A geometric characterization of sensitivity analysis in monomial models

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Abstract
Sensitivity analysis in probabilistic discrete graphical models is usually conducted by varying one probability value at a time and observing how this affects output probabilities of interest. When one probability is varied then others are proportionally covaried to respect the sum-to-one condition of probability laws. The choice of proportional covariation is justified by a variety of optimality conditions, under which the original and the varied distributions are as close as possible under different measures of closeness. For variations of more than one parameter at a time proportional covariation is justified only in some special cases only. In this work, for the large class of discrete statistical models entertaining a regular monomial parametrisation, we demonstrate the optimality of newly defined proportional multi-way schemes with respect to an optimality criterion based on the notion of I-divergence. We demonstrate that there are varying parameters choices for which proportional covariation is not optimal and identify the sub-family of model distributions where the distance between the original distribution and the one where probabilities are covaried proportionally is minimum. This is shown by adopting a new formal, geometric characterization of sensitivity analysis in monomial models, which include a wide array of probabilistic graphical models. We also demonstrate the optimality of proportional covariation for multi-way analyses in Naive Bayes classifiers.

Keywords: Bayesian network classifiers, Covariation, I-projections, Monomial models, Sensitivity analysis.

1. Introduction

The assessment of the validity of a statistical model’s outputs, usually referred to as model validation, is a critical task of any applied analysis. This consists of checking that a model produces outputs that are in line with current understanding, following a defensible and expected mechanism [17]. Useful techniques to do so are nowadays well-established for various probabilistic graphical models, in particular for discrete Bayesian networks (BNs) [9, 28]. For such models the validation process can be broken down into two steps: the first concerns the auditing of the validity of the conditional independences implied by the underlying graphical structure; the second, assuming the graph is valid, checks the impact of the numerical elicited probabilities on outputs of interest. Our focus lies in this second validation phase, usually called sensitivity analysis.

Because of both its simplicity and its proven theoretical justifications, the most common investigation is the so-called one-way sensitivity analysis, where the impacts of changes made to a single probability parameter are studied. When one parameter is varied, then others are required to be adjusted, or covaried, to respect the sum-to-one condition of probabilities. Although there are various ways to covary probabilities, the most common covariation scheme is the proportional one, where, after a change to a parameter, the covarying parameters have the same proportion of the residual probability mass as they originally had. Proportional covariation in one-way analyses is “optimal” since it minimizes a large array of divergences between the original and the varied probability distributions amongst any valid covariation scheme [6, 26].

Multi-way methods, where two or more parameters are varied contemporaneously, have not been extensively studied in the literature [see 3, 7, 18 for some exceptions]. This is not only because they require a much more intensive computational power, but also, and more critically, because there are very little, if no, theoretical justifications in using...
a covariation scheme over another. Output probabilities have been shown to heavily depend on the covariation scheme used \[26, 29\]. In \[26\] it has been recently proven that for specific multi-way analyses called full single conditional probability table (CPT) analyses, proportional covariation is optimal. However, such analyses highly restrict the parameters that can be varied, for instance only one parameter from the CPT of a vertex in a BN conditionally on every parents’ configuration.

In this paper we demonstrate the optimality of a variety of newly defined, flexible multi-way covariation schemes where individual parameters are proportionally covaried. The optimality criterion is based on the minimization of the I-divergence, also known as Kullback-Leibler divergence \[24\]. By taking an information geometry approach \[1, 12\], we provide a new formal, geometric characterization of sensitivity analysis in terms of distances and projections over a probability simplex. We demonstrate that the probability distributions resulting from our multi-way schemes correspond to the I-projection of the original distribution, i.e. the distribution with smallest I-divergence from the original one within a well-specified subset of the probability simplex. More generally, we derive the condition which specifies the family of distributions for which proportional covariation is the I-projection. For our covariation schemes such condition is void, thus implying that these schemes are optimal. But in general the condition restricts the family of distributions usually investigated in sensitivity analysis and therefore, depending on the choice of parameters varied, proportional covariation is not always optimal.

As a consequence of these results, we are able to prove the optimality of proportional covariation in naive Bayes classifiers (e.g. \[2\]) for any combination of probabilities associated to feature variables. Naive Bayes models are a specific type of BN classifiers often used to assign instances to a specific class in a classification problem. The tuning of the feature’s probabilities is often critical to ensure that the classifier works reliably \[3\].

As in \[26\] we consider models where the probability of any element of the sample space is represented by a monomial. They are called monomial discrete parametric models (MDPMs). Specifically we focus on a subclass of MDPMs, formally defined below and called monomial models (MMs), where specific subsets of parameters need to respect the sum-to-one condition. We show below that BNs and staged trees \[30\] can be seen as a specific instance of such models. Many other well-known statistical models can be represented as a MM \[19, 26\]: for instance chain event graphs \[31\], context-specific BNs \[4\], decomposable Markov networks and probabilistic chain graphs \[25\].

2. Monomial discrete parametric models

Let \(Y\) be a finite set with \(q\) elements and \(P\) a strictly positive probability density function for \(Y\). We write \(#Y = q\), call \(y \in Y\) an atom and \(P(y)\) the atomic probability of \(y\). The generic probability \(P\) can be seen as a point in the interior of the \(q\)-dimensional simplex and we write \(P \in \Delta_{q-1}\). Next, to \(Y\) we associate a particular class of parametric statistical models, called monomial discrete parametric models, in short MDPMs.

Let \(k = \{1, 2, \ldots, k\}\). A MDPM is defined through a \(q \times k\) matrix \(A\) with non-negative integer entries and a \(k\)-dimensional parameter vector \(\theta\) with positive real entries. We write \(A \in M_{q,k}(\mathbb{Z}_{\geq 0})\) and \(\theta = (\theta_i)_{i \in [k]} \in \mathbb{R}^k_{>0}\). There is a row of \(A\) for each atom \(y\) and \(A_y\) indicates the \(y\)-th row of \(A\). The atomic probability of \(y \in Y\) given \(\theta\) and \(A\) is defined as 
\[
P(y) = \prod_{i \in [k]} \theta_i^{A_{yi}} = \theta^{A_y}
\]

**Definition 1.** The MDPM model over \(Y\) associated to \(\theta\) and \(A\) is defined as 
\[
\text{MDPM}(A, \theta) = \{P \in \Delta_{q-1} : P(y) = \theta^{A_y} \text{ for } y \in Y \text{ and } \theta \in \mathbb{R}^k_{>0}\}
\]

By definition a MDPM \((A, \theta)\) is the image of a map from \(\mathbb{R}^k_{>0}\) to \(\Delta_{q-1}\). As the entries of \(A\) are taken to be non-negative integer numbers, the atomic probabilities are monomials in the \(\theta_i\’s\) and as \(\theta\) varies in \(\mathbb{R}^k_{>0}\) the model describes an algebraic variety in \(\Delta_{q-1}\) \[16, 30\]. The assumption of strictly positive probabilities is often met in practice, for instance for models learnt with complete data \[24\]. The condition is imposed here to ensure the I-divergence exists and is finite \[24\]. Degenerate cases are avoided by requiring \(\theta \in \mathbb{R}^k_{>0}\) and in particular \(A\) cannot have all elements of a row equal to one or all equal to zero.

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Example 1. Let \( \mathcal{Y} = \{0, 1\} \) and \( A \) be the \( 4 \times 3 \) matrix with rows \( A_1 = (1, 0, 0) \), \( A_2 = (0, 1, 0) \), \( A_3 = (0, 0, 1) \) and \( A_4 = (0, 1, 1) \). The MDPM\((A, \theta)\) is defined by \( P(y) = \theta_y \) for \( y \in \{0, 1\} \) and \( P(y) = \theta_2 \theta_0 \) for \( y = 4 \), entailing \( P(4) = P(3) P(2) \). This MDPM is represented by the intersection of \( \Delta_3 \) with the affine variety defined by the polynomial \( \theta_0 + \theta_2 + \theta_3 + \theta_2 \theta_1 - 1 \) and is represented by the surface in Figure 1.

Example 2. Log-linear models for the analysis of contingency tables \([13, 15]\) are MDPMs. For a finite sample space \( \mathcal{Y} \) consider an exponential family with non-negative integer valued sufficient statistics. This can be expressed as a log-linear model with canonical parameter vector \( \xi = (\xi_i)_{i \in [k]} \in \mathbb{R}^k \), for example

\[
P(y) = Z(\xi) \exp \left( \sum_{i \in [k]} \xi_i T_i(y) \right) = Z(\xi) \prod_{i \in [k]} \theta_i^{T_i(y)}, \quad y \in \mathcal{Y},
\]

where \( T_i = (T_i)_i \in [k] \) is the sufficient statistic and \( Z(\xi) \) a normalizing constant and \( \theta_i = \exp(\xi_i) \). Consider \( \mathcal{Y} = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \) and the log-linear model \( P(y) = Z(\xi_1, \xi_2) \exp \{\xi_1 y_1 + \xi_2 y_2\} \), with sufficient statistic \( T(y) = (y_1, y_2) \). Its intrinsic monomial structure is evident in the parameterization \( \theta_{00} = Z(\xi_1, \xi_2), \theta_{10} = \exp[\xi_1], \theta_{01} = \exp[\xi_2] \) and namely it is \( P(y) = \theta_{00} \theta_0 \theta_1 + \theta_{01} \), for all \( y \in \mathcal{Y} \).

