New Properties of Besov and Triebel-Lizorkin Spaces on RD-Spaces

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Abstract An RD-space $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling property holds in $\mathcal{X}$. In this paper, the authors first give several equivalent characterizations of RD-spaces and show that the definitions of spaces of test functions on $\mathcal{X}$ are independent of the choice of the regularity $\epsilon \in (0,1)$; as a result of this, the Besov and Triebel-Lizorkin spaces on $\mathcal{X}$ are also independent of the choice of the underlying distribution space. Then the authors characterize the norms of inhomogeneous Besov and Triebel-Lizorkin spaces by the norms of homogeneous Besov and Triebel-Lizorkin spaces together with the norm of local Hardy spaces in the sense of Goldberg. Also, the authors obtain the sharp locally integrability of elements in Besov and Triebel-Lizorkin spaces.

1 Introduction

The theory of Besov and Triebel-Lizorkin spaces plays an important role in various fields of mathematics such as harmonic analysis, partial differential equations, geometric analysis and etc; see, for example, [20, 17, 18, 19]. Recently, Besov and Triebel-Lizorkin spaces on metric measure spaces obtained a rapid development; see [9, 13, 5, 20].

We first recall the definitions of spaces of homogenous type in [3, 4] and RD-spaces in [9]. In this paper, we always assume that $(\mathcal{X},d)$ is a metric space with a regular Borel measure $\mu$ such that all balls defined by $d$ have finite and positive measures. In what follows, set $\text{diam}(\mathcal{X}) \equiv \sup\{d(x,y) : x, y \in \mathcal{X}\}$ and for any $x \in \mathcal{X}$ and $r > 0$, set $B(x,r) \equiv \{y \in \mathcal{X} : d(x,y) < r\}$.

Definition 1.1. (i) The triple $(\mathcal{X},d,\mu)$ is called a space of homogeneous type if there exists a constant $C_0 \in (1,\infty)$ such that for all $x \in \mathcal{X}$ and $r > 0$,

\[
\mu(B(x,2r)) \leq C_0 \mu(B(x,r)) \quad \text{(doubling property).}
\]
(ii) The triple \((X, d, \mu)\) is called an RD-space if it is a space of homogeneous type and there exist constants \(a_0, C_0 \in (1, \infty)\) such that for all \(x \in X\) and \(0 < r < \text{diam} (X)/a_0,$

\[
\mu(B(x, a_0r)) \geq C_0 \mu(B(x, r)) \quad (\text{reverse doubling property}).
\]

It is easy to see that \(X\) is an RD-space if and only if it is a space of homogeneous type and there exists a constant \(a_0 \in (1, \infty)\) such that for all \(x \in X\) and \(0 < r < \text{diam} (X)/a_0,$

\[
\mu(B(x, a_0r)) \geq 2 \mu(B(x, r)).
\]

We will establish several other equivalent characterizations of RD-spaces in Proposition 2.1 below.

The following spaces of test functions play a key role in the theory of function spaces on RD-spaces; see [8, 9]. In what follows, for any \(x, y \in X\) and \(r > 0\), set \(V(x, y) \equiv \mu(B(x, d(x, y)))\) and \(V_r(x) \equiv \mu(B(x, r)).$

**Definition 1.2.** Let \(x_1 \in X, r \in (0, \infty), \beta \in (0, 1]\) and \(\gamma \in (0, \infty).\) A function \(\varphi\) on \(X\) is called a test function of type \((x_1, r, \beta, \gamma)\) if there exists a nonnegative constant \(C\) such that

\[
(i) \quad |\varphi(x)| \leq C \frac{1}{V(x_1) + V(x)} \left[ \frac{r}{r + d(x_1, x)} \right]^{\gamma} \quad \text{for all } x \in X;
\]

\[
(ii) \quad |\varphi(x) - \varphi(y)| \leq C \left[ \frac{d(x, y)}{r + d(x_1, x)} \right]^{\beta} \left[ \frac{r}{r + d(x_1, x)} \right]^{\gamma} \quad \text{for all } x, y \in X \text{ satisfying } d(x, y) \leq [r + d(x_1, x)]/2.
\]

The space \(\mathcal{G}(x_1, r, \beta, \gamma)\) of test functions is defined to be the set of all test functions of type \((x_1, r, \beta, \gamma).\) If \(\varphi \in \mathcal{G}(x_1, r, \beta, \gamma),\) its norm is defined by \(\|\varphi\|_{\mathcal{G}(x_1, r, \beta, \gamma)} \equiv \inf \{C : (i) \text{ and } (ii) \text{ hold}\}.$

Throughout the whole paper, we fix \(x_1 \in X\) and let \(\mathcal{G}(\beta, \gamma) \equiv \mathcal{G}(x_1, 1, \beta, \gamma).\) It is easy to see that \(\mathcal{G}(\beta, \gamma)\) is a Banach space.

For any given \(\epsilon \in (0, 1],\) let \(\mathcal{G}_{0}^\epsilon(\beta, \gamma)\) be the completion of the space \(\mathcal{G}(\epsilon, \epsilon)\) in \(\mathcal{G}(\beta, \gamma)\) when \(\beta, \gamma \in (0, \epsilon].\) Obviously, \(\mathcal{G}_{0}^\epsilon(\epsilon, \epsilon) = \mathcal{G}(\epsilon, \epsilon).\) Moreover, it is easy to see that \(\varphi \in \mathcal{G}_{0}^\epsilon(\beta, \gamma)\) if and only if \(\varphi \in \mathcal{G}(\epsilon, \epsilon)\) and there exists \(\{\phi_i\}_{i \in \mathbb{N}} \subset \mathcal{G}(\epsilon, \epsilon)\) such that \(|\varphi - \phi_i|_{\mathcal{G}(\beta, \gamma)} \rightarrow 0\) as \(i \rightarrow \infty.\) If \(\varphi \in \mathcal{G}_{0}^\epsilon(\beta, \gamma),\) define \(\|\varphi\|_{\mathcal{G}_{0}^\epsilon(\beta, \gamma)} \equiv \|\varphi\|_{\mathcal{G}(\beta, \gamma)}.\) Obviously, \(\mathcal{G}_{0}^\epsilon(\beta, \gamma)\) is a Banach space and \(\|\varphi\|_{\mathcal{G}_{0}^\epsilon(\beta, \gamma)} = \lim_{i \rightarrow \infty} \|\phi_i\|_{\mathcal{G}(\beta, \gamma)}\) for the above chosen \(\{\phi_i\}_{i \in \mathbb{N}}.\) Let \((\mathcal{G}_{0}^\epsilon(\beta, \gamma))^\prime\) be the set of all bounded linear functionals \(f\) from \(\mathcal{G}_{0}^\epsilon(\beta, \gamma)\) to \(\mathbb{C}.)\) Denote by \(\langle f, \varphi \rangle\) the natural pairing of elements \(f \in (\mathcal{G}_{0}^\epsilon(\beta, \gamma))^\prime\) and \(\varphi \in \mathcal{G}_{0}^\epsilon(\beta, \gamma).$

In what follows, we define

\[
\mathcal{G}(x_1, r, \beta, \gamma) \equiv \left\{ f \in \mathcal{G}(x_1, r, \beta, \gamma) : \int_X f(x) d\mu(x) = 0 \right\}.
\]

The space \(\mathcal{G}_{0}^\epsilon(\beta, \gamma)\) is defined to be the completion of the space \(\mathcal{G}(\epsilon, \epsilon)\) in \(\mathcal{G}(\beta, \gamma)\) when \(\beta, \gamma \in (0, \epsilon).\) Moreover, if \(f \in \mathcal{G}_{0}^\epsilon(\beta, \gamma),\) we then define \(\|f\|_{\mathcal{G}_{0}^\epsilon(\beta, \gamma)} = \|f\|_{\mathcal{G}(\beta, \gamma)}.$

One of the main targets of this paper is to show that spaces of test functions are independent of the choices of \(\epsilon \in (0, 1)\) via the continuous Calderón reproducing formulae.

**Theorem 1.1.** Let \(\epsilon, \bar{\epsilon} \in (0, 1)\) and \(0 < \beta, \gamma < (\epsilon \wedge \bar{\epsilon}).\) Then \(\mathcal{G}_{0}^\epsilon(\beta, \gamma) = \mathcal{G}_{0}^{\bar{\epsilon}}(\beta, \gamma)\) and \(\mathcal{G}_{0}^\epsilon(\beta, \gamma) = \mathcal{G}_{0}^{\bar{\epsilon}}(\beta, \gamma).$
The proof of Theorem 1.1 will be given in Section 3.

Based on the above spaces of test functions, the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathcal{X})$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathcal{X})$, and the inhomogeneous Besov spaces $B^s_{p,q}(\mathcal{X})$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathcal{X})$ on RD-spaces were introduced in [9]; see also Definitions 4.1 through 4.3 below. As a corollary of Theorem 1.1, we see that the Besov and Triebel-Lizorkin spaces are independent of the regularity of the underlying distribution space; see Corollary 4.1 below.

Recall that the local Hardy space $h^p(\mathcal{X})$ with $p \in (n/(n+1), \infty)$ is just the inhomogeneous Triebel-Lizorkin space $F^0_{p,2}(\mathcal{X})$; see [9, Theorem 5.42].

Recently, Koskela and Saksman [11] characterized the Hardy-Sobolev space by using some Hajlasz-Sobolev space on $\mathbb{R}^n$. Motivated by this, as another main target of this paper, in Theorem 1.2 below, we characterize the norms of inhomogeneous Besov spaces and Triebel-Lizorkin spaces by the norms of homogeneous Besov and Triebel-Lizorkin spaces together with the norm of local Hardy spaces in the sense of Goldberg (see [6]).

**Theorem 1.2.** Let $\mu(\mathcal{X}) = \infty$ and $\mathcal{X}$ have the “dimension” $n$ as in (2.2) below. Let $s \in (0,1)$ and $p \in (n/(n+s), \infty)$. Let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation of the identity of order 1 with bounded support as in Definition 3.1 and set $D_k \equiv S_k - S_{k-1}$ for all $k \in \mathbb{Z}$.

(i) If $q \in (0,\infty]$, then $f \in B^s_{p,q}(\mathcal{X})$ if and only if $f \in h^p(\mathcal{X})$ and

$$
(1.2) \quad \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k(f)\|^q_{L^p(\mathcal{X})} \right\}^{1/q} \equiv J_1 < \infty;
$$

moreover, $\|f\|_{B^s_{p,q}(\mathcal{X})}$ is equivalent to $\|f\|_{h^p(\mathcal{X})} + J_1$.

(ii) If $q \in (n/(n+s), \infty]$, then $f \in \dot{F}^s_{p,q}(\mathcal{X})$ if and only if $f \in h^p(\mathcal{X})$ and

$$
\left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q \right\}^{1/q} \equiv J_2 < \infty;
$$

moreover, $\|f\|_{\dot{F}^s_{p,q}(\mathcal{X})}$ is equivalent to $\|f\|_{h^p(\mathcal{X})} + J_2$.

The proof of Theorem 1.2 is given in Section 4.

**Remark 1.1.** (i) It is known that when $p \in (1,\infty)$, $h^p(\mathcal{X}) = L^p(\mathcal{X})$ and when $p \in (n/(n+1),1)$, $(h^p(\mathcal{X}) \cap L^p_{\text{loc}}(\mathcal{X})) \subseteq L^p(\mathcal{X})$; see [9]. Thus, when $p \in (n/(n+1),1]$, even for the Euclidean space $\mathbb{R}^n$, Theorem 1.2 is also an improvement of the known classical results; see [18, Theorem 2.3.3].

(ii) Theorem 1.2 with $p \in [1,\infty]$ and $h^p(\mathcal{X})$ replaced by $L^p(\mathcal{X})$ was obtained in [9, Proposition 5.39]. However, differently from the Euclidean space (see [18, Theorem 2.3.3]), it is unclear if Theorem 1.2 is still true if $h^p(\mathcal{X})$ is replaced by $L^p(\mathcal{X})$ when $p \in (n/(n+s),1)$.

(iii) When $p = \infty$, it is known that $F^0_{p,2}(\mathcal{X}) = \text{bmo}(\mathcal{X})$; see [9, Theorem 6.28]. In this case, if we replace $h^p(\mathcal{X})$ by $\text{bmo}(\mathcal{X})$ in Theorem 1.2, all conclusions of Theorem 1.2 are
still true. This can be deduced from the corresponding conclusions described in (ii) of this remark and an easy argument as in the proof of Theorem 1.2.

(iv) Usually, it makes no sense to write the conclusions of Theorem 1.2 into $\dot{B}^s_{p,q}(\mathcal{X}) = (h^p(\mathcal{X}) \cap \dot{B}^s_{p,q}(\mathcal{X}))$ and $F^s_{p,q}(\mathcal{X}) = (h^p(\mathcal{X}) \cap F^s_{p,q}(\mathcal{X}))$, since homogeneous and inhomogeneous spaces are defined via different kinds of spaces of distributions, which was pointed out by Professor Hans Triebel to the first author.

In Section 5 of this paper, via the discrete Calderón reproducing formulae in [9], we study the locally integrability of elements in Besov and Triebel-Lizorkin spaces; see Propositions 5.2 and 5.3 below. Such results on $\mathbb{R}^n$ were obtained by Sickel and Triebel [16]. However, the method used in [16] is not valid for RD-spaces $\mathcal{X}$, since there exists none counterpart on $\mathcal{X}$ of the embedding theorems for different metrics as in Triebel [17, p. 129] on $\mathbb{R}^n$; see the proof of [16, Theorem 3.3.2].

The results in this paper apply in a wide range of settings, for instance, to Ahlfors $n$-regular metric measure spaces (see [10]), $d$-spaces (see [20]), Lie groups of polynomial volume growth (see [21, 15, 1]), (compact) Carnot-Carathéodory (also called sub-Riemannian) manifolds with doubling measures (see [15, 7]) and to boundaries of certain unbounded model domains of polynomial type in $\mathbb{C}^N$ appearing in the work of Nagel and Stein (see [14, 15]).

