Review of the Onsager “Ideal Turbulence” Theory

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In his famous undergraduate physics lectures, Richard Feynman remarked about the problem of fluid turbulence: “Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences” [1]. This statement was already false when Feynman made it. Unbeknownst to him, Lars Onsager decades earlier had made an exact mathematical analysis of the high Reynolds-number limit of incompressible fluid turbulence, using a method that would now be described as a non-perturbative renormalization group analysis and discovering the first “conservation-law anomaly” in theoretical physics. Onsager’s results were only cryptically announced in 1949 and he never published any of his detailed calculations. Onsager’s analysis was finally rescued from oblivion and reproduced by this author in 1992. The ideas have subsequently been intensively developed in the mathematical PDE community, where deep connections emerged with John Nash’s work on isometric embeddings. Furthermore, Onsager’s method has more recently been successfully applied to new physics problems, such as compressible fluid turbulence and relativistic fluid turbulence, yielding many novel testable predictions. This note will explain Onsager’s exact analysis of incompressible turbulence using modern ideas on renormalization group and conservation-law anomalies, and it will also very briefly review subsequent developments.

I. INTRODUCTION

Onsager’s several contributions to the theory of turbulence have already been reviewed from a history of science point of view [2]. This note is instead intended to give a busy, working physicist a concise, accurate and painless explanation of Onsager’s theory of “ideal turbulence” for a low Mach-number fluid, described by the incompressible Navier-Stokes equation

\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \]  (I.1)

For previous physics explanations of the Onsager theory, see [3] and, for an extended, pedagogical presentation, the course notes of [4]. None of these works, however, explain the theory systematically from the point of view of renormalization group, as we do here. We shall avoid full rigorous details (which can be readily found in the cited papers) and instead focus on the intuitive ideas, exact calculations, and essential estimates.

II. DIVERGENCES AND REGULARIZATION

The key empirical fact underlying the Onsager theory is the non-vanishing of turbulent energy dissipation in the zero-viscosity limit. This was first suggested on a semi-phenomenological basis by Taylor [3], who argued that energy could be dissipated “in fluid of infinitesimal viscosity”. More properly, the phenomenon occurs in the limit of high Reynolds numbers. When the equations are rescaled by characteristic large length \( L \) and velocity \( U \), then in terms of the dimensionless variables

\[ \hat{x} = x/L, \quad \hat{t} = t/(L/U), \quad \hat{\mathbf{u}} = \mathbf{u}/U, \quad \hat{p} = p/U^2, \]  (II.1)

the Navier-Stokes equation assume the similarity form

\[ \partial_t \hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \hat{\nabla}) \hat{\mathbf{u}} = -\hat{\nabla} \hat{p} + \frac{1}{Re} \hat{\nabla}^2 \hat{\mathbf{u}}, \quad \hat{\nabla} \cdot \hat{\mathbf{u}} = 0 \]  (II.2)

with \( Re = U L/\nu = 1/\hat{\nu} \) the Reynolds number. Hereafter we omit the hats \( (\hat{\cdot}) \) and understand that the limit \( \nu \to 0 \) is really to be interpreted as the limit \( Re \to \infty \). Laboratory experiments [6, 7] and numerical simulations [8, 9] both confirm that the kinetic energy dissipation rate

\[ \varepsilon(x, t) := \nu |\nabla_x \hat{\mathbf{u}}(x, t)|^2 \]  (II.3)

has space-average converging in the limit as \( \nu \to 0 \) and not vanishing: \( \langle \varepsilon(t) \rangle \to \langle \varepsilon_\ast(t) \rangle > 0 \). It is furthermore observed in experiment that when integrated over small balls or cubes in space the high-\( Re \) limit \( \epsilon_\ast(x, t) \) defines a positive measure with multifractal scaling [10, 11].

