RECTIFIABILITY OF THE FREE BOUNDARY AND DENSITY SET FOR VARIFOLDS

LUIGI DE MASI

ABSTRACT. We establish a partial rectifiability result for the free boundary of a $k$-varifold $V$ and an upper bound on the Hausdorff dimension of the set where the $k$-density of $V$ is infinite or does not exist. We first refine a theorem of Grüter and Jost by showing that the first variation of a general varifold with free boundary is a Radon measure. Next we show that if the mean curvature $H$ of $V$ is in $L^p$ for some $p \in [1,k]$, then the set of points where the $k$-density of $V$ is infinite or does not exists has Hausdorff dimension at most $k-p$. We use this result to prove that the part of the first variation of $V$ with positive and finite $(k-1)$-density is $(k-1)$-rectifiable.

1. Introduction

1.1. Motivations. The main goals of this paper are to study the rectifiability of the free boundary for a $k$-varifold $V$ in a compact domain $\mathcal{M} \subset \mathbb{R}^n$ with smooth boundary $\partial \mathcal{M}$ and to give an estimate of the Hausdorff dimension of the set of points where the $k$-density $\Theta^k(\|V\|, \cdot)$ does not exist or is infinite, where $\|V\|$ is the mass of $V$.

We say that $V$ has free boundary at $\partial \mathcal{M}$ if the following first variation formula holds for every vector field $X$ that is tangent to $\partial \mathcal{M}$ (see next section for more detailed definitions):

$$\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x) \, dV(x,S) = - \int_{\mathcal{M}} \langle X, H \rangle \, d\|V\|,$$

where $H \in L^1(\mathcal{M}, \|V\|)$.

If $V$ is the varifold induced by a smooth $k$-surface $\Sigma$ with smooth boundary $\partial \Sigma$, $H$ is the mean curvature vector of $\Sigma$ and (1.1) implies that $\partial \Sigma \subset \partial \mathcal{M}$ and that $\Sigma$ meets $\partial \mathcal{M}$ orthogonally: that is the unit conormal $\eta$ to $\partial \Sigma$ coincides with the exterior unit normal vector $N$ to $\partial \mathcal{M}$. So, varifolds with free boundary generalize in a weak sense the idea of surfaces that meet $\partial \mathcal{M}$ orthogonally.

If $\Sigma \subset \mathcal{M}$ is a smooth $k$-surface with smooth boundary $\partial \Sigma$ and meets $\partial \mathcal{M}$ orthogonally, we can test the first variation formula

$$\int_{\Sigma} \operatorname{div}_{T_{\Sigma}X} X(x) \, d\mathcal{H}^k(x) = - \int_{\Sigma} \langle X, H \rangle \, d\mathcal{H}^k + \int_{\partial \Sigma} \langle X, N \rangle \, d\mathcal{H}^{k-1},$$

where $\mathcal{H}^s$ is the $s$-dimensional Hausdorff measure) with a smooth vector field $X$ such that $X(x) = N(x)$ on $\partial \mathcal{M}$, obtaining the estimate

$$\mathcal{H}^{k-1}(\partial \Sigma) \leq c \left( \frac{\mathcal{H}^k(\Sigma)}{R(\mathcal{M})} + \int_{\Sigma} |H| \, d\mathcal{H}^k \right),$$

where $c = c(k, \mathcal{M})$ and $R(\mathcal{M})$ is the minimum radius of curvature of $\partial \mathcal{M}$. This bound can be easily localized to any ball $B_r(x)$ where $x \in \partial \mathcal{M}$. (In particular, if $\mathcal{M} = B_1 \subset \mathbb{R}^n$ is the unit ball with center 0 and $\Sigma \subset B_1$ is a minimal $k$-surface $\Sigma$ that meets $\partial B_1$ orthogonally, choosing $X = x$ we obtain the nice identity $\mathcal{H}^{k-1}(\partial \Sigma) = k \mathcal{H}^k(\Sigma)$.)

The simple proofs of these a-priori bounds strongly rely on the fact that $\Sigma$ and $\partial \Sigma$ are assumed to be smooth (that is the first variation of $\Sigma$ is assumed to be bounded) and that the conormal $\eta$ of $\partial \Sigma$ points outside $\mathcal{M}$. It is natural to ask if similar estimates hold also when $V$ is a general varifold with free boundary: that is if $V$ has bounded first variation and if its unit conormal on $\partial \mathcal{M}$ is orthogonal to $\partial \mathcal{M}$ and points outside $\mathcal{M}$.

We answer this question refining a result stated by Grüter and Jost in [9] and by Edelen in [6], proving that if $V$ satisfies (1.1), then it has bounded first variation: namely there exists a positive
Radon measure $\sigma_V$ such that, for each smooth vector field $X$ on $\mathcal{M}$ we have
\begin{equation}
\int_{G_k(\mathcal{M})} \text{div}_S X(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle X, H + \tilde{H} \rangle \, d\|V\| + \int_{\partial \mathcal{M}} \langle X, N \rangle \, d\sigma_V,
\end{equation}
where $\tilde{H} \in L^\infty(\partial \mathcal{M}, \|V\|)$ is orthogonal to $\partial \mathcal{M}$ and $N$ is the exterior unit normal vector to $\partial \mathcal{M}$. Moreover we prove a local bound on $\sigma_V$ similar to (1.3).

Clearly (1.4) is analogous to (1.2): by comparison we have that if $V$ is induced by $\Sigma$, then $\sigma_V = \mathcal{H}^{k-1,1}(\partial \Sigma)$. It is natural to ask also for a general $k$-varifold with free boundary, if $\sigma_V$ is singular with respect to $\|V\|$ or, more precisely, if $\sigma_V$ is $(k-1)$-rectifiable. As far as we know, this question has not been investigated. Under suitable assumptions, we are able to show a rectifiability result for $\sigma_V$.

To prove it, we analyze tangent cones to $V$ at points on $\partial \mathcal{M}$; tangent varifolds to $V$ exist if the upper $k$-density of $V$ is finite. When the mean curvature $H$ of $V$ is in $L^p(\mathcal{M}, \|V\|)$ for some $p > k$, it is well-known that the density of $V$ exists and is finite for every point; whereas if $p \leq k$, the finiteness of the upper density is guaranteed just $\|V\|$-a.e. (and, by standard arguments, $\mathcal{H}^{k}$-a.e.) by monotonicity formula and differentiation theorems, which is not enough to prove the rectifiability result on $\sigma_V$.

In order to deal with this case, we establish an estimate of the size of the set where the $k$-density of the varifold is infinite or does not exist in terms of Hausdorff measures: if $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \in [1, k]$, then this set has Hausdorff dimension at most $k - p$.

1.2. Background and main results. Allard studied the first variation of a varifold in the two seminal papers [1] and [2]. In the former he considers the interior case, while in the latter he studies the behavior of a $k$-varifold $V$ assuming it has, as a boundary, a smooth $(k-1)$-dimensional submanifold $\Gamma$, i.e. he assumes that $V$ has generalized mean curvature with respect to vector fields that vanish on $\Gamma$.

An $\varepsilon$-regularity theorem similar to the ones by Allard is proved by Gruter and Jost in [3] for varifolds with free boundaries. Moreover they prove in [3, 4.11(ii)] that a varifold with free boundary with $\|V\|_0 = 0$ has bounded first variation $\delta V$; this is also proved by Edelen in [6, Proposition 3.2] removing the hypothesis that $\|V\|_0 = 0$, but assuming that $V$ is rectifiable.

We refine these boundedness results, extending them to general varifolds and removing the assumption $\|V\|_0 > 0$. We state it in a slightly more general setting: if $V$ has generalized mean curvature with respect to vector fields that vanish on $\partial \mathcal{M}$, then it has bounded first variation with respect to vector fields that are orthogonal to $\partial \mathcal{M}$ (see next section for precise definitions):

**Theorem 1.1.** Let $V \in \mathcal{V}_k(\mathcal{M})$ be a $k$-varifold with generalized mean curvature $H$ with respect to $\mathcal{X}_0(\mathcal{M})$ and $H \in L^1(\mathcal{M}, \|V\|)$. Then there exists a positive Radon measure $\sigma_V$ on $\partial \mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ on $\partial \mathcal{M}$ such that
\begin{equation}
\int_{G_k(\mathcal{M})} \text{div}_S X(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle X, H + \tilde{H} \rangle \, d\|V\| + \int_{\partial \mathcal{M}} \langle X, N \rangle \, d\sigma_V \quad \forall X \in \mathcal{X}_\perp(\mathcal{M}),
\end{equation}
where $\tilde{H}$ is orthogonal to $\partial \mathcal{M}$ for $\|V\|$-a.e. $x \in \partial \mathcal{M}$, $\tilde{H} \in L^\infty(\partial \mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends on the second fundamental form of $\partial \mathcal{M}$. In particular, $V$ has bounded first variation with respect to $\mathcal{X}_\perp(\mathcal{M})$. Moreover, the following estimate holds:
\begin{equation}
\sigma_V(B_r(x_0)) \leq \frac{2}{r} \|V\|(B_r(x_0)) + \int_{B_r(x_0)} (c + |H|) \, d\|V\| \quad \forall x_0 \in \partial \mathcal{M}, \forall r \leq R(\mathcal{M})
\end{equation}
where $R(\mathcal{M})$ is such that the distance function from $\partial \mathcal{M}$ is of class $C^2$ in $U_R(\partial \mathcal{M})$ and the constant $c$ depends on the second fundamental form of $\partial \mathcal{M}$.

In other words, this theorem states that the component of the first variation of $V$ on $\partial \mathcal{M}$ that is orthogonal to $\partial \mathcal{M}$ is the sum of two terms:

- An absolutely continuous part with respect to $\|V\|$ given by $\tilde{H}\|V\|$, which takes into account the fact that $V$ can “lean” on $\partial \mathcal{M}$: indeed, if $V$ is induced by a smooth surface $\Sigma$ with constant multiplicity, then $\tilde{H}$ is the mean curvature of $\Sigma$ where $\Sigma$ lean on $\partial \mathcal{M}$. This is
orthogonal to ∂M and depends on T_xΣ and of the second fundamental form of ∂M, see (3.12).

• A part given by Nσ_V, which “points outward M” and is bounded. Roughly speaking, we expect that this is the “transversal boundary” of V at ∂M. On the other hand, V can have unbounded first variation on ∂M only where “V meets ∂M tangentially”.

Looking at the case of varifolds with free boundary, since the tangent part to ∂M of the first variation of such a varifold is controlled by definition, Theorem 1.1 easily implies that varifolds with free boundary have bounded first variation (Corollary 4.4).

As we said before, Corollary 4.4 was already proved by Grüter and Jost when ||V|| (∂M) = 0, and Edelen extended the result to ||V|| (μ) > 0 but assuming that V is rectifiable. In Edelen’s proof, the rectifiability is used to show that for ||V||-a.e. x ∈ ∂M the only planes charged by V are those included in T_x∂M. We are able to remove the rectifiability assumption by the use of Lemma 3.1 which is a form of the Constancy Theorem (see [11, Theorem 41.1]) and asserts that on ∂M, even in the non-rectifiable case, V charges only planes that are included in T_x∂M.

We have already stated above that, if the varifold is induced by a smooth surface with free boundary at ∂M, then σ_V = H^{k-1,∂}Σ. It is then natural to ask if σ_V is (k-1)-rectifiable in a more general case as well.

To prove Theorem 1.2, we make an analysis of the blow-ups of V on ∂M similar to the one performed in [2]. The study of blow-ups of V allows us to deduce, for σ_V\*-a.e. x ∈ ∂M, that every (k-1)-blow-up of σ_V\* at x is of the form βH^{k-1,S} for some (k-1)-dimensional plane S and β > 0. The Marstrand-Mattila Rectifiability Criterion (Theorem 6.1), then implies that σ_V\* is (k-1)-rectifiable.

As we show at the beginning of section 6, if H ∈ L^p(M, ||V||) for some p > k, then the condition Θ^{(k-1)}(σ_V, x) < +∞ is not restrictive, since it holds for every point x, basically by (1.6) and by the monotonicity formula for ||V|| that we prove in Corollary 4.5. Thus in this case, Theorem 1.2 can be read in the following way:

Theorem 1.3. Let V ∈ V_k(M) be a rectifiable k-varifold with free boundary at ∂M such that H ∈ L^p(M, ||V||) for some p > k and Θ^{k}(|V||, x) ≥ 1 for ||V||-a.e. x ∈ M. Then σ_{V\*{}}\{x | Θ^{k-1}(σ_V, x) > 0\} is (k-1)-rectifiable.

To perform the analysis of the blow-ups of V in the proof of Theorem 1.2 we have to distinguish two cases:

• If H ∈ L^p(M, ||V||) for some p > k, then the monotonicity formula (Corollary 4.5) assures the existence of the k-density Θ^{k}(|V||, ·) < +∞ and of tangent cones to V at every point in ∂M.

• The situation is more delicate when p ≤ k, because the Lebesgue-Besicovitch differentiation Theorem applied to the classical monotonicity identity for varifolds with bounded first variation guarantees (see e.g. [11, Lemma 40.5]) the existence and finiteness of the k-density and of tangent cones just ||V||-a.e.. Since ||V|| (μ) = 0 may hold true, it was in principle possible that Θ^{k}(|V||, x) = +∞ for every point in ∂M ∩ supp||V||, and this would stop our analysis. To our knowledge, no better results in this direction were known.

To overcome this difficulty, in section 5 we study more carefully the set of points where the k-density of V exists and is finite also when p ≤ k. More precisely we define the density set of V (Definition 5.1), denoted by Dens(V), and we prove the following result.
Theorem 1.4. Let $V \in \mathcal{V}_k(M)$ be a varifold with free boundary at $\partial M$ such that $H \in L^p(M, \|V\|)$ for some $1 \leq p \leq k$. Then

$$\mathcal{H}^{n}(M \setminus \text{Dens}(V)) = 0 \quad \forall s > k - p$$

and, for every $x_0 \in \text{Dens}(V)$ there exists an increasing function $\varphi_{x_0} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|V\|(B_{r}(x_0)) \leq \|V\|(B_{t}(x_0)) + \varphi_{x_0}(t) \quad \forall 0 < r < t, \quad \lim_{t \to 0} \varphi_{x_0}(t) = 0.$$  

In particular, the $k$-density $\Theta^k(\|V\|, x_0)$ exists and is finite for every $x_0 \in \text{Dens}(V)$; moreover, the restrictions of $\Theta^k(\|V\|, \cdot)$ to $\text{Dens}(V) \cap M^0$ and to $\text{Dens}(V) \cap \partial M$ are upper semi-continuous.

Thus the set of points where $\Theta^k(\|V\|, \cdot)$ is infinite or does not exist has Hausdorff dimension at most $k - p$. $\text{Dens}(V)$ is the “good set” where we are able to study the blow-ups of $\|V\|$ and $\sigma^V$, to conclude the proof of Theorem 1.2.

Besides its application in the proof of Theorem 1.2 Theorem 1.4 is also interesting in itself, since it is a natural counterpart of the similar result for the set of Lebesgue points of a Sobolev function proved by Federer and Ziemer in [8]: if $f \in W^{1,p}(M)$ for $p \in [1, n]$ and if $\text{Leb}(f)$ is the set of Lebesgue points of $f$, then $M \setminus \text{Leb}(f)$ has Hausdorff dimension at most $n - p$.

1.3. Outline of the paper. In section 2 we recall the notations and the definitions used throughout the paper.

In section 3 we first prove Lemma 3.1 which concerns the behavior of $V$ on $\partial M$; next we move on to the proof of Theorem 1.1.

In section 4 we describe some consequences of Theorem 1.1, we notice that, if $k = n - 1$, $\hat{H}$ coincides with the mean curvature of $\partial M$; next we adapt Theorem 1.1 to an other class of varifolds; we move on to varifolds with free boundary, proving that they have bounded first variation, establishing some monotonicity formulae and refining Lemma 3.1 in this framework.

In section 5 we first derive a monotonicity identity for distorted balls and next we prove Theorem 1.3.

In section 6 we prove Theorem 1.2, this is obtained by the study of tangent varifolds to $V$ at points in $\text{Dens}(V)$, to prove that $\sigma^V$ satisfies the hypotheses of the Marstrand-Mattila Rectifiability Criterion.

1.4. Acknowledgments. I am grateful to Guido De Philippis for his invaluable help and to Carlo Gasparetto for the useful discussions and comments.

2. Notations

2.1. Basic notations. For a fixed orthonormal system of coordinates, we denote by $e_i$ the $i$-th coordinate unit vector. If $x \in \mathbb{R}^n$, we denote by $B_r(x)$ the closed ball with center $x$ and radius $r$ and by $\omega_n$ the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Moreover we set $B_r := B_r(0)$. $c$ and $c'$ are generic positive constants, unless otherwise specified. If $v \in \mathbb{R}^n$ we call $L_v$ the translation

$$L_v : x \mapsto x + v.$$ 

For each $A \subset \mathbb{R}^n$, we denote by $1_A$ the indicator function of $A$, by $\overline{A}$ and $A^0$ respectively the closure and the interior of $A$ in the euclidean topology. If $A \subset \mathbb{R}^n$, and $r > 0$ we write $U_r(A)$ for the tubular neighborhood of $A$, i.e.

$$U_r(A) = \bigcup_{x \in A} B_r(x).$$

We denote by $S$ a generic $k$-dimensional linear subspace (or $k$-plane) of $\mathbb{R}^n$ and we write $S^\perp$ for the orthogonal complement of $S$ in $\mathbb{R}^n$. We denote by $P_S$ the orthogonal projection on $S$. If $X$ is a $C^1$ vector field, we call $\text{div}_S X$ the scalar product $P_S \cdot DX$. If $\tau_1, \ldots, \tau_k$ is an orthonormal basis of $S$, by simple computations one has

$$\text{div}_S X(x) = \sum_{i=1}^{k} D_{\tau_i} \langle X(x), \tau_i \rangle.$$
Throughout the paper, \( \gamma \in C^\infty([0, \infty)) \) denotes a cut-off function such that
\[
\begin{align*}
\gamma(t) &= 1 \text{ for each } t \in [0, \frac{1}{2}]; \\
\gamma(t) &= 0 \text{ for each } t \geq 1; \\
\gamma'(t) &\leq 0 \text{ and } |\gamma'(t)| \leq 3 \text{ for every } t \in \mathbb{R}.
\end{align*}
\]

For each \( r > 0 \) and \( x \in \mathbb{R}^n \) we consider the dilation map
\[
(2.2) \quad \tau_{x,r}(y) = \frac{1}{r}(y - x).
\]

If \( \Gamma \) is a \( C^1 \) \( k \)-dimensional sub-manifold in \( \mathbb{R}^n \) and \( x \in \Gamma \), we write \( T_x\Gamma \) for the tangent space to \( \Gamma \) at \( x \). We see \( T_x\Gamma \) as an immersed \( k \)-plane in \( \mathbb{R}^n \); more precisely, we see \( T_x\Gamma \) as the blow-up of \( \Gamma \) at the point \( x \). If \( \Gamma \) has non-empty boundary \( \partial \Gamma \) of class \( C^1 \) and if \( x \in \partial \Gamma \), we see \( T_x\Gamma \) as containing \( T_x\partial \Gamma \), which divide \( T_x\Gamma \) into two half-spaces. We call these two parts \( T_x^+\Gamma \) and \( T_x^-\Gamma \) and we set \( T_x^+\Gamma \) to be the blow-up at \( x \) of the interior part of \( \Gamma \).

We work on a compact domain \( M \subset \mathbb{R}^n \) with \( C^2 \) boundary \( \partial M \). We write \( N(x) \) for the exterior unit normal vector to \( \partial M \) at \( x \). In the following, \( d \) denotes the signed distance function from \( \partial M \) (with the exception of Section 5 where it denotes the distorted distance function from the origin) such that \( d > 0 \) in \( M^o \), that is
\[
(2.3) \quad d(x) = \begin{cases} 
\inf \{|x - y| \mid y \in \partial M\} & \text{if } x \in M \\
-\inf \{|x - y| \mid y \in \partial M\} & \text{if } x \in \mathbb{R}^n \setminus M.
\end{cases}
\]

Unless otherwise specified, we denote by \( R = R(M) > 0 \) a number such that \( d \) is \( C^2 \) in \( U_R(\partial M) \). Thus \( \nabla d \) exists in \( U_R(\partial M) \) and points inside \( M \).

