Success and failure of the plasma analogy for Laughlin states on a torus

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Abstract
We investigate the nature of the plasma analogy for the Laughlin wave function on a torus describing the quantum Hall plateau at \( \nu = \frac{1}{q} \). We first establish, as expected, that the plasma is screening if there are no short nontrivial paths around the torus. We also find that when one of the handles has a short circumference—i.e. the thin-torus limit—the plasma no longer screens. To quantify this we compute the normalization of the Laughlin state, both numerically and analytically. In the thin torus limit, the analytical form of the normalization simplify and we can reconstruct the normalization and analytically extend it back into the 2D regime. We find that there are geometry dependent corrections to the normalization, and this in turn implies that the plasma in the plasma analogy is not screening when in the thin torus limit. Despite the breaking of the plasma analogy in this limit, the analytical approximation is still a good description of the normalization for all tori, and also allows us to compute hall viscosity at intermediate thickness.

Keywords: quantum hall effect, Laughlin wave function, conformal field theory, condensed matter

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(Some figures may appear in colour only in the online journal)

1. Introduction

The Laughlin wave function is the drosophila of representative trial wave functions for the fractional quantum Hall effect (FQHE). It was introduced in a planar geometry by Laughlin
[Lau83], generalized to the sphere by Haldane [Hal83a], to the torus by Haldane and Rezayi [HR85], and has has been extensively studied on these geometries ever since. Most remarkably, its elementary excitations—its quasi-particles—have fractional charges [Lau83], and are believed to obey anyonic statistics [ASW84]. The latter, although not proven experimentally, is strongly suggested by extensive numerical studies [KL99, KM99]. Recently, matrix product states have been successfully used to probe the anyonic statistics [CV13, ZMP13].

The theoretical argument for fractional statistics is based on the so called plasma analogy, first introduced by Laughlin [Lau83]. The main observation is that, introducing a particular factor dependent on the quasi-particle positions into the normalization of the wave functions, the corresponding full normalization constant can be expressed as the partition function for a classical screening plasma. In this analogy the quasi-particles, at positions $\eta_i$, appear as test charges with a strength one $q$th of the electron charge. As the plasma is screening, the partition function does not depend on the positions of the quasi-particles, provided they are separated much further than the screening length. From this, one concludes that the full $\eta$-dependence of the normalization is accounted for, and from this it is straightforward to calculate the fractional statistics parameter.

Another way of saying this, is that the ‘holonomy’ i.e. the phase related to adiabatically dragging one quasi-particle at $\eta_1$ around another at $\eta_2$ will equal the ‘monodromy’, i.e. the explicit phase obtained from the normalization factor. For this to occur, no additional contribution from any Berry phase related to the adiabatic change must be accumulated.

Since screening is essential for the argument of vanishing extra Berry phases, we may ask the contrary question: When does the plasma not screen? Can the screening properties of the plasma change if the quasi-particles move too close to the edge of the sample or if there are non-trivial loops on the spatial surface with short length scales [KL99, KM99]? A related question that is also important for the Laughlin states is the following: How does the free energy of the plasma depend on the topology, geometry and size of the surface on which the QH liquid resides? This is particularly interesting in the case of a torus where the normalization can depend on the aspect ratio of the two principal axes, $\tau$. $\tau$ is a complex valued parameter that can be changed while still keeping the density and number of electrons fixed. In the context of the plasma analogy, this corresponds to asking if the plasma is screening for all $\tau$.

Recently the $\tau$-dependence of the normalized Laughlin wave function, $\psi_L$, has been determined using techniques from conformal field theory [Rea09]. Later numerical work [RR11] confirmed that there where no extra $\tau$-dependent contributions to this normalization. The numerical analysis was however only performed for special values of $\tau$, so the question remains if there is need for corrections when considering general $\tau$.

It is fruitful to think of $\tau$ as a parameter that can be adiabatically changed, in the same way as the quasi-particle positions can be adiabatically dragged around each other. Under such an adiabatic change analogous questions appear as for the quasi-particles, i.e. is the Berry phase the same as the monodromy of changing $\tau$. In fact, for the Laughlin state this computation can be considered even more interesting as changing $\tau$ can transform the $q$ degenerate Laughlin states into each other, giving rise to Berry matrices. These matrices, which are well defined even without the presence of quasi-particles, would form a non-abelian representation of the modular group, the mapping class group of the torus, while exchanges of quasi-holes only give rise to phase factors.

