Perturbation Theory for the Multidimensional Schrödinger Operator with a Periodic Potential

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Abstract

In this paper we obtain asymptotic formulas of arbitrary order for the Bloch eigenvalue and the Bloch function of the periodic Schrödinger operator $-\Delta + q(x)$, of arbitrary dimension, when corresponding quasi-momentum lies near a diffraction hyperplane. Moreover, we estimate the measure of the isoenergetic surfaces in the high energy region.

Besides, writing the asymptotic formulas for the Bloch eigenvalue and the Bloch function, when corresponding quasimomentum lies far from the diffraction hyperplanes, obtained in my previous papers in improved and enlarged form, we obtain the complete perturbation theory for the multidimensional Schrödinger operator with a periodic potential.

1 Introduction

In this paper we consider the operator

$$L(q(x)) = -\Delta + q(x), \ x \in \mathbb{R}^d, \ d \geq 2$$

(1)

with a periodic (relative to a lattice $\Omega$) potential $q(x) \in W^s_2(F)$, where

$$s \geq s_0 = \frac{3d-1}{2}(3d^2+d+2) + \frac{1}{4}d^3 + d + 6, \ F \equiv \mathbb{R}^d/\Omega$$

is a fundamental domain of $\Omega$. Without loss of generality it can be assumed that the measure $\mu(F)$ of $F$ is 1 and $\int_F q(x)dx = 0$. Let $L_t(q(x))$ be the operator generated in $F$ by (1) and the conditions:

$$u(x + \omega) = e^{i(t,\omega)}u(x), \ \forall \omega \in \Omega,$$

(2)

where $t \in F^* \equiv \mathbb{R}^d/\Gamma$ and $\Gamma$ is the lattice dual to $\Omega$, that is, $\Gamma$ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $(\gamma, \omega) \in 2\pi\mathbb{Z}$ for all $\omega \in \Omega$. It is well-known that ( see [2]) the spectrum of the operator $L_t(q(x))$ consists of the eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq \ldots$. The function $\Lambda_n(t)$ is called $n$th band function and its range $A_n = \{\Lambda_n(t) : t \in F^*\}$ is called the $n$th band of the spectrum $Spec(L)$ of
$L$ and $\text{Spec}(L) = \bigcup_{n=1}^{\infty} A_n$. The eigenfunction $\Psi_{n,t}(x)$ of $L_t(q(x))$ corresponding to the eigenvalue $\Lambda_n(t)$ is known as Bloch functions. In the case $q(x) = 0$ these eigenvalues and eigenfunctions are $|\gamma + t|^2$ and $e^{i(\gamma + t,x)}$ for $\gamma \in \Gamma$.

This paper consists of 6 section. First section is the introduction, where we describe briefly the scheme of this paper and discuss the related papers.

In papers [13-17] for the first time the eigenvalues $|\gamma + t|^2$, for big $\gamma \in \Gamma$, were divided into two groups: non-resonance ones and resonance ones and for the perturbations of each group various asymptotic formulae were obtained. Let the potential $q(x)$ be a trigonometric polynomial

$$\sum_{\gamma \in Q} q_{\gamma} e^{i(\gamma,x)},$$

where $q_{\gamma} = (q(x), e^{i(\gamma,x)}) = \int_{F} q(x) e^{-i(\gamma,x)} dx$, and $Q = \{ \gamma \in \Gamma : q_{\gamma} \neq 0 \}$ consists of a finite number of vectors $\gamma$ from $\Gamma$. Then the eigenvalue $|\gamma + t|^2$ is called a non-resonance eigenvalue if $\gamma + t$ does not belong to any of the sets $W_{b,\alpha_1} = \{ x \in \mathbb{R}^d : |x|^2 - |x + b|^2 < |x|^\alpha_1 \}$, that is, if $\gamma + t$ lies far from the diffraction hyperplanes $D_b = \{ x \in \mathbb{R}^d : |x|^2 = |x + b|^2 \}$, where $\alpha_1 \in (0,1)$, $b \in Q$ (see [15-17]). The idea of the definition of the non-resonance eigenvalue $|\gamma + t|^2$ is the following. If $\gamma + t \notin W_{b,\alpha_1}$ then the influence of $q_{\gamma} e^{i(\gamma,x)}$ to the eigenvalue $|\gamma + t|^2$ is not significant. If $\gamma + t$ does not belong to any of the sets $W_{b,\alpha_1}$ for $b \in Q$ then the influence of the trigonometric polynomial $q(x)$ to the eigenvalue $|\gamma + t|^2$ is not significant. Therefore the corresponding eigenvalue of the operator $L_t(q(x))$ is close to the eigenvalue $|\gamma + t|^2$ of $L_t(0)$.

If $q(x) \in W^2_{2}(F)$, then to describe the non-resonance and resonance eigenvalues $|\gamma + t|^2$ of the order of $\rho^2$ (written as $|\gamma + t|^2 \sim \rho^2$) for big parameter $\rho$ we write the potential $q(x) \in W^2_{2}(F)$ in the form

$$q(x) = \sum_{\gamma \in \Gamma(\rho^\alpha)} q_{\gamma} e^{i(\gamma,x)} + O(\rho^{-p\alpha}),$$

(3)

where $\Gamma(\rho^\alpha) = \{ \gamma \in \Gamma : 0 < |\gamma| < \rho^\alpha \}$, $p = s - d$, $\alpha = \frac{1}{q}$, $q = 3^d + d + 2$, and the relation $|\gamma + t|^2 \sim \rho^2$ means that $c_1 \rho < |\gamma + t| < c_2 \rho$. Here and in subsequent relations we denote by $c_i$ ($i = 1, 2, \ldots$) the positive, independent of $\rho$ constants whose exact values are inessential. Note that $q(x) \in W^2_{2}(F)$ means that $\sum_{\gamma} |q_{\gamma}|^2 (1 + |\gamma|^2 s) < \infty$. If $s \geq d$, then

$$\sum_{\gamma} \sum_{\gamma \in \Gamma(\rho^\alpha)} |q_{\gamma}| \leq \sum_{|\gamma| \geq c_1 \rho^\alpha} |q_{\gamma}| = O(\rho^{-p\alpha}),$$

(4)
i.e., (3) holds. It follows from (4) that the influence of $\sum_{\gamma \in \Gamma(\rho^\alpha)} q_{\gamma} e^{i(\gamma,x)}$ to the eigenvalue $|\gamma + t|^2$ is $O(\rho^{-p\alpha})$. If $\gamma + t$ does not belong to any of the sets $W_{b,\alpha_1} = \{ x \in \mathbb{R}^d : |x|^2 - |x + b|^2 < c_2 \} = \{ b \in \Gamma(\rho^\alpha) \}$, then the influence of the trigonometric polynomial $P(x) = \sum_{\gamma \in \Gamma(\rho^\alpha)} q_{\gamma} e^{i(\gamma,x)}$ to the eigenvalue $|\gamma + t|^2$ is not significant. Thus the corresponding eigenvalue of the
operator \( L_t(q(x)) \) is close to the eigenvalue \( | \gamma + t |^2 \) of \( L_t(0) \). Note that changing the values of \( c_1 \) and \( c_2 \) in the definitions of \( W_{b,\alpha_1}(c_2) \) and \( P(x) \) we obtain the different definitions of the non-resonance eigenvalues. However, in any case we obtain the same asymptotic formulas and the same perturbation theory, that is, this changing does not change anything for asymptotic formulas. Therefore we can define the non-resonance eigenvalue in different way. In accordance with the case of the trigonometric polynomial it is natural to say that the eigenvalue \( | \gamma + t |^2 \) is a non-resonance eigenvalue if \( \gamma + t \) does not belong to any of the sets \( W_{b,\alpha_1}(c_2) \) for \( b \mid c_1 p | \gamma + t |^\alpha \). However, for simplicity, we give the definitions as follows. By definition, put \( \alpha_k = 3^k \alpha \) for \( k = 1, 2, \ldots \) and introduce the sets

\[
E_k(\rho^{\alpha_1}, p) = \bigcup_{\gamma_1 \in \Gamma(p^{\rho^{\alpha_1}})} V_{\gamma_1}(\rho^{\alpha_1}), \quad U(\rho^{\alpha_1}, p) = (R(\frac{3}{2} \rho) \setminus R(\frac{1}{2} \rho)) \setminus E_1(\rho^{\alpha_1}, p),
\]

where \( R(\rho) = \{ x \in \mathbb{R}^d : | x | < \rho \} \), \( \rho \) is a big parameter and the intersection \( \bigcap_{k=1}^{\infty} V_{\gamma_k} \) in the definition of \( E_k \) is taken over \( \gamma_1, \gamma_2, \ldots, \gamma_k \), that are linearly independent. The set \( U(\rho^{\alpha_1}, p) \) is said to be a non-resonance domain and the eigenvalue \( | \gamma + t |^2 \) is called a non-resonance eigenvalue if \( \gamma + t \in U(\rho^{\alpha_1}, p) \). The domains \( V_{\gamma_1}(\rho^{\alpha_1}) \) for \( \gamma_1 \in \Gamma(p^{\rho^{\alpha_1}}) \) are called resonance domains and \( | \gamma + t |^2 \) is called a resonance eigenvalue if \( \gamma + t \in V_{\gamma_1}(\rho^{\alpha_1}) \). In Remark 1 we will discuss the relations between sets \( W_{b,\alpha_1}(c_2) \) and \( V_{\rho^{\alpha_1}}(\gamma_k) \).

In section 2 we prove that for each \( \gamma + t \in U(\rho^{\alpha_1}, p) \) there exists an eigenvalue \( \Lambda_N(t) \) of the operator \( L_t(q(x)) \) satisfying the following formula

\[
\Lambda_N(t) = | \gamma + t |^2 + F_{k-1}(\gamma + t) + O(| \gamma + t |^{-3k\alpha})
\]

for \( k = 1, 2, \ldots, [\frac{1}{3}(p-\frac{1}{2}q(d-1))] \), where \([a]\) denotes the integer part of \( a \), \( F_0 = 0 \), and \( F_{k-1} \) (for \( k > 1 \)) is expressed by the potential \( q(x) \) and eigenvalues of \( L_t(0) \). Besides, we prove that if the conditions

\[
| \Lambda_N(t) - | \gamma + t |^2 | < \frac{1}{2} \rho^{\alpha_1},
\]

\[
| b(N, \gamma) | > c_4 \rho^{-c_3 \alpha}
\]

hold, where \( b(N, \gamma) = (\Psi_{N,t}, e^{i(\gamma + t, x)} \), \( \Psi_{N,t}(x) \) is a normalized eigenfunction of \( L_t(q(x)) \) corresponding to \( \Lambda_N(t) \), then the following statements are valid:

(a) if \( \gamma + t \) is in the non-resonance domain, then \( \Lambda_N(t) \) satisfies (5) for \( k = 1, 2, \ldots, [\frac{1}{3}(p-c)] \) (see Theorem 1);

(b) if \( \gamma + t \in E_s \setminus E_{s+1} \), where \( s = 1, 2, \ldots, d-1 \), then

\[
\Lambda_N(t) = \lambda_j(\gamma + t) + O(| \gamma + t |^{-k\alpha}),
\]

where \( \lambda_j \) is an eigenvalue of the matrix \( C(\gamma + t) \) (see (27) and Theorem 2). Moreover, we prove that every big eigenvalue of the operator \( L_t(q(x)) \) for all values of \( t \) satisfies one of these formulae.

3
The results of section 2 (see Theorem 1, 2) is considered in [17]. However, in that paper these results are written only briefly. The enlarged variant is written in [19, 21] which can not be used as a reference. Here we write the non-resonance case in an improved and enlarged form and so that it can easily be used in the next sections. Moreover it helps to read section 3, where we consider in detail the single resonance domains $V_{\gamma_1}(\rho^{a_1}) \subseteq V_{\gamma_1}(\rho^{a_1}) \setminus E_2$, i.e., the part of the resonance domains $V_{\gamma_1}(\rho^{a_1})$, which does not contain the intersection of two resonance domains. Namely, for this case we obtain asymptotic formulas of arbitrary order for the eigenvalues of the $d$ dimensional periodic Schrödinger operator $L(q)$ for arbitrary dimension $d$. This case is connected with Sturm-Liouville operators. In the next papers, which use this, we will constructively determine a family of spectral invariants by given Floquet spectrum and give an algorithm for finding the potential $q(x)$ by these spectral invariants. Thus the new results about asymptotic formulas for eigenvalues are the results corresponding to the single resonance case (section 3). It follows from (5) that the non-resonance eigenvalue of $L_t(q(x))$ is close to the eigenvalue of the Laplace operator $L_t(0)$. So the influence of the potential $q(x)$ is not significant. To obtain the asymptotic formula for the non-resonance eigenvalues we take the operator $L_t(0)$ for an unperturbed operator and $q(x)$ for a perturbation. We use the formula (16) connecting the eigenvalues and eigenfunctions of $L_t(q(x))$ and $L_t(0)$. We call (16) binding formula for $L_t(q(x))$ and $L_t(0)$. In section 2 substituting the decomposition (3) of $q(x)$ into (16) and iterating it several times we get the asymptotic formulas for the non-resonance eigenvalues.

The resonance eigenvalues corresponds to the quasimomentum $\gamma + t$ lying near the diffraction hyperplanes $\{x : |x - \delta|^2 = |x + \delta|^2\}$. In this case the influence of the directional potential $q^\delta(x) = \sum_{n \in \mathbb{Z}} q_n \delta e^{in(\delta,x)} = Q(\zeta)$, $\zeta = (\delta, x)$

(9)
is significant, but the influence of $q(x) - q^\delta(x)$ is not significant. Therefore these eigenvalues of $L_t(q(x))$ is close to the eigenvalues of $L_t(q^\delta(x))$. Hence in section 3 for investigation of the resonance eigenvalues we take the operator $L_t(q^\delta(x))$ for an unperturbed operator and $q(x) - q^\delta(x)$ for a perturbation. To prove the asymptotic formulas for the resonance eigenvalues first we consider the eigenvalues and eigenfunctions of $L_t(q^\delta(x))$ (see Lemma 2). Then we iterate the formula (53) connecting the eigenvalues and the eigenfunctions of $L_t(q(x))$ and $L_t(q^\delta(x))$. We call (53) binding formula for $L_t(q(x))$ and $L_t(q^\delta(x))$. The formulas (16), (53) and their iterations are similar. In the resonance case (in section 3) we use the ideas of the non-resonance case (of section 2) and replace the perturbation $q(x)$ by the perturbation $q(x) - q^\delta(x)$, the eigenvalues $|\gamma + t|^2$ and eigenfunctions $e^{i(\gamma + t, x)}$ of the unperturbed (for non-resonance case) operator $L_t(0)$ by the eigenvalues and eigenfunctions of the unperturbed (for resonance case) operator $L_t(q^\delta(x))$ respectively. Therefore the simple iterations of (16) helps to read the complicated iteration of (53).

For investigation of the Bloch function in the non-resonance domain, in section 4, we find the values of quasimomenta $\gamma + t$ for which the corresponding
eigenvalues are simple, namely we construct the subset \( B \) of \( U(\rho^{\alpha_1}, p) \) with the following properties:

Pr.1. If \( \gamma + t \in B \), then there exists a unique eigenvalue, denoted by \( \Lambda(\gamma + t) \), of the operator \( L_t(q(x)) \) satisfying (5). This is a simple eigenvalue of \( L_t(q(x)) \). Therefore we call the set \( B \) the simple set of quasimomenta.

Pr.2. The eigenfunction \( \Psi_N(\gamma + t)(x) \equiv \Psi_{\gamma + t}(x) \) corresponding to the eigenvalue \( \Lambda(\gamma + t) \) is close to \( e^{i(\gamma + t,x)} \), namely

\[
\Psi_N(x) = e^{i(\gamma + t,x)} + O(|\gamma + t|^{-\alpha_1}), \tag{10}
\]

\[
\Psi_{\gamma + t}(x) = e^{i(\gamma + t,x)} + \Phi_{k-1}(x) + O(|\gamma + t|^{-k\alpha_1}), \quad k = 1, 2, \ldots ,
\]

where \( \Phi_{k-1} \) is expressed by \( q(x) \) and the eigenvalues of \( L_0(0) \).

Pr.3. The set \( B \) contains the intervals \( \{a + sb : s \in [-1, 1]\} \) such that \( \Lambda(a - b) < \rho^2 \), \( \Lambda(a + b) > \rho^2 \), and \( \Lambda(\gamma + t) \) is continuous on these intervals. Hence there exists \( \gamma + t \) such that \( \Lambda(\gamma + t) = \rho^2 \) for \( \rho \gg 1 \). It implies that there exist only a finite number of gaps in the spectrum of \( L \), that is, it implies the validity of Bethe-Sommerfeld conjecture for arbitrary dimension and for arbitrary lattice.

Construction of the set \( B \) consists of two steps.

Step 1. We prove that all eigenvalues \( \Lambda_N(t) \sim \rho^2 \) of the operator \( L_t(q(x)) \) lie in the \( \varepsilon_1 = \rho^{-d-2\alpha} \) neighborhood of the numbers

\[
F(\gamma + t) = |\gamma + t|^2 + \mathcal{K}_k - 1(\gamma + t), \quad (\text{see (5), (8)}), \quad k = 1, 2, \ldots.
\]

We call these numbers as the known parts of the eigenvalues. Moreover, for \( \gamma + t \in U(\rho^{\alpha_1}, p) \) there exists \( \Lambda_N(t) \) satisfying \( \Lambda_N(t) = F(\gamma + t) + o(\varepsilon_1) \).

Step 2. By eliminating the set of quasimomenta \( \gamma + t \), for which the known parts \( F(\gamma + t) \) of \( \Lambda_N(t) \) are situated from the known parts \( F(\gamma' + t) \), \( \Lambda_j(\gamma' + t) \) \( (\gamma' \neq \gamma) \) of other eigenvalues at a distance less than \( 2\varepsilon_1 \), we construct the set \( B \) with the following properties: if \( \gamma + t \in B \), then the following conditions (called simplicity conditions for \( \Lambda_N(t) \)) hold

\[
|F(\gamma + t) - F(\gamma' + t)| \geq 2\varepsilon_1 \tag{12}
\]

for \( \gamma' \in K \setminus \{\gamma\} \), \( \gamma' + t \in U(\rho^{\alpha_1}, p) \) and

\[
|F(\gamma + t) - \lambda_j(\gamma' + t)| \geq 2\varepsilon_1 \tag{13}
\]

for \( \gamma' \in K \), \( \gamma' + t \in E_k \setminus E_{k+1} \), \( j = 1, 2, \ldots \), where \( K \) is the set of \( \gamma' \in \Gamma \) satisfying \( |F(\gamma + t) - |\gamma' + t|^2| < \frac{1}{2}\rho^{\alpha_1} \). Thus \( B \) is the set of \( x \in U(\rho^{\alpha_1}, p) \cap (R(3/2\rho - \rho^{\alpha_1-1}) \setminus R(3/2\rho + \rho^{\alpha_1-1})) \) such that \( x = \gamma + t \), where \( \gamma \in \Gamma, t \in F^* \), and the simplicity conditions (12), (13) hold. As a consequence of these conditions the eigenvalue \( \Lambda_N(t) \) does not coincide with other eigenvalues. To prove this, namely to prove the Pr.1 and (10), we show that for any normalized eigenfunction \( \Psi_N(x) \) corresponding to \( \Lambda_N(t) \) the following equality holds:

\[
\sum_{\gamma' \in \Gamma \setminus \gamma} |b(N, \gamma')|^2 = O(\rho^{-2\alpha_1}). \tag{14}
\]
For the first time in [15-17] we constructed the simple set \( B \) with \( \text{Pr.1} \) and \( \text{Pr.3.} \), though in those papers we emphasized the Bethe-Sommerfeld conjecture. Note that for this conjecture and \( \text{Pr.1, Pr.3.} \) it is enough to prove that the left-hand side of (14) is less than \( \frac{1}{4} \) (we proved this inequality in [15-17] and as noted in Theorem 3 of [16] and in [18] the proof of this inequality does not differ from the proof of (14)). From (10) we got (11) by iteration (see [18]). But in those papers these results are written briefly. The enlarged variant is written in [19,22] which cannot be used as reference. In this paper we write these results in improved and enlarged form. The main difficulty and the crucial point of papers [15-17] were the construction of the simple set \( B \) with \( \text{Pr.1, Pr.3.} \). This difficulty of the perturbation theory of \( L(q(x)) \) is of a physical nature and it is connected with the complicated picture of the crystal diffraction. If \( d = 2,3 \), then \( F(\gamma + t) = |\gamma + t|^2 \) and the matrix \( C(\gamma + t) \) corresponds to the Schrödinger operator with directional potential (9) (see [16]). So for construction of the simple set \( B \) of quasimomenta we eliminated the vicinities of the diffraction planes and the sets connected with directional potential (see (12), (13)). Besides, for nonsmooth potentials \( q(x) \in L_2(\mathbb{R}^2/\Omega) \), we eliminated a set, which is described in the terms of the number of states (see [15, 19, 20]). The simple sets \( B \) of quasimomenta for the first time is constructed and investigated (hence the main difficulty and the crucial point of perturbation theory of \( L(q) \) is investigated) in [16] for \( d = 3 \) and in [15,17] for the cases:

1. \( d = 2, q(x) \in L_2(F) \);
2. \( d > 2, q(x) \) is a smooth potential.

Then, Yu.E. Karpeshina proved (see [7-9]) the convergence of the perturbation series of two and three dimensional Schrödinger operator \( L(q) \) with a wide class of nonsmooth potential \( q(x) \) for a set, that is similar to \( B \), of quasimomenta. In papers [3,4] the asymptotic formulas for the eigenvalues and Bloch function of the two and three dimensional operator \( L_1(q(x)) \) were obtained. In [5] the asymptotic formulae for the eigenvalues of \( L_0(q(x)) \) were obtained.

In section 5 we consider the geometrical aspects of the simple sets. We prove that the simple sets \( B \) has asymptotically full measure on \( \mathbb{R}^d \). Moreover, we construct a part of isoenergetic surfaces corresponding to \( \rho^2 \), which is smooth surfaces and has the measure asymptotically close to the measure of the isoenergetic surfaces of the operator \( L(0) \). The nonemptiness of the isoenergetic surfaces for \( \rho \gg 1 \) implies the validity of the Bethe-Sommerfeld conjecture. Note that one can read Section 4 and Section 5 without reading Section 3.

For the first time M.M. Skriganov [11,12] proved the validity of the Bethe-Sommerfeld conjecture for the Schrodinger operator for dimension \( d = 2,3 \) for arbitrary lattice, for dimension \( d > 3 \) for rational lattice. The Skriganov’s method is based on the detail investigation of the arithmetic and geometric properties of the lattice. B.E.J.Dahlberg and E.Trubowits [1] using an asymptotic of Bessel function, gave the simple proof of this conjecture for the two dimensional Schrodinger operator. Then in papers [15-17] we proved the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice and for arbitrary dimension by using the asymptotic formulas and by construction of the simple set \( B \), that is, by the method of perturbation theory. Yu.E. Karpeshina (see
[7-9]) proved this conjecture for two and three dimensional Schrodinger operator $L(q)$ for a wide class of singular potentials $q(x)$, including Coulomb potential, by the method of perturbation theory. B. Helffer and A. Mohamed [6], by investigations the integrated density of states, proved the validity of the Bethe-Sommerfeld conjecture for the Schrodinger operator for $d \leq 4$ for arbitrary lattice. Recently L. Parnovski and A. V. Sobolev [10] proved this conjecture for $d \leq 4$. The method of this paper and papers [15-17] is a first and unique, for the present, by which the validity of the Bethe-Sommerfeld conjecture for arbitrary lattice and for arbitrary dimension is proved.

In section 6 we construct simple sets in the resonance domain and obtain the asymptotic formulas of arbitrary order for the Bloch functions of the $d$ dimensional Schrodinger operator $L(q(x))$, where $q(x) \in W_2^2(F)$, $s \geq 6(3^d(d+1)^2)+d$, when corresponding quasimomentum lies in these simple sets. Note that we construct the simple sets in the non-resonance domain so that it contains a big part of the isoenergetic surfaces of $L(q)$. However in the case of resonance domain we construct the simple set so that it can be easily used for the constructive determination (in next papers) a family of the spectral invariants by given Floquet spectrum.

In this paper for the different types of the measures of the subset $A$ of $\mathbb{R}^d$ we use the same notation $\mu(A)$. By $|A|$ we denote the number of elements of the set $A \subset \Gamma$ and use the following obvious fact. If $a \sim \rho$, then the number of elements of the set $\{ \gamma + t : \gamma \in \Gamma \}$ satisfying $| | \gamma + t | - a | < 1$ is less than $c_5 \rho^{d-1}$. Therefore the number of eigenvalues of $L_1(q)$ lying in $(a^2 - \rho, a^2 + \rho)$ is less than $c_5 \rho^{d-1}$. Besides, we use the inequalities:

$$
\alpha_1 + d \alpha < 1 - \alpha, \quad d\alpha < \frac{1}{2} d\alpha, \quad k_1 \leq \frac{1}{3} (p - \frac{1}{2} q(d - 1)),
$$

$$
\alpha_1 \alpha \geq p\alpha, \quad 3k_1 \alpha > d + 2\alpha, \quad k_1 \leq \frac{1}{3} (p - \frac{1}{2} q(d - 1))
$$

(15)

$$
\alpha_1 \alpha \geq p\alpha, \quad 3k_1 \alpha > d + 2\alpha, \quad k_1 \leq \frac{1}{3} (p - \frac{1}{2} q(d - 1))
$$

for $k = 1, 2, ..., d$, which follow from the definitions $p = s - d$, $a_k = 3^k \alpha$, $\alpha = \frac{1}{p}$, $q = 3^d + d + 2$, $k_1 = \left[\frac{d}{d+1}\right] + 2$, $p_1 = \left[\frac{d}{3}\right] + 1$ of the numbers $p, q, a_k, \alpha, k_1, p_1$.