We work with several classes of vector fields on \( M \), which we denote with the letter \( \chi \) with subscripts based on their behavior on \( \partial M \):
\[
\begin{align*}
\chi(M) &= C^1(\overline{M}, \mathbb{R}^{n+1}), \\
\chi_x(M) &= \{X \in \chi(M) \mid X(x) \in T_x\partial M, \forall x \in \partial M\}, \\
\chi_{\perp}(M) &= \{X \in \chi(M) \mid X(x) \in (T_x\partial M)^\perp, \forall x \in \partial M\}, \\
\chi_{0}(M) &= \{X \in \chi(M) \mid X(x) = 0, \forall x \in \partial M\}.
\end{align*}
\]

If \( \Gamma \) is a \( C^2 \) submanifold of \( \mathbb{R}^n \), by slight abuse of notation we write \( \chi_x(\Gamma) \) (respectively \( \chi_{0}(\Gamma) \)) for the set of compactly supported \( C^1 \) vector fields on \( \mathbb{R}^n \) that are tangent to \( \Gamma \) (respectively that vanish on \( \Gamma \)).

2.2. Measures, rectifiable sets. If \( A \subset \mathbb{R}^n \), \( \mathbb{M}(A, \mathbb{R}^m) \) is the space of \( \mathbb{R}^m \)-valued Radon measures on \( A \) and \( \mathbb{M}^+(A, \mathbb{R}^m) \) is the space of positive Radon measures on \( A \). If \( \mu \in \mathbb{M}(A, \mathbb{R}^m) \) we denote by \( |\mu| \) the total variation measure of \( \mu \). If \( B \subset \mathbb{R}^n \) is Borel, we write \( \mu, B \) for the restriction of the measure \( \mu \) to \( B \). If \( A \subset \mathbb{R}^n \) has non-empty interior, we endow \( \mathbb{M}(A, \mathbb{R}^m) \) with the weak*-topology: i.e. we say that a sequence of Radon measures \( \{\sigma_j\}_j \) converges to \( \mu \) (\( \sigma_j \rightharpoonup \mu \)) if
\[
\lim_{j \to \infty} \int_A f \, d\sigma_j = \int_A f \, d\mu \quad \forall f \in C_c(A, \mathbb{R}^m).
\]

If \( \mu \in \mathbb{M}^+(\mathbb{R}^n), x \in \mathbb{R}^n, k \in \mathbb{N} \), we define the upper and the lower \( k \)-densities of \( \mu \) at \( x \):
\[
\Theta^k(\mu, x) = \limsup_{r \to 0} \frac{\mu(B_r(x))}{r^k} \quad \Theta_k^-(\mu, x) = \liminf_{r \to 0} \frac{\mu(B_r(x))}{r^k}.
\]

If the above limits coincide, then we define the \( k \)-density of \( \mu \) at \( x \) as their common value, which we denote by \( \Theta^k(\mu, x) \). The \( k \)-singular set \( \text{Sing}^k(\mu) \) is defined as
\[
\text{Sing}^k(\mu) = \{x \in M \mid \Theta^k(\mu, x) = +\infty\}.
\]

If \( \mu \in \mathbb{M}(A, \mathbb{R}^m) \) and \( f : A \to \mathbb{R}^N \) is proper, we define the push-forward \( f_#\mu \) of \( \mu \) through \( f \) as the Radon measure in \( \mathbb{M}(\mathbb{R}^N, \mathbb{R}^m) \) defined by
\[
f_#\mu(B) = \mu(f^{-1}(B)) \quad \forall B \subset \mathbb{R}^N \text{ Borel}.
\]
If \( \mu \in \mathcal{M}^+(\mathbb{R}^n) \), we say that \( \nu \) is a \( k \)-blow-up of \( \mu \) at \( x \) or a \( k \)-tangent measure to \( \mu \) in \( x \) if there exists a sequence \( r_j \downarrow 0 \) such that

\[
\mu_j := \frac{1}{r_j^k} (r_{x,j})_\# \mu \xrightarrow{\ast} \nu.
\]

We denote by \( \text{Tan}^k(\mu, x) \) the (possibly empty) set of \( k \)-blow-ups of \( \mu \) at the point \( x \). If \( \Theta^k(\mu, x) < \infty \) then, by Banach-Alaoglu Theorem, \( \text{Tan}^k(\mu, x) \) is non-empty. If \( \Theta^k(\mu, x) > 0 \), then every \( k \)-blow-up of \( \mu \) at \( x \) is non-trivial; indeed, if \( \nu \in \text{Tan}^k(\mu, x) \) and \( \mu_j \xrightarrow{\ast} \nu \) as in (2.5), then

\[
\nu(B_1) \geq \limsup_j \mu_j(B_1) = \limsup_k \frac{\mu(B_{r_j}(x))}{r_j^k} \geq \Theta^k(\mu, x) > 0.
\]

For each \( s > 0 \), we denote by \( \mathcal{H}^s \) the \( s \)-dimensional Hausdorff measure and, if \( A \subset \mathbb{R}^N \), \( \mathcal{H}_{\text{dim}}(A) \) denotes the Hausdorff dimension of \( A \). We say that a Borel set \( M \subset \mathbb{R}^n \) is \( k \)-rectifiable if there exist \( M_0 \subset \mathbb{R}^n \) with \( \mathcal{H}^k(M_0) = 0 \) and a countable family of \( C^1 \) \( k \)-submanifolds \( \{M_j\}_{j=1}^\infty \) such that

\[
M \subset \bigcup_{i=0}^\infty M_j.
\]

We say that a measure \( \mu \in \mathcal{M}^+(\mathbb{R}^n) \) is \( k \)-rectifiable if there exist a \( k \)-rectifiable set \( M \) and a positive function \( \theta \in L^1_{\text{loc}}(M, \mathcal{H}^k) \) such that \( \mu = \theta \mathcal{H}^k \llcorner M \).

2.3. Varifolds. If \( 1 \leq k \leq n \) we call \( G(k, n) \) the Grassmannian of the un-oriented \( k \)-dimensional linear subspaces (or \( k \)-planes) of \( \mathbb{R}^n \). If \( A \subset \mathbb{R}^n \) we denote by \( G_k(A) := A \times G(k, n) \) the trivial Grassmannian bundle over \( A \).

A \( k \)-varifold on \( A \) is a positive Radon measure on \( G_k(A) \). We denote by \( \mathcal{V}_k(A) \) the set of all \( k \)-varifolds on \( A \) and we endow \( \mathcal{V}_k(A) \) with the topology of the weak*-convergence of Radon measures, i.e. we say that \( V_j \rightharpoonup V \) if

\[
\lim_{j \to \infty} \int_{G_k(A)} \varphi(x, S) \, dV_j(x, S) = \int_{G_k(A)} \varphi(x, S) \, dV(x, S) \quad \forall \varphi \in C_c(G_k(A)).
\]

A \( k \)-rectifiable measure \( \mu = \theta \mathcal{H}^k \llcorner M \) in \( \mathbb{R}^n \) induces the \( k \)-varifold

\[
V = \theta \mathcal{H}^k \llcorner M \otimes \delta_{T_x M},
\]

where \( T_x M \) is the approximate tangent space of \( M \) at \( x \). A varifold that is induced by a rectifiable set is called a rectifiable varifold. If the multiplicity function assumes only integer values, we say that the varifold is integer rectifiable. If \( V \) is a \( k \)-varifold on \( \Omega \), the mass \( \|V\| \) (or total variation) of \( V \) is the positive Radon measure defined as

\[
\|V\|(A) = V(G_k(A)) \quad \forall A \subset \Omega \text{ Borel.}
\]

If \( V \) is the \( k \)-varifold induced by the rectifiable measure \( \mu = \theta \mathcal{H}^k \llcorner M \), then

\[
\|V\|(B) = \int_B \theta(x) \, d\mathcal{H}^k(x).
\]

By slight abuse of notation, we often denote \( \text{supp}\|V\| \) by \( \text{supp}V \). If \( \Omega \subset \mathbb{R}^n \) is a domain, \( V \in \mathcal{V}_k(\Omega) \) and if \( \psi : \Omega \to \mathbb{R}^n \) is a diffeomorphism, the push forward \( \psi_* V \) of \( V \) through \( \psi \) is the varifold in \( \mathcal{V}_k(\psi(\Omega)) \) such that, \( \forall \varphi \in C_c(G_k(\psi(\Omega))) \),

\[
\int_{G_k(\psi(\Omega))} \varphi(y, T) \, d\psi_* V(y, T) = \int_{G_k(\Omega)} J_S \psi(x) \varphi(\psi(x), d\psi_x(S)) \, dV(x, S),
\]

where \( J_S \psi(x) \) is the Jacobian of \( \psi \) relative to the \( k \)-plane \( S \), i.e.

\[
J_S \psi(x) = \sqrt{\det ((d\psi_x)^*_S \circ ((d\psi_x)_S)).}
\]

We notice that this is not the push forward of measures previous defined (which is denoted by the different symbol \( f_\# \mu \)). In fact, the push forward of varifolds is defined in this way in order to ensure the validity of the area formula: indeed if \( V \) is induced by a rectifiable set \( M \), then \( \psi_* V \) is induced
by $\psi(M)$. If $V \in \mathcal{V}_k(\mathbb{R}^n)$, we say that $C \in \mathcal{V}_k(\mathbb{R}^n)$ is a blow-up of $V$ at $x$ or a tangent varifold to $V$ at $x$ if there exists a sequence of radii $r_j \downarrow 0$ such that

$$(\tau_{x,r_j})_t^\sim V \to C.$$

We write $\text{Tan}(V, x)$ for the set of tangent varifold to $V$ at the point $x$.

If $V \in \mathcal{V}_k(\mathcal{M})$ and if $X \in \mathcal{X}(\mathcal{M})$ the first variation $\delta V(X)$ of $V$ with respect to $X$ is

$$\delta V(X) = \frac{d}{dt} \left( \| (\psi_t)_t^\sim V \|_{\mathbb{R}^n} \right)_{t=0}$$

where $\psi_t$ is the flow map of $X$ at the time $t$. The following first variation formula holds:

$$\delta V(X) = \int_{G_k(\mathcal{M})} \text{div}_S X(x) \, dV(x, S).$$

We now define the class of varifolds with bounded first variation:

**Definition 2.1.** We say that a varifold $V$ has bounded first variation in $\mathcal{M}$ if

$$\sup \{|\delta V(X)| \mid X \in \mathcal{X}(\mathcal{M}), \max |X| \leq 1\} < +\infty.$$  

If (2.8) holds with a proper subset of $\mathcal{X}(\mathcal{M})$ (e.g. $\mathcal{X}_c(\mathcal{M})$, $\mathcal{X}_0(\mathcal{M})$...) in place of $\mathcal{X}(\mathcal{M})$, we say that $V$ has bounded first variation with respect to this subset.

Therefore $V$ has bounded first variation if $\delta V \in \mathcal{M}(\mathcal{M}, \mathbb{R}^n)$. If $V$ has bounded first variation, then by Lebesgue decomposition there exist $|\delta^s V| \in \mathcal{M}^+(\mathcal{M})$, a $|\delta^s V|$-measurable function $\eta : \mathcal{M} \to \mathbb{R}^n$ and a $\|V\|$-measurable function $H : \mathcal{M} \to \mathbb{R}^n$ such that

$$\delta V(X) = -\int_{\mathcal{M}} \langle H, X \rangle \, d\|V\| + \int_{\mathcal{M}} \langle X, \eta \rangle \, d|\delta^s V| \quad \forall X \in \mathcal{X}(\mathcal{M})$$

where $|\delta^s V|$ is the singular part of $|\delta V|$ with respect to $\|V\|:

$$|\delta^s V| = |\delta V|_k Z \quad Z = \left\{ x \in \mathcal{M} \mid \limsup_{r \to 0} \frac{|\delta V|(B_r(x))}{\|V\|(B_r(x))} = +\infty \right\}.$$ 

Since the previous formula is similar to the corresponding one for smooth surfaces, we call $H$ the generalized mean curvature of $V$, $|\delta^s V|$ the boundary measure of $V$, the set $Z$ is the boundary of $V$ and $\eta$ is the unit co-normal of $V$.

We now define the class of varifolds with generalized mean curvature:

**Definition 2.2.** We say that $V \in \mathcal{V}_k(\mathcal{M})$ has generalized mean curvature $H$ with respect to $\mathcal{X}_c(\mathcal{M})$ (respectively $\mathcal{X}_0(\mathcal{M})$, $\mathcal{X}_l(\mathcal{M})$) if for each $X \in \mathcal{X}_c(\mathcal{M})$ (respectively $\mathcal{X}_0(\mathcal{M})$, $\mathcal{X}_l(\mathcal{M})$) the following formula holds:

$$\int_{G_k(\mathcal{M})} \text{div}_S X(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle H, X \rangle \, d\|V\|.$$ 

Thus $V$ has generalized mean curvature with respect to $\mathcal{X}_c(\mathcal{M})$ (respectively $\mathcal{X}_0(\mathcal{M})$, $\mathcal{X}_l(\mathcal{M})$) if has bounded variation with respect to $\mathcal{X}_c(\mathcal{M})$ (respectively $\mathcal{X}_0(\mathcal{M})$, $\mathcal{X}_l(\mathcal{M})$) and $\delta V$ has no singular part with respect to $\|V\|$ when we test with vector fields in $\mathcal{X}_c(\mathcal{M})$ (respectively $\mathcal{X}_0(\mathcal{M})$, $\mathcal{X}_l(\mathcal{M})$).

**Definition 2.3** (Varifold with free boundary). If $V \in \mathcal{V}_k(\mathcal{M})$ has generalized mean curvature with respect to $\mathcal{X}_l(\mathcal{M})$, we say that $V$ has free boundary at $\partial \mathcal{M}$.

As we mentioned in the introduction, a varifold with free boundary meets $\partial \mathcal{M}$ orthogonally in a weak sense: indeed, if $V$ is the varifold induced by a surface $\Sigma$ that is smooth up to $\partial \mathcal{M}$, (2.9) implies that the conormal $\eta$ to $\Sigma$ is orthogonal to $\partial \mathcal{M}$. Since in the definition of varifold with free boundary we test only with vector fields that are tangent to $\partial \mathcal{M}$, we assume that the mean curvature vector $H$ is tangent to $\partial \mathcal{M}$. 

3. Proof of Theorem 1.1

3.1. Constancy Lemma. The proof of the Theorem 1.1 is based on the following lemma, which is a form of the Constancy Theorem [11, Theorem 41.1] with weaker hypotheses (and conclusion) and it is interesting in itself. The result is used in the proof of the Theorem 1.1 to deal with the case \( \|V\| \langle \partial M \rangle > 0 \).

**Lemma 3.1.** Let \( \Gamma \subset \mathbb{R}^n \) be a \( C^2 \)-hypersurface without boundary, let \( V \in \mathcal{V}_k(\mathbb{R}^n) \) have generalized mean curvature \( H \) with respect to \( X_0(\Gamma) \) and \( H \in L^1_{\text{loc}}(\mathbb{R}^n, \|V\|) \). Then

\[
V \{ \{(x, S) \in G_k(\mathbb{R}^n) \mid x \in \Gamma, S \not\subset T_\gamma \Gamma \} \} = 0.
\]

**Proof.** By a simple covering argument it is enough to prove the result locally: that is, that, for each \( x_0 \in \Gamma \), there exists \( r = r(x_0) > 0 \) and a ball \( B_r(x_0) \) such that

\[
V \{ \{(x, S) \in G_k(\mathbb{R}^n) \mid x \in \Gamma \cap B_r(x_0), S \not\subset T_\gamma \Gamma \} \} = 0.
\]

We fix \( x_0 \in \Gamma \) and without loss of generality we can assume that \( x_0 = 0 \).

Since \( \Gamma \) is locally-orientable, there exists \( r' > 0 \) and a ball \( B_{r'} \) such that \( B_{r'} \setminus \Gamma \) is made of two connected components \( D^+ \) and \( D^- \) separated by \( \Gamma \). Only in this proof, \( d \) denotes a fixed one of the two signed distance function from \( \Gamma \) in \( B_{r'} \), that is

\[
d(x) = \begin{cases} 
\inf \{|x - y| \mid y \in \Gamma\} & \text{if } x \in D^+ \\
-\inf \{|x - y| \mid y \in \Gamma\} & \text{if } x \in D^-.
\end{cases}
\]

Since \( \Gamma \) is of class \( C^2 \), there exists \( r \leq r' \) such that \( d \in C^2(\overline{B_r}) \). We set \( r(x_0) = r \).

We recall that \( \gamma \) is the cut-off function defined in section 2. Since \( \nabla d(x) \) is orthogonal to \( \Gamma \) for every \( x \in \Gamma \), we have \( |P_S \nabla d(x)|^2 = 0 \) if and only if \( S \subset T_\gamma \Gamma \). Therefore, to get the conclusion, it is enough to prove that

\[
\int_{G_k(\mathbb{R}^n)} \gamma \left( \frac{|x|}{r} \right) |P_S \nabla d(x)|^2 \, dV(x, S) = 0.
\]

To do so, we test (2.9) with a suitable vector field \( X \in \mathcal{X}_0(\mathcal{M}) \). If \( \rho < r \), we choose \( X(x) = d(x) \gamma \left( \frac{|x|}{\rho} \right) \nabla d(x) \). We clearly have \( X \in \mathcal{X}_0(\mathcal{M}) \) and

\[
\text{div}_S X(x) = \gamma \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) |P_S \nabla d(x)|^2 + \frac{d(x)}{\rho} \gamma' \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) \langle \frac{x}{|x|} , P_S \nabla d(x) \rangle + \frac{d(x)}{\rho} \gamma' \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) |P_S \nabla d(x)|^2 + d(x) \gamma \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) \text{div}_S \nabla d(x).
\]

Since \( V \) has generalized mean curvature with respect to \( X_0(\Gamma) \), testing (2.9) with \( X \) we obtain

\[
\left| \int_{G_k(\mathbb{R}^n)} \gamma \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) |P_S \nabla d(x)|^2 \, dV(x, S) \right| \\
\leq \left| \int_{G_k(\mathbb{R}^n)} \frac{d(x)}{\rho} \gamma' \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) \langle \frac{x}{|x|} , P_S \nabla d(x) \rangle \, dV(x, S) \right| \\
+ \left| \int_{G_k(\mathbb{R}^n)} \frac{d(x)}{\rho} \gamma \left( \frac{|x|}{r} \right) \gamma' \left( \frac{d(x)}{\rho} \right) |P_S \nabla d(x)|^2 \, dV(x, S) \right| \\
+ \left| \int_{G_k(\mathbb{R}^n)} d(x) \gamma \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) \text{div}_S \nabla d(x) \, dV(x, S) \right| \\
+ \left| \int_{\mathbb{R}^n} (X(x), H(x)) \, d\|V\|(x) \right|
\]

For the left-hand side of the above inequality, if we let \( \rho \to 0 \), by dominated convergence we have

\[
\lim_{\rho \to 0} \int_{G_k(\mathbb{R}^n)} \gamma \left( \frac{|x|}{r} \right) \gamma \left( \frac{d(x)}{\rho} \right) |P_S \nabla d(x)|^2 \, dV(x, S) = \int_{G_k(\Gamma)} \gamma \left( \frac{|x|}{r} \right) |P_S \nabla d(x)|^2 \, dV(x, S).
\]
Therefore, to show (3.1), we have to prove that the terms on the right-hand side of (3.2) go to 0 as \( \rho \to 0 \):

1. Since \( \gamma' \) is bounded and \( d(x) \leq \rho \) by cut-off, for the first term we have

\[
\lim_{\rho \to 0} \left| \int_{G_{k}(\mathbb{R}^n)} \frac{d(x)}{r} \gamma \left( \frac{|x|}{r} \right) \frac{d(x)}{\rho} \gamma \left( \frac{d(x)}{\rho} \right) \left( x, P_{S} \nabla d(x) \right) dV(x, S) \right| = 0.
\]

2. Since \( |\frac{d}{\rho}| \leq 1 \) and \( \gamma'(s) \neq 0 \) only if \( s \in (1/2, 1) \), for the second term we have

\[
\lim_{\rho \to 0} \int_{G_{k}(\mathbb{R}^n)} \frac{d(x)}{r} \gamma \left( \frac{|x|}{r} \right) \frac{d(x)}{\rho} \gamma \left( \frac{d(x)}{\rho} \right) |P_{S} \nabla d(x)|^2 dV(x, S) = 0.
\]

3. By the choice of \( r \) we have that \( \text{div}_S \nabla d(x) \leq c \) in \( B_r \); thus

\[
\lim_{\rho \to 0} \int_{G_{k}(\mathbb{R}^n)} \frac{d(x)}{r} \gamma \left( \frac{|x|}{r} \right) \frac{d(x)}{\rho} \gamma \left( \frac{d(x)}{\rho} \right) \text{div} \nabla d(x) dV(x, S) = 0.
\]

4. For the last term, since \( H \in L^1(\mathbb{R}^n, ||V||) \), it holds

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^n} (X(x), H(x)) d||V||(x) = 0.
\]

This completes the proof. \( \square \)

3.2. Proof of Theorem 1.1.1 We can now prove Theorem 1.1.1.

Proof of Theorem 1.1.1. Let us fix \( R > 0 \) (as in section 2) so that the distance function \( d \) from \( \partial M \) defined in (2.23) is of class \( C^2 \) in \( U_R(\partial M) \).

