Mass gap for a monopole interacting with different nonlinear spinor fields

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We study the properties of the mass gap for monopole solutions in SU(2) Yang-Mills theory with a source of the non-Abelian gauge field in the form of a spinor field described by the nonlinear Dirac equation. Different types of nonlinearities parameterized by the parameter \( \lambda \) are under consideration. It is shown that for the range of values of this parameter studied in the present paper the value of the mass gap depends monotonically on this parameter, and the position of the mass gap does not practically depend on \( \lambda \).

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I. INTRODUCTION

The problem of a mass gap occupies a special place in quantum chromodynamics. The significance of this problem is related to the fact that its solution requires the development of nonperturbative quantization methods in field theory, which are absent at present. It should be emphasized that the nonperturbative quantization is carried out numerically in lattice calculations. However, such calculations do not enable one to have a full understanding of the nonperturbative quantization, and some questions of principle still remain to be clarified: What are the properties of operators of a strongly interacting field? How one can determine a quantum state of such a field? Also, there are many other unclear questions for which there are answers in the case of perturbative quantization using Feynman diagrams.

In this connection, the question arises as to the existence of a mass gap in other theories. The presence of a mass gap in any theory is of interest by itself, but it is also possible that such a theory may serve as some approximate description of nonperturbative effects in quantum theory involving a strong interaction. In Refs. 1,2, it is shown that in SU(2) gauge theory with a source in the form of a nonlinear spinor field, there exist topologically trivial solutions with a radial magnetic field (“monopoles”). Such “monopoles” possess the following features: (a) the radial field decreases as \( r^{-3} \) at infinity (this is the reason why we use quotation marks for the word monopole); (b) the solutions are topologically trivial, unlike those describing the ’t Hooft-Polyakov monopole; (c) the most interesting fact is that the energy spectrum of such solutions has an absolute minimum, which we can refer to as a mass gap.

In the absence of a SU(2) gauge field, the corresponding mass gap was found in Refs. 3,4 in studying a nonlinear spinor field. These references also consider a generalization of nonlinear spinor field when one introduces some constant in the nonlinear term for the spinor field. In the present paper we study effects of the presence of a mass gap and its properties in a system containing such a nonlinear spinor field.

The nonlinear Dirac equation was introduced by W. Heisenberg as an equation describing the properties of an electron. The classical properties of this equation, in particular, its soliton-like solutions, were investigated in Refs. 3,4. Subsequently one of variants of the nonlinear Dirac equation was employed for an approximate description of the properties of hadrons (this approach is called the Nambu-Jona-Lasinio model 5; for a review, see Ref. 6).

In the present paper we study the dependence of the size of a mass gap, its position, etc., on the value of a parameter describing the nonlinear self-interaction potential of the spinor field.

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II. EQUATIONS AND ANSÄTZE FOR YANG-MILLS FIELDS COUPLED TO A NONLINEAR DIRAC FIELD

In this section we closely follow Refs. [1, 2]. The Lagrangian describing a system consisting of a non-Abelian SU(2) field \( A^a_\mu \) interacting with nonlinear spinor field \( \psi \) can be taken in the form

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i \hbar c \bar{\psi} \gamma^\mu D_\mu \psi - m_f c^2 \bar{\psi} \psi + \frac{\Lambda}{2} g \hbar c V (\bar{\psi}, \psi) .
\]  

(1)

Here \( m_f \) is the mass of the spinor field; \( D_\mu = \partial_\mu - i \frac{\sigma^a}{2} A^a_\mu \) is the gauge-covariant derivative, where \( g \) is the coupling constant and \( \sigma^a \) are the SU(2) generators (the Pauli matrices); \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g e_{abc} A^b_\mu A^c_\nu \) is the field strength tensor for the SU(2) field, where \( e_{abc} \) (the completely antisymmetric Levi-Civita symbol) are the SU(2) structure constants; \( \Lambda \) is a constant; \( \gamma^\mu \) are the Dirac matrices in the standard representation; \( \bar{\psi} \psi \) interact with nonlinear spinor field \( \psi \). According to Ref. [4], this potential is a linear combination of scalar and pseudoscalar interactions:

\[
V (\bar{\psi}, \psi) = \alpha (\bar{\psi} \psi)^2 + \beta (\bar{\psi} \gamma^5 \psi)^2 ,
\]

(2)

where \( \alpha \) and \( \beta \) are constants.

