Gravitational microlensing source limb-darkening and limb-polarization, I: angle-averaged amplification functions

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ABSTRACT
There is increasing interest in extended source effects in microlensing events, as probes of the unresolved sources. Previous work has either presumed a uniform source, or else required an approximate or numerical treatment of the amplification function averaged over the source disk. In this paper, I present analytic expressions for the angle-averaged amplification functions for the rotationally-symmetric intensity and polarization cases.

These integrals will allow us to use the technology of inverse problems to study the source limb-darkening and limb-polarization functions.

Key words: gravitational lensing – methods: data analysis

1 INTRODUCTION
The most common interest in microlensing events is as a probe of the lensing object itself. Recently, however, there has been increasing interest in such events as probes of the otherwise unresolved sources (Valls-Gabaud 1998; Mao and Witt 1998; Gaudi and Gould 1999). When the lens transits the source (or nearly so), it breaks any rotational symmetry, and this gives us access to the surface brightness, as well as polarization (Simmons et al. 1995; Simmons et al. 1995) and chromaticity (Valls-Gabaud 1995; Valls-Gabaud 1998) information. Previous work on extended source effects has concentrated on the forward problem and generally either performed the required calculations numerically rather than analytically, or used an approximate form of the amplification function. Also, much of the work on source effects has relied on the high amplification provided by binary lens caucic crossings, rather than the amplification of a single lens.

The gravitational lensing forward problem – that of predicting centroid motion and magnification for a given set of source parameters – is relatively easy. The problem is also, however, typically poorly conditioned, in the sense that there will be a broad range of limb-darkening or limb-polarization functions which could plausibly correspond to the observed signal in a microlensing event. This means that a parameter-fitting approach to recovering these functions is very dangerous.

We can make progress by expressing the problem explicitly as an inverse problem (IP), and using the technology of IP methods to analyse precisely what information can be recovered for a given set of observations.

That is the subject of, and motivation for, a forthcoming paper (Gray and Coleman 2000); here I am concerned with identifying the angle-averaged amplification functions as IP kernels, and obtaining analytic expressions for them. As well as facilitating the IP analysis of the problem, these analytic kernels can help in the treatment of the forward problem, since they can be evaluated more efficiently than by a numerical integration, and with high accuracy over their entire domain.

In Sect. 3, I define the amplification functions as IP kernels, in Sect. 4, I integrate them about the source’s centre, and in Sect. 5, I present the results of that integration.

2 AMPLIFICATION FUNCTIONS AS INVERSE PROBLEM KERNELS
The geometry of a microlensing event is as shown in Fig. 1.

The gravitational lens amplification function is (Schneider et al. 1992)

$$A(\xi) = \frac{1}{2} \left( \xi + \frac{1}{\xi} \right),$$

where

$$\xi = \left( 1 + \frac{4}{\zeta^2} \right)^{1/2}, \quad \zeta^2 = r^2 + s^2 - 2rs \cos(\chi - \phi).$$

Denote the intensity by $I(r)$ and the Stokes parameter by $Q(r, \chi) = -P(r) \cos 2\chi$, where $P(r)$ is the polarization of the stellar surface, and we are assuming that the surface brightness is rotationally symmetric. In the case of a microlensing event, we cannot resolve details of the lensed source, and must therefore measure integrals over the source surface. We immediately obtain

$$I(s(t), \phi(t)) = \int_0^\infty I(r) A_I(r; s, \phi) \, dr$$

\[\sum\]
Figure 1. Geometry of a lensing event. The projected path of the lens has impact parameter \( l \), and the path is parameterised by polar coordinates \( s(t) \) and \( \phi(t) \), relative to the centre of the source, projected into the lens plane. Any point in that plane can be given in polar coordinates \( r \) and \( \chi \), and this point is a distance \( \zeta \) from the centre of the lens. All dimensions are normalised to the Einstein radius in the source plane. The angles \( \phi \) and \( \chi \) are taken with respect to the line joining the source to the lens’ point of closest approach.

