Ostrowski type inequalities for $p$-convex functions

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Abstract: In this paper, we give a different version of the concept of $p$-convex functions and obtain some new properties of $p$-convex functions. Moreover we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $p$-convex.

Keywords: $p$-convex function, Ostrowski type inequality, hypergeometric function.

1 Introduction

Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in $I^0$ (the interior of $I$) and let $a,b \in I^0$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a,b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t) dt \right| \leq \frac{M}{b - a} \left[ \frac{(x-a)^2 + (b-x)^2}{2} \right].$$

(1)

for all $x \in [a,b]$. In the literature, the inequality (1) is known as Ostrowski inequality (see [18]), which gives an upper bound for the approximation of the integral average $\frac{1}{b - a} \int_a^b f(t) dt$ by the value $f(x)$ at point $x \in [a,b]$. In [3,5,6,9,10,11], the reader can find generalizations, improvements and extensions for the inequality (1).

For $p \in \mathbb{R}$ the power mean $M_p(a,b)$ of order $p$ of two positive numbers $a$ and $b$ is defined by

$$M_p = M_p(a,b) = \begin{cases} \left( \frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0 \\ \frac{a + b}{2}, & p = 0 \end{cases}.$$

It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$.

Let $L = L(a,b) = (b-a)/|\ln b - \ln a|$, $I = I(a,b) = \frac{1}{L}(a^p/b^p)^{1/a-b}$, $A = A(a,b) = (a+b)/2$, $G = G(a,b) = \sqrt{ab}$, and $H = H(a,b) = 2ab/(a+b)$ be the logarithmic, identric, arithmetic, geometric, and harmonic means of two positive real numbers $a$ and $b$ with $a \neq b$, respectively. Then

$$\min \{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) < I(a,b) < A(a,b) = M_1(a,b) < \max \{a,b\}.$$

Let $\mathfrak{M}$ be the family of all mean values of two numbers in $\mathbb{R}_+ = (0,\infty)$. Given $M,N \in \mathfrak{M}$, we say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $(M,N)$-convex if $f(M(x,y)) \leq N(f(x), f(y))$ for all $x,y \in \mathbb{R}_+$. The concept of $(M,N)$-convexity has been studied extensively in the literature from various points of view (see e.g. [1,4,12,15]).
Let $A(a,b;t) = ta + (1-t)b$, $G(a,b;t) = a^t b^{1-t}$, $H(a,b;t) = ab/(ta + (1-t)b)$ and $M_p(a,b;t) = (ta^p + (1-t)b^p)^{1/p}$ be the weighted arithmetic, weighted geometric, weighted harmonic, weighted power of order $p$ means of two positive real numbers $a$ and $b$ with $a \neq b$ for $t \in [0,1]$, respectively. $M_p(a,b;t)$ is continuous and strictly increasing with respect to $t \in \mathbb{R}$ for fixed $p \in \mathbb{R} \setminus \{0\}$ and $a, b > 0$ with $a > b$. See [8, 14] for some kinds of convexity obtained by using weighted means.

In [8], the author gave definition Harmonically convex and concave functions as follow.

**Definition 1.** Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

(2)

for all $x, y \in I$ and $t \in [0,1]$. If the inequality (2) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds for harmonically convex functions.

**Theorem 1**[8]. Let $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a \leq b$. If $f \in L[a, b]$ then the following inequalities hold

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{f(a) + f(b)}{2}.$$  

The above inequalities are sharp.

### 2 The Definition of $p$-convex Function

In [19], Zhang and Wan give the definition of $p$-convex function as follows:

**Definition 2.** Let $I$ be a $p$-convex set. A function $f : I \rightarrow \mathbb{R}$ is said to be a $p$-convex function or belongs to the class $PC(I)$, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

**Remark**[19]. An interval $I$ is said to be a $p$-convex set if $[tx^p + (1-t)y^p]^{1/p} \in I$ for all $x, y \in I$ and $t \in [0,1]$, where $p = 2k+1$ or $p = n/m, n = 2r+1, m = 2s+1$ and $k, r, t \in \mathbb{N}$.

