The General Induction Functors for the Category of Entwined Hom-Modules

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Abstract. We find a sufficient condition for the category of entwined Hom-modules to be monoidal. Moreover, we introduce morphisms between the underlying monoidal Hom-algebras and monoidal Hom-coalgebras, which give rise to functors between the category of entwined Hom-modules, and we study tensor identities for monodial categories of entwined Hom-modules. Finally, we give necessary and sufficient conditions for the general induction functor from $\mathcal{H}(M_k)(\psi)$ to $\mathcal{H}(M_k)(\psi')$ to be separable.

1. Introduction

Entwining modules were introduced in \cite{1}, which arise from noncommutative geometry, are modules of an algebra and comodules of a coalgebra such that the action and the coaction satisfy a certain compatibility condition. Unlike Doi-Hopf modules, entwined modules are defined purely using the properties of an algebra and a coalgebra combined into an entwining structure. There is no need for a “background” bialgebra, which is an indispensable part of the Doi-Hopf construction. Entwining modules are more general and easier to deal with, and provide new fields of applications. It is well-known that entwining modules unify modules, comodules, Sweedler’s Hopf modules, Takeuchi’s relative Hopf modules, graded modules, modules graded by $G$-sets, Long dimodules, Yetter-Drinfeld modules and Doi-Hopf modules \cite{4}.

Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov in \cite{16} as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity and similar for Hom-coassociativity. They also described the structures of Hom-bialgebras and Hom-Hopf algebras, and extended some important theories from ordinary Hopf algebras to Hom-Hopf algebras in \cite{17} and \cite{18}. Recently, many more properties and structures of Hom-Hopf algebras have been developed, see \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9}, \cite{10}, \cite{11}, \cite{12} and references cited therein.

Caenepeel and Goyvaerts studied in \cite{3} Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. In \cite{15}, Makhlouf...
and Panaite defined Yetter-Drinfeld modules over Hom-bialgebras and shown that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. Also Liu and Shen [13] studied Yetter-Drinfeld modules over monoidal Hom-bialgebras and called them Hom-Yetter-Drinfeld modules, and shown that the category of Hom-Yetter-Drinfeld modules is a braided monoidal categories. Chen and Zhang [7] defined the category of Hom-Yetter-Drinfeld modules in a slightly different way to [13], and shown that it is a full monoidal subcategory of the left center of left Hom-module category. We have defined in [9] the category of Doi Hom-Hopf modules and we prove there that the category of Hom-Yetter-Drinfeld modules is a subcategory of our category of Doi Hom-Hopf modules.

As a generalization of entwining modules in a Hopf algebra setting, entwined Hom-modules were introduced by Karacuba [11]. It is natural to ask the following question: can we prove a Maschke type theorem for entwined Hom-modules under more general assumptions? This is the motivation of this paper.

In this paper, we discuss the following questions: how do we make the category of entwined Hom-modules into monoidal? We show in Section 3 that it is sufficient that \((A, \beta)\) and \((C, \gamma)\) are monoidal Hom-bialgebras with some extra conditions. As an example, we consider the category of Doi Hom-Hopf modules [9], which is well known to be a monoidal category, this category is a special of our theory.

In Section 4, we first give the maps between the underlying Hom-comodule algebras and Hom-module coalgebras, which give rise to functors between the category of entwined Hom-modules. Moreover, we study tensor identities for monoidal categories of entwined Hom-modules. As an application, we prove that the category of entwined Hom-modules has enough injective objects.

In Section 5, let \((\Phi, \Psi): (A, C, \psi) \to (A', C', \psi')\) be a morphism of (right-right) Hom-entwining structures. The results of [9] can be extended to the general induction functor

\[
F : \mathcal{H}(A_1)(\psi)_A^C \to \mathcal{H}(A_1)(\psi')_{A'}^{C'}.
\]

In order to avoid technical complications, we will assume that the Hom-entwining map \(\psi\) is bijective, and write \(\psi^{-1} = \delta\).

2. Preliminaries

Throughout this paper we work over a commutative ring \(k\), we recall from [3] and [9] for some informations about Hom-structures which are needed in what follows.

Let \(C\) be a category. We introduce a new category \(\mathcal{H}(C)\) as follows: objects are couples \((M, \mu)\), with \(M \in C\) and \(\mu \in \text{Aut}_C(M)\). A morphism \(f : (M, \mu) \to (N, \nu)\) is a morphism \(f : M \to N\) in \(C\) such that \(\nu \circ f = f \circ \mu\).

Let \(\mathcal{M}_k\) denotes the category of \(k\)-modules. \(\mathcal{H}(\mathcal{M}_k)\) will be called the Hom-category associated to \(\mathcal{M}_k\). If \((M, \mu) \in \mathcal{M}_k\), then \(\mu : M \to M\) is obviously a morphism in \(\mathcal{H}(\mathcal{M}_k)\). It is easy to show that \(\mathcal{H}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (1, 1), \alpha, I, \tau)\) is a monoidal category by Proposition 1.1 in [3]: the tensor product of \((M, \mu)\) and \((N, \nu)\) in \(\mathcal{H}(\mathcal{M}_k)\) is given by the formula \((M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)\).

Assume that \((M, \mu), (N, \nu), (\rho, \pi) \in \mathcal{H}(\mathcal{M}_k)\). The associativity and unit constraints are given by the formulas

\[
\begin{align*}
\alpha_{MN, P}((m \otimes n) \otimes p) &= \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\
\tau_M(x \otimes m) &= \tau_M(m \otimes x) = x\mu(m).
\end{align*}
\]

An algebra in \(\mathcal{H}(\mathcal{M}_k)\) will be called a monoidal Hom-algebra.

**Definition 2.1.** A monoidal Hom-algebra is an object \((A, \alpha) \in \mathcal{H}(\mathcal{M}_k)\) together with a \(k\)-linear map \(m_A : A \otimes A \to A\) and an element \(1_A \in A\) such that

\[
\begin{align*}
\alpha(ab) &= \alpha(a)\alpha(b); & \alpha(1_A) &= 1_A, \\
\alpha(a)(bc) &= (ab)\alpha(c); & a1_A &= 1_A a = a(a),
\end{align*}
\]

for all \(a, b, c \in A\). Here we use the notation \(m_A(a \otimes b) = ab\).
Definition 2.2. A monoidal Hom-coalgebra is an object $(C, \gamma) \in \mathcal{H}(\mathcal{M}_k)$ together with $k$-linear maps $\Delta : C \rightarrow C \otimes C$, $\varepsilon : C \rightarrow k$ such that
\[
\Delta(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}), \quad \varepsilon(\gamma(c)) = \varepsilon(c),
\]
and
\[
\gamma^{-1}(c_{(1)}) \otimes c_{(2)(3)} \otimes c_{(2)(3)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)}), \quad \varepsilon(c_{(1)})c_{(2)} = \varepsilon(c) = \gamma^{-1}(c)
\]
for all $c \in C$.

