that

$$\min_{\omega \in \mathcal{K}} |\Psi_1(E_{1}^+(\omega))| \geq c > 0. \quad (5.2.6)$$

For every $\omega \in X$ Birkhoff regular for the Teichmüller geodesic flow and Oseledets regular for the Kontsevich–Zorich cocycle, and for every forward return time $t > 0$ of the Teichmüller geodesic flow to the compact set $\mathcal{K} \subset X$, we let

$$W(g_t \omega) := \Psi_1^{-1}\left\{ \Psi_1(E_{1}^+(g_t \omega)) \right\}. \quad (5.2.7)$$

We note that $W(g_t \omega)$ is a real analytic submanifold of (real) codimension 2 which contains the point $E_{1}^+(g_t \omega) \subset \mathbb{P}(H_{g_t \omega})$. We then let, for all $\omega \in X$,

$$\mathcal{V}(\omega) = \bigcap_{s \geq 0} \bigcup_{\tau \geq s} \{ g_{-t}(W(g_t \omega)) \mid g_t \omega \in \mathcal{K} \}. $$
We remark that since $\lambda_1 > 0$, by Oseledets theorem the set $\mathcal{V}(\omega)$ is contained in a finite union of $g_t$-invariant subspaces.

By definition, for $\nu$-almost all $\omega \in X$, the set $\mathcal{V}(\omega) \subset \mathbb{P}(H_\omega)$ is closed and non-empty, since it contains $E_1^+(\omega)$. It is straightforward to derive from its definition that $\mathcal{V}$ is a (measurable) $\{g_t\}$-invariant subset.

The crucial point of the argument is to prove that $\mathcal{V}$ is invariant under the stable Teichmüller horocycle flow $\{h^-\}$.

Let $v \in g_{-t}(\mathcal{W}(g_t\omega))$. By definition, $\Psi_1(g_t(v)) \in \Psi_1(E_1^+(g_t(\omega)))$. There exists a constant $C_K$ such that for any fixed $r > 0$ the distance
\[ d(g_t(\omega), g_t(h^-_r\omega)) = d(g_t(\omega), h^-_{r-2t}g_t(\omega)) \leq C_K r e^{-2t}, \]
hence by the Lipschitz property of the function $\Psi_1 \circ E_1^+$ (which holds by Lemma 5.5) we have that there exists a constant $C'_K$ such that
\[ \| (\Psi_1 \circ E_1^+)(g_t(\omega)) - (\Psi_1 \circ E_1^+)(g_t(h^-_r\omega)) \| \leq C'_K r e^{-2t}. \]
We also have (with respect to the Hodge metric)
\[ d \left( (h^-_r \omega), g_t(v) \right) \leq r e^{-2t}, \]
so that we have the estimate
\[
\begin{align*}
\| (\Psi_1(g_t(h^-_r v)) - \Psi_1(E_1^+(g_t h^-_r \omega)) &\| \\
\leq \| \Psi_1(g_t(h^-_r v)) - \Psi_1(g_t(v)) \| + \| \Psi_1(g_t(v)) - \Psi_1(E_1^+(g_t h^-_r \omega)) \|| \\
&= \| \Psi_1(g_t(h^-_r v)) - \Psi_1(g_t(v)) \| + \| \Psi_1(E_1^+(g_t(\omega)) - \Psi_1(E_1^+(g_t(h^-_r \omega))) \| \\
&\leq D\Psi_1\|K\|d \left( (h^-_r \omega), g_t(v) \right) + \| \Psi_1(E_1^+(g_t(\omega)) - \Psi_1(E_1^+(g_t h^-_r \omega)) \|. 
\end{align*}
\]
hence there exists a constant $C''_K > 0$ so that we have the inequality
\[ \| (\Psi_1(g_t(h^-_r v)) - \Psi_1(E_1^+(g_t h^-_r \omega)) \| \leq C''_K r e^{-2t}. \]