## 2 Asymptotic Formulae for the Eigenvalues

First we obtain the asymptotic formulas for the non-resonance eigenvalues by iteration of the formula

$$
(\Lambda_N - | \gamma + t |^2) b(N, \gamma) = (\Psi_{N,t}(x)q(x), e^{i(\gamma + t,x)}),
$$

(16)

which is obtained from equation $-\Delta \Psi_{N,t}(x) + q(x)\Psi_{N,t}(x) = \Lambda_N \Psi_{N,t}(x)$ by multiplying by $e^{i(\gamma + t,x)}$, where $\gamma + t \in U(\rho^{\alpha_1}, p)$. Introducing into (16) the expansion (3) of $q(x)$, we get

$$
(\Lambda_N - | \gamma + t |^2) b(N, \gamma) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-\alpha_1}).
$$

(17)
From the relations (16), (17) it follows that

\[ b(N, \gamma') = \frac{(\Psi_{N,t} q(x), e^{i(\gamma'+t,x)})}{A_N - |\gamma'+t|^2} = \sum_{\gamma_1 \in \Gamma(p^\alpha)} \frac{q_{\gamma_1} b(N, \gamma' - \gamma_1)}{A_N - |\gamma' + t|^2} + O(p^{-\alpha}) \quad (18) \]

for all vectors \( \gamma' \in \Gamma \) satisfying the inequality

\[ |A_N - |\gamma' + t|^2| > \frac{1}{2} p^{\alpha_1}. \quad (19) \]

If (6) holds and \( \gamma + t \in U(p^{\alpha_1}, p) \), then

\[ ||\gamma + t|^2 - |\gamma - \gamma_1 + t|^2|| > p^{\alpha_1}, \quad |A_N - |\gamma - \gamma_1 + t|^2| > \frac{1}{2} p^{\alpha_1} \quad (20) \]

for all \( \gamma_1 \in \Gamma(p^p p^\rho) \). Hence the vector \( \gamma - \gamma_1 \) for \( \gamma + t \in U(p^{\alpha_1}, p) \) and \( \gamma_1 \in \Gamma(p^p p^\rho) \) satisfies (19). Therefore, in (18) one can replace \( \gamma' \) by \( \gamma - \gamma_1 \) and write

\[ b(N, \gamma - \gamma_1) = \sum_{\gamma_1 \in \Gamma(p^\alpha)} \frac{q_{\gamma_1} q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{A_N - |\gamma - \gamma_1 + t|^2} + O(p^{-\alpha}). \]

Substituting this for \( b(N, \gamma - \gamma_1) \) into the right-hand side of (17) and isolating the terms containing the multiplicand \( b(N, \gamma) \), we get

\[ (A_N - |\gamma + t|^2) b(N, \gamma) = \sum_{\gamma_1, \gamma_2 \in \Gamma(p^\alpha)} \frac{|q_{\gamma_1}|^2 b(N, \gamma)}{A_N - |\gamma - \gamma_1 + t|^2} + \sum_{\gamma_1, \gamma_2 \in \Gamma(p^\alpha), \gamma_1 + \gamma_2 \neq 0} \frac{q_{\gamma_1} q_{\gamma_2} b(N, \gamma - \gamma_1 - \gamma_2)}{A_N - |\gamma - \gamma_1 + t|^2} + O(p^{-\alpha}), \]

since \( q_{\gamma_1} q_{\gamma_2} = |q_{\gamma_1}|^2 \) for \( \gamma_1 + \gamma_2 = 0 \) and the last summation is taken under the condition \( \gamma_1 + \gamma_2 \neq 0 \). Repeating this process \( p_1 \equiv \lceil \frac{d}{2} \rceil + 1 \) times, i.e., in the last summation replacing \( b(N, \gamma - \gamma_1 - \gamma_2) \) by its expression from (18) (in (18) replace \( \gamma' \) by \( \gamma - \gamma_1 - \gamma_2 \)) and isolating the terms containing \( b(N, \gamma) \) etc., we obtain

\[ (A_N - |\gamma + t|^2) b(N, \gamma) = A_{p_1}(A_N, \gamma + t) b(N, \gamma) + C_{p_1} + O(p^{-\alpha}), \quad (21) \]

where

\[ A_{p_1}(A_N, \gamma + t) = \sum_{k=1}^{p_1} S_k(A_N, \gamma + t), \]

\[ S_k(A_N, \gamma + t) = \sum_{\gamma_1, \ldots, \gamma_k \in \Gamma(p^\alpha)} \frac{q_{\gamma_1} q_{\gamma_2} \ldots q_{\gamma_k} q_{\gamma_1 - \gamma_2 - \ldots - \gamma_k}}{\prod_{j=1}^k (A_N - |\gamma + t - \sum_{i=1}^j \gamma_i|^2)}, \]

\[ C_{p_1} = \sum_{\gamma_1, \ldots, \gamma_{p_1 + 1} \in \Gamma(p^\alpha)} \frac{q_{\gamma_1} q_{\gamma_2} \ldots q_{\gamma_{p_1 + 1}} b(N, \gamma - \gamma_1 - \gamma_2 - \ldots - \gamma_{p_1 + 1})}{\prod_{j=1}^{p_1} (A_N - |\gamma + t - \sum_{i=1}^j \gamma_i|^2)}. \]
Here the summations for $S_k$ and $C_{p_1}$ are taken under the additional conditions $\gamma_1 + \gamma_2 + \ldots + \gamma_s \neq 0$ for $s = 1, 2, \ldots, k$ and $s = 1, 2, \ldots, p_1$ respectively. These conditions and the inclusion $\gamma_i \in \Gamma(\rho^a)$ for $i = 1, 2, \ldots, p_1$ imply the relation $\sum_{i=1}^{j} \gamma_i \in \Gamma(p_1 \rho^a)$. Therefore from the second inequality in (20) it follows that the absolute values of the denominators of the fractions in $S_k$ and $C_{p_1}$ are greater than $(\frac{1}{2} \rho^{a_1})^k$ and $(\frac{1}{2} \rho^{a_1})^{p_1}$ respectively. Hence the first inequality in (4) and $p_1 \alpha_1 \geq p \alpha$ (see the fourth inequality in (15)) yield

$$C_{p_1} = O(\rho^{-p_1 \alpha_1}) = O(\rho^{-p \alpha}), \quad S_k(\Lambda_N, \gamma + t) = O(\rho^{-k \alpha_1}), \forall k = 1, 2, \ldots, p_1.$$  \hspace{1cm} (22)

Since we used only the condition (6) for $\Lambda_N$, it follows that

$$S_k(a, \gamma + t) = O(\rho^{-k \alpha_1})$$  \hspace{1cm} (23)

for all $a \in \mathbb{R}$ satisfying $|a| > g + t |^2| < \frac{1}{2} \rho^{a_1}$. Thus finding $N$ such that $\Lambda_N$ is close to $|\gamma + t |^2$ and $b(N, \gamma)$ is not very small, then dividing both sides of (21) by $b(N, \gamma)$, we get the asymptotic formulas for $\Lambda_N$.

**Theorem 1** (a) Suppose $\gamma + t \in U(\rho^{a_1}, p)$. If (6) and (7) hold, then $\Lambda_N$ satisfies formulas (5) for $k = 1, 2, \ldots, \lceil \frac{1}{2} (p - c) \rceil$, where

$$F_s = O(\rho^{-a_1}), \forall s = 0, 1, \ldots,$$  \hspace{1cm} (24)

and $F_0 = 0$. $F_s = A_s(|\gamma + t |^2 + F_{s-1}, \gamma + t)$ for $s = 1, 2, \ldots$.

(b) For $\gamma + t \in U(\rho^{a_1}, p)$ there exists an eigenvalue $\Lambda_N$ of $L_t(q(x))$ satisfying (5).

**Proof.** (a) To prove (5) in case $k = 1$ we divide both side of (21) by $b(N, \gamma)$ and use (7), (22). Then we obtain

$$\Lambda_N - |\gamma + t |^2 = O(\rho^{-a_1}).$$  \hspace{1cm} (25)

This and $\alpha_1 = 3 \alpha$ (see the end of the introduction) imply that formula (5) for $k = 1$ holds and $F_0 = 0$. Hence (24) for $s = 0$ is also proved. Moreover, from (23), we obtain $S_k(|\gamma + t |^2 + O(\rho^{-a_1}), \gamma + t) = O(\rho^{-k \alpha_1})$ for $k = 1, 2, \ldots$.

Therefore (24) for arbitrary $s$ follows from the definition of $F_s$ by induction. Now we prove (5) by induction on $k$. Suppose (5) holds for $k = j$, that is,

$$\Lambda_N = |\gamma + t |^2 + F_{k-1}(\gamma + t) + O(\rho^{-3k \alpha}).$$

Substituting this into $A_{p_1}(\Lambda_N, \gamma + t)$ in (21) and dividing both sides of (21) by $b(N, \gamma)$, we get

$$\Lambda_N = |\gamma + t |^2 + A_{p_1}(|\gamma + t |^2 + F_{j-1} + O(\rho^{-j \alpha_1}), \gamma + t) + O(\rho^{-(p-c)\alpha}) =$$

$$|\gamma + t |^2 + A_{p_1}(|\gamma + t |^2 + F_{j-1} + O(\rho^{-j \alpha_1}), \gamma + t) -$$

$$A_{p_1}(|\gamma + t |^2 - F_{j-1}, \gamma + t) + A_{p_1}(|\gamma + t |^2 + F_{j-1}, \gamma + t) + O(\rho^{-(p-c)\alpha}).$$

To prove (a) for $k = j + 1$ we need to show that the expression in curly brackets is equal to $O(\rho^{-(j+1) \alpha_1})$. It can be checked by using (4), (20), (24) and the
obvious relation

\[
\prod_{j=1}^n (|\gamma + t|^2 + F_j - 1 + O(\rho^{-\alpha_1})) - (\gamma + t - \sum_{i=1}^s \gamma_i)^2
\]

\[
= \prod_{j=1}^n (|\gamma + t|^2 + F_j - 1 - |\gamma + t - \sum_{i=1}^s \gamma_i|^2) \left(1 - \frac{1}{1 - O(\rho^{-(j+1)\alpha_1})} - 1\right)
\]

\[
= O(\rho^{-(j+1)\alpha_1}) \quad \text{for } s = 1, 2, \ldots, p_1.
\]

Let A be the set of indices N satisfying (6). Using (16) and Bessel inequality, we obtain

\[
\sum_{N \notin A} |b(N, \gamma)|^2 = \frac{1}{A_N - |\gamma + t|^2} = O(\rho^{-2\alpha_1})
\]

Hence, by the Parseval equality, we have \(\sum_{N \in A} |b(N, \gamma)|^2 = 1 - O(\rho^{-2\alpha_1})\). This and the inequality \(|A| < c_5 \rho^{-d-1} = c_5 \rho^{-d-1}q_0\) (see the end of the introduction) imply that there exists a number N satisfying \(|b(N, \gamma)| > \frac{1}{2}(c_5)^{-1} \rho^{d+1}q_0\), that is, (7) holds for \(c = \frac{(d+1)q_0}{2}\). Thus \(A_N\) satisfies (5) due to (a) \(\blacksquare\)

Theorem 1 shows that in the non-resonance case the eigenvalue of the perturbed operator \(L_t(q(x))\) is close to the eigenvalue of the unperturbed operator \(L_t(0)\). However, in Theorem 2 we prove that if \(\gamma + t \in \cap_{i=1}^k \mathcal{V}_i(\rho^{a_2})\setminus E_{k+1}\) for \(k \geq 1\), where \(\gamma_1, \gamma_2, \ldots, \gamma_k\) are linearly independent vectors of \(\Gamma(\rho\alpha)\), then the corresponding eigenvalue of \(L_t(q(x))\) is close to the eigenvalue of the matrix constructed as follows. Introduce the sets:

\[
B_k \equiv B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{b : b = \sum_{i=1}^k n_i \gamma_i, n_i \in \mathbb{Z}, |b| < \frac{1}{2} \rho^{\frac{1}{2} + k+1}\},
\]

\[
B_k(\gamma + t, p_1) = \{\gamma + t + b : b \in B_k, |b| < p_1 \rho^{\alpha}, a \in \Gamma\}.
\]

Denote by \(b_i + t\) for \(i = 1, 2, \ldots, b_k\) the vectors of \(B_k(\gamma + t, p_1)\), where \(b_k \equiv b_k(\gamma_1, \gamma_2, \ldots, \gamma_k)\) is the number of the vectors of \(B_k(\gamma + t, p_1)\). Define the matrix \(C(\gamma + t, \gamma_1, \gamma_2, \ldots, \gamma_k) \equiv (c_{i,j})\) by the formulas

\[
c_i,j = |h_i + t|^2, \quad c_{i,j} = q_{h_i - h_j}, \quad \forall i \neq j,
\]

where \(i, j = 1, 2, \ldots, b_k\). We consider the resonance eigenvalue \(|\gamma + t|^2\) for \(\gamma + t \in (\cap_{i=1}^k \mathcal{V}_i(\rho^{a_2}))\) by using the following Lemma.

**Lemma 1** If \(\gamma + t \in \cap_{i=1}^k \mathcal{V}_i(\rho^{a_2})\setminus E_{k+1}\), \(h + t \in B_k(\gamma + t, p_1)\), \((h - \gamma' + t) \notin B_k(\gamma + t, p_1)\), then

\[
||\gamma + t|^2 - |h - \gamma' - \gamma_1 - \gamma_2 - \ldots - \gamma_s + t|^2| > \frac{1}{5} \rho^{a_{k+1}},
\]

where \(\gamma' \in \Gamma(\rho^{\alpha})\), \(\gamma_j \in \Gamma(\rho^{\alpha})\), \(j = 1, 2, \ldots, s\) and \(s = 0, 1, \ldots, p_1 - 1\).
The inequality $p > 2p_1$ (see the end of the introduction) and the conditions of Lemma 1 imply that

$$h - \gamma' - \gamma_1' - \gamma_2' - \ldots - \gamma_s' + t \in B_k(\gamma + t, p) \setminus B_k(\gamma + t) \text{ for all } s = 0, 1, \ldots, p_1 - 1.$$  

It follows from the definitions of $B_k(\gamma + t, p), B_k$ that (see (26))

$$h - \gamma' - \gamma_1' - \gamma_2' - \ldots - \gamma_s' + t = \gamma + t + b + a,$$

where

$$| b | < \frac{1}{2} \rho^{\frac{1}{2} \alpha + 1}, | a | < p \rho^\alpha, \gamma + t + b + a \notin B_k. \quad (29)$$

Then (28) has the form

$$\| \gamma + t + a + b \|^2 = | \gamma + t |^2 > \frac{1}{5} \rho^{\alpha + 1}. \quad (30)$$

To prove (30) we consider two cases:

Case 1. $a \in P$, where $P = \text{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. Since $b \in B_k \subset P$, we have $a + b \in P$. This with the third relation in (29) imply that $a + b \in P \setminus B_k$ i.e.,

$$| a + b | \geq \frac{1}{2} \rho^{\frac{1}{2} \alpha + 1}.$$  

Consider the orthogonal decomposition $\gamma + t = y + v$ of $\gamma + t$, where $v \in P$ and $y \perp P$. First we prove that the projection $v$ of any vector $x \in \cap_{i=1}^k V_i(\rho^{\alpha s})$ on $P$ satisfies

$$| v | = O(\rho^{(k-1)\alpha + \alpha}). \quad (31)$$

For this we turn the coordinate axis so that $\text{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ coincides with the span of the vectors $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_k$. Then

$$\gamma_s = \sum_{i=1}^k \gamma_{s,i} e_i \text{ for } s = 1, 2, \ldots, k.$$  

Therefore the relation $x \in \cap_{i=1}^k V_i(\rho^{\alpha s})$ implies that

$$\sum_{i=1}^k \gamma_{s,i} x_i = O(\rho^{\alpha s}), s = 1, 2, \ldots, k; \quad x_n = \frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})}, n = 1, 2, \ldots, k,$$

where $x = (x_1, x_2, \ldots, x_d), \gamma_{j,i} = (\gamma_{j,1}, \gamma_{j,2}, \ldots, \gamma_{j,k}, 0, 0, \ldots, 0), b_{j,i}^n = \gamma_{j,i}$ for $n \neq j$ and $b_{j,i}^n = O(\rho^{\alpha s})$ for $n = j$. Taking into account that the determinant $\det(\gamma_{j,i})$ is the volume of the parallelepiped $\{\sum_{i=1}^k b_i \gamma_{i} : b_i \in [0, 1], i = 1, 2, \ldots, k\}$ and using $| \gamma_{j,i} | < p \rho^\alpha$ (since $\gamma_{j} \in \Gamma(p^\alpha)$), we get the estimations

$$x_n = O(\rho^{\alpha s + (k-1)\alpha}), \forall n = 1, 2, \ldots, k; \forall x \in \cap_{i=1}^k V_i(\rho^{\alpha s}). \quad (32)$$

Hence (31) holds. Therefore, using the inequalities $| a + b | \geq \frac{1}{2} \rho^{\frac{1}{2} \alpha + 1}$ (see above), $\alpha_k + 1 > 2(\alpha_k + (k-1)\alpha)$ (see the seventh inequality in (15)), and the obvious equalities $(y, v) = (y, a) = (y, b) = 0,$

$$| \gamma + t + a + b |^2 - | \gamma + t |^2 = | a + b + v |^2 - | v |^2, \quad (33)$$

we obtain the estimation (30).

Case 2. $a \notin P$. First we show that

$$\| \gamma + t + a \|^2 - | \gamma + t |^2 \geq \rho^{\alpha_k + 1}. \quad (34)$$
Suppose, to the contrary, that it does not hold. Then $\gamma + t \in V_{\alpha}(\rho^{\alpha k + 1})$. On the other hand $\gamma + t \in \bigcap_{i=1}^{k} V_{\gamma_{i}}(\rho^{\alpha k + 1})$ (see the conditions of Lemma 1). Therefore we have $\gamma + t \in E_{k+1}$ which contradicts the conditions of the lemma. So (34) is proved. Now, to prove (30) we write the difference $|\gamma + t + a + b|^{2} - |\gamma + t|^{2}$ as the sum of $d_{1} \equiv |\gamma + t + a + b|^{2} - |\gamma + t + b|^{2}$ and $d_{2} \equiv |\gamma + t + b|^{2} - |\gamma + t|^{2}$. Since $d_{1} = |\gamma + t + a|^{2} - |\gamma + t|^{2} + 2(a, b)$, it follows from the inequalities (34), (29) that $d_{1} > \frac{2}{3} \rho^{\alpha k + 1}$. On the other hand, taking $a = 0$ in (33), we have $d_{2} = |b + v|^{2} - |v|^{2}$. Therefore (31), the first inequality in (29) and the seventh inequality in (15) imply that $d_{2} < \rho^{\alpha k + 1}$, $|d_{1}| - |d_{2}| > \frac{1}{3} \rho^{\alpha k + 1}$, that is, (30) holds.

**Theorem 2** (a) Suppose $\gamma + t \in \bigcap_{i=1}^{k} V_{\gamma_{i}}(\rho^{\alpha k}) \setminus E_{k+1}$, where $k = 1, 2, ..., d - 1$. If (6) and (7) hold, then there is an index $i$ such that

$$\Lambda_{N}(t) = \lambda_{i}(\gamma + t) + O(\rho^{-(n-c-\frac{4}{3}d+1)|a|}),$$

where $\lambda_{1}(\gamma + t) \leq \lambda_{2}(\gamma + t) \leq ... \leq \lambda_{b}(\gamma + t)$ are the eigenvalues of the matrix $C(\gamma + t, \gamma_{1}, \gamma_{2}, ..., \gamma_{k})$ defined in (27).

(b) Every eigenvalue $\Lambda_{N}(t)$ of the operator $L_{1}(q(x))$ satisfies either (5) or (35) for $c = \frac{2(d-1)}{3}$.

**Proof.** (a) Writing the equation (17) for all $h_{k} + t \in B_{k}(\gamma + t, p_{1})$, we obtain

$$(\Lambda_{N} - |h_{i} + t|^{2})b(N, h_{i}) = \sum_{\gamma' \in \Gamma(\rho^{\alpha})} q_{\gamma'} b(N, h_{i} - \gamma') + O(\rho^{-p_{\alpha}}) \tag{36}$$

for $i = 1, 2, ..., b_{k}$ (see (26) for definition of $B_{k}(\gamma + t, p_{1})$). It follows from (6) and Lemma 1 that if $(h_{i} - \gamma' + t) \notin B_{k}(\gamma + t, p_{1})$, then

$$|\Lambda_{N} - |h_{i} - \gamma' - \gamma_{1} - \gamma_{2} - ... - \gamma_{s} + t|^{2}| > \frac{1}{6} \rho^{\alpha k + 1},$$

where $\gamma' \in \Gamma(\rho^{\alpha}), \gamma_{j} \in \Gamma(\rho^{\alpha}), j = 1, 2, ..., s$ and $s = 0, 1, ..., p_{1} - 1$. Therefore, applying the formula (18) $p_{1}$ times, using (4) and $p_{1} \alpha_{k+1} > p_{1} \alpha_{1} \geq p_{\alpha}$ (see the fourth inequality in (15)), we see that if $(h_{i} - \gamma' + t) \notin B_{k}(\gamma + t, p_{1})$, then

$$b(N, h_{i} - \gamma') = \sum_{\gamma_{1}, ..., \gamma_{p_{1}-1} \in \Gamma(\rho^{\alpha})} q_{\gamma_{1}}q_{\gamma_{2}}...q_{\gamma_{p_{1}}} b(N, h_{i} - \gamma' - \sum_{i=1}^{p_{1}} \gamma_{i} + t - \sum_{i=1}^{2} \gamma_{i})^{2}$$

$$+ O(\rho^{-p_{\alpha}}) = O(\rho^{p_{1}\alpha_{k+1}}) + O(\rho^{-p_{\alpha}}) = O(\rho^{-p_{\alpha}}).$$

Hence (36) has the form

$$(\Lambda_{N} - |h_{i} + t|^{2})b(N, h_{i}) = \sum_{\gamma'} q_{\gamma'} b(N, h_{i} - \gamma') + O(\rho^{-p_{\alpha}}), i = 1, 2, ..., b_{k},$$

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where the summation is taken under the conditions \( \gamma' \in \Gamma(\rho^{\alpha}) \) and \( h_1 - \gamma + t \in B_k(\gamma + t, p_1) \). It can be written in matrix form

\[
(C - \Lambda_N I)(b(N, h_1), b(N, h_2), \ldots b(N, h_{b_k})) = O(\rho^{-\alpha}),
\]

where the right-hand side of this system is a vector having the norm \( \| O(\rho^{-\alpha}) \| = O(\sqrt{b_k} \rho^{-\alpha}) \). Now, taking into account that \( \gamma + t \in \{ h_1 + t : i = 1, 2, \ldots, b_k \} \) and (7) holds, we have

\[
c_4 \rho^{-c\alpha} < \left( \sum_{i=1}^{b_k} | b(N, h_i) |^2 \right)^{\frac{1}{2}} \leq \| (C - \Lambda_N I)^{-1} \| \sqrt{b_k} \rho^{-\alpha},
\]

\[
\max_{i=1, 2, \ldots, b_k} \| \Lambda_N - \lambda_i \|^{-1} = \| (C - \Lambda_N I)^{-1} \| > c_4 b_k^{-\frac{1}{2}} \rho^{-c\alpha + \alpha_k}.
\]

Since \( b_k \) is the number of the vectors of \( B_k(\gamma + t, p_1) \), it follows from the definition of \( B_k(\gamma + t, p_1) \) (see (26)) and the obvious relations \( \| B_k \| = O(\rho^{\frac{d}{2} + \alpha + 1}) \), \( \| \Gamma(p_1 \rho^\alpha) \| = O(\rho^{d_{\alpha}}) \) and \( d_{\alpha} < \frac{1}{4} d \) (see the second paragraph of introduction), we get

\[
b_k = O(\rho^{d_{\alpha} + \frac{d}{2} + \alpha_k}) = O(\rho^{d_{\alpha}}), \forall k = 1, 2, \ldots, d - 1
\]

Thus formula (35) follows from (39) and (40).

(b) Let \( \Lambda_N(t) \) be any eigenvalue of the operator \( L_t(q(x)) \) such that \( \sqrt{\Lambda_N(t)} \in \left( \frac{1}{4} \rho, \frac{1}{2} \rho \right) \). Denote by \( D \) the set of all vectors \( \gamma \in \Gamma \) satisfying (6). From (16), arguing as in the proof of Theorem 1(b), we obtain

\[
\sum_{\gamma \in D} | b(N, \gamma) |^2 = 1 - O(\rho^{-2\alpha}).
\]

Since \( D = O(\rho^{-d+1}) \) (see the end of introduction), there exists \( \gamma \in D \) such that

\[
| b(N, \gamma) | > c_7 \rho^{-\frac{d+1}{2}} = c_7 \rho^{-\frac{d+1}{2} \alpha},
\]

that is, condition (7) for \( c = \frac{d_{\alpha}}{2} \) holds. Now the proof of (b) follows from Theorem 1(a) and Theorem 2(a), since either \( \gamma + t \in U(\rho^{\alpha_1}, p) \) or \( \gamma + t \in E_k \backslash E_{k+1} \) for \( k = 1, 2, \ldots, d - 1 \) (see (43)).

**Remark 1** Here we note that the non-resonance domain

\[
U(c \rho^{\alpha_1}, p) = (R(\frac{1}{2} \rho) \backslash R(\frac{1}{2} \rho)) \cup \bigcup_{\gamma \in \Gamma(\rho^{\alpha_1})} V_{\gamma_1}(c \rho^{\alpha_1}),
\]

where

\[
V_{\gamma_1}(c \rho^{\alpha_1}) \equiv \{ x \mid x^2 - | x + \gamma_1 |^2 < c \rho^{\alpha_1} \} \cap (R(\frac{1}{2} \rho) \backslash R(\frac{1}{2} \rho)),
\]

has an asymptotically full measure on \( \mathbb{R}^d \) in the sense that

\[
\frac{\mu(U \cap B(\rho))}{\mu(B(\rho))} \to 1 \quad \text{as} \quad \rho \to \infty,
\]

where \( B(\rho) = \{ x \in \mathbb{R}^d \mid | x | = \rho \} \). Clearly, \( B(\rho) \cap V_{\gamma_1}(c \rho^{\alpha_1}) \) is the part of sphere \( B(\rho) \), which is contained between two parallel hyperplanes

\[
\{ x \mid x^2 - | x + b |^2 = -c \rho^{\alpha_1} \} \quad \text{and} \quad \{ x \mid x^2 - | x + b |^2 = c \rho^{\alpha_1} \}.
\]

The distance of these hyperplanes from origin is \( O(\frac{c^{\alpha}}{\rho^{\alpha}}) \). Therefore, the relations

\[
| \Gamma(p \rho^{\alpha_1}) | = O(\rho^{d_{\alpha}}), \quad \text{and} \quad \alpha_1 + d_{\alpha} < 1 - \alpha \quad \text{see the first inequality in (15)}
\]

imply

\[
\mu(B(\rho) \cap V_{\gamma_1}(c \rho^{\alpha_1})) = O(\frac{\rho^{d_{\alpha}+d-2}}{| b |}), \quad \mu(E_1 \cap B(\rho)) = O(\rho^{d_{\alpha}+d-1}),
\]

\[
\mu(U(c \rho^{\alpha_1}, p) \cap B(\rho)) = (1 + O(\rho^{-\alpha})) \mu(B(\rho)).
\]

If \( x \in \bigcap_{i=1}^{d} V_{\gamma_i}(\rho^{\alpha_1}) \), then (32) holds for \( k = d \) and \( n = 1, 2, \ldots, d \). Hence we have

\[
| x | = O(\rho^{d_{\alpha}+(d-1)\alpha}) \quad \text{It is impossible, since} \quad \alpha_1 + (d-1)\alpha < 1 \quad \text{see the sixth}
\]
inequality in (15) and \( x \in B(\rho) \). It means that \( (\cap_{i=1}^{d} V_{i}(\rho^{\alpha}) \cap B(\rho) = \emptyset \) for \( \rho \gg 1 \). Thus for \( \rho \gg 1 \) we have

\[
R(\frac{3}{2}\rho) \setminus R(\frac{1}{2}\rho) = (U(\rho^{\alpha_1}, p) \cup (\bigcup_{s=1}^{d-1}(E_{s} \setminus E_{s+1}))).
\]

(43)

Note that everywhere in this paper we use the big parameter \( \rho \). All considered eigenvalues \( |\gamma + t|^2 \) of \( L_{t}(0) \) satisfy the relations \( \frac{1}{2} \rho < |\gamma + t| < \frac{3}{2} \rho \). Therefore in the asymptotic formulas instead of \( O(\rho^8) \) one can take \( O(|\gamma + t|^8) \). For simplicity, we often use \( O(\rho^8) \). It is clear that the asymptotic formulas hold true if we replace \( U(\rho^{\alpha_1}, p) \) by \( U(c_8 \rho^{\alpha_1}, p) \). Since \( V_{b}(\frac{3}{2} \rho^{\alpha_1}) \subset (R(\frac{3}{2} \rho) \setminus R(\frac{1}{2} \rho)) \cap W_{b, \alpha_1}(1) \subset V_{b}(\frac{3}{2} \rho^{\alpha_1}) \), in all considerations the resonance domain \( V_{b}(\rho^{\alpha_1}) \) can be replaced by \( W_{b, \alpha_1}(1) \cap (R(\frac{3}{2} \rho) \setminus R(\frac{1}{2} \rho)) \).

Remark 2 Here we note some properties of the known parts of the non-resonance and resonance eigenvalues of \( L_{t}(q(x)) \). Denoting \( \gamma + t \) by \( x \), where \( \gamma + t \in U(\rho^{\alpha_1}, p) \), we prove

\[
\frac{\partial F_{k}(x)}{\partial x_{i}} = O(\rho^{-2\alpha_1 + \alpha}), \forall i = 1, 2, \ldots, d; \forall k = 1, 2, \ldots
\]

(44)

by induction on \( k \). For \( k = 1 \) the formula (44) follows from (4) and

\[
\frac{\partial}{\partial x_{i}} \left( \frac{1}{x^2} - \frac{|x - \gamma_{1}|^2}{\rho^2} \right) = \frac{-2\gamma_{1}(i)}{(|x|^2 - |x - \gamma_{1}|^2)^2} = O(\rho^{-2\alpha_1 + \alpha}),
\]

(45)

where \( \gamma_{1}(i) \) is the \( i \)th component of the vector \( \gamma_{1} \in \Gamma(\rho^a) \) hence is equal to \( O(\rho^a) \). Now suppose that (44) holds for \( k = s \). Using this and (24), replacing \( |x|^2 \) by \( |x|^2 + F_{s}(x) \) in (45) and evaluating as above we obtain

\[
\frac{\partial}{\partial x_{i}} \left( \frac{1}{|x|^2 + F_{s}} - \frac{|x - \gamma_{1}|^2}{\rho^2} \right) = \frac{-2\gamma_{1}(i) + \frac{\partial F_{s}(x)}{\partial x_{i}}}{(|x|^2 + F_{s} - |x - \gamma_{1}|^2)^2} = O(\rho^{-2\alpha_1 + \alpha}).
\]

This formula together with the definition of \( F_{k} \) give (44) for \( k = s + 1 \).

