GRADIENT ESTIMATES FOR THE WEIGHTED POROUS MEDIUM EQUATION ON GRAPHS

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Abstract. In this paper, we study the gradient estimates for the positive solutions of the weighted porous medium equation
\[ \Delta u^m = \delta(x)u_t + \psi u^m \]
on graphs for \( m > 1 \), which is a nonlinear version of the heat equation. Moreover, as applications, we derive a Harnack inequality and the estimates of the porous medium kernel on graphs. The obtained results extend the results of Y. Lin, S. Liu and Y. Yang for the heat equation [8, 9].

1. Introduction

The porous medium type equations have been extensively studied. For example, in (11), Felix Otto studied the following porous medium equation
\[ \frac{\partial \rho}{\partial t} - \nabla^2 \rho = 0 \]
where \( \nabla^2 \) denotes the Laplacian, \( \rho \geq 0 \) and \( \frac{\partial \rho}{\partial t} = -gradE|\rho| \), and showed that the porous medium equation has a gradient flow structure. In (15), Guangyue Huang, Zhijie Huang and Haizhong Li studied the gradient estimates for the positive solutions of the porous medium equation
\[ \Delta u^m = u_t \]
on Riemannian Manifolds, where \( m > 1 \). Many papers are devoted to this kind of equations such as (3, 4, 6, 7, 8, 9) and the references therein.

In this paper, we consider the porous medium type equation on graphs. Suppose that \( G = (V,E) \) is a graph with the vertex set \( V \) and the edge set \( E \). We investigate the following weighted porous medium equation on graph \( G = (V,E) \),
\[ \Delta u^m = \delta(x)u_t + \psi u^m, \]
where \( m > 1 \), \( \Delta \) is the usual graph Laplacian, \( \delta(x) : V \rightarrow \mathbb{R} \) is a weight and \( \psi(x,t) : V \times (-\infty, +\infty) \rightarrow \mathbb{R} \) is a given function. We get gradient estimates for the positive solutions of the equation (13) and derive a Harnack inequality and the estimates of the porous medium kernel (the solution of the porous medium equation) on graph \( G \).

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As is known to all, following from the chain rule, one has
\[
\Delta u^p = pu^{p-1}\Delta u + \frac{p-1}{p}u^{-p}\|\nabla u^p\|^2.
\] (1.4)

But generally (1.4) doesn’t hold in the discrete settings because the chain rule fails. In [1], Frank Bauer, Paul Horn, Yong Lin made the following crucial observation: while there exists no chain rule in the discrete setting, quite remarkably, identity (1.4) for \( p = \frac{1}{2} \) still holds on graphs. Indeed, this idea starts their work in paper [1]. This together with the maximum principle, they proved the Li-Yau gradient estimates for the heat kernel on graphs.

In this paper, we also make a observation. The following identity holds in the discrete settings for any \( m \in \mathbb{R} \) and \( u(x) > 0 \)
\[
\Delta u^m(x) = 2u^\frac{m}{2}(x)\Delta u^\frac{m}{2}(x) + 2\Gamma(u^\frac{m}{2}(x)).
\] (1.5)

This idea starts our work to get gradient estimates for the porous medium type equation on graph \( G \).

This paper is organized as follows. In Section 2, we introduce some notations and state our main results on graph \( G \). In Section 3, Section 4 and Section 5, we prove our main theorems.

2. Preliminaries and main results

Let \( G = (V,E) \) be a graph where \( V \) denotes the vertex set and \( E \) denotes the edge set. We denote \( x \sim y \) if vertex \( x \) is adjacent to vertex \( y \). We use \( xy \) to denote an edge in \( E \) connecting vertices \( x \) and \( y \). Let \( \omega_{xy} \) be the edge weight satisfying \( \omega_{xy} = \omega_{yx} > 0 \). The degree of vertex \( x \), denoted by \( \deg(x) = \sum_{y \sim x} \omega_{xy} \), is the number of edges connected to \( x \). If for every vertex \( x \) of \( V \), \( \deg(x) \) is finite, we say that \( G \) is a locally finite graph.