**Remark.** If $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$, then

$$[tx^p + (1-t)y^p]^{1/p} \in I \text{ for all } x, y \in I \text{ and } t \in [0,1].$$

According to Remark 2, we can give a different version of the definition of $p$-convex function as follows:

**Definition 3.** Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f : I \rightarrow \mathbb{R}$ is said to be a $p$-convex function, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq tf(x) + (1-t)f(y)$$

(3)

for all $x, y \in I$ and $t \in [0,1]$. If the inequality (3) is reversed, then $f$ is said to be $p$-concave.

According to Definition 3, It can be easily seen that for $p = 1$ and $p = -1$, $p$-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.
Example 1. Let \( f : (0, \infty) \to \mathbb{R}, \ f(x) = x^p, p \neq 0, \) and \( g : (0, \infty) \to \mathbb{R}, \ g(x) = c, \ c \in \mathbb{R}, \) then \( f \) and \( g \) are both \( p \)-convex and \( p \)-concave functions.

In [7, Theorem 5], if we take \( I \subset (0, \infty), \ h(t) = t \) and \( p \in \mathbb{R} \setminus \{0\}, \) then we have the following theorem.

**Theorem 2.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a \( p \)-convex function, \( p \in \mathbb{R} \setminus \{0\}, \) and \( a, b \in I \) with \( a < b. \) If \( f \in L[a, b] \) then we have

\[
\frac{p}{b^p - a^p} \int_a^b f(x) \frac{dx}{x^{1-p}} \leq \frac{f(a) + f(b)}{2}.
\]  

(4)

**Remark.** The inequalities (4) are sharp. Indeed we consider the function \( f : (0, \infty) \to \mathbb{R}, \ f(x) = 1. \) Thus

\[
1 = f \left( \left[ t a^p + (1 - t) b^p \right]^{1/p} \right) = t f(y) + (1 - t) f(x) = 1
\]

for all \( x, y \in (0, \infty) \) and \( t \in [0, 1]. \) Therefore \( f \) is \( p \)-convex on \((0, \infty). \) We also have

\[
f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) = 1, \quad \frac{p}{b^p - a^p} \int_a^b f(x) \frac{dx}{x^{1-p}} = 1,
\]

and

\[
\frac{f(a) + f(b)}{2} = 1
\]

which shows us that the inequalities (4) are sharp.

For some results related to \( p \)-convex functions and its generalizations, we refer the reader to see [7, 8, 17, 19].

### 3 Main Results

**Proposition 1.** Let \( I \subset (0, \infty) \) be a real interval, \( p \in \mathbb{R} \setminus \{0\} \) and \( f : I \to \mathbb{R} \) is a function, then:

1. If \( p \leq 1 \) and \( f \) is convex and nondecreasing function then \( f \) is \( p \)-convex.
2. If \( p \geq 1 \) and \( f \) is \( p \)-convex and nondecreasing function then \( f \) is convex.
3. If \( p \leq 1 \) and \( f \) is \( p \)-concave and nondecreasing function then \( f \) is concave.
4. If \( p \geq 1 \) and \( f \) is concave and nondecreasing function then \( f \) is \( p \)-concave.
5. If \( p \geq 1 \) and \( f \) is convex and nonincreasing function then \( f \) is \( p \)-convex.
6. If \( p \leq 1 \) and \( f \) is \( p \)-convex and nonincreasing function then \( f \) is \( p \)-concave.
7. If \( p \geq 1 \) and \( f \) is \( p \)-concave and nonincreasing function then \( f \) is concave.
8. If \( p \leq 1 \) and \( f \) is concave and nonincreasing function then \( f \) is \( p \)-concave.