Now denoting \( \lambda_{i}(\gamma + t) - |\gamma + t|^2 \) by \( r_{i}(\gamma + t) \) we prove that

\[
|r_{i}(x) - r_{i}(x')| \leq 2\rho^{\alpha_4} |x - x'|, \forall i.
\]

(46)

Clearly \( r_{1}(x) \leq r_{2}(x) \leq \ldots \leq r_{b_{k}}(x) \) are the eigenvalue of the matrix \( C(x) - |x|^2 I \equiv C'(x) \), where \( C(x) \) is defined in (27). By definition, only the diagonal elements of the matrix \( C'(x) \) depend on \( x \) and they are

\[
|a_{i} + h_{i}| - |x|^2 = 2(x, a_{i}) + |a_{i}|^2, \text{ where } a_{i} = h_{i} + t - \gamma - t \text{ and } h_{i} + t \in B_{k}(\gamma + t, p_{1}). \text{ It follows from the definitions of } B_{k}(\gamma + t, p_{1}) \text{ for } k < d \text{ (see (26)) and } a_{i,d} \text{ (see introduction) that } |a_{i}| < \frac{1}{\rho^{\alpha_4}} + p_{1} \rho^a < \rho^{\alpha_4}. \text{ Using this and taking into account that } C'(x) - C'(x') = (a_{i,j}), \text{ where } a_{i,j} = 2(x - x', a_{j}), a_{i,j} = 0 \text{ for } i \neq j, \text{ we obtain } ||C'(x) - C'(x')|| \leq 2\rho^{\alpha_4} |x - x'| \text{ from which follows (46).}
3 Bloch Eigenvalues near the Diffraction Planes

In this section we obtain the asymptotic formulae for the eigenvalues corresponding to the quasimomentum $\gamma + t$ lying near the diffraction hyperplane

$$D_\delta = \{ x \in \mathbb{R}^d : |x|^2 = |x - \delta|^2 \}. \quad \text{In section 2 to obtain the asymptotic formula for the non-resonance eigenvalues (that is, for the eigenvalues corresponding to the quasimomentum $\gamma + t$ lying far from the diffraction planes) we considered the operator $L_t(q(x))$ as perturbation of the operator $L_t(0)$ with $q(x)$. As a result the asymptotic formulas for these eigenvalues of $L_t(q(x))$ is expressed in the term of the eigenvalues of $L_t(0)$. To obtain the asymptotic formulae for the eigenvalues corresponding to the quasimomentum $\gamma + t$ lying near the diffraction plane $D_\delta$ we consider the operator $L_t(q(x))$ as the perturbation of the operator $L_t(q^\delta(x))$, where the directional potential $q^\delta(x)$ is defined in (9), with $q(x) - q^\delta(x)$. So it is natural that the asymptotic formulas, which will be obtained in this section, for these eigenvalues will be expressed in the term of the eigenvalues of the operator $L_t(q^\delta(x))$. Therefore first of all we need to investigate the eigenvalues and eigenfunctions of the operator $L_t(q^\delta(x))$. Without loss of generality it can be assumed that $\delta$ is the maximal element of $\Gamma$, that is, $\delta$ is the element of $\Gamma$ of minimal norm belonging to the line $\delta \mathbb{R}$, since it is easy to verify that $V_{\delta\delta}(\rho^\omega) \subset V_{\delta}(\rho^\omega)$ for $k = \pm 2, \pm 3, \ldots$. Note that $\delta$ is the maximal element of $\Gamma$ if $\{(\delta, \omega) : \omega \in \Omega \} = 2\pi Z$. Let $\Omega_\delta$ be the sublattice $\{ h \in \Omega : (h, \delta) = 0 \}$ of $\Omega$ in the hyperplane $H_\delta = \{ x \in \mathbb{R}^d : (x, \delta) = 0 \}$, and $\Gamma_\delta \equiv \{ a \in H_\delta : (a, k) \in 2\pi Z, \forall k \in \Omega_\delta \}$ be the dual lattice of $\Omega_\delta$. Denote by $F_\delta$ the fundamental domain $H_\delta/\Gamma_\delta$ of $\Gamma_\delta$. Then $t \in F^*$ has a unique decomposition

$$t = a + \tau + |\delta|^{-2} (t, \delta) \delta, \quad (47)$$

where $a \in \Gamma_\delta$, $\tau \in F_\delta$. Define the sets $\Omega^\prime$ and $\Gamma^\prime$ by $\Omega^\prime = \{ h + i\delta^* : h \in \Omega_\delta, i \in \mathbb{Z} \}$, and by $\Gamma^\prime = \{ b + (p - (2\pi)^{-1} (b, \delta^*)) \delta : b \in \Gamma_\delta, p \in \mathbb{Z} \}$, where $\delta^*$ is the element of $\Omega$ satisfying $(\delta^*, \delta) = 2\pi$.

Lemma 2 (a) The following relations hold: $\Omega = \Omega^\prime$, $\Gamma = \Gamma^\prime$.

(b) The eigenvalues and eigenfunctions of the operator $L_t(q^\delta(x))$ are

$$\lambda_{j,\beta}(v, \tau) = |\beta + \tau|^2 + \mu_j(v(\beta, t)), \quad \Phi_{j,\beta}(x) = e^{i(\beta + \tau, x)} \varphi_{j,\beta}(\beta, t)(\zeta)$$

for $j \in \mathbb{Z}$, $\beta \in \Gamma_\delta$, where $v(\beta, t)$ is the fractional part of $|\delta|^{-2} (t, \delta) - (2\pi)^{-1} (\beta - a, \delta^*)$, $\tau$ and $a$ are uniquely determined from decomposition (47), $\zeta = (\delta, x)$, and $\mu_j(v(\beta, t)), \varphi_{j,\beta}(\beta, t)(\zeta)$ are eigenvalues and corresponding normalized eigenfunctions of the operator $T_{v(\beta, t)}(Q(\zeta))$ (see (9)) generated by the boundary value problem

$$-|\delta|^2 y''(\zeta) + Q(\zeta)y(\zeta) = \mu y(\zeta), \quad y(\zeta + 2\pi) = e^{i2\pi v}y(\zeta),$$

where, for simplicity of the notation, instead of $v(\beta, t)$ we write $v(\beta)$ (or $v$) if $t$ (or $t$ and $\beta$), for which we consider $v(\beta, t)$, is unambiguous.
Proof. (a) For each vector $\omega$ of the lattice $\Omega$ assign $h = \omega - (2\pi)^{-1}(\omega, \delta)\delta^*$. Using the relations $(\omega, \delta) = 2\pi l \in 2\pi Z$, and $(\delta^*, \delta) = 2\pi$ we see that $h \in \Omega$ and $(h, \delta) = 0$, i.e., $h \in \Omega_\delta$. Hence $\Omega \subset \Omega'$. Now for each vector $\gamma$ of the lattice $\Gamma$ assign $b = \gamma - |\delta|^{-2}(\gamma, \delta)\delta$. It is not hard to verify that $b \in H_\delta$ and $(b, \omega) = (\gamma, \omega) \in 2\pi Z$ for $\omega \in \Omega_\delta \subset \Omega$. Therefore $b \in \Gamma_\delta$. Moreover $(b, \delta^*) = (\gamma, \delta^*) - 2\pi(\gamma, \delta) |\delta|^{-2}$. Since $(\gamma, \delta^*) \in 2\pi Z$, that is, $(\gamma, \delta^*) = 2\pi n$, where $n \in Z$, we have $(\gamma, \delta) |\delta|^{-2} = n - (2\pi)^{-1}(b, \delta^*)$. Therefore we obtain an orthogonal decomposition

$$\gamma = b + (\gamma, \delta) \delta = b + (n - (2\pi)^{-1}(b, \delta^*))\delta$$

(48)

of $\gamma \in \Gamma$, where $b \in \Gamma_\delta$, and $n \in Z$. Hence $\Gamma \subset \Gamma'$. On the other hand if $b \in \Gamma_\delta$, $h \in \Omega_\delta$ and $n, l \in Z$, then $(h + \delta^*, b + (n - (2\pi)^{-1}(b, \delta^*))\delta) = (h, b) + (2\pi n)l \in 2\pi Z$. Thus, we have the relations (see definition of the sets $\Omega', \Gamma'$)

$$\Omega \subset \Omega', \Gamma \subset \Gamma', (\omega', \gamma') \in 2\pi Z, \forall \omega' \in \Omega', \forall \gamma' \in \Gamma'. \quad (49)$$

Since $\Omega$ is the set of all vectors $\omega \in \mathbb{R}^d$ satisfying $(\omega, \gamma) \in 2\pi Z$ for all $\gamma \in \Gamma$ and $\Gamma$ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $(\omega, \gamma) \in 2\pi Z$ for all $\omega \in \Omega$ the relations in (49) imply $\Omega' \subset \Omega$, $\Gamma' \subset \Gamma$ and hence $\Omega = \Omega'$, $\Gamma = \Gamma'$.

(b) Since $\beta + \tau$ is orthogonal to $\delta$, turning the coordinate axis so that $\delta$ coincides with one of the coordinate axis and taking into account that the Laplace operator is invariant under rotation, one can easily verify that

$$(-\Delta + q^d(x))\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x)$$

Now using the relation $(\delta, \omega) = 2\pi l$, where $\omega \in \Omega$, $l \in Z$, and the definitions of $\Phi_{j,\beta}(x), \varphi_{j,\nu}(\delta, x)$ we obtain

$$\Phi_{j,\beta}(x + \omega) = e^{i(\beta + \tau, x + \omega)}\varphi_{j,\nu}(\delta, x + \omega) = \Phi_{j,\beta}(x)e^{i(\beta + \tau, \omega) + i2\pi l\nu(\beta, t)}.$$

Replacing $\tau$ and $\omega$ by $t - a - |\delta|^{-2}(t, \delta)\delta$ and $h + \delta^*$, where $h \in \Omega_\delta, l \in Z$, (see (47) and the first equality of (a)) respectively, and then using $(h, \delta) = 0$, $(\delta^*, \delta) = 2\pi$ one can easily verify that

$$(\beta + \tau, \omega) = (t, \omega) + (\beta - a, h) - 2\pi l[|\delta|^{-2}(t, \delta) - (2\pi)^{-1}(\beta - a, \delta^*)].$$

From this using that $(\beta - a, h) \in 2\pi Z$, (since $\beta - a \in \Gamma_\delta$, $h \in \Omega_\delta$), and $\nu(\beta, t)$ is a fractional part of the expression in the last square bracket, we infer $\Phi_{j,\beta}(x + \omega) = e^{i(\omega, \nu(\beta, t))}\Phi_{j,\beta}(x)$. Thus $\Phi_{j,\beta}(x)$ is an eigenfunction of $L_{\ell}(q^d(x))$.

Now we prove that the system $\{\Phi_{j,\beta}(x) : j \in Z, \beta \in \Gamma_\delta\}$ contains all eigenfunctions of $L_{\ell}(q^d(x))$. Assume the converse. Then there exists a nonzero function $f(x) \in L_2(F)$, which is orthogonal to all elements of this system. Using (47), (48) of and the definition of $\nu(\beta, t)$ (see Lemma 2(b)) we get

$$\gamma + t = \beta + \tau + (j + \nu)\delta,$$

(50)

where $\beta \in \Gamma_\delta, \tau \in F_\delta, j \in Z$, and $\nu = \nu(\beta, t)$. Since $e^{i(j + \nu)\zeta}$ can be decomposed by basis $\{\varphi_{j,\nu(\beta, t)}(\zeta) : j \in Z\}$ the function $e^{i(\gamma + t, x)} = e^{i(\beta + \tau, x)}e^{i(j + \nu)\zeta}$ (see
(50)) can be decomposed by system \( \{ \Phi_{j,\beta}(x) = e^{i(\beta + \tau, x)} \varphi_{j,\beta}(\beta, t)}(\zeta) : j \in \mathbb{Z} \). Then the above assumption \( \{ \Phi_{j,\beta}(x), f(x) = 0 \) for \( j \in \mathbb{Z}, \beta \in \Gamma_\delta \) implies that \( (f(x), e^{i(\gamma + t, x)}) = 0 \) for all \( \gamma \in \Gamma \). This is impossible, since the system \( \{ e^{i(\gamma + t, x)} : \gamma \in \Gamma \} \) is a basis of \( L_2(F) \). 

**Remark 3** Clearly every vectors \( x \) of \( \mathbb{R}^d \) has decompositions \( x = \gamma + t \), where \( \gamma \in \Gamma, t \in F \), and \( x = \beta + \tau + (j + v)\delta \), where \( \beta \in \Gamma_\delta, \tau \in F_\delta, j \in \mathbb{Z}, v \in [0,1). \)

We say that the first and second decompositions are \( \Gamma \) and \( \Gamma_\delta \) decompositions respectively. Writing \( \gamma + t = \beta + \tau + (j + v(\beta, t))\delta \) (see (50)) we mean the \( \Gamma_\delta \) decomposition of \( \gamma + t \). As it is noted in lemma 2 instead of \( v(\beta, t) \) we write \( v(\beta) \) (or \( v \)) if \( t \) or \( \tau \) and \( \beta \), for which we consider \( v(\beta, t) \), is unambiguous.

In section 2 we proved that if \( \gamma + t \notin V_\delta(\rho^{\alpha_1}) \) for all \( \delta \in \Gamma(\rho^{\alpha}) \) then there is an eigenvalue \( \Lambda_N \) of \( L_t(q(x)) \), which is close to the eigenvalue \( | \gamma + t |^2 \) of \( L_t(0) \), that is, the influence of the perturbation \( q(x) \) is not significant. If \( \gamma + t \in V_\delta(\rho^{\alpha_1}) \backslash E_2 \), then using (50) and \( \alpha_1 = 3\alpha \), we get

\[
| (j + v)\delta | < r_1, \quad | j\delta | < r_1, \quad r_1 > 2\rho^\alpha, \quad (51)
\]

where \( r_1 = \frac{\rho^\alpha}{2|\delta|} + |2\delta| \). To the eigenvalue \( | \gamma + t |^2 = | \beta + \tau |^2 + | (j + v)\delta |^2 \) (see (50)) of \( L_t(0) \) assign the eigenvalue \( \lambda_{j,\beta}(v, \tau) = | \beta + \tau |^2 + \mu_j(v) \) of \( L_t(q^\delta(x)) \), where \( | (j + v)\delta |^2 \) for \( j \in \mathbb{Z} \) is the eigenvalue of \( T_v(Q(\zeta)) \) (see lemma 2(b)) satisfying

\[
| \mu_j(v) - | (j + v)\delta |^2 | \leq \sup | Q(\zeta) |, \quad \forall j \in \mathbb{Z}. \quad (52)
\]

The eigenvalue \( \lambda_{j,\beta}(v, \tau) \) of \( L_t(q^\delta(x)) \) can be considered as the perturbation of the eigenvalue \( | \gamma + t |^2 = | \beta + \tau |^2 + | (j + v)\delta |^2 \) of \( L_t(0) \) by \( q^\delta(x) \). Lemma 2(b) shows that for this perturbation the influence of \( q^\delta(x) \) is significant for small value of \( j \). Now we prove that there is an eigenvalue \( \Lambda_N \) of \( L_t(q(x)) \) which is close to the eigenvalue \( \lambda_{j,\beta}(v, \tau) \) of \( L_t(q^\delta(x)) \), that is, we prove that the influence of \( q(x) - q^\delta(x) \) is not significant. To prove this we consider the operator \( L_t(q(x)) \) as perturbation of the operator \( L_t(q^\delta(x)) \) with \( q(x) - q^\delta(x) \) and use the formula

\[
(\Lambda_N - \lambda_{j,\beta})(\Psi_N,\beta(x), \Phi_{j,\beta}(x)) = (\Psi_N,\beta(x), (q(x) - q^\delta(x))\Phi_{j,\beta}(x)), \quad (53)
\]

called binding formula for \( L_t(q(x)) \) and \( L_t(q^\delta(x)) \), which can be obtained from

\[
(L_t(q^\delta(x)) + (q(x) - q^\delta(x)))\Psi_N,\beta(x) = \Lambda_N \Psi_N,\beta(x)
\]

by multiplying by \( \Phi_{j,\beta}(x) \) and using \( L_t(q^\delta(x))\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x) \). Note that the binding formula (53) can be obtained from the binding formula (16) by replacing the perturbation \( q(x) \) by the perturbation \( q(x) - q^\delta(x) \), the eigenvalues \( | \gamma + t |^2 \) and eigenfunctions \( e^{i(\gamma + t, x)} \) of \( L_t(0) \) by the eigenvalues and eigenfunctions of the operator \( L_t(q^\delta(x)) \) respectively. Recall that we obtained the asymptotic formulas for the perturbation of the non-resonance eigenvalue \( | \gamma + t |^2 \) by iteration the binding formula (16) for the unperturbed operator \( L_t(0) \)
and the perturbed operator $L_t(q(x))$ (see section 2). Similarly, now to obtain the asymptotic formulas for resonance eigenvalue we iterate the binding formula (53) for the unperturbed operator $L_t(q^0(x))$ and perturbed operator $L_t(q(x))$. For this (as in the non-resonance case) we decompose $(q(x) - q^0(x))\Phi_j,\beta(x)$ by the basis $\{\Phi_{j',\beta'}(x) : j' \in Z, \beta' \in \Gamma_b\}$ and put this decomposition into (53). Let us find this decomposition. Using the decomposition (48) of $\gamma_1 \in \Gamma(\rho^\alpha)$ and (3), we get

$$
\gamma_1 = \beta_1 + (n_1 - (2\pi)^{-1}(\beta_1, \delta^*))\delta, \quad e^{i(\gamma_1, x)} = e^{i(\beta_1, x)}e^{i(n_1 - (2\pi)^{-1}(\beta_1, \delta^*))}\zeta,
$$

$$
q(x) - Q(\zeta) = \sum_{(n_1, \beta_1) \in \Gamma'(\rho^\alpha)} c(n_1, \beta_1)e^{i(\beta_1, x)}e^{i(n_1 - (2\pi)^{-1}(\beta_1, \delta^*))}\zeta + O(\rho^{-\rho^\alpha}),
$$

$$(q(x) - Q(\zeta))\Phi_j,\beta(x) = \sum_{(n_1, \beta_1) \in \Gamma'(\rho^\alpha)} c(n_1, \beta_1)e^{i(\beta_1, x)}e^{i(n_1 - (2\pi)^{-1}(\beta_1, \delta^*))}\zeta \varphi_{j,v}(\gamma_1) + O(\rho^{-\rho^\alpha}),
$$

where $c(n_1, \beta_1) = q_{\gamma_1}$,

$$
\Gamma'(\rho^\alpha) = \{(n_1, \beta_1) : \beta_1 \in \Gamma_b \setminus \{0\}, n_1 \in Z, \beta_1 + (n_1 - (2\pi)^{-1}(\beta_1, \delta^*))\delta \in \Gamma(\rho^\alpha)\}.
$$

Note that if $(n_1, \beta_1) \in \Gamma'(\rho^\alpha)$, then $|\beta_1 + (n_1 - (2\pi)^{-1}(\beta_1, \delta^*))\delta| < \rho^\alpha$ and

$$
|\beta_1| < \rho^\alpha, \quad |(n_1 - (2\pi)^{-1}(\beta_1, \delta^*))\delta| < \rho^\alpha < \frac{1}{2}r_1,
$$

since $\beta_1$ is orthogonal to $\delta$ and $r_1 > 2\rho^\alpha$ (see (51)). To decompose the right-hand side of (54) by basis $\{\Phi_{j',\beta'}(x)\}$ we use the following lemma

**Lemma 3** (a) If integers $j, m$ satisfy the inequalities $|m| > 2 |j|, |m\delta| \geq 2r$, then

$$
(\varphi_{j,v}(\zeta), e^{i(m+v)\zeta}) = O(|m\delta|^{-s-1}) = O(\rho^{-(s+1)\alpha}),
$$

$$
(\varphi_{m,v}, e^{i(j+v)\zeta}) = O(|m\delta|^{-s-1}).
$$

where $r \geq r_1$ and $r_1$ is defined in (51), $\varphi_{j,v}(\zeta)$ is the eigenfunctions of the operator $T_v(Q(\zeta))$, and $Q(\zeta) \in W_2(0, 2\pi)$.

(b) The set $W(\rho) = \{v \in (0, 1) : |\mu_j(v) - \mu_{j'}(v)| > \frac{1}{2m}, \forall j, j' \in Z, j' \neq j\}$ contains a set $A(\varepsilon(\rho)) = (\varepsilon(\rho), \frac{1}{2} - \varepsilon(\rho)) \cup (\frac{1}{2} + \varepsilon(\rho), 1 - \varepsilon(\rho))$, where $\varepsilon(\rho) > 0$ and $\varepsilon(\rho) \to 0$ as $\rho \to \infty$.

**Proof.** (a). To prove (56) we iterate the formula

$$
(\mu_j(v) - |(m + v)\delta|^{\frac{1}{2}})(\varphi_{j,v}(\zeta), e^{i(m+v)\zeta}) = (\varphi_{j,v}(\zeta)Q(\zeta), e^{i(m+v)\zeta}),
$$

by using the decomposition

$$
Q(\zeta) = \sum_{|l_1| < \frac{m}{2\rho}} q_{l_1,\delta}e^{i\xi_1\zeta} + O(|m\delta|^{-(s-1)}).
$$
Note that (58), (59) is one dimensional case of (16), (3) and the iteration of (58) is very simple. If \( |j| < \frac{|m|}{2} \), and \( l_i < \frac{|m|}{2} \) for \( i = 1, 2, ... k \equiv \left[ \frac{x}{2} \right] \), then the inequalities:

\[
| m + v - l_1 - l_2 - ... - l_q | - | j | \frac{1}{m} m,
\]

\[
| m | - | j + v - l_1 - l_2 - ... - l_q | > \frac{1}{2} m | m | \text{ hold for } q = 0, 1, ..., k. \]

Therefore by (52), we have

\[
\begin{align*}
( \mu_j - | | m - l_1 - l_2 - ... - l_q + v) \delta | |^2 | |^{-1} &= O( | m \delta |^{-2}), \\
( \mu_m - | | j - l_1 - l_2 - ... - l_q + v) \delta | |^2 | |^{-1} &= O( | m \delta |^{-2}),
\end{align*}
\]

for \( q = 0, 1, ..., k \). Iterating (58) \( k \) times, by using (60), (61), we get

\[
(\varphi_j, e^{i(m+v)\zeta}) = \sum_{|l_1\delta|, |l_2\delta|, ..., |l_{k+1}\delta| < \frac{|m\delta|}{2}} q_{l_1\delta}q_{l_2\delta}...q_{l_{k+1}\delta} \times
\]

\[
\prod_{p=0}^{\ell} (\varphi_j, e^{i(m-l_1-l_2-...-l_{k+1}+v)\zeta}) + O( | m \delta |^{-s-1}).
\]

Now (56) follows from (60), (62), and the first inequality in (4). Formula (57) can be proved in the same way by using (61) instead of (60). Note that in (56), and (57) instead of \( O( | m \delta |^{-s-1}) \) we can write \( O(\rho^{-s+1/2}) \), since

\[
| m \delta | \geq r \geq r_1 > 2\rho^s \quad \text{(see (51)).}
\]

(b). During the proof of (b) we numerate the eigenvalue of \( T_v(Q(\zeta)) \) in non-decreasing order, i.e., \( \mu_1(v) \leq \mu_2(v) \leq \ldots \). It is well-known that the spectrum of Hill’s operator \( T(Q(\zeta)) \) consists of the intervals

\[
[\mu_{2j-1}(0), \mu_{2j-1}(\frac{1}{2})], [\mu_{2j}(\frac{1}{2}), \mu_{2j}(1)] \text{ for } j = 1, 2, ..., \]

The length of the \( j \)th interval \( \Delta_j \) of the spectrum tends to infinity as \( j \) tends to infinity. The distance between neighbouring intervals, that is the length of gaps in spectrum, tends to zero. The eigenvalues \( \mu_{2j-1}(v) \) and \( \mu_{2j}(v) \) are increasing continuous functions in the intervals \((0, \frac{1}{2})\) and \((\frac{1}{2}, 1)\) respectively and \( \mu_j(1+v) = \mu_j(v) = \mu_j(1-v) \).

Since \( (\ln \rho)^{-1} \to 0 \) as \( \rho \to \infty \), the length of the interval \( \Delta_j \) is sufficiently greater than \( (\ln \rho)^{-1} \) for \( \rho > 1 \) and there are numbers \( \varepsilon_j'(\rho), \varepsilon_j''(\rho) \) in \((0, \frac{1}{2})\) such that

\[
\begin{align*}
\mu_{2j-1}(\varepsilon_{2j-1}(\rho)) &= \mu_{2j-1}(0) + (\ln \rho)^{-1}, \\
\mu_{2j-1}(\frac{1}{2} - \varepsilon_j''(\rho)) &= \mu_{2j-1}(\frac{1}{2}) - (\ln \rho)^{-1}, \\
\mu_{2j}(\frac{1}{2} + \varepsilon_j'(\rho)) &= \mu_{2j}(\frac{1}{2}) + (\ln \rho)^{-1}, \\
\mu_{2j}(1 - \varepsilon_j''(\rho)) &= \mu_{2j}(1) - (\ln \rho)^{-1}.
\end{align*}
\]

Denote \( \varepsilon'(\rho) = \sup_j \varepsilon_j'(\rho), \quad \varepsilon''(\rho) = \sup_j \varepsilon_j''(\rho), \quad \varepsilon(\rho) = \max\{\varepsilon'(\rho), \varepsilon''(\rho)\} \). To prove that \( \varepsilon(\rho) \to 0 \) as \( \rho \to \infty \) we show that both \( \varepsilon'(\rho) \) and \( \varepsilon''(\rho) \) tends to zero as \( \rho \to \infty \). If \( \rho_1 < \rho_2 \) then \( \varepsilon_j'(\rho_2) < \varepsilon_j'(\rho_1) \) and \( \varepsilon_j(\rho_2) < \varepsilon_j(\rho_1) \), since \( \mu_{2j-1}(v) \) and \( \mu_{2j}(v) \) are increasing functions in intervals \((0, \frac{1}{2})\) and \((\frac{1}{2}, 1)\) respectively.

Hence \( \varepsilon'(\rho) \to a \in [0, \frac{1}{2}) \) as \( \rho \to \infty \). Suppose that \( a > 0 \). Then there is sequence \( \rho_k \to \infty \) as \( k \to \infty \) such that \( \varepsilon'(\rho_k) > \frac{a}{2} \) for all \( k \). This implies that there is
a sequence \( \{k_i\} \) and without loss of generality it can be assumed that there is a sequence \( \{2j_k - 1\} \) of odd numbers such that \( \varepsilon_{2j_k-1}(\rho_k) > \frac{\rho}{4} \) for all \( k \). Since 

\[
\mu_{2j_k-1}(v) \text{ increases in } (0, \frac{1}{2}) \text{ and } \mu_{2j_k-1}\left(\varepsilon_{2j_k-1}(\rho_k)\right) - \mu_{2j_k-1}(0) = (\ln \rho_k)^{-1}
\]

we have

\[
\left| \mu_{2j_k-1}\left(\frac{\rho}{4}\right) - \mu_{2j_k-1}(0) \right| \leq (\ln \rho_k)^{-1} \to 0 \text{ as } k \to \infty, \text{ which contradicts the well-known asymptotic formulas for eigenvalues } \mu_j(v), \text{ for } v = 0 \text{ and } v = \frac{\rho}{4},
\]

where \( a \in (0, \frac{1}{2}) \). Thus we proved that \( \varepsilon'(\rho) \to 0 \) as \( \rho \to \infty \). In the same way we prove this for \( \varepsilon''(\rho) \), and hence for \( \varepsilon(\rho) \). Now suppose \( v \in A(\varepsilon(\rho)) \). Using (63), the definition of \( \varepsilon(\rho) \), and taking into account that \( \mu_{2j_k-1}(v) \) and \( \mu_{2j}(v) \) increase in \((0, \frac{1}{2}) \) and \((\frac{1}{2}, 1) \) respectively, we obtain that the eigenvalues \( \mu_1(v), \mu_2(v), \ldots \), are contained in the intervals

\[
[\mu_{2j_k-1}(0) + (\ln \rho_k)^{-1}, \mu_{2j_k-1}\left(\frac{1}{2}\right) - (\ln \rho_k)^{-1}], [\mu_{2j_k}\left(\frac{1}{2}\right) - (\ln \rho_k)^{-1}, \mu_{2j_k}(1) - (\ln \rho_k)^{-1}]
\]

for \( j = 1, 2, \ldots \), and in each interval there exists a unique eigenvalue of \( T_v \). Therefore the distance between eigenvalues of \( T_v \) for \( v \in A(\varepsilon(\rho)) \) is not less than the distance between these intervals, which is not less than \( 2(\ln \rho_k)^{-1} \). Hence the inequality in the definition of \( W(\rho) \) holds, that is, \( A(\varepsilon(\rho)) \subset W(\rho) \).

