Elliptic problems with growth in nonreflexive Orlicz spaces 
and with measure or \( L^1 \) data

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Abstract

We investigate solutions to nonlinear elliptic Dirichlet problems of the type

\[
\begin{align*}  
- \text{div} A(x, u, \nabla u) &= \mu \quad \text{in} \quad \Omega, \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega, 
\end{align*}
\]

where \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^N \) and \( A(x, z, \xi) \) is a Carathéodory’s function. The growth of the monotone vector field \( A \) with respect to the \((z, \xi)\) variables is expressed through some \( N \)-functions \( B \) and \( P \). We do not require any particular type of growth condition of such functions, so we deal with problems in nonreflexive spaces. When the problem involves measure data and weakly monotone operator, we prove existence. For \( L^1 \)-data problems with strongly monotone operator we infer also uniqueness and regularity of solutions and their gradients in the scale of Orlicz-Marcinkiewicz spaces.

Keywords: existence, measure-data problems, regularity, Orlicz-Sobolev spaces, Orlicz-Marcinkiewicz spaces

1. Introduction

The main aim of this paper is to present the study of boundary value problems for a class of nonlinear elliptic equations. More precisely, we consider elliptic operators whose nonlinearity is expressed through \( N \)-functions which do not need to satisfy any particular growth condition. Since admitted data are merely integrable or in the space of measures, in general they do not belong to natural dual space and we do not study energy solutions but the more delicate notion of solution.

So far the effort in the research on Dirichlet problems associated to nonlinear elliptic equations concentrates mainly on the case when modular function has a growth comparable with a polynomial or trapped between two power-type functions. This includes the well understood case when both the modular function and its conjugate satisfy the so-called \( \Delta_2 \) (or doubling) condition necessary for an Orlicz space to be reflexive. Example 3.1 below indicates that \( \Delta_2 \)-condition is stronger than requirement that the growth is trapped between two power-type functions. Otherwise, i.e. when a modular function grows too slowly, too fastly, or not regularly enough, the analytical difficulties appear and significantly restrict good properties of the underlying functional space. We avoid this kind of growth restrictions and thus work in the nonreflexive space. Although this case requires an approach alternative to the classical one, we make an attempt to convince that the basic toolkit is small and easy to handle.
For the foundations of nonlinear boundary value problems in non-reflexive Orlicz-Sobolev-type setting we refer to Donaldson [20], Gossez [23, 24], and [29] by Mustonen and Tienari. In particular, in [20], the coefficients are assumed coercive, monotone with respect to $u$ and its derivatives, and the $N$-functions controlling their growth have conjugates satisfying the $\Delta_2$ condition. In [23, 24, 29], the authors removed or weakened previous assumptions. Nonetheless, these research was focused on energy solutions.

In the present paper we consider elliptic Dirichlet problems of the type

$$
\begin{aligned}
-\text{div}A(x,u,\nabla u) &= f & \text{in } \Omega, \\
 u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

where the monotone operator $A(x,z,\xi)$ has a growth with respect to the $(z,\xi)$ variables performed by general $N$-functions and where the right-hand side data is merely integrable and further also in the space of measures.

It is worth pointing out that if the datum $f$ only belongs to $L^1(\Omega)$ or to the set of Radon measure with finite total variation on $\Omega$, $M(\Omega)$, a special notion of solutions has to be considered. Indeed belonging to the duals of the natural Orlicz-Sobolev energy spaces associated with problems (1) is the minimal assumption on $f$ for weak solutions to be well defined. Our idea will be to get a solution $u$ that is the limit of a sequence of weak solutions to problems whose right-hand sides converge to $f$. More precisely, following [15], we choose the notion of approximable solutions somehow combining the notion of solutions obtained as a limit of approximation (SOLA for short) and entropy solutions, see [16, 8, 6]. When the problem involves measure data and the operator is weakly monotone, we prove existence. For $L^1$-data problems with strongly monotone operator we infer also uniqueness and regularity in the Orlicz-Marcinkiewicz spaces.

Elliptic differential equations with the right-hand side which is less regular than naturally belonging to the dual space to the one of the leading part of the operator, have received special attention and a few main ideas of relevant notions of solution, cf. [11, Section 3] and references therein. The key property we expect from this special notions of solutions is uniqueness, which is not shared by distributional solutions. The classical example of Serrin [33] is a linear homogeneous equation of the type $\text{div}(A(x)Du) = 0$ defined on a ball, with strongly elliptic and bounded, measurable matrix $A(x)$, that has at least two distributional solutions, among which only one belongs to the natural energy space $W^{1,2}(B)$. The problem of uniqueness of very weak solutions to measure-data equations is, to our best knowledge, an open problem.

There are at least three different and already classical approaches to this kind of problems keeping uniqueness even under weak assumptions on the data. The notion of renormalized solutions appeared first in [19], whereas the entropy solutions comes from [6]. The SOLA were introduced in [8, 16]. See also [21, 17] for other classical results. Under certain restrictions the mentioned notions coincide, [31, 26]. Following [15], we investigate the already mentioned approximable solutions, which for $L^1$-data are unique.

On the other hand, regularity for $L^1$ or measure data is deeply investigated in the Sobolev setting, e.g. [18, 27, 28], but besides little is known in general Orlicz spaces, especially outside $\Delta_2$-family, where we want to contribute. To our best knowledge, gradient estimates provided to elliptic problems posed in the Orlicz setting are restricted to [3, 10, 15]. None of this results however concerns the class of operators $A$ depending also on the solution itself, as we do here.

We underline we relax the growth restrictions of [15] allowing to study spaces equipped with modular functions with $L \log L$ or exponential growth. To obtain existence we need to by-pass tools working in the reflexive spaces only, employing some ideas of [25] in the Musielak-Orlicz setting. The powerful tool we use and find particularly useful is the modular approximation in the classical Orlicz version of Gossez [24] (see Definition 3.2 and Theorem 3) recently adapted to the Musielak-Orlicz case in [2].

To establish regularity results we need to apply the embeddings of Orlicz-Sobolev spaces into some Orlicz space, see Section 4. As a tool we provide inequalities of modular Sobolev-Poincaré-type and Poincaré-type, holding with a modular function of arbitrary growth, see Proposition 4.1 and Corollary 4.1, respectively. Once these inequalities are available, we are able to obtain two types of level sets estimates giving regularity properties for the solutions in the scale of Orlicz-Marcinkiewicz spaces.

Since many parts of our framework (in particular the approximation method) require $\Omega$ to have a regular boundary, we present all of the results on a bounded Lipschitz domain.
2. Statements of main results

For brevity we skip listing here full notation, presented in detail in Section 3.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded Lipschitz domain and a function $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$. We shall consider the following set of assumptions.

(A1) $A(x, z, \xi)$ is a Carathéodory’s function, i.e. measurable w.r.t. to $x$ and continuous w.r.t. $z$, as well as w.r.t. $\xi$;

(A2) for a.e. $x \in \Omega$ and for all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the following growth conditions hold

\begin{align}
\frac{d\mu(|\xi|)}{d\nu} & \leq A(x, z, \xi) \cdot \xi, \\
|A(x, z, \xi)| & \leq \frac{1}{3d} \left[ B^{-1}(B(|\xi|)) + D^{-1}(B(z)) + K(x) \right],
\end{align}

where $B : [0, \infty) \to [0, \infty)$ and $P : [0, \infty) \to [0, \infty)$ are two N-functions such that $P << B$, $\tilde{B}$ is the conjugate of $B$, $B^{-1}$ is the inverse of $B$ and $K(x)$ is a function belonging to $E_{\tilde{B}}(\Omega)$, the closure of $L^\infty$ in the $L_{\tilde{B}}$-norm.

(A3)\textsubscript{w} $A(x, z, \xi)$ is monotone in the last variable, i.e.

$$(A(x, z, \xi) - A(x, z, \eta)) \cdot (\xi - \eta) \geq 0$$

for a.e $x \in \Omega$, for every $z \in \mathbb{R}$ and all $\xi, \eta \in \mathbb{R}^N$;

(A3)\textsubscript{s} $A(x, z, \xi)$ is strictly monotone in the last variable, i.e.

$$(A(x, z, \xi) - A(x, z, \eta)) \cdot (\xi - \eta) > 0$$

for a.e $x \in \Omega$, for every $z \in \mathbb{R}$ and all $\xi \neq \eta \in \mathbb{R}^N$;

(A4) for a.e $x \in \Omega$ and for $z \in \mathbb{R}$, it holds

$$A(x, z, 0) = 0$$

Note that conditions (A1)–(A3) are generalizations of the classical Leray-Lions conditions to the Orlicz-Sobolev space setting.

We consider the problem

\begin{align}
\left\{ \begin{array}{ll}
-\text{div}A(x, u, \nabla u) = \mu & \text{in} \quad \Omega, \\
u(x) = 0 & \text{on} \quad \partial\Omega,
\end{array} \right.
\end{align}

where $\mu \in \mathcal{M}(\Omega)$ is a Radon measure with bounded total variation $|\mu|(\Omega) < \infty$ or $\mu$ is replaced by $f \in L^1(\Omega)$.

To define the solution we need to recall the truncation $T_k(u)$ defined as

$$T_k(u) = \left\{ \begin{array}{ll}
u & |u| \leq k, \\
k \frac{u}{|u|} & |u| \geq k,
\end{array} \right.$$ 

and the following notation

$$\mathcal{T}^{1,B}(\Omega) = \{ u \text{ is measurable in } \Omega : T_t(u) \in W^{1,B}(\Omega) \text{ for every } t > 0 \}. \quad (6)$$

The Orlicz-Sobolev space $W^{1,B}$ is defined in Section 3.

**Definition 2.1.** A function $u \in \mathcal{T}^{1,B}(\Omega)$ is called an **approximable solution** to the Dirichlet problem (4) with given $\mu \in \mathcal{M}(\Omega)$, if there exist a sequence $\{f_k\}_k \subset L^1(\Omega)$ such that $f_k \rightharpoonup^* \mu$ weakly-* in the space of measures, namely that it holds

$$\lim_{k \to \infty} \int_\Omega \varphi f_k \, dx = \int_\Omega \varphi \, d\mu$$

for every $\varphi \in C_c(\Omega)$ and a sequence of weak solutions $\{u_k\}_k \subset W^{1,B}_0(\Omega)$ to problem (4) with $\mu$ replaced by $f_k$, satisfying $u_k \to u$ a.e. in $\Omega$. 

3
**Definition 2.2.** A function \( u \in T^{1,B}(\Omega) \) is called an **approximable solution** to the Dirichlet problem (4) with \( \mu \) replaced by \( f \in L^1(\Omega) \), if there exist a sequence \( \{f_k\}_k \subset L^1(\Omega) \cap (W^{1,B}_0(\Omega))' \) such that \( f_k \to f \) in \( L^1(\Omega) \) and a sequence of weak solutions \( \{u_k\}_k \subset W^{1,B}_0(\Omega) \) to problem (4) with \( \mu \) replaced by \( f_k \), satisfying \( u_k \to u \) a.e. in \( \Omega \).

