The large quark mass expansion of
\[ \Gamma(Z^0 \rightarrow \text{hadrons}) \] and
\[ \Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons}) \] in the order \( \alpha_s^3 \)

S.A. Larin\(^1\), T. van Ritbergen, J.A.M. Vermaseren

NIKHEF-H, P.O. Box 41882, 1009 DB, Amsterdam

Abstract

We present the analytical \( \alpha_s^3 \) correction to the \( Z^0 \) decay rate into hadrons. We calculate this correction up to (and including) terms of the order \( (m_{Z}^2/m_{t_{\text{top}}}^2)^3 \) in the large top quark mass expansion. We rely on the technique of the large mass expansion of individual Feynman diagrams and treat its application in detail. We convert the obtained results of six flavour QCD to the results in the effective theory with five active flavours, checking the decoupling relation of the QCD coupling constant. We also derive the large charm quark mass expansion of the semihadronic \( \tau \) lepton decay rate in the \( \alpha_s^3 \) approximation.

\(^1\)On leave from the Institute for Nuclear Research (INR) of the Russian Academy of Sciences, Moscow 117312.
1 Introduction

Precision measurements of the $Z^0$ decay rate into hadrons at LEP\cite{1} provide precise means to extract the QCD coupling constant from experiment. This is a very clean process from a theoretical point of view since its calculation can be reduced to the calculation of the $Z$ boson propagator within the standard model. The status of electroweak corrections to $Z^0$ decay can be found in ref. \cite{2}. Now the calculational techniques of Feynman diagrams have advanced so far that the calculation of the $\alpha_s^3$ order (=4 loop approximation of the $Z$ boson propagator) is feasible. The $\alpha_s^3$ approximation to the $Z^0$ decay rate into hadrons is important for an accurate determination of the QCD coupling constant $\alpha_s$, or equivalently the fundamental scale of QCD, $\Lambda_{QCD}$.

The hadronic $Z^0$ decay rate is a sum of vector and axial vector contributions of which the vector contribution is known to order $\alpha_s^3$ from the calculation of $\sigma_{tot}(e^+e^- \rightarrow \gamma \rightarrow hadrons)$ \cite{3}. This calculation was performed in the approximation of effective QCD with five massless quarks which involved the calculation of only massless diagrams. The correctness of this calculation is strongly supported by \cite{4} where the non-trivial connection between the result \cite{3} and the $\alpha_s^3$ approximation \cite{5} to deep inelastic sum rules was established.

The calculation of the axial vector part of the hadronic $Z^0$ decay rate is more involved than that of the vector part. This is because the heavy quark does not decouple in the axial vector part and one cannot avoid to calculate massive diagrams, even in the leading order of the large mass expansion. The axial vector part was calculated to order $\alpha_s^2$ in \cite{6} and confirmed in \cite{7} where the operator product expansion technique was used to sum up the massive logarithmic terms. The $Z^0$ decay into 3 gluons in order $\alpha_s^3$ has been calculated in \cite{8}. The $\alpha_s^3$ correction to the axial vector part of the hadronic $Z^0$ decay rate in the leading order of the large top mass expansion was presented in \cite{9,10}. In this paper we elaborate the details of the calculation in \cite{9} and extend the large mass expansion of both the vector and axial vector parts to the order $(m_Z^2/m_{top}^2)^3$. This calculation allows us also to check the decoupling mechanism at the next-next-to-leading order.

Another prominent process (beside $Z$ boson decay) for the extraction of $\alpha_s$ from experiment is semihadronic $\tau$ lepton decay. In the last section of our article we convert the obtained result for $\Gamma(Z^0 \rightarrow hadrons)$ to the large charm quark mass expansion for $\Gamma(\tau^- \rightarrow \nu_\tau + hadrons)$ in the order $\alpha_s^3$.

2 Preliminaries

For the $Z^0$ decay rate into hadrons, the quantity to be determined is the squared matrix element summed over all final hadronic states. One can express this quantity as the
imaginary part of a current correlator in the standard way

\[ \sum_h <0|J^\mu|h><h|J^\nu|0> = 2Im\Pi^{\mu\nu}, \]  

(1)

\[ \Pi^{\mu\nu} = i \int d^4z e^{iq\cdot z} <0|T(J^\nu(z)J^\mu(0))|0> = -g^{\mu\nu} q^2 \Pi_1(q^2) - q^{\mu}q^{\nu} \Pi_2(q^2). \]  

(2)

Here \( J^\mu = \frac{g}{2cos\theta_W} \sum_{i=1}^{6} \overline{\psi}_i \gamma^\mu (g^V_i - g^A_i \gamma^5) \psi_i \) is the neutral weak quark current coupled to the \( Z^0 \) boson in the Lagrangian of the Standard Model, where we use the notations as given in [11] \( g^V_i \equiv t_3^{L}(i) - 2q_i \sin^2 \theta_W \) and \( g^A_i \equiv t_3^{L}(i) \).

The hadronic \( Z^0 \) decay width is expressed as

\[ \Gamma_{had} \equiv \Gamma_{V,had}^{V} + \Gamma_{A,had}^{A} = m_Z Im\Pi_1(m_Z^2 + i\epsilon) \]  

(3)

with the indicated decomposition into vector and axial vector parts imposed by the structure of the neutral current. We will calculate \( Im\Pi_1 \) in the order \( g^2\alpha_s^3 \). It is a calculation within perturbative QCD except for two weak current vertex insertions (i.e. the weak current is considered as an external current for QCD).

Throughout this paper we use dimensional regularization [12] in \( D = 4 - 2\varepsilon \) space-time dimensions and the standard modification of the minimal subtraction scheme [13], the \( \overline{MS} \) scheme [14]. For the treatment of the \( \gamma_5 \) matrix in dimensional regularization we use the technique described in [13] which is based on the original definition of \( \gamma_5 \) in [12]. We work in the approximation of 5 massless quark flavours and the top quark mass large compared to the \( Z^0 \) mass. We should stress that the top quark does not decouple [16] from the axial vector part due to diagrams of the axial anomaly type.

It is convenient to split the vector and axial vector contribution in non-singlet and singlet parts

\[ \Gamma_{had}^{V} = \Gamma_{had}^{V,NS} + \Gamma_{had}^{V,S}, \quad \Gamma_{had}^{A} = \Gamma_{had}^{A,NS} + \Gamma_{had}^{A,S}. \]  

(4)

The non-singlet parts come from Feynman diagrams where both weak current vertices are located in one fermion loop. The singlet contributions come from diagrams where each weak current vertex is located in a separate fermion loop. The massive non-singlet diagrams are presented in figure 1, the singlet diagrams are presented in figures 2,3.

The \( \alpha_s^3 \) approximation for the vector part in effective QCD with 5 active massless quark flavours in the \( \overline{MS} \) scheme was calculated in [3] (in the leading order of the large top quark mass expansion). This calculation used the fact that the top quark decouples for the vector part in the leading order of the large quark mass expansion. Therefore within effective 5 flavour QCD this calculation involved only massless diagrams and the result reads

\[ \Gamma_{had}^{V,NS} = \frac{G_F m_Z^2}{2\pi\sqrt{2}} \sum_{i=1}^{5} (g^V_i)^2 \left[ 1 + \frac{\alpha_s^{(5)}}{\pi} + 1.40923 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \right] \]
\[ \Gamma_{\text{had}}^{V,S} = \frac{G_F m_Z^3}{2\pi \sqrt{2}} \left( \sum_{i=1}^{5} g_i \right)^2 \left[ -0.41318 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \right] \]  

with the Fermi constant \( G_F = \frac{g^2 \sqrt{2}}{8 \cos^2(\theta_W) m_Z^2} \). Here \( \alpha_s^{(5)}(m_Z) \) is the coupling constant in effective QCD with 5 active flavours. The coupling constant of effective 5 flavour QCD, \( \alpha_s^{(5)} \) and the coupling constant of full 6 flavour QCD, \( \alpha_s^{(6)} \) both obey the renormalization group equation (with \( n_f = 5 \) for \( \alpha_s^{(5)} \) and \( n_f = 6 \) for \( \alpha_s^{(6)} \))

\[ \frac{\partial \alpha_s / \pi}{\partial \ln Q^2} = \beta(\frac{\alpha_s}{\pi}) \]

\[ = -\beta_0 (\frac{\alpha_s}{\pi})^2 - \beta_1 (\frac{\alpha_s}{\pi})^3 - \beta_2 (\frac{\alpha_s}{\pi})^4 + O(\alpha_s)^5 \]  

(6)

where

\[ \beta_0 = \frac{1}{4} \left( \frac{11}{3} C_A - \frac{4}{3} T_F n_f \right) \]

\[ \beta_1 = \frac{1}{16} \left( \frac{34}{3} C_A^2 - 4 C_F T_F n_f - \frac{20}{3} C_A T_F n_f \right) \]

\[ \beta_2 = \frac{1}{64} \left( \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_f - \frac{205}{9} C_F C_A T_F n_f - \frac{1415}{27} C_A^2 T_F n_f + \frac{44}{9} C_F T_F^2 n_f^2 + \frac{158}{27} C_A T_F^2 n_f^2 \right) \]  

(7)

The three loop QCD beta function in the \( \overline{\text{MS}} \)-scheme was calculated in [17]. \( C_F = \frac{4}{3} \) and \( C_A = 3 \) are the Casimir operators of the fundamental and adjoint representation of the colour group \( SU(3) \), \( T_F = \frac{1}{2} \) is the trace normalization of the fundamental representation.