It follows that, by the lower bound in formula (5.2.6) and by Lemma 3.6, for sufficiently large $t > 0$, there exists a constant $C^{(3)}_K > 0$ such that
\[ d \left( (h^-_r v), g_{-t} \mathcal{W}(g_t h^-_r \omega) \right) \leq C^{(3)}_K r e^{-2t}. \]
Next we claim that there exists $\lambda < 1$ such that the above estimate implies that there exists a constant $C^{(4)}_K > 0$ such that
\[ d \left( (h^-_r v), g_{-t} \mathcal{W}(g_t h^-_r \omega) \right) \leq C^{(4)}_K r e^{-2(1-\lambda)t}. \]
The above conclusion follows from the fact that there exists $\lambda \in (0, 1)$ such that, on the symplectic orthogonal of the tautological bundle, the Lyapunov spectrum is contained in the interval $(-\lambda, \lambda)$. It then follows by Oseledets theorem that for every Birkhoff generic and regular $\omega \in X$ and for every $v$, $w \in \mathbb{P}(H_\omega)$,
\[ \limsup_{t \to +\infty} \frac{1}{t} \log d(g_t(v), g_t(w)) \leq 2\lambda. \]
We have thus proved that, for every $s > 0$,
\[
    h_r^-(v) \in \bigcup_{t \geq s} \{ g_{-t} (W(g_t \omega)) \mid g_t \omega \in K \},
\]

hence $h_r^-(v) \in \mathcal{V}(h_r^- \omega)$, for all $v \in g_{-t}(W(g_t \omega))$. Since $\mathcal{V}(h_r^- \omega)$ is closed, it follows that, for every $r \in \mathbb{R}$,
\[
    h_r^-(\mathcal{V}(\omega)) \subset \mathcal{V}(h_r^- \omega),
\]

and, since the reverse inclusion can be proved by reversing the time in the horocycle flow, we have proved the invariance of the bundle $\mathcal{V}$ under the unstable Teichmüller horocycle flow.

We claim that by the construction of the bundle $\mathcal{V}$, the unstable bundle $E^+_1 \subset \mathcal{V}$. In fact, by the definition in formula (5.2.7) we have that, for almost all $\omega \in X$ and for all $t \in \mathbb{R}$,
\[
    E^+_1(g_t \omega) \subset W(g_t \omega),
\]

and since $E^+_1$ is a $\{g_t\}$-invariant bundle, it follows that, for all $t \geq 0$,
\[
    E^+_1(\omega) \subset g_{-t}W(g_t \omega)
\]

which implies the claim.

We can then define an $\text{SL}(2, \mathbb{R})$-invariant subbundle as follows. Let $\mathcal{E}$ denote the smallest measurable forward $\{h_r^-\}$-invariant bundle which contains $E^+_1$. In other terms, for almost all $\omega \in X$, let
\[
    \mathcal{E}(\omega) := \sum_{r \geq 0} h_r^- E^+_1(h_r^- \omega).
\]

We note that, by the above definition, $\mathcal{E} \subset \mathcal{V}$ since $E^+_1 \subset \mathcal{V}$, and the latter bundle is $\{h_r^-\}$-invariant (as well as $\{g_t\}$-invariant).

We then prove that the bundle $\mathcal{E}$ is $\text{SL}(2, \mathbb{R})$-invariant. It is clearly forward $\{h_r^-\}$-invariant by definition. Let us prove that it is $\{g_t\}$-invariant.

By the commutation relation and by the $\{g_t\}$-invariance of the bundle $E^+_1$, for almost all $\omega \in X$ and for all $t, r \in \mathbb{R}$, we have
\[
    g_t (h_r^- E^+_1(h_r^- \omega)) = (h_{e^{-r}t} \circ g_t) E^+_1(h_r^- \omega) = h_{e^{-r}t} (E^+_1(g_t \circ h_r^- \omega)) = h_{e^{-r}t} E^+_1(h_r^- \circ g_t \omega),
\]

which immediately implies that, for all $t \in \mathbb{R}$,
\[
    g_t \mathcal{E}(\omega) = \mathcal{E}(g_t \omega),
\]

hence the bundle $\mathcal{E}$ is (forward and backward) $\{g_t\}$-invariant.

