LARGE TIME BEHAVIOR OF SOLUTION TO QUASILINEAR CHEMOTAXIS SYSTEM WITH LOGISTIC SOURCE

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Abstract. This paper deals with the quasilinear parabolic-elliptic chemotaxis system

\[
\begin{aligned}
&\frac{\partial u}{\partial t} = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u \nabla v) + \mu u - \mu u^r, & x \in \Omega, & t > 0, \\
&\tau \frac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega, & t > 0,
\end{aligned}
\]

under homogeneous Neumann boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary, where \( \tau \in \{0, 1\} \), \( \chi > 0 \), \( \mu > 0 \) and \( r \geq 2 \). \( D(u) \) is supposed to satisfy

\[ D(u) \geq (u + 1)\alpha \] with \( \alpha > 0 \).

It is shown that when \( \mu > \frac{\chi^2}{16} \) and \( r \geq 2 \), then the solution to the system exponentially converges to the constant stationary solution \((1,1)\).

1. Introduction. In this paper, we consider the following chemotaxis-growth system

\[
\begin{aligned}
&\frac{\partial u}{\partial t} = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u \nabla v) + \mu u - \mu u^r, & x \in \Omega, & t > 0, \\
&\tau \frac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega, & t > 0, \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, & t > 0, \\
&u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega,
\end{aligned}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary, where \( u(x, t) \) and \( v(x, t) \) represent the density of the cells population and the concentration of the chemoattractant, respectively. \( \chi > 0 \) is a parameter referred to as chemosensitivity and the parameter \( \mu \) is assumed to be positive. We suppose that the diffusion function \( D(u) \) fulfills

\[ D(u) \in C^2([0, \infty)), \]

and

\[ (u + 1)^3 \geq D(u) \geq (u + 1)^\alpha \] for all \( u > 0 \),

where \( \beta > 0 \), \( \alpha > 0 \) and \( \alpha \neq 1 \). \( \partial / \partial \nu \) denotes the derivative with respect to the outer normal of \( \partial \Omega \).

In the last four decades, many biological phenomena such as angiogenesis, morphogenesis, immune system response have been described using mathematical model. As a subsystem, \((1.1)\) contains the classical chemotaxis system obtained from \((1.1)\)
by setting $\mu = 0$, which was established by Keller and Segel in 1970 [9] (see also [8, 10]) to describe the collective behavior of cells type as follows

$$
\begin{align*}
\left\{ \begin{array}{ll}
  u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u \nabla v), & x \in \Omega, \ t > 0, \\
  \tau v_t &= \Delta v - v + u, & x \in \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega.
\end{array} \right.
\end{align*}
$$

(1.4)

Since then, from a mathematical point of view, the system has been widely studied in the literature. For instance, when $D(u) \equiv 1$, it was shown in [17] that the system (1.4) admits a unique global solution under the condition $n = 1$; when $\tau = 1$ and $n = 2$, it was known that the system (1.4) has a global bounded solution provided that $\int_{\Omega} u_0 < 4\pi$ (see [16]); for the case $\tau = 1$, $n \geq 3$, there is no such threshold, when $\Omega$ is a disk, relying on a Lyapunov function, Winkler [29] showed that there exists radially symmetric solution blowing up in finite time with proper initial conditions.

In general, the chemical substances are much smaller than the cells and therefore the chemicals diffuse much faster than cells. In this view, system (1.4) can be simplified to the case of a parabolic-elliptic system, i.e. $(\tau = 0)$ Nagai (see [14, 15]) found a critical mass which determines the behavior of the solution. Namely, there is a critical mass $m_c > 0$ such that the system (1.4) possesses a global and bounded solution if $\int_{\Omega} u_0 < m_c$, whereas finite time blow up occurs when $\int_{\Omega} u_0 > m_c$.

In view of the underlying biological background, cell motility should be regarded as movement in a porous medium, for this reason, the cell movement can be described by a nonlinear function $D(u) \sim u^\alpha$. The number $1 - \frac{2}{n}$ was detected to be the critical exponent for system (1.4). More precisely, the influence of the diffusion term is greater than the attractive term and hence the solution to system (1.4) exists globally (see [2, 6, 20]) when $\alpha > 1 - \frac{2}{n}$. However, the attractive drift term prevails in the sense that $\alpha < 1 - \frac{2}{n}$ and this leads to the solution to (1.4) blow-up in finite time (see [1, 28]).

When $\mu > 0$, a number of dynamical properties have been detected both numerically and also analytically. For example, the exceedance of corresponding carrying capacities seems possible (see [7, 11, 12, 19, 23, 31, 34, 39]). For example, when $\mu > 0$ is small, the chemotactic cross diffusion was shown to enforce the occurrence of solutions which attain possibly finite but arbitrarily large values (see [31]). In [19], the authors have proved that the population as a whole always persists in the sense that for any nonnegative global classical solution, there exists a lower bound for mass. In [23], it is showed that logistic dampening may prevent blow-up of solutions. Besides, when $\tau = 0$, $r = 2$ and $D(u) \equiv 1$, the main results in [22] showed the prevention of blow up under the conditions $n \leq 2$, $\mu > 0$ or $n \geq 3$, $\mu > \frac{n-2}{2}\chi$. For the case $\tau = 0$ and $D(u) \geq u^\alpha$, Wang et al. [24] established the boundedness of the solution to system (1.1). However, it is shown that in presence of merely certain subquadratic (generalized) logistic-type dampenings, even finite-time blow up will occur (see [32, 35, 39]). For instance, the author in [32] proved that when $\tau = 0$ and $r < \frac{2}{n}$ for $n = 3, 4$ or $r < 1 + \frac{1}{2(n-1)}$ for $n \geq 5$, the solution to system (1.1) blows up at finite time. Under the assumptions $\tau = 1$, $D(u) \equiv 1$, $n \geq 3$ and $\Omega$ is a smooth bounded convex domain, Winkler [30] proved that sufficiently large $\mu$ ensures the global existence and boundedness of the solution to system (1.1). For more results on the classical Keller-Segel model and its variants, we refer the readers to [3–5, 21, 26, 27, 33, 35–37].

Recently, there is an increasing interest in studying the Keller-Segel model and it is worthwhile and challenging to investigate the large time asymptotic behavior.
of the solution. Tello and Winkler [22] showed that the solution to system (1.1) fulfilling
\[
\lim_{t \to \infty} \|u - 1\|_{L^\infty(\Omega)} + \lim_{t \to \infty} \|v - 1\|_{L^\infty(\Omega)} = 0
\]
provided that \(\tau = 0, D(u) \equiv 1, r = 2\) and \(\mu > 2\chi\). Furthermore, Wang et al. [24] extended the results given in [22] and they proved that the quasilinear parabolic-elliptic system (1.1) admits the positive constant equilibrium \((1, 1)\) as a global attractor when \(\mu > 2\chi\). For the case \(\tau = 1\), Lin [13] established the large time asymptotic behavior of the solution to the parabolic-parabolic system (1.1). When \(D(u) \equiv 1\), by constructing a Lyapunov function, the main results in [4] showed that the solution to the system (1.1) exponentially converges to the constant stationary solution \((1, 1)\).

Due to the lack of an effective way, the large time behavior, especially the convergence rate of the solution to quasilinear system (1.1) is still open, our first aim in the present paper is to obtain the condition about the convergence rate for solution to parabolic-elliptic quasilinear system (1.1) i.e. \((\tau = 0)\). In order to prove our main result in this direction, we construct two new Lyapunov functions
\[
F(t) = \int_{\Omega} (\alpha u - u^\alpha + 1 - \alpha)
\]
and
\[
G(t) = \int_{\Omega} (u^\alpha - \alpha u - 1 + \alpha)
\]
for \(\alpha \in (0, 1)\) and \(\alpha > 1\), respectively. Relying on an estimate of the corresponding energy inequality, we can first obtain the convergence of \((u, v)\) in \(L^2(\Omega)\) as well as \(L^\infty(\Omega)\). Finally, we we establish the convergence rate of \((u, v)\) by means of some inequalities.

