Supercongruences arising from hypergeometric series identities

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Abstract. By using some hypergeometric series identities, we prove two supercongruences on truncated hypergeometric series, one of which is related to a modular Calabi–Yau threefold, and the other is regarded as $p$-adic analogue of an identity due to Ramanujan.

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1 Introduction

Let

\[ f(z) := \eta^4(2z)\eta^4(4z) = \sum_{n=1}^{\infty} a(n)q^n, \]

where $q = e^{2\pi i z}$ and the Dedekind eta function is given by

\[ \eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n). \]

For odd primes $p$, let $N(p)$ denote the number of solutions to the modular Calabi–Yau threefold:

\[ x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0 \]

over the finite field with $p$ elements. Ahlgren and Ono [1], van Geemen and Nygaard [12], and Verrill [14] showed by different methods that

\[ a(p) = p^3 - 2p^2 - 7 - N(p). \]

In 2006, Kilbourn [6] proved that for any odd prime $p$,

\[ a(p) \equiv 4F_3 \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1 \right] _{p-1} \pmod{p^3}. \]
Here the truncated hypergeometric series are given by
\[ _rF_s \left[ \begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array} \right]_n = \sum_{k=0}^{n} \frac{(a_1)_k(a_2)_k \cdots (a_r)_k}{(b_1)_k(b_2)_k \cdots (b_s)_k} \frac{z^k}{k!}, \]
where \((a)_0 = 1\) and \((a)_k = a(a + 1) \cdots (a + k - 1)\) for \(k \geq 1\).

The first aim of this paper is to prove another supercongruence for \(a(p)\).

**Theorem 1.1** For any prime \(p \geq 5\), we have
\[
a(p) \equiv p \cdot 4F_3 \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, \frac{5}{4}, 1 \end{array} \right]_{p-1} \quad (\text{mod } p^3). \tag{1.2}
\]

In 1997, Van Hamme [13, (A.2)] proposed the following supercongruence conjecture.

**Conjecture 1.2** (Van Hamme, 1997) For any odd prime \(p\), we have
\[
6F_5 \left[ \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1 \\ \frac{1}{4}, 1, 1, 1 \end{array} \right]_{p-\frac{1}{2}} \equiv \begin{cases} -p\Gamma_p \left( \frac{1}{4} \right)^4 & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases} \tag{1.3}
\]
where \(\Gamma_p(\cdot)\) denotes the \(p\)-adic Gamma function.

The above supercongruence was regarded as \(p\)-adic analogue of the following identity due to Ramanujan (announced in his second letter to Hardy on February 27):
\[
6F_5 \left[ \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1 \\ \frac{1}{4}, 1, 1, 1 \end{array} \right] = \frac{2}{\Gamma \left( \frac{3}{4} \right)^4},
\]
which was later proved by Hardy [5] and Watson [15]. The supercongruence (1.3) was first confirmed by McCarthy and Osburn [9].

In 2015, Swisher [11, Theorem 1.5] also showed that (1.3) holds modulo \(p^5\) for primes \(p \equiv 1 \pmod{4}\). Recently, Guo and Schlosser [4, Theorem 2.2] established an interesting \(q\)-analogue of a supercongruence closely related to (1.3). By using the software package Sigma due to Schneider [10], the author [7, Theorem 1.3] extended the case \(p \equiv 3 \pmod{4}\) in (1.3) as follows.

**Theorem 1.3** Let \(p \geq 5\) be a prime. For \(p \equiv 3 \pmod{4}\), we have
\[
6F_5 \left[ \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1 \\ \frac{1}{4}, 1, 1, 1 \end{array} \right]_{p-\frac{1}{2}} \equiv -\frac{p^3}{16} \Gamma_p \left( \frac{1}{4} \right)^4 \quad (\text{mod } p^4). \tag{1.4}
\]
However, the proof of (1.4) in [7] is based on software package and seems unnatural. The second aim of this paper is to provide a human proof of (1.4) by hypergeometric series identities, which seems to be more natural.

The rest of this paper is organized as follows. Section 2 is devoted to recalling some properties of Gamma function and $p$-adic Gamma function. We prove Theorems 1.1 and 1.3 in Sections 3 and 4, respectively.

