Full counting statistics for noninteracting fermions: Exact results and the Levitov-Lesovik formula

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Exact numerical results for the full counting statistics (FCS) for a one-dimensional tight-binding model of noninteracting electrons are presented without using an idealized measuring device. The two initially separate subsystems are connected at \( t = 0 \) and the exact time evolution for the large but finite combined system is obtained numerically via a finite dimensional determinant. Even for surprisingly short times the approximate description of the time evolution with the help of scattering states agrees well with the exact result for the local current matrix elements and the FCS. An additional approximation has to be made to recover the Levitov-Lesovik formula in the limit where the system size becomes infinite and afterwards the long time limit is addressed. The new derivation of the Levitov-Lesovik formula is generalized to more general geometries like a Y-junction enclosing a magnetic flux.

I. INTRODUCTION

The theory of noise in quantum transport in mesoscopic systems is a very active field of research. In addition to the first few moments of the transmitted charge the full probability distribution can be studied, called full counting statistics (FCS). Neglecting the electron-electron interaction Levitov and Lesovik presented an analytical result in the long time limit using the cumulant generating function. It was later questioned pointing out that different measurement procedures may lead to different answers for the distribution function. Recently also results for models including the e-e-interaction locally were obtained.

For the case of perfect transmission the Levitov-Lesovik formula yields at zero temperature a delta function for the probability distribution \( w(t, \mathcal{Q}) \) of the transmitted charge with the position increasing linearly with time. This is in obvious contradiction to the fact that \( w \) should only have weight at integer values. In order to obtain the correct answer also in this limit it is desirable to start from a formally exact expression. Such a starting point was presented by Klich with the help of a formula to calculate the trace over Fock space for products of exponentials of one-particle operators.

Klich’s formula can be viewed as the general expression for the counting statistics of noninteracting fermions, at any given time, and without any approximations. Nevertheless Klich’s attempt to derive the Levitov-Lesovik formula from his Eq. (13) is not really satisfying. He performs the infinite time limit to introduce the scattering matrix and notices that therefore the information about the length of the time interval has to be put in by hand. As an alternative Muzykantskii and Adamov start from Klich’s formula and use results from the theory of singular integral equations to proceed. They restrict themselves to zero temperature and switch on the bias adiabatically. For the special case of an energy independent transmission probability they recover the \( T = 0 \) version of the Levitov-Lesovik formula from the solution of a matrix Riemann-Hilbert problem. From the approximate solution of a more difficult Riemann-Hilbert problem they also obtain the subleading correction for their special case. An extension of this approach to finite temperatures was presented by Braun.

The present paper takes a new look at the problem in several ways. We first describe the “experimental” setup and its formal theoretical description at zero and finite temperatures. For noninteracting fermions the central object is the time dependent projection operator \( P_L(t) \) onto one of the subsystems in the Hilbert space of one particle. By numerically calculating \( P_R(t) \) in section III we present exact results for \( w(t, \mathcal{Q}) \) at \( T = 0 \) for a one-dimensional tight-binding model. For perfect transmission the distribution has a nonzero width increasing logarithmically with time. In section IV we introduce scattering states to describe the long time limit. By using the current operator at the connection of the subsystems we obtain an excellent approximation for \( P_L(t) \). The new derivation of the Levitov-Lesovik formula from Klich’s exact formal expression for arbitrary temperatures presented in section V clearly shows the approximations involved. We also generalize the derivation to more complex geometries like a Y-junction which encloses a magnetic flux. In order to make the paper self-contained we present an alternative proof of Klich’s formula using Wick’s theorem in Appendix A.

II. COUNTING STATISTICS FOR NONINTERACTING ELECTRONS

A. Zero temperature limit

In the following we consider a system which consists of two initially separate subsystems described by the Hamiltonians \( H_{0,a} \) with \( a = L, R \). The labels stand for the “left” and “right” one-dimensional subsystems treated later. The number of electrons in the subsystems in the initial state are \( N_{0,a} \) and the total number is given by \( N_{\text{tot}} = N_{0,L} + N_{0,R} \). We assume the initial state \( |\Phi(0)\rangle \)
to be an eigenstate of the $H_{0,a}$

$$|\Phi(0)\rangle = |E^{N_{0,L}}_m\rangle \otimes |E^{N_{0,R}}_n\rangle .$$  \hspace{1cm} (1)$$

The time evolution for times greater than zero is described by the Hamiltonian

$$H = H_{0,L} + H_{0,R} + V_{LR} \equiv H_0 + V_{LR} .$$  \hspace{1cm} (2)$$

The term $V_{LR}$ which couples the two subsystems will be specified later. The probability distribution that $Q$ electrons are transferred to the right system after time $t$ is given by

$$w_R(t,Q) = \langle \Phi(t)|\delta(Q - (N_R - N_{0,R})|\Phi(0)\rangle$$

$$= \langle \Phi(0)|\delta(Q - (N_R(t) - N_{0,R})|\Phi(0)\rangle .$$  \hspace{1cm} (3)$$

Here $N_R$ is the particle number operator of the right system and $N_R(t)$ is the corresponding operator in the Heisenberg picture. This probability distribution can be measured (conceptually simple by “weighing”) when the connection between the subsystems is cut again at time $t$.

The integral representation of the delta function yields a form more convenient for the actual calculation

$$w_R(t,Q) = \frac{1}{2\pi} \int d\lambda e^{-i\lambda Q} g_R(t,\lambda) ,$$  \hspace{1cm} (4)$$

where the information about the FCS is encoded in the characteristic function

$$g_R(t,\lambda) = \langle \Phi(0)|e^{i\lambda N_R(t)}|\Phi(0)\rangle e^{-i\lambda N_{0,R}} .$$  \hspace{1cm} (5)$$

This result is valid also for interacting electrons.

The evaluation simplifies considerably for noninteracting electrons as the expectation value in Eq. (5) can be expressed as a $N_{0,R} \times N_{0,R}$ determinant in the space of the initially occupied one-particle states. As $N_R(t)$ is a one-particle operator $e^{i\lambda N_R(t)}$ acts on the one-particle states occupied in the initial Slater determinant as $e^{i\lambda P_R(t)}$ where $P_R(t)$ is the time dependent projection operator

$$P_R(t) = e^{iht}P_Re^{-iht} = P^2_R(t)$$  \hspace{1cm} (6)$$

with $P_R$ the projection operator in the Hilbert space of one particle onto the states in the right reservoir

$$P_R = \sum_{j} |k_j^R\rangle \langle k_j^R| = \sum_{m} |m, R\rangle \langle m, R|$$  \hspace{1cm} (7)$$

and $h$ is the full Hamiltonian in the Hilbert space of a single particle. The $|k_j^R\rangle$ describe standing waves in the right reservoir and the $m$ in $|m, R\rangle$ denote the site indices.