The solution of eq.(3) in the next-next-to-leading order has the standard form

\[ \frac{\alpha_s}{\pi} = \frac{1}{\beta_0 \ln(\frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2})} - \frac{\beta_1 \ln \ln(\frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2})}{\beta_0^2 \ln^2(\frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2})} \]

\[ + \frac{1}{\beta_0^3 \ln^3(\frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2})} \left( \beta_1^2 \ln \ln(\frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2}) - \beta_1 \ln \ln(\frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2}) + \beta_2 \beta_0 - \beta_1^2 \right) \]  

(8)

where it is understood that the scale \( \Lambda_{\overline{\text{MS}}} \) also depends on the number of active flavours.
The (top quark mass dependent) relation between \( \alpha_s^{(6)} \) and \( \alpha_s^{(5)} \) is called the decoupling relation and will be discussed later in more detail but for completeness we give the NNL order expression

\[
\frac{\alpha_s^{(6)}}{\pi} = \frac{\alpha_s^{(5)}}{\pi} + \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 \frac{T_F}{3} \ln \left( \frac{\mu^2}{m_t^2(\mu)} \right) + \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \left( \frac{T_F^2}{9} \ln^2 \left( \frac{\mu^2}{m_t^2(\mu)} \right) + \frac{5C_A T_F - 3C_F T_F}{12} \ln \left( \frac{\mu^2}{m_t^2(\mu)} \right) + \frac{13}{48} T_F C_F - \frac{2}{9} T_F C_A \right) + \mathcal{O}(\alpha_s^4). 
\]  

(9)

where \( \mu \) is the renormalization scale and \( m_t(\mu) \) is the top quark mass in the \( \overline{\text{MS}} \) scheme. Please note that the term \( \frac{13}{48} T_F C_F \) that we found is slightly different from the one in ref. \[18\]. Substituting expression (8) for \( \alpha_s^{(5)} \) and \( \alpha_s^{(6)} \) one can find the connection between \( \Lambda_{\overline{\text{MS}}}^{(6)} \) and \( \Lambda_{\overline{\text{MS}}}^{(5)} \) via \( m_t(\mu) \).

\( m_t(\mu) \) obeys the renormalization group equation

\[
\frac{\partial \ln(m_t(\mu))}{\partial \ln(\mu^2)} = -\gamma_m(\alpha_s) = -\gamma_0 \left( \frac{\alpha_s}{\pi} \right) - \gamma_1 \left( \frac{\alpha_s}{\pi} \right)^2 - \gamma_2 \left( \frac{\alpha_s}{\pi} \right)^3 + \mathcal{O}(\alpha_s^4)
\]  

(10)

where

\[
\begin{align*}
\gamma_0 &= \frac{1}{4} 3C_F \\
\gamma_1 &= \frac{1}{16} \left( \frac{3}{2} C_F^2 + \frac{97}{6} C_F C_A - \frac{10}{3} C_F T_F n_f \right) \\
\gamma_2 &= \frac{1}{64} \left( \frac{129}{2} C_F^3 - \frac{129}{4} C_F^2 C_A + \frac{11413}{108} C_F C_A^2 + C_F^2 T_F n_f (48 \zeta_3 - 46) + C_F C_A T_F n_f (-48 \zeta_3 - \frac{556}{27}) - \frac{140}{27} C_F^2 T_F^2 n_f^2 \right)
\end{align*}
\]

and \( \zeta \) is the Riemann zeta-function. The three loop quark mass anomalous dimension was calculated in ref. \[19\].

Let us quote the existing results for the axial vector contribution. The axial vector non-singlet part can be reduced to the vector case by using the effective anticommutation property of the \( \gamma_5 \) matrix in the prescription that we use. More strictly, it can be done only in the limit of massless light quarks. Thus the non-singlet axial part coincides with the vector non-singlet part up to a change of the weak coupling constants and reads (in the leading order of the large top quark mass expansion)

\[
\begin{align*}
\Gamma_{\text{had}}^{\text{A,NS}} &= \frac{G_F m_t^3}{2 \pi \sqrt{2}} \sum_{i=1}^{5} (g_A^i)^2 \left[ 1 + \frac{\alpha_s^{(5)}}{\pi} + 1.40923 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 - 12.76706 \left( \frac{\alpha_s^{(5)}}{\pi} \right)^3 \right]
\end{align*}
\]  

(12)
Let us turn to the singlet axial vector part. In the Standard Model quarks in a weak doublet couple with opposite sign to the $Z^0$ boson in the axial vector part of the neutral current. That is why the contributions from light doublets add up to zero in the massless limit for axial vector singlet diagrams. The only non-zero contribution comes from the top-bottom doublet due to the large mass difference between top and bottom quarks.

The axial vector singlet part in the leading order of the large top quark mass expansion has recently been calculated in refs. [9, 10]. The result in the effective theory with 5 active massless quark flavours is

$$
\Gamma_{A,S}^{\text{had}} = \frac{G_F m_Z^3}{2\pi \sqrt{2}} (g_A^{\text{bot}})^2 \left[ \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 \left( \frac{17}{6} \right) + \frac{4673}{144} + \frac{67}{12} \zeta_3 + \frac{23}{36} \pi^2 - \frac{1}{36} \ln \left( \frac{m_Z^2}{m_t^2} \right) - \frac{1}{6} \ln^2 \left( \frac{m_Z^2}{m_t^2} \right) \right] + \frac{1}{4} \ln \left( \frac{m_Z^2}{m_t^2} \right) + \frac{2717}{432} + \frac{55}{12} \zeta_3 + \frac{7}{4} \ln \left( \frac{m_Z^2}{m_t^2} \right) - \frac{25}{12} \ln \left( \frac{m_Z^2}{m_t^2} \right) \right]
$$

$$
= \frac{G_F m_Z^3}{2\pi \sqrt{2}} \left[ \left( \frac{\alpha_s^{(5)}}{\pi} \right)^2 \left( \frac{1}{4} - \ln \left( \frac{m_Z^2}{m_t^2} \right) \right) + \frac{1}{4} \ln \left( \frac{m_Z^2}{m_t^2} \right) + \frac{18.65440}{18} \ln \left( \frac{m_Z^2}{m_t^2} \right) + \frac{31}{12} \ln \left( \frac{m_Z^2}{m_t^2} \right) + \frac{23}{12} \ln^2 \left( \frac{m_Z^2}{m_t^2} \right) \right],
$$

where we separated the two weak coupling structures (which was not present in refs. [9, 10]) and used the notation $g_A^{\text{bot}} \equiv g_A^5$ and $g_A^{\text{top}} \equiv g_A^6$. Here and below $m_t \equiv m_t(m_Z)$ is the $\overline{\text{MS}}$ top mass at the scale $m_Z$. One may relate it to the pole mass through the expression

$$
m_t(m_Z) = m_{\text{pole}} \left[ 1 - \frac{\alpha_s(m_Z)}{\pi} \left( \ln \left( \frac{m_Z^2}{m_{\text{pole}}^2} \right) + \frac{4}{3} \right) + O(\alpha_s^2) \right]
$$

which is known in the NNL approximation [10] or relate it to $m_t(m_t)$ through the expression

$$
m_t(m_Z) = m_t(m_t) \left[ 1 - \frac{\alpha_s(m_Z)}{\pi} \ln \left( \frac{m_Z^2}{m_t^2} \right) + O(\alpha_s^2) \right].
$$

This would correspondingly modify the coefficients of the $\alpha_s^3$ term in (13) but we prefer to use the $\overline{\text{MS}}$ top quark mass, $m_t(\mu)$ (at $\mu = m_Z$) which is the original mass from the QCD Lagrangian.
In the present paper we present the power suppressed top quark mass corrections for both the vector and axial vector contributions. The Feynman diagrams that we have to calculate to obtain the power suppressed top quark mass corrections are given in figures 1,2,3.

**Figure 1.** Massive diagrams (= with top quark loops) contributing to the vector non-singlet part, $\Gamma_{had}^{V, NS}$ and axial non-singlet part, $\Gamma_{had}^{A, NS}$. The symbol $\otimes$ is used to indicate an external vertex of the neutral weak vector current for $\Gamma_{had}^{V, NS}$ and an axial vector vertex for $\Gamma_{had}^{A, NS}$. It is understood that for each diagram at least one fermion loop has to be a massive top quark loop and a loop that contains the external vertices is always a massless quark loop. The massless diagrams that were already calculated in ref. are not considered here.
Figure 2. Massive diagrams (= with top quark loops) contributing to the vector singlet part, $\Gamma_{\text{had}}^{V,S}$. The symbol $\otimes$ is used to indicate a vector current vertex. It is understood that for each diagram at least one fermion loop has to be a massive top quark loop. The massless diagrams (of the same topologies) were already calculated in ref. [3] and are not considered here.

Figure 3. Massive and massless diagrams contributing to the axial vector singlet part, $\Gamma_{\text{had}}^{A,S}$. The symbol $\otimes$ is used to indicate an axial vector current vertex. For each fermion loop both massless quarks and massive top quarks are considered.
3 The calculation of the massless diagrams

In this section we will treat the calculation of the four-loop massless diagrams from figure 3 that contribute to $\Gamma_{A,S}^{4,\text{had}}$. We will illustrate the techniques by considering the most difficult diagram (the first one on the second line in figure 3).

Since we are only interested in the structure function $\Pi_1(q^2)$ (see equations (2,3)), we contract the diagrams with the projector $(g^{\mu\nu} - q^{\mu}q^{\nu}/q^2)$ which reduces the diagram to a scalar integral. In figure 4 we present diagrammatically the renormalization of ultraviolet divergences of this diagram.

![Figure 4](image-url)

**Figure 4.** The ultraviolet renormalization of the four-loop diagram. The ultraviolet counterterms are presented between round brackets.

At present, we do not have a technique which would allow a direct calculation of this 4-loop diagram. But we can use the fact that we need to know only poles in $\varepsilon$ for this diagram, since only these pole terms generate terms containing $\ln(Q^2/\mu^2)$ which produce non-zero imaginary parts (logarithms come from $\frac{1}{\varepsilon}(\frac{Q^2}{\mu^2})^{\varepsilon} = \frac{1}{\varepsilon} + \ln(\frac{Q^2}{\mu^2}) + O(\varepsilon)$ and give imaginary parts through $\ln(-s - i\varepsilon) = \ln(s) - i\pi$, where $s = -Q^2 = q^2 = m_Z^2$). One can see from figure 4 that it is sufficient to calculate all renormalization terms (2nd - 4th terms in figure 4), including the 4-loop counterterm, in order to restore the pole terms for the diagram itself (because the sum is finite). All renormalization terms except the 4-loop counterterms can be directly calculated with the help of the package MINCER [21] written for the symbolic manipulation program FORM [22]. This package calculates analytically 3-loop massless propagator diagrams using the integration by part algorithm of ref. [23] for dimensionally regularized diagrams.

