Equality conditions for internal entropies of certain classical and quantum models

Peter Gmeiner*

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Abstract

Mathematical models use information from past observations to generate predictions about the future. If two models make identical predictions the one that needs less information from the past to do this is preferred. It is already known that certain classical models (certain Hidden Markov Models called $\epsilon$-machines which are often optimal classical models) are not in general the preferred ones. We extend this result and show that even optimal classical models (models with minimal internal entropy) in general are not the best possible models (called ideal models). Instead of optimal classical models we can construct quantum models which are significantly better but not yet the best possible ones (i.e. they have a strictly smaller internal entropy). In this paper we show conditions when the internal entropies between classical models and specific quantum models coincide. Furthermore it turns out that this situation appears very rarely. An example shows that our results hold only for the specific quantum model construction and in general not for alternative constructions. Furthermore another example shows that classical models with minimal internal entropy need not to be related to quantum models with minimal internal entropy.

1 Introduction

Mathematical modeling of natural and technological systems plays an important role in modern science. In general, there are many ways to model a system mathematically. One possibility is to view the system of interest as an information processing black box generating an observable output from given past observations. The observed data can be treated as a stochastic process and we try to find models which are called Hidden Markov Models, that generate the same statistical behaviour and that are denoted as classical models. We prefer models which predict future data from past observations in an optimal way, i.e. they need as little memory as possible to do this. The amount of information the past contains about the future is measured by the mutual information between past and future data. This quantity is known as excess entropy [Cru83]. A model that should be able to predict future data in an optimal way has at least to store this amount of information to do this. One method to construct such a model in a systematic way is used in computational mechanics and called (classical) $\epsilon$-machine. $\epsilon$-machines are the optimal classical models for a certain subset in the set of all possible alternative Hidden Markov Models but not the optimal classical models in general. The optimality

*Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Bismarckstraße 1 1/2, D-91054 Erlangen, Germany. E-mail: gmeiner@mi.uni-erlangen.de
of a classical model is quantified by the classical internal state entropy of the model. Usually this is the Shannon entropy and for an optimal classical model the internal state entropy is called *generative complexity* $C_{\mathcal{C}}$. Instead of considering classical models one can think about analog quantum models (called *Hidden Quantum Markov Models*). Recent results show that if the classical $\epsilon$-machine is not already the best possible model (called *ideal model*), it is always possible to find a quantum model that needs less memory than the classical $\epsilon$-machine to reconstruct the statistical behaviour of the stochastic process \cite{GuW11}. Usually the internal state entropy $C_q$ of the quantum model is strictly greater than the excess entropy $E$ and there remains room for improvement. We extend this results for all optimal classical models.

The Hidden Quantum Markov Model induced from a classical Hidden Markov Model, can be formulated in the setting of a *quantum channel*. The initial distribution and the transition probabilities of a classical Hidden Markov Model (Definition 1) can be used to calculate the mutual information $I(X;Y)$ between a specific classical input random variables $X$ and a classical output random variables $Y$ related to the classical model. We achieve the following inequality chain in the subsequent sections

$$E \leq I(X;Y) \leq C_q \leq C_{\mathcal{C}}.$$ 

In this paper we investigate for a specific quantum model construction equality conditions for the last two inequalities above. We will see that in general there remains a gap between the different internal state entropies for the suggested quantum model construction introduced in \cite{GuW11} and that the last two inequalities are strict in most cases. Furthermore for $\epsilon$-machines we prove that $E = I(X;Y)$ hold and show with an example that a quantum model induced by a minimal classical model is not the minimal quantum model. The relationship between minimal classical models and minimal quantum models remains an open question.

This paper is organized as follows. In Section 2 some basic notations and definitions are introduced. Section 3 introduces $\epsilon$-machines, restates a recently proved theorem and extends this theorem to minimal Hidden Markov Models. Section 4 introduces Hidden Quantum Markov Models. Furthermore two well-known propositions applied to our context are presented and we generalize a further theorem from $\epsilon$-machines to minimal Hidden Markov Models. The example which shows that minimal classical models do not correspond to minimal quantum models is also presented here. In Section 5 we prove the equality conditions for the internal entropies and in Section 6 we present a calculation example and verify the proven results. Section 7 describes an alternative construction of a quantum model to model a stochastic process and shows that the equality conditions in Section 5 in general cannot be extended to other quantum model constructions than the suggested one in Section 4.

### 2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a metric space $\Omega$, a $\sigma$-algebra $\mathcal{F}$ and a probability measure $P$. For random variables $X, Y : \Omega \to \Sigma$ mapping to a finite alphabet $\Sigma$ the Shannon entropy is defined by

$$H(X) := - \sum_{x \in \Sigma} P(X = x) \log P(X = x),$$
and the conditioned Shannon entropy by
\[ H(X|Y) := - \sum_{x,y \in \Sigma} P(X = x, Y = y) \log P(X = x|Y = y), \]
where \( P(X = x) := P(\{\omega \in \Omega | X(\omega) = x\}) \) denotes the probability that the random variable \( X \) is equal to \( x \in \Sigma \), \( P(X = x, Y = y) \) is the joint probability between \( X \) and \( Y \) and for \( P(Y = y) > 0 \) the conditional probability is \( P(X = x|Y = y) := \frac{P(X=x,Y=y)}{P(Y=y)} \).

In the definitions the convention \( 0 \log(0) = 0 \) is used. Given a distribution \( \mu \) of a random variable \( X \) we sometimes write \( H(\mu) \) instead of \( H(X) \). The mutual information between two random variables is
\[ I(X;Y) := H(X) - H(X|Y). \]

The mutual information is non negative (\( I(X;Y) \geq 0 \)) and equals zero if and only if \( X \) and \( Y \) are independent random variables [Cov06].