Our first result reads as follows:

**Theorem 1.1.** Let \(n \geq 2, \tau = 0, D(u) fulfills (1.2) and (1.3). If \(\mu > \frac{\chi^2}{16}\), then for any initial data \(u_0 \in C^0(\Omega)\), the solution \((u, v)\) of model (1.1) is global bounded and satisfies
\[
\|u - 1\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty,
\]
\[
\|v - 1\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.
\]
Moreover, we can find two positive constants \(c(p)\) and \(\delta(p)\) such that for all \(p \in [1, \infty)\), the classical solution of (1.1) satisfies
\[
\|u - 1\|_{L^p(\Omega)} + \|v - 1\|_{L^p(\Omega)} \leq ce^{-\delta t}, \quad t > 0.
\]

**Remark 1.1.** Theorem 1.1 removes the assumption that \(r = 2\), which was essential in [22, 24].

**Remark 1.2.** Our result in this paper, together with the previous results in [24], show that the solution to system (1.1) enjoys the property (1.5) when \(r = 2\) and \(\mu > \min\{2\chi, \frac{\chi^2}{16}\}\).

As an interesting by-product of Theorem 1.1, let us write down

**Corollary 1.1.** Under the same conditions of Theorem 1.1, for all \(p \in [1, \infty]\), there exist positive constants \(c > 0\) and \(a > 0\) independent of \(p\) such that the component solution \(v\) of (1.1) satisfying
\[
\|v - 1\|_{L^p(\Omega)} \leq ce^{-at}.
\]
To the best of our knowledge, few rigorous result seems to be known about the large time behavior and convergence rate of the solution to quasilinear parabolic-parabolic model (1.1). With regard to this, the goal of the present work is to make a substantial step forward towards the large time behavior and convergence rate in quasilinear parabolic-parabolic setting. Motivated by the arguments in [21], fortunately, we construct new functions

\[ H(t) = F(t) + \frac{\chi^2\alpha(1 - \alpha)}{8} \int_\Omega (v - 1)^2 \]

and

\[ I(t) = G(t) + \frac{\chi^2\alpha(1 - \alpha)}{8} \int_\Omega (v - 1)^2 \]

which act as Lyapunov functions \( \alpha \in (0, 1) \) and \( \alpha > 1 \), respectively. By means of the energy function above, we can establish following convergence

\[ \int_\Omega (u - 1)^2 \]

and hence obtain

\[ \|u - 1\|_{L^\infty(\Omega)} \text{ as } t \to \infty. \]

In light of these premises, it seems natural and inevitable that our second result addressing asymptotic homogenization of the solution can be proven.

Now we state the second result as follows.

**Theorem 1.2.** Let \( \tau = 1 \), the diffusion function \( D(u) \) satisfies (1.2) and (1.3). Assume the parameter \( \mu \) fulfills \( \mu > \frac{\chi^2}{16} \) and

\[
\begin{cases}
\alpha > \frac{n-2}{n}, & \text{if } \frac{n+2}{2} \geq r \geq 2, \\
\alpha > 0, & \text{if } r > \frac{n+2}{2},
\end{cases}
\]

then for any initial data \( u_0 \in C^0(\Omega) \), the solution \( (u, v) \) of model (1.1) is global bounded and satisfies

\[
\|u - 1\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty,
\]

\[
\|v - 1\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.
\]

In addition, for \( p \in [1, \infty) \), the classical solution of (1.1) enjoys the following property

\[
\|u - 1\|_{L^p(\Omega)} + \|v - 1\|_{L^p(\Omega)} \leq ce^{-\delta t}, \quad t > 0,
\]

where \( c := c(p) \) and \( \delta := \delta(p) \) are two positive constants.

**Remark 1.3.** Theorem 1.2 is an improvement of the result in [4,13].

**Remark 1.4.** Due to the lack of an effective way, it is not clear about the behavior of the solution when \( \alpha = 1 \) and we have to leave it as an open problem.
Similarly, from Theorem 1.2, we can obtain the following corollary.

**Corollary 1.2.** Let the assumptions of Theorem 1.2 hold, for all \( p \in [1, \infty] \), there exist positive constants \( c > 0 \) and \( b > 0 \) independent of \( p \) such that

\[
\|v - 1\|_{L^p(\Omega)} \leq ce^{-bt}.
\]

2. **Preliminaries.** In this section, we recall some preliminary estimates and some results which will be used in our proof. We begin with some basic results of the solution to model (1.1), the results are quite standard and for details we refer the readers to [22, 24, 25, 37, 38].

**Lemma 2.1.** Suppose that \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with smooth boundary and \( u_0 \in C^0(\Omega) \), \( v_0 \in W^{1,\infty}(\Omega) \) are non-negative functions, \( \tau = \{0, 1\} \), \( D(u) \) fulfills (1.2) and (1.3), then problem (1.1) possesses a local-in-time non-negative classical solution \( u \in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})) \), \( v \in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})) \), where \( T_{\text{max}} \) denotes the maximal existence time. Moreover, if \( T_{\text{max}} < \infty \), then

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\text{max}}.
\]

**Lemma 2.2.** (see [24, 37]) Suppose that \( D(u) \) fulfills (1.2) and (1.3). \( \tau = 0, n \geq 2 \), \( \alpha > 0 \), then for \( r > 2 \) or \( r = 2 \) and \( \mu > \mu_0 \), where

\[
\mu_0 := \begin{cases} 
0, & \text{if } 1 - \alpha < \frac{2}{n}, \\
\frac{n(1-\alpha) - 2}{n(1-\alpha)} \chi, & \text{if } 1 - \alpha \geq \frac{2}{n},
\end{cases}
\]

the system (1.1) possesses a classical bounded solution \((u, v)\) satisfying

\[
\|u\|_{L^\infty(\Omega)} \leq c, \quad \text{for all } t \in (0, \infty),
\]

where \( c = c(\|u_0\|_{L^\infty(\Omega)}) \).

**Lemma 2.3.** (see [38]) Assume that \( \tau = 1, n \geq 2 \) and \( \mu > 0 \). Let the diffusion function \( D(u) \) fulfills (1.2) and (1.3) with

\[
\alpha + 1 \begin{cases} 
> 2 - \frac{2}{n}, & \text{if } 1 < r < \frac{n+2}{n}, \\
> 1 + \frac{(N+2-2r)^+}{N}, & \text{if } \frac{N+2}{2} \geq r \geq \frac{N+2}{N}, \\
\geq 1, & \text{if } r > \frac{N+2}{2}.
\end{cases}
\]

Then for any nonnegative initial data \( u_0 \in C^0(\Omega) \) and \( v_0 \in W^{1,\infty}(\Omega) \), the solution \((u, v)\) of model (1.1) is global bounded.

In the proof of the main result, we will frequently use the following two inequalities.