In this way we have reduced the problem of calculating the imaginary part of the 4-loop diagram of figure 4 to the problem of calculating its 4-loop ultraviolet counterterm. The last problem can be reduced to the calculation of 3-loop propagator type diagrams (which are calculable with the package MINCER) using the infrared rearrangement method [24, 25]. This method relies heavily on the fact that in the $\overline{\text{MS}}$-scheme ultraviolet counterterms are polynomials in momenta and masses [26], i.e. do not contain logarithms or inverse powers of momenta and masses. In our case the 4-loop counterterm has dimension two which means that it is simply proportional to $Q^2$. If we take the d’Alembertian in $Q$ of this counterterm we get a dimensionless quantity which is just a Laurent series in $\varepsilon$. To obtain this Laurent series we need to calculate counterterms of dimensionless diagrams. These diagrams are produced after applying the d’Alembertian in $Q$ to the 4-loop diagram of figure 4. In fact, the application of the d’Alembertian produces several dimensionless
4-loop diagrams after differentiation of the lines of the original diagram; some of these diagrams are presented in figure 5.

**Figure 5.** Diagrammatic representation of applying the d’Alembertian in $Q$ to the 4-loop counterterm. A prime on a line denotes the differentiation of this line in its momentum. The round brackets denote the ultraviolet counterterms of the diagrams inside the brackets.

Since counterterms of dimensionless diagrams do not depend on the external momentum $Q$, we can change the route of this momentum through these diagrams as we wish, e.g. as it is chosen in the r.h.s. of figure 5. The choice of a new route for the external momentum can generate infrared poles (even though the original diagram did not have infrared divergences) which then essentially complicates the extraction of the ultraviolet poles. For example, nullifying the momentum $Q$ will nullify the whole diagram because of new infrared divergences. The chosen momentum route in the r.h.s. of figure 5 reduces the calculation of the 4-loop diagrams in figure 5 to the calculation of simpler 4-loop topologies. These topologies have the form of a 3-loop propagator type subdiagram (which can be done with the package MINCER) inserted in a trivial one-loop topology.

The essential complication now is that the second diagram in the r.h.s. of figure 5 contains infrared divergences. For this diagram it is impossible to choose the route of the external momentum in a way that avoids infrared divergences and simultaneously reduces the calculation of the corresponding diagram to a 3-loop propagator insertion. In this case we need to apply the technique of the $R^*$ operation \cite{27} which allows to calculate ultraviolet counterterms of dimensionally regularized diagrams even in the presence of infrared singularities. It is interesting to note that the diagram of figure 4 was the only diagram that needed the application of the $R^*$ operation in the calculation of $\Gamma_{\text{had}}^{A,S}$. 

10
4 The expansion of massive diagrams

We will now treat the large mass expansion of the individual massive diagrams. We calculate all integrals in Euclidean momentum space. The general theory of Euclidean asymptotic expansions was developed in [28, 29]. For practical purpose we use the techniques developed in [30, 31].

Let’s go through some simple ideas (see e.g. [32]) that generalise to a recipe of expanding individual dimensionally regularized diagrams. This recipe can then also be used to expand $\overline{MS}$ renormalized diagrams since it can be applied to each term of the renormalized expression for a given diagram (i.e. the expression after application of the ultraviolet R-operation to this diagram).

A simple scalar diagram containing both massless lines and massive lines is shown in the l.h.s. of figure 6a.

Figure 6a. Thick lines denote massive scalar propagators, thin lines denote massless scalar propagators. The symbol $\times$ indicates the insertion of the small momentum expansion of the massive triangle.

We are going to expand this diagram in a large mass which is equivalent to the expansion in a small (in comparison to the mass) external momentum, Q. Please note that we can not simply expand the integrand of the corresponding Feynman integral as a Taylor series in Q, because putting Q = 0 generates infrared divergences when one integrates over momentum k. Let us therefore first consider the expansion of the massive one-loop subgraph as a Taylor series in its external momenta, k and Q. This expansion can be obtained by a simple Taylor expansion of its integrand in k and Q because it does not generate infrared divergences when one integrates over l. More generally, the expansion of diagrams with only massive propagators in terms of (small) external momenta can be safely done by making Taylor expansions in these momenta in the integrands. The Taylor expansions of integrands are generated by simple expansions of propagators in small momenta Q

\[
\frac{1}{(P + Q)^2 + M^2} = \frac{1}{P^2 + M^2} \sum_{i=0}^{N} \frac{(-2P \cdot Q - Q^2)^i}{(P^2 + M^2)^i} + O\left(\frac{1}{(P^2 + M^2)^{N+2}}\right)
\]

where N is the desired order in the expansion. We then add and subtract the expansion of the massive one-loop subgraph as is indicated in the r.h.s. of figure 6a.

The first term in the r.h.s. of figure 6a is a massless one-loop integral with the expanded massive subintegral inserted in the integrand and it can be evaluated without difficulty, keeping Q finite.
The second term in the r.h.s. of figure 6a (the combination in the square brackets) has a vanishing contribution from the integration region where \( k \) is small because the behaviour of the massive subintegral for small \( k \) is subtracted off until the necessary order (that is determined by the depth of the expansion). One may then Taylor expand the integrand of the second term around \( Q = 0 \) to obtain an integral without an external momentum (a tadpole integral) that can also be evaluated without difficulty. It should be noted that massless tadpole diagrams vanish in dimensional regularization which means that after the Taylor expansions in the integrand of the second term in \( Q \), only the two loop massive tadpole contribution survives.

Finally we get the large mass expansion as it is presented in figure 6b where we used the notation that a box around a (sub)graph indicates that the integrand of this (sub)graph is Taylor expanded up to the desired order in its external momenta. This will be the standard notation in the following. Please note that the nullified term in figure 6b is a massless tadpole. Although the expansion procedure produces new ultraviolet and infrared divergences in separate terms of the r.h.s. in figure 6b, these terms are well defined for non-zero \( \varepsilon \) and these new divergences cancel in the sum.

\textbf{Figure 6b.} The diagrammatic large mass expansion of the diagram 6a.

Let us consider another diagram presented in figure 7a. The same reasoning as for diagram 6a holds but now we should first add and subtract the small momentum expansion of the massive propagator as it is shown in the r.h.s. of figure 7a.

\textbf{Figure 7a.} A scalar triangle diagram with one massive line

After this we can expand the expression in the square brackets in its external momenta. Using again the property that massless tadpoles vanish in dimensional regularization we are left with the diagrammatic expansion in figure 7b. Please note that the nullified term in figure 7b is a massless tadpole.
Figure 7b. The diagrammatic large mass expansion of the diagram 7a.

The triangle diagram of figure 7 can be part of a larger diagram such as the scalar diagram presented in figure 8. For figure 8, the same reasoning as for the diagrams in figures 6 and 7 holds and we are left with the indicated diagrammatic expansion.

Figure 8 The diagrammatic large mass expansion of a 2-loop diagram.

The large mass expansion of two loop propagator type diagrams is also treated in the recent article ref. [33]. It interesting to note that an analogous reasoning (i.e. focusing on infrared regions) can be applied to understand the related problem of small mass expansions for which a recipe is given in [30, 34].

Let us now formulate the recipe for the large mass expansion of diagrams. A line with a large mass, M, will be called a heavy line.

First we have to find all asymptotically irreducible (sub)graphs. An asymptotically irreducible (sub)graph, $g_{AI}$, is a connected subgraph which contains at least one heavy line (and all heavy lines connected to this one via heavy lines) and which can not be made disconnected by cutting a non-heavy line. We have to expand each asymptotically irreducible (sub)graph as a Taylor series in terms of its external momenta. We graphically indicate this by drawing a box around the asymptotically irreducible (sub)graph. Then the large mass expansion of the whole Feynman diagram is the sum over all combinations of non-overlapping boxes that can be drawn in this diagram so that all heavy lines are in boxes. Note that a box does never cut a heavy line.

Or in a symbolic form:

$$G = \sum_{\{g_{AI}\}} G/\{g_{AI}\} * T^{(N)} \{g_{AI}\} + O(1/M^N),$$

where $G$ is the Feynman graph to be expanded. The sum goes over all sets, $\{g_{AI}\}$, of non-overlapping asymptotically irreducible (sub)graphs comprising all heavy lines. $T^{(N)} \{g_{AI}\}$
denotes the Taylor expansion of (sub)graphs from the set \( \{ g_{AI} \} \) in their external momenta until the necessary order. \( G/\{ g_{AI} \} \) is the graph obtained from \( G \) by shrinking the subgraphs from \( \{ g_{AI} \} \) into points.

As a practical example we will now treat the diagrams that contribute to the axial vector part of the Z boson decay rate. At the 3-loop level the only Feynman diagrams that contribute are so called ‘double triangle diagrams’ (the first topology in figure 3) with the triangles formed by bottom and top fermion loops. This correction was originally calculated in [6] and confirmed in [7]. The diagram with two massive triangles is zero since the only physical cut is through two gluon propagators and \( Z^0 \) decay in two gluons is kinematically forbidden (Landau-Yang theorem). We are therefore left with two diagrams to be calculated, one massive and one massless diagram. The massless diagram can be directly calculated with the package MINCER. The massive diagram is calculated with the above recipe.

A diagrammatic representation of the ultraviolet R operation followed by the asymptotic expansion procedure, applied to the massive double triangle diagram is given in figure 9.

**Figure 9.** Thick lines indicate top quark propagators, thin lines indicate the massless quark propagators and spiral lines - gluon propagators. Ultraviolet counterterms are indicated between round brackets. The asymptotically irreducible (sub)graphs are surrounded by boxes with the corresponding tadpole topologies indicated below.

