Private Algorithms with Private Predictions

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Abstract

When applying differential privacy to sensitive data, a common way of getting improved performance is to use external information such as other sensitive data, public data, or human priors. We propose to use the algorithms with predictions framework—previously applied largely to improve time complexity or competitive ratios—as a powerful way of designing and analyzing privacy-preserving methods that can take advantage of such external information to improve utility. For four important tasks—quantile release, its extension to multiple quantiles, covariance estimation, and data release—we construct prediction-dependent differentially private methods whose utility scales with natural measures of prediction quality. The analyses enjoy several advantages, including minimal assumptions about the data, natural ways of adding robustness to noisy predictions, and novel “meta” algorithms that can learn predictions from other (potentially sensitive) data. Overall, our results demonstrate how to enable differentially private algorithms to make use of and learn noisy predictions, which holds great promise for improving utility while preserving privacy across a variety of tasks.

1 Introduction

Algorithms with predictions, also known as learning-augmented algorithms, is an expanding field of algorithm design that constructs methods whose instance-dependent performance improves with the accuracy of some prediction about the instance [50]. The goal is to bound the cost—e.g. computational complexity—$C_t(w)$ of running an algorithm on instance $t$ given a prediction $w$ by some metric $U_t(w)$ of the quality of the prediction on that instance. Motivated by practical success [40] and as a type of beyond-worst-case analysis [52], learning-augmented algorithms have been introduced for a wide variety of problems targeting competitive ratios in online algorithms [5, 9, 20, 26, 32, 34, 41, 47, 51], improved space complexity in streaming algorithms [24], and improved time complexity in graph algorithms [22, 17, 53] and distributed systems [42, 45, 54].

Departing from metrics that encode computation and communication costs, we propose to design algorithms with predictions that target utility loss due to imposition of differential privacy (DP) [27]. Here, each instance $t$ corresponds to a dataset $X_t$ and the algorithm must estimate some quantity (for example, the median) while making it information-theoretically hard to use the output to learn anything about any specific element of $X_t$. These algorithms are necessarily randomized and typically involve adding some noise to the output. Therefore, DP algorithms often come with worst-case high-probability bounds on how far away the released estimate is from the truth; these utility guarantees are what we will seek to improve using predictions $w$. This idea is well-motivated by the abundance of related datasets, public information, and human knowledge that can be leveraged to inform a DP algorithm, and indeed there are many related approaches such as public-private methods [3, 10, 11, 44, 46] and private posterior inference [21, 55]. Our approach has numerous comparative advantages, including proving utility guarantees with minimal assumptions and adapting existing algorithms with predictions tools to study the robustness of the algorithms [47] and the learning of the predictions [38].

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1.1 Overview of related work

To improve the accuracy of differentially private methods, there has been significant work on incorporating external information into DP algorithms. A major line of work in this area is the public-private framework, in which a method is given access to some amount of public data that is related in some way to the private data [10, 11, 46]. Here, the use of this public data can be viewed as using a prediction, but the guarantees for these approaches generally assume the public and private data are both drawn i.i.d. from related distributions, whereas we make no distributional assumptions on either. Furthermore, our framework allows us to ensure robustness to poor predictions without making assumptions about the data distribution and to derive learning algorithms even for the setting where the training data itself may be sensitive. Another approach is to treat DP mechanisms (e.g. the exponential mechanism) as Bayesian posterior sampling [21, 55], although we are not aware of any prior-dependent utility guarantees from this literature. We adapt this approach for quantile estimation, although the predictions-based framework that we advocate is much broader, as many DP methods combine multiple queries that must be considered jointly.

Our approach for augmenting DP algorithms with external information is centered around the algorithms with predictions framework [50]. Past work in this setting has largely focused on using predictions to improve metrics related to time, space, and communication complexity. Generally, data-driven tuning of DP algorithms has been an important topic in private machine learning, e.g. for hyperparameter tuning [16] and federated learning [6], but these have not to our knowledge considered incorporating per-instance predictions. Our work does incorporate existing techniques from algorithms with predictions, including robustness-consistency tradeoffs [47] and the online learning of predictions [38]. The latter we are able to extend to the DP setting by building upon the extensive work on private online learning [33, 57, 2, 35], where we make a contribution of independent interest: a DP online convex optimization method that can be customized using any strongly-convex regularizer to obtain better regret guarantees on different geometries (c.f. Theorems 2.5 and 5.1).

2 Overview of results

We begin with a section reviewing our key results, grouped by the main aspects of the framework: developing prediction-dependent upper bounds on the error, ensuring robustness, and learning predictions. We then give the complete technical details, formal statements, and proofs in subsequent sections. While our focus is on four specific applications, we believe the algorithms with predictions setup has great potential as a general approach to designing and analyzing privacy-preserving algorithms that incorporate external information.

2.1 Preliminaries

We use \([n]\) to denote the sequence \((1, \cdots, n)\), \(x_{[i]}\) to denote the \(i\)th element of a vector \(x\), and \(X_{[i,j]}\) to denote the \(j\)th element of the \(i\)th row of a matrix \(X\). For both vectors and matrices we will use \(\|\cdot\|_p\) to denote the entry-wise \(p\)-norm, and for the latter we will use \(\|\cdot\|_2\) to denote the Schatten \(p\)-norm; thus \(\|\cdot\|_2 = \|\cdot\|_F\) is the Frobenius norm, \(\|\cdot\|_\infty\) is the spectral norm, and \(\|\cdot\|_1 = \|\cdot\|_\text{Tr}\) is the trace or nuclear norm. We will use \(\|\cdot\|_*\) to refer to the dual norm of \(\|\cdot\|\), and for any dataset \(X\) we use \(|X|\) to denote the number of entries. We use \(0_d\) and \(1_d\) to denote the \(d\)-dimensional all-zero and all one vectors, \(1_S\) to denote the indicator on set \(S\), \(x \sim \text{Lap}(b)\) to denote sampling a Laplace r.v. with mean zero and scale \(b\), \(x \sim \mathcal{N}(0_d, \sigma^2)\) to denote sampling a Gaussian vector with mean \(0_d\) and covariance \(\sigma^2I_d\), and \(\Delta_d\) to denote the simplex on \(d\) elements. For simplicity, for any probability measure \(\mu : (a, b) \mapsto \mathbb{R}_{\geq 0}\) we use \(\mu(I) = \int_I \mu(dx)\) to denote the probability it assigns to any interval \(I \subset (a, b)\). Unless otherwise specified, \(\tilde{O}\) will be used to ignore logarithmic factors in standard asymptotic notation.
2.2 Prediction-dependent utility bounds

As noted in the introduction, the basic requirement for a learning-augmented algorithm $A$ is that the cost $C_t(w)$ of running it on an instance $t$ with prediction $w$ should be upper bounded—usually up to constant or reasonable polylogarithmic factors—by a metric $U_t(w)$ of the quality of the prediction on the instance. Beyond consistency—making the cost small when the metric is small—it is also desirable for $U_t(w)$ to encode insightful information about what makes a useful prediction. The first of these properties is shared by existing guarantees in the DP literature, e.g. via the KL-divergence or TV-distance between the source and target distributions in the public-private literature [11, 46]. However, the metrics we provide will have fewer assumptions and will also be more amenable to robustness and learning analysis.

Unlike past work on algorithms with predictions, in our work the cost $C_t(w)$ will be the error of a privately released statistic, as compared to some ground truth. While the framework does not depend on it, in this work we will mainly focus on the following standard notion of privacy:

**Definition 2.1 ([27]).** Algorithm $A$ is $(\varepsilon, \delta)$-differentially private (DP) if for all subsets $S$ of its range we have $\Pr\{A(X) \in S\} \leq e^{\varepsilon} \Pr\{A(\tilde{X}) \in S\} + \delta$ for all neighboring $X$ and $\tilde{X}$, denoted $X \sim \tilde{X}$.

Note that we will call an algorithm $\varepsilon$-DP if it is $(\varepsilon, 0)$-DP according to the above definition. For easier comparison to past prediction-free results, we will define neighboring datasets differently depending on the application; specifically, for quantile release we use add-remove privacy, where $X$ can be obtained from $\tilde{X}$ by adding or removing an entry, while for covariance estimation and data release we use swap privacy, in which $X$ can be obtained from $\tilde{X}$ by replacing one entry with another. A general way of preserving privacy can be obtained using the exponential mechanism:

**Theorem 2.1 ([49]).** If the utility $u(X, o)$ of an outcome $o$ of a query over dataset $X$ has sensitivity $\max_{o, X-\tilde{X}} |u(X, o) - u(\tilde{X}, o)| \leq \Delta$ then the exponential mechanism (EM), which releases $o$ w.p. $\propto \exp(-\frac{\varepsilon}{2\Delta} u(X, o)) \mu(o)$ for some base measure $\mu$ over the outcome domain, is $\varepsilon$-DP.

2.2.1 Single-quantile estimation

The base measure $\mu$ of DP mechanisms such as the exponential is the starting point of many approaches to incorporating external information, especially ones focused on Bayesian posterior sampling [21, 55]; while it is also our approach to single-quantile estimation with predictions, a key difference is that the focus on utility guarantees depending on both the prediction and instance, which is missing from this past work. In the quantile problem, given a quantile $q$ and a sorted dataset $x \in \mathbb{R}^n$ of $n$ distinct points, the goal is to release a number $o$ that upper bounds exactly $\lfloor qn \rfloor$ of the entries. A natural error metric, $\text{Gap}_q(x, o)$, is the number of entries between the released number $o$ and $\lfloor qn \rfloor$.

A straightforward application of EM with utility $-\text{Gap}_q$ outputs $o$ that satisfies w.p. $\geq 1 - \beta$

$$\text{Gap}_q(x, o) \leq \frac{2}{\varepsilon} \log \frac{1}{\beta \Psi_q^{(e)}(x, \mu)} \leq \frac{2}{\varepsilon} \log \frac{1}{\beta \Psi_q^*(x, \mu)} \tag{1}$$

where $\Psi_q^{(e)}$ is the inner product $\int \exp(-\frac{\varepsilon}{2} \text{Gap}_q(x, o)) \mu(o) do$ between the prior and the EM score and $\Psi_q^*$ is the probability $\mu([x_{\lfloor qn \rfloor}, x_{\lfloor qn \rfloor + 1}])$ that the prior assigns to the optimal interval. Both make intuitive sense as metrics of prediction quality: we expect predictions that assign a high probability to intervals that the EM score weighs heavily to perform well, and the heaviest weight is of course assigned to the optimal interval. They thus suggest two candidates for prediction-dependent error bounds $U_t(\mu)$: $-\log \Psi_q^{(e)}$ and its upper bound $-\log \Psi_q^*$. While the former is tighter...
and more amenable for both learning and robustness, the latter is useful for a quick comparison with prediction-free methods, and to highlight the insight our approach provides.

In particular, when the data is in a finite interval \((a, b)\) and \(\mu(o) = 1_{o \in (a, b)}/(b - a)\) is the uniform measure then \(\Psi^*(x, \mu) \geq \psi_x/(b - a)\), where \(\psi_x = \min_k x_{[k+1]} - x_{[k]}\) is the minimum distance between entries; thus we recover past guarantees, e.g. [36, Lemma A.1], that implicitly use the uniform measure to guarantee \(\text{Gap}_q(x, o) \leq \frac{2}{\varepsilon} \log \frac{b - a}{\beta \psi_x}\). However, analyzing \(- \log \Psi^*_q\) also allows us a novel way of removing the boundedness assumption. In particular, if we suspect the data lies in an interval \((a, b)\) we can set \(\mu\) to be the Cauchy distribution with location \((a + b)/2\) and scale \((b - a)/2\). Even if we are wrong about the interval, there must be a scalar \(R > 0\), s.t. the data lies in the interval \(((a + b)/2 \pm R(b - a)/2)\), so using this Cauchy prior yields \(\Psi^*_q(x, \mu) \geq \frac{2\psi_x/(b - a)}{\pi(1 + R^2)}\) and thus the following guarantee:

**Corollary 2.1** (of Lem. 3.1). If the data lies in the interval \((\frac{a+b}{2} \pm \frac{R}{2}(b - a))\) and we set \(\mu\) to be the Cauchy measure with location \(\frac{a+b}{2}\) and scale \(\frac{b-a}{2}\) then the output of the exponential mechanism satisfies \(\text{Gap}_q(x, o) \leq \frac{2}{\varepsilon} \log \frac{\pi(1 + R^2)/(b - a)}{\beta \psi_x} w.p. \geq 1 - \beta\).

If \(R = 1\), i.e. we get the interval exactly right, then this yields a bound on \(\text{Gap}_q\) only an additive factor \(\frac{2}{\varepsilon} \log \pi\) worse than the uniform prior, but if we are wrong then performance degrades gracefully as \(O(\log \frac{1+R^2}{\beta \psi_x})\), unlike the unbounded error of the uniform prior. Notably the use of a heavy-tailed distribution is crucial here; using one with sub-exponential tails in place of the Cauchy would lead to a linear rather than logarithmic dependence on \((1 + R^2)/\psi_x\). While this first result can be viewed as designing a better prediction-free algorithm, it can also be viewed as making more robust use of the external information about the interval containing the data.

### 2.2.2 Releasing multiple quantiles

For simultaneously estimating multiple quantiles \(q_1, \ldots, q_m\) we adapt the ApproximateQuantiles method of [36], a recursive approach that assigns each \(q_i\) to a node in a binary tree and, starting from the root, uses EM with the uniform prior to estimate a quantile before sending the data below the outcome o to its left child and the data above o to its right child. This way each data point is only involved in \([\log_2 m]\) exponential mechanisms, and so for data known to lie in a finite interval \((a, b)\) the maximum \(\text{Gap}_{q_i}\) across quantiles is \(O\left(\frac{\log^2 m}{\varepsilon} \log \frac{m(b - a)}{\beta \psi_x}\right)\), which is much more than the naive bound of a linear function of \(m\).

While we can extend our earlier analysis to get similar polylog\((m)\) bounds—notably we can extend our Cauchy-based unbounded interval result to multiple quantiles—the associated error metric \(U_t\) is non-deterministic: it depends on the execution of the DP-algorithm via inner products between the priors and EM scores computed over recursive splits of the data. This makes it less desirable because it does not encode a relationship between the prediction and instance data, and it also makes it less amenable to learning.

We would thus like to derive guarantees depending on a more natural metric, e.g. one aggregating \(\Psi_{q_i}(x, \mu_i)\) from the previous section across (quantile, prior) pairs \((q_i, \mu_i)\). The core issue with this is that the recursive splitting makes the probability assigned by a prior \(\mu_i\) to data outside the interval induced by the outcomes of quantiles earlier in the tree not affect its own outcome. One way to handle this is to assign this probability mass to the edge of this induced interval, rather than the more natural conditioning approach taken by ApproximateQuantiles. We refer to this as “edge-based prior adaptation” and use it to show a guarantee on the maximum \(\text{Gap}_{q_i}\) that depends on the harmonic mean \(\bar{\Psi}\) of the inner products \(\Psi_{q_i}(x, \mu_i)\):
**Theorem 2.2** (c.f. Thm. 3.1). If \( q_1, \ldots, q_m \) are uniform negative powers of two and for each we have a prior \( \mu_i : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \) then running \textit{ApproximateQuantiles} with edge-based prior adaptation is \( \varepsilon \)-DP and guarantees the following w.p. \( \geq 1 - \beta \):

\[
\max_i \text{Gap}_{q_i}(x, o_i) \leq \frac{2}{\varepsilon} \phi^{|\log_2 m| \log_2 m} \log \frac{m}{\beta \Psi} \quad \text{for} \quad \Psi = \left( \sum_{i=1}^{m} \frac{1/m}{\Psi_{q_i}(x, \mu_i)} \right)^{-1}
\]

(2)

Here \( o_i \) is the estimate released for quantile \( q_i \), \( \varepsilon \) = \( \varepsilon /|\log_2 m| \), and \( \phi = \frac{1+\sqrt{5}}{2} \) is the golden ratio.

Note that golden ratio appears due to the Fibonacci-type recurrence bounding the maximum \( \text{Gap}_{q_i} \), across depths of the binary tree. While the error metric now depends solely on the predictions and the instance data, the dependence of the error on \( m \) is now worse than that of \textit{ApproximateQuantiles}, as \( \phi^{|\log_2 m|} \) is roughly \( \mathcal{O}(m^{0.7}) \). Note however that it is still sublinear and thus better than the naive baseline of running \( m \) independent exponential mechanisms.

This worse dependence on \( m \) results from error compounding across depths of the tree, so by reducing the depth one might hope to do better. We can do this by increasing the branching factor from \( K = 2 \) to arbitrary \( K \), thus going from a binary tree to a \( K \)-ary tree. Doing so involves running EM \( K - 1 \) times at each node—and paying \( K - 1 \) more in privacy budget—in order to split the data into \( K \) subsets; the approach also introduces the possibility of estimates being out-of-order at each node. However, by showing that sorting them back into order does not increase the error and then controlling the maximum \( \text{Gap}_{q_i} \) at each depth via another recurrence relation we can show the following result:

**Theorem 2.3** (c.f. Thm. 3.2). For any \( q_1, \ldots, q_m \in (0, 1) \), running \textit{ApproximateQuantiles} with branching factor \( K = \lfloor \exp(\sqrt{\log m \log 2}) \rfloor \), edge-based prior adaptation, and appropriate budgets \( \varepsilon_i \) satisfies \( \varepsilon \)-DP and w.p. \( \geq 1 - \beta \) guarantees \( \max_i \text{Gap}_{q_i}(x, o_i) \leq \frac{2\alpha^2}{\varepsilon} \exp \left( 2\sqrt{\log(2) \log(m)} \right) \log \frac{m}{\beta \Psi} \).

The rate in \( m \) of this result is both sub-polynomial and super poly-logarithmic (both \( o(m^\alpha) \) and \( \omega(\log^\alpha m) \forall \alpha > 0 \); thus while still asymptotically worse than the prediction-free original result [36], for almost any practical value of \( m \) the bound does not exceed a small constant times \( \log^2 m \). Therefore if the error \(-\log \Psi\) of the prediction is small—i.e. the inner product between the prior and the EM score is large on (harmonic) average—then we may hope to do much better with this approach.

### 2.2.3 Covariance estimation

While encoding predictions in via base measures of DP mechanisms is a natural starting point, it is not the only way of doing so. In our last two applications we instead start with existing algorithms whose errors have explicit or implicit dependence on some measure of complexity of the data and use this to convert them into algorithms with predictions. The errors will then have an (explicit) dependence on a related measure of the error between the data and a point (rather than distributional) prediction, leading to highly interpretable bounds \( U_i(w) \) on the utility loss.

For covariance estimation specifically we take advantage of recent “trace-sensitive” results, which bound the Frobenius error between the covariance matrix \( C = XX^T/n \) of a dataset \( X \in \mathbb{R}^{d \times n} \) by some function of its trace [4, 23]. Since the core component of these algorithms is a DP estimate of a symmetric \( d \times d \) matrix, if we have a symmetric prediction \( W \in \mathbb{R}^{d \times d} \) we can try to use the methods to instead privately estimate the error \( C - W \) and then add \( W \) to the result; we can then hope to show that the error depends on the trace norm \( \|C - W\|_{\text{Tr}} \) of the error rather than the trace of \( C \). We achieve exactly this by extending the analysis in this prior work to the negative spectrum, in order to handle the possibly negative eigenvalues of \( C - W \). The result below, for the learning-augmented extension of the state-of-the-art \textit{SeparateCov} algorithm [23], is characteristic of these results, and indeed recovers their bound for \( W = 0_{d \times d} \):
Theorem 2.4 (c.f. Thm. 3.3). If $X \in \mathbb{R}^{d \times n}$ has columns bounded by 1 in $\ell_2$-norm then applying SeparateCov to $C - W$ and obtaining $\hat{C}$ by adding $W$ to the result is $\varepsilon$-DP and w.p. $\geq 1 - \beta$ guarantees

$$\|\hat{C} - C\|_F^2 \leq \mathcal{O}\left(\frac{d}{\varepsilon^2 n^2} + \frac{d\sqrt{d}}{\varepsilon n} \|C - W\|_F\right)$$

2.2.4 Data release

In our last application we study private data release, where we seek to construct a synthetic dataset $\hat{x} \in \mathbb{R}^d$ using sensitive data $x \in \mathbb{Z}_d$ such that the maximum error of a finite set $Q$ of linear queries $q \in [-1,1]^d$ is bounded. To do so we use the well-known MWEM method of [30], which has an implicit dependence on the KL-divergence $D_{KL}(x/n||1_d/d)$ between the data distribution and the uniform distribution it uses to initialize its iterative approach; by instead initializing with a prediction $w \in \triangle_d$ in the $d$-dimensional simplex one can instead obtain a dependence on $D_{KL}(x/n||w)$:

Lemma 2.1 (c.f. Lem. 3.4). Initializing MWEM with $w \in \triangle_d$ and running it for $m$ iterations on dataset $x$ is $\varepsilon$-DP and w.p. $\geq 1 - \beta$ produces a synthetic dataset $s.t. the largest mean squared error of any linear query in $Q$ is bounded by $O\left(\frac{d}{m} D_{KL}(x||w) + \frac{m^2}{\varepsilon n} \log^2 \frac{m}{\varepsilon} \log |Q|\right)$, where $n = \|x\|_1$.

