WRITING PERIODIC MAPS AS WORDS IN DEHN TWISTS

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Abstract. In this paper, we will discuss procedures for expressing a finite order mapping class as a product of Dehn twists, up to conjugacy. These algorithms are based on the chain and star relations in the mapping class group, the geometric realizations of torsion elements of the mapping class group, and the symplectic representations of the mapping class group. As an application, we develop an algorithm to write some of the roots of Dehn twists as words in Dehn twists.

1. Introduction

Let \( S_g \) be a closed orientable surface of genus \( g \geq 1 \), and let \( \text{Mod}(S_g) \) denote the mapping class group of \( S_g \). Dehn twist is named after the German mathematician Max Dehn, who proved \([5]\) that a finite set of Dehn twists in \( S_g \) generates \( \text{Mod}(S_g) \). Lickorish \([14]\) proved that \( \text{Mod}(S_g) \) is generated by \( 3g-1 \) Dehn twists about non-separating curves, and subsequently Humphries \([11]\) showed that \( \text{Mod}(S_g) \) is generated by a minimal generating set comprising \( 2g+1 \) Dehn twists about non-separating curves. Thus, it is a natural question to ask whether we can derive methods for representing an arbitrary periodic \( F \in \text{Mod}(S_g) \) as a word \( W(F) \) in Dehn twists, up to conjugacy. In this paper, we develop algorithms to write a word \( W(F) \) for arbitrary periodic mapping class \( F \in \text{Mod}(S_g) \) in Dehn twists.

In their seminal paper \([3]\), Birman-Hilden derived an expression for \( W(F) \) when \( F \) is of (largest possible) order \( 4g+2 \) in \( \text{Mod}(S_g) \), and consequently for \( F^{2g+1} \), the hyperelliptic involution. This problem was solved for involutions in \( \text{Mod}(S_2) \) by Matsumoto \([17]\), which was later generalized to \( g \geq 2 \) by Korkmaz \([13]\). Using techniques of algebraic geometry, Hirose \([10]\) derived expressions for \( W(F) \) for every periodic \( F \in \text{Mod}(S_g) \), for \( 2 \leq g \leq 4 \). However, these specialized techniques of Hirose do not easily generalize for \( g \geq 5 \). In \([20]\), a method was described to decompose an arbitrary periodic mapping class \( F \in \text{Mod}(S_g) \) into irreducible components. We use this decomposition to develop various methods in this paper for deriving \( W(F) \), for an arbitrary periodic mapping class \( F \in \text{Mod}(S_g) \).

In Section \([3]\) we provide a method for deriving \( W(F) \) when \( F \) is rotational (i.e. realizable a rotation of \( S_g \)), which include all type of involutions. For hyperelliptic involution \( F \in \text{Mod}(S_g) \), we have following.

Corollary 1.1. Let \( F \in \text{Mod}(S_g) \) be a hyperelliptic involution then up to conjugacy \( W(F) = \prod_{j=1}^{g}(T_{a_j}T_{b_j}T_{a_j})^{2(-1)^j+1} \).

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Where, \(a_j, b_j\), for \(1 \leq i \leq g\), are the standard generators of \(H_1(S_g, \mathbb{Z})\) and the \(\{a_j, a'_j\}\) form a bounding pair for \(2 \leq j \leq g - 1\).

In Section 4 we use the well known chain relation in \(\text{Mod}(S)\) to develop a method (that we call the \textit{chain method}) see \[3, 4\] for representing a large family of periodic mapping classes (that we will call \textit{chain-realizable} periodics) as words in Dehn twists. As an immediate application of the chain method, we represent the torsion elements in \(\text{Mod}(S_g)\) (up to conjugacy) as words in Dehn twists. In Section 5 we apply a well known generalization \[26, 16\] of the standard star relation in \(\text{Mod}(S^3)\) to \(g \geq 2\), to develop a method (that we call the \textit{star method}) for representing a even larger family of periodic mapping classes (that encompasses \textit{chain-realizable} periodics) as words in Dehn twists. For an \(F \in \text{Mod}(S_g)\) that is neither rotational nor star-realizable, in Section 6 we formulate a method for deriving \(\mathcal{W}(F)\) that we call the \textit{symplectic method}, see algorithm 6.3. As the name suggests, this method uses the symplectic representations of periodic maps as described in [20].

In Section 7 we provide two applications of our methods. In Subsection 7.1, we give an algorithm to write certain roots of Dehn twists as words in Dehn twists. In particular we give the word for the highest root of non-separating curve, which is not conjugate to Margalit-Schleimer root.

\textbf{Example 1.} Consider a root \(F \in \text{Mod}(S_g)\) of degree \(2g - 1\), where \(D_F = (2g - 1, 0; (g, 2g - 1), (g, 2g - 1), (2g - 2, 2g - 1))\). Since \(\mathcal{W}(F) = W^2_{2g-1}\), by applying Algorithm 7.2 we get

\[
\mathcal{W}(F) = T_{a_1}^{-1}(T_{c_1} T_{a_2} \prod_{i=2}^{g-1} (T_{b_i} T_{c_i}) T_{b_g} T_{a_g})^2.
\]

where, \(a_j, b_j\), and \(c_j\) are the curve used for Lickorish generators. By applying the star and symplectic methods, in Subsection 7.2 we obtain representations for the torsion elements in \(\text{Mod}(S_3)\) (up to conjugacy) as words in Dehn twists.

2. Preliminaries

2.1. Periodic mapping classes. For \(g \geq 1\), let \(F \in \text{Mod}(S_g)\) be of order \(n\). By the Nielsen-Kerckhoff theorem \[12, 19\], \(F\) is represented by a \textit{standard representative} \(\mathcal{F} \in \text{Homeo}^+(S_g)\) of order \(n\). Let \(\mathcal{O}_F := S_g/\langle \mathcal{F} \rangle\) be the \textit{corresponding orbifold} of \(F\) of genus \(g_0\) (say). Each cone point \(x_i \in \mathcal{O}_F\) lifts under the branched cover \(S_g \to S_g/\langle \mathcal{F} \rangle\) to an orbit of size \(n/n_i\) on \(S_g\), where the local rotation induced by \(\mathcal{F}\) is given by \(2 \pi c_i^{-1}/n_i\). If \(c_i c_i^{-1} \equiv 1 \pmod{n_i}\).

The tuple \(\Gamma(\mathcal{O}_F) := (g_0; n_1, \ldots, n_\ell)\), is called the signature of \(\mathcal{O}_F\). By the theory of group actions on surfaces (see [9] and the references therein), we obtain an exact sequence:

\[
1 \to \pi_1(S_g) \to \pi_1^{\text{orb}}(\mathcal{O}_F) \xrightarrow{\phi_F} \langle \mathcal{F} \rangle \to 1,
\]

where the \(\phi_F(\xi_i) = \mathcal{F}^{n/n_i} c_i\), for \(1 \leq i \leq \ell\). We will now define a tuple of integers that will encode the conjugacy class of a periodic mapping class \(F \in \text{Mod}(S_g)\) of order \(n\).
Definition 2.1. A data set of degree \( n \) is a tuple
\[
D = (n, g_0, r; (c_1, n_1), \ldots, (c_\ell, n_\ell)),
\]
where \( n \geq 2, g_0 \geq 0, \) and \( 0 \leq r \leq n - 1 \) are integers, and each \( c_i \in \mathbb{Z}_{n_i}^* \) such that:

(i) \( r > 0 \) if and only if \( \ell = 0 \) and \( \gcd(r, n) = 1 \), whenever \( r > 0 \),
(ii) each \( n_i | n \),
(iii) \( \text{lcm}(n_1, \ldots, n_i, \ldots, n_\ell) = N \), for \( 1 \leq i \leq r \), where \( N = n, \) if \( g_0 = 0 \), and
(iv) \( \sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \pmod{n} \).

The number \( g \) determined by the Riemann-Hurwitz equation
\[
\frac{2 - 2g}{n} = 2 - 2g_0 + \sum_{j=1}^{\ell} \left( \frac{1}{n_j} - 1 \right)
\]
is called the genus of the data set, denoted by \( g(D) \).

The following proposition, which allows us to use data sets to represent the conjugacy classes of cyclic actions on \( S_g \), follows from [23, Theorem 3.8] and [9].

Proposition 2.2. For \( g \geq 1 \) and \( n \geq 2 \), data sets of degree \( n \) and genus \( g \) correspond to conjugacy classes of \( \mathbb{Z}_n \)-actions on \( S_g \).

We will denote the data set encoding the conjugacy class of a periodic mapping class \( F \) by \( D_F \). The parameter \( r \) (in Definition 2.1) will come into play in a data set \( D_F \) only when \( F \) is a free rotation of \( S_g \) by \( 2\pi r/n \), in which case, \( D_F \) will take the form \( (n, g_0, r; ) \). We will avoid including \( r \) in the notation of a data set \( D_F \), whenever \( F \) is non-free. Furthermore, for compactness of notation, we also write a data set \( D \) (as in Definition 2.1) as
\[
D = (n, g_0, r; ((d_1, m_1), \alpha_1), \ldots, ((d_\ell, m_\ell), \alpha_\ell)),
\]
where \((d_i, m_i)\) are the distinct pairs in the multiset \( S = \{(c_1, n_1), \ldots, (c_\ell, n_\ell)\} \), and the \( \alpha_i \) denote the multiplicity of the pair \((d_i, m_i)\) in the multiset \( S = \{(c_1, n_1), \ldots, (c_\ell, n_\ell)\} \).

Let \( F \in \text{Mod}(S_g) \) be of order \( n \). Then \( F \) is said to be rotational if \( F \) is a rotation of the \( S_g \) through an axis by \( 2\pi r/n \), where \( \gcd(r, n) = 1 \). It is apparent that \( F \) is either has no fixed points, or \( 2k \) fixed points which are induced at the points of intersection of the axis of rotation with \( S_g \). Moreover, these fixed points will form \( k \) pairs of points \((x_i, x_i')\), for \( 1 \leq i \leq k \), such that the sum of the angles of rotation induced by \( F \) around \( x_i \) and \( x_i' \) add up to \( 0 \) modulo \( 2\pi \). Consequently, we have the following:

Proposition 2.3. Let \( F \in \text{Mod}(S_g) \) be a rotational mapping class of order \( n \).

(i) When \( F \) is a non-free rotation, then \( D_F \) has the form \( (n, g_0; (s, n), (n - s, n), \ldots, (s, n), (n - s, n)) \)
for integers \( k \geq 1 \) and \( 0 < s \leq n - 1 \) with \( \gcd(s, n) = 1 \), and \( k = 1 \) if \( n > 2 \).
(ii) When $F$ is a free rotation, then $D_F$ has the form
\[(n, \frac{g-1}{n}+1, r; ).\]

We say $F$ is of Type 1 if $\Gamma(O_F)$ has the form $(g_1; n_1, n_2, n)$, and $F$ is said to be of Type 2 if $F$ is neither rotational nor of Type 1. Gilman \[8\] showed that a periodic mapping class $F \in \text{Mod}(S_g)$ is irreducible if and only if $O_F$ is a sphere with three cone points. Thus, $F$ is an irreducible Type 1 mapping class if and only if $O_F$ has the form $(0; n_1, n_2, n)$.