![Diagram of lensing event](image)

Equations (2) and (3) are in the form of an inverse problem. We address the inverse problem in a forthcoming paper (Gray and Coleman 2000). The evaluation of the integrals \( \tilde{A}_I, \tilde{Q} \) is rather hard, and I describe it in this paper.

3 INTEGRATION OF THE AMPLIFICATION FUNCTIONS

Write

\[
z = \exp i(\chi - \phi), \quad dz = i zd\chi,
\]

so that

\[
\zeta^2 = r^2 + s^2 - rs \left( z + \frac{1}{z} \right).
\]

Define

\[
a_1 = \frac{r^2 + s^2}{2rs}, \quad a_2 = \frac{4 + r^2 + s^2}{2rs}
\]

and

\[
x_1 = a_1 + \sqrt{a_1^2 - 1} = \begin{cases} r/s, & r \geq s \\ s/r, & r < s \end{cases},
\]

\[
x_2 = a_2 + \sqrt{a_2^2 - 1}.
\]

Defining \( \tau_i = 1/x_i \), we have

\[
Q(s(t), \phi(t)) = \int_0^\infty P(r)\tilde{A}_Q(r; s, \phi) \, dr
\]

where the amplification kernels are

\[
\tilde{A}_I = r \int_0^{2\pi} A(r, \chi; s, \phi) \, d\chi
\]

\[
\tilde{A}_Q = -r \int_0^{2\pi} \cos 2\chi A(r, \chi; s, \phi) \, d\chi.
\]

Note that the kernel \( \tilde{A}_I \) is a factor \( 2\pi \) times the angular average of the amplification function, and the functions \( I(s, \phi) \) and \( Q(s, \phi) \) have the dimensions of flux rather than intensity.

Equations (4) and (5) are in the form of an inverse problem. We address the inverse problem in a forthcoming paper (Gray and Coleman 2000). The evaluation of the integrals \( \tilde{A}_I, \tilde{Q} \) is rather hard, and I describe it in this paper.

Figure 2. The contour for the integral in Eqn. (11), showing the cuts between the poles at \( \tau_2, \tau_1, x_1 \) and \( x_2 \).

\[
x_1 + \tau_i = 2a_i, \quad x_i, \tau_i = 1.
\]

It is easy to show that

\[
0 < \tau_2 < \tau_1 \leq 1 \leq x_1 < x_2.
\]

Rewriting the expression for \( \xi \), we obtain

\[
\zeta \approx \left( 1 + \frac{4}{\xi^2} \right)^{1/2} = \left( \frac{z - x_2}{z - x_1} \right)^{1/2}.
\]

We now evaluate the integral

\[
\tilde{I} = \oint_{C \tau} \frac{\xi + 1/\xi}{\xi} \, dz,
\]

where \( C \) is the contour shown in Fig. 2. The integrand has poles at \( z = x_{1,2} \) and \( z = 0 \).

The contour encloses a single singularity at \( z = 0 \). We have \( \xi(z = 0) = 1 \), so that

\[
\tilde{I} = 2\pi i \times \text{Res}(\tilde{I}(z = 0)) = 4\pi i.
\]

For contour 1, substitute \( z = e^{i\psi} \), for \( \psi \in [0, 2\pi] \). Then

\[
\tilde{I}^{(1)} = i \int_0^{2\pi} (\xi + 1/\xi) \, d\psi,
\]

and the substitution \( \psi = \chi - \phi \) produces

\[
\tilde{I}^{(1)} = 2i \int_0^{\phi + 2\pi} A(\xi(\chi, \phi)) \, d\chi = \frac{2i}{r} \tilde{A}_I(r; s, \phi).
\]

Contours 5 and 6 cancel, and with the substitution \( z = \tau_2 + \rho e^{i\phi} \), it is clear that \( \tilde{I}^{(2)} \rightarrow 0 \) as \( \rho \rightarrow 0 \).

