QUANTUM-CLASSICAL INTERACTIONS
AND GALOIS TYPE EXTENSIONS *

Władysław Marcinek
Institute of Theoretical Physics, University of Wrocław,
Pl. Maxa Borna 9, 50-204 Wrocław,
Poland

Abstract

An algebraic model for the relation between a certain classical particle system and the quantum environment is proposed. The quantum environment is described by the category of possible quantum states. The initial particle system is represented by an associative algebra in the category of states. The key new observation is that particle interactions with the quantum environment can be described in terms of Hopf-Galois theory. This opens up a possibility to use quantum groups in our model of particle interactions.

1 Introduction

The study of highly organized structures of matter leads to the investigation of some non-standard physical particle systems and effects. The fractional quantum Hall effect provides an example of a system with a well-defined internal order [8, 7, 10, 27]. Other interesting structures appear in the so-called 1/2-electronic magnetotransport anomaly [1, 2, 3, 4], high temperature super-conductors or laser excitations of electrons. In these cases, anomalous behaviour of electrons occurs. An example is also given by the concept of statistical-spin liquids (see [5] and references therein).

It seems interesting to develop an algebraic approach to the unified description of all these new structures and effects. To this purpose, it is natural to assume that the whole world is divided into two parts: a classical particle system and its quantum environment. The classical system represents the observed reality, particles that really exist. The quantum environment represents all quantum possibilities that can become part of reality in the future [6]. The goal of this paper is to sketch a proposal of an algebraic model to

*This work was partially sponsored by the Polish Committee for Scientific Research (KBN) under Grant No. 5P03B05620.
describe interactions responsible for the appearance of the aforementioned highly organized structures. This model is based on a general algebraic formalism of Hopf algebras and Galois extensions of rings \[24\].

Our construction can be described in two steps. The first step concerns the transformation of the initial particles under interaction into composite systems consisting of quasi-particles and quanta. Such systems represent possible results of interactions \[17, 20, 22, 21\]. In the second step, we describe the algebra of realizations of quantum possibilities. This step is connected with the construction of an algebra extension and with a ‘decision’ which possibility can be realized and which one cannot. The problem how such ‘decisions’ are made was solved in \[19\] with the help of quantum commutativity and generalized Pauli exclusion principle. Our approach is based on the previously developed concept of particle systems with generalized statistics and quantum symmetries \[15, 17, 20, 22, 21, 23\].

The paper is organized as follows. In Section 2, we propose our general model within the framework of Hopf algebras and Galois extensions of rings. Then, in the subsequent section, we review Hopf-algebraic generalities. This recalls mathematical concepts employed in our proposal and allows us to specialize in the final section to the setting already appearing in some physical models. Since there are many interesting finite quantum groups related to spin coverings (e.g., \[3, 5\]) and the appearance of quantum symmetry in physics is more and more pronounced (e.g., \[4\]), our hope is that our general model will help us to understand some physical phenomena that cannot be adequately described by earlier methods.

## 2 The main idea

Let us consider a system of charged particles interacting with an external quantum environment. We assume that every charge is equipped with the ability to absorb and emit quanta of a certain nature. A system that contains a charge and a certain number of quanta as a result of interaction with the quantum environment is said to be a dressed particle \[16, 18\]. A particle dressed with a single quantum is a fictitious particle called quasi-particle. Our model is based on the assumption that every charged particle transforms under interaction into a composite system consisting of quasi-particles and quanta \[20\]. This system represents possible results of interactions. Note that the process of absorption of quanta by a charged particle should be described as the creation of quasi-particles, whereas the emission as the annihilation of quasi-particles.

In our model the quantum environment is represented by a tensor category \(\mathcal{C} = \mathcal{C}(\otimes, k)\) with duals \[13\]. All possible physical processes are represented as arrows of the category \(\mathcal{C}\). If \(f: U \rightarrow V\) is an arrow from \(U\) to \(V\), then the object \(U\) represents physical objects before interactions and \(V\) represents possible results of interactions. Different objects of the category \(\mathcal{C}\) describe physical objects of different nature, charged particles, quasi-particles or different species of quanta of an external field, etc. If \(U\) is an object of \(\mathcal{C}\)
representing particles, quasi-particles or quanta, then the object $\mathcal{U}^*$ corresponds to antiparticles, or quasi-holes or dual fields, respectively. In the same fashion, if $\mathcal{U}$ represents charged particles and $\mathcal{V}$ describes certain quanta, then the product $\mathcal{U} \otimes \mathcal{V}$ encodes a composite system containing both particles and quanta. An arrow $\mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{V}$ means an interaction causing the passage from a single particle state to a composite quantum system. Thus the arrow $\mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{V}$ describes a process of absorption. Much in the same way, we conclude that the arrow $\mathcal{U} \otimes \mathcal{V} \rightarrow \mathcal{U}$ describes a process of emission.

