Federated Learning with Nesterov Accelerated Gradient Momentum Method

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Abstract—Federated learning (FL) is a fast-developing technique that allows multiple workers to train a global model based on a distributed dataset. Conventional FL employs gradient descent algorithm, which may not be efficient enough. It is well known that Nesterov Accelerated Gradient (NAG) is more advantageous in centralized training environment, but it is not clear how to quantify the benefits of NAG in FL so far. In this work, we focus on a version of FL based on NAG (FedNAG) and provide a detailed convergence analysis. The result is compared with conventional FL based on gradient descent. One interesting conclusion is that as long as the learning step size is sufficiently small, FedNAG outperforms FedAvg. Extensive experiments based on real-world datasets are conducted, verifying our conclusions and confirming the better convergence performance of FedNAG.

Index Terms—Distributed machine learning, Federated learning, Edge computing, Momentum gradient descent, Nesterov accelerated gradient

I. INTRODUCTION

Recent advances in edge computing drastically motivate Federated Learning (FL). Since tremendous data are now generated at the network edge, conventional centralized machine learning is insufficient when a large volume and sensitive data are required to be uploaded to datacenters. Meanwhile, the development of hardware of edge devices (i.e., workers) improves their computing capability, allowing more complicated computing operations at edge devices. Conventional Federated Learning is based on gradient descent [1] at each edge device (worker): In each round, each worker locally updates its weights by gradient descent for a number of times by its local dataset, and then the aggregator averages the weights from all workers and distribute them to the workers again. The above process is repeated for multiple rounds. Such implementation is referred to as FedAvg [2]. However, one disadvantage of gradient descent is its low efficiency and potential in oscillations. Momentum is able to improve the situation by adding inertia to accelerate the convergence by dampening oscillations and causing the algorithm to barrel through narrow valleys, small humps, and local minima [3]–[8].

In this paper, we focus on the convergence analysis of FL where each worker updates its weights based on Nesterov Accelerated Gradient (NAG) [9] instead of gradient descent. NAG is known to be an advantageous form of momentum, compared with Polyak’s momentum [4] and gradient descent in centralized machine learning. Some existing papers [10]–[13] focused on the convergence analysis based on gradient descent in FL environment. Some papers [14]–[17] analyzed NAG in centralized learning environment. Some other papers [18], [19] studied applications of Polyak’s momentum in FL environment. However, no prior work has rigorously analyzed convergence of NAG in the FL environment. It is also not clear how to quantify the performance gap between gradient descent and NAG in the FL environment.

In this paper, we focus on a version of FL based on NAG, namely FedNAG: (1) Each worker locally updates its weights and momenta using NAG for \( \tau \) iterations on its local dataset; (2) the aggregator collects and averages the weights and momenta from all workers and distribute them to the workers again; (1) and (2) are repeated for multiple rounds. \( \tau \) is a factor to trade off local update and global aggregation. Larger \( \tau \) reduces the frequency of aggregations and thus reduces communication overhead, but it also lowers the efficiency of local updates, causing worse convergence performance.

We theoretically provide a detailed convergence analysis for FedNAG. The progress mainly includes three steps: (1) We define virtual update as if centralized NAG is conducted between two global aggregations; (2) We bound the gap of weights \( w \) between FedNAG update and virtual update; and (3) We bound the values of global loss functions \( F(w) \) between FedNAG and the optimal solution. Since the convergence analysis of FedAvg is provided in [13], we compare the convergence performance of FedNAG and FedAvg and derive the conditions that FedNAG outperforms FedAvg.

Experimentally, we use different models such as linear regression, logistic regression, and convolutional neural network (CNN) based on MNSIT and CIFAR-10 datasets, to test the performance of FedNAG. We analyze the impact of different factors such as number of workers, number of local updates between two global aggregations \( \tau \), and momentum coefficient. The experiment shows that FedNAG outperforms FedAvg under a wide range of settings.

II. SYSTEM MODEL AND PRELIMINARIES

A. Overview

In the context of federated edge learning, there are \( N \) workers, located at different sites and communicating with an aggregator to learn a model \( w^* \) which is a solution to the following problem

\[
\min_{w \in \mathbb{R}^d} F(w) \triangleq \frac{1}{D} \sum_{i=1}^{N} D_i F_i(w),
\]
where $D_i$ is the number of data samples in worker $i$; $D = \sum_{i=1}^N D_i$ is the total number of data samples; and $d$ is the dimension of $w_i$. $F_i(\cdot)$ is the local loss function at worker $i$ and $F(\cdot)$ is the global loss function. We assume $F_i(\cdot)$ satisfies the following conditions.

1) $F_i(w)$(w) is convex.
2) $F_i(w)(w)$ is $\rho$-Lipschitz, i.e., $\|F_i(w_1) - F_i(w_2)\| \leq \rho \|w_1 - w_2\|$ for any $w_1, w_2$.
3) $F_i(w)(w)$ is $\beta$-smooth, i.e., $\|\nabla F_i(w_1) - \nabla F_i(w_2)\| \leq \beta \|w_1 - w_2\|$ for any $w_1, w_2$.

The above assumptions are widely adopted in a range of literature [13], [19]–[21].

B. Algorithm

**Algorithm 1: FedNAG**

Input: $\tau$, $T = K \tau$  
Output: Global model parameter $w^f$

1. Initialize $v_i(0) = 0$, and $w_i(0)$ as same value for all $i$
2. for $t = 1, 2, \ldots, T$ do
3. For each worker $i$ in parallel, compute its local update as (2) and (3).
4. if $t = k \tau$ where $k$ is a positive integer then
5. Aggregate $v(t)$ and $w(t)$ as (4) and (5).
6. Set $v_i(t)$ into $v(t)$ and $w_i(t)$ into $w(t)$ for all $i$
7. end
8. end
9. Set $w^f$ as (6).

Algorithm 1 shows NAG in Federated Learning. We use $w_i(t)$ and $v_i(t)$ to denote the model parameter and momentum parameter in worker $i$ at $t$th iteration. Initially, at $t = 0$, we set $v_i(0) = 0$ and a same $w_i(0)$ for all $i$. Each $\tau$ iterations will lead to a global aggregation.

Each iteration includes a local update, followed by a global aggregation if $t = k \tau$, $k = 1, 2, \ldots$

1) **Local Updates:** In each iteration, the following update is conducted in each worker $i$,

$$v_i(t) \leftarrow \gamma v_i(t-1) - \eta \nabla F_i(w_i(t-1)),$$

$$w_i(t) \leftarrow w_i(t-1) - \gamma v_i(t-1) + (1 + \gamma) v_i(t) = w_i(t-1) + \gamma v_i(t) - \eta \nabla F_i(w_i(t-1)).$$

Fig. 1 gives us a graphic view of local update and red line shows the illustration for $w_i(t)$. $v_i(t)$ are momentum terms. The above updates follow [22], [23].

2) **Global Aggregation:** If $t = k \tau$, $k = 1, 2, \ldots$, all workers will send $v_i(t)$ and $w_i(t)$ values to the aggregator and the aggregator calculates $v(t)$ and $w(t)$ as follows:

$$v(t) \leftarrow \frac{\sum_{i=1}^N D_i v_i(t)}{D},$$

$$w(t) \leftarrow \frac{\sum_{i=1}^N D_i w_i(t)}{D}.$$  

Then aggregator will send back $v(t)$ and $w(t)$ to each worker $i$ to update $v_i(t) \leftarrow v(t)$ and $w_i(t) \leftarrow w(t)$.

Note that only if $t = k \tau$, $v(t)$ and $w(t)$ are aggregated in (4) and (5). For the purpose of analysis, we define $v(t) = \sum_{i=1}^N D_i v_i(t)$ and $w(t) = \sum_{i=1}^N D_i w_i(t)$ at any iteration $t$ so that $v(t)$ and $w(t)$ can be used for convergence analysis.

After $T = K \tau$ iterations, the output $w^f$ is computed as follows:

$$w^f \triangleq \arg \min_{w \in \{w(k\tau); k = 1, 2, \ldots, K\}} F(w).$$

C. Preliminary Analysis

We present some simple preliminary analyses, which will be used in the rest of the paper. We also list important notations in Table I.

1) **Property of $F(w)$:** First, according to the assumptions, it is straightforward to show that $F(w)$ is convex, $\rho$-Lipschitz and $\beta$-smooth by applying triangle inequalities.
2) Divergence of Gradient: The divergence of gradient, which is commonly adopted in convergence analysis [13], [19], [21] can be defined as follows.

**Definition 1. (Gradient Divergence)** For \( i \) and \( \forall w \), we define \( \delta_i \) as the upper bound between \( \nabla F_i(w) \) and \( \nabla F(w) \), i.e.,

\[
\|\nabla F_i(w) - \nabla F(w)\| \leq \delta_i.
\]

We also define

\[
\delta \triangleq \sum_i D_i \delta_i / D.
\]

Please note that \( \delta_i \) is different at different workers, indicating the datasets at different workers may not be independent and identically distributed [13].

3) Virtual Updates: We use \([k]\) to denote interval \( t \in [(k - 1)\tau, k\tau] \) for \( k = 1, 2, 3, \ldots, K \). It shows \( \tau \) iterations within two global aggregations.