We consider a time-discrete stationary stochastic process \( \mathbf{X} := (X_t)_{t \in \mathbb{Z}} \) with random variables \( X_t : \Omega \rightarrow \Sigma \) for all \( t \in \mathbb{Z} \). We define the semi-infinite processes \( \mathbf{X} := (X_t)_{t \in \mathbb{N}} \) interpreted as past and \( \mathbf{X} := (X_t)_{t \in \mathbb{N}_0} \) interpreted as future respectively. Blocks of random variables with finite length are denoted by \( X_a^b := (X_k)_{k \in [a,b]} \) for \( -\infty < a \leq b < \infty \). The one-sided sequence space is \( \Sigma^\mathbb{N} := \times_{i \in \mathbb{N}} \Sigma \) and in the same way the two-sided sequence space \( \Sigma^\mathbb{Z} \) is defined. We introduce the shift function \( \sigma : \Sigma^\mathbb{Z} \rightarrow \Sigma^\mathbb{Z} \) by \( \sigma(x)_i := x_{i+1} \). At any time \( t \in \mathbb{Z} \) we have random variables \( X_{t-\infty}^t := (X_k)_{k \leq t} \) and \( X_{t+1}^\infty := (X_k)_{k \geq t+1} \) that govern the systems observed behaviour respectively in the shifted past and the shifted future. The mutual information between these two variables is the well-known excess entropy [Cru83, Cru03]
\[ E := \lim_{L \rightarrow \infty} I(X_0^{L-1};X_{-L}^{-1}). \]  
(1)

In general, it is not clear if the limit in (1) exists. We will see later that in the setting of this paper \( E \) always exists. With the assumption that the limit in (1) exists as a finite number the following equality holds: \( E = I(\mathbf{X};\bar{\mathbf{X}}) \), see Chapter 2.2 in [Pin64].

The stochastic process generates a sequence of output symbols which represents the observed behaviour of a system for which we construct a mathematical model in a discretized fashion.

We use a Hidden Markov Model (HMM) to model a given stochastic process. In general there are different kinds of HMMs. For our purpose we use a transition-emitting HMM and use the same terminology as in [Lec10, Lec09a, Lec09b].

**Definition 1** With \( \mathcal{P}(A) \) we denote the space of all probability measures on a set \( A \). A transition-emitting HMM consists of a set \( S \) of internal states and a pair \((T,\mu)\) with an initial distribution \( \mu \in \mathcal{P}(S) \) and a measurable function \( T : S \rightarrow \mathcal{P}(S \times \Sigma) \), called generator. We say that \((T,\mu)\) is an HMM of \( \mathbf{X} \) if the output-distribution which is determined by the output kernel \( K_s(.) := T(s)(S \times .) \), \( s \in S \) of the HMM coincide with the distribution of \( \mathbf{X} \).

In the following we abbreviate transition-emitting HMM with HMM. Since we are considering stationary stochastic processes we require that the HMM is invariant in the following sense.
Definition 2 A HMM \((T, \mu)\) is invariant, if \(\mu\) is \(T\)-invariant i.e.
\[\mu(G) = \int_S T(s)(G \times \Sigma)d\mu(s), \quad \forall G \in S.\]

We are interested in HMMs with minimal internal state entropy \(H(\mu)\) which can be considered as a complexity measure of the process generated by the HMM. Following Löhr [Loe09c, Loe10] we define the generative complexity.

Definition 3 The (classical) generative complexity of a stationary stochastic process \(\vec{X}\) is the infimum of the entropies of internal states
\[C_{CI} := \inf \left\{ H(\mu) \mid (T, \mu) \text{ is an invariant HMM of } \vec{X} \right\}.\]

Löhr showed that for every stationary stochastic process there exists an invariant HMM \((T, \mu)\) such that \(H(\mu) = C_{CI}\) hold and the infimum in Definition 3 is actually a minimum (Corollary 4.14 in [Loe10]). In the following we denote this invariant HMM as minimal HMM.

The generative complexity is an upper bound for the excess entropy \(E \leq C_{CI}\). (2)

In this paper we only consider processes which can be modeled by a minimal HMM with finitely many internal states. Markov processes of finite order are examples for processes with a finite set of internal states. Assuming finitely many internal states \(S = \{S_1, \ldots, S_n\}\), we can write the initial distribution as a probability vector \(\mu := (p_i)_{i=1}^n\) and the generator as a set of substochastic \(n \times n\) matrices \(T^{(r)}\) with entries \(T^{(r)}_{ij} := T(S_i)(S_j, r)\) for all \(r \in \Sigma\). Since we are considering only a finite set of internal states, \(C_{CI}\) is always finite and with (2) the excess entropy (1) is also finite.

3 \(\epsilon\)-Machines and minimal HMMs

The following construction of a transition-emitting HMM is often regarded in the literature and the resulting HMM coincide in many cases with a minimal HMM. Unfortunately not in any case this construction leads to a minimal HMM as often wrongly claimed in the literature (see [Loe10, Loe09b, Loe09c] for counterexamples). On the set \(\Sigma^\mathbb{N}\) of all past trajectories of the process \(\vec{X}\) we define an equivalence relation \([\text{Sha01}]\)
\[x \sim x' : \iff P(\vec{X} \in \vec{\mathcal{C}} | \vec{X} = x) = P(\vec{X} \in \vec{\mathcal{C}} | \vec{X} = x'), \quad \forall \vec{\mathcal{C}} \in \vec{\mathcal{C}},\]
(3)
where \(x, x' \in \Sigma^\mathbb{N}, \vec{\mathcal{C}}\) is the product \(\sigma\)-algebra generated by cylinder sets on \(\Sigma^\mathbb{N}\) and \(P(\vec{X} \in \vec{\mathcal{C}} | \vec{X} = x)\) is a regular version \(^1\) of the conditional expectation. The equivalence classes \(S(x) := \{x' \in \Sigma^\mathbb{N} | x' \sim x\}\) of relation (3) are called causal states and are the internal states of the constructed HMM. The set of all causal states is denoted by \(S := \{S(x)| x \in \Sigma^\mathbb{N}\}\) and is measurable (Lemma

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\(^1\) \(P(\vec{X} \in \vec{\mathcal{C}} | \vec{X} = x)\) is called a regular version if it is a Markov kernel.
3.18 in [Loe10]). In general there can be uncountably many causal states [Cru94, Loe10, Loe09b] and the causal states depend on the version of conditional probability used in the definition [Loe10]. We say that the number of causal states is finite if there exists a version of conditional probability such that there are only finitely many equivalence classes. A characteristic property of causal states is that they induce a minimal sufficient memory.\footnote{A memory kernel is a Markov kernel $\gamma : \Sigma^\infty \to \mathcal{P}(S)$. The associated random variable $\mathcal{M}$ is called memory variable or simply memory. A memory variable is called sufficient if $P(\vec{X} \in A, \vec{X} \in B|\mathcal{M}) = P(\vec{X} \in A|M)P(\vec{X} \in B|M)$ a.s. for all measurable sets $A, B$. A memory is minimal if every other measurement memory has at least the same number of internal states and the corresponding memory variable has at least the same entropy, Corollary 3.21 in [Loe10].}