Note that the observation that MWEM can be initialized non-uniformly is not novel, having been used by both the original authors and by subsequent public-private work [46]. However, our analysis in subsequent sections reveals interesting aspects of our framework, especially w.r.t. learning, that this prior work does not consider as closely, such as how the optimal choice for other parameters of the algorithm are influenced by the prediction quality.

2.3 Robustness-consistency tradeoffs

While we incorporate external information into DP-algorithms because we hope to improve performance, if not done carefully it may also lead to worse results. This is also true of the algorithms we have introduced thus far; as discussed before if we specify a quantile prior that is concentrated far away from the data then our bound depends linearly on the reciprocal of the minimum gap between data points. As another example, for covariance matrices we might specify a prediction matrix $W$ with arbitrarily high trace distance to the ground truth. Ideally an algorithm that uses a prediction will be robust, i.e. revert back to worst-case guarantees if the prediction is poor, without significantly sacrificing consistency, i.e. performing well if the prediction is good. Conveniently, the algorithms with prediction framework provides a formal way of quantifying this tradeoff [47, 50]:

Definition 2.2 ([50]). An algorithm with predictions is $r$-robust and $c$-consistent if for all predictions its cost is at most $r$ and as the prediction error tends to zero its cost is at most $c$.

In the context of differential privacy, costs measure the quality of the released statistic (e.g. Gap$_q$ for quantile release or Frobenius error for covariance estimation) while prediction error measures the quality of our prior knowledge (e.g. $-\log \Psi_q^{(e)}$ or trace error). A common variant of this tradeoff is the parameterized robustness-consistency tradeoff, for a parameter $\lambda \in [0,1]$ the method is $r_\lambda$-robust and $c_\lambda$-consistent [9, 50]. In all four settings that we study, the metrics $U_i$ that we develop are amenable to this robustness-consistency analysis. These results demonstrate the advantage of our framework in providing ways to design algorithms that can make robust use of possibly noisy predictions. Notably, the only related public-private work that we know of that studies robustness still assumes source and target data are Gaussian [11], whereas we make no distributional assumptions.
For both single and multiple quantile release, interpolating the prediction-based prior with a "robust" prior allows for an elegant parameterized tradeoff between robustness and consistency. In particular, since $\Psi_q^\varepsilon(x, \mu)$ is linear we can pick $\rho$ to be some fixed "robust" prior such as the uniform or Cauchy and for any prediction $\mu$ use EM with the prior $\mu(\lambda) = (1 - \lambda)\mu + \lambda\rho$. Substituting $\Psi_q^\varepsilon(x, \mu(\lambda)) = (1 - \lambda)\Psi_q^\varepsilon(x, \mu) + \lambda\Psi_q^\varepsilon(x, \rho)$ into the bound yields the following tradeoff:

**Corollary 2.2** (of Lem. 3.1; c.f. Cor. 4.1). For quantile $q$ over dataset $x$, applying EM with prior $\mu(\lambda) = (1 - \lambda)\mu + \lambda\rho$ is $\left(\frac{2}{\varepsilon} \log \frac{1/\beta}{\lambda\Psi_q^\varepsilon(x, \rho)}\right)$-robust and $\left(\frac{2}{\varepsilon} \log \frac{1/\lambda}{1 - \lambda}\right)$-consistent.

Thus we simultaneously perform at most an additive factor $\frac{2}{\varepsilon} \log \frac{1/\lambda}{1 - \lambda}$ worse than that of only using the robust prior $\rho$ and w.p. $\geq 1 - \beta$ only have error $\frac{2}{\varepsilon} \log \frac{1/\lambda}{1 - \lambda}$ if the prediction $\mu$ is perfect, i.e. if it assigns all probability mass to the optimal interval. This result is straightforward to extend to the multiple-quantile metric $-\log \Psi$. In fact, we can go further and interpolate between the polylog($m$) prediction-free guarantee of past work and our learning-augmented guarantee with the worse dependence on $m$; thus if the prediction is not good enough to overcome the worse rate in $m$ we can still ensure that we do not do much worse than the polylog($m$) guarantee.

**Corollary 2.3** (of Lem. 3.2 & Thm. 3.1; c.f. Cor. 4.2). If we run ApproximateQuantiles over data in the interval $((a + b)/2 \pm R(b - a)/2)$ for unknown $R > 0$ and we use the prior $\mu_i(\lambda) = (1 - \lambda)\mu_i + \lambda\rho$ for each quantile $q_i$, where $\rho$ is the Cauchy prior with location $(a + b)/2$ and scale $(b - a)/2$, then the algorithm is $\left(\frac{2}{\varepsilon} \log_2 m\right)^2 \log \frac{2m(b - a)}{\lambda(1 + R^2)} log_2 \exp\left(\frac{1}{1 - \lambda}\right)$-robust and $\left(\frac{2}{\varepsilon} \log_2 m\right) \log m/\beta$-consistent.

We take a different approach ensuring robustness for covariance estimation, for which we privately check if the quality metric $U_t(W) = \|XX^T/n - W\|_F$ of the prediction $W \in \mathbb{R}^{d \times d}$ is better than $U_t(0_{d \times d})$, i.e. that of the default. In doing so we pay for robustness by a factor of $\sqrt{d}$ in the leading (non-trace-sensitive) term, although as we discuss later this may be an artifact of the setting, e.g. for zCDP we do not pay any (asymptotic) cost.

**Corollary 2.4** (of Thm. 3.3; c.f. Cor. 4.3). Running SeparateCov with the prediction $W$ only if its trace distance $\|XX^T/n - W\|_F$ is smaller than $\|XX^T/n\|_F$ according to the Laplace mechanism is $\tilde{O}\left(\frac{d \sqrt{d}}{\varepsilon n} \left(\frac{1}{\varepsilon n} + \|XX^T/n\|_F\right)\right)$-robust and $\tilde{O}\left(\frac{d \sqrt{d}}{\varepsilon n^2}\right)$-consistent.

Lastly, for data release we go back to an interpolation-based approach, although here we are mixing finite-dimensional vectors rather than probability distributions. Note that using the uniform prior guarantees $\tilde{O}\left(3\sqrt{n \log^2 d \varepsilon^{-2}}\right)$ error, so since the data-dimension $d$ can be very large in this application, if we use small enough $\lambda$ we can obtain a strong advantage under perfect predictions while ensuring performance similar to the prediction-free guarantee.

**Corollary 2.5** (of Lem. 3.4; c.f. Cor. 4.4). There exists a fixed number of iterations s.t. using $w(\lambda) = (1 - \lambda)w + \lambda 1_d/d$ instead of the prediction $w \in \Delta_d$ to initialize MWEM is $\tilde{O}\left(3\sqrt{n \log d \varepsilon^{-2}}\right)$-robust, and $\tilde{O}\left(\lambda^2 n \log^2 d \varepsilon^{-2}\right)$-consistent, where $n$ is the number of records.
2.4 Learning predictions, privately

As with public-private and other approaches, our starting focus is on designing DP algorithms that make use of external information and providing utility guarantees. However, while past approaches depend largely on data drawn i.i.d. from source and target distributions—which our methods can also use—the minimal assumptions made by our framework allows us to study how to obtain predictions from past instances, using the quality metrics we derive as objective functions. Notably, there is an additional challenge as the training instances can themselves contain sensitive data. As far as we are aware, learning upper bounds of DP (learning) algorithms across sensitive tasks has only been previously done by [43], who consider the meta-learning setting and largely focus on a federated setup motivated by training deep nets.

Specifically we target the online learning setting, in which we sequentially see $T$ datasets $X_1, \ldots, X_T$ and at each time $t = 1, \ldots, T$ must choose a prediction $w_t \in W$ to provide to a DP algorithm on task $X_t$. The goal is to have low-regret $\sum_{t=1}^{T} U_t(w_t) - \min_{w \in W} \sum_{t=1}^{T} U_t(w)$ w.r.t. any fixed $w$ in the prediction domain $W$, where each $U_t(w)$ bounds the cost $C_t(w)$ of running an algorithm with prediction $w$ on instance $t$. This goal aligns well with the important scenario in which one is faced with sequentially estimating statistics from many related datasets. Using online-to-batch conversion one can also convert regret bounds to excess risk guarantees for the case where the DP tasks are drawn i.i.d. from a fixed distribution.

We make the important stipulation that the online learner must be private, i.e. the predictions themselves cannot reveal anything about previous datasets. This allows us to budget for both running the DP algorithm on $X_t$—where we run a $(\varepsilon', \delta')$-DP online learner—and using the dataset’s data for learning. It also makes our problem more challenging, since as we will see some online learning techniques for handling more challenging losses are not obviously transferable to the private setting. One important tool that we do extend—and which is a contribution of independent interest—is adaptation to non-Euclidean geometries. In particular, similar to past work on learning predictions [38], the domains our predictions are learned over are not $\ell_2$-balls but rather simplices or trace-norm balls. On the other hand, most DP online learning results do not adapt to such domains, so to obtain better dimension-dependence—and sometimes even better rates in $T$—we extend the recent DP-FTRL method of [35] to non-Euclidean regularizers:

**Theorem 2.5** (c.f. Thm. 5.1). Suppose we run DP-FTRL (Algorithm 3) with step-size $\eta > 0$ and regularizer $\phi : \Theta \to \mathbb{R}$ that is strongly-convex w.r.t. $\| \cdot \|$ on losses that are $L$-Lipschitz w.r.t. $\| \cdot \|$ and whose gradients have $\ell_2$-sensitivity $\Delta_2$. Then the regret w.r.t. any $\theta^* \in \Theta$ is bounded w.h.p. by

$$\frac{\phi(\theta^*) - \phi(\theta_1)}{\eta} + \tilde{O}(\eta L (L + G\sigma \Delta_2) T)$$

(4)

where $\sigma$ is the noise used by tree aggregation and $G$ is the Gaussian width of the unit $\| \cdot \|$-ball. Furthermore, for any $\varepsilon' \leq 2 \log \frac{1}{\delta'}$, setting $\sigma = \frac{1}{\sqrt{2\log T} \log \frac{1}{\delta'}}$ makes the algorithm $(\varepsilon', \delta')$-DP.

While the result still has some dependence on the Euclidean norm due to the $\ell_2$-sensitivity $\Delta_2$, we can still achieve strong improvements via the non-Euclidean Lipschitz constants $L$ and by measuring the noise due to privacy using a different norm; as an example, while the Gaussian width of the $\ell_2$-ball is $\mathcal{O}(\sqrt{d})$, that of the $\ell_1$-ball is $\mathcal{O}(\sqrt{\log d})$. We believe this is the first DP online convex optimization method that can be customized to different geometries on $\mathbb{R}^d$ using any strongly-convex regularizer: past work we are aware of make significant restrictions on either the losses [2] or the regularizer [10]. Besides the interest in this contribution on its own, we also believe this extension of DP-FTRL will be useful in future work on private algorithms with predictions, as its non-Euclidean variant leads to better bounds in all of the applications we consider.
2.4.1 Learning discretized priors for quantile estimation

Equipped with this result, we start with the most difficult learning problem from among our settings: learning priors for quantile estimation. In this overview we only discuss the single quantile setting, as our results for multiple quantiles are largely natural generalizations. The key issue we face is that optimizing over spaces of probability measures is highly nontrivial, especially because the target function

\[
U_t(\mu) = -\log \psi \left( \frac{1}{T} \sum_{o=1}^{T} \text{Gap}_q(x, o) \right) \mu(o) \, do
\]

is unbounded when the overlap between the prior and the EM score is small. We will thus instead optimize over a simpler class consisting of discrete, robust priors \( \mu_w \) that are piecewise constant over each element of a uniform partition of the finite interval \((a, b)\); these can be defined using vectors \( w \in \Delta_d \) in the \( d \)-dimensional simplex whose entries are lower-bounded by \( \lambda \). This converts the objectives to

\[
U_t(\mu_w) = -\log \langle g_t, w \rangle;
\]

the discretization thus allows finite-dimensional methods from DP online learning to be applied while the robustness keeps the loss Lipschitz over the optimization domain.

By running DP-FTRL with entropic regularizer

\[
\phi(w) = w \log w,
\]

effectively DP exponentiated gradient—we have the following guarantee:

**Corollary 2.6** (of Thm. 5.2; c.f. Cor. 5.1). There exists an \( (\varepsilon', \delta') \)-DP online learner whose regret w.r.t. all \( \lambda \)-robust and discrete \( d \)-dimensional priors is \( \tilde{O} \left( \frac{d}{\lambda} \left( 1 + \frac{\min \{1, d \}}{\varepsilon} \right) T \right) \) w.h.p.

Note that while this statement makes a strong restriction on the class of priors, by bounding the approximation error and allowing the dimension to be set depending on the time horizon \( T \) we also show how to use this approach to compete with any (non-robust) Lipschitz prior over the bounded interval \((a, b)\), albeit with a worse regret rate. Moreover, even the restricted class of priors includes the uniform distribution, ensuring that we asymptotically perform at least as well as a prior-free approach. An interesting aspect of the result is that for small enough \( \varepsilon \)—the privacy budget allocated to running the target DP-algorithm—the regret *improves* with additional privacy. This is due to a refined analysis of the \( \ell_2 \)-sensitivity that shows that the gradient is less sensitive under higher instance-privacy.

2.4.2 Learning to predict covariance matrices over the trace ball

For covariance estimation we are faced with a more standard learning problem: compete with the optimal matrix \( W \in \mathbb{R}^{d \times d} \) over the functions \( U_t(W) = \|X_tX_t^T/n - W\|_T \). Nevertheless, we can obtain a \( \sqrt{d} \)-factor improvement in the regret—and a corresponding \( d \)-factor improvement in sample complexity—by applying DP-FTRL with a Schatten \( p \)-norm regularizer, which applies \( p \)-norm regularization to the spectrum of the matrix [25]. We provide the result here, along with an example of how it extends to the statistical setting where the datasets \( X_t \) are all drawn i.i.d. from the same distribution:

**Theorem 2.6** (c.f. Thm. 5.3). There exists an \( (\varepsilon', \delta') \)-DP online learner whose regret w.r.t. all symmetric \( W \in \mathbb{R}^{d \times d} \) is bounded w.h.p. by \( \tilde{O} \left( \sqrt{(1 + d/\varepsilon')T} \right) \). Furthermore, if the datasets \( X_t \) are all drawn i.i.d. from the same distribution and we set \( \hat{W} = \frac{1}{T} \sum_{t=1}^{T} W_t \) to be the average iterate then \( T = \Omega \left( \frac{1 + d/\varepsilon'}{\alpha^2} \right) \) samples suffice to ensure that w.h.p. its excess risk is at most \( \alpha \).
2.4.3 Learning to initialize and set the iteration count for data release

Our last learning-theoretic analysis is again in a setting where we are optimizing a non-Lipschitz function—this time the KL-divergence—over the simplex. As we did for quantiles, we approach this using entropic DP-FTRL and a restriction to optimizing over robust priors. However, a key aspect here is that the utility bounds \( U_t \) have the form

\[
\sum_{d=1}^{\infty} D_{KL}(x_{t}/n_t||w) + \tilde{O}\left(\frac{m^2}{\varepsilon n_t}\right),
\]

where \( n_t \) is the number of examples in \( x_t \) and \( m \) is the number of iterations. Thus the optimal setting of \( m \) depends on the similarity between instances: if \( \min_{w} \sum_{t=1}^{T} n_t D_{KL}(x_{t}/n_t||w) \), i.e. the entropy of the average distribution \( \frac{\sum_{t=1}^{T} x_t}{\sum_{t=1}^{T} n_t} \), is small then we can take advantage of this by taking fewer iterations. However, we do not know this entropy \textit{a priori}, so we can instead compete with the best step-size—which will encode the unknown entropy—by simultaneously running both the online learner for \( w \) and the one for \( m \), with the optimization domain being the \( m \)-simplex \( \Delta_m \).

\textbf{Theorem 2.7} (c.f. Thm. 5.4). There exists an \((\varepsilon', \delta')\)-DP algorithm that adaptively sets the initializations \( w_t \in \Delta_d \) and number of iterations \( m_t > 0 \) s.t. the regret w.r.t. the optimal (initialization, iteration) pair \((w, m)\) is \( \tilde{O}\left(\frac{dN^{\frac{1}{2}}}{\lambda \min\{1, \varepsilon'\}} \sqrt{T/\varepsilon^2}\right) \), where \( N = \max_t n_t \) is the maximum number of entries in any dataset \( x_t \).

2.5 Discussion

This work introduces the framework of private algorithms with private predictions, an application of the algorithms with predictions setup to DP methods. We provide extensive evidence of its utility as a way of integrating external information in privacy-preserving algorithms. In particular, our work highlights multiple ways of incorporating predictions—as priors in DP mechanisms, as offsets to be corrected using sensitive data, or as initializations for iterative methods—as well as two ways of making the methods robust to noisy predictions: interpolating with a default prediction or privately checking whether the quality of the default prediction is better. Lastly, we provide a new tool for learning the predictions by targeting measures of prediction quality using DP online convex optimization, and we show its effectiveness on our problems.

We believe this way of studying DP methods is highly applicable and will see a great deal of future work in finding new applications for incorporating predictions or improving the approaches described here. Some specific areas to explore include other forms of iterative data analysis beyond MWEM [28, 29] and other important dataset statistics [12]. An important goal for the applications we discuss is finding other ways of integrating priors into (multiple) quantile release while obtaining useful (deterministic and learnable) measures of prediction quality. Can we find metrics for quantiles (and data release) that are Lipschitz in addition to convex and are thus easier to learn? Overall, we believe these methods also hold great promise for improving utility while preserving privacy on practical, real-world problems.

3 Prediction-dependent utility bounds

Our goal in this section is to demonstrate different approaches for designing DP algorithms that incorporate predictions \( w \), such that their cost \( C_t(w) \) on instance \( t \) is controlled by a measure \( U_t(w) \) of how good the prediction is on that instance. We start with the straightforward approach of incorporating prediction-dependent priors in DP mechanisms, discuss the challenges that arise when combining multiple such mechanisms, and finally show a different approach that estimates the additive error between the prediction and ground truth.
3.1 Quantile estimation via a prediction-dependent prior

Given a quantile \( q \in (0, 1) \) and a sorted dataset \( x \in \mathbb{R}^n \) of \( n \) distinct points, we are interested in releasing a number \( o \in [x_{[qn]}, x_{[qn]+1}] \), i.e. such that the proportion of entries in the dataset less than \( o \) is \( q \). Following prior work \[36\], the error of the released number will be the number of points between it and the desired interval:

\[
\text{Gap}_q(x,o) = |\{i : x_i < o\} - |q|x|| = \max_{x_i < o} i - |q|x| \tag{5}
\]

Note that \( \text{Gap}_q(x,o) \) is constant on the interior of all intervals \( I_k = [x_{[k]}, x_{[k+1]}] \) in the partition induced by \( x \) of \( \mathbb{R} \) (let \( I_0 = (-\infty, x_{[1]}) \) and \( I_n = [x_{[n]}, \infty) \), so we will abuse notation to also say that \( \text{Gap}_q(x,I_k) \) is the same as \( \text{Gap}_q(x,o) \) for some \( o \) in the interior of \( I_k \).

