Perverse sheaves and graphs on surfaces

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1 Introduction

The aim of this note is to propose a combinatorial description of the categories \( \text{Perv}(S, N) \) of perverse sheaves on a Riemann surface \( S \) with possible singularities at a finite set of points \( N \). Here we consider the sheaves with values in the category of vector spaces over a fixed base field \( k \). We allow (in fact, require) the surface \( S \) to have a boundary. Categories of the type \( \text{Perv}(S, N) \) can be used as inductive building blocks in the study of any category of perverse sheaves, see [GMV]. Therefore explicit combinatorial descriptions of them are desirable.

We proceed in a manner similar to [KS1]. To define a combinatorial data we need to fix a Lagrangian skeleton of \( S \). In our case it will be a spanning graph \( K \subset S \) with the set of vertices \( \text{Vert}(K) = N \) (for the precise meaning of the word "spanning" see Section 3B). (In the case of a hyperplane arrangement in \( \mathbb{C}^n \) discussed in [KS1] the Lagrangian skeleton was \( \mathbb{R}^n \subset \mathbb{C}^n \).) We denote by \( \text{Ed}(K) \) the set of edges of \( K \). We suppose for simplicity in this Introduction that \( K \) has no loops. As any graph embedded into an oriented surface, \( K \) is naturally a ribbon graph, i.e., it is equipped with a cyclic order on the set of edges incident to any vertex.

To any ribbon graph \( K \) we associate a category \( \mathcal{A}_K \) whose objects are collections \( \{ E_x, E_e \in \text{Vect}(k), x \in \text{Vert}(K), e \in \text{Ed}(K) \} \) together with linear maps

\[
E_x \xrightarrow{\gamma} E_e,
\]

given for each couple \((x, e)\) with \( x \) being a vertex of an edge \( e \). Here \( \text{Vect}(k) \) denotes the category of finite dimensional \( k \)-vector spaces. These maps must
satisfy the relations which use the ribbon structure on \( K \) and are listed in Section 3C below.

If \( K \subset S \) is a spanning graph as above, then our main result (see Theorem 3.6) establishes an equivalence of categories

\[
Q_K : \text{Perv}(S, N) \rightarrow A_K
\]

For \( F \in \text{Perv}(S, N) \) the vector spaces \( Q_K(F)_x, Q_K(F)_v \) are the stalks of the constructible complex \( R_K(F) = R\Gamma_K(F)[1] \) on \( K \) which, as we prove, is identified with a constructible sheaf in degree 0.

A crucial particular case is \( S = \) the unit disc \( D \subset \mathbb{C} \), \( N = \{0\} \). Take for the skeleton a corolla \( K_n \) with center at 0 and \( n \) branches. Thus, the same category \( \text{Perv}(D, 0) \) has infinitely many incarnations, being equivalent to \( A_n := A_{K_n}, n \geq 1 \).

The corresponding equivalence \( Q_n \) is described in Section 2, see Theorem 2.1. The special cases of this equivalence are:

(i) \( Q_1 : \text{Perv}(D, 0) \rightarrow A_1 \): this is a classical theorem, in the form given in [GGM].

(ii) \( Q_2 : \text{Perv}(D, 0) \rightarrow A_2 \) is a particular case of the main result of [KS1], see op. cit., §9A. In loc. cit. we have also described the resulting equivalence

\[
A_1 \rightarrow A_2
\]

explicitly. In a way, objects of \( A_2 \) are ”square roots” of objects of \( A_1 \), in the same manner as the Dirac operator is a square root of the Schrödinger operator. That is why we call \( A_n \) a ”1/n-spin (parafermionic) incarnation” of \( \text{Perv}(D, 0) \).

Finally, in Section 3D we give (as an easy corollary of the previous discussion) a combinatorial description of the category \( \text{PolPerv}(S, N) \) of polarized perverse sheaves, cf. [S]. These objects arise ”in nature” as decategorified perverse Schobers, cf. [KS2], the polarization being induced by the Euler form \( (X, Y) \mapsto \chi(R\text{Hom}(X, Y)) \).

The idea of localization on a Lagrangian skeleton was proposed by M. Kontsevich in the context of Fukaya categories. The fact that it is also applicable to the problem of classifying perverse sheaves (the constructions of [GGM] and [KS1] can be seen, in retrospect, as manifestations of this
idea) is a remarkable phenomenon. It indicates a deep connection between Fukaya categories and perversity. A similar approach will be used in [DKSS] to construct the Fukaya category of a surface with coefficients in a perverse Schober.

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2 The “fractional spin” description of perverse sheaves on the disk

A. Statement of the result. Let $X$ be a complex manifold. By a perverse sheaf on $X$ we mean a $C$-constructible complex $\mathcal{F}$ of sheaves of $k$-vector spaces on $X$, which satisfies the middle perversity condition, normalized so that a local system in degree 0 is perverse. Thus, if $k = \mathbb{C}$ and $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $R\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is a perverse sheaf.

Let $D = \{|z| < 1\} \subset \mathbb{C}$ be the unit disk. Let $k$ be a field. We denote by $\text{Perv}(D, 0)$ the abelian category of perverse sheaves of $k$-vector spaces on $D$ which are smooth (i.e., reduce to a local system in degree 0) outside $0 \in D$.

Let $n \geq 1$ be an integer. Let $\mathcal{A}_n$ be the category of diagrams of finite-dimensional $k$-vector spaces (quivers) $Q$, consisting of spaces $E_0, E_1, \cdots, E_n$ and linear maps

$$E_0 \xrightarrow{\gamma_i} E_i, \quad i = 1, \cdots, n,$$

satisfying the conditions (for $n \geq 2$):

(C1) $\gamma_i \delta_i = \text{Id}_{E_i}$.

(C2) The operator $T_i := \gamma_i + 1 \delta_i : E_i \to E_{i+1}$ (where $i+1$ is considered modulo $n$), is an isomorphism for each $i = 1, \cdots, n$.