Definition 2.3. A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the symmetric monoidal category $\mathcal{H}(\mathcal{M}_k)$. This means that $(H, \alpha, m, \eta)$ is a monoidal Hom-algebra, $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra and that $\Delta$ and $\varepsilon$ are morphisms of Hom-algebras, that is,
\[
\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}; \quad \Delta(1_H) = 1_H \otimes 1_H,
\]
\[
\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(1_H) = 1_H.
\]

Definition 2.4. A monoidal Hom-Hopf algebra is a monoidal Hom-bialgebra $(H, \alpha)$ together with a linear map $S : H \rightarrow H$ in $\mathcal{H}(\mathcal{M}_k)$ such that
\[
S \ast I = I \ast S = \eta \varepsilon, \quad S \alpha = \alpha S.
\]

Definition 2.5. Let $(A, \alpha)$ be a monoidal Hom-algebra. A right $(A, \alpha)$-Hom-module is an object $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ consists of a $k$-module and a linear map $\mu : M \rightarrow M$ together with a morphism $\psi : M \otimes A \rightarrow M, \psi(m \cdot a) = m \cdot a$, in $\mathcal{H}(\mathcal{M}_k)$ such that
\[
(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab); \quad m \cdot 1_A = \mu(m),
\]
for all $a \in A$ and $m \in M$. The fact that $\psi \in \mathcal{H}(\mathcal{M}_k)$ means that
\[
\mu(m \cdot a) = \mu(m) \cdot \alpha(a).
\]

A morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\mathcal{H}(\mathcal{M}_k)$ is called right $A$-linear if it preserves the $A$-action, that is, $f(m \cdot a) = f(m) \cdot a$. $\mathcal{H}(\mathcal{M}_k)^A$ will denote the category of right $(A, \alpha)$-Hom-modules and $A$-linear morphisms.

Definition 2.6. Let $(C, \gamma)$ be a monoidal Hom-coalgebra. A right $(C, \gamma)$-Hom-comodule is an object $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ together with a $k$-linear map $\rho_M : M \rightarrow M \otimes C$ notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\mathcal{H}(\mathcal{M}_k)$ such that
\[
m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]}))) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); \quad m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),
\]
for all $m \in M$. The fact that $\rho_M \in \mathcal{H}(\mathcal{M}_k)$ means that
\[
\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).
\]
Morphisms of right $(C, \gamma)$-Hom-comodules are defined in the obvious way. The category of right $(C, \gamma)$-Hom-comodules will be denoted by $\mathcal{H}(\mathcal{M}_k)^C$.

Definition 2.7. A right-right Hom-entwining structure is a triple $(A, C, \psi)$, where $(A, \beta)$ is a monoidal Hom-algebra and $(C, \gamma)$ is a monoidal Hom-coalgebra with a linear map $\psi : C \otimes A \rightarrow A \otimes C$ such that $\psi \circ (\gamma \otimes \beta) = (\beta \otimes \gamma) \circ \psi$ satisfying the following conditions:
\[
(ab)_{(\psi)} \otimes c_{(\psi)} = a_{(\psi)}b_{(\psi)} \otimes \gamma((\gamma^{-1}(c_{(\psi)})))_{(\psi)}),
\]
\[
\psi(c \otimes 1_A) = 1_A \otimes c_{(\psi)},
\]
\[
a_{(\psi)} \otimes \Delta(c_{(\psi)}) = \beta(\beta^{-1}(a_{(\psi)})) \otimes (c_{(1)}_{(\psi)} \otimes c_{(2)}_{(\psi)}),
\]
\[
\varepsilon(c_{(\psi)})a_{(\psi)} = \varepsilon(c)a_{(\psi)}.
\]
Over a Hom-entwining structure \((A, C, \psi)\), a right-right entwined Hom-module \((M, \mu)\) is both a right \((C, \gamma)\)-Hom-comodule and a right \((A, \beta)\)-Hom-module such that
\[
\rho_M(m \cdot a) = \mu(m_{[0]} \cdot \psi(m_{[1]} \otimes \beta^{-1}(a))) = m_{[0]} \cdot \beta^{-1}(a)_{[0]} \otimes \gamma(m_{[1]}^{\psi}),
\]
for all \(a \in A\) and \(m \in M\). \(\overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\) will denote the category of right entwined Hom-modules and morphisms between them.

A morphism between right-right entwined Hom-modules is a \(k\)-linear map which is a morphism in the categories \(\overline{\mathcal{H}}(\mathcal{M}_k)_A\) and \(\overline{\mathcal{E}}(\mathcal{M}_k)^C\) at the same time. \(\overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\) will denote the category of right-right entwined Hom-modules and morphisms between them.

3. Making the Category of Entwined Hom-Modules into a Monoidal Category

Now suppose that \((A, \beta)\) and \((C, \gamma)\) are both monoidal Hom-bialgebras.

**Proposition 3.1.** Let \((M, \mu) \in \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\), \((N, \nu) \in \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\). Then we have \(M \otimes N \in \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\) with structures:
\[
(m \otimes n) \cdot a = m \cdot a_{(1)} \otimes n \cdot a_{(2)},
\]
\[
\rho_{M \otimes N}(m \otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]}^{\mu} n_{[1]}^{\nu},
\]
if and only if the following condition holds:
\[
a_{(1)}^{\psi} \otimes a_{(2)}^{\psi} \otimes e^{cd} = a_{\psi(1)} \otimes a_{\psi(2)} \otimes (cd)^{\psi},
\]
(3.1)
for all \(a \in A\) and \(c, d \in C\). Furthermore, the category \(C = \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\) is a monoidal category.

**Proof.** It is easy to see that \(M \otimes N\) is a right \((A, \beta)\)-module and that \(M \otimes N\) is a right \((C, \gamma)\)-comodule. Now we check that the compatibility condition holds:
\[
\rho_{M \otimes N}((m \otimes n) \cdot a) = (m \cdot a_{(1)})_{[0]} \otimes (n \cdot a_{(2)})_{[0]} \otimes (m \cdot a_{(1)})_{[1]} (n \cdot a_{(2)})_{[1]}
\]
\[
= m_{[0]} \cdot \beta^{-1}(a_{(1)})_{[0]} \otimes n_{[0]} \cdot \beta^{-1}(a_{(2)})_{[0]} \otimes (\gamma(m_{[1]}^{\psi} n_{[1]}^{\psi}))
\]
\[
= m_{[0]} \cdot \beta^{-1}(a_{(1)})_{[0]} \otimes n_{[0]} \cdot \beta^{-1}(a_{(2)})_{[0]} \otimes \gamma((m_{[1]}^{\mu} n_{[1]}^{\nu}))
\]
\[
= (m_{[0]} \otimes n_{[0]}) \cdot \beta^{-1}(a_{[0]})_{[0]} \otimes \gamma((m_{[1]}^{\mu} n_{[1]}^{\nu})).
\]
(3.1)
So \(M \otimes N \in \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\).

Conversely, one can easily check that \(A \otimes C \in \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\), let \(m = 1 \otimes c\) and \(n = 1 \otimes d\) for any \(c, d \in C\) and easily get (3.1).

