Non-stationary temperature field of the heating device in the conditions of unsteady thermal field of the space

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Abstract. The trend of the modern world is aimed at improving the energy efficiency of buildings and reducing unnecessary costs of heat and electricity. In view of this, heating systems are used with a periodic operating time, in order to reduce the cost of heat energy during periods of absence of people. As a result, this paper will consider a method for determining the temperature of a heating device, taking into account the heat accumulation by the metal body of the device and the heat carrier inside it, with changing environmental parameters, which will allow taking into account the accumulated heat during the heating system operation.

1. Introduction
This scientific work is aimed at determining the wall temperature of a heating device with changing parameters that directly affect heat exchange with the environment. Since 95% of building heating systems in the Russian Federation are water-based, the processes of changing the temperature of the wall of the heating device will directly depend on the heat carrier inside the device - water. In this regard, in order to obtain the most accurate results, it is necessary to take into account the heat flow from the heat carrier. In view of the large number of water parameters that affect heat exchange, to simplify the equation, this article introduces the concept of "volume density of heat sources", which takes into account the transfer of heat from water to the heating device.

The volume density function of heat sources describes the heat exchange of water with the inner wall of the heater. When cooling down, the heat exchange inside the device will be characterized by free convection. The movement of air washing the walls of the heating device will be accompanied by cooling the surface of the device and thereby drive the heat carrier, due to the difference in water temperature inside its volume. Thus, the volume density function of heat sources expresses the heat flow from the heat carrier at each time, while taking into account the structural dimensions of the heating device and the physical parameters of water that characterize the internal heat transfer coefficient.

Despite numerous works in the field of temperature fields, that is, consideration of heating and cooling of various objects of various shapes and sizes [6-22]. In this paper, various types of heating devices are considered and this method is applicable to all types of water radiators. One of the areas of applicability of this method is the calculation of the processes of heating and cooling of the building, taking into account the accumulative heat – the mass of metal of the heating device body and the heat
carrier inside it. These processes can occur during temporary or emergency shutdown of the building's heat supply, operation of heating devices in the "on-duty" or intermittent heating system.

The purpose of on-duty heating is to maintain the set room temperature during the absence of heat access to compensate for heat losses, to prevent accidents caused by cooling the building(room). Taking into account all parameters that affect the temperature field of the heating device, it will allow you to identify: the time of reaching the operating temperatures of the coolant of the standby heating system; it will reduce the operating time of the system by taking into account the thermal potential of the heat carrier filling the heating device and the device itself, which will maintain the set temperature with residual thermal inertia until heat is available. Based on the above, the standby heating system will be designed in a more balanced way.

We would like to pay special attention to the applicability of this method to individual temperature control of heating devices. In the case of automatic shut-off of the coolant flow in cases of compensation for heat losses by heat input from the environment, according to computer modeling, it is ineffective if the heat storage capacity of the heating device is high. Thus, accounting for accumulated heat with a heating device would not be completely correct or inappropriate at all.

2. Materials and methods
First, let's find a solution to a one-dimensional problem. The influence of water in the heating device is replaced by the volume density of heat sources.

So the problem statement will look like this:

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2} + \frac{q(x, \tau)}{c \rho}$$

(1)

Here $T$ - the temperature of the wall of the heating device, $a = \frac{\lambda}{c \rho}$ - thermal diffusivity, $q(x, \tau)$ - volume density of heat sources from the liquid in the heating device, $c, \rho, \lambda$ - accordingly, the specific heat capacity, density and coefficient of thermal conductivity of the heating device material.

As an initial condition, we will set a certain temperature distribution:

$$T(x, 0) = T_0$$

(2)

Next, we define the boundary conditions. The heating device is located in the room, so the heat transfer will occur in two ways: convection and radiation.

Let's write down the effect of convection and radiation in general form:

$$-\lambda \frac{\partial T}{\partial n} = \alpha(\tau)(T(\delta, \tau) - T_{env}(\tau)) + \sigma(\tau)\left(T^4(\delta, \tau) - T_{rad}^4(\tau)\right)$$

$$-\lambda \frac{\partial T}{\partial n} = \alpha(\tau)(T(-\delta, \tau) - T_{env}(\tau)) + \sigma(\tau)\left(T^4(-\delta, \tau) - T_{rad}^4(\tau)\right)$$

(3)

Where $\alpha(\tau)$ - the heat transfer coefficient of the heating device - air system, $\sigma$ - Stefan-Boltzmann constant, $T_{rad}(\tau)$ - average radiation temperature of the surfaces around the heater.