**Proof.** Since \( g(x) = x^p, \ p \in (-\infty, 0] \cup [1, \infty), \) is a convex function on \((0, \infty)\) and \( g(x) = x^p, \ p \in [0, 1], \) is a concave function on \((0, \infty)\), the proof is obvious from the following power mean inequalities

\[
[t x^p + (1 - t) y^p]^{1/p} \geq tx + (1 - t)y, \ p \geq 1,
\]

and

\[
[t x^p + (1 - t) y^p]^{1/p} \leq tx + (1 - t)y, \ p \leq 1,
\]

for all \( x, y \in (0, \infty) \) and \( t \in [0, 1]. \)

According to above Proposition, we can give the following examples for \( p \)-convex and \( p \)-concave functions.
Example 2. Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$, then $f$ is $p$-convex function for $p \leq 1$ and $f$ is $p$-concave function for $p \geq 1$.

Example 3. Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{-p}$, $p \geq 1$, then $f$ is $p$-convex function.

Example 4. Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$ and $p \geq 1$, then $f$ is $p$-convex function.

Example 5. Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$ and $p \geq 1$, then $f$ is $p$-concave function.

The following proposition is obvious.

Proposition 2. If $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ and we consider the function $g : [a^p, b^p] \to \mathbb{R}$, defined by $g(t) = f(t^{1/p})$, $p \in \mathbb{R} \setminus \{0\}$, then $f$ is $p$-convex on $[a, b]$ if and only if $g$ is convex on $[a^p, b^p]$, $p > 0$ (or $[b^p, a^p]$, $p < 0$).

Remark. According to Proposition 2, as examples of $p$-convex functions we can take $f(t) = g(t^p)$, $p \in \mathbb{R} \setminus \{0\}$, where $g$ is any convex function on $[a^p, b^p]$. Thus, we can obtain the inequality (4) in a different manner as follows:

If $f$ is a is $p$-convex on $[a, b]$ then we write the Hermite-Hadamard inequality for the convex function $g(t) = f(t^{1/p})$ on the closed interval $[a^p, b^p]$ as follows

$$g \left( \frac{a^p + b^p}{2} \right) \leq \frac{1}{b^p - a^p} \int_{a^p}^{b^p} g(t)dt \leq \frac{g(a^p) + g(b^p)}{2},$$

that is equivalent to

$$f \left( \left[ \frac{a^p + b^p}{2} \right]^{1/p} \right) \leq \frac{1}{b^p - a^p} \int_{a^p}^{b^p} f(t^{1/p})dt \leq \frac{f(a) + f(b)}{2}. \tag{5}$$

Using the change of variable $x = t^{1/p}$, then

$$\int_{a^p}^{b^p} f(t^{1/p})dt = p \int_{a}^{b} \frac{f(x)}{x^{1-p}}dx$$

and we get the inequality (4) by using the inequality (5).

Lemma 1. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I$ and $a, b \in I$ with $a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $f' \in L[a, b]$ then

$$f(x) = \frac{p}{b^p - a^p} \int_{a}^{b} \frac{f(u)}{u^{1-p}}du = \frac{1}{p(b^p - a^p)} \left\{ \left( x^p - a^p \right)^2 \int_{0}^{1} \frac{t}{(tx^p + (1-t)a^p)^{1-1/p}}f'' \left( [tx^p + (1-t)a^p]^{1/p} \right)dt \right\} - \left( b^p - x^p \right)^2 \int_{0}^{1} \frac{t}{(tx^p + (1-t)b^p)^{1-1/p}}f'' \left( [tx^p + (1-t)b^p]^{1/p} \right)dt \right\}.$$
Proof. Integrating by part and changing variables of integration yields

\[
\frac{1}{p(b^p-a^p)} \left\{ (x^p-a^p)^2 \int_0^1 \frac{t}{(tx^p+(1-t)a^p)^{1-1/p}} f' \left( \frac{tx^p+(1-t)a^p}{p} \right) dt \right\} - \left\{ (b^p-x^p)^2 \int_0^1 \frac{t}{(tx^p+(1-t)b^p)^{1-1/p}} f' \left( \frac{tx^p+(1-t)b^p}{p} \right) dt \right\} \\
= \frac{1}{(b^p-a^p)} \left\{ \frac{1}{p} \int_0^1 t df \left( \frac{tx^p+(1-t)a^p}{p} \right) + (b^p-x^p) \int_0^1 t df \left( \frac{tx^p+(1-t)b^p}{p} \right) \right\} \\
= \frac{1}{(b^p-a^p)} \left\{ \frac{1}{p} \int_0^1 t df \left( \frac{tx^p+(1-t)a^p}{p} \right) \right\} + \frac{1}{(b^p-a^p)} \left\{ \frac{1}{p} \int_0^1 t df \left( \frac{tx^p+(1-t)b^p}{p} \right) \right\} \\
= f(x) - \frac{p}{(b^p-a^p)} \int_a^b \frac{f(u)}{u^{1-1/p}} du.
\]