Furthermore, \(k\) is an object in \(\overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A\) with structures:
\[
x \cdot a = \varepsilon_A(a)x, \quad \rho(x) = x \otimes 1_C,
\]
for all \(x \in k\) if and only if the following condition holds:
\[
\varepsilon_A(a) 1_C = \varepsilon_A(\beta^{-1}(a)_{[0]} \gamma(1^C_{[0]})),
\]
(3.2)
for all \(a \in A\). Then it is easy to get that \((C = \overline{\mathcal{H}}(\mathcal{M}_k)(\psi)^C_A) \otimes (k, \alpha, \tilde{r})\) is a monoidal category, where \(\alpha, \tilde{r}\) are given by the formulas:
\[
\alpha_{M,N,P}(m \otimes n) \cdot p = \mu(m) \otimes (n \otimes \pi^{-1}(p)),
\]
Example 3.3. Let \( \mathcal{M} \) be a monoidal Hom-bialgebra, and \( A, C \) are monoidal Hom-bialgebras with the additional compatibility relations (3.1) and (3.2). If \( (A, C, \psi) \) is a monoidal Hom-entwining structure, then \( (A, \beta) \) and \( (C, \gamma) \) can be made into objects of \( \mathcal{H} (\mathcal{M}_k)(\psi)^C_A \) and \( \mathcal{H} (\mathcal{M}_k)(\psi)^A_C \). Proposition 3.2. Let \( (A, C, \psi) \) be a monoidal Hom-entwining structure. On \( (A, \beta) \) and \( (C, \gamma) \), we consider the following right \( (A, \beta) \)-action and right \( (C, \gamma) \)-coaction:

\[
 b \cdot a = ba \quad \text{and} \quad \rho^r(b) = \psi(1_C \otimes b) = \beta^{-1}(b) \otimes 1^C, \\
 c \cdot a = \varepsilon_A(a_{(1)}) \gamma(c^0) \quad \text{and} \quad \rho^r(c) = c_{(1)} \otimes c_{(2)}.
\]

Then \( (A, \beta) \) and \( (C, \gamma) \) are entwined Hom-modules.

Proof. We will show \( (A, \beta) \in \mathcal{H} (\mathcal{M}_k)(\psi)^C_A \) and leave the other statement to the reader. First, \( (A, \beta) \) is a right \( (C, \gamma) \)-comodule, since

\[
 (id_A \otimes \varepsilon_C) \rho^r(b) = \varepsilon_C(1^C) \beta^{-1}(b) = \varepsilon_C(1_C) \beta^{-1} = b,
\]

\[
 (\beta^{-1} \otimes \Delta_C) \rho^r(b) = \beta^{-2}(b_{\psi})(1_C \otimes \Delta_C(1^C)) = \beta^{-2}(b_{\psi}) \otimes 1^C = (\rho^r(b) \otimes \gamma^{-1}) \rho^r(b),
\]

and

\[
 b_{(1)} \beta^{-1}(a_{(1)}) \otimes \gamma(b_{(1)}^r) = \beta^{-1}(b_{\psi}) \beta^{-1}(a_{(1)}) \otimes \gamma(1^C_{(1)}) = \beta^{-1}(ba_{(1)}) \otimes \gamma(1^C_{(1)}) = \rho^r(ba).
\]

Thus \( (A, \beta) \in \mathcal{H} (\mathcal{M}_k)(\psi)^C_A \). Example 3.3. Let \( (H, \alpha) \) be a monoidal Hom-Hopf algebra, \( (C, \gamma) \) a right \( (H, \alpha) \)-Hom module bialgebra, and that \( (H, \alpha) \) acts on \( (C, \gamma) \) in such a way that \( (C, \gamma) \) is an \( (H, \alpha) \)-Hom module algebra and \( (H, \alpha) \)-Hom module coalgebra. Now let \( (A, \beta) \) be a monoidal Hom-bialgebra and a right \( (H, \alpha) \)-Hom comodule algebra such that the following compatibility relation holds, for all \( a \in A \):

\[
 a_{(1)[0]} \otimes a_{(1)[1]} \otimes a_{(2)[1]} \otimes a_{(2)[2]} = a_{(0)[0]} \otimes a_{(0)[1]} \otimes a_{(1)[1]} \otimes a_{(1)[2]}.
\]

We know that \( (H, A, C) \) is a right-right Doi Hom-Hopf datum in \( \mathcal{D} \), and we have a corresponding right-right Hom-entwining structure \( (A, C, \psi) \). It is straightforward to check that \( (A, C, \psi) \) is monoidal.

4. Tensor Identities

Theorem 4.1. Given two Hom-entwining structures \( (A, C, \psi) \) and \( (A', C', \psi') \), suppose that two maps \( \Phi : A \rightarrow A' \) and \( \Psi : C \rightarrow C' \) which are respectively monodial Hom-algebra and monodial Hom-coalgebra maps satisfying

\[
 \Phi(a_{(1)}) \otimes \Psi(c_{(1)}) = \Phi(a_{(1)}) \otimes \Psi(c_{(1)}', \psi'),
\]

then the induction functor \( F : \mathcal{H} (\mathcal{M}_k)(\psi)^C_A \rightarrow \mathcal{H} (\mathcal{M}_k)(\psi)^C_{A'} \), defined as follows:

\[
 F(M) = M \otimes_A A',
\]

where \( (A', \beta') \) is a left \( (A, \beta) \)-module via \( \Phi \) and with structure maps defined by

\[
 (m \otimes a') \cdot b' = m \otimes a' \beta^{-1}(b'),
\]

\[
 \rho_F(m \otimes a') = m_{(0)} \otimes (\beta^{-1}(a'))_{\psi'} \otimes \Psi(\gamma^{-1}(m_{(1)})),
\]

for all \( a', b' \in A' \) and \( m \in M \).
Proof. Let us show that $M \otimes_A A'$ is an object of $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi')^C$. It is routine to check that $F(M)$ is a right $(A', \beta')$-module. For this, we need to show that $M \otimes_A A'$ is a right $(C', \gamma')$-comodule and satisfy the compatible condition, for any $m \in M$ and $a', b' \in A'$, we have

$$\rho_{F(M)}((m \otimes_A a') \cdot b') = \rho_{F(M)}(\mu(m) \otimes_A a' \beta^{-1}(b'))$$

$$= \mu(m_{[0]}) \otimes_A (\beta^{-1}((a' \beta^{-1}(b'))))_{\psi} \otimes \Psi(m_{[1]}'_{\psi})$$

$$= [m_{[0]} \otimes_A (\beta^{-1}(a'))_{\psi} \otimes \Psi(\gamma^{-1}(m_{[1]}'_{\psi})))b'$$

$$= \rho_{F(M)}(m \otimes_A a')b',$$

i.e., the compatible condition holds. It remains to prove that $M \otimes_A A'$ is a right $(C', \gamma')$-comodule. For any $m \in M$ and $a' \in A'$, we have

$$(\rho_{F(M)} \otimes \text{id}_C)\rho_{F(M)}(m \otimes_A a')$$

$$= (\rho_{F(M)} \otimes \text{id}_C)(m_{[0]} \otimes_A (\beta^{-1}(a'))_{\psi} \otimes \Psi(\gamma^{-1}(m_{[1]}'_{\psi})))$$

$$= m_{[0]} \otimes_A (\beta^{-2}(a'))_{\psi_{\psi'}} \otimes \Psi(\gamma^{-1}(m_{[1]}'_{\psi})) \otimes \Psi(\gamma^{-1}(m_{[1]}_{\psi}))$$

$$= [m_{[0]} \otimes_A (\beta^{-1}(a'))_{\psi} \otimes \Psi(\gamma^{-1}(m_{[1]}'_{\psi}))) \otimes \Psi(\gamma^{-1}(m_{[1]}_{\psi}))_{(1)}$$

$$= \rho_{F(M)}(m \otimes_A a'),$$

as desired. This completes the proof. \qed

Theorem 4.2. Under the assumptions of Theorem 4.1, we have a functor $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi')^C \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi')^C$ which is right adjoint to $F$. $G$ is defined by

$$G(M') = M' \Box_C C,$$

with structure maps

$$(m' \otimes c) \cdot a = m' \cdot \beta^{-1}(a)_{\psi} \otimes \gamma(c_{\psi}), \quad (4.3)$$

$$\rho_{G(M')}(m' \otimes c) = \mu^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)}), \quad (4.4)$$

for all $a \in A$.