Let us then prove that the bundle $\mathcal{E}$ is forward $\{h^+_r\}$-invariant. For the unstable horocycle flow $\{h^+_r\}$ we have the following commutation relations. For every $r, s \in \mathbb{R}$, with $rs \neq -1$, let
\[
    \rho(r, s) = \frac{r}{1 + rs}, \quad \sigma(r, s) = s(1 + rs), \quad \tau(r, s) = \log(1 + rs).
\]

We then have the commutation relations:

\[ h_s^+ h_r^- = h_r^- h_s^+ g_r. \]

Since \( E_1^+ \) is \( \{g_t\}\)-invariant and \( \{h_s^+\}\)-invariant, it follows that we have

\[
h_s^+ (h_r^- E_1^+ (h_r^- \omega)) = h_r^- h_s^+ g_r (E_1^+ (h_r^- \omega))
= h_r^- (E_1^+ (h_s^+ g_r h_r^- \omega)) = h_r^- (E_1^+ (h_s^+ (h_r^- \omega))),
\]

which immediately implies that \( E \) is forward \( \{h_s^+\}\)-invariant. Finally, since \( E \) is \( \{g_t\}\)-invariant and forward \( \{h_s^+\}\)-invariant, it follows that it is \( SL(2, \mathbb{R}) \)-invariant as claimed.

Finally we remark that by the condition that the Lyapunov spectrum is simple, it follows that \( V \neq \mathbb{P}(H) \), hence \( E \subset V \neq \mathbb{P}(H) \). In fact, by definition, since for almost all \( \omega \in X \) and for all \( t \geq 0 \), the real analytic sets \( \mathcal{W}(g_t \omega) \) have positive codimension equal to 2, by Oseledets theorem, for almost all \( \omega \in X \), the subset \( V \) is contained in the union of finitely many proper \( g_t \)-invariant sub-bundles of \( \mathbb{P}(H) \), given by the Oseledets decomposition, namely all the codimension 2 sums of the one-dimensional Oseledets subspaces, in contradiction with the hypothesis that \( H \) is strongly irreducible. □

6. A Central Limit Theorem for Generic Sections

Since our main results randomizes both the Abelian differential \( \omega \in X \) and vector \( v_\omega \in \mathbb{P}(H^{(k)}_\omega) \) with respect to the measure \( \hat{\nu} \), it is natural to ask if our results also hold for (suitably defined) sections of \( \mathbb{P}(H^{(k)}) \). Following [AHF24], we introduce the following

**Definition 6.1.** We say that \( v = (\omega, v_\omega) \) is future-Oseledets-generic for \( \nu \) a.e. \( \omega \in X \) if \( v \) is a (measurable) section \( v : X \to \mathbb{P}(H^{(k)}) \) of \( \mathbb{P}(H^{(k)}) \) such that, for \( \nu \)-a.e. \( \omega \in X \),

\[
\lim_{T \to \infty} \frac{1}{T} \sigma_k(g_T, v_\omega) = \sum_{i=1}^{k} \lambda_i.
\]

In particular, for the deterministic cocycle, it is straightforward to derive

**Corollary 6.2.** Under the hypothesis of Theorem 2.1, there exists a real number \( V_{g_\infty}^{(k)} \geq 0 \) such that for any future-Oseledets-generic section \( v = (\omega, v_\omega) \) of \( \mathbb{P}(H^{(k)}) \), we have

\[
\lim_{T \to \infty} \nu \left( \left\{ \omega \in X : a \leq \frac{1}{\sqrt{T}} \left( \sigma_k(g_T, v_\omega) - T \left( \sum_{i=1}^{k} \lambda_i \right) \right) \leq b \right\} \right)
= \frac{1}{\sqrt{2\pi V_{g_\infty}^{(k)}}} \int_a^b \exp(-x^2/V_{g_\infty}^{(k)}) dx.
\]
Moreover, if the Lyapunov spectrum is simple, then  \( V_{g}^{(1)} > 0 \).

**Proof.** By Oseledets’ theorem, for \( \nu \)-a.e. \( \omega \), for all \( \mathbf{v} \in \mathbb{P}(H^{(k)}) \), there exist constants \( C > 0 \) and \( \lambda > 0 \) such that, for all \( t > 0 \),

\[
\text{dist}_{g\omega}(\mathbf{v}_{g\omega}, E_{k}^{+}(g\omega)) \leq Ce^{-\lambda t}.
\]

Since the logarithm of the Hodge norm is Lipschitz, we observe that

\[
\sigma_{k}(gT, \mathbf{v}_{\omega}) = \sigma_{k}(gT, \mathbf{v}_{\omega}) - \sigma_{k}(gT, E_{k}^{+}(\omega)) + \log(1 + O(\nu_{\omega} e^{(\lambda' - \lambda)T})) + O(\nu_{\omega}(1))
\]

for \( 0 < \lambda' < \lambda \), and this gives us the conclusion of this corollary. \( \Box \)

**Remark 6.3.** See also [AHF24, Theorems 3.10, 3.11, 3.21 and 3.22] for an abstract central limit theorem for generic sections that implies our corollary. See also [AHF24, Theorem 4.25] for a stronger result that proves a mixing CLT for Oseledets-generic sections of \( \mathbb{P}(H^{(k)}) \) (based on our Theorem 2.1 as a crucial input).

