A N-dimensional elastic\viscoelastic transmission problem with Kelvin–Voigt damping and non smooth coefficient at the interface

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Abstract
We investigate the stabilization of a multidimensional system of coupled wave equations with only one Kelvin–Voigt damping. Using a unique continuation result based on a Carleman estimate and a general criteria of Arendt–Batty, we prove the strong stability of the system in the absence of the compactness of the resolvent without any geometric condition. Then, using a spectral analysis, we prove the non uniform stability of the system. Further, using frequency domain approach combined with a multiplier technique, we establish some polynomial stability results by considering different geometric conditions on the coupling and damping domains. In addition, we establish two polynomial energy decay rates of the system on a square domain where the damping and the coupling are localized in a vertical strip.

Keywords Wave equation · Kelvin–Voigt damping · Semigroup · Non uniform stability · Polynomial stability

Mathematics Subject Classification 35B35 · 93B52 · 93C20

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary $\Gamma$. We consider the following two wave equations coupled through velocities with a viscoelastic damping:

$$
\begin{align*}
\begin{cases}
    u_{tt} - \text{div}(a \nabla u + b(x) \nabla u_t) + c(x)y_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\
y_{tt} - \Delta y - c(x)u_t = 0 & \text{in } \Omega \times \mathbb{R}^+, 
\end{cases}
\end{align*}
$$

(1.1)

with the following initial conditions:

$$
u(x, 0) = u_0(x), \quad y(x, 0) = y_0(x), \quad u_t(x, 0) = u_1(x), \quad y_t(x, 0) = y_1(x) \quad x \in \Omega,$n

(1.2)

and the following boundary conditions:

$$u(x, t) = y(x, t) = 0 \quad \text{on } \Gamma \times \mathbb{R}^+. $$

(1.3)

The functions $b, c \in L^\infty(\Omega)$ such that

$$b(x) \geq b_0 > 0 \quad \text{in } \omega_b \quad \text{and} \quad b(x) = 0 \quad \text{in } \Omega \setminus \omega_b$$

(1.4)

and

$$|c(x)| \geq c_0 > 0 \quad \text{in } \omega_c \quad \text{and} \quad c(x) = 0 \quad \text{in } \Omega \setminus \omega_c$$

(1.5)

where $\omega_b$ and $\omega_c$ are two non-empty open subsets in $\Omega$. The constant $a$ is a strictly positive constant.

The stabilization of the wave equation with localized damping has received a special attention since the seventies (see [6, 14, 15, 36]). The stabilization of a material composed of two parts: one that is elastic and the other one that is a Kelvin–Voigt type viscoelastic material was studied extensively. This type of material is encountered in real life when one uses patches to suppress vibrations, the modeling aspect of which may be found in [5]. This type of damping was examined in the one-dimensional setting in [24, 25, 28]. Later on, the wave equation with Kelvin–Voigt damping in the multidimensional setting was studied. Let us consider the wave equation with Kelvin–Voigt damping given in the following system

$$
\begin{align*}
\begin{cases}
    u_{tt} - \text{div}(a \nabla u + b(x) \nabla u_t) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\
u(x, t) = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\
u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, & \text{in } \Omega.
\end{cases}
\end{align*}
$$

(1.6)

In [19], the author proved that when the Kelvin–Voigt damping $\text{div}(b(x) \nabla u_t)$ is globally distributed, i.e. $b(x) \geq b_0 > 0$ for almost all $x \in \Omega$, the wave equation generates an analytic semi-group. In [27], the authors considered the wave equation with local visco-elastic damping distributed around the boundary of $\Omega$. They proved that the energy of the system decays exponentially to zero as $t$ goes to infinity for all usual initial data under the assumption that the damping coefficient satisfies: $b \in C^{1,1}(\Omega), \Delta b \in L^\infty(\Omega)$ and $|\nabla b(x)|^2 \leq M_0 b(x)$ for almost every $x$ in $\Omega$ where $M_0$ is a positive constant. On the other hand, in [36], the author
studied the stabilization of the wave equation with Kelvin–Voigt damping. He established a polynomial energy decay rate of type $t^{-1}$ provided that the damping region is localized in a neighborhood of a part of the boundary and verifies certain geometric condition. Also, in [30], under the same assumptions on $b$, the authors established the exponential stability of the wave equation with local Kelvin–Voigt damping localized around a part of the boundary and an extra boundary with time delay where they added an appropriate geometric condition. Later on, in [3], the wave equation with Kelvin–Voigt damping localized in a subdomain $\omega$ far away from the boundary without any geometric conditions was considered. The authors established a logarithmic energy decay rate for smooth initial data. In [13], the authors proved an exponential decay of the energy of a wave equation with two types of locally distributed mechanisms; a frictional damping and a Kelvin–Voigt damping where the location of each damping is such that none of them is able to exponentially stabilize the system. Under an appropriate geometric condition, piecewise multiplier geometric condition in short PMGC introduced by K. Liu in [23], on a subset $\omega$ of $\Omega$ where the dissipation is effective, they proved that the energy of the system decays polynomially of type $t^{-1}$ in the absence of regularity of the Kelvin–Voigt damping coefficient $b$. In [1], the authors considered a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain and they proved stability results under geometric control condition (GCC in short, see Definition 4.1). In [2], the author established a polynomial energy decay rate of type $t^{-1}$ for smooth initial data under some geometric conditions. Also, they proved a general polynomial energy decay estimate on a bounded domain where the geometric conditions on the localized viscoelastic damping are violated and they applied it on a square domain where the damping is localized in a vertical strip. Also, in [34], the authors analyzed the long time behavior of the wave equation with local Kelvin–Voigt damping where they showed the logarithmic decay rate for energy of the system without any geometric assumption on the subdomain on which the damping is effective. Further, in [10], the author showed how perturbative approaches and the black box strategy allow to obtain decay rates for Kelvin–Voigt damped wave equations from quite standard resolvent estimates (for example Carleman estimates or geometric control estimates). Recently, in [11], the authors studied the energy decay rate of the Kelvin–Voigt damped wave equation with piecewise smooth damping on the multi-dimensional domain. Under suitable geometric assumptions on the support of the damping, they obtained an optimal polynomial decay rate. In 2021, in [12], they studied the decay rates for Kelvin–Voigt damped wave equations under a geometric control condition. When the damping coefficient is sufficiently smooth they showed that exponential decay follows from geometric control conditions.

Over the past few years, the coupled systems received a vast attention due to their potential applications. The system of coupled wave equations with only one Kelvin–Voigt damping was considered in [31]. The authors considered the damping and the coupling coefficients to be constants and they established a polynomial energy decay rate of type $t^{-1/2}$ and an optimality result. In [26], exponential stability for the wave equations with local Kelvin–Voigt damping was considered where the local viscoelastic damping distributed around the boundary of the domain. They showed that the energy of the system goes uniformly and exponentially to zero for all initial data of finite energy. In [38], the author considered the wave equation with Kelvin–Voigt damping in a non empty bounded convex domain $\Omega$ with partition $\Omega = \Omega_1 \cap \Omega_2$ where the viscoelastic damping is localized in $\Omega_1$, the coupling is through a common interface. Under the condition that the damping coefficient $b$ is non smooth, she established a polynomial energy decay rate of type $t^{-1}$ for smooth initial data. Also, in [16], the authors studied the stability of coupled wave equations under Geometric Control Condition (GCC in short) where they considered one viscous damping. Finally, in
[17], the authors considered a system of weakly coupled wave equations with one or two locally internal Kelvin–Voigt damping and non-smooth coefficient at the interface. They established some polynomial energy decay estimates under some geometric condition. The stability of wave equations coupled through velocity and with non-smooth coupling and damping coefficients is not considered yet. Also, the study of the coupled wave equations under several geometric condition is not covered. In this work, we consider the coupled system represented in (1.1)–(1.3) by considering several geometric conditions (H1), (H2), (H3), (H4), and (H5) (see Sect. 4) where the coupling is made via velocities and with non smooth coupling and damping coefficients. In addition, this work is a generalization of the work in [37] where the system is described by

\[
\begin{align*}
  u_{tt} - (au_x + b(x)u_{tx})_x + c(x) y_t &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \\
  y_{tt} - y_{xx} - c(x) u_t &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, 
\end{align*}
\]

with fully Dirichlet boundary conditions and with the following initial data

\[
\begin{align*}
  u(0, t) &= u(L, t) = y(0, t) = y(L, t) = 0, \quad \forall \ t \in \mathbb{R}^+, \\
  u(\cdot, 0) &= u_0(\cdot), \ u_t(\cdot, 0) = u_1(\cdot), \ y(\cdot, 0) = y_0(\cdot) \quad \text{and} \quad y_t(\cdot, 0) = y_1(\cdot), 
\end{align*}
\]

where

\[
b(x) = \begin{cases} b_0 & \text{if } x \in (\alpha_1, \alpha_3) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0 & \text{if } x \in (\alpha_2, \alpha_4) \\ 0 & \text{otherwise} \end{cases}
\]

and \(a > 0, b_0 > 0, c_0 \in \mathbb{R}^*, \) and \(0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L.\) The authors considered that both the damping and the coupling coefficients are non smooth and showed that the energy of the smooth solutions of the system decays polynomially of type \(t^{-1}.\) We generalize this work to a multidimensional case and we study the stability of the system (1.1)–(1.3) under several geometric control conditions. We establish polynomial stability when there is an intersection between the damping and the coupling regions. Also, when the coupling region is a subset of the damping region and under Geometric Control Condition GCC. Moreover, in the absence of any geometric condition, we study the stability of the system on the 2-dimensional square domain.

The paper is organized as follows: first, in Sect. 2, we show that the system (1.1)–(1.3) is well-posed using semi-group approach. Then, using a unique continuation result based on a Carleman estimate and a general criteria of Arendt–Batty, we prove the strong stability of the system in the absence of the compactness of the resolvent and without any geometric condition. In Sect. 3, using a spectral analysis, we prove the non uniform stability of the system in the case where \(b(x) = b \in \mathbb{R}^*_+\) and \(c(x) = c \in \mathbb{R}^*.\) In Sect. 4, we establish some polynomial energy decay rates under several geometric conditions by using a frequency domain approach combined with a multiplier method. In addition, we establish two polynomial energy decay rates on a square domain where the damping and the coupling are localized in a vertical strip.

### 2 Well-posedness and strong stability

#### 2.1 Well posedness

In this part, using a semigroup approach, we establish the well-posedness result for the system (1.1)–(1.3).
Let \((u, u_t, y, y_t)\) be a regular solution of the system (1.1)–(1.3). The energy of the system is given by

\[
E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + |y_t|^2 + a|\nabla u|^2 + |\nabla y|^2) \, dx.
\] (2.1)

A straightforward computation gives

\[
E'(t) = - \int_{\Omega} b(x)|\nabla u_t|^2 \, dx \leq 0.
\]

Thus, the system (1.1)–(1.3) is dissipative in the sense that its energy is a non increasing function with respect to the time variable \(t\). We define the energy Hilbert space \(H\) by

\[
H = (H^1_0(\Omega) \times L^2(\Omega))^2
\]
equipped with the following inner product

\[
\langle U, \bar{U} \rangle = \int_{\Omega} \left( a \nabla u \cdot \nabla \bar{u} + \nabla y \cdot \nabla \bar{y} + v \bar{v} + z \bar{z} \right) \, dx,
\]

for all \(U = (u, v, y, z)^T \in H\) and \(\bar{U} = (\bar{u}, \bar{v}, \bar{y}, \bar{z})^T \in H\). Finally, we define the unbounded linear operator \(A\) by

\[
D(A) = \left\{ U = (u, v, y, z) \in H : v, z \in H^1_0(\Omega), \text{ div}(a(x)\nabla u + b(x)\nabla v) \in L^2(\Omega), y \in H^2(\Omega) \cap H^1_0(\Omega) \right\},
\]

and for all \(U = (u, v, y, z) \in D(A),\)

\[
A(u, v, y, z)^T = \begin{pmatrix} v \\ c(x)z \\ \Delta y + c(x)v \end{pmatrix}.
\]

If \(U = (u, u_t, y, y_t)^T\) is a regular solution of system (1.1)–(1.3), then we rewrite this system as the following evolution equation

\[
U_t = AU, \quad U(0) = U_0
\] (2.2)

where \(U_0 = (u_0, u_1, y_0, y_1)^T\).

**Proposition 2.1** The unbounded linear operator \(A\) is m-dissipative in the energy space \(H\).

**Proof** For all \(U = (u, v, y, z)^T \in D(A)\), we have

\[
\Re (AU, U)_H = - \int_{\Omega} b(x)|\nabla v|^2 \, dx \leq 0,
\] (2.3)

which implies that \(A\) is dissipative. Now, let \(F = (f_1, f_2, f_3, f_4)^T \in H\), we prove the existence of

\[
U = (u, v, y, z)^T \in D(A)
\]

unique solution of the equation

\[
- AU = F.
\] (2.4)
Equivalently, we have the following system

\[ \begin{align*}
- v &= f_1, \\
- \text{div}(a \nabla u + b(x) \nabla v) + c(x) z &= f_2, \\
- z &= f_3, \\
- \Delta y - c(x) v &= f_4.
\end{align*} \]

Inserting (2.5), (2.7) into (2.6) and (2.8), we get

\[ \begin{align*}
- \text{div}(a \nabla u - b(x) \nabla f_1) &= f_2 + c(x) f_3, \\
- \Delta y &= f_4 - c(x) f_1.
\end{align*} \]

Let \((\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)\). Multiplying (2.9) and (2.10) by \(\bar{\varphi}\) and \(\bar{\psi}\) respectively, and integrate over \(\Omega\), we obtain

\[ \Lambda((u, v), (\varphi, \psi)) = L(\varphi, \psi), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \]

where

\[ \Lambda((u, v), (\varphi, \psi)) = \int_\Omega (a \nabla u \cdot \nabla \varphi + \nabla y \cdot \nabla \psi) \, dx \]

and

\[ L(\varphi, \psi) = \int_\Omega (f_2 + c(x) f_3) \varphi \, dx + \int_\Omega b(x) \nabla f_1 \cdot \nabla \varphi \, dx + \int_\Omega (f_4 - c(x) f_1) \psi \, dx. \]

Thanks to (2.12), (2.13), we have that \(\Lambda\) is a sesquilinear, continuous and coercive form on \((H_0^1(\Omega) \times H_0^1(\Omega))^2\), and \(L\) is an antilinear continuous form on \(H_0^1(\Omega) \times H_0^1(\Omega)\). Then, using Lax-Milgram theorem, we deduce that there exists \((u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)\) unique solution of the variational problem (2.11). By using the classical elliptic regularity, we deduce that (2.9)–(2.10) admits a unique solution \((u, y) \in H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))\) such that \(\text{div}(a \nabla u - b(x) \nabla f_1) \in L^2(\Omega)\). By taking \(F = (0, 0, 0, 0)^\top\) in (2.4) it is easy to see that \(\ker(\mathcal{A}) = \{0\}\). Consequently, we get \(U = \left( u, -f_1, y, -f_3 \right)^\top \in D(\mathcal{A})\) is a unique solution of (2.4). Then, \(\mathcal{A}\) is an isomorphism and since \(\rho(\mathcal{A})\) is open set of \(\mathbb{C}\) (see Theorem 6.7 (Chapter III) in [21]), we easily get \(R(\lambda I - \mathcal{A}) = \mathcal{H}\) for a sufficiently small \(\lambda > 0\). This, together with the dissipativeness of \(\mathcal{A}\), imply that \(D(\mathcal{A})\) is dense in \(\mathcal{H}\) and that \(\mathcal{A}\) is m-dissipative in \(\mathcal{H}\) (see Theorem 4.5, 4.6 in [32]). The proof is thus complete.