It may happen that an approximable solution is not weakly differentiable. However, it is associated with a vector-valued function on \( \Omega \) playing the role of its gradient on every level of truncation and therefore, with some abuse of notation, we will still use the symbol \( \nabla u \). More details on this issue can be found in Section 3.

Our main results state as follows.

**Theorem 1.** Consider a measure \( \mu \in \mathcal{M}(\Omega) \) and a function \( A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) satisfying assumptions (A1), (A2), (A3), and (A4). Then there exists an approximable solution \( u \in T^{1,B}(\Omega) \) to the problem (4) and moreover

\[
A(u,T_t u,k_N T_t u)) \xrightarrow[k \to \infty]{} A(x,T_t u,k_N T_t u) \quad \text{in} \quad L_\mathcal{B}.
\]

**Theorem 2.** Assume \( f \in L^1(\Omega) \) and \( A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) a function satisfying assumptions (A1), (A2), (A3), and (A4). Then there exists a **unique** approximable solution \( u \in T^{1,B}(\Omega) \) to the problem (4) with \( \mu \) replaced by \( f \) and (8) holds.

Uniqueness in this context means that the solution does not depend on the choice of approximate problems. Consequently, for the problem with regular data the unique approximable solution agrees with the weak solution, which is trivially also an approximable solution.

As announced in the Introduction, we shall also obtain some regularity results for the solution and its gradient. For their statements and proofs we refer to Section 6 since they can be deduced by propositions which are interesting by themselves.

3. Preliminaries

3.1. Notation and basic lemmas

Throughout the paper \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^N \), \( N \geq 1 \). We shall use the notation \(| \cdot |\) for the absolute value, as well as for the norm in \( \mathbb{R}^N \) (for gradient norm) and denote by \( \mathbf{1}_A \) the characteristic function of a set \( A \).

Let us start with two useful results.

**Lemma 3.1** (e.g. Lemma 9.1, [22]). If \( g_n : \Omega \to \mathbb{R} \) are measurable functions converging to \( g \) almost everywhere, then for each regular value \( t \) of the limit function \( g \) we have \( \mathbf{1}_{\{t < |g_n|\}} \xrightarrow[n \to \infty]{} \mathbf{1}_{\{t < |g|\}} \) a.e. in \( \Omega \).

**Lemma 3.2.** Suppose \( w_n \xrightarrow[n \to \infty]{} w \) in \( L^1(\Omega) \), \( v_n,v \in L^\infty(\Omega) \), and \( v_n \xrightarrow[n \to \infty]{} v \). Then

\[
\int_\Omega w_n v_n \, dx \xrightarrow[n \to \infty]{} \int_\Omega w v \, dx.
\]

3.2. The Orlicz setting

We refer the interested reader to [32] for an exhaustive treatment of the theory of Orlicz spaces and to [1] for compact, though capturing the point, description of the necessary properties of the Orlicz-Sobolev spaces.

Recall that a function \( B : [0,\infty) \to [0,\infty) \) is called an \( N \)-function if \( B \) is a strictly increasing convex function satisfying

\[
\lim_{r \to 0} \frac{B(r)}{r} = 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{B(r)}{r} = \infty.
\]

Its conjugate function \( \tilde{B} : [0,\infty) \to [0,\infty) \) is defined by

\[
\tilde{B}(s) = \sup_{t > 0} (t \cdot s - B(t)) \quad (9)
\]

and is an \( N \)-function as well.

Given two \( N \)-functions \( P \) and \( B \), we shall write \( P < \!< \! B \) in order to mean that for each \( \varepsilon > 0 \), \( P(t)/B(\varepsilon t) \to 0 \) as \( t \to \infty \).

Observe that one has \( P < \!< \! B \) if and only if \( \tilde{B} < \!< \! \tilde{P} \), see [23].
Definition 3.1 (The function spaces). Let $B$ be an $N$-function. We deal with the three Orlicz classes of functions.

i) $\mathcal{L}_B(\Omega)$ - the generalised Orlicz class is the set of all measurable functions $\xi$ defined on $\Omega$ such that

$$\int_{\Omega} B(|\xi(x)|) \, dx < \infty.$$ 

ii) $L_B(\Omega)$ - the generalised Orlicz space is the smallest linear space containing $\mathcal{L}_B(\Omega)$, equipped with the Luxemburg norm

$$\|\xi\|_{L_B} = \inf \left\{ \lambda > 0 : \int_{\Omega} B\left(\frac{|\xi(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$ 

iii) $E_B(\Omega)$ - the closure in $L_B$-norm of the set of bounded functions.

Then $E_B(\Omega) \subset L_B(\Omega) \subset L_B(\Omega)$ and without growth restrictions the inclusions can be proper.

Remark 3.1. If $B$ is an $N$-function and $\tilde{B}$ its conjugate, we have

- the Fenchel-Young inequality

$$|\xi \cdot \eta| \leq B(|\xi|) + \tilde{B}(|\eta|) \quad \text{for all } \xi, \eta \in \mathbb{R}^N. \quad (10)$$

- the generalized Hölder’s inequality

$$\left| \int_{\Omega} \xi \cdot \eta \, dx \right| \leq 2\|\xi\|_{L_B}\|\eta\|_{L_{\tilde{B}}} \quad \text{for all } \xi \in L_B(\Omega), \eta \in L_{\tilde{B}}(\Omega). \quad (11)$$

Moreover, we shall consider the Orlicz-Sobolev space $W^{1,B}(\Omega)$ defined as follows

$$W^{1,B}(\Omega) = \{ u \in W^{1,1}(\Omega) : u, \nabla u \in L_B(\Omega) \},$$

where $\nabla$ denotes the distributional gradient. The space $W^{1,B}(\Omega)$ is endowed with the Luxemburg norm

$$\|u\|_{W^{1,B}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} B\left(\frac{|u|}{\lambda}\right) \, dx + \int_{\Omega} B\left(\frac{|
abla u|}{\lambda}\right) \, dx \leq 1 \right\}. \quad (12)$$

The space $W^{1,0}_0(\Omega)$ is defined as a closure of smooth functions, see (17) below.

If $B$ is an $N$-function, then $(W^{1,B}(\Omega), \|u\|_{W^{1,B}(\Omega)})$ is a Banach space.

The space $E_B(\Omega)$ is separable and due to [1, Theorem 8.19] we have the duality

$$(E_B(\Omega))' = L_{\tilde{B}}(\Omega).$$

Recall the space $\mathcal{T}^{1,B}(\Omega)$ defined in (6). For every $u \in \mathcal{T}^{1,B}(\Omega)$ there exists a unique measurable function $Z_u : \Omega \to \mathbb{R}^N$ such that

$$\nabla(T_t(u)) = 1_{\{|u| < t\}} Z_u \quad \text{a.e. in } \Omega, \text{ for every } t > 0,$$

see [6, Lemma 2.1]. Since

$$u \in W^{1,B}(\Omega) \iff u \in \mathcal{T}^{1,B}(\Omega) \cap L_B(\Omega) \text{ and } |Z_u| \in \mathcal{T}^{1,B}(\Omega),$$

for such $u$, we have $Z_u = \nabla u$ a.e. in $\Omega$. Thus, we call $Z_u$ the generalized gradient of $u$ and, abusing the notation, for $u \in \mathcal{T}^{1,B}(\Omega)$ we write simply $\nabla u$ instead of $Z_u$.

For the spaces $E_B$ and $L_B$ to coincide, and consequently for their reflexivity, one has to impose $\Delta_2$-condition on $B$ close to infinity (denoted $\Delta_2^\infty$). Namely, it has to be assumed that there exists a constant $c_{\Delta_2} > 0$ such that

$$B(2s) \leq c_{\Delta_2} B(s) \quad \text{for } s > s_0. \quad (14)$$
The spaces equipped with the modular functions satisfying $\Delta_2$-condition close to infinity have strong properties. In particular, we have

$$E_B \overset{B \in \Delta_2^\infty}{\longrightarrow} L_B = (E_B)' \overset{B \in \Delta_2^\infty}{\longrightarrow} (L_B)'$$

Moreover, when $B \in \Delta_2^\infty$, then modular and strong convergence coincide.

We would like to stress that we face the problem without this structure. This allows us to deal with a broader class of modular functions. Let us discuss the typical assumption of $\Delta_2$-condition, which we do not impose.

It describes the speed and the regularity of the growth of the function. For example, when we take $B(s) = (1 + |s|) \log(1 + |s|) - |s|$, its conjugate function is given by $\tilde{B}(s) = \exp(|s|) - |s| - 1$. Then $B \in \Delta_2$ and $\tilde{B} \notin \Delta_2$.

We point out that despite the typical condition

$$\Delta$$

we get that

$$\tilde{B}(s) = \exp(|s|) - |s| - 1$$

is often treated as equivalent to $B, \tilde{B} \in \Delta_2$, as well as to comparison with power-type functions. Nonetheless, the assumption (15) is more restrictive, as it requires both regularity of the growth and restricts its speed. Indeed, if $i_B > 1$ then $B \in \Delta_2$, whereas $s_B < \infty$ entails the $\Delta_2$-condition imposed on $\tilde{B}$. When $B$ satisfies (15), then

$$\frac{B(t)}{t^{i_B}} \text{ is non-decreasing and } \frac{B(t)}{t^{s_B}} \text{ is non-increasing.}$$

Moreover, $B(s) = (1 + |s|) \log(1 + |s|) \in \Delta_2$, but $i_B = 1$. On the other hand, the following example shows that comparison with two power-type functions is not enough for $\Delta_2$-condition. Another construction can be found in [9].

**Example 3.1.** For arbitrary $1 < p < q < \infty$, there exists a continuous, increasing, and convex function $B : [0, \infty) \to [0, \infty)$ which is trapped between power type functions $t^p$ and $t^q$ and $B$ does not satisfy $\Delta_2$-condition, nor (15).

Proof. We shall construct $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ such the desired function is given by the following formula

$$B(t) = \begin{cases} \text{affine} & t \in (a_i, b_i), \\ t^p & \text{otherwise.} \end{cases}$$

To describe $\{a_i\}_{i \in \mathbb{N}}$ let us introduce yet another sequence $\{k_i\}_{i \in \mathbb{N}}$ and fix $a_i = 2^{k_i}$ for every $i \in \mathbb{N}$. Let $k_1 \in \mathbb{N}$ be large enough to satisfy both

$$k_1 > 2^p \quad \text{and} \quad \left(\frac{k_1 - 1}{q}\right)^{\frac{1}{k_1}} \leq 2^{q-p}. \quad (16)$$

Define

$$B(t) = 2^{p k_1} + 2^{(p-1) k_1}(k_1 - 1)(t - 2^{k_1}) \quad \text{for} \quad t \in (a_1, b_1),$$

where $b_1 > a_1$ is an intersection point of chord $f_1(t) = 2^{p k_1} + 2^{(p-1) k_1}(k_1 - 1)(t - 2^{k_1})$ and $t \mapsto t^p$. Note that (16) ensures that

$$2^{p k_1} + 2^{(p-1) k_1}(k_1 - 1)(2^{k_1+1} - 2^{k_1}) = k_1 2^{p k_1} > (2^{k_1+1})^p,$$

so in particular $2^{k_1+1} < b_1$ and $f(k_1+1) = k_1 2^{p k_1}$. On the other hand, (16) implies that the slope of the line given by $f_1$ equals $2^{(p-1) k_1}(k_1 - 1)$ and is smaller than the derivative of $t^q$ in $a_1$. Combining it with $t^p|_{2^{k_1}} < t^q|_{2^{k_1}}$, we get that $B(t) < t^q$ on $(a_1, b_1)$.