For quantiles we choose perhaps the most natural—but not the only nor always the easiest—way of specifying a prediction for a DP algorithm, which is to run the exponential mechanism so since this is constant on each interval \( I_k \) the mechanism here is equivalent to sampling \( k \) w.p. \( \propto \exp(-\varepsilon \text{Gap}_q(x,I_k)/2)\mu(I_k) \) and then sampling \( o \) from \( I_k \) w.p. \( \propto \mu(o) \). While the idea of specifying a prior for EM is well-known, the key idea here is to obtain a prediction-dependent bound on the error that reveals a useful measure of the quality of the prediction:

**Lemma 3.1.** Releasing \( o \in \mathbb{R} \) w.p. \( \propto \exp(-\varepsilon \text{Gap}_q(x,o)/2)\mu(o) \) is \( \varepsilon \)-DP, and w.p. \( 1 - \beta \)

\[
\text{Gap}_q(x,o) \leq \frac{2}{\varepsilon} \left( \log \frac{1}{\beta} - \log \Psi_q^c(x,\mu) \right) \leq \frac{2}{\varepsilon} \left( \log \frac{1}{\beta} - \log \Psi_q^*(x,\mu) \right) \tag{6}
\]

where \( \Psi_q^c(x,\mu) = \sum_{i=0}^{n} \exp(-\varepsilon \text{Gap}_q(x,I_i)/2)\mu(I_i) = \exp(-\varepsilon \text{Gap}_q(x,o)/2)\mu(o) \) is the inner product between \( \mu \) and the exponential score while \( \Psi_q^*(x,\mu) = \mu(I_{[qn]}) \) is the measure of the optimal interval (note \( \max_k u_q(x,I_k) = -\text{Gap}_q(x,I_{[qn]}) = 0 \) and so \( \Psi_q^*(x,\mu) \leq \Psi_q^c(x,\mu) \) for all \( \varepsilon > 0 \).

**Proof.** \( \varepsilon \)-DP follows from \( u_q \) having sensitivity one and the guarantee of EM with base measure \( \mu \) \[49, Theorem 6\]. For the error, since we sample an interval \( I_k \) and then sample \( o \in I_k \) we have

\[
\Pr\{\text{Gap}_q(x,o) \geq \gamma\} = \Pr\{u_q(x,I_k) \leq -\gamma\} = \sum_{j=0}^{n} \Pr\{k = j\}1_{u_q(x,I_j) \leq -\gamma} \leq \sum_{j=0}^{n} \exp(-\frac{\varepsilon \gamma}{2})\mu(I_j) \leq \exp\left(\frac{-\varepsilon \gamma}{2}\right) \Psi_q^c(x,\mu) \tag{7}
\]

The result follows by substituting \( \beta \) for the failure probability and solving for \( \gamma \).

Lemma 3.1 suggests two metrics of the quality of the prediction \( \mu \): the negative log-inner-products \( -\log \Psi_q^c(x,\mu) \) and \( -\log \Psi_q^*(x,\mu) \). There are several reasons why these metrics are useful. For one, in the case of perfect prediction—i.e. if \( \mu \) assigns probability one to the optimal interval \( I_{[qn]} \)—then \( \Psi_q^c(x,\mu) = \Psi_q^*(x,\mu) = 1 \) and so both metrics are zero, yielding an upper bound on the error of only \( \frac{2}{\varepsilon} \log \frac{1}{\beta} \). As we will see later, both are also amenable for analyzing robustness and learning. Between the two, the first metric provides a tighter bound on the utility loss while the second does not depend on \( \varepsilon \), which may be desirable. A final and important quality is that the guarantees using these metrics hold under no extra assumptions on the data.

We can also analyze these error metrics for specific measures \( \mu \). In particular, if the points are a bounded interval \((a,b)\) and we use the uniform measure \( \mu(o) = 1_{o \in (a,b)}/(b-a) \) then \( \Psi_q^*(x,\mu) \geq \frac{\Psi_q}{b-a} \).
where $\psi_\chi = \min_k, x_{[k+1]} - x_{[k]}$, and we exactly recover the standard bound of $\frac{2}{\log(b-a)}$, e.g. the one in [36, Lemma A.1] (indeed their analysis implicitly uses this measure). However, our approach also allows us to remove the boundedness assumption, which itself can be viewed as a type of prediction, as one needs external information to assume that the data, or at least the quantile, lies within the interval $(a, b)$. Taking this view, we can use the prediction to set the location $x \in \mathbb{R}$ and scale $\gamma > 0$ of a Cauchy prior $\mu(x) = \gamma / (\pi (\gamma^2 + (a-x)^2))$ without committing to $(a, b)$ actually containing the data. Since we know that the optimal interval $[x_{[q_n]}, x_{[q_n]+1}]$ is a subset of $((a+b)/2 \pm R(b-a)/2)$ for some $R > 0$, setting $x = (a+b)/2$ and $\gamma = (b-a)/2$ yields

$$\Psi_q(x, \mu) \geq \frac{\gamma}{\pi} \min_k (x_{[q_n]} - x_{[q_n]+1]) \gamma^2 + \min_k \in [q_n, q_n] (x - x^2) \gamma^2 + R^2 (b-a)^2/4 \geq 2\psi_\chi(b-a)/\pi (1+R^2) \tag{8}$$

If $R = 1$, i.e. we get the interval containing the data correct, then substituting the above into Lemma 3.1 recovers the guarantee of the uniform prior up to an additive factor $\frac{2}{\log \pi}$. However, whereas for the uniform prior we have no performance guarantees if the interval is incorrect, using the Cauchy prior the performance degrades gracefully as the error $(R)$ grows. Note that the use of a heavy-tailed distribution is crucial here; a sub-exponential density would decay too quickly and lead to the error depending on $b-a$ rather than $\log(b-a)$. We can also adapt this technique if we know only a single-sided bound, e.g. if values must be positive, by using an appropriate half-Cauchy distribution.

### 3.2 Multiple-quantile release using multiple priors

To estimate $m > 1$ quantiles $q_1, \ldots, q_m$ at once, we adapt the recursive approach of [36], whose method ApproximateQuantiles implicitly constructs a binary tree with a quantile $q_i$ at each node and uses the exponential mechanism to compute the quantile $\hat{q}_i = (q_i - q_{i-1})/(\hat{q}_i - q_{i-1})$ of the dataset $\hat{x}_i$ of points in the original dataset $x$ restricted to the interval $\hat{x}_i$. Here $q_{i-1} < q_i$ and $q_i > q_{i-1}$ are quantiles appearing earlier in the tree whose respective estimates $\hat{a}_i$ and $\hat{b}_i$ determine the sub-interval (if there is no earlier quantile on the left and/or right of $q_i$ we use $q_{i-1} = 0, \hat{a}_i = a$ and/or $q_i = 1, \hat{b}_i = b$). Because each datapoint only participates in $O(\log_2 m)$ exponential mechanisms, the approach is able to run each mechanism with budget $O(\varepsilon/\log_2 m)$ and thus only suffer error logarithmic in the number of quantiles $m$, a significant improvement upon running one EM with budget $\varepsilon/M$ on the entire dataset for each quantile, which has error $O(M)$ in the number of quantiles.

We can apply prior-dependent guarantees to ApproximateQuantiles—pseudocode for a generalized version of which is provided in Algorithm 4—by recognizing that implicitly the method assigns a uniform prior $\mu_i$ to each quantile $q_i$ and then running EM with the conditional prior $\mu_i$ restricted to the interval $[\hat{a}_i, \hat{b}_i]$ determined by earlier quantiles in the binary tree. An extension of the argument in Equation 7 (c.f. Lemma 3.2) then yields a bound on the error of the estimate $\hat{o}_i$ returned for quantile $q_i$ in terms of the prior-EM inner-product computed with this conditional prior $\mu_i$ over the subset $\hat{x}_i$:

$$\Pr\{\text{Gap}_q(x, o_i) \geq \gamma \} \leq \frac{\exp \left( \frac{\gamma (\hat{y}_i - \gamma) }{\Psi_{\hat{q}_i}(\hat{x}_i, \mu_i) } \right) }{\Psi_{\hat{q}_i}(\hat{x}_i, \mu_i) } \quad \text{for} \quad \hat{y}_i = (1 - \hat{q}_i) \text{Gap}_{\hat{q}_i}(x, \hat{a}_i) + \hat{q}_i \text{Gap}_{\hat{q}_i}(x, \hat{b}_i) \tag{9}$$

Note that the error is offset by a weighted combination $\hat{y}_i$ of the errors of the estimates of quantiles earlier in the tree. Controlling this error allows us to bound the maximum error of any quantile via the harmonic mean of the inner products between the exponential scores and conditional priors:

**Lemma 3.2.** Algorithm 4 with $K = 2$ and $\varepsilon = \bar{c} = \varepsilon/|\log_2 m|$ $\forall i$ is $\varepsilon$-DP and w.p. $\geq 1 - \beta$ has

$$\max_i \text{Gap}_q(x, o_i) \leq \frac{2}{\bar{c}^2} |\log_2 m|^2 \log \frac{m}{\beta \Psi} \quad \text{for} \quad \hat{\Psi} = \left( \sum_{i=1}^m \frac{1/m}{\Psi_{\hat{q}_i}(\hat{x}_i, \mu_i)} \right)^{-1} \tag{10}$$

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Proof. The privacy guarantee follows as in [36, Lemma 3.1]. Setting the above probability bound (9) to \( \frac{\beta}{m \Psi_q^{(\varepsilon)}(X, \hat{\mu}_i)} \) for each \( i \) we have w.p. \( \geq 1 - \beta \) that \( \text{Gap}_{\hat{q}_i}(x, o_i) \leq \frac{2}{\varepsilon} \log \frac{m}{\beta \Psi} + \tilde{\gamma}_i \forall i \). Now let \( k_i \) be the depth of quantile \( q_i \) in the tree. If \( k_i = 1 \) then \( i \) is the root node so \( \tilde{\gamma}_i = 0 \) and we have \( \text{Gap}_{q_i}(x, o_i) \leq \frac{2}{\varepsilon} \log \frac{m}{\beta \Psi} \). To make an inductive argument, we assume \( \text{Gap}_{q_i}(x, o_i) \leq \frac{2k}{\varepsilon} \log \frac{m}{\beta \Psi} \forall i \) s.t. \( k_i \leq k \), and so for any \( i \) s.t. \( k_i = k + 1 \) we have that

\[
\text{Gap}_{q_i}(x, o_i) \leq \frac{2}{\varepsilon} \log \frac{m}{\beta \Psi} + (1 - \tilde{q}_i) \text{Gap}_{\hat{q}_i}^2(x, \hat{a}_i) + \tilde{q}_i \text{Gap}_{\hat{q}_i}^2(x, \hat{b}_i) \leq \frac{2(k + 1)}{\varepsilon} \log \frac{m}{\beta \Psi}
\]

Thus \( \text{Gap}_{q_i}(x, o_i) \leq \frac{2k}{\varepsilon} \log \frac{m}{\beta \Psi} \forall i \), so using \( k_i \leq \lceil \log_2 m \rceil \) and \( \varepsilon = \frac{\beta}{\log_2 m} \) yields the result. \( \square \)

Setting \( \hat{\mu}_i \) to be uniform on \( [\hat{a}_i, \hat{b}_i] \) exactly recovers both the algorithm and guarantee of [36, Theorem 3.3]. As before, we can also extend the algorithm to the infinite interval.

**Corollary 3.1.** If all priors are Cauchy with location \( \frac{2k + b - a}{2} \) and scale \( \frac{b - a}{2} \) and the data lies in the interval \( (\frac{2k + b}{2} - \frac{R}{2}(b - a)) \) then w.p. \( \geq 1 - \beta \) the maximum error is at most \( \frac{2}{\varepsilon} \log_2 m |\log_2 m | \log_2 \frac{2 \pi m(b - a)}{(1 + \delta^2) \Psi} \).

However, while this demonstrates the usefulness of Lemma 3.2 for obtaining robust priors on infinite intervals, the associated prediction measure \( \Psi \) is imperfect because it is non-deterministic: its value depends on the random execution of the algorithm, specifically on the data subsets \( \hat{x}_i \) and priors \( \hat{\mu}_i \), which for \( i \) not at the root of the tree are affected by the DP mechanisms of \( i \)'s ancestor nodes. In addition to not being given fully specified by the prediction and data, this makes \( \Psi \) difficult to use as an objective for learning. A natural more desirable prediction metric is the harmonic mean of the inner products between the exponential scores and original priors \( \mu_i \) over the original dataset \( x \), i.e. the direct generalization of our approach for single quantiles.

Unfortunately, the conditional restriction of \( \mu_i \) to the interval \([\hat{a}_i, \hat{b}_i]\) removes the influence of probabilities assigned to intervals between points not in this interval. To solve this, we propose a different edge-restriction of \( \mu_i \) that assigns probabilities \( \mu_i((\infty, \hat{a}_i]) \) and \( \mu_i([\hat{b}_i, \infty)) \) of being outside the interval \([\hat{a}_i, \hat{b}_i]\) to atoms on its edges \( \hat{a}_i \) and \( \hat{b}_i \), respectively. Despite not using any information from points outside \( \hat{x}_i \), this approach puts probabilities assigned to intervals outside \([\hat{a}_i, \hat{b}_i]\) to the edge closest to them, allowing us to extend the previous probability bound (9) to depend on the original prior-EM inner-product (c.f. Lemma A.3):

\[
\Pr\{\text{Gap}_{q_i}(x, o_i) \geq \gamma\} \leq \exp(\varepsilon(\tilde{\gamma}_i - \gamma/2))/\Psi_{\hat{q}_i}^{(\varepsilon)}(x, \mu_i)
\]

However, the stronger dependence of this bound on errors \( \tilde{\gamma}_i \) earlier in the tree lead to an \( \tilde{O}(\phi \log m) = \tilde{O}(m^{0.7}) \) dependence on \( m \), where \( \phi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio:

**Theorem 3.1.** If the quantiles are uniform negative powers of two then Algorithm 4 with \( K = 2 \), edge-based prior adaptation, and \( \varepsilon_i = \varepsilon = \varepsilon/|\log_2 m| \) \( \forall i \) is \( \varepsilon \)-DP and w.p. \( \geq 1 - \beta \) has

\[
\max_i \text{Gap}_{q_i}(x, o_i) \leq \frac{2}{\varepsilon} \hat{\Phi}^{(\varepsilon)} m \log_2 m \log \frac{m}{\beta \Psi} \quad \text{for} \quad \hat{\Phi} = \left( \sum_{i=1}^{m} \frac{1/m}{\Psi_{\hat{q}_i}^{(\varepsilon)}(x, \mu_i)} \right)^{-1}
\]

**Proof.** Since \( \tilde{\gamma}_i = 1/2 \forall i \), setting the new probability bound equal to \( \frac{\beta}{m \Psi_{\hat{q}_i}^{(\varepsilon)}(x, \mu_i)} \) yields that \( \text{Gap}_{q_i}(x, o_i) \leq \frac{2}{\varepsilon} \log \frac{m}{\beta \Psi} + 2 \tilde{\gamma}_i = \frac{2}{\varepsilon} \log \frac{m}{\beta \Psi} + \text{Gap}_{\hat{q}_i}^2(x, \hat{a}_i) + \text{Gap}_{\hat{q}_i}^2(x, \hat{b}_i) \forall i \) w.p. \( \geq 1 - \beta \). If for each \( k \leq \lceil \log_2 m \rceil \) we define \( E_k \) to be the maximum error of any quantile of at most depth \( k \) in
the tree then since one of $q_i$ and $\overline{q}_i$ is at depth at least one less than $q_i$ and the other is at depth at least two less than $q_i$ we have $E_k \leq \frac{2A_k}{\varepsilon} \log \frac{m}{\beta \Psi}$ for recurrent relation $A_k = 1 + A_{k-1} + A_{k-2}$ with $A_0 = 0$ and $A_1 = 1$. Since $A_k = F_k - 1$ for Fibonacci sequence $F_j = \frac{\phi^j - (1 - \phi)^j}{\sqrt{5}}$, we have $\max_i \text{Gap}_{q_i}(x, o_i) = \max_k E_k \leq \frac{2\log m^{\max k+1}}{\varepsilon \sqrt{5}} \log \frac{m}{\beta \Psi} \leq \frac{2\log m^{\max k+1}}{\varepsilon \sqrt{5}} [\log_2 m] \log \frac{m}{\beta \Psi}$. □

Thus while we have obtained a performance guarantee depending only on the prediction and the data via the harmonic mean $\overline{\Psi}$ of the true prior-EM inner-products, the dependence on $m$ is now polynomial. Note that it is still sublinear, which means it is better than the naive baseline of running $m$ independent exponential mechanisms. Still, we can do much better—in fact asymptotically better than any power of $m$—by recognizing that the main issue is the compounding errors induced by successive errors to the boundaries of sub-intervals. We can reduce this by reducing the depth of the tree using a $K$-ary rather than binary tree and instead paying $K-1$ times the privacy budget at each depth in order to naively release values for $K-1$ quantiles. This can introduce out-of-order quantiles, but by Lemma A.4 swapping any two out-of-order quantiles does not increase the maximum error and so this issue can be solved by sorting the $K-1$ quantiles before using them to split the data. We thus have the following prediction-dependent performance bound for multiple quantiles:

**Theorem 3.2.** If we run Algorithm 4 with $K = \lceil \exp(\sqrt{\log m \log 2}) \rceil$, edge-based prior adaptation, and $\varepsilon_i = \frac{\varepsilon}{K}$ for $k_i$ the depth of $q_i$ in the $K$-ary tree and $\varepsilon = \frac{\varepsilon}{K-1} \left( \sum_{k=1}^{[\log_K m]} \frac{1}{k^2} \right)^{-1}$, then the result satisfies $\varepsilon$-DP and w.p. $\geq 1 - \beta$ we have $\max_i \text{Gap}_{q_i}(x, o_i) \leq \frac{2\varepsilon^2}{\varepsilon} \exp \left( 2 \sqrt{\log(2) \log(m)} \right) \log \frac{m}{\beta \Psi}$. □

Proof. The privacy guarantee follows as in [36, Lemma 3.1] except before each split we compute $K-1$ quantiles with $K-1$ times less budget. As in the previous proof, we have w.p. $\geq 1 - \beta$ that $\text{Gap}_{q_i}(x, o_i) \leq \frac{2}{\varepsilon_i} \log \frac{m}{\beta \Psi} + 2\gamma_i = \frac{2k_i^2}{\varepsilon} \log \frac{m}{\beta \Psi} + 2(1 - \overline{q}_i) \text{Gap}_{\overline{q}_i}(x, \overline{a}_i) + 2\overline{q}_i \text{Gap}_{\overline{q}_i}(x, \overline{b}_i) \forall i$. If for each $k \leq \lceil \log_K m \rceil$ we define $E_k$ to be the maximum error of any quantile of at most depth $k$ in the tree then since both $q_i$ and $\overline{q}_i$ are at depth at least one less than $q_i$ we have $E_k \leq \frac{2A_k}{\varepsilon} \log \frac{m}{\beta \Psi}$, where $A_k = k^2 + 2A_{k-1}$ and $A_1 = 1$. Since $A_k \leq 6 \cdot 2^k$ and $1/\varepsilon = (K-1) \sum_{k=1}^{[\log_K m]} \frac{1}{k^2} \leq \frac{\varepsilon^2}{6} (K-1)$ we have that $\max_i \text{Gap}_{q_i}(x, o_i) = \max_k E_k \leq \frac{12 \log m}{\varepsilon \sqrt{5}} \log \frac{m}{\beta \Psi} \leq \frac{2\varepsilon^2}{\varepsilon} (K-1)^2 \log m \log \frac{m}{\beta \Psi}$. Substituting $K = \lceil \exp(\sqrt{\log m \log 2}) \rceil$ and simplifying yields the result. □

Similarly to Theorem 3.1, the proof establishes a recurrence relationship between the maximum errors at each depth. Note that in addition to the $K$-ary tree this bound uses depth-dependent budgeting to remove a $O(\log m)$-factor. As discussed before, the new dependence $\tilde{O} \left( \exp \left( 2 \sqrt{\log(2) \log(m)} \right) \right)$ on $m$ is sub-polynomial, i.e $o(m^\alpha) \forall \alpha > 0$. While it is also superpolylogarithmic, its shape for any practical value of $m$ is roughly $O(\log^2 m)$, making the result of interest as a justification for the negative log-inner-product performance metric.