According to Lumer–Phillips Theorem (see [32]), Proposition 2.1 implies that the operator \(\mathcal{A}\) generates a \(C_0\)-semigroup of contractions \(e^{t \mathcal{A}}\) in \(\mathcal{H}\) which gives the well-posedness of (2.2). Then, we have the following result

**Theorem 2.2** For any \(U_0 \in \mathcal{H}\), Problem (2.2) admits a unique weak solution satisfying

\[ U(t) \in C^0(\mathbb{R}^+; \mathcal{H}). \]

Moreover, if \(U_0 \in D(\mathcal{A})\), then Problem (2.2) admits a unique strong solution \(U\) satisfying

\[ U(t) \in C^1(\mathbb{R}^+; \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})). \]
2.2 Strong stability

This subsection is devoted to study the strong stability of System (1.1)–(1.3) in the sense that its energy converges to zero when \( t \) goes to infinity for all initial data in \( \mathcal{H} \). The proof will be done using the unique continuation theorem based on a Carleman estimate and a general criteria of Arendt–Batty [4]. In this part, we prove that the energy of the System (1.1)–(1.3) decays to zero as \( t \) tends to infinity if one of the following assumptions hold:

(A1) Assume that \( \omega_b \) and \( \omega_c \) are non-empty open subsets of \( \Omega \) such that \( \omega_c \subset \omega_b \) and \( \text{meas}(\omega_b \cap \Gamma) > 0 \) (see Figs. 1, 2, 3).

(A2) Assume that \( \omega_b \) and \( \omega_c \) are non-empty open subsets of \( \Omega \) such that \( \omega = \omega_b \cap \omega_c \neq \emptyset \). Also, assume that \( \omega \) satisfies \( \text{meas}(\omega \cap \Gamma) > 0 \) (see Figs. 4, 5, 6).

(A3) Assume that \( \omega_b \) and \( \omega_c \) are non-empty open subset of \( \Omega \) such that \( \omega = \omega_b \cap \omega_c \neq \emptyset \), \( \text{meas}(\omega_b \cap \Gamma) > 0 \) and \( \omega_c \) not near the boundary (see Fig. 7).

(A4) Assume that \( \omega_b \) is non-empty open subsets of \( \Omega \) and \( c(x) = c_0 \in \mathbb{R}^+ \) in \( \Omega \). Also, assume that \( \omega_b \) is not near the boundary (see Fig. 8).

(A5) Assume that \( \omega_b \) and \( \omega_c \) are non-empty open subset of \( \Omega \) such that \( \omega = \omega_b \cap \omega_c \neq \emptyset \), \( \text{meas}(\omega_b \cap \Gamma) = 0 \) and \( \text{meas}(\omega_c \cap \Gamma) = 0 \) (see Fig. 9).

We note that some of these figures were also mentioned because they are examples of the geometric conditions we consider in Sect. 4, where we study the polynomial stability of the system.

Before stating the main theorem of this section, we will give the proof of a local unique continuation result for a coupled system of wave equations.

We define the following elliptic operator \( P \) defined on a product space by

\[
P : H^2(V) \times H^2(V) \to L^2(V) \times L^2(V)
\]

\[
(u, y) \to (\Delta u, \Delta y)
\]

(2.14)

and the following function \( g \) defined by

\[
g : L^2(V) \times L^2(V) \to L^2(V) \times L^2(V)
\]

\[
(u, y) \to \left( \frac{1}{\lambda}(-\lambda^2 u + c(x)i\lambda y), -\lambda^2 y - c(x)i\lambda u \right)
\]

(2.15)
Lemma 2.3 (See [18] also [17]) Let $V$ be a bounded open set in $\mathbb{R}^N$ and let $\varphi = e^{\rho \psi}$ with $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$; $|\nabla_x \psi| > 0$ and $\rho > 0$ large enough. Then, there exist $\tau_0$ large enough and $C > 0$ such that

$$
\tau^3 \|e^{\tau \varphi} u\|^2_{L^2(V)} + \tau \|e^{\tau \varphi} \nabla_x u\|^2_{L^2(V)} \leq C \|e^{\tau \varphi} \Delta u\|^2_{L^2(V)}, \quad \text{for all } u \in H^2_0(V) \text{ and } \tau > \tau_0.
$$

(2.16)

Proposition 2.4 Let $\Omega_1$ be a bounded open set in $\mathbb{R}^N$ and $x_0$ be a point in $\Omega$. In a neighborhood $V$ of $x_0 \in \Omega$, we take a function $f$ such that $\nabla f \neq 0$ in $\overline{V}$. Moreover, let $(u, y) \in H^2(V) \times H^2(V)$ be a solution of $P(u, y) = g(u, y)$. If $u = y = 0$ in $\{x \in V; f(x) \geq f(x_0)\}$ then $u = y = 0$ in a neighborhood of $x_0$.

Proof We call $W$ the region $\{x \in V; f(x) \geq f(x_0)\}$. We choose $V'$ and $V''$ neighborhoods of $x_0$ such that $V'' \subseteq V' \subseteq V$, and we choose a function $\chi \in C^\infty_c(V')$ such that $\chi = 1$ in $V''$. Set $\tilde{u} = \chi u$ and $\tilde{y} = \chi y$. Then, $(\tilde{u}, \tilde{y}) \in H^2_0(V) \times H^2_0(V)$. Let $\psi = f(x) - c|x - x_0|^2$. 

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and set $\varphi = e^{\rho \psi}$. Then, apply the Carleman estimate of Lemma 2.3 to $\tilde{u}$ and $\tilde{y}$ respectively, then sum the two inequalities we obtain

$$\tau^3 \int_{V'} e^{2\tau \varphi} \left( |\tilde{u}|^2 + |\tilde{y}|^2 \right) dx + \tau \int_{V''} e^{2\tau \varphi} \left( |\nabla \tilde{u}|^2 + |\nabla \tilde{y}|^2 \right) dx \leq C \int_{V'} e^{2\tau \varphi} \left( |\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2 \right) dx$$

(2.17)

As $V'' \subseteq V'$ and $\chi \in C^\infty_c (V')$ such that $\chi = 1$ in $V''$, we get

$$\tau^3 \int_{V''} e^{2\tau \varphi} \left( |u|^2 + |y|^2 \right) dx + \tau \int_{V'''} e^{2\tau \varphi} \left( |\nabla u|^2 + |\nabla y|^2 \right) dx \leq C \int_{V''} e^{2\tau \varphi} \left( |\Delta u|^2 + |\Delta y|^2 \right) dx$$

$$+ C \int_{V' \setminus V''} e^{2\tau \varphi} \left( |\Delta u|^2 + |\Delta y|^2 \right) dx.$$  

(2.18)
Fig. 6 Model satisfying (A2) and (H1)

Fig. 7 Model satisfying (A3)

Fig. 8 Model satisfying (A4)
This implies that,
\[
\tau_3 \int_{\Omega'} e^{2\tau \varphi} (|u|^2 + |y|^2) \, dx \leq C \int_{\Omega'} e^{2\tau \varphi} (|\Delta u|^2 + |\Delta y|^2) \, dx + C \int_{\Omega' \setminus \Omega''} e^{2\tau \varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) \, dx.
\] (2.19)

We have that, \( a \Delta u = -\lambda^2 u + c(x)\bar{\lambda} y \) and \( \Delta y = -\lambda^2 u - c(x)\bar{\lambda} u \). Then, there exists \( C_{\lambda, c, a} > 0 \) such that
\[
(\tau_3 - C_{\lambda, c, a}) \int_{\Omega''} e^{2\tau \varphi} (|u|^2 + |y|^2) \, dx \leq C \int_{\Omega' \setminus \Omega''} e^{2\tau \varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) \, dx.
\] (2.20)

Then, there exists \( \tau > 0 \) large enough and \( C > 0 \) such that
\[
\tau^3 \int_{\Omega''} e^{2\tau \varphi} (|u|^2 + |y|^2) \, dx \leq C \int_{\Omega' \setminus \Omega''} e^{2\tau \varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) \, dx.
\] (2.21)

By using that \( u = y = 0 \) in \( W \), we obtain
\[
\tau^3 \int_{\Omega''} e^{2\tau \varphi} (|u|^2 + |y|^2) \, dx \leq C \int_{S} e^{2\tau \varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) \, dx.
\] (2.22)

where \( S = V' \setminus (V'' \cup W) \).

For all \( \varepsilon \in \mathbb{R} \), we set \( V_{\varepsilon} = \{ x \in V; \varphi(x) \leq \varphi(x_0) - \varepsilon \} \) and \( V'_{\varepsilon} = \{ x \in V; \varphi(x) \geq \varphi(x_0) - \frac{\varepsilon}{2} \} \). There exists \( \varepsilon \) such that \( S \subset V_{\varepsilon} \). Then, choose a ball \( B_0 \) with center \( x_0 \) such that \( B_0 \subset V'' \setminus V'_{\varepsilon} \). Hence, using (2.22), we have
\[
\int_{B_0} (|u|^2 + |y|^2) \, dx \leq \frac{Ce^{-\tau \varepsilon}}{\tau^3} \int_{S} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) \, dx.
\] (2.23)

Letting \( \tau \) tends to infinity, we obtain \( u = y = 0 \) in \( B_0 \). Hence, we reached our desired result.

\[ \square \]

**Theorem 2.5** (Calderón Theorem) Let \( \Omega \) be a connected open set in \( \mathbb{R}^N \) and let \( \omega \subset \Omega \), with \( \omega \neq \emptyset \). If \((u, y) \in H^2(\Omega) \times H^2(\Omega)\) satisfies \( P(u, y) = g(u, y) \) in \( \Omega \) and \( u = y = 0 \) in \( \omega \), then \( u \) and \( y \) vanishes in \( \Omega \).

**Proof** By setting \( F = \text{supp} u \cup \text{supp} y \) and using Proposition 2.4 instead of Proposition 4.1 in the proof of the Theorem 4.2 in \cite{22} the result holds.

\[ \square \]
Theorem 2.6 Assume that either (A1), (A2) (A3), (A4) or (A5) holds. Then, the $C_0$-semigroup $e^{tA}$ is strongly stable in $\mathcal{H}$ in the sense that for all $U_0 \in \mathcal{H}$, the solution $U(t) = e^{tA}U_0$ of (2.2) satisfies

$$\lim_{t \to \infty} \|e^{tA}U_0\|_{\mathcal{H}} = 0.$$  

For the proof of Theorem 2.6, the resolvent of $A$ is not compact. Then, in order to prove this Theorem we will use a general criteria Arendt–Batty. We need to prove that the operator $A$ has no pure imaginary eigenvalues and $\sigma (A) \cap i\mathbb{R}$ contains only a countable number of continuous spectrum of $A$. The argument for Theorem 2.6 relies on the subsequent lemmas.

