Passivity-Based Analysis of Sampled and Quantized Control Implementations

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Abstract

This paper studies the performance of a continuous controller when implemented on digital devices via sampling and quantization, by leveraging passivity analysis. Degradation of passivity indices from a continuous-time control system to its sampled, input and output quantized model is studied using a notion of quasi-passivity. Based on that, the passivity property of a feedback-connected system where the continuous controller is replaced by its sampled and quantized model is studied, and conditions that ensure the state boundedness of the interconnected system are provided. Additionally, the approximate bisimulation-based control implementation where the controller is replaced by its approximate bisimilar symbolic model whose states are also quantized is analyzed. Several examples are provided to illustrate the theoretical results.

1 Introduction

Cyber-physical control systems consist of software-based controllers interacting with physical processes (i.e., the plant to be controlled). While the control design can be done using continuous-time and continuous-space methods, in order to guarantee the desirable operation of the closed-loop system, it is important to keep in mind the restrictions the software implementation imposes. In particular, sampling and quantization are prevalent in real control implementations. Therefore, sampled-data control and quantized control have attracted the attention of researchers for decades. However, many problems are still open, especially when dynamics of the system are nonlinear or when both sampling and quantization are considered [5, 3, 8, 9]. This work presents a unified framework for analyzing control implementations using tools from passivity theory. In addition to analyzing the effects of sampling and quantization of the input and output signals within this framework, we also consider quantization of the internal states of the controller. This enables analysis of controllers implemented with finitely many bits, and can be relevant in extremely resource-constrained settings like micro-robotics or applications where a computer that can do floating-point arithmetic is not feasible, which constitute the motivation for our work.

The concept of passivity emerged from the study of energy dissipation in circuit analysis, which, roughly speaking, means that a system can not generate “internal energy” on its own. In the paper [43], Willems systematized the theory of dissipativity (with passivity as its special case) using the concepts of storage function and supply rate, where dissipativity is defined as the property that the rate of increase of the storage function is not larger than the supply rate. Since then, the fundamental connections between dissipativity, stability (either in the sense

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of Lyapunov or input/output) and optimality for control systems have been explored, making passivity/dissipativity theory a widely-used tool in control theory [1, 20, 37]. One important property of passive systems is the compositionality - the parallel or feedback connection of two passive systems remains passive. This property offers an effective method for analyzing the stability of large-scale, interconnected systems in a compositional way [26, 42]. Several challenging problems in cyber-physical systems, such as quantization, time delay and packet drops, which are induced by the digital devices or communication networks, have also been studied under the passivity/dissipativity-based framework (see [16, 39, 36, 10, 11, 2] and references therein).

A general form of dissipativity that has been well-studied is the QSR-dissipativity whose supply rate is a quadratic function of the input and output. A special case of the QSR-dissipativity allows one to use two real numbers, which are termed passivity indices (or passivity levels), to characterize the shortage/excess of passivity for a system. These two numbers can be either positive or negative, where a negative value implies a shortage of passivity while a positive value implies an excess. Passivity indices are rather useful in the analysis of nonlinear and interconnected systems; for instance, [14, 15] showed that the feedback interconnection of two non-passive systems is finite gain $\mathcal{L}_2$ stable provided that the shortage of passivity of one component can be compensated for by the excess of passivity of another component, [28] provided a controller design method for asymptotic stabilization of a class of nonlinear systems using passivity indices, [38] and [10] presented conditions on passivity indices of a closed-loop system in the continuous-time setting and with the event-triggered mechanism, respectively.

It is known that passivity is not preserved under sampling or quantization in general (e.g., see [22, 50, 2, 32]). However, passivity indices, as an quantitative “abstraction” of the passivity property of a system, can be used to study how the properties of two related systems (e.g., a continuous-time system and its sampled or quantized system) differ from each other. Some interesting results have been obtained along this direction; for example, [32] analyzed the passivity degradation for a system under time sampling with the zero-order hold and the ideal sampler, [44] showed that passivity indices of one system can be inferred from an approximate model of that system when the norm of the model error is small (in some appropriate sense) and the approximate model has an excess of passivity. In spite of these interesting results, passivity analysis for systems that are both sampled and quantized, as well as its control implementations in a feedback loop, are still lacking, to the best of our knowledge.

Symbolic models for dynamical systems provide a unified framework for studying the interactions of software and physical phenomena [40]. Such symbolic models are sampled and quantized (in state, input and output), which are used in control software synthesis from high-level specifications [33]. The basic workflow in these approaches is to compute an approximate symbolic model of the plant based on (bi)simulation relations at first, then synthesize a discrete controller for the symbolic plant model, and finally refine that controller and compose it with the plant. The constructed controller can be implemented in software and the closed-loop system is guaranteed to satisfy the desired high-level specifications. A major limitation of these techniques is the curse of dimensionality: the complexity of the symbolic plant model grows exponentially with the dimension of the state-space of the plant (when quantized uniformly). To tackle this computational problem, compositional approaches have been utilized. For instance, [29] proposed a compositional symbolic abstraction method for a class of continuous-time nonlinear control systems using the notion of approximate disturbance bisimulation and discussed the related controller synthesis problem, [6] provided a compositional control design method, which is inspired by two classes of compositional techniques (small gain theorem and assume-guarantee reasoning), for the feedback composition of two discrete systems with a persistency specification. As passivity has the nice property of compositionality, it has been used to tame the computational complexity of symbolic control. For instance, by using simulation functions and the dissipativity-based approach, [35, 47] investigated compositional schemes for abstracting a network of continuous-time control systems to a lower dimensional, continuous-time intercon-
nected system. However, results that combine the bisimulation-based finite state abstraction and the passivity/dissipativity-based approach are still rare.

In this paper, we analyze the passivity property of sampled and quantized controller implementations with passivity indices as the main tool. Specifically, the question we consider is - suppose that a continuous, dynamic controller has been designed to ensure specified passivity indices for the closed-loop system, what guarantees on passivity can be inherited if a sampled and quantized controller is implemented? and under what conditions the closed-loop system is (ultimately) bounded? To this end, we first propose the notion of quasi-passivity and the strong detectability for discrete-time control systems, and give several lemmas that relate ultimate boundedness, strong detectability and passivity indices. Then, we explore how passivity degrades from a continuous-time system to its sampled, input and output quantized model, where a set of results quantifying the passivity degradation are derived, which relate the passivity indices, the sampling time and the input/output quantization precisions. Based on that, we study the implementation of the controller by replacing the continuous controller with its sampled, input and output quantized model, and we provide conditions (on the passivity indices and the strong detectability of the interconnected components) under which the state of the closed-loop system eventually converges into a compact set whose size can be made arbitrarily small by choosing the input/output quantization precisions small enough. Finally, we consider implementing an approximate bisimulation-based symbolic controller whose states are also quantized, in the closed-loop system, and give conditions that guarantee the ultimate boundedness of the closed-loop system.

A preliminary version of the paper appeared in the conference publication [46]. In this paper, we extend the results in [46] in the following important ways: new lemmas that relate quasi-passivity and boundedness are included, several new theorems about passivity analysis of the closed-loop system are given by using relaxed assumptions, all the complete proofs are included, and a few new examples are provided. The remainder of the paper is organized as follows. In Section 2 some preliminaries are introduced and the problem formulation is given. In Section 3 some lemmas about quasi-passivity and ultimate boundedness of compositional systems are provided. Passivity degradation of continuous systems under sampling, input and output quantization is considered in Section 4 and passivity analysis of the closed-loop system with such control implementations is discussed in Section 5. Results are extended to the analysis of the approximate bisimulation-based control implementation in Section 6. Finally, some concluding remarks are given in Section 7.

Nomenclature. The set of integer, non-negative integer, real, and non-negative real numbers are denoted as \( Z, Z_\geq, \mathbb{R}, \mathbb{R}_\geq, \) respectively. The \( \ell_2 \) and \( \ell_\infty \) norm of a vector \( x \in \mathbb{R}^n \) are denoted as \( |x| \) and \( |x|_\infty \), respectively; the \( \ell_\infty \) norm of a function \( \phi : Z_\geq \to \mathbb{R}^n \) is denoted as \( ||\phi|| := \sup_{k \in Z_\geq} |\phi[k]| \). Given a matrix \( M, M > 0 \) (or \( M \prec 0 \)) denotes \( M \) is positive definite (or negative definite). and \( \text{diag}(a_1, ..., a_n) \) denotes the diagonal matrix with \( a_1, ..., a_n \) the diagonal entries. For any \( A \subseteq \mathbb{R}^n \) and \( \mu > 0 \), \([A]_\mu := \{a \in A | a_i = k_i \mu, k_i \in Z, i = 1, 2, ..., n \}\). A relation \( R \subseteq A \times B \) is identified with the map \( R : A \to 2^B \), which is defined by \( b \in R(a) \) if and only if \( (a,b) \in R \). For a set \( S \), the set \( R(S) \) is defined as \( R(S) = \{b \in B : \exists a \in S, (a,b) \in R \} \). Given a relation \( R \subseteq A \times B \), \( R^{-1} \) denotes the inverse relation of \( R \), i.e., \( R^{-1} := \{(b,a) \in B \times A : (a,b) \in R \} \). A continuous function \( f : \mathbb{R}_\geq \to \mathbb{R}_\geq \) belongs to class \( K \) if it is strictly increasing and \( f(0) = 0 \); \( f \) belongs to class \( K_\infty \) if \( f \in K \) and \( f(r) \to \infty \) as \( r \to \infty \); a continuous function \( f : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n \) belongs to class \( KL \) if for each fixed \( s \), function \( f(s,r) \in K_\infty \) with respect to \( r \) and for each fixed \( r \), function \( f(r,s) \) is decreasing with respect to \( s \) and \( f(r,s) \to 0 \) as \( s \to \infty \).
2 Preliminaries and Problem Statement

2.1 Transition Systems and Approximate Bisimulation

A continuous-time control system is a tuple $\Sigma = (X, U, Y, f, h)$ where $X \subseteq \mathbb{R}^n$ is a set of states, $U \subseteq \mathbb{R}^m$ is a set of inputs, $Y \subseteq \mathbb{R}^s$ is a set of outputs, $f : X \times U \to \mathbb{R}^n$ is Lipschitz continuous, and $h : X \times U \to \mathbb{R}^s$ is continuous. The state, input and output of $\Sigma$ at time $t \in \mathbb{R}_{\geq 0}$ are denoted by $x(t)$, $u(t)$, $y(t)$, respectively, and their evolution is governed by:

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \\
y(t) &= h(x(t), u(t)), \quad \forall t \in \mathbb{R}_{\geq 0}.
\end{align*}
\]

(1)

We assume that $f(0, 0) = 0$ and $h(0, 0) = 0$. We also assume that given any sufficiently regular control input signal $u : [0, T] \to U$ with $T > 0$ and any initial condition $x_0 \in X$, there exist a unique state trajectory $\mathbf{x}$ and a corresponding output trajectory $\mathbf{y}$ defined on $[0, T]$ satisfying $x(0) = x_0$ and Eq. (1). Denote by $\mathbf{x}(\tau, x_0, u)$ the state reached at time $\tau$ under the input $u$ from the initial state $x_0$ of $\Sigma$.

A discrete-time control system is a tuple $\Sigma_d = (X, U, Y, f_d, h_d)$ where $X \subseteq \mathbb{R}^n$ is a set of states, $U \subseteq \mathbb{R}^m$ is a set of inputs, $Y \subseteq \mathbb{R}^s$ is a set of outputs, $f_d : X \times U \to X$ and $h_d : X \times U \to \mathbb{R}^s$ are both continuous maps. The state, input and output of $\Sigma_d$ at time step $k \in \mathbb{Z}_{\geq 0}$ are denoted by $x[k]$, $u[k]$, $y[k]$, respectively, and their evolution is governed by:

\[
\begin{align*}
x[k+1] &= f_d(x[k], u[k]), \\
y[k] &= h_d(x[k], u[k]), \quad \forall k \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

(2)

The state and output trajectories of the system $\Sigma_d$ are discrete-time signals satisfying Eq. (2).

Definition 1. A transition system is a quintuple $T = (Q, L, O, \rightarrow, H)$, where:

- $Q$ is a set of states;
- $L$ is a set of inputs;
- $O$ is a set of outputs;
- $\rightarrow \subseteq Q \times L \times Q$ is the transition relation;
- $H : Q \times L \to O$ is the output function.

In the following, we denote an element $(q, \ell, p) \in \rightarrow$ in a transition relation by $q \xrightarrow{\ell} p$ where $p, q \in Q$, $\ell \in L$.

Given a continuous-time system $\Sigma = (\mathbb{R}^n, U, \mathbb{R}^s, f, h)$ and a sampling time $\tau$, we suppose that the control inputs of $\Sigma$ are piecewise constant, that is, $u(t) = u((k-1)\tau) \text{ for any } t \in [(k-1)\tau, k\tau), k \in \mathbb{Z}_{\geq 0}$. Then, similar to [33], we define a transition system $T_\tau(\Sigma) = (X_1, U_1, Y_1, \xrightarrow{1}, H_1)$ associated with the time-sampling of $\Sigma$ as follows:

- $X_1 = \mathbb{R}^n$;
- $U_1 = U$;
- $Y_1 = \mathbb{R}^s$;
- $p \xrightarrow{u_{i-1}} q$ if $\mathbf{x}(\tau, p, u) = q$ where $u : [0, \tau) \to \{u\}$, $u \in U_1$;
- $H_1(x, u) = h(x, u)$.


We interpret the trajectories of $T_r(\Sigma)$ in discrete-time, that is, it has an equivalent representation in terms of a discrete-time control system as in [2], where its state, input and output at time step $k \in \mathbb{Z}_{\geq 0}$ are denoted by $x[k]$, $u[k]$, $y[k]$, respectively. Note that $T_r(\Sigma)$ can be obtained by putting $\Sigma$ between a zero-order hold device (D/A) and an uniform sampler (A/D).

By further quantizing the state and input spaces of $T_r(\Sigma)$, we obtain an infinitely countable transition system $T_{r,\mu,\eta}(\Sigma) = (X_2, U_2, Y_2, H_2)$ for some $r, \mu, \eta > 0$ as follows (see also [33]):

- $X_2 = [\mathbb{R}^n]_\eta$;
- $U_2 = [U]_\mu$;
- $Y_2 = \mathbb{R}^r$;
- $p \xrightarrow{u} q$ if $|x(\tau, p, u) - q|_\infty \leq \eta/2$ where $u : [0, \tau) \to \{u\}$, $u \in U_2$;
- $H_2(x, u) = h(x, u)$.

Bisimulation is a binary relation between two transition systems, which, roughly speaking, requires the two systems match each other’s behavior [40]. In [12], the exact bisimulation was generalized to $\epsilon$-approximate bisimulation, which allows the states of two transition systems to be within certain bounds. To further capture the input and output behaviors of transition systems, we consider the following $(\epsilon, \mu)$-approximate bisimulation relation adopted from [19].

