Decision Making, Strategy dynamics, and Crowd Formation in Agent-based models of Competing Populations

K.P. Chan*, Pak Ming Hui*, and Neil F. Johnson+

*Department of Physics, The Chinese University of Hong Kong
Shatin, New Territories, Hong Kong
+ Department of Physics, University of Oxford, Oxford, OX1 3PU, UK

Abstract

The Minority Game (MG) is a basic multi-agent model representing a simplified and binary form of the bar attendance model of Arthur. The model has an informationally efficient phase in which the agents lack the capability of exploiting any information in the winning action time series. We illustrate how a theory can be constructed based on the ranking patterns of the strategies and the number of agents using a particular rank of strategies as the game proceeds. The theory is applied to calculate the distribution or probability density function in the number of agents making a particular decision. From the distribution, the standard deviation in the number of agents making a particular choice (e.g., the bar attendance) can be calculated in the efficient phase as a function of the parameter $m$ specifying the agent’s memory size. Since situations with tied cumulative performance of the strategies often occur in the efficient phase and they are critical in the decision making dynamics, the theory is constructed to take into account the effects of tied strategies. The analytic results are found to be in better agreement with numerical results, when compared with the simplest forms of the crowd-anticrowd theory in which cases of tied strategies are ignored. The theory is also applied to a version of minority game with a networked population in which connected agents may share information.

Paper to be presented in the 10th Annual Workshop on Economic Heterogeneous Interacting Agents (WEHIA 2005), 13-15 June 2005, University of Essex, UK.


I. INTRODUCTION

Agent-based models represent an efficient way in exploring how individual (microscopic) behaviour may affect the global (macroscopic) behaviour in a competing population. This theme of relating macroscopic to microscopic behaviour has been the focus of many studies in physical systems, e.g., macroscopic magnetic properties of a material stem from the local microscopic interactions of magnetic moments between atoms making up of the material. In recent years, physicists have constructed interesting models for non-traditional systems and established new branches in physics such as econophysics and sociophysics. The Minority Game (MG) proposed by Challet and Zhang [1, 2] and the Binary-Agent-Resource (B-A-R) model proposed by Johnson and Hui [3-7], for example, represent a typical physicists’ binary abstraction of the bar attendance problem proposed by Arthur [8, 9]. In MG, agents repeatedly compete to be in a minority group. The agents have similar capabilities, but are heterogeneous in that they use different strategies in making decisions. Decisions are made based on the cumulative performance of the strategies that an agent holds. The performance is a record of the correctness of the predictions of a strategy on the winning action which, in turn, is related to the collective behaviour of the agents. Thus, the agents interact through their decision-making process, creation of the record of winning actions, and strategy selection process. Interesting quantities for investigations include the statistics of the fraction of agents making a particular choice $A(t)$ every time step and the variance or standard deviation (SD) $\sigma$ of this number [1, 4]. These quantities are related in that knowing the distribution of $A$, one may obtain $\sigma$. The MG, suitably modified, can be used to model financial markets and reproduce stylized facts. The variance, for example, is a quantity related to the volatility in markets [4].

Recently, we proposed a theory of agent-based models based on the consideration of decision-making and strategy dynamics [10]. The importance of the strategy selection dynamics has been pointed out by D’Hulst and Rodgers [11]. This approach [10, 11], which we refer to as the strategy-ranking theory (SRT), emphasizes on how the strategies performance ranking pattern changes as the game proceeds and the number of agents using a strategy in a certain rank for making decisions. It is recognized that the SRT has the advantages of including tied strategies into consideration and avoiding the troublesome in considering each strategy’s performance separately. The theory, thus, represents a generalization of the
crowd-anticrowd theory to cases with tied strategies and strategy ranking evolutions – two factors that are particularly important in the so-called informationally efficient phase of the MG. The theory has been applied successfully to explain non-trivial features in the mean success rate of the agents in (i) MG with a population of non-networked or networked agents, (ii) MG with some randomly participating agents, and (iii) B-A-R model with a tunable resource level. In this conference paper, we aim to illustrate the basic ideas of SRT. In particular, we present results based on SRT in evaluating the distribution of $A(t)$ and $\sigma$, in the efficient phase of MG in non-networked and networked populations. Validity of the results of our theory is tested against results obtained by numerical simulations. While the SRT was developed within the context of MG, many of the ideas are should also be appliable to a wide range of agent-based models.

