Re-summation of QED radiative corrections in a strong constant crossed field

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ABSTRACT: By considering radiative corrections of up to 3rd-loop order, Ritus and Narozhny conjectured that the proper expansion parameter for QED perturbation theory in a strong constant crossed field is $g = \alpha \chi^{2/3}$. Here we present and discuss the first result beyond the 3rd loop order in this context, the re-summed bubble-type polarization corrections to the electron mass operator in a constant crossed field. Our analysis confirms the importance and provides deeper insights into the relevance of the parameter $g$ for such kind of corrections. In particular, we identify two contributions to the on-shell amplitude with different accumulation ranges and asymptotic behavior for $g \gg 1$. The developed tools will be useful for elaborating definite predictions about the non-perturbative regime $g \gtrsim 1$.

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1 Introduction

Strong electromagnetic fields show up in atomic physics (including heavy ion collisions and passage of ultrarelativistic particles through crystals), astrophysics of compact objects, at the interaction point of future lepton colliders, and in interaction of high-power lasers with matter. Within a strong field approach, originated in the works of Furry [1], Sokolov and Ternov [2], and Keldysh [3], one considers a field as external (classical and given), but takes its action into account non-perturbatively.

A very important case is a constant crossed ($\vec{E} \cdot \vec{H} = 0$ and $E = H$) field (CCF), for which both field invariants are zero. It is a very good ‘instantaneous’ approximation in many situations involving ultra-relativistic particles. Even in very first considerations of the basic QED processes (photon emission, pair photoproduction) in such a field it was observed [4] that asymptotically, for $\chi \gg 1$, the probability rates are proportional to the combination $g = \alpha \chi^{2/3}$, where $\alpha = e^2/4\pi$ is the fine structure constant and $\chi = (e/m^3)\sqrt{-(F_{\mu\nu}p^\nu)^2}$ is so-called dynamical quantum parameter, of the meaning of a ratio of the field strength to Schwinger critical field $F_0 = m^2/e$ in a rest frame of the initial particle. Later, the same was found also for the simplest one-loop polarization [5] and mass [6] radiative corrections, related to the probability rates by the optical theorem.

\footnote{We use units such that $\hbar = c = \varepsilon_0 = 1$, electron mass and charge are $m$ and $-e$ with $e > 0$, and Minkowski metrics signature is $(+, -, -, -)$.}
In a series of papers [7–12] published in 1972-1985, by considering higher order radiative corrections of up to 3-loop order, it was eventually conjectured that $g$ might replace $\alpha$ as an effective expansion parameter for the QED perturbation theory in a strong CCF. Nowadays, this conjecture is known as the Ritus-Narozhny conjecture [13], see Appendix A for details. More precisely, the conjecture states that for $\chi \gg 1$: (i) the radiation probability and radiative corrections are enhanced by powers of $\chi$; (ii) the ratio of the dominant contributions to the $(n+1)$th and the $n$th orders of perturbation theory scales proportional to $g$, in this sense $g$ represents the effective expansion parameter for perturbation theory in a strong CCF; (iii) the corrections growing as the highest power of $g$ in each order of the perturbative expansion are those accommodating the maximal number of the successive polarization loops (bubbles) as shown in Figure 1. Note that this is in sharp contrast to ordinary (field-free) QED, where the expansion parameter $\alpha$ is small and the effect of higher-order vacuum polarization corrections, after renormalization, is a logarithmic growth of the effective, so-called running coupling constant, still remaining small for all reasonable energies below the electro-weak unification scale.

Re-summation of radiative corrections in a non-perturbative regime was already discussed for the case of a supercritical magnetic field [15, 16] in the context of spontaneous chiral symmetry breaking [17]. By noting that the ground Landau energy level $\varepsilon_{LLL} \propto \sqrt{B/F_0}$ [18], the corresponding value $\chi \simeq (B/F_0) \cdot (\varepsilon_{LLL}/m) \simeq (B/F_0)^{3/2}$ effectively maps into $g \simeq \alpha B/F_0$ (c.f. [19]). However, the supercritical magnetic field case differs qualitatively from the CCF case considered here, as the virtual electrons and positrons are mostly confined to the ground Landau level, while they remain semiclassical in a CCF. This implies that effective dimensional reduction is only applicable in the case of a supercritical magnetic field. For example, the one-loop mass operator is enhanced in a CCF [6, 7] and not in a supercritical magnetic field [20].

For the CCF case further studies stopped for decades until very recently, when it was proposed that due to mitigation of rapid radiation losses the value $g \simeq 1$ might be attained in beam-beam collisions at a near-future lepton collider [21] or, alternatively, for electrons either colliding with strong attosecond pulses generated by reflection of the high power optical laser pulses from a solid target [22], or propagating through aligned crystals [23], or passing through solid targets irradiated from the back side with ultraintense laser pulses [24]. It was also claimed that values up to $g \simeq 0.1$ might show up in oblique collisions of electrons with next-generation high-power optical laser pulses [25].

It is worth to emphasize that the idealized situation of a constant crossed field discussed here is applicable to realistic field configurations if the so-called locally constant field approximation (LCFA) is valid. Recent discussion (see, e.g., [26–28]) revealed that
the LCFA is valid under the conditions $a_0 \gg 1$ and $a_0 \gg \chi^{1/3}$, where $a_0 = eF\tau/m$ is the classical non-linearity parameter, $F$ and $\tau$ are the typical field strength and variation time (e.g., $\tau \sim \omega^{-1}$ in case of laser field, where $\omega$ is the laser carrier frequency). These conditions actually ensure that the typical formation length for strong-field processes like photon emission or pair production is smaller than the length scale over which the field changes significantly [21]. Note that while the importance of the former condition ($a_0 \gg 1$) was realized and stated explicitly already in the initial considerations (see, e.g., [4, 12]), the necessity of the latter one ($a_0 \gg \chi^{1/3}$) was not widely known, as it was commonly implied that $\chi \lesssim 1$ for foreseeable realistic setups. The resulting domains in the plane of parameters $(\chi,a_0)$ are shown in Figure 2, where the domain of validity of the LCFA is painted blue and the location of the non-perturbative regime in question is red hatched.

Figure 2. The domain of validity of LCFA $a_0 \gg \max\{1,\chi^{1/3}\}$ (blue) in a $(\chi,a_0)$ plane and the subdomain of non-perturbative regime $g = \alpha \chi^{2/3} \gtrsim 1$ (red hatched).

