Adaptive invariant Kalman filtering for attitude estimation on $SO(3)$ thorough feedback calibration of prior error covariance

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Abstract
For invariant attitude dynamics evolving on matrix Lie groups, by proposing the stochastic feedback–based covariance calibration scheme, an adaptive invariant Kalman filter (AIKF) is elaborated to deal with the attitude estimation problems corrupted by unknown or inaccurate process noise statistics. The invariant Kalman filter (IKF) takes into account the geometry property of attitude dynamics and can boost the estimation performance; however, IKF requires accurate knowledge of the noise statistics and an incorrect noise parameter is prone to deteriorating the precision of final estimates. To eliminate this impact, instead of using the original covariance propagation step of IKF, the prior error covariance of the proposed AIKF is online calibrated based on the posterior information of the feedback stochastic sequence. As the main advantage, the statistics parameter of system process noise is no longer required in the proposed AIKF and the negative influence by unknown/incorrect noise parameters can be reduced significantly. The mathematical foundation for the new adaption scheme of AIKF is also presented. The AIKF’s advantage in filtering adaptability and simplicity is further demonstrated by numerical simulations.

1 INTRODUCTION

Motivated by the navigation and control of aerospace engineering and robotics applications, the dynamics and kinematics of Lie groups have been actively studied for the past decade [1–6]. The attitude estimation problem whose states evolve into Lie groups with invariance properties is a benchmark and motivating application, and various Kalman type filtering methods have been developed by researchers for attitude estimation [7, 8], target tracking and so on [9, 10]. Note that, in robotics or satellite engineering the attitude kinematic models might be corrupted by noisy disturbances and the accurate noise statistics is usually unavailable [9, 10], which is prone to degrade the performance of Kalman type attitude estimators [11, 12]. So, for attitude models on matrix Lie groups, this work focuses on the Kalman filtering problems with no accurate noise statistics.

Researchers and practitioners have typically used unconstrained three-set parameterizations of the rotation matrix or unit quaternion. The multiplicative extended Kalman filter [13] corresponds to an ad hoc modification of usual EKF and takes into account the constraint of a unit quaternion. Nevertheless, it only recovers the convergence property at equilibrium points and has to identify antipodal quaternions with a single attitude [14]. In recent years there has been significant interest in performing estimation and control directly on matrix Lie groups $SO(3)$ [15]. Some state observers and filters for attitude estimation on $SO(3)$ and $SE(3)$ have been reported in [16–19] and they do not suffer from kinematic singularities and unwinding [14]. Note that, these researches try to devise general adaptations of Kalman theory to matrix Lie groups but they still do not exploit the geometrical properties of the specific invariant case [20, 21].

Taking into account the geometry property of a Lie group system usually leads to well-posed problems and can boost the performance of estimation algorithms [17]. For attitude estimation of Lie groups, the invariant extended Kalman filter (IEKF) was originally introduced in [22] and continued in [13, 16, 23, 24]. In invariant Kalman filter (IKF), the prediction and correction steps are derived based on the linear invariant error and then the gain and covariance equations will converge to constant values on a much bigger set of trajectories, which means a better convergence property [20]. The use of Lie groups for state estimation has recently spanned a
range of applications and a rich stream of theoretical results [25, 26] have demonstrated the importance of using probability distributions on Lie groups notably for attitude estimation [27–30].

According to invariant Kalman filtering theory, Lie group attitude estimation problems can be projected into Euclidean space and resolved by mimicking classical Kalman filtering steps [31]. However, Kalman theory’s prior constraint on the knowledge of noise statistics is inevitably inherited by IKF and its filtering performance is rather sensitive to the accuracy of noise statistics. For aerospace and robotic applications, the Lie groups kinematic models of a robot or a satellite are usually corrupted by noises and disturbances [8, 13, 16]; the noise statistics parameter of onboard sensors might be offline tuned by experimental tests, but in practice it is too difficult to isolate process noise from attitude model and determine the covariance parameter in advance [32]. In Euclidean space, various adaptive approaches have been studied to deal with the trouble of inaccurate noise parameters [33], but until now there is no special study on developing an adaptive version of IKF in R^d cannot be directly applied in Lie group space G, the multiplication form of concentrated Gaussian distribution has been defined [31, 34] to describe the probability distribution on Lie groups, i.e. for the random variable \( \mathbf{X} \in G \),

