WEAKLY ALMOST PERIODIC FUNCTIONS AND FINITE VON NEUMANN ALGEBRAS

PAUL JOLISSAINT

Abstract. Let $M \subset B(\mathcal{H})$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. We prove that $M$ is finite if and only if, for every $x \in M$ and for all vectors $\xi, \eta \in \mathcal{H}$, the coefficient function $u \mapsto \langle uxu^* \xi | \eta \rangle$ is weakly almost periodic on the topological group $U_M$ of unitaries in $M$ (equipped with the weak operator topology). If it is the case, there exists a conditional expectation $E_M : C^*(M, M') \to M'$ such that the restriction of $E_M$ to $M$ coincides with the canonical central trace on $M$. The existence of $E_M$ is given by the unique invariant mean on the $C^*$-algebra $WAP(U_M)$ of weakly almost periodic functions on $U_M$.

1. Introduction

The present work is inspired by the article [2] where P. de la Harpe proved that, if $M$ is a von Neumann algebra with separable predual, then it is Approximately Finite Dimensional (AFD) if and only if there exists a left invariant mean on the $C^*$-algebra $C_{b,r}(U_M)$ of right uniformly continuous functions on the unitary group $U_M$ of $M$; in other words, $M$ is AFD if and only if the Polish group $U_M$ is amenable. In fact, he used the existence of a left invariant mean on $C_{b,r}(U_M)$ to show that $M$ has Schwartz’s property P [11, Definition 1], the latter being equivalent to injectivity hence to Approximate Finite Dimensionality.

For every topological group $G$, there is a space of continuous functions (in fact a $C^*$-algebra) on which a unique bi-invariant mean always exists: it is the set of all weakly almost periodic functions on $G$. The aim of the present note is then to exploit the existence of such a mean on the group $U_M$. It turns out that it provides a characterization of finite von Neumann algebras.

In order to present the content of this article, let us recall some definitions and fix notation.

First, let $G$ be a topological group. We denote by $C_b(G)$ the $C^*$-algebra of all bounded, continuous, complex-valued functions on $G$ equipped with the uniform norm $\|f\|_\infty := \sup_{s \in G} |f(s)|$. For $g \in G$ and $f : G \to \mathbb{C}$, we

Date: July 1, 2020.

2010 Mathematics Subject Classification. Primary 46L10, 11K70, 22D25; Secondary 22A10.

Key words and phrases. Almost periodic functions, unitary group, finite von Neumann algebra, invariant mean, conditional expectation.
denote by \( g \cdot f : G \to \mathbb{C} \) (resp. \( f \cdot g \)) the left (resp. right) translate of \( f \) by \( g \), i.e.

\[(g \cdot f)(s) = f(g^{-1}s) \quad \text{and} \quad (f \cdot g)(s) = f(gs)\]

for all \( f : G \to \mathbb{C} \) and \( g, s \in G \). The corresponding left (resp. right) orbit is denoted by \( Gf \) (resp. \( fG \)). A function \( f \in C_b(G) \) is right uniformly continuous if \( \|g \cdot f - f\|_\infty \to 0 \) as \( g \to 1 \). The subset of all right uniformly continuous functions \( f \in C_b(G) \) is a sub-\( C^* \)-algebra of \( C_b(G) \) denoted by \( C_{b,r}(G) \), and it contains all right translates of all its elements.

A function \( f \in C_{b,r}(G) \) is weakly almost periodic if its orbit \( Gf \) is weakly relatively compact in \( C_{b,r}(G) \). It follows from [7, Proposition 7] that \( Gf \) is weakly relatively compact if and only if \( fG \) is (see also Proposition 4.3 in the appendix). The set of all weakly almost periodic functions \( f \in C_b(G) \) is a sub-\( C^* \)-algebra of \( C_b(G) \) denoted by \( C_{b,r}(G) \), and its main feature, which will play a central role here, is the existence of a unique left and right \( G \)-invariant mean \( m \) on \( WAP \). We are indebted to S. Knudby for having indicated to us that Greenleaf’s monograph [6] contains a proof of that result for locally compact groups which relies on Ryll-Nardzewski Theorem, but we think that it is worth presenting a self-contained proof for arbitrary topological groups in the appendix of the present note. Our proof also uses Ryll-Nardzewski Theorem.

Next, let \( M \subset B(H) \) be a von Neumann algebra acting on the Hilbert space \( H \). Our references for von Neumann algebras are the monographs [3] and [12]. We denote by \( U_M \) the group of unitary elements of \( M \). It is a topological group when endowed with any of the following equivalent topologies on it: the weak, ultraweak, strong and ultrastrong operator topologies.

For \( T \in B(H) \) and \( \xi, \eta \in \mathcal{H} \), we define the associated coefficient function \( \xi \cdot T \cdot \eta \) on \( U_M \) as follows:

\[\xi \cdot T \cdot \eta(u) := \langle uTu^* \xi | \eta \rangle \]

for every \( u \in U_M \).

Here is our main result, whose proof is contained in §3.

**Theorem 1.1.** Let \( M \subset B(H) \) be a von Neumann algebra. Then it is finite if and only if, for every \( x \in M \), all its coefficient functions \( \xi \cdot x \cdot \eta \) are weakly almost periodic. If it is the case, there exists a conditional expectation \( \mathcal{E}_M : C^*(M,M') \to M' \) whose restriction to \( M \) coincides with the canonical centre-valued trace on \( M \).

The next section deals with an arbitrary von Neumann algebra \( M \subset B(H) \); we present a general study of the space \( \text{wap}_H(M) \) of operators on \( H \) whose all coefficient functions are weakly almost periodic with respect to \( M \).

---

1Our definition follows that of P. de la Harpe [2] and Greenleaf [6], but not that of Eymard [4] or Paterson [8] for instance.
Theorem 1.2. The set \( \text{wap}_H(M) \) is a norm-closed, unital selfadjoint subspace of \( B(H) \) which contains the commutant \( M' \) of \( M \) and the ideal \( K(H) \) of compact operators, and it is an \( M' \)-bimodule. In particular, it is spanned by its positive elements. Moreover, there exists a linear, positive unital map \( E : \text{wap}_H(M) \to M' \) such that:

1. For all \( T \in \text{wap}_H(M) \) and \( x',y' \in M' \), one has \( E(x'Ty') = x'E(T)y' \).
2. For every \( T \in \text{wap}_H(M) \) and every \( v \in U_M \), the element \( vTv^* \) belongs to \( \text{wap}_H(M) \) and \( E(vTv^*) = E(T) \).

As will be seen in §2, the existence of \( E \) comes from the unique bi-invariant mean \( m \) on \( \text{WAP}(U_M) \).

Remark 1.3. Contrary to what was stated in a previous version of the present article, there is a priori no reason that \( \text{wap}_H(M) \) be a \( C^* \)-algebra.

