COMPRESSIBLE-INCOMPRESSIBLE TWO-PHASE FLOWS  
WITH PHASE TRANSITION: MODEL PROBLEM  

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Abstract. We study the compressible and incompressible two-phase flows separated by a sharp interface with a phase transition and a surface tension. In particular, we consider the problem in $\mathbb{R}^N$, and the Navier-Stokes-Korteweg equations is used in the upper domain and the Navier-Stokes equations is used in the lower domain. We prove the existence of $\mathcal{R}$-bounded solution operator families for a resolvent problem arising from its model problem. According to Shibata [13], the regularity of $\rho_+$ is $W^{1,q}_t$ in space, but to solve the kinetic equation: $\mathbf{u}\cdot\mathbf{n} = [[\rho\mathbf{u}]]/[[\rho]]$ on $\Gamma_t$ we need $W^{2-1/q}_t$ regularity of $\rho_+$ on $\Gamma_t$, which means the regularity loss. Since the regularity of $\rho_+$ dominated by the Navier-Stokes-Korteweg equations is $W^{1,q}_t$ in space, we eliminate the problem by using the Navier-Stokes-Korteweg equations instead of the compressible Navier-Stokes equations.

1. Introduction

This paper deals with compressible-incompressible two-phase flows separated by a sharp interface. In particular, we consider the phase transition at the interface. Our problem is formulated as follows: Let $\Omega_{t^+}$ and $\Omega_{t^-}$ be two time dependent domains, and $\Gamma_t$ be the common boundary of $\Omega_{t^+}$ and $\Omega_{t^-}$. We assume that $\Omega_{t^+} \cap \Omega_{t^-} = \emptyset$ and $\Omega_{t^+} \cup \Gamma_t \cup \Omega_{t^-} = \mathbb{R}^N$, where $\mathbb{R}^N$ denotes the $N$-dimensional Euclidean space. Furthermore, we assume that $\Omega_{t^+}$ and $\Omega_{t^-}$ are occupied by a compressible viscous fluid and an incompressible viscous fluid, respectively. For example, $\Omega_{t^-}$ is corresponding to an ocean of infinite extent without bottom, $\Omega_{t^+}$ the atmosphere, and $\Gamma_t$ the surface of the ocean. Let $\mathbf{n}_t$ be the unit outer normal to $\Gamma_t$ pointed from $\Omega_{t^+}$ to $\Omega_{t^-}$. For any $x_0 \in \Gamma_t$ and function $f$ defined on $\Omega_{t^+} \cup \Omega_{t^-}$, we set 

$$[[f]](x_0,t) = \lim_{x \to x_0} f(x,t) - \lim_{x \to x_0^-} f(x,t),$$

which is the jump quantity of $f$ across $\Gamma_t$. Let $\hat{\Omega}_t = \Omega_{t^+} \cup \Omega_{t^-}$, and for any function $f$ defined on $\hat{\Omega}_t$, we write $f_{\pm} = f|_{\Omega_{t^+}}$. In the following, we use the following symbols:

- $\rho : \hat{\Omega}_t \to [0,\infty)$ the density,
- $\mathbf{u} : \hat{\Omega}_t \to \mathbb{R}^N$ the velocity field,
- $\mathbf{u}_t : \Gamma_t \to \mathbb{R}^N$ the interfacial velocity field,
- $\pi : \hat{\Omega}_t \to \mathbb{R}$ the pressure field,
- $\mathbf{T} : \hat{\Omega}_t \to \{ A \in GL_N(\mathbb{R}) \ | \ A = A \}$ the stress tensor field,
- $\theta : \hat{\Omega}_t \to (0,\infty)$ the thermal field,
- $e : \hat{\Omega}_t \to \mathbb{R}_+$ the internal energy density,
- $\eta : \hat{\Omega}_t \to \mathbb{R}$ the entropy density,
- $\psi : \hat{\Omega}_t \to \mathbb{R}$ the Helmholtz free energy function,
- $\mathbf{q} : \hat{\Omega}_t \to \mathbb{R}^N$ the energy flux,
- $f : \hat{\Omega}_t \to \mathbb{R}^N$ the external body force per unit mass.

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Here and in the following, $\nabla^T M$ denotes the transposed $M$. And then, our problem is:

$$
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div} T &= \rho f, \\
\partial_t (\frac{1}{2} |u|^2 + \rho e) + \text{div} \left( (\frac{1}{2} |u|^2 + \rho e) u \right) - \text{div} (Tu - q) &= \rho f \cdot u + \rho r,
\end{align*}
$$

for $x \in \hat{\Omega}_t$, $t > 0$,

subject to the interface conditions:

$$
\begin{align*}
|u| &= j[(1/\rho)] n_t, \\
|u| - |(u)| n_t &= -\sigma H_{\Gamma}, n_t, \\
|\theta| &= 0, \\
\theta|\eta| - [(d \nabla \theta) \cdot n_t] &= 0, \\
\frac{j^2}{2}\left[\frac{1}{\rho^2}\right] + [\psi] - \left[\frac{1}{\rho} (T n_t \cdot n_t)\right] &= 0, \\
V_t &= u_T \cdot n_t = \left[\frac{[\rho u] - [n_t]}{|\rho|}\right], \\
(\nabla \rho_+) \cdot n_t &= 0.
\end{align*}
$$

where $\Omega_{t \pm}$ and $\Gamma_{t}$ are given by

$$
\Omega_{t \pm} = \left\{ (x', x_N) \in \mathbb{R} \times \mathbb{R}^{N-1} \mid \pm (x_N - h(x', t)) > 0, \ t \geq 0 \right\}, \\
\Gamma_{t} = \left\{ x \in \mathbb{R}^N \mid x_N = h(x', t) \quad \text{for} \ x' \in \mathbb{R}^{N-1} \right\},
$$

respectively, with unknown function $h(x', t)$. Above, $H_{\Gamma}$ is the $N - 1$ times mean curvature of $\Gamma_{t}$, $\sigma$ a positive constant describing the coefficient of the surface tension, and $V_t$ the velocity of evolution of $\Gamma_{t}$ with respect to $n_t$. Furthermore, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, for any matrix field $K$ with $(i, j)^{th}$ component $K_{ij}$, the quantity $\text{div} K$ is the $N$-vector with $i^{th}$ component $\sum_{j=1}^{N} \partial_j K_{ij}$, and for any $N$ vector fields, $a = \nabla^T(a_1, \ldots, a_N)$, $b = \nabla^T(b_1, \ldots, b_N)$, we set $a \cdot b = \sum_{j=1}^{N} a_j b_j$, $\text{div} a = \sum_{j=1}^{N} \partial_j a_j$, and $a \otimes b$ denotes the $N \times N$ matrix with $(i, j)_{th}$ component $a_i b_j$. The interface condition (1.2) is explained in more detail in Sect. 2 below. To describe the motion of incompressible viscous flow occupying $\Omega_{t \pm}$ we use the usual Navier-Stokes equations, and so we set

$$
\rho_- = \rho_+, \quad T_- = \mu_- D(u_-) - \pi_- I, \quad q_- = -d_\pm \nabla \theta_-,
$$

where $\rho_+$ is a positive constant describing the mass density of the reference body $\Omega_+$, $\mu_+$ is the viscosity coefficient, $D(u) = (1/2)(\nabla u + \nabla^T u)$, is the deformation tensor with $i^{th}$ component $D_{ij}(u) = \partial_i u_j + \partial_j u_i$ for $\partial_t = \partial/\partial x_i$ and $u = \nabla^T(u_1, \ldots, u_N)$, and $D$ is the coefficient of the heat flux. In particular, the equation of mass conservation: $\partial_t \rho_- + \text{div} (\rho_- u_-) = 0$ leads to $\text{div} u_- = 0$ in $\Omega_{t \pm}$.

On the other hand, to describe the motion of a compressible viscous fluid occupying $\Omega_+$, we adopt the Navier-Stokes-Korteweg tensor of the following form: $T_+ = S_+ + K_+ - \pi_+ I$ with

$$
S_+ = \mu_+ D(u_+) + (\nu_+ - \mu) \text{div} u_+ I - \pi_+, \\
K_+ = \left( k_+ + \rho_+ j_+ \right) |\nabla \rho_+|^2 + k_+ \rho_+ \Delta \rho_+ \right) I - \kappa_+ \nabla \rho_+ \otimes \nabla \rho_+.
$$

Here, $K_+$ is called the Korteweg tensor (cf. Dunn and Serrin [9] and Kotsche [17]). According to Dunn and Serrin [9], in view of the second law of thermodynamics the energy flux includes not only a classical contribution corresponding to the Fourier law but also a nonclassical contribution, which we now call the interstitial working. In this sense, the energy flux $q_+$ is given by

$$
q_+ = -d_+ \nabla \theta_+ + (\kappa_+ \rho_+ \text{div} u_+) \nabla \rho_+
$$

when we use the Navier-Stokes-Korteweg equations, where $d_+$ is the coefficient of the heat flux.

We assume that $\rho_+ = \rho_+(\rho_+, \theta_+)$, $\nu_+ = \nu_+(\rho_+, \theta_+)$, $k_+ = k_+(\rho_+, \theta_+)$, $\kappa_+ = \kappa_+(\rho_+, \theta_+)$, $\epsilon_+ = \epsilon_+(\rho_+, \theta_+)$, $\eta_+ = \eta_+(\rho_+, \theta_+)$ and $\eta_- = \eta_-(\rho_-, \theta_-)$ are positive $C^\infty$ functions with respect to $(\rho_+, \theta_+) \in (0, \infty) \times (0, \infty)$, and $\psi_+ = \psi_+(\rho_+, \theta_+)$ and $\psi_- = \psi_-(\rho_-, \theta_-)$ are positive $C^\infty$ functions with respect to $\theta_- \in (0, \infty)$. Moreover, we assume that
\[ \partial e_\pm / \partial \theta_\pm > 0, \quad e'_- > 0, \quad \text{and} \quad \pi_+ \text{ is given by} \quad \pi_+ = P_+(\rho_+, \theta_+), \quad \text{where} \quad P_+ \text{ is some} \ C^\infty \text{ function with} \\
\text{respect to} \quad (\rho_+, \theta_+) \in (0, \infty) \times (0, \infty). \]

We now explain why the Navier-Stokes-Korteweg equations is used in \( \Omega_{t^+} \) to describe the motion of the compressible viscous fluid. Shibata [28] used the Navier-Stokes-Fourier equations for \( \Omega_{t^+} \), that is \( \mathbf{T}_+ = \mathbf{S}_+ \) and \( \mathbf{q}_+ = -a_+ \nabla \theta_+ \), to formulate the compressible-incompressible two-phase flows separated by a sharp interface with the phase transition. He proved the existence of \( \mathcal{R} \) bounded solution operators for the model problem that derives the maximal \( L_p-L_q \) regularity of solutions to the linearized equations automatically with the help of Weis’s operator valued Fourier multiplier theorem [35]. According to Shibata [13], the regularity of \( \rho_+ \) is \( W^1_q \) in space, but to solve the kinetic equation:
\[ u_t \cdot n_t = \|\pi_t\| \cdot n_t / \|\rho\| \quad \text{on} \quad \Gamma_t \]
we need \( W^{2-1/q}_q \) regularity of \( \rho_+ \) on \( \Gamma_t \), which means the regularity loss. On the other hand, the regularity of \( \rho_+ \) dominated by the Navier-Stokes-Korteweg equations is \( W^3_q \) in \( \Omega_{t^+} \) (cf. Kotschote [16, 17] and Saito [25]), which is enough to solve the kinetic equation. In addition, quite recently Gorban and Karlin [12] proved that the Navier-Stokes-Korteweg equations is implied by the Boltzmann equation that describes the statistical behavior of a gas. In this sense, to use the Navier-Stokes-Korteweg equations to describe the motion of compressible viscous fluid flow is meaningful. Furthermore, we would like to add some comments about the Navier-Stokes-Korteweg equations. More than one hundred years ago, Korteweg [15] derived the Navier-Stokes-Korteweg equations to describe the two phase problem with diffused interface like liquid and vapor flows with phase transition, which was based on the gradient theory for the interface developed by van der Waals [33]. In 1985, Dunn and Serrin [9] studied the Navier-Stokes-Korteweg equations with the second law of thermodynamics. As equations describing the two-phase flows with diffused interface, we also know the Navier-Stokes-Allen-Chan equations and the Navier-Stokes-Chan-Hilliard equations (cf. [2]), but they can be reduced to the Navier-Stokes-Korteweg equations, which is quite recently proved by Freisthüler and Kotschote [11]. Thus, our formulation (1.1) and (1.2) includes the following situation: The ocean and atmosphere are separated by a sharp interface and on this interface the phase transition occurs. In addition, the atmosphere part is two-phase flows with diffused interface like the mixture of gas and ice. Thus, we totally treat three phase problem, and liquid and gas-solid are separated by a sharp interface with phase transition and gas-solid part has diffused interface with phase transition.

Finally, let us mention related results about the initial-boundary value problem for the Navier-Stokes-Korteweg equations. In 2008 and 2010, Kotschote [16, 17] proved the existence and uniqueness of local strong solutions for an isothermal and non-isothermal model of capillary compressible fluids derived by Dunn and Serrin [9]. Recently, Tsuda [32] studied the existence and stability of time periodic solution to the Navier-Stokes-Korteweg equations in \( \mathbb{R}^3 \), and Saito [25] proved the existence of \( \mathcal{R} \) bounded solution operators for the model problem of the Navier-Stokes-Korteweg equations with free boundary conditions.

The two phase problem has been studied by Abels [1], Denisova [3, 4], Denisova and Solonnikov [6, 7], Giga and Takahashi [14], Maryani and Saito [19], Nouri and Poupaud [20], Prüss et al. [21, 22, 23], Shibata and Shimizu [30], etc. Although these works dealt with the two phase problem for the incompressible-incompressible case, as far as the author knows, the compressible-incompressible case is few. The compressible-incompressible case was studied by Denisova [5], Denisova and Solonnikov [8], Kubo, Shibata, and Soga [18], and Shibata [28]. In particular, Denisova [5] and Denisova and Solonnikov [8] studied the compressible-incompressible case in the \( L_2 \) Sobolev-Sobodetski space. Denisova [5] proved the energy inequality without surface tension. Denisova and Solonnikov [8] proved the global-in-time solvability without surface tension under the assumption that the data are small. On the other hand, Kubo, Shibata, and Soga [18] and Shibata [28] studied the compressible-incompressible case in the \( L_p \) in time and \( L_q \) in space frame work. Kubo, Shibata, and Soga [18] proved the existence of \( \mathcal{R} \)-bounded solution operators to the corresponding generalized resolvent problem without surface tension and without phase transition and Shibata [28] prove it with surface tension and phase transition. However, the work in [18, 28] included the problem about the regularity of density, which we mentioned above. In this paper, we eliminate this problem by using the Navier-Stokes-Korteweg equations instead of the compressible Navier-Stokes equations.
Our goal is to prove the local well-posedness and for this purpose, the key step is to prove the maximal $L_p$-$L_q$ regularity of the model problem. Let
\[
\mathbb{R}^N_\pm = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N | \pm x_N > 0 \}, \quad \mathbb{R}^0_0 = \{ x \in \mathbb{R}^N | x_N = 0 \}, \quad \mathbb{R}^N = \mathbb{R}^N_+ \cup \mathbb{R}^N_-
\]
and for any $x_0 \in \mathbb{R}^0_0$, we define $f|_{\pm}(x_0)$ by
\[
f|_{\pm}(x_0) = \lim_{x \to x_0 \in \mathbb{R}^N_{\pm}} f(x),
\]
Let $\rho_{\pm}$ and $\theta_{\pm}$ be the mass density and the absolute temperature of the reference domain: $\Omega_{\pm}|_{t = 0}$, all of which are positive constants. Transforming $\tilde{\Omega}_t$ and $\Gamma_t$ to $\mathbb{R}^N$ and $\mathbb{R}^N_0$, respectively, and linearizing the problem at $\rho_{\pm}$ and $\theta_{\pm}$, we have following two model problems. One is the following system:

\begin{align}
(1.6) \quad \partial_t \rho_+ + \rho_+ \partial_t u_+ - \mu_+ \Delta u_+ - \nu_+ \nabla \nabla \nabla u_+ - \rho_+ \kappa_+ \nabla \Delta \rho_+ = f_2 & \quad \text{in } \mathbb{R}^N_+ \times (0, T), \\
\rho_+ \partial_t u_+ - \mu_+ \Delta u_+ - \nu_+ \nabla \nabla \nabla u_+ = f_3 & \quad \text{in } \mathbb{R}^N_+ \times (0, T), \\
\rho_+ \partial_t u_- - \mu_+ \Delta u_- - \nabla \nabla \nabla u_- = f_5 & \quad \text{in } \mathbb{R}^N_- \times (0, T),
\end{align}

subject to the interface condition: for $x_0 \in \mathbb{R}^N_0$ and $t \in (0, T)$

\begin{align}
(1.7) \quad \partial_t h - \left( \frac{\rho_+}{\rho_+ - \rho_+} u_{N-1}(x_0) - \frac{\rho_+}{\rho_+ - \rho_+} u_{N+1}(x_0) \right) = d, \\
\mu_+ D_{NN}(u_+)(x_0) - \mu_+ D_{NN}(u_+)(x_0) = g_m, \\
\mu_+ D_{NN}(u_-)(x_0) - \mu_+ D_{NN}(u_-)(x_0) = g_m, \\
\frac{1}{\rho_+} \{ \mu_+ D_{NN}(u_-) - \mu_+ \} \nabla \nabla \nabla u_+ = f_5, \\
\frac{1}{\rho_+} \{ \mu_+ D_{NN}(u_-) - \mu_+ \} \nabla \nabla \nabla u_+ = f_5,
\end{align}

and initial condition:

\begin{align}
(1.8) \quad u_\pm|_{t=0} = u_0\pm & \quad \text{in } \mathbb{R}^N_\pm, \\
\rho_+|_{t=0} = \rho_+ + \rho_+ & \quad \text{in } \mathbb{R}^N_+, \\
h|_{t=0} = h_0 & \quad \text{in } \mathbb{R}^N_0,
\end{align}

where $m$ ranges from 1 to $N-1$ and we have set $u_\pm = (u_1, \ldots, u_N), \rho_+ = \mu_+(\rho_+, \theta_+), \kappa_+ = \kappa_+(\rho_+, \theta_+), \nu_+ = \nu_+(\rho_+, \theta_+), \mu_- = \mu_-(\theta_-), \Delta h = \sum_{j=1}^{N-1} \partial_j^2 h$.

The other is the heat equations:

\begin{align}
(1.9) \quad \rho_+ \kappa_{\nu+} \partial_t \theta_+ - d_+ \Delta \theta_+ = \tilde{f}_+ & \quad \text{in } \mathbb{R}^N_+ \times (0, T), \\
\rho_+ \kappa_{\nu-} \partial_t \theta_- - d_- \Delta \theta_- = \tilde{f}_- & \quad \text{in } \mathbb{R}^N_- \times (0, T),
\end{align}

subject to the interface condition: for $x_0 \in \mathbb{R}^N_0$ and $t \in (0, T)$

\begin{align}
(1.10) \quad \theta_+ - \theta_+(x_0) = 0, \quad d_+ \partial_N \theta_+ - d_+ \partial_N \theta_+(x_0) = \tilde{g}, \\
\theta_- - \theta_- + \theta_+(x_0) = 0, \quad d_- \partial_N \theta_- - d_+ \partial_N \theta_+(x_0) = \tilde{g},
\end{align}

and the initial condition:

\begin{align}
(1.11) \quad \theta_+|_{t=0} = \theta_0 & \quad \text{in } \mathbb{R}^N_+, \\
\theta_-|_{t=0} = \theta_0 & \quad \text{in } \mathbb{R}^N_-
\end{align}

where we have set $d_+ = d(\rho_+, \theta_+), \kappa_{\nu+} = \kappa_{\nu+}(\rho_+, \theta_+)$, and $d_- = d_-(\theta_-), \kappa_{\nu-} = \kappa_{\nu-}(\theta_-)$. Here, the right-hand sides of (1.6), (1.7), (1.9), and (1.10) are nonlinear terms.

We note that the interface condition (1.7) can be rewritten as follows: for $x_0 \in \mathbb{R}^N_0$ and $t \in (0, T)$

\[
\partial_t h - \left( \frac{\rho_+}{\rho_+ - \rho_+} u_{N-1}(x_0) - \frac{\rho_+}{\rho_+ - \rho_+} u_{N+1}(x_0) \right) = d,
\]
\[ \mu_+ - D_m N(u_+)(x) - \mu_+ + D_m N(u_+)(x) = g_m, \]

\[ \{\mu_+ - D_N N(u_-) - \mu_+\} - (x) = \sigma_+ \Delta h + g_N, \]

\[ \{\mu_+ - D_N N(u_+) + (\nu_+ - \mu_+ + \kappa_+ \Delta \rho_+)\} + (x) = \sigma_+ \Delta h + g_{N+1}, \]

\[ u_{m_-} - (x) - u_{m_+}(x) = h_m, \]

\[ \partial_N \rho_+ + (x) = k, \]

with

\[ \sigma_+ = \frac{\rho_+ - \sigma}{\rho_+ - \rho_+}, \quad g_N = \frac{\rho_+ - \rho_+}{\rho_+ - \rho_+}(f_6 - \rho_+ - f_7), \quad g_{N+1} = \frac{\rho_+ - \rho_+}{\rho_+ - \rho_+}(f_6 - \rho_+ - f_7). \]

As in Shibata [26, 29], the maximal \( L_p L_q \) regularity and the generation of \( C^0 \) analytic semigroup follow automatically from the existence of \( \mathcal{R} \) bounded solution operator families of the corresponding generalized resolvent problem. Hence, in this paper we concentrate on the existence of \( \mathcal{R} \)-bounded solution operator families for the resolvent problem arising from model problem with the interface condition:

(1.12) \[
\lambda \rho_+ + \rho_+ \text{div} u_+ = f_1 \quad \text{in} \, \mathbb{R}^N_+, \\
\rho_+ \lambda u_+ - \rho_+ \Delta u_+ - \nu_+ \nabla \text{div} u_+ - \rho_+ \kappa_+ \nabla \Delta \rho_+ = f_2 \quad \text{in} \, \mathbb{R}^N, \\
\text{div} u_+ = f_3 = \text{div} f_4, \quad \rho_+ \lambda u_- - \rho_+ \Delta u_- + \nabla \pi_- = f_5 \quad \text{in} \, \mathbb{R}^N, \\
\lambda H - \left( \frac{\rho_+ - \rho_+}{\rho_+ - \rho_+} u_{N_-} - (x) \right) \left( \rho_+ - \rho_+ \right) u_{N_+}(x) = g_m, \\
\{\mu_+ - D_m N(u_-) - \mu_+\} - (x) = \sigma_+ \Delta h + g_N, \\
\{\mu_+ - D_N N(u_+) + (\nu_+ - \mu_+ + \kappa_+ \Delta \rho_+)\} + (x) = \sigma_+ \Delta h + g_{N+1}, \]

\[ u_{m_-} - (x) - u_{m_+}(x) = h_m, \]

\[ \partial_N \rho_+ + (x) = k, \]

which is corresponding to the time dependent problem (1.6), (1.7), and (1.8). Here, \( H(x,t) \) is an extension of \( h(x',t) \) such that \( H = h \) on \( \mathbb{R}^N_0 \). In this paper, we do not consider (1.6), (1.7), and (1.8) anymore, and so we use the same symbols in the right-hand side of (1.12) as used in (1.6) and (1.7) below.

In order to state our main results precisely we introduce function spaces and some more symbols which will be used in the paper. For any scalar field \( \theta \) we set \( \nabla \theta = (\partial_\theta \theta, \ldots, \partial_N \theta) \), and for any \( N \)-vector field \( u = (u_1, \ldots, u_N) \), \( \nabla u \) is the \( N \times N \) matrix with \( (i,j) \)th component \( \partial_i u_j \). For any domain \( D \) in \( \mathbb{R}^N \), integer \( m_+ \), and \( 1 \leq q \leq \infty \), \( L_q(D) \) and \( W^m_q(D) \) denote the usual Lebesgue space and Sobolev space of functions defined on \( D \) with norms: \( \| \cdot \|_{L_q(D)} \) and \( \| \cdot \|_{W^m_q(D)} \), respectively. We set \( W^m_0(D) = L_q(D) \). The \( \hat{W}^m_q(D) \) is a homogeneous space defined by \( \hat{W}^m_q(D) = \{ f \in L_{q,\text{loc}}(D) \mid \nabla f \in L_q(D) \} \). For any Banach space \( X \), interval \( I \), integer \( m_+ \), and \( 1 \leq p \leq \infty \), \( L_p(I, X) \) and \( W^m_p(I, X) \) denote the usual Lebesgue space and Sobolev space of the \( X \)-valued functions defined on \( I \) with norms: \( \| \cdot \|_{L_p(I, X)} \) and \( \| \cdot \|_{W^m_p(I, X)} \), respectively. For any Banach space \( X, X^N \) denote the \( N \)-product space of \( X \), that is \( X^N = \{ (f_1, \ldots, f_N) \mid f_i \in X (i = 1, \ldots, N) \} \). The norm of \( X^N \) is also denoted by \( \| \cdot \|_X \) for simplicity and \( \| f \|_X = \sum_{j=1}^N \| f_j \|_X \) for \( f = (f_1, \ldots, f_N) \in X^N \). For any two Banach spaces \( X \) and \( Y \), \( \mathcal{L}(X,Y) \) denotes the space of all bounded linear operators from \( X \) to \( Y \), and \( \mathcal{L}(X) \) is the abbreviation of \( \mathcal{L}(X,X) \). Let \( U \) be a subset of \( \mathbb{C} \). Then \( \text{Ana}(U, \mathcal{L}(X,Y)) \) denotes the set of all \( \mathcal{L}(X,Y) \)-valued analytic functions defined on \( U \). Throughout in this paper, the letter \( C \) denotes generic constants and \( C_{a,\beta,\gamma,\ldots} \) means that the constant depends on the quantities \( \alpha, \beta, \gamma, \ldots \). The values of constants \( C \) and \( C_{a,\beta,\gamma,\ldots} \) may change from line to line.