**Definition 2.** Given two transition systems $T_1 = (Q_1, L, O, \rightarrow_1, H_1)$ and $T_2 = (Q_2, L, O, \rightarrow_2, H_2)$ where the state sets $Q_1, Q_2$ are equipped with the same metric $d_1$ and the input set $L$ is equipped with the metric $d_2$, for any $\epsilon, \mu \in \mathbb{R}^+$, a relation $R \subset Q_1 \times Q_2$ is said to be an $(\epsilon, \mu)$-approximate bisimulation relation between $T_1$ and $T_2$, if for any $(q_1, q_2) \in R$:

(i) $d_1(q_1, q_2) \leq \epsilon$;

(ii) $q_1 \xrightarrow{l_1} p_1$ implies the existence of $l_2 \in L$ such that $d_1(l_1, l_2) \leq \mu$, $q_2 \xrightarrow{l_2} p_2$ and $(p_1, p_2) \in R$;

(iii) $q_2 \xrightarrow{l_2} p_2$ implies the existence of $l_1 \in L$ such that $d_1(l_1, l_2) \leq \mu$, $q_1 \xrightarrow{l_1} p_1$ and $(p_1, p_2) \in R$.

If there exists an $(\epsilon, \mu)$-approximate bisimulation relation $R$ between $T_1$ and $T_2$ such that $R(Q_1) = Q_2$ and $R^{-1}(Q_2) = Q_1$, $T_1$ is said to be $(\epsilon, \mu)$-bisimilar to $T_2$, which is denoted as $T_1 \approx_{(\epsilon, \mu)} T_2$.

The $(\epsilon, \mu)$-approximate bisimulation defined above is slightly different from that in [19]. The two definitions coincide if we define an observation map (in the language of [19]) that is chosen to be an identity mapping from $Q$ to $O$ for $\Sigma$. In fact, this is the case in [33]: the observation maps of $\Sigma$, $T_r(\Sigma)$ and $T_{r,\mu,\eta}(\Sigma)$ are all chosen to be identity functions. In our definitions above, the output functions $h, H_1, H_2$ are used for the passivity analysis, not in the definition of $(\epsilon, \mu)$-approximate bisimulation.

The continuous-time system $\Sigma$ in [1] is called incrementally input-to-state stable (δ-ISS) if it is forward complete and there exist functions $\beta_1 \in \mathcal{KL}$ and $\beta_2 \in \mathcal{K}_\infty$ such that for any $t \in \mathbb{R}_{\geq 0}$, any initial state $x_1, x_2 \in \mathbb{R}^n$ and any input $u, v$, it holds that $|x(t, x_1, u) - x(t, x_2, v)| \leq \beta_1(|x_1 - x_2|, t) + \beta_2(||u - v||)$ [1].

1Note that the sets of state, input and output of $T_{r,\mu,\eta}(\Sigma)$ are all countable.
Lemma 1. Consider the continuous-time control system $\Sigma$ in [1] and any desired precision $\epsilon > 0$. If $\Sigma$ is $\delta$-ISS satisfying $|x(t, x_1, u) - x(t, x_2, v)| \leq \beta_1(|x_1 - x_2|, t) + \beta_2(\|u - v\|)$ and parameters $\tau, \eta, \mu > 0$ satisfy the inequality

$$\beta_1(\epsilon, \tau) + \beta_2(\mu) + \eta/2 \leq \epsilon,$$

then $T_\tau(\Sigma) \cong^{(\epsilon, \mu)} T_{\tau\mu\eta}(\Sigma)$.

Lemma 1 can be proved following the proof of Theorem 5.1 in [33]. The key step is to show (ii) and (iii) in Definition 2 by the following fact: for any input $u \in U_1$ of $T_\tau(\Sigma)$, we can choose input $\ell_2 \in U_2$ of $T_{\tau\mu\eta}(\Sigma)$ such that $|\ell_1 - \ell_2| \leq \mu$, and vice versa. Specifically, let $\ell_2 = Q(\ell_1)$ where the quantization function $Q(\cdot)$ is defined entry-wisely as in (12).

We call the transition system $T_{\tau\mu\eta}(\Sigma)$ a symbolic model for $\Sigma$ where $T_\tau(\Sigma) \cong^{(\epsilon, \mu)} T_{\tau\mu\eta}(\Sigma)$.

One nice property of this symbolic model is that its evolution can be chosen to be deterministic [11], which is appropriate for discrete software-based implementation. Particularly, if the state space and input space of $T_{\tau\mu\eta}(\Sigma)$ are chosen to be bounded sets, then the resulting $T_{\tau\mu\eta}(\Sigma)$ will be a finite transition system.

2.2 Passivity Indices

Definition 3. [4] A continuous-time control system $\Sigma$ as in [1] is called dissipative with respect to a supply function $w(u, y)$ if there exists a positive semi-definite storage function $V(x)$ such that the following (integral) dissipation inequality is satisfied for any $x(t_0) \in \mathbb{R}^n$ with $t_0, t_1 \in \mathbb{R}_{\geq 0}, t_0 < t_1$, and any admissible inputs $u$:

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} w(u(s), y(s)) \, ds.$$

(4)

If $V(x)$ is differentiable, an equivalent (differential) form of (4) is $\dot{V}(x(t)) \leq w(u(t), y(t)), \forall t \in \mathbb{R}_{\geq 0}$.

Definition 4. [4] A discrete-time control system $\Sigma_d$ as in [2] is called dissipative with respect to the supply function $w(u, y)$ if there exists a positive semi-definite storage function $V(x)$ such that the following dissipation inequality is satisfied for any $x[k_0] \in \mathbb{R}^n$ with $k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1$, and any admissible input $u$:

$$V(x[k_1]) - V(x[k_0]) \leq \sum_{k=k_0}^{k_1-1} w(u[k], y[k]).$$

(5)

It is known that the summation-form of the dissipation inequality [5] is equivalent to the one-step form: $V(x[k + 1]) - V(x[k]) \leq w(u[k], y[k]), \forall k \in \mathbb{Z}_{\geq 0}$ (see [4]).

In the rest of the paper, we assume that the input and the output of $\Sigma$ (or $\Sigma_d$) have the same dimension, i.e., $m = s$.

Definition 5. A system $\Sigma$ (or $\Sigma_d$) is called input feedforward output feedback passive (IF-OFP) with passivity indices $(\nu, \rho)$ if it is dissipative with respect to the supply function $w(u, y) = u^T y - \nu u^T u - \rho y^T y$

for some $\nu, \rho \in \mathbb{R}$, denoted as IF-OFP$(\nu, \rho)$.

Passivity indices $\nu, \rho$ reflect the excess or shortage of passivity of a system. A system that is IF-OFP$(\nu, \rho)$ with $\nu > 0, \rho > 0$ has an excess of passivity, and therefore, is passive. The indices $\nu, \rho$ are not unique for a given system, and they are always in the set $\{\nu, \rho \in \mathbb{R} \mid \nu \rho < 1/4\} \cup \{\nu, \rho \in \mathbb{R} \mid \nu \rho = 1/4, \rho > 0\}$ [45]. For a system that is IF-OFP$(\nu, \rho)$, it is also IF-OFP$(\nu', \rho')$ for any $\nu' < \nu, \rho' < \rho$.
2.3 Problem Statement

We will consider the closed-loop configurations shown in Fig. 1 where the feedback connection of $T_r(P)$ and $\hat{T}_r(\Sigma)$ is denoted as $T_r(P) \times_{\mathcal{F}} \hat{T}_r(\Sigma)$, and the feedback connection of $T_r(P)$ and $\check{T}_{\tau \mu \eta}(\Sigma)$ is denoted as $T_r(P) \times_{\mathcal{F}} \check{T}_{\tau \mu \eta}(\Sigma)$. In these configurations, $P$ and $\Sigma$ are both continuous models, which can be considered as the original plant and the controller, respectively; $T_r(P)$ (resp. $T_r(\Sigma)$) is the time-sampled model of $P$ (resp. $\Sigma$), which consists of $P$ (resp. $\Sigma$), a zero-order hold device and an uniform sampler; $T_r(\Sigma)$ (resp. $\hat{T}_r(\Sigma)$) is the model that consists of $T_r(\Sigma)$ (resp. $\check{T}_{\tau \mu \eta}(\Sigma)$), the input quantizer $Q_1$ and the output quantizer $Q_2$. The setup in Fig. 1 (a) can be considered as replacing $\Sigma$ with $\hat{T}_r(\Sigma)$, the sampled and quantized model of $\Sigma$; the setup in Fig. 1 (b) can be considered as replacing $\Sigma$ with $\check{T}_{\tau \mu \eta}(\Sigma)$, the approximate bisimilar symbolic model of $T_r(\Sigma)$. We assume that 1) the external reference inputs $r_i$, $i=1,2$ to the closed-loop system are discrete-time signals; 2) the sampling times of $T_r(P)$ and $T_r(\Sigma)$ are both $\tau$; 3) all the discrete-time signals in the feedback loop are synchronized.

![Diagram](image1.png)

Figure 1: Two closed-loop system setups considered.

The problems that will be studied are the following: Given the passivity indices of $P$ and $\Sigma$ and the setup of Fig. 1 (a), what passivity property can be preserved for $T_r(P) \times_{\mathcal{F}} \hat{T}_r(\Sigma)$, and under what conditions the states of $T_r(P) \times_{\mathcal{F}} \hat{T}_r(\Sigma)$ are bounded? Similarly, given the passivity indices of $P$ and $\Sigma$ and the setup of Fig. 1 (b), under what conditions the states of $T_r(P) \times_{\mathcal{F}} \check{T}_{\tau \mu \eta}(\Sigma)$ are bounded?

We will study the passivity of $T_r(P) \times_{\mathcal{F}} \hat{T}_r(\Sigma)$ and give a boundedness result for it in Section 5 based on the passivity degradation results in Section 4. After that, we will investigate the state boundedness of $T_r(P) \times_{\mathcal{F}} \check{T}_{\tau \mu \eta}(\Sigma)$ in Section 6. The main theorems of the paper are
summarized in Table 1

| Theorem 1 | passivity degradation from $\Sigma$ to $T_\tau(\Sigma)$ |
| Theorem 2 | passivity degradation from $\Sigma_r$ to $T_\tau(\Sigma)$ |
| Theorem 3 | passivity inequality of $T_\tau(P) \times_T T_\tau(\Sigma)$ |
| Theorem 4 | state boundedness of $T_\tau(\Sigma)$ |
| Corollary 6 | state boundedness of $T_\tau(P) \times_T T_{\tau,\mu,\eta}(\Sigma)$ |

Table 1: Summary of the main theorems.

3 Quasi-Passivity and Strong Detectability

In this section, we introduce quasi-passivity, strong detectability and some relevant lemmas, which are of interest by their own and will be used in later sections.

Inspired by the notion of quasi-dissipative systems (also known as almost-dissipative systems), which allows for “internal energy generation” (see [7, 34] for more details), we define a quasi-passive system as follows.

**Definition 6.** The continuous (resp. discrete) time system $\Sigma$ in (1) (resp. $\Sigma_d$ in (2)) is called input feedforward output feedback quasi-passive (IF-OFQP) if it is dissipative with respect to the supply function $w(u,y) = u^T y - \nu u^T u - \rho y^T y + \delta$, denoted as IF-OFQP$(\nu, \rho, \delta)$.

A discrete-time system $\Sigma_d$ that is IF-OFQP$(\nu, \rho, \delta)$ satisfies the following inequality

$$V(x[k_1]) - V(x[k_0]) \leq \sum_{i=k_0}^{k_1-1} \left( u[i]^T y[i] - \nu |u[i]|^2 - \rho |y[i]|^2 + \delta \right)$$

for any $k_0, k_1 \in \mathbb{Z}_{\geq 0}$, $k_0 < k_1$ and any admissible input $u[i]$ where $V(x)$ is a positive semi-definite function.

Consider the feedback-connection of two discrete-time systems $\Sigma_1$, $\Sigma_2$ shown in Fig. 2. When $\Sigma_1$ and $\Sigma_2$ are both IF-OFP, the passivity indices of their feedback-connected system was discussed in [48]. Similarly, we have the following Lemma 2 for the feedback connection of two IF-OFQP systems. We define $x_1$ and $x_2$ as the states of $\Sigma_1$ and $\Sigma_2$, respectively, $V_1(x_1)$ and $V_2(x_2)$ as the storage functions of $\Sigma_1$ and $\Sigma_2$, respectively, $r := (r_1^T, r_2^T)$ as the external input, and $y := (y_1^T, y_2^T)^T$ as the output.

![Figure 2: Feedback connection of two discrete-time systems](image)

**Lemma 2.** Consider the feedback connection of two discrete-time systems $\Sigma_1, \Sigma_2$ as shown in Fig. 2. If $\Sigma_i (i = 1, 2)$ is IF-OFQP$(\nu_i, \rho_i, \delta_i)$, then the feedback-connected system is IF-OFQP$(\hat{\nu}, \hat{\rho}, \hat{\delta})$ with respect to the input $r$ and the output $y$ where $\hat{\delta} = \delta_1 + \delta_2$ and $\hat{\nu}, \hat{\rho}$ can be chosen as

$$\begin{cases} 
\hat{\nu} < \min\{\nu_1, \nu_2\}, \\
\hat{\rho} \leq \min\{\rho_1 - \frac{\dot{\nu}_2}{\nu_2 - \hat{\nu}} \rho_2 - \frac{\dot{\nu}_1}{\nu_1 - \hat{\nu}} \nu_1\}
\end{cases}$$

(7)
The proof of this lemma is similar to that of Theorem 6 in [38], and its sketch is given in Appendix A.

Next we define the strong detectability of discrete-time systems.

**Definition 7.** The discrete-time system (2) is said to be N-step strongly detectable (SD) if there exist two numbers \( N \in \mathbb{Z}_{>0} \), \( \theta \geq 0 \) and a positive definite matrix \( M > 0 \) such that for any \( k_0 \in \mathbb{Z}_{\geq 0} \), any initial state \( x[k_0] \in \mathbb{R}^n \) and any admissible input \( u[k] \), the following holds:

\[
\sum_{k=k_0}^{k_0+N} \vartheta |u[k]|^2 + |y[k]|^2 \geq x[k_0]^{\top} M x[k_0],
\]

(8)

The intuition behind the strong detectability is that large initial states must dictate large input-output signals. Similar definitions were also given in [34, 27, 38]. The strong detectability in Def. 7 implies the zero-state observability given in Def. 6.5 of [20], because \( u[k] = 0, y[k] = 0, k = k_0, ..., k_0 + N_0 \), implies \( x[k_0] = 0 \).

**Remark 1.** We can show that a linear discrete-time system that is observable with an observability index \( v \) is \( v \)-step SD. Indeed, suppose that the system is given as \( x[k + 1] = A_d x[k] + B_d u[k], y[k] = C_d x[k] + D_d u[k] \), and define \( U = [u[k_0]^\top, ..., u[k_0 + v - 1]^\top]^\top \), \( Y = [y[k_0]^\top, ..., y[k_0 + v - 1]^\top]^\top \). Then, we can express \( Y \) as \( Y = Ox[k_0] + HU \) where \( O \) is the observability matrix and \( H \) is a matrix that can be constructed easily from \( A_d, B_d, C_d, D_d \). Hence, \( \sum_{k=k_0}^{k_0+v} \vartheta |u[k]|^2 + |y[k]|^2 = \vartheta U^\top U + x[k_0]^\top O^\top Ox[k_0] + U^\top H^\top HU + 2x[k_0]^\top O^\top HU \). Since \( \text{rank}(O^\top O) = \text{rank}(O) = n \), \( O^\top O \succ 0 \). Then it is easy to see that there always exist \( \vartheta > 0 \) and \( M > 0 \) such that (8) holds.

**Example 1.** Consider the double integrator system \( x_1[k + 1] = x_2[k], x_2[k + 1] = u[k], y[k] = x_1[k] + u[k] \) where \( x_1, x_2, y, u \in \mathbb{R} \). It is easy to verify that the system is not 0-step SD, but 1-step SD with \( \theta = 2 \) and \( M = \text{diag}(0.5, 0.5) \).