The most obvious requirement for such a non-vanishing limit of dissipation is that space-gradients of velocity must diverge, \( \nabla_x \hat{\mathbf{u}} \to \infty \), as \( \nu \to 0 \). This is a short-distance/ultraviolet (UV) divergence in the language of quantum field-theory, or what Onsager himself termed a “violet catastrophe” [12]. The inviscid limit for turbulent fluids is analogous to a “continuum” or “critical” limit in quantum field-theory, where a scale-invariant regime is expected and similar UV divergences are encountered. Since the fluid equations of motion [13] contain diverging gradients, they become ill-defined in the limit. In order to develop a dynamical description which can be valid even as \( \nu \to 0 \), some regularization of this divergence must be introduced. Here we shall employ a simple coarse-graining or “block-spin” regularization, defining a coarse-grained velocity field at length-scale \( \ell \) by local space-averaging:

\[ \mathbf{u}_\ell(x, t) = \int d^3 r \, G_\ell(r) \, \mathbf{u}(x + r, t). \]  (II.4)
The coarse-graining kernel \( G_\ell(r) = \ell^{-3}G(r/\ell) \) may be chosen rather freely, subject to general constraints that the function \( G \) be

\[
\begin{align*}
G(r) &\geq 0 \quad \text{(non-negative)} \\
\int d^3r \ G(r) &= 1 \quad \text{(normalized)} \\
\int d^3r \ r \ G(r) &= 0 \quad \text{(centered)} \\
\int d^3r \ |r|^2 G(r) &= 1 \quad \text{(unit variance)} \quad (\text{II.5})
\end{align*}
\]

Also, we require that \( G \) be both smooth and rapidly decaying in space, e.g. \( G \in C_0^\infty(\mathbb{R}^3) \), the space of infinitely-differentiable functions with compact support. We also for convenience assume isotropy, or \( G = G(r) \) with \( r = |r| \), so that \( \int d^3r \ r_i r_j G(r) = (1/3)\delta_{ij} \). We note in passing that an identical coarse-graining operation is employed in the “filtering approach” to turbulence advocated by Germano \cite{13}, but with a different motivation than regularization of divergences. For us, coarse-graining is a convenient choice as a regularizer for many purposes, but not the only one possible and not always even adequate (e.g. see section VII-A of \cite{14}).

The coarse-graining operation \( \text{II.4} \) clearly regularizes gradients, so that \( \nabla_x \bar{u} \) remains finite as \( \nu \to 0 \) for any fixed length \( \ell > 0 \). This may be shown formally using the simple identity

\[
\nabla_x \bar{u}(x, t) = -\frac{1}{\ell} \int d^3r \ (\nabla G)_\ell(r) \ u(x + r, t), \quad (\text{II.6})
\]

which by Cauchy-Schwartz inequality yields the bound

\[
|\nabla_x \bar{u}(x, t)| \leq \frac{1}{\ell} \sqrt{C_\ell \int d^3r |u(r, t)|^2}
\]

with constant \( C_\ell = \int d^3r \ |(\nabla G)_\ell(r)|^2 \). Thus, the coarse-grained gradient is bounded as long as the total energy remains finite as \( \nu \to 0 \) (which, for instance, is necessarily true for freely-decaying turbulence with no stirring). The price of this regularization is that a new, arbitrary length-scale \( \ell \) has been introduced. Although the description of turbulent phenomena will depend very strongly on a particular choice, it is clear that no objective physical fact can depend upon the arbitrary scale \( \ell \). The coarse-graining performed in \( \text{II.4} \) is a purely passive operation, which corresponds to observing the fluid at spatial resolution \( \ell \), whereas objective facts cannot depend upon the “eyesight” of the observer. This situation exactly parallels that in quantum field-theory, where regularization of ultraviolet divergences in weak-coupling perturbation theory introduces a new arbitrary momentum scale \( \mu \) at which coefficients of the renormalized theory are defined and upon which objective physics cannot depend. This is the statement of what is called “renormalization-group invariance” in quantum field-theory and condensed matter physics and renormalization group (RG) methods can be understood as the systematic exploitation of the invariance of the physics to changes of this arbitrary regularization scale (see \cite{15}, section 4).