After the Taylor expansions the subgraphs in boxes are reduced to massive vacuum integrals and the resulting master topologies are indicated under the boxes. When the boxed subgraphs are integrated out we are left with massless diagrams that can be calculated with the package MINCER. We have written efficient FORM procedures to perform the necessary massive vacuum integrals. The procedures include 3-loop topologies of the type Benz and non-planar and use recursions based on the recursion scheme of [35]. They are essential for the large mass expansion of 4-loop diagrams. We should emphasise that only with very efficient massive vacuum procedures, that can deal with tensor numerators, the 4-loop calculation is feasible.
Making the Taylor expansions deep enough, we find the results for the contributions in figure 9. Adding the massless double triangle diagram and taking the imaginary part by applying \( \ln(-s - i\epsilon) = \ln(s) - i\pi \) (remember that \( s = m_Z^2 \) is the squared 4-momentum of the Z boson) we reproduced the large mass expansion of the known result \[6\]

\[
\Gamma_{A,S}^{had} = \frac{G_F m_Z^3}{8\sqrt{2\pi}} \left( \frac{\alpha_s}{\pi} \right)^2 \left( d_0^2 + d_1^1 \frac{s}{m_t^2} + d_2^2 \frac{s^2}{m_t^4} + d_3^3 \frac{s^3}{m_t^6} \right)
\] (15)

with \( d_0^2 = T_F^2 D \left[ -\frac{37}{24} + \frac{1}{2} \ln\left( \frac{s}{m_t^2} \right) \right] \), \( d_1^1 = T_F^2 D \left( -\frac{7}{162} \right) \), \( d_2^2 = T_F^2 D \left( \frac{7}{2400} \right) \), \( d_3^3 = T_F^2 D \left( \frac{52}{165375} \right) \), \( D = n^2 - 1 \) is the number of generators of the colour group \( SU(n) \) (\( D = 8 \) for QCD).

We will now show that the same method can be used to compute the \( \alpha_3^3 \) correction to \( \Gamma_{had} \). In contrast to the previous 3-loop case where we could determine both the real and imaginary parts of a diagram, we are now able to compute only the necessary imaginary part of diagrams with the present available techniques (we don’t have analytical results for general 4-loop massive tadpoles). In order to find the imaginary part of a diagram we need to calculate only terms that contain logarithms of \( s \) (since we take the imaginary part by applying \( \ln(-s - i\epsilon) = \ln(s) - i\pi \)). In dimensional regularization every massless propagator type integral receives a factor \( \left( \frac{\mu^2}{s} \right)^\varepsilon = 1 + \varepsilon \ln\left( \frac{\mu^2}{s} \right) + O(\varepsilon^2) \). In contrast, vacuum massive integrals produce factors \( \left( \frac{\mu^2}{m_t^2} \right)^\varepsilon \) and do not give logarithms of \( s \). This means that contributions without massless propagator type integrals do not have to be considered for our purpose. The expansion of one renormalized 4-loop diagram that contributes to \( \Gamma_{A,S}^{had} \) is presented in figure 10.

**Figure 10.** The large mass expansion of a renormalized 4-loop diagram

In figure 10 we have only presented terms which contribute to the imaginary part of the diagram. All 4-loop massive diagrams contributing to \( \Gamma_{had} \) give after the large mass expansion topologies that can be calculated with MINCER (the massless parts) and with the tadpole procedures (the vacuum massive parts).
5 The results for $\Gamma_{had}$

The results of the $\alpha_s^3$ approximation for $\Gamma_{had}^{V,NS}$, $\Gamma_{had}^{A,NS}$, $\Gamma_{had}^{V,S}$ and $\Gamma_{had}^{A,S}$ in the order $1/m_t^6$ of the large top quark mass expansion are presented below. Note that these results are in 6-flavour QCD, $N_f = 6$ and the decoupling relation (5) has not been applied yet.

$$\Gamma_{had}^{V(A),NS} = \frac{G_F m_Z^3}{2 \pi \sqrt{2}} \sum_{i=1}^{5} (g_{\chi_i}^{(A)})^2 \left( \frac{n_i}{3} \right) \left[ 1 + \left( \frac{\alpha_s^6}{\pi} \right) b_1^0 + \left( \frac{\alpha_s^6}{\pi} \right)^2 \left( b_2^0 + b_2^1 \frac{s}{m_t^2} + b_2^2 \frac{s^2}{m_t^4} + b_2^3 \frac{s^3}{m_t^6} \right) \right] + \left( \frac{\alpha_s^6}{\pi} \right)^3 \left( b_3^0 + b_3^1 \frac{s}{m_t^2} + b_3^2 \frac{s^2}{m_t^4} + b_3^3 \frac{s^3}{m_t^6} \right),$$

(16)

$$b_1^0 = C_F \left( \frac{3}{4} \right),$$

$$b_2^0 = N_f T_F C_F \left[ -\frac{11}{8} \zeta_3 + \frac{1}{4} \ln \left( \frac{s}{m_t^2} \right) \right] + T_F C_F \left[ \frac{11}{8} - \zeta_3 - \frac{1}{4} \ln \left( \frac{s}{m_t^2} \right) \right] + C_A C_F \left[ \frac{123}{32} - \frac{11}{4} \zeta_3 - \frac{11}{16} \ln \left( \frac{s}{m_t^2} \right) \right] + C_F^2 \left( -\frac{3}{32} \right),$$

$$b_2^1 = T_F C_F \left[ \frac{22}{27} - \frac{1}{2} \ln \left( \frac{s}{m_t^2} \right) \right],$$

$$b_2^2 = T_F C_F \left[ -\frac{1303}{705600} + \frac{1}{160} \ln \left( \frac{s}{m_t^2} \right) \right],$$

$$b_2^3 = T_F C_F \left[ -\frac{1643}{17860500} - \frac{1}{28350} \ln \left( \frac{s}{m_t^2} \right) \right],$$

$$b_3^0 = N_f T_F C_A C_F \left[ -\frac{485}{27} + \frac{112}{9} \zeta_3 + \frac{5}{6} \zeta_5 - \frac{11}{12} \pi^2 - \frac{11}{3} \zeta_3 \ln \left( \frac{s}{m_t^2} \right) + \frac{259}{48} \ln \left( \frac{s}{\mu^2} \right) - \frac{11}{24} \ln^2 \left( \frac{s}{\mu^2} \right) \right] + C_A^2 C_F \left[ \frac{6045}{353} - \frac{2747}{144} \zeta_3 - \frac{55}{24} \zeta_5 - \frac{124}{27} \pi^2 + \frac{124}{27} \zeta_3 \ln \left( \frac{s}{\mu^2} \right) - \frac{485}{64} \ln \left( \frac{s}{\mu^2} \right) + \frac{124}{192} \ln^2 \left( \frac{s}{\mu^2} \right) \right] + N_f T_F^2 C_F \left[ \frac{151}{94} - \frac{10}{9} \zeta_3 - \frac{1}{3} \pi^2 + \frac{2}{3} \zeta_3 \ln \left( \frac{s}{\mu^2} \right) - \frac{11}{12} \ln \left( \frac{s}{m_t^2} \right) + \frac{11}{12} \ln^2 \left( \frac{s}{m_t^2} \right) \right] + T_F C_A C_F \left[ \frac{979}{54} - \frac{112}{9} \zeta_3 - \frac{5}{6} \zeta_5 - \frac{11}{12} \pi^2 + \frac{11}{6} \zeta_3 \ln \left( \frac{s}{m_t^2} \right) \right] + \frac{11}{3} \zeta_3 \ln \left( \frac{s}{\mu^2} \right) + \frac{11}{24} \ln \left( \frac{s}{m_t^2} \right) \ln \left( \frac{s}{\mu^2} \right) - \frac{23}{8} \ln \left( \frac{s}{m_t^2} \right) - \frac{259}{48} \ln \left( \frac{s}{\mu^2} \right) + N_f T_F^2 C_F \left[ -\frac{151}{27} + \frac{38}{9} \zeta_3 + \frac{1}{18} \pi^2 - \frac{1}{6} \zeta_3 \ln \left( \frac{\mu^2}{m_t^2} \right) \right] - \frac{11}{3} \zeta_3 \ln \left( \frac{\mu^2}{m_t^2} \right) - \frac{1}{6} \ln \left( \frac{\mu^2}{m_t^2} \right) \ln \left( \frac{s}{\mu^2} \right) + \frac{11}{12} \ln \left( \frac{m_t^2}{\mu^2} \right) + \frac{11}{6} \ln \left( \frac{\mu^2}{m_t^2} \right) \right] + T_F C_F \left[ \frac{1}{4} - \frac{10}{4} \zeta_3 + 5 \zeta_5 + \frac{1}{4} \ln \left( \frac{m_t^2}{m_t^2} \right) - \frac{1}{8} \ln \left( \frac{\mu^2}{m_t^2} \right) \right] + C_A^2 C_F \left[ -\frac{127}{64} - \frac{144}{9} \zeta_3 + \frac{55}{6} \zeta_5 + \frac{11}{16} \ln \left( \frac{s}{\mu^2} \right) \right] + N_f T_F C_F \left[ -\frac{29}{64} + \frac{19}{4} \zeta_3 - 5 \zeta_5 + \frac{1}{8} \ln \left( \frac{s}{\mu^2} \right) \right] + C_F^3 \left( -\frac{69}{128} \right),
where $n$ is the parameter of the colour group $SU(n)$ ($n = 3$ for QCD). Note that the coefficient $b_1^2$ agrees with ref. [36] and $b_2^2$, $b_3^2$ agree with the expansion of the exact result ref. [37].
\[
\Gamma^A_{\text{had}} = \frac{G_F m_Z^3}{8\sqrt{2}\pi} \left[ \left( \frac{\alpha_s}{\pi} \right)^2 (d_0^0 + d_1^1 \frac{s}{m_t^2} + d_2^2 \frac{s^2}{m_t^4} + d_3^3 \frac{s^3}{m_t^6}) + \left( \frac{\alpha_s}{\pi} \right)^3 (d_0^0 + d_1^1 \frac{s}{m_t^2} + d_2^2 \frac{s^2}{m_t^4} + d_3^3 \frac{s^3}{m_t^6}) \right],
\]

