Kundt spacetimes

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Abstract

Kundt spacetimes are of great importance in general relativity in four dimensions and have a number of physical applications in higher dimensions in the context of string theory. The degenerate Kundt spacetimes have many special and unique mathematical properties, including their invariant curvature structure and their holonomy structure. We provide a rigorous geometrical kinematical definition of the general Kundt spacetime in four dimensions; essentially a Kundt spacetime is defined as one admitting a null vector that is geodesic, expansion-free, shear-free and twist-free. A Kundt spacetime is said to be degenerate if the preferred kinematic and curvature null frames are all aligned. The degenerate Kundt spacetimes are the only spacetimes in four dimensions that are not \textit{I}-non-degenerate, so that they are not determined by their scalar polynomial curvature invariants. We first discuss the non-aligned Kundt spacetimes, and then turn our attention to the degenerate Kundt spacetimes. The degenerate Kundt spacetimes are classified algebraically by the Riemann tensor and its covariant derivatives in the aligned kinematic frame; as an example, we classify Riemann type D degenerate Kundt spacetimes in which $\nabla(Riem), \nabla^{(2)}(Riem)$ are also of type D. We discuss other local characteristics of the degenerate Kundt spacetimes. Finally, we discuss degenerate Kundt spacetimes in higher dimensions.

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1. Introduction

Kundt spacetimes have very interesting mathematical properties and are of great importance in general relativity in four dimensions and have a number of topical applications in higher
dimensions in the context of string theory\(^3\). The Kundt spacetimes, and especially the degenerate Kundt subclass (to be defined below), have many special and unique mathematical properties. The degenerate Kundt metrics are the only metrics not determined by their scalar curvature invariants [1], and they have extraordinary holonomy structure [2]. In particular, it is not possible to define a unique timelike curvature operator in a degenerate Kundt spacetime and consequently there is no unique, preferred timelike direction associated with its geometry (through the curvature and its covariant derivatives). Therefore, there is no intrinsic 1+3 split of the degenerate Kundt spacetime in general relativity (or a 1 + (n − 1) split in general).

In particular, for any given Lorentzian spacetime \((M, g)\) we denote by, \(I\), the set of all scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives. If there does not exist a metric deformation of \(g\) having the same set of invariants as \(g\), then we call the set of invariants \(I\)-non-degenerate, and the spacetime metric \(g\) is called \(I\)-non-degenerate. This means that for a metric which is \(I\)-non-degenerate, the invariants locally characterize the spacetime uniquely. In [1] it was proven that a four-dimensional (4D) Lorentzian spacetime metric is either \(I\)-non-degenerate or degenerate Kundt. This striking result implies that metrics not determined by their scalar polynomial curvature invariants (at least locally) must be of degenerate Kundt form.

We first present a rigorous geometrical kinematical definition of the general Kundt spacetime in four dimensions, and discuss some of its (local) invariant characteristics. We present some necessary conditions, in terms of the curvature components, for a space to define (locally) a Kundt geometry and discuss the issue of complete gauge fixing. Although section 2.1 is essentially review, a number of new results are presented in sections 2.2 and 2.3.

The 4D Kundt class is defined as those spacetimes admitting a null vector \(\ell\) that is geodesic, expansion-free, shear-free and twist-free [3, 4]; it follows that in 4D there exists a kinematic frame with \(\kappa = \sigma = \rho = \epsilon = 0\). We first discuss the non-aligned (non-degenerate) Kundt spacetimes (and show that this set is non-trivial). We then turn to the degenerate Kundt spacetimes. All of the results presented in sections 3–6 are new.

In a degenerate Kundt spacetime, the kinematical frame and the Riemann \(\nabla^k(\text{Riem})\) type II aligned null frame are all aligned. We algebraically classify the degenerate Kundt spacetimes in terms of their Riemann type in the aligned kinematic frame, or more finely by their Ricci and Weyl types separately. Within each algebraic type, it is also useful to classify the covariant derivative(s) of the Riemann tensor (and particularly \(\nabla(\text{Riem})\) and \(\nabla^2(\text{Riem})\)) in terms of their algebraic types. As an example, we classify \(\nabla(\text{Riem}), \nabla^{(2)}(\text{Riem})\) type D degenerate Kundt spacetimes\(^4\).

We then discuss the Kundt spacetimes and their invariant properties. Degenerate Kundt spacetimes in 4D are not \(I\)-non-degenerate, so that the scalar polynomial curvature invariants do not locally characterize the spacetime uniquely [1]. We discuss the equivalence problem for degenerate Kundt spacetimes, focussing on the special type D subclass, and discuss Kundt spacetimes and scalar invariants.

Finally, we turn to higher dimensions. The (general) higher-dimensional Kundt class is defined in a very similar manner to the 4D class [3]. We review important examples of degenerate Kundt spacetimes in higher dimensions and discuss applications in the context of string theory in sections 7.1 and 7.2. We note that many of the results discussed in this paper can be generalized to higher dimensions; in particular, two important higher-dimensional theorems are presented in section 7.3.

\(^3\) The primary focus of this paper is on the mathematical properties of Kundt spacetimes. The physical motivation for studying these spacetimes, particularly in higher dimensions, has been discussed previously (however, see the discussion and references in section 7.2).

\(^4\) This is the first time the covariant derivative(s) of the Riemann tensor have been explicitly classified algebraically.
Let us remark on the technical assumptions made in this paper. The following theorems hold on neighbourhoods where the Riemann, Weyl and Segre types do not change. In the algebraically special cases we also need to assume that the algebraic type of the higher-derivative curvature tensors also do not change, up to the appropriate order. Most crucial is the definition of the curvature operators and in order for these to be well defined the algebraic properties of the curvature tensors need to remain the same over a neighbourhood. Finally, we note that extensive use is made of the Newman–Penrose (NP) formalism [4, 5] and many calculations are done using GRTensor II5.

2. Kundt geometries in four dimensions

It is the purpose of this section to provide a complete local account of the Kundt geometries in all their generality. The first subsection contains a kinematical definition which implies a proposition which, in turn, implies a lemma. All three can be used to invariantly describe/characterize (locally) a Kundt geometry in kinematical terms. The second subsection gives some necessary conditions, in terms of the curvature components, for a space to define (locally) a Kundt geometry. The third subsection attacks the same problem, but from the point of view of complete gauge fixing.

2.1. Definition, proposition and lemma

The Kundt geometry is defined (cf [4, 6]) as

**Definition 2.1.** A space $S$ defines (locally) a Kundt geometry in four dimensions if there exists (locally) a null congruence of which the tangent vector field is hypersurface orthogonal$^6$ (and therefore geodesic), non-diverging$^7$ and non-shearing.

In the original paper [6], and within the context of the NP formalism, appeal to the energy conditions is made: a space $S$ defines (locally) a Kundt geometry in four dimensions if there exists (locally) a null congruence $\ell (\equiv \ell^a \partial_a)$, of which the tangent vector field is hypersurface orthogonal (and therefore geodesic), non-diverging and obeying the energy condition $T_{ab}l^a l^b \geq 0$; the latter, through the Ricci identities and Einstein’s equations, implies that it will be non-shearing as well$^8$.

By virtue of the Ricci identities one could replace the energy condition by the vanishing of the shear, thus providing a completely kinematic definition (i.e., that given at the beginning). In fact, this attitude is partially adopted in [4] (chapter 31, section 31.2; but immediately after the authors consider further restrictions, like the conditions of the Goldberg–Sachs theorem). Based on the definition (2.1) one can prove the following.

**Proposition 2.2.** A space $S$ defines (locally) a Kundt geometry in four dimensions if and only if there exists (locally) a family of frames $\{e_A\} = \{\ell, n, m, \bar{m}\}$ (i.e., defined up to null rotations about $\ell$) characterized by the conditions that

$$\epsilon = \kappa = \rho = \sigma = 0$$

5 This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World Wide Web from the address [http://grtensor.org](http://grtensor.org).

6 Other terms are also in use such as non-rotating, normal, non-twisting—although the latter is reserved for the null congruences only.

7 Or non-expanding.

8 A null congruence of which the tangent vector field is hypersurface orthogonal (and therefore geodesic), and non-shearing defines a ray congruence—see [6] and the references therein.
for the spin connection coefficients. Also, the generic line element is given, in a local coordinate system \( \{x^a\} = \{u, v, \xi, \bar{\xi}\} \) (coordinates \( u \) and \( v \) are real, while coordinate \( \xi \) is complex):

\[
\mathrm{d}s^2 = 2\mathrm{d}u(H\,\mathrm{d}u + \mathrm{d}v + W\,\mathrm{d}\xi + \bar{W}\,\mathrm{d}\bar{\xi}) - 2P^{-2}\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}, \quad \partial_i P = 0 \tag{1}
\]

**Proof.** Let \( S \) be a space which defines (locally) a Kundt geometry. The optimal way towards the proof of the assertion would perhaps be provided by the observation that the NP equations are invariant under arbitrary frame transformations (for the case under consideration, Lorentz transformations). Therefore, this freedom should be implemented in order to gauge fix (even partially) the frame relative to the geometric structure(s) assumed for the space.

Indeed, let \( \ell' \) denoting the vector field which is tangent to the congruence under consideration. The effect of a boost transformation \( [7] \):

\[
\ell' \rightarrow \ell, \quad \ell = z^{-1}\ell'
\]

with a proper value for the complex parameter \( z \), can make the vector field \( \ell \) to be not only a geodesic (as a consequence of being both null and normal) but also affinely parametrized; i.e., \( \kappa = 0 \), and \( \epsilon + \bar{\epsilon} = 0 \), respectively. In this gauge, the properties for being normal, non-diverging and non-shearing correspond to the vanishing of \( \rho - \bar{\rho}, \rho + \bar{\rho} \) and \( \sigma \), respectively (see mainly \([8]\), but also \([5, 9]\) as well). Therefore,

\[
\begin{align*}
\epsilon & = 0, \quad \kappa = 0, \quad \rho = 0, \\
\sigma & = 0. \tag{2a}
\end{align*}
\]

Since \( \ell \) is normal, the corresponding covector (also denoted by \( \ell \)) complies with all the conditions of Frobenius’ theorem. The latter guarantees (at least locally) that there exist two non-constant, non-zero, real functions \( F \), and \( u' \) such that \( \ell = F\,\mathrm{d}u' \). Another non-constant, non-zero, real function \( u \) can be defined such that \( \mathrm{d}u = F\,\mathrm{d}u' \). In addition, if \( v \) denotes an affine parameter, then a local coordinate system \( \{y^a\} = \{u, v, \chi^1, \chi^2\} \) can be adopted, where all coordinates are real, such that \( \ell = \partial_u \) or, alternatively, \( l^a \equiv \mathrm{d}y^a/\mathrm{d}v = (0, 1, 0, 0) \).

Then

\[
\begin{align*}
\ell &= l^a\partial_a, \\
l^a &= (0, 1, 0, 0),
\end{align*}
\]

(\( \text{where } A \in \{1, 2\} \)) by completing the frame.

In this coordinate system, the line element assumes the form,

\[
\mathrm{d}s^2 = 2\mathrm{d}u((N_0 - M_0M_\bar{A})\,\mathrm{d}u + \mathrm{d}v + (N_A - M_0M_A - M_\bar{A}\木M_{\bar{A}})\,\mathrm{d}\chi^A) - 2M_A\overline{M_B}\,\mathrm{d}\chi^A\,\mathrm{d}\chi^B. \tag{3a}
\]

But, \( \sigma = -i\ast(\ell \wedge m \wedge \mathrm{d}m) \), and \( \rho = \frac{i}{2} \ast(\ell \wedge (\overline{m} \wedge \mathrm{d}m - m \wedge \overline{\mathrm{d}m} - n \wedge \overline{\mathrm{d}l})) \) (the \( \ast \) denotes the Hodge dual) as the first Cartan structure equations imply. The implication of the vanishing of these two spin connection coefficients results in \( \partial_A M_A = 0 \). A general coordinate transformation, which as such leaves all the NP quantities invariant, of the form:

\[
[u, v, \chi^A] \rightarrow [u, v, \psi^A] \equiv [u, v, \xi, \overline{\xi}], \quad \xi = \frac{1}{\sqrt{2}}(f^1(u, \chi^A) + i f^2(u, \chi^A))
\]

where \( f^A \) are real functions, renders the two-dimensional part of the previous line element, \( M_A\overline{M_B}\,\mathrm{d}\chi^A\,\mathrm{d}\chi^B \) apparently conformally flat\(^9\); say \( P^{-2}\,\mathrm{d}\xi\,\mathrm{d}\overline{\xi}, \) with \( \partial_i P = 0 \).

---

\(^9\) Because, any two-dimensional metric is conformally flat.
Hence, in the local coordinate system \( \{ x^a \} = \{ u, v, \zeta, \bar{\zeta} \} \), the line element (4) assumes the form:

\[
d s^2 = 2 du (H du + dv + W d\zeta + \bar{W} d\bar{\zeta}) - 2P^{-3} d\zeta d\bar{\zeta}, \quad \partial \mu P = 0, \tag{5}
\]

where the coordinates \( u \) and \( v \) and the functions \( H \) and \( P \) are real, while the coordinate \( \zeta \) and the function \( W \) are complex.

Finally, the effect of a spin transformation [7], which preserves (2), can be used to set the imaginary \( \epsilon \) equal to zero. Then, the conditions:

\[
\begin{align*}
\epsilon &= 0, \quad (6a) \\
\kappa &= 0, \quad (6b) \\
\rho &= 0, \quad (6c) \\
\sigma &= 0 \quad (6d)
\end{align*}
\]

are invariant under null rotations about the \( \ell \) vector field, defining thus a family of frames. This completes the first logical direction of the proposition. The other direction is straightforward to prove. \( \square \)

Based on proposition (2.2), it is possible to prove the following.

**Lemma 2.3.** A space \( S \) defines (locally) a Kundt geometry in four dimensions if and only if there exists (locally) a family of frames \( \{ e_A \} = \{ \ell, n, m, \bar{m} \} \) (i.e., defined up to null rotations about \( \ell \) and spin-boosts) characterized by the conditions that

\[
\kappa = \rho = \sigma = 0
\]

for the spin connection coefficients.

**Proof.** Indeed, a spin-boost transformation, which preserves the vanishing of the coefficients \( \kappa, \rho, \sigma \), can be used to convert a non-vanishing \( \epsilon \) to a vanishing quantity and vice versa. \( \square \)

### 2.2. Necessary conditions

Based on the proposition and the lemma, in principle we can deduce some necessary conditions which must be met in order for a general spacetime, given in an arbitrary frame, to (locally) define a Kundt geometry.

Indeed, let \( S \) be a general space defined locally in terms of the completely arbitrary local frame \( \{ e_A \} = \{ \ell, n, m, \bar{m} \} \). Then, according to the proofs of proposition 2.1 and lemma 2.1, it is easily inferred that \( S \) defines (locally) a Kundt geometry if and only if a null rotation about the \( n \) (co)vector (with a proper complex parameter \( z \)) suffices to make the three connection coefficients \( \kappa', \rho' \) and \( \sigma' \) in the primed frame vanish. However, in that frame these conditions, by virtue of the NP equations, correspond to the following curvature conditions (cf the following subsection):

\[
\begin{align*}
\Psi'_0 &= \Phi'_{00} = 0, \quad (7a) \\
\Psi'_i &= \Phi'_{0i} \quad (7b)
\end{align*}
\]

An observation is pertinent at this point. These are not all the curvature conditions implied by the vanishing of the spin connection coefficients; in general, we need to completely gauge fix the frame in order for all the curvature conditions to emerge. Then, we would have to establish
the one-to-one correspondence amongst the spin connection coefficient conditions and the ensuing curvature conditions. The equations above comprise just a subset of the totality of the curvature conditions. Moreover, the previously mentioned one-to-one correspondence has not been established. From this point of view, these conditions are simply necessary but not sufficient.