From [10], for any function \( u : V \to \mathbb{R} \), the \( \vartheta \)-Laplacian of \( u \) is defined as
\[
\Delta u(x) = \frac{1}{\vartheta(x)} \sum_{y \sim x} \omega_{xy}[u(y) - u(x)].
\] (2.1)

The associated gradient form reads
\[
\Gamma(u,v)(x) = \frac{1}{2} \left\{ \Delta(u(x)v(x)) - u(x)\Delta v(x) - v(x)\Delta u(x) \right\},
\]
\[
= \frac{1}{2\vartheta(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x))
\] (2.2)

We denote \( \Gamma(u) = \Gamma(u,u) \) for short, and we set
\[
\tilde{\sum}_{y \sim x} h(y) := \frac{1}{\vartheta(x)} \sum_{y \sim x} \omega_{xy}h(y)
\] (2.3)

for convenience of computations. So we can rewrite (2.1) as
\[
\Delta u(x) = \tilde{\sum}_{y \sim x} (u(y) - u(x)).
\] (2.4)

For the \( \vartheta \)-Laplacian \( \Delta \), assume \( \omega_{\min} := \inf_{e \in E} \omega_e > 0 \), \( D_\omega := \max_{y \sim x} \frac{\deg(x)}{\omega_{xy}} < +\infty \), \( D_\vartheta := \max_{x \in V} \frac{\deg(x)}{\vartheta(x)} < +\infty \).
Now we introduce the following lemma from \[1\] which will be used to prove our main theorems.

**Lemma 2.1.** (\[1\]) For any constant \(c, \alpha > 0\) and any functions \(\gamma, \psi_1, \psi_2 : [T_1, T_2] \rightarrow \mathbb{R}\), we have

\[
\min_{s \in (T_1, T_2)} \gamma(s) - \frac{1}{c} \int_{T_1}^{T_2} \gamma^2(t) dt + \alpha \int_{T_1}^{T_2} \psi_1(t) dt + \alpha \int_{T_1}^{T_2} \psi_2(t) dt \leq \frac{c}{(T_2 - T_1)} + \alpha \int_{T_1}^{T_2} \psi_1(t) dt + \frac{\alpha}{(T_2 - T_1)^2} \int_{T_1}^{T_2} (t - T_2)^2 (\psi_2(t) - \psi_1(t)) dt. \tag{2.5}
\]

Now, we state our main results.

**Lemma 2.2.** We assume \(G\) is a finite (or infinite) graph and \(\Delta\) is the Laplace operator on \(G\). Suppose \(u : V \rightarrow \mathbb{R}\) satisfies \(u(x) > 0\). Then for any \(m \in \mathbb{R}\) we have

\[
\Delta u^m(x) = 2u^m(x)\Delta \underline{u}(x) + 2\Gamma(u^m(x)). \tag{2.6}
\]

**Proof.**

\[
\Delta u^m(x) = \sum_{y \sim x} [u^m(y) - u^m(x)]
\]

\[
= \sum_{y \sim x} [(u^m)^2(y) - 2u^m(y)u^m(x) + (u^m)^2(x) + 2u^m(y)u^m(x) - 2(u^m)^2(x)]
\]

\[
= \sum_{y \sim x} [(u^m(y) - u^m(x))^2 + \sum_{y \sim x} 2u^m(x)|u^m(y) - u^m(x)|]
\]

\[
= 2\Gamma(u^m(x)) + 2u^m(x)\Delta \underline{u}(x).
\]

\[
\square
\]

**Theorem 2.3.** Let \(G = (V, E)\) be a finite or locally finite graph and \(u = u(x, t)\) be a positive solution to the equation

\[
\Delta u^m = \delta(x)u_t + \psi u^m,
\]

where \(m > 1\), \(\delta(x) : V \rightarrow \mathbb{R}\) is a weight and \(\psi(x, t) : V \times (-\infty, +\infty) \rightarrow \mathbb{R}\) is a given function. Suppose that \(D_\delta < +\infty\). Then we have

\[
\frac{\Gamma(u^m(x))}{u^m(x)} - \frac{\delta(x)u_t(x) + \psi u^m(x)}{2u^m(x)} \leq D_\delta, \forall x \in V.
\]