Remark 1.2. We point out that in the original definition of spaces of homogeneous type in [3, 4], $d$ is only assumed to be a quasi-metric instead of a metric. As pointed out by [9, Remark 1.4], all the results in this paper are still true for quasi-metrics having some regularity. Moreover, Macías and Segovia [12] proved that if $d$ is a quasi-metric, then there exists a quasi-metric $\tilde{d}$, which is equivalent to $d$ and has some regularity. Thus, all the results of this paper are still true if $d$ is only known to be a quasi-metric since they are invariant under equivalent quasi-metrics.

Finally, we state some conventions. Throughout the paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as $C_0$, do not change in different occurrences. The symbol $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For any $p \in [1, \infty]$, we denote by $p'$ its conjugate index, namely, $1/p + 1/p' = 1$. If $E$ is a subset of $\mathcal{X}$, we denote by $\chi_E$ the characteristic function of $E$ and define $\text{diam } E \equiv \sup_{x, y \in E} d(x, y)$. We also set $\mathbb{N} \equiv \{1, 2, \cdots \}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For any $a, b \in \mathbb{R}$, we denote $\min\{a, b\}$, $\max\{a, b\}$, and $\max\{a, 0\}$ by $a \wedge b$, $a \vee b$ and $a_+$, respectively. For any measurable set $E$ and locally integrable function $f$, we set $m_E(f) \equiv \frac{1}{\mu(E)} \int_E f(x) \, d\mu(x)$.

2 Characterizations of RD-spaces

In this section, we establish several equivalent characterizations of RD-spaces in Proposition 2.1 below, which should be useful in applications.

Proposition 2.1. The following statements are equivalent.

(i) The triple $(\mathcal{X}, d, \mu)$ is an RD-space.
(ii) The triple \((\mathcal{X}, d, \mu)\) is a space of homogeneous type and there exists a constant \(a_0 > 1\) such that for all \(x \in \mathcal{X}\) and \(0 < r < \text{diam}(\mathcal{X})/a_0\),
\[
B(x, a_0 r) \setminus B(x, r) \neq \emptyset \quad \text{(geometrical property)}.
\]

(2.1)

(iii) The triple \((\mathcal{X}, d, \mu)\) is an \((\kappa, n)\)-space for some \(0 < \kappa \leq n\), which means that there exist constants \(C_1 \geq 1\) and \(0 < C_2 \leq 1\) such that for all \(x \in \mathcal{X}\), \(0 < r < 2 \text{diam}(\mathcal{X})/r\) and \(1 \leq \lambda < 2 \text{diam}(\mathcal{X})/r\),
\[
\mu(B(x, \lambda r)) \leq C_1 \lambda^n \mu(B(x, r))
\]
and
\[
\mu(B(x, \lambda r)) \geq C_2 \lambda^n \mu(B(x, r)).
\]

Recall that a metric space satisfying (2.1) is usually called uniformly perfect; see, for example, [10, p. 88]. Moreover, parts of the proof of Proposition 2.1 can be found in [10], for example, the proof of “(ii) implies (iii)” in Proposition 2.1 is just [10, p. 31, (4.6) and p. 102, Exercise 13.1]. (We thank the referees to point out these facts to us.) However, for the convenience of readers and its importance in applications, we would like to give a detailed proof of Proposition 2.1. Indeed, we conclude Proposition 2.1 from Lemmas 2.1 through 2.4 below.

**Lemma 2.1.** The following statements are equivalent.

(D1) There exist constants \(C_1 \geq 1\), \(\varepsilon_0 > 1\) and \(n > 0\) such that (2.2) holds for all \(x \in \mathcal{X}\), \(0 < r < \varepsilon_0 \text{diam}(\mathcal{X})\) and \(1 \leq \lambda < \varepsilon_0 \text{diam}(\mathcal{X})/r\).

(D2) There exist constants \(C_1 \geq 1\), \(\varepsilon_0 > 1\) and \(n > 0\) such that (2.2) holds for all \(x \in \mathcal{X}\), \(0 < r < \varepsilon_0 \text{diam}(\mathcal{X})\) and \(\lambda \geq 1\).

(D3) There exist constants \(C_1 \geq 1\) and \(n > 0\) such that (2.2) holds for all \(x \in \mathcal{X}\), \(r > 0\) and \(\lambda \geq 1\).

(D4) \((\mathcal{X}, d, \mu)\) is a space of homogeneous type.

**Proof.** If \(\text{diam}(\mathcal{X}) = \infty\), then (D1), (D2) and (D3) of Lemma 2.1 are the same; otherwise, it is easy to see that (D1) implies (D2), (D2) implies (D3) and (D3) implies (D1). Moreover, (D3) implies (D4) with the doubling constant \(C_0 \equiv C_1 2^n\) and (D4) implies (D3) with \(n \equiv \log_2 C_0\) and \(C_1 \equiv C_0\), which completes the proof of Lemma 2.1. \(\square\)

**Lemma 2.2.** The following statements are equivalent.

(RD1) There exist constants \(\varepsilon_1 > 0\), \(0 < C_2 \leq 1\) and \(\kappa > 0\) such that (2.3) holds for all \(x \in \mathcal{X}\), \(0 < r < \varepsilon_1 \text{diam}(\mathcal{X})\) and \(1 \leq \lambda < \varepsilon_1 \text{diam}(\mathcal{X})/r\).

(RD2) For any \(\varepsilon_1 > 0\), there exist constants \(0 < C_2 \leq 1\) and \(\kappa > 0\) such that (2.3) holds for all \(x \in \mathcal{X}\), \(0 < r < \varepsilon_1 \text{diam}(\mathcal{X})\) and \(1 \leq \lambda < \varepsilon_1 \text{diam}(\mathcal{X})/r\).

(RD3) There exist constants \(\varepsilon_1 > 0\), \(\widetilde{C}_2 > 1\) and \(a_1 \in (1, \infty) \cap [\varepsilon_1, \infty)\) such that for all \(x \in \mathcal{X}\) and \(0 < r < \varepsilon_1 \text{diam}(\mathcal{X})/a_1\),
\[
\mu(B(x, a_1 r)) \geq \widetilde{C}_2 \mu(B(x, r)).
\]

(RD4) For any \(\varepsilon_1 > 0\), there exist constants \(\widetilde{C}_2 > 1\) and \(a_1 \in (1, \infty) \cap [\varepsilon_1, \infty)\) such that (2.4) holds for all \(x \in \mathcal{X}\) and \(0 < r < \varepsilon_1 \text{diam}(\mathcal{X})/a_1\).
Lemma 2.2. still need to prove that if $\epsilon$.

In fact, if $0 < r < \epsilon_1 \text{diam}(\mathcal{X})$ and $1 \leq \lambda < \epsilon_1 \text{diam}(\mathcal{X})/r$, we have

$$\mu(B(x, \lambda r)) \geq \mu(B(x, \epsilon_1^2 r)) \geq \tilde{C}^2_2 \mu(B(x, r)) \geq \tilde{C}^{-1}_2 \lambda^\kappa \mu(B(x, r)).$$

By the same proof as above, we also have that (RD4) implies (RD2).

Now we prove (RD1) implies (RD2). In fact, if $0 < \epsilon_2 \leq \epsilon_1$, then (RD2) holds for $\epsilon_2$. If $\epsilon_2 > \epsilon_1$, then since (RD1) holds for $\epsilon_1$, to prove that (RD2) also holds for $\epsilon_2$, we still need to prove that if $\epsilon_1 \text{diam}(\mathcal{X}) \leq r < \epsilon_2 \text{diam}(\mathcal{X})$ and $1 \leq \lambda < \epsilon_2 \text{diam}(\mathcal{X})/r$, or if $0 < r < \epsilon_1 \text{diam}(\mathcal{X})$ and $\epsilon_1 \text{diam}(\mathcal{X})/r \leq \lambda < \epsilon_2 \text{diam}(\mathcal{X})/r$, then

$$\mu(B(x, \lambda r)) \geq C_2 \lambda^\kappa \mu(B(x, r)).$$

In fact, if $\epsilon_1 \text{diam}(\mathcal{X}) \leq r < \epsilon_2 \text{diam}(\mathcal{X})$ and $1 \leq \lambda < \epsilon_2 \text{diam}(\mathcal{X})/r$, then $1 \leq \lambda < \epsilon_2/\epsilon_1$, and hence

$$\mu(B(x, \lambda r)) \geq [\epsilon_2/\epsilon_1]^{-\kappa} \lambda^\kappa \mu(B(x, r)).$$

If $0 < r < \epsilon_1 \text{diam}(\mathcal{X})$ and $\epsilon_1 \text{diam}(\mathcal{X})/r \leq \lambda < \epsilon_2 \text{diam}(\mathcal{X})/r$, choosing

$$\tilde{\lambda} \equiv \max\{\epsilon_1 \text{diam}(\mathcal{X})/(2r), 1\},$$

then $\tilde{\lambda}/\lambda \geq \epsilon_1/(2\epsilon_2)$. In fact, if $r \geq \epsilon_1 \text{diam}(\mathcal{X})/2$, then $\tilde{\lambda} = 1$ and hence $\tilde{\lambda}/\lambda \geq \epsilon_1/\epsilon_2$,

and if $r < \epsilon_1 \text{diam}(\mathcal{X})/2$, then $\tilde{\lambda} = \epsilon_1 \text{diam}(\mathcal{X})/(2r)$ and hence $\tilde{\lambda}/\lambda \geq \epsilon_1/(2\epsilon_2)$. Since $B(x, \lambda r) \subset B(x, \lambda r)$ and $\lambda r < \epsilon_1 \text{diam}(\mathcal{X})$, we have

$$\mu(B(x, \lambda r)) \geq \mu(B(x, \tilde{\lambda} r)) \geq C_2 (\tilde{\lambda})^\kappa \mu(B(x, r)) = C_2 (\tilde{\lambda}/\lambda)^\kappa \lambda^\kappa \mu(B(x, r)) \geq C_2 (\epsilon_1/(2\epsilon_2))^\kappa \lambda^\kappa \mu(B(x, r)),$$

which is desired.

By the same proof as above, (RD3) also implies (RD4), which completes the proof of Lemma 2.2. \qed

Lemma 2.3. The following geometric properties are equivalent.

(i) There exist constants $\epsilon_0 > 0$ and $a_0 \in (1, \infty) \cap [\epsilon_0, \infty)$ such that (2.1) holds for all $x \in \mathcal{X}$ and $0 < r < \epsilon_0 \text{diam}(\mathcal{X})/a_0$.

(ii) For any $\epsilon_0 > 0$, there exist constants $a_0 \in (1, \infty) \cap [\epsilon_0, \infty)$ such that (2.1) holds for all $x \in \mathcal{X}$ and $0 < r < \epsilon_0 \text{diam}(\mathcal{X})/a_0$.

Proof. (G2) obviously implies (G1). On the other hand, if $\epsilon_1 \leq \epsilon_0$, since (G1) holds for $\epsilon_0$, then (G2) holds for $\epsilon_1$ with the same $a_0$. If $\epsilon_1 > \epsilon_0$, taking $a_1 \equiv a_0 \epsilon_1/\epsilon_0$, we know that (G2) holds for $\epsilon_1$ and $a_1$, which completes the proof of Lemma 2.3. \qed

Lemma 2.4. (i) (RD3) implies (G1) with the same constants.

(ii) If $\epsilon_0 > 1$, then (G1) and (D3) imply (RD3) with $\epsilon_1 \equiv \epsilon_0/2$ and $a_1 \equiv 1 + 2a_0$. 
Proof. (i) obviously holds. To see (ii), if \(0 < r < \epsilon_1 \text{diam}(\mathcal{X})/a_0\), we then have \(0 < 2r < \epsilon_0 \text{diam}(\mathcal{X})/a_0\). Thus, by (G1), we have \(B(x, 2a_0r) \setminus B(x, 2r) \neq \emptyset\). Choose \(y \in B(x, 2a_0r) \setminus B(x, 2r)\). Then \(B(y, r) \cap B(x, r) = \emptyset\). Notice that

\[
B(y, r) \subset B(x, \lfloor 1 + 2a_0 \rfloor r) \subset B(y, \lceil 1 + 4a_0 \rfloor r).
\]

By (D3), we have

\[
\mu(B(x, \lfloor 1 + 2a_0 \rfloor r)) \geq \mu(B(x, r)) + \mu(B(y, r))
\geq \mu(B(x, r)) + C_1^{-1}(1 + 4a_0)^{-n}\mu(B(y, \lfloor 1 + 4a_0 \rfloor r))
\geq \mu(B(x, r)) + C_1^{-1}(1 + 4a_0)^{-n}\mu(B(x, \lceil 1 + 2a_0 \rfloor r)),
\]

which implies that \(\mu(B(x, a_1 r)) \geq \tilde{C}_2 \mu(B(x, r))\) with \(a_1 \equiv 1 + 2a_0\) and

\[
\tilde{C}_2 \equiv [1 - C_1^{-1}(1 + 4a_0)^{-n}]^{-1} > 1.
\]

This implies that (RD3) holds and hence finishes the proof of Lemma 2.4.

Proof of Proposition 2.1. Notice that \(\mathcal{X}\) is an RD-space if and only if \(\mathcal{X}\) satisfies (D4) and (RD3) with \(\epsilon_1 = 1, \tilde{C}_2 = 2\) and \(a_0 = a_1\); \(\mathcal{X}\) is an \((\kappa, n)\)-space if and only if \(\mathcal{X}\) satisfies (D1) and (RD1) with \(\epsilon_1 \equiv \epsilon_0 \equiv 2\); \(\mathcal{X}\) satisfies (ii) of Proposition 2.1 if and only if \(\mathcal{X}\) satisfies (D4) and (G1) with \(\epsilon_0 \equiv 1\). Then Proposition 2.1 follows from Lemmas 2.1 through 2.4, which completes the proof of Proposition 2.1.

By the well-known fact that connected spaces are uniformly perfect (see, for example, [10, p. 88]), as a corollary of Proposition 2.1, we know that all connected spaces of homogeneous type are RD-spaces.