Proof. We first show that $G(M')$ is an object of $\widetilde{\mathcal{H}}(\mathcal{M}_k)(\psi')^C$. It is not hard to check that $G(M')$ is a right $(A', \beta')$-module. Now we check that $G(M')$ is a right $(C, \gamma)$-comodule and satisfy the compatible condition. For any $m' \in M'$ and $a, c \in C$, we have

$$\rho_{G(M')}(m' \otimes c) \cdot a = \rho_{G(M')}(m' \cdot \beta^{-1}(a)_{\psi} \otimes \gamma(c_{\psi}))$$

$$= \mu^{-1}(m') \cdot \beta^{-2}(a_{\psi}) \otimes \gamma(c_{(1)}) \otimes \gamma(\gamma(c_{(2)}))$$

$$= (\mu^{-1}(m') \otimes c_{(1)} \otimes \gamma(c_{(2)}))a$$

$$= \rho_{G(M')}(m' \otimes c)a,$$

i.e., the compatible condition holds. It remains to prove that $M' \Box_C C$ is a right $(C, \gamma)$-comodule. For any
\[ m' \in M' \text{ and } a \in A, \text{ we have} \]
\[
(\rho_{GM} \otimes id_C)\rho_{GM}(m' \otimes_A c) \\
= (\rho_{GM} \otimes id_C)(\mu^{-1}(m') \otimes (c(1) \otimes \gamma(c(2)))) \\
= \mu^{-2}(m') \otimes (c(1(1)) \otimes \gamma(c(1(2)) \otimes \gamma(c(2))) \\
= \mu^{-1}(m') \otimes c(1) \otimes \gamma(c(2(1))) \otimes \gamma^2(c(2(2))) \\
= (id_{GM} \otimes \Delta_C)\rho_{GM}(m' \otimes c),
\]
and
\[
(id_{GM} \otimes \iota)\rho_{GM}(m' \otimes c) \\
= (id_{GM} \otimes \iota)(\mu^{-1}(m') \otimes (c(1) \otimes \gamma(c(2)))) \\
= \mu^{-1}(m') \otimes c(1) \otimes \gamma^2(c(2)) \otimes 1_C = m' \otimes c,
\]
as required.

\(G(M') \in \hat{H}(\mathcal{M}_k)(\psi)^C_A\) and the functorial properties can be checked in a straightforward way. Finally, we show that \(G\) is a right adjoint to \(F\). Take \((M, \mu) \in \hat{H}(\mathcal{M}_k)(\psi)^C_A\), define \(\eta_M : M \to GF(M) = (M \otimes_A A') \Box C\) as follows: for all \(m \in M\),
\[
\eta_M(m) = m_{[0]} \otimes_A 1_A' \otimes m_{[1]}.
\]
It is easy to see that \(\eta_M \in \hat{H}(\mathcal{M}_k)(\psi)^C_A\). Take \((M', \mu') \in \hat{H}(\mathcal{M}_k)(\psi)^C_{A'}\), define \(\delta_M : FG(M') \to M'\), where \(\delta_M((m' \otimes c) \otimes_A a') = \epsilon_C(c)\mu'(m') \cdot a'\).

It is easy to check that \(\delta_M\) is \((A, \beta)\)-linear and therefore \(\delta_M \in \hat{H}(\mathcal{M}_k)(\psi)^C_{A'}\). We can also verify \(\eta\) and \(\delta\) defined above are all natural transformations and satisfy
\[
G(\delta_M) \circ \eta_{GM} = \text{Id}, \quad \delta_{GM} \circ F(\eta_M) = \text{Id},
\]
for all \(M \in \hat{H}(\mathcal{M}_k)(\psi)^C_A\) and \(M' \in \hat{H}(\mathcal{M}_k)(\psi')^C_{A'}\). And this completes the proof. \(\square\)

A morphism \((\Phi, \Psi)\) between two monoidal Hom-entwining structures is called monoidal if \(\Phi\) and \(\Psi\) are monoidal Hom-bialgebra maps. We now consider the particular situation where \(A = A'\) and \(\Phi = I_A\). The following result is a generalization of [4].

**Theorem 4.3.** Let \((I_A, \Psi) : (A, C, \psi) \to (A, C', \psi')\) be a monoidal morphism of monoidal Hom entwining structures. Then
\[
G(C') = C. \tag{4. 5}
\]
Let \((M, \mu) \in \hat{H}(\mathcal{M}_k)(\psi)^C_A\) be flat as a k-module, and take \((N, v) \in \hat{H}(\mathcal{M}_k)(\psi')^C_{A'}\). If \((C, \gamma)\) is a monoidal Hom-Hopf algebra, then
\[
M \otimes_G N = G(F(M) \otimes N) \text{ in } \hat{H}(\mathcal{M}_k)(\psi')^C_{A'} \tag{4. 6}
\]
If \((C, \gamma)\) has a twisted antipode \(\tilde{S}\), then
\[
G(N) \otimes M = G(N \Box_F M) \text{ in } \hat{H}(\mathcal{M}_k)(\psi)^C_A \tag{4. 7}
\]
**Proof.** We know that \(\epsilon_C \otimes id_C : C' \Box_C C \to C\) is an isomorphism; the inverse map is \((\Psi \otimes id_C)\Delta_C : C \to C' \Box_C C\). It is clear that \(\epsilon_C \otimes id_C\) is \((A, \beta)\)-linear and \((C, \gamma)\)-colinear. And this prove (4.5).

Now we define the map
\[
\Gamma : M \otimes G(N) = M \otimes (N \Box_C C) \to G(F(M) \otimes N) = (F(M) \otimes N) \Box_C C,
\]
which is given by
\[ \Gamma(m \otimes (n_i \otimes c_i)) = (m_{[0]} \otimes n_i) \otimes m_{[1]} c_i. \]
Recall that \( F(M) = M \) as an \((A, \beta)\)-module, with \((C', \gamma')\)-coaction given by
\[ \rho_{F(M)}(m) = m_{[0]} \otimes \Psi(m_{[1]}). \]

(1) \( \Gamma \) is well-defined, we have to show that
\[ \Gamma(m_i \otimes (n_i \otimes c_i)) \in (F(M) \otimes N) \otimes'_C C. \]

This may be seen as follows: for any \( m \in M \) and \( n_i \otimes c_i \in N \otimes C \), we have
\[
\begin{align*}
(\rho_{F(M) \otimes N} \otimes \text{id}_C)(m_{[0]} \otimes n_i) & \otimes m_{[1]} c_i \\
= (m_{[0][0]} \otimes n_{[0]}) & \otimes \Psi(m_{[0][1]} n_{[1]} \otimes m_{[1]} c_i) \\
= (\mu(m_{[0]}) \otimes \nu(n_i)) & \otimes \Psi(m_{[0][1]} \Psi(c_{[1]}) \otimes \gamma^{-1}(m_{[1]} c_{[2]})) \\
= (m_{[0]} \otimes n_i) & \otimes [\phi(m_{[0][1]}) \Psi(c_{[1]}) \otimes m_{[1]} c_{[2]}] \\
= (\text{id}_{F(M) \otimes N} \otimes \rho_C)(m_{[0]} \otimes n_i) & \otimes m_{[1]} c_i.
\end{align*}
\]