In each interval \([k]\), first, at \((k - 1)\tau\), we set

\[
\begin{align*}
    v_{[k]}((k - 1)\tau) &\leftarrow v((k - 1)\tau), \\
    w_{[k]}((k - 1)\tau) &\leftarrow w((k - 1)\tau).
\end{align*}
\]

\( v_{[k]}((k - 1)\tau) \) and \( w_{[k]}((k - 1)\tau) \) are set as the aggregated values right after the global aggregation is conducted.

Second, starting from the aggregated values, we consider virtual updates as if centralized NAG is adopted. In iterations \((k - 1)\tau < t \leq k\tau\), we conduct

\[
\begin{align*}
    v_{[k]}(t) &\leftarrow \gamma v_{[k]}(t - 1) - \eta \nabla F(w_{[k]}(t - 1)), \\
    w_{[k]}(t) &\leftarrow w_{[k]}(t - 1) - \gamma v_{[k]}(t - 1) + (1 + \gamma) v_{[k]}(t) \\
    &\quad - \eta \nabla F(w_{[k]}(t - 1)).
\end{align*}
\]

We repeat the above process for each \([k]\). These \( w_{[k]}(t) \) and \( v_{[k]}(t) \) are virtual values assuming there is a centralized update. They are used to bound the gap to prove the convergence shortly. Please note that \( w_{[k]}(k\tau) \) and \( w_{[k+1]}(k\tau) \) are different. \( w_{[k]}(k\tau) \) is calculated from \( w_{[k]}((k - 1)\tau) \) after \( \tau \) iterations of centralized update, and \( w_{[k+1]}(k\tau) \) is directly given by \( w(k\tau) \). Figs. 2 and 3 illustrate the evolution of \( w_{[k]}(k\tau) \) and \( F(w_{[k]}(k\tau)) \) respectively.

### III. CONVERGENCE ANALYSIS OF FedNAG

In this section, we provide detailed convergence analysis of FedNAG. This includes two steps: We first bound the gap of the weight \( w \) between FedNAG and virtual updates; Then we bound the loss function \( F(w) \) between FedNAG and the optimal solution.

A. Bounding \( \|w(t) - w_{[k]}(t)\| \)

We firstly analyze the upper bound between \( w(t) \) and \( w_{[k]}(t) \), leading to the following theorem.

**Theorem 1.** For any interval \([k]\), \( \forall t \in [k] \), we have:

\[
\|w(t) - w_{[k]}(t)\| \leq h(t - (k - 1)\tau),
\]

where we define

\[
\begin{align*}
    A &\triangleq \frac{(1 + \eta\beta)(1 + \gamma) + \sqrt{(1 + \eta\beta)^2(1 + \gamma)^2 - 4\gamma(1 + \eta\beta)}}{2\gamma}, \\
    B &\triangleq \frac{(1 + \eta\beta)(1 + \gamma) - \sqrt{(1 + \eta\beta)^2(1 + \gamma)^2 - 4\gamma(1 + \eta\beta)}}{2\gamma}, \\
    E &\triangleq \frac{\gamma A + A - 1}{(A - B)(\gamma A - 1)}, \\
    F &\triangleq \frac{\gamma B + B - 1}{(A - B)(1 - \gamma B)},
\end{align*}
\]

and \( h(x) \) yields

\[
\begin{align*}
    h(x) &= \eta\delta \left[ E(\gamma A)^x + F(\gamma B)^x - \frac{1}{\eta\beta} \right. \\
    &\left. - \gamma^2(\gamma^x - 1) - (\gamma - 1)x \right] / (\gamma - 1)^2.
\end{align*}
\]

for \( 0 < \gamma < 1 \) and any \( x = 0, 1, 2, \ldots \)

We note that \( F(w) \) is \( \rho \)-Lipschitz, so we also have:

\[
F(w(t)) - F(w_{[k]}(t)) \leq \rho h(t - (k - 1)\tau).
\]

**Proof.** See Appendix for detailed proof.

We have the following observations on Theorem 1.

1) **Monotone of \( h(x) \).** \( h(0) = h(1) = 0 \) and \( h(x) \) increases with respect to integer \( x \) for \( x \geq 1 \). See Appendix for detailed proof.

2) **Property of \( h(0) \).** When \( x = 0 \), we have \( t = (k - 1)\tau \) (the beginning of interval \([k]\)) and the upper bound in (13) is 0. This is consistent with (9) and (10) for any \( k \).
Then we also define with 

\[ B. \text{Bounding gap of } h. \]

\[ \text{Property of } h, \] when \( x = 1, \) we have \( t = (k - 1)\tau + 1 \) (the beginning of second iteration of interval \([k]\)) and the upper bound in (13) is still zero. It is easy to verify that if all workers conduct global aggregation right after the end of the first local iteration, there is no gap between FedNAG and centralized NAG.

\[ \text{Property of } \tau = 1. \] When \( \tau = 1, \) we have \( t - (k - 1)\tau = 0 \) or 1. Thus, for any interval \( k \) and \( t \in [k], \) the gap in (13) and (15) is always zero. This means that FedNAG is equivalent to centralized NAG when there is only one local update step between two global aggregation steps. See Appendix for detailed discussion.

\[ \text{Property of } \tau > 1. \] When \( \tau > 1, \) because \( t \in [(k - 1)\tau, k\tau], \) we have \( x = t - (k - 1)\tau \in [0, \tau]. \) Thus, the value of \( x \) could be larger when \( \tau \) is large. According to the definitions of \( A, B, E, \) and \( F, \) we can see that \( \gamma A > 1, 0 < \gamma B < 1, E > 0, F > 0. \) When \( x \) is large, because \( 0 < \gamma < 1, \) the last term in (14) will linearly decrease with respect to \( x. \) Therefore, for (14), \( E(\gamma A)^x \) dominates when \( x \) is large that means the upper bound in (13) will be exponentially increased with \( t \in [k]. \)

\[ \text{Impact of } \delta, \] \( h(x) \) increases linearly with respect to \( \delta. \) The value of \( \delta \) reflects the difference of data distribution in each worker. Larger divergence of data distribution leads to larger gap of \( h(x). \)

\[ \text{B. Bounding } F(w(T)) - F(w^*). \]

For convenience, we use \( \theta_{[k]}(t) \) to denote the angle between vector \( -\nabla F(w_{[k]}(t)) \) and \( v_{[k]}(t) \) for \( t \in [k], \)

\[
\cos \theta_{[k]}(t) \triangleq \frac{-\nabla F(w_{[k]}(t))^T v_{[k]}(t)}{\|\nabla F(w_{[k]}(t))\| \|v_{[k]}(t)\|}.
\]

\( \theta \) is defined as the maximum value of \( \theta_{[k]}(t) \) for \( k \in [1, K] \) with \( t \in [k] \)

\[
\theta \triangleq \max_{k \in [1, K], t \in [k]} \theta_{[k]}(t).
\]

Then we also define

\[
p \triangleq \max_{k \in [1, K], t \in [k]} \frac{\|\gamma v_{[k]}(t)\|}{\|\eta \nabla F(w_{[k]}(t))\|},
\]

\[
q \triangleq \min_{k \in [1, K], t \in ((k - 1)\tau, k\tau]} \frac{\|\nabla F(w_{[k]}(t - 1))\|}{\|\nabla F(w_{[k]}(t))\|},
\]

\[
\omega \triangleq \min_{k \in [1, K], t \in [k]} \frac{1}{\|w_{[k]}(t) - w^*\|^2}.
\]

We can obtain the following theorem to get the upper bound as follows.

**Theorem 2.** When all the following conditions are satisfied:

1. \( \cos \theta \geq 0, 0 < \beta \eta (\gamma + 1) \leq 1 \) and \( 0 \leq \gamma < 1, \)
2. \( \omega \alpha - \frac{\rho h(\tau)}{\tau^2} > 0, \)
3. \( F(w_{[k]}(k\tau)) - F(w^*) \geq \epsilon \) for all \( k, \)
4. \( F(w(T)) - F(w^*) \geq \epsilon, \)

for some \( \epsilon > 0, \) the convergence upper bound of Algorithm 1 after \( T \) iterations is given by

\[
F(w(T)) - F(w^*) \leq \frac{1}{T} \left( \omega \alpha - \frac{\rho h(\tau)}{\tau} \right),
\]

where we define

\[
\alpha \triangleq \eta (\gamma + 1) \left( 1 - \frac{\beta \eta (\gamma + 1)}{2} - \frac{\beta \eta^2 \gamma^2 \rho^2}{2} + \frac{\gamma^2 \eta (1 - \beta \eta (\gamma + 1)) \cos \theta}{2} \right).
\]

**Proof.** See Appendix for detailed proof.

Through Theorem 2, we can further obtain the following bound between \( F(w^f) \) and \( F(w^*). \)

**Theorem 3.** When \( \cos \theta \geq 0, 0 < \beta \eta (\gamma + 1) \leq 1, \) and \( 0 \leq \gamma < 1, \) we have

\[
F(w^f) - F(w^*) \leq \frac{1}{2T} \omega \alpha + \sqrt{\frac{1}{4T^2 \omega^2 \alpha^2} + \frac{\rho h(\tau)}{\omega \alpha \tau} + \rho h(\tau)}.
\]

**Proof.** See Appendix for detailed proof.

Please note we have the following observations on Theorem 3.

\[ \text{Effect of } \tau. \] From Appendix, we have known that \( h(\tau) \geq 0 \) and increases with integer \( \tau. \) Thus, for a given \( T, \) the convergence upper bound becomes larger when \( \tau \) is larger.

\[ \text{Property of } \tau = 1. \] When \( \tau = 1, \) we have \( h(\tau) = 0. \) We can observe that the gap converges to zero when \( T \to \infty. \) This means if we conduct global aggregation after every local update, \( F(w(t)) \) will converge to the optimal solution.

\[ \text{Property of } \tau > 1. \] When \( \tau > 1, \) we have \( h(\tau) > 0. \) We can observe that the gap converges to a non-zero gap \( \sqrt{\frac{\rho h(\tau)}{\omega \alpha \tau} + \rho h(\tau)} \) when \( T \to \infty. \) This means if we conduct global aggregation after multiple local updates, there is a non-zero gap to the optimal solution.

\[ \text{Tradeoff between communication and convergence.} \]

Based on the Observations 2 and 3 above, \( \tau = 1 \) gives the best convergence performance. However, by doing so, it will increase the communication frequency. This will lead to a tradeoff between communication overhead and convergence performance. In this paper, we do not model the costs and utilities of communication overhead (in different types of distributed systems) and convergence performance, so that the optimal tradeoff is left for future work.

\[ \text{Effect of } \delta. \] Following the Observation 6 of Theorem 1, the convergence upper bound will be increased when \( \delta \) is getting larger.