Before we state the main theorem, we first introduce the definition of \( \mathcal{R} \)-boundedness.

**Definition 1.1.** \( \mathcal{R} \)-boundedness Let \( X \) and \( Y \) be Banach spaces. A set of operators \( \mathcal{T} \subset \mathcal{L}(X,Y) \) is called \( \mathcal{R} \)-bounded, if there is a constant \( 0 < C < \infty \) and \( 1 \leq p < \infty \) such that, for all \( T_1, \ldots, T_m \in \mathcal{T} \)
and \( x_1, \ldots, x_m \in X \) with \( m \in \mathbb{N} \), we have
\[
\left( \int_0^1 \left\| \sum_{n=1}^m r_n(t) x_n \right\|_{Y}^p dt \right)^{1/p} \leq C \left( \int_0^1 \left\| \sum_{n=1}^m r_n(t) x_n \right\|_{X}^p dt \right)^{1/p}
\]
where \( r_n(t) = \text{sign} \sin(2^n \pi t) \) are the Rademacher functions on \([0, 1] \). The smallest such \( C \) is called \( R \)-bound of \( T \) on \( L(X,Y) \), which is denoted by \( R_{L(X,Y)}(T) \).

Let \( \eta_* \) be a constant given by
\[
\eta_* = \left( \frac{\mu_{**} + \nu_{**}}{2\kappa_{**}} \right)^2 - \frac{1}{\kappa_{**}}
\]
and let \( \varepsilon_* \in [0, \pi/2) \) be some angle that is given precisely in Lemma 6.1 below. In this paper, we assume \( \eta_* \neq 0 \) and \( \kappa_{**} \neq \mu_{**} + \nu_{**} \). We discuss these conditions in more detail in Remark 5.2 below (cf. Saito [25, Remark 3.3]).

The following theorem is a main theorem in this paper.

**Theorem 1.2.** Let \( 1 < q < \infty \) and \( \varepsilon_* < \varepsilon < \pi/2 \). Assume that \( \rho_{**} \neq \rho_{**} \), \( \eta_* \neq 0 \), and \( \kappa_{**} \neq \mu_{**} + \nu_{**} \). Set
\[
\Sigma_{\varepsilon} = \{ \lambda = \gamma + i \tau \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \varepsilon \},
\Sigma_{\varepsilon, \lambda_0} = \{ \lambda \in \Sigma_{\varepsilon} \mid |\lambda| > \lambda_0 \} \quad (\lambda_0 \geq 0), \quad \partial_{\varepsilon} = \partial / \partial \tau,
\]
\[
X_q = \{(f_1, f_2, f_3, f_4, f_5, g, h, k, d) \mid f_1 \in W^1_q(\mathbb{R}^N_+), \ f_2 \in L_q(\mathbb{R}^N_+), \ f_3 \in W^1_q(\mathbb{R}^N_+), \ f_4 \in L_q(\mathbb{R}^N_+), \ f_5 \in L_q(\mathbb{R}^N_+), \ g = (g_1, \ldots, g_{N+1}) \in W^1_q(\mathbb{R}^N)^{N+1}, \ h = (h_1, \ldots, h_{N-1}) \in W^2_q(\mathbb{R}^N), \ k \in W^2_q(\mathbb{R}^N), \ d \in W^2_q(\mathbb{R}^N) \},
\]
\[
X_q = \{(F_1, \ldots, F_{15}) \mid F_1 \in W^1_q(\mathbb{R}^N_+), \ F_2 \in L_q(\mathbb{R}^N_+), \ F_3 \in L_q(\mathbb{R}^N_+), \ F_4, F_5, F_6 \in L_q(\mathbb{R}^N_+), \ F_7 \in L_q(\mathbb{R}^N_+), \ F_8 \in L_q(\mathbb{R}^N_+), \ F_9 \in L_q(\mathbb{R}^N)^{N-1}, \ F_{10} \in L_q(\mathbb{R}^N)^{N-1}, \ F_{11} \in L_q(\mathbb{R}^N)^{N-1}, \ F_{12}, F_{13} \in L_q(\mathbb{R}^N)^{N-1}, \ F_{14} \in L_q(\mathbb{R}^N)^{N-2}, \ F_{15} \in W^2_q(\mathbb{R}^N) \}.
\]
Then, there exist a positive constant \( \lambda_0 \) and operator families \( A_\pm(\lambda) \), \( B_\pm(\lambda) \), \( P_- \), and \( H(\lambda) \) with
\[
A_\pm(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(X_q, W^2_q(\mathbb{R}^N_+))), \quad B_\pm(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(X_q, W^3_q(\mathbb{R}^N_+))),
\]
\[
P_- \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(X_q, W^2_q(\mathbb{R}^N))), \quad H(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(X_q, W^2_q(\mathbb{R}^N))),
\]
\[
\text{such that for any } \lambda \in \Sigma_{\varepsilon, \lambda_0} \text{ and } F = (f_1, f_2, f_3, f_4, f_5, g, h, k, d) \in X_q, \ u_\pm = A_\pm(\lambda) F, \ \rho_+ = B_+(\lambda) F, \ \pi_- = P_- F, \ \text{and } H(\lambda) F \text{ are unique solutions of problem (1.12). Furthermore, for } s = 0, 1, \text{ we have}
\]
\[
R_{L(X_q, L_q(\mathbb{R}^N)^{N+2}+\mathbb{R}^N)}((\tau \partial_{\varepsilon})^s (G^1_\pm A_\pm(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \leq c_0,
\]
\[
R_{L(X_q, L_q(\mathbb{R}^N)^{N+2}+\mathbb{R}^N)}((\tau \partial_{\varepsilon})^s (G^2_\pm B_\pm(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \leq c_0,
\]
\[
R_{L(X_q, L_q(\mathbb{R}^N)^{N+2}+\mathbb{R}^N)}((\tau \partial_{\varepsilon})^s (\nabla P_- F) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \leq c_0,
\]
\[
R_{L(X_q, W^2_q(\mathbb{R}^N)^{N+1})}((\tau \partial_{\varepsilon})^s (\nabla H(\lambda) F) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \leq c_0,
\]
with some positive constant \( c_0 \). Here, \( G^1_\pm A_\pm(\lambda) = (\lambda A_\pm(\lambda), \lambda^{1/2} \nabla A_\pm(\lambda), \nabla^2 A_\pm(\lambda)) \), \( G^2_\pm B_- \) \( = (\lambda B_-(\lambda), \lambda^{1/2} \nabla^2 B_-(\lambda), \nabla^2 B_-(\lambda)) \), \( G^3_\pm H(\lambda) = (\lambda H(\lambda), \nabla H(\lambda)) \), and
\[
\bar{F} = (f_1, f_2, \lambda^{1/2} f_3, \nabla f_3, \lambda f_4, f_5, \lambda^{1/2} g, \nabla g, \lambda^{1/2} \nabla h, \nabla^2 h, \lambda k, \lambda^{1/2} \nabla k, \nabla^2 k, d).
\]

**Remark 1.3.** (1) The uniqueness of solutions of problem (1.12) follows from the existence of solutions for a dual problem in a similar way to Shibata and Shimizu [31, Sect. 3], so that we omit its proof.
(2) It is easy to show the existence of $\mathcal{R}$-bounded solution operator families for the resolvent problem arising from (1.9), (1.10), and (1.11). In fact, when we employ the similar argumentation to that in the proof of Theorem 1.2 given in the sequel. Hence, we do not consider problem (1.9), (1.10), and (1.11) in this paper.

(3) We can show the maximal $L_p$-$L_q$ regularity theorem for (1.6), (1.7), (1.8), (1.9), (1.10), and (1.11) due to the same theory as in Shibata [28] with the help of the $\mathcal{R}$-bounded solution operator and the operator valued Fourier multiplier theorem of Weis [35].

This paper is organized as follows. In Sect. 2, according to the argument due to Prüss et al. [21] (cf. Prüss and Simonett [24] and Shibata [27]) we explain the interface condition (1.2) in more detail from the point of view of conservation of mass, conservation of momentum, conservation of energy and increment of entropy and we show the complete model. In Sect. 3 we introduce some results of half spaces. From Sect. 4 to Sect. 6 we consider the problem without the surface tension. In Sect. 4, by the partial Fourier transform, we have ordinary differential equations with respect to $x_N$. Then, we solve them and apply the inverse partial Fourier transform to its solution in order to obtain exact solution formulas to the resolvent problem. In Sect. 5 we introduce some technical lemmas and give some estimates for the multipliers appearing in the solution formula. In Sect. 6 we analyze the Lopatinski determinant appearing in the solution formula. Finally in Sect. 7 we prove the main theorem for the $\mathcal{R}$-bounded solution operator families.

2. Derivation of interface conditions

In this section, assuming that the equation (1.11) holds in the bulk $\dot{\Omega}_t$, we derive interface conditions (1.2) under which balance of mass, balance of momentum, balance of energy, and entropy production hold. We follow the argument due to Prüss et al. in [21]. Our model is, however, different from Prüss et al. [21], and so we give a detailed explanation.

For our purpose we may assume that integration appearing below is finite. In this sense, our argument below is rather formal from the integrability point of view. In addition, we assume that there exists a smooth diffeomorphism $\phi_t : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\hat{\Omega}_t = \{ x = \phi_t(y) | y \in \Omega_0 \}, \quad \Gamma_t = \{ x = \phi_t(y) | y \in \Gamma_0 \}.$$ 

for $t > 0$. Let $v = (\partial_\nu \phi_t)(\phi_t^{-1}(x), t)$ and then, it follows from the Reynolds transport theorem that

$$\frac{d}{dt} \int_{\dot{\Omega}_t} f \, dx = \int_{\dot{\Omega}_t} (\partial_t f + \text{div} (fv)) \, dx.$$ 

In particular, $u_t \cdot n_t = v \cdot n_t$ on $\Gamma_t$. We assume that

$$\begin{align*}
(2.2) \quad & \partial_t \rho + \text{div} (\rho u) = 0, \\
(2.3) \quad & \partial_t (\rho u) + \text{div} (\rho u \otimes u) - \text{div} T = \rho f, \\
(2.4) \quad & \partial_t \left( \frac{\rho}{2} |u|^2 + \rho e \right) + \text{div} \left( \left( \frac{\rho}{2} |u|^2 + \rho e \right) u \right) - \text{div} (Tu - q) = \rho f \cdot u + \rho r
\end{align*}$$

hold in the bulk $\dot{\Omega}_t$. And then, we look for the interface conditions under which the following formulas hold:

$$\begin{align*}
(2.5) \quad & \frac{d}{dt} \int_{\dot{\Omega}_t} \rho \, dx = 0 \quad \text{(Balance of Mass)}, \\
(2.6) \quad & \frac{d}{dt} \int_{\dot{\Omega}_t} \rho u \, dx = \int_{\dot{\Omega}_t} \rho f \, dx \quad \text{(Balance of Momentum)}, \\
(2.7) \quad & \frac{d}{dt} \left\{ \int_{\dot{\Omega}_t} \left( \frac{\rho}{2} |u|^2 + \rho e \right) \, dx + \sigma |\Gamma_t| \right\} = \int_{\dot{\Omega}_t} (\rho f \cdot u + \rho r) \, dx \quad \text{(Balance of Energy)}, \\
(2.8) \quad & \frac{d}{dt} \int_{\dot{\Omega}_t} \rho \theta r \, dx \geq \int_{\dot{\Omega}_t} \rho \theta^{-1} r \, dx \quad \text{(Entropy Production)}.
\end{align*}$$

*In order to make the discussion in this section rigorous, it is enough to assume that the domain is bounded and the outer boundary conditions are imposed like Prüss et al. [21].
Since \( n = 1 \), by (2.1), (2.2), and the divergence theorem of Gauss we have

\[
\frac{d}{dt} |\Gamma_t| = - \int_{\Gamma_t} H_{\Gamma_t} u_{\Gamma_t} \cdot n_t \, d\tau.
\]

Here and in the sequel, \( d\tau \) denotes the surface element of \( \Gamma_t \).

**Balance of Mass:**

By (2.11), (2.2), and the divergence theorem of Gauss we have

\[
\frac{d}{dt} \int_{\Omega_t} \rho \, dx = \int_{\Omega_t} \left( \partial_t \rho + \text{div} (\rho \mathbf{v}) \right) \, dx = \int_{\Omega_t} \text{div} (\rho (\mathbf{v} - \mathbf{u})) \, dx = \int_{\Gamma_t} \left[ \rho (\mathbf{u}_F - \mathbf{u}) \right] \cdot n_t \, d\tau.
\]

Thus, to obtain (2.5) it is sufficient to assume that

\[
\left[ \rho (\mathbf{u}_F - \mathbf{u}) \right] \cdot n_t = 0 \quad \text{on} \ \Gamma_t,
\]

and so we define \( j \) be

\[
\mathbf{j} = \rho_+ (\mathbf{u}_+ - \mathbf{u}_F) \cdot n_t = \rho_- (\mathbf{u}_- - \mathbf{u}_F) \cdot n_t,
\]

which is called the phase flux, more precisely, the interfacial mass flux. Since \( \mathbf{u}_F \cdot n_t = \mathbf{u}_+ \cdot n_t - j / \rho_+ = \mathbf{u}_- \cdot n_t - j / \rho_- \), we have

\[
\mathbf{j} \equiv \left[ \rho \right] \cdot \frac{\mathbf{n}_t}{\left[1/\rho\right]}.
\]

A phase transition takes place if \( j \neq 0 \).

Furthermore, by (2.10), we have

\[
V_{\Gamma_t} = \mathbf{u}_F \cdot \mathbf{n}_t = \left( \frac{\rho_+ - \rho_-}{\rho_+ - \rho_-} \mathbf{u}_+ - \frac{\rho_-}{\rho_+ - \rho_-} \mathbf{u}_- \right) \cdot \mathbf{n}_t = \left[ \rho \mathbf{u} \right] \cdot \frac{\mathbf{n}_t}{\left[\rho\right]}.
\]

which is the kinetic condition in the case that \( j \neq 0 \).

On the other hand, if \( j = 0 \), then we have \( \mathbf{u}_+ \cdot \mathbf{n}_t = \mathbf{u}_F \cdot \mathbf{n}_t = \mathbf{u}_- \cdot \mathbf{n}_t \) on \( \Gamma_t \). Thus, we have a usual kinetic condition: \( V_{\Gamma_t} = \mathbf{u} \cdot \mathbf{n}_t \). If \( \Gamma_t \) is defined by \( F(x,t) = 0 \) locally, then \( F(\phi_t(y),t) = 0 \) for \( y \in \Omega_0 \). Thus, we have

\[
0 = \frac{d}{dt} F(\phi_t(y),t) = \partial_t F + (\nabla F) \cdot \mathbf{u}_F.
\]

Since \( \mathbf{n}_t \) is parallel to \( \nabla F \), we have \( (\nabla F) \cdot \mathbf{u}_F = (\nabla F) \cdot \mathbf{u}_F = 0 \) on \( \Gamma_t \), and so we have

\[
\partial_t F + \mathbf{u} \cdot \nabla F = 0 \quad \text{on} \ \Gamma_t.
\]

This is a different representation formula of kinetic condition when the phase transition does not take place.

**Balance of Momentum:**

We will prove that it follows from the balance of momentum that

\[
\begin{cases}
\rho (\partial \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \text{div} \mathbf{T} = \rho \mathbf{f} & \text{in} \ \Omega_t, \\
j[\mathbf{u}] - [\mathbf{T} \mathbf{n}_t] = -\sigma H_{\Gamma_t} \mathbf{n}_t & \text{on} \ \Gamma_t.
\end{cases}
\]

In fact, we write (2.3) componentwise as

\[
\partial_t (\rho u_i) + \text{div} (\rho u_i \mathbf{u}) = \sum_{j=1}^{N} \partial_j T_{ij} = \rho f_i \quad \text{in} \ \Omega_t.
\]

And then, by (2.1) we have

\[
\frac{d}{dt} \int_{\Omega_t} \rho u_i \, dx = \int_{\Omega_t} \left( \partial_t (\rho u_i) + \text{div} (\rho u_i \mathbf{u}) \right) \, dx
\]

\[
= \int_{\Omega_t} \left( \text{div} (\rho u_i (\mathbf{v} - \mathbf{u})) + \sum_{j=1}^{N} \partial_j T_{ij} + \rho f_i \right) \, dx
\]

\[
= \int_{\Gamma_t} \left[ \rho (\mathbf{u}_F - \mathbf{u}) \right] \cdot \mathbf{n}_t + (i^{\text{th}} \text{ component of } [\mathbf{T} \mathbf{n}_t]) \, d\tau + \int_{\Omega_t} \rho f_i \, dx.
\]
By (2.10), we have \([\rho u_i (u_i - u)] \cdot n_i = -j[[u]]\) on \(\Gamma_t\), and so in order that (2.13) holds it is sufficient to assume that
\[
\int_{\Gamma_t} (-j[[u]] + [[Tn]]) \, dt = 0,
\]
which leads to
\[
-j[[u]] + [[Tn]] = \text{div}_\Gamma T_{\Gamma_t} \quad \text{on} \, \Gamma_t,
\]
where \(T_{\Gamma_t}\) denotes surface stress and \(\text{div}_\Gamma\) denotes the surface divergence. When we consider surface tension on \(\Gamma_t\), we have
\[
\text{div}_\Gamma T_{\Gamma_t} = \sigma \text{div}_\Gamma n = \sigma \Delta x = \sigma H_{\Gamma_t} \, n_t
\]
where \(x\) is a position vector of \(\Gamma_t\). Thus, we have the interface condition:
\[
j[[u]] - [[Tn]] = -\sigma H_{\Gamma_t} \, n_t. \quad \text{on} \, \Gamma_t.
\]
Moreover, by (2.2) we have
\[
\partial_t (\rho u) + \text{div} (\rho u \otimes u) = (\partial_t \rho + \text{div} (\rho u)) u + \rho (\partial_t u + u \cdot \nabla u) = \rho (\partial_t u + u \cdot \nabla u),
\]
and so we can rewrite (2.3) as
\[
\rho (\partial_t u + u \cdot \nabla u) - \text{div} T = \rho f \quad \text{in} \, \Omega_t.
\]

**Balance of Energy:**

We will prove that it follows from the balance of energy that
\[
(2.14) \begin{cases}
\rho (\partial_t e + u \cdot \nabla e) + \text{div} q - T : \nabla u = \rho r & \text{in} \, \Omega_t, \\
\frac{1}{2} [[|u - u_i|^2]] + j[e] - [[(T(u - u_i))] \cdot n_i + [[q]] \cdot n_i = 0 & \text{on} \, \Gamma_t.
\end{cases}
\]

In fact, by (2.21), (2.9) and (2.14), we have
\[
\begin{align*}
\frac{d}{dt} \left\{ \int_{\Omega_t} \left( \frac{\rho}{2} |u|^2 + \rho e \right) \, dx + \sigma |\Gamma_t| \right\} \\
= \int_{\Omega_t} \left\{ \partial_t \left( \frac{\rho}{2} |u|^2 + \rho e \right) + \text{div} \left( \left( \frac{\rho}{2} |u|^2 + \rho e \right) v \right) \right\} \, dx - \sigma \int_{\Gamma_t} H_{\Gamma_t} u_t \cdot n_t \, dt \\
= \int_{\Omega_t} \left\{ \text{div} \left( \left( \frac{\rho}{2} |u|^2 + \rho e \right) (v - u) \right) + \text{div} (Tu - q) + \rho f \cdot u + \rho r \right\} \, dx - \sigma \int_{\Gamma_t} H_{\Gamma_t} u_t \cdot n_t \, dt \\
= \int_{\Gamma_t} \left\{ \left[ \left( \frac{\rho}{2} |u|^2 + \rho e \right) (u_t - u) \right] \cdot n_t + \left[ (Tu - q) \cdot n_t - \sigma H_{\Gamma_t} u_t \cdot n_t \right] \right\} \, dt + \int_{\Omega_t} (\rho f \cdot u + \rho r) \, dx.
\end{align*}
\]

If we assume that
\[
\left[ \left( \frac{\rho}{2} |u|^2 + \rho e \right) (u_t - u) \right] \cdot n_t + \left[ (Tu - q) \cdot n_t - \sigma H_{\Gamma_t} u_t \cdot n_t \right] = 0,
\]
then we have the balance of energy (2.7). By (2.10)
\[
\left[ \left( \frac{\rho}{2} |u|^2 + \rho e \right) (u_t - u) \right] \cdot n_t = -\frac{1}{2} [[|u|^2]] - j[e] = -\frac{1}{2} [[|u - u_i|^2]] - j[[u]] \cdot u_t - j[[e]].
\]

Noting that \(T\) is a symmetric matrix, by (2.13) we have
\[
\left[ (Tu - q) \cdot n_t - \sigma H_{\Gamma_t} u_t \cdot n_t \right] = \left[ (Tu - u_i) \right] \cdot n_t + \left[ [[q]] \cdot n_t - \sigma (H_{\Gamma_t} n_t) \cdot u_t \right] = \left[ (Tu - u_i) \right] \cdot n_t + \left[ [[q]] \cdot n_t + j[[u]] \cdot u_t \right].
\]

Putting these formulas together gives the following interface condition:
\[
\frac{1}{2} [[|u - u_i|^2]] + j[e] - [[(T(u - u_i))] \cdot n_t + [[q]] \cdot n_t = 0 \quad \text{on} \, \Gamma_t.
\]

By (2.22) and (2.23), we rewrite (2.24) as
\[
\partial_t \left( \frac{\rho}{2} |u|^2 + \rho e \right) + \text{div} \left( \left( \frac{\rho}{2} |u|^2 + \rho e \right) u \right) - \text{div} (Tu - q)
\]
\[
= \left( \frac{1}{2} |u|^2 + e \right) (\partial_t \rho + \text{div} (\rho u)) + \rho (u \cdot u_t + e_t) + \rho u \cdot \nabla \left( \frac{1}{2} |u|^2 + e \right) - (\text{div} T) \cdot u - T : \nabla u + \text{div} q
\]
\[
= \rho (\partial_t e + u \cdot \nabla e) + \rho f \cdot u - T : \nabla u + \text{div} q.
\]
where we have set $\mathbf{T} : \nabla \mathbf{u} = \sum_{i,j=1}^{N} T_{ij} \partial_i u_j$. Putting this and (2.4) together gives
\[
\rho (\partial_t e + \mathbf{u} \cdot \nabla e) + \text{div} \mathbf{q} - \mathbf{T} : \nabla \mathbf{u} = \rho r \quad \text{in } \Omega_t.
\]

**Entropy Production:**

We now introduce the fundamental thermodynamic relations which read
\[
(2.15) \quad \frac{\partial e}{\partial \eta} = \theta, \quad e = \psi + \theta \eta, \quad \eta = - \frac{\partial \psi}{\partial \theta}, \quad \kappa_v = \frac{\partial e}{\partial \theta}, \quad \ell = \theta \left[ \frac{\partial \psi}{\partial \theta} \right] = - \theta [\eta].
\]

The quantities $\kappa_v$ and $\ell$ are called heat capacity and latent heat, respectively. We assume that
\[
(2.16) \quad \kappa_v = \frac{\partial e}{\partial \theta} > 0, \quad \theta > 0.
\]

As constitutive laws in the phases, for the compressible viscous fluid part, $\Omega_{t+}$, we employ the Korteweg’s law for the stress tensor and the Dunn-Serrin law for the energy flux, while for the incompressible viscous fluid part, $\Omega_{t-}$, we employ the Newton’s law for the stress tensor and Fourier’s law for the energy flux. Namely, we assume that

**Constitutive Law in the Phases:**

\[
\mathbf{T}_+ = \mathbf{S}_+ + \mathbf{K}_+ - \pi_+ \mathbf{I}, \quad \mathbf{S}_+ = \mu_+ \mathbf{D} (\mathbf{u}_+) + (\nu_+ - \mu_+) \text{div} \mathbf{u}_+ - \pi_+ \mathbf{I},
\]
\[
\mathbf{K}_+ = (\alpha_0 (\rho_+)) |\nabla \rho_+|^2 + \alpha_1 (\rho_+) \Delta \rho_+ \mathbf{I} + \beta (\rho_+) \nabla \rho_+ \otimes \nabla \rho_+,
\]
\[
\mathbf{q}_+ = -d_+ \nabla \theta_+ + \left( \kappa_+ (\rho_+) \right) \text{div} \mathbf{u}_+ \nabla \rho_+ \quad \text{in } \Omega_{t+},
\]
\[
\mathbf{T}_- = \mathbf{S}_- - \pi_- \mathbf{I}, \quad \mathbf{S}_- = \mu_- \mathbf{D} (\mathbf{u}_-) - \pi_- \mathbf{I}, \quad \mathbf{q} = -d_- \nabla \theta_- \quad \text{in } \Omega_{t-}.
\]

Here, $\mathbf{D} (\mathbf{u}) = (1/2) (\nabla \mathbf{u} + \mathbf{T} \nabla \mathbf{u})$. To ensure non-negative entropy production in the bulk $\Omega_{t\pm}$, we assume that
\[
(2.17) \quad \alpha_0 = \frac{\kappa_+ + \rho_+ \kappa_+'}{2} |\nabla \rho_+|^2, \quad \alpha_1 = \kappa_+ \rho_+, \quad \beta = -\kappa_+ + \rho_+ \beta',
\]
\[
\mu_\pm > 0, \quad \nu_+ > \frac{N-1}{N} \mu_+, \quad d_\pm > 0.
\]

Assuming $\mathbf{q}_+ = -d_+ \nabla \theta_+ + \left( \kappa_+ (\rho_+) \right) \text{div} \mathbf{u}_+ \nabla \rho_+$, Dunn and Serrin [9] proved that $\mathbf{K}_+$ should equal $(1/2) (\kappa_+ + \rho_+ \kappa_+') |\nabla \rho_+|^2 + \kappa_+ \rho_+ \Delta \rho_+ \mathbf{I} - \kappa_+ \nabla \rho_+ \otimes \nabla \rho_+$ to ensure non-negative entropy production. But, in the following, assuming that $\mathbf{K}_+ = (\alpha_0 (\rho_+)) |\nabla \rho_+|^2 + \alpha_1 (\rho_+) \Delta \rho_+ \mathbf{I} + \beta (\rho_+) \nabla \rho \otimes \nabla \rho$, we prove that $\mathbf{q}_+ = -d_+ \nabla \theta_+ + \left( \kappa_+ (\rho_+) \right) \text{div} \mathbf{u}_+ \nabla \rho_+$, and $\alpha_0, \alpha_1$ and $\beta$ should be given as in (1.4) to ensure non-negative entropy production. Namely, our argument below is just opposite direction to Dunn and Serrin [9].