**Lemma 2.4.** Suppose the assumptions as that in Theorem 1.1 hold, then the solution to system (1.1) fulfills (1.5).

**Proof.** Let

\[
A(t) = \int_\Omega (u - 1 - \ln u) \quad (2.1)
\]
and it is easy to verify that \( s - 1 - \ln s \geq 0 \) for \( s > 0 \). We collect (2.1) and the first equation of (1.1) to see that

\[
\frac{d}{dt} A(t) = \int_{\Omega} \left( 1 - \frac{1}{u} \right) u_t
= \int_{\Omega} \left( 1 - \frac{1}{u} \right) (\nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u \nabla v) + \mu u - \mu u^r)
\]

(2.2)

for all \( t \in (0, \infty) \). Due to (1.3), we arrive at

\[
\frac{d}{dt} A(t) \leq -\mu \int_{\Omega} (u - 1)(u^{r-1} - 1) - \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u}
\]

(2.3)

for all \( t \in (0, \infty) \). Employing the Young’s inequality, we see

\[
\chi \int_{\Omega} \frac{\nabla u \cdot \nabla v}{u} \leq \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{4} \int_{\Omega} |\nabla v|^2,
\]

(2.4)

on another hand, we can find

\[
(u - 1)(u^{r-1} - 1) \geq (u - 1)^2
\]

(2.5)

hold for all \( u \geq 0 \) and \( r \geq 2 \). A combination of (2.3)-(2.5) yields

\[
\frac{d}{dt} A(t) \leq -\mu \int_{\Omega} (u - 1)^2 + \frac{\chi^2}{4} \int_{\Omega} |\nabla v|^2
\]

(2.6)

for all \( t \in (0, \infty) \). To estimate \( \int_{\Omega} |\nabla v|^2 \), we test the second equation of (1.1) by \((v - 1)\) and integrate by parts to compute

\[
0 = \int_{\Omega} \Delta v (v - 1) + \int_{\Omega} u (v - 1) - \int_{\Omega} v (v - 1)
= -\int_{\Omega} |\nabla v|^2 + \int_{\Omega} (u - 1) (v - 1) - \int_{\Omega} (v - 1)^2.
\]

(2.7)

Applying the Young’s inequality again, we discover

\[
\int_{\Omega} |\nabla v|^2 \leq \frac{1}{4} \int_{\Omega} (u - 1)^2 + \int_{\Omega} (v - 1)^2 - \int_{\Omega} (v - 1)^2
\leq \frac{1}{4} \int_{\Omega} (u - 1)^2.
\]

(2.8)

Applying (2.6) and (2.8), we deduce

\[
\frac{d}{dt} A(t) \leq \left( \frac{\chi^2}{16} - \mu \right) \int_{\Omega} (u - 1)^2
\]

for all \( t \in (0, \infty) \). Since \( \mu > \frac{\chi^2}{16} \), we have

\[
\frac{d}{dt} A(t) \leq -\epsilon \int_{\Omega} (u - 1)^2
\]

(2.9)

for all \( t \in (0, \infty) \), where \( \epsilon := \mu - \frac{\chi^2}{16} > 0 \). Integrating (2.9) from \( t_0 > 0 \) to \( t \), we infer that

\[
A(t) - A(t_0) \leq -\epsilon \int_{t_0}^t \int_{\Omega} (u - 1)^2 \quad \text{for all } t > t_0 > 0.
\]
Thanks to $A(t) \geq 0$ and the boundedness of $u$, we conclude
\[
\int_{t_0}^\infty \int_\Omega (u - 1)^2 \leq \frac{A(t_0)}{\epsilon},
\]
for all $t > t_0 > 0$, from [18] and the condition (1.3), we have $u$ and $u - 1$ are uniform continue and hence
\[
\int_\Omega (u - 1)^2 \to 0 \text{ as } t \to \infty.
\]
According to (2.7) and using the Young’s inequality, we arrive at
\[
\int_\Omega |\nabla v|^2 = \int_\Omega (u - 1)(v - 1) - \int_\Omega (v - 1)^2 
\leq \frac{1}{2} \int_\Omega (u - 1)^2 + \frac{1}{2} \int_\Omega (u - 1)^2 - \int_\Omega (v - 1)^2 
\leq \frac{1}{2} \int_\Omega (u - 1)^2 - \frac{1}{2} \int_\Omega (v - 1)^2,
\]
and hence
\[
\int_\Omega (v - 1)^2 \leq \int_\Omega (u - 1)^2 \to 0 \text{ as } t \to \infty.
\]
If (1.5) was false, we could find a positive constant $l > 0$, \{t_k\}_{k \in \mathbb{N}} \subset (1, \infty)$, and \{x_k\}_{k \in \mathbb{N}} \subset \Omega such that $t_k \to \infty$ as $k \to \infty$ and
\[
|u(x_k, t_k) - 1| \geq l \text{ for all } k \in \mathbb{N}.
\]
Since $u$ and $\nabla v$ are bounded in $\Omega \times (1, \infty)$ according to Lemma 2.2, from two straightforward applications of well-known H"{o}lder estimates for scalar parabolic problems (see [18]), we can claim that $u$ and $u - 1$ are uniformly continuous in $\Omega \times (1, \infty)$. Therefore, there exist two positive constants $m$ and $\rho$ fulfill
\[
|u(x, t) - 1| \geq \frac{l}{2} \text{ for all } x \in B_{\rho}(x_k) \cap \Omega \text{ and } t \in (t_k, t_k + m)
\]
for arbitrary $k \in \mathbb{N}$. Due to the smoothness of the boundary $\partial \Omega$, we can pick a positive constant $c := \inf_{k \in \mathbb{N}} |B_{\rho}(x_k) \cap \Omega|$, from this we infer that
\[
\int_{t_k}^{t_k + m} \int_\Omega (u(x, t) - 1)^2 \, dx \, dt \geq \int_{t_k}^{t_k + m} \int_{B_{\rho}(x_k) \cap \Omega} (u(x, t) - 1)^2 \, dx \, dt \geq \frac{l^2mc}{4}
\]
for all $k \in \mathbb{N}$.

However, on the other hand, from Lemma 3.1, we must have
\[
\frac{l^2mc}{4} \leq \int_{t_k}^{t_k + m} \int_\Omega (u(x, t) - 1)^2 \, dx \, dt \leq \int_{t_k}^{\infty} \int_\Omega (u(x, t) - 1)^2 \, dx \, dt \to 0
\]
as $k \to \infty$, which is absurd and hence establishes (1.5).

**Lemma 2.5.** Suppose the assumptions as that in Theorem 1.2 hold, then the solution to system (1.1) fulfills (1.8).
Proof. We text the second function in (1.1) with $\chi_2^4 (v-1)$ to obtain

$$\frac{\chi_2^2}{4} \int_{\Omega} (v-1)v_t = \int_{\Omega} (v-1)(\Delta v - (v-1) + (u-1))$$

$$\leq - \frac{\chi_2^2}{4} \int_{\Omega} |\nabla v|^2 - \frac{\chi_2^2}{4} \int_{\Omega} (v-1)^2 + \frac{\chi_2^2}{4} \int_{\Omega} (v-1)(u-1)$$

$$\leq - \frac{\chi_2^2}{4} \int_{\Omega} |\nabla v|^2 - \frac{\chi_2^2}{4} \int_{\Omega} (v-1)^2 + \frac{\chi_2^2}{4} \int_{\Omega} (u-1)^2$$

$$\leq - \frac{\chi_1^2}{4} \int_{\Omega} |\nabla v|^2 + \frac{\chi_1^2}{16} \int_{\Omega} (u-1)^2.$$  \hfill (2.10)

From (2.6) and (2.11), define $B(t) := A(t) + \frac{\chi_2^2}{8} \int_{\Omega} (v-1)^2$, we see $B(t) \geq 0$ for all $t > 0$ and

$$\frac{d}{dt} B(t) \leq - \left( \mu - \frac{\chi_2^2}{16} \right) \int_{\Omega} (u-1)^2$$  \hfill (2.11)

for all $t > 0$. Thanks to $\mu > \frac{\chi_2^2}{16}$, for any $t_0 \geq 0$, we have

$$B(t) - B(t_0) \leq - \epsilon \int_{t_0}^t \int_{\Omega} (u-1)^2$$

for all $t > t_0 > 0$. Thanks to $B(t) \geq 0$, the boundedness and uniform continuity of $u$ and $v$ (from Lemma 2.3, (1.3) and [18]), we conclude

$$\int_{t_0}^{\infty} \int_{\Omega} (u-1)^2 \leq \frac{B(t_0)}{\epsilon}$$

for all $t > t_0 > 0$, this implies

$$\int_{\Omega} (u-1)^2 \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Same arguments give

$$\int_{\Omega} (v-1)^2 \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Similar to the proof of Lemma 2.4, we arrive at (1.8) immediately.