We follow a similar strategy here. An energy dissipation rate which is non-vanishing in the limit as \( \nu \to 0 \) implies that energy decrease over a fixed interval of time \([0, \ell]\) will persist in the inviscid limit. On the other hand, an observer at the coarse-grained scale \( \ell \) can only miss some kinetic energy of smaller eddies, since by convexity

\[
\frac{1}{2} |\bar{u}(x, t)|^2 \leq \frac{1}{2} (|u(x, t)|^2), \quad (\text{II.7})
\]

and it then follows that \( E_\ell(t) := (1/2) \int d^3x \ |u(x, t)|^2 \leq (1/2) \int d^3x |u(x, t)|^2 = E(t) \). If kinetic energy continues to decay even in the limit as \( \nu \to 0 \), then such persistent energy decay must also be seen by the “myopic” observer who observes fluid features only at space-resolution \( \ell \). As we now show, however, the persistent energy decay observed at the fixed length-scale \( \ell \) with \( \nu \to 0 \) is not due to molecular viscosity acting directly at those scales.

III. COARSE-GRAINED EULER AND ENERGY CASCADE

The equation obeyed by the coarse-grained velocity field defined in \( \text{II.4} \) is easily found to be

\[
\partial_t \bar{u}_\ell + \nabla_x (uu)_\ell = -\nabla_x p_\ell + \nu \Delta \bar{u}_\ell, \quad \nabla_x \cdot \bar{u}_\ell = 0, \quad (\text{III.1})
\]

because the coarse-graining operation commutes with all space- and time-derivatives. By writing

\[
\nu \Delta \bar{u}_\ell(x, t) = -\frac{\nu}{\ell} \int d^3r \ (\nabla G)_\ell(r) \nabla_x u(x + r, t), \quad (\text{III.2})
\]

and by again applying Cauchy-Schwartz inequality, one obtains \( \text{II.6} \)

\[
|\nu \Delta \bar{u}_\ell(x, t)| \leq \sqrt{\nu C_\ell \int d^3r |(\nabla G)_\ell(r)|^2 \epsilon |x + r|}. \quad (\text{III.3})
\]

Since the integral \( \int d^3r |(\nabla G)_\ell(r)|^2 \epsilon |x + r| \) as \( \nu \to 0 \) converges to \( \int d^3r |(\nabla G)_\ell(r)|^2 \epsilon |x + r|, \) then the viscous diffusion term in \( \text{III.1} \) has an upper bound \( G(\sqrt{\nu}) \) and thus vanishes for fixed \( \ell \) in the limit \( \nu \to 0 \). The result is a simplified set of dynamical equations

\[
\partial_t \bar{u}_\ell + \nabla_x (uu)_\ell = -\nabla_x \bar{p}_\ell, \quad \nabla_x \cdot \bar{u}_\ell = 0, \quad (\text{III.4})
\]

at the range of scales \( \ell \) where the viscosity term is negligible. This set of length-scales \( \ell \) constitute what is called the “inertial-range” in the turbulence literature, where the direct action of viscosity is vanishingly small. The dynamical equations \( \text{III.4} \) are what we shall term “coarse-grained Euler equations” at the length-scale \( \ell \). This notion formalizes the intuitive idea that the inertial-range eddies should have their dynamics governed by the ideal Euler equations \( \text{II.7} \). If a strong limit \( \bar{u}_\ell = \lim_{\nu \to 0} u \) exists, then the coarse-grained equations \( \text{III.4} \) hold for \( \bar{u}_\ell \) at any length-scale \( \ell > 0 \) and the inviscid limit field is what in mathematics is called a “distributional” or “weak” solution of the incompressible Euler equations. See Propositions 1 and 2 in \( \text{II.16} \).
Although Euler equations hold in the coarse-grained sense within the inertial-range of scales, the energy contained within those eddies is not conserved in time. As the argument below \((\text{III.7})\) demonstrates, if energy dissipation persists in the limit as \(\nu \to 0\), then energy must also decay for the observer with space-resolution \(\ell\), even though for sufficiently small \(\nu\) the fluid motions at the fixed scale \(\ell\) are described by “Euler equations”. The resolution of this seeming paradox is that the statement that the velocity field \(u\) satisfies the “coarse-grained Euler equations” \((\text{III.3})\) at scale \(\ell\) is quite different from the statement that the coarse-grained velocity \(\bar{u}\) satisfies the Euler equations in the naive sense. To make this point very clearly, we introduce the “turbulent” or “sub-scale” stress-tensor