2.2. Decomposing periodic maps into irreducibles. In \[2, 20\], a method was described to decompose an arbitrary non-rotational periodic element $F \in \text{Mod}(S_g)$, for $g \geq 2$, into irreducible Type 1 components, which are realized as rotations of certain unique hyperbolic polygons with side-pairings.

**Theorem 2.4.** For $g \geq 2$, consider an irreducible Type 1 action $F \in \text{Mod}(S_g)$ with
\[D_F = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).\]
Then $F$ can be realized explicitly as the rotation $\theta_F = 2\pi c_3^{-1}/n$ of a hyperbolic polygon $P_F$ with a suitable side-pairing $W(P_F)$, where $P_F$ is a hyperbolic $k(F)$-gon with
\[k(F) := \begin{cases} 2n, & \text{if } n_1, n_2 \neq 2, \\ n, & \text{otherwise}, \end{cases}\]
and for $0 \leq m \leq n - 1$,
\[W(P_F) = \begin{cases} \prod_{i=1}^{n} a_{2i-1}^{-1}a_{2i}, & \text{with } a_{2n+1}^{-1} \sim a_{2z}, \text{ if } k(F) = 2n, \text{ and} \\ \prod_{i=1}^{n} a_i, & \text{with } a_{m+1}^{-1} \sim a_z, \text{ otherwise}, \end{cases}\]

where $z \equiv m + qj \pmod{n}$ with $q = (n/n_2)c_3^{-1}$ and $j = n_2 - c_2$.

Consequently, the process yielded explicit solutions to the Nielsen realization problem \[12, 19\] for the cyclic case. Further, it was shown that the process of decomposition can be reversed by piecing together the irreducible Type 1 components (described in Theorem 2.4) using the methods that we will now describe in Constructions 2.5 and 2.7.

**Construction 2.5** ($k$-compatibility). For $i = 1, 2$, let $F_i \in \text{Mod}(S_{g_i})$ be of order $n$. Suppose that the actions of $\langle F_i \rangle$ on $S_{g_i}$ induces a pair of orbits $O_i$ such that $|O_1| = |O_2|$ and the rotation angles induced by the $\langle F_i \rangle$-action around points in the $O_i$ add up to $0$ (mod $2\pi$). Then we remove (cyclically permuted) $\langle F_i \rangle$-invariant disks around points in the $O_i$ and then attach $k$-annuli $A_i$ connecting the resulting boundary components, to obtain an $F \in \text{Mod}(S_g)$ of order $n$, where $g(F) := g = g_1 + g_2 + k - 1$. This method of constructing $F$ is called a $k$-compatibility, and we say that $F$ is realizable as a $k$-compatible pair $(F_1, F_2)$ of genus $g(F)$. Further, we denote $A(F) := \cup_{i=1}^{k} A_i$. A typical 1-compatibility between irreducible Type 1 maps is illustrated in Figure 5 below. (For a visualization of a $k$-compatibility for $k > 2$, see Figure 6.)
Let $\Sigma_i(F) := S_{g_i} \setminus A(F)$. Then by construction, the maps $F|_{\Sigma_i(F)}$ and $F|_{A(F)}$ commute with each other.

If in the construction above, the orbits $O_i$ are induced by a single action on a surface $S_g$, then the method is called a \textit{self $k$-compatibility}, wherein the resultant action is on $S_{g+k}$.

Generalizing the ideas in Construction 2.5, we have the following.

\textbf{Definition 2.6.} Let $F \in \text{Mod}(S_g)$ be of order $n$. We say $F$ is a \textit{linear $s$-tuple} $(F_1, F_2, \ldots, F_s)$ of degree $n$ and genus $g$ if for $1 \leq i \leq s-1$, there exists $F_i \in \text{Mod}(S_{g_i})$ of order $n$ satisfying the following conditions.

(i) $F_{i,i+1} := (F_i, F_{i+1})$ is realizable as a $k_i$-compatible pair of genus $g(F_{i,i+1})$.

(ii) Let

$$\Sigma_i(F) := \begin{cases} S_{g_i} \setminus A(F_{i,i+1}), & \text{if } i = 1, \\
S_{g_i} \setminus A(F_{i-1,i}), & \text{if } i = s, \\
S_{g_i} \setminus (A(F_{i-1,i}) \sqcup A(F_{i,i+1})), & \text{for } 2 \leq i \leq s-1.
\end{cases}$$

Then $F|_{\Sigma_i(F)} = F|_{\Sigma_i(F)}$, for $1 \leq i \leq s$.

(iii) $S_g = \bigsqcup_{i=1}^s \Sigma_i(F) \sqcup \bigsqcup_{i=1}^{s-1} A(F_{i,i+1})$, where

$$g = \sum_{i=1}^s g_i + \sum_{i=1}^{s-1} (k_i - 1).$$

Given a linear $s$-tuple $F = (F_1, F_2, \ldots, F_s)$ as in Definition 2.6 we denote $g(F) := g$, and further, we fix the following notation that for $1 \leq i < j - 1 \leq s$, we denote

$$F_{i,j} := (F_i, F_{i+1}, \ldots, F_j) \text{ and } \Sigma_{i,j}(F) := \bigsqcup_{k=i}^j \Sigma_k(F).$$

\textbf{Construction 2.7} (Permutation additions and deletions). The \textit{addition of a $g'$-permutation component} to a periodic map $F$ is a process that involves the removal of (cyclically permuted) invariant around points in an orbit of size $n$ and then pasting $n$ copies of $S_{g'}$ (i.e. $S_{g'}$ with one boundary component) to the resultant boundary components. This realizes an action on $S_{g+ng'}$ with the same fixed point and orbit data as $F$. A visualization of
a permutation addition to an irreducible Type 1 map is shown in Figure 2 below.

**Figure 2.** Addition of a $g'$-permutation component to an irreducible Type 1 map.

The reversal of this process, wherein a $g'$-permutation is removed from $F$ (when possible), is called the deletion of $g'$-permutation component.

The upshot of the discussion above is the following:

**Theorem 2.8.** For $g \geq 2$, an arbitrary non-rotational periodic mapping class in $\text{Mod}(S_g)$ can be constructed through finitely many $k$-compatibilities, permutation additions, and permutation deletions on irreducible Type 1 mapping classes.

**Definition 2.9.** Given integers $s > 0$, $u, v, w \geq 0$, an admissible $(s, u, v, w)$-tuple $T$ is a tuple of integers of the form

$$T = [(i_1, j_1), k_1; (i_u, g'_1), \ldots, (i'_u, g''_u); (i''_1, g''_1), \ldots, (i''_w, g''_w)],$$

where for each $q$, $1 \leq i_q < j_q \leq s$, $k_q \geq 1$, $1 \leq i'_q \leq s$, $1 \leq i''_q < j''_q \leq s$, and $g'_q, g''_q > 0$.

**Definition 2.10.** Given a linear $s$-tuple $(F_1, \ldots, F_s)$ of degree $n$ and genus $g$ as in Definition 2.6 and an admissible $(s, u, v, w)$-tuple $T$ as in Definition 2.9, we construct a compatible $(F, T)$-tuple $F_T$ of degree $n$ and genus $g(F_T)$ through the constructions in the following sequence of steps.

**Step 1.** If $u = 0$, then we skip this step. Otherwise, for $1 \leq q \leq u$, we perform a self $k_q$-compatibility in $F_{i_q,j_q}$, if $F_{i_q,j_q}$ admits such a compatibility.

**Step 2.** If $v = 0$, then we skip this step. Otherwise, for $1 \leq q \leq v$, we perform a $g'_q$-permutation addition on the $F_q$.

**Step 3.** If $w = 0$, then we skip this step. Otherwise, for $1 \leq q \leq w$, we perform a $g''_q$-permutation deletion on the $F_q$, if $F_q$ admits such a deletion.

Note that a compatible tuple $(F, T)$-tuple, where $T$ is an admissible $(s, 0, 0, 0)$-tuple simply refers to the linear $s$-tuple $F$. With this notation in place, Theorem 2.8 can be now restated as follows.
Theorem 2.11. Given an arbitrary non-rotational periodic mapping class in $G \in \text{Mod}(S_g)$, for $g \geq 2$, there exists a linear $s$-tuple $F \in \text{Mod}(S_g)$ of irreducible Type 1 actions and an admissible $(s,u,v,w)$-tuple $T$ of integers such that $G = F_T$.

2.3. Symplectic representations of periodic mapping classes. For $g \geq 1$, let $\Psi : \text{Mod}(S_g) \to \text{Sp}(2g;\mathbb{Z})$ be the surjective representation afforded by the action of $\text{Mod}(S_g)$ on $H_1(S_g,\mathbb{Z})$. In this subsection, we will state some results from [20, Section 4] that are relevant to this paper.

Let $F \in \text{Mod}(S_g)$ be an irreducible Type 1 action that is realized by the rotation of a hyperbolic polygon $P_F$ with a boundary word $W(P_F)$ (when read counterclockwise) as in Theorem 2.4. An application of the handle normalization algorithm detailed in [24, Section 3.4] shows that $P_F$ is of the form $QaRbSa^{-1}Tb^{-1}U$, for some words $Q, R, S, T, U$ (possibly empty), and letters $a, b$, and we have the following.

Proposition 2.12. Let $W(P_F) = QaRbSa^{-1}Tb^{-1}U$. Suppose that $P'$ is the polygon with boundary word $W(P') = QTSRUx^{-1}y^{-1}$ obtained by applying the handle normalization algorithm once to $P_F$. Then $x$ and $y$ are homotopically equivalent to $QTb^{-1}U$ and $U^{-1}R^{-1}a^{-1}Tb^{-1}U$, respectively.

We fix the following notation.

(a) We denote by $N^i(P_F)$, the polygon obtained from $P_F$ after $i$ successive applications of the normalization procedure described in Proposition 2.12.

(b) We denote by $L(P_F)$, the set of distinct letters in $W(P_F)$.

(c) We denote by $B(P_F)$, the set of standard generators of $H_1(S_g;\mathbb{Z})$ expressed in terms of elements in $L(P_F)$.

Let $W = W(P_F)$ and $W' = W(P')$ be as in Proposition 2.12. Then the map $B(P') \to B(P_F) : x \mapsto QTb^{-1}U, y \mapsto U^{-1}R^{-1}a^{-1}Tb^{-1}U, z \mapsto z,$ for all $z \in B(P') \setminus \{x,y\}$, uniquely determines an isomorphism on $H_1(S_g;\mathbb{Z})$, which we denote by $f_{P',P_F}$, which leads us to the following lemma.