IV. COMPARISON BETWEEN FEDAVG AND FEDNAG

In this section, we compare the performance between FedAvg and FedNAG. The convergence upper bound of FedAvg has been derived in Theorem 2 in [13] as follows:

\[
F(\hat{w}) - F(w^*) \leq \frac{1}{2T} \omega \alpha + \sqrt{\frac{1}{4T^2 \omega^2 \alpha^2} + \frac{\rho h(\tau)}{\omega \alpha \tau} + \rho h(\tau)},
\]

where we define

\[
\alpha \triangleq \eta (\gamma + 1) \left( 1 - \frac{\beta \eta (\gamma + 1)}{2} - \frac{\beta \eta^2 \gamma^2 \rho^2}{2} + \frac{\gamma^2 \eta (1 - \beta \eta (\gamma + 1)) \cos \theta}{2} \right).
\]

**Proof.** See Appendix for detailed proof.
where
\[
\hat{h}(\tau) = \frac{\delta}{\beta} ((\eta\beta + 1)^{\tau} - 1) - \eta\delta\tau, \\
\alpha = \eta \left(1 - \frac{\beta\eta}{2}\right).
\]

Please note that \(\rho, \beta, \tau, \omega,\) and \(\eta\) are defined the same way as those in FedNAG in this paper. \(\alpha\) and \(h(\cdot)\) are defined differently, but with similar meanings as \(\alpha\) and \(h(\cdot)\) in this paper.

In order to make a fair comparison, we let FedAvg and FedNAG trained under the same environment using the same configuration. Here, we note that \(\delta\) and \(\omega\) reflect the properties of data distribution. We assume the dataset is distributed in the same way in FedAvg and FedNAG, so that the values of \(\omega\) and \(\delta\) are same. The loss function \(F_i(\cdot), F(\cdot),\) constants \(\rho\) and \(\beta,\) and hyper-parameters \(\tau\) and \(\eta\) are the same. We also set the same initial value for \(w^1, w_i(0)\) for FedAvg and FedNAG. The only new term in FedNAG is \(v_i(t),\) and we set \(v_i(0) = 0.\)

We use \(f_1(T)\) and \(f_2(T)\) to define the convergence upper bound of FedNAG and FedAvg respectively. Small function value implies better convergence performance.

\[
f_1(T) \triangleq \frac{1}{2T\omega\alpha} + \sqrt{\frac{1}{4T^2\omega^2\alpha^2} + \frac{\rho h(\tau)}{\omega\alpha\tau} + \rho h(\tau)}, \\
f_2(T) \triangleq \frac{1}{2T\omega\alpha} + \sqrt{\frac{1}{4T^2\omega^2\alpha^2} + \frac{\rho h(\tau)}{\omega\alpha\tau} + \rho h(\tau)}.
\]

To prevent the gradient descent from overshooting the minimum or failing to converge [24], we choose a sufficiently small \(\eta\) to guarantee the convergence of FedNAG and FedAvg. The following conclusion is made when \(\eta \rightarrow 0^+.\)

**Theorem 4.** When \(\cos \theta \geq 0, 0 < \beta\eta(\gamma + 1) \leq 1\) and \(0 < \gamma < 1,\) FedNAG outperforms FedAvg, i.e.,

\[f_1(T) < f_2(T)\]

for any \(T\) and an arbitrarily small \(\eta \rightarrow 0^+.\)

**Proof.** See Appendix for detailed discussion.

Please note we have the following observations on Theorem 4.

1. **Simplified conditions.** As mentioned in previous section, \(\theta\) is the maximum value of angle between descent direction and momentum (velocity) direction. From Appendix, \(\cos \theta \geq 0\) is always true if \(\eta \rightarrow 0^+\) (since \(v_i(0) = 0\)). Also, \(0 < \beta\eta(\gamma + 1) \leq 1\) is true when \(\eta \rightarrow 0^+\) as \(\beta\) is a positive finite number. Therefore, as long as \(\eta \rightarrow 0^+\) and \(0 < \gamma < 1,\) FedNAG convergence performance is better than FedAvg.

2. **Discussion of \(\eta.\)** In Theorem 4, we set \(\eta \rightarrow 0^+.\) Actually, there exists a threshold value for \(\eta\) called \(\tilde{\eta}\). If \(\eta < \tilde{\eta},\) \(\cos \theta \geq 0, 0 < \beta\eta(\gamma + 1) \leq 1,\) and \(0 < \gamma < 1,\) then \(f_1(T) < f_2(T)\) is still true. Numerical method can be used to calculate the value of \(\tilde{\eta}\).

**V. Experiments**

In this section we evaluate the convergence performance of FedNAG compared with benchmark algorithms including FedAvg, centralized SGD (cSGD) and centralized NAG (cNAG) by real-world experiments. We then discuss the impacts of hyper-parameters, including global aggregation frequency \(\tau,\) momentum coefficient \(\gamma,\) and number of workers \(N.\)

**A. Experimental Setup**

In order to evaluate the convergence performance of FedNAG, we employ two real-world datasets including “MNIST” [25] and “CIFAR-10” [26] for image classification. While MNIST dataset contains gray-scale images of 70,000 samples (60,000 for training and 10,000 for testing), CIFAR-10 contains 60,000 color images (50,000 for training and 10,000 for testing). In our experiment, all samples in MNIST and CIFAR-10 are evenly distributed in each worker. We implement FedNAG and other benchmarks using PySyft library [27] based on the PyTorch framework. PySyft can emulate various virtual workers to process federated learning jobs. The training process is run on a CPU tower server (Intel(R) Xeon(R) Gold 6252 CPU @2.10GHz 24 cores, and 32GB DDR4 2933 MHz memory, running Ubuntu 18.04.5 LTS).

We use three models including linear regression model, logistic regression model, and Convolutional Neural Network (CNN) model. Linear regression uses mean squared error loss, and logistic regression uses cross-entropy loss. The CNN model’s structure is similar to the classic one in [28], which has two \(5 \times 5\) convolutional layers with 32 and 64 channels respectively. In each convolutional layer, \(2 \times 2\) max pooling is used. The last two following layers are ReLu activation and softmax. We use mini-batch in all experiments, and the batch size is 64. We set the default learning step size \(\eta = 0.01.\) Other parameters will be specified in each experiment. We also note that the number of global aggregation iterations is \(K = T/\tau.\)

**B. Performance Evaluation**

1. **Convergence Performance:** In Fig. 4, we compare the convergence performance of FedNAG with other three benchmarks. The experiment is performed on two datasets. MNIST is trained by linear regression, logistic regression, and CNN; and CIFAR-10 is trained by CNN. The setting in this experiment is \(\tau = 4, \gamma = 0.9, N = 4.\) For MNIST, the total number of iterations \(T\) is 1000. For CIFAR-10, \(T\) is set to 10000.

Figs. 4(a), 4(b), 4(c), and 4(d) show the values of the global loss function and accuracy trained under different models and datasets respectively. In general, we have cNAG > FedNAG > cSGD > FedAvg.

For centralized approaches, we can see cNAG performs better than cSGD in all cases. For distributed approaches, FedNAG also performs better than FedAvg. It confirms that NAG is more advantageous compared with gradient descent for both centralized and federated learning environment.

For cNAG and FedNAG, we can find FedNAG performs worse. This follows our expectation shown in Theorem 3. FedNAG performs \(\tau\) local updates before a global aggregation,
(a) Linear regression on MNIST

(b) Logistic regression on MNIST

(c) CNN on MNIST

(d) CNN on CIFAR-10

Fig. 4. Convergence performance with benchmark algorithms

(a) Effect of aggregation frequency $\tau$

(b) Total iterations when global loss reaches $0.5$ for different $\tau$

(c) Total iterations when accuracy reaches $85\%$ for different $\tau$

(d) Effect of momentum coefficient $\gamma$

(e) Global loss when $T$ reaches $500$ and $1000$ when $0 < \gamma < 1$

(f) Effect of momentum coefficient $\gamma$ when $\gamma = 1$

(g) Effect of number of workers $N$

Fig. 5. Effect of $\tau$, $\gamma$, and $N$ when CNN trained on MNIST
causing less efficient updates and thus decreases the convergence performance.

Another interesting observation is that FedNAG can perform better than cSGD in the four cases: The benefits of the momentum method can outweigh the performance loss by federated learning.