If we introduce the projection operator $\tilde{n}_0$

$$\tilde{n}_0 = \sum_{k,a} |k^a\rangle f_{k,a} (k^a) \equiv \bar{n}_{0,L} + \bar{n}_{0,R} ,$$

where the $f_{k,a}$ are the ($T = 0$) Fermi functions of the subsystems in the initial state the operator

$$\tilde{P}_R(t) \equiv \tilde{n}_0 P_R(t) \tilde{n}_0 .$$  \hspace{1cm} (8)$$

determines the zero temperature FCS. Using $e^{i\lambda P_R(t)} = 1 + (e^{i\lambda} - 1)P_R(t)$ we obtain

$$g_R(t,\lambda) = e^{-i\lambda N_{0,R}} det_{\tilde{n}_0} [\tilde{n}_0 + (e^{i\lambda} - 1)\tilde{P}_R(t)] ,$$  \hspace{1cm} (9)$$

where $det_{\tilde{n}_0}$ is a determinant in the subspace of the initially occupied one particle states. If we denote the eigenvalues of $\tilde{P}_R(t)$ by $p_m(t)$ the characteristic function takes the form

$$g_R(t,\lambda) = e^{-i\lambda N_{0,R}} \prod_{m=1}^{N_{0,R}} \left[ 1 + (e^{i\lambda} - 1)p_m(t) \right]$$

$$= \sum_{n=0}^{N_{0,R}} c_n(t) e^{i(n - N_{0,R})\lambda} .$$  \hspace{1cm} (10)$$

The coefficients $c_n(t)$ can be calculated recursively from the product representation. This leads to the probability distribution

$$w_R(t,Q) = \sum_{n=0}^{N_{0,R}} c_n(t) \delta (Q - (n - N_{0,R})) .$$  \hspace{1cm} (11)$$

To obtain exact results for the FCS one first has to calculate $\tilde{P}_R(t)$ and then obtain its eigenvalues $p_m(t)$.

\section*{B. Finite temperatures}

The starting point for finite temperatures is again given by Eq. (5). We take the factor $e^{-i\lambda N_{0,R}}$ inside the expectation value and replace it by $e^{-i\lambda N_R(t)}$. We then perform a grand canonical averaging corresponding to the statistical operator

$$\rho_0 = e^{-\beta L \langle H_{0,L} - \mu L N_L \rangle} \otimes e^{-\beta R \langle H_{0,R} - \mu R N_R \rangle}$$

$$\equiv e^{-\tilde{H}_0/Z_0}$$  \hspace{1cm} (12)$$

The probability distribution $w$ is then given by

$$w_R^{gc}(t,Q) = \frac{1}{2\pi} \int d\lambda e^{-i\lambda Q} \langle e^{i\lambda N_R(t)} e^{-i\lambda N_R} \rangle ,$$  \hspace{1cm} (13)$$

where $\langle \cdot \rangle$ denotes the averaging with the statistical operator $\rho_0$. The information about the FCS is therefore encoded in the characteristic function

$$g_R(t,\lambda) = \langle e^{i\lambda N_R(t)} e^{-i\lambda N_R} \rangle .$$  \hspace{1cm} (14)$$

Again this result is valid also for interacting electrons.

For noninteracting fermions the characteristic function can again be expressed as a determinant but now in the full one particle Hilbert space. One can either use Klich’s
trace formula (Eq. [A13]) or use the proof using Wick’s theorem presented in appendix A. This yields
\[ g_R(t, \lambda) = \det \left[ 1 + \left( e^{i\lambda P_R(t)} e^{-i\lambda P_R} - \hat{1} \right) \tilde{n}_0 \right], \quad (15) \]
where the Fermi functions in the definition of \( \tilde{n}_0 \) are now the ones for finite temperatures. This implies that \( \tilde{n}_0 \) is no longer a projection operator. At \( T = 0 \) Eq. (15) reduces to Eq. (9).

As the next step we rewrite Eq. (15) in the form
\[ g_R(t, \lambda) = \det \left[ 1 - \tilde{n}_0 + e^{i\lambda P_R(t)} (\tilde{n}_0 - e^{-i\lambda} \tilde{n}_0 R) \right] \]

In contrast to statements in the literature it is useful to write \( P_R(t) \) as a sum
\[ P_R(t) = P_R(0) + \int_0^t \dot{P}_R(t') dt' = P_R + \delta P_R(t). \quad (16) \]

With \( e^{i\lambda P_R(t)} = 1 + (e^{i\lambda} - 1)(P_R + \delta P_R(t)) \) we obtain
\[ e^{i\lambda P_R(t)} (\tilde{n}_0 - e^{-i\lambda} \tilde{n}_0 R) \]
\[ = \delta P_R(t) \left[ \tilde{n}_0 - e^{-i\lambda} \tilde{n}_0 R \right] \]
\[ = \delta P_R(t) \left[ \tilde{a}_L(\lambda) - \tilde{a}_R(\lambda) \right]. \]

The general result for the characteristic function then takes the form
\[ g_R(t, \lambda) = \det \left[ 1 + \delta P_R(t) (\tilde{a}_L(\lambda) - \tilde{a}_R(\lambda)) \right]. \quad (18) \]

As shown in section VI this is the starting point for a transparent derivation of the approximate (long time limit) Levitov-Lesovik formula
\[ \ln g_R^{(L)}(\lambda) = \frac{t}{2\pi} \int d\epsilon \ln \left[ 1 + T(\epsilon) \left\{ (e^{i\lambda} - 1) f_L(\epsilon)(1 - f_R(\epsilon)) + (e^{-i\lambda} - 1) f_R(\epsilon)(1 - f_L(\epsilon)) \right\} \right], \quad (19) \]

where \( T(\epsilon) \) is the transmission probability for the scattering of a single particle between the two subsystems.

Note that compared to the zero temperature result the operator added to the unit operator in the determinant in Eq. (18) cannot be written as a product of a function of \( \lambda \) and an operator independent of \( \lambda \). Therefore the numerical effort for an exact numerical calculation of \( w_R(t, Q) \) is much higher than at \( T = 0 \).

III. EXACT NUMERICAL RESULTS FOR \( T = 0 \)

A. The model

In this section we present exact numerical results for the probability distribution \( w(t, Q) \) for a one dimensional tight binding model. The generalization for the model with more than two leads discussed in section VI is straightforward.

The unperturbed one particle Hamiltonians of the subsystems are given by
\[ h_{0L} = - \sum_{m=-\left( M_L - 1 \right)}^{-1} \tau_m \langle m \rangle \langle m + 1 \rangle + H.c. \]
\[ + v_0 |0\rangle \langle 0| \]

\[ h_{0R} = - \frac{B}{4} \sum_{m=1}^{M_R - 1} \langle m \rangle \langle m + 1 \rangle + H.c. \]. \quad (20) \]

The number of sites \( m \) in the subsystems are given by \( M_a \). In the explicit calculations we later specialize to \( \tau_m = B/4 \) for \( m = -(M_L - 1), ... , -2 \) and \( \tau_0 = t_L \).

The one particle eigenstates \( |k_j^R\rangle \) of the right subsystem are standing waves
\[ \langle m | k_j^R \rangle = \sqrt{\frac{2}{M_R + 1}} \sin(k_j^R m), \quad k_j^R = \frac{j\pi}{M_R + 1} \quad (21) \]

with the integer \( j \in [1, M_R] \). The corresponding energies are given by \( \epsilon_{k_j^R} = -(B/2) \cos k_j^R \), i.e. the total band width is given by \( B \).

In order to obtain a stationary current flow in the long time limit \( t \to \infty \) one first has to take the limit \( M_a \to \infty \). In this limit the quantum numbers \( k_j \) become continuous with \( k \in [0, \pi] \) and one has to go over to the delta function normalization of the states
\[ \sqrt{\frac{M_R + 1}{\pi}} |k_j^R \rangle \to |k, R \rangle \quad (22) \]

with \( \langle k, R|k', R \rangle = \delta(k - k') \).