3. Proof of Theorem 1.1. In this section, we treat the asymptotic behavior of the solution to system (1.1) under the condition $\tau = 0$. Motivated by the methods in [4, 21], we can derive the following estimates of the solution to (1.1) by making full use of the new Lyapunov function $F(t)$ and $G(t)$ as follows.

We first consider the case $\alpha \in (0, 1)$.

Lemma 3.1. Let $n \geq 2$, $\tau = 0$, $\alpha \in (0, 1)$, $r \geq 2$, $\mu > \frac{\chi_2^2}{16} \geq \frac{\chi_2^2}{16(r-1)}$, the diffusion function $D(u)$ satisfies (1.2) and (1.3). Then for any initial data $u_0 \in C^0(\overline{\Omega})$, the corresponding solution of (1.1) fulfills (1.6)

Proof. We construct a new Lyapunov function

$$F(t) = \int_{\Omega} (\alpha u - u^\alpha + 1 - \alpha)$$  \hfill (3.1)
and it is easy to verify that $\alpha s - s^{\alpha} + 1 - \alpha \geq 0$ for $s > 0$, thus we have $F(t) \geq 0$ for all $t > 0$. We collect (1.2), (1.3), (3.1) and the first equation of (1.1) to see that

$$
\frac{d}{dt} F(t) = \int_\Omega \alpha u_t - \alpha \int_\Omega u^{\alpha-1} u_t
$$

$$
= \alpha \mu \int_\Omega u - \alpha \mu \int_\Omega u^r - \int_\Omega \alpha u^{\alpha-1} \big( \nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u \nabla v) + \mu u - \mu u^r \big)
$$

$$
= \alpha \mu \int_\Omega (-u^r + u^{r+\alpha-1} + u - u^\alpha) - \alpha (1 - \alpha) \int_\Omega u^{\alpha-2} D(u) |\nabla u|^2
$$

$$
+ \chi \alpha (1 - \alpha) \int_\Omega u^{\alpha-1} \nabla u \cdot \nabla v
$$

$$
\leq - \alpha \mu \int_\Omega (u^{r-1} - 1)(u - u^\alpha) - \alpha (1 - \alpha) \int_\Omega u^{2\alpha-2} |\nabla u|^2
$$

$$
+ \chi \alpha (1 - \alpha) \int_\Omega u^{\alpha-1} \nabla u \cdot \nabla v
$$

for all $t > 0$. Employing Young’s inequality, we arrive at

$$
\chi \alpha (1 - \alpha) \int_\Omega u^{\alpha-1} \nabla u \cdot \nabla v \leq \alpha (1 - \alpha) \int_\Omega u^{2\alpha-2} |\nabla u|^2 + \frac{\chi^2 \alpha (1 - \alpha)}{4} \int_\Omega |\nabla v|^2
$$

for all $t > 0$. A combination of (3.2) and (3.3) yields

$$
\frac{d}{dt} F(t) \leq \frac{\chi^2 \alpha (1 - \alpha)}{4} \int_\Omega |\nabla v|^2 - \alpha \mu \int_\Omega (u^{r-1} - 1)(u - u^\alpha)
$$

for all $t > 0$. To estimate $\int_\Omega |\nabla v|^2$, we test the second equation of (1.1) by $(v - 1)$ and integrate by parts to compute

$$
0 = \int_\Omega \Delta v (v - 1) + \int_\Omega u (v - 1) - \int_\Omega v (v - 1)
$$

$$
= - \int_\Omega |\nabla v|^2 + \int_\Omega (u - 1) (v - 1) - \int_\Omega (v - 1)^2
$$

for all $t > 0$. Applying Young’s inequality again, we discover

$$
\int_\Omega |\nabla v|^2 \leq \frac{1}{4} \int_\Omega (u - 1)^2 + \int_\Omega (v - 1)^2 - \int_\Omega (v - 1)^2
$$

$$
\leq \frac{1}{4} \int_\Omega (u - 1)^2
$$

for all $t > 0$. Recalling (3.2)-(3.5), we obtain

$$
\frac{d}{dt} F(t) \leq - \alpha \mu \int_\Omega (u^{r-1} - 1)(u - u^\alpha) + \frac{\chi^2 \alpha (1 - \alpha)}{16} \int_\Omega (u - 1)^2
$$

for all $t > 0$. Making use of the fact

$$
\lim_{s \to 1} \frac{(u^{r-1} - 1)(u - u^\alpha)}{(u - 1)^2} = (r - 1)(1 - \alpha)
$$

and

$$
\mu > \frac{\chi^2}{16(r - 1)},
$$

(3.8)
there exist positive constants $t_1 > 0$ and $\epsilon_1 > 0$ such that
\[
\mu(r-1)(1-\epsilon_1) > \frac{\chi^2}{16}
\]
and
\[(1-\epsilon_1)(r-1)(1-\alpha)(u-1)^2 \leq (u^{-\alpha} - 1)(u-u^\alpha) \leq (1 + \epsilon_1)(r-1)(1-\alpha)(u-1)^2
\]
for all $t > t_1$. Combine (3.6)-(3.9), we deduce
\[
\frac{d}{dt} F(t) \leq -\epsilon_2 \int_{\Omega} (u-1)^2
\]
where $\epsilon_2 := \mu(r-1)(1-\alpha)\alpha (1-\epsilon_1) - \frac{\chi^2}{16} \alpha (1-\alpha) > 0$. Since
\[
\lim_{u \to 1} \frac{\alpha u - u^\alpha - \alpha + 1}{(u-1)^2} = \frac{(1-\alpha)\alpha}{2}.
\]
A combination of Lemma 2.4 and (3.11) yields a positive constant $t_2 > t_1 > 0$ such that
\[
\frac{(1-\alpha)\alpha}{4} (u-1)^2 \leq \alpha u - u^\alpha - \alpha + 1 \leq (1-\alpha)\alpha (u-1)^2
\]
for all $t > t_2$. Combining (3.10) and (3.12), it shows that
\[
\frac{d}{dt} F(t) \leq -\epsilon_3 F(t) \text{ for all } t > t_2
\]
if we let $\epsilon_3 := \frac{\epsilon_2}{\alpha (1-\alpha)}$ and hence we conclude
\[
F(t) \leq F(t_2) e^{-\epsilon_3(t-t_2)} \text{ for all } t > t_2.
\]
Recalling (3.12) and (3.14), it follows that
\[
\int_{\Omega} (u-1)^2 \leq \frac{4}{(1-\alpha)\alpha} F(t_2) e^{-\epsilon_3(t-t_2)} \text{ for all } t > t_2
\]
and this ensures that
\[
\|u-1\|_{L^2(\Omega)} \leq \left( \frac{4}{(1-\alpha)\alpha} F(t_2) e^{-\epsilon_3(t-t_2)} \right)^{\frac{1}{2}} \leq \left( \frac{4}{(1-\alpha)\alpha} F(t_2) \right)^{\frac{1}{2}} e^{-\frac{\epsilon_3}{2}(t-t_2)}
\]
for all $t > t_2$. Applying the interpolation inequality, for any $p \in (2, \infty)$, we deduce
\[
\|u-1\|_{L^p(\Omega)} \leq \|u-1\|_{L^2(\Omega)}^{1-a} \|u-1\|_{L^\infty(\Omega)}^{a} \leq \|u-1\|_{L^2(\Omega)}^{a} e^{-\frac{\epsilon_3}{2}(t-t_2)}
\]
where $a = \frac{2}{p} \in (0, 1)$ is a positive constant. According to Lemma 2.2 we know that $u$ and hence also $u-1$ is uniformly boundedness in $\Omega \times (0, \infty)$ and this readily entails a constant $c > 0$ such that
\[
\|u-1\|_{L^p(\Omega)} \leq c \|u-1\|_{L^2(\Omega)} \leq c \left( \frac{4}{(1-\alpha)\alpha} F(t_2) \right)^{\frac{1}{2}} e^{-\frac{\epsilon_3}{2}(t-t_2)}.
\]
On the other hand, by Holder’s inequality, we can conclude
\[
\|u-1\|_{L^p(\Omega)} \leq |\Omega|^{\frac{2p}{p-2}} \|u-1\|_{L^2(\Omega)} \leq |\Omega|^{\frac{2p}{p-2}} \left( \frac{4}{(1-\alpha)\alpha} F(t_2) \right)^{\frac{1}{2}} e^{-\frac{\epsilon_3}{2}(t-t_2)}
\]
where $p \in [1, 2)$. Similar arguments give the desired estimates of $v$, together with (3.15) and (3.16), we obtain (1.6) immediately.