**Lemma 4** If \( |j\delta| < r \) and \( (n_1, \beta_1) \in \Gamma(\rho^\alpha) \), then

\[
e^{i(n_1-(2\pi)^{-1}(\beta_1, \delta^\ast))}\varphi_{j,v(\beta)}(\zeta) = \sum_{|j\delta|<9r} a(n_1, \beta_1, j, \beta, j + j_1, \beta + \beta_1)\varphi_{j+j_1,v(\beta+\beta_1)}(\zeta) + O(\rho^{-(s-1)\alpha}), \tag{64}\]

where \( r, \Gamma(\rho^\alpha) \) are defined in Lemma 3(a), (54), and

\[
a(n_1, \beta_1, j, \beta, j + j_1, \beta + \beta_1) = e^{i(n_1-(2\pi)^{-1}(\beta_1, \delta^\ast))}\varphi_{j,v(\beta)}(\zeta), \varphi_{j+j_1,v(\beta+\beta_1)}(\zeta)\).\tag{65}\]

**Proof.** Since \( e^{i(n_1-(2\pi)^{-1}(\beta_1, \delta^\ast))}\varphi_{j,v(\beta)}(\zeta) \) is equal to its Fourier series with the orthonormal basis \( \{\varphi_{j,v(\beta)}(\zeta): j_1 \in Z\} \) it suffices to show that

\[
\sum_{j_1,|j_1\delta|\geq9r} |a(n_1, \beta_1, j, \beta, j + j_1, \beta + \beta_1)| = O(\rho^{-(s-1)\alpha}).\tag{66}\]

for all \( j_1 \) satisfying \( |j_1\delta| \geq 9r \) and take into account that \( r \geq r_1 > \rho^\alpha \) (see the last inequality in (51)). Decomposing \( \varphi_{j,v(\beta)} \) over \( \{e^{i(m+v)\zeta}: m \in Z\} \) and using the last inequality in (55), we have

\[
e^{i(n_1-(2\pi)^{-1}(\beta_1, \delta^\ast))}\varphi_{j}(\zeta) = \sum_{m\in Z} (\varphi_{j}, e^{i(m+v)\zeta}) e^{i(m+n+v(\beta+\beta_1))\zeta}, \tag{66}\]

where \( n \in Z \) and \( |n\delta| < r \). This and the decomposition

\[
\varphi_{j+j_1}(\zeta) = \sum_{m\in Z} (\varphi_{j+j_1}, e^{i(m+v(\beta+\beta_1))\zeta}) e^{i(m+n+v(\beta+\beta_1))\zeta}
\]

imply that

\[
a(n_1, \beta_1, j, \beta, j + j_1, \beta + \beta_1) = \sum_{m\in Z} (\varphi_{j}, e^{i(m-n+v)\zeta})(\varphi_{j+j_1}, e^{i(m+v(\beta+\beta_1))\zeta}) \tag{67}\]
Consider two cases: 1. \(|m\delta| > \frac{1}{3} |j_1\delta| \geq 3r\) and 2. \(|m| \leq \frac{1}{3} |j_1|\). In the first case we have \(|(m-n)\delta| > 2r\). Therefore (66) implies that
\[
(\varphi_{j,v}(\zeta), e^{i(m-n+v)\zeta}) = O(|m\delta|^{-s-1}),
\]
\[
\sum_{|m|>|j_1|} |(\varphi_{j,v}, e^{i(m-n+v)\zeta})| = O(|j_1\delta|^{-s}).
\]

In the second case using \(|j_1\delta| \geq 9r, |j\delta| < r\), (57), we get \(|j_1 + j| > 2 |m|\),
\[
(\varphi_{j_{+j_1}}, e^{i(m+v(\beta+\beta_1))\zeta}) = O(|(j_1 + j)\delta|^{-(s-1)}) = O(|j_1\delta|^{-s-1}),
\]
\[
\sum_{|m| \leq \frac{1}{3}|j_1|} |(\varphi_{j_{+j_1}}(\zeta), e^{i(m+v(\beta+\beta_1))\zeta})| = O(|j_1\delta|^{-s}).
\]

These estimations for these two cases together with (67) yield (65).

Now it follows from (54) and (64) that
\[
(q(x) - Q(\zeta))\Phi_{j',\beta'}(x) = O(\rho^{-p\alpha}) + \sum_{(n_1,j_1,\beta_1) \in G(p^\alpha, 9r)} c(n_1, \beta_1) \times
\]
\[
a(n_1, \beta_1, j', j_1, \beta' + \beta_1)e^{i(\beta_1 + \beta' + \tau, x)}\varphi_{j_1, n(\beta_1)}(\zeta)
\]
\[
(68)
\]
for all \(j'\) satisfying \(|j'\delta| < r\), where \(G(p^\alpha, 9r) = \{(n,j,\beta) : |j| < 9r, (n,\beta) \in \Gamma'(p^\alpha), \beta \neq 0\}\). In (68) the multiplicant \(e^{i(\beta_1 + \beta' + \tau, x)}\varphi_{j_1, n(\beta_1)}(\zeta) = \Phi_{j_1, j_1, \beta_1, \beta_1}(x)\) does not depend on \(n_1\). Its coefficient is
\[
A(j', \beta', j' + j_1, \beta' + \beta_1) = \sum_{n_1 : (n_1, \beta_1) \in \Gamma'(p^\alpha)} c(n_1, \beta_1) a(n_1, \beta_1, j', j_1, \beta', \beta_1).
\]
\[
(69)
\]

**Lemma 5** If \(|\beta'| \sim \rho^\alpha|\beta| < r\), where \(r\) is defined in lemma 3(a), then
\[
(q(x) - Q(\zeta))\Phi_{j',\beta'}(x) = \sum_{(j_1, \beta_1) \in Q(p^\alpha, 9r)} A(j', \beta', j_1, \beta' + \beta_1)\Phi_{j_1, j_1, \beta' + \beta_1}(x) + O(\rho^{-p\alpha}),
\]
\[
(70)
\]
where \(Q(p^\alpha, 9r) = \{(j,\beta) : |j\delta| < 9r, 0 < |\beta| < \rho^\alpha\}\). Moreover,
\[
\sum_{(j_1, \beta_1) \in Q(p^\alpha, 9r)} |A(j', \beta', j_1, \beta' + \beta_1)| < c_9,
\]
\[
(71)
\]
where \(c_9\) does not depend on \((j', \beta')\).

**Proof.** The formula (70) follows from (68), (69). Now we prove (71). Since
\[
\sum_{(n_1, \beta_1) \in \Gamma'(\rho^\alpha)} \left| c(n_1, \beta_1) \right| \leq \sum_{\gamma} \left| q_{\gamma} \right| < c_3 \quad \text{(see definition of } c(n_1, \beta_1) \text{ and } (4)) \text{, it follows from } (69) \text{ that we need to prove }
\sum_{j_1} \left| a(n_1, \beta_1, j', \beta', j_1, \beta_1) \right| < c_9(c_3)^{-1}.
\]

For this we use (67) and prove the inequalities:

\[
\sum_{m \in Z} \left| (\varphi_{j'}, e^{i(m-n+v(\beta'))}) \right| < c_{10}, \quad (73)
\]

\[
\sum_{j_1 \in Z} \left| (\varphi_{j'+j_1}, e^{i(m+n+v(\beta_1+\beta'))}) \right| < c_{11}. \quad (74)
\]

Since the distance between numbers \(| v \delta |^2, (1 + v)\delta |^2, ..., \text{ for } v \in [0,1] \text{ and similarly the distance between numbers } |(-1 + v)\delta |^2, (-2 + v)\delta |^2, ..., \text{ is not less than } c_{12}, \text{ it follows from } (52) \text{ that the number of elements of the sets } \\
A = \{ m : (m - n + v(\beta')) \delta |^2 \in [\mu_{j'}(v(\beta')) - 1, \mu_{j'}(v(\beta')) + 1] \} \text{ and } \\
B = \{ j_1 : \mu_{j'}(v(\beta_1 + \beta')) \delta |^2 - 1, \mu_{j'}(v(\beta_1 + \beta')) \delta |^2 \} \text{ is less than } c_{13}. \text{ Now in } (73) \text{ and } (74) \text{ isolating the term with } m \in A \text{ and } j_1 \in B \text{ respectively, applying } (58) \text{ to other terms and then using } \\
\sum_{m \not\in A} \frac{1}{| \mu_{j'}(v') - (m - n + v')\delta |^2} < c_{14}, \\
\sum_{j_1 \not\in B} \frac{1}{| \mu_{j'}(v_1') - (m + v_1')\delta |^2} < c_{14}
\]

we get (73), (74), and hence (64). Clearly the constants \(c_{14}, c_{13}, c_{12}, c_{11}, c_{10}\) can be chosen independently on \((j', \beta')\). Therefore \(c_9\) does not depend on \((j', \beta')\). ■

Replacing \((j, \beta)\) by \((j', \beta')\) in (53) and using (70), we get

\[
(A_N - \lambda_{j', \beta'})b(N, j', \beta') = (\Psi_N(x), (q(x) - Q(\zeta))\Phi_{j', \beta'}(x)) = O(\rho^{-p\alpha})
\]

\[
+ \sum_{(j_1, \beta_1) \in Q(\rho^\alpha, 9r)} A(j', \beta', j_1, \beta_1 + \beta_1)b(N, j_1, \beta_1 + \beta_1) \quad (75)
\]

for \(| \beta' | \sim \rho \text{ and } | j' \delta | < r, \text{ where } b(N, j, \beta) = (\Psi_N(x), \Phi_j(x)) \text{. We would like to emphasize that if } | j' \delta | < r, \text{ then the summation in } (75) \text{ is taken over } Q(\rho^\alpha, 9r) \text{. Therefore if } | j \delta | < r_1 \text{ (see (51))}, \text{ then we have the formula } \\
(A_N - \lambda_{j, \beta})b(N, j, \beta) = O(\rho^{-p\alpha})
\]

\[
+ \sum_{(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)} A(j, \beta, j_1, \beta + \beta_1)b(N, j_1, \beta + \beta_1). \quad (76)
\]

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So (76) is obtained from (75) by interchanging $j', \beta', r$, and $j, \beta, r_1$. Besides (76) is obtained from (53) by applying lemma 5. Now to find the eigenvalue $\Lambda_N$, which is close to $\lambda_{j,\beta}$, where $|j\delta| < r_1$ (see (51)) we are going to iterate (76) as follows. Since $|j\delta| < r_1$ and $(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)$, we have $|(j_1+1)\delta| < 10r_1$. Therefore in (75) interchanging $j', \beta', r$, and $j + j_1, \beta + \beta_1, 10r_1$ and introducing the notations $r_2 = 10r_1$, $j^2 = j + j_1 + j_2$, $\beta^2 = \beta + \beta_1 + \beta_2$, we obtain

$$(\Lambda - \lambda_{j+j_1,\beta+\beta})b(N, j + j_1, \beta + \beta_1) = O(\rho^{-\alpha_1}) +$$

$$\sum_{(j_2, \beta_2) \in Q(\rho^\alpha, 9r_2)} b(N, j^2, \beta^2)A(j + j_1, \beta + \beta_1, j^2, \beta^2). \quad (77)$$

Clearly, there exist an eigenvalue $\Lambda_N(t)$ satisfying $|\lambda_{j,\beta} - \Lambda_N| \leq 2M$, where $M = \text{sup} |q(x)|$. Moreover, we will prove that if $|\beta| \sim \rho$, and

$$(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1) \quad \text{(see Lemma 6)},$$

and introducing $r_2 = 10r_1$, $j^2 = j + j_1 + j_2$, $\beta^2 = \beta + \beta_1 + \beta_2$, we obtain

$$|\lambda_{j,\beta} - \lambda_{j+j_1,\beta+\beta_1}| > \frac{5}{9}\rho^\alpha, \quad |\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}| > \frac{1}{9}\rho^\alpha. \quad (78)$$

Then dividing both side of (77) by $\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}$ and using (78), we get

$$b(N, j + j_1, \beta_1 + \beta) = O(\rho^{-\alpha_1}) +$$

$$\sum_{(j_2, \beta_2) \in Q(\rho^\alpha, 9r_2)} \frac{A(j + j_1, \beta + \beta_1, j^2, \beta^2) b(N, j^2, \beta^2)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta}}. \quad (79)$$

Putting the obtained formula for $b(N, j + j_1, \beta_1 + \beta)$ into (76), we obtain

$$(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = O(\rho^{-\alpha_1}) +$$

$$\sum_{(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)} \sum_{(j_2, \beta_2) \in Q(\rho^\alpha, 9r_2)} \frac{A(j, \beta, j + j_1, \beta + \beta_1) A(j + j_1, \beta + \beta_1, j^2, \beta^2) b(N, j^2, \beta^2)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta}}. \quad (80)$$

Thus we got the one time iteration of (76). It will give the first term of asymptotic formula for $\Lambda_N$. For this we find the index $N$ such that $b(N, j, \beta)$ is not very small (see Lemma 7) and (78) is satisfied, i.e., the denominator of the fraction in (80) is a big number. Then dividing both sides of (80) by $b(N, j, \beta)$, we get the asymptotic formula for $\Lambda_N$ (see Theorem 3).

**Lemma 6** Let $\gamma + t \equiv \beta + \tau + (j + v)\delta \in V'_g(\rho^\alpha_1)$ (see (50) and Remark 3), and $(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)$, $(j_k, \beta_k) \in Q(\rho^\alpha, 9r_k)$, where $r_1$ is defined in (51) and $r_k = 10r_{k-1}$ for $k = 2, 3, ..., p - 1$. Then

$$|j\delta| = O(\rho^\alpha_1), \quad |j_k\delta| = O(\rho^\alpha_1), \quad |\beta_k| < \rho^\alpha, \forall k = 1, 2, ..., p - 1, \quad (81)$$

$$|\lambda_{j,\beta}(v, \tau) - \lambda_{j',\beta}| > 2(\ln \rho)^{-1}, \forall j' \neq j, \forall v(\beta) \in W(\rho). \quad (82)$$

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Moreover if \( |j' \delta| < \frac{1}{9} \rho^2, |\beta' - \beta| < (p-1) \rho^2, \beta' \in \Gamma_\delta, j^k = j + j_1 + \ldots + j_k, \beta^k = \beta + \beta_1 + \ldots + \beta_k, \) where \( k = 1, 2, \ldots, p - 1, \) then

\[
|\lambda_{j, \beta} - \lambda_{j', \beta'}| > \frac{5}{9} \rho^2, \quad \forall \beta' \neq \beta,
\]

\[
|\lambda_{j, \beta}(v, \tau) - \lambda_{j^k, \beta^k}| > \frac{5}{9} \rho^2, \quad \forall \beta^k \neq \beta.
\]

**Proof.** The relations in (81) follows from (51) and the definitions of \( r_1, r_k, Q(\rho^\alpha, 9r_k) \) (see Lemma 5). Inequality (82) is a consequence of the definition of \( W(\rho) \) (see Lemma 3(b)). Inequality (84) follows from (83) and (81). It remains to prove (83). Since

\[
|\lambda_{j, \beta} - \lambda_{j', \beta'}| \geq ||\beta' + \tau| - |\beta + \tau|| - |\mu_j - \mu_{j'}|,
\]

it is enough to prove the following two inequalities \( |\mu_j - \mu_{j'}| < \frac{1}{9} \rho^2; \)

\[
||\beta' + \tau| - |\beta + \tau|| > 8 \frac{\rho^2}{9}.
\]

The first inequality follows from \( |j' \delta| < \frac{1}{9} \rho^2, |\beta' - \beta| < (p-1) \rho^2, |\delta| < \rho^2 \) imply that there exist \( n \in \mathbb{Z} \) and \( \gamma' \in \Gamma \) such that

\[
\gamma' = \beta' - \beta + (n + (2\pi)^{-1} (\beta - \beta, \delta)) \delta \in \Gamma(p\rho^\alpha).
\]

Since \( \beta' - \beta \) is nonzero element of \( \Gamma_\delta \) we have \( \gamma' \in \Gamma(p\rho^\alpha) \setminus \delta R. \) This together with the inclusion \( \gamma + t = \beta + \tau + (j + v) \delta \in V_\delta(\rho^\alpha) = V_\delta(\rho^\alpha) \setminus E_2 \) (see assumption of the lemma and definition of \( E_2 \)) implies that \( \gamma + t \notin V_\gamma(\rho^\alpha), \)

that is, \( ||\gamma + t| - |\gamma + t + \gamma'||^2 \geq \rho^2. \) From this using the orthogonal decompositions (50), (87) of \( \gamma + t, \gamma \) and taking into account that \( \beta, \tau, \beta' \) are orthogonal to \( \delta, |j \delta| = O(\rho^\alpha) \) (see (81)), \( |n + (2\pi)^{-1} (\beta - \beta, \delta)\delta| = O(\rho^\alpha), \alpha_2 > 2\alpha, \) we obtain (86).

**Lemma 7.** Suppose \( h_1(x), h_2(x), \ldots, h_m(x) \in L_2(F), \) where \( m = p_1 - 1, \)

\[
p_1 = \left\lfloor \frac{q}{d} \right\rfloor + 1.
\]

Then for every eigenvalue \( \lambda_{j, \beta} \sim \rho^2 \) of the operator \( L_q(q^2(x)) \) there exists an eigenvalue \( \Lambda_N \) and a corresponding normalized eigenfunction \( \Psi_N \) of the operator \( L_q(q^2(x)) \) such that:

(i) \( |\lambda_{j, \beta} - \Lambda_N| \leq 2M, \) where \( M = \sup \left\{ q(x) \right\}, \)

(ii) \( |b(N, j, \beta)| > \frac{1}{4} (c_5)^{-\frac{1}{2}} \rho^{-\frac{1}{2}} \rho^{d-1} \rho^\alpha, \)

(iii) \( |b(N, j, \beta)|^2 > \frac{1}{2M} \sum_{i=1}^{m} (\Psi_N, \frac{h_i}{||h_i||})^2 > \frac{1}{2M} (\Psi_N, \frac{h_i}{||h_i||})^2, \forall i. \)

**Proof.** Let \( A, B, C \) be the set of indexes \( N \) satisfying (i), (ii), (iii) respectively. Using (53) and the Bessel inequality we obtain

\[
\sum_{N \notin A} |b(N, j, \beta)|^2 = \sum_{N \notin A} \left| \frac{(\Psi_N(x), (q(x) - Q(\xi)) \Phi_{j, \beta}(x))}{\Lambda_N - \lambda_{j, \beta}} \right|^2
\]

\[
< (2M)^{-2} \left| q(x) - Q(\xi)) \Phi_{j, \beta}(x) \right| \leq \frac{1}{4}.
\]

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This and the inequality $| A | < c_5 \rho^{(d-1) \rho \alpha}$ (see the end of the introduction) imply that $\sum_{i=1}^{n} \sum_{\alpha} | b(N, j, \beta) |^2 < \frac{1}{2}$. Therefore using the Parseval equality, we obtain

$$\sum_{N \in A} | b(N, j, \beta) |^2 \geq \frac{3}{4} \sum_{N \in A \cap B} | b(N, j, \beta) |^2 > \frac{1}{2}.$$ 

Now to prove the lemma we show that there exists $N \in A \cap B$ satisfying (iii). Assume the converse, that is, assume that the condition (iii) does not hold for all $N \in A \cap B$. Then using the Bessel inequality, and the last inequalities we get the contradiction $\frac{1}{2} < \sum_{N \in A \cap B} | b(N, j, \beta) |^2 < \frac{1}{2m} \sum_{i=1}^{m} \sum_{N \in A} | (\Psi_N, \frac{h_i}{\|h_i\|}) |^2$.

Theorem 3 For every eigenvalue $\lambda_{j, \beta}(v, \tau)$ of $L_i(q^\delta(x))$ such that $\beta + \tau + (j + \nu) \delta \in V_0^{(\rho \alpha)}$, there exists an eigenvalue $\Lambda_N$, denoted by $\Lambda_N(\lambda_{j, \beta}(v, \tau))$, of $L_i(q(x))$ satisfying

$$\Lambda_N(\lambda_{j, \beta}(v, \tau)) = \lambda_{j, \beta}(v, \tau) + O(\rho^{-\alpha_2}).$$

Proof. By Lemma 7 there is an eigenvalue $\Lambda_N$ satisfying (i)-(iii) for

$$h_i(x) = \sum_{(j_1, \beta_1) \in 2^m, (\beta_2) \in 2^m} \frac{A(j_1, \beta_1, j_2, \beta_2) \Phi_{j_2, \beta_2}(x)}{(\lambda_{j, \beta} - \lambda_{j_1, \beta_1})},$$

where $i = 1, 2, \ldots, m; m = p_1 - 1$. Since $\beta_1 \neq 0$, the inequality (83) and condition (i) of lemma 7 yield (78). Hence, in brief notations $a = \lambda_{j, \beta}, z = \lambda_{j, \beta}, z = \lambda_{j_1, \beta_1}$, we have $| \Lambda_N - a | < 2M, | z - a | > \frac{1}{2} \rho^{\alpha_2}$. Using the relations

$$\frac{1}{\Lambda_N - z} = -\sum_{i=1}^{\infty} \frac{(\Lambda_N - a)^{i-1}}{(z - a)^i} = -\sum_{i=1}^{m} \frac{(\Lambda_N - a)^{i-1}}{(z - a)^i} + O(\rho^{-p_1 \alpha_2})$$

and $p_1 \alpha_2 > \rho \alpha$ we can see that formula (80) can be written as

$$(\Lambda_N - \lambda_{j, \beta})b(N, j, \beta, \alpha) = \sum_{i=1}^{m} (\Lambda_N - a)^{i-1} (\Psi_N, \frac{h_i}{\|h_i\|}) \| h_i \| + O(\rho^{-\rho \alpha}).$$

Dividing both sides by $b(N, j, \beta)$, using (ii), (iii) of lemma 7, and the obvious inequality $p \alpha - \frac{1}{2} (d - 1) q \alpha > \alpha_2$ (see the third inequality in (15) and the definitions of $k_1, \alpha, \alpha_2$ in the end of introduction), we get

$$| (\Lambda_N - \lambda_{j, \beta}) | < (2m)^{\frac{1}{2}} \sum_{i=1}^{m} | \Lambda_N - a |^{i-1} \| h_i \| + O(\rho^{-\alpha_2})$$

On the other hand the inequalities (71) and (84) imply that $\| h_i \| = O(\rho^{-\alpha_2})$. Therefore (89) and $| \Lambda_N - a | < 2M$, yield the proof of the theorem.

It follows from formulas (82), (84), and (88) that

$$| \Lambda_N(\lambda_{j, \beta}) - \lambda_{j, \beta}(v, \tau) | > c(\beta^k, \rho), \forall v(\beta) \in W(\rho),$$

(90)
where \((j_k, \beta_k) \in Q(\rho^a, 9r_k), k = 1, 2, ..., p - 1; c(\beta^k, \rho) = (\ln \rho)^{-1}\) when \(\beta^k = \beta, j^k \neq j\) and \(c(\beta^k, \rho) = \frac{1}{2}\rho^{2\epsilon}\) when \(\beta^k \neq \beta\). We iterated (76) one time and got (80) from which the formula (88) is obtained. Now to obtain the asymptotic order for \(\Lambda_N\) we repeat this iteration \(2p_1\) times as follows. Since \(|j^2| < r_1\) (see (51)), \((j_1, \beta_1) \in Q(\rho^a, 9r_1), (j_2, \beta_2) \in Q(\rho^a, 9r_2)\) (see (80)) and \(j^2 = j + j_1 + j_2\) (see definition of \(j^2\) in (77)), we have \(|j^2| < 10r_2\).

Therefore in (75) interchanging \(j', \beta', r\), and \(j^2, \beta^2, 10r_2\) and using the notations \(r_3 = 10r_2, j^3 = j^2 + j_3, \beta^3 = \beta^2 + \beta_3\) (see Lemma 6), we obtain

\[
(A_N - \lambda_{j^2, \beta^2})b(N, j^2, \beta^2) = O(\rho^{-p_0}) +
\]

\[
\sum_{(j_3, \beta_3) \in Q(\rho^a, 9r_3)} b(N, j^3, \beta^3)A(j^2, \beta^2, j^3, \beta^3).
\]

Dividing both side of (91) by \(A_N - \lambda_{j^2, \beta^2}\) and using (90), we get

\[
b(N, j^2, \beta^2) = O(\rho^{-p_0}(c(\beta^2, \rho))^{-1}) + \sum_{(j_3, \beta_3) \in Q(\rho^a, 9r_3)} \frac{b(N, j^3, \beta^3)A(j^2, \beta^2, j^3, \beta^3)}{A_N - \lambda_{j^2, \beta^2}}.
\]

(92)

for \((j^2, \beta^2) \neq (j, \beta)\). In the same way we obtain

\[
b(N, j^k, \beta^k) = O(\rho^{-p_0}(c(\beta^k, \rho))^{-1}) + \sum_{(j_{k+1}, \beta_{k+1}) \in Q(\rho^a, 9r_{k+1})} \frac{b(N, j^{k+1}, \beta^{k+1})A(j^k, \beta^k, j^{k+1}, \beta^{k+1})}{A_N - \lambda_{j^k, \beta^k}}.
\]

(93)

for \((j^k, \beta^k) \neq (j, \beta)\), \(k = 3, 4, ...\). Now we isolate the terms in the right-hand side of (80) with multipicand \(b(N, j, \beta)\), i.e., the case \((j^2, \beta^2) = (j, \beta)\), and replace \(b(N, j^2, \beta^2)\) in (80) by the right-hand side of (92) when \((j^2, \beta^2) \neq (j, \beta)\) and use (78), (90) to get

\[
(A_N - \lambda_{j, \beta})b(N, j, \beta) = S'_1(A_N, \lambda_{j, \beta})b(N, j, \beta) + O(\rho^{-p_0}) + \sum_{(j_1, \beta_1) \in Q(\rho^a, 9r_1), (j_2, \beta_2) \in Q(\rho^a, 9r_2), (j^2, \beta^2) \neq (j, \beta)} \frac{A(j_1, \beta_1, j^2, \beta^2)b(N, j^3, \beta^3)}{(A_N - \lambda_{j_1, \beta_1 + \beta_1})(A_N - \lambda_{j^2, \beta^2})},
\]

(94)

where

\[
S'_1(A_N, \lambda_{j, \beta}) = \sum_{(j_1, \beta_1) \in Q(\rho^a, 9r_1)} \frac{A(j_1, \beta_1 + \beta_1 + \beta_1, j + j_1, \beta_1 + \beta_1, \beta_1, \beta_1, \beta_1)}{A_N - \lambda_{j_1, \beta_1 + \beta_1}}.
\]

(95)

The formula (94) is the two times iteration of (76). Repeating these process \(2p_1\) times, i.e., in (94) isolating the terms with multipicand \(b(N, j, \beta)\) (i.e., the case
\((j^3, \beta^3) = (j, \beta)\) and replacing \(b(N, j^3, \beta^3)\) by the right-hand side of (93) for \(k = 3\) when \((j^3, \beta^3) \neq (j, \beta)\) etc., we obtain

\[
(\Lambda_N - \lambda_{j,\beta})b(N, j, \beta) = (\sum_{k=1}^{2p_1} S_k'(\Lambda_N, \lambda_{j,\beta}))b(N, j, \beta) + C_{2p_1}' + O(\rho^{-pa}), \tag{96}
\]

where

\[
S_k'(\Lambda_N, \lambda_{j,\beta}) = \sum_{i=1}^{k} \left(\prod_{i=1}^{k} A\left(\frac{j^{i-1} \beta^{i-1}}{\Lambda_N - \lambda_j, \beta}\right) A(j^k, \beta^k, j, \beta),
\]

\[
C_k' = \sum_{i=1}^{k} \left(\prod_{i=1}^{k} A\left(\frac{j^{i-1} \beta^{i-1}}{\Lambda_N - \lambda_j, \beta}\right) A(j^k, \beta^k, j^{k+1}, \beta^{k+1})b(N, j^{k+1}, \beta^{k+1})
\]

Here \(j^0 = j, \beta^0 = \beta\) and the summation for \(S_k'\), and \(C_k'\) are taken under the conditions \((j_i, \beta_i) \in Q(\rho^a, 9r_1), (j^i, \beta^i) \neq (j, \beta)\), for \(i = 2, 3, ..., k\) and for \(i = 2, 3, ..., k + 1\) respectively. Since \(\beta_k \neq 0\) for every integer \(k\), the relation \(\beta_1 + ... + \beta_i = 0\) implies that \(\beta_1 + ... + \beta_{i+1} \neq 0\). Hence \(\beta^i \neq \beta\). Moreover if \(\beta^i = \beta\), then \(\beta^{i+1} \neq \beta\). Therefore the number of the multiplicands \(\Lambda_N - \lambda_j, \beta\) in the denominators of \(S_k\) and \(C_{2p_1}'\), satisfying \(|\Lambda_N(\lambda_{j,\beta}) - \lambda_j, \beta| > \frac{1}{2} \rho^{a^2}\) (see 90) is not less than \(\frac{k}{2}\) and \(p_1\) respectively. Hence using (71) and \(p_1a_2 > pa\), we obtain

\[
C_{2p_1}' = O((\rho^{-a_2} \ln \rho)^{p_1}) = O(\rho^{-pa}), \quad S_k'(\Lambda_N, \lambda_{j,\beta}) = O(\rho^{-a_2}), \quad \forall k = 2, 3, ..., 2p_1.
\]

To prove this estimation we used (90). Moreover, if a real number \(a\) satisfies \(|a - \lambda_{j,\beta}| < (\ln \rho)^{-1}\) then, by (82), (84) we have \(|a - \lambda_j, \beta| > c(\beta, \rho)|. Therefore using this instead of (90) and repeating the proof of (97) we obtain

\[
S_k'(a, \lambda_{j,\beta}) = O(\rho^{-a_2}), \quad S_k'(a, \lambda_{j,\beta}) = O(\rho^{-a_2} \ln \rho)^{\frac{k}{2}}, \quad \forall k = 2, 3, ..., 2p_1.
\]

**Theorem 4** For every eigenvalue \(\lambda_{j,\beta}(v, \tau)\) of the operator \(L_4(q^6(v))\) such that \(\beta + \tau + (j + v) \delta \in V^3_4(\rho^a), \quad v(\beta) \in W(\rho)\) there exists an eigenvalue \(\Lambda_N\), denoted by \(\Lambda_N(\lambda_{j,\beta}(v, \tau))\), of \(L_4(q(x))\) satisfying the formulas

\[
\Lambda_N(\lambda_{j,\beta}(v, \tau)) = \lambda_{j,\beta}(v, \tau) + E_{k-1}(\lambda_{j,\beta}) + O(\rho^{-k\alpha}(\ln \rho)^{2k}), \tag{99}
\]

where \(E_0 = 0, \quad E_s = \sum_{k=1}^{2p_1} S_k'(\lambda_{j,\beta} + E_{s-1}, \lambda_{j,\beta})\), for \(s = 1, 2, ..., \) and

\[
E_k(\lambda_{j,\beta}) = O(\rho^{-a_2}(\ln \rho)), \tag{100}
\]

for \(k = 1, 2, ..., \lceil\frac{1}{2}(p - \frac{1}{2}q(d - 1))\rceil\)