2. The operator system \( \text{wap}_H(M) \)

Let \( H \) be a Hilbert space and let \( M \subset B(H) \) be a von Neumann algebra acting on \( H \).

The next lemma contains general properties of coefficient functions, in particular the left and right translates of by elements of \( U_M \). Its proof uses straightforward computations which are left to the reader.

Lemma 2.1. Let \( T \in B(H) \) and \( \xi, \eta \in H \) and \( v \in U_M \). Then we have the following formulas:

(2.1) \( \xi \star T^* \star \eta = \overline{\eta \star T \star \xi} \),

(2.2) \( v \cdot (\xi \star T \star \eta) = (v\xi) \star T \star (v\eta) \)

and

(2.3) \( (\xi \star T \star \eta) \cdot v = \xi \star (vTv^*) \star \eta \).

Let furthermore \( x',y' \in M' \). Then

(2.4) \( \xi \star (x'Ty') \star \eta = (y'\xi) \star T \star (x'^*\eta) \).

Our next lemma is [2, Lemme 1], but we provide a proof for the sake of completeness.

Lemma 2.2. Let \( T \in B(H) \) and \( \xi, \eta \in H \). Then \( \xi \star T \star \eta \in C_{b,r}(U_M) \). Moreover, \( T \in M' \) if and only if all its associated coefficient functions are constant.

Proof. Let us set \( \varphi := \xi \star T \star \eta \) for short. Then one has, for every \( u \in U_M \), by equality (2.2):

\[
|\langle v \cdot \varphi \rangle(u) - \varphi(u) | = |\langle uTu^*v\xi|v\eta \rangle - \langle uTu^*\xi|\eta \rangle | \\
\leq |\langle uTu^*(v\xi - \xi)|v\eta \rangle| + |\langle uTu^*\xi|v\eta - \eta \rangle | \\
\leq \| T \| \| v\xi - \xi \| |\eta \| + \| T \| \| \xi \| |v\eta - \eta \|.
\]
hence
\[ \|v \cdot \varphi - \varphi\|_\infty \leq \|T\| \max(\|\xi\|, \|\eta\|)(\|v\xi - \xi\| + \|v\eta - \eta\|) \]
which proves that \( \|v \cdot \varphi - \varphi\|_\infty \rightarrow 0 \) as \( v \rightarrow 1 \) in the strong operator topology. The last assertion is obvious. \( \square \)

**Definition 2.3.** A linear, bounded operator \( T \in B(\mathcal{H}) \) is weakly almost periodic with respect to \( M \) if, for all \( \xi, \eta \in \mathcal{H} \), the coefficient function \( \xi \ast T \ast \eta \) belongs to \( \text{WAP}(U_M) \). The set of all weakly almost periodic operators with respect to \( M \) is denoted by \( \text{wap}_M(M) \).

The existence of a unique invariant mean \( m \) on \( \text{WAP}(U_M) \) implies the existence of a positive, \( M' \)-bimodular map \( E \) from \( \text{wap}_M(M) \) onto \( M' \). More precisely, one has the following result.

**Theorem 2.4.** The set \( \text{wap}_M(M) \) has the following properties:

(a) \( \text{wap}_M(M) \) is a norm-closed, unital operator system in the sense of Chapter 2: it is a closed, selfadjoint subspace of \( B(\mathcal{H}) \), thus it is spanned by its positive elements.

(b) For every \( T \in \text{wap}_M(M) \) and for all \( x', y' \in M' \), one has \( x' T y' \in \text{wap}_M(M) \). In particular, \( M' \subseteq \text{wap}_M(M) \).

(c) The ideal \( K(\mathcal{H}) \) of compact operators on \( \mathcal{H} \) is contained in \( \text{wap}_M(M) \). Furthermore, there exists a linear, bounded, and unital map
\[ E : \text{wap}_M(M) \rightarrow B(\mathcal{H}) \]
which is characterized by the equality
\[ \langle E(T)\xi|\eta \rangle = m(\xi \ast T \ast \eta) \quad (\xi, \eta \in \mathcal{H}), \]
and which possesses the following properties:

1. \( E \) is a positive map.
2. For every \( T \in \text{wap}_M(M) \), one has \( E(T) \in M' \).
3. For every \( T \in \text{wap}_M(M) \) and all \( x', y' \in M' \), one has \( E(x' T y') = x'E(T)y' \).
4. For every \( T \in \text{wap}_M(M) \), \( E(T) \in K_T \), where the latter denotes the weakly closed convex hull of the orbit \( \{u T^* u : u \in U_M\} \).
5. For every \( T \in \text{wap}_M(M) \) and every \( v \in U_M \), the operator \( v T v^* \) belongs to \( \text{wap}_M(M) \) and \( E(v T v^*) = E(T) \).
6. For every \( C^* \)-subalgebra \( A \) of \( \text{wap}_M(M) \), the restriction of \( E \) to \( A \) is completely positive.

**Proof.** (a) As \( \text{WAP}(U_M) \) is a \( C^* \)-subalgebra of \( C_{b,r}(U_M) \), \( \text{wap}_M(M) \) is a norm-closed subspace of \( B(\mathcal{H}) \) because, if \( \|T_n - T\| \rightarrow n \rightarrow \infty 0 \) then
\[ \|\xi \ast T_n \ast \eta - \xi \ast T \ast \eta\|_\infty \leq \|T_n - T\| \|\xi\| \|\eta\| \rightarrow n \rightarrow \infty 0. \]
The fact that \( T^* \in \text{wap}_M(M) \) for every \( T \in \text{wap}_M(M) \) follows from equation \[2.1\] of Lemma 2.1.