\[
\mathbf{X} \sim \mathcal{N}_L (\mathbf{\hat{X}}, \mathbf{P}) \Rightarrow \mathbf{X} = \mathbf{\hat{X}} \exp_G (\mathbf{\xi}), \xi \sim \mathcal{N} (0, \mathbf{\Sigma}_X, \mathbf{P}) \quad \text{(left multiplication)}
\]

\[
\mathbf{X} \sim \mathcal{N}_R (\mathbf{\hat{X}}, \mathbf{P}) \Rightarrow \mathbf{X} = \exp_G (\mathbf{\xi}), \xi \sim \mathcal{N} (0, \mathbf{\Sigma}_X, \mathbf{P}) \quad \text{(right multiplication)}
\]

where \( \mathcal{N}_L \) and \( \mathcal{N}_R \) denote the distributions on matrix Lie groups based on left and right multiplication, respectively; \( \mathcal{N} \) is the Gaussian distribution in \( \mathbb{R}^d \); \( \mathbf{\hat{X}} \in G \) is the expectation of \( \mathbf{X} \) in Lie group space \( G, \mathbf{\Sigma}_X \in \mathbb{R}^{d \times d} \) is a zero-mean Gaussian noise vector; in the definition of concentrated Gaussian distribution, noise \( \mathbf{\xi} \) and all the eigenvalues of covariance \( \mathbf{P} \in \mathbb{R}^{d \times d} \) are assumed to be small enough [31], thus, almost all the mass of the distribution is concentrated in a small neighbourhood around \( \mathbf{\hat{X}} \). Owing to the noncommutativity of matrix multiplication, the left multiplication version \( \mathcal{N}_L \) is different from \( \mathcal{N}_R \) and, in this work, the right multiplication probability definition \( \mathcal{N}_R \) is employed in the attitude estimation problem.

2.2 Attitude estimation problem on Lie group SO(3)

Consider the discrete attitude dynamics on \( SO(3) \) associated with observations as follows [17, 30]:

\[
R_k = \exp_G (w_{k-1}) R_{k-1} \Omega_{k-1} \quad k = 1, 2, \ldots \quad (1)
\]

\[
Y_k = \begin{pmatrix} y_{k}^l \\ y_{k}' \\ y_{k}'' \end{pmatrix} = \begin{pmatrix} R_k^l \hat{b} + l_k' \\ R_k^l \hat{b}' + l_k'' \end{pmatrix} \quad (2)
\]

where \( R_k \in G \equiv SO(3) \) represents the rotation matrix that maps the body frame to the earth-fixed frame at time instant \( k \), \( \Omega_k \in G \) is the control input encoding the instantaneous rotation according to attitude evolution model, the Gaussian white
Invariant Kalman filtering theory

For attitude estimation, the invariant Kalman filter (IKF) is to recursively predict and correct the estimate for the Lie group state and its covariance. Given the initial $R_0|0| \sim N_0(R_0, P_0|0|)$, the state prediction using the deterministic part of dynamics (1) is

$$\hat{R}_{k|k-1} = \hat{R}_{k-1|k-1} \Omega_{k-1}$$

where $\hat{R}_{k|k-1}$ and $\hat{R}_{k-1|k-1}$ denote the prior and posterior error estimate for true state. Resorting to the equivalence of $R_k$ and $\hat{R}_k$ in probability distribution, the prior error covariance $P_{k|k-1}$ for $\hat{R}_{k|k-1}$ is propagated according to the evolution of $\hat{R}_k$ in (4)

$$P_{k|k-1} = P_{k-1|k-1} + Q_w$$

where $P_{k-1|k-1}$ denotes the covariance of the posterior estimate $\hat{R}_{k|k-1}$ as $\exp(\hat{R}_{k|k-1} R_k^1)$ for $\xi_k$ and also the covariance of $\hat{R}_{k|k-1}$.