2) Effects of Global Aggregation Frequency $\tau$: In Figs. 5(a), 5(b) and 5(c), we evaluate the impact of $\tau$ based on global loss and accuracy using the same CNN model and MNIST dataset. The setting for this experiment is $\gamma = 0.5, T = 1000, N = 4$.

From Fig. 5(a), we can observe when $\tau$ is increased, the convergence performance is reduced. With the same $T$, loss is larger and accuracy is lower. This matches $\frac{1}{2}$ of Theorem 3. The convergence upper bound increases with $\tau$.

In Figs. 5(b) and 5(c), we observe the impact of $\tau$ in a wider range $[5, 640]$. In Fig. 5(b), we plot the number of iterations when the global loss reaches the target value 0.5. In Fig. 5(c), we plot the number of iterations when the accuracy reaches the target value 85%. Since the global loss and accuracy may oscillate during the training process, the target global loss and accuracy may be reached several times. The red horizontal lines indicate the first and last iterations when the target values are reached, and the bar indicates the mean of the iterations when the targets are reached.

The outcome shows that larger $\tau$ causes more iterations for convergence. If we double $\tau$ when $\gamma$ is small, the number of iterations to reach the targets does not increase much. However, if we double $\tau$ when $\gamma$ is larger (e.g., $\gamma \geq 0.8$), then the number of iterations to reach the targets substantially increases. This matches $\frac{5}{2}$ of Theorem 1, which concludes that larger $\tau$ leads to exponential increase of $h(\cdot)$. Therefore, increasing $\tau$ will more significantly delay the training process when $\tau$ is large.

3) Effects of Momentum Coefficient $\gamma$: In Figs. 5(d), 5(e) and 5(f), we evaluate the effects of $\gamma$. The setting for this experiment is $\tau = 4, T = 1000, N = 4$. We use the same CNN model trained on the same MNIST dataset.

Fig. 5(d) shows the global loss and accuracy under $\gamma = 0.1, 0.3, 0.6, 0.9$ respectively. It shows that $\gamma$ can increase the convergence performance (smaller global loss value and higher accuracy).

For Fig. 5(e), we evaluate the global loss at $T = 500$ and $T = 1000$ respectively, when $\gamma$ ranges from $[0, 0.99]$. Two horizontal lines are the benchmarks where only FedAvg is used. For both $T = 500$ and $T = 1000$, we can see the global loss decreases when $\gamma$ is getting large. Accuracy is also increased at the same time. However, from Fig. 5(f), when $\gamma = 1$, the global loss cannot converge due to the prerequisite where $0 < \gamma < 1$.

4) Effects of Number of Workers $N$: In Fig. 5(g), we evaluate the global loss and accuracy based on different number of workers $N$ using the same CNN model and MNIST dataset. The experiment setting is $\tau = 4, \gamma = 0.5, T = 2000$. From Fig. 5(g), we can see that increasing $N$ will cause a decline of convergence performance. This follows our expectation because more workers cause more divergence among the workers and thus decrease convergence performance. However, after a sufficient number of iterations, the global loss and accuracy with more workers will be closer to those with fewer workers. It shows that FedNAG is applicable when there are more workers in the system.

VI. CONCLUSION

In this paper, we focus on FedNAG with detailed convergence analysis. FedNAG allows each worker to update its weights and momenta by its local dataset for a number of local iterations between two global aggregations. On the global aggregation step, the aggregator collects and averages the weights and momenta from all workers and distributes them to the workers. The convergence analysis shows the upper bound of the gap between the global loss function derived by FedNAG at iteration $T$ and the optimal solution.

We compare FedNAG and FedAvg and conclude that as long as the learning step size is sufficiently small, FedNAG performs better than FedAvg. Using the PyTorch with PySyft framework, we validate theoretical results by experiments on real-world datasets.

APPENDIX

A. FedNAG vs. Centralized NAG (Observation $\frac{3}{2}$ in Theorem 1)

Proposition 1. When $\tau = 1$, FedNAG is equivalent to centralized NAG. The update rules of FedNAG yield as follows:

$$v(t) = \gamma v(t-1) - \eta \nabla F(w(t-1)),$$

$$w(t) = w(t-1) - \gamma v(t-1) + (1+\gamma)v(t)$$

$$= w(t-1) + \gamma v(t) - \eta \nabla F(w(t-1)).$$

Proof. When $\tau = 1$, we have $v_i(t) = v(t)$ and $w_i(t) = w(t)$ for all $t$. Thus,

$$v(t) = \frac{\sum_{i=1}^{N} D_i v_i(t)}{D} = \frac{\sum_{i=1}^{N} D_i (\gamma v_i(t-1) - \eta \nabla F_i(w_i(t-1)))}{D}$$

$$= \gamma v(t-1) - \frac{\sum_{i=1}^{N} D_i \nabla F_i(w_i(t-1))}{D}$$

$$= \gamma v(t-1) - \eta \nabla F(w(t-1)),$$

where the last term in the last equality is because

$$\frac{\sum_{i=1}^{N} D_i \nabla F_i(w)}{D} = \nabla \left( \frac{\sum_{i=1}^{N} D_i F_i(w)}{D} \right) = \nabla F(w)$$

based on the linearity of the gradient operator. Then,

$$w(t) = \frac{\sum_{i=1}^{N} D_i w_i(t)}{D} = \frac{\sum_{i=1}^{N} D_i (w_i(t-1) - \gamma v_i(t-1) + (1+\gamma)v_i(t))}{D}$$

$$= w(t-1) - \gamma v(t-1) + (1+\gamma)v(t).$$

Therefore, Proposition 1 has been proven. □
B. Proof of Theorem 1

To prove Theorem 1, the progress mainly includes four steps. (1) We first introduce an important equality in Lemma 1, which will be used later. (2) We bound \( \|w_i(t) - w_{[k]}(t)\| \) in Lemma 2 based on Lemma 1. (3) Based on the result of Lemma 2, we then bound \( \|v(t) - v_{[k]}(t)\| \) in Lemma 3. (4) Finally, based on the result of Lemma 3, we bound \( \|w(t) - w_{[k]}(t)\| \), which concludes Theorem 1.

**Lemma 1.** Given

\[
a_t = \frac{\delta_i}{\beta} \left( \frac{1 + \eta^2 \beta^2 + \gamma \eta^2}{\gamma^2} A - \frac{1 + \eta^2 \beta^2 + \gamma \eta^2}{\gamma^2} A - B^t \right),
\]

\[
A + B = \frac{1 + \eta \beta + \eta \beta \gamma + \gamma}{\gamma} (1 + \gamma \beta) (1 + \gamma),
\]

\[
AB = \frac{1 + \eta \beta \gamma}{\gamma},
\]

where \( t = 0, 1, 2, ..., 0 < \gamma < 1, \eta \beta > 0 \), we have

\[
(1 + \eta \beta) a_{t-1} + \eta \beta \gamma t \sum_{i=0}^{t-1} a_i = \gamma a_t.
\]

**Proof of Lemma 1.** For convenience, we define

\[
C \triangleq \frac{1 + \eta^2 \beta^2 + \gamma \eta^2}{\gamma^2} A - \frac{1 + \eta^2 \beta^2 + \gamma \eta^2}{\gamma^2} A - B^t
\]

\[
D \triangleq \frac{A - \frac{1 + \eta^2 \beta^2 + \gamma \eta^2}{\gamma^2}}{A - B}
\]

Therefore,

\[
a_t = \frac{\delta_i}{\beta} (CA^t + DB^t).
\]

According to the inverse theorem of Vieta’s formulas, we have

\[
\gamma x^2 - (1 + \eta \beta + \eta \beta \gamma + \gamma) x + \eta \beta + 1 = 0,
\]

where \( x \) values are the roots of quadratic equation. Here, the discriminant of the quadratic equation is positive.

\[
\Delta = (1 + \eta \beta + \eta \beta \gamma + \gamma)^2 - 4(1 + \eta \beta) \gamma
\]

\[
> (1 + \eta \beta + \gamma)^2 - 4(1 + \eta \beta) \gamma
\]

\[
= ((1 + \eta \beta) - \gamma)^2 > 0.
\]

Thus, \( A \) and \( B \) (roots) can be expressed as follows:

\[
A = \frac{(1 + \eta \beta)(1 + \gamma) + \sqrt{(1 + \eta \beta)^2(1 + \gamma)^2 - 4(1 + \eta \beta) \gamma}}{2 \gamma},
\]

\[
B = \frac{(1 + \eta \beta)(1 + \gamma) - \sqrt{(1 + \eta \beta)^2(1 + \gamma)^2 - 4(1 + \eta \beta) \gamma}}{2 \gamma}.
\]

Then we have

\[
(1 + \eta \beta) a_{t-1} + \eta \beta \gamma \sum_{i=0}^{t-1} a_i = \gamma a_t,
\]

\[
= (1 + \eta \beta) \frac{\delta_i}{\beta} \left( CA^{t-1} + DB^{t-1} \right) + \eta \beta \gamma \frac{\delta_i}{\beta} C A^t - \frac{1}{A - 1}
\]

\[
+ \eta \beta \gamma \frac{\delta_i}{\beta} D B^t - \frac{1}{B - 1} - \gamma \frac{\delta_i}{\beta} C A^t - \gamma \frac{\delta_i}{\beta} D B^t
\]

\[
= \frac{\delta_i}{\beta} \left[ \frac{A^{t-1} C}{1 - A} \left( \gamma A^2 - (1 + \eta \beta + \eta \beta \gamma + \gamma) A + 1 + \eta \beta \right)
\]

\[
+ \frac{B^{t-1} D}{1 - B} \left( \gamma B^2 - (1 + \eta \beta + \eta \beta \gamma + \gamma) B + 1 + \eta \beta \right) - \gamma \frac{\delta_i}{\beta} \eta \beta \gamma \left( \frac{C}{A - 1} + \frac{D}{B - 1} \right)
\]

\[
= 0 - \eta \delta_i \gamma \left( \frac{C}{A - 1} + \frac{D}{B - 1} \right)
\]

\[
= 0.
\]

(because \( A, B \) satisfy (26)).