(b) We have \( M' \subseteq \text{wap}_M(M) \) thanks to Lemma 2.2. Moreover, \( \text{wap}_M(M) \) is an \( M' \)-bimodule by equation \[2.4\] of Lemma 2.4.
(c) As $\text{wap}_H(M)$ is a closed subspace of $B(H)$, in order to show that $K(H) \subset \text{wap}_H(M)$, it suffices to prove that $\text{wap}_H(M)$ contains all rank one operators. Thus, let $\zeta, \omega \in H$ and let $T_{\zeta,\omega}$ be the rank one operator defined by
\[ T_{\zeta,\omega}(\xi) := \langle \xi | \omega \rangle \zeta \quad (\xi \in H). \]
Then we have for all $\xi, \eta \in H$ and $u, v \in U_M$:
\[
\begin{align*}
v \cdot (\xi \ast T_{\zeta,\omega} \ast \eta)(u) &= \langle \xi \ast T_{\zeta,\omega} \ast \eta \rangle(u^* v) = \langle v^* u T_{\zeta,\omega}(u^* v \xi) \rangle \eta \\
&= \langle v^* u^* v \xi | \omega \rangle \zeta \eta = \langle u^* v \xi \rangle \langle v^* u \zeta | \eta \rangle \\
&= \langle v^* u \zeta | \xi \rangle \langle v^* u \xi | \eta \rangle \\
&= v \cdot (\overline{\varphi_{\omega,\xi} \varphi_{\zeta,\eta}})(u)
\end{align*}
\]
where we set $\varphi_{\xi,\eta} : u \mapsto \langle u \xi | \eta \rangle$ for all $\xi, \eta \in H$. As $\text{WAP}(U_M)$ is a $\ast$-algebra, in order to show that the orbit $U_M(\xi \ast T_{\zeta,\omega} \ast \eta)$ is relatively weakly compact, it suffices to prove that, for all $\xi, \eta \in H$, the orbit $U_M \varphi_{\xi,\eta}$ is relatively weakly compact. Thus, let us fix $\xi, \eta \in H$. One has for all $u, v \in U_M$:
\[
(v \cdot \varphi_{\xi,\eta})(u) = \langle v^* u \xi | \eta \rangle = \langle u \xi | v \eta \rangle = \varphi_{\xi,\eta}(u).
\]
This implies in particular that $\varphi_{\xi,\eta} \in C_{b,r}(U_M)$, as the proof of Lemma 2.2 shows. As the orbit $\{v \eta, v \in U_M\}$ is weakly relatively compact in $H$, it suffices to prove that the map $\eta \mapsto \varphi_{\xi,\eta}$ is continuous when $H$ and $C_{b,r}(U_M)$ are equipped with their respective weak topologies. Thus, let $\mu$ be a continuous linear functional on $C_{b,r}(U_M)$. The sesquilinear form $(\zeta, \omega) \mapsto \mu(\varphi_{\zeta,\omega})$ satisfies the following inequality: $|\mu(\varphi_{\zeta,\omega})| \leq \|\mu\| \|\zeta\| \|\omega\|$. Hence there exists a unique operator $T_\mu \in B(H)$ such that $\mu(\varphi_{\zeta,\omega}) = \langle T_\mu \zeta | \omega \rangle$ for all $\zeta, \omega \in H$.

Now, if $(\eta_n) \subset H$ converges weakly to $\eta$, we have
\[
\mu(\varphi_{\zeta,\eta_n}) = \langle T_\mu \zeta | \eta_n \rangle \rightarrow_{n \rightarrow \infty} \langle T_\mu \zeta | \eta \rangle = \mu(\varphi_{\zeta,\eta}).
\]
This ends the proof of the fact that all rank one operators (hence all compact operators) belong to $\text{wap}_H(M)$.

Let us now prove the existence of the map $E$ and all its stated properties. For $T \in \text{wap}_H(M)$, the sesquilinear form $(\xi, \eta) \mapsto m(\xi \ast T \ast \eta)$ is continuous since one has $\|\xi \ast T \ast \eta\| \leq \|T\| \|\xi\| \|\eta\|$. Hence this proves the existence and uniqueness of $E(T)$ for every $T \in \text{wap}_H(M)$, as well as its linearity and boundedness.

1. If $T \in \text{wap}_H(M)$ is a positive operator and if $\xi \in H$, then
\[
\xi \ast T \ast \xi(u) = \langle u T u^* \xi | \xi \rangle = \langle T u^* \xi | u^* \xi \rangle \geq 0,
\]
which implies that $E(T) \geq 0$ since $m$ is a positive functional.

2. Let $T \in \text{wap}_H(M)$, $v \in U_M$ and $\xi, \eta \in H$. Then, using equality (2.2) and left invariance of $m$, we get
\[
\begin{align*}
\langle v^* E(T) | v \xi | \eta \rangle &= \langle E(T) | v \xi | v \eta \rangle = m((v \xi) \ast T \ast (v \eta)) \\
&= m(v \cdot (\xi \ast T \ast \eta)) = m(\xi \ast T \ast \eta) \\
&= \langle E(T) | \xi | \eta \rangle,
\end{align*}
\]
which shows that \( v^* \mathcal{E}(T)v = \mathcal{E}(T) \) for every \( v \in U_M \), thus \( \mathcal{E}(T) \in M' \).

(3) follows from equality 2.4.

(4) We could reproduce the proof of statement (iii) of [2, Lemme 2], but we present a different one, based on the following property of the mean \( m \) (property (d) of Theorem 4.5): For every weakly almost periodic function \( f \) on \( U_M \), its mean \( m(f) \) belongs to the norm-closed convex hull of its right orbit \( fU_M \). Thus, if \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H} \) and \( \varepsilon > 0 \) are given, there exist \( s_1, \ldots, s_m > 0 \), \( \sum_i s_i = 1 \), and \( v_1, \ldots, v_m \in U_M \) such that

\[
(2.6) \quad \left\| m \left( \sum_j \xi_j \ast T \ast \eta_j \right) - \sum_i s_i \left( \sum_j \xi_j \ast T \ast \eta_j \right) \cdot v_i \right\|_\infty \leq \varepsilon.
\]

By equality (2.3), one has

\[
\sum_i s_i \left( \sum_j \xi_j \ast T \ast \eta_j \right) \cdot v_i = \sum_i s_i \left( \sum_j \xi_j \ast (v_i T v_i^*) \ast \eta_j \right),
\]

which yields, when evaluated at \( u = 1 \),

\[
\sum_i s_i \left( \sum_j \xi_j \ast (v_i T v_i^*) \ast \eta_j \right) (1)
\]

\[
= \sum_i s_i \left( \sum_j \langle v_i T v_i^* \xi_j | \eta_j \rangle \right)
\]

\[
= \sum_j \left( \sum_i s_i v_i T v_i^* \right) \xi_j | \eta_j \rangle
\]

As \( m \left( \sum_j \xi_j \ast T \ast \eta_j \right) = \sum_j \langle \mathcal{E}(T) \xi_j | \eta_j \rangle \), we get

\[
\left| \sum_j \langle \mathcal{E}(T) \xi_j | \eta_j \rangle - \sum_j \langle \left( \sum_i s_i v_i T v_i^* \right) \xi_j | \eta_j \rangle \right| \leq \varepsilon
\]

which proves the claim.

(5) For \( \xi, \eta \in \mathcal{H} \), equality (2.3) shows that the right orbit \( (\xi \ast (v T v^*) \ast \eta) U_M = ((\xi \ast T \ast \eta) \cdot v) U_M \) is weakly relatively compact, hence that \( v T v^* \in \text{wap}_\mathcal{H}(M) \).