### 3.3 Covariance estimation by estimating the prediction error

Encoding predictions as priors for EM and other mechanisms is a natural starting point for integrating external information into DP algorithms, but one might also wish to use a point prediction directly and hope to perform well if some distance measure between it and the output is small. While this is a less natural requirement for quantile release, where errors are measured using data points rather than metrics over the domain they live in, we show how this is easily achievable for the important problem of covariance estimation. In this setting we have a dataset $X \in \mathbb{R}^{d \times n}$, where each of $n$ records is a $d$-dimensional column with $\ell_2$-norm bounded by 1, and we want to privately
Algorithm 1: SeparateCov with predictions

Input: data $X \in \mathbb{R}^{d\times n}$, symmetric prediction matrix $W \in \mathbb{R}^{d\times d}$, privacy parameter $\varepsilon > 0$

$U\Lambda U^T \leftarrow XX^T/n - W$

$\hat{\Lambda} \leftarrow \Lambda + \text{diag}(z)$ where $z[i] \sim \text{Lap}\left(\frac{4\varepsilon}{n}\right)$ // add noise to prediction error eigenvalues

$\hat{C} \leftarrow XX^T/n + Z$ for $Z[i,j] = Z[j,i] \sim \text{Lap}\left(\frac{24\sqrt{2}}{\varepsilon n}\right)$

$\hat{U}\hat{\Lambda}\hat{U}^T \leftarrow \hat{C} - Z$ // get eigenvectors of noised prediction error

Output: $\hat{C} = \hat{U}\hat{\Lambda}\hat{U}^T + W$ // combine to estimate $XX^T/n - W$, then add $W$

release an approximation $\hat{C}$ of its covariance matrix $C = XX^T/n$ such that the Frobenius distance between the two is small.

Given a prediction $W \in \mathbb{R}^{d\times d}$ of $C$, one can immediately construct the trivial, private, prediction-sensitive algorithm of just releasing $W$, which has the obvious prediction-dependent performance guarantee of $\|W - C\|_F$. However, we can hope to use the data to get an error that both decreases with $n$ and is small if some distance between the prediction and ground truth is small. To do so, we make use of recent approaches that enjoy trace-sensitive guarantees, i.e. their utility improves if $\text{Tr}(XX^T)$ is small [4, 23]; for example, the state-of-the-art method SeparateCov returns $\hat{C}$ that is $\varepsilon$-DP and satisfies $\|\hat{C} - C\|_F^2 = \tilde{O}(\frac{d^4}{\varepsilon^4 n^2} + \frac{d^2}{\varepsilon^2 n} \text{Tr}(XX^T/n))$ w.h.p. [23, Lemma 18]. This suggests a natural way to incorporate a symmetric prediction matrix $W$: use the existing algorithm to privately estimate its difference $C - W$ with the ground truth, and then add $W$ to the result; since $C - W$ is no longer PSD, the hope would be to obtain error that scales with its trace norm.

We do exactly this in Algorithm 1, which uses the SeparateCov approach of separately estimating and combining eigenvalues and eigenvectors but applies it to $C - W$. The one potential issue is showing that their main error bound holds for symmetric matrices with negative eigenvalues, but this follows in Lemma 3.3 by applying their argument to both sides of the spectrum (c.f. Appendix B.2):

Lemma 3.3. For $X \in \mathbb{R}^{d\times n}$ and symmetric $W \in \mathbb{R}^{d\times d}$, if $\hat{U}\hat{\Lambda}\hat{U}^T = XX^T/n - W + Z$ for some symmetric $Z \in \mathbb{R}^{d\times d}$ and $\hat{\Lambda} = \Lambda + \text{diag}(z)$ for $U\Lambda U^T = XX^T/n - W$ and some vector $z \in \mathbb{R}^d$ then

$$\|\hat{U}\hat{\Lambda}\hat{U}^T + W - XX^T/n\|_F^2 \leq 4 \left(\|z\|_2^2 + \|Z\|_F^2\right) \|XX^T/n - W\|_{\text{Tr}}$$ (14)

Our performance-dependent guarantee then follows via Laplace concentration in Theorem 3.3, which recovers the guarantee of [23, Lemma 18] when $W = 0_{d\times d}$.\(^1\) The result shows that if we have a good guess of the prediction matrix in terms of trace distance then the error can be made to depend mostly on the first term—which has a better dependence on both $d$ and $n$—without sacrificing privacy. Note that the algorithm requires the same number of eigen-decompositions as the one without predictions [23] and only requires some extra matrix additions to implement.

Theorem 3.3. If $X$ has columns bounded by 1 in $\ell_2$-norm then Algorithm 1 is $\varepsilon$-DP and w.p. $\geq 1 - \beta$

$$\|\hat{C} - XX^T/n\|_F^2 \leq \frac{144d + O(\log^2 \frac{d}{\beta} \log^2 d)}{\varepsilon^2 n^2} + \frac{48d\sqrt{2}d + O(d \log \frac{1}{\beta} \log d)}{\varepsilon n} \|XX^T/n - W\|_{\text{Tr}}$$ (15)

Proof. Following the analysis in [4, Theorem 1] (c.f. Lemma B.1) the $\ell_1$-sensitivity of the eigenvalues of $XX^T/n - W$ is $2/n$, and upper-bounding the $\ell_2$-sensitivity of the covariance $XX^T/n$ of $\sqrt{2}/n$ [12, ...]

\(^1\)Unlike [23] we square the Frobenius norm for the purposes of learning predictions later; in the single-instance setting this is immaterial. Whether one is more interested in one or the other is application-dependent.
we will assume an indexing that allows us to specify datasets as vectors. We also show that the result holds for the older algorithm of [4] (c.f. Appendix B.4), which requires MWEM. Indeed this observation has been made in both the original work and by [46], who adapt the method of [30], which uses multiplicative weights to iteratively update a distribution 

\[ w_{i+1} \leftarrow w_i \odot \exp \left( \frac{\langle q_i, x - n w_i / |w_i| \rangle + \text{Lap}(4m/\varepsilon) q_i}{2n} \right) \] // multiplicative weights update

Lemma 3.2] shows that its \( \ell_1 \)-sensitivity is \( d \sqrt{2}/n \). Thus the privacy guarantee follows from the composition of two Laplace mechanisms with budget \( \varepsilon/2 \) each. For the utility guarantee we use concentration of \( \|Y\|_2 \leq 3\sqrt{d}/2 + O \left( \log \frac{1}{\beta} \log d \right) \) w.p. \( \geq 1 - \beta/2 \) for i.i.d. \( Y[i] \sim \text{Lap}(1) \) [23, Lemma 15] and \( \|Y\|_\infty \leq 3\sqrt{d}/2 + O \left( \log \frac{1}{\beta} \log d \right) \) w.p. \( \geq 1 - \beta/2 \) for i.i.d. \( Y[i,j] \sim \text{Lap}(1) \) for \( i \geq j \) and \( Y[i,j] = Y[j,i] \) for \( i < j \) [23, Lemma 16]. Substituting \( z = \frac{4}{\varepsilon n} y \) and \( Z = \frac{2d \sqrt{2}}{\varepsilon n} Y \) into Lemma 3.3 yields the result.

While our paper focuses on pure DP (except for learning), the main analysis of [23] is in the zCDP setting; in Appendix B.3 we show that similar guarantees hold there, albeit with a constant-factor worse sensitivity (unlike the \( \varepsilon \)-DP analysis). To demonstrate the generality of this approach, we also show that the result holds for the older algorithm of [4] (c.f. Appendix B.4), which requires a two-sided algorithm and more involved sensitivity analysis.

### 3.4 Initializing synthetic dataset construction with a predicted dataset

Our final application is to private data release, in which the goal is to privately respond to queries of a dataset, with the latter being defined via counts of items from some finite universe. For simplicity we will assume an indexing that allows us to specify datasets as vectors \( x \in \mathbb{Z}_{\geq 0}^d \), and we will consider a finite set \( Q \) of linear queries, i.e. ones that can be defined as an inner product of \( x \) with a vector \( q \in [-1,1]^d \). Here again we will incorporate a prediction into an existing algorithm, specifically the MWEM method of [30], which uses multiplicative weights to iteratively update a distribution over the data domain and to construct a synthetic dataset \( \hat{x} \in \mathbb{R}_{\geq 0}^d \) such that the maximum error \( \max_{q \in Q} |\langle q, x - \hat{x} \rangle| \) of all queries is small. The natural approach here is to assume the prediction can be written as a distribution \( w \in \Delta_d \) and use it instead of the uniform initialization used by [30]. Indeed this observation has been made in both the original work and by [46], who adapt the method to only operate over the support of a source dataset. A prediction-dependent guarantee also follows in a straightforward manner from the original analysis:2

\[ \max_{q \in Q} \frac{|\langle q, x - \hat{x} \rangle|^2}{n} \leq 8n m D_{KL} \left( \frac{X}{n} \mid \mid w \right) + \frac{16m^2}{\varepsilon^2 n} \left( 3 \log \frac{2m}{\beta} + 2 \log^2 |Q| \right)^2 \] \hspace{1cm} (16)

Our main purpose with this application is thus to discuss interesting issues arising in its robustness and especially in learning the prediction. We also conclude by noting the similarity of deriving prediction-based guarantees for all four methods—finding algorithms that implicitly use a default prediction such as a uniform distribution or zero matrix—even while the actual algorithms and uses of the predictions are quite different.

\[ ^2 \text{Similar to covariance estimation, we consider the mean squared error for the purposes of learning the prediction.} \]
4 Robustness-consistency tradeoffs

While prediction-dependent guarantees work well if the prediction is accurate, without safeguards they may perform catastrophically poorly if the prediction is incorrect. In this section we provide robust alternatives to the methods we derived in the previous section, demonstrate the usefulness of the algorithms with predictions framework for understanding robustness when incorporating external information into DP algorithms.

4.1 Quantile estimation

Quantiles provide a prime demonstration of the importance of robustness, as using priors allows for approaches that may assign very little probability to the interval containing the quantile. For example, if one is confident that it has a specific value \( x \in (a, b) \) one can specify a more concentrated prior, e.g. the Laplace distribution around \( x \). Alternatively, if one believes the data is drawn i.i.d. from some known distribution then \( \mu \) can be constructed via its CDF using order statistics [19, Equation 2.1.5]. These reasonable approaches can result in distributions with exponential or high-order-polynomial tails, using which directly may work poorly if the prediction is incorrect.

Luckily, for our negative log-inner-product error metric it is straightforward to show a parameterized robustness-consistency tradeoff by simply mixing the prediction prior \( \mu \) with a robust prior \( \rho \):

\[\text{Corollary 4.1.} \text{ For any prior } \mu : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, \text{ robust prior } \rho : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, \text{ and robustness parameter } \lambda \in [0, 1], \text{ releasing } o \in \mathbb{R} \text{ w.p. } \alpha \exp(-\varepsilon \text{Gap}_q(x, o)/2)\mu^{(\lambda)}(o) \text{ for } \mu^{(\lambda)} = (1 - \lambda)\mu + \lambda\rho \text{ is } \left(\frac{2}{\varepsilon} \log \frac{1/\beta}{\lambda \Psi_q^{(\lambda)}(x, \rho)}\right)\text{-robust and } \left(\frac{2}{\varepsilon} \log \frac{1/\beta}{1 - \lambda}\right)\text{-consistent w.p. } \geq 1 - \beta.\]

Proof. Apply Lemma 3.1 and linearity of \( \Psi_q^{(\lambda)}(x, \mu^{(\lambda)}) = (1 - \lambda)\Psi_q^{(\lambda)}(x, \mu) + \lambda\Psi_q^{(\lambda)}(x, \rho). \)

Thus if the interval is finite and we set \( \rho \) to be the uniform prior, using \( \mu^{(\lambda)} \) in the algorithm will have a high probability guarantee at most \( \frac{2}{\varepsilon} \log \frac{1}{\lambda} \)-worse than the prediction-free guarantee of [36, Lemma A.1], no matter how poor \( \mu \) is for the data, while also guaranteeing w.p. \( \geq 1 - \beta \) that the error will be at most \( \frac{2}{\varepsilon} \log \frac{1/\beta}{1 - \lambda} \) if \( \mu \) is perfect. A similar result holds for the case of an infinite interval if we instead use a Cauchy prior. Corollary 4.1 demonstrates the usefulness of the algorithms with predictions framework for not only quantifying improvement in utility using external information but also for making the resulting DP algorithms robust to prediction noise.

4.2 Multiple-quantile estimation

The above argument for single-quantiles is straightforward to extend to the negative log of the harmonic means of the inner products. In fact for the binary case with uniform quantiles we can tradeoff between polylog \((m)\)-guarantees similar to those of [36] and our prediction-dependent bounds:

\[\text{Corollary 4.2.} \text{ Consider priors } \mu_1, \ldots, \mu_m : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, \text{ Cauchy prior } \rho : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \text{ with location } \frac{a + b}{2} \text{ and scale } \frac{b - a}{2}, \text{ and robustness parameter } \lambda \in [0, 1]. \text{ Then running Algorithm 4 on quantiles that are uniform negative powers of two with } K = 2, \text{ edge-based prior adaptation, } \varepsilon_i = \bar{\varepsilon} = \varepsilon/[\log_2 m] \forall i, \text{ and priors } \mu_i^{(\lambda)} = \lambda\rho + (1 - \lambda)\mu_i \forall i \text{ is } \left(\frac{2}{\varepsilon} \log^2 m \log \frac{2\pi m (b - a)}{\lambda(1 + R^2)\beta \Phi_x}\right)\text{-robust and } \left(\frac{2}{\varepsilon} \log^2 m \log \frac{m/\beta}{\lambda\Phi_x}\right)\text{-consistent w.p. } \geq 1 - \beta.\]

Proof. Apply Lemma 3.2, Theorem 3.1, and the linearity of inner products making up \( \hat{\Psi} \) and \( \Psi \).
4.3 Covariance estimation

We take a different approach to making our prediction-based covariance estimation method robust to matrices \( \mathbf{W} \) with large trace distance to \( \mathbf{X} \mathbf{X}^T/n \). Instead of combining the prediction with a robust default, we simply spend some privacy to check whether \( \| \mathbf{X} \mathbf{X}^T/n - \mathbf{W} \|_{\text{Tr}} \) is larger than \( \| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}} \) and if so run Algorithm 1 with the zero matrix instead. This has the following guarantee:

**Corollary 4.3.** Pick \( \lambda \in (0, 1) \) and run Algorithm 1 with privacy \((1 - \lambda)\varepsilon \) and symmetric prediction matrix \( \mathbf{W} \) if \( \| \mathbf{X} \mathbf{X}^T/n - \mathbf{W} \|_{\text{Tr}} + z \leq \| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}} \) and \( \mathbf{0}_{d \times d} \) otherwise, where \( z \sim \text{Lap}(\frac{1}{\lambda \varepsilon n}) \). This procedure is \( \varepsilon\)-DP, \( \bar{O}\left(\frac{\sqrt{d} \varepsilon n}{\varepsilon} + \| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}}\right) \)-robust, and \( \bar{O}\left(\frac{d \varepsilon n^{3/2}}{\varepsilon} \right) \)-consistent w.h.p.

**Proof.** By Lemma B.1 the difference \( \| \mathbf{X} \mathbf{X}^T/n - \mathbf{W} \|_{\text{Tr}} - \| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}} \) has sensitivity \( 4/n \), so the comparison of \( \| \mathbf{X} \mathbf{X}^T/n - \mathbf{W} \|_{\text{Tr}} + z \) and \( \| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}} \) is equivalent to using the Laplace mechanism with \( \lambda \varepsilon\)-DP to estimate this difference and then taking the sign. Composing this with the privacy guarantee of Theorem 3.3 yields \( \varepsilon\)-DP. Since \( \Pr\{|z| \geq \frac{1}{\lambda \varepsilon n} \log \frac{2}{\beta} \} \leq \beta/2 \), the matrix \( \mathbf{W}_z \in \{\mathbf{W}, \mathbf{0}_{d \times d}\} \) passed to Algorithm 1 satisfies \( \| \mathbf{X} \mathbf{X}^T/n - \mathbf{W}_z \|_{\text{Tr}} \leq \min\{\| \mathbf{X} \mathbf{X}^T/n - \mathbf{W} \|_{\text{Tr}}, \| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}}\} + \frac{1}{\lambda \varepsilon n} \log \frac{2}{\beta} \) w.p. \( \geq 1 - \beta/2 \). Applying the utility guarantee of Theorem 3.3 w.p. \( 1 - \beta/2 \) for constant \( \lambda \in (0, 1) \) yields the result. \( \Box \)

Adding this check for robustness make the data-independent term worse by a factor of \( \sqrt{d} \); note that the data-dependent term can still be up to \( \bar{O}(\varepsilon n\| \mathbf{X} \mathbf{X}^T/n \|_{\text{Tr}}) \) times larger, so this does not remove the usefulness of the prediction guarantee. The additional cost results from the large dependence on \( d \) of the latter term in the original bound, which is itself can be caused by a mismatch between the \( \ell_1 \)-sensitivity measure and the \( \ell_2 \)-bound on the columns. Specifically, if instead the \( \ell_1 \)-norms of the columns are assumed bounded by one then the \( \ell_1 \)-sensitivity of \( \mathbf{X} \mathbf{X}^T/n \) is \( 2/n \), making the numerator of the second term in Theorem 3.3 be \( \bar{O}(\sqrt{d}) \) and thus causing no (asymptotic) cost due to robustness.\(^3\) Similarly, under the original assumption the corresponding term in the \( \ell_2 \)-sensitivity-based zCDP guarantee is also \( \bar{O}(\sqrt{d}) \) (c.f. Theorem B.1) and also does not lead to a cost due to robustness.

4.4 Data release

As with quantiles, a natural approach to making data release robust is to mix the initialization with the default uniform distribution, achieving a tunable tradeoff. In the following result we specify the number of steps based on the worst-case guarantees for a prediction-free algorithm and obtain a favorable tradeoff that allows for very small values of \( \lambda \) for high consistency while still maintaining robustness due to the latter’s \( \log \frac{d}{\lambda} \) dependence.

**Corollary 4.4.** For \( d \geq 2 \) and any \( \mathbf{w} \in \Delta_d \), running Algorithm 2 with \( m = \frac{3}{\varepsilon^2 n^2 \log d} \) and initialization \( \mathbf{w}^{(\lambda)} = (1 - \lambda)\mathbf{w} + \lambda \mathbf{1}_{d/d} \) is \( \varepsilon\)-DP, \( \bar{O}\left((1 + \log^{4/3}|Q|)\sqrt{\frac{n}{\varepsilon^2 \log d} \log \frac{d}{\lambda}}\right) \)-robust, and \( \bar{O}\left(\lambda (1 + \log^{4/3}|Q|)\sqrt{\frac{n \log^2 d}{\varepsilon^2}}\right) \)-consistent w.h.p., where \( \bar{O} \) hides poly-log terms in \( \varepsilon, n, \log d, \log |Q| \).

**Proof.** If \( \mathbf{w} = \frac{\mathbf{x}}{n} \) then we have \( D_{KL}(\frac{\mathbf{x}}{n} \| \mathbf{w}^{(\lambda)} \) \) \( \leq (1 - \lambda)D_{KL}(\frac{\mathbf{x}}{n} \| \mathbf{w} + \lambda D_{KL}(\frac{\mathbf{x}}{n} \| \mathbf{w} \leq \lambda \log d \) by joint convexity of \( D_{KL} \). On the other hand \( D_{KL}(\frac{\mathbf{x}}{n} \| \mathbf{w}^{(\lambda)} \) \) \( \leq \langle \frac{\mathbf{x}}{n}, \log \frac{\mathbf{w}^{(\lambda)}}{\mathbf{w}} \rangle \leq \log \frac{d}{\lambda} \). Substituting into Lemma 3.4 and simplifying the result. \( \Box \)

\(^3\)It is not as clear that the \( \ell_1 \)-sensitivity of the eigenvalues would be as affected by the different assumption.
Algorithm 3: Non-Euclidean DP-FTRL

**Input:** Datasets $X_1, \ldots, X_T$ arriving in a stream in arbitrary order, domain $\Theta \subset \mathbb{R}^p$, step-size $\eta > 0$, noise scale $\sigma > 0$, $\ell_2$-sensitivity $\Delta_2 > 0$, regularizer $\phi : \Theta \rightarrow \mathbb{R}$

\[ g_1 \leftarrow 0_p, \]
\[ T \leftarrow \text{InitializeTree}(T, \sigma^2, \Delta_2) \quad // \text{start tree aggregation [35, Section B.1]} \]
\[ \text{for } t = 1, \ldots, T \text{ do} \]
\[ \theta_t \leftarrow \arg \min_{\theta \in \Theta} \phi(\theta) + \eta \langle g_t, \theta \rangle \]
\[ \text{suffer } \ell(\theta_t; X_t) \]
\[ T \leftarrow \text{AddToTree}(T, t, \nabla_{\theta} \ell(\theta_t; X_t)) \quad // \text{add gradient to tree [35, Section B.1]} \]
\[ g_{t+1} \leftarrow \text{GetSum}(T, t) \quad // \text{estimate } \sum_{s=1}^{t} \nabla_{\theta} \ell(\theta_s; X_s) \text{ [35, Section B.1]} \]

5 Learning predictions, privately

Our last objective will be to learn predictions that do well according to the quality metrics we have defined, which themselves control the utility loss of running the DP algorithms. Past work, e.g. the public-private framework [46, 10, 11], has often focused on domain adaptation-type learning where we adapt a public source to private target. We avoid assuming access to large quantities of i.i.d. public data and instead assume numerous tasks that can have sensitive data and may be adversarially generated. As discussed before, this is the online setting where we see loss functions defined by a sequence of datasets $X_1, \ldots, X_T$ and aim to compete with best fixed prediction in hindsight. Note such a guarantee can also be converted into excess risk bounds (c.f. Appendix D.1).