Thus, Lemma 1 has been proven.

**Lemma 2.** For any interval \([k], \forall t \in [(k - 1) \tau, k \tau]\), we have

\[
\|w_i(t) - w_{[k]}(t)\| \leq f_i(t - (k - 1) \tau),
\]

where we define the function \( f_i(x) \) as

\[
f_i(x) \triangleq \frac{\delta_i}{\beta} (\gamma (CA^x + DB^x) - 1).
\]

**Proof of Lemma 2.** When \( t = (k - 1) \tau \), we know \( w_i(t) = w_{[k]}(t) \) by the definition of \( w_{[k]}(t) \) and aggregation rules. Hence, we have \( \|w_i(t) - w_{[k]}(t)\| = 0 \). Meanwhile, when \( t = (k - 1) \tau, x = 0 \) and \( f_i(0) = 0 \). Thus, Lemma 2 holds.

When \( t \in ((k - 1) \tau, k \tau] \), we bound the momentum gap

\[
\|v_i(t) - v_{[k]}(t)\|
\]

\[
= \|\gamma v_i(t - 1) - \eta \nabla F_i(w_i(t - 1)) - \eta v_{[k]}(t - 1) - \eta \nabla F(w_{[k]}(t - 1))\|
\]

\[
= \|\gamma (v_i(t - 1) - v_{[k]}(t - 1)) - \eta \nabla F_i(w_i(t - 1)) - \nabla F_i(w_i(t - 1)) - \nabla F(w_i(t - 1))\|
\]

\[
= \|\gamma (v_i(t - 1) - v_{[k]}(t - 1)) - \eta \nabla F_i(w_i(t - 1)) - \nabla F(w_i(t - 1))\|
\]

\[
\leq \gamma \|v_i(t - 1) - v_{[k]}(t - 1)\|
\]

\[
+ \eta \|\nabla F_i(w_i(t - 1)) - \nabla F(w_i(t - 1))\|
\]

\[
+ \eta \|\nabla F_i(w_{[k]}(t - 1)) - \nabla F(w_{[k]}(t - 1))\|
\]

\[
= \|v_i(t - 1) - v_{[k]}(t - 1)\| + \eta \beta \|w_i(t - 1) - w_{[k]}(t - 1)\| + \eta \delta_i.
\]

(from triangle inequality)

\[
\leq \gamma \|v_i(t - 1) - v_{[k]}(t - 1)\|
\]

\[
+ \eta \beta \|w_i(t - 1) - w_{[k]}(t - 1)\| + \eta \delta_i,
\]

(from \( \beta \)-smoothness and (7))
We use \( \gamma_0, \gamma_1, \ldots, \gamma^{t-(k-1)\tau-1} \) as multipliers to multiply (31) when \( t, t-1, \ldots, (k-1)\tau + 1 \), respectively.

\[
\|v_i(t) - v_{|k|}(t)\| \leq \gamma \|v_i(t-1) - v_{|k|}(t-1)\| + \eta \beta \|w_i(t-1) - w_{|k|}(t-1)\| + \eta \delta_i,
\]
\[
\gamma \|v_i(t-1) - v_{|k|}(t-1)\| \leq \gamma (\|v_i(t-2) - v_{|k|}(t-2)\| + \eta \beta \|w_i(t-2) - w_{|k|}(t-2)\| + \eta \delta_i),
\]
\[
\ldots
\]
\[
\gamma^{t-(k-1)\tau-1} \|v_i((k-1)\tau + 1) - v_{|k|}((k-1)\tau + 1)\| \\
\leq \gamma^{t-(k-1)\tau-1} \|v_i((k-1)\tau) - v_{|k|}((k-1)\tau)\| + \eta \beta \|w_i((k-1)\tau) - w_{|k|}((k-1)\tau)\| + \eta \delta_i.
\]

For convenience, we define \( G_i(t) \triangleq \|w_i(t) - w_{|k|}(t)\| \). Summing up all of the above inequalities, we have

\[
\|v_i(t) - v_{|k|}(t)\| \leq \eta \beta (G_i(t-1) + \gamma G_i(t-2) + \gamma^2 G_i(t-3) + \cdots + \gamma^{t-(k-1)\tau-1} G_i((k-1)\tau)) + \eta \delta_i (1 + \gamma + \gamma^2 + \cdots + \gamma^{t-(k-1)\tau-1}) + \gamma^{t-(k-1)\tau} \|v_i((k-1)\tau) - v_{|k|}((k-1)\tau)\|. \tag{32}
\]

When \( t = (k-1)\tau \), we know \( v_i(t) = v(t) = v_{|k|}(t) \) by the definition of \( v_{|k|}(t) \) and aggregation rules. Then we have

\[
\|v_i((k-1)\tau) - v_{|k|}((k-1)\tau)\| = 0,
\]
so that the last term of above inequality is zero and

\[
\|v_i(t) - v_{|k|}(t)\| \leq \eta \beta (G_i(t-1) + \gamma G_i(t-2) + \gamma^2 G_i(t-3) + \cdots + \gamma^{t-(k-1)\tau-1} G_i((k-1)\tau)) + \eta \delta_i (1 + \gamma + \gamma^2 + \cdots + \gamma^{t-(k-1)\tau-1}). \tag{32}
\]

Now, we can bound the gap between \( w_i(t) \) and \( w_{|k|}(t) \). When \( t \in ((k-1)\tau, k\tau] \), we have

\[
\|w_i(t) - w_{|k|}(t)\| = \|w_i(t-1) + \gamma v_i(t) - \eta \nabla F_i(w_i(t-1)) \] 
\[= \|w_{|k|}(t-1) + \gamma v_{|k|}(t) - \eta \nabla F_i(w_{|k|}(t-1))\| - \|\nabla F_i(w_i(t-1)) - \nabla F_i(w_{|k|}(t-1))\|\] (from (3) and (12))
\[= \|w_i(t-1) - w_{|k|}(t-1) + \gamma(v_i(t) - v_{|k|})(t) - \eta(\nabla F_i(w_i(t-1)) - \nabla F_i(w_{|k|}(t-1)))\|] (adding a zero term)
\[\leq \|w_i(t-1) - w_{|k|}(t-1)\| + \gamma \|v_i(t) - v_{|k|}(t)\| + \eta \beta \|w_i(t-1) - w_{|k|}(t-1)\| + \eta \delta_i\]
(\text{from triangle inequality, } \beta\text{-smoothness and (7)})
\[= (\eta \beta + 1) \|w_i(t-1) - w_{|k|}(t-1)\| + \gamma \|v_i(t) - v_{|k|}(t)\| + \eta \delta_i. \tag{33}
\]

Substituting inequality (32) into (33) and using \( G_i(t) \) to denote \( \|w_i(t) - w_{|k|}(t)\| \) for \( t, t-1, \ldots, 1 + (k-1)\tau \), we have

\[
G_i(t) \leq (\eta \beta + 1) G_i(t-1) + \eta \beta G_i(t-1) + \gamma G_i(t-2) + \gamma^2 G_i(t-3) + \cdots + \gamma^{t-(k-1)\tau-1} G_i((k-1)\tau)) + \eta \delta_i (1 + \gamma + \gamma^2 + \cdots + \gamma^{t-(k-1)\tau-1}) + \eta \delta_i.
\]

For convenience, we define \( g_i(x) \triangleq \frac{\delta_i}{\beta} (CA^t + DB^t) \), where \( A \) and \( B \) are defined in Theorem 1; \( C^i \) and \( D \) are defined in Lemma 1. We have

\[
f_i(x) = \gamma^t g_i(x) - \frac{\delta_i}{\beta}. \tag{35}
\]

Next, we use induction to prove \( G_i(t) \leq f_i(t - (k-1)\tau) \). For the induction, we assume that

\[
G_i(p) \leq f_i(p - (k-1)\tau) \tag{36}
\]
holds for some \( p \in ((k-1)\tau, t) \). Thus, we have

\[
G_i(t) \leq f_i(t - 1 - (k-1)\tau) + \eta \beta (f_i(t - 1 - (k-1)\tau) + \gamma f_i(t - 2 - (k-1)\tau) + \cdots + \gamma^{t-(k-1)\tau} f_i(0)) + \eta \delta_i (1 + \gamma + \gamma^2 + \cdots + \gamma^{t-(k-1)\tau})\]
\[\leq \gamma^{t-1-(k-1)\tau} g_i(t - 1 - (k-1)\tau) - \delta_i \beta \]
\[+ \eta \beta (\gamma^{t-1-(k-1)\tau} g_i(t - 1 - (k-1)\tau) - \delta_i \beta) \]
\[+ \gamma \gamma^{t-1-(k-1)\tau} g_i(t - 1 - (k-1)\tau) - \gamma \delta_i \beta \]
\[+ \gamma^2 \gamma^{t-2-(k-1)\tau} g_i(t - 2 - (k-1)\tau) - \gamma^2 \delta_i \beta \]
\[\cdots \]
\[+ \gamma^{t-(k-1)\tau} g_i(0) - \gamma^{t-(k-1)\tau} \delta_i \beta \eta \delta_i \]
\[= \gamma^{t-1-(k-1)\tau} g_i(t - 1 - (k-1)\tau) + \eta \beta g_i(t - 1 - (k-1)\tau) + \gamma g_i(t - 2 - (k-1)\tau) + \cdots + g_i(0)) - \frac{\delta_i}{\beta} \]
\[= f_i(t - (k-1)\tau).
\]

Thus, Lemma 2 has been proven. \( \square \)
2) Bounding $\|w(t) - w_{[k]}(t)\|$: Based on the result of Lemma 2, we first bound the gap of $\|v(t) - v_{[k]}(t)\|$ in Lemma 3. Based on the result of Lemma 3, we then bound the gap of $\|w(t) - w_{[k]}(t)\|$, which concludes Theorem 1.

