Thurston Geometries from Eleven Dimensions

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ABSTRACT

In three dimensions, a ‘master theory’ for all Thurston geometries requires imaginary flux. However, these geometries can be obtained from physical three-dimensional theories with various additional scalar fields, which can be interpreted as moduli in various compactifications of a higher-dimensional ‘master theory’. Three Thurston geometries are of the form \( N_2 \times S^1 \), where \( N_2 \) denotes a two-dimensional Riemannian space of constant curvature. This enables us to twist these spaces, via T-duality, into other Thurston geometries as a \( U(1) \) bundle over \( N_2 \). In this way, Hopf T-duality relates all but one of the geometries in the higher-dimensional M-theoretic framework. The exception is the ‘Sol geometry,’ which results from the dimensional reduction of the decoupling limit of the D3-brane in a background \( B \) field.

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1 Introduction

There is a powerful but under-utilized technology available for dealing with three-manifolds, due to the work of mathematicians in the 1970’s and 1980’s and culminating in Thurston’s Geometrization Conjecture \[1\]. This conjecture states that a three-manifold with a given topology has a canonical decomposition into a connected sum of ‘simple three-manifolds,’ each of which admits one, and only one, of eight homogeneous geometries: \(H^3\), \(S^3\), \(E^3\), \(S^2 \times S^1\), \(H^2 \times S^1\), Sol, Nil and SL(2,R). The conjecture has not been completely proven but few in the field doubt its veracity.

To see why this technology would be important in fundamental physics, consider the case of two-dimensional physics, such as conformal field theories or the world-volume of string theory. In the path-integral formalism, one must sum over two-dimensional topologies and geometries. This procedure is executable for two-dimensional Riemannian manifolds because of the existence of a geometrization theorem \[13\]. Namely, a closed two-dimensional manifold with handle number (i.e. topology) \(h = 0, 1\) or \(\geq 2\), respectively, admits the spherical, torus, or hyperbolic geometry. Hence, in the path integral, we can sum over deformations of each of these geometries, then sum over the handle number.

In (super)membrane physics \[14\] or in three-dimensional quantum gravity we should be able to perform path-integral quantization via a similar procedure to that in two dimensions. However, it is crucial to understand the connection between three-dimensional topology and M-theory in an analogous manner to the two-dimensional case, where two-dimensional topology provides the stringy analogue of Feynman diagrams. We are a long way from this but as a first step we search for a theory which admits the eight Thurston spaces as solutions.

The plan of this paper is as follows. In section 2, we discuss the difficulties involved in attempting to obtain all eight Thurston spaces from a single ‘master theory’ in three dimensions. This serves as motivation to search for a ‘master theory’ in higher dimensions. In section 3, we show how three of the Thurston geometries are of the form \(N_2 \times S^1\), where \(N_2\) denotes \(H^2\), \(E^2\) or \(S^2\). In an M-theoretic context, these geometries can be T-dualized into other Thurston geometries which are a \(U(1)\) bundle over \(N_2\). The only Thurston space

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\[1\] A fairly clear exposition of this can be found in the review article by P. Scott \[2\]. There has been some use made of the Geometrization Conjecture in cosmology, beginning with the paper by Fagundes \[3\]. A non-exhaustive list of other work along these lines is in \[4, 5, 6, 7, 8\]. Examples of attempts to understand the proof of the conjecture using techniques from gravitational and particle physics are \[9, 10\]. The role of the conjecture in high-energy physics is explored in \[1, 2\].
which does not naturally fit into this scheme is Sol, which results from the dimensional reduction of the decoupling limit of the D3-brane in a background \( B \) field. We present conclusions in section 4.