\[
\tau_\ell(u, u) = (uu)_\ell - \bar{u}_\ell \bar{u}_\ell, \tag{III.5}
\]

in terms of which the “coarse-grained Euler equations” \((\text{III.4})\) may be equivalently written as \([9]\)

\[
\partial_t \bar{u}_\ell + \nabla_x \cdot (\bar{u}_\ell \bar{u}_\ell + \tau_\ell) = - \nabla_x \bar{p}_\ell + \nu \Delta \bar{u}_\ell, \quad \nabla_x \cdot \bar{u}_\ell = 0, \tag{III.6}
\]

Only for \(\tau_\ell \equiv 0\) does \((\text{III.6})\) correspond to \(\bar{u}\) satisfying the incompressible Euler equations in the naive sense. Once one understands that the inertial-range eddies satisfy the “Euler equations” only in the coarse-grained sense of \((\text{III.3})\) or \((\text{III.6})\), then it is no mystery how the energy is dissipated at those scales. The local kinetic energy balance at length-scales \(\ell\) in the inertial-range is easily calculated to be

\[
\partial_t \left( \frac{1}{2} |\bar{u}|^2 \right) + \nabla_x \cdot \left( \frac{1}{2} |\bar{u}|^2 + \bar{p}_\ell \right) \bar{u}_\ell + \tau_\ell \bar{u}_\ell = - \Pi_\ell, \tag{III.7}
\]

where the quantity on the right side of the equation,

\[
\Pi_\ell(x, t) = - \nabla_x \bar{u}_\ell(x, t) \cdot \tau_\ell(x, t), \tag{III.8}
\]

is the “deformation work” \([19]\) of the large-scale strain acting against small-scale stress, or the “energy flux” from resolved scales > \(\ell\) to unresolved scales < \(\ell\). The mechanism of loss of energy by the inertial-range eddies is thus “energy cascade”, a term first used in this connection by Onsager \([12]\).

Note that the energy flux defined in \((\text{III.8})\) is a spatially local version of the standard concept of “spectral energy flux” \(\Pi(k, t)\) (e.g. see Frisch (1995), section 6.2.2). Indeed, it is not difficult to show \([20]\) that the two fluxes are related by

\[
\frac{1}{|V|} \int_V d^3 x \, \Pi_\ell(x, t) = \int_0^\infty dk \, P_\ell(k) \Pi(k, t) \tag{III.9}
\]

where, for any isotropic kernel \(G(r)\) with 3D Fourier transform \(\hat{G}(k)\), the formula

\[
P_\ell(k) = - \frac{d}{dk} |\hat{G}(k\ell)|^2 \tag{III.10}
\]

defines a distribution function satisfying \(\int_0^\infty dk \, P_\ell(k) = 1\) and \(P_\ell(k) \geq 0\) for standard kernels with \(|\hat{G}(k)|^2\) decaying monotonically in wave-number. Intuitively, the flux \(\Pi_\ell(x, t)\) is well-localized in physical space and \(\Pi(k, t)\) is well-localized in Fourier space, but their respective averages over space and wavenumber agree. Note that the width of distribution \(P_\ell(k)\) in -space is \(\Delta k \sim 1/\ell\), consistent with the uncertainty principle of Fourier analysis, \(\Delta k \Delta x \sim 1\). When time-average spectral flux is constant in a long range of wavenumbers \(k\) where \(\langle \Pi(k) \rangle = \varepsilon\), then identity \((\text{III.9})\) implies \(\langle \Pi_\ell \rangle = \varepsilon\) for \(\ell \sim 1/k\).