The strong detectability of a feedback-connected system can be derived by the strong detectability of each subsystem, which is shown by the following Lemma 3. The proof of this lemma is given in Appendix B.

**Lemma 3.** Consider the feedback connection of two discrete-time systems \( \Sigma_1, \Sigma_2 \) as shown in Fig. 2. If \( \Sigma_i (i = 1, 2) \) is \( n_i \)-step SD, then the feedback-connected system is \( N \)-step SD with respect to the input \( r \) and the output \( y \) where \( N = \max\{N_1, N_2\} \).

The next lemma shows that a discrete-time system \( \Sigma_d \) that is SD and IF-OFQP with \( \rho > 0 \) has the (uniform) bounded-input-bounded-state property. The proof of this lemma is given in Appendix C.

**Lemma 4.** Suppose that the system \( \Sigma_d \) given in (2) is 1) N-step SD satisfying (8), 2) IF-OFQP(\( \nu, \rho, \delta \)) satisfying (6) with \( \rho > 0, \delta \geq 0 \) and a function \( V(x) \) that is continuous, positive semi-definite, radially unbounded. Let \( \lambda \) be a number such that \( 0 < \lambda < \rho \), and define \( \eta_1 = \frac{1}{\lambda} - \nu > 0 \), \( \eta_2 = \rho - \lambda > 0 \). Then, 1) for any \( k \in \mathbb{Z}_{\geq 0} \), it holds that \( x[k] \in \mathcal{D}_1 \) where

\[
\mathcal{D}_1 := \{ z \mid V(z) \leq c_1(x[0]) + c_3 \}
\]

(9)

with

\[
\begin{align*}
c_1 &= \max_{z \in \mathcal{C}_1} V(z), \\
c_2 &= (N + 1)\left(\eta_1 + \vartheta \eta_2\right)\|u\|^2 + \delta, \\
c_3 &= (N + 1)(\eta_1\|u\|^2 + \delta), \\
C_1 &= \{ z \mid z^{\top} M z \leq \max\{x[0]^{\top} M x[0], c_2/\eta_2\} \}.
\end{align*}
\]

(10)
2) there exists $K \in \mathbb{Z}_{\geq 0}$ such that $x[k] \in D_2$ for all $k \geq K$ where

$$D_2 := \{ x \mid V(x) \leq c_2 + c_4 \}$$

with $c_2$ given in (10),

$$c_4 = \max_{z \in c_2} V(z), \quad C_2 = \{ z \mid z^T M z \leq (c_2 + c_5)/\eta_2 \},$$

and $c_5 > 0$ a positive number.

**Remark 2.** Lemma 6.7 in [20] shows the zero-input Lyapunov stability for continuous-time, output strictly passive systems with the zero-state observability assumption. Lemma 4 is a complement to Lemma 6.7 in the discrete-time, quasi-passivity setting.

The following two definitions are taken from [25].

**Definition 8.** The discrete-time system $\Sigma_d$ given in (2) is called (globally) input-to-state practically stable (ISpS) if there exist a $K\mathcal{L}$-function $\beta_1$, a $K$-function $\beta_2$ and a positive constant $d$ such that, for any $u[k]$ with $\|u\| < \infty$ and any $x_0 \in \mathbb{R}^n$, it holds that

$$|x[k]| \leq \beta_1(|x_0|, k) + \beta_2(\max_{0 \leq j \leq k - 1} |u[j]|) + d, \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

**Definition 9.** A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called the ISpS-Lyapunov function for $\Sigma_d$ if there exist $d_1, d_2, a, b, c, \lambda > 0$ with $c \leq b$ and a $K$-function $\sigma$, such that the following hold: 1) $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) + d_1, \forall x \in \mathbb{R}^n$ where $\alpha_1(s) = as^\lambda$, $\alpha_2(s) = bs^\lambda$; 2) $V(f_d(x, u)) - V(x) \leq -\alpha_3(|x|) + \sigma(|u|) + d_2, \forall u \in \mathbb{R}^m, \forall x \in \mathbb{R}^n$.

Clearly, $\Sigma_d$ is input-to-state stable (ISS) when $d = 0$ in Definition 8. In Theorem 2.5 in [25], it was shown that the discrete-time ISpS-Lyapunov function is a sufficient condition for $\Sigma_d$ to be ISpS.

The following Lemma 5 shows conditions under which a discrete-time, quasi-passive system $\Sigma_d$ is ISpS. Its proof is shown in Appendix 1.

**Lemma 5.** If the system $\Sigma_d$ given in (2) is 1) 0-step SD satisfying (5), 2) IF-OFQP($\nu, \rho, \delta$) satisfying (6) with $\rho > 0, \delta \geq 0$ and a continuous function $V(x)$ that satisfies $a|x|^2 \leq V(x) \leq b|x|^2 + d_1$ for some $a, b > 0, d_1 \geq 0$, then $\Sigma_d$ is ISpS.

Lemma 5 establishes the connection between quasi-passivity and input-to-state practically stability, which enables us to use the results of ISpS (or ISS) to analyze IF-OFQP systems under certain circumstances [17, 18, 25]. For instance, if $V(x)$ in Lemma 4 satisfies $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \forall x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2$ are $K_\infty$-functions, then the ultimate bound of $\Sigma_d$ can be derived similar to the proof of Lemma 3.5 in [18]. Specifically, the set $D_2$ can be given as

$$D_2 = \{ x \mid V(x) \leq \alpha_4^{-1} \circ \alpha_5^{-1}((\eta_1 + \vartheta \eta_2)\|u\|^2 + \delta) \}$$

where $\alpha_4 \in K_\infty$ satisfies $\alpha_4 \leq \alpha_3 \circ \alpha_2^{-1}$, $id - \alpha_4 \in K$, $id$ is the identity function, $\alpha_3(x) = x^T M x \in K_\infty$, and $\alpha_5 \in K_\infty$ satisfies $id - \alpha_5 \in K_\infty$.

We point out that the set $D_2$ expressed in (11) or (13) can be made arbitrarily small by choosing $\|u\|$ and $\delta$ small enough.

**Remark 3.** In [13], the input-to-state stability was discussed by a relaxed ISS-Lyapunov function, which satisfies a weaker decrease condition defined over a bounded time interval. Because of the N-step strongly detectability condition, the proof of ultimate boundedness in Lemma 4 also has to consider the decrease of $V$ every $N+1$ steps; however, it is different from the proof of [13].
4 Passivity Degradation Under Sampling and Quantization

In this section, we study the degradation of passivity indices from a continuous-time system $\Sigma$ to the time-sampled system $T_\tau(\Sigma)$ and the time-sampled, input/output quantized system $\hat{T}_\tau(\Sigma)$. Configurations of $\Sigma$, $T_\tau(\Sigma)$ and $T_\tau(\Sigma)$ are shown in Fig. 3.

![Diagram](image)

(a) $\Sigma$ is a continuous time model.

(b) $T_\tau(\Sigma)$ is a time-sampled model of $\Sigma$, which consists of $\Sigma$, a ZOH device and an uniform sampler.

(c) $\hat{T}_\tau(\Sigma)$ is the model that consists of $T_\tau(\Sigma)$, the input quantizer $Q_1$ and the output quantizer $Q_2$.

Figure 3: Three system setups considered.

We first make the following two assumptions on the output function $h$ of $\Sigma$.

**Assumption 1.** The output function $h$ has the additive form $h(x, u) = h_1(x) + h_2(u)$ for some functions $h_1, h_2$.

**Assumption 2.** There exist a constant $\gamma > 0$ and a function $\beta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $\beta(0) = 0$ such that for any initial state $x_0 \in \mathbb{R}^n$, $h_1$ satisfies the following inequality for any $T > 0$ and any admissible $u(t)$:

$$
\int_0^T |\dot{h}_1(x(t))|^2 \, dt \leq \gamma^2 \int_0^T |u(t)|^2 \, dt + \beta(x_0). \tag{14}
$$

**Remark 4.** Similar to (14), a gain assumption without considering the initial state was made in [22, 44]: for any $T > 0$ and any admissible $u(t)$, it holds that

$$
\int_0^T |\dot{h}(t)|^2 \, dt \leq \gamma^2 \int_0^T |u(t)|^2 \, dt. \tag{15}
$$

But for the inequality (15) to hold, the function $h$ in [22, 44] needs to be independent of $u$ and additionally, the initial state is assumed to be zero (i.e., $x_0 = 0$). In comparison, Assumption 1 and Assumption 2 are much less restrictive by considering the additive form and the initial condition.

**Remark 5.** If there exist a positive semi-definite function $\beta(x)$ and a positive number $\gamma$ such that

$$
\dot{\beta}(x(t)) \leq \gamma^2 |u(t)|^2 - |\dot{h}_1(t)|^2, \; \forall t \in \mathbb{R}_{\geq 0}, \tag{16}
$$
then Assumption 2 holds. Particularly, for a stable LTI system \( \dot{x} = Ax + Bu \) with \( h_1(x) = Cx \), if there exist a matrix \( P \succeq 0 \) and a number \( \gamma > 0 \) such that
\[
\begin{pmatrix}
A^\top P + PA + A^\top C^\top CA \\
B^\top P + B^\top C^\top CA
\end{pmatrix}
\begin{pmatrix}
PB + A^\top C^\top CB \\
-\gamma^2 I + B^\top C^\top CB
\end{pmatrix} \preceq 0
\]
then one can verify that the function \( \beta(x) = x^\top Px \) satisfies (16). For nonlinear systems whose dynamics are polynomial, the sum-of-squares optimization can be used to search for \( \beta(x) \) that satisfies (16).

4.1 Passivity Degradation of \( T_\tau(\Sigma) \)

In this subsection, we consider deriving passivity indices of \( T_\tau(\Sigma) \) from those of \( \Sigma \). As shown in Fig. 3(a), \( \Sigma \) is a continuous-time system whose input and output are \( u(t) \) and \( y(t) \), respectively, and \( T_\tau(\Sigma) \) is a discrete-time system whose input and output are \( u[k] \) and \( y[k] \), respectively. The following theorem shows quantitatively how passivity indices degrade from \( \Sigma \) to \( T_\tau(\Sigma) \). The proof of the theorem is given in Appendix E.

**Theorem 1.** Suppose that \( \Sigma \) satisfies Assumption 2 and Assumption 3, and is IF-OFP(\( \nu, \rho \)) with a positive semi-definite storage function \( V(x) \). Then \( T_\tau(\Sigma) \) satisfies the following inequality for any \( k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1 \) and admissible inputs \( u \in U_1 \):
\[
\dot{V}(x[k_1]) - \dot{V}(x[k_0]) \leq \delta(x[k_0]) + \sum_{k=k_0}^{k_1-1} \left( u[k]^\top y[k] - \nu'|y[k]|^2 - \rho'|y[k]|^2 \right) \tag{17}
\]
where \( \dot{V}(x) = \frac{1}{2} V(x) \) and
\[
\begin{aligned}
\nu' &= \nu - \tau \gamma - \tau^2 \gamma^2 (1 + \lambda_1)|\rho|, \\
\rho' &= \rho - |\rho|/\lambda_1, \\
\delta(x[k_0]) &= w \beta(x[k_0]), \quad w = |\rho|/\tau (1 + \lambda_1) + \frac{1}{\gamma},
\end{aligned}
\tag{18}
\]
with \( \lambda_1 > 0 \) an arbitrary positive number.

Although the inequality (17) has an additional bias term \( \delta \) on its right-hand side, we still call (17) the passivity inequality satisfied by \( \Sigma_d \), with some abuse of language. Note that this bias term \( \delta \) will not affect the summation of the supply function, and \( \delta = 0 \) when \( x[0] = 0 \). In Subsection 4.3, we will discuss circumstances when the bias term can be removed.

Equation (18) indicates that \( \nu' < \nu, \rho' < \rho \), and when \( \lambda_1 \) is fixed, a smaller sampling time \( \tau \) can result in a smaller passivity degradation (i.e., larger \( \nu', \rho' \)), which was also shown in [32, 44]. Theorem 1 also generalizes the corresponding results of [32, 44] in several ways. Firstly, a more general gain assumption (14) that takes into account the initial condition is used, which induces a bias term \( \delta(x[0]) \) on the right hand side of (17). Secondly, the positive number \( \lambda_1 \) in (18) provides additional flexibility in balancing the passivity indices \( \nu', \rho' \) and the term \( \delta \). Furthermore, to obtain the degraded indices in [44], it was assumed that \( \nu > 0 \) and an inequality about \( \tau, \nu, \rho \) needs to be satisfied (see Corollary 6 in [44]), but neither of them is required in Theorem 1.

**Remark 6.** The system \( \Sigma \) in Theorem 2 can be nonlinear control systems in general. When \( \Sigma \) is a continuous-time LTI system, dynamics of \( T_\tau(\Sigma) \) can be given explicitly as \( x[k+1] = A_d x[k] + B_d u[k], y[k] = C_d x[k] + D_d u[k] \) where \( A_d, B_d, C_d, D_d \) are matrices of proper dimensions. Then it is possible to obtain passivity indices of \( T_\tau(\Sigma) \) directly by solving an LMI. Specifically,
\( T_r(\Sigma) \) is IF-OFP(\( \nu_0, \rho_0 \)) if and only if there exists \( P = P^\top > 0 \) such that the following holds [21]:

\[
\begin{bmatrix}
A_d^\top P A_d - P + \hat{Q}, & A_d^\top P B_d - \hat{S} \\
(A_d^\top P B_d - \hat{S})^\top, & R + B_d^\top P B_d
\end{bmatrix} \preceq 0,
\]

where \( \hat{Q} = \rho_0 C_d^\top C_d, \hat{S} = C_d^\top / 2 - \rho_0 C_d^\top D_d, R = \rho_0 D_d^\top D_d + \nu_0 I - (D_d + D_d^\top) / 2 \). By fixing \( \nu \) (resp. \( \rho \)), \( [19] \) is an LMI with decision variables \( \rho \) (resp. \( \nu \)) and \( P \). The passivity indices computed this way results in a passivity inequality without the bias term \( \delta \).

**Example 2.** Consider the following LTI system:

\[
\Sigma :\begin{cases}
\dot{x}_2 = \begin{bmatrix} -1.8 & -1.3 \\ 1.2 & -2.5 \end{bmatrix} x_2 + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} u_2, \\
y_2 = \begin{bmatrix} 0.2 & -0.3 \\ 0.3 & 0.15 \end{bmatrix} x_2 + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} u_2,
\end{cases}
\]

where \( x_2(t), u_2(t), y_2(t) \in \mathbb{R}^2 \). Here, we use the notations for \( \Sigma \) in Fig. \( 7 \) (a), which have subscript indices. One can verify that \( \Sigma \) is IF-OFP(0.3, 0.5628) with a positive definite storage function \( V_2(x_2) = 0.23(x_{21}^2 + x_{22}^2) \) where \( x_2 = (x_{21}, x_{22})^\top \). Furthermore, Assumption 1 holds and \( h_1 \) satisfies Assumption 2 with \( \gamma_2 = 0.2 \) and \( \beta_2(x_2) = 0.2187(x_{21}^2 + x_{22}^2) \).