In what follows we are going to repeatedly use the decomposition of a vector field \( X \in \mathfrak{X}(\mathcal{M}) \) we now present; within \( U_R(\partial M) \), we can decompose \( X \) in its normal and tangent component: there exists a scalar function \( \chi(x) \) such that \( X = X^\perp + X^T \) with \( X^\perp(x) = \chi(x) \nabla d(x) \) and \( \langle X^T(x), \nabla d(x) \rangle = 0 \) for all \( x \in U_R(\partial M) \).

Step 1: We begin by cut-offing a vector field in its “interior” and “boundary” part. For every \( X \in \mathfrak{X}_\perp(\mathcal{M}) \) and \( \rho < R \) one has

\[
X(x) = \gamma \left( \frac{d(x)}{\rho} \right) X(x) + \left( 1 - \gamma \left( \frac{d(x)}{\rho} \right) \right) X(x).
\]

Therefore

\[
\int_{G_{k}(\mathcal{M})} \text{div}_S X(x) dV(x, S) = \int_{G_{k}(\mathcal{M})} \text{div}_S \left[ \gamma \left( \frac{d(x)}{\rho} \right) X(x) \right] dV(x, S) + \int_{G_{k}(\mathcal{M})} \text{div}_S \left[ X(x) \right] dV(x, S).
\]

Thus we have splitted \( X \) in the “interior” and the “boundary part” by cut-offing with \( \gamma(d/\rho) \) and the idea is to send \( \rho \to 0 \).

Step 2: For the interior part, Since \( 1 - \gamma \left( \frac{d(x)}{\rho} \right) X(x) \in \mathfrak{X}_c(\mathcal{M}) \) (that is it is compactly supported in the interior of \( \mathcal{M} \), by (2.9) and dominated convergence we have

\[
\lim_{\rho \to 0} \int_{G_{k}(\mathcal{M})} \text{div}_S \left[ \left( 1 - \gamma \left( \frac{d(x)}{\rho} \right) \right) X(x) \right] dV(x, S)
\]

\[
= \lim_{\rho \to 0} \int_{\mathcal{M}} \left( 1 - \gamma \left( \frac{d(x)}{\rho} \right) \right) \langle X(x), H(x) \rangle d||V||(x)
\]

\[
= - \int_{\mathcal{M}} \langle X(x), H(x) \rangle d||V||(x).
\]
Step 3: Since the limit in (3.4) exists, also for the “boundary part” of $X$ the limit

$$
\lim_{\rho \to 0} \int_{G_k(M)} \text{div}_S \left[ \gamma \left( \frac{d(x)}{\rho} \right) X(x) \right] dV(x, S)
$$

exists. We want now to compute (3.5) and to show the existence of $\tilde{H}$ and $\sigma_V$ on $\partial M$. We begin by writing

$$
\int \text{div}_S \left[ \gamma \left( \frac{d(x)}{\rho} \right) X(x) \right] dV(x, S) = \int \gamma \left( \frac{d(x)}{\rho} \right) \text{div}_S X(x) dV(x, S)
$$

(3.6)

$$
+ \int \frac{1}{\rho} \gamma \left( \frac{d(x)}{\rho} \right) \langle P_S \nabla d(x), X(x) \rangle dV(x, S).
$$

Step 4: At this point, we want to study separately the limit as $\rho \to 0$ of each term in the right-hand side of the above equation.

- For the first term of the right-hand side in (3.6), we expect that, as $\rho \to 0$, it will give a sort of mean curvature of $\partial M$. This expectation is justified by the fact that on $\partial M$ $V$ charges only planes that are tangent to $\partial M$ by Lemma 3.1 and because $X$ is orthogonal to $\partial M$ on $\partial M$. More precisely, we are going to prove that there exists a $\|V\|$-measurable vector field $\tilde{H}$ such that

$$
\lim_{\rho \to 0} \int_{G_k(M)} \gamma \left( \frac{d(x)}{\rho} \right) \text{div}_S X(x) dV(x, S) = - \int_{\partial M} \langle \tilde{H}(x), X(x) \rangle d\|V\|(x),
$$

(3.7)

which is an extension of $H$ on $\partial M$ and is orthogonal to $\partial M$ for $\|V\|$-a.e. $x \in \partial M$. To this aim, we first observe that by dominated convergence we have

$$
\lim_{\rho \to 0} \int_{G_k(M)} \gamma \left( \frac{d(x)}{\rho} \right) \text{div}_S X(x) dV(x, S) = \int_{G_k(\partial M)} \text{div}_S X(x) dV(x, S)
$$

(3.8)

We now have to compute the right-hand side of (3.5). To do this, we decompose $\text{div}_S X(x) = \text{div}_S X^\perp(x) + \text{div}_S X^T(x)$. For the tangent part, we claim that

$$
\text{div}_S X^T(x) = 0 \quad \forall x \in \partial M, \forall S \subset T_x \partial M.
$$

This is true since $X^T \equiv 0$ on $\partial M$, therefore $D\tau \langle X(x), \tau \rangle = 0$ for each $\tau \in T_x \partial M$. Thus the claim follows by definition of tangential divergence (2.4). Since for $x \in \partial M$ by Lemma 3.1 $V$ charges only planes $S \subset T_x \partial M$, we have

$$
\int_{G_k(\partial M)} \text{div}_S X^T(x) dV(x, S) = 0.
$$

(3.9)

For the orthogonal component we get

$$
\text{div}_S X^\perp(x) = \langle \nabla \chi, P_S \nabla d(x) \rangle + \chi(x) \text{div}_S \nabla d(x).
$$

Since $\nabla d(x)$ is orthogonal to $\partial M$, by Lemma 3.1 again we have $\langle \nabla \chi(x), P_S \nabla d(x) \rangle = 0$ for $V$-a.e. $(x, S) \in G_k(\partial M)$. Hence

$$
\int_{G_k(\partial M)} \langle P_S \nabla \chi(x), \nabla d(x) \rangle dV(x, S) = 0.
$$

Since $\nabla d(x) = -N(x)$ for every $x \in \partial M$, where $N(x)$ is the unit normal vector to $\partial M$ at $x$, we obtain

$$
\int_{G_k(\partial M)} \chi(x) \text{div}_S \nabla d(x) dV(x, S) = \int_{G_k(\partial M)} \langle X(x), N(x) \rangle \text{div}_S N(x) dV(x, S)
$$

(3.10)

Thus, combining (3.9)–(3.10), we obtain

$$
\int_{G_k(\partial M)} \text{div}_S X(x) dV(x, S) = \int_{G_k(\partial M)} \langle X(x), N(x) \rangle \text{div}_S N(x) dV(x, S).
$$

(3.11)

We are now going to define $\tilde{H}$ and write the last integral in terms of it. To do so, by disintegration of $V$ we can write

$$
V = \|V\| \otimes \sigma_x
$$
where $\sigma_x$ is a probability measure on $G(k, n)$ for $\|V\|$-a.e. $x$. Hence the right-hand side of (3.11) can be written as

$$\int_{G_k(\partial M)} \langle X(x), N(x) \rangle \, \text{div}_S N(x) \, dV(x, S)$$

$$= \int_{\partial M} \langle X(x), N(x) \rangle \left( \int_{G(k,n)} \text{div}_S N(x) \, d\sigma_x(S) \right) \, d\|V\|(x).$$

If we define

$$(3.12) \quad \hat{H}(x) := -N(x) \int_{G(k,n)} \text{div}_S N(x) \, d\sigma_x(S) \quad \text{for } \|V\|$-a.e. $x \in \partial M,$

we can write (3.11) as

$$(3.13) \quad \int_{G_k(\partial M)} \text{div}_S X(x) \, dV(x, S) = -\int_{\partial M} \langle \hat{H}(x), X(x) \rangle \, d\|V\|(x).$$

Loosely speaking, $\hat{H}(x)$ can be interpreted as the “mean curvature of $\partial M$ weighted according to the planes charged by $V$ at $x$”; in fact, in co-dimension 1, $\hat{H}$ turns out to be precisely the mean curvature of $\partial M$ (see Corollary 4.1). By its definition, it is clear that $\hat{H}$ is orthogonal to $\partial M$ for $\|V\|$-a.e. $x \in \partial M$, that $\hat{H} \in L^\infty(\partial M, \|V\|)$ and that $\|\hat{H}\|_{L^\infty(\partial M, \|V\|)}$ depends only on the second fundamental form of $\partial M$. Gathering (3.13) and (3.14) we finally get (3.7).

- We now have to study the second term in the right-hand side of (3.6). Roughly speaking, we can see it as “the mean orthogonal part to $\partial M$ of $V$” on the tubular neighborhood $U_\rho(\partial M)$.

When $\rho \to 0$, we expect that this term takes into account the singular part of the first variation of $V$ on $\partial M$.

More precisely, we are going to show the existence of a positive Radon measure $\sigma_V$ (as expressed in the statement of the theorem), such that

$$(3.14) \quad \lim_{\rho \to 0} \frac{1}{\rho} \int_{G_k(\partial M)} \frac{\gamma'(d(x)/\rho)}{\rho} \langle P_S \nabla d(x), X(x) \rangle \, dV(x, S) = \int_{\partial M} \langle X(x), N(x) \rangle \, d\sigma_V(x),$$

where $N(x)$ is the exterior unit normal vector to $\partial M$. We first of all remark that the above limit exists by the existence of limits of the other two terms in (3.13).

To compute the limit in (3.14), we use again the decomposition $X = X^T + X^\perp = X^T + \nabla^d$. As $\rho \to 0$, we expect that the contribution of $X^T$ is zero. Indeed, since $X \in \mathcal{X}_\perp(\mathcal{M})$ and $X$ is of class $C^1$, there exists a constant $c > 0$ such that $\|X^T(x)\| \leq c d(x)$. Therefore, since $\gamma'(s) \neq 0$ only for $s \in (1/2, 1)$, we obtain

$$(3.15) \quad \lim_{\rho \to 0} \frac{1}{\rho} \int_{G_k(\partial M)} \left| \gamma'(d(x)/\rho) \langle P_S \nabla d(x), X^T(x) \rangle \right| \, dV(x, S) \leq \lim_{\rho \to 0} 3c \|V\| \left( U_\rho(\partial M) \setminus U_{\rho/2}(\partial M) \right) = 0.$$

This proves the existence of the limit for the orthogonal part

$$(3.16) \quad \lim_{\rho \to 0} \frac{1}{\rho} \int_{G_k(\partial M)} \gamma'(d(x)/\rho) \langle P_S \nabla d(x), X^\perp(x) \rangle \, dV(x, S)$$

$$= \lim_{\rho \to 0} \frac{1}{\rho} \int_{G_k(\partial M)} \chi(x) \gamma'(d(x)/\rho) \|P_S \nabla d(x)\|^2 \, dV(x, S).$$

(3.15) and (3.16) yield

$$(3.17) \quad \lim_{\rho \to 0} \frac{1}{\rho} \gamma'(d(x)/\rho) \langle P_S \nabla d(x), X(x) \rangle \, dV(x, S)$$

$$= \lim_{\rho \to 0} \frac{1}{\rho} \int_{G_k(\partial M)} \chi(x) \gamma'(d(x)/\rho) \|P_S \nabla d(x)\|^2 \, dV(x, S).$$
We now observe that the map
\[
T: X \in \mathcal{H}(\mathcal{M}) \mapsto \lim_{\rho \to 0} \frac{1}{\rho} \int_{G_{k}(\mathcal{M})} \chi(x)\gamma'\left(\frac{d(x)}{\rho}\right)|P_{S}V d(x)|^{2} dV(x, S).
\]
is a well-defined distribution and, by its definition, supp\(T \subseteq \partial \mathcal{M}\). Again by definition, if \(\chi(x) \leq 0\) for every \(x \in \partial \mathcal{M}\) (i.e. if \(T\) point outward \(\partial \mathcal{M}\)), then \(T(X) \geq 0\). Therefore \(T\) is a signed distribution and by Riesz Representation Theorem there exists a positive Radon measure \(\sigma_{V}\) such that
\[
\langle T, X \rangle = \int_{\partial \mathcal{M}} \langle X, N \rangle d\sigma_{V} \quad \forall X \in \mathcal{H}(\mathcal{M}).
\]
This completes the proof of (3.14).

**Step 5:** We now gather the previous computations to get (1.5).

By (3.16), (3.7) and (3.14), we can rewrite (3.16) as
\[
\lim_{\rho \to 0} \int_{G_{k}(\mathcal{M})} \text{div}_{S} \left[ \gamma\left(\frac{d(x)}{\rho}\right)X(x)\right] dV(x, S) = -\int_{\partial \mathcal{M}} \langle \tilde{H}(x), X(x) \rangle d\|V\|(x) + \int_{\partial \mathcal{M}} \langle X, N \rangle d\sigma_{V}.
\]
Going back to (3.3), by (3.4) and (3.18) we finally get
\[
\int_{G_{k}(\mathcal{M})} \text{div}_{S} X(x) dV(x, S) = -\int_{\mathcal{M}} \langle X(x), H(x) + \tilde{H}(x) \rangle d\|V\|(x) + \int_{\partial \mathcal{M}} \langle X, N \rangle d\sigma_{V}.
\]
which completes the proof of (1.5).

**Step 6:** We are left with the proof of (1.6). We fix \(x_{0} \in \partial \mathcal{M}\) and \(r \leq R\). Without loss of generality we can assume that \(x_{0} = 0\). To prove the estimate, we test (1.5) with \(X(x) = -\gamma\left(\frac{|x|}{r}\right)\nabla d(x)\) which clearly belongs to \(\mathcal{H}(\mathcal{M})\). We have
\[
\text{div}_{S} X(x) = -\gamma'\left(\frac{|x|}{r}\right)\frac{1}{r}P_{S}\frac{x}{|x|} \nabla d(x) - \gamma\left(\frac{|x|}{r}\right) \text{div}_{S} \nabla d(x).
\]
Therefore
\[
\sigma_{V}(B_{r/2}(x)) \leq \int_{\partial \mathcal{M}} \gamma\left(\frac{|x|}{r}\right) d\sigma_{V}
\]
\[
= \int_{\partial \mathcal{M}} \langle X, N \rangle d\sigma_{V}
\]
\[
= \int_{G_{k}(\mathcal{M})} \text{div}_{S} X(x) dV(x, S) + \int_{\mathcal{M}} \langle X, H + \tilde{H} \rangle d\|V\|
\]
\[
= \int_{G_{k}(\mathcal{M})} \left[ -\gamma'\left(\frac{|x|}{r}\right)\frac{1}{r}P_{S}\frac{x}{|x|} \nabla d(x) - \gamma\left(\frac{|x|}{r}\right) \text{div}_{S} \nabla d(x) \right] dV(x, S)
\]
\[
- \int_{\mathcal{M}} \gamma\left(\frac{|x|}{r}\right)\langle \nabla d(x), H(x) + \tilde{H}(x) \rangle d\|V\|(x)
\]
We want to estimate the last member of the above inequality. To do so, we choose a constant \(c = c(\partial \mathcal{M}, R)\) such that
\[
|\text{div}_{S} \nabla d(x)| \leq c \quad \forall x \in U_{R}(\partial \mathcal{M}).
\]
By (3.12), the choice of \(c\) provides also \(|\tilde{H}(x)| \leq c\) for \(\|V\|\)-a.e. \(x \in \partial \mathcal{M}\). Thus
\[
- \int_{G_{k}(\mathcal{M})} \gamma\left(\frac{|x|}{r}\right) \text{div}_{S} \nabla d(x) dV(x, S) - \int_{\mathcal{M}} \gamma\left(\frac{|x|}{r}\right)\langle \nabla d(x), H(x) + \tilde{H}(x) \rangle d\|V\|(x)
\]
\[
\leq \int_{B_{r}(x)} (c + |H(x)|) d\|V\|(x).
\]
Substituting in (3.19) and since we can modify \(\gamma\) so that \(\|\gamma'\|_{\infty}\) is arbitrarily close to 2, we get
\[
\sigma_{V}(B_{r/2}(x)) \leq \frac{2}{r}\|V\|(B_{r}(x)) + \int_{B_{r}(x)} (c + |H(x)|) d\|V\|(x).
\]
4. Consequences of Theorem 1.1

In this section we clarify some consequences of Theorem 1.1.

- In subsection 4.1 we prove that if $k = n - 1$, then the vector field $\tilde{H}$ of Theorem 1.1 is the mean curvature vector of $\partial M$ (Corollary 4.1).
- In subsection 4.2 we extend Theorem 1.1 to varifolds with generalized mean curvature with respect to $\mathcal{X}_0(M)$, i.e. vector fields compactly supported in the interior of $\mathcal{M}$ (see (2.1)), assuming that $\|V\|(\partial M) = 0$ (Corollary 4.2). As a consequence, such varifolds have generalized mean curvature with respect to the larger class of vector fields $\mathcal{X}_0(M)$.
- In subsection 4.3 we show that varifolds with free boundary have bounded first variation (Corollary 4.4); next we prove a monotonicity formula at points lying on $\partial M$ (Corollary 4.5) without the reflections used by Gr"uter and Jost in [9]. Lastly, we prove a refined version of Lemma 3.1 for varifolds with free boundary with codimension 1: if $k = n - 1$, then the restriction of $V$ to $\partial M$ is $(n-1)$-rectifiable (Corollary 4.7).

4.1. Case $k = n - 1$. If $k = n - 1$, then we can characterize $\tilde{H}$ in a simpler way: if $x \in \partial M$, then $\tilde{H}(x)$ is the mean curvature vector of $\partial M$.

**Corollary 4.1.** Let $V \in \mathcal{V}_{n-1}(\mathcal{M})$ has generalized mean curvature $H$ with respect to $\mathcal{X}_0(M)$ with $H \in L^1(\mathcal{M}, \|V\|)$. Then there exists a positive Radon measure $\sigma_V$ on $\partial \mathcal{M}$ such that

$$\int_{G_{n-1}(\partial \mathcal{M})} \text{div}_S X(x) \, dV(x, S) = - \int_{\mathcal{M}} (X, H + \tilde{H}) \, d\|V\| + \int_{\partial \mathcal{M}} (X, N) \, d\sigma_V \quad \forall X \in \mathcal{X}_\perp(\mathcal{M})$$

where $\tilde{H}$ is the mean curvature vector of $\partial \mathcal{M}$, that is $\tilde{H}(x) := -N(x)\big(\text{div}_{T_x\partial \mathcal{M}} N(x)\big)$ for $x \in \partial \mathcal{M}$. Moreover, (1.6) holds true.

**Proof.** If $S$ is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, then $S \subset T_x\partial \mathcal{M}$ if and only if $S = T_x\partial \mathcal{M}$. Therefore, if $k = n - 1$ and $V \in \mathcal{V}_{n-1}(\mathcal{M})$ with generalized mean curvature with respect to $\mathcal{X}_0(M)$, Lemma 3.1 yields

$$(4.1) \quad V\left(\{(x, S) \in G_{n-1}(\partial \mathcal{M}) \mid S \neq T_x\Gamma\}\right) = 0.$$

Hence, for $X \in \mathcal{X}_\perp(\mathcal{M})$,

$$\int_{G_{n-1}(\partial \mathcal{M})} X(x) \, \text{div}_S N(x) \, dV(x, S) = \int_{\partial \mathcal{M}} X(x) \, \text{div}_{T_x\partial \mathcal{M}} N(x) \, d\|V\|,$$

thus (3.11) becomes

$$\int_{G_{n-1}(\partial \mathcal{M})} \text{div}_S X(x) \, dV(x, S) = \int_{\partial \mathcal{M}} \langle N(x), X(x) \rangle \, \text{div}_{T_x\partial \mathcal{M}} N(x) \, d\|V\|(x).$$

Thus, defining

$$\tilde{H}(x) := -N(x)\text{div}_{T_x\partial \mathcal{M}} N(x),$$

We get (3.13). \qed

4.2. Varifolds with mean curvature with respect to $\mathcal{X}_c(M)$. The analogous of Theorem 1.1 holds, adding the extra hypothesis $\|V\|(\partial \mathcal{M}) = 0$, even if $V$ has generalized mean curvature with respect to $\mathcal{X}_c(M)$, i.e. the vector fields with compact support in the interior of $\mathcal{M}$. In fact, if we analyze the proof of Theorem 1.1 we can see that the only point where we used the existence of generalized mean curvature with respect $\mathcal{X}_0(M)$ is to obtain (3.11), that is to characterize

$$\int_{G_k(\partial \mathcal{M})} \text{div}_S X(x) \, dV(x, S) = - \int_{G_k(\partial \mathcal{M})} (X(x), N(x)) \, \text{div}_S N(x) \, dV(x, S)$$

by the use of Lemma 3.1 whereas if $\|V\|(\partial \mathcal{M}) = 0$ then obviously we have

$$\int_{G_k(\partial \mathcal{M})} \text{div}_S X(x) \, dV(x, S) = 0.$$
Since the remaining arguments remain valid also if $V$ has mean curvature with respect to $\mathcal{X}_c(\mathcal{M})$, we have proved the following corollary.