Here we revisit the CCF case by presenting and discussing the first result beyond the 3rd loop order, the re-summed on-shell bubble-type polarization corrections to the electron mass operator in a CCF shown in Figure 1. Note that a similar re-summation of the 1-loop radiative corrections to external electron and photon lines in a laser field was previously discussed in [29, 30], see also [31] for more details. Our consideration not only confirms the importance of the parameter $g$ for such kind of corrections, but also provides further insights into its nature and significance. It was recently demonstrated that the Ritus-Narozhny conjecture is irrelevant to the fundamental ultraviolet behavior of QED, i.e., to the limit of large energies for fixed field strength [32, 33]. The essential argument is that for a realistic localized field of fixed strength and duration (given $a_0$) a high energy limit means motion rightwards along a horizontal line in Figure 2, which inevitably leaves the domain of validity of the LCFA. Here we strengthen this conclusion by pointing out that the running coupling constant exhibits a logarithmic dependence on the field strength parameter $\chi$ even in a pure CCF. This agrees with the fact that the compound parameter $g$ originates in a growth of the effective photon mass in a strong field, hence is associated with the corresponding spatio-temporal and energy scales.
2 Bubble-type polarization corrections to the mass operator in a constant crossed field

In the Ritus $E_p$-representation [6, 7, 12] the correction to the mass operator in a CCF depicted in Figure 1 reads

$$-iM(p', p) = \int d^4x\, d^4x' \, \tilde{E}_{\mu
u}(x'')(i\epsilon\gamma^\nu)S^e(x', x)(i\epsilon\gamma^\mu)E_p(x)D^e_{\mu\nu}(x', x)$$

$$= \int \frac{d^4l}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \Gamma^\mu(l; p', q) \frac{i(q + m)}{q^2 - m^2 + i0} \Gamma^\nu(-l; q, p) D^e_{\mu\nu}(l).$$  \hspace{1cm} (2.1)

Here $S^e$ denotes the tree-level dressed electron propagator and $D^e$ is the photon propagator with account for 1-loop polarization operator in a CCF [5], $l^\mu$ and $q^\mu$ are 4-momenta of the virtual photons and the electron in the outer loop, $q = \gamma^\mu q_\mu$, $E_p(x)$ is a matrix solution to the Dirac equation in a CCF reducing to unity matrix upon switching the field off adiabatically [6] and

$$\Gamma^\mu(l; p, q) = \int d^4x\, e^{-i\epsilon x} \tilde{E}_\mu(x)(i\epsilon\gamma^\mu)E_q(x),$$  \hspace{1cm} (2.2)

with Dirac conjugation of a matrix $\tilde{E}_\mu = \gamma^0 E_\mu^\dagger \gamma^0$ denoted by bar, is the dressed vertex in a CCF [7]. For the sake of clarity Eq. (2.1) is given in two ways: the interior in the upper line is written in a coordinate representation, whereas in the lower one it is expressed in terms of electron propagator in the $E_p$-representation and photon propagator in the momentum representation.

The dressed photon propagator in a CCF with account for 1-loop polarization correction reads [5, 7, 12]

$$D^e_{\mu\nu}(l) = D_0(l^2, \chi_l)g_{\mu\nu} + D_1(l^2, \chi_l)\varepsilon_\mu(l)\varepsilon_\nu(l) + D_2(l^2, \chi_l)\varepsilon_\mu^*(l)\varepsilon_\nu^*(l),$$  \hspace{1cm} (2.3)

where $\chi_l = (e/m^3)\sqrt{-(F_{\mu\nu}l^\nu)^2}$ is the dynamical quantum parameter of the virtual photon and $\varepsilon_\mu(l) = eF_{\mu\nu}l^\nu/(m^3\chi_l)$, $\varepsilon_\mu^*(l) = eF_{\mu\nu}^* l^\nu/(m^3\chi_l)$ are the normalized field-induced transverse 4-vectors, $F_{\mu\nu}^* = \frac{i}{2}\varepsilon_{\mu\lambda\sigma}F^{\lambda\sigma}$ is the dual field strength tensor. The longitudinal component in (2.3)

$$D_0(l^2, \chi_l) = \frac{-iZ}{l^2 + i0},$$  \hspace{1cm} (2.4)

differs from the free one only by a finite renormalization factor $Z(l^2, \chi_l)$, whereas the transverse components

$$D_{1,2}(l^2, \chi_l) = \frac{iZ^2 \Pi_{1,2}}{(l^2 + i0)(l^2 - Z\Pi_{1,2})} = \frac{-iZ}{l^2 + i0} - \frac{-iZ}{l^2 - Z\Pi_{1,2}},$$  \hspace{1cm} (2.5)

possess additional poles corresponding to two effective photon masses (one per each photon polarization state) given by the renormalized eigenvalues $\Pi_{1,2}(l^2, \chi_l)$ of the on-shell polarization operator. Overall, the only effect of the factor $Z(l^2, \chi_l)$ is passing to a running coupling $\alpha \mapsto \alpha(l^2, \chi_l) = Z(l^2, \chi_l)\alpha$, however $Z$ remains very close to unity for all reasonable values of $\chi_l$ (see Appendix B for the explicit expressions and further discussion). We will ignore this logarithmic correction and set $Z = 1$ in the following.
Let us briefly describe the next steps of our calculation, starting with the expression in the second line of (2.1), which closely follows Ref. [7] (further details can be found there). Since $E_p(x)$ differs from the plane wave $e^{-ipx}$ by a factor depending solely on $\varphi = kx$ ($k^\mu$ is directed along the Poynting 4-vector of the CCF and its normalization is arbitrary), the dressed vertex (2.2) in a CCF can be written in the form

$$\Gamma^\mu(l; p, q) = \int_{-\infty}^{\infty} d\nu \delta^{(4)}(p - q - l - \nu k) \tilde{\Gamma}^\mu(\nu; p, q),$$

(2.6)

where $\nu k^\mu$ has the meaning of energy-momentum transferred to the external field. $\tilde{\Gamma}^\mu(\nu; p, q)$ is expressed in terms of the Airy function [34]

$$\text{Ai}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma \ e^{-i(t\sigma + \sigma^3/3)},$$

(2.7)

and due to the transversality of CCF the dressed vertex remains unchanged under the shifts of its arguments $p$ and $q$ by 4-vectors proportional to $k^\mu$. The 4-dimensional $\delta$-function, isolated in Eq. (2.6), expresses energy-momentum conservation with the external CCF included. In the presence of two such $\delta$-functions (one from each dressed vertex) in Eq. (2.1) $p'$ can actually differ from $p$ only by a 4-vector proportional to $k^\mu$, hence we can replace $\tilde{\Gamma}^\mu(-\nu'; p', q) \mapsto \tilde{\Gamma}^\mu(-\nu'; p, q)$. Then one of the two arising 4-dimensional $\delta$-functions in (2.1) removes the integration over $d^4q$, after which only 6 integrations remain: over $d^4l$, $d\nu$ and $d\nu'$. It is convenient to change variables $l^\mu \mapsto \{l^2, u, \rho, \tau\}$ and $\nu \mapsto \mu$, where $u = \chi l / \chi q$, $\rho = p^\mu \varepsilon_\mu(l)/m$, $\tau = p^\mu \varepsilon_\mu^*(l)/m$ and $\mu = q^2 - m^2$ is the electron virtuality. After the substitutions one finds that the integrals over $\rho$ and $\nu'$ are trivial, and that the residual 4-dimensional $\delta$-function provides the diagonality of the mass operator in $E_p$-representation, $M(p', p) = (2\pi)^4 \delta^{(4)}(p' - p) M(p)$. This diagonality is expected due to the translational symmetry of the constant field, as $M(p', p)$ is gauge invariant. Finally, the variable $\tau$ can also be integrated out by employing the formula

$$\int_{-\infty}^{\infty} d\tau \text{Ai}^2(a + \tau^2) = \frac{1}{2} \text{Ai}_1(2^{2/3}a),$$

