Realizations of observables in Hamiltonian systems with first class constraints.

A.V. Bratchikov
Kuban State Technological University,
2 Moskovskaya Street, Krasnodar, 350072, Russia
E-mail: bratchikov@kubstu.ru

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Abstract

In a Hamiltonian system with first class constraints observables can be defined as elements of a quotient Poisson bracket algebra. In the gauge fixing method observables form a quotient Dirac bracket algebra. We show that these two algebras are isomorphic. A new realization of the observable algebras through the original Poisson bracket is found. Generators, brackets and pointwise products of the algebras under consideration are calculated.
In a Hamiltonian system with the first class constraints \( \varphi_j(p,q) \), \( j = 1\ldots J \),

\[
\left[ \varphi_i, \varphi_j \right]|_{\varphi=0} = 0, \tag{1}
\]

physical functions are elements of a Poisson bracket algebra of the first class functions

\[
P = \{ f(p,q) \mid [f, \varphi_j]|_{\varphi=0} = 0 \}. \tag{2}
\]

Here \( p = (p_1, \ldots, p_n) \), \( q = (q_1, \ldots, q_n) \) are the canonical coordinates. Observables are elements of the algebra \( P/I \), where

\[
I = \{ u(p,q) \mid u|_{\varphi=0} = 0 \}
\]

(see e.g. [1] and references therein). These definitions correspond to the Dirac quantization [2] without gauge fixing.

In the gauge fixing method [3] the gauge functions \( \chi_i(p,q) \), \( i = 1\ldots J \), are introduced which serve as auxiliary constraints. \( \chi_i \) are supposed to satisfy the conditions

\[
\det([\chi_i, \varphi_j]|_{\varphi=\chi}=0) \neq 0
\]

and the constraints \((\pi_\alpha) = (\varphi_1, \ldots, \varphi_J, \chi_1, \ldots, \chi_J)\) are second class. Then the original Poisson bracket is replaced by the Dirac one

\[
[g, h]_D = [g, h] - [g, \pi_\alpha]c_{\alpha\beta}[\pi_\beta, h], \quad c_{\alpha\beta}[\pi_\beta, \pi_\gamma] = \delta_{\alpha\gamma}.
\]

The constraints \((\pi_\alpha)\) are first class with respect to the Dirac bracket and physical functions are defined by the equations

\[
[f, \pi_\alpha]_D|_{\pi=0} = 0
\]

which are satisfied identically. Let \( A \) be the space of all the functions on phase space and

\[
\Phi = \{ v(p,q) \mid v|_{\pi=0} = 0 \}.
\]

The algebra of observables in the gauge fixing method is the Dirac bracket algebra \( A/\Phi \). In fact, \( A/\Phi \) is a Poisson algebra, as well as \( P/I \). A connection between the classical Hamilton equations which correspond to these two methods is described in [1].
In the present paper we show that $P/I$ and $A/\Phi$ are isomorphic as Poisson algebras.

In a recent article [4] a family of the new algebras with respect to the original Poisson bracket was constructed which are isomorphic to a Dirac bracket algebra. The algebras which are isomorphic to $A/\Phi$ give us realizations of $P/I$.

Using gauge fixing functions we find a new realization of $P/I$. It looks like $Q/K$, where $Q$ and $K$ are Poisson subalgebras of $P$ and $I$ respectively.

To describe elements of $P/I$ and prove the isomorphisms we find a local solution to equations (2). Similar equations determine elements of $Q$. Explicit expressions for generators enable us to calculate brackets and pointwise products for the observable algebras $P/I$ and $Q/K$.

We shall assume that all the quantities vanishing on a constraint surface are linear functions of the constraints.

2. To find elements of $P$ explicitly let us consider the equations

$$[f, \varphi_j] \in I. \quad (3)$$

with the initial condition

$$f(p, q) \in \{f_0(p, q)\}. \quad (4)$$

Here $\{f_0\} \in A/\Phi$ is the coset represented by $f_0 \in A$.

Due to (4)

$$f = f_0 + r_i \varphi_i + s_j \chi_j \quad (5)$$

for some $r_i = r_i(p, q), s_j = s_j(p, q)$. Substituting this into equation (3), we get

$$[f_0, \varphi_i] + \chi_j[s_j, \varphi_i] + s_j[\chi_j, \varphi_i] \in I$$

or

$$\psi_k + (Bs)_k \in I.$$  

Here

$$\psi_k = [f_0, \varphi_i]b_{ik}, \quad (Bs)_k = s_k + \chi_j[s_j, \varphi_i]b_{ik}, \quad [\chi_i, \varphi_j]b_{jk} = \delta_{ik}.$$
We assume that the operator $B$ is locally invertible. For $u_i \in I$ we have $(B^{-1}u)_i \in I$. Hence

$$s_j = -(B^{-1}\psi)_j + s_{jk}\varphi_k$$

for some functions $s_{jk}(p, q)$. Expressions (5, 6) give us a solution to equations (3) with initial condition (4).

We have shown that for any $f_0 \in A$ the set $\{f_0\} \cap P$, consists of all the expressions

$$f = L(f_0) + r_i\varphi_i.$$  

Here $r_i(p, q)$ are arbitrary functions and

$$L(f_0) = f_0 - \chi_j(B^{-1}\psi)_j.$$  

From this it follows

$$\{f_0\} \cap P = \{L(f_0)\}_{P/I}.  \tag{7}$$

Here $\{L(f_0)\}_{P/I} \in P/I$ denotes the coset represented by $L(f_0) \in P$.