## 2 Preliminary results

We first recall some properties of Gamma function. The Gamma function $\Gamma(z)$ is an extension of the factorial function, which satisfies the functional equation:

$$\Gamma(z + 1) = z\Gamma(z). \quad (2.1)$$

From the above equation, we immediately deduce that for complex numbers $z$ and positive integers $n$,

$$(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)}. \quad (2.2)$$

It also satisfies the following reflection formula and duplication formula:

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}, \quad (2.3)$$

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1 - 2z}\sqrt{\pi}\Gamma(2z). \quad (2.4)$$

We next recall the definition and some basic properties of $p$-adic Gamma function. For more details, we refer to [3, Section 11.6]. Let $p$ be an odd prime and $\mathbb{Z}_p$ denote the set of all $p$-adic integers. For $x \in \mathbb{Z}_p$, the $p$-adic Gamma function is defined as

$$\Gamma_p(x) = \lim_{m \to x} (g)^m \prod_{0 < k < m \atop (k, p) = 1} k,$$

where the limit is for $m$ tending to $x$ $p$-adically in $\mathbb{Z}_{\geq 0}$.

We require several properties of $p$-adic Gamma function.

**Lemma 2.1** (See [3, Section 11.6].) For any odd prime $p$ and $x, y \in \mathbb{Z}_p$, we have

$$\Gamma_p(1) = -1, \quad (2.5)$$

$$\Gamma_p(x)\Gamma_p(1 - x) = (-1)^{s_p(x)}, \quad (2.6)$$

$$\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p} \text{ for } x \equiv y \pmod{p}, \quad (2.7)$$

where $s_p(x) \in \{1, 2, \cdots, p\}$ with $s_p(x) \equiv x \pmod{p}$.

**Lemma 2.2** (See [3, Lemma 17, (4)].) Let $p$ be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a + 1, \cdots, a + n - 1$ in $p\mathbb{Z}_p$, then

$$(a)_n = (-1)^n \frac{\Gamma_p(a + n)}{\Gamma_p(a)}. \quad (2.8)$$
3 Proof of Theorem 1.1

Let \( \omega \) be any primitive 3th root of unity. Letting \( a = \frac{1}{2}, b = \frac{1-\omega p}{2}, c = \frac{1-\omega^2 p}{2}, k = \frac{3}{2}, m = \frac{p-1}{2} \) in [2] (1), page 32, we obtain

\[
{4F_3} \left[ \frac{1}{2}, \frac{1-\omega p}{2}, \frac{1-\omega^2 p}{2}, \frac{1-p}{2}; 1 \right] = \frac{p\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-\omega p}{2}\right)_{\frac{p-1}{2}}}{(1+\omega p)_{\frac{p-1}{2}} \left(1+\omega^2 p\right)_{\frac{p-1}{2}}} {4F_3} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1 \right]. \tag{3.1}
\]

By the fact that

\[(u + vp)(u + vp\omega)(u + vp\omega^2) = u^3 + v^3 p^3,
\]

we have

\[(u + vp)_k(u + vp\omega)_k(u + vp\omega^2)_k \equiv (u)_k^3 \pmod{p^3}. \tag{3.2}
\]

It follows from (3.1) and (3.2) that

\[
{4F_3} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; 1 \right]_{\frac{p-1}{2}} \equiv \frac{p\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-\omega p}{2}\right)_{\frac{p-1}{2}}}{(1+\omega p)_{\frac{p-1}{2}} \left(1+\omega^2 p\right)_{\frac{p-1}{2}}} {4F_3} \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; 1 \right]_{\frac{p-1}{2}} \pmod{p^3}. \tag{3.3}
\]

In order to prove (1.2), by (1.1) and (3.3) it suffices to show that

\[
\frac{p\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-\omega p}{2}\right)_{\frac{p-1}{2}}}{(1+\omega p)_{\frac{p-1}{2}} \left(1+\omega^2 p\right)_{\frac{p-1}{2}}} \equiv 1 \pmod{p^3}. \tag{3.4}
\]

By (3.2), we have

\[
\left(1+\frac{p}{2}\right)_{\frac{p-1}{2}} \left(1+\frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1+\frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}} \equiv (1)_{\frac{p-1}{2}} \pmod{p^3},
\]