We are only considering stationary stochastic processes with a finite set of causal states $S = \{S_1, \ldots, S_n\}$. Given a past observation of infinite length $x^{t}_{\infty} \in \Sigma^t$ at time $t \in \mathbb{Z}$ using stationarity we identify this shifted past with a causal state $S(\sigma^{-t-1}(x^{t}_{\infty})) \in S$. Together with the next symbol $x_{t+1}$ generated by the process the next causal state $S(\sigma^{-t-2}(x^{t-1}_{\infty}, x_{t+1})) \in S$ is uniquely determined and the causal states are Markov [Sha01, Loe10]. We define the Markov kernels between two causal states $S_i, S_j \in S$ emitting an output symbol $r \in \Sigma$ for any $t \in \mathbb{Z}$ as follows

$$T_{i,j}^{(r)} := T(S_i)(S_j, r) = P\left(S(\sigma^{-t-2}(x_{\infty}^{t-1}, x_{t+1})) = S_j \text{ and } X_{t+1} = r \mid S(\sigma^{-t-1}(x_{\infty}^{t})) = S_i \right).$$

The probability of a causal state $S_i \in S$ is denoted by $p_i := P(S_i)$. The ordered pair $(T, (p_1, \ldots, p_n))$ is called $\epsilon$-machine. The $\epsilon$-machine is a transition-emitting HMM and a model for the original stochastic process [Loe10, Loe09a].

**Remark 1** In general the $\epsilon$-machine is not the HMM with minimal number of internal states and also not the one with minimal classical internal state entropy. To be precise Löhr proved in [Loe10] that for a countable alphabet $\Sigma$ the $\epsilon$-machine is the minimal partially deterministic HMM\footnote{An invariant HMM $(T, \mu)$ with measurable spaces $(\Sigma, D)$ and $(S, \mathcal{G})$ is called partially deterministic if there is a measurable function $f : \Sigma \times \Sigma \to S$ (transition function), such that for $\mu$-almost all $s \in S$ we have $T(s)(G \times D) = K_s(D \cap f(s, \cdot)^{-1}(G)) \quad \forall D \in D, G \in \mathcal{G}$, where $K_s(\cdot) := T(s)(\mathcal{S} \times \cdot)$ is the output kernel.} of the process $\vec{X}$.\footnote{\[E \leq C_\epsilon.\]}

The $\epsilon$-machine has classical internal state entropy

$$C_\epsilon := H(S) = -\sum_{j=1}^{n} p_j \log p_j,$$

which is also known as statistical complexity [Gra86, Sha01]. Since the generative complexity is an upper bound for the excess entropy, the statistical complexity is also an upper bound for the excess entropy [Sha01, Cru03]

$$E \leq C_\epsilon. \quad (4)$$

The next theorem gives a characterization when (4) is strict.

**Theorem 1** Given a stationary stochastic process $\vec{X}$ with excess entropy $E$ and statistical complexity $C_\epsilon$. Let its corresponding $\epsilon$-machine have transition probabilities $T^{(r)}_{i,j}$. Then $C_\epsilon > E$ if and only if there exists a non-zero probability that two different causal states $S_j$ and $S_k$ will both make a transition to a coinciding causal state $S_i$ upon emission of a coinciding output $r \in \Sigma$, i.e., $T^{(r)}_{j,i}, T^{(r)}_{k,i} \neq 0$.\footnote{\[A memory kernel is a Markov kernel $\gamma : \Sigma^\infty \to \mathcal{P}(S)$. The associated random variable $\mathcal{M}$ is called memory variable or simply memory. A memory variable is called sufficient if $P(\vec{X} \in A, \vec{X} \in B|\mathcal{M}) = P(\vec{X} \in A|M)P(\vec{X} \in B|M)$ a.s. for all measurable sets $A, B$. A memory is minimal if every other measurement memory has at least the same number of internal states and the corresponding memory variable has at least the same entropy, Corollary 3.21 in [Loe10].}
Proof. Theorem 1 in [GuW11].

As a next step we extend the last theorem from $\epsilon$-machines to minimal HMMs. We want now to return to the general case and consider minimal HMMs which we denote as minimal classical models. From the definitions of the internal entropies it is clear that

$$E \leq C_{CI} \leq C_{\epsilon}. \tag{5}$$

There exists examples such that $C_{CI} < C_{\epsilon}$ holds and it is known that [Loe10]

$$C_{CI} < C_{\epsilon} \Rightarrow E < C_{CI},$$

or the negation of this

$$E = C_{CI} \Rightarrow C_{CI} = C_{\epsilon}. \tag{6}$$

With this fact it is possible to generalize Theorem 1.

**Theorem 2** Given a stationary stochastic process $\mathbf{X}$ with excess entropy $E$ and generative complexity $C_{CI}$. Let its corresponding minimal HMM have transition probabilities $T^{(r)}_{ij}$. Then $C_{CI} > E$ if and only if there exists a non-zero probability that two different internal states $S_j$ and $S_k$ will both make a transition to a coinciding internal state $S_l$ upon emission of a coinciding output $r \in \Sigma$, i.e. $T^{(r)}_{jl}, T^{(r)}_{kl} \neq 0$.

Proof. With (6) and (5) we get $E = C_{CI} \iff C_{CI} = C_{\epsilon}$. With Theorem 1 and the negation of the last expression we yield the result. \hfill \Box

**Remark 2** Theorem 2 shows that there is a kind of redundance in the minimal HMM producing the gap between $E$ and $C_{CI}$. This redundance is an indicator for a possible improvement of the classical minimal HMM, see Theorem 3.

4 Hidden Quantum Markov Models and Holevo-Bound

Based on the classical minimal HMM introduced in Section 2 it is possible to define quantum models with the same statistical behaviour. In the spirit of classical HMM we define a quantum version of such models introduced in [Mon11] to reproduce a given stochastic process.