Now turn to contours 2 and 4. By substituting \( z = \tau_2 + \sigma, \sigma = 0 \rightarrow (\tau_1 - \tau_2) \), into \( \tilde{I}^{(4)} \), substituting \( z = \tau_2 + \sigma e^{i\phi} \), \( \sigma = (\tau_1 - \tau_2) \), into \( \tilde{I}^{(2)} \), and noting that \( 0 < \sigma < \tau_1 - \tau_2 \) (so that \( |\sigma| < |\tau_2 - x_2|, |\tau_2 - x_1|, \) and \( |\tau_2 - \tau_1| \), so that the phases of the corresponding square-rooted factors in \( \tilde{I}^{(2)} \) are unaffected by the factor of \( e^{i2\phi} \), we can see that

\[
\tilde{I}^{(2)} = \tilde{I}^{(4)}.
\]

Now substituting \( z = x \) (\( x \) real) directly into Eqn. (11), we obtain

\[
\tilde{I}^{(4)} = -i(I_1 - I_2),
\]

where

\[
I_1 = \int_{\tau_2}^{x_1} \frac{1}{x} \left( \frac{(x_2 - x)(x - \tau_2)}{(x_1 - x)(x - \tau_2)} \right)^{1/2} \, dx
\]

\[
I_2 = \int_{\tau_2}^{x_2} \frac{1}{x} \left( \frac{(x_2 - x)(x - \tau_2)}{(x_1 - x)(x - \tau_2)} \right)^{-1/2} \, dx.
\]
Using the notation of Carlson (1988),

\[ [p_1, \ldots, p_n] \equiv \int \prod_{i=1}^{n} (a_i + b_i t)^{n/2} \, dt, \]

we may rewrite these as

\[ I_1 = [+1, -1, -1, 1, -2] \quad \text{(18)} \]
\[ I_2 = [-1, +1, +1, -1, -2] \quad \text{(19)} \]

with \( a_i = (-2, 2, x_1, x_2, 0) \) and \( b_i = (+1, -1, -1, 1, +1) \) in both cases. We will evaluate these integrals when we return to them below, in Sect. 4.

Thus, collecting Eqns. (12), (13), (14) and (15), we obtain the angle-averaged amplification function as

\[ \tilde{A}_I(r, s) = r (2\pi + I_1 - I_2). \quad \text{(20)} \]

This has been confirmed by direct numerical integration of the integrand.

Turning now to Eqn. (11), we may again substitute \( z = \exp(i\chi - \phi) \), and obtain

\[ \cos 2\chi = \frac{1}{2} \left( z^2 e^{i2\phi} + z^{-2} e^{-i2\phi} \right). \]

Now evaluate the integral

\[ \tilde{Q} = \int_{C} \left( z e^{i2\phi} + z^{-3} e^{-i2\phi} \right) \left( \xi + \frac{1}{\xi} \right) \, dz, \]

with the same contour as above, and with \( \xi \) as in Eqn. (16). This has a third-order pole at \( z = 0 \), which means that the residue is

\[ \text{Res}(\tilde{Q}) = a_{-1} = \frac{1}{2} \frac{d^2}{dz^2} \left( \xi + \frac{1}{\xi} \right)_{z=0} = (a_1 - a_2)^2 \quad \text{(22)} \]

with \( a_i \) as defined in Eqn. (17) above. Thus

\[ \tilde{Q} = 2\pi i e^{-i2\phi} (a_1 - a_2)^2. \quad \text{(23)} \]

Much of the calculation goes through as before. Substituting \( z = e^{i\phi} \), we obtain

\[ \tilde{Q}^{(1)} = -\frac{4\pi}{r} \tilde{A}_Q(r, s, \phi). \quad \text{(24)} \]

Contours 5 and 6 cancel, and contour 3 makes zero contribution in the \( \rho \to 0 \) limit. Similarly,

\[ \tilde{Q}^{(2)} = \tilde{Q}^{(4)} = ie^{i2\phi} (-Q_1 + Q_2) + ie^{-i2\phi} (-Q_3 + Q_4), \quad \text{(25)} \]

where

\[ Q_1 = [+1, -1, -1, +1, +2] \quad \text{(26)} \]
\[ Q_2 = [-1, +1, +1, -1, +2] \quad \text{(27)} \]
\[ Q_3 = [+1, -1, -1, +1, -6] \quad \text{(28)} \]
\[ Q_4 = [-1, +1, +1, -1, -6] \quad \text{(29)} \]

with \( a_i \) and \( b_i \) as above.