Lemma 2.7 Assume that (A1) holds. Then, we have

$$\ker (i\lambda I - A) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$  

Proof From Proposition 2.1, $0 \in \rho(A)$. We still need to show the result for $\lambda \in \mathbb{R}^\star$. Suppose that there exists a real number $\lambda \neq 0$ and $U = (u, v, y, z)^\top \in D(A)$ such that

$$AU = i\lambda U.$$  

From (2.3) and (2.24), we have

$$0 = \Re (i\lambda \|U\|_{\mathcal{H}}^2) = \Re ((AU, U)_{\mathcal{H}}) = - \int_{\Omega} b(x)|\nabla v|^2 dx.$$  

Using the condition (1.4) and Poincaré’s inequality implies that

$$b(x)v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ in } \partial \Omega.$$  

Detailing (2.24) and using (2.26), we get the following system

$$v = i\lambda u \quad \text{in } \Omega,$$  

$$a\Delta u - c(x)z = i\lambda v \quad \text{in } \Omega,$$  

$$z = i\lambda y \quad \text{in } \Omega,$$  

$$\Delta y + c(x)v = i\lambda z \quad \text{in } \Omega.$$  

From (2.26), (2.27) and (2.28) with the assumption (A1), we get

$$z = 0 \text{ on } \partial \Omega.$$  

Inserting (2.27) into (2.28), and using (2.26) we get

$$\begin{cases} \lambda^2 u + a\Delta u = 0 \text{ in } \Omega, \\ u = 0 \text{ in } \partial \Omega. \end{cases}$$  

Using the unique continuation theorem, we get

$$u = 0 \text{ on } \Omega.$$  

Now, substituting (2.29) in (2.30), using (2.31) and the definition of $c(x)$, we get

$$\begin{cases} \lambda^2 y + \Delta y = 0 \text{ in } \Omega, \\ y = 0 \text{ in } \partial \Omega. \end{cases}$$  

Using the unique continuation theorem, we get

$$y = 0 \text{ on } \Omega.$$  

Therefore, $U = 0$. thus the proof is complete. □
Lemma 2.8 Assume that either (A2), (A3), (A4) or (A5) holds. Then, we have
\[ \text{ker } (i\lambda I - A) = \{0\}, \quad \forall \lambda \in \mathbb{R}. \]

Proof From Proposition 2.1, 0 \in \rho(A). We still need to show the result for \lambda \in \mathbb{R}^+. Suppose that there exists a real number \lambda \neq 0 and \(U = (u, v, y, z)\) \(\top \in D(A)\) such that
\[ AU = i\lambda U. \] (2.36)
From (2.3) and (2.36), we have
\[ 0 = \Re(i\lambda\|U\|_{H^2}^2) = \Re((AU, U)_{H^2}) = -\int_{\Omega} b(x)|\nabla v|^2dx. \] (2.37)
Condition (1.4) implies that
\[ \nabla v = 0 \quad \text{in} \quad \omega_b. \] (2.38)
Detailing (2.36) and using (2.38), we get the following system
\[
\begin{align*}
v &= i\lambda u \quad \text{in} \quad \Omega, \\
a\Delta u - c(x)z &= i\lambda v \quad \text{in} \quad \Omega, \\
z &= i\lambda y \quad \text{in} \quad \Omega, \\
\Delta y + c(x)v &= i\lambda z \quad \text{in} \quad \Omega.
\end{align*}
\] (2.39 - 2.42)
Now, we will distinguish between the following three cases:

Case 1. If (A2) holds. Then, by using Poincaré’s inequality we get
\[ v = 0 \quad \text{on} \quad \omega. \] (2.43)
From (2.40), (2.43) and using (2.41) we get
\[ y = 0 \quad \text{on} \quad \omega. \] (2.44)
Inserting (2.39) and (2.41) into (2.40) and (2.42) respectively, we get
\[
\begin{align*}
\begin{cases}
\lambda^2 u + a\Delta u - i\lambda c(x)y = 0 \quad \text{in} \quad \Omega, \\
\lambda^2 y + \Delta y + i\lambda c(x)u = 0 \quad \text{in} \quad \Omega, \\
u = y = 0 \quad \text{in} \quad \omega.
\end{cases}
\end{align*}
\] (2.45)
Then, using Theorem 2.5 we get that \(u = y = 0\) in \(\Omega\). Thus, we deduce that \(U = 0\) in \(\Omega\) and we reached our desired result.

Case 2. If (A3) holds. Then, by using Poincaré’s inequality we get
\[ v = 0 \quad \text{on} \quad \omega_b. \] (2.46)
Proceeding in the same way as in Case 1., we get
\[
\begin{align*}
\begin{cases}
\lambda^2 u + a\Delta u - i\lambda c(x)y = 0 \quad \text{in} \quad \Omega, \\
\lambda^2 y + \Delta y + i\lambda c(x)u = 0 \quad \text{in} \quad \Omega, \\
u = y = 0 \quad \text{in} \quad \omega.
\end{cases}
\end{align*}
\] (2.47)
Then, using Theorem 2.5 we get that \(u = y = 0\) in \(\Omega\). Thus, we deduce that \(U = 0\) in \(\Omega\) and we reached our desired result.

Case 3. Assume that (A4) holds. By differentiating (2.40) and using the fact that \(c(x) = c_0\) in \(\Omega\), we obtain
\[ \partial_j y = 0 \quad \text{in} \quad \omega_b, \quad \forall j = 1 \ldots, N. \]
Then, for all \( j = 1, \ldots, N \), we have the following system

\[
\begin{align*}
\lambda^2\partial_j u + a\Delta \partial_j u - i\lambda c_0 \partial_j y &= 0 \quad \text{in } \Omega, \\
\lambda^2\partial_j y + \Delta \partial_j y + i\lambda c_0 \partial_j u &= 0 \quad \text{in } \Omega, \\
\partial_j u &= \partial_j y = 0 \quad \text{in } \omega_b.
\end{align*}
\] (2.48)

By applying Theorem 2.5, we obtain

\[\partial_j u = \partial_j y = 0 \quad \text{in } \Omega, \quad \forall \ j = 1 \ldots, N.\]

Using the fact that \( u = y = 0 \) on \( \Gamma \), we get \( u = y = 0 \) in \( \Omega \). Consequently, \( U = 0 \) in \( \Omega \).

**Case 4.** Assume that (A5) holds.

From (2.38), we have

\[u = k \quad \text{in } \omega_b.\] (2.49)

Inserting (2.39) and (2.41) into (2.40) and (2.42) respectively, we get

\[
\begin{align*}
\lambda^2 u + a\Delta u - i\lambda c(x)y &= 0 \quad \text{in } \Omega, \\
\lambda^2 y + \Delta y + i\lambda c(x)u &= 0 \quad \text{in } \Omega,
\end{align*}
\] (2.50)

From the first equation (2.50), we have

\[\lambda^2 u + a\Delta u = 0 \quad \text{in } \omega_{b\setminus c} = \omega_b - \omega_c.\]

From the above equation and (2.49), we get \( u = 0 \) in \( \omega_{b\setminus c} \). Now, since \( u \in H^2(\Omega) \) then, \( u = 0 \) on \( \partial\omega_{b\setminus c} \).

Using the fact that \( u = k \) in \( \omega \) thus we get \( u = 0 \) in \( \omega_b \). Now, from the first equation of (2.50), we get \( y = 0 \) in \( \omega \).

Inserting (2.39) and (2.41) into (2.40) and (2.42) respectively, we get

\[
\begin{align*}
\lambda^2 u + a\Delta u - i\lambda c(x)y &= 0 \quad \text{in } \Omega, \\
\lambda^2 y + \Delta y + i\lambda c(x)u &= 0 \quad \text{in } \Omega, \\
u &= y = 0 \quad \text{in } \omega.
\end{align*}
\] (2.51)

Then, using Theorem 2.5 we get that \( u = y = 0 \) in \( \Omega \). Thus, we deduce that \( U = 0 \) in \( \Omega \) and we reached our desired result.

**Lemma 2.9** Assume that either (A1), (A2), (A3), (A4) or (A5) holds. Then, we have

\[R(i\lambda I - \mathcal{A}) = \mathcal{H}, \quad \text{for all } \lambda \in \mathbb{R}.\]

**Proof** From Proposition 2.1, we have \( 0 \in \rho(\mathcal{A}) \). We still need to show the result for \( \lambda \in \mathbb{R}^* \). Set \( F = (f_1, f_2, f_3, f_4) \in \mathcal{H} \), we look for \( U = (u, v, y, z) \in D(\mathcal{A}) \) solution of

\[ (i\lambda I - \mathcal{A})U = F. \] (2.52)

Equivalently, we have

\[
\begin{align*}
v &= i\lambda u - f_1, \\
n\lambda v - \text{div}(a \nabla u + b(x) \nabla v) + c(x)z &= f_2, \\
z &= i\lambda y - f_3, \\
i\lambda z - \Delta y - c(x)v &= f_4.
\end{align*}
\] (2.53)-(2.56)
Let \((\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)\), multiplying Eqs. (2.54) and (2.56) by \(\tilde{\varphi}\) and \(\tilde{\psi}\) respectively and integrating over \(\Omega\), we obtain

\[
\int_{\Omega} i \lambda v \tilde{\varphi} dx + \int_{\Omega} a \nabla u \nabla \tilde{\varphi} dx + \int_{\Omega} b(x) \nabla v \nabla \tilde{\varphi} dx + \int_{\Omega} c(x) \tilde{\varphi} dx = \int_{\Omega} f_2 \tilde{\varphi} dx, \tag{2.57}
\]

\[
\int_{\Omega} i \lambda \tilde{\psi} dx + \int_{\Omega} \nabla v \nabla \tilde{\psi} dx - \int_{\Omega} c(x) v \tilde{\psi} dx = \int_{\Omega} f_4 \tilde{\psi} dx. \tag{2.58}
\]

Substituting \(v\) and \(z\) in (2.53) and (2.55) into (2.57) and (2.58) and taking the sum, we obtain

\[
\vartheta ((u, y), (\varphi, \psi)) = L(\varphi, \psi), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \tag{2.59}
\]

where

\[
\vartheta ((u, y), (\varphi, \psi)) = \vartheta_1 ((u, y), (\varphi, \psi)) + \vartheta_2 ((u, y), (\varphi, \psi))
\]

with

\[
\begin{align*}
\vartheta_1 ((u, y), (\varphi, \psi)) &= \int_{\Omega} \left( a \nabla u \nabla \varphi + \nabla y \nabla \psi \right) dx + i \lambda \int_{\Omega} b(x) \nabla u \nabla \varphi dx, \\
\vartheta_2 ((u, y), (\varphi, \psi)) &= -\lambda^2 \int_{\Omega} \left( u \varphi + y \psi \right) dx + i \lambda \int_{\Omega} c(x) \left( y \varphi - u \psi \right) dx,
\end{align*}
\]

and

\[
L(\varphi, \psi) = \int_{\Omega} \left( f_2 + c(x) f_3 + i \lambda f_1 \right) \varphi dx + \int_{\Omega} \left( f_4 - c(x) f_1 + i \lambda f_3 \right) \psi dx + \int_{\Omega} b(x) \nabla f_1 \varphi dx.
\]

Let \(V = H_0^1(\Omega) \times H_0^1(\Omega)\) and \(V' = H^{-1}(\Omega) \times H^{-1}(\Omega)\) the dual space of \(V\). Let us consider the following operators,

\[
\begin{align*}
A : V & \to V' & A_1 : V & \to V' & A_2 : V & \to V' \\
(u, y) & \to A(u, y) & (u, y) & \to A_1(u, y) & (u, y) & \to A_2(u, y)
\end{align*}
\]

such that

\[
\begin{align*}
(A(u, y)) (\varphi, \psi) &= \vartheta ((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \\
(A_1(u, y)) (\varphi, \psi) &= \vartheta_1 ((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \\
(A_2(u, y)) (\varphi, \psi) &= \vartheta_2 ((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{2.60}
\end{align*}
\]

Our goal is to prove that \(A\) is an isomorphism operator. For this aim, we divide the proof into three steps.

**Step 1.** In this step, we prove that the operator \(A_1\) is an isomorphism operator. For this aim, following the second equation of (2.60) we can easily verify that \(\vartheta_1\) is a bilinear continuous coercive form on \(H_0^1(\Omega) \times H_0^1(\Omega)\). Then, by Lax-Milgram Lemma, the operator \(A_1\) is an isomorphism.

**Step 2.** In this step, we prove that the operator \(A_2\) is compact. According to the third equation of (2.60), we have

\[
|\vartheta_2 ((u, y), (\varphi, \psi))| \leq C \|(u, y)\|_{L^2(\Omega)} \|(\varphi, \psi)\|_{L^2(\Omega)}.
\]

Finally, using the compactness embedding from \(H_0^1(\Omega)\) to \(L^2(\Omega)\) and the continuous embedding from \(L^2(\Omega)\) into \(H^{-1}(\Omega)\) we deduce that \(A_2\) is compact.

From steps 1 and 2, we get that the operator \(A = A_1 + A_2\) is a Fredholm operator of index zero. Consequently, by Fredholm alternative, to prove that operator \(A\) is an isomorphism it is enough to prove that \(A\) is injective, i.e. \(\ker \{A\} = \{0\}\).
Step 3. In this step, we prove that $\ker\{A\} = \{0\}$. For this aim, let $(\bar{u}, \bar{y}) \in \ker\{A\}$, i.e.

$$\vartheta((\bar{u}, \bar{y}), (\varphi, \psi)) = 0, \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega).$$

Equivalently, we have

$$-\lambda^2 \int_\Omega (\bar{u}\varphi + \bar{y}\psi) \, dx + i\lambda \int_\Omega c(x) (\bar{y}\varphi - \bar{u}\psi) \, dx + \int_\Omega (a\nabla\bar{u}\nabla\varphi + \nabla\bar{y}\nabla\psi) \, dx$$

$$+ i\lambda \int_\Omega b(x) \nabla\bar{u}\nabla\varphi \, dx = 0. \quad (2.61)$$

Taking $\varphi = \bar{u}$ and $\psi = \bar{y}$ in Eq. (2.61), we get

$$-\lambda^2 \int_\Omega |\bar{u}|^2 \, dx - \lambda^2 \int_\Omega |\bar{y}|^2 \, dx + a \int_\Omega |\nabla\bar{u}|^2 \, dx + \int_0^L |\nabla\bar{y}|^2 \, dx - 2\lambda \Im\left(\int_\Omega c(x) \bar{y}\bar{u} \, dx\right)$$

$$+ i\lambda \int_\Omega b(x) |\nabla\bar{u}|^2 \, dx = 0.$$

Taking the imaginary part of the above equality, we get

$$\int_\Omega b(x) |\nabla\bar{u}|^2 \, dx = 0,$$

we get,

$$\nabla\bar{u} = 0, \quad \text{in } \omega_b. \quad (2.62)$$

Then, we find that

$$\begin{cases}
-\lambda^2 \bar{u} - a \Delta\bar{u} + i\lambda c(x) \bar{y} = 0, & \text{in } \Omega \\
-\lambda^2 \bar{y} - \Delta\varphi - i\lambda c(x) \bar{u} = 0, & \text{in } \Omega \\
\bar{u} = \bar{y} = 0. & \text{in } \omega
\end{cases}$$

Now, it is easy to see that the vector $\bar{U}$ defined by $\bar{U} = (\bar{u}, i\lambda\bar{u}, \bar{y}, i\lambda\bar{y})$ belongs to $D(A)$ and we have $i\lambda\bar{U} - A\bar{U} = 0$. Therefore, $\bar{U} \in \ker(i\lambda I - A)$, then by Lemmas 2.7 and 2.8, we get $\bar{U} = 0$, this implies that $\bar{u} = \bar{y} = 0$. Consequently, $\ker\{A\} = \{0\}$. Thus, from step 3 and Fredholm alternative, we get that the operator $A$ is an isomorphism. It is easy to see that the operator $L$ is continuous from $V$ to $L^2(\Omega) \times L^2(\Omega)$. Consequently, Eq. (2.59) admits a unique solution $(u, \bar{y}) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Thus, using $v = i\lambda u - f_1$, $z = i\lambda y - f_3$ and using the classical regularity arguments, we conclude that Eq. (2.52) admits a unique solution $U \in D(A)$. The proof is thus complete. \hfill \Box

Proof of Theorem 2.6 Using Lemmas 2.7 and 2.8, we have that $A$ has non pure imaginary eigenvalues. According to Lemmas 2.7, 2.8, 2.9 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Thus, we get the conclusion by applying Arendt–Batty Theorem. The proof of the theorem is thus complete. \hfill \Box

3 Non uniform stability

In this section, our aim is to prove the non-uniform stability of the system (1.1)–(1.3).