and so

\[
\frac{\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-\omega p}{2}\right)_{\frac{p-1}{2}}}{(1+\omega p)_{\frac{p-1}{2}} \left(1+\omega^2 p\right)_{\frac{p-1}{2}}} \equiv \frac{\left( \frac{1}{2} \right)_{\frac{p-1}{2}} \left( \frac{1-\omega p}{2} \right)_{\frac{p-1}{2}} \left( \frac{1-\omega^2 p}{2} \right)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{2}} \left( \frac{p-1}{2} \right)} \pmod{p^3}. \tag{3.5}
\]
Furthermore, we have
\[
\left( \frac{1 - p}{2} \right) \frac{p-1}{2} = (-1)^{\frac{p-1}{2}} \left( 1 \right)^{\frac{p-1}{2}},
\]
and
\[
\left( \frac{1}{2} \right) \frac{p-1}{2} = (-1)^{\frac{p-1}{2}} \left( 1 - \frac{p}{2} \right)^{\frac{p-1}{2}}.
\]
Thus,
\[
\frac{\left( \frac{1}{2} \right) \frac{p-1}{2} \left( \frac{1 - p}{2} \right) \frac{p-1}{2}}{\left( 1 + \frac{p}{2} \right)^{\frac{p-1}{2}} \left( 1 + \frac{\omega p}{2} \right)^{\frac{p-1}{2}} \left( 1 + \frac{\omega^2 p}{2} \right)^{\frac{p-1}{2}}} \equiv \frac{\left( \frac{1}{2} \right) \frac{p-1}{2} \left( 1 - \frac{p}{2} \right) \frac{p-1}{2}}{\left( 1 \right)^{\frac{p-1}{2}} \left( 1 \right)^{\frac{p-1}{2}}} \pmod{p^3}. \quad (3.5)
\]
We next evaluate the product on the right-hand side of (3.5) modulo \( p^4 \):
\[
\frac{\left( 1 + \frac{p}{2} \right) \frac{p-1}{2} \left( 1 - \frac{p}{2} \right) \frac{p-1}{2}}{\left( 1 \right)^{\frac{p-1}{2}}} = \prod_{j=1}^{\frac{p-1}{2}} \left( 1 - \frac{p^2}{4j^2} \right).
\]
From the following Taylor expansion:
\[
\prod_{j=1}^{\frac{p-1}{2}} \left( a_j + b_j x^2 \right) = \prod_{j=1}^{\frac{p-1}{2}} a_j \cdot \left( 1 + x^2 \sum_{j=1}^{\frac{p-1}{2}} b_j \frac{1}{a_j} \right) + \mathcal{O}(x^4),
\]
we deduce that
\[
\frac{\left( 1 + \frac{p}{2} \right) \frac{p-1}{2} \left( 1 - \frac{p}{2} \right) \frac{p-1}{2}}{\left( 1 \right)^{\frac{p-1}{2}}} \equiv 1 - \frac{p^2}{4} \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} \pmod{p^4}.
\]
By Wolstenholme’s theorem, we have
\[
\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} \equiv \frac{1}{2} \left( \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} + \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{(p-j)^2} \right) \equiv 0 \pmod{p}.
\]
It follows that
\[
\frac{\left( 1 + \frac{p}{2} \right) \frac{p-1}{2} \left( 1 - \frac{p}{2} \right) \frac{p-1}{2}}{\left( 1 \right)^{\frac{p-1}{2}}} \equiv 1 \pmod{p^3}. \quad (3.6)
\]
Combining (3.5) and (3.6), we complete the proof of (3.4).
4 A human proof of Theorem 1.3

Letting $a = \frac{1}{2}$, $x = 2n + \frac{3}{2}$ in [16] (14.1), we obtain

\[
\begin{align*}
_6F_5 \left[ \begin{array}{cccc}
\frac{5}{7}, & \frac{1}{2}, & -2n - 1, & 2n + 2, & \frac{1}{2} + y, & \frac{1}{2} - y \\
\frac{1}{4}, & 2n + \frac{5}{2}, & -2n - \frac{1}{2}, & 1 - y, & 1 + y; & -1
\end{array} \right]
&= \frac{\pi \Gamma \left( -2n - \frac{1}{2} \right) \Gamma \left( 2n + \frac{5}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right) \Gamma \left( n + \frac{9}{2} + \frac{3}{2} \right) \Gamma \left( n - \frac{9}{2} + \frac{3}{2} \right) \Gamma \left( -n + \frac{9}{2} \right) \Gamma \left( -n - \frac{9}{2} \right)}.
\end{align*}
\]

Note that

\[
\Gamma \left( -2n - \frac{1}{2} \right) \Gamma \left( 2n + \frac{5}{2} \right) = \frac{(4n + 3)\pi}{2 \sin \left( (2n + \frac{3}{2}) \pi \right)}
\]

and

\[
\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right) = \frac{\pi}{2}.
\]

Also,

\[
\Gamma(1 + y)\Gamma(1 - y) = \frac{1}{\pi} \Gamma \left( \frac{1 + y}{2} \right) \Gamma \left( \frac{1 - y}{2} \right) \Gamma \left( 1 + \frac{y}{2} \right) \Gamma \left( 1 - \frac{y}{2} \right).
\]

