Refined algebraic quantisation with the triangular subgroup of SL(2, \mathbb{R})

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Abstract

We investigate refined algebraic quantisation with group averaging in a constrained Hamiltonian system whose gauge group is the connected component of the lower triangular subgroup of SL(2, \mathbb{R}). The unreduced phase space is $T^*\mathbb{R}^{p+q}$ with $p \geq 1$ and $q \geq 1$, and the system has a distinguished classical $\mathfrak{o}(p, q)$ observable algebra. Group averaging with the geometric average of the right and left invariant measures, invariant under the group inverse, yields a Hilbert space that carries a maximally degenerate principal unitary series representation of $O(p, q)$. The representation is nontrivial iff $(p, q) \neq (1, 1)$, which is also the condition for the classical reduced phase space to be a symplectic manifold up to a singular subset of measure zero. We present a detailed comparison to an algebraic quantisation that imposes the constraints in the sense $\hat{H}_a \Psi = 0$ and postulates self-adjointness of the $\mathfrak{o}(p, q)$ observables. Under certain technical assumptions that parallel those of the group averaging theory, this algebraic quantisation gives no quantum theory when $(p, q) = (1, 2)$ or $(2, 1)$, or when $p \geq 2$, $q \geq 2$ and $p + q \equiv 1 \pmod{2}$.

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1 Introduction

In quantisation of constrained systems, an elegant idea for constructing the physical Hilbert space is to average states in an auxiliary Hilbert space $\mathcal{H}_{aux}$ over a suitable action of the gauge group $[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]$. For a noncompact gauge group, the averaging need not converge on all of $\mathcal{H}_{aux}$, but when the averaging is formulated within refined algebraic quantisation $[4, 8, 12]$, convergence on a suitable linear subspace will suffice, and such convergence has been found to occur in concrete examples $[9, 10, 14]$. Results on the equivalence of refined algebraic quantisation with other methods $[13, 15]$ further show that group averaging provides considerable control over the quantisation.

In this paper we study group averaging in a system with a nonunimodular gauge group. The interest of this situation arises from the rather different senses in which group averaging satisfies the Dirac constraints for unimodular and nonunimodular groups $[8]$. To guarantee that the would-be inner product provided by group averaging is real, the averaging measure needs to be invariant under the group inverse. For a unimodular group, the left and right invariant Haar measure has this property. For a nonunimodular group, the left and right invariant Haar measures do not coincide, and neither is invariant under the group inverse, but their geometric average $d_0 g$ is. Suppose $\hat{H}_a$ are the constraint operators that generate the unitary gauge group action on the auxiliary Hilbert space, with the commutators

$$[\hat{H}_a, \hat{H}_b] = iC_{c}^{ab}\hat{H}_c ,$$

where $C_{c}^{ab}$ are the structure constants of the Lie algebra. As was shown in $[8]$ and will be reviewed in section 5 below, group averaging with $d_0 g$ gives physical states $\Psi$ that satisfy

$$\hat{H}_a \Psi = -\frac{i}{2}C_{c}^{ab}\hat{H}_c \Psi ,$$

which agrees with the naïve Dirac quantum constraints, $\hat{H}_a \Psi = 0$, if and only if the group is unimodular. The term on the right-hand side of (1.2) leads to no known inconsistencies: For systems amenable to geometric quantisation in both reduced and unreduced phase space, (1.2) is in fact the form of Dirac constraints equivalent to reduced phase space quantisation $[16, 17, 18]$. Related observations for first class constrained systems with one constraint quadratic in the momenta and several constraints linear in the momenta were made in $[19]$.

We shall consider a system obtained by replacing a unimodular gauge group $G_u$ by its nonunimodular subgroup $G$. The effect of this replacement on the constraints $\hat{H}_a \Psi = 0$ of $G_u$ is not just that some of the constraint equations are dropped: In the constraints that remain, (1.2) shows that new terms appear on the right-hand side. An observable algebra that commutes strongly with the old constraints is still represented on the solution space to the new constraints, but the new representation need not be isomorphic to the old representation. In group averaging, where no observables need
be explicitly constructed, these changes are encoded in the integration measures on $G_u$ and $G$. The integrals over $G_u$ and $G$ may also have differing convergence properties, and it may hence be necessary to choose the test spaces differently even when $H_{\text{aux}}$ is unchanged and the representation of $G$ is obtained from that of $G_u$ by restriction. Our system will indeed exemplify all these phenomena.

Our unimodular gauge system [14] has gauge group $G_u \simeq \text{SL}(2, \mathbb{R})$, unreduced phase space $T^* \mathbb{R}^{p+q} \simeq \mathbb{R}^{2(p+q)}$ with $p \geq 1$ and $q \geq 1$ and a distinguished classical $\mathfrak{o}(p,q)$ observable algebra. The reduced phase space is a symplectic manifold up to a singular subset of measure zero if and only if $p \geq 2$ and $q \geq 2$: This manifold has dimension $2(p+q-3)$ and is separated by the $\mathfrak{o}(p,q)$ observables. The system was quantised with group averaging in [10, 14], recovering a quantum theory with a nontrivial representation of the quantum $\mathfrak{o}(p,q)$ observables when $p \geq 2$, $q \geq 2$ and $p + q \equiv 0 \pmod{2}$. The quadratic $\mathfrak{o}(p,q)$ Casimir was found to take the value $-\frac{1}{4}(p+q)(p+q-4)$. Quantisations of this system for special values of $p$ and $q$ by a variety of other methods can be found in [10, 14, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

Our nonunimodular gauge group $G \subset G_u$ is the connected component of the lower triangular subgroup of $\text{SL}(2, \mathbb{R})$. $G$ is two-dimensional and nonabelian, and hence isomorphic to every two-dimensional connected nonabelian group. The reduced phase space is a symplectic manifold up to a singular subset of measure zero if and only if $(p,q) \neq (1,1)$: This manifold is symplectomorphic to $T^*(S^{p-1} \times S^{q-1})$, and it is separated by the $\mathfrak{o}(p,q)$ observables up to a set of measure zero. Quantisation with group averaging gives a quantum theory with a nontrivial representation of the quantum $\mathfrak{o}(p,q)$ observables for all $(p,q) \neq (1,1)$. This representation is the end-point of the maximally degenerate principal unitary series [32], and the quadratic Casimir takes the value $-\frac{1}{4}(p + q - 2)^2$.

For comparison, we quantise the system also with the constraints $\hat{H}_a \Psi = 0$, adopting the algebraic quantisation framework of [33, 34] and requiring the $\mathfrak{o}(p,q)$ observables to become self-adjoint operators. For $(p,q) = (1,3), (3,1)$ and $(2,2)$, the group averaging theory and the algebraic quantisation theory are qualitatively fairly similar, with minor differences in the spectra of certain operators. Our algebraic quantisation results for general $(p,q)$ remain incomplete, but we show that under certain technical assumptions that parallel those of the group averaging theory, the algebraic quantisation gives no quantum theory for $(p,q) = (1,2)$ and $(2,1)$, or for $p \geq 2$, $q \geq 2$ and $p + q \equiv 1 \pmod{2}$.

The rest of the paper is as follows. Section 2 introduces and analyses the classical system. Sections 3 and 4 discuss algebraic quantisation, with respect to the constraints $1.2$ and $\hat{H}_a \Psi = 0$. Refined algebraic quantisation with group averaging is briefly reviewed in section 5 and carried out in section 6 for $(p,q) \neq (1,1)$ and in section 7 for $p = q = 1$. Section 8 presents a summary and concluding remarks.

Appendix A collects some basic properties of the group $G$. The separation properties of the $\mathfrak{o}(p,q)$ observables on the reduced phase space are verified in appendix B and certain technical results for refined algebraic quantisation with $(p,q) = (1,1)$ are proved in appendix C.
2 Classical system

In this section we introduce a classical constrained system with the unreduced phase space \( T^* \mathbb{R}^{p+q} \), where \( p \geq 1 \) and \( q \geq 1 \). The system is obtained from the SL(2, \( \mathbb{R} \)) system of [14] by dropping the constraint \( H_1 \) therein and relabelling one of the canonical pairs by \((v, \pi) = (\varpi, w)\). The gauge transformations and the \( \mathfrak{o}(p, q) \) observables are therefore obtained directly from [14]. We shall however see that the structure of the reduced phase space differs markedly from that in [14].

2.1 The system

The system is defined by the action

\[
S = \int dt \left( \mathbf{p} \cdot \dot{\mathbf{u}} + \varpi \cdot \dot{w} - MH - \lambda D \right),
\]

where \( \mathbf{u} \) and \( \mathbf{p} \) are real vectors of dimension \( p \geq 1 \), \( \mathbf{w} \) and \( \varpi \) are real vectors of dimension \( q \geq 1 \) and the overdot denotes differentiation with respect to \( t \). The symplectic structure is

\[
\Omega = \sum_{i=1}^{p} dp_i \wedge du_i + \sum_{j=1}^{q} d\varpi_j \wedge dw_j,
\]

and the phase space is \( \Gamma := T^* \mathbb{R}^{p+q} \simeq \mathbb{R}^{2(p+q)} \). \( M \) and \( \lambda \) are Lagrange multipliers associated with the constraints

\[
H := \frac{1}{2}(w^2 - u^2),
D := \mathbf{u} \cdot \mathbf{p} + \mathbf{w} \cdot \varpi.
\]

The Poisson algebra of the constraints is

\[
\{H, D\} = 2H,
\]

and the system is a first class constrained system [35] [36]. The finite gauge transformations on \( \Gamma \) generated by the constraints are

\[
\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} \mapsto g \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{w} \\ \varpi \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \varpi \end{pmatrix},
\]

where \( g \in \text{SL}(2, \mathbb{R}) \) is a lower diagonal matrix with positive diagonal elements. Relevant properties of the gauge group \( G \) and its Lie algebra are collected in appendix A.

2.2 Classical observables

Recall that an observable is a function on \( \Gamma \) whose Poisson brackets with the first class constraints vanish when the first class constraints hold [36]. As discussed in [14], the
system has the observables
\[
A_{ij} := u_i p_j - u_j p_i , \quad 1 \leq i \leq p, \quad 1 \leq j \leq p ;
\]
\[
B_{ij} := w_i \varpi_j - w_j \varpi_i , \quad 1 \leq i \leq q, \quad 1 \leq j \leq q ;
\]
\[
C_{ij} := - u_i \varpi_j - p_i w_j , \quad 1 \leq i \leq p, \quad 1 \leq j \leq q ,
\]
whose Poisson brackets close as the \( \mathfrak{o}(p, q) \) Lie algebra, \( \{ A_{ij} \} \) and \( \{ B_{ij} \} \) spanning respectively the \( \mathfrak{o}(p) \) and \( \mathfrak{o}(q) \) Lie subalgebras. We denote the algebra generated by the observables \( \text{(2.6)} \) by \( \mathcal{A}_{\text{class}} \). The finite transformations that \( \mathcal{A}_{\text{class}} \) generates on \( \Gamma \) are
\[
\begin{pmatrix} u \\ w \end{pmatrix} \mapsto R \begin{pmatrix} u \\ w \end{pmatrix} , \quad \begin{pmatrix} p \\ \varpi \end{pmatrix} \mapsto (R^T)^{-1} \begin{pmatrix} p \\ \varpi \end{pmatrix} ,
\]
where \( R \) is a matrix in the connected component \( \text{O}_c(p, q) \) of \( \text{O}(p, q) \).