**Definition 4** ([Mon11]) A quantum operation $K_r : \text{Mat}(d, \mathbb{C}) \rightarrow \text{Mat}(d, \mathbb{C})$ is a completely positive, trace non-increasing linear map on the space of complex $d \times d$-matrices $\text{Mat}(d, \mathbb{C})$. A Hidden Quantum Markov Model (HQMM) is a density matrix $\rho \in \text{Mat}(d, \mathbb{C})$ together with a set of quantum operations $K_r, \forall r \in \Sigma$ such that $\sum_{r \in \Sigma} K_r$ is trace-preserving. At every time step a symbol $r \in \Sigma$ is generated with probability $P(r) = \text{Tr}(K_r \rho)$ and the state vector is updated to $\rho_r = K_r \rho / P(r)$.

There is an analogy between classical HMM and HQMM, for example the quantum operation $K_r$ plays the role of a substochastic matrix $T^{(r)}$ and the density matrix corresponds to the probability vector $(p_1, \ldots, p_n)$, see [Mon11] for more details. Furthermore it can be proved that for every transition-emitting HMM it is possible to construct a HQMM with the same statistical behaviour, i.e. the HQMM generates the same stochastic process.
Consider a finite input alphabet $\mathcal{X}$ and a finite output alphabet $\mathcal{Y}$. Further let $\mathcal{H}$ and $\mathcal{J}$ be the input and output Hilbert spaces. We want to transmit classical input data via a quantum channel. For this we introduce the general setting of a quantum channel.

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The probability that $y \in \mathcal{Y}$ is the output symbol, given $x \in \mathcal{X}$ as input is

$$T_{y,x} := \text{Tr}_{\mathcal{J}}(\mathcal{E}(\rho_x) \mathcal{K}_y),$$

and the output distribution takes the form

$$\tilde{p}_y := \sum_{x \in \mathcal{X}} \text{Tr}_{\mathcal{J}}(p_x \mathcal{E}(\rho_x) \mathcal{K}_y), \quad \text{for every } y \in \mathcal{Y}.$$
Remark 3 In [GuW11] this construction is applied to classical ε-machines and the HQMM is called quantum ε-machine. Since we are considering minimal HMMs which need not to be ε-machines we call the defined HQMM a quantum model induced by a minimal HMM.

The quantum internal state entropy of a HQMM is the von Neumann entropy

\[ C_q := S(\rho) := -\text{Tr}\rho \log \rho. \]

\( C_q \) is the quantum version of the classical internal state entropy \( H(\mu) \) and is bounded by this internal state entropy and especially by the generative complexity \( C_{\text{Cl}} \).

**Proposition 1** Suppose \( \rho = \sum_{j=1}^{n} p_j \rho_j \) where \( p = (p_j)_{j=1}^{n} \) is a probability vector with \( \sum_{j=1}^{n} p_j = 1 \) and the \( \rho_j := |S_j\rangle\langle S_j| \) are density operators for every \( j \in \{1, \ldots, n\} \). Then

\[ C_q \leq H(p), \]

with equality if and only if the quantum internal states \( |S_j\rangle \) are mutually orthogonal. In especially given a minimal HMM \((T, p)\) with an induced quantum model \( \rho \) we have

\[ C_q \leq C_{\text{Cl}}. \]

**Proof.** Theorem 11.10 in [Nie00] or alternatively an adaption of Theorem 3.7 in [Pet08]. □

Remark 4 In the case that the classical minimal HMM coincide with the classical ε-machine it is not clear if the quantum internal states of the induced quantum model share the same properties as the classical causal states, i.e. the question if quantum internal states are minimal sufficient in the sense of quantum mechanics is not yet answered.

The next proposition is the well-known Holevo-Bound and gives an upper bound for the mutual information between classical input and classical output data.

**Proposition 2 (Holevo-Bound)** Given the setting above with classical input random variable \( X \) and classical output random variable \( Y \), the following bound holds

\[ I(X; Y) \leq S(\rho) - \sum_{i=1}^{n} p_i S(\rho_i), \]

where \( \rho = \sum_{i=1}^{n} p_i \rho_i \) and with equality if and only if all \( \rho_i \) commute.

**Proof.** Theorem 12.1 in [Nie00] or Theorem 7.3 in [Pet08]. For the equality condition see for example [Rus02]. □

In the case that the HMM is an ε-machine the lefthand side of (8) is the excess entropy.

**Proposition 3** Let \((T, (p_1, \ldots, p_n))\) be an ε-machine then given the setting above it holds that

\[ I(X; Y) = I(\tilde{X}; \tilde{X}) = E. \]
Proof. To prove the proposition we use a four variable mutual information introduced in [Yen91] and follow the same strategy as in [Cru10]. For random variables $X, Y, Z, U$ we define

\[
I(X; Y; Z; U) := I(X; Y; Z) - I(X; Y; Z|U),
\]

\[
I(X; Y; Z) := I(X; Y) - I(X; Y|Z),
\]

with $I(X; Y|Z) := H(X|Z) - H(X|Y, Z)$,

\[
I(X; Y; Z|U) := I(X; Y|U) - I(X; Y|Z; U),
\]

with $I(X; Y|Z; U) := H(X|Z, U) - H(X|Z; U, Y)$.

Furthermore we use the following two identities which hold for a measurable function $f$ of a random-variable $X$ ([Gra11], Lemma 3.12)

\[
H(f(X)|X) = 0, \quad H(X, f(X)) = H(X).
\]  

(9)

We define mappings $g : \Sigma^N \rightarrow \mathcal{X}$, with $g(\sigma) := j$ if $\sigma \in S_j$ and $f : \Sigma^{-N_0} \rightarrow \mathcal{Y}$, with $f(\sigma\sigma_0) := (i, \sigma_0)$ if $\sigma \in S_i$. Since we are considering $\epsilon$-machines $g$ and $f$ are well-defined and measurable. Thus we can write $X = g(\overline{X}), Y = f(\overline{X})$ and using (9) we get

\[
H(Y|\overline{X}) = 0, \quad H(X|\overline{X}) = 0, \quad H(\overline{X}, Y) = H(\overline{X}), \quad H(\overline{X}, X) = H(\overline{X}),
\]  

(10)\(11\)

\[
H(\overline{X}, \overline{X}, Y) = H(\overline{X} | Y), \quad H(\overline{X}, \overline{X}) = H(\overline{X} | X).
\]  

(12)