In this paper, as in [20], we consider a specific subclass of MDPMs, called monomial models (MMs). As shown below, many statistical graphical models can be framed as MMs. In a MM the parameters are grouped in such a way that those in a group sum to one. For a subset \( S \subseteq [k] \), the notation \( \theta_S = (\theta_i)_{i \in S} \) indicates the sub-vector of elements of \( \theta \) indexed by \( S \) and \( \theta_{i:S} = \prod_{j \in S} \theta_{i:j} \) denotes the monomial associated to an event \( y \in \mathcal{Y} \) where only parameters \( \theta_i \) for \( i \in S \) can have non-zero exponent.

Definition 2. Let \( \mathcal{Y} \) be a finite set and \( A \in \mathcal{M}_{k \times d}(\mathbb{Z}_{\geq 0}) \). Let \( S = \{S_1, \ldots, S_n\} \) be a partition of \( [k] \) such that \( \theta_{S_i} \in \Delta_{A_{S_i}, -1} \) for all \( i \in [n] \).

- A MDPM\((A, \theta)\) over \( \mathcal{Y} \) is called MM\((A, S)\) if for all \( y \in \mathcal{Y} \) it holds

\[
P(y) = \prod_{i \in [n]} \prod_{j \in S_i} \theta_{i:j}^{A_{i:j}} = \prod_{i \in [n]} \theta_{A_{S_i}}^{A_{S_i}}
\]

- A MM\((A, S)\) is said to be multilinear if \( A \in \mathcal{M}_{k \times d}([0, 1]) \).

- A multilinear MM\((A, S)\) is called regular if for all \( y \in \mathcal{Y} \) and all \( i \in [n], A_{i:j} = 1 \) for one \( j \in S_i \) and \( A_{i:k} = 0 \) for all \( k \in S_i \setminus \{j\} \).

A MM model is such that \( \theta = (\theta_{S_i})_{i \in [n]} \in \times_{i \in [n]} \Delta_{A_{S_i}, -1} \) and a MM is multilinear if all its monomials are square free, i.e. the exponents of the parameters are either zero or one.
Example 3. The simplest example of a MM(A, S) over a finite set \( \mathcal{Y} \) is the saturated model where \( P(y) = \theta_i \) for all \( y \in \mathcal{Y} \), i.e. one parameter is associated to the probability of each atomic event. In this case \( \theta \in \Delta_{q-1} \) and \( A \) is the \( q \)-by-\( q \) identity matrix. The model in Example 1 is not a MM.

Henceforth we work with regular multilinear MMs. The authors have not been able to find a multilinear MM model which is not regular nor to prove that any multilinear square-free MM is regular. There are commonly used models, often ones where probabilities are recursively defined, that have a MM representation but whose atomic probabilities are not multilinear [5][19]. BNs, staged trees that admit a multilinear monomial representation as well as decomposable Markov networks and context specific BNs are regular. Conversely, the model in Example 1 which is not a MM, does not respect the property of regularity since \( P(4) = \theta_2\theta_3 \) whilst these parameters belong to \( \Delta_2 \).

2.1. Bayesian networks

Many discrete statistical problems in a variety of domains are often modeled using BNs and there are now thousands of practical applications of these models [22]. A BN expresses graphically a collection of conditional independence [25][27]. For a random vector \( Y = (Y_i)_{i \in [m]} \) taking values in the Cartesian product \( \mathcal{Y} = \prod_{i \in [m]} \mathcal{Y}_i \) and three disjoint subsets \( B, C, D \) of \([m]\), the marginal vector \( Y_B \) is said to be conditionally independent of \( Y_C \) given \( Y_D \) if \( P(Y_B = b|Y_C = c, Y_D = d) = P(Y_B = b|Y_D = d) \), for all \( b \in \mathcal{Y}_B, c \in \mathcal{Y}_C \) and \( d \in \mathcal{Y}_D \).

A BN over a discrete random vector \( Y = (Y_i)_{i \in [m]} \) is given by

- \( m - 1 \) conditional independence statements of the form \( Y_i \perp Y_{[i-1]} | Y_{\Pi_i} \), where \( \Pi_i \subseteq [i \uparrow] \);
- a directed acyclic graph \( G = (\mathcal{V}, \mathcal{E}) \) with vertex set \( \mathcal{V} = \{ Y_i : i \in [m] \} \) and edge set \( \mathcal{E} = \{(Y_i, Y_j) : j \in [m], i \in \Pi_j \} \);
- conditional probabilities \( P(Y_i = y_i|Y_{\Pi_i} = y_{\Pi_i}) \) for every \( y_i \in \mathcal{Y}_i, y_{\Pi_i} \in \prod_{j \in \Pi_i} \mathcal{Y}_j \) and \( i \in [m] \).

The components of the vector \( Y_{\Pi_i} \) are said to be the parents of the vertex \( Y_i \), and in the graphical representation of a BN there is an arrow from each component of \( Y_{\Pi_i} \) pointing into \( Y_i \). For a vertex \( Y_i \) with parents \( Y_{\Pi_i} \), let \( \theta_{i|y_{\Pi_i}} = P(Y_i = y_i|Y_{\Pi_i} = y_{\Pi_i}) \). The probability of any atom \( y = (y_1, \ldots, y_m) \in \mathcal{Y} \) can then be written as the monomial \( P(Y = y) = \prod_{i \in [m]} \theta_{i|y_{\Pi_i}} \). Notice that \( \{\theta_{i|y_{\Pi_i}} \}_{i \in [m]} \in \Delta_{\mathcal{Y}_i-1} \) for all \( i \in [m] \) and any possible value of the parent set of node \( i \), namely \( y_{\Pi_i} \in \mathcal{Y}_{\Pi_i} \). Thus a BN is a multilinear regular MM where parameters associated to each vertex conditionally to each combination of parents need to respect the sum-to-one condition.

Example 4. Suppose we are interested in studying how a population’s health (Y1) is affected by both sports activity (Y2) and alcoholic drinking habits (Y3). Suppose these three variables can be categorized into high, medium and low, coded with 3, 2 and 1 respectively. Suppose that health’s levels are a function of both sports activity and drinking habits and that people who choose to work out a lot tend to drink less alcohol. This situation can be depicted by the complete BN in Figure 2 with probabilities

\[
P(Y_1 = i) = \theta_i, \quad P(Y_2 = j|Y_1 = i) = \theta_{ij}, \quad P(Y_3 = l|Y_2 = j, Y_1 = i) = \theta_{ijkl}
\]

where \( \sum_{i \in [3]} \theta_i = 1, \sum_{j \in [3]} \theta_{ij} = 1 \) and \( \sum_{l \in [3]} \theta_{ijkl} = 1 \) for all \( i, j \in [3] \). The associated MM is given in Table 1 where the monomial representation of the 27 atomic probabilities is listed. Of course the parameters \( \theta_k, \theta_{ij} \) and \( \theta_{ijkl} \) can be renamed to give some \( \{\theta_l\}_{l \in [39]} \). The A matrix has dimension \( 27 \times 39 \) and is very sparse: in each row there is a one in three positions and zero otherwise.
Edges emanating from vertex $v_2$. In Figure 3 the transition probabilities from vertex $v_2$ denote the different levels of sports activity, whilst the edges emanating from $v_1$, $v_2$ and $v_3$ denote the levels of alcohol consumption conditional on the level of activity. Edges emanating from $v_4$, . . . , $v_{12}$ are associated to the population’s health conditional on both preceding variables. A (conditional/transition) probability is associated to each edge and probabilities from edges emanating from the same non-leaf vertex must sum to one. The atomic probabilities are then simply given by the product of the edge probabilities along a root-to-leaf path.

Staged trees [20,31,32] are a particular class of trees where conditional probability distributions emanating from different vertices are identified. This is denoted by framing vertices whose distributions are identified by the same shape. In Figure 3 the transition probabilities from vertex $v_7$ to $v_{22}$ and from $v_9$ to $v_{25}$ are equal because $v_7$ and $v_8$ are in the same stage, i.e. vertices whose distributions are identified. Setting transition probabilities equal can be thought of as representing context-specific conditional independence information. Staged trees are capable of representing all conditional independence hypotheses within discrete BNs [31]. At the same time they are a larger class of statistical models, as illustrated next.

In Example 4 suppose the following equalities are believed to hold:

$$P(Y_1 = y_1|Y_2 = 3, Y_1 = 2) = P(Y_3 = y_3|Y_2 = 2, Y_1 = 2),$$  

(1)

$$P(Y_3 = y_3|Y_2 = 3, Y_1 = 1) = P(Y_3 = y_3|Y_2 = 2, Y_1 = 1),$$  

(2)

for all $y_3 \in [3]$. For instance, equation (2) states that the probability distribution of health for individuals with high alcohol consumption and low physical activity is equal to that of individuals with medium alcohol consumption and low physical activity. Such context-specific independence constraints cannot be explicitly represented in a BN model.
Table 2: Monomial atomic probabilities of the staged tree in Figure 3.