**Proof.** The proof of this Theorem is similar to the proof of Theorem 1(a). By Theorem 3 formula (99) for the case \(k = 1\) is proved and \(E_0 = 0\). Hence
(100) for \( k = 1 \) is also proved. The proof of (100), for arbitrary \( k \), follows from (98) and the definition of \( E_s \) by induction. Now we prove (99) by induction. Assume that (99) is true for \( k = s < \left[ \frac{1}{2}(p - \frac{1}{2}q(d - 1)) \right] \), i.e.,

\[
\Lambda_N = \lambda_{j, \beta} + E_{s-1} + O(\rho^{-s\alpha_2}(\ln \rho)^{2s}).
\]

Putting this expression for \( \Lambda_N \) into \( \sum_{k=1}^{2p_1} S'_k(\Lambda_N, \lambda_{j, \beta}) \), dividing both sides of (96) by \( b(N, j, \beta) \), using (97), (98), condition (ii) of Lemma 7 and the equality \( \alpha_2 = 9\alpha \), we get

\[
\Lambda_N = \lambda_{j, \beta} + \sum_{k=1}^{2p_1} S'_k(\lambda_{j, \beta} + E_{s-1} + O(\frac{\ln \rho}{\rho})^{2s}), \lambda_{j, \beta}) + O(\rho^{-\frac{1}{2}(p - \frac{1}{2}q(d-1))\alpha_2})
\]

\[
= O(\rho^{-\frac{1}{2}(p - \frac{1}{2}q(d-1))\alpha_2}) + \lambda_{j, \beta} + \sum_{k=1}^{2p_1} S'_k(\lambda_{j, \beta} + E_{s-1}, \lambda_{j, \beta}) +
\]

\[
\sum_{k=1}^{2p_1} S'_k(\lambda_{j, \beta} + E_{s-1} + O(\rho^{-s\alpha_2}(\ln \rho)^{2s}), \lambda_{j, \beta}) - \sum_{k=1}^{2p_1} S'_k(\lambda_{j, \beta} + E_{s-1}, \lambda_{j, \beta})\}.
\]

To prove (99) for \( k = s + 1 \) we need to show that the expression in the curly brackets is equal to \( O((\rho^{-(s+1)\alpha_2}(\ln \rho)^{2s+1}) \). This can be checked by using the estimations (71), (100), (82), (84) and the obvious relation

\[
\frac{1}{\prod_{i=1}^{p_1} (\lambda_{j, \beta} + E_{s-1} + O(\rho^{-s\alpha_2}(\ln \rho)^{2s}) - \lambda_{j', \beta'})} - \frac{1}{\prod_{i=1}^{p_1} (\lambda_{j, \beta} + E_{s-1} - \lambda_{j', \beta'})} = O(\rho^{-s\alpha_2}(\ln \rho)^{2s}(\ln \rho) - 1)
\]

\[
= O(\rho^{-(s+1)\alpha_2}(\ln \rho)^{2(s+1)}) \quad \text{for all } n = 1, 2, ..., 2p_1. \quad \text{The theorem is proved} \]

Remark 4 Here we note some properties of the known parts \( \lambda_{j, \beta} + E_k \) (see (99)), where \( \lambda_{j, \beta} = \mu_j(v) + \beta + \tau \| \beta \|^2 \) (see Lemma 2(b)), of the eigenvalues of \( L_t(q(x)) \). We prove the equality

\[
\frac{\partial}{\partial \tau_i} (E_k(\mu_j(v) + \beta + \tau \| \beta \|^2)) = O(\rho^{-2\alpha_2 + \alpha} \ln \rho), \quad (101)
\]

for \( i = 1, 2, ..., d - 1 \), where \( \tau = (\tau_1, \tau_2, ..., \tau_{d-1}) \), \( k = 1, 2, ..., \left[ \frac{1}{2}(p - \frac{1}{2}q(d - 1)) \right] \), and \( v(\beta) \in W(\rho) \). To prove (101) for \( k = 1 \) we calculate the derivatives of the expression \( H(\beta^k, \tau) = (\mu_j + \beta + \tau \| \beta \|^2 - \mu_j - \beta^k + \tau \| \beta \|^2)^{-1} \). Since \( \mu_j \) and \( \mu_j' \) do not depend on \( \tau_i \), the function \( H(\beta^k, \tau) \) for \( \beta^k = \beta \) do not depend on \( \tau_i \) and it follows from Lemma 3(b) that \( H(\beta, \tau) = O(\ln \rho) \). For \( \beta^k \neq \beta \) using (84), and equality \( |\beta^k - \beta| = |\beta_1 + \beta_2 + ... + \beta_{d-1} + \beta| = O(\rho^\alpha) \) (see last inequality in (81)) we obtain that the derivatives of \( H(\beta^k, \tau) \) is equal to \( O(\rho^{-2\alpha_2 + \alpha}) \). Therefore using (71) and the definition of \( E_k(\lambda_{j, \beta}) \) (see (99) and (96)), by direct calculation, we get (101) for \( k = 1 \). Now suppose that (101) holds for \( k = s - 1 \). Using this, replacing \( \mu_j + \beta + \tau \| \beta \|^2 \) by \( \mu_j + \beta + \tau \| \beta \|^2 + E_{s-1} \) in \( H(\beta^k, \tau) \), arguing as above we get (101) for \( k = s \).
4 Asymptotic Formulas for the Bloch Functions

In this section using the asymptotic formulas for the eigenvalues and the simplicity conditions (12), (13), we prove the asymptotic formulas for the Bloch functions with a quasimomentum of the simple set $B$.

**Theorem 5** If $\gamma + t \in B$, then there exists a unique eigenvalue $\Lambda_N(t)$ satisfying (5) for $k = 1, 2, ..., \lfloor \frac{d}{2} \rfloor$, where $p$ is defined in (3). This is a simple eigenvalue and the corresponding eigenfunction $\Psi_{N,t}(x)$ of $L(q(x))$ satisfies (10) if $q(x) \in W_2^{s_0}(F)$, where $s_0 = \frac{3d-1}{2}(3^d + d^2 + \frac{1}{4}d3^d + d + 6)$.

**Proof.** By Theorem 1(b) if $\gamma + t \in B \subset U(\rho^{\alpha_1}, p)$, then there exists an eigenvalue $\Lambda_N(t)$ satisfying (5) for $k = 1, 2, ..., \lfloor \frac{1}{3}(p - \frac{1}{2}q(d - 1)) \rfloor$. Since $k_1 = \lfloor \frac{1}{3}(p - \frac{1}{2}q(d - 1)) \rfloor + 2 \leq \frac{1}{3}(p - \frac{1}{2}q(d - 1))$ (see the third inequality in (15)) formula (5) holds for $k = k_1$. Therefore using (5), the relation $3k_1\alpha > d + 2\alpha$ (see the fifth inequality in (15)), and notations $F(\gamma + t) = |\gamma + t|^2 + F_{k_1-1}(\gamma + t)$, $\varepsilon_1 = \rho^{-d-2\alpha}$ (see Step 1 in introduction), we obtain

$$
\Lambda_N(t) = F(\gamma + t) + o(\varepsilon_1). \tag{102}
$$

Let $\Psi_N$ be any normalized eigenfunction corresponding to $\Lambda_N$. Since the normalized eigenfunction is defined up to constant of modulus 1, without loss of generality it can be assumed that arg $b(N, \gamma) = 0$, where $b(N, \gamma) = (\Psi_N, e^{i(\gamma + t,x)})$. Therefore to prove (10) it suffices to show that (14) holds. To prove (14) first we estimate $\sum_{\gamma' \notin K} |b(N, \gamma')|^2$ and then $\sum_{\gamma' \in K \setminus \gamma} |b(N, \gamma')|^2$, where $K$ is defined in (12), (13). Using (102), the definition of $K$, and (16), we get

$$
\sum_{\gamma' \notin K} |b(N, \gamma')|^2 \geq \|q(x)\Psi_N\|^2 O(\rho^{-2\alpha_1}) = O(\rho^{-2\alpha_1}). \tag{103}
$$

If $\gamma' \in K$, then by (102) and by definition of $K$, it follows that

$$
|\Lambda_N - |\gamma' + t|^2| < \frac{1}{4}\rho^{\alpha_1}. \tag{104}
$$

Now we prove that the simplicity conditions (12), (13) imply

$$
|b(N, \gamma')| < c_4\rho^{-\alpha}, \forall \gamma' \in K \setminus \gamma, \tag{105}
$$

where $c = p - dq - \frac{1}{2}d3^d - 3$. The conditions $\gamma' \in K$, $\gamma + t \in B$ and (24) imply the inclusion $\gamma' + t \in R(\frac{3}{2}p)(R(\frac{3}{2}p))$. If for $\gamma' + t \in U(\rho^{\alpha_1}, p)$ and $\gamma' \in K \setminus \gamma$ the inequality in (105) is not true, then by (104) and Theorem 1(a), we have

$$
\Lambda_N = |\gamma' + t|^2 + F_{k_1-1}(\gamma' + t) + O(\rho^{-3\alpha_1}) \tag{106}
$$

for $k = 1, 2, ..., \lfloor \frac{1}{3}(p - c) \rfloor = \lfloor \frac{1}{3}(dq + \frac{1}{4}d3^d + 3) \rfloor$. Since $\alpha = \frac{1}{q}$ and
Therefore arguing as in the prove of (102), we get \(\Lambda_N\) holds. Therefore, using (105) holds. Therefore, using $|p|$ where $\gamma \in \Gamma((n, b))$ for $n = \left\lfloor \frac{2p - (3d - 1)q - \frac{1}{2}d^3 - 6}{d} \right\rfloor$. Hence we have

\[
\Lambda_N - \lambda_j(\gamma' + t) = o(\varepsilon_1).
\]

This with (102) contradicts (13). So the inequality in (105) holds. Therefore, using $|K| = O(p^{d-1})$, $\gamma = 1$, we get

\[
\sum_{\gamma' \in K \setminus (\gamma)} |b(N, \gamma')|^2 = O(\rho^{-2(c-q(d-1)\alpha)}) = O(\rho^{-2(p-(3d-1)q-\frac{1}{2}d^3-6)\alpha}).
\]

If $s = s_0$, that is, $p = s_0 - d$, then $2p - (3d - 1)q - \frac{1}{2}d^3 - 6 = 6$. Since $\alpha_1 = 3\alpha$, the equality (108) and the equality in (103) imply (14). Thus we proved that the equality (10) holds for any normalized eigenfunction $\Psi_N$ corresponding to any eigenvalue $\Lambda_N$ satisfying (5). If there exist two different eigenvalues or multiple eigenvalue satisfying (5), then there exist two orthogonal normalized eigenfunction satisfying (10), which is impossible. Therefore $\Lambda_N$ is a simple eigenvalue. It follows from Theorem 1(a) that $\Lambda_N$ satisfies (5) for $k = 1, 2, ..., \left\lfloor \frac{n}{3} \right\rfloor$, since the inequality (7) holds for $c = 0$ (see (10)). \(\blacksquare\)

**Remark 5** Since for $\gamma + t \in B$ there exists a unique eigenvalue satisfying (5), (102) we denote this eigenvalue by $\Lambda(\gamma + t)$. Since this eigenvalue is simple, we denote the corresponding eigenfunction by $\Psi_{\gamma+t}(x)$. By Theorem 5 this eigenfunction satisfies (10). Clearly, for $\gamma + t \in B$ there exists a unique index $N = N(\gamma + t)$ such that $\Lambda(\gamma + t) = \Lambda_N(\gamma + t)$ and $\Psi_{\gamma+t}(x) = \Psi_{N(\gamma + t)}(x)$.

Now we prove the asymptotic formulas of arbitrary order for $\Psi_{\gamma+t}(x)$.

**Theorem 6** If $\gamma + t \in B$, then the eigenfunction $\Psi_{\gamma+t}(x) = \Psi_{N(\gamma + t)}(x)$ corresponding to the eigenvalue $\Lambda_N = \Lambda(\gamma + t)$ satisfies formulas (11), for $k = 1, 2, ..., n$, where $n = \left\lfloor \frac{2p - (3d - 1)q - \frac{1}{2}d^3 - 6}{d} \right\rfloor$.

\[
\Phi_0(x) = 0, \quad \Phi_1(x) = \sum_{\gamma_1 \in \Gamma((n, \rho))} \frac{q_{\gamma_1} e^{i(\gamma + (\gamma_1 + t))}}{|\gamma + t|^2 - |\gamma + \gamma_1 + t|^2},
\]

and $\Phi_{k-1}(x)$ for $k > 2$ is a linear combination of $e^{i(\gamma + \gamma' + x)}$ for $\gamma' \in \Gamma((k - 1)\rho^n) \cup \{0\}$ with coefficients (114), (115).

**Proof.** By Theorem 5, formula (11) for $k = 1$ is proved. To prove formula (11) for arbitrary $k \leq n$ we prove the following equivalent relations

\[
\sum_{\gamma' \in \Gamma((k-1)\rho^n)} |b(N, \gamma + \gamma')|^2 = O(\rho^{-2k\alpha_1}),
\]

**\(\Psi_N = b(N, \gamma)e^{i(\gamma + x)} + \sum_{\gamma' \in \Gamma((k-1)\rho^n)} b(N, \gamma + \gamma')e^{i(\gamma + \gamma' + x)} + H_k(x), \)**

30
where \( \Gamma'(m) = \Gamma'(m) \setminus \{ \Gamma(m \rho^\alpha) \} \) and \( \| H_k(x) \| = O(\rho^{-k\alpha_1}) \). The case \( k = 1 \) is proved due to (14). Assume that (109) is true for \( k = m \). Then using (110) for \( k = m, \) and (3), we have \( \Psi_N(x)(q(x)) = H(x) + O(\rho^{-m\alpha_1}), \) where \( H(x) \) is a linear combination of \( e^{i(\gamma + t + \gamma' x : x)} \) for \( \gamma' \in \Gamma(m \rho^\alpha) \cup \{0\} \). Hence (110) for \( k = m \) and (113), we get

\[
\sum_{\gamma'} | b(N, \gamma + \gamma') |^2 = \sum_{\gamma'} \left| \frac{(O(\rho^{-m\alpha_1}), e^{i(\gamma + t + \gamma' x : x)})}{\Lambda_N - | \gamma + \gamma' + t |^2} \right|^2 = O(\rho^{-2(m+1)\alpha_1}), \tag{111}
\]

where the summation is taken under conditions \( \gamma' \in \Gamma'(m) \), \( \gamma + \gamma' \notin K \). On the other hand, using \( \alpha_1 = 3\alpha \), (108), and the definition of \( n \), we obtain

\[
\sum_{\gamma' \in \mathcal{K} \setminus \{ \gamma \}} | b(N, \gamma') |^2 = O(\rho^{-2n\alpha_1}).
\]

This with (111) implies (109) for \( k = m + 1 \). Thus (110) is also proved. Here \( b(N, \gamma) \) and \( b(N, \gamma + \gamma') \) for \( \gamma' \in \Gamma((n - 1)\rho^\alpha) \) can be calculated as follows. First we express \( b(N, \gamma + \gamma') \) by \( b(N, \gamma) \). For this we apply (18) for \( b(N, \gamma + \gamma') \), where \( \gamma' \in \Gamma((n - 1)\rho^\alpha) \), that is, in (18) replace \( \gamma \) by \( \gamma' \). Iterate it \( n \) times and every time isolate the terms with multiplicand \( b(N, \gamma) \). In other word apply (18) for \( b(N, \gamma + \gamma') \) and isolate the terms with multiplicand \( b(N, \gamma) \). Then apply (18) for \( b(N, \gamma + \gamma' - \gamma) \) when \( \gamma' - \gamma \neq 0 \). Then apply (18) for

\[
b(N, \gamma + \gamma' - \sum_{i=1}^j \gamma_i) \text{ when } \gamma' - \sum_{i=1}^j \gamma_i \neq 0, \text{ etc.}
\]

Apply (18) for \( b(N, \gamma + \gamma' - \sum_{i=1}^j \gamma_i) \) when \( \gamma' - \sum_{i=1}^j \gamma_i \neq 0 \), where \( \gamma_i \in \Gamma(\rho^\alpha) \), \( j = 3, 4, ..., n - 1 \). Then using (4) and the relations

\[
| \Lambda_N - | \gamma + t + \gamma' - \sum_{i=1}^j \gamma_i |^2 | > \frac{3}{2} \rho^\alpha \text{ (see (20) and take into account that } \gamma' - \sum_{i=1}^j \gamma_i \in \Gamma(p\rho^\alpha), \text{ since } p > 2n, \text{ then use (18) and (108), we get }
\]

\[
b(N, \gamma + \gamma') = \sum_{k=1}^{n-1} A_k(\gamma') b(N, \gamma) + O(\rho^{-n\alpha_1}), \tag{112}
\]

where

\[
A_1(\gamma') = \frac{q_{\gamma'} \gamma'}{P(\gamma + t) - | \gamma + t + \gamma' |^2} = \frac{q_{\gamma'}}{| \gamma + t + \gamma' |^2} + O(\rho^{-\alpha_1}),
\]

\[
A_k(\gamma') = \sum_{\gamma_1, ..., \gamma_{k-1}} \frac{q_{\gamma_1} q_{\gamma_2} ... q_{\gamma_{k-1}} q_{\gamma_{k-1}} - q_{\gamma_1 - \gamma_{k-1} - ... - \gamma_{k-1}}}{P(\gamma + t) - | \gamma + t + \gamma' - \sum_{i=1}^{k-1} \gamma_i |^2} = O(\rho^{-k\alpha_1}),
\]

\[
\sum_{\gamma' \in \Gamma'((n - 1)\rho^\alpha)} | A_1(\gamma') |^2 = O(\rho^{-2\alpha_1}), \sum_{\gamma' \in \Gamma'((n - 1)\rho^\alpha)} | A_k(\gamma') | = O(\rho^{-k\alpha_1}), \tag{113}
\]
for \( k > 1 \). Now from (110) for \( k = n \) and (112), we obtain
\[
\Psi_N(x) = b(N, \gamma) e^{i(\gamma t,x)} + \sum_{\gamma^* \in \Gamma((n-1)\rho^\alpha)} A_k(\gamma^*) b(N, \gamma) + O(\rho^{-n\alpha_1}) e^{i(\gamma t + \gamma^*,x)} + H_n(x).
\]
Using the equalities \( \| \Psi_N \| = 1 \), \( \arg b(N, \gamma) = 0 \), \( \| H_n \| = O(\rho^{-n\alpha_1}) \) and taking into account that the functions \( e^{i(\gamma t,x)}, H_n(x), e^{i(\gamma t + \gamma^*,x)} \) are orthogonal, we get
\[
1 = | b(N,\gamma) |^2 + \sum_{k=1}^{n-1} (\sum_{\gamma^* \in \Gamma((n-1)\rho^\alpha)} | A_k(\gamma^*) b(N, \gamma) |^2 + O(\rho^{-n\alpha_1})),
\]
(see the second equality in (113)). Thus from (112), we obtain
\[
b(N, \gamma + \gamma') = (\sum_{k=1}^{n-1} A_k(\gamma')) (1 + \sum_{k=1}^{n-1} \sum_{\gamma^*} | A_k(\gamma^*) |^2)^{-\frac{1}{2}} + O(\rho^{-n\alpha_1}). \tag{115}
\]
Consider the case \( n = 2 \). By (114), (113), (115) we have \( b(N,\gamma) = 1 + O(\rho^{-2\alpha_1}) \),
\[
b(N, \gamma + \gamma') = A_1(\gamma') + O(\rho^{-2\alpha_1}) = \frac{q_{\gamma'}}{|\gamma + t|^2 - |\gamma + \gamma' + t|^2} + O(\rho^{-2\alpha_1})
\]
for all \( \gamma' \in \Gamma(\rho^\alpha) \). These and (110) for \( k = 2 \) imply the formula for \( \Phi_1 \). \( \blacksquare \)

## 5 Simple Sets and Isoenergetic Surfaces

In this section we consider the simple sets \( B \) and construct a big part of the isoenergetic surfaces corresponding to \( \rho^2 \) for big \( \rho \). The isoenergetic surfaces of \( L(q) \) corresponding to \( \rho^2 \) is the set \( I_\rho(q(x)) = \{ t \in F^* : \exists \gamma, \Lambda_N(t) = \rho^2 \} \). In the case \( q(x) = 0 \) the isoenergetic surface \( I_\rho(0) = \{ t \in F^* : \exists \gamma \in \Gamma, |\gamma + t|^2 = \rho^2 \} \) is the translation of the sphere \( B(\rho) = \{ \gamma + t : t \in F^*, \gamma \in \Gamma, |\gamma + t|^2 = \rho^2 \} \) by the vectors \( \gamma \in \Gamma \). We call \( B(\rho) \) the translated isoenergetic surfaces of \( L(0) \) corresponding to \( \rho^2 \). Similarly, we call the sets \( P_\rho = \{ \gamma + t : \Lambda(\gamma + t) = \rho^2 \} \) and
\[
P_{\rho}'' = \{ t \in F^* : \exists \gamma \in \Gamma, \Lambda(\gamma + t) = \rho^2 \},
\]
defined in Remark 5, the parts of translated isoenergetic surfaces and isoenergetic surfaces of \( L(q) \). In this section we construct the subsets \( I_\rho' \) and \( I_\rho'' \) of \( P_\rho \) and \( P_\rho'' \) respectively and prove that the measures of these subsets are asymptotically equal to the measure of the isoenergetic surfaces \( I_\rho(0) \) of \( L(0) \). In other word we construct a big part (in some sense) of isoenergetic surfaces \( I_\rho(q(x)) \) of \( L(q) \).

As we see below the set \( I_\rho'' \) is a translation of \( I_\rho' \) by vectors \( \gamma \in \Gamma \) to \( F^* \) and the set \( I_\rho \) lies in \( \varepsilon \) neighborhood of the surface \( S_\rho = \{ x \in U(2\rho^{\alpha_1}, p) : F(x) = \rho^2 \} \), where \( F(x) \) is defined in Step 1 of introduction. Due to (102) it is natural to call
S_{\rho}$ the approximated isoenergetic surfaces in the non-resonance domain. Here we construct a part of the simple set $B$ in neighborhood of $S_{\rho}$ that contains $I_{\rho}$. For this we consider the surface $S_{\rho}$. As we noted in introduction (see Step 2 and (12)) the non-resonance eigenvalue $\Lambda(\gamma + t)$ does not coincide with other non-resonance eigenvalue $\Lambda(\gamma + t + b)$ if $| F(\gamma + t) - F(\gamma + t + b) | > 2\varepsilon_1$ for $\gamma + t + b \in U(\rho^{\alpha_1}, p)$ and $b \in \Gamma \setminus \{0\}$. Therefore we eliminate

$$P_b = \{ x : x, x + b \in U(\rho^{\alpha_1}, p), | F(x) - F(x + b) | < 3\varepsilon_1 \} \quad (116)$$

for $b \in \Gamma \setminus \{0\}$ from $S_{\rho}$. Denote the remaining part of $S_{\rho}$ by $S'_{\rho}$. Then we consider the $\varepsilon$ neighbourhood $U_\varepsilon(S'_{\rho}) = \bigcup_{a \in S'_{\rho}} U_\varepsilon(a)$ of $S'_{\rho}$,

where $\varepsilon = \frac{\rho}{\rho^{\alpha_1}}$, $U_\varepsilon(a) = \{ x \in \mathbb{R}^d : | x - a | < \varepsilon \}$. In this set the first simplicity condition (12) holds (see Lemma 8(a)). Denote by

$$Tr(E) = \{ \gamma + x \in U_\varepsilon(S'_{\rho}) : \gamma \in \Gamma, x \in E \}$$

and

$$Tr_\rho(E) = \{ \gamma + x \in F^{\ast} : \gamma \in \Gamma, x \in E \}$$

the translations of $E \subset \mathbb{R}^d$ into $U_\varepsilon(S'_{\rho})$ and $F^{\ast}$ respectively. In order that the second simplicity condition (13) holds, we discard from $U_\varepsilon(S'_{\rho})$ the translation $Tr(A(\rho))$ of

$$A(\rho) = \bigcup_{k=1}^{d-1} (\cup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p^{2\alpha})} (\cup_{i=1}^{b_k} A_{k,i}(\gamma_1, \gamma_2, \ldots, \gamma_k))) \quad (117)$$

where $A_{k,i}(\gamma_1, \ldots, \gamma_k) = \{ x \in (\cap_{k=1}^{b_k} V_{\gamma_k}(\rho^{\alpha_2})) \cap K_\rho : \lambda_i(x) \in (\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1) \}$, $\lambda_i(x)$, $b_k$ is defined in Theorem 2 and

$$K_\rho = \{ x \in \mathbb{R}^d : | x |^2 - \rho^2 < \rho^{\alpha_1} \}. \quad (118)$$

As a result we construct the part $U_\varepsilon(S'_{\rho}) \setminus Tr(A(\rho))$ of the simple set $B$ (see Theorem 7(a)) which contains the set $I_{\rho}$ (see Theorem 7(c)). For this we need the following lemma.

Lemma 8 (a) If $x \in U_\varepsilon(S'_{\rho})$ and $x + b \in U(\rho^{\alpha_1}, p)$, where $b \in \Gamma$, then

$$| F(x) - F(x + b) | > 2\varepsilon_1, \quad (119)$$

where $\varepsilon = \frac{\rho}{\rho^{\alpha_2}}\varepsilon_1 = \rho^{-d-2\alpha}$, $F(x) = | x |^2 + F_{k_{1,-1}}(x)$, $k_1 = \left[ \frac{d}{2\alpha} \right] + 2$, hence for $\gamma + t \in U_\varepsilon(S'_{\rho})$ the simplicity condition (12) holds.

(b) If $x \in U_\varepsilon(S'_{\rho})$, then $x + b \notin U_\varepsilon(S'_{\rho})$ for all $b \in \Gamma$.

(c) If $E$ is a bounded subset of $\mathbb{R}^d$, then $\mu(Tr(E)) \leq \mu(E)$.

(d) If $E \subset U_\varepsilon(S'_{\rho})$, then $\mu(Tr_\rho(E)) = \mu(E)$.

Proof. (a) If $x \in U_\varepsilon(S'_{\rho})$, then there exists a point $a$ in $S'_{\rho}$ such that $x \in U_\varepsilon(a)$. Since $S'_{\rho} \cap P_b = \emptyset$ (see (116) and the definition of $S'_{\rho}$), we have

$$| F(a) - F(a + b) | \geq 3\varepsilon_1 \quad (120)$$

On the other hand, using (44) and the obvious relations
Moreover, $U$ denotes smooth surfaces and has the measure $|x| < \rho + 1, |x - a| < \varepsilon, |x + b - a| < \varepsilon$, we obtain

$$|F(x) - F(a)| < 3\rho\varepsilon, |F(x + b) - F(a + b)| < 3\rho\varepsilon$$

These inequalities together with (120) give (119), since $6\rho\varepsilon < \varepsilon_1$.

(b) If $x$ and $x + b$ lie in $U_\varepsilon(S'_\rho)$, then there exist points $a$ and $c$ in $S'_\rho$ such that $x \in U_\varepsilon(a)$ and $x + b \in U_\varepsilon(c)$. Repeating the proof of (121), we get

$$|F(c) - F(x + b)| < 3\rho\varepsilon.$$ This, the first inequality in (121), and the relations $F(a) = \rho^2, F(c) = \rho^2$ (see the definition of $S'_\rho$) give

$$|F(x) - F(x + b)| < \varepsilon_1,$$ which contradicts (119).

(c) Clearly, for any bounded set $E$ there exist only a finite number of vectors $\gamma_1, \gamma_2, ..., \gamma_s$ such that $E(k) \equiv (E + \gamma_k) \cap U_\varepsilon(S'_\rho) \neq \emptyset$ for $k = 1, 2, ..., s$ and $Tr(E)$ is the union of the sets $E(k)$. For $E(k) - \gamma_k$ we have the relations $\mu(E(k) - \gamma_k) = \mu(E(k)), E(k) - \gamma_k \subset E$. Moreover, by (b)

$$E(k) - \gamma_k \cap (E(j) - \gamma_j) = \emptyset$$ for $k \neq j$. Therefore (c) is true.

(d) Now let $E \subset U_\varepsilon(S'_\rho)$. Then by (b) the set $E$ can be divided into a finite number of the pairwise disjoint sets $E_1, E_2, ..., E_n$ such that there exist the vectors $\gamma_1, \gamma_2, ..., \gamma_n$ satisfying $(E_k + \gamma_k) \subset F^*, (E_k + \gamma_k) \cap (E_j + \gamma_j) \neq \emptyset$ for $k, j = 1, 2, ..., n$ and $k \neq j$. Using $\mu(E_k + \gamma_k) = \mu(E_k)$, we get the proof of (d), since $TrF_0(E)$ and $E$ are union of the pairwise disjoint sets $E_k + \gamma_k$ and $E_k$ for $k = 1, 2, ..., n$ respectively.