Lemma 2. Let \(0 < a \leq x \leq b\), \(p \in \mathbb{R} \setminus \{0\}\), \(\lambda \geq 0\), \(\mu \geq 0\) and \(\eta \geq 1\). Then

\[
\int_0^1 \frac{t^\lambda (1-t)\mu}{(tx^p+(1-t)a^p)^{\eta-\eta/p}} dt = C_{a,p}(x,\lambda,\mu,\eta) = \begin{cases} B_{a,p}(x,\lambda,\mu,\eta), p < 0 \\ A_{a,p}(x,\lambda,\mu,\eta), p > 0 \end{cases},
\]

\[
\int_0^1 \frac{t^\lambda (1-t)\mu}{(tx^p+(1-t)b^p)^{\eta-\eta/p}} dt = S_{b,p}(x,\lambda,\mu,\eta) = \begin{cases} T_{b,p}(x,\lambda,\mu,\eta), p < 0 \\ U_{b,p}(x,\lambda,\mu,\eta), p > 0 \end{cases},
\]

where

\[
B_{a,p}(x,\lambda,\mu,\eta) = \frac{\beta(\lambda+1,\mu+1)}{b^{\eta p-\eta}} 2F_1 \left( \frac{\eta-\eta/p, \lambda+1; \lambda+\mu+2; 1 - \left( \frac{\lambda}{\eta} \right)^p }{b} \right),
\]

\[
A_{a,p}(x,\lambda,\mu,\eta) = \frac{\beta(\mu+1,\lambda+1)}{a^{\eta p-\eta}} 2F_1 \left( \frac{\eta-\eta/p, \mu+1; \lambda+\mu+2; 1 - \left( \frac{\mu}{\eta} \right)^p }{a} \right),
\]

\[
T_{b,p}(x,\lambda,\mu,\eta) = \frac{\beta(\mu+1,\lambda+1)}{x^{\eta p-\eta}} 2F_1 \left( \frac{\eta-\eta/p, \mu+1; \lambda+\mu+2; 1 - \left( \frac{\mu}{\eta} \right)^p }{x} \right),
\]

\[
U_{b,p}(x,\lambda,\mu,\eta) = \frac{\beta(\lambda+1,\mu+1)}{b^{\eta p-\eta}} 2F_1 \left( \frac{\eta-\eta/p, \lambda+1; \lambda+\mu+2; 1 - \left( \frac{\lambda}{\eta} \right)^p }{b} \right),
\]

\(\beta\) is Euler Beta function defined by

\[
\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \text{ } x, y > 0,
\]
and \( \mathbf{2F}_1 \) is hypergeometric function defined by

\[
\mathbf{2F}_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-z)^{-a} \, dt, \quad c > b > 0, \quad |z| < 1 \,(\text{see [2]}).
\]

**Proof.** (i) Let \( p > 0 \). Then

\[
\frac{1}{t^p + (1-t)^p} \frac{1}{\int_0^1 t^\mu (1-t)^\lambda \, dt} = \frac{1}{t^p + (1-t)^p} \int_0^1 t^\mu (1-t)^\lambda \, dt = \frac{\beta(\mu + 1, \lambda + 1)}{\lambda} \mathbf{2F}_1\left( \eta - \eta/p, \mu + 1; \lambda + \mu + 2; 1 - \left( \frac{a}{x} \right)^p \right)
\]

and

\[
\frac{1}{t^p + (1-t)^p} \frac{1}{\int_0^1 t^\mu (1-t)^\lambda \, dt} = \frac{1}{t^p + (1-t)^p} \int_0^1 t^\mu (1-t)^\lambda \, dt = \frac{\beta(\lambda + 1, \mu + 1)}{\mu} \mathbf{2F}_1\left( \eta - \eta/p, \lambda + 1; \lambda + \mu + 2; 1 - \left( \frac{a}{x} \right)^p \right).
\]