Suppose that the sampling time \( \tau = 0.3 \). Letting \( \lambda_1 = 10 \) in [18], we obtain \( \nu_2 = 0.2177, \rho_2 = 0.5065 \) and \( \delta_2 = 6.8572/\beta_2(x_2) \). By Theorem 1, \( T_r(\Sigma) \) satisfies the following inequality for any \( k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1 \):

\[
\dot{V}_2(x_2[k_1]) - \dot{V}_2(x_2[k_0]) \leq 1.4997(x_{21}^2[k_0] + x_{22}^2[k_0]) + \sum_{k=k_0}^{k_1-1} (u_2[k]^\top y_2[k] - 0.2177|u_2[k]|^2 - 0.5065|y_2[k]|^2)
\]

(20)

where \( \dot{V}_2(x_2) = \frac{1}{\tau} V_2(x_2) = 0.7667(x_{21}^2 + x_{22}^2) \).

Dynamics of the time-sampled system \( T_r(\Sigma) \) can be expressed explicitly as

\[
T_r(\Sigma) :\begin{cases}
x_2[k+1] = \begin{bmatrix} 0.5450 & -0.2002 \\ 0.1848 & 0.4372 \end{bmatrix} x_2[k] + \begin{bmatrix} 0.0455 & -0.0114 \\ 0.0070 & 0.0620 \end{bmatrix} u_2[k], \\
y_2[k] = \begin{bmatrix} 0.2 & -0.3 \\ 0.3 & 0.15 \end{bmatrix} x_2[k] + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} u_2[k].
\end{cases}
\]

By solving [19], one can verify that \( T_r(\Sigma) \) is IF-OFP(0.20, 0.9803) with a positive definite storage function \( V_2(x_2) = 0.23(x_{21}^2 + x_{22}^2) \), which is deliberately chosen the same as above. Then, \( T_r(\Sigma) \) satisfies the following inequality for any \( k \in \mathbb{Z}_{\geq 0} \):

\[
\dot{V}_2(x_2[k+1]) - \dot{V}_2(x_2[k]) \leq u_2[k]^\top y_2[k] - 0.2|u_2[k]|^2 - 0.9803|y_2[k]|^2.
\]

(21)

where \( \dot{V}(x_2) = \frac{1}{\tau} V_2(x_2) = 0.7667(x_{21}^2 + x_{22}^2) \). From inequalities (20) and (21), one can observe that passivity indices derived from [18] and from solving [19] are not comparable directly. Whether the equation (18) is strictly conservative than solving for the passivity indices by the LMI [19] or not is left for future work.

### 4.2 Passivity Degradation of \( \bar{T}_r(\Sigma) \)

In this subsection, we consider deriving passivity indices of \( \bar{T}_r(\Sigma) \) from those of \( T_r(\Sigma) \). Recall that the passivity indices of \( T_r(\Sigma) \) not only can be obtained from a continuous-time system \( \Sigma \)
by Theorem 1 in which case $T_r(\Sigma)$ satisfies (17) with $\nu', \rho', \delta$ given in (18), but also may be obtained directly from the dynamics of $T_r(\Sigma)$, in which case the bias term $\delta$ in (17) does not exist.

Note that the system $\tilde{T}_r(\Sigma)$ shown in Fig. 3 consists of $T_r(\Sigma)$ and two uniform quantizers $Q_1, Q_2$. The (uniform) quantization function $Q(\cdot)$ with a quantization precision $\mu$ is defined as

$$Q(s) := \begin{cases} \frac{s}{\mu} & \text{if } s \geq 0; \\ \frac{s}{\mu} + \frac{1}{2} & \text{if } s < 0, \end{cases}$$

(22)

where $[\cdot]$ is the floor function (i.e., $[x]$ is the greatest integer no larger than $x$) and $\lceil \cdot \rceil$ is the ceiling function (i.e., $\lceil x \rceil$ is the least integer no less than $x$). If $s$ is a vector, then $Q$ is implemented entry-wisely.

The input of $\tilde{T}_r(\Sigma)$, $\tilde{u}[k]$, and the quantized input, $u[k]$, are related by $u[k] = Q_1(\tilde{u}[k])$ where the function $Q_1(\cdot)$ is given in (22) with a quantization precision $\mu_1$. It is easy to see that for any $k \in \mathbb{Z}_{\geq 0}$, $|u[k]| \leq |\tilde{u}[k]|$ and $|u[k] - \tilde{u}[k]| \leq \sqrt{\mu_1}$, which results from the facts that $|u[k] - \tilde{u}[k]| \leq \sqrt{\mu_1} |u[k] - \tilde{u}[k]|_\infty$ and $|u[k] - \tilde{u}[k]|_\infty \leq \mu_1$. The output of $\tilde{T}_r(\Sigma)$, $\tilde{y}[k]$, and the output before quantization, $\hat{y}[k]$, are related by $\tilde{y}[k] = Q_2(y[k])$ where the function $Q_2(\cdot)$ is given in (22) with a precision $\mu_2$. It is clear that $|\tilde{y}[k] - \hat{y}[k]| \leq \sqrt{\mu_2}$ for any $k \in \mathbb{Z}_{\geq 0}$.

The following theorem shows how passivity indices degrade from $T_r(\Sigma)$ to $\tilde{T}_r(\Sigma)$. The proof of the theorem is given in Appendix E.

**Theorem 2.** Suppose that $T_r(\Sigma)$ is IF-OFPP($\nu', \rho'$) that satisfies (17) with a positive semidefinite storage function $\hat{V}(x)$. Then $\tilde{T}_r(\Sigma)$ satisfies the following passivity inequality for any $k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1$ and any admissible inputs $u \in U_1$:

$$\hat{V}(x[k_1]) - \hat{V}(x[k_0]) \leq \delta(x[k_0]) + \sum_{k=k_0}^{k_1-1} \left( \tilde{u}[k]^T \tilde{y}[k] - \tilde{v}[\tilde{u}[k]]^2 - \tilde{\rho}[\tilde{y}[k]]^2 + \tilde{\delta} \right)$$

(23)

where

$$\begin{cases} \tilde{v} = \frac{\nu'}{|\nu'|/\lambda_3 - 1/4\lambda_4}, \\ \tilde{\rho} = \frac{\rho'}{|\rho'|/2 - 1/4\lambda_5}, \\ \tilde{\delta} = ||\rho'|/(1 + \lambda_2) + \lambda_4|m\mu_2^2 + ||\nu'|/(1 + \lambda_3) + \lambda_5|m\mu_1^2, \end{cases}$$

(24)

and $\lambda_2, \ldots, \lambda_5 > 0$ are arbitrary positive numbers.

In (23), the term $\tilde{\delta}$ exists due to the input and output quantizations of $\tilde{T}_r(\Sigma)$, and the term $\delta = 0$ if it is 0 for $T_r(\Sigma)$. The following observations can also be made from (23). 1) When $\mu_1$ and $\mu_2$, the quantization precisions, are fixed, there is a trade-off in choosing $\lambda_i (2 \leq i \leq 5)$: larger $\lambda_i$ result in larger $\tilde{\nu}$ and $\tilde{\rho}$ (i.e., more “excess of passivity”), while smaller $\lambda_i$ result in smaller $\tilde{\delta}$ (smaller “internal energy generation”). 2) If $\nu' > 0$ (resp. $\rho' > 0$), then it is always possible to choose $\lambda_2, \lambda_3$ (resp. $\lambda_4, \lambda_5$) large enough such that $\tilde{\nu} > 0$ (resp. $\tilde{\rho} > 0$). 3) With fixed $\lambda_i$, smaller $\mu_1, \mu_2$ (i.e., more precise quantizations) result in smaller $\tilde{\delta}$ (i.e., less quantization effect), and $\tilde{\delta} \rightarrow 0$ when $\mu_1, \mu_2 \rightarrow 0$. Therefore, if $\nu' > 0, \rho' > 0$, then it is always possible to choose $\lambda_i (2 \leq i \leq 5)$ large enough and $\mu_1, \mu_2$ small enough such that $\tilde{\nu} > 0, \tilde{\rho} > 0$ and $\tilde{\delta}$ arbitrarily small.

**Example 3.** Consider the system $T_r(\Sigma)$ in Example 2 again, which is IF-OFPP(0.20, 0.9803) with a storage function $\hat{V}_2(x_2) = 0.7667(x_2^2_1 + x_2^2_2)$ when $\tau = 0.3$. Suppose that $\mu_1 = \mu_2 = 0.01$, and let $\lambda_i = 20$ for $i = 2, 3, 4, 5$ in (24). Then, by equation (24), $T_r(\Sigma)$ is IF-OFQP(0.1775, 0.9188, 0.0130). That is, for any $k \in \mathbb{Z}_{\geq 0}$,

$$\hat{V}_2(x_2[k+1]) - \hat{V}_2(x_2[k]) \leq \tilde{u}_2[k]\tilde{y}_2[k] - 0.1775|\tilde{u}_2[k]|^2 - 0.9188|\tilde{y}_2[k]|^2 + 0.0130.$$ 

(25)
4.3 Discussion

In this subsection, we will discuss conditions under which the term \( \delta \), if it exists, in (17) and (23), can be removed, so that the passivity inequalities can be simplified. Recall that \( \delta \) appears only when Assumption 2 is used to derive the passivity indices degradation from \( \Sigma \) to \( T_r(\Sigma) \).

We suppose that the function \( \beta \) in Assumption 2 is upper bounded by a quadratic function.

**Assumption 3.** For the function \( \beta \) in Assumption 2, there exists a positive semi-definite matrix \( S \) such that

\[
\beta(x) \leq x^\top Sx, \forall x \in \mathbb{R}^n.
\]

**Remark 7.** As discussed in Remark 2, function \( \beta \) can be chosen to be a positive semi-definite function that satisfies (16). When \( \beta(x) \) is found to be a quadratic function of \( x \), then Assumption 3 holds automatically. Furthermore, if \( \delta \) does not exist, then we can consider \( S = 0 \).

Suppose that conditions in Theorem 1 hold, Assumption 3 holds and \( T_r(\Sigma) \) is 0-step SD. Because \( S \leq \lambda_{\max}(SM^{-1})M \), by (17) the following holds for any \( k \in \mathbb{Z}_{\geq 0} \):

\[
\dot{V}(x[k+1]) - \dot{V}(x[k]) \leq wx[k]^\top Sx[k] + u[k]^\top y[k] - \nu'[u[k]]^2 - \rho'|y[k]|^2
\]

\[
\leq \dot{w}x[k]^\top Mx[k] + u[k]^\top y[k] - \nu'[u[k]]^2 - \rho'|y[k]|^2
\]

\[
\leq u[k]^\top y[k] - \nu'[u[k]]^2 - \rho'|y[k]|^2.
\]

where

\[
\begin{aligned}
\dot{\nu'} &= \nu' - \partial \dot{\nu}, \\
\dot{\rho'} &= \rho' - \dot{\omega}, \\
\dot{\omega} &= w\lambda_{\max}(SM^{-1}).
\end{aligned}
\]

and \( w \) is given in (18). That is, \( T_r(\Sigma) \) is IF-OFP(\( \dot{\nu}', \dot{\rho}' \)). Moreover, if \( \rho' > 0 \) and \( \lambda \) is a number satisfying \( 0 < \lambda < \rho' \), then we have

\[
\dot{V}(x[k+1]) - \dot{V}(x[k]) \leq wx[k]^\top Sx[k] + \eta_1|u[k]|^2 - \eta_2|y[k]|^2
\]

\[
\leq (\eta_1 + \partial \eta_2)|u[k]|^2 - x[k]^\top (\eta_2 M - wS)x[k]
\]

where

\[
\begin{aligned}
\eta_1 &= \frac{1}{4\lambda} - \nu', \\
\eta_2 &= \rho' - \lambda.
\end{aligned}
\]

If the matrix \( \eta_2 M - wS \) is positive definite, then \( T_r(\Sigma) \) is ISS with respect to the input \( u \). We summarize the discussion above in the following corollary.

**Corollary 1.** Suppose that conditions in Theorem 1 hold, Assumption 3 holds and \( T_r(\Sigma) \) is 0-step SD. Then, \( T_r(\Sigma) \) is IF-OFP(\( \dot{\nu}', \dot{\rho}' \)) where \( \dot{\nu}', \dot{\rho}' \) are given in (26). Furthermore, if \( \rho' > 0 \) and \( \eta_2 M - wS \succ 0 \), then \( T_r(\Sigma) \) is ISS with respect to \( u \).

For \( \tilde{T}_r(\Sigma) \), since \( |\tilde{u}[k] - u[k]| \leq \sqrt{m}\mu_1 \) for any \( k \in \mathbb{Z}_{\geq 0} \), we have \( |\tilde{u}[k]|^2 \geq \frac{1}{2}|u[k]|^2 - |\tilde{u}[k] - u[k]|^2 \geq \frac{1}{2}|u[k]|^2 - m\mu_1^2 \); similarly, we have \( |\tilde{y}[k]|^2 \geq \frac{1}{2}|y[k]|^2 - m\mu_2^2 \). Hence,

\[
\dot{\nu}|\tilde{u}[k]|^2 + |\tilde{y}[k]|^2 \geq \frac{1}{2}(\dot{\nu}|u[k]|^2 + |y[k]|^2) - m(\dot{\nu}\mu_1^2 + \mu_2^2)
\]

\[
\geq \frac{1}{2}x[k]^\top Mx[k] - m(\dot{\nu}\mu_1^2 + \mu_2^2).
\]

By (23) the following holds for any \( k \in \mathbb{Z}_{\geq 0} \):

\[
\dot{V}(x[k+1]) - \dot{V}(x[k]) \leq wx[k]^\top Sx[k] + \tilde{u}[k]^\top \tilde{y}[k] - \nu|\tilde{u}[k]|^2 - \rho|\tilde{y}[k]|^2 + \tilde{\delta}
\]
\[ \leq \bar{u}[k]^\top \bar{y}[k] - \bar{v}'|\bar{u}[k]|^2 - \bar{\rho}'|\bar{y}[k]|^2 + \bar{\delta}' \]

where

\[
\begin{cases} 
    \bar{v}' = \nu' - 2\vartheta \hat{w}, \\
    \bar{\rho}' = \rho' - 2\bar{w}, \\
    \bar{\delta}' = \bar{\delta} + 2m\vartheta(\vartheta \mu_1^2 + \mu_2^2), \\
    \hat{w} = w\lambda_{\text{max}}(SM^{-1}),
\end{cases}
\]

and \( w \) is given in (18). That is, \( \hat{T}_r(\Sigma) \) is IF-OFP(\( \bar{v}' \), \( \bar{\rho}' \), \( \bar{\delta}' \)). Similarly, it is easy to show that if \( \bar{\rho} > 0 \) and \( \frac{\bar{\rho}}{2} M - wS \succ 0 \), then \( \hat{T}_r(\Sigma) \) is ISpS with respect to the input \( \bar{u} \). This is summarized by the following corollary.

**Corollary 2.** Suppose that conditions in Theorem 3 hold, Assumption 3 holds and \( T_r(\Sigma) \) is 0-step SD. Then, \( \hat{T}_r(\Sigma) \) is IF-OFP(\( \bar{v}' \), \( \bar{\rho}' \), \( \bar{\delta}' \)) where \( \bar{v}' \), \( \bar{\rho}' \), \( \bar{\delta}' \) are given in (29). Furthermore, if \( \bar{\rho} > 0 \) and \( \frac{\bar{\rho}}{2} M - wS \succ 0 \), then \( \hat{T}_r(\Sigma) \) is ISpS with respect to \( \bar{u} \).