For this aim, assume that

$$b(x) = b \in \mathbb{R}^n_+ \quad \text{and} \quad c(x) = c \in \mathbb{R}^*.$$  \hfill (3.1)

Our main result in this section is the following theorem.
Theorem 3.1 Under condition (3.1). Then, the energy of the system (1.1)–(1.3) does not decay uniformly in the energy space $H$.

For the proof of Theorem 3.1, we aim to study the asymptotic behavior of the eigenvalues of the operator $A$ near the imaginary axis. First, we will determine the characteristic equation satisfied by the eigenvalues of $A$. So, let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ and let $U = (u, v, y, z)^\top \in D(A)$ be an associated eigenvector, i.e,

$$AU = \lambda U,$$

Equivalently,

$$v = \lambda u,$$  \hspace{1cm} (3.2)

$$\text{div}(a \nabla u + b \nabla v) - cz = \lambda v,$$  \hspace{1cm} (3.3)

$$z = \lambda y,$$  \hspace{1cm} (3.4)

$$\text{div}(\nabla y) + cv = \lambda z.$$  \hspace{1cm} (3.5)

Inserting (3.2) and (3.4) into (3.3) and (3.5) respectively, we get

$$\lambda^2 u - (a + \lambda b) \Delta u + c\lambda y = 0,$$  \hspace{1cm} (3.6)

$$\lambda^2 y - \Delta y - c\lambda u = 0.$$  \hspace{1cm} (3.7)

From (3.7), we have

$$u = -\frac{1}{\lambda c} \left[ \lambda^2 y - \Delta y \right].$$  \hspace{1cm} (3.8)

Substitute (3.8) in (3.6), we get

$$\begin{cases}
(a + \lambda b) \Delta^2 y - [(1 + a)\lambda^2 + b\lambda^3] \Delta y + \lambda^2 (\lambda^2 + c^2) y = 0, \text{ in } \Omega \\
y = \Delta y = 0 \text{ on } \Gamma.
\end{cases}$$  \hspace{1cm} (3.9)

Now, let $(\mu_k, \varphi_k)$ be, respectively, the sequence of the eigenvalues and the eigenvectors of the Laplace operator with fully Dirichlet boundary conditions in $\Omega$, i.e,

$$\begin{cases}
-\Delta \varphi_k = \mu_k^2 \varphi_k \text{ in } \Omega, \\
\varphi_k = 0 \text{ on } \Gamma.
\end{cases}$$  \hspace{1cm} (3.10)

Then by taking $y = \varphi_k$ in (3.9), we deduce the following characteristic equation

$$P(\lambda) = \lambda^4 + b\mu_k^2 \lambda^3 + [(1 + a)\mu_k^2 + c^2] \lambda^2 + b\mu_k^4 \lambda + a\mu_k^4 = 0.$$  \hspace{1cm} (3.11)

We note that we use conventional asymptotic notation, including ‘little o’ (limit tends to zero) and ‘big O’ (bounded), and we occasionally write $p \lesssim q$ to indicate that $p \leq Cq$ for some (implicit) constant $C > 0$.

Proposition 3.2 There exists $k_0 \in \mathbb{N}^*$ sufficiently large and two sequences $(\lambda_{1,k})_{|k| \geq k_0}$ and $(\lambda_{2,k})_{|k| \geq k_0}$ of simple roots of $P$ satisfying the following asymptotic behavior

$$\lambda_{1,k} = i\mu_k - \frac{c^2}{2b\mu_k^2} + o \left( \frac{1}{\mu_k^3} \right)$$  \hspace{1cm} (3.12)

and

$$\lambda_{2,k} = -i\mu_k - \frac{c^2}{2b\mu_k^2} + o \left( \frac{1}{\mu_k^3} \right).$$  \hspace{1cm} (3.13)
Proof Set $\xi = \frac{2}{\mu_k}$ and $\zeta_k = \frac{1}{\mu_k}$ in (3.11), we obtain
\[ h(\xi) = b\xi^3 + b\xi + a\xi^3 + c^2\xi^3 + (1 + a)\xi^2 \xi_k + \xi^4 \xi_k = 0. \quad (3.14) \]
Now, in order to find the eigenvalues of the operator $A$ we need to give the roots of $h$. For this aim, we will proceed in the following two steps.

**Step 1.** Let
\[ f(\xi) = b(\xi^3 + \xi) \quad \text{and} \quad f_1(\xi) = a\xi^3 + c^2\xi^3 + (1 + a)\xi^2 \xi_k + \xi^4 \xi_k. \]
We look for $r_k$ sufficiently small such that
\[ |f| > |h - f| = |f_1| \quad \text{on} \quad \partial D, \]
where $D = \{ \xi \in \mathbb{C}; |\xi - i| \leq r_k \}$.
Let $\xi \in \partial D(i, r_k)$, then $\xi = i + r_k e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. We have
\[ f(\xi) = b(\xi^3 + \xi) = b\xi r_k \left( 2ie^{i\theta} + r_k e^{2i\theta} \right). \]
But, if $r_k \leq \frac{1}{2}$ then
\[ |\xi| \geq |1 - r_k| \geq \frac{1}{2} \]
and
\[ |2ie^{i\theta} + r_k e^{2i\theta}| \geq |2ie^{i\theta}| - r_k \geq \frac{3}{2}. \]
This implies that
\[ |f| = |b(\xi^3 + \xi)| \geq \frac{3br_k}{4}, \quad \text{if} \quad r_k \leq \frac{1}{2}. \]
On the other hand, since $\xi$ is bounded in $D$ and $\xi_k \to 0$ we have,
\[ |f_1(\xi)| \leq c \zeta_k, \quad \text{for some constant} \quad c > 0. \]
So, it is enough to choose $r_k = \frac{4c}{3br_k}\zeta_k$.
Similarly, we can find $r_k$ sufficiently small such that
\[ |f| > |h - f| = |f_1| \quad \text{on} \quad \partial D' = \partial \{ \xi \in \mathbb{C}; |\xi + i| \leq r_k \}. \]

**Step 2.** By using Step 1. and Rouche's Theorem, there exists $k_0$ large enough such that for all $|k| \geq k_0$ the roots of the polynomial $h(\xi) = b(\xi^3 + \xi)$. Then,
\[ \xi_k^+ = i + \varepsilon_k^+ \quad \text{and} \quad \xi_k^- = -i + \varepsilon_k^-, \quad \text{with} \quad \lim_{|k| \to \infty} \varepsilon_k^\pm = 0. \quad (3.15) \]
Inserting Eq. (3.15) in Eq. (3.14) and using the fact that $\lambda_k^\pm = \mu_k \xi_k^\pm$, we get
\[ \varepsilon_k^\pm = o \left( \frac{1}{\mu_k} \right) \quad \text{and} \quad \lambda_k^\pm = \pm i\mu_k + \varepsilon_k, \quad \text{where} \quad \lim_{|k| \to +\infty} \varepsilon_k = 0. \quad (3.16) \]
Multiplying Eq. (3.11) by $\frac{1}{\mu_k^2}$, we get
\[ \frac{1}{\mu_k^4} \lambda^4 + \frac{b}{\mu_k^2} \lambda^3 + \frac{(1 + a)}{\mu_k^2} \lambda^2 + \frac{c^2}{\mu_k^2} \lambda + b \lambda + a = 0. \quad (3.17) \]
Inserting Eq. (3.16) in Eq. (3.17), we get
\[
\tilde{\varepsilon}_k = -\frac{c^2}{2b\mu_k^2} + o\left(\frac{1}{\mu_k^2}\right). \tag{3.18}
\]

The proof is thus complete.

**Proof of Theorem 3.1** From Proposition 3.2 the large eigenvalues in (3.12)–(3.13) approach the imaginary axis and therefore the system (1.1)–(1.3) is not uniformly stable in the energy space $\mathcal{H}$. □

### 4 Polynomial stability

In this section, we will study the polynomial energy decay rate of the system (1.1)–(1.3). First, we present the definition of some geometric conditions that we encounter in this work.

**Definition 4.1** For a subset $\omega$ of $\Omega$ and $T > 0$, we shall say that $(\omega, T)$ satisfies the Geometric Control Condition if there exists $T > 0$ such that every geodesic traveling at speed one in $\Omega$ meets $\omega$ in time $t < T$.

In order to study the energy decay rate of the system, we consider the following geometric assumptions on $\omega_b$, $\omega_c$ and $\omega = \omega_b \cap \omega_c$:

(H1) The open subset $\omega$ verifies the GCC such that $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$ (see Figs. 6 and 10).

(H2) Assume that $\text{meas}(\overline{\omega_c} \cap \Gamma) > 0$ and $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$. Also, assume that $\omega_c \subseteq \omega_b$ and $\omega_c$ satisfies the GCC (see Fig. 3).

(H3) Assume that $\omega_b \subseteq \Omega$, $\overline{\omega_c} \subseteq \omega_b$ such that $\Omega$ is a non-convex open set and $\omega_c$ satisfies GCC (see Fig. 11).

(H4) Assume that $\Omega = (0, L) \times (0, L), \omega_c \subseteq \omega_b$ such that $\omega_b = \{(x, y) \in \mathbb{R}^2; \varepsilon_1 < x < \varepsilon_4$ and $0 < y < L\}$, $\omega_c = \{(x, y) \in \mathbb{R}^2; \varepsilon_2 < x < \varepsilon_3$ and $0 < y < L\}$ for $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < L$ (see Fig. 12).

(H5) Assume that $\Omega = (0, L) \times (0, L), \omega_c \subseteq \omega_b$ such that $\omega_b = \{(x, y) \in \mathbb{R}^2; 0 < x < \varepsilon_2$ and $0 < y < L\}$ and $\omega_c = \{(x, y) \in \mathbb{R}^2; 0 < x < \varepsilon_1$ and $0 < y < L\}$ for $0 < \varepsilon_1 < \varepsilon_2 < L$ (see Fig. 13).
Remark 4.2 (About Geometric Conditions)

(1) In (H1), if $\Omega$ is a convex domain. Then, the subset $\omega$ satisfies the GCC condition means that $\text{meas}(\omega \cap \Gamma) > 0$ (i.e $\text{meas}(\omega_b \cap \Gamma) > 0$ and $\text{meas}(\omega_c \cap \Gamma) > 0$).

(2) In (H1), if $\Omega$ is a non convex domain. When $\omega$ satisfies the GCC, we study the case when $\text{meas}(\omega_b \cap \Gamma) > 0$ and without the condition $\text{meas}(\omega_c \cap \Gamma) > 0$. The case when both $\text{meas}(\omega_b \cap \Gamma) = 0$ and $\text{meas}(\omega_c \cap \Gamma) = 0$, we studied only the strong stability (see assumption (A5)). Polynomial stability in this case remains an open problem.

(3) In (H4) and (H5), $\omega_b$ and $\omega_c$ does not satisfy any geometric condition.

One of the main tools to prove the polynomial stability of (1.1)–(1.3) such that the assumption (H1) holds and such that $c \in W^{1,\infty}(\Omega)$ is to use the exponential energy decay of the coupled wave equations via velocities with two viscous dampings. We consider the following system...
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Fig. 13 Model satisfying (H5)

\[
\begin{cases}
\phi_{tt} - a\Delta\phi + d(x)\phi_t + c(x)\psi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\
\psi_{tt} - \Delta\psi + d(x)\psi_t - c(x)\phi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\
\phi(x, t) = \psi(x, t) = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\
(\phi(x, 0), \psi(x, 0)) = (\phi_0, \psi_0) & \text{and} \\
(\phi_t(x, 0), \psi_t(x, 0)) = (\phi_1, \psi_1) & \text{in } \Omega.
\end{cases}
\]

(4.1)

where \(d \in W^{1,\infty}(\Omega)\) such that

\[d(x) = 0 \quad \text{in } \Omega \setminus \omega_e \quad \text{and} \quad d(x) \geq 0 \text{ in } \omega_e \subset \omega.\]

The energy of System (4.1) is given by

\[E_{aux}(t) = \frac{1}{2} \int_{\Omega} \left( |\phi_t|^2 + a|\nabla\phi|^2 + |\psi_t|^2 + |\nabla\psi|^2 \right) dx\]

and by a straightforward calculation, we have

\[
\frac{d}{dt} E_{aux}(t) = - \int_{\Omega} d(x)|\phi_t|^2 dx - \int_{\Omega} d(x)|\psi_t|^2 dx \leq 0.
\]

Thus, System (4.1) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable \(t\). The auxiliary energy Hilbert space of Problem (4.1) is given by

\[\mathcal{H}_{aux} = \left( H^1_0(\Omega) \times L^2(\Omega) \right)^2.\]

We denote by \(\eta = \phi_t\) and \(\xi = \psi_t\). The auxiliary energy space \(\mathcal{H}_{aux}\) is endowed with the following norm

\[\|\Phi\|_{\mathcal{H}_{aux}}^2 = \|\eta\|^2 + a\|\nabla\phi\|^2 + \|\xi\|^2 + \|\nabla\psi\|^2,\]
where \( \Phi = (\varphi, \eta, \psi, \xi) \) and \( \| \cdot \| \) denotes the norm of \( L^2(\Omega) \). We define the unbounded linear operator \( A_{aux} \) by

\[
D(A_{aux}) = \left( (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \right)^2,
\]

and

\[
A_{aux}(\varphi, \eta, \psi, \xi)^\top = \begin{pmatrix}
\eta \\
\Delta \varphi - d(x) \eta - c(x) \xi \\
\Delta \psi - d(x) \xi + c(x) \eta
\end{pmatrix}.
\]

If \( \Phi = (\varphi, \psi, \eta, \xi) \) is the state of System (4.1), then this system is transformed into a first order evolution equation on the auxiliary Hilbert space \( H_{aux} \) given by

\[
\Phi_t = A_{aux} \Phi, \quad \Phi(0) = \Phi_0,
\]

where \( \Phi_0 = (\varphi_0, \eta_0, \psi_0, \xi_0) \). It is easy to see that \( A_{aux} \) is \( m \)-dissipative and generates a \( C_0 \)-semigroup of contractions \( (e^{tA_{aux}})_{t \geq 0} \).