We also assume the following.

**Constitutive Law on the Interface $\Gamma_t$:**

\[
(2.18) \quad [\theta] = 0, \quad [\mathbf{u} - (\mathbf{u} \cdot \mathbf{n}_t) \mathbf{n}_t] = 0, \quad (\nabla \rho_+) \cdot \mathbf{n}_t = 0 \quad \text{on } \Gamma_t.
\]

Hence in our model, the temperature and the tangential part of velocity field are continuous across the interface, and the interstitial working does not take place in the normal direction of boundary $\Gamma_t$. In addition, the third boundary condition in (2.18) ensures the Fourier law and the generalized Gibbs-Thomson law, which we will explain below.

We first consider
\[
\frac{d}{dt} \int_{\Omega_t} \rho_+ \eta_+ \, dx.
\]

We assume that $e_+ = e_+ (\eta_+, \rho_+, \nabla \rho_+)$. Let $z_j$ be a variable corresponding to $\partial_j \rho_+$. We then have
\[
\partial_t e_+ + \mathbf{u}_+ \cdot \nabla e_+ = \frac{\partial e_+}{\partial \eta_+} (\partial_\eta_+ \eta_+ + \mathbf{u}_+ \cdot \nabla \eta_+) + \frac{\partial e_+}{\partial \rho_+} (\partial_\rho_+ \rho_+ + \mathbf{u}_+ \cdot \nabla \rho_+)
\]
\[+ \sum_{j=1}^{N} \frac{\partial e_+}{\partial z_j} (\partial_j \eta_+ + \mathbf{u}_+ \cdot \nabla \eta_+).\]
By (2.21), we have
\begin{equation}
\partial_t \rho_+ + u_+ \cdot \nabla \rho_+ = -\rho_+ \text{div} u_+.
\end{equation}
Differentiating (2.19) by $x_j$, we have
\begin{equation}
\partial_i \partial_j \rho_+ + u_+ \cdot \nabla \partial_j \rho_+ = -\partial_j (u_+ \cdot (\nabla \rho_+)) - \partial_j (\rho_+ \text{div} u_+).
\end{equation}
Inserting (2.19) and (2.20) and using the assumption: $\partial e_+ / \partial \eta_+ = \theta_+$, we have
\begin{equation*}
\partial_t e_+ + u_+ \cdot \nabla e_+ = \theta_+ (\partial_t \eta_+ + u_+ \cdot \nabla \eta_+) - \frac{\partial e_+}{\partial \rho_+} \rho_+ \text{div} u_+ \\
- \sum_{j=1}^N \frac{\partial e_+}{\partial z_j} ((\partial_j u_+) \cdot (\nabla \rho_+) + \partial_j (\rho_+ \text{div} u_+)).
\end{equation*}
On the other hand, by (2.14), we have
\begin{equation*}
\rho_+ (\partial_t e_+ + u_+ \cdot \nabla e_+) = -\text{div} q_+ + T_+ : \nabla u_+ + \rho_+ r_+,
\end{equation*}
and so
\begin{equation}
\rho_+ \theta_+ (\partial_t \eta_+ + u_+ \cdot \nabla \eta_+) = -\text{div} q_+ + T_+ : \nabla u_+ + \frac{\partial e_+}{\partial \rho_+} \rho_+^2 \text{div} u_+ \\
+ \sum_{j=1}^N \frac{\partial e_+}{\partial z_j} \rho_+ ((\partial_j u_+) \cdot (\nabla \rho_+) + \partial_j (\rho_+ \text{div} u_+)) + \rho_+ r_+.
\end{equation}
Using (2.2), we have $\partial_t (\rho \eta) + \text{div} (\rho \eta \mathbf{v}) = \rho (\partial_t \eta + \mathbf{u} \cdot \nabla \eta)$, and so by (2.1) we have
\begin{equation}
\frac{d}{dt} \int_{\Omega} \rho \eta \, dx = \int_{\Omega} (\partial_t (\rho \eta) + \text{div} (\rho \eta \mathbf{v})) \, ds \\
= \int_{\Omega} \rho (\partial_t \eta + \mathbf{u} \cdot \nabla \eta) \, dx + \int_{\Omega} \text{div} (\rho \eta (\mathbf{v} - \mathbf{u})) \, dx.
\end{equation}
From (2.21) we have
\begin{equation}
\rho_+ (\partial_t \eta_+ + u_+ \cdot \nabla \eta_+) - \theta_+^{-1} \rho_+ r_+ \\
= \theta_+^{-1} \left\{ -\text{div} q_+ + T_+ : \nabla u_+ + \frac{\partial e_+}{\partial \rho_+} \rho_+^2 \text{div} u_+ + \sum_{j=1}^N \frac{\partial e_+}{\partial z_j} \rho_+ ((\partial_j u_+) \cdot (\nabla \rho_+) + \partial_j (\rho_+ \text{div} u_+)) \right\}.
\end{equation}
We now assume that
\begin{align*}
\mathbf{S}_+ &= \mu_+ \mathbf{D}(u_+) + (\nu_+ - \mu_+) \text{div} u_+ \mathbf{I} - \pi_+ \mathbf{I}, \\
\mathbf{K}_+ &= (\alpha_0(\rho_+))^2 \nabla \rho_+^2 + \alpha_1(\rho_+) \Delta \rho_+ \mathbf{I} + \beta(\rho_+) \nabla \rho_+ \otimes \rho_+.
\end{align*}
Since $\mathbf{D}(\mathbf{u}) = (D_{ij}(\mathbf{u}))$ is symmetric, setting $\mathbf{u} = \begin{pmatrix} u_1, \ldots, u_N \end{pmatrix}$, we have
\begin{equation*}
\mathbf{D}(\mathbf{u}) : \nabla \mathbf{u} = \sum_{i,j=1}^N D_{ij}(\mathbf{u}) \partial_i u_j = \frac{1}{2} \sum_{i,j=1}^N D_{ij}(\mathbf{u}) (\partial_i u_j + \partial_j u_i) = |\mathbf{D}(\mathbf{u})|^2.
\end{equation*}
In the same manner, we have
\begin{align*}
\nabla \rho_+ \otimes \nabla \rho_+ : \nabla u_+ &= \sum_{i,j=1}^N \frac{1}{2} (\partial_i \rho_+)(\partial_j \rho_+)(\partial_i u_{j+}) \\
&= \sum_{i,j=1}^N D_{ij}(u_+)(\partial_i \rho_+)(\partial_j \rho_+).
\end{align*}
We then have
\[ T_+ : \nabla u_+ \]
\[ = \sum_{i,j=1}^{N} \left\{ \mu_i D_{ij}(u_+) + (\nu_+ - \mu_+) \text{div } u_+ - \pi_+ + \alpha_0 |\nabla \rho_+|^2 + \alpha_1 \Delta \rho_+ \delta_{ij} + \beta (\partial_i \rho_+)(\partial_j \rho_+) \right\} \partial_i u_{ij} \\
= \mu_+ |D(u_+)|^2 + (\nu_+ - \mu_+) (\text{div } u_+)^2 + (-\pi_+ + \alpha_0 |\nabla \rho_+|^2 + \alpha_1 \Delta \rho_+) \text{div } u_+ \\
+ \beta \sum_{i,j=1}^{N} D_{ij}(u_+)(\partial_i \rho_+)(\partial_j \rho_+). \]

Using chain rule and the fact that \((\nabla \alpha_1) \cdot (\nabla \rho_+) = \alpha'_1 |\nabla \rho_+|^2\), we have
\[ \theta_+^{-1} \alpha_1(\Delta \rho_+) \text{div } u_+ = \text{div} (\theta_+^{-1} \alpha_1(\nabla \rho_+ \text{div } u_+) - (\nabla (\theta_+^{-1} \alpha_1 \text{div } u_+)) \cdot \nabla \rho_+ \\
= \text{div} (\theta_+^{-1} \alpha_1(\nabla \rho_+ \text{div } u_+) + \theta_+^{-2} \alpha_1(\nabla \rho_+ \text{div } u_+)) \cdot (\nabla \theta_+ ) \\
- \theta_+^{-1} \alpha'_1 |\nabla \rho_+|^2 \text{div } u_+ - \theta_+^{-1} \alpha_1 (\nabla \text{div } u_+) \cdot \nabla \rho_+ , \]
\[ \theta_+^{-1} \text{div } q_+ = \text{div} (\theta_+^{-1} q_+) + \theta_+^{-2} q_+ \cdot (\nabla \theta_+). \]

Inserting the formulas above into (2.23), we have
\[ (2.24) \quad \rho_+ (\partial_i \eta_+ + u_+ \cdot \eta_+) - \theta_+^{-1} \rho_+ \eta_+ = \\
= -\text{div} (\theta_+^{-1} q_+) - \theta_+^{-2} q_+ \cdot (\nabla \theta_+) + \text{div} \left( \theta_+^{-1} \alpha_1(\nabla \rho_+ \text{div } u_+) + \theta_+^{-2} \alpha_1(\nabla \rho_+ \text{div } u_+) \cdot (\nabla \theta_+ ) \right) \\
+ \theta_+^{-1} \left\{ \mu_+ |D(u_+)|^2 + (\nu_+ - \mu_+) (\text{div } u_+)^2 + (-\pi_+ + (\alpha_0 - \alpha'_1)|\nabla \rho_+|^2) \text{div } u_+ \right\} \\
+ \beta \sum_{i,j=1}^{N} D_{ij}(u_+)(\partial_i \rho_+)(\partial_j \rho_+) + \frac{\partial e_+}{\partial \rho_+} \rho_+^2 \text{div } u_+ \\
+ \sum_{j=1}^{N} \frac{\partial e_+}{\partial z_j} \rho_+ ((\partial_j u_+) \cdot (\nabla \rho_+) + (\partial_j \rho_+) \text{div } u_+ + \rho_+ (\partial_j \text{div } u_+)) - \alpha_1 (\nabla \text{div } u_+) \cdot \nabla \rho_+ \right\}. \]

We now assume that
\[ (2.25) \quad \frac{\partial e_+}{\partial z_j} = \frac{\alpha_1}{\rho_+^2} \partial_j \rho_+, \]

to obtain
\[ \sum_{j=1}^{N} \frac{\partial e_+}{\partial z_j} \rho_+^2 \partial_j \text{div } u_+ = \alpha_1 (\nabla \text{div } u_+) \cdot (\nabla \rho_+). \]

In particular, we have
\[ (2.26) \quad e_+ = \frac{\alpha_1}{2 \rho_+^2} |\nabla \rho_+|^2 + e'_1(\eta_+, \rho_+). \]

with some function \(e'_1(\eta_+, \rho_+)\). Thus, we have
\[ \frac{\partial e_+}{\partial \rho_+} = \frac{\alpha'_1}{2 \rho_+^2} |\nabla \rho_+|^2 - \frac{\alpha_1}{\rho_+^3} |\nabla \rho_+|^2 + \frac{\partial e'_1}{\partial \rho_+}, \]

and so we have
\[ \frac{\partial e_+}{\partial \rho_+} \rho_+^2 \text{div } u_+ + \sum_{j=1}^{N} \frac{\partial e_+}{\partial z_j} \rho_+ ((\partial_j u_+) \cdot (\nabla \rho_+) + (\partial_j \rho_+) \text{div } u_+ + \rho_+ (\partial_j \text{div } u_+)) - \alpha_1 (\nabla \text{div } u_+) \cdot \nabla \rho_+ \]
\[ = \frac{1}{2} \alpha'_1 |\nabla \rho_+|^2 \text{div } u_+ - \alpha_1 \rho_+^{-1} |\nabla \rho_+|^2 \text{div } u_+ + \frac{\partial e'_1}{\partial \rho_+} \rho_+^2 \text{div } u_+ \\
+ \sum_{i,j=1}^{N} \frac{\alpha_1}{\rho_+} D_{ij}(u_+)(\partial_i \rho_+)(\partial_j \rho_+) + \frac{\alpha_1}{\rho_+} |\nabla \rho_+|^2 \text{div } u_+, \]
which, combined with (2.24), leads to
\[
\rho_+ (\partial_t \eta_+ + \mathbf{u}_+ \cdot \mathbf{n}_+) - \theta_+^{-1} \rho_+ r_+
\]
\[
= -\text{div} (\theta_+^{-1} \mathbf{q}_+) - \theta_+^{-2} \mathbf{q}_+ \cdot \nabla \theta_+ + \text{div} (\theta_+^{-1} \alpha_1 (\nabla \rho_+) \text{div} \mathbf{u}_+) + \theta_+^{-2} (\alpha_1 (\nabla \rho_+) \text{div} \mathbf{u}_+) \cdot (\nabla \theta_+)
\]
\[
+ \theta_+^{-1} \left\{ \mu_+ |\mathbf{D}(\mathbf{u}_+)|^2 + (\nu_+ - \mu_+) (\text{div} \mathbf{u}_+)^2 \right\}
\]
\[
+ \left( - \pi_+ + \left( \alpha_0 - \frac{1}{2} \alpha'_0 \right)|\nabla \rho_+|^2 + \frac{\partial e'_+}{\partial \rho_+} \rho_+^2 \right) \text{div} \mathbf{u}_+ + \left( \beta + \frac{\alpha_1}{\rho_+} \right) \sum_{i,j=1}^N D_{ij} (\mathbf{u}_+) (\partial_i \rho_+) (\partial_j \rho_+) \right}.\]

We now assume that \( \mathbf{q}_+ = -d_+ \nabla \theta_+ + \alpha_1 \text{div} \mathbf{u}_+(\nabla \rho_+), \) that is we employ the Dunn-Serrin law for the energy flux. Furthermore, we assume that
\[
(2.27)\quad -\pi_+ + \left( \alpha_0 - \frac{1}{2} \alpha'_0 \right) + \frac{\partial e'_+}{\partial \rho_+} \rho_+^2 = 0, \quad \beta + \frac{\alpha_1}{\rho_+} = 0.
\]
We then have
\[
(2.28)\quad \rho_+ (\partial_t \eta_+ + \mathbf{u}_+ \cdot \nabla \eta_+) = \text{div} (\theta_+^{-1} d_+ \nabla \theta_+) + \theta_+^{-2} d_+ |\nabla \theta_+|^2
\]
\[
+ \theta_+^{-1} \left\{ \mu_+ |\mathbf{D}(\mathbf{u}_+)|^2 + (\nu_+ - \mu_+) (\text{div} \mathbf{u}_+)^2 \right\} + \theta_+^{-1} \rho_+ r_+ \quad \text{in} \; \Omega_{t+}.
\]
If we choose
\[
(2.29)\quad \beta = -\kappa, \quad \alpha_1 = \rho_+ \kappa, \quad \alpha_0 = \frac{1}{2} \alpha'_0 = \frac{1}{2} (\kappa + \rho_+ \kappa'), \quad \frac{\partial e'_+}{\partial \rho_+} \rho_+^2 = \pi_+,
\]
then, the formulas in (2.27) hold. In particular, we have the Korteweg tensor given in (1.4).

Since \( \mathbf{T}_- = \mathbf{S}_- \), assuming that \( e_- = e_-(\rho_-, \eta_-), \) \( \mathbf{q}_- = -d_- \nabla \theta_- \) and \( \frac{\partial e_-}{\partial \rho_-} = \theta_- \), and \( \frac{\partial e_-}{\partial \rho_-} \rho_-^2 = \pi_- \), we also have
\[
(2.30)\quad \rho_- (\partial_t \eta_- + \mathbf{u}_- \cdot \nabla \eta_-) = \text{div} (\theta_+^{-1} d_- \nabla \theta_-) + \theta_+^{-2} d_- |\nabla \theta_-|^2
\]
\[
+ \theta_+^{-1} \left\{ \mu_+ |\mathbf{D}(\mathbf{u}_-)|^2 + (\nu_- - \mu_-) (\text{div} \mathbf{u}_-)^2 \right\} + \theta_+^{-1} \rho_- r_- \quad \text{in} \; \Omega_{t-}.
\]
Putting (2.22), (2.28), and (2.30) together and assuming that \([\theta] = 0\), by the divergence theorem of Gauss we have
\[
(2.31)\quad \frac{d}{dt} \int_{\Omega_t} \rho \eta \, dx = \int_{\Omega_t} (\theta^{-2} d |\nabla \theta|^2 + \mu |\mathbf{D}(\mathbf{u})|^2 + (\nu - \mu) (\text{div} \mathbf{u})^2) \, dx
\]
\[
+ \int_{\Gamma_t} (\theta^{-1} [(d \nabla \theta)] \cdot \mathbf{n}_t - [(\rho \eta (\mathbf{u} - \mathbf{u}_t))] \cdot \mathbf{n}_t) \, d\tau + \int_{\Gamma_t} \theta^{-1} \rho r \, dx.
\]
Since \( (\text{div} \mathbf{u})^2 \leq N |\mathbf{D}(\mathbf{u})|^2 \), if we assume that
\[
(2.32)\quad \mu_\pm > 0, \quad \nu_+ \geq \frac{N - 1}{N} \mu_+,
\]
then we have
\[
\int_{\Omega_t} (\theta^{-2} d |\nabla \theta|^2 + \mu |\mathbf{D}(\mathbf{u})|^2 + (\nu - \mu) (\text{div} \mathbf{u})^2) \, dx \geq 0
\]
because \( \text{div} \mathbf{u}_- = 0 \) in \( \Omega_{t-} \). Since \([\rho \eta (\mathbf{u} - \mathbf{u}_t)] \cdot \mathbf{n}_t = j[\eta] \) as follows from (2.10), to obtain Entropy Production:
\[
\frac{d}{dt} \int_{\Omega_t} \rho \eta \, dx \geq \int_{\Omega_t} \rho \theta^{-1} r \, dx,
\]
it is sufficient to assume that
\[
(2.33)\quad j[\eta] - [(d \nabla \theta)] \cdot \mathbf{n}_t = 0 \quad \text{on} \; \Gamma_t,
\]
which is called the Stefan law.

We now derive the generalized Gibbs-Thomson law:
\[
(2.34)\quad ([\psi]) + j^2 \left[ \frac{1}{2 \rho^2} \right] - \left[ \frac{1}{\rho} \mathbf{n}_t \cdot (\mathbf{T} \mathbf{n}_t) \right] = 0 \quad \text{on} \; \Gamma_t.
\]
provided that \( j \neq 0 \). Let \( \tau_i \) \((i = 1, \ldots, N-1)\) be the tangent vectors of \( \Gamma_i \). We then write

\[
\mathbf{u} - \mathbf{u}_\Gamma = \sum_{i=1}^{N-1} \mathbf{n}_i \cdot (\mathbf{u} - \mathbf{u}_\Gamma, \tau_i > \tau_i).
\]

Using the orthogonality of \( \{\tau_1, \ldots, \tau_{N-1}, \mathbf{n}_i\} \), we have

\[
|\mathbf{u} - \mathbf{u}_\Gamma|^2 = |(\mathbf{u} - \mathbf{u}_\Gamma) \cdot \mathbf{n}_i|^2 + \sum_{i=1}^{N-1} |(\mathbf{u} - \mathbf{u}_\Gamma) \cdot \tau_i|^2.
\]

Since \([\mathbf{u} - \mathbf{u}_\Gamma, \mathbf{n}_i > \mathbf{n}_j]\) = 0 as follows from (2.18), we have

\[
(2.35) \quad [(\mathbf{u} - \mathbf{u}_\Gamma, \tau_i > 0, 0), 0 = 0,
\]

and so by (2.10) we have

\[
\frac{1}{2}[(\mathbf{u} - \mathbf{u}_\Gamma)^2] = \frac{1}{2}[(\mathbf{u} - \mathbf{u}_\Gamma) \cdot \mathbf{n}_i]^2 = \frac{1}{2}[((\mathbf{u} + \mathbf{u}_\Gamma) \cdot \mathbf{n}_i)^2 - (\mathbf{u} - \mathbf{u}_\Gamma) \cdot \mathbf{n}_i]^2
\]

Inserting this formula into the second formula in (2.14), using the relation: \( \psi = \phi - \theta \eta \), and recalling the formulas: \( \mathbf{q}_+ = -d_+ \nabla \theta + (\kappa + \rho \div \mathbf{u}_+) \nabla \rho_+ \) and \( \mathbf{q}_- = -d_- \nabla \theta_- \), by (2.18) and (2.33) we have

\[
(2.36) \quad 0 = \frac{3}{2}
\]

We write the last term as

\[
[[((\mathbf{u} - \mathbf{u}_\Gamma) \cdot (\nabla \mathbf{t}_\mathbf{n}_\mathbf{t}))] = [(\mathbf{u} - \mathbf{u}_\Gamma, \mathbf{n}_i > \mathbf{n}_j, \mathbf{T}_\mathbf{n}_\mathbf{t})] + \sum_{i=1}^{N-1} [(\mathbf{u} - \mathbf{u}_\Gamma, \tau_i > \tau_i, \mathbf{T}_\mathbf{n}_\mathbf{t})]
\]

By (2.10), we have

\[
\psi \mathbf{u}_\Gamma, \mathbf{n}_i > \mathbf{n}_j, \mathbf{T}_\mathbf{n}_\mathbf{t} = \psi \mathbf{u}_\Gamma, \mathbf{n}_i > \mathbf{n}_j, \mathbf{T}_\mathbf{n}_\mathbf{t} = 0
\]

On the other hand, by (2.18), (2.13), and (2.30), we have

\[
\psi \mathbf{u}_\Gamma, \tau_i > \tau_i, \mathbf{T}_\mathbf{n}_\mathbf{t} = \psi \mathbf{u}_\Gamma, \tau_i > \tau_i, \mathbf{T}_\mathbf{n}_\mathbf{t} = 0
\]

Summing up, we have obtained

\[
[[((\mathbf{u} - \mathbf{u}_\Gamma) \cdot (\nabla \mathbf{t}_\mathbf{n}_\mathbf{t}))] = \frac{1}{2}[(\mathbf{n}_\mathbf{t} \cdot (\nabla \mathbf{t}_\mathbf{n}_\mathbf{t}))],
\]

which, combined with (2.35), leads to (2.34). Notice that if \( j = 0 \), we do not have (2.34).

Finally, using \( \theta \) we rewrite the first equation in (2.14). Since

\[
div (\theta^{-1} \mathbf{D} \mathbf{u}) + \theta^{-1} |\nabla \theta|^2 = \theta^{-1} \mathbf{D} \mathbf{u}
\]

by (2.28) and (2.30), we have

\[
(2.37) \quad \rho \mathbf{t} \div (\mathbf{D} \mathbf{u}) = div (\mathbf{D} \mathbf{u}) + \mu |\mathbf{D} \mathbf{u}|^2 + (\nu - \mu) \mathbf{D} \mathbf{u}^2 + \rho r \quad \text{in} \quad \Omega_t.
\]

By (2.15),

\[
\kappa = \frac{\partial \psi}{\partial \theta} = \frac{\partial}{\partial \theta} (\psi + \theta \eta) = \frac{\partial \psi}{\partial \theta} + \eta + \theta \frac{\partial \eta}{\partial \theta} = \theta \frac{\partial \eta}{\partial \theta}, \quad \frac{\partial \eta}{\partial \rho} = \frac{\partial^2 \psi}{\partial \rho \partial \theta}.
\]
and so, by (2.2) we have
\[
\rho \theta (\partial_t \eta + u \cdot \nabla \eta) = \rho \theta \frac{\partial \eta}{\partial \theta} (\partial_t \theta + u \cdot \nabla \theta) + \rho \theta \frac{\partial \eta}{\partial \rho} (\partial_t \rho + u \cdot \nabla u)
\]
\[
= \rho \kappa (\partial_t \theta + u \cdot \nabla \theta) + \theta \rho^2 \frac{\partial^2 \psi}{\partial \rho \partial \theta} \text{div} u,
\]
which, combined with (2.3) leads to
\[
\rho \kappa (\partial_t \theta + u \cdot \nabla \theta) = \text{div} (d \nabla \theta) + \mu |D(u)|^2 + (\nu - \mu) (\text{div} u)^2 - \theta \rho^2 \frac{\partial^2 \psi}{\partial \rho \partial \theta} \text{div} u + \rho r \quad \text{in } \Omega_t.
\]
Summing up, we have obtained the following complete model.