In the left subsystem the eigenstates \( |k_j^L \rangle \) are pure sine-waves only in the special case \( v_0 = 0 \) and \( \tau_m = const. \).

The coupling between the two subsystems is described by the hopping term
\[ v_{LR} = -t_R |0\rangle \langle 1| + H.c. \]. \quad (23) \]

The special case \( v_0 = 0 \) and \( \tau_m = const. \) corresponds to two subsystems which, apart from their initial filling, are identical. The numerical results presented in this section are for this simplest case. The model with \( \tau = t_L \) and \( v_0 \) taking arbitrary values is a model which allows resonant scattering through the “quantum dot” at site zero. Some properties of this model simplify in the “wide band limit” \( B \to \infty \).

B. Results for finite times

As discussed in the previous section one has to address the eigenvalue problem of the operator \( \hat{P}_R(t) \) in order to calculate the probability distribution \( w(t, Q) \). For the calculation of the eigenvalues \( p_m(t) \) of \( \hat{P}_R(t) \) we use
\[ \langle k^{a}_{\mu} | \hat{P}_R(t) | k^{a'}_{\mu} \rangle = \sum_{m=1}^{M_R} \langle k^{a}_{\mu} | m \rangle \langle m | k^{a'}_{\mu} \rangle (t), \quad (24) \]
where $\mu, \mu'$ label the occupied one particle states. With the help of the spectral representation of the full Hamiltonian $h$

$$h = \sum_i |k_i\rangle \epsilon_{k_i} \langle k_i|$$

the time dependent quantities in Eq. (24) can be expressed as sums

$$\langle m|k_{\mu'}^a(t)\rangle = \sum_i \langle m|k_i\rangle e^{-i\epsilon_{k_i}t} \langle k_i|k_{\mu'}^a\rangle$$.

In order to obtain the eigenvalues the resulting $N_{\text{tot}} \times N_{\text{tot}}$ matrix has to be diagonalized numerically.

As discussed in more detail in the next section the results are almost independent of the size of the system and depend only on the values of the chemical potential $\mu_L$ and $\mu_R$ for times smaller than the time it takes the “charge fronts” which move into the subsystems after connecting them to return to the connection point after having been reflected at the boundaries of the system.

We always consider the case $N_{0,L} \geq N_{0,R}$, i.e. the net particle flow is from left to right. As $\bar{P}_R(t)$ is a projection operator the eigenvalues $p_m(t)$ of $\bar{P}_R(t)$ obey the bounds $0 \leq p_m(t) \leq 1$. In the figures we show the eigenvalues $p_m(t)$ in descending order.

For $t = 0$ the matrix $\bar{P}_R(0)$ is diagonal with $N_{0,R}$ eigenvalues 1 and $N_{0,L}$ eigenvalues 0. For $t > 0$ the $N_{0,R}$ eigenvalues almost identical to 1 persist. As shown in Fig. $\bar{P}_R(0)$ for perfect transmission $N_1$ additional eigenvalues almost identical to 1 occur where the value $N_1 \sim t$ is discussed in detail later. The crossover from eigenvalues close to one to those close to zero is smooth. For non-perfect transmission the eigenvalues which significantly differ from 1 and 0 start at $N_{R,0} + 1$ and approximately extend to $N_{R,0} + N_1$.

If one has a look at the Levitov-Lesovik formula Eq. (19) for $T = 0$

$$\ln g_R^{(L)}(t, \lambda) = \frac{t}{2\pi} \int_{\mu_R}^{\mu_L} d\lambda \ln [1 + T(\epsilon)(e^{i\lambda} - 1)]$$

one expects the states below $\mu_R$ which are initially occupied in both subsystems to be irrelevant. We therefore also calculated the eigenvalues of the $N_{\text{diff}} \times N_{\text{diff}}$ submatrix $\bar{P}_{R,\text{diff}}(t)$ in the space of the $N_{\text{diff}} = N_{0,L} - N_{0,R}$ states of the left system in the energy window $[\mu_R, \mu_L]$. As can be seen in Fig. 1 these eigenvalues deviate little from the eigenvalues of the full $N_{\text{tot}} \times N_{\text{tot}}$ matrix.

The weights of the probability distribution corresponding to the parameters of Fig. 1 are shown in Fig. 2. For the non-ideal transmission ($t_R \equiv \Delta t_B = 0.5$) the distribution calculated from the “extra electrons” only, agrees very well with the one from the full calculation. For the perfect transmission case slight deviations show up.

If one doubles the size of the system and the number of electrons but leaves $t$ fixed, the result for $w_R(t, Q)$ is almost identical to the one shown in Fig. 2. Our approach using finite systems and nevertheless see “asymptotic” properties in a time window is very useful also for interacting fermions for which effective methods have been worked out recently to describe the time dependence for finite systems. Promising techniques are the time dependent density matrix renormalization group (DMRG) and the time dependent numerical renormalization group (NRG).

Note that the $T = 0$ Levitov-Lesovik formula predicts zero width of the probability distribution for the perfect transmission case which is in contrast to the exact nu-
The circles show the result using the matrix \( \bar{T} = 0 \) as a function of time for the system parameters of Figs. 1 and 2. The squares are the results for the matrix \( \bar{R} = \bar{T} \) which only treats the electrons in the energy window \([\mu_R, \mu_L]\).

C. Low order cumulants

The low order cumulants \( \kappa_i \) of \( W_R(t, Q) \) are obtained from the Taylor expansion of \( \ln q_R(t, Q) \). Using \( \det a = \exp(\text{tr} \ln a) \) in Eq. (29) one obtains for \( i \) up to 4

\[
\begin{align*}
\kappa_1 &= \text{tr} \bar{P}_R - N_R^0 \\
\kappa_2 &= \text{tr} \bar{P}_R - \text{tr} \bar{P}_R^2 \\
\kappa_3 &= \text{tr} \bar{P}_R - 3\text{tr} \bar{P}_R^2 + 2\text{tr} \bar{P}_R^3 \\
\kappa_4 &= \text{tr} \bar{P}_R - 7\text{tr} \bar{P}_R^2 + 12\text{tr} \bar{P}_R^3 - 6\text{tr} \bar{P}_R^4,
\end{align*}
\]

In the long time limit discussed in the next section analytical results for the low order cumulants can be obtained.

In Fig. 3 we show results for \( \kappa_2 \) for perfect transmission and the parameters of Fig. 1. The circles correspond to the full calculation and the squares to the result using \( \bar{P}_{R,\text{diff}}(t) \). It will be shown analytically in the next section that \( \kappa_2 \) increases logarithmically at \( T = 0 \) in the long time limit for perfect transmission. The numerical results for \( \kappa_i \) with \( i \geq 3 \) show no logarithmic increase. Apart from small transient effects these cumulants stay close to zero.

The Levitov-Lesovik formula for \( T = 0 \) predicts zero width of the probability distribution also for the zero bias case \( \mu_L = \mu_R \) for arbitrary transmission. Absolutely no charge transport is predicted in contrast to the exact numerical results not shown here.