Next, making full use of a function $G(t) = u^α - αu + α - 1$, we consider the case $α > 1$.

**Lemma 3.2.** Let $n \geq 2$, $τ = 0$, $α > 1$, $r ≥ 2$ and $μ > \frac{χ^2}{16} ≥ \frac{χ^2}{16(τ-1)}$. Then for any initial data $u_0 ∈ C^0(Ω)$, the corresponding solution of (1.1) fulfills (1.6).

**Proof.** Let

$$G(t) = u^α - αu + α - 1$$

(3.17)

and it is easy to see that $s^α - αs + α - 1 ≥ 0$ for $s > 0$, thus we have $G(t) ≥ 0$ for all $t > 0$. By (1.1)-(1.3) and (3.17), we observe that

$$\frac{d}{dt}G(t) = α \int_Ω u^{α-1}u_t - \int_Ω αu_t$$

$$= \int_Ω αu^{α-1}(∇ \cdot (D(u)∇u) - ∇ \cdot (χu∇v) + μu - μu_r) - αμ \int_Ω u + αμ \int_Ω u_r$$

$$= -αμ \int_Ω (-u_r + u^{r+α-1} + u - u^α) - α(α - 1) \int_Ω u^{α-2}D(u)|∇u|^2$$

$$+ χα(α - 1) \int_Ω u^{α-1}∇u \cdot ∇v$$

$$≤ -αμ \int_Ω (u^{r-1} - 1)(u^α - u) - α(α - 1) \int_Ω u^{2α-2}|∇u|^2$$

$$+ χα(α - 1) \int_Ω u^{α-1}∇u \cdot ∇v$$

(3.18)

for all $t > 0$. In light of Young’s inequality, we thereby conclude

$$χα(α - 1) \int_Ω u^{α-1}∇u \cdot ∇v ≤ α(α - 1) \int_Ω u^{2α-2}|∇u|^2 + \frac{χ^2α(α - 1)}{4} \int_Ω |∇v|^2$$

(3.19)

for all $t > 0$. A combination of (3.18) and (3.19) yields

$$\frac{d}{dt}G(t) ≤ -αμ \int_Ω (u^{r-1} - 1)(u^α - u) + \frac{χ^2α(α - 1)}{4} \int_Ω |∇v|^2$$

(3.20)

for all $t > 0$. Next, we obtain from (3.5) and (3.20) that

$$\frac{d}{dt}G(t) ≤ -αμ \int_Ω (u^{r-1} - 1)(u^α - u) + \frac{χ^2α(α - 1)}{16} \int_Ω (u - 1)^2$$

(3.21)

for all $t > 0$. By L’Hospital’s rule, we can see

$$\lim_{u \to 1} \frac{(u^{r-1} - 1)(u^α - u)}{(u - 1)^2} = (r - 1)(α - 1),$$

(3.22)

combine with the condition $μ > \frac{χ^2}{16(τ-1)}$, it immediately implies the existence of $t_3 > 0$ and $ε_4 > 0$ such that

$$(1 - ε_4)(r - 1)(α - 1)(u - 1)^2 ≤ (u^{r-1} - 1)(u^α - u) ≤ (1 + ε_4)(r - 1)(α - 1)(u - 1)^2$$

(3.23)
for all \( t > t_3 \) and \( \mu(r - 1)(1 - \epsilon_4) > \frac{x^2}{16} \). Thus, from (3.21)-(3.23), we arrive at
\[
\frac{d}{dt} G(t) \leq - \left( \mu\alpha(\alpha - 1)(r - 1)(1 - \epsilon_4) - \frac{\chi^2\alpha(\alpha - 1)}{16} \right) \int_{\Omega} (u - 1)^2 \\
\leq - \epsilon_5 \int_{\Omega} (u - 1)^2,
\]
where \( \epsilon_5 \) is a positive constant. Using L’Hospital’s rule again, we can find the fact that
\[
\lim_{u \to 1} \frac{u^\alpha - \alpha u + \alpha - 1}{(u - 1)^2} = \frac{(\alpha - 1)\alpha}{2} > 0. \tag{3.25}
\]
(1.5) and (3.25) ensures the existence of \( t_4 > 0 \) such that
\[
\frac{(\alpha - 1)\alpha}{4}(u - 1)^2 \leq u^\alpha - \alpha u + \alpha - 1 \leq (\alpha - 1)\alpha(u - 1)^2 \quad \text{for all } t > t_4
\]
and this entails
\[
\frac{(\alpha - 1)\alpha}{4} \int_{\Omega} (u - 1)^2 \leq G(t) \leq (\alpha - 1)\alpha \int_{\Omega} (u - 1)^2 \quad \text{for all } t > t_4. \tag{3.26}
\]
Recalling (3.24) and (3.26), we derive
\[
\frac{d}{dt} G(t) \leq - \epsilon_5 \int_{\Omega} (u - 1)^2 \leq - \frac{\epsilon_5}{(\alpha - 1)\alpha} G(t) \quad \text{for all } t > t_4 \tag{3.27}
\]
and consequently,
\[
G(t) \leq G(t_4)e^{-\frac{\epsilon_5}{(\alpha - 1)\alpha}(t-t_4)} \quad \text{for all } t > t_4. \tag{3.28}
\]
Collecting (3.26) and (3.28), we get
\[
\int_{\Omega} (u - 1)^2 \leq \frac{4}{(\alpha - 1)\alpha} G(t) \leq \frac{4}{(\alpha - 1)\alpha} G(t_4)e^{-\frac{\epsilon_5}{(\alpha - 1)\alpha}(t-t_4)} \quad \text{for all } t > t_4
\]
and it follows that
\[
\|u - 1\|_{L^2(\Omega)} \leq \left( \frac{4}{(\alpha - 1)\alpha} G(t) \right)^{\frac{1}{2}} \leq \left( \frac{4}{(\alpha - 1)\alpha} G(t_4) \right)^{\frac{1}{2}} e^{-\frac{\epsilon_5}{2(\alpha - 1)\alpha}(t-t_4)} \tag{3.29}
\]
for all \( t > t_4 \). Using the interpolation inequality and the uniformly boundedness of \( u \) to obtain
\[
\|u - 1\|_{L^p(\Omega)} \leq \|u - 1\|_{L^\infty(\Omega)}^{1-a} \|u - 1\|_{L^2(\Omega)}^{a} \leq c \left( \frac{4}{(\alpha - 1)\alpha} G(t_4) \right)^{\frac{1}{2}} \frac{4}{(\alpha - 1)\alpha} e^{-\frac{\epsilon_5}{2(\alpha - 1)\alpha}(t-t_4)},
\]
where \( p \in (2, \infty) \) and \( a = \frac{2}{p} \in (0, 1) \). By means of Holder’s inequality, we can conclude
\[
\|u - 1\|_{L^p(\Omega)} \leq \left| \Omega \right|^{\frac{2-p}{p}} \|u - 1\|_{L^2(\Omega)}^{\frac{2-p}{p}} \left( \frac{4}{(\alpha - 1)\alpha} G(t_4) \right)^{\frac{1}{2}} e^{-\frac{\epsilon_5}{2(\alpha - 1)\alpha}(t-t_4)}, \tag{3.30}
\]
where \( p \in [1, 2) \). Similar arguments give the desired estimates of \( v \), combine with (3.30) and (3.31), this immediately leads to (1.6).