(17)

\[
d_2^0 = T_F^2 D \left[-\frac{37}{24} + \frac{1}{2} \ln \left( \frac{s}{m_t} \right) \right],
\]

\[
d_2^1 = T_F^2 D \left( \frac{7}{163} \right),
\]

\[
d_2^2 = T_F^2 D \left( \frac{7}{2400} \right),
\]

\[
d_2^3 = T_F^2 D \left( \frac{52}{105375} \right),
\]

\[
d_3^0 = N_f T_F^3 D \left[ \frac{25}{36} - \frac{1}{18} \pi^2 + \frac{1}{6} \ln \left( \frac{\mu^2}{m_t^2} \right) - \frac{11}{6} \ln^2 \left( \frac{\mu^2}{m_t^2} \right) - \frac{11}{12} \ln \left( \frac{s}{\mu} \right) + \frac{1}{6} \ln^2 \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_A T_F^2 D \left[ -\frac{215}{48} + \frac{5}{12} \zeta_3 + \frac{11}{12} \pi^2 + \frac{19}{36} \ln \left( \frac{\mu^2}{m_t^2} \right) + \frac{11}{24} \ln^2 \left( \frac{\mu^2}{m_t^2} \right) + \frac{161}{48} \ln \left( \frac{s}{\mu} \right) - \frac{11}{24} \ln^2 \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_F T_F^2 D \left[ -\frac{2}{3} + \frac{2}{9} \zeta_3 - \frac{1}{3} \ln \left( \frac{\mu^2}{m_t^2} \right) \right]
= T_F^3 D \left[ -\frac{157}{108} + \frac{1}{18} \pi^2 + \frac{1}{12} \ln \left( \frac{s}{m_t} \right) - \frac{1}{6} \ln^2 \left( \frac{s}{m_t} \right) \right],
\]

\[
d_3^1 = N_f T_F^3 D \left[ \frac{19188}{3888} - \frac{17}{48} \ln \left( \frac{\mu^2}{m_t^2} \right) - \frac{1}{162} \ln \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_A T_F^2 D \left[ -\frac{3983}{31104} + \frac{13}{256} \zeta_3 + \frac{187}{1944} \ln \left( \frac{\mu^2}{m_t^2} \right) + \frac{19}{648} \ln \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_F T_F^2 D \left[ -\frac{1109}{5184} + \frac{3}{112} \zeta_3 - \frac{7}{108} \ln \left( \frac{\mu^2}{m_t^2} \right) \right]
= T_F^3 D \left[ -\frac{29309}{1458000} + \frac{7}{108} \zeta_3 + \frac{1}{270} \pi^2 + \frac{128}{2025} \ln \left( \frac{s}{m_t} \right) - \frac{1}{90} \ln^2 \left( \frac{s}{m_t} \right) \right],
\]

\[
d_3^2 = N_f T_F^3 D \left[ \frac{14321}{585200} - \frac{259}{9720} \ln \left( \frac{\mu^2}{m_t^2} \right) - \frac{7}{9720} \ln \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_A T_F^2 D \left[ -\frac{5465477}{463490000} + \frac{290451}{6035290} \zeta_3 + \frac{2849}{388800} \ln \left( \frac{\mu^2}{m_t^2} \right) + \frac{77}{388800} \ln \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_F T_F^2 D \left[ -\frac{12499297}{24883200} + \frac{463087}{1105920} \zeta_3 - \frac{7}{400} \ln \left( \frac{\mu^2}{m_t^2} \right) \right]
= T_F^3 D \left[ -\frac{1672471517}{64012032000} + \frac{1519}{1105920} \zeta_3 + \frac{1}{2520} \pi^2 + \frac{13163}{2381400} \ln \left( \frac{s}{m_t} \right) - \frac{1}{840} \ln^2 \left( \frac{s}{m_t} \right) \right],
\]

\[
d_3^3 = N_f T_F^3 D \left[ \frac{2588111}{10001880000} - \frac{4973}{15876000} \ln \left( \frac{\mu^2}{m_t^2} \right) - \frac{47}{453600} \ln \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_A T_F^2 D \left[ -\frac{279724766851}{490670840000} + \frac{50083309}{8847300000} \zeta_3 + \frac{54703}{63504000} \ln \left( \frac{\mu^2}{m_t^2} \right) + \frac{517}{1814400} \ln \left( \frac{s}{\mu} \right) \right]
\]

\[
+ C_F T_F^2 D \left[ -\frac{1237708334993}{4877107200000} + \frac{994661}{13762560} \zeta_3 - \frac{26}{18375} \ln \left( \frac{\mu^2}{m_t^2} \right) \right]
= T_F^3 D \left[ -\frac{17132293649}{3292014796000} + \frac{4873}{14745600} \zeta_3 + \frac{1}{17010} \pi^2 + \frac{21353}{28576800} \ln \left( \frac{s}{m_t} \right) - \frac{1}{5670} \ln^2 \left( \frac{s}{m_t} \right) \right].
\]
For completeness we also present separately the bottom-bottom and top-top contributions to $\Gamma_{\text{had}}^{A,S}$. ($\Gamma_{\text{had}}^{A,S} = \Gamma_{\text{had}}^{A,S,bb} + \Gamma_{\text{had}}^{A,S,tt} + \Gamma_{\text{had}}^{A,S,bb}$ is the sum over three possible pairs of the weak coupling constants: the bottom-bottom, top-top and top-bottom contributions.)

\[ \Gamma_{\text{had}}^{A,S,bb} = \frac{G_F m_Z^2}{2\sqrt{2}\pi} \left( g_A^{\text{top}} \right)^2 \left( \frac{\alpha_s^{(6)}}{\pi} \right)^2 d_2^{b,b} + \left( \frac{\alpha_s^{(6)}}{\pi} \right)^3 \left( d_2^{b,b} + d_3^{b,s} \frac{s}{m_t^2} + d_2^{b,b} \frac{s^2}{m_t^2} + d_2^{3,b,s^3} \right) \]  

(18)

\[ d_2^{b,b} = T_F^2 D \left[ -\frac{17}{12} + \frac{1}{2} \ln\left( \frac{s}{\mu^2} \right) \right], \]

\[ d_3^{b,b} = C_F T_F^2 D \left[ -\frac{53}{96} + \frac{1}{2} C_3 \right] \]

\[ \quad + C_A T_F^2 D \left[ \frac{939}{648} - \frac{17}{12} \pi^2 - \frac{11}{12} \ln\left( \frac{s}{\mu^2} \right) + \frac{1}{6} \ln^2\left( \frac{s}{\mu^2} \right) \right] \]

\[ \quad + C_A T_F^2 D \left[ -\frac{1621}{2592} + \frac{17}{24} \pi^2 + \frac{211}{48} \ln\left( \frac{s}{\mu^2} \right) - \frac{11}{24} \ln^2\left( \frac{s}{\mu^2} \right) \right] \]

\[ \quad + T_F^3 D \left[ -\frac{211}{108} + \frac{1}{18} \pi^2 + \frac{11}{12} \ln\left( \frac{s}{m_t^2} \right) - \frac{1}{96} \ln^2\left( \frac{s}{m_t^2} \right) \right], \]

\[ d_3^{1,b} = T_F^3 D \left[ -\frac{134}{1125} + \frac{1}{270} \pi^2 + \frac{77}{1350} \ln\left( \frac{s}{m_t^2} \right) - \frac{1}{96} \ln^2\left( \frac{s}{m_t^2} \right) \right], \]

\[ d_3^{2,b} = T_F^3 D \left[ -\frac{673639}{7408000} + \frac{1}{2520} \pi^2 + \frac{53}{11025} \ln\left( \frac{s}{m_t^2} \right) - \frac{1}{840} \ln^2\left( \frac{s}{m_t^2} \right) \right], \]

\[ d_3^{3,b} = T_F^3 D \left[ -\frac{2679041}{2250423000} + \frac{1}{17010} \pi^2 + \frac{2299}{3572100} \ln\left( \frac{s}{m_t^2} \right) - \frac{1}{5670} \ln^2\left( \frac{s}{m_t^2} \right) \right]. \]

Note that except for the massless diagrams there is only one massive diagram that contributes to $\Gamma_{\text{had}}^{A,S,tt}$ (the $3^{rd}$ topology in the $3^{rd}$ line of figure 3).

\[ \Gamma_{\text{had}}^{A,S,tt} = \frac{G_F m_Z^2}{2\sqrt{2}\pi} \left( g_A^{\text{top}} \right)^2 \left( \frac{\alpha_s^{(6)}}{\pi} \right)^3 \left( d_3^{0,t} + d_3^{1,t} \frac{s}{m_t^2} + d_3^{2,t} \frac{s^2}{m_t^2} + d_3^{3,t} \frac{s^3}{m_t^2} \right), \]

(19)

\[ d_3^{0,t} = d_3^{1,t} = 0, \]

\[ d_3^{2,t} = T_F^3 D(N_f - 1) \left( \frac{1}{8640} \right), \]

\[ d_3^{3,t} = T_F^3 D(N_f - 1) \left( \frac{13}{583200} \right). \]

Note also that although separate diagrams that contribute to $\Gamma_{\text{had}}^{A,S,tt}$ are generally non-zero, they cancel in the sum except for the one diagram (the $3^{rd}$ topology in the $3^{rd}$ line of figure 3) that has a colour factor proportional to $T_F^3 D(N_f - 1)$. This fact can be understood through the operator product expansion technique ref. [31]. In addition note
that the imaginary part of the diagram with three top quark loops (the $3^{rd}$ topology in the $3^{rd}$ line of figure 3) vanishes according to the Landau-Yang theorem (which we checked explicitly).

The result for the vector singlet part reads

$$\Gamma_{V,S}^{3\text{had}} = \frac{G_F m_Z^2}{2\pi \sqrt{2}} \left( \sum_{i=1}^{5} g_V^i \right)^2 \left( \frac{\alpha_s^{(6)}}{\pi} \right)^3 c_3^0$$

$$+ \frac{G_F m_Z^2}{2\pi \sqrt{2}} \sum_{i=1}^{5} g_V^i \left( \frac{\alpha_s^{(6)}}{\pi} \right)^3 \left( c_3^1 \frac{s}{m_t^2} + c_3^2 \frac{s^2}{m_t^4} + c_3^3 \frac{s^3}{m_t^6} \right), \quad (20)$$

$c_3^0 = T_F^2 d^{abc} d_{abc} \left( \frac{11}{144} - \frac{1}{2} \zeta_3 \right)$

$c_3^1 = T_F^2 d^{abc} d_{abc} \left( -\frac{37}{576} + \frac{13}{216} \zeta_3 \right)$

$c_3^2 = T_F^2 d^{abc} d_{abc} \left( -\frac{110401}{2332800} + \frac{29}{720} \zeta_3 \right)$

$c_3^3 = T_F^2 d^{abc} d_{abc} \left( -\frac{110401}{1143072000} + \frac{5009}{151200} \zeta_3 \right)$,

where $d^{abc}$ are the symmetrical structure constants of the $SU(n)$ colour group, $d^{abc} d_{abc} = 40/3$ for QCD. Note that the constant $c_3^0$ comes from massless diagrams only (although separate massive diagrams are non-zero in the leading order of the large top mass expansion, they add up to zero) as it is required by the decoupling mechanism for the vector part. It is interesting to mention that the diagrams with two massive top quark loops are non-zero separately but they add up to zero in all orders of the large mass expansion.