(2.8)

where

$$\text{Ai}_1(t) = \int_{-\infty}^{\infty} \text{Ai}(x) \ dx = \frac{-i}{2\pi} \int_{-\infty}^{\infty} \frac{d\sigma}{\sigma - i0} \ exp \left[ -i t\sigma - i\sigma^3/3 \right]$$

(2.9)

is the Aspnes function, see section 3.5.2 and Eq. (3.105) in [34]. Thus the final expression contains three outer integrations over $u$ and the virtualities $l^2$ and $\mu$. In addition, several inner integrations are ‘hidden’ in the definition of the Airy functions and in the final form of the dressed photon propagator (2.4). Starting from this point, we process them differently than in [7].

In this paper we are focusing on an asymptotic behaviour at $\chi \gg 1$ of the on-shell invariant amplitude $M(\chi) \equiv \bar{u}_{p,\lambda} M(p) u_{p,\lambda}$, where $u_{p,\lambda}$ is the free Dirac spinor, $p^2 = m^2$ and $\lambda$ indicates a spin state. However, for $\chi \gg 1$ the terms depending on electron
spin are relatively small (see [7, 10, 11] for details), hence are neglected in what follows. Furthermore, it is natural to split the amplitude into two terms,

$$\mathcal{M}(\chi) = \mathcal{M}_0(\chi) + \delta \mathcal{M}(\chi),$$

(2.10)

where

$$\mathcal{M}_0(\chi) = \frac{\alpha m^2}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{du}{(1+u)^2} \int_{-\infty}^{+\infty} \frac{d\mu}{\mu + i0} D_0(t^2, \chi) \times \left\{ \left( 2 + \frac{l^2}{m^2} \right) A_1(t) + 2 \frac{u^2 + 2u + 2}{1 + u} \left( \frac{\chi}{u} \right)^{2/3} A'_1(t) \right\},$$

(2.11)

$$t = \left( \frac{u}{\chi} \right)^2 \left( 1 + \frac{1 + u}{1} \frac{l^2}{m^2} + \frac{1 + u}{u} \frac{m}{m^2} \right),$$

(2.12)

$$\chi_l = \frac{u\chi}{1 + u},$$

(2.13)

corresponds to a 1-loop contribution (i.e. the one containing no vacuum polarization insertions, see the first diagram in Figure 1), having been already calculated and discussed by Ritus [7], whereas $\delta \mathcal{M}(\chi)$ represents the remainder affected by vacuum polarization.

The multi-loop contributions of our interest are subject to renormalization, to be processed by successive steps following from inner to outer loops. If we use the renormalized polarization operator from the beginning, only the outer (photon) loop remains to be renormalized. When dealing with an on-shell radiative correction in the presence of an external field, a standard way to achieve this is to subtract the field-free amplitude $\mathcal{M}(F = 0)$, which is renormalized in the standard way and vanishes on-shell [7]. In case of $\mathcal{M}_0(\chi)$ this implies that we have to replace the function $A_1(t)$ in (2.11) with

$$A_1^{(\text{ren})}(t) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{d\sigma}{\sigma} e^{-i\sigma} \left( e^{-i\sigma^{3/3}} - 1 \right).$$

(2.14)

In the sequel we assume such a replacement in $\mathcal{M}_0(\chi)$ by default without introducing a special notation. The thus renormalized term $\mathcal{M}_0$ behaves for $\chi \gg 1$ as\(^2\)

$$\mathcal{M}_0(\chi \gg 1) \approx e^{-i\pi/3} \frac{28\sqrt{3}}{27} \Gamma \left( \frac{2}{3} \right) \alpha \chi^{2/3} m^2 \simeq 0.843(1 - i\sqrt{3}) \alpha \chi^{2/3} m^2$$

(2.15)

(this result is listed in Appendix A in the first row on the right of Table 1).

The non-trivial contribution $\delta \mathcal{M}(\chi) = \delta \mathcal{M}_1(\chi) + \delta \mathcal{M}_2(\chi)$ in Eq. (2.10) provides a higher-order vacuum polarization correction, where

$$\delta \mathcal{M}_{1,2}(\chi) = -\frac{\alpha m^2}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{du}{(1+u)^2} \int_{-\infty}^{+\infty} \frac{d\mu}{\mu + i0} D_{1,2}(t^2, \chi) \times \left\{ \left[ 1 + \frac{l^2}{m^2} \frac{u^2 + 2u + 2}{2u^2} \right] A_1(t) + \left( \frac{u^2 + 2u + 2}{1 + u} \pm 1 \right) \left( \frac{\chi}{u} \right)^{2/3} A'_1(t) \right\},$$

(2.16)

\(^2\)See Eq. (72) in Ref. [7]. Note that the invariant amplitude $\mathcal{M}$ is related to the elastic scattering amplitude $T_s$ by $T_s(p) = -\mathcal{M}(\chi)/(2p^0)$. 

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- 6 -
and \( t, \chi \) are defined in Eqs. (2.12) and (2.13). Unlike \( \mathcal{M}_0 \), these terms vanish as the field is switched off, hence remain unaffected by renormalization.

In fact, our expression (2.16) is equivalent\(^3\) to Eq. (42) in [11], where also the renormalization factor \( Z \) was set to unity and the spin-dependent terms were neglected. As we discuss in the next section, the extra terms proportional to \( \mu \) inside the coefficients of the Airy and Aspnes functions in Eq. (42) of [11] actually vanish after integration.

However, it is worth stressing that, in contrast to previous considerations [7–11], our further derivation is based neither on truncating the perturbative expansion

\[
\Pi_{1,2}(l^2, \chi_l) = \sum_{n=0}^{\infty} \left( \frac{\Pi_{1,2}(l^2, \chi_l)}{l^2 + i0} \right)^{n+1}
\]  

(2.17)

in the dressed propagator (2.5), nor on the evaluation of separate fixed loop order contributions shown in Figure 1. Rather, we are interested in a nonperturbative large-\( \chi \) asymptotic behavior of the whole amplitude (2.16), or, equivalently, in re-summation of all the bubble-type diagrams in Figure 1. In order to achieve this goal, we process the outer integrals in a different order than in [7, 11].