IV. THE LONG TIME LIMIT

A. Introduction of scattering states

We now examine if \( P_R(t) \) can be discussed analytically in the long time limit. For the calculation of the matrix elements of \( P_R(t) \) it is useful to split off a phase factor in the time dependent one particle states and define

\[
\langle k_i^a(t) | = e^{i(h - e_k^0)t} \langle k_j^a | (30)
\]

It is known from scattering theory that in the long time limit (after taking the limit \( M_a \to \infty \)) the states \( |k, a\rangle \) converge to the scattering states. This can be seen using an Abelian limit procedure

\[
\lim_{t \to \infty} e^{-i(h - \epsilon_{k,a})t} |k, a\rangle = \lim_{\eta \to 0} \int_0^\infty e^{-\eta t} e^{-i(h - \epsilon_{k,a})t} |k, a\rangle dt
\]

\[
= \lim_{\eta \to 0} \epsilon_{k,a} - h + i\eta |k, a\rangle
\]

\[
= |k, a+\rangle
\]

The state on the rhs of the last equality is just the scattering state with outgoing scattered waves.

The explicit form of the scattering states for our Hamiltonian \( h \) with \( \tau_m = B/4 \) for \( m \leq -1 \) is given by

\[
\langle m | k, L+ \rangle = \frac{1}{\sqrt{2\pi}} e^{ikm} + r_k e^{-ikm}, \quad m \leq -1
\]

\[
= \frac{1}{\sqrt{2\pi}} t_k e^{ikm}, \quad m \geq 1
\]

with \( r_k \) and \( t_k \) the reflection and transmission amplitude. Similarly the scattering states with incoming wave from the right are given by

\[
\langle m | k, R+ \rangle = \frac{1}{\sqrt{2\pi}} e^{-ikm} + \bar{r}_k e^{ikm}, \quad m \geq 1
\]

\[
= \frac{1}{\sqrt{2\pi}} \bar{t}_k e^{-ikm}, \quad m \leq -1.
\]

If one returns to the description of the finite system at finite times the question arises how well the approximation

\[
\langle m | k_j^a(t) \rangle \approx \sqrt{\frac{\pi}{N_a + 1}} \langle m | k_j^a, a+ \rangle e^{-i\epsilon_{k_j^a}t}
\]

works.

In Fig. 4 we show \( 2(N_L + 1) |\langle m | k_j^+ \rangle|^2 \) for different values \( m > 0 \) of the site index for identical subsystems with \( M_L = M_R = 100 \), hopping matrix element \( \bar{t}_R = 4t \eta / B = 0.5 \) and a state in the middle of the band. In the approximation of Eq. (29) this quantity should be time independent and given by the transmission probability \( |t_k|^2 \). The figure shows that the larger the value of \( m \) the later the oscillatory convergence sets in.
It would be totally wrong to use Eq. \(33\) in Eq. \(24\) for all sites of the subsystems. The only time dependence would then result from the factors \(e^{i\epsilon_{k,a} t} e^{-i\epsilon_{k',a} t}\). They could be gauged away by absorbing them in the definition of the basis states making the eigenvalues \(p_m(t)\) time independent. This would imply that \(g_m(t,\lambda)\) and with it the FCS is time independent. This shows clearly that one should apply Eq. \(33\) only to the calculation of observables localized in the neighborhood of the junction.

It is therefore useful to express \(P_R(t)\) as an integral over the current operator between the two subsystems as in Eq. \(16\)

\[
P_R(t) = P_R(0) + \int_0^t j_{0\to1}(t')dt' \equiv P_R + \delta P_R(t) \tag{34}
\]

with the current operator given by

\[
j_{0\to1} = it_R \langle \{1\}|0\rangle - |0\rangle\langle 1| \tag{35}
\]

The matrix elements of \(P_R(t)\) are therefore given by

\[
\langle k_j^a | P_R(t) | k_j^b \rangle = \delta_{a,a'} \delta_{a,R} \delta_{j,j'} + \int_0^t \langle k_j^a(t')|j_{0\to1}|k_j^b(t')\rangle dt' \tag{36}
\]

As expected from Fig. 4, the use of the approximation in Eq. \(33\) to evaluate the matrix elements of the local observable \(j_{0\to1}\) works surprisingly well even for rather short times. With this replacement the time integration in Eq. \(33\) can be carried out and one obtains the very useful approximation also for finite systems

\[
\langle k_j^a | P_R(t) | k_j^b \rangle \approx \delta_{a,a'} \delta_{a,R} \delta_{j,j'} + \int_0^t \langle k_j^a(t)|j_{0\to1}|k_j^b(t')\rangle dt' \tag{37}
\]

For the calculation of the current matrix elements we also have to know how \(|0\rangle |k, a\rangle\) is related to the scattering amplitudes in Eqs. \(37\) and \(38\). The Schrödinger equation \(\langle 1|h |k, a\rangle = \epsilon_{k,a} \langle 1 |k, a\rangle\) yields

\[
t_R \langle 0|k, L\rangle = \frac{t_{k} B}{\sqrt{2\pi}4}, \quad t_R \langle 0|k, R\rangle = \frac{1 + \tilde{r}_k B}{\sqrt{2\pi}4}. \tag{38}
\]

This allows to express the current matrix elements in Eq. \(37\) in terms of the \(t_k\) and \(\tilde{r}_k\), e.g.

\[
\langle k, L | j_{0\to1} | k', L\rangle \approx \frac{i}{2\pi} t_{k}^* t_{k'} (e^{-ik} - e^{ik'}) \frac{B}{4}. \tag{39}
\]

and

\[
\langle k, L | j_{0\to1} | k', R\rangle \approx \frac{i}{2\pi} t_{k}^* \tilde{r}_{k'} (e^{-ik} - e^{ik'}) \frac{B}{4}. \tag{40}
\]

These expressions simplify in the in the wide band limit \(|\mu_L - \mu_R| \ll B\) for the scattering states in the energy window between \(\mu_L\) and \(\mu_R\). Then one can replace \(k\) and \(k'\) by \(k_F \equiv (k_{F,L} + k_{F,R})/2\). This yields

\[
\langle k, L | j_{0\to1} | k', L\rangle \approx \frac{v(\tilde{k}_F)}{2\pi} t_{k} t_{k'} \tag{41}
\]

with the Fermi velocity \(v_F \equiv v(\tilde{k}_F) = 2B \sin \tilde{k}_F\). As the transmission amplitudes may have a rapid variation in this energy window the replacement \(k, k' \to \tilde{k}_F\) has not been made for the factors involving the scattering amplitudes.

### B. Low order cumulants

As for the Hamiltonian in Eq. \(24\) the transmission and reflection amplitudes can easily be calculated (see the appendix), the matrix elements of \(P_R(t)\) in the scattering approximation Eq. \(37\), called \(P_R^s(t)\) in the following, are known explicitly. Without further approximations the eigenvalues of this matrix have to be calculated numerically. The low order cumulants \(\kappa_i\) of \(w_R(t, Q)\) on the other hand can be discussed analytically. We begin with the \(T = 0\) limit where the cumulants have the simple form presented in Eq. \(25\).
1. Results for $T = 0$

Using Eqs. (28) and (37) the explicit form of the time derivative of the first order cumulant for $T = 0$ is given by

$$\dot{\kappa}_1(t) = \sum_{\mu,a} \langle k^a_{\mu}(t)|j_{0-1}|k^a_{\mu}(t)\rangle$$

$$\approx \sum_{\mu,a} \frac{\pi}{N_a + 1} \langle k_{\mu},a + |j_{0-1}|k_{\mu},a+ \rangle$$

For symmetric subsystems ($\tau_m = B/4$, $v_0 = 0$ and $M_L = M_R$) the contributions of the states with energy smaller than $\mu_R(< \mu_L)$ which are occupied in both subsystems exactly cancel and only the states $|k^a_{\mu}\rangle$ in the energy window $[\mu_R, \mu_L]$ contribute. In the thermodynamic limit $\pi/(N_a + 1)/\sum_{\mu}(\cdot) \rightarrow \int(\cdot)dk = \int(\cdot)de/\nu(\epsilon)$ the cancellation of the contributions from the states below $\mu_R$ holds generally in the approximation of using the scattering states and one obtains using Eq. (39)

$$\dot{\kappa}_1(t) \approx \frac{1}{2\pi} \int_{\mu_R}^{\mu_L} T(\epsilon)de$$

This is the zero temperature version of the Landauer-Büttiker formula with $T(\epsilon) = |t_{k(\epsilon)}|^2$ the transmission probability.