### 2 Difficulties finding ‘master theory’ in three dimensions

One can construct three-dimensional Chern-Simons gauge theories in which each gauge group is the group of isometries of a given Thurston geometry. This is well-known for the isotropic geometries \( E^3, S^3 \) and \( H^3 \), and can also be done for the anisotropic Thurston geometries\(^2\). Of the Thurston spaces, only \( E^3, S^3 \) and \( H^3 \) are solutions of Einstein gravity. In search of a single theory from which all eight of the Thurston geometries arise, we next turn to the low-energy limit of three-dimensional string theory, which has a metric \( g_{\mu\nu} \), dilaton \( \phi \), Abelian 2-form potential \( B(2) \) with field strength \( H(3) = dB(2) \) and a ‘constant’ term in the level of the original sigma model \( 15, 16 \). Finding that this theory fares no better than pure gravity as a ‘master theory,’ we consider a hypothetical three-dimensional theory which is identical to the low-energy limit of three-dimensional string theory, except for an additional Abelian gauge field \( A(1) \). This hypothetical theory will serve as a straw man to be knocked down, in order to effectively demonstrate the difficulties encountered in our search for a three-dimensional ‘master theory’. The corresponding action is given by

\[
S = \int d^3x \sqrt{g} e^{-2\phi} \left( \frac{4}{k} R + 4|\nabla \phi|^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right) + \frac{e}{3} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho}, \tag{2.1}
\]

where the last term is the Abelian Chern-Simons term for the field \( A(1) \), and \( F(2) = dA(1) \). All the Thurston geometries are solutions of the equations of motion of this theory for various values of the level \( k \) and the constant \( e \), as well as the other fields.

With the exceptions of Sol and \( H^3 \), the Thurston spaces can be characterized topologically as Seifert fibre bundles \( \eta \) over an orbifold \( Y \). The topology of a Seifert fibre bundle is determined by the Euler number \( \chi(Y) \) of \( Y \) and the Euler number \( e(\eta) \) of the bundle \( \eta \).\(^1\)\(^2\)\(^3\)\(^4\)\(^5\)\(^6\). It turns out that the level \( k \) is related to \( \chi(Y) \) by \( \chi(Y) = 4/k \) for all the Thurston spaces except Sol and \( H^3 \). In addition, the constant \( e \) turns out to be precisely the Euler number \( e(\eta) \).

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\(^2\)For example, the isometry group of Sol is the solvable Lie Group with the three generators \( T, L, M \) which satisfy the algebra \( [T, L] = -L; [T, M] = M; [L, M] = 0 \). The gauge potential is \( A = A^1 T + A^2 M + A^3 L \). The Chern-Simons equation of motion is \( F(A) = dA + \frac{1}{2}[A, A] = 0 \), which has a solution \( A^1 = dx; A^2 = e^{-x} dz; A^3 = e^x dy \). These are the frame-fields for metric corresponding to Sol.
However, this is a ‘master theory’ only formally since not all of the Thurston geometries arise as physical solutions. The three-form field strength $H_{(3)}$ is real only for the Thurston spaces which are Seifert fibre bundles. Specifically, Sol and $H^3$ require an imaginary $H_{(3)}$. In addition, $H^2 \times S^1$, Nil and SL(2,R) require an imaginary $F_{(2)}$.

In order for all Thurston solutions to have real fields in a hypothetical three-dimensional theory of sufficient generality, we could embark on the guessing game of adding, for example, more scalar fields. For instance, Sol arises as a physical solution when there are three scalars within the theory. However, at the end of this laborious procedure, one is left with a model that is not well-motivated from the three-dimensional viewpoint. We have arbitrarily added fields and do not even know the precise way in which they interact with the already-present fields. However, one could quite naturally regard the additional scalars as being the moduli of compact spaces, due to various reductions from a ‘master theory’ in higher dimensions. Also, the curvatures of the compact spaces yield effective cosmological terms.

3 Thurston geometries from eleven dimensions

3.1 Twisted spaces

Various $d$-dimensional geometries $M_d$ can be expressed as $U(1)$ bundle over $(d-1)$-dimensional geometries $N_{d-1}$. We will consider the case of $d = 3$, for which the metric of the twisted space has the form

$$dM_3^2 = dN_2^2 + (dz + A_{(1)})^2,$$

with $N_{(2)} = dA_{(1)}$ the volume form on $N_2$.