**The Complete Model:**

In the bulk:
\[
\begin{aligned}
\partial_t \rho_+ + \text{div} (\rho_+ u_+) &= 0, \\
\rho_+ (\partial_t u_+ + u_+ \cdot \nabla u_+) - \text{div} (S_+ + \mathbf{T}_+ - \pi_+ \mathbf{I}) &= \rho_+ \mathbf{f}_+ \\
\rho_+ \kappa_+ (\partial_t \theta_+ + u_+ \cdot \nabla \theta_+) - \text{div} (d_+ \nabla \theta_+) - \mu_+ |D(u_+)|^2 &= 0 \\
- (\nu_+ - \mu_+) (\text{div} u_+)^2 + \theta_+ \rho_+^2 \frac{\partial^2 \psi_+}{\partial \rho_+ \partial \theta_+} \text{div} u_+ &= \rho_+ r_+
\end{aligned}
\]
\[
\text{div} u_- = 0 \\
\rho_- (\partial_t u_- + u_- \cdot \nabla u_-) - \text{div} (S_- - \pi_- \mathbf{I}) &= \rho_- \mathbf{f}_- \\
\rho_- \kappa_- (\partial_t \theta_- + u_- \cdot \nabla \theta_-) - \text{div} (d_- \nabla \theta_-) - \mu_- |D(u_-)|^2 &= \rho_- r_-
\]

On the interface \( \Gamma_1 \):
\[
\begin{aligned}
[[u]] &= \left[ \left[ \frac{1}{\rho} \right] \right] n_t, \\
[[\theta]] &= 0, \\
[[\psi]] &= 0,
\end{aligned}
\]
\[
V_t = u_+ \cdot n_t = \frac{[[\rho u]] \cdot n_t}{[[\rho]]}, \\
(\nabla \rho_+) \cdot n_t = 0.
\]

3. **Results of Half Spaces**

In this section, we introduce some results of half spaces. To prove Theorem 1.2 we consider the following systems:
\[
\begin{aligned}
\lambda \rho_+ + \rho_+ \text{div} u_+ &= f_1 \quad \text{in } \mathbb{R}_+^N, \\
\rho_+ \kappa_+ \Delta u_+ - \nu_+ \nabla \text{div} u_+ - \rho_+ \kappa_+ \Delta \nabla \rho_+ &= f_2 \quad \text{in } \mathbb{R}_+^N,
\end{aligned}
\]
\[
\mu_+ D_{mN}(u_+) \big|_+ = g_m, \\
\{ \mu_+ D_{NN}(u_+) \} + (\nu_+ - \mu_+) \text{div} u_+ + \rho_+ \kappa_+ \Delta \rho_+ \big|_+ = g_{N+1},
\]
\[
\partial_{N} \rho_+ \big|_+ = k
\]
and
\[
\begin{aligned}
\text{div} u_- &= f_3 = \text{div} f_3, \\
\rho_- \kappa_- \Delta u_- - \nu_- \nabla \text{div} u_- - \rho_- \kappa_- \Delta \nabla \rho_- &= f_5 \quad \text{in } \mathbb{R}_-^N,
\end{aligned}
\]
\[
\mu_- D_{mN}(D(u_-)) \big|_- = 0, \\
\{ \mu_- D_{NN}(u_-) - \pi_- \} \big|_- = g_N.
\]

The existence of \( \mathcal{R} \)-bounded solution operators of (3.1) and (3.2) are proved by Saito [25] and Shibata [26], respectively. In fact, we know the following two lemmas.
Lemma 3.1. \([25]\) Let \(1 < q < \infty, \varepsilon < \varepsilon < \pi/2\). Assume that \(\rho_{*+} \neq \rho_{*+}, \eta_{*} \neq 0\), and \(k_{*+} \neq \mu_{*+} + \nu_{*+}\). Set

\[
Y_{q+} = \{(f_1, f_2, g, k) \mid f_1 \in W^1_q(\mathbb{R}^N_+), \ f_2 \in L_q(\mathbb{R}^N_+)^N, \ g = (g_1, \ldots, g_{(N-1)}g_N+1) \in W^1_q(\mathbb{R}^N_+)^N, \ k \in W^2_q(\mathbb{R}^N_+)^N\},
\]
\[Y_{q-} = \{(f_3, f_4, f_5, g) \mid f_3 \in W^1_q(\mathbb{R}^N_-), \ f_4, f_5 \in L_q(\mathbb{R}^N_-)^N, \ g_N \in W^1_q(\mathbb{R}^N_-)^N\},
\]
\[
\mathcal{Y}_{q+} = \{(F_1, F_2, F_3, F_4, F_5, F_6, F_{\tau+}, F_{\tau-}) \mid F_1, f_2, F_3, F_4, F_5 \in L_q(\mathbb{R}^N_+)^N, \ F_6, F_{\tau+}, F_{\tau-} \in L_q(\mathbb{R}^N_+)^N, \ F_{\tau+} = L_q(\mathbb{R}^N_+)^N, \ F_{\tau-} = L_q(\mathbb{R}^N_-)^N\}.
\]

Then, there exists a positive constant \(\lambda_0\) and operator families \(A^*_+ (\lambda)\) and \(B^*_+ (\lambda)\) with

\[
A^*_+ (\lambda) \in \text{Anal} (\Sigma_{\varepsilon, \lambda_0}, L(\mathcal{Y}_{q+}, W^2_q(\mathbb{R}^N_+)^N)),
\]
\[
B^*_+ (\lambda) \in \text{Anal} (\Sigma_{\varepsilon, \lambda_0}, L(\mathcal{Y}_{q+}, W^2_q(\mathbb{R}^N_+)^N)),
\]
such that for any \(\lambda \in \Sigma_{\varepsilon, \lambda_0}\) and \(F_+ = (f_1, f_2, g, k) \in Y_{q+}\), \(u_+ = A^*_+ (\lambda) F_+\) and \(\rho_+ = B^*_+ (\lambda) F_+\) are unique solutions of problem \([3.7]\). Furthermore, for \(s = 0, 1\), we have

\[
R_{L(\mathcal{Y}_{q+}, L_q(\mathbb{R}^N_+)^{N^3+N^2+N})} \{((\tau \partial_{\tau})^s (G_1^1 A^*_+ (\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \} \leq 0,
\]
\[
R_{L(\mathcal{Y}_{q+}, L_q(\mathbb{R}^N_+)^{N^3+N^2+2W^1_q(\mathbb{R}^N_+)^N})} \{((\tau \partial_{\tau})^s (G_2^1 B^*_+ (\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \} \leq 0,
\]

with some positive constant \(c_0\). Here, \(F_+ = (f_1, f_2, \lambda^{1/2} g, \nabla g, \lambda, \lambda^{1/2} \nabla k, \nabla^2 k)\).

Lemma 3.2. \([26]\) Let \(1 < q < \infty, 0 < \varepsilon < \pi/2\). Set

\[
Y_{q-} = \{(f_3, f_4, f_5, g) \mid f_3, f_4, f_5 \in L_q(\mathbb{R}^N_-)^N, \ g_N \in W^1_q(\mathbb{R}^N_-)^N\},
\]
\[
\mathcal{Y}_{q-} = \{(F_3, F_4, F_5, F_6, F_{\tau+}, F_{\tau-}) \mid F_3, F_4, F_5 \in L_q(\mathbb{R}^N_-)^N, \ F_6, F_{\tau+}, F_{\tau-} \in L_q(\mathbb{R}^N_-)^N, \ F_{\tau+} = L_q(\mathbb{R}^N_-)^N, \ F_{\tau-} = L_q(\mathbb{R}^N_-)^N\}.
\]

Then, there exists operator families \(A_-(\lambda)\) and \(P_-(\lambda)\) with

\[
A_-(\lambda) \in \text{Anal} (\Sigma_{\varepsilon, L(\mathcal{Y}_{q-}, W^2_q(\mathbb{R}^N_-)^N))),
\]
\[
P_-(\lambda) \in \text{Anal} (\Sigma_{\varepsilon, L(\mathcal{Y}_{q-}, \dot{W}^1_q(\mathbb{R}^N_-)^N))},
\]
such that for any \(\lambda \in \Sigma_{\varepsilon}\) and \(F_- = (f_3, f_4, f_5, g_N) \in Y_{q-}\), \(u_- = A_- (\lambda) F_-\) and \(\pi_- = P_- (\lambda) F_-\) are unique solutions of problem \([3.8]\). Furthermore, for \(s = 0, 1\), we have

\[
R_{L(\mathcal{Y}_{q-}, L_q(\mathbb{R}^N_-)^{N^3+N^2+N})} \{((\tau \partial_{\tau})^s (G_1^1 A_- (\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \} \leq 0,
\]
\[
R_{L(\mathcal{Y}_{q-}, L_q(\mathbb{R}^N_-)^N)} \{((\tau \partial_{\tau})^s (\nabla P_-(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \} \leq 0,
\]

with some positive constant \(c_0\). Here, \(F_- = (\lambda^{1/2} f_3, \nabla f_3, \lambda f_4, f_5, \lambda^{1/2} g_N, \nabla g_N)\).

Thus, it is sufficient to consider the problem \([1.12]\) with \(f_1 = 0, f_2 = 0, f_3 = 0, f_5 = 0, g_k = 0\) \((k = 1, \ldots, N + 1)\). Finally, we consider one more auxiliary problem:

\[
\begin{align*}
\lambda \rho_{*+} + \rho_{*+} \text{div} u_{*+} &= 0 \quad \text{in} \ \mathbb{R}^N_+,
\rho_{*+} \lambda u_{*+} - \mu_{*+} \Delta u_{*+} - \nu_{*+} \nabla \text{div} u_{*+} - \rho_{*+} \kappa_{*+} \Delta \nabla \rho_{*+} &= 0 \quad \text{in} \ \mathbb{R}^N_+,
\text{div} u_{*+} &= 0 \quad \text{in} \ \mathbb{R}^N_+,
\rho_{*+} \lambda u_{*-} - \mu_{*+} \Delta u_{*-} + \nabla \pi_{*-} &= 0 \quad \text{in} \ \mathbb{R}^N_-,
\mu_{*+} D_{MN}(D(u_{*+})) \big|_{*+} - \mu_{*+} D_{MN}(u_{*+}) \big|_{*+} &= 0,
\{\mu_{*+} D_{NN}(u_{*+}) \big(\pi_{*+} \big) \} \big|_{*+} &= 0,
\{\mu_{*+} D_{NN}(u_{*+}) + (\nu_{*+} - \mu_{*+}) \text{div} u_{*+} + \rho_{*+} \kappa_{*+} \Delta \rho_{*+} \} \big|_{*+} &= 0,
u_{*+} u_{*+} = h_{*+}, \partial_N \rho_{*+} \big|_{*+} &= 0.
\end{align*}
\]

From Sect. [4] to Sect. [9] we prove the following theorem.
Theorem 3.3. Let $1 < q < \infty$ and $\varepsilon < \varepsilon < \pi/2$. Assume that $\rho_{\ast+} \neq \rho_{\ast-}$, $\eta_{\ast} \neq 0$, and $\kappa_{\ast+} \neq \kappa_{\ast+} + \nu_{\ast+}$. Set

$$Z_q = \{ h = (h_1, \ldots, h_{N-1}) \mid h \in W^2_q(\mathbb{R}^{N-1}) \},$$

$$Z_q = \{ (F_0, F_{10}, F_{11}) \mid F_0 \in L_q(\mathbb{R}^N)^{N-1}, \ F_{10} \in L_q(\mathbb{R}^{(N-1)N}), \ F_{11} \in L_q(\mathbb{R}^{(N-1)N^2}) \}.$$

Then, there exist a positive constant $\lambda_0$ and operator families $\mathbf{A}_{\pm}(\lambda)$, $\mathbf{B}_{\pm}(\lambda)$, and $\mathcal{P}_{-}(\lambda)$ with

$$\mathbf{A}_{\pm}^2(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(Z_q, W^2_q(\mathbb{R}^N)^N),$$

$$\mathbf{B}_{\pm}^2(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(Z_q, W^2_q(\mathbb{R}^N))),$$

$$\mathcal{P}_{-}^2(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(Z_q, W^1_q(\mathbb{R}^N))),$$

such that for any $\lambda \in \Sigma_{\varepsilon, \lambda_0}$ and $h \in Z_q$, $u_{\pm} = \mathbf{A}_{\pm}^2(\lambda)\tilde{h}$, $\rho_{\ast+} = \mathbf{B}_{\pm}^2(\lambda)\tilde{h}$, and $\pi_{\ast-} = \mathcal{P}_{-}^2(\lambda)\tilde{h}$ are solutions of problem (4.3). Furthermore, for $s = 0, 1$, we have

$$\mathcal{R}_L(z_q, L_q(\mathbb{R}^{N^3+N^2+N})) \{ (\tau \partial \tau)^{s}(G^1_{\pm}(\mathbf{A}_{\pm}^2(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \} \leq c_0,$$

with some positive constant $c_0$. Here, $\tilde{h} = (\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h)$.}

4. Solution formulas without surface tension

In this section, we consider the following equations:

$$\lambda \rho_{\ast+} + \rho_{\ast+} \text{div} u_{\ast+} = 0 \quad \text{in } \mathbb{R}^N_+,$$

$$\rho_{\ast+} \lambda u_{\ast+} + (\rho_{\ast+} + \kappa_{\ast+} + \nu_{\ast+}) \nabla \rho_{\ast+} = 0 \quad \text{in } \mathbb{R}^N_+,$$

$$\text{div} u_{\ast+} = 0, \ \rho_{\ast-} \lambda u_{\ast-} - \rho_{\ast+} \lambda u_{\ast+} + \nabla \pi_{\ast-} = 0 \quad \text{in } \mathbb{R}^N_+,$$

$$\mu_{\ast+} D_{mN}(D(u_{\ast-}))|_{\ast-} = \mu_{\ast+} D_{mN}(u_{\ast+})|_{\ast+} = \sigma_{\ast-} \Delta' H, \ \mu_{\ast-} D_{mN}(D(u_{\ast+}))|_{\ast+} = \mu_{\ast+} D_{mN}(u_{\ast-})|_{\ast-} = \sigma_{\ast+} \Delta' H,$$

$$u_{m\ast+\ast-} = h_m, \ \partial N \rho_{\ast-} = 0,$$

where we have added $\sigma_{\ast+} \Delta' H$ with $\sigma_{\pm} = (\rho_{\ast+} + \sigma_{\ast-} + \rho_{\ast-})$ to (3.3) for the latter use. Let $\tilde{u} = F_{x'}[v](\xi^l, x_N)$, the partial Fourier transform with respect to the tangential variable $x' = (x_1, \ldots, x_{N-1})$ with $\xi^l = (\xi_1, \ldots, \xi_{N-1})$ defined by $F_{x'}[v](\xi^l, x_N) = \int_{\mathbb{R}^{N-1}} e^{-i\xi^l \cdot x'} v(x', x_N) dx'$. Applying the partial Fourier transforms to yields ordinary differential equations with respect to $x_N \not= 0$:}

$$\lambda \bar{\rho}_{\ast+} + \rho_{\ast+} \text{div} \bar{u}_{\ast+} = 0 \quad \text{for } x_N > 0,$$

$$\rho_{\ast+} \lambda \bar{u}_{\ast+} - \rho_{\ast+} \nabla \bar{u}_{\ast+} - \nabla \rho_{\ast+} \text{div} \bar{u}_{\ast+} = 0 \quad \text{for } x_N > 0,$$

$$\rho_{\ast+} \lambda \bar{u}_{\ast-} - \rho_{\ast+} \nabla \bar{u}_{\ast-} - \nabla \rho_{\ast-} \text{div} \bar{u}_{\ast-} = 0 \quad \text{for } x_N > 0,$$

$$\text{div} \bar{u}_{\ast+} = 0 \quad \text{for } x_N < 0,$$

$$\rho_{\ast-} \lambda \bar{u}_{\ast-} - \rho_{\ast-} \nabla \bar{u}_{\ast-} - \nabla \rho_{\ast-} \text{div} \bar{u}_{\ast-} = 0 \quad \text{for } x_N < 0,$$

$$\rho_{\ast+} \lambda \bar{u}_{\ast+} - \rho_{\ast+} \nabla \bar{u}_{\ast+} - \nabla \rho_{\ast+} \text{div} \bar{u}_{\ast+} = 0 \quad \text{for } x_N < 0,$$

subject to the interface condition:

$$\mu_{\ast-} \partial N \bar{u}_{\ast-} + i \xi_m \bar{u}_{\ast-} = 0 - \mu_{\ast+} \partial N \bar{u}_{\ast+} + i \xi_m \bar{u}_{\ast+} = 0,$$

$$\{2 \mu_{\ast+} \partial N \bar{u}_{\ast-} - \bar{\pi}_{\ast-} = \sigma_{\ast-} |\xi|^2 \tilde{H}(0),$$
\[
(4.10)\quad \{2\mu_+\partial_N \tilde{u}_{N+} + (\nu_+ - \mu_+)\text{div}\ u_+ - \rho_+\kappa_+ + (\partial_N^2 - |\xi'|^2)\tilde{p}_+\} + \sigma_+|\xi'|^2 \tilde{H}(0),
\]
\[
(4.11)\quad \tilde{u}_{m-} - \tilde{u}_{m+} = \tilde{h}_m(0),
\]
\[
(4.12)\quad \partial_N \tilde{p}_+ = 0.
\]

Here and in the sequel, \(j\) runs from 1 to \(N - 1\). According to Saito \(25\), from \((4.2)\), \((4.3)\), and \((4.3)\), we obtain
\[
(4.13)\quad (\partial^2_N - B^2_+)P_\lambda(\partial_N)\tilde{u}_{J+} = 0 \quad (J = 1, \ldots, N - 1)
\]
with
\[
B_+ = \sqrt{|\xi'|^2 + \rho_+\mu_-^{-1}\lambda} \quad (\text{Re} B_+ > 0),
\]
\[
P_\lambda(t) = \rho_+\lambda^2 - \lambda(\mu_+ + \nu_+)(t^2 - |\xi'|^2) + (t^2 - |\xi'|^2)\{\rho_+\kappa_+(t^2 - |\xi'|^2)\}.
\]

The roots of \(P_\lambda(t) = 0\) are \(t = \pm \sqrt{|\xi'|^2 + s_i\lambda} (i = 1, 2)\) and \(s_i(i = 1, 2)\) are the root of the following equation:
\[
(4.14)\quad z^2 - \left(\frac{\mu_+ + \nu_+}{\kappa_+}\right)z + \frac{1}{\kappa_+} = 0.
\]

Here, \(t_i\) are defined by \(t_i = \sqrt{|\xi'|^2 + s_i\lambda}\), whose detail will be discussed in Sect. \(5\). As seen in Sect. \(5\) we have three roots \(B_+\), \(t_1\), and \(t_2\) with positive real parts different from each other.

On the other hand, according to Shibata \(25\), from \((4.5)\), \((4.6)\), and \((4.7)\), we obtain
\[
(4.15)\quad (\partial^2_N - A^2)(\partial_N^2 - B^2)\tilde{u}_{J-} = 0,
\]
\[
(4.16)\quad (\partial^2_N - A^2)^2\tilde{p}_- = 0
\]
with
\[
A = |\xi'|, \quad B_- = \sqrt{|\xi'|^2 + \rho_+\mu_-^{-1}\lambda} \quad (\text{Re} B_- > 0).
\]

In view of \((4.13)\), \((4.16)\), and \((4.14)\), we look for solutions \(\tilde{u}_{J\pm}\) and \(\tilde{p}_-\) of the forms:
\[
(4.17)\quad \tilde{u}_{J+}(x_N) = \alpha_{J+}e^{-B_+x_N} + \beta_{J+}(e^{-t_1x_N} - e^{-B_+x_N})
\]
\[
+ \gamma_{J+}(e^{-t_2x_N} - e^{-B_+x_N}),
\]
\[
(4.18)\quad \tilde{u}_{J-}(x_N) = \alpha_{J-}e^{-B_-x_N} + \beta_{J-}(e^{B_-x_N} - e^{Ax_N}),
\]
\[
(4.19)\quad \tilde{p}_-(x_N) = \gamma e^{Ax_N}.
\]

Using \(A\) and \(B_\pm\), we rewrite \((4.13)\), \((4.16)\), \((4.14)\), and \((4.17)\) as follows:
\[
(4.20)\quad \mu_+\lambda(\partial^2_N - B^2_+)\tilde{u}_{J+} + i\xi_+\{\nu_+\lambda - \rho_+\kappa_+ + (\partial^2_N - A^2)\}\text{div}\ u_+ = 0,
\]
\[
(4.21)\quad \mu_+\lambda(\partial^2_N - B^2_+)\tilde{u}_{N+} + \partial_N\{\nu_+\lambda - \rho_+\kappa_+ + (\partial^2_N - A^2)\}\text{div}\ u_+ = 0,
\]
\[
(4.22)\quad \mu_+\lambda(\partial^2_N - B^2_+)\tilde{u}_{J-} + \partial_N\{\nu_+\lambda - \rho_+\kappa_+ + (\partial^2_N - A^2)\}\text{div}\ u_+ = 0,
\]
\[
(4.23)\quad \mu_+\lambda(\partial^2_N - B^2_+)\tilde{u}_{N-} - \partial_N\tilde{p}_- = 0.
\]

To state our solution formulas of equations: \((4.2)\) - \((4.12)\), we introduce some classes of multipliers.

**Definition 4.1.** Let \(0 < \varepsilon < \pi/2\), \(\lambda_0 \geq 0\), and let \(s\) be a real number. Set
\[
\tilde{\Sigma}_{\varepsilon,\lambda_0} = \{(\lambda, \xi') | \lambda = \gamma + i\tau \in \Sigma_{\varepsilon,\lambda_0}, \xi' = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{R}^{N-1}\} \setminus \{0\}.
\]

Let \(m(\lambda, \xi')\) be a function defined on \(\tilde{\Sigma}_{\varepsilon,\lambda_0}\) which is infinitely times differentiable with respect to \(\tau\) and \(\xi'\) when \((\lambda, \xi') \in \tilde{\Sigma}_{\varepsilon,\lambda_0}\).

(1) If there exists a real number \(s\) such that for any multi-index \(\alpha' = (\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{N}_0^{N-1}\) and \((\lambda, \xi') \in \tilde{\Sigma}_{\varepsilon,\lambda_0}\) there hold the estimates:
\[
(4.24)\quad |\partial^{\alpha'}_{\xi'} m(\lambda, \xi')| \leq C_{\alpha'}(|\lambda|^{1/2} + A)^{s-|\alpha'|},
\]
\[
|\partial^{\alpha'}_{\xi'} (\tau \partial_{\tau} m(\lambda, \xi'))| \leq C_{\alpha'}(|\lambda|^{1/2} + A)^{s-|\alpha'|}
\]
for some constant \(C_{\alpha'}\) depending on \(s, \alpha', \varepsilon, \mu_{s+}, \nu_{s+}, \kappa_{s+},\) and \(\rho_{s+}\). Then, \(m(\lambda, \xi')\) is called a multiplier of order \(s\) with type 1.

(2) If there exists a real number \(s\) such that for any multi-index \(\alpha' = (\alpha_1, \ldots, \alpha_{N-1}) \in \mathbb{N}^{N-1}\) and \((\lambda, \xi') \in \Sigma_{s, \lambda_0}\) there hold the estimates:

\[
|\partial^\alpha_\xi' m(\lambda, \xi')| \leq C_{\alpha'}(|\lambda|^{1/2} + A)^s A^{-|\alpha'|},
\]

\[
|\partial^\alpha_\xi' \left(t \partial_r m(\lambda, \xi')\right)| \leq C_{\alpha'}(|\lambda|^{1/2} + A)^s A^{-|\alpha'|}
\]

for some constant \(C_{\alpha'}\) depending on \(s, \alpha', \varepsilon, \mu_{s+}, \nu_{s+}, \kappa_{s+},\) and \(\rho_{s+}\). Then, \(m(\lambda, \xi')\) is called a multiplier of order \(s\) with type 2.

In what follows, we denote the set of multipliers defined on \(\Sigma_{s, \lambda_0}\) of order \(s\) with type \(l\) \((l = 1, 2)\) by \(M_{s,l, c, \lambda_0}\).

Obviously, \(M_{s,l, c, \lambda_0}\) are the vector spaces on \(\mathbb{C}\). Furthermore, by the fact \(|\lambda|^{1/2} + A|^{-|\alpha'|} \leq A^{-|\alpha'|}\) and the Leibniz rule, we have the following lemma immediately.

