Orbifold Kähler–Einstein metrics on projective toric varieties

Lukas Braun

Mathematisches Institut,
Albert-Ludwigs-Universität Freiburg,
Freiburg, Germany

Correspondence
Lukas Braun, Mathematisches Institut,
Albert-Ludwigs-Universität Freiburg,
Ernst-Zermelo-Strasse 1, 79104 Freiburg,
Germany.
Email: lukas.braun@math.uni-freiburg.de

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Abstract
In this short note, we investigate the existence of orbifold Kähler–Einstein metrics on toric varieties. In particular, we show that every $\mathbb{Q}$-factorial normal projective toric variety allows an orbifold Kähler–Einstein metric. Moreover, we characterize $K$-stability of $\mathbb{Q}$-factorial toric pairs of Picard number one in terms of the log Cox ring and the universal orbifold cover.

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1 | INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers. In contrast to the case of negative or zero first Chern class, where Kähler–Einstein metrics are known to always exist, due to the confirmation of the Yau–Tian–Donaldson conjecture, we know that in the case of Fano manifolds, the existence of a Kähler–Einstein metric is equivalent to the algebraic notion of $K$-polystability [6–8, 15]. This purely smooth setting was extended in the last years to the case of klt log Fano pairs $(X, \Delta)$ culminating in the analogous statement for such pairs [12, Theorem 1.6]: the existence of a singular Kähler–Einstein metric being equivalent to $K$-polystability of the pair $(X, \Delta)$. In the case of toric varieties, this is equivalent to the corresponding polytope having its barycenter at the origin [2–4, 16].
It was conjectured in [10, Conjecture 1] that for non-$K$-stable Fano manifolds, a Kähler–Einstein metric with certain \textit{cone singularities} should exist. Partial results in this direction can, for example, be found in [13], but there also have been found counterexamples to the original version of the conjecture [14, Theorem 1]. In fact, the counterexamples given are toric Gorenstein del Pezzo surfaces. On the other hand, a modified version of the conjecture, see [4, Conjecture 7.4], was proven in [12, Theorem 1.8], stating that for a log Fano pair $(X, \Delta)$, there exists a natural number $m$ and a $\mathbb{Q}$-divisor $D$ in the linear system $\frac{1}{m} | - m(K_X + \Delta)|$, such that $(X, \Delta + D)$ is $K$-polystable. Before that, in [4, Theorem 7.10], the authors showed that in the toric case one can find a torus invariant boundary with these properties.

However, as Donaldson remarks [10], the only singular metrics for which we know that “a great deal of the standard theory can be brought to bear” are orbifold metrics. Singular (or weak) Kähler–Einstein metrics on orbifolds are smooth orbifold metrics, see [11]. To have an orbifold structure on our variety $X$, a necessary but not sufficient criterion is that $X$ has quotient singularities and $\Delta$ is snc on the smooth locus with so-called \textit{standard coefficients} of the form $1 - \frac{1}{m_i}$. In the case of $\mathbb{Q}$-factorial toric varieties, an orbifold structure exists if the boundary is torus invariant and has standard coefficients, see Proposition 3.1.

## 1.1 Toric varieties with orbifold Kähler–Einstein metrics

As mentioned above, a toric boundary with standard coefficients indeed provides an orbifold metric. Our first result says that one can always find such a boundary.

**Theorem 1.** Let $X$ be a normal projective toric variety. Then $X$ allows a toric boundary $\Delta$ with standard coefficients, such that $(X, \Delta)$ is $K$-polystable. In particular, if $X$ is $\mathbb{Q}$-factorial, it allows an orbifold Kähler–Einstein metric.

While this can be deduced from the existence of some (possibly nonstandard) toric boundary $\Delta'$ with $(X, \Delta')$ $K$-polystable due to [4, Theorem 7.10], our proof of Theorem 1 also provides an alternative purely convex geometric proof for [4, Theorem 7.10].

## 1.2 $K$-stability in terms of the (log) Cox ring

As all toric varieties have a polynomial Cox ring, the grading of this ring by the divisor class group alone must encode the $K$-polystability of a toric variety. In [5], the authors introduced the notion of the \textit{log Cox ring} of a pair $(X, \Delta)$, which is the right object to study in this context, as it takes into account the boundary $\Delta$. For a toric orbifold boundary $\Delta = \sum_{i=1}^{N} (1 - \frac{1}{m_i}) \Delta_i$, where $\Delta_1, \ldots, \Delta_N$ are the prime components of $\Delta$, the \textit{log divisor class group} $\text{Cl}(X, \Delta)$ is the quotient of orbifold Weil divisors ($\mathbb{Q}$-divisors that become integral on orbifold charts) by linear equivalence. The log Cox ring is the associated divisorial algebra. Its spectrum $\tilde{X}_\Delta$, the \textit{log characteristic space}, allows a good quotient $\tilde{X}_\Delta \to X$ by the diagonalizable group $H_{(X, \Delta)} := \text{Spec} \mathbb{C}[\text{Cl}(X, \Delta)]$ that ramifies over $\Delta_i$ with order $m_i$. In this setting, we have the following characterization of $K$-polystability:

**Theorem 2.** Let $X$ be a $\mathbb{Q}$-factorial toric variety of Picard number one and dimension $n$ and let $\Delta = \sum_{\rho \in \Sigma(1)} (1 - 1/m_\rho) D_\rho$ be a toric orbifold boundary. Then the following are equivalent.
(1) \((X, \Delta)\) is K-polystable.

(2) The barycenter of \(P^\vee_{-(K_X+\Delta)} = \text{conv}(m_\rho u_\rho)_{\rho \in \Sigma(1)} \subseteq N_Q\) is 0.

(3) The orbifold universal cover of \((X_{\text{reg}}, \Delta)\) is \((\mathbb{P}^n, \emptyset)\).

(4) There is a subgroup \(\mathbb{Z} \leq \text{Cl}(X, \Delta)\) such that \(\text{Spec} \mathbb{C}[\mathbb{Z}] \cong \mathbb{C}^*\) acts with weights \((1, \ldots, 1)\) on \(\tilde{X}_\Delta\).

Unfortunately, this characterization breaks down for higher Picard numbers. This is partly because the dual of a polytope (which is not a simplex) with barycenter at the origin may have its barycenter away from the origin.

2 | PRELIMINARIES

2.1 | Log pairs and their singularities

Let \(X\) be a normal variety and \(\Delta\) be an effective \(\mathbb{Q}\)-divisor. We call \((X, \Delta)\) a log pair if \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. In case \(0 \leq \Delta \leq 1\), we call \(\Delta\) a boundary. Then for a log resolution \(f : Y \to X\), we define the discrepancies of \(K_X + \Delta\) to be the coefficients at exceptional prime divisors of the divisor \(K_Y - f^*(K_X + \Delta)\). We say that \((X, \Delta)\) is a klt pair, if \(\Delta < 1\) and the discrepancies are greater than \(-1\). We call \((X, \Delta)\) log Fano, if it is klt and \(-K_X + \Delta\) is ample. Moreover, we say that \(X\) is of klt type (Fano type), if there exists a boundary \(\Delta\) with \((X, \Delta)\) klt (log Fano).

2.2 | Toric geometry

We follow [9]. Let \(X\) be a toric variety with acting torus \(T\). As usual, by \(M\) and \(N\) we denote the dual lattices of characters and one-parameter subgroups of \(T\), respectively. Then \(X = X_\Sigma\) for some polyhedral fan \(\Sigma\) in \(N_{\mathbb{Q}}\). Every ray \(\rho\) of \(\sigma\) is associated with a \(T\)-invariant prime divisor \(D_\rho\), and these generate the group of \(T\)-invariant Weil divisors. We denote the primitive ray generators by \(u_\rho\). Elements \(m \in M_{\mathbb{Q}}\) define \(T\)-invariant \(\mathbb{Q}\)-principal divisors \(D_m\) in the following way:

\[ D_m = \sum_{\rho \in \Sigma} -\langle m, u_\rho \rangle D_\rho. \]

As the Picard group of affine toric varieties \(X_\sigma\) is trivial, consequently invariant Cartier divisors on \(X_\Sigma\) are just given by collections \((m_\sigma)_{\sigma \subseteq \Sigma}\) such that \(\langle m_\sigma, u_\rho \rangle = \langle m_\tau, u_\rho \rangle\) whenever \(\rho\) is a common ray of \(\sigma\) and \(\tau\) [9, Theorem 4.2.8]. Obviously, it suffices to specify \(m_\sigma\) for the maximal cones of \(\Sigma\). The situation gets even simpler if we consider ample divisors. For those, the \(m_\sigma\) are pairwise distinct and form the vertices of a convex polytope \(P_D \subseteq M_\mathbb{Q}\) [9, Corollary 6.1.16]. Moreover, the normal fan of \(P_D\) is \(\Sigma\) and the vertices of the dual polytope \(P^\vee_D\) are supported on the rays of \(\Sigma\). If we denote such a vertex supported on \(\rho\) by \(v_\rho\), then the value \(a_\rho\) of \(D\) at \(D_\rho\) is the rational number satisfying \(a_\rho v_\rho = u_\rho\). In particular, \(P^\vee_D\) is a lattice polytope if and only if the \(a_\rho\) are of the form \(1/k\) with \(k \in \mathbb{Z}\).