By Theorem 2.3, we can easily get the following Theorem 2.4

**Theorem 2.4.** Let \(G = (V, E)\) be a finite or locally finite graph and \(u = u(x, t)\) be a solution to the equation

\[
\begin{cases}
\Delta u^m = \delta(x)u_t + \psi u^m, \\
u_t > 0, u > 0, \delta(x) < 0,
\end{cases}
\]

where \(m > 1\), \(\delta(x) : V \rightarrow \mathbb{R}\) is a weight and \(\psi(x, t) : V \times (-\infty, +\infty) \rightarrow \mathbb{R}\) is a given function. Suppose that \(D_\delta < +\infty\). Then we have

\[
\frac{\Gamma(u^m(x))}{u^m(x)} - \frac{u_t}{u} + \frac{\psi}{2} \leq D_\delta, \forall x \in V.
\]
**Theorem 2.5.** Let $G = (V, E)$ be a finite or locally finite graph and $u = u(x, t)$ be a solution to the equation
\[
\begin{align*}
\begin{cases}
\Delta u^m = \delta(x)u_t + \psi u^m, \\
u_t > 0, \ u > 0, \ \delta(x) < 0,
\end{cases}
\end{align*}
\]
where $m > 1$, $\delta(x) : V \to \mathbb{R}$ is a weight and $\psi(x, t) : V \times (-\infty, +\infty) \to \mathbb{R}$ is a given function. Suppose that $D_\phi < +\infty, \ \vartheta_{\max} = \sup_{x \in V} \vartheta(x) < +\infty$. Then for any $(x, T_1)$ and $(y, T_2)$, $T_1 < T_2$, we have
\[
u(x, T_1) \leq \nu(y, T_2) \exp\{D_\phi(T_2 - T_1) + \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}(T_2 - T_1)} + \min \Phi(x, y, T_1, T_2)\},
\]
where
\[
\Phi(x, y, T_1, T_2) = \sum_{k=0}^{\eta-1} \left(\frac{\omega}{2} \int_{t_k}^{t_{k+1}} \psi(x_k, t) \, dt\right)
\]
\[
+ \frac{(\text{dist}(x, y))^2}{2(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2(\psi(x_{k+1}, t) - \psi(x_k, t)) \, dt.
\]
In particular, if there exists some constant $C_0$ such that $|\psi(x, t)| \leq C_0$ for all $(x, t)$, then for any $x, y \in V$ and $T_1 < T_2$, there holds
\[
u(x, T_1) \leq \nu(y, T_2) \exp\{D_\phi + \frac{5C_0}{6}(T_2 - T_1) + \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}(T_2 - T_1)}\}.
\]

**Theorem 2.6.** Let $G = (V, E)$ be a finite or locally finite graph and $p(t, x, y)$ be the porous medium kernel to the equation
\[
\begin{align*}
\begin{cases}
\Delta u^m = \delta(x)u_t + \psi u^m, \\
u_t > 0, \ u > 0, \ \delta(x) < 0,
\end{cases}
\end{align*}
\]
where $m > 1$, $\delta(x) : V \to \mathbb{R}$ is a weight and $\psi(x, t) : V \times (-\infty, +\infty) \to \mathbb{R}$ is a given function. Suppose that $D_\phi < +\infty, \ \vartheta_{\max} = \sup_{x \in V} \vartheta(x) < +\infty$. Then for any $(x, T_1)$ and $(y, T_2)$, $T_1 < T_2$, we have
(i) For any $t > 0$, $x, y \in V$, there holds
\[
p(t, x, y) \leq \frac{\exp\left\{4\sqrt{\frac{(6D_\phi + 5C_0)\vartheta_{\max}}{6m^2\omega_{\min}}}t\right\}}{V^*\text{Vol}(x, \sqrt{t})}.
\]
(ii) If we further assume $\vartheta(x) = \deg(x)$ for all $x \in V$, then
\[
p(t, x, y) \geq \frac{1}{\deg(y)} \exp\left\{-\left(1 + \frac{5C_0}{6}\right)t - \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}}t\right\}.
\]

3. **The Proof of Theorem 2.3**

By Lemma 2.2, we give the proof of Theorem 2.3 now.