Now we are in a position to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** With the aid of Lemma 2.4, we can find that the solution to system (1.1) fulfills (1.5). In addition, by Lemma 3.1 and Lemma 3.2, we obtain the desired result (1.6).
4. **Proof of Theorem 1.2.** This section deals with the parabolic-parabolic case \( \tau = 1 \) and we prove the following inequality, which plays an important role to obtain the convergence rate of the solution to (1.1).

**Lemma 4.1.** Let \( n \geq 2, \tau = 1, \alpha \in (0,1), \mu > \frac{\chi^2}{16} \geq \frac{\chi^2}{16(\tau-1)}, \) the diffusion function \( D(u) \) satisfies (1.2) and (1.3), parameters \( r \) and \( \alpha \) satisfy (1.7). Then for any initial data \( u_0 \in C^0(\Omega) \), the corresponding solution of (1.1) fulfills the property (1.9).

**Proof.** We construct a new function

\[
H(t) = F(t) + \frac{\chi^2\alpha(1-\alpha)}{8} \int_{\Omega} (v-1)^2
\]

and it is easy to observe that \( H(t) = 0 \) for all \( t > 0 \). It follows from (1.2), (1.3), (4.1) and the first equation of (1.1) that

\[
\frac{d}{dt} H(t) = \int_{\Omega} \alpha u_t + u^2 - \int_{\Omega} \alpha u^2 \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\chi u \nabla v) + \mu u - \mu u^r
\]

\[
= -\alpha \mu \int_{\Omega} (u^r - 1)(u - u^o) - \alpha (1 - \alpha) \int_{\Omega} u^{2\alpha-2} |\nabla u|^2
\]

\[
+ \chi \alpha (1 - \alpha) \int_{\Omega} u^{\alpha-1} \nabla u \cdot \nabla v - \frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} |\nabla v|^2
\]

\[
- \frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} (v-1)^2 + \frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} (v-1)(u-1)
\]

(4.2)

for all \( t > 0 \). We invoke Young’s inequality to see that

\[
\chi \alpha (1 - \alpha) \int_{\Omega} u^{\alpha-1} \nabla u \cdot \nabla v \leq \alpha (1 - \alpha) \int_{\Omega} u^{2\alpha-2} |\nabla u|^2 + \frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} |\nabla v|^2
\]

(4.3)

for all \( t > 0 \). Inserting (4.3) back into (4.2), we conclude

\[
\frac{d}{dt} H(t) \leq -\alpha \mu \int_{\Omega} (u^r-1)(u-u^o) - \frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} (v-1)^2
\]

\[
+ \frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} (v-1)(u-1)
\]

(4.4)

for all \( t > 0 \). (3.7) and \( \mu > \frac{\chi^2}{16(\tau-1)} \) ensure the existence of positive constant \( t_5, \epsilon_6 \) and \( \epsilon \) such that

\[
\mu(r-1) - \frac{\chi^2}{16} > \frac{\chi^2}{4} \epsilon > 0,
\]

\[
\frac{\chi^2\alpha(1-\alpha)}{4} \int_{\Omega} (v-1)(u-1) \leq \frac{\chi^2\alpha(1-\alpha)}{4} \left( \frac{1}{4} + \epsilon \right) \int_{\Omega} (u-1)^2
\]

\[
+ \frac{\chi^2\alpha(1-\alpha)}{4} (1 - C(\epsilon)) \int_{\Omega} (v-1)^2
\]

(4.5)
and
\[(r-1)(1-\alpha)(1-\epsilon_6)(u-1)^2 \leq (u^{r-1}-1)(u-u^\alpha) \leq (r-1)(1-\alpha)(1+\epsilon_6)(u-1)^2,\]
\[\mu(1-\alpha)(r-1)(1-\epsilon_6) - \frac{\chi^2\alpha(1-\alpha)}{4} \left(\frac{1}{4} + \epsilon\right) > 0\] (4.6)
for all \(t > t_5\). Together (4.4), (4.5) and Young’s inequality, we show
\[
\frac{d}{dt}H(t) \leq -\alpha \mu \int_\Omega (u^{r-1}-1)(u-u^\alpha) - \frac{\chi^2\alpha(1-\alpha)}{4} C(\epsilon) \int_\Omega (u-1)^2
\]
\[+ \frac{\chi^2\alpha(1-\alpha)}{4} \left(\frac{1}{4} + \epsilon\right) \int_\Omega (u-1)^2.\] (4.7)
Applying (4.6), we have
\[-\mu(1-\alpha)(r-1)(1-\epsilon_6) \int_\Omega (u-1)^2 \leq -\mu(1-\alpha)(r-1)(1-\epsilon_6) \int_\Omega (u-1)^2,\] (4.8)
combine with (4.7) and (4.8), we conclude
\[
\frac{d}{dt}H(t) \leq -\left(\alpha \mu(1-\alpha)(r-1)(1-\epsilon_6) - \frac{\chi^2\alpha(1-\alpha)}{4} \left(\frac{1}{4} + \epsilon\right)\right) \int_\Omega (u-1)^2
\]
\[-\frac{\chi^2\alpha(1-\alpha)}{4} C(\epsilon) \int_\Omega (v-1)^2
\]
\[\leq -\epsilon_7 \int_\Omega (u-1)^2 - \frac{\chi^2\alpha(1-\alpha)}{4} C(\epsilon) \int_\Omega (v-1)^2.\] (4.9)
where \(\epsilon_7\) is a positive constant. According to (1.8), there exists a positive constant \(t_6 > t_5 > 0\) such that
\[
\frac{(1-\alpha)\alpha}{4} (u-1)^2 \leq \alpha u - u^\alpha - \alpha + 1 \leq (1-\alpha)\alpha(u-1)^2 \text{ for all } t > t_6,
\]
and
\[
\frac{(1-\alpha)\alpha}{4} \int_\Omega (u-1)^2 + \frac{\chi^2\alpha(1-\alpha)}{8} \int_\Omega (v-1)^2 \leq H(t) \leq H(t_6)e^{-\chi_8(t-t_6)} \text{ for all } t > t_6.
\] (4.10)
From (4.11), we derive
\[
\int_\Omega (u-1)^2 \leq \frac{4}{(1-\alpha)\alpha} H(t) \leq \frac{4}{(1-\alpha)\alpha} H(t_6)e^{-\chi_8(t-t_6)} \text{ for all } t > t_6,\] (4.12)
\[
\int_\Omega (v-1)^2 \leq \frac{8}{\chi^2\alpha(1-\alpha)} H(t) \leq \frac{8}{\chi^2\alpha(1-\alpha)} H(t_6)e^{-\chi_8(t-t_6)} \text{ for all } t > t_6\] (4.13)
and thus we deduce
\[
\|(u-1)\|_{L^2(\Omega)} \leq \sqrt{\frac{4}{(1-\alpha)\alpha} H(t_6)e^{-\chi_8(t-t_6)}} \leq \left(\frac{4}{(1-\alpha)\alpha} H(t_6)\right)^{\frac{1}{2}} e^{-\frac{\chi_8}{2}(t-t_6)}\] (4.14)
Lemma 4.2. Let \( n \geq 2, \tau = 1, r \geq 2 \) and \( \mu > \frac{\chi^2}{16(r-1)} \), the diffusion function \( D(u) \) satisfies (1.2) and (1.3) with \( \alpha > 1 \). Then for any initial data \( u_0 \in C^6(\Omega) \), the corresponding solution of (1.1) fulfills the property (1.9).