**Corollary 4.2.** Let $V \in \mathcal{V}_k(\mathcal{M})$ with generalized mean curvature $H$ with respect to $\mathcal{X}_c(\mathcal{M})$ with $H \in L^1(\mathcal{M}, \|V\|)$ and $\|V\|(\partial \mathcal{M}) = 0$. Then there exists a positive Radon measure $\sigma_V$ on $\partial \mathcal{M}$ such that

\[
\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle X, H \rangle \, d\|V\| + \int_{\partial \mathcal{M}} \langle X, N \rangle \, d\sigma_V \quad \forall X \in \mathcal{X}_\perp(\mathcal{M}),
\]

In particular, $V$ has bounded first variation with respect to $\mathcal{X}_\perp(\mathcal{M})$ and the estimate (1.6) on $\sigma_V$ holds true.

**Remark 4.1.** If we remove the hypothesis $\|V\|(\partial \mathcal{M}) = 0$ nothing can be said about the behavior of $V$ on $\partial \mathcal{M}$, because any vector field in $\mathcal{X}_c(\mathcal{M})$ has first derivatives compactly supported in the interior of $\mathcal{M}$. So we have a lack of test vector fields to establish any property of $V$ on $\partial \mathcal{M}$: e.g. take a smooth surface in $\mathcal{M}^0$ and add any varifold $W \in \mathcal{V}_k(\partial \mathcal{M})$ with unbounded first variation with respect $\mathcal{X}_\perp(\mathcal{M})$.

Since $\mathcal{X}_0(\mathcal{M}) \subset \mathcal{X}_\perp(\mathcal{M})$, the following result follows.

**Corollary 4.3.** Let $V \in \mathcal{V}_k(\mathcal{M})$ with generalized mean curvature $H$ with respect to $\mathcal{X}_c(\mathcal{M})$, with $H \in L^1(\mathcal{M}, \|V\|)$ and $\|V\|(\partial \mathcal{M}) = 0$. Then $V$ has generalized mean curvature $H$ with respect to $\mathcal{X}_0(\mathcal{M})$.

### 4.3. Varifolds with free boundaries.

As an immediate corollary of Theorem [1.1] varifolds with free boundaries have bounded first variation.

**Corollary 4.4.** Let $V \in \mathcal{V}_k(\mathcal{M})$ with free boundary at $\partial \mathcal{M}$ and such that $H \in L^1(\mathcal{M}, \|V\|)$. Then $V$ has bounded first variation. More precisely, Then there exists a positive Radon measure $\sigma_V$ on $\partial \mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ such that

\[
\int_{G_k(\mathcal{M})} \operatorname{div}_S X(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle X, H + \tilde{H} \rangle \, d\|V\| + \int_{\partial \mathcal{M}} \langle X, N \rangle \, d\sigma_V \quad \forall X \in \mathcal{X}(\mathcal{M}),
\]

where $\tilde{H}$ is defined as in (3.12). In particular $\tilde{H}$ is orthogonal to $\partial \mathcal{M}$, $\tilde{H} \in L^\infty(\partial \mathcal{M}, \|V\|)$ and $\|\tilde{H}\|_\infty$ depends on the second fundamental form of $\partial \mathcal{M}$. Moreover, (1.6) holds true.

**Proof.** If $X \in \mathcal{X}(\mathcal{M})$, there exist $X^T, X^\perp$ such that $X = X^T + X^\perp$, with $X^T \in \mathcal{X}_c(\mathcal{M})$ and $X^\perp \in \mathcal{X}_\perp(\mathcal{M})$ (see at the beginning of the proof of Theorem [1.1]). We have

\[
\operatorname{div}_S X(x) = \operatorname{div}_S X^T(x) + \operatorname{div}_S X^\perp(x).
\]

Hence, by Theorem [1.1], there exists a positive Radon measure $\sigma_V$ on $\partial \mathcal{M}$ and a $\|V\|$-measurable vector field $\tilde{H}$ such that

\[
\int_{G_k(\mathcal{M})} \operatorname{div}_S X^\perp(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle X^\perp, H + \tilde{H} \rangle \, d\|V\| + \int_{\partial \mathcal{M}} \langle X, N \rangle \, d\sigma_V.
\]

with the same estimate (1.6) on $\sigma_V$. For what concerns the tangent part $X^T$, the definition of varifold with free boundary yields

\[
\int_{G_k(\mathcal{M})} \operatorname{div}_S X^T(x) \, dV(x, S) = -\int_{\mathcal{M}} \langle X^T, H \rangle \, d\|V\|.
\]

This shows the conclusion.

Grüter and Jost established in [9] several properties of varifolds with free boundaries: monotonicity formulae for $\|V\|$ at the boundary [9, Theorem 3.1], which imply existence of $\Theta_k(\|V\|, x)$ for every point $x$.

The monotonicity results are obtained by reflecting the balls across $\partial \mathcal{M}$, i.e. they have monotonicity of the sum of the masses in the ball and in the reflected ball [9, Theorem 3.1]. Using Corollary [4.4] it is possible to obtain the monotonicity of the mass in $B_r(x)$, without reflecting the balls.
Corollary 4.5. Let $V \in \mathcal{V}_k(M)$ with free boundary at $\partial M$, with $H \in L^p(M, \|V\|)$ for some $p \in (k, +\infty)$. Then there exists $\Lambda = \Lambda(k, p, M, \|H\|_{L^p}) > 0$ such that, for all $x_0 \in \partial M$ the function

$$
\rho \mapsto e^{\Lambda \rho} \left( \frac{\|V\|(B_\rho(x_0))}{\rho^k} \right)^{\frac{1}{p}} + \Lambda e^{\Lambda \rho} \rho^{-\frac{1}{p}}
$$

is monotone increasing.

Proof. Without loss of generality we can suppose $x_0 = 0$. Since for large $\rho$ the statement is obvious, we have to prove it only for $0 < \rho < R(M)$, where $R(M)$ is defined in section 2. We want to bound from below the following derivative:

$$
\frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right) = \frac{1}{\rho^{k+1}} \int_{G_k(M)} k \gamma \left( \frac{|x|}{\rho} \right) + \left| \frac{x}{\rho} \right| \gamma' \left( \frac{|x|}{\rho} \right) dV(x, S).
$$

To do so, we want to bound from below the derivative in the right-hand side to get a differential inequality. We have

$$
\frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) \right) = -\frac{1}{\rho^{k+1}} \int_{G_k(M)} k \gamma \left( \frac{|x|}{\rho} \right) + \left| \frac{x}{\rho} \right| \gamma' \left( \frac{|x|}{\rho} \right) dV(x, S).
$$

Let us choose $X(x) = \gamma \left( \frac{|x|}{\rho} \right) x$. Then

$$
div_X X(x) = k \gamma \left( \frac{|x|}{\rho} \right) + \left| \frac{x}{\rho} \right| \gamma' \left( \frac{|x|}{\rho} \right) \left| P_S \frac{x}{|x|} \right|^2.
$$

We use Corollary 4.4 by testing (4.3) with $X$ to get the following monotonicity identity:

$$
\frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) \right) = -\frac{1}{\rho^{k+1}} \int_{G_k(M)} div_X X(x) dV(x, S)
$$

$$
= \frac{1}{\rho^{k+1}} \int_{G_k(M)} \langle X, H + \tilde{H} \rangle \|V\| - \frac{1}{\rho^{k+1}} \int_{\partial M} \langle X, N \rangle d\sigma V
$$

$$
- \frac{1}{\rho^{k+1}} \int_{G_k(M)} \left| \frac{x}{\rho} \right| \gamma' \left( \frac{|x|}{\rho} \right) \left| P_S \frac{x}{|x|} \right|^2 dV(x, S).
$$

We have to estimate from below the last member of the above identity.

Since $\gamma' \leq 0$, we can neglect the last integral and, since $|x| \leq \rho$ by cut-off, we obtain

$$
\frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) \right) \geq -\frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) |H + \tilde{H}| d\|V\|
$$

$$
- \frac{1}{\rho^k} \int_{\partial M} \gamma \left( \frac{|x|}{\rho} \right) \left| \frac{x}{|x|}, N \right| d\sigma V.
$$

We have now to bound the two terms in the right-hand side of (4.7).

- For the first one, since $H \in L^p(M, \|V\|)$ and $\tilde{H} \in L^\infty(\partial M, \|V\|)$, also $H + \tilde{H} \in L^p(M, \|V\|)$ and $|H + \tilde{H}| \leq c + |H|$ where $c$ depends on the second fundamental form of $\partial M$. Therefore

$$
\frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) |H + \tilde{H}| d\|V\| \leq \frac{1}{\rho^k} \|H + \tilde{H}\|_{L^p} \left( \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right)^{\frac{1}{p}}
$$

$$
\leq \left( \|H\|_{L^p} + c \right) \rho^{-\frac{1}{p}} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right)^{\frac{1}{p}}
$$

- We now move on the estimate of the second integral in the right-hand side of (4.7). Since $\partial M$ is of class $C^2$ and since $0 \in \partial M$, there exists a constant $c$ such that

$$
\left| \frac{x}{|x|}, N(x) \right| \leq c|x|.
$$
This yields

\[(4.10)\quad \frac{1}{\rho^{k-1}} \int_{\partial \Omega} \gamma \left(\frac{|x|}{\rho}\right) \left|\nabla \frac{x}{|x|}, N\right| \, d\sigma_V \leq \frac{c}{\rho^{k-1}} \int \gamma \left(\frac{|x|}{\rho}\right) \, d\sigma_V.\]

We have to further estimate the right-hand side of this inequality. This is done by testing \((4.3)\) with \(X(x) = -\gamma \left(\frac{|x|}{\rho}\right) \nabla d(x)\); as in \((3.19)\) we get

\[
\frac{1}{\rho^{k-1}} \int \gamma \left(\frac{|x|}{\rho}\right) \, d\sigma_V = -\frac{1}{\rho^{k-1}} \int_{G_k(\Omega)} \gamma \left(\frac{|x|}{\rho}\right) \left(\frac{1}{\rho} \langle P_S \frac{x}{|x|}, \nabla d(x) \rangle \right) \, dV(x, S)
- \frac{1}{\rho^{k-1}} \int_{G_k(\Omega)} \gamma \left(\frac{|x|}{\rho}\right) \text{div}_S \nabla d(x) \, dV(x, S)
- \frac{1}{\rho^{k-1}} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \left(\nabla d(x), H + \hat{H}\right) \, d\|V\|.
\]

Since \(\gamma'(s) = 0\) if \(s \in (0, 1/2)\), we have \(\frac{|x|}{\rho} \geq \frac{1}{2}\). Moreover, using \(|\text{div}_S \nabla d| \leq c\) (because \(\rho < R\) and \(d\) is of class \(C^2\) in \(\overline{U_R(\partial \Omega)}\)) and \((1.8)\), we obtain

\[
\frac{1}{\rho^{k-1}} \int \gamma \left(\frac{|x|}{\rho}\right) \, d\sigma_V \leq -\frac{2}{\rho^{k-1}} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \left|\langle P_S \frac{x}{|x|}, \nabla d(x) \rangle \right| \, d\|V\| + \frac{c}{\rho^{k-1}} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|(x)
+ \rho^{1 - \frac{1}{p}} \left(\|H\|_{L^p(B_\rho)} + c\right) \left(\frac{1}{\rho^k} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|\right)^{1 - \frac{1}{p}}
\]

\[(4.11)\]

Concerning the last member, we now observe that

\[
\frac{2}{\rho^{k-1}} \frac{d}{d\rho} \left(\int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|\right) - \frac{2(k - 1)}{\rho^k} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|(x)
= \frac{d}{d\rho} \left(\frac{1}{\rho^{k-1}} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|\right).
\]

Substituting in \((4.11)\) we get

\[
\frac{1}{\rho^{k-1}} \int \gamma \left(\frac{|x|}{\rho}\right) \, d\sigma_V \leq \frac{2}{\rho^{k-1}} \frac{d}{d\rho} \left(\frac{1}{\rho^{k-1}} \int \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|(x)\right)
+ \frac{2(k - 1) + c\rho}{\rho^k} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|(x)
+ \rho^{1 - \frac{1}{p}} \||H\|_{L^p(B_\rho)} \left(\frac{1}{\rho^k} \int \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|\right)^{1 - \frac{1}{p}}.
\]

Taking into account that

\[
\frac{d}{d\rho} \left(\frac{1}{\rho^{k-1}} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|\right)
= \frac{1}{\rho^k} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|(x) + \rho \frac{d}{d\rho} \left(\frac{1}{\rho^k} \int_{\mathcal{M}} \gamma \left(\frac{|x|}{\rho}\right) \, d\|V\|\right),
\]
we obtain
\[
\frac{1}{\rho^{k-1}} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\sigma_V \leq \frac{2k + c\rho}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) + 2\rho \frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) \right) \\
+ \rho^{1-\frac{k}{p}} \|H\|_{L^p(B_\rho)} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right)^{1-\frac{1}{p}}.
\]
(4.12)

This complete the estimate of the second term in the right-hand side of (4.17).

Gathering (4.18), (4.10) and (4.12) in (4.17), we can estimate (4.8) as follows:
\[
\frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right) \frac{1}{p} = \frac{d}{d\rho} \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right) \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\| \right)^{\frac{1-k}{p}} \\
\geq -\rho^{\frac{k}{p}} \left( ||H||_{L^p(B_\rho)} + c \right) - c \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) \right)^{\frac{1}{p}} \\
- c\rho^{1-\frac{k}{p}} \|H\|_{L^p(B_\rho)}.
\]

Therefore, rearranging we obtain the existence of $\Lambda > 0$, which depends on $||H||_{L^p(M)}$ and on the second fundamental form of $\partial M$, such that
\[
\frac{d}{d\rho} e^{\Lambda \rho} \left[ \left( \frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|(x) \right)^{\frac{1}{p}} + \Lambda \rho^{1-\frac{k}{p}} \right] \geq 0.
\]

Since the previous estimates are independent on the choice of $\gamma$ (unless that $\gamma'(s) = 0$ for $s \in (0, 1/2)$), letting $\gamma$ increase to $1_{[0,1]}$ we have that the function
\[
\rho \mapsto e^{\Lambda \rho} \left[ \left( \frac{||V||_{(B_\rho)} \|B_\rho(x)\|}{\rho^k} \right)^{\frac{1}{p}} + \Lambda \rho^{1-\frac{k}{p}} \right]
\]
is monotone increasing. \(\square\)

If $H \in L^\infty(M, ||V||)$ in (4.8) we simply have
\[
\frac{1}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) |H + \hat{H}| d\|V\| \leq \frac{||H||_{\infty} + c}{\rho^k} \int_M \gamma \left( \frac{|x|}{\rho} \right) d\|V\|
\]
where $c$ depends on the second fundamental form of $\partial M$. Therefore, by repeating the previous proof we have the following result:

**Corollary 4.6.** Let $V \in \mathcal{V}_k(M)$ with free boundary at $\partial M$, with $H \in L^\infty(M, ||V||)$. Then there exists $\Lambda = \Lambda(k, M, ||H||_{\infty})$ such that, for all $x \in \partial M$ the function
\[
\rho \mapsto e^{\Lambda \rho} \left[ \frac{||V||_{(B_\rho(x))}}{\rho^k} \right]^{\frac{1}{p}} + \Lambda \rho^{1-\frac{k}{p}} \]
is monotone increasing.

If $k = n - 1$ and $V$ has free boundary at $\partial M$, we can strengthen the conclusion of Lemma 3.1 $V \cup G_{n-1}(\partial M)$ is $(n - 1)$-rectifiable.

**Corollary 4.7.** Let $V \in \mathcal{V}_{n-1}(M)$ with free boundary at $\partial M$ and $H \in L^p(M, ||V||)$ for some $p \geq 1$. Then $V \cup G_{n-1}(\partial M)$ is an $(n - 1)$-rectifiable varifold. More precisely, if $\varphi \in C_c(G_{n-1}(M))$, then
\[
\int_{G_{n-1}(M)} \varphi(x, S) dV(x, S) = \int_{\partial M} \varphi(x, T_x \partial M) \theta(x) d\mathcal{H}^{n-1}(x) + \int_{G_{n-1}(M^o)} \varphi(x, S) dV(x, S),
\]
where $\omega_{n-1}(x) = \Theta^{n-1}(||V||, x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial M$. 

Proof. By \(\text{(4.1)}\), if \(\varphi \in C_c(G_{n-1}(M))\), we have
\[
\int_{G_{n-1}(M)} \varphi(x, S) dV(x, S) = \int_{\partial M} \varphi(x, T_{x}\partial M) d\|V\|(x) + \int_{G_{n-1}(M')} \varphi(x, S) dV(x, S).
\]
By \(\text{(11, Lemma 40.5)}\), for \(\|V\|\)-a.e. \(x \in \partial M\) there exists the density \(\Theta^{n-1}(\|V\|, x) < \infty\). Hence, since \(\partial M\) is of class \(C^2\), for \(\|V\|\)-a.e. \(x \in \partial M\) we define
\[
\theta(x) := \lim_{r \to 0} \frac{\|V\|(B_r(x))}{H^{n-1}\partial M(B_r(x))} = \Theta^{n-1}(\|V\|, x).
\]
By Radon-Nikodym Theorem \(\text{(11, Theorem 4.7)}\), since the singular set \(\{x \in M \mid \theta(x) = +\infty\}\) is \(\|V\|\)-negligible, the conclusion follows. \(\square\)

5. Density set and proof of Theorem 1.4

In order to prove Theorem \(\text{(1.2)}\) when \(p \in (1, k]\), we have to show that, for every varifold \(V\) with free boundary at \(\partial M\) with \(H \in L^p(M, \|V\|)\), the Hausdorff dimension of the set of points where the \(k\)-density \(\Theta^k(\|V\|, \cdot)\) does not exist is at most \(k - p\).

We introduce the definition of \(density set\) for a \(k\)-varifold \(V\).

**Definition 5.1** (Density set). Let \(V \in V_k(M)\) has free boundary at \(\partial M\) and \(H \in L^p(M, \|V\|)\) for some \(p \geq 1\). Then the density set for \(V\) is defined as
\[
\text{Dens}(V) = \bigcup_{s \in (k-p, k]} \left\{ x \in M \mid \limsup_{r \to 0} \frac{1}{r^s} \int_{B_r(x)} (|H|^p + 1) d\|V\| = 0 \right\}.
\]
We notice that, if \(p > k\), then \(\text{Dens}(V) = M\).

This section is devoted to the proof of Theorem \(\text{(1.2)}\) which states the properties of \(\text{Dens}(V)\) when \(p \leq k\). The proof of the theorem is achieved in three steps:

1. We first derive a monotonicity inequality (Lemma \(\text{(5.2)}\)) at points on \(\partial M\) that does not involve estimates on \(\sigma_V\);
2. Next we adapt a measure theoretic lemma to \(\|V\|\), which is essential to study the size of \(\mathcal{M}\setminus\text{Dens}(V)\);
3. Finally, we estimate the right-hand side of the monotonicity inequality \(\text{(5.1)}\) for points in \(\text{Dens}(V)\), to show that it remains bounded as \(r \to 0\) and conclude the proof of Theorem \(\text{(1.2)}\).

5.1. Monotonicity formula at points on \(\partial M\). In order to obtain the monotonicity formula, we introduce the notion of distortion of distance.

If \(\partial M\) is not flat, the vector field used to obtain the monotonicity formula in Section 4 is not tangent to \(\partial M\). Since \(\nabla x = \frac{x}{|x|}\), we can try to modify the function \(x\) in order to obtain that its gradient is tangent to \(\partial M\), with small error. This motivates the following definition.