Remark 2.1. (i) The numbers \(\kappa\) and \(n\) appearing in the definition of \((\kappa, n)\)-space here measure the “dimension” of \(\mathcal{X}\) in some sense.

(ii) We point out that the definition of \((\kappa, n)\)-spaces \(\mathcal{X}\) is slightly different from that of [9, Definition 1.1], where (2.3) and (2.2) are assumed to hold only when \(0 < r < \text{diam}(\mathcal{X})/2\) and \(1 \leq \lambda < \text{diam}(\mathcal{X})/(2r)\), from which it is still unclear how to deduce that \(\mathcal{X}\) is a space of homogeneous type when \(\text{diam}(\mathcal{X}) < \infty\).

(iii) By Proposition 2.1, the definition of RD-spaces in [9, Definition 1.1] is equivalent to Definition 1.1.

3 Properties of spaces of test functions

We prove Theorem 1.1 in this section. To this end, we need the homogeneous and inhomogeneous Calderón reproducing formulae established in Theorems 3.10 and 3.26 of [9], respectively.

We begin with the following notion of approximations of the identity on RD-spaces introduced in [9]; see [9, Theorem 2.6] for its existence.
Lemma 3.1. (I) Let $\epsilon_1 \in (0, 1]$. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an approximation of the identity of order $\epsilon_1$ with bounded support, if there exist constants $C_3, C_4 > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y$ and $y' \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of $S_k$ is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(i) $S_k(x, y) = 0$ if $d(x, y) > C_4 2^{-k}$ and $|S_k(x, y)| \leq C_3 \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$;

(ii) $|S_k(x, y) - S_k(x', y)| \leq C_3 2^{k+1} \frac{|d(x, x')|^{\epsilon_1} + |d(y, y')|^{\epsilon_1}}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$ for $d(x, x') \leq \max\{C_4, 1\} 2^{1-k}$;

(iii) Property (ii) holds with $x$ and $y$ interchanged;

(iv) $|S_k(x, y) - S_k(x', y') - S_k(x', y) - S_k(x, y')| \leq C_3 2^{2k+1} \frac{|d(x, x')|^{\epsilon_1} |d(y, y')|^{\epsilon_1}}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$ for $d(x, x') \leq \max\{C_4, 1\} 2^{1-k}$;

(v) $\int_{\mathcal{X}} S_k(x, w) d\mu(w) = 1 = \int_{\mathcal{X}} S_k(w, y) d\mu(w)$.

(II) A sequence $\{S_k\}_{k \in \mathbb{Z}^+}$ of linear operators is called an inhomogeneous approximation to the identity of order $\epsilon_1$ with bounded support, if $S_k$ for $k \in \mathbb{Z}^+$ satisfies (I).

Lemma 3.1. Let $\epsilon \in (0, 1)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation of the identity of order $\epsilon$ with bounded support. Set $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then there exists a family $\{\tilde{D}_k\}_{k \in \mathbb{Z}}$ of linear operators such that for all $f \in G_0^\epsilon(\beta, \gamma)$ with $\beta, \gamma \in (0, \epsilon)$,

$$f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k(f),$$

where the series converges in both the norm of $G_0^\epsilon(\beta, \gamma)$ and the norm of $L^p(\mathcal{X})$ for $p \in (1, \infty)$. Moreover, for any $\epsilon' \in (\epsilon, 1)$, there exists a positive constant $C_{\epsilon'}$ such that the kernels, denoted by $\{\tilde{D}_k(x, y)\}_{k \in \mathbb{Z}}$, of the operators $\{\tilde{D}_k\}_{k \in \mathbb{Z}}$ satisfy that for all $x, x', y \in \mathcal{X}$,

(i) $|\tilde{D}_k(x, y)| \leq C_{\epsilon'} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} 2^{-k \epsilon'}$;

(ii) $|\tilde{D}_k(x, y) - \tilde{D}_k(x', y)| \leq C_{\epsilon'} \frac{|d(x, x')|^{\epsilon'} 2^{-k \epsilon'} + |d(y, y')|^{\epsilon'} 2^{-k \epsilon'}}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)}$ for $d(x, x') \leq [2^{-k} + d(x, y)]/2$;

(iii) $\int_{\mathcal{X}} \tilde{D}_k(x, w) d\mu(w) = 0 = \int_{\mathcal{X}} \tilde{D}_k(w, y) d\mu(w)$.

The following is the inhomogeneous version of Lemma 3.1.

Lemma 3.2. Let $\epsilon \in (0, 1)$ and $\{S_k\}_{k \in \mathbb{Z}^+}$ be an inhomogeneous approximation of the identity of order $1$ with bounded support. Set $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 \equiv S_0$. Then there exists a family $\{\tilde{D}_k\}_{k \in \mathbb{Z}^+}$ of linear operators such that for all $f \in G_0^\epsilon(\beta, \gamma)$ with $\beta, \gamma \in (0, \epsilon)$,

$$f = \sum_{k=0}^{\infty} \tilde{D}_k D_k(f),$$

where the series converges in both the norm of $G_0^\epsilon(\beta, \gamma)$ and the norm of $L^p(\mathcal{X})$ for $p \in (1, \infty)$. Moreover, for any $\epsilon' \in (\epsilon, 1)$, there exists a positive constant $C_{\epsilon'}$ such that the kernels, denoted by $\{\tilde{D}_k(x, y)\}_{k \in \mathbb{Z}^+}$, of the operators $\{\tilde{D}_k\}_{k \in \mathbb{Z}^+}$ satisfy (i) and (ii) of Lemma 3.1 and
Proof of Theorem 1.1. Without loss of generality, we may assume that $\varepsilon < \bar{\varepsilon}$. Since $G(\varepsilon, \bar{\varepsilon}) \subset G(\varepsilon, \varepsilon)$ and $G(\bar{\varepsilon}, \varepsilon) \subset G(\varepsilon, \varepsilon)$, we then have $G_0^\varepsilon(\beta, \gamma) \subset G_0^\varepsilon(\beta, \gamma)$ and $G_0^\bar{\varepsilon}(\beta, \gamma) \subset G_0^\varepsilon(\beta, \gamma)$.

We now show that $G_0^\varepsilon(\beta, \gamma) \subset G_0^\beta(\beta, \gamma)$. To this end, it suffices to prove that

$$\mathcal{G}(\varepsilon, \varepsilon) \subset G_0^\beta(\beta, \gamma).$$

To do so, choose a radial function $\phi \in C^1(\mathbb{R})$ such that $\text{supp} \phi \subset (-2, 2)$ and $\phi(x) = 1$ if $x \in (-1, 1)$. Fix any $\psi \in \mathcal{G}(\varepsilon, \varepsilon)$. For all $N \in \mathbb{N}$ and all $x \in \mathcal{X}$, let $\psi_N(x) \equiv \psi(x)\phi\left(\frac{d(x, x_1)}{N}\right)$.

We first show that as $N \to \infty$,

$$\|\psi - \psi_N\|_{\mathcal{G}(\beta, \gamma)} \to 0.$$  

In fact, for all $x \in \mathcal{X}$, we have

$$\|\psi_N(x)\| = \left|\psi(x)\left[1 - \phi\left(\frac{d(x, x_1)}{N}\right)\right]\right| \leq \|\psi\|_{\mathcal{G}(\varepsilon, \varepsilon)} \frac{1}{V_1(x_1) + V(x_1, x)} \left[\frac{1}{1 + d(x, x_1)}\right]^\varepsilon x_{x \in \mathcal{X}: d(x, x_1) > N} \lesssim \frac{1}{N^{\varepsilon - 1}}\|\psi\|_{\mathcal{G}(\varepsilon, \varepsilon)} \frac{1}{V_1(x_1) + V(x_1, x)} \left[\frac{1}{1 + d(x, x_1)}\right]^\gamma .$$

Notice that if $d(x, y) \leq \frac{1}{2}[1 + d(x, x_1)]$ and $d(x, x_1) \leq 2N$, then $V_1(x_1) + V(x_1, y) \gtrsim V_1(x_1) + V(x_1, x)$ and $d(y, x_1) \leq 3d(x, x_1)$, which imply that for all $x, y \in \mathcal{X}$ with $d(y, x_1) \leq [1 + d(y, x_1)]/2$,

$$\|\psi(x) - \psi_N(x)\| = \left|\psi(x)\left[1 - \phi\left(\frac{d(x, x_1)}{N}\right)\right]\right| + \left|\psi(y)\left[\phi\left(\frac{d(y, x_1)}{N}\right) - \phi\left(\frac{d(x, x_1)}{N}\right)\right]\right| \lesssim \|\psi\|_{\mathcal{G}(\varepsilon, \varepsilon)} \left\{\frac{1}{N^{1 - \beta}} + \frac{1}{N\beta^{1 - \beta} \wedge (\varepsilon - \gamma)}\right\}\times \left[\frac{d(x, y)}{1 + d(x, x_1)}\right]^{1 - \beta} \frac{1}{V_1(x_1) + V(x_1, x)} \left[\frac{1}{1 + d(x, x_1)}\right]^\gamma .$$

Combining the estimates (3.3) with (3.4) yields

$$\|\psi - \psi_N\|_{\mathcal{G}(\beta, \gamma)} \lesssim \|\psi\|_{\mathcal{G}(\varepsilon, \varepsilon)} \left\{\frac{1}{N^{1 - \beta}} + \frac{1}{N\beta^{1 - \beta} \wedge (\varepsilon - \gamma)}\right\} \to 0,$$
New Properties of Besov and Triebel-Lizorkin Spaces on RD-spaces

as $N \to \infty$. It is easy to see that $\psi_N \in \mathcal{G}(\epsilon, \epsilon')$ for any $\epsilon' > 0$ by noticing that $\psi_N$ has bounded support. In particular, $\psi_N \in \mathcal{G}_0^0(\beta, \gamma)$. With the notation same as in Lemma 3.2, by Lemma 3.2, we have

$$\psi_N = \sum_{k=0}^{\infty} \bar{D}_k D_k(\psi_N)$$

in $\mathcal{G}_0^0(\beta, \gamma)$, which means that as $L \to \infty$,

$$\left\| \psi_N - \sum_{k=0}^{L} \bar{D}_k D_k(\psi_N) \right\|_{\mathcal{G}(\beta, \gamma)} \to 0,$$

where $\bar{D}_k(\cdot, y) \in \mathcal{G}(y, 2^{-k}, \epsilon', \epsilon')$ for any $\epsilon' \in (\bar{\epsilon}, 1)$. To finish the proof of (3.1), it suffices to show that

$$(3.5) \sum_{k=0}^{L} \bar{D}_k D_k(\psi_N) \in \mathcal{G}(\bar{\epsilon}, \bar{\epsilon})$$

with its norm depending on $L$ and $N$. To see this, for any $\epsilon' > 0$ and $k \in \{0, 1, \cdots, L\}$, we have that for all $x \in X$,

$$(3.6) |D_k(\psi_N)(x)| = \left| \int_{d(x,z)<C_42^{-k+1}} D_k(x,z)\psi_N(z) \, d\mu(z) \right|$$

$$\lesssim \int_X \frac{1}{V_1(x) + V(x,z)} \left[ \frac{1}{1 + d(x,z)} \right]^{\epsilon'} d\mu(z)$$

$$\times \frac{1}{V_1(x_1) + V(x_1,z)} \left[ \frac{1}{1 + d(x_1,z)} \right]^{\epsilon'} d\mu(z)$$

$$\lesssim \frac{1}{V_1(x_1) + V(x_1,x)} \left[ \frac{1}{1 + d(x_1,x)} \right]^{\epsilon'},$$

and that for all $x, y \in X$ with $d(x,y) \leq (C_4 \vee 1)2^{1-k}$,

$$(3.7) |D_k(\psi_N)(x) - D_k(\psi_N)(y)|$$

$$= \left| \int_{d(x,z)<(C_4 \vee 1)2^{-k}} [D_k(x,z) - D_k(y,z)]\psi_N(z) \, d\mu(z) \right|^{1+\epsilon'}$$

$$\lesssim d(x,y) \int_X \frac{1}{V_1(x) + V(x,z)} \left[ \frac{1}{1 + d(x,z)} \right]^{1+\epsilon'} d\mu(z)$$

$$\times \frac{1}{V_1(x_1) + V(x_1,z)} \left[ \frac{1}{1 + d(x_1,z)} \right]^{1+\epsilon'} d\mu(z)$$

$$\lesssim \frac{d(x,y)}{1 + d(x_1,x)} \frac{1}{V_1(x_1) + V(x_1,x)} \left[ \frac{1}{1 + d(x_1,x)} \right]^{\epsilon'}.$$
where the implicit constants depend on $L$ and $N$. The estimates (3.6) and (3.7) further imply that if $d(x, y) \leq [1 + d(x, x_1)]/2$, then

\begin{equation}
|D_k(\psi_N)(x) - D_k(\psi_N)(y)| \lesssim \frac{d(x, y)}{1 + d(x_1, x)} \frac{1}{V_1(x) + V(x, x)} \left[ \frac{1}{1 + d(x_1, x)} \right]^{\epsilon'}
\end{equation}

with the implicit constant depending on $L$ and $N$.