Explicit checks show that the coefficients of the logarithms in eqs. (16) and (17) are in agreement with the required renormalization group invariance of the physical quantity which in the $\alpha_s^3$ approximation reads

$$\mu^2 \frac{d}{d\mu^2} \Gamma(\mu^2, \alpha_s(\mu), m_t(\mu)) = O(\alpha_s^4). \quad (21)$$

Of course the true physical quantity is $\Gamma_{\text{had}} = \Gamma_{\text{had}}^{V,NS} + \Gamma_{\text{had}}^{V,S} + \Gamma_{\text{had}}^{A,NS} + \Gamma_{\text{had}}^{A,S}$. But from a theoretical point of view each of the four separate parts is renormalized independently and is therefore renormalization group invariant by itself.

The results that are presented in this section were obtained in an arbitrary covariant gauge for the gluon fields i.e. keeping the gauge parameter as a free parameter in the calculations. The explicit cancellation of the gauge dependence in the physical quantities gives a good check of the results. Individual diagrams contain $\zeta_2$, and $\zeta_4$ but these contributions add up to zero in the total results.
6 Final results (in a decoupled form)

It is known that the Appelquist-Carazzone theorem \[38\] about the decoupling of a heavy particle in quantum field theory does not work in its naive form for the \(\overline{MS}\) renormalization scheme and one should make an extra shift in the coupling constant (see eq.\((9)\) ) to make the decoupling explicit (i.e. to kill the large-mass logarithms) \[39, 18\]. However in the presence of an axial vector current the decoupling does not work (even after the shift in the coupling constant) due to the presence of axial anomaly type diagrams as is the case for the axial vector singlet contribution to \(\Gamma_{\text{had}}\).

Since we have explicitly calculated the top quark mass terms in the order \(\alpha_s^3\) for \(\Gamma_{\text{had}}\), we can derive the decoupling relation for the QCD coupling constant in the next-to-next-to-leading (NNL) order. Because the vector contribution to \(\Gamma_{\text{had}}\) should obey the decoupling mechanism, the use of the decoupling relation should convert the new 6 flavour result for \(\Gamma_{\text{had}}^{V,NS}\) (see eq.\((13)\)) to the previously known result in effective massless 5 flavour QCD (see eq.\((5)\)). One can see that the NNL order decoupling relation obtained in this way (see eq.\((9)\)) slightly differs from the one known in literature \[18\].

In order to settle this discrepancy, we performed an independent calculation. We have calculated the 3-loop massless quark propagator with a zero momentum \(\overline{\psi}\psi\) operator insertion (where \(\psi\) is the quark field)

\[
G_{\overline{\psi}[\psi\psi]\psi}(Q^2) = \int e^{iq\cdot x}dx dy < 0|T\left\{ \overline{\psi}_\alpha(x)\psi_\beta(y)\psi_\beta(y)\psi_\alpha(0)e^{iS}\right\}|0 > \tag{22}
\]

We will derive the decoupling relations from this (gauge dependent) Green function. One can also use the normal quark propagator for this purpose but one has to evaluate it at 4-loop level to derive the decoupling relation for the coupling constant in the NNL order. This is because at the one loop level the quark propagator is proportional to the gauge parameter.

For a physical quantity the decoupling mechanism consists of a shift in the coupling constants (and in the light masses, if present). The decoupling mechanism for the Green function is more complicated than that for a physical quantity because the renormalization of the Green function involves an overall renormalization constant, \(G_{\text{ren}} = ZG_{\text{Bare}}\), and one should also obtain the decoupling relation for \(Z\). To avoid this small complication, we prefer to take the quantity

\[
\frac{d}{d\ln(Q^2)} \ln\left(G_{\overline{\psi}[\psi\psi],\text{ren}}(Q^2)\right)
\]

that imitates a physical quantity in the sense that it has no overall renormalization constant \(Z\). The result of our calculation in the leading order of the large top quark mass expansion reads
\[
\frac{d}{d \ln(Q^2)} \ln \left( G_{\psi^3 \psi^3}^{\text{ren}}(Q^2) \right) = \left( \frac{\alpha_s^{(6)}}{\pi} \right) \left( C_f(1 - \frac{1}{3} \xi) \right) \\
+ \left( \frac{\alpha_s^{(6)}}{\pi} \right)^2 \left( N_f T_F C_F \left[ -\frac{5}{9} + \frac{1}{3} \ln(\frac{Q^2}{\mu^2}) \right] + T_F C_F \left[ -\frac{1}{3} \ln(\frac{\mu^2}{m_t^2}) \right] \right) \\
+ C_A C_F \left\{ \frac{295}{144} - \frac{11}{12} \ln(\frac{Q^2}{\mu^2}) + \xi \left[ -\frac{1}{6} + \frac{1}{8} \ln(\frac{Q^2}{\mu^2}) \right] + \xi^2 \left[ -\frac{1}{48} + \frac{1}{24} \ln(\frac{Q^2}{\mu^2}) \right] \right\} + C_F^2 \frac{1}{4} \\
+ \left( \frac{\alpha_s^{(6)}}{\pi} \right)^3 \left( N_f T_F^2 C_F \left[ -\frac{2}{9} \ln(\frac{\mu^2}{m_t^2}) \ln(\frac{Q^2}{\mu^2}) + \frac{10}{27} \ln(\frac{\mu^2}{m_t^2}) \right] \right) \\
+ T_F C_A C_F \left\{ \frac{2}{9} + \frac{11}{18} \ln(\frac{\mu^2}{m_t^2}) \ln(\frac{Q^2}{\mu^2}) - \frac{385}{216} \ln(\frac{\mu^2}{m_t^2}) \right\} \\
+ \xi \left\{ -\frac{89}{1728} - \frac{1}{24} \ln(\frac{\mu^2}{m_t^2}) \ln(\frac{Q^2}{\mu^2}) + \frac{13}{144} \ln(\frac{\mu^2}{m_t^2}) - \frac{1}{48} \ln(\frac{\mu^2}{m_t^2}) \right\} \\
+ T_F C_F^2 \left[ -\frac{13}{48} + \frac{1}{12} \ln(\frac{\mu^2}{m_t^2}) + T_F C_F^2 \left( \frac{1}{9} \ln(\frac{\mu^2}{m_t^2}) \right) + \left( \frac{\alpha_s^{(6)}}{\pi} \right)^3 \left( \text{massless contrib.} \right) \right], \tag{23}
\]

up to an overall normalization. \( \xi \) is the gauge parameter that appears in the gluon propagator as \( \frac{i}{q^{\mu+\nu} - (1 - \xi) q^{\mu+\nu}} \). For the \( \alpha_s^3 \) term only contributions that contain one or more top quark loops are presented. These massive contributions of the order \( \alpha_s^3 \) should disappear after the application of the decoupling relations for the coupling constant and for the gauge parameter \( \xi \). From this requirement we find the decoupling relations in the NNL order for the coupling constant presented in eq.\([9]\) and for the gauge parameter presented below.

\[
\xi^{(6)}(\mu) = \xi^{(5)}(\mu) \left\{ \left( \frac{\alpha_s^{(5)}(\mu)}{\pi} \right) \frac{T_F}{3} \ln(\frac{\mu^2}{m_t^2}(\mu)) + \left( \frac{\alpha_s^{(5)}(\mu)}{\pi} \right)^2 \left[ -\frac{1}{16} T_F C_A \ln^2(\frac{\mu^2}{m_t^2}(\mu)) \right] \right. \\
+ \left( \frac{1}{4} T_F C_F - \frac{5}{16} T_F C_A \right) \ln(\frac{\mu^2}{m_t^2}(\mu)) + \frac{13}{192} T_F C_A - \frac{13}{48} T_F C_F \right\} + O(\alpha_s^3). \tag{24}
\]

Please note that the term \( \frac{13}{48} T_F C_F \) that we found in eq.\([24]\) for the gauge parameter (as well as the analogous term for the coupling constant) is slightly different from the one in ref.\([18]\). Note that the decoupling relation for the coupling constant, eq.\([9]\), is derived by us in two independent ways: from the 4-loop calculation of \( \Gamma_{\text{had}} \) and the 3-loop calculation of \( G_{\psi^3 \psi^3} \).
We will now give the results for $\Gamma_{\text{had}}$ in effective QCD with 5 active massless flavours at the renormalization scale $\mu = m_Z$. These results are obtained by substitution of the decoupling relation, eq. (9), into the results for 6 flavour QCD of eqs. (16), (17) and (20). We will use the notation $x \equiv \frac{m_Z^2}{m_t^2}$.

$$\Gamma_{\text{had}} = \Gamma_{\text{had}}^{V,NS} + \Gamma_{\text{had}}^{A,NS} + \Gamma_{\text{had}}^{V,S} + \Gamma_{\text{had}}^{A,S}.$$ 

$$\Gamma_{\text{had}}^{V(A),NS} = \frac{G_F m_Z^3}{2 \pi \sqrt{2}} \left[ 1 + \left( \frac{\alpha_s}{\pi} \right)^2 b_1 + b_2 + \left( \frac{\alpha_s}{\pi} \right)^3 b_3 + O\left( \alpha_s^4 \right) \right],$$