**Remark 6.4.** By applying the results in [CE15, CF20, ASAE+21], Arana-Herrera and the second-named author derive in [AHF24, Theorem 4.32] that (measurable) \( \text{SO}(2, \mathbb{R}) \)-invariant sections of \( \mathbb{P}(H^{(k)}) \) are future-Oseledets-generic, and we refer to Section 4 of their paper for a thorough discussion on Oseledets genericity of sections and other related results.

**Appendix A. Solving Poisson’s equation**

For every \( k \in \{1, \ldots, h\} \), the group \( \text{SL}(2, \mathbb{R}) \) acts on the projective bundle \( \mathbb{P}(H^{(k)}) \) by parallel transport on the fibers. It is then possible to define a stochastic process on \( \mathbb{P}(H^{(k)}) \) as a lift by parallel transport of the foliated Brownian motion on \( X \).

It is possible to find the solution to the Poisson equation for the generator \( \Delta \) based on the semi-group of the stochastic process. Indeed, if \( \{P_{t}^{(k)}\} \) denotes the semi-group of the stochastic process on the projectivized bundle \( \mathbb{P}(H^{(k)}) \) over \( X \), given by the formula, for every \( f \in L^{\infty}(\mathbb{P}(H^{(k)})) \),

\[
P_{t}^{(k)}(f)(x, \mathbf{v}) := \int_{D} p_{t}(x, y) f(y, \mathbf{v}) dy
\]
where \( p_t(x, y) \) denotes the hyperbolic (heat) kernel on the Poincaré disk \( \mathbb{D} \), then formally the Green operator of \( \mathcal{L} \) is given by the formula

\[
(Gf)(x, v) := \int_0^{+\infty} P_t^{(k)}(f)dt = \int_0^{+\infty} \int_{\mathbb{D}} p_t(x, y) f(y, v) dy dt.
\]

In fact, by a formal calculation

\[
\Delta(Gf)(x, v) = \int_{\mathbb{D}} \int_0^{+\infty} \Delta_{L_t} p_t(x, y) f(y, v) dt dy
\]

\[
= -\int_{\mathbb{D}} \int_0^{+\infty} \frac{\partial}{\partial t} p_t(x, y) f(y, v) dt dy
\]

\[
= \lim_{t \to 0} \int_{\mathbb{D}} p_t(x, y) f(y, v) dy - \lim_{t \to +\infty} \int_{\mathbb{D}} p_t(x, y) f(y, v) dy
\]

\[
= f(x, v) - \lim_{t \to +\infty} \int_{\mathbb{D}} p_t(x, y) f(y, v) dy.
\]

The formula gives a Green operator under the conditions that

\[
\int_0^{+\infty} \left\| \int_{\mathbb{D}} p_t(x, y) f(y, v) dy \right\|_{L^2(\mathbb{P}(\mathbf{H}^{(k)}), d\nu)} dt < +\infty;
\]

\[
\lim_{t \to +\infty} \left\| \int_{\mathbb{D}} p_t(x, y) f(y, v) dy \right\|_{L^2(\mathbb{P}(\mathbf{H}^{(k)}), d\nu)} = 0.
\]

**Lemma A.1.** Let \( P_t^{(k)} \) denote the semigroup of the lift of the Brownian motion to \( \mathbb{P}(\mathbf{H}^{(k)}) \). Assume that the Kontsevich-Zorich Lyapunov exponents satisfy the strict inequality \( \lambda_k > \lambda_{k+1} \). Then there exist constants \( C > 0 \) and \( \lambda > 0 \) such that, for every Lipschitz function (with respect to the Hodge metric on the bundle) \( f \in L^\infty(\mathbb{P}(\mathbf{H}^{(k)})) \) and for all \( t > 0 \),

\[
\left\| P_t^{(k)}(f) - \int_{\mathbb{P}(\mathbf{H}^{(k)})} f d\nu \right\|_{L^2(\mathbb{P}(\mathbf{H}^{(k)}))} \leq C \| f \|_{\text{Lip}} e^{-\lambda t}.
\]

**Proof.** Let \( f \) be a Lipschitz \( L^\infty \) function on \( \mathbb{P}(\mathbf{H}^{(k)}) \) and let

\[
P_t^{(k)}(f)(\omega, v) = \int_{\mathbb{D}} p_t(x, y) f(y, v) dy
\]

This function is \( \text{SO}(2, \mathbb{R}) \)-invariant since \( p_t(x, y) \) is radial, \( dy \) is rotationally invariant and \((y, v)\) is obtained by parallel transport of \( v \) at \( y \in D \).