Lemma 2.13. Let $F \in \text{Mod}(S_g)$ be an irreducible Type 1 action that is realized by the rotation of a hyperbolic polygon $P_F$ as in Theorem 2.4. Then for $i = 0, 1, \ldots, g$, we have

$$W(N^i(P_F)) = \prod_{i=1}^g [x_i, y_i],$$

and the mapping

$$f_{P_F} = \prod_{i=1}^{g-1} f_{N^i(P_F), N^{i+1}(P_F)}$$

defines an isomorphism of the homology group $H_1(S_g;\mathbb{Z})$ such that $B(P_F) \xrightarrow{f_{P_F}} B(P_F)$.

For an isomorphism $\varphi : H_1(S_g;\mathbb{Z}) \to H_1(S_g;\mathbb{Z})$, let $M_\varphi$ denote the matrix of $\varphi$ with respect to the standard homology generators. The following theorem describes the structure of $\Psi(F)$, up to conjugacy.
Theorem 2.14. Let $F \in \Mod(S_g)$ be an irreducible Type 1 with $D_F = ((n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)))$. Then up to conjugacy, $\Psi(F) = M_\varphi$, where $\varphi = f_{p_F}^{-1} \phi_{p_F} f_{p_F}$, with $f_{p_F}$ as in Lemma 2.13, and $B(p_F) \phi_{p_F} B(p_F)$ is induced by $a_i \mapsto a_j$, where

$$j \equiv \begin{cases} i + 2c_3^{-1} \mod 2n, & \text{if } n_1, n_2 \neq 2, \text{ and} \\ i + c_3^{-1} \mod n, & \text{otherwise}. \end{cases}$$

We conclude this subsection with the following remark.

Remark 2.15. By Theorem 2.8, an arbitrary non-rotational periodic mapping class $F \in \Mod(S_g)$ can be decomposed into irreducible Type 1 mapping classes. This decomposition induces a decomposition of $\Psi(F)$ (up to conjugacy) into a block-diagonal matrix, where each diagonal block is of one of the following types.

(i) The image under $\Psi$ of an irreducible Type 1 component (of $F$) as described in Theorem 2.14.

(ii) Let $F'$ be a component of $F$ resulting from a $k$-compatibility (or a self $k$-compatibility), and let $S_{g,b}$ denote the surface of genus $g$ with $b$ boundary components. Then there exists a subsurface $S$ (of $S_{g,b}$) homeomorphic to $S^{2k-1}_0$ (shown in Figure 3 below) in which $F'$ cyclically permutes the disjoint union of the $k$ annuli $A_F \subset S$ involved in the construction.

The diagonal block is obtained from the well-defined action of such an $F'$ on $H_1(S, \mathbb{Z})$.

(iii) The image under $\Psi$ under a permutation component of $F$, which permutes $n$ subsurfaces of $S_g$ homeomorphic to some $S^{1}_{g'}$ as in Construction 2.7.

Note that as the blocks of type (ii) and (iii) are simple permutation blocks, one can obtain a complete description of $\Psi(F)$ (up to conjugacy).

2.4. Relations involving Dehn twists. Let $i(c, d)$ denote the geometric intersection number of simple closed curves $c$ and $d$ in $S_g$. A collection $\mathcal{C} = \{c_1, \ldots, c_k\}$ of simple closed curves in $S_g$ is said to form a chain if $i(c_i, c_{i+1}) = 1$, for $1 \leq i \leq k - 1$, and $i(c_i, c_j) = 0$, if $|i - j| > 1$. We state the following basic fact [6, Section 1.2] about chains.

Lemma 2.16. For $g \geq 1$, there is a unique chains in $S_g$ of size $2g$, up to homeomorphism. Moreover, when $g > 1$, there is a unique chain in $S_g$ of size $2g + 1$, up to homeomorphism.
A closed regular neighborhood of the union of curves in $C$ is a subsurface $S$ of $S_g$ that has one or two boundary components, depending on whether $k$ is even or odd. Let the isotopy classes of $\partial S$ be represented by the curves $d$ (resp. $d_1, d_2$) when $k$ is even (resp. odd). Let $T_c$ denote the left-handed Dehn twist about a simple closed curve $c$ in $S_g$. We will make extensive use of the well known chain relation involving Dehn twists.

**Proposition 2.17 (Chain relation).** Let $C = \{c_1, \ldots, c_k\}$ be a chain in $S_g$. Then:

\[
\begin{align*}
(T_{c_1}T_{c_2}\ldots T_{c_k})^{2k+2} &= T_d, & \text{when } k \text{ is even, and} \\
(T_{c_1}T_{c_2}\ldots T_{c_k})^{k+1} &= T_{d_1}T_{d_2}, & \text{when } k \text{ is odd.}
\end{align*}
\]

Equivalently, we have

\[
\begin{align*}
(T_{c_1}^2T_{c_2}\ldots T_{c_k})^{2k} &= T_d, & \text{when } k \text{ is even, and} \\
(T_{c_1}^2T_{c_2}\ldots T_{c_k})^{k} &= T_{d_1}T_{d_2}, & \text{when } k \text{ is odd.}
\end{align*}
\]

In each case of Proposition 2.17 we will denote the word enclosed within the parentheses on the left hand side by $W_C$. Note that for all $i < k$, $W_C(c_i) = c_{i+1}$. We will also use the following relation also known as the hyperelliptic relation (see [6, Chapter 5]) in $\text{Mod}(S_g)$, for $g \geq 2$.

**Proposition 2.18.** For $g \geq 2$, let $\{c_1, \ldots, c_{2g+1}\}$ be a chain in $S_g$. Then:

\[
\begin{align*}
(T_{c_{2g+1}}\ldots T_{c_1}T_{c_1}\ldots T_{c_{2g+1}})^2 &= 1, & \text{and} \\
[T_{c_{2g+1}}\ldots T_{c_1}T_{c_1}\ldots T_{c_{2g+1}}, T_{c_{2g+1}}] &= 1,
\end{align*}
\]

where $T_{c_{2g+1}}\ldots T_{c_1}T_{c_1}\ldots T_{c_{2g+1}}$ represents the conjugacy class of the standard hyperelliptic involution in $\text{Mod}(S_g)$ encoded by $(2,0;((1,2),2g+2))$.

Let $c_1, c_2, c_3, d_1, d_2, d_3$, and $b$ be the curves in $S^3_1$, as indicated in Figure 4 below.

![Figure 4](image)

**Figure 4.** The curves involved in the star relation in $\text{Mod}(S^3_1)$.

We will use the following relation in $\text{Mod}(S^3_1)$ due to Gervais [7], also known as the star relation, to develop a method for writing periodic mapping classes of order 3 as words in Dehn twists.

**Proposition 2.19 (Star relation).** Let $c_1, c_2, c_3, d_1, d_2, d_3$, and $b$ be the curves in $S^3_0$, as indicated in Figure 4. Then:

\[
(T_{c_1}T_{c_2}T_{c_3}T_b)^3 = T_{d_1}T_{d_2}T_{d_3}.
\]
In Section 5, we will derive a generalization of this relation, which we will apply to develop a method for obtaining the word representations for a larger family of periodic maps. The final result we state in this subsection pertains to the Burkhardt handle swap map $H_i$ which swaps the $i^{th}$ handle in $S_g$ (for $g \geq 2$) with the $(i + 1)^{st}$ handle.

Proposition 2.20. For $g \geq 2$, let $a_1, b_1, \ldots, a_g, b_g$ be the curves that represent the standard generators of $H_1(S_g, \mathbb{Z})$. Then for $1 \leq i \leq g - 1$, the "$i^{th}$ handle" swap map is given by

$$H_{i,i+1} := (T_{a_i+1}T_{b_i}x_iT_{a_i}T_{b_i})^3,$$

where $x_i$ is a simple closed curve that represents the homology class $a_i+1+b_i$.

2.5. Periodic maps on the torus as words in Dehn twists. Since $\text{Mod}(S_1) \cong \text{SL}(2, \mathbb{Z}) = \mathbb{Z}_4 \ast_{\mathbb{Z}_2} \mathbb{Z}_6$, any periodic element in $\text{Mod}(S_1)$ is of order 2, 3, 4, or 6. Moreover, since $\{a, b\}$ (as indicated in Figure 5 below) is a chain in $S_1$, it follows by Proposition 2.17 that $T_a T_b$ is of order 6, and $T_a^2T_b$ is of order 4 in $\text{Mod}(S_1)$. Note that $T_a T_b$ (resp. $T_a^2T_b$) is represented by a rotation of a regular hexagon (resp. square) with opposite sides identified, by $2\pi/6$ (resp. $\pi/2$).

![Figure 5. A chain in the torus.](image)

Taking the powers of these maps, we obtain a word $W(F)$ (in Dehn twists) representing the conjugacy class of each periodic element $F \in \text{Mod}(S_2)$.

| $F$   | $D_F$       | $W(F)$          |
|-------|-------------|-----------------|
| 6     | $6, 0; (1, 2), (1, 3), (1, 6)$ | $T_a T_b$       |
| 6     | $6, 0; (1, 2), (2, 3), (5, 6)$ | $(T_a T_b)^3$   |
| 4     | $4, 0; (1, 2), (1, 4), (1, 4)$ | $T_a^2 T_b$     |
| 4     | $4, 0; (1, 2), (3, 4), (3, 4)$ | $(T_a^2 T_b)^3$ |
| 3     | $3, 0; (1, 3), (1, 3), (1, 3)$ | $(T_a T_b)^2$   |
| 3     | $3, 0; (2, 3), (2, 3), (2, 3)$ | $(T_a T_b)^3$   |
| 2     | $2, 0; ((1, 2), 4))$          | $(T_a T_b)^4$   |

Table 1. Words (in Dehn twists) representing the conjugacy classes of periodic elements in $\text{Mod}(S_1)$.

3. Rotational mapping classes as words in Dehn twists

In this section, we will provide a method for writing rotational mapping classes as products of Dehn twists. The key idea is to write given rotational
mapping class as a product of two involutions, whose representations (as words) will be discussed in the following subsection.

3.1. Non-free involutions as words in Dehn twists. By Proposition 2.3 given an arbitrary involution \( F \in \text{Mod}(S_g) \), \( D_F \) has one of the following forms:

\[
(2, g_0; ((1, 2), 2k)) \text{ or } (2, (g + 1)/2, 1);
\]

depending on whether \( F \) is non-free or free. First, we consider the cases \( g = 1, 2 \), where there are three possible conjugacy classes of involutions.

(a) The hyperelliptic involution in \( \text{Mod}(S_1) \): \( (0; ((1, 2), 4)) \).
(b) The hyperelliptic involution in \( \text{Mod}(S_2) \): \( (0; ((1, 2), 6)) \).
(c) The rotation of \( S_2 \) with two fixed points: \( (1; ((1, 2), 2)) \).

The word representation for the involution in (a) was featured in Table 1, while the word for (b), the hyperelliptic involution, is known from Proposition 2.18. Since (c) swaps the two genera of \( S_2 \), it is the map \( H_{2,1} \) from Proposition 2.20. We will collectively call these involutions the fundamental involutions. We will show that an arbitrary involution can be obtained by piecing together the fundamental involutions via 1-compatibilities. We will require the following lemmas, which are simple consequences of Proposition 2.17.