IV. VELOCITY-INCREMENTS AND SINGULARITIES

The turbulent stress tensor \(\tau_\ell(u, u)\) defined in \((\text{III.5})\) is not a simple functional of the resolved velocity \(\bar{u}\). This can be seen by recasting the dynamical equation \((\text{I.1})\) as a path-integral over an ensemble of velocities \(u\), by assuming either random initial data \(u_0\) or by adding to the right-hand side of the momentum balance a random stirring force \(f\). Writing \(u = \bar{u} + u'\) and integrating out the small-scale field \(u'\) yields a reduced path-integral for \(\bar{u}\). This new path-integral corresponds exactly to the coarse-grained equation \((\text{III.6})\), where the stress \(\tau_\ell\) produced by integrating out \(u'\) is a highly complicated functional of \(\bar{u}\), with transcendental nonlinearity, long-term memory, and intrinsic stochasticity (e.g. see \([21]\)). This is not surprising, since Wilson-Kadanoff RG procedures typically lead to highly complicated effective actions in the path-integrals for “block-spin” fields \(\bar{u}\). This lack of a simple expression for \(\tau_\ell(u, u)\) in terms of the resolved velocity \(\bar{u}\) is what is termed the “closure problem” of turbulence theory. For engineering modelling by the “Large-Eddy Simulation” (LES) method, the primary problem is to develop suitable model expressions \(\tau_\ell^{\text{mod}}(\bar{u})\) that are closed in terms of \(\bar{u}\) and that are amenable to numerical integration of \((\text{III.6})\) on a coarse mesh with grid-length \(\Delta \sim \ell\) (e.g. see \([22]\)).

Onsager did not tackle this “closure problem” directly, but instead found a way to by-pass it. We discuss below his original approach using a “point-splitting regularization,” but within our “block-spin” regularization an analogous strategy may be followed. A key observation is that the stress-tensor \(\tau_\ell(u, u)\) may be rewritten in terms of velocity-increments \(\delta u\) defined in \((\text{IV.1})\) as

\[
\tau_\ell(u, u) = \langle \delta u \delta u \rangle_\ell - \langle \delta u \rangle_\ell \langle \delta u \rangle_\ell, \tag{IV.1}
\]

where \(\langle f \rangle = \langle f(x, t) \rangle := \int d^3 r G_\ell(r) f(r, x, t)\). This formula was originally obtained in Constantin et al. \([23]\) in a slightly different form, and as above in \([8]\) as a re-interpretation of their result. Equation \((\text{IV.1})\) is easy to verify by direct calculation, but it can be simply understood as the due to the invariance of the 2nd-order cumulant \(\tau_\ell(u, u)\) to shifts of \(u\) by vectors that are
“non-random” with respect to the average \( \langle \cdot \rangle_\ell \) over displacements \( r \), i.e., that are independent of \( r \). This allows \( u(x + r, t) \) in the definition (IV.3) of \( \tau_\ell(u, u) \) to be replaced with \( \delta u(r; x, t) \), yielding the formula (IV.1). Similarly, one may rewrite eq. (I.5) for coarse-grained velocity-gradients in terms of increments as

\[
\nabla_x \mathbf{u}_\ell(x, t) = -\frac{1}{\ell} \int d^3r \left( \nabla G_\ell(r) \right) \delta u(r; x, t), \quad (IV.2)
\]

using the fact that \( \int d^3r \left( \nabla G_\ell(r) \right) = 0 \). The formulas (IV.1) and (IV.2), together with the expression (III.8) for local energy flux, are the main tools in the Onsager “ideal turbulence” theory for incompressible fluids.

As an immediate application of these formulas, we can rederive the prediction of Onsager \cite{24} that \( H^\infty = h^\infty \leq 1 \), by rewriting eq. (IV.2) in the definition (III.5) of \( \tau_\ell(u, u) \) to \( \Pi_\ell(u, x, t) \) of all \( x \) with \( \Pi_\ell(u, x, t) \) hold, and then subsequently further decrease \( \ell \). If the H"older regularity (IV.3) hold for all \( (x, t) \) with \( h > 1/3 \), then clearly by (IV.4) it would follow that \( \int d^3x \Pi_\ell(x, t) \to 0 \) as \( \ell \to 0 \). This is a contradiction, since the rate of decay of energy must be independent of the arbitrary length-scale of resolution \( \ell \) as \( \ell \to 0 \). Just as Onsager did, we thus infer that somewhere in the flow there must appear H"older singularities \( h \leq 1/3 \) in the limit as \( \nu \to 0 \) or \( Re \to \infty \).