The quadratic Casimir element in \( \mathcal{A}_{\text{class}} \) is
\[
\mathcal{C} := \sum_{i<j} (A_{ij})^2 + \sum_{i<j} (B_{ij})^2 - \sum_{i,j} (C_{ij})^2 .
\]

When the constraints hold, \( \mathcal{C} \) vanishes.

### 2.3 Reduced phase space

Let \( \overline{\Gamma} \) be the subset of \( \Gamma \) where the constraints hold. The reduced phase space, denoted by \( \mathcal{M} \), is the quotient of \( \overline{\Gamma} \) under the gauge action \( \text{(2.5)} \). We now discuss the structure of \( \mathcal{M} \).

Note first that as the Hamiltonian is a linear combination of the constraints, there is no dynamics on \( \mathcal{M} \); and \( \mathcal{M} \) can be identified with the space of classical solutions. As the functions in \( \mathcal{A}_{\text{class}} \) are gauge invariant, they project to functions on \( \mathcal{M} \).

\( \overline{\Gamma} \) is clearly connected. Hence also \( \mathcal{M} \) is connected.

Let \( \overline{\Gamma}_0 = \{ q_0 \} \), where \( q_0 \) is the origin of \( \overline{\Gamma} \), \( u = p = 0 = w = \varpi \). Let \( \overline{\Gamma}_{\text{ex}} \) contain all other points on \( \overline{\Gamma} \) at which \( u = w = 0 \), and let \( \overline{\Gamma}_{\text{reg}} \) contain the points on \( \overline{\Gamma} \) at which \( u \neq 0 \neq w \). By the constraint \( H = 0 \), \( \overline{\Gamma} = \overline{\Gamma}_0 \cup \overline{\Gamma}_{\text{ex}} \cup \overline{\Gamma}_{\text{reg}} \). We refer to \( \overline{\Gamma}_{\text{ex}} \) and \( \overline{\Gamma}_{\text{reg}} \) as respectively the “exceptional” and “regular” parts of \( \overline{\Gamma} \).

As \( \overline{\Gamma}_0, \overline{\Gamma}_{\text{ex}} \) and \( \overline{\Gamma}_{\text{reg}} \) are preserved by the gauge transformations, they project onto disjoint subsets of \( \mathcal{M} \). We denote these sets respectively by \( \mathcal{M}_0, \mathcal{M}_{\text{ex}} \) and \( \mathcal{M}_{\text{reg}} \) and analyse each in turn.

#### 2.3.1 \( \mathcal{M}_0 \)

\( \mathcal{M}_0 \) contains only one point, the projection of \( q_0 \). All observables in \( \mathcal{A}_{\text{class}} \) vanish on \( \mathcal{M}_0 \).
2.3.2 \( \mathcal{M}_{\text{ex}} \)

Each point in \( \Gamma_{\text{ex}} \) is gauge-equivalent to a unique point that satisfies

\[
\mathbf{u} = 0 = \mathbf{w} \ , \ p^2 + \varpi^2 = 1 \ .
\] (2.9)

\( \mathcal{M}_{\text{ex}} \) has thus topology \( S^{p+q-1} \). By (2.2), the projection of \( \Omega \) vanishes on \( \mathcal{M}_{\text{ex}} \). All observables in \( \mathcal{A}_{\text{class}} \) vanish on \( \mathcal{M}_{\text{ex}} \).

2.3.3 \( \mathcal{M}_{\text{reg}} \)

Each point in \( \Gamma_{\text{reg}} \) is gauge-equivalent to a unique point that satisfies

\[
\mathbf{u}^2 = \mathbf{w}^2 = 1 \ , \ \mathbf{u} \cdot \mathbf{p} = \mathbf{w} \cdot \varpi = 0 \ .
\] (2.10)

\( \mathcal{M}_{\text{reg}} \) can therefore be represented as the set (2.10), which is the cotangent bundle over \( S^{p-1} \times S^{q-1} \), with \( (\mathbf{u}, \mathbf{w}) \) forming the base space and \( (\mathbf{p}, \varpi) \) the fibres. By (2.2), the symplectic structure on \( \mathcal{M}_{\text{reg}} \) induced from \( \Gamma \) is precisely the symplectic structure of this cotangent bundle description. \( \mathcal{M}_{\text{reg}} \) is connected when \( p > 1 \) and \( q > 1 \), and it has two connected components when exactly one of \( p \) and \( q \) equals 1. When \( p = q = 1 \), \( \mathcal{M}_{\text{reg}} \) contains just four points.

\( \mathcal{A}_{\text{class}} \) does not separate all of \( \mathcal{M}_{\text{reg}} \). However, we show in appendix B that when \( p > 1 \) and \( q > 1 \), \( \mathcal{A}_{\text{class}} \) separates the subset of \( \mathcal{M}_{\text{reg}} \) in which \( 0 \neq p^2 \neq \varpi^2 \neq 0 \), in the gauge (2.10), up to the twofold degeneracy

\[
(\mathbf{u}, \mathbf{w}, \mathbf{p}, \varpi) \mapsto (-\mathbf{u}, -\mathbf{w}, -\mathbf{p}, -\varpi) \ .
\] (2.11)

We also show that when \( p = 1 \) and \( q > 1 \) (respectively \( p > 1 \) and \( q = 1 \)), \( \mathcal{A}_{\text{class}} \) separates the subset of \( \mathcal{M}_{\text{reg}} \) in which \( \varpi^2 \neq 0 \) (\( p^2 \neq 0 \)), again up to the degeneracy (2.11). When \( p = q = 1 \), all observables in \( \mathcal{A}_{\text{class}} \) vanish on \( \mathcal{M}_{\text{reg}} \).

3 Algebraic quantisation with the constraints (1.2)

In this section we quantise the system by the algebraic techniques of [33, 34], imposing the constraints in the form (1.2), promoting the classical \( \mathfrak{o}(p,q) \) observables into an \( \mathfrak{o}(p,q) \) algebra of operators, and seeking an inner product in which these quantum observables are self-adjoint. Subsection 3.1 shows that this procedure leads to quantum theories for arbitrary \( (p,q) \): It will be seen in sections 6 and 7 that one of these quantum theories is isomorphic to that emerging from refined algebraic quantisation with group averaging. The remaining subsections analyse the quantum theories in detail for \( p + q \leq 4 \). The use of the classical \( \mathfrak{o}(p,q) \) observables is motivated by their separation properties on the reduced phase space (see subsection 2.3 above).
3.1 General \((p, q)\)

Treating \(u\) and \(w\) as the ‘configuration’ variables, we represent the elementary operators as

\[
\hat{u} \Psi = u \Psi, \quad \hat{p} \Psi = -i \nabla_u \Psi, \\
\hat{w} \Psi = w \Psi, \quad \hat{w} \Psi = -i \nabla_w \Psi, 
\]

where the class of ‘functions’ \(\Psi(u, w)\) will be specified shortly. For the quantum constraint operators, we take

\[
\hat{H} := \frac{1}{2} (w^2 - u^2), \\
\hat{D} := -i \left( u \cdot \nabla_u + w \cdot \nabla_w + \frac{p+q}{2} \right),
\]

so that the commutator algebra reads

\[
[\hat{H}, \hat{D}] = 2i \hat{H}. 
\]

The constraints (3.2) are symmetric in the inner product in which the integration measure is \(dp \, du \, dq \, dw\), as motivated by comparison with refined algebraic quantisation in sections 6 and 7. Note however that algebraic quantisation does not introduce an inner product at this stage, and the physical inner product will not be integration in the measure \(dp \, du \, dq \, dw\).

The quantum constraints (1.2) read

\[
\hat{H} \Psi = 0, \\
(\hat{D} - i) \Psi = 0. 
\]

As the only continuous solution to (3.4a) is \(\Psi = 0\), we seek solutions in the space of (say, Schwarz) distributions with the integration measure \(dp \, du \, dq \, dw\). Equation (3.4b) is equivalent to the homogeneity condition \(\Psi(r u, r w) = r^{-(p+q+2)/2} \Psi(u, w)\) for \(r > 0\). The set (3.4) is thus satisfied by

\[
\Psi(u, w) = \delta(u^2 - w^2) f(u, w), 
\]

where \(u := \sqrt{u^2}\) and \(w := \sqrt{w^2}\), \(\delta\) is the Dirac delta-distribution, and \(f(u, w)\) is smooth for \((u, w) \neq (0, 0)\) and homogeneous of degree \(-(p + q - 2)/2\),

\[
f(r u, r w) = r^{-(p+q-2)/2} f(u, w) \quad \text{for} \quad r > 0.
\]

We denote the vector space of the solutions (3.5) by \(V_{\text{phys}}\).

We define the quantum counterparts of the classical observables (2.6) as

\[
\hat{A}_{ij} := -i (u_i \partial_{u_j} - u_j \partial_{u_i}), \quad 1 \leq i \leq p, \quad 1 \leq j \leq p; \\
\hat{B}_{ij} := -i (w_i \partial_{w_j} - w_j \partial_{w_i}), \quad 1 \leq i \leq q, \quad 1 \leq j \leq q; \\
\hat{C}_{ij} := i (u_i \partial_{w_j} + w_j \partial_{u_i}), \quad 1 \leq i \leq p, \quad 1 \leq j \leq q,
\]

\(\hat{A}_{ij} := -i (u_i \partial_{u_j} - u_j \partial_{u_i}), \quad 1 \leq i \leq p, \quad 1 \leq j \leq p; \\
\hat{B}_{ij} := -i (w_i \partial_{w_j} - w_j \partial_{w_i}), \quad 1 \leq i \leq q, \quad 1 \leq j \leq q; \\
\hat{C}_{ij} := i (u_i \partial_{w_j} + w_j \partial_{u_i}), \quad 1 \leq i \leq p, \quad 1 \leq j \leq q,
\]
These operators commute with the quantum constraints \((3.2)\) and are thus quantum observables, and their commutator algebra closes as \((i\text{ times})\) the \(\mathfrak{o}(p,q)\) Lie algebra. As in \([14]\), we define the full star-algebra \(\mathcal{A}_{\text{phys}}^\epsilon\) as the algebra generated by \((3.6)\), the antilinear star-relation being defined so that it leaves the observables \((3.6)\) invariant.

The quantum quadratic Casimir observable is \([37]\)

\[
\hat{C} := \sum_{i<j} (\hat{A}_{ij})^2 + \sum_{i<j} (\hat{B}_{ij})^2 - \sum_{i,j} (\hat{C}_{ij})^2.
\]  

On states satisfying \((3.4)\), \(\hat{C}\) takes the value \(-\frac{1}{4}(p+q-2)^2\) \([14]\).