In the correction step of IKF, the innovation vector $z_k \in \mathbb{R}^6$ is redefined and linearized as [25, 30]

$$z_k = \hat{R}_{k|k-1} Y_k - (b')^{'}(b'')^{''} = \left(\hat{R}_{k|k-1} Y_k - (b')^{'}(b'')^{''}\right) = \left(H_k \hat{\xi}_{k|k-1} + V_k\right)$$

where $H_k = \left(b_1', b_2''\right)$ is the converted observation matrix; $V_k = \left(\text{Var}(v_k')\right)$ and $Q_k = \text{Cov}(V_k) = \left(\text{Var}(v_k'')\right)$

since $v_1'$ and $v_2''$ are independent isotropic and $R_{k|k-1} R_{k|k-1}^{1} = I_{3\times 3}$. Then the posterior estimate $\hat{R}_{k|k}$ and covariance $P_{k|k}$ can be obtained as

$$\hat{R}_{k|k} = \exp\left(\Delta \hat{\xi}_k\right) \hat{R}_{k|k-1}$$

$$\Delta \hat{\xi}_k = K_k z_k$$

$$K_k = P_{k|k-1} H_k^T \left(H_k P_{k|k-1} H_k^T + Q_k\right)^{-1}$$

$$P_{k|k} = P_{k|k-1} - K_k H_k P_{k|k-1}$$

where $\Delta \hat{\xi}_k \in \mathbb{R}^3$ is the correction term. Note that, to update the state estimate $\hat{R}_{k|k}$ of the Lie group $G$, the vector space term

$$\hat{R}_{k|k} = \exp\left(\Delta \hat{\xi}_k\right) \hat{R}_{k|k-1}$$

and the model parameter $\Omega_{k-1}$ in (1) have been cancelled out in the derivation process and, although $\hat{R}_k = \exp(\hat{R}_{k|k-1} R_k^1)$ is defined using the estimate $\hat{R}_k$ and true state $R_{k-1}$, the evolution of the projected invariant error $\xi_k$ is independent of the true trajectory of state $R_{k-1}$ as well as the parameter $\Omega_{k-1}$. More theoretical details can be found in [20, 30].

As the consequence of the above theorem, the probability distribution of the Lie group state $R_k$ based on observations is equivalent to that of error $\xi_k$ in $\mathbb{R}^3$, which actually constitutes the theoretical basis of the invariant Kalman filter.
$\Delta \xi$ is projected into the Lie group space with a Lie exponential operation as in (8).

**Remark 2.** According to Theorem 1, in the above IKF, the original system (1) and (2) for Lie group state $R_k$ is converted to the vector space system (3) and (7) for error $\xi$ and, thus, the calculation of covariance $P_{\xi|k-1}$, $P_{\hat{\xi}|k}$ and Kalman gain $K_k$ for IKF can mimic that of the classical Kalman filter. More detailed results about IKF can be found in [25, 26, 30].

**Problem.** Note that IKF mimics the Kalman filter for vector space system (3) and (6) but, inevitably, it also inherits the critical requirement on the a priori and accurate knowledge of noise statistics, i.e. the $Q_w$ in (5) and $Q_v$ in (9). For some aerospace and robotic applications, the parameter $Q_v$ of onboard observation sensors usually can be determined through offline tests, but it is impractical to obtain the accurate $Q_w$ [32, 33], which makes some trouble for the practical implementation of IKF and even deteriorates the final estimate results.

So this work focuses on the invariant Kalman filtering problems with an inaccurate/unknown process noise covariance and tries to relax IKF’s constraint on the a priori and accurate $Q_w$ without sacrificing IKF’s simplicity and elegance.

## 3 INVARIANT KALMAN FILTERING BASED ON FEEDBACK CALIBRATION OF PRIOR ERROR COVARIANCE

### 3.1 New adaptive invariant Kalman filter

According to the above steps of IKF, an incorrect covariance $Q_w$ would mislead the propagation of prior error covariance $P_{\xi|k-1}$ which is a critical parameter for the calculation of the Kalman gain matrix $K_k$.