**Definition 5.2** (distortion of distance). Suppose \(x_0 \in \partial M\). We call a function \(d_{x_0}(x)\) a *distortion of the distance* adapted to \(\partial M\) if the following two conditions hold:

1. \(d_{x_0} \in C^2(\mathbb{R}^{n+1} \setminus \{x_0\})\) and \(D^j d_{x_0}(x) = D^j |x - x_0| + o(|x - x_0|^{1-j+\alpha})\) with \(\alpha \in (0, 1]\) and for \(j = 0, 1, 2\);
2. \(\nabla d_{x_0}(x)\) is tangent to \(\partial M\).

A distortion of distance exists if \(\partial M\) is of class \(C^3\), as stated in \(\text{(4, Lemma 4.25)}\):

**Lemma 5.1.** If \(\partial M\) is of class \(C^3\), then for every \(x_0 \in \partial M\) there exists a distortion of the distance \(d_{x_0}\) adapted to \(\partial M\) and moreover \(\alpha\) can be taken equal to 1.

For a fixed distortion of distance \(d_{x_0}(x)\), we denote by \(\mathcal{B}\) the distorted balls, that is if \(t > 0\) then \(\mathcal{B}_t(x_0) = \{x \in \mathbb{R}^{n+1} \mid d_{x_0}(x) \leq t\}\). If \(x_0 = 0\), we drop the subscripts and the center of the balls, i.e. \(d_0 = d\) and \(\mathcal{B}_t(0) = \mathcal{B}_t\) (this is an abuse of notation with respect to the distance function \(d\) from \(\partial M\) defined in Section 2; we set that only in Section 5 \(d\) denotes the distorted distance function from 0).
**Lemma 5.2** (Monotonicity inequality for distorted balls). Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial \mathcal{M}$. There exist constants $\kappa(\mathcal{M}) > 0$ and $c(\mathcal{M}) > 0$ such that if $x_0 \in \partial \mathcal{M}$ and $0 < r < t < \kappa$ then the following formula holds:

$$\frac{\|V\|^{(B_r(x_0))}}{r^k} \leq \frac{\|V\|^{(B_t(x_0))}}{t^k} + \frac{1}{kr^k} \int_{B_r(x_0)} \left( \frac{d_x \nabla d_x}{|\nabla d_x|^2} \cdot H - cd_{x_0} \right) d\|V\|$$

where $\gamma$ is a constant and $\kappa$ can be chosen independently of $x_0$. For the same reason there exists $\kappa(\mathcal{M}) \leq \kappa'$ such that for every $x_0 \in \partial \mathcal{M}$, the distorted ball $B_{r}(x_0)$ is well defined.

Without loss of generality we can assume $x_0 = 0$. For $\rho \in (0, \kappa)$ we consider the vector field

$$X(y) = \gamma\left(\frac{d(y)}{\rho}\right) \frac{\nabla d(y)}{|\nabla d(y)|^2},$$

where $\gamma$ is the cut-off function defined in section 2. $X$ is clearly compactly supported in $B_\rho$ and by the definition of distortion of the distance we have $X \in \mathcal{X}_k(\mathcal{M})$. By computation we get

$$\frac{d}{d\rho} \left( \int \left( \gamma\left(\frac{d(y)}{\rho}\right) d\|V\| \right) \right) = \gamma\left(\frac{d(y)}{\rho}\right) \left( x - x_0 \right) + \gamma\left(\frac{x - x_0}{\rho}\right) d\|V\|$$

and

$$DX(y) = \gamma\left(\frac{d(y)}{\rho}\right) \left( \frac{\nabla d(y)}{|\nabla d(y)|} \cdot \nabla d(y) \right) + \gamma\left(\frac{d(y)}{\rho}\right) \frac{\nabla d(y)}{|\nabla d(y)|} \left( \frac{\nabla d(y)}{|\nabla d(y)|} \cdot \nabla d(y) \right) + \gamma\left(\frac{d(y)}{\rho}\right) \frac{d^2 d(y)}{|\nabla d(y)|^2} - \frac{2d(y)}{|\nabla d(y)|^2} \nabla d(y) \cdot \left( D^2 d \cdot \nabla d \right)$$

If $S$ is a $k$-plane and $\tau_1, \ldots, \tau_k$ is an orthonormal basis of $S$, then

$$\text{div}_S X(y) = \sum_{i=1}^k (\tau_i, DX \tau_i) = \gamma\left(\frac{d(y)}{\rho}\right) \left( P_{S^+} \frac{\nabla d(y)}{|\nabla d(y)|} \right) + \gamma\left(\frac{d(y)}{\rho}\right) \left( 1 + o(|y|) \right)$$

By testing with $X$ and multiplying by $-\frac{1}{\rho^{k+1}}$, taking into account (5.2) we have

$$\frac{d}{d\rho} \left( \int \left( \gamma\left(\frac{d(y)}{\rho}\right) d\|V\| \right) \right) \geq \frac{1}{\rho^{k+1}} \int_{\mathcal{M}} \gamma\left(\frac{d(y)}{\rho}\right) \left( \frac{d\nabla d(y)}{|\nabla d(y)|^2} \cdot H - cd \right) d\|V\|$$

where the constant $c > 0$ comes out by the estimates $D^3 d_{x_0}(x) = D^3 |x - x_0| + o(|x - x_0|^{3-j})$. Since these estimates depends only on the geometry of $\mathcal{M}$ and, more precisely, by the $C^3$-norm of $\partial \mathcal{M}$, we can choose $c = c(\mathcal{M})$ independently of $x_0$. 

If $0 < r < t < \kappa$, integrating between $r$ and $t$ the above inequality, we obtain
\[
\frac{1}{tk} \int \gamma \left( \frac{d}{t} \right) d\|V\| - \frac{1}{tr} \int \gamma \left( \frac{d}{r} \right) d\|V\| \geq \int_r^t \left( \frac{1}{\rho^{k+1}} \int_M \gamma \left( \frac{d}{\rho} \right) \left( \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) d\|V\| d\rho \left( \frac{d}{\rho} \right) d\rho.
\]

Using Fubini Theorem on the right-hand side we get
\[
\frac{1}{tk} \int \gamma \left( \frac{d}{t} \right) d\|V\| - \frac{1}{tr} \int \gamma \left( \frac{d}{r} \right) d\|V\| \geq \int_M \left( \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) \left( \int_t^r \frac{1}{\rho^{k+1}} \gamma \left( \frac{d}{\rho} \right) d\rho \right) d\|V\| + \int_{G_k(M)} \left| P_{S^+} \frac{\nabla d}{|\nabla d|} \right|^2 \left( \frac{1}{tk} \gamma \left( \frac{d}{t} \right) - \frac{1}{rk} \gamma \left( \frac{d}{r} \right) + \int_t^r \frac{k}{\rho^{k+1}} \gamma \left( \frac{d}{\rho} \right) d\rho \right) dV(y, S).
\]

Letting $\gamma$ to increase to $1_{(-\infty, 1]}$, by dominated convergence we obtain
\[
\frac{\|V\|((B_r))}{tk} - \frac{\|V\|((B_r))}{rk} \geq \int_M \left( \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) \left( \int_t^r \frac{1}{\rho^{k+1}} \gamma \left( \frac{d}{\rho} \right) d\rho \right) d\|V\| + \int_{G_k(M)} \left| P_{S^+} \frac{\nabla d}{|\nabla d|} \right|^2 \left( \frac{1}{tk} \gamma \left( \frac{d}{t} \right) - \frac{1}{rk} \gamma \left( \frac{d}{r} \right) + \int_t^r \frac{k}{\rho^{k+1}} \gamma \left( \frac{d}{\rho} \right) d\rho \right) dV(y, S),
\]

which becomes
\[
\frac{\|V\|((B_r))}{tk} - \frac{\|V\|((B_r))}{rk} \geq -\frac{1}{k} \int_{B_t} \left( \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) \left( \frac{1}{tk} - \frac{1}{\max\{d, r\}^k} \right) d\|V\| + \int_{G_k(M)} \left| P_{S^+} \frac{\nabla d}{|\nabla d|} \right|^2 \frac{dV(y, S)}{dk}.
\]

Since the last term in the above inequality is non-negative we can neglect it and, by re-arranging we obtain
\[
\frac{\|V\|((B_r))}{tk} \leq \frac{\|V\|((B_r))}{tk} + \frac{1}{k} \int_{B_t} \left( \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) d\|V\| - \frac{1}{k} \int_{B_r} \left( \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) d\|V\| - \int_{B_t \setminus B_r} \frac{d\nabla d}{|\nabla d|^2}, H \right) - cd \right) d\|V\|.
\]

\[
\square
\]

5.2. Dimension of the set where the $s$-density of a function $f \in L^1(\mathcal{M}, \|V\|)$ is non-zero.

**Lemma 5.3.** Let $V \in \mathcal{V}_k(\mathcal{M})$ be a general varifold with free boundary at $\partial \mathcal{M}$ and $H \in L^1(\mathcal{M}, \|V\|)$, suppose $f \in L^1(\mathcal{M}, \|V\|)$ and let us assume $s \in [0, k)$. Then
\[
\limsup_{r \to 0} \frac{1}{r^s} \int_{B_r(x)} \|f\| d\|V\| = 0. \quad \text{for } \mathcal{H}^s\text{-a.e. } x \in \mathcal{M}.
\]

The proof of Lemma 5.3 relies on the following measure theoretic lemma on densities, which is an adaptation of [2, Theorem 2.10].

**Lemma 5.4.** Let $\mu$ be a positive Radon measure on $\mathbb{R}^n$ such that $\Theta^s(\mu, x) < \infty$ for $\mu$-a.e. $x \in \mathbb{R}^n$, suppose $f \in L^1(\mathbb{R}^n, \mu)$ and let us assume $0 \leq s < k$. Let us define
\[
\Lambda_s = \left\{ x \in \mathbb{R}^n \mid \limsup_{r \to 0} \frac{1}{r^s} \int_{B_r(x)} |f(y)| \, d\mu(y) > 0 \right\}.
\]

Then $\mathcal{H}^s(\Lambda_s) = 0$. 
Proof of Lemma 5.4. Since $f \in L^1(\mathbb{R}^n, \mu)$, by Lebesgue-Besicovitch Differentiation Theorem it follows that for $\mu$-a.e. $x \in \mathbb{R}^n$

$$
\lim_{r \to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)| \, d\mu(y) = |f(x)|.
$$

Thus, for $\mu$-a.e. $x \in \mathbb{R}^n$

$$
\limsup_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |f(y)| \, d\mu(y) = \limsup_{r \to 0} \frac{\mu(B_r(x))}{r^k} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)| \, d\mu(y) \\
\leq \Theta^{\kappa}(\mu, x) |f(x)| < \infty.
$$

Hence, since $0 \leq s < k$,

$$
(5.4) \quad \lim_{r \to 0} \frac{1}{r^n} \int_{B_r(x)} |f(y)| \, d\mu(y) = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n.
$$

Thus $\mu(\Lambda_s) = 0$. For every $\varepsilon > 0$ we define

$$
\Lambda^\varepsilon_s = \left\{ x \in \mathbb{R}^n \mid \limsup_{r \to 0} \frac{1}{r^s} \int_{B_r(x)} |f(y)| \, d\mu(y) > \varepsilon \right\}.
$$

Since $\Lambda_s = \bigcup_{\varepsilon > 0} \Lambda^\varepsilon_s$ and $\Lambda^\varepsilon_s \subset \Lambda^\tau_s$ if $\varepsilon \leq \tau$, to obtain the conclusion it is enough to show that $\mathcal{H}^s(\Lambda^\varepsilon_s) = 0$ for every $\varepsilon > 0$. Therefore we fix $\varepsilon > 0$.

Since $\Lambda^\varepsilon_s \subset \Lambda_s$, by $\mu(\Lambda_s) = 0$ it follows $\mu(\Lambda^\varepsilon_s) = 0$. For any $\eta > 0$, by absolute continuity of the integral, there exists $\zeta > 0$ such that, if $U$ is an open set with $\mu(U) \leq \zeta$, then $\int_U |f(y)| \, dy \leq \eta$. Therefore let us fix an open set $U$ such that

$$
\Lambda^\varepsilon_s \subset U, \quad \mu(U) \leq \zeta \quad \Rightarrow \quad \int_U |f(y)| \, dy \leq \eta.
$$

For any $\delta > 0$ we define

$$
\mathcal{F}_\delta = \left\{ B_r(x) \subset U \mid x \in \Lambda^\varepsilon_s, \ 0 < r < \delta, \int_{B_r(x)} |f(y)| \, d\mu(y) > \varepsilon r^s \right\}.
$$

By Vitali Covering Theorem [11, Theorem 1.24] there exists a countable subcollection $\mathcal{G} = \{ B_j \}_{j \in \mathbb{N}}$ of $\mathcal{F}_\delta$ made of disjoint balls, such that $B_j = B_{r_j}(x_j)$ for each $j \in \mathbb{N}$ and

$$
\Lambda^\varepsilon_s \subset \bigcup_{j=1}^\infty B_{5r_j}(x_j).
$$

Hence

$$
\mathcal{H}^s(\Lambda^\varepsilon_s) \leq c5^s \sum_{j \in \mathbb{N}} (r_j)^s \leq c5^s \int_{B_j} |f(y)| \, d\mu(y) \leq \frac{c5^s}{\varepsilon} \int_U |f(y)| \, d\mu(y) \leq \frac{c5^s}{\varepsilon} \eta.
$$

Letting $\delta \to 0$ we obtain $\mathcal{H}^s(\Lambda^\varepsilon_s) \leq \frac{cN}{\varepsilon} \eta$; then letting $\eta \to 0$ we get $\mathcal{H}^s(\Lambda^\varepsilon_s) = 0$. \hfill \Box

We can now proceed with the proof of Lemma 5.3.

Proof of Lemma 5.3. By Corollary 4.3 $V$ has bounded first variation; thus by Lebesgue-Besicovitch Differentiation Theorem, for $\|V\|$-a.e. $x \in \mathcal{M}$ there exists a constant $c(x) > 0$ and a radius $\rho(x) > 0$ such that

$$
|\delta V|(B_r(x)) \leq c \|V\|(B_r(x)) \quad \forall r \in (0, \rho(x)).
$$

This allows to apply the standard monotonicity formula for general varifolds to conclude (see e.g. [11, Lemma 40.5]) that $\Theta^k(\|V\|, x)$ exists and is finite for $\|V\|$-a.e. $x \in \mathcal{M}$. Thus we can apply Lemma 5.3 to $\mu = \|V\|$. \hfill \Box
5.3. Proof of Theorem 1.4

Proof of Theorem 1.4. Step 1: For any \( s \in (k - p, k) \), since

\[
\mathcal{M} \setminus \text{Dens}(V) \subset \left\{ x \in \mathcal{M} \mid \limsup_{r \to 0} \frac{1}{r^s} \int_{B_r} (|H|^p + 1) d\|V\| > 0 \right\},
\]

Lemma 5.3 yields \( \mathcal{H}^s(\mathcal{M} \setminus \text{Dens}(V)) = 0 \) and this proves

\[
\mathcal{H}_{\text{dim}}(\mathcal{M} \setminus \text{Dens}(V)) \leq k - p.
\]

Step 2: Since \( \rho \mapsto d_{\mathcal{M}}(x_0, \partial \mathcal{M}) \) is increasing and \( \rho \mapsto \rho^k \) is continuous, it is enough to prove the remaining statements of the theorem just for small radii of the balls. More precisely it is enough to show the existence of \( \varphi_{x_0}(t) \) and the validity of (1.9) only for \( 0 < r < t < \zeta(x_0) \) where

\[
(5.5) \quad \zeta(x_0) = \begin{cases} 
\text{dist}(x_0, \partial \mathcal{M}) & \text{if } x_0 \in \mathcal{M}^o; \\
\frac{\kappa(\mathcal{M})}{2} & \text{if } x_0 \in \partial \mathcal{M},
\end{cases}
\]

with that being said, we will prove the conclusion of the theorem only for \( x_0 \in \partial \mathcal{M} \cap \text{Dens}(V) \), because the proof for \( x_0 \in \mathcal{M}^o \) is exactly the same; the only difference is that, in place of (5.1), one has just to use the standard monotonicity identity for varifolds with bounded first variation (see e.g. [11, eq. 17.4] for the rectifiable case, but the formula is similar for general varifolds).

As mentioned above, our strategy is to send \( r \downarrow 0 \) in (5.1) and to estimate the terms in the right-hand side, showing that they remain bounded for each \( x_0 \in \text{Dens}(V) \cap \partial \mathcal{M} \). This will prove that

\[
\Theta^k(\|V\|, x_0) < +\infty \quad \forall x_0 \in \text{Dens}(V) \cap \partial \mathcal{M}.
\]

By this, we will deduce the existence of \( \varphi_{x_0} \) for every \( x_0 \in \text{Dens}(V) \cap \partial \mathcal{M} \) as in the statement of the theorem, and this will prove the existence of \( \Theta^k(\|V\|, x_0) \).

Let us fix \( x_0 \in \text{Dens}(V) \cap \partial \mathcal{M} \). Without loss of generality we can assume \( x_0 = 0 \). Since \( 0 \in \text{Dens}(V) \), there exists \( s \in (k - p, k] \) such that

\[
(5.6) \quad \limsup_{r \to 0} \frac{1}{r^s} \int_{B_r} (|H|^p + 1) d\|V\| = 0.
\]

Let us fix such \( s \) and let us call \( \alpha = s - (k - p) > 0 \). By the estimate \( d_{x_0}(x) = |x - x_0| + o(|x - x_0|^2) \), (5.6) holds for distorted balls \( B_r \) as well.

Throughout this proof \( c = c(\mathcal{M}) \) denotes the (fixed) constant in (5.1) and \( c' \) denotes a generic positive constant that can vary from line to line.

Step 3: We begin by deriving the following dyadic estimate:

\[
(5.7) \quad \frac{\|V\|(B_{2^{-1}r})}{(2^{-1}r)^k} \leq \left(1 + c' \varepsilon^{\frac{1}{p-1}}\right) \frac{\|V\|(B_r)}{r^k} + \frac{c'}{\varepsilon^\alpha},
\]

where \( \varepsilon > 0 \) and \( c' \) depends only on \( x_0 \) and is independent of \( r \).
In order to prove (5.7), let us consider \(0 < r < \kappa(M)\) where \(\kappa(M)\) is the constant in Lemma 5.2. By \(D^j d(x) = D^j |x| + o(|x|^{2-j})\) (for \(j = 0, 1, 2\)) we get

\[
\frac{\|V\|(B_{2^{-r}})}{(2^{-r})^k} \leq \frac{\|V\|(B_r)}{r^k} + \frac{1}{k r^k} \int_{B_r} \left( \frac{dV}{d|z|^k} H - cd \right) d\|V\| \\
- \frac{1}{k(2^{-r})^k} \int_{B_{2^{-r}}} \left( \frac{dV}{d|z|^k} H - cd \right) d\|V\| - \int_{B_r \setminus B_{2^{-r}}} \left( \frac{dV}{d|z|^k} H - cd \right) \frac{k d^k}{k d^k} d\|V\|
\]

(5.8)

We now have to estimate the integral in the last member of the above inequality. For every \(r > 0\) we have

\[
\frac{1}{r^{k-1}} \int_{B_r} (|H| + c) d\|V\| \leq \frac{1}{r^{k-1}} \left( \int_{B_r} (|H| + c)^p d\|V\| \right)^{\frac{1}{p}} \left( \|V\|(B_r) \right)^{1 - \frac{1}{p}}
\]

(5.9)

\[
= \left( \frac{r^{k(p-1)}}{\varepsilon^p r^{k(p-1)}} \int_{B_r} (|H| + c)^p d\|V\| \right)^{\frac{1}{p}} \left( \frac{\|V\|(B_r)}{r^k} \right)^{1 - \frac{1}{p}}
\]

\[
\leq \frac{c' r^\alpha}{\varepsilon^p r^\alpha} \int_{B_r} (|H| + c)^p d\|V\| + c' \varepsilon^{p-1} \frac{\|V\|(B_r)}{r^k},
\]

where in the first inequality we used Hölder inequality, in the second inequality we used Young’s inequality and \(\varepsilon > 0\) is a fixed parameter. Since \(0 \in \text{Dens}(V)\) and the distorted balls are comparable with the euclidean ones (e.g. \(B_r(x_0) \subset B_{2r}(x_0)\) for \(r\) sufficiently small), we have

\[
\sup_{0 < t < \kappa(M)} \frac{1}{t^k} \int_{B_t} (|H| + c)^p d\|V\| \leq c'(x_0) < \infty.
\]

Hence

\[
\frac{1}{r^{k-1}} \int_{B_r} (|H| + c) d\|V\| \leq \frac{c' r^\alpha}{\varepsilon^p r^\alpha} + c' \varepsilon^{p-1} \frac{\|V\|(B_r)}{r^k}
\]

(5.8)

(where we have substituted \(\varepsilon = \frac{1}{N^2(p-1)}\)). Putting this into the last member of (5.8), we obtain (5.7).