Observe that for $k \in \{0, 1, \cdots, L\}$, $\tilde{D}_k(\cdot, z) \in G(z, \tilde{\epsilon}, \epsilon)$ with its norm depending on $L$. By (3.6) with $\epsilon' = \epsilon$, we obtain that for all $x \in \mathcal{X}$,

\begin{equation}
|\tilde{D}_k D_k(\psi_N)(x)| \lesssim \int_{\mathcal{X}} \frac{1}{\tilde{V}_1(x) + \tilde{V}(x, z)} \left[ \frac{1}{1 + d(x, z)} \right]^\tilde{\epsilon} \times \frac{1}{\tilde{V}_1(x_1) + \tilde{V}(x_1, x)} \left[ \frac{1}{1 + d(x_1, x)} \right]^\tilde{\epsilon} \tilde{d}\mu(z)
\end{equation}

with the implicit constant depending on $L$ and $N$, and that for all $x, y \in \mathcal{X}$ with $d(x, y) \leq [1 + d(x, x_1)]/4$,

\begin{equation}
|D_k D_k(\psi_N)(x) - \tilde{D}_k D_k(\psi_N)(y)|
= \left| \int_{\mathcal{X}} [\tilde{D}_k(x, z) - \tilde{D}_k(y, z)] [D_k(\psi_N)(z) - D_k(\psi_N)(x)] d\mu(z) \right|
\end{equation}

\begin{equation}
= \sum_{i=1}^{3} \left| \int_{W_i} [\tilde{D}_k(x, z) - \tilde{D}_k(y, z)] [D_k(\psi_N)(z) - D_k(\psi_N)(x)] d\mu(z) \right| \equiv I_1 + I_2 + I_3,
\end{equation}

where $W_1 \equiv \{ z \in \mathcal{X} : d(x, y) \leq [1 + d(x, z)]/2 \leq [1 + d(x, x_1)]/4 \}$,

\begin{equation}
W_2 \equiv \{ z \in \mathcal{X} : d(x, y) \leq [1 + d(x, x_1)]/4 \leq [1 + d(x, z)]/2 \},
\end{equation}

and $W_3 \equiv \{ z \in \mathcal{X} : d(x, y) > [1 + d(x, z)]/2 \}$. For $I_1$, by (3.6) and (3.8), we have

\begin{equation}
I_1 \lesssim \int_{\mathcal{X}} \left[ \frac{d(x, y)}{1 + d(x, z)} \right]^\tilde{\epsilon} \frac{1}{\tilde{V}_1(x) + \tilde{V}(x, z)} \left[ \frac{1}{1 + d(x, z)} \right]^\tilde{\epsilon} \times \left[ \frac{d(x, z)}{1 + d(x_1, x)} \right]^\tilde{\epsilon} \frac{1}{\tilde{V}_1(x_1) + \tilde{V}(x_1, x)} \left[ \frac{1}{1 + d(x_1, x)} \right]^\tilde{\epsilon} \tilde{d}\mu(z)
\end{equation}

\begin{equation}
\lesssim \left[ \frac{d(x, y)}{1 + d(x_1, x)} \right]^\tilde{\epsilon} \frac{1}{\tilde{V}_1(x_1) + \tilde{V}(x_1, x)} \left[ \frac{1}{1 + d(x_1, x)} \right]^\tilde{\epsilon},
\end{equation}

where all implicit constants depend on $L$ and $N$, and we used the following estimate that for all $x \in \mathcal{X}$,

\begin{equation}
\int_{\mathcal{X}} \frac{1}{\tilde{V}_1(x) + \tilde{V}(x, z)} \left[ \frac{1}{1 + d(x, z)} \right]^\tilde{\epsilon} d\mu(z) \lesssim 1.
\end{equation}
To estimate $I_2$, by (3.6) and (3.10), we obtain
\[
I_2 \lesssim \int_\mathcal{X} \left[ \frac{d(x, y)}{1 + d(x, z)} \right]^{\tilde{\tau}} \frac{1}{V_1(x) + V(x, z)} \left[ \frac{1}{1 + d(x, z)} \right]^{\tilde{\tau}} \times \left\{ \frac{1}{V_1(x_1) + V(x_1, z)} \left[ 1 + \frac{1}{1 + d(x_1, z)} \right]^{1 + \tilde{\tau}} + \frac{1}{V_1(x_1) + V(x_1, x)} \left[ 1 + \frac{1}{1 + d(x_1, x)} \right]^{1 + \tilde{\tau}} \right\} d\mu(z)
\]
\[
\lesssim \left[ \frac{d(x, y)}{1 + d(x_1, x)} \right]^{\tilde{\tau}} \frac{1}{V_1(x_1) + V(x_1, x)} \left[ 1 + \frac{1}{1 + d(x_1, x)} \right]^{\tilde{\tau}},
\]
where the implicit constants depend on $L$ and $N$.

If $z \in W_3$, then $d(x, z) < 2d(x, y) \leq [1 + d(x, x_1)]/2$, which together with (3.8) and (3.10) implies that
\[
I_3 \lesssim \int_\mathcal{X} \left[ |\tilde{D}_k(x, z)| + |\tilde{D}_k(y, z)| \right] \frac{d(x, z)}{1 + d(x_1, x)} \frac{1}{V_1(x_1) + V(x_1, x)} \left[ 1 + \frac{1}{1 + d(x_1, x)} \right]^{\tilde{\tau}} d\mu(z)
\]
\[
\lesssim \left[ \frac{d(x, y)}{1 + d(x_1, x)} \right]^{\tilde{\tau}} \frac{1}{V_1(x_1) + V(x_1, x)} \left[ 1 + \frac{1}{1 + d(x_1, x)} \right]^{\tilde{\tau}},
\]
where the implicit constants depend on $L$ and $N$.

Thus, for all $x, y \in \mathcal{X}$ with $d(x, y) \leq [1 + d(x, x_1)]/4$,
\[
\left| \tilde{D}_k D_k(\psi_N)(x) - \tilde{D}_k D_k(\psi_N)(y) \right|
\]
\[
\lesssim \left[ \frac{d(x, y)}{1 + d(x_1, x)} \right]^{\tilde{\tau}} \frac{1}{V_1(x_1) + V(x_1, x)} \left[ 1 + \frac{1}{1 + d(x_1, x)} \right]^{\tilde{\tau}},
\]
which together with (3.9) further implies that this estimate also holds for all $x, y \in \mathcal{X}$ with $d(x, y) \leq [1 + d(x, x_1)]/2$. Thus, \( \{D_k D_k(\psi_N)\}_{k=0}^\infty \in \mathcal{G}(\tilde{\tau}, \epsilon) \) with their norms depending on $L$ and $N$ and hence (3.1) holds. This shows that $\mathcal{G}_0^0(\beta, \gamma) = \mathcal{G}_0^0(\beta, \gamma)$.

To finish the proof of Theorem 1.1, we still need to show that
\[
(3.11) \quad \mathcal{G}_0^0(\beta, \gamma) \subset \mathcal{G}_0^0(\beta, \gamma).
\]
To this end, fix any $\psi \in \mathcal{G}(\epsilon, \epsilon)$ and let $\phi$ be as in (3.2). For any $N \in \mathbb{N}$, let
\[
\psi_N(x) \equiv \psi(x) \phi \left( \frac{d(x, x_1)}{N} \right).
\]
Let $\{S_k\}_{k \in \mathbb{Z}}$ be as in Lemma 3.1. We then define
\[
g_N(x) \equiv \psi_N(x) - \left\{ \int_\mathcal{X} \psi_N(z) d\mu(z) \right\} S_0(x, x_1).
\]
Then
\[
(3.12) \quad \int_\mathcal{X} g_N(x) d\mu(x) = 0,
\]
and since \( \int_X \psi(x) \, d\mu(x) = 0 \), we have

\[
\| g_N \|_{\mathcal{G}(\beta, \gamma)} \lesssim \left\| \psi \right\|_{\mathcal{G}(\epsilon, \epsilon)} \left\{ \frac{1}{N^{\epsilon - \beta}} + \frac{1}{N^{1 - \beta}} \right\} \to 0,
\]

as \( N \to \infty \). By (3.12) and noticing that \( g_N \) has bounded support, it is easy to see that \( g_N \in \mathcal{G}(\epsilon, \epsilon') \) for any \( \epsilon' > 0 \). In particular, \( g_N \in \mathcal{G}_0(\beta, \gamma) \). With the notation same as in Lemma 3.1, by Lemma 3.1, we have

\[
g_N = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k(g_N)
\]
in \( \mathcal{G}_0(\beta, \gamma) \), which means that as \( L \to \infty \),

\[
\left\| g_N - \sum_{k=-L}^{L} \tilde{D}_k D_k(g_N) \right\|_{\mathcal{G}(\beta, \gamma)} \to 0,
\]

where \( \tilde{D}_k(\cdot, y) \in \hat{\mathcal{G}}(y, 2^{-k}, \epsilon', \epsilon') \) for any \( \epsilon' \in (\epsilon, 1) \). An argument similar to (3.5) gives that \( \sum_{k=-L}^{L} \tilde{D}_k D_k(g_N) \in \mathcal{G}(\epsilon, \epsilon) \) with its norm depending on \( N \) and \( L \), which completes the proof of (3.11) and hence the proof of Theorem 1.1.

\section{New characterizations of \( B_{p,q}^s(\mathcal{X}) \) and \( F_{p,q}^s(\mathcal{X}) \)}

This section is devoted to the proof of Theorem 1.2. We first recall the notions of homogeneous Besov spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \) and Triebel-Lizorkin spaces \( \dot{F}_{p,q}^s(\mathcal{X}) \), and the inhomogeneous Besov spaces \( B_{p,q}^s(\mathcal{X}) \) and Triebel-Lizorkin spaces \( F_{p,q}^s(\mathcal{X}) \) introduced in [9]. We point out that when we mention the homogeneous Besov spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \) and Triebel-Lizorkin spaces \( \dot{F}_{p,q}^s(\mathcal{X}) \), we always assume that \( \mu(\mathcal{X}) = \infty \) since they are well-defined only when \( \mu(\mathcal{X}) = \infty \). Denote by \( n \) the “dimension” of \( \mathcal{X} \); see Section 2.

\begin{definition}
Let \( \mu(\mathcal{X}) = \infty \), \( \epsilon \in (0, 1) \) and \( \{S_k\}_{k \in \mathbb{Z}} \) be an approximation of the identity of order \( \epsilon \) with bounded support. For \( k \in \mathbb{Z} \), set \( D_k \equiv S_k - S_{k-1} \). Let \( 0 < s < \epsilon \).

(i) Let \( n/(n + \epsilon) < p \leq \infty \) and \( 0 < q \leq \infty \). The \textit{homogeneous Besov space} \( \dot{B}_{p,q}^s(\mathcal{X}) \) is defined to be the set of all \( f \in (\mathcal{G}_0(\beta, \gamma))' \) for some \( \beta, \gamma \) satisfying

\[
s < \beta < \epsilon \quad \text{and} \quad \max\{s - \kappa/p, (n/p - 1)_+\} < \gamma < \epsilon
\]

\end{definition}
such that
\[ \|f\|_{B^s_{p,q}(X)} \equiv \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k(f)\|_{L^p(X)}^q \right\}^{1/q} < \infty \]
with the usual modifications made when \( p = \infty \) or \( q = \infty \).

(ii) Let \( n/(n + \epsilon) < p < \infty \) and \( n/(n + \epsilon) < q \leq \infty \). The homogeneous Triebel-Lizorkin space \( \dot{F}^s_{p,q}(X) \) is defined to be the set of all \( f \in (\dot{G}^s_0(\beta, \gamma))' \) for some \( \beta, \gamma \) satisfying (4.1) such that
\[ \|f\|_{\dot{F}^s_{p,q}(X)} \equiv \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q \right\}^{1/q} < \infty \]
with the usual modification made when \( q = \infty \).

To define the inhomogeneous Besov and Triebel-Lizorkin spaces, we need to first recall the following construction given by Christ in [2], which provides an analogue of the set of Euclidean dyadic cubes on spaces of homogeneous type.

**Lemma 4.1.** Let \( X \) be a space of homogeneous type. Then there exists a collection \( \{Q^k_{\alpha} \subset X : k \in \mathbb{Z}, \alpha \in I_k \} \) of open subsets, where \( I_k \) is some index set, and constants \( \delta \in (0,1) \) and \( C_5, C_6 > 0 \) such that

(i) \( \mu(X \setminus \bigcup_{\alpha} Q^k_{\alpha}) = 0 \) for each fixed \( k \) and \( Q^k_{\alpha} \cap Q^k_{\beta} = \emptyset \) if \( \alpha \neq \beta \);

(ii) for any \( \alpha, \beta, k \) and \( l \) with \( l \geq k \), either \( Q^l_{\beta} \subset Q^k_{\alpha} \) or \( Q^l_{\beta} \cap Q^k_{\alpha} = \emptyset \);

(iii) for each \( (k, \alpha) \) and each \( l < k \), there exists a unique \( \beta \) such that \( Q^k_{\alpha} \subset Q^l_{\beta} \);

(iv) \( \text{diam}(Q^k_{\alpha}) \leq C_5 \delta^k \);

(v) each \( Q^k_{\alpha} \) contains some ball \( B(z^k_{\alpha}, C_6 \delta^k) \), where \( z^k_{\alpha} \in \mathcal{X} \).

In fact, we can think of \( Q^k_{\alpha} \) as being a **dyadic cube** with diameter rough \( \delta^k \) and centered at \( z^k_{\alpha} \). In what follows, to simplify our presentation, we always suppose \( \delta = 1/2 \); see [9] for more details.

In the following, for \( k \in \mathbb{Z} \) and \( \tau \in I_k \), we denote by \( Q^k_{\tau,\nu} \), \( \nu = 1, 2, \cdots, N(k, \tau) \), the set of all cubes \( Q^k_{\tau,\nu} \subset Q^k_{\tau} \), where \( Q^k_{\tau} \) is the dyadic cube as in Lemma 4.1 and \( j \) is a fixed positive large integer such that \( 2^{-j}C_5 < 1/3 \). Denote by \( z^k_{\tau,\nu} \) the “center” of \( Q^k_{\tau,\nu} \) as in Lemma 4.1 and by \( y^k_{\tau,\nu} \) a point in \( Q^k_{\tau,\nu} \).

**Definition 4.2.** Let \( \epsilon \in (0,1) \) and \( \{S_k\}_{k \in \mathbb{Z}} \) be an inhomogeneous approximation of the identity of order \( \epsilon \) with bounded support as in Definition 3.1. Set \( D_0 \equiv S_0 \) and \( D_k \equiv S_k - S_{k-1} \) for \( k \in \mathbb{N} \). Let \( \{Q^0_{\tau,\nu} : \tau \in I_0, \nu = 1, \cdots, N(0, \tau) \} \) with a fixed large \( j \in \mathbb{N} \) be dyadic cubes as above. Let \( 0 < s < \epsilon \).