In Fig. 5 we show how the long time result for $\dot{\kappa}_1$ is approached at finite times. From the results in Fig. 4 one expects also an oscillatory approach of the exact numerical result to the Landauer-Büttiker formula. This is confirmed in Fig. 5 for different values of the hopping parameter between the two subsystems.

![Graph showing $\dot{\kappa}_1(t)$ vs. $tB/4$](image)

FIG. 5: Comparison of the exact and the approximate long time result for $d\kappa_1/dt$ given in Eq. (43) for a system with $M_L = M_R = 200$, $N_{a,L} = 120$, $N_{a,R} = 100$ and three different values of $t_R$. For the smaller values of $t_R$ the convergence to the approximate constant result (dotted curves) is faster.

For perfect transmission the long time approximation for the first cumulant is given by

$$\kappa_1 = \frac{t}{2\pi} \Delta \mu$$

with $\Delta \mu \equiv \mu_L - \mu_R$. Neglecting the transition region in Fig. 1 this is realized by $N_t$ eigenvalues 1 of $P_{R,\text{diff}}$ which implies

$$N_t \approx \frac{t}{2\pi} \Delta \mu$$

(45)

The numerical evidence that the number of eigenvalues $p_{m,\text{diff}}(t)$ which are different from zero is in general approximately given by $N_t$ yields for integer $N_t$

$$\kappa_1 = \sum_{m=1}^{N_{\text{diff}}} p_{m,\text{diff}}(t) \approx \sum_{m=1}^{N_t} p_{m,\text{diff}}(t)$$

(46)

This can be compared to the approximate evaluation of Eq. (43) using the trapezoidal rule for $N_t$ subintervals. Together with Eq. (45) one obtains

$$\kappa_1 \approx \sum_{m=1}^{N_t} T(\mu_R + m - 1/2) \Delta \mu = \sum_{m=1}^{N_t} T_m(t)$$

(47)

It is therefore tempting to conclude

$$p_{m,\text{diff}}(t) \approx T(\mu_R + m - 1/2) \Delta \mu$$

(48)

for $m = 1, \ldots, N_t$ and zero otherwise. With Eq. (48) the argument of $T$ in this equation can also be written as $\mu_R + 2\pi(m - 1/2)/t$. To really justify the guess in Eq. (48) one has to examine also the higher order cumulants or compare with the exact numerical results.

For $T = 0$ the explicit form of the second order cumulant is

$$\kappa_2 = \sum_{\mu,a} \sum_{\mu,a'} \langle k^a_{\mu} | P_R(t) | k^{a'}_{\mu} \rangle \langle k^{a'}_{\mu} | P_R(t) | k^a_{\mu} \rangle$$

$$\approx \sum_{k_{\mu},a_{\alpha,a'}} \frac{\pi^2}{(N_a + 1)(N_a' + 1)} \langle k_{\mu,a'} + |j_{0-1}|k_{\mu},a+ \rangle \left( \frac{\sin[(\epsilon_{k_{\mu}} - \epsilon_{k_{\mu}})t]/2}{(\epsilon_{k_{\mu}} - \epsilon_{k_{\mu}})/2} \right)^2$$

(49)

where $\mu$ labels occupied and $\alpha$ unoccupied one particle states.

If one performs the thermodynamic limit and replaces the last factor on the rhs of Eq. (49) by $2\pi t \delta(\epsilon_{k_{\mu}} - \epsilon_{k_{\mu}})$ and uses Eq. (40) one obtains Lesovik’s shot noise result:

$$\kappa_2 \approx \frac{t}{2\pi} \int_{\mu_R}^{\mu_L} T(\epsilon)(1 - T(\epsilon))de$$

(50)

which is the result for $\kappa_2$ which holds for the zero temperature Levitov-Lesovik distribution in Eq. (27). It is consistent with the guess in Eq. (48). To further test this approximation for the eigenvalues $p_{m,\text{diff}}(t)$ we show a comparison with exact results in Fig. 6. Note that the approximation $p_{m,\text{diff}}(t) \approx T_m(t)$ fails to describe the transition region to the zero eigenvalues. This is in contrast to the eigenvalues $P_{m,\text{diff}}(t)$ using the scattering
approximation Eq. (37) which at \( t = 80 \) show an overall good agreement with the exact \( p_m(t) \).

For the special case of perfect transmission \( T(\epsilon) \equiv 1 \) the approximate result in Eq. (50) yields zero width for the probability distribution \( w_R(t, Q) \) in contrast to the exact numerical result shown in Fig. 4.

In order to obtain a non zero result for \( \kappa_2 \) for the perfect transmission case at \( T = 0 \) the integrations have to be performed more carefully. In order to keep the calculation as simple as possible we use the approximation to only treat the electrons in the energy window \([\mu_R, \mu_L]\) which corresponds to \( N_{0,R} = 0 \). In addition we work in the wide band limit and assume an energy independent transmission probability \( T \), i.e. approximate the current matrix element \( \langle k, L + | j_{0-1}|k', L+ \rangle \) by \( T v_F/(2\pi) \) (see Eq. (11)). This yields for the trace of \( \bar{P}_R(t) \) entering the expression for \( \kappa_2 \) in Eq. (28).

\[
\text{tr} \bar{P}_R(t) = \left(\frac{T}{2\pi}\right)^2 \int_0^{\Delta \mu} \int_0^{\Delta \mu} \left( \sin\left(\left(\epsilon - \epsilon'\right)t/2\right) / \left(\epsilon - \epsilon'\right)/2 \right)^2 d\epsilon d\epsilon' = \frac{T^2}{2\pi^2} \int_0^{\Delta \mu} d\epsilon \left(\Delta \mu - \epsilon\right) \left(\sin\left(\epsilon t/2\right) / \epsilon/2\right)^2.
\]

The remaining integral is logarithmically divergent in the long time limit \( t \gg 1/B \) and one obtains

\[
\kappa_2 \approx \frac{t \Delta \mu}{2\pi} \bar{T}(1 - \bar{T}) + \frac{T^2}{\pi^2} \ln(t \Delta \mu/2).
\]

This agrees with the corresponding result from the approximate solution of the matrix Riemann-Hilbert problem in Ref. [8]. The logarithmic correction term is only important for the case of (nearly) perfect transmission.

As already mentioned for the numerical result using the exact \( p_m(t) \) the cumulants \( \kappa_i \) for \( i \geq 3 \) vanish (apart from small transients) for perfect transmission. Therefore, the values of \( w_R(t, Q) \) lie on a Gaussian of width \( \sim |\ln(t \Delta \mu/2)|^{1/2} \) to an excellent approximation. This is in agreement with the general result for the subleading correction for the special case considered in Ref. [8].

