ON THE QUANTUM INVARIANTS FOR THE SPHERICAL SEIFERT MANIFOLDS

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ABSTRACT. We study the Witten–Reshetikhin–Turaev SU(2) invariant for the Seifert manifolds $S^3/\Gamma$ where $\Gamma$ is a finite subgroup of SU(2). We show that the WRT invariants can be written in terms of the Eichler integral of modular forms with half-integral weight, and we give an exact asymptotic expansion of the invariants by use of the nearly modular property of the Eichler integral. We further discuss that those modular forms have a direct connection with the polyhedral group by showing that the invariant polynomials of modular forms satisfy the polyhedral equations associated to $\Gamma$.

1. INTRODUCTION

Since the Witten invariant for 3-manifold was introduced [55] as the Chern–Simons path integral, studies of the quantum invariants have been much developed. The Witten invariant was later redefined mathematically rigorously by Reshetikhin and Turaev [47] by use of the surgery description of the 3-manifold and the colored Jones polynomial for links.

As was already pointed out in Witten’s original paper [55] (see also Ref. 2), it is expected that classical topological invariants for 3-manifold $\mathcal{M}$ could be extracted from asymptotic behavior of the Witten–Reshetikhin–Turaev (WRT) partition function $Z_k(\mathcal{M})$ due to that the saddle point of the Chern–Simons path integral corresponds to the flat connection. Explicitly the SU(2) WRT invariant could behave as [2, 11, 55]

$$Z_k(\mathcal{M}) \sim \frac{1}{2} e^{-\frac{3}{4}\pi i} \sum_{\alpha} \sqrt{T_{\alpha}(\mathcal{M})} e^{-\frac{2\pi i}{4} I_{\alpha}} e^{2\pi i(k+2) \text{CS}(A_{\alpha})}$$

in large $k$ limit. Here $T_{\alpha}$, $I_{\alpha}$, and $\text{CS}(A_{\alpha})$, respectively denote the Reidemeister–Ray–Singer torsion, spectral flow, and the Chern–Simons invariant. By this observation, much attention has been paid on analysis of the WRT invariants [24, 35–37, 49–52].

Recently it was clarified that the WRT invariant for the Poincaré homology sphere can be rewritten in terms of the Eichler integral of the modular form with half-integral weight [38] (see, e.g., Ref. 34 for classical definition of the Eichler integral of modular form with integral weight). As a consequence a “nearly” modular property of the Eichler integral enables us to compute

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an exact asymptotic expansion of the WRT invariant. Later it was shown [17–19] that the WRT invariants for the Seifert homology spheres can also be written in terms of the Eichler integrals of the half-integral weight modular forms. Based on this intriguing structure, topological invariants such as the Reidemeister torsion, spectral flow, the Casson invariant, and the Ohtsuki invariant, can be reinterpreted from the viewpoint of modular forms.

\[ a_1/b_1 \quad \quad -b \quad \quad a_3/b_3 \]

\[ a_2/b_2 \]

**Figure 1:** Surgery description of the Seifert manifold with three singular fibers \( M(b; (a_1, b_1), (a_2, b_2), (a_3, b_3)) \).

In this article as a continuation of Ref. 17, we study the WRT invariant for the Seifert manifold with 3 singular fibers (the Brieskorn manifold), \( \mathcal{M} = M(b; (a_1, b_1), (a_2, b_2), (a_3, b_3)) \) [40]. This 3-manifold has a surgery description as in Fig. 1, and throughout this article for our convention we depict it as

\[ a_1/b_1 \quad -b \quad a_3/b_3 \]

\[ a_2/b_2 \]

The fundamental group of the Seifert manifold \( \mathcal{M} \) is written as (see, e.g., Ref. 41).

\[ \pi_1(\mathcal{M}) = \langle x_1, x_2, x_3, h \mid h \text{ is center}, x_i^{a_i} = h^{-b_i}, x_1 x_2 x_3 = h^b \rangle \]

and it is a homology sphere iff \( a_i \) are pairwise coprime integers.

Hereafter, among the 3-fibered Seifert manifolds (1.1), we study the spherical Seifert manifolds \( S^2/\Gamma \) where \( \Gamma \) is a finite subgroup of SU(2) [40]. We define \( M(p_1, p_2, p_3) \) by

\[ M(p_1, p_2, p_3) = \text{SU}(2)/\Gamma \]

where \( \Gamma \) is a discrete subgroup classified as in Table 1. The triples \( (p_1, p_2, p_3) \) are solutions of inequality

\[ \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1 \]
with \( p_j \in \mathbb{Z}_{\geq 2} \). The order of \( \Gamma \) is given by
\[
4 \left( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 1 \right)^{-1}
\]
Only the manifold \( M(2, 3, 5) \) in Table 1 is homology 3-sphere, i.e., the Poincaré homology sphere. Note that the manifold \( M(p_1, p_2, p_3) \) is obtained by intersecting the Brieskorn surface
\[
z_1^{p_1} + z_2^{p_2} + z_3^{p_3} = 0 \tag{1.5}
\]
with unit sphere \( |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \).

\[
\begin{array}{c|c|c|c|c}
\mathcal{M} = M(p_1, p_2, p_3) & \text{Seifert invariant} & \Gamma & \text{type} & \text{order} \\
& (b; (a_1, b_1), (a_2, b_2), (a_3, b_3)) & & & \\
M(2, 2, K_{\geq 2}) & (-1; (2, 1), (2, 1), (K, 1)) & \text{binary dihedral} & D_{K+2} & 4 K \\
M(2, 3, 3) & (-1; (2, 1), (3, 1), (3, 1)) & \text{binary tetrahedral} & E_6 & 24 \\
M(2, 3, 4) & (-1; (2, 1), (3, 1), (4, 1)) & \text{binary octahedral} & E_7 & 48 \\
M(2, 3, 5) & (-1; (2, 1), (3, 1), (5, 1)) & \text{binary icosahedral} & E_8 & 120 \\
\end{array}
\tag{1.6}
\]

Table 1: The Seifert manifolds \( S^3/\Gamma \) where \( \Gamma \) is a finite subgroup of SU(2).

For the 3-manifold \( \mathcal{M} \) in Table 1, the fundamental group becomes
\[
\pi_1(\mathcal{M}) \cong \Gamma = \langle x_1, x_2, x_3 \mid x_1^{p_1} = x_2^{p_2} = x_3^{p_3} = x_1 x_2 x_3 = 1 \rangle
\]
This group is the \((p_1, p_2, p_3)\)-triangle group \( T_{p_1, p_2, p_3} \), and it corresponds to a spherical tessellation due to a condition (1.4) [40]. It is well known that the group \( \Gamma \) is the symmetry group of a Platonic solid [32]. According to Klein [32], the \( \Gamma \)-invariant polynomials on \( \mathbb{C}^2 \) are generated by three fundamental invariants, \( x, y, \) and \( z \), and they satisfy \( R(x, y, z) = 0 \) (Table 2), which basically comes from (1.5) after suitable change of variables. The hypersurface \( R(x, y, z) = 0 \) has a singularity only at the origin. This singularity is the quotient singularity of the hypersurface \( \mathbb{C}^2/\Gamma \) in \( \mathbb{C}^3 \), and resolving these simple singularities gives configuration of rational curves whose weighted dual graph coincides with the Dynkin diagram of the Lie algebra as in Table 2 (see, e.g., Ref. 53).

Our purpose is two-fold. First we show that the WRT invariant for the spherical Seifert manifolds \( S^3/\Gamma \) can be written in terms of the Eichler integrals of modular forms with half-integral weight. This result was first demonstrated by Lawrence and Zagier in the case of the Poincaré homology sphere. Based on this correspondence, we shall give an exact asymptotic expansion of the WRT invariant and study a correspondence with other topological invariants. In the second
part we show that those modular forms are related to the polyhedral group associated to \( \Gamma \), and that they construct a solution of the polyhedral equations. It suggests that the WRT invariant knows the fundamental group in some sense. This type of correspondence was conjectured in Ref. 13, and it was checked for a case of lens space [56].

This article is constructed as follows. In Section 2, we present properties of the modular form. We define the modular form with weight \( \frac{3}{2} \), and give a nearly modular property of the Eichler integral thereof. In Section 3 we give an explicit form of the WRT invariant for the Seifert manifolds following Ref. 37. In Section 4, we show that the WRT invariant for the spherical Seifert manifolds \( SU(2)/\Gamma \) in Table 1 can be written in terms of the Eichler integrals of the modular form with half-integral weight. We shall also give an exact asymptotic expansion in \( N \to \infty \), and discuss the classical topological invariants that appear in this limit. In Section 5, we study the congruence subgroup, and we shall reveal that the modular form is related to the polyhedral group. The last section is devoted to concluding remarks and discussions.

### 2. Preliminaries

Throughout this article, we set

\[
q = \exp \left( 2 \pi i \tau \right)
\]

where \( \tau \) is in the upper half plane, \( \tau \in \mathbb{H} \). We use the Dedekind \( \eta \)-function,

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)
\]

which is a modular form with weight 1/2 satisfying

\[
\eta(-1/\tau) = \sqrt{\frac{\tau}{i}} \eta(\tau)
\]

\[
\eta(\tau + 1) = e^{\pi i} \eta(\tau)
\]
Another important family of the modular forms is the (normalized) Eisenstein series (see, e.g., Ref. 33)

\[ E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m\tau + n)^k} \quad (2.3) \]

Here \( k \) is even integer greater than 2, and the Riemann \( \zeta \)-function is

\[ \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \]

Note that the \( \zeta \)-function for even \( k \) is given by

\[ \zeta(k) = \frac{(2\pi i)^k}{2k!} B_k \quad \text{for even } k \geq 2 \]

where \( B_k \) is the \( k \)-th Bernoulli number,

\[ \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \]

The Eisenstein series is a modular form with weight \( k \),

\[ E_k(-1/\tau) = \tau^k E_k(\tau) \quad (2.4) \]

and it has a Fourier expansion

\[ E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (2.5) \]

Here the arithmetic function \( \sigma_k(n) \) is a sum of the \( k \)-th powers of the positive divisors of \( n \),

\[ \sigma_k(n) = \sum_{d|n} d^k \]

and the Fourier expansion can be rewritten in the form of the Lambert series

\[ E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n} \]

The cusp form with weight 12 is the Ramanujan \( \Delta \)-function

\[ \Delta(\tau) = (\eta(\tau))^{24} \quad (2.6) \]

and it is given from the Eisenstein series as

\[ (E_4(\tau))^3 - (E_6(\tau))^2 = 1728 \Delta(\tau) \quad (2.7) \]
Besides the Dedekind $\eta$-function (2.1), we make use of another family of the modular form with half-integral weight [16]. For $P \in \mathbb{Z}_{>0}$ and $a \in \mathbb{Z}$ we set

$$\Psi^{(a)}_P (\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \psi^{(a)}_{2P} (n) q^{\frac{n^2}{4P}}$$

(2.8)

where $\psi^{(a)}_{2P} (n)$ is an odd periodic function with modulus $2P$;

$$\psi^{(a)}_{2P} (n) = \begin{cases} 
\pm 1 & \text{for } n \equiv \pm a \mod 2P \\
0 & \text{otherwise}
\end{cases}$$

(2.9)

These $q$-series are related to the characters of the affine Lie algebra $\widehat{su}(2)$ (see, e.g., Refs. 9, 25). We see that this family of $q$-series is a vector modular form with weight $3/2$; under the modular $S$- and $T$-transformations,

$$S : \tau \rightarrow -\frac{1}{\tau}$$

$$T : \tau \rightarrow \tau + 1$$

satisfying

$$S^2 = (ST)^3 = 1$$

it transforms as

$$\Psi^{(a)}_P (\tau) = \left( \frac{i}{\tau} \right)^{\frac{P-1}{2}} \sum_{b=1}^{P-1} M(P)_b^a \Psi^{(b)}_P (-1/\tau)$$

(2.10)

$$\Psi^{(a)}_P (\tau + 1) = e^{\frac{a^2}{2} \pi i} \Psi^{(a)}_P (\tau)$$

Here $M(P)$ is a $(P-1) \times (P-1)$ matrix whose elements are

$$M(P)_b^a = \sqrt{\frac{2}{P}} \sin \left( \frac{ab}{P} \pi \right)$$

(2.11)

Following Ref. 38, we define the Eichler integral of this family of the modular forms with half-integral weight by (see also Ref. 16)

$$\tilde{\Psi}^{(a)}_P (\tau) = \sum_{n=0}^{\infty} \psi^{(a)}_{2P} (n) q^{\frac{n^2}{4P}}$$

(2.12)