(C3) For $i \neq j, j + 1 \mod n$, we have $\gamma_i \delta_j = 0$.

For $n = 1$ we impose the standard relation:

(C) The operator $T = \text{Id}_{E_1} - \gamma_1 \delta_1 : E_1 \to E_1$ is an isomorphism.

Theorem 2.1. For each $n \geq 1$, the category $\text{Perv}(D, 0)$ is equivalent to $\mathcal{A}_n$. 

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For $n = 1$ this is the standard $(\Phi, \Psi)$ description of perverse sheaves on the disk [GGM], [Be]. For $n = 2$ this is a particular case of the description in [KS1] (§9 there).

**B. Method of the proof.** For the proof we consider a star shaped graph $K = K_n \subset D$ obtained by drawing $n$ radii $R_1, \ldots, R_n$ from 0 in the counterclockwise order, see Fig. 1. Then $D - K$ is the union of $n$ open sectors $U_1, \ldots, U_n$ numbered so that $U_\nu$ is bordered by $R_\nu$ and $R_{\nu + 1}$.

**Proposition 2.2.** For any $F \in \text{Perv}(D, 0)$ we have $H^j_K(F) = 0$ for $i \neq 1$. Therefore the functor 
$$R : \text{Perv}(D, 0) \longrightarrow \text{Sh}_K, \quad F \mapsto R(F) = R_K(F) := H^1_K(F)$$

is an exact functor of abelian categories.

**Proof:** Near a point $x \in K$ other than 0, the graph $K$ is a real codimension 1 submanifold in $D$, and $F$ is a local system in degree 0, so the statement is obvious (“local Poincaré duality”). So we really need only to prove that the space $H^j_{K_n}(F)_0 = H^j_{K_n}(D, F)$ vanishes for $j \neq 1$. The case $n = 0$ is known, $H^1_{K_1}(D, F)$ being identified with $\Phi(F)$, see [GGM]. The general case is proved by induction on $n$. We consider an embedding of $K_n$ into $K_{n + 1}$ so that the new radius $R_{n + 1}$ subdivides the sector $U_n$ into two. This leads to a morphism between the long exact sequences relating hypercohomology with and without support in $K_n$ and $K_{n + 1}$:

$$\cdots \longrightarrow H^j(D - K_n, F) \longrightarrow H^{j + 1}_{K_n}(D, F) \longrightarrow H^{j + 1}(D, F) \longrightarrow \cdots$$

For $j \neq 0$ the map $\alpha_j$ is an isomorphism because its source and target are 0. Indeed, $D - K_n$ as well as $D - K_{n + 1}$ is the union of contractible sectors, and $F$ is a local system in degree 0 outside 0, so the higher cohomology of each sector with coefficients in $F$ vanishes. This means that

$$H^{j + 1}_{K_n}(D, F) = H^{j + 1}_{K_{n + 1}}(D, F) = 0, \quad j \neq 0$$
and so by induction all these spaces are equal to 0. □

**Remark 2.3.**  An alternative proof of the vanishing of $H^{\neq 1}_{K_n}(\mathcal{F})_0$ can be obtained by noticing that $R\Gamma_{K_n}(\mathcal{F})_0[1]$ can be identified with $\Phi_{z^n}(\mathcal{F})$, the space of vanishing cycles with respect to the function $z^n$. It is known that forming the sheaf of vanishing cycles with respect to any holomorphic function preserves perversity.

The graph $K = K_n$ is a regular cellular space with cells being \{0\} and the open rays $R_1, \ldots, R_n$. For $\mathcal{F} \in \text{Perv}(D, 0)$ the sheaf $\mathcal{R}(\mathcal{F})$ is a cellular sheaf on $K$ and as such is completely determined by the linear algebra data of:

1. Stalks at the (generic point of the) cells, which we denote:
   
   $E_0 = E_0(\mathcal{F}_n) := \mathcal{R}(\mathcal{F})_0 = \text{stalk at } 0$

   $E_i = E_i(\mathcal{F}) = \mathcal{R}(\mathcal{F})_{R_i} = \text{stalk at } R_i, \ i = 1, \ldots, n.$

2. Generalization maps corresponding to inclusions of closures of the cells, which we denote

   $\gamma_i : E_0 \longrightarrow E_i, \ i = 1, \ldots, n.$

This gives “one half” of the quiver we want to associate to $\mathcal{F}$.

**C. Cousin complex.** In order to get the second half of the maps (the $\delta_i$), we introduce, by analogy with [KS1], a canonical “Cousin-type” resolution of any $\mathcal{F} \in \text{Perv}(D, 0)$.

Denote by

$i : K \hookrightarrow D, \ j : D - K \hookrightarrow D$

the embeddings of the closed subset $K$ and of its complement $D - K = \bigsqcup_{\nu=1}^n U_{\nu}$. For any complex of sheaves $\mathcal{F}$ on $D$ (perverse or not) we have a canonical distinguished triangle in $D^b \text{Sh}_{D}$:

\[(2.4) \quad i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_* j^* \mathcal{F} \xrightarrow{\delta} i_* i^! \mathcal{F}[1] \]

(here and elsewhere $j_*$ means the full derived direct image). Recalling that $i^!$ has the meaning of cohomology with support, and denoting $j_{\nu} : U_{\nu} \hookrightarrow D$ the embeddings of the connected components of $D - K$, we conclude, from Proposition 2.2.2
Corollary 2.5. Let $\mathcal{F} \in \text{Perv}(D, 0)$. Then $\mathcal{F}$ is quasi-isomorphic (i.e., can be thought of as represented by) the following 2-term complex of sheaves on $D$:

$$\mathcal{E}^\bullet(\mathcal{F}) = \left\{ \bigoplus_{\nu=1}^n j_{\nu*}(\mathcal{F}|_{U_\nu}) \xrightarrow{\delta} \mathcal{R}(\mathcal{F}) \right\}.$$ 

Here the grading of the complex is in degrees 0, 1 and the map $\delta$ is induced by the boundary morphism $\delta$ from (2.4).