Proof. We construct a function

\[
I(t) = G(t) + \frac{\chi^2}{16(r-1)} \int_\Omega (v - 1)^2
\]

(4.20)

Similarly, we have \( I(t) \geq 0 \) for all \( t \geq 0 \). Differentiate \( I(t) \) and integrate by parts to see that

\[
\frac{d}{dt} I(t) = \alpha \int_\Omega u^{\alpha-1}u_t - \int_\Omega \alpha u_t + \frac{\chi^2}{16(r-1)} \int_\Omega (v - 1) v_t
\]

\[
= \int_\Omega \alpha u^{\alpha-1}(\nabla \cdot (D(u) \nabla u) - \nabla \cdot (\chi u \nabla v) + \mu u - \mu u^r) - \alpha \mu \int_\Omega u + \alpha \mu \int_\Omega u^r
\]

\[
+ \frac{\chi^2}{16(r-1)} \int_\Omega (v - 1) (\Delta v - (v - 1) + (u - 1))
\]

\[
\leq - \alpha \mu \int_\Omega (u^{r-1} - 1)(u^\alpha - u) - \alpha (\alpha - 1) \int_\Omega u^{2\alpha-2} |\nabla u|^2
\]

and

\[
\|v - 1\|_{L^2(\Omega)} \leq \left(\frac{8}{\chi^2(1-\alpha)} H(t_0) e^{-\alpha(t-t_0)}\right)^{\frac{1}{2}} \leq \left(\frac{8}{\chi^2(1-\alpha)} H(t_0)\right)^{\frac{1}{2}} e^{-\frac{8t}{\alpha}(t-t_0)}
\]

(4.15)

for all \( t > t_6 \). In view of the interpolation inequality and the boundedness of \( u \), for any \( p \in (2, \infty) \) there is a positive constant \( c \) such that

\[
\|u - 1\|_{L^p(\Omega)} \leq \|u - 1\|_{L^\infty(\Omega)} \|u - 1\|_{L^2(\Omega)} \leq c \|u - 1\|_{L^2(\Omega)}
\]

\[
\leq c \left(\frac{8}{\chi^2(1-\alpha)} H(t_0)\right)^{\frac{1}{2}} e^{-\frac{8t}{\alpha}(t-t_0)}
\]

(4.16)

for all \( t > t_6 \), where \( a = \frac{2}{p} \in (0, 1) \) and \( c = c(\|u\|_{L^\infty(\Omega)}) \). Noting the Holder’s inequality, for the case \( p \in (1, 2) \), we can conclude

\[
\|u - 1\|_{L^p(\Omega)} \leq \|\Omega\|^{\frac{2-p}{16}} \|u - 1\|_{L^2(\Omega)} \leq \|\Omega\|^{\frac{2-p}{16}} \left(\frac{4}{\chi^2(1-\alpha)} H(t_0)\right)^{\frac{1}{2}} e^{-\frac{8t}{\alpha}(t-t_0)}
\]

(4.17)

for all \( t > t_6 \). Similarly, we have

\[
\|v - 1\|_{L^p(\Omega)} \leq \|\Omega\|^{\frac{2-p}{16}} \|v - 1\|_{L^2(\Omega)} \leq \|\Omega\|^{\frac{2-p}{16}} \left(\frac{8}{\chi^2(1-\alpha)} H(t_0)\right)^{\frac{1}{2}} e^{-\frac{8t}{\alpha}(t-t_0)}
\]

(4.18)

for all \( p \geq 2, a = \frac{2}{p} \), \( t > t_6 \) and

\[
\|v - 1\|_{L^p(\Omega)} \leq \|\Omega\|^{\frac{2-p}{16}} \|v - 1\|_{L^2(\Omega)} \leq \|\Omega\|^{\frac{2-p}{16}} \left(\frac{8}{\chi^2(1-\alpha)} H(t_0)\right)^{\frac{1}{2}} e^{-\frac{8t}{\alpha}(t-t_0)}
\]

(4.19)

for \( p \in (1, 2) \) and \( t > t_6 \). Collecting (4.16)-(4.19), we arrive at (1.9). \( \square \)

Next, we investigate the case \( \alpha > 1 \).

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Together with (4.25) to obtain
\[ I(t) \leq - \epsilon_0 \int \Omega (u-1)^2 \leq \frac{\lambda^2\alpha(a-1)}{4} C(\xi) \int \Omega (v-1)^2 \]
for all \( t > t_7 \), to \( t \), where \( \epsilon_{10} \) is a positive constant. Using (1.8) and (3.25), we can pick a positive constant \( t_8 > t_7 > 0 \) such that
\[ \frac{(a-1)\alpha}{4} (u-1)^2 \leq a^\alpha - au + \alpha - 1 \leq (a-1)\alpha (u-1)^2 \] for all \( t > t_8 \). Together with (4.25) to obtain
\[ \frac{d}{dt} I(t) \leq - \epsilon_{10} \int \Omega (u-1)^2 - \frac{\lambda^2\alpha(a-1)}{4} C(\xi) \int \Omega (v-1)^2 \]
\[ \leq - \frac{\epsilon_{10}}{a(a-1)} G(t) - \frac{\lambda^2\alpha(a-1)}{4} C(\xi) \int \Omega (v-1)^2 \]
\[ \leq - \epsilon_{11} I(t) \]
(4.26)
for all \( t > t_8 \), where \( c_{11} \) is a positive constant and hence we deduce
\[
\frac{(\alpha - 1)\alpha}{4} \int_{\Omega} (u - 1)^2 + \frac{\lambda^2\alpha(\alpha - 1)}{8} \int_{\Omega} (v - 1)^2 \leq I(t) \leq I(t_8) e^{-c_{11}(t-t_8)} \tag{4.27}
\]
for all \( t > t_8 \). From (4.27), we deduce
\[
\int_{\Omega} (u - 1)^2 \leq \frac{4}{(\alpha - 1)\alpha} I(t) \leq \frac{4}{(\alpha - 1)\alpha} I(t_8) e^{-c_{11}(t-t_8)} \text{ for all } t > t_8, \tag{4.28}
\]
\[
\int_{\Omega} (v - 1)^2 \leq \frac{8}{\lambda^2\alpha(\alpha - 1)} I(t) \leq \frac{8}{\lambda^2\alpha(\alpha - 1)} I(t_8) e^{-c_{11}(t-t_8)} \text{ for all } t > t_8. \tag{4.29}
\]
Therefore, we have
\[
\|u - 1\|_{L^2(\Omega)} \leq \left( \frac{4}{(\alpha - 1)\alpha} I(t_8) e^{-c_{11}(t-t_8)} \right)^\frac{1}{2} \leq \left( \frac{4}{(\alpha - 1)\alpha} I(t_8) \right)^\frac{1}{2} e^{-\frac{c_{11}}{2}(t-t_8)} \tag{4.30}
\]
and
\[
\|v - 1\|_{L^2(\Omega)} \leq \left( \frac{8}{\lambda^2\alpha(\alpha - 1)} I(t_8) e^{-c_{11}(t-t_8)} \right)^\frac{1}{2} \leq \left( \frac{8}{\lambda^2\alpha(\alpha - 1)} I(t_8) \right)^\frac{1}{2} e^{-\frac{c_{11}}{4}(t-t_8)} \tag{4.31}
\]
for all \( t > t_8 \). According to the interpolation inequality and the boundedness of \( u \), we have
\[
\|u - 1\|_{L^p(\Omega)} \leq \|u - 1\|_{L^{\infty}(\Omega)}^{1-a} \|u - 1\|_{L^2(\Omega)}^a \leq c \|u - 1\|_{L^2(\Omega)} \leq c \left( \frac{4}{(\alpha - 1)\alpha} I(t_8) \right)^\frac{1}{2} e^{-\frac{c_{11}}{2}(t-t_8)}, \tag{4.32}
\]
where \( p \geq 2, a = \frac{2}{p} \in (0,1) \) and \( c := c(\|u\|_{L^{\infty}(\Omega)}) \). For the case \( p \in (0,1) \), in light of Holder’s inequality, we arrive at
\[
\|u - 1\|_{L^p(\Omega)} \leq |\Omega|^{\frac{2-p}{2p}} \|u - 1\|_{L^2(\Omega)} \leq |\Omega|^{\frac{2-p}{2p}} \left( \frac{4}{(\alpha - 1)\alpha} I(t_8) \right)^\frac{1}{2} e^{-\frac{c_{11}}{2}(t-t_8)} \tag{4.33}
\]
for all \( t > t_8 \). Similar arguments give the desired estimates of \( v \), combine with (4.32) and (4.33), we obtain (1.9) immediately. \( \square \)