In our approach a unital and associative algebra $\mathcal{A}$ in the category $\mathcal{C}$ represents the classical states of a system. The multiplication $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is a morphism in this category representing the creation of a single object of reality from a composite system of objects of the same species. Quanta are encoded in a finitely generated coquasitriangular Hopf algebra $\mathcal{H}$. Quasi-particles are described by a new algebra $\mathcal{A}^{\text{ext}}$, which is an extension of $\mathcal{A}$. Interactions are described by a right action and coaction of $\mathcal{H}$ on the algebra $\mathcal{A}^{\text{ext}}$. It is natural to assume that the algebra $\mathcal{A}$ is invariant and coinvariant with respect to the action and coaction of $\mathcal{H}$, respectively, i.e., $\mathcal{A} = (\mathcal{A}^{\text{ext}})^{\mathcal{H}} = (\mathcal{A}^{\text{ext}})^{\text{co}\mathcal{H}}$. (Here $(\mathcal{A}^{\text{ext}})^{\mathcal{H}}$ is the set of $\mathcal{H}$-invariants and $(\mathcal{A}^{\text{ext}})^{\text{co}\mathcal{H}}$ the set of $\mathcal{H}$-coinvariants.)

We would like to represent the interaction of a charged particle with external quanta as a process of creation or annihilation of quasi-particles. A composite system of quasi-particles and quanta is described by a tensor product $\mathcal{A}^{\text{ext}} \otimes \mathcal{H}$ representing all possible quantum configurations coming as a result of the quantum absorption process. On the other hand, a composite system of two quasi-particles (related to the same particle) is described by a tensor product $\mathcal{A}^{\text{ext}} \otimes \mathcal{A}^{\text{ext}}$. When $\mathcal{H}$ is a group-ring Hopf algebra $\mathbf{k}G$, our model assumes that an element of $G$ can be understood as a specific charge characterizing an internal degree of freedom of a quasiparticle. We also assume that these charges are additive. Mathematically, this means that the algebra $\mathcal{A}^{\text{ext}}$ is $G$-graded, and that this grading is strong. As explained in the subsequent sections, the strongness of the $G$-grading of $\mathcal{A}^{\text{ext}}$ is known to be equivalent to the bijectivity of the canonical map

$$\beta : \mathcal{A}^{\text{ext}} \otimes \mathcal{A}^{\text{ext}} \rightarrow \mathcal{A}^{\text{ext}} \otimes \mathcal{H}. \quad (1)$$

The bijectivity of this map means that the coaction $\mathcal{A}^{\text{ext}} \rightarrow \mathcal{A}^{\text{ext}} \otimes \mathcal{H}$ is Galois. ($\mathcal{A}^{\text{ext}}$ is a Hopf-Galois $\mathcal{H}$-extension of $\mathcal{A}$.)

An advantage of the above Galois condition is that it makes sense for an arbitrary Hopf algebra $\mathcal{H}$ and does not force $\mathcal{A}^{\text{ext}}$ to be a crossed-product algebra $\mathcal{A}^{\text{ext}} \rtimes \mathcal{H}$ [4]. Thus, if we think of $\mathcal{H}$ as ‘the group algebra of a quantum group’, we have a rather general mathematical formalism capable of describing quasi-particles with the possible charges that are labeled by ‘the elements of a quantum group’ and additive according to the multiplication of $\mathcal{H}$. This way the Galois condition corresponds to the additivity of charges.
3 Quantum commutativity and Hopf-Galois extensions