The main difficulty of the solution is the nonlinearity of the part responsible for radiation. As part of this task, we propose to use linearization of boundary conditions. Taking into account linearization, the system will look like:

$$-\lambda \frac{\partial T}{\partial x} = \alpha(\tau)(T(\delta, \tau) - T_{env}(\tau)) + 4\sigma T_{rad}^3(\tau)\left(T(\delta, \tau) - T_{rad}(\tau)\right)$$

$$-\lambda \frac{\partial T}{\partial x} = \alpha(\tau)(T(-\delta, \tau) - T_{env}(\tau)) + 4\sigma T_{rad}^3(\tau)\left(T(-\delta, \tau) - T_{rad}(\tau)\right)$$

(4)

To simplify the problem, we reduce the problem to a dimensionless form:
\[ \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\delta^2 q(X,Fo)}{ac \rho(T_0 - T_{0,env})} \]

(5)

\[ \frac{\partial \theta(1,Fo)}{\partial X} = Bi(Fo)\left[ \theta(1,Fo) - \theta_{env}(Fo) \right] + 4Sk(Fo)\left[ \theta(1,Fo) - \theta_{rad}(Fo) \right] \]

(6)

Where \( \theta = \frac{T - T_{0,env}}{T_0 - T_{0,env}} \) - dimensionless excess temperature of the heating device, \( \theta_{env} = \frac{T_{env} - T_{0,env}}{T_0 - T_{0,env}} \) - dimensionless excess temperature of the heating device, \( \theta_{rad} = \frac{T_{rad} - T_{0,env}}{T_0 - T_{0,env}} \) - dimensionless radiation temperature of indoor surfaces, \( Bi(Fo) = \frac{\alpha(Fo) \cdot \delta}{\lambda} \) - Bio number, \( Sk(Fo) = \frac{\sigma \cdot T^4_{rad}(Fo) \cdot \delta}{\lambda} \) - Stark number, \( \delta \) - half the thickness of the heater/plate (depends on the geometry in question and will be discussed later).

We obtain a solution to equation (5) with the initial condition (6) taken into account. To begin with, we will look for a solution, assuming that the heating device is located in infinite space. We use the Fourier transform with the core \( e^{ikx} \). Let’s denote the Fourier images of functions \( u(X,Fo) = \theta(X,Fo) - 1 \) and \( f(X,Fo) = \frac{\delta^2 q(X,Fo)}{ac \rho(T_0 - T_{0,env})} \) respectively through \( U(k,Fo) \) and \( F(k,Fo) \):

\[ U(k,Fo) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(X,Fo) e^{ikx} dX \]

(9)

\[ F(k,Fo) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(X,Fo) e^{ikx} dX \]

(10)

For the existence of the Fourier integral, we assume that the function \( u(X,Fo) \) and its partial derivatives quickly tend to zero at \( x \to \pm \infty \). Also, we assume that the integral in (9) can be differentiated under the sign of the integral.

Multiply both parts of equation (5) by \( \frac{1}{2\pi} e^{ikx} \) and integrate from \( -\infty \) to \( +\infty \). After that we get:

\[ U_{Fo} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u_{xx}(X,Fo) e^{ikx} dX + F(k,Fo) \]

(11)

Integrating by parts the right side of the formula (11), taking into account that the substitutions will turn to zero. Taking into account a homogeneous initial condition, we obtain that the problem in the original space corresponds to the following Cauchy problem in the image space:

\[ U_{Fo} + k^2 U = F \]

(12)

\[ U(k,0) = 0 \]

Let’s write down the solution of the problem using the pulse function:

\[ U(k,Fo) = \int_{0}^{Fo} e^{-k^2(Fo-Fo')} F(k,Fo') d(Fo') = \frac{1}{2\pi} \int_{0}^{Fo} e^{-k^2(Fo-Fo')} \left( \int_{-\infty}^{+\infty} f(X',Fo') e^{ikx} d(X') \right) d(Fo') \]