## 5 Passivity Analysis of the Closed-Loop System

### 5.1 Passivity Property of \( T_r(P) \times F \hat{T}_r(\Sigma) \)

In this subsection, we analyze the passivity of the system \( T_r(P) \times F \hat{T}_r(\Sigma) \) shown in Fig. 1 (a).

Suppose that \( P \) satisfies Assumption 1 and Assumption 2 with constant \( \gamma_1 \) and function \( \beta_1 \). Suppose also that \( P \) is IF-OFP(\( \nu_1 \), \( \rho_1 \)) with a continuous, positive definite, radially unbounded storage function \( V_1 \). Denote the state, input and output of \( T_r(P) \) by \( x_1[k] \in \mathbb{R}^{n_1}, u_1[k] \in \mathbb{R}^{m_1}, y_1[k] \in \mathbb{R}^{m_2} \), respectively. By Theorem 1, \( T_r(P) \) satisfies the following inequality for any \( k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1 \):

\[
\frac{1}{\tau} V_1(x_1[k_1]) - \frac{1}{\tau} V_1(x_1[k_0]) \leq \delta_1(x_1[k_0]) + \sum_{k=k_0}^{k_1-1} (u_1[k] \top y_1[k] - \nu_1' |u_1[k]|^2 - \rho_1' |y_1[k]|^2)
\]

where

\[
\begin{cases} 
    \nu_1' = \nu_1 - \tau \gamma_1 - \tau^2 \gamma_1^2 (1 + \lambda_{11}) |\rho_1|, \\
    \rho_1' = \rho_1 - |\rho_1|/\lambda_{11}, \\
    \delta_1 = w_1 \beta_1(x_1[0]), \\
    w_1 = |\rho_1|\tau(1 + \lambda_{11}) + 1/\gamma_1,
\end{cases}
\]

and \( \lambda_{11} > 0 \) is a positive number.

Suppose that \( \Sigma \) satisfies Assumption 1 and Assumption 2 with constant \( \gamma_2 \) and function \( \beta_2 \). Suppose also that \( \Sigma \) is IF-OFP(\( \nu_2 \), \( \rho_2 \)) with a continuous, positive definite, radially unbounded storage function \( V_2 \). Denote the state, input and output of \( \hat{T}_r(\Sigma) \) by \( x_2[k] \in \mathbb{R}^{n_2}, \bar{u}_2[k] \in \mathbb{R}^{m_2}, \bar{y}_2[k] \in \mathbb{R}^{m_2} \), respectively. By Theorem 1 and Theorem 2, \( \hat{T}_r(\Sigma) \) satisfies the following inequality for any \( k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1 \):

\[
\frac{1}{\tau} V_2(x_2[k_1]) - \frac{1}{\tau} V_2(x_2[k_0]) \leq \sum_{k=k_0}^{k_1-1} (\bar{u}_2[k] \top \bar{y}_2[k] - \bar{v}_2|\bar{u}_2[k]|^2 - \bar{\rho}_2|\bar{y}_2[k]|^2 + \bar{\delta}_2) + \delta_2(x_2[k_0])
\]
where
\[
\begin{align*}
\hat{v}_2 &= \nu_2' - |\nu_2'|/\lambda_{23} - 1/4\lambda_{24}, \\
\hat{\rho}_2 &= \rho_2' - |\rho_2'|/\lambda_{22} - 1/4\lambda_{25}, \\
\delta_2 &= \|\rho_2'(1 + \lambda_{22}) + \lambda_{24}\| \mu_2^2 + |\nu_2'(1 + \lambda_{23}) + \lambda_{25}| \mu_1^2, \\
\nu_2' &= \nu_2 - \tau \gamma_2 - \tau^2 \gamma_2^2 (1 + \lambda_{21}) |\rho_2|, \\
\rho_2' &= \rho_2 - |\rho_2|/\lambda_{21}, \\
\delta_2 &= w_2 \beta_2(x_2[0]), \\
w_2 &= |\rho_2| \tau (1 + \lambda_{21}) + 1/\gamma_2,
\end{align*}
\]
and \(\lambda_{21}, \ldots, \lambda_{25}\) are positive real numbers.

Define
\[
\begin{align*}
x &= \begin{pmatrix} x_1 \\
x_2 \end{pmatrix}, & r &= \begin{pmatrix} r_1 \\
r_2 \end{pmatrix}, & y &= \begin{pmatrix} y_1 \\
y_2 \end{pmatrix}, & u &= \begin{pmatrix} u_1 \\
u_2 \end{pmatrix}, \\
V(x) &= \frac{1}{\tau} V_1(x_1) + \frac{1}{\tau} V_2(x_2).
\end{align*}
\]
The passivity property of the system \(T_r(P) \times_{\mathcal{F}} \tilde{T}_r(\Sigma)\) in Fig. 1(a) is given by the following theorem.

**Theorem 3.** Suppose that \(P\) (resp. \(\Sigma\)) satisfies Assumption 1 and Assumption 3 with constant \(\gamma_1\) and function \(\beta_1\) (resp. with constant \(\gamma_2\) and function \(\beta_2\)). Suppose also that \(P\) (resp. \(\Sigma\)) is IF-OFP(\(\nu_1, \rho_1\)) (resp. IF-OFP(\(\nu_2, \rho_2\))) with a continuous, positive definite, radially unbounded storage function \(V_1\) (resp. \(V_2\)). Then the system \(T_r(P) \times_{\mathcal{F}} \tilde{T}_r(\Sigma)\) in Fig. 1(a) satisfies the following passivity inequality for any \(k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1\) and any admissible input \(r[k] \in \mathbb{R}^{2m} \) :
\[
V(x[k_1]) - V(x[k_0]) \leq \delta(x[k_0]) + \sum_{k=k_0}^{k_1-1} \left( r[k] \mathbf{y}[k] - \hat{\nu}[k] \mathbf{y}[k] \right)^2 - \hat{\rho}[k] |\mathbf{y}[k]|^2 + \delta_2
\]
where
\[
\begin{align*}
\hat{\nu} &< \min\{\nu_1', \hat{\nu}_2\}, \\
\hat{\rho} &< \min\{\rho_1' - \frac{\hat{\nu}_2}{\nu_2 - \hat{\nu}}, \hat{\rho}_2 - \frac{\hat{\nu}_1'}{\nu_1' - \hat{\nu}}\},
\end{align*}
\]
and \(\delta_2, \delta_2, \nu_1', \nu_2', \rho_1', \rho_2, \hat{\rho}_2\) are given in (31) and (33).

The proof of Theorem 3 can be easily obtained by Lemma 2. Note that \(\delta_1 + \delta_2\) on the right-hand side of (36) is a bias term, which only depends on the initial state \(x[k_0]\) and is equal to 0 when \(x[k_0] = 0\).

The following corollary shows the passivity inequality of \(T_r(P) \times_{\mathcal{F}} \tilde{T}_r(\Sigma)\) for a particular case when \(r[k] \equiv 0\).

**Corollary 3.** Suppose that conditions in Theorem 3 hold. When \(r[k] \equiv 0\), the system \(T_r(P) \times_{\mathcal{F}} \tilde{T}_r(\Sigma)\) in Fig. 1(a) satisfies the following inequality
\[
V(x[k_1]) - V(x[k_0]) \leq \delta + \sum_{k=k_0}^{k_1-1} \left[ (\hat{\nu}_2 + \rho_1') [y_1[k]]^2 - (\nu_1' + \rho_2) [\tilde{y}_2[k]]^2 + \delta_2 \right], \quad \forall k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1,
\]
where \(\delta(x[k_0]) = \delta_1(x_1[k_0]) + \delta_2(x_2[k_0]), \) and \(\delta_1, \delta_2, \nu_1', \nu_2', \rho_1', \rho_2, \hat{\rho}_2\) are given in (31) and (33).

From the discussion at the end of Subsection 4.2, it is obvious that when \(r[k] \equiv 0\), if \(\nu_2' + \rho_1' > 0\) and \(\nu_1' + \rho_2 > 0\), then it is always possible to choose \(\lambda_{22}, \ldots, \lambda_{25}\) large enough and \(\mu_1, \mu_2\) small enough such that \(\nu_2 + \rho_1' > 0, \nu_1' + \rho_2 > 0\) and \(\delta_2\) arbitrarily small. On the contrary, (37) implies that \(\hat{\rho}\) might be negative even if \(\nu_1', \nu_2', \rho_1', \rho_2\) are all positive. Therefore, the particular case \(r[k] \equiv 0\) requires a less restrictive condition for the closed-loop system to have positive indices.
5.2 Ultimate Boundedness Of The Closed-Loop System

In this subsection, we will study conditions under which the states of $T_r(P) \times \tilde{T}_r(\Sigma)$ is (ultimately) bounded, using the notations in the preceding subsection. We suppose that $T_r(P)$ and $\tilde{T}_r(\Sigma)$ satisfy the passivity inequalities (30) and (32), respectively.

Suppose that $T_r(P)$ is SD with $N_1, \vartheta_1, M_1$ such that

$$\sum_{k=k_0}^{k_0+N_1} \vartheta_1 |u_1[k]|^2 + |y_1[k]|^2 \geq x_1[k_0]^\top M_1 x_1[k_0]$$

(39)

for any $x_1[k_0], k_0 \in \mathbb{Z}_{\geq 0}$; suppose that $T_r(\Sigma)$ is SD with $N_2, \vartheta_2, M_2$ such that

$$\sum_{k=k_0}^{k_0+N_2} \vartheta_2 |u_2[k]|^2 + |y_2[k]|^2 \geq x_2[k_0]^\top M_2 x_2[k_0]$$

(40)

for any $x_2[k_0], k_0 \in \mathbb{Z}_{\geq 0}$.

Then we have the following theorem as the main result of this subsection. Its proof is given in Appendix \textsuperscript{G}.

**Theorem 4.** Consider the system $T_r(P) \times \tilde{T}_r(\Sigma)$ in Fig. 1 (a). Suppose that 1) $T_r(P)$ and $\tilde{T}_r(\Sigma)$ satisfy passivity inequalities (30) and (32), respectively; 2) $T_r(P)$, $T_r(\Sigma)$ are SD that satisfy (39) and (40), respectively; 3) functions $\beta_1, \beta_2$ satisfy Assumption 3 with positive semi-definite matrices $S_1, S_2$, respectively; 4) $\rho$ shown in (37) can be chosen as $\rho > 0$; 5) the matrix $G$ defined as $G := \eta_2 M - S$ is positive definite where $\eta_1 = \frac{1}{4M} - \hat{\rho} > 0$, $\eta_2 = \hat{\rho} - \lambda > 0$ with $\lambda$ a number such that $0 < \lambda < \hat{\rho}$, and

$$M = (1 - \vartheta) \text{diag}\{M_1, \frac{1}{2} M_2\},$$

$$S = \text{diag}\{w_1 S_1, w_2 S_2\},$$

$$\vartheta = \max\{\frac{2\vartheta_1}{2\vartheta_1 + 1}, \frac{2\vartheta_2}{2\vartheta_2 + 1}\}.$$  

(41)

(42)

(43)

Then, 1) for any $k \in \mathbb{Z}_{\geq 0}$, $x[k] \in D_1$ where $D_1$ is a compact set defined as $D_1 := \{z \mid V(z) \leq \max\{V(x[0]), V(x[1]), \ldots, V(x[N]), d_1 + d_2 + d_4\}\}$, $V$ is defined in (35), and

$$\begin{cases}
  d_1 = (N + 1)(\eta_1 + \vartheta \eta_2)||r||^2 + \delta_2, \\
  d_2 = m(1 - \vartheta)(N_2 + 1)(\vartheta_2 \mu_2^2 + \mu_2^2), \\
  d_3 \text{ is a positive real number}, \\
  d_4 = \max_{z \in C} V(z), \\
  C = \{z \mid z^\top Gz \leq d_1 + d_2 + d_3\}, \\
  N = \max\{N_1, N_2\}\}
\end{cases}$$

(44)

2) there exists $K \in \mathbb{Z}_{\geq 0}$ such that $x[k] \in D_2$ for all $k \geq K$ where $D_2$ is a compact set defined as $D_2 := \{x \mid V(x) \leq d_1 + d_2 + d_4\}$.

**Remark 8.** Conditions 3) and 5) in Theorem 4 will be satisfied automatically when $\delta_1 = 0$ in (30) and $\delta_2 = 0$ in (32). Moreover, Lemma 7 and Theorem 4 altogether provide conditions under which the feedback-connected system is ISpS with respect to $r$.

When $r[k] \equiv 0$, conditions in Theorem 4 can be simplified, as shown by the following corollary. The proof of this corollary is given in Appendix \textsuperscript{H}.
Corollary 4. Consider the system $T_r(P) \times \mathcal{F} \tilde{T}_r(\Sigma)$ in Fig. 1 where $r[k] \equiv 0$. Suppose that conditions 1)-3) in Theorem 4 hold, and 4) $\tilde{v}_2 + \rho_1 > 0$, $\nu_1' + \tilde{\rho}_2 > 0$; 5) the matrix $G$ defined as $G := M - S$ is positive definite where $M = \text{diag}\{a_1M_1, \frac{\nu_1}{r}M_2\}$, $S = \text{diag}\{w_1S_1, w_2S_2\}$ and $a_1, a_2$ are two positive number such that $a_1 \vartheta_1 + a_2 \leq \nu_1' + \tilde{\rho}_2$, $a_2 \vartheta_2 + a_1 \leq \tilde{\nu}_2 + \rho_1'$. Then 1) there exists a compact set $D_1$ such that $x[k] \in D_1$ for any $k \in \mathbb{Z}_{\geq 0}$; 2) there exist a compact set $D_2$ and a constant $K \in \mathbb{Z}_{\geq 0}$ such that $x[k] \in D_2$ for all $k \geq K$ where $D_2 := \{ x \mid V(x) \leq d_1 + d_2 + d_4 \}$, $d_1 = N\delta_2$, $d_2 = m(N_2 + 1)(\vartheta_2\mu_1^2 + \mu_2^2)$, $d_3$ is a positive real number, $d_4 = \max_{z \in \mathcal{C}} V(z)$, $\mathcal{C} = \{ z \mid z^T G z \leq d_1 + d_2 + d_3 \}$, $N = \max\{N_1, N_2\}$.

Corollary 4 shows that the state of the system $T_r(P) \times \mathcal{F} \tilde{T}_r(\Sigma)$ with $r[k] \equiv 0$ will be (ultimately) bounded, if the shortage of passivity of one component can be compensated for by the excess of passivity of another component in the feedback connection, and in addition, each component is SD.

When the dynamical models of a plant and a controller are known exactly, it might be easier to analyze the closed-loop system using these models directly. However, it is almost impossible to obtain an exact model of a plant or a controller. In contrast, the passivity-based analysis shown above enables us to compute the passivity degradation from the original indices and derive the property of the closed-loop system (under sampling and quantization) without knowing the model at all. Therefore, the passivity-based analysis is robust because it holds for a family of plants and controllers. Although the ultimate bounds given in Theorem 4 and Corollary 4 may not be tight, we point out that the ultimate bound given in Theorem 4 will decrease as $\|r\|$, $\mu_1, \mu_2$ decrease, and the ultimate bound given in Corollary 4 can be made arbitrarily small when $\mu_1, \mu_2$ are chosen small enough.

Figure 4: Trajectories of $x_1[k]$, $x_2[k]$ using three different sets of $\mu_1, \mu_2$, all from $x_1[0] = (1.6, -2.9)^T$, $x_2[0] = (1.9, -1.5)^T$.