This can be regarded as a half-integration of $\Psi^{(a)}_P (\tau)$ with respect to $\tau$. A limiting values of the Eichler integral in $\tau \rightarrow \frac{M}{N} \in \mathbb{Q}$ can be computed by use of the Mellin transformation, and we
have [16]

\[ \tilde{\Psi}_P^{(a)}(1/N) = -\sum_{k=0}^{2PN} \psi_{2P}^{(a)}(k) e^{\frac{a^2}{2P} \pi i} B_1 \left( \frac{k}{2PN} \right) \tag{2.13} \]

\[ \tilde{\Psi}_P^{(a)}(N) = \left( 1 - \frac{a}{P} \right) e^{\frac{a^2}{2P} \pi i N} \tag{2.14} \]

where \( N \in \mathbb{Z} \), and \( B_k(x) \) denotes the \( k \)-th Bernoulli polynomial defined by

\[ \frac{t e^t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k \]

From the topological viewpoint, a limiting value (2.13) in \( \tau \to 1/N \) is related to the specific value of the \( N \)-colored Jones polynomial for torus links \( T_{2,2P} \) with \( P > 0 \) [16]. Explicitly we have

\[ \langle T_{2,2P} \rangle_N = PN e^{-\frac{(P-1)^2}{2PN} \pi i} \tilde{\Psi}_P^{(P-1)}(1/N) \tag{2.15} \]

Here \( \langle K \rangle_N \) is Kashaev’s invariant for a knot \( K \) [27], and it coincides with a specific value of the \( N \)-colored Jones polynomial \( J_N(q; K) \) as [43]

\[ \langle K \rangle_N = J_N \left( \frac{2 \pi i}{N}; K \right) \tag{2.16} \]

where we have normalized the quantum invariant s.t.

\[ \langle \text{unknot} \rangle_N = 1 \]

Topological meaning of other Eichler integrals \( \tilde{\Psi}_P^{(a)}(1/N) \) with \( a \neq P - 1 \) is not clear, and we show hereafter that some of them are related to the WRT invariant for the spherical Seifert manifolds \( M(p_1, p_2, p_3) \).

A crucial property of the Eichler integral (2.12) is that it is nearly modular [38, 58]. For \( N \in \mathbb{Z}_{>0} \), we have an exact asymptotic expansion in \( N \to \infty \) as

\[ \tilde{\Psi}_P^{(a)}(1/N) + \sqrt{\frac{N}{i}} \sum_{b=1}^{P-1} M(P)_b \tilde{\Psi}_P^{(b)}(-N) \sim \sum_{k=0}^{\infty} \frac{L(-2k; \psi_{2P}^{(a)})}{k!} \left( \frac{\pi i}{2PN} \right)^k \tag{2.17} \]

where \( \tilde{\Psi}_P^{(a)}(1/N) \) and \( \tilde{\Psi}_P^{(b)}(N) \) are given in (2.13) and (2.14). The Dirichlet \( L \)-function \( L\left( s, \psi_{2P}^{(a)} \right) \) at negative integers \( s = -k \) is given by

\[ L\left( -k, \psi_{2P}^{(a)} \right) = \frac{(2P)^k}{k+1} \sum_{n=1}^{2P} \psi_{2P}^{(a)}(n) B_{k+1} \left( \frac{n}{2P} \right) \tag{2.18} \]
See Refs. 16, 38, 58 for proof. We should remark that the generating function of the $L$-functions at negative integers, $L\left(-2k, \psi_{2p}^{(a)}\right)$ for $0 < a < P$, is given by

$$
\frac{\text{sh}((P - a) z)}{\text{sh}(P z)} = \sum_{k=0}^{\infty} \frac{L\left(-2k, \psi_{2p}^{(a)}\right)}{(2k)!} z^{2k}
$$

(2.19)

To close this section, we recall the Gauss sum reciprocity formula [5, 24],

$$
\sum_{n \mod N} e^{\frac{2\pi i}{N} n^2 + 2\pi i k n} = \sqrt{\frac{N}{M}} e^{\frac{\pi i}{4} \text{sign}(M)} \sum_{n \mod M} e^{-\frac{\pi i}{4} N(n+k)^2}
$$

(2.20)

where $N, M \in \mathbb{Z}$ with $Nk \in \mathbb{Z}$ and $NM$ being even. This can be derived based on the transformation law of the theta series.

### 3. WITTEN–RESHETIKHIN–TURAEV INVARIANT

The explicit form of the WRT invariant for the Seifert manifolds is given by the method of Reshetikhin and Turaev [47]. Based on a surgery description of 3-manifold $M$, we can compute the SU(2) WRT invariant using the colored Jones polynomial for link. The SU(2) WRT invariant for the Seifert manifolds has been extensively studied (see, e.g., Refs. 35–37, 49–52), and we note the known result as follows;

**Proposition 1** ([37]). Let $M$ be the Seifert manifold $M(0; (p_1, q_1), (p_2, q_2), (p_3, q_3))$. Then we have

$$
e^{\frac{2\pi i}{N} \left(\frac{\phi}{2} - \frac{1}{2}\right)} \left(e^{\frac{2\pi i}{N}} - 1\right) \cdot \tau_N(M)
$$

$$=
\frac{e^{\frac{\pi i}{2}}}{\sqrt{2N p_1 p_2 p_3}} \sum_{k_0=1}^{N-1} \sum_{n_j \mod p_j} \frac{1}{e^{\frac{\pi i}{N} k_0} - e^{-\frac{\pi i}{N} k_0}}
$$

$$\times \prod_{j=1}^{3} e^{-\frac{2\pi i}{N} q_j (k_0 + 2Nn_j)^2} \left(e^{\frac{2\pi i}{N} p_j} (k_0 + 2Nn_j) - e^{-\frac{2\pi i}{N} p_j} (k_0 + 2Nn_j)\right)
$$

(3.1)

Here we have set

$$
\phi = \sum_{j=1}^{3} \left(12 s(q_j, p_j) - \frac{q_j}{p_j}\right) + 3
$$

(3.2)

where $s(b, a)$ is the Dedekind sum (see, e.g., Ref. 46)

$$
s(b, a) = \text{sign}(a) \sum_{k=1}^{\lfloor a \rfloor - 1} \left(\left\langle \frac{k}{a}\right\rangle \cdot \left\langle \frac{kb}{a}\right\rangle\right)
$$

(3.3)
with
\[(x) = \begin{cases} \frac{x - \lfloor x \rfloor}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}\]
and \(\lfloor x \rfloor\) is the greatest integer not exceeding \(x\).

It is known that the SU(2) WRT invariant for a 3-manifold \(M\) can be factorized [28] as
\[
\tau_N(M) = \begin{cases} \tau_3(M) \tau_N^{SO(3)}(M) & \text{for } N = 3 \mod 4 \\ \frac{\tau_3(M)}{\tau_3(M)} \tau_N^{SO(3)}(M) & \text{for } N = 1 \mod 4 \end{cases}
\] (3.4)
where \(\tau_N^{SO(3)}(M)\) is the SO(3) WRT invariant, and
\[
\tau_3(M) = (1 + i)^{\sigma^+} (1 - i)^{\sigma^-} \sum_{x \in (\mathbb{Z}/2\mathbb{Z})^4} i^{e(xL)}
\] (3.5)
where \(L\) is the linking matrix of a link which presents a surgery description of \(M\), and \(\sigma_{\pm}\) denote the number of positive/negative eigenvalues of the linking matrix \(L\). Studied in detail [28] is the condition for the manifold \(M\) that \(\tau_3(M) = 0\).

It is remarked that the Dedekind sum can also be written as [46]
\[
s(b, a) = \frac{1}{4|a|} \sum_{k=1}^{|a|-1} \cot \left( \frac{k}{a} \pi \right) \cot \left( \frac{k b}{a} \pi \right)
\]
and that it satisfies
\[
s(-b, a) = -s(b, a) \quad s(b, a) = s(b', a) \quad \text{for } b b' \equiv 1 \pmod{a}
\]
It is well known that the Dedekind sum is related to the Casson–Walker invariant, which naively denotes the number of the irreducible SU(2) representation of the fundamental group \(\pi_1(M)\). Explicitly the Casson–Walker invariant \(\lambda_{CW}(M)\) for \(M = M(b; (a_1, b_1), (a_2, b_2), (a_3, b_3))\) is given by [8, 12]
\[
\lambda_{CW}(M) = \frac{a_1 a_2 a_3}{8} \left( \frac{\text{sign}(e(M))}{3} \left( -1 + \sum_{j=1}^{3} \frac{1}{a_j^2} \right) + \frac{e(M) |e(M)|}{3} - e(M) - 4 |e(M)| \sum_{j=1}^{3} s(b_j, a_j) \right)
\] (3.6)
where \( e(M) \) is the Euler characteristic

\[
e(M) = b + \sum_{j=1}^{3} \frac{b_j}{a_j}
\]

Interest in asymptotic behavior of the WRT invariant is motivated by Witten’s original results that the saddle point of the Chern–Simons path integral in the large \( N \) limit is given by contribution coming solely from the flat connections, and that (classical) topological invariants should appear in this limit. The asymptotic behavior of the WRT invariant for 3-manifold \( M \) is expected to be [11, 49]

\[
Z_k(M) \sim \frac{1}{2} e^{-\frac{3}{2} \pi i} \sum_\alpha \sqrt{T_\alpha(M)} e^{-\frac{2\pi i}{3} I_\alpha} e^{2\pi i (k+2) \text{CS}(A_\alpha)}
\] (3.7)

Here \( Z_k(M) \) is the partition function due to Witten’s normalization,

\[
Z_k(M) = \frac{\tau_{k+2}(M)}{\tau_{k+2}(S^2 \times S^1)}
\] (3.8)

where

\[
\tau_N(S^2 \times S^1) = \sqrt{\frac{N}{2}} \frac{1}{\sin(\pi/N)}
\]

The index \( \alpha \) ranges over all gauge equivalence classes of flat connections. The Reidemeister torsion, spectral flow, and the Chern–Simons invariant are respectively denoted by \( T_\alpha, I_\alpha, \) and \( \text{CS}(A) \).

In the case of the Seifert manifold \( M(p_1, p_2, p_3) \), the explicit values of the torsion and the Chern–Simons invariant are known. The Reidemeister torsion is given by [10]

\[
\sqrt{T_\alpha} = \prod_{j=1}^{3} \frac{2}{\sqrt{p_j}} \left| \sin \left( \frac{q_j' \ell_j}{p_j \pi} \right) \right|
\] (3.9)

where \( q_j q_j' \equiv 1 \mod p_j \). An integer \( \ell_j \) satisfying \( 0 < \ell_j < p_j \) parametrizes the irreducible SU(2) representation \( \rho \) of the fundamental group (1.6), and we have

\[
\rho(x_j) \sim \left( \begin{array}{c}
\begin{array}{c}
\ell_j e^{\pi i}

\ell_j e^{-\pi i}
\end{array}
\end{array}
\right)
\]

up to conjugation. Corresponding to this representation \( \rho \), the Chern–Simons invariant is given by [3, 8, 30, 44]

\[
\text{CS}(A_\alpha) = -\frac{1}{4} \sum_{j=1}^{3} \frac{q_j}{p_j} \ell_j^2 \mod 1
\] (3.10)
4. WRT INVARIANT FOR THE SPHERICAL SEIFERT MANIFOLDS

We shall clarify the relationship between the WRT invariant for the spherical Seifert manifolds $S^3/\Gamma$ in Table 1 and the Eichler integral $\tilde{\Psi}_p^{(a)}(1/N)$ of modular forms with weight 3/2. The expression (3.1) can be simplified into that with a unique sum for a case of the homology sphere [37], but in our case it is necessary to treat each case one by one.

4.1 $M(2, 3, 3)$

Let $\mathcal{E}_6$ be $M(2, 3, 3)$. The Euler characteristic is given by

$$e(\mathcal{E}_6) = \frac{1}{6}$$

The surgery description given in (1.1) can be transformed into the following form by the Kirby move:

$$-2 \quad -2 \quad -2 \quad -2 \quad -2$$

This is nothing but the Dynkin diagram for the Lie algebra $E_6$.