(13)

Using the inverse Fourier transform formula, we obtain:
\[ u(X, Fo) = \int_{-\infty}^{+\infty} U(k, Fo) e^{ikX} dk = \int_{0}^{+\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(X'\cdot X)} dk \right] f(X', Fo') d(X') d(Fo') \] (14)

Let’s introduce the Green’s function:

\[ G(X, X', Fo, Fo') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik(X'\cdot X)} dk = \frac{1}{2\sqrt{\pi}} \frac{(X-X')^2}{4(Fo-Fo')} \] (15)

Then the solution of the problem will look like:

\[ u(X, Fo) = \int_{0}^{+\infty} \frac{1}{\sqrt{\pi}} e^{\frac{(X-X')^2}{4(Fo-Fo')}} f(X', Fo') d(X') d(Fo') \] (16)

But for the problem under consideration, it would be incorrect to assume that the heating device is located in an infinite unlimited space, so we introduce two more thermal potentials. Then, with this condition, the solution of equation (5) under the initial condition (6) can be represented as the sum of the Duhamel integral and two thermal potentials.

\[ \theta(X, Fo) = 1 + \frac{\delta^2}{\alpha c \rho(T_0 - T_{0,env})} \int_{0}^{+\infty} \frac{1}{\sqrt{\pi}} e^{\frac{(X-X')^2}{4(Fo-Fo')}} \cdot q(X', Fo') d(X') d(Fo') + \]

\[ + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{\sqrt{Fo-Fo'}} e^{\frac{(X-X')^2}{4(Fo-Fo')}} dFo' + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_2(Fo')}{\sqrt{Fo-Fo'}} e^{\frac{(X-X')^2}{4(Fo-Fo')}} dFo' \] (17)

Consider the properties of thermal potentials:

\[ \lim_{X \to X_{0,rad}} \frac{\partial}{\partial X} \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{\sqrt{Fo-Fo'}} e^{\frac{(X-X')^2}{4(Fo-Fo')}} dFo' = \pm \varphi(Fo) \] (18)

Let’s prove the equality of thermal potentials \( \varphi_1(Fo') = \varphi_2(Fo) \). For this purpose, let us put (17) in boundary conditions (7), (8), taking into account (18). For simplicity, let us denote Duhamel's integral:

\[ Y(X, Fo) = \frac{\delta^2}{\alpha c \rho(T_0 - T_{0,env})} \int_{0}^{+\infty} \frac{1}{\sqrt{\pi}} e^{\frac{(X-X')^2}{4(Fo-Fo')}} \cdot q(Fo') d(X') d(Fo') \] (19)

After substitution and taking into account (19) we will have:

\[ - \frac{\partial Y(1, Fo)}{\partial X} - \varphi_1(Fo) + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{(Fo-Fo')} e^{-\frac{(X-X')^2}{4(Fo-Fo')}} dFo' + \]

\[ + Bi(Fo) \left[ Y(1, Fo) + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{\sqrt{Fo-Fo'}} dFo' + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_2(Fo')}{\sqrt{Fo-Fo'}} e^{-\frac{(X-X')^2}{4(Fo-Fo')}} dFo' - T_{env}(Fo) \right] + \]

\[ + 4Sk(Fo) \left[ Y(1, Fo) + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{\sqrt{Fo-Fo'}} dFo' + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_2(Fo')}{\sqrt{Fo-Fo'}} e^{-\frac{(X-X')^2}{4(Fo-Fo')}} dFo' - T_{rad}(Fo) \right] = 0 \] (20)

\[ - \frac{\partial Y(-1, Fo)}{\partial X} - \varphi_2(Fo) + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{(Fo-Fo')} e^{-\frac{(X-X')^2}{4(Fo-Fo')}} dFo' + \]

\[ + Bi(Fo) \left[ Y(-1, Fo) + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{\sqrt{Fo-Fo'}} dFo' + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_2(Fo')}{\sqrt{Fo-Fo'}} e^{-\frac{(X-X')^2}{4(Fo-Fo')}} dFo' - T_{env}(Fo) \right] + \]