Conversely they have a straightforward representation in the staged tree reported in Figure 3 which is stratified according to the definition of 11. The atomic probabilities of this staged tree are multilinear monomials in the parameters associated to edges, where pairs of parameters from two vertices in the same stage are identified. Letting \( \theta_{ij} \) denote the transition probability from \( v_i \) to \( v_j \), the staged tree in Figure 3 has the following probabilities identified:

\[
\begin{align*}
\theta_{v_1 v_2} &= \theta_{v_3 v_2}, & \theta_{v_1 v_3} &= \theta_{v_3 v_3}, & \theta_{v_1 v_4} &= \theta_{v_3 v_4}, & \theta_{v_1 v_5} &= \theta_{v_3 v_5}, \\
\theta_{v_2 v_5} &= \theta_{v_3 v_5}, & \theta_{v_3 v_2} &= \theta_{v_3 v_3}, & \theta_{v_3 v_3} &= \theta_{v_3 v_4}, & \theta_{v_3 v_4} &= \theta_{v_3 v_5}. 
\end{align*}
\]

(3)

The parameter equalities in the first row of equation (3) derive from equation (1). Similarly, the bottom row of (3) is associated to (2). Given these constraints, the atomic probabilities of the staged tree in Figure 3 are those reported in Table 2. For a formal derivation see [19]. The A matrix has dimensions 27 \( \times \) 33, is very sparse (again in each row there are 3 ones and 24 zeros) and there are 11 elements in the partition of 27 of Definition 2.

In general staged trees may not be multilinear MMVs since two vertices in the same stage can possibly be along a same root-to-leaf path. However, staged trees that are multilinear are also regular since no parameters associated to emating edges from vertices in the same stage appears in one atomic probability monomial.

3. I-projections

As a measure of closeness of two distributions we consider the I-divergence and in the sequel we follow [12] Chapter 3.

**Definition 3.** Let \( P \) and \( Q \) be two probability distributions over a finite space \( \mathcal{Y} \). The I-divergence (or Kullback-Leibler divergence) from \( P \) to \( Q \) is defined as

\[
\mathcal{D}(Q \parallel P) = \sum_{y \in \mathcal{Y}} Q(y) \ln \frac{Q(y)}{P(y)}
\]

It is often of interest to find the distribution that, within a given set, is closest to a given \( P \). I-projections formalize this idea. I-projections are used e.g. for maximum likelihood estimation in the context of the exponential families.

**Definition 4.** Let \( L \) be a closed, convex set in the pointwise topology of distributions over \( \mathcal{Y} \). The I-projection of a distribution \( P \) over \( \mathcal{Y} \) onto \( L \) is a distribution \( P^* \in L \) such that

\[
\mathcal{D}(P^* \parallel P) = \min_{Q \in L} \mathcal{D}(Q \parallel P).
\]

If \( P \in L \) then \( P^* = P \). The fact that \( L \) is closed and convex guarantees that \( P^* \) exists in \( L \), and for strictly positive probabilities \( P^* \) is unique.

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Theorem 1. Let $P'$ be the I-projection of $P$ in $L$. For all $Q \in L$ it holds
\[ \mathcal{D}(Q \parallel P) \geq \mathcal{D}(Q \parallel P') + \mathcal{D}(P' \parallel P). \]

As a straightforward consequence of Theorem 1 if the Pythagorean identity
\[ \mathcal{D}(Q \parallel P) = \mathcal{D}(Q \parallel R) + \mathcal{D}(R \parallel P) \]  
holds for all $Q \in L$ and a specific $R \in L$ then $R = P'$. Theorem 1 and equation (4) are used extensively in Section 5 to prove the optimality of the new multi-way covariation schemes introduced in Section 4.4.

4. Sensitivity analysis

4.1. Covariation

When some parameters of a (conditional) probability distribution are varied to a new specific value, then the remaining parameters need to be adjusted (or to covary) to respect the sum-to-one condition of probability measures. In the binary case when one of the two parameters is varied this is straightforward, since the second parameter will be equal to one minus the other. But in generic discrete finite cases there are various considerations to be taken into account, as reviewed below.

We start by giving an alternative definition of a covariation scheme to [29]. Our definition of covariation allows more than one or no parameters to be varied and maps into a probability simplex, i.e. the scheme is valid [29]. Let $k$ be the number of parameters in the model, $\emptyset$ the empty set and let $|v|$ denote the sum of the elements of a vector $v$. 

Definition 5. For $\emptyset \neq V \subseteq [k]$, let $\theta_S \in \Delta_{k-1}$ be partitioned as $\theta_S = (\theta_V, \theta_{S \setminus V})$ and let $\hat{\theta}_V$ be such that $|\hat{\theta}_V| \in (0, 1)$. A $\hat{\theta}_V$-covariation scheme is a function $\sigma$ from $\Delta_{k-1}$ to $\Delta_{k-1}$ which fixes the subvector $\theta_V$ of $\theta_S$ to $\hat{\theta}_V$, i.e.
\[ \sigma : \Delta_{k-1} \rightarrow \Delta_{k-1} \]
\[(\theta_V, \theta_{S \setminus V}) \mapsto (\hat{\theta}_V, \cdot). \]

When $V = \emptyset$, the $\hat{\theta}_V$-covariation scheme is the identity function.

Thus $\hat{\theta}_S$ denotes a vector of parameters that need to respect the sum to one condition, $\hat{\theta}_V$ denotes the new numerical specification of the parameters varied, i.e. those with index in a set $V$, and the values of the parameters with index in $[k] \setminus S$ do not vary. Below we generalise some frequently applied covariation schemes.

Definition 6. In the notation of Definition 5

- the $\hat{\theta}_V$-proportional covariation scheme $\sigma_{\text{pro}}(\theta_S) = (\hat{\theta}_V, \hat{\theta}_{S \setminus V})$ is defined by setting
  \[ \hat{\theta}_j = \frac{1 - |\hat{\theta}_V|}{1 - |\hat{\theta}_V|} \theta_j \quad \text{for all } j \in S \setminus V. \]

- The $\hat{\theta}_V$-uniform covariation scheme, $\sigma_{\text{uni}}(\theta_S) = (\hat{\theta}_V, \hat{\theta}_{S \setminus V})$ is defined by setting
  \[ \hat{\theta}_j = \frac{1 - |\hat{\theta}_V|}{|S| - |V|} \theta_j \quad \text{for all } j \in S \setminus V. \]

Different covariation schemes may entertain different properties which, depending on the domain of application, might be more or less desirable [see 26, 29, for a list]. Definition 6 extends the proportional and uniform covariation schemes given in [29] to cases where one or more parameters are varied.

Example 5. Consider $\theta_S = (\theta_1, \theta_2, \theta_3) \in \Delta_2$, $V = \{1\}$ and $\hat{\theta}_V = 0.4$. The simplex $\Delta_2$ is given by the surface in Figure 4 whilst the dark full line denotes the image of any $\hat{\theta}_V$-covariation scheme $\sigma$ which fixes $\hat{\theta}_1 = 0.4$, i.e. the set defined by the intersection of the simplex with the affine variety defined by $\theta_1 = 0.4$. That is, this line describes all possible ways $\theta_2$ and $\theta_3$ can be covaried. When $\sigma$ is the uniform covariation scheme any $\theta_S \in \Delta_2$ is projected to the same point, as illustrated by the dotted lines in Figure 4. Conversely, the dashed lines refer to the proportional covariation scheme which can project points $\theta_S \in \Delta_2$ to different elements.
The order-preserving covariation scheme in Definition 7 follows from changing one parameter and its definition is slightly convoluted. Let us now take an instance, in a BN model the sets $S$, $V$ may be any proper subset of $\{1, \ldots, n\}$ and $\theta$ is a parameter vector of a MM. For $V \subset [k]$, let $V_i = S_i \cap V$ and $\sigma_i$ a $\theta_{V_i}$-covariation scheme for each $i \in [n]$. Then a $\theta_V$-covariation scheme for $\theta$ is a function $\sigma : \bigtimes_{i \in [n]} \Delta_{\theta_{S_i}, -1} \rightarrow \bigtimes_{i \in [n]} \Delta_{\theta_{S_i}, -1}$ such that $\sigma|_{\delta i} = \sigma_i$, where $\sigma|_{\delta i}$ denotes the restriction of $\sigma$ over $\Delta_{\theta_{S_i}, -1}$.

Definition 8. A $\theta_V$-covariation scheme for $\theta$ is called proportional if $\sigma_i$ is a $\theta_{V_i}$-proportional covariation scheme whenever $V_i \neq \emptyset$.

Definition 9. A $\theta_V$-covariation scheme for $\theta$ is called proportional if $\sigma_i$ is a $\theta_{V_i}$-proportional covariation scheme whenever $V_i \neq \emptyset$.

A public web app has been developed to intuitively perform such covariations and is available at the link https://manueleleonelli.shinyapps.io/covariation/.

Definitions 5 to 7 assume $\theta_S \in \Delta_{\theta_{S}, -1}$. Next we specialize them to apply to the parameter vector $\theta$ of a MM.
Theorem 2. If $P \in \text{MM}(A, S)$ then $\sigma(P) \in \text{MM}(A, S)$ for all covariation schemes $\sigma$ in Definition 8.

Proof. The result follows by noting that the probability $\sigma(P)(y) = \prod_{i \in [k]} \prod_{j \in S_i} \tilde{\theta}_{i,j}^{\lambda_{i,j}}$ for all $y \in Y$. \qed

4.2. Sensitivity functions

Sensitivity functions [10, 26] are frequently used during model validation to investigate how an output probability of interest varies as one (or possibly more) model’s parameter is allowed to change. They are particularly useful since, for instance, the conditional specification of probabilities in a BN might imply a marginal probability which appears to be unreasonable to a user, although being a coherent consequence of his/her beliefs. Sensitivity functions depict the required change of a parameter that would give a reasonable marginal probability. In essence they represent the functional relationship between a parameter being varied and the output probability of an event of interest.