**Proposition 2.** The $SU(2)$ WRT invariant for $\mathcal{E}_6$ is written as a sum of the Eichler integrals $\tilde{\Psi}_p^{(a)}(1/N)$ as

$$e^{\frac{1}{12}\pi i} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N (\mathcal{E}_6)$$

$$= \frac{1 + 2 e^{\frac{2\pi i N}{3}}}{\sqrt{3}} e^{\frac{\pi i}{12 N}} - \frac{1 + 2 e^{\frac{2\pi i N}{3}}}{2\sqrt{3}} \left( \tilde{\Psi}_6^{(1)}(1/N) + \tilde{\Psi}_6^{(5)}(1/N) \right) - \frac{1 - e^{\frac{2\pi i N}{3}}}{\sqrt{3}} \tilde{\Psi}_6^{(3)}(1/N) \quad (4.1)$$

**Proof.** In the case of the manifold $\mathcal{E}_6$ we have $\vec{p} = (2, 3, 3)$ and $\vec{q} = (-1, 1, 1)$ in (3.1). To rewrite this expression in terms of the Eichler integrals, we note that the summand of the right hand side of (3.1) is invariant [37] under

- $k_0 \to k_0 + 2N$ and $\forall n_j \to n_j - 1$
- $n_j \to n_j + p_j$
With above symmetries, the sum \( \sum_{k_0=1}^{N-1} \sum_{n_j \mod p_j} \) can be replaced with a sum \( \sum_{k_0=a+2Nn}^{1 \leq a \leq N-1} \sum_{0 \leq n \leq 5} \), and \( n_1 = n_2 = 0 \). After taking a sum over \( n_3 \) explicitly, we get

\[
\text{l.h.s. of (4.1)} = \frac{e^{\frac{\pi i}{4}}}{6 \sqrt{N}} \sum_{k_0=a+2Nn}^{1 \leq a \leq N-1} \sum_{0 \leq n \leq 5} \frac{\left( e^{\frac{\pi i}{2N} k_0} - e^{-\frac{\pi i}{2N} k_0} \right) \left( e^{\frac{\pi i}{2N} N k_0} - e^{-\frac{\pi i}{3N} k_0} \right)}{e^{\frac{\pi i}{N} k_0} - e^{-\frac{\pi i}{5} k_0}} e^{-\frac{\pi i}{12N} k_0^2} \\
\times \left( e^{\frac{\pi i}{N} k_0} \left( 1 + e^{\frac{\pi i}{2} (1-N-k_0)} + e^{\frac{\pi i}{2} (1-N+k_0)} \right) \right)
\]

As the summand of the above expression is invariant under \( k_0 \rightarrow 12N - k_0 \), we obtain

\[
\text{l.h.s. of (4.1)} = \frac{e^{\frac{\pi i}{4}}}{12 \sqrt{N}} \sum_{k_0=0}^{12N-1} \sum_{N/k_0}^{1 \leq a \leq N-1} \frac{\left( e^{\frac{\pi i}{2N} k_0} - e^{-\frac{\pi i}{2N} k_0} \right) \left( e^{\frac{\pi i}{2N} N k_0} - e^{-\frac{\pi i}{3N} k_0} \right)}{e^{\frac{\pi i}{N} k_0} - e^{-\frac{\pi i}{5} k_0}} e^{-\frac{\pi i}{12N} k_0^2} \\
\times \left( e^{\frac{\pi i}{N} k_0} \left( 1 + e^{-\frac{\pi i}{3N} k_0} (3 \delta_3(k_0-1) - 1) \right) \right) - e^{-\frac{\pi i}{3N} k_0} \left( 1 + e^{-\frac{\pi i}{3N} (3 \delta_3(k_0+1) - 1) \right)}
\]

where we have used

\[
1 + e^{\frac{2\pi i}{3}} + e^{-\frac{2\pi i}{3}} = 3 \delta_3[n]
\]

We then introduce the even periodic function \( \chi_{12}(n) \) with modulus 12 by

\[
\chi_{12}(n) = \begin{array}{c|cccccc}
\mod\,12 & 1 & 5 & 7 & 11 & \text{others} \\
\hline
n \chi_{12}(n) & 1 & -1 & -1 & 1 & 0
\end{array}
\]

whose generating function is

\[
\frac{(t^2 - t^{-2}) (t^3 - t^{-3})}{t^6 - t^{-6}} = - \sum_{n=0}^{\infty} \chi_{12}(n) t^n
\]  

(4.2)

Using this, we have

\[
\text{l.h.s. of (4.1)} = -\frac{e^{\frac{\pi i}{4}}}{12 \sqrt{N}} \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \chi_{12}(n) e^{-nt} \sum_{k_0=0}^{12N-1} \sum_{N/k_0}^{1 \leq a \leq N-1} \frac{\left( e^{\frac{\pi i}{2N} k_0} - e^{-\frac{\pi i}{2N} k_0} \right) \left( e^{\frac{\pi i}{2N} N k_0} - e^{-\frac{\pi i}{3N} k_0} \right)}{e^{\frac{\pi i}{N} k_0} - e^{-\frac{\pi i}{5} k_0}} e^{-\frac{\pi i}{12N} k_0^2} \\
\times \left( e^{\frac{\pi i}{N} k_0} \left( 1 + e^{-\frac{\pi i}{3N} k_0} (3 \delta_3(k_0-1) - 1) \right) \right) - e^{-\frac{\pi i}{3N} k_0} \left( 1 + e^{-\frac{\pi i}{3N} (3 \delta_3(k_0+1) - 1) \right)}
\]

where we have used a fact that the sum for \( N \mid k_0 \) is zero. We apply the Gauss sum reciprocity formula (2.20), and then use an identity

\[
\sum_{k \mod 3} e^{\frac{\pi i}{3N} k (k+\frac{1}{N})^2} = e^{\frac{\pi i}{12N}} \left( 1 + e^{-\frac{\pi i}{3N} (3 \delta_3[k] - 1) \right)}
\]
for $x \in \mathbb{Z}$. After some computations, we find

l.h.s. of (4.1)
\[
= -\frac{1}{\sqrt{12}} \lim_{t \to 0^+} \sum_{n=0}^{\infty} e^{-nt} \chi_{12}(n) \left( e^{\frac{(n+1)^2}{12N}\pi i} \left( \left( 1 + 2 e^{-\frac{4}{3}\pi iN} \right) \delta_{3(n+1)} + \left( 1 - e^{-\frac{4}{3}\pi iN} \right) \delta_{3(n+2)} \right) \right) \\
- e^{\frac{(n-2)^2}{12N}\pi i} \left( \left( 1 - e^{-\frac{4}{3}\pi iN} \right) \delta_{3(n+1)} + \left( 1 + 2 e^{-\frac{4}{3}\pi iN} \right) \delta_{3(n+2)} \right)
\]
\[
= \frac{1}{\sqrt{12}} \sum_{k=0}^{12N-1} B_1 \left( \frac{k}{12N} \right) \left( e^{\frac{(k+2)^2}{12N}\pi i} \chi_{12}^{(-)}(k) \left( 1 + 2 e^{-\frac{4}{3}\pi iN} \right) + e^{\frac{(k+2)^2}{12N}\pi i} \chi_{12}^{(+)}(k) \left( 1 - e^{-\frac{4}{3}\pi iN} \right) \right) \\
- e^{\frac{(k-2)^2}{12N}\pi i} \chi_{12}^{(-)}(k) \left( 1 - e^{-\frac{4}{3}\pi iN} \right) - e^{\frac{(k-2)^2}{12N}\pi i} \chi_{12}^{(+)}(k) \left( 1 + 2 e^{-\frac{4}{3}\pi iN} \right)
\]

Here we have defined the periodic functions $\chi_{12}^{(\pm)}(k)$ by

| $n \mod 12$ | 1 | 7 | others |
|------------|---|---|-------|
| $\chi_{12}^{(+)}(n)$ | $1$ | $-1$ | $0$ |

| $n \mod 12$ | 5 | 11 | others |
|------------|---|---|-------|
| $\chi_{12}^{(-)}(n)$ | $-1$ | $1$ | $0$ |

which satisfy $\chi_{12}(k) = \chi_{12}^{(+)}(k) + \chi_{12}^{(-)}(k)$. Finally we shift a sum $k \to k \pm 2$, and use a relationship between the periodic functions $\chi_{12}^{(\pm)}(n)$ and $\psi_{12}^{(\pm)}(n)$, such as $\chi_{12}^{(\pm)}(n+2) = \pm \psi_{12}^{(3)}(n)$, $\chi_{12}^{(+)}(n) = \chi_{12}^{(-)}(n-2)$, and $\chi_{12}^{(+)}(n) - \chi_{12}^{(-)}(n) = \psi_{12}^{(1)}(n) + \psi_{12}^{(5)}(n)$. Reforming a sum and using an expression (2.13), we get the assertion of the proposition. \hfill \Box

**Corollary 3.** Exact asymptotic expansion of the WRT invariant in $N \to \infty$ is given by

\[
e^{\frac{1}{12N}\pi i} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N(\mathcal{E}_6) \simeq \sqrt{\frac{N}{4}} e^{-\frac{\pi i}{12N}N} + \frac{1 + 2 e^{\frac{2\pi i}{3N}N}}{\sqrt{3}} e^{\frac{\pi i}{12N}} \\
- \sum_{k=0}^{\infty} \frac{1}{k!} \left( 1 + 2 e^{\frac{2\pi i}{3N}N} \right) L \left( -2k, \psi_{12}^{(1)} + \psi_{12}^{(5)} \right) + \frac{1 - e^{\frac{2\pi i}{3N}N}}{\sqrt{3}} L \left( -2k, \psi_{12}^{(3)} \right) \left( \frac{\pi i}{12N} \right)^k
\] (4.3)
where the generating functions of the $L$-functions are given by

\[
\frac{\text{ch}(2z)}{\text{ch}(3z)} = \sum_{k=0}^{\infty} \frac{L(-2k, \psi^{(1)}_{12} + \psi^{(5)}_{12})}{(2k)!} z^{2k}
\]

\[
\frac{1}{2\ ch(3z)} = \sum_{k=0}^{\infty} \frac{L(-2k, \psi^{(3)}_{12})}{(2k)!} z^{2k}
\]

We thus have a dominating term of the Witten partition function in $N \to \infty$ as

\[
Z_{N-2}(E_6) \sim \frac{1}{2} e^{-\frac{3}{4} \pi i} \cdot \sqrt{2} e^{-\frac{1}{24} \pi i N}
\]

**Proof.** We apply (2.17), and we obtain (4.3) immediately. As a result, the dominating terms of the partition function $Z_k(E_6)$, which is defined in (3.8), can be given.

The torsion and the Chern–Simons invariant are respectively computed from (3.9) and (3.10) by setting surgery data, $\vec{p} = (2, 3, 3)$ and $\vec{q} = (-1, 1, 1)$, for $E_6$. By choosing $\vec{\ell} = (1, 1, 1)$, we have

\[
\sqrt{T_\alpha} = \sqrt{2}
\]

\[
\text{CS}(A_\alpha) = -\frac{1}{24} \mod 1
\]

This result with an asymptotic behavior (4.4) coincides with an ansatz (3.7).

### 4.2 $M(2, 3, 4)$

Let $E_7$ be $M(2, 3, 4)$, and we have

\[
e(E_7) = \frac{1}{12}
\]

The linking matrix (1.1) can be transformed by the Kirby move to a form of the Coxeter–Dynkin diagram for an exceptional Lie algebra $E_7$;
Proposition 4. The WRT invariant for \( \mathcal{E}_7 \) is written in terms of a limiting value of the Eichler integrals \( \tilde{\Psi}_{12}^{(0)}(1/N) \) as
\[
e^{\frac{\pi i}{4N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N(\mathcal{E}_7)
= \frac{\sqrt{2}}{4} \left( 1 + (-1)^N \right) \left( 2 e^{\frac{ni}{2N}} - \tilde{\Psi}_{12}^{(1)}(1/N) - \tilde{\Psi}_{12}^{(5)}(1/N) - \tilde{\Psi}_{12}^{(7)}(1/N) - \tilde{\Psi}_{12}^{(11)}(1/N) \right)
\tag{4.6}
\]