Lemma 3.1. Let \( H_{2,1} \) be the restriction of \( H_{2,1} \) on \( S_2^2 \). Then,

\[
\overline{H_{2,1}^2} = T_{d_1}T_{d_2},
\]

where \( d_1, d_2 \) are the boundary curves of \( S_2^2 \) as shown in Figure 6 below.

\[ \text{Figure 6. The boundary curves } d_1, d_2 \text{ in } S_2^2. \]

Lemma 3.2. Let \( a, b, a', d_1, d_2 \) be the curves in \( S_1^2 \) as indicated in the Figure 7 below.

\[ \text{Figure 7. The curves } a, b, a', d_1, d_2 \text{ in } S_1^2. \]

Then

\[
(T_aT_bT_{a'})^4 = T_{d_1}T_{d_2}.
\]
We will now provide an algorithm for writing involutions as words in the Dehn twists.

**Algorithm 3.3.** Let $F \in \text{Mod}(S_g)$ be a non-free involution with $D_F = ((2,g_0;((1,2),2k))$ (by virtue of Lemma 2.3).

**Step 1.** If $k = 1$, then:

1a. We decompose $F$ into fundamental involutions as shown in the Figure 8.

$$W(F) = \prod_{i=1}^{g_0} (H_{2i,2i-1})^{(-1)^{i-1}}.$$ 

**Step 2.** If $k > 1$, then:

1a. We decompose $F$ into fundamental involutions as shown in the Figure 9.

$$W(F) = \prod_{i=1}^{g_0} (H_{2i,2i-1})^{(-1)^{i-1}} \prod_{j=2g_0+1}^{g} (T_{a_j'}T_{b_j}T_{a_j})^{2(-1)^{i+c}},$$

where $c = \begin{cases} 1, & \text{if } g_0 \text{ is even, and} \\ 0, & \text{if } g_0 \text{ is odd.} \end{cases}$

**Step 3.** By Proposition 2.20 and Lemmas 3.1-3.2, $W(F)$ is the desired representation of $F$ as a word in Dehn twists, up to conjugacy.
3.2. **Surface rotations as words in Dehn twists.** For \( g \geq 2 \), any rotation of \( S_g \) (that is free or non-free) of order \( n \geq 2 \) can be written as a product of two involutions, as illustrated in Figure 10 below. This leads to

![Figure 10. A surface rotation as a product of two involutions.](image)

the following method for writing surface rotations as words in Dehn twists.

**Algorithm 3.4.** Let \( F \in \text{Mod}(S_g) \) be realized as a rotation \( \mathcal{F} \) of order \( n > 2 \). Then by Proposition 2.3, \( D_F \) has the form

\[
(n, g_0; (s, n), (n - s, n), \ldots, (s, n), (n - s, n)) \text{ or } (n, \frac{g - 1}{n} + 1, 1;)
\]

depending on whether \( \mathcal{F} \) is free rotation or not.

**Step 1.** Consider an embedding of \( S_g \) in \( \mathbb{R}^3 \), as indicated in Figure 10, where there is a “genus in the middle”, only when \( \mathcal{F} \) is free.

**Step 2.** For \( i = 1, 2 \), let the reflection along the axis \( X_i \) (as shown in the figure) be \( \Theta_i \), where \( \Theta_i \) is a non-free involution determined by Algorithm 3.3. We set \( R_g = \Theta_1 \cdot \Theta_2 \).

**Step 3.** If \( \mathcal{F} \) is free, then we set \( W(F) = R_g^{(g-1)/n} \), else we set \( W(F) = \frac{R_g^{g^{-1}}}{n} \).

**Step 4.** \( W(F) \) is the desired representation of \( F \) as a word in Dehn twists, up to conjugacy.

4. **Chain method**

In this section, we provide a method by which one can write certain periodic mapping classes as words in Dehn twists by repeated application of the chain relation. Let \( F \in \text{Mod}(S_g) \) be an irreducible Type 1 mapping class. Let \( F \in \text{Mod}(S_g) \) be of order \( n \), and let \( (1, n) \) be a pair in \( D_F \) representing a fixed point of the \( \langle \mathcal{F} \rangle \)-action on \( S_g \). Now consider the mapping class \( F^m \), for some integer \( 1 \leq m \leq |F| \). Then in \( D_{F^m} \), there exists a pair \( (c', n') \) (representing a fixed point of the \( \langle F^m \rangle \)-action on \( S_g \)) that originated from the pair \( (1, n) \) such that \( n' = |F^m| = n / \gcd(m, n) \) and \( (c')^{-1} \equiv m / \gcd(m, n) \) (mod \( n' \)). We will denote this pair \( (c', n') \) in \( D_{F^m} \) by \( (1, n)_{m,F} \).
Definition 4.1. Let $F \in \text{Mod}(S_g)$ be realizable as a linear $s$-tuple $(F_1, \ldots, F_s)$ of degree $n$ and genus $g$ as in Definition 2.6. Then $F$ is said to be chain-realizable if $F$ admits a realization as a linear $s$-tuple $(F_1, \ldots, F_s)$ of genus $g$ such that the following conditions hold.

(i) For $1 \leq i \leq s$, there exists an irreducible Type 1 mapping class $\tilde{F}_i \in \text{Mod}(S_{g_i})$, a filling chain $C(F_i)$ in $S_{g_i}$, and an $m_i > 1$ such that $F_i$ is conjugate to $(W_C(F_i))^{m_i}$. Then:

(a) For each $i$, $D_{\tilde{F}_i}$ has one of the following forms on $S_{g_i}$:
   1. $(2g_i + 1, 0; (2g_i, 1, 2g_i + 1), (1, 2g_i + 1, 1, 2g_i + 1))$
   2. $(2g_i + 2, 0; (g_i, 1, g_i + 1), (1, 2g_i + 2, 1, 2g_i + 2))$
   3. $(4g_i, 0; (1, 2), (1, 4g_i), (2g_i, 1, 4g_i))$
   4. $(4g_i + 2, 0; (1, 2), (g_i, 2g_i + 1), (1, 4g_i + 2))$

(b) For $1 \leq i \leq s - 1$, $k_i = 1$, and for each pair $(F_i, F_{i+1})$, the $1$-compatibility is across a pair of fixed points represented by pairs of the form $(c_i, n)$ (in $D_{\tilde{F}_i}$) and $(n - c_i, n)$ (in $D_{\tilde{F}_{i+1}}$), where $(c_i, n) = (1, |F_i|)_{m_i, F_i}$ and $(n - c_i, n) = (1, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}$.

Definition 4.2. A periodic mapping class $G \in \text{Mod}(S_g)$ is said to be chain-realizable if there exists a chain-realizable linear $s$-tuple $F \in \text{Mod}(S_g)$ and a nonzero integer $q$ such that $G = F^q$.

Given $c \in \mathbb{Z}_n^s$, we will fix the following notation.

(a) $c^+ = c(+1) := \{d \in \mathbb{Z} : cd \equiv 1 \pmod{n}\} \cap [0, n]$.
(b) $c^- = c(-1) := \{d \in \mathbb{Z} : cd \equiv 1 \pmod{n}\} \cap [-n, 0]$.

Lemma 4.3. Let $F \in \text{Mod}(S_g)$ be realizable as a chain-realizable $s$-tuple of degree $n$ and genus $g$ as in Definition 4.1. For all $i$, let $W(F_i) = (W^\beta_{C(F_i)})^{\tilde{c}_i}$, where $\tilde{c}_i = c_i((-1)^{i+1})$ and $\beta_i = \frac{|F_i|}{|F_i|}$. Then:

$$W(F) = \prod_{i=1}^{s} W(F_i)$$

is conjugate to $F$.

Proof. By Proposition 2.17 for each $i$, $((W^\beta_{C(F_i)})^{\tilde{c}_i})^{\tilde{c}_i}$ equals either $(T_{d_1} T_{d_2})^{\tilde{c}_i}$ or $(T_{d_1})^{\tilde{c}_i}$, depending upon whether $|C(F_i)|$ is odd or even. Thus, $\tilde{c}_i$ measures the amount of twisting along the boundary of a closed neighborhood of the chain $C(F_i)$. Thus, by Construction 2.5 and Definition 4.1, we have that $W(F)$ is conjugate of $F$. Finally, since $W(F_i)$ commutes with $W(F_j)$ for all $1 \leq i, j \leq s$, we have

$$(W(F))^n = \left(\prod_{i=1}^{s} W(F_i)\right)^n = \prod_{i=1}^{s} W(F_i)^n = \prod_{i=1}^{s} (T_{d_{i1}} T_{d_{i2}})^{\tilde{c}_i} = 1,$$

where $d_{i2}$ is taken to be the trivial curve when $i = 1, s$. \qed

We will now provide an algorithm for representing a chain-realizable periodic mapping classes as words in Dehn twists.

Algorithm 4.4 (Chain method). Let $G \in \text{Mod}(S_g)$ be a chain-realizable periodic mapping class.
Step 1. Write \( G = F^q \), where \( F \) is a compatible chain-realizable \( s \)-tuple \((F_1, \ldots, F_s)\) of degree \( n \) and genus \( g \) as in Definition 4.1.

Step 2. Set \( W(F_i) = (W_{C(F_i)})^{\bar{c}_i} \), where \( \bar{c}_i = c_i((-1)^i + 1) \), and set
\[
W(F) = \prod_{i=1}^{s} W(F_i).
\]

Step 3. By Lemma 4.3, \( W(G) = W(F)^q \) is the desired representation of \( G \) as a word in Dehn twists, up to conjugacy.

Example 4.5. For \( i = 1, 2 \), consider the order 6 mapping classes \( F_i \in \text{Mod}(S_2) \) with \( D_{F_1} = (6, 0; (1, 2), (1, 3), (1, 6)) \) and \( D_{F_2} = (6, 0; (1, 2), (2, 3), (5, 6)) \). The \( F_i \) admit a 1-compatibility along a pair of compatible fixed points that correspond to the pairs \((1, 6)\) and \((5, 6)\) in the \( D_{F_i} \) where the induced rotation angles are \( 2\pi/6 \) and \( 10\pi/6 \), respectively. This 1-compatibility yields an \( F = (F_1, F_2) \in \text{Mod}(S_2) \) with \( D_F = (6, 0; (1, 2), (1, 2), (1, 3), (2, 3)) \). If \( C(F_1) = \{a_1, b_1\} \) and \( C(F_2) = \{a_2, b_2\} \), then by Table 1 and Algorithm 4.4 and \( F \) is represented up to conjugacy by the word
\[
W(F) = (T_{a_1}T_{b_1})(T_{a_2}T_{b_2})^{-1}.
\]

4.1. Periodic maps on \( S_2 \) as words in Dehn twists. Let \( a_1, b_1, c_1, a_2, b_2, \) and \( x \) be curves in \( S_2 \), as indicated in Figure 11 below.