Our purpose is to study monopole-like solutions of these equations. To do this, we use the standard SU(2) monopole Ansatz

\[
A_i^a = \frac{1}{\theta} [1 - f(r)] \begin{pmatrix} \sin \varphi & \sin \theta \cos \theta \cos \varphi \\ - \cos \varphi & \sin \theta \cos \theta \sin \varphi \\ 0 & - \sin^2 \theta \end{pmatrix} , \quad i = r, \theta, \varphi \quad \text{(in polar coordinates),}
\]

(3)

\[
A_i^a = 0 ,
\]

(4)

and the Ansatz for the spinor field from Refs. [7, 8]

\[
\psi^T = \frac{e^{-i \frac{4 \pi}{3}}}{g \sqrt{2}} \left\{ \begin{array}{c} 0 \\ -u \\ v \end{array} \right\} \left\{ \begin{array}{c} i v \sin \theta e^{-i \varphi} \\ - i v \cos \theta \\ - i v \sin \theta e^{i \varphi} \end{array} \right\} ,
\]

(5)

where \( E/\hbar \) is the spinor frequency and the functions \( u \) and \( v \) depend on the radial coordinate \( r \) only.

We will seek a solution to the Euler-Lagrange equations coming from the variation of the Lagrangian (1). Substituting in these equations the Ansätze (3)-(5), the expression (5), and integrating over the angles \( \theta, \varphi \) (for details see Ref. [3]), the resulting equations for the unknown functions \( f, u, \) and \( v \) are

\[
-f'' + \frac{f (f^2 - 1)}{x^2} + \frac{\tilde{g} \tilde{u} \tilde{v}}{x} = 0 ,
\]

(6)

\[
v' + \frac{f \tilde{v}}{x} = \tilde{u} \left( -1 - \tilde{E} + \frac{\tilde{u}^2 - \lambda \tilde{v}^2}{x^2} \right) ,
\]

(7)

\[
u' - \frac{f \tilde{u}}{x} = \tilde{v} \left( -1 - \tilde{E} + \frac{\tilde{u}^2 - \lambda \tilde{v}^2}{x^2} \right) .
\]

(8)

Here, for convenience of making numerical calculations, we have introduced the following dimensionless variables:

\[
x = r/\lambda_c , \quad \tilde{u} = u \sqrt{\lambda_c / \lambda} , \quad \tilde{v} = v \sqrt{\lambda_c / \lambda} , \quad \tilde{E} = \lambda_c E / (\hbar c) , \quad \tilde{g} = g \hbar c \lambda_c^2 / \lambda ,
\]

where \( \lambda_c = \hbar / (m_f c) \) is the Compton wavelength and \( \lambda \) is some combination of the constants \( \alpha, \beta \). The prime denotes differentiation with respect to \( x \).

The total energy density of the monopole under consideration is

\[
\tilde{\epsilon} = \tilde{\epsilon}_m + \tilde{\epsilon}_s = \frac{1}{\tilde{g}^2} \left[ \frac{f^2}{x^2} + \frac{(f^2 - 1)^2}{2x^4} \right] + \left[ \tilde{E} \tilde{u}^2 + \tilde{v}^2 + \frac{(\tilde{u}^2 - \lambda \tilde{v}^2)^2}{2x^4} \right] .
\]

(9)

Here the expressions in the square brackets correspond to the dimensionless energy densities of the non-Abelian gauge fields, \( \tilde{\epsilon}_m = \frac{\lambda_c^2 g^2}{\tilde{g}^2} \epsilon_m \), and of the spinor field, \( \tilde{\epsilon}_s = \frac{\lambda_c^2 g^2}{\tilde{g}^2} \epsilon_s \).

Correspondingly, the total energy of the monopole is calculated using the formula

\[
\tilde{W}_i = \frac{\lambda_c g^2}{\tilde{g}^2} W_i = 4\pi \int_0^\infty x^2 \tilde{\epsilon} dx = \left( \tilde{W}_i \right)_m + \left( \tilde{W}_i \right)_s ,
\]

(10)

where the energy density \( \tilde{\epsilon} \) is taken from Eq. (9), and it is the sum of the energies of the magnetic field and the nonlinear spinor field.
III. NUMERICAL RESULTS

Numerical integration of Eqs. (6)-(8) is carried out using the shooting method with the boundary conditions given for \( x = \delta \ll 1 \) (for details see Refs. [1,2]):

\[
f(\delta) = 1 + \frac{f_2}{2}\delta^2, \quad f'(\delta) = f_2\delta, \quad \tilde{u}(\delta) = \tilde{u}_1\delta, \quad \tilde{v}(\delta) = \frac{\tilde{v}_2}{2}\delta^2.
\]

The value of the parameter \( \tilde{v}_2 \) can be found from Eqs. (6)-(8), \( \tilde{v}_2 = 2\tilde{u}_1 \left( \tilde{E} - 1 + \tilde{\Lambda}\tilde{u}_1^2 \right) / 3 \). Adjusting the values of the parameters \( f_2, \tilde{u}_1 \), one can find the required regular solutions.