(i) Let \( n/(n + \epsilon) < p \leq \infty \) and \( 0 < q \leq \infty \). The Besov space \( B^s_{p,q}(X) \) is defined to be the set of all \( f \in (G^s_0(\beta, \gamma))' \) for some \( \beta, \gamma \) satisfying
\[ s < \beta < \epsilon \text{ and } n(1/p - 1)_+ < \gamma < \epsilon \]
such that

$$
\|f\|_{B^{p,q}_s(X)} \equiv \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau) \left[ m^{Q^{0,\nu}_\tau}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\{ \sum_{k=1}^{\infty} 2^{ksq} \|D_k(f)\|_{L^p(X)}^q \right\}^{1/q} < \infty
$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) Let $n/(n+\epsilon) < p < \infty$ and $n/(n+\epsilon) < q \leq \infty$. The Triebel-Lizorkin space $F^s_{p,q}(X)$ is defined to be the set of all $f \in (G^\beta_0(\beta, \gamma))'$ for some $\beta, \gamma$ satisfying (4.2) such that

$$
\|f\|_{F^s_{p,q}(X)} \equiv \left\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau) \left[ m^{Q^{0,\nu}_\tau}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\{ \sum_{k=1}^{\infty} 2^{ksq} \|D_k(f)\|_{L^p(X)}^q \right\}^{1/q} < \infty
$$

with the usual modification made when $q = \infty$.

Recall that the local Hardy spaces $h^p(X) \equiv F^0_{p,2}(X)$ when $p \in (n/(n+1), \infty)$; see [9].

**Definition 4.3.** Let $\epsilon \in (0,1)$ and $s \in (0, \epsilon)$.

(i) Assume that $\mu(X) = \infty$, and let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation of the identity of order $\epsilon$ with bounded support. For $k \in \mathbb{Z}$, set $D_k \equiv S_k - S_{k-1}$. Let $n/(n+\epsilon) < q \leq \infty$. The Triebel-Lizorkin space $F^s_{\infty,q}(X)$ is defined to be the set of all $f \in (G^\beta_0(\beta, \gamma))'$ for some $\beta, \gamma$ satisfying $s < \beta, \gamma < \epsilon$ such that

$$
\|f\|_{F^s_{\infty,q}(X)} \equiv \sup_{l \in \mathbb{Z}} \sup_{\alpha \in I_l} \left\{ \frac{1}{\mu(Q^\alpha_\tau)} \int_{Q^\alpha_\tau} \sum_{k=l}^{\infty} 2^{ksq} |D_k(f)(x)|^q d\mu(x) \right\}^{1/q} < \infty,
$$

where the supremum is taken over all dyadic cubes as in Lemma 4.1 and the usual modification is made when $q = \infty$.

(ii) Let $\{S_k\}_{k \in \mathbb{Z}^+}$ be an inhomogeneous approximation of the identity of order $\epsilon$ with bounded support. Set $D_k \equiv S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 \equiv S_0$. Let $Q^\nu_\tau : \tau \in I_0$, $\nu = 1, \cdots, N(0,\tau)$} with a fixed large $j \in \mathbb{N}$ be dyadic cubes as above. Let $0 < s < \epsilon$ and $n/(n+\epsilon) < q \leq \infty$. The Triebel-Lizorkin space $F^s_{\infty,q}(X)$ is defined to be the set of all $f \in (G^\beta_0(\beta, \gamma))'$ for some $\beta, \gamma$ satisfying $s < \beta < \epsilon$ and $0 < \gamma < \epsilon$ such that

$$
\|f\|_{F^s_{\infty,q}(X)} \equiv \max_{\tau \in I_0} \sup_{\nu=1, \cdots, N(0,\tau)} m^{Q^{0,\nu}_\tau}(|D_0(f)|),
$$
where the supremum is taken over all dyadic cubes as in Lemma 4.1, and the usual modification is made when \( q = \infty \).

For a given \( \epsilon \in (0, 1) \), it was proved in [9] that the definitions of the spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \), \( \dot{F}_{p,q}^s(\mathcal{X}) \), \( \dot{F}_{\infty,q}^s(\mathcal{X}) \), \( B_{p,q}^s(\mathcal{X}) \), \( F_{p,q}^s(\mathcal{X}) \) and \( F_{\infty,q}^s(\mathcal{X}) \) are independent of the choices of the approximation of the identity and the distribution space, \((\mathcal{G}_0^s(\beta,\gamma))'\) with suitable \( \beta, \gamma \), respectively, the inhomogeneous approximation of the identity and the distribution space, \((\mathcal{G}_0^s(\beta,\gamma))'\) with suitable \( \beta, \gamma \) as in the above definitions. From Theorem 1.1, it is easy to see that the definitions of these spaces are also independent of \( \epsilon \).

**Corollary 4.1.** Let \( p \in (n/(n+1), \infty] \) and \( s \in (0,1) \).

(i) If \( \epsilon \in (\max\{s,n(1/p-1)_+, 1\}, 1) \) and \( q \in (0, \infty] \), then the definitions of the spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \) and \( B_{p,q}^s(\mathcal{X}) \) are independent of the choices of \( \epsilon \) as above, the approximation of the identity and the distribution space, \((\mathcal{G}_0^s(\beta,\gamma))'\) with \( \beta, \gamma \) as in (4.1), respectively, the inhomogeneous approximation of the identity and the distribution space, \((\mathcal{G}_0^s(\beta,\gamma))'\) with \( \beta, \gamma \) as in (4.2).

(ii) If \( q \in (n/(n+1), \infty] \) and \( \epsilon \in (\max\{s,n(1/p-1)_+, n(1/q-1)_+, 1\}, 1) \), then the definitions of the spaces \( \dot{F}_{p,q}^s(\mathcal{X}) \) and \( F_{p,q}^s(\mathcal{X}) \) are independent of the choices of \( \epsilon \) as above, the approximation of the identity and the distribution space, \((\mathcal{G}_0^s(\beta,\gamma))'\) with \( \beta, \gamma \) as in (4.1), respectively, the inhomogeneous approximation of the identity and the distribution space, \((\mathcal{G}_0^s(\beta,\gamma))'\) with \( \beta, \gamma \) as in (4.2).

**Remark 4.1.** (i) In the definitions of the spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \), \( \dot{F}_{p,q}^s(\mathcal{X}) \), \( \dot{F}_{\infty,q}^s(\mathcal{X}) \), \( B_{p,q}^s(\mathcal{X}) \), \( F_{p,q}^s(\mathcal{X}) \) and \( F_{\infty,q}^s(\mathcal{X}) \) as in Definitions 4.1, 4.2 and 4.3, the approximations of the identity are not necessary to have bounded support; see [9]. All the conclusions in Corollary 4.1 are still true.

(ii) When \( s \in (-1,0] \), the spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \) and \( B_{p,q}^s(\mathcal{X}) \) with \( p \in (n/(n+1), \infty] \) and \( q \in (0, \infty] \) and the spaces \( \dot{F}_{p,q}^s(\mathcal{X}) \) and \( F_{p,q}^s(\mathcal{X}) \) with \( p, q \in (n/(n+1), \infty] \) are also well defined; see [9]. Moreover, some conclusions similar to Corollary 4.1 are also true for these spaces.

(iii) From now on, when we mention the spaces \( \dot{B}_{p,q}^s(\mathcal{X}) \), \( \dot{F}_{p,q}^s(\mathcal{X}) \), \( B_{p,q}^s(\mathcal{X}) \) and \( F_{p,q}^s(\mathcal{X}) \) with \( s \in (0,1) \), we always mean that we choose \( \epsilon \) as in Corollary 4.1 and then define these spaces as in Definitions 4.1, 4.2 and 4.3.

To prove Theorem 1.2, we need the following Calderón reproducing formula established in [9, Theorem 4.14].

**Lemma 4.2.** Let \( \epsilon \in (0,1) \) and \( \{S_k\}_{k \in \mathbb{Z}_+} \) be an inhomogeneous approximation of the identity of order 1 with bounded support. Set \( D_0 \equiv S_0 \) and \( D_k \equiv S_k - S_{k-1} \) for \( k \in \mathbb{N} \). Then for any fixed \( j \) large enough, there exists a family \( \{D_k(x,y)\}_{k \in \mathbb{Z}_+} \) of functions such that
for any fixed \( y_r^{k,\nu} \in Q_r^{k,\nu} \) with \( k \in \mathbb{N} \), \( \tau \in I_k \) and \( \nu = 1, \cdots, N(k, \tau) \) and all \( f \in (\mathcal{G}_0^\varepsilon(\beta, \gamma))' \) with \( 0 < \beta, \gamma < \varepsilon \) and \( x \in \mathcal{X} \),

\[
 f(x) = \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0, \tau)} \int_{Q_r^{k,\nu}} \tilde{D}_0(x, y) \, d\mu(y) m_{Q_r^{0,\nu}}(D_0(f)) + \sum_{k = 1}^\infty \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q_r^{k,\nu}) \tilde{D}_k(x, y_r^{k,\nu}) D_k(f)(y_r^{k,\nu}),
\]

where the series convergence in \( (\mathcal{G}_0^\varepsilon(\beta, \gamma))' \). Moreover, for any \( \varepsilon' \in (\varepsilon, 1) \), there exists a positive constant \( C \) depending on \( \varepsilon' \) such that the function \( \tilde{D}_k(x, y) \) satisfies (i) and (ii) of Lemma 3.1 and (iii)' of Lemma 3.2 for \( k \in \mathbb{Z}_+ \).

Proof of Theorem 1.2. For \( k \in \mathbb{Z} \), let \( D_k \equiv S_k - S_{k-1} \). We also choose \( \varepsilon \in (s, 1) \). We first show (i). If \( f \in h^p(\mathcal{X}) \) and (1.2) holds, by Definition 4.2, there exist \( \beta \) and \( \gamma \) as in (4.2) such that \( f \in (\mathcal{G}_0^\varepsilon(\beta, \gamma))' \) and

\[
 \|f\|_{h^p(\mathcal{X})} \equiv \left\{ \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0, \tau)} \mu(Q_r^{0,\nu}) \left[ m_{Q_r^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\{ \sum_{k = 1}^\infty |D_k(f)|^2 \right\}^{1/2}_{L^p(\mathcal{X})}.
\]

From this and Definition 4.2 together with Corollary 4.1, it follows that

\[
 \|f\|_{B_{p,q}^s(\mathcal{X})} \equiv \left\{ \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0, \tau)} \mu(Q_r^{0,\nu}) \left[ m_{Q_r^{0,\nu}}(|D_0(f)|) \right]^p \right\}^{1/p} + \left\{ \sum_{k = 1}^\infty 2^{ksq} \|D_k(f)\|_{L^p(\mathcal{X})}^q \right\}^{1/q} \lesssim \|f\|_{h^p(\mathcal{X})} + J_1 < \infty,
\]

where \( J_1 \) is as in Theorem 1.2. Thus, \( f \in B_{p,q}^s(\mathcal{X}) \).

Conversely, assume that \( f \in B_{p,q}^s(\mathcal{X}) \). By Definition 4.2 again, we know that \( f \in (\mathcal{G}_0^\varepsilon(\beta, \gamma))' \) for some \( \beta, \gamma \) as in (4.2). Recall that for all \( \{a_j\}_j \subset \mathbb{C} \) and \( r \in (0, 1] \),

\[
 (4.3) \quad \left( \sum_j |a_j| \right)^r \lesssim \sum_j |a_j|^r.
\]

If \( p/2 \leq 1 \), by (4.3), we have

\[
 \left\| \left\{ \sum_{k = 1}^\infty |D_k(f)|^2 \right\}^{1/2}_{L^p(\mathcal{X})} \right\| \lesssim \left\{ \sum_{k = 1}^\infty \|D_k(f)\|_{L^p(\mathcal{X})}^p \right\}^{1/p}.
\]
From this, (4.3) when $q/p \leq 1$ and the Hölder inequality when $q/p > 1$ together with the assumption that $s > 0$, it further follows that

$$
\left\| \left\{ \sum_{k=1}^{\infty} |D_k(f)|^2 \right\} \right\|_{L^p(\mathcal{X})}^{1/2} \leq \left\{ \sum_{k=1}^{\infty} 2^{ksq} \|D_k(f)\|_{L^p(\mathcal{X})}^q \right\}^{1/q} \lesssim \|f\|_{B^s_{p,q}(\mathcal{X})},
$$

which together with Definition 4.2 shows that $f \in h^p(\mathcal{X})$ and $\|f\|_{h^p(\mathcal{X})} \lesssim \|f\|_{B^s_{p,q}(\mathcal{X})}$.

If $p/2 > 1$, by the Hölder inequality and the assumption that $s > 0$, we have

$$
\left\| \left\{ \sum_{k=1}^{\infty} |D_k(f)|^2 \right\} \right\|_{L^p(\mathcal{X})}^{1/2} \lesssim \left\{ \sum_{k=1}^{\infty} 2^{ksq/2} \|D_k(f)\|_{L^p(\mathcal{X})}^p \right\}^{1/p}.
$$

Then an argument similar to the case $p/2 \leq 1$ also yields that $f \in h^p(\mathcal{X})$ and $\|f\|_{h^p(\mathcal{X})} \lesssim \|f\|_{B^s_{p,q}(\mathcal{X})}$.