![Figure 11](image-url)

Using Algorithms 3.3 and 4.4 in Table 2 below, we provide a word \( W(F) \) (in Dehn twists) representing the conjugacy class of each periodic element \( F \in \text{Mod}(S_2) \).

5. Generalized star method

In this section, we first derive a generalization of Proposition 2.19 for \( g \geq 2 \). Using this result, we will develop a method to represent a much larger family of periodic mapping classes as words in Dehn twists, as compared with the chain method. As we will see, this family will also encompass the family of periodics described in Definition 4.1. Let
\[
a_1', a_1, \ldots, a_1g, b_1, \ldots, b_g, c_1, \ldots, c_{g-1}, d_1, d_2, \text{ and } d_3
\]
be the isotopy classes of the simple closed curves in \( S_2^3 \), as shown in Figure 12 below.
Module relations hold in generalization of the star relation.

When \( k = 1 \), we have:

\[
(T_1 T_1') \prod_{i=1}^{g-1} (T_{c_i} T_{b_i}) T_{b_1} = T_{d_2}^{(2g-1)^+} T_{d_1},
\]

where \( 2g - 1 \in \mathbb{Z}^+ \).

(iii) When \( k = 3 \), we have:

\[
(T_1 T_1') \prod_{i=1}^{g-1} (T_{b_i} T_{c_i} T_{b_{2i}}) T_{b_1} = T_{d_3}^{(2g-1)^+} T_{d_1} T_{d_3},
\]

Note that the curves \( a_1 \) and \( a_1' \) are isotopic in the surface \( \approx S^2_g \) obtained by capping off the boundary curve \( d_2 \). Further, we consider the surface \( S^2_g \) obtained by capping off the boundary curve \( d_3 \). We have the following generalization of the star relation.

**Theorem 5.1** (Generalized star relation). For \( g \geq 2 \) and \( k = 2, 3 \), the following relations hold in \( \text{Mod}(S^k_g) \).

(i) When \( k = 2 \), we have:

\[
(T_1 T_1') \prod_{i=1}^{g-1} (T_{b_i} T_{c_i}) T_{b_1} = T_{d_2}\end{array}^{(2g-1)^+} T_{d_1},
\]

where \( 2g - 1 \in \mathbb{Z}^+ \).

(ii) When \( k = 3 \), we have:

\[
(T_1 T_1') \prod_{i=1}^{g-1} (T_{b_i} T_{c_i} T_{b_{2i}}) T_{b_1} = T_{d_3}\end{array}^{(2g-1)^+} T_{d_1} T_{d_3},
\]
where $2g - 1 \in \mathbb{Z}_{2g+1}$.

Proof. To prove the result, we will use the well known Alexander method (see [6, Proposition 2.8]). For simplicity, we will only consider the case when $g = 2$, as our arguments easily generalize for $g > 2$. We provide our proofs for $k = 2$ and $k = 3$ through a series of pictures shown in Figures 14-15 below. The filling we consider (for the application of the Alexander method) is indicated in Figure 13.

Figure 13. Fillings of $S^2_2, S^3_2$ under consideration

Figure 14. Proof of the generalized star relation in Mod($S^2_2$).

Figure 15. Proof of the generalized star relation in Mod($S^3_2$).

□
Clearly, Theorem 5.1 is a generalization of Proposition 2.19. Moreover, by capping the boundary curve $d_2$, we can also recover Proposition 2.17. Following the notation from Section 1, we will now introduce a family of periodic mapping classes for which we form which the method we will develop in this section can be applied.

**Definition 5.2.** Let $F \in \text{Mod}(S_g)$ be realizable as a linear s-tuple $(F_1, \ldots, F_s)$ of degree $n$ and genus $g$ as in Definition 2.6. Then $F$ is said to be star-realizable if $F$ admits a realization as a linear s-tuple $(F_1, \ldots, F_s)$ of genus $g$ such that the following conditions hold.

(i) For $1 \leq i \leq s$, there exists an irreducible Type 1 mapping class $\tilde{F}_i \in \text{Mod}(S_g)$, a filling chain $\mathcal{C}(F_i)$ in $S_g$, and an $m_i > 1$ such that $F_i$ is conjugate to $(W_{\mathcal{C}(F_i)})^{m_i}$. Then:

(a) For each $i$, $D_{\tilde{F}_i}$ has one of the following forms on $S_g$,

- $1. (2g_i + 2, 0; (g_i - 1, g_i + 1), (1, 2g_i + 2), (1, 2g_i + 2))$,
- $2. (4g_i, 0; (1, 2), (1, 4g_i), (2g_i - 1, 4g_i))$,
- $3. (4g_i + 2, 0; (1, 2), (g_i - 2g_i + 1), (1, 4g_i + 2))$,
- $4. (2g_i + 1, 0; (2g_i - 1, 2g_i + 1), (1, 2g_i + 1), (1, 2g_i + 1))$

(b) For $1 \leq i \leq s - 1$, $k_i = 1$, and for each pair $(F_i, F_{i+1})$, the 1-compatibility is across a pair fixed points represented by pairs of the form $(c_i, n) (in D_{\tilde{F}_i})$ and $(n - c_i, n)$ (in $D_{\tilde{F}_{i+1}}$), where $(c_i, n) \in \{(1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i|/2 - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i| - 2, |\tilde{F}_i|)_{m_i, \tilde{F}_i}\}$ and $(n - c_i, n) \in \{(1, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}, (|\tilde{F}_{i+1}|/2 - 1, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}, (|\tilde{F}_{i+1}| - 2, |\tilde{F}_{i+1}|)_{m_{i+1}, \tilde{F}_{i+1}}\}$.

We will now fix the following notation:

$$W_i := \begin{cases} T_{a_i} \prod_{i=1}^{i-1} (T_{b_i} T_{c_i}) T_{b_i}, & \text{if } i = 4g + 2, \\ T_{a_i} T_{d_i} \prod_{i=1}^{i-1} (T_{b_i} T_{c_i}) T_{b_i}, & \text{if } i = 4g, \\ T_{a_i} \prod_{i=1}^{i-1} (T_{b_i} T_{c_i}) T_{b_i} T_{a_i}, & \text{if } i = 2g + 2, \text{ and} \\ T_{a_i} T_{d_i} \prod_{i=1}^{i-1} (T_{b_i} T_{c_i}) T_{b_i} T_{a_i}, & \text{if } i = 2g + 1, \end{cases}$$

where

$$D_{W_i} = \begin{cases} (4g + 2, 0; (1, 2), (g, 2g + 1), (1, 4g + 2)), & \text{if } i = 4g + 2, \\ (4g, 0; (1, 2), (1, 4g), (2g - 1, 4g)), & \text{if } i = 4g, \\ (2g + 2, 0; (g - 1, g + 1), (1, 2g + 2), (1, 2g + 2)), & \text{if } i = 2g + 2, \text{ and} \\ (2g + 1, 0; (1, 2g + 1), (1, 2g + 1), (2g - 1, 2g + 1)), & \text{if } i = 2g + 1. \end{cases}$$

Let $d_i$ denote the boundary curve of $\Sigma_i$ involved in the 1-compatibility of $F_i$ with $F_{i+1}$, and let $\gamma_i$ represent the isotopy class of $d_i$ in $S_g$ after the compatibility. Let $c^+$ denote the unique integer in $[0, n]$ representing the
Lemma 5.3. Let $F \in \text{Mod}(S_g)$ be a star-realizable linear $s$-tuple of degree $n$ as in Definition 5.2. For all $i$, let $\mathcal{W}(F_i) = (W_{|\tilde{F}_i|})^{m_i}$. Then:

$$\mathcal{W}(F) = \left( \prod_{i=1}^{s} \mathcal{W}(F_i) \right)^{s-1} \prod_{i=1}^{s} (T_{\gamma_i})^{-\eta_i},$$

where $\eta_i = \frac{\mu_{z_i,i} + \mu_{z_{i+1},i+1}}{n}$, is conjugate to $F$.

Proof. Since $\mathcal{W}(F_i)$ commutes with $\mathcal{W}(F_j)$ for $1 \leq i, j \leq s$, we have

$$\left( \prod_{i=1}^{s} \mathcal{W}(F_i) \right)^n = \prod_{i=1}^{s} (\mathcal{W}(F_i))^n.$$

Since $\mathcal{W}(F_i) = (W_{|\tilde{F}_i|})^{m_i}$, the fact that

$$(c_i, n) \in \{(1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i|/2 - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (2|\tilde{F}_i| - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}\}$$

implies that

$$\prod_{i=1}^{s} (\mathcal{W}(F_i))^n = \prod_{i=1}^{s} (W_{|\tilde{F}_i|})^{m_i \gcd(m_i, |\tilde{F}_i|)}.$$

By Theorem 5.1 depending on $|\tilde{F}_i|$ and $D_{\tilde{F}_i}$, $(W_{|\tilde{F}_i|})^{F_i}$ one of:

$$T_{d_1}, T_{d_1}^{-((|\tilde{F}_i|)/2-1)^+}, \text{ or } T_{d_1}^{-((|\tilde{F}_i|)/2-1)^+} T_{d_3}.$$  

By the definition of $\mu_{z_i,i}$, we have

(*)

$$\prod_{i=1}^{s} (\mathcal{W}(F_i))^n = \prod_{i=1}^{s} (\mathcal{W}(F_i))^n = \prod_{i=1}^{s-1} \left( T_{\gamma_i}^{\mu_{z_i,i} + \mu_{z_{i+1},i+1}} \right)$$

As each $T_{\gamma_i}$ commutes with every other Dehn twist appearing in (*) and $\mu_{z_i,i} + \mu_{z_{i+1},i+1} \equiv 0 \pmod{n}$, we get

$$\mathcal{W}(F)^n = \left( \prod_{i=1}^{s} \mathcal{W}(F_i) \right)^{n-1} \prod_{i=1}^{s} (T_{\gamma_i})^{-\eta_i} = \left( \prod_{i=1}^{s} \mathcal{W}(F_i) \right)^{s-1} \prod_{i=1}^{s} (T_{\gamma_i})^{-\eta_i} = 1,$$

from which the assertion follows. \qed
We will describe an algorithm to write a star-realizable linear $s$-tuple $F \in \text{Mod}(S_g)$ as a word in Dehn twists (up to conjugacy).

**Algorithm 5.4.** Let $F \in \text{Mod}(S_g)$ be a star-realizable linear $s$-tuple of degree $n$ and genus $g$.

**Step 1.** Write $F = (F_1, \ldots, F_s)$ as in Definition 5.2.

**Step 2.** For each $i$, we set $W(F_i) = W(F_i^{m_i})$, after appropriately relabeling the curves in $\Sigma_i$ in order to ensure consistency with the (assumed) labeling in Theorem 5.7.