For \( \nu \)-almost all \( x \in X \) let \( \rho_x \) denote the hyperbolic Brownian motion starting at \( x \) on the leaf \( \text{SO}(2, \mathbb{R})/\text{SL}(2, \mathbb{R})x \). Since \( \lambda_k > \lambda_{k+1} \), the unstable Oseledets space \( E_k^+ \) is well-defined \( \nu \)-almost everywhere on \( X \). For all \( t > 0 \), let then \( E_k^+(\rho_x(t)) \) denote the unstable space evaluated at the radial outward unit vector in the circle orbit above \( \rho_x(t) \). By the Oseledets theorem, for all \( v \in \mathbb{P}(\mathbf{H}^{(k)}) \), there exist constants \( C > 0 \) and \( \lambda > 0 \) such that, for all \( t > 0 \),

\[
\text{dist}_{\rho_x(t)}(v, E_k^+(\rho_x(t))) \leq Ce^{-\lambda t}.
\]
Since the function $f$ is Lipschitz, it follows that the function $P_t^{(k)} f(x, v)$ approaches almost everywhere exponentially fast for diverging $t > 0$ the function, defined for $\nu$-almost all $x \in X$, 

$$F_t(x) = \int_{\mathbb{D}} p_t(x, y) f(y, E^+(y)) dy$$

which is $SO(2, \mathbb{R})$ invariant and $L^\infty$. By the spectral gap property of the $SL(2, \mathbb{R})$ action, which follows from [AGY06] and [AG13], there exist constants $C' > 0$ and $\lambda' > 0$ such that, for any $SO(2, \mathbb{R})$-invariant $L^2$ function $F$ on $X$ (in particular to $F_t$), for all $s > 0$,

$$\|P_s^{(k)}(F) - \int_X F d\nu\|_{L^2(X, d\nu)} \leq C' e^{-\lambda' s} \|F\|_{L^2(X, d\nu)}.$$

Therefore we have that

$$P_{2t}^{(k)}(f) - \int_{\mathbb{P}(H^{(k)})} f d\hat{\nu} = P_{2t}^{(k)}(f) - P_t^{(k)}(F_t) + P_t^{(k)}(F_t) - \int_{\mathbb{P}(H^{(k)})} f d\hat{\nu}$$

$$= P_{2t}^{(k)}(f) - P_t^{(k)}(F_t) + P_t^{(k)}(F_t) - \int_X F_t d\nu$$

converges to zero exponentially in $L^2(\mathbb{P}(H^{(k)}), d\hat{\nu})$ since, for all $t > 0$,

$$\|P_{2t}^{(k)}(f) - P_t^{(k)}(F_t)\|_{L^2(\mathbb{P}(H^{(k)}), d\hat{\nu})} = \|P_t^{(k)}(P_t^{(k)}(f) - F_t)\|_{L^2(\mathbb{P}(H^{(k)}), d\hat{\nu})}$$

$$\leq \|P_t^{(k)}(f) - F_t\|_{L^\infty(\mathbb{P}(H^{(k)}), d\hat{\nu})} \leq C' \|f\|_{L^\infty} e^{-\lambda t},$$

and

$$\|P_t^{(k)}(F_t) - \int_X F_t d\nu\|_{L^2(\mathbb{P}(H^{(k)}), d\hat{\nu})} = \|P_t^{(k)}(F_t) - \int_X F_t d\nu\|_{L^2(X, d\nu)}$$

$$\leq C' \|F_t\|_{L^2(X, d\nu)} e^{-\lambda' t} \leq C' \|f\|_{L^\infty(\mathbb{P}(H^{(k)}), d\hat{\nu})} e^{-\lambda' t}.$$ 