**Lemma 3.** For any interval $[k, \forall t \in ((k-1)\tau, k\tau]$, we have:

$$\|v(t) - v_{[k]}(t)\| \leq \eta \delta \left( \frac{C(\gamma A)^{\tau_0} + D(\gamma B)^{\tau_0}}{\gamma(A-1) + \gamma(B-1)} - \frac{\gamma^{\tau_0} - 1}{\gamma - 1} \right),$$  
(37)

where $\tau_0 = t - (k-1)\tau$.

**Proof of Lemma 3.** For convenience, we define

$$p(t) \triangleq \gamma'(CA^t + DB^t) - 1.$$  
(38)

Therefore, we get

$$f_i(t) = \frac{\delta_i}{\beta} p(t).$$  
(39)

From (2) and (4), we have

$$v(t) = \gamma v(t-1) - \sum_{i=1}^{N} D_i \nabla F_i(w_i(t-1)) / D$$  
(40)

For $t \in ((k-1)\tau, k\tau]$, we have

$$\|v(t) - v_{[k]}(t)\| = \|\gamma v(t-1) - \sum_{i=1}^{N} D_i \nabla F_i(w_i(t-1)) / D - \gamma v_{[k]}(t-1) + \eta \nabla F(w_{[k]}(t-1))\|$$  

(from (40) and (11))

$$\leq \gamma \|v(t-1) - v_{[k]}(t-1)\| + \eta \sum_{i=1}^{N} D_i \|\nabla F_i(w_i(t-1)) - \nabla F_i(w_{[k]}(t-1))\| / D$$  

$$\leq \gamma \|v(t-1) - v_{[k]}(t-1)\| + \eta \sum_{i=1}^{N} D_i f_i(t-1 - (k-1)\tau) / D$$  

(from $\beta$-smoothness and Lemma 2)

$$= \gamma \|v(t-1) - v_{[k]}(t-1)\| + \eta \delta p(t-1 - (k-1)\tau).$$  
(41)

(\text{from (39) and (8)})

We use $\gamma_0, \gamma_1, \ldots, \gamma^{t-(k-1)\tau-1}$ as multipliers to multiply (41) when $t, t-1, \ldots, (k-1)\tau + 1$, respectively.

$$\|v(t) - v_{[k]}(t)\| \leq \gamma \|v(t-1) - v_{[k]}(t-1)\| + \eta \delta p(t-1 - (k-1)\tau),$$  

$$\gamma \|v(t-1) - v_{[k]}(t-1)\| \leq \gamma^2 \|v(t-2) - v_{[k]}(t-2)\| + \gamma \delta p(t-2 - (k-1)\tau),$$  

$$\ldots$$  

$$\gamma^{t-(k-1)\tau-1} \|v((k-1)\tau + 1) - v_{[k]}((k-1)\tau + 1)\| \leq \gamma^{t-(k-1)\tau} \|v((k-1)\tau) - v_{[k]}((k-1)\tau)\| + \gamma^{t-1-(k-1)\tau} \eta \delta p(0).$$

Summing up all of the above inequalities, we have

$$\|v(t) - v_{[k]}(t)\| \leq \eta \delta (\gamma^{t-1-(k-1)\tau} p(0) + \ldots + \gamma p(t - 2 - (k-1)\tau) + p(t - 1 - (k-1)\tau))$$  
(42)

(because $\|v((k-1)\tau) - v_{[k]}((k-1)\tau)\| = 0$ from (9))

$$= \eta \delta (\gamma^{t-1-(k-1)\tau} C(1 + A + A^{t-1-(k-1)\tau}) + \gamma^{t-1-(k-1)\tau} D(1 + B + B^{t-1-(k-1)\tau})$$

$$- (1 + \gamma + \ldots + \gamma^{t-1-(k-1)\tau}))$$

$$= \eta \delta (\frac{C(\gamma A)^{\tau_0} - 1}{\gamma(A-1) + \gamma(B-1)} - \frac{\gamma^{\tau_0} - 1}{\gamma - 1})$$

$$- \eta \delta \gamma^{\tau_0-1} \left( \frac{C}{A-1} + \frac{D}{B-1} \right)$$

$$= \eta \delta (\frac{C(\gamma A)^{\tau_0} - 1}{\gamma(A-1) + \gamma(B-1)} - \frac{\gamma^{\tau_0} - 1}{\gamma - 1})$$

(38)

where $\tau_0 = t - (k-1)\tau$. Thus, Lemma 3 has been proven. $\Box$

Based on the result in Lemma 3, we can now bound $\|w(t) - w_{[k]}(t)\|$.

**Proof of Theorem 1.** From (3), (4), and (5), we have

$$w(t) = w(t-1) + \gamma v(t) - \sum_{i=1}^{N} D_i \nabla F_i(w_i(t-1)) / D.$$  
(44)

From (12) and (44), we have

$$\|w(t) - w_{[k]}(t)\|$$

$$= \|w(t-1) + \gamma v(t) - \sum_{i=1}^{N} D_i \nabla F_i(w_i(t-1)) / D - w_{[k]}(t) - \gamma v_{[k]}(t) + \eta \nabla F(w_{[k]}(t-1))\|$$

$$\leq \|w(t-1) - w_{[k]}(t)\| + \gamma \|v(t) - v_{[k]}(t)\| + \eta \delta p(t-1 - (k-1)\tau).$$

(from $\beta$-smoothness, Lemma 2, (39), and (8))

Thus, according to Lemma 3, we have

$$\|w(t) - w_{[k]}(t)\| - \|w(t-1) - w_{[k]}(t-1)\|$$

$$\leq \eta \delta \left( \frac{C(\gamma A)^{\tau_0} - 1}{\gamma(A-1) + \gamma(B-1)} - \frac{\gamma^{\tau_0} - 1}{\gamma - 1} \right)$$

$$+ \eta \delta \left( \frac{C(\gamma A)^{\tau_0} - 1}{A-1} \right)$$

$$+ \frac{D(\gamma B)^{\tau_0-1} - 1}{B-1} \left( \gamma B + B - 1 \right).$$

(45)

(46)
When \( t = (k-1)\tau \), we have \( \|w(t) - w_k(t)\| = 0 \). When \( t \in ((k-1)\tau, k\tau) \), we sum up (46) for \( t, t-1, \ldots, (k-1)\tau+1 \). Then we have
\[
\|w(t) - w_k(t)\|
\leq \sum_{x=1}^{t_0} \eta \delta \left( \frac{C(\gamma A)^{x-1}}{A - 1} (\gamma A + A - 1) + \frac{D(\gamma B)^{x-1}}{B - 1} (\gamma B + B - 1) - \frac{\gamma^{x+1} - 1}{\gamma - 1} \right) = \eta \delta \left( E (\gamma A)^{t_0} - 1 + F (\gamma B)^{t_0} - 1 - \frac{\gamma^2 (\gamma A)^{t_0} - (\gamma A) t_0}{(\gamma - 1)^2} \right)
= h(t_0),
\]
where \( E = \frac{\gamma A + A - 1}{(A - 1)(A - \gamma B)} \) and \( F = \frac{\gamma B + B - 1}{(B - 1)(\gamma A - B)} \) (as defined in Theorem 1). \( E + F = \frac{1}{\eta \delta}; \quad t_0 = t - (k-1)\tau \). Thus, Theorem 1 has been proven.

C. Proof of Monotone of \( h(x) \) (Observation 1 in Theorem 1)

We first introduce following Lemma 4 for later use.

**Lemma 4.** Given \( A, B, C, \) and \( D \) according to their definitions, then we have
\[
C(\gamma A)^i + D(\gamma B)^i \geq (1 + \eta \beta + \eta \beta \gamma)^i
\]
holds for \( i = 0, 1, 2, 3, \ldots \)

**Proof.** We note that according to the definitions of \( A, B, C, \) and \( D \), we know that \( \gamma A > 1, 0 < \gamma B < 1, \frac{1}{\gamma A} < B < 1, C > 0, D > 0, E > 0, \) and \( F > 0 \). We also have \( C + D = 1 \).