**Lemma 4.2.** Let \(s_1, s_2 \in \mathbb{R}, 0 < \varepsilon < \pi/2, \lambda \in \Sigma_{c, \lambda_0}\).

1. Given \(m_i \in M_{s_1, c, \lambda_0} \ (i = 1, 2)\), we have \(m_1 m_2 \in M_{s_1 + s_2, c, \lambda_0}\).
2. Given \(l_i \in M_{s_1, c, \lambda_0} \ (i = 1, 2)\), we have \(l_1 l_2 \in M_{s_1 + s_2, c, \lambda_0}\).
3. Given \(n_i \in M_{s_1, c, \lambda_0} \ (i = 1, 2)\), we have \(n_1 n_2 \in M_{s_1 + s_2, c, \lambda_0}\).

**Remark 4.3.** We easily see that \(i\xi_j \in M_{1, 2, c, 0} \ (j = 1, \ldots, N - 1)\), and \(A \in M_{1, 2, c, 0}\). Especially, \(i\xi_j / A \in M_{0, 2, c, 0} \ (j = 1, \ldots, N - 1)\). In addition, \(M_{s_1, c, \lambda} \subset M_{s_2, c, \lambda}\) for any \(s \in \mathbb{R}\).

Then, we arrive at the following solution formulas for equations (1.22) - (1.24):

\[
\hat{u}_{j+} = \sum_{i=1}^{4} \hat{u}_{j_{i+}}, \quad \hat{u}_{j-} = \sum_{i=1}^{3} \hat{u}_{j_{i-}},
\]

\[
\hat{\rho}_+ = AM_{0+}(x_N) \left\{ \sum_{m=1}^{N-1} P_{m,1}^+ \hat{h}_m(0) + AP_{N,1}^+ \hat{H}(0) \right\},
\]

\[
+ Ae^{-t_{1,X_N}} \left\{ \sum_{m=1}^{N-1} P_{m,2}^+ \hat{h}_m(0) + AP_{N,2}^+ \hat{H}(0) \right\},
\]

\[
\hat{\pi}_- = e^{A_{X_N}} \left\{ \sum_{m=1}^{N-1} P_{m}^- \hat{h}_m(0) + AP_{N}^- \hat{H}(0) \right\},
\]

\[
\hat{u}_{j1}^+ = AM_{1+}(x_N) \left\{ \sum_{m=1}^{N-1} Q_{j1}^+ \hat{h}_m(0) + AQ_{jN,1}^+ \hat{H}(0) \right\},
\]

\[
\hat{u}_{j2}^+ = AM_{2+}(x_N) \left\{ \sum_{m=1}^{N-1} Q_{j2}^+ \hat{h}_m(0) + + AQ_{jN,2}^+ \hat{H}(0) \right\},
\]

\[
\hat{u}_{j3}^+ = Ae^{-B_{+X_N}} \left\{ \sum_{m=1}^{N-1} R_{j1}^+ \hat{h}_m(0) + AR_{jN}^+ \hat{H}(0) \right\},
\]

\[
\hat{u}_{j4}^+ = e^{-B_{+X_N}} S_{j}^+ \hat{h}_j,
\]

\[
\hat{u}_{N4}^+ = 0,
\]

\[
\hat{\tilde{u}}_{j1} = AM_{-}(x_N) \left\{ \sum_{m=1}^{N-1} Q_{j1}^- \hat{h}_m(0) + AQ_{jN}^- \hat{H}(0) \right\},
\]

\[
\hat{\tilde{u}}_{j2} = Ae^{B_{-X_N}} \left\{ \sum_{m=1}^{N-1} R_{j1}^- \hat{h}_m(0) + AR_{jN}^- \hat{H}(0) \right\},
\]

\[
\hat{\tilde{u}}_{j3} = e^{B_{-X_N}} S_{j}^- \hat{h}_j,
\]

\[
\hat{\tilde{u}}_{N3} = 0,
\]
with
\begin{align}
\begin{aligned}
P_{m,1}^+ &\in \mathbb{M}_{-1,2,\epsilon,0}, & P_{m,1}^- &\in \mathbb{M}_{-1,2,\epsilon,0}, & P_{m,2}^+ &\in \mathbb{M}_{-1,2,\epsilon,0}, \\
P_{N,2}^+ &\in \mathbb{M}_{-1,2,\epsilon,0}, & P_{m}^- &\in \mathbb{M}_{1,2,\epsilon,0}, & P_{N}^- &\in \mathbb{M}_{1,2,\epsilon,0}, \\
Q_{Jm,1}^+ &\in \mathbb{M}_{0,2,\epsilon,0}, & Q_{Jm}^+ &\in \mathbb{M}_{0,2,\epsilon,0}, & Q_{Jm,2}^- &\in \mathbb{M}_{0,2,\epsilon,0}, \\
Q_{Jm,2}^+ &\in \mathbb{M}_{0,2,\epsilon,0}, & R_{Jm}^+ &\in \mathbb{M}_{-1,2,\epsilon,0}, & R_{Jm}^- &\in \mathbb{M}_{-1,2,\epsilon,0}, \\
S_j^+ &\in \mathbb{M}_{0,1,\epsilon,0}, & Q_{Jm}^- &\in \mathbb{M}_{0,2,\epsilon,0}, & Q_{Jm}^- &\in \mathbb{M}_{0,2,\epsilon,0}, \\
R_{Jm}^+ &\in \mathbb{M}_{-1,2,\epsilon,0}, & S_j^- &\in \mathbb{M}_{0,1,\epsilon,0}.
\end{aligned}
\end{align}

Here and in the following, \( J \) runs from 1 through \( N \). Recall that \( j \) and \( m \) run from 1 through \( N - 1 \), respectively. Furthermore, we define \( M_{0+}(x_N), M_{1+}(x_N), M_{2+}(x_N) \), and \( M_{-}(x_N) \) as follows:
\begin{align}
M_{0+}(x_N) &= e^{-t_{2x_N}} - e^{-t_{1x_N}}, \\
M_{1+}(x_N) &= \frac{e^{-t_{1x_N}} - e^{-B_+x_N}}{t_1 - B_+} \quad (i = 1, 2), \\
M_{-}(x_N) &= \frac{e^{B_-x_N} - e^{A_{x_N}}}{B_- - A}.
\end{align}

From now on, we prove (4.26). On the other hand, we prove (4.27) in Sect 5. By (4.19), we obtain
\[
\text{div} \tilde{u}_+ = (\xi', \alpha' + \xi' \cdot \beta' + \xi' \cdot \gamma' - B_+ \alpha_{N+} + B_+ \beta_{N+} + B_+ \gamma_{N+}) e^{-B_+ x_N} \nonumber
\]
\[
+ (\xi' \cdot \beta' - t_1 \beta_{N+}) e^{-t_1 x_N} + (\xi' \cdot \beta' - t_2 \beta_{N+}) e^{-t_2 x_N}
\]
with
\[
\sum_{j=1}^{N-1} \xi_j v_j \quad \text{for} \quad \mathbf{v} = (v_1, \ldots, v_{N-1}, v_N).
\]

Then, from (4.20) and (4.21), we have
\begin{align}
\begin{aligned}
\mu_{+} + \lambda \beta_{+} + (t_1^2 - B_{+}^2) + i \xi_{j} (\xi' \cdot \beta_{+} - t_1 \beta_{N+}) (\nu_{+} + \lambda - \rho_{+} + \kappa_{+} (t_1^2 - A^2)) &= 0, \\
\mu_{+} + \lambda \gamma_{+} + (t_1^2 - B_{+}^2) + i \xi_{j} (\xi' \cdot \gamma_{+} - t_2 \gamma_{N+}) (\nu_{+} + \lambda - \rho_{+} + \kappa_{+} (t_2^2 - A^2)) &= 0, \\
\mu_{+} + \lambda \beta_{N+} + (t_2^2 - B_{-}^2) - t_1 (i \xi' \beta_{+} - t_1 \beta_{N+}) (\nu_{+} + \lambda - \rho_{+} + \kappa_{+} (t_2^2 - A^2)) &= 0, \\
\mu_{+} + \lambda \gamma_{N+} + (t_2^2 - B_{-}^2) - t_2 (i \xi' \gamma_{+} - t_2 \gamma_{N+}) (\nu_{+} + \lambda - \rho_{+} + \kappa_{+} (t_2^2 - A^2)) &= 0,
\end{aligned}
\end{align}

which furnishes that
\begin{align}
\begin{aligned}
\text{div} \tilde{u}_+ &= (\xi' \cdot \beta_{+} - t_1 \beta_{N+}) e^{-t_{1x_N}} + (\xi' \cdot \gamma_{+} - t_2 \gamma_{N+}) e^{-t_{2x_N}}, \\
\beta_{j+} &= -\frac{i \xi_j}{t_1} \beta_{N+}, \quad \gamma_{j+} = \frac{i \xi_j}{t_2} \gamma_{N+}.
\end{aligned}
\end{align}

By (4.25) and (4.22), we have
\begin{align}
\begin{aligned}
i \xi' \cdot \alpha'_{+} + i \xi' \cdot \beta_{-} + i \xi' \cdot \gamma_{+} - B_{-} \alpha_{N-} + B_{-} \beta_{N-} &= 0, \\
\mu_{-} - (B_{-}^2 - A^2) \beta_{-} + i \xi_{j} \gamma_{-} &= 0, \\
\mu_{-} - (B_{-}^2 - A^2) \beta_{N-} - A \gamma_{-} &= 0.
\end{aligned}
\end{align}

Combining (4.31) and (4.32), we deduce \( i \xi' \cdot \beta_{-} + A \beta_{N-} = 0 \). Hence, by (4.31), (4.32), and (4.33), we observe
\begin{align}
\begin{aligned}
i \xi \cdot \beta_{-} &= \frac{A}{B_{-} - A} (i \xi' \cdot \alpha'_{+} - B_{-} \alpha_{N-}), \\
\beta_{N-} &= \frac{1}{A - B_{-}} (i \xi' \cdot \alpha'_{+} + B_{-} \alpha_{N-}),
\end{aligned}
\end{align}
\( (4.36) \quad \gamma_- = -\frac{\mu_{r-}(A + B_1)}{A}(i\xi' \cdot \alpha_- + B_\alpha N_-). \)

Next, we consider the interface condition. From \((4.12)\) and \((4.10)\), we have
\[
(4.37) \quad \lambda_+ A^2 \hat{H}(0) = 2\mu_{r+} \lambda \{-B_+ \alpha N_+ - (t_1 - B_+)\beta_1 N_+ - (t_2 - B_+)\gamma_{N_+}\}
\]
\[
\quad + \{\lambda(t_+ - \mu_{r+}) - \rho_{r+}\kappa_{r+}(t_1^2 - A^2)\}(i\xi' \cdot \beta_- - t_1\beta_1 N_+)
\]
\[
\quad + \{\lambda(t_+ - \mu_{r+}) - \rho_{r+}\kappa_{r+}(t_2^2 - A^2)\}(i\xi' \cdot \gamma_- - t_2\gamma_{N_+}).
\]
Substituting \((4.28)\) and \((4.30)\) into \((4.37)\) to obtain
\[
(4.38) \quad \alpha_{N_+} = -\frac{\sigma_+ A^2 \hat{H}(0)}{2\mu_{r+} B_1} + \frac{1}{2t_1 B_1} (2t_1 B_1 - B^2_1 - A^2) \beta_{N_+} + \frac{1}{2t_2 B_1} (2t_2 B_1 - B^2_1 - A^2) \gamma_{N_+}
\]
because \(\lambda \mu_{r+} \neq 0\). In addition, by \((4.28)\) and \((4.38)\), it follows that
\[
(4.39) \quad i\xi' \cdot \alpha_- = \frac{-\sigma_+ A^2 \hat{H}(0)}{2\mu_{r+}} + \frac{A^2 - B^2_1}{2t_1} \beta_{N_+} + \frac{A^2 - B^2_1}{2t_2} \gamma_{N_+}.
\]
Together with \((4.11)\) and \((4.39)\), this shows
\[
(4.40) \quad i\xi' \cdot \alpha_- = \frac{-\sigma_+ A^2 \hat{H}(0)}{2\mu_{r+}} + i\xi' \cdot \hat{H}(0) + \frac{A^2 - B^2_1}{2t_1} \beta_{N_+} + \frac{A^2 - B^2_1}{2t_2} \gamma_{N_+}.
\]
By \((4.19)\), we have
\[
2\mu_{r-} \{B_\alpha N_- + (B_1 - A)\beta_{N_-}\} - \gamma_- = \sigma_- A^2 \hat{H}(0).
\]
Combining with \((4.39)\) and \((4.36)\), this yields
\[
(4.41) \quad \alpha_{N_+} = \frac{-\sigma_+ A^2 \hat{H}(0)}{\mu_{r-}(A + B_1)B_1} + \frac{A - B_1}{(A + B_1)B_1} i\xi' \cdot \alpha_-.
\]
From \((4.8)\), we have
\[
(4.42) \quad 0 = \mu_{r-} (B_\alpha n_- + (B_1 - A)\beta_{n_-} + i\xi_m \alpha_{N_-})
\]
\[
\quad - \mu_{r+} (-B_\alpha m_+ + (t_1 - B_1)\beta_{m_+} - (t_2 - B_1)\gamma_{m_+} + i\xi_m \alpha_{N_+}),
\]
which implies
\[
(4.43) \quad 0 = \mu_{r-} (B_\alpha i\xi' \cdot \alpha_- + (B_1 - A)i\xi' \cdot \beta_- - A^2 \alpha_{N_-})
\]
\[
\quad - \mu_{r+} (-B_\alpha i\xi' \cdot \alpha_- + (t_1 - B_1)i\xi' \cdot \beta_- - (t_2 - B_1)i\xi' \cdot \gamma_- - A^2 \alpha_{N_+}).
\]
Substituting \((4.30)\), \((4.31)\), \((4.38)\), \((4.39)\), \((4.40)\), and \((4.41)\) into \((4.43)\), we obtain
\[
(4.44) \quad t_2(Dt_1 - E)\beta_{N_+} + t_1(Dt_2 - E)\gamma_{N_+} = 2t_1 t_2 \{A^2 F \hat{H}(0) - G i\xi' \cdot \hat{h}(0)\}
\]
with
\[
(4.45) \quad D = 4\mu_{r+} A^2 B_1 B_1 (A + B_1),
\]
\[
E = \mu_{r+} B_1 (A + B_1)(A^2 + B^2_1)^2
\]
\[
\quad + \mu_{r+} B_1 (A^2 - B^2_1)(A^2 - 3AB_1 - AB_1 B_1 - B^2_1),
\]
\[
F = \frac{\sigma_+ B_1}{2\mu_{r+}} \{\mu_{r+} (A + B_1)(A^2 + B_1 B_1)
\]
\[
\quad + \mu_{r+} (B_1^2 + B^2_1 A + 3B_1 A^2 - A^3)\} - \sigma_- A^2 B_1 (B_1 - A),
\]
\[
G = \mu_{r+} B_1^2 (A + B_1) + \mu_{r+} B_1 (B_1^3 + B_1^3 + 2B_1^2 A + 3B_1 A^2 - A^3).
\]
By \((4.12)\) and \((4.17)\), we have
\[
(4.46) \quad \rho_{r+} (t_1^2 - A^2) \beta_{N_+} + \rho_{r+} (t_2^2 - A^2) \gamma_{N_+} = 0.
\]
Consequently, by \((4.34)\) and \((4.46)\), we have
\[
(4.47) \quad L \left( \frac{\beta_{N_+}}{\gamma_{N_+}} \right) = \left( \begin{array}{c} 2t_1 t_2 \{A^2 F \hat{H}(0) - G i\xi' \cdot \hat{h}(0)\} \\ 0 \end{array} \right),
\]
where

\[ L = \begin{pmatrix} t_2(Dt_1 - E) & t_1(Dt_2 - E) \\ \rho_+(t_1^2 - A^2) & \rho_+(t_2^2 - A^2) \end{pmatrix}. \]

By direct calculations, we have

(4.48) \( \det L = \rho_+(t_1 - t_2)\{E(t_1^2 + t_1t_2 + t_2^2 - A^2) - Dt_1t_2(t_1 + t_2)\}. \)

According to Saito [25], we have following formula:

\[ (A^2 + B^2)(t_1^2 + t_1t_2 + t_2^2 - A^2) - 4A^2B_+(t_1t_2(t_1 + t_2) \]

\[ = \frac{\lambda m_1(\lambda, \xi')}{t_1(t_1 + B_+)} = \frac{\lambda m_2(\lambda, \xi')}{t_2(t_2 + B_+)} =: \lambda n(\lambda, \xi') \]

with

\[ m_i = \rho_+^2 + \rho_+^{-2} \lambda \eta_i(t_i + B_+)(t_i^2 + t_1t_2 + t_2^2 - A^2) \]

\[ + 4A^2B_+ \{s_1(t_1B_+(t_1 + B_+) - (s_i - \rho_+^2 + \rho_+^{-1})t_1t_2(t_1 + t_2)\} \quad (i = 1, 2) \]

Then, we rewrite (4.48) as follows:

(4.49) \( \det L = \rho_+\lambda(t_1 - t_2)\left(\mu_++B_-(A + B_-)n(\lambda, \xi') \]

\[ - \rho_+^{-2} \lambda \eta_i(t_i + B_+)(A^2 - 3A^2B_- - AB_-^2 - B_+^2)(t_1^2 + t_1t_2 + t_2^2 - A^2) \]

\[ =: \rho_+\lambda(t_1 - t_2)I(\lambda, \xi'). \]

If \( \det L \neq 0 \), the inverse of \( L \) exists and we see

(4.50) \( \begin{pmatrix} \beta_{N^+} \\ \gamma_{N^+} \end{pmatrix} = L^{-1} \begin{pmatrix} 2t_1t_2\{A^2F \hat{H}(0) - Gi\xi' \cdot \hat{h}(0)\} \\ 0 \end{pmatrix} \)

with

(4.51) \( L^{-1} = \frac{1}{\det L} \begin{pmatrix} \rho_+(t_1^2 - A^2) & -t_1(Dt_2 + E) \\ -\rho_+(t_2^2 - A^2) & t_2(Dt_1 + E) \end{pmatrix} =: \frac{1}{\det L} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}. \)

In this section, we assume \( \det L \neq 0 \) and continue to obtain the solution formula. We shall prove \( \det L \neq 0 \) when \((\lambda, \xi') \in \Sigma_{\epsilon, \lambda_0} \) in Sect. [6]. By (4.50) and (4.51), we obtain

\[ \beta_{N^+} = \frac{2t_1t_2G_{111}i\xi' \cdot \hat{h}(0)}{\det L} + \frac{2t_1t_2A^2F \hat{H}(0)}{\det L}, \]

\[ \gamma_{N^+} = \frac{2t_1t_2G_{211}i\xi' \cdot \hat{h}(0)}{\det L} + \frac{2t_1t_2A^2F \hat{H}(0)}{\det L}. \]

Setting

(4.52) \( Q_{N^+}^{+1} = \frac{2i\xi m_1t_1G_{111}(t_1 - B_+)}{A \det L}, \quad Q_{NN^+}^{+1} = \frac{2t_1t_2F \hat{H}(0)}{\det L}, \)

\[ Q_{N^+}^{+2} = \frac{2i\xi m_1t_1G_{211}(t_2 - B_+)}{A \det L}, \quad Q_{NN^+}^{+2} = \frac{2t_1t_2F \hat{H}(0)}{\det L}, \]

with \( m = 1, \ldots, N - 1 \), we have

(4.53) \( \beta_{N^+} = \frac{A}{t_1 - B_+} \left\{ \sum_{m=1}^{N-1} Q_{Nm^+}^{+1} \hat{h}_m(0) + AQ_{NN^+}^{+1} \hat{H}(0) \right\}, \)

\[ \gamma_{N^+} = \frac{A}{t_2 - B_+} \left\{ \sum_{m=1}^{N-1} Q_{Nm^+}^{+2} \hat{h}_m(0) + AQ_{NN^+}^{+2} \hat{H}(0) \right\}. \]

From (4.30), we have

\[ \beta_{j^+} = \frac{A}{t_1 - B_+} \left\{ \sum_{m=1}^{N-1} Q_{jm^+}^{+1} \hat{h}_m(0) + AQ_{jN^+}^{+1} \hat{H}(0) \right\}, \]
\[ \gamma_{j+} = \frac{A}{t_2 - B_+} \left\{ \sum_{m=1}^{N-1} Q_{j+m,2}^+ \tilde{h}_m(0) + AQ_{j,N,2}^+ \tilde{H}(0) \right\}. \]

with
\[ Q_{j,m,1}^+ = -\frac{2\xi_m \zeta t_2 G L_{11}(t_1 - B_+)}{A \det L}, \quad Q_{j,N,1}^+ = -\frac{2i \xi t_2 F L_{11}(t_1 - B_+)}{\det L}, \]
\[ Q_{j,m,2}^+ = -\frac{2\xi_m \zeta t_1 G L_{21}(t_2 - B_+)}{A \det L}, \quad Q_{j,N,2}^+ = -\frac{2i \xi t_2 F L_{21}(t_2 - B_+)}{\det L}. \]

Furthermore, combined with (4.24), (4.30), and (4.33), we have
\[ R_{N,m}^+ = \frac{P_{m,1}^+}{(l_{11})_N, \xi}, \quad \overline{R}_{N,m}^+ = \frac{P_{m,2}^+}{(l_{11})_N, \xi}. \]

By (4.38), we have
\[ \alpha_{N+} = A \left\{ \sum_{m=1}^{N-1} \overline{R}_{N,m}^+ \tilde{h}_m(0) + A \overline{R}_{N,N}^+ \tilde{H}(0) \right\} \]
with
\[ R_{N,m}^+ = \frac{\overline{\rho}_+ s_1 s_2 i \xi_m t_1 G}{A(l_{11})_N, \xi}, \quad P_{N,1}^+ = \frac{2 \rho_+ s_1 s_2 t_1 F}{(l_{11})_N, \xi}, \]
\[ P_{N,2}^+ = \frac{2 \rho_+ s_1 s_2 (t_1 - t_2) F}{(l_{11})_N, \xi}. \]

Substituting (4.53) into (4.40) to obtain
\[ \alpha_{N-} = A \left\{ \sum_{m=1}^{N-1} R_{N,m}^- \tilde{h}_m(0) + A R_{N,N}^- \tilde{H}(0) \right\} \]
with
\[ R_{N,m}^- = \frac{\overline{\rho}_- s_1 s_2 i \xi_m t_1 G}{2(t_1 - t_2) B_+}, \quad P_{N,1}^- = \frac{A(2A^2 - B_+^2)}{2t_1(t_1 - B_+)}, \]
\[ R_{N,N}^- = \frac{\overline{\rho}_- s_1 s_2 i \xi_m t_1 G}{2(t_1 - t_2) B_+}. \]

Substituting (4.55) and (4.56) into (4.57) and (4.58), we have
\[ \gamma_+ = \sum_{m=1}^{N-1} P_{m,1}^+ \tilde{h}_m(0) + AP_{N,2}^+ \tilde{H}(0), \]
\[ \gamma_- = \sum_{m=1}^{N-1} P_{m,2}^- \tilde{h}_m(0) + AP_{N,2}^- \tilde{H}(0), \]
respectively. Here we set
\[
Q_{Nm}^- = -\frac{1}{A + B_-} \left( \frac{A^2}{2t_1(t_1 - B_+)} Q_{Nm,1}^+ + \frac{A^2}{2t_2(t_2 - B_+)} Q_{Nm,2}^+ + i\xi_m \right),
\]
\[
Q_{NN}^- = -\frac{1}{A + B_-} \left( \frac{A^2}{2t_1(t_1 - B_+)} Q_{NN,1}^+ + \frac{A^2}{2t_2(t_2 - B_+)} Q_{NN,2}^+ - \frac{\sigma_A}{2\mu_+} \right) - \frac{a_A}{\mu_-(A + B_-)},
\]
\[
P_m^- = -\mu_-(A + B_-) Q_{Nm}^-,\]
\[
P_N^- = -\mu_-(A + B_-) Q_{NN}^-,
\]
for short. From (4.32) and (4.58), we have
\[
(4.59) \quad \beta_{j-} = \frac{A}{B_- - A} \left\{ \sum_{m=1}^{N-1} Q_j m \hat{h}_m(0) + AQ_{jN}^- \hat{H}(0) \right\}
\]
with
\[
Q_j m = -\frac{i\xi_j}{A} Q_{Nm}^-, \quad Q_{jN}^- = -\frac{i\xi_j}{A} Q_{NN}^-.
\]
Accordingly, by (4.11), (4.42), and (4.59), we obtain
\[
\alpha_{j+} = A \left\{ \sum_{m=1}^{N-1} R_j m \hat{h}_m(0) + AR_{jN}^- \hat{H}(0) \right\} + S_j \hat{h}_j,
\]
\[
\alpha_{j-} = A \left\{ \sum_{m=1}^{N-1} R_j m \hat{h}_m(0) + AR_{jN}^- \hat{H}(0) \right\} + S_j \hat{h}_j,
\]
with
\[
R_j m = -(\mu_+ B_+ + \mu_- B_-)^{-1} \left( (\mu_+ + AQ_{jN,1}^- + AQ_{jN,2}^- - 3i\xi_j R_{jN}^+ ) + \mu_-(Q_{jm} + \xi_j R_{jN}^-) \right),
\]
\[
R_j N = -(\mu_+ B_+ + \mu_- B_-)^{-1} \left( (\mu_+ + AQ_{jN,1}^- + AQ_{jN,2}^- - 3i\xi_j R_{jN}^+ ) + \mu_-(Q_{jN} + \xi_j R_{jN}^-) \right),
\]
\[
S_j = -\mu_+ B_+ (\mu_+ + B_+ + \mu_- B_-)^{-1}, \quad R_j m = R_j m, \quad R_j N = R_j N,
\]
This completes the proof of (4.26).