(25)

$$b_1 = 1,$$

$$b_2 = 1.4092$$

$$\begin{align*}
+ [0.065185 - 0.014815 \ln(x)] x \\
+ [-0.0012311 + 0.00039683 \ln(x)] x^2 \\
+ [0.00006127 - 0.00023516 \ln(x)] x^3 + O(x^4),
\end{align*}$$

$$b_3 = -12.767$$

$$\begin{align*}
+ [-0.17374 + 0.21242 \ln(x) - 0.037243 \ln^2(x)] x \\
+ [-0.0075218 - 0.00058859 \ln(x) + 0.00038305 \ln^2(x)] x^2 \\
+ [0.00050411 - 0.00012099 \ln(x) + 0.000031419 \ln^2(x)] x^3 + O(x^4).
\end{align*}$$

$$\Gamma_{\text{had}}^{V,S} = \frac{G_F m_Z^3}{2 \pi \sqrt{2}} \left[ \sum_{i=1}^{5} \left( g_{V(A)}^i \right)^2 \left( \frac{\alpha_s}{\pi} \right)^3 c_3 + g_{V}^{\text{top}} \left( \sum_{i=1}^{5} g_{V}^i \right) \left( \frac{\alpha_s}{\pi} \right)^3 c_{3,\text{top}} + O\left( \alpha_s^4 \right) \right],$$

(26)

$$c_3 = -0.41318,$$

$$c_{3,\text{top}} = 0.027033x + 0.0036355x^2 + 0.00058874x^3 + O(x^4).$$

$$\Gamma_{\text{had}}^{A,S} = \frac{G_F m_Z^3}{2 \pi \sqrt{2}} \left( \frac{1}{4} \right) \left[ \left( \frac{\alpha_s}{\pi} \right)^2 d_2 + \left( \frac{\alpha_s}{\pi} \right)^3 d_3 + O\left( \alpha_s^4 \right) \right],$$

(27)

$$d_2 = -3.0833 + \ln(x) + 0.086420x + 0.0058333x^2 + 0.00062887x^3 + O(x^4),$$

$$d_3 = -18.654 + 1.7222 \ln(x) + 1.9167 \ln^2(x)$$

$$\begin{align*}
+ [-0.12585 + 0.28646 \ln(x) - 0.011111 \ln^2(x)] x
\end{align*}$$
We want to stress that the results show an excellent convergence of the large top quark mass expansion which can be seen from the fast decrease of the coefficients with the order of the expansion parameter \( x = \frac{m_Z^2}{m_t^2} \).

Note that the massive logarithms \( \ln \left( \frac{m_Z^2}{m_t^2} \right) \) are present in the leading order of the large mass expansion in \( \Gamma_{A,S}^{\text{had}} \) only (the violation of decoupling of the top quark). These logarithms can be summed up using the operator product expansion technique to produce a result that is finite in the limit of an infinitely large top quark mass as it was done in ref. \([7]\) for the order \( \alpha_s^2 \). However, for realistic values of \( m_t \) the above expressions can be trusted and are quite stable with respect to a change in the renormalization parameter \( \mu \) around the natural scale for this process \( \mu = m_Z \), as for example can be seen for \( \Gamma_{A,S}^{\text{had}} \) in figure 11.

**Figure 11.** The \( \mu \) dependence of \( \Gamma_{A,S}^{\text{had}} \) around \( \mu = m_Z \) for \( m_t(m_Z) = 140 \text{ GeV} \) and \( \Lambda_{QCD}^{(5)} = 0.2 \text{ GeV} \). The dotted line indicates the 3-loop result, the solid line indicates the result up to (and including) 4-loops.

One can use \( \Gamma_{V,NS}^{\text{had}} \) from the present paper to obtain the hadronic decay width \( \Gamma(W^\pm \to \text{hadrons}) \) (by a standard change of the weak coupling constants) and to obtain the large mass expansion in the \( \alpha_s^3 \) order for \( \Gamma(\tau^- \to \nu_\tau + \text{hadrons}) \) (as we do it in the next section). From the results of this paper one can also straightforwardly obtain the \( \alpha_s^3 \) approximation to the total cross section of electron-positron annihilation \( \sigma_{\text{tot}}(e^+e^- \to \gamma, Z^0 \to \text{hadrons}) \) in the necessary energy range, e.g. below, or above the \( Z^0 \) peak.
7 The large charm quark mass expansion for $\tau$ lepton decay

Another prominent process (beside Z boson decay) to extract the value of $\alpha_s$ from experiment is semihadronic $\tau$ lepton decay. Although the scale of this process is relatively low, non-perturbative corrections turn out to be small and perturbative QCD can be used to calculate the $\tau$ lepton decay rate, or the ratio

$$ R_\tau = \frac{\Gamma(\tau^- \to \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} $$

The ratio $R_\tau$ is expressed through the imaginary part of the W boson current correlator as (modulo a contribution that vanishes for massless active flavours)

$$ R_\tau = \frac{12\pi}{m_\tau^2} \int_0^{m_\tau^2} ds \left( 1 - \frac{s}{m_\tau^2} \right)^2 \left( 1 + \frac{2s}{m_\tau^2} \right) \text{Im}\Pi_1(s + i\epsilon) $$

where the integration is over the invariant mass of the hadrons, $s$, and we use the normalizations of ref. \cite{12}. $\Pi_1$ is the transverse part of the W boson current correlator analogous to the Z boson current correlator of eq.(2) (in eq.(2) the neutral weak quark current should be replaced by the charged weak quark current $J^\mu = \frac{g}{\sqrt{2}} \sum_{i,j} u_i \gamma^\mu (1 - \gamma^5) V_{ij} d_j$ with $u_i = (u,c,t)$ and $d_j = (d,s,b)$).

We work in the approximation of massless $u,d,s$ quarks and a heavy $c$ quark within perturbative QCD omitting non-QCD corrections. The $\alpha_s^3$ approximation in the leading order of a large mass expansion was calculated in ref. \cite{3}. The first charm quark mass suppressed term of the order $\alpha_s^2$ is obtained in ref. \cite{36}. We will obtain the large $c$ quark mass expansion of $R_\tau$ in the parameter $m_c^2/m_\tau^2$ within effective QCD with 3 massless flavours. This is the correct expansion parameter since the mass of the $\tau$ lepton is below the threshold for the production of charmed hadrons. Therefore a charm quark appears only in internal fermion loops, the effective expansion parameter appears to be $m_c^2/(4m_\tau^2)$ and a large $c$ quark mass expansion is justified (although the charm quark mass $m_c$ is smaller that the tau lepton mass $m_\tau$). The Feynman diagrams that contribute to $\text{Im}\Pi_1$ for the W boson are of the non-singlet type only (see figure 1) and were already calculated for the case of Z boson decay (see eq.(13)). After performing the integration in eq.(29) with the results of eq.(16) we have to apply the decoupling relation to go to effective QCD with 3 active flavours. Putting $\mu = m_\tau$ we obtain

$$ R_\tau = n \left( |V_{ud}|^2 + |V_{us}|^2 \right) \left[ 1 + \left( \frac{\alpha_s^{(3)}}{\pi} \right) r_1^0 + \left( \frac{\alpha_s^{(3)}}{\pi} \right)^2 \left( r_2^0 + r_2^1 \frac{m_c^2}{m_\tau^2} + r_2^2 \frac{m_c^4}{m_\tau^4} + r_2^3 \frac{m_c^6}{m_\tau^6} + O\left( \frac{m_c^8}{m_\tau^8} \right) \right) \right] $$

25
\[ + \left( \frac{\alpha_s^{(3)}}{\pi} \right)^3 \left( r_3^0 + r_3^1 m_c^2 m_{\tau}^2 + r_3^2 m_{\tau}^4 + r_3^3 m_{\tau}^6 + O\left( \frac{m_\tau^8}{m_c^8} \right) + O(\alpha_s^4) \right), \quad (30) \]

\[ r_1^0 = C_F \left( \frac{3}{4} \right) \]

\[ r_2^0 = C_A C_F \left( \frac{947}{192} - \frac{11 \pi}{32} \right) + C_F^2 \left( - \frac{3}{32} \right) + n_f T_F C_F \left( - \frac{85}{48} \right) + \zeta_3 \]

\[ r_2^1 = T_F C_F \left[ \frac{107}{300} - \frac{1}{150} \ln \left( \frac{m_\tau^2}{m_c^2} \right) \right] \]

\[ r_2^2 = T_F C_F \left[ - \frac{1597}{52920000} + \frac{1}{12600} \ln \left( \frac{m_\tau^2}{m_c^2} \right) \right] \]

\[ r_2^3 = T_F C_F \left[ \frac{3991}{500094000} - \frac{1}{3969000} \ln \left( \frac{m_\tau^2}{m_c^2} \right) \right] \]

\[ r_3^0 = n_f T_F C_A C_F \left( - \frac{24359}{8624} + \frac{73}{4} \zeta_4 + \frac{53}{6} \zeta_5 + \frac{11}{72} \pi^2 \right) + n_f T_F C_F^2 \left[ - \frac{125}{192} + \frac{19 \pi}{36} + 5 \zeta_5 \right] + n_f T_F C_F \left( - \frac{1733}{768} - \frac{143}{16} \zeta_3 + \frac{55}{4} \pi^2 \right) + C_A C_F^2 \left( - \frac{557915}{13824} - \frac{2591}{96} \zeta_3 - \frac{55}{27} \pi^2 + \frac{121}{1536} \pi^2 \right) + C_F^3 \left( - \frac{69}{128} \right) \]

\[ r_3^1 = n_f T_F^2 C_F \left[ - \frac{2633}{81050000} + \frac{1}{1350} \pi^2 - \frac{1}{250} \ln \left( \frac{m_\tau^2}{m_c^2} \right) + \frac{1}{450} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_A C_F \left[ - \frac{808427}{38880000} + \frac{53}{2880} \zeta_3 - \frac{11}{10800} \pi^2 + \frac{31309}{648000} \ln \left( \frac{m_\tau^2}{m_c^2} \right) - \frac{11}{1200} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_F^2 \left[ \frac{512251}{9720000} - \frac{209}{7200} \zeta_3 + \frac{1}{11} \pi^2 - \frac{5181}{40500} \ln \left( \frac{m_\tau^2}{m_c^2} \right) + \frac{7}{1800} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_F \left( - \frac{23}{3240} \right) \]