Because the optimization domains are not well-described by the $\ell_2$-ball, throughout our applications we are able to obtain significant savings in dependence on the dimension and in some cases even in the number of instances $T$ by extending the DP-FTRL algorithm of [35] to use non-Euclidean regularizers, as in Algorithm 3. For this we prove the following regret guarantee:

**Theorem 5.1.** Let $\theta_1, \ldots, \theta_T$ be the outputs of Algorithm 3 using a regularizer $\phi : \Theta \rightarrow \mathbb{R}$ that is strongly-convex w.r.t. $\| \cdot \|$. Suppose $\forall \; t \in [T]$ that $\ell(\cdot; X_t)$ is $L$-Lipschitz w.r.t. $\| \cdot \|$ and its gradient has $\ell_2$-sensitivity $\Delta_2$. Then w.p. $\geq 1 - \beta'$ we have $\forall \; \theta^* \in \Theta$ that

\[ \sum_{t=1}^{T} \ell(\theta_t; X_t) - \ell(\theta^*; X_t) \leq \frac{\phi(\theta^*) - \phi(\theta_1)}{\eta} + \eta L \left( L + \left( G + C \sqrt{2 \log \frac{T}{\beta'}} \right) \sigma \Delta_2 \sqrt{\log_2 T} \right) T \quad (17) \]

where $G = \mathbb{E}_{z \sim \mathcal{N}(0_p, I_p)} \sup_{y \in \mathcal{Y}} \langle z, y \rangle = \mathbb{E}_{z \sim \mathcal{N}(0_p, 1)} \| z \|_*$ is the Gaussian width of the unit $\| \cdot \|$-ball and $C$ is the Lipschitz constant of $\| \cdot \|_*$ w.r.t. $\| \cdot \|_2$. Furthermore, for any $\varepsilon' \leq 2 \log \frac{1}{\beta'}$, setting $\sigma = \frac{1}{\varepsilon'} \sqrt{2 \log_2 T} \log \frac{1}{\beta'}$ makes the algorithm $(\varepsilon', \delta')$-DP.
We will focus on minimizing regret \( \max \phi \) of two other classes of interest: the class \( \phi \) of convex \( \min \)imization \([58]\), the second by the regret guarantee for online mirror descent \([56, \text{Theorem 2.15}]\), where \( \phi = \eta \sum_{s=1}^{t} \langle \nabla_s, \hat{\theta}_t - \theta \rangle \). Then

\[
\sum_{t=1}^{T} \ell(\theta_t; X_t) - \ell(\theta^*; X_t) \leq \sum_{t=1}^{T} \langle \nabla_{\theta_t}, \hat{\theta}_t - \theta^* \rangle = \sum_{t=1}^{T} \langle \nabla_{\theta_t}, \hat{\theta}_t - \theta \rangle + \sum_{t=1}^{T} \langle \nabla_{\theta_t}, \theta - \hat{\theta}_t \rangle \\
\leq \frac{\phi(\theta^*) - \phi(\theta_1)}{\eta} + \eta \sum_{t=1}^{T} \|\nabla_{\theta_t}\|^2 + \sum_{t=1}^{T} \|\nabla_{\theta_t}\| \|\hat{\theta}_t - \theta\| \\
\leq \frac{\phi(\theta^*) - \phi(\theta_1)}{\eta} + \eta \|L\| T \sum_{t=1}^{T} \|\theta_t\|_* 
\]

(18)

where the first inequality follows from the standard linear approximation in online convex optimization \([58]\), the second by the regret guarantee for online mirror descent \([56, \text{Theorem 2.15}]\), and the last by applying \([48, \text{Lemma 7}]\) with \( \phi_1(\cdot) = \phi(\cdot) + \eta \sum_{s=1}^{t} \langle \nabla_{\theta_t}, \cdot \rangle \), \( \psi(\cdot) = \eta \langle \theta_t, \cdot \rangle \), and \( \phi_2(\cdot) = \phi(\cdot) + \eta \langle \theta_t, \cdot \rangle \), yielding \( \|\hat{\theta}_t - \theta_t\| \leq \eta \|\theta_t\|_* \) \( \forall t \in [T] \). The final guarantee follows by observing that the tree aggregation protocol adds noise \( \theta_t \sim N(0, \sigma^2 \Delta_t^2 \log_2 t) \) to each prefix sum and applying the Gaussian concentration of Lipschitz functions \([13, \text{Theorem 5.6}]\). \( \square \)

The above proof of this result follows that of the Euclidean case, which can be recovered by setting \( G = O(\sqrt{d}) \), \( C = 1 \), and \( \Delta_2 = O(L) \). \(^4\) In addition to the Lipschitz constants \( L \), a key term that can lead to improvement is the Gaussian width \( G \) of the unit \( \cdot \cdot\cdot \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\..
measures that are constant on each of the $d$ intervals. The latter can be parameterized by $W \in \Delta^m_d$, so that the losses have the form $U_t(\mu W) = \log \sum_{i=1}^m \langle s_t, i \rangle W[i] \cdot |p^{t-1}$ for $s_t, i \in \mathbb{R}^{d \geq 0}$. This can be seen by setting $s_t[i] = \frac{d}{b-a} \int a + \frac{b-a}{d(j-1)} \exp(-x \cdot \text{Gap}_q(x_t, o)/2) dx$ and $w_{t[i]}(o) = \frac{d}{b-a} W[i,j]$ over the interval $[a + \frac{b-a}{d(j-1)}, a + \frac{b-a}{d(j-1)}]$. Finally, for $\lambda \in [0, 1]$ we also let $\mathcal{F}(\lambda) = \{(1-\lambda)\mu + \frac{\lambda}{b-a} : \mu \in \mathcal{F}\}$ denote the class of mixtures of measures $\mu \in \mathcal{F}$ with the uniform measure.

As detailed in Appendix D.2, losses of the form $-\log \langle s_t, \cdot \rangle$, i.e. those above when $m = 1$, have been studied in (non-private) online learning [31, 8]. However, specialized approaches, e.g. those taking advantage exp-concavity, are not obviously implementable via prefix sums of gradients, the standard approach to private online learning [57, 2, 35]. Still, we can at least use the fact that we are optimizing over a product of simplices to improve the dimension-dependence by applying Non-Euclidean DP-FTRL with entropic regularizer $\phi(W) = m \langle W, \log W \rangle$, which yields an $m$-way exponentiated gradient (EG) update [39]. To apply its guarantee for the problem of learning priors for quantile estimation, we need to bound the sensitivity of the gradients $\nabla_W U_t(\mu W)$ to changes in the underlying datasets $x_t$. This is often done via a bound on the gradient norm, which in our case is $U_t(\mu W)$ for $t$ uniformly measure.

We thus have a bound of $2d/\gamma$ on the $\ell_2$-sensitivity of the gradients. However, this may be too loose since it allows for changing the entire dataset $x_t$, whereas we are only interested in changing one of its entries. Indeed, the following lemma shows that for small $\varepsilon$ we can obtain a tighter bound:

**Lemma 5.1.** The $\ell_2$-sensitivity of $\nabla_W U_t(\mu W)$ is $\frac{d}{\gamma} \min \{2, e^{\varepsilon m} - 1\}$, where $\varepsilon_m = (1 + 1_{m>1}) \max_i \varepsilon_i$.

**Proof for $m = 1$; c.f. Appendix D.2.1.** Let $\tilde{x}_t$ be a neighboring dataset of $x_t$ and let $\tilde{U}_t(\mu W) = -\log \langle \tilde{s}_t, w \rangle$ be the corresponding loss. Note that $\max_{o \in [a,b]} |\text{Gap}_q(x_t, o) - \text{Gap}_q(\tilde{x}_t, o)| \leq 1$ so

$$\tilde{s}_{t[j]} = \int_{a + \frac{b-a}{d(j-1)}}^a \exp \left( -\frac{\varepsilon}{2} \text{Gap}_q(\tilde{x}_t, o) \right) dx \in e^{\pm \frac{1}{2} \varepsilon} \int_{a + \frac{b-a}{d(j-1)}}^a \exp \left( -\frac{\varepsilon}{2} \text{Gap}_q(x_t, o) \right) dx = e^{\pm \frac{1}{2} \varepsilon} s_{t[j]}$$

Therefore since $m = 1$ we denote $w = W[1], s_t = s_{t,1}$, and $\tilde{s}_t = \tilde{s}_{t,1}$ and have

$$\| \nabla_W U_t(\mu W) - \nabla_W \tilde{U}_t(\mu W) \|_2 = \sqrt{\sum_{j=1}^d \left( \frac{\tilde{s}_{t[j]} - \tilde{s}_{t[j]}}{\langle \tilde{s}_t, w \rangle} \right)^2} = \sqrt{\sum_{j=1}^d \frac{s^2_{t[j]}}{\langle s_t, w \rangle^2} \left( 1 - \frac{\tilde{s}_{t[j]} \langle s_t, w \rangle}{\tilde{s}_{t[j]} \langle s_t, w \rangle} \right)^2}$$

$$\leq \| \nabla_W U_t(\mu W) \|_1 \max_j |1 - \kappa_j|$$

where $\kappa_j = \frac{\tilde{s}_{t[j]} \langle s_t, w \rangle}{s_{t[j]} \langle s_t, w \rangle} \in \frac{s_{t[j]} \exp(\pm \varepsilon)}{s_{t[j]} \exp(\pm \varepsilon)} \in \exp(\pm \varepsilon)$ by Equation 20. The result follows by taking the minimum with the bound on the Euclidean norm of the gradient (Lemma D.1). \qed

Since $e^\varepsilon - 1 \leq 2e$ for $\varepsilon \in (0, 1.25)$, for small $\varepsilon$ this allows us to add less noise in DP-FTRL. With this sensitivity bound, we apply Algorithm 3 using the entropic regularizer to obtain the following result:
Theorem 5.2. For \( d \geq 2, \gamma \in (0, 1/2] \) if we run Algorithm 3 on \( U_t(\mu W) = \log \sum_{i=1}^{m} \frac{1}{\psi_{i,r}(x_t, \mu W)} \) over \( \gamma \)-robust priors with step-size \( \eta = \frac{2\eta}{d} \sqrt{\log(d)/T} \) and regularizer \( \phi(W) = m \langle W, \log W \rangle \) then for any \( V > 0, \lambda \in [0, 1], \) and \( \beta' \in (0, 1] \) we will have regret

\[
\max_{\mu_{(t)} \in F_{V,d}} \sum_{t=1}^{T} U_t(\mu W_t) - U_t(\mu) \leq \frac{V m T}{\gamma d \psi} (b - a)^3 + 2 \max\{\gamma - \lambda, 0\} T \log 2 + \frac{2m \max\{\gamma - \lambda, 0\} \log 2}{\gamma d \psi} \]  

w.p. \( \geq 1 - \beta' \), where \( \bar{\psi} \) is the harmonic mean of \( \psi_{x_t} = \min_k x_{t[k+1]} - x_{t[k]} \) and \( \bar{\varepsilon}_m = (1 + 1_{m>1}) \max_i \varepsilon_i \). For any \( \varepsilon' \leq 2 \log \frac{1}{\beta'} \) setting \( \sigma = \frac{1}{\beta'} \sqrt{2 \log \log T} \) makes this procedure \((\varepsilon', \delta')\)-DP.

Proof. For set of \( \gamma \)-robust priors \( \rho \) s.t. \( \rho[i] = \min\{1 - \gamma + \lambda, 1\} \mu[i] + \frac{\max\{\gamma - \lambda, 0\} \mu[i]}{b - a} \) and \( W \in \Delta_d^m \) s.t. \( W[i,j] = \frac{b-a}{d} \int_a^{a+b-a/j} \rho[i](o) do \) we can divide the regret into three components:

\[
\sum_{t=1}^{T} U_t(\mu W_t) - U_t(\mu) = \sum_{t=1}^{T} U_t(\mu W_t) - U_t(\mu W) + \sum_{t=1}^{T} U_t(\mu W) - U_t(\rho) + \sum_{t=1}^{T} U_t(\rho) - U_t(\mu) \tag{23}
\]

The first summation is the regret of DP-FTRL with regularizer \( \phi \), which is strongly convex w.r.t. \( \| \cdot \|_1 \). The Gaussian width of its unit ball is \( 2 \sqrt{\log(md)} \), by Lemma D.1 the losses are \( \frac{d}{\gamma} \)-Lipschitz w.r.t. \( \| \cdot \|_1 \), and by Lemma 5.1 the \( \ell_2 \)-sensitivity is \( \Delta_2 = \frac{d}{\gamma} \min\{2, \varepsilon_m^2 - 1\} \leq \frac{2d}{\gamma} \min\{1, \varepsilon_m\} \), so applying Theorem 5.1 yields the bound \( \frac{m^2 \log d}{\eta} + \frac{m^3 T}{\gamma} \left( 1 + \left( 4 \sqrt{\log d + 2 \log \frac{T}{\sigma^2}} \right) \sigma \right) \min\{1, \varepsilon\} \). The second summation is a sum over the errors due to discretization, where we have

\[
\sum_{t=1}^{T} U_t(\mu W) - U_t(\rho) = \sum_{t=1}^{T} \log \sum_{i=1}^{m} \frac{\exp(-\varepsilon_i \psi_{x_t}(o))}{\rho[i](o) do} \rho[i](o) do - \sum_{i=1}^{m} \frac{1}{\gamma \psi_{x_t}(o) do} \left( \frac{b-a}{d} \int_a^{\psi_{x_t}(o)} \frac{\rho[i](o) - \mu[i](o) do}{\gamma \psi_{x_t}(o) do} \right) \tag{24}
\]

where the first inequality follows by concavity, the second by using the definition of \( W \) to see that \( \langle s_{t,i}, W[i] \rangle = \int_a^{b} \exp(-\frac{\varepsilon_i}{\psi_{x_t}(o)} \mu[i](o) do \geq \gamma \psi_{x_t}(o) \frac{b-a}{d} \), the third by Hölder’s inequality and the mean value theorem for some \( a_{i,j} \in (a + \frac{b-a}{d}(j-1), a + \frac{b-a}{d}) \), and the fourth by the Lipschitzness of \( \rho[i] \in F_{V,d}^{(\gamma)} \). The third summation is a sum over the errors due to \( \gamma \)-robustness, with the result following by \( U_t(\rho) - U_t(\mu) \leq U_t(\mu) - \log(1 - \max\{\gamma - \lambda, 0\}) - U_t(\mu) \leq 2 \max\{\gamma - \lambda, 0\} \log 2. \)

Note that in the case of $V > 0$ or $\lambda = 0$ we will need to set $d = \omega_T(1)$ or $\gamma = \sigma_T(1)$ in order to obtain sublinear regret. Thus for these more difficult classes our extension of DP-FTRL to non-Euclidean regularizers yields improved rates, as in the Euclidean case the first term has an extra $\sqrt{d}$-factor. The following provides some specific upper bounds derived from Theorem 5.2:

**Corollary 5.1.** For each of the following classes of priors there exist settings of $d$ (where needed) and $\gamma > 0$ in Theorem 5.2 that guarantee the following regret w.p. $\geq 1 - \beta'$:

1. $\lambda$-robust and discrete $\mu[i] \in \mathcal{F}_{0,d}^{(\lambda)}$:
   \[
   \hat{\Omega} \left( \frac{dm}{\lambda} \sqrt{\left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) T} \right)
   \]

2. $\lambda$-robust and $V$-Lipschitz $\mu[i] \in \mathcal{F}_{V,1}^{(\lambda)}$:
   \[
   \hat{\Omega} \left( \frac{m V^4}{\lambda} \sqrt{\left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) T^3} \right)
   \]

3. discrete $\mu[i] \in \mathcal{F}_{0,d}$:
   \[
   \hat{\Omega} \left( \sqrt{dm^4 \frac{V^4}{\psi} \left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) T^3} \right)
   \]

4. $V$-Lipschitz $\mu[i] \in \mathcal{F}_{V,1}$:
   \[
   \hat{\Omega} \left( \sqrt{m^4 \frac{V^4}{\psi} \left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) T^3} \right)
   \]

Thus competing with $\lambda$-robust priors with discrete PDFs enjoys the fastest regret rate of $\hat{\Omega}(\sqrt{T})$, while either removing robustness or competing with any $V$-Lipschitz prior has regret $\hat{\Omega}(T^{3/4})$, and doing both has regret $\hat{\Omega}(T^{7/8})$. When comparing to Lipschitz priors we also incur a dependence on the inverse of minimum datapoint separation, which may be small. A notable aspect of all the bounds is that the regret improves with small $\varepsilon$ due to the sensitivity analysis in Lemma 5.1; indeed for $\varepsilon = O(\varepsilon')$ the regret bound only has a $O(\log \frac{1}{\varepsilon'})$-dependence on the privacy guarantee. Finally, for $\lambda$-robust priors we can also apply the log $\frac{1}{\varepsilon}$-boudedness of $- \log \frac{\psi_{\mu}(x, \mu)}{\psi_{\bar{\mu}}(x, \bar{\mu})}$ and standard online-to-batch conversion (e.g. [14, Proposition 1]) to obtain the following sample complexity guarantee:

**Corollary 5.2.** For any $\alpha > 0$ and distribution $D$ over finite datasets $x$ of $\psi$-separated points from $(a, b)$, if we run the algorithm in Theorem 5.2 on $T = \Omega \left( \frac{\log \frac{1}{\alpha^2} \left( \frac{d^2 m^2}{\lambda} \left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) + \log^2 \frac{1}{\lambda \psi} \right)}{\log \frac{1}{\alpha^2} \left( \frac{d^2 m^2}{\lambda} \left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) + \log^2 \frac{1}{\lambda \psi} \right)} \right)$

i.i.d. samples from $D$ then w.p. $\geq 1 - \beta'$ the average $\hat{W} = \frac{1}{T} \sum_{t=1}^{T} W_t$ of the resulting iterates satisfies $\mathbb{E}_{x \sim D} \log \sum_{i=1}^{m} \frac{1}{\psi_{\mu[i]}(x, \mu[i])} \leq \min_{\mu[i] \in \mathcal{F}_{0,d}^{(\lambda)}} \mathbb{E}_{x \sim D} \log \sum_{i=1}^{m} \frac{1}{\psi_{\mu[i]}(x, \mu[i])} + \alpha$. For $\alpha$-suboptimality w.r.t. $\mu[i] \in \mathcal{F}_{V,1}^{(\lambda)}$ the sample complexity is $\Omega \left( \frac{\log \frac{1}{\alpha^2} \left( \frac{d^2 m^2}{\lambda} \left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) + \log^2 \frac{1}{\lambda \psi} \right)}{\log \frac{1}{\alpha^2} \left( \frac{d^2 m^2}{\lambda} \left(1 + \frac{\min(1, \bar{\varepsilon} m)}{\varepsilon} \right) + \log^2 \frac{1}{\lambda \psi} \right)} \right)$.