**Remark 4.3** From [16], we know that when \( \omega \) satisfies the GCC condition and under the equality speed condition we have that the system of two wave equations coupled through velocity with one viscous damping is exponentially stable (see Theorem 3.1 in [16]). Taking this result into consideration and the fact that our system is considered with two viscous dampings and that \( d, c \in W^{1,\infty}(\Omega) \), by proceeding with a similar proof with \( a \in \mathbb{R}_+^{*} \) as in Theorem 3.1 in [16], we can reach that the system (4.1) decays exponentially such that there exists \( M \geq 1 \) and \( \theta > 0 \) such that for all initial data \( U_0 \in H_{aux} \), the energy of the system (4.1) satisfies the following estimation

\[
E_{aux}(t) \leq Me^{-\theta t} E(0), \quad \forall t > 0.
\]

Now, we will state the main theorems in this section.

**Theorem 4.4** Assume that the boundary \( \Gamma \) is of class \( C^3 \). Also, assume that assumption (H1) holds and that \( c \in W^{1,\infty}(\Omega) \). Then, for all initial data \( U_0 \in D(A) \), there exists a constant \( C > 0 \) independent of \( U_0 \), such that the energy of the system (1.1)–(1.3) satisfies the following estimation

\[
E(t, U) \leq \frac{C}{t} \| U_0 \|_{D(A)}^2, \quad \forall t > 0.
\]

According to Theorem 4.27 of Borichev and Tomilov (see Appendix), by taking \( \ell = 2 \), the polynomial energy decay (4.3) holds if the following conditions

\[
i \mathbb{R} \subset \rho(A),
\]

are satisfied. Since Condition (C1) is already proved in Lemmas 2.7 and 2.8. We will prove condition (C2) by an argument of contradiction. For this purpose, suppose that (C2) is false, then there exists \( \{ (\lambda_n, U_n := (u_n, v_n, y_n, z_n)^\top) \} \subset \mathbb{R}^{*} \times D(A) \) with

\[
|\lambda_n| \to +\infty \quad \text{and} \quad \| U_n \|_{\mathcal{H}} = \| (u_n, v_n, y_n, z_n) \|_{\mathcal{H}} = 1,
\]

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such that
\[(\lambda_n^2)(i\lambda_n I - A)U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \to 0 \text{ in } \mathcal{H}. \tag{4.5}\]

For simplicity, we drop the index $n$. Equivalently, from (4.5), we have
\[i\lambda u - v = \frac{f_1}{\lambda^2} \text{ in } H_0^1(\Omega), \tag{4.6}\]
\[i\lambda v - \text{div}(a\nabla u + b(x)\nabla v) + c(x)z = \frac{f_2}{\lambda^2} \text{ in } L^2(\Omega), \tag{4.7}\]
\[i\lambda y - z = \frac{f_3}{\lambda^2} \text{ in } H_0^1(\Omega), \tag{4.8}\]
\[i\lambda z - \text{div}(\nabla y) - c(x)v = \frac{f_4}{\lambda^2} \text{ in } L^2(\Omega). \tag{4.9}\]

Here we will check the condition (C2) by finding a contradiction with (4.4) such as $\|U\|_{\mathcal{H}} = o(1)$. From equations (4.4), (4.6) and (4.8) we obtain
\[\|u\|_{L^2(\Omega)} = O\left(\frac{1}{\lambda}\right) \text{ and } \|y\|_{L^2(\Omega)} = O\left(\frac{1}{\lambda}\right). \tag{4.10}\]

For clarity, we will divide the proof into several lemmas.

**Lemma 4.5** Assume that the assumption (H1) holds. Then, we have that the solution $(u, v, y, z) \in D(A)$ of (4.6)–(4.9) satisfies the following estimations
\[\|\nabla v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda}, \quad \|v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda}, \quad \|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2} \text{ and } \|\nabla u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \tag{4.11}\]

**Proof** Taking the inner product of (4.5) with $U$ in $\mathcal{H}$, we get
\[\int_{\Omega} b(x)|\nabla v|^2 \, dx = -\Re \langle AU, U \rangle_{\mathcal{H}} = \Re \langle ((i\lambda I - A)U, U \rangle_{\mathcal{H}} = \frac{o(1)}{\lambda^2}. \tag{4.12}\]

Then,
\[\int_{\omega_b}|\nabla v|^2 \, dx = \frac{o(1)}{\lambda^2}. \tag{4.13}\]

By using Poincaré inequality and Eq. (4.13), we get the second estimation in (4.11).

From Eq. (4.6) and the second estimation in (4.11), we obtain
\[\|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \]

By using (4.6) and the first estimation in (4.11), we get the last estimation. \[\square\]

Inserting Eqs. (4.6) and (4.8) into (4.7) and (4.9), we get
\[\lambda^2 u + \text{div}(a\nabla u + b(x)\nabla v) - i\lambda c(x)y = -\frac{f_2}{\lambda^2} - c(x)\frac{f_3}{\lambda^2} - \frac{if_1}{\lambda}, \tag{4.14}\]
\[\lambda^2 y + \Delta y + i\lambda c(x)u = -\frac{f_4}{\lambda^2} + c(x)\frac{f_1}{\lambda^2} - \frac{if_3}{\lambda}. \tag{4.15}\]
Lemma 4.6 Assume that the assumption (H1) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.6)–(4.9) satisfies the following estimation
\[
\int_{\omega_e} |\lambda y|^2 \, dx = o(1)
\]  
where \(\omega_e \subset \omega\) such that \(\omega_e\) satisfies the GCC condition.

**Proof** First, we define the function \(\zeta \in C_c^\infty(\mathbb{R}^N)\) such that
\[
\zeta(x) = \begin{cases} 
1 & \text{if } x \in \omega_e, \\
0 & \text{if } x \in \Omega \setminus \omega, \\
0 & \text{elsewhere},
\end{cases}
\]
such that \(\omega_e \subset \omega\) satisfies the GCC condition. Multiply Eq. (4.14) by \(\lambda \zeta \bar{y}\) and integrate over \(\Omega\) and using Green’s formula, and using Eq. (4.10) and the fact that \(\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)\), we get
\[
\lambda^3 \int_{\Omega} u \zeta \bar{y} \, dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\bar{y} \nabla \zeta + \zeta \nabla \bar{y}) \, dx - i \int_{\Omega} c(x) \zeta |\lambda y|^2 \, dx = \frac{o(1)}{\lambda}.
\]  

Estimation of the first term in (4.18). Using Cauchy–Schwarz inequality, (4.11) and (4.10) we get
\[
\left| \lambda^3 \int_{\Omega} u \zeta \bar{y} \, dx \right| \leq \lambda^3 \|u\|_{L^2(\omega)} \cdot \|y\|_{L^2(\omega)} = o(1).
\]  

Estimation of the second term in (4.18). Using Cauchy–Schwarz inequality, (4.10), (4.11), the fact that \(\text{supp} \, \zeta \subset \omega\), and that \(\|U\|_{\mathcal{H}} = 1\), we obtain the following estimations
\[
\lambda \int_{\Omega} a \nabla u \cdot \nabla \zeta \bar{y} \, dx \leq \lambda \|\nabla u\|_{L^2(\omega)} \cdot \|\nabla y\|_{L^2(\omega)} = \frac{o(1)}{\lambda^2},
\]
\[
\lambda \int_{\Omega} a \nabla u \zeta \nabla \bar{y} \, dx \leq \lambda \|\nabla u\|_{L^2(\omega)} \cdot \|\nabla y\|_{L^2(\omega)} = \frac{o(1)}{\lambda},
\]
\[
\lambda \int_{\Omega} b(x) \nabla v \zeta \nabla \bar{y} \, dx \leq \lambda \|\nabla v\|_{L^2(\omega)} \cdot \|\nabla y\|_{L^2(\omega)} = o(1),
\]
\[
\lambda \int_{\Omega} b(x) \nabla v \cdot \nabla \zeta \bar{y} \, dx \leq \lambda \|\nabla v\|_{L^2(\omega)} \cdot \|y\|_{L^2(\omega)} = \frac{o(1)}{\lambda}.
\]

Inserting Eqs. (4.19)–(4.23) in Eq. (4.18), we get that
\[
i \int_{\Omega} c(x) \zeta |\lambda y|^2 \, dx = o(1).
\]  

Using the definition of the function \(c\) and \(\zeta\), we obtain our desired result. □

Lemma 4.7 For any \(\lambda \in \mathbb{R}\), the solution \((\varphi, \psi) \in ((H^2(\Omega) \cap H^1_0(\Omega))^2\) of the system
\[
\begin{aligned}
\begin{cases}
\lambda^2 \varphi + a \Delta \varphi - i \lambda d(x) \varphi - i \lambda c(x) \psi = u, & \text{in } \Omega \\
\lambda^2 \psi + \Delta \psi - i \lambda d(x) \psi + i \lambda c(x) \varphi = y, & \text{in } \Omega
\end{cases}
\end{aligned}
\]
\[
\varphi = \psi = 0, \quad \text{on } \Gamma
\]  

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satisfies the following estimation
\[
\|\lambda \varphi\|_{L^2(\Omega)}^2 + \|\nabla \varphi\|_{L^2(\Omega)}^2 + \|\lambda \psi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega)}^2 \leq M \left( \|u\|_{L^2(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2 \right).
\]  
(4.26)

where \( M > 0 \).

**Proof** Using Remark 4.3, then the resolvent set of the associated operator \( A_{aux} \) contains \( i\mathbb{R} \) and \((i\lambda I - A_{aux})^{-1}\) is uniformly bounded on the imaginary axis. Consequently, there exists \( M > 0 \) such that
\[
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A_{aux})^{-1} \|_{\mathcal{L}(\mathcal{H}_{aux})} \leq M < +\infty.
\]  
(4.27)

Now, since \((u, y) \in H^1_0(\Omega) \times H^1_0(\Omega)\), then \((0, -u, 0, -y)\) belongs to \( \mathcal{H}_{aux} \), and from (4.27), there exists \((\varphi, \eta, \psi, \xi) \in D(A_{aux})\) such that \((i\lambda I - A_{aux})(\varphi, \eta, \psi, \xi) = (0, -u, 0, -y)^T\) i.e.
\[
i\lambda \varphi - \eta = 0, \tag{4.28}
i\lambda \eta - a\Delta \varphi + d(x)\eta + c(x)\xi = -u, \tag{4.29}
i\lambda \psi - \xi = 0, \tag{4.30}
i\lambda \xi - \Delta \psi + d(x)\xi - c(x)\eta = -y, \tag{4.31}
\]
such that
\[
\| (\varphi, \eta, \psi, \xi) \|_{\mathcal{H}_{aux}} \leq M \left( \|u\|_{L^2(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2 \right).
\]  
(4.32)

From Eqs. (4.28)–(4.32), we deduce that \((\varphi, \psi)\) is a solution of (4.25) and we have
\[
\|\lambda \varphi\|_{L^2(\Omega)}^2 + \|\nabla \varphi\|_{L^2(\Omega)}^2 + \|\lambda \psi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega)}^2 \leq M \left( \|u\|_{L^2(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2 \right).
\]  
(4.33)

Thus, we get our desired result. \(\square\)

**Lemma 4.8** Assume that the assumption (H1) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.6)–(4.9) satisfies the following estimations
\[
\int_{\Omega} |\lambda u|^2 \, dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 \, dx = o(1).
\]  
(4.34)

**Proof** For clarity, we will divide the proof of this Lemma into two steps.

**Step 1.** Multiply (4.14) by \(\lambda^2 \overline{\varphi}\) and integrate over \(\Omega\), and using Green’s formula, Eq. (4.26), and the fact that \(\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)\), we obtain
\[
\int_{\Omega} (\lambda^2 \overline{\varphi} + a\Delta \overline{\varphi}) \lambda^2 u \, dx - \lambda^2 \int_{\Omega} b(x) \nabla u \cdot \nabla \overline{\varphi} \, dx - \int_{\Omega} i\lambda^3 c(x) y \overline{\varphi} \, dx = \frac{o(1)}{\lambda}. \tag{4.35}
\]

From Eqs. (4.11) and (4.26), we obtain
\[
\left| \lambda^2 \int_{\Omega} b(x) \nabla u \cdot \nabla \overline{\varphi} \, dx \right| = o(1).
\]  
(4.36)

By using Eq. (4.36) in (4.35), we get
\[
\int_{\Omega} (\lambda^2 \overline{\varphi} + a\Delta \overline{\varphi}) \lambda^2 u \, dx - \int_{\Omega} i\lambda^3 c(x) y \overline{\varphi} \, dx = o(1). \tag{4.37}
\]
Now, from System (4.25), we have that
\[ \lambda^2 \psi + a \Delta \psi = -i \lambda d(x) \overline{\psi} - i \lambda c(x) \overline{\psi} + \overline{\eta}. \] (4.38)

Inserting Eq. (4.38) into (4.37), we obtain
\[ \int_{\Omega} |\lambda u|^2 \, dx - i \lambda^3 \int_{\Omega} d(x) u \overline{\psi} \, dx - i \lambda^3 \int_{\Omega} c(x) u \overline{\psi} \, dx - \int_{\Omega} i \lambda^3 c(x)y \overline{\psi} \, dx = o(1). \] (4.39)

By using (4.11) and (4.26), we get
\[ \left| i \lambda^3 \int_{\Omega} d(x) u \overline{\psi} \, dx \right| \leq |\lambda|^3 \|u\|_{L^2(\omega_\epsilon)} \cdot \|\psi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda}. \] (4.40)

Now, inserting Eq. (4.40) into (4.39), we get
\[ \int_{\Omega} |\lambda u|^2 \, dx - i \lambda^3 \int_{\Omega} c(x) y \overline{\psi} \, dx - i \lambda^3 \int_{\Omega} c(x) y \overline{\psi} \, dx = o(1). \] (4.41)

Step 2.