II. MODEL: THE MINORITY GAME

The basic MG comprises of $N$ agents competing to be in a minority group at each time step. The only information available to the agents is the history. The history is a bit-string of length $m$ recording the minority (i.e., winning) option for the most recent $m$ time steps. There are a total of $2^m$ possible history bit-strings. For example, $m = 2$ has $2^2 = 4$ possible histories of the winning outcomes: 00, 01, 10 and 11. At the beginning of the game, each agent picks $s$ strategies, with repetition allowed. They make their decisions based on their strategies. A strategy is a look up table with $2^m$ entries giving the predictions for all possible history bit-strings. Since each entry can either be ‘0’ or ‘1’, the full strategy pool contains $2^{2^m}$ strategies. Adaptation is built in by allowing the agents to accumulate a merit (virtual) point for each of her $s$ strategies as the game proceeds, with the initial merit points set to zero for all strategies. Strategies that predicted the winning (losing) action at a given time step, are assigned (deducted) one virtual point. At each turn, the agent follows the prediction of her best-scoring strategy. In case of tied best-scoring strategies, a random choice will be made to break the tie.

In the present work, we will focus on the regime where $2 \cdot 2^m \ll N \cdot s$, i.e., the efficient phase. In MG literature, a parameter $\alpha = 2^m / N$ is defined with $\alpha < \alpha_c \approx 0.34$ characterizing the efficient phase. Features in this regime is known to be dominated by the crowd effect. A quantitative theory in this regime would have to include the consideration of
frequently occurred tied strategies into account, as the dynamics in this regime is highly sensitive to the agents’ strategy selection. In what follows, we introduce the basic physical picture of the strategy ranking theory and apply it to evaluate the distribution in the fraction of agents making a particular choice $P(A)$ and the variance $\sigma^2$ from an analytic expression for non-networked and networked populations.

III. NUMERICAL AND ANALYTICAL RESULTS: NON-NETWORKED AGENTS

To put our discussions into proper context, we will first present the numerical results of the quantities that we are focusing on. Let $A(t)$ be the fraction of agents taking the action “1” (or “0”) at time step $t$. As the game proceeds, there will be a time series $A(t)$. We may then analyze these values of $A(t)$ by considering the distribution or probability density function $P(A)$, where $P(A)dA$ is the probability of having a value within the interval $A$ to $A + dA$. In using the MG for market modelling, $A(t)$ can be taken to be the fraction of agents deciding to buy (or sell) an asset at time $t$. In the context of the El Farol bar attendance problem \cite{8, 9}, $A(t)$ may be taken to be the fraction of agents attending the bar. Note that every realization of the MG may have a different distribution of strategies among the agents and a different initial bit-string to start the game. These details do not affect the main results reported here, especially when we consider cases deep into the efficient phase, i.e., when $2 \cdot 2^m \ll N$. To illustrate the point, we have carried out detailed numerical simulations for the simplest case of $m = 1$ and $s = 2$. Figure 1 shows the numerical results (squares) of $P(A)$ for systems with two different sizes ($N = 129$ and $N = 4097$), with the aim of emphasizing the size effect on $P(A)$. Notice that the distribution consists of a few peaks (five peaks for the case of $m = 1$ and $s = 2$), indicating that as the game proceeds the number $A(t)$ jumps among values characterized by these peak values. For larger population, the peaks are sharper. Also shown in Fig.1 are the results of the strategy ranking theory (lines). The theoretical results are in reasonably agreement with numerical results. We defer the discussion on obtaining the theoretical results to the next section.

Besides the typical results shown in Fig.1, we have studied the variance $\sigma^2$ in the following way. We carried out numerical simulations in many realizations using different values of $m$ and $N$, with $N$ up to 8193 and $m$ up to 8. For each run, a value of $\sigma^2$ is obtained. To
FIG. 1: The probability density function $P(A)$ of the fraction of agents making a particular decision for a typical run with $s = 2$ and $m = 1$: (a) $N=129$ and (b) $N=4097$. The symbols are results obtained by numerical simulations. The lines give the results of the strategy ranking theory. The peak around $A = 0.5$ comes from the time steps that we have classified as even time steps in the theory. The peaks on the two sides come from the odd time steps, where there are registered virtual points in the strategies for the history bit-string concerned. For a system of larger size, the peaks are sharper.