\[ + 4Sk(Fo) \left[ Y(-1, Fo) + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_1(Fo')}{\sqrt{Fo-Fo'}} dFo' + \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \frac{\varphi_2(Fo')}{\sqrt{Fo-Fo'}} e^{-\frac{(X-X')^2}{4(Fo-Fo')}} dFo' - T_{rad}(Fo) \right] = 0 \] (21)
We get that the difference $\varphi_1 - \varphi_2$ satisfies the condition of the Volterra integral equation, therefore $\varphi_1 = \varphi_2 = \varphi$. By rearranging the terms and performing a few simple algebraic operations, we get an expression for the heat potential density:

$$
\phi(Fo) = \psi(Fo) + \frac{1}{\sqrt{\pi}} \int_0^R K(Fo, Fo')\phi(Fo')d(Fo') = -\frac{\partial Y(1, Fo)}{\partial X} + Bi(Fo)[Y(1, Fo) - T_{\text{env}}(Fo)] +
$$

$$
+ 4Sk(Fo)[Y(1, Fo) - T_{\text{env}}(Fo)] + \frac{1}{\sqrt{\pi}} \int_0^R \left\{ \frac{1}{Fo - Fo'} Bi(Fo)(Fo - Fo')^{-\frac{1}{2}} + 4Sk(Fo)(Fo - Fo')^{-\frac{1}{2}} +
$$

$$
+ (Fo - Fo')^{-\frac{1}{2}} e^{-\frac{1}{Fo-Fo'}} \left[ Bi(Fo) + 4Sk(Fo) + \frac{1}{Fo - Fo'} \right] \phi(Fo')d(Fo')
$$

(22)

Where $K(Fo, Fo')$ - core of the Volterra integral equation.

Due to the complexity of the obtained expression, its analytical solution is not possible. For the solution presented above it is proposed to use numerical methods that can be easily implemented in Maplesoft Maple, Wolfram Mathematica, MathWorks MATLAB, and other software packages.

In this way, we have determined the temperature of the infinite plate, but it is obvious that such a solution will not accurately describe the problem. It is known that in the considered heating systems with a periodic operating time, fluctuations in the temperature of the wall of the heating device occur in the range of $\Delta t \approx 4 \div 6 \ ^\circ \text{C}$. As a result, it is obvious that the temperature around the heater varies in a small range. Then we can use the multiplication theorem for approximate analytical solutions:

$$
\theta = \theta_x \theta_y \theta_z
$$

(23)

$\theta_x, \theta_y$ - dimensionless temperatures along the other two axes of the heater/plate.

Speaking about the geometry of the heating device, we can assume that the radiator has the shape of a parallelepiped. But this topology will not show the exact distribution of heat along the surface of the radiator. According to the above solution for a parallelepiped, we can assume that the radiator consists of the corresponding primitives, that is, we can divide it into smaller parallelepipeds/plates. This geometry will most accurately describe the heating device. It is currently not possible to consider a more precise geometry analytically.

Often, the most necessary task is to find the released amount of heat from the cooling heater. During the heating device will give the amount of heat equal to:

$$
Q = cm(T_0 - T_{\text{env}})(1 - \bar{\theta})
$$

(24)

Where $\bar{\theta}$ - average dimensionless temperature of the heating device.

$$
\bar{\theta} = \frac{1}{0} \theta_x dX \cdot \frac{1}{0} \theta_y dY \cdot \frac{1}{0} \theta_z dZ
$$

(25)

3. Conclusion and discussions

Thus, analytically, using the methods described in [1-5], a technique was obtained that allows us to obtain the most accurate temperature values at different times under different environmental conditions. Direct account of volume density of heat sources and thermal inertia of the heater in this methodology will allow to determine the time of operation of the heating system, which includes the compensation of heat losses of the room accumulated heat capacity of the device when it is disconnected, thereby the heat consumption becomes more appropriate, and the heating system more energy efficient.

As a continuation of the initial study in the future it is necessary to obtain a similar formula for objects having a cylindrical shape, this geometry will describe the pipeline systems located in the
building. This will take into account the thermal inertia of the heating system pipelines in calculating the cooling time of the building, as well as the time of exit to the operating temperature of the coolant.

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