\[ r_3^2 = n_f T_F^2 C_F \left[ - \frac{76567}{555660000} - \frac{1}{13140} \pi^2 + \frac{37}{1984500} \ln \left( \frac{m_\tau^2}{m_c^2} \right) - \frac{1}{37800} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_A C_F \left[ - \frac{93273701}{80015040000} + \frac{271}{138240} \zeta_3 - \frac{43}{1814400} \pi^2 + \frac{424327}{381024000} \ln \left( \frac{m_\tau^2}{m_c^2} \right) + \frac{131}{604800} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_F^2 \left[ - \frac{3331729}{6667920000} + \frac{289}{2192000} \zeta_3 + \frac{43}{907200} \pi^2 + \frac{6887}{2976750} \ln \left( \frac{m_\tau^2}{m_c^2} \right) - \frac{23}{60480} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F^2 C_F \left( - \frac{767}{48096000} - \frac{1}{20250} \ln \left( \frac{m_\tau^2}{m_c^2} \right) \right) \]

\[ r_3^3 = n_f T_F^2 C_F \left[ - \frac{1094017}{472588830000} + \frac{3572109}{7501440000} \ln \left( \frac{m_\tau^2}{m_c^2} \right) + \frac{1}{1190700} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_A C_F \left[ - \frac{13293894937}{16803158400000} - \frac{6271}{9676800} \zeta_3 - \frac{1}{142884000} \pi^2 + \frac{260809}{1778112000} \ln \left( \frac{m_\tau^2}{m_c^2} \right) + \frac{73}{158760000} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F C_F^2 \left[ - \frac{3881354513}{7561422800000} + \frac{121}{3763200} \zeta_3 + \frac{7}{7144200} \pi^2 - \frac{10469}{24045120} \ln \left( \frac{m_\tau^2}{m_c^2} \right) + \frac{269}{23814000} \ln^2 \left( \frac{m_\tau^2}{m_c^2} \right) \right] + T_F^2 C_F \left( - \frac{1661}{2500470000} + \frac{1}{411000} \ln \left( \frac{m_\tau^2}{m_c^2} \right) \right) \]

where \( \alpha_s^{(3)}(m_\tau) \), \( m_c \equiv m_c(m_\tau) \) is the \( \overline{MS} \) charm quark mass and \( n = 3 \) is the number of quark colours. We neglected terms that are suppressed by the bottom and top quark masses (which in principle are present after the decoupling). Substitution of the QCD colour factors and \( n_f = 3 \) gives
\[ r_1^0 = 1, \quad r_2^0 = 5.2023, \]
\[ r_1^1 = 0.023778 - 0.0044444 \ln \left( \frac{m^2}{\tau m_c^2} \right), \]
\[ r_2^2 = -0.00020118 + 0.000052910 \ln \left( \frac{m^2}{\tau m_c^2} \right), \]
\[ r_2^3 = 0.0000053203 - 0.0000016797 \ln \left( \frac{m^2}{\tau m_c^2} \right), \]
\[ r_3^0 = 26.3659, \]
\[ r_3^1 = -0.057156 + 0.079881 \ln \left( \frac{m^2}{\tau m_c^2} \right) - 0.012654 \ln^2 \left( \frac{m^2}{\tau m_c^2} \right), \]
\[ r_3^2 = -0.00099668 - 0.00016858 \ln \left( \frac{m^2}{\tau m_c^2} \right) + 0.0000687096 \ln^2 \left( \frac{m^2}{\tau m_c^2} \right) + 0.00000168435 \ln^2 \left( \frac{m^2}{\tau m_c^2} \right), \]

Note that the coefficient \( r_1^2 \) agrees with ref. [36].

Although the expansion parameter \( \frac{m^2}{\tau m_c^2} \approx \left( \frac{1.777 \pm 0.3}{1.340 \pm 0.3} \right)^2 \) is slightly larger than 1, the fast decrease of the coefficients ensures a good convergence of the large charm quark mass expansion.

We conclude that the large mass expansion converges fast for both \( Z \) boson and \( \tau \) lepton decays and the obtained \( \alpha_s^3 \) approximations can be trusted.

8 Acknowledgements

We are grateful to D.Yu. Bardin, D.J. Broadhurst, K.G. Chetyrkin, A.I. Davydychev, A.L. Kataev, R. Kleiss and P.J. Nogueira for helpful discussions. One of us, S.L., is grateful to the theory group of HIKHEF-H for its kind hospitality. This work is supported in part by INTAS, Grant no. 93-1180.

References

[1] The LEP Collaborations: ALEPH, DELPHI, L3 and OPAL, Phys. Lett. B276 (1992) 247; The LEP Collaborations ALEPH, DELPHI, L3, OPAL and The LEP Electroweak Working Group, preprint CERN/PPE/93-157 (1993).

[2] D. Bardin et al., preprint CERN-TH. 6443/92 (1992); B.A. Kniehl, preprint KEK-TH-412 (1994).

[3] S.G. Gorishny, A.L. Kataev, S.A Larin, Phys. Lett. B212 (1988) 238; ibid. B259 (1991) 144; L.R. Surguladze, M.A. Samuel, Phys. Rev. Lett. 66 (1991) 560, erratum ibid, 2416.
[4] D.J. Broadhurst, A.L. Kataev, Phys. Lett. B315 (1993) 179.
[5] S.A. Larin, J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345.
[6] B.A. Kniehl, J.H. Kühn, Nucl. Phys. B329 (1990) 547.
[7] K.G. Chetyrkin, J.H. Kühn, Phys. Lett. B 308 (1993) 127.
[8] J.J. van der Bij, E.W.N. Glover, Nucl. Phys. B313 (1989) 237; R. Hopker, J.J. van der Bij, preprint THEP 93/3 (Freiburg, 1993).
[9] S.A. Larin, T. van Ritbergen, J.A.M. Vermaseren, Phys. Lett. B 320 (1994) 159.
[10] K.G. Chetyrkin, O.V. Tarasov, Phys. Lett. B 327 (1994) 114.
[11] Review of particle properties, Particle Data Group, Phys. Rev. D50 (1994) nr. 3.
[12] G.’t Hooft, M. Veltman, Nucl. Phys. B44 (1972) 189; for a review see G. Leibbrandt, Rev. Mod. Phys. 47 (1975) 849.
[13] G. ’t Hooft, Nucl. Phys. B61 (1973) 455.
[14] W.A. Bardeen, A.J. Buras, D.W. Duke, T. Muta, Phys. Rev. D18 (1978) 3998.
[15] S.A. Larin, Phys. Lett. B303 (1993) 113.
[16] J.C. Collins, F. Wilczek, A. Zee, Phys. Rev. D18 (1978) 242.
[17] O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, Phys. Lett. B93 (1980) 429; S.A. Larin, J.A.M. Vermaseren, Phys. Lett. B 303 (1993) 334.
[18] W. Bernreuther, W. Wetzel, Nucl. Phys. B197 (1982) 228; W. Bernreuther, Annals of Physics, 151 (1983) 127.
[19] O.V. Tarasov, Preprint JINR P2-82-900, (Dubna, 1982); S.A. Larin, Preprint NIKHEF-H/92-18, (Amsterdam, 1992), hep-ph/9302240.
[20] N. Gray, D.J. Broadhurst, W. Grafe, K. Schilcher, Z. Phys. C 48 (1990) 673.
[21] S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, preprint NIKHEF-H/91-18 (Amsterdam, 1991).
[22] J.A.M. Vermaseren, Symbolic Manipulation with Form, published by CAN (Computer Algebra Nederland), Kruislaan 419, 1098 VA Amsterdam, 1991, ISBN 90-74116-01-9.
[23] F.V. Tkachov, Phys. Lett. 100B (1981) 65; K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B192 (1981) 159.

[24] A.A. Vladimirov, Theor. Mat. Fiz. 43 (1980) 210.

[25] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Nucl. Phys. B 174 (1980) 345.

[26] J.C. Collins, Nucl. Phys. B 80 (1974) 341.

[27] K.G. Chetyrkin, F.V. Tkachov, Phys. Lett. B 114 (1982) 340; K.G. Chetyrkin, V.A. Smirnov, Phys. Lett. B 144 (1984) 419.

[28] F.V. Tkachov, preprint INR P-358 (Moscow, 1984); Int. Journ. Mod. Phys. A8 (1993) 2047; G.B. Pivovarov, F.V. Tkachov, ibid. A8 (1993) 2241.

[29] K.G. Chetyrkin, V.A. Smirnov, preprint INR P-0518 (Moscow, 1987); K.G. Chetyrkin, Teor. Mat. Fiz. 76 (1988) 207; V.A. Smirnov, Commun. Math. Phys. 134 (1990) 109.

[30] S.G. Gorishny, Nucl. Phys. B319 (1989) 633.

[31] S.G. Gorishny, S.A. Larin, Nucl. Phys. B283 (1987) 452.

[32] J.C. Collins, Renormalization, Cambridge University Press, 1984.

[33] F.A. Berends, A.I. Davydychev, V.A. Smirnov, J.B. Tausk, preprint INLO-PUB-15/94 (Leiden 1994).

[34] A.I. Davydychev, V.A. Smirnov, J.B. Tausk, Nucl. Phys. B 410 (1993) 325.

[35] D.J. Broadhurst, Z. Phys. C54 (1992) 599.

[36] K.G. Chetyrkin, Phys. Lett. B 307 (1993) 169.

[37] B.A. Kniehl, Phys. Lett. B 237 (1990) 127; A.H. Hoang, M. Jezabek, J.H. Kühn, T. Teubner, preprint TTP94-11 (Karlsruhe, 1994).

[38] T. Appelquist, J. Carazzone, Phys. Rev. D11 (1975) 2856.

[39] B. Ovrut, H. Schnitzer, Nucl. Phys. B189 (1981) 509.

[40] K. Schilcher, M.D. Tran, Phys. Rev. D29 (1984) 570.

[41] E. Braaten, Phys. Rev. Lett. 60 (1988) 1606.

[42] E. Braaten, S. Narison, A. Pich, Nucl. Phys. B 373 (1992) 581.