We remark that by the Fubini theorem and by the $SL(2, \mathbb{R})$-invariance of the measure $\nu$, for all $t > 0$ we have

$$\int_X F_t d\nu = \int_X \int_{\mathbb{D}} p_t(x, y) f(y, E^+(y)) dy d\nu$$

$$= \int_X \int_0^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} p_t(r) f(g, r_\theta(x), E^+(r_\theta(x))) d\theta dr d\nu$$

$$= \frac{1}{2\pi} \int_0^{+\infty} p_t(r) \int_X f((x, E^+(r_\theta(x)))) d\nu dr = \int_{\mathbb{P}(H^{(k)})} f d\hat{\nu}$$

The argument is complete as the stated exponential $L^2$ convergence follows. \qed

**Question A.2.** Is the projective Kontsevich-Zorich cocycle on $\mathbb{P}(H^{(k)})$ above the Teichmüller geodesic flow exponentially mixing (under the hypothesis of Lemma A.1 on the Kontsevich–Zorich spectrum)? Our result establishes exponential mixing for the projective Kontsevich-Zorich cocycle
above Brownian motion, but our argument breaks down for the deterministic cocycle (above the Teichmüller geodesic flow).

Let $\Delta$ the generator of the foliated Brownian motion on $\text{SO}(2, \mathbb{R}) \setminus X$. The operator $\Delta$ extends naturally to an operator on $\text{SO}(2, \mathbb{R}) \setminus \mathbb{P}(\mathbb{H}^{(k)})$ which is the generator of the lift of the Brownian motion by parallel transport. Let $W^{2,2} \subset L^2(\mathbb{P}(\mathbb{H}^{(k)}), d\hat{\nu})$ denote the domain of $\Delta$ which consists of all square-integrable functions which are differentiable up to second order along the leaves of the foliation of $\text{SO}(2, \mathbb{R}) \setminus \mathbb{P}(\mathbb{H}^{(k)})$ by hyperbolic disks.

We can now derive the following corollary (by an argument, given below, analogous to the proof of [Led95, Corollary 1]):

**Lemma A.3.** For any $\text{SO}(2, \mathbb{R})$-invariant $L^\infty$ zero-average Lipschitz function $f : (\mathbb{P}(\mathbb{H}^{(k)}), d\hat{\nu}) \to \mathbb{C}$, the Poisson equation

$$\Delta u = f$$

has a unique solution $u \in W^{2,2}$.

**Proof.** Let $u \in L^2(\mathbb{P}(\mathbb{H}^{(k)}), d\hat{\nu})$ be defined as

$$u(x, v) = \int_0^{+\infty} P^k_t(f)(x, v)dt := \int_0^{+\infty} \int_{\mathbb{D}} p_t(x, y)f(y, v)dydt.$$ 

The improper integral converges in $L^2$ by Lemma A.1 and by the assumption that $f$ has zero average. It is a solution of the Poisson equation by the calculation in formula (A.0.1) and since, again by Lemma A.1,

$$P^k_t(f) \to 0 \quad \text{in} \quad L^2(\mathbb{P}(\mathbb{H}^{(k)}), d\hat{\nu}).$$

The solution of the Poisson equation is unique up to additive constants since, by ergodicity of the Teichmüller geodesic flow and by the Oseledets theorem, the foliation of $\text{SO}(2, \mathbb{R}) \setminus \mathbb{P}(\mathbb{H}^{(k)})$, with leaves given by the projections of $\text{SL}(2, \mathbb{R})$ orbits, is ergodic (with respect to the measure $\hat{\nu}$). In fact, by [Gar83], Theorem 1 (b), any bounded Borel function which is harmonic on each leaf must be constant on almost all leaves, relative to any finite harmonic measure. (Note that the theorem is stated for compact manifolds, but proved, as remarked in [Gar83], for any manifold provided that the foliation satisfies a condition of bounded geometry for the leaves, stated in [Gar83], §2, which holds in our case).

Finally, since the operator $\mathcal{L}$ is elliptic along the leaves of the foliation, which are of the form $\text{SO}(2, \mathbb{R}) \setminus \text{SL}(2, \mathbb{R})(x, v)$ with $(x, v) \in \mathbb{P}(\mathbb{H}^{(k)})$, and since the function $f$ is bounded, it follows that the solution $u$ belongs to the space $W^{2,2}$ of function locally $L^2$ with all derivatives up to second order along the leaves of the foliation. \qed

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