We further identify $\mathcal{F}|_{U_\nu} \simeq E_{\nu U_\nu} = \text{constant sheaf on } U_\nu$ with stalk $E_\nu$ as follows. As $\mathcal{F}$ is locally constant (hence constant) on $U_\nu$, it is enough to specify an isomorphism

$$H^0(U_\nu, \mathcal{F}) \xrightarrow{\alpha_\nu} E_\nu = H^1_{K}(\mathcal{F})_x,$$

where $x$ is any point on $R_\nu$. Taking a small disk $V$ around $x$, we have $E_\nu = H^1_{V \cap K}(V, \mathcal{F})$. This hypercohomology with support is identified, via the coboundary map of the standard long exact sequence relating cohomology with and without support), with the cokernel of the map

$$H^0(V, \mathcal{F}) \longrightarrow H^0(V \cap U_\nu, \mathcal{F}) \oplus H^0(V \cap U_{\nu-1}, \mathcal{F}) = H^0(V - K, \mathcal{F})$$

Since $\mathcal{F}$ is constant on $V$, the projection of $H^0(V \cap U_\nu, \mathcal{F}) = H^0(U_\nu, \mathcal{F})$ to the cokernel, i.e., to $E_\nu$, is an isomorphism. We define $\alpha_\nu$ to be this projection.

We now assume $n \geq 2$. Then the closure of $U_\nu$ is a proper closed sector $\overline{U}_\nu$, and so we can rewrite the complex $\mathcal{E}^\bullet(\mathcal{F})$ as

$$\mathcal{E}^\bullet(\mathcal{F}) = \left\{ \bigoplus_{\nu=1}^n E_{\nu \overline{U}_\nu} \xrightarrow{\delta} \mathcal{R}(\mathcal{F}) \right\}.$$ 

Indeed,

$$j_{\nu*}(\mathcal{F}|_{U_\nu}) = j_{\nu*}(E_{\nu U_\nu}) = E_{\nu \overline{U}_\nu}.$$

D. Analysis of the morphism $\delta$. We now analyze the maps of stalks over various points induced by $\delta$. Since the target of $\delta$ is supported on $K$, it is enough to consider two cases:
Stalks over 0. We get maps of vector spaces 
\[ \delta_\nu : (\mathcal{E}_\nu_{U_\nu})_0 = E_\nu \rightarrow \mathcal{R}(\mathcal{F})_0 = E_0, \quad \nu = 1, \cdots, n. \]
These maps, together with the generalization maps \( \gamma_\nu \), form the quiver 
\[ Q = Q(\mathcal{F}) = \{ E_0 \xrightarrow{\gamma_i} E_i \delta_i \} \]
which we associate to \( \mathcal{F} \).

Stalks over a generic point \( x \in R_\nu \). As \( x \) lies in two closed subsets \( U_\nu \) and \( U_{\nu-1} \), we have two maps 
\[ \delta_{U_\nu, R_\nu} : (\mathcal{E}_\nu_{U_\nu})_x = E_\nu \rightarrow \mathcal{R}(\mathcal{F})_x = E_\nu, \]
\[ \delta_{U_{\nu-1}, R_\nu} : (\mathcal{E}_{\nu-1_{U_{\nu-1}}})_x = E_{\nu-1} \rightarrow \mathcal{R}(\mathcal{F})_x = E_\nu. \]

**Proposition 2.6.** We have the following relations: 
\[ \delta_{U_\nu, R_\nu} = \text{Id}_{E_\nu}, \quad \delta_{U_{\nu-1}, R_\nu} = -T_{\nu-1}, \]
where 
\[ T_{\nu-1} : E_{\nu-1} = h^0(U_{\nu-1}, \mathcal{F}) \rightarrow E_\nu = H^0(U_\nu, \mathcal{F}) \]
is the counterclockwise monodromy map for the local system \( \mathcal{F}|_{D\setminus\{0\}} \).

**Proof:** We start with the first relation. Recall that the identification \( \alpha_\nu : H^0(U_\nu, \mathcal{F}) \rightarrow E_\nu \) was defined in terms of representation of \( E_\nu \) as a quotient, i.e., in terms of the coboundary map in the LES relating hypercohomology with and without support. So the differential \( \delta \) in \( \mathcal{E}^* (\mathcal{F}) \), applied to a section \( s \in H^0(U_\nu, \mathcal{F}) \), gives precisely \( \alpha_\nu (s) \), so after the identification by \( \alpha_\nu \), the map on stalks over \( x \in R_\nu \), becomes the identity.

We now prove the second relation. Representing \( e \in E_{\nu-1} \) by a section \( s \in H^0(U_{\nu-1}, \mathcal{F}) \), we see that \( \delta_{U_{\nu-1}, R_\nu} (e) \) is represented by the image of 
\[ (0, s) \in H^0(U_\nu, \mathcal{F}) \oplus H^0(U_{\nu-1}, \mathcal{F}) \]
in the quotient 
\[ (H^0(U_\nu, \mathcal{F}) \oplus H^0(U_{\nu-1}, \mathcal{F}))/H^0(V, \mathcal{F}). \]
But the identification of this quotient with \( E_\nu \) is via the projection to the first, not second, summand, i.e., to \( H^0(V \cap U_\nu, \mathcal{F}) = H^0(U_\nu, \mathcal{F}) \). The element \( (t, 0) \) projecting to the same element of the quotient as \( (0, s) \), has \( t = -T_{\nu-1}(s) = \text{minus} \) the analytic continuation of \( s \) to \( U_\nu \).  \( \square \)
Proposition 2.7. The maps $\gamma_\nu, \delta_\nu$ in the diagram $Q(\mathcal{F})$ satisfy the conditions (C1)-(C3), i.e., $Q(\mathcal{F})$ is an object of the category $A_n$.