### 3 Integration over the electron and dressed photon virtualities

Let us continue the evaluation of Eq. (2.16) by using the integral representations for the Airy (2.7) and the Aspnes function (2.9), taking the involved integral over \( \sigma \) out and focusing attention on the integrals over the virtualities \( \mu \) and \( l^2 \). Then, with the argument \( t \) given explicitly in Eq. (2.12), the integral over \( \mu \) reduces to the textbook form

\[
\int_{-\infty}^{\infty} d\mu \frac{e^{-i\mu s}}{\mu + i0} = -2\pi i \theta (\text{Re } s),
\]  

(3.1)

where \( \theta \) is the Heaviside step function and \( s = \sigma (u/\chi)^{2/3} (1 + u)/(\mu m^2) \) has dimension of inverse mass squared and is proportional to the proper time of the electron in the outer loop. We treat the parameter \( s \) complex-valued if \( u \) is negative. Note that even if the coefficients of the Airy and Aspnes functions in Eq. (2.16) contained additional terms proportional to \( \mu \) as in [11], their integration over \( \mu \) would result in

\[
\int_{-\infty}^{\infty} d\mu \frac{\mu e^{-i\mu s}}{\mu + i0} = 2\pi \delta(s),
\]  

(3.2)

hence their contribution vanishes after integration over \( \sigma \) discussed below.

Next we consider the integral over \( l^2 \), which is more tricky and requires an approximation. After substituting (2.5) into (2.16) we are faced by two kinds of integrals:

\[
J_{1,2}(\tau) = \int_{-\infty}^{+\infty} d\tau \left\{ \frac{l^2}{m^2}, 1 \right\} \frac{\Pi(l^2, \chi_l)e^{-il^2\tau}}{(l^2 + i0)((l^2 - \Pi(l^2, \chi_l))^2)}
\]  

(3.3)

\[
\tau = \tau(\sigma, u) = \frac{\sigma}{m^2} \left( \frac{u}{\chi} \right)^{2/3} \frac{(1 + u)}{u^2},
\]  

(3.4)

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\(^3\)Apart from an extra overall factor \( 1/|\chi| \) present in [11], which is most likely a typo.
where $\Pi(t^2, \chi_l)$ is either $\Pi_1$ or $\Pi_2$ and $\tau$ is proportional to the dressed photons proper time. Note that the components of the polarization operator admit a one-sided Fourier integral representation (see Appendix B)

$$\Pi(t^2, \chi_l) = \int_0^\infty d\tau \tilde{\Pi}(\tau, \chi_l) e^{i t^2 \tau},$$  \hspace{1cm} (3.5)

where $\tilde{\Pi}_{1,2}(t^2, \chi_l)$ are given in Eq. (B.7) and for $\chi_l \gtrsim 1$ the value of the integral is effectively accumulated at $\tau \approx \tau_{\text{eff}}^{(1)} = 1/(m^2 \chi_l^{2/3})$. In order to formulate a reasonable approximation, let us for definiteness consider $J_1$ and for the moment combine Eq. (3.5) with the full-length perturbative expansion (2.17):

$$J_1(\tau) = \frac{-2\pi i}{m^2} \sum_{n=0}^\infty \frac{(-i)^n}{n!} \left( \prod_{l=1}^{n+1} \int_0^\infty d\tau_l \tilde{\Pi}(\tau_l, \chi_l) \right) \left( \tau - \sum_{l=1}^{n+1} \tau_l \right)^n \theta \left( \text{Re}(\tau) - \sum_{l=1}^{n+1} \tau_l \right) \theta \left( \tau - \sum_{l=1}^{n+1} \tau_l \right).$$  \hspace{1cm} (3.6)

Note that previous derivations [7, 11] were based on an accurate direct evaluation of separate low-order terms of the expansion (3.6) for $n \leq 1$. This, however, becomes hardly possible at higher orders, when the overall number of nested integrals grows substantially.

Here we proceed with an alternative approximation tailored for a correct account of higher orders. Note that in virtue of $\int_0^\infty d\tau \tilde{\Pi}(\tau, \chi_l) = \Pi(0, \chi_l)$, upon neglecting $\sum_a \tau_a$ against $\tau$, the series in Eq. (3.6) is easily re-summed to an exponential. In the non-perturbative domain $g = \alpha \chi^{2/3} \gg 1$ this simplification is formalized by the expectation that the contribution to the outer integrals is dominated by [see Eq. (B.5)]

$$\tau \sim \tau_{\text{eff}} = \Pi^{-1}(0, \chi_l) \sim \left( \alpha \chi_l^{2/3} m^2 \right)^{-1} \gg \tau_{\text{eff}}^{(1)}. \hspace{1cm} (3.7)$$

However, we have to be careful and should additionally preserve that $J_{1,2}(\tau)$ vanish at $\tau \to 0$, which can be seen from Eq. (3.6). Otherwise, we would introduce an artificial divergence in the outer integral over $\sigma$. Motivated by this reasoning, let us write

$$J_1(\tau) \approx -2\pi i \theta \left( \text{Re}(\tau) - \tau_{\text{eff}}^{(1)} \right) \frac{\Pi(0, \chi_l)}{m^2} e^{-i \Pi(0, \chi_l) \tau},$$  \hspace{1cm} (3.8)

and, following the same reasoning,

$$J_2(\tau) \approx -2\pi i \theta \left( \text{Re}(\tau) - \tau_{\text{eff}}^{(1)} \right) \left( e^{-i \Pi(0, \chi_l) \tau} - 1 \right),$$  \hspace{1cm} (3.9)

where unlike Eq. (3.8) the insertion of the $\theta$-function is actually not mandatory but with the same accuracy was made for the sake of uniformity.

Since the approximations (3.8), (3.9) are actually at the core of our approach, we checked the validity of the above analytical argument (3.7) by comparing our approximation (3.8) with a direct numerical calculation of $J_1(\tau)$ based on the definition (3.3). The result is shown in Figure 3, where we assumed $\chi_l = 10^4$ but scaled the axes so that the pattern gets frozen in the limit $\chi_l \to \infty$. The figure clearly demonstrates that Eq. (3.8) works perfectly for $\tau \gg \tau_{\text{eff}}^{(1)} = 1/(m^2 \chi_l^{2/3})$ and also ensures (due to the proposed insertion of the $\theta$-function) that $J_1(\tau)$ vanishes at $\tau \to 0$. As it is illuminated by the log-log scale in use, the relative size of the range $\tau \lesssim \tau_{\text{eff}}^{(1)}$ of the quantitative disagreement is of the order of 1%, hence with about the same accuracy this interval can be ignored in processing the outer integrals.
Figure 3. A test of the approximation (3.8) for $J_1(\tau)$ (dashed line) against its direct numerical evaluation (solid line) shown in a double-logarithmic scale (the inset shows the same in a linear scale).