**Step 3.** Set

$$W(F) = \left( \prod_{i=1}^{s} W(F_i) \right) \prod_{i=1}^{s-1} (T_{\gamma_i})^{-\eta_i}.$$

**Step 4.** By Lemma 5.3, $W(F)$ is the desired representation of $F$ as a word in Dehn twists, up to conjugacy.

The method described in Algorithm 5.4 can be generalized to certain types of $(F, T)$-tuples.

**Definition 5.5.** A compatible $(F, T)$-tuple as in Definition 2.10 is said to be **star-realizable** if the following conditions hold.

(i) $v = w = 0$.

(ii) $F$ is star-realizable.

(iii) For $1 \leq q \leq u$, $k_q = 1$.

(iv) For $1 \leq q \leq u$, suppose the self $1$-compatibility in $F_{i_q,j_q}$ is along fix points represented by $(c_{iq}, n)$ (in $D_{F_{iq}}$) and $(n - c_{iq}, n)$ (in $D_{F_{iq}}$), then

$$(c_{iq}, n) \in \{(1, |\tilde{F}_{iq}|)_{m_{iq}, \tilde{F}_{iq}}, (|\tilde{F}_{iq}|/2 - 1, |\tilde{F}_{iq}|)_{m_{iq}, \tilde{F}_{iq}}, (|\tilde{F}_{iq}| - 2, |\tilde{F}_{iq}|)_{m_{iq}, \tilde{F}_{iq}}\}$$

and

$$(n - c_{iq}, n) \in \{(1, |\tilde{F}_{jq}|)_{m_{jq}, \tilde{F}_{jq}}, (|\tilde{F}_{jq}|/2 - 1, |\tilde{F}_{jq}|)_{m_{jq}, \tilde{F}_{jq}}, (|\tilde{F}_{jq}| - 2, |\tilde{F}_{jq}|)_{m_{jq}, \tilde{F}_{jq}}\}.$$

**Definition 5.6.** A periodic mapping class $G \in \text{Mod}(S_g)$ is said to be **star-realizable** if there exists a star-realizable compatible $(F, T)$-tuple $F_T \in \text{Mod}(S_g)$ and a nonzero integer $q$ such that $G = F_T^q$.

We will now extend Algorithm 5.4 to this broader class of periodic mapping classes. While doing so, we will retain the notation for $\gamma_i$ and $\eta_i$, for $1 \leq i \leq s$ (for $F$) from Algorithm 5.4. To further simplify notation, we will denote the additional curves involved in the additional self 1-self compatibilities (of $F_T$) by $\{\gamma_{jq}\}_{j_q=s+1}^{u+s-1}$ and also extend the earlier definition of $\eta_j$ to $s \leq j \leq u + s - 1$.

**Algorithm 5.7.** Let $G \in \text{Mod}(S_g)$ be a star-realizable periodic mapping class.

**Step 1.** Write $G = F_T^q$, where $F_T$ is a compatible $(F, T)$-tuple as in Definition 2.10.

**Step 2.** By Algorithm 5.4, we obtain

$$W(F) = \left( \prod_{i=1}^{s} W(F_i) \right) \prod_{i=1}^{s} (T_{\gamma_i})^{-\eta_i}.$$
Step 3. We set
\[ W(F_T) = W(F) \prod_{i=s+1}^{u+s-1} (T_{\gamma_i})^{-\eta_i}. \]

Step 4. By the same arguments from Lemma 5.3, \( (W(F_T))^q \) is the desired representation of \( G \) as a word in Dehn twists (after an appropriate relabeling of curves to ensure consistency with Theorem 5.7).

We will now give three examples to demonstrate the application of Algorithms 5.4 and 5.7.

**Example 5.8.** Consider an \( F \in \text{Mod}(S_7) \) with
\[ D_F = (6, 0; (1, 2), (1, 3), (2, 3), (1, 6), (5, 6)). \]
Then \( F \) is a star-realizable linear 3-tuple \((F_1, F_2, F_3)\), where
\[
F_1 = (6, 0; (1, 2), (1, 6), (5, 6)) \quad \text{with} \quad g_1(D_{F_1}) = 3 \\
F_2 = (6, 0; (1, 3), (5, 6), (5, 6)) \quad \text{with} \quad g_2(D_{F_2}) = 2, \quad \text{and} \\
F_3 = (6, 0; (2, 3), (1, 6), (1, 6)) \quad \text{with} \quad g_3(D_{F_3}) = 2.
\]
Note that the 1-compatibility of \( F_1 \) with \( F_2 \) is along fixed points represented by the pairs \((1, 6)\) (in \( D_{F_1} \)) and \((5, 6)\) (in \( D_{F_2} \)), while the compatibility of \( F_2 \) with \( F_3 \) is along the pairs \((5, 6)\) (in \( D_{F_2} \)) and \((1, 6)\) (in \( D_{F_3} \)). By Algorithm 5.4, we have \( W(F_1) = W_{F_1}^2, W(F_2) = W_{2+2}^5, W(F_3) = W_{2+2}, \) and \( \eta_1 = 1 = \eta_2 \). Therefore, we have
\[
W(F) = (T_{a_1}T_{b_1}T_{c_1}T_{b_2}T_{c_2}T_{b_3})^2(T_{a_4}T_{b_4}T_{c_4}T_{b_5}T_{c_5})^3(T_{a_6}T_{b_6}T_{c_6}T_{b_7}T_{c_7})(T_{\gamma_1}T_{\gamma_2})^{-1}.
\]

**Example 5.9.** Consider a periodic mapping class \( G \in \text{Mod}(S_9) \) with \( D_G = (2g - 2, 1; (1, 2), (1, 2)) \). Then \( G \) is a star-realizable mapping class \( F_T \), where \( T = (1, 1, 0, 0) \) and \( F = (F_1) \) with \( D_{F_1} = (2g - 2, 0, (1, 2), (1, 2), (1, 2g - 2), (2g - 1, 2g - 2)) \) and \( g_1 = g(F) = g - 1 \). Note that the self 1-compatibility of \( F \) is along a pair of fixed points of the \( (F) \)-action represented by the pairs \((1, 2g - 2)\) and \((2g - 1, 2g - 2)\) (in \( D_{F_1} \)). By Algorithm 5.7, we have \( W(F_1) = W_{2+1}^2 \), and so
\[
W(G) = (T_{a_2}^2 \prod_{i=1}^{g-1} (T_{c_i}T_{b_{i+1}}))^{2T_{a_1}^{-1}},
\]
where we have relabeled \( \gamma_1 \) as \( a_1 \), and \( a_1' \) as \( c_1 \), so as to ensure consistency with Theorem 5.7.

**Example 5.10.** Consider a periodic mapping class \( F \in \text{Mod}(S_{10}) \) with \( D_F = (7, 1; (1, 7), (3, 7), (3, 7)) \). Then \( F \) is star-realizable linear 2-tuple \((F_1, F_2)\), where \( D_{F_1} = (7, 1; (3, 7), (4, 7)) \) with \( g_1 = g(D_{F_1}) = 7 \) and \( D_{F_2} = (7, 0; (1, 7), (3, 7), (3, 7)) \) with \( g_2 = g(D_{F_2}) = 3 \). Note that \( F_1 \) is a rotational mapping class, and the 1-compatibility of \( F_1 \) with \( F_2 \) is along fixed points represented by the pairs \((4, 7)\) (in \( D_{F_1} \)) and \((3, 7)\) (in \( D_{F_2} \)). Following Algorithm 5.3, we have \( W(F_1) = W_{4+7}^2, W(F_2) = W_{2+3+1}^5, \) and \( \eta_1 = 1 \). Consequently,
\[
W(F) = (T_{a_1}^2 \prod_{i=1}^{6} (T_{b_i}T_{c_i})T_{b_7})^8(T_{a_8}T_{b_8}T_{c_8}T_{b_9}T_{c_9}T_{b_{10}}T_{a_{10}}^2)^5T_{\gamma_1}^{-1},
\]
where \( \gamma_1 \) is the separating curve involved in the 1-compatibility \( F_1 \) with \( F_2 \).
In general, the addition of a \(g'\)-permutation component to a periodic mapping class \(F_2 \in \text{Mod}(S_g)\) of order \(n\) (as in Construction 2.7) can also be viewed as 1-compatibility of \(F_2\) with the rotational mapping class \(F_1 \in \text{Mod}(S_{n'g})\) with \(D_{F_1} = (n, g'; (1, n), (n - 1, n))\). (Note that this compatibility is along fixed points represented by \((n - 1, n)\) (in \(D_{F_1}\) and \((1, n)\) in \(D_{F_2}\)).) Moreover, it is not hard to see that \(W(F_1) = W_4^{4n}\). Thus, the ideas in Example 5.10 easily generalize to yield the following.

**Proposition 5.11.** Let \(F_2 \in \text{Mod}(S_g)\) be a periodic star-realizable mapping class of odd order with \(D_{F_1} = (n, g'; (c_1, n_1)(c_2, n_2), \ldots, (c_r, n_r))\). Let \(F\) be obtained from \(F_1\) through the addition of a 1-permutation component to \(F_2\). Then viewing \(F\) as a 1-compatible pair \((F_1, F_2)\) along a separating curve \(\gamma\), where \(D_{F_1} = (n, 1; (c, n), (n - c, n))\), we have

\[
W(F) = W(F_2)W^{4c+4T_\gamma - \eta},
\]

where \(\eta\) is defined along the same lines as in Lemma 5.3.

### 6. Symplectic method

Let \(F \in \text{Mod}(S_g)\) be of order \(n\). In this section, we give a method by which one can use \(\Psi(F)\) for finding a representation of \(F\) as a word \(W(F)\), up to conjugacy. (Here, we compute \(\Psi(F)\) using Theorem 2.14 and Remark 2.15.)

In other words, we have to find a suitable candidate for \(W(F)\) in

\[
\mathcal{M}_F := \{G \in \text{Mod}(S_g) : \Psi(G) \text{ is conjugate to } \Psi(F)\}.
\]

Let \(\Psi_m\) denote the composition of \(\Psi\) with the canonical projection \(\text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}_m)\). It is well known that \(\ker \Psi_m\) (also known as the level-\(m\) subgroup \(\text{Mod}(S_g)[m]\)) is torsion-free for \(m \geq 3\) (see [6, Theorem 6.9]). Considering that the conjugacy class of \(\Psi(F)\) can be infinite in \(\text{Sp}(2g, \mathbb{Z})\), for computational purposes, we consider the set

\[
\widetilde{\mathcal{M}}_F := \{G \in \text{Mod}(S_g) : \Psi_3(G) \text{ is conjugate to } \Psi_3(F)\}
\]

in place of \(\mathcal{M}(F)\). The key idea behind our method is to provide a systematic procedure for carefully and efficiently sifting through the elements in set \(\widetilde{\mathcal{M}}_F\) to find a suitable candidate for \(W(F)\).