Recall from Section 9.2.9 of \([32]\) that \(\mathcal{V}_{\text{phys}}\) carries a representation \(T\) of \(O_c(p,q)\) given by

\[
[T(R)\Psi](\mathbf{u}, \mathbf{w}) = \Psi(R^{-1}(\mathbf{u}, \mathbf{w}))
\]

where \(R \in O_c(p,q)\) acts on the configuration space as in \([27]\). Writing \(\Psi = \delta f\) as in \((3.5)\), and noting that \(u^2 - w^2\) is invariant under \(O_c(p,q)\), this representation reads

\[
[T(R)\delta f](\mathbf{u}, \mathbf{w}) = \delta(u^2 - w^2)f(R^{-1}(\mathbf{u}, \mathbf{w})).
\]

\(T\) is thus isomorphic to the representation of \(O_c(p,q)\) on homogeneous functions of degree \(-\frac{p+q-2}{2}\) on the light cone of \(\mathbb{R}^{p,q}\). It was observed in \([32]\) that \(T\) is unitary in the inner product

\[
(\Psi_1, \Psi_2) = \int_{S^{p-1}_+ \times S^{q-1}} d\mathbf{u} \, d\mathbf{w} \, f_1(\mathbf{u}, \mathbf{w}) \, f_2(\mathbf{u}, \mathbf{w}),
\]

where the overline denotes complex conjugation and the integration is over the product of the unit spheres in \(\mathbf{u}\) and \(\mathbf{w}\), \(u^2 = w^2 = 1\). Completion of \(\mathcal{V}_{\text{phys}}\) in the inner product \((3.10)\) therefore gives a physical Hilbert space on which \(T\) is unitary. The infinitesimal generators of \(T\) are given by \((3.6)\), and they are densely-defined self-adjoint operators on the physical Hilbert space.

Now, in algebraic quantisation the task is to seek vector spaces on which the representation of \(\mathcal{A}_{\text{phys}}^\epsilon\) is (algebraically) irreducible and on which there exists an inner product in which the generators \((3.6)\) are self-adjoint. Each such subspace yields a Hilbert space and an independent quantum theory by Cauchy completion. The properties of \(T\) show that each component in the decomposition of \(T\) into unitary irreducible representations gives such a quantum theory, the vector space being a suitable domain of the infinitesimal generators of \(T\). In the following subsections we shall examine this issue in detail for \(p+q \leq 4\), showing that an appropriate subspace of \(\mathcal{V}_{\text{phys}}\) will provide the requisite vector space.

We record here some properties of \(T\) \([32]\). For any \(p\) and \(q\), \(T\) reduces into subrepresentations \(T^\epsilon\) on functions of parity \(\epsilon \in \{0, 1\}\); \(f(-\mathbf{u}, -\mathbf{w}) = (-1)\epsilon f(\mathbf{u}, \mathbf{w})\). In the terminology of \([32]\), \(T^\epsilon\) is the end-point of the maximally degenerate principal unitary series of representations of \(O_c(p,q)\), denoted by \(T_{pq}^{(p+q-2)/2, \epsilon}\). \(T_{pq}^{(p+q-2)/2, \epsilon}\) is nontrivial iff \((p,q) \neq (1,1)\). \(T_{pq}^{(p+q-2)/2, \epsilon}\) is known to be irreducible when \(p\) and \(q\) are of opposite parity, and also when \(p\) and \(q\) are both even and \((p+q)/2 + \epsilon \equiv 0 \pmod{2}\).
3.2 (p, q) = (1, 1)

When (p, q) = (1, 1), $V_{\text{phys}}$ has dimension four, corresponding to the four branches of the 1+1 light cone. The representation of $A^{(p,q)}_{\text{phys}}$ on $V_{\text{phys}}$ is trivial and does not restrict the choice of an inner product on $V_{\text{phys}}$.

3.3 (p, q) = (2, 1)

When (p, q) = (2, 1), we consider the subspaces $V^\kappa := \text{span}\{\psi^\kappa_m\} \subset V_{\text{phys}}$, where $\kappa \in \{1, -1\}$,

$$\psi^\kappa_m := (-i\kappa)^m \delta(u^2 - w^2)\theta(kw_1)u^{-1/2}e^{im\alpha} , \quad (3.11)$$

$m \in \mathbb{Z}$, $\theta$ is the Heaviside function, and we have written $u_1 + iu_2 = ue^{i\alpha}$. Writing $\hat{C}_\pm := \hat{C}_{11} \pm i\hat{C}_{21}$, the set $\{\hat{A}_{12}, \hat{C}_+, \hat{C}_-\}$ forms a standard raising and lowering operator basis for $\mathfrak{o}(2, 1)$ [38, 39], and we find

$$\hat{A}_{12}\psi^\kappa_m = m\psi^\kappa_m, \quad (3.12a)$$

$$\hat{C}_\pm\psi^\kappa_m = (m \pm \frac{1}{2})\psi^\kappa_{m \pm 1}. \quad (3.12b)$$

Each $V^\kappa$ hence carries an irreducible representation of $A^{(p,q)}_{\text{phys}}$. This means in particular that the branches $w_1 > 0$ and $w_1 < 0$ of the light cone decouple. Requiring $\hat{A}_{12}$, $\hat{C}_{11}$ and $\hat{C}_{21}$ to be self-adjoint determines the inner product on each $V^\kappa$ to be such that $\{\psi^\kappa_m\}$ is an orthonormal set up to an overall scale [38]. Note that this agrees with the inner product (3.10). The Hilbert space is obtained by Cauchy completion. The representation of $O_c(2, 1)$ on each of the two Hilbert spaces is isomorphic to the principal series irreducible unitary representation denoted in [38] by $C^0_{1/4}$. The representations $T^{1/4}_2$ arise by starting from the parity $\epsilon$ subspaces $\text{span}\{\psi^1_m + (-1)^\epsilon\psi^{-1}_m\} \subset V^1 \oplus V^{-1}$ and are each isomorphic to $C^0_{1/4}$.

The spectra of the quantum observables in $A_{\text{phys}}^{(p,q)}$ are in qualitative agreement with the ranges of the corresponding classical observables. In particular, both $A_{12}$ and the spectrum of $\hat{A}_{12}$ are unbounded above and below.

3.4 (p, q) = (2, 2)

When (p, q) = (2, 2), we write [10]

$$\hat{\tau}^\eta_0 := \frac{1}{2}(\hat{A}_{12} + \eta\hat{B}_{12}) ,$$

$$\hat{\tau}^\eta_1 := \frac{1}{2}(\hat{C}_{11} - \eta\hat{C}_{22}) ,$$

$$\hat{\tau}^\eta_2 := \frac{1}{2}(\hat{C}_{21} + \eta\hat{C}_{12}) , \quad (3.13)$$

where $\eta \in \{1, -1\}$ and

$$\hat{\tau}^\eta_\pm := \hat{\tau}^\eta_1 \pm i\hat{\tau}^\eta_2 . \quad (3.14)$$
The commutators of the $\hat{\tau}$s are

\[
\begin{align*}
[\hat{\tau}^\eta_0, \hat{\tau}^\eta_\pm] &= \pm \delta^{\eta\eta'} \hat{\tau}^\eta_\pm , \\
[\hat{\tau}^\eta_+, \hat{\tau}^\eta_-] &= -2\delta^{\eta\eta'} \hat{\tau}^\eta_0 ,
\end{align*}
\]  

(3.15a)

(3.15b)

which shows that the $\hat{\tau}$s provide standard $\mathfrak{sl}(2, \mathbb{R})$ raising and lowering operator bases in the decomposition $\mathfrak{sl}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. In the polar coordinates $u_1 + iu_2 = we^{i\beta}$, $w_1 + iw_2 = we^{i\beta}$, we find

\[
\begin{align*}
\hat{\tau}^\eta_0 &= -\frac{1}{2}i (\partial_{\alpha} + \eta \partial_{\beta}) , \\
\hat{\tau}^\eta_\pm &= \frac{1}{2}ie^{\pm i(\alpha+\eta\beta)} \left\{ u[\partial_{\nu} \pm \eta(i/w)\partial_{\beta}] + w[\partial_{\alpha} \pm (i/u)\partial_{\alpha}] \right\} .
\end{align*}
\]  

(3.16a)

(3.16b)

Let $\mathcal{V} \subset \mathcal{V}_{\text{phys}}$ be the subspace spanned by the vectors $\delta(u^2 - w^2)u^{-1}e^{i(m\alpha + n\beta)}$, where $m$ and $n$ are integers. We label these vectors by $\mu := \frac{1}{2}(m + n)$ and $\nu := \frac{1}{2}(m - n)$, defining

\[
\psi_{\mu\nu} := (-i)^{\mu+\nu} \delta(u^2 - w^2)u^{-1}e^{i[(\mu+\nu)\alpha + (\mu-\nu)\beta]} ,
\]  

where $\mu$ and $\nu$ are either both integers or both half-integers. A direct computation gives

\[
\begin{align*}
\hat{\tau}^\pm_0 \psi_{\mu\nu} &= \mu \psi_{\mu\nu} , \\
\hat{\tau}^\pm_{\pm} \psi_{\mu\nu} &= (\mu \pm \frac{1}{2}) \psi_{\mu\pm1, \nu} , \\
\hat{\tau}^-_0 \psi_{\mu\nu} &= \nu \psi_{\mu\nu} , \\
\hat{\tau}^-_{\pm} \psi_{\mu\nu} &= (\nu \pm \frac{1}{2}) \psi_{\mu, \nu\pm1} ,
\end{align*}
\]  

(3.18)

which shows that $\mathcal{V}$ carries a representation of $\mathcal{A}_{\text{phys}}^{(s)}$.

We decompose $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}^e \oplus \bigoplus_{\epsilon_1 \epsilon_2} \mathcal{V}^s_{\epsilon_1 \epsilon_2}$, where

\[
\begin{align*}
\mathcal{V}^e &:= \text{span}\{\psi_{\mu\nu} \mid \mu, \nu \in \mathbb{Z}\} , \\
\mathcal{V}^s_{\epsilon_1 \epsilon_2} &:= \text{span}\{\psi_{\mu\nu} \mid \mu, \nu \in \mathbb{Z} + \frac{1}{2} ; \ sgn(\mu) = \epsilon_1; \ sgn(\nu) = \epsilon_2\} ,
\end{align*}
\]  

(3.19)

and $\epsilon_i \in \{1, -1\}$. Equations (3.18) show that each of the five spaces in (3.19) carries a representation of $\mathcal{A}_{\text{phys}}^{(s)}$. These representations are irreducible: Given a nonzero vector in one of the spaces, acting on this vector repeatedly with $\hat{\tau}^\eta_0$ and taking suitable linear combinations generates some $\psi_{\mu\nu\rho\sigma}$, and repeatedly acting on this $\psi_{\mu\nu\rho\sigma}$ by $\hat{\tau}^\eta_{\pm}$ and taking linear combinations generates all of the space. Note that the difference between integer and half-integer indices in (3.19) arises because in the latter case the numerical coefficients in the raising and lowering operator action in (3.18) may vanish.

On each of the five spaces in (3.19), the $\mathfrak{sl}(2, \mathbb{R})$ analysis of (3.8) in each index shows that the adjoint relations

\[
(\hat{\tau}^\eta_0)^\dagger = (\hat{\tau}^\eta_0)^\dagger , \quad (\hat{\tau}^\eta_{\pm})^\dagger = (\hat{\tau}^\eta_{\pm})^\dagger ,
\]  

(3.20)
determine uniquely an inner product in which \(\{\psi_{\mu\nu}\}\) is an orthonormal set up to an overall scale. This agrees with the inner product (3.10). The Hilbert spaces \(H^e\) and \(H^o_{\epsilon_1 \epsilon_2}\) are obtained by Cauchy completion. In the terminology of [38], the representations of \(\text{O}_c(2,2) \simeq (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2\) are respectively \(C^0_{1/4} \times C^0_{1/4}\) and \(D^{\epsilon_1}_{1/2} \times D^{\epsilon_2}_{1/2}\). The first of these is \(T^{1,0}_{2,2}\) and the other four constitute \(T^{1,1}_{2,2}\).