**Step 4:** We now want to iterate (5.7) to obtain that \(\Theta^k(\|V\|, 0) < \infty\).

To simplify the notations, if \(0 < t < \kappa(M)\), we call

\[
D(t) = \frac{\|V\|(B_t)}{t^k}
\]

and let us consider \(N \in \mathbb{N}\). Iterating (5.7) (choosing \(\varepsilon = \frac{1}{N^2(p-1)}\), \(\frac{1}{(N-1)^2(p-1)}\), \ldots) we get

\[
D(2^{-N} r) \leq \left( 1 + \frac{1}{N^2} \right) D(2^{-N+1} r) + c' N^{2(p-1)}(2^{-N+1} r)^\alpha
\]

\[
\leq \left( 1 + \frac{1}{N^2} \right) \left( 1 + \frac{1}{(N+1)^2} \right) D(2^{-N+2} r) + \left( 1 + \frac{1}{N^2} \right) c'(N-1)^{2(p-1)}(2^{-N+2} r)^\alpha
\]

\[
+ c' N^{2(p-1)}(2^{-N+1} r)^\alpha
\]

\[
\leq \ldots
\]

\[
\leq D(r) \prod_{j=1}^N \left( 1 + \frac{1}{j^2} \right) + \sum_{i=0}^{N-1} \left( \prod_{j=i+2}^N \left( 1 + \frac{1}{j^2} \right) c'(i+1)^{2(p-1)}(2^{-i} r)^\alpha.
\]
Since \( \prod_{j=1}^{\infty} (1 + 1/j^2) = \beta < \infty \), we have
\[
D(2^{-N} r) \leq \beta \left[ D(r) + c'r^\alpha \sum_{j=1}^{N-1} \frac{(i+1)^{2(p-1)}}{2^\alpha} \right]
\]
and the last sum converges as \( N \to \infty \). Hence
\[
D(2^{-N} r) \leq \beta \left[ D(r) + c'r^\alpha \right] \quad \forall N \in \mathbb{N}.
\]
(5.10)

Since \( t > 0 \) we have
\[
\sup_{r \in [t,2t]} \frac{\|V\|(B_r)}{r^k} < \infty,
\]
(5.10) yields
\[
\limsup_{r \to 0} \frac{\|V\|(B_r)}{r^k} < \infty.
\]
(5.11)

**Step 5:** We now have to prove the existence of \( \varphi_0 \) and of the density \( \Theta^k(\|V\|,0) \). In order to do this, let us notice that the previous arguments imply
\[
\int_{B_r} \frac{|H| + c}{d^{k-1}} d\|V\| < \infty;
\]
Indeed, the choice of \( s \) made at the beginning of the proof and (5.11) imply
\[
\int_{B_r} \frac{|H| + c}{d^{k-1}} d\|V\| = \sum_{i=0}^{2^{k-1}} \int_{B_{r/2^i}} \frac{|H| + c}{d^{k-1}} d\|V\|
\]
\[
\leq 2^{k-1} \sum_{i=0}^{\infty} \left( \frac{2^i}{r} \right) \int_{B_{r/2^i}} (|H| + c) d\|V\|
\]
\[
\leq 2^{k-1} \sum_{i=0}^{\infty} \left( \frac{2^i}{r} \right)^s \int_{B_{r/2^i}} (|H| + c)^p d\|V\| \left( \frac{\|V\|(B_{r/2^i})}{2^{-ik}r} \right)^{1-\frac{1}{p}}
\]
\[
\leq c' \sum_{i=0}^{\infty} 2^{-ia} r^{-p} < +\infty.
\]
Moreover (5.12) yields
\[
\limsup_{r \to 0} \int_{B_r} \frac{|H| + c}{d^{k-1}} d\|V\| = 0, \quad \limsup_{r \to 0} \frac{1}{r^{k-1}} \int_{B_r} (|H| + c) d\|V\| = 0.
\]
This implies that the function \( \varphi_0: (0,\kappa(M)) \to \mathbb{R}^+ \) defined as
\[
\varphi_0(t) = \int_{B_t} \frac{|H| + c}{d^{k-1}} d\|V\| + \frac{1}{t^{k-1}} \int_{B_t} (|H| + c) d\|V\| + \sup_{r \in (0,t)} \frac{1}{r^{k-1}} \int_{B_r} (|H| + c) d\|V\|
\]
is well-defined and increasing. Moreover, by (5.13), it satisfies
\[
\lim_{t \to 0} \varphi_0(t) = 0.
\]
By comparing with the right-hand side of (5.11) we obtain
\[
\frac{\|V\|(B_r)}{r^k} \leq \frac{\|V\|(B_t)}{t^k} + \varphi_0(t), \quad 0 < r < t < \kappa(M)
\]
By the estimate \( d(x) = |x| + o(|x|^2) \), up to a small modification of \( \varphi_0 \), we have that a similar statement holds for the euclidean balls as well if \( 0 < r < t < \kappa(M)/2 \), thus the validity of (1.9).

The existence of \( \Theta^k(\|V\|,\cdot) \) on \( \text{Dens}(V) \) and the upper semi-continuity of the restrictions of \( \Theta^k(\|V\|,\cdot) \) to \( \text{Dens}(V) \cap M^0 \) and to \( \text{Dens}(V) \cap \partial M \) are easy consequences of (1.9).
\[\square\]
6. Proof of Theorem 1.2

Throughout this section, unless otherwise specified, \( V \in \mathcal{V}_k(\mathcal{M}) \) is a rectifiable \( k \)-varifold with free boundary at \( \partial \mathcal{M} \), with generalized mean curvature \( H \in L^p(\mathcal{M}, \|V\|) \) for some \( p > 1 \) and \( \Theta^k(\|V\|, x) \geq 1 \) for \( |V| \)-a.e. \( x \in \mathcal{M} \). We remark that Corollary 4.4 establish the existence of the measure \( \sigma_V \) and that \( V \) has bounded first variation.

We first recall the definition of \( \sigma^*_V \), the \((k - 1)\)-dimensional part of \( \sigma_V \):

\[
\sigma^*_V = \sigma_{V \setminus E}, \quad E = \{x \mid 0 < \Theta^{k-1}_*(\sigma_V, x) \leq \Theta^{(k-1)}(\sigma_V, x) < +\infty\}.
\]

As stated in the introduction, we recall that if \( H \in L^p(\mathcal{M}, \|V\|) \) for some \( p > k \), then the condition \( \Theta^{(k-1)}(\sigma_V, x) < +\infty \) is not restrictive, since it holds for every point \( x \in \partial \mathcal{M} \); indeed (1.0) and Hölder inequality yield

\[
\frac{\sigma_V(B_r(x))}{r^{k-1}} \leq 2 \left\| \frac{\|V\|(B_r(x))}{r^k} + \frac{1}{r^{k-1}} \int_{B_r(x)} (c + |H|) \, d\|V\| \right. \\
\left. \leq 2 \left( \int_{B_r(x)} (c + |H|)^p \, d\|V\| \right)^{\frac{1}{p}} \left( \frac{\|V\|(B_r(x))}{r^k} \right)^{\frac{1}{p}}.
\]

By Lemma 4.5 and \( p > k \), the last member of the above inequality is bounded as \( r \to 0 \), therefore it follows \( \Theta^{(k-1)}(\sigma_V, x) < +\infty \). This proves that Theorem 1.3 is a corollary of Theorem 1.2.

Since in the proof of Theorem 1.2 we have to deal with the blow-ups of \( V \) at points on \( \partial \mathcal{M} \), we introduce a notation for the scalings of \( V \) at a point \( x_0 \in \partial \mathcal{M} \) that we use throughout this section.

**Notation** (Scalings). If \( V \in \mathcal{V}_k(\mathcal{M}) \) has free boundary at \( \partial \mathcal{M} \), if \( x_0 \in \partial \mathcal{M} \) is a fixed point and \( r_j \downarrow 0 \) is a fixed sequence, we use the following notations:

\[
M_j := \tau_{x_0, r_j}(\mathcal{M}) \quad V_j := (\tau_{x_0, r_j})_*^*V \in \mathcal{V}_k(M_j) \quad \sigma_j := \sigma_{V_j};
\]

where \( \tau_{x_0, r_j} \) is the dilation function defined in (2.2) and \( f_j^*V \) denotes the push-forward of \( V \) through \( f \) defined in (2.7). In Lemma 6.3 we show that each \( V_j \) has free boundary at \( \partial M_j \), thus Corollary 4.4 states the existence of \( \sigma_{V_j} \), which we call \( \sigma_j \). Moreover, we denote by \( H_j \) the generalized mean curvature of \( V_j \) and \( H_j \) the vector field provided by Corollary 4.4 relative to \( V_j \). If \( x \in \partial M_j \), we denote by \( N_j(x) \) the exterior unit normal to \( M_j \) at \( x \). As \( j \to \infty \), according to the definition of tangent space given in section 2, we say that \( M_j \to T_{x_0}^+ \mathcal{M} \) and that \( \partial M_j \to T_{x_0} \partial \mathcal{M} \).

The idea of the proof is to study the blow-ups of \( V \) at points on \( \partial \mathcal{M} \), to prove that the at \( \sigma^*_V \)-a.e. point on \( \partial \mathcal{M} \), every \((k - 1)\)-blow-up of \( \sigma^*_V \) is of the form \( \alpha \mathcal{H}^{k-1} S \) for some \((k - 1)\)-dimensional linear subspace \( S \). This allows us to apply the Marstrand-Mattila Rectifiability Criterion [3, Theorem 5.1] to \( \sigma^*_V \). We state explicitly the criterion for the reader convenience:

**Theorem 6.1** (Marstrand-Mattila Rectifiability Criterion). Let \( m \leq n \) be a natural number and let \( \mu \) be a positive Radon measure on \( \mathbb{R}^n \) such that, for \( \mu \)-a.e. \( x \in \mathbb{R}^n \), we have

1. \( 0 < \Theta^m_*(\mu, x) \leq \Theta^m_*(\mu, x) < \infty \);
2. Every \( m \)-tangent measure of \( \mu \) at \( x \) is of the form \( \beta \mathcal{H}^m S \) for some \( m \)-dimensional linear subspace \( S \).

Then \( \mu \) is \( m \)-rectifiable.

We begin in subsection 6.1 where we adapt to \( \sigma^*_V \) two well-known facts about measures:

- in Lemma 6.2 we show that \( \sigma^*_V(\mathcal{M} \setminus \text{dens}(V)) = 0 \); this allows us to check the conditions of the Marstrand-Mattila criterion just on \( \text{dens}(V) \).
- In Lemma 6.3 we prove that for \( \sigma^*_V \)-a.e. \( x \in \partial \mathcal{M} \), \( \sigma_V \) and \( \sigma^*_V \) have the same \((k - 1)\)-tangent measures.

Since by definition

\[
0 < \Theta^{k-1}_*(\sigma^*_V, x) \leq \Theta^{(k-1)}(\sigma^*_V, x) < +\infty \quad \text{for } \sigma^*_V \text{-a.e. } x \in \partial \mathcal{M},
\]

to apply Marstrand-Mattila Rectifiability Criterion it remains to show that for \( \sigma^*_V \)-a.e. \( x_0 \in \text{dens}(V) \), every tangent measure to \( \sigma_V \) is a \((k - 1)\)-dimensional plane. The rest of the section is devoted to prove this, which is achieved in several steps. The outline of the proof is the following:
In Lemma 6.4 we first prove that at every $x_0 \in \text{Dens}(V) \cap \partial \mathcal{M}$, tangent varifolds to $V$ are cones that are stationary with respect to $X_1(\mathcal{M})$ in a half-space. Thus, if $C \in \text{Tan}(V, x_0)$ for some $x_0 \in \text{Dens}(V) \cap \partial \mathcal{M}$, Corollary 4.4 provides the existence of the measure $\sigma_C$. We also prove that the measures $\sigma_j$ relative to the scalings of $V$ converge weakly to $\sigma_C$; This imply that (Corollary 6.5)

\[
\text{Tan}^{k-1}(\sigma^*_V, x_0) = \{\sigma_C \mid C \in \text{Tan}(V, x_0)\} \quad \text{for } \sigma^*_V\text{-a.e. } x_0 \in \partial \mathcal{M}.
\]

(2) In Lemma 6.6 we prove that each tangent cone $C$ has an invariant linear subspace $D_C$ that coincides with the set of points where the $k$-density $\Theta^k(\|C\|)$ of $C$ attains its maximum;

(3) In Lemma 6.7 we show that for $\sigma^*_V$-a.e. $x_0 \in \partial \mathcal{M}$, if $C \in \text{Tan}(V, x_0)$, then $\sigma_C$ is concentrated on $D_C$;

(4) In Lemma 6.8 we prove that for $\sigma^*_V$-a.e. $x_0 \in \partial \mathcal{M}$, if $C \in \text{Tan}(V, x_0)$, then $D_C$ is $(k-1)$-dimensional and $\sigma_C = \alpha \mathcal{H}^{k-1} \upharpoonright_{D_C}$ for some constant $\alpha = \alpha(x_0, C)$;

(5) In subsection 6.3 we summarize all these facts to conclude the proof of Theorem 1.2 by Lemma 6.8 and (6.1) it follows that, for $\sigma^*_V$-a.e. $x_0 \in \partial \mathcal{M}$, every $(k-1)$-tangent measure to $\sigma_V$ at $x_0$ is a $(k-1)$-plane.

6.1. Summary of well-known facts. We begin with a measure-theoretic lemma applied to $\sigma^*_V$.

Lemma 6.2. Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial \mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$. Then $\sigma^*_V \ll \mathcal{H}^{k-1}$. In particular, if $p > 1$, we have

\[
\sigma^*_V(\partial \mathcal{M} \setminus \text{Dens}(V)) = 0
\]

Proof. The proof is an easy consequence of [10, Theorem 6.9] which states that if $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ is a positive Radon measure and if

\[
A \subset \{x \in \mathbb{R}^n \mid \Theta^{(k-1)}(\mu, x) \leq \lambda\},
\]

then $\mu(A) \leq 2^{k-1} \lambda \mathcal{H}^{k-1}(A)$. Let us assume $\mathcal{H}^{k-1}(A) = 0$; since by definition $\Theta^{(k-1)}(\sigma^*_V, x) < \infty$ for $\sigma^*_V$-a.e. $x \in \mathcal{M}$ we have that

\[
\sigma^*_V(A) = \sigma^*_V\left(\bigcup_{i \in \mathbb{N}} A \setminus \{\Theta^{(k-1)}(\sigma^*_V, x) < i\}\right) \leq \sum_{i \in \mathbb{N}} i 2^{k-1} \mathcal{H}^{k-1}(A) = 0.
\]

(6.2) clearly follows by $\sigma^*_V \ll \mathcal{H}^{k-1}$ and (1.8). \hfill \square

Since $\sigma^*_V$ is concentrated on $E$, we expect that $\text{Tan}^{k-1}(\sigma^*_V, x) = \text{Tan}^{k-1}(\sigma_V, x)$ for every point of $E$ that has density 1 with respect to $\sigma_V$. This is a well-known fact, see e.g. [2, Remark 3.13], but we recall the proof for the reader convenience.

Lemma 6.3. Let $V \in \mathcal{V}_k(\mathcal{M})$ has free boundary at $\partial \mathcal{M}$ with $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$, let $\sigma_V$ be the measure provided by Corollary 6.4 and let $E$ be defined as in (1.7). Then

\[
\text{Tan}^{k-1}(\sigma^*_V, x_0) = \text{Tan}^{k-1}(\sigma_V, x_0) \quad \forall x_0 \in E \mid \lim_{r \to 0} \frac{\sigma_V(E \cap B_r(x))}{\sigma_V(B_r(x))} = 1.
\]

In particular, (6.3) holds $\sigma^*_V$-a.e.

Proof. Let us fix $x_0 \in \partial \mathcal{M}$ such that

\[
\lim_{r \to 0} \frac{\sigma_V(E \cap B_r(x))}{\sigma_V(B_r(x))} = \lim_{r \to 0} \frac{\sigma^*_V(B_r(x))}{\sigma_V(B_r(x))} = 1,
\]

let $\nu \in \text{Tan}^{k-1}(\sigma_V, x_0)$ and $r_j \downarrow 0$ such that

\[
\sigma_j := \frac{1}{r_j^{k-1}}(\tau_{x_0, r_j})^* \sigma_V \overset{\ast}{\rightarrow} \nu.
\]

We are going to prove that

\[
\sigma^*_j := \frac{1}{r_j^{k-1}}(\tau_{x_0, r_j})^* \sigma^*_V \overset{\ast}{\rightarrow} \nu.
\]
Without loss of generality we can assume \( x_0 = 0 \). Let us consider \( f \in C_c(\mathbb{R}^n) \). We have
\[
\int f(x) \, d\nu(x) = \lim_{j \to \infty} \frac{1}{r_j^{k-1}} \int f\left(\frac{x}{r_j}\right) \, d\sigma_V(x)
= \lim_{j \to \infty} \frac{1}{r_j^{k-1}} \left( \int_{E} f\left(\frac{x}{r_j}\right) \, d\sigma_V(x) + \int_{\mathbb{R}^n \setminus E} f\left(\frac{x}{r_j}\right) \, d\sigma_V(x) \right).
\]
For the last term there exists \( c > 0 \) such that
\[
\frac{1}{r_j^{k-1}} \int_{\mathbb{R}^n \setminus E} \left| f\left(\frac{x}{r_j}\right) \right| \, d\sigma_V(x) \leq \frac{c}{r_j^{k-1}} \sigma_V(B_{cr_j} \setminus E) \leq c \frac{\sigma_V(B_{cr_j})}{r_j^{k-1}} \frac{\sigma_V(B_{cr_j} \setminus E)}{\sigma_V(B_{cr_j})} \xrightarrow{j \to \infty} 0
\]
since \( E \) has density 1 with respect to \( \sigma_V \) and \( \sigma_V(B_{cr_j})/r_j^{k-1} \) remains bounded by \( \sigma_j \rightharpoonup \nu \). Hence
\[
\int f(x) \, d\nu(x) = \lim_{j \to \infty} \frac{1}{r_j^{k-1}} \int f\left(\frac{x}{r_j}\right) \, d\sigma_V'(x),
\]
thus
\[
\sigma_j^* = \frac{1}{r_j^{k-1}}(\tau_0, r_j) \# \sigma_V \rightharpoonup \nu.
\]
This proves \( \text{Tan}^{k-1}(\sigma_V, x_0) \subset \text{Tan}^{k-1}(\sigma_V', x_0) \); the reverse inclusion can be proved in a similar way. \( \square \)

6.2. Proof of the second condition of the Marstrand-Mattila Criterion. We begin by studying tangent varifolds to \( V \) in a point \( x \in \partial M \).

**Lemma 6.4.** Let \( V \in \mathcal{V}_k(M) \) has free boundary at \( \partial M \) with \( H \in L^p(M, \|V\|) \) for some \( p \geq 1 \). Then, for every \( x_0 \in \text{Dens}(V) \cap \partial M \) the following statements hold: let \( r_j \downarrow 0 \) be fixed and let us use the notations for the scalings defined on p. 23.\( \square \) then

1. every \( V_j \) has free boundary at \( \partial M_j \) with \( \tilde{H}_j(x) = r_j \tilde{H}(r_jx) \) for every \( x \in \partial M_j \)

\[
(6.6) \quad \sigma_{V_j} = \frac{1}{r_j^{k-1}}(\tau_0, r_j) \# \sigma_V;
\]

2. there exist a subsequence of \( r_j \), not relabeled, and a \( k \)-varifold \( C \) such that

\[
V_j \rightharpoonup \sigma \stackrel{\#}{\rightarrow} C.
\]

In particular, \( \text{Tan}(V, x_0) \neq \emptyset \).