**Proof.** Since $m > 1$ and $u(y) > 0$ for any $y \in V$, we have
\[
-\Delta u_\tilde{\varphi}(x) = \sum_{y \sim x} [u_\tilde{\varphi}(x) - u_\tilde{\varphi}(y)]
\leq \sum_{y \sim x} u_\tilde{\varphi}(x) = \frac{\deg(x)}{\vartheta(x)} u_\tilde{\varphi}(x).
\]
On the other hand, by Lemma 2.2 and (5.1), we have
\[
\Gamma(\varphi^m(x)) = \frac{1}{2} \Delta u^m(x) - u^m(x) \Delta \varphi(x)
\]
\[
\leq \frac{1}{2} \Delta u^m(x) + \frac{\deg(x)}{\vartheta(x)} u^m(x)
\]
\[
= \frac{\delta(x) u_t(x) + \psi u^m_t}{2} + \frac{\deg(x)}{\vartheta(x)} u^m(x).
\]
(3.2)

By (3.2) we have
\[
\frac{\Gamma(\varphi^m(x))}{u^m(x)} - \frac{\delta(x) u_t(x) + \psi u^m_t}{2u^m(x)} \leq \frac{\deg(x)}{\vartheta(x)} \leq D_0.
\]
(3.3)

4. THE PROOF OF THEOREM 2.5

By Lemma 2.1 and the result of Theorem 2.4, we prove Theorem 2.5 now.

Proof. Let \( u \) be a positive solution to the weighted porous medium equation \( \Delta u^m = \delta(x)u_t + \psi u^m \). By Theorem 2.4 we have for any \( m > 1 \),
\[
-\frac{\partial_t \log u}{u^m} \leq D_0 + \frac{\psi}{2} - \frac{\Gamma(\varphi^m(x))}{u^m(x)}.
\]
(4.1)

We derive the estimates of the terms in (4.2) respectively. It is obvious that
\[
-\int_{T_1}^{T_2} \frac{\Gamma(\varphi^m(x,t))}{u^m(x,t)} dt \leq 0.
\]
(4.3)

On the other hand, since
\[
\Gamma(\varphi^m(y,t)) = \sum_{z \sim y} [\varphi^m(z,t) - \varphi^m(y,t)]^2
\]
\[
\geq \frac{\omega_{\min}}{\vartheta_{\max}} [\varphi^m(x,t) - \varphi^m(y,t)]^2,
\]
(4.4)
we have
\[
-\int_{T_1}^{T_2} \frac{\Gamma(\varphi^m(y,t))}{u^m(y,t)} dt \leq -\int_{T_1}^{T_2} \frac{\omega_{\min}}{\vartheta_{\max}} \left[ \frac{[\varphi^m(x,t) - \varphi^m(y,t)]^2}{\varphi^m(y,t)} \right] dt.
\]
Arranging (4.8), we get
\[
\gamma(x, y, t) = \frac{u(x, t) - u(y, t)}{u(y, t)}. 
\] (4.6)

Using an elementary inequality \( \log r \leq r - 1, \forall r > 0 \), we have
\[
\log \frac{u(x, s)}{u(y, s)} \leq \gamma(x, y, s). 
\] (4.7)

Inserting (4.3), (4.5) and (4.7) into (4.2), and by using Lemma 2.1 we get
\[
\log u(x, T_1) - \log u(y, T_2) \leq \frac{m}{2} D_\theta(T_2 - T_1) + \frac{m}{4} \int_{T_1}^{T_2} \psi(x, t) dt + \frac{m}{4} \int_s^{T_2} \psi(y, t) dt + \gamma(x, y) - \frac{m \omega_{\min}}{2 \theta_{\max}} \int_s^{T_2} \gamma^2(x, y, t) dt
\]
\[
\leq \frac{m}{2} D_\theta(T_2 - T_1) + \frac{2 \theta_{\max}}{m \omega_{\min}(T_2 - T_1)} + \frac{m}{4} \int_{T_1}^{T_2} \psi(x, t) dt
\]
\[
+ \frac{m}{4(T_2 - T_1)^2} \int_{T_1}^{T_2} (t - T_2)^2 (\psi(y, t) - \psi(x, t)) dt.
\] (4.8)

Arranging (4.8), we get
\[
\log u(x, T_1) - \log u(y, T_2) \leq D_\theta(T_2 - T_1) + \frac{4 \theta_{\max}}{m^2 \omega_{\min}}(T_2 - T_1) + \frac{1}{2} \int_{T_1}^{T_2} \psi(x, t) dt
\]
\[
+ \frac{1}{2(T_2 - T_1)^2} \int_{T_1}^{T_2} (t - T_2)^2 (\psi(y, t) - \psi(x, t)) dt.
\]