(2) \( \Gamma \) is \((A, \beta)\)-linear. Indeed, for any \( a \in A, m \in M \) and \( n_i \otimes c_i \in N \otimes C \), we have
\[
\begin{align*}
\Gamma((m \otimes (n_i \otimes c_i)) \cdot a) & = \Gamma(m \cdot a_{(1)} \otimes (n_i \cdot \beta^{-1}(a_{(2)} \otimes \gamma(c_i^{(2)}))) \\
& = (m_{[0]} \cdot \beta^{-1}(a_{(1)}^{(0)}) \otimes n_i \cdot \beta^{-1}(a_{(2)}^{(0)}) \otimes \gamma(m_{[1]}^{(1)} \gamma(c_i^{(2)})) \\
& = (m_{[0]} \cdot \beta^{-1}(a_{(1)}^{(0)}) \otimes n_i \cdot \beta^{-1}(a_{(2)}^{(0)}) \otimes \gamma(m_{[1]}^{(1)} c_i^{(1)})) \\
& = (m_{[0]} \otimes n_i) \cdot \beta^{-1}(a_{(0)}) \otimes \gamma((m_{[1]} c_i)^{(0)}) \\
& = \Gamma((m \otimes (n_i \otimes c_i)) \cdot a).
\end{align*}
\]

(3) \( \Gamma \) is \((C, \gamma)\)-colinear. Indeed, for any \( m \in M \) and \( n_i \otimes c_i \in N \otimes C \), we have
\[
\begin{align*}
\rho \circ \Gamma & = \rho((m \otimes (n_i \otimes c_i)) \\
& = \rho((m_{[0]} \otimes n_i) \otimes m_{[1]} c_i) \\
& = (\mu^{-1}(m_{[0]}) \otimes \nu^{-1}(n_i) \otimes m_{[1]}^{(1)} c_{[1]} \otimes \gamma(m_{[1]}^{(1)} \gamma(c_{[2]}))) \\
& = (m_{[0]} \otimes n_i) \otimes m_{[0][1]}^{(1)} c_{[1]} \otimes m_{[1]}^{(1)} \gamma(c_{[2]})) \\
& = (\Gamma \otimes \text{id}_C)(m_{[0]} \otimes (\nu^{-1}(n_i) \otimes c_{[1]})) \otimes m_{[1]} \gamma(c_{[2]}) \\
& = (\Gamma \otimes \text{id}_C) \circ \rho(m \otimes (n_i \otimes c_i)).
\end{align*}
\]

Assume \((C, \gamma)\) has an antipode and define
\[
\begin{align*}
\Theta : (F(M) \otimes N) \otimes C & \rightarrow M \otimes (N \otimes C), \\
\Theta((m_i \otimes n_i) \otimes c_i) & = \mu^2(m_{[0]}) \otimes (n_i \otimes S(m_{[1]})) \gamma^{-2}(c_i).
\end{align*}
\]

We have to show that \( \Psi \) is well-defined. \((M, \mu)\) is flat, so \( M \otimes (N \otimes C) \) is the equalizer of the maps
\[ id_M \otimes id_N \otimes \rho_C : M \otimes N \otimes C \rightarrow M \otimes N \otimes C' \otimes C, \]
and
\[ id_M \otimes \rho_N \otimes id_C : M \otimes N \otimes C \rightarrow M \otimes N \otimes C' \otimes C. \]

Now take \((m_i \otimes n_i) \otimes c_i \in (F(M) \otimes N) \otimes C\), then
\[ (m_{[0]} \otimes n_{[0]}) \otimes \phi(m_{[1]}) n_{[1]} \otimes c_i = (\mu^{-1}(m_i) \otimes \nu^{-1}(n_i)) \otimes \Psi(c_{[1]}) \otimes \gamma(c_{[2]}). \quad (4.8) \]
Therefore, we get
\[
\begin{align*}
\text{id}_M \otimes \text{id}_N \otimes \rho_C (\mu^2(m_{00}) \otimes (n_i \otimes S(m_{11}))^{-2}(c_i)) \\
= \mu^2(m_{00}) \otimes (n_i \otimes \Psi(S(m_{11})^{-2}(c_{i1})) \otimes S(m_{11}))^{-2}(c_{i2}) \\
= m_{00} \otimes \nu^{-1}(n_i) \otimes \Psi(S(\gamma(m_{11}))^{-1}(c_{i1})) \otimes S(\gamma^2(m_{11}))c_{i2},
\end{align*}
\]
and
\[
\begin{align*}
\text{id}_M \otimes \rho_N \otimes \text{id}_C (\mu^2(m_{00}) \otimes (n_i \otimes S(m_{11}))^{-2}(c_i)) \\
= \mu^2(m_{00}) \otimes (n_i \otimes \Psi(S(m_{11})^{-2}(c_{i1})) \otimes S(m_{11}))^{-2}(c_{i2}) \\
= m_{00} \otimes \nu^{-1}(n_i) \otimes \Psi(\gamma(m_{11}))^{-1}(c_i).
\end{align*}
\]

Applying \((\text{id}_M \otimes \Psi \otimes \text{id}_C) \circ (\text{id}_M \otimes (\Delta_C \otimes S_C)) \circ \rho_M\) to the first factor of (4.8), we obtain
\[
\begin{align*}
m_{00} \otimes \Psi(S(m_{11})) \otimes S(m_{11}) \otimes n_{00} \otimes \Psi(m_{11})n_{11} \otimes c_i \\
= \mu^{-1}(m_{00}) \otimes \Psi(S(\gamma^{-1}(m_{11})) \otimes S(\gamma^{-1}(m_{11})) \otimes \nu^{-1}(n_i) \otimes \phi(c_{i1}) \otimes \gamma(c_{i2}).
\end{align*}
\]

Applying \(\text{id}_M \otimes S^2 \otimes \text{id}_C \otimes \text{id}_N \otimes \nu^{-1} \otimes \gamma^{-1} \otimes \gamma^{-1}\) to the above identity, we have
\[
\begin{align*}
m_{00} \otimes \Psi(S(\gamma^2(m_{11})) \otimes S(m_{11}) \otimes n_{00} \otimes \gamma^{-1}(\phi(m_{11})n_{11}) \otimes \gamma^{-1}(c_i) \\
= \mu^{-1}(m_{00}) \otimes \Psi(S(\gamma^2(m_{11})) \otimes S(\gamma^{-1}(m_{11})) \otimes \nu^{-1}(n_i) \otimes \phi(\gamma^{-1}(c_{i1})) \otimes \gamma(c_{i2}).
\end{align*}
\]

Multiplying the second and the fifth factor, and also the third and sixth factor, we have
\[
\begin{align*}
\mu(m_{00}) \otimes n_{00} \otimes \gamma(n_{11}) \otimes \gamma^{-1}(c_i) \\
= \mu(m_{00}) \otimes \nu^{-1}(n_i) \otimes \Psi(S(\gamma^{-1}(m_{11})) \otimes \gamma^{-1}(c_{i1})) \otimes \gamma^{-1}(m_{11})c_{i2},
\end{align*}
\]
and applying \(\mu^{-1} \otimes \text{id}_N \otimes \text{id}_C \otimes \text{id}_C\) to the above identity, we obtain
\[
\begin{align*}
m_{00} \otimes n_{00} \otimes \gamma(n_{11}) \otimes S(\gamma(m_{11}))^{-1}(c_i) \\
= m_{00} \otimes \nu^{-1}(n_i) \otimes \Psi(S(\gamma^{-1}(m_{11})) \otimes \gamma^{-1}(c_{i1})) \otimes S(\gamma^2(m_{11}))c_{i2},
\end{align*}
\]
or
\[
\text{id}_M \otimes \rho_N \otimes \text{id}_C \circ (\Theta((m_i \otimes n_i) \otimes c_i)) = \text{id}_M \otimes \text{id}_N \otimes \rho_C \circ (\Theta((m_i \otimes n_i) \otimes c_i)).
\]