To show that $P/I$ is isomorphic to $A/\Phi$ let us define the linear function $T : P/I \to A/\Phi$

$$T(\{f\}_{P/I}) = \{f\}.  \tag{8}$$

Due to (7) the inverse function $T^{-1} : A/\Phi \to P/I$ is given by

$$T^{-1}(\{f_0\}) = \{L(f_0)\}_{P/I}.$$  

To show that $T$ is a homomorphism let us compute

$$T([\{f\}_{P/I}, \{g\}_{P/I}]) = T([\{f,g\}_{P/I}]).$$

Due to (11) for $f, g \in P$

$$[f,g] - [f,g]_D \in I.$$  

From this and definition (8) it follows

$$T([\{f,g\}_{P/I}]) = T([\{f,g\}_D]_{P/I}) = [[f,g]_D] = [[f],[g]]_D = [T(\{f\}_{P/I}), T(\{g\}_{P/I})]_D.$$
Hence the Dirac bracket algebra $A/\Phi$ is isomorphic to the Poisson bracket algebra $P/I$.

It is easy to check that $T$ is also an isomorphism with respect to the pointwise multiplication. We get

$$T(\{f\}_{P/I}\{g\}_{P/I}) = T(\{fg\}_{P/I}) = \{fg\} = \{f\}\{g\} = T(\{f\}_{P/I})T(\{g\}_{P/I}).$$

Thus we have shown that $A/\Phi$ and $P/I$ are isomorphic as Poisson algebras.

3. Let us define the space

$$Q = \{ F \in P \mid \{\chi_j, F\} \in I \}.$$  \hspace{1cm} (9)

One can check that $Q$ is a Poisson algebra and $K = Q \cap I$ is an ideal of $Q$.

Let $F$ be a solution to equations (9) with the initial condition

$$F(p,q) \in \{F_0(p,q)\}_{P/I}.$$  

The function $F$ can be represented in the form

$$F = F_0 + \nu_i \phi_i,$$  \hspace{1cm} (10)

for some $\nu_i = \nu_i(p,q)$. Substituting (10) into equations (9) we get

$$[\chi_j, F_0] + \nu_i [\chi_j, \phi_i] \in I.$$  

A solution to these equations is

$$\nu_i = -b_{ij} [\chi_j, F_0] + \nu_{ij} \phi_j.$$  

Here $\nu_{ij} = \nu_{ij}(p,q)$ are arbitrary functions.

We have proven that for any $F_0 \in P$ the set $\{F_0\}_{P/I} \cap Q$ consists of all the expressions

$$F = R(F_0) + \nu_{ij} \phi_i \phi_j,$$

where

$$R(F_0) = F_0 - b_{ij} [\chi_j, F_0] \phi_i.$$

From this it follows

$$\{F_0\}_{P/I} \cap Q = \{R(F_0)\}_{Q/K}.$$  \hspace{1cm} (11)
Here \( \{R(F_0)\}_{Q/K} \in Q/K \) is the coset represented by \( R(F_0) \in Q \).

Our aim is to show that \( Q/K \) is isomorphic to \( P/I \). Let us define the linear function \( S : Q/K \rightarrow P/I \)
\[
S(\{F\}_{Q/K}) = \{F\}_{P/I}.
\]
Due to equation (11), the inverse function \( S^{-1} : P/I \rightarrow Q/K \) is given by
\[
S^{-1}(\{F_0\}_{P/I}) = \{R(F_0)\}_{Q/K}.
\]
To show that \( S \) is a homomorphism we have the following computations
\[
S(\{[F,G]_{Q/K}\}) = \{[F,G]_{P/I}\} = \{S(\{F\}_{Q/K}), S(\{G\}_{Q/K})\}.
\]
Hence \( Q/K \) and \( P/I \) are isomorphic as Poisson bracket algebras. It is easy to check that \( S \) is also a homomorphism with respect the pointwise multiplication
\[
S(\{F\}_{Q/K}, \{G\}_{Q/K}) = S(\{F\}_{Q/K})S(\{G\}_{Q/K}).
\]
This tells us that \( Q/K \) and \( P/I \) are isomorphic as Poisson algebras.

Brackets and pointwise products for observables can be calculated as follows. One can check that for \( f = L(f_0) \in P \) and \( g = L(g_0) \in P \) the functions \([f,g] \) and \( fg \) satisfy equations (9) with the initial conditions \([f,g] \in \{[f_0,g_0]_D\} \) and \( fg \in \{f_0g_0\} \) respectively. Due to (11)
\[
[f,g] = L([f_0,g_0]_D) + \tilde{u}, \quad fg = L(f_0g_0) + \tilde{w},
\]
where \( \tilde{u}, \tilde{w} \in I \). From this it follows
\[
\{[f]_{P/I}, \{g\}_{P/I}\} = \{L([f_0,g_0]_D)\}_{P/I},
\]
\[
\{f\}_{P/I}\{g\}_{P/I} = \{L(f_0g_0)\}_{P/I}.
\]
Consider the algebra \( Q/K \). For \( F = R(F_0) \in Q \) and \( G = R(G_0) \in Q \) the functions \([F,G] \) and \( FG \) satisfy equations (12) with the initial conditions \([F,G] \in \{[F_0,G_0]_{P/I}\} \) and \( FG \in \{F_0G_0\}_{P/I} \) respectively. According to (11)
\[
[F,G] = R([F_0,G_0]) + \tilde{U}, \quad FG = R(F_0G_0) + \tilde{W},
\]
where \( \tilde{U}, \tilde{W} \in K \). Therefore, we have
\[
\{F\}_{Q/K}, \{G\}_{Q/K} = \{R([F_0,G_0])\}_{Q/K},
\]
\[
\{F\}_{Q/K}\{G\}_{Q/K} = \{R(F_0G_0)\}_{Q/K}.
\]

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