In the next step we show $I(\overline{X}; \overline{X}; X; Y) = I(\overline{X}; \overline{X}) = E$. Consider

\[
I(\overline{X}; \overline{X}; X|Y) = I(\overline{X}; \overline{X}|Y) - I(\overline{X}; \overline{X}|X; Y),
\]  

(13)

then the first term disappear because with (12) it holds

\[
I(\overline{X}; \overline{X}|Y) = H(\overline{X}|Y) - H(\overline{X}|X, Y) \overset{12}{=} 0.
\]

The second term of (13) is also zero, since

\[
I(\overline{X}; \overline{X}|X; Y) = H(\overline{X}|X, Y) - H(\overline{X}|X, Y, \overline{X}) \overset{12}{=} 0.
\]

Putting all together we yield

\[
I(\overline{X}; \overline{X}; X|Y) = 0.
\]

Furthermore we have

\[
I(\overline{X}; \overline{X}; X) = I(\overline{X}; \overline{X}) - I(\overline{X}; \overline{X}|X) = I(\overline{X}; \overline{X}),
\]

since $I(\overline{X}; \overline{X}|X) = H(\overline{X}|X) - H(\overline{X}|\overline{X}, X) \overset{12}{=} 0$. Finally we get

\[
I(\overline{X}; \overline{X}; X; Y) = I(\overline{X}; \overline{X}).
\]

In a second step we show $I(\overline{X}; \overline{X}; X; Y) = I(X; Y)$. As in the first step the following term vanish

\[
I(X; Y; \overline{X}|\overline{X}) = I(X; Y|\overline{X}) - I(X; Y|\overline{X}; \overline{X}) = 0,
\]  

(14)

9
since $I(X; Y|\tilde{X}) = H(Y|\tilde{X}) - H(Y|X, \tilde{X}) \equiv 0$ and
\[
I(X; Y|\tilde{X}; \tilde{X}) = H(X|\tilde{X}, \tilde{X}) - H(X|Y, \tilde{X}, \tilde{X}) \equiv 0.
\]
Consider now
\[
I(X; Y; \tilde{X}) = I(X; Y) - I(X; Y|\tilde{X}),
\]
then the second term disappear, since
\[
I(X; Y|\tilde{X}) = H(X|\tilde{X}) - H(X|Y, \tilde{X}) \equiv 0.
\]
Thus we yield
\[
I(\tilde{X}; \tilde{X}; X; Y) = I(X; Y),
\]
and finally we get
\[
E = I(\tilde{X}; \tilde{X}) = I(X; Y).
\]

The converse of Proposition 3 is not true as can be seen in the example treated in Section 6.

**Remark 5** In general it is difficult to calculate the excess entropy of a given stationary stochastic process. If one has given an $\epsilon$-machine for a process it is easy to calculate $I(X; Y)$ which coincide with the excess entropy $E$. Compared to the method in [Ell09], which uses the structure of the $\epsilon$-machine, this is an alternative method to calculate $E$.

Since $I(X; Y)$ depends on the classical HMM we sometimes write $I_{HMM}(X; Y)$ if a distinction is necessary. For general HMMs and especially for minimal HMMs which are not an $\epsilon$-machine the excess entropy is in general smaller than $I(X; Y)$ as the next example shows. This example can be found in [Loe09c].

**Example 1** Let $\Sigma := \{0, 1\}$ and consider a stationary Markov process generated by the $\epsilon$-machine $(T, (p_0, p_1))$ with $p_0 = p_1 = \frac{1}{2}$ and
\[
T^{(0)} = \begin{pmatrix}
\frac{1}{2}(1 + \epsilon) & 0 \\
\frac{1}{2}(1 - \epsilon) & 0
\end{pmatrix}, \quad T^{(1)} = \begin{pmatrix}
0 & \frac{1}{2}(1 - \epsilon) \\
0 & \frac{1}{2}(1 + \epsilon)
\end{pmatrix},
\]
where $0 < \epsilon \leq 1$. The statistical complexity is $C_\epsilon = 1$ for $\epsilon > 0$ and the excess entropy amounts to
\[
E = \frac{1}{2} \left((1 + \epsilon) \log(1 + \epsilon) + (1 - \epsilon) \log(1 - \epsilon)\right),
\]
and coincide with $I_{Markov}(X; Y)$. We give now a HMM which generates the same process (see [Loe09c]), but with three internal states and smaller internal state entropy than $C_\epsilon$. Let $S := \{0, 1, 2\}$ with
\[
T^{(0)} = \begin{pmatrix}
\epsilon & 0 & 1 - \epsilon \\
0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2 - \epsilon}
\end{pmatrix}, \quad T^{(1)} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \epsilon & 1 - \epsilon \\
\frac{1}{2} & 0 & \frac{1}{2 - \epsilon}
\end{pmatrix},
\]

10
and initial distribution \((p_0, p_1, p_2)\)

\[
p_i = \begin{cases} 
\varepsilon^2, & \text{if } i \in \{0, 1\} \\
1 - \varepsilon, & \text{if } i = 2 
\end{cases}
\]

The internal state entropy of this HMM is given by

\[
H(p) = -(1 - \varepsilon) \log(1 - \varepsilon) - \varepsilon \log \left(\frac{\varepsilon}{2}\right).
\]

It is easy to calculate the left hand side of the Holevo-Bound

\[
I_{3\text{state}}(X; Y) = \varepsilon.
\]

For \(\varepsilon \in (0, 1)\) the excess entropy is always strictly smaller than \(I_{3\text{state}}(X; Y)\). Especially for \(\varepsilon\) small enough the three state HMM has smaller internal state entropy \(H(p)\) than the \(\varepsilon\)-machine as can be seen in Figure 1.