4 Asymptotic behavior of $\delta M$ for $g \gg 1$

Note that the double $\theta$-function arising due to Eqs. (3.1) and (3.8), (3.9) can be transformed into

$$\theta(\text{Re}(s))\theta\left(\text{Re}(\tau) - \tau_{\text{eff}}^{(1)}\right) = \theta(u)\theta(\sigma - \sigma_0(u)), \quad \sigma_0(u) = \left[u^2/(1 + u)\right]^{1/3}. \tag{4.1}$$

Next, let us split the target amplitude (2.16) into three parts:

$$\delta M_i(\chi) = \delta M_i^{(I)}(\chi) + \delta M_i^{(II)}(\chi) + \delta M_i^{(III)}(\chi), \quad i = 1, 2, \tag{4.2}$$

where

$$\delta M^{(I)}_{1,2}(\chi) = \frac{\alpha m^2}{2\pi} \int_0^\infty \frac{du}{(1 + u)^2} \int_{\sigma_0(u)}^\infty d\sigma \frac{e^{-i\sigma^3/3 - i\sigma(u/\chi)^{2/3}}}{\sigma} e^{-ig\varphi_{1,2}(u)} \left(e^{-i\sigma(\chi/\chi_1)} - 1 \right), \tag{4.3}$$

$$\delta M^{(II)}_{1,2}(\chi) = \frac{\alpha m^2}{2\pi} \int_0^\infty \frac{du}{(1 + u)^2} \left(\frac{\chi}{u}\right)^{2/3} \left(\frac{u^2 + 2u + 2}{1 + u} \mp 1\right) \times \int_{\sigma_0(u)}^\infty d\sigma e^{-i\sigma^3/3 - i\sigma(u/\chi)^{2/3}} e^{-ig\varphi_{1,2}(u)} \left(e^{-i\sigma(\chi/\chi_1)} - 1 \right), \tag{4.4}$$

$$\delta M^{(III)}_{1,2}(\chi) = \frac{\alpha g m^2}{4\pi} \int_0^\infty \frac{du}{(1 + u)^2} \left(\frac{\chi}{u}\right)^{2/3} \left(\frac{2 + 2u + u^2}{1 + u} \varphi_{1,2}(u) \right) \times \int_{\sigma_0(u)}^\infty d\sigma e^{-i\sigma^3/3 - i\sigma(u/\chi)^{2/3}} e^{-ig\varphi_{1,2}(u)}, \tag{4.5}$$
with the abbreviations \( \varphi_i(u) = (1 + u)\pi_i(\chi_l)/(\chi_u)^{4/3} \) and \( \Pi_i(l^2 = 0, \chi_l) = \alpha m^2\pi_i(\chi_l) \). Note that in dealing with Eqs. (4.3) and Eqs. (4.4) it is convenient to swap the order of integrals

\[
\int_0^\infty du \int_0^\infty d\sigma \ldots = \int_0^\infty d\sigma \int_0^{u_0(\sigma)} du \ldots, \quad u_0(\sigma) = \frac{\sigma^3}{2} + \sqrt{\frac{\sigma^6}{4} + \sigma^3}. \tag{4.6}
\]

Below we evaluate separately the asymptotic behavior for \( \chi \gg 1 \) of each isolated contribution (4.3), (4.4) and (4.5). As we will see shortly, they essentially differ in their accumulation ranges, hence they should also differ by physical meaning.

**4.1 Contribution \( \delta \mathcal{M}^{(I)} \)**

Starting with Eq. (4.3), we swap the order of the integrals according to (4.6) and note that the effective values of these integrals are formed where \( \sigma \simeq \sigma_{\text{eff}} = 1 \) and \( u \simeq u_{\text{eff}} = 1 \) (to be justified a posteriori). This further implies that \( \chi_l \approx u\chi \sim \chi \gg 1 \) [see (2.13)] and thus \( \pi_i(\chi_l) \approx K_i\chi_l^{2/3} \), where \( K_i \) are the numerical coefficients defined in Eq. (B.6) of Appendix B.

In virtue of the above we can neglect \( \sigma(u/\chi)^{2/3} = O(\chi^{-2/3}) \) and retain just the first non-vanishing term of the expansion over the small argument \( g\sigma\varphi_i(u) \approx \alpha K_i \ll 1 \) of the exponential, thus arriving at

\[
\delta \mathcal{M}_i^{(I)}(\chi) \simeq -C^{(I)} K_i\alpha^2 m^2, \quad i = 1, 2; \tag{4.7}
\]

where the coefficient

\[
C^{(I)} = \frac{i}{2\pi} \int_0^\infty d\sigma e^{-i\sigma^3/3} \int_0^{u_0(\sigma)} \frac{du}{u^{2/3}(1 + u)^{5/3}} \approx 0.256 + 0.325i, \tag{4.8}
\]

is easily evaluated numerically. The structure of the double integral (4.8) obviously justifies our assumptions on the effective scales of the integration variables.

The resulting contribution \( \delta \mathcal{M}^{(I)} = \sum_{i=1}^2 \delta \mathcal{M}_i^{(I)} = O(\alpha^2) \) contains no enhancement for \( \chi \gg 1 \), hence it is actually perturbative\(^4\) and subleading. It is worth noting that since \( \tau_{\text{eff}} \simeq 1/\left(m^2\chi_l^{2/3}\right) \), its accurate evaluation actually requires going beyond the approximation (3.8), (3.9). Fortunately, retaining this contribution against the others would be after all an excess of accuracy, hence we simply ignore it in the sequel.

**4.2 Contribution \( \delta \mathcal{M}^{(II)} \)**

Next let us proceed with the evaluation of Eq. (4.4). It is again convenient to swap the order of integrals according to Eq. (4.6), this time the resulting integral is accumulated at \( \sigma \simeq \sigma_{\text{eff}} = 1 \) but in contrast to \( \delta \mathcal{M}^{(I)} \) at small \( u \simeq u_{\text{eff}} = \alpha^{3/2} \ll 1 \). Assuming \( g = \alpha^{2/3} \gg 1 \), this still implies \( \chi_l \approx u\chi \simeq \alpha^{3/2} \gg 1 \) [see Eq. (2.13)] and thus \( \pi_i(\chi_l) \simeq K_i\chi_l^{2/3} \).

The approximations (3.8), (3.9) are valid, since

\[
\tau_{\text{eff}} \simeq \tau(\sigma_{\text{eff}}, u_{\text{eff}}) \simeq \frac{1}{\alpha gm^2} \gg \frac{1}{m^2\chi_l^{2/3}}, \tag{4.9}
\]

\(^4\)An expansion of the exponential over \( g\sigma\varphi_i(u) \) is clearly the perturbative expansion.
where \( \tau(\sigma,u) \) is defined by Eq. (3.4). As in the previous case we can again neglect the term \( \sigma(u/\chi)^{2/3} = O(\alpha \chi^{-2/3}) \).