As \( m \) is right invariant, we get

\[
\langle \mathcal{E}(v T v^*) \xi | \eta \rangle = m(\langle \xi \ast T \ast \eta \rangle \cdot v) = m(\xi \ast T \ast \eta) = \langle \mathcal{E}(T) \xi | \eta \rangle
\]

which proves (5).

(6) By [12, Corollary 3.4, Chapter V], it suffices to prove that

\[
\sum_{i,j} \left\langle y_i^* \mathcal{E}(a_i^* a_j) y_i^* \xi | \xi \right\rangle \geq 0
\]
for all \( a_1, \ldots, a_n \in C^*(M, M') \), all \( y'_1, \ldots, y'_n \in M' \) and every \( \xi \in \mathcal{H} \). We have

\[
\sum_{i,j} \langle y'_i \mathcal{E}(a_i^* a_j) y'_j | \xi \rangle = \sum_{i,j} \langle \mathcal{E}([a_i y'_i]^* [a_j y'_j]) | \xi \rangle = m \left( \sum_{i,j} \xi^* [(a_i y'_i)^* (a_j y'_j)] * \xi \right) \geq 0
\]

because

\[
\left( \sum_{i,j} \xi^* [(a_i y'_i)^* (a_j y'_j)] * \xi \right) (u) \geq 0
\]

for every \( u \in U_M \). Indeed, put

\[
C = \begin{pmatrix}
a_1 y'_1 & 0 & \ldots & 0 \\
a_2 y'_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_n y'_n & 0 & \ldots & 0
\end{pmatrix} \in M_N(B(\mathcal{H})).
\]

Then

\[
C^* C = \begin{pmatrix}
\sum_{i,j} (a_i y'_i)^* (a_j y'_j) & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{pmatrix},
\]

and if

\[
\zeta_u = \begin{pmatrix} u^* \xi \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]

we have

\[
\left( \sum_{i,j} \xi^* [(a_i y'_i)^* (a_j y'_j)] * \xi \right) (u) = \langle C^* C \zeta_u | \zeta_u \rangle \geq 0
\]

for every \( u \in U_M \). \( \square \)

Remark 2.5. (1) If \( M \) is such that \( \text{wap}_\mathcal{H}(M) = B(\mathcal{H}) \), then \( M \) is Approximately Finite Dimensional. Indeed, if it is the case, as in [2], the map \( \mathcal{E} \) is a conditional expectation from \( B(\mathcal{H}) \) onto \( M' \), and \( M \) has Schwartz’s property P. Moreover, as will be proved in §3, \( M \) is finite since it is then contained in \( \text{wap}_\mathcal{H}(M) \). In particular, if \( \text{wap}_\mathcal{H}(B(\mathcal{H})) = B(\mathcal{H}) \) then \( \mathcal{H} \) is finite-dimensional.

(2) The next remark is inspired by [2, Remarques (i)]. Let \( \mathcal{H} \) be an infinite dimensional Hilbert space and let \( A \subset B(\mathcal{H}) \) be a unital \( C^* \)-algebra which has no tracial states. Let \( M \subset B(\mathcal{H}) \) be a von Neumann algebra containing \( A \). Then \( A \subset \text{wap}_\mathcal{H}(M) \). Indeed, otherwise, we would have \( \mathcal{E}(uau^*) = \mathcal{E}(a) \) for all \( a \in A \) and \( u \in U_M \). This would imply that \( \mathcal{E}(xy - yx) = 0 \) for all
3. The case of finite von Neumann algebras

In this section, we consider a von Neumann algebra $M \subset B(\mathcal{H})$ and we denote by $M_*$ its predual. We denote by $\text{Int}(M)$ the group of its inner automorphisms and, for $v \in U_M$, we denote by $\text{Ad}(v) \in \text{Int}(M)$ the automorphism given by $\text{Ad}(v)(x) = vxv^*$ for every $x \in M$.

Let $B(M)$ (resp. $B_*(M)$) denote the Banach space of all bounded (resp. ultraweakly continuous) linear operators on $M$. The weak* topology on $B(M)$ is the $\sigma(B(M), M \otimes \gamma, M_*)$-topology, where $M \otimes \gamma, M_*$ is the projective tensor product of $M$ and $M_*$ (see [12, Chapter IV]). In fact, if $(\Phi_i) \subset B(M)$ is finite, we denote by $\text{Int}_p(M)$ for every $x \in M$ and $\varphi \in M_*$.

If $M$ is finite, we denote by $\text{Ctr}_M$ its canonical centre-valued trace. In this case, it follows from (the proof of) [12, Theorem V.2.4] that the weak* closure of $\text{Int}(M)$ in $B(M)$ is contained in $B_*(M)$. Thus, as $\text{Int}(M)$ is bounded, it is relatively weakly* compact in $B_*(M)$ (see [12, Exercise 6, p. 333]).

This allows us to prove the following theorem.

**Theorem 3.1.** The von Neumann algebra $M$ is finite if and only if $M \subset \text{wap}_\mathcal{H}(M)$. In other words, $M$ is finite if and only if, for every $x \in M$, all its coefficient functions $\xi \star x \star \eta$ are weakly almost periodic. If it is the case, $\text{wap}_\mathcal{H}(M)$ contains $C^*(M, M')$, the $C^*$-algebra generated by $M$ and $M'$ in $B(\mathcal{H})$, and the restriction $\mathcal{E}_M$ of $\mathcal{E}$ to $C^*(M, M')$ has the additional properties:

(i) The restriction of $\mathcal{E}_M$ to $M$ coincides with the canonical centre-valued trace $\text{Ctr}_M$.

(ii) $\mathcal{E}_M : C^*(M, M') \to M'$ is a conditional expectation.

**Proof.** If $M \subset \text{wap}_\mathcal{H}(M)$, then the restriction of $\mathcal{E}$ to $C^*(M, M')$ satisfies condition (5) in Theorem 2.4.1 and this implies that $\mathcal{E}(xy) = \mathcal{E}(yx)$ for all $x, y \in M$. Furthermore, if $x \in M$ and $u' \in U_M$, one has by property (3) in Theorem 2.4.1 $u^*\mathcal{E}(x)u' = \mathcal{E}(u^*xu') = \mathcal{E}(x)$, which means that the restriction of $\mathcal{E}$ to $M$ maps $M$ onto its centre $Z_M$. Then [3, Corollary 3, Part III, Chapter 8] shows that $M$ is a finite von Neumann algebra.