![Figure 1: Excess entropy \(E\), \(I_{3\text{state}} := I_{3\text{state}}(X; Y)\), \(C^3_{\text{state}}\), \(I_{\text{Markov}} := I_{\text{Markov}}(X; Y)\), \(C^\text{Markov}\), internal state entropy \(H(p)\) of the three state HMM described in Example 1 and statistical complexity of the \(\varepsilon\)-machine.](image)

Furthermore Löhr showed in [Loe09c] that the internal state entropy of the minimal HMM is bounded from below by

\[
C_{\text{Cl}} \geq -(1 - \varepsilon/2) \log(1 - \varepsilon/2) - \varepsilon/2 \log(\varepsilon/2),
\]

where the lower bound coincide with the internal state entropy \(C^3_{\text{state}}\) of the quantum model induced by the three state HMM. This example shows that it is possible that the excess entropy is smaller than the lower-bound \(I_{3\text{state}}(X; Y)\) of \(C^3_{\text{state}}\) given by the Holevo-Bound. Furthermore it also shows that even if the three state HMM has smaller internal entropy for sufficient small \(\varepsilon\), the internal state entropy \(C^\text{Markov}\) of the quantum model induced by the markov model is strictly smaller than \(C^3_{\text{state}}\) and especially smaller than \(I_{3\text{state}}(X; Y)\), see Figure 1. So it is not clear at all how minimal classical models and minimal quantum models are related to each other.

Since the states \(\rho_i = |S_i\rangle\langle S_i|\) are pure, we have \(S(\rho_i) = 0\) so that Proposition 1 and Proposition 2 imply that in general

\[
E \leq I(X; Y) \leq C_q \leq C_{\text{Cl}},
\]

holds.
Remark 6 Inequality (15) allows us to compare the information stored in a classical minimal HMM and an induced quantum model which generate the same stochastic process. In order to compare the quantum internal state entropies of different HQMM constructions with the internal state entropy of a given classical minimal HMM we have to ensure that (15) hold. Considering the right hand side of (15) the second term has to vanish and the internal states of such a HQMM has to fulfill

$$S(\rho_i) = 0, \quad \forall i \in \{1, \ldots, n\}.$$

Gu et al. proved in [GuW11] a remarkable theorem for classical $\epsilon$-machines that shows that if $C_\epsilon > E$ holds then the induced quantum model (7) has internal state entropy strictly smaller than the internal state entropy of the classical $\epsilon$-machine $C_\epsilon < C_{C_{\epsilon}}$. We extend this result to classical minimal HMMs.

Theorem 3 Given a stationary stochastic process $\mathbf{X}$ with excess entropy $E$ and generative complexity $C_{C_{\epsilon}}$ and $C_{C_{\epsilon}} > E$. Then there exists a quantum system that exhibits identical statistics with internal state entropy $C_q < C_{C_{\epsilon}}$.

Proof. Use Theorem 2 instead of Theorem 1 in the proof of Theorem 2 in [GuW11]. □

In the next section we investigate equality conditions for these different internal state entropies.

5 Equality conditions

The next two propositions deliver a characterization when equality in the last two inequalities of (15) holds.

Proposition 4 Given a stationary stochastic process $\mathbf{X}$ with excess entropy $E$ and generative complexity $C_{C_{\epsilon}}$. Let the corresponding induced quantum model defined in (7) have quantum internal state entropy $C_q$. Then it holds that $E = I(X;Y) = C_q = C_{C_{\epsilon}}$ if and only if all quantum internal states are mutually orthogonal.

Proof. $\Rightarrow$: It holds that $E = I(X;Y) = C_q = C_{C_{\epsilon}}$. Theorem 2 gives us that for each output $r \in \Sigma$, each index $l \in \{1, \ldots, n\}$ and each pair of indices $j \neq k$ it holds that one of the transition probabilities $T_{j,l}^{(r)}, T_{k,l}^{(r)}$ is zero. With the definition of the quantum internal states (7) this implies $\langle S_j | S_k \rangle = 0$ for all indices $j \neq k$.

$\Leftarrow$: The definition of the scalar product and $\langle S_j | S_k \rangle = 0$ for all indices $j \neq k$ imply that one of $T_{j,l}^{(r)}, T_{k,l}^{(r)}$ is zero for each output $r \in \Sigma$, index $l \in \{1, \ldots, n\}$ and pair of indices $j \neq k$. Again with Theorem 2 we get $E = C_{C_{\epsilon}}$. Together with (15) it follows that $E = I(X;Y) = C_q = C_{C_{\epsilon}}$. □

Proposition 5 Given a stationary stochastic process $\mathbf{X}$ with excess entropy $E$. For a given classical HMM generating $\mathbf{X}$ with internal state entropy $H(\mu)$ let the corresponding induced quantum model defined in (7) have quantum internal state entropy $C_q$. Then it holds that $E \leq I(X;Y) = C_q < H(\mu)$ if and only if there exist at least two quantum internal states which are identical and all other quantum internal states are mutually orthogonal or also identical (i.e. $\exists k \neq i : \langle S_k | S_i \rangle = 1, \langle S_j | S_l \rangle = 0$ or 1 for all other indices $l \neq j$).
Proof. ”⇒”: Since $C_q < H(\mu)$ it follows from Proposition 1 that not all quantum internal states are mutually orthogonal, i.e. there exist at least one pair of indices $i \neq k$ such that $\langle S_i | S_k \rangle \neq 0$. Furthermore Proposition 2 implies that $I(X; Y) = C_q$ if and only if all density operators $\rho_i = |S_i \rangle \langle S_i|$ commute. It is easy to prove that all $\rho_i$ commute if and only if $\langle S_i | S_k \rangle = 0$ or $|S_i \rangle \langle S_k| = 1$ for all indices $i, k \in \{1, \ldots, n\}$. From this equivalence relation the claim follows.

”⇐”: There exist at least one pair of indices $i \neq k$ such that $\langle S_i | S_k \rangle = 1$. Together with the definition of quantum internal states there is an $r \in \Sigma$ and an index $l \in \{1, \ldots, n\}$ such that $T_{k,l}^{(r)} \neq 0$ and $T_{i,l}^{(r)} \neq 0$. Since not all quantum internal states are mutually orthogonal it follows from Proposition 1 that $C_q < H(\mu)$. From the Holevo-Bound (Proposition 2) we know that $I(X; Y) \leq C_q$ with equality if and only if all density operators $\rho_i = |S_i \rangle \langle S_i|$ commute which is again equivalent to the condition that $\langle S_i | S_k \rangle = 1$ or $|S_i \rangle \langle S_k| = 0$ for all indices $i, k \in \{1, \ldots, n\}$. Hence $I(X; Y) = C_q$ follows. \hfill \Box

A direct consequence of Proposition 3 is that if $E \leq I(X; Y) = C_q < H(\mu)$ there exist two identical quantum internal states $|S_i \rangle = |S_k \rangle$, $i \neq k$. This implies that for all $r \in \Sigma$ and all indices $l \in \{1, \ldots, n\}$ it holds that $T_{k,l}^{(r)} = T_{i,l}^{(r)}$. Which means that in the corresponding classical HMM there are two states which are redundant and can be merged to one state. This HMM is not a classical minimal HMM for the underlying stochastic process as the next proposition shows.