Proof. The method is essentially same with Prop. 2, although we have \( \tilde{\rho} = (2, 3, 4) \) and \( \tilde{q} = (-1, 1, 1) \) in this case. Using symmetries of the summand of (3.1), the sum \( \sum_{k_0=1}^{N-1} \sum_{n_j \mod p_j} \) with \( \tilde{\rho} = (2, 3, 4) \) can be replaced with a sum \( \sum_{k_0=0}^{N-1} \sum_{n_3=0}^{3} \) and \( n_1 = n_2 = 0 \). Taking a sum of \( n_3 \) and using a symmetry of the summand, we obtain
\[
\text{l.h.s. of (4.6)} = \frac{e^{\frac{\pi i}{4N}}}{4 \sqrt{3} N} \sum_{k_0=0}^{12N} \left( \frac{\tilde{\rho}_{2N}^k - e^{-\frac{\pi i}{2N} k_0}}{e^{\frac{\pi i}{2N} k_0} - e^{-\frac{\pi i}{2N} k_0}} \right) e^{-\frac{\pi i}{2N} k_0^2} \\
\times \left( e^{\frac{\pi i}{4N} k_0} \left( 1 - \delta_2 k_0 + e^{-\frac{\pi i}{2N}} (-1)^{k_0+1} \delta_2(k_0-1) \right) \\
- e^{-\frac{\pi i}{4N} k_0} \left( 1 - \delta_2 k_0 + e^{-\frac{\pi i}{2N}} (-1)^{k_0+1} \delta_2(k_0+1) \right) \right)
\]
We introduce an infinitesimal variable \( t \) in the fraction, and apply (4.2). We then get
\[
\text{l.h.s. of (4.6)} = \frac{e^{\frac{\pi i}{4N}}}{4 \sqrt{3} N} \lim_{t \to 0} \sum_{n=0}^{\infty} \chi_{12}(n) e^{-nt} \sum_{k=0}^{6N} e^{\frac{\pi i}{2N}(2k+1)n-\frac{\pi i}{2N}(2k+1)^2} \\
\times \left( e^{\frac{\pi i}{4N}(2k+1)} \left( 1 + e^{-\frac{\pi i}{2N}} (-1)^k \right) - e^{-\frac{\pi i}{4N}(2k+1)} \left( 1 + e^{-\frac{\pi i}{2N}} (-1)^{k+1} \right) \right)
\]
In this computation, we need to subtract a sum over \( N \mid k_0 \), but it is proved to vanish. We apply the Gauss reciprocity formula (2.20), and then obtain
\[
\text{l.h.s. of (4.6)} = \frac{\sqrt{2}}{4} \left( 1 + (-1)^N \right) e^{\frac{\pi i}{2N}} \lim_{t \to 0} \sum_{n=0}^{\infty} e^{-nt} \chi_{12}(n) e^{\frac{\pi i}{2N} n^2} \left( e^{\frac{\pi i}{2N} n} - e^{-\frac{\pi i}{2N} n} \right) \\
= \frac{\sqrt{2}}{4} \left( 1 + (-1)^N \right) \sum_{k=1}^{12N} \chi_{12}(k) B_1 \left( \frac{k}{12N} \right) \left( e^{\frac{\pi i}{12N}(k+\frac{1}{2})^2} - e^{\frac{\pi i}{12N}(k-\frac{1}{2})^2} \right)
\]
Finally we replace a sum of \( k \) by \( n = 2k \pm 3 \). After some algebra, we obtain (4.6). \( \square \)

This result proves that the SU(2) WRT invariant \( \tau_N(\mathcal{E}_7) \) vanishes when \( N \) is odd. Due to the factorization property (3.4), this indicates that
\[
\tau_3(\mathcal{E}_7) = 0 \tag{4.7}
\]
which can be directly checked from (3.5) using the $E_7$ Dynkin matrix as the linking matrix.

Asymptotic expansion of $\tau_N(E_7)$ in $N \to \infty$ directly follows from (4.6) with a help of (2.17).

**Corollary 5.** Exact asymptotic expansion of the WRT invariant for $E_7$ in $N \to \infty$ is

$$e^{\frac{37\pi i}{24N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N(E_7) \simeq 1 + \frac{(-1)^N}{\sqrt{2}} \sqrt{\frac{N}{i}} e^{-\frac{1}{24} \pi i N} + \frac{1 + (-1)^N}{\sqrt{2}} e^{\frac{i}{24N}}$$

$$- \frac{\sqrt{2}}{4} \left( 1 + (-1)^N \right) \sum_{k=0}^{\infty} \frac{L \left( -2k, \psi_2^{(1)} + \psi_4^{(5)} + \psi_4^{(7)} + \psi_4^{(11)} \right)}{k!} \left( \frac{\pi i}{24N} \right)^k$$

(4.8)

Here the $L$-function is given by

$$2 \frac{\text{ch}(3z) \text{ch}(2z)}{\text{ch}(6z)} = \sum_{k=0}^{\infty} \frac{L \left( -2k, \psi_2^{(1)} + \psi_4^{(5)} + \psi_4^{(7)} + \psi_4^{(11)} \right)}{(2k)!} z^{2k}$$

Then an asymptotic behavior of the partition function $Z_{N-2}(E_7)$ in $N \to \infty$ is

$$Z_{N-2}(E_7) \sim \frac{1}{2} e^{-\frac{1}{48} \pi i} \left( e^{-\frac{1}{24} \pi i N} + e^{\frac{25}{24} \pi i N} \right)$$

(4.9)

By setting $\vec{p} = (2, 3, 4)$ and $\vec{q} = (-1, 1, 1)$ in (3.9) and (3.10), we obtain the torsion and the Chern–Simons invariant as follows:

$$\begin{array}{c|cc}
\vec{\ell} & \sqrt{T_\alpha} & \text{CS}(A_\alpha) \\
(1, 1, 1) & 1 & -\frac{1}{48} \\
(1, 1, 3) & 1 & -\frac{25}{48} \\
\end{array}$$

(4.10)

Substituting this result for (3.7), we recover (4.9).

### 4.3 Poincaré Homology Sphere $M(2, 3, 5)$

Let $E_8$ be $M(2, 3, 5)$, i.e., the Poincaré homology sphere, which has a following $E_8$ Coxeter–Dynkin diagram as a surgery description:

```
-2 -2 -2 -2 -2 -2
```

The Euler characteristic is

$$e(E_8) = \frac{1}{30}$$
As was demonstrated by Lawrence and Zagier, the WRT invariant for \( \mathcal{E}_8 \) can be written in the following form:

**Proposition 6 (\[38\], also Ref. 17).** The WRT invariant for the Poincaré homology sphere is written as

\[
e^{\frac{12\pi i}{60N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N(\mathcal{E}_8)
= e^{\frac{\pi i}{60N}} - \frac{1}{2} \left( \Psi_{30}^{(1)}(1/N) + \Psi_{30}^{(11)}(1/N) + \Psi_{30}^{(19)}(1/N) + \Psi_{30}^{(29)}(1/N) \right)
\]

(4.11)

**Proof.** We omit the proof. See Refs. 17, 38.

\(\square\)

Applying (2.17), we obtain the asymptotic expansion of the WRT invariant for the Poincaré homology sphere in \( N \to \infty \).

**Corollary 7 (\[38\]).** Exact asymptotic expansion of the WRT invariant for the Poincaré homology sphere in \( N \to \infty \) is

\[
e^{\frac{12\pi i}{60N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N(\mathcal{E}_8)
\approx \sqrt{\frac{N}{\pi}} \frac{2}{\sqrt{5}} \left( \sin \left( \frac{\pi}{5} \right) e^{-\frac{1}{60\pi i N}} + \sin \left( \frac{2\pi}{5} \right) e^{-\frac{49}{60\pi i N}} \right) + e^{\frac{\pi i}{60N}}
+ \frac{1}{2} \sum_{k=0}^{\infty} \frac{L \left( -2k, -\psi_6^{(1)} - \psi_6^{(11)} - \psi_6^{(19)} - \psi_6^{(29)} \right)}{k!} \left( \frac{\pi i}{60N} \right)^k
\]

(4.12)

where we have the generating function of the \( L \)-function as

\[
2 \frac{\text{ch}(5z) \text{ch}(9z)}{\text{ch}(15z)} = - \sum_{k=0}^{\infty} \frac{L \left( -2k, -\psi_6^{(1)} - \psi_6^{(11)} - \psi_6^{(19)} - \psi_6^{(29)} \right)}{(2k)!} z^{2k}
\]

Then we have an asymptotic behavior of the partition function \( Z_{N-2}(\mathcal{E}_8) \) in \( N \to \infty \) as

\[
Z_{N-2}(\mathcal{E}_8) \sim \frac{1}{2} e^{-\frac{\pi i}{120}} \left( \sqrt{\frac{5 - \sqrt{5}}{5}} e^{-\frac{1}{60\pi i N}} + \sqrt{\frac{5 + \sqrt{5}}{5}} e^{-\frac{49}{60\pi i N}} \right)
\]

(4.13)

The torsion and the Chern–Simons invariant are given from (3.9) and (3.10) by setting \( \vec{p} = (2, 3, 5) \) and \( \vec{q} = (-1, 1, 1) \):

\[
\begin{array}{c|cc}
\ell & \sqrt{T_\alpha} & \text{CS}(A_\alpha) \\
\hline
(1, 1, 1) & 2 \sqrt{\frac{2}{5}} \sin \left( \frac{\pi}{5} \right) & -\frac{1}{120} \\
(1, 1, 3) & 2 \sqrt{\frac{2}{5}} \sin \left( \frac{3\pi}{5} \right) & -\frac{49}{120}
\end{array}
\]

(4.14)
which supports (3.7).

4.4 $M(2, 2, K)$

Let $\mathcal{D}_K$ be the prism manifold $M(2, 2, K)$ where we assume $K \geq 2$. This manifold, which is defined by (1.1), also has the following surgery description as of the Dynkin diagram for $D_{K+2}$:

where we have $K + 2$ vertices $\bullet$. Note that the Euler characteristic is

$$e(\mathcal{D}_K) = \frac{1}{K}$$

**Proposition 8.** The WRT invariant for $\mathcal{D}_K$ is written as a sum of the Eichler integrals, $\widetilde{\Psi}_K^{(1)}(1/N)$ and $\widetilde{\Psi}_K^{(K-1)}(1/N)$, as follows.

- $K$ is even;

$$e^{\frac{1}{2N}(\sqrt{K} - \sqrt{K})^2 \pi i} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N (\mathcal{D}_K)$$

$$= \left( 1 + (-1)^N(1+\sqrt{K}) \right) \left( e^{\frac{\pi i}{2N}} - \widetilde{\Psi}_K^{(1)}(1/N) - \widetilde{\Psi}_K^{(K-1)}(1/N) \right)$$

(4.15)

- $K$ is odd;

$$e^{\frac{1}{2N}(\sqrt{K} - \sqrt{K})^2 \pi i} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N (\mathcal{D}_K)$$

$$= \left( 1 + e^{-\frac{KN}{2N} \pi i} \right) e^{\frac{\pi i}{2N}} - \widetilde{\Psi}_K^{(1)}(1/N) - \widetilde{\Psi}_K^{(K-1)}(1/N)$$

$$+ e^{-\frac{KN}{2N} \pi i} \left( -\widetilde{\Psi}_K^{(1)}(1/N) + \widetilde{\Psi}_K^{(K-1)}(1/N) \right)$$

(4.16)

**Proof.** Using symmetries of the summand of (3.1), the sum $\sum_{k_0=1}^{N-1} \sum_{n_j \mod p_j} \text{ with } \vec{p} = (2, 2, K)$ can be replaced with a sum $\sum_{k_0=a+2Nn}^{\sum_{k_0=1}^{a+2Nn} \sum_{n_j=0}^{1} \sum_{n_2=0}^{1} \sum_{n_1=0}^{1} \text{ with } n_3 = 0$. Taking sums
over \( n_1 \) and \( n_2 \), and using a symmetry of the summand under \( k_0 \to 2Nk - k_0 \), we get

l.h.s. of (4.15)

\[
= \frac{e^{\frac{ni}{2}}}{\sqrt{8KN}} \sum_{\substack{k_0=0 \\
N \nmid k_0}}^{2KN} \left( e^{\frac{ni}{2N}k_0} - e^{-\frac{ni}{2N}k_0} \right) \left( e^{\frac{ni}{N}k_0} - e^{-\frac{ni}{N}k_0} \right) e^{-\frac{ni}{2N}k_0^2} (1 - (-1)^{N+k_0})
\]

We first assume that \( K \) is even. In this case, we use the periodic function

\[
\varphi^{(e)}_{2K}(n) = \begin{cases} 
K/2 - 1 & 1 \\
K/2 + 1 & -1 \\
3K/2 - 1 & -1 \\
3K/2 + 1 & 1 \\
others & 0
\end{cases}
\]

which satisfies

\[
\frac{z - z^{-1}}{z^{K/2} + z^{K/2}} = -\sum_{n=0}^{\infty} \varphi^{(e)}_{2K}(n) z^n
\]  

(4.17)

We get

l.h.s. of (4.15)

\[
= -\frac{e^{\frac{ni}{2}}}{\sqrt{8KN}} \lim_{t \to 0} \sum_{n=0}^{\infty} \varphi^{(e)}_{2K}(n) e^{-nt} \sum_{\substack{k_0=0 \\
N \nmid k_0}}^{2KN} e^{\frac{ni}{N}k_0n - \frac{ni}{2N}k_0^2} \left( e^{\frac{ni}{2N}k_0} - e^{-\frac{ni}{2N}k_0} \right) (1 - (-1)^{N+k_0})
\]

We can check that the sum over \( N \mid k_0 \) vanishes. By applying the Gauss reciprocity formula (2.20) and taking a limit \( t \to 0 \), we get

l.h.s. of (4.15)

\[
= 1 + (-1)^{N+K} \sum_{n=0}^{2KN} \varphi^{(e)}_{2K}(n) B_1 \left( \frac{n}{2K} \right) \left( e^{\frac{ni}{2N}(n+K/2)^2} - e^{\frac{ni}{2N}(n-K/2)^2} \right)
\]

As we have \( \varphi^{(e)}_{2K}(n \pm K/2) = \mp \left( \psi^{(1)}_{2K}(n) + \psi^{(K-1)}_{2K}(n) \right) \), we obtain (4.15) after some manipulations.