Using the well-known transformation laws of the NP variables under the action of the Lorentz group, and specifically here the null rotations about the (co)vector $n$, conditions (7) assume the following form (in the initial, arbitrary frame):

\[
\Psi_0 + 4z\Psi_1 + 6z^2\Psi_2 + 4z^3\Psi_3 + z^4\Psi_4 = 0, \hspace{1cm} (8a)
\]
\[
\Phi_{00} + 2z\Phi_{10} + 2z\Phi_{10} + 4z\tilde{\Phi}_{11} + z^2\Phi_{02} + z^2\Phi_{20} + 2z^2\tilde{\Phi}_{21} + 2z^2z\Phi_{12} + z^2z\tilde{\Phi}_{22} = 0, \hspace{1cm} (8b)
\]
\[
\Psi_1 + 3z\Psi_2 + 3z^2\Psi_3 + z^3\Psi_4 = \Phi_{01} + 2z\Phi_{11} + z\Phi_{02} + 2z\tilde{\Phi}_{12} + z^2\Phi_{12} + z^2\tilde{\Phi}_{22}. \hspace{1cm} (8c)
\]

The objective here is to eliminate the Lorentz parameter $z$ from this system, but the complexity of the general system renders this goal unrealistic. However, an algorithmic procedure could be established: in the initial, arbitrary frame, one tries to solve algebraically one of the first two equations (the choice has to do with the ‘extremal’ cases of conformally or Ricci flat spaces). As the fundamental theorem of algebra states, any of these two is susceptible to four solutions—say \{\(z_1, z_2, z_3, z_4\)\}; the degeneracy of this set reflects the Petrov or the Petrov–Plebański type, respectively. Suppose that one chooses to solve, for example, the first equation. If the result is a set of four independent solutions (i.e., Petrov type I), then the second and the third equations must be satisfied separately for each of these four solutions. Therefore, in that case eight independent (in general) syzygies amongst the components of the Weyl and Ricci tensors will emerge. If the result is three independent solutions (i.e., Petrov type II), then there emerges six independent syzygies (in general), etc.

Of course, the entire analysis above comprises one of the two possibilities, which corresponds to the alignment of the generic frame in such a way that the $\ell'$ vector is aligned with the characteristic Kundt vector. The other possibility concerns the alignment of the generic frame in such a way that the $n'$ vector is aligned with the characteristic Kundt vector. In order to get the set of necessary conditions for the latter case, one has simply to use the $\circ$ operation which interchanges the role of the $\ell$ and $n$ (co)vectors:

$$\circ : \ell \longleftrightarrow n$$

along with changes like $\Psi_0 \longleftrightarrow \Psi_4$, etc (see, the standard literature on the NP formalism).

### 2.3. Complete gauge fixing and necessary curvature conditions

It is expected that the special features which define, in kinematical terms, a Kundt geometry will impose some constraints upon the curvature components; i.e., not all the components of the Riemann tensor (and the components of its covariant derivatives) will be independent. A standard approach towards these conditions is to implement the entire Lorentz freedom in order to completely gauge fix the frame relative to the geometric structure(s) assumed for the space. Of course, there is no ‘recipe’ for finding the optimal gauge fixing; i.e., that fixing which will exhibit as many characteristics as possible. However, it is possible to postulate some criteria. For instance, a gauge which renders many of the NP equations purely algebraic is better (more useful) than another one which will give fewer algebraic equations. Thus, a suggested way is the following.

Let \{\(e^m_A\) \} = \{\(\ell'^m, n'^m, m'^m, \bar{m}'^m\)\} be a family of local frames, such that the characteristic vector field (in the definition of the Kundt geometries) is a linear combination of the frame...
vectors. First, a null rotation about $n''''$ can be used to align the vector $\ell''''$ with the characteristic vector field. Then, the covector $\ell''''$ satisfies the condition $\ell'''' \wedge d\ell'''' = 0 \iff \ell'''' = F du$ locally, for some non-constant, non-zero, real functions $F, u$ (Poincaré's lemma). Second, a boost in the $\ell'''' - n''''$ plane can be used to affinely parametrize the vector field $\ell'': \ell'' = du$. This in turn implies

$$\kappa'' = 0, \quad (10a)$$
$$\epsilon'' + \check{\epsilon}'' = 0, \quad (10b)$$
$$\rho'' - \check{\rho}'' = 0, \quad (10c)$$
$$\tau'' - \check{\tau}'' - \beta'' = 0 \quad (10d)$$

by virtue of $d\ell'' = 0$. The properties of being geodesic, affinely parametrized, normal, non-diverging and non-shearing correspond to the conditions:

$$\kappa'' = 0, \quad (11a)$$
$$\epsilon'' + \check{\epsilon}'' = 0, \quad (11b)$$
$$\rho'' = 0, \quad (11c)$$
$$\sigma'' = 0. \quad (11d)$$

Third, a spin in the $m'''' - \bar{m}''''$ plane, which preserves all the previous restrictions, can be used to set

$$\epsilon' = 0. \quad (12)$$

Finally, a null rotation about $\ell'$, preserving again all the previous restrictions, is implemented to set

$$\gamma = 0. \quad (13)$$

Now, the completely gauged system reads

$$\epsilon = 0, \quad (14a)$$
$$\gamma = 0, \quad (14b)$$
$$\kappa = 0, \quad (14c)$$
$$\rho = 0, \quad (14d)$$
$$\sigma = 0, \quad (14e)$$
$$\alpha = \tau - \bar{\tau}. \quad (14f)$$

Substitution of (14) into the system of the NP and eliminant equations (cf [9]) plus some algebraic manipulation result in the following curvature conditions of zero order:

$$\Psi_0 = 0, \quad (15a)$$
$$\Phi_{00} = 0, \quad (15b)$$
$$\Phi_{01} = \Psi_1 = 0, \quad (15c)$$
$$\Lambda - \beta \pi - \tau \check{\pi} + \check{\tau} \beta - \beta \tau - \tau \check{\tau} + \check{\beta} \pi - \check{\tau} \pi - \Phi_{11} - \Psi_2 = 0. \quad (15d)$$
2\Lambda - 2\beta \tau - 2\tau \beta + 2\tau \pi - \Phi_{02} + \Psi_2 = 0, \quad (15e)
-2\Lambda - 2\beta \pi + 2\tau \beta - 2\pi \tau + 2\beta \pi + \Phi_{02} - \Psi_2 = 0. \quad (15f)

The presence of the spin connection coefficients \( \beta, \tau, \pi \), should not concern us since in a completely gauged system everything is invariant. So, all +2 boost weight terms vanish, there is only one independent +1 boost weight term, while not all 0 boost weight terms are independent. Unfortunately, it seems that an elimination of the spin connection coefficients, which would lead to gauge independent syzygies, is not possible.

3. Kundt spacetimes

The 4D Kundt spacetimes are those spacetimes admitting a null vector \( \ell \) that is geodesic, expansion-free, shear-free and twist-free [3, 4]. Since \( \ell \) is geodesic, without loss of generality we can always choose a complete NP tetrad with \( \ell \) (so that the NP spin coefficient \( \kappa = 0 \)) in which \( \epsilon = 0 \) and hence the geodesic \( \ell \) is affinely parametrized. Demanding that \( \ell \) is expansion-free, shear-free and twist-free in this frame, then implies that \( \sigma = \rho = 0 \). Hence in the canonical Kundt frame, or kinematic frame, \( \kappa = \sigma = \rho = \epsilon = 0 \). \( \text{^{10}} \) It follows that in 4D there exists a kinematic frame with

\[ \kappa = \sigma = \rho = \epsilon = 0, \quad (16) \]

in which the (general) Kundt metric can always be written as

\[ d\tilde{x}^2 = 2 du[H(v, u, x^k) du + W_i(v, u, x^k) dx^i] + g_{ij}(u, x^k) dx^i \cdot dx^j. \quad (17) \]

The metric functions \( H, W_i \) and \( g_{ij} \) \( (i, j = 2, 3) \) are real and will satisfy additional conditions if the Einstein field equations are applied \( \text{^{11}} \).

We note that in the above kinematic frame, the positive boost weight +2 terms of the Riemann tensor are automatically zero. However, in the above kinematic frame, if \( W_{vv} \neq 0 \), then the Ricci tensor (and Riemann tensor) have positive boost weight terms, and if \( H_{vv} \neq 0 \) the covariant derivatives of the Riemann tensor have positive boost weight terms.

Higher dimensions. The (general) higher-dimensional Kundt class is defined in a very similar manner (and is consistent with the definition of generalized Kundt spacetimes in \( n \)-dimensions given, for example, in [3]); namely, the spacetimes admit a null vector \( \ell \) which satisfies \( \ell_{a:b} \ell^b \propto \ell_a \) (which is zero for an affine parametrization), \( \ell^a_{,a} = 0, \ell_{[a:b]} \ell^a_{[b]} = 0 \) and \( \ell_{[a:b]} \ell^a_{[b]} = 0 \), so that there exists a kinematic frame in which \( \ell \) is geodesic, expansion-free, shear-free and twist-free. In particular, the \( n \)-dimensional Kundt metric can be written as in (17), where now \( i, j = 2, \ldots, n - 1 \). In the kinematic Kundt null frame

\[ \ell = du, \quad n = dv + H \, du + W'_i \, dx^i, \quad m'_j = m''_{ij} \, dx^j, \quad (18) \]

such that \( g_{ij} = m^k_{ij} m_{kj} \) and where \( m''_{ij} \) can be chosen to be in upper triangular form by an appropriate choice of frame. The metric (17) possesses a null vector field \( \ell = \frac{d}{d u} \) which is geodesic, non-expanding, shear-free and non-twisting; i.e., \( \ell_{i;j} = 0 \) [3]. Since \( \ell \) is affinely parametrized, the Ricci rotation coefficients in the kinematic frame are thus given by

\[ \ell_{[a:b]} = L_a \ell_b + L_b \ell_a (\ell_{a} m''_{b} + \ell_{b} m''_{a}), \quad (19) \]

\( \text{^{10}} \) For \( \ell \) to be affinely parametrized it is only necessary for \( \epsilon + \tau = 0 \). However, since all of the calculations that follow are done in the frame (gauge) in which \( \epsilon = 0 \) for convenience (it is always possible to choose such a gauge without loss of generality), and this choice of gauge does not affect the definitions of algebraic types (of, for example, the Riemann tensor), for simplicity of presentation we have defined the kinematic frame so that \( \epsilon = 0 \) in addition to \( \kappa = \sigma = \rho = 0 \).

\( \text{^{11}} \) It is this real form for the Kundt metric which is more appropriate for generalizing to higher dimensions.
3.1. Non-aligned Kundt metrics

Consider (17) with \( g_{ij}(\alpha, x^k) = \delta_{ij} \) and define the standard Newman–Penrose (NP) tetrad for the Kundt metric having an \( \ell \) with \( \kappa = \sigma = \rho = \epsilon = 0 \) (i.e., the kinematic frame). It immediately follows from the NP equations that in the kinematic frame

\[
\Psi_0 = \Phi_{00} = 0, \quad \Psi_1 = \Phi_{01}, \tag{20}
\]

Moreover, the two complex Weyl invariants \( I \) and \( J \) are nonzero, and \( I^3 - 27J^2 \neq 0 \). Therefore (17) is Petrov type I (and hence from (20) also Riemann type I).

We shall now consider the possibility that the Weyl tensor is Petrov type II with a Weyl aligned null direction (WAND) \( k \) not equal to \( \ell \). Clearly, if \( \Psi_1 = 0 \) then \( k = \ell \) and the Petrov type II Weyl aligned frame coincides with the kinematic frame; henceforth, we assume \( \Psi_1 \neq 0 \). The following Lorentz transformations are applied successively in order to normalize the Weyl tensor and determine conditions so that \( k \) is a Petrov type II WAND.

1. Null rotations about \( \ell \) (c; \( \Psi_A^{(3)} \));
2. Spin-boost \((a, \theta; \Psi_A^{(2)})\);
3. Null rotations about \( n \) \((b; \Psi_A^{(3)})\)

(where in (·) we give the corresponding Lorentz parameters and denote the Weyl scalars \( \Psi_A^{(i)}, A = 0, \ldots, 4 \), resulting from transformation (·)). By applying (1) \( c \) can always be chosen such that

\[
\Psi_4^{(1)} = \Psi_4 + 4c\Psi_5 + 6c^2\Psi_2 + 4c^3\Psi_1 = 0. \tag{21}
\]

Next, apply (2) so that \( \Psi_1^{(2)} = 1 \). Since the kinematic frame is invariant under (1) and (2) we still have \( \ell^{(2)} \) with \( \kappa^{(2)} = \sigma^{(2)} = \rho^{(2)} = 0 \). Lastly, apply (3) and requiring that \( \ell^{(3)} = k \) is a Petrov type II WAND gives

\[
\Psi_0^{(3)} = 2b[2 + 3b\Psi_2^{(2)]} + 2b^2\Psi_3^{(2)}] = 0, \tag{22}
\]
\[
\Psi_1^{(3)} = 1 + 3b\Psi_2^{(2)} + 3b^2\Psi_3^{(2)} = 0, \tag{23}
\]
\[
\Psi_2^{(3)} = \Psi_2^{(2)}, \tag{24}
\]
\[
\Psi_3^{(3)} = \Psi_3^{(2)}, \tag{25}
\]
\[
\Psi_4^{(3)} = 0. \tag{26}
\]

Note, \( \Psi_2^{(2)} \neq 0 \) otherwise equations (22) and (23) give a contradiction. Solving \( \Psi_0^{(3)} = \Psi_1^{(3)} = 0 \) gives that the null rotation parameter, \( b \), is the root of \( 3b\Psi_2^{(2)} + 4 = 0 \) subject to the constraint \( 9\Psi_2^{(2)} = 16\Psi_3^{(2)} \). Therefore, if there exists a Petrov type II WAND \( k = \ell^{(3)} = \ell \) then the nonzero Weyl scalars are

\[
\Psi_2^{(3)} = -\frac{1}{2}\Psi_2^{(2)}, \quad \Psi_3^{(3)} = \Psi_3^{(2)}. \tag{27}
\]

and satisfy the following condition in the Weyl aligned frame:

\[
9\Psi_2^{(3)} = 4\Psi_3^{(3)}. \tag{28}
\]

Transforming (28) to the kinematic frame gives

\[
16\Psi_1\Psi_3 - 9\Psi_2^2 + 12c\Psi_1\Psi_2 + 12c^2\Psi_1^2 = 0. \tag{29}
\]

Since \( c \) is a root of (21), we obtain the necessary condition

\[
9\Psi_2^2 - 16\Psi_1\Psi_2\Psi_3 + 6\Psi_1^2\Psi_4 + 2c\Psi_1[3\Psi_2^2 - 4\Psi_1\Psi_3] = 0, \tag{30}
\]

which implies the following two cases.
Case 1: $3 \Psi_2^2 - 4 \Psi_4 \Psi_3 = 0$. From (29) it is easily shown that

$$\Psi_4^{(1)} = \Psi_3 + 3c\Psi_2 + 3c^2 \Psi_1 = 0.$$  \hfill (31)

In this case, equation (30) reduces to the constraint

$$2 \Psi_2^2 \Psi_4 - \Psi_2^2 = 0,$$ \hfill (32)

and since $\Psi_2^{(1)} = 0$ we obtain the null rotation parameter, $c = -\Psi_2/(2\Psi_4)$. Therefore, $\Psi_2^{(1)} = 0$ with $\Psi_4^{(1)}$ as the only remaining nonzero Weyl scalar; thus $k = n^{(1)}$ is the WAND for a Petrov type III Weyl tensor in the kinematic frame $\ell^{(1)} = \ell$.