3. \( C \) is stationary with respect to \( \mathcal{K}(T^{+1}_{20}M) \);

4. if \( \sigma_C \) is the measure given by Corollary 4.4 relative to \( C \), we have

\[
(6.7) \quad \sigma_j := \sigma_{V_j} \rightharpoonup \sigma \rightharpoonup \sigma_C;
\]

5. if in addition \( \Theta^k(\|V\|, x) \geq 1 \) for \( \|V\| \cdot \text{a.e.} \ x \in M \), then \( C \) is a rectifiable cone and \( \sigma_C \) is scaling invariant, that is

\[
\frac{1}{r^{k-1}}(\tau_0, r) \# \sigma_C = \sigma_C \quad \forall r > 0.
\]

**Proof.** To simplify the notations, we assume without loss of generality that \( x_0 = 0 \in \partial M \cap \text{Dens}(V) \) and \( T_0^{+}M = \{x_n \geq 0\} \).
Step 1: We first prove that every $V_j$ has free boundary at $\partial M_j$. For every $X \in \mathfrak{X}_t(M_j)$ the vector field $x \mapsto X(x)$ belongs to $\mathfrak{X}_t(M)$, thus we can apply (6.8) to $V$ and get

$$
\int_{G_k(M_j)} \text{div}_S X(x) \, dV_j(x, S) = \frac{1}{r_j^{k-1}} \int_{G(M)} \text{div}_S X(x) \, dV(x, S) = -\frac{1}{r_j^{k-1}} \int_M \langle X(x), H(x) \rangle \, d\|V\|(x).
$$

If we define $H_j(x) = r_j H(r_j x)$ for all $x \in M_j$, we have

$$
\int_{G_k(M_j)} \text{div}_S X(x) \, dV_j(x, S) = -\int_{M_j} \langle X(x), H_j(x) \rangle \, d\|V\|(x),
$$

that is $V_j$ has free boundary at $\partial M_j$, with $H_j(x) = r_j H(r_j x)$. By Corollary (6.11) $V_j$ has bounded first variation and there exists $\sigma_j = \sigma_{V_j}$ and $\tilde{H}_j$. By a similar argument and by definition of $\tilde{H}$ it easily seen that

$$
\tilde{H}_j(x) = r_j \tilde{H}(r_j x) \quad \text{for } \|V_j\|\text{-a.e. } x \in \partial M_j.
$$

If $X \in \mathfrak{X}(M_j)$, then

$$
\int_{G_k(M_j)} \text{div}_S X(x) \, dV_j(x, S) = \frac{1}{r_j^{k-1}} \int_{G_k(M)} \text{div}_S X(x) \, dV(x, S) = -\frac{1}{r_j^{k-1}} \int_M \langle X(x), H(x) + \tilde{H}_j(x) \rangle \, d\|V\|(x)
$$

$$
+ \frac{1}{r_j^{k-1}} \int_{\partial M} \langle X(x), N(x) \rangle \, d\sigma_V(x)
$$

$$
= -\int_{M_j} \langle X, H_j + \tilde{H}_j \rangle \, d\|V\| + \frac{1}{r_j^{k-1}} \int_{\partial M_j} \langle X, \tau_0 \rangle \, d(\tau_0, x) \# \sigma_V
$$

If we compare this equality with (6.3) for $V_j$ tested with $X$, we get

$$
\sigma_j := \sigma_{V_j} = \frac{1}{r_j^{k-1}} (\tau_0, x) \# \sigma_V,
$$

that is $\sigma_j$ is obtained by scaling $\sigma_V$.

Step 2: We now want to study the limit of the sequence $V_j$. Since $0 \in \text{Dens}(V)$, by Theorem (6.14) there exists a constant $c > 0$ such that

$$
\|V_j\|(B_1) = \|(\tau_0, r_j)\#V\|(B_1) = \frac{\|V\|(B_{r_j})}{r_j^k} \leq c \quad \forall j \in \mathbb{N}.
$$

Thus by compactness of Radon measures, there exist $C \in \mathcal{K}(\mathbb{R}^n)$ and a subsequence of $r_j$, not relabeled, such that

$$
V_j \overset{\lambda}{\rightharpoonup} j \rightarrow \infty C.
$$

Step 3: Clearly $C \subseteq T_{0}^+ M$. To show that $C$ is stationary with respect to $\mathfrak{X}_t(T_{0}^+ M)$, we test with a vector field $X \in \mathfrak{X}_t(T_{0}^+ M)$. We first need the following estimate on $H_j$ (exactly the same relation holds for $\tilde{H}_j$):

$$
\|H_j\|_{L^p(B_1)} = \left( \int_{B_1} |H_j|^p \, d\|V\| \right)^{\frac{1}{p}} = \left( \frac{1}{r_j^{k-1}} \int_{B_{r_j}} |H|^p \, d\|V\| \right)^{\frac{1}{p}} \rightarrow 0.
$$

where in the second equality we have used $H_j(x) = r_j H(r_j x)$ and the limit follows by $0 \in \text{Dens}(V)$.

To prove that $C$ is stationary, let us pick $X \in \mathfrak{X}_t(T_{0}^+ M)$ with compact support; there exists a sequence of vector fields $X_j \in \mathfrak{X}_t(M_j)$ with compact support such that $X_j \rightarrow X$ in the $C^1$ topology.
Hence, using (6.9) and (6.10) we get
\[
\int_{G_k(T_0^+M)} \text{div}_S X(x) \, dC(x, S) = \lim_{j \to \infty} \int_{G_k(M_j)} \text{div}_S X_j(x) \, dV_j(x, S)
\]
\[
= \lim_{j \to \infty} \int_{M_j} \langle X_j(x), H_j(x) \rangle \, d\|V_j\|(x)
\]
\[
= 0
\]
that is, by definition, \( C \) is stationary with respect to \( X(T_0^+M) \).

**Step 4:** Corollary 4.4 provides that \( C \) has bounded first variation and the existence of \( \sigma_C \). We want to prove that

\[
\sigma_j \xrightarrow{j \to \infty} \sigma_C.
\]

To this aim, we first remark that \( \text{supp} \sigma_C \subset T_0^+\partial M \) and, since \( T_0^+\partial M \) is flat, we have that \( \tilde{H}_C = 0 \).

We next need an uniform bound on \( \sigma_j(B_1) \) and we use (1.6): since the second fundamental forms of \( \partial M_j \) go to 0 as \( r_j \to 0 \), when we apply (1.6) to \( V_j \) in \( B_1 \), the constant \( c \) in (1.6) is bounded uniformly in \( j \). This implies that there exists an uniform constant \( c \) such that for each \( j \in \mathbb{N} \)

\[
\sigma_j(B_1) = \frac{\sigma_V(B_j)}{r_j^k-1} \leq \frac{c\|V\|(B_2r_j)}{r_j^k} + \frac{1}{r_j^{k-1}} \int_{B_2r_j} |H| \, d\|V\|.
\]

and the last member of (6.12) is uniformly bounded in \( j \) since \( 0 \in \text{Dens}(V) \), by Theorem 1.4 and (5.13). This proves the uniform bound on \( \sigma_j(B_1) \).

To complete the proof of (6.11), let \( X \in \mathcal{X}(\mathbb{R}^n) \) be a vector field with compact support. If \( e_n \) is the \( n \)-th coordinate unit vector (that is the interior unit normal vector to \( \partial T_0^+M \)), we have

\[
- \int_{\partial \partial M} \langle X, e_n \rangle \, d\sigma_C = \int_{G_k(T_0^+M)} \text{div}_S X(x) \, dC(x, S)
\]

\[
= \lim_{j \to \infty} \int_{G_k(M_j)} \text{div}_S X_j(x) \, dV_j(x, S)
\]

\[
= - \lim_{j \to \infty} \left( \int_{M_j} \langle X(x), H_j(x) + \tilde{H}_j(x) \rangle \, d\|V_j\|(x) + \int_{\partial M_j} \langle X(x), N_j(x) \rangle \, d\sigma_j(x) \right)
\]

\[
= \lim_{j \to \infty} \int_{\partial M_j} \langle X(x), N_j(x) \rangle \, d\sigma_j(x),
\]

where the first identity follows by \( H_C = \tilde{H}_C = 0 \) and the last one follows by (6.10). Thus \( N_j \sigma_j \rightharpoonup -e_n \sigma_C \) is proved, by (6.13), the uniform bound on \( \sigma_j(B_1) \) (6.12) and because \( \mathcal{X}(\mathbb{R}^n) \) is dense in \( C_c(\mathbb{R}^n, \mathbb{R}^n) \). To prove (6.11), it is enough to observe that, if \( f \in C_c(\mathbb{R}^n, \mathbb{R}) \), then

\[
\int f \, d\sigma_C = \int \langle f e_n, e_n \rangle \, d\sigma_C = - \lim_{j \to \infty} \int \langle f e_n, N_j \rangle \, d\sigma_j = \lim_{j \to \infty} \int f \, d\sigma_j,
\]

by the uniform bound on \( \sigma_j(B_1) \) and since \( \partial M_j \to \partial T_0^+M \) in the \( C^1 \) topology.

**Step 5:** We are left to prove that, if \( \Theta^k(\|V\|, x) \geq 1 \) for \( \|V\|\text{-a.e.} \, x \in M \), then \( C \) is a rectifiable cone. The scaling invariance of \( \sigma_C \) will be an easy consequence of this. We first claim that \( C \) is rectifiable. In fact, we have:

- \( \Theta^k(\|V\|, x) \geq 1 \) for \( \|V\|\text{-a.e.} \, x \in M_j \), since \( \Theta^k(\|V\|, x) \geq 1 \) for \( \|V\|\text{-a.e.} \, x \in M \);
- \( \sup_{j \in \mathbb{N}} \|V_j\|(B_1) < \infty \) by (6.8);
- the \( V_j \)'s have locally uniformly bounded first variations because \( 0 \in \text{Dens}(V) \), by (6.10) and the uniform bound on (6.12).
Thus we can apply to the sequence $V_j$ the compactness theorem for rectifiable varifolds \cite{11}, Theorem 42.7, which proves that $C$ is rectifiable and that $\Theta^k(||C||, x) \geq 1$ for $||C||$-a.e. $x \in M$. In particular, there exists a $k$-rectifiable set $\Gamma \subset T_0^+ M$ such that

$$C = \theta(x) H^k \Gamma$$

with $\theta(x) = \Theta^k(||C||, x) \geq 1$ for $H^k$-a.e. $x \in \Gamma$.

It remains to show that $C$ is a cone. The argument is exactly the same as \cite{11}, Theorem 19.3, but we recall it for the reader convenience. Since $C$ is rectifiable, to prove

$$\langle \nabla (6.14) \rangle (\tau_{0, \lambda}) C = C \quad \forall \lambda > 0,$$

(that is the fact that $C$ is a cone), it is enough to show that

$$\Theta^k(||C||, \lambda x) = \Theta^k(||C||, x) \quad \forall x \in \mathbb{R}^n, \forall \lambda > 0,$$

that is $\theta$ is homogeneous of degree 0. This is clearly implied by

$$||C||((\lambda A) = \lambda^k ||C|| (A) \quad \forall A \subset \mathbb{R}^n \text{ Borel}, \forall \lambda > 0.$$ 

By approximation, it is enough to prove that, for every 0-homogeneous function $h \in C^1(\mathbb{R}^n)$, one has

$$\frac{d}{d\lambda} \left( \frac{1}{\lambda^k} \int_M \gamma \left( \frac{|x|}{\lambda} \right) h(x) d||V||(x) \right) = 0,$$

where $\gamma$ is the cut-off function defined in Section 2. To this aim, let $\lambda > 0$ be such that $||C||((\partial B_\lambda)) = 0$, we observe that

$$\frac{||C||((B_\lambda))}{\lambda^k} = \lim_{j \to \infty} \frac{||V_j||((B_\lambda))}{\lambda^k} = \lim_{j \to \infty} \frac{||V||((B_\lambda r_j))}{\lambda^k r_j^k} = \Theta^k(||V||, 0).$$

Since there exists at most a countable number of radii $\lambda_j$ such that $||C||((\partial B_{\lambda_j})) > 0$ by approximation it follows that

$$\frac{||C||((B_\lambda))}{\lambda^k} = \Theta^k(||V||, 0) \quad \forall \lambda > 0.$$

If we write the monotonicity identity (4.6) for $C$, since $H_C = H^k C = 0$ and $\partial(T_0^+ M)$ is flat, integrating between $\sigma, \lambda$ we obtain

$$\frac{1}{\lambda^k} \int_W \gamma \left( \frac{|x|}{\lambda} \right) d||C||(x) - \frac{1}{\sigma^k} \int_W \gamma \left( \frac{|x|}{\sigma} \right) d||C||(x)$$

$$= \frac{1}{\lambda^k} \int_{G_k(W)} \gamma \left( \frac{|x|}{\lambda} \right) |P_{S^+} \nabla |x||^2 dC(x, S)$$

$$- \frac{1}{\sigma^k} \int_{G_k(W)} \gamma \left( \frac{|x|}{\sigma} \right) |P_{S^+} \nabla |x||^2 dC(x, S)$$

$$+ \int_{G_k(W)} |P_{S^+} \nabla |x||^2 \left( \int_{|x|}^{\lambda} \frac{k}{p^{n+1}} \gamma \left( \frac{|x|}{p} \right) d\rho \right) dC(x, S)$$

Letting $\gamma$ increase to $1_{[0,1]}$ in (6.19), by dominated convergence and (6.18) we get

$$0 = \frac{||C||((B_\lambda))}{\lambda^k} - \frac{||C||((B_\sigma))}{\sigma^k} = \int_{G_k(B_\lambda \setminus B_\sigma)} \frac{|P_{S^+} \nabla |x||^2}{|x|^k} dC(x, S).$$

Since the last term is non-negative, we have

$$|P_{S^+} \nabla |x||^2 = \frac{|P_{S^+} x|}{|x|^k} > 0$$

for $C$-a.e. $(x, S) \in G_k(T_0^+ M)$, that is

$$C((x, S) | P_S(x) \neq x) = C((x, S) | x \notin S) = 0.$$

If $h$ is 0-homogeneous, then $\langle \nabla h(x), x \rangle = 0$ and (6.22) implies

$$\langle \nabla h(x), P_S x \rangle = \langle \nabla h(x), x \rangle = 0$$

for $C$-a.e. $(x, S) \in G_k(T_0^+ M)$. 

Since $C$ has free boundary at $\partial T_0^+M$, by testing (2.20) for $C$ with $X(x) = h(x)\gamma\left(\frac{|x|}{\lambda}\right)x \in \mathcal{X}_l(T_0^+M)$, one obtains
\[
\frac{d}{d\lambda} \left( \frac{1}{\lambda^k} \int_{M} \gamma\left(\frac{|x|}{\lambda}\right)h(x) d\|V\|(x) \right) = -\frac{1}{\lambda^{k+1}} \int_{G_k(T_0^+M)} \text{div}_S X(x) dV(x, S) + \frac{1}{\lambda^{k+1}} \int_{G_k(T_0^+M)} \gamma\left(\frac{|x|}{\lambda}\right)(\nabla h(x), PS x) dV(x, S) = 0,
\]
where the last equality follows by $H_C = 0$ and (6.23). This proves (6.17), thus $C$ is a cone. The scaling invariance of $\sigma_C$ is a trivial consequence of (4.3) applied to $C$ and (6.14).

Before going on, we highlight an easy consequence of Lemma 6.2, Lemma 6.3 and Lemma 6.4

**Corollary 6.5.** Let $V \in \mathcal{V}_k(M)$ has free boundary at $\partial M$ with $H \in L^p(M, \|V\|)$ for some $p > 1$ and let $\sigma_V$ be the measure provided by Corollary 4.4. Then
\[
\text{Tan}^{k-1}(\sigma_V, x_0) = \{\sigma_C: C \in \text{Tan}(V, x_0)\} \quad \text{for } \sigma_V^{k-1}\text{-a.e. } x_0 \in \partial M.
\]
**Proof.** By Lemma 6.4 it clearly follows that for each $x_0 \in \text{Dens}(V) \cap \partial M$,
\[
\text{Tan}^{k-1}(\sigma_V, x_0) = \{\sigma_C: C \in \text{Tan}(V, x_0)\}.
\]
By Lemma 6.2 and Lemma 6.3 we have the conclusion. \qed

It is well-known that, for a cone, points with maximal density form a linear subspace and that the cone is invariant by translation with respect to these points (see e.g. [12, Section 3.3] and [13, Theorem 3.1, Example (4) of Section 4]). We report here the simple proof of this fact for the sake of completeness.

**Lemma 6.6.** Let $V \in \mathcal{V}_k(M)$ a rectifiable varifold with free boundary at $\partial M$, $H \in L^p(M, \|V\|)$ for some $p > 1$ and $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|\text{-a.e. } x \in M$. Let $x_0 \in \text{Dens}(V) \cap \partial M$ and let $C$ be a tangent cone to $V$ at $x_0$. Then the set
\[
D_C = \{y \in T_{x_0}\partial M | \Theta^k(\|C\|, y) = \Theta^k(\|C\|, 0)\}
\]
is a linear subspace of $\mathbb{R}^n$. Moreover the translated cone $(L_y)_*C$ coincides with $C$ for all $y \in D_C$. In particular, if $\sigma_C$ is the measure given by Corollary 4.4 relative to $C$, then $(L_y)_\#\sigma_C = \sigma_C$.

**Definition 6.1.** We say that $D_C$ is the invariant subspace of the cone $C$.

**Proof of Lemma 6.6.** Without loss of generality we can assume that $x_0 = 0$. Let us call $\theta_0 = \Theta^k(\|V\|, 0)$. By Lemma 6.4 $C$ is rectifiable and it holds $\Theta^k(\|V\|, y) = \Theta^k(\|C\|, y)$. Since $C$ is a cone, for $y \in T_z\partial M$ we have
\[
\theta_0 \overset{(6.25)}{=} \lim_{r \to +\infty} \frac{\|C\|(B_r)}{r^k} \geq \lim_{r \to +\infty} \frac{\|C\|(B_{r-|y|}(y))}{r^k} \frac{(r-|y|)^k}{|y|^k} \geq \lim_{r \to +\infty} \frac{\|C\|(B_r(y))}{r^k} \geq \Theta^k(\|C\|, y).
\]
The last inequality is given by the monotonicity identity for $C$: in the last member of (4.6), the first two integrals disappear (since $H + \hat{H} = 0$ and $\langle N(x), x \rangle = 0$) and the last term is non-negative. (6.25) shows that
\[
\Theta^k(\|C\|, 0) \geq \Theta^k(\|C\|, y) \quad \forall y \in T_0\partial M.
\]
If $y \in D_C$, then (6.25) yields
\[
\frac{\|C\|'(B_r(y))}{r^k} = \theta_0 \quad \forall r > 0.
\]
By the same arguments of Lemma 6.3, it follows that $C$ is a rectifiable cone also with respect to $y$. By rectifiability, in order to show that $(L_y)_*C = C$, it is enough to prove that
\[
\Theta^k(C, z) = \Theta^k(C, y + z) \quad \forall z \in T_0^+M.
\]
To this aim, let \( z \in T_0^+\mathcal{M} \) be an arbitrary point. Since \( C \) is a cone with respect to \( y \), we have that
\[
\Theta^k(C, z) = \Theta^k\left(C, y + \frac{1}{2}(z - y)\right) = \Theta^k\left(C, \frac{1}{2}(y + z)\right).
\]
On the other hand, since \( C \) is a cone we have
\[
\Theta^k(\|C\|, y + z) = \Theta^k\left(C, \frac{1}{2}(y + z)\right).
\]
This shows \( 6.27 \) and hence \( (L_y)_C C = C \). The translation invariance of \( \sigma_C \) is a trivial consequence of \( 6.3 \) applied to \( C \) and of \( (L_y)_C C = C \).

It remains to show that \( D_C \) is a linear subspace of \( \mathbb{R}^n \). Since \( C \) is a cone, if \( y \in D_C \), then \( \lambda y \in D_C \) for each \( \lambda > 0 \). Since \( C \) is a cone also with respect to \( y \), then \( \lambda y \in D_C \) also if \( \lambda < 0 \). If \( y, z \in D_C \), it follows from the previous discussion that also \( y + z \in D_C \) and this proves that \( D_C \) is a linear subspace. \( \square \)

Before going on, we first recall the definition of approximate continuity:

**Definition 6.2.** (Approximate continuity). If \( \mu \) is a positive Radon measure and \( f : \mathbb{R}^n \to \mathbb{R} \) is a Borel function, we say that \( f \) is approximate continuous at \( x \in \mathbb{R}^n \) with respect to \( \mu \) if for every \( \varepsilon > 0 \)
\[
\lim_{r \to 0} \frac{\mu\left(\{z \in B_r(x) \mid |f(z) - f(x)| > \varepsilon\}\right)}{\mu(B_r(x))} = 0.
\]

**Remark 6.1.** It is well-known that, if \( \mu \) is a Radon measure, then every \( \mu \)-measurable function is approximate continuous at \( \mu \)-a.e. point (see e.g. [7, Theorem 1.37] where the proof is done for the Lebesgue measure, but the same arguments can be applied to any Radon measure).

The following lemma states that there exists a set \( F \) of full \( \sigma_V \)-measure with respect to \( \text{Dens}(V) \) such that for every \( x \in F \) the invariant subspace \( D_C \) of any cone \( C \in \text{Tan}(V, x) \) coincides with \( \text{supp} \sigma_C \).