**Proposition 6** Given a stationary stochastic process $\widehat{X}$ with excess entropy $E$. For a given classical HMM generating $\widehat{X}$ with internal state entropy $H(\mu)$ let the corresponding induced quantum model defined in (7) have quantum internal state entropy $C_q$. The classical HMM corresponding to the induced quantum model in the case $E \leq I(X; Y) = C_q < H(\mu)$ is not a classical minimal HMM and therefore has not minimal classical internal state entropy.

**Proof.** Suppose that the classical HMM corresponding to the induced quantum model is a minimal HMM (i.e. $H(\mu) = C_{Cl}$), then one can remove all redundant states in this classical HMM and in the resulting induced quantum model there remains only orthogonal quantum internal states. With Proposition 3 we have $E = I(X; Y) = C_q = C_{Cl}$ and the reduced classical HMM is in fact the minimal HMM which is an ideal model. So the not reduced classical HMM cannot be the minimal HMM which is a contradiction to the assumption and the claim is proved. \hfill \Box

The last proposition implies that the case $E \leq I(X; Y) = C_q < C_{Cl}$ cannot exist.

**Remark 7** The case $E \leq I(X; Y) < C_q = C_{Cl}$ does not exist. Suppose this case exists. Then Proposition 1 would imply that all quantum internal states are mutually orthogonal and Proposition 2 implies $E = I(X; Y) = C_q = C_{Cl}$ which is a contradiction to the assumption.

That is given a minimal classical HMM one is either in the case that the classical HMM is as good as the induced quantum model or the induced quantum model has a quantum internal state entropy $C_q$ strictly smaller than $C_{Cl}$ and strictly greater than $I(X; Y)$. We summarize the different cases:

(i) $E = I(X; Y) = C_q = C_{Cl} \iff$ the classical HMM and the induced quantum model are both optimal and all quantum internal states are mutually orthogonal.
(ii) \( E \leq I(X;Y) = C_q < C_{Cl} \) is not possible.

(iii) \( E \leq I(X;Y) = C_q < H(\mu) \iff \) the corresponding classical model contains redundant states and is not a minimal HMM and the induced quantum model contains only orthogonal or identical states but at least two identical states.

(iv) \( E \leq I(X;Y) < C_q = C_{Cl} \) is not possible.

(v) \( E \leq I(X;Y) < C_q < C_{Cl} \iff \) the classical HMM can be optimal and there exists quantum internal states which are not orthogonal and not identical.

So if one chooses an optimal classical HMM which is not an ideal classical model, there is always an induced quantum model which is nearer to an ideal model but never achieve such an ideal model.

6 Calculation Example

The following example illustrates the propositions shown in the preceding sections. We consider the Random Noisy Copy HMM (RnC) \([E109]\). This HMM generates a binary stochastic output process. It is given by a binary alphabet \( \Sigma = \{0, 1\} \), the internal states \( S = \{A, B, C\} \) (which are also the causal states) and the Markov kernels

\[
T^{(0)} = \begin{pmatrix}
0 & p & 0 \\
1 & 0 & 0 \\
q & 0 & 0
\end{pmatrix}, \quad T^{(1)} = \begin{pmatrix}
0 & 0 & 1 - p \\
0 & 0 & 0 \\
1 - q & 0 & 0
\end{pmatrix},
\]

with \( 0 \leq p, q \leq 1 \). Figure 2 (a) shows a graphical representation of the RnC HMM.

![Minimal HMM for the RnC process.](image)

Figure 2: (a) Minimal HMM for the RnC process. Nodes denoting the internal states of the HMM and edges labels \( t|x \) give the probability \( t = T_{S,S'}^{(x)} \) of making a transition from \( S \) to \( S' \) and seeing symbol \( x \). (b) Minimal HMM for the underlying process in the case \( q = 1 \).

The RnC HMM coincide with the classical \( \epsilon \)-machine. The left eigenvector of the stochastic matrix \( T^{(0)} + T^{(1)} \) gives us the stationary distribution over the internal states

\[
P(S) = \frac{1}{2} \begin{pmatrix} 1 & p & 1 - p \end{pmatrix}.
\]
This allows us to calculate the generative complexity (which is identical with the statistical complexity)

\[ C_{Cl} = 1 + \frac{H(p)}{2}, \]

where \( H(p) = -p \log(p) - (1 - p) \log(1 - p) \) is the binary entropy function. In this section logarithm is taken to the base 2. With more sophisticated techniques (see [Ell09] for calculation details) or with Proposition 5 one can also calculate the excess entropy directly

\[ E = I(X; Y) = 1 + \frac{H(p)}{2} - \frac{p + q(1 - p)}{2} H \left( \frac{p}{p + q(1 - p)} \right). \]

The quantum internal states defined in (7) are

\[ |A\rangle = \begin{pmatrix} 0 \\ \sqrt{p} \\ 0 \\ \sqrt{1 - p} \end{pmatrix}, \quad |B\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |C\rangle = \begin{pmatrix} \sqrt{q} \\ 0 \\ \sqrt{1 - q} \\ 0 \end{pmatrix}. \]

The eigenvalues of \( \rho = \frac{1}{4}(|S_0\rangle\langle S_0| + p|S_1\rangle\langle S_1| + (1 - p)|S_2\rangle\langle S_2|) \) are

\[ \left\{ \frac{1}{2}, \frac{1}{4} \left( 1 \pm \sqrt{1 - 4p + 4p^2 + 4pq - 4p^2q} \right) \right\}. \]

Setting \( \eta(x) := -x \log(x) \) the internal entropy of the induced quantum model amounts to

\[ C_q = \eta \left( \frac{1}{2} \right) + \eta \left( \frac{1}{4} \left( 1 + \sqrt{1 - 4p + 4p^2 + 4pq - 4p^2q} \right) \right) + \eta \left( \frac{1}{4} \left( 1 - \sqrt{1 - 4p + 4p^2 + 4pq - 4p^2q} \right) \right). \]

Fixing the parameter \( q \) to certain values and varying \( p \) we obtain the different cases described in Section 5. For this we calculate the scalar product between the quantum internal states \( \langle A|B \rangle = \langle A|C \rangle = 0 \) and \( \langle B|C \rangle = \sqrt{q} \). Setting \( q = 0 \) all quantum internal states are mutually orthogonal and we are in case (i) which is shown in Figure 3 (a).