On \(H^e\) the spectra of the quantum observables \(\hat{A}_{12} \pm \hat{B}_{12}\) are unbounded both above and below. This might have been expected on the grounds that the range of the classical observables \(A_{12} \pm B_{12}\) are unbounded both above and below. By contrast, on each \(H^o_{\epsilon_1 \epsilon_2}\) the spectra of \(\hat{A}_{12} \pm \hat{B}_{12}\) have a definite sign. That quantum theories with this property can arise may be related to the failure of \(A\) to separate the subsets of \(\mathcal{M}_{\text{reg}}\) where one of \(A_{12} \pm B_{12}\) vanishes.

3.5 \((p, q) = (3, 1)\)

When \((p, q) = (3, 1)\), the value of the quadratic Casimir operator \((3.7)\) in the representations \(T^{2,\epsilon}_{3,1}\) is \(-1\), and a direct computation shows that the Casimir operator \(\hat{A}_{12} \hat{C}_{31} + \hat{A}_{23} \hat{C}_{11} + \hat{A}_{31} \hat{C}_{21}\) has value zero. It follows that the representations \(T^{2,\epsilon}_{3,1}\) are each isomorphic to the principal series unitary irreducible representation \(\Theta_{0,0}\) of \(\text{O}_c(3,1)\) [40]. Proceeding as in subsection 3.3 yields a decomposition of \(T^{2,0}_{3,1} \oplus T^{2,1}_{3,1}\) in which each irreducible component is supported on its own branch of the light cone.

4 Algebraic quantisation with \(\hat{H}_a \Psi = 0\)

In this section we discuss how the algebraic quantisation of section 3 is modified when the constraints (1.2) are replaced by \(\hat{H}_a \Psi = 0\). We give a complete analysis for \(p + q \leq 4\) and partial results for other values of \(p\) and \(q\).

4.1 General \((p, q)\)

The quantum constraints (3.4) are replaced by

\[
\begin{align*}
\hat{H} \Psi &= 0, \\
\hat{D} \Psi &= 0.
\end{align*}
\]

(4.1a) (4.1b)

Proceeding as in subsection 3.1 the exponent of \(r\) in (3.5b) is replaced by \(-(p + q - 4)/2\), and the value of the quadratic Casimir (3.7) is \(-\frac{1}{4}(p + q)(p + q - 4)\) [14].

The representation of \(\text{O}_c(p, q)\) generated by the quantum observables is now isomorphic to the representation on homogeneous functions of degree \(-(p + q - 4)/2\) on the light cone of \(\mathbb{R}^{p,q}\). The outstanding question is whether this representation or some subrepresentation thereof is unitary in some inner product.

We analyse different ranges of \((p, q)\) in the following subsections. By interchange of \(p\) and \(q\), it suffices to consider \(p \geq q\).
4.2 \((p, q) = (1, 1)\)

When \((p, q) = (1, 1)\), the discussion follows subsection 3.2. The representation of \(A_{\text{phys}}^{(s)}\) is trivial.

4.3 \((p, q) = (2, 1)\)

When \((p, q) = (2, 1)\), the factor \(u^{-1/2}\) in (3.11) is replaced by \(u^{1/2}\), and the numerical factor on the right-hand side of (3.12b) is replaced by \(m \mp \frac{1}{2}\). The representations of \(A_{\text{phys}}^{(s)}\) on the counterparts of \(V^\kappa\) are irreducible, but there is no inner product in which \(\hat{A}_{12}, \hat{C}_{11}\) and \(\hat{C}_{21}\) would be self-adjoint [38]. No quantum theory is recovered.

4.4 \((p, q) = (2, 2)\)

When \((p, q) = (2, 2)\), (3.17) is replaced by

\[
\tilde{\psi}_{\mu\nu} := (-i)^{\mu+\nu} \delta(u^2 - w^2) e^{[(\mu+\nu)\alpha + (\mu-\nu)\beta]} ,
\]

where \(\mu\) and \(\nu\) are again either both integers or both half-integers. We write \(\tilde{\mathcal{V}} := \text{span}\{\tilde{\psi}_{\mu\nu}\}\). A direct computation gives

\[
\hat{\tau}_0^+ \tilde{\psi}_{\mu\nu} = \mu \tilde{\psi}_{\mu\nu} ,
\hat{\tau}_0^- \tilde{\psi}_{\mu\nu} = \nu \tilde{\psi}_{\mu\nu} ,
\hat{\tau}_\pm \tilde{\psi}_{\mu\nu} = \nu \tilde{\psi}_{\mu,\nu \pm 1} .
\]

Hence \(\tilde{\mathcal{V}}\) carries a representation of \(A_{\text{phys}}^{(s)}\).

We decompose \(\tilde{\mathcal{V}}\) as \(\tilde{\mathcal{V}} = \tilde{\mathcal{V}}^o \oplus \bigoplus_{\epsilon_1, \epsilon_2} \tilde{\mathcal{V}}^e_{\epsilon_1 \epsilon_2}\), where

\[
\tilde{\mathcal{V}}^o := \text{span}\{\tilde{\psi}_{\mu\nu} | \mu, \nu \in \mathbb{Z} + \frac{1}{2}\} ,
\tilde{\mathcal{V}}^e_{\epsilon_1 \epsilon_2} := \text{span}\{\tilde{\psi}_{\mu\nu} | \mu, \nu \in \mathbb{Z}; \text{sgn}(\mu) = \epsilon_1; \text{sgn}(\nu) = \epsilon_2\} ,
\]

and \(\epsilon_i \in \{1, 0, -1\}\). Equations [133] show that each of the ten spaces in (4.4) carries a representation of \(A_{\text{phys}}^{(s)}\), given by [133] except that whenever a raising (respectively lowering) operator raises (lowers) the index \(-1\) (+1) to zero, the vector on the right-hand side is replaced by the zero vector. It can be verified as in subsection 3.4 that these representations are irreducible. The \(\mathfrak{sl}(2, \mathbb{R})\) analysis of [38] in each index then shows that there is no inner product on \(\tilde{\mathcal{V}}^o\) compatible with the adjoint relations (3.20), whereas on each \(\tilde{\mathcal{V}}^e_{\epsilon_1 \epsilon_2}\) these relations determine an inner product that is unique up to an overall scale. For \(\epsilon_1 \neq 0 \neq \epsilon_2\), this inner product reads \((\tilde{\psi}_{\mu,\nu'}, \tilde{\psi}_{\mu\nu}) = |\mu\nu| \delta_{\mu\mu'} \delta_{\nu\nu'}\), while the formulas for \(\epsilon_1 = 0 = \epsilon_2\) and \(\epsilon_1 = 0 \neq \epsilon_2\) are respectively \((\tilde{\psi}_{\mu,0}, \tilde{\psi}_{\mu0}) = |\mu| \delta_{\mu\mu'}\)
and $(\tilde{\psi}_{00}, \tilde{\psi}_{00}) = |\nu|\delta_{\nu\nu'}$. On the one-dimensional space $\tilde{V}_{00}$ the representation of $A_{\text{phys}}^{(s)}$ is trivial, and we have $(\tilde{\psi}_{00}, \tilde{\psi}_{00}) = 1$.

The Hilbert spaces $\tilde{H}_{\epsilon_1\epsilon_2}$ are obtained by Cauchy completion. In the terminology of [38], the representation of $O_c(2,2) \simeq (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2$ on $\tilde{H}_{\epsilon_1\epsilon_2}$ is $D_{\epsilon_1} \times D_{\epsilon_2}$ for $\epsilon_1 \neq 0 \neq \epsilon_2$, and for $\epsilon_i = 0$, $D_{\epsilon_i}$ is replaced by the trivial representation.

The quantum theories on $\tilde{H}_{\epsilon_1\epsilon_2}$ with $\epsilon_1 \neq 0 \neq \epsilon_2$ are qualitatively similar to the theories on $H_{\epsilon_1\epsilon_2}$ obtained in subsection 3.4, with the roles of integer and half-integer eigenvalues of $\hat{\tau}_{\eta_0}$ interchanged. The remaining five theories are degenerate in that at least one $\hat{\tau}_{\eta_0}$ annihilates the whole space. Note that in contrast to subsection 3.4, we now obtained no theory in which the representation of $A_{\text{phys}}^{(s)}$ would be irreducible and the spectra of $\hat{A}_{12} \pm \hat{B}_{12}$ would be unbounded both above and below.

4.5 $(p, q) = (3, 1)$

When $(p, q) = (3, 1)$, the branches $w_1 > 0$ and $w_1 < 0$ of the light cone decouple and give isomorphic quantum theories. For concreteness, we consider states whose support is on the $w_1 > 0$ branch. Isomorphic theories in which the states have support on both branches and have respectively even or odd parity could be constructed as in subsections 3.3 and 3.5.

We set $\tilde{V} := \text{span}\{\tilde{\psi}_{lm}\}$, where

$$\tilde{\psi}_{lm} := \delta(u^2 - w^2)\theta(w_1)Y_{lm} ,$$

(4.5)

$Y_{lm}$ are the usual spherical harmonics on unit $S^2$ in $u$ [41] and $\theta$ is the Heaviside function as in (3.1). The action of the operators (3.6) on $\tilde{V}$ can be computed from standard properties of the spherical harmonics [41, 42] and is displayed in Table 1. This shows that $\tilde{V}$ carries a representation of $A_{\text{phys}}^{(s)}$.

We decompose $\tilde{V}$ as $\tilde{V} = \tilde{V}_0 \oplus \tilde{V}_+$, where

$$\tilde{V}_0 := \text{span}\{\tilde{\psi}_{00}\} ,$$

$$\tilde{V}_+ := \text{span}\{\tilde{\psi}_{lm} | l > 0\} .$$

(4.6)

Table 1 shows that $A_{\text{phys}}^{(s)}$ is represented trivially on $\tilde{V}_0$, while $\tilde{V}_+$ carries a representation that is in the Table except that any term on the right-hand side with the first index taking the value zero is replaced by the zero vector. Comparison with the infinitesimal representations of $O_c(3, 1)$ [40], section 8.3) shows that the representation on $\tilde{V}_+$ is isomorphic to the principal series irreducible representation $S_{2,0}$, which is unitary precisely when the inner product is

$$(\tilde{\psi}_{lm'}, \tilde{\psi}_{lm}) = l(l+1)\delta_{\nu\nu'}\delta_{mm'} ,$$

(4.7)

up to an overall multiple. The Casimir operator $\hat{A}_{12}\hat{C}_{31} + \hat{A}_{23}\hat{C}_{11} + \hat{A}_{31}\hat{C}_{21}$ takes value zero.
\[ \hat{A}_{12} \tilde{\psi}_{lm} = m \tilde{\psi}_{lm} \]
\[ (\hat{A}_{23} + i \hat{A}_{31}) \tilde{\psi}_{lm} = \sqrt{(l+m+1)(l+m)} \tilde{\psi}_{l,m+1} \]
\[ \hat{C}_{31} \tilde{\psi}_{lm} = -i l \sqrt{\frac{(l+m)(l-m+1)}{(2l+1)(2l-1)}} \tilde{\psi}_{l-1,m} \]
\[ (\hat{C}_{11} + i \hat{C}_{21}) \tilde{\psi}_{lm} = \pm i l \sqrt{\frac{(l+m)(l+m+2)}{(2l+1)(2l+3)}} \tilde{\psi}_{l,m+1} \]
\[ \pm i (l+1) \sqrt{\frac{(l+m)(l+m+1)}{(2l+1)(2l-1)}} \tilde{\psi}_{l-1,m+1} \]

Table 1: The action of \( \mathcal{A}^{(\ast)}_{\text{phys}} \) on \( \tilde{\mathcal{V}} \) for \( (p, q) = (3, 1) \).