Furthermore, due to \( u_{\text{eff}} \ll 1 \), we can also neglect \( u \) in the integrand where possible and replace the upper limit of the inner integral over \( u \) by infinity,

\[
\delta M_{1,2}^{(\Pi)} \approx \frac{(2 \pm 1)\alpha \chi^{2/3} m^2}{2\pi} \int_0^\infty d\sigma \sigma e^{-i\sigma \chi/3} \int_0^\infty \frac{du}{u^{2/3}} \left( e^{-iK_1 \sigma \chi/u^{2/3}} - 1 \right). \tag{4.10}
\]

The integrals involved are reduced to

\[
\int_0^\infty \frac{du}{u^{2/3}} \left( e^{-i\zeta/u^{2/3}} - 1 \right) = 3e^{\frac{5\pi}{12}} \sqrt{\pi \zeta}, \quad \zeta = K_{1,2} \alpha \sigma, \tag{4.11}
\]

\[
\int_0^\infty d\sigma \sigma^{3/2} e^{-i\sigma \chi/3} = e^{-i\frac{5\pi}{12}} 3^{\frac{1}{3}} \Gamma \left( \frac{5}{6} \right), \tag{4.12}
\]

where \( \Gamma(\zeta) \) is the Euler Gamma function, and we arrive at

\[
\delta M^{(\Pi)} = \sum_{i=1}^2 \delta M_i^{(\Pi)} \simeq e^{\frac{5\pi}{12}} \frac{3^{\frac{5}{3}}}{2\sqrt{\pi}} \left( \frac{5}{6} \right) \left( 3\sqrt{K_1} + \sqrt{K_2} \right) \alpha^{3/2} \chi^{2/3} m^2
\]

\[
= (-0.995 + 1.72i) \alpha^{3/2} \chi^{2/3} m^2 \tag{4.13}
\]

The aforementioned scales \( \sigma \simeq 1 \) and \( u \simeq \zeta^{3/2} \simeq \alpha^{3/2} \) of the variables for this case are fixed by Eqs. (4.12) and (4.11).

The obtained Eq. (4.13) is tested against a direct numerical evaluation of Eq. (4.4) in the left upper panel of Figure 4. One can see that the asymptotics (4.13) is indeed eventually achieved, though for extremely high values \( \chi \gtrsim 10^6 \div 10^7 \) corresponding to \( g = \alpha \chi^{2/3} \sim 50 \div 300 \). For smaller \( \chi \) Eq. (4.13) notably overestimates the values of the correction, especially for its real part, which actually alternates sign from plus to minus at \( \chi \simeq 8 \cdot 10^3 \).

### 4.3 Contribution \( \delta M^{(\text{III})} \)

Finally, consider the last contribution (4.5). Here it is convenient to preserve the order of integrals but replace the integration variable \( u \) with \( \chi \) and replace the upper limit of the inner integral over \( u \) by infinity,

\[
\delta M_i^{(\text{III})} = \frac{\alpha^2 m^2 \chi}{2\pi} \int_0^\infty \frac{d\chi_i}{\chi_i^2} \pi_i(\chi_i) \int_1^{\infty} \frac{d\tilde{\sigma}}{\tilde{\sigma}} e^{-i(\chi_i / \chi)^2 \tilde{\sigma}^{3/3} - i \alpha \tilde{\sigma} \pi_i(\chi_i) / \chi_i^{2/3}}. \tag{4.14}
\]

In virtue of \( \pi_i(\chi_i) \gg 1 \simeq K_i \chi_i^{2/3} \), for \( g \gg 1 \) the integrals are effectively truncated from above at \( \chi_i \simeq (\chi_i)^{\text{eff}} = 1 \) and \( \tilde{\sigma} \simeq \alpha^{-1} \). This implies that \( u_{\text{eff}} = \chi^{-1} \ll 1 \) (as expected), \( \sigma_{\text{eff}} = g^{-1} \ll 1 \) and our approximations (3.8), (3.9) are justified by

\[
\tau_{\text{eff}} \simeq \tau(\sigma_{\text{eff}}, u_{\text{eff}}) \simeq \frac{1}{\alpha m^2} > \frac{1}{m^2 (\chi_i)^{2/3}}. \tag{4.15}
\]
Now, neglecting also the first term $O(g^{-3})$ in the exponential in Eq. (4.14), we arrive at

$$\delta M_{\text{III}} \approx C_{i}^{(\text{III})} \alpha^{2} \chi m^{2},$$

(4.16)

where the arising numerical factors $C_{i}^{(\text{III})}$ being expressed in terms of the exponential integral $E_1(\zeta) = \int_{1}^{\infty} \frac{dte^{-\zeta t}}{t}$ by

$$C_{1,2}^{(\text{III})} = \frac{1}{2\pi} \int_{\chi_{l}}^{\infty} \frac{d\chi}{\chi^{2}} \pi_{1,2}(\chi) E_1 \left( i\alpha \pi_{1,2}(\chi) / \chi^{2/3} \right) = \left\{ -0.0395 - 0.472i, -0.0634 - 0.703i \right\},$$

(4.17)

were computed numerically. As a result, we obtain

$$\delta M^{(\text{III})} = \sum_{i=1}^{2} \delta M_{i}^{(\text{III})} = -(0.103 + 1.18i) \alpha^{2} \chi m^{2}.$$  (4.18)

The resulting Eq. (4.18) is tested against a direct numerical evaluation of Eq. (4.5) in the left lower panel of Figure 4. Obviously, the asymptotics (4.18) is achieved starting with the same values $\chi \sim 10^{6} \div 10^{7}$ as for $\delta M^{(\text{II})}$. But unlike for the latter, Eq. (4.18) remains a good order-of-magnitude estimation even for smaller $\chi$.

5 Summary and discussion

Motivated by the Ritus-Narozhny conjecture, we have computed the high-$\chi$ asymptotics of the on-shell electron mass radiative correction in a constant crossed field with the photon propagator fully incorporating the one-loop vacuum polarization operator. This is obviously equivalent to re-summation of the bubble-type radiative corrections in Figure 1.
Recall that previously in this context only diagrams of up to 3rd loop order have been studied, see Appendix A. Let us now discuss the meaning and practical implications of our results.

The running coupling constant $\alpha(l^2, \chi_l) = Z(l^2, \chi_l)\alpha$, naturally arising in our calculation, has a log-dependence on $\chi_l$, hence also on energy, of the same kind as in conventional field-free QED. This justifies and even strengthens the recent claims [32, 33] that the Ritus-Narozhny conjecture (formulated for fixed energy, large field limit) does not affect the fundamental high-energy limit of strong-field QED. In contrast to [32, 33], where this point was proved by considering a more realistic localized pulsed field of fixed strength and duration, here we obtained the same result directly for a constant crossed field (see also [35]). Being focused on the on-shell mass correction, we could not examine here the impact of a strong CCF on mass renormalization. However, our derivation strongly implies that the same should be true for it as well.