Let $k_2$ be the smallest natural number such that $a_2 = 2^{k_2} \geq b_1$ and set $B(t) = t^p$ on $(b_1, a_2)$. We repeat the construction of chord. Note that since $k_2 > k_1$, the condition (16) with $k_1$ substituted with $k_2$ is satisfied. Thus, the chord is trapped between $t^p$ and $t^q$. Iterating further the construction we obviously obtain a continuous, increasing, and convex function, whose graph lies between the same power-type functions. Moreover, we also get the sequences $\{a_i\}, \{b_i\},$ and $\{k_i\}$, such that $k_i \to \infty$, $2a_i < b_i \leq a_{i+1}$ and

$$B(a_i) = a_i^p \quad \text{and} \quad B(2a_i) = k_i a_i^p = k_i B(a_i),$$

which contradicts with $\Delta_2$-condition. Moreover, one can check that $i_B = 1$, which violates (15).
3.3. The topologies

We shall distinguish topology \( \sigma(L_B, L_{\delta B}) \) from weak-* topology in \( L_B \), namely \( \sigma(L_B, E_{\delta B}) \).

We say that \( \{u_n\} \subset L_B \) is \( \sigma(L_B, L_{\delta B}) \)-convergent to \( u \in L_B \), if for any \( v \in L_{\delta B} \)

\[
\int u_n v \, dx \xrightarrow{n \to \infty} \int u v \, dx.
\]

We say that \( \{u_n\} \subset L_B \) is weakly-* convergent to \( u \in L_B \), if for any \( v \in E_{\delta B} \)

\[
\int u_n v \, dx \xrightarrow{n \to \infty} \int u v \, dx.
\]

We say that \( \{u_n\}_n \) converges to \( u \) in norm (strongly) in \( L_B(\Omega) \), if \( \|u_n - u\|_{L_B(\Omega)} \to 0 \) as \( n \to \infty \).

Obviously strong convergence implies both weak-type convergences above, but there is one more intermediate type of convergence more relevant in this setting.

**Definition 3.2** (Modular convergence). A sequence \( \{u_\delta\}_\delta \) is said to converge modularly to \( u \) in \( L_B(\Omega) \)

if there exists a parameter \( \lambda > 0 \) such that

\[
\int_B \frac{|u_\delta - u|}{\lambda} \, dx \to 0 \quad \text{as} \quad \delta \to 0,
\]

equivalently

if there exists a parameter \( \lambda > 0 \) such that \( \{B(|u_\delta|/\lambda)\}_\delta \) is uniformly integrable in \( L^1(\Omega) \) and \( u_\delta \xrightarrow{\delta \to 0} u \) in measure.

Following Gossez [24], we define the space

\[
W^{1,B}_0(\Omega) = \overline{C_0^\infty(\Omega)}^{\sigma(L_B, E_{\delta B})}
\]

\[\text{(17)}\]

i.e. the closure of \( C_0^\infty(\Omega) \) in \( W^{1,B}_0(\Omega) \) with respect to the topology \( \sigma(L_B, E_{\delta B}) \). Naturally \( T^{1,B}_0(\Omega) \) is defined as \( T^{1,B}(\Omega) \) in (6) replacing \( W^{1,B}(\Omega) \) with \( W^{1,B}_0(\Omega) \).

We write

\[
u_\delta \xrightarrow{\text{mod} \ \delta \to 0} u \text{ in } W^{1,B}(\Omega) \quad \iff \quad \begin{cases}
u_\delta \xrightarrow{\text{mod} \ \delta \to 0} u & \text{ and } \nabla u_\delta \xrightarrow{\text{mod} \ \delta \to 0} \nabla u \text{ in } L_B(\Omega)\end{cases}
\]

We will use the following approximation in the modular topology due to Gossez. Note that the final boundedness of the norm results from the original proof.

**Theorem 3** (cf. [24], Theorem 4). Suppose \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \) is a bounded Lipschitz domain and \( u \in W^{1,B}_0(\Omega) \). Then there exists a sequence \( \{u_\delta\}_\delta \subset C_0^\infty(\Omega) \) such that \( u_\delta \xrightarrow{\text{mod} \ \delta \to 0} u \) in \( W^{1,B}(\Omega) \).

Moreover, if \( u \in L^\infty(\Omega) \), then \( \|u_\delta\|_{L^\infty(\Omega)} \leq c(\Omega)\|u\|_{L^\infty(\Omega)} \).

Because of the notion of the modular convergence the fundamental role in the theory is played by the following classical results.

**Theorem 4** (Vitali Convergence Theorem). Let \( (X, \mu) \) be a positive measure space, \( \mu(X) < \infty \), and \( 1 \leq p < \infty \). If \( \{u_n\} \) is uniformly integrable in \( L^p_\mu \), \( u_n \to u \) in measure and \( |u(x)| < \infty \) a.e. in \( X \), then \( u \in L^p_\mu(X) \) and \( u_n \to u \) in \( L^p_\mu(X) \).

**Theorem 5** (de la Vallet Poussin Theorem). Let \( B \) be an \( N \)-function and \( \{u_n\} \) be a sequence of measurable functions such that \( \sup_{n \in \mathbb{N}} \int \Omega B(|u_n(x)|) \, dx < \infty \). Then the sequence \( \{u_n\}_n \) is uniformly integrable.

In general, if \( u_\delta \xrightarrow{\delta \to 0} u \) in norm in \( L_B \), then \( u_\delta \xrightarrow{\text{mod} \ \delta \to 0} u \) and not conversely. Nonetheless, the reverse implication can be obtained via the following lemma.

**Lemma 3.3**. Let \( B \) be an \( N \)-function and \( u_n \xrightarrow{n \to \infty} u \) in \( L_B(\Omega) \) with every \( \lambda > 0 \), then \( u_n \xrightarrow{n \to \infty} u \) in the norm topology in \( L_B(\Omega) \).
Proof. We present the proof for \(u \equiv 0\) only.

If \(\int_\Omega B(\lambda u_n)dx \xrightarrow{n \to \infty} 0\), then for every \(\lambda > 0\) there exists \(n_\lambda\), such that for every \(n > n_\lambda\) we have \(\int_\Omega B(\lambda u_n)dx \leq 1/\lambda\). Therefore, for every \(n > n_\lambda\) also \(\|u_n\|_{L^p(\Omega)} \xrightarrow{n \to \infty} 0\). On the other hand, if \(\|u_n\|_{L^p(\Omega)} \xrightarrow{n \to \infty} 0\), then for any fixed \(\lambda > 0\) we get \(\|\lambda u_n\|_{L^p(\Omega)} \xrightarrow{n \to \infty} 0\). This means that for every \(\varepsilon \in (0,1)\) there exists \(n_\varepsilon\), such that for every \(n > n_\varepsilon\) it holds that \(\|\lambda u_n\|_{L^p(\Omega)} < \varepsilon < 1\).

Since for arbitrary \(\xi \in L^p(\Omega)\) with \(\|\xi\|_{L^p} \leq 1\), we have \(\int_\Omega B(\xi(x))dx \leq \|\xi\|_{L^p(\Omega)}\). Therefore, \(\int_\Omega B(\lambda u_n)dx \leq \|\lambda u_n\|_{L^p(\Omega)} < \varepsilon\) for every \(n > n_\varepsilon\), which implies the claim.

**Lemma 3.4** (Lemma 6, [24]). Let \(u_n, u \in L^p(\Omega)\). If \(u_n \xrightarrow{n \to \infty} u\) modularly, then \(u_n \to u\) in \(\sigma(L^p, L^\hat{p})\).

Note nonetheless, that for \(B \in \Delta_2\), the weak and modular closures are equal.

**Lemma 3.5** (Weak-strong convergence). Assume that \(\{u_n\}_n \subset E_B\) and \(\{v_n\}_k \subset L_B\) are sequences such that

\[
\begin{align*}
\int_\Omega & u_n v_n dx \to \int_\Omega uv dx.
\end{align*}
\]

Then

**Proof.** We write

\[
\int_\Omega (u_n v_n - uv) dx = \int_\Omega (u_n - u)v_n dx + \int_\Omega uv_n - u dx.
\]

Then, by Hölder’s inequality (11) we have

\[
\left| \int_\Omega (u_n v_n - uv) dx \right| \leq \|u_n - u\|_{L^\hat{p}} \|v_n\|_{L^p} + \int_\Omega |u(v_n - v)| dx,
\]

and therefore, the result follows observing that \(\|v_n\|_{L^\hat{p}}\) is uniformly bounded due to the assumption \(v_n \xrightarrow{\ast} v \in L_B\) and that \((E_B')' = L_B\).

4. Sobolev-type Embeddings

To establish regularity result we need to apply the results on embedding of the Orlicz-Sobolev spaces into some Orlicz space, namely

\[W^{1,B}_0(\Omega) \hookrightarrow L^\hat{B}_0(\Omega),\]

with \(\hat{B}\) growing in a certain sense faster than \(B\). We use two types of results, which – to be distinguished – will be roughly called the optimal and the easy one. The optimal embedding proven by Cianchi [13] distinguishes two cases: of quickly and slowly growing modular function \(B\), corresponding to the cases of \(p\)-Laplacian with \(p > n\) and \(p \leq n\). The easy embedding, which yields that \(W^{1,B}_0(\Omega) \hookrightarrow L^{\hat{B}_0}(\Omega)\) is provided below the optimal one. It is weaker than the optimal, but it is easy and captures a general \(N\)-function \(B\) independently of any growth conditions. Let us stress that since the rest of our framework requires \(\Omega\) to be a Lipschitz bounded domain, we present all of the results on such domains. See e.g. [15] for an overview on the issue of the regularity of the boundary in relation to the embedding.