Note that the results of Section 4 and Section 5 also apply to the case when $\Sigma$ is a static output feedback function.

Remark 9. In [43], modifying the passivity indices of a system by simply adding a feedforward loop and/or a feedback loop was discussed, where the achievable bounds of the modified passivity indices were given explicitly. When condition 4) in Theorem 4 (or Corollary 4) is not satisfied,
the results in [43] may be used to modify the passivity indices of the feedback-connected system without interfering the existing structure of the system.

**Remark 10.** The emulation-based design is a general framework for the controller design under time sampling: a controller is designed in the continuous time domain at first, and then sampled and implemented using a sampler and hold device so that certain property can be preserved under the time sampling (see [23, 24, 30, 31] and references therein). The notion of (V, w)-dissipativity was used to characterize the property of the system for preservation. There are two main differences between the emulation-based approach and the method used here: 1) the sampling time in the emulation-based approach is an implicit parameter that needs to be chosen sufficiently small but whose explicit value is hard to compute in general (see [31] for more details), while the sampling time in the results above is an explicit parameter whose effect on the degradation of the passivity indices is explicitly known; 2) the input and output quantizations are not considered in the emulation-based approach. In the next section, we will show that the symbolic control implementation can be also studied in our framework.

**Example 4.** Consider a continuous-time system P whose time-sampled system T_s(P) is expressed as

\[
T_s(P) : \begin{cases} 
    x_1[k + 1] &= \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 1 \end{bmatrix} x_1[k] + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u_1[k], \\
    y_1[k] &= \begin{bmatrix} 0.35 & 0 \\ 0.18 & 0.27 \end{bmatrix} x_1[k], \\
\end{cases}
\]

where \( x_1[k], u_1[k], y_1[k] \in \mathbb{R}^2 \). One can verify that \( T_s(P) \) is zero-input unstable, and it is IF-OFP\((-0.5, -0.0771)\) with a positive definite storage function \( V_1(x_1) = 0.23(x_{11}^2 + x_{12}^2) \) where \( x_1 := (x_{21}, x_{22})^T \). Then \( \dot{V}_1(x_1) = \frac{1}{2} V_1(x_1) = 0.7667(x_{11}^2 + x_{12}^2) \). Consider the feedback connection of \( T_s(P) \) and \( \tilde{T}_s(\Sigma) \) as shown in Fig. 4 (a) where \( \Sigma \) is the system given in Example 3.

Suppose that \( v[k] = 0 \) and choose \( \mu_1 = \mu_2 = 0.01 \) as in Example 3. One can verify that conditions of Corollary 2 are all satisfied: \( T_s(P) \) is 0-step SD with \( \tilde{\vartheta}_1 = 0, M_1 = \text{diag}\{0.0159, 0.0159\} \), and \( \tilde{T}_s(\Sigma) \) is 0-step SD with \( \tilde{\vartheta}_2 = 1, M_2 = \text{diag}\{0.0503, 0.0503\} \); functions \( \beta_2, \beta_\rho \) are all 0, which means that \( S_1 = S_2 = 0 \); it is clear that \( \tilde{\vartheta} + \tilde{\rho}_1 = 0.1004 > 0 \), \( \nu_1 + \tilde{\rho}_2 = 0.4188 > 0 \); choose \( a_1 = 0.06, a_2 = 0.04 \) so that \( a_1 \tilde{\vartheta}_1 + a_2 \leq 0.4188, a_2 \tilde{\vartheta}_2 + a_1 \leq 0.1004 \).

Then, by Corollary 4, the state of the system \( T_s(P) \times \tilde{T}_s(\Sigma) \) will be ultimately bounded.

When the quantization precisions \( \mu_1, \mu_2 \) are chosen as \( \mu_1 = \mu_2 = 0.001 \) or \( \mu_1 = \mu_2 = 0.05 \), it is easy to show that the feedback-connected system is also ultimately bounded. In Fig. 4, the trajectories of \( x_1[k], x_2[k] \) are shown for these three different choices of \( \mu_1, \mu_2 \), all from the initial states \( x_1[0] = (1.6, -2.9)^T, x_2[0] = (1.9, -1.5)^T \). From these figures, it can be observed that the smaller the (input and output) quantization precisions are, the smaller the ultimate bounds would be, which verify our theoretical results. We also emphasized that the ultimate bound can be made arbitrarily small when \( \mu_1, \mu_2 \) are chosen small enough.

### 6 Analysis of the Approximate Bisimulation-Based Control Implementation

In this section, we study state boundedness of the system \( T_s(P) \times \tilde{T}_s(\Sigma) \) shown in Fig. 4 (b) where a symbolic controller \( T_s(\Sigma) \) approximately bisimilar to \( \Sigma \) is implemented. To that end, we introduce an auxiliary configuration, denoted by \( T_s(P) \times \tilde{T}_s(\Sigma) \), as shown in Fig. 5 where an external bounded disturbance \( w \) is added to \( \tilde{y}_2 \) in the system \( T_s(P) \times \tilde{T}_s(\Sigma) \). In what follows, we will first show the passivity property and state boundedness of \( T_s(P) \times \tilde{T}_s(\Sigma) \) by assuming that \( w \) is bounded, and based on that, we will show the state boundedness of
Denote the states of \( T_r(P), T_r(\Sigma) \) and \( T_{\tau\mu}(\Sigma) \) by \( x_1[k], x_2[k] \) and \( x_3[k] \), respectively (“s” stands for “symbolic”). Suppose that \( P \) (resp. \( \Sigma \)) satisfies Assumption 1 with \( y_1(x_1, u_1) = h_{11}(x_1) + h_{12}(u_1) \) (resp. with \( y_2(x_2, u_2) = h_{21}(x_2) + h_{22}(u_2) \)) and Assumption 2 with constant \( \gamma_1 \) and function \( \beta_1 \) (resp. constant \( \gamma_2 \) and function \( \beta_2 \)). Suppose also that \( P \) (resp. \( \Sigma \)) is IF-OFP(\( \nu_1, \rho_1 \)) (resp. IF-OFP(\( \nu_2, \rho_2 \))) with a continuous, positive definite, radially unbounded storage function \( V_1 \) (resp. \( V_2 \)). Furthermore, suppose that there exist positive numbers \( \epsilon, \mu, \eta > 0 \) such that \( T_r(\Sigma) \cong^{(\epsilon, \mu)} T_{\tau\mu}(\Sigma) \) holds where \( \mu = \mu_1 \) (recall that \( \mu_1 \) is the quantization level of the quantizer \( Q_1 \)), and there exists \( L > 0 \) such that for all \( z_1, z_2 \in \mathbb{R}^n \),

\[
|h_{21}(z_1) - h_{21}(z_2)| \leq L|z_1 - z_2|_{\infty}. \tag{45}
\]

Define

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\]

for signals depicted in Fig. 5. Next, we show the passivity property of \( T_r(P) \times_{\mathcal{F}} \tilde{T}_r(\Sigma) \) by assuming that

\[
|w[k]| \leq L \epsilon + 2\sqrt{m}\mu_2, \quad \forall k \in \mathbb{Z}_{\geq 0}, \tag{46}
\]

where \( \mu_2 \) is the quantization level of quantizer \( Q_2 \). Recall that for any \( k \in \mathbb{Z}_{\geq 0} \), \( |u_2[k]| \leq |\tilde{u}_2[k]| \), \( |u_2[k] - \tilde{u}_2[k]| \leq \sqrt{m}\mu_1 \) and \( |\tilde{y}_2[k] - y_2[k]| \leq \sqrt{m}\mu_2 \). In addition, since \( \tilde{y}_2[k] = w[k] + \tilde{y}_2[k] \), it follows that

\[
|\tilde{y}_2[k] - y_2[k]| \leq |\tilde{y}_2[k] - \tilde{y}_2[k]| + |\tilde{y}_2[k] - y_2[k]| \leq L \epsilon + 3\sqrt{m}\mu_2, \forall k \in \mathbb{Z}_{\geq 0}.
\]

Similar to Theorem 2, it is easy to show that the following inequality holds for any \( k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1 \), and any admissible inputs \( \tilde{u}_2 \in U_1 \):

\[
\hat{V}_2(x_2[k_1]) - \hat{V}_2(x_2[k_0]) \leq \delta_2(x_2[k_0]) + \sum_{k=k_0}^{k_1-1} \left( \tilde{u}_2[k]^{\top} \tilde{y}_2[k] - \tilde{v}_2[k]^{\top} \tilde{u}_2[k] \right)^2 - \bar{\rho}_2[y_2[k]]^2 + \bar{\delta}_2 \tag{47}
\]

where \( \hat{V}_2(x_2) = \frac{1}{2} V_2(x_2) \), \( \delta_2, \tilde{\delta}_2, \bar{\rho}_2 \) are given by the equations in (33), respectively, and

\[
\bar{\delta}_2 = ||\rho'_2(1 + \lambda_{22}) + \lambda_{24}|| (L \epsilon + 3\sqrt{m}\mu_2)^2 + ||\nu'_2(1 + \lambda_{23}) + \lambda_{25}|| \mu_1^2 \tag{48}
\]
sections. Suppose that \( \delta(x) \). Then, similar to Theorem 3, it is easy to show that the closed-loop system with \( x \geq 0 \) and satisfies
\[
V(x[k]) - V(x[k_0]) \leq \delta(x[k_0]) + \sum_{k=k_0}^{k_1-1} \left( r[k]^T y[k] - \tilde{\nu} |r[k]|^2 - \tilde{\rho} |y[k]|^2 + \delta_2 \right)
\]
where \( V(x) = \frac{1}{2} V_1(x_1) + \frac{1}{2} V_2(x_2) \),
\[
\tilde{\nu} < \min\{\nu_1', \tilde{\nu}_2\},
\]
\[
\tilde{\rho} \leq \min\{\rho_1' - \tilde{\nu}_2 - \tilde{\rho}_2 - \tilde{\nu}_1', \rho_2' - \tilde{\rho}_2 - \tilde{\nu}_1'\},
\]
\( \delta(x[0]) = \delta_1(x_1[0]) + \delta_2(x_2[0]) \), and \( \delta_1, \delta_2, \tilde{\nu}_1', \rho_1', \tilde{\rho}_2 \) are those in (31) and (47).

Now let us consider the SD property of \( T_r(\Sigma) \) similar to the discussion in preceding sections. Suppose that \( T_r(P) \) and \( T_r(\Sigma) \) are both SD such that inequalities (39) and (40) hold. Note that \( \|u_2^k\|^2 \geq \frac{1}{2}\|u_2\|^2 - \|u_2 - u_2^k\|^2 \geq \frac{1}{2}\|u_2\|^2 - \|u_2^k\|^2 \geq \frac{1}{2}\|y_2\|^2 \) and \( \|y_2^k\|^2 \geq \frac{1}{2}\|y_2\|^2 \). Then, we have
\[
\|y_2^k\|^2 \leq \frac{1}{2}\|u_2^k\|^2 + \|y_2\|^2 - \|y_2^k\|^2 - (L + 3\|\mu_2\|^2)^2.
\]
which implies that
\[
\sum_{k=k_0}^{k_0+N} \|y_2\|^2 \geq \frac{1}{2}\|u_2^k\|^2 + \|y_2\|^2 - (N_2 + 1)\|\mu_2\|^2 + (L + 3\|\mu_2\|^2)^2.
\]

Then, similar to the proof of Lemma 3, it is easy to show that for any \( x[k_0] \) and any \( r[k] \), we have
\[
\sum_{k=k_0}^{k_0+N} \|y_2\|^2 \geq \frac{1}{2}\|u_2^k\|^2 + \|y_2\|^2 - (N_2 + 1)\|\mu_2\|^2 + (L + 3\|\mu_2\|^2)^2.
\]
where \( N = \max\{N_1, N_2\} \), \( M \) and \( \vartheta \) are given by (41) and (43), respectively.

With notations above, we have the following theorem and corollary for the system \( T_r(P) \) shown in Fig. 5. The proof of Theorem 5 and Corollary 5 is similar to that of Theorem 4 and Corollary 4, respectively, and is omitted here.

**Theorem 5.** Consider the system \( T_r(P) \) shown in Fig. 5 where \( r[k] \) is bounded and satisfies (46) with \( L \) given in (45) and \( \epsilon, \mu_2 > 0 \). Suppose that 1) inequalities (30) and (47) hold; 2) \( T_r(P) \) and \( T_r(\Sigma) \) are SD that satisfy (39) and (40), respectively; 3) functions \( \beta_1, \beta_2 \) satisfy Assumption 3 with positive semi-definite matrices \( S_1, S_2 \), respectively; 4) \( \rho \) shown in (50) can be chosen as \( \rho > 0 \); 5) the matrix \( G \) defined as \( G := \eta_2 M - S \) is positive definite where \( M, S, \eta_2 \) are given in (41)-43, respectively. Then, for any \( w[k] \) satisfies (46). 1) there exists a compact set \( D_1 \) such that \( x[k] \in D_1 \) for any \( k \in \mathbb{Z}_{\geq 0} \); 2) there exist a compact set \( D_2 \) and a constant \( K \in \mathbb{Z}_{\geq 0} \) such that \( x[k] \in D_2 \) for all \( k \geq K \).

**Corollary 5.** Consider the system \( T_r(P) \) shown in Fig. 5 where \( r[k] \equiv 0 \) and \( w[k] \) is bounded and satisfies (46) with \( L \) given in (45) and \( \epsilon, \mu_2 > 0 \). Suppose that conditions 1)-3) in Theorem 5 hold, and 4) \( \tilde{\nu}_2 + \rho_1' > 0, \nu_1' + \rho_2 > 0 \); 5) the matrix \( G \) defined as \( G := M - S \) is positive definite where \( M = \text{diag}\{a_1 M_1, a_2 M_2\} \), \( S = \text{diag}\{w_1 S_1, w_2 S_2\} \) and \( a_1, a_2 \) are two positive number such that \( a_1 \nu_1 + a_2 \leq \nu_1' + \rho_2, a_2 \nu_2 + a_1 \leq \tilde{\rho}_2 + \rho_1' \). Then, for any \( w[k] \) that satisfies (46). 1) there exists a compact set \( D_1 \) such that \( x[k] \in D_1 \) for any \( k \in \mathbb{Z}_{\geq 0} \); 2) there exist a compact set \( D_2 \) and a constant \( K \in \mathbb{Z}_{\geq 0} \) such that \( x[k] \in D_2 \) for all \( k \geq K \).
The following corollary provides conditions for $T_r(P) \times T_{\tau \mu}(\Sigma)$ to be ultimately bounded.

**Corollary 6.** Consider the system $T_r(P) \times T_{\tau \mu}(\Sigma)$ shown in Fig. 4 (b). Suppose that all the conditions in Theorem 3 hold and (13) holds. Suppose that the initial conditions $x_2[0]$ and $x_2^*[0]$ satisfy $|x_2[0] - x_2^*[0]| \leq \epsilon$. Then, 1) there exists a compact set $D_1$ such that $(x_1[k], x_2^*[k]) \in D_1$ for all $k \in \mathbb{Z}_{\geq 0}$; 2) there exist a compact set $D_2$ and a constant $K \in \mathbb{Z}_{\geq 0}$ such that $(x_1[k], x_2^*[k]) \in D_2$ for all $k \geq K$.