Let us point out that \(\Gamma\) and \(\Theta\) are each other’s inverses. In fact,
\[
\begin{align*}
\Theta \circ \Gamma(m_i \otimes n_i) \otimes c_i) \\
= \Theta((m_i \otimes n_i) \otimes S(m_{11}))^{-2}(c_i)) \\
= (\mu^2(m_{00}) \otimes n_i) \otimes \gamma^2(m_{11})S(m_{11}))^{-2}(c_i)) \\
= (\mu^2(m_{00}) \otimes n_i) \otimes [\gamma^{-1}(m_{11})^{-2}(c_{i1})] \\
= \mu(m_{00}) \otimes n_i) \otimes [\gamma^{-1}(m_{11})^{-2}(c_{i1})] \\
= (m_i \otimes n_i) \otimes c_i,
\end{align*}
\]
and
\[
\begin{align*}
\Theta \circ \Gamma(m_i \otimes n_i) \otimes c_i) \\
= \Theta((m_i \otimes n_i) \otimes S(m_{11}))^{-2}(c_i)) \\
= (\mu^2(m_{00}) \otimes n_i) \otimes [\gamma^2(m_{11})S(m_{11}))^{-2}(c_i)) \\
= \mu(m_{00}) \otimes n_i) \otimes [\gamma^{-1}(m_{11})^{-2}(c_{i1})] \\
= m \otimes n_i \otimes c_i.
\end{align*}
\]

The proof of (4.7) is similar and left to the reader. □
Corollary 4.4. Let \((A, C, \psi)\) be a monoidal Hom-entwining structure, \(\Lambda : \mathcal{H}(\mathcal{M}_k)(\psi)_A^C \to \mathcal{H}(\mathcal{M}_k)\) the functor forgetting the \((C, \gamma)\)-coaction. For any flat entwined Hom-module \((M, \mu)\), we have an isomorphism
\[
M \otimes C \cong \Lambda(M) \otimes C
\]
in \(\mathcal{H}(\mathcal{M}_k)(\psi)_A^C\). If \(k\) is a field, then \(\mathcal{H}(\mathcal{M}_k)(\psi)_A^C\) has enough injective objects, and any injective object in \(\mathcal{H}(\mathcal{M}_k)(\psi)_A^C\) is a direct summand of an object of the form \(I \otimes C\), where \(I\) is an injective \((A, \beta)\)-module.

We have already proved that the category of Doi Hom-Hopf modules may be viewed as the category of entwined Hom-modules corresponding to a monoidal Hom-entwining structure. Then we have the following corollary.

Corollary 4.5. Let \((H, A, C)\) be a monoidal Doi Hom-Hopf Datum. If \(k\) is a field, then \(\mathcal{H}(\mathcal{M}_k)(H)_A^C\) has enough injective objects, and any injective object in \(\mathcal{H}(\mathcal{M}_k)(H)_A^C\) is a direct summand of an object of the form \(I \otimes C\), where \(I\) is an injective \((A, \beta)\)-module.

We continue with the dual version of Theorem 4.3.

Theorem 4.6. Let \((\Phi, I_c) : (A, C, \psi) \to (A', C, \psi)\) be a monoidal morphism of monoidal Hom-entwining structures. Then
\[
F(A) = A'.
\]
(4.9)

Let \((M, \mu) \in \mathcal{H}(\mathcal{M}_k)(\psi)_A^C\) be flat as a \(k\)-module, and take \((N, \nu) \in \mathcal{H}(\mathcal{M}_k)(\psi)_A^C\). If \((A', \beta')\) is a monoidal Hom-Hopf algebra, then
\[
F(M) \otimes N \cong F(M \otimes G(N)) \text{ in } \mathcal{H}(\mathcal{M}_k)(\psi)_A^C.
\]
(4.10)

If \((A', \beta')\) has a twisted antipode \(\bar{S}\), then
\[
N \otimes F(M) \cong F(G(N) \otimes M) \text{ in } \mathcal{H}(\mathcal{M}_k)(\psi)_A^C.
\]
(4.11)

Proof. We only prove (4.10) and similar for (4.9) and (4.11). Assume that \((A', \beta')\) is a monoidal Hom-Hopf algebra and define
\[
\Gamma : F(M \otimes G(N)) = M \otimes G(N) \otimes_A A' \to F(M) \otimes N = (M \otimes_A A') \otimes N
\]
by
\[
\Gamma((m \otimes n) \otimes a') = (m \otimes a'_1) \otimes n \cdot a'_{(2)},
\]
for all \(a' \in A', m \in M\) and \(n \in N\). \(\Gamma\) is well-defined since
\[
\Gamma((m \otimes n) \otimes \Phi(a)a') = (m \otimes \Phi(a_1)a'_1) \otimes n \cdot \Phi(a_2)a'_2
\]
\[
= (m \cdot a_1 \otimes a'_1) \otimes n \cdot \Phi(a_2) a'_2
\]
\[
= \Gamma((m \cdot a_1) \otimes n \cdot \Phi(a_2)) \otimes a'
\]
\[
= \Gamma((m \otimes n) \cdot a \otimes a').
\]

It is easy to check that \(\Gamma\) is \((A', \beta')\)-linear. Now we shall verify that \(\Gamma\) is \((C, \gamma)\)-colinear based on (3.1). For any \(a' \in A', m \in M\) and \(n \in N\), we have
\[
\rho(\Gamma((m \otimes n) \otimes a')) = \rho((m \otimes a'_1) \otimes n \cdot a'_{(2)})
\]
\[
= \rho((m_0 \otimes \beta'^{-1}(a'_1)) \otimes (n_0 \cdot \beta'^{-1}(a'_{(2)})) \otimes \gamma(m_1)_\psi \gamma(n_1)_\psi
\]
(3.1)
\[
= (m_0 \otimes \beta'^{-1}(a'_1)) \otimes (n_0 \cdot \beta'^{-1}(a'_{(2)})) \otimes \gamma(m_1\cdot n_1)_\psi
\]
\[
= (\Gamma \otimes id_c)((m_0 \otimes n_0) \otimes \beta'^{-1}(a'_{(2)}) \otimes \gamma(m_1\cdot n_1)_\psi)
\]
\[
= (\Gamma \otimes id_c)\rho((m \otimes n) \otimes a').
\]
The inverse of $\Gamma$ is given by
\[
\Pi((m \otimes a') \otimes n) = (m \otimes v^{-2}(n)S^{-1}(a'_{(2)})) \otimes \beta^2(a'_{(1)})
\]
for all $a' \in A'$, $m \in M$ and $n \in N$. One can check that $\Pi$ is well-defined similar to $\Gamma$. Finally, we have
\[
\Pi(\Gamma((m \otimes n) \otimes a')) = \Pi((m \otimes a'_{(1)}) \otimes n \cdot a'_{(2)})
= (m \otimes v^{-2}(n \cdot a'_{(2)})S(a'_{(1)(2)})) \otimes \beta^2(a'_{(1)})
= (m \otimes v^{-1}(n) \cdot [\beta^{-1}(a'_{(2)(2)})S^{-1}(\beta^{-1}(a'_{(2)(1)}))] \otimes \beta^2(a'_{(1)}))
= (m \otimes n) \otimes a',
\]
and
\[
\Gamma(\Pi((m \otimes a') \otimes n)) = \Gamma(m \otimes v^{-2}(n)S^{-1}(a'_{(2)})) \otimes \beta^2(a'_{(1)})
= (m \otimes \beta^2(a'_{(1)(1)})) \otimes v^{-2}(n) \cdot S^{-1}(a'_{(2)})) \otimes \beta^2(a'_{(1)(2)})
= ([\beta^2(a'_{(1)}) \otimes m] \otimes v^{-1}(n) \cdot [S^{-1}(\beta^2(a'_{(2)(2)}))] \otimes \beta^2(a'_{(2)(1)}))
= (m \otimes a') \otimes n,
\]
as needed. The proof is completed.