To apply the optimal embeddings we employ, we note that in [13] the Sobolev inequality is proven under the restriction

\[
\int_\Omega \left( \frac{t}{B(t)} \right)^{\frac{1}{s-1}} dt < \infty,
\]

concerning the growth of \(B\) in the origin. Nonetheless, the properties of \(L_B\) depend on the behaviour of \(B(s)\) for large values of \(s\) and (18) can be easily by-passed in application. Indeed, if it would be necessary for (18) we shall substitute \(B(t)\) by \(B^s(t) = tB(1)1_{[0,1]}(t) + B(t)1_{(1,\infty)}(t)\).
The conditions
\[
\int_0^\infty \left( \frac{t}{B(t)} \right)^{\frac{1}{N-1}} dt = \infty \quad \text{and} \quad \int_0^\infty \left( \frac{t}{B(t)} \right)^{\frac{1}{N}} dt < \infty, \tag{19}
\]
roughly speaking, describe slow and fast growth of $B$ at infinity respectively.

For $N' = N/(N - 1)$, we consider
\[
H_N(s) = \left( \int_0^s \left( \frac{t}{B(t)} \right)^{\frac{1}{N-1}} dt \right)^{\frac{1}{N}}, \quad B_N(t) = B(H_N^{-1}(t)), \quad \text{and} \quad \phi_N(s) = (H_N(s))^{N'}. \tag{20}
\]

When the integrability in the origin condition (18) is satisfied and the growth of $B$ at infinity is slow, that is when (19)$_1$ holds, then [13, Theorem 3] provides the following continuous embedding
\[
W_0^{1,B} \hookrightarrow L_{B_N}(\Omega), \tag{21}
\]
where $B_N$ is given by (20). Otherwise, when the growth of $B$ at infinity is fast, that is when (19)$_2$ holds, then we have the following continuous embedding
\[
W_0^{1,B} \hookrightarrow L^\infty(\Omega). \tag{22}
\]
This result was proven first in [34], see also [12].

In the general case, independently of the growth conditions we provide the easy embedding
\[
W_0^{1,B} \hookrightarrow L_{B_N'}(\Omega).
\]

More precisely, we prove the following

**Proposition 4.1** (The Sobolev-Poincaré inequality without growth restrictions). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$ and $B$ be an $N$-function. There exist constants $c_1, c_2 > 0$ depending on $\Omega$, such that for every $u \in W_0^{1,B}(\Omega)$
\[
\left( \int_\Omega B_N'(c_1|u|)dx \right)^{\frac{1}{N'}} \leq c_2 \int_\Omega B(|\nabla u|)dx.
\]

Before giving the proof of the above Proposition, let us observe that as a direct consequence, by the use of the Hölder inequality, we can easily obtain the following Poincaré-type inequality

**Corollary 4.1** (The modular Poincaré inequality). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$ and $B$ be an $N$-function. There exist constants $c_1, c_2 > 0$ depending on $\Omega$, such that for every $u \in W_0^{1,B}(\Omega)$
\[
\int_\Omega B(c_1|u|)dx \leq c_2 \int_\Omega B(|\nabla u|)dx.
\]

In the proof of Proposition 4.1 we will use the following version of the Hölder inequality.

**Lemma 4.1.** Suppose $Q^N = [-1, 1]^N$ and $f_i \in L^{N-1}(Q^{N-1})$, then
\[
\int_{Q^N} \prod_{i=1}^N |f_i|dx \leq \prod_{i=1}^N \left( \int_{Q^{N-1}} |f_i|^{N-1}dx' \right)^{\frac{1}{N-1}}.
\]

Proof of Proposition 4.1. The proof consists of three steps starting with the case of smooth and compactly supported functions on small cube, then turning to the Orlicz class and concluding the claim on arbitrary bounded set.

**Step 1.** We start the proof for $u \in C_0^\infty(\Omega)$ with supp $u \subset \subset [-1/4, 1/4]^N$. Let $u$ be extended by 0 outside $\Omega$. Note that for every $j = 1, \ldots, N$
\[
|u(x)| \leq \int_\frac{1}{4}^\frac{1}{2} |\partial_j u(x)|dx_j.
\]
Applying \( B^{1/(N-1)} \), which is increasing, to both sides above and using Jensen’s inequality, we get
\[
B^{\frac{N}{N-1}}(|u(x)|) \leq B^{\frac{N}{N-1}} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} |\partial_j u(x)|\,dx \right) \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} B^{\frac{N}{N-1}} (|\partial_j u(x)|)\,dx.
\]

When we multiply \( N \) copies of the above inequality, integrate over \( \Omega \), and apply Lemma 4.1, we obtain
\[
\int_{\Omega} B^{\frac{N}{N-1}}(|u(x)|)\,dx = \int_{\Omega} B^{\frac{N}{N-1}}(|u(x)|)\,dx \leq \prod_{i=1}^{N} \int_{Q^{N-1}} B^{\frac{N}{N-1}} (|\partial_j u(x)|)\,dx \,dx
\]
\[
\leq \prod_{i=1}^{N} \left( \int_{Q^{N-1}} B^{\frac{N}{N-1}} (|\partial_j u(x)|)\,dx \,dx \right) \leq \left( \int_{\Omega} B^{\frac{N}{N-1}} (|\nabla u(x)|)\,dx \right)^{\frac{N}{N-1}}.
\]

**Step 2.** Let \( u \in W^{1,B}_0(\Omega) \). Then by Theorem 3 there exists a sequence \( \{u_\delta\}_\delta \subset C_0^\infty(\Omega) \) such that
\[
u_\delta \xrightarrow{\text{mod} \, \delta \to 0} u \text{ in } W^{1,B}(\Omega).
\]
Note that \( \{u_\delta\}_\delta \) is a Cauchy sequence in the modular topology in \( W^{1,B}_0(\Omega) \) and the inequality obtained above holds for every \( u_\delta \). Moreover, \( \{u_\delta\}_\delta \) is also a Cauchy sequence in the modular topology in \( L_{B,N}(\Omega) \).

Due to the modular convergence we get \( \nabla u_\delta \to \nabla u \) in measure. Jensen’s inequality and properties of modular convergence together with the Lebesgue Dominated Convergence Theorem enable to pass to the limit with \( \delta \to 0 \) to get the final claim on the small set \( \Omega \).

**Step 3.** Suppose that \( \Omega \) is arbitrary bounded set containing \( 0 \). It is contained in the cube of the edge \( D = \text{diam} \Omega \). Then \( \tilde{u}(x) = u(4dx) \) has \( \text{supp} \tilde{u} \subset \Omega_1 \subset \left[ -\frac{1}{4}, \frac{1}{4} \right]^N \). We have
\[
\left( \int_{\Omega} B^{N}(|u|)\,dx \right)^{\frac{N}{N}} = \left( (4D)^N \int_{\Omega_1} B^{N}(|u|)\,dx \right)^{\frac{N}{N}} \leq (4D)^N \int_{\Omega_1} B(|\nabla \tilde{u}|)\,dx = \frac{1}{4D} \int_{\Omega} B(4D|\nabla \tilde{u}|)\,dx.
\]
To obtain the estimate on an arbitrary domain we need only to observe that the Lebesgue measure is translation-invariant.

5. Main proofs

This section is devoted to the proofs of our main results which will be splitted into different steps. We start with showing the existence of solutions \( u_k \) to problems with regular and bounded data by using the general known theory. In the second and in the third steps we respectively obtain uniform a priori estimates for such weak solutions and almost every where convergence of \( u_k \) to some \( u \). Step 4 provides that this limit \( u \) is the desired approximable solution. Finally in Step 5 we pass to measure data.

The first subsection is dedicated to the monotonicity trick which will be instrumental for our arguments.

5.1. Monotonicity trick

Note that the idea of this trick was used in [23, 29] in a very general situation. We present it together with the proof for the sake of completeness.

**Proposition 5.1** (Monotonicity trick). Suppose \( A \) satisfies conditions (A1) and (A2).

Assume further that there exists \( A \in L_B(\Omega) \) such that for some \( v \in W^{1,B}_0(\Omega) \cap L^\infty(\Omega) \) it holds
\[
\int_{\Omega} (A(x,v,\zeta) - A) \cdot (\zeta - \nabla v)\,dx \geq 0 \quad \forall \zeta \in L_B(\Omega).
\]
Then
\[
A(x,v,\nabla v) = A \quad \text{a.e. in } \Omega.
\]
Proof. Let us define
$$\Omega_m = \{ x \in \Omega : \| \nabla v \| \leq m \}.$$  
Fix arbitrary $0 < j < i$ and notice that $\Omega_j \subset \Omega_i$.

We consider (23) with
$$\zeta = (\nabla v) 1_{\Omega_i} + h\vec{w} 1_{\Omega_j},$$
where $h > 0$ and $\vec{w} \in L^\infty(\Omega; \mathbb{R}^N)$, namely
$$\int_{\Omega} \bigg( A(x, v, (\nabla v) 1_{\Omega_i} + h\vec{w} 1_{\Omega_j}) - A \bigg) \cdot \bigg( (\nabla v) 1_{\Omega_i} + h\vec{w} 1_{\Omega_j} - \nabla v \bigg) \, dx \geq 0.$$  
Notice that it is equivalent to
$$-\int_{\Omega \setminus \Omega_i} (A(x, v, 0) - A) \cdot \nabla v \, dx + h \int_{\Omega_j} (A(x, v, \nabla v + h\vec{w}) - A) \cdot \vec{w} \, dx \geq 0. \quad (24)$$
The first integral above tends to zero when $i \to \infty$. Indeed $A(x, v, 0) = 0$, $A \in L_B$, $\nabla v \in L_B$ and therefore Hölder’s inequality gives the boundedness of the integrands in $L^1$. The convergence to zero follows taking into account the shrinking domains of integration.

It follows that
$$h \int_{\Omega_j} (A(x, v, \nabla v + h\vec{w}) - A) \cdot \vec{w} \, dx \geq 0$$
and obviously that
$$\int_{\Omega_j} (A(x, v, \nabla v + h\vec{w}) - A) \cdot \vec{w} \, dx \geq 0. \quad (25)$$
Note that $\nabla v + h\vec{w} \to \nabla v$ in $L^\infty(\Omega_j)$ as $h \to 0$ and thus
$$A(x, v, \nabla v + h\vec{w}) \xrightarrow[h \to 0]{} A(x, v, \nabla v) \quad \text{a.e. in } \Omega_j.$$  
Moreover, $A(x, v, \nabla v + h\vec{w})$ is bounded on $\Omega_j$. Let $\bar{c} = h\| \vec{w} \|_\infty$. Using (A2) and Jensen’s inequality we have that in $\Omega_j$
$$\bar{B} (d|A(x, v, \nabla v + h\vec{w})|) \leq \bar{B} \left( \frac{1}{3} |K(x) + \bar{P}^{-1}(|v|) + \bar{B}^{-1}(B(\bar{c}))| \right)$$
$$\leq \frac{1}{3} \bar{B} (K(x)) + \frac{1}{3} \bar{B} \left( \bar{P}^{-1}(B(\|v\|_{L^\infty(\Omega_j)})) \right) + \frac{1}{3} \bar{B} \left( B(\bar{c}) \right) \in L^1(\Omega_j).$$
Hence, we have uniform boundedness of $\bar{B}(A(x, v, \nabla v + h\vec{w}))$ in $L^1(\Omega)$ and by Theorem 5 we deduce the uniform integrability of $(A(x, v, \nabla v + h\vec{w}))_h$. Since $|\Omega_j| < \infty$ and (A1) implies continuity with respect to the last variable, we can apply Theorem 4 to get
$$A(x, v, \nabla v + h\vec{w}) \xrightarrow[h \to 0]{} A(x, v, \nabla v) \quad \text{in } L^1(\Omega_j; \mathbb{R}^N).$$
Thus
$$\int_{\Omega_j} (A(x, v, \nabla v + h\vec{w}) - A) \cdot \vec{w} \, dx \xrightarrow[h \to 0]{} \int_{\Omega_j} (A(x, v, \nabla v) - A) \cdot \vec{w} \, dx.$$  
Taking into account (25), it follows that
$$\int_{\Omega_j} (A(x, v, \nabla v) - A) \cdot \vec{w} \, dx \geq 0,$$
for any $\vec{w} \in L^\infty(\Omega; \mathbb{R}^N)$.