2. Finite temperature results

As the finite temperature result for \( g_R(t, \lambda) \) in Eq. (18) contains two different functions of \( \lambda \) the calculation of the cumulants is more tedious than for \( T = 0 \).

\[
\begin{align*}
\kappa_1(t) &= \text{tr} \left[ \delta P_R(t) \bar{n}_0 \right] \\
\kappa_2(t) &= \text{tr} \left[ \delta P_R(t) \bar{n} - \delta P_R(t) \bar{n}_0 \delta P_R(t) \bar{n}_0 \right] \\
&= \text{tr} \left[ \delta P_R(t) \left( \bar{1} - \bar{n}_0 \right) \delta P_R(t) \bar{n}_0 \right] \quad (54)
\end{align*}
\]

with \( \bar{n} \equiv \bar{n}_{0,L} - \bar{n}_{0,R} \). In the second equality for \( \kappa_2 \) we have used the fact that \( P_R(t) = P_R + \delta P_R(t) \) is a projection operator which reads for \( \delta P_R(t) \)

\[
\delta P_R(t) = P_R \delta P_R(t) + \delta P_R(t) P_R + \delta P_R^2(t) \quad (55)
\]

The long time result for \( \kappa_1 \) is just the Landauer-Büttiker formula integrated over time

\[
\kappa_1 = \sum_{i,a} \langle k_i^a | \delta P_R(t) | k_i^a \rangle f_{k_i}^a \quad (56)
\]

\[
\approx \frac{t}{2\pi} \int T(\epsilon) \left( f_L(\epsilon) - f_R(\epsilon) \right) d\epsilon.
\]

The second form for \( \kappa_2 \) shows explicitly its positivity

\[
\begin{align*}
\kappa_2 &= \sum_{i,a} \sum_{j,a'} \langle |k_j^{a'}| \delta P_R(t) | k_j^{a'} \rangle^2 f_{k_j}^a \left(1 - f_{k_j}^{a'}\right) \\
&\approx \frac{t}{2\pi} \int \left| T(\epsilon) \right|^2 \left( f_L(\epsilon) \left(1 - f_L(\epsilon)\right) + f_R(\epsilon) \left(1 - f_R(\epsilon)\right) \right) \\
&+ T(\epsilon) \left(1 - T(\epsilon)\right) \left[ f_L(\epsilon) \left(1 - f_R(\epsilon)\right) + f_R(\epsilon) \left(1 - f_L(\epsilon)\right) \right].
\end{align*}
\]

The energy integral is the contribution linear in time and is obtained by the generalization of the steps to derive Eq. (50). At finite temperatures there is a contribution linear in \( t \) even for the perfect transmission case, which masks the logarithmic correction in Eq. (52) for long enough times.

We show in the next section that the Levitov-Lesovik approximation Eq. (19) amounts to include the linear in \( t \) contributions in all orders.

V. DERIVATION OF THE LEVITOV-LESOVIK FORMULA

A. The one-dimensional model

After we have shown how the use of scattering states simplifies the theoretical description in the long time
In a 2-dimensional space spanned by the quantum numbers $k$, the summations over the quantum numbers $k$ have to be performed. They can be expressed as a trace over the Hilbert space

$$\text{tr} b^m = \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \text{tr} b^m \right)$$

If one inserts this into Eq. (63), this completes the derivation of the Levitov-Lesovik formula. Note that in the derivation it was not necessary to invoke the wide band limit.

If one is interested in logarithmic corrections in time to $\ln g_R(t, \lambda)$, the approximation in Eq. (64) should not be made for carrying out the energy integrations (see the discussion of $\kappa_2$ leading to Eq. (52) in section IV b).

### B. Multi-lead geometries

The FCS in multiterminal circuits was addressed by various authors. For noninteracting electrons the Levitov-Lesovik formula for the quantum wire discussed in the previous subsection can easily be generalized to such more complex geometries. As an example we show in Fig. 7 a Y-junction pierced by a magnetic flux. The zero temperature limit of a Y-junction was already discussed in ref. 3. The proof given below works for the general case of $M$ legs at finite temperatures.

We denote the states on the central “ring” as $|0, a\rangle$ with $a = 1, \ldots, M$ and the states in the legs in the site representation as $|m, a\rangle$ with $m = 1, 2, \ldots, \infty$. The connection of the legs with the ring generalizes Eq. (23) to

$$v = -\sum_{a=1}^{M} \tau_a |0, a\rangle\langle 1, a| + \text{H.c.},$$

where the $\tau_a$ can be assumed to be real. In the legs we assume nearest neighbor hopping $\tau = 1$. At time $t = 0$ the subsystems with different chemical potentials (and temperatures) are connected via the hopping term $v$ in Eq. (65). Instead of the particle number in the right part of the system we now monitor the particle number in leg 2.

![FIG. 7: Junction of $M = 3$ quantum wires connected to a “ring” indicated by the filled circles, which is pierced by a magnetic flux. In the initial state the hopping matrix elements between the ring sites and the adjacent wire sites vanish.](image-url)
We then obtain as the generalization of Eq. (61) we can neglect the ring state contribution to the trace.

\[
\det(\hat{1}_M + c_{(3)}) = (1 + D)(1 + t^*_a a_1 + t^*_a a_2 + r^* a_3 t^*_a a_1 + t^*_a a_2).
\]

(74)

For arbitrary values of \( M \) the sum over the \( t^*_a a \) extends to \( M - 1 \). Therefore the generalization to arbitrary values of \( M \) reads with \( d_{\pm} \equiv \varepsilon \pm \lambda - 1 \) and using \( d_+ d_- = -(d_+ + d_-) \)

\[
det(\hat{1}_M + c_{(3)}) = 1 + d_+ (1 - f_M) \sum_{a=1}^{M-1} |t^*_{M-a}|^2 f_a
\]

(75)

\[
1 - |r_{M,k}|^2 = \sum_{a=1}^{M-1} \sum_{a=1}^{M-1} |t^*_{k-M-a}|^2 |t^*_{k-a}|^2
\]

(76)

As generally \( |t^*_{k-M-a}|^2 \neq |t^*_{k-M-a}|^2 \) it requires an additional argument to bring Eq. (74) into its final form. It is shown in the appendix B that

\[
\sum_{a=1}^{M-1} |t^*_{k-M-a}|^2 = \sum_{a=1}^{M-1} |t^*_{k-a}|^2
\]

(77)

also holds in the presence of a magnetic flux. Using Eq. (63) for arbitrary values of \( M \) we obtain the generalized Levitov-Lesovik formula. With \( T_a(\varepsilon) \equiv |t^*_{k-a}|^2 \) it reads

\[
\ln g_M(t, \lambda) \rightarrow \frac{t}{2\pi} \int d\varepsilon \ln [1 + A(\varepsilon, \lambda)]
\]

(78)

with

\[
A(\varepsilon, \lambda) = \sum_{a=1}^{M-1} T_a(\varepsilon) [(\varepsilon^2 - 1)f_a(\varepsilon)(1 - f_M(\varepsilon)) + (\varepsilon^{-2} - 1)f_M(\varepsilon)(1 - f_a(\varepsilon))]
\]

(79)

Again it was not necessary to invoke the wide band limit.