To prove Theorem 3.3, we consider problem (3.3), namely, problem (4.1) with $H = 0$. First of all, we define our solution operators $A_{j1}^{\pm} (\lambda)$ ($i = 1, 2, 3, 4$), $A_{j2}^{\pm} (\lambda)$ ($i = 1, 2, 3$), $B_1^{\pm} (\lambda)$, and $P_1^{\pm} (\lambda)$ of problem (4.26) such that
\[
u_i^{+} = A_{j1}^{\pm} (\lambda) h \quad \text{on } \mathbb{R}_+^N \quad (i = 1, 2, 3, 4),
\]
\[
u_i^{+} = A_{j1}^{\pm} (\lambda) h \quad \text{on } \mathbb{R}_-^N \quad (i = 1, 2, 3),
\]
\[
\rho_+ = B_1^{\pm} (\lambda) h \quad \text{on } \mathbb{R}_+^N,
\]
\[
\rho_- = P_1^{\pm} (\lambda) h \quad \text{on } \mathbb{R}_-^N
\]
with
\[
A_{j1}^{\pm} (\lambda) h = \sum_{m=1}^{N-1} F_{\xi_j}^{-1} \left[ Q_{j1,1}^{\pm} A M_{11} + (x_N) \hat{h}_m(0) \right](x'),
\]
\[
A_{j2}^{\pm} (\lambda) h = \sum_{m=1}^{N-1} F_{\xi_j}^{-1} \left[ Q_{j1,2}^{\pm} A M_{12} + (x_N) \hat{h}_m(0) \right](x'),
\]
\[
A_{j2}^{\pm} (\lambda) h = \sum_{m=1}^{N-1} F_{\xi_j}^{-1} \left[ R_{j1,m} A e^{-B_+ x_N} \hat{h}_m(0) \right](x'),
\]
\[
A_{j2}^{\pm} (\lambda) h = \sum_{m=1}^{N-1} F_{\xi_j}^{-1} \left[ S_j^+ e^{-B_+ x_N} \hat{h}_j(0) \right](x'),
\]
\[
A_{j2}^{\pm} (\lambda) h = 0,
\]
\[ A_{j1}^3(\lambda) h = \sum_{m=1}^{N-1} F_{\xi'}^{-1} \left[ Q_m \lambda AM_-(x_N) \tilde{h}_m(0) \right] (x'), \]
\[ A_{j2}^3(\lambda) h = \sum_{m=1}^{N-1} F_{\xi'}^{-1} \left[ R_m \lambda e^{B_{x_N}} \tilde{h}_m(0) \right] (x'), \]
\[ A_{j3}^3(\lambda) h = F_{\xi'}^{-1} \left[ S_j e^{B_{x_N}} \tilde{h}_j(0) \right] (x'), \]
\[ A_{N}^3(\lambda) h = 0, \]
\[ B_{\xi}^3(\lambda) h = \sum_{m=1}^{N-1} \left[ P_m^+ AM_0 + (x_N) \tilde{h}_m(0) \right] (x') + F_{\xi'}^{-1} \sum_{m=1}^{N-1} \left[ P_m^+ e^{-t_{1x_N}} \tilde{h}_m(0) \right] (x'), \]
\[ P_{\xi}^3(\lambda) h = \sum_{m=1}^{N-1} F_{\xi'}^{-1} \left[ P_m \lambda e^{A_{x_N}} \tilde{h}_m(0) \right] (x'). \]

In order to prove Theorem 3.3, we introduce following lemma and corollary.

**Lemma 4.4.** Let \( 1 < q < \infty, \lambda_0 \geq 0, \varepsilon_0 < \varepsilon < \pi/2. \) Assume that \( \rho_{s+} \neq \rho_{s-}, \eta_{s} \neq 0, \) and \( \kappa_{s+} \neq \mu_{s+} + \nu_{s+}. \) For \( m(\lambda, \xi') \in M_{0,2,\varepsilon,\lambda_0}, i = 0, 1, 2, 3, \) and \( j = 1, 2, \) we define operators \( J_i(\lambda), K_i(\lambda), \) and \( L_j(\lambda) \) by

\[
\begin{align*}
|J_i(\lambda)f(x)| &= \int_0^\infty F_{\xi'}^{-1} \left[ m(\lambda, \xi') (x + y_N) \tilde{f}(\xi', y_N) \right] (x') dy_N, \\
|K_i(\lambda)f(x)| &= \int_0^\infty F_{\xi'}^{-1} \left[ m(\lambda, \xi') (x + y_N) \tilde{f}(\xi', y_N) \right] (x') dy_N, \\
|L_j(\lambda)f(x)| &= \int_0^\infty F_{\xi'}^{-1} \left[ m(\lambda, \xi') A_{x_N} \tilde{f}(\xi', y_N) \right] (x') dy_N.
\end{align*}
\]

Then, for \( i = 0, 1, 2, j = 1, 2, \) and \( s = 0, 1, \) the sets \( \{(\tau \partial_{r})^s J_i(\lambda)\}, \{(\tau \partial_{r})^s K_i(\lambda)\}, \) and \( \{(\tau \partial_{r})^s L_j(\lambda)\} \) are \( R \)-bounded families in \( L_q(R_N^+), \) whose \( R \)-bounds do not exceed some constant \( C_{N,q,\lambda_0,\varepsilon,\mu_{s+},\nu_{s+},\kappa_{s+}} \) depending essentially only on \( N, q, \lambda_0, \varepsilon, \mu_{s+}, \nu_{s+}, \) and \( \kappa_{s+}. \)

**Proof.** First we consider \( K_0(\lambda). \) Setting

\[ k_{0,\lambda}(x) = F_{\xi'}^{-1} \left[ m(\lambda, \xi') A^2 M_0 + (x_N) \right] (x'), \]

we have

\[ |K_0(\lambda)f(x)| = \int_{R_N^+} k_{0,\lambda}(x' - y', x_N + y_N) f(y) dy. \]

Employing the same argumentation due to Shibata and Shimizu [31, Lemma 5.4], it is sufficient to prove

\[ |(\tau \partial_{r})^s k_{0,\lambda}(x)| \leq C_{N,\lambda_0,\varepsilon,\mu_{s+},\nu_{s+},\kappa_{s+}} |x|^{-N} \quad (s = 0, 1). \]

According to Saito [25, Lemma 4.8], we have

\[ |(\tau \partial_{r})^s \{m(\lambda, \xi') A^2 M_0 + (x_N)\}| \leq C_{x_N} (|\lambda|^{1/2} + A)^{-|\alpha'|} e^{-b(|\lambda|^{1/2} + A) x_N} \quad (s = 0, 1). \]

Here and in the sequel, \( b \) is a positive constant depending on \( \varepsilon, \mu_{s+}, \nu_{s+}, \) and \( \kappa_{s+}. \) On the other hand, \( C \) is a positive constant depending on \( \alpha', \lambda_0, \varepsilon, \mu_{s+}, \nu_{s+}, \) and \( \kappa_{s+}. \) Then, by the Leibniz rule and the assumption we have

\[ |(\tau \partial_{r})^s \{m(\lambda, \xi') A^2 M_0 + (x_N)\}| \leq C_{\alpha',\lambda_0,\varepsilon,\mu_{s+},\nu_{s+},\kappa_{s+}} A^{1-|\alpha'|} e^{-b(|\lambda|^{1/2} + A) x_N} \quad (s = 0, 1), \]

which, combined with Theorem 3.6 in Shibata and Shimizu [31], furnishes that

\[ |(\tau \partial_{r})^s k_{0,\lambda}(x)| \leq C_{N,\lambda_0,\varepsilon,\mu_{s+},\nu_{s+},\kappa_{s+}} |x'|^{-N} \quad (s = 0, 1). \]
On the other hand, using (4.61) with $s = 0$ and $\alpha' = 0$, for $s = 0, 1$, we have
\[
|\tau \partial_x^s k_{0\lambda}(x)| \leq C_{\lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} \left( \frac{1}{2\pi} \right)^{N-1} \int_{\mathbb{R}^{N-1}} A e^{-(b/2)A x_N} d\xi'
\leq |x_N|^{-N} C_{\lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} \left( \frac{1}{2\pi} \right)^{N-1} \int_{\mathbb{R}^{N-1}} |\eta'| e^{-(b/2)|\eta'|} d\eta',
\]
which, combined with (4.61), implies (4.60). Thus this completes the case $K_0(\lambda)$.

Next we consider $K_1(\lambda)$ and $K_2(\lambda)$. By the identities:
\[
(4.62) \quad B_2^2 = A^2 + \frac{\rho_+ \lambda}{\mu_+} - \sum_{m=1}^{N-1} (i \xi_m)^2 + \frac{\rho_+ \lambda}{\mu_+},
\]
we have for $j = 1, 2$
\[
M_j + (x_N) = \tau_j(\lambda, \xi') \left( e^{t_j x_N} - e^{B x_N} \right), \quad \tau_j(\lambda, \xi') = \left( \frac{s_j - \rho_+ \mu_+^{-1}}{s_j - \rho_+}(t_1 + t_2) \right).
\]
Since $M_j + (x_N) = -\tau_j(\lambda, \xi') x_N \int_0^1 e^{-(\theta t_j + (1-\theta)B) x_N} d\theta$ and $\tau_j(\lambda, \xi') \in \mathcal{M}_{0, 1, \varepsilon, 0}$ by (5.6) below, we can prove the required properties in the same manner as the case $K_0(\lambda)$. In addition, we can prove the case $J_j(\lambda) (i = 0, 1, 2)$ in the same manner as the case $K_0(\lambda)$, so that we may omit those proof.

Then, we consider $L_1(\lambda)$. If we set
\[
l_{1, \lambda}(x) = \mathcal{F}_{\xi'}^{-1} [m(\lambda, \xi') A e^{-t_1 x_N}](x'),
\]
the operator $L_1(\lambda)$ is given by the formula:
\[
[L_1(\lambda)f](x) = \int_{\mathbb{R}^N} l_{1, \lambda}(x' - y', x_N + y_N) f(y) dy,
\]
so that to prove that $L_1$ has the required properties it is sufficient to prove
\[
|\tau \partial_x^s l_{1, \lambda}(x)| \leq C_{N, \lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} |x|^{-N} \quad (s = 0, 1).
\]
According to Saito [25, Lemma 4.5] and Shibata and Shimizu [31, Lemma 5.3], we have
\[
|\partial_x^s \tau \partial_x^s (e^{-t x_N})| \leq C(|\lambda|^{1/2} + A)^{-|\alpha'|} e^{-b(|\lambda|^{1/2} + A)x_N} \quad (s = 0, 1).
\]
Then, by the Leibniz rule and the assumption we have
\[
|\partial_x^s \tau \partial_x^s m(\lambda, \xi') A e^{-t_1 x_N}|
\leq C_{\alpha', \lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} A^{1-|\alpha'|} e^{-b(|\lambda|^{1/2} + A)x_N},
\]
which, combined with Theorem 3.6 in Shibata and Shimizu [31], furnishes that
\[
|\tau \partial_x^s l_{1, \lambda}(x)| \leq C_{N, \lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} |x|^{-N} \quad (s = 0, 1).
\]
On the other hand, using (4.64) with $s = 0$ and $\alpha' = 0$, for $s = 0, 1$, we have
\[
|\tau \partial_x^s l_{1, \lambda}(x)| \leq C_{\lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} \left( \frac{1}{2\pi} \right)^{N-1} \int_{\mathbb{R}^{N-1}} A e^{-bA x_N} d\xi'
\leq |x_N|^{-N} C_{\lambda_0, \varepsilon, \mu_+, \nu_+, \kappa_+} \left( \frac{1}{2\pi} \right)^{N-1} \int_{\mathbb{R}^{N-1}} |\eta'| e^{-b|\eta'|} d\eta',
\]
which, combined with (4.64), implies (4.63). Hence, this completes the case $L_1(\lambda)$. Furthermore, we can prove the case $L_2(\lambda)$ in the same manner as the case $L_1(\lambda)$, so that we omit its proof.

**Corollary 4.5.** Let $1 < q < \infty$ and $\varepsilon < \varepsilon < \pi/2$. Assume that $\rho_+ \neq \rho_{-\varepsilon}$, $\eta_+ \neq 0$, and $\kappa_+ \neq \mu_++\nu_+$. Set $N_1 = N + 1$ and $N_2 = N^2 + N + 1$. 


(1) Let \( r = 1, 2 \). For \( k_r,1(\lambda, \xi') \in M_{r-3,2,\varepsilon,0} \) and \( l_r(\lambda, \xi') \in M_{r-4,2,\varepsilon,0} \), we define operators \( K_0(\lambda) \), \( L_j^r(\lambda) \) \((j = 1, 2)\) by

\[
\begin{align*}
[K_0^r(\lambda)f](x) &= \mathcal{F}^{-1}_{\xi'}[k_r,1(\lambda, \xi') AM_0+(x_N)\tilde{f}(\xi', 0)](x'), \\
[L_j^r(\lambda)f](x) &= \mathcal{F}^{-1}_{\xi'}[l_r(\lambda, \xi') Ae^{-t_1 x_N}\tilde{f}(\xi', 0)](x') \quad (j = 1, 2),
\end{align*}
\]

for \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \), and \( f \in W_q^r(\mathbb{R}_+^N) \). Then, there exist operators \( \tilde{K}_0^r(\lambda), \tilde{L}_j^r(\lambda) \), with

\[
\tilde{K}_0^r(\lambda), \tilde{L}_j^r(\lambda) \in \text{Anal} (\Sigma_{\varepsilon, \lambda_0}, \mathcal{L}(L_q(\mathbb{R}_+^N), W_0^3(\mathbb{R}_+^N))),
\]

such that for any \( g \in W_q^1(\mathbb{R}_+^N) \) and any \( h \in W_q^2(\mathbb{R}_+^N) \)

\[
\begin{align*}
K_0^r(\lambda)g &= \tilde{K}_0^r(\lambda)(\lambda^{1/2}g, \nabla g), & K_0^r(\lambda)h &= \tilde{K}_0^r(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h), \\
L_j^r(\lambda)g &= \tilde{L}_j^r(\lambda)(\lambda^{1/2}g, \nabla g), & L_j^r(\lambda)h &= \tilde{L}_j^r(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h) \quad (j = 1, 2).
\end{align*}
\]

Furthermore, for \( s = 0, 1 \), \( j = 1, 2 \)

\[
\begin{align*}
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^N, L_q(\mathbb{R}_+^N)^N)}\left \{ (\tau \nabla)^t g \mathcal{A} \tilde{K}_0^r(\lambda) \right \} \leq C, \\
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^N, L_q(\mathbb{R}_+^N)^N)}\left \{ (\tau \nabla)^t g \mathcal{A} \tilde{L}_j^r(\lambda) \right \} \leq C,
\end{align*}
\]

with a positive constant \( C = C_{N, q, \varepsilon, \lambda_0, \mu_r, \nu_r, \nu_\varepsilon, \nu_\lambda, \nu_\varepsilon, \nu_\lambda} \).

(2) Let \( i, r = 1, 2 \). For \( k_r,2(\lambda, \xi') \in M_{r-2,2,\varepsilon,0} \), we define operators \( K_0^r(\lambda) \) by

\[
[K_0^r(\lambda)f](x) = \mathcal{F}^{-1}_{\xi'}[k_r,2(\lambda, \xi') AM_1+(x_N)\tilde{f}(\xi', 0)](x'),
\]

for \( \lambda \in \Sigma_{\varepsilon, 0} \), and \( f \in W_q^r(\mathbb{R}_+^N) \). Then, there exist operators \( \tilde{K}_0^r(\lambda), \tilde{L}_j^r(\lambda) \), with

\[
\tilde{K}_0^r(\lambda), \tilde{L}_j^r(\lambda) \in \text{Anal} (\Sigma_{\varepsilon, 0}, \mathcal{L}(L_q(\mathbb{R}_+^N), W_0^3(\mathbb{R}_+^N))),
\]

such that for any \( g \in W_q^1(\mathbb{R}_+^N) \) and any \( h \in W_q^2(\mathbb{R}_+^N) \)

\[
\begin{align*}
K_1^r(\lambda)g &= \tilde{K}_1^r(\lambda)(\lambda^{1/2}g, \nabla g), & K_1^r(\lambda)h &= \tilde{K}_1^r(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h), \\
L_j^r(\lambda)g &= \tilde{L}_j^r(\lambda)(\lambda^{1/2}g, \nabla g), & L_j^r(\lambda)h &= \tilde{L}_j^r(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h) \quad (j = 1, 2).
\end{align*}
\]

Furthermore, for \( s = 0, 1 \),

\[
\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N)^N, L_q(\mathbb{R}_+^N)^N)}\left \{ (\tau \nabla)^t g \mathcal{A} \tilde{K}_0^r(\lambda) \right \} \leq C,
\]

with a positive constant \( C = C_{N, q, \varepsilon, \mu_r, \nu_r, \nu_\varepsilon, \nu_\lambda} \).

Proof. We only prove the case \( K_0^r(\lambda) \). The proof of other cases are similar to \( K_0^r(\lambda) \), so that we may omit the detailed proof (cf. Saito [20] Corollary 4.9] and Shibata and Shimizu [31] Lemma 5.6, Lemma 5.7).

By the definition of \( M_{0+}(x_N) \), we have

\[
\partial_N M_{0+}(x_N) = -t_2 M_{0+}(x_N) - e^{-t_1 x_N},
\]

which, combined with \((4.02)\), implies that

\[
\begin{align*}
K_0^r(\lambda)g &\quad = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[ \frac{\mu_+ + \lambda^{1/2}t_2 k_{1,1}(\lambda, \xi')}{\mu_+ + B_+^2} AM_{0+}(x_N + y_N)\tilde{\lambda}(\lambda^{1/2}g(\lambda, \xi')) \right](x') \, dy_N \\
&\quad + \sum_{m=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[ \frac{ik_m k_{1,1}(\lambda, \xi')}{B_+^2} AM_{0+}(x_N + y_N)\tilde{\partial}_m g(\xi', y_N) \right](x') \, dy_N \\
&\quad + \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[ \frac{\mu_+ + \lambda^{1/2}k_{1,1}(\lambda, \xi')}{\mu_+ + B_+^2} Ae^{-t_1(x_N + y_N)}\tilde{\lambda}(\lambda^{1/2}g(\xi', y_N)) \right](x') \, dy_N \\
&\quad - \sum_{m=1}^{N-1} \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[ \frac{ik_m k_{1,1}(\lambda, \xi')}{B_+^2} Ae^{-t_1(x_N + y_N)}\tilde{\partial}_m g(\xi', y_N) \right](x') \, dy_N \\
&\quad - \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[ k_{1,1}(\lambda, \xi') AM_{0+}(x_N + y_N)\tilde{\partial}_N g(\xi', y_N) \right](x') \, dy_N
\end{align*}
\]
\[\begin{align*}
&= \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\rho_{+}^2 \lambda_2 k_{1,1}(\lambda, \xi')}{\mu_{+}^2 + B_{+}^2} \lambda^{1/2} A M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N \\
&+ \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\rho_{+} \lambda^{1/2} m k_{2} k_{r,1}(\lambda, \xi')}{\mu_{+} + B_{+}^2} \lambda^{1/2} A M_{0+}(x_N + y_N) \lambda^{1/2} \partial \overrightarrow{m} g(\xi', y_N) \right] (x') \, dy_N \\
&+ \sum_{m=1}^{N-1} \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\rho_{+} \lambda^{1/2} m k_{2} k_{r,1}(\lambda, \xi')}{\mu_{+} + B_{+}^2} \lambda^{1/2} A M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N \\
&+ \sum_{m=1}^{N-1} \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{i \xi_m \lambda^{1/2} m k_{2} k_{r,1}(\lambda, \xi')}{\mu_{+} + B_{+}^2} \lambda^{1/2} A M_{0+}(x_N + y_N) \lambda^{1/2} \partial \overrightarrow{m} g(\xi', y_N) \right] (x') \, dy_N \\
&+ \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\rho_{+} \lambda^{1/2} k_{1,1}(\lambda, \xi')}{\mu_{+} + B_{+}^2} A e^{-t_1(x_N + y_N)} \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N \\
&- \sum_{m=1}^{N-1} \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{i \xi_m \lambda^{1/2} k_{1,1}(\lambda, \xi')}{B_{+}^2} A e^{-t_1(x_N + y_N)} \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N \\
&- \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\rho_{+} \lambda^{1/2} k_{1,1}(\lambda, \xi')}{\mu_{+} + B_{+}^2} \lambda^{1/2} A M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N \\
&- \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{A k_{1,1}(\lambda, \xi')}{B_{+}^2} A^{2} M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N \\
&= : \sum_{l=1}^{8} \tilde{K}_{0, l}(\lambda)(\lambda^{1/2}, \nabla g).
\end{align*}\]

Here, we use Volovich’s formula \[34\] :

\[f(\xi', x_N) \tilde{g}(0) = - \int_{0}^{\infty} \{ (\partial_N f)(\xi', x_N + y_N) \tilde{g}(y_N) + f(\xi', x_N + y_N) \partial \overrightarrow{n} g(\xi', y_N) \} \, dy_N\]

with \(f(\xi', x_N + y_N) \tilde{g}(y_N) \to 0\) as \(y_N \to \infty\).

First, we estimate \(\tilde{K}_{0, 0}(\lambda)\) (\(n = 1, 2, 3, 4, 7, 8\)). Here, we consider \(\tilde{K}_{0, 0}(\lambda)\) only, because we can treat other terms similarly. Let \(j, k, l = 1, \ldots, N - 1\). By (4.69), \(\tilde{K}_{0, 0}(\lambda)\) can be written as

\[\begin{align*}
\lambda \tilde{K}_{0, 0}(\lambda)(\lambda^{1/2}, \nabla g) &= \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\lambda A k_{1,1}(\lambda, \xi')}{B_{+}^2} A^{2} M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N, \\
(\lambda \partial_j, \lambda^{1/2} \partial_j \partial_k, \partial_j \partial_k \partial_l) \tilde{K}_{0, 0}(\lambda)(\lambda^{1/2}, \nabla g) &= \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\lambda A k_{1,1}(\lambda, \xi')}{B_{+}^2} A^{2} M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N, \\
(\lambda, \lambda^{1/2} \partial_j, \partial_j \partial_k, \partial_j \partial_k \partial_l) \tilde{K}_{0, 0}(\lambda)(\lambda^{1/2}, \nabla g) &= \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\lambda A k_{1,1}(\lambda, \xi')}{B_{+}^2} A^{2} M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N, \\
(\lambda^{1/2}, \partial_j) \partial_j^2 \tilde{K}_{0, 0}(\lambda)(\lambda^{1/2}, \nabla g) &= \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\lambda A k_{1,1}(\lambda, \xi')}{B_{+}^2} A^{2} M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N, \\
(\lambda^{1/2}, \partial_j) \partial_j^2 \tilde{K}_{0, 0}(\lambda)(\lambda^{1/2}, \nabla g) &= \int_{0}^{\infty} \mathcal{F}^{-1}_{x'} \left[ \frac{\lambda A k_{1,1}(\lambda, \xi')}{B_{+}^2} A^{2} M_{0+}(x_N + y_N) \lambda^{1/2} / (\lambda, \xi') \right] (x') \, dy_N.
\end{align*}\]
below, we have for \( (4.67) \)

\[
- \int_0^\infty \mathcal{F}_{\xi}^{-1} \left[ \frac{(\lambda^{1/2}, i\xi_j)(t_1 + t_2)A^2 k_{1,1}(\lambda, \xi')}{B_+^2} \right] A e^{-t_1(x_N + y_N)\bar{\partial}_N g(\xi', y N)} (x') \, dyN,
\]

\[
\bar{\partial}_N^1 \bar{K}_{0,8}^1 (\lambda)(\lambda^{1/2} g, \nabla g)
\]

\[
= \int_0^\infty \mathcal{F}_{\xi}^{-1} \left[ \frac{t_3^2 A k_{1,1}(\lambda, \xi')}{B_+^2} \right] A^2 M_0^+ (x_N + y_N)\bar{\partial}_N g(\xi', y N) (x') \, dyN
\]

\[
+ \int_0^\infty \mathcal{F}_{\xi}^{-1} \left[ \left( t_1^2 + t_1 t_2 + \frac{t_1^2}{2} t_2 \right) A^2 k_{1,1}(\lambda, \xi') \right] A e^{-t_1(x_N + y_N)\bar{\partial}_N g(\xi', y N)} (x') \, dyN.
\]

Then, by Lemma 4.2 the assumption for \( k_{1,1}(\lambda, \xi') \), and \( (4.65) \) below, we have

\[
\frac{\lambda A k_{1,1}(\lambda, \xi')}{B_+^2} \in \mathbb{M}_{-1,2,\varepsilon,0} \subset \mathbb{M}_{0,2,\varepsilon,\lambda_0},
\]

\[
\frac{(\lambda^{1/2}, i\xi_j, -\lambda^{1/2} x_N k_{1,1}(\lambda, \xi'))}{B_+^2} \in \mathbb{M}_{0,2,\varepsilon,0},
\]

\[
\frac{(\lambda, \lambda^{1/2} i\xi_j, -\lambda^{1/2} x_N k_{1,1}(\lambda, \xi'))}{B_+^2} \in \mathbb{M}_{0,2,\varepsilon,0},
\]

\[
\frac{\lambda^{1/2}, i\xi_j, -\lambda^{1/2} x_N k_{1,1}(\lambda, \xi')}{B_+^2} \in \mathbb{M}_{0,2,\varepsilon,0},
\]

\[
\frac{(\lambda^{1/2}, i\xi_j)(t_1 + t_2)A^2 k_{1,1}(\lambda, \xi')}{B_+^2} \in \mathbb{M}_{0,2,\varepsilon,0},
\]

\[
\frac{t_3^2 A k_{1,1}(\lambda, \xi')}{B_+^2} \in \mathbb{M}_{0,2,\varepsilon,0},
\]

\[
\frac{(t_1^2 + t_1 t_2 + \frac{t_1^2}{2} t_2) A^2 k_{1,1}(\lambda, \xi')}{B_+^2} \in \mathbb{M}_{0,2,\varepsilon,0}.
\]

Combining these properties with Lemma 4.3 furnishes for \( s = 0, 1 \)

\[
(4.66) \quad \mathcal{R}_c \left( L_q (\mathbb{R}^N)^{N_1}, L_q (\mathbb{R}^N)^{N_3+N_2} \times W_2^q (\mathbb{R}^N) \right) \left( (\tau \partial_\tau) \ast (G_s^2 \bar{K}_{0,8}^1 (\lambda)) \mid \lambda \in \Sigma_{c,\lambda_0} \right) \leq C,
\]

with some positive constant \( C = C_{N,q,\varepsilon,\lambda_0,\mu_{\varepsilon,\lambda_0},\kappa_{\varepsilon,\lambda_0}} \). Analogously, we have

\[
(4.67) \quad \mathcal{R}_c \left( L_q (\mathbb{R}^N)^{N_1}, L_q (\mathbb{R}^N)^{N_3+N_2} \times W_2^q (\mathbb{R}^N) \right) \left( (\tau \partial_\tau) \ast (G_s^2 \bar{K}_{0,n_1}^1 (\lambda)) \mid \lambda \in \Sigma_{c,\lambda_0} \right) \leq C
\]

for \( n_1 = 1, 2, 3, 4, 7 \).