Conversely, suppose that $M$ is finite. Let $x \in M$ and $\xi, \eta \in \mathcal{H}$. We must prove that the coefficient function $\xi \star x \star \eta$ belongs to $\text{WAP}(U_M)$. It follows from [7, Proposition 7] or by Proposition 4.4 of the appendix that it suffices to prove that the right orbit $(\xi \star x \star \eta)U_M$ is relatively weakly compact in $C_{b, r}(U_M)$. But we have, by equation (2.3):

$$(\xi \star x \star \eta)U_M = \{\xi \star vxv^* \star \eta : v \in U_M \} = \{\xi \star \theta(x) \star \eta : \theta \in \text{Int}(M)\}.$$
Thus, from Grothendieck’s Theorem 4.2 in the appendix, it suffices to prove that if \((u_i) \subset U_M\) and \((\theta_j) \subset \text{Int}(M)\) are sequences such that the following double limits

\[
\ell_1 := \lim \left( \lim_{i} \langle \xi \ast \theta_j(x) \ast \eta \rangle (u_i) \right) \quad \text{and} \quad \ell_2 := \lim \left( \lim_{i} \langle \xi \ast \theta_j(x) \ast \eta \rangle (u_i) \right)
\]

exist, then they are equal. Set \(\psi_i = \text{Ad}(u_i)\) for every \(i\). Then \((\xi \ast \theta_j(x) \ast \eta)(u_i) = \langle u_i \theta_j(x) \ast u_i^* \xi \ast \eta \rangle = \langle \psi_i(\theta_j(x)) \langle \xi \ast \eta \rangle \rangle \) for all \(i, j\). Extracting subsequences if necessary, we assume that \((\theta_j)\) converges weakly* to some limit \(\theta \in \text{Int}(M) \subset B_u(M)\), and that \((\psi_i)\) converges weakly* to some limit \(\psi \in \text{Int}(M) \subset B_u(M)\). Denoting by \(\omega_{\xi, \eta} \in M_u\) the normal linear functional \(\omega_{\xi, \eta} : y \mapsto \langle y \xi | \eta \rangle\), one has for every \(i\) on the one hand, because \(\psi_i\) is a normal map,

\[
\lim_{i} \langle \psi_i(\theta_j(x)) \langle \xi \ast \eta \rangle \rangle = \langle \psi(\theta(x)) \langle \xi \ast \eta \rangle \rangle = \langle \psi \circ \theta(x) \langle \xi \ast \eta \rangle \rangle = \langle \psi, \theta(x) \otimes \omega_{\xi, \eta} \rangle.
\]

Hence \(\ell_1 = \lim_i \langle \psi_i(\theta_j(x)) \langle \xi \ast \eta \rangle \rangle = \langle \psi \circ \theta(x) \langle \xi \ast \eta \rangle \rangle\).

On the other hand, one has for every \(j\)

\[
\lim_{i} \langle \psi_i(\theta_j(x)) \langle \xi \ast \eta \rangle \rangle = \langle \psi(\theta_j(x)) \langle \xi \ast \eta \rangle \rangle.
\]

As \(\psi\) is normal (that is where we use finiteness of \(M\), we get

\[
\ell_2 = \lim_j \langle \psi(\theta_j(x)) \langle \xi \ast \eta \rangle \rangle = \langle \psi \circ \theta(x) \langle \xi \ast \eta \rangle \rangle = \ell_1.
\]

From now on, we assume that \(M\) is finite.

(i) For every \(x \in M\), the map \(E\) is constant on the convex hull of \(\{uxu^* : u \in U_M\}\), hence, by its boundedness, it is constant on the norm closure \(K'_x\) of the latter. By \([3\text{ Theorem 1, Part III, Chapter 5}]\), \(E(\text{Ctr}_M(x)) = E(x)\), and as \(\text{Ctr}_M(x) = M \cap M'\), we have \(E(x) = \text{Ctr}_M(x)\). This proves (i).

(ii) follows from property (3) of Theorem 2.4.

Here is another property of \(E\) when \(M\) is diffuse.

**Proposition 3.2.** Let \(M \subset B(H)\) be a finite and diffuse von Neumann algebra, and let \(E : \text{wap}(H) \to M'\) be the positive map of Theorem 2.4. Then \(K(H) \subset \ker E\).

**Proof.** Since \(E\) is a continuous, positive map, it suffices to prove that, for all \(\zeta, \xi \in H\), \(\langle E(T_{\zeta, \xi}) \xi | \zeta \rangle = 0\) where, as in the proof of Theorem 2.4, \(T_{\zeta, \xi}\) is the rank one operator defined by \(T_{\zeta, \xi}(\xi) = \langle \xi | \zeta \rangle \zeta\) for every \(\xi \in H\).

Thus, fix \(\zeta, \xi \in H\); we have for every \(u \in U_M\):

\[
\xi \ast T_{\zeta, \xi} \ast \xi(u) = \langle uT_{\zeta, \xi}u^* \xi | \zeta \rangle = \langle u^* \xi | \zeta \rangle u\xi = \langle \xi | u\xi \rangle \langle u\xi | \zeta \rangle = \langle u\xi | \zeta \rangle^2.
\]

The function \(\varphi\) on \(U_M\) defined by \(\varphi(u) := \langle u\xi | \zeta \rangle\) for every \(u \in U_M\) belongs to \(\text{WAP}(U_M)\) by the proof of property (c) of Theorem 2.4. It is a coefficient
function associated to the unitary representation \( \pi : U_M \to U(\mathcal{H}) \) given by \( \pi(u) = u \) for every \( u \in U_M \). As

\[
|\langle u \zeta | \xi \rangle|^2 \leq \|\zeta\| \|\xi\| \langle u \zeta | \xi \rangle|
\]

for every \( u \in U_M \), it suffices to prove that \( m(|\varphi|) = 0 \). As in the proof of [1] Theorem 1.3, let \( \varepsilon > 0 \). By condition (d) of Theorem 4.5 there exist \( v_1, \ldots, v_m \in U_M \) and \( t_1, \ldots, t_m > 0 \) such that \( \sum_j t_j = 1 \) and

\[
\left| \sum_j t_j |\varphi(v_j^* u)| - m(|\varphi|) \right| < \varepsilon/2
\]

for every \( u \in U_M \). As \( M \) is diffuse, there exists a sequence \( (u_n) \subset U_M \) such that \( u_n \to 0 \) weakly. Thus, there exists \( n \) such that \( |\varphi(v_j^* u_n)| < \varepsilon/2 \) for every \( j \), so that

\[
0 \leq \sum_j t_j |\varphi(v_j^* u_n)| < \varepsilon/2.
\]

This implies that

\[
0 \leq m(|\varphi|) \leq \left| \sum_j t_j |\varphi(v_j^* u_n)| - m(|\varphi|) \right| + \sum_j t_j |\varphi(v_j^* u_n)| \leq \varepsilon.
\]

The following example shows that the hypothesis that \( M \) is diffuse cannot be removed.