Let $\text{MM}(A, S)$ be a multilinear model over a finite set $Y$ whose parameter $\theta$ is a $k$-dimensional vector partitioned in $S_1, \ldots, S_n$ as in Definition 8. In the setting of Definition 8 we fix $V \subset [k]$ and consider $V_i = V \cap S_i$ for $i \in [n]$. Now we allow the $\tilde{\theta}_{i,j}$, $i \in [n]$, to vary so that $|\tilde{\theta}_{i,j}| \in (0, 1)$ and for each value of $\tilde{\theta}_{i,j}$ consider a $\tilde{\theta}_{i,j}$-covariation scheme. For this family of covariation schemes and for an event $E \subset Y$ of interest, Definition 10 gives the probability of $E$ under the different covariation schemes in the family.

Definition 10. Let $\sigma$ be a $\tilde{\theta}_{i,j}$-covariation scheme. For $P \in \text{MM}(A, S)$ the probability $\sigma(P)(E)$ read as function of $\tilde{\theta}_{i,j}$ is called sensitivity function associated to the (family of) $\tilde{\theta}_{i,j}$-covariation schemes.

By Theorem 2 if $P \in \text{MM}(A, S)$ then $\sigma(P) \in \text{MM}(A, S)$, and in particular all $\sigma(P)(E)$ respect the monomial structure of $P$. The fact that the resulting probability of an event of interest $\sigma(P)(E)$ depends on the covariation scheme used is illustrated by the following example.

Example 6. The BN in Figure 2 is refined with the numerical specification of its parameters given in Appendix A. The probability of an individual being healthy is the event of interest, i.e. $P(Y_3 = 3)$. Given the elicited probabilities, this is equal to 0.343. However, for the population investigated $P(Y_3 = 3)$ is known to be lower than 0.3. To achieve this upper bound it is decided to try varying the parameter $\theta_2 \in (0, 1)$. Figure 5 reports the sensitivity functions for three families of covariation schemes. In a family all $\tilde{\theta}_{2,j}$-covariation schemes are proportional (full line), in another family they are uniform (dashed line) and in the third family they are order-preserving (dotted line). For proportional and uniform covariation, $\theta_2$ needs to be varied to around 0.6, whilst for order-preserving covariation the required bound cannot be achieved, indeed order-preserving covariation restricts the values the varied parameter can take.

As highlighted by Example 6 a parameter variation might be enforced to entertain some specific bounds on probabilities of interest. However these probabilities are affected by the choice of the covariation scheme. In some simple cases proportional covariation has been demonstrated to be optimal, in the sense that the original and the resulting probability distributions are as close as possible.
4.3. Global dissimilarity

The closeness of the original and varied distributions can be quantified using different distances or divergences. The most commonly used distance in sensitivity studies is the so called CD distance [8]. The CD distance is defined as the DeRobertis distance, which has been used for quite some time in the Bayesian inference literature [21]. For two probability distributions $P$ and $Q$ over a finite space $Y$ this is

$$D_{CD}(P, Q) = \ln \max_{y \in Y} \left( \frac{P(y)}{Q(y)} \right) - \ln \min_{y \in Y} \left( \frac{P(y)}{Q(y)} \right) = \max_{y, y' \in Y} \ln \left( \frac{P(y)}{Q(y')} \frac{Q(y')}{P(y')} \right).$$

Until recently, proportional covariation had a theoretical justification only for one-way analyses in BN models, since this scheme minimizes the CD distance between the original and the varied distributions [6]. In [26] it is proven that this is also true for full single CPT analyses in any multilinear MM.

Proportional covariation also minimizes the $\phi$-divergence from the original to the varied distributions in full single CPT analyses [26]. The $\phi$-divergence between from $P$ to $Q$ is defined as

$$D_\phi(Q || P) = \sum_{y \in Y} P(y)\phi \left( \frac{Q(y)}{P(y)} \right), \quad \phi \in \Phi,$$

where $\Phi$ is the class of convex functions $\phi(x), x \geq 0$, such that $\phi(1) = 0, \phi(0/0) = 0$ and $\phi(x/0) = \lim_{x \to \infty} \phi(x)/x$. The I-divergence in Definition [3] can be seen as a special instance of $\phi$-divergences for $\phi(x) = x \ln(x)$.

Example 7. Figure [6] reports the CD distance and I-divergence for the BN in Example [4] under different covariation schemes. These plots show the optimality of proportional covariation which for both metrics takes smaller values than the other schemes.

4.4. Multi-way sensitivity analyses

Full single CPT analyses, for which the optimality of proportional covariation has been already proven, highly restrict the parameters that can be varied. To our knowledge, the only attempt in defining other multi-way schemes is given in [3], where balanced schemes are introduced. In a nutshell, these reduce a multi-way problem into a one-way analysis by restricting the possible parameter variations.

Because of the monomial structure of their atomic probabilities, more general covariation schemes can be defined for MMs. These new multi-way analyses depend on the partition $\{S_1, \ldots, S_n\}$ of the $k$ parameters of a MM and on two sets $V_i = V \cap S_i$, where $V \subset [k] \text{ and } i \in [n]$. Let $C = \bigcup_{i \in [n], V_i \neq \emptyset} S_i$ be the union of all $S_i$ for which $V_i$ is not empty and $F = [k] \setminus C$. The set $F$ includes the indices of the parameters that do not need to be covaried, whilst $C$ is the index set of the (co)varied parameters. Definition [11] gives special ways to choose $V$ which depend on the model structure.

Definition 11. A sensitivity analysis is said to be
dependent analysis all sets

For a conditionally dependent analysis suppose monomials \( \theta \) in a conditionally dependent analysis di

\[ \{S_{k_1} \setminus V\} \cup \{V_1 \times [S_{2_1} \setminus V]\} \cup \{V_1 \times V_2 \times [S_{3_1} \setminus V]\} \cup \{V_1 \times V_2 \times V_3\}. \] (5)

The set \( C \) in an independent sensitivity analysis includes the indices of the varied or covaried parameters. In a fully dependent analysis all sets \( H \in X_{i \in [n]} V_i \neq \varnothing \) include the same number of indices, equal to \( \sum_{i \in [n]} 1_{\{S_i \cap V \neq \emptyset\}} \). Conversely in a conditionally dependent analysis different sets \( H \) include a different number of indices. This implies that all monomials \( \theta_H \) have the same degree in fully independent analyses, whilst they have different degrees in conditionally dependent ones.

**Example 8.** For a conditionally dependent analysis suppose \( \{S_{k_1}, \ldots, S_{k_l}\} = \{S_1, S_2, S_3\} \). Then equation (5) becomes

\[ \{S_1 \setminus V\} \cup \{V_1 \times [S_2 \setminus V]\} \cup \{V_1 \times V_2 \times [S_3 \setminus V]\} \cup \{V_1 \times V_2 \times V_3\}. \]

Although the definition of such new schemes may appear obscure, these have a very simple graphical interpretation and include some well-known sensitivity analyses. In an independent sensitivity analysis no (co)varied parameters appear in the same monomial. It thus includes the following analyses in BN models:

- one-way sensitivity analyses: one parameter of a CPT of a vertex is varied
- full single CPT analyses: one parameter from each CPT of a vertex is varied
- multi-way analyses where two or more parameters from one CPT are varied
- multi-way analyses where parameters from CPTs associated to incompatible parent configurations are varied.

To see this consider the BN in Figure 2 and suppose the parameters \( \theta_{22} \) and \( \theta_{11} \) are varied, implying respectively that \( Y_1 = 2 \) and \( Y_1 = 1 \). From Table 1 we can see that no two parameters from the associated conditional probability laws appear in the same monomial

- any combination of the four above.

For staged trees, an independent analysis varies parameters associated to edges emanating from nodes that do not appear in the same root-to-leaf path.

Fully dependent sensitivity analyses are such that all possible combinations of (co)varied parameters in different sets \( S_i \), for \( i \in [n] \) such that \( V_i \neq \emptyset \), appear in at least one monomial. The simplest possible example of such analyses is in the case of a BN consisting of two independent random variables where one parameter from each distribution is varied. But more generally such analyses are varied to parameters in CPTs implying disjoint parent sets. To illustrate this consider the BN in Figure 2. The variation of one parameter from the distribution of \( Y_3|Y_1 \) and another from the distribution of \( Y_5|Y_2 \) would give a fully dependent sensitivity analysis since the conditioning variables are different. For staged trees, Figure 3 illustrates a fully dependent analysis. Varying one parameter from each of the two distributions defined over this tree would give a fully dependent analysis. This is true since this tree implies the underlying two random variables are independent.