Multiply Eq. (4.15) by \( \lambda^2 \psi \), integrate over \( \Omega \), using Green’s formula, and the fact that
\[ \|F\|_{H^ \mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{H^ \mathcal{H}} = o(1), \] we obtain
\[ \int_{\Omega} (\lambda^2 \psi + \Delta \psi) \lambda^2 y \, dx + i \lambda^3 \int_{\Omega} c(x) y \overline{\psi} \, dx = \frac{o(1)}{\lambda}. \] (4.42)

From System (4.25), we have
\[ \lambda^2 \psi + \Delta \psi = -i \lambda d(x) \overline{\psi} + i \lambda c(x) \overline{\psi} + \overline{\eta}. \] (4.43)

Inserting (4.43) into (4.42), we get
\[ \int_{\Omega} |\lambda y|^2 \, dx - i \lambda^3 \int_{\Omega} d(x) y \overline{\psi} \, dx + i \lambda^3 \int_{\Omega} c(x) y \overline{\psi} \, dx + i \lambda^3 \int_{\Omega} c(x) y \overline{\psi} \, dx = \frac{o(1)}{\lambda}. \] (4.44)

Using Cauchy–Schwarz inequality, Lemma 4.6, and Eq. (4.26)
\[ \left| i \lambda^3 \int_{\Omega} d(x) y \overline{\psi} \right| = o(1). \] (4.45)

Inserting (4.45) into (4.44), we get
\[ \int_{\Omega} |\lambda y|^2 \, dx + i \lambda^3 \int_{\Omega} c(x) y \overline{\psi} \, dx + i \lambda^3 \int_{\Omega} c(x) u \overline{\psi} \, dx = o(1). \] (4.46)

Summing Eqs. (4.41) and (4.46), we get
\[ \int_{\Omega} |\lambda u|^2 \, dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 \, dx = o(1). \] (4.47)

Thus, the proof of the Lemma is completed. \( \Box \)

**Lemma 4.9** Assume that the assumption (H1) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.6)-(4.9) satisfies the following estimations
\[ \int_{\Omega} |\nabla u|^2 \, dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 \, dx = o(1). \] (4.48)
Proof Multiply Eq. (4.14) by \( \overline{u} \), integrating over \( \Omega \), Green’s formula, Eq. (4.10) and the fact that \( \|F\|_{H} = \|(f_1, f_2, f_3, f_4)\|_{H} = o(1) \), we obtain

\[
\int_{\Omega} |\lambda u|^2 \, dx - a \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} b(x) \nabla v \cdot \nabla \overline{u} \, dx - \int_{\Omega} i \lambda c(x) y \overline{u} \, dx = o(1) \quad (4.49)
\]

Using Eq. (4.11) and Lemma 4.8, we obtain

\[
\int_{\Omega} |\nabla u|^2 \, dx = o(1). \quad (4.50)
\]

Multiplying Eq. (4.15) by \( \overline{y} \) and proceeding in a similar way as above, we get

\[
\int_{\Omega} |\nabla y|^2 \, dx = o(1). \quad (4.51)
\]

\( \Box \)

Proof of Theorem 4.4 Consequently, from the results of Lemmas 4.8 and 4.9, we obtain

\[
\int_{\Omega} (|v|^2 + |z|^2 + a |\nabla u|^2 + |\nabla y|^2) \, dx = o(1). \]

Hence \( \|U\|_{H} = o(1) \), which contradicts (4.4). Consequently, condition (C2) holds. This implies that the energy decay estimation (4.3). The proof is thus complete. \( \Box \)

**Remark 4.10** In the case when \( d, c \in L^\infty(\Omega) \) such that they are discontinuous functions, we didn’t find any result on the stability of the system (4.1). But, we can conjecture that the system (4.1) is exponentially stable. Further, we have that the system (4.1) with \( d, c \) are discontinuous functions is exponentially stable in the dimension 1 (see [37]).

One of the main tools to prove the polynomial stability of the system (1.1)–(1.3) when one of the assumptions (H2), (H3), (H4) or (H5) holds is to use the exponential or polynomial decay of the wave equation with viscous damping. We consider the following system

\[
\begin{cases}
\varphi_{tt} - \Delta \varphi + \mathbb{1}_{\omega_c}(x)\varphi_t = 0 & \text{in } \Omega \times (0, +\infty) \\
\varphi = 0 & \text{in } \Gamma^1 \times (0, +\infty) \\
\varphi(\cdot, 0) = \varphi_0, \quad \varphi_t(\cdot, 0) = \varphi_1.
\end{cases} \quad (4.52)
\]

**Remark 4.11** (About System (4.52))

1. If (H2) or (H3) holds, system (4.52) is exponentially stable (see [9] and Lemma 3.8 in [2]).
2. If (H4) holds, the energy of the wave equation (4.52) with local viscous damping decays polynomially as \( t^{-1} \) for smooth initial data (see Example 3 in [28]).
3. If (H5) holds, the energy of the wave equation (4.52) with local viscous damping decays polynomially as \( t^{-\frac{5}{3}} \) for smooth initial data (see [35]).

**Theorem 4.12** Assume that the boundary \( \Gamma \) is of class \( C^3 \). Also, assume that assumption (H2) or (H3) holds. Then, for all initial data \( U_0 \in D(A) \), there exists a constant \( C_2 > 0 \) independent of \( U_0 \), such that the energy of the system (1.1)–(1.3) satisfies the following estimation

\[
E(t, U) \leq \frac{C_2}{t} \|U_0\|_{D(A)}^2, \quad \forall t > 0. \quad (4.53)
\]
Following Theorem 4.27 of Borichev and Tomilov (see Appendix), the polynomial energy decay (4.53) holds if (C1) and

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - A)^{-1}\|_{L(\mathcal{H})} = O(|\lambda|^2) \quad \text{(C3)}$$

holds. Since Condition (C1) is already proved. We will prove condition (C3) by an argument of contradiction. For this purpose, suppose that (C3) is false, then there exists \( \{(\lambda_n, U_n := (u_n, v_n, y_n, z_n)^\top)\} \subset \mathbb{R}^n \times D(A) \) with

$$|\lambda_n| \to +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1,$$  \quad \text{(4.54)}$$

such that

$$\lambda_n^2 (i\lambda_n I - A) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \to 0 \text{ in } \mathcal{H}. \quad \text{(4.55)}$$

For simplicity, we drop the index \( n \). Equivalently, from (4.55), we have

$$i\lambda u - v = \lambda^{-2} f_1 \text{ in } H^1_0(\Omega), \quad \text{(4.56)}$$
$$i\lambda v - \text{div}(a \nabla u + b(x) \nabla v) + c(x) z = \lambda^{-2} f_2 \text{ in } L^2(\Omega), \quad \text{(4.57)}$$
$$i\lambda y - z = \lambda^{-2} f_3 \text{ in } H^1_0(\Omega), \quad \text{(4.58)}$$
$$i\lambda z - \text{div}(\nabla y) - c(x) v = \lambda^{-2} f_4 \text{ in } L^2(\Omega). \quad \text{(4.59)}$$

Here we will check the condition (C3) by finding a contradiction with (4.54) such as \( \|U\|_{\mathcal{H}} = o(1) \). From Eqs. (4.54), (4.56) and (4.58) we obtain

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda} \text{ and } \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}. \quad \text{(4.60)}$$

Lemma 4.13 Assume that the assumption (H2) or (H3) holds. We have that the solution \((u, v, y, z) \in D(A)\) of (4.56)-(4.59) satisfies the following estimations

$$\|\nabla v\|_{L^2(\omega_b)} = o(\lambda) \text{ and } \|\nabla u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad \text{(4.61)}$$

The proof of this Lemma is similar to that of Lemma 4.5.

Lemma 4.14 Under the assumption (H2). We have that the solution \((u, v, y, z) \in D(A)\) of (4.56)-(4.59) satisfies the following estimations

$$\|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad \text{(4.62)}$$

Proof By using Poincaré inequality and equation (4.61), we get

$$\|v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda}. \quad \text{(4.63)}$$

From equation (4.56) and (4.63), we obtain

$$\|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad \text{(4.64)}$$
Inserting Eqs. (4.56) and (4.58) into (4.57) and (4.59), we get
\[
\lambda^2 u + \text{div}(a \nabla u + b(x) \nabla v) - i \lambda c(x) y = -\frac{f_2}{\lambda^2} - c(x) \frac{f_3}{\lambda^2} - \frac{if_1}{\lambda}, \quad (4.65)
\]
\[
\lambda^2 y + \Delta y + i \lambda c(x) u = -\frac{f_4}{\lambda^2} + c(x) \frac{f_1}{\lambda^2} - \frac{if_3}{\lambda}. \quad (4.66)
\]

**Lemma 4.15** Assume that assumption (H3) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.56)–(4.59) satisfies the following estimation
\[
\int_{\tilde{\omega}_b} |\lambda u|^2 dx = o(1). \quad (4.67)
\]
such that \(\omega_c \subset \tilde{\omega}_b \subset \omega_b\).

**Proof** Let a non-empty open subset \(\tilde{\omega}_b\) such that \(\omega_c \subset \tilde{\omega}_b \subset \omega_b\). Then, we define the function \(h_1 \in C_\infty_c(\mathbb{R}^N)\) such that
\[
h_1(x) = \begin{cases} 
1 & \text{if } x \in \tilde{\omega}_b, \\
0 & \text{if } x \in \Omega \setminus \omega_b, \\
0 \leq h_1 \leq 1 & \text{elsewhere.}
\end{cases} \quad (4.68)
\]
Multiply (4.65) by \(h_1 \bar{u}\) and integrate over \(\Omega\), we get
\[
\int_{\Omega} h_1 |\lambda u|^2 dx - \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (h_1 \nabla \bar{u} + \nabla h_1 \bar{u}) dx - i \lambda \int_{\Omega} h_1 c(x) y \bar{u} dx = \frac{o(1)}{\lambda^2}. 
\]
Using (4.60) and (4.61), we have
\[
\left| \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (h_1 \nabla \bar{u} + \nabla h_1 \bar{u}) dx \right| = \frac{o(1)}{\lambda^2} \quad (4.70)
\]
and
\[
\left| i \lambda \int_{\Omega} h_1 c(x) y \bar{u} dx \right| = \frac{O(1)}{\lambda}. \quad (4.71)
\]
Thus, by using Eqs. (4.70) and (4.71) in (4.69), we obtain
\[
\int_{\Omega} h_1 |\lambda u|^2 dx = \frac{O(1)}{\lambda}. \quad (4.72)
\]
Thus, we reach our desired result. \(\square\)

**Lemma 4.16** Assume that assumption (H2) or (H3) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.56)–(4.59) satisfies the following estimation
\[
\int_{\omega_c} |\lambda y|^2 dx = o(1). \quad (4.73)
\]

**Proof** **Case I.** Assume that assumption (H2) holds, we define the function \(\rho \in C_\infty_c(\mathbb{R}^N)\) such that
\[
\rho(x) = \begin{cases} 
1 & \text{if } x \in \omega_c, \\
0 & \text{if } x \in \Omega \setminus \omega_b, \\
0 \leq \rho \leq 1 & \text{elsewhere.}
\end{cases} \quad (4.74)
\]
Now, multiplying (4.65) by \( \lambda \rho \overline{y} \), integrating over \( \Omega \) and using Green’s formula, (4.60) and the fact that \( \| F \|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1) \), we get
\[
\lambda^3 \int_{\Omega} u \rho \overline{y} dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \overline{y} + \rho \nabla \overline{y}) dx - i \int_{\Omega} c(x) \rho |\lambda y|^2 dx = \frac{o(1)}{\lambda}.
\]
(4.75)
Using (4.60), (4.61) and Cauchy-Schwarz we obtain
\[
\left| \lambda^3 \int_{\Omega} u \rho \overline{y} dx \right| \leq \lambda^3 \| u \|_{L^2(\omega_b)} \cdot \| y \|_{L^2(\omega_b)} = o(1).
\]
(4.76)
and
\[
\left| \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \overline{y} + \rho \nabla \overline{y}) dx \right| = o(1).
\]
(4.77)
Thus, using equations (4.76) and (4.77) in (4.75) we obtain our desired result for the first case.