#### 6.1. Structured searching for \(W(F)\)

To standardize our procedure, we consider the Lickorish [14] generating set \(\mathcal{L}_g\) for \(\text{Mod}(S_g)\) and assume that each element in \(\mathcal{M}_F\) is a word in \(\mathcal{L}_g\). To fix notation, let

\[
\mathcal{L}_g = \{T_{a_1}, T_{b_1}, T_{c_1}, T_{b_2}, T_{a_2}, T_{c_2}, \ldots, T_{c_{g-1}}, T_{b_g}, T_{a_g}\}
\]

with the \(a_i\), the \(b_i\) and the \(c_i\) are indicated in Figure 16 below.
In order to make our search for $W(F)$ in $\tilde{M}_F$ more efficient, we first have to ensure the implementation of a well-structured search process. For achieving this, we introduce the notion of the depth of a word. Let $W$ be a reduced word in $L_g$, and let $n_i$ be the number of times the $i^{th}$ generator in $L_g$ appears in $W$. Then the depth $d(W)$ of the word $W$ is defined by $d(W) = \max\{n_i : 1 \leq i \leq 3g-1\}$. For example, for the word $W = T_{a_1}^4 T_{a_2}^3 T_{b_2}^2 T_{a_2}^4 T_{b_1}$, $d(W) = 2$. Further, we denote the largest power (in absolute value) of a Dehn twist in $L_g$ appearing in a word $W$ by $p(W)$, and fix the notation $\tilde{M}_{i,j}^F := \{W \in \tilde{M}_F : d(W) = i \text{ and } p(W) = j\}$. Thus, we will begin our search for $W(F)$ in $\tilde{M}_{1,1}^F$, and then gradually broaden our search in an incremental manner to $\tilde{M}_{i,j}^F$ for $i, j > 1$.

6.2. Discarding redundant words in $M_F$. To begin with, we apply the basic property that Dehn twists about isotopically disjoint curves in $S_g$ commute, we would like to discard redundant variants of words that are equivalent up to commutativity of the twists in $L_g$. For this reason, we assign numbers 1 through $3g-1$. A permutation $\sigma$ of $\{1, \ldots, 3g-1\}$ is said to be good if for $1 \leq i \leq 3g-1$, either $\sigma(i+1) - \sigma(i) \leq 1$ or $(\sigma(i), \sigma(i+1)) = (3k-2, 3k)$ for some $k$. Thus, in our process, we will filter out many (redundant) words in $\tilde{M}_F$ by considering only words as arise as good permutations of the (powers of the) Dehn twists appearing in $L_g$. We will further discard several non-periodic words in $\tilde{M}_F$ by applying the Penner’s construction [21] of pseudo-Anosov mapping classes.

**Theorem 6.1.** Let $C = \{\alpha_1, \ldots, \alpha_n\}$ and $D = \{\alpha_{n+1}, \ldots, \alpha_{n+m}\}$ be multicurves in $S_g$ that together fill $S_g$. Then any product of positive powers of the $T_{\alpha_i}$, for $i = 1, \ldots, n$ and negative powers of the $T_{\alpha_{n+j}}$, for $j = 1, \ldots, m$, where each $\alpha_i$ and each $\alpha_{n+j}$ appears at least once, is pseudo-Anosov.

6.3. Searching for periodics. Let $i(\alpha, \beta)$ be the geometric intersection number of essential simple closed curves $\alpha, \beta$ in $S_g$. In order to identify the periodics in $M_F$, we will (in general) use the well-known Bestvina-Handel algorithm [1], which provides an effective way of identifying them. This brings us to the following remark

**Remark 6.2.** When $F$ is irreducible, the elements $\tilde{M}_F$ are either irreducible periodics or pseudo-Anosovs. In this context, we have observed that it is easier to identify the periodics by simply determining whether the orbits (under $F$) of certain appropriately chosen curves are finite. To this
end, a software named Teruaki for Mathematica 7 (or TKM7) by Sakasai-Suzuki [25] designed for the visualization of actions of Dehn twits on curves (in $S_g$) really comes in handy.

This finally bring us to our method.

**Algorithm 6.3** (Symplectic method). Let $F \in \text{Mod}(S_g)$ be of order $n$.

**Step 1.** Compute $\Psi(F)$ (up to conjugacy) using Theorem 2.14 and Remark 2.15

**Step 2.** Set $i = 1$, $j = 1$, and flag = 0

**Step 3.** Repeat Steps 4-5 until flag = 1.

**Step 4.** Repeat Steps 4a – 4f, while $j \leq n$.

**Step 4a.** Compute the elements in $\tilde{M}^i_{ij}$ up to good permutations.

**Step 4b.** Discard the words in $\tilde{M}^i_{ij}$ that are pseudo-Anosovs using Theorem 6.7 (or Remark 6.3 when $F$ is irreducible).

**Step 4c.** If $|\tilde{M}^i_{ij}| \geq 0$, repeat Steps 4d – 4e, for each $W \in \tilde{M}^i_{ij}$. Else, proceed to Step 4f.

**Step 4d.** Apply the Bestvina-Handel algorithm to determine the mapping class type of $W$.

**Step 4e.** If $W$ is periodic, then set $\tilde{W}(F) = W$, flag = 1, and then proceed to Step 5. Else, set $\tilde{M}^i_{ij} = \tilde{M}^i_{ij} \setminus \{W\}$ and proceed to Step 4c.

**Step 4f.** Set $j = j + 1$ and proceed to Step 4.

**Step 5.** If flag = 0, set $i = i + 1$ and proceed to Step 3. Else, proceed to Step 6.

**Step 6.** $\tilde{W}(F)$ is the desired representation of $F$ as a word in Dehn twists, up to conjugacy.

**Example 6.4.** Consider the irreducible Type 1 mapping class $F \in \text{Mod}(S_3)$ with $D_F = (9, 0; (1, 3), (1, 9), (5, 9))$. Clearly, $F$ is neither rotational nor star realizable, so we will use Algorithm 6.3 to find $\tilde{W}(F)$. We will follow the notation from Subsections 2.2-2.3. By Theorem 2.4 $F$ is realized as a rotation of the polygon $P_F$, as shown in Figure 17 below, with

\[
L(P_F) = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}
\]

and

\[
W(P_F) = a_1a_2a_3a_4a_5a_6a_7a_4^{-1}a_8a_6^{-1}a_9a_7^{-1}a_1^{-1}a_8^{-1}a_3^{-1}a_9^{-1}a_5^{-1}.
\]

Let $P'$ be the standard 12-gon (realizing the surface $S_3$) with

\[
L(P_3) = \{x_1, y_1, x_2, y_2, x_3, y_3\} \quad \text{and} \quad W(P_3) = [x_1, y_1][x_2, y_2][x_3, y_3].
\]

Denoting $\phi = \phi_{P_F}$ and $f = f_{P_F}$, we obtain $\varphi = f^{-1}\phi f$ (as in Theorem 2.14) in the following manner.

\[
\begin{align*}
[x_1] & \xrightarrow{f} [b_1] + [b_2] & \xrightarrow{\phi} & [b_1] - [b_3] - [b_6] & \xrightarrow{f^{-1}} & [-y_1] + [x_2] - [y_2] \\
[y_1] & \xrightarrow{f} [b_1] + [b_3] & \xrightarrow{\phi} & [b_2] - [b_3] + [b_4] & \xrightarrow{f^{-1}} & [x_1] - [y_1] + [x_3] - [y_3] \\
[x_2] & \xrightarrow{f} [b_2] + [b_4] + [b_5] & \xrightarrow{\phi} & [-b_4] - [b_6] & \xrightarrow{f^{-1}} & [x_2] - [y_2] - [x_3] + [y_3] \\
[y_2] & \xrightarrow{f} [b_4] + [b_5] + [b_6] + [b_2] & \xrightarrow{\phi} & [b_3] - [b_4] + [b_5] - [b_6] & \xrightarrow{f^{-1}} & [-x_1] + [y_1] + 2[x_2] - [y_2] - 2[x_3] + 2[y_3] \\
[x_3] & \xrightarrow{f} [-b_5] & \xrightarrow{\phi} & [b_4] & \xrightarrow{f^{-1}} & [x_3] - [y_3] \\
[y_3] & \xrightarrow{f} [-b_4] - [b_5] & \xrightarrow{\phi} & [b_4] - [b_1] & \xrightarrow{f^{-1}} & [-x_1] + [x_2] + [x_3]
\end{align*}
\]
Here, \([b_1] = [a_8^{-1}a_3^{-1}], [b_2] = [a_9^{-1}a_5^{-1}], [b_3] = [a_2^{-1}a_1^{-1}], [b_4] = [a_7^{-1}a_9], [b_5] = [a_7^{-1}a_2], \) and \([b_6] = [a_4^{-1}a_8] \). Thus, the matrix \(M_\varphi\) representing the conjugacy class of \(\Psi(F)\) in \(\text{Sp}(2g,\mathbb{Z})\) is given by

\[
M_\varphi = \begin{pmatrix}
0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\
-1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 2 & 0 & 1 & 0 & -1 \\
-1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\
0 & -1 & -1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 2 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 2 & 1 & 0 \\
2 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
0 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 2 & 2 & 0 & 0 & 0
\end{pmatrix},
\]

and so we have

\[
\Psi_3(F) = \begin{pmatrix}
0 & 1 & 0 & 2 & 0 & 2 \\
2 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 0 & 0 \\
0 & 2 & 2 & 2 & 1 & 1 \\
0 & 1 & 1 & 2 & 2 & 0
\end{pmatrix}.
\]

Following Algorithm 6.3, we begin our search for \(\mathcal{W}(F)\) in \(\widetilde{M}_F^{1,1}\). As it turns out, even after considering only the words among the good permutations that are not Penner-type pseudo-Anosovs, we were still left with numerous (at least 150) possible candidates for \(\mathcal{W}(F)\) in \(\widetilde{M}_F^{1,1}\). For brevity, we will demonstrate the algorithm on a small subcollection of words

\[
\widetilde{M}_F^{1,1}(\sigma) = \{T_{a_1}^{-1}T_{c_1}^{-1}T_{a_2}^{-1}T_{b_2}^{-1}T_{b_2}^{-1}T_{b_1}^{-1}, T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1}, T_{c_1}T_{a_2}T_{b_3}T_{a_3}T_{c_2}T_{b_2}T_{b_1}, T_{c_1}^{-1}T_{a_2}^{-1}T_{b_2}^{-1}T_{b_2}^{-1}T_{b_1}^{-1}, T_{c_1}^{-1}T_{a_2}^{-1}T_{b_2}^{-1}T_{b_2}^{-1}T_{b_1}^{-1}\}
\]

corresponding to the permutation of \(\sigma = (2 3 5 8)(4 7)\) on \(L_3\). Using Remark 6.2 (and the Teruaki software [25]), we can easily deduce that all words in \(\widetilde{M}_F^{1,1}(\sigma)\) are finite order. Thus, we may choose \(\mathcal{W}(F) = T_{a_1}T_{c_1}T_{a_2}T_{b_3}T_{c_2}T_{b_2}T_{b_1}\).