In this paper we study the normalization properties of $\psi_L$, with no quasi-particles present. For general $\tau$ we obtain the normalization as a finite—but intractably large—sum in powers of $e^{i\tau\pi}$. We expand $\psi_L$ in $\tau$ by rewriting it in a basis of single particle orbitals i.e. a Fock basis. Progress has been made on the plane and sphere by identifying the Laughlin states with Jack polynomials [BH08]. The Jack polynomials can be recursively calculated [LLM00] by...
starting with the root partition in the orbital occupation basis and applying squeezing rules. The root partition is the configuration of electrons where the distance between all occupied orbitals is maximized. For the Laughlin state at $\nu = \frac{1}{3}$, the root partition is the orbital occupation pattern $\ldots 1001001001 \ldots$.

On the torus, no known squeezing rules exist and the Laughlin state can not be written in terms of Jack polynomials. On the other hand, there is a well defined limit, the Tau-Thouless (TT) limit [BK08], in which the Laughlin state is precisely a single slater determinant corresponding to the root partition. From the TT-limit it is possible to generate the Laughlin state at a general aspect ratio $\tau$ using a differential equation in this parameter [ZNS13]. The differential operator acts on the Fock expansion of the Laughlin state. In the TT-limit the analytical expression for the normalization also simplifies greatly can be expanded perturbatively in powers of $e^{i\pi \tau}$.

To find the Fock expansion of the wave function is useful since it would allow for a direct computation of the normalization of the Laughlin state as well as many other interesting quantities! It is conjectured that for a torus large enough, the $\tau$-dependence of the Laughlin state is fully captured by the normalization factor proposed by Read [Rea09]. In that case, there would be no extra dependence on the precise geometry of the torus, just as the normalization is insensitive to the positions of well separated quasi-holes of the Laughlin state [Lau83], if the factors mentioned before are taken into account.

For small system sizes the normalization of the Laughlin state can be computed exactly for any value of $\tau$. The rough picture is that when the torus is large enough in both directions then there is no $\tau$-dependence in the normalization. However, if one of the torus handles becomes too thin there is $\tau$-dependence in the normalization. See e.g. figure 2 in section 2. This is consistent with the intuitive picture that at some length scale the screening properties of the Laughlin plasma analogy break down.

In this paper we first analytically extract the Fock coefficients for the Laughlin state in a generic torus geometry $\tau$. We obtain a sum that is finite, but untractably large for practical purposes. From the analytical expressions we approximate the leading behavior of the Fock coefficients in the TT-limit, and using this leading order expansion we comment on the validity of the plasma analogy in the TT-limit and on the universality of the Hall viscosity. We find that the plasma analogy is no longer screening when the circumference of the torus becomes sufficiently small, and we compute the leading corrections to the normalization in this limit. We conclude that although the plasma is screening in the thermodynamic limit, it will always cease to screen if there is a loop around the torus shorter than some characteristic length scale. Intuitively, this distance, which we find to be roughly 6 magnetic lengths, should be on par with the screening length of the plasma.

This paper is organized as follows. Section 2 contains a brief summary of relevant notation. In section 4 we introduce the analytical Fock expansion for the Laughlin state on the torus. In section 5 we use the Fock expansion to study the Laughlin state in the TT-limit. We address both the proper norm in the TT-limit as well as extrapolate back to the thick torus. We also show that the quantum Hall viscosity of the Laughlin state is different in the TT-limit than in the thermodynamic limit. Finally, in section 6 we sketch the route to obtain a Fock expansion for the chiral Haldane–Halperin hierarchy states [Hal83a, Hal83b] constructed using CFT techniques [FHS14].

2. Lowest Landau level wave functions

For self consistency, but also to define our notation, we give the basic formula for the lowest Landau level (LLL) wave functions and translation operators on the torus.
In this paper we will exclusively work in the dimensionless coordinates where $x, y < 1$, defined on a unit square. They are related to the dimensionful coordinate $z$ as $(\tau, \gamma) = (z L_x, y)$. See figure 1. The complex parameter $\tau = \gamma + i \gamma_2$ parametrizes the torus geometry and is defined in the complex upper half plane.