Case 2: $3 \Psi_2^2 - 4 \Psi_4 \Psi_3 \neq 0$. The null rotation parameter, $c$, is the root of (30) which upon substitution into (21) and (29) results in

$$F := 27 \Psi_2^4 - 108 \Psi_1 \Psi_2 \Psi_3 \Psi_4 + 64 \Psi_1 \Psi_3^2 + 54 \Psi_2 \Psi_4^2 - 36 \Psi_2^2 \Psi_2^2 = 0.$$ \hfill (33)

If there exists a Petrov type II WAND $k = \ell^{(3)} \neq \ell$, then in the kinematic frame the Weyl scalars satisfy (33) which is equivalent to the Weyl aligned frame condition (28). Since there was no remaining freedom left at equation (28) it must invariantly define Petrov type II; a similar conclusion holds for $F = 0$. More precisely, in the kinematic frame we have a factorization of the invariant

$$I^3 - 27J^2 = -\Psi_1 F = 0,$$ \hfill (34)

whose vanishing is the well-known result that the Weyl tensor is Petrov type II or D (assuming $I$ and $J$ are nonzero). By assuming $\Psi_1 \neq 0$, it was possible to normalize $\Psi_1^{(2)}$ in transformation 2). However, it is known [4] that in an NP tetrad where $\Psi_0 = \Psi_1 = 0$ the Weyl tensor is Petrov type D if and only if $3 \Psi_2 \Psi_4 = 2 \Psi_2^2$ is satisfied. Evidently, assuming $\Psi_1 \neq 0$ implicitly excludes Petrov type D.

In addition to a Weyl tensor of Petrov type II we also require Riemann type II, which implies that the Ricci tensor is PP-type II or less and the Weyl and Ricci aligned frames coincide but differ from the kinematic frame. An invariant characterization of a PP-type II Ricci is given by the following syzygy:

$$r_2^2 \left( 4r_1^3 - 6r_1 r_3 + r_2^2 \right) - r_3^2 \left( 3r_1^2 - 4r_3 \right) = 0,$$ \hfill (35)

which is expressed in terms of the Carminati–Zakhary (CZ) Ricci invariants [11] (see definitions later). Thus if Riemann type II, then in the kinematic frame (34) and (35) are satisfied and the alignment of the Weyl and Ricci frames will impose further constraints relating the Weyl and Ricci scalars through syzygies among the mixed invariants. A solution with the property that the Riemann type II aligned frame does not coincide with the kinematic frame can be found by noting that $\Psi_4 \neq 0$ implies $\Phi_{01} \neq 0$; we can thus regard (34) and (35) as polynomials in $\Psi_1, \Phi_{01}$ and $\Phi_{10}$ with constraints given by the vanishing of their coefficients. We obtain $\Psi_4 = \Psi_3 = 0$ and $\Phi_{12} = \Phi_{22} = 0$ from equations (34) and (35), respectively. Consequently, the Riemann tensor has vanishing negative boost weight components and $\mathbf{n}$ is the Riemann aligned null direction in the kinematic frame. If the null congruence defined by $\mathbf{n}$ is geodesic, expansion-free, shear-free and twist-free, then by applying null rotations in the following order (3) $b = i$, (2) $c = -i$ and (3) $b = i$ results in an NP tetrad $\ell = n, \ell' = \ell, \ell'' = \ell, \ell''' = \ell$. Thereby interchange $\ell$ and $\mathbf{n}$ the Riemann tensor has vanishing positive boost weight components which gives a Riemann type II aligned null direction in the kinematic frame defined by $\ell'$. To avoid this possibility, we require that at least one of the kinematic scalars associated with the null congruence defined by $\mathbf{n}$ is...
non-vanishing. \( \mathbf{n} \) is geodesic if \( \nu = 0 \) and affinely parametrized if \( \gamma + \overline{\gamma} = 0 \), shear-free if \( \lambda = 0 \), expansion-free and twist-free if the real and imaginary parts of \( \mu \) vanish, respectively. Therefore, \( \nu, \lambda, \mu \) are the analogues of \( \kappa, \sigma, \rho \) for \( \mathbf{n} \). In section 3.1.1 we provide an example of this type of solution.

3.1.1. Subcases. There are a number of algebraically special (Riemann) subcases. First, there is the algebraically special Riemann type II subcase, in which there is a frame in which all positive boost weight components of the Riemann tensor are zero. Second, there is the aligned subcase (of this subcase) in which the frame in which positive boost weight terms are zero and the kinematic frame are aligned. This is a distinct subcase to the first subcase because there exist algebraically special Riemann type II Kundt spacetimes which are not aligned (and occurs when the metric function \( W,vv = 0 \) in the kinematic frame). For example, suppose that \( \mathbf{k} \) is a shear-free, geodesic Riemann type I null vector, and that \( \mathbf{n} \) is an aligned Riemann type II null vector. Suppose that \( \mathbf{k} \) and \( \mathbf{n} \) are not aligned (i.e., subcase (1) but not subcase (2)). Choose a null frame \((\mathbf{k}, \mathbf{n}, \mathbf{m}_1, \mathbf{m}_2)\); in this frame the Riemann tensor has no boost weight +2 and no boost weight \(-1\) and \(-2\) terms. It is possible to show there are non-trivial spacetimes in this class (satisfying the Bianchi identities, etc), in which boost weight +1 terms are non-zero (and the zero boost weight terms are non-zero) and \( \mathbf{n} \) is not geodesic, expansion-free, shear-free and twist-free. Since the components of the Ricci tensor have positive boost weight terms, this does not violate the Goldberg–Sachs theorem [4].

3.1.2. Example. We now give an example where the Riemann type II aligned frame does not coincide with the kinematic frame, in the sense that \( \mathbf{n} \) is the Riemann-aligned null direction having non-vanishing expansion and shear. We define the following NP tetrad for the Kundt metric:

\[
\ell = du, \quad \mathbf{n} = \left[ H + \frac{1}{2} (W_1^2 + W_2^2) \right] du + dv, \quad \mathbf{m} = \frac{1}{\sqrt{2}} \left[ (W_1 - iW_2) du - dx + i dy \right]
\]

(36)

(where \( g_{ij} = \delta_{ij} \) in this example) with metric functions

\[
H = -\frac{1}{8} J^2 v^4 - \frac{1}{2} J F_1 v^3 - \frac{1}{2} \left[ (F_2 x + F_3) J + F_1^2 \right] v^2 - \left[ (F_2 x + F_3) F_1 + F_2 + \frac{F_2^2}{F_2} \right] v \\
- \frac{1}{2} \left[ (F_2 x + F_3)^2 + \frac{F_2^2}{J} \right] + \frac{1}{J} [F_1^2 + 2 F_1 F_2],
\]

(37)

\[
W_1 = \frac{1}{2} J v^2 + F_1 v + F_2 x + F_3, \quad W_2 = F_4
\]

where

\[
J(u) = \exp \left[ - \int \left( \frac{F_2}{F_2} + 3 F_2 \right) du \right]
\]

(38)

and the arbitrary functions \( F_1, \ldots, F_4 \) only depend on \( u \). (We note that coordinate transformations can still be used to simplify these metric functions). \( \ell \) has \( \kappa = \sigma = \rho = 0 \) and the only non-vanishing curvature scalars are \( \Psi_1, \Psi_2, \Phi_{01}, \Phi_{02}, \Phi_{11} \) and \( \Lambda \) (all negative boost weight components vanish)—therefore (34) and (35) are satisfied. Moreover, \( \nu = 0 \). However, \( \gamma + \overline{\gamma} \neq 0 \), so that \( \mathbf{n} \) is geodesic but not affinely parametrized. By performing a boost \( \ell \to A \ell, \mathbf{n} \to A^{-1} \mathbf{n} \) where

\[
A(u) = J^{-1} \exp \left( -2 \int F_2 du \right),
\]

(39)
we can affinely parametrize n (while also maintaining ℓ affinely parametrized). As a result we find that n is geodesic and twist-free; however, λ + λ and μ + μ are non-zero so that it is expanding and shearing. We have shown that there exists a Riemann type II aligned frame, with aligned null direction n, that differs from the kinematic frame determined by ℓ.

Note that there are further subcases depending upon whether ∇(Riem) is algebraically special and aligned. We shall discuss this in detail in the following section. Essentially, for the ‘zeroth order’ degenerate Kundt spacetimes (with \( W_{vv} = 0 \)), in the kinematic frame \( \nabla(\text{Riem}) \) can have positive boost weight terms. In particular, the boost weight +1 component \( R_{0101} \sim D/\Phi_1 \) \( 1 \sim H_{vv} \) is non-zero in general. However, if \( H_{vv} = 0 \), \( \nabla(\text{Riem}) \) has no positive boost weight terms in the kinematic frame and the Kundt spacetime is ‘first-order’ degenerate.

4. Degenerate Kundt spacetimes

In the aligned-algebraically special Riemann type II subcase the boost weight +1 terms of the Riemann tensor are also zero (in the kinematic frame). In the analysis of [1] it was found that a Kundt metric is \( I \)-non-degenerate if the metric functions \( W(v, u, x) \) and \( H(v, u, x) \) in the kinematic Kundt null frame satisfy \( W_{vv} \neq 0 \) and \( H_{vv} \neq 0 \). Hence the exceptional spacetimes are the aligned algebraically special Riemann type II Kundt spacetimes and, from the fact that \( H_{vv} = 0 \), in this frame \( \nabla(\text{Riem}) \) (and, as we shall show later, all covariant derivatives) do not have any positive boost weight terms. In short (and consistent with the terminology of the above theorem), we shall call such spacetimes degenerate Kundt spacetimes, in which there exists a common null frame in which the geodesic, expansion-free, shear-free and twist-free null vector \( \ell \) is also the null vector in which all positive boost weight terms of the Riemann tensor and its covariant derivatives are zero (i.e., the kinematic Kundt frame and the Riemann type II aligned null frame are all aligned). The degenerate Kundt spacetimes are the only spacetimes in 4D that are not \( I \)-non-degenerate, and their metrics are the only metrics not determined by their curvature invariants [1]. We note that the important CSI and VSI spacetimes are degenerate Kundt spacetimes. We also note that the degenerate Kundt spacetimes are the original 4D spacetimes satisfying the energy conditions and for particular physically motivated energy–momentum tensor (e.g., Einstein–Maxwell fields, with a restricted Ricci type satisfying \( W_{vv} = 0 \) and \( H_{vv} = 0 \)) [4] studied by Kundt [6].

4.1. Analysis

Consider the 4D Kundt metric (17) and a kinematically aligned NP tetrad, so that \( \ell = du \) has \( \kappa = \sigma = \rho = \epsilon = 0 \). In order to delineate among the Kundt metrics those that are \( I \)-non-degenerate we make the following distinction:

**Definition 4.1.** Let \( K_n \) denote the subclass of Kundt metrics such that there exists a kinematic frame in which Riemann up to and including its nth covariant derivative have vanishing positive boost weight components. We call \( K_n \) the nth-order degenerate Kundt class.

Therefore, for every metric in \( K_n \) there exists an NP tetrad in which \( \kappa = \sigma = \rho = \epsilon = 0 \) and \( R_{abcd}, \ldots, \nabla^{(n)} R_{abcd} \) are type II or less. In general the Kundt metric (17) is Riemann type I; imposing \( K_0 \) is equivalent to \( \Psi_1 = \Phi_0 = 0 \), which in terms of metric functions gives \( W_{vv} = 0 \). It now follows that \( K_0 \) does not imply \( K_1 \) since \( \nabla R_{abcd} \) has a boost weight +1 component linear in the scalars \( D\Psi_2, D\Phi_1 \) and \( D\Lambda \). Requiring \( K_1 \) is equivalent to \( H_{vv} = 0 \) (in addition to \( W_{vv} = 0 \)) and consequently \( D\Psi_2 = D\Phi_1 = D\Lambda = 0 \). A direct calculation
now shows that $K_1$ implies $K_2$, leading us to the following result (the converse is trivial by definition of $K_n$)\textsuperscript{12}.

**Theorem 4.2.** In the Kundt class, $K_1$ implies $K_n$ for all $n \geq 2$.

**Proof.** In four dimensions the transverse metric of \textsuperscript{(17)} is conformally flat so that $g_{ij} \, dx^i \, dx^j = -P^{-2}(dx^2 + dy^2)$, where $P = P(u, x, y)$ \textsuperscript{[4]}. We choose an NP tetrad of the form

$$
\ell = du, \quad n = \left[ H + \frac{p^2}{2}(W_1^2 + W_2^2) \right] du + dv,
$$

(40)

and impose $K_1$ to find that the boost weight 0 components of $R_{abcd}$ all contain $W_i$ with at least one derivative with respect to $v$, and $H$ with at least two derivatives with respect to $v$. In addition, the boost weight $-1$ components have $H$ appearing with at least one derivative with respect to $v$. It now follows by calculation that $R_{abcd}$ satisfies $K_2$ (positive boost weight components vanish) and its boost weight 0 and $-1$ components have the same dependence as $R_{abcd}$ with respect to the derivatives of $W$ and $H$ with respect to $v$. Proceeding by induction, we consider the $n$th covariant derivative of Riemann and denote its components of boost weight $b$ by $(\nabla^n R)_b$. Assume that $K_n$ holds with the property that the occurrence of $W_i$ and $H$ in the boost weight 0 components, $(\nabla^n R)_0$, only depends\textsuperscript{13} on $W_{i, v}$ and $H_{i, v}$ (where $\cdots$ refers to derivatives with respect to $u$, $x$, or $y$) and $(\nabla^n R)_{-1}$ components have a dependence on $H_{i, v}$ but not $H$.

Before considering the $n+1$ derivative of Riemann, we note that since $\kappa = \sigma = \rho = \epsilon = 0$ for (40) then the covariant derivatives of the NP tetrad, sorted according to decreasing boost weights, are

$$
\ell_{a, c} = -\left( (\bar{\alpha} + \bar{\beta})\ell_{c} - (\alpha + \beta)\ell_{a} \right) \ell_{c} - (\alpha + \beta)\ell_{a} \ell_{c} + (\gamma + \bar{\gamma})\ell_{a} \ell_{c},
$$

(41)

$$
n_{a, c} = \left( (\alpha + \beta)\ell_{c} + (\bar{\alpha} + \bar{\beta})n_{a} \ell_{c} - (\alpha + \beta)n_{a} \ell_{c} - (\gamma + \bar{\gamma})n_{a} \ell_{c} - (\alpha + \beta)n_{a} \ell_{c} - (\gamma + \bar{\gamma})n_{a} \ell_{c} \right),
$$

(42)

$$
m_{a, c} = \left( (\alpha + \beta)\ell_{c} + (\bar{\alpha} + \bar{\beta})m_{a} \ell_{c} - (\alpha + \beta)m_{a} \ell_{c} + (\gamma + \bar{\gamma})m_{a} \ell_{c} + (\alpha + \beta)m_{a} \ell_{c} - (\gamma + \bar{\gamma})m_{a} \ell_{c} - (\alpha + \beta)m_{a} \ell_{c} - (\gamma + \bar{\gamma})m_{a} \ell_{c} \right),
$$

(43)

Therefore, the covariant derivative of any outer product of NP tetrad vectors gives components of equal or lesser boost weight. Suppose an outer product has boost weight $b$; then applying a covariant derivative gives components of boost weight $b$ depending on $\{\tau, \pi, \alpha, \beta\}$, $b - 1$ depending on $\{\gamma, \mu, \lambda\}$ and $b - 2$ depending on $\{v\}$, along with their complex conjugates.

Consider the $n+1$ covariant derivative of Riemann. Then the boost weight $-1$ components, $(\nabla^{n+1} R)_{-1}$, can only arise by applying $D = \partial_v$ to the components of $(\nabla^n R)_0$. By hypothesis $(\nabla^n R)_0$ only depends on $W_{i, v}$ and $H_{i, v}$; therefore, from the $K_1$ conditions, $D W_i = D H = 0$, $(\nabla^{n+1} R)_{-1}$ vanishes whereby we obtain that the $n+1$ covariant derivative of Riemann is type II.

Furthermore, the boost weight 0 components of $(\nabla^{n+1} R)_0$ arise from applying $\delta$ or $\bar{\delta}$ to the components of $(\nabla^n R)_0$. By assumption $(\nabla^n R)_0$ has the property that they only depend

\textsuperscript{12} A similar theorem is valid in arbitrary dimensions—see theorem 7.1.