**Theorem 7** (a) The set $U_\varepsilon(S'_\rho) \setminus Tr(A(\rho))$ is a subset of $B$. For every connected open subset $E$ of $U_\varepsilon(S'_\rho) \setminus Tr(A(\rho))$ there exists a unique index $N$ such that $\Lambda_N(t) = \Lambda(\gamma + t)$ for $\gamma + t \in E$, where $\Lambda(\gamma + t)$ is defined in Remark 5. Moreover,

$$|\partial_{\gamma(j)}(\Lambda(\gamma + t))| = |\partial_{\gamma(j)}(\gamma + t)|^2 + O(\rho^{1-2\alpha_1}), \forall j = 1, 2, ..., d.$$ (122)

(b) For the part $V_\rho \equiv S'_\rho \setminus U_\varepsilon(Tr(A(\rho)))$ of the approximated isoenergetic surface $S'_\rho$, the following holds

$$\mu(V_\rho) > (1 - c_{15}\rho^{-\alpha})\mu(B(\rho)).$$ (123)

Moreover, $U_\varepsilon(V_\rho)$ lies in the subset $U_\varepsilon(S'_\rho) \setminus Tr(A(\rho))$ of the simple set $B$.

(c) The isoenergetic surface $I(\rho)$ contains the set $I''_\rho$, which consists of the smooth surfaces and has the measure

$$\mu(I''_\rho) = \mu(I'_\rho) > (1 - c_{16}\rho^{-\alpha})\mu(B(\rho)),$$ (124)

where $I'_\rho$ is a part of the translated isoenergetic surfaces of $L(q)$, which is contained in the subset $U_\varepsilon(S'_\rho) \setminus Tr(A(\rho))$ of the simple set $B$. In particular the number $\rho^2$ for $\rho \gg 1$ lies in the spectrum of $L(q)$, that is, the number of the gaps in the spectrum of $L(q)$ is finite, where $q(x) \in W_2^m(\mathbb{R}^d/\Omega), d \geq 2, s_0 = \frac{3d-1}{2}(3^d + d + 2) + \frac{1}{4}d^2 + d + 6, and \Omega$ is an arbitrary lattice.
Proof. (a) To prove that \( U_\varepsilon(S'_\rho) \backslash \text{Tr}(A(\rho)) \subset B \) we need to show that for each point \( \gamma + t \) of \( U_\varepsilon(S'_\rho) \backslash \text{Tr}(A(\rho)) \) the simplicity conditions (12), (13) hold and \( U_\varepsilon(S'_\rho) \backslash \text{Tr}(A(\rho)) \subset U(\rho^{a_1}, p) \). By Lemma 8(a), the condition (12) holds. Now we prove that (13) holds too. Since \( \gamma + t \in U_\varepsilon(S'_\rho) \), there exists \( a \in S'_\rho \) such that \( \gamma + t \in U_\varepsilon(a) \). The inequality (121) and equality \( F(a) = \rho^2 \) imply
\[
F(\gamma + t) \in (\rho^2 - \varepsilon_1, \rho^2 + \varepsilon_1)
\] for \( \gamma + t \in U_\varepsilon(S'_\rho) \). On the other hand, \( \gamma + t \notin \text{Tr}(A(\rho)) \). It means that for any \( \gamma' \in \Gamma \), we have \( \gamma' + t \notin A(\rho) \). Therefore for each \( \gamma' \in \Gamma \), we have \( \lambda(\gamma' + t) \notin (\rho^2 - 3\varepsilon_1) \) for \( \gamma' \in K \) and \( \gamma' + t \in E_k \setminus E_{k+1} \). Thus (13) follows from (125). Moreover, it is clear that the inclusion \( S'_\rho \subset U(2\rho^{a_1}, p) \) (see definition of \( S'_\rho \) and \( S'_\rho \)) implies that \( U_\varepsilon(S'_\rho) \subset U(\rho^{a_1}, p) \). Thus \( U_\varepsilon(S'_\rho) \backslash \text{Tr}(A(\rho)) \subset B \).

Now let \( E \) be a connected open subset of \( U_\varepsilon(S'_\rho) \backslash \text{Tr}(A(\rho)) \subset B \). By Theorem 5 and Remark 5 for \( a \in E \subset U_\varepsilon(S'_\rho) \backslash \text{Tr}(A(\rho)) \) there exists a unique index \( N(a) \) such that \( \Lambda(a) = \Lambda_{N(a)}(a) \), \( \Psi_a(x) = \Psi_{N(a),a}(x) \), \( |(\Psi_{N(a),a}(x), e^{ii(x)})|^2 > \frac{1}{2} \) and \( \Lambda(a) \) is a simple eigenvalue. On the other hand, for fixed \( N \) the functions \( \Lambda_N(t) \) and \( \Psi_{N,t}(x), e^{ii(x)} \) are continuous in a neighborhood of \( a \) if \( \Lambda_N(a) \) is a simple eigenvalue. Therefore for each \( a \in E \) there exists a neighborhood \( U(a) \subset E \) of \( a \) such that \( |(\Psi_{N,a}(y,x), e^{ii(x)})|^2 > \frac{1}{2} \), for \( y \in U(a) \). Since for \( y \in E \) there is a unique integer \( N(y) \) satisfying \( |(\Psi_{N,y}(y,x), e^{ii(x)})|^2 > \frac{1}{2} \), we have \( N(y) = N(a) \) for \( y \in U(a) \). Hence we proved that
\[
\forall a \in E, \exists U(a) \subset E : N(y) = N(a), \forall y \in U(a).
\]

Now let \( a_1 \) and \( a_2 \) be two points of \( E \), and let \( C \subset E \) be the arc that joins these points. Let \( U(y_1), U(y_2), \ldots, U(y_k) \) be a finite subcover of the open cover \( \cup_{a \in C} U(a) \) of the compact \( C \), where \( U(a) \) is the neighborhood of \( a \) satisfying (126). By (126), we have \( N(y) = N(y_i) = N_i \) for \( y \in U(y_i) \). Clearly, if \( U(y_i) \cap U(y_j) \neq \emptyset \), then \( N_i = N_j \), for \( z \in U(y_i) \cap U(y_j) \). Thus \( N_1 = N_2 = \ldots = N_k \) and \( N(a_1) = N(a_2) \).

To calculate the partial derivatives of the function \( \Lambda(\gamma + t) = \Lambda_N(t) \) we write the operator \( L_t \) in the form \(-\Delta - (2it, \nabla) + (t, t)\). Then, it is clear that
\[
\frac{\partial}{\partial t_j} \Lambda_N(t) = 2t_j(\Phi_{N,t}(x), \Phi_{N,t}(x)) - 2i(\frac{\partial}{\partial x_j} \Phi_{N,t}(x), \Phi_{N,t}(x)),
\]
\[
\Phi_{N,t}(x) = \sum_{\gamma' \in \Gamma} b(N, \gamma') e^{i(\gamma', x)},
\]
where \( \Phi_{N,t}(x) = e^{-it(x)} \Psi_{N,t}(x) \). If \( |\gamma'| \geq 2\rho \), then using
\[
\Lambda_N \equiv \Lambda(\gamma + t) = \rho^2 + O(\rho^{-1}), \quad \text{(see (102), (125))},
\]
and the obvious inequality
of $S \in \text{set} V$ (see below, Estimations 1, 2, 3). The estimation (123) of the measure of $U$ if

$$\sum_{i} \mu_{i} = \mu(\Lambda_{N}) \geq c_{17} |\gamma' - c_{17}| < \frac{1}{4} |\gamma'|$$

and, iterating (18) $p$ times by using the decomposition $q(x) = \sum_{|\gamma'| < \frac{1}{4} |\gamma'|} q_{\gamma} e^{i(\gamma_{1} \cdot x)} + O(|\gamma'|^{-p})$, we get

$$b(N, \gamma') = \sum_{\gamma_{1}, \gamma_{2}, \ldots} q_{\gamma_{1}} q_{\gamma_{2}} \cdots q_{\gamma_{p}} b(N, \gamma' - \sum_{i=1}^{p} \gamma_{i}) + O(|\gamma'|^{-p}),$$

(129)

$$b(N, \gamma') = O(|\gamma'|^{-p}), \quad \forall |\gamma'| \geq 2\rho$$

(130)

By (130) the series in (128) can be differentiated term by term. Hence

$$-i \frac{\partial}{\partial x_{j}} f_{N,t}(\Phi_{N,t}) = \sum_{\gamma' \in \Gamma} \gamma' \mu \frac{\partial b}{\partial x_{j}}(N, \gamma') \mu(N, \gamma') b(N, \gamma')^{2} + \Sigma_{1} + \Sigma_{2},$$

(131)

where $\Sigma_{1} = \sum_{|\gamma'| \geq 2\rho} \gamma' \mu \frac{\partial b}{\partial x_{j}}(N, \gamma')^{2}, \Sigma_{2} = \sum_{|\gamma'| < 2\rho} \gamma' \mu \frac{\partial b}{\partial x_{j}}(N, \gamma')^{2}$.

By (14), $\Sigma_{2} = O(\rho^{-2\alpha+1})$, and $\gamma' \mu \frac{\partial b}{\partial x_{j}}(N, \gamma')^{2} = \gamma'(1 + O(\rho^{-2\alpha}))$, and by (130), $\Sigma_{1} = O(\rho^{-2\alpha})$. Therefore (127) and (131) imply (122).

To prove (123) first we estimate the measure of $S_{\rho}, S'_{\rho}, U_{2\rho}(A(\rho))$, namely we prove

$$\mu(S_{\rho}) > (1 - c_{18}\rho^{-\alpha})\mu(B(\rho)),$$

(132)

$$\mu(S'_{\rho}) > (1 - c_{19}\rho^{-\alpha})\mu(B(\rho)),$$

(133)

$$\mu(U_{2\rho}(A(\rho))) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon$$

(134)

(see below, Estimations 1, 2, 3). The estimation (123) of the measure of the set $V_{\rho}$ is done in Estimation 4 by using Estimations 1, 2, 3.

(c) In Estimation 5 we prove the formula (124). The Theorem is proved.

In Estimations 1-5 we use the notations: $G(+i, a) = \{x \in G, x_{1} > a\}$, $G(-i, a) = \{x \in G, x_{1} < -a\}$, where $x = (x_{1}, x_{2}, \ldots, x_{d}), a > 0$. It is not hard to verify that for any subset $G$ of $U_{\varepsilon}(S'_{\rho}) \cup U_{2\rho}(A(\rho))$, that is, for all considered sets $G$ in these estimations, and for any $x \in G$ the followings hold

$$\rho - 1 < |x| < \rho + 1, \quad G \in \cup_{i=1}^{d} (G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1}))$$

(135)

Indeed, if $x \in S_{\rho}$, then $F(x) = \rho^{2}$ and by definition of $F(x)$ (see Lemma 8) and (24) we have $|x| = \rho + O(\rho^{1-\alpha})$. Hence the inequalities in (135) hold for $x \in U_{\varepsilon}(S'_{\rho})$. If $x \in A(\rho)$, then by definition of $A(\rho)$ (see (117), (118)), we have $x \in K_{\rho}$, and hence $|x| = \rho + O(\rho^{1+\alpha})$. Thus the inequalities in (135) hold for $x \in U_{2\rho}(A(\rho))$ too. The inclusion in (135) follows from these inequalities.

If $G \subseteq S_{\rho}$, then by (44) we have $\frac{\partial F(x)}{\partial x_{k}} > 0$ for $x \in G(+k, \rho^{-\alpha})$. Therefore to calculate the measure of $G(+k, a)$ for $a \geq \rho^{-\alpha}$ we use the formula

$$\mu(G(+k, a)) = \int_{Pr_{k}(G(+k, a))} \left(\frac{\partial F}{\partial x_{k}}\right)^{-1} |\text{grad}(F)| \, dx_{1} \cdots dx_{k-1} dx_{k+1} \cdots dx_{d},$$

(136)
where \( \Pr_k(G) \equiv \{(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, x_{k+2}, \ldots, x_d) : x \in G \} \) is the projection of \( G \) on the hyperplane \( x_k = 0 \). Instead of \( \Pr_k(G) \) we write \( \Pr(G) \) if \( k \) is unambiguous. If \( D \) is \( m \)-dimensional subset of \( \mathbb{R}^m \), then to estimate \( \mu(D) \), we use the formula

\[
\mu(D) = \int_{\Pr(D)} \mu(D(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)) dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_m, \tag{137}
\]

where \( D(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) = \{x_k : (x_1, x_2, \ldots, x_m) \in D \} \).

ESTIMATION 1. Here we prove (132) by using (136). During this estimation the set \( S_\rho \) is redenoted by \( G \). First we estimate \( \mu(G(+1, a)) \) for \( a = \rho^{1-\alpha} \) by using (136) for \( k = 1 \) and the relations

\[
\frac{\partial F}{\partial x_1} > \rho^{1-\alpha}, \quad (\frac{\partial F}{\partial x_1})^{-1} \mid \grad(F) \mid = \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - \ldots - x_d^2}} + O(\rho^{-\alpha}), \tag{138}
\]

\[
\Pr(G(+1, a)) \supset \Pr(A(+1, 2a)), \tag{139}
\]

where \( x \in G(+1, a) \), \( A = B(\rho) \cap U(3\rho^{\alpha}, p) \). Here (138) follows from (44), (24), and the definitions of \( F(x) \), \( S_\rho \). Now we prove (139). If

\[
(x_2, \ldots, x_d) \in \Pr_1(A(+1, 2a)) \), then by definition of \( A(+1, 2a) \) there exists \( x_1 \) such that

\[
x_1 > 2a = 2\rho^{1-\alpha}, \quad x_1^2 + x_2^2 + \ldots + x_d^2 = \rho^2, \quad \mid \sum_{i \geq 1} (2x_ib_i - b_i^2) \mid \geq 3\rho^{\alpha}, \tag{140}
\]

for all \((b_1, b_2, \ldots, b_d) \in \Gamma(\rho \rho^{\alpha})\). Therefore for \( h = \rho^{-\alpha} \) we have

\[
(x_1 + h)^2 + x_2^2 + \ldots + x_d^2 > \rho^2 + \rho^{-\alpha}, \quad (x_1 - h)^2 + x_2^2 + \ldots + x_d^2 < \rho^2 - \rho^{-\alpha}.
\]

This and (24) give \( F(x_1 + h, x_2, \ldots, x_d) > \rho^2 \), \( F(x_1 - h, x_2, \ldots, x_d) < \rho^2 \). Since \( F \) is a continuous function there is \( y_1 \in (x_1 - h, x_1 + h) \) such that (see (140))

\[
y_1 > a, \quad F(y_1, x_2, \ldots, x_d) = \rho^2, \quad \mid 2y_1b_1 - b_1^2 + \sum_{i \geq 2} (2x_i b_i - b_i^2) \mid > \rho^{\alpha}, \tag{141}
\]

since the expression under the absolute value in (141) differs from the expression under the absolute value in (140) by \( 2(y_1 - x_1) b_1 \), where \( \mid y_1 - x_1 \mid < h = \rho^{-\alpha} \), \( \mid b_1 \mid < \rho^{\alpha} \), \( \mid 2(y_1 - x_1) b_1 \mid < 2\rho \rho^{\alpha} < \rho^{\alpha} \). The relations in (141) mean that \((x_2, \ldots, x_d) \in \Pr G(+1, a) \). Hence (139) is proved. Now (136), (138), and the obvious relation \( \mu(\Pr G(+1, a)) = O(\rho^{d-1}) \) (see (135)) imply that

\[
\mu(G(+1, a)) = \int_{\Pr(G(+1, a))} \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - \ldots - x_d^2}} dx_2 dx_3 \ldots dx_d + O(\frac{1}{\rho^{\alpha}}) \mu(B(\rho))
\]

\[
\geq \int_{\Pr(A(+1, 2a))} \frac{\rho}{\sqrt{\rho^2 - x_2^2 - x_3^2 - \ldots - x_d^2}} dx_2 dx_3 \ldots dx_d - c_{20} \rho^{-\alpha} \mu(B(\rho)) = \mu(A(+1, 2a)) - c_{20} \rho^{-\alpha} \mu(B(\rho)).
\]
Similarly, \( \mu(G(-1, a)) \geq \mu(A(-1, 2a)) - c_{20} \rho^{-\alpha} \mu(B(\rho)) \). Now using the inequality \( \mu(G) \geq \mu(G(1, a)) + \mu(G(-1, a)) \) we get
\[
\mu(G) \geq \mu(A(-1, 2a)) + \mu(A(1, 2a)) - 2c_{20} \rho^{-\alpha} \mu(B(\rho)).
\]
On the other hand it follows from the obvious relation
\[
\mu(\{x \in B(\rho) : -2a \leq x_1 \leq 2a\}) = O(\rho^{-\alpha}) \mu(B(\rho))
\]
that
\[
\mu(A(-1, 2a)) + \mu(A(1, 2a)) \geq \mu(A) - 3c_{20} \rho^{-\alpha} \mu(B(\rho)).
\]
Therefore (132), since
\[
\mu(A) = (1 + O(\rho^{-\alpha})) \mu(B(\rho))
\]
we obtain (133). For this we estimate the measure of the set \( S_\rho \cap P_k \) by using (136). During this estimation the set \( S_\rho \cap P_k \) is redenoted by \( G \). We choose the coordinate axis so that the direction of \( b \) coincides with the direction of \((1, 0, 0, ..., 0)\), i.e., \( b = (b_1, 0, 0, ..., 0) \) and \( b_1 > 0 \). It follows from the definitions of \( S_\rho, P_k \) and \( F(x) \) (see the beginning of this section, (116), and Lemma 8(a)) that if \((x_1, x_2, ..., x_d) \in G\), then
\[
\begin{align*}
x_1^2 + x_2^2 + ... + x_d^2 + F_{k_1-1}(x) &= \rho^2, \\
(x_1 + b_1)^2 + x_2^2 + x_3^2 + ... + x_d^2 + F_{k_1-1}(x + b) &= \rho^2 + h,
\end{align*}
\]
where \( h \in (-3\varepsilon_1, 3\varepsilon_1) \). Subtracting (142) from (143) and using (24), we get
\[
(2x_1 + b_1)b_1 = O(\rho^{-\alpha_1}).
\]
This and the inequalities in (135) imply
\[
|b_1| < 2\rho + 3, \quad x_1 = \frac{b_1}{2} + O(\rho^{-\alpha_1}b_1^{-1}), \quad |x_1| - (\frac{b_1}{2})^2 = O(\rho^{-\alpha_1}).
\]  
Consider two cases. Case 1: \( b \in \Gamma_1 \), where \( \Gamma_1 = \{ b \in \Gamma : |\rho^2 - \frac{b}{\rho}^2| < 3d\rho^{-2\alpha} \} \).

In this case using the last equality in (145), (142), (24), and taking into account that \( b = (b_1, 0, 0, ..., 0) \), \( \alpha_1 = 3\alpha \), we obtain
\[
x_1^2 = \rho^2 + O(\rho^{-2\alpha}), \quad |x_1| = \rho + O(\rho^{-2\alpha-1}), \quad x_2^2 + x_3^2 + ... + x_d^2 = O(\rho^{-2\alpha}).
\]
Therefore \( G \subset G(1, a) \cup G(-1, a) \), where \( a = \rho - \rho^{-1} \). Using (136), the obvious relation \( \mu(Pr_{\Gamma_1}(G(\pm 1, a)) = O(\rho^{-(d-1)\alpha}) \) (see (146)) and taking into account that the expression under the integral in (136) for \( k = 1 \) is equal to \( 1 + O(\rho^{-\alpha}) \) (see (146)), we get
\[
\mu(G(\pm 1, a)) = O(\rho^{-(d-1)\alpha}).
\]
Thus \( \mu(G) = O(\rho^{-(d-1)\alpha}) \). Since \( |\Gamma_1| = O(\rho^{d-1}) \), we have
\[
\mu(\cup_{b \in \Gamma_1}(S_\rho \cap P_k)) = O(\rho^{-(d-1)\alpha+d-1}) = O(\rho^{-\alpha}) \mu(B(\rho)).
\]  
Case 2: \( |\rho^2 - \frac{b}{\rho}^2| \geq 3d\rho^{-2\alpha} \). Repeating the proof of (146), we get
\[
|\rho^2 - \rho^{-\alpha}| > 2d\rho^{-2\alpha}, \quad \sum_{k=2}^{d} x_k^2 > d\rho^{-2\alpha}, \quad \max_{k \geq 2} |x_k| > \rho^{-\alpha}.
\]
Therefore \( G \subset \cup_{k \geq 2}(G(\pm k, \rho^{-\alpha}) \cup G(-k, \rho^{-\alpha})) \). Now we estimate \( \mu(G(\pm d, \rho^{-\alpha})) \) by using (136). Redenote by \( D \) the set \( Pr_{\Gamma_1}G(\pm d, \rho^{-\alpha}) \). If \( x \in G(\pm d, \rho^{-\alpha}) \), then
Then by (32), we have
\[ G, \text{ give the proof of } (133). \]
According to (142) and (44) the under integral expression in (136) for \( k = d \) is \( O(\rho^{1+\alpha}) \). Therefore the first equality in
\[ \mu(D) = O(\varepsilon_1 | b |^{-1}) \rho^{d-2}, \quad \mu(G(+d, \rho^{-\alpha})) = O(\rho^{d-1+\alpha} \varepsilon_1 | b |^{-1}) \] (149)
implies the second equality in (149). To prove the first equality in (149) we use (137) for \( m = d - 1 \) and \( k = 1 \) and prove the relations \( \mu(\text{Pr}_1 D) = O(\rho^{d-2}), \)
\[ \mu(D(x_2, x_3, \ldots, x_{d-1})) < 6 \varepsilon_1 | b |^{-1} \] (150)
for \((x_2, x_3, \ldots, x_{d-1}) \in \text{Pr}_1 D\). First relation follows from the inequalities in (135). So we need to prove (150). If \( x_1 \in D(x_2, x_3, \ldots, x_{d-1}) \) then (142) and (143) hold. Subtracting (142) from (143), we get
\[ 2x_1b_1 + (b_1)^2 + F_{k-1}(x - b) - F_{k-1}(x) = h, \] (151)
where \( x_2, x_3, \ldots, x_{d-1} \) are fixed. Hence we have two equations (142) and (151) with respect two unknown \( x_1 \) and \( x_d \). Using (44), the implicit function theorem, and the inequalities \( |x_d| > \rho^{-\alpha}, \alpha_1 > 2\alpha \) from (142), we obtain
\[ x_d = f(x_1), \quad \frac{df}{dx_1} = -\frac{2x_1 + O(\rho^{-2\alpha_1+\alpha})}{2x_d + O(\rho^{-2\alpha_1+\alpha})}. \] (152)
Substituting this in (151), we get
\[ 2x_1b_1 + b_1^2 + F_{k-1}(x_1+b_1, x_2,\ldots, x_{d-1}, f(x_1)) - F_{k-1}(x_1,\ldots, x_{d-1}, f) = h. \] (153)
Using (44), (152), the first equality in (145), and \( x_d > \rho^{-\alpha} \) we see that the absolute value of the derivative (w.r.t. \( x_1 \)) of the left-hand side of (153) satisfies the inequality
\[ |2b_1 + O(\rho^{-2\alpha_1+\alpha}) + O(\rho^{-2\alpha_1+\alpha}) x_1 + O(\rho^{-2\alpha_1+\alpha})| > b_1 \]
for \( x_1 = \frac{b_1}{2} + O(\rho^{-\alpha_1}) \) (see (145)). Therefore from (153) by implicit function theorem, we get \( |\frac{df}{dx_1}| < \frac{1}{|b_1|} \). This inequality and relation \( h \in (-3 \varepsilon_1, 3 \varepsilon_1) \) imply (150). Thus (149) is proved. In the same way we get the same estimation for \( G(+\alpha, \rho^{-\alpha}) \) and \( G(-\alpha, \rho^{-\alpha}) \) for \( k \geq 2 \). Hence
\[ \mu(S_{\rho} \cap P_{\varepsilon_1}) = O(\rho^{d-1+\alpha} \varepsilon_1 | b |^{-1}), \text{ for } b \notin \Gamma_1. \] Since \( |b| < 2\rho + 3 \) (see (145)) and \( \varepsilon_1 = \rho^{-d-2\alpha} \), taking into account that the number of the vectors of \( \Gamma \) satisfying \( |b| < 2\rho + 3 \) is \( O(\rho^d) \), we obtain
\[ \mu(S_{\rho} \cap P_{\varepsilon_1}) = O(\rho^{d-1+\alpha} \varepsilon_1) = O(\rho^{-\alpha}) \mu(\text{Pr}(\rho)). \] This, (147) and (132) give the proof of (133).

ESTIMATION 3. Here we prove (134). Denote \( U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, \ldots, \gamma_k)) \) by \( G \), where \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(\rho^\alpha), k \leq d - 1 \), and \( A_{k,j} \) is defined in (117). We turn the coordinate axis so that
\[ \text{Span}\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = \{ x = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) : x_1, x_2, \ldots, x_k \in \mathbb{R} \}. \]
Then by (32), we have \( x_n = O(\rho^{\alpha_k + (k-1)\alpha}) \) for \( n \leq k, x \in G \). This, (135), and \( \alpha_k + (k-1)\alpha < 1 \) (see the sixth inequality in (15)) give
\[ G \subset (\bigcup_{i \geq k} G(+i, \rho d^{-1}) \cup G(-i, \rho d^{-1})), \]
\[ \mu(Pr_1(G(\pm i, \rho d^{-1}))) = O(\rho^{k(\alpha_0 + (k-1)\alpha) + (d-1-k)}) \] for \( i > k \). Now using this and (137) for \( m = d \), we prove that
\[ \mu(G(\pm i, \rho d^{-1})) = O(\varepsilon \rho^{k(\alpha_0 + (k-1)\alpha) + (d-1-k)}), \forall i > k. \] (154)

For this we redenote by \( D \) the set \( G(\pm i, \rho d^{-1}) \) and prove that
\[ \mu(D(x_1, x_2, \ldots, x_{i-1}, x_i + 1, \ldots, x_d)) \leq (42d^2 + 4)\varepsilon \] (155)
for \((x_1, x_2, \ldots, x_{i-1}, x_i + 1, \ldots, x_d) \in Pr_1(D) \) and \( i > k \). To prove (155) it is sufficient to show that if both \( x = (x_1, x_2, \ldots, x_i, \ldots, x_d) \) and \( x' = (x_1, x_2, \ldots, x_i', \ldots, x_d) \) are in \( D \), then \(| x_i - x_i' | \leq (42d^2 + 4)\varepsilon \). Assume the converse. Then
\[ | x_i - x_i' | > (42d^2 + 4)\varepsilon. \]
Without loss of generality it can be assumed that \( x_i > x'_i \). So \( x_i > x'_i > \rho d^{-1} \) (see definition of \( D \)). Since \( x \) and \( x' \) lie in the \( 2\varepsilon \) neighborhood of \( A_{k,j} \), there exist points \( a \) and \( a' \) in \( A_{k,j} \) such that \(| x - a | < 2\varepsilon \) and \(| x' - a' | < 2\varepsilon \). It follows from the definitions of the points \( x, x', a, a' \) that the following inequalities hold:
\[ \rho d^{-1} - 2\varepsilon < a_i < a'_i, \quad a'_i - a_i > 42d^2\varepsilon, \]
\[ (a'_i)^2 - (a_i)^2 > 2(\rho d^{-1} - 2\varepsilon)(a'_i - a_i), \]
\[ ||a_s|| - ||a'_s|| < 4\varepsilon, \forall s \neq i. \] (156)

On the other hand the inequalities in (135) hold for the points of \( A_{k,j} \), that is, we have \(| a_s | < \rho + 1, \quad | a'_s | < \rho + 1 \). Therefore these inequalities and the inequalities in (156) imply \(| a_s |^2 - | a'_s |^2 | < 12\rho \varepsilon \) for \( s \neq i \), and hence
\[ \sum_{s \neq i} | a_s |^2 - | a'_s |^2 | < 12d\rho \varepsilon < \frac{3}{2}\rho d^{-1}(a'_i - a_i), \]
\[ || a - | a' |^2 || > \frac{3}{2}\rho d^{-1} | a'_i - a_i |. \] (157)

Now using the inequality (46), the obvious relation \( \frac{1}{3}\alpha_0 \rho < 1 \) (see the end of the introduction), the notations \( r_j(x) = \lambda_j(x) - | x |^2 \) (see Remark 2), \( \varepsilon_1 = 7\rho \varepsilon \) (see Lemma 8(a)), and (157), (156), we get
\[ | r_j(a) - r_j(a') | < \frac{1}{3}\rho d^{-1} | a - a' |, \]
\[ | \lambda_j(a) - \lambda_j(a') | \geq || a |^2 - | a' |^2 || = | r_j(a) - r_j(a') | > \rho d^{-1} | a'_i - a_i | > 42d\rho \varepsilon > 6\varepsilon_1. \]
The obtained inequality \( | \lambda_j(a) - \lambda_j(a') | > 6\varepsilon_1 \) contradicts with the inclusions \( a \in A_{k,j}, \quad a' \in A_{k,j} \), since by definition of \( A_{k,j} \) (see (117)) both \( \lambda_j(a) \) and \( \lambda_j(a') \) lie in \( (\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1) \). Thus (155), hence (154) is proved. In the same way we get the same formula for \( G(-i, \frac{7}{\rho}) \). So
\[ \mu(U_2(A_{k,j}(\gamma_1, \gamma_2, \ldots, \gamma_k))) = O(\varepsilon \rho^{k(\alpha_0 + (k-1)\alpha) + (d-1-k)}) \]
Now taking into account that \( U_2(A_{k,j}(\gamma_1, \gamma_2, \ldots, \gamma_k)) \) for \( k = 1, 2, \ldots, d-1 \; j = 1, 2, \ldots, b_k(\gamma_1, \gamma_2, \ldots, \gamma_k) \), and \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(\rho \alpha^a) \) (see (117)) and using that \( b_k = O(\rho^{d\alpha + \frac{1}{2}\alpha_k + 1}) \) (see (40)) and the number of the vectors \( (\gamma_1, \gamma_2, \ldots, \gamma_k) \) for \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(\rho \alpha^a) \) is \( O(\rho^{d\alpha \alpha_k}) \), we obtain
\[ \mu(U_2(A(\rho))) = O(\varepsilon \rho^{d\alpha + \frac{1}{2}\alpha_k + 1 + d\alpha + k(\alpha_k + (k-1)\alpha) + (d-1-k)}). \]
Therefore to prove (134), it remains to show that
\[ d\alpha + \frac{k}{2}\alpha_{k+1} + dK\alpha + k(\alpha_k + (k-1)\alpha) + d - 1 - k \leq d - 1 - \alpha \]
\[ (d + 1)\alpha + \frac{k}{2}\alpha_{k+1} + dK\alpha + k(\alpha_k + (k-1)\alpha) \leq k \]  
(158)

for \(1 \leq k \leq d - 1\). Dividing both side of (158) by \(k\alpha\) and using \(\alpha_k = 3^d\alpha\), \(\alpha = \frac{1}{d}\), 
\(q = 3^d + d + 2\) (see the end of the introduction) we see that (158) is equivalent to
\[ \frac{d+1}{2} + \frac{3^d}{2} + 3^k + k - 1 \leq 3^d + 2. \]
The left-hand side of this inequality gets its maximum at \(k = d - 1\). Therefore we need to show that
\[ \frac{d+1}{2} + \frac{3^d}{2} + d \leq 3^d + 4, \]
which follows from the inequalities \(\frac{d+1}{2} \leq 3, d < \frac{1}{3}3^d + 1\) for \(d \geq 2\).