On $J_1$, we have

$$
J_1 \lesssim \left\{ \sum_{k=-\infty}^{0} 2^{ksq} \|D_k(f)\|_{L^p(\mathcal{X})}^q \right\}^{1/q} + \left\{ \sum_{k=1}^{\infty} 2^{ksq} \|D_k(f)\|_{L^p(\mathcal{X})}^q \right\}^{1/q} \lesssim Z_1 + \|f\|_{B^s_{p,q}(\mathcal{X})},
$$

where

$$
Z_1 \equiv \left\{ \sum_{k=-\infty}^{0} 2^{ksq} \|D_k(f)\|_{L^p(\mathcal{X})}^q \right\}^{1/q}.
$$

Using the notation as in Lemma 4.2, since $f \in (\mathcal{G}_0^s(\beta, \gamma))^\prime$, by Lemma 4.2, for all $z \in \mathcal{X}$, we have

$$
f(z) = \sum_{\tau' \in I_0} \sum_{\nu' = 1}^{N(0,\tau')} \int_{Q_{\tau',\nu'}^0} \tilde{D}_0(z, y) \, d\mu(y) m_{\tau',\nu'}(D_0(f)) + \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu' = 1}^{N(k',\tau')} \mu(Q_{\tau',\nu'}^{k',\nu'}) \tilde{D}_{k'}(z, y_{k',\nu'}) D_{k'}(f)(y_{k',\nu'})
$$

in $(\mathcal{G}_0^s(\beta, \gamma))^\prime$. Obviously, $D_k(x, \cdot) \in \mathcal{G}_0^s(\beta, \gamma)$. Thus, we obtain that for all $x \in \mathcal{X}$,

$$
D_k(f)(x) = \sum_{\tau' \in I_0} \sum_{\nu' = 1}^{N(0,\tau')} (D_k \tilde{D}_0)(x, y) \, d\mu(y) m_{\tau',\nu'}(D_0(f)) + \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu' = 1}^{N(k',\tau')} \mu(Q_{\tau',\nu'}^{k',\nu'}) (D_k \tilde{D}_{k'})(x, y_{k',\nu'}) D_{k'}(f)(y_{k',\nu'})
$$

\( \equiv I_1 + I_2 \).
For any $\varepsilon' \in (\varepsilon, 1)$, $k \leq 0$ and $y_{r'}^0 \in Q_{r'}^0$, by the size conditions of $\tilde{D}_0$ and $S_k$, for all $x, y \in X$, we have

$$|(D_k \tilde{D}_0)(x, y)| = \left| \int_X D_k(x, z) \tilde{D}_0(z, y) \, d\mu(z) \right| \leq \int_{d(x, z) \geq d(x, y)/2} \frac{1}{V_2^{-k}(x) + V(x, z)} \frac{1}{V_1(z) + V(z, y)} \frac{2^{-ke'}}{[2^{-k} + d(x, z)]^{e'}} \, d\mu(z) + \int_{d(z, y) > d(x, y)/2} \ldots$$

From this and Lemma 5.3 of [9] with $n/(n + \varepsilon') < r \leq 1$, it follows that for all $x \in X$,

$$|I_1| \lesssim \sum_{\tau' \in I_0} \sum_{\nu' = 1}^{N(0, \tau')} \mu(Q_{\tau'}^{0, \nu'}) m_{Q_{\tau'}^{0, \nu'}}(|S_0(f)|) \leq 2^{kn(1-1/r)} \left\{ M \left( \sum_{\tau' \in I_0} \sum_{\nu' = 1}^{N(0, \tau')} \left[ m_{Q_{\tau'}^{0, \nu'}}(|S_0(f)|) \right]^{r} \chi_{Q_{\tau'}^{0, \nu'}} \right) (x) \right\}^{1/r}.$$
From (4.7), (4.8), (4.3) and Lemma 5.2 of [9], it follows that when
\[ x \in Q_{r}^{k,\nu}, \]

\[ -d(x, y_{r}^{k,\nu}) \leq 2^{-k} + |d(x, y_{r}^{k,\nu})| \leq 2^{-k} + d(y_{r}^{k,\nu}, y_{r}^{k,\nu}). \]

Notice that if \( x \in Q_{r}^{k,\nu}, \) then
\[ V_{2^{-k}}(y_{r}^{k,\nu}) + V(x, y_{r}^{k,\nu}) \sim V_{2^{-k}}(y_{r}^{k,\nu}) + V(y_{r}^{k,\nu}, y_{r}^{k,\nu}) \]
and
\[ 2^{-k} + d(x, y_{r}^{k,\nu}) \sim 2^{-k} + d(y_{r}^{k,\nu}, y_{r}^{k,\nu}). \]

From (4.7), (4.8), (4.3) and Lemma 5.2 of [9], it follows that when \( n/(n+s) < p \leq 1, \)
\[
\lambda \left\{ \sum_{k=-\infty}^{0} 2^{kq} \left| I_{2} \right|_{L^{p}(X)} \right\}^{1/q} \leq \lambda \left\{ \sum_{k=-\infty}^{0} 2^{kq} \left[ \sum_{\tau \in I_{k}} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_{r}^{k,\nu}, \lambda) \left( \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{\infty} 2^{-(k'-k)\nu'} \mu(Q_{r}^{k',\nu'}) \right)^{p} \right] \right\}^{1/q} \]
\[
\times \left[ V_{2^{-k}}(y_{r}^{k,\nu}) + V(\lambda, y_{r}^{k,\nu}) \right]^{(1-p)/p} \}
\[
\times \left[ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} N(k',\tau') \mu(Q_{r}^{k',\nu'}) \left( \sum_{\nu'=1}^{\infty} \mu(Q_{r}^{k',\nu'}) \right)^{p} \right]^{1/q} \]
\[
\lambda \left\{ \sum_{k=-\infty}^{0} 2^{kq} \left[ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{\infty} 2^{-k'\nu'} \mu(Q_{r}^{k',\nu'}) \left( \sum_{\nu'=1}^{\infty} \mu(Q_{r}^{k',\nu'}) \right)^{p} \right] \right\}^{1/q} \]
\[
\times \left\{ \sum_{k=-\infty}^{0} 2^{kq} \lambda \left\{ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{\infty} 2^{-k'\nu'} \mu(Q_{r}^{k',\nu'}) \left( \sum_{\nu'=1}^{\infty} \mu(Q_{r}^{k',\nu'}) \right)^{p} \right\}^{1/q} \right\} \]
\[
\lambda \left\{ \sum_{k=-\infty}^{0} 2^{kq} \lambda \left\{ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{\infty} 2^{-k'\nu'} \mu(Q_{r}^{k',\nu'}) \left( \sum_{\nu'=1}^{\infty} \mu(Q_{r}^{k',\nu'}) \right)^{p} \right\}^{1/q} \right\} \]
\[
\lambda \left\{ \sum_{k=-\infty}^{0} 2^{kq} \lambda \left\{ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{\infty} \mu(Q_{r}^{k',\nu'}) \left( \sum_{\nu'=1}^{\infty} \mu(Q_{r}^{k',\nu'}) \right)^{p} \right\}^{1/q} \right\} \].
where in the third-to-last inequality, we used the fact that

\[ V_{2^{-k}}(y_{\tau'}) \lesssim 2^{(k'-k)n} V_{2^{-k'}}(y_{\tau'}) \sim 2^{(k'-k)n} \mu(Q_{\tau'}^{k',\nu'}), \]

and in the penultimate inequality, we used the arbitrariness of \( y_{\tau'}^{k',\nu'} \in Q_{\tau'}^{k',\nu'} \), (4.3) when \( q/p \leq 1 \), or the Hölder inequality when \( q/p > 1 \).

From the arbitrariness of \( y_{\tau'}^{k',\nu'} \in Q_{\tau'}^{k',\nu'} \) again, it is easy to see that

\[ \sum_{k'=1}^{\infty} 2^{-(k'-k)(\epsilon'+s)} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) \frac{1}{V_{2^{-k}}(x) + V(x, y_{\tau'}) [2^{-k} + d(x, y)]^{\epsilon'}} \leq 1. \]

By this estimate, the Hölder inequality, (3.7) and the arbitrariness of \( y_{\tau'}^{k',\nu'} \in Q_{\tau'}^{k',\nu'} \), we obtain that when \( p \in (1, \infty) \),

\[ \left\{ \sum_{k=-\infty}^{\infty} 2^{kq} \| I_2 \|_{L^p(X)} \right\}^{1/q} \leq \left\{ \sum_{k=-\infty}^{\infty} \left[ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{-(k'-k)(\epsilon'+s)} 2^{k'sp} \mu(Q_{\tau'}^{k',\nu'}) |D_k(f)(y_{\tau'})|^p \right] \times \int_X \frac{1}{V_{2^{-k}}(y_{\tau'}) + V(x, y_{\tau'}) [2^{-k} + d(x, y)]^{\epsilon'}} \, d\mu(x) \right\}^{1/q} \]

\[ \lesssim \left\{ \sum_{k=-\infty}^{\infty} \left[ \sum_{k'=1}^{\infty} 2^{-(k'-k)(\epsilon'+s)} 2^{k'sp} \| D_k(f) \|_{L^p(X)}^p \right]^{1/q} \right\} \lesssim \| f \|_{B_{p,q}^s(X)}, \]

where in the penultimate inequality, we used the arbitrariness of \( y_{\tau'}^{k',\nu'} \in Q_{\tau'}^{k',\nu'} \), (4.3) when \( q/p \leq 1 \), or the Hölder inequality when \( q/p > 1 \).

Thus, \( Z_1 \lesssim \| f \|_{B_{p,q}^s(X)} \) and therefore, \( J_1 \lesssim \| f \|_{B_{p,q}^s(X)} \), which completes the proof of Theorem 1.2(i).

To show (ii) of Theorem 1.2, if \( f \in h^p(X) \) and (4.2) holds, by an argument similar to (i), then it is easy to see that \( f \in F_{p,q}^s(X) \) and \( \| f \|_{F_{p,q}^s(X)} \lesssim \| f \|_{h^p(X)} + J_2 \), where \( J_2 \) is as in Theorem 1.2.

Conversely, if \( f \in F_{p,q}^s(X) \), since

\[ F_{p,q}^s(X) \subset B_{p,\max(p,q)}^s(X) \]
where in the last inequality, we used the arbitrariness of \( y \).

Assume that \( f \in (G^0(\beta,\gamma))' \) for some \( \beta, \gamma \) as in (4.2). Write \( D_k(f) \) as in (4.4). By (4.5) with \( n/(n+\epsilon') < r \leq 1, \epsilon' \in (\epsilon,1) \) and the \( L^{p/r}(\mathcal{X}) \)-boundedness of \( M \) with \( r < p \), we obtain

\[
\left\| \left\{ \sum_{k=0}^{0} 2^{kq}|D_k(f)|^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \approx Z_2 + \|f\|_{F^p_{p,q}(\mathcal{X})},
\]

where

\[
Z_2 \equiv \left\| \left\{ \sum_{k=-\infty}^{0} 2^{kq}|D_k(f)|^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})}.
\]

To estimate \( I_2 \), by (4.6) and Lemma 5.3 of [9] with \( n/(n+\epsilon') < r \leq 1, \) for all \( x \in \mathcal{X} \), we have

\[
|I_2| \lesssim \sum_{k'=1}^{\infty} 2^{-(k'-k)\epsilon'} \left\{ M \left( \sum_{k \in I_{k'}} \sum_{\nu' = 1}^{N(0,\tau')} |D_k(f)(y_{k',\nu'}^{k'})|^r \chi_{Q_{k',\nu'}} \right) (x) \right\}^{1/r}
\]

\[
\lesssim \sum_{k'=1}^{\infty} 2^{-(k'-k)\epsilon'} \left\{ M (|D_k(f)|^r) (x) \right\}^{1/r},
\]

where in the last inequality, we used the arbitrariness of \( y_{k',\nu'}^{k'} \in Q_{k',\nu'}^{k'} \). From this, Lemma 3.14 of [9], (4.3) when \( q \leq 1 \), or the Hölder inequality when \( q > 1 \), it follows that

\[
\left\| \left\{ \sum_{k=-\infty}^{0} 2^{kq}|I_2|^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})}
\]
Proof. It was proved in [13, (4.4)] that \( \dot{\mathcal{B}}_{s,p}^q (\mathcal{X}) \) and \( \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \) were essentially given in the proof of Proposition 4.2 in [13]. Here we sketch it for the convenience of readers.

Let \( f = \sum_{k=1}^\infty 2^{ks/q} |D_k (f^\prime (r))|^q \) for \( f \in \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \), then, \( f \) is \( \mathcal{F}_{s,p}^q (\mathcal{X}) \)-integrable of elements in Besov spaces and Triebel-Lizorkin spaces, which were essentially given in the proof of Theorem 1.2.

Thus, \( Z_2 \lesssim \| f \|_{\mathcal{F}_{s,p}^q (\mathcal{X})} \) and therefore, \( J_2 \lesssim \| f \|_{\mathcal{F}_{s,p}^q (\mathcal{X})} \), which completes the proof of Theorem 1.2.

\[ \]

5 Local integrability of \( \mathcal{B}_{s,p}^q (\mathcal{X}), \mathcal{F}_{s,p}^q (\mathcal{X}), \dot{\mathcal{B}}_{s,p}^q (\mathcal{X}) \) and \( \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \)

We first recall the following (locally) \( \mathcal{L}^p (\mathcal{X}) \)-integrability of elements in Besov spaces and Triebel-Lizorkin spaces, which were essentially given in the proof of Proposition 4.2 in [13]. Here we sketch it for the convenience of readers.

Proposition 5.1. Let \( s \in (0, 1) \) and \( p \in (n/(n + 1), \infty] \). Then,
(i) \( \dot{\mathcal{B}}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^p_{\text{loc}} (\mathcal{X}) \) for \( q \in (0, \infty] \) and \( \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^p_{\text{loc}} (\mathcal{X}) \) for \( q \in (n/(n + 1), \infty] \);
(ii) \( \mathcal{B}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^p (\mathcal{X}) \) for \( q \in (0, \infty] \) and \( \mathcal{F}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^p (\mathcal{X}) \) for \( q \in (n/(n + 1), \infty] \).

Proof. It was proved in [13, (4.4)] that \( \dot{\mathcal{B}}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^p_{\text{loc}} (\mathcal{X}) \) for \( s, p, q \) as in the proposition, which together with \( \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \subset \mathcal{B}_{\text{max}(p,q)}^q (\mathcal{X}) \) for \( s \in (0, 1) \) and \( p, q \in (n/(n + 1), \infty] \) (see [9, Proposition 5.10(ii)] and [9, Proposition 6.9(ii)]) further implies that \( \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^p_{\text{loc}} (\mathcal{X}) \).