Proof: We spell out the conditions that the differential $\delta$ in the Cousin complex $\mathcal{E}^\bullet(\mathcal{F})$ is a morphism of sheaves. More precisely, both terms of the complex are cellular sheaves on $D$ with respect to the regular cell decomposition given by $0, R_1, \ldots, R_n, U_1, \ldots, U_n$. So the maps of the stalks induced by $\delta$ must commute with the generalization maps.

Consider the generalization maps from $0$ to $R_\nu$. In the following diagram the top row is the stalk of the complex $\mathcal{E}^\bullet(\mathcal{F})$ over $0$, the bottom row is the stalk over $R_\nu$, and the vertical arrows are the generalization maps:

$$
\begin{array}{ccc}
\bigoplus_{\mu=1}^n E_\mu & \xrightarrow{\sum \delta_\mu} & E_0 \\
p_{\nu, \nu-1} & & \downarrow \gamma_\nu \\
E_\nu \oplus E_{\nu-1} & \xrightarrow{\Id - T_{\nu-1}} & E_\nu,
\end{array}
$$

the lower horizontal arrow having been described in Proposition 2.6. We now spell out the condition of commutativity on each summand $E_\mu$ inside $\bigoplus_{\mu=1}^n E_\mu$.

- **Commutativity on $E_\nu$:** this means $\gamma_\nu \delta_\nu = \Id$.
- **Commutativity on $E_{\nu-1}$:** this means $\gamma_\nu \delta_{\nu-1} = -T_{\nu-1}$, in particular, this composition is an isomorphism.
- **Commutativity on $E_\mu$ for $\mu \neq \nu, \nu - 1$:** This means that $\gamma_\nu \delta_\mu = 0$, since the projection $p_{\nu, \nu-1}$ annihilates $E_\mu$. The proposition is proved.

E. $Q(\mathcal{F})$ and duality. Recall that the category Perv($D, 0$) has a perfect duality

$$
(2.8) \quad \mathcal{F} \mapsto \mathcal{F}^\star = \mathbb{D}(\mathcal{F})[2],
$$

i.e., the shifted Verdier duality normalized so that for $\mathcal{F}$ being a local system (in our case, constant sheaf) in degree 0, we have that $\mathcal{F}^\star$ is the dual local system in degree 0. We will use the notation (2.8) also for more general complexes of sheaves on $D$.

On the other hand, the category $A_n$ also has a perfect duality

$$
Q = \left\{ E_0 \xrightarrow{\gamma_\nu} E_\nu \right\}_{\nu=1}^n \quad \mapsto \quad Q^\star = \left\{ E_0^\star \xrightarrow{\delta_\nu} E_\nu^\star \right\}_{\nu=1}^n.
$$
Proposition 2.9. The functor \( Q : \text{Perv}(D, 0) \to \mathcal{A}_n \) commutes with duality, i.e., we have canonical identifications \( Q(\mathcal{F}^\bullet) \simeq Q(\mathcal{F})^* \).

Proof: We modify the argument of [KS1], Prop. 4.6. That is, we think of \( K \) as consisting of \( n \) “equidistant” rays \( R_\nu \), joining \( 0 \) with \( \zeta^{\nu-1} \), \( \zeta = e^{2\pi i/n} \), \( \nu = 1, 2, \ldots, n \).

We consider another star-shaped graph \( K' \) formed by the radii \( R_1', \ldots, R_n' \) so that \( R_\nu' \) is in the middle of the sector \( U_\nu \). Thus, the rotation by \( e^{i\pi/n} \) identifies \( R_\nu \) with \( R_\nu' \) and \( K' \) with \( K \).

We can use \( K' \) instead of \( K \) to define \( R(\mathcal{F}) \) and \( R(\mathcal{F}^\bullet) \). We will denote the corresponding sheaves \( R_{K'}(\mathcal{F}) = H^1_{K'}(\mathcal{F}) \), and similarly for \( R_{K'}(\mathcal{F}^\bullet) \).

Since Verdier duality interchanges \( i^! \) and \( i^\ast \) (for \( i : K' \to D \) being the embedding), we have

\[
R_{K'}(\mathcal{F}^\bullet)^\ast \simeq \mathcal{F}|_{K'}
\]

(usual restriction). To calculate this restriction, we use the Cousin resolution of \( \mathcal{F} \) defined by using \( K \) and the \( U_\nu \):

\[
\mathcal{F} \simeq \mathcal{E}^\bullet = \left\{ \bigoplus_{\nu=1}^n E_{\nu, U_\nu} \xrightarrow{\delta} R_K(\mathcal{F}) \right\}.
\]

So we restrict \( \mathcal{E}^\bullet \) to \( K' \). Since \( K' \cap K = \{0\} \) and \( R_K(\mathcal{F}) \) is supported on \( K \), the restriction \( R_K(\mathcal{F})|_{K'} = E_{0, 0} \) is the skyscraper sheaf at 0 with stalk \( E_0 \).

So

\[
\mathcal{F}|_{K'} \simeq \left\{ \bigoplus_{\nu=1}^n (E_{\nu, U_\nu})_{R_\nu'} \xrightarrow{\delta'} \sum_{\nu=1}^n \delta_\nu (E_0)_{0} \right\}.
\]

On the other hand, the shifted Verdier dual to \( R_{K'}(\mathcal{F}^\bullet) \), as a sheaf on \( K' \) is identified by, e.g., [KS1], Prop. 1.11 with the complex of sheaves

\[
\left\{ \bigoplus_{\nu \subset K'} E_{\nu} (\mathcal{F}^\bullet)^\ast \otimes \text{or}(C) \right\}.
\]