\textsuperscript{13} Although the components also depend on $P(u, x, y)$, this is of no consequence since $DP = 0$ and hence $P$ will not appear in a boost weight $+1$ component.
on $W_{i,v\cdots}$ and $H_{vv\cdots}$. Since $\delta = P(\partial_{i} - i\partial_{i})/\sqrt{2}$ then $(\nabla^{n+1} R)_{0}$ will also depend on $W_{i,v\cdots}$ and $H_{vv\cdots}$. In addition, $(\nabla^{n+1} R)_{0}$ also arise from $(\nabla^{n} R)_{0}$ by taking the covariant derivative of the outer product of NP tetrad vectors whose total boost weight is zero. From (41)–(43) we showed that these contributions will have the form of $\tau$, $\pi$, $\alpha$ or $\beta$ multiplied by the components of $(\nabla^{n} R)_{0}$. Since these spin coefficients only depend on $W_{i,v}$ (and $P$) it follows that the components of $(\nabla^{n+1} R)_{0}$ depend on $W_{i,v\cdots}$ and $H_{vv\cdots}$. Lastly, $(\nabla^{n+1} R)_{0}$ also have contributions from applying $D = \partial_{v}$ to $(\nabla^{n} R)_{-1}$. Since it is assumed that $(\nabla^{n} R)_{-1}$ does not depend on $H$ (i.e., $H$ appears with at least one derivative with respect to $v$), then we conclude that $(\nabla^{n+1} R)_{0}$ will only depend on $W_{i,v\cdots}$ and $H_{vv\cdots}$.

Next consider the boost weight $-1$ components $(\nabla^{n+1} R)_{-1}$ which arise from either applying $\delta$ or $\delta$ to $(\nabla^{n} R)_{-1}$ or as products of $\tau$, $\pi$, $\alpha$ or $\beta$ with $(\nabla^{n} R)_{-1}$. Since $(\nabla^{n} R)_{-1}$ has $H$ appearing with at least one derivative with respect to $v$, then from the same argument used above in the case of $(\nabla^{n+1} R)_{0}$ we have that $(\nabla^{n+1} R)_{-1}$ will also have the same $H_{vv\cdots}$ dependence (no dependence on $H$). Moreover, $(\nabla^{n+1} R)_{-1}$ also arises from applying $D$ to $(\nabla^{n} R)_{-2}$; however, no matter what functional dependence we have in $(\nabla^{n} R)_{-2}$ we again obtain that $H$ appears with at least one derivative with respect to $v$ in $(\nabla^{n+1} R)_{-1}$. Finally, we have contributions to $(\nabla^{n+1} R)_{-1}$ occurring through the application of $\Delta$ to $(\nabla^{n} R)_{0}$ or as products of $\gamma$, $\mu$ or $\lambda$ with $(\nabla^{n} R)_{0}$. In the first case, since

$$
\Delta = \partial_{v} - \left[ H + \frac{p^{2}}{2} \left( W_{1}^{2} + W_{2}^{2} \right) \right] \partial_{v} + p^{2} W_{1} \partial_{i} + p^{2} W_{2} \partial_{j}
$$

then when applied to $(\nabla^{n} R)_{0}$ we have, by virtue of the $K_{1}$ conditions, that $\partial_{v}$ on the components of $(\nabla^{n} R)_{0}$ vanish. Consequently, by hypothesis on $(\nabla^{n} R)_{0}$ then $(\nabla^{n+1} R)_{-1}$ has $H$ appearing with at least one derivative of $v$. In the second case, we note that $(\gamma$, $\mu$, $\lambda)$ only has $H$ appearing with one derivative of $v$ in $\gamma$, and it follows that the product of these spin coefficients with $(\nabla^{n} R)_{0}$ will give rise to $H$ appearing with at least one derivative of $v$ in $(\nabla^{n+1} R)_{-1}$.

We can now conclude that the $n + 1$ covariant derivative of Riemann is type II, the boost weight 0 components, $(\nabla^{n+1} R)_{0}$, only depend on $W_{i,v\cdots}$ and $H_{vv\cdots}$, and the boost weight $-1$ components, $(\nabla^{n+1} R)_{-1}$, contain $H$ with at least one derivative with respect to $v$; therefore, $K_{n+1}$ holds.

We note that the VSI (or proper CSI) spacetimes satisfy $K_{0}$ and have vanishing (or constant) boost weight 0 components for all orders $n$. This result partially characterizes the metrics not determined by their scalar polynomial curvature invariants, namely all $Z$-degenerate metrics must satisfy $K_{1}$. We note that there may exist cases where Riemann and all of its covariant derivatives are type D (and hence $K_{n}$ for all $n$) but a sufficient number of independent curvature invariants can be constructed such that Riemann and its derivatives can be determined in some sense. We return to this in the following subsection.

We also note that the Goldberg–Sachs theorem (GS: theorem 7.1 in [4]) states that a spacetime with a shear-free, geodesic null congruence $k$ ($\kappa = \sigma = 0$) satisfying $R_{ab}k^{a}k^{b} = 0$, $R_{ab}k^{a}m^{b} = 0$, $R_{ab}m^{a}m^{b} = 0$ (what we might call a spacetime admitting an aligned algebraically special Ricci tensor), necessarily has $\Psi_{0} = \Psi_{1} = 0$ (aligned algebraically special Weyl tensor). When applied to the Kundt class, GS implies $K_{0}$. However, the conditions of GS on the Ricci tensor are slightly stronger than what are required for $K_{0}$ to be satisfied; GS also imposes $R_{ab}m^{a}m^{b} = \Phi_{02} = 0$. Based on the known solutions [4] of the Kundt class satisfying GS, these conditions and $\rho = 0$ imply $K_{0}$ which in turn implies $K_{n}$ for all $n \geq 1$. 14
Table 1. Within the degenerate Kundt class we list the possible algebraic types of $\nabla(\text{Riem})$ corresponding to the vanishing of the appropriate boost weight (bw) components. In each of these types it is assumed that either there is no remaining frame freedom or Riem and $\nabla(\text{Riem})$ have an isotropy group consisting of two-dimensional null rotations about $\ell$ (or subgroup thereof). A simplified notation, as defined in the text, is often used; for example, $D := (\text{II,II})$, $\text{III} := (\text{III,G})$ or $(\text{III,H})$ and $N := (\text{N,G})$ or $(\text{N,H})$.

| bw  | II,G | II,H | II,I | D   | III,G | III,H | III,I | N,G  | N,H  | O,G  | O   |
|-----|------|------|------|-----|------|------|------|------|------|------|-----|
| 0   | *    | *    | *    | 0   | 0    | 0    | 0    | 0    | 0    | 0    |     |
| −1  | *    | *    | *    | 0   | *    | *    | 0    | 0    | 0    |     |     |
| −2  | *    | 0    | 0    | *   | *    | 0    | *    | 0    |     |     |     |
| −3  | 0    | 0    | 0    | *   | 0    | *    | 0    |     |     |     |     |

5. Classification of degenerate Kundt spacetimes

The degenerate Kundt spacetimes are classified algebraically by their Riemann type (II, III, N, D or O) in the aligned kinematic frame, or more finely by their Ricci and Weyl types separately. We are only interested here in types II and D, since otherwise the degenerate Kundt spacetime is VSI [12]. Within each algebraic type, it may also be useful to classify the covariant derivative(s) of Riemann tensor (and particularly $\nabla(\text{Riem})$ and $\nabla^2(\text{Riem})$) in terms of their algebraic types.

In the analysis below it is important to fix the frame in each algebraic class completely. We shall begin with the classification of $\nabla(\text{Riem})$.

Classification of $\nabla(\text{Riem})$ for Riemann type II. The degenerate Kundt spacetimes of proper Riemann type II (i.e., with some non-zero boost weight zero terms but not type D) are further classified in the aligned kinematic frame by the algebraic type of $\nabla(\text{Riem})$ (and, for example, $\nabla^2(\text{Riem})$). In general, $\nabla(\text{Riem})$ in a degenerate Kundt spacetime is of type (II,G). In table 1, we list the possible algebraic types of $\nabla(\text{Riem})$ corresponding to the vanishing of the appropriate boost weight components in the degenerate Kundt class. Further subcases consist of types (II,H), (II,I), (II,II), (III/N,H), (III/N,I) and (III/N,II), etc. Further subclasses exist, including those where certain contractions of $\nabla^n(\text{Riem})$ are separately classified algebraically. The conditions for each of these subclasses can be presented in a similar manner to those of the following section.

Classification of $\nabla(\text{Riem})$ type III, N or O. Of particular interest, perhaps, is the case where $\nabla(\text{Riem})$ (and $\nabla^n(\text{Riem})$) are of type III, N or O, and hence all terms in $\nabla(\text{Riem})$ ($\nabla^n(\text{Riem})$) are of negative boost weight and hence do not contribute to any scalar invariants containing covariant derivative(s) of the Riemann tensor (i.e., the only non-vanishing polynomial scalar invariants are the zeroth-order ones constructed from the Riemann tensor alone).

Let us present the calculation of Riemann type II and $\nabla(\text{Riem})$ of type (III,G) or more specialized. We begin by assuming that there exists an NP tetrad in which Riemann is type II (therefore $\Psi_0 = F_1 = \Phi_{00} = \Phi_{01} = 0$) and simultaneously in this frame $\kappa = \sigma = \rho = \epsilon = 0$. Recall that this is the definition of a zeroth-order degenerate Kundt spacetime which we denoted as $K_0$. The remaining frame freedom consists of a two-dimensional group of null rotations.

14 The notation is consistent with that of [12, 13]. Since, by definition, $\nabla(\text{Riem})$ has no positive boost weight terms, its principal type is II or more special. In general its secondary type is G, but if there are no boost weight $−3$ ($−2$) terms it is of type II (I), etc. If $\nabla(\text{Riem})$ is of type (II,II), and consequently has only boost weight zero terms, it is said to be of type D.
about $\ell$, to be used later to simplify the Weyl tensor and thus completely fix the frame. Calculating $\nabla(Riem)$ and employing the Bianchi identities results in components of boost weight $\leq 1$. Since this case considers $\nabla(Riem)$ of type III or less we require the vanishing of boost weight +1 and 0 components in this frame. All components of $\nabla(Riem)$ of boost weight +1 reduce to a constant multiple of $DR$; hence, a necessary and sufficient condition for $\nabla(Riem)$ to be type II is $DR = 0$. The Bianchi equations provide further conditions showing that $D$ of all boost weight 0 components of $Riem$ vanish

$$D\Psi_2 = D\Phi_{11} = D\Phi_{02} = DR = 0. \tag{45}$$

Note that the boost weight +1 components of $\nabla(Riem)$ are invariant with respect to null rotations about $\ell$; therefore, conditions (45) are invariant conditions with respect to this remaining frame freedom. Assuming boost weight +1 components of $\nabla(Riem)$ vanish, the necessary and sufficient conditions for the boost weight 0 components to vanish are then

$$\tau\Psi_2 = 0, \tag{46}$$
$$2\tau\Phi_{11} + \tau\Phi_{02} = 0, \tag{47}$$
$$D\Psi_3 = 3\pi\Psi_2, \quad D\Phi_{12} = 2\pi\Phi_{11} + \pi\Phi_{02}, \tag{48}$$
$$\delta R = 0, \tag{49}$$
$$\delta\Phi_{02} = 2(\alpha - \beta)\Phi_{02}, \quad \delta\Phi_{02} = -2(\alpha - \beta)\Phi_{02}. \tag{50}$$

Again, (46)–(50) follow after (45) has been imposed. As above, if the boost weight +1 components vanish, then the boost weight 0 components are invariant with respect to null rotations about $\ell$; hence (46)–(50) provide invariant conditions for the vanishing of the boost weight 0 components. The remaining non-vanishing components of $\nabla(Riem)$ have boost weights $-1, -2$ and $-3$. Therefore, in a $K_0$ spacetime, $\nabla(Riem)$ is of type (III,G) if and only if equations (45) and (46)–(50) hold. Consequently, the Bianchi equations give some additional useful relations: $\delta\Psi_2 = \delta\Psi_2 = \delta\Phi_{11} = 0$.

We now specialize by assuming that Weyl is proper type II; i.e., $\Psi_2 \neq 0$ and the type does not reduce to another more algebraically special type. Using a null rotation about $\ell$ to set $\Psi_3 = 0$, the frame is fixed with $\Psi_2$ and $\Psi_4$ nonzero. If $\Psi_3$ was zero initially, then the frame is already fixed since any non-trivial null rotation about $\ell$ results in a nonzero $\Psi_3$. If under the null rotation about $\ell$ both $\Psi_3$ and $\Psi_4$ become zero then we obtain Weyl type D, a case which is excluded since we have assumed proper type II. In this fixed NP tetrad, equations (45) and (46)–(50) simplify, giving a number of cases. Here, we consider one of these.

**Case 1**: $\tau = 0, \Psi_2 \neq 0, \Psi_3 = 0, \Psi_4 \neq 0$. $\nabla(Riem)$ is type (III,G) if and only if

$$\tau = \pi = 0, \tag{51}$$
$$\Psi_2 - \frac{R}{12} = \Phi_{02} = 0, \tag{52}$$
$$D\Psi_2 = D\Phi_{11} = D\Phi_{12} = DR = 0, \tag{53}$$
$$\delta\Psi_2 = \delta\Phi_{11} = \delta R = 0. \tag{54}$$

Note that the first equation of (52) implies that $\Psi_2$ is real and, in addition, since $\Psi_2 \neq 0$ in this case, $R \neq 0$. Further algebraically special types of $\nabla(Riem)$ assume (51)–(54) are
satisfied. Implementing the NP and Bianchi equations, we consider the vanishing of $\nabla(Riem)$ components at each boost order separately.

$$(\nabla R)_{-1} = 0:
\begin{align*}
\mu &= \lambda = 0, \\
D\Psi_4 &= D\Phi_{22} = 0, \\
\Delta R &= 0, \\
Dv &= \Phi_{12} = 0,
\end{align*}$$

$$(55) \quad (56) \quad (57) \quad (58) \quad (59)$$

In (58) and (59) we list some of the consequences of the NP and Bianchi equations. In particular, (59) implies that all zeroth-order scalar polynomial invariants of the Riemann tensor are constant.

$$(\nabla R)_{-2} = 0:
\begin{align*}
v &= 0, \\
\delta\Phi_{22} &= -2(\bar{\alpha} + \beta)\Phi_{22}, \\
\bar{\delta}\Psi_4 &= -4\alpha\Psi_4, \\
\delta\Psi_4 &= -4\beta\Psi_4, \\
\Phi_{12} &= 0,
\end{align*}$$

$$(60) \quad (61) \quad (62) \quad (63)$$

Therefore, in this case 1, $\nabla(Riem)$ is at most type (III,G) and is of more special type if the following conditions are satisfied:

(III,H) : (64), (65)
(III,I) : (64), (65) and (60)–(63); here (64) simplifies to $\Delta\Phi_{22} = -2(\gamma + \bar{\gamma})\Phi_{22}$.
(N,G) : (55)–(59)
(N,H) : (55)–(59) and (64), (65); here (64) simplifies to $\Delta\Phi_{22} = -2(\gamma + \bar{\gamma})\Phi_{22}$.

For algebraic type (O,G), where (55)–(63) hold, it follows that $v = \mu = \lambda = 0$. Therefore, in addition to $\Phi_{12} = 0$, the NP equations give $\Psi_4 = \Phi_{22} = 0$; however, in case 1 $\Psi_4 \neq 0$ and thus type (O,G) is excluded. By a similar argument type O is also excluded; i.e., a symmetric space cannot exist in case 1.