Whereas it is well known that the one-loop correction $M_0(\chi) = \mathcal{O}(g)$ (see Eq. (2.11) and Ref. [7]), we have identified two essentially different cumulative higher-order contributions (4.13) and (4.18) for $g \gg 1$. Both are of the same general form $\delta M(\chi) \simeq \sqrt{\alpha} F(\chi)$, but with $F^{(II)}(g) = \mathcal{O}(g)$ (see Eq. (4.13)) and $F^{(III)}(g) = \mathcal{O}(g^{3/2})$ (see Eq. (4.18)). This is consistent with the high-$\chi$ behavior of the known leading lowest-order mass corrections marked bold in the right column of Table 1 of Appendix A and confirms the Ritus-Narozhny conjecture in its part of the proposed significance of the compound parameter $g = \alpha \chi^{2/3}$ in shaping the structure of high-order radiative corrections. The non-perturbative nature of the re-summed corrections is manifested by their non-analytical dependence on $\chi$: $\delta M^{(II)}$ involves a half-integer power of $\alpha$, whereas $\delta M^{(III)}$ implicitly contains $\log \alpha$ in the definition (4.16) of $C_i^{(III)}$.

For $g \gg 1$ the isolated contributions $\delta M^{(II)}$ and $\delta M^{(III)}$ differ not only in growth with $g$, but also by their formation scales. As we have seen in our derivation, in contrast to the perturbative contributions (such as $M_0$ or $\delta M^{(I)}$) being formed at $\chi_l \simeq \chi$ and $\sigma \simeq 1$, the obtained $\delta M^{(II)}$ is formed still at $\sigma \simeq 1$ but at $\chi_l \simeq (\chi_l)_{\text{eff}} = g^{3/2}$, such that $1 \ll (\chi_l)_{\text{eff}} \ll \chi$, whereas $\delta M^{(III)}$ is formed at $\chi_l \simeq 1 \ll \chi$ and $\sigma \simeq g^{-1} \ll 1$. The respective dressed photon proper times, exceeding the electron proper time by factor $u^{-1} \gtrsim 1$ and thus estimating the whole proper time loop durations, are $m\tau^{(I)} \simeq (m\chi^{2/3})^{-1}$ (cf. [21, 36]), $m\tau^{(II)} \simeq (amg)^{-1}$ and $m\tau^{(III)} \simeq (am)^{-1}$, respectively. However, it is worth to emphasize that, as we demonstrate in the left panel of Figure 4, the obtained asymptotic expressions under discussion (hence also the above scalings) are actually reached only for extremely high values of $\chi$ corresponding to $g \gtrsim 50 \div 300$. Most probably, this happens because the high-$\chi_l$ asymptotics of $\Pi_i(0, \chi_l)$, essentially used in our derivation, is in turn reached for rather high values of $\chi_l$, in that case because the high-$\chi_l$ expansion actually runs in increment powers of $\chi_l^{-1/3}$ and involves large numerical coefficients.

It is known that radiative corrections are closely related to the total probabilities of branching processes [18]. First, the optical theorem connects the imaginary part of a correction to the contribution into the total probability of the processes obtained by all

\footnote{Apart from occasional appearance of the log $\chi$-factors in perturbative calculations.}
possible cuts (picturing going the cut lines on-shell) of the given initial diagram. Second, due to unitarity, the correction can be reconstructed back from its imaginary part by using a suitable dispersion relation. In the presence of an external field this procedure has been explicitly illustrated, e.g., in Ref. [6]. Being distinguished by formation scales as discussed, the two isolated contributions (4.13) and (4.18) should naturally differ in physical interpretation as well. Namely, as of now we have quite good reasons to believe that \( \delta M^{(II)} \) and \( \delta M^{(III)} \) correspond to higher-order corrections to photon emission (with the cut shown in left panel in Figure 5) and to trident pair production (center panel in Figure 5), respectively.

Our interpretation is based on the coincidence of their dependence on \( \chi \) (apart from possible occurrence of \( \log \chi \)-factors) with the corresponding probabilities in the lowest order, which are known to be the same as the marked bold 1- and 2-loop contributions in the right column of Table 1 in Appendix A. It is further supported by the normal and abnormal signs of the imaginary parts of the corrections \( \delta M^{(III)} \) and \( \delta M^{(II)} \), respectively, which in virtue of the optical theorem can be only explained by the observation that the former corresponds to a whole process, while the latter to just a correction to another process. The only candidate for its bare part would be the 1-loop contribution \( M_0 \) related to photon emission. Hence we can interpret our results also in terms of potentially observable high-\( \chi \) modifications of the total probabilities for photon emission and trident pair production.

The cumulative non-perturbative contributions into the real and imaginary parts of the on-shell mass operator are illustrated in the right panels of Figure 4. According to the interpretation suggested above, the solid yellow line \( M_0(\chi) + \delta M^{(II)}(\chi) \) (to be compared with the dash-dot blue one \( M_0(\chi) \)) separately indicates the impact of higher-order corrections on photon emission only. One can see that after our asymptotics sets in, the net effect of higher orders is an about \( \sqrt{\alpha} \approx 10\% \) reduction of both the real part and the magnitude of the imaginary part of the invariant amplitude. In general, however, the contribution \( \delta M^{(III)}(\chi) \) (additionally included into the solid green curve) totally dominates and results in a rather substantial suppression of the total real part and enhancement of the magnitude of the total imaginary part. A region of recent special interest with \( g \approx 1 \) [21–23, 25] is shown separately in the insets. Since our asymptotics has not set in yet, here the curves are plotted by direct numerical evaluation of the integrals in Eqs. (4.4) and (4.5). As one can see here, the higher-order corrections to photon emission are suppressed (they are at the level of 0.1\% and few percents for the real part and the magnitude of the imaginary parts, respectively), in accordance with the left panel.

*Figure 5.* The cuts of the bubble diagram for corrections to photon emission (left) and to trident pair production (center). Right: additional dressing due to electron mass corrections.
In summary, the diagrams considered here only have a substantial effect in the fully nonperturbative regime, i.e., for very large values of $\chi$ ($\chi \gtrsim 10^3 \div 10^4$ for $\delta M^{(III)}$ and $\chi \gtrsim 10^4 \div 10^6$ for $\delta M^{(II)}$, see Figure 4). Nevertheless, our calculations could be tested experimentally, as the regime $\chi \sim 10^3$ is accessible in the mid-term future [21–24]. Furthermore, we believe that future studies are both prospective and required before a final conclusion can be reached. In particular, it would be highly instructive though challenging to check the actual validity of our main approximation (3.8), (3.9) by a bypassing direct numerical evaluation of the integrals in Eq. (2.16) and provide a better understanding why it requires very high values of $\chi$ to observe the the derived asymptotic behavior, e.g., by deriving the next-to-leading terms of the expansion in $g^{-1}$. Furthermore, it would be useful to examine how the expressions (4.4), (4.5) are composed directly from the contributions of individual loop orders beyond our approximation. Also, even though our calculation permits conclusions about the total probability for photon emission and trident pair production via the optical theorem, it is certainly instructive to calculate higher-order loop corrections to these processes as well as pair photoproduction directly, in order to clarify also the modifications of their energy distributions, which are most important for practical applications.