In the case that \( K \) is odd, we use the periodic function

\[
\varphi^{(o)}_{4K}(n) = \begin{cases} 
K-2 & 1 \\
K+2 & -1 \\
3K-2 & -1 \\
3K+2 & 1 \\
others & 0
\end{cases}
\]

which has the following generating function;

\[
\frac{z^2 - z^{-2}}{z^K + z^{-K}} = -\sum_{n=0}^{\infty} \varphi^{(o)}_{4K}(n) z^n
\]  

(4.18)
Using the same method, we obtain

\[
\text{l.h.s. of (4.16)} = \frac{1}{2} \sum_{n=0}^{4NK-1} \varphi^{(o)}_{4K}(n) B_1 \left( \frac{n}{4NK} \right) \\
\times \left( e^{\pi i (n+K)^2} \left( 1 - (-1)^N e^{\frac{n+K}{2} i + \frac{K}{2} \pi i} \right) - e^{\pi i (n-K)^2} \left( 1 - (-1)^N e^{-\frac{n-K}{2} i + \frac{K}{2} \pi i} \right) \right)
\]

When we use a relationship between \( \varphi^{(o)}_{4K}(n \pm K) \) and \( \psi^{(o)}_{2K}(n) \) as in the case of even \( K \), we obtain (4.16).

This proposition indicates that the WRT invariant \( \tau_N(D_K) \) vanishes if \( N \) is odd and \( K \equiv 0 \mod 4 \). By use of (3.5) with the Coxeter–Dynkin type linking matrix, we can check directly

\[
\tau_3(D_K) = 0 \quad \text{if} \quad K \equiv 0 \mod 4 \quad (4.19)
\]

The factorization property (3.4) proves this fact.

The exact asymptotic expansion of the WRT invariant in \( N \to \infty \) simply follows from (2.17).

**Corollary 9.** Exact asymptotic expansion of the WRT invariant \( \tau_N(D_K) \) in \( N \to \infty \) is given as follows.

- for even \( K \):

\[
e^{\frac{1}{2N} \left( \sqrt{K} - \frac{1}{\sqrt{K}} \right)^2} \left( e^{\frac{2\pi i}{N}} - 1 \right) \cdot \tau_N(D_K) \simeq \left( 1 + (-1)^{N+\frac{N}{2}} \right) \left( \frac{N}{i} \sqrt{\frac{1}{2}} \right) \\
\times \left( \sin \left( \frac{K}{4} \pi \right) \cos \left( \frac{K-2}{4} \pi \right) e^{-\frac{K}{2} \pi i} + 2 \sum_{b=1}^{K-1} \sin \left( \frac{b}{2} \pi \right) \cos \left( \frac{K-2}{2K} b \pi \right) e^{-\frac{b}{2K} \pi i N} \right) \\
+ e^{\pi i N} \sum_{n=0}^{\infty} \frac{L(-2n, \psi^{(1)}_{2K} + \psi^{(K-1)}_{2K})}{n!} \left( \frac{\pi i}{2KN} \right)^n \right)
\]

(4.20)
• for odd $K$:

\[
\frac{1}{2} e^{\frac{3\pi i}{4}} \left( \sqrt{\sqrt{\frac{K-1}{K}}} \right)^2 \left( e^{2\pi i K} - 1 \right) \cdot \tau_N(D_K)
\]

\[
\approx \frac{1}{i} \sum_{k=1}^{K-1} \sin \left( \frac{b}{2} \pi \right) \cos \left( \frac{K - 2}{2K} b \pi \right) - e^{-\frac{KN}{2K} \pi i} \cos \left( \frac{b}{2} \pi \right) \sin \left( \frac{K - 2}{2K} b \pi \right) e^{-\frac{K}{2K} \pi i N}
\]

\[
= \frac{1}{\sqrt{N}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{m=0}^{\left\lfloor \frac{K}{2} \right\rfloor} 4 \left( \begin{array}{c} 2m + 1 \end{array} \right) \sin \left( \frac{2m + 1}{K} \pi \right) e^{-\frac{(2m+1)^2}{2K} \pi i N} \right)
\]

Here the $L$-functions are computed from the generating functions as

\[
\frac{\text{ch} \left( \frac{K}{2} z \right)}{\text{ch} \left( \frac{K}{2} z \right)} = \sum_{k=0}^{\infty} \frac{L \left( -2k, \psi^{(1)}_{2K} + \psi^{(K-1)}_{2K} \right)}{(2k)!} z^{2k}
\]

\[
\frac{\text{sh} \left( \frac{K}{2} z \right)}{\text{sh} \left( \frac{K}{2} z \right)} = \sum_{k=0}^{\infty} \frac{L \left( -2k, \psi^{(1)}_{2K} - \psi^{(K-1)}_{2K} \right)}{(2k)!} z^{2k}
\]

Thus a dominating term of the partition function $Z_{N-2}(D_K)$ in $N \to \infty$ is summarized by

\[
Z_{N-2}(D_K) \sim \frac{1}{2} e^{\frac{3\pi i}{4}} \sum_{m=0}^{\left\lfloor \frac{K}{2} \right\rfloor} 4 \sqrt{\frac{K}{2}} \sin \left( \frac{2m + 1}{K} \pi \right) e^{-\frac{(2m+1)^2}{4K} \pi i N} \]  

(4.22)

For the manifold $D_K$, the torsion (3.9) and the Chern–Simons invariant (3.10) can be computed by setting $\vec{p} = (2, 2, K)$ and $\vec{q} = (-1, 1, 1)$. When we choose $\vec{\ell} = (1, 1, 2m + 1)$ with $0 \leq m < \frac{K-1}{2}$, we obtain

\[
\sqrt{T_\alpha} = \frac{4}{\sqrt{K}} \sin \left( \frac{2m + 1}{K} \pi \right)
\]

\[
\text{CS}(A_\alpha) = -\frac{(2m+1)^2}{4K} \mod 1
\]

(4.23)

which supports (3.7).

4.5 Comments

To close this section, we shall give several relations among the SU(2) quantum invariants for the 3-manifolds and links. First of all we see that the WRT invariants for manifolds $E_6$ and $D_6$
are related to each other, and we have
\[ e^{\pi i \frac{N}{N}} \cdot \tau_N (D_6) + \frac{e^{-2\pi i \frac{N}{N^2}}}{e^{2\pi i \frac{N}{N}} - 1} = 4 \sqrt{3} \tau_N (E_6) \quad \text{for} \ 3 \mid N \quad (4.24) \]

Recalling a result (2.15) in Ref. 16, the WRT invariant for \( D_2 \) is related to Kashaev’s invariant for the torus link \( T_{2,4} \):
\[ \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N (D_2) = 2 \left( 1 - \frac{1}{N} \langle T_{2,4} \rangle_N \right) \quad (4.25) \]

Topologically this coincidence may be explained from a fact that the Seifert manifold \( M(0; (k, -1), (k, 1), (k, 1)) \) is constructed from 0-framed surgery of the torus link \( T_{2,2k} \), and the manifold \( D_2 \) is given from \( T_{2,4} \):

Furthermore we find that the WRT invariant for \( D_K \) with odd \( K \) is related to Kashaev’s invariant for torus link \( T_{2,2K} \) as
\[ \left( e^{\frac{2\pi i}{K}} - 1 \right) \tau_N (D_K) = -\frac{2}{K N} \langle T_{2,2K} \rangle_N \quad \text{for odd} \ K \text{ and} \ N \equiv 2 \mod 4 \quad (4.26) \]

We do not know a precise meaning of this relation, but a connection between \( D_K \) and \( T_{2,2K} \) might be explained as follows. The triangle group \( T_{2,2,K} \) is a subgroup of \( T_{2,2,2K} \), and the manifold \( D_{2K} \) is homeomorphic to the 2-fold cyclic branched covering of \( S^3 \), branched along a torus link \( T_{2,2K} \) [40].

5. RELATIONSHIP WITH THE PLATONIC SOLIDS

We have clarified that the WRT invariant for the spherical Seifert manifolds \( S^3/\Gamma \) with the finite subgroup \( \Gamma \) of SU(2) is written in terms of a limiting value of the Eichler integrals of the half-integral weight modular forms. As the fundamental group (1.6) of these manifolds is related to the polyhedral group as in Table 1 and the manifold is a spherical neighborhood of the Kleinian singularities associated to hypersurface in Table 2, one may expect that the modular forms, whose Eichler integrals denote the WRT invariant of the manifolds, have a connection with the Platonic solid. This type of relationship was conjectured in Ref. 13, and established was the connection between the absolute value of the WRT invariant and the fundamental group for a case of lens space [56].
Here we shall demonstrate this connection for several cases. This may be compared with the ADE classification of the modular invariant partition function of the conformal field theory [4] and the classification of the rational conformal field theories based on the Fuchsian differential equation [29, 39].

5.1 Tetrahedral Group

Our result (4.1) indicates that the WRT invariant for $E_6 = M(2, 3, 3)$ is regarded as a sum of the Eichler integral of modular forms $\Psi_E^{(1)}(\tau) + \Psi_E^{(5)}(\tau)$ and $\Psi_E^{(3)}(\tau)$. These two $q$-series span a two-dimensional vector modular form with weight $3/2$, and when we set

$$\Psi_{E_6}(\tau) \equiv \left( \begin{array}{c} X(\tau) \\ Y(\tau) \end{array} \right) = \left( \begin{array}{c} \frac{1}{\sqrt{2}} \left( \Psi_E^{(1)}(\tau) + \Psi_E^{(5)}(\tau) \right) \\ \Psi_E^{(3)}(\tau) \end{array} \right)$$

we have the transformation property by use of (2.10)

$$\Psi_{E_6}(\tau) = \frac{1}{\sqrt{3}} \left( \begin{array}{cc} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{array} \right) \Psi_{E_6}(-1/\tau)$$

$$\Psi_{E_6}(\tau + 1) = \left( \begin{array}{cc} e^{-\frac{1}{3}\pi i} & 0 \\ 0 & e^{\frac{1}{3}\pi i} \end{array} \right) \Psi_{E_6}(\tau)$$

(5.2)

We note that we have divided the vector modular form by powers of Dedekind $\eta$-function for our physical convention, and that this is the character for $k = 1$ SU(3) WZW model up to constant which can be checked from the Verlinde formula [54].

We now look for invariant polynomials of $X$ and $Y$ under the modular group (5.2). As homogeneous polynomials of $X$ and $Y$, we define three polynomials by [6]

$$V_T(X, Y) = X^4 + 2\sqrt{2}XY^3$$

$$F_T(X, Y) = Y^4 - 2\sqrt{2}X^3Y$$

$$E_T(X, Y) = X^6 - 5\sqrt{2}X^3Y^3 - Y^6$$

(5.3)

Based on the transformation laws (5.2), we can check directly that these polynomials are invariant under the modular group. Furthermore the modular transformation property shows that they can
be written as

\[(\eta(\tau))^8 \cdot V_T(\tau) = \frac{1}{4} E_4(\tau)\]

\[F_T(\tau) = -3\]

\[(\eta(\tau))^{12} \cdot E_T(\tau) = \frac{1}{8} E_6(\tau)\]  \hspace{1cm} (5.4)

where we have used the Eisenstein series (2.3) and the Dedekind \(\eta\)-function (2.1). By definition of the invariant polynomials (5.3), those 3 polynomials satisfy the tetrahedral equation \[32\]

\[V_T^3 + F_T^3 = E_T^2\]  \hspace{1cm} (5.5)

which reduces to

\[R(x, y, z) = x^3 + y^4 + z^2 = 0\]

in Table 2 by setting \(x = -4^{1/3} V_T F_T, y = E_T,\) and \(z = i (V_T^3 - F_T^3).\)

The symmetry group of the tetrahedron can be derived by considering the principal congruence subgroup of \(SL(2; \mathbb{Z})\). We note that the vector modular form \(\Psi_{E_6}(\tau)\) is written as the theta series on the root lattice of \(A_2;\)

\[(\eta(\tau))^2 \cdot \Psi_{E_6}(\tau) = \left( \frac{1}{\sqrt{2}} \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2-xy+y^2} \right)^{1/2} \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2-xy+y^2+x-y} \]  \hspace{1cm} (5.6)

We see from the transformation law of the right hand side that it is a modular form with weight 1 for the group \(\Gamma(3)\), where \(\Gamma(M)\) is the principal congruence subgroup of level \(M\) of \(SL(2; \mathbb{Z})\) (see, e.g., Refs. 6, 33, 57)

\[\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}) \left| \begin{array}{c} a \equiv d \equiv 1 \mod M \\ b \equiv c \equiv 0 \mod M \end{array} \right. \right\}\]  \hspace{1cm} (5.7)

This fact is based on that the level of the root lattice \(A_2\) is 3. The group \(PSL(2; \mathbb{Z})/\Gamma(3)\) is isomorphic to the symmetry group of the tetrahedron [6]. Then we have a mapping

\[\Psi_{E_6} : \mathbb{H}/\Gamma(3) \rightarrow \mathbb{P}^1\]

where \(\mathbb{H}/\Gamma(3)\) means a compactification of \(\mathbb{H}/\Gamma(3)\) by adding a point \(\infty\), and the tetrahedral group acts on the tetrahedron in the Riemann sphere \(\mathbb{P}^1\).