If we consider
$$\vec{w} = \begin{cases} \frac{A(x, v, \nabla v) - A}{\| A(x, v, \nabla v) - A \|} & \text{if } A(x, v, \nabla v) - A \neq 0, \\ 0 & \text{if } A(x, v, \nabla v) = A, \end{cases}$$
we obtain
\[ \int_{\Omega} |A(x,v,\nabla v) - A| \, dx \leq 0, \]
and hence
\[ A(x,v,\nabla v) = A \quad \text{a.e. in } \Omega. \]
Since \( j \) is arbitrary, we have the equality a.e. in \( \Omega \) and (23) is satisfied.

\[ \square \]

5.2. Proof of Theorem 1

**Step 1. Existence of \( u_k \) solving approximate problem**

Let us consider \( \{ f_k \}_{k \in \mathbb{N}} \subset \mathcal{C}^\infty_0(\Omega) \), such that
\[ f_k \to f \quad \text{in } L^1(\Omega) \quad \text{and} \quad f_k(x) \leq 2f(x) \quad \text{a.e. in } \Omega. \] (26)
We are going to show the existence of a weak solution \( u_k \) to the problem
\[ \left\{ \begin{array}{ll}
-\text{div} A(x,u_k,\nabla u_k) = f_k & \text{in } \Omega, \\
u_k(x) = 0 & \text{on } \partial\Omega,
\end{array} \right. \] (27)
Recall that \( u_k \in W^{1,B}_0(\Omega) \) is a weak solution to the problem (1) if
\[ \int_{\Omega} A(x,u_k,\nabla u_k) \nabla \varphi \, dx = \int_{\Omega} f_k \varphi \, dx \]
for every \( \varphi \in W^{1,B}_0(\Omega) \). Since (A1)-(A3) hold, by Theorem 4.3 in [29] we have that the operator is pseudomonotone and therefore, by Theorem 5.1 in [29], we get the existence of a distributional solution.

Then, due to the modular approximation (see Theorem 3), we obtain the existence of \( u_k \in W^{1,B}_0(\Omega) \), such that
\[ \int_{\Omega} A(x,u_k,\nabla u_k) \cdot \nabla \varphi \, dx = \lim_{\delta \to 0} \int_{\Omega} A(x,u_k,\nabla u_k) \cdot \nabla \varphi_{\delta} \, dx \]
for every \( \varphi \in W^{1,B}_0(\Omega) \).

**Step 2. A priori estimates**

In order to obtain uniform integrability of sequences \( \{ A(x,T_1u_k,\nabla T_1u_k) \}_{k} \) and \( \{ \nabla T_1u_k \}_{k} \) we will prove the two following a priori estimates. For \( u_k \) being a weak solution to (27) and \( f \in L^1(\Omega) \), we will have for any \( t > 0 \)
\[ \int_{\Omega} B(|\nabla T_1u_k|) \, dx \leq c_0 t \| f \|_{L^1(\Omega)}, \] (29)
\[ \int_{\Omega} \tilde{B} \left( \frac{1}{d} |A(x,T_1u_k,\nabla T_1u_k)| \right) \, dx \leq c_0 t \| f \|_{L^1(\Omega)} + c_1 B_s(t) + c_2, \] (30)
where \( B_s << B \) and the constants \( c_0, c_1, c_2 \) depend only on the growth condition (A2). More precisely, \( c_0 = 2/d_0, \)
\( c_1 = c(B,P,\Omega), \) and \( c_2 = c(K). \)

Due to (28), we get
\[ \int_{\Omega} A(x,T_1u_k,\nabla T_1u_k) \nabla T_1u_k \, dx = \int_{\Omega} A(x,u_k,\nabla u_k) \nabla T_1u_k \, dx = \int_{\Omega} f_k T_1u_k \, dx \leq 2t \| f \|_{L^1(\Omega)}. \] (31)
Observe that we used that \( A(x,u_k,\nabla u_k) \in L^\infty_B \) and estimate at (26). Estimate (29) immediately follows by using (2)
\[ d_0 \int_{\Omega} B(|\nabla T_1u_k|) \, dx \leq \int_{\Omega} A(x,T_1u_k,\nabla T_1u_k) \nabla T_1u_k \, dx \leq 2t \| f \|_{L^1(\Omega)}. \] (32)
On the other hand, if we use (3), Jensen’s inequality and (29), we have
\[ \int_{\Omega} \tilde{B}(d|A(x,T_1u_k,\nabla T_1u_k)|) \, dx \leq \int_{\Omega} \tilde{B} \left( \frac{1}{d} \left( \tilde{B}^{-1}(B(|\nabla T_1u_k|)) + \tilde{P}^{-1}(B(|T_1u_k|)) + K(x) \right) \right) \, dx \]
\[ \leq \frac{1}{3} \int_{\Omega} \tilde{B} \left( \tilde{B}^{-1}(B(|\nabla T_i u_k|)) \right) + \tilde{B} \left( \tilde{P}^{-1}(B(|T_i u_k|)) \right) + \tilde{B}(K(x)) \, dx \]

\[ \leq \frac{1}{3} \int_{\Omega} B(|\nabla T_i u_k|) + c(B, P)B(t) + \tilde{B}(K(x)) \, dx \]

\[ \leq c_0 t||f||_{L^1(\Omega)} + c(B, P, \Omega)B_\varepsilon(t) + c(K). \]

Note that we have used that the assumption \( P << B \) is equivalent to \( \tilde{B} << \tilde{P} \), estimate \((32)\) and that \( K \in E_\tilde{B} \).

**Step 3. Convergence** \( u_k \overset{a.e.}{\longrightarrow} u \)

The a priori estimates \((29)\), the Banach-Alaoglu theorem combined with Dunford-Pettis theorem, and the fact that \( B \) is an \( N \)-function imply that for each \( t > 0 \) the sequence \( \{T_i u_k\}_k \) is bounded in \( W_0^{1,1}(\Omega) \). Moreover, the Poincaré inequality from Corollary 4.1 and estimate \((29)\) ensure that \( \{T_i u_k\}_k \) is bounded in \( W_0^{1,B}(\Omega) \). Hence, the embedding imply that there exists a function \( u \) such that

\[
\begin{align*}
T_i u_k & \xrightarrow[k \to \infty]{\text{strongly in}} L^1(\Omega), \\
T_i u_k & \xrightarrow[k \to \infty]{\text{a.e.}} L^1(\Omega), \\
\nabla T_i u_k & \xrightarrow[k \to \infty]{\text{weakly in}} L^1(\Omega), \\
\nabla T_i u_k & \xrightarrow[k \to \infty]{\text{weakly-*}} L^B(\Omega). 
\end{align*}
\]

Since truncated functions converge \( a.e. \), for every \( t \) fixed and for every \( \varepsilon \) there exists \( \tau \) such that for \( k, m \) sufficiently large

\[
||\{T_i u_k - T_i u_m| > \tau\}|| \leq \varepsilon. \tag{34}
\]

Now observe that for given \( t, \tau > 0 \) we have

\[
||\{u_k - u_m| > \tau\}|| \leq ||\{u_k| > t\}|| + ||\{u_m| > t\}|| + ||\{T_i u_k - T_i u_m| > \tau\}||
\]

for \( k, m \in \mathbb{N} \).

On the other hand, since \( B \) is increasing we get for every \( l > 0 \)

\[
||\{u_k| \geq l\}|| = ||\{T_i u_k| = l\}|| = ||\{T_i u_k| \geq l\}|| = ||B(c_l|T_i u_k|)|| \geq B(c_l)||,
\]

therefore

\[
||\{u_k| \geq l\}|| \leq \int_{\Omega} B(|c_l T_i u_k|) \frac{dx}{B(c_l)} \leq \frac{c(N, \Omega)}{B(l)} \int_{\Omega} B(|\nabla T_i u_k|) \, dx \\
\leq \frac{C(N, \Omega)}{B(l)} \cdot l \|f\|_{L^1(\Omega)} \leq \frac{C(f, B, N, \Omega)}{B(l)} \xrightarrow[l \to \infty]{l} 0. \tag{35}
\]

In the above estimates we apply \((\text{respective ly})\) the Chebyshev inequality, Corollary 4.1, the a priori estimate \((29)\).

The limit results from the superlinear growth in the infinity of \( N \)-function \( B \).