Note that the symplectic method can be applied to any periodic mapping class. However, as the method is computationally intense, we recommend its...
application only for non-rotational periodics that are neither star-realizable nor chain-realizable. It is apparent that while the earlier methods were more restrictive in terms of their applicability, they work quite efficiently for the specific families of periodic mapping classes they were designed for.

7. Applications

7.1. Roots of Dehn twists. For $g \geq 2$, let $c$ be a non-separating curve in $S_g$. A root of $T_c$ of degree $n$ is an $F \in \text{Mod}(S_g)$ such that $F^n = T_c$. Margalit-Schleimer [13] gave the first example of such a root of degree $2g - 1$ in $\text{Mod}(S_g)$. A complete classification of such roots was obtained [13], where it was also shown that the Margalit-Sceimer root (of degree $2g - 1$) had the largest possible degree in $\text{Mod}(S_g)$. A periodic mapping class $F \in \text{Mod}(S_{g-1})$ is said to be root-realizing if the $(\bar{F})$-action on $S_{g-1}$ has two distinguished fixed points where the induced local rotation angles add up to $2\pi/n \pmod{2\pi}$. Given a root-realizing $F \in \text{Mod}(S_{g-1})$ of order $n$ with distinguished fixed points $P_1$ and $P_2$, one can remove $(\bar{F})$-invariant neighborhoods around the $P_i$ and then attach an annulus $A$ with an $(1/n)\theta$-twist connecting the resulting boundary components to realize a root $F \in \text{Mod}(S_g)$ of a Dehn twist $T_c$ about the non-separating curve $c$ in $A$. Conversely, for $g \geq 2$, given a root $F \in \text{Mod}(S_g)$ of $T_c$ of degree $n$, one can reverse this process to recover a root-realizing periodic mapping class $\bar{F} \in \text{Mod}(S_{g-1})$.

Thus, the conjugacy class of a typical root realizing $\bar{F} \in \text{Mod}(S_g)$ that corresponds to the conjugacy class of a root $F \in \text{Mod}(S_g)$ of degree $n$ has the form

$$D_{\bar{F}} = (n, g_0; (a, n), (b, n), (c_1, n_1), \ldots, (c_\ell, n_\ell)),$$

where $a + b \equiv ab \pmod{n}$. Here the pairs $(a, n)$ and $(b, n)$ represent the fixed points of the $(\bar{F})$-action involved in the construction of the root $F$.

We will now describe a family of roots that can be represented as words in Dehn twists by using a minor modification of the method described in Algorithm 6.7.

**Definition 7.1.** A root $F \in \text{Mod}(S_g)$ of $T_c$ of degree $n$ is said to be star-realizable if the following conditions hold.

(i) The root-realizing periodic mapping class $\bar{F} \in \text{Mod}(S_{g-1})$ with

$$D_{\bar{F}} = (n, g_0; (a, n), (b, n), (c_1, n_1), \ldots, (c_\ell, n_\ell)),$$

$a + b \equiv ab \pmod{n}$, is star-realizable.

(ii) Suppose that $\bar{F} = F_T^q$, for some star-realizable $F_T$ as in Definition 5.6, so that the pairs $(a, n)$ (resp. $(b, n)$) belong to $D_{F_{Tq}}$ (resp. $D_{F_{Th}}$). Then

$$(a, n) \in \{(1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i|/2 - 1, |\tilde{F}_i|)_{m_i, \tilde{F}_i}, (|\tilde{F}_i| - 2, |\tilde{F}_i|)_{m_i, \tilde{F}_i}\}$$

and

$$(b, n) \in \{(1, |\tilde{F}_j|)_{m_j, \tilde{F}_j}, (|\tilde{F}_j|/2 - 1, |\tilde{F}_j|)_{m_j, \tilde{F}_j}, (|\tilde{F}_j| - 2, |\tilde{F}_j|)_{m_j, \tilde{F}_j}\}$$

Denoting $\eta = \frac{\mu_{m_i, \tilde{F}_i} + \mu_{m_j, \tilde{F}_j} - 1}{n}$, we will now give an algorithm to represent a star-realizable root as a word in Dehn twists.

**Algorithm 7.2.** Consider a star-realizable root $F \in \text{Mod}(S_g)$ of $T_c$ as in Description 7.1.
Step 1. Apply Algorithm 5.7 to obtain $W(\bar{F})$.

Step 2. Set

$$W(F) = W(\bar{F})(T_c)^{-\eta}.$$

Step 3. By Lemma 5.3, $W(F)$ is the desired representation of $F$ as a word in Dehn twists, up to conjugacy.

Let $G \in \text{Mod}(S_g)$ be the Margalit-Schleimer root (of $T_c$) of degree $2g - 1$. In [15] an expression for $W(G)$ was derived using the chain relation, and in [18] it was shown that $D_G = (2g - 1, 0; (2, 2g - 1), (2, 2g - 1), (-4, 2g - 1))$. In the following example, we will apply Algorithm 7.2 to derive $W(G)$ for a root $F \in \text{Mod}(S_g)$ of degree $2g - 1$ for which $D_F$ is different from $D_G$.

**Example 7.3.** Consider a root $F \in \text{Mod}(S_g)$ of degree $2g - 1$, where $D_F = (2g - 1, 0; (g, 2g - 1), (2g - 2, 2g - 1))$. Since $W(\bar{F}) = W^2_{2g-1}$, by applying Algorithm 7.2 we get

$$W(F) = T_{a_1}^{-1}(T_{c_1}T_{a_2}\prod_{i=2}^{g-1}(T_{b_i}T_{c_i}T_{b_0}T_{a_0})^2).$$

For $g \geq 2$, a fractional root of $T_c$ of degree $(m, n)$ is an $F \in \text{Mod}(S_g)$ such that $F^n = T_c^m$. It is known [22] that such a root of $T_c$ may either preserve or reverse the two sides of $c$, and for a side-preserving root of degree $(m, n)$, $n \leq 4g$. Further, a side-preserving fractional root of degree $(2g, 4g)$ always exists in $\text{Mod}(S_g)$. As in the case of roots of Dehn twists, a side-preserving fractional root $F \in \text{Mod}(S_g)$ of degree $(m, n)$ corresponds to an $\bar{F} \in \text{Mod}(S_g)$ of order $n$ such that the $\langle F \rangle$-action has two distinguished fixed points where the induced rotation angles add up to $2\pi m/n$ (mod $2\pi$). This brings us to the final result in this subsection, in which we assume the notation of Theorem 5.1.

**Proposition 7.4.** Let $F \in \text{Mod}(S_g)$ be a side-preserving fractional root of $T_{a_1}$ of degree $(2g, 4g)$. Then

$$F = T_{a_2}\prod_{i=1}^{g-1}(T_{c_i}T_{b_{i+1}}).$$

**Proof.** Assume that $F$ is realized from $\bar{F} \in \text{Mod}(S_{g-1})$ by attaching an annulus (with a $2\pi m/n$-twist) connecting the two boundary components $(d_2$ and $d_3$) of the subsurface $S^2_{g-1}$. By Theorem 5.1 (for $k = 2$), we have

$$(T_{a_2}\prod_{i=1}^{g-1}(T_{c_i}T_{b_{i+1}}))^{4g} = T_{a_1}^{-1}T_{a_1}^{2g-1}.$$ But, as $2g - 1 \equiv 2g - 1$ (mod $4g$), this further simplifies to

$$(T_{a_2}\prod_{i=1}^{g-1}(T_{c_i}T_{b_{i+1}}))^{4g} = T_{a_1}^{2g}.$$
7.2. Representing periodic elements in \( \text{Mod}(S_3) \) as words in Dehn twists. Using Algorithms 3.3, 4.4, 5.7, and 6.3 in Table 3 below, we provide a word \( W(F) \) (in Dehn twists) representing the conjugacy class of each periodic mapping class \( F \in \text{Mod}(S_3) \).

| \( F \) | \( D_F \) | \( W(F) \) | Algorithm |
|-----|-----|-----|--------|
| 14 | (14,0; (1,2), (3,7), (1,14)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.3 |
| 14 | (14,0; (1,2), (2,7), (3,14)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.3 |
| 14 | (14,0; (1,2), (1,7), (5,14)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.3 |
| 14 | (14,0; (1,2), (6,7), (9,14)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.3 |
| 14 | (14,0; (1,2), (5,7), (11,14)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 14 | (14,0; (1,2), (4,7), (13,14)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 12 | (12,0; (1,2), (5,12), (1,12)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 12 | (12,0; (1,2), (7,12), (11,12)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 12 | (12,0; (2,3), (1,4), (1,12)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 12 | (12,0; (1,3), (1,4), (5,12)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 12 | (12,0; (2,3), (3,4), (7,12)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 12 | (12,0; (1,3), (3,4), (11,12)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 9 | (9,0; (1,3), (5,9), (1,9)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 9 | (9,0; (2,3), (1,9), (2,9)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 9 | (9,0; (1,3), (2,9), (4,9)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 9 | (9,0; (2,3), (7,9), (5,9)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 9 | (9,0; (1,3), (8,9), (7,9)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 9 | (9,0; (2,3), (4,9), (8,9)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 8 | (8,0; (3,4), (1,8), (1,8)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 8 | (8,0; (1,4), (3,8), (3,8)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 8 | (8,0; (3,4), (5,8), (5,8)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 8 | (8,0; (1,4), (7,8), (7,8)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 8 | (8,0; (1,4), (5,8), (5,8)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 8 | (8,0; (3,4), (7,8), (7,8)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (5,7), (1,7), (1,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (3,7), (2,7), (2,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (1,7), (3,7), (3,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (6,7), (4,7), (4,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (4,7), (5,7), (5,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (2,7), (6,7), (6,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (4,7), (2,7), (1,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (3,7), (5,7), (6,7)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 7 | (7,0; (1,7), (2,7), (1,6)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 6 | (6,0; (1,2), (1,2), (1,6), (5,6)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 6 | (6,0; (1,2), (2,3), (2,3), (1,6)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 6 | (6,0; (1,2), (1,3), (1,3), (5,6)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 6 | (6,0; (1,2), (1,3), (1,2), (1,2)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 4 | (4,0; (1,2), (1,2), (1,4), (1,4)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 4 | (4,0; (1,2), (1,2), (3,4), (3,4)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 4 | (4,0; (1,4), (1,4), (3,4), (3,4)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 4 | (4,0; (1,4), (1,4), (1,4), (1,4)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 4 | (4,0; (3,4), (3,4), (3,4), (3,4)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 3 | (3,0; (1,3), (1,3), (1,3), (1,3), (1,3), (2,3)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 3 | (3,0; (2,3), (2,3), (2,3), (2,3), (2,3), (2,3)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 3 | (3,1; (1,3), (2,3)) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 2 | (2,2; 1) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 2 | (2; 1) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |
| 2 | (2; 0) | \( (T_1 T_2 T_3 T_4 T_5 T_6) \) | 1.4 |

Table 3. Words (in Dehn twists) representing the conjugacy classes of periodic elements in \( \text{Mod}(S_3) \).
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