The single particle Hamiltonian is the Landau Hamiltonian

$$H = \frac{1}{2m} \left( \frac{1}{L_x^2} (p_x - eA_x)^2 + \frac{1}{2m\gamma_2 L_y^2} (p_y - \gamma_1 p_x + \gamma_2 A_y)^2 \right).$$

In these units the vector potential is $(A_x, A_y) = (2\pi N_\ell L_x^2 B_y, 0) = (\gamma_1 L_x^2 B_y, 0)$. We have here chosen to use, what we call the $\tau$-gauge, as in the physical coordinates $z = x + iy$ the vector potential is perpendicular to $\tau = (\gamma, \gamma_2)$. The physical area of the torus is $A = \gamma L_x^2 = 2\pi N_\ell L_x^2$ and is penetrated by $N_\ell$ magnetic fluxes; we also define $\epsilon = \frac{1}{N_\ell}$. We will throughout this paper let the magnetic length be $\ell_\ell = 1$. The Landau Hamiltonian can be brought to the form of a harmonic oscillator $H = \hbar \omega \left( a^+ a + \frac{1}{2} \right)$ by introducing ladder operators

$$a = \sqrt{2} \left( \frac{\partial_x - \frac{\gamma_2}{2} L_y}{2} \right),$$

$$a^+ = -\sqrt{2} \left( \frac{\partial_x + \frac{\gamma_2}{2} L_y}{2} \right).$$

The energy levels are called Landau levels (LL) and are all $N_\ell$-fold degenerate. Acting with $a$ on a state in the lowest Landau level (LLL) gives the equation $\left( \partial_x + \frac{\gamma_2}{2} L_y \right) \psi_{\text{LLL}} = 0$. The solution to this equation shows that all LLL wave functions have the form

$$\psi_{\text{LLL}} = e^{i\pi N_\ell \gamma f(x)} f(z),$$

where $f(z)$ is a holomorphic function.

The translation operators that commute with the Hamiltonian are finite magnetic translation operators. There are two minimal translation operators $t_x$ and $t_y$, that move a state a finite distance along the principal axes of the torus, i.e. in the directions $x$ and $y$. The distance

\[ L_x = L \]

\[ L_y = \tau \]

\[ L_{\Delta} \]

\[ \tau = \frac{L_{\Delta} + i L_y}{L_x} \]

\[ L_{\Delta} = L \]

Figure 1. The relationship between the Cartesian coordinates $(\tilde{x}, \tilde{y})$ and the dimensionless coordinates $(x, y)$. In the figure one can also see that $\gamma = L_x$ is interpreted as the skewness, and $\gamma_2 = \frac{L_x}{L_y}$ as the aspect ratio, of the torus. The area of the torus is fixed to be $L_x L_y = 2\pi N_\ell \ell_\ell$. In this paper we will exclusively work in the dimensionless coordinates where $0 < x, y < 1$, defined on a unit square. They are related to the dimensionful coordinate $z$ as $z = L(x + \gamma y)$. See figure 1. The complex parameter $\tau = \gamma + i \gamma_2$ parametrizes the torus geometry and is defined in the complex upper half plane.
translated by \( t_s \) is \( \epsilon \) and so \( t_x^N \) makes a full revolution around the torus. The periodic boundary conditions are then defined through the equation
\[
\psi = e^{\phi t_x} \psi,
\]
where the \( \phi_a \) can be thought of as fluxes threading the two handles of the torus. The translation operators expressed in the \( \tau \)-gauge are
\[
t_x = \pi \partial_+ e^{-\epsilon t_x} \quad \text{and} \quad t_y = \pi \partial_+ e^{-\epsilon t_y}.
\]
From these two minimal translations we construct the full set of translation operators as
\[
t_{m,n} = e^{i ( m \phi + n \phi ) + 2 \pi n x}.
\]

The minimal translations are identified as \( t_x = t_{0,0}, \ t_y = t_{0,1} \). The operator \( t_{m,n} \) moves a wave function a distance \( z \rightarrow z + e L ( m + \pi n ) \) and has commutation relation
\[
t_{m,a',a} = e^{2 \pi ( m' - m_n ) t_{a',a} t_{m,n}}.
\]

We choose to diagonalize the single particle wave functions in the LLL with respect to \( t_x \).