(ii) Let \( p < 0 \). Then

\[
\frac{1}{t^p + (1-t)^p} \frac{1}{\int_0^1 t^\mu (1-t)^\lambda \, dt} = \frac{1}{t^p + (1-t)^p} \int_0^1 t^\mu (1-t)^\lambda \, dt = \frac{\beta(\mu + 1, \lambda + 1)}{\lambda} \mathbf{2F}_1\left( \eta - \eta/p, \mu + 1; \lambda + \mu + 2; 1 - \left( \frac{a}{x} \right)^p \right)
\]

and

\[
\frac{1}{t^p + (1-t)^p} \frac{1}{\int_0^1 t^\mu (1-t)^\lambda \, dt} = \frac{1}{t^p + (1-t)^p} \int_0^1 t^\mu (1-t)^\lambda \, dt = \frac{\beta(\lambda + 1, \mu + 1)}{\mu} \mathbf{2F}_1\left( \eta - \eta/p, \lambda + 1; \lambda + \mu + 2; 1 - \left( \frac{a}{x} \right)^p \right).
\]

By using Lemma 1 and Lemma 2, we obtained the following some new Ostrowski type inequalities for \( p \)-convex functions.

**Theorem 3.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^p \), \( a, b \in I^p \) with \( a < b \), \( p \in \mathbb{R} \backslash \{0\} \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( p \)-convex on \( [a, b] \) for \( q \geq 1 \), then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1/p}} \, du \right| \leq \frac{1}{p (b^p - a^p)} \left( x^p - a^p \right) \left( \left| f'(x) \right|^q + \left| f'(a) \right|^q \right)^{1/q} \times \left\{ \left( x^p - a^p \right)^2 C_{a,p}^{1/2} \left| f'(x) \right|^q + C_{a,p} \left| f'(x) \right|^q \right\}^{1/q}.
\]
Proof. From Lemma 1, Power mean integral inequality and the $p$-convexity of $|f'|^q$ on $[a, b]$, we have

$$\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1-p}} \, du \right| \leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 \int_0^t \frac{t}{(tx^p + (1-t)a^p)^{1-p}} \left| f'(tx^p + (1-t)a^p)^{1/p} \right| \, dt \right.$$  \
$$+ \left. (b^p - x^p)^2 \int_0^t \frac{t}{(tx^p + (1-t)b^p)^{1-p}} \left| f'(tx^p + (1-t)b^p)^{1/p} \right| \, dt \right\}^{1-1/q}$$

$$\leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 \left[ \int_0^t \frac{t}{(tx^p + (1-t)a^p)^{1-p}} \, dt \right]^{1-1/q} \times \left[ \int_0^t \frac{t}{(tx^p + (1-t)a^p)^{1-1/p}} \left| f'(tx^p + (1-t)a^p)^{1/p} \right| \, dt \right]^{1/q} \right.$$  \
$$+ \left. (b^p - x^p)^2 \left[ \int_0^t \frac{t}{(tx^p + (1-t)b^p)^{1-p}} \, dt \right]^{1-1/q} \times \left[ \int_0^t \frac{t}{(tx^p + (1-t)b^p)^{1-1/p}} \left| f'(tx^p + (1-t)b^p)^{1/p} \right| \, dt \right]^{1/q} \right\}$$

$$\leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 \left[ \int_0^t \frac{t}{(tx^p + (1-t)a^p)^{1-p}} \, dt \right]^{1-1/q} \times \left[ \int_0^t \frac{t}{(tx^p + (1-t)a^p)^{1-1/p}} \left| f'(tx^p + (1-t)a^p)^{1/p} \right| \, dt \right]^{1/q} \right.$$  \
$$+ \left. (b^p - x^p)^2 \left[ \int_0^t \frac{t}{(tx^p + (1-t)b^p)^{1-p}} \, dt \right]^{1-1/q} \times \left[ \int_0^t \frac{t}{(tx^p + (1-t)b^p)^{1-1/p}} \left| f'(tx^p + (1-t)b^p)^{1/p} \right| \, dt \right]^{1/q} \right\}.$$  

(7)

Hence, If we use (7) and the equalities in Lemma 2, we obtain the desired result. This completes the proof.