**Case 2** If \( x \) is not adjacent to \( y \), we have the following results.
Let \( \text{dist}(x, y) = \eta \). Take a shortest path \( x = x_0, x_1, \cdots, x_\eta = y \). Set \( T_1 = t_0 < t_1 < \cdots < t_\eta = T_2, t_k = t_{k-1} + (T_2 - T_1)/\eta, k = 1, \cdots, \eta \). By the result of Case 1, we have
\[
\log u(x, T_1) - \log u(y, T_2) = \sum_{k=0}^{\eta-1} (\log u(x_k, t_k) - \log u(x_{k+1}, y_{k+1}))
\]
\[
\leq \sum_{k=0}^{\eta-1} (D_\theta(t_{k+1} - t_k) + \frac{4 \theta_{\max}}{m^2 \omega_{\min}}(t_{k+1} - t_k))
\]
\[
+ \sum_{k=0}^{\eta-1} \frac{1}{2} \int_{t_k}^{t_{k+1}} \psi(x_k, t) dt + \frac{\eta^2}{2(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (\psi(x_{k+1}, t) - \psi(x_k, t)) dt.
\]
\[
\leq D_\theta(T_2 - T_1) + \frac{4 \theta_{\max} \eta^2}{m^2 \omega_{\min}(T_2 - T_1)}
\]
\[
+ \sum_{k=0}^{\eta-1} \frac{1}{2} \int_{t_k}^{t_{k+1}} \psi(x_k, t) dt + \frac{\eta^2}{2(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (\psi(x_{k+1}, t) - \psi(x_k, t)) dt.
\]
\[
\leq D_\theta(T_2 - T_1) + \frac{4 \theta_{\max}(\text{dist}(x, y))^2}{m^2 \omega_{\min}(T_2 - T_1)} + \min \Phi(x, y, T_1, T_2),
\] (4.9)
that is
\[ u(x, T_1) \leq u(y, T_2) \exp\left\{ D_\vartheta(T_2 - T_1) + \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}(T_2 - T_1)} + \min \Phi(\psi)(x, y, T_1, T_2) \right\}, \]
(4.10)

where
\[
\Phi(\psi)(x, y, T_1, T_2) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \psi(x_k, t)\, dt 
+ \frac{\eta^2}{2(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (\psi(x_{k+1}, t) - \psi(x_k, t))\, dt,
\]
(4.11)
and the minimum takes over all shortest paths connecting \( x \) and \( y \).

Moreover, if \( |\psi(x, t)| \leq C_0 \) for all \((x, t)\), then we have
\[
\Phi(\psi)(x, y, T_1, T_2) \leq \frac{5C_0(T_2 - T_1)}{6} \tag{4.12}
\]
By (4.9) and (4.12), we have
\[
\log u(x, T_1) - \log u(y, T_2) \leq (D_\vartheta + \frac{5C_0}{6})(T_2 - T_1) + \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}(T_2 - T_1)}. \tag{4.13}
\]
That is
\[
u(x, T_1) \leq u(y, T_2) \exp\left\{ (D_\vartheta + \frac{5C_0}{6})(T_2 - T_1) + \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}(T_2 - T_1)} \right\}. \tag{4.14}
\]

5. The Proof of Theorem 2.6

The proof of Theorem 2.6 is very similar to the proof of Theorem 4 in [8]. We give the proof here for the readers’ convenience.

**Proof.** Let \( P(t, x, y) \) be the minimal heat kernel of the graph \( G = (V, E) \). Let \( t > 0 \) be fixed. For any \( t' > t \) and \( x, y, z \in V \), by Theorem 3.2 we get
\[
p(t, x, y) \leq p(t', z, y)\exp\left\{ (D_\vartheta + \frac{5C_0}{6})(t' - t) + \frac{4\vartheta_{\max}(\text{dist}(x, y))^2}{m^2\omega_{\min}(t' - t)} \right\}. \tag{5.1}
\]
For \( z \in B(x, \sqrt{t}) \), by (5.1) we obtain
\[
p(t, x, y) \leq \frac{(\sum_{z \in B(x, \sqrt{t})} \var\var(z)p(t', z, y))\exp\left\{ (D_\vartheta + \frac{5C_0}{6})(t' - t) + \frac{4\vartheta_{\max}(t' - t)}{m^2\omega_{\min}(t' - t)} \right\}}{VolB(x, \sqrt{t})}. \tag{5.2}
\]
Since
\[
\sum_{z \in B(x, \sqrt{t})} \var\var(z)p(t', z, y) \leq 1, \tag{5.3}
\]
and
\[
\inf_{t' > t} (D_\vartheta + \frac{5C_0}{6})(t' - t) + \frac{4\vartheta_{\max}t}{m^2\omega_{\min}(t' - t)} = 4\sqrt{(6D_\vartheta + 5C_0)\var\var_{\max}t} \tag{5.4}
\]
By (5.2), we get
\[
p(t, x, y) \leq \frac{\exp\{4 \sqrt{(6D_0 + 5C_0)\vartheta_{\text{max}}t}}{\text{VolB}(x, \sqrt{t})}. \tag{5.5}
\]
and thus (i) holds.