Here \( C \) runs over all cells of the cell complex \( K' \), and \( E_{\nu} (\mathcal{F}^\bullet)^\ast \) is the stalk of the cellular sheaf \( R_{K'}(\mathcal{F}^\bullet) \) at the cell \( C \). Explicitly, \( C \) is either 0 or one of the \( R'_{\nu} \), so

\[
R_{K'}(\mathcal{F}^\bullet)^\ast = \left\{ \bigoplus_{\nu=1}^n E_{\nu} (\mathcal{F}^\bullet)^\ast \xrightarrow{\delta} \sum_{\nu=1}^n \delta_\nu (E_0)_{0} \right\}.
\]
By the above, this complex is quasi-isomorphic to
\[
\mathcal{F}|_{K'} = \left\{ \bigoplus_{\nu=1}^{n} E_{\nu}(\mathcal{F}) \xrightarrow{\sum \delta_{\nu}} E_{0}(\mathcal{F})_{0} \right\}.
\]
So we conclude that
\[
E_{\nu}(\mathcal{F}^{\bullet}) = E_{\nu}(\mathcal{F})^{*}, \quad \gamma_{\nu}^{\bullet} = (\delta_{\nu}^{\mathcal{F}})^{*}.
\]
This proves the proposition.

**Proof of Theorem 2.1** We already have the functor
\[
Q : \text{Perv}(D, 0) \rightarrow \mathcal{A}_{n}, \quad \mathcal{F} \mapsto Q(\mathcal{F}).
\]
Let us define a functor \( \mathcal{E} : \mathcal{A}_{n} \rightarrow D^{b}\text{Sh}_{D} \). Suppose we are given
\[
Q = \left\{ E_{0} \xrightarrow{\gamma_{\nu}} E_{\nu} \right\}_{\nu=1}^{n} \in \mathcal{A}_{n}.
\]
We associate to it the Cousin complex
\[
\mathcal{E}^{\bullet}(Q) = \left\{ \bigoplus_{\nu=1}^{n} E_{\nu}(\mathcal{F})_{\nu} \xrightarrow{\delta} \mathcal{R}(Q) \right\}.
\]
Here \( \mathcal{R}(Q) \) is the cellular sheaf on \( K \) with stalk \( E_{0} \) at 0, stalk \( E_{\nu} \) at \( R_{\nu} \) and the generalization map from 0 to \( R_{\nu} \) given by \( \gamma_{\nu} \). The map \( ul\delta \) is defined on the stalks as follows:

**Over 0:** the map
\[
(E_{\nu}(\mathcal{F}))_{0} = E_{\nu} \rightarrow \mathcal{R}(Q)_{0} = E_{0}
\]
is given by \( \delta_{\nu} : E_{\nu} \rightarrow E_{0} \).

**Over \( R_{\nu} \):** The map
\[
\left( \bigoplus_{\mu=1}^{n} E_{\mu}(\mathcal{F}) \right)_{R_{\nu}} = E_{\nu} \oplus E_{\nu-1} \rightarrow \mathcal{R}(Q)_{R_{\nu}} = E_{\nu}
\]
is given by
\[
\text{Id} - T_{\nu-1} : E_{\nu} \oplus E_{\nu-1} \rightarrow E_{\nu}.
\]
Reading the proof of Proposition 2.6 backwards, we see that the conditions (C1)-(C3) mean that in this way we get a morphism \( \delta \) of cellular sheaves on \( D \), so \( \mathcal{E}^{\bullet}(Q) \) is an object of \( D^{b}\text{Sh}_{D} \).

Further, similarly to Proposition 2.9 we see that \( \mathcal{E}^{\bullet}(Q^{*}) \simeq \mathcal{E}(Q)^{*} \).
**Proposition 2.10.** \( \mathcal{E}^\bullet(Q) \) is constructible with respect to the stratification \((\{0\}, D - \{0\})\) and is perverse.

**Proof:** Constructibility. It is sufficient to prove the following:

(a) The sheaf \( H^0(\mathcal{E}^\bullet(Q))|_{D-\{0\}} \) is locally constant.

(b) The sheaf \( H^1(\mathcal{E}^\bullet(Q))|_{D-\{0\}} \) is equal to 0.

To see (a), we look at the map of stalks over \( R_\nu \):

\[
\text{Id} - T_{\nu-1} : E_\nu \oplus E_{\nu-1} \to E_\nu
\]
given by the differential \( \delta \) in \( \mathcal{E}^\bullet(Q) \). So, by definition, \( H^0(\mathcal{E}^\bullet(Q))_{R_\nu} \) and \( H^1(\mathcal{E}^\bullet(Q))_{R_\nu} \) are the kernel and cokernel of this map.

Now, since \( T_{\nu-1} \) is an isomorphism, \( \text{Ker}(\text{Id} - T_{\nu-1}) \) projects to both \( E_\nu \) and \( E_{\nu-1} \) isomorphically. This means that \( H^0(\mathcal{E}^\bullet(Q)) \) is locally constant over \( R_\nu \); the stalk at \( R_\nu \) projects (“generalizes”) to the stalks at \( U_\nu \) and \( U_{\nu-1} \) in an isomorphic way.

To see (b), we notice that \( \text{Id} - T_{\nu-1} \) is clearly surjective and so \( H^1(\mathcal{E}^\bullet(Q))_{R_\nu} = 0 \). Since \( \mathcal{E}^1(Q) = \mathcal{R}(Q) \) is supported on \( K \), this means that \( H^1(\mathcal{E}^\bullet(Q)) \) is supported at 0, and its restriction to \( D - \{0\} \) vanishes. So \( H^1(\mathcal{E}^\bullet(Q)) \) are \( \mathbb{C} \)-constructible as claimed.

Perversity. By the above, \( \mathcal{E}^\bullet(Q) \) is semi-perverse, i.e., lies in the non-positive part of the perverse t-structure, that is, \( H^i(\mathcal{E}^\bullet(Q)) \) is supported on complex codimension \( \geq i \). Further, \( \mathcal{E}^\bullet(Q)^\bullet \) also satisfies the same semi-perversity since it is identified with \( \mathcal{E}^\bullet(Q^*) \). This means that \( \mathcal{E}^\bullet(Q) \) is fully perverse. Proposition 2.10 is proved.