To eliminate the impact of an incorrect $Q_w$, the conventional step (5) requiring $Q_w$ should be removed and, to calculate the prior error covariance $P_{\xi|k-1}$, this work proposes the feedback calibration based adaption scheme:

$$P_{\xi|k-1} = P_{\xi|k-2} + \Delta P_k$$

$$\Delta P_k = \left( \Delta \xi_{k-1} \Delta \xi_{k-1}^T - K_k H \xi_{k-1} \right) (k-1)$$

where $P_{\xi|k-1}$ denotes the estimate for true prior error covariance $P_{\xi|k-1}$ and $\Delta P_k$ denotes the feedback correction term. Note that, the Kalman gain $K_k$ is still calculated as (10) but the original $P_{\xi|k-1}$ should be replaced by the $P_{\xi|k-1}$ estimated using (11); the posterior error covariance $P_{\hat{\xi}|k}$ is not used in AIKF and related calculations can be dropped off. The main steps of the proposed adaptive invariant Kalman filter (AIKF) are concluded in the iteration part of Algorithm 1 and the diagram is given in Figure 1.

![Algorithm 1](image)

**Algorithm 1** The detailed steps of proposed AIKF

**Preparation ($k < k_0$)**

- Initialize $R_{0|0}$ and $R_{0|0}$.
- Use the steps (4)–(10) of IKF with the available inaccurate $Q_w$ to help achieve a rough steady filtering before $k_0$.
- Initialize the estimated prior error covariance $\hat{P}_{k_0|k_0-1} = P_{k_0|k_0-1}$.

**Iteration ($k \geq k_0$)**

$$R_{k|k-1} = \hat{R}_{k|k-1} + \Omega_{k-1}$$

$$\Delta P_k = (\Delta \xi_{k-1} \Delta \xi_{k-1}^T - K_k H \xi_{k-1}) (k-1)$$

$$P_{\xi|k-1} = P_{\xi|k-2} + \Delta P_k$$

$$K_k = P_{\xi|k-1} H \xi^T (H P_{\xi|k-1} H \xi^T + Q_v)$$

$$\Delta \xi_{k} = K_k \xi$$

$$R_{k\xi|k-1} = \text{Exp}_{\xi} (\Delta \xi_{k}) R_{k\xi|k-1}$$

**FIGURE 1** The filtering diagram of the proposed AIKF

The key innovation of the proposed AIKF lies in that the prior error covariance is directly modelled using a posteriori sequence. Obviously, $Q_w$ no longer appears in the filtering diagram of AIKF as in Figure 1, therefore IKF’s critical constraint on $Q_w$ is relaxed by the proposed scheme (11) and (12) for AIKF.

**Remark 3.** Since proposed scheme (11) and (12) depends on the filter parameters of the last instant such as $P_{\xi|k-1}$ and $\Delta \xi_{k-1}$, for practical implementation, it is better to use IKF with the available inaccurate $Q_w$ to help achieve a roughly stable state for the initialization of $P_{\xi|k-2}$, $\Delta \xi_{k-1}$ as the preparation in Algorithm 1. Note that, without the accurate $Q_w$ the optimal filtering state is almost impossible, but the rough steady and stable filtering can be achieved after some time instants (denoted as $k_0$) by tuning the initial guess for true $Q_w$. The parameter $k_0$ depends on the detailed problem at hand and is usually set to a value between two and three times of the state dimension (e.g. $k_0 = 8$ for attitude estimation in Section 4).
Theoretical derivation of the covariance adaption scheme

Theorem 1. For matrix Lie groups attitude model (1) and (2), if invariant Kalman filter is proceeding at its optimal steady state, according to Theorem 1, the matrix Lie groups system (1) and (2) can be regarded as a constant in optimal Kalman filtering will converge component-wise to the constant solution of all time instants could be regarded as a constant in optimal Kalman filter in [35] or constant gain Kalman filter in [36].