Assembling equations (22), (24) and (25), we obtain

\[ \tilde{A}_Q(r, s, \phi) = \frac{r}{2} \left[ e^{i2\phi} (-Q_1 + Q_2) + e^{-i2\phi} (-Q_3 + Q_4) \right. \\
\left. - \pi e^{-i2\phi} (a_1 - a_2)^2 \right]. \]

However, this integral should be real, so the imaginary part must be zero:

\[ -Q_1 + Q_2 + Q_3 - Q_4 + \pi (a_1 - a_2)^2 \equiv 0. \]

Figure 3. The amplification kernel \( \tilde{A}_I(r, s) \), plotted as a function of \( r \) for a selection of values of the distance \( s \), and for an impact parameter \( t = 0.1 \) (see Fig. 2). There is a singularity along the line \( r = s \), where the integration includes the point \( \zeta = 0 \).

Thus, the final expression for the angle-averaged polarization amplification function is

\[ \tilde{A}_Q(r, s, \phi) = -r \cos 2\phi (Q_1 - Q_2) \]
\[ -r \left( \frac{2}{\pi} - 1 \right) (Q_1 - Q_2). \quad \text{(30)} \]

This also has been confirmed by numerical integration.

Eqn. (20) and Eqn. (30) are the principal results of this paper.

In Fig. 3, I show the amplification function \( \tilde{A}_I(r, s) \), with a singularity along the line \( r = s \). Note particularly the breadth of the kernel after the singularity: although the angle-averaged amplification is close to 1 away from a well-defined peak at \( r = s \), the factor of \( r \) in Eqn. (2) means that there are contributions to \( I(t) \) in Eqn. (2) from a broad range of \( I(r) \), a situation especially severe for cases where the source function \( I(r) \) extends significantly beyond \( r = 1 \) (i.e. those cases where \( R_s > R_E \)). The breadth and asymmetry of the kernel is what makes the recovery of the source function so problematic. Similarly, Fig. 4 shows the amplification kernel \( \tilde{A}_Q(r, s, \phi) \). This has the same factor of \( r \) as \( A_I(r, s) \), but because the underlying angle-averaged function is close to zero away from \( r = s \), this has a less damaging effect, so that the polarization signal should be easier to recover (which is fortunate, since that signal is so much harder to detect than the intensity signal). The kernel is still, however, both broad and asymmetric.

4 AMPLIFICATION FUNCTIONS AS ELLIPTIC INTEGRALS

The integrals defined by equations (8) and (9), and equations (21) to (24), are elliptic integrals. Carlson (1988) provides a set of recurrence relations to reduce such integrals to a small set of elementary integrals, which are in turn expressed in terms of a set of functions \( RF, R_{FP} \) and \( R_{DP} \) which are more symmetrical alternatives to the traditional Legend-
reduction mechanically, to obtain (4.1 Singularies and asymptotic behaviour of \( \tilde{A}_J \) and \( \tilde{A}_Q \)). We can write Eqn. (32) in terms of Heuman’s Lambda function (Abramowitz and Stegun 1965, 17.4.39) as follows:

\[
I_1 - I_2 = 2\frac{x_2 - x_1}{\sqrt{x_1 x_2} K(\kappa)} + 2\pi \left[ A_0 \left( \arcsin(1 + \lambda_1)^{-1/2} \right) - \lambda_1 \right].
\]