Let $\mathcal{H}$ be a Hopf algebra over a ground field $k$, and let $m, \eta, \Delta, \varepsilon, S$ denote its multiplication, unit, comultiplication, counit and antipode, respectively. We use the following notation for the coproduct in $\mathcal{H}$. If $h \in \mathcal{H}$, then $\Delta(h) := \sum h_{(1)} \otimes h_{(2)} \in \mathcal{H} \otimes \mathcal{H}$. We assume that $\mathcal{H}$ is a coquasitriangular Hopf algebra (CQTHA) (e.g., see [24, p.184]). This means that $\mathcal{H}$ is equipped with a convolution invertible homomorphism $b \in \text{Hom}(\mathcal{H} \otimes \mathcal{H}, k)$ such that

\begin{align}
\sum b(h_{(1)}, k_{(1)}) k_{(2)} h_{(2)} &= \sum h_{(1)} k_{(1)} b(h_{(2)}, k_{(2)}), \\
b(h, kl) &= \sum b(h, k_{(2)}) b(k_{(1)}, l), \\
b(hk, l) &= \sum b(h, l_{(2)}) b(k, l_{(1)}),
\end{align}

for every $h, k, l \in \mathcal{H}$. We call such a bilinear form $b$ a coquasitriangular structure on $\mathcal{H}$.

Another ingredient of our model is the concept of quantum commutativity [2, 19]. Let $\mathcal{A}$ be a unital and associative algebra and $\mathcal{H}$ be a Hopf algebra. If $\mathcal{A}$ is a right $\mathcal{H}$-comodule such that the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit map $\eta : k \to \mathcal{A}$ are $\mathcal{H}$-comodule maps, then we say that it is a right $\mathcal{H}$-comodule algebra. The algebra $\mathcal{A}$ is said to be quantum commutative with respect to the coaction of $\mathcal{H}$ and its coquasitriangular structure $b$ if an only if we have the relation

$$ab = \sum b(a_{(1)}, b_{(1)}) b_{(0)} a_{(0)}.$$  

(5)

Here $\rho(a) = \sum a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes \mathcal{H}$, and $\rho(b) = \sum b_{(0)} \otimes b_{(1)} \in \mathcal{A} \otimes \mathcal{H}$ for every $a, b \in \mathcal{A}$. The Hopf algebra $\mathcal{H}$ is called a quantum symmetry of $\mathcal{A}$.

Finally, let us recall the definition of a Hopf-Galois extension. An algebra extension $\mathcal{A}^{ext}$ of $\mathcal{A}$ such that it is a right $\mathcal{H}$-comodule algebra and $\mathcal{A}$ is its coinvariant subalgebra

$$\mathcal{A} \equiv (\mathcal{A}^{ext})^{co\mathcal{H}} := \{ a \in \mathcal{A}^{ext} : \delta(a) = a \otimes 1 \}$$

(6)

is said to be an $\mathcal{H}$-extension. If in addition the map $\beta : \mathcal{A}^{ext} \otimes \mathcal{A}^{ext} \to \mathcal{A}^{ext} \otimes \mathcal{H}$ defined by

$$\beta(a \otimes b) := (a \otimes 1) \delta(b)$$

(7)

is bijective, then the $\mathcal{H}$-extension is called Hopf-Galois. If $\mathcal{A}^{ext}$ is a Hopf-Galois $\mathcal{H}$-extension, then there is also a bijection

$$\beta^n : \underbrace{\mathcal{A}^{ext} \otimes \cdots \otimes \mathcal{A}^{ext}}_{n+1} \leftrightarrow \underbrace{\mathcal{A}^{ext} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_n$$

(8)

given by

$$\beta^n := (\beta \otimes id) \circ \cdots \circ (id \otimes \mathcal{A} \otimes \beta \otimes id) \circ (id \otimes \mathcal{A} \otimes \beta).$$

(9)

In our physical interpretation, the one-to-one correspondence $\beta^n$ means that the $n$-th $\mathcal{A} \otimes$-tensor product representing a composite system of $n$ quasi-particles also corresponds to a system of a single quasi-particle and $n$ quanta.
4 Strongly $G$-graded quantum-commutative algebras

Recall first that the group algebra $kG$ is a Hopf algebra for which the comultiplication, the counit, and the antipode are given by the formulae

$$\Delta(g) := g \otimes g, \quad \varepsilon(g) := 1, \quad S(g) := g^{-1},$$

respectively. The coquasitriangular structure on $kG$ is given by a commutation factor $b : G \times G \to k \setminus \{0\}$ \cite{24, 25, 14, 25}, and the category of right $H$-comodules is equivalent to the category of $G$-graded vector spaces.