Theorem 4. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I$, $a, b \in I$ with $a < b$, $p \in \mathbb{R} \backslash \{0\}$ and $f' \in L[a, b]$. If $|f'|^q$ is $p$-convex on $[a, b]$ for $q \geq 1$, then for all $x \in [a, b]$, we have

$$\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1-p}} \, du \right| \leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 C_{a,p}^{1/q}(x, 0, 1) \right.$$  \
$$\times \left[ C_{a,p}(x, q + 1, 0, 1) \int_0^t \left| f'(x) \right|^q + C_{a,p}(x, q, 1, 1) \left| f'(a) \right|^q \right]^{1/q} \right.$$  \
$$+ \left. (b^p - x^p)^2 S_{b,p}^{1/q}(x, 0, 1, 1) \left| f'(x) \right|^q + S_{b,p}(x, q, 1, 1) \left| f'(b) \right|^q \right\}. \right.$$  

(8)

Proof. From Lemma 1 and Lemma 2, Power mean integral inequality and the $p$-convexity of $|f'|^q$ on $[a, b]$, we have

$$\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1-p}} \, du \right| \leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 \left[ \int_0^t \frac{t}{(tx^p + (1-t)a^p)^{1-p}} \left| f'(tx^p + (1-t)a^p)^{1/p} \right| \, dt \right] \right.$$  \
$$+ \left. (b^p - x^p)^2 \left[ \int_0^t \frac{t}{(tx^p + (1-t)b^p)^{1-p}} \left| f'(tx^p + (1-t)b^p)^{1/p} \right| \, dt \right] \right\}.$$
\[
\begin{align*}
&\leq \frac{(x^p - a^p)^2}{p(b^p - a^p)} \left( \frac{1}{(tx^p + (1-t)a)^{1-1/p}} \right) \int_0^1 \frac{1}{(tx^p + (1-t)a)^{1-1/p}} dt \left( f \left( \int_0^1 \frac{t^q}{(tx^p + (1-t)a)^{1-1/p}} \right) \right) \left( \frac{1}{q} \right)^{1/q} \\
&+ \frac{(b^p - x^p)^2}{p(b^p - a^p)} \left( \frac{1}{(tx^p + (1-t)a)^{1-1/p}} \right) \int_0^1 \frac{1}{(tx^p + (1-t)a)^{1-1/p}} dt \left( f \left( \int_0^1 \frac{t^q}{(tx^p + (1-t)a)^{1-1/p}} \right) \right) \left( \frac{1}{q} \right)^{1/q} \\
&\leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 C_{a,p}^{1/q} |x, 0, 0, 1| f(x)^q + C_{a,p} |x, q + 1, 1, 1| f(a)^q \right\}^{1/q} \\
&+ (b^p - x^p)^2 S_{b,p}^{1/q} |x, 0, 0, 1| f(x)^q + S_{b,p} |x, q + 1, 1, 1| f(b)^q \right\}^{1/q}.
\end{align*}
\]

This completes the proof.

For \( q \geq 1 \), we can give the following result:

**Corollary 1.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), \( p \in \mathbb{R} \setminus \{0\} \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( p \)-convex on \([a, b] \) for \( q \geq 1 \). If \( |f'(x)| \leq M, x \in [a, b] \) then

\[
\left| \frac{f(x) - \frac{p}{b^p - a^p} \int_a^b f(u) du}{b - a} \right| \leq \frac{M}{p(b^p - a^p)} \min \{ I_1, I_2 \}
\]

where

\[
I_1 = \left\{ (x^p - a^p)^2 C_{a,p}^{1/q} (x, 1, 0, 1) |C_{a,p} |x, 2, 0, 1| + C_{a,p} |x, 1, 1, 1| \right\}^{1/q} \\
I_2 = \left\{ (b^p - x^p)^2 S_{b,p}^{1/q} (x, 0, 0, 1) |C_{a,p} |x, q + 1, 0, 1| + C_{a,p} |x, q, 1, 1| \right\}^{1/q}.
\]