This action can be seen immediately by studying the zeros of the homogeneous polynomials, \(V_T, F_T,\) and \(E_T\) [6]. We consider the regular tetrahedron in \(\mathbb{P}^3\), which is inscribed in the unit sphere \(S^2\) around the origin with the south pole \((0, 0, -1)\) as one of vertices. We identify \(S^2\) with \(\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}\) by the projection from the north pole \((0, 0, 1)\) to the equatorial plane, and we
regard \((X : Y)\) as the homogeneous coordinates of \(\mathbb{P}^1\) identifying \((1 : 0)\) with \(\infty\). Then we see that the four zeros of the homogeneous polynomial \(V_T\) (5.3) denote the vertices of the regular tetrahedron. In the same manner, the zeros of the polynomial \(F_T\) are the mid-points of faces, \textit{i.e.}, the intersections of the unit sphere \(S^2\) and the straight line which connects the origin and each vertex of the tetrahedron. The zeros of the homogeneous polynomial \(E_T\) denote the mid-points of edges, \textit{i.e.}, the intersections of the sphere and the straight line which connects the middle point of edges which do not share the vertex of the tetrahedron. As a consequence the polynomials \(V_T\), \(F_T\), and \(V_T\) are invariant under the tetrahedral group.

5.2 Octahedral Group

We see from (4.6) that the WRT invariant for \(\mathcal{E}_7 = M(2, 3, 4)\) can be regarded as the Eichler integrals of the \(q\)-series \(\Psi_{12}^{(1)}(\tau) + \Psi_{12}^{(5)}(\tau) + \Psi_{12}^{(7)}(\tau) + \Psi_{12}^{(11)}(\tau)\). This function with 2 more functions spans a 3-dimensional space of the modular form with weight \(3/2\); when we define the vector modular form \(\Psi_{E_7}(\tau)\) by

\[
\Psi_{E_7}(\tau) = \frac{1}{(\eta(\tau))^3} \begin{pmatrix}
\frac{1}{2} \left( \Psi_{12}^{(1)}(\tau) + \Psi_{12}^{(5)}(\tau) + \Psi_{12}^{(7)}(\tau) + \Psi_{12}^{(11)}(\tau) \right) \\
\frac{1}{\sqrt{2}} \left( \Psi_{12}^{(4)}(\tau) + \Psi_{12}^{(8)}(\tau) \right) \\
\frac{1}{2} \left( \Psi_{12}^{(1)}(\tau) - \Psi_{12}^{(5)}(\tau) + \Psi_{12}^{(7)}(\tau) - \Psi_{12}^{(11)}(\tau) \right)
\end{pmatrix}
\]

\[\equiv \begin{pmatrix}
X(\tau) \\
Y(\tau) \\
Z(\tau)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} q^{-\frac{5}{24}} \left( 1 + 5 q^{\frac{1}{2}} + 10 q + 15 q^2 + \cdots \right) \\
2 \sqrt{2} q^{\frac{5}{24}} \left( 1 + 5 q + 15 q^2 + 40 q^3 + \cdots \right) \\
\frac{1}{2} q^{-\frac{5}{24}} \left( 1 - 5 q^2 + 10 q - 15 q^3 + \cdots \right)
\end{pmatrix}
\]

the modular transformation (2.10) reduces to

\[
\Psi_{E_7}(\tau) = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix} \Psi_{E_7}(-1/\tau)
\]

\[\Psi_{E_7}(\tau + 1) = \begin{pmatrix}
e^{-\frac{5}{24} \pi i} & e^{\frac{5}{12} \pi i} & e^{-\frac{5}{24} \pi i} \\
e^{\frac{5}{12} \pi i} & e^{-\frac{5}{24} \pi i}
\end{pmatrix} \Psi_{E_7}(\tau)
\]
We consider the homogeneous polynomials which are invariant under the modular group (5.9). Empirically we define polynomials as follows;

\[ V_C(X, Y, Z) = (XY Z)^2 \]

\[ F_C(X, Y, Z) = X^8 - Y^8 - Z^8 \]

\[ (E_C(X, Y, Z))^2 = X^{24} - Y^{24} - Z^{24} - \frac{3351}{4} (XY Z)^8 \]

\[ - 3 \left( Y^8 + Z^8 \right) \left( X^{16} + Y^8 Z^8 \right) + 3 \left( Y^{16} + Z^{16} \right) X^8 \]

(5.10)

Using modular transformation laws (5.9) and recalling properties of the space of the modular form, we find that these invariant polynomials can be written as

\[ V_C(\tau) = \frac{1}{2} \]

\[ (\eta(\tau))^8 F_C(\tau) = \frac{5}{16} E_4(\tau) \]

\[ (\eta(\tau))^{12} E_C(\tau) = \frac{5 \sqrt{5}}{64} E_6(\tau) \]

(5.11)

The invariant polynomials \( V_C, F_C, \) and \( E_C \) satisfy the cubic equation [32]

\[ E_C^2 = F_C^3 - \frac{3375}{4} V_C^4 \]

(5.12)

which follows from the definition (5.10), and coincides with the identity (2.7). After setting \( x = \frac{153}{2} V_C^2, y = -F_C, \) and \( z = \frac{153}{4} E_C V_C \), we recover the hypersurface for \( E_7; \)

\[ R(x, y, z) = x^3 + x y^3 + z^2 = 0 \]

To discuss the modular group, we recall the Jacobi theta functions

\[ \theta_{00}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2} n^2} = \frac{(\eta(\tau^2))^2}{\eta(\tau)} \]

\[ \theta_{10}(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2} (n + \frac{1}{2})^2} = \frac{(\eta(2 \tau))^2}{\eta(\tau)} \]

\[ \theta_{01}(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} n^2} = \frac{(\eta(\tau^2))^2}{\eta(\tau)} \]

(5.13)

When we set the vector as

\[ \Theta(\tau) = \begin{pmatrix} \theta_{00}(\tau) \\ \theta_{10}(\tau) \\ \theta_{01}(\tau) \end{pmatrix} \]

(5.14)
this becomes a vector modular form with weight 1/2 (see, e.g., Ref. 42);

\[
\Theta(\tau) = \sqrt{\frac{i}{\tau}} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Theta(-1/\tau)
\]

(5.15)

\[
\Theta(\tau + 1) = \begin{pmatrix} e^{\frac{1}{8} \pi i} & 1 \\ 1 & 1 \end{pmatrix} \Theta(\tau)
\]

With these modular transformation formulae, we find that the functions \(X, Y, \) and \(Z\) in (5.8), are written in terms of the Jacobi theta functions

\[
\begin{pmatrix} X^2 \\ Y^2 \\ Z^2 \end{pmatrix} = \frac{1}{4(\eta(\tau))^3} \begin{pmatrix} (\theta_{00})^5 \\ (\theta_{10})^5 \\ (\theta_{01})^5 \end{pmatrix}
\]

(5.16)

The transformation properties (5.15) show that the theta functions defined by \(\Theta^2(\tau) = ((\theta_{00})^2, (\theta_{10})^2, (\theta_{01})^2)^T\) is a modular form with weight 1 for the group \(\Gamma(4)\). We then have a map

\[
(\Psi_{E7})^4 : \mathbb{H}/\Gamma(4) \to \mathbb{P}^2
\]

and the group \(PSL(2; \mathbb{Z})/\Gamma(4)\) denotes the symmetry of the cube.

Simple explanation of connection with the octahedral group is as follows. We reconsider the modular group acting on \((x, y, z) \equiv (X^{24}, Y^{24}, Z^{24})\). Under the action of \(S\) and \(R = TS\), we have

\[
S : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ z \\ y \end{pmatrix} \quad R = TS : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} -z \\ -x \\ y \end{pmatrix}
\]

When we interpret these actions on \(\mathbb{R}^3\) with coordinates, \(x, y,\) and \(z\), the action \(S\) can be regarded as a reflection at the plane \(y = z\). In the same way, the action \(R\) denotes a \(\frac{2\pi}{3}\) rotation around an axis which passes both the origin and \((-1, 1, 1)\). As a result the cube whose vertices are on \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\) with \(\varepsilon_i = \pm 1\) is invariant under the modular group.

### 5.3 Icosahedral Group

We have seen that the WRT invariant (4.11) for the Poincaré homology sphere \(E_8 = M(2, 3, 5)\) is regarded as the Eichler integral of \(\Psi^{(1)}_{30}(\tau) + \Psi^{(11)}_{30}(\tau) + \Psi^{(19)}_{30}(\tau) + \Psi^{(20)}_{30}(\tau)\) of weight 3/2. As
was pointed out in Ref. 38, it spans a two-dimensional vector modular form; when we define

\[
\Psi_{E_8}(\tau) = \frac{1}{(\eta(\tau))^2} \left( \Psi^{(1)}_{30}(\tau) + \Psi^{(11)}_{30}(\tau) + \Psi^{(19)}_{30}(\tau) + \Psi^{(29)}_{30}(\tau) \right)
\]

\[
\equiv \begin{pmatrix} X(\tau) \\ Y(\tau) \end{pmatrix} = \begin{pmatrix} q^{-\frac{7}{30}} (1 + 14 q + 42 q^2 + 140 q^3 + \cdots) \\ q^{\frac{17}{30}} (7 + 34 q + 119 q^2 + 322 q^3 + \cdots) \end{pmatrix}
\]

we have under the \(S\)- and \(T\)-transformations

\[
\Psi_{E_8}(\tau) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \left( \frac{\pi}{5} \right) & \sin \left( \frac{2\pi}{5} \right) \\ \sin \left( \frac{2\pi}{5} \right) & -\sin \left( \frac{\pi}{5} \right) \end{pmatrix} \Psi_{E_8}(-1/\tau)
\]

\[
\Psi_{E_8}(\tau + 1) = \left( e^{-\frac{7}{30} \pi i} e^{\frac{17}{30} \pi i} \right) \Psi_{E_8}(\tau)
\]

We notice that in the vector modular form \(\Psi_{E_8}(\tau)\) the subscript 30 is the Coxeter number of the Lie algebra \(E_8\), and that a set of superscripts, \(\{1, 7, 11, 13, 17, 19, 23, 29\}\), also coincides with the exponents of \(E_8\) (see, e.g., Ref. 22). So we may expect that the vector modular form \(\Psi_{E_8}(\tau)\) has a connection with the exceptional Lie algebra \(E_8\). Although, we note that the modular form \(\Psi_{E_8}(\tau)\) denotes the character of the \(k = 1 G_2\) WZW model [39, 54].

To find a more explicit and geometrical relationship with the \(E_8\) algebra, we define three homogeneous polynomials of \(X\) and \(Y\) following Klein [32] by

\[
V_I(X, Y) = XY \left( X^{10} + 11X^5Y^5 - Y^{10} \right)
\]

\[
F_I(X, Y) = X^{20} + Y^{20} - 228X^5Y^5 \left( X^{10} - Y^{10} \right) + 494X^{10}Y^{10}
\]

\[
E_I(X, Y) = X^{30} + Y^{30} + 522X^5Y^5 \left( X^{20} - Y^{20} \right) - 10005X^{10}Y^{10} \left( X^{10} + Y^{10} \right)
\]

We can check that these are invariant polynomials under the modular group (5.18), and by investigating the modular properties these polynomials are written in terms of the Eisenstein series as

\[
27 \Delta(\tau) V_I = 125 \left( E_4 \right)^3 + 64 \left( E_6 \right)^2
\]

\[
2916 \left( \eta(\tau) \right)^{56} F_I = E_4 \left( -3125 \left( E_4 \right)^6 + 9625 \left( E_4 \right)^3 \left( E_6 \right)^2 - 3584 \left( E_6 \right)^4 \right)
\]

\[
157464 \left( \eta(\tau) \right)^{84} E_I
\]

\[
= E_6 \left( 546875 \left( E_4 \right)^9 - 931875 \left( E_4 \right)^6 \left( E_6 \right)^2 + 575232 \left( E_4 \right)^3 \left( E_6 \right)^4 - 32768 \left( E_6 \right)^6 \right)
\]

We see by definition (5.19) that these invariant polynomials satisfy the icosahedron equation

\[
1728 V_I^5 + F_I^3 = E_I^2
\]
which reduces to the hypersurface for \( E_8 \),
\[
R(x, y, z) = x^3 + y^5 + z^2 = 0
\]
when we set \( x = -F_I \), \( y = -12^{1/5} V_I \), and \( z = E_I \).