It is also worth stressing that the validity of the last part (iii) of the Ritus-Narozhny conjecture regarding the presumed dominance of the bubble-type diagrams (which we took here for granted) still remains unexplored. In particular, the observed here dominance of $\delta M^{(III)}$ over $\delta M^{(II)}$ may indicate that other yet missing (i.e., rainbow) types of diagrams with higher multiplicity in the virtual channel may be equally (if not even more) important. Also, since in a CCF the 1-loop corrections to field-induced masses of dressed electron and photon are of the same order, it is worth to examine how our results will be modified by re-summing additionally the electron mass corrections, see right panel in Figure 5. The observed relative suppression of the 3rd loop correction of this kind (see Table 1 in Appendix A) could be peculiar to this order. Note that since the electron on-shell condition is modified by inclusion of such corrections, more care should be taken when proceeding with mass renormalization. Finally, the presumption of dominance of the bubble-type diagrams is essentially related to an expected suppression of the vertex corrections. Notably, unlike the magnetic field case, for which the latter expectation was indeed confirmed [37], it was in fact questioned for the CCF case in the later work of the Ritus group on the subject [38], and hence should be revisited. We believe that our results, as well as the approach and methods developed here will be useful for further studies on the topic.

A Essence of the Ritus-Narozhny conjecture

The Ritus-Narozhny conjecture is based on the results of a series of calculations of radiative corrections in a CCF of up to 3-loop order, see Table 1. The first (longtime and now well-known) observation is that for $\chi \gg 1$ the 1-loop corrections to both the polarization and mass operators are proportional to $g = \alpha \chi^{2/3}$, see the first row. Though this already indicates that $g$ regulates the importance of radiative corrections, it was still instructive to inspect the impact of this parameter on the assembly of the loop expansion.
The next advance was the calculation of the two-loop corrections to the mass operator. It turned out that the diagram with 1-bubble polarization correction carries an additional factor $\alpha \chi^{1/3}$ (neglecting logarithmic corrections) over the 1-loop contribution, see the second row in the right column\(^6\). This initially caused Ritus to identify $\alpha \chi^{1/3}$ as the effective expansion parameter of the loop expansion in a strong CCF. Moreover, this conclusion was still supported by the next calculated 3-loop corrections to the mass and polarization operators including the 1-bubble corrections to one of two involved photon propagators [10, 11], see the last row.

However, later consideration [11] identified the mass correction due to 2-bubble insertion as the leading one in the 3-loop order, hence also $g = \alpha \chi^{2/3}$ as a refined candidate for an expansion parameter. Hence it was finally conjectured that this parameter should serve as the ratio of the next-order to previous-order leading radiative corrections in all the following successive orders, presumably due to the diagrams accumulating the maximal number of bubble insertions, with an exceptional failure in the transition from 1- to 2-loop order. Note that corrections of this kind to the polarization operator (left column) are upshifted by one loop order, as the polarization operator already contains an extra

\(^6\)Note that the 2- and 3-loop contributions containing vertex corrections are missing in Table 1 as have been not yet considered. However, they were believed to be subleading [10, 11], although cf. [38].
outer fermion loop. It is therefore believed that $g$ also represents the effective expansion parameter of the polarization operator starting from 4th-loop order, yet to be calculated.

B One-loop polarization operator in a constant crossed field

For completeness, here we provide the explicit expressions for the components of the renormalized one-loop polarization operator in a CCF [5, 6, 12].

The renormalization constant $Z$ in Eqs. (2.4), (2.5) is given by

$$Z^{-1}(l^2, \chi_l) - 1 = \frac{4\alpha}{\pi} \int_4^\infty \frac{dv}{v^{5/2}\sqrt{v - 4}} \left[f_1(\zeta) - \log \left(1 - \frac{1}{v/m^2}\right)\right], \quad (B.1)$$

and the polarization operator eigenvalues read

$$\Pi_{1,2}(l^2, \chi_l) = \frac{4\alpha\chi_l^{2/3} m^2}{3\pi} \int_4^\infty \frac{dv}{v^{13/6}} \frac{v + 0.5 \mp 1.5}{\sqrt{v - 4}} f'(\zeta). \quad (B.2)$$

Here

$$\zeta = \left(\frac{v}{\chi_l}\right)^{2/3} \left(1 - \frac{l^2}{vm^2}\right), \quad (B.3)$$

and

$$f(\zeta) = i \int_0^\infty d\sigma e^{-i(\zeta\sigma + \sigma^3/3)} , \quad f_1(\zeta) = \int_\zeta^\infty dz \left(f(z) - \frac{1}{z}\right) \quad (B.4)$$

are the Ritus functions [12].

Note that for $\chi_l \gg 1$ and on bare mass shell $l^2 = 0$ we have

$$Z^{-1}(0, \chi_l) - 1 \simeq -\frac{\alpha}{3\pi} \log \chi_l^{2/3}, \quad \Pi_i(0, \chi_l) \simeq \alpha m^2 \pi_i(\chi_l), \quad \pi_i(\chi_l) = K_i\chi_l^{2/3}, \quad (B.5)$$

where

$$K_1 = 0.175(1 - i\sqrt{3}), \quad K_2 = 1.5K_1. \quad (B.6)$$

The on-shell dependence of the expressions (B.2) and (B.1) on $\chi_l$ is shown in Figure 6. One can see that the asymptotics (B.5) are achieved for $\chi_l \gtrsim 10^3$ and that $Z^{-1} - 1 = \mathcal{O}(10^{-2})$ for all reasonable values of $\chi_l$. 

Figure 6. Dependence on $\chi_l$ of the real (left) and imaginary (center) parts of the polarization operator eigenvalues $\pi_{1,2}$ on the bare mass shell $l^2 = 0$, along with the corresponding asymptotics (B.5). Right: the same dependence for the real and imaginary parts of $Z^{-1} - 1$. 


Finally, in virtue of Eqs. (B.2), (B.3) and (B.4) $\pi_i$ can be represented by a one-sided Fourier integral (3.5), where

$$\tilde{\Pi}_{1,2}(\tau, \chi) = \frac{4\alpha}{3\pi} \chi m^6 \int_4^\infty \frac{dv}{v^{3/2}} \frac{v + 0.5 \mp 1.5}{\sqrt{v - 4}} e^{-im^2v(\tau + m^4\chi^23/3)}, \quad (B.7)$$

and the characteristic values of the variables around which the integral is formed are obviously $v \simeq 1$ and $\tau \simeq \min \left\{ \frac{1}{m^2}, \frac{1}{m^2\chi^2} \right\}$.

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