A \(T\)-invariant canonical divisor on a toric variety is given by \(K_{X_\Sigma} = -\sum_{\rho \in \Sigma} D_\rho\). So for a boundary \(\Delta = \sum_{\rho \in \Sigma} a_\rho D_\rho\), the vertices of the polytope \(P^\vee_{-(K_X+\Delta)}\) are given by \(v_\rho = -\frac{1}{1-a_\rho} u_\rho\).

Depending on the needs, people in toric geometry either work with the polytope \(P_D \subseteq M_\mathbb{Q}\), the dual \(P^\vee_D \subseteq N_{\mathbb{Q}}\), or both. K-(poly/semi)stability is equivalent to the barycenter of \(P_D \subseteq M_\mathbb{Q}\) lying at the origin [2–4, 16].
3 | PROOFS OF THE MAIN STATEMENTS

We start with observing that a toric boundary with standard coefficients on a \( \mathbb{Q} \)-factorial (not necessarily complete) toric variety induces an orbifold structure, a statement that should be very well known to experts but which we have not found in the literature.

**Proposition 3.1.** Let \( X \) be a normal toric \( \mathbb{Q} \)-factorial variety and \( \Delta \) a toric boundary with standard coefficients. Then the pair \((X, \Delta)\) is an orbifold.

**Proof.** Let \( X = X_\Sigma \). We have to show that locally \((X, \Delta = \sum_{\rho \in \Sigma(1)}(1 - 1/m_\rho)D_\rho)\) is a finite quotient ramifying over \( D_\rho \) of order \( m_\rho \). For the toric canonical divisor \( K_X \), the formula from Subsection 2.2 gives

\[-(K_X + \Delta) = \sum_{\rho \in \Sigma(1)} \frac{1}{m_\rho}D_\rho.\]

Take a maximal cone \( \sigma \subseteq \Sigma \) with extremal rays \( \rho_1, \ldots, \rho_n \) and corresponding primitive ray generators \( v_1, \ldots, v_n \). Here \( n = \dim(X) \) by \( \mathbb{Q} \)-factoriality. The toric log Cox construction, see [5, section 3.1], is given by the map of lattices \( \mathbb{Z}^n \rightarrow N; e_i \mapsto v_i \), that is, by the multiplication with the matrix \( P \) having the \( v_i \) as columns. In particular, the grading of the log Cox ring \( \mathbb{C}[x_1, \ldots, x_n] \) is given by the matrix \( Q \) Gale dual to \( P \) [1, section 2.2]. This is a toric morphism from a smooth variety, in particular a finite quotient by a finite abelian group, which ramifies over \( D_\rho \) of the right order \( m_\rho \), see [9, chapter 3.3].

**Proof of Theorem 1.** Let \( X = X_\Sigma \) be a normal projective toric variety. Choose some ample toric \( \mathbb{Q} \)-divisor \( L = \sum a_\rho D_\rho \). As \( L \) is ample, it corresponds to a full-dimensional rational convex polytope

\[ P_L = \{ u \in M_\mathbb{Q} | \langle u, v_\rho \rangle \geq -a_\rho \ \forall \rho \in \Sigma(1) \} \subseteq M_\mathbb{Q}, \]

not necessarily containing the origin. We denote by \( u_{p_L} \) the barycenter of \( P_L \). As \( P_L \) is full-dimensional and convex, we have \( u_{p_L} \in P_L^o \). Now denote by \( P' \) the translation of \( P_L \) by \(-u_{p_L} \):

\[ P' := P_L - u_{p_L} = \{ u \in M_\mathbb{Q} | \langle u, v_\rho \rangle \geq -a_\rho + \langle u_{p_L}, v_\rho \rangle \forall \rho \in \Sigma(1) \}. \]

The polytope \( P' = P_{l'} \) has its barycenter at the origin and corresponds to an ample \( \mathbb{Q} \)-divisor \( L' := \sum (a_\rho - \langle u_{p_L}, v_\rho \rangle)D_\rho \), which, as \( u_{p_L} \) was the interior of \( P_L \), now is effective and fully supported on \( \sum D_\rho \). That is, the coefficients \( b_\rho := (a_\rho - \langle u_{p_L}, v_\rho \rangle) \) are strictly positive rational numbers. We write \( b_\rho = p_\rho / q_\rho \) with natural numbers \( p_\rho \) and \( q_\rho \) and denote \( r := \lcm(p_\rho)_{\rho \in \Sigma(1)} \) and \( m_\rho := l \cdot b_\rho^{-1} \). Then scaling \( P_{l'} \) by \( 1/l \) yields another polytope

