A simple Discrete-Element-Model of Brazilian Test

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We present a statistical model which is able to capture some interesting features exhibited in the Brazilian test. The model is based on breakable elements which break when the force experienced by the elements exceed their own load capacity. In this model when an element breaks, the capacity of the neighboring elements are decreased by a certain amount assuming weakening effect around the defected zone. We numerically investigate the stress-strain behavior, the strength of the system, how it scales with the system size and also it’s fluctuation for both uniformly and weibull distributed breaking threshold of the elements in the system. We find that the strength of the system approaches it’s asymptotic value $\sigma_c = 1/6$ and $\sigma_c = 5/18$ for uniformly and Weibull distributed breaking threshold of the elements respectively. We have also shown the damage profile right at the point when the stress-strain curve reaches at it’s maximum and then it is compared with our experimental observations.

I. INTRODUCTION

The tensile strength of the rock is one of the important parameters that influences the rock crushing and rock blasting results. To measure the tensile strength of the rock, concrete, ceramics the Brazilian test [1–4] is performed. This test is also named as the Diametral Compression test and the Indirect Tensile test. The test was introduced by Carniero[5] in Brazil, and Akazawa[6] in Japan in 1943.

In the Brazilian test a circular disk is diametrically compressed and due to this a tensile stress induces perpendicular to the direction of the applied force and is proportional to the force. When the induced tensile stress crosses certain limit, fracture develops mainly in the middle zone of the sample. In Fig. 1 four typical loading configurations has been shown. The indirect tensile strength of a cylindrical sample with radius $R$ and thickness $T$ is given by [1],

$$\sigma_t = \frac{P}{\pi RT}$$

where $P$ is the load at failure point of the material. However, this equation was obtained analytically based on the assumption that the rock to be isotropic and homogeneous [2, 3]. But nature is more complex - therefore we must take into account anisotropy and heterogeneity issues.

Besides the tensile strength determination of materials, one of the widely experimentally investigated studies is the crack initiation and crack propagation in the Brazilian disk test [10, 11]. In this report, we present a very simple statistical model which is able to capture some features exhibited in the Brazilian test.

In Sec. II we describe the experimental procedure of our Brazilian test followed by some experimental results in Sec. III. We elaborately describe our model in Sec. IV and present numerical results obtained from this model in Sec. V. Here we investigate stress-strain behavior, damage profile critical strength of the system. Finally we make some concluding remarks in Sec. VI.

II. EXPERIMENTAL PROCEDURE

The Brazilian(splitting) indirect tensile strength test [9] was performed on two rock types - Castlegate sandstone (CG) and Mons chalk (MC). Both rocks have a fairly homogeneous and isotropic(with no distinct layering) structure. The specimens were cored perpendicular to bedding and cut into the discs of approximately 22 mm thickness and 52 mm diameter. Each disc was wrapped around the
circumference with a masking tape. This softened the contact between the rock and the steel curved jaws (Fig. 1d) within which the disc was clamped during the test. This curved jaws is then placed on a loading apparatus having a base and a cross head. We then continuously increase the externally applied load at a very slow but constant rate until the failure of the sample occurs. In our experiment the loading was applied with a MTS load frame at the rate 0.003 mm/s, until the failure. The reaction force given by the sample is recorded at every time. Fracture development, indicated by the sudden drop in loading, was captured using a digital camera.

III. EXPERIMENTAL RESULTS

In the previous section it is already mentioned that we experimentally examined the strength of two cylindrical samples CG and MC. The details about the samples are enumerated in the Table I. In our experiment we increase the externally applied load on the material and measure the reaction force with time. In Fig. 2 the load-time data has been plotted for both of the samples. We define the tensile strength of material as the maximum load that a material can sustain without failure. For CG sample the first drop in the measured load around time 195sec. signifies the material failure. When a fracture opens the near by sand grains occupy that fractured region and also the contact area to the curved jaws increase and thus again the reaction force increases. Due to the brittleness of the chalk sample this is not observed in case of chalk. From the load-time plot it is obvious that the tensile strength of CG is greater than that of MC. In Fig. 3 the damage zone has been shown for both of the samples after their failure.