This prediction can be easily generalized within the Parisi-Frisch “multifractal model” for the turbulent velocity field \cite{23, 26}. Using similar arguments as above, one can easily show that \( p \)-th order scaling exponents for “velocity-structure functions”

\[
S_p(r) = \frac{1}{|V|} \int d^3x |\delta u(r; x, t)|^p = C_p U^p |(r/L)|^{\zeta_p}, \quad (IV.5)
\]

must satisfy \( \zeta_p \leq p/3 \) for \( p \geq 3 \). See \cite{3, 23} who took the limit \( \nu \to 0 \) first before then taking \( \ell \to 0 \), and the more recent analysis of \cite{27} who take \( \nu > 0 \) small but non-zero and exploit the arbitrariness of \( \ell \) to derive \( \zeta_p \leq p/3 \) as a result on “quasi-singularities” of Navier-Stokes solutions. Recall within the multifractal framework that \( h_p = d\zeta_p/dp \) gives the H"older exponent that contributes dominantly to \( \zeta_p = \inf_h (h_p + (3 - D(h))) \), with \( D(h) \) the fractal dimension of the singularity set \( S(h) \) on which H"older exponent \( h \) occurs. Because of the concavity of \( \zeta_p \)

in \( p \) \cite{4, 26}, one therefore concludes that \( h_p \leq \zeta_p/p \leq 1/3 \) for all \( p \geq 3 \). Onsager’s original result corresponds to the prediction that \( h_{\min} = h^\infty \leq 1/3 \). These detailed predictions have been confirmed by laboratory experiments and numerical simulations. E.g. see \cite{26} for a survey or \cite{28} for more recent numerical results.

It should be emphasized that the singularities inferred by this argument need not develop in finite time for Euler solutions starting from smooth initial data. The most common experiments study turbulent flows produced downstream of wire-mesh grids in wind-tunnels or turbulent flows generated by flows past other solid obstacles, such as plates, cylinders, etc. \cite{6, 7}. The generation of turbulence is associated to vorticity fed into these flows by viscous boundary layers that detach from the walls. Since the boundary layers become thinner as \( \nu = 1/Re \) decreases, the initial data of these experiments cannot be considered to be smooth uniformly in \( \nu > 0 \). Similar comments apply to numerical simulations. Long-time steady states with external body forcing correspond to taking first a limit \( \nu \to 0 \) before subsequently taking \( \nu \to 0 \). In that case, singularities have an infinite amount of time to reach the small “scales” where viscosity is important. Only subsequently does one take \( \nu = 1/Re \to 0 \) so that the dissipation length shrinks to zero and the singularity becomes exact.

In practice, some numerical simulations, such as that of \cite{9}, show evidence that energy dissipation is anomalous when time-averaged over only a few large-eddy turnover times. However, a close examination reveals that those studies also do not employ initial data that is uniformly smooth as \( Re \to 0 \). A standard practice is to initialize the simulation at high \( Re \) by \( u_\nu(0) = u_\nu(0, T') \), where the second velocity field is the final state at time \( T' \) of a smaller Reynolds-number \( Re' < Re \) simulation performed at lower resolution and interpolated onto the finer grid of the \( Re \)-simulation (e.g. see p.121 of \cite{3}). This practice of “nested” initialization means that initial conditions \( u_\nu(0) \) have Kolmogorov-type spectra over increasing ranges of scales as \( \nu \) decreases.