**Example 3.3.** Set \( \mathcal{H} = l^2(\mathbb{N}) \), let \( (\delta_k)_{k \in \mathbb{N}} \) be the natural orthonormal basis of \( \mathcal{H} \) and let \( M = A = l^\infty(\mathbb{N}) \) be the atomic maximal abelian \(*\)-subalgebra of \( B(\mathcal{H}) \) acting by pointwise multiplication on \( \mathcal{H} \) so that \( a \delta_k = a(k) \delta_k \) for all \( a \in A \) and \( k \in \mathbb{N} \). We claim that \( \text{wap}_M(A) = B(\mathcal{H}) \) and that \( E(T) \in A \) is the function \( k \mapsto \langle T \delta_k | \delta_k \rangle \) for every \( T \in B(\mathcal{H}) \).

Indeed, let \( T \in B(\mathcal{H}) \). In order to prove that it belongs to \( \text{wap}_M(A) \), it suffices to verify that \( \delta_k \ast T \ast \delta_\ell \) is weakly almost periodic for all \( k, \ell \in \mathbb{N} \). Thus, let us fix integers \( k \) and \( \ell \). As \( U_A = \mathbb{T}^\mathbb{N} \), where \( \mathbb{T} \) denotes the unit circle, we have for all \( u \in U_A \):

\[
\delta_k \ast T \ast \delta_\ell(u) = \langle Tu(k) \delta_k | u(\ell) \delta_\ell \rangle = \langle Tu(k) \delta_k | u(k) \rangle \langle u(k) | u(\ell) \delta_\ell \rangle = \langle u \ast \delta_k | \delta_\ell \rangle.
\]

But

\[
u(k)u(\ell) = \langle \delta_k | u \delta_k \rangle \langle u \delta_\ell | \delta_\ell \rangle = \langle \delta_k | u \delta_k \rangle \langle u \delta_\ell \rangle = \langle u \ast \delta_k | \delta_\ell \rangle
\]

where we use the same notation as in the proof of Theorem 2.4(c) for rank one operators. Hence \( T \in \text{wap}_M(A) \) and

\[
\langle E(T) \delta_k | \delta_\ell \rangle = \langle T \delta_k | \delta_\ell \rangle \cdot m(\delta_k \ast T \delta_\ell \ast \delta_\ell).
\]

Set \( \varphi_{k, \ell} = \delta_k \ast T \delta_\ell \ast \delta_\ell \) for short so that \( \varphi_{k, \ell}(u) = u(k)u(\ell) \) for every \( u \in U_A \).
If \( k = \ell \), then \( \varphi_{k,k}(u) = 1 \) for every \( u \) and \( \mathfrak{m}(\varphi_{k,k}) = 1 \). If \( k \neq \ell \), then we can view \( \varphi_{k,\ell} \) as the continuous function on \( \mathbb{T}^2 \) defined by \( \varphi_{k,\ell}(z,w) = \bar{zw} \).

As \( \mathbb{T}^2 \) is a compact group, one has \( C(\mathbb{T}^2) = \text{WAP}(\mathbb{T}^2) \) and the invariant mean on the latter coincides with the Haar measure. Hence, by property (d) of Theorem 4.5, we have

\[
\mathfrak{m}(\varphi_{k,\ell}) = \int_{\mathbb{T}^2} \tilde{z}wdzdw = 0.
\]

4. Appendix: Weakly almost periodic functions on topological groups

As promised in §1, the aim of this appendix is to give a proof of the existence of a unique invariant mean on \( \text{WAP}(G) \), where \( G \) is an arbitrary topological group. We keep notation that were settled in §1, except that we denote by \( e \) the unit element of \( G \).

Let us recall first the following two theorems of A. Grothendieck.

**Theorem 4.1.** [7, Théorème 5] Let \( \Omega \) be a compact space and let \( A \subset C(\Omega) \) be a bounded set. Then \( A \) is relatively weakly compact if and only if, for every sequence \( (f_n) \subset A \), there exists a subsequence \( (f_{n_k}) \) and an element \( h \in C(\Omega) \) such that

\[
\lim_k f_{n_k}(\omega) = h(\omega)
\]

for every \( \omega \in \Omega \).

**Proof.** (Sketch) The proof rests on Eberlein-Smulian theorem [5, Theorem A.12] which states that if \( A \) is a bounded subset of a Banach space \( X \), then \( A \) is relatively weakly compact if and only if every sequence in \( A \) has a subsequence which converges weakly in \( X \).

Thus, if a bounded sequence \( (f_n) \) converges pointwise to the limit \( h \), then, by Lebesgue theorem, \( \int \! f_n d\mu \to \int \! h d\mu \) for every regular complex measure \( \mu \) on \( \Omega \). Compactness of \( \Omega \) implies that every continuous linear functional on \( C(\Omega) \) is such a measure, hence relative compactness in the pointwise convergence topology implies relatively weak compactness. The converse is obvious, as every linear form of the type \( f \mapsto f(\omega) \) is weakly continuous. \( \square \)

Using the Stone-Cech compactification of \( G \), A. Grothendieck gets the following theorem.

**Theorem 4.2.** [7, Théorème 6] Let \( G \) be an arbitrary topological group. Then a bounded subset \( A \) of \( C_0(G) \) is weakly relatively compact if and only if there do not exist sequences \( (x_i) \subset G \) and \( (f_j) \subset A \) such that

\[
\lim_i (\lim_j f_j(x_i)) \quad \text{and} \quad \lim_j (\lim_i f_j(x_i))
\]

both exist and are different.

As a consequence, he gets the following criterium for weakly almost periodic functions on topological groups.
Proposition 4.3. Let $G$ be a topological group and let $f \in C_{b,r}(G)$. Then $f$ is weakly almost periodic if and only if there do not exist sequences $(x_i), (y_j)$ in $G$ such that

$$\lim_{i} \lim_{j} f(x_iy_j) \quad \text{and} \quad \lim_{j} \lim_{i} f(x_iy_j)$$

both exist and are different. In particular, the left orbit $Gf$ is relatively weakly compact if and only if the right orbit $fG$ is.

Proposition 4.4. Let $G$ be as above. The set $WAP(G)$ is a unital $C^*$-subalgebra of $C_{b,r}(G)$ which is left- and right-invariant under translations of $G$.