**Theorem 5.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), \( r \in \mathbb{R} \setminus \{0\} \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( p \)-convex on \([a, b] \) for \( q > 1 \), \( \frac{1}{q} + \frac{1}{q} = 1 \), then

\[
\left| \frac{f(x) - \frac{p}{b^p - a^p} \int_a^b f(u) du}{b - a} \right| \leq \frac{1}{p(b^p - a^p)} \left( \frac{1}{q + 2} \right)^{1/q} \left\{ (x^p - a^p)^2 C_{a,p}^{1/q} (x, 0, 0, r) \\
\times \left[ f(x)^q + \frac{1}{q + 1} |f(a)|^q \right]^{1/q} \\
+ (b^p - x^p)^2 S_{b,p}^{1/q} (x, 0, 0, r) \left[ f(x)^q + \frac{1}{q + 1} |f(b)|^q \right]^{1/q} \right\}.
\]
**Proof.** From Lemma 1 and Lemma 2, Hölder’s inequality and the $p$-convexity of $|f'|^q$ on $[a, b]$, we have

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1-p}} du \right| \leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 \left( \int_0^1 \frac{t^r}{(tx^p + (1-t)a^p)^{r/p}} dt \right)^{1/r} \left( \int_0^1 t^q \left[ f'(tx^p + (1-t)a^p)^{1/p} \right]^q dt \right)^{1/q} \right. \\
+ \left. (b^p - x^p)^2 \left( \int_0^1 \frac{1}{(tx^p + (1-t)b^p)^{r/p}} dt \right)^{1/r} \left( \int_0^1 t^q \left[ f'(tx^p + (1-t)b^p)^{1/p} \right]^q dt \right)^{1/q} \right\} \\
\leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 C_{a,p}^{1/r}(x, r, 0, r) \times \left[ \frac{|f'(x)|^q + |f'(a)|^q}{2} \right]^{1/q} \right. \\
+ \left. (b^p - x^p)^2 S_{b,p}^{1/r}(x, r, 0, r) \left[ \frac{|f'(x)|^q + |f'(b)|^q}{2} \right]^{1/q} \right\}. 
\]

This completes the proof.

**Theorem 6.** Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I$, $a, b \in I$ with $a < b$, $r \in \mathbb{R} \setminus \{0\}$ and $f' \in L[a, b]$. If $|f'|^q$ is $p$-convex on $[a, b]$ for $q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1-p}} du \right| \leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 C_{a,p}^{1/r}(x, r, 0, r) \times \left[ \frac{|f'(x)|^q + |f'(a)|^q}{2} \right]^{1/q} \right. \\
+ \left. (b^p - x^p)^2 S_{b,p}^{1/r}(x, r, 0, r) \left[ \frac{|f'(x)|^q + |f'(b)|^q}{2} \right]^{1/q} \right\}. 
\]

**Proof.** From Lemma 1 and Lemma 2, Hölder’s inequality and the $p$-convexity of $|f'|^q$ on $[a, b]$, we have

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b \frac{f(u)}{u^{1-p}} du \right| \leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 \left( \int_0^1 \frac{t^r}{(tx^p + (1-t)a^p)^{r/p}} dt \right)^{1/r} \left( \int_0^1 t^q \left[ f'(tx^p + (1-t)a^p)^{1/p} \right]^q dt \right)^{1/q} \right. \\
+ \left. (b^p - x^p)^2 \left( \int_0^1 \frac{1}{(tx^p + (1-t)b^p)^{r/p}} dt \right)^{1/r} \left( \int_0^1 t^q \left[ f'(tx^p + (1-t)b^p)^{1/p} \right]^q dt \right)^{1/q} \right\} \\
\leq \frac{1}{p(b^p - a^p)} \left\{ (x^p - a^p)^2 C_{a,p}^{1/r}(x, r, 0, r) \times \left[ \frac{|f'(x)|^q + |f'(a)|^q}{2} \right]^{1/q} \right. \\
+ \left. (b^p - x^p)^2 S_{b,p}^{1/r}(x, r, 0, r) \left[ \frac{|f'(x)|^q + |f'(b)|^q}{2} \right]^{1/q} \right\}. 
\]