**ESTIMATION 4.** Here we prove (123). During this estimation we denote by \(G\) the set \(S_\rho' \cap U_\varepsilon(Tr(A(\rho)))\). Since \(V_\rho = S_\rho' \setminus G\) and (133) holds, it is enough to prove that \(\mu(G) = O(\rho^{-\alpha})\mu(B(\rho))\). For this we use (135) and prove \(\mu(G(+i, pd^{-1})) = O(\rho^{-\alpha})\mu(B(\rho))\) for \(i = 1, 2, ..., d\) by using (136) (the same estimation for \(G(-i, pd^{-1})\) can be proved in the same way). By (44), if \(x \in G(+i, pd^{-1})\), then the under integral expression in (136) for \(k = i\) and \(a = pd^{-1}\) is less than \(d + 1\). Therefore it is sufficient to prove
\[ \mu(Pr(G(+i, pd^{-1}))) = O(\rho^{-\alpha})\mu(B(\rho)) \]  
(159)

Clearly, if \((x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_d) \in Pr(G(+i, pd^{-1}))\), then
\[ \mu(U_{\varepsilon}(G)(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_d)) \geq 2\varepsilon \]  
and by (137), it follows that
\[ \mu(U_{\varepsilon}(G)) \geq 2\varepsilon \mu(Pr(G(+i, pd^{-1}))). \]  
(160)

Hence to prove (159) we need to estimate \(\mu(U_{\varepsilon}(G))\). For this we prove that
\[ U_{\varepsilon}(G) \subset U_{\varepsilon}(S_\rho'), U_{\varepsilon}(G) \subset U_{2\varepsilon}(Tr(A(\rho))), U_{\varepsilon}(G) \subset Tr(U_{2\varepsilon}(A(\rho))). \]  
(161)

The first and second inclusions follow from \(G \subset S_\rho'\) and \(G \subset U_{\varepsilon}(Tr(A(\rho)))\) respectively (see definition of \(G\)). Now we prove the third inclusion in (161). If \(x \in U_{\varepsilon}(G)\), then by the second inclusion of (161) there exists \(b\) such that \(b \in Tr(A(\rho)), x - b < 2\varepsilon\). Then by the definition of \(Tr(A(\rho))\) there exist \(\gamma \in \Gamma\) and \(c \in A(\rho)\) such that \(b = \gamma + c\). Therefore \(x - \gamma - c = x - b < 2\varepsilon\).

\[ x - \gamma \in U_{2\varepsilon}(c) \subset U_{\varepsilon}(A(\rho)). \]
This together with \(x \in U_{\varepsilon}(G) \subset U_{\varepsilon}(S_\rho')\) (see the first inclusion of (161)) give \(x \in Tr(U_{2\varepsilon}(A(\rho)))\) (see the definition of \(Tr(E)\) in the beginning of this section), i.e., the third inclusion in (161) is proved. The third inclusion, Lemma 8(c), and (134) imply that
\[ \mu(U_{\varepsilon}(G)) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon. \]  
This and (160) imply the proof of (159).

**ESTIMATION 5** Here we prove (124). Divide the set \(V_\rho = V\) into pairwise disjoint subsets \(V'(\pm 1, pd^{-1}) \equiv V(\pm 1, pd^{-1})\), 
\[ V'(\pm 1, pd^{-1}) \equiv V(\pm 1, pd^{-1}) \setminus (U_{j-1}^i(\pm j, pd^{-1})), \]  
for \(i = 2, 3, ..., d\). Take any point \(a \in V'(\pm i, pd^{-1}) \subset S_\rho\) and consider the function \(F(x)\) (see Lemma 8(a)) on the interval \([a - \varepsilon], a + \varepsilon]\), where \(e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ...\)
By the definition of $S_\rho$ we have $F(\alpha) = \rho^2$. It follows from (44) and the definition of $V'(\pm i, \rho d^{-1})$ that $\frac{\partial F(x)}{\partial x_1} > \rho d^{-1}$ for $x \in [a - \varepsilon e_i, a + \varepsilon e_i]$. Therefore

$$F(a - \varepsilon e_i) < \rho^2 - c_{21} \varepsilon_1, \quad F(a + \varepsilon e_i) > \rho^2 + c_{21} \varepsilon_1.$$  

(162)

Since $[a - \varepsilon e_i, a + \varepsilon e_i] \subset U\varepsilon(a) \subset U\varepsilon(V_\rho) \subset U\varepsilon(S_\rho') \setminus \text{Tr}(A(\rho))$ (see Theorem 7(b)), it follows from Theorem 7(a) that there exists index $N$ such that $\Lambda(y) = \Lambda_N(y)$ for $y \in U\varepsilon(a)$ and $\Lambda(y)$ satisfies (102) (see Remark 5). Hence (162) implies that

$$\Lambda(a - \varepsilon e_i) < \rho^2, \quad \Lambda(a + \varepsilon e_i) > \rho^2.$$  

(163)

Moreover it follows from (122) that the derivative of $\Lambda(y)$ with respect to $i$th coordinate is positive for $y \in [a - \varepsilon e_i, a + \varepsilon e_i]$. So $\Lambda(y)$ is a continuous and increasing function in $[a - \varepsilon e_i, a + \varepsilon e_i]$. Therefore (163) implies that there exists a unique point $y(a, i) \in [a - \varepsilon e_i, a + \varepsilon e_i]$ such that $\Lambda(y(a, i)) = \rho^2$. Define $I'_\rho(+i)$ by $I'_\rho(+i) = \{ y(a, i) : a \in V'(\pm i, \rho d^{-1}) \}$. In the same way we define $I'_\rho(-i) = \{ y(a, i) : a \in V'(\pm i, \rho d^{-1}) \}$ and put $I'_\rho = \bigcup_{i=1}^d (I'_\rho(+i) \cup I'_\rho(-i))$. To estimate the measure of $I'_\rho$ we compare the measure of $V'(\pm i, \rho d^{-1})$ with the measure of $I'_\rho(\pm i)$ by using the formula (136) and the obvious relations

$$\Pr(V'(\pm i, \rho d^{-1})) = \Pr(I'_\rho(\pm i)), \quad \mu(\Pr(I'_\rho(\pm i))) = O(\rho^{d-1}),$$  

(164)

$$\frac{\partial F}{\partial x_1} \mid_{\text{grad}(F)} - \frac{\partial \Lambda}{\partial x_1} \mid_{\text{grad}(\Lambda)} = O(\rho^{-2\alpha_1}).$$  

(165)

Here the first equality in (164) follows from the definition of $I'_\rho(\pm i)$. The second equality in (164) follows from the inequalities in (135), since $I'_\rho \subset U\varepsilon(S_\rho')$. Formulas (44), (122) imply (165). Clearly, using (164), (165), and (136) we get

$$\mu(V'(\pm i, \rho d^{-1})) - \mu(I'_\rho(\pm i)) = O(\rho^{d-1-2\alpha_1}).$$  

On the other hand if

$$y = (y_1, y_2, \ldots, y_d) \in I'_\rho(+i) \cap I'_\rho(+j) \quad \text{for} \quad i < j \quad \text{then there are} \quad a \in V'(\pm i, \rho d^{-1})$$

and $a' \in V'(\pm j, \rho d^{-1})$ such that $y = y(a, i) = y(a', j)$ and $y \in [a - \varepsilon e_i, a + \varepsilon e_i]$. These inclusions imply that $\rho d^{-1} - \varepsilon \leq y_i \leq \rho d^{-1}$. Therefore

$$\mu(I'_\rho(\pm i) \cap I'_\rho(\pm j)) = O(\varepsilon \rho^{d-2}).$$  

This equality, (136) and (122) imply that

$$\mu((I'_\rho(+i) \cap I'_\rho(+j))) = O(\varepsilon \rho^{d-2})$$

for all $i$ and $j$. Similarly

$$\mu((I'_\rho(+i) \cap I'_\rho(-j))) = O(\varepsilon \rho^{d-2})$$

for all $i$ and $j$. Thus

$$\mu(I'_\rho) = \sum_i \mu(I'_\rho(+i)) + \sum_i \mu(I'_\rho(-i)) + O(\varepsilon \rho^{d-2}) =$$

$$\sum_i \mu(V'(\pm i, \rho d^{-1})) + \sum_i \mu(V'(\pm i, \rho d^{-1})) + O(\rho^{d-1-2\alpha_1}) =$$

$$\mu(V_\rho) + O(\rho^{-2\alpha_1})\mu(B(\rho)).$$  

This and (123) yield the inequality (124) for $I'_\rho$. Now we define $I'_{\rho, k}$ as follows. If $\gamma + t \in I'_\rho$, then $\Lambda(\gamma + t) = \rho^2$, where $\Lambda(\gamma + t)$ is a unique eigenvalue satisfying (5) (see Remark 5). Since

$$\Lambda(\gamma + t) = \gamma^2 + O(\rho^{-\alpha_1})$$

(see (5) and (24)), for fixed $t$ there exist only a finite number of vectors $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$ satisfying $\Lambda(\gamma_k + t) = \rho^2$. Hence $I'_\rho$ is the union of pairwise disjoint subsets $I'_{\rho, k} = \{ \gamma_k + t \in I'_\rho : \Lambda(\gamma_k + t) = \rho^2 \}$ for
The translation $I''_{p,k} = I'_{p,k} - \gamma_k = \{ t \in F^* : \gamma_k + t \in I'_{p,k} \}$ of $I'_{p,k}$ is a part of the isoenergetic surfaces $I_p$ of $L(q(x))$. Put $I'_\rho = \bigcup_{k=1}^n I''_{p,k}$. If $t \in I''_{p,k} \cap I''_{p,m}$ for $k \neq m$, then $\gamma_k + t \in I'_\rho \subset U_\varepsilon(S'_\rho)$ and $\gamma_m + t \in U_\varepsilon(S'_\rho)$, which contradict Lemma 8(b). So $I'_\rho$ is union of the pairwise disjoint subsets $I'_{p,k}$ for $k = 1, 2, ..., s$. Thus

$$\mu(I'_\rho) = \sum_k \mu(I''_{p,k}) = \sum_k \mu(I'_{p,k}) = \mu(I'_\rho) > (1 - c_1 \rho^{-\alpha}) \mu(B(\rho)) \diamond$$

6 Bloch Functions near the Diffraction Planes

In this section we obtain the asymptotic formulas for the Bloch function corresponding to the quasimomentum lying near the diffraction hyperplanes. Here we assume that $q(x) \in W^{2,1}(F)$, where $s \geq 6(3^d(d + 1)^2) + d$. In this section we define the number $q$ by $q = 4(3^d(d + 1))$. The other numbers $p, \alpha_k, \alpha, k_1, p_1$ are defined as in the end of introduction. Clearly these numbers satisfy all inequalities in (15). Therefore the formulas obtained in previous sections hold in this notations too. Moreover the following relations hold

$$k_2 < \frac{1}{9} (p - \frac{1}{2} q(d - 1), \ k_2 \alpha_2 > d + 2 \alpha, \ 4(d + 1) \alpha_d = 1, \ (166)$$

where $k_2 = \lfloor \frac{d}{\log 2} \rfloor + 2$. In this section we construct a subset $B_\delta$ of $V_\rho(\rho^{\alpha_1})$ such that if $\gamma + t \equiv \beta + \tau + (j + \nu)\delta \in B_\delta$ (see Remark 3 for this notations), then there exists a unique eigenvalue $\Lambda_N(\lambda_{j,\beta}(v, \tau))$ satisfying (99). This is a simple eigenvalue. We say that the set $B_\delta$ is a simple set in the resonance domain $V_\delta(\rho^{\alpha_1})$. Then we obtain the asymptotic formulas of arbitrary order for the eigenfunction $\Psi_N(x)$ corresponding to the eigenvalue $\Lambda_N(\lambda_{j,\beta}(v, \tau))$. At the end of this section we prove that $B_\delta$ has asymptotically full measure on $V_\delta(\rho^{\alpha_1})$. The construction of the simple set $B_\delta$ in the resonance domain $V_\delta(\rho^{\alpha_1})$ is similar to the construction of the simple set $B$ in the non-resonance domain (see Step 1 and Step 2 in introduction). So as in Step 2 we need to find the simplicity conditions for the eigenvalue $\Lambda_N(\lambda_{j,\beta})$. Since the first inequality in (166) holds, $\Lambda_N(\lambda_{j,\beta})$ satisfies the formula (99) for $k = k_2$. Therefore it follows from the second inequality of (166) that $\Lambda_N(\lambda_{j,\beta})$ lies in the $\varepsilon_1 = \rho^{d-2\alpha}$ neighbourhood of $E(\lambda_{j,\beta}(v, \tau))$, where $E(\lambda_{j,\beta}(v, \tau)) = \lambda_{j,\beta}(v, \tau) + E_{k_2-1}(\lambda_{j,\beta}(v, \tau))$. Note that we have the relations

$$E_{k_2-1}(\lambda_{j,\beta}) = O(\rho^{-\alpha_2}(\ln \rho)), \ \lambda_{j,\beta}(v, \tau) \sim \rho^2 \ \ (167)$$

$$\lambda_{j,\beta}(v, \tau) = |\beta + \tau|^2 + \mu_j(v) = |\beta + \tau|^2 + O(\rho^{2\alpha_1}) \ \ (168)$$

(see (100), Lemma 2(b), (52), (51)). We call $E(\lambda_{j,\beta}(v, \tau))$ the known part of $\Lambda_N(\lambda_{j,\beta}(v, \tau))$. Since known parts of the other eigenvalues are $\lambda_{1}(\gamma' + t), \ F(\gamma' + t)$ (see Step 1), that is, the other eigenvalues lie in the $\varepsilon_1$ neighbourhood of $\lambda_{1}(\gamma' + t), \ F(\gamma' + t)$, in order that $\Lambda_N(\lambda_{j,\beta}(v, \tau))$ does not coincide with any other eigenvalue we use the following two simplicity conditions

$$|E(\lambda_{j,\beta}(v, \tau)) - F(\gamma' + t)| \geq 2\varepsilon_1, \forall \gamma' \in M_1$$

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\[ | E(\lambda_{j,\beta}) - \lambda_i(\gamma' + t) | \geq 2\varepsilon_1, \forall \gamma' \in M_2; i = 1, 2, \ldots, b_k, \]  

(169)

where \( M \) is the set of \( \gamma' \in \Gamma \) satisfying \(| E(\lambda_{j,\beta}(v, \tau)) - | \gamma' + t |^2 | < \frac{1}{2} \rho^{\alpha_1} \); \( M_1 \) is the set of \( \gamma' \in M \) satisfying \( \gamma' + t \in U(\rho^{\alpha_1}, p) \); \( M_2 \) is the set of \( \gamma' \in M \) such that \( \gamma' + t \notin U(\rho^{\alpha_1}, p) \) and \( \gamma' + t \) has the \( \Gamma_\delta \) decomposition

\[ \gamma' + t \equiv \beta' + \tau + (j' + v(\beta'))\delta \ (\text{see Remark 3)} \] with \( \beta' \neq \beta \). Let \( B_\delta \) be the set of \( x \in V'_\delta(\rho^{\alpha_1}) \cap (R(\frac{1}{2}\rho - \rho^{\alpha_1}) \backslash R(\frac{1}{2}\rho + \rho^{\alpha_1}^{-1})) \) such that \( x = \gamma + t \), where \( \gamma \in \Gamma, t \in F^* \) (it is \( \Gamma \) decomposition of \( x \)), and \( x = \beta + \tau + (j + v(\beta))\delta \), where \( \beta \in \Gamma_\delta, \tau \in F_\delta, j \in Z, v(\beta) \in W(\rho) \) (it is \( \Gamma_\delta \) decomposition of \( x \)), and (168), (169) hold. Using this conditions we prove that \( \Lambda_N(\lambda_{j,\beta}(v, \tau)) \) does not coincide with other eigenvalues if \( \beta + \tau + (j + v(\beta))\delta \in B_\delta \). The existence and properties of the sets \( B_\delta \), will be considered in the end of this section. Recall that in Section 4 the simplicity conditions (12), (13) implied the asymptotic formulas for the Bloch functions in the non-resonance domain. Similarly, here the simplicity conditions (168), (169) imply the asymptotic formula for the Bloch function in the resonance domain \( V'_\delta(\rho^{\alpha_1}) \). For this we use the following lemma.

**Lemma 9** Let \( \Lambda_N(\lambda_{j,\beta}(v, \tau)) \) be the eigenvalue of the operator \( L_i(q(x)) \) satisfying (99), where \( \beta + \tau + (j + v(\beta))\delta = \gamma + t \in B_\delta \). If for \( \gamma' + t = \beta' + \tau + (j' + v(\beta'))\delta \) at least one of the following conditions:

- **Cond.1:** \( \gamma' \in M, \beta' \neq \beta \),
- **Cond.2:** \( | \beta - \beta' | > (p - 1)\rho^{\alpha_1} \),
- **Cond.3:** \( | \beta - \beta' | \leq (p - 1)\rho^{\alpha_1}, | j' \delta | \geq h \) hold, then

\[ | b(N, \gamma') | \leq c_4\rho^{-c_0}, \]  

(170)

where \( h \equiv 10^{-p}\rho^{\frac{1}{2}c_2}, c = p - dq - \frac{1}{4}d^3 \varepsilon_1 - 3, b(N, \gamma') = (\Psi_{N,t}, e^{i(\gamma' + t, x)}), \Psi_{N,t}(x) \) is any normalized eigenfunction of \( L_i(q(x)) \) corresponding to the eigenvalue \( \Lambda_N(\lambda_{j,\beta}(v, \tau)) \).

**Proof.** Repeating the proof of the inequality in (105) and instead of the simplicity conditions (12), (13) and the set \( K \) using the simplicity conditions (168), (169), and the set \( M \) we obtain the proof of (170) under Cond.1. Suppose Cond.2 holds. Take \( n \) vectors \( \gamma_1, \gamma_2, \ldots, \gamma_n \), where \( n \leq p - 1 \), from \( \Gamma(\rho^{\alpha_1}) \). Using the decomposition \( \gamma_i = \beta_i + a_i\delta \) (see (48)), where \( \beta_i \in \Gamma_\delta, a_i \in \mathbb{R}, 1 \leq i \leq n \leq (p - 1) \), and Cond.2 we have

\[ | \beta_i | < \rho^{\alpha_1}, | a_i\delta | < \rho^{\alpha_1}, \beta' - \sum_{i=1}^{n} \beta_i \neq \beta \]  

(171)

\[ \gamma' + t - \sum_{i=1}^{n} \gamma_i = \beta' - \sum_{i=1}^{n} \beta_i + \tau + (j' + v(\beta')) - \sum_{i=1}^{n} a_i\delta. \]  

(172)

If \( \gamma' - \sum_{i=1}^{n} \gamma_i \in M \), then (171) and (170) under Cond.1. imply that
\[ b(N, \gamma - \sum_{i=1}^{n} \gamma_i) = O(\rho^{-\alpha_1}). \] If \( \gamma' \notin M \) then applying (17) \((p-1)\)-times and each time instead of \( b(N, \gamma' - \sum_{i=1}^{j} \gamma_i) \) for \( \gamma' - \sum_{i=1}^{j} \gamma_i \in M \) writing \( O(\rho^{-\alpha_1}) \), we obtain

\[ b(N, \gamma') = \sum_{\gamma_1, \gamma_2, \ldots, \gamma_{p-1}} q_{\gamma_1} q_{\gamma_2} \ldots q_{\gamma_{p-1}} b(N, \gamma' - \sum_{i=1}^{p-1} \gamma_i) + o(\rho^{-\alpha_1}), \quad (173) \]

where the summation is taken under the conditions \( \gamma' - \sum_{i=1}^{j} \gamma_i \notin M \), for \( j \leq p - 1 \). These conditions, the definition of \( M \) and (99) imply that

\[ |\Lambda_N - |\gamma' - \sum_{i=1}^{j} \gamma_i + t|^2| > \frac{1}{4} \rho^{\alpha_1}. \] Therefore (173) and (4) give (170).

Now assume that Cond.3. holds. First we prove that if \( \gamma' - \sum_{i=1}^{s} \gamma_i \in M \), where \( s = 0, 1, \ldots, n \) and \( n, \gamma_i \) are defined in the case of Cond.2., then \( \beta' - \sum_{i=1}^{s} \beta_i \neq \beta \). Assume the converse, i.e., \( \beta' - \sum_{i=1}^{s} \beta_i = \beta \). Then the relations \((\beta + \tau, \delta) = 0, |a_i \delta| < \rho^\alpha, \) (see (171)) \(|j \delta| \geq h, \ |v(\beta')| \leq 1 \) and (172) imply that

\[ |\gamma' + t - \sum_{i=1}^{s} \gamma_i|^2 \geq |\beta + \tau|^2 + \frac{1}{2} h^2. \quad (174) \]

On the other hand the definition of \( E(\lambda_{j, \beta}(v, \tau)) \), and (167) yield

\[ E(\lambda_{j, \beta}(v, \tau)) = |\beta + \tau|^2 + O(\rho^{2\alpha_1}) \quad (175) \]

Therefore using (174), (175) and the relations \( \frac{1}{2} h^2 > c_{22}\rho^{\alpha_2} \), \( \alpha_2 = 3\alpha_1 \) we get

\[ |E(\lambda_{j, \beta}(v, \tau)) - |\gamma' + t - \sum_{i=1}^{s} \gamma_i|^2| > \rho^{\alpha_1} \]

which contradicts \( \gamma' - \sum_{i=1}^{s} \gamma_i \in M \). Thus we proved that if \( \gamma' - \sum_{i=1}^{s} \gamma_i \in M \), then \( \beta' - \sum_{i=1}^{s} \beta_i \neq \beta \). This relation for \( s = 0 \) means that \( \beta' \neq \beta \) if \( \gamma' \notin M \). Therefore if Cond.3. holds and \( \gamma' \in M \), then Cond.1. holds too and hence (170) holds. To prove (170) under Cond.3. in case \( \gamma' \notin M \) we repeat the prove of the case of Cond.2. that is, use (173) and etc.

**Theorem 8** If \( \gamma + t = \beta + \tau + (j + v(\beta))\delta \in B_\delta \), then there exists a unique eigenvalue \( \Lambda_N(\lambda_{j, \beta}(v, \tau)) \) satisfying (99). This is a simple eigenvalue and the corresponding eigenfunction \( \Psi_N \) satisfies the asymptotic formula

\[ \Psi_N = \Phi_{j, \beta}(x) + O(\rho^{-\alpha_2} \ln \rho). \quad (176) \]

**Proof.** The proof is similar to the proof of Theorem 5. Arguing as in the proof of the Theorem 5 we see that to prove this theorem it is enough to show that for any normalized eigenfunction \( \Psi_N \) corresponding to any eigenvalue \( \Lambda_N \) satisfying (99) the following equality holds

\[ \sum_{(j', \beta') \in K_0} |b(N, j', \beta')|^2 = O(\rho^{-2\alpha_2}(\ln \rho)^2), \quad (177) \]
where $b(N, j', \beta') = (\Psi_N(x), \Phi_{j', \beta'}(x))$,

$$K_0 = \{(j', \beta') : j' \in Z, \beta' \in \Gamma \setminus \{(j', \beta') \neq (j, \beta)\}$$

To prove (177) we divide the set $K_0$ into subsets: $K_0^1, K_0^2 \subset S(p-1)$, where

$$K_0^1 = K_0 \setminus K_0^2, S_c(n) = K_0 \setminus S(n), K_0 = \{(j', \beta') : j', \beta' \in K_0 \setminus \Lambda_N - \Lambda_{j', \beta'} < h^2\}$$

If $(j', \beta') \in K_0^1$, then using (53), the definitions of $K_0^1$ and $h$, we have

$$\sum_{(j', \beta') \in K_0^1} |b(N, j', \beta')|^2 = \sup |q - q^\delta|^2 O(\rho^{-2\alpha_2}) = O(\rho^{-2\alpha_2}). \quad (178)$$

To consider the set $K_1 \cap S(p-1)$ we prove that

$$K_1 \cap S(n) = K_1 \cap \{(j', \beta') : j' \in Z\} \subset \{(j', \beta') : |j'\delta| < 2h\}, \quad (179)$$

for $n = 1, 2, \ldots, p - 1$. Take any element $(j', \beta)$ from $K_1 \cap \{(j', \beta') : j' \in Z\}$. Since $\lambda_{j', \beta}(v, \tau) = \beta + |\tau|^2 + \mu_{j'}(v) = \beta + |(j' + v)\delta|^2 + O(1)$, where $v \in [0, 1]$ (see Lemma 2(b) and (52)), using the definition of $K_1$, (99) (175) and the relations $h^2 \sim \rho^2$, $\alpha_2 = 3\alpha_1$, we obtain

$$|O(\rho^{-2\alpha}) - |(j' + v)\delta|^2| < 2h^2, |j'\delta| < 2h.$$ 

Hence the inclusion in (179) is proved and $K_1 \cap \{(j', \beta) : j' \in Z\} \subset K_1 \cap S(n)$. If the inclusion $K_1 \cap S(n) \subset K_1 \cap \{(j', \beta) : j' \in Z\}$ does not hold then there is an element $(j', \beta')$ of $K_1 \cap S(n)$ such that

$$0 < |\beta - \beta'| \leq n\rho^\alpha \leq (p - 1)\rho^\alpha, |j'\delta| < 10^n h < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}.$$ 

These inequalities and the inequality

$$|\Lambda_N - \Lambda_{j', \beta'}| > \frac{1}{2}\rho^{\alpha_2}, \quad (180)$$

for $0 < |\beta - \beta'| \leq (p - 1)\rho^\alpha, |j'\delta| < \frac{1}{2}\rho^{\alpha_2}$, which follows from (83), (88), imply that $|\Lambda_N - \Lambda_{j', \beta'}| > h^2$. This contradicts $(j', \beta') \in K_1$. So (179) is proved. Therefore

$$\sum_{(j', \beta') \in K_1 \cap S(p-1)} |b(N, j', \beta')|^2 \leq \sum_{j' \neq j, \|j'\delta| < 2h} |b(N, j', \beta')|^2 \quad (181)$$

For estimation of $b(N, j', \beta)$ for $|j'\delta| < 2h$, we use (75) as follows. In (75) replacing $\beta'$ and $\bar{r}$ by $\beta$ and $2h$, we obtain

$$(\Lambda_N - \Lambda_{j', \beta'})b(N, j', \beta) = O(\rho^{-\alpha_2}) +$$

$$+ \sum_{(j_1, \beta_1) \in Q(\rho^\alpha, 18h)} A(j', \beta, j_1 + \beta + \beta_1) b(N, j_1, \beta + \beta_1). \quad (182)$$

By definition of $Q(\rho^\alpha, 18h)$ we have $|\beta_1| < \rho^\alpha$, $|j_1\delta| < 18h$, and hence

$$|j' + j_1\delta| < 20h < \frac{1}{2}\rho^{\alpha_2}.$$ 

Therefore in the right-hand side of (182) the multiplicand $b(N, j_1, \beta + \beta_1)$ for $(j' + j_1, \beta + \beta_1) \in D(\beta)$ takes part, where
\[ D(\beta) = \{(j, \beta + \beta_0) : |j\delta| < \frac{1}{2} \rho^{1/2} \alpha, 0 < |\beta| < \rho^\alpha\}. \] Put

\[ |b(N, j_0, \beta + \beta_0)| = \max_{(j, \beta + \beta_0) \in D(\beta)} |b(N, j, \beta + \beta_1)|. \]

By definition of \( D(\beta) \) and by (180) we have \( |\Lambda_N - \lambda_{j_0, \beta + \beta_0}| > \frac{1}{2} \rho^{\alpha^2} \). This together with (53) gives \( |b(N, j_0, \beta + \beta_0)| = O(\rho^{-\alpha^2}) \). Using this, (182) and (71), we get

\[ |b(N, j, \beta)| < c_{23} |\Lambda_N - \lambda_j, \beta|^{-1} \rho^{-\alpha^2} \tag{183} \]

for \( j \neq j' |j'\delta| < 2h \), where

\[ \Lambda_N - \lambda_{j', \beta} = \lambda_j, \beta - \lambda_{j', \beta} + O(\rho^{-\alpha^2}) = \mu_j(v) - \mu_{j'}(v) + O(\rho^{-\alpha^2}) \tag{see (88) and Lemma 2(b)} \]

and \( v \in W(\rho) \) (see the definition of \( B_3 \)). Now using the definition of \( W(\rho) \) (see Lemma 3(b)) and (52) we obtain \( \sum_{j' \neq j} |\Lambda_N - \lambda_{j', \beta}|^{-1} = O(\ln \rho) \). This with (183) and (181) yield

\[ \sum_{(j', \beta') \in K_1 \cap S^c(p - 1)} |b(N, j', \beta')|^2 = O(\rho^{-2\alpha^2}(\ln \rho)^2) \tag{184} \]

It remains to consider \( K_1 \cap S^c(p - 1) \). We prove that

\[ b(N, j', \beta') = O(\rho^{-\alpha^2}) \tag{185} \]

for \( (j', \beta') \in K_1 \cap S^c(p - 1) \), where the number \( c \) is defined in Lemma 9. For this using the decomposition of \( \varphi_{j', x(\beta')}(s) \) by \( \{e^{i(m+v(\beta'))s} : m \in \mathbb{Z}\} \), we get

\[ b(N, j', \beta') = \sum_m (\varphi_{j', x}(s), e^{i(m+v)s})(\Psi_N(x), e^{i(\beta + \tau + (m+v)x)}). \tag{186} \]

If \( |\beta - \beta'| > (p-1)\rho^\alpha \) then Lemma 9 (see Cond. 2) and (73), (186) give the proof (185). So we need to consider the case \( |\beta - \beta'| \leq (p-1)\rho^\alpha \). Then by definition of \( S^c(p - 1) \) we have \( |j'\delta| > 10^{p-1}h \). Write the right-hand side of (186) as \( \sum_1 + \sum_2 \), where the summations in \( \sum_1 \) and \( \sum_2 \) are taken under conditions \( |m\delta| \geq h \) and \( |m\delta| < h \) respectively. By (73) and Lemma 9 (see Cond. 3) we have \( \sum_1 = O(\rho^{-\alpha^2}) \). If \( |m\delta| < h \), then the inequality \( |j'| > 2 |m| \) holds. Therefore using (57), taking into account that \( |j'\delta| \sim \rho^{\alpha^2} \) and the number of summand in \( \sum_2 \) is less than \( \rho^{\alpha^2} \), we get \( \sum_2 = O(\rho^{-\alpha^2}) \). The estimations for \( \sum_1, \sum_2 \) give (185). Now using \( |K_1| = O(\rho^{d-1}) \), we get

\[ \sum_{(j', \beta') \in K_1 \cap S^c(p - 1)} |b(N, j', \beta')|^2 = O(\rho^{-2(c-(d-1)q^\alpha)}) \tag{187} \]

This, (178), (184) give the proof of (177), since \( 2(c-(d-1)q^\alpha) > \alpha_2 \). The theorem is proved.