Facilitate comparison with theory, we select those data that are deep in the efficient phase, i.e., with $2 \cdot 2^m / N < 0.125$ and plotted them (black dots) in Fig. 2 to show the dependence of $\sigma^2 / N^2$ on $m$. The data points do not show significant scatter, and essentially fall on a line. Also included in the figure are two (dashed) lines corresponding to two approximations within the crowd-anticrowd theory $[6, 12, 13]$. These approximations assume that all the strategies can be ranked at every time step without tied virtual points. One of them assumes that the popularity rankings, i.e., ranking based on the number of agents using a strategy, of a strategy and its anti-correlated partner are uncorrelated and gives an expression for $\sigma^2 / N^2$ for cases with $s = 2$ as $[12]

$$
\frac{\sigma_{flat}^2}{N^2} = \frac{1}{24 \times 2^m} \left[ \frac{1}{2(m+1)^2} \right].
$$

(1)

Another approximation is that the ranking of strategies are highly correlated. For example, the anti-correlated partner of the momentarily most-popular strategy is the least-popular
FIG. 2: The variance $\sigma^2/N^2$ as a function of $m$, $N$ ranging from 129 to 8193. Only data points satisfying $2 \cdot 2^m/N < 1/8$ are shown. The dashed lines give the two approximations within the crowd-anticrowd theory. The open squares give the results, which are in good agreement with numerical results in the small $m$ regime, of the strategy ranking theory.

one, and so on. This leads to another expression within the crowd-anticrowd theory \cite{12}:

$$\frac{\sigma^2_{\text{delta}}}{N^2} = \frac{1}{12} \times 2^m \left[ 1 - \left( \frac{1}{2^m+1} \right)^2 \right].$$

(2)

We note that for small values of $m$, the numerical data fall within the two crowd-anticrowd approximations, with neither of the approximations capturing the $m$-dependence of $\sigma^2/N^2$. As will be discussed later, the strategy ranking theory gives an analytic expression for $\sigma^2/N^2$ that captures the $m$-dependence very well in the small $m$ regime where the criteria $2 \cdot 2^m/N \ll 1$ is satisfied to a fuller extent.

IV. STRATEGY RANKING THEORY: KEY IDEAS

We proceed to discuss how we could obtain the analytic results shown in Figs. 1 and 2, within the strategy ranking theory. Details of the theory can be found in \cite{10,11,15}. Here we briefly summarize the key ideas, with the aim to make the theory physically transparent. We note that in MG and other agent-based models of competing populations, it is the interplay between decision-making, strategy selections, and collective response that leads to the non-trivial and often interesting global behaviour of a system. With this in mind,
the strategy performance ranking pattern is of crucial importance. At any time step, the strategies can be classified into $\kappa + 1$ ranks, according to the virtual points of the strategies. The momentarily best-performing strategy (or strategies) belongs (belong) to rank-1, and so on. At the beginning of the game, all strategies are tied that thus they all belong to the same rank. This is also the case when the strategies are all tied during the game. Thus, the lower bound of $\kappa$ is zero. It is also noted that there are two different kinds of behaviour in the ranking pattern after a time step: (i) the number of different ranks increases and such a time step is called an “even” time step, and (ii) the number of different ranks decreases and such a time step is called an “odd” time step. Take, for example, a time step at which the strategies are all tied before decision. Regardless of the history based on which the agents decide and the winning outcome after the agents decided, the strategies split into two ranks, i.e., $\kappa$ increases from 0 to 1 after the time step. Half of the strategies belong to the better rank and half to the worse rank, as half of the strategies would have predicted the correct winning outcome for the history concerned. Generally speaking, the underlying mechanism for this splitting is that there is no registered virtual point or stored information in the strategies for the history concerned. We call this kind of time steps “even” time steps because this is what would happen when the population encounters a history for decision that had occurred an even number of times since the beginning of the game, not counting the one that is currently in use for decisions.