Proof. It is clear that $WAP(G)$ contains all constant functions, and that $f \in WAP(G)$ if $f \in WAP(G)$. Moreover, for fixed $g \in G$, the maps $f \mapsto g \cdot f$ and $f \mapsto f \cdot g$ are clearly weakly continuous, hence $g \cdot f, f \cdot g \in WAP(G)$ if $f \in WAP(G)$

In order to prove that $WAP(G)$ is a $C^*$-algebra, we are going to apply Theorem 4.1. In order to do that, let $\Omega$ be the Gelfand spectrum of the unital $C^*$-algebra $C_{b,r}(G)$. It is a compact space, and Gelfand transform $f \mapsto \hat{f} : \chi \mapsto \chi(f)$ is a $\ast$-isomorphism from $C_{b,r}(G)$ onto $C(\Omega)$, and the weak topology on $C_{b,r}(G)$ corresponds to that on $C(\Omega)$. Furthermore, $G$ acts continuously on $\Omega$ by $g \cdot \chi(f) = \chi(g^{-1} \cdot f)$. Indeed, if $\chi_i \mapsto \chi$ and $g_j \rightarrow e$, one has

$$|\chi(f) - \chi_i(g_j^{-1} \cdot f)| \leq |\chi(f) - \chi_i(f)| + |\chi_i(f) - \chi_i(g_j^{-1} \cdot f)|$$

$$\leq |\chi(f) - \chi_i(f)| + \|f - g_j^{-1} \cdot f\|_{\infty}$$

$$= |\chi(f) - \chi_i(f)| + \|g_j \cdot f - f\|_{\infty} \rightarrow 0$$

as $i, j \rightarrow \infty$. Thus the image $\overline{WAP(G)}$ of $WAP(G)$ under Gelfand transform is exactly the set of elements $f \in C(\Omega)$ for which $Gf$ is relatively weakly compact. The fact that $WAP(G)$ is a $\ast$-algebra is a straightforward consequence of Theorem 4.1.

Let us prove finally that $WAP(G)$ closed in $C_{b,r}(G)$ or, what amounts to be the same, that $\overline{WAP(G)}$ is closed in $C(\Omega)$: let $(f_n)_{n \geq 1} \subset \overline{WAP(G)}$ be a sequence which converges to $f \in C(\Omega)$. Let us show that $Gf$ is relatively weakly compact. In order to do that, let $(g_k)_{k \geq 1} \subset G$ be a sequence. We are going to prove that there exists a subsequence $(g_{k_j})_{j \geq 1} \subset (g_k)$ and an element $h \in C(\Omega)$ such that $g_{k_j} \cdot f(\omega) \rightarrow h(\omega)$ for every $\omega \in \Omega$. By the standard diagonal process, there exist a subsequence $(g_{k_j}) \subset (g_k)$ and a sequence $(h_\ell)_{\ell \geq 1} \subset C(\Omega)$ such that

$$\lim_{j \rightarrow \infty} g_{k_j} \cdot f(\omega) = h_\ell(\omega)$$
for every $\omega \in \Omega$ and every $\ell \geq 1$. Let us prove that $(h_\ell)$ is a Cauchy sequence in $C(\Omega)$. Let $m, \ell > 0$; for every $\omega \in \Omega$, one has

$$
|h_\ell(\omega) - h_m(\omega)| \leq |h_\ell(\omega) - f_\ell(g_{k_j}^{-1}\omega)| + |f_\ell(g_{k_j}^{-1}\omega) - f_m(g_{k_j}^{-1}\omega)|
$$

$$
+ |f_m(g_{k_j}^{-1}\omega) - h_m(\omega)|
$$

$$
\leq |h_\ell(\omega) - f_\ell(g_{k_j}^{-1}\omega)|
$$

$$
+ \|f_\ell - f_m\|_\infty + |f_m(g_{k_j}^{-1}\omega) - h_m(\omega)|
$$

for every $j$. At the limit $j \to \infty$, this gives

$$
|h_\ell(\omega) - h_m(\omega)| \leq \|f_\ell - f_m\|_\infty
$$

and then $(h_\ell)$ converges in norm to a limit denoted by $h \in C(\Omega)$. Let us fix $\omega \in \Omega$ and show that $g_{k_j} \cdot f(\omega) \to_{j \to \infty} h(\omega)$. Let $\varepsilon > 0$. There exists $\ell$ such that

$$
\|f - f_\ell\|_\infty + \|h - h_\ell\|_\infty \leq \frac{2\varepsilon}{3}.
$$

Next, there exists $J$ such that, for every $j \geq J$, one has

$$
|g_{k_j} \cdot f_\ell(\omega) - h_\ell(\omega)| \leq \frac{\varepsilon}{3}.
$$

Hence,

$$
|g_{k_j} \cdot f(\omega) - h(\omega)| \leq \|f - f_\ell\|_\infty + |g_{k_j} \cdot f_\ell(\omega) - h_\ell(\omega)| + \|h_\ell - h\|_\infty \leq \varepsilon
$$

for every $j \geq J$. This proves that $f \in \text{WAP}(G)$, which is then closed. \hfill \square

Here is now the promised theorem on the existence of the unique bi-invariant mean on $\text{WAP}(G)$.

**Theorem 4.5.** There exists a unique linear functional $m : \text{WAP}(G) \to \mathbb{C}$ with the following properties:

(a) $m(f) \geq 0$ for every $f \geq 0$;

(b) $m(1) = 1$;

(c) $m(g \cdot f) = m(f \cdot g) = m(f)$ for all $f \in \text{WAP}(G)$ and $g \in G$;

(d) for every $f \in \text{WAP}(G)$ and every $\varepsilon > 0$, there exists a convex combination $\psi := \sum_{j=1}^{m} t_j g_j \cdot f$ (with $g_j \in G$ and $t_j \geq 0$, $\sum_j t_j = 1$) such that

$$
\|\psi - m(f)\|_\infty < \varepsilon,
$$

and there exists a convex combination $\varphi := \sum_i s_i f \cdot h_i$ such that

$$
\|\varphi - m(f)\|_\infty < \varepsilon.
$$

**Proof.** For $f \in \text{WAP}(G)$, let us denote by $Q_l(f) = \overline{\text{conv}}(Gf)$ the norm closed convex hull of $Gf$ and similarly $Q_r(f) = \overline{\text{conv}}(fG)$. The group $G$ acts by left translations on $Q_l(f)$ which are affine transformations and are weakly continuous since, for every fixed $g$, the map $f \mapsto g \cdot f$ is linear and isometric. Moreover, if $\psi_1, \psi_2 \in Q_l(f)$ are such that $\psi_1 \neq \psi_2$, then

$$
\|g \cdot \psi_1 - g \cdot \psi_2\|_\infty = \|\psi_1 - \psi_2\|_\infty > 0
$$
and thus $0 \notin \{g \cdot \psi_1 - g \cdot \psi_2 : g \in G\}^{1,1}_{\infty}$ and the action of $G$ on $Q_l(f)$ is distal. By Ryll-Nardzewski Theorem, there exists $c_l(f) \in Q_l(f)$ such that $g \cdot c_l(f) = c_l(f)$ for every $g \in G$. This means that $c_l(f)$ is constant.

Similarly, $G$ acts on the right on $Q_r(f)$, and by the same arguments, $Q_r(f)$ contains a constant $c_r(f)$. It will be proved that both $Q_l(f)$ and $Q_r(f)$ contain the same and unique constant.