\section{The General Induction Functor}

Let $\left(\Phi, \Psi\right) : (A, C, \psi) \to (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. The results of \cite{9} can be extended to the general induction functor

\[
F : \mathcal{H}(A, \Psi) \rightarrow \mathcal{H}(A', \Psi')
\]
and its right adjoint $G$ (see Theorem 4.2). In order to avoid technical complications, we will assume that the Hom-entwining map $\psi$ is bijective, and write $\psi^{-1} = \delta$.

\textbf{Proposition 5.1.} Let $(\Phi, \Psi) : (A, C, \psi) \rightarrow (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. With $\psi$ invertible, and $\delta : A \otimes C \rightarrow C \otimes A$ its inverse. Let $V_2$ consist of all left and right $(A, \beta)$-linear maps $\lambda : GF(C \otimes A) \rightarrow A$ satisfying

\[
\lambda((\gamma^{-1}(c_i) \otimes a') \otimes d_{(1)}) \otimes \gamma(d_{(2)}) = \sum f_i \lambda(c_i \otimes a') \otimes \gamma^{-1}(d_i) \otimes \gamma^2(c_i)
\]
(5. 1)
for all $(c_i \otimes a') \otimes d_i \in GF(C \otimes A)$. We have a $k$-linear isomorphism

\[
f_1 : V_1 = GF(C \otimes A) \rightarrow V_2, \quad f_1(\overline{\psi}) = (\varepsilon \otimes I_A) \circ \overline{\psi}.
\]

\textbf{Proof.} $\lambda = f_1(\overline{\psi})$ is left and right $(A, \beta)$-linear since $\overline{\psi}$ and $\varepsilon \otimes I_A$ are left and right $(A, \beta)$-linear. Take $\sum (c_i \otimes a'_i) \otimes d_i \in GF(C \otimes A)$, and we write

\[
\overline{\psi}(\sum (c_i \otimes a'_i) \otimes d_i) = \sum c_j \otimes a_j.
\]
Using the left $(C, \gamma)$-co-linearity of $\overline{\psi}$, we have

\[
\gamma^2(c_{i(1)}) \otimes \overline{\psi}(\sum (c_{i(2)} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(d_i)) = \sum \gamma(c_{j(1)}) \otimes (c_{j(2)} \otimes \beta^{-1}(a_j)),
\]
and applying $\varepsilon_C$ to the second factor

$$\gamma^2(c_{(1)}) \otimes \overline{\lambda}(\sum_i (c_{(2)} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(d_i)) = \sum_i c_i \otimes \beta^{-1}(a_i),$$

$\overline{\varepsilon}$ is also right $(C, \gamma)$-colinear, hence

$$\overline{\varepsilon}(\sum_i (\gamma^{-1}(c_i) \otimes \beta^{-1}(a'_i)) \otimes d_{(1)}) \otimes \gamma(d_{(2)}) = \sum_i [c_{(1)} \otimes \beta^{-1}(a_{(a'_i)})] \otimes \gamma(c_{(2)}).$$

Applying $\varepsilon_C$ to the first factor, we obtain

$$\overline{\lambda}(\sum_i (\gamma^{-1}(c_i) \otimes \beta^{-1}(a'_i)) \otimes d_{(1)}) \otimes \gamma(d_{(2)}) = \sum_i \beta^{-1}(a_{(a'_i)}) \otimes c_{(2)}^{\psi},$$

and we have shown that $\overline{\lambda}$ satisfies (5.1), and $f_1$ is well-defined. The inverse of $f_1$ is given by

$$g_1(\sum_i (c_i \otimes a'_i) \otimes d_i) = \sum_i \gamma^2(c_{(1)}) \otimes \lambda(\sum_i (c_{(2)} \otimes \beta^{-1}(a_{(a'_i)})) \otimes \gamma^{-1}(d_i)).$$

It is obvious that $\overline{\varepsilon} = g_1(\lambda)$ is left $(C, \gamma)$-colinear and right $(A, \beta)$-linear. $\overline{\varepsilon}$ is right $(C, \gamma)$-colinear since

$$\overline{\varepsilon}(\sum_i (\gamma^{-1}(c_i) \otimes \beta^{-1}(a'_i) \otimes d_{(1)}) \otimes \gamma(d_{(2)}))$$

$$= \sum_i \gamma(c_{(1)}) \otimes \overline{\lambda}(\sum_i (\gamma^{-1}(c_{(2)}) \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(d_{(1)}) \otimes \gamma(d_{(2)}))$$

$$= \sum_i \gamma(c_{(1)}) \otimes \overline{\lambda}(\sum_i (c_{(2)(2)} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-2}(d_{(1)}) \otimes \gamma^2(c_{(2)(1)}))$$

$$= \rho(\sum_i \gamma^2(c_{(1)}) \otimes \overline{\lambda}((c_{(2)} \otimes \beta^{-1}(a'_i)) \otimes \gamma^{-1}(d_{(1)})))$$

$$= \rho(\overline{\varepsilon}(c_i \otimes a'_i \otimes d_i)),$$

and $\overline{\varepsilon}$ is left $(A, \beta)$-linear since

$$\overline{\lambda}(\sum_i (c_i \otimes a'_i) \otimes d_i))$$

$$= \overline{\lambda}(\sum_i (\gamma(c_{(1)}) \otimes \Phi(\beta^{-2}(a_{(a'_i)})) \otimes \gamma(d_i)))$$

$$= \gamma^3(c_{(1)}) \otimes \lambda((\gamma(c_{(2)}) \otimes \Phi(\beta^{-3}(a_{(a'_i)})) \otimes \gamma(d_i)))$$

$$= \gamma^3(c_{(1)}) \otimes \lambda((\gamma(c_{(2)}) \otimes \Phi(\beta^{-3}(a_{(a'_i)})) \otimes \gamma(d_i)))$$

$$= \gamma^3(c_{(1)}) \otimes \lambda((\gamma(c_{(2)}) \otimes \Phi(\beta^{-3}(a_{(a'_i)})) \otimes \gamma(d_i)))$$

$$= \rho(\overline{\varepsilon}(c_i \otimes a'_i \otimes d_i)),$$

We have it to the reader to show that $g_1 = f_1^{-1}$.