There are three compact, locally homogeneous Riemannian spaces in two dimensions, which we label as $N_2$ in order to represent $S^2$, $H^2$ or $T^2$. In three dimensions, according to Thurston’s Geometrization Conjecture [1], there are eight of such spaces. A direct product of $N_2 \times S^1$ produces three of the Thurston spaces. A $U(1)$ bundle along $S^1$ and around $N_2$ twists these previous direct product spaces into four other Thurston spaces [1, 3, 12].

To be explicit, $S^3$ as a $U(1)$ bundle over $S^2$, is expressed as

$$ds_3^2 = \frac{1}{4}\left(d\theta^2 + \sin^2 \theta d\phi^2 + (d\psi + \cos \theta d\phi)^2\right).$$

(3.2)

Nil, as a bundle over $T^2$, is given by

$$ds_3^2 = dx^2 + dy^2 + (dz - x dy)^2.$$  

(3.3)
$H^3$ can be expressed as twisted $H^2 \times S^1$ only in the unphysical case of an imaginary $U(1)$ field:

$$ds^2_3 = \frac{1}{4} \left( \cosh^2 \rho dt^2 + d\rho^2 + (dz + i \sinh \rho dt)^2 \right).$$

(3.4)

On the other hand, SL(2,R) can be expressed as twisted $H^2 \times S^1$, as in the above equation, except with a real $U(1)$ field. This means that T-duality can physically relate the geometry $H^2 \times S^1$ with SL(2,R) but not with $H^3$. Thus, in particular, from a supergravity geometry which contains $T^2 \times S^1$ or $H^2 \times S^1$, one can T-dualize to obtain a solution which contains the less common geometries Nil or SL(2,R), respectively.

Direct products of hyperbolic, spherical and Euclidean spaces frequently result as vacua in the near-horizon regions of $p$-brane solutions in string/M-theory. As an example, the near-horizon region geometry of a standard M2/M2/M5/M5 brane intersection can be expressed as $(AdS_2 \times S^1) \times (S^2 \times S^1) \times (T^2 \times S^1) \times T^2$. With the appropriate $D = 11$ three-form field, T-duality serves to twist the above product spaces in the parentheses, in order to obtain other Thurston spaces embedded within the M-theoretic context. T-duality along a $U(1)$ fibre is known as Hopf T-duality [17, 18]. The connection between the twisted spaces and direct product spaces is summarized in Table 1.

| Product Space ($N_2 \times S^1$) | Hopf T-duality | Twisted Space ($M_3$) |
|----------------------------------|---------------|----------------------|
| $S^2 \times S^1$                | $\leftrightarrow$ | $S^3$                |
| $T^2 \times S^1$                | $\leftrightarrow$ | Nil                  |
| $H^2 \times S^1$                | $\leftrightarrow$ | $H^3$, SL(2,R)       |

Table 1: Twisted Thurston spaces

It is interesting to note that the Thurston spaces which are related to each other by Hopf T-duality share the same value for the level $k$ in the hypothetical three-dimensional theory of Section 2. Only Sol does not appear in Table 1.

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$^3H^3$ expressed as an imaginary bundle over $H^2$ can be obtained as a Wick rotation of $AdS_3$ expressed as a bundle over $AdS_2$. The alternative Euclideanization of $g_{tt} \rightarrow -g_{tt}$ produces SL(2,R) as a real bundle over $H^2$. However, since Wick rotations do not ensure that Majorana or self-duality constraints are compatible with the Euclidean signature, it is best to begin with a solution which contains a (Riemannian) hyperbolic factor from the start [19].

$^4$Hopf T-duality untwists Thurston spaces into the form $N_2 \times S^1$ only when there are only R-R charges present in the ten-dimensional theory [30].