The advantage of this is that we can now easily isolate the singularity at \( r = s \), where we have \( x_1 = \frac{1}{\sqrt{2}}, x_2 = 1, \) and thus \( \kappa = 1 \). The \( A_0 \) function has no singularities, and the coefficient of \( K(\kappa) \) is finite there, so the only singularity is at \( K(\kappa = 1) \) where (Abramowitz and Stegun 1965, 17.3.26)

\[
\lim_{\kappa \to 1} \left( K - \frac{1}{2} \ln \frac{16}{1 - \kappa^2} \right) = 0.
\]

Thus the leading order term in Eqn. (32) at \( r = s \) is

\[
\tilde{A}_J(r \sim s; s) \sim \frac{x_2 - x_1}{\sqrt{x_1 x_2} K(\kappa)} + \frac{2\pi}{\sqrt{x_1 x_2}} \frac{16}{1 - \kappa^2}.
\]

We can also confirm the behaviour as \( r \to 0 \) and \( r \to \infty \). For \( r > s \), we have

\[
\frac{x_2}{x_1} = 1 + \frac{1}{2} \left( \frac{4 + s^2}{r^2} - \frac{4 - s^2}{r^2} \right) + O(r^{-4})
\]

and

\[
\frac{\sqrt{x_2}}{x_1} = \frac{1}{2} \left( \frac{4 + s^2}{r^2} - \frac{4 - s^2}{r^2} \right) + O(r^{-4}).
\]
It follows from Eqn. (37) that both \(1 + \lambda_1\) and \(1 + \lambda_2\) go to 1 as \(r \to \infty\), so that the difference of Lambda functions in Eqn. (35) goes to zero. The factor \(\kappa\) \(\to\) 0 in this limit, and \(K(0)\) is finite, but the coefficient of \(K\) goes to zero, from Eqn. (38), so \(I_1 - I_2 \to 0\), and \(\tilde{A}_I \to 2\pi r\), as expected.

For \(r < s\), both \(x_1\) and \(x_2\) diverge as \(r \to 0\), but \(x_2/x_1 = 1 + 4/s^2 + O(r^2)\), so that \(\kappa \to 0\). Both \(K\) and \(\Lambda_0\) are finite here, so that \(I_1 - I_2\) does not diverge, and the singularity is confined to the coordinates \(x_i\).

We may now move on to the difference \(Q_1 - Q_2\). After rewriting \(\Pi_2\) in terms of \(K\) and \(\Lambda_0\), the coefficient of \(K(\kappa)\) in Eqn. (34) is \((q_k + q_\pi/(1+\lambda_2))/2(x_1 x_2)^{3/2}\). This is a rather messy expression in general, but at \(r = s\), where \(x_1 = 1\), its value is \(2(\sqrt{x_2} - 1/\sqrt{x_2})\), so that the leading order term in Eqn. (34) is

\[
\tilde{A}_Q(r \sim s; s, \phi) = Q_1 - Q_2(r \sim s) \\
\sim \left(\sqrt{x_2} - \frac{1}{\sqrt{x_2}}\right) \ln \frac{16}{1 - \kappa^2}.
\] (39)

One can draw the same conclusion directly from Eqn. (33) by using the useful asymptotic expansions in Carlson and Gustafson 1994, specifically relations (26), (34) and (44) in that paper.

Since the singularity in \(\tilde{A}_I\) and \(\tilde{A}_Q\) is no worse than logarithmic, we may numerically evaluate integrals involving these by using Gaussian quadrature with a log weight.

5 CONCLUSION

I have obtained analytic angular integrals of the microlensing amplification function, for the case of a rotationally symmetric source. This avoids the need to use approximate methods to obtain this expression, and means that they can be evaluated more efficiently than using general numerical integrations. Also, we are able to make analytic statements about the leading-order behaviour of the integrals along their \(r = s\) singularity, and so use such asymptotic approximations in further treatments.

This also means that the dependence of the observed flux on the limb-darkening function, and of the observed polarization on the limb-polarization function, can be expressed as integral equations. Thus the problem of recovering the latter from the former can be viewed as a classic inverse problem, which can be analysed in detail using the sophisticated techniques developed for such problems. This is the subject of a forthcoming paper (Gray and Coleman 2000).

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