This completes the proof.
Theorem 7. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), \( r \in \mathbb{R} \setminus \{0\} \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( p \)-convex on \([a, b]\) for \( q > 1 \), \( \frac{1}{r} + \frac{1}{q} = 1 \), then

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b f(u) \frac{du}{u^{1-p}} \right| \leq \frac{1}{p(b^p - a^p)} \left( \frac{1}{r + 1} \right)^{1/q} \left\{ \left( x^p - a^p \right)^2 \left[ C_{a,p}(x, 1, 0, q) |f'(x)|^q + C_{a,p}(x, 0, 1, q) |f'(a)|^q \right]^{1/q} \right. \\
+ \left. \left( b^p - x^p \right)^2 \left[ S_{b,p}(x, 1, 0, q) |f'(x)|^q + S_{b,p}(x, 0, 1, q) |f'(b)|^q \right]^{1/q} \right\}.
\]

Proof. From Lemma 1 and Lemma 2, Hölder’s inequality and the \( p \)-convexity of \(|f'|^q\) on \([a, b]\), we have

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b f(u) \frac{du}{u^{1-p}} \right| \leq \frac{1}{p(b^p - a^p)} \left( \frac{1}{r + 1} \right)^{1/q} \left\{ \left( x^p - a^p \right)^2 \left[ C_{a,p}(x, 1, 0, q) |f'(x)|^q + C_{a,p}(x, 0, 1, q) |f'(a)|^q \right]^{1/q} \right. \\
+ \left. \left( b^p - x^p \right)^2 \left[ S_{b,p}(x, 1, 0, q) |f'(x)|^q + S_{b,p}(x, 0, 1, q) |f'(b)|^q \right]^{1/q} \right\}.
\]

This completes the proof.

For \( q > 1 \), we can give the following result:

Corollary 2. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \), \( a, b \in I \) with \( a < b \), \( r \in \mathbb{R} \setminus \{0\} \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( p \)-convex on \([a, b]\) for \( q > 1 \), \( \frac{1}{r} + \frac{1}{q} = 1 \), if \( |f'(x)| \leq M, x \in [a, b] \) then

\[
\left| f(x) - \frac{p}{b^p - a^p} \int_a^b f(u) \frac{du}{u^{1-p}} \right| \leq \frac{M}{p(b^p - a^p)} \min \{ J_1, J_2, J_3 \}
\]  

(10)

where

\[
J_1 = \left( \frac{1}{q + 1} \right)^{1/q} \left\{ \left( x^p - a^p \right)^2 C_{a,p}^{1/q}(x, 0, 0, r) + \left( b^p - x^p \right)^2 S_{b,p}^{1/q}(x, 0, 0, r) \right\},
\]

\[
J_2 = \left( x^p - a^p \right)^2 C_{a,p}^{1/q}(x, r, 0, r) + \left( b^p - x^p \right)^2 S_{b,p}^{1/q}(x, r, 0, r),
\]

\[
J_3 = \left( \frac{1}{r + 1} \right)^{1/q} \left\{ \left( x^p - a^p \right)^2 \left[ C_{a,p}(x, 1, 0, q) + C_{a,p}(x, 0, 1, q) \right]^{1/q} \right. \\
+ \left. \left( b^p - x^p \right)^2 \left[ S_{b,p}(x, 1, 0, q) + S_{b,p}(x, 0, 1, q) \right]^{1/q} \right\}.
\]

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4 Conclusion

The paper deals with Ostrowski type inequalities for p-convex functions. Firstly, we give a different version of the concept of p-convex functions and get some new properties of p-convex functions. Later, by using a new identity, we obtain several new Ostrowski type inequalities for this class of functions via hypergeometric functions.

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