Now we assume \(\vartheta(x) = \deg(x)\) and thus \(D_\vartheta = 1\). Let \(p(x, y) = \omega_{xy}/\deg(x), p_0(x, y) = \chi_{xy}\), where \(\chi_{xy} = 1\) if \(x = y\), and \(\chi_{xy} = 0\) if \(x \neq y\). Set \(p_{k+1}(x, z) = \sum_{y \in V} p(x, y)p_k(y, z)\).

It is well known (see for example [2]) that the kernel \(p(t, x, y)\) can be written as
\[
p(t, x, y) = e^{-t} \sum_{k=0}^{+\infty} \frac{t^k p_k(x, y)}{k! \deg(y)}. \tag{5.6}
\]
which leads to
\[
p(t, y, y) \geq \frac{e^{-t}}{\deg(y)}, \forall t > 0. \tag{5.7}
\]

We now distinguish two cases of \(t\) to proceed.

**Case 1.** \(t > 1\). For any \(0 < \tau < 1\), by (5.7) and Theorem 3.2 we have that
\[
\frac{e^{-\tau t}}{\deg(y)} \leq p(\tau t, y, y) \leq p(t, x, y)\exp\{(1 + \frac{5C_0}{6})(t - \tau) + \frac{4\vartheta_{\text{max}}(\text{dist}(x, y))^2}{m^2\omega_{\text{min}}(t - \tau)}\}. \tag{5.8}
\]
Since \(t - \tau \geq (1 - \tau)t\), we have
\[
p(t, x, y) \geq \frac{e^{-\tau t}}{\deg(y)}\exp\{- (1 + \frac{5C_0}{6})(t - \tau) - \frac{4\vartheta_{\text{max}}(\text{dist}(x, y))^2}{m^2\omega_{\text{min}}(1 - \tau)t}\}. \tag{5.9}
\]

**Case 2.** \(0 < t \leq 1\). For any \(0 < \tau < 1\), by (5.7) and Theorem 3.2 we have
\[
\frac{e^{-\tau t}}{\deg(y)} \leq p(\tau t, y, y) \leq p(t, x, y)\exp\{(1 + \frac{5C_0}{6})(1 - \tau)t + \frac{4\vartheta_{\text{max}}(\text{dist}(x, y))^2}{m^2\omega_{\text{min}}(1 - \tau)t}\}. \tag{5.10}
\]
Thus, by (5.10) we get
\[
p(t, x, y) \geq \frac{e^{-\tau t}}{\deg(y)}\exp\{- (1 + \frac{5C_0}{6})(1 - \tau)t - \frac{4\vartheta_{\text{max}}(\text{dist}(x, y))^2}{m^2\omega_{\text{min}}(1 - \tau)t}\}. \tag{5.11}
\]

By (5.9) and (5.11), we get
\[
p(t, x, y) \geq \frac{e^{-\tau t}}{\deg(y)}\exp\{- (1 + \frac{5C_0}{6})t - \frac{4\vartheta_{\text{max}}(\text{dist}(x, y))^2}{m^2\omega_{\text{min}}(1 - \tau)t}\}. \tag{5.12}
\]

Let \(\tau \to 0^+\), and we have
\[
p(t, x, y) \geq \frac{1}{\deg(y)}\exp\{- (1 + \frac{5C_0}{6})t - \frac{4\vartheta_{\text{max}}(\text{dist}(x, y))^2}{m^2\omega_{\text{min}}t}\}. \tag{5.13}
\]
This gives the desired result. \(\Box\)

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