IV. MODEL

Our model consists of Hookean elements, placed on each lattice point of a regular square lattice, with their

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TABLE I: Details about the samples that we examined. The thickness/diameter ratio is 0.42 for both of the samples.

| Sample name            | Weight (gm) | Diameter (mm) | Thickness (mm) |
|------------------------|-------------|----------------|----------------|
| Castlegate sandstone   | 91.61       | 52.26          | 21.78          |
| Mons chalk             | 71.73       | 51.75          | 21.73          |

FIG. 2: The load-time behaviour of Castlegate sandstone (CG) and Mons chalk (MC) samples.

FIG. 3: Fracture patterns after failure. Top one is for Castlegate sandstone (CG) and the lower one is for Mons chalk (MC) samples.
This breaking strength distribution makes the system in-
tutive distribution of which is given by
\[ P \]
numerically examined.

In our model the breaking properties of the system is nu-
vistically in either sides from the point of loading. From
the applied force diminishes linearly as we move on hor-
point of the system and it is assumed that the effect of
In our model the external load is applied at the middle
viving elements as in the case of fiber bundle model [12].
think of redistribution of released load among all the sur-
the weakening effect around the defected zone. One may
its neighboring intact elements is decreased indicating
our model when an element breaks the load capacity of
ceed its breaking thresholds drawn from a given probabil-
ity distribution. Once the load though each element ex-
ceeds its breaking threshold, it breaks immediately. In
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y distribution. Once the load though each element ex-
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\[ F(x) = mx + F_0 \]  \hspace{1cm} (2)

and since it is symmetric about the point of loading so we
take \( F(-x) = F(x) \). In the above Eq. \( F_0 \) is the amplitude
of the applied force at the point \( x = 0 \) and \( m \) defines
how far the effect of this applied force should extend. In
our model we prefix a cut off length \( L_c \) beyond which
\( F(x) = 0 \), this actually defines the value of \( m(=F_0/L_c) \).
This is depicted in the Fig. 4(b). We also assume that
the load through all the elements of a particular column
is same i.e., \( F(x,y) = F(x) \).

To model the Brazilian test we apply a force at the
middle point of the lattice from the top of it as shown in
the Fig. 4(a). We assume that the effect of this applied
force extends within a certain region from the point of
application of the force. We consider the effect of this
force decreases linearly and this force profile is symmet-
ric about the point of loading. The force profile at the
positive part of the lattice is of the form

\[ F(x) = mx + F_0 \]  \hspace{1cm} (2)

and since it is symmetric about the point of loading so we
take \( F(-x) = F(x) \). In the above Eq. \( F_0 \) is the amplitude
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the load through all the elements of a particular column
is same i.e., \( F(x,y) = F(x) \).

In our model when the force through an element ex-
ceeds it’s breaking threshold it breaks irreversibly and it’s
neighbourhood gets affected. When an element breaks,
the breaking strength of four of it’s surviving nearest
neighbour is decreased by a factor \( \alpha (0 \leq \alpha \leq 1) \), i.e.,
their new thresholds are assigned a value equal to the \( \alpha \)
times of their own previous thresholds. This is actually
the sense of weakening around the defected zone. The force profile remains unaltered in this model.

When a force is applied to the system, the breakdown
of the elements occur from the middle part(close to the
x = 0) of the system since the force field is much stronger
in this region. So, the local failures will grow from the
middle zone of the system. This local failures decrease
the strength of the elements around the defected zone. Sometimes a local failure occurs due to the fact that the
element being weak at that point, other times the fail-
FIG. 6: Crack profile right at the point where the stress-strain curve reached at its maximum. The black colour represents the intact elements and the white patches at the middle of it are the broken elements within the system of size $L = 64$. Here the thresholds are distributed uniformly between 0.1 to 1 and in simulation we used $L_c = L/12$.

ure occurs due to the decrement of the strength of the elements. This leads to the development of spatial correlation. Since the force field decreases linearly in the horizontal direction, it is more probable that failure will occur in the vertical direction. This makes cracks to propagate in the vertical up or vertical down direction rather than horizontal direction from the defected zone, which is seen in the experiment of Brazilian disk test.