Since $\nabla(Riem)$ is at most type (III,G) it has only negative boost weight components; therefore, all first-order scalar polynomial invariants vanish. Taking another covariant derivative we observe that the boost weight 0 components of $\nabla^{(2)}(Riem)$ can only arise from taking $D$ of the boost weight $-1$ components of $\nabla(Riem)$. After simplifying we note that every boost weight $-1$ component of $\nabla(Riem)$ is a sum of the terms $D\Psi_4, D\Phi_{22}, \Delta R, \mu R$ or $\lambda R$. Using the NP equations

$$D\lambda = D\mu = D\alpha = D\beta = 0,$$

and (53), we have that $D(\mu R) = D(\lambda R) = 0$. In addition, consider the following Bianchi equations and commutation relations:
to obtain the class of metrics corresponding to Riemann type D with a given algebraic type of equations nor the commutation relations. In principle, these would have to be solved in order

Our analysis takes into account the Bianchi equations; however, we shall not consider the NP

aligned frame coincide. In particular,

we shall begin by choosing to use the NP tetrad where the kinematic frame and the Riemann

order covariant derivatives of the Riemann tensor will also have vanishing boost weight 0 components so that $\nabla^{(0)}(\text{Riem})$ is type (III,G) for all $n \geq 1$, thus giving an $I$-symmetric space [1].

5.1. Classification of $\nabla(\text{Riem})$: type D

In order to perform an algebraic classification of $\nabla(\text{Riem})$ within the degenerate Kundt class we shall begin by choosing to use the NP tetrad where the kinematic frame and the Riemann aligned frame coincide. In particular, $\Psi_0 = \Psi_1 = \Phi_{00} = \Phi_{01} = 0$ so that Riemann is type II or less and in this frame $\kappa = \sigma = \rho = \epsilon = 0$. Moreover, since degenerate Kundt has $\nabla^{(0)}(\text{Riem})$ type (II,G) or less for all $n \geq 0$, one consequence was found to be $D\Psi_2 = D\Phi_{11} = D\Phi = 0$. By fixing $\epsilon = 0$ using a spin-boost then the only remaining freedom preserving the kinematic and Riemann aligned frames are null rotations about $\ell$. The classification of $\nabla(\text{Riem})$ will, in general, be subject only to this two-dimensional group of null rotations.

5.1.1. Riemann type D. The only non-vanishing curvature scalars are $\Psi_2$, $\Phi_{11}$, $\Phi_{02}$, and $R$, and hence Riemann has only boost weight 0 components. Since any further null rotations about $\ell$ do not preserve the Riemann type D aligned frame, the frame is thus completely fixed and the classification of $\nabla(\text{Riem})$ will depend on the vanishing of its boost weight components at various orders. For Riemann type D we find that, in general, $\nabla(\text{Riem})$ is type (II,H) (boost weight $-3$ vanish) and must satisfy the Bianchi equations

$$D\Psi_4 = \tilde{\delta}\Phi_{21} = \frac{\lambda R}{4} - 2\lambda \Phi_{11} + 2\alpha \Phi_{21}.$$  
(67)

$$D\Phi_{22} = \delta\Phi_{12} = \frac{\mu R}{4} - 2\mu \Phi_{11} + 2\beta \Phi_{12}.$$  
(68)

$$[\Delta, D] = (\gamma + \bar{\gamma}) D, \quad [\tilde{\delta}, D] = (\alpha + \bar{\beta}) D.$$  
(69)

The first equation of (69) and equation (53) give $D\Delta R = 0$. Next, we take $D$ of (67) and (68) and apply the second equation of (69), (66) and (53) to obtain $D^2\Psi_4 = D^2\Phi_{22} = 0$. Therefore all boost weight 0 components of $\nabla^{(2)}(\text{Riem})$ vanish, and the algebraic type is at most (III,G) and all second-order scalar polynomial invariants vanish. We expect that higher-order covariant derivatives of the Riemann tensor will also have vanishing boost weight 0 components so that $\nabla^{(n)}(\text{Riem})$ is type (II,G) for all $n \geq 1$, thus giving an $I$-symmetric space [1].

Calculating the components of $\nabla(\text{Riem})$ shows that the Bianchi equations (70), (71)–(73) and (74)–(76) provide constraints for the boost weight $-2$, $-1$ and 0 components, respectively. Our analysis takes into account the Bianchi equations; however, we shall not consider the NP equations nor the commutation relations. In principle, these would have to be solved in order to obtain the class of metrics corresponding to Riemann type D with a given algebraic type of $\nabla(\text{Riem})$. As an example, using Bianchi equation (70) all boost weight $-2$ components of
∇(Riem) are proportional to $\nu \Psi_2$. Solving for the vanishing of all components of $\nabla (\text{Riem})$ at each boost order separately gives

$$\nabla (\nabla (\text{Riem}))_{-2} = 0:$$

$$\nu \Psi_2 = 0,$$  \hspace{1cm} (77)

$$2\nu \Phi_{11} + \bar{\nu} \Phi_{20} = 0.$$ \hspace{1cm} (78)

$$\nabla (\nabla (\text{Riem}))_{-1} = 0:$$

$$\Delta \Psi_2 = \Delta \Phi_{11} = \Delta R = 0,$$  \hspace{1cm} (79)

$$\lambda \Psi_2 = 0, \hspace{0.5cm} \mu \Psi_2 = 0,$$ \hspace{1cm} (80)

$$2\bar{\mu} \Phi_{11} + \lambda \Phi_{02} = 0, \hspace{0.5cm} 2\bar{\lambda} \Phi_{11} + \mu \Phi_{02} = 0,$$ \hspace{1cm} (81)

$$\Delta \Phi_{02} = 2(\gamma - \bar{\gamma}) \Phi_{02}.$$ \hspace{1cm} (82)

$$\nabla (\nabla (\text{Riem}))_0 = 0:$$

$$\delta \Psi_2 = \bar{\delta} \Psi_2 = \delta \Phi_{11} = \delta R = 0,$$ \hspace{1cm} (83)

$$\pi \Psi_2 = 0, \hspace{0.5cm} \tau \Psi_2 = 0,$$ \hspace{1cm} (84)

$$2\bar{\pi} \Phi_{11} + \pi \Phi_{02} = 0, \hspace{0.5cm} 2\tau \Phi_{11} + \bar{\tau} \Phi_{02} = 0,$$ \hspace{1cm} (85)

$$\bar{\delta} \Phi_{02} = 2(\alpha - \bar{\beta}) \Phi_{02}, \hspace{0.5cm} \delta \Phi_{02} = -2(\bar{\alpha} - \beta) \Phi_{02}.$$ \hspace{1cm} (86)

Here we have also included the reduced Bianchi equations relevant to each boost weight.

Evidently, from equations (77), (80) and (84) the algebraic classification of $\nabla (\text{Riem})$ depends on whether $\Psi_2$ vanishes or not. If $\Psi_2 = 0$, then equations (78), (81) and (85) give rise to additional subcases dependent on whether $4\Phi_{11}^2 - \Phi_{02} \Phi_{20}$ vanishes or not, thus determining if there exists non-trivial algebraic solutions of the spin coefficients. Since the boost weight $-3$ components of $\nabla (\text{Riem})$ vanish, we use the simplified notation for the algebraic types given below by $\text{III} := (\text{III,H})$ and $\text{N} := (\text{N,H})$.

**Case 1**: $\Psi_2 \neq 0$. $\nabla (\text{Riem})$ is type

$$(\text{II}, \text{I}) : \nu = 0$$

$$(\text{D}) : \nu = \lambda = \mu = 0, \text{ and } (79) \text{ and } (82)$$

$$(\text{III}) : \tau = \pi = 0, \text{ and } (83) \text{ and } (86)$$

$$(\text{N}) : \lambda = \mu = \tau = \pi = 0, \text{ and } (79), (82), (83) \text{ and } (86)$$

$$(\text{O}) : \text{ same as type N and also } \nu = 0.$$

**Case 2**: $\Psi_2 = 0$ (conformally flat). In this case boost weight $-2$ components of $\nabla (\text{Riem})$ always vanish; therefore, $\nabla (\text{Riem})$ is, in general, type $(\text{II}, \text{I})$.

**Case 2.1**: $4\Phi_{11}^2 - \Phi_{02} \Phi_{20} \neq 0$. For this subcase $\nu = 0$, and the rest of the conditions for $\nabla (\text{Riem})$ types are the same as in case 1 except type N is excluded.

**Case 2.2**: $4\Phi_{11}^2 - \Phi_{02} \Phi_{20} = 0$. Admits non-trivial solutions for the spin coefficients and $\nu$ always satisfies (78). Then $\nabla (\text{Riem})$ is type

$$(\text{II}, \text{I}) : \text{ no further conditions}$$

$$(\text{D}) : (79), (82) \text{ and } \lambda, \mu \text{ satisfy } (81)$$

$$(\text{III}) : (83), (86) \text{ and } \tau, \pi \text{ satisfy } (85)$$

$$(\text{O}) : \text{ union of type D and III conditions}.$$
Vacuum can only occur in case 1 whereas in 2.2 it reduces to flat space. All vacuum Riemann type D (hence Petrov type D) solutions are known (see (31.41) of [4]). In both cases 1 and 2, if $\nabla(Riem)$ is type O then we obtain a symmetric space. It is important to note that the non-vanishing boost weight components of a given $\nabla(Riem)$ type possess additional constraints through Bianchi which are not included above. For example, in $\nabla(Riem)$ type D all boost weight 0 components are also subject to (74)–(76).

5.2. Classification of $\nabla^2(Riem)$

We can also algebraically classify $\nabla^2(Riem)$. We shall just present type D here.

5.2.1. Classification of $\nabla^{(2)}(Riem)$: type D. In principle, an algebraic classification of $\nabla^{(2)}(Riem)$ can be performed for any fixed algebraic types of Riem and $\nabla(Riem)$. We shall present a partial classification of $\nabla^{(2)}(Riem)$ when Riem and $\nabla(Riem)$ are both of type D.

For case 1, that is assuming $\Psi_2 \neq 0$, we obtain the following boost weight $+1$ and $-1$ components, up to a constant factor, after using the Bianchi equations and commutator relations

$$\nabla^2 R)_{+1}:
\begin{align*}
D\pi(3\Psi_2 - 2\Phi_{11}) - D\pi\Phi_{20} &= 0, \\
D\pi(3\Psi_2 + 2\Phi_{11}) + D\pi\Phi_{20} &= 0.
\end{align*}
$$

$$\nabla^2 R)_{-1}:
\begin{align*}
[\Delta \tau - (\gamma - \bar{\gamma})\tau][3\Psi_2 + 2\Phi_{11}] + [\Delta \tau - (\bar{\gamma} - \gamma)\bar{\tau}]\Phi_{02} &= 0, \\
[\Delta \tau - (\gamma - \bar{\gamma})\tau][-3\Psi_2 + 2\Phi_{11}] + [\Delta \bar{\tau} - (\bar{\gamma} - \gamma)\bar{\tau}]\Phi_{02} &= 0.
\end{align*}
$$

along with their complex conjugates. In addition, there are boost weight 0 components in $\nabla^{(2)}(Riem)$ that are not given here. Since $\Psi_2 \neq 0$ and recalling that the Riemann type D aligned frame is completely fixed, then the vanishing of boost weight $+1$ components implies (91) and vanishing of boost weight $-1$ components implies (92):

$$D\pi = 0,$$

$$\Delta \tau - (\gamma - \bar{\gamma})\tau = 0.$$ 

Therefore, if Riem and $\nabla(Riem)$ are of type D and $\Psi_2 \neq 0$ then, in general, $\nabla^{(2)}(Riem)$ will be of type (I,I) (i.e., has only non-vanishing boost weight $+1$, $0$, $-1$ components). $\nabla^{(2)}(Riem)$ is type (II,I) if (91) is satisfied and type D if (91) and (92) are satisfied. In fact, these conditions also characterize the same algebraic types of $\nabla^{(2)}(Riem)$ for case 2.1. This follows from the definition of case 2.1; since it has the same conditions as case 1 except with the additional constraints $\Psi_2 = 0, 4\Phi_{02}^1 \neq 0$ we obtain precisely the same components of $\nabla^{(2)}(Riem)$. If $\Psi_2 = 0, 4\Phi_{02}^1 \neq 0$ then the vanishing of boost weight $+1$ components, (87) and its complex conjugate, gives again (91). In addition, the vanishing of boost weight $-1$ components, (89) and its complex conjugate, again results in (92). We can now characterize when Riemann and all of its covariant derivatives will be type D in cases 1 and 2.1.

**Theorem 5.2.** In the Kundt class, if Riem and $\nabla(Riem)$ are type D and $\Psi_2 \neq 0$ or $\Psi_2 = 0, 4\Phi_{02}^1 \neq 0$, then $D\pi = 0$ and $\Delta \tau - (\gamma - \bar{\gamma})\tau = 0$ if and only if $\nabla^{(n)}(Riem)$ is type D for all $n \geq 2$. 

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**Proof.** In the above, we showed that (91) and (92) are necessary and sufficient conditions for $\nabla^{(2)}(\text{Riem})$ to be type D. Suppose $\nabla^{(n-1)}(\text{Riem})$ and $\nabla^{(n)}(\text{Riem})$ are type D for a fixed $n \geq 2$; then there exists an NP tetrad in which the only non-vanishing components are boost weight 0. Symbolically, we shall write a representative term of $(\nabla^{(n)} R)_0$ as

$$\nabla^{(n)}(\text{Riem}) = R_0^n S,$$

(93)

where $R_0^n$ is an NP tetrad component of boost weight 0 of the $n$th covariant derivative of Riemann. In general, $S = S(p, q, r, s)$, is a rank $n + 4 = p + q + r + s$ tensor representing the outer product of $n + 4$ tetrad vectors with $p, q, r, s$ counting the number of $\ell, n, m$ and $\bar{n}$ vectors, respectively. Hence, $S(p, q, r, s)$ is associated with tetrad components of boost weight $q - p$; in particular, (93) has $S = S(q, q, r, s)$.

Taking a covariant derivative of (93) and applying (41)–(43) gives the following non-vanishing components of boost weight +1, 0 and −1:

$$\nabla^{(n+1)}(\text{Riem}) = DR_0^n Sn + (\nabla^n R)_0 + \left[ \Delta R_0^n + (r - s)(\gamma - \bar{\gamma}) R_0^n \right] S\ell. \tag{94}$$

Therefore, $\nabla^{(n+1)}(\text{Riem})$ is type D if the following conditions hold for all boost weight 0 components of $\nabla^{(n)}(\text{Riem})$:

$$D R_0^n = 0, \tag{95}$$

$$\Delta R_0^n + (r - s)(\gamma - \bar{\gamma}) R_0^n = 0. \tag{96}$$

To proceed, we must determine the form of the $R_0^n$ tetrad components. Since it is assumed that $\nabla^{(n-1)}(\text{Riem})$ is type D then we also have

$$\nabla^{(n-1)}(\text{Riem}) = R_0^{n-1} \hat{S}(\bar{q}, \bar{q}, \bar{r}, \bar{s}), \tag{97}$$

where $n + 3 = 2q + r + s$. By assumption, the covariant derivative of (97) gives rise only to boost weight 0 components, $R_0^n$.

$$\nabla^{(n)}(\text{Riem}) = \left[ - \delta R_0^{n-1} + (\bar{r} - \bar{s})(\bar{a} - \bar{\beta}) R_0^{n-1} \right] \hat{S}(\bar{q}, \bar{q}, \bar{r}, \bar{s}) \hat{n}$$

$$+ \left[ - \delta R_0^{n-1} - (\bar{r} - \bar{s})(\bar{a} - \bar{\beta}) R_0^{n-1} \right] \hat{S}(\bar{q}, \bar{q}, \bar{r}, \bar{s}) \hat{m}$$

$$+ R_0^{n-1} \left[ \bar{q} \left[ - \bar{r} \hat{S}(\bar{q} - 1, \bar{q}, \bar{r} + 1, \bar{s}) \ell - \tau \hat{S}(\bar{q} - 1, \bar{q}, \bar{r}, \bar{s} + 1) \ell \right.ight.$$  

$$+ \tau \hat{S}(\bar{q} - 1, \bar{q}, \bar{r}, \bar{s} + 1) \ell \right] + R_0^{n-1} \left[ \bar{q} \left[ - \bar{r} \hat{S}(\bar{q} - 1, \bar{q}, \bar{r} + 1, \bar{s}) \ell - \tau \hat{S}(\bar{q} - 1, \bar{q}, \bar{r}, \bar{s} + 1) \ell \right.$$  

$$+ \tau \hat{S}(\bar{q} - 1, \bar{q}, \bar{r}, \bar{s} + 1) \ell \right] + R_0^{n-1} \left[ \bar{q} \left[ - \bar{r} \hat{S}(\bar{q} - 1, \bar{q}, \bar{r} + 1, \bar{s}) \ell - \tau \hat{S}(\bar{q} - 1, \bar{q}, \bar{r}, \bar{s} + 1) \ell \right.$$  

$$+ \tau \hat{S}(\bar{q} - 1, \bar{q}, \bar{r}, \bar{s} + 1) \ell \right] \tag{98}$$

Furthermore, equations (95) and (96) are identically satisfied under replacement $n, r, s \mapsto n - 1, \bar{r}, \bar{s}$; i.e., $\nabla^{(n)}(\text{Riem})$ is type D.