Proof. According to Theorem 1, the matrix Lie groups system (1) and (2) can be equivalently converted to Euclidean space model (3) and (6) as described in Section II, so the filtering parameters of invariant Kalman filter for model (1) and (2) are the same as classical Kalman filter for (3) and (6). Then, with Lemma 1 and 2, the assumptions about $P_{jk|k-1}$ being constant and $z_j$ being Gaussian white are valid for optimal filtering with (1) and (2).

Let $e_k = \{z_0, z_1, \ldots, z_{k-1}\}$ be the set of innovation sequences before instant $k$ and $\hat{P}_{jk|k-1}$ denote an estimate for the unknown $P_{jk|k-1}$ based on the set $e_k$, according to the above two lemmas the maximum likelihood function can be formulated as follows:

$L(\hat{P}_{jk|k-1}) = ln p(\xi | \hat{P}_{jk|k-1}) = ln \prod_{j=1}^{k-1} p(\zeta_j | \hat{P}_{jk|k-1})$

$= \sum_{j=1}^{k-1} ln p(\zeta_j | \hat{P}_{jk|k-1}) j = 1, 2, \ldots, k-1.$ (13)
Then the proof for the proposed covariance adaption scheme (11) and (12) is completed.

**Remark 6.** Although the two assumptions of constant $P_{j|j-1}$ and Gaussian white noise $z_j$ are strictly valid for the optimal steady filtering state, in [39, 40] they have already been used to seek an explicit and simple approximation for unknown parameters. To improve the quality of these approximations, the IKF is employed before starting the new approach as in Algorithm 1 to help the filter achieve a rough convergence, and the validity of this approach is demonstrated by the following simulations.

**4 | NUMERICAL SIMULATIONS**

To investigate the performance of the proposed AIKF for attitude estimation, the attitude model (1) and (2) is simulated using...
the parameter $R_{00} = 0.5236^2 I_{3x3}, Q_w = 0.01745^2 I_{3x3}, b' = [1, 0, 0]^T, b'' = [0, 1, 0]^T, Q_w = 0.01745^2 I_{3x3}, Q_w = 0.0873^2 I_{3x3}$. The total steps are set to 10,000 and the results of 1000 random simulation runs are evaluated. Note that, true attitude trajectories are generated with the real value of $Q_w$, but in practical applications there is no access to the accurate $Q_w$. In order to study the filtering adaptability to $Q_w$ of various quality, the filtering tests of AIKF, the QeIKF [38, 40] and IKF using the rough $\tilde{Q}_w = Q_w \times \text{diag}([\alpha, 1/\alpha, 1])$ with $\alpha$ ranging from 0.1 to 10 are implemented. To implement the AIKF and QeIKF, the filtering instants before $k_0$ is initialized using the IKF with the inaccurate $\tilde{Q}_w$ and, for this attitude model, at the time step $k_0 = 8$ the covariance parameters of IKF gradually converges. The error variable is expressed in Lie algebra and the root mean square error $\text{RMSE}_k$ at time instant $k$ and the average root mean square error $\text{ARMSE}$ is calculated as the performance metric, i.e.

$$\text{RMSE}_k = \sqrt{\frac{1}{1000} \sum_{l=1}^{1000} \left[ \left( \xi_{k,l}^{1} - \xi_{k,l}^{1*} \right)^2 + \left( \xi_{k,l}^{2} - \xi_{k,l}^{2*} \right)^2 \right]}$$

$$\text{ARMSE} = \sqrt{\frac{1}{10000 \times 10000} \sum_{k=1}^{10000} \sum_{l=1}^{10000} \left[ \left( \xi_{k,l}^{1} - \xi_{k,l}^{1*} \right)^2 + \left( \xi_{k,l}^{2} - \xi_{k,l}^{2*} \right)^2 + \left( \xi_{k,l}^{3} - \xi_{k,l}^{3*} \right)^2 \right]}$$

where the subscript $l$ marks the corresponding Monte Carlo simulation run and $k$ denotes the particular time instant, $\xi_{k,l}^{i}$.
is the first element of the estimated Lie algebra state vector, $\xi_{k,i}^*$ refers to the $i$th component of reference true state.