4.6 \( 2 \leq q \leq p, (p, q) \neq (2, 2) \)

When \( 2 \leq q \leq p \) and \( (p, q) \neq (2, 2) \) we set \( \tilde{\mathcal{V}} := \text{span}\{ \tilde{\psi}_{ljk, u, k, w} \} \), where
\[ \tilde{\psi}_{ljk, u, k, w} := \delta(u^2 - w^2) u^{-(p+q-4)/2} Y_{lk}(\theta^{(u)}) Y_{jk}(\theta^{(w)}) \] \( l \) and \( j \) are non-negative integers and \( Y_{lk} \) (respectively \( Y_{jk} \)) are the spherical harmonics on unit \( S^{p-1} \) in \( u \) (\( S^{q-1} \) in \( w \)) \[32, 41\]. The notation for the angular coordinates \( \theta^{(u)} \) and \( \theta^{(w)} \) and the spherical harmonics follows \[14\]. The construction of the spherical harmonics implies that \( \tilde{\mathcal{V}} \) carries a representation of \( \mathcal{A}^{(\ast)}_{\text{phys}} \).

We seek a linear subspace \( \tilde{\mathcal{V}}_0 \subset \tilde{\mathcal{V}} \) with the following properties:

1. \( \tilde{\mathcal{V}}_0 := \text{span}\{ \tilde{\psi}_{ljk, k, w} \mid (l, j) \in I \} \), where \( I \) is some nonempty index set.

2. \( \tilde{\mathcal{V}}_0 \) carries a representation of \( \mathcal{A}^{(\ast)}_{\text{phys}} \).

3. The generators \( \tilde{\mathcal{A}}^{(\ast)}_i \) of \( \mathcal{A}^{(\ast)}_{\text{phys}} \) are self-adjoint in an inner product of the form
\[ (\tilde{\psi}_{l', j', k', u'}, \tilde{\psi}_{ljk, k, w}) = K_{l'l} \delta_{j'j} \delta_{k'u} \delta_{k'w} \] \( \text{where the positive numbers } K_{lj} \text{ depend only on } l \text{ and } j \).

By the properties of the spherical harmonics, the rotation generators \( \hat{A}_{ij} \) and \( \hat{B}_{ij} \) in \([3, 6]\) leave \( \tilde{\mathcal{V}}_0 \) invariant and are self-adjoint in the inner product \([4, 9]\). What remains is to examine the boost generators \( \hat{C}_{ij} \).
Let $Y_{l0}$ (respectively $Y_{j0}$) denote the zonal spherical harmonics, which can be expressed in terms of Gegenbauer polynomials of argument $u_p/u$ \cite{41}. The recursion relations of the Gegenbauer polynomials \cite{43} allow an explicit computation of the action of $\hat{C}_{pq}$ on $\psi_{l,j00}$. Suppressing the indices $k_u = k_w = 0$, we find

$$4i\hat{C}_{pq}\tilde{\psi}_{lj} = \left[l + j + \frac{1}{2}(p + q - 4)\right] \left(W_{p,l+1}W_{q,j+1}\tilde{\psi}_{l+1,j+1} - W_{p,l}W_{q,j}\tilde{\psi}_{l-1,j-1}\right) + \left[l - j + \frac{1}{2}(p - q)\right] \left(W_{p,l+1}W_{q,j}\tilde{\psi}_{l+1,j-1} - W_{p,l}W_{q,j+1}\tilde{\psi}_{l-1,j+1}\right),$$

(4.10)

where

$$W_{aq} := 2\left[\frac{j(j + q - 3)}{(2j + q - 2)(2j + q - 4)}\right]^{1/2} \text{ for } j > 0, (q, j) \neq (2, 1),$$

$$W_{21} := \sqrt{2},$$

$$W_{q0} := 0.$$  \hspace{1cm} (4.11)

By (4.9) and (4.10), self-adjointness of $\hat{C}_{pq}$ implies the recursion relations

$$\left[l + j + \frac{1}{2}(p + q - 4)\right]K_{l+1,j+1} = \left[l + j + \frac{1}{2}(p + q)\right]K_{lj},$$

(4.12a)

$$\left[l - j - 1 + \frac{1}{2}(p - q)\right]K_{l+1,j} = \left[l - j + 1 + \frac{1}{2}(p - q)\right]K_{l,j+1}.$$  \hspace{1cm} (4.12b)

Note that the coefficients in (4.12a) are always positive.

Suppose first that $p + q$ is odd. From (4.10) it follows that the index set $I$ must contain all pairs $(l, j)$ where $l + j$ is odd or all pairs where $l + j$ is even. The coefficients in (4.12a) are always nonzero, but the coefficients on the two sides have opposite sign for $j - l = \frac{1}{2}(p - q \pm 1)$. Hence there are both positive and negative $K_{lj}$, and the inner product does not exist. We have proved:

**Theorem 4.1** Let $p \geq 2$, $q \geq 2$ and $p + q \equiv 1 \pmod{2}$. Then there is no $\tilde{V}_0$ satisfying 1–3.

Suppose then that $p + q$ is even. If $\tilde{V}_0$ contains a vector for which $l - j + \frac{1}{2}(p - q)$ is odd, (4.10) shows that it must contain all such vectors, and examination of the signs in (4.12a) shows that the inner product does not exist. Hence $\tilde{V}_0$ can contain only vectors for which $l - j + \frac{1}{2}(p - q)$ is even. From (4.10) and (4.12) we obtain:

**Theorem 4.2** Let $p \geq 2$, $q \geq 2$, $(p,q) \neq (2,2)$ and $p + q \equiv 0 \pmod{2}$. For a $\tilde{V}_0$ satisfying 1–3, $I$ is either $I_0 := \{ (l,j) \mid l - j + \frac{1}{2}(p - q) = 0 \}$ or one of $I_+ := \{ (l,j) \mid l - j + \frac{1}{2}(p - q) = 2k, k \in \mathbb{Z} \}$ or a union of two or all of these. For indices within each of $I_0$, $I_+$ and $I_-$, (4.12) determines the positive numbers $K_{lj}$ uniquely up to an overall multiple.
For \( p + q \) even, Theorem 4.2 severely restricts the possible candidates for \( \tilde{V}_0 \). A more complete understanding of the \( \hat{C}_{ij} \) action would be required to determine whether the candidate spaces of Theorem 4.2 indeed have properties 2 and 3. It is known ([32], Section 9.2.10) that there exist subspaces of \( \tilde{V} \) carrying an irreducible representation of \( A_{\text{phys}}^{(\star)} \) such that the generators (3.6) are self-adjoint in an appropriate inner product, and these representations are equivalent to certain discrete series representations of \( O_c(p,q) \). We shall not examine correspondences between these subspaces and the spaces \( \tilde{V}_0 \) of Theorem 4.2 here.

### 4.7 \( p \geq 4, \ q = 1 \)

When \( p \geq 4 \) and \( q = 1 \), the branches \( w_1 > 0 \) and \( w_1 < 0 \) of the light cone decouple and lead to isomorphic situations. We consider the branch \( w_1 > 0 \). We set \( \tilde{V} := \text{span}\{\tilde{\psi}_{lk_u}\} \), where

\[
\tilde{\psi}_{lk_u} := \delta(u^2 - w^2)\theta(w_1)u^{-(p-3)/2}Y_{lk_u},
\]

(4.13)

\( \theta \) is the Heaviside function as in (3.11) and (4.5) and \( Y_{lk_u} \) are the spherical harmonics on unit \( S^{p-1} \) in \( u \) as in subsection 4.6. We seek an inner product of the form \( (\tilde{\psi}_{lk'}, \tilde{\psi}_{lk_u}) = K_l \delta_{ll'}\delta_{k_u k'_u} \), where the positive numbers \( K_l \) depend only on \( l \). From the action of \( \hat{C}_{p1} \) on the \( k_u = 0 \) states it then follows as in subsection 4.6 that a necessary condition for the operators (3.6) to be self-adjoint is \( K_l = r\left[ l + \frac{1}{2}(p-1) \right]\left[ l + \frac{1}{2}(p-3) \right] \), where \( r \) is a positive constant. This fixes the candidate inner product uniquely up to an overall multiple. The rotation generators \( \hat{A}_{ij} \) are clearly self-adjoint in this inner product, but further analysis would be required to see whether the same holds for all the boost generators \( \hat{C}_{i1} \).

### 5 Refined algebraic quantisation with a nonunimodular gauge group

In this section we give a brief outline of refined algebraic quantisation (RAQ) with group averaging when the gauge group is a connected but not necessarily unimodular Lie group. We follow [8] but include the possibility of non-unimodularity at the outset and take the opportunity to clarify in subsection 5.2 certain technical issues that arise from the antilinearity of the RAQ observable action on the RAQ physical Hilbert space, on comparison with quantisations in which the action of the observables on the physical states is linear. A recent status report of refined algebraic quantisation can be found in [12].

#### 5.1 Refined algebraic quantisation

RAQ begins by choosing an auxiliary Hilbert space \( H_{\text{aux}} \) and implementing the quantum constraints as self-adjoint operators on it, such that the commutators of the constraints
close as a Lie algebra and the constraints exponentiate into a unitary representation $U$ of a corresponding connected Lie group $G$. We refer to $G$ as the gauge group of the quantum theory. We denote the left and right Haar measures on $G$ by respectively $d_{LG}$ and $d_{RG}$, and we denote their geometric average by $d_{0}g$. These measures are related by $d_{0}g = [\Delta(g)]^{1/2}d_{LG} = [\Delta(g)]^{-1/2}d_{RG}$, where $\Delta(g) := \det(Ad_g)$ is the modular function.

We wish to solve the constraints in an enlargement of $\mathcal{H}_{aux}$. We introduce a space of test states, a dense linear subspace $\Phi \subset \mathcal{H}_{aux}$ such that the operators $U(g)$ map $\Phi$ to itself. The desired enlargement is the algebraic dual of $\Phi$, denoted by $\Phi^{*}$ and topologised by the topology of pointwise convergence. For $f \in \Phi^{*}$ and $\phi \in \Phi$, we denote the dual action of $f$ on $\phi$ by $f[\phi]$. $\Phi^{*}$ carries a representation $U^{*}$ of $G$ defined by the dual action: For $f \in \Phi^{*}$, $(U^{*}(g)f)[\phi] = f[U(g^{-1})\phi]$ for all $\phi \in \Phi$. Solutions to the quantum constraints are then defined to be the elements $f \in \Phi^{*}$ for which $U^{*}(g)f = [\Delta(g)]^{1/2}f$ for all $g \in G$. We return to the reason for including the factor $[\Delta(g)]$ in subsection 5.2.