First consider equation (95). From (98), a component $R_0^n$ is proportional to $R_0^{n-1} \pi$; therefore, $D R_0^n = D \left( R_0^{n-1} \pi \right) = R_0^{n-1} D \pi = 0$ which is satisfied since $\nabla^{(2)}(\text{Riem})$ type D implies $D \pi = 0$. Another boost weight 0 component is proportional to $R_0^{n-1} \tau$, then $D(R_0^{n-1} \tau) = R_0^{n-1} D \tau = 0$ holds as a consequence of an NP equation. Similar conclusions hold for $R_0^{n-1} \bar{\pi}$ and $R_0^{n-1} \bar{\tau}$. Next, using the NP equations $D \alpha = D \bar{\beta} = 0$ and commutator relation we find that

$$D \left[ - \delta R_0^{n-1} + (\bar{r} - \bar{s})(\bar{a} - \bar{\beta}) R_0^{n-1} \right] = - \delta D R_0^{n-1} + (\bar{a} + \beta - \bar{\pi}) D R_0^{n-1} = 0. \tag{99}$$

The same conclusion holds for the second component of (98). It now follows that $D \pi = 0$ is the necessary and sufficient condition for $\nabla^{(n+1)}(\text{Riem})$ to have vanishing boost weight +1 components.
Last, consider equation (96). Using the NP equation $\Delta \pi = -(\gamma - \tilde{\gamma})\pi$, the component $R_0^{\alpha-1}_0$ gives

$$\Delta (R_0^{\alpha-1}_0) + (r - s)(\gamma - \tilde{\gamma}) R_0^{\alpha-1}_0 = (r - s - \tilde{\tau} + \tilde{s} - 1)(\gamma - \tilde{\gamma}) R_0^{\alpha-1}_0 = 0,$$  
(100)

where the last equality follows from the fact that $R_0^{\alpha-1}_0$ occurs in (98) with either $r = \tilde{r} + 1, s = \tilde{s}$ or $r = \tilde{r}, s = \tilde{s} - 1$. For the boost weight 0 component $R_0^{\alpha-1}_0$, equation (96) becomes

$$[\Delta \tau + (r - s - \tilde{\tau} + \tilde{s})(\gamma - \tilde{\gamma})\tau] R_0^{\alpha-1}_0 = 0. \tag{101}$$

Since (98) implies that $R_0^{\alpha-1}_0$ occurs with either $r = \tilde{r}, s = \tilde{s} + 1$ or $r = \tilde{r} - 1, s = \tilde{s}$, then (101) reduces to $\Delta \tau - (\gamma - \tilde{\gamma})\tau = 0$ which is satisfied as a consequence of having $\nabla^{(2)}(\text{Riem})$ type D. Again, similar arguments apply to the components $R_0^{\alpha-1}_0$ and $R_0^{\alpha-1}_0$. Next, we consider the $R_0^{\alpha}_0$ component of (98) proportional to $-\delta R_0^{\alpha-1}_0 + (\tilde{r} - \tilde{s})(\tilde{\alpha} - \beta) R_0^{\alpha-1}_0$ and substitute it into (96). By applying the commutator relation

$$\Delta \delta (R_0^{\alpha-1}_0) = \delta \Delta (R_0^{\alpha-1}_0) - (\tau - \tilde{\alpha} - \beta) \Delta R_0^{\alpha-1}_0 + (\gamma - \tilde{\gamma}) \delta R_0^{\alpha-1}_0, \tag{102}$$

the identity $\Delta R_0^{\alpha-1}_0 = - (\tilde{\tau} - \tilde{s})(\gamma - \tilde{\gamma}) R_0^{\alpha-1}_0$, and the NP equations

$$\Delta \alpha = \tilde{\delta} \gamma + \tilde{\gamma} \alpha + (\tilde{\beta} - \tilde{\tau}) \gamma, \tag{103}$$
$$\Delta \beta = \tilde{\delta} \gamma + (\tilde{\alpha} + \tilde{\beta} - \tilde{\tau}) \gamma + (\gamma - \tilde{\gamma}) \beta, \tag{104}$$

we simplify to get

$$(\tilde{r} - \tilde{s} - r - s + 1)(\gamma - \tilde{\gamma})[\delta R_0^{\alpha-1}_0 - (\tilde{\tau} - \tilde{s})(\tilde{\alpha} - \beta) R_0^{\alpha-1}_0] = 0. \tag{105}$$

Equality in (105) follows from (98), which implies $r = \tilde{r}, s = \tilde{s} + 1$. The second component of (98) also identically satisfies (96), except here $r = \tilde{r} + 1, s = \tilde{s}$. Thus, $\Delta \tau - (\gamma - \tilde{\gamma})\tau = 0$ is the necessary and sufficient condition for $\nabla^{(\alpha+1)}(\text{Riem})$ to have vanishing boost weight $-1$ components.

Therefore, we have shown that (91) and (92) are the necessary and sufficient conditions for $\nabla^{(\alpha+1)}(\text{Riem})$ to have vanishing boost weight $+1$ and $-1$ components for any $n \geq 2$, and hence to be of type D.

Now consider case 2.2, defined by $\Psi_2 = 0, 4\Phi^2_{11} - \Phi_{02}\Phi_{20} = 0$ with $\nu, \lambda$ and $\mu$ satisfying (78) and (81). In addition to (79) and (82) holding, the Bianchi equations also give

$$D\Phi_{11} = D\Phi_{02} = DR = 0. \tag{106}$$

Clearly, $\Phi_{11} = 0$ if and only if $\Phi_{02} = 0$, in which case $R$ is the only non-vanishing curvature scalar. However, the Bianchi equations consequently give $\delta R = 0$ so that (79) and (106) imply that $R$ is a constant. Thus we obtain a space of constant curvature having Riem type D and $\nabla(\text{Riem})$ type O since it vanishes (a symmetric space). In the remainder of this section we shall assume $\Phi_{11} \neq 0$.

Using the NP equations, Bianchi equations and commutation relations to simplify the components of $\nabla^{(2)}(\text{Riem})$ we find that, up to a constant factor, all boost weight $+1$ components reduce to $2D\pi \Phi_{11} + D\bar{\pi} \Phi_{20}$ or its complex conjugate. Taking the conjugate of (75) and subtracting (74) gives the identity

$$2(\tau + \pi) \Phi_{11} + (\bar{\tau} + \bar{\pi}) \Phi_{02} = 0, \tag{107}$$

and applying $D$ to (107) and using (106) and the NP equation $D\tau = 0$ gives

$$2D\bar{\pi} \Phi_{11} + D\pi \Phi_{02} = 0; \tag{108}$$
therefore, all boost weight +1 components vanish. All other positive boost weight components of $\nabla^{(2)}(\text{Riem})$ vanish. We note that similar relations to (108) can be obtained by taking $D$ of (78) and (81).

Since $\nabla(\text{Riem})$ is type D, all boost weight $-3$ components of $\nabla^{(2)}(\text{Riem})$ must vanish. Evaluating these components we find that they can all be made to vanish by applying the following identities. Taking $\Delta$ of (78) gives

$$2\Delta \bar{\nu} \Phi_{11} + [\Delta \nu + 2(\gamma - \bar{\gamma}) \nu] \Phi_{02} = 0,$$

(109)

similar relations to (109) can be obtained by taking $\Delta$ of (81). Assuming $\nu \neq 0$, then from (78), its conjugate and the second equation of (81) we obtain

$$\nu^2 \Phi_{02} - \bar{\nu}^2 \Phi_{20} = 0, \quad \bar{\nu} \nu - \mu \bar{\nu} = 0.$$  

(110)

If $\nu = 0$, then all boost weight $-3$ components vanish identically in $\nabla^{(2)}(\text{Riem})$.

Next, we list some of the identities that are useful in the simplification of the remaining non-vanishing components of $\nabla^{(2)}(\text{Riem})$ at boost weight $-2, -1$ and $0$:

\begin{align*}
\mu \lambda \Phi_{02} - \bar{\mu} \bar{\lambda} \Phi_{20} &= 0, \quad \mu \bar{\mu} - \lambda \bar{\lambda} = 0, \quad (111) \\
\lambda^2 \Phi_{02} - \bar{\mu}^2 \Phi_{20} &= 0, \quad \nu \lambda \Phi_{02} - \bar{\nu} \bar{\mu} \Phi_{20} = 0, \quad (112) \\
\lambda(\bar{\tau} + \pi) \Phi_{02} - \bar{\mu}(\tau + \bar{\tau}) \Phi_{20} &= 0, \quad \bar{\lambda}(\tau + \bar{\pi}) - \mu(\tau + \bar{\tau}) = 0, \quad (113) \\
\nu(\bar{\tau} + \pi) \Phi_{02} - \bar{\nu}(\tau + \bar{\pi}) \Phi_{20} &= 0, \quad \nu(\tau + \bar{\pi}) - \bar{\nu}(\tau + \bar{\pi}) = 0, \quad (114) \\
(\bar{\tau} + \pi)^2 \Phi_{02} - (\tau + \bar{\pi})^2 \Phi_{20} &= 0, \quad (115) \\
\Delta^2 \Phi_{02} - 2[\Delta \gamma - \Delta \bar{\gamma} + 2(\nu - \bar{\nu})^2] \Phi_{02} &= 0, \quad (116) \\
2\bar{\delta} \nu \Phi_{11} + \delta \nu \Phi_{02} - \frac{1}{12} \nu \delta R - \frac{1}{12} \bar{\nu} \bar{\delta} R + 2(\nu - \bar{\nu})(\nu \Phi_{11} + 2\nu(\alpha - \bar{\beta}) \Phi_{02} = 0, \quad (117) \\
2\bar{\delta} \nu \Phi_{11} + \delta \nu \Phi_{02} + \nu \delta \Phi_{02} - \frac{1}{12} \nu \delta R + 2(\nu - \bar{\nu}) \Phi_{02} &= 0, \quad (118) \\
(\bar{\tau} + \pi)^2 \delta R + (\tau + \bar{\pi})(\bar{\gamma} R) &= 0. \quad (119)
\end{align*}

Other identities also follow by taking $D$, $\Delta$, $\delta$ or $\bar{\delta}$ of known identities and applying the NP equations, Bianchi equations and commutation relations. We find that the boost weight $-1$ and $-2$ components of $\nabla^{(2)}(\text{Riem})$ reduce to

$$\nabla^{(2)} R_{-1}:$$

$$\mu \delta R + \bar{\lambda} \bar{\delta} R,$$  

(120)

$$\nabla^{(2)} R_{-2}:$$

$$\nu \delta R + \bar{\nu} \delta R.$$  

(121)

As in the previous cases, the boost weight 0 components, $\nabla^{(2)} R_0$, are, in general, non-zero (but we do not give them here). Therefore, if Riem and $\nabla(\text{Riem})$ are type D and $\Psi_2 = 0, 4 \Phi_{11}^2 - \Phi_{02} \Phi_{20} = 0$ then, in general, $\nabla^{(2)}(\text{Riem})$ is type (II,H) (i.e., has non-vanishing components of boost weight 0, $-1$ and $-2$). $\nabla^{(2)}(\text{Riem})$ is type (II) if (121) vanishes and type D if (120) and (121) vanish. Note that the type D conditions are the requirements for (119) and the second equations of (113) and (114) to admit a non-trivial solution; $\tau + \bar{\tau} \neq 0$. 

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5.2.2. Final comments. Suppose we impose the requirement that $\nabla^{(2)}(\text{Riem})$ is type D, so that (120) and (121) vanish. Calculating $\nabla^{(3)}(\text{Riem})$, we expect all positive boost weight components to vanish, which would follow from (106), the NP equations $D\tau = D\alpha = D\beta = 0$, and a higher-order identity on $D\tau$ similar to equation (108). It is likely that these conditions will result in the vanishing of all positive boost weight components of $\nabla^{(n)}(\text{Riem})$ for all $n \geq 3$, thus leading in general to primary alignment type II. In addition to boost weight 0 components, $\nabla^{(3)}(\text{Riem})$ may contain components of boost weight $-1$ and $-2$. One possibility is that the boost weight $-1$, $-2$ components vanish identically, as a result of (120) and (121) vanishing. In this case, $\nabla^{(3)}(\text{Riem})$ is type D and one might expect that all higher covariant derivatives of the Riemann tensor are also type D, so that the vanishing of (120) and (121) provide necessary and sufficient conditions for the Riemann tensor and all of its covariant derivatives to be type D within case 2.2. Another possibility is that components of $\nabla^{(3)}(\text{Riem})$ of boost weight $-1$, $-2$ do not vanish. Requiring boost weight $-1$ and $-2$ to vanish would provide additional constraints on $\mu$, $\lambda$, and $\nu$, respectively. By requiring $\nabla^{(n)}(\text{Riem})$ be type D for all $n \geq 3$ we would obtain a sequence of constraints on $\mu$, $\lambda$, and $\nu$, and it is plausible that this will reduce to the trivial solution $\mu = \lambda = \nu = 0$. It should be noted that this is not in general a specialization of case 1 or 2.1, since throughout 2.2 we require $\Psi_1 = 0$ and $\Phi_1 = 0$, $\Phi_1 = 0$.

6. Kundt spacetimes and their invariant classification

It is of interest to provide invariant conditions to distinguish the degenerate Kundt class from the remaining Kundt metrics. We define the following invariant and covariant quantities involving the second and third Lie derivatives with respect to $\ell$:

$$I_0 = R^{abcd} R_{ef} \epsilon^a \epsilon^c \ell_{e} \ell_{f} \ell_{g} \ell_{h}, \quad K_{ab} = \ell_{e} \ell_{f} \ell_{g} \ell_{h}.$$

(122)

From the Kundt metric we find that $I_0$ is proportional to $P^4(W_{1,v}^2 + W_{2,v}^2)$ and $K_{ab}$ has components proportional to the third derivative of $H$, $W_1$ and $W_2$ with respect to $v$, thus establishing the following result\(^{15}\).

**Proposition 6.1.** Within the Kundt class

(i) $I_0 = 0$ if and only if $K_0$ is satisfied.

(ii) $I_0 = K_{ab} = 0$ if and only if $K_1$ is satisfied.

The covariant condition in (ii) can be made invariant by replacing it with the vanishing of the trace of $K_{ab}$.

**Equivalence problem for degenerate Kundt spacetimes.** The degenerate Kundt spacetimes are not completely characterized by their scalar polynomial curvature invariants [1]. However, they are completely characterized by their algebraic properties (as we have discussed above in detail). Moreover, the degenerate Kundt spacetimes are, of course, uniquely characterized by their Cartan invariants.

This work is of importance to the equivalence problem of characterizing Lorentzian spacetimes (in terms of their Cartan invariants) [4]. By knowing which spacetimes can be characterized by their scalar curvature invariants alone, the computations of the invariants (i.e., simple scalar invariants) are much more straightforward and can be done algorithmically (i.e., the full complexity of the equivalence method is not necessary). On the other hand, the Cartan equivalence method also contains, at least in principle, the conditions under which the classification is complete (although in practice carrying out the classification for the more

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15 We recall that the $n$th-order degenerate Kundt class, $K_n$, is defined in definition 4.1.
general spacetimes is difficult, if not impossible). Therefore, in a sense, the full machinery of the Cartan equivalence method is only necessary for the classification of the degenerate Kundt spacetimes.