Therefore, using \((35)\), for every \( \varepsilon \) we can choose \( t \) so large that

\[
||\{u_k| > t\}|| < \varepsilon \quad \text{and} \quad ||\{u_m| > t\}|| < \varepsilon
\]

and then, recalling also \((34)\), we obtain that \( u_k \) is a Cauchy sequence in measure. It follows that , up to a subsequence,

\[
u_k \xrightarrow[k \to \infty]{\text{a.e. in}} \Omega, \tag{36}
\]

that is \( u \) is an approximable solution to our problem.
Step 4. Convergence $A(u, T_i u_k, \nabla T_i (u_k)) \xrightarrow{\ast} A(x, T_i u, \nabla T_i u)$ in $L^p_B$

Since by (30) we have that there exists $A_t \in L^p_B(\Omega)$ such that

$$A(x, T_i u_k, \nabla T_i (u_k)) \xrightarrow{\ast} A_t \text{ weakly} - \ast \text{ in } L^p_B(\Omega),$$

our first aim is to prove that

$$\limsup_{k \to \infty} \int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) \nabla T_i u_k \, dx = \int_{\Omega} A_t \cdot \nabla T_i u \, dx,$$  \hspace{1cm} (38)

which will be instrumental in order to use the monotonicity trick. By Theorem 3 we can take an approximating sequence $(T_i u)_\delta$ of smooth functions such that $\nabla (T_i u)_\delta \xrightarrow{\operatorname{mod} \delta \to 0} \nabla T_i (u)$ in $L^p_B$ and write

$$\int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) \nabla T_i u_k \, dx = \int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) \nabla (T_i u)_\delta \, dx + \int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) (\nabla T_i u_k - \nabla (T_i u)_\delta) \, dx$$

Therefore, if we take into account Lemma 3.4 and that (37) holds, in order to get (38), it suffices to show that

$$\lim_{\delta \to 0} \limsup_{k \to \infty} \int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) [T_i u_k - (T_i u)_\delta] \, dx = 0.$$  \hspace{1cm} (39)

Let us define the cut-off function $\psi_l : \mathbb{R} \to \mathbb{R}$ by

$$\psi_l(x) := \min\{(l + 1 - |x|)^+, 1\}. \hspace{1cm} (40)$$

Observe that since $A(x, z, 0) = 0$, $A(x, T_i u_k, \nabla T_i u_k)$ is not zero provided $|u_k| \leq t$. Then for $l > t$, it is $\psi_l(u_k) = 1$ and hence

$$\int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) [T_i u_k - (T_i u)_\delta] \, dx = \int_{\Omega} A(x, T_i u_k, \nabla T_i u_k) [T_i u_k - (T_i u)_\delta] \psi_l(u_k) \, dx$$

$$= \int_{\Omega} A(x, u_k, \nabla u_k) [T_i u_k - (T_i u)_\delta] \psi_l(u_k) \, dx$$

$$- \int_{\Omega} (A(x, u_k, \nabla u_k) - A(x, T_i u_k, \nabla T_i u_k)) [T_i u_k - (T_i u)_\delta] \psi_l(u_k) \, dx$$

$$= \int_{\Omega} A(x, u_k, \nabla u_k) \nabla (\psi_l(u_k) (T_i u_k - (T_i u)_\delta)) \, dx$$

$$- \int_{\Omega} (A(x, u_k, \nabla u_k) \nabla \psi_l(u_k)) (T_i u_k - (T_i u)_\delta) \, dx$$

$$- \int_{\Omega} (A(x, u_k, \nabla u_k) - A(x, T_i u_k, \nabla T_i u_k)) [T_i u_k - (T_i u)_\delta] \psi_l(u_k) \, dx$$

$$= I_1 + I_2 + I_3 \hspace{1cm} (41)$$

In order to show (39), it will be enough to prove that each of the integrals in the right hand side of last equality goes to zero as $\delta \to 0$ and $k \to \infty$.

Note that $\varphi = \psi_l(u_k) (T_i u_k - (T_i u)_\delta)$ is a legitimate test function for the equation (28) because of (31). It follows that for $I_1$ we have

$$I_1 = \int_{\Omega} A(x, u_k, \nabla u_k) \nabla (\psi_l(u_k) (T_i u_k - (T_i u)_\delta)) \, dx = \int_{\Omega} f_k \psi_l(u_k) (T_i u_k - (T_i u)_\delta) \, dx$$
and, for every \( l \) fixed, we have
\[
\lim_{\delta \to 0} \lim \sup_{k \to \infty} \int_{\Omega} f_k \psi_l(u_k)(T_i u_k - (T_i u)_\delta) dx = 0.
\]
To this end, observe that
\[
\lim_{\delta \to 0} \lim \sup_{k \to \infty} \left| \int_{\Omega} f_k \psi_l(u_k)(T_i u_k - (T_i u)_\delta) dx \right|
\leq \lim_{\delta \to 0} \lim \sup_{k \to \infty} \int_{\Omega} |f_k - f| |T_i u_k - (T_i u)_\delta| dx + \lim \sup_{k \to \infty} \int_{\Omega} |f| |T_i u_k - (T_i u)_\delta| dx
\leq \lim_{\delta \to 0} \int_{\Omega} |f| |T_i u - (T_i u)_\delta| dx = 0.
\]
Note that the first limit in the second line vanishes since \( u_k \to u \text{ a.e.}, f_k \to f \) in \( L^1(\Omega) \) and \((T_i u)_\delta \to T_i u \) modularly (so in \( L^1 \)). On the other hand, the last equality holds thanks to the Lebesgue Dominated Convergence Theorem that is legitimate to be used since \((T_i u)_\delta\) are uniformly bounded.

It follows that
\[
\lim_{\delta \to 0} \lim \sup_{k \to \infty} \int_{\Omega} A(x, u_k, \nabla u_k) \nabla \left( \psi_l(u_k)(T_i u_k - (T_i u)_\delta) \right) dx = 0 \tag{42}
\]
To deal with \( I_2 \) we need to show that
\[
\lim_{l \to \infty} \lim_{\delta \to 0} \lim \sup_{k \to \infty} \int_{\Omega} \left( A(x, u_k, \nabla u_k) \nabla \psi_l(u_k) \right) (T_i u_k - (T_i u)_\delta) dx = 0 \tag{43}
\]
By the definition of \( \psi_l \) we first obtain that
\[
\left| \int_{\Omega} \left( A(x, u_k, \nabla u_k) \nabla \psi_l(u_k) \right)(T_i u_k - (T_i u)_\delta) dx \right| \leq \int_{\Omega} \left| A(x, u_k, \nabla u_k) \nabla \psi_l(u_k) \right| |T_i u_k - (T_i u)_\delta| dx \tag{44}
\]
Since \((T_i u)_\delta\) is uniformly bounded and for \( l > t \), it is \(|u_k| > \delta|t|\) on the set \( \{ l < |u_k| < l + 1 \} \), the integral in the right hand side of previous inequality can be estimated by
\[
c \lim_{l \to \infty} \lim_{\delta \to 0} \lim \sup_{k \to \infty} \int_{\{ l < |u_k| < l + 1 \}} A(x, u_k, \nabla u_k) \nabla u_k dx =
\lim_{l \to \infty} \lim_{k \to \infty} \int_{\Omega} A(x, u_k, \nabla u_k)(\nabla T_{l+1} u_k - \nabla T_l u_k) dx =
c \lim_{l \to \infty} \lim_{k \to \infty} \int_{\Omega} f_k(T_{l+1} u_k - T_l u_k) dx \tag{45}
\]
where we also used that \( u_k \) is a solution of (28). Then, recalling the pointwise inequality at (26), the fact that \( f \in L^1 \) and that (36) holds, we obtain from previous calculations that
\[
\lim_{l \to \infty} \lim_{\delta \to 0} \lim \sup_{k \to \infty} I_2 \leq c \lim_{l \to \infty} \lim_{k \to \infty} \int_{\{ l < |u_k| < l + 1 \}} |f_k| dx \leq
c \lim_{l \to \infty} \lim_{k \to \infty} \int_{\{ l < |u_k| \}} |f_k| dx = 0. \tag{46}
\]
Now we concentrate on \( I_3 \) and show that
\[
\lim_{l \to \infty} \lim_{\delta \to 0} \lim \sup_{k \to \infty} \int_{\Omega} \left( A(x, u_k, \nabla u_k) - A(x, T_i u_k, \nabla T_i u_k) \right) \nabla [T_i u_k - (T_i u)_\delta] \psi_l(u_k) dx = 0 \tag{47}
\]
Recalling that \( A(x, z, 0) = 0 \) and the definition of the function \( \psi_t \) at (40), we have for \( t > t \)

\[
\int_{\Omega} (A(x, u_k, \nabla u_k) - A(x, T_t u_k, \nabla T_t u_k)) \nabla [T_t u_k - (T_t u)_{\delta}] \psi_l(u_k) \, dx \\
= \int_{\{t < |u_k| < t + 1\}} (A(x, T_{t+1} u_k, \nabla T_{t+1} u_k) - A(x, T_t u_k, \nabla T_t u_k)) \nabla [T_t u_k - (T_t u)_{\delta}] \psi_l(u_k) \, dx \\
= -\int_{\{t < |u_k| < t + 1\}} A(x, T_{t+1} u_k, \nabla T_{t+1} u_k) \nabla (T_t u)_{\delta} \psi_l(u_k) \, dx \\
= -\int_{\Omega} A(x, T_{t+1} u_k, \nabla T_{t+1} u_k) \nabla (T_t u)_{\delta} \psi_l(u_k) \mathbb{I}_{\{t < |u_k| < t + 1\}} \, dx
\]

Since

\[
A(x, T_t u_k, \nabla T_t u_k) \rightharpoonup A_t \text{ weakly } \ast \text{ in } L_B(\Omega) \quad \text{as } k \to \infty,
\]

we have

\[
A(x, T_{t+1} u_k, \nabla T_{t+1} u_k) \nabla (T_t u)_{\delta} \to A_{t+1} \nabla (T_t u)_{\delta} \text{ in } L^1
\]

Therefore we have

\[
\lim_{t \to \infty} \lim_{\delta \to 0} \lim_{k \to \infty} \int_{\Omega} \left( A(x, u_k, \nabla u_k) - A(x, T_t u_k, \nabla T_t u_k) \right) \nabla [T_t u_k - (T_t u)_{\delta}] \psi_l(u_k) \, dx
\]

\[
= \lim_{t \to \infty} \lim_{\delta \to 0} \lim_{k \to \infty} \int_{\Omega} A(x, T_{t+1} u_k, \nabla T_{t+1} u_k) \nabla (T_t u)_{\delta} \psi_l(u_k) \mathbb{I}_{\{t < |u_k| < t + 1\}} \, dx
\]

\[
= \lim_{\delta \to 0} \int_{\Omega} |A_{t+1}| |\nabla T_t u| \mathbb{I}_{\{t < |u|\}} \, dx
\]

\[
= 0.
\]

where we used Lemma 3.2 and Lemma 3.1.

Combining (42), (43) and (47), we infer that (39) is true and therefore (38) holds.

Now, using the monotonicity trick, we identify the limit \( A_t \). More precisely, now our aim is to show that in (37)

\[
A_t(x) = A(x, T_t u(x), \nabla T_t u(x)) \quad \text{a.e. in } \Omega.
\]

Monotonicity of \( A \) results in

\[
\int_{\Omega} A(x, T_t u_k, \nabla T_t u_k) \nabla T_t u_k \, dx \geq \int_{\Omega} A(x, T_t u_k, \nabla T_t u_k) \eta \, dx + \int_{\Omega} A(x, T_t u_k, \eta) (\nabla T_t u_k - \eta) \, dx
\]

for any \( \eta \in \mathbb{R}^N \). Taking the upper limit with \( k \to \infty \) above, we have in the left hand side

\[
\int_{\Omega} A_t \cdot \nabla T_t u \, dx
\]

and for the first term in the right hand side

\[
\int_{\Omega} A_t \cdot \eta \, dx
\]

respectively thanks to (38) and (37).