Next, we estimate \( \bar{K}_{0,n_2}^1 (\lambda) \) \((n_2 = 5, 6)\). By Lemma 4.2, the assumption for \( k_{1,1}(\lambda, \xi') \), and \( (5.5) \) below, we have for \( m = 1, \ldots, N - 1 \)

\[
\frac{\rho_+ + \lambda^{1/2} k_{1,1}(\lambda, \xi')}{\mu_+ B_+^2}, \quad \frac{i\xi_m k_{1,1}(\lambda, \xi')}{B_+^2} \in \mathbb{M}_{-3,2,\varepsilon,0}.
\]

Accordingly, if we employ the same argument as in proving \( (4.66) \), we obtain

\[
\mathcal{R}_c \left( L_q (\mathbb{R}^N)^{N_1}, L_q (\mathbb{R}^N)^{N_3+N_2} \times W_2^q (\mathbb{R}^N) \right) \left( (\tau \partial_\tau) \ast (G_s^2 \bar{K}_{0,n_2}^1 (\lambda)) \mid \lambda \in \Sigma_{c,\lambda_0} \right) \leq C
\]

for \( s = 0, 1 \) and \( n_2 = 3, 4 \), with \( C = C_{N,q,\varepsilon,\lambda_0,\mu_{\varepsilon,\lambda_0},\kappa_{\varepsilon,\lambda_0}} \), which, combined with \( (4.66) \) and \( (4.67) \), complete the proof. \( \square \)

Employing the argument in Shibata [28 Sect. 4], by \( (4.26) \) and \( (4.27) \), there exist operator families \( \overline{A}_{j+1}^j(\lambda) \) \((j = 1, 2, 3, 4)\), \( \overline{A}_{j+1}^j(\lambda) \) \((j = 1, 2, 3)\), and \( \bar{P}_s^3(\lambda) \) such that \( \bar{A}^3_1(\lambda) h = \bar{A}^3_1(\lambda) \bar{H}, \quad \bar{A}^3_2(\lambda) h = \bar{A}^3_2(\lambda) \bar{H}, \quad \bar{P}_s^3(\lambda) h = \bar{P}_s^3(\lambda) \bar{H} \), respectively. Using \( (4.26) \), \( (4.27) \), and Corollary 4.5, there exists operator family \( \bar{B}_s^3(\lambda) \) such that \( \bar{B}_s^3(\lambda) h = \bar{B}_s^3(\lambda) \bar{H} \).
Consequently, if we define operators $\mathcal{A}_+^2(\lambda)$, $\mathcal{B}_+^2(\lambda)$, and $\mathcal{P}_+^2(\lambda)$ by

$$
\mathcal{A}_+^2(\lambda)\mathbf{H} = \sum_{i=1}^4 (A_{i1}^2(\lambda)\mathbf{H}, \ldots, A_{iN}^2(\lambda)\mathbf{H}),
$$

$$
\mathcal{A}_-^2(\lambda)\mathbf{H} = \sum_{i=1}^3 (A_{i1}^2(\lambda)\mathbf{H}, \ldots, A_{iN}^2(\lambda)\mathbf{H}),
$$

$$
\mathcal{B}_+^2(\lambda)\mathbf{H} = \mathcal{B}_+^2(\lambda)\mathbf{H},
$$

$$
\mathcal{P}_+^2(\lambda)\mathbf{H} = \mathcal{P}_+^2(\lambda)\mathbf{H},
$$

respectively, from (4.2) we have

$$
u_+ = A_+^2(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h),
$$

$$
\rho_+ = B_+^2(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h),
$$

$$
\pi_- = \mathcal{P}_+^2(\lambda)(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h).
$$

Combining Corollary 4.5 and the argument in Shibata [28 Sect. 4], we have (3.3), so that we have completed the proof of Theorem 3.3.

### 5. Analysis of Multipliers

In this section, we estimate several multipliers. To this end, we start with the following wildly known estimate:

$$
|\alpha\lambda + \beta| \geq \left( \sin \frac{\xi}{2} \right)(\alpha|\lambda| + \beta)
$$

for any $\lambda \in \Sigma_\epsilon$ and positive numbers $\alpha$ and $\beta$.

First, we estimate $B_+^2$ and $(\mu_+ B_+ + \mu_- B_-)$. For this purpose, we use the estimates:

$$
c_1(|\lambda|^{1/2} + A) \leq \text{Re } B_\pm \leq |B_\pm| \leq c_2(|\lambda|^{1/2} + A)
$$

for any $(\lambda, \xi) \in \tilde{\Sigma}_{\epsilon,0}$ with some positive constant $c_1$ and $c_2$, which immediately follows from (5.1). Here and in the sequel, $c_1$ and $c_2$ denote some positive constants essentially depending on $\epsilon$, $\mu_\pm$, $\nu_\pm$, $\kappa_\pm$, and $\rho_\pm$. In particular, we have

$$
c_1(|\lambda|^{1/2} + A) \leq \text{Re } (\mu_+ B_+ + \mu_- B_-)
$$

$$
\leq |(\mu_+ B_+ + \mu_- B_-)| \leq c_2(|\lambda|^{1/2} + A)
$$

for any $(\lambda, \xi) \in \tilde{\Sigma}_{\epsilon,0}$. As shown in Enomoto and Shibata [10 Lemma 4.3], using (5.2), (5.3) and the Bell’s formula:

$$
\partial_{\xi'}^\alpha f(g(\xi')) = \sum_{|\alpha'| \geq 1} f^{(i)}(g(\xi')) \sum_{\alpha_1 + \cdots + \alpha_l = \alpha'} \Gamma_{\alpha_1, \ldots, \alpha_l}(\partial_{\xi'}^{\alpha_1} g(\xi')) \cdots (\partial_{\xi'}^{\alpha_l} g(\xi'))
$$

with suitable coefficients $\Gamma_{\alpha_1, \ldots, \alpha_l}$, where $f^{(i)}(t) = d^i f(t)/dt^i$, we see that

$$
(M_i)^s \in \mathbb{M}_{s, 1, \epsilon, 0} \quad (M_1 = B_\pm, \mu_+ B_+ + \mu_- B_-).
$$

Second, we estimate $(t_i)^s$, $t_i + B_+$, and $t_i B_+ + A^2 (i = 1, 2)$. As seen in Saito [24], we have the following lemma.

**Lemma 5.1.** Let $i = 1, 2$. Then, the roots $s_i$ of (4.14) are given by

$$
s_1 = \begin{cases} 
\frac{\mu_+ + \nu_+}{2\kappa_+} + \sqrt{\eta_*} & (\eta_* > 0), \\
\frac{\mu_+ + \nu_+}{2\kappa_+} + i\sqrt{|\eta_*|} & (\eta_* < 0),
\end{cases}
$$

$$
s_2 = \begin{cases} 
\frac{\mu_+ + \nu_+}{2\kappa_+} - \sqrt{\eta_*} & (\eta_* > 0), \\
\frac{\mu_+ + \nu_+}{2\kappa_+} - i\sqrt{|\eta_*|} & (\eta_* < 0)
\end{cases}
$$

for any $(\eta_*)$.
with
\[ \eta_* = \left( \frac{\mu_++\nu_+}{2\kappa_+} \right)^2 - \frac{1}{\kappa_+}. \]

In addition, there exist positive constants \(c_1\) and \(c_2\) such that
\[ c_1(|\lambda|^{1/2} + A) \leq \text{Re} t_i \leq |t_i| \leq c_2(|\lambda|^{1/2} + A) \quad (i = 1, 2) \]
for any \((\lambda, \xi') \in \bar{\Sigma}_{\varepsilon,0}.

**Remark 5.2.** We have in general the following situations concerning roots with positive real parts for the characteristic equation of (4.13):

1. When \(\eta_* < 0\), it holds that \(B_+ \neq t_1, B_+ \neq t_2,\) and \(t_1 \neq t_2.\)
2. When \(\eta_* = 0\), there are two cases: \(B_+ \neq t_1\) and \(t_1 = t_2; B_+ = t_1 = t_2.\)
3. When \(\eta_* > 0,\) there are three cases: \(B_+ \neq t_1, B_+ \neq t_2,\) and \(t_1 \neq t_2; B_+ = t_1\) and \(t_1 \neq t_2; B_+ = t_2\) and \(t_1 \neq t_2.\)

We assume \(\eta_* \neq 0\) and \(\kappa_+ \neq \nu_++\mu_++.\) Under these assumptions, we have the three roots with positive real parts different from each other. We consider, however, that our technique in this paper can be applied to the case of equal roots.

From the Bell’s formula and Lemma 5.1 for \(i = 1, 2,\) we have
\[ (M_2)^* \in M_{\varepsilon,1,\varepsilon,0} \quad (M_2 = t_i, t_i + B_+), \quad n(\lambda, \xi') \in M_{\varepsilon,1,\varepsilon,0}. \]

Then, by (4.45), (4.49), (4.51), (4.52), and Lemma 5.1 we have
\[ Q_{N,m}^+ = -\frac{2i\xi_n t_1 t_2 G \rho_+((t_2^2 - A^2)(t_2^2 - B_+^2))(t_1 + t_2)}{A \rho_+ \lambda (t_1^2 - t_2^2)|I(\lambda, \xi')(t_1 + B_+)} = -\frac{2(s_1 - \rho_+ \mu_+^+ s_2 \xi_n t_1 t_2 (t_1 + t_2))G}{(s_3 - s_2) A(t_1 + B_+)|I(\lambda, \xi')} \]
with \(s_1 - \rho_+ \mu_+^+ \neq 0.\) Hence, by Remark 4.3 (5.5), Lemma 5.1 (5.6), and Lemma 6.1 below, we have \(Q_{N,m}^+ \in M_{\varepsilon,2,\varepsilon,0} \). Analogously, we have \(Q_{J,m}^+ \in M_{\varepsilon,2,\varepsilon,0} \) \((j = 1, \ldots, N - 1),\) which yields \(Q_{J,m}^+ \in M_{\varepsilon,2,\varepsilon,0} \) \((J = 1, \ldots, N).\) Furthermore, employing the same argument as \(Q_{J,m}^+ \), we have other assertions in (4.27). Then, we finish the estimate of multipliers.

**6. Analysis of Lopatinski determinant**

**Lemma 6.1.** Let \(\varepsilon_* < \varepsilon < \pi/2\) and \((\lambda, \xi')\) be defined in (4.45). Assume that \(\rho_+ \neq \rho_-, \eta_* \neq 0,\) and \(\kappa_+ \neq \mu_+ + \nu_+.\) Then, there exists a positive constant \(C\) such that
\[ |I(\lambda, \xi')| \geq C(|\lambda|^{1/2} + A)^6 \]
for any \((\lambda, \xi') \in \bar{\Sigma}_{\varepsilon,0} \). Here, positive constant \(C_{\alpha'}\) is depending on \(\alpha', \varepsilon, \mu_{\pm}, \nu_{\pm}, \kappa_{\pm},\) and \(\rho_{\pm}.\)

In addition, we have
\[ |\partial_{\xi'}^{(s)} (|\lambda(\xi')^{-1}|^-1)| \leq C(|\lambda|^{1/2} + A)^{-6}\! A^{-|\alpha'|} \quad (s = 0, 1) \]
for any multi-index \(\alpha' \in \mathbb{N}^{N-1}_{\varepsilon} \) and \((\lambda, \xi') \in \bar{\Sigma}_{\varepsilon,0},\) that is, \(I(\lambda, \xi') \in M_{-6,2,\varepsilon,0}.\)

Proof. We can prove (6.2) by using (6.1) with the Leibniz rule and the Bell’s formula (5.4) with \(f(t) = t^{-1}\) and \(g(\xi') = I(\lambda, \xi').\)

In order to prove (6.1), we consider the three cases: (1) \(R_1|\lambda|^{1/2} \leq A,\) (2) \(R_2A \leq |\lambda|^{1/2},\) (3) \(R_2^{-1}|\lambda|^{1/2} \leq A \leq |\lambda|^{1/2}\) for large \(R_1 \geq 1\) and \(R_2 \geq 1.\)

First, we consider the case: \(R_1|\lambda|^{1/2} \leq A\) with large \(R_1 \geq 1.\) In this case, we set \(\delta_1 = \lambda^{1/2}/A\) and see that
\[ B_+ = A(1 + O(\delta_1)), \quad B_- = A(1 + O(\delta_1)), \quad t_i = A(1 + O(\delta_1)) \quad (i = 1, 2), \]
which imply that
\[ m_i(\lambda, \xi') = 8\rho_{++}\mu_+^{-1}A^6(1 + O(\delta_1)) \quad (i = 1, 2), \]
\[ n(\lambda, \xi') = 4\rho_{++}\mu_+^{-1}A^4(1 + O(\delta_1)). \]
Here, by Lemma 5.1, we have \( s_i - \rho_+ + \mu^{-1}_+ \neq 0 \). Then, we obtain
\[
I(\lambda, \xi') = (24 \rho_+ + 8 \rho_+ + \mu_+ + \mu_-) A^6 (1 + O(\delta_1)), =: \omega_1 A^6 (1 + O(\delta_1)).
\]
Summing up, there exists a positive constant \( C_1 := \omega_1/2 \) such that
\[
|I(\lambda, \xi')| \geq C_1 |(\lambda|1/2 + A)|^6.
\]

Second, we consider the case \( R_2 A \leq |\lambda|1/2 \) for large \( R_2 \). In this case, we set \( \delta_2 = A/|\lambda|^{1/2} \) and see that
\[
B_+ = \sqrt{\rho_+ + \mu_+ \lambda(1 + O(\delta_2))}, \quad B_- = \sqrt{\rho_- + \mu_- \lambda(1 + O(\delta_2))},
\]
which imply that
\[
m_i = \rho_+^2 + \mu_+^2 \sqrt{s_i (s_1 + \rho_+^{1/2} \mu_+^{1/2}) (s_1 + \sqrt{s_2} + s_2) \lambda^2 (1 + O(\delta_2))} \quad (i = 1, 2),
\]
\[
n = \rho_+^2 + \mu_+^2 \left( \sqrt{s_1} + \frac{s_2}{\sqrt{s_1}} (\sqrt{s_1} + \sqrt{s_2}) \right) \lambda^2 (1 + O(\delta_2))
\]
\[
= \rho_+^2 + \mu_+^2 \left( \sqrt{s_2} + \frac{s_1}{\sqrt{s_2}} (\sqrt{s_1} + \sqrt{s_2}) \right) \lambda^2 (1 + O(\delta_2)).
\]
Then, we obtain
\[
I(\lambda, \xi') = \frac{(s_1 + \sqrt{s_1 s_2} + s_2)}{\sqrt{s_1}} \left( \rho_+ + \mu_+ \frac{1}{(s_1 + \sqrt{s_1 s_2} + s_2)} \right) \lambda^2 (1 + O(\delta_2))
\]
\[
= \frac{(s_1 + \sqrt{s_1 s_2} + s_2)}{\sqrt{s_2}} \left( \rho_+ + \mu_+ \frac{1}{(s_1 + \sqrt{s_1 s_2} + s_2)} \right) \lambda^2 (1 + O(\delta_2))
\]
\[
= : \omega_2 \lambda^2 (1 + O(\delta_2)).
\]
From Lemma 5.1, we obtain \( \omega_2 \neq 0 \). Summing up, there exists a positive constant \( C_2 := |\omega_2|/2 \) such that
\[
|I(\lambda, \xi')| \geq C_2 |(\lambda|1/2 + A)|^6.
\]

Third, we consider the case \( R_2^{-1} |\lambda|1/2 \leq A \leq R_1 |\lambda|1/2 \). Here, we set
\[
\bar{\lambda} = \frac{A}{(\lambda|1/2 + A)}, \quad \bar{A} = \frac{\lambda}{(\lambda|1/2 + A)^2},
\]
\[
\bar{\xi}' = \frac{\xi'}{|\lambda|1/2 + A}, \quad \bar{\xi}' = \frac{\xi}{|\lambda|1/2 + A}, \quad \bar{\lambda} = \frac{\lambda}{(\lambda|1/2 + A)^2},
\]
\[
\bar{B}_\pm = \sqrt{\bar{A}^2 + \rho_+ (\mu_+)^{-1} \bar{\lambda}}, \quad \bar{r}_i = \sqrt{\bar{A}^2 + s_i \bar{\lambda}},
\]
\[
D_\varepsilon(R_1, R_2) = \{(\bar{\lambda}, \bar{A}) | (1 + R_1)^{-2} \leq |\bar{\lambda}| \leq R_2^2 (1 + R_2)^2,
\]
\[
(1 + R_2)^{-1} \leq \bar{A} \leq R_1 (1 + R_1)^{-1}, \quad \bar{\lambda} \in \Sigma_\varepsilon \}.
\]
If \((\lambda, \xi')\) satisfies the condition: \( R_2^{-1} |\lambda|1/2 \leq A \leq R_1 |\lambda|1/2 \) and \( \lambda \in \Sigma_\varepsilon \), then \((\bar{\lambda}, \bar{A}) \in D_\varepsilon(R_1, R_2)\). We also define \( I(\hat{\lambda}, \hat{\xi}') \) by replacing \( A, B_\pm \) and \( t_i \) \((i = 1, 2)\) by \( \bar{A}, \bar{B}_\pm \) and \( \bar{r}_i \), respectively. Then, we have
\[
I(\lambda, \xi') = (|\lambda|1/2 + A)^6 I(\hat{\lambda}, \hat{\xi}').
\]
We prove that \( I(\lambda, \xi') \neq 0 \) provided that \((\bar{\lambda}, \bar{A}) \in D_\varepsilon(R_1, R_2)\) by contradiction. Suppose that \( I(\lambda, \xi') = 0 \), namely, \( \det L = 0 \). In this case, in view of (4.47) we assume that there exist \( u_{\pm}(x_N) = (u_{1\pm}(x_N), \ldots, u_{N\pm}(x_N)) \neq 0, \rho_+(x_N) \neq 0, \) and \( \pi_-(x_N) \neq 0 \) satisfying (4.12) with \( \hat{d}(0) = 0, \hat{H}(0) = 0, \hat{h}_m(0) = 0 \), and \( \rho\neq 0 \), that is, \( u_{\pm}(x_N) \neq 0, \rho_+(x_N) \neq 0, \pi_-(x_N) \neq 0 \) satisfy the following homogeneous equations: for \( x_N \neq 0 \) and \( w_+ = \sum_{j=1}^{N-1} i \xi_j u_{j+} + \partial_N u_{j+} + i \xi_j u_{j+} \)
\[
\lambda \rho_+ + \rho_+ w_+ = 0 \quad \text{for} \quad x_N > 0,
\]
\[
\rho_+ \lambda u_{j+} - \mu_+ \sum_{k=1}^{N-1} i \xi_k (\xi_k u_{j+} + i \xi_j u_{j+}) - \mu_+ \partial_N (\partial_N u_{j+} + i \xi_j u_{j+})
\]
\[
-(\rho_+ - \mu_+) \xi_j w_+ - i \xi_j \rho_+ \kappa_+(\partial_N^2 - |\xi'|^2) \rho_+ = 0 \quad \text{for} \quad x_N > 0,
\]
\[
\rho_+ \lambda u_{N+} - \mu_+ + (\partial_N^2 - |\xi'|^2)u_{N+} - \mu_+ \sum_{k=1}^{N-1} i\xi_k (i\xi_k u_{N+} + \partial_N u_{k+}) \\
-2\mu_+ \partial_N^2 u_{N+} - (\nu_+ - \mu_+) \partial_N w_+ - \rho_+ \kappa_+ \partial_N (\partial_N^2 - |\xi'|^2) \rho_+ = 0 \quad \text{for} \quad x_N > 0,
\]
\[
\sum_{j=1}^{N-1} i\xi_j u_{j+} + \partial_N u_{N-} = 0 \quad \text{for} \quad x_N < 0,
\]
\[
\rho_- \lambda u_{j-} - \mu_- \sum_{k=1}^{N-1} i\xi_k (i\xi_k u_{j-} + i\xi_j u_{k-}) - \mu_- \partial_N (\partial_N u_{j-} + i\xi_j u_{N-}) - i\xi_j \pi_- = 0 \quad \text{for} \quad x_N < 0,
\]
\[
\rho_- \lambda u_{N-} - \mu_- \sum_{k=1}^{N-1} i\xi_k (i\xi_k u_{N-} + \partial_N u_{k-}) - 2\mu_- \partial_N^2 u_{N-} - \partial_N \pi_- = 0 \quad \text{for} \quad x_N < 0,
\]
\[
\mu_-(\partial_N u_{m-} + i\xi_m u_{N-})|_- - \mu_+(\partial_N u_{m+} + i\xi_m u_{N+})|_+ = 0,
\]
\[
\{2\mu_+ \partial_N u_{N-} - \pi_-\}|_- - \{2\mu_+ \partial_N u_{N+} + (\nu_- + \mu_-) (i\xi' \cdot u' + \partial_N u_{N+}) \}
\]
\[
-\rho_+ \kappa_+ (\partial_N^2 - |\xi'|^2) \rho_+ \}|_+ = 0,
\]
\[
\partial_N \rho_+|_+ = 0.
\]

Set \((f, g)_\pm = \pm \int_0^\infty f(x_N)\overline{g(x_N)} dx_N\) and \(\|f\|_\pm = (f, f)_\pm^{1/2}\). Multiplying the equations in (6.6) by \(u_{j\pm}\) and using integration by parts and interface conditions in (6.6), we have
\[
(6.7) \quad 0 = \lambda \left( \rho_+ \sum_{j=1}^{N} \|u_{j+}\|^2 + \rho_- \sum_{j=1}^{N} \|u_{j-}\|^2 \right) \\
+ \mu_+ \left( \sum_{j=1}^{N-1} \|i\xi_j u_{j+}\|^2 + \left\| \sum_{j=1}^{N-1} i\xi_j u_{j+} \right\|^2 + \sum_{k=1}^{N-1} \|\partial_N u_{j+} + i\xi_j u_{N+}\|^2 + 2\|\partial_N u_{N+}\|^2 \right) \\
+ (\nu_+ - \mu_+) \|w_+\|^2 + \rho_+ \kappa_+ \left( \|\partial_N w_+\|^2 + |\xi'|^2 \|w_+\|^2 \right) \\
+ \mu_- \left( \sum_{j=1}^{N-1} \|i\xi_j u_{j-}\|^2 + \left\| \sum_{j=1}^{N-1} i\xi_j u_{j-} \right\|^2 + \sum_{k=1}^{N-1} \|\partial_N u_{j-} + i\xi_j u_{N-}\|^2 + 2\|\partial_N u_{N-}\|^2 \right).
\]