Substituting (4.2)–(4.4) into the right-hand side of (4.1) gives

\[
\begin{align*}
_6F_5 \left[ \begin{array}{cccc}
\frac{5}{7}, & \frac{1}{2}, & -2n - 1, & 2n + 2, & \frac{1}{2} + y, & \frac{1}{2} - y \\
\frac{1}{4}, & 2n + \frac{5}{2}, & -2n - \frac{1}{2}, & 1 - y, & 1 + y; & -1
\end{array} \right]
&= -(4n + 3) \frac{\Gamma \left( \frac{1+y}{2} \right) \Gamma \left( \frac{1-y}{2} \right) \Gamma \left( 1 + \frac{y}{2} \right) \Gamma \left( 1 - \frac{y}{2} \right)}{\Gamma \left( n + \frac{9}{2} + \frac{3}{2} \right) \Gamma \left( n - \frac{9}{2} + \frac{3}{2} \right) \Gamma \left( -n + \frac{9}{2} \right) \Gamma \left( -n - \frac{9}{2} \right)}
\end{align*}
\]

\[
\begin{align*}
&= -(4n + 3) \frac{\left( -n + \frac{9}{2} \right)_{n+1} \left( -n - \frac{9}{2} \right)_{n+1}}{\left( \frac{1+y}{2} \right)_{n+1} \left( \frac{1-y}{2} \right)_{n+1}}
\end{align*}
\]

\[
= -(4n + 3) \frac{\left( \frac{9}{2} \right)_{n+1} \left( -n + \frac{9}{2} \right)_{n+1}}{\left( -n + \frac{9}{2} - \frac{1}{2} \right)_{2n+2}}.
\]
Let \( i \) be any primitive 4th root of unity. Setting \( n = \frac{p^3}{4} \) and \( y = -\frac{ip}{2} \) in (4.3) yields

\[
\binom{\frac{5}{4}, \frac{1}{2}, \frac{1-p}{2}, \frac{1+p}{2}, \frac{1+ip}{2}, \frac{1-ip}{2}}{\frac{1}{4}, 1 - \frac{p}{2}, 1 + \frac{p}{2}, 1 - \frac{ip}{2}, 1 + \frac{ip}{2}; -1}
\]

By the fact that

\[
(u + vp)(u - vp)(u + vpi)(u - vpi) = u^4 - v^4 p^4,
\]

we have

\[
(u + vp)_k(u - vp)_k(u + vpi)_k(u - vpi)_k \equiv (u)_k^4 \pmod{p^4}.
\]

In order to prove (1.4), by (4.6) and (4.7) it suffices to show that

\[
-p \left( -\frac{ip}{4} \right)^{p+1} \left( \frac{3-(i+1)p}{4} \right)^{p+1} \equiv \frac{-p^3}{16} \Gamma_p \left( \frac{1}{4} \right)^4 \pmod{p^4}.
\]

Note that

\[
\left( -\frac{ip}{4} \right)^{p+1} \left( \frac{3-(i+1)p}{4} \right)^{p+1} = -\frac{p^3}{16} \prod_{j=1}^{p+3} \left( \frac{1}{16} - \frac{p^2}{16} - j^2 \right),
\]

and

\[
\left( \frac{1 - (i+1)p}{4} \right)^{p+1} = \prod_{j=1}^{p+1} \left( -\frac{p^2}{16} - \left( \frac{1}{2} + j \right)^2 \right).
\]

It follows that

\[
-p \left( -\frac{ip}{4} \right)^{p+1} \left( \frac{3-(i+1)p}{4} \right)^{p+1} \equiv \frac{-p^3}{16} \prod_{j=1}^{p+3} \left( \frac{1}{16} - \frac{p^2}{16} - j^2 \right) \prod_{j=1}^{p+1} \left( -\frac{p^2}{16} - \left( \frac{1}{2} + j \right)^2 \right)
\]

\[
\equiv \frac{-p^3}{16} \cdot \frac{(1)^{p+1}}{(\frac{1}{2})^{p+1}} \pmod{p^4}.
\]

\[
\equiv \frac{p^3}{16} \cdot \frac{\Gamma_p \left( \frac{p+1}{4} \right)^2 \Gamma_p \left( \frac{1}{2} \right)^2}{\Gamma_p \left( 1 \right)^2 \Gamma_p \left( \frac{p+3}{4} \right)^2}.
\]
Furthermore, by (2.5)–(2.7), we have
\[
-p \left( -\frac{i^p}{4} \right)^{p+1} \frac{\left( \frac{3-(i+1)p}{4} \right)^{p+1}}{\left( \frac{1-(i+1)p}{4} \right)^{\frac{p+1}{2}}} \equiv -\frac{p^3}{16} \cdot \frac{\Gamma_p \left( \frac{1}{4} \right)^2}{\Gamma_p \left( \frac{3}{4} \right)^2} \pmod{p^4} = -\frac{p^3}{16} \Gamma_p \left( \frac{1}{4} \right)^4.
\]
This completes the proof of (4.8).

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