The modular transformation property (5.18) proves that the modular form \((\eta(\tau))^{14/5} \cdot \Psi_{E_8}(\tau)\) with rational weight \(7/5\) is on the group \( \Gamma(5) \). As was studied in Ref. 23, the polynomial ring of the group \( \Gamma(5) \) is known to be spanned by modular forms \((\eta(\tau))^{2/5} \Phi_1(\tau)\) and \((\eta(\tau))^{2/5} \Phi_2(\tau)\) with weight \(1/5\), where we use
\[
\chi_{2,5}(\tau) = \left( \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{5}{20}(10n+1)^2} \right) \left( \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{5}{20}(10n+3)^2} \right)
\]
(5.22)
\[
= \left( q^{\frac{1}{60}} (1 + q + q^2 + q^3 + \cdots) \right) \left( q^{\frac{11}{60}} (1 + q^2 + q^3 + \cdots) \right)
\]
The transformation laws of these \(q\)-series are given by
\[
\chi_{2,5}(\tau) = \left( \frac{2}{\sqrt{5}} \left( \sin \left( \frac{2}{5} \pi \right) \sin \left( \frac{1}{5} \pi \right) \right) \right) \chi_{2,5}(-1/\tau)
\]
and
\[
\chi_{2,5}(\tau + 1) = \left( e^{-\frac{i}{30} \pi i} \right) \chi_{2,5}(\tau)
\]
(5.23)
It should be noted that the rational weight plays a crucial role in studying congruence subgroup in Ref. 23, but here we choose \( \chi_{2,5}(\tau) \) to be weight-zero modular form from the point of view of the conformal field theory, because these are the Virasoro characters of the minimal model \( \mathcal{M}(2, 5) \), or the Lee–Yang theory [48]. They can be written as follows due to the Rogers–Ramanujan identity and the Jacobi triple product formula;
\[
q^{\frac{1}{60}} \Phi_1(\tau) = \prod_{n=0}^{\infty} \frac{1}{1 - q^{5n+1}} \frac{1}{(1 - q^{5n+4})}
\]
\[
= \sum_{n=0}^{\infty} q^{n^2} \prod_{k=1}^{n} (1 - q^k)
\]
\[
q^{\frac{11}{60}} \Phi_2(\tau) = \prod_{n=0}^{\infty} \frac{1}{1 - q^{5n+2}} \frac{1}{(1 - q^{5n+3})}
\]
\[
= \sum_{n=0}^{\infty} q^{n^2+n} \prod_{k=1}^{n} (1 - q^k)
\]
See Refs. 14, 21 for recent studies on the Rogers–Ramanujan type generating function of the $L$-function as a generalization of Zagier’s identity [1, 45, 57]. We should remark that the Eichler integral of the modular form $\eta(\tau) \cdot \Phi_2(\tau)$ with weight $1/2$ coincides with Kashaev’s invariant for torus knot $T_{2,5}$ [15, 20].

To see a relationship between these two bases, $\Psi_E(\tau)$ and $\chi_{2,5}(\tau)$, of the group $\Gamma(5)$, we recall that invariant polynomials for the vector modular form $\chi_{2,5}(\tau)$ have the same form with (5.19) replacing $(X,Y)$ with $(\Phi_2, \Phi_1)$. Explicitly they are computed as

$$V_I(\Phi_2, \Phi_1) = -1$$

$$\left(\eta(\tau)\right)^8 \cdot F_I(\Phi_2, \Phi_1) = E_4$$

Equating (5.20) with (5.24), we find that

$$X = \Phi_1^2 \left(\Phi_1^5 + 7 \Phi_2^5\right)$$

$$Y = \left(7 \Phi_1^5 - \Phi_2^5\right) \Phi_2^2$$

which coincides with one of solutions in Ref. 26. Therein the Fuchsian differential equation [29, 39] was investigated, and given explicitly were homogeneous polynomials of $\Phi_1$ and $\Phi_2$ which constitute the two dimensional vector modular space.

To conclude we have a mapping

$$\Psi_E : \mathbb{H}/\Gamma(5) \to \mathbb{P}^1$$

and the modular form $\Psi_E$ is related to the icosahedral group.

It may help our understanding to discuss a direct connection between the homogeneous polynomials $V_I$, $F_I$, and $E_I$ (5.19) and the regular icosahedron. We consider the unit sphere $S^2$ in $\mathbb{R}^3$, and inscribe the regular icosahedron in it with the north and south poles as two of vertices thereof. We identify $S^2$ with $\mathbb{P}^1$ as before, and take the coordinates of $\mathbb{P}^1$ as $(X : Y)$. Then the zeros of the polynomial $V_I$ coincides with the vertices of the icosahedron while the zeros of the polynomials $F_I$ and $E_I$ denote the vertices of the dual dodecahedron, or the mid-points of face of the icosahedron, and the mid-edge points respectively. Therefore these polynomials are invariant under the icosahedral group.

5.4 $D_3$

A realization of the cube in $\mathbb{P}^1$ appears in the modular forms for the manifold $D_3$. As pointed out in (4.26), the WRT invariant for the manifold $D_3$ is related to Kashaev’s invariant for torus knot $T_{2,5}$ [15, 20].
link $T_{2,6}$. From the viewpoint of modular forms, these two quantum invariants can be regarded as the Eichler integrals of the following two-dimensional vector modular form:

$$
\Psi_{D_5}(\tau) = \frac{1}{(\eta(\tau))^3} \begin{pmatrix} \Psi_3^{(1)}(\tau) \\ \Psi_3^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} (\eta(2\tau))^5 \\ (\eta(\tau))^3 (\eta(4\tau))^2 \end{pmatrix}
$$

$$
\equiv \begin{pmatrix} X(\tau) \\ Y(\tau) \end{pmatrix} = \begin{pmatrix} q^{-\frac{1}{12}} (1 + 3q + 4q^2 + 7q^3 + \cdots) \\ 2q^{\frac{5}{24}} (1 + q + 3q^2 + 4q^3 + \cdots) \end{pmatrix}
$$

This transforms as

$$
\Psi_{D_5}(\tau) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Psi_{D_5}(-1/\tau)
$$

$$
\Psi_{D_5}(\tau + 1) = \begin{pmatrix} e^{-\frac{1}{12} \pi i} \\ e^{\frac{5}{24} \pi i} \end{pmatrix} \Psi_{D_5}(\tau)
$$

The homogeneous invariant polynomials of $X$ and $Y$ are then given by [32]

$$
F_C = XY \left( X^4 - Y^4 \right)
$$

$$
V_C = X^8 + 14X^4Y^4 + Y^8
$$

$$
E_C = X^{12} - 33X^8Y^4 - 33X^4Y^8 + Y^{12}
$$

From the transformation law under the modular group, we see that these polynomials can be written in terms of the Eisenstein series as

$$
F_C(\tau) = 2 (\eta(\tau))^8 V_C(\tau) = E_4(\tau)
$$

$$
(\eta(\tau))^{12} E_C(\tau) = E_6(\tau)
$$

By definition (5.28) the invariant polynomials fulfill the cubic equation

$$
V_C^3 - 108F_C^4 = E_C^2
$$

If we set $x = 12\sqrt{3}E_C F_C^2$, $y = V_C^2$, and $z = i \left( E_C^2 - 108F_C^4 \right)$, we recover

$$
R(x, y, z) = x^2 y + y^4 + z^2 = 0
$$
From the viewpoint of the principal congruence subgroup we note that the vector modular form is written as

\[ \eta(\tau) \Psi_{D_5}(\tau) = \left( \sum_{n \in \mathbb{Z}} q^{\frac{1}{4} n^2} \right) \left( \sum_{n \in \mathbb{Z}+1} q^{\frac{1}{4} n^2} \right) \]

The first component in the right hand side denotes the theta function on the root lattice \( A_1 \), i.e., the lattice \( \sqrt{2} \mathbb{Z} \), and the sum of two components becomes a theta series on the dual lattice \( \frac{1}{\sqrt{2}} \mathbb{Z} \).

As the level of the lattice \( \sqrt{2} \mathbb{Z} \) is 4, the right hand side is the modular form for the subgroup \( \Gamma(4) \). The group \( \text{PSL}(2; \mathbb{Z})/\Gamma(4) \) is isomorphic to the cubic group, and we have a mapping

\[ \Psi_{D_5} : \mathbb{H}/\Gamma(4) \rightarrow \mathbb{P}^1 \]

This correspondence may be explained simply as follows [6]. We consider the unit sphere \( S^2 \) around the origin, and draw a cube inscribed therein with faces perpendicular to the coordinate axes. As before, we identify \( S^2 \) with \( \mathbb{P}^1 \), and set \( (X : Y) \) as the homogeneous coordinates of \( \mathbb{P}^1 \). Then the zeros of the polynomial \( V_C \) denote the eight vertices of the cube. Correspondingly the zeros of \( F_C \) and \( E_C \) are mid-points of faces and edges respectively, and it is natural that the invariant polynomials have a form of (5.28).

### 5.5 \( D_2 \)

As a final example, we briefly study the manifold \( D_2 \). The WRT invariant for this manifold is the Eichler integral of the modular form with weight \( 3/2 \)

\[ \Psi_2^{(1)}(\tau) = (\eta(\tau))^3 \] (5.31)

In terms of the Jacobi theta function (5.13), this modular form can be factorized as

\[ 2 (\eta(\tau))^3 = \theta_{00}(\tau) \theta_{01}(\tau) \theta_{10}(\tau) \] (5.32)

These theta functions satisfy

\[ (\theta_{10}(\tau))^4 + (\theta_{01}(\tau))^4 = (\theta_{00}(\tau))^4 \] (5.33)

This algebraic equation may be identified with that in Table 2;

\[ R(x, y, z) = x^2 y + y^3 + z^2 = 0 \]

after setting \( x = -i (\theta_{01}^4 - \theta_{10}^4), \ y = -\theta_{01}^4, \) and \( z = 2 (\theta_{00} \theta_{01} \theta_{10})^2 \).
6. Conclusions and Discussions

We have revealed the connection between the SU(2) WRT invariants and modular forms. We have shown that the WRT invariant for the spherical Seifert manifolds $S^3/\Gamma$ with a finite subgroup $\Gamma$ can be written in terms of the Eichler integrals of modular forms with weight $3/2$. Explicit forms are given in (4.1), (4.6), (4.11), (4.15), and (4.16), and they suggest that the WRT invariants may be decomposed by the torsion linking pairing, $\lambda : \text{Tors } H_1(M; \mathbb{Z}) \otimes \text{Tors } H_1(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$, as

$$\tau_N(M) = \sum_{\lambda} e^{2\pi i \lambda N} \tau_N^{(\lambda)}(M)$$

where $\tau_N^{(\lambda)}(M)$ is a limiting value of holomorphic $q$-series at the $N$-th root of unity. We can check from a result of Ref. 24 that this decomposition is fulfilled for lens space.