Figure 3: Generative complexity \( C_{Cl} \), quantum internal entropy \( C_q \) and excess entropy \( E = I(X; Y) \) for the RnC process with different \( q \)-values.

For \( q = 1 \) the quantum internal states \( |B\rangle \) and \( |C\rangle \) are identical while \( |A\rangle \) and \( |B\rangle \) are orthogonal. Thus we are in case (iii) as seen in Figure 3 (b). For \( 0 < q < 1 \) we are in case (v) and have a gap between \( E, C_q \) and \( C_{Cl} \) as depicted in Figure 3 (c) for \( q = 0.7 \).
For $q = 1$ the states $|B\rangle$ and $|C\rangle$ are identical and the corresponding classical HMM is not an $\epsilon$-machine but still $E = I(X;Y)$ holds for this model. This shows that the converse of Proposition 3 is not true. In the corresponding classical model (Fig. 2 (a)) the states $B$ and $C$ can be merged to a state $BC$ (see Fig. 2 (b)). This is the classical minimal HMM for the underlying process.

7 Alternative HQMMs

The induced quantum model (7) introduced in Section 4 is not the only possible HQMM construction that models a given stochastic process. In this section we present an alternative HQMM construction which is also able to model a stochastic process generated by a corresponding classical minimal HMM. For this we follow the construction suggested in [Mon11]. Given a classical minimal HMM $(T, (p_1, \ldots, p_n))$ with internal states $S = \{S_1, \ldots, S_n\}$ we define internal states of the quantum model as $|i\rangle$ for $i \in \{1, \ldots, n\}$.

Furthermore we have $\rho_i := |i\rangle \langle i|$ and define quantum operations with a sum representation

$$K_r \rho := \sum_{i,j=1}^n K_r^{ij} \rho (K_r^{ij})^*,$$

$$K_r^{ij} := \sqrt{T_{j,i}} |i\rangle \langle j|,$$

for every symbol $r \in \Sigma$. With $K_r \rho_j = \sum_{i=1}^n T_{j,i}^{(r)} \rho_i$ we get

$$P(X_0 = r|S_j) = \text{Tr}(K_r \rho_j) = \sum_{i=1}^n T_{j,i}^{(r)} = \sum_{i=1}^n P(X_0 = r; S_i|S_j),$$

and thus have the same transition probabilities as in the classical minimal HMM.

The quantum internal state entropy $\tilde{C}_q$ of this quantum model always coincide with the generative complexity of the process

$$\tilde{C}_q = S \left( \sum_{i=1}^n p_i \rho_i \right) = H(\{p_i\}_{i=1}^n) = C_{\text{Cl}}.$$ 

In the next example treated in [Mon11] we will see that in general $I(X;Y)$ is strictly smaller than $\tilde{C}_q$ and Proposition 4 is not true for this type of HQMM construction. Consider the stochastic process generated by a classical 4-symbol HMM (which is minimal and coincide with the classical $\epsilon$-machine) with internal states $S = \{U, D, R, L\}$ and transition matrices

$$T^{(0)} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \end{pmatrix}, \quad T^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \end{pmatrix},$$

$$T^{(2)} = \begin{pmatrix} 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \quad (16)$$
Figure 4: Classical 4-symbol HMM defined by equations (16).

Figure 4 shows a graphical representation of this HMM.

We obtain as a stationary distribution

\[ P(S) = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \]

and the generative complexity calculates to \( C_{CI} = 2 \). With the framework introduced in Section 4 it is possible to calculate \( I(X;Y) \) which is the left hand side in (8) and amounts to \( I(X;Y) = \frac{1}{2} \). Since \( \tilde{C}_q = C_{CI} = 2 \) Proposition 4 holds not in this situation. The quantum internal state entropy of the induced quantum model defined in Section 4 is (logarithm is taken to the base 2)

\[
C_q = \frac{1}{8} \left( \log(64) + \left( -3 + 2\sqrt{2} \right) \log \left( \frac{1}{8} (3 - 2\sqrt{2}) \right) \right. \\
\left. - \left( 3 + 2\sqrt{2} \right) \log \left( \frac{1}{8} (3 + 2\sqrt{2}) \right) \right) \\
\approx 1.2018.
\]

Monras et al. suggest in [Mon11] another quantum model for this process which is only a 2-level quantum system instead of the 4-level quantum system given above. Given the internal states \(|\uparrow\rangle, |\downarrow\rangle, |+\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \) and \(|-\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} \) and quantum operations \( K_r \rho = K_r \rho K_r^\dagger \) for \( r \in \{0, 1, 2, 3\} \) with

\[
K_0 = \frac{1}{\sqrt{2}} |\uparrow\rangle \langle \uparrow|, \quad K_2 = \frac{1}{\sqrt{2}} |+\rangle \langle +|,
\]

\[
K_1 = \frac{1}{\sqrt{2}} |\downarrow\rangle \langle \downarrow|, \quad K_3 = \frac{1}{\sqrt{2}} |\downarrow\rangle \langle \downarrow|,
\]

it can be derived from this HQMM the same statistical behaviour as the classical HMM. The quantum internal state entropy of this quantum model is smaller than \( C_q \) and amounts to

\[ S(\rho) = 1, \]

\[^4\text{The Stinespring-Kraus Theorem shows that every completely positive map admits a (nonunique) operator-sum representation, so that can be written as } K_\rho = \sum_i K_i \rho K_i^\dagger \text{ where } K_i \text{ are linear operators on a Hilbert space. } [\text{Kra83}].\]
with $\rho = \frac{1}{4} |\uparrow\rangle\langle\uparrow| + \frac{1}{4} |\downarrow\rangle\langle\downarrow| + \frac{1}{4} |+\rangle\langle+| + \frac{1}{4} |-\rangle\langle-|$. 

This example shows that in general the induced quantum model (7) is not the one with minimum quantum internal state entropy. The structure of quantum models with minimal internal state entropy is an open question.

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