For periodic boundary conditions \( ( \phi_x = \phi_y = 0 ) \), the single particle basis states are
\[
\eta_k(z) = \frac{1}{\sqrt{L \sqrt{\pi}}} e^{\pi t \pm \eta_0 t} \sum_{t \in \mathbb{Z}} e^{i \pi \eta_0 t} e^{x \xi + y \eta_0 t} \left( \begin{array}{c} \eta_k \ \eta_k \\ \eta_{k+1} \ \eta_{k+1} \end{array} \right),
\]
with properties \( t_x \eta_k = e^{2 \pi \eta_0 t} \) and \( t_y \eta_k = \eta_{k+1} \). The eigenstates for generic boundary conditions are trivially obtained by acting with \( t_x, t_y \) on \( \eta_k \). There are \( N_0 \) single particle orbitals, one for each flux quantum. Since the LLL has a non-commutative geometry there is a connection between the momentum \( k \) and the expectation value of the position in \( y \)-direction. This relation is simply \( \langle y \rangle = - \eta_0 k \). As such, the momentum label can alternatively be thought of as a physical displacement label of a one-dimensional system. This view is especially fruitful in the TT-limit where the width of the orbitals is much smaller than the inter-orbital distance, \( \sigma \ll \epsilon \). Here \( \sigma_0 = \sqrt{\langle y^2 \rangle - \langle y \rangle^2} \) is the standard deviation measure of uncertainty in the \( y \)-direction.

The \( \vartheta \)-function appearing in the above formula is defined as
\[
\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (z; \tau) = \sum_{t=-\infty}^{\infty} e^{i \pi ( t \pm a )^2} e^{i \pi ( t \pm a ) ( z + b )},
\]
and is frequently encountered when considering wave functions on the torus.

Note that, in (1) due to the holomorphic structure of \( \psi_{\text{LLL}} \), the exponentials linear in \( x \) and \( y \) are locked to form the object \( e^{i \pi t / \tau} \). We will utilize this fact when extracting Fock coefficients for the Laughlin state in the next section. For future convenience we also define the short hand notation
\[
\zeta_k(z) = e^{i \eta_0 t} e^{i \pi t \pm \eta_0 t} e^{i \pi \eta_0 t} e^{i \pi \eta_0^2} e^{i \pi \eta_0^3} e^{i \pi \eta_0^4} \left( \begin{array}{c} \zeta_k \\ \zeta_{k+1} \end{array} \right),
\]
for the terms entering (1). This allows us to write the basis wave functions (1) as
\[
\eta_k(z) = \frac{1}{\sqrt{L \sqrt{\pi}}} \sum_{t \in \mathbb{Z}} \zeta_k (z),
\]
In this manner we can isolate the Fourier factors $\zeta(z)$ that make up the single particle orbitals $\eta_k$. Note that if we set $\tau = \frac{L}{2\pi}$ and $p_k = \frac{2\pi}{L} k$, then $\zeta(z)$ can be identified with the single particle eigenstates on a cylinder as $\zeta(z) = e^{ip_k z} e^{-\frac{iL}{2} \tau + p_k \tau}$ in the ordinary Landau gauge and physical coordinates.

2.1. The Fock basis

In this section we summarize the essential features of the occupation basis, or Fock basis, for many-body quantum Hall states. The single particle orbitals defined in equation (1) are labeled by the momentum label $k$. A Fock state is defined as a state with a definite set $k$ of single particle orbitals $\eta_k$ occupied. We write this state as

$$\mathfrak{F}_k(z) = A \prod_{i=1}^N \eta_k(z_i),$$

where $A$ is an anti-symmetrization operator over the different coordinates. The anti-symmetrization operator is efficiently implemented by constructing the slater determinant of $\eta_k(z_i)$. A generic many-body state can thus be written as

$$\psi_{MB}(z) = \sum_k a_k \mathfrak{F}_k(z),$$

where the sum is constrained to $1 \leq k_1 < k_2 < \ldots < k_N \leq N_\text{c}$.