To show (ii), by \( \mathcal{F}_{s,p}^q (\mathcal{X}) \subset \mathcal{B}_{\text{max}(p,q)}^q (\mathcal{X}) \) (see [9, Proposition 5.31(iii)]), it suffices to establish the conclusion for Besov spaces. Moreover, this was given in the proof of Proposition 4.2 in [13], which completes the proof of Proposition 5.1.

As a corollary of Proposition 5.1 and the Hölder inequality, we have the following obvious conclusions.

Corollary 5.1. Let \( s \in (0, 1) \) and \( p \in [1, \infty] \). Then,
(i) \( \dot{\mathcal{B}}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^1_{\text{loc}} (\mathcal{X}) \) for \( q \in (0, \infty] \) and \( \dot{\mathcal{F}}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^1_{\text{loc}} (\mathcal{X}) \) for \( q \in (n/(n + 1), \infty] \);
(ii) \( \mathcal{B}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^1 (\mathcal{X}) \) for \( q \in (0, \infty] \) and \( \mathcal{F}_{s,p}^q (\mathcal{X}) \subset \mathcal{L}^1 (\mathcal{X}) \) for \( q \in (n/(n + 1), \infty] \).

Comparing Corollary 5.1 with the corresponding conclusions of Besov and Triebel-Lizorkin spaces on \( \mathbb{R}^n \) in [16, Theorem 3.3.2], the corresponding conclusions for \( \mathcal{X} \) when \( p \in (n/(n + s), 1) \) are missed. To obtain these cases, the method in [16] strongly depends on the embedding theorems for different metrics on \( \mathbb{R}^n \) in [17, p. 29]. However, such embedding conclusions are not available for \( \mathcal{X} \) due to the fact that for an RD-space \( \mathcal{X} \), its “local”
dimension may strictly less than its global dimension such as classes of nilpotent groups; see also [9]. But, using the inhomogeneous discrete Calderón reproducing formula, Lemma 4.2, and some basic properties of Besov and Triebel-Lizorkin spaces, we can improve Corollary 5.1 into the following proposition, which, according to [16, Theorem 3.3.2], is sharp even for Euclidean spaces.

In what follows, for \(|s| < 1\), let

\[ p(s) \equiv \max\{n/(n+1), \, n/(n+1+s)\}. \]

The properties of Besov and Triebel-Lizorkin spaces on RD-spaces in the following Lemma 5.1 can be found in [9].

**Lemma 5.1** ([9]). Let \(|s| < 1\).

(i) For \(p(s) < p \leq \infty\), \(B^s_{p,q_0}(\mathcal{X}) \subset B^s_{p,q_1}(\mathcal{X})\) when \(0 < q_0 \leq q_1 \leq \infty\,\), and \(F^s_{p,q_0}(\mathcal{X}) \subset F^s_{p,q_1}(\mathcal{X})\) when \(p(s) < q_0 \leq q_1 \leq \infty\).

(ii) Let \(-1 < s + \theta < 1\) and \(\theta > 0\). Then for \(p(s) < p \leq \infty\), \(B^{s+\theta}_{p,q_0}(\mathcal{X}) \subset B^{s+\theta}_{p,q_1}(\mathcal{X})\) when \(0 < q_0 \leq q_1 \leq \infty\), and \(F^{s+\theta}_{p,q_0}(\mathcal{X}) \subset F^{s+\theta}_{p,q_1}(\mathcal{X})\) when \(p(s) < q_0, \, q_1 \leq \infty\).

(iii) If \(p(s) < p, \, q \leq \infty\), then \(B^s_{p,\min(p,q)}(\mathcal{X}) \subset F^s_{p,q}(\mathcal{X}) \subset B^s_{p,\max(p,q)}(\mathcal{X})\).

(iv) \(F^0_{p,2}(\mathcal{X}) = L^p(\mathcal{X})\) for \(p \in (1, \infty)\), \(F^0_{1,2}(\mathcal{X}) = h^1(\mathcal{X})\) and \(F^0_{\infty,2}(\mathcal{X}) = \text{bmo}(\mathcal{X})\) with equivalent norms.

With the aid of Lemma 5.1, we further have the following conclusions.

**Proposition 5.2.** Let \(s \in [0,1]\). Then

(i) \(B^s_{p,q}(\mathcal{X}) \subset L^1_{\text{loc}}(\mathcal{X})\) if either \(p \in (n/(n+1), \infty)\), \(s \in (n(1/p-1)+1,1)\), \(q \in (0,\infty)\) or \(p \in (n/(n+1),1)\), \(s = n(1/p-1)+1\), \(q \in (0,1)\) or \(p \in (1,\infty)\), \(s = 0\), \(q \in (0,\min(p,2)]\);

(ii) \(F^s_{p,q}(\mathcal{X}) \subset L^1_{\text{loc}}(\mathcal{X})\) if either \(p \in (n/(n+1),1)\), \(s = n(1/p-1)+1\), \(q \in (n/(n+1),1)\) or \(p \in (n/(n+1),1)\), \(s \in (n(1/p-1),1)\), \(q \in (n/(n+1),\infty)\) or \(p \in [1,\infty)\), \(s \in (0,1)\), \(q \in (n/(n+1),\infty)\) or \(p \in [1,\infty)\), \(s = 0\), \(q \in (n/(n+1),2]\).

**Remark 5.1.** Comparing Proposition 5.2(ii) with the sharp result on \(\mathbb{R}^n\) in [16, Theorem 3.3.2(i)], the conclusion that \(F^s_{p,q}(\mathcal{X}) \subset L^1_{\text{loc}}(\mathcal{X})\) when \(p \in (n/(n+1),1)\), \(s = n(1/p-1)\) and \(q \in (1,\infty)\) is still unknown. But it is easy to show that this is true if \(\mathcal{X}\) is an Ahlfors \(n\)-regular metric measure space, by using the embedding theorem in [22].

**Proof of Proposition 5.2.** To show (i), we consider the following several cases. Case (i)_{1} \(p \in [1,\infty)\), \(s \in (0,1)\) and \(q \in (0,\infty)\). In this case, by (ii) and (iii) of Lemma 5.1, we have \(B^s_{p,q}(\mathcal{X}) \subset B^0_{p,\min(p,2)}(\mathcal{X}) \subset F^0_{p,2}(\mathcal{X})\), which together with Lemma 5.1(iv) and the known facts that \(L^p(\mathcal{X})\) for \(p \in (1,\infty)\), \(h^1(\mathcal{X})\) and \(\text{bmo}(\mathcal{X})\) are all subspaces of \(L^1_{\text{loc}}(\mathcal{X})\) further implies that in this case, \(B^s_{p,q}(\mathcal{X}) \subset L^1_{\text{loc}}(\mathcal{X})\). (This case is also included in Corollary 5.1(ii).)

Case (i)_{2} \(p \in (n/(n+1),1)\), \(s > n(1/p-1) > 0\) and \(q \in (0,\infty)\) or \(p \in (n/(n+1),1)\), \(s = n(1/p-1)\) and \(q \in (0,1)\). In this case, we need to use Lemma 4.2. Let all the notation be as in there. By Lemma 4.2, we know that for all \(x \in \mathcal{X}\)

\[ f(x) = \sum_{\tau \in T_0} \sum_{\nu=1}^{N(0,\tau)} m_{Q^{0,\nu}_\tau}(D_0(f)) \int_{Q^{0,\nu}_\tau} \tilde{D}_0(x,y) \, d\mu(y), \]

where \(Q^{0,\nu}_\tau\) is an \(n\)-cube containing \(x\), \(D_0(f)\) is the \(n\)-dimensional Radon measure of \(f\) over \(\mathcal{X}\), \(\mu\) is the \(n\)-dimensional Hausdorff measure on \(\mathcal{X}\), \(\tilde{D}_0(x,y)\) is the \(n\)-dimensional characteristic function of \(x\) over \(\mathcal{X}\), and \(\tau\) is a parameter in \(T_0\).
To this end, set 

\[ E_{\tau,0} \equiv \{ (\tau, \nu) : y_{\tau}^{k,\nu} \in B(x_1, 2^{-l_0+2}) \} \]

and for \( l \in \mathbb{N} \),

\[ E_{\tau,l} \equiv \{ (\tau, \nu) : y_{\tau}^{k,\nu} \in B(x_1, 2^{-l_0+2+l}) \setminus B(x_1, 2^{-l_0+1+l}) \} . \]

Then by the size condition of \( \tilde{D}_k \), for all \( x \in \mathcal{X} \), we have

\[ |f(x)| \lesssim \sum_{\tau \in \mathcal{I}_0} \sum_{\nu=1}^{N(\tau,\nu)} \mu(Q_{\tau}^{0,\nu}) m_{Q_{\tau}^{0,\nu}}(|D_0(f)|) \frac{1}{V_1(x) + V_1(y_{\tau}^{0,\nu}) + V(x, y_{\tau}^{0,\nu})} \]

\[ \times \frac{1}{[1 + d(x, y_{\tau}^{0,\nu})]^{\epsilon'}} + \sum_{k=1}^{\infty} \sum_{\tau \in \mathcal{I}_k} \sum_{\nu=1}^{N(\tau,\nu)} \mu(Q_{\tau}^{k,\nu}) \left| D_k(f)(y_{\tau}^{k,\nu}) \right| \]

\[ \times \frac{1}{V_{2-k}(x) + V_{2-k}(y_{\tau}^{k,\nu}) + V(x, y_{\tau}^{k,\nu}) \left[ 2^{-k} + d(x, y_{\tau}^{k,\nu}) \right]^{\epsilon'}} 2^{-ke'} \]

\[ \approx \sum_{l=0}^{\infty} \sum_{\tau \in \mathcal{I}_0} \sum_{\nu=1}^{N(\tau,\nu)} \chi_{E_{\tau,l}}(\tau, \nu) \mu(Q_{\tau}^{0,\nu}) m_{Q_{\tau}^{0,\nu}}(|D_0(f)|) \]

\[ \times \frac{1}{V_1(x) + V_1(y_{\tau}^{0,\nu}) + V(x, y_{\tau}^{0,\nu}) \left[ 1 + d(x, y_{\tau}^{0,\nu}) \right]^{\epsilon'}} 1 \]

\[ + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{\tau \in \mathcal{I}_k} \sum_{\nu=1}^{N(\tau,\nu)} \chi_{E_{\tau,l}}(\tau, \nu) \mu(Q_{\tau}^{k,\nu}) \left| D_k(f)(y_{\tau}^{k,\nu}) \right| \]

\[ \times \frac{1}{V_{2-k}(x) + V_{2-k}(y_{\tau}^{k,\nu}) + V(x, y_{\tau}^{k,\nu}) \left[ 2^{-k} + d(x, y_{\tau}^{k,\nu}) \right]^{\epsilon'}} 2^{-ke'} , \]

where \( \epsilon' \in (\epsilon, 1) \). For any \( (\tau, \nu) \in E_{\tau,0}^{k,\nu} \), it is easy to see that

\[ |f(x)| \lesssim \int_{B(x, 2^{-l_0})} \frac{1}{V_2(x)} d\mu(x) \]

\[ \lesssim \int_{\mathcal{X}} \frac{1}{V_{2-k}(y_{\tau}^{k,\nu}) + V(x, y_{\tau}^{k,\nu})} \frac{2^{-ke'}}{[2^{-k} + d(x, y_{\tau}^{k,\nu})]^{\epsilon'}} d\mu(x) \lesssim 1 . \]
and that
\[(5.3) \mu(B(x_1, 2^{-l_0})) \leq \mu(B(y^{k,\nu}_{\tau}, 2^{-l_0+3})) \lesssim 2^{(k-l_0)n}\mu(Q^{k,\nu}_{\tau}).\]

For any \((\tau, \nu) \in E_{\tau,l}^{k,\nu}\) with \(l \in \mathbb{N}\) and \(x \in B(x_1, 2^{-l_0})\), we have
\[(5.4) d(x, y^{k,\nu}_{\tau}) \geq d(x_1, y^{k,\nu}_{\tau}) - d(x, x_1) > d(x_1, y^{k,\nu}_{\tau})/2 \geq 2^{l-l_0}\]
and \(B(x_1, 2^{-l_0}) \subset B(y^{k,\nu}_{\tau}, 2^{l+3-l_0})\), which both imply that
\[(5.5) V(x, y^{k,\nu}_{\tau}) \sim V(y^{k,\nu}_{\tau}, x) \gtrsim V(y^{k,\nu}_{\tau}, x_1) \gtrsim \mu(B(x_1, 2^{l-l_0})) \gtrsim 2^{l_0} \mu(B(x_1, 2^{-l_0}))\]
and
\[(5.6) \mu(B(x_1, 2^{-l_0})) \lesssim 2^{(k+l-l_0)n}\mu(Q^{k,\nu}_{\tau}).\]

From the estimates (5.4) and (5.5), it further follows that for any \((\tau, \nu) \in E_{\tau,l}^{k,\nu}\) with \(l \in \mathbb{N}\),
\[(5.7) \int_{B(x_1, 2^{-l_0})} \frac{1}{V_{2-k}(x) + V_{2-k}(y^{k,\nu}_{\tau}) + V(x, y^{k,\nu}_{\tau})} \frac{2^{-k\epsilon'}}{2^{-l_0} \epsilon'} d\mu(x) \lesssim \frac{2^{-k\epsilon'}}{2^{(l-l_0)\epsilon'}}.\]

The estimates (5.2), (5.3), (5.6) and (5.7) together with (5.1) and (4.3) yield that
\[
\int_{B(x_1, 2^{-l_0})} |f(x)| d\mu(x) \\
\lesssim \sum_{l=0}^{\infty} \frac{1}{2^{l(\epsilon' + \kappa)}} \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \chi_{E_{\tau,l}^{0,\nu}}(\tau, \nu) \mu(Q^{0,\nu}_{\tau}) m_{Q^{0,\nu}_{\tau}}(|D_0(f)|) \\
+ \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{1}{2^{l(\epsilon' + \kappa)}} \sum_{\tau \in I_0} \sum_{\nu=1}^{N(k,\tau)} \chi_{E_{\tau,l}^{k,\nu}}(\tau, \nu) \mu(Q^{k,\nu}_{\tau}) |D_k(f)(y^{k,\nu}_{\tau})| \\
\lesssim \left\{ \sum_{l=0}^{\infty} \frac{2^n (1/p-1)}{2^{l(\epsilon' + \kappa)}} \left[ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_{\tau}) \left[ m_{Q^{0,\nu}_{\tau}}(|D_0(f)|) \right]^{1/p} \right]^{1/p} \right. \\
+ \left. \sum_{k=1}^{\infty} 2^{kn (1/p-1)} \left[ \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_{\tau}) |D_k(f)(y^{k,\nu}_{\tau})| \right]^{1/p} \right\}^{1/p} \\
\lesssim \|f\|_{B_{p,q}^s} + \sum_{k=1}^{\infty} 2^{k[n(1/p-1)-s]} 2^{k^2} \|D_k(f)\|_{L^p(\mathbb{R})},
\]
where in the last inequality we used the fact that \(\epsilon' + \kappa > n(1/p - 1)\) and the arbitrariness of \(y^{k,\nu}_{\tau} \in Q^{k,\nu}_{\tau}\). Now if \(s > n(1/p - 1)\), by the Hölder inequality when \(q \in [1, \infty]\) or by
When $q \in (0, 1)$, and if $s = n(1/p - 1)$ and $q \in (0, 1]$, by (4.3), we obtain from the last inequality that
\[
\int_{B(x_1, 2^{-1}L)} |f(x)| \, d\mu(x) \lesssim \|f\|_{B_{p,q}^s(X)} < \infty,
\]

namely, $f \in L^1_{\text{loc}}(X)$ in this case.