When \( i = 0 \), \( C(\gamma A)^i + D(\gamma B)^i = (1 + \eta \beta + \eta \beta \gamma)^i = 1 \), so the inequality holds. When \( i = 1 \), we have
\[
C(\gamma A)^i + D(\gamma B)^i = \gamma \frac{A - 1}{A - B} \frac{A + 1 - B}{A - B} = \gamma (A + B - 1) = 1
\]
so the inequality still holds. When \( i > 1 \), according to Jensen inequality, and \( f(x) = x^i \) is convex, we have
\[
C(\gamma A)^i + D(\gamma B)^i \geq (\gamma CA + \gamma DB)^i = (1 + \eta \beta + \eta \beta \gamma)^i
\]
so the inequality still holds. When \( i > 1 \), according to Jensen inequality, and \( f(x) = x^i \) is convex, we have
\[
C(\gamma A)^i + D(\gamma B)^i \geq (\gamma CA + \gamma DB)^i = (1 + \eta \beta + \eta \beta \gamma)^i.
\]
To conclude, Lemma 4 has been proven.

Then we can prove the monotone of \( h(x) \).

**Proof.** It is equivalent to prove
\[
h(x) - h(x - 1) \geq 0
\]
for all integer \( x \geq 1 \). When \( x = 0 \) or \( x = 1 \), we have
\[
h(0) = \eta \delta (E + F - \frac{1}{\eta \beta}) = 0,
\]
\[
h(1) = \eta \delta \left( \gamma (EA + FB) - \frac{1}{\eta \beta} - \gamma - 1 \right) = 0,
\]
because \( EA + FB = \frac{1 + \eta \beta + \eta \beta \gamma}{\eta \beta \gamma} \). Therefore, when \( x = 1 \), \( h(x) - h(x - 1) = 0 \).

When \( x > 1 \), according to Lemma 4 and (38), we have \( p(x) = C(\gamma A)^x + D(\gamma B)^x - 1 \geq (1 + \eta \beta + \eta \beta \gamma)^x - 1 > 0 \). Then we have
\[
h(x) - h(x - 1) = \eta \delta \left( \frac{C(\gamma A)^x (\gamma A + (1 + \gamma B)^x - 1)}{\gamma (A + (1 + \gamma B)^x - 1)} - \gamma (\gamma A + (1 + \gamma B)^x - 1) \right)
= \eta \delta (\gamma x^2 - 1) + \eta \delta (\gamma x^{-1} (CA^{-1} + DB^{-1}) - 1)
= \eta \delta (\gamma x^{-1} p(x + 1) + \cdots + \eta \delta x^2 - 1 + p(x - 1))
+ \eta \delta p(x - 1)
\]
(because (46) equals (45))
\[
\geq \eta \delta (\gamma x^{-1} p(x + 1) + \cdots + \eta \delta x^2 - 1 + p(x - 1))
\]
(because (43) equals (42), \( x = t - (k-1)\tau \), and (38))
\[
> 0.
\]
Thus, we have proven that \( h(0) = h(1) = 0 \) and \( h(x) \) increases with \( x \) when \( x \geq 1 \).

D. Proof of Theorem 2

For convenience, we define \( c_{[k]}(t) \triangleq F(w_{k}(t)) - F(w^*) \) for a given interval \( [k] \), where \( t \in [(k-1)\tau, k\tau] \).

**Proof.** According to the convergence lower bound of any gradient descent methods given in Theorem 3.14 in [30], we always have
\[
c_{[k]}(t) > 0 \quad (47)
\]
for any \( t \) and \( k \).

Then we derive the upper bound of \( c_{[k]}(t + 1) - c_{[k]}(t) \).

Because \( F(\cdot) \) is \( \beta \)-smooth, according to Lemma 3.4 in [30], we have
\[
F(x) - F(y) \leq \nabla F(y)^T (x - y) + \frac{\beta}{2} \|x - y\|^2
\]
for arbitrary x and y. Thus,
\[ c_{[k]}(t+1) - c_{[k]}(t) = F(w_{[k]}(t+1)) - F(w_{[k]}(t)) \]
\leq \nabla F(w_{[k]}(t))^T (w_{[k]}(t+1) - w_{[k]}(t))
+ \frac{\beta}{2} \| w_{[k]}(t+1) - w_{[k]}(t) \|^2
\leq \gamma \nabla F(w_{[k]}(t))^T v_{[k]}(t+1) - \eta \| \nabla F(w_{[k]}(t)) \|^2
+ \frac{\beta}{2} \| \gamma v_{[k]}(t+1) - \eta \nabla F(w_{[k]}(t)) \|^2
= - \eta (\gamma + 1) \left( 1 - \frac{\beta \eta (\gamma + 1)}{2} \right) \| \nabla F(w_{[k]}(t)) \|^2
+ \frac{\beta \gamma^4}{2} \| v_{[k]}(t) \|^2 + \gamma^2 (1 - \beta \eta (\gamma + 1)) \nabla F(w_{[k]}(t))^T v_{[k]}(t)
\]
(48)
where the second term in (48) is because \( \| \gamma v_{[k]}(t) \| \leq \frac{p}{\eta} \| \nabla F(w_{[k]}(t)) \| \) with the definition of p. Since \( 0 < \beta \eta (\gamma + 1) \leq 1 \), the third term in (48) is because
\[ - \nabla F(w_{[k]}(t))^T v_{[k]}(t) \]
\[ = \| \nabla F(w_{[k]}(t)) \| \| v_{[k]}(t) \| \cos \theta_{[k]}(t) \]
\[ = \| \nabla F(w_{[k]}(t)) \| \| \gamma v_{[k]}(t+1) - \eta \nabla F(w_{[k]}(t+1)) \| \cos \theta_{[k]}(t) \]
\[ \geq \eta \| \nabla F(w_{[k]}(t)) \| \| \nabla F(w_{[k]}(t)) \| \cos \theta_{[k]}(t) \]
(49)
According to the convexity condition and Cauchy-Schwarz inequality, we have:
\[ c_{[k]}(t) = F(w_{[k]}(t)) - F(w^*) \leq \| \nabla F(w_{[k]}(t)) \| \| w_{[k]}(t) - w^* \| . \]
Equivalently,
\[ \| \nabla F(w_{[k]}(t)) \| \geq \frac{c_{[k]}(t)}{\| w_{[k]}(t) - w^* \|} \]
(50)
Substituting (50) into (49), and noting \( \omega \leq \frac{1}{\| w_{[k]}(t) - w^* \|^2} \) by
the definition of \( \omega \), we get
\[ c_{[k]}(t+1) - c_{[k]}(t) = \frac{\omega c_{[k]}(t)^2}{\| w_{[k]}(t) - w^* \|^2} \leq c_{[k]}(t) - \omega c_{[k]}(t)^2. \]
Because \( \alpha > 0 \), \( c_{[k]}(t) > 0 \) in (47), and (49), we have \( 0 < c_{[k]}(t+1) \leq c_{[k]}(t) \). Dividing both side by \( c_{[k]}(t+1) c_{[k]}(t) \), we get
\[ \frac{1}{c_{[k]}(t)} \leq \frac{1}{c_{[k]}(t+1)} - \omega \frac{c_{[k]}(t)}{c_{[k]}(t+1)}. \]
We note that \( \frac{c_{[k]}(t)}{c_{[k]}(t+1)} \geq 1. \) Thus,
\[ \frac{1}{c_{[k]}(t+1)} - \frac{1}{c_{[k]}(t)} \geq \omega \frac{c_{[k]}(t)}{c_{[k]}(t+1)} \geq \omega. \]
(51)
Summing up the above inequality by \( t \in [(k-1)\tau, k\tau - 1] \), we have
\[ \frac{1}{c_{[k]}(k\tau)} - \frac{1}{c_{[k]}((k-1)\tau)} = \sum_{t=(k-1)\tau}^{k\tau-1} \left( \frac{1}{c_{[k]}(t+1)} - \frac{1}{c_{[k]}(t)} \right) \]
\[ \geq \sum_{t=(k-1)\tau}^{k\tau-1} \omega \alpha = T \omega \alpha. \]
(52)
Then, we sum up the above inequality by \( k \in [1, K] \), after rearranging the left-hand side and noting that \( T = K \tau \), we can get
\[ \sum_{i=1}^{K} \left( \frac{1}{c_{[k]}(i\tau)} - \frac{1}{c_{[k]}((i-1)\tau)} \right) \]
\[ = \sum_{i=1}^{K-1} \left( \frac{1}{c_{[k]}(i\tau)} - \frac{1}{c_{[k]}(i\tau+1)} \right) \]
\[ = \frac{1}{c_{[k]}(T)} - \frac{1}{c_{[k]}(1)} - \sum_{k=1}^{K-1} \left( \frac{1}{c_{[k]}(k\tau)} - \frac{1}{c_{[k]}((k-1)\tau)} \right) \]
\[ \geq K \tau \omega \alpha = T \omega \alpha. \]
(53)
Here, we note that
\[ \frac{1}{c_{[k]}(k\tau)} - \frac{1}{c_{[k]}((k+1)\tau)} = \frac{c_{[k]}((k+1)\tau) - c_{[k]}(k\tau)}{c_{[k]}((k+1)\tau) c_{[k]}(k\tau)} \]
\[ = \frac{F(w_{[k]}((k+1)\tau)) - F(w_{[k]}(k\tau))}{c_{[k]}((k+1)\tau) c_{[k]}(k\tau)} \]
\[ \leq -\rho h(\tau) \frac{c_{[k]}(k\tau) c_{[k]}((k+1)\tau)}{c_{[k]}((k+1)\tau) c_{[k]}(k\tau)} \]
(54)
where the last inequality is because \( w_{[k]}((k+1)\tau) = w_{[k]}(k\tau) \) in (10), and (15) in Theorem 1.
From (49), we can get \( F(w_{[k]}(t)) \geq F(w_{[k]}(t+1)) \) for any \( t \in [(k-1)\tau, k\tau) \). Recalling condition 3 in Theorem 2, where \( F(w_{[k]}(k\tau)) - F(w^*) \geq \varepsilon \) for all \( k \), we can obtain \( c_{[k]}(t) = F(w_{[k]}(t)) - F(w^*) \geq \varepsilon \) for all \( t \in [(k-1)\tau, k\tau) \) and \( k \). Thus,
\[ c_{[k]}(k\tau) c_{[k]}(k\tau+1) \geq \varepsilon^2. \]
(55)
According to Appendix C, we have \( h(\tau) \geq 0. \) Then substituting (55) into (54), we have
\[ \frac{1}{c_{[k]}(k\tau)} - \frac{1}{c_{[k]}(k\tau+1)} \geq \frac{-\rho h(\tau)}{\varepsilon^2}. \]
(56)
Substituting (56) into (53) and rearrange, we get
\[
\frac{1}{c[|K|](T)} - \frac{1}{c[|1|](0)} \geq T\omega_\alpha - (K - 1)\frac{\rho h(\tau)}{\varepsilon^2}.
\] (57)
Recalling condition 4 in Theorem 2, where \( F(w(T)) - F(w^*) \geq \varepsilon \), and noting that \( c[|K|](T) \geq \varepsilon \), we get
\[
(F(w(T)) - F(w^*))c[|K|](T) \geq \varepsilon^2
\] (58)
Thus,
\[
\frac{1}{F(w(T)) - F(w^*)} = \frac{1}{c[|K|](T) - (F(w(T)) - F(w^*))c[|K|](T)}
\]
\[
= \frac{F(w(T)) - F(w^*)c[|K|](T)}{(F(w(T)) - F(w^*))c[|K|](T)}
\]
\[
\geq -\rho h(\tau)
\]
\[
\geq -\frac{\rho h(\tau)}{\varepsilon^2},
\] (59)
where the first inequality is because (15) in Theorem 1 when \( t = K\tau \) in interval \([K]\). Combining (57) with (59), we get
\[
\frac{1}{F(w(T)) - F(w^*)} - \frac{1}{c[|1|](0)} \geq T\omega_\alpha - K\frac{\rho h(\tau)}{\varepsilon^2}
\]
\[
= T\omega_\alpha - \frac{T\rho h(\tau)}{\varepsilon^2}
\]
\[
= T\left(\omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2}\right).
\]
Noting that \( c[|1|](0) = F(w[|1|](0)) - F(w^*) > 0 \), the above inequality can be expressed as
\[
\frac{1}{F(w(T)) - F(w^*)} \geq T\left(\omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2}\right).
\] (60)
Recalling condition 2 in Theorem 2, where \( \omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2} > 0 \), we obtain that the right-hand side of above inequality is greater than zero. Therefore, taking the reciprocal of the above inequality, we finally get the result
\[
F(w(T)) - F(w^*) \leq \frac{1}{T\left(\omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2}\right)}.
\]