### 5.2 Learning to estimate covariance matrices

We next study how to learn prediction matrices for DP covariance estimation by targeting the trace distance between them and the ground truth. This is a more straightforward learning task, with Lipschitz losses over a finite-dimensional domain. Indeed, we could apply standard DP-FTRL and obtain regret $\hat{\Omega}(\sqrt{1 + \frac{d}{\varepsilon} dT})$ w.r.t. any symmetric matrix $W$ because the losses $U_t(W) = ||X_t X_t^T - W||_F$ are $\sqrt{d}$-Lipschitz w.r.t. the Frobenius norm. However, we can reduce the dependence on the dimension by a $\sqrt{d}$-factor by combining our non-Euclidean DP-FTRL algorithm with the well-known matrix-learning technique of using Schatten $p$-norm regularization [25]:
Theorem 5.3. Let $X_1, \ldots, X_T$ be a sequence of datasets with $d$-dimensional columns bounded by 1 in the $\ell_2$-norm. If we run Algorithm 3 on losses $U_t(W) = \|X_tX_t^T/|X_t| - W\|_{\text{Tr}}$, with step-size $\eta = \frac{6\log(d)/T}{1 + (\sqrt{d} + \sqrt{2\log \frac{1}{\beta'}})\sigma \sqrt{d\log_2 T}}$ and regularizer $\phi(\cdot) = \frac{2}{d} \log d \cdot \|\cdot\|_p^2$, then we will have regret

$$\max_{W \in \mathbb{R}^{d \times d}} \sum_{t=1}^T U_t(W_t) - U_t(W) \leq \mathcal{O} \left( \sqrt{\left(1 + \left(\sqrt{d} + \sqrt{\frac{T}{\beta'}}\right)\sigma \sqrt{d\log_2 T}\right) T \log d} \right)$$

w.p. $\geq 1 - \beta'$. For any $\varepsilon' \leq 2 \log \frac{1}{\beta'}$ setting $\sigma = \frac{1}{\varepsilon'} \sqrt{2\log_2 T} \log \frac{1}{\beta'}$ makes this procedure $(\varepsilon', \delta')$-DP. Furthermore, suppose the datasets are drawn i.i.d. from some distribution $D$. If we run the same algorithm and return the average prediction $\hat{W} = \frac{1}{T} \sum_{t=1}^T W_t$ then $T = \tilde{\Omega} \left( \frac{1+d/d'}{\alpha^2} \log \frac{1}{\beta'} \right)$ samples suffice to guarantee that w.p. $1 - \beta'$

$$\mathbb{E}_{X \sim D} \|XX^T/|X| - \hat{W}\|_{\text{Tr}} \leq \min_W \mathbb{E}_{X \sim D} \|XX^T/|X| - W\|_{\text{Tr}} + \alpha$$

Proof. The loss functions $\|X_tX_t^T/|X_t| - W\|_{\text{Tr}}$ have gradients $-U_tS_tU_t^T$, where $U_t$ is the matrix of eigenvectors of $X_tX_t^T/|X_t| - W$ and $S_t$ is the diagonal matrix of the signs of its eigenvalues; the losses are thus $\sqrt{d}$-Lipschitz w.r.t. the Frobenius norm and 1-Lipschitz w.r.t. the trace norm. Note that these gradients can be computed in polynomial time via eigendecomposition and used in DP-FTRL with the Schatten-$p$ norm regularizer $\frac{2}{d} \log d \cdot \|\cdot\|_p$ for $p = 1 + 1/\log d$, which is strongly-convex w.r.t. the trace norm $\|\cdot\|_{\text{Tr}}$ [25]. Since the Gaussian width of the (symmetric) trace ball is $\mathcal{O}(\sqrt{d})$ [1] and the spectral norm is 1-Lipschitz w.r.t. the Frobenius norm, applying Theorem 5.1 yields the bound

$$\frac{3\log d \|W\|_p^2}{2\eta} + \eta \left( 1 + \left(\mathcal{O}(\sqrt{d}) + \sqrt{\frac{T}{\beta'}}\right)\sigma \sqrt{d\log_2 T}\right) T$$

For any optimal $W$ we have

$$\|W\|_p \leq \frac{\|W\|_{\text{Tr}}}{\sqrt{T}} \leq \frac{1}{T} \sum_{t=1}^T \|W - X_tX_t^T/|X_t|\|_{\text{Tr}} + \|X_tX_t^T/|X_t|\|_{\text{Tr}} \leq \frac{2}{T} \sum_{t=1}^T \text{Tr}(X_tX_t^T)/|X_t| \leq 2$$

so the regret follows by substituting $\eta$. The sample complexity result follows from online-to-batch conversion (c.f. Appendix D.1). 

Thus prediction matrices for covariance estimation are efficiently and privately learnable, in both the online and distributional settings. Moreover, for both our extension to non-Euclidean DP-FTRL is critical for obtaining a weaker dependence on the dimension. One limitation of the analysis is that, unlike for quantiles, we did not conduct a refined analysis by studying how swapping single columns of $X_t$ rather than the entire dataset affects the gradient of $U_t$. It is not immediately clear that an improvement is possible, with the difficulty being the gradient’s dependence on the signs of the eigenvalues.
5.3 Learning the initialization and number of iterations for data release

Finally, we learn to initialize MWEM-based data release. Here we are faced with optimizing

\[ U_t(w, m) = \frac{8n_t}{m} D_{KL} \left( \frac{x_t}{n_t} || w \right) + \frac{16m^2}{\epsilon^2 n + t} \left( 3 \log \frac{2m}{\beta} + 2 \log^2 |Q| \right)^2 \]  

(29)

Notably, unlike the past learning settings, this function is parameterized by both a prediction \( w \) and the number of steps \( m \), which we will also set online. The reason for this is that the optimal step-size depends on the similarity between instances: if for the optimal \( w \) the measure \( D_{KL}(x_t/n_t||w) \) is small for most datasets \( x_t \) then it is better to set a small \( m \) above, whereas if it is usually large it should be counter-acted with a larger \( m \). Our goal will thus be to set \( w_t \) and \( m_t \) together in an online fashion so as to simultaneously compete with the optimal \( \lambda \)-robust \( w \in \Delta_d \) for some \( \lambda > 0 \) and the optimal number of steps \( m > 0 \). To do so we will run DP-FTRL with the entropic regularizer, i.e. private exponentiated gradient (EG), to set both the initialization from the simplex \( \Delta_d \) and to set the number of iterations \( m_t \) at step \( t \) by sampling from a categorical distribution. This has the following regret guarantee:

**Theorem 5.4.** Let \( x_1, \ldots, x_T \in \mathbb{Z}_{\geq 0}^d \) be a sequence of datasets with \( n_t = |x_t| \) entries each, let \( N = \max_t n_t \), and consider any \( \gamma \in (0, 1] \) and \( \lambda \in (0, 1] \). Then there exists \( M \in \mathbb{Z}_{\geq 0} \) and \( \eta_t, \sigma_t, \eta_w, \sigma_w > 0 \) s.t. running DP-FTRL with regularizer \( \phi(w) = \langle w, \log w \rangle \), step-size \( \eta_w \), and noise \( \sigma_w \) on the losses \( n_t D_{KL}(x_t/n_t||w) \) over the domain \( w_{[t]} \geq \gamma/d \) to set \( w_t \) and simultaneously running DP-FTRL with regularizer \( \phi(\theta) = \langle \theta, \log \theta \rangle \), step-size \( \eta_\theta \), and noise \( \sigma_\theta \) on the losses \( \mathbb{E}_{w \sim \theta} U_t(w, m) \) over the domain \( \Delta_M \) and setting \( m_t \) using the categorical distribution defined by \( \theta_t \) over the \( M \)-simplex such that the entire scheme is \((\epsilon', \delta')\)-DP and w.h.p. has regret

\[ \tilde{O} \left( \left( \frac{N^{\frac{4}{d}} + N^{\frac{3}{d}}/\min\{1, \epsilon^2\}}{\min\{1, \epsilon\}} + \frac{dN}{\gamma} + \frac{d}{\gamma} \sqrt{\frac{N}{\epsilon \gamma}} \right) \sqrt{T + \max\{\gamma - \lambda, 0\} NT} \right) \]  

(30)

w.r.t. any \( m > 0 \) and \( w \in \Delta_d \) satisfying \( w_{[t]} \geq \gamma/d \). For \( \lambda > 0 \) setting \( \gamma = \lambda \) yields regret

\[ \tilde{O} \left( \frac{d}{\lambda \min\{1, \epsilon^2\}} N^{\frac{4}{d}} \sqrt{T/\epsilon \gamma} \right) \]  

for \( \lambda = 0 \) and \( T \geq d^2 \), setting \( \gamma = \frac{d^2}{\lambda T^2} \) has regret \( \tilde{O} \left( \frac{N^{\frac{4}{d}}}{\min\{1, \epsilon^2\}} \sqrt{d/\epsilon \gamma T^2} \right) \).

As in quantile learning, we suffer a strong dependence on the dimension here, and the rate is worse if we try to compete the non-robust initializations (\( \lambda = 0 \)). It thus remains an open question whether either a better learning result or upper bound is possible. Nevertheless, to interpret this guarantee, note that for \( H_\lambda = \min_{w_{[t]} \geq \lambda/d} \left( \sum_{t=1}^T n_t D_{KL}(x_t/n_t||w) \right) / \left( \sum_{t=1}^T n_t \right) \) and if \( n_t = n \forall t \) then we have that the optimum-in-hindsight for the average upper bound is

\[ \min_{m > 0, w_{[t]} \geq \lambda/d} \frac{1}{T} \sum_{t=1}^T U_t(w, m) = \tilde{O} \left( \frac{\log^4 |Q|}{\epsilon \gamma^2} \left( \frac{1}{\lambda} \sum_{t=1}^T n_t \right) \frac{4}{\gamma} H_\lambda^2 \right) = \tilde{O} \left( \frac{\sqrt{n \log^4 |Q|}}{\epsilon^2 \gamma^2} \right) H_\lambda^2 \]  

(31)

Since \( H_\lambda \) approximates the entropy of the aggregate distribution across instances \( x_1, \ldots, x_T \)—indeed for \( \lambda = 0 \) it is exactly the entropy of the average distribution \( \left( \sum_{t=1}^T x_t \right) / \sum_{t=1}^T n_t \)—the regret guarantee shows that we will do well asymptotically if the entropy is small. Note that being able to choose \( m \) in addition to \( w \) is crucial to adapting to this entropy, and is closely related to the problem of choosing the step-size in meta-learning, where similar aggregate measures appear as forms of task-similarity [37, 7].
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A  Multiple quantiles

Lemma A.1. In Algorithm 4, for any \(i \in \{m\}\) we have

1. \(\text{Gap}_{\tilde{\gamma}}(\hat{x}_i, o) \leq \text{Gap}_{\hat{x}}(x, o) + \gamma_i \forall o \in \mathbb{R}\)

2. \(\text{Gap}_{\tilde{\gamma}}(x, o) \leq \text{Gap}_{\hat{x}}(\hat{x}_i, o) + \gamma_i \forall o \in [\hat{a}_i, \hat{b}_i]\)

where \(\gamma_i = (1 - \tilde{\gamma}_i) \text{Gap}_{\hat{x}}(x, \hat{a}_i) + \tilde{\gamma}_i \text{Gap}_{\hat{\gamma}}(x, \hat{b}_i)\).

Proof. For \(o \in [\hat{a}_i, \hat{b}_i]\) we apply the triangle inequality twice to get

\[
\text{Gap}_{\tilde{\gamma}}(\hat{x}_i, o) = |\max_{\hat{x}_i < o} j - [\tilde{q}_i \hat{n}_i]| \\
= |\max_{\hat{x}_i < o} j + \max_{\hat{x}_i < \hat{a}_i} j - [q_i n] + [q_i n] - \max_{\hat{x}_i < \hat{a}_i} j - [\tilde{q}_i \hat{n}_i]| \\
\leq \text{Gap}_{\hat{x}}(x, o) + \left| [\tilde{q}_i ([\hat{r}_i n] - [q_i n])] + [q_i n] - \max_{\hat{x}_i < \hat{a}_i} j - [\tilde{q}_i (\max_{\hat{x}_i < \hat{b}_i} j - \max_{\hat{x}_i < \hat{a}_i} j)] \right| \\
\leq \text{Gap}_{\hat{x}}(x, o) + (1 - \tilde{\gamma}_i) \text{Gap}_{\hat{x}}(x, \hat{a}_i) + \tilde{\gamma}_i \text{Gap}_{\hat{\gamma}}(x, \hat{b}_i)
\]

and again to get

\[
\text{Gap}_{\tilde{\gamma}}(x, o) = |\max_{\hat{x}_i < o} j - [q_i n]| \\
= |\max_{\hat{x}_i < o} j + \max_{\hat{x}_i < \hat{a}_i} j - [\tilde{q}_i \hat{n}_i] - [q_i n]| \\
\leq \text{Gap}_{\hat{x}}(\hat{x}_i, o) + \left| \max_{\hat{x}_i < \hat{a}_i} j - [\tilde{q}_i (\max_{\hat{x}_i < \hat{b}_i} j + \max_{\hat{x}_i < \hat{a}_i} j)] - [\tilde{q}_i ([\hat{r}_i n] - [q_i n])] - [q_i n] \right| \\
\leq \text{Gap}_{\hat{x}}(\hat{x}_i, o) + (1 - \tilde{\gamma}_i) \text{Gap}_{\hat{x}}(x, \hat{a}_i) + \tilde{\gamma}_i \text{Gap}_{\hat{\gamma}}(x, \hat{b}_i)
\]

For \(o < \hat{a}_i\) we use the fact that \(\max_{\hat{x}_i < o} j \leq \max_{\hat{x}_i < \hat{a}_i} j\) and the triangle inequality to get

\[
\text{Gap}_{\tilde{\gamma}}(\hat{x}_i, o) = [\tilde{q}_i \hat{n}_i] \\
= [\tilde{q}_i (\max_{\hat{x}_i < \hat{b}_i} j - \max_{\hat{x}_i < \hat{a}_i} j)] \\
\leq [\tilde{q}_i \max_{\hat{x}_i < \hat{b}_i} j + (1 - \tilde{\gamma}_i) \max_{\hat{x}_i < \hat{a}_i} j - \max j] \\
= [\tilde{q}_i \max_{\hat{x}_i < \hat{b}_i} j + (1 - \tilde{\gamma}_i) \max_{\hat{x}_i < \hat{a}_i} j - [q_i n] - [\tilde{q}_i ([\hat{r}_i n] - [q_i n])] - [q_i n]] \\
\leq \text{Gap}_{\hat{x}}(x, o) + (1 - \tilde{\gamma}_i) \text{Gap}_{\hat{x}}(x, \hat{a}_i) + \tilde{\gamma}_i \text{Gap}_{\hat{\gamma}}(x, \hat{b}_i)
\]

For \(o > \hat{b}_i\) we use the fact that \(\max_{\hat{x}_i < o} j \leq \max_{\hat{x}_i < \hat{a}_i} j\) and the triangle inequality to get

\[
\text{Gap}_{\tilde{\gamma}}(\hat{x}_i, o) = [1 - \tilde{\gamma}_i] \hat{n}_i] \\
= [(1 - \tilde{q}_i) (\max_{\hat{x}_i < \hat{b}_i} j - \max_{\hat{x}_i < \hat{a}_i} j)] \\
\leq \max_{\hat{x}_i < o} j - [\tilde{q}_i \max_{\hat{x}_i < \hat{b}_i} j - (1 - \tilde{\gamma}_i) \max_{\hat{x}_i < \hat{a}_i} j] \\
= \max_{\hat{x}_i < o} j - [\tilde{q}_i \max_{\hat{x}_i < \hat{b}_i} j + (1 - \tilde{\gamma}_i) \max_{\hat{x}_i < \hat{a}_i} j - [q_i n] + [\tilde{q}_i ([\hat{r}_i n] - [q_i n])] + [q_i n]) \\
\leq \text{Gap}_{\hat{x}}(x, o) + (1 - \tilde{\gamma}_i) \text{Gap}_{\hat{x}}(x, \hat{a}_i) + \tilde{\gamma}_i \text{Gap}_{\hat{\gamma}}(x, \hat{b}_i)
\]
Algorithm 4: ApproximateQuantiles with predictions

Input: sorted unrepeated data $\mathbf{x} \in (a, b)^n$, ordered quantiles $q_1, \ldots, q_m \in (0, 1)$, priors $\mu_1, \ldots, \mu_m : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, prior adaptation rule $r \in \{\text{conditional}, \text{edge}\}$, privacy parameters $\varepsilon_1, \ldots, \varepsilon_m > 0$, branching factor $K \geq 2$

// runs single-quantile algorithm on datapoints $\hat{x}$

Method $\text{quantile}(\hat{x}, q, \varepsilon, \mu)$:
- Output: $o \in (a, b)$ w.p. $\propto \exp(-\varepsilon \text{Gap}_q(\hat{x}, o)/2)\mu(o)$

Method $\text{recurse}(j, q, \hat{q}, \hat{a}, \hat{b})$:
- // determines $K - 1$ indices $i$ whose quantiles to compute at this node
- if $|i| \geq K$ then
-   $i \leftarrow ([j]\lfloor|j|/K]\rceil, \cdots, [j]\lfloor(K-1)|j|/K}\rceil)$
- else
-   $i \leftarrow j$

- // restricts dataset to the interval $(\hat{a}, \hat{b})$
- $k_{\hat{a}} \leftarrow \min_{x[i]} > \hat{a} k$
- $k_{\hat{b}} \leftarrow \max_{x[i]} < \hat{b} k$
- $\hat{x}_i \leftarrow (x_{[k_{\hat{a}]}, \cdots, x_{[k_i]}])$

- // sets relative quantiles $\hat{q}_i$ and restricts priors to the interval $[\hat{a}, \hat{b}]$
- for $j = 1, \ldots, |i|$ do
-   $\hat{q}_{i[j]} \leftarrow (q_{i[j]} - q)/\hat{q} - q$
-   if $r = \text{conditional}$ then
-     $\hat{\mu}_{i[j]}(o) \leftarrow \frac{\mu_{i[j]}(o)}{\mu_{i[j]}([\hat{a}, \hat{b}]), o \in [\hat{a}, \hat{b}]}$
-   else
-     $\hat{\mu}_{i[j]}(o) \leftarrow \mu_{i[j]}(o) \delta(o - \hat{a}) + \mu_{i[j]}([\hat{b}, \infty)) \delta(o - \hat{b})$

- // computes $K - 1$ quantiles $o_i$ and sorts the results
- $o_i \leftarrow \text{quantile}(\hat{x}_i, \hat{q}_{i[j]}, \hat{a}, \hat{b})$, $\cdots$, $\text{quantile}(\hat{x}_i, \hat{q}_{i[j]}, \hat{a}, \hat{b})$
- $o_i \leftarrow \text{sort}(o_i)$

- // recursively computes remaining indices on the $K$ intervals induced by $o_i$
- if $|i| < K$ then
-   $o \leftarrow o_i$
- else
-   $o \leftarrow \text{concat}(\text{recurse}(j_1, \cdots, j\lfloor|j|/K\rceil - 1) ; q, q_{i[j]}, \hat{a}, o_{i[j]}), (o_{i[j]})$
-   for $j = 2, \ldots, |i|$ do
-     $o \leftarrow \text{concat}(o, \text{recurse}(j\lfloor|j|/K\rceil + 1, \cdots, j\lfloor|j|/K\rceil - 1) ; q_{i[j-1]}, q_{i[j]}, o_{i[j]})$
-     $o \leftarrow \text{concat}(o, (o_{i[j]}))$
-   $o \leftarrow \text{concat}(o, \text{recurse}(j\lfloor|j|/K\rceil - 1, \cdots, j\lfloor|j|/K\rceil + 1) ; q_{i[K-1]}, \hat{a}, o_{[K-1], \hat{b}})$
- Output: $o$

Output: $\text{recurse}((1, \cdots, m), 0, 1, -\infty, \infty)$
Lemma A.2. For any $\gamma > 0$ the estimate $o_i$ of the quantile $q_i$ by Algorithm 4 satisfies

$$
Pr\{\text{Gap}_q(x, o_i) \geq \gamma\} \leq \frac{\exp(\varepsilon_\gamma - \gamma/2)}{\Psi^{(e)}_{\tilde{q}_i}(\tilde{x}_i, \tilde{\mu}_i)} \quad (36)
$$

Proof. We use $k_i$ to denote the interval $\tilde{j}_k^{(i)}$ sampled at index $i$ in the algorithm and note that $o_i$ corresponds to the released number $o$ at that index. Since $o_i \in [\hat{a}_i, \hat{b}_i]$, applying Lemma A.1 yields

$$
\Pr\{\text{Gap}_q(x, o_i) \geq \gamma\} = \sum_{j=0}^{n_i} \Pr\{k_i = j\} \frac{\exp(-\varepsilon \text{Gap}_q(x, \tilde{j}_j^{(i)})/2)\tilde{\mu}_i(\tilde{j}_j^{(i)})}{\Psi^{(e)}_{\tilde{q}_i}(\tilde{x}_i, \tilde{\mu}_i)} \text{Gap}_q(x, \tilde{j}_j^{(i)}) \geq \gamma
$$

$$
= \sum_{j=0}^{n_i} \sum_{l=0}^{\hat{n}_i} \exp(-\varepsilon \text{Gap}_q(x, \tilde{j}_j^{(i)})/2)\tilde{\mu}_i(\tilde{j}_j^{(i)}) \text{Gap}_q(x, \tilde{j}_j^{(i)}) \geq \gamma
$$

$$
\leq \exp(\varepsilon_\gamma/2) \exp(-\varepsilon \text{Gap}_q(x, \tilde{j}_j^{(i)})/2)\tilde{\mu}_i(\tilde{j}_j^{(i)}) \text{Gap}_q(x, \tilde{j}_j^{(i)}) \geq \gamma
$$

$$
\leq \exp(\varepsilon_\gamma - \gamma/2) \Psi^{(e)}_{\tilde{q}_i}(\tilde{x}_i, \tilde{\mu}_i)
$$