For bosons, the anti-symmetrization operator is replaced by a symmetrization operator $S$ and the determinant becomes a permanent, which is harder to evaluate. Also, the set of momentum labels need not all be distinct, so the $<$ is replaced by $\leq$ in the sum over $k$.

2.2. The Laughlin wave function

The original construction of the torus Laughlin state goes back to Haldane and Rezayi [HR85] by generalizing the short distance behavior on the plane to also be true on the torus. The complication on the torus was to find a proper center of mass (CoM) piece $F_s(z, \tau)$ that together with the Jastrow factor and Gaussian piece would give the desired single particle periodic boundary conditions. The resulting wave function for the Laughlin state at $\nu = \frac{1}{q}$ is

$$\psi_s(z) = N(z) e^{i\tau N_\text{c}} \sum_{i=1}^q \prod_{i<j} \vartheta_1 \left( \frac{z_{ij}}{L} \right) \vartheta \left( \frac{Z}{L}, \tau \right).$$

where $Z = \sum_i z_i$ and $z_{ij} = z_i - z_j$. The function $\vartheta$ is related to the generalized $\vartheta$-function in (2) as $\vartheta_1(z, \tau) = \vartheta \left[ \frac{1}{2}, \frac{1}{2} \right] (z, \tau)$, and has zeros at $z = n + m\tau$.

The label $s$ enumerates the $q$ different degenerate ground states that exists on the torus [Hal85]. These states only differ in the CoM function $F_s$, given as

$$F_s(Z, \tau) = \vartheta \left[ \frac{s}{q}, \frac{N_\text{c} - 1}{2} \right] (qZ, q\tau).$$
The \( q \) states are all related by rigid magnetic translations of all the particles. As such, the different states are transformed into each other as \( \prod_{j=1}^{N} \psi_{j} = \psi_{\vartheta} \). Note that \( \psi_{\vartheta+q} = \psi_{\vartheta} \) since under a rigid translation of \( q \) steps \( Z \rightarrow Z + \tau q \frac{N}{N_c} \rightarrow Z + \tau \), and \( \vartheta \) is quasi-periodic under this shift. The CoM label \( s \) is related to the total momentum of the state as \( \sum_{j} N \psi_{j} \equiv \tau N \), and \( \vartheta \) is quasi-periodic under this shift. The CoM label \( s \) is related to the total momentum of the state as \( \sum_{j} N \psi_{j} \equiv \tau N \), which also shows that there are precisely \( q \) independent values of \( s \). To specify periodic boundary conditions, \( s \) is an (half) integer when \( N_c \) is (even) odd.

The normalization factor \( N(\tau) \) comes from constructing the Laughlin state using conformal field theory and is

\[
N(\tau) = \left[ \sqrt{\frac{2}{\pi}} \frac{\eta(\tau)}{\eta(\tau + 1)} \right] \frac{\vartheta N}{\sqrt{\tau N}},
\]

(9)

where \( \eta(\tau) \) is Dedekind’s \( \eta \)-function. This normalization was first introduced by Read [Rea09] and later generalized to the full chiral abelian hierarchy in [FHS14]. The normalization coefficient \( N(\tau) \) has a particular \( \tau \)-dependence that ensures that \( \psi_{\vartheta} \) transforms as

\[
\psi_{\vartheta} \rightarrow S \sum_{s} S_{s',s} \psi_{s}',
\]

\[
\psi_{\vartheta} \rightarrow T \sum_{s} T_{s',s} \psi_{s}',
\]

(10)

under the modular \( S \) transformation \( \tau \rightarrow -\frac{1}{\tau} \) and \( T \) transformation \( \tau \rightarrow \tau + 1 \). The matrices \( S_{s',s} = \frac{1}{\sqrt{q}} e^{-i2\pi \vartheta s'/s} \) and \( T_{s',s} = \delta_{s,s'} e^{i2\pi(s'/s)} \) are the modular \( S \) and \( T \) matrices of the CFT for the Laughlin state.