E. Proof of Theorem 3

**Proof.** At the beginning, we see that condition 1 in Theorem 2 always holds due to the conditions in Theorem 3, where \( \cos \theta \geq 0 \), \( 0 < \beta_0(\gamma + 1) \leq 1 \), and \( 0 \leq \gamma < 1 \).

When \( \rho h(\tau) = 0 \), there is always an arbitrarily small \( \varepsilon \) but great than zero that let conditions 2–4 in Theorem 2 hold. Under this circumstance, Theorem 2 holds. We also note that the right-hand side of (17) is equivalent to the right-hand side of (16) when \( \rho h(\tau) = 0 \). Moreover, according to the definition of \( w^f \) in (6), we have
\[
F(w^f) = F(w(T)) - F(w^*) \leq \frac{1}{T\omega_\alpha},
\] which satisfies the result in Theorem 2 directly. Thus, Theorem 3 holds when \( \rho h(\tau) = 0 \).

When \( \rho h(\tau) > 0 \), considering the right-hand side of (16) and let
\[
\varepsilon_0 = \frac{1}{T\left(\omega_\alpha - \frac{\rho h(\tau)}{\tau^2}\right)}.
\] (61)
Rearranging and calculating \( \varepsilon_0 \), we get
\[
\varepsilon_0 = \frac{1}{2T\omega_\alpha} + \sqrt{\frac{1}{4T^2\omega^2\alpha^2} + \frac{\rho h(\tau)}{T\omega_\alpha\tau}}.
\] (62)
Here, we take the positive solution \( \varepsilon_0 \) because \( \varepsilon > 0 \) in Theorem 2. Considering above two equations for \( \varepsilon_0 \), we get \( \varepsilon_0 > 0 \) and the denominator in (61) is greater than zero. We also note that \( \omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2} \) increases with \( \varepsilon \). Thus, when \( \varepsilon \geq \varepsilon_0 \), condition 2 in Theorem 2 holds. Under this circumstance, we assume that there exists \( \varepsilon > \varepsilon_0 \) that satisfies both condition 3 and 4 in Theorem 2 at the same time, so that Theorem 2 holds. Then we get,
\[
F(w(T)) - F(w^*) \leq \frac{1}{T\left(\omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2}\right)}
\]
\[
= \frac{1}{T\left(\omega_\alpha - \frac{\rho h(\tau)}{\varepsilon^2}\right)} = \varepsilon_0,
\]
which contradicts the condition 4 in Theorem 2. Using the proof by contradiction, we conclude that there does not exist \( \varepsilon > \varepsilon_0 \) that satisfies both condition 3 and 4 in Theorem 2 at the same time. Equivalently, it happens either (1) \( \exists k \in [1, K] \) allows \( F(w[k](K\tau)) = F(w^*) \leq \varepsilon_0 \) or (2) \( F(w(T)) \leq F(w^*) \leq \varepsilon_0 \), which follows
\[
\min \left\{ \min_{k \in [1, K]} F(w[k](K\tau)): F(w(T)) \right\} - F(w^*) \leq \varepsilon_0.
\] (63)
Recalling (15) in Theorem 1, when \( t = K\tau \), we have
\[
F(w[k](K\tau)) \leq F(w[k](K\tau)) + \rho h(\tau) \text{ for any interval } [k].
\]
Combining it with (63), we have
\[
\min_{k \in [1, K]} F(w[k](K\tau)) - F(w^*) \leq \varepsilon_0 + \rho h(\tau).
\]
Recalling the definition of \( w^f \) in (6), \( T = K\tau \), and combining \( w^f \) with above inequality, we get
\[
F(w^f) = F(w(T)) - F(w^*) \leq \varepsilon_0 + \rho h(\tau).
\]
Substituting (62) into above inequality, we finally get the result in (17), which proves the Theorem 3.
F. Proof of Theorem 4

Proof. When $\eta \to 0$, we have $\gamma A \simeq 1$, $\gamma B \simeq \gamma$, and $F \simeq \frac{\gamma^2}{(1-\gamma^2)}$. Therefore,

\[
\lim_{\eta \to 0} h(\tau) = \lim_{\eta \to 0} \eta \delta \left[ E(\gamma A)^\tau + F(\gamma B)^\tau - \frac{1}{\eta^2} \right] = \frac{\gamma^2(\gamma^2 - 1) - (\gamma - 1)\tau}{(\gamma - 1)^2}.
\]

Also, we have $\alpha > \hat{\alpha}$. Theorem 4, where $0 < \eta \to 0$. Rewrite $f_1(T)$ and $f_2(T)$, we have

\[
f_1(T) = \frac{1}{2T\omega_0} + \frac{1}{4T^2\omega^2\alpha^2} + \rho h(\tau) \\
\simeq \frac{1}{2T\omega_0} + \frac{1}{4T^2\omega^2\alpha^2} \equiv \frac{1}{T\omega_0},
\]

\[
f_2(T) = \frac{1}{2T\omega_0} + \frac{1}{4T^2\omega^2\alpha^2} + \rho h(\tau) \\
\simeq \frac{1}{2T\omega_0} + \frac{1}{4T^2\omega^2\alpha^2} \equiv \frac{1}{T\omega_0}.
\]

According to the definition of $\alpha$, and condition 2 of Theorem 2 with $h(\cdot) \geq 0$, we have $\alpha > 0$. Based on the conditions in Theorem 4, where $0 < \beta(\gamma + 1) \leq 1$ and the definition of $\alpha$, we have $\alpha > 0$. Furthermore, for any $0 < \gamma < 1$ and $\eta \to 0^+$, we have $\alpha > \hat{\alpha}$. Therefore, we get $f_1(T) < f_2(T)$. \qed

G. Proof of $\cos \theta \geq 0$ when $\eta \to 0^+$ (Observation 1 in Theorem 4)

Proof. When $\eta \to 0^+$, recalling the definition of $p$, we can get $\|\eta F(w[k](t))\| \leq p\|\eta F(w[k](t))\| \simeq 0$. Then, We have $\|v[k](t)\| \simeq 0, v[k](t) \simeq 0$. Thus, we have

\[
-2\nabla F(w[k](t))^T v[k](t) = ||\nabla F(w[k](t))||^2 + ||v[k](t)||^2 \equiv 0.
\]

Therefore, $\cos \theta[k](t) \simeq 0$ for any $k \in [1, K]$ and $t \in [k]$, which proves $\cos \theta \geq 0$.

\[ \]

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