It remains to show that the functors

\[
Perv(D, 0) \xrightarrow{Q} \mathcal{A}_n
\]

are quasi-inverse to each other. This is done in a way completely parallel to [KS1], Prop. 6.2 and Lemma 6.3 (“orthogonality relations”). Theorem 2.1 is proved. □

See Appendix for some further study of the categories \( \mathcal{A}_n \).
3 The graph description of perverse sheaves on an oriented surface

A. Generalities. The purity property. Let $S$ be a compact topological surface, possibly with boundary $\partial S$; we denote $S^\circ = S - \partial S$ the interior. Let $N \subset S^\circ$ be a finite subset. We then have the category $\text{Perv}(S, N)$ formed by perverse sheaves of $k$-vector spaces on $S$, smooth outside $N$.

By a graph we mean a topological space obtained from a finite 1-dimensional CW-complex by removing finitely many points. Thus we do allow edges not terminating in a vertex on some side ("legs"), as well as 1-valent and 2-valent vertices as well as loops. For a vertex $x$ of a graph $K$ we denote by $H(x)$ the set of half-edges incident to $x$. We can, if we wish, consider any point $x \in K$ as a vertex: if it lies on an edge, we consider it as a 2-valent vertex, so $H(x)$ is this case is the set of the two orientations of the edge containing $x$. Further, for a graph $K$ we denote by $\text{Vert}(K)$ and $\text{Ed}(K)$ the sets of vertices and edges of $K$.

We denote by $\mathcal{C}_K$ the cell category of $K$ defined as follows. The set $\text{Ob}(\mathcal{C}_K)$ is $\text{Vert}(K) \sqcup \text{Ed}(K)$ ("cells"). Non-identity morphisms can exist only between a vertex $x$ and an edge $e$, and

$$\text{Hom}(x, e) = \{ \text{half-edges } h \in H(x) \text{ contained in } e \}.$$ 

So $|\text{Hom}(x, e)|$ can be 0, 1 or 2 (the last possibility happens when $e$ is a loop beginning and ending at $x$). If $K$ has no loops, then $\mathcal{C}_K$ is a poset. We denote by $\text{Rep}(\mathcal{C}_K) = \text{Fun}(\mathcal{C}_K, \text{Vect}_k)$ the category of representations of $\mathcal{C}_K$ over $k$.

**Proposition 3.1.** The category of cellular sheaves on $K$ is equivalent to $\text{Rep}(\mathcal{C}_K)$.

**Proof:** This is a particular case of general statement [1] which describes constructible sheaves on any stratified space in terms of representations of the category of exit paths. 

Let now $K \subset S$ be any embedded graph (possibly passing through some points of $N$). We allow 1-valent vertices of $K$ to be situated inside $S$, as well as on $\partial S$.

**Proposition 3.2.** For $\mathcal{F} \in \text{Perv}(S, N)$ we have $\mathbb{H}^i_K(\mathcal{F}) = 0$ for $i \neq 1$.
Proof: This follows from Proposition 2.2. Indeed, the statement is local on \( S \), and the graph \( K \) is modeled, near each of its points, by a star shaped graph in a disk. 

We denote 
\[
\mathcal{R}(\mathcal{F}) = \mathcal{R}_K(\mathcal{F}) := H^1_K(\mathcal{F}),
\]
this is a cellular sheaf on \( K \).

B. Spanning ribbon graphs. From now on we assume that \( S \) is oriented. Graphs embedded into \( S \) have, therefore, a canonical ribbon structure, i.e., a choice of a cyclic ordering on each set \( H(x) \). See, e.g., [DK] for more background on this classical concept.

For a ribbon graph \( K \) we have a germ of an oriented surface with boundary \( \text{Surf}(K) \) obtained by thickening each edge to a ribbon and gluing the ribbons at vertices according to the cyclic order. In the case of a 1-valent vertex \( x \) we take the ribbon to contain \( x \), so that \( x \) will be inside \( \text{Surf}(K) \).

By a spanning graph for \( S \) we mean a graph \( K \), embedded into \( S^\circ \) as a closed subset, such that the closure \( \overline{K} \subset S \) is a graph embedded into \( S \), and the embedding \( K \hookrightarrow S^\circ \) is a homotopy equivalence. Thus we allow for legs of \( K \) to touch the boundary of \( S \).

C. The category associated to a ribbon graph.

Definition 3.3. Let \( K \) be a graph, and \( C_K \) be its cell category. By a double representation of \( C_K \) we mean a datum \( Q \) of:

1. For each \( x \in \text{Vert}(K) \), a vector space \( E_x \).
2. For each \( e \in \text{Ed}(K) \), a vector space \( E_e \).
3. For each half-edge \( h \) incident to a vertex \( x \) and an edge \( e \), linear maps 
\[
E_x \xrightarrow{\gamma_h} E_e.
\]

Let \( \text{Rep}^{(2)}C_K \) be the category of double representations of \( C_K \).

Let now \( K \) be a ribbon graph. Denote by \( A_K \) the full subcategory in \( \text{Rep}^{(2)}C_K \) formed by double representations \( Q = (E_x, E_e, \gamma_h, \delta_h) \) such that for each vertex \( x \in K \) the following conditions are satisfied (depending on the valency of \( x \):
• If \( x \) is 1-valent, then we require:

\[
(C_x) \quad \text{Id}_{E_e} - \gamma_h \delta_h : E \to E \text{ is an isomorphism.}
\]

• If the valency of \( x \) is \( \geq 2 \), then we require:

\[
(C1_x) \quad \text{For each half-edge } h \text{ incident to } x, \text{ we have } \gamma_h \delta_h = \text{Id}_{E_e}.
\]

\[
(C2_x) \quad \text{Let } h, h' \text{ be any two half-edges incident to } x \text{ such that } h' \text{ immediately follows } h \text{ in the cyclic order on } H(x). \text{ Let } e, e' \text{ be the edges containing } h, h'. \text{ Then } \gamma_{h'} \delta_h : E_e \to E_{e'} \text{ is an isomorphism.}
\]

\[
(C3_x) \quad \text{If } h, h' \text{ are two half edges incident to } x \text{ such that } h \neq h' \text{ and } h' \text{ does not immediately follow } h, \text{ then } \gamma_{h'} \delta_h = 0.
\]

**Example 3.4.** If \( K = K_n \) is a “ribbon corolla” with one vertex and \( n \) legs, then \( \mathcal{A}_K = \mathcal{A}_n \) is the category from \( \S 2 \). 