Lemma A.3. For any $\gamma > 0$ the estimate $o_i$ of the quantile $q_i$ by Algorithm 4 with edge-based prior adaptation satisfies

$$
Pr\{\text{Gap}_q(x, o_i) \geq \gamma\} \leq \frac{\exp(\varepsilon_\gamma - \gamma/2)}{\Psi^{(e)}_{\tilde{q}_i}(x, \mu_i)} \quad (38)
$$

Proof. Applying Lemma A.1 yields the following lower bound on $\Psi^{(e)}_{\tilde{q}_i}(\tilde{x}_i, \tilde{\mu}_i)$:

$$
\sum_{l=0}^{\hat{n}_i} \exp(\varepsilon u_{\tilde{q}_i}(\tilde{x}_i, \tilde{j}_j^{(i)})/2)\tilde{\mu}_i(\tilde{j}_j^{(i)}) = \exp(\varepsilon u_{\tilde{q}_i}(\tilde{x}_i, \tilde{j}_j^{(i)})/2)\mu_i((-\infty, \hat{a}_i]) + \exp(\varepsilon u_{\tilde{q}_i}(\tilde{x}_i, \tilde{j}_j^{(i)})/2)\mu_i([\hat{b}_i, \infty))
$$

$$
+ \sum_{l=0}^{\hat{n}_i} \exp(\varepsilon u_{\tilde{q}_i}(\tilde{x}_i, \tilde{j}_j^{(i)})/2)\mu_i(\tilde{j}_j^{(i)})
$$

$$
+ \sum_{l=0}^{\hat{n}_i} \max_{x_{[l]} < \hat{a}_i, j} \exp(-\varepsilon \text{Gap}_q(\tilde{x}_i, I_l \cap (-\infty, \hat{a}_i])/2)\mu_i(I_l \cap (-\infty, \hat{a}_i])
$$

$$
+ \sum_{l=0}^{\hat{n}_i} \max_{x_{[l]} < \hat{b}_i, j} \exp(-\varepsilon \text{Gap}_q(\tilde{x}_i, I_l \cap [\hat{b}_i, \infty))/2)\mu_i(I_l \cap [\hat{b}_i, \infty))
$$

$$
+ \sum_{l=0}^{\hat{n}_i} \max_{x_{[l]} < \hat{a}_i, j} \exp(-\varepsilon \text{Gap}_q(\tilde{x}_i, I_l \cap [\hat{a}_i, \hat{b}_i])\mu_i(I_l \cap [\hat{a}_i, \hat{b}_i])
$$

$$
\geq \Psi^{(e)}_{\tilde{q}_i}(x, \mu_i) \exp(-\varepsilon_\gamma/2)
$$

Substituting into Lemma 3.2 yields the result. \qed
Lemma A.4. Suppose $q_0 < q_1$ are two quantiles and $a_0 > o_1$. Then
\[
\max_{i=0,1} \text{Gap}_{q_i}(x, o_i) \geq \max_{i=0,1} \text{Gap}_{q_i}(x, o_{1-i})
\] (40)

Proof. We consider four cases. If $|q_0|x| \leq \max_{x_{[i]} < o_1} j$ and $|q_1|x| \leq \max_{x_{[j]} < a_0} j$ then
\[
|q_0|x| \leq \min\{|q_1|x|, \max_{x_{[i]} < o_1} j\} \leq \max\{|q_1|x|, \max_{x_{[j]} < a_0} j\} \leq \max j
\] (41)
and so
\[
\max_{i=0,1} \text{Gap}_{q_i}(x, o_i) = \max j - |q_0|x| \geq \max_{i=0,1} \text{Gap}_{q_i}(X, o_{i-1})
\] (42)
If $|q_0|x| \leq \max_{x_{[i]} < o_1} j$ and $|q_1|x| \leq \max_{x_{[j]} < a_0} j$ then
\[
|q_0|x| \leq \max j \leq \max_{x_{[j]} < a_0} j \leq |q_1|x|
\] (43)
and so both improve after swapping. If $|q_0|x| > \max_{x_{[i]} < o_1} j$ and $|q_1|x| > \max_{x_{[j]} < a_0} j$ then
\[
\max j \leq \min\{|q_0|x|, \max_{x_{[i]} < o_1} j\} \leq \max|\{q_0|x|, \max_{x_{[j]} < a_0} j\} \leq |q_1|x|
\] (44)
and so
\[
\max_{i=0,1} \text{Gap}_{q_i}(x, o_i) = \max j - |q_1|x| \geq \max_{i=0,1} \text{Gap}_{q_i}(X, o_{i-1})
\] (45)
Finally, if $|q_0|x| > \max_{x_{[i]} < o_1} j$ and $|q_1|x| \leq \max_{x_{[j]} < a_0} j$ then
\[
\max j < |q_0|x| \leq |q_1|x| \leq \max j
\] (46)
so swapping will make the new largest error for each quantile at most as large as the other quantile’s current error. \qed

B Covariance estimation

B.1 Sensitivity results

Lemma B.1. The eigenvalues $\Lambda$ of $XX^T/n - W = U\Lambda U^T$ for $X \in \mathbb{R}^{d \times n}$ with 1-bounded columns and symmetric $W \in \mathbb{R}^{d \times d}$ has $\ell_1$-sensitivity $2/n$, as does its trace norm $\|\Lambda\|_1 = \|XX^T/n - W\|_T$.

Proof. Consider two datasets $X, \tilde{X} \in \mathbb{R}^{d \times n}$ that share the same first $n-1$ columns $Z \in \mathbb{R}^{d \times n-1}$ but have different respective last columns $x, \tilde{x} \in \mathbb{R}^d$. For any vector $v \in \mathbb{R}^d$ we have
\[
v^T(XX^T/n - W)v = v^TZZ^Tv/n + v^Txx^Tv/n - v^TWv \geq v^T(ZZ^T/n - W)v
\] (47)
so for $\tilde{U}\tilde{\Lambda}\tilde{U}^T = ZZ^T/n - W$ we have
\[
\Lambda_{[i]} \geq \tilde{\Lambda}_{[i]} \forall i \in [d]
\] (48)
Thus
\[
\|\Lambda - \tilde{\Lambda}\|_1 = \text{Tr}(XX^T/n - W) - \text{Tr}(ZZ^T/n - W) = \text{Tr}(xx^T/n) \leq 1/n
\] (49)
The same argument holds when replacing $X$ by $\tilde{X}$, so the result for the eigenvalues follows by the triangle inequality.

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For the trace norm we have that
\[
\|XX^T/n - W\|_{\text{Tr}} - \|ZZ^T/n - W\|_{\text{Tr}} = \left| \sum_{i=1}^{d} |\Lambda[i]| - |\hat{\Lambda}[i]| \right| \leq \sum_{i=1}^{d} |\Lambda[i]| - |\hat{\Lambda}[i]| \leq \sum_{i=1}^{d} |\Lambda[i] - \hat{\Lambda}[i]| \leq 1/n
\] (50)

where the second inequality holds trivially when \(\Lambda[i]\) and \(\hat{\Lambda}[i]\) have the same sign and otherwise the latter is negative (48) so we have \(|\Lambda[i] - \hat{\Lambda}[i]| = |\Lambda[i] + \hat{\Lambda}[i]| \leq |\Lambda[i] - \hat{\Lambda}[i]|\), and the third is by Equation 49. This also holds when replacing \(X\) by \(\tilde{X}\), so the result follows by the triangle inequality. \(\square\)

**Lemma B.2.** The \(\ell_2\)-sensitivity of \(XX^T/n - W\) is \(\sqrt{2}/n\) and that of its eigenvalues \(\Lambda\) is \(\frac{4\sqrt{2}}{n}\).

**Proof.** The first result follows directly by [12, Lemma 3.2] For the second, consider two datasets \(X, \tilde{X} \in \mathbb{R}^{d \times n}\) that share the same first \(n - 1\) columns \(Z \in \mathbb{R}^{d \times n-1}\) but have different respective last columns \(x, \tilde{x} \in \mathbb{R}^d\). Applying [23, Fact 1] and Lemma B.3 yields
\[
\sum_{i=1}^{d} \left( \max\{\Lambda[i], 0\} - \max\{\hat{\Lambda}[i], 0\} \right)^2 \leq \left( \sum_{i=1}^{d} \max\{\Lambda[i], 0\} - \max\{\hat{\Lambda}[i], 0\} \right)^2 + \left( \sum_{i=1}^{d} \max\{\hat{\Lambda}[i], 0\} - \max\{\hat{\Lambda}[i], 0\} \right)^2 \leq 8/n^2
\] (51)

As a similar fact holds for the negative spectrum, we have by triangle inequality that the Frobenius norm of the difference is bounded by \(4\sqrt{2}/n\). \(\square\)

**Lemma B.3.** If \(W \in \mathbb{R}^{d \times d}\) is symmetric, \(XX^T/n = ZZ^T/n + xx^T\), \(\Lambda\) is the matrix of eigenvalues of \(XX^T/n - W\), and \(\hat{\Lambda}\) is the matrix of eigenvalues of \(ZZ^T/n - W\) then
\[
\sum_{i=1}^{d} \left| \max\{\Lambda[i], 0\} - \max\{\hat{\Lambda}[i], 0\} \right| \leq 2/n \quad \text{and} \quad \sum_{i=1}^{d} \left| \max\{-\Lambda[i], 0\} - \max\{-\hat{\Lambda}[i], 0\} \right| \leq 2/n
\] (52)

**Proof.** By Equation 47 we have
\[
\sum_{i=1}^{d} \left| \max\{\Lambda[i], 0\} - \max\{\hat{\Lambda}[i], 0\} \right| \leq \sum_{i=1}^{d} |\Lambda[i] - \hat{\Lambda}[i]| = \text{Tr}(XX^T/n - W) - \text{Tr}(ZZ^T/n - W) \leq \text{Tr}(xx^T/n) \leq 1/n
\] (53)

where the first inequality hold trivially when \(\Lambda[i]\) and \(\hat{\Lambda}[i]\) have the same sign and when they are different the latter is the negative one and so \(\max\{\Lambda[i], 0\} - \max\{\hat{\Lambda}[i], 0\}\) is positive. The same argument holds when replacing \(X\) by \(\tilde{X}\), so the result follows by triangle inequality. Similarly,
\[
\sum_{i=1}^{d} \left| \max\{-\Lambda[i], 0\} - \max\{-\hat{\Lambda}[i], 0\} \right| \leq \sum_{i=1}^{d} |\min\{\Lambda[i], 0\} - \min\{\hat{\Lambda}[i], 0\}| \leq \sum_{i=1}^{d} |\Lambda[i] - \hat{\Lambda}[i]| \leq 1/n
\] (54)

where the first inequality hold trivially when \(\Lambda[i]\) and \(\hat{\Lambda}[i]\) have the same sign and when they are different the latter is the negative one and so \(\min\{\Lambda[i], 0\} - \min\{\hat{\Lambda}[i], 0\}\) is negative. The same argument holds when replacing \(X\) by \(\tilde{X}\), so the result follows by triangle inequality. \(\square\)
B.2 Proof of Lemma 3.3

Proof. Let $C = XX^T/n$. Then by triangle inequality and the orthonormality of $\tilde{U}$ we have

$$
\|C - W - \tilde{U} \hat{\Lambda} \tilde{U}^T\|_F = \|C - W - \tilde{U} \hat{\Lambda} \tilde{U}^T\|_F \\
= \|C - W - \tilde{U} \Lambda \tilde{U}^T - \tilde{U} (\hat{\Lambda} - \Lambda) \tilde{U}^T\|_F \\
\leq \|C - W - \tilde{U} \Lambda \tilde{U}^T\|_F + \|\tilde{U} (\hat{\Lambda} - \Lambda) \tilde{U}^T\|_F \\
\leq \sqrt{\|C - W\|_F^2 - 2 \text{Tr}((C - W)(\hat{\Lambda} \tilde{U}^T))} + \|\tilde{U} \Lambda \tilde{U}^T\|_F + \|\tilde{U}\|_2
$$

(55)

\[
= \sqrt{\sum_{j=1}^{d} \Lambda_{[j]}^2 - 2 \text{Tr}(\Lambda \tilde{U}^T (C - W) \tilde{U}) + \|\tilde{U}\|_2}
\]

Therefore

\[
\tilde{U}_{[j]}^T (XX^T - W) \tilde{U}_{[j]} = \tilde{U}_{[j]}^T (XX^T - W + Z) \tilde{U}_{[j]} - \tilde{U}_{[j]}^T Z \tilde{U}_{[j]} \\
\geq \tilde{U}_{[j]}^T (XX^T - W + Z) \tilde{U}_{[j]} - \|Z\|_\infty \\
\geq u_j^T (C - W + Z) u_j - \|Z\|_\infty \\
\geq u_j^T (C - W) u_j - 2 \|Z\|_\infty \geq \Lambda_{[j]} - 2 \|Z\|_\infty 
\]

(57)

Similarly, for $j \in [d]$ s.t. $\Lambda_{[j]} \leq 0$ let $u_j = \arg \max_{u \in \mathbb{R}^{(d-j+1)} | u_2 = 1, \tilde{U}_{[1:j]} u = 0} u^T (C - W) u$. By the Courant-Fischer-Weyl min-max principle we have that

\[
\Lambda_{[j]} = \min_{\mathbf{v} \in \mathbb{R}^{(d-j) \times d}} \max_{\|u\|_2 = 1, \mathbf{v} u = 0} u^T (C - W) u \leq \max_{\|u\|_2 = 1} u^T (C - W) u = u_j^T (C - W) u_j
\]

(56)

Therefore

\[
\tilde{U}_{[j]}^T (C - W) \tilde{U}_{[j]} = \tilde{U}_{[j]}^T (C - W + Z) \tilde{U}_{[j]} - \tilde{U}_{[j]}^T Z \tilde{U}_{[j]} \\
\leq \tilde{U}_{[j]}^T (C - W + Z) \tilde{U}_{[j]} + \|Z\|_\infty \\
\leq u_j^T (C - W + Z) u_j + \|Z\|_\infty \\
\leq u_j^T (C - W) u_j + 2 \|Z\|_\infty \leq \Lambda_{[j]} + 2 \|Z\|_\infty
\]

(59)

Substituting the bounds (57) and (59) in for the appropriate $j$ terms in the summation in Equation 55 yields $\|C - W - \tilde{U} \hat{\Lambda} \tilde{U}^T\|_F \leq 2 \|Z\| + 2 \sqrt{\|Z\|_\infty \sum_{j=1}^{d} \Lambda_{[j]}^2} = 2 \left( \|Z\|_2 + \sqrt{\|Z\|_\infty \|C - W\|_F} \right)$. \hfill \Box
Algorithm 5: SeparateCov with predictions (zCDP)

Input: data $X \in \mathbb{R}^{d \times n}$, symmetric prediction matrix $W \in \mathbb{R}^{d \times d}$, privacy parameter $\rho > 0$

$U \Lambda U^T \leftarrow X X^T / n - W$

$\tilde{\Lambda} \leftarrow \Lambda + \text{diag}(z)$ where $z \sim \mathcal{N}(0, \frac{4 \alpha^2}{n \rho})$ // add prediction noise to error eigenvalues

$\tilde{C} \leftarrow X X^T / n + Z$ for $Z_{[i,j]} = Z_{[j,i]} \sim \mathcal{N}\left(0, \frac{4 \alpha^2}{n \rho}\right)$

$\tilde{U} \tilde{\Lambda} \tilde{U}^T \leftarrow \tilde{C} - Z$ // get eigenvectors of noised prediction error

Output: $\hat{C} = \tilde{U} \tilde{\Lambda} \tilde{U}^T + W$ // combine to estimate $X X^T / n - W$, then add $W$

B.3 zCDP guarantees for SeparateCov with predictions

**Definition B.1.** Algorithm $A$ is $\rho$-zCDP if $D_\alpha(A(X)||A(\tilde{X})) \leq \rho \alpha \forall \alpha > 1$ whenever $X$ and $\tilde{X}$ differ in a single element, where $D_\alpha$ is the $\alpha$-Rényi divergence.

**Theorem B.1.** If $X$ has columns bounded by 1 in $\ell_2$-norm then Algorithm 5 is $\rho$-zCDP and w.p. $\geq 1 - \beta$

$$\| \hat{C} - X X^T / n \|_F^2 \leq O \left( \frac{d}{n^2 \rho} + \frac{\| C - W \|_{Tr}}{n} \sqrt{\frac{d}{\rho}} \right)$$

**Proof.** The privacy guarantee follows from the composition of two Gaussian mechanisms with the sensitivities of Lemma B.2. The utility guarantee is due to substituting Gaussian concentration from [23, Lemmas 5 and 6] into Lemma 3.3. \qed

B.4 IterativeEigenvectorSampling with predictions

**Lemma B.4.** Given a dataset $X \in \mathbb{R}^{d \times n}$ with 1-bounded columns, any orthonormal basis $P \in \mathbb{R}^{k \times d}$, and a symmetric matrix $W \in \mathbb{R}^{d \times d}$ the queries $u^T P (X X^T / n - W) P^T u$ and $u^T P (M - X X^T / n) P^T u$—where $\{A\}_+$ denotes taking only the components of $A$ with positive eigenvalues—have sensitivity $2/n$.

**Proof.** Consider two datasets $X, \tilde{X} \in \mathbb{R}^{d \times n}$ that share the same first $n - 1$ columns $Z \in \mathbb{R}^{d \times n - 1}$ but have different respective last columns $x, \tilde{x} \in \mathbb{R}^d$. Let $P_+$ and $Q_+ \in \mathbb{R}^{d \times d}$ be projection matrices removing the negative components of $X X^T - W$ and $Z Z^T - W$, respectively. Then we have

$$\| [X X^T / n - W]^+ - [Z Z^T / n - W]^+ \|_2$$

$$= \| P_+ (X X^T / n - W) P_+ - Q_+ (Z Z^T / n - W) Q_+ \|_2$$

$$= \max_{|v|_2 = 1} v^T P_+ (Z Z^T / n - W) P_+ v + v^T P_+ x x^T P_+ v / n - v^T Q_+ (Z Z^T / n - W) Q_+ v$$

$$\leq 1 / n + \max_{|v|_2 = 1} v^T Q_+ (Z Z^T / n - W) Q_+ v - v^T Q_+ (Z Z^T / n - W) Q_+ v / n = 1 / n$$

(61)

where the second equality follows by (47) and the definition of the spectral norm. The same argument holds when replacing $X$ by $\tilde{X}$, so we can bound the sensitivity by the triangle inequality:

$$|u^T P (X X^T / n - W) P^T u - u^T P (\tilde{X} \tilde{X}^T / n - W) P^T u|$$

$$\leq \| P ([X X^T / n - W]^+ - [\tilde{X} \tilde{X}^T / n - W]^+) P^T \|_2$$

$$\leq \| [X X^T / n - W]^+ - [X X^T / n - W]^+ \|_2 \leq 2 / n$$

(62)
Similarly, for $P_-$ and $Q_-$ $\in \mathbb{R}^{d \times d}$ the projection matrices removing the negative components of $XX^T - W$ and $ZZ^T - W$, respectively, we have

$$
\|\{W - XX^T/n\}^+ - \{W - ZZ^T/n\}^+\|_2
= \|P_-(W - XX^T/n)P_+ - Q_-(W - ZZ^T/n)Q_+\|_2 \\
= \max_{\|v\|_2 = 1} v^TQ_-(W - XX^T/n)Q_+v + v^TQ_-XX^TQ_+v/n - v^TP_-(W - XX^T/n)P_+v \\
\leq 1/n + \max_{\|v\|_2 = 1} v^TP_-(W - XX^T/n)P_+v - v^TP_-(W - XX^T/n - W)P_+v = 1/n
$$

where the second equality follows by (47) and the definition of the spectral norm. The same argument holds when replacing $X$ by $\hat{X}$, so as before we can obtain the sensitivity via the triangle inequality.

\[ \Box \]

**Theorem B.2.** Algorithm 6 preserves $(\sum s \sum_{i=0}^d \varepsilon^{(s)}_i)\cdot DP$ and the output $\hat{C}$ satisfies w.p. $\geq 1 - \beta$

$$
|XX^T/n - \hat{C}|_F^2 \leq \tilde{O}\left(\frac{d}{n} \left( \sum_{s \in \{\pm 1\}} \frac{1}{\varepsilon^{(s)}_0} - \frac{1}{\varepsilon^{(s)}_1} \right) \right)
$$

where $\Lambda[i]$ is the matrix of eigenvalues of $XX^T/n - W$, $S[i]$ is the matrix of its signs, and $\tilde{O}$ hides logarithmic factors in $d$, $|XX^T/n - W|_2$, and $\beta$.