To justify that

\[
A(x, T_t u_k, \eta) \to A(x, T_t u, \eta) \text{ strongly in } L_B
\]

we recall that \( P << B, A \) is continuous with respect to the second variable, and we have almost everywhere convergence of \( T_t u_k \). Altogether, we infer uniform boundedness of \( \{ P(\|A(x, T_t u_k, \eta)/\lambda\|) \} \) in \( L^1 \) for arbitrary \( \lambda > 0 \). Further, via Theorem 5, we get uniform integrability of \( \{ B(\|A(x, T_t u_k, \eta)/\lambda\|) \} \) in \( L^1 \) and due to Lemma 3.3 we get the desired limit.
Then, recalling that, by (33), we have
\[
\nabla T_t u_k \rightharpoonup_{k \to \infty} \nabla T_t u \quad \text{weakly-* in } L_B(\Omega),
\]
(48)
thank to Lemma 3.5 with \(A(x, T_t u_k, \eta) \in E_B\) and to the continuity of \(A\), we get
\[
\lim_{k \to \infty} \int_\Omega A(x, T_t u_k, \eta)(\nabla T_t u_k - \eta) \, dx = \int_\Omega A(x, T_t u, \eta)(\nabla T_t u - \eta) \, dx.
\]
In conclusion, we have
\[
\int_\Omega A_t \cdot \nabla T_t u \, dx \geq \int_\Omega A_t \cdot \eta \, dx + \int_\Omega A(x, T_t u, \eta)(\nabla T_t u - \eta) \, dx
\]
that it is equivalent to
\[
\int_\Omega (A_t - A(x, T_t u, \eta))(\nabla T_t u - \eta) \, dx \geq 0.
\]
(49)
Then the monotonicity trick (see Proposition 5.1) implies
\[
A(x, T_t u, \nabla T_t u) = A_t \quad \text{a.e.}
\]
(50)
The convergence of the left-hand side of (27) follows from the facts that \(u \in W^{1,B}(\Omega)\), \(\nabla u\) can be understood as the generalized gradient in the sense of (13), and (50), whereas the right-hand side of (27) converges due to (26).

**Step 5. Measure data problem**

To study measure-data problems let us consider \(f_k \in L^1(\Omega) \cap (W^{1,B}(\Omega))^\prime\), given by
\[
f_k(x) = \int_\Omega \rho(|y-x|/k) \, d\mu(y) \quad x \in \Omega,
\]
where \(\rho : \mathbb{R}^N \to [0, \infty)\) is a standard mollifier (i.e. smooth function compactly supported in the unit ball with \(\|\rho\|_{L^1(\Omega)} = 1\)). Note that
\[
\|f_k\|_{L^1(\Omega)} \leq 2\|\mu\|_{(\Omega)}
\]
and
\[
\lim_{k \to \infty} \int_\Omega \varphi \, f_k \, dx = \int_\Omega \varphi \, d\mu
\]
for every \(\varphi \in C_c(\Omega)\). Then, for the problem (27) under such a choice of \(f_k\) the above proof still hold. \(\Box\)

### 5.3. Uniqueness for \(L^1\)-data problem with strongly monotone operator

**Proof of Theorem 2.** To complete the proof of Theorem 2 having Theorem 1 it suffices to infer uniqueness. We suppose \(u\) and \(\bar{u}\) are approximable solutions to problem (4) with the same \(L^1\)-data but which are obtained as limits of different approximate problems and prove that they have to be equal almost everywhere. By Definition 2.2 there exist sequences \(\{f_k\}\) and \(\{\bar{f}_k\}\) in \(L^1(\Omega) \cap (W^{1,B}(\Omega))^\prime\), such that \(f_k \to f\) and \(\bar{f}_k \to f\) in \(L^1(\Omega)\) and weak solutions \(u_k\) to (27) and \(\bar{u}_k\) to
\[
\begin{cases}
-\text{div}A(x, \bar{u}_k, \nabla \bar{u}_k) = \bar{f}_k & \text{in } \Omega, \\
\bar{u}_k(x) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(51)
such that for a.e. in \(\Omega\) we have both \(u_k \to u\) and \(\bar{u}_k \to \bar{u}\). We fix arbitrary \(t > 0\), use \(\varphi = T_t(u_k - \bar{u}_k)\) as a test function in (27) and (51), and subtract the equations to obtain
\[
\int_{\{|u_k - \bar{u}_k| \leq t\}} (A(x, u_k, \nabla u_k) - A(x, \bar{u}_k, \nabla \bar{u}_k)) \cdot (\nabla u_k - \nabla \bar{u}_k) \, dx = \int_\Omega (f_k - \bar{f}_k) T_t(u_k - \bar{u}_k) \, dx \quad \text{for every } k \in \mathbb{N}.
\]
(52)
The right-hand side above tends to 0, because $|T_t(u_k - \bar{u}_k)| \leq t$ and for $k \to \infty$ we have $f_k - \bar{f}_k \to 0$ in $L^1(\Omega)$. The left-hand side is convergent due to Step 4, (A3), and Fatou’s Lemma. We get
\[
\int_{\{|u - \bar{u}| \leq t\}} (A(x, u, \nabla u) - A(x, \bar{u}, \nabla \bar{u})) \cdot (\nabla u - \nabla \bar{u}) \, dx = 0.
\]
Consequently, $\nabla u = \nabla \bar{u}$ a.e. in $\{|u - \bar{u}| \leq t\}$ for every $t > 0$, and so
\[
\nabla u = \nabla \bar{u} \quad \text{a.e. in } \Omega. \tag{53}
\]

Then, using the Poincaré inequality (Corollary 4.1) with $T_r(u - T_t(\bar{u}))$, for a fixed $r > 0$, in place of $u$, we get
\[
\int_{\Omega} B(c_1|T_r(u - T_t(\bar{u})))|) \, dx \leq c_2 \int_{\{|u - t| \leq r\}} B(|\nabla u|) \, dx.
\]
We will prove that the left-hand side above tends to zero with $t \to \infty$.

By using (A2) we have that
\[
\int_{\{|u - t| \leq r\}} B(|\nabla u|) \, dx \leq c_2 \liminf_{k \to \infty} \int_{\{|u_k - t| \leq r\}} A(x, u_k, \nabla u_k) \nabla u_k \, dx
\]
\[
= c_2 \liminf_{k \to \infty} \int_{\{|u_k - t| \leq r\}} A(x, u_k, \nabla u_k) \nabla T_{t+r}(u_k) \, dx
\]
\[
= 2rc_2 \liminf_{k \to \infty} \int_{\Omega} A(x, u_k, \nabla u_k) \nabla (1 - \psi_{t,r}(u_k)) \, dx.
\]
where we introduced the notation
\[
\psi_{t,r}(s) = \frac{1}{2r} \min\{1, t + r - |s|\}.
\]

Now, using weak formulation of the problem, we get
\[
\int_{\Omega} B(c_1|T_r(u - T_t(\bar{u})))|) \, dx \leq c \int_{\{|t-r|<|u|\}} |f| \, dx \xrightarrow{t \to \infty} 0.
\]
Fatou’s Lemma enables to pass to the limit to get
\[
\int_{\Omega} B(c_1|T_r(u - \bar{u})|) \, dx = 0 \quad \text{for every } r > 0.
\]
Therefore, $B(c_1|T_r(u - \bar{u})|) = 0$ a.e. in $\Omega$ for every $r > 0$, and consequently $u = \bar{u}$ a.e. in $\Omega$. \hfill \Box

6. Regularity

Our next aim is to provide some regularity results in the Orlicz-Marcinkiewicz scale for the solutions of problem (4) with measure data. Note that the key estimates of the proof are interesting by themselves, see Propositions 6.1, 6.2, and 6.3. It is worth pointing out that we get the regularity of the whole function $u$ and of its full gradient $\nabla u$, not only of the truncation $T_k(u)$ and its gradient.

The classical way of introducing the Orlicz-Marcinkiewicz spaces goes via rearrangement approach, see e.g. [14, 30]. The decreasing rearrangement $f^* : [0, \infty) \to [0, \infty]$ of a measurable function $f : \Omega \to \mathbb{R}$ is the unique right-continuous, non-increasing function equidistributed with $f$, namely,
\[
f^*(s) = \inf\{t \geq 0 : |\{|f| > t\}| \leq s\} \quad \text{for } s \geq 0.
\]
Its maximal rearrangement is defined as follows
\[
f^{**}(x) = \frac{1}{x} \int_0^x f^*(t) \, dt \quad \text{and} \quad f^{**}(0) = f^*(0). \tag{54}
\]
**Definition 6.1** (The Orlicz–Marcinkiewicz-type spaces). Let \( \varphi : (0, |\Omega|) \rightarrow (0, \infty) \) be a Young function. We define the Orlicz–Marcinkiewicz-type spaces

\[
\mathcal{M}^\varphi(\Omega) := \left\{ f \text{ measurable in } \Omega : \|f\|_{\mathcal{M}^\varphi(\Omega)} := \sup_{s \in (0, |\Omega|)} \frac{f^*(s)}{s^{1/\varphi^{-1}(1/s)}} < \infty \right\}
\]

and

\[
\mathcal{M}_w^\varphi(\Omega) := \left\{ f \text{ measurable in } \Omega : \|f\|_{\mathcal{M}_w^\varphi(\Omega)} := \lim_{t \to \infty} \sup_{t \in (0, \infty)} \frac{t}{\varphi^{-1}(1/\{[f] > t\})} < \infty \right\}.
\]

While treated as a case of the Lorentz-type space the notation \( \mathcal{M}^\varphi(\Omega) = L^{\varphi, \infty}(\Omega) \) can be also used.

**Remark 6.1.** It can be shown that \( \| \cdot \|_{\mathcal{M}^\varphi(\Omega)} \) defines a norm, while \( \| \cdot \|_{\mathcal{M}_w^\varphi(\Omega)} \) only a quasi-norm.

Orlicz–Marcinkiewicz spaces are intermediate to Orlicz spaces in the sense that

\[ L^\varphi(\Omega) \subset \mathcal{M}^\varphi(\Omega) \subset \mathcal{M}_w^\varphi(\Omega) \subset L^\varphi_{1-\varepsilon}(\Omega) \quad \text{for all } \varepsilon \in (0, 1). \]

Let us stress that \( \mathcal{M}^\varphi(\Omega) = \mathcal{M}_w^\varphi(\Omega) \) if and only if \( \varphi \) satisfies

\[
\int_0^s \varphi^{-1}\left(\frac{1}{r}\right) dr \leq c\varphi^{-1}(1/s).
\]

(**57**) In particular, if \( p > 1 \) and \( \beta \geq 0 \), the function \( \varphi(t) = t^p \log^\beta(1 + t) \) satisfies (**57**) and hence \( \mathcal{M}^\varphi(\Omega) = \mathcal{M}_w^\varphi(\Omega) \).

We provide two types of level-sets estimates resulting from the different embeddings discussed in Section 4.