**D. Description of \( \text{Perv}(S, N) \) in terms of spanning graphs.** Let \( S \) be an oriented surface, \( K \subset S \) be a spanning graph and \( N = \text{Vert}(K) \). For \( F \in \text{Perv}(S, N) \) we have the sheaf \( R_K(F) \) on \( K \), cellular with respect to the cell structure given by the vertices and edges. Therefore by Proposition 3.1 it gives the representation of \( \mathcal{C}_K \) which, explicitly, consisting of:

- The stalks \( E_x, E_e \) at the vertices and edges of \( K \). We write \( E_x(F), E_e(F) \) if needed.
- The generalization maps \( \gamma_h : E_x \to E_e \) for any incidence, i.e. half-edge \( h \) containing \( x \) and contained in \( e \).

**Proposition 3.5.** For the Verdier dual perverse sheaf \( F^\bullet \) we have canonical identifications

\[
E_x(F^\bullet) \simeq E_x(F)^*, \quad E_e(F^\bullet) \simeq E_e(F)^*.
\]

**Proof:** Follows from the local statement for a star shaped graph in a disk, Prop. 2.9

So we define

\[
\delta_h = (\gamma_h^\bullet)^* : E_e \to E_x.
\]
Theorem 3.6. (a) The data $Q = Q(F) = (E_x, E_e, \gamma_h, \delta_h)$ form an object of the category $\mathcal{A}_K$.

(b) If $K$ is a spanning graph for $S$, and $N = \text{Vert}(K)$, then the functor

$$Q_K : \text{Perv}(S, N) \to \mathcal{A}_K, \quad F \mapsto Q(F)$$

is an equivalence of categories.

Proof: (a) The relations $(C1_x)$-$$(C3_x)$ resp. $(C_x)$ defining $\mathcal{A}_K$, are of local nature, so they follow from the local statement (Proposition 2.6) about a star shaped graph in a disk.

(b) This is obtained by gluing the local results (Theorem 2.1). More precisely, perverse sheaves smooth outside $N$, form a stack $\mathfrak{P}$ of categories on $S$. We can assume that $S = \text{Surf}(K)$, so $\mathfrak{P}$ can be seen as a stack on $K$, and $\text{Perv}(S, N) = \Gamma(K, \mathfrak{P})$ is the category of global sections of this stack. Similarly, $\mathcal{A}_K$ also appears as $\Gamma(K, \mathfrak{A})$, where $\mathfrak{A}$ is the stack of categories on $K$ given by $K' \mapsto \mathcal{A}_K'$ (here $K'$ runs over open subgraphs of $K$). Our functor $Q$ comes from a morphism of stacks $Q : \mathfrak{P} \to \mathfrak{A}$, so it is enough to show that $Q$ is an equivalence of stacks. This can be verified locally, at the level of stalks at arbitrary points $x \in K$, where the statement reduces to Theorem 2.1.

Cf. [KS1], §9B for a similar argument.

D. Polarized sheaves. Let us call a polarized space an object $E \in \text{Vect}(k)$ equipped with a nondegenerate $k$-bilinear form

$$\langle \ , \ \rangle : E \times E \to k,$$

not necessarily symmetric. A linear map $f : E \to E'$ between polarized spaces has two adjoints: the left and the right one, $^tf, f^\top : E' \to E$, defined by

$$\langle ^tf(x), y \rangle = \langle x, f(y) \rangle; \quad \langle y, f^\top(x) \rangle = \langle f(y), x \rangle.$$

Polarized spaces give rise to several interesting geometric structures motivated by category theory, see [Bon].

Let us call a polarized perverse sheaf over $S$ an object $\mathcal{F} \in \text{Perv}(S, N)$ equipped with an isomorphism with its Verdier dual $B : \mathcal{F} \to \mathcal{F}^\ast$. This concept can be compared with that of [S]; however we do not require any symmetry of $B$. Polarized perverse sheaves on $S$ with singularities in $N$ form a category $\text{PolPerv}(S, N)$ whose morphisms are morphisms of perverse sheaves commuting with the isomorphisms $B$. 

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Definition 3.7. Given a ribbon graph $K$, we define the category $p\mathcal{A}_K$ (resp. $A^p_K$) whose objects are collections $Q = (E_x, E_e, \gamma_h, \delta_h)$ as in $\mathcal{A}_K$, together with an additional data of polarizations on all spaces $E_e, E_x$, subject to an additional condition: for each $h$, we have $\delta_h = \gamma_h^\top$ (resp. $\delta_h = \gamma_h^\top$).

Since by definition the equivalence $Q_K$ from Theorem 3.6 commutes with the duality, we obtain:

**Corollary 3.8.** If $K$ is a spanning graph for $S$ and $N = \text{Vert}(K)$, then we have two equivalences

$$pQ_K : \text{PolPerv}(S, N) \sim \rightarrow p\mathcal{A}_K; \quad Q^p_K : \text{PolPerv}(S, N) \sim \rightarrow A^p_K. \quad \square$$

### A Appendix. Coboundary actions and helices

Since the same marked surface $(S, N)$ has many spanning graphs $K \supset N$, the corresponding categories $\mathcal{A}_K$ are all equivalent to each other, being identified with $\text{Perv}(S, N)$. One can continue the analysis of this paper by constructing a system of explicit identifications $\mathcal{A}_K \rightarrow \mathcal{A}_K'$ for pairs of spanning graphs $K, K' \supset N$ connected by “elementary moves”, in the spirit of [DK]. To keep the paper short, we do not do it here, but discuss a local aspect of this issue: the action of $\mathbb{Z}/n$ on $\mathcal{A}_n$.