The RAQ observable algebra is determined by the above structure. An operator $\mathcal{O}$ on $\mathcal{H}_{aux}$ is called gauge invariant if the domains of $\mathcal{O}$ and $\mathcal{O}^{\dagger}$ include $\Phi$, $\mathcal{O}$ and $\mathcal{O}^{\dagger}$ map $\Phi$ to itself, and $\mathcal{O}$ commutes with the $G$-action on $\Phi$. Note that if $\mathcal{O}$ is gauge invariant, so is $\mathcal{O}^{\dagger}$. The algebra of gauge invariant operators is called the observable algebra and denoted by $\mathcal{A}_{obs}$. $\mathcal{A}_{obs}$ has on $\Phi^{*}$ an antilinear representation defined by the dual action $[15]$: For $f \in \Phi^{*}$, $(\mathcal{O}f)[\phi] := f[\mathcal{O}^{\dagger}\phi]$ for all $\phi \in \Phi$.

The last ingredient in RAQ is a rigging map, which is an antilinear map $\eta : \Phi \to \Phi^{*}$ satisfying four postulates:

1. The image of $\eta$ solves the constraints.
2. $\eta$ is real: $\eta(\phi_{1})[\phi_{2}] = \eta(\phi_{2})[\phi_{1}]$ for all $\phi_{1}, \phi_{2} \in \Phi$.
3. $\eta$ is positive: $\eta(\phi)[\phi] \geq 0$ for all $\phi \in \Phi$.
4. $\eta$ intertwines with the representations of the observable algebra on $\Phi$ and $\Phi^{*}$: $\mathcal{O}(\eta \phi) = \eta(\mathcal{O}\phi)$ for all $\mathcal{O} \in \mathcal{A}_{obs}$ and all $\phi \in \Phi$.

Now, the RAQ physical Hilbert space $\mathcal{H}_{RAQ}$ is defined to be the completion of the image of $\eta$ in the Hermitian inner product

$$
(\eta(\phi_{1}), \eta(\phi_{2}))_{RAQ} := \eta(\phi_{2})[\phi_{1}].
$$

$\mathcal{H}_{RAQ}$ carries an antilinear representation of $\mathcal{A}_{obs}$, and the adjoint map in this representation is that induced from the adjoint map on $\mathcal{H}_{aux}$.

### 5.2 Group averaging

The aim of group averaging is to provide a rigging map.

We define on $\Phi$ the group averaging sesquilinear form

$$
(\phi_{2}, \phi_{1})_{ga} := \int_{G} d_{0}g(\phi_{2}, U(g)\phi_{1})_{aux},
$$

assuming that the integral on the right-hand side converges in absolute value for all $\phi_{1}$ and $\phi_{2}$ in $\Phi$. The group averaging rigging map candidate is

$$
\eta(\phi_{1})[\phi_{2}] := (\phi_{1}, \phi_{2})_{ga}.
$$
This map satisfies (i) because \( d_0(gh) = [\Delta(h)]^{-1/2}d_0(g) \) for constant \( h \in G \) and (ii) because \( d_0g = d_0(g^{-1}) \). Property (iv) is clear. If \( \eta \) further satisfies (iii), and if \( \eta \) is not identically zero, the group averaging rigging map candidate is a rigging map. If the convergence of the averaging is sufficiently strong, in the sense explained in \[3\], the group averaging rigging map is the unique map satisfying the axioms (i)–(iv).

The antilinearity of the rigging map is a technical advantage in practical computations on the test space, but a complication when one wishes to compare the results to quantisations in which the constraints and the observables act on physical states linearly as on \( \mathcal{H}_{\text{aux}} \). With the group averaging rigging map, the antilinear isomorphism required for the comparison is provided by complex conjugation of the group averaging sesquilinear form. We denote this antilinear map by \( J: \mathcal{H}_{\text{RAQ}} \rightarrow J(\mathcal{H}_{\text{RAQ}}) \). Adopting Dirac’s bra-ket notation, if \( |\phi\rangle \) is in \( \Phi \) and \( \langle \phi| \) is its Hilbert dual state, we have \( \langle \langle \psi| := \eta(|\phi\rangle) = \int_G d_0g \langle \phi|U(g) \rangle \), where the rightmost expression is understood in the sense of taking matrix elements from the right with states in \( \Phi \). The gauge invariance of \( \langle \langle \psi| \) reads in this language \( \langle \langle \psi|U(g) = \langle \langle \psi|[\Delta(g)]^{1/2} \). Now, by unitarity of \( U \) and the invariance of \( d_0g \) under the group inverse, we have \( |\psi\rangle := J(\langle \langle \psi|) = \int_G d_0g U(g)|\phi\rangle \), where the rightmost expression is understood in the sense of taking matrix elements from the left with states in the Hilbert dual image of \( \Phi \). The gauge invariance of \( |\psi\rangle \) reads

\[
U(g)|\psi\rangle = [\Delta(g)]^{-1/2}|\psi\rangle ,
\]

and the linear representation of \( \mathcal{A}_{\text{obs}} \) on \( J(\eta(\Phi)) \) is induced from the representation of \( \mathcal{A}_{\text{obs}} \) on \( \Phi \). Although \( |\psi\rangle \) may not be in \( \mathcal{H}_{\text{aux}} \), in situations where the generators of \( U \) are interpretable as (say) differential operators on smooth functions on a manifold, the infinitesimal version of (5.4) is well-defined as a differential equation and given by (1.2)\(^1\).

6 Refined algebraic quantisation for \( (p,q) \neq (1,1) \)

In this section we carry out refined algebraic quantisation of our system for \( (p,q) \neq (1,1) \). The special case \( p = q = 1 \) will be treated in section 7.

6.1 Auxiliary Hilbert space and representation of the gauge group

We use the auxiliary Hilbert space \( \mathcal{H}_{\text{aux}} \simeq L^2(\mathbb{R}^{p+q}) \) of square integrable functions \( \Psi(u,v) \) in the inner product

\[
\langle \Psi_1, \Psi_2 \rangle_{\text{aux}} := \int d^p u \, d^q v \, \overline{\Psi_1} \Psi_2 .
\]  

\(^1\)In [8], equation (4.2) should be understood as our (5.4). Consequently, equation (B6) in [8] should be understood as \( \langle \mathcal{J}_a|\phi\rangle = -(i/2)\text{tr}(ad_a)|\phi\rangle \), which agrees with equation (6.4) in [8] and with our (12). We thank Nico Giulini and Don Marolf for discussions and correspondence on this point.
The algebra of the quantum constraints (3.2) exponentiates to a unitary representation \( U \) of \( G \) on \( \mathcal{H}_{\text{aux}} \). In the decomposition (A.3) of appendix A, the group elements are represented by

\[
U(\exp(\mu e^{-})) = \exp(-i\mu \hat{H}),
\]

\[
U(\exp(\lambda h)) = \exp(-i\lambda \hat{D}),
\]

where

\[
[\exp(-i\mu \hat{H}) \Psi](u, w) = \exp\left[\frac{i\mu (u^2 - w^2)}{2}\right] \Psi(u, w),
\]

\[
[\exp(-i\lambda \hat{D}) \Psi](u, w) = \exp\left[-\frac{\lambda(p + q)}{2}\right] \Psi(e^{-\lambda}u, e^{-\lambda}w).
\]

### 6.2 Test space

Let

\[
\Psi_{ljmnk_u^kw_w}(u, w) := u^{l+2m}w^{j+2n}e^{-\frac{1}{2}(u^2+w^2)}Y_{lk_u}(\theta(u))Y_{jk_w}(\theta(w)),
\]

where \( l, j, m \) and \( n \) are non-negative integers and the spherical harmonics are as in subsection 4.6. \( p = 1 \) is covered as the special case in which \( \theta(u) := u_1/u \in \{1, -1\} \), \( l \in \{0, 1\} \), the index \( k_u \) takes only a single value and can be dropped and \( Y_l(\theta(u)) := (\theta(u))^l/\sqrt{2} \), and similarly for \( q = 1 \).

We set \( \Phi_0 := \text{span}\{\Psi_{ljmnk_u^kw_w}\} = \{P(u, w) \exp\left[-\frac{1}{2}(u^2 + w^2)\right] \mid P(u, w) \text{ is a polynomial in } \{u_i\} \text{ and } \{w_i\}\} \). \( \Phi_0 \) is dense in \( \mathcal{H}_{\text{aux}} \) and mapped to itself by \( \mathcal{A}^{(\text{phys})} \). We adopt as our test space \( \Phi \) the closure of \( \Phi_0 \) under the algebra generated by \( \{U(g) \mid g \in G\} \).

### 6.3 \((\cdot, \cdot)_{\text{ga}}\)

We parametrise the elements in \( G \) as in (A.3) and normalise the symmetric measure so that \( d\sigma g = e^{\lambda}d\lambda d\mu \). We shall show that the group averaging sesquilinear form \((\cdot, \cdot)_{\text{ga}}\) is well defined and evaluate it explicitly.

From (A.3), (6.2) and (6.3) we find

\[
[U(g)\Psi_{ljmnk_u^kw_w}](u, w) = z^{-\frac{1}{2}(p+q)+\frac{1}{2}(l+j)+m+n}u^{l+2m}w^{j+2n}Y_{lk_u}(\theta(u))Y_{jk_w}(\theta(w))
\]

\[
\times \exp\left[-\frac{1}{2}\left(1 - i\mu\right)u^2 - \frac{1}{2}\left(1 + i\mu\right)w^2\right],
\]

where \( z := e^{2\lambda} \). In \((\Psi_{ljmnk_u^kw_w}, U(g)\Psi_{ljmnk_u^kw_w})_{\text{aux}} \), the angular integrals give the factor

\[
d\sigma_{j}d\delta_{k_u^k_u^'}d\delta_{k_w^k_w^'}
\]

and the integrals over \( u \) and \( w \) give Gamma-functions [44], with the
where we have written \( \mu = \omega (1 + z) / z \). An elementary analysis shows that necessary and sufficient conditions for (6.6) to be integrable in absolute value in the measure \( d_0 g = \frac{1}{2} z^{-3/2} (1 + z) d\omega dz \) are
\[
\frac{1}{2} (p + q - 2) + \frac{1}{2} (l + j) + m + n > 0 , \\
\frac{1}{2} (p + q - 2) + \frac{1}{2} (l + j) + m' + n' > 0 .
\] (6.7)

As \( p + q > 2 \) by assumption, (6.7) is satisfied, and the integral of (6.6) in the measure \( d_0 g \) is well defined. To evaluate this integral, we perform first the \( dz \) integral by 3.194.3 in [41]. If one of the exponents in the remaining \( d\omega \) integral is less than unity, a contour deformation brings the integral to a form to which 3.194.3 in [41] applies, and an analytic continuation in the exponents shows that the result is valid also for larger exponents.