**Proof.** The privacy result follows from Lemma B.4 applied to Algorithm 6’s release of $\lambda^{(s)}$ and $\bar{u}^{(s)}$ using the Laplace and Exponential mechanisms, respectively, followed by basic composition. For utility, since $XX^T/n - \hat{C} = XX^T - W + C - W = \{XX^T/n - W\}^+ - \{C - W\}^+ - \{W - XX^T/n\}^+ - \{W - \hat{C}\}^+$ we have by the triangle inequality, the fact that $(a + b)^2 \leq 2(a^2 + b^2)$ $\forall$ $a, b \in \mathbb{R}$, and the utility guarantee (squared and normalized by $n$) of IterativeEigenvectorSampling [4, Theorem 1] applied to $\{XX^T/n - W\}^+$ and $\{W - XX^T/n\}^+$ that

$$
|XX^T/n - \hat{C}|_F^2 \leq 2|XX^T/n - W|^2 + 2|W - XX^T/n|^2 + |W - \hat{C}|_F^2 \\
\leq \tilde{O}\left(\frac{d}{n} \left( \sum_{i=1}^{d} \frac{\max\{\varLambda^{(i)}\}, 0\}}{\varepsilon^{(i)}_1} + \frac{1}{\varepsilon^{(i)}_1} \right) \right) + \tilde{O}\left(\frac{d}{n} \left( \sum_{i=1}^{d} \frac{\max\{-\Lambda^{(i)}\}, 0\}}{\varepsilon^{(-i)}_1} \right) \right)
$$

$$
= \tilde{O}\left(\frac{d}{n} \left( \sum_{s \in \{\pm 1\}} \frac{1}{\varepsilon^{(s)}_0} + \frac{1}{\varepsilon^{(s)}_1} \right) \right)
$$

\[ \Box \]
Algorithm 6: IterativeEigenvectorSampling with predictions

Input: data matrix \( X \in \mathbb{R}^{d \times n} \), symmetric prediction \( W \in \mathbb{R}^{d \times d} \), privacy parameters \( \varepsilon_0^{(\pm 1)}, \ldots, \varepsilon_d^{(\pm 1)} \)

initialize \( \hat{C} \leftarrow W \)

for \( s, C \in \{ (1, \{XX^T - W\}_+), (-1, \{W - XX^T\}_+) \} \) do

// run IterativeEigenvectorSampling [4] on \( C \), add \( s \) to \( \hat{C} \)

initialize \( C_1 \leftarrow C \) and \( P_1 \leftarrow I_d \)

\( UAU^T \leftarrow C \) // get eigenvalues of \( C \)

for \( i=1, \ldots, d \) do

\( \lambda_i^{(s)} \leftarrow \Lambda_{[i]} + \text{Lap}(2/\varepsilon_0^{(s)}) \)

\( \hat{\theta}_i^{(s)} \leftarrow P_i^T \hat{u}_i^{(s)} \) for \( \hat{u}_i^{(s)} \) sampled w.p. \( \propto f_{C_i}(u) = \exp \left( \frac{\varepsilon_i^{(s)}}{4} u^T C_i u \right) \)

set \( P_{i+1} \in \mathbb{R}^{(d-i) \times d} \) to be an orthonormal basis orthogonal to \( \hat{\theta}_1^{(s)}, \ldots, \hat{\theta}_i^{(s)} \)

\( C_{i+1} \leftarrow P_{i+1} C P_{i+1}^T \in \mathbb{R}^{(d-i) \times (d-i)} \)

\( \hat{C} \leftarrow \hat{C} + s \sum_{i=1}^d \lambda_i^{(s)} \hat{\theta}_i^{(s)} \hat{\theta}_i^{(s)T} \)

Output: \( \hat{C} \)

Corollary B.1. Suppose for \( s \in \{ \pm 1 \} \) we set \( \varepsilon_0^{(s)} = \varepsilon/4 \) and \( \varepsilon_i^{(s)} = \varepsilon/4d \) \( \forall i \in [d] \), where \( \varepsilon > 0 \) is the overall privacy budget. Then w.p. \( 1 - \beta \) the output \( \hat{C} \) of Algorithm 6 satisfies

\[
\|XX^T/n - \hat{C}\|_F^2 \leq \tilde{O} \left( \frac{d}{\varepsilon n} \left( \frac{1}{\varepsilon n} + d \|XX^T/n - W\|_{Tr} \right) \right)
\]  

(66)

C Data release

Proof of Lemma 3.4. Follow the proof of the original [30] but replace Fact A.3 by \( \Psi_0 \leq D_{KL}(\frac{\varepsilon}{2} \|w\|) \), upper bound the square of the result by twice the sum of the squares of the two terms, and obtain guarantees w.p. \( \geq 1 - \beta \) by solving \( \frac{2m}{|Q|} = \beta \) for \( c \) and substituting the solution into the bound. \( \square \)
D  Online learning

D.1  Online-to-batch conversion

Theorem D.1. Suppose an online algorithm sees a sequence \( \ell(\cdot; X_1), \ldots, \ell(\cdot; X_T) : \Theta \rightarrow [0, B] \) of convex losses whose data \( X_1, \ldots, X_T \) are drawn i.i.d. from some distribution \( D \), and let \( \theta_1, \ldots, \theta_T \) be its predictions. If \( \max_{\theta \in \Theta} \sum_{t=1}^{T} \ell(\theta; X_t) - \ell(\theta; X_t) \leq R_T, \hat{\theta} = \frac{1}{T} \sum_{t=1}^{T} \theta_t, \) and \( T = \Omega \left( T_\alpha + \frac{R_T^2}{\alpha} \log \frac{1}{\beta'} \right) \) for \( T_\alpha = \min_{2R_T \leq T_\alpha} T, \) then w.p. \( \geq 1 - \beta' \)

\[
\mathbb{E}_{X \sim D} \ell(\hat{\theta}; X) \leq \min_{\theta \in \Theta} \mathbb{E}_{X \sim D} \ell(\theta; X) + \alpha
\]

Proof. This is a formalization of a standard procedure; we follow the argument in [38, Lemma A.1]. Applying Jensen’s inequality, [14, Proposition 1], the assumption that regret is \( \leq R_T \), and Hoeffding’s inequality yields

\[
\mathbb{E}_{X \sim D} \ell(\hat{\theta}; X) \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{X \sim D} \ell(\theta_t; X_t) \leq \frac{1}{T} \sum_{t=1}^{T} \ell(\theta_t; X_t) + B \sqrt{\frac{2}{T} \log \frac{2}{\beta'}}
\]

\[
\leq \min_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \ell(\theta_t; X_t) + B \sqrt{\frac{2}{T} \log \frac{2}{\beta'}}
\]

\[
\leq \min_{\theta \in \Theta} \mathbb{E}_{X \sim D} \ell(\theta; X) + \frac{R_T}{T} + 2B \sqrt{\frac{2}{T} \log \frac{2}{\beta'}}
\]

w.p. \( \geq 1 - \beta' \). Substituting the lower bound on \( T \) yields the result. \( \square \)

D.2  Negative log-inner-product losses

For functions of the form \( f_t(\mu) = -\log \int_a^b s_t(o) \mu(o) \, do \), [8] showed \( \hat{O}(T^{3/4}) \) regret for the case \( s_t(o) \in \{0, 1\} \) \( \forall o \in [a, b] \) using a variant of exponentiated gradient with a dynamic discretization. Notably their algorithm can be extended to (non-privately) learn \( -\log \Psi_t(x_t, \mu) \), since \( s_t \) in this case is one on the optimal interval and zero elsewhere. However, the changing discretization and dependence of the analysis on the range of \( s_t \) suggests it may be difficult to privatize their approach. The discretized form \( -\log \langle s_t, w \rangle \) is more heavily studied, arising in portfolio management [18]. It enjoys the exp-concavity property, leading to \( O(d \log T) \) regret using the EWOO method [31]. However, EWOO requires maintaining and sampling from a distribution defined by a product of inner products, which is inefficient and similarly difficult to privatize. Other algorithms, e.g. adaptive FTAL [31], also attain logarithmic regret for exp-concave functions, but the only private variant we know of is non-adaptive and only guarantees \( O(\sqrt{T}) \)-regret for non-strongly-convex losses [57]. The adaptivity, which is itself data-dependent, seems critical for taking advantage of exp-concavity.

Lemma D.1. If \( f_t(\mu w) = -\log \sum_{i=1}^{m} \frac{1}{s_{t,i} W[i]} \) for \( s_{t,i} \in \mathbb{R}_{\geq 0}^d \) then \( \nabla w f_t(\mu w) \|_1 \leq d/\gamma \) \( \forall \mu W \in \Delta_d^m \) s.t. \( W_{[i,j]} \geq \gamma/d \) \( \forall i, j \) for some \( \gamma \in (0, 1] \).

Proof.

\[
\|\nabla w f_t(\mu w)\|_1 = \sum_{i=1}^{m} \|\nabla W[i] f_t(\mu w)\|_1 = \left( \sum_{i=1}^{m} \frac{1}{s_{t,i} W[i]} \right)^{-1} \sum_{i=1}^{m} \sum_{j=1}^{d} \frac{s_{t,i} W[i]}{s_{t,i} W[i]}^2 
\]

\[
\leq \left( \sum_{i=1}^{m} \frac{1}{s_{t,i} W[i]} \right)^{-1} \sum_{i=1}^{m} \frac{1}{s_{t,i} W[i]} \leq d/\gamma
\]

where the first inequality follows by Sedrakyan’s inequality and the second by \( W_{[i,j]} \geq \gamma/d \). \( \square \)
D.2.1 Proof of Lemma 5.1 for $m > 1$

Proof. Let $\tilde{x}_t$ be a neighboring dataset of $x_t$ constructed by adding or removing a single element, and let $\tilde{U}_t$ be the corresponding loss function. We note that changing from $x_t$ to $\tilde{x}_t$ changes the value of $\text{Gap}_{q_t}(x_t, o)$ at any point $o \in [a, b]$ by at most $\pm 1$ and so the value of the exponential score at any point $o \in [a, b]$ is changed by at most a multiplicative factor $\exp(-\varepsilon_t/2)$ in either direction. Therefore

$$\tilde{s}_{t,i[j]} = \int_{a+b-o(j-1)}^{a+b-o(j)} \exp(-\varepsilon_t \text{Gap}_{q_t}(\tilde{x}_t, o)/2)do$$

$$\in \exp(\pm \varepsilon_t/2)\int_{a+b-o(j-1)}^{a+b-o(j)} \exp(-\varepsilon_t \text{Gap}_{q_t}(x_t, o)/2)do = \exp(\pm \varepsilon_t/2)s_{t,i[j]}$$

where $\pm$ indicates the interval between values.

$$\|\nabla_w u_t(W) - \nabla_w \tilde{u}_t(W)\|_F$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{d} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} s_{t,i}[j] - \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \tilde{s}_{t,i}[j]$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{d} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1} \left( \frac{1}{\langle s_{t,i'}, W[i'] \rangle} \right)^{-1}$$

where we have

$$\kappa_{i,j} = \frac{\langle s_{t,i}, W[i] \rangle^2}{\langle \tilde{s}_{t,i}, W[i] \rangle^2} \sum_{i'=1}^{m} \frac{s_{t,i}[j]}{\langle s_{t,i'}, W[i'] \rangle} \in \frac{\langle s_{t,i}, W[i] \rangle^2}{\langle \tilde{s}_{t,i}, W[i] \rangle^2} \exp(\pm \varepsilon_t) \frac{\sum_{i'=1}^{m} s_{t,i}[j] \exp(\pm \varepsilon_t)}{\sum_{i'=1}^{m} \langle s_{t,i'}, W[i'] \rangle \exp(\pm \varepsilon_t)} = \exp(\pm 2 \max_i \varepsilon_i)$$

Substituting into the previous inequality and taking the minimum with the $\ell_1$ bound on the gradient of the losses from Lemma D.1 yields the result. \[\square\]
D.2.2 Settings of $\gamma$ and $d$ for Corollary 5.1

1. $\lambda$-robust and discrete $\mu[i] \in \mathcal{F}_{0,d}^{(\lambda)}$: $\gamma = \lambda$

2. $\lambda$-robust and $V$-Lipschitz $\mu[i] \in \mathcal{F}_{V,1}^{(\lambda)}$: $\gamma = \lambda$ and $d = \sqrt{V(b-a)^d/\psi \sqrt{(1 + \min[1,\epsilon_m] / \epsilon')} T}$

3. discrete $\mu[i] \in \mathcal{F}_{0,d}$: $\gamma = \sqrt{md} \sqrt{4/(1+\min[1,\epsilon_m]/\epsilon')}$

4. $V$-Lipschitz $\mu[i] \in \mathcal{F}_{V,1}$: $\gamma = \sqrt{m} \sqrt{V(b-a)^d/\psi \sqrt{4/(1+\min[1,\epsilon_m]/\epsilon')}}$ and $d = \sqrt{V(b-a)^d/\psi \sqrt{(1 + \min[1,\epsilon_m] / \epsilon')} T}$

D.3 Data release

Lemma D.2. For $w \in \Delta_d$ s.t. $w[i] \geq \gamma/d \forall i \in [n]$ the gradient $\nabla_w D_{KL}(x^n||w)$ of the KL divergence w.r.t. its second argument is bounded in $\ell_\infty$-norm by $d/\gamma$ and has $\ell_2$-sensitivity $\frac{d\sqrt{2}}{\gamma n}$.

Proof. We have $\nabla_w D_{KL}(x^n||w) = -\nabla_w \langle x^n, \log w \rangle = \frac{x^n}{nw}$ so since $w[i] \geq \gamma/d$ we have that $\|\nabla_w D_{KL}(x^n||w)\|_{\infty} = \|x^n / nw\|_{\infty} \leq \frac{d_{\max, x^n}}{\gamma n} \leq d/\gamma$. Furthermore, for neighboring datasets $x$ and $\tilde{x}$ we have

$$\|\nabla_w D_{KL}(x^n||w) - \nabla_w D_{KL}(\tilde{x}^n||w)\|_2 = \frac{x^n}{nw} - \frac{\tilde{x}^n}{\tilde{n}w} \leq \frac{d\sqrt{2}}{\gamma n} \quad (73)$$

Lemma D.3. For $w \in \Delta_d$ s.t. $w[i] \geq \gamma/d \forall i \in [n]$ the gradient $\nabla_{\theta} E_{m\sim \theta} U_t(w, m)$ has $\ell_2$-sensitivity at most $7\pi \log \frac{d}{\gamma}$.

Proof. For any $x_t$ and neighboring $\tilde{x}_t$ that replaces one element we have

$$|D_{KL}(x_t / n_t || w) - D_{KL}(\tilde{x}_t / n_t || w)| = |\langle x_t - \tilde{x}_t, \log w \rangle | / n_t \leq \frac{2}{n_t} \log \frac{d}{\gamma} \quad (74)$$

Therefore

$$\|\nabla_{\theta} E_{m \sim \theta} U_t(w, m) - \nabla_{\theta} E_{m \sim \theta} \tilde{U}_t(w, m)\|_2 \leq 8n_t \sum_{m=1}^{\infty} \left( \frac{D_{KL}(x_t / n_t || w) - D_{KL}(\tilde{x}_t / n_t || w)}{m} \right)^2 \quad (75)$$

$$\leq 7\pi \log \frac{d}{\gamma} \quad \square$$
D.4 Proof of Theorem 5.4

Proof. Let \( M = \left\lfloor \sqrt{\frac{\epsilon^2 N^2 \log \frac{d}{M}}{16 \log^2 \frac{d}{M}}} \right\rfloor \) and note that the \( m \) minimizing \( \sum_{t=1}^{T} U_t(w_t, m) \) is in \([M]\) and also

\[
\max_{t,m} U_t(w_t, m) \leq 8N \log \frac{d}{\gamma} + 54 \log \frac{2}{\gamma} \left( \left( \frac{N^2}{\epsilon} \right)^{\frac{1}{2}} + 1/\epsilon^2 \right) \left( \log 2 \frac{\epsilon N \log \frac{d}{\beta}}{\gamma} + \log^4 |Q| \right)
\]

\[
= \tilde{O} \left( \frac{N^{\frac{1}{2}} \log^4 |Q|}{\min\{1, \epsilon^2\}} \frac{d}{\log \gamma} \right)
\]

(76)

Letting \( A = \mathcal{O} \left( \log \frac{d}{\gamma} \right) \) and \( B = \tilde{O} \left( \frac{N^{\frac{1}{2}} \log^4 |Q|}{\min\{1, \epsilon^2\}} \log \frac{d}{\gamma} \right) \) we have

\[
\sum_{t=1}^{T} U_t(w_t, m_t) \leq \frac{\log M}{\eta_\theta} + \eta_\theta \left( B \left( B + \left( 2 \sqrt{\log M} + \sqrt{2 \log \frac{3T}{\beta'}} \right) \sigma_\theta A \sqrt{\log_2 T} \right) \right) T
\]

\[
+ B \sqrt{\frac{T}{2} \log \frac{3}{\beta'}} + \min_{\theta \in \Theta} \sum_{t=1}^{T} U_t(w_t, \theta)
\]

\[
\leq \frac{\log M}{\eta_\theta} + \eta_\theta \left( B \left( B + \left( 2 \sqrt{\log M} + \sqrt{2 \log \frac{3T}{\beta'}} \right) \sigma_\theta A \sqrt{\log_2 T} \right) \right) T
\]

\[
+ B \sqrt{\frac{T}{2} \log \frac{3}{\beta'}} + \min_{m \in [M]} \frac{8n_t}{m} D_{KL} \left( \frac{x_t}{n_t} \parallel w_t \right) + \frac{16m^2}{\epsilon^2 n_t} \left( 3 \log \frac{2m}{\beta} + 2 \log^2 |Q| \right) T
\]

\[
\leq \frac{\log M}{\eta_\theta} + \eta_\theta \left( B \left( B + \left( 2 \sqrt{\log M} + \sqrt{2 \log \frac{3T}{\beta'}} \right) \sigma_\theta A \sqrt{\log_2 T} \right) \right) T
\]

\[
+ 8 \left( \frac{\log d}{\eta_w} + \eta_w \frac{Nd}{\gamma} \left( \frac{Nd}{\gamma} + \left( 2 \sqrt{\log d} + \sqrt{2 \log \frac{3T}{\beta'}} \right) \sigma_w d \frac{\sqrt{2}}{\gamma} \sqrt{\log_2 T} \right) \right) T
\]

\[
+ B \sqrt{\frac{T}{2} \log \frac{3}{\beta'}} + \min_{m>0, w_t \parallel \gamma/d} \sum_{t=1}^{T} U_t(w, m)
\]

\[
\leq \tilde{O} \left( \sqrt{(B + A\sigma_\theta)BT} \right) + \tilde{O} \left( \frac{d}{\gamma} \sqrt{(N + \sigma_w)NT} \right) + B \sqrt{\frac{T}{2} \log \frac{3}{\beta'}}
\]

\[
+ \tilde{O} \left( \max \{\gamma - \lambda, 0\} NT \right) + \min_{m>0, w_t \parallel \lambda/d} \sum_{t=1}^{T} U_t(w, m)
\]

\[
= \tilde{O} \left( \left( \frac{N^{\frac{1}{2}}}{\min\{1, \epsilon^2\}} + \frac{N^{\frac{1}{2}}}{\min\{1, \epsilon^2\}} + \frac{dN}{\gamma} + \frac{d}{\gamma} \sqrt{\frac{N}{\epsilon'}} \right) \sqrt{T} + \max \{\gamma - \lambda, 0\} NT \right)
\]

\[
+ \min_{m>0, w_t \parallel \lambda/d} \sum_{t=1}^{T} U_t(w, m)
\]

(77)

where the first inequality follows by the regret of DP-FTRL w.r.t. \( \theta \) together with [15, Lemma 4.1], the second by noting the definition of \( U_t \) and restricting to integer \( m \), the third by the guarantee of DP-FTRL w.r.t. \( w \), and the fourth by joint-convexity of \( D_{KL} \) and simplifying terms.\[\square\]