Let $G$ be a discrete group which we consider as a category with one object $pt$. Recall (see, e.g., [De], [GK]) that a category with $G$-action, or a categorical representation of $G$ is a lax 2-functor $F : G \rightarrow \text{Cat}$ from $G$ to the 2-category of categories.

Explicitly, it consists of the following data (plus the data involving the unit of $G$, see [GK]):

1. A category $\mathcal{C} = F(pt)$.
2. For each $g \in G$, a functor $g_* = F(g) : \mathcal{C} \rightarrow \mathcal{C}$.
3. For each $h, g \in G$, an isomorphism of functors $\alpha_{h,g} = F(h,g) : h_*g_* \Rightarrow (hg)_*$.
4. It is required that for any three elements $h, g, f \in G$ the square

$$
\begin{array}{ccc}
h_*g_*f_* & \xrightarrow{\alpha_{h,g}} & (hg)_*f_* \\
\alpha_{g,f} \downarrow & & \downarrow \alpha_{hg,f} \\
h_*(gf)_* & \xrightarrow{\alpha_{h,gf}} & (hg f)_*
\end{array}
$$


is commutative, i.e.
\[ \alpha_{h,g,f} = \alpha_{h,g} \alpha_{f}. \]

**Example A.1.** If \( F(g) = \text{Id}_C \) for all \( g \in G \), then \( F \) is the same as a 2-cocycle \( \alpha \in Z^2(G; Z(C)) \) where \( Z(C) \) is the center of \( C := \text{the group of automorphisms of the identity functor Id}_C \), the action of \( G \) on \( Z(C) \) being trivial.

We say that the action is **strict** if \( (gf)_* = g_* f_* \), and \( \alpha_{g,f} = \text{Id}_{g_* f_*} \) for all composable \( g, f \). In other words, a strict action is simply a group homomorphism \( F : G \to \text{Aut}(C) \).

All actions of \( G \) on a given category \( C \) form themselves a category, denoted \( \text{Act}(G, C) \). It has a distinguished object \( I \), the trivial action, with all \( g_* = \text{Id}_C \) and all \( \alpha_{h,g} = \text{Id} \). Given an action \( F \) as above, a **coboundary structure** on \( F \) is an isomorphism \( \beta : I \to F \) in \( \text{Act}(G, C) \). Explicitly, it consists of:

- A collection of natural transformations \( \beta_g : \text{Id}_C \xrightarrow{\sim} g_* \), given for all \( g \in G \) such that
- For each \( g, f \in G \) the square
  \[
  \begin{array}{ccc}
  \text{Id}_C & \xrightarrow{\beta_f} & f_* \\
  \beta_{gf} & \downarrow & \downarrow \beta_g \\
  (gf)_* & \xrightarrow{\alpha_{g,f}^{-1}} & g_* f_*
  \end{array}
  \]
  is commutative, in other words, \( \alpha_{g,f} = \beta_{gf} \beta_f^{-1} \beta_g^{-1} \).

**Examples A.2.** (a) In the situation of Example A.1, a coboundary structure on \( F \) is the same as a 1-cochain
\[
\beta \in C^1(G; Z(C)) = \text{Hom}_{\text{Set}}(G, Z(C)), \quad d \beta = \alpha.
\]

(b) If our action is strict, then a coboundary structure on it is a collection of natural transformations \( \{ \beta_g \} \) as above, such that \( \beta_{gf} = \beta_g \beta_f \).

Returning now to the situation of §2, we have a strict action of \( Z/n \) on \( A_n \) such that \( k \in Z/n \) acts by rotation by \( 2\pi k/n \). More precisely, for
\[
x = (E_0, E_1, \ldots, E_n; \gamma_i, \delta_i) \in \text{Ob}(A_n)
\]
we define
\[
k_* x = (E_0, E_1 + k, \ldots, E_{n+k}; \gamma_{i+k}, \delta_{i+k}), \quad k \in Z/n
\]
where the indices (except for \( E_0 \)) are understood modulo \( n \).
Proposition A.3. The strict action of \( \mathbb{Z} \) on \( \mathcal{A}_n \) induced by the composition
\[
\mathbb{Z} \to \mathbb{Z}/n \to \text{Aut}(\mathcal{A}_n)
\]
is coboundary.

Proof: For \( x \in \mathcal{A}_n \) as above we have an arrow \( \beta_1(x) : x \to 1_* x \) in \( \mathcal{A}_n \), induced by the fractional monodromies
\[
T_i = \gamma_{i+1} \delta_i : E_i \to E_{i+1}, \ i \in \mathbb{Z}/n,
\]
which give rise to natural isomorphisms \( \beta_1 : \text{Id}_{\mathcal{A}_n} \to 1_* \) (here \( 1 \in \mathbb{Z}/n \) is the generator). More generally, putting \( \beta_k := (\beta_1)^k, \ k \in \mathbb{Z} \), we get a coboundary structure on the composed action. \( \square \)

Note that in particular the global monodromy \( T = \beta_n \) is a natural transformation \( \text{Id}_{\mathcal{A}_n} \to \text{Id}_{\mathcal{A}_n} \), so it is an element of \( Z(\mathcal{A}_n) \). For an object \( x \in \mathcal{A}_n \) the sequence
\[
\cdots (-1)_* x, x, 1_* x, \cdots, n_* x, \cdots
\]
can be seen as a decategorified analog of a helix, see \[BP\], with the monodromy \( T \) playing the role of the Serre functor.

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