Our purpose is to construct a spectrum of solutions for different values of \( \lambda \). To do this, we find regular solutions for different values of the frequency \( \tilde{E} \) with a fixed value of the parameter \( \lambda \). For every value of \( \tilde{E} \), we calculate the total energy \( \tilde{W}_t \) given by the expression (10). Then we construct the dependence \( \tilde{W}_t(\tilde{E}) \), using which it is possible to find a mass gap as a minimum of the function \( \tilde{W}_t(\tilde{E}) \). Such a procedure is repeated for different values of the parameter \( \lambda \). In the present study we investigate the dependence of the value of the mass gap \( \Delta(\lambda) \) on the parameter \( \lambda \). Also, we study the dependence \( \tilde{E}_\Delta(\lambda) \) on the same parameter, where the quantity \( \tilde{E}_\Delta \) is defined as the value of the frequency \( \tilde{E} \) for which the energy spectrum has a minimum, i.e., it is the position of the mass gap on the axis \( \tilde{E} \).

The typical solution of the equations (6)-(8) is given in Fig. 1, where the profiles of the functions \( f(x), \tilde{u}(x), \) and \( \tilde{v}(x) \) are given. Fig. 2 shows the typical distribution of the energy density \( \tilde{\epsilon} \) from Eq. (9) of the “monopole” solution for \( \tilde{g}_\Lambda = 0.3354, \lambda = 0.8, \tilde{E} = 0.99, f_2 = -0.0042694, u_1 = 0.46657, \).

![Fig. 1](image1.png)

**FIG. 1.** The typical behavior of the functions \( f, \tilde{u}, \) and \( \tilde{v} \) of the “monopole” solution for \( \tilde{g}_\Lambda = 0.3354, \lambda = 0.8, \tilde{E} = 0.99, f_2 = -0.0042694, u_1 = 0.46657 \).

![Fig. 2](image2.png)

**FIG. 2.** The typical behavior of the magnetic fields \( \tilde{H}_r, \tilde{H}_\varphi, \) and \( \tilde{H}_\theta \) given by Eqs. (12)-(14) and the energy density \( \tilde{\epsilon} \) from Eq. (9) of the “monopole” solution for \( \tilde{g}_\Lambda = 0.3354, \lambda = 0.8, \tilde{E} = 0.99, f_2 = -0.0042694, u_1 = 0.46657 \).

Fig. 3 shows the dependence of the energy spectrum \( \tilde{W}_t(\tilde{E}) \) on the parameter \( \lambda \). From this figure, one can conclude that when \( \lambda \) increases, the energy spectrum rises, that is the corresponding values of the energy \( \tilde{W}_t \) increase.
Our main purpose here is to study the dependence of the value of the mass gap $\Delta$ on the nonlinearity parameter $\lambda$ characterizing the properties of the nonlinear potential of the spinor field $\psi$. The corresponding results of computations are given in Fig. 4. In turn, Fig. 5 shows the dependence of the position of the mass gap on the nonlinearity parameter $\lambda$. By the term a position of the mass gap, we mean the value of the frequency $\tilde{E}_\Delta$ for which the energy spectrum has a minimum. It is of interest to note that the position of the mass gap $\tilde{E}_\Delta$ does not practically depend on the nonlinearity parameter $\lambda$. One can assume that in fact this is the case, and deviations from this value are related to errors in numerical calculations.

In the present paper we have studied the dependence of the value of the mass gap and its position on the parameter describing the nonlinearity of the spinor field in the Dirac equation. We have found out that the value of the mass gap, which is assumed to be a global minimum of the energy spectrum of the “monopole” solution in SU(2) Yang-Mills theory with a source in the form of a nonlinear spinor field, depends smoothly on the nonlinearity parameter of the spinor field. We use the word “monopole” in quotation marks because the magnetic field decreases asymptotically as $r^{-3}$, unlike the ’t Hooft-Polyakov monopole solution where the magnetic field decreases according to the Coulomb law: $r^{-2}$. It is of interest that the position of the mass gap does not practically depend on the value of the nonlinearity parameter, at least in the range of values of the spinor field nonlinearity parameter considered here.

Thus, the study indicates that the mass gap in SU(2) Yang-Mills theory with a source in the form of a nonlinear spinor field does exist for different types of nonlinearity and depends smoothly on the nonlinearity parameter.
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