The first step is to completely fix the frame in all algebraic classes (which is what was actually done in the computations above). In such a fixed frame all of the remaining components are, in fact, Cartan scalars. This is the easy part; the complete characterization depends on all of the different branches that occur. We hope to return to this problem in the future.

Even though the degenerate Kundt metrics are not determined by their scalar polynomial curvature invariants, a special result is possible. Suppose there exists a frame in which all of the positive boost weight terms of the Riemann tensor and all of its covariant derivatives $\nabla^{(k)}(\text{Riem})$ are zero (in this frame). It follows from [1] that in 4D the resulting spacetime is degenerate Kundt. It is of interest to prove that such a spacetime is degenerate Kundt in arbitrary dimensions (i.e., the appropriate Ricci rotation coefficients $L_{ij}$ are zero). We shall prove this in the final section.

In [14] it was shown that a 4D type $D^k$ CSI spacetime, in which the Riemann tensor and $\nabla^k(\text{Riem})$ are all simultaneously of type $D$, is locally homogeneous. Thus a CSI spacetime in which the Riemann tensor and all of its covariant derivatives are aligned and of algebraic type $D$, even though they are not $I$-non-degenerate, are in some sense ‘characterized’ by their constant curvature invariants, at least within the class of type $D^k$ CSI spacetimes. In general, there are many degenerate Kundt CSI metrics (that are not type $D^k$) with the same set of constant invariants, $I$. In this case there is at least one $\nabla^k(\text{Riem})$ which is proper type II and thus has negative boost weight terms; this Kundt CSI metric will have precisely the same scalar curvature invariants as the corresponding type $D^k$ CSI metric (which has no negative boost weight terms). Therefore, there is a distinguished or a ‘preferred’ metric with the same set of constant invariants, $I$; namely, the corresponding type $D^k$ locally homogeneous CSI metric, which is distinguished within the class of algebraic type $D^k$ CSI spacetimes. Similar properties are likely true for all $D^k$ spacetimes.

6.1. Kundt spacetimes and scalar invariants

Let us consider to what extent the class of degenerate Kundt spacetimes can be characterized by their scalar polynomial curvature invariants. Clearly such spacetimes are algebraically special and of Riemann type II. In particular, the Weyl tensor is of type II (or more special) and hence $27J^2 = I^3$ (see (34)). If $I = J = 0$, then the spacetime is of Weyl type III, N or O. If the spacetime is of Weyl type N, then $I_1 = I_2 = 0$ if and only if $\kappa = \rho = \sigma = 0$ from the results in [12] (the definitions of the invariants $I_1$ and $I_2$ are given therein). Similar results follow for Weyl type III spacetimes (in terms of invariants $I_1$ and $I_2$) and in the conformally flat (but non-vacuum) case (in terms of similar invariants $I_1$ and $I_2$ constructed from the Ricci tensor [12]; also see equations (35)). These conditions, the list of conditions on the scalar invariants of the Weyl tensor and its covariant derivatives summarized in [1] (and, indeed, all of the conditions discussed below), are necessary conditions in order for a spacetime not to be $I$-non-degenerate (i.e., if any of these necessary conditions are not satisfied, the spacetime cannot be degenerate Kundt). For example, if $27J^2 \neq I^3$, then the spacetime is of Petrov type I, and the spacetime is $I$-non-degenerate.

In the case that $27J^2 = I^3 \neq 0$ (Weyl types II or D), in [1] two higher-order invariants were given as necessary conditions for $I$-non-degeneracy (if $27J^2 = I^3$, but $S_1 \neq 0$ or $S_2 \neq 0$, then the spacetime is $I$-non-degenerate). Indeed, if $27J^2 = I^3 \neq 0$ (Weyl types II and D), essentially if $\kappa = \rho = \sigma \neq 0$, we can construct positive boost weight terms in the derivatives.
of the curvature and determine an appropriate set of scalar curvature invariants. For example, consider the positive boost weight terms of the first covariant derivative of the Riemann tensor, $\nabla Riem$. If the spacetime is $I$-non-degenerate, then each component of $\nabla Riem$ is related to a scalar curvature invariant. In this case, in principle we can solve (for the positive boost weight components of $\nabla Riem$) to uniquely determine $\kappa, \rho, \sigma$ in terms of scalar invariants, and we can therefore find necessary conditions for the spacetime to be degenerate Kundt (there are two cases to consider, corresponding to whether $\Psi_2 + \frac{1}{3} \Phi_{11}$ is zero or non-zero). We note that even if the invariants exist in principle, it may not be possible to construct them in practice.

Further necessary conditions can be obtained from the fact that the Ricci tensor is of type II or D (or more special) (see equations (35)/(126) and (128)).

It is useful to express the conditions (34) in non-NP form. The syzygy $I_3 - 27 I_2 = 0$ is complex, whose real and imaginary parts can be expressed using invariants of Weyl not containing duals. The real part is equivalent to

$$-11 W_2^3 + 33 W_2 W_4 - 18 W_6 = 0, \quad (123)$$

and the imaginary part is equivalent to

$$(W_2^2 - 2 W_4)(W_2^2 + W_4)^2 + 18 W_2^2 (6 W_6 - 2 W_3^2 - 9 W_2 W_4 + 3 W_2^3) = 0, \quad (124)$$

where

$$W_2 = \frac{1}{4} C_{abcd} C^{abcd}, \quad W_3 = \frac{1}{10} C_{abcd} C^{ced} p q C^{pqab},$$

$$W_4 = \frac{1}{32} C_{abcd} C^{ced} p q C^{pq} r s C^{rstab},$$

$$W_6 = \frac{1}{128} C_{abcd} C^{ced} p q C^{pq} r s C^{stu} t u C^{vw} C^{vwab}. \quad (125)$$

6.1.1. A set of necessary conditions for degenerate Kundt. For a degenerate Kundt spacetime there exists a frame in which the Riemann tensor and all of its covariant derivatives have no positive boost weight components (i.e., all of $\nabla Riem$, $\nabla^2 Riem$, ... are of ‘type II’) (in addition, for example, to being Kundt). This means that every tensor, $T$, constructed from the Riemann tensor and its covariant derivatives by contractions, additions and products, are of type II (i.e., have no positive boost weight components). We can generate a set of necessary conditions from these type II conditions.

(i) IIs: all symmetric trace-free $(0,2)$ tensors, $S = T'$, give rise to necessary conditions of the form (35), which we shall denote by $I'_{s}$, where

$$s_1^2 (4 s_1^3 - 6 s_1 s_3 + s_2^2) - s_2^2 (3 s_1^2 - 4 s_3) = 0, \quad (126)$$

for each $S_i$ and $s_\alpha$ ($\alpha = 1, 2, 3$) are defined by

$$s_1 = \frac{1}{17} S_a^b S_b^a,$$

$$s_2 = \frac{1}{33} S_a^b S_b^c S_c^a,$$

$$s_3 = \frac{1}{38} \left[ S_a^b S_b^c S_c^d S_d^a - \frac{1}{4} (S_a^b S_b^a)^2 \right]. \quad (127)$$

For example, in the case of $(S_{ab}$ given by) the trace-free Ricci tensor $R_{ab}, s_a \equiv r_a$ and $I'_{1}$ is given by equation (35) in terms of the CZ Ricci invariants $r_a$. Other examples include the symmetric trace-free parts of $R_{ab}, R_{cd,a} R_{cd}.b, \ldots$.

We note that not all such invariants are independent due to the symmetries of the curvature tensor and the Bianchi identities. For example, the necessary condition obtained from the trace-free part of $\Box (R_{ab})$ would be equivalent to derivatives of the condition obtained from the trace-free part of the Ricci tensor itself.
(ii) \textit{IIw}: all completely trace-free (0,4) tensors with the same symmetries as the Riemann tensor, \(W = T^w\), give rise to necessary conditions of the form (34), denoted by \(I^w_i\). For example, for the case of \((W_{abcd} \text{ given by})\) the Weyl tensor, \(I^w_{1,2}\) are defined by equations (123)/(124) (the real and imaginary parts of (34)). Other examples can be constructed from the covariant derivatives of the Weyl tensor.

(iii) \textit{IIo}: any other invariants, \(I^o_i\), that can be constructed in a similar way\textsuperscript{16}. In particular, the invariants \(I^1_1\) and \(I^o_{1,2}\) arise from the properties of the eigenvalues in the degenerate cases of the eigenvalue (eigenbivalue) problems associated with the Ricci and the Weyl tensors (defined as curvature operators acting on tangent vectors and bivectors, respectively [1]). Any curvature operator, constructed from the Riemann tensor and its covariant derivatives, will give rise to an eigenvalue problem which will give rise to invariants in a similar way.

In this way we generate a set of invariants:

\[
I = \{I^1_1, \ldots, I^1_i, \ldots, I^w_1, \ldots, I^w_i, \ldots, I^o_1, \ldots, I^o_i, \ldots\},
\]

which, in turn, generates a set of necessary conditions (each member of the set must vanish) for a degenerate Kundt spacetime. Indeed, if any \(I_i \neq 0\), the spacetime cannot be degenerate Kundt.

6.1.2. Comments. There are a number of questions that arise: (1) can we find an independent subset of \(I\)? (2) Can we find a minimal subset of \(I\)? (3) Does there exist a finite independent and minimal subset of \(I\)? (4) Does there exist such a subset of \(I\) that is complete in the sense that if all invariants are zero, the spacetime is degenerate Kundt (i.e., the conditions are sufficient as well as necessary) for some appropriate subclass?

This last possibility is not guaranteed, since it is not immediately clear that the conditions of alignment and Kundt (in the definition of degenerate Kundt) are contained in this list. And the converse is false, in general, since there exist counterexamples that were presented in [1]. In addition, consider, for example, a symmetric space in which \(\nabla Riem = 0\). This means that we can construct no invariants using the covariant derivatives of the Riemann tensor; i.e., we have \(I^s_1\) (35) constructed from the Ricci tensor, \(I^w_{1,2}\) (123) and (124) constructed from the Weyl tensor, and possibly an invariant \(I^s_1\) constructed from the zeroth-order mixed invariants. Therefore, the spacetime is very restricted. Clearly, there is a relationship between such spacetimes and symmetric and \(k\)-symmetric spacetimes. For example, in [15] the set of 2-symmetric spacetimes were investigated. It was found that a 2-symmetric spacetime is either CSI, or there exists a covariantly constant null vector (CCNV) (a similar result was conjectured for the set of \(k\)-symmetric spacetimes).

Another approach in determining an invariant characterization of the degenerate Kundt class (or Kundt class) would be to start by considering an independent set of differential invariants of the Riemann tensor and its covariant derivatives. By requiring the metric to be degenerate Kundt would result in syzygies among the set of differential invariants; thus necessary conditions can be derived. In [16], the authors have developed the \textit{Invar} tensor package and performed a detailed study of the differential invariants of the Riemann tensor. By using their set of independent differential invariants of the Riemann tensor and expressing them in terms of NP scalars, the degeneracies that occur within this set once the specialization \(\kappa = \sigma = \rho = 0\), the Riemann tensor and \(\nabla(Riem)\) are type II are imposed, can be investigated.

\textsuperscript{16}For example, if a spacetime is Riemann type II, then not only do the Weyl type II and Ricci type II syzygies hold, but there are additional alignment conditions; e.g., \(C_{abcd}R^{bd}, C_{abcd}R^{ae}R_{e}^{d}\) are of type II.
7. Higher dimensions

The higher-dimensional Kundt metrics are given by (17), with $i, j = 2, \ldots, n - 1$, in the kinematic frame (18), (19).\textsuperscript{17} From equations (49) and (54) in [19] it follows that if $W_{i,vv} = 0$ for all $i$, then all positive boost weight terms of the Riemann tensor are zero in the kinematic frame, and the spacetime is of aligned algebraically special Riemann type II. From the equations for $\nabla(Riem)$ in this frame (cf equations (55) and (60)–(63) in [19]), it then follows that if $H_{vvv} = 0$, then all of the positive boost weight terms of the covariant derivative of the Riemann tensor are zero. By a direct calculation, it then follows that all of the positive boost weight terms of $\nabla^2 (Riem)$ are zero in this aligned frame. It then follows from an argument similar to that of theorem 3.1 that all of the positive boost weight terms of all of the covariant derivatives of the Riemann tensor are zero. Hence, if $W_{i,vv} = 0$ and $H_{vvv} = 0$, the higher-dimensional Kundt metric is degenerate.

7.1. Examples of degenerate Kundt spacetimes in higher dimensions

Spacetimes which are CSI, VSI or CCNV are degenerate-Kundt spacetimes.

7.1.1. Constant scalar curvature invariants. Lorentzian spacetimes for which all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant (CSI spacetimes) were studied in [19]. If a spacetime is CSI, the spacetime is either locally homogeneous or belongs to the higher-dimensional Kundt CSI class (the CSI$_K$ conjecture) and can be constructed from locally homogeneous spaces and VSI spacetimes. The CSI conjectures were proven in four dimensions in [14].

In [19] it was shown that for Kundt CSI metric of the form (17) there exists (locally) a coordinate transformation such that the transverse is independent of $u$ and a locally homogeneous space. The remaining CSI conditions then imply that

\begin{equation}
W_i(v, u, x^k) = v W_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k),
\end{equation}

\begin{equation}
H(v, u, x^k) = \frac{v^2}{8} \left[ 4\sigma + \left( W_i^{(1)}(W_i^{(1)}) \right) \right] + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k),
\end{equation}

where $\sigma$ is a constant [19].

7.1.2. Vanishing scalar curvature invariants. All curvature invariants of all orders vanish in an $n$-dimensional Lorentzian spacetime if and only if there exists an aligned non-expanding, non-twisting, shear-free geodesic null direction $\ell^a$ along which the Riemann tensor has negative boost order [3]. Thus the Riemann tensor, and consequently the Weyl and Ricci tensors, are of algebraic type III, N or O [13], and VSI spacetimes belong to the Kundt class [4]. It follows that any VSI metric can be written in the form (17), where local coordinates can be chosen so that the transverse metric is flat; i.e., $g_{ij} = \delta_{ij}$ [19]. The metric functions $H$ and $W_i$ in the Kundt metric (which can be obtained by substituting $\sigma = 0$ in equations (128) and (129)), satisfy the remaining vanishing scalar invariant conditions and any relevant Einstein field equations.

\textsuperscript{17}In a very recent preprint [17], building upon the earlier work of [18], the class of higher-dimensional Kundt spacetimes has been analysed, with an emphasis on studying exact solutions of the higher-dimensional Einstein–Maxwell equations for a vacuum or aligned Maxwell field with and without a cosmological constant.
7.1.3. Covariantly constant null vector. The aligned, repeated, null vector \( \ell \) of (17) is a null Killing vector (KV) if and only if \( H_v = 0 \) and \( W_{i,v} = 0 \), and it then follows that \( \ell \) is also covariantly constant. Therefore, the most general metric that admits a covariantly constant null vector is (17) with \( H = H(u, x^k) \) and \( W_i = W_i(u, x^k) \) and is of Ricci and Weyl type II [13]. In 4D the CCNV spacetimes are the well-known pp-wave type N VSI spacetimes.

7.2. Discussion: supersymmetry and holonomy
Supersymmetric solutions of supergravity theories have played an important role in the development of string theory. The existence of parallel (Killing) spinor fields, plays a central role in supersymmetry. In the physically important dimensions below 12 the maximal indecomposable Lorentzian holonomy groups admitting parallel spinors are known [20]. A systematic classification of supersymmetric solutions in \( M \)-theory was provided in [21]. There are two classes of solutions. If the spacetime admits a covariantly constant timelike vector, the spacetime is static and its classification reduces to the classification of ten-dimensional Riemannian manifolds.