Based on the nearly modular property of the Eichler integrals, we have obtained the exact asymptotic expansion of the WRT invariants in $N \rightarrow \infty$. We have checked that a dominating term of the Witten partition function $Z_{N-2}(M)$ can be written in terms of the classical topological invariants as (3.7). Our results are summarized in Table 3. We see that an inverse of the Euler characteristic coincides with subscript $P$ of the Eichler integrals $\tilde{\Psi}^{(a)}_P(1/N)$. As seen from the nearly modular transformation formula (2.17), the Chern–Simons invariant is related to an exponential factor of a limiting value of the Eichler integrals at integers $\tilde{\Psi}^{(a)}_P(-N)$, while both the absolute value of the Eichler integrals at integers and matrix elements of the modular $S$-matrix are related to the torsion.

$$\begin{array}{|c|c|c|c|}
\hline
\mathcal{M} & e(M) & \text{Eichler integrals} & \text{Chern–Simons invariant } \text{CS}(A_\alpha) \\
\hline
M(2, 3, 3) & \frac{1}{6} & \tilde{\Psi}^{(1),(3),(5)}_6 & \left\{ -\frac{1}{24} \right\} \\
M(2, 3, 4) & \frac{1}{12} & \tilde{\Psi}^{(1),(5),(7),(11)}_{12} & \left\{ -\frac{1}{48}, -\frac{25}{48} \right\} \\
M(2, 3, 5) & \frac{1}{30} & \tilde{\Psi}^{(1),(11),(19),(29)}_{30} & \left\{ -\frac{1}{120}, -\frac{49}{120} \right\} \\
M(2, 2, K) & \frac{1}{K} & \tilde{\Psi}^{(1),(K-1)}_{K} & \left\{ \frac{(2m+1)^2}{4K} \right\} \text{ for } 0 \leq m < \frac{K-1}{2} \\
\hline
\end{array}$$

Table 3: Relationship between the WRT invariants and the Eichler integrals is given. The Eichler integrals $\tilde{\Psi}^{(a)}_P(1/N), \tilde{\Psi}^{(b)}_P(1/N), \ldots$ means that the WRT invariant $\tau_N(M)$ is written as a linear combination of the Eichler integrals $\tilde{\Psi}^{(a)}_P(1/N), \tilde{\Psi}^{(b)}_P(1/N), \ldots$

*This observation is due to K. Habiro. The author thanks him for pointing out.
Moreover we have clarified that the modular forms, whose Eichler integrals contribute to the quantum invariants, have connections with the polyhedral group. We have studied the invariant polynomials of the modular group, and we have found that they construct the polyhedral equations (see Tables 1 and 2). Pointed out in Ref. 13 is that the absolute value of the WRT invariant depends on the fundamental group. Our results prove that the WRT invariant has some informations about the fundamental group of manifolds. As the WRT invariant for the Seifert homology spheres can be written in terms of the Eichler integrals of half-integral weight modular forms as was studied in Refs. 17–19, studies on geometry of modular forms will bring us fruitful insights on geometry of the quantum invariants even though the fundamental group is no longer finite.

We take the Brieskorn homology sphere \( \Sigma(2, 3, 7) \) as an example. The fundamental group is not finite, and it corresponds to a hyperbolic tessellation [40]. The WRT invariant for \( \Sigma(2, 3, 7) \) is identified with \( \tilde{\Phi}^{(1,1,1)}_{2,3,7}(1/N) \) which is the Eichler integral of modular form with weight \( 3/2 \) [17]. This modular form spans a 3-dimensional space with two more \( q \)-series; we introduce the vector modular form \( \Phi_{2,3,7}(\tau) \) by

\[
\Phi_{2,3,7}(\tau) = \frac{1}{(\eta(\tau))^2} \begin{pmatrix}
\phi_{2,3,7}^{(1,1,1)}(\tau) \\
\phi_{2,3,7}^{(1,1,2)}(\tau) \\
\phi_{2,3,7}^{(1,1,3)}(\tau)
\end{pmatrix}
\]

\[
\equiv \begin{pmatrix}
X(\tau) \\
-Y(\tau) \\
-Z(\tau)
\end{pmatrix} = \begin{pmatrix}
qu^{-\frac{5}{12}} (1 - 10q - 30q^2 - 95q^3 - \cdots) \\
-q^{\frac{1}{12}} (5 + 15q + 64q^2 + 190q^3 + \cdots) \\
-q^{\frac{25}{12}} (11 + 50q + 150q^2 + 420q^3 + \cdots)
\end{pmatrix}
\]

where each element is defined by

\[
\phi_{2,3,7}^{\ell}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \chi_{84}^{\ell}(n) q^{\frac{n^2}{168}}
\]

for a triple \( \ell = (\ell_1, \ell_2, \ell_3) \), and \( \chi_{84}(n) \) is the odd periodic function with modulus 84;

\[
\chi_{84}^{(1,1,1)}(n) = \psi_{84}^{(1)}(n) - \psi_{84}^{(13)}(n) - \psi_{84}^{(29)}(n) + \psi_{84}^{(41)}(n)
\]

\[
\chi_{84}^{(1,1,2)}(n) = -\psi_{84}^{(5)}(n) - \psi_{84}^{(19)}(n) - \psi_{84}^{(23)}(n) + \psi_{84}^{(37)}(n)
\]

\[
\chi_{84}^{(1,1,3)}(n) = -\psi_{84}^{(11)}(n) - \psi_{84}^{(17)}(n) - \psi_{84}^{(25)}(n) + \psi_{84}^{(31)}(n)
\]
The transformation properties are given by [17]

\[ \Phi_{2,3,7}(\tau) = \frac{-2}{\sqrt{7}} \begin{pmatrix} \sin \left( \frac{\pi}{7} \right) & \sin \left( \frac{2\pi}{7} \right) & \sin \left( \frac{3\pi}{7} \right) \\ \sin \left( \frac{2\pi}{7} \right) & -\sin \left( \frac{2\pi}{7} \right) & \sin \left( \frac{3\pi}{7} \right) \\ \sin \left( \frac{3\pi}{7} \right) & -\sin \left( \frac{3\pi}{7} \right) & \sin \left( \frac{\pi}{7} \right) \end{pmatrix} \Phi_{2,3,7}(\tau) \]

\[ \Phi_{2,3,7}(\tau + 1) = \begin{pmatrix} e^{-\frac{3\pi i}{7}} & e^{\frac{1}{2\pi i}} & e^{\frac{2\pi i}{7}} \end{pmatrix} \Phi_{2,3,7}(\tau) \] 

Due to that coefficients of \( q \)-series \( X(\tau) \) have both positive and negative integers (constant term is \( +1 \) while coefficients of positive powers of \( q \) are negative), we are not sure whether this vector modular form is related to character of the conformal field theory as in the case of the Poincaré homology sphere.

Following results on the Klein quartic [31], we define the homogeneous polynomials \( F_Q, G_Q, \) and \( H_Q \) by

\[ F_Q(X, Y, Z) = X^3 Y + Y^3 Z + Z^3 X \]
\[ G_Q(X, Y, Z) = XY^5 + YZ^5 + ZX^5 - 5X^2Y^2Z^2 \]
\[ H_Q(X, Y, Z) = X^{14} + Y^{14} + Z^{14} - 34(X^{11}Y^2Z + X^2YZ^{11} + XY^{11}Z^2) 
- 250(X^9YZ^4 + X^4Y^4Z^9 + X^4Y^9Z) 
+ 375(X^8Y^4Z^2 + X^4Y^2Z^8 + X^2Y^8Z^4) 
+ 18(X^7Y^7 + Y^7Z^7 + Z^7X^7) - 126(X^6Y^3Z^5 + X^3Y^5Z^6 + X^5Y^6Z^3) \] 

(6.3)

We can check that these are invariant polynomials under (6.2), and by use of the \( q \)-series expansion we find that they are given in terms of the Eisenstein series and the Dedekind \( \eta \)-function as

\[ (\eta(\tau))^8 F_Q = 5 E_4(\tau) \]
\[ G_Q = 3136 \] 
\[ (\eta(\tau))^{40} H_Q = \frac{1}{27}(E_4)^2(21832(E_4)^3 - 21805(E_6)^2) \] 

(6.4)

As a consequence of (2.7), we obtain an algebraic equation of the invariant polynomials as

\[ G_Q H_Q = \frac{3136}{3125} F_Q^5 + \frac{89}{5} F_Q^2 G_Q^2 \] 

(6.5)

The modular transformation law (6.2) shows that the modular form \( (\eta(\tau))^{20/7} \Psi_{2,3,7}(\tau) \) with rational weight \( 10/7 \) is on the group \( \Gamma(7) \). Previously known basis of polynomial ring associated
to the group $\Gamma(7)$ is the modular form $(\eta(\tau))^{4/7} \chi_{2,7}(\tau)$ with weight $2/7$ where we mean [23]

$$\chi_{2,7}(\tau) =\left(\begin{array}{c}
-x(\tau) \\
y(\tau) \\
z(\tau)
\end{array}\right) = \frac{1}{\eta(\tau)} \left(\begin{array}{c}
q^{\frac{12}{7}} (1 + q^2 + q^3 + \cdots) \\
q^{\frac{5}{7}} (1 + q + q^2 + 2q^3 + \cdots) \\
q^{-\frac{1}{7}} (1 + q + 2q^2 + 2q^3 + \cdots)
\end{array}\right)$$

(6.6)

It should be noted that the weight-zero vector $\chi_{2,7}(\tau)$ coincides with the character of the Virasoro minimal model $\mathcal{M}(2,7)$ [48]. In general the theta function basis in Ref. 23 is the character of the Virasoro minimal model $\mathcal{M}(2,N)$ for odd $N$ up to fractional powers of the Dedekind $\eta$-function, and as was shown in Refs. 15, 20 their Eichler integrals are proportional to Kashaev’s invariant for torus knot $T_{2,N}$. The modular transformation of $\chi_{2,7}(\tau)$ is given as

$$\chi_{2,7}(\tau) = \frac{2}{\sqrt{7}} \left(\begin{array}{ccc}
\sin\left(\frac{2\pi}{7}\right) & -\sin\left(\frac{3\pi}{7}\right) & \sin\left(\frac{\pi}{7}\right) \\
-\sin\left(\frac{3\pi}{7}\right) & -\sin\left(\frac{\pi}{7}\right) & \sin\left(\frac{2\pi}{7}\right) \\
\sin\left(\frac{\pi}{7}\right) & \sin\left(\frac{2\pi}{7}\right) & \sin\left(\frac{3\pi}{7}\right)
\end{array}\right) \chi_{2,7}(-1/\tau)$$

(6.7)

$$\chi_{2,7}(\tau + 1) = \left(\begin{array}{ccc}
e^{\frac{2\pi i}{7}} & e^{\frac{5\pi i}{7}} & e^{-\frac{1\pi i}{7}}
\end{array}\right) \chi_{2,7}(\tau)$$

The invariant polynomials under these transformations have the same form with (6.3) and (6.9) replacing $(X, Y, Z)$ with $(x, y, z)$ in (6.6), and by simple computations we obtain [7]

$$F_Q(x, y, z) = 0$$

$$G_Q(x, y, z) = 1$$

$$(\eta(\tau))^8 H_Q(x, y, z) = E_4(\tau)$$

$$(\eta(\tau))^{12} W_Q(x, y, z) = E_6(\tau)$$

(6.8)

where we have used one more invariant polynomial of order 21 defined by the Jacobian

$$W_Q(x, y, z) = \frac{1}{14} \frac{\partial (F_Q, G_Q, H_Q)}{\partial (x, y, z)}$$

(6.9)
We have an algebraic relation between these 4 invariant polynomials as
\[
W_Q^2 = H_Q^3 - 1728 G_Q^7 + 1008 F_Q G_Q^4 H_Q - 32 F_Q^2 G_Q H_Q^2 + 19712 F_Q^3 G_Q^5
- 1152 F_Q^4 G_Q^3 H_Q + 11264 F_Q^6 G_Q^3 - 256 F_Q^7 H_Q + 12288 F_Q^9 G_Q^3
\]  
(6.10)

Result of Ref. 23 shows that the polynomial ring of \((x, y, z)\) is on \(\Gamma(7)\), and we find that our basis \((X, Y, Z)\) is in fact given by
\[
X = z^5 - 10 x^2 y z^2 + 5 x y^4
\]  
\[
Y = x^5 - 10 x^2 y^2 z + 5 y z^4
\]  
\[
Z = y^5 - 10 x y^2 z^2 + 5 x^4 z
\]  
(6.11)

It is interesting to study, as a generalization of Ref. 26, the third-order Fuchsian differential equation, and to find the homogeneous polynomials which constitute the three dimensional vector modular space.

As a result, the modular form \(\Phi_{2,3,7}(\tau)\) is on \(\Gamma(7)\), and we have a mapping
\[
\Phi_{2,3,7} : \overline{\mathbb{H}/\Gamma(7)} \rightarrow \mathbb{P}^2
\]
Furthermore in the basis of \((x, y, z)\) an algebraic equation (6.10) reduces to
\[
W_Q^2 + 1728 G_Q^7 = H_Q^3
\]  
(6.12)
due to a condition of the Klein quartic \(F_Q = 0\) (6.8). This equation has the \(E_{12}\)-type exceptional singularity of Arnold, and it is obtained by hyperbolic tessellation of a triangle \((\pi/2, \pi/3, \pi/7)\).

The algebraic equation (6.12) should be compared with (1.5), and the fundamental group of the Brieskorn sphere \(\Sigma(2, 3, 7)\) indeed denotes the reflection group of this hyperbolic triangle.

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