The parameter $\kappa$ has another physical meaning. It is the number of history bit-strings that have occurred an odd number of times since the beginning of the game, regardless the current history in use for decisions. Since there are at most $2^m$ history bit-strings for a given $m$, the upper bound of $\kappa$ is $2^m$. Thus we have $0 \leq \kappa \leq 2^m$. Therefore, every time step as the game proceeds can be classified as “even” or “odd”, together with a parameter $\kappa$. For $\kappa = 0$ when all the strategies are tied, the time step is necessarily an even time step. For $\kappa = 2^m$ where there are $2^m + 1$ ranks, the time step is necessarily an odd time step since all the histories have occurred an odd number of times, including the current history in use for decisions. Noting that the total number of strategies is $2^{2^m}$, there are in general several strategies in a certain ranking. In this way, the theory takes explicit account of cases of tied strategies.

For even time steps (regardless of the value of $\kappa$), there is no registered virtual points in the strategies for the current history. Therefore, even time steps are characterized by agents
making random decisions\textsuperscript{10,11,14,15}. Using a random walk argument, the distribution $P_{\text{even},\kappa}(A) = P_{\text{even}}(A)$ is a normal distribution independent of $\kappa$, with a mean $\mu_{\text{even}} = 0.5$ and a variance $\sigma^2_{\text{even}} = 1/(4N)$, i.e.,

$$P_{\text{even}}(A) = \frac{1}{\sqrt{2\pi}\sigma_{\text{even}}} \exp\left(-\frac{(A - \mu_{\text{even}})^2}{2\sigma^2_{\text{even}}}\right).$$

(3)

It turns out that the part of the distribution around $A = 0.5$ shown in Fig.1 originates from the even time steps.

For odd time steps, there are registered virtual points or stored information in the strategies for the current history. This is the origin of the crowd effect\textsuperscript{6,12,13}, which is fundamental to the understanding of collective response in the class of agent-based models based on MG. In this case, the momentarily better performing strategies have predicted the correct action in the last occurrence of the current history in use for decision. There will then be more agents using these better-performing strategies for decisions. However, the number is too large, hence forming a crowd, that the winning action in the last occurrence becomes the losing action in this turn. This is the anti-persistent nature or double periodicity of MG\textsuperscript{16,17,18,19,20,21}. Using the strategy ranking theory, we know that there are $(\kappa + 1)$ ranks among the strategies for time steps labelled $\kappa$. The ratio of the fractions of strategies in different ranks is given by\textsuperscript{10} $C^\kappa_0 : C^\kappa_1 : \cdots : C^\kappa_\ell : \cdots : C^\kappa_\kappa$, which are simply the numbers in the Pascal triangles. Given that the agents use their best-performing strategy for decision, we can readily count the number of agents using a strategy in a particular rank. As mentioned, the better-performing strategies are more likely to lead to wrong predictions at odd time steps. This can be modelled by a winning probability at odd time steps of the form of $(\ell - 1)/\kappa$ for a strategy belonging to rank-$\ell$, for a given value of $\kappa$\textsuperscript{10,11}. Putting the information together, we arrive at the probability density function $P_{\text{odd},\kappa}(A)$ for $1 \leq \kappa \leq 2^m$.

The distribution $P_{\text{odd},\kappa}(A)$ is given by normal distributions centered at the mean values of

$$\mu_{\text{odd},\kappa}^\pm = 0.5 \pm \frac{C^\kappa_{\kappa-1}}{2\kappa},$$

(4)

with a variance

$$\sigma^2_{\text{odd},\kappa} = \frac{C^\kappa_{\kappa-2}}{2^{2\kappa-1}} \frac{1}{4N}.$$  

(5)

Applying Eq. (4) to the results for $m = 1$ in Fig.1, we immediately identify that the peaks in $P(A)$ at $A = 1/4$ and $A = 3/4$ are originated from odd time steps corresponding to $\kappa = 1$ and the peaks at $A = 0.6875$ and $0.3125$ are originated from odd time steps corresponding.
to $\kappa = 2$. These peaks are more noticeable in Fig. 1(b) when the population size is large. In Eq. (5), the binomial coefficients should formally be expressed in terms of Gamma functions, so that when the lower index in the coefficient becomes negative, $\sigma_{\text{odd},\kappa}$ vanishes. This is the case for $\kappa = 1$, and the corresponding distribution will then be very sharp. This is, for example, the case for the sharp peaks at $A = 1/4$ and $A = 3/4$ in Fig. (1).