The second class of solutions consists of spacetimes which are not static but which admit a covariantly constant null vector. The isotropy subgroup of a null spinor is contained in the isotropy subgroup of the null vector, which in arbitrary dimensions is isomorphic to the spin cover of \( ISO(n - 2) \subset SO(n - 1, 1) \). For \( n \leq 5 \) this means the holonomy group is \( \mathbb{R}^{n-2} \), which implies that the metric is Ricci-null. This leads to the \( n \)-dimensional Kundt spacetimes (see equation (17), with \( H_v = 0 \) and \( W_{i,v} = 0 \), whence the metric no longer has any \( v \) dependence) [13, 23]. This class includes Kundt-CSI and Kundt-VSI spacetimes as special cases. The VSI and CSI spacetimes are of fundamental importance since they are solutions of supergravity or superstring theory, when supported by appropriate bosonic fields [24]. Supersymmetry in VSI and CSI type IIB supergravity solutions was studied in [24].

The classification of holonomy groups in Lorentzian spacetimes is quite different from the Riemannian case since the de Rham decomposition theorem does not apply without modification. For a Lorentzian manifold \( M \) there are the following two possibilities [25]—completely reducible: here \( M \) decomposes into irreducible or flat Riemannian manifolds and a manifold which is an irreducible or a flat Lorentzian manifold or \((\mathbb{R}, -dt)\). The irreducible Riemannian holonomies are known, as well as the irreducible Lorentzian one, which has to be the whole of \( SO(1, n - 1) \). Not completely reducible: this is equivalent to the existence of a degenerate invariant subspace and entails the existence of a holonomy invariant lightlike subspace. The Lorentzian manifold decomposes into irreducible or flat Riemannian manifolds and a Lorentzian manifold with indecomposable, but non-irreducible holonomy representation; i.e., with (a one-dimensional) invariant lightlike subspace. These are the CCNV and RNV (Kundt) spacetimes, which contain the VSI and CSI subclasses as special cases [2].

Therefore, the Kundt spacetimes that are of particular physical interest are degenerately reducible, which leads to complicated holonomy structure and various degenerate mathematical properties. Such spacetimes have a number of other interesting and unusual properties, which may lead to novel and fundamental physics. Indeed, a complete understanding of string theory is not possible without a comprehensive knowledge of the properties of the Kundt spacetimes [2]. For example, in general a Lorentzian spacetime is completely classified by its set of scalar polynomial curvature invariants. However, this is not true for the degenerate Kundt spacetimes [19] (i.e., they have important geometrical information that is not contained in the scalar invariants). This leads to interesting problems...

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18 The null vector of a metric with \( \text{Sim}(n - 2) \) holonomy is a recurrent null vector (RNV) and the metric belongs to the class of Kundt metrics ((17) with \( W_i = W_i(u, x^k) \)), are also of interest [22].
with any physical property that depends essentially on scalar invariants, and may lead to ambiguities and pathologies in models of quantum gravity or string theory.

As an illustration, in many theories of fundamental physics there are geometric classical corrections to general relativity. Different polynomial curvature invariants (constructed from the Riemann tensor and its covariant derivatives) are required to compute different loop orders of renormalization of the Einstein–Hilbert action. In specific quantum models such as supergravity there are particular allowed local counterterms \[26\]. In particular, a classical solution is called \textit{universal} if the quantum correction is a multiple of the metric. In \[22\] metrics of holonomy Sim\((n − 2)\) were investigated, and it was found that all four-dimensional Sim\((2)\) metrics are universal and consequently can be interpreted as metrics with vanishing quantum corrections and are automatically solutions to the quantum theory. The RNV and CCNV (Kundt) spacetimes therefore play an important role in the quantum theory, regardless of what the exact form of this theory might be.

7.3. Results

Many of the results in this paper (for 4D spacetimes) can be generalized to higher dimensions. In particular, we would like to prove that the degenerate Kundt metrics are the only metrics not determined by their curvature invariants (i.e., not \(I\)-non-degenerate) in any dimension. There are higher-dimensional generalizations to theorem 4.2 and the type D result, which we will present in sections 7.3.2 and 7.3.3.

7.3.1. Partial converse result. In the analysis in 4D it was determined for which Segre types for the Ricci tensor the spacetime is \(I\)-non-degenerate (similar results were obtained for the Weyl tensor). In each case, it was found that the Ricci tensor, considered as a curvature operator, admits a timelike eigendirection. Therefore, if a spacetime is not \(I\)-non-degenerate, its Ricci tensor must be of a particular Segre type (corresponding to the non-existence of a unique timelike direction). Therefore, it is plausible that if the algebraic type of the Ricci tensor (or any other \((0, 2)\) curvature operator written in ‘Segre form’) is not of one the following types:

\begin{enumerate}
  \item \{211 \cdots\}, \{2(11) \cdots\}, \{2(111) \cdots\}, \ldots, \{2(111 \cdots)\},
  \item \{(21)11 \cdots\}, \{(21)(11) \cdots\}, \{(21)(111) \cdots\}, \ldots, \{(21)(111 \cdots)\},
  \item \{(211)11 \cdots\}, \{(211)(11) \cdots\}, \{(211)(111) \cdots\}, \ldots, \{(211)(111 \cdots)\},
  \item \{311 \cdots\}, \{3(11) \cdots\}, \{3(111) \cdots\}, \ldots, \{3(111 \cdots)\},
\end{enumerate}

and so on, then the spacetime is \(I\)-non-degenerate. (Similar results in terms of the Weyl tensor in bivector form are possible). It remains to prove the converse; namely, if it is of one of these types it must be degenerate Kundt.

7.3.2. Kundt theorem. Let us show that theorem 4.2 is valid in any dimension. The proof of theorem 4.2 was computational in nature and specific to four dimensions only; here we will give an alternative proof, valid in any dimension.

\textbf{Theorem 7.1.} In the higher-dimensional Kundt class, \(K_1\) implies \(K_n\) for all \(n \geq 2\).

\textbf{Proof.} First, \(K_1\) implies that there exists a frame such that all positive boost weight components of the connection coefficients are zero (this is precisely the Kundt condition in higher dimensions), with respect to which all positive boost weight components of Riemann and \(\nabla(Riem)\) are zero. In order to show that this implies \(K_n\) for all \(n\), we will use the two
identities:

\[ R_{abcd} = 0, \quad \text{(Bianchi identity)} \quad (130) \]

\[ \nabla_a \nabla_b T_{cd} = \sum_{i=1}^{k} T_{a \cdot \cdot c \cdot d} R^i_{d,ab} \quad \text{(gen. Ricci identity).} \quad (131) \]

Let us first assume only \( K_0 \). The covariant derivative consists of a partial derivative and an algebraic piece; symbolically we can write:

\[ \nabla T = \partial T - \sum \Gamma \ast T. \quad (132) \]

Since the connection coefficients do not contain positive boost weight components the algebraic piece, \( \sum \Gamma \ast T \), cannot raise the boost weight. Let us, for simplicity, denote components with \( a = 0, 1, i \), where \( \ell^a T_{a,b} = T_0 b, n^a T_{a,b} = T_1 b, \) and \( m^a T_{a,b} = T_i b \). The components 0, 1 and \( i \) (‘downstairs’) would therefore carry a boost weight +1, −1 and 0, respectively. The only way that the boost weight can be raised (since the connection coefficients are of boost order 0) is through the partial derivatives. If we consider, for example, the +1 component:

\[ \nabla_i R_{jk0} = \partial_i (R_{jk0}) = 0, \]

since \( R_{ij0} = 0 \) by the \( K_0 \) assumption. Analogously, by considering all +1 components of the form \( \nabla_i R_{abcd} \), we consequently find that they are all zero. Similarly, we cannot obtain any +1 components by applying \( n^a \nabla_a \); hence, positive boost weight components of \( \nabla R \) can only come from the covariant derivative with respect to \( \ell \).

Consider the positive boost weight components of the form

\[ \ell^a \nabla_a T_{bc\cdot\cdot} \equiv T_{bc\cdot\cdot}. \]

The possible (independent) boost weight +1 components of \( \nabla R \) are

\[ R_{j01;0} = R_{0ij;0}, R_{0i1;j;0}, R_{ijk;0}, R_{0101;0}. \]

For the first covariant derivative of the Riemann tensor, we can use the Bianchi identity:

\[ R_{abcd;0} = -R_{ab0c;d} + R_{ab0d;c}. \]

Using the Bianchi identity and the above results, we see that all of the positive boost weight components have to be zero, except possibly \( R_{0101;0} \) (this component corresponds to the term \( H_{vvv} \) and is, in general, non-zero). However, by also assuming \( K_1 \), \( R_{0101;0} = 0 \) (and consequently, \( H_{vvv} = 0 \)). Therefore, let us assume that \( (\nabla R)_{b;0} = 0 \) and \( (\nabla \nabla R)_{b;0} = 0 \), and consider \( \nabla \nabla R \). By the argument above, using \( T = \nabla R \), we see that the only possible contributors to the positive boost weight components are \( \ell^a \nabla_a (\nabla R) \). The right-hand side of the generalized Ricci identity is purely algebraic and hence, because of the \( K_0 \) assumption, the positive boost weight components of \( [\nabla, \nabla]R \) must be zero. Consider, for example, the component

\[ R_{01ij;0} = R_{01ij;0;0} \quad \text{+ (alg. piece)}. \]

So, \( R_{01ij;0} = R_{01ij;0;0} = \nabla_i (R_{01ij;0}) = 0 \). For all other components, except for \( R_{abcd;00} \), we can use the same trick to show that the positive boost weight components are also zero. For \( R_{abcd;00} \) consider, for example, the boost weight +1 component

\[ R_{01ij;00} = -(R_{010c;ij})_0 + (R_{010j;ic})_0 \]

(using the Bianchi identity); consequently, by the Ricci identity this component is also zero. In the same way, all of the positive boost weight components can be shown to be zero also. Thus, \( (\nabla \nabla R)_{b;0} = 0 \), and the spacetime is \( K_2 \).
We have now shown that \( K_1 \) implies \( K_2 \). In the same manner, we can use these arguments recursively for any \( V^a R \) by using the Ricci identity and the Bianchi identity. Therefore, we cannot acquire positive boost weight components by taking the covariant derivatives of the Riemann tensor. Hence, the spacetime must be \( K_n \) for all \( n \).

### 7.3.3. Type D theorem

Suppose, in higher dimensions, there exists a frame in which all of the positive and negative boost weight components of the Riemann tensor and all of its covariant derivatives \( \nabla^{(k)}(\text{Riem}) \) are zero (in this frame). Let us prove that the resulting spacetime is Kundt (this was shown to be true in 4D in [1]).

**Theorem 7.2.** If for a spacetime \((\mathcal{M}, \mathbf{g})\), the Riemann tensor and all of its covariant derivatives \( \nabla^{(k)}(\text{Riem}) \) are simultaneously of type D (in the same frame), then the spacetime is degenerate Kundt.

**Proof.** If the Riemann tensor and all of its covariant derivatives are of type D in the same frame; i.e., there exists a frame such that

\[
R = (R)_0, \quad \nabla^{(k)} R = (\nabla^{(k)} R)_0,
\]

then at every point all the curvature tensors are boost invariant. In particular, the curvature tensors experience a boost isotropy. Therefore, consider a point \( p \) and assume this is regular\(^{19}\). At this point there is a one-parameter family of boosts \( B_t : T_p \mathcal{M} \mapsto T_p \mathcal{M} \), which maps the frame onto another frame with identical components of the curvature tensors. Explicitly, we can define the boost to be over the neighbourhood \( U \) relative to the canonical type \( D^5 \) frame:

\[
\ell \mapsto e^{\psi(t;x^i)} \ell, \quad n \mapsto e^{-\psi(t;x^i)} n, \quad m^i \mapsto m^i',
\]

for any \( \psi(t; x^i) \) such that when restricted to \( p \) gives \( B_t \). Note that any such boost will leave the curvature tensors invariant; hence, the Cartan scalars of the transformed tensor are identical to the original ones. By the equivalence principle, since this is an isometry of the curvature tensors at the point \( p \), there exists an isometry \( \phi_t \) on a neighbourhood \( U \) of \( p \) such that it induces the map \( B_t \) at the tangent space \( T_p \mathcal{M} \) \([4, 10]\). Furthermore, we can choose \( \phi_t(p) = p \); hence, the isometry \( \phi_t \) is in the isotropy group at \( p \). Define also the map \( M \) as the induced map (the push-forward) of \( \phi_t \) acting on \( TM \) over \( U \); i.e., in components, \( \phi_t(e\sigma) = M^\nu_\mu e_\nu \) over \( U \). Note that at \( p, M \) coincides with \( B_t \), and hence the map \( M \) must act, up to conjugation, as a boost. Since \( \phi_t \) is an isometry, this boost must also be in the stabilizer of the curvature tensor. Therefore, align the null-frame over \( U \) such that this boost acts as \( (\ell, n, m^i) \mapsto (e^\ell \ell, e^{-\lambda} n, m^i) \) (note that the curvature tensors must still be of type D with respect to this frame).

Since an isometry leaves the connection invariant \([4, 10]\) (i.e., if \( \Omega \) is the connection form, then \( \phi_t^* \Omega = \Omega \), where \( \phi_t \) is the induced transformation on the frame bundle), we get over \( U \):

\[
\Gamma^\mu_{\alpha\beta} = (M^{-1})^\mu_\nu [M^\nu_\alpha \phi_t^* \Gamma^\nu_{\gamma\beta} + M^\nu_{\alpha,\delta} \Gamma^\delta_\beta].
\]

Furthermore, since \( p = \phi_t(p) \), we have \( \Gamma^\mu_{\gamma\beta} = \phi_t^* \Gamma^\nu_{\gamma\beta} \) at \( p \). Moreover, in the aforementioned type D frame, we have \( M^0\alpha_\mu = -M^1\alpha,\mu \), while all other components of \( M^\nu_{\alpha,\beta} \) are zero. Consequently, the following components of \( \Gamma^\mu_{\alpha\beta} \) must vanish:

\[
\begin{align*}
\Gamma^i_{00} &= \Gamma^i_{10} \equiv \Gamma^i_{0j} \equiv \Gamma^i_{ij} \equiv \Gamma^i_{j0} = 0 \quad \text{(boost weight + 2, +1)} \\
\Gamma^i_{11} &= \Gamma^0_{i0} \equiv \Gamma^0_{ij} \equiv \Gamma^0_{0j} = 0 \quad \text{(boost weight − 2, −1)}.
\end{align*}
\]

Note that the only remaining components of positive/negative boost weights are \( \Gamma^0_{00} = -\Gamma^1_{10} \) (boost weight +1) and \( \Gamma^0_{01} = -\Gamma^1_{11} \) (boost weight −1) (note that these are the components

\(^{19}\) In the sense of \([4]\), i.e., the number of independent Cartan invariants does not change at \( p \).
that do not transform algebraically but rather through derivatives of the boost parameter). At least one of these can be set to zero by a boost, which means that there exists a frame such that all positive boost weight components of the connection coefficients are zero. The vanishing of these positive boost weight components precisely corresponds to the existence of a shear-free, expansion-free, twist-free, geodesic null congruence, at the point \( p \). However, since this is valid at any point \( p \), this means that the spacetime is Kundt; there exists a null vector field over \( U \) which is geodesic, expansion-free, shear-free and twist-free.

Actually, by the argument of the proof we can see that there are always two such null vector-fields for these type \( D^k \) spacetimes. They are, in general:

\[
\begin{align*}
    k_1 &= f \ell, \\
    k_2 &= gn.
\end{align*}
\]

It is always possible to boost so that either \( f = 1 \) or \( g = 1 \). However, if \( k_1^\mu k_{2\mu} = fg \neq 1 \), it is not possible to boost so that both \( f = 1 \) and \( g = 1 \). Therefore, there are two null-vector fields that are shear-free, expansion-free, twist-free and geodesic; one is aligned with \( \ell \), the other is aligned with \( n \).

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