Conditionally dependent analyses imply an order over the varied parameters. A varying parameter needs to be a probability which is conditional on the events associated to preceding varying parameters in this order. An example from Figure 7 illustrates this for BNs. Suppose the parameter associated to \( P(Y_2 = y_2) \) is varied. Then in a conditionally dependent analysis any parameter from \( P(Y_5 = y_5|Y_2 = y_2) \) can be varied and, if so, also any parameter from \( P(Y_6 = y_6|Y_5 = y_5, Y_2 = y_2) \). For an illustration of conditionally dependent analyses in staged trees consider Figure 3. If the probability \( \theta_{01} \) associated to the edge \((v_0, v_1)\) is varied, then one of the probabilities \( \theta_{14}, \theta_{15} \) or \( \theta_{16} \) can be varied.
5. A geometric characterization of sensitivity analysis

After the variation of some parameters of a MM with indices in $V$ to a value $\tilde{\theta}_V$, a $\tilde{\theta}_V$-covariation scheme needs to be applied to respect all sum-to-one conditions of the model. In this section, given $P \in \text{MM}(A,S)$ and its $\tilde{\theta}_V$-proportional covariation $\tilde{P}$ we first determine a family $L$ of $Q$ densities for which the Pythagorean equality $\mathcal{D}(Q || P) = \mathcal{D}(Q || \tilde{P}) + \mathcal{D}(\tilde{P} || Q)$ holds.

In the notation of Definition 8 let $\emptyset \neq V \subseteq [k]$, $C = \bigcup_{i \in |V|} V_i$ and $F = [k] \setminus C$. Let $\Delta^F = \prod_{i \in |V| \setminus V \cap \emptyset} \Delta_{S_i, \mathbf{1}}$ and $\Delta^C = \prod_{i \in |V| \setminus V \cap \emptyset} \Delta_{S_i, -\mathbf{1}}$. The set $[k]$ is so partitioned into $V, C \setminus V$ and $F$, namely the index set of the varied, covaried and fixed parameters. A generic parameter vector can be written as $\theta = (\theta_F, \theta_V, \theta_{C \setminus V})$ and for $P \in \text{MM}(A,S)$ the atomic probability of $y \in Y$ can be written as $P(y) = \theta_{F}^{y_F} \theta_V^{y_V} \theta_{C \setminus V}^{y_{C \setminus V}}$. For any given $\theta_F \in \Delta^F$, $\text{Slice}(\theta_F)$ is the subset of densities in $\text{MM}(A,S)$ for which the parameters indexed by $F$ take value $\theta_F$, namely

$$\text{Slice}(\theta_F) = \{P \in \text{MM}(A,S) : P(y) = \theta_{F}^{y_F} \theta_V^{y_V} \theta_{C \setminus V}^{y_{C \setminus V}} \text{ for all } \theta_C \in \Delta^C \text{ and } y \in Y\}.$$  

It holds $\text{MM}(A,S) = \bigcup_{\theta_F \in \Delta^F} \text{Slice}(\theta_F)$. Theorem 3 shows that $P \in \text{MM}(A,S)$ and its $\tilde{\theta}_V$-proportional covariation density belong to the same slice.

**Theorem 3.** Let $\theta = (\theta_F, \theta_V, \theta_{C \setminus V})$ be the parameter vector of $P \in \text{MM}(A,S)$ and let $\tilde{\theta} = (\tilde{\theta}_F, \tilde{\theta}_V, \tilde{\theta}_{C \setminus V})$ be the parameter vector of the $\tilde{\theta}_V$-proportional covariation of $P$ called $\tilde{P}$. Then $\tilde{P} \in \text{Slice}(\theta_F)$, that is $\tilde{\theta}_F = \theta_F$.

**Proof.** The proof follows straightforward from Definition 9. Indeed $\tilde{\theta}_F = \theta_F$, $\tilde{\theta}_V$ is given and $\tilde{\theta}_{C \setminus V} = \left( \left( \frac{1 - \theta_{C \setminus V}}{\theta_{C \setminus V}} \right)_{j \in |S|} \right)_{i \in |A|}$.

As customary in sensitivity analysis, we focus on $\text{Slice}(\theta_F)$ and characterize the family of distribution describing ways parameters can be covaried. To this end, let

$$L_{\text{semi}} = \text{Slice}(\theta_F) \cap \{Q \in \text{MM}(A,S) : \tilde{\theta}_V = \tilde{\theta}_V\},$$

denote the family of distributions where only the parameters $\theta_{C \setminus V}$ can vary. It follows from Theorem 3 that $\tilde{P} \in L_{\text{semi}}$.

The following example demonstrate that given a $\text{MM}(A,S)$ the choice of parameters varied affects the form of the family of densities for which the Pythagorean equality holds.
Example 9. Consider a MM(A, S) with parameters \((\theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3)\) such that \(|\{\theta_1, \theta_2, \theta_3\}| = 1\) and \(|\{\psi_1, \psi_2, \psi_3\}| = 1\), and matrix \(A\)

\[
\begin{array}{cccccc}
\theta_1 & \theta_2 & \theta_3 & \psi_1 & \psi_2 & \psi_3 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\(P(y)\)

where for clarity we labelled the columns and the rows with the associated parameters and events, respectively, and reported the atomic probabilities. This model can be depicted graphically by the staged tree in Figure 9. Parameter vectors \(\theta_i\) to be considered are \((\theta_1), (\theta_2), (\psi_1), (\psi_1)\) and \((\theta_2, \psi_1)\). All other subvectors \(\theta_i\) of \((\theta_1, \psi_1)_{i\in[3]}\) can be dealt with as one of the cases above by symmetry. For \(P, \tilde{P}, Q \in \text{MM}(A, S)\), we denote with \((\theta_1, \psi_1)_{i\in[3]}, (\tilde{\theta}_1, \tilde{\psi}_1)_{i\in[3]}\) and \((\theta, \tilde{\psi}_1)_{i\in[3]}\) the parameter vectors of \(P, \tilde{P}\) and \(Q\) respectively. In general, the I-divergence from \(P\) to \(Q\) takes the form

\[
\mathcal{D}(Q \parallel P) = \sum_{i \in [3]} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\bar{\theta}_i} \right) + \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right).
\]

The Pythagorean equality in equation (4) can then be written as

\[
\sum_{i \in [3]} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\bar{\theta}_i} \right) + \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) - \sum_{i \in [3]} \tilde{\theta}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) - \tilde{\psi}_i \ln \left( \frac{\tilde{\theta}_i}{\bar{\theta}_i} \right) - \sum_{i \in [3]} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\bar{\theta}_i} \right) - \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) = 0. \quad (6)
\]

Next we look at the form of the above equality for each of the possible varied parameter choices. For each case, we consider only densities \(Q \in L_{\text{sensi}}\) that are usually investigated in sensitivity analysis after a parameter variation.

1. For \(\tilde{\theta}_1 = \bar{\theta}_1\), we consider \(Q \in \text{Slice}(\psi_1, \psi_2, \psi_3)\) and \(\tilde{\psi}_1 = \bar{\psi}_1\). Then \(\tilde{P}\) has parameter vector \((\tilde{\theta}_1, \bar{\psi}_1)_{i\in[3]}\), whilst \(Q\) has parameters \((\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3, \bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)\). Under these conditions, equation (6) can be simplified to

\[
\sum_{i = 2, 3} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\bar{\theta}_i} \right) - \sum_{i = 2, 3} \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) = 0.
\]

By substituting \(\tilde{\theta}_i = \bar{\theta}_i(1 - \bar{\theta}_1)/(1 - \bar{\theta}_1)\) into the logarithms the above equation reduces to

\[
\ln \left( \frac{1 - \bar{\theta}_1}{1 - \bar{\theta}_1} \right) \sum_{i = 2, 3} (\tilde{\theta}_i - \bar{\theta}_i) = 0,
\]

which holds for all \(Q \in L_{\text{sensi}}\) since \(\sum_{i = 2, 3} \bar{\theta}_i = \sum_{i = 2, 3} \bar{\psi}_i = 1 - \bar{\theta}_1\).

2. For \(\tilde{\theta}_2 = \bar{\theta}_2\), we consider \(Q \in \text{Slice}(\psi_1, \psi_2, \psi_3)\) and \(\tilde{\psi}_2 = \bar{\psi}_2\). Then \(\tilde{P}\) has parameter vector \((\bar{\theta}_1, \tilde{\theta}_2, \bar{\theta}_3, \bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)\). Under these conditions, equation (6) can be written as

\[
\sum_{i = 1, 3} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\bar{\theta}_i} \right) - \sum_{i = 1, 3} \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) = 0,
\]

which can be simplified as in the previous case to show that the equality holds for all \(Q \in L_{\text{sensi}}\).

3. For \(\tilde{\theta}_3 = \bar{\theta}_3\), we consider \(Q \in \text{Slice}(\theta_1, \theta_2, \theta_3)\) and \(\tilde{\psi}_3 = \bar{\psi}_3\). Then \(\tilde{P}\) has parameter vector \((\theta_1, \tilde{\theta}_2, \theta_3, \bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)\). Under these conditions, equation (6) can be written as

\[
\theta_1 \left( \sum_{i = 2, 3} \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) - \sum_{i = 2, 3} \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\bar{\psi}_i} \right) \right) = 0,
\]

which can be simplified as in the two previous cases to show that the equality holds for all \(Q \in L_{\text{sensi}}\).
and the varied distribution, as reported in Figure 10. Consider now case 5 of Example 9 and suppose \( \tilde{\theta}_1 = \tilde{\theta}_1 \) and \( \tilde{\psi}_1 = \tilde{\psi}_1 \), since there are no parameters with index in \( F \). Then \( P \) has parameter vector \( (\tilde{\theta}_1, \psi_j)_{j \in [3]} \), whilst \( Q \) has parameters \( (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \psi_1, \psi_2, \psi_3) \). Under these conditions, equation (6) can be written as

\[
\sum_{i=2,3} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\tilde{\theta}_i} \right) - \sum_{i=2,3} \tilde{\theta}_i \ln \left( \frac{\tilde{\theta}_i}{\tilde{\theta}_i} \right) + \tilde{\theta}_1 \left( \sum_{i=2,3} \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\tilde{\psi}_i} \right) - \sum_{i=2,3} \tilde{\psi}_i \ln \left( \frac{\tilde{\psi}_i}{\tilde{\psi}_i} \right) - \tilde{\psi}_1 \ln \left( \frac{\tilde{\psi}_1}{\tilde{\psi}_1} \right) \right) = 0.
\]

The above equation can be simplified as in the previous cases to show that the equality holds for all \( Q \in L_{\text{sensi}} \).