To obtain an expression for the overall $P(A)$, including both even and odd time steps and all possible values of $\kappa$, we need to take a weighted average over the occurrence of odd and even time steps [10]. The resulting expression is

$$P(A) = \sum_{\kappa=0}^{2^m} \frac{C_{2^m}^{2m}}{2^m} \{\left(\frac{\kappa}{2^m}\right)P_{\text{odd},\kappa}(A) + (1 - \frac{\kappa}{2^m})P_{\text{even}}(A)\},$$

(6)

where the factor $\frac{C_{2^m}^{2m}}{2^m}$ is the probability of having $\kappa$ history bit-strings occurred an odd number of times. The factor $\frac{\kappa}{2^m}$ is the probability that given $\kappa$, the time step is odd. Applying Eq. (6) to the case of $m = 1$, we obtain the results (lines) shown in Fig. 1. We note that the expression in Eq. (6) is also applicable to $m > 1$, as long as the efficient phase criteria is satisfied.

The calculation of the variance follows from the definition

$$\sigma^2 = N^2\langle(A - \overline{A})^2\rangle_t,$$

(7)

where $\overline{A} = 0.5$ is the mean value of $A$ and the average $\langle\cdots\rangle_t$ represents a time average. Replacing the time average by invoking the probability density function $P(A)$, we have

$$\frac{\sigma^2}{N^2} = \int_0^1 (A - 0.5)^2 P(A)dA$$

$$= \sum_{\kappa=0}^{2^m} \frac{C_{2^m}^{2m}}{2^m} \left\{\left(\frac{\kappa}{2^m}\right)\left[\frac{1}{2}(0.5 - \mu_{\text{odd},\kappa})^2 + \frac{1}{2}(0.5 - \mu_{\text{odd},\kappa})^2 + \sigma_{\text{odd},\kappa}^2\right] + (1 - \frac{\kappa}{2^m})\sigma_{\text{even}}^2\right\}$$

(8)

$$\approx \sum_{\kappa=0}^{2^m} \frac{C_{2^m}^{2m}}{2^m} \left(\frac{\kappa}{2^m}\right)\left(\frac{C_{2^m-1}^{2^m-1}}{2^m}\right)^2$$

$$= \sum_{\kappa=0}^{2^m} \frac{C_{2^m}^{2m}}{2^m} \left(\frac{\kappa}{2^m}\right)\left(\frac{1}{2} \prod_{q=1}^{\kappa} (1 - \frac{1}{2q})\right)^2,$$

(9)

where the approximation is valid for $2.2^m/N << 1$. Eq. (9) is an analytic expression for $\sigma^2$. The last two expressions are equivalent and one may use whichever convenient in obtaining numerical values from Eq. (9).

Several remarks are worth mentioning. Firstly, we note that the expression of $\sigma^2$ is closely related to the analytic expression for the winning probability reported in [10], from which
an alternative approach arriving at the same result is possible [22]. Secondly, the results from Eq. (9) are plotted (open squares) in Fig. 2. We note that the strategy ranking theory does capture the $m$-dependence of $\sigma^2/N^2$, with good agreement with numerical simulation results in the range where the criteria $2 \cdot 2^m/N \ll 1$ is better fulfilled [11]. The success of the theory stems from the inclusion of tied strategies, as each rank typically consists of a number of strategies. In the simplest case of $m = 1$, for example, there are tied strategies in every time step of the game. The better agreement with numerical results when compared with the crowd-anticrowd approximations is thus an indication of the importance of (i) the tied strategies and (ii) the time evolution of the ranking pattern from time step to time step. In MG, both the number of tied strategies, i.e., number of strategies belonging to the same rank, and the time evolution of strategy ranking pattern can be readily found.

Thirdly, the result Eq. (9) is interesting in that there have been much effort in trying to re-scale numerical results of $\sigma^2$ as a function of the parameter $\alpha = 2^m/N$ so that results from systems of different values of $N$ and $m$ can be collapsed onto a single curve. Eq. (9) suggests that $\sigma^2/N^2$ is a complicated function of $m$, deep in the efficient phase. In particular, as one increases the population size at fixed and small $m$, one should approach the result given by Eq. (9) assuming a uniform initial distribution of strategies to the agents. It is, in fact, possible to include the effects of a finite population size $N$ into the strategy-ranking theory starting from Eq. (8) by incorporating the so-called market impact effects [11, 14, 22].