VI. SUMMARY

The aim of this paper was to clarify the validity of the Levitov-Lesovik formula for the FCS of noninteracting fermions and to present a simple derivation which clearly shows the approximations involved. As a starting point we used the formula for the characteristic function in Eq. (15) first derived by Klich. No measuring device is used in this description. As Klich noted his formal result should be viewed as the general expression for the counting statistics of noninteracting fermions, at any
given time and without any approximation. Our alternative derivation of Klich’s formula for finite temperatures presented in Appendix A uses Wick’s theorem.

The first step in the actual calculation of the characteristic function is to obtain an explicit result for the time dependent projection operator \( P_R(t) \). This was done exactly for a microscopic lattice model using a numerical approach and analytically in the long time limit using scattering states. It was shown by a comparison of the two approaches that the analytical expression in Eq. 67 works very well after transients with a time scale \( \sim 1/B \) have died out. At zero temperature the eigenvalues \( p_m(t) \) of the operator \( P_R(t) \) directly determine the probability distribution \( w(Q,t) \). The fact that the Levitov-Lesovik formula provides only an approximate description of the FCS was studied in detail using low order cumulants. A comparison to results obtained by different methods\(^2\) was given.

In section VI we presented a straightforward derivation of the Levitov-Lesovik formula summing all linear in \( t \) contributions to \( \ln g_R(t, \lambda) \) using Eqs. 11 and 57. An important step in the derivation was to first bring Eq. 11 into the form Eq. 18 introducing the operator \( \delta P_R(t) \). In the derivation no assumptions about the band width and the energy dependence of the transmission probability were necessary. The derivation applied to the simple “experimental” setup studied in this paper shows that the Levitov-Lesovik formula provides an excellent approximation in the long time limit, except in the case of (near) perfect transmission at very low temperatures. An extension of this new derivation to multi-lead circuits with possible broken time reversal symmetry was also given.

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APPENDIX A: KLICH’S FORMULA USING WICK’S THEOREM

In this appendix we discuss the expectation value of an exponential of a selfadjoint one-particle operator \( A \)

\[
A = \sum_{i,j} \langle i| \hat{a}_j^\dagger \hat{a}_j |j \rangle c_i^\dagger c_j = \sum_\lambda \alpha_\lambda c_\lambda^\dagger c_\lambda
\]

(A1)

where the \( c_i^\dagger \) (as well as the \( c_i^\dagger \)) obey the usual anticommutation relations \( \{ c_i, c_j \} = \delta_{ij}, \{ c_i, c_j \} = 0 \). In order to avoid mathematical subtleties we assume the dimension of the one-particle Hilbert space to be finite \( (M) \). The creation operators for the two different one-particle basis states obey the usual relation

\[
c_\lambda^\dagger = \sum_i c_i^\dagger \langle i| \lambda \rangle.
\]

(A2)

We consider expectation values with a statistical operator in Fock space

\[
\rho_0 = e^{-\hat{H}_0}/\text{Tr}e^{-\hat{H}_0} \equiv e^{-\hat{H}_0}/Z_0,
\]

(A3)

with

\[
\hat{H}_0 = \sum_i c_i^\dagger c_i.
\]

(A4)

In the section II we use \( \hat{H}_0 = \beta L(\hat{H}_{0L} - \mu_L \mathcal{N}_L) + \beta R(\hat{H}_{0R} - \mu_R \mathcal{N}_R) \), where the \( \hat{H}_{0L,a} \) are one-particle operators and the \( \mathcal{N}_a \) are the particle number operators of the left and right part of the system. The grand canonical partition function \( Z_0 \) is given by

\[
Z_0 = \prod_i (1 + e^{-\epsilon_i}) = \text{det}(1 + e^{-\hat{h}_0}).
\]

(A5)

With the operator in the one particle Hilbert space

\[
\hat{n}_0 \equiv (\hat{n}_0 + 1)^{-1} = \sum_i |i\rangle \frac{1}{e^{\epsilon_i} + 1} \langle i| \]

(A6)

the expectation values of bilinear operators in the \( \lambda \)-representation can be written as

\[
\langle c_\lambda^\dagger c_\mu \rangle = \sum_{i,j} (\mu |j\rangle \langle j| i \rangle \langle i| \lambda \rangle = \langle \mu | \hat{n}_0 | \lambda \rangle .
\]

(A7)

We now consider the expectation value of \( e^{\alpha A} \), where \( \alpha \) is an arbitrary complex number

\[
\langle e^{\alpha A} \rangle = \langle e^{\alpha \sum_\lambda \alpha_\lambda c_\lambda^\dagger c_\lambda} \rangle = \prod_\lambda \langle 1 + (e^{\alpha \lambda_\lambda} - 1)c_\lambda^\dagger c_\lambda \rangle .
\]

(A8)

The product runs over the \( M \) different eigenstates labeled by \( \lambda \). If one multiplies out the product one has to calculate expectation values of products of different \( \hat{n}_\lambda = c_\lambda^\dagger c_\lambda \). They are given by Wick’s theorem\(^\text{22}\) as a determinant of the bilinear expectation values in Eq. 17

\[
\langle c_{\lambda_1}^\dagger c_{\lambda_2}^\dagger ... c_{\lambda_m}^\dagger \rangle = \begin{vmatrix}
\langle c_{\lambda_1}^\dagger c_{\lambda_1} \rangle & \cdots & \langle c_{\lambda_1}^\dagger c_{\lambda_m} \rangle \\
\vdots & \ddots & \vdots \\
\langle c_{\lambda_m}^\dagger c_{\lambda_1} \rangle & \cdots & \langle c_{\lambda_m}^\dagger c_{\lambda_m} \rangle
\end{vmatrix}
\]

(A9)

This expectation value is multiplied by \( \prod_{\lambda=1}^m (e^{\alpha \lambda} - 1) \). These factors can be incorporated into the matrix elements on the rhs of Eq. 10 using Eq. 17

\[
(e^{\alpha \lambda_\lambda} - 1) \langle c_{\lambda_1}^\dagger c_{\lambda_1} \rangle = \langle \lambda_1 | (e^{\alpha \hat{n}_0} - 1) \hat{n}_0 | \lambda_1 \rangle .
\]

(A10)

A compact expression for \( \langle e^{\alpha A} \rangle \) is finally obtained by comparison with the formula for the determinant of \( 1 + \hat{b} \), where \( \hat{b} \) is an arbitrary linear operator in the \( M \)-dimensional Hilbert space

\[
\det(1 + \hat{b}) = 1 + \text{tr} \hat{b} + \sum_{i<j} \det^{(2)} \hat{b}^{(2)} + \sum_{i<j<k} \det^{(3)} \hat{b}^{(3)} + \ldots + \det(\hat{b}),
\]

(A11)
where e.g. \( \det^{(3)} b^{(3)} \) denotes a 3 \( \times \) 3 subdeterminant of \( \hat{b} \) with the indices given by the summation variables. The comparison with the calculation of \( \langle e^{\alpha A} \rangle \) yields

\[
\langle e^{\alpha A} \rangle = \det \left[ \hat{1} + (e^{\alpha \hat{a}} - 1) \hat{n}_0 \right].
\]  

(A12)

This result also follows from Klich’s elegant proof. As Klich’s elegant proof involves a result not widely known and it is not available in a journal publication, we presented the alternative proof of Eq. (A12) using Wick’s theorem to make this paper self-contained. In section II we use a slight generalization of Eq. (A12).