4. For \( \tilde{\theta}_V = (\tilde{\theta}_1, \tilde{\psi}_1) \), we consider \( Q \) such that \( \tilde{\theta}_1 = \tilde{\theta}_1 \) and \( \tilde{\psi}_1 = \tilde{\psi}_1 \), since there are no parameters with index in \( F \). Then \( P \) has parameter vector \( (\tilde{\theta}_1, \psi_j)_{j \in [3]} \), whilst \( Q \) has parameters \( (\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \psi_1, \psi_2, \psi_3) \). Under these conditions, equation (6) can be written as

\[
\sum_{i=1,3} \left( \tilde{\theta}_1 \ln \left( \frac{\tilde{\theta}_1}{\tilde{\theta}_1} \right) - \tilde{\theta}_1 \ln \left( \frac{\tilde{\theta}_1}{\tilde{\theta}_1} \right) + \tilde{\theta}_1 \ln \left( \frac{\tilde{\theta}_1}{\tilde{\theta}_1} \right) \right) + \tilde{\theta}_1 \left( \tilde{\psi}_1 \ln \left( \frac{\tilde{\psi}_1}{\tilde{\psi}_1} \right) - \tilde{\psi}_1 \ln \left( \frac{\tilde{\psi}_1}{\tilde{\psi}_1} \right) \right) = 0.
\]

The first summation in the above equality is zero as in the previous cases. Rearranging and using the definition of proportional covariation, we can rewrite the equality as

\[
(\tilde{\theta}_1 - \tilde{\theta}_1) \left( \tilde{\psi}_1 \ln \left( \frac{\tilde{\psi}_1}{\tilde{\psi}_1} \right) + (1 - \tilde{\psi}_1) \ln \left( \frac{1 - \tilde{\psi}_1}{1 - \tilde{\psi}_1} \right) \right) = 0.
\]

Equation (6) cannot be simplified any further and consequently for the choice \( \tilde{\theta}_V = (\tilde{\theta}_1, \tilde{\psi}_1) \) the family of distributions for which the Pythagorean identity holds is restricted.

Notice that the first four choices of parameters corresponded to the analyses introduced in Definition [K], namely independent in the first three cases and conditionally independent in the fourth, whilst the last choice of parameters does not correspond to any of the newly introduced schemes.

**Example 10.** For the staged tree in Figure 9 we refine the model definition with the following probability specifications:

\( \theta_1 = 0.2, \theta_2 = 0.5, \theta_3 = 0.3, \psi_1 = 0.4, \psi_2 = 0.4 \) and \( \psi_3 = 0.4 \). Next consider the choices of parameters varied in points 4 and 5 of Example 9. First suppose that \( \tilde{\theta}_1 = 0.4 \) and \( \tilde{\psi}_1 = 0.2 \). Example 9 demonstrated that the Pythagorean identity holds for all \( Q \in L_{\text{sensi}} \). In this case proportional covariation minimizes the I-divergence between the original and the varied distribution, as reported in Figure 10. Consider now case 5 of Example 9 and suppose \( \tilde{\theta}_2 = 0.3 \) and \( \tilde{\psi}_1 = 0.2 \). Example 9 demonstrated that the Pythagorean identity does not hold for all \( Q \in L_{\text{sensi}} \). The identity holds in the restricted family of distributions characterized by equation (7). In this case the I-divergence is not minimized by proportional covariation as reported in Figure 11.

Given a \( P \in \text{MM}(A, S) \) and its \( \tilde{\theta}_V \)-proportional covariation \( \tilde{P} \), Theorem 4 identifies the set of distributions \( Q \in L_{\text{sensi}} \) that satisfy the Pythagorean identity in (4). For \( \emptyset \neq H \subseteq C \) define

\[
Y_H = \{ y \in Y : A_{i,j} = 1 \text{ for all } i \in H \text{ and } A_{j,i} = 0 \text{ for all } i \in C \setminus H \}
\]
Corollary 1. In the notation of Theorem 4, the density equation (8) has one term only depending on the density $Q$, namely $\sum_{H \subseteq H \neq \emptyset} \hat{\theta}_{\emptyset \setminus \emptyset \setminus \emptyset \setminus \emptyset} \ln \left( \frac{\hat{\theta}_{\emptyset \setminus \emptyset \setminus \emptyset \setminus \emptyset}}{\hat{\theta}_{\emptyset \setminus \emptyset \setminus \emptyset \setminus \emptyset}} \right) \hat{\theta}_{\emptyset \setminus \emptyset \setminus \emptyset \setminus \emptyset} \geq 0.$

If and only if $D(Q \| P) \geq D(Q \| \hat{P}) + D(\hat{P} \| Q).$
This result follows by substituting the equalities in the proof of Theorem 4 with inequalities.

Since for \( \bar{P} \), \( P \) and all the distributions \( Q \) characterized by equation (9) the Pythagorean disequality holds, then it can be proven that \( \bar{P} \) is the I-projection of \( P \) into this well-specified family of distributions. Let

\[
L_{\text{constr}} = L_{\text{sensi}} \cap \left\{ Q \in \text{MM}(A,S) : \sum_{H \subseteq C \setminus Y \neq \emptyset} \begin{array}{l}
\tilde{\theta}_{\mid Y \mid H} (\tilde{\theta}_{(Y \setminus H) \setminus Y} - \hat{\theta}_{(Y \setminus H) \setminus Y}) \ln \left( \frac{\hat{\theta}_{(Y \setminus H) \setminus Y}}{\tilde{\theta}_{\mid Y \mid H}} \right) + \alpha \sum_{y \in Y} \theta_{F}^{y} \geq 0 \end{array} \right\}.
\]

**Corollary 2.** In the notation of Theorem 4, \( \bar{P} \) is the I-projection of \( P \) in the set \( L_{\text{constr}} \).

**Proof.** Let \( L \) be the smallest convex and closed subset of \( \Delta_{q-1} \) which includes \( L_{\text{constr}} \). From Section 5 there exists a unique \( P' \in L \) such that \( D(Q \parallel P) \geq D(Q \parallel P') + D(P' \parallel P) \). But since \( \bar{P} \in L_{\text{constr}} \subseteq L \) satisfies the Pythagorean identity then \( \bar{P} = P' \).

Csiszár and Shields [12] proved that the I-projection satisfies the Pythagorean identity using the fact that \( L \) is closed and convex. Here we took a different approach by taking advantage of the specific monomial form of the statistical models we study. By characterizing the class of distributions for which the Pythagorean identity holds, we have then been able to prove that proportional covariation is the I-projection within this family.

Although Corollary 2 demonstrates that proportional covariation minimizes the I-divergence between the original distribution and those in the set \( L_{\text{constr}} \), it does not provide information on whether \( L_{\text{constr}} \) includes all distributions of interest in sensitivity analysis or not. More explicitly, Corollary 2 does not specify whether, given \( P \in \text{MM}(A,S) \) and \( \theta_{V} \), \( \bar{P} \) is the I-projection of \( P \) in \( L_{\text{sensi}} \). This is the case if and only if \( L_{\text{sensi}} = L_{\text{constr}} \), i.e. if for all \( Q \in L_{\text{sensi}} \) the condition in equation (7) holds. Theorem 5 below states that for the covariation schemes in Definition 11 proportional covariation is indeed the I-projection of the original distribution in the set of all distributions usually considered in sensitivity analysis. Namely for such schemes \( L_{\text{sensi}} = L_{\text{constr}} \).

**Theorem 5.** In the notation of Theorem 4, if \( \hat{\theta}_{V} \) is chosen according to an independent, fully dependent or conditionally dependent sensitivity analyses, then \( \bar{P} \) is the I-projection of \( P \) in \( L_{\text{sensi}} \).

The proof is given in Appendix B.2. Notice that the result holds for regular MMs and in particular it holds for all the already mentioned graphical models entertaining a monomial parameterization. Illustrations of this result were given in Example 9 in the first four cases, corresponding to independent or conditionally dependent analyses, the Pythagorean identity holds for all \( Q \in L_{\text{sensi}} \) and thus the \( \theta_{V} \)-proportional covariation scheme is the I-projection over the set of all distribution of interest. Conversely, in the fifth case of Example 9 which does not correspond to any of the new covariation schemes of Definition 11 the Pythagorean identity holds in a restricted set of distributions. Thus, as specified by Corollary 2 proportional covariation is the I-projection over this restricted space only.