Next, assume that an algebra $A^{ext}$ is an object of this category. This means that it is a $G$-graded algebra. Now we come to the crucial theorem \cite[p.126]{24} stating that, for an arbitrary $G$-graded algebra and $kG$-coaction compatible with the grading ($\rho(a) = a \otimes g$ for $a \in A^{ext}_g$), the coaction is Galois if and only if the algebra is strongly $G$-graded. The latter means that

$$A^{ext} = \bigoplus_{g \in G} A^{ext}_g, \quad A^{ext}_g A^{ext}_h = A^{ext}_{gh}, \quad A^{ext}_e \equiv A, \quad \text{(10)}$$

where $e$ is the neutral element of $G$.

As an example, let us consider a $G$-graded $b$-commutative $\mathbb{C}$-algebra $A^{ext}$ with the so-called standard gradation \cite{14}. This means that we take as the strongly grading group $Z^N := Z \oplus \ldots \oplus Z$ and assume

$$b(\xi^i, \xi^j) =: b^{ij} = (-1)^{\Sigma_{ij}} q^{\Omega_{ij}}. \quad \text{(11)}$$

Here $\xi^i := (0, \ldots, 1, \ldots, 0)$ (1 on the $i$-th place) is the set of generators of $Z^N$, $\Sigma := (\Sigma_{ij})$ and $\Omega := (\Omega_{ij})$ are integer-valued matrices such that $\Sigma_{ij} = \Sigma_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$, and $q \in \mathbb{C} \setminus \{0\}$ is a parameter \cite{22}. Since our Hopf algebra is a group ring, the equation \cite{3} for the bilinear form $b$ is automatically satisfied, whereas the equations \cite{3, 4} uniquely determine $b$ once we set its value on the generators. The convolution-invertibility of $b$ follows from the fact that $b^{ij}$ are always non-zero. (Notice that $b(\xi^i, (\xi^j)^n) = (b^{ij})^n$, so that for $q = exp(\frac{2\pi i}{n})$ and $n$ even, the grading group $Z^N$ can be reduced to $Z_n \oplus \ldots \oplus Z_n$.) Combining \cite{3} with \cite{11}, we obtain the following quantum commutativity relations:

$$a_{\xi^i} a_{\xi^j} = b^{ij} a_{\xi^j} a_{\xi^i}, \quad \text{where} \quad a_{\xi^i} \in A^{ext}_{\xi^i}, \quad a_{\xi^j} \in A^{ext}_{\xi^j}. \quad \text{(12)}$$

It is the behaviour of $b^{ij}$ that determines whether we obtain a system with the $q$-statistics, or Fermi statistics and the Pauli exclusion principle, or whether we obtain bosons. On the other hand, the strong gradation ensures that the internal degrees of freedom of a quasi-particle are labeled by charges ($N$-tuples of integers), and that these charges are additive.

Acknowledgments. It is a pleasure to thank Cezary Juszczak for his help with typesetting this article.
References

[1] K. BYCZUK AND J. SPALEK, Universality classes, statistical exclusion principle, and properties of interacting fermions, Phys. Rev. B51, 7934 (1995).

[2] M. COHEN AND S. WESTRICH, Quantum commutative algebras, J. Alg. 168, 1 (1994).

[3] A. CONNES, Gravity coupled with matter and the foundation of non-commutative geometry, Comm. Math. Phys. 182 (1996), 155–176.

[4] A. CONNES AND D. KREIMER, From local perturbation theory to Hopf and Lie algebras of Feynman graphs. Lett. Math. Phys. 56 (2001), 3–15.