Now using Theorem 8 we obtain the asymptotic formulas of arbitrary order.
Theorem 9  The eigenfunction \( \Psi_N \) defined in Theorem 8, besides formula (175), satisfies the following asymptotic formulas

\[
\Psi_N = \Phi_{j,\beta}(x) + \Phi_{k-1}(x) + O(\rho^{-k\alpha_2} \ln \rho)
\] (188)

for \( k = 1, 2, \ldots, n_1 \), where \( n_1 = \left[ \frac{1}{4}(p - q(\frac{3d-1}{2}) - \frac{1}{3}d^3 - 3) \right] \). \( \Phi_0(x) = 0, \Phi_{k-1}(x) \) is a linear combination of \( \Phi_{j,\beta}(x) \) and \( \Phi_{j',\beta'}(x) \) for \( (j, \beta) \in S(k - 1) \) with expressed by \( q(x), \lambda_{j',\beta'} \), \( \Phi_{j',\beta'}(x) \) coefficients.

**Proof.** By Theorem 8 the formula (188) for \( k = 1 \) is proved. To prove it for arbitrary \( k \ (k \leq n_1) \) we prove the following equivalent formulas

\[
\sum_{(j',\beta') \in S^c(k-1)} |b(N, j', \beta')|^2 = O(\rho^{-2\alpha_2}(\ln \rho)^2),
\] (189)

\[
\Psi_N = \sum_{(j',\beta') \in S^c(k-1) \cup \{j,\beta\}} b(N, j', \beta') \Psi_{j',\beta'} + O(\rho^{-k\alpha_2} \ln \rho). \tag{190}
\]

First consider the set \( S^c(k-1) \cap K_1 \). It follows from the relations

\[
S(k-1) \cap K_1 = S(p-1) \cap K_1 \quad (\text{see (179)}) \quad \text{and} \quad S(k-1) \subset S(p-1) \quad (\text{see definition of } S(k-1));
\]

\[
S^c(k-1) \cap K_1 = S^c(p-1) \cap K_1, \quad \text{Therefore using (179), the equalities}
\]

\[
c = p - dq - \frac{1}{4}d^3 - 3 \quad (\text{see Lemma 9}), \quad \alpha_2 = 9\alpha, \quad n_1 = \left[ \frac{1}{4}(p - q(\frac{3d-1}{2}) - \frac{1}{3}d^3 - 3) \right]
\]

(see Theorem 9) we have

\[
\sum_{(j',\beta') \in S^c(k-1) \cap K_1} |b(N, j', \beta')|^2 = O(\rho^{-2n_1\alpha_2}).
\]

Thus it remains to prove

\[
\sum_{(j',\beta') \in S^c(k-1) \cap K_1^c} |b(N, j', \beta')|^2 = O(\rho^{-2\alpha_2}(\ln \rho)^2) \tag{191}
\]

for \( k = 2, 3, \ldots, n_1 \). We prove this by induction. By formula (70) and (176) we have \( \Psi_N(x)(q(x) - Q(x)) = H(x) + O(\rho^{-\alpha_2} \ln \rho) \), where \( H(x) \) is a linear combination of \( \Phi_{j,\beta}(x) \) and \( \Phi_{j',\beta'}(x) \) for \( (j, \beta) \in S(1) \), since \( |j^\delta| < r_1 < h \) (see (51)). Hence \( H(x) \) orthogonally to \( \Phi_{j',\beta'}(x) \) for \( (j', \beta') \in S'(1) \). Therefore using (75) and the definition of \( K_1^c \) we have

\[
\sum_{(j',\beta') \in S'(1) \cap K_1^c} |b(N, j', \beta')|^2 = \sum \left| \frac{(O(\rho^{-\alpha_2} \ln \rho), \Phi_{j',\beta'})}{\Lambda_N - \lambda_{j',\beta'}} \right|^2
\]

\[
= O(\rho^{-4\alpha_2}(\ln \rho)^2).
\]

Hence (191) for \( k = 2 \) is proved. Assume that this is true for \( k = m \). Then (190) for \( k = m \) holds too. This and (70) for \( r = 10^{m-1}h \) give
\[ \Psi_N(x)(q(x) - Q(s)) = H(x) + O(\rho^{-m\alpha_2}\ln\rho), \] where \( H(x) \) is a linear combination of \( \Phi_j,\beta(x) \) and \( \Psi_{j',\beta'}(x) \) for \((j',\beta') \in S(m)\). Thus \( H(x) \) is orthogonal to \( \Psi_{j',\beta'}(x) \) for \((j',\beta') \in S^c(m)\). Using this and repeating the proof of (191) for \( k = 2 \) we obtain the proof of (191) for \( k = m + 1 \). Thus (189) and (190) are proved. Here \( b(N,j,\beta) \) and \( b(N,j',\beta') \) for \((j',\beta') \in S(k-1)\) can be calculated in the same way as we found \( b(N,\gamma) \) and \( b(N,\gamma + \gamma') \) for \( \gamma' \in \Gamma((n-1)\rho^\alpha) \) in Theorem 5. Namely we apply the formula (75) \( 2k+2 \) times and each time isolate the terms with multiplicand \( b(N,j,\beta) \). Then in the obtained expression instead of \( \Lambda_N \) writing the right side of (99) for \( k = k_3 \), where \( k_3 = \lceil \frac{1}{2}(\rho - \frac{1}{2}q(d - 1)) \rceil \) we write \( b(N,j',\beta') \) in term of \( b(N,j,\beta) \). Substituting the obtained formula for \( b(N,j',\beta') \) into (190), taking into account that \( \| \Psi_N \| = 1, \) \( \arg b(N,j,\beta) = 0 \) (it can be assumed without loss of generality) we find \( b(N,j,\beta) \) and then \( b(N,j',\beta') \).

Now we consider the simple set \( B_\delta \) in the resonance domain \( V_\delta(\rho^{\alpha_1}) \). As we noted in Remark 3 every vectors \( w \) of \( \mathbb{R}^d \) has decomposition \( \equiv \beta + \tau + (j + v)\delta \), where \( \beta \in \Gamma_\delta, \tau \in F_\delta, j \in Z, v \in [0,1) \). Hence the space \( \mathbb{R}^d \) is the union of the pairwise disjoint sets \( P(\beta,j) \equiv (\beta + \tau + (j + v)\delta) : \tau \in F_\delta, v \in [0,1) \) for \( \beta \in \Gamma_\delta, j \in Z \). To prove that \( B_\delta \) has an asymptotically full measure on \( V_\delta(\rho^{\alpha_1}) \), that is,

\[
\lim_{\rho \to \infty} \frac{\mu(B_\delta)}{\mu(V_\delta(\rho^{\alpha_1}))} = 1
\]

we define the following sets: \( R_1(\rho) = \{ j \in Z : |j| < \frac{\rho^{\alpha_1}}{\sqrt{d+1}} + \frac{3}{2} \} \), \( R_2(\rho) = \{ j \in \mathbb{Z} : j < \frac{\rho^{\alpha_1}}{\sqrt{d+1}} + \frac{3}{2} \} \).

Moreover we define a subset \( P'(\beta,j) \) of \( P(\beta,j) \) as follows. Introduce the sets \( A(\beta,b,\rho) = \{ v \in [0,1) : 3j \in Z, |2(\beta,b) + |b|^2 + |j + v\delta|^2 < 4d\rho^{\alpha_2} \} \), \( A(\beta,b,\rho) = \bigcup_{b \in \Gamma_\delta(\rho^{\alpha_2})} A(\beta,b,\rho), S_\beta(\beta,\rho) = W(\rho) \setminus A(\beta,\rho) \) and put \( S_\delta(\beta,j,v,\rho) = \{ v \in F_\delta : \beta + \tau + (j + v)\delta \in B_\delta \} \) for \( j \in S_1, \beta \in S_2, v \in F_\delta \).

Then define \( P'(\beta,j) \) by \( P'(\beta,j) = \{ \beta + \tau + (j + v)\delta : v \in S_\delta(\beta,\rho), \tau \in S_\delta(\beta,j,v,\rho) \} \). It is not hard to see that (193) follows from the following relations:

\[
\lim_{\rho \to \infty} \frac{|S_i(\rho)|}{R_i(\rho)} = 1, \quad \forall i = 1, 2, \quad (194)
\]

\[
B_\delta \supset \bigcup_{j \in S_1, \beta \in S_2} P'(\beta,j), \quad (195)
\]

\[
V_\delta(\rho^{\alpha_1}) \subset \bigcup_{j \in R_1, \beta \in R_2} P(\beta,j), \quad (196)
\]

\[
\lim_{\beta \to \infty} \frac{\mu(P'(\beta,j))}{\mu(P(\beta,j))} = 1. \quad (197)
\]

To prove these relations we use the following lemma.
Lemma 10 Let \( w \equiv \beta + \tau + (j + v)\delta \). Then the following implications:

(a) \( w \in V_\delta(\rho^{\alpha_1}) \Rightarrow j \in R_1, \beta \in R_2, \)

(b) \( j \in S_1, \beta \in S_2 \Rightarrow w \in V_\delta(\rho^{\alpha_1}) \cap (R(\frac{3}{2}\rho - \rho^{\alpha_1-1}) \setminus R(\frac{1}{2}\rho + \rho^{\alpha_1-1})), \)

(c) \( j \in S_1, \beta \in S_2 \Rightarrow w \in V_\delta(\rho^{\alpha_1}) \cap (R(\frac{3}{2}\rho - \rho^{\alpha_1-1}) \setminus R(\frac{1}{2}\rho + \rho^{\alpha_1-1})) \) hold.

The relations (195), (196) and the equality (194) are true.

**Proof.** Since \( (\beta + \tau, \delta) = 0 \) the inclusion \( \omega \in V_\delta(\rho^{\alpha_1}) \) means that
\[
\| (j + v + 1)\delta \|^2 - \| (j + v)\delta \|^2 < \rho^{\alpha_1}, \quad \text{where} \quad v < 1, \quad \text{and}
\]
\[
(\frac{3}{2}\rho)^2 < \| \beta + \tau \|^2 + \| (j + v)\delta \|^2 < (\frac{3}{2}\rho)^2, \quad \text{where} \quad | \tau | \leq \delta_3 = O(1) \quad \text{(see the definition of} \ V_\delta(\rho^{\alpha_1}) \text{in introduction).}
\]

Therefore by direct calculation we get the proof of the implications (a) and (b).

Now we prove (c). It follows from (b) and the definition of \( V_\delta(\rho^{\alpha_1}) \) (see introduction) that it is enough to show the relation \( w \notin V_\alpha(\rho^{\alpha_1}) \) for \( \alpha \in \Gamma(\rho^{\alpha_1}) \setminus \delta \). Using \( a = a_1 + a_2 \delta \) (see (48)), where \( a_1 \in \Gamma_\delta, a_2 \in \mathbb{R} \) and \( |a_1| < \rho^{\alpha_1}, \)
\|a_2\delta \| < \rho^{\alpha_1},\) we obtain \( |w + a|^2 - |w|^2 = d_1 + d_2, \) where \( d_1 = |\beta + a_1|^2 - |\beta|^2, \)
\[d_2 = (|j + a_2\delta| - |j + v\delta|)^2 - (|j + (v + 1)\delta|^2 - 2(a_1, \tau)).\] The requirements on \( j, a_1, a_2 \) imply that \( d_2 = O(\rho^{\alpha_1}). \) On the other hand the condition \( \beta \in S_2 \) gives \( \beta \notin V_\delta(\rho^{\alpha_1}), \) i.e., \( |d_1| \geq \rho^{\frac{3}{2}}. \) Since \( 2\alpha_1 < \frac{1}{2} \) we have \( |w + a|^2 - |w|^2 > \frac{1}{2}\rho^{\frac{3}{2}}, \) \( w \notin V_\alpha(\rho^{\alpha_1}). \) So (c) is proved.

The inclusion (196) follows from the implication (a). If
\[w \equiv \beta + \tau + (j + v)\delta \] belongs to the right-hand side of (195) then using the implication (c) we obtain \( w \in V_\delta(\rho^{\alpha_1}). \) Therefore (195) follows from the definitions of \( P(\beta, j) \) and \( S_1(\beta, j, v, \rho). \) It remains to prove the equality (194). Using the definitions of \( R_1, S_1 \) and inequalities \( |\delta| < \rho^{\alpha_1}, \) \( \alpha_1 > 2\alpha \) we obtain that (194) for \( i = 1 \) holds. If \( \beta \in R_2 \) then \( \beta + F_3 \subset R_3(\frac{3}{2}\rho + 2d_3 + 1) \setminus R_3(\frac{1}{2}\rho - 2d_3 - 1). \)

This implies that,
\[|R_2| < (\mu(F_3))^{-1} \mu(R_3(\frac{3}{2}\rho + 2d_3 + 1) \setminus R_3(\frac{1}{2}\rho - 2d_3 - 1)), \]

since the translations \( \beta + F_3 \) of \( F_3 \) for \( \beta \in \Gamma_\delta \), are pairwise disjoint sets having measure \( \mu(F_3) \).

Suppose \( \beta + \tau \in D(\rho), \) where \( D(\rho) = (R_3(\frac{3}{2}\rho - 1) \setminus R_3(\frac{1}{2}\rho + 1)) \cup \bigcup_{\delta \in \Gamma_\delta(\rho^{\alpha_2})} V_\delta(2\rho^{\frac{1}{2}}) \). Then \( \frac{3}{2}\rho - 1 < |\beta + \tau| < \frac{3}{2}\rho + 1, \) \| \beta + \tau + b \|^2 - |\beta + \tau|^2 \geq 2\rho^{\frac{1}{2}} \) for \( b \in \Gamma_\delta(\rho^{\alpha_2}). \)

Therefore using \( |\tau| \leq \delta_3 \) it is not hard to verify that \( \beta \in S_2. \) Hence the sets \( \beta + F_3 \) for \( \beta \in S_2 \) is cover of \( D(\rho). \) Thus \( |S_2| \geq (\mu(F_3))^{-1} \mu(D(\rho)). \) This, the estimation for \( |R_2| \), and the obvious relations \( |\Gamma_\delta(\rho^{\alpha_2})| = O(\rho^{d-1}) \)
\[
\mu((R_3(\frac{3}{2}\rho - 1) \setminus R_3(\frac{1}{2}\rho + 1))) = O(\rho^{d-1}),
\]
\[
\mu((R_3(\frac{3}{2}\rho - 1) \setminus R_3(\frac{1}{2}\rho + 1)) \cap V_\delta(2\rho^{\frac{1}{2}})) = O(\rho^{d-2}) \rho^{\frac{1}{2}}, \quad (d - 1)\alpha d < \frac{1}{2} \quad \text{(see the equality in (166))},
\]

\[
\lim_{\rho \to \infty} \frac{\mu((R_3(\frac{3}{2}\rho - 1) \setminus R_3(\frac{1}{2}\rho + 1)))}{\mu(R_3(\frac{3}{2}\rho + 2d_3 + 1) \setminus R_3(\frac{1}{2}\rho - 2d_3 - 1))} = 1,
\]

\( S_2(\rho) \subset R_2(\rho) \) imply (194) for \( i = 2 \)

**Theorem 10** The simple set \( B_2 \) has an asymptotically full measure in the resonance set \( V_\delta(\rho^{\alpha_2}) \) in the sense that (193) holds.
Proof. The proof of the Theorem follows from (194)-(197). By Lemma 10 we need to prove (197). Since the translations $P(\beta,j) - \beta - j\delta$ and $P'(\beta,j) - \beta - j\delta$ of $P(\beta,j)$ and $P'(\beta,j)$ are \{ $\tau + v\delta : v \in [0,1), \tau \in F_3$ \} and
\{ $\tau + v\delta : v \in S_3(\beta,\rho), \tau \in S_4(\beta,j,v,\rho)$ \} respectively, it is enough to prove
\[
\lim_{\rho \to \infty} \mu(S_3(\beta,\rho)) = 1, \quad \mu(S_4(\beta,j,v,\rho)) = \mu(F_3)(1 + O(\rho^{-\alpha})),
\]  
(198)
where $j \in S_1, \beta \in S_2, v \in S_3(\beta,\rho)$, and $O(\rho^{-\alpha})$ does not depend on $v$. To prove the first equality in (198) it is enough to show that
\[
\mu(A(\beta,\rho)) = O(\rho^{-\alpha}),
\]
(199)
since $W(\rho) \supset A(\rho)$ and $\mu(A(\rho)) \to 1$ as $\rho \to \infty$ (see Lemma 3(b)). Using the definition of $A(\beta,\rho)$ and the obvious relation $|\Gamma(\rho^\alpha)| = O(\rho^{\frac{d-1}{2}\alpha})$ we see that (199) holds if $\mu(A(\beta,b,\rho)) = O(\rho^{-\alpha_\delta})$. In other word we need to prove that
\[
\mu\{s \in \mathbb{R} : |f(s)| < 4d\delta\rho^\alpha\} = O(\rho^{-\alpha_\delta}),
\]
(200)
where $f(s) = 2(\beta,b) + \beta + b^2 + \delta^2, \beta \in S_2, b \in \Gamma(\rho^\alpha)$. The last inclusions yield $|2(\beta,b) + b + b^2| \geq \rho^{d/2}$ for $b < \rho^{d/2}$. This and the inequalities $|f(s)| < 4d\delta\rho^\alpha$ (see (200)), $\alpha_\delta < \frac{1}{2}$ (see the equality in (166)) imply that $s^2 \beta^2 > \frac{1}{2}\rho^{d}$ from which we obtain $|f(s)| > |\delta| \rho^{d/2}$. Therefore (200) follows from the equality in (166). Thus (199) and hence the first equality in (198) is proved.

Now we prove the second equality in (198). For this we consider the set $S_4(\beta,j,v,\rho)$ for $j \in S_1, \beta \in S_2, v \in S_3(\beta,\rho)$. By the definitions of $S_4$ and $B_3$ the set $S_4(\beta,j,v,\rho)$ is the set of $\tau \in F_3$ such that $E(\lambda_j,\beta(\tau,\rho))$ satisfies the conditions (168), (169). So we need to consider these conditions. For this we use the decompositions $\gamma + t = \beta + \tau + (j + v)\delta, \gamma + t = \beta + \tau + (j + v(\beta,t))\delta$ (see Remark 3) and the notations $\lambda_{j,\beta}(v,\tau) = \mu_j(v) + |\beta + \tau|^2, \lambda_j(\gamma + t) = |\gamma + t|^2 + r_j(\gamma + t), \lambda_{j,\beta}(v,\tau)$ (see Lemma 2(b) and Remark 2). Denoting by $b$ the vector $\beta - \beta$ we write the decomposition of $\gamma + t$ in the form $\gamma + t = \beta + b + \tau + (j + v(\beta,t))\delta$. Then to every $\gamma \in \Gamma$ there corresponds $b = b(\gamma) \in \Gamma_3$. For $\gamma \in M_1$ denote by $B(\beta,b(\gamma),j,v)$ the set of $\tau$ not satisfying (168). For $\gamma \in M_2$ denote by $B^2(\beta,b(\gamma),j,v)$ the set of $\tau$ not satisfying (169), where $M_i$ for $i = 1,2$ is defined in (168), (169). Clearly, if $\tau \in F_3 \setminus \cup_{s=1,2}(\cup_{\gamma \in M_s}(B^s(\beta,b(\gamma),j,v)))$ then the inequalities (168), (169) hold, that is, $\tau \in S_4(\beta,j,v,\rho)$. Therefore using $\mu(F_3) \sim 1$ and proving that
\[
\mu(\cup_{s=1,2}(B^s(\beta,b(\gamma),j,v))) = O(\rho^{-\alpha}), \quad \forall s = 1,2,
\]
(201)
we get the proof of the second equality in (198). Now we prove (201). Using the above notations and the notations of (168), (169) it is not hard to verify that if $\tau \in B^s(\beta,b(\gamma),j,v)$, then
\[
|2(\beta,b) + b + (j + v(\beta + b))\delta|^2 + 2(b,\tau) - \mu_j(v) + h_s(\gamma + t) | < 2\varepsilon_1,
\]
(202)
where $h_1 = F_{k_1 - 1} - E_{k_1 - 1}$, $h_2 = r_1 - E_{k_2 - 1}$, $\gamma' \in M_s$, $s = 1, 2$. First we prove that (202) for $s = 1, 2$ and $b \equiv b(\gamma') \in \Gamma_\delta(\rho^\alpha)$ does not hold. The assumption $v \in S_3(\beta, \rho)$ implies that $v \not\in A(\beta, \rho)$. This means that

$$|2(\beta, b) + |b|^2 + (j' + v(\beta + b))\delta|^2| \geq 4\delta\rho^\alpha.$$ 

Therefore if

$$|2(b, \tau) - \mu_j(v) + h_s(\gamma' + t)| < 3\delta\rho^\alpha,$$  

then (202) does not hold. Now we prove (203). The relations $b \in \Gamma_\delta(\rho^\alpha)$, $\tau \in F_\delta$ imply that $|2(b, \tau)| < 2\delta\rho^\alpha$. The inclusion $j \in S_1$ and (52) imply that

$$\mu_j(v) = O(\rho^{2\alpha_1}).$$ By (24) and (100), $h_1 = O(\rho^{\alpha_1})$. Now we prove that $r_i = O(\rho^{\alpha_1})$ which implies that $h_2 = O(\rho^{\alpha_1})$ and hence ends the proof of (203). The inclusion $\tau \in B^2(\beta, b(\gamma'), j, v)$, means that (169) does not hold, that is,

$$|E(\lambda_{i, \beta}(v, \tau)) - \lambda_i(\gamma' + t)| < 2\varepsilon_1.$$ On the other hand the inclusion $\gamma' \in M_s$ implies that $\gamma' \in M$ (see the definitions of $M_2$, and $M$) and hence

$$|E(\lambda_{i, \beta}(v, \tau)) - \gamma' + t| \leq \frac{1}{2}\rho^\alpha.$$ The last two inequalities imply that $r_i(\gamma' + t) = O(\rho^{\alpha_1})$. Thus (203) is proved. Hence (202) for $b \in \Gamma_\delta(\rho^\alpha)$ does not hold. It means that the sets $B^1(\beta, b, j, v)$ and $B^2(\beta, b, j, v)$ for $|b| < \rho^\alpha$ are empty.

To estimate the measure of the set $B^*(\beta, b(\gamma'), j, v)$ for $\gamma' \in M_s$, $|b(\gamma')| \geq \rho^\alpha$, $b \in \Gamma_\delta$ we choose the coordinate axis so that the direction of $b$ coincides with the direction of $(1, 0, 0, ..., 0)$, i.e., $b = (b_1, 0, 0, ..., 0), b_1 > 0$ and the direction of $\delta$ coincides with the direction of $(0, 0, ..., 0, 1)$. Then $H_\delta$ and $B^*(\beta, b, j, v)$ can be considered as $\mathbb{R}^{d-1}$ and as a subset of $F_\delta \subset \mathbb{R}^{d-1}$ respectively. We estimate the measure of $B^*(\beta, b, j, v)$ by using (137) for

$$D = B^*(\beta, b, j, v), m = d - 1, k = 1.$$ For this we prove that

$$\mu((B^*(\beta, b, j, v))(\tau_2, \tau_3, ..., \tau_{d-1})) < 4\varepsilon_1 |b|^{-1},$$ 

for all fixed $(\tau_2, \tau_3, ..., \tau_{d-1})$. Assume the converse. Then there are two points $\tau = (\tau_1, \tau_2, \tau_3, ..., \tau_{d-1}) \in F_\delta$, $\tau' = (\tau_1', \tau_2, \tau_3, ..., \tau_{d-1}') \in F_\delta$ of $B^*(\beta, b, j, v)$, such that $|\tau_1 - \tau_1'| \geq 4\varepsilon_1 |b|^{-1}$. Since (202) holds for $\tau$ and $\tau'$ we have

$$|2\delta_1(\tau_1 - \tau_1') + g_2(\tau) - g_2(\tau')| < 4\varepsilon_1,$$ 

where $g_2(\tau) = h_2(\beta' + \tau + (j' + v(\beta + b))\delta).$ Using (44), (46), (101), and the inequality $|b| \geq \rho^\alpha$, we obtain

$$|g_1(\tau) - g_1(\tau')| < \rho^{-\alpha_1} |\tau_1 - \tau_1'| < b_1 |\tau_1 - \tau_1'|,$$ 

$$|g_2(\tau) - g_2(\tau')| < 3\rho^{2\alpha_1} |\tau_1 - \tau_1'| < b_1 |\tau_1 - \tau_1'|.$$ 

The above inequality $b_1 |\tau_1 - \tau_1'| \geq 4\varepsilon_1$ together with (206) and (207) contradicts (204) for $s = 1$ and for $s = 2$ respectively. Hence (204) is proved. Since $B^*(\beta, b, j, v) \subset F_\delta$ and $d_\delta = O(1)$, we have $\mu(\text{Pr } B^*(\beta, b, j, v)) = O(1).$ Therefore formula (137), the inequalities (204) and $|b| \geq \rho^\alpha$ yield

$$\mu((B^*(\beta, b(\gamma'), j, v)) = O(\varepsilon_1 |b(\gamma')|^{-1}) = O(\rho^{-\alpha_1} \varepsilon_1) \text{ for } \gamma' \in M_s \subset M$$ 

and $s = 1, 2$. This implies (201), since $|M| = O(\rho^{d-1})$, $\varepsilon_1 = \rho^{-d-2\alpha}$ and $O(\rho^{d-1-\alpha_1}) = O(\rho^{-\alpha})$.
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