**Case 2.** Assume that assumption (H3) holds. Define the function \( h_2 \in C^\infty_c(\mathbb{R}^N) \) such that
\[
h_2(x) = \begin{cases} 
1 & \text{if } x \in \omega_c, \\
0 & \text{if } x \in \Omega \setminus \hat{\omega}_b, \\
0 \leq h_2 \leq 1 & \text{elsewhere}.
\end{cases}
\]
(4.78)
Multiply (4.65) by \( \lambda h_2 \overline{y} \) and integrate over \( \Omega \), and using Green’s formula, (4.60) and the fact that \( \| F \|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1) \), we get
\[
\lambda^3 \int_{\Omega} u h_2 \overline{y} dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla h_2 \overline{y} + h_2 \nabla \overline{y}) dx - i \int_{\Omega} c(x) h_2 |\lambda y|^2 dx = \frac{o(1)}{\lambda}.
\]
(4.79)
Using (4.60), (4.61) and Cauchy-Schwarz, we obtain
\[
\left| \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla h_2 \overline{y} + h_2 \nabla \overline{y}) dx \right| = o(1).
\]
(4.80)
By using Eq. (4.80) in (4.79), we obtain
\[
\lambda^3 \int_{\Omega} u h_2 \overline{y} dx - i \int_{\Omega} c(x) h_2 |\lambda y|^2 dx = o(1).
\]
(4.81)
Now, multiply (4.66) by \( \lambda h_2 \overline{u} \) and integrate over \( \Omega \), and using Green’s formula, (4.60) and the fact that \( \| F \|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1) \), we get
\[
\lambda^3 \int_{\Omega} y h_2 \overline{u} dx - \lambda \int_{\Omega} \nabla y (\nabla h_2 \overline{u} + h_2 \nabla \overline{u}) dx + i \int_{\omega_c} c(x) |\lambda u|^2 dx = \frac{o(1)}{\lambda}.
\]
(4.82)
Using (4.54), (4.60), (4.61), (4.67) and Cauchy-Schwarz, we obtain
\[
\left| \lambda \int_{\Omega} \nabla y \cdot (\nabla h_2 \overline{u} + h_2 \nabla \overline{u}) dx \right| \leq \| \nabla y \|_{L^2(\Omega)} \cdot \| \lambda u \|_{L^2(\omega_b)} + |\lambda| \| \nabla y \|_{L^2(\Omega)} \| \nabla u \|_{L^2(\omega_b)} = o(1).
\]
(4.83)
Inserting (4.83) into (4.82), we get
\[
\lambda^3 \int_{\Omega} y h_2 \overline{u} dx + i \int_{\omega_c} c(x) |\lambda u|^2 dx = o(1).
\]
(4.84)
Summing Eqs. (4.81) and (4.84), and taking the imaginary part and using Eq. (4.67), we obtain our desired result.
\[\square\]
Lemma 4.17 Assume that assumption (H2) or (H3) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.56)–(4.59) satisfies the following estimations

\[
\int_{\Omega} |u|^2 \, dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 \, dx = o(1).
\]  

(4.85)

Proof Let \(\phi, \psi \in H^2(\Omega) \cap H^1_0(\Omega)\) be the solution of the following system

\[
\begin{align*}
\lambda^2 \phi + aA \phi - i\lambda L_{\omega_c}(x)\phi &= u, \quad \text{in} \ \Omega \\
\lambda^2 \psi + A \psi - i\lambda L_{\omega_c}(x)\psi &= y, \quad \text{in} \ \Omega \\
\phi &= \psi = 0, \quad \text{on} \ \Gamma
\end{align*}
\]

(4.86)

where \((u, v, y, z)\) is the solution of (4.56)–(4.59). Since either (H2) or (H3) holds, then system (4.52) is exponentially stable. Thus, there exists \(M > 0\) such that system (4.86) satisfies the following estimation

\[
\|\lambda \phi\|_{L^2(\Omega)} + \|\nabla \phi\|_{L^2(\Omega)} + \|\lambda \psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \leq M \left( \|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)} \right).
\]

(4.87)

Case 1. Under the assumption (H2). Multiply (4.65) by \(\lambda^2 \psi\) and integrate over \(\Omega\), and using Green’s formula, Eq. (4.87), and the fact that \(\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3)\|_{\mathcal{H}} = o(1)\), we obtain

\[
\int_{\Omega} (\lambda^2 \psi + aA \psi - i\lambda L_{\omega_c}(x)\psi) \lambda^2 u \, dx - \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \psi \, dx - \int_{\Omega} i\lambda^3 c(x) y \psi \, dx = o(1) / \lambda.
\]

(4.88)

From Eq. (4.61) and (4.87), we obtain

\[
\left| \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \psi \, dx \right| = o(1).
\]

(4.89)

Now, using System (4.86) and Eq. (4.89) in (4.88), we get

\[
\int_{\Omega} |\lambda u|^2 \, dx - i\lambda^3 \int_{\Omega} L_{\omega_c}(x)u \psi \, dx - i\lambda^3 \int_{\Omega} c(x) y \psi \, dx = o(1).
\]

(4.90)

By using (4.62), (4.87) and the fact that \(\omega_c \subset \omega_b\)

\[
\left| i\lambda^3 \int_{\Omega} L_{\omega_c}(x)u \psi \, dx \right| \leq \lambda^3 \left( \|u\|_{L^2(\omega_c)} \cdot \|\psi\|_{L^2(\Omega)} \right) = o(1) / \lambda.
\]

(4.91)

Using (4.73) and (4.87), we get that

\[
\left| i\lambda^3 \int_{\Omega} c(x) y \psi \, dx \right| = o(1).
\]

(4.92)

Now, inserting Eqs. (4.91) and (4.92) into (4.90), we get

\[
\int_{\Omega} |\lambda u|^2 \, dx = o(1).
\]

(4.93)

Multiply (4.66) by \(\lambda^2 \psi\) and integrate over \(\Omega\), and using Green’s formula, Eq. (4.87), and the fact that \(\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)\), we obtain

\[
\int_{\Omega} (\lambda^2 \psi + A \psi - i\lambda L_{\omega_c}(x)\psi) \lambda^2 y \, dx + i\lambda^3 \int_{\Omega} c(x) u \psi \, dx = o(1) / \lambda.
\]

(4.94)

By using System (4.86) in (4.94), we get

\[
\int_{\Omega} |\lambda y|^2 \, dx - i\lambda^3 \int_{\Omega} L_{\omega_c}(x) y \psi \, dx + i\lambda^3 \int_{\Omega} c(x) u \psi \, dx = o(1).
\]

(4.95)
Using (4.62), (4.73), and (4.87) and the fact that \( \omega_c \subset \omega_b \), we get

\[
\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) y \psi dx \right| = o(1).
\]  

(4.96)

and

\[
\left| i \lambda^3 \int_{\Omega} c(x) u \psi dx \right| = \frac{o(1)}{\lambda}.
\]  

(4.97)

Inserting (4.96) and (4.97) into (4.95) we obtain

\[
\int_{\Omega} |\lambda y|^2 dx = o(1).
\]  

(4.98)

**Case 2.** Under the assumption (H3). We proceed in the same way as in Case 1., the only change is that we have the following two estimations instead of (4.91) and (4.97), by using (4.67) and (4.87) we get

\[
\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) u \varphi dx \right| = o(1)
\]  

(4.99)

and

\[
\left| i \lambda^3 \int_{\Omega} c(x) u \varphi dx \right| = o(1).
\]  

(4.100)

Thus, we reach our desired result.

\[
\square
\]

**Lemma 4.18** Assume that either the assumption (H2) or (H3) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.56)-(4.59) satisfies the following estimations

\[
\int_{\Omega} |\nabla u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1).
\]  

(4.101)

**Proof** The proof is similar to the proof of Lemma 4.9.

\[
\square
\]

**Proof of Theorem 4.12** Consequently, from the results of Lemmas 4.17, and 4.18, we obtain that \( \|U\|_{H^2} = o(1) \), which contradicts (4.54). Consequently, condition (C3) holds. This implies, from Theorem 4.27, the energy decay estimation (4.53). The proof is thus complete.

\[
\square
\]

**Theorem 4.19** Assume that assumption (H4) or (H5) holds. Then, for all initial data \( U_0 \in D(A) \), there exists a constant \( C_3 > 0 \) independent of \( U_0 \), the energy of the system (1.1)–(1.3) satisfies the following estimation

\[
E(t, U) \leq \frac{C_3}{t^\frac{2}{2+3\beta}} \|U_0\|_{D(A)}, \quad \forall t > 0.
\]  

(4.102)

where

\[
\beta = \begin{cases} 
2 & \text{if } (H4) \text{ holds} \\
3 & \text{if } (H5) \text{ holds} 
\end{cases}
\]  

(4.103)
Following Theorem 4.27 of Borichev and Tomilov (see Appendix), the polynomial energy decay (4.102) holds if (C1) and
\[ \sup_{\lambda \in \mathbb{R}} \| (i \lambda I - A)^{-1} \|_{L(H)} = O(|\lambda|^{2+4\beta}) \] (C4)
are satisfied. Since Condition (C1) is already satisfied (see Lemmas 2.7 and 2.8). We will prove condition (C4) by an argument of contradiction. For this purpose, suppose that (C4) is false, then there exists \( \{ (\lambda_n, U_n) := (u_n, v_n, y_n, z_n) \} \subset \mathbb{R}^* \times \mathbb{D}(A) \) with
\[ |\lambda_n| \to +\infty \quad \text{and} \quad \| U_n \|_{H^1} = \|(u_n, v_n, y_n, z_n)\|_{H^1} = 1, \] (4.54)
such that
\[ \lambda_n^{2+4\beta} (i \lambda_n I - A) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^T \to 0 \text{ in } H^1. \] (4.55)

For simplicity, we drop the index \( n \). Equivalently, from (4.55), we have
\[ i \lambda u - v = \frac{f_1}{\lambda^{2+4\beta}} \text{ in } H^1_0(\Omega), \] (4.56)
\[ i \lambda v - \text{div}(a \nabla u + b(x) \nabla v) + c(x)z = \frac{f_2}{\lambda^{2+4\beta}} \text{ in } L^2(\Omega), \] (4.57)
\[ i \lambda y - z = \frac{f_3}{\lambda^{2+4\beta}} \text{ in } H^1_0(\Omega), \] (4.58)
\[ i \lambda z - \text{div}(\nabla y) - c(x)v = \frac{f_4}{\lambda^{2+4\beta}} \text{ in } L^2(\Omega). \] (4.59)

Here we will check the condition (C4) by finding a contradiction with (4.54) such as \( \| U \|_{H^1} = o(1) \). From Eqs. (4.54), (4.56) and (4.58), we obtain
\[ \| u \|_{L^2(\Omega)} = \frac{O(1)}{\lambda} \quad \text{and} \quad \| y \|_{L^2(\Omega)} = \frac{O(1)}{\lambda}. \] (4.60)

**Lemma 4.20** Under the assumptions (H4) or (H5). We have that the solution \((u, v, y, z) \in D(A) \) of (4.56)–(4.59) satisfies the following estimations
\[ \| \nabla v \|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{1+2\beta}} \quad \| \nabla u \|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{2+2\beta}} \quad \text{and} \quad \| u \|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{2+2\beta}}. \] (4.61)

The proof of this Lemma is similar to that of Lemma 4.5.

Inserting Eqs. (4.56) and (4.58) into (4.57) and (4.59), we get
\[ \lambda^2 u + \text{div}(a \nabla u + b(x) \nabla v) - i \lambda c(x)y = -
\frac{f_2}{\lambda^{2+4\beta}} - c(x) \frac{f_3}{\lambda^{2+4\beta}} - \frac{i f_1}{\lambda^{1+4\beta}}, \] (4.62)
\[ \lambda^2 y + \Delta y + i \lambda c(x)u = -
\frac{f_4}{\lambda^{2+4\beta}} + c(x) \frac{f_1}{\lambda^{2+4\beta}} - \frac{i f_3}{\lambda^{1+4\beta}}. \] (4.63)

**Lemma 4.21** Assume that assumption (H4) or (H5) holds. Then, the solution \((u, v, y, z) \in D(A) \) of (4.56)–(4.59) satisfies the following estimation
\[ \int_{\omega_c} |\lambda y|^2 \, dx = \frac{o(1)}{\lambda^{2\beta}}. \] (4.64)

**Proof** Assume that either assumption (H4) or (H5) holds. Define the function \( \rho \in C^\infty_0(\mathbb{R}^N) \) such that
\[ \rho(x) = \begin{cases} 
1 & \text{if } x \in \omega_c, \\
0 & \text{if } x \in \Omega \setminus \omega_b, \\
0 \leq \rho \leq 1 & \text{elsewhere.}
\end{cases} \] (4.65)
Now, multiply (4.62) by $\lambda \rho \overline{y}$, integrate over $\Omega$ and using Green’s formula, (4.60) and the fact that $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$, we get

$$
\lambda^3 \int_{\Omega} u \rho \overline{y} \, dx - \lambda \int_{\Omega} \left[(a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \overline{y} + \rho \nabla \overline{y})\right] \, dx - i \int_{\Omega} c(x) \rho |y|^2 \, dx = \frac{o(1)}{\lambda^{1+4\beta}}.
$$

(4.66)

Using (4.60), (4.61) and Cauchy–Schwarz we obtain

$$
\left| \lambda^3 \int_{\Omega} u \rho \overline{y} \, dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_0)} \cdot \|\overline{y}\|_{L^2(\omega_0)} = \frac{o(1)}{\lambda^{2\beta}}
$$

(4.67)

and

$$
\left| \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \overline{y} + \rho \nabla \overline{y}) \, dx \right| = \frac{o(1)}{\lambda^{2\beta}}.
$$

(4.68)

Thus, using Eq. (4.67) and (4.68) in (4.66) we obtain our desired result.