4
3.2 Sol from D3-brane in $B$ field

Consider the decoupling limit of a D3-brane with a constant NS $B$ field, for which the $B$ field goes to infinity. The metric solution is given by \[20, 21\]

\[
ds^2_{10} = \left(\frac{u}{R}\right)^2 \left( -dt^2 + dx_1^2 + \frac{1}{1 + b^2(u/R)}(dx_2^2 + dx_3^2) \right) + \left(\frac{R}{u}\right)^2 du^2 + R^2 d\Omega_5^2, \tag{3.5}
\]

where $R$ corresponds to the D3-brane charge and $b$ is proportional to the $B$ field, which we take to both be unity for simplicity. This is the metric of the supergravity dual to a Yang-Mills gauge theory with noncommuting $x_2, x_3$ coordinates. For small $u$, the above geometry reduces to $AdS_5 \times S^5$, corresponding to ordinary Yang-Mills occupying the IR region of the dual field theory. Using the metric Ansatz \[22\]

\[
ds^2_D = e^{-2\alpha \phi} ds_d^2 + e^{2(d-2)\alpha \phi} dz^2, \tag{3.6}
\]

where $\alpha = -1/\sqrt{2(d-2)(D-2)}$, we dimensionally reduce on $S^5$, $t$ and $x_3$. Approaching the boundary at $u = \infty$ yields the metric for Sol, given by

\[
ds^2_3 = u^2 dx_1^2 + \frac{dx_2^2}{u^2} + \frac{du^2}{u^2}. \tag{3.7}
\]

Note that all the fields in our three-dimensional hypothetical model of Section 2, for which Sol is a solution, are accounted for via the above dimensional reduction from ten dimensions. In particular, the five-form field strength and $B$ field dimensionally reduce to three and two-form field strengths respectively. Also, the ten-dimensional dilaton and moduli of $t$ and $x_3$ correspond to the three scalar fields in our hypothetical model.

Since we took the large $u$ limit in (3.5), Sol originates from the dual gravity description of the UV regime of non-commutative Yang-Mills.

4 Conclusions

Thurston spaces can be obtained as solutions to various Chern-Simons gauge theories. The Euclidean path-integral approach to three-dimensional quantum gravity and (super) membrane physics motivated us to search for a single ‘master theory’ for all of the Thurston spaces. In this vein, we constructed a model theory by adding a gauge field to low-energy string theory in three dimensions. We found that some of the Thurston solutions required an imaginary flux unless there are additional scalar fields in the theory.

Although such a three-dimensional theory does not give a complete picture of the Thurston geometries, it yields some interesting hints of the underlying structure. This
is contained in the correspondence between the orbifold topology of six of the eight geometries that are Seifert fibre bundles and the parameters $e$ and $k$ of the string theory/sigma model whose low-energy effective action is (2.1). It would also appear that the fact that Sol and $H^3$ do not admit a Seifert fibre bundle structure is related to the non-existence of a real 2-form potential, unless there are additional scalar fields.

Additional scalars arise naturally as the moduli of compact dimensions. Also, various cosmological terms required by most of the Thurston solutions could be considered as the curvatures of compact spaces. We were thus motivated to look to higher dimensions for a ‘master theory’ for the Thurston solutions. Two and three-dimensional Euclidean, spherical and hyperbolic spaces frequently occur within M-theoretic vacuum solutions. T-dualizing such spaces yields additional Thurston geometries expressed as a $U(1)$ bundle over a two-dimensional Einstein space. In particular, T-duality twists $T^2 \times S^1$ and $H^2 \times S^1$ into Nil and SL(2,R) respectively. The only Thurston space which does not fit into this scheme is Sol, which is derivable from the decoupling limit of a D3-brane in a background $B$ field.

Sol originates from the dimensional reduction of a dual gravity description of the UV regime of non-commutative Yang-Mills. Whether this indicates that Sol may be a useful toy model for the study of holography is still in the realm of speculation. The scalar wave equation on the Sol background, as for the ten-dimensional origin, is a Mathieu equation. This would indicate that the correlation functions in our toy model would have the same form as that derived from ten-dimensional theory. Sol has boundaries at $u = 0$ and $u = \infty$, and holographical dual field theory would reside at the latter boundary.

Perhaps understanding how the Thurston spaces arise from the deep well of M-theory may indicate whether the Euclidean path integral approach to quantum gravity lies within a corner of M-theory. We are currently exploring issues involving the topology of the Thurston spaces and the structure of the fields in the three and higher-dimensional theories of origin.

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