If a linear form $m : \text{WAP}(G) \to \mathbb{C}$ satisfies (a) and (b) and if $m(g \cdot f) = m(f)$ for all $f \in \text{WAP}(G)$ and $g \in G$, then $m$ is constant on $Q_l(f)$ and we infer that $m(f) = c_l(f)$. If $m'$ is another left-invariant mean, one has necessarily $m'(f) = m(f)$ by the above remarks. This proves uniqueness of $m$. Furthermore, the fact that $m(f)$ belongs to $Q_l(f) \cap Q_r(f)$ implies (d) and then the right invariance of $m$.

It remains to prove the existence of a left $G$-invariant mean on $\text{WAP}(G)$. Let us prove first that for $f \in \text{WAP}(G)$, the sets $Q_l(f)$ and $Q_r(f)$ contain the same unique constant.

Indeed, let $a \in Q_l(f)$ and $b \in Q_r(f)$ be constants, which exist by the previous discussion. If $\varepsilon > 0$ is fixed, there exist $s_1, \ldots, s_m > 0$, $t_1, \ldots, t_n > 0$ such that $\sum_i s_i = \sum_j t_j = 1$ and $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that

$$\left| \sum_i s_i f(g_i^{-1} g) - a \right| < \frac{\varepsilon}{2} \quad \text{et} \quad \left| \sum_j t_j f(gh_j) - b \right| < \frac{\varepsilon}{2}$$

for every $g \in G$. One has:

$$|a - b| = \left| \sum_j t_j a - \sum_i s_i b \right| \leq \left| \sum_j t_j \left( a - \sum_i s_i f(g_i^{-1} h_j) \right) \right| + \sum_i s_i \left( \sum_j t_j f(g_i^{-1} h_j) - b \right)$$

$$\leq \sum_j t_j \max_i \left| a - \sum_i s_i f(g_i^{-1} h_j) \right| + \sum_i s_i \max_j \left| \sum_j t_j f(g_i^{-1} h_j) - b \right| < \varepsilon.$$

We define then $m(f)$ as the unique constant in $Q_l(f) \cap Q_r(f)$. Properties (a), (b), (c) and (d) are obvious, as well as the fact that $m(\alpha f) = \alpha m(f)$ for all $\alpha \in \mathbb{C}$ and $f \in \text{WAP}(G)$. We are left to prove that, for $f_1, f_2 \in \text{WAP}(G)$, one has $m(f_1 + f_2) = m(f_1) + m(f_2)$. Let us fix $\varepsilon > 0$. There exist $s_1, \ldots, s_m > 0$ such that $\sum_i s_i = 1$, and $g_1, \ldots, g_m \in G$ such that

$$\left\| \sum_i s_i g_i \cdot f_1 - m(f_1) \right\|_{\infty} \leq \frac{\varepsilon}{2},$$

for every $g \in G$. But $m(f_2) = m \left( \sum_i s_i g_i \cdot f_2 \right)$. Indeed, $\sum_i s_i g_i \cdot f_2 \in Q_l(f_2)$, and as the latter set is convex, we have

$$Q_l \left( \sum_i s_i g_i \cdot f_2 \right) \subset Q_l(f_2).$$
Hence the constant in the left-hand convex set is equal to the one in $Q_l(f_2)$. Thus there exist $t_1, \ldots, t_n > 0$ such that $\sum_j t_j = 1$, and $h_1, \ldots, h_n \in G$ such that

$$\left\| m(f_2) - \sum_{i,j} s_i t_j (h_j g_i) \cdot f_2 \right\|_\infty = \left\| m\left( \sum_i s_i g_i \cdot f_2 \right) - \sum_j t_j h_j \cdot \left( \sum_i s_i g_i \cdot f_2 \right) \right\|_\infty \leq \frac{\varepsilon}{2}.$$ 

As $\sum_{i,j} s_i t_j = 1$, one has

$$\sum_{i,j} s_i t_j (h_j g_i) \cdot [f_1 + f_2] \in Q_l(f_1 + f_2)$$

and

$$\left\| \sum_{i,j} s_i t_j (h_j g_i) \cdot [f_1 + f_2] - m(f_1) - m(f_2) \right\|_\infty \leq \left\| \sum_{i,j} s_i t_j (h_j g_i) \cdot f_1 - m(f_1) \right\|_\infty + \left\| \sum_{i,j} s_i t_j (h_j g_i) \cdot f_2 - m(f_2) \right\|_\infty$$

$$\leq \sum_j t_j \left\| \sum_i s_i h_j \cdot g_i \cdot f_1 - m(f_1) \right\|_\infty + \frac{\varepsilon}{2}$$

$$= \sum_j t_j \left\| h_j \cdot \left( \sum_i s_i g_i \cdot f_1 - m(f_1) \right) \right\|_\infty + \frac{\varepsilon}{2} \leq \varepsilon.$$ 

This shows that $m(f_1) + m(f_2) \in Q_l(f_1 + f_2)$. The proof is now complete. \hfill \square

References

[1] V. Bergelson and J. Rosenblatt. Mixing actions of groups. Illinois J. Math., 32:65–80, 1988.
[2] P. de la Harpe. Moyennabilité du groupe unitaire et propriété P de Schwarz des algèbres de von Neumann. In P. de la Harpe, editor, Algèbres d’Opérateurs, volume 725 of Lecture Notes in Mathematics, pages 220–227. Springer-Verlag, Berlin, 1979.
[3] J. Dixmier. Von Neumann Algebras. North-Holland, Amsterdam, New York, Oxford, 1981.
[4] P. Eymard. Initiation à la théorie des groupes moyennables. Lecture Notes in Math., 497, 1975.
[5] E. Glasner. Ergodic Theory via Joinings. Amer. Math. Soc., 1981.
[6] F.P. Greenleaf. Invariant means on topological groups. Van Nostrand, New York, 1969.
[7] A. Grothendieck. Critères de compacité dans les espaces fonctionnels généraux. Amer. J. Math., 74:168–186, 1952.
[8] A.L. Paterson. Amenability. American Mathematical Society, Providence, Rhode Island, 1988.
[9] V.I. Paulsen. Completely bounded maps and dilations. Pitman Research Notes in Mathematics, New York, 1986.
[10] C. Pop. Finite sums of commutators. Proceedings of the Amer. Math. Soc., 130:3039–3041, 2002.
[11] J. Schwartz. Two finite, non-hyperfinite, non isomorphic factors. *Comm. Pure Appl. Math.*, 16:19–26, 1963.

[12] M. Takesaki. *Theory of Operator Algebras I*. Springer Verlag, New York, 1979.

Université de Neuchâtel, Institut de Mathématiques, E.-Argand 11, 2000 Neuchâtel, Switzerland

E-mail address: paul.jolissaint@unine.ch