**Theorem 5.2.** Let $(\Phi, \Psi) : (A, C, \psi) \to (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. With $\psi$ invertible, and $\delta : A \otimes C \to C \otimes A$ its inverse. Define the $A$-action on $C \otimes A'$ by

$$a \cdot (c \otimes b') = \sum \gamma^{-1}(c_{(a)} \otimes \beta^{-1}(a_{(a'_i)})) \otimes b', \quad \text{where } a \in A, \ c \in C, \ b' \in B'.$$

If $(C, \gamma)$ is left $(C', \gamma')$-coflat, then $V_1$ and $V_2$ are isomorphic as $k$-modules.
Proof. In view of the previous results, it suffices to show that \( f \circ f_1 : V \to V_2 \) is surjective. Starting from \( \lambda \in V_2 \), we have to construct a natural transformation \( v \), that is, for all \((M, \mu) \in \mathcal{H}(\mathcal{M})(\psi)_{A}^{C}\), we have to construct a morphism

\[
v_{M} : GF(M) = (M \otimes_{A} A') \square_{C} C \to M.
\]

First we remark that the map

\[
\phi : M \otimes_{A} A' \to M \otimes_{A} (C \otimes A'), \quad \phi(m \otimes_{A} a') = \mu(m_{[0]} \otimes (m_{[1]} \otimes \beta^{-1}(a'))) 
\]

is well-defined. Indeed,

\[
\phi(ma \otimes_{A} a') = \mu((ma)_{[0]} \otimes ((ma)_{[1]} \otimes \beta^{-1}(a'))) \\
= \sum \mu(m_{[0]}) \cdot \beta(\beta^{-1}(a')_{\psi}) \cdot \gamma(m_{[1]} \psi) \otimes \beta^{-1}(a') \\
= \sum \mu(m_{[0]}) \otimes_{A} \beta(\beta^{-1}(a')_{\psi}) \cdot (\gamma(m_{[1]} \psi) \otimes \beta^{-1}(a')) \\
= \mu(m_{[0]}) \otimes_{A} (m_{[1]} \otimes \beta^{-1}(aa')) = \phi(m \otimes_{A} aa').
\]

From the fact that \((C, \gamma)\) is left \((C', \gamma')\)-coflat, so we have

\[
(M \otimes_{A} (C \otimes A')) \square_{C} C \cong M \otimes_{A} ((C \otimes A') \square_{C} C),
\]

and we consider the map

\[
v_{M} = (I_{M} \otimes_{A} \lambda) \circ \tilde{a} \circ (\phi \square_{C} I_{C}) : GF(M) \to M \otimes_{A} A \cong M
\]

given by

\[
v_{M}(\sum (m_{i} \otimes a'_{i} \otimes c_{i})) = \sum \mu^{2}(m_{[0]}) \cdot \lambda((m_{[1]} \otimes \beta^{-1}(a'_{i})) \otimes \gamma^{-1}(c_{i})).
\]

Let us first show that \( v \) is right \( A \)-linear.

\[
v_{M}(\sum (m_{i} \otimes a'_{i} \otimes c_{i}) \cdot a) \\
= v_{M}(\sum (\mu(m_{i}) \otimes a'_{i} \beta^{-1}(a(\beta^{-1}(a'_{i})))) \otimes \gamma(c_{i})) \\
= \sum \mu^{2}(m_{[0]}) \cdot \lambda((\gamma(m_{[1]} \otimes \beta^{-1}(a'_{i})) \beta^{-2}(a(\beta^{-1}(a'_{i})))) \otimes \gamma^{-1}(c_{i})) \\
= \sum \mu^{2}(m_{[0]}) \cdot \lambda((\gamma(m_{[1]} \otimes \beta^{-1}(a'_{i})) \beta^{-1}(a(\beta^{-1}(a'_{i})) \otimes \gamma(\gamma^{-1}(c_{i})))) \\
= \sum \mu^{2}(m_{[0]}) \cdot \lambda((m_{[1]} \otimes \beta^{-1}(a'_{i})) \otimes \gamma^{-1}(c_{i})) \cdot \beta^{-1}(a) \\
= v_{M}(\sum (m_{i} \otimes a'_{i} \otimes c_{i}) \cdot a) \\
= v_{M}(\sum (m_{i} \otimes a'_{i} \otimes c_{i}) \cdot a).
\]
Let us show that $\nu$ is natural. Let $g : (M, \mu) \to (N, \nu)$ be a morphism in $\mathcal{H}(\mathcal{M})(\psi)_A^C$, and take $x = \sum (m_i \otimes a'_i) \otimes c_i \in (M \otimes A') \otimes C$. Then
\[
\nu_N(GF(g))(x) = \sum \nu_N((g(m_i) \otimes a'_i) \otimes c_i) = \sum \mu^2(g(m_{i0})) \cdot \lambda((m_{i1} \otimes \beta^{-1}(a'_i) \otimes \gamma^{-1}(c_i))) = \sum g(\mu^2(m_{i0}) \cdot \lambda((m_{i1} \otimes \beta^{-1}(a'_i) \otimes \gamma^{-1}(c_i))) = \sum g(\nu_M(x)).
\]

Finally, we have to show that $f_1(\nu) = \lambda$. Indeed, we have
\[
(\overline{I}_A \circ (\epsilon_C \otimes I_A))(\nu_{\mathcal{C} \mathcal{C} A}(\sum (c_i \otimes 1_A) \otimes d_i)) = (\overline{I}_A \circ (\epsilon_C \otimes I_A))(\sum (\gamma^2(c_{i1}) \otimes 1_A) \cdot \lambda((c_{i2} \otimes \beta^{-1}(a'_i) \otimes \gamma^{-1}(d_i)))) = \sum 1_A \lambda((\gamma^{-1}(c_i) \otimes \beta^{-1}(a'_i) \otimes \gamma^{-1}(d_i))) = \sum \lambda((c_i \otimes a'_i) \otimes d_i)),
\]
as needed.

**Corollary 5.3.** Let $(\Phi, \Psi) : (A, C, \psi') \to (A', C', \psi')$ be a morphism of (right-right) Hom-entwining structures. with $\psi$ invertible, and $\mathcal{A} : A \otimes C \to C \otimes A$ its inverse. If $(C, \gamma)$ is left $(C', \gamma')$-coflat, then induction functor $F : \mathcal{H}(\mathcal{M})(\psi)_A^C \to \mathcal{H}(\mathcal{M})(\psi')_{A'}^C$ is separable if and only if there exists $\lambda \in V_2$ such that
\[
\lambda((\gamma^{-1}(c_{i1}) \otimes 1_{A'}) \otimes c_{i2}) = \epsilon(c)1_A
\]
for all $c \in C$ and $a \in A$. $F$ is full and faithful if and only if $\eta_{\mathcal{C} \mathcal{C} A}$ is an isomorphism.

**Proof.** If $F$ is separable, then there exists $\nu \in V$ such that $\nu \circ \eta$ is the identity natural transformation, in particular
\[
\nu_{\mathcal{C} \mathcal{C} A} \circ \eta_{\mathcal{C} \mathcal{C} A} = I_{\mathcal{C} \mathcal{C} A}.
\]
Write $\overline{\nu} = f(\nu)$ and $\lambda = f_1(\overline{\nu})$, and apply both sides to $c \otimes 1_A$:
\[
\overline{\nu}((\gamma^{-1}(c_{i1}) \otimes \Phi((1_A)^\psi)) \otimes c_{i2}) = c \otimes 1_A.
\]
and (5.2) follows after we apply $\varepsilon$ to the first factor. Conversely, if $\lambda \in V_2$ satisfies (5.2), and $v$ is the natural transformation corresponding to $\lambda$, then

$$
v_M(\eta_M(m)) = v_M((\mu^{-1}(m_{[0]} \otimes 1_A) \otimes m_{[1]}))
= \mu(m_{[0]0}) \otimes \lambda((\gamma^{-1}(m_{[0]11}) \otimes 1_A) \otimes \gamma^{-1}(m_{[1]}))
= m_{[0]} \otimes \lambda((\gamma^{-1}(m_{[11]}1) \otimes 1_A) \otimes m_{[1]}2))
= m_{[0]} \cdot \lambda((\gamma^{-1}(m_{[1]}1) \otimes 1_A) \otimes m_{[1]}2))
= m_{[0]} \cdot \epsilon(m_{[1]}1)1_A = m.
$$

The second statement is proved in the same way.

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