**Proof.** Consider the system $T_r(P) \times T_{\tau \mu}(\Sigma)$ shown in Fig. 5 where the disturbance $w[k]$ is chosen as

$$w[k] = \bar{y}_2[k] - \tilde{y}_2[k], \forall k \in \mathbb{Z}_{\geq 0}. \quad (51)$$

Because $|y_2[k] - \bar{y}_2[k]| = |h_21(x_2[k]) - h_21(x_2^*[k])| \leq L|x_2[k] - x_2^*[k]| \leq L\epsilon$ by the Lipschitz property of $h_{21}$, and $|\bar{y}_2[k] - \tilde{y}_2[k]| \leq \sqrt{m} \mu_2$ by the property of the quantizer $Q_2$, it follows that

$$|w[k]| \leq |y_2[k] - \tilde{y}_2[k]| + |y_2[k] - \bar{y}_2[k]| + |\bar{y}_2[k] - \tilde{y}_2[k]| \leq L\epsilon + 2\sqrt{m} \mu_2, \forall k \in \mathbb{Z}_{\geq 0}.$$

Hence, the particular choice $w[k]$ shown in (51) satisfies the assumption (40). Moreover, if both $T_r(P) \times T_{\tau \mu}(\Sigma)$ and $T_r(P) \times T_{\tau \mu}(\Sigma)$ are driven by the same $r[k]$, then $w[k]$ shown in (51) ensures that $\bar{y}_2[k] = \tilde{y}_2[k]$ and $u_2[k] = u_2^*[k], \forall k \in \mathbb{Z}_{\geq 0}$. By the definition of approximate bisimulation (cf. Definition 2), $x_2[k]$ and $x_2^*[k]$ are related by

$$|x_2[k] - x_2^*[k]| \leq \epsilon, \forall k \in \mathbb{Z}_{\geq 0}. \quad (52)$$

Therefore, the state boundedness of $T_r(P) \times T_{\tau \mu}(\Sigma)$ can be derived immediately by Theorem 5.

When $r[k] \equiv 0$, state boundedness of $T_r(P) \times T_{\tau \mu}(\Sigma)$ can also be derived using Corollary 5 and the same argument as Corollary 6.

Similar to the discussion in Section 3, the bounds of $x_1, x_2^*$ can be made arbitrarily small by letting the parameters $\mu_1, \mu_2, \eta, \epsilon$ small enough. Furthermore, the bounds of $x_2^*$ can be used to determine how the state space and input space of $T_{\tau \mu}(\Sigma)$ can be chosen as compact sets; the resulting $T_{\tau \mu}(\Sigma)$ with bounded state space and input space is a finite transition system that can be implemented with finite precision.

Above result formalizes the intuition that when a controller guaranteeing stability in some robust way is replaced by its symbolic bisimilar version, the feedback-connected system should be “somewhat stable”. The key challenge to keep in mind when replacing the controller in the feedback loop is the fact that the internal signals driving the plant changes therefore, within the feedback-loop, symbolic model and the actual controller can be driven by totally different inputs even if they are initialized with the same initial conditions. The external bounded signal $w$ we introduce in our analysis captures this additional robustness required. Moreover, passivity indices provide us a way to explicitly compute global and ultimate bounds on the states of the closed-loop system with the symbolic controller implementation.

**Remark 11.** There is a trade-off between the passivity degradation and the construction of the approximately bisimilar model $T_{\tau \mu}(\Sigma)$: on one hand, a smaller sampling time $\tau$ can result in a smaller passivity degradation; on the other hand, given a precision $\epsilon$, the sampling time $\tau$ should be chosen large enough in order to make (3) holds and $T_r(\Sigma) \equiv_{(\epsilon, \mu)} T_{\tau \mu}(\Sigma)$.

**Example 5.** Consider the system $P$ and $\Sigma$ given in Example 4 and its feedback connection shown in Fig. 4 (b). Suppose that $r[k] \equiv 0$ and choose $\mu_1 = \mu_2 = 0.01$. Given a precision $\epsilon = 0.25$, parameters $\tau = 0.3, \mu = \mu_1, \eta = 0.1$ can be chosen such that $T_r(\Sigma) \equiv_{(\epsilon, \mu)} T_{\tau \mu}(\Sigma)$. To
Figure 6: Trajectories of $x_1[k], x_2[k]$ using three different $\eta$, all from initial states $x_1[0] = (-0.7, -2.5)^T$, $x_2[0] = (1.5, -1.6)^T$.

demonstrate the effect of the state quantization of $T_{\tau \mu \eta}(\Sigma)$, we also consider using quantization precisions $\eta = 0.05$ and $\eta = 0.01$, both of which also guarantee that $T_\tau(\Sigma) \cong (\epsilon, \mu) T_{\tau \mu \eta}(\Sigma)$. One can verify that conditions of Corollary 5 hold and thus $x_1[k], x_2[k]$ are (ultimately) bounded. In Fig. 6, the trajectories of $x_1[k], x_2[k]$ are shown for three different choices of $\eta$, all from the same initial states $x_1[0] = (-0.7, -2.5)^T$, $x_2[0] = (1.5, -1.6)^T$. From these figures, it can be observed that the state trajectories are eventually bounded, and the smaller $\eta$ (i.e., the state quantization precision) is, the smaller the ultimate bound would be.

7 Conclusions

In this paper, we analyzed the passivity degradation from a continuous-time control system to its sampled, input and output quantized model, and used these results to analyze the feedback-connected system when a continuous controller is implemented via sampling and quantization. We also considered bisimulation-based symbolic implementations, where in addition to inputs and outputs, the internal states of the controller is also quantized, therefore the controller has a fully discrete representation. We derived conditions under which the closed-loop system with those control implementations is (ultimately) bounded, where explicit bounds are presented and ways to render these bounds arbitrarily small are discussed. Our method provides a novel perspective for the analysis of heterogeneous, compositional systems, and can be used in the analysis of potentially complex control systems with various implementation artifacts due to resource constraints.

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Then \( \nu \in \mathbb{R} \) such that the following inequalities hold for any \( k \in \mathbb{Z}^+ \),

\[
\nu_k \geq \nu_{k+1} - \frac{\nu_k}{1+\nu_k} - \frac{\nu_k}{\nu_k+1} - \frac{\nu_k}{\nu_k+2},
\]

which implies that \( M_1 - M_2 \geq 0 \) where

\[
M_1 := \text{diag} \left( \begin{bmatrix} \nu & -\nu \ \\ -\nu & \nu + \rho \end{bmatrix}, \begin{bmatrix} -\nu & -\nu \\ -\nu & \nu + \rho \end{bmatrix} \right), \quad M_2 := \text{diag} \left( \nu, \hat{\nu}, \hat{\rho}, \hat{\rho} \right).
\]

Noting that \( r_1 = u_1 + y_2 \) and \( r_2 = y_2 - y_1 \), for any \( k_0, k_1 \in \mathbb{Z}^+ \), we have \( V(x[k_1]) - V(x[k_0]) \leq \sum_{k=k_0}^{k_1-1} \left( u_1[k]^\top y_1[k] - \nu_1 u_1[k]^2 - \rho_1 y_1[k]^2 + \delta_1 + u_2[k]^\top y_2[k] - \nu_2 y_2[k]^2 - \rho_2 y_2[k]^2 + \delta_2 \right) \leq \sum_{k=k_0}^{k_1-1} \left( r_1[k]^\top y_1[k] - z[k] M_1 z[k]^\top + \delta \right) \leq \sum_{k=k_0}^{k_1-1} \left( r_1[k]^\top y_1[k] - y_1[k] M_1 y_1[k]^\top + \delta \right) \leq \sum_{k=k_0}^{k_1-1} \left( r_1[k]^\top y_1[k] - y_1[k] M_1 y_1[k]^\top + \delta \right). \]

This completes the proof. \( \square \)

## B Proof of Lemma 3

Since \( \Sigma_1, \Sigma_2 \) are SD, there exist numbers \( N_1, N_2 \in \mathbb{Z}^+ \), \( \vartheta_1, \vartheta_2 \geq 0 \) and matrices \( M_1, M_2 > 0 \) such that the following inequalities hold for any \( k_0 \in \mathbb{Z}^+ \):

\[
\sum_{k=k_0}^{k_0+N_1} \vartheta_1 |u_1[k]|^2 + |y_1[k]|^2 \geq x_1[k_0]^\top M_1 x_1[k_0],
\]

\[
\sum_{k=k_0}^{k_0+N_2} \vartheta_2 |u_2[k]|^2 + |y_2[k]|^2 \geq x_2[k_0]^\top M_2 x_2[k_0].
\]

Define

\[
\vartheta = \max \left\{ \frac{2 \vartheta_1}{2 \vartheta_1 + 1}, \frac{2 \vartheta_2}{2 \vartheta_2 + 1} \right\},
\]

\[
M = (1 - \vartheta) \text{diag} \{ M_1, M_2 \}.
\]

Then \( N \in \mathbb{Z}^+ \), \( M > 0 \) and

\[
0 \leq \max \{ 2 \vartheta_1 (1 - \vartheta), 2 \vartheta_2 (1 - \vartheta) \} \leq \vartheta < 1.
\]
Recall that \( r_1[k] = y_2[k] + u_1[k], r_2[k] = u_2[k] - y_1[k]. \) For any \( x[k_0] \) and any \( r[k], \) we have

\[
\sum_{k=k_0}^{k_0+N} \vartheta |r[k]|^2 + |y[k]|^2
= \sum_{k=k_0}^{k_0+N} \vartheta ((y_2[k] + u_1[k])^2 + |u_2[k] - y_1[k]|^2)
+ |y_1[k]|^2 + |y_2[k]|^2
= \sum_{k=k_0}^{k_0+N} (1 - \vartheta) (|y_1[k]|^2 + |y_2[k]|^2) + \frac{\vartheta}{2} (|u_1[k]|^2 + |u_2[k]|^2) + \vartheta \left( \sum_{k=k_0}^{k_0+N} \frac{|u_1[k]|^2}{2} + 2|y_2[k]|^2 + 2u_1[k]^\top y_2[k] \right)
+ \vartheta \left( \frac{|u_2[k]|^2}{2} + 2|y_1[k]|^2 - 2u_2[k]^\top y_1[k] \right)
\geq \sum_{k=k_0}^{k_0+N} (1 - \vartheta) (|y_1[k]|^2 + |y_2[k]|^2) + \frac{\vartheta}{2} (|u_1[k]|^2 + |u_2[k]|^2)
\geq (1 - \vartheta) \sum_{k=k_0}^{k_0+N} |y_1[k]|^2 + |y_2[k]|^2 + \vartheta_1 |u_1[k]|^2 + \vartheta_2 |u_2[k]|^2
\geq (1 - \vartheta) (x_1[k_0]^\top M_1 x_1[k_0] + x_2[k_0]^\top M_2 x_2[k_0])
= x[k_0]^\top M x[k_0].
\]

This completes the proof by definition. \( \square \)

## C Proof of Lemma 4

Because \( \nu \rho \leq \frac{1}{4}, \) we have \( \eta_1 > \frac{1}{4\rho} - \nu \geq 0. \) Because for any \( k \in \mathbb{Z}_{\geq 0}, \) \( u^\top[k] y[k] \leq \frac{1}{4\lambda} |u[k]|^2 + \lambda |y[k]|^2, \) from (6) we have

\[
V(x[k+1]) - V(x[k]) \leq \frac{1}{4\lambda} |u[k]|^2 + \lambda |y[k]|^2 - \nu |u[k]|^2 - \rho |y[k]|^2 + \delta
= \eta_1 |u[k]|^2 - \eta_2 |y[k]|^2 + \delta
\leq \eta_1 |u[k]|^2 + \delta.
\]

Therefore, for any \( k_0 \in \mathbb{Z}_{\geq 0}, \)

\[
V(x[k_0 + N + 1]) - V(x[k_0]) \leq \sum_{k=k_0}^{k_0+N} \left( \eta_1 |u[k]|^2 - \eta_2 |y[k]|^2 + \delta \right)
\leq \sum_{k=k_0}^{k_0+N} \left[ (\eta_1 + \vartheta \eta_2) |u[k]|^2 + \delta \right] - \eta_2 x[k_0]^\top M x[k_0]
\leq c_2 - \eta_2 x[k_0]^\top M x[k_0],
\]

where the second inequality is from (8).

1) Since \( V(x) \) is radially unbounded, \( C_1, D_1 \) are compact sets and \( c_1 < \infty. \) Clearly, \( x[0] \in D_1. \) For \( k \in \{1, \ldots, N, N + 1\}, \) (55) implies

\[
V(x[k]) \leq V(x[0]) + k(\eta_1 |u[k]|^2 + \delta) \leq c_1 + c_3.
\]

Therefore, \( x[k] \in D_1 \) for any \( k \in \{1, \ldots, N, N + 1\}. \) Consider now \( k \in \{N + 2, \ldots, 2N + 2\}. \) If there exists some \( k^* \in \{N + 2, \ldots, 2N + 2\} \) such that \( x[k^*] \notin D_1, \) or equivalently,

\[
V(x[k^*]) > c_1 + c_3.
\]
then by (55) we have $V(x[k^* - N - 1]) \geq V(x[k^*]) - (N + 1)(\eta_1||u||^2 + \delta) > c_1$, implying that $x[k^* - N - 1] = \eta_2 x[k^* - N - 1] + c_2$. Then,

$$V(x[k^*]) \leq V(x[k^* - N - 1]) + c_2 - \eta_2 x[k^* - N - 1] + c_2 \leq V(x[k^* - N - 1]) \leq c_1 + c_3 \quad (59)$$

where the first inequality is from (56), and the third inequality is from (57). Noting that (59)

contradicts with (58), we conclude that $x[k] \in D_1$ for $k \in \{N + 2, ..., 2N + 2\}$. By induction, $x[k] \in D_1$ for any $k \in \mathbb{Z}_{\geq 0}$.