V. NETWORKED AGENTS

Systems in the real world are characterized by connected agents [23]. The connections are often used for collecting information from the neighbours. Recently, several interesting attempts [10, 14, 24, 25, 26] have been made to incorporate information sharing mechanisms among the agents into MG and B-A-R models. As an illustration of the application of SRT to networked MG, we focus on the model proposed by Anghel et al. [14, 24]. As in the MG, Anghel et al.’s model [24] features $N$ agents who repeatedly compete to be in a minority group. Communications between agents are introduced by assuming that the agents are connected by an undirected random network, i.e., classical random graph, with a connectivity $p$ being the probability that a link between two randomly chosen agents exists. The links are used as follows. Each agent compares the cumulated performance
FIG. 3: The probability density function $P(A)$ of the fraction of agents making a particular decision for a typical run with $s = 2$, $m = 1$ and $N = 1001$ in the networked model of Anghel et al. [24] for two different values of the connectivity $p = 0.01$ (upper panel) and $p = 0.02$ (lower panel). The symbols are results obtained by numerical simulations. The lines give the results of the strategy ranking theory.

of his predictor, which is the suggested action from his own best-performing strategy at each time step, with that of his neighbours, and then follows the suggested action of the best performing predictor among his neighbours and himself. The $p = 0$ limit of the model reduces to the MG. Note that the identity of the best-performing strategy changes over time. For $p > 0$ the predictor’s performance is generally different from the agent’s performance. It has been reported that the efficiency of the population as a whole, characterized by either $\sigma^2$ [24] or by the average winning probability per agent per turn [14], shows a non-monotonic dependence on the connectivity $p$ with the most efficient performance occurring at a small but finite value of $p$. In other words, a small fraction of links is beneficial but too many of them are bad. We have explained the feature successfully within the framework of SRT [14]. The most important point is that, from our understanding of the non-networked MG (e.g., see Fig. 1), the performance of an agent actually depends on how similar the $s = 2$ strategies that he is holding, with the best performing ones holding two identical strategies. The links then act in two ways depending on the connectivity. For low connectivity, the links bring
the agents with two anti-correlated strategies to have the chance to use other strategies so that these agents will not always join the crowd at odd time steps and hence with their winning probability enhanced. For high connectivity, there are so many links that many agents are linked to the momentarily best-performing predictor or predictors. As discussed in previous section, the higher ranking strategies have a smaller chance of predicting the correct minority outcome. When the connectivity is high, there are many links so that agents have access to strategies that are more likely to lose. This leads to a drop in the average winning probability of the agents [14].

Figure 3 shows how the distribution $P(A)$ changes with the connectivity $p$ at two small values of $p$. The range of small $p$ is particularly of interest since for a large population ($N = 1001$) the non-monotonic feature occurs for $p < 0.01$. The symbols (open circles) give the results from numerical simulations. The peaks of the distribution $P(A)$ shifts as $p$ is varied. Applying SRT and incorporating the effects of the presence of links, we found that $P(A)$ can again be represented by a weighted sum of distributions characterized by different kinds of time steps. In particular, for $p = 0.01$, the parameters of the distributions in Eq. (4) can be found [14, 22] to be $\mu_{\text{odd},k=1} = 0.207$, $\mu_{\text{odd},k=2} = 0.278$, and $\mu_{\text{odd},k=1,2}^+ = 1 - \mu_{\text{odd},k=1,2}^-$. The variances are given by Eq. (5) as $\sigma_{\text{odd},k=1}^2 = 0$ and $\sigma_{\text{odd},k=2}^2 = 1/32N$. Similarly for $p = 0.02$, we have $\mu_{\text{odd},k=1} = 0.075$ and $\mu_{\text{odd},k=2} = 0.160$, with the same variances. The values of these parameters are obtained by considering the different winning probabilities of the strategies in different ranks and the change in the number of agents using a strategy of a certain rank due to the presence of the links. The solid lines in Fig. 3 show the distributions obtained by SRT. The theory captures the shifts in $P(A)$ with the connectivity $p$.