6. **BN classifiers**

BN classifiers are BNs whose graph entertains some specific properties designed for classification problems. BN classifiers have been successfully used in a wide array of real-world applications, with a competitive predictive performance against other classification techniques, despite their intuitiveness and computational efficiency [see e.g. 24 for a review]. A BN classifier is defined by partitioning the BN vertex set into the set of features \( F \) and the classes \( C \), so that \( V = \{ Y_i : i \in F \} \cup \{ Y_i : i \in C \} \). Its edge set is such that feature variables are not allowed to have class children. For simplicity here we focus on univariate classification problems where there is a single class variable. However our results apply to multidimensional classes since in a BN classifier the class variables can be collapsed into a unique vertex.

BN classifiers range from the simplest Naive Bayes classifier where the features are conditionally independent given the class variable (given in Figure 12a), to generic dependence structures between the features (as for example in Figure 12b). A BN classifier of interest is the super-parent-one-dependence estimator (SPODE) where all features depend on one specific feature called super-parent (see Figure 12c).

Since BN classifiers are BN models, they can be represented as MMs as shown in Example 4. This is not the only representation a BN classifier can have as a MDPM [see e.g. 33, 34]. Since BN classifiers are MMs, we can apply our methodology and deduce the following result.
Theorem 6. Consider a Naive Bayes classifier with features $Y_{Fe_1}, \ldots, Y_{Fe_m}$. In the notation of Theorem 4, if $\tilde{\theta}_V$ is chosen so that $V \subset \times_{i \in [m]} Y_{Fe_i}$, then $\tilde{P}$ is the I-projection of $P$ in $\text{sens}$.

Proof. This result follows from Theorem 5 by noticing that (co)varied parameters conditionally on different values of $Y_{Cl}$ never appear in the same monomial, thus giving an independent sensitivity analysis. For each instantiation of $Y_{Cl}$, the feature variables are independent, thus giving a totally dependent sensitivity analysis. Since for these two analyses the $\tilde{\theta}_V$-proportional covariation scheme is optimal, the result then follows.

Thus in a Naive Bayes classifier for any choice of conditional probabilities from the feature variables to be varied, proportional covariation is optimal. This result can be extended straightforwardly to SPODE classifiers by excluding the super-parent node from the feature variables set. Then for any variation of probabilities of the other features, proportional covariation is optimal. For generic BN classifiers the optimality of proportional covariation holds for the cases formalized in Theorem 4 and Theorem 5.

7. Discussion

The representation of a wide array of statistical models in terms of the defining atomic monomial probabilities has proven useful for a variety of applications, including sensitivity analysis. In this paper, we took advantage of this representation to develop a formal geometric approach for sensitivity analysis which uses elements of information geometry. This approach has enabled us to demonstrate the optimality of a variety of multi-way schemes defined by the characteristics of the monomial atomic probabilities. Furthermore, Theorem 4 gives the necessary condition that any choice of parameters to vary needs to respect for optimal proportional covariation. Attention was devoted to BN classifiers where the tuning of the feature probabilities is often critical to ensure the classifier works reliably.

Although in this work we focused on models having multilinear atomic probabilities, our geometric approach could be used to investigate more general classes of models, for instance dynamic BNs whose atomic probabilities are not necessarily multilinear. Preliminary results suggest that the I-divergence exhibit different properties than in the multilinear case, with the potential of even more informative sensitivity investigations.

We concentrated on I-divergences but other measures of closeness between distributions could have been considered, for instance the already mentioned $\phi$-divergences and CD distances. It is yet unknown whether our newly introduced covariation schemes would be optimal under these other measures.

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Appendix A. Probabilities associated to the BN in Figure 2

| $\theta_{11}$ | $\theta_{12}$ | $\theta_{13}$ | $\theta_{14}$ | $\theta_{15}$ | $\theta_{31}$ | $\theta_{32}$ | $\theta_{33}$ | $\theta_{34}$ | $\theta_{35}$ |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0.2           | 0.3           | 0.5           | 0.2           | 0.3           | 0.5           | 0.2           | 0.3           | 0.5           | 0.2           |
| 0.3           | 0.2           | 0.4           | 0.7           | 0.2           | 0.3           | 0.6           | 0.5           | 0.2           | 0.1           |
| 0.1           | 0.2           | 0.3           | 0.5           | 0.1           | 0.3           | 0.2           | 0.4           | 0.5           | 0.3           |
| 0.2           | 0.3           | 0.5           | 0.1           | 0.1           | 0.1           | 0.1           | 0.2           | 0.2           | 0.1           |
| 0.8           | 0.1           | 0.5           | 0.1           | 0.7           | 0.2           | 0.3           | 0.5           | 0.2           | 0.1           |
| 0.3           | 0.5           | 0.1           | 0.1           | 0.2           | 0.2           | 0.5           | 0.1           | 0.1           | 0.1           |
Appendix B. Proofs

Appendix B.1. Proof of Theorem 4

Substituting \( \bar{\theta}_V = \bar{\theta}_V \) and \( Q \in \text{Slice}(\theta_F) \), we can write \( D(Q|P) \) as

\[
D(Q|P) = \sum_{y \in \mathbb{Y}} \theta_f^{A_{i,v}} \frac{\theta_{C_{i,v}}}{\theta_{C_{i,v}}} \ln \frac{\theta_{A_{i,v}}^{A_{i,v}} \theta_{C_{i,v}}^{A_{i,v}}}{\theta_{C_{i,v}}^{A_{i,v}}} = \sum_{y \in \mathbb{Y}} \theta_f^{A_{i,v}} \frac{\theta_{C_{i,v}}}{\theta_{C_{i,v}}} \ln \frac{\theta_{A_{i,v}}^{A_{i,v}} \theta_{C_{i,v}}^{A_{i,v}}}{\theta_{C_{i,v}}^{A_{i,v}}}. \tag{B.1}
\]

For all \( \emptyset \neq H \subset [k] \), define \( \mathbb{Y}_H^n = \{y \in \mathbb{Y} : A_{i,v} = 0, \text{ for all } i \in H\} \). Equation (B.1) can be split as

\[
D(Q|P) = \sum_{y \in \mathbb{Y}_C} \theta_f^{A_{i,v}} \frac{\theta_{C_{i,v}}}{\theta_{C_{i,v}}} \ln \frac{\theta_{A_{i,v}}^{A_{i,v}} \theta_{C_{i,v}}^{A_{i,v}}}{\theta_{C_{i,v}}^{A_{i,v}}} + \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \frac{\theta_{C_{i,v}}}{\theta_{C_{i,v}}} \ln \frac{\theta_{A_{i,v}}^{A_{i,v}} \theta_{C_{i,v}}^{A_{i,v}}}{\theta_{C_{i,v}}^{A_{i,v}}}. \tag{B.2}
\]

but since for all \( y \in \mathbb{Y}_C \) and \( i \in C \), \( A_{i,v} = 0 \), the second term on the rhs of equation (B.2) is equal to zero. The set \( \mathbb{Y} \setminus \mathbb{Y}_C \) includes all events \( y \) for which \( A_{i,v} = 1 \) for at least one \( i \in C \). Thus \( \mathbb{Y} \setminus \mathbb{Y}_C = \bigcup_{i \in C, \emptyset \neq H \subseteq C} \mathbb{Y}_H \), recalling that \( \mathbb{Y}_H \) is the set of events \( y \) for which \( A_{i,v} = 1 \) for \( i \in H \) and \( A_{i,v} = 0 \) for \( i \notin H \). Furthermore since these sets \( \mathbb{Y}_H \), for \( H \subseteq C \), are disjoint we have that

\[
D(Q|P) = \sum_{H \subseteq C, \emptyset \neq H \subseteq C} \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \frac{\theta_{C_{i,v}}}{\theta_{C_{i,v}}} \ln \frac{\theta_{A_{i,v}}^{A_{i,v}} \theta_{C_{i,v}}^{A_{i,v}}}{\theta_{C_{i,v}}^{A_{i,v}}} \tag{B.3}
\]

where terms in the internal sum are only for \( \mathbb{Y}_H \neq \emptyset \). For any \( H \subseteq C, \ P \in \text{MM}(A,S) \) and \( y \in \mathbb{Y}_H \), by multilinearity it holds

\[
\theta_{i,v}^{A_{i,v}} = \prod_{i \in V \cap H} \theta_i = \theta_{i,v}^{A_{i,v}}, \quad \theta_{C_{i,v}}^{A_{i,v}} = \prod_{i \in V \cap C \setminus H} \theta_i = \theta_{C_{i,v}}^{A_{i,v}}. \tag{B.4}
\]

Substituting equation (B.4) and using properties of the logarithm, equation (B.3) simplifies to

\[
D(Q|P) = \sum_{H \subseteq C, \emptyset \neq H \subseteq C} \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \bar{\theta}_V \bar{\theta}_{C_{i,v}} \ln \bar{\theta}_{C_{i,v}^{A_{i,v}}} \bar{\theta}_{C_{i,v}^{A_{i,v}}} \tag{B.5}
\]