V. WEAK EULER SOLUTIONS AND DISSIPATIVE ANOMALY

A further observation of Onsager \cite{24} was that any suitable (strong) limit \( u_* = \lim_{\nu \to 0} u \) of Navier-Stokes solutions with persistent energy dissipation as \( \nu \to 0 \) must correspond to a “generalized” Euler solution that dissipates kinetic energy. The notion of “generalized” solution proposed by Onsager corresponds exactly to the modern notion of a “weak” or “distributional” solution \cite{29, 50}. From our RG point of view, these are “ultra-violet fixed-point solutions” that are obtained by taking first the limit \( \nu \to 0 \) in the regularized equations (III.1) to obtain (III.4) for \( u = u_* \), and then taking the UV limit \( \ell \to 0 \) so that \( \mathbf{u}_\ell \to u_* \), \( \tau_\ell \to 0 \), and

\[
\partial_\tau u_* + \nabla_x u_*(u_*, u_*) = -\nabla_x p_*, \quad \nabla_x u_* = 0 \quad (V.1)
\]
in the sense of distributions. Such “weak” or “distributional” Euler solutions possess the same self-similarity under rescalings \( x' = \lambda x, t' = \lambda^{1-h} t, u' = \lambda^h u \) as do ordinary smooth Euler solutions \([2, 20]\). However, the kinetic energy balance for such dissipative weak solutions is modified by an “anomaly term”. This result can be derived from the regularized energy balance \([\text{III.7}]\) by taking the double limit first \( \nu \to 0 \) and then \( \ell \to 0 \) to obtain in the sense of distributions \([31]\)

\[
\partial_t \left( \frac{1}{2} |u_\star|^2 \right) + \nabla_x \cdot \left( \frac{1}{2} |u_\star|^2 + p_\star \right) u_\star = -\Pi_\star \tag{V.2}
\]

with

\[
\Pi_\star = -\lim_{\ell \to 0} \nabla_x \overline{u}_\ell : \tau_\ell (u_\star, u_\star) \tag{V.3}
\]

The anomaly term is non-vanishing, \( \Pi_\star \neq 0 \), when there is nonlinear energy flux \( \Pi_\ell \) even as length-scale \( \ell \to 0 \).

As first noted by Polyakov \([32, 33]\), there is a striking analogy to conservation-law anomalies in quantum field-theory, where terms similar to \( \Pi_\star \) appear that vitiate conservation laws which hold classically. The most standard example is axial charge conservation which holds for a classical electrodynamic field coupled to a classical spinor field, but which is violated in quantum electrodynamics (QED). The source of that anomaly is a flux of axial/chiral charge produced at the ultraviolet cut-off momentum \( \Lambda \) and which is transferred through momentum space to finite momentum values even as \( \Lambda \to \infty \) (see Gribov \([24]\)). As remarked by Polyakov \([33]\), “in Kolmogorov’s case the same happens with enstrophy or with energy.” As \([33]\) also observed, the anomaly of turbulent “dissipative anomalies” with the axial anomaly in QED is made more striking by the fact that Schwinger \([34]\) originally obtained the axial anomaly by a “point-splitting regularization” of UV divergences in QED, with a calculation formally very similar to that used by Kolmogorov \([33]\) in his derivation of the “4/5th-law” within his statistical theory of turbulence. Remarkably, we now know that Onsager in 1945 had performed a very similar “point-splitting regularization” of kinetic energy density

\[
\frac{1}{2} \overline{u}(x,t) : \overline{G}(x,t) = \int d^3r \, G_t(r) \overline{u}(x,t) : \overline{u}(x+r,t) \tag{V.4}
\]

and took its time-derivative to derive a deterministic analogue of the “4/5th-law” (see \([2]\) for a historical review). This calculation recovers the anomalous energy balance \([\text{V.2}]\) with an expression for the anomaly term that corresponds to the anisotropic version of the Kolmogorov “4/5th-law”:

\[
\Pi_\star = \lim_{\ell \to 0} \frac{1}{4\ell} \int d^3r \, (\nabla G)_t(r) \cdot \delta u_\star(r) \cdot \delta u_\star(r) \tag{V.5}
\]

See Duchon & Robert \([31]\) for a complete derivation of \([\text{V.2}], \text{[V.3]}\) where it is also shown under reasonable assumptions that the anomaly term \( \Pi_\star(x,t) \) coincides with the zero-viscosity limit \( \varepsilon_\star(x,t) \) of the viscous energy dissipation \([\text{II.3}]\). Note that there is no statistical averaging over ensembles of velocities in the formula \([\text{V.5}]\), which gives a deterministic and space-time local version of the “4/5th-law” \([37, 38]\). This calculation was presumably the basis of the claims made about dissipative Euler solutions by Onsager \([24]\).