Lemma 4.22 Assume that the assumption (H4) or (H5) holds. Then, the solution $(u, v, y, z) \in D(A)$ of (4.56)-(4.59) satisfies the following estimations

$$
\int_{\Omega} |\lambda u|^2 \, dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 \, dx = o(1).
$$

(4.69)

Proof Let $\varphi, \psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the following system

$$
\begin{cases}
\lambda^2 \varphi + a \Delta \varphi - i \lambda \mathbb{1}_{\omega_0}(x) \varphi = u, & \text{in } \Omega \\
\lambda^2 \psi + a \Delta \psi - i \lambda \mathbb{1}_{\omega_0}(x) \psi = y, & \text{in } \Omega \\
\varphi = \psi = 0, & \text{on } \Gamma
\end{cases}
$$

(4.70)

where $(u, v, y, z)$ is the solution of (4.56)–(4.59). We suppose that the energy of the System (4.52) satisfies the following estimate

$$
E(t, U) \leq \frac{C}{t^{\beta}} \|U_0\|^2_{D(A)}, \quad \forall t > 0.
$$

When assumption (H4) holds, we have that System (4.52) is polynomially stable with an energy decay rate $t^{-1}$, i.e. $\beta = 2$. However, when assumption (H5) holds then we have that System (4.52) is polynomially stable with an energy decay rate $t^{-\frac{3}{2}}$, i.e. $\beta = \frac{3}{2}$. Thus, there exists $M > 0$ such that system (4.70) satisfies the following estimation

$$
\|\lambda \varphi\|_{L^2(\Omega)} + \|\nabla \varphi\|_{L^2(\Omega)} + \|\lambda \psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \leq M |\lambda|^{\beta} \left(\|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}\right)
$$

(4.71)

Assuming that (H4) or (H5) holds. Multiply (4.62) by $\lambda^2 \overline{\varphi}$ and integrate over $\Omega$, and using Green’s formula, Eq. (4.87), and the fact that $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$, we obtain

$$
\int_{\Omega} \left(\lambda^2 \overline{\varphi} + a \Delta \overline{\varphi}\right) \lambda^2 u \, dx - \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \overline{\varphi} \, dx - i \lambda^3 \int_{\Omega} c(x) y \overline{\varphi} \, dx = \frac{o(1)}{\lambda^{1+3\beta}}.
$$

(4.72)

From Eqs. (4.61) and (4.71), we obtain

$$
\left| \lambda^2 \int_{\Omega} b(x) \nabla v \nabla \overline{\varphi} \, dx \right| = \frac{o(1)}{\lambda^{\beta}}.
$$

(4.73)
Now, using System (4.70) and Eq. (4.73) in (4.72), we get
\[
\int_{\Omega} |\lambda u|^2 \, dx - i \lambda^3 \int_{\Omega} \mathbb{I}_{\omega_b}(x) u \overline{\varphi} \, dx - i \lambda^3 \int_{\Omega} c(x) y \overline{\varphi} \, dx = \frac{o(1)}{\lambda^\beta}.
\] (4.74)

By using (4.61), (4.71) and the fact that \(\omega_c \subset \omega_b\)
\[
\left| i \lambda^3 \int_{\Omega} \mathbb{I}_{\omega_c}(x) u \overline{\varphi} \, dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_c)} \cdot \|\varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{1+\beta}}.
\] (4.75)

Using (4.64) and (4.71), we get that
\[
\left| i \lambda^3 \int_{\Omega} c(x) y \overline{\varphi} \, dx \right| = o(1).
\] (4.76)

Now, inserting Eqs. (4.75) and (4.76) into (4.74), we get
\[
\int_{\Omega} |\lambda u|^2 \, dx = o(1).
\] (4.77)

Multiply (4.63) by \(\lambda^2 \overline{\psi}\) and integrate over \(\Omega\), and using Green’s formula, Eq. (4.71), and the fact that \(\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)\), we obtain
\[
\int_{\Omega} (\lambda^2 \overline{\psi} + \Delta \overline{\psi}) \lambda^2 y \, dx + i \lambda^3 \int_{\Omega} c(x) u \overline{\psi} \, dx = \frac{o(1)}{\lambda^{1+3\beta}}.
\] (4.78)

By using System (4.70) in (4.78) we get
\[
\int_{\Omega} |\lambda y|^2 \, dx - i \lambda^3 \int_{\Omega} \mathbb{I}_{\omega_c}(x) y \overline{\psi} \, dx + i \lambda^3 \int_{\Omega} c(x) u \overline{\psi} \, dx = \frac{o(1)}{\lambda^{1+3\beta}}.
\] (4.79)

Using (4.61), (4.64), and (4.71) and the fact that \(\omega_c \subset \omega_b\), we get
\[
\left| i \lambda^3 \int_{\Omega} \mathbb{I}_{\omega_c}(x) y \overline{\psi} \, dx \right| = o(1).
\] (4.80)

and
\[
\left| i \lambda^3 \int_{\Omega} c(x) u \overline{\psi} \, dx \right| = \frac{o(1)}{\lambda^{1+\beta}}.
\] (4.81)

Inserting (4.80) and (4.81) into (4.79) we obtain
\[
\int_{\Omega} |\lambda y|^2 \, dx = o(1).
\] (4.82)

**Lemma 4.23** Assume that either the assumption (H4) or (H5) holds. Then, the solution \((u, v, y, z) \in D(A)\) of (4.56)–(4.59) satisfies the following estimations
\[
\int_{\Omega} |\nabla u|^2 \, dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 \, dx = o(1).
\] (4.83)

**Proof** The proof is similar to the proof of Lemma 4.9. □

**Proof of Theorem 4.19** Consequently, from the results of Lemma 4.22, 4.23, we obtain that \(\|U\|_{\mathcal{H}} = o(1)\), which contradicts (4.54). Consequently, condition (C4) holds. This implies, from Theorem 4.27, the energy decay estimation (4.102). The proof is thus complete.
Appendix

In this section, we introduce the notions of stability that we encounter in this work.

**Definition 4.24** Assume that \( A \) is the generator of a \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \) on a Hilbert space \( \mathcal{H} \). The \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is said to be

1. strongly stable if
   \[
   \lim_{t \to +\infty} \| e^{tA} x_0 \|_H = 0, \quad \forall \ x_0 \in H;
   \]
2. exponentially (or uniformly) stable if there exist two positive constants \( M \) and \( \epsilon \) such that
   \[
   \| e^{tA} x_0 \|_H \leq M e^{-\epsilon t} \| x_0 \|_H, \quad \forall \ t > 0, \ \forall \ x_0 \in H;
   \]
3. polynomially stable if there exists two positive constants \( C \) and \( \alpha \) such that
   \[
   \| e^{tA} x_0 \|_H \leq C t^{-\alpha} \| x_0 \|_H, \quad \forall \ t > 0, \ \forall \ x_0 \in D(A).
   \]

In that case, one says that the semigroup \( (e^{tA})_{t \geq 0} \) decays at a rate \( t^{-\alpha} \). The \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is said to be polynomially stable with optimal decay rate \( t^{-\alpha} \) (with \( \alpha > 0 \)) if it is polynomially stable with decay rate \( t^{-\alpha} \) and, for any \( \epsilon > 0 \) small enough, the semigroup \( (e^{tA})_{t \geq 0} \) does not decay at a rate \( t^{-(\alpha-\epsilon)} \).

To show the strong stability of a \( C_0 \)-semigroup of contraction \( (e^{tA})_{t \geq 0} \) we rely on the following result due to Arendt–Batty [4].

**Theorem 4.25** Assume that \( A \) is the generator of a \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \) on a Hilbert space \( \mathcal{H} \). If

1. \( A \) has no pure imaginary eigenvalues,
2. \( \sigma(A) \cap i\mathbb{R} \) is countable,

where \( \sigma(A) \) denotes the spectrum of \( A \), then the \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is strongly stable.

Concerning the characterization of exponential stability of a \( C_0 \)-semigroup of contraction \( (e^{tA})_{t \geq 0} \) we rely on the following result due to Huang [20] and Prüss [33].

**Theorem 4.26** Let \( A : D(A) \subset H \to H \) generate a \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \) on \( H \). Assume that \( i\lambda \in \rho(A), \ \forall \lambda \in \mathbb{R} \). Then, the \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is exponentially stable if and only if

\[
\lim_{\lambda \in \mathbb{R}, \ |\lambda| \to +\infty} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(H)} < +\infty.
\]

Also, concerning the characterization of polynomial stability of a \( C_0 \)-semigroup of contraction \( (e^{tA})_{t \geq 0} \) we rely on the following result due to Borichev and Tomilov [8] (see also [7, 29]).

**Theorem 4.27** Assume that \( A \) is the generator of a strongly continuous semigroup of contractions \( (e^{tA})_{t \geq 0} \) on \( H \). If \( i\mathbb{R} \subset \rho(A) \), then for a fixed \( \ell > 0 \) the following conditions are equivalent

\[
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(H)} = O \left( |\lambda|^\ell \right), \quad (5.1)
\]
\[
\| e^{tA} U_0 \|_H^2 \leq \frac{C}{t^d} \| U_0 \|_{D(A)}^2, \quad \forall t > 0, \ U_0 \in D(A), \text{ for some } C > 0. \quad (5.2)
\]

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References

1. Akil, M., Wehbe, A.: Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions. Math. Control Relat. Fields 8, 1–20 (2018)
2. Ali Wehbe, N.N., Nasser, R.: Stability of n-d transmission problem in viscoelasticity with localized kelvin-voigt damping under different types of geometric conditions. Math. Control Relat. Fields 11(4), 885–904 (2021)
3. Ammari, K., Hassine, F., Robbiano, L.: Stabilization for the wave equation with singular kelvin-voigt damping. Arch. Ration. Mech. Anal. 236, 577–601 (2019)
4. Arendt, W., Batty, C.J.K.: Tauberian theorems and stability of one-parameter semigroups. Trans. Am. Math. Soc. 306(2), 837–852 (1988)
5. Banks, H., Smith, R., Wang, Y.: The modeling of piezoceramic patch interactions with shells, plates, and beams. Q. Appl. Math. 53, 353–381 (1995)
6. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. 30(5), 1024–1065 (1992)
7. Batty, C.J.K., Duyckaerts, T.: Non-uniform stability for bounded semi-groups on Banach spaces. J. Evol. Equ. 8(4), 765–780 (2008)
8. Borichev, A., Tomilov, Y.: Optimal polynomial decay of functions and operator semigroups. Math. Ann. 347(2), 455–478 (2010)
9. Burq, N.: Contrôlabilité exacte des ondes dans des ouverts peu réguliers. Asymptot. Anal. 14, 157–191 (1997)
10. Burq, N.: Decays for kelvin-voigt damped wave equations I: the black box perturbative method. SIAM J. Control Optim. 58(4), 1893–1905 (2020)
11. Burq, N., Sun, C.: Decay for the kelvin-voigt damped wave equation: piecewise smooth damping. arXiv:AnalysisofPDEs (2020)
12. Burq, N., Sun, C.: Decays rates for kelvin-voigt damped wave equations ii: the geometric control condition. Proc. Amer. Math. Soc. 150, 1021–1039 (2022)
13. Cavalcanti, M., Cavalcanti, V.D., Tebou, L.: Stabilization of the wave equation with localized compensating frictional and kelvin-voigt dissipating mechanisms. Electron. J. Differ. Equ. 2017, 1–18 (2017)
14. Dafermos, C.: Asymptotic behavior of solutions of evolution equations. In: Crandall, M.G. (ed.) Nonlinear Evolution Equations, pp. 103–123. Academic Press, Singapore (1978)
15. Enrike, Z.: Exponential decay for the semilinear wave equation with locally distributed damping. Commun. Partial Differ. Equ. 15(2), 205–235 (1990)
16. Gerbi, S., Kassem, C., Mortada, A., Wehbe, A.: Exact controllability and stabilization of locally coupled wave equations: theoretical results. Zeitschrift für Analysis und ihre Anwendungen 40, 67–96 (2021)
17. Hayek, A., Nicaise, S., Salloum, Z., Wehbe, A.: A transmission problem of a system of weakly coupled wave equations with kelvin–voigt dampings and non-smooth coefficient at the interface. SeMA J. 77(3), 305–338 (2020)
18. Hörmander, L.: Linear Partial Differential Operators. Springer, Berlin (1969)
19. Huang, F.: On the mathematical model for linear elastic systems with analytic damping. SIAM J. Control Optim. 26(3), 714–724 (1988)
20. Huang, F.L.: Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. Ann. Differ. Equ. 1(1), 43–56 (1985)
21. Kato, T.: Perturbation Theory for Linear Operators. Die Grundlehren der mathematischen Wissenschaften, vol. 132. Springer, New York (1966)
22. Le Rousseau, J., Lebeau, G.: On carleman estimates for elliptic and parabolic operators applications to unique continuation and control of parabolic equations. ESAIM COCV 18(3), 712–747 (2012)
23. Liu, K.: Locally distributed control and damping for the conservative systems. SIAM J. Control Optim. 35(5), 1574–1590 (1997)
24. Liu, K., Liu, Z.: Exponential decay of energy of the Euler–Bernoulli beam with locally distributed kelvin-voigt damping. SIAM J. Control Optim. 36, 1086–1098 (1998)
25. Liu, K., Liu, Z.: Exponential decay of energy of vibrating strings with local viscoelasticity. Zeitschrift für Angewandte Mathematik Und Physik ZAMP 53, 265–280 (2002)
26. Liu, K., Rao, B.: Stabilité exponentielle des équations des ondes avec amortissement local de kelvin-voigt. Comptes Rendus Math. 339(11), 769–774 (2004)
27. Liu, K., Rao, B.: Exponential stability for the wave equations with local kelvin-voigt damping. Zeitschrift für angewandte Mathematik und Physik 57, 419–432 (2006)
28. Liu, Z., Rao, B.: Characterization of polynomial decay rate for the solution of linear evolution equation. Zeitschrift für angewandte Mathematik und Physik 56(4), 630–644 (2005)
29. Liu, Z., Rao, B.: Characterization of polynomial decay rate for the solution of linear evolution equation. Z. Angew. Math. Phys. 56(4), 630–644 (2005)
30. Nicaise, S., Pignotti, C.: Stability of the wave equation with localized kelvin-voigt damping and boundary delay feedback. Discrete Contin. Dyn. Syst. Ser. S 9, 791–813 (2016)
31. Oquendo, H.P., Pacheco, P.S.: Optimal decay for coupled waves with kelvin-voigt damping. Appl. Math. Lett. 67, 16–20 (2017)
32. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, vol. 44. Springer, New York (1983)
33. Prüss, J.: On the spectrum of $C_0$-semigroups. Trans. Am. Math. Soc. 284(2), 847–857 (1984)
34. Robbiano, L., Zhang, Q.: Logarithmic decay of wave equation with kelvin-voigt damping. Mathematics 8(5), 715, 1–19 (2020)
35. Stahn, R.: Optimal decay rate for the wave equation on a square with constant damping on a strip. Zeitschrift für angewandte Mathematik und Physik 68(2), 1–10 (2017)
36. Tebou, L.: A constructive method for the stabilization of the wave equation with localized kelvin-voigt damping. Comptes Rendus Math. 350, 603–608 (2012)
37. Wehbe, A., Issa, I., Akil, M.: Stability results of an elastic/viscoelastic transmission problem of locally coupled waves with non smooth coefficients. Acta Appl. Math. 171(1), 1–46 (2021)
38. Zhang, Q.: Polynomial decay of an elastic/viscoelastic waves interaction system. Zeitschrift für angewandte Mathematik und Physik 69(4), 1–10 (2018)

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