Analogously

\[
D(P|Q) = \sum_{H \subseteq C, \emptyset \neq H \subseteq C} \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \bar{\theta}_V \bar{\theta}_{C_{i,v}} \ln \bar{\theta}_{C_{i,v}^{A_{i,v}}} \bar{\theta}_{C_{i,v}^{A_{i,v}}} \tag{B.6}
\]

\[
D(\bar{P}|\bar{Q}) = \sum_{H \subseteq C, \emptyset \neq H \subseteq C} \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \bar{\theta}_V \bar{\theta}_{C_{i,v}} \ln \bar{\theta}_{C_{i,v}^{A_{i,v}}} \bar{\theta}_{C_{i,v}^{A_{i,v}}} \tag{B.7}
\]

In equation (B.6) we used the assumption that \( \bar{\theta}_V = \bar{\theta}_V \) and in equations (B.6) and (B.7) we used Theorem 3. Next we use the fact that \( \theta_{C_{i,v}^{A_{i,v}}} = \theta_{C_{i,v}^{A_{i,v}}} \) is computed via proportional covariation. For \( H \subseteq C \) it holds that

\[
\bar{\theta}_{C_{i,v}^{A_{i,v}}} = \prod_{i \in V \cap H} \prod_{j \in S \cap H} \left[ \frac{1 - |\theta_i|}{1 - |\theta_i|} \right] \theta_{C_{i,v}^{A_{i,v}}} = \alpha \theta_{C_{i,v}^{A_{i,v}}} \tag{B.8}
\]

Substituting equation (B.8) into the logarithms in equations (B.6) and (B.7) and rearranging the factors yields

\[
D(Q|P) = \sum_{H \subseteq C, \emptyset \neq H \subseteq C} \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \bar{\theta}_V \bar{\theta}_{C_{i,v}^{A_{i,v}}} \ln \frac{\bar{\theta}_{C_{i,v}^{A_{i,v}}} \bar{\theta}_{C_{i,v}^{A_{i,v}}}}{\theta_{C_{i,v}^{A_{i,v}}} \theta_{C_{i,v}^{A_{i,v}}}} - \sum_{H \subseteq C, \emptyset \neq H \subseteq C} \sum_{y \in \mathbb{Y}_H} \theta_f^{A_{i,v}} \bar{\theta}_V \bar{\theta}_{C_{i,v}^{A_{i,v}}} \ln \alpha \tag{B.9}
\]

with \( \alpha = \left[ \frac{1 - |\theta_i|}{1 - |\theta_i|} \right] \theta_{C_{i,v}^{A_{i,v}}} \).
By rearranging the terms in equation (B.11) we have that

$$\sum_{H \in \mathbb{C} \neq \emptyset} \frac{\partial \theta}{\partial \theta} \left( \hat{\theta}_{C|V|\bar{H}} \ln \frac{\hat{\theta}_{V|H}}{\theta_{V|H}} + \hat{\theta}_{C|V|\bar{H}} \ln \frac{\hat{\theta}_{V|H}}{\theta_{V|H}} + \hat{\theta}_{C|V|\bar{H}} \ln \alpha \right) = 0. \quad (B.11)$$

By rearranging the terms in equation (B.11) we have that

$$\sum_{H \in \mathbb{C} \neq \emptyset} \frac{\partial \theta}{\partial \theta} \left( \hat{\theta}_{C|V|\bar{H}} \ln \frac{\hat{\theta}_{V|H}}{\theta_{V|H}} + \hat{\theta}_{C|V|\bar{H}} \ln \frac{\hat{\theta}_{V|H}}{\theta_{V|H}} + \hat{\theta}_{C|V|\bar{H}} \ln \alpha \right) = 0. \quad (B.12)$$

which yields

$$\sum_{H \in \mathbb{C} \neq \emptyset} \frac{\partial \theta}{\partial \theta} \left( \hat{\theta}_{C|V|\bar{H}} \ln \left( \frac{\hat{\theta}_{V|H}}{\theta_{V|H}} \right) \right) = 0. \quad (B.13)$$

Noticing that equation (B.13) equals equation (8), since only $\frac{\partial \theta}{\partial \theta}$ depends on the event $y \in \mathbb{Y}_H$, proves the result.

**Appendix B.2. Proof of Theorem 3**

The result is proven if the condition in equation (5) holds for all $Q \in L$. For an independent analysis, for all $i, j \in C$, the monomial $\theta_i \theta_j$ does not divide $\theta^k$ for any $y \in \mathbb{Y}$. Thus all sets $H$ in condition (3) that need to be considered, i.e. those such that $\mathbb{Y}_H$ is non-empty, have one element only because of regularity. If $H$ is an element of $V$ then $\hat{\theta}_{C|V|\bar{H}} - \hat{\theta}_{C|V|\bar{H}} = 0$ by construction and the result thus follows. Conversely, if $H$ is an element of $C \setminus V$, condition (3) holds if and only if

$$\sum_{j \in C \setminus V} \theta_j = \hat{\theta}_j = 0. \quad (B.14)$$

Equation (B.14) can be rewritten as

$$\sum_{j \in [n], V \neq \emptyset} 1 - |\theta_j| - |\hat{\theta}_j| = 0,$$

which is always true. This proves Theorem 3 for independent analyses.

In a totally dependent sensitivity analysis all sets $H$ in condition (3) that need to be considered, i.e. those such that $\mathbb{Y}_H$ is non-empty, are in $\times_{i \in [n], V \neq \emptyset} S_i$, by regularity. Thus equation (8) can be written as

$$\sum_{H \in \mathbb{C} \neq \emptyset} \frac{\partial \theta}{\partial \theta} \left( \hat{\theta}_{C|V|\bar{H}} \ln \left( \frac{\hat{\theta}_{V|H}}{\theta_{V|H}} \right) \right) \sum_{y \in \mathbb{Y}_n} \frac{\partial \theta}{\partial \theta} = 0. \quad (B.15)$$

Suppose with no loss of generality that the sets $S_i$, such that $V_i \neq \emptyset$ are those with index in the set $[r], r \leq n$. Notice that

$$\bigwedge_{i \in [r]} S_i = \bigcup_{R \subseteq [r]} \left( \bigwedge_{i \in [r]} S_i \right) \left( \bigcap_{i \in [r], R} [C_i \setminus V_i] \right). \quad (B.16)$$
Thus the result is proven if the equality in (B.15) holds for each $R \subseteq [r]$, i.e. if
\[
\sum_{H \in R} \sum_{J \in [r] \setminus R} \tilde{\theta}_{V \setminus H} (\tilde{\theta}_{C \setminus V \setminus J} - \tilde{\theta}_{C \setminus V \setminus J}) \ln \left( \frac{\tilde{\theta}_{V \setminus H}}{\tilde{\theta}_{V \setminus H}} \right) \sum_{y \in \mathcal{Y}_{(V \setminus J) \setminus (H)}} \theta_F^{h,y} = 0, \tag{B.17}
\]
for $H$ and $J$ such that $\mathcal{Y}_{(H \cup J) \setminus \emptyset} \neq \emptyset$. First notice that if $R = \{r\}$, then $\bar{\theta}_{C \setminus V \setminus J} - \tilde{\theta}_{C \setminus V \setminus J} = 0$ by construction and the result follows. Now fix an $R \subset [r]$ and suppose $k \in [r] \setminus R$. Equation (B.17) can be written as
\[
\sum_{H \in R} \sum_{J \in [r] \setminus R} \tilde{\theta}_{V \setminus H} (\tilde{\theta}_{C \setminus V \setminus J} - \tilde{\theta}_{C \setminus V \setminus J}) \ln \left( \frac{\tilde{\theta}_{V \setminus H}}{\tilde{\theta}_{V \setminus H}} \right) \sum_{y \in \mathcal{Y}_{(V \setminus J) \setminus (H)}} \theta_F^{h,y} = 0. \tag{B.18}
\]
Noticing that $\sum_{j \in C \setminus V} \tilde{\theta}_j = \sum_{j \in C \setminus V} \bar{\theta}_j = 1 - |\tilde{\theta}_V|$, equation (B.18) can be rearranged as
\[
\sum_{H \in R} \sum_{J \in [r] \setminus R} \tilde{\theta}_{V \setminus H} (\tilde{\theta}_{C \setminus V \setminus J} (1 - |\tilde{\theta}_V|) - \tilde{\theta}_{C \setminus V \setminus J} (1 - |\tilde{\theta}_V|)) \ln \left( \frac{\tilde{\theta}_{V \setminus H}}{\tilde{\theta}_{V \setminus H}} \right) \sum_{y \in \mathcal{Y}_{(V \setminus J) \setminus (H)}} \theta_F^{h,y} = 0. \tag{B.19}
\]
By applying the same steps as in equations (B.18)-(B.19) for all $k \in [r] \setminus R$, we have that
\[
\sum_{H \in R} \tilde{\theta}_{V \setminus H} \left( \prod_{k \in [r] \setminus R} (1 - |\tilde{\theta}_V|) - \prod_{k \in [r] \setminus R} (1 - |\tilde{\theta}_V|) \right) \ln \left( \frac{\tilde{\theta}_{V \setminus H}}{\tilde{\theta}_{V \setminus H}} \right) \sum_{y \in \mathcal{Y}_{(V \setminus J) \setminus (H)}} \theta_F^{h,y} = 0, \tag{B.20}
\]
which always holds, thus proving Theorem 5 for fully dependent analyses.

The proof for conditionally dependent analyses follows from the one of fully dependent sensitivity analyses by noticing that the sets $R \subseteq [r]$ for which condition (B.17) needs to hold is a subset of those already demonstrated in the fully dependent case.