**Proposition 6.1.** Let \( B \) be an \( N \)-function. Suppose \( v \in T_0^{1,B}(\Omega) \) and constants \( K > 0 \) and \( r_0 \geq 0 \) are such that

\[
\int_{\{|v| < r\}} |\nabla v| dx \leq Kr \quad \text{for } r > r_0.
\]

Then

\[
v \in \mathcal{M}_w^{\Phi_1}(\Omega) \quad \text{and} \quad \nabla v \in \mathcal{M}_w^{\Psi_1}(\Omega),
\]

where

\[
\Phi_1(r) = \left( \frac{B(c_1 r)}{K r} \right)^{N'} \quad \text{and} \quad \Psi_1(r) = \frac{B(r)}{K \Phi^{-1}(B(r))},
\]

with \( c_1 = c_1(\text{diam} \, \Omega) \) and \( K = 2 \max\{K, K^{N'}\} \).

Proof. First of all we notice that since \( v \in T_0^{1,B}(\Omega) \), then of course \( T_r(v) \in W_0^{1,B}(\Omega) \) for every \( r > 0 \). Therefore by the Sobolev-Poincaré inequality at Proposition 4.1, we get

\[
\left( \int_\Omega B^{N'}(c_1 |T_r(v)|) dx \right)^\frac{1}{N'} \leq c_2 \int_{\{|v| < r\}} B(|\nabla v|) dx.
\]

To estimate the left-hand side from below we note that for every \( r > 0 \)

\[
\{c_1 |T_r(v)| > c_1 r\} = \{|v| > r\}.
\]

Thus

\[
|\{|v| > r\}|B^{N'}(c_1 r) \leq \int_{\{|v| < r\}} B^{N'}(c_1 |T_r(v)|) dx.
\]

Summing up the above observations and taking into account (**58**) we obtain

\[
|\{|v| > r\}|B^{N'}(c_1 r) \leq \left( \int_{\{|v| < r\}} B(|\nabla v|) dx \right)^{N'} \leq (Kr)^{N'} \quad \text{for } r > r_0,
\]

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implying
\[
|\{|v| > r\}| \leq \left( \frac{Kr}{B(c_1 r)} \right)^{N'} \quad \text{for } r > r_0.
\] (62)

By using (62) we deduce that
\[
|\{|B(|\nabla v|) > s\}| \leq |\{|v| > r\}| + |\{|B(|\nabla v|) > s, |v| \leq r\}| \leq (Kr/B(r))^{N'} + Kr/s \quad \text{for } r > r_0, s > 0.
\]

Now recall the definition of $\phi$ and set $r = \phi^{-1}(s)$. Then, for $s > \phi(r_0)$, we get
\[
|\{|B(|\nabla v|) > s\}| \leq \frac{K\phi^{-1}(s)}{s}
\]
and it suffices to take $\theta = B^{-1}(s)$ to ensure that for $s > \phi(r_0)$
\[
|\{|\nabla v| > \theta\}| \leq \frac{K\phi^{-1}(B(\theta))}{B(\theta)}.
\] (63)

Taking into account (62) and (63) we get the claim. \hfill \square

Further we employ the following estimates by Cianchi and Mazya [15]. Note that this result follows independently of the type of the growth of $B$.

**Proposition 6.2** (cf. Lemma 4.1, [15]). Let $B$ be an $N$-function and $\Omega$ is a Lipschitz bounded domain. Suppose $v \in T_0^{1,B}(\Omega)$ and there exist constants $K > 0$ and $r_0 \geq 0$, such that (58) is satisfied.

(a) If $(19)_1$, then there exists a constant $c = c(N)$ such that
\[
|\{|v| > r\}| \leq \frac{Kr}{B_N(c^{r/N}/K)} \quad \text{for } r > r_0.
\] (64)

(b) If $(19)_2$, then there exists a constant $r_1 = r_1(r_0, N, M)$ such that
\[
|\{|v| > r\}| = 0 \quad \text{for } r > r_1.
\] (65)

Despite [15, Lemma 4.1] is formulated assuming (18), it is explained in [15] that it is not necessary. Moreover, the proof admits to consider functions from $T_0^{1,B}(\Omega)$ not only $W_0^{1,B}(\Omega)$ as in the statement therein.

Now we can infer gradient estimates.

**Proposition 6.3.** Let $B$ be an $N$-function satisfying (18) and recall $\phi_N$ and $B_N$ given by (20). Suppose $v \in T_0^{1,B}(\Omega)$ and constants $K > 0$ and $r_0 \geq 0$ are such that (58) holds.

(a) If $(19)_1$, then $v \in M_0^{B_2}(\Omega)$ and $\nabla v \in M_0^{B_3}(\Omega)$, where
\[
\Phi_2(r) = \frac{B_N(\tilde{c}r^{(N-1)/N})}{r} \quad \text{and} \quad \Psi_2(r) = \frac{B(r)}{\phi_N(r)}
\] (66)

with a constant $\tilde{c} = \tilde{c}(N, K)$.

(b) If $(19)_2$, then $v \in L^\infty(\Omega)$ and $\nabla v \in M_0^B(\Omega)$.

Proof. We notice first that (58) implies
\[
|\{|B(|\nabla v|) > s, |v| \leq r\}| \leq \frac{1}{s} \int_{\{|B(|\nabla v|) > s, |v| \leq r\}} B(|\nabla v|) \, dx \leq K\frac{r^2}{s} \quad \text{for } r > r_0 \text{ and } s > 0.
\]

Let us concentrate on (a). We infer that $v \in M_0^{B_3}(\Omega)$ directly from (64). Furthermore, since
\[
|\{|B(|\nabla v|) > s\}| \leq |\{|v| > r\}| + |\{|B(|\nabla v|) > s, |v| \leq r\}|,
\] (67)
then by (64) we get
\[ |\{B(\nabla v) > s\}| \leq \frac{Kr}{B_N(c \frac{r}{\sqrt{s}}/K^N)} + K^r \frac{r}{s} \quad \text{for } r > r_0, \ s > 0. \]

Substitute \( r = (K^{1/N}B_N^{-1}(s)/c)^N \) and consider \( s \geq B_N(c \frac{r^{1/N}}{K^{1/N}}) \). Then
\[ |\{B(\nabla v) > s\}| \leq 2 \left( \frac{K^r}{c} \right)^N \frac{(B_N^{-1}(s))^N}{s}. \]

Taking \( \theta = B^{-1}(s) \) we obtain that there exists a constant \( K_1 = K_1(N, K) \) such that
\[ |\{|\nabla v| > \theta\}| \leq K_1 \frac{(B_N^{-1}(B(\theta)))^N}{B(\theta)} \quad \text{for } \theta > 0 \]
implies (a).

Now we turn to prove (b). Boundedness of \( v \) results directly from (65). For estimating super-level set of its gradient we use again (67) to get
\[ |\{B(\nabla v) > s\}| \leq K^r \frac{r}{s} \quad \text{for } r > r_2 = \max\{r_0, r_1\} \quad \text{and} \quad s > 0. \]

Taking \( \theta = B^{-1}(s) \) we obtain
\[ |\{|\nabla v| > \theta\}| \leq K^r_3 \frac{r}{B(\theta)} \quad \text{for } \theta > 0 \]
for a constant \( r_2 = r_2(r_0, K, N) \), implying (b).

Let us carry on by giving regularity results of approximable solutions to (4) in the scale of Orlicz-Marcinkiewicz spaces. Let us mention that results of this type were already obtained recently in the reflexive case in [15, 10] and in nonreflexive spaces [3, 5].

**Theorem 6** (Estimates on approximable solutions). Assume \( \mu \in \mathcal{M}(\Omega) \) and \( A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) a function satisfying assumptions (A1), (A2), (A3)\(_w\), and (A4). Recall function \( \Phi_1, \Psi_1, \Phi_2, \Psi_2 \), given by (59) and (66), respectively. Then every approximable solution \( u \in \mathcal{T}^{1,B}(\Omega) \) to (4) satisfies
\[ u \in \mathcal{M}^{\Phi_1}_w(\Omega) \quad \text{and} \quad \nabla u \in \mathcal{M}^{\Phi_2}_w(\Omega). \]

Moreover,
(a) if \( B \) satisfies (19)\(_1\), then \( u \in \mathcal{M}^{\Phi_2}_w(\Omega) \) and \( \nabla u \in \mathcal{M}^{\Phi_2}_w(\Omega); \)
(b) if \( B \) satisfies (19)\(_2\), then \( u \in L^\infty(\Omega) \) and \( \nabla u \in \mathcal{M}^B_w(\Omega). \)

**Proof.** Since the approximable solutions to (4) satisfy (58), as a direct consequence of Propositions 6.1, 6.2, and 6.3, we infer the following information on their regularity.

According to definition in (17) the space \( W^{1,B}_0(\Omega) \) is closed in weak-* topology and in (33) we infer that
\[ \nabla T_k u_k \overset{*}{\longrightarrow} \nabla T_k u \quad \text{weakly-* in } L^B(\Omega). \]

Recall that under condition (57) on \( \varphi \), we can substitute above each \( \mathcal{M}^{\Phi}_w(\Omega) \) with \( \mathcal{M}^\varphi(\Omega) \), cf. Definition 6.1.

We give examples related to the case of the Zygmund-type modular functions and extending this setting.

**Example 6.1** (Zygmund-type functions). Consider \( B(t) \sim t^p \log^\beta(1 + t) \) with \( p > 1 \) and \( \beta \geq 0 \) near infinity. Then \( B \in \Delta_2 \). Our framework admits to use [15, Lemma 4.5] to get estimates for approximable solutions to (4) with \( L^1 \)-data as in [15, Example 3.4], in particular implying what follows.
If $1 < p < n$, then $u \in \mathcal{M}^\Phi(\Omega) = L^{\frac{n(1-p)+p}{n-p}}(\log L)^{\frac{1}{n-p}}(\Omega)$ and $\nabla u \in \mathcal{M}^\Psi(\Omega) = L^{\frac{n(1-p)+p}{n-p}}(\log L)^{\frac{1}{n-p}}(\Omega)$.

If $p > n$, or $p = n$ and $\beta > n - 1$, then $u \in L^\infty$ and $\nabla u \in \mathcal{M}^\beta(\Omega) = L^{n(1-p)+p}(\log L)^{\frac{n(1-p)+p}{n-p}}(\Omega)$.

Let us point out that there is a misprint in powers in [15, Example 3.4].

Example 6.2 (Outside $\Delta_2$ or polynomial control). We have the following examples.

If $B(t) = t \log(1+t) \in \Delta_2$, but is not controlled by two power functions greater than 1. Indeed, $B(t) \not\geq t^{1+\epsilon}$ for any $\epsilon > 0$. Then $u \in \mathcal{M}^{\log^N(1+t)}(\Omega)$ and $\nabla u \in L \log L(\Omega)$.

If $B(t) = t \exp t \in \Delta_2$, it grows faster than any power and then $u \in L^\infty$ and $\nabla u \in \mathcal{M}^{\exp t}(\Omega)$.

We can also infer estimates of the Orlicz-Marcinkiewicz-type, when the modular function is irregular: trapped between two power-type functions, but does not satisfy $\Delta_2$-condition, see Example 3.1 or [9].

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