Case (i)$_3$ $p = 1$, $s = 0$ and $q \in (0, 1]$. In this case, by (i) and (iv) of Lemma 5.1, we have $B^0_{1,q}(X) \subset B^0_{1,1}(X) \subset F^0_{1,2}(X) = L^1(X) \subset L^1_{\text{loc}}(X)$.

Case (ii)$_4$ $p \in (1, \infty]$, $s = 0$, and $q \in (0, \min(p, 2)]$. In this case, (iii) and (iv) of Lemma 5.1 and the Hölder inequality yield that $B^0_{p,q}(X) \subset F^0_{p,2}(X) = L^p(X)$ when $p \in (1, \infty)$, $= \text{bmo}(X)$ when $p = \infty \subset L^1_{\text{loc}}(X)$. All these cases complete the proof of (i).

To prove (ii), we also consider the following four cases. Case (ii)$_1$ $p \in (n/(n+1), 1)$, $s = n(1/p - 1) > 0$, and $q \in (n/(n+1), 1]$. In this case, by (iii) and (i) of Lemma 5.1, and the corresponding conclusion on $B^s_{p,q}(X)$, we immediately obtain that $F^0_{p,q}(X) \subset L^1_{\text{loc}}(X)$.

Case (ii)$_2$ $p \in (n/(n+1), 1)$, $s \in (n(1/p - 1), 1)$, and $q \in (n/(n+1), \infty]$. In this case, Lemma 5.1(ii) and the conclusion in Case (ii)$_1$ imply that $F^s_{p,q}(X) \subset F^0_{p,1}(X) \subset L^1_{\text{loc}}(X)$.

Case (ii)$_3$ $p \in [1, \infty]$, $s \in (0, 1)$, and $q \in (n/(n+1), \infty]$. Lemma 5.1(i) yields that $F^s_{p,q}(X) \subset F^0_{p,2}(X)$, which together with Lemma 5.1(iv) and the known facts that $L^p(X)$ for $p \in (1, \infty)$, $h^1(X)$ and bmo($X$) are all subspaces of $L^1_{\text{loc}}(X)$ further implies that in this case, $F^s_{p,q}(X) \subset L^1_{\text{loc}}(X)$.

Case (ii)$_4$ $p \in [1, \infty]$, $s = 0$, and $q \in (n/(n+1), 2]$. In this case, Lemma 5.1(i) implies that $F^s_{p,q}(X) \subset F^0_{p,2}(X)$, which together with Lemma 5.1(iv) and the known facts that $L^p(X)$ for $p \in (1, \infty)$, $h^1(X)$ and bmo($X$) are all subspaces of $L^1_{\text{loc}}(X)$ again further implies that in this case $F^0_{p,q}(X)$ is a subspace of $L^1_{\text{loc}}(X)$. This finishes the proof of Proposition 5.2.

To obtain some conclusions similar to Proposition 5.2 for the spaces $\dot{B}^s_{p,q}(X)$ and $\dot{F}^s_{p,q}(X)$, we need to overcome another difficulty, namely, there exists no counterpart to Lemma 5.1(ii) for the spaces $\dot{B}^s_{p,q}(X)$ and $\dot{F}^s_{p,q}(X)$. However, via Lemma 5.2, we can still improve Corollary 5.1(i) in this case into the following conclusions.

**Proposition 5.3.** Let $s \in [0, 1]$. Then

(i) $\dot{B}^s_{p,q}(X) \subset L^1_{\text{loc}}(X)$ if either $p \in (n/(n+1), \infty]$, $s \in (n(1/p - 1) + 1, q \in (0, \infty]$ or $p \in (n/(n+1), 1]$, $s = n(1/p - 1)$, $q \in (0, 1]$ or $p \in (1, \infty]$, $s = 0$, $q \in (0, \min(p, 2)];$

(ii) $\dot{F}^s_{p,q}(X) \subset L^1_{\text{loc}}(X)$ if either $p \in (n/(n+1), 1)$, $s = n(1/p - 1)$, $q \in (n/(n+1), 1]$ or $p \in (n/(n+1), 1)$, $s \in (n(1/p - 1), 1)$, $q \in (n/(n+1), \infty]$ or $p \in [1, \infty]$, $s \in (0, 1)$, $q \in (n/(n+1), \infty]$ or $p \in [1, \infty]$, $s = 0$, $q \in (n/(n+1), 2]$.

To prove Proposition 5.3, we still need the following discrete homogeneous Calderón reproducing formula established in Theorems 4.11 of [9].

**Lemma 5.2.** Let $\varepsilon \in (0, 1)$ and $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation of the identity of order 1 with bounded support. For $k \in \mathbb{Z}$, set $D_k \equiv S_k - S_{k-1}$. Then, for any fixed $j \in \mathbb{N}$ large enough,
there exists a family \( \{ \widetilde{D}_k \}_{k \in \mathbb{Z}} \) of linear operators such that for any fixed \( y_{\tau}^{k,\nu} \in Q_\tau^{k,\nu} \) with \( k \in \mathbb{Z}, \tau \in I_k \) and \( \nu = 1, \ldots, N(k,\tau) \), and all \( f \in (\dot{G}_0^s(\beta,\gamma))' \) with \( 0 < \beta, \gamma < \epsilon \) and \( x \in \mathcal{X} \),

\[
    f(x) = \sum_{k=\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x,y_{\tau}^{k,\nu}) D_k(f)(y_{\tau}^{k,\nu}),
\]

where the series converge in \((\dot{G}_0^s(\beta,\gamma))'\). Moreover, for any \( \epsilon' \in (\epsilon,1) \), there exists a positive constant \( C_{\epsilon'} \) such that the kernels, denoted by \( \widetilde{D}_k(x,y) \), of the operators \( \widetilde{D}_k \) satisfy (i), (ii) and (iii) of Lemma 3.1.

**Proof of Proposition 5.3.** To establish (i) of Proposition 5.3, by Corollary 5.1(i), some properties similar to Lemma 5.1 (except (ii); see [9]) for the space \( \dot{B}_{sp,q}^s(\mathcal{X}) \), we only need to consider the cases when \( p \in (n/(n+1),1) \), \( s \in (n(1/p-1),1) \) and \( q \in (0,\infty] \) or \( p \in (n/(n+1),1) \), \( s = n(1/p-1) \) and \( q \in (0,1) \). Let \( f \in \dot{B}_{sp,q}^s(\mathcal{X}) \) with \( s, p \) and \( q \) as above. We show that for any \( l_0 \in \mathbb{Z} \),

\[
    \int_{B(x_1,2^{-l_0})} |f(x)| \, d\mu(x) \lesssim 2^{-l_0[s-n(1/p-1)]} \|f\|_{\dot{B}_{sp,q}^s(\mathcal{X})}.
\]

Let \( \{D_k\}_{k \in \mathbb{Z}} \) and other notation be as in Lemma 5.2. Since \( f \in \dot{B}_{sp,q}^s(\mathcal{X}) \), by Definition 4.1, we know that \( f \in (\dot{G}_0^s(\beta,\gamma))' \) with \( \beta, \gamma \) as in (4.1), where \( \epsilon \in (s,1) \) and \( p \in (n/(n+\epsilon),1) \). Thus, for any \( f \in \dot{G}_0^s(\beta,\gamma) \), since \( \int_\mathcal{X} g(x) \, d\mu(x) = 0 \), by Lemma 5.2, we have

\[
    \langle f, g \rangle = \sum_{k=-\infty}^{l_0-1} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(f)(y_{\tau}^{k,\nu}) \left[ \widetilde{D}_k(\cdot,y_{\tau}^{k,\nu}) - \widetilde{D}_k(x_1,y_{\tau}^{k,\nu}) \right] g(x_1)
\]



Thus, f is given by the (in \((\dot{G}_0^s(\beta,\gamma))'\) with \( \beta, \gamma \) as in (4.1)) convergent series of functions

\[
    \sum_{k=-\infty}^{l_0-1} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(f)(y_{\tau}^{k,\nu}) \left[ \widetilde{D}_k(x,y_{\tau}^{k,\nu}) - \widetilde{D}_k(x_1,y_{\tau}^{k,\nu}) \right]
\]

\[
    + \sum_{k=l_0}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(f)(y_{\tau}^{k,\nu}) \widetilde{D}_k(x,y_{\tau}^{k,\nu}) \equiv Y_1 + Y_2.
\]

We now prove that (5.8) holds for \( Y_1 \) and \( Y_2 \).

By the regularity of \( \widetilde{D}_k \) with \( \epsilon' \in (\epsilon,1) \), for all \( x \in \mathcal{X} \), we have

\[
    |Y_1| \lesssim \sum_{k=-\infty}^{l_0-1} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left| D_k(f)(y_{\tau}^{k,\nu}) \right| \left[ \frac{d(x_1)}{2^{-k} + d(x_1,y_{\tau}^{k,\nu})} \right]^{\epsilon'}
\]
Since $p \in (n/(n + 1), 1)$, for $k \leq l_0 - 1$, it is easy to see that

$$
\frac{1}{[\mu(Q^k_{\tau \nu})]^{1-p}} \left[ \frac{1}{V_{2^{-k}(x_1)} + V_{2^{-k}(y^k_{\tau \nu})} + V(x_1, y^k_{\tau \nu})} \right]^p \frac{2^{-k\epsilon p}}{2^{-k+d(x_1, y^k_{\tau \nu})}\epsilon^p} \leq \left\{ \begin{array}{ll}
\frac{1}{V_{2^{-l_0}(x_1)}}, & d(x, y^k_{\tau \nu}) \leq 2^k \\
\frac{1}{V_{2^{-l_0}(x_1)}} \frac{2^{ln(1-p)}}{2^{\epsilon \rho}}, & 2^{-k+l} < d(x, y^k_{\tau \nu}) \leq 2^{-k+l+1}, \ l \in \mathbb{Z}_+
\end{array} \right.
\quad \frac{1}{V_{2^{-l_0}(x_1)}}.
$$

Using this estimate, we further obtain that for all $x \in \mathcal{X}$,

$$
|Y_1| \lesssim [d(x, x_1)]^\epsilon' \sum_{k = -\infty}^{l_0-1} 2^{k\epsilon'} \left\{ \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q^k_{\tau \nu}) \left| D_k(f)(y^k_{\tau \nu}) \right|^p \right\}^{1/p} \lesssim [d(x, x_1)]^\epsilon' \sum_{k = -\infty}^{l_0-1} 2^{k(\epsilon' - s)} 2^{ks} \|D_k(f)||_{L^p(\mathcal{X})} \lesssim [d(x, x_1)]^\epsilon' \|f\|_{B^s_{p,q}(\mathcal{X})},
$$

where in the penultimate inequality, we used the arbitrariness of $y^k_{\tau \nu} \in Q^k_{\tau \nu}$ and in the last inequality, we used the Hölder inequality when $q \in (1, \infty)$, or (4.3) when $q \in (0, 1]$ together with $\epsilon' > s$. Thus,

$$
\int_{B(x_1, 2^{-l_0})} |Y_1| \, d\mu(x) \lesssim \|f\|_{B^s_{p,q}(\mathcal{X})} < \infty.
$$

The estimate for $Y_2$ is similar to the proof of Case (i)$_2$ in the proof of Proposition 5.2(i). Let $E_{k,l}^f$ for $l \in \mathbb{Z}_+$ be the same as in the proof of Proposition 5.2(i). By the size condition of $D_k$, we have

$$
\int_{B(x_1, 2^{-l_0})} |Y_2| \, d\mu(x)
$$
Proof of Proposition 5.2. This finishes the proof of Proposition 5.3 (i).

Similarly to Remark 5.1, if $X$ is an Ahlfors $n$-regular metric measure space, $p \in (n/(n+1), 1)$, $s = n(1/p - 1)$ and $q \in (n/(n+1), \infty]$, then $\dot{F}^{s}_{p,q}(X) \subset \dot{B}^{s}_{p,\max(p,q)}(X) \subset L^{1}_{\text{loc}}(X)$. The other cases are similar to the proof of Proposition 5.2. We omit the details, which completes the proof of Proposition 5.3.

\begin{flushright}
\Box
\end{flushright}

**Remark 5.2.** Similarly to Remark 5.1, if $X$ is an Ahlfors $n$-regular metric measure space, $p \in (n/(n+1), 1)$, $s = n(1/p - 1)$ and $q \in (1, \infty]$, then $\dot{F}^{s}_{p,q}(X) \subset L^{1}_{\text{loc}}(X)$. We omit the details.

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