\[ P'' := \frac{1}{l}P' = \{ u \in M_\mathbb{Q} | \langle u, v_\rho \rangle \geq -1/m_\rho \ \forall \rho \in \Sigma(1) \}, \]

still having its barycenter at the origin and corresponding to the ample divisor \( L'' \). Writing

\[ L'' = \sum_{\rho \in \Sigma(1)} \frac{1}{m_\rho}D_\rho = -(K_X + \sum_{\rho \in \Sigma(1)} (1 - 1/m_\rho)D_\rho) =: \Delta. \]
yields a $K$-polystable pair $(X, \Delta)$ with $\Delta$ having standard coefficients. This proves the first statement of the theorem. The second statement then follows from Proposition 3.1 and, for example, the considerations in [11]. □

The following is another easy but useful observation concerning barycenters of dual simplices, that we have not found in the literature either.

**Lemma 3.2.** Let $P \subseteq \mathbb{Q}^n$ be a simplex. Then $b_P = 0$ if and only if $b_{P^\vee} = 0$.

**Proof.** As $(P^\vee)^\vee = P$, we only have to prove that $b_{P^\vee} = 0$ if $b_P = 0$. So, assuming $b_P = 0$ and applying a change of basis, we are in the situation that the vertices of $P$ are $a_i = e_i$ for $1 \leqslant i \leqslant n$ (where $(e_i)$ is the standard basis), and $a_{n+1} = -\sum e_i$.

The facets of $P^\vee$ are given by the $n + 1$ hyperplanes $\{x_i = -1\} (1 \leqslant i \leqslant n)$ and $\{\sum x_i = 1\}$. Thus, in the dual basis $(e^i)$, the vertices of $P^\vee$ are given by

$$b_k := ne^k + \sum_{k \neq i=1}^n -e^i \quad \text{for} \quad 1 \leqslant k \leqslant n, \quad \text{and} \quad b_{k+1} := \sum_{i=1}^n -e^i.$$

Thus, $(n + 1)b_{P^\vee} = \sum_{k=1}^{n+1} b_k = 0$ and the claim is proven. □

**Proof of Theorem 2.** The equivalence of (1) and (2) follows from [3, Theorem 1.2] and Lemma 3.2.

Now assume that (2) holds, that is, the barycenter of $P := \text{conv}(m_\rho u_\rho)$ is zero, where $\Delta = \sum (1 - 1/m_\rho) D_\rho$ and $u_\rho$ are the primitive lattice generators of $\Sigma_X$. Choose some numbering $\rho_0, \ldots, \rho_n$ of the columns of $\Sigma_X$. Then the matrix with columns $m_{\rho_1}u_{\rho_1}, \ldots, m_{\rho_n}u_{\rho_n}$ yields a lattice homomorphism (and a vector space isomorphism) that, as $b_P = 0$, maps the cones of the fan $\Sigma_{\mathbb{P}^n}$ to the cones of $\Sigma_X$ and thus by [9, Theorem 3.3.4] yields a toric morphism $\mathbb{P}^n \to X$. This morphism ramifies over $D_\rho$ exactly with order $m_\rho$ and thus due to (the log version of) [9, Theorem 12.1.10] corresponds to the orbifold universal cover of $(X, \Delta)$. So, (3) follows from (2).

Again by [9, Theorem 12.1.10], as $\pi_1^{\text{orb}}(X_{\text{reg}}, \Delta) = \text{N}/N_{(\Sigma, \Delta)}$, this group is a quotient of $\text{Cl}(X, \Delta)$ by some subgroup $H$, such that $X_\Delta/H = \tilde{X}_\Delta$ is the orbifold universal cover of $(X, \Delta)$. Now this group is isomorphic to $\mathbb{Z}$ and acts with weights $(1, \ldots, 1)$ if and only if $\tilde{X}_\Delta = (\mathbb{P}^n, \emptyset)$. So (3) and (4) are equivalent.

Finally assume that (3) holds. Then the covering $\mathbb{P} \to X$ is toric and again by [9, Theorem 3.3.4] yields a lattice homomorphism (and a vector space isomorphism) mapping the cones of $\Sigma_{\mathbb{P}^n}$ to the cones of $\Sigma_X$. This homomorphism maps the barycenter of $P_{\mathbb{P}^n}^\vee$, which is the origin, to the barycenter of $P_{(K_X + \Delta)}^\vee$, which therefore is the origin. Thus, (2) follows from (3) and the claim is proven. □

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ORCID

Lukas Braun https://orcid.org/0000-0002-2407-3656

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