The width of disorder, the value of $L_c$ and the strength decrement factor $\alpha$ play a crucial role in the crack propagation. For a fixed window of disorder, smaller the value of $\alpha$ and $L_c$, lead failure to be more and more localized and directed in the vertical direction i.e., crack propagation zone become more and more narrower.

V. NUMERICAL RESULTS

We investigate the stress-strain behaviour of the system which is one of the most important studies in the Brazilian test. We observed the stress-strain behaviour, the system size dependence of the maximum value of the stress and it’s fluctuation when the threshold values are distributed uniformly between the range 0.1 to 1. We put a lower cut off in the distribution, assuming that the strength of the elements can not be lower than a particular level.

To obtain the stress-strain curve we increase the external load quasistatically. We call this external load as strain ($\varepsilon$). We define stress ($\sigma$) as the fraction of intact elements at the column $x = 0$ times the strain, i.e.,

$$\sigma = \frac{N_i \varepsilon}{2L}$$ (3)
where, $N_i$ is the number of intact elements at that particular column.

Numerically, the stress-strain curve is obtained in the following way. Initially we start with a fully intact system and then we apply a load such that the weakest element in the system breaks. The weakest surviving element is determined by calculating the maximum of the ratio between force field to the breaking threshold i.e., $\max \{ F(x,y)/t(x,y) \}$, which is equal to $\max \{ F(x)/t(x,y) \}$. Once it is removed, the breaking strength of it’s nearest surviving neighbour is reduced by a factor $\alpha$. Due to this, it may so happen that the load through some of the elements exceed their breaking threshold and we break them simultaneously, again we reduce the strength of their neighbouring elements, again there might be breakdown of few elements and so on. This process stops when all the remaining intact elements have their breaking strength greater than their respective experienced force. We then increase the external applied load in such a way that the weakest element among the remaining intact system reaches at it’s breaking threshold and then the entire procedure is repeated. This procedure is continued until we are at a point, well above the point where the maximum of the stress-strain curve takes place. In Fig. 5 the stress-strain curve is shown for a particular configuration for different values of $\alpha$ and in Fig. 6 the defected zone right at the point where the stress-strain curve reaches at it’s maximum has been shown by the white colour. For every results obtained from our simulation we have used $L_c = L/12$ and $\alpha = 0.70$.

Once we generate the stress-strain curve, we then calculate the critical strength of the system. We define critical strength $\sigma_c^\beta(L)$ for a particular sample $\beta$ of size $L$ with a particular set of breaking thresholds $\{ t_{x,y} \}$ as the maximum value of the stress in the stress-strain curve. For each sample $\beta$ we numerically calculate $\sigma_c^\beta(L)$ and then it is repeated over a large number of un-correlated samples to obtain the average value of the critical strength $\sigma_c(L) = \langle \sigma_c^\beta(L) \rangle$ for the system of size $L$. We then repeat the entire calculation for different values of $L$. In Fig. 7(a) $\sigma_c(L)$ has been plotted with $L$.

We assume that the average value $\sigma_c(L)$ for a system of size $L$ converges to it’s asymptotic value $\sigma_c = \sigma_c(\infty)$ as $L \rightarrow \infty$ according to the following form,

$$\sigma_c(L) = \sigma_c + A L^{-1/\nu} \quad (4)$$

The $\sigma_c(L)$ values are plotted in Fig. 7(b) as a function of $L^{-1/\nu}$ and taking $1/\nu = 0.964$, we observed that it fits very nicely by a straight line. The least-squares fitted straight line cuts the vertical axis and has the form $\sigma_c(L) = 0.168 + 1.21 L^{-1/\nu}$. From here we can conclude that $\sigma_c = 1/6$ and $\nu = 1.0(3)$. 