Collecting, we find
\[
(\Psi^{j'm'k'k'}_{j'k'}, \Psi^{lmnkukw}_{lmnkukw})_{ga} = \frac{\pi}{2} \delta_{ll'} \delta_{kk'} \delta_{rr'} (1 + i\omega)^{\frac{1}{2} (p + q) + 1 + j + m + m' + n + n' - 2} \Gamma\left(\frac{1}{4} p + l + j + m + m' + n + n' - 2\right) \Gamma\left(\frac{1}{4} q + j + n + n' - 2\right) \frac{z^{(1 + i\omega)^{\frac{1}{2} (p + q) + 1 + j + m + m' + n + n' - 2}}}{(1 - i\omega)^{\frac{1}{2} (p + q) + 1 + j + m + m' + n + n' - 2}} ,
\] (6.6)

where \( \eta \) is well defined on \( \Phi \) and given by (6.8). From the relations [8] \( d_0 (hg) = [\Delta (h)]^{1/2} d_0 g \) and \( d_0 (gh) = [\Delta (h)]^{-1/2} d_0 g \) it follows that \( (\cdot, \cdot)_{ga} \) is well defined on all of \( \Phi \) and given by (6.8) and
\[
(\phi_1, U(g)\phi_2)_{ga} = (U(g)\phi_1, \phi_2)_{ga} = [\Delta (g)]^{1/2} (\phi_1, \phi_2)_{ga} .
\] (6.9)

### 6.4 Rigging map

If follows from (6.8) that the map \( \eta \) defined by (5.3) has nontrivial image. We now give an explicit characterisation of the image of \( \eta \) and evaluate \( (\cdot, \cdot)_{RAQ} \), showing that this sesquilinear form defines an inner product.

By (6.9) we have \( \eta (\phi_1) [U(g)\phi_2] = \eta (U(g)\phi_1) [\phi_2] = [\Delta (g)]^{1/2} \eta (\phi_1) [\phi_2] \), and hence \( \eta (U(g)\phi) = [\Delta (g)]^{1/2} \eta (\phi) \). It therefore suffices to evaluate \( \eta (\phi_1) [\phi_2] \) for \( \phi_1, \phi_2 \in \Phi \).

Let
\[
\chi_{ijkukw} (u, w) := \delta (u^2 - w^2) u^{-(p+q-2)/2} Y_{ikw} (\theta (u)) Y_{jkw} (\theta (w)) .
\] (6.10)
We interpret $\chi_{ljkuw}$ as an element of $\Phi^*$, acting on states $\phi \in \Phi$ by

$$\chi_{ljkuw}^*[\phi] = \int d^p u \, d^q w \, \chi_{ljkuw}(u, w) \phi(u, w) \ .$$

(6.11)

An explicit computation gives

$$\chi_{ljkuw}^*[\Psi_{ljmnuw}] = \frac{1}{4} \delta_{ll'} \delta_{jk} \delta_{k'k} \delta_{k_wk_w} \Gamma \left( \frac{1}{4} (p + q - 2) + \frac{1}{2} (l + j) + m + n \right) \ .$$

(6.12)

From (5.3), (6.8) and (6.12) we find

$$\eta(\Psi_{ljmnuw}) = 2\pi \Gamma \left( \frac{1}{4} (p + q - 2) + \frac{1}{2} (l + j) + m + n \right) \chi_{ljkuw} \ .$$

(6.13)

Hence the image of $\eta$ is spanned by $\{\chi_{ljkuw}\}$. From (5.1), (6.8), (6.12) and (6.13) we obtain

$$\left( \chi_{ljkuw}, \chi_{ljkuw}^* \right)_{\text{RAQ}} = (8\pi)^{-1} \delta_{ll'} \delta_{jk} \delta_{k'k} \delta_{k_wk_w} \ .$$

(6.14)

We see that $(\cdot, \cdot)_{\text{RAQ}}$ is positive definite. It follows that $\eta$ is a rigging map and the physical Hilbert space $\mathcal{H}_{\text{RAQ}}$ is the Cauchy completion of the image of $\eta$ in $(\cdot, \cdot)_{\text{RAQ}}$.

The representation (3.6) of $\mathcal{A}_{\text{phys}}$ on $\mathcal{H}_{\text{aux}}$ leaves $\Phi$ invariant and commutes with $U(g)$, and the star-relation in this representation coincides with the adjoint map on $\mathcal{H}_{\text{aux}}$. It follows that $\mathcal{A}_{\text{phys}} \subset \mathcal{A}_{\text{obs}}$. Comparison of (6.10) and (6.11) to (3.5) shows that the representation of $\mathcal{A}_{\text{phys}}$ on $\mathcal{H}_{\text{RAQ}}$ is antilinearly isomorphic to the representation of $\mathcal{A}_{\text{phys}}$ obtained in the algebraic quantisation in subsection 3.1.

7 Refined algebraic quantisation for $p = q = 1$

In this section we carry out refined algebraic quantisation for $p = q = 1$.

When $p = q = 1$, the convergence conditions (3.7) fail to hold when $l = j = 0$ and at least one of the pairs $(m, n)$ and $(m', n')$ equals $(0, 0)$. Further, the integral of (6.6) is unambigiously divergent for $l = l' = j = j' = m = m' = n = n' = 0$. We shall remedy this problem by modifying the test space.

Dropping the redundant indices $k_u$ and $k_w$, we write

$$\phi_{ljmno}(u_1, w_1) := \Psi_{ljmno0} = u^{l+2m} w^{j+2n} e^{-\frac{1}{2}(u^2+w^2)} Y_l(\theta(u)) Y_j(\theta(w)) \ ,$$

(7.1)

where $l, j \in \{0, 1\}$. We also define

$$\psi_{mn} := -2\phi_{00,m+1,n+1} + (2m + 1)\phi_{00,m,n+1} + (2n + 1)\phi_{00,m+1,n} \ .$$

(7.2)

$$\Phi_0^{\text{mod}} := \text{span\{ } \phi_{ljmn} | l + j > 0 \} \cup \{ \psi_{mn} \} \ .$$

(7.3)

An explicit computation shows

$$\hat{C}_{11}\phi_{00mn} = 2i(-\phi_{11mn} + m\phi_{11,m-1,n} + n\phi_{11,m,n-1}) \ ,$$

(7.4a)

$$\hat{C}_{11}\phi_{01mn} = i[-2\phi_{01,m,n+1} + 2m\phi_{01,m-1,n+1} + (2n + 1)\phi_{01mn}] \ ,$$

(7.4b)

$$\hat{C}_{11}\phi_{10mn} = i[-2\phi_{01,m+1,n} + 2n\phi_{01,m+1,n-1} + (2m + 1)\phi_{01mn}] \ ,$$

(7.4c)

$$\hat{C}_{11}\phi_{11mn} = i\psi_{mn} \ ,$$

(7.4d)
where $\hat{C}_{11}$ is the single generator of $A^{(s)}_{\text{phys}}$. Hence $\Phi^\text{mod}_0$ is invariant under $A^{(s)}_{\text{phys}}$. We show in appendix C that $\Phi^\text{mod}_0$ is dense in $H_{\text{aux}}$.

We denote by $\Phi^\text{mod}$ the closure of $\Phi^\text{mod}_0$ under the algebra generated by $\{U(g) \mid g \in G\}$ and adopt $\Phi^\text{mod}$ as the test space. By construction, $\Phi^\text{mod}$ is invariant under $A^{(s)}_{\text{phys}}$. From (6.7), (7.2) and (7.3) it follows that the integral in (5.2) converges in absolute value, and $(\cdot, \cdot)_{g^a}$ is hence well defined. The evaluation of $(\cdot, \cdot)_{g^a}$ for $l + j > 0$ proceeds as in section 6, while $(\psi^{m'n'}, \psi^{mn})_{g^a} = 0$ by an explicit computation using (6.8) and (7.2), in the index range where (6.8) is valid. Hence the formulas of section 6 hold for $l + j > 0$, while $\eta$ sends the whole $l = j = 0$ sector of $\Phi^\text{mod}$ to zero.

$H_{\text{RAQ}}$ has dimension three. The representation of $A^{(s)}_{\text{phys}}$ on $H_{\text{RAQ}}$ is trivial.

8 Discussion

In this paper we have discussed refined algebraic quantisation with group averaging in a constrained Hamiltonian system with unreduced phase space $\mathbb{R}^{2(p+q)}$, where $p \geq 1$ and $q \geq 1$, and a nonunimodular gauge group. The system arose by restricting the gauge transformations in a system with the unimodular gauge group $\text{SL}(2, \mathbb{R})$, previously analysed in [10, 14], to the connected two-dimensional nonunimodular subgroup $G$ that consists of lower triangular matrices with positive diagonal elements. We obtained a Hilbert space with a nontrivial representation of the distinguished $\mathfrak{o}(p,q)$ observable algebra for $(p,q) \neq (1,1)$, which is precisely the condition for the classical reduced phase space to be a symplectic manifold up to a singular subset of measure zero. The representation was found to be the end-point of the maximally degenerate principal unitary series [32], with the quadratic Casimir taking the value $-\frac{1}{4}(p+q-2)^2$.

By contrast, the reduced phase space of the $\text{SL}(2, \mathbb{R})$ system is a symplectic manifold up to a singular subset of measure zero for $\min(p,q) \geq 2$, but group averaging in this system gives a Hilbert space with a nontrivial representation of the $\mathfrak{o}(p,q)$ observables only when $\min(p,q) \geq 2$ and $p + q \equiv 0 \pmod{2}$ [10, 14]. The quadratic Casimir in this representation takes the value $-\frac{1}{4}(p+q)(p+q-4)$. The difference in the Casimirs in the $G$ system and the $\text{SL}(2, \mathbb{R})$ system arises from the different senses in which the physical states produced by group averaging satisfy the constraints for unimodular and nonunimodular gauge groups [8]. The physical states in the $\text{SL}(2, \mathbb{R})$ system are invariant under $\text{SL}(2, \mathbb{R})$, in a representation induced from the unitary representation on the auxiliary Hilbert space, while the physical states in the $G$ system are not invariant under the corresponding representation of $G$.

As neither $\text{SL}(2, \mathbb{R})$ nor $G$ is compact, convergence of the group averaging was an issue in both systems. We found for each system a test space on which the convergence is sufficiently strong for the group averaging formulation of [8] to apply, and we could further choose this space so that the $\mathfrak{o}(p,q)$ observables are included in the physical observable algebra. In the $\text{SL}(2, \mathbb{R})$ system the uniqueness theorem of Giulini and Marolf [8] implied that the group averaging rigging map is the only rigging map admitted by the auxiliary Hilbert space, the representation of the gauge group and the test space. In
the $G$ system the number of rigging maps admitted by the auxiliary Hilbert space, the representation of the gauge group and the test space in our $G$ system remains open, even though the test spaces in the two systems are very similar. The reason for this difference is that for a nonunimodular gauge group the uniqueness theorem of Giulini and Marolf assumes absolute convergence not just in the measure $d_0 g$ in which the actual averaging is performed, but in the whole family of measures $\{\Delta^{n/2}(g) d_0 g \mid n \in \mathbb{Z}\}$, and for our test space this convergence fails for large $|n|$.

We also quantised the nonunimodular gauge system in the algebraic quantisation scheme of [33, 34], seeking an inner product in which the classical $\mathfrak{o}(p,q)$ observables are promoted into an irreducibly-represented algebra of self-adjoint operators. When we took the physical states to satisfy the constraints in a sense that corresponds to that in group averaging, we found quantum theories that are isomorphic to the irreducible representations of the $\mathfrak{o}(p,q)$ algebra in the group averaging quantum theory, and we displayed these theories explicitly for $p + q \leq 4$. When we took the physical states to be strictly annihilated by the generators of the $G$-action, we found qualitatively similar quantum theories for $(p,q) = (1,3), (3,1)$ and $(2,2)$, but for $(p,q) = (1,2)$ and $(2,1)$, and for $p \geq 2, q \geq 2$ and $p + q \equiv 1 \pmod{2}$, algebraic quantisation produced no quantum theory, under certain technical assumptions that parallel those of the group averaging theory. It would be of interest to understand to what extent these phenomena, in particular our failure to find quantum theories for some $(p,q)$, result from the technical input we used and to what extent they arise from deeper reasons in the representation theory of $O(p,q)$. Similar questions arose in algebraic and refined algebraic quantisation of the $\text{SL}(2,\mathbb{R})$ system in [14].