It is still an open question in the mathematical foundations of Onsager’s theory whether suitable limits \( u_\star = \lim_{\nu \to 0} u \) exist, which will yield the conjectured dissipative Euler solutions. Reasonable conditions which guarantee the existence of such Euler solutions as inviscid limits are verified over accessible ranges of Reynolds numbers \([27, 33, 40]\). Furthermore, in very deep mathematical work, dissipative, Hölder-continuous Euler solutions \( u_\star \) have been constructed by “convex integration” methods, using ideas originating in the Nash-Kuiper theorem and Gromov’s “h-principle” \([30, 41]\). This circle of ideas led recently to a proof that Onsager’s 1/3 Hölder exponent is sharp and that dissipative Euler solutions exist with spatial Hölder exponent \((1/3) - \epsilon \) for any \( \epsilon > 0 \) \([42, 43]\). These dissipative Euler solutions \( u_\star \), are not constructed by zero-viscosity limits but instead by an “inverse RG” procedure in which \([\text{III.6}]\) is solved for some specified \( \overline{u}_{\ell_{k-1}} \) and \( \tau_{\ell_{k-1}} \) and then one proceeds to a new length-scale \( \ell_k \ll \ell_{k-1} \) by adding small-scale modes to the velocity field in such a way that \( \overline{u}_k \) and \( \tau_k \) again satisfy \([\text{III.6}]\) but with \( |\tau_{\ell_k}| \ll |\tau_{\ell_{k-1}}| \). Iterating this construction, \( \tau_{\ell_k} \to 0 \) as \( k \to \infty \) and the limit \( u_\star = \lim_{k \to \infty} u_k \) is a weak Euler solution. Further mathematical work along these lines will hopefully lead to more complete understanding of the inviscid limit solutions \( u_\star = \lim_{\nu \to 0} u \) which can describe the infinite Reynolds-number limit of physical turbulent flows, providing additional computational and theoretical tools.

It should be strongly emphasized, however, that much of the Onsager theory does not depend upon the assumption that limits \( u_\star = \lim_{\nu \to 0} u \) exist with viscosity taken to zero and the most significant empirical consequences follow whenever the Reynolds number is sufficiently large, but finite. This should be clear from the derivation of the bound \( \zeta_p \leq p/3 \) presented above, which never required the hypothesis that \( u_\star = \lim_{\nu \to 0} u \) must exist (see also \([27]\)). The Onsager theory provides exact, non-perturbative tools for the analysis of fluid turbulence at very large (but finite) \( Re \). For example, the formulas \([\text{IV.1}], \text{IV.2}\) are the basis for a demonstration of the scale-locality of turbulent energy cascade at \( Re \gg 1 \), whenever \( 0 < \zeta_p < p \) for any \( p \geq 3 \) \([12]\). As emphasized by Wilson \([12]\, Section VI), the property of locality by itself can provide very effective tools for systematic approximation. In critical phenomena and quantum field-theory it was space-locality rather than scale-locality, but the basic principle is the same. In \([46, 47]\) the scale-locality of the turbulent stress \( \tau/\nu \) was exploited to develop a “multi-scale gradient expansion” which yields systematic approximations to the turbulent stress tensor that can be the basis of practical closures and give phys-
Onsager’s pioneering ideas on “ideal turbulence” thus continue to stimulate new developments and provide, in the opinion of this author, the current “standard model” of high-Reynolds turbulence. In a subject where non-trivial exact results are rare and where much work involves ad hoc closures and hand-waving phenomenology, it remains a central pillar of our understanding of fully-developed turbulent flows.
true, however, for the direct viscous dissipation of kinetic energy at inertial-range scales, whereas the energy in eddies at those scales, in fact, must be dissipated. The rate of decrease of energy for free-decay or rate of power input for forced turbulence are objective facts that cannot depend upon the resolution of eddies in the inertial-range.

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