FIG. 8: (a) Log-log plot of $W(L)$ with $L$ for system size up to $L = 512$. For each data point we averaged over 1000 independent configurations. (b) The same data has been plotted with $L^{-0.786}$ in linear scale and this gives a nice straight line. From here our conclusion is $W(t) \sim t^{-0.786}$.

FIG. 9: Plot of $\sigma_c(L)$ with $L^{-0.817}$ gives a nice straight line, the extrapolation of which gives the value of $\sigma_c = 0.2779$ and our conclusion is $\sigma_c = 5/18$. Here the breaking threshold of the elements are drawn from the Weibull distribution of the form of Eqn. (6). (b) The fluctuation $W(L)$ fits nicely with a straight line when it is plotted against $L^{-0.817}$ indicates $W(L) \sim L^{-0.817}$. 


We also measure the fluctuation of the critical stress. The expression of the fluctuation is given by

\[ W(L) = \sqrt{\langle \sigma_c(L)^2 \rangle - \langle \sigma_c(L) \rangle^2} \]  

(5)

In Fig. 8(a) we plot this \( W(L) \) for different values of \( L \) in log-log scale and we observe that it decreases with increase in \( L \). We then plot \( W(L) \) against \( L^{-0.786} \) in Fig. 8(b). This fits with a very nice straight line, implies the fluctuation diminishes with \( L \) as \( W(L) \sim L^{-0.786} \).

We then repeat our entire set of calculations for elements with Weibull breaking strength distribution of the form

\[ P(t) = 1 - e^{-t^2} \]  

(6)

We have estimated the values of \( \sigma_c(L) \) numerically for five different system sizes from 32 to 512 increased by a factor of 2 at every step. Plotting them against \( L^{-0.917} \) and on extrapolation as \( L \to \infty \) we have obtained \( \sigma_c = 0.2779 \). Our conclusion is that for the Weibull distribution of breaking thresholds of the form of Eq. 6, the value of \( \sigma_c = 5/18 \) and \( \nu = 1.0(9) \). In Fig. 9(a) the plot of \( \sigma_c(L) \) with \( L^{-0.917} \) has been shown.

We calculate \( W(L) \) for different values of \( L \) in this case also. In Fig. 9(b) we plot the \( W(L) \) with \( L^{-0.817} \) in linear scale and this is fitted with a straight line.

**VI. CONCLUSION**

To model the Brazilian test we presented a model and have studied numerically the strength of the material. We have examined the stress-strain behaviour of the system and found that it has a maxima. We investigated the critical strength of the system and also the system size dependence of the critical strength for both uniformly and Weibull distributed breaking threshold of the elements in the system. We have also shown the damage profile right at the critical point in Fig. 6 which resembles the nature of the damage profile observed in Brazilian test experimentally.

For breaking thresholds distributed uniformly and with Weibull distribution it has been observed that the critical strength \( \sigma_c(L) \) of a system of size \( L \) approaches it’s asymptotic values of \( 1/6 \) and \( 5/18 \) respectively. All the results obtained from our model we have taken the strength decrement factor \( \alpha = 0.7 \) and the cut-off length of the applied force \( L_c = L/12 \) beyond which the force has no effect.

The stress-strain curve is very sensitive to the value of \( \alpha \). In Fig. 5 the stress-strain behaviour of a system with \( L = 64 \) has been shown for three different values of \( \alpha \). Lower the value of \( \alpha \), more and more narrower will be the damage zone and also directed in the vertical direction, shown in Fig. 6 and as an effect the strength of the system will decrease. For \( \alpha = 1 \) the behaviour of this model is exactly same as the Fiber Bundle model [12, 13].

Our future plan is to investigate extensively how the two factors \( \alpha \) and \( L_c \) affect the strength of the system.

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