Finally, we are in a position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** With the aid of Lemma 2.5, we can find that the solution to system (1.1) fulfills (1.8). In addition, by Lemma 4.1 and Lemma 4.2, we obtain the desired result (1.9). \( \square \)

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**REFERENCES**

[1] T. Cieślak and C. Stinner, *Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions*, *J. Differential Equations*, 252 (2012), 5832–5851.
[2] T. Cieślak and M. Winkler, *Finite-time blow-up in a quasilinear system of chemotaxis*, *Nonlinearity*, 21 (2008), 1057–1076.
[3] E. Galakhov, O. Salieva and J. I. Tello, *On a parabolic-elliptic system with chemotaxis and logistic type growth*, *J. Differential Equations*, 261 (2016), 4631–4647.
[4] X. He and S. N. Zheng, *Convergence rate estimates of solutions in a higher dimensional chemotaxis system with logistic source*, *J. Math. Anal. Appl.*, 436 (2016), 970–982.
[5] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations*, **215** (2005), 52–107.

[6] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains, *J. Differential Equations*, **256** (2014), 2993–3010.

[7] K. Kang and A. Stevens, Blowup and global solutions in a chemotaxis-growth system, *Nonlinear Anal. TMA*, **135** (2016), 57–72.

[8] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399–415.

[9] E. F. Keller and L. A. Segel, Model for chemotaxis, *J. Theoret. Biol.*, **30** (1971), 225–234.

[10] E. F. Keller and L. A. Segel, Traveling bands of chemotactic bacteria: A theoretical analysis, *J. Theoret. Biol.*, **30** (1971), 235–248.

[11] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, *J. Differential Equations*, **258** (2015), 1158–1191.

[12] J. Lankeit, Chemotaxis can prevent thresholds on population density, *Discrete Contin. Dyn. Syst. Ser. B*, **20** (2015), 1499–1527.

[13] K. Lin and C. L. Mu, Global dynamics in a fully parabolic chemotaxis system with logistic source, *Discrete Contin. Dyn. Syst.*, **36** (2016), 5025–5046.

[14] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, *Adv. Math. Sci. Appl.*, **5** (1995), 581–601.

[15] T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, *J. Inequal. Appl.*, **6** (2001), 37–55.

[16] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkc. Ekvacioj*, **40** (1997), 411–433.

[17] K. Osaki and A. Yagi, Finite dimensional attractors for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.*, **44** (2001), 441–469.

[18] M. M. Porzio and V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differential Equations*, **103** (1993), 146–178.

[19] Y. S. Tao and M. Winkler, Persistence of mass in a chemotaxis system with logistic source, *J. Differential Equations*, **259** (2015), 6142–6161.

[20] Y. S. Tao and M. Winkler, Boundedness in a quasilinear parabolic-elliptic Keller-Segel system with subcritical sensitivity, *J. Differential Equations*, **252** (2012), 692–715.

[21] Y. S. Tao and M. Winkler, Large time behavior in a multi-dimensional chemotaxis-haptotaxis model with slow signal diffusion, *SIAM J. Math. Anal.*, **47** (2015), 4229–4250.

[22] J. I. Tello and M. Winkler, A chemotaxis system with logistic source, *Comm. Partial Differential Equations*, **32** (2007), 849–877.

[23] G. Viglialoro and T. E. Woolley, Eventual smoothness and asymptotic behaviour of solutions to a chemotaxis system perturbed by a logistic growth, *Discrete Contin. Dyn. Syst. Ser. B*, **23** (2018), 3023–3045.

[24] J. C. Wang, C. L. Mu and P. Zheng, On a quasilinear parabolic-elliptic chemotaxis system with logistic source, *J. Differential Equations*, **256** (2014), 1847–1872.

[25] Z. A. Wang and T. Xiang, A class of chemotaxis systems with growth source and nonlinear secretion, arXiv:1510.07204v1.

[26] L. C. Wang, C. L. Mu, X. G. Hu and P. Zheng, Boundedness and asymptotic stability of solutions to a two-species chemotaxis system with consumption of chemotactant, *J. Differential Equations*, **254** (2018), 3369–3401.

[27] L. C. Wang, J. Zhang, C. L. Mu and X. G. Hu, Boundedness and stabilization in a two-species chemotaxis system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 191–221.

[28] M. Winkler, Does a ‘volume-filling effect’ always prevent chemotactic collapse?, *Math. Methods Appl. Sci.*, **33** (2010), 12–24.

[29] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, **100** (2013), 748–767.

[30] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations*, **35** (2010), 1516–1537.

[31] M. Winkler, Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 2777–2793.

[32] M. Winkler, Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation, *Z. Angew. Math. Phys.*, **69** (2018), Art. 69, 40 pp.
[33] M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, *Nonlinearity*, **31** (2018), 2031–2056.

[34] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities?, *J. Nonlinear Sci.*, **24** (2014), 809–855.

[35] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, *J. Math. Anal. Appl.*, **384** (2011), 261–272.

[36] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, *J. Differential Equations*, **257** (2014), 1056–1077.

[37] J. Zhao, C. L. Mu, L. C. Wang and K. Lin, A quasilinear parabolic-elliptic chemotaxis-growth system with nonlinear secretion, *Appl. Anal.*, (2018).

[38] J. S. Zheng and Y. F. Wang, Boundedness and decay behavior in a higher-dimensional quasilinear chemotaxis system with nonlinear logistic source, *Comput. Math. Appl.*, **72** (2016), 2604–2619.

[39] P. Zheng, C. L. Mu and X. G. Hu, Boundedness and blow-up for a chemotaxis system with generalized volume-filling effect and logistic source, *Discrete Contin. Dyn. Syst.*, **35** (2015), 2399–2423.

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