Here, we use the identity
\[
\sum_{j,k=1}^{N-1} (i\xi_k u_{j\pm} + i\xi_j u_{k\pm}, i\xi_k u_{j\pm})_\pm = \sum_{j,k=1}^{N-1} \|i\xi_k u_{j\pm}\|^2_\pm + \left\| \sum_{j=1}^{N-1} i\xi_j u_{j\pm} \right\|^2_\pm,
\]
\[
\sum_{j=1}^{N-1} (\partial_N u_{j\pm} + i\xi_j u_{N\pm}, \partial_N u_{j\pm})_\pm + \sum_{j=1}^{N-1} (i\xi_j u_{N\pm} + \partial_N u_{k\pm}, i\xi_k u_{N\pm})_\pm = \sum_{j=1}^{N-1} \|\partial_N u_{j\pm} + i\xi_j u_{N\pm}\|^2_\pm.
\]

Taking the real part of (6.7) to obtain
\[
0 = \text{Re}(\lambda) \left( \rho_+ \sum_{j=1}^{N} \|u_{j+}\|^2 + \rho_- \sum_{j=1}^{N} \|u_{j-}\|^2 \right) \\
+ \mu_+ \left( \sum_{j=1}^{N-1} \|i\xi_j u_{j+}\|^2 + \left\| \sum_{j=1}^{N-1} i\xi_j u_{j+} \right\|^2 + \sum_{k=1}^{N-1} \|\partial_N u_{j+} + i\xi_j u_{N+}\|^2 + 2\|\partial_N u_{N+}\|^2 \right) \\
+ (\nu_+ - \mu_+) \|w_+\|^2 + \frac{\rho_+ \kappa_+ \text{Re}(\lambda)}{|\lambda|^2} \left( \|\partial_N w_+\|^2 + |\xi'|^2 \|w_+\|^2 \right) \\
+ \mu_- \left( \sum_{j=1}^{N-1} \|i\xi_j u_{j-}\|^2 + \left\| \sum_{j=1}^{N-1} i\xi_j u_{j-} \right\|^2 + \sum_{k=1}^{N-1} \|\partial_N u_{j-} + i\xi_j u_{N-}\|^2 + 2\|\partial_N u_{N-}\|^2 \right).
\]
which, combined with the inequality:

$$\|w_+\|_+^2 \leq \sum_{j,k=1}^{N-1} \|i\xi_k u_{j+}\|_+^2 + \sum_{j=1}^{N-1} i\xi_j u_{j+}\|_+^2 + 2\|\partial_N u_{N+}\|_+^2,$$

furnishes that

$$0 \geq (\text{Re} \lambda) \left( \rho_+ \sum_{j=1}^N \|u_{j+}\|_+^2 + \rho_- \sum_{j=1}^{N-1} \|u_{j-}\|_+^2 \right) + \nu_+ \|w_+\|_+^2$$

$$+ \mu_+ \sum_{k=1}^{N-1} \|\partial_N u_{j+k} + i\xi_j u_{N+}\|_+^2 + \mu_- \sum_{k=1}^{N-1} \|\partial_N u_{j-k} + i\xi_j u_{N-}\|_+^2$$

$$+ \frac{\rho_+ \kappa_+ \text{Re} \lambda}{|\lambda|^2} (\|\partial_N w_+\|_+^2 + |\xi'|^2 \|w_+\|_+^2).$$

When $\text{Im} \lambda = 0$, we have $\lambda > 0$ because $\lambda \in \Sigma_z$. However, this contradicts to (6.8). Summing up, we have $I(\bar{\lambda}, \bar{\xi}') \neq 0$ for $(\bar{\lambda}, \bar{A}) \in D(R_1(R_2))$, where

$$D(R_1, R_2) = \{(\bar{\lambda}, \bar{A}) \mid (1 + R_1)^{-2} \leq |\bar{\lambda}| \leq R_2^2(1 + R_2)^2, \ (1 + R_2)^{-1} \leq \bar{A} \leq R_1(1 + R_1)^{-1}, \ \text{Re} \bar{\lambda} \geq 0\}.$$

Then, there exists a positive constant $C_3$ such that

$$\inf_{(\bar{\lambda}, \bar{A}) \in D(R_1, R_2)} \|I(\bar{\lambda}, \bar{\xi}')\| = 2C_3 > 0$$

because $D(R_1, R_2)$ is compact. Since $I(\lambda, \xi')$ is continuous in $\Sigma_{\varepsilon, \lambda_0} \times \mathbb{R}^{N-1}$ for $0 < \varepsilon < \pi/2$ and $D_{\varepsilon}(R_1, R_2)$ is compact, $I(\bar{\lambda}, \bar{\xi}')$ is uniformly continuous in $D_{\varepsilon}(R_1, R_2) \times \mathbb{R}^{N-1}$ for $0 < \varepsilon < \pi/2$. Then, there exist a constant $\varepsilon_0 \in (0, \pi/2)$ such that

$$\|I(\lambda, \xi')\| \geq C_3 \quad \text{for} \quad (\bar{\lambda}, \bar{A}) \in D_{\varepsilon}(R_1, R_2), \ \varepsilon \in (\varepsilon_0, \pi/2),$$

which, combined with (6.5), furnishes that

$$\|I(\lambda, \xi')\| \geq C_3(|\lambda|^{1/2} + A)^6$$

provided that $R_2^{-1}|\lambda|^{1/2} \leq A \leq R_1|\lambda|^{1/2}$ and $\lambda \in \Sigma_z$.

Summing up, setting $C = \min(C_1, C_2, C_3)$, by (6.3), (6.4), and (6.9), we have (6.1), which completes the proof of Lemma 6.1.

## 7. Problem with Surface Tension and Height Function

In this final section, we consider the following problem:

$$\lambda \rho_+ + \rho_+ \text{div} u_+ = 0 \quad \text{in} \ \mathbb{R}^N_+,$$

$$\rho_+ \lambda u_+ - \mu_+ \Delta u_+ - \nu_- \nabla \text{div} u_+ - \kappa_+ \Delta \nabla \rho_+ = 0 \quad \text{in} \ \mathbb{R}^N_+,$$

$$\text{div} u_- = 0, \ \rho_- \lambda u_- - \mu_- \Delta u_- + \nabla \pi_- = 0 \quad \text{in} \ \mathbb{R}^N_-,$$

$$\mu_- D_{mN}(D(u_-))_{-} - \mu_+ D_{mN}(u_+)_{+} = 0,$$

$$\{\mu_+ D_{NN}(u_-) - \pi_-\}_{-} = \sigma_- \Delta' H,$$

$$\{\mu_+ D_{NN}(u_+) + (\nu_+ + \mu_+) \text{div} u_+ + \kappa_+ \Delta \nabla \rho_+\}_{+} = \sigma_+ \Delta' H,$$

$$u_{m-} - u_{m+} = 0, \ \partial_N \rho_+ = 0,$$

$$\lambda H - \left( \frac{\rho_-}{\rho_+ - \rho_+} u_{N-} - \frac{\rho_+}{\rho_+ - \rho_-} u_{N+} \right) = d,$$

where $\sigma_\pm = \rho_\pm \sigma / (\rho_- - \rho_+)$, and prove the following theorem.

**Theorem 7.1.** Let $1 < q < \infty$ and $\varepsilon_0 < \varepsilon < \pi/2$. Assume that $\rho_+ \neq \rho_-, \eta_0 \neq 0$, and $\kappa_+ \neq \mu_+ + \nu_+$ Then, there exist a positive constant $\lambda_0$ and operator families $A^1_\pm(\lambda), B^1_\pm(\lambda), \mathcal{P}^1_\pm(\lambda)$, and $\mathcal{H}(\lambda)$ with

$$A^1_\pm(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, \mathcal{L}(W^1_q(\mathbb{R}^N)), W^1_q(\mathbb{R}^N)^N),$$

$$B^1_\pm(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, \mathcal{L}(W^1_q(\mathbb{R}^N)), W^1_q(\mathbb{R}^N)),$$
\[ P^+(\lambda) \in \text{Anal} (\Sigma_{\varepsilon, \lambda_0}, \mathcal{L}(W^2_q(\mathbb{R}^N), W^1_q(\mathbb{R}^N))), \]
\[ \mathcal{H}(\lambda) \in \text{Anal} (\Sigma_{\varepsilon, \lambda_0}, \mathcal{L}(W^2_q(\mathbb{R}^N), W^1_q(\mathbb{R}^N))), \]

such that for any \( \lambda \in \Sigma_{\varepsilon, \lambda_0} \) and \( d \in W^2_q(\mathbb{R}^N) \), \( u_+ = A^+_{\lambda}(\lambda)d, \rho_+ = B^+_{\lambda}(\lambda)d, \pi_- = P^+(\lambda)d, \) and \( H = \mathcal{H}(\lambda)d \) are solutions of problem \((7.1)\). Furthermore, for \( s = 0, 1, \) we have
\[
\mathcal{R}_{\mathcal{L}(W^2_q(\mathbb{R}^N), L^4_q(\mathbb{R}^N)^{N^2 + N^2})} \{ \{ (\tau \partial_t)^s(G_{\lambda}^3 A^3_{\lambda}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \} \} \leq c_0, 
\]
\[
\mathcal{R}_{\mathcal{L}(W^2_q(\mathbb{R}^N), L^4_q(\mathbb{R}^N)^{N^2 + N^2})} \{ \{ (\tau \partial_t)^s(G_{\lambda}^2 B^3_{\lambda}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \} \} \leq c_0, 
\]
\[
\mathcal{R}_{\mathcal{L}(W^2_q(\mathbb{R}^N), L^4_q(\mathbb{R}^N)^{N^2})} \{ \{ (\tau \partial_t)^s(\nabla P^3_{\lambda}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \} \} \leq c_0, 
\]
\[
\mathcal{R}_{\mathcal{L}(W^2_q(\mathbb{R}^N), W^2_q(\mathbb{R}^N)^{N^2 + 1})} \{ \{ (\tau \partial_t)^s(G_{\lambda}^3 \mathcal{H}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0} \} \} \leq c_0, 
\]

with some constant \( c_0 \).

**Remark 7.2.** Combining Theorem \((7.1)\) with Theorem \((5.1)\) Theorem \((3.2)\) and Theorem \((3.3)\) we have Theorem \((1.2)\) immediately.

As discussed in Sect. \( 4 \) applying the partial Fourier transform to \((7.1)\), we have
\[
\lambda \mathcal{H} + \rho_+ \text{div} \mathcal{H} + \rho_+ \text{div} \mathcal{H} = 0 \quad \text{for } x_N > 0, 
\]
\[
\rho_+ \lambda \mathcal{H} + \rho_+ \partial_N^2 - |\xi'|^2 \mathcal{H} + \nu_+ \xi j \text{div} \mathcal{H} + \nu_+ i \xi j \text{div} \mathcal{H} = 0 \quad \text{for } x_N > 0, 
\]
\[
\rho_+ \lambda \mathcal{H} + \rho_+ \partial_N^2 - |\xi'|^2 \mathcal{H} + \nu_+ \partial_N \text{div} \mathcal{H} = 0 \quad \text{for } x_N > 0, 
\]
\[
\rho_+ \lambda \mathcal{H} + \rho_+ \partial_N^2 - |\xi'|^2 \mathcal{H} + \nu_+ \partial_N \text{div} \mathcal{H} = 0 \quad \text{for } x_N < 0, 
\]
\[
\rho_+ \lambda \mathcal{H} + \rho_+ \partial_N^2 - |\xi'|^2 \mathcal{H} + \nu_+ \partial_N \text{div} \mathcal{H} = 0 \quad \text{for } x_N < 0, 
\]

with the interface condition:
\[
\mu_- (\partial_N \mathcal{H} + i \xi m \mathcal{H}) = - \mu_+ (\partial_N \mathcal{H} + i \xi m \mathcal{H}) \big|_{+} = 0, 
\]
\[
\{ 2 \mu_- \partial_N \mathcal{H} + \partial_N \mathcal{H} \} = \sigma - |\xi'|^2 \mathcal{H}, 
\]
and the resolvent equation for \( H \):

\[
\lambda \mathcal{H} = \left( \frac{\rho_-}{\rho_+ - \rho_-} \mathcal{H} \right)_{-} - \left( \frac{\rho_+}{\rho_+ - \rho_-} \mathcal{H} \right)_{+} = \tilde{d}(0), 
\]

Our task is to represent \( \tilde{H} \) in terms of \( \tilde{d}(0) \), so that we look for solutions \( \tilde{u}_J^\pm \) and \( \tilde{p}_- \) of the form \((4.26)\) with \( g_j = h_j = k = 0 \). In view of \((4.17)\), \((4.18)\), and \((4.19)\), when \( g_j = h_j = k = 0 \), we have
\[
\tilde{u}_{N+}^J(0) = \alpha_{N+} = AR_{NN}^+ \mathcal{H}(0), 
\]
\[
\tilde{u}_{N-}^J(0) = \alpha_{N-} = AR_{NN}^\epsilon \mathcal{H}(0). 
\]

Inserting these formulas into \((7.2)\), we have

\[
(\lambda + K_H) \mathcal{H}(0) = \tilde{d}(0), 
\]

with

\[
K_H = \frac{\rho_-}{\rho_+} AR_{NN}^\epsilon - \frac{\rho_+}{\rho_+ - \rho_-} AR_{NN}^+. 
\]

We now prove the following lemma.
Lemma 7.3. Let $\varepsilon_* < \varepsilon < \pi/2$ and let $K_H$ be the function defined in (7.4). Assume that $\rho_{*+} \neq \rho_{*-}$, $\eta_* \neq 0$, and $\kappa_{*+} \neq \mu_{*+}\nu_{*+}$. Then there exists a positive constant $\lambda_0$ depending on $\varepsilon$, $\mu_{*\pm}$, $\nu_{*+}$, $\kappa_{*+}$, and $\rho_{*\pm}$ such that

$$|\partial^{\alpha'}_\xi \{(\tau \partial_\tau)^s (\lambda + K_H)^{-1}\}| \leq C_{\alpha'}(|\lambda|^{1/2} + A)^{-1} A^{-(\alpha')}, \quad (s = 0, 1)$$

for any multi-index $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \bar{\Sigma}_{\varepsilon, \lambda_0}$ with some constant $C_{\alpha'}$ depending on $\alpha'$, $\lambda_0$, $\varepsilon$, $\mu_{*\pm}$, $\nu_{*+}$, $\kappa_{*+}$, and $\rho_{*\pm}$.

Proof: To prove (7.5) with $\alpha' = 0$ and $s = 0$, first we consider the case where $R_1|\lambda|^{1/2} \leq A$ with large $R_1$. In the following, $\delta_1$ is the same small number as in the proof of Lemma 6.1. In this case, we see that

$$AR_{NN}^+ = -\frac{\sigma_+}{2\mu_{*+}} A(1 + O(\delta_1)), \quad AR_{NN}^- = \frac{-\sigma_-}{2\mu_{*-}} A(1 + O(\delta_1)).$$

Then, we have

$$K_H = \left[ \left( \frac{\rho_{*+}}{\rho_{*+} - \rho_{*+}} \right)^2 \frac{\sigma_+}{2\mu_{*+}} + \left( \frac{\rho_{*-}}{\rho_{*+} - \rho_{*+}} \right)^2 \frac{\sigma_-}{2\mu_{*-}} \right] A(1 + O(\delta_1))$$

$$=: \omega_3 A(1 + O(\delta_1)) \quad (\omega_3 > 0).$$

Since $\lambda \in \Sigma_{\varepsilon}$, by (6.1) and (7.6), we have

$$|\lambda + K_H| \geq \left( \sin \frac{\varepsilon}{2} \right) \left( |\lambda|^{1/2} + \omega_3 A \right) - \omega_3 A O(\delta_1).$$

If we choose $\delta_1$ so small that $O(\delta_1) \leq (\sin(\varepsilon/2))/2$, we have

$$|\lambda + K_H| \geq \left( \frac{1}{2} \sin \frac{\varepsilon}{2} \right) \left( |\lambda|^{1/2} + \omega_3 A \right)$$

provided that $R_1|\lambda|^{1/2} \leq A$ with large $R_1 > 0$ and $\lambda \in \Sigma_{\varepsilon}$.

Next we consider the case where $A \leq R_1|\lambda|^{1/2}$. Here and in the sequel, $C$ denotes a generic constant depending on $R_1$, $\varepsilon$, $\mu_{*\pm}$, $\nu_{*+}$, $\kappa_{*+}$, and $\rho_{*\pm}$. From (4.53), (4.49), (4.51), and (4.52), we have

$$|Q_{NN,1}^+| \leq C|\lambda|^{1/2}, \quad |Q_{NN,2}^+| \leq C|\lambda|^{1/2}.$$ 

Then, by (4.51) and (4.57), we easily see

$$|R_{NN}^+| \leq C, \quad |R_{NN}^-| \leq C,$$

which, combined with (6.3), furnishes that

$$|K_H| \leq C|\lambda|^{1/2}$$

for any $(\lambda, \xi') \in \bar{\Sigma}_{\varepsilon, 0}$ provided that $A \leq R_1|\lambda|$. Thus we have $|\lambda + K_H| \geq |\lambda|^{1/2}(|\lambda|^{1/2} - C)$. Consequently, when we take $\lambda_0 > 0$ so large that $\lambda_0^{1/2}/2 \geq C$, we have

$$|\lambda + K_H| \geq \frac{1}{2}|\lambda|$$

for any $(\lambda, \xi') \in \bar{\Sigma}_{\varepsilon, 0}$ provided that $A \leq R_1|\lambda|$. Since $A \leq R_1|\lambda|^{1/2}$, we have

$$|\lambda + K_H| \geq \frac{1}{4}|\lambda| + \frac{1}{4}|\lambda| \geq \frac{1}{4}|\lambda| + \frac{\lambda_0^{1/2}}{4}|\lambda|^{1/2} \geq \frac{1}{4} \left( |\lambda| + \frac{\lambda_0^{1/2} A}{R_1} \right).$$

Choosing $R_1$ so large that $\lambda_0^{1/2} R_1^{-1} \leq \omega_3$, we obtain

$$|\lambda + K_H| \geq \frac{1}{4} \left( |\lambda|^{1/2} + \omega_3 A \right)$$

for any $(\lambda, \xi') \in \bar{\Sigma}_{\varepsilon, 0}$ provided that $A \leq R_1|\lambda|$. Accordingly, by $\Sigma_{\varepsilon, 0} \subset \Sigma_{\varepsilon}$, combining (7.7) and (7.8), we obtain

$$|\lambda + K_H| \geq \omega_4(|\lambda|^{1/2} + A)$$

for any $(\lambda, \xi') \in \bar{\Sigma}_{\varepsilon, 0}$ with

$$\omega_4 = \min \left( \frac{1}{4}, \frac{\omega_3}{4}, \frac{\omega_3 \sin(\varepsilon/2)}{2}, \frac{\omega_3 \sin(\varepsilon/2)}{2} \right).$$
Finally, we prove (7.5) for any multi-index $\alpha' \in \mathbb{N}_0^{N-1}$. By Lemma 4.2, 4.27, and Lemma 6.1, we have $K_H \in M_{1,2,\varepsilon,0}$, so that by the Bell’s formula (6.4) with $f(t) = (\lambda + t)^{-1}$ and $g = K_H$, we have

$$|\partial_{\xi'}^{\alpha'}(\lambda + K_H)^{-1}| \leq C_{\alpha'} \sum_{l=1}^{\alpha'} |\lambda + K_H|^{-(l+1)}(|\lambda|^{1/2} + A)^l A^{-|\alpha'|} \leq C_{\alpha'}(|\lambda|^{1/2} + A)^{-1} A^{-|\alpha'|}.$$

Analogously, we have

$$|\partial_{\xi'}^{\alpha'}(\tau\partial_{\tau}(\lambda + K_H)^{-1})| \leq C_{\alpha'}(|\lambda|^{1/2} + A)^{-1} A^{-|\alpha'|}.$$

Summing up, we have (7.5). \hfill \Box

From (7.4) and Lemma 7.3 we have

$$\tilde{\mathcal{H}}(\xi', 0) = (\lambda + K_H)^{-1} \tilde{d}(\xi', 0),$$

so that we define $\tilde{\mathcal{H}}(\xi', x_N)$ by $\tilde{\mathcal{H}}(\xi', x_N) = e^{-(1+\lambda^2)^{1/2}x_N}(\lambda + K_H)^{-1} \tilde{d}(\xi', 0)$. The following lemma was proved in Shibata [28].

**Lemma 7.4.** Let $1 < q < \infty$, $\varepsilon_* < \varepsilon < \pi/2$ and let $\lambda_0$ be the same constant as in Lemma 7.3. Assume that $\rho_+ \neq \rho_-$, $\eta_* \neq 0$, and $\kappa_+ \neq \mu_++\nu_+$. Given that the operator $\mathcal{H}(\lambda)$ is defined by

$$[\mathcal{H}(\lambda)]d(x) = \mathcal{F}_\xi' e^{-\lambda^2 x_N}(\lambda + K_H)^{-1} \tilde{d}(\xi', 0)(x')$$

for any $d \in W^2_q(\mathbb{R}^N)$, $\mathcal{H}(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(W^2_q(\mathbb{R}^N), W^3_q(\mathbb{R}^N)))$, and

$$\mathcal{R}_{\lambda}(W^2_q(\mathbb{R}^N), W^3_q(\mathbb{R}^N)) \{(\tau\partial_{\tau}^{\alpha'}(\lambda, \nabla)\mathcal{H}(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \leq \gamma \}$$

with some constant $\gamma$ depending on $\lambda_0, \varepsilon_*, \mu_*, \nu_*, \kappa_+$, and $\rho_\pm$.

We extend $\tilde{\mathcal{H}}(\lambda)d$ to $x_N < 0$, namely, we define $\mathcal{H}(\lambda)$ by

$$[\mathcal{H}(\lambda)]d(x) = \begin{cases} \frac{\tilde{\mathcal{H}}(\lambda)}{4} d(x), & (x_N > 0), \\ \sum_{j=1}^{4} a_j [\tilde{\mathcal{H}}(\lambda)]d(x', -jx_N), & (x_N < 0), \end{cases}$$

where $a_j$ are constants satisfying the equations: $\sum_{j=1}^{4} a_j (-j)^k = 1$ for $k = 0, 1, 2, 3$. By Lemma 7.4, we have the following corollary of Lemma 7.4 immediately.

**Corollary 7.5.** Let $1 < q < \infty$, $\varepsilon_* < \varepsilon < \pi/2$ and let $\lambda_0$ be the same constant as in Lemma 7.3. Assume that $\rho_+ \neq \rho_-$, $\eta_* \neq 0$, and $\kappa_+ \neq \mu_++\nu_+$. Then, there exists an operator family

$$\mathcal{H}(\lambda) \in \text{Anal}(\Sigma_{\varepsilon, \lambda_0}, L(W^2_q(\mathbb{R}^N), W^3_q(\mathbb{R}^N)))$$

such that for any $\lambda \in \Sigma_{\varepsilon, \lambda_0}$ and $d \in W^2_q(\mathbb{R}^N)$, $\mathcal{F}_\xi' d(\xi', 0)/(\lambda + K_H)(x') = \mathcal{H}(\lambda)d_{x_N=0}$ and

$$\mathcal{R}_{\lambda}(W^2_q(\mathbb{R}^N), W^3_q(\mathbb{R}^N)) \{(\tau\partial_{\tau}^{\alpha'}(\lambda, \nabla)\mathcal{H}(\lambda) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}) \leq c_0 \}$$

with some constant $c_0$ depending on $\lambda_0, \varepsilon, \mu_*, \nu_*, \kappa_+$, and $\rho_\pm$.

Then, we construct solution operators of (7.1). Using (4.20) and (7.3), we have

$$\hat{u}_{j+} = \frac{AQ^+_{JN,1}}{\lambda + K_H} AM_{1+}(x_N) \hat{d}(0) + \frac{AQ^+_{JN,2}}{\lambda + K_H} AM_{2+}(x_N) \hat{d}(0) + \frac{AR^+_{JN}}{\lambda + K_H} A e^{-B_{+x_N}} \hat{d}(0),$$

$$\hat{u}_{j-} = \frac{AQ^+_{JN}}{\lambda + K_H} AM_{-}(x_N) \hat{d}(0) + \frac{AR^+_{JN}}{\lambda + K_H} A e^{-B_{-x_N}} \hat{d}(0),$$

$$\hat{\rho}_+ = \frac{AP^+_{N,1}}{\lambda + K_H} AM_{0+}(x_N) \hat{d}(0) + \frac{AP^+_{N,2}}{\lambda + K_H} A e^{-\lambda x_N} \hat{d}(0),$$

$$\hat{\rho}_- = \frac{AP^+_{N}}{\lambda + K_H} e^{\lambda x_N} \hat{d}(0),$$

which yields

$$u_{j+} = \mathcal{F}_\xi' \left[ \frac{AQ^+_{JN,1}}{\lambda + K_H} AM_{1+}(x_N) \hat{d}(0) \right](x') + \mathcal{F}_\xi' \left[ \frac{AQ^+_{JN,2}}{\lambda + K_H} AM_{2+}(x_N) \hat{d}(0) \right](x')$$
Accordingly, if we define the operator $A^j_\pm(\lambda)$ by
\[ A^j_\pm = (A^{1\pm}_1, \ldots, A^{1\pm}_N)d, \]
by (1.22), Corollary 1.3, Lemma 7.3, and the argument in Shibata [28, Sect. 6], we have Theorem 7.1, which completes the proof of Theorem 1.2.

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