To study the performance of the proposed adaptation scheme (11), the covariance error index $\tilde{\lambda}_k$ are also calculated as

$$\tilde{\lambda}_k = \left( \frac{1}{n^2 5000} \sum_{i=1}^{5000} \left\| \hat{P}_{k|i-1} - \hat{P}_{\omega, k|i-1} \right\|^2 \right)^{1/4}$$

where $\|D\| = \text{tr}(DD^T)$, $\hat{P}_{k|i-1}$ is the prior error covariance $P_{k|i-1}$ in IKF and $QeIKF$ or the estimated $\hat{P}_{k|i-1}$ in AIKF at the $i$th simulation, and $\hat{P}_{\omega, k|i-1}$ is optimal $P_{k|i-1}$ of IKF with the real value of $Q_{\omega}$. It is obvious that, with a smaller covariance error index $\tilde{\lambda}_k$, the propagated $\hat{P}_{k|i-1}$ is closer to the optimal one and the estimation accuracy of the corresponding filtering approach is sure to be better. Generally, the covariance error index of an optimal IKF with accurate $Q_{\omega}$ will gradually converge to zero.

As the simulation result given in Figure 2, when the parameter $\alpha$ is around 1 the available $Q_{\omega}$ is close to the true value of $Q_{\omega}$, so the ARMSE of both IKF and AIKF are better than QeIKF and near the optimal; but, when $\alpha$ becomes larger or smaller, the accuracy of $Q_{\omega}$ is gradually destroyed and the ARMSE result of IKF increases significantly, certifying IKF's critical dependency on accurate $Q_{\omega}$. Although the ARMSE data of QeIKF is smaller than IKF for $\alpha < 0.5$ and $\alpha > 2$, but it is still prone to be influenced by the quality of the initial guess $\hat{Q}_{\omega}$. As to the new AIKF, although its ARMSE data is still related to $\alpha$, it is obvious that the ARMSE result of AIKF is rather less sensitive than other filters, which certifies AIKF's advantage in filtering adaptability to inaccurate $Q_{\omega}$.

For the cases of $\alpha = 10, 6, 0.4$ and 0.1, the process noise covariance $\hat{Q}_{\omega}$ of different accuracy are constructed; the RMSE$_{\omega}$ results during whole filtering processes of IKF, QeIKF and AIKF are displayed in Figure 3. Obviously, the smaller RMSE$_{\omega}$ data of AIKF shows that its filtering precision is better than QeIKF and IKF for all cases, which certifies AIKF's superior filtering adaptability to inaccurate $Q_{\omega}$.

The covariance error index $\tilde{\lambda}_k$ during the filtering process for $\alpha = 10, 6, 0.4$ and 0.1 are also presented in Figure 4. Note that, if given the accurate $Q_{\omega}$ the prior error covariance of IKF will converge to the optimal value and its covariance error index $\tilde{\lambda}_k$ will then become zero; but, with the $Q_{\omega}$ of different accuracy as $\alpha = 10, 6, 0.4$ and 0.1, the covariance error index $\tilde{\lambda}_k$ of IKF converge to some nonzero value as in Figure 4. The covariance error index $\tilde{\lambda}_k$ of QeIKF for $\alpha = 10, 6, 0.4$ and 0.1 gradually decreases as the filtering processes proceed, but in the four cases in Figure 4 the covariance error index $\tilde{\lambda}_k$ of the new approach AIKF is always the least, which again demonstrates that the prior error covariance estimated by the proposed scheme (11) significantly improves the filtering adaptability of AIKF to the $Q_{\omega}$ of different accuracy.

## 5 CONCLUSION

For Lie group attitude estimation problems with no accurate process noise statistics, this paper proposes an adaptive version of the invariant Kalman filtering method. To reduce the negative impact of inaccurate covariance parameters, a new covariance propagation scheme was elaborated based on feedback evaluation of the posterior sequence, which significantly improves the adaptability of the invariant Kalman filter and constitutes the main contribution of this work. The mathematical derivation of the proposed approach is also presented and the numerical simulations further certify AIKF's superior adaptability and accuracy.

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