Let \( C \) be a one particle operator in Fock space which commutes with \( \hat{H}_0 \). With \( \hat{H}_0 = \hat{H}_0 - \gamma C \) we obtain

\[
\langle e^{\alpha A} e^{\beta B} \rangle = \frac{\text{Tr} e^{\alpha A} e^{\beta B} e^{-\hat{H}_0}}{\text{Tr} e^{-\hat{H}_0}}
\]

\[
= \det \left[ \hat{1} + \hat{n}_0 + e^{\alpha \hat{a}} \hat{n}_0 \right] \det(\hat{1} + e^{-\hat{H}_0})
\]

\[
= \det \left[ \hat{1} + (e^{\alpha \hat{a}} e^{\gamma C} - 1) \hat{n}_0 \right].
\]  

(A14)

**APPENDIX B: PROOF OF EQ. (77)**

The scattering states can be expressed via the full resolvent \( g(z) = (z - \hat{h})^{-1} \) and the unperturbed resolvent \( g_0(z) = (z - \hat{h}_0)^{-1} \) which are related by \( g = g_0 + g \hat{g} g_0 \). From the definition in Eq. (30) one obtains for \( a' \neq a \)

\[
\langle m, a|k, a' \rangle = \langle m, a|g_0(\epsilon k + i0) 0, \alpha \rangle \tau_a \times \langle 0, 0|g_0(\epsilon k + i0) 0, a' \rangle \tau_{a'} \langle 0, a'|k, a' \rangle.
\]  

(B1)

The resolvent matrix element of the semi-infinite chain is given by

\[
\langle m, a|g_0(\epsilon k + i0) 0, \alpha \rangle = -e^{-ikm}.
\]  

With Eqs. (21), (22) and (67) the transmission probabilities follow as

\[
|t_k^{a' a}|^2 = 4 \tau_a^2 \sin^2 k |0, 0|]\langle 0, \alpha'||\epsilon k + i0)0, a \rangle|^2.
\]  

(B2)

Using the projection \( P_r \) onto the states on the ring the full resolvent matrix elements in Eq. (B2) can be written as

\[
\langle 0, a'|g(z)|0, a \rangle = \langle 0, a'|[z P_r - \hat{h}_r - \gamma(z)]^{-1} 0, a \rangle.
\]  

(B3)

with

\[
\gamma(z) = g_0^{(1)}(z) \sum_{s=1}^{M} \tau_o^2 |0, a \rangle \langle 0, a|,
\]  

(B4)

where \( \hat{h}_r \) is the one particle Hamiltonian on the ring and \( g_0^{(1)}(z) \) is the diagonal element of the resolvent of the semi-infinite chain at the boundary. For \( z = \epsilon + i0 \) it is given by

\[
g_0^{(1)}(\epsilon + i0) = (\epsilon + i \sqrt{4 - \epsilon^2})/2.
\]  

(B5)

In order to prove the relation between the transmission probabilities in Eq. (77) we us a simple operator identity. Let \( a \) and \( b \) operators such that the inverses of \( a \pm b \) exist. The one has

\[
\frac{1}{a - b} =\ \frac{1}{a + b}\ \frac{1}{a + b} - \frac{1}{a - b} \frac{1}{a - b}
\]

\[
= \frac{1}{a + b} \frac{1}{a - b}.
\]  

(B6)

If one puts the unit operators \( (a \pm b)(a \pm b)^{-1} \) on the opposite sides one obtains by comparison the operator relation

\[
\frac{1}{a - b} b \frac{1}{a + b} = \frac{1}{a + b} b \frac{1}{a - b}.
\]  

(B7)

which we now use for operators acting in the subspace of the ring states. With \( a = \epsilon I - \hat{h}_r - \text{Re}\gamma(\epsilon + i0) \) and \( b = \text{Im}\gamma(\epsilon + i0) \equiv \gamma_1(\epsilon) \) the relation reads using \( \text{Re}\gamma(\epsilon + i0) \) and \( \text{Im}\gamma(\epsilon + i0) = -\text{Im}\gamma(\epsilon - i0) \)

\[
g(\epsilon + i0) \gamma_1(\epsilon) g(\epsilon - i0) = g(\epsilon - i0) \gamma_1(\epsilon) g(\epsilon + i0).
\]  

(B8)

We next take the expectation value in the state \( |0, M \rangle \) and use the explicit form of \( \gamma(z) \) in Eq. (B4). The \( M = a \) term of the sum can be omitted on both sides of the equation as they are identical. Dividing by \( g_0^{(1)}(\epsilon + i0) \) yields

\[
\sum_{a=1}^{M-1} \tau_a^2 |0, M| g(\epsilon + i0) 0, a \rangle|^2 = \sum_{a=1}^{M-1} \tau_a^2 |0, a| g(\epsilon + i0) 0, M \rangle|^2
\]

Together with the explicit result for the transmission probabilities in Eq. (B2) this completes the proof of the relation between the transmission probabilities in Eq. (77).

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1 Ya.M. Blanter and M. Büttiker, Phys. Rep. 336, 1 (2000).
2 *Quantum noise in Mesoscopic Physics*, NATO Science series II, vol 97, ed. by Yu.V. Nazarov (Kluwer, Dordrecht
3 Boston London 2003).
4 L. S. Levitov and G. B. Lesovik, JETP Lett. 58, 230 (1993).
4 L. S. Levitov, H.-W. Lee and G. B. Lesovik, J. Math. Phys. (NY), 37, 10 (1996)
5 G. B. Lesovik and N.M. Chtchelkatchev, JETP Lett. 77, 393 (2003).
6 A. Gogolin and A. Komnik, Phys. Rev. B 73, 195301 (2006).
7 A. Braggio, J. König, and R. Fazio, Phys. Rev. Lett. 96, 026805, (2006).
8 I. Klich, “Full counting statistics: An elementary derivation of Levitov’s formula”, in ref. 2
9 B.A. Muzykantskii and Y. Adamov, Phys. Rev. B 68, 155304 (2003).
10 B. Braunecker, Phys. Rev. B 73, 075122 (2006).
11 M. Combescot and P. Noziéres, J. Phys. (Paris) 32, 913 (1971).
12 Similar numerical results are presented in S. Riecke, Diploma thesis, U Göttingen, (2006)
13 C. Kollath, U. Schollwöck, and W. Zwerger Phys. Rev. Lett. 95, 176401 (2005).
14 D. Bohr, P. Schmitteckert, and P. Wölle, Europhys. Lett. 73, 246 (2006).
15 F.B. Anders and A. Schiller Phys. Rev. B 74, 245113 (2006).
16 J. R. Taylor, Scattering Theory, Wiley, New York (1972).
17 R. Landauer, Phil. Mag. 21, 863 (1970), M. Büttiker, Phys. Rev. Lett. 57, 1761 (1986).
18 G. B. Lesovik, JETP Lett. 49, 592 (1989).
19 Yu. V. Nazarov and D. A. Bagrets, Phys. Rev. Lett. 88, 196801 (2002).
20 J. Börlin, W. Belzig, and C. Bruder, Phys. Rev. Lett. 88, 197001 (2002).
21 X. Barnabé-Thériault, A. Sedeki, V. Meden, and K. Schönhammer, Phys. Rev. Lett. 94, 136405 (2005).
22 A.L. Fetter and J.D. Walecka, Quantum Theory of Many Particle Systems, Mc Graw Hill, New York, 1971).
23 T. Enns, V. Meden, S. Andergassen, X. Barnabé-Thériault, W. Metzner, and K. Schönhammer, Phys. Rev. B 71, 155401 (2005).