**Lemma 6.7.** Let \( V \in \mathcal{V}_k(\mathcal{M}) \) a rectifiable varifold with free boundary at \( \partial \mathcal{M} \) with \( H \in L^p(\mathcal{M}, \|V\|) \) for some \( p \geq 1 \) and \( \Theta^k(\|V\|, x) \geq 1 \) for \( \|V\| \)-a.e. \( x \in \mathcal{M} \). Then there exists a set \( F \subset \text{Dens}(V) \cap \partial \mathcal{M} \) that satisfies
\[
\sigma_V(\text{Dens}(V) \setminus F) = 0
\]
and the following property: for every \( x_0 \in F \) and for every \( C \in \text{Tan}(V, x_0) \) we have \( \sigma_C \neq 0 \) and
\[
\Theta^k(\|C\|, y) = \Theta^k(\|C\|, 0) \quad \forall y \in \text{supp} \sigma_C.
\]
In particular, either \( \sigma_C = 0 \) or \( \text{supp} \sigma_C = D_C \).

**Proof.** The idea of the proof is the following: we first define the “good” set \( F \) of full \( \sigma_V \)-measure with respect to \( \text{Dens}(V) \) where \( \Theta^k(\|V\|, \cdot) \) exists and is approximate continuous with respect to \( \sigma_V \). Next, we fix \( x_0 \in F, r_j \to 0 \) and, using the notations for the scalings,
\[
V_j \xrightarrow{\Lambda} C \subset \text{Tan}(V, x_0).
\]
We fix \( y \in \text{supp} \sigma_C \) and we assume by contradiction that the statement is false. We find a tiny ball \( B_r(y) \) where \( \Theta^k(\|V_j\|, \cdot) \) is close to \( \Theta^k(\|C\|, y) \) for \( j \) sufficiently large. Since \( \Theta^k(\|C\|, y) \) is close to \( \|C\|(B_\rho(y))/\rho^k \) for small \( \rho \), this is achieved by weak convergence \( V_j \xrightarrow{\Lambda} C \) and using the properties of \( \varphi_{x_0} \) stated in Theorem 1.3. Since \( \sigma_j(B_r(y)) > \beta > 0 \) for large \( j \), this contradicts the approximate continuity of \( \Theta^k(\|V\|, \cdot) \) at \( x_0 \) with respect to \( \sigma_V \).

**Step 1:** We are going first to define the set \( F \) of full \( \sigma_V \)-measure with respect to \( \text{Dens}(V) \) and next we will prove that the conclusion of the theorem holds for every \( x \in F \).

We call \( A \) the set of points \( x \in \partial \mathcal{M} \) that satisfy all the following conditions:
1. \( x \in \text{Dens}(V) \cap \text{supp} \sigma_V \);
2. \( x \) is a point of approximate continuity for \( \Theta^k(\|V\|, \cdot) \) (which is a well-defined Borel function in \( \text{Dens}(V) \)) with respect to \( \sigma_V \).
We have
\begin{equation}
\sigma_V(Dens(V) \setminus A) = 0; \tag{6.28}
\end{equation}
This is true because since by Remark 6.1 \( \sigma_V \text{-a.e. } x \in Dens(V) \) is of approximate continuity for \( \Theta^k(\|V\|, \cdot) \) with respect to \( \sigma_V \). In addition, by Theorem 1.4 the maps \( \{ \varphi_x(\rho) \}_{\rho > 0} \) converge pointwise to 0 as \( \rho \downarrow 0 \), that is
\[ \lim_{\rho \to 0} \varphi_x(\rho) = 0 \quad \forall x \in A. \]
Since for every \( x \in A \) the map \( \rho \mapsto \varphi_x(\rho) \) is monotone increasing, by Egoroff’s Theorem for every \( h \in \mathbb{N} \) there exists a set \( F_h \subset A \) such that
\begin{equation}
\sigma_V(A \setminus F_h) \leq 1/h, \quad \varphi_x(\rho) \xrightarrow{\text{unif } \rho \to 0} 0 \text{ on } F_h. \tag{6.29}
\end{equation}
Up to removing sets of \( \sigma_V \)-measure 0, we can assume that every \( x \in F_h \) is a point of density 1 with respect to \( \sigma_V \), that is
\begin{equation}
\lim_{\rho \to 0} \frac{\sigma_V(F_h \cap B_\rho(x))}{\sigma_V(B_\rho(x))} = 1 \quad \forall x \in F_h. \tag{6.30}
\end{equation}
We now define
\[ F = A \cap \left( \bigcup_{h \in \mathbb{N}} F_h \right). \]
By (6.28) and (6.29) it follows
\[ \sigma^*_V(Dens(V) \setminus F) = 0. \]

**Step 2:** Let us fix \( x_0 \in F \) and let us consider \( h \in \mathbb{N} \) such that \( x_0 \in F_h \). Without loss of generality we can assume \( x_0 = 0 \).

Now let us fix \( r_j \downarrow 0 \). Using the notations for the scalings, since \( F \subset Dens(V) \), by Lemma 6.4 there exists a subsequence, not relabeled, such that \( V_j \xrightarrow{\text{scal}} C \) with \( C \in \text{Tan}(V, 0) \) and \( C \) is a rectifiable cone with \( \Theta^k(\|C\|, y) \geq 1 \) for \( \|C\| \text{-a.e. } y \in \mathbb{R}^n \).

We now need a technical remark that is useful in the rest of the proof: recalling that \( H_j(y) = r_j H(r_j y) \), by a simple change of variables one obtains
\[ r_j x \in Dens(V) \iff x \in Dens(V_j). \]
More precisely, if we call \( V_j^\# \) the function for \( V_j \) defined in Theorem 1.4, we get
\begin{equation}
\varphi^+_j(\rho) = \varphi_{r_j x}(r_j \rho). \tag{6.31}
\end{equation}

**Step 3:** We can now begin with the proof. If \( \sigma_C = 0 \) there is nothing to prove. Thus we can assume \( \sigma_C \neq 0 \) and \( 0 \neq \text{supp } \sigma_C \subset \partial T^+_0 \mathcal{M} \). Since \( C \) is a cone, By (6.26) we have
\[ \Theta(\|C\|, y) \leq \Theta(\|C\|, 0) \quad \forall y \in \partial T^+_0 \mathcal{M}. \]
By contradiction, let us assume that there exists \( y \in \text{supp } \sigma_C \) and \( \varepsilon > 0 \) such that
\begin{equation}
\Theta^k(\|C\|, y) < \Theta^k(\|C\|, 0) - \varepsilon. \tag{6.32}
\end{equation}
Since \( \sigma_C \) is scaling invariant by Lemma 6.3 without loss of generality we can assume that \( y \in B_{1/2} \). By the uniform convergence (6.29) and by definition of density, there exists \( \rho \in (0, 1/2) \) such that
\begin{equation}
\frac{\|C\|(B_\rho(y))}{\rho^k} + \varphi_\varepsilon(\rho) \leq \Theta^k(\|C\|, y) + \frac{\varepsilon}{8} \quad \forall z \in F_h. \tag{6.33}
\end{equation}
Since \( V_j \xrightarrow{\text{scal}} C \), without loss of generality we can choose \( \rho \) so that there exists \( J \in \mathbb{N} \) and a small \( 0 < r < \rho \) for which
\begin{equation}
\left| \frac{\|C\|(B_\rho(y))}{\rho^k} - \frac{\|V_j\|(B_\rho(y))}{(\rho - r)^k} \right| < \frac{\varepsilon}{8} \quad \forall j > J. \tag{6.34}
\end{equation}
Let us choose $j > J$; for every $z \in B_r(y)$ such that $r_j z \in F_h$ we have

$$\Theta^k(\|V_j\|, z) \leq \frac{\|V_j\|(B_{\rho - r}(z))}{(\rho - r)^k} + \varphi_z^j(\rho - r)$$

$$\leq \frac{\|V_j\|(B_{\rho}(y))}{(\rho - r)^k} + \varphi_{r_j z}(r_j \rho)$$

$$\leq \frac{\|C\|(B_{\rho}(y))}{\rho^k} + \varphi_{r_j z}(r_j \rho) + \frac{\varepsilon}{8}$$

$$\leq \Theta^k(\|C\|, y) + \frac{\varepsilon}{4}$$

$$\leq \Theta^k(\|C\|, 0) - \frac{3\varepsilon}{4}$$

$$= \Theta^k(\|V_j\|, 0) - \frac{3\varepsilon}{4}.$$ 

where we used the fact that every $\varphi_z$ is increasing and (6.31). This shows that, for $j > J$,

$$B_r(y) \cap \frac{1}{r_j} F_h \subseteq \left\{ z \in B_1 \mid \Theta^k(\|V_j\|, z) - \Theta^k(\|V_j\|, 0) \geq \frac{\varepsilon}{2} \right\}.$$

**Step 4:** We now want to estimate from below the measure of this set to get a contradiction with the approximate continuity of $\Theta^k(\|V\|, \cdot)$ in 0.

By approximate continuity of the $\Theta^k(\|V\|, \cdot)$ in 0 with respect to $\sigma_V$ we have

$$0 = \limsup_{j \to \infty} \frac{\sigma_V(\{z \in B_{r_j} \mid \Theta^k(\|V_j\|, z) - \Theta^k(\|V_j\|, 0) \geq \frac{\varepsilon}{2}\})}{\sigma_V(B_{r_j})}$$

$$\geq \limsup_{j \to \infty} \frac{\sigma_V(B_{r_{r_j}}(r_j y) \cap F_h)}{\sigma_V(B_{r_j})}$$

$$= \limsup_{j \to \infty} \frac{\sigma_V(B_{r_{r_j}}(r_j y))}{\sigma_V(B_{r_j})}$$

$$= \limsup_{j \to \infty} \frac{\sigma_j(B_r(y))}{\sigma_j(B_1)},$$

where the second to last identity is consequence of (6.30) and the last one follows by (6.6). We want to estimate from below the last term to get a contradiction. To do so, let us notice that, by $\sigma_j \searrow \sigma_C$ and $y \in \text{supp} \sigma_C$, there exist two constants $c, \beta > 0$ such that, for $j$ sufficiently large,

$$\sigma_j(B_1) \leq c, \quad \sigma_j(B_r(y)) \geq \beta,$$

which contradicts (6.35).

This also prove the inclusion $\text{supp} \sigma_C \subseteq D_C$. To prove the other inclusion, let us notice that the scaling invariance of $\sigma_C$ and $\text{supp} \sigma_C \neq \emptyset$ imply $0 \in \text{supp} \sigma_C$. Since $\sigma_C$ is invariant by translations along $D_C$ by Lemma 6.6 we have the opposite inclusion and $\text{supp} \sigma_C = D_C$.

We next prove that, for every $x \in F$ (where $F$ is defined in the previous Lemma) such that $\Theta^{k-1}_*(\sigma_V, x) > 0$ and for every $C \in \text{Tan}(V, x)$, $\sigma_C$ is the surface measure of a $(k - 1)$-plane, which coincides with $D_C$.

**Lemma 6.8.** Let $V \in \mathcal{V}_k(\mathcal{M})$ be a rectifiable varifold with free boundary at $\partial \mathcal{M}$ with generalized mean curvature $H \in L^p(\mathcal{M}, \|V\|)$ for some $p \geq 1$, $\Theta^k(\|V\|, x) \geq 1$ for $\|V\|\text{-}a.e. x \in \mathcal{M}$ and let $F$ be the set defined in Lemma 6.4. Then for every $x_0 \in F$ such that $\Theta^{k-1}_*(\sigma_V, x) > 0$ and for every $C \in \text{Tan}(V, x_0)$, the invariant subspace $D_C$ of $C$ is $(k - 1)$-dimensional; moreover there exists $\alpha_0 > 0$ such that

$$\sigma_C = \alpha_0 \mathcal{H}^{k-1 \cap D_C}.$$
Proof. Without loss of generality we can assume that \( x = 0 \) and that \( T_0 \partial \mathcal{M} \) is the subspace \( \{ x_n = 0 \} \). Using the notations for the scalings, since \( F \subset \text{Dens}(V) \), by Lemma 6.4 there exists a subsequence, not relabeled, such that \( V_j \to^* C \) with \( C \in \text{Tan}(V,0) \); we fix such \( C \in \text{Tan}(V,0) \). Moreover Lemma 6.4 asserts that \( \sigma_j \overset{\ast}{\to} \sigma_C \), where \( \sigma_C \) is the measure relative to \( C \) given by Corollary 4.1.

We first recall that the condition
\[
\Theta_k^j(\sigma_V, 0) = \liminf_{r \to 0} \frac{\sigma_V(B_r(x_0))}{r^k} > 0,
\]
together with \( \sigma_j \overset{\ast}{\to} \sigma_C \), implies that \( \sigma_C \neq 0 \). Thus, Lemma 6.7 provides \( \text{supp} \sigma_C = D_C \), where \( D_C \) is the invariant subspace \( D_C \) of \( C \), given by Lemma 6.6. \( D_C \) is a linear subspace of \( \mathbb{R}^n \) and throughout this proof we call \( m = \dim D_C \) its dimension. By definition of \( D_C \), we clearly have \( D_C \subset \{ x_n = 0 \} \).

For any \( y \in D_C \) and any \( r > 0 \), we denote by \( Q_{D_C}(y, r) \) the closed cube included in \( D_C \) with center \( y \), side of length \( r \) and faces parallel to the coordinate vectors \( e_1, \ldots, e_{n-1} \). If we set
\[
\alpha_0 = \liminf_{j \to \infty} \frac{\sigma_V(B_{r_j})}{\omega_{k-1} r_j^{k-1}},
\]
we have \( \alpha_0 > 0 \), by (6.36). Since \( \sigma_C \) is invariant by scalings (by Lemma 6.4) and translations in \( D_C \) (by Lemma 6.6), there exists a fixed \( \beta_0 > 0 \) such that
\[
\sigma_C(Q_{D_C}(y, r)) = \beta_0 r^{-k-1} \quad \forall y \in D_C \quad \forall r > 0.
\]
We are going to show that (6.37) implies \( m = k - 1 \). We argue by contradiction and by cases:

- Let us assume, by contradiction, that \( m < k - 1 \). For each \( l \in \mathbb{N} \), there exists a covering \( \{Q^l_i\}_{i=1}^{2^m} \) of \( Q_{D_C}(0,1) \) such that each \( Q^l_i \) is a cube included in \( D_C \) and of side length \( 2^{-l} \). Therefore
\[
\sigma_C(Q_{D_C}(0,1)) \leq \sum_{i=1}^{2^m} \sigma_C(Q^l_i) = 2^m \beta_0 2^{-l(k-1)} \leq \beta_0 2^{-l} \quad \lim_{l \to \infty} 0.
\]
Thus \( \sigma_C(Q_{D_C}(0,1)) = 0 \). By translation invariance of \( \sigma_C \), it follows that \( \sigma_C = 0 \), which is a contradiction.

- Let us assume now that \( m \geq k \). We first observe that by approximation, (6.35) holds also for cubes that are open in \( D_C \). Hence, taking for every \( l \in \mathbb{N} \) a covering \( \{Q^l_i\}_{i=1}^{2^m} \) of \( Q_{D_C}(0,1) \) of cubes included in \( D_C \) with disjoint interiors and of side length \( 2^{-l} \), we have
\[
\sigma_C(Q_{D_C}(0,1)) \geq \sum_{i=1}^{2^m} \sigma_C((Q^l_i)^o) = 2^m \beta_0 2^{-l(k-1)} \geq \beta_0 2^l \quad \lim_{l \to \infty} +\infty,
\]
where \( (Q^l_i)^o \) is intended in the topology of \( D_C \) which is a contradiction.

This shows that \( \dim D_C = k - 1 \). Since \( \sigma_C \) is invariant by scalings and translations in \( D_C \), we have
\[
\frac{\sigma_C(B_{r}(y))}{\mathcal{H}^{k-1}(D_C \cap B_{r}(y))} = \alpha_0 \quad \forall y \in D_C, \forall r > 0.
\]
(This shows, in particular, that the lim inf in (6.37) is in fact a limit.) By Radon-Nikodym Theorem [11, Theorem 4.7], we obtain that \( \sigma_C = \alpha_0 \mathcal{H}^{k-1} \downarrow D_C \). 
\( \square \)

6.3. Proof of Theorem 1.2 We can now prove Theorem 1.2

Proof of Theorem 1.2 By definition of \( \sigma_V^* \) we have
\[
0 < \Theta_k^{s-1}(\sigma_V^*, x) \leq \Theta_k^{s(k-1)}(\sigma_V^*, x) < +\infty \quad \text{for } \sigma_V^* \text{-a.e. } x \in \partial \mathcal{M},
\]
thus \( \sigma_V^* \) satisfies the first condition of the Marstrand-Mattila Rectifiability Criterion (Theorem 6.1).

To check the second condition of the criterion, let us notice that Corollary 5.5 yield
\[
\text{Tan}^{k-1}(\sigma_V^*, x) = \{ \sigma_C \mid C \in \text{Tan}(V, x) \} \quad \text{for } \sigma_V^* \text{-a.e. } x \in \partial \mathcal{M}.
\]
Since $\Theta^{k-1}_x(\sigma_V, x) > 0$ for every $x \in E$, Lemma 6.7 and Lemma 6.8 yield
\begin{equation}
\{\sigma_C \mid C \in \text{Tan}(V, x)\} \subset \{\alpha H^{k-1} \setminus D_C \mid D_C \text{ (}\ k-1\text{-dimensional plane)} \} \quad \forall x \in F \cap E.
\end{equation}
Moreover
\begin{equation}
\sigma^*_V(\partial M \setminus (F \cap E)) \leq \sigma^*_V(\partial M \setminus \text{Dens}(V)) + \sigma^*_V(\text{Dens}(V) \setminus F) + \sigma^*_V(\partial M \setminus E) = 0,
\end{equation}
where $F$ is the set defined in Lemma 6.7 and $E$ is defined in (1.7). Every set in the right-hand side is $\sigma^*_V$-negligible: $\sigma^*_V(\partial M \setminus \text{Dens}(V)) = 0$ by Lemma 6.2 and $p > 1$; $\sigma^*_V(\text{Dens}(V) \setminus F) = 0$ by Lemma 6.7 and $\sigma^*_V \ll \sigma_V$; $\sigma^*_V(\partial M \setminus E) = 0$ by definition of $\sigma^*_V$.

Summarizing (6.40), (6.41) and (6.42) we obtain
\[ \text{Tan}^{k-1}(\sigma^*_V, x) \subset \{\alpha H^{k-1} \setminus D_C \mid D_C \text{ (}\ k-1\text{-dimensional plane)} \} \quad \text{for } \sigma^*_V\text{-a.e. } x \in \partial M. \]
Since this is the second condition for the Mastrand-Mattila Rectifiability Criterion, we have that $\sigma^*_V$ is $(k-1)$-rectifiable.

\begin{thebibliography}{99}
\item [1] W. K. Allard. “On the First Variation of a Varifold”. In: Annals of Mathematics (1972), pp. 417–491.
\item [2] W. K. Allard. “On the First Variation of a Varifold: Boundary Behavior”. In: Annals of Mathematics (1975), pp. 418–446.
\item [3] C. De Lellis. Rectifiable Sets, Densities and Tangent Measures. Zurich lectures in advanced mathematics. European Mathematical Society, 2008.
\item [4] C. De Lellis et al. On the boundary behavior of mass-minimizing integral currents. 2018. arXiv: 1809.09457 [math.AP].
\item [5] G. De Philippis, A. De Rosa, and F. Ghiraldin. “Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies”. In: Comm. Pure App. Math. (2016).
\item [6] N. Edelen. “The free-boundary Brakke flow”. In: Journal für die reine und angewandte Mathematik (Crelles Journal) 2020 (2018).
\item [7] L.C. Evans and R.F. Gariepy. Measure Theory and Fine Properties of Functions, Revised Edition. Textbooks in Mathematics. CRC Press, 2015.
\item [8] H. Federer and W. P. Ziemer. “The Lebesgue set of a function whose distribution derivatives are p-th power summable”. In: Indiana University Mathematics Journal 22.2 (1972), pp. 139–158.
\item [9] M. Grütter and J. Jost. “Allard type regularity results for varifolds with free boundaries”. In: Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 13.1 (1986), pp. 129–169.
\item [10] P. Mattila. Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability. Cambridge University Press, 1995.
\item [11] L. Simon. Lectures on geometric measure theory. Proceedings of the Centre for Mathematical Analysis. Australian National University, 1984.
\item [12] L. Simon. Theorems on Regularity and Singularity of Energy Minimizing Maps. Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 1996.
\item [13] B. White. “Stratification of minimal surfaces, mean curvature flows, and harmonic maps.” In: Journal für die reine und angewandte Mathematik 488 (1997), pp. 1–36.
\end{thebibliography}