Our analysis adapts readily to the system in which $G$ is replaced by the full lower triangular subgroup $G_{\text{ext}}$ of $\text{SL}(2,\mathbb{R})$. Let $\text{Id}$ stand for the $2 \times 2$ identity matrix. $G_{\text{ext}}$ consists of two connected components, the component of identity is $G$, $\{\text{Id}, -\text{Id}\} \cong \mathbb{Z}_2$ is a normal Abelian subgroup of $G_{\text{ext}}$, and $G_{\text{ext}}/\{\text{Id}, -\text{Id}\} \cong G$. From (2.5) it follows that the reduced phase space of the $G_{\text{ext}}$ system is the quotient of $\mathcal{M}$ under the $\mathbb{Z}_2$ group generated by the involutive map $P : (u, w, p, \varpi) \mapsto (-u, -w, -p, -\varpi)$. From subsection 2.3 it is seen that this quotient acts individually on each of $\mathcal{M}_0$, $\mathcal{M}_{\text{ex}}$ and $\mathcal{A}_{\text{class}}$, the quotient of $\mathcal{M}_{\text{reg}}$ is a manifold with the induced symplectic structure, and $\mathcal{A}_{\text{class}}$ separates the quotient of $\mathcal{M}_{\text{reg}}$ up to a set of measure zero. Two choices to promote $P$ into a quantum operator are $\hat{P}_\epsilon : \Psi(u, w) \mapsto (-1)^\epsilon \Psi(-u, -w)$, where the index $\epsilon \in \{0, 1\}$ labels the choice. Given $\epsilon$, both algebraic and refined algebraic quantisations are then modified by restriction to parity $\epsilon$ states.

Like previous work [9, 10, 14], our work displayed the dual role of the test space in refined algebraic quantisation. This space is a fundamental technical ingredient in achieving convergence of the group averaging with a noncompact gauge group, and it has a deep physical significance in that it determines the algebra of physical observables. In specific systems like ours, these issues of principle become however intertwined with the practical issue of finding a test space for which the rigging map can be analysed and evaluated at a sufficient level of detail. General theorems on the existence of test
spaces that lead to physically reasonable quantisations are still very much lacking. One
expects this question to remain significant when techniques akin to group averaging are
extended to systems in which the constraint algebra closes with nonconstant structure
functions [13].

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A Appendix: Gauge group

In this appendix we collect some relevant properties of the gauge group $G$. The notation
follows the SL(2, $\mathbb{R}$) notation of [45].

$G$ is the lower triangular subgroup of SL(2, $\mathbb{R}$) with positive diagonal elements. $G$
is two-dimensional, nonabelian and connected, which properties characterise $G$ uniquely
up to isomorphisms.

The Lie algebra $\mathfrak{g}$ is spanned by the matrices

\[
\begin{align*}
h & := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

(A.1)

whose commutator is

\[
[h, e^-] = -2e^-.
\]

(A.2)

Elements of $G$ can be written uniquely as

\[
g = \exp(\mu e^-) \exp(\lambda h)
\]

\[
= \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix},
\]

(A.3)

where $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

From (A.3) we have $g^{-1}dg = h\lambda + e^{-2\lambda}d\mu$ and $dgg^{-1} = h\lambda + e^- (d\mu + 2\mu d\lambda)$. The left and right invariant Haar measures are thus respectively $d_Lg = e^{2\lambda}d\lambda d\mu$ and $d_Rg = d\lambda d\mu$.

The adjoint action of $G$ on $\mathfrak{g}$ reads $\text{Ad}_g(h) = ghg^{-1} = h + 2\mu e^-$, $\text{Ad}_g(e^-) = ge^- g^{-1} = e^{-2\lambda}e^-$. Hence the modular function is $\Delta(g) := \det(\text{Ad}_g) = e^{-2\lambda}$. The symmetric measure, invariant under $g \mapsto g^{-1}$, is $d_0g = [\Delta(g)]^{1/2}d_Lg = [\Delta(g)]^{-1/2}d_Rg = e^{\lambda}d\lambda d\mu$. 

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Appendix: Separation of $\mathcal{M}_{\text{reg}}$ by $\mathcal{A}_{\text{class}}$

In this appendix we verify the separation properties of $\mathcal{A}_{\text{class}}$ on $\mathcal{M}_{\text{reg}}$ stated in subsection 2.3. We represent $\mathcal{M}_{\text{reg}}$ as the set $\mathcal{N}_{\text{reg}}$ defined by (2.10).

**Theorem B.1** Let $p > 1$ and $q > 1$, and let $\mathcal{N}_{\text{reg}}^+ \subset \mathcal{N}_{\text{reg}}$ where $0 \neq p^2 \neq q^2 \neq 0$. Then $\mathcal{A}_{\text{class}}$ separates $\mathcal{N}_{\text{reg}}^+$ up to the twofold degeneracy (2.17).

**Proof.** Let $a = (u, p, w, \varpi) \in \mathcal{N}_{\text{reg}}^+$ and $b = (u', p', w', \varpi') \in \mathcal{N}_{\text{reg}}^+$ such that $A(a) = A(b)$ for all $A \in \mathcal{A}_{\text{class}}$. We shall show that $a = \pm b$. We use the basis (2.6).

Using (2.10), the condition $A_{ij}(a) = A_{ij}(b)$ implies

\[
\begin{align*}
    u' &= \cos(\theta)u + \sin(\theta)|p|^{-1}p, \quad (B.1a) \\
    p' &= \cos(\theta)p - \sin(\theta)|p|u, \quad (B.1b)
\end{align*}
\]

where $0 \leq \theta < 2\pi$. Similarly, the condition $B_{ij}(a) = B_{ij}(b)$ implies

\[
\begin{align*}
    w' &= \cos(\varphi)w + \sin(\varphi)|\varpi|^{-1}\varpi, \quad (B.2a) \\
    \varpi' &= \cos(\varphi)\varpi - \sin(\varphi)|\varpi|w, \quad (B.2b)
\end{align*}
\]

where $0 \leq \varphi < 2\pi$. With (B.1) and (B.2), the condition $C_{ij}(a) = C_{ij}(b)$ reads

\[
0 = p_i w_j \left[ \cos(\theta) \cos(\varphi) - 1 + \sin(\theta) \sin(\varphi) |\varpi| |p|^{-1} \right] \\
- u_i \varpi_j \left[ \cos(\theta) \cos(\varphi) - 1 + \sin(\theta) \sin(\varphi) |p| |\varpi|^{-1} \right] \\
+ u_i w_j \left[ \cos(\theta) \sin(\varphi) |\varpi| - \sin(\theta) \cos(\varphi) |p| \right] \\
+ p_i \varpi_j \left[ \cos(\theta) \sin(\varphi) |\varpi|^{-1} - \sin(\theta) \cos(\varphi) |p|^{-1} \right]. \quad (B.3)
\]

Contracting (B.3) with $u_i w_j$, $u_i \varpi_j$, $p_i w_j$, and $p_i \varpi_j$ shows that each of the four terms must vanish individually. Using $|p| \neq |\varpi|$, elementary algebra shows that the only solutions are $\theta = \varphi = 0$ and $\theta = \varphi = \pi$.

**Remark.** The assumption $|p| \neq |\varpi|$ is necessary, for otherwise (B.3) is satisfied by $\theta = \varphi$.

**Theorem B.2** Let $p > 1$ and $q > 1$ (respectively $p > 1$ and $q = 1$), and let $\mathcal{N}_{\text{reg}}^+$ be the subset of $\mathcal{N}_{\text{reg}}$ where $\varpi^2 \neq 0$ ($p^2 \neq 0$). Then $\mathcal{A}_{\text{class}}$ separates $\mathcal{N}_{\text{reg}}^+$ up to the twofold degeneracy (2.17).

**Proof.** It suffices to consider $p = 1$ and $q > 1$.

Proceeding as in the proof of Theorem B.1, the condition $A_{ij}(a) = A_{ij}(b)$ is an identity, the condition $B_{ij}(a) = B_{ij}(b)$ leads to (B.2), and the condition $C_{ij}(a) = C_{ij}(b)$ reads $u_1 \varpi = u'_1 \varpi'$. As $(u'_1)^2 = (u_1)^2 = 1$, the result follows by contracting (B.2b) with $\varpi$. ■
Appendix: $\Phi_0^{\text{mod}}$ is dense in $\mathcal{H}_{\text{aux}}$

**Theorem C.1** For $p = q = 1$, the test space $\Phi_0^{\text{mod}}$ defined in section 7 is dense in $\mathcal{H}_{\text{aux}}$.

**Proof.** Let $\mathcal{H}_{\text{aux}}^{++} \subset \mathcal{H}_{\text{aux}}$ be the subspace in which the functions are even both in $u_1$ and in $w_1$. By (7.3), it suffices to show that $\Phi_0^{\text{mod}++} := \text{span}\{\psi_{mn}\}$ is dense in $\mathcal{H}_{\text{aux}}^{++}$.

Let

$$\tilde{\phi}_{mn} := 2^{-(m+n)/2}(\pi m!)^{-1/2}H_m(u_1)H_n(w_1)\exp\left[-\frac{1}{2}(u_1^2 + w_1^2)\right], \quad (C.1)$$

where the $H$s are the Hermite polynomials [13]. \{$\tilde{\phi}_{mn}$\} is an orthonormal basis of $\mathcal{H}_{\text{aux}}$, \{$\tilde{\phi}_{2m,2n}$\} is an orthonormal basis of $\mathcal{H}_{\text{aux}}^{++}$, and the recursion relations of the Hermite polynomials imply

$$\hat{C}_{11}\tilde{\phi}_{mn} = i\left(-\sqrt{(m+1)(n+1)}\tilde{\phi}_{m+1,n+1} + \sqrt{mn}\tilde{\phi}_{m-1,n-1}\right). \quad (C.2)$$

As $\text{span}\{\phi_{1mn}\} = \text{span}\{\tilde{\phi}_{2m+1,2n+1}\}$, (7.4d) shows that $\Phi_0^{\text{mod}++} = \text{span}\{\hat{C}_{11}\tilde{\phi}_{2m+1,2n+1}\}$.

Suppose now that $\Phi_0^{\text{mod}++}$ is not dense in $\mathcal{H}_{\text{aux}}^{++}$. Then there exists a nonzero vector $y = \sum_{mn} a_{mn} \tilde{\phi}_{2m,2n} \in \mathcal{H}_{\text{aux}}^{++}$ that is orthogonal to each $\hat{C}_{11}\tilde{\phi}_{2m'+1,2n'+1}$. By (C.2), the orthogonality implies

$$a_{m+1,n+1} = \sqrt{\frac{(m + \frac{1}{2})(n + \frac{1}{2})}{(m + 1)(n + 1)}} a_{mn}, \quad (C.3)$$

from which it follows by elementary analysis that $y$ has finite norm only if it is the zero vector. Hence $\Phi_0^{\text{mod}++}$ is dense in $\mathcal{H}_{\text{aux}}^{++}$, and $\Phi_0^{\text{mod}}$ is dense in $\mathcal{H}_{\text{aux}}$. □
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