VI. SUMMARY

In the present work, we illustrated the basic ideas in constructing a strategy ranking theory for a class of multi-agent models incorporating the effects of tied strategies and strategy selections. We showed how the theory can be applied to MG in the efficient phase to evaluate the distribution $P(A)$ in the fraction of agents making a particular decision and the associated variance $\sigma^2$. In particular, an analytic expression is given for $\sigma^2$ in a non-networked population. The theory is also applied to a version of networked MG in which there exists non-trivial dependence on the performance of the agents as a function.
of the connectivity. Besides $P(A)$ and $\sigma^2$, the theory can also be applied to evaluate other quantities such as the average winning probability of the agents. In closing, while SRT is developed with models based on the MG in mind, the general approach, namely that of focusing on the ranking pattern of the strategies and how the pattern evolves in time, should be a key ingredient in the construction of theories for a large class of agent-based models.

Acknowledgments

Work at CUHK was supported in part by a Grant from the Research Grants Council of the Hong Kong SAR Government. K.P. Chan acknowledges the support of a conference grant from the Graduate School at CUHK for attending WEHIA 2005.

[1] D. Challet and Y.C. Zhang, Physica A 246, 407(1997); 256, 514(1998).
[2] D. Challet, M. Marsili and Y.C. Zhang, Minority Games and Beyond (Oxford University Press, Oxford, 2004).
[3] N.F. Johnson, P.M. Hui, Dafang Zheng, and C.W. Tai, Physica A 269 (2-4): 493-502 (1999).
[4] N.F. Johnson, P. Jefferies, P.M. Hui, Financial Market Complexity (Oxford University Press, 2003).
[5] N.F. Johnson, S.C. Choe, S. Gourley, T. Jarret, and P.M. Hui, in Advances in Solid State Physics 44, edited by B. Kramer (Springer-Verlag, Heidelberg, 2004), p.427.
[6] N.F. Johnson and P.M. Hui, e-print cond-mat/0306516 at http://arxiv.org/.
[7] H.Y. Chan, T.S. Lo, P.M. Hui, and N.F. Johnson, e-print cond-mat/0408557 at http://arxiv.org/.
[8] B. Arthur Amer. Econ. Rev. 84, 406 (1994); Science 284, 107 (1999).
[9] N.F. Johnson, S. Jarvis, R. Jonson, P. Cheung, Y. Kwong and P.M. Hui, Physica A 258, 230 (1998).
[10] T.S. Lo, H.Y. Chan, P.M. Hui, and N.F. Johnson, Phys. Rev. E 70, 056102 (2004).
[11] R. D’Hulst and G.J. Rodgers, Physica A 278, 579 (2000).
[12] N.F. Johnson, M. Hart and P.M. Hui, Physica A 269, 1 (1999).
[13] M. Hart, P. Jefferies, N.F. Johnson and P.M. Hui, Physica A 298, 537 (2001).
[14] T.S. Lo, K.P. Chan, P.M. Hui, and N.F. Johnson, Phys. Rev. E 71, 050101(R) (2005).
[15] K.F. Yip, T.S. Lo, P.M. Hui, and N.F. Johnson, Phys. Rev. E 69, 046120 (2004).
[16] D. Challet and M. Marsili, Phys. Rev. E 60, R6271 (1999).
[17] D. Challet, M. Marsili, and R. Zecchina, Phys. Rev. Lett. 85, 5008 (2000).
[18] M. Marsili, D. Challet, and R. Zecchina, Physica A 280, 522 (2000).
[19] R. Savit, R. Manuca, and R. Riolo, Phys. Rev. Lett. 82, 2203 (1999).
[20] P. Jefferies, M.L. Hart, and N.F. Johnson, Phys. Rev. E 65, 016105 (2002).
[21] D. Zheng and B.H. Wang, Physica A 301, 560 (2001).
[22] K.P. Chan, MPhil Dissertation, The Chinese University of Hong Kong (August 2005, unpublished).
[23] S.N. Dorogovtsev and J.F.F. Mendes, Evolution of Networks: From Biological Nets to the Internet and WWW (Oxford University Press, Oxford, 2002).
[24] M. Anghel, Z. Toroczkai, K.E. Bassler, G. Kroniss, Phys. Rev. Lett. 92, 058701 (2004).
[25] S. Gourley, S.C. Choe, P.M. Hui, and N.F. Johnson, Europhys. Lett. 67, 867 (2004).
[26] S.C. Choe, N.F. Johnson, and P.M. Hui, Phys. Rev. E 70, 055101(R) (2004).