[5] L. DĄBROWSKI AND C. REINA, Quantum spin coverings and statistics, math.QA/0208088

[6] R.R. DU, H.L. STORMER, D.C. TSUI, A.S. YEH, L.N. PFEIFFER AND K.W. WEST, Drastic Enhancement of Composite Fermion Mass near Landau Level Filling v= \( \frac{1}{2} \), Phys. Rev. Lett. 73, 3274 (1994)

[7] Z.F. EZAWA AND H. HOTTA, Field theory of anyons and the fractional quantum Hall effect, Phys. Rev. B 46, 7765 (1992)

[8] A.C. GOSSARD, H.L. STORMER AND D.C. TSUI, Two-dimensional magnetotransport in the extreme quantum limit, Phys. Rev. Lett. 48 1559 (1982)

[9] R. HAAG, An evolutionary picture for quantum physics, Commun. Math. Phys. 180, 733 (1996).

[10] B.I. HALPERIN, P.A. LEE AND N. READ, Theory of the half-filled Landau level, Phys. Rev. B 46, 7312 (1993)

[11] J.K. JAIN, Composite-fermion approach for the fractional quantum Hall effect, Phys. Rev. Lett. 63, 199 (1989)

[12] J.K. JAIN, Incompressible quantum Hall states, Phys. Rev. B 40, 8079 (1989)

[13] J.K. JAIN, Theory of the fractional quantum Hall effect, Phys. Rev. B 41, 7653 (1990)

[14] W. MARCINEK, On unital braidings and quantization, Rep. Math. Phys. 34, 325 (1994).

[15] W. MARCINEK, Categories and quantum statistics, in Proceedings of the symposium: Quantum Groups and their Applications in Physics, Poznań, 17–20 October 1995, Poland, Rep. Math. Phys. 38, 149–179 (1996)

[16] W. MARCINEK, Topology and quantization, in Proceeding of the IVth International School on Theoretical Physics, Symmetry and Structural Properties, Zajączkowo k. Poznań, 29 August – 4 September 1996, Poland, ed. by B. Lulek, T. Lulek and R. Florek, pp. 415–124, World Scientific, Singapore 1997, hep-th/9705098.
[17] W. Marcinek, Remarks on Quantum Statistics, in Proceedings of the Conference “Particles, Fields and Gravitation”, 15–19 April 1998, Lódź, Poland, ed. by J. Rembieliński, World Scientific, Singapore 1998, and math.QA/9806158.

[18] W. Marcinek, On Commutation Relations for Quons, Rep. Math. Phys. 41, 155 (1998).

[19] W. Marcinek, Particles and quantum symmetries, in Proceedings of the XVI Workshop on Geometric Methods in Physics, 1–7 July 1997, Białowieża, Poland, math.QA/9805124, Rep. Math. Phys. 43, 239 1999.

[20] W. Marcinek, On composite systems and quantum statistics, in the Proceedings of the Vth International School on Theoretical Physics, Symmetry and Structural Properties, Zajączkowki Poznań, 27 August – 2 September 1998, Poland, ed. by B. Lulek, T. Lulek and A. Wal, World Scientific, Singapore 1999, math.QA/9810060.

[21] W. Marcinek, On generalized statistics and one dimensional systems, in Proceedings of the III International Seminar “Hidden Symmetry”, Rzeszów, 20–22 October 1998, Poland, Mol. Phys. Rep. 23, 170–173 (1999).

[22] W. Marcinek, On generalized quantum statistics, in Proceedings of the XII-th Max Born Symposium, Wrocław, 23–26 September 1998, Poland, ed. by A. Borowiec et al, Theoretical Physics Fin de Siecle, Springer 2000.

[23] W. Marcinek, On generalized statistics and interactions, in Proceedings of the XVI Workshop on Geometric Methods in Physics, 1–7 July 1998, Białowieża, Poland, in Coherent States, Quantization and Gravity, ed. by M. Schlichenmaier et al, Wydawnictwa Uniwersytetu Warszawskiego, Warszawa 2001, and math.QA/990029.

[24] S. Montgomery, Hopf algebras and their actions on rings, Regional Conference Series in Mathematics, No 82, AMS 1993.

[25] Z. Oziewicz, Lie algebras for arbitrary grading group, in Differential Geometry and Its Applications, ed. by J. Janyska and D. Krupka, World Scientific, Singapore 1990.

[26] M. Scheunert, Generalized Lie algebras, J. Math. Phys. 20, 712, (1979).

[27] A. Zee, Quantum Hall fluids, in Field Theory, Topology and Condensed Matter Physics, ed. by H.D. Geyer, Lecture Notes in Physics, Springer 1995.