2). We claim that $x[k] \in D_2$ for some $k \in \mathbb{Z}_{\geq 0}$ implies that $x[k + n(N + 1)] \in D_2$ for any $n \in \mathbb{Z}_{\geq 0}$. Indeed, $x[k] \in D_2$ implies that $V(x[k]) \leq c_2 + c_4$. If $x[k + N + 1] \notin D_2$, then $V(x[k]) \geq c_2 + c_4$. From (56), we have $V(x[k]) \geq V(x[k] + N) - c_2 > c_4$, which implies that $\eta_2 x[k^* - N] > c_2 + c_4$ by the definition of $c_4$. Then, again from (56), we have $V(x[k + N + 1]) \leq V(x[k]) + c_2 - \eta_2 x[k^* - N] < V(x[k]) \leq c_2 + c_4$, which conflicts with the assertion that $V(x[k]) > c_2 + c_4$. Define $j_s = \min\{k \in \mathbb{Z}_{\geq 0} \mid k \equiv s \ (\text{mod } N + 1), V(x[k]) \in D_2\} \leq \infty$ for $s = 0, 1, ..., N$ where “mod” denotes the modulo. The claim above shows that $x[k] \in D_2$ for any $k \leq j_s$ where $k \equiv s \ (\text{mod } N + 1)$. For any $k + N + 1 < j_s$ where $k \equiv s \ (\text{mod } N + 1), k \in \mathbb{Z}_{\geq 0}$, (56) implies that $V(x[k + N + 1]) - V(x[k]) \leq c_2 - \eta_2 x[k^* - N]$. Since $x[k] \notin D_2$, $V(x[k]) > c_2 + c_4$, which implies that $\eta_2 x[k^* - N] > c_2 + c_4$. Therefore, $V(x[k + N + 1]) - V(x[k]) \leq -c_3$. Hence, $j_s \leq (V(x[0] - c_2 - c_4))/c_3 < \infty$. Choose $K = \max\{j_0, j_1, ..., j_N\}$. Then, $x[k] \in D_2$ for any $k \geq K, k \in \mathbb{Z}_{\geq 0}$. This completes the proof. $\square$

D Proof of Lemma 5

The system $\Sigma_d$ is 0-step SD implies that $\nu \|u[k]\|^2 + |y[k]|^2 \geq x[k]^\top M x[k], \forall k \in \mathbb{Z}_{\geq 0}$. As in Lemma 4, we choose $\lambda$ such that $0 < \lambda < \rho$, and define $\eta_1 = \frac{1}{\lambda} - \nu > 0, \eta_2 = \rho - \lambda > 0$. Then we have

$$V(x[k + 1]) - V(x[k]) \leq \eta_1 \|u[k]\|^2 + \eta_2 |y[k]|^2 + \delta
\leq -\eta_2 x[k]^\top M x[k] + (\eta_1 + \delta \eta_2) \|u[k]\|^2 + \delta
\leq -\eta_2 \lambda \min\{M\} \|x[k]\|^2 + (\eta_1 + \delta \eta_2) \|u[k]\|^2 + \delta.$$  Define $K_\infty$-functions $\alpha_1(s) = as^2$, $\alpha_2(s) = b s^2$ where $b = \max\{\hat{b}, \eta_2 \lambda \min\{M\}\}$, and $\alpha_3(s) = \eta_2 \lambda \min\{M\} s^2$. Define a $K$-function $\sigma(s) = (\eta_1 + \delta \eta_2)s^2$ and let $d_2 = \delta$. Then $V$ is an ISpS-Lyapunov function of $\Sigma_d$ by definition. The conclusion follows by Theorem 2.5 in [25]. $\square$

E Proof of Theorem 1

Let the inputs of $\Sigma$ to be piecewise constant, that is, $u(t) = u[k]$ for any $t \in [k\tau, (k + 1)\tau)$. Since $x[k] = x(k\tau)$, we have

$$|y(t) - y[k]| e^{[h_1(x(t)) - h_1(x(k\tau)))]} \leq \int_{k\tau}^{(k + 1)\tau} |h_1(s)| ds \leq \sqrt{\tau} \int_{k\tau}^{(k + 1)\tau} |h_1(s)|^2 ds, \quad (60)$$
Furthermore, it is clear that the fourth inequality is from (14), the fifth inequality is because 
where the second inequality is from (60), the third inequality is from the Cauchy-Schwarz inequality. For any 
where the last inequality is from the Cauchy-Schwarz inequality. For any $k_0, k_1 \in \mathbb{Z}_{\geq 0}, k_0 < k_1$
and for any admissible $u(t)$,

$$
\left| \int_{k_0 \tau}^{k_1 \tau} u(t)^\top y(t) \, dt - \tau \sum_{k=k_0}^{k_1-1} u[k]^\top y[k] \right| 
\leq \sum_{k=k_0}^{k_1-1} \int_{k \tau}^{(k+1) \tau} |u[k]| |y(t) - y[k]| \, dt 
\leq \tau \sqrt{\tau} \sum_{k=k_0}^{k_1-1} |u[k]| \sqrt{\int_{k \tau}^{(k+1) \tau} |\dot{h}_1(s)|^2 \, ds} 
\leq \tau \sqrt{\tau} \sqrt{\sum_{k=k_0}^{k_1-1} |u[k]|^2 \int_{k \tau}^{(k+1) \tau} |\dot{h}_1(s)|^2 \, ds} 
\leq \tau \sqrt{\tau} \sqrt{\sum_{k=k_0}^{k_1-1} |u[k]|^2 \int_{k \tau}^{k_1 \tau} |u(s)|^2 \, ds + \beta(x[0])} 
\leq \tau^2 \gamma \sum_{k=k_0}^{k_1-1} |u[k]|^2 + \frac{\beta(x[0])}{\gamma^2} = \tau^2 \gamma \sum_{k=k_0}^{k_1-1} |u[k]|^2 + \frac{\beta(x[0])}{\gamma},
$$

where the second inequality is from (60), the third inequality is from the Cauchy-Schwarz inequality, the fourth inequality is from (14), the fifth inequality is because $\frac{\beta(x[k_0])}{\gamma^2} > 0$. Therefore,

$$
\int_{k_0 \tau}^{k_1 \tau} u(t)^\top y(t) \, dt \leq \tau \sum_{k=k_0}^{k_1-1} u[k]^\top y[k] + \tau^2 \gamma \sum_{k=k_0}^{k_1-1} |u[k]|^2 + \frac{\beta(x[0])}{\gamma}.
$$

(61)

It is clear that

$$
- \nu \int_{k_0 \tau}^{k_1 \tau} |u(t)|^2 \, dt = - \tau \nu \sum_{k=k_0}^{k_1-1} |u[k]|^2.
$$

(62)

Furthermore,

$$
\left| \int_{k_0 \tau}^{k_1 \tau} y(t)^\top y(t) \, dt - \tau \sum_{k=k_0}^{k_1-1} y[k]^\top y[k] \right| 
\leq \sum_{k=k_0}^{k_1-1} \int_{k \tau}^{(k+1) \tau} |y(t) - y[k]|^2 \, dt + 2 \sum_{k=k_0}^{k_1-1} \int_{k \tau}^{(k+1) \tau} |y[k]| |y(t) - y[k]| \, dt 
\leq \sum_{k=k_0}^{k_1-1} \int_{k \tau}^{(k+1) \tau} |y(t) - y[k]|^2 \, dt + \sum_{k=k_0}^{k_1-1} \int_{k \tau}^{(k+1) \tau} \lambda_1 |y(t) - y[k]|^2 + \frac{1}{\lambda_1} |y[k]|^2 \, dt 
\leq \tau^2 (1 + \lambda_1) \left( \gamma^2 \int_{k_0 \tau}^{k_1 \tau} |u(t)|^2 \, dt + \beta(x[0]) \right) + \frac{\tau}{\lambda_1} \sum_{k=k_0}^{k_1-1} |y[k]|^2 
= \tau^3 \gamma^2 (1 + \lambda_1) \sum_{k=k_0}^{k_1-1} |y[k]|^2 + \frac{\tau}{\lambda_1} \sum_{k=k_0}^{k_1-1} |y[k]|^2 + \tau^2 (1 + \lambda_1) \beta(x[k_0])
$$
where \( \lambda_1 \) is an arbitrary positive number, the third inequality is from (14). Therefore,

\[-\rho \int_{k_0 \tau}^{k_1 \tau} y(t) \top y(t) \, dt \leq \tau^3 \gamma^2 (1 + \lambda_1) |\rho| \sum_{k=k_0}^{k_1-1} |u[k]|^2 + \tau \left( \frac{|\rho|}{\lambda_1} - \rho \right) \sum_{k=k_0}^{k_1-1} |y[k]|^2 + \tau^2 |\rho|(1 + \lambda_1) \beta(x[k_0]). \]

(63)

Since \( V(x[k_0 \tau]) = V(x(k_0 \tau)), V(x[k_1]) = V(x(k_1 \tau)) \) and

\[V(x(k_1 \tau)) - V(x(k_0 \tau)) \leq \int_{k_0 \tau}^{k_1 \tau} u(t) \top y(t) - \nu |u(t)|^2 - \rho |y(t)|^2 \, dt, \]

adding (61), (62), (63) together we obtain the inequality (17). This completes the proof. \( \square \)

F Proof of Theorem 2

For any \( k \in \mathbb{Z}_{\geq 0} \), since \(|y[k] - \tilde{y}[k]| \leq \sqrt{m_2} \), we have

\[|y[k] - \tilde{y}[k]| \leq 2 |\tilde{y}[k]| |y[k] - \tilde{y}[k]| + |y[k] - \tilde{y}[k]|^2 \leq \frac{1}{\lambda_2} |\tilde{y}[k]|^2 + (1 + \lambda_2) |y[k] - \tilde{y}[k]|^2 \leq \frac{1}{\lambda_2} |\tilde{y}[k]|^2 + (1 + \lambda_2) \mu_2^2 \]

where \( \lambda_2 \) is an arbitrary positive number. Hence,

\[-\rho_1 |\tilde{y}[k]|^2 \leq (\frac{|\rho_1|}{\lambda_2} - \rho_1) |\tilde{y}[k]|^2 + |\rho_1|(1 + \lambda_2) \mu_2^2. \]

(65)

Similarly, since \(|u[k] - \tilde{u}[k]| \leq \sqrt{m_1} \), we have

\[|u[k] - \tilde{u}[k]| \leq 2 |\tilde{u}[k]| |u[k] - \tilde{u}[k]| + |u[k] - \tilde{u}[k]|^2 \leq \frac{1}{\lambda_3} |\tilde{u}[k]|^2 + (1 + \lambda_3) |u[k] - \tilde{u}[k]|^2 \leq \frac{1}{\lambda_3} |\tilde{u}[k]|^2 + (1 + \lambda_3) \mu_1^2 \]

where \( \lambda_3 \) is an arbitrary positive number. Hence,

\[-\nu_1 |u[k]|^2 \leq (\frac{|\nu_1|}{\lambda_3} - \nu_1) |\tilde{u}[k]|^2 + |\nu_1|(1 + \lambda_3) \mu_1^2. \]

(66)

Since

\[|u[k] \top y[k] - \tilde{u}[k] \top \tilde{y}[k]| \leq |\tilde{u}[k]| |y[k] - \tilde{y}[k]| + |\tilde{y}[k]| |u[k] - \tilde{u}[k]| \leq \sqrt{m_2} |\tilde{u}[k]| + \sqrt{m_1} |\tilde{y}[k]| \leq \lambda_4 \mu_2^2 + \frac{1}{4 \lambda_4} |\tilde{u}[k]|^2 + \lambda_5 \mu_2^2 + \frac{1}{4 \lambda_5} |\tilde{y}[k]|^2, \]

where \( \lambda_4, \lambda_5 \) are arbitrary positive numbers, we have

\[u[k] \top y[k] \leq \tilde{u}[k] \top \tilde{y}[k] + \frac{1}{4 \lambda_4} |\tilde{u}[k]|^2 + \frac{1}{4 \lambda_5} |\tilde{y}[k]|^2 + \mu_4 \mu_2^2 + \lambda_5 \mu_1^2. \]

(67)

Then the inequality (23) follows immediately from (17), (65), (66) and (67). \( \square \)
G Proof of Theorem 4

Similar to the discussion in Subsection 4.3 and the inequality (28), we have

\[ \sum_{k=k_0}^{k_0+N} \vartheta_2 |\tilde{u}_2[k]|^2 + |\tilde{y}_2[k]|^2 \geq \frac{1}{2} x_2[k_0] \top M_2 x_2[k_0] - m(N_2 + 1)(\vartheta_2 \mu_2^2 + \mu_2^2). \]  

(68)

Then, similar to the proof of Lemma 3, it is easy to show that \( \sum_{k=k_0}^{k_0+N} \vartheta |r[k]|^2 + |y[k]|^2 \geq x[k_0] \top M x[k_0] - d_2 \) for any \( x[k_0] \) and any \( r[k] \).

By Theorem 3, the feedback-connected system satisfies inequality (36). It is clear that \( V(x) \) is positive definite, radially unbounded. By Assumption 3, we have

\[ V(x[k_0 + N + 1]) - V(x[k_0]) \leq \delta_1(x[k_0]) + \delta_2(x[k_0]) + \sum_{k=k_0}^{k_0+N} (\eta_1 |r[k]|^2 - \eta_2 |y[k]|^2 + \tilde{\delta}_2) \]

\[ \leq w_1 x_1[k_0] \top S_1 x_1[k_0] + w_2 x_2[k_0] \top S_2 x_2[k_0] \]

\[ + \sum_{k=k_0}^{k_0+N} [(\eta_1 + \vartheta \eta_2) |r[k]|^2 + \tilde{\delta}_2] - \eta_2 x[k_0] \top M x[k_0] + d_2 \]

\[ = d_1 + d_2 - x[k_0] \top G x[k_0]. \]  

(69)

Define \( j_s = \min \{ k \in \mathbb{Z}_{\geq 0} | k \equiv s \pmod{N + 1}, V(x[k]) \in D_2 \} \) \( \leq \infty \) for \( s = 0, 1, \ldots, N - 1 \). As in part 2) of the proof of Lemma 4 for any \( k \geq j_s \) where \( k \equiv s \pmod{N + 1} \), it can be shown that \( x[k] \in D_2 \), and for any \( k_0 + N < j_s \) where \( k_0 \equiv s \pmod{N + 1} \), it can be shown that \( V(x[k_0 + N + 1]) - V(x[k_0]) \leq -d_3 < 0 \). Then the conclusion follows immediately. \( \square \)

H Proof of Corollary 4

When \( r[k] \equiv 0 \), \( u_1[k] = -\tilde{y}_2[k] \) and \( \tilde{u}_2[k] = y_1[k] \). Recalling (39) and (68), we have

\[ \sum_{k=k_0}^{k_0+N_1} \vartheta_1 |\tilde{y}_2[k]|^2 + |y_1[k]|^2 \geq x_1[k_0] \top M_1 x_1[k_0] , \]

\[ \sum_{k=k_0}^{k_0+N_2} \vartheta_2 |y_1[k]|^2 + |\tilde{y}_2[k]|^2 \geq \frac{1}{2} x_2[k_0] \top M_2 x_2[k_0] - m(N_2 + 1)(\vartheta_2 \mu_2^2 + \mu_2^2) . \]

Since \( \tilde{\nu}_2 + \rho_1 > 0 \), \( \nu'_1 + \tilde{\rho}_2 > 0 \), it is clear that \( a_1, a_2 \) satisfying \( a_1 \vartheta_1 + a_2 \leq \nu'_1 + \tilde{\rho}_2 \) and \( a_2 \vartheta_2 + a_1 \leq \nu'_2 + \tilde{\rho}_2 \) always exist. Letting \( N = \max \{ N_1, N_2 \} \), we have

\[ \sum_{k=k_0}^{k_0+N} [(\tilde{\nu}_2 + \rho_1') |y_1[k]|^2 + (\nu'_1 + \tilde{\rho}_2) |\tilde{y}_2[k]|^2] \geq \sum_{k=k_0}^{k_0+N} [(a_2 \vartheta_2 + a_1) |y_1[k]|^2 + (a_1 \vartheta_1 + a_2) |\tilde{y}_2[k]|^2] \]

\[ \geq x[k] \top M x[k] - m(N_2 + 1)(\vartheta_2 \mu_2^2 + \mu_2^2) . \]

By Corollary 3, when \( r[k] \equiv 0 \), the feedback-connected system satisfies the inequality shown in (38). Then,

\[ V(x[k_0 + N + 1]) - V(x[k_0]) \leq w_1 x_1[k_0] \top S_1 x_1[k_0] + w_2 x_2[k_0] \top S_2 x_2[k_0] + \sum_{k=k_0}^{k_0+N} \left[ - (\tilde{\nu}_2 + \rho_1') |y_1[k]|^2 - (\nu'_1 + \tilde{\rho}_2) |\tilde{y}_2[k]|^2 + \tilde{\delta}_2 \right] \]

\[ \leq (N + 1) \tilde{\delta}_2 + m(N_2 + 1)(\vartheta_2 \mu_2^2 + \mu_2^2) - x[k] \top G x[k] . \]

The following proof is then the same as that of Theorem 4 after obtaining (69). \( \square \)