The modular covariance of (10) is related of the redundancy in the parametrization of the torus using \( \tau \). All \( \tau \) related by \( S \) and \( T \) transformations are identical and this has to be reflected in the transformation properties of the wave functions under such transformations. In particular, the space spanned by the \( q \)-fold degenerate \( \psi_{\vartheta} \) should be invariant.

The real space wave function \( \psi_{\vartheta} \) is however not properly normalized. There exists an extra—unknown—normalization factor \( N_0 \) that can not be obtained by CFT arguments. The normalized Laughlin wave function \( \psi_{L} \) is related to \( \psi_{\vartheta} \) as

\[
\psi_{L} = N_0^{-1} \psi_{\vartheta},
\]

(11)

and the general consensus is that in the thermodynamic limit (at fixed \( \tau \) ) \( N_0 \) is constant—indepedent of \( \tau \)—and only depends on the particle number \( N_c \). To say that \( N_0 \) is constant is analogous to claiming that the Laughlin plasma [Lau83] is in a screening phase. Several works have added to the consensus of screening [BBR12, RR11] but they have not considered the TT-limit.

From the analysis of the modular properties of \( \psi_{\vartheta} \) we can reassuringly see that the proposed normalization (9) ensures that the density of \( \psi_{\vartheta} \) is modularly covariant. This means that

\[
|\psi_{\vartheta}(z, \tau)| = |\psi_{\vartheta}(z, \tau + 1)| = |\psi_{\vartheta}\left(\frac{\tau}{|\tau|}, \frac{z}{|\tau|} - \frac{1}{\tau}\right)|,
\]

which is also a property that we request from \( \psi_{L} \). As a byproduct we note that the unknown normalization factor \( N_0 \) must satisfy the same modular covariance
Although unknown, $N_0$ can be computed numerically for small systems by using that $\psi_2$ is the normalized ground state of the Haldane pseudo-potential [Hal83a]. By comparing the analytical $\psi_1$ in (7) to the numerically obtained $\psi_2$, $N_0$ can be extracted. In figure 2, $N_0$ is shown for $N_e = 6$ particles in the entire $\tau$-plane. As expected $N_0$ is constant and independent of $\tau$ in a large section of the $\tau$-plane. However, when $\tau \gtrsim 10$ then $N_0$ develops clear

$$|N_0(\tau)| = |N_0(\tau + 1)| = \left| N_0 \left( -\frac{1}{\tau} \right) \right|.$$  \hspace{1cm} (12)
τ-dependence, see figure 2(a). This change in $N_0$ signals that we are entering the region where the thinness of the torus becomes noticeable. Physically this means that one of the torus handles is so thin that the plasma stops screening. We will return to this in section 5 where we will use the analytic Fock expansion developed in section 4 to analyze the behavior of $N_0$ in the TT-limit.

3. Normalization and Berry phases

We can indirectly gauge the τ-dependence of $N_0$ by computing the Berry curvature of (7) as a function of the geometry parameter τ. Computing the Berry curvature is analogous to computing the quantum hall viscosity [ASZ95, Rea09]. Just as in [Rea09] and [FHS14] we make the ansatz that the normalized Laughlin wave function can be written on the form

$$\hat{\psi}(z; \tau) = N_0^{-1/2} e^{i \phi(z) \tau},$$

(13)

where $\hat{\psi}$ is holomorphic in $z$ and $\tau$, and $N_0$ is constant. The power of $\tau$ is then $P = \frac{4N_0}{2\pi}$. Note that (7) has precisely this form. After a short calculation, one finds that the Berry potential $A_\tau = \langle \psi_\tau | \partial_\tau \psi_\tau \rangle$ is $A_\tau = A_r = -\frac{P}{2\tau} = -\frac{4N_0}{8\tau}$. In terms of $A_r$ and $A_{\tau_1}$ this means that

$$A_{\tau_1} = A_r + A_{\tau} = -\frac{P}{\tau_2} = -\frac{4N_0}{4\tau_2}$$

(14)

$$A_{\tau_2} = \tau A_r - iA_r = 0.$$

We note that $A_r$ is a gauge dependent quantity and will be sensitive to the τ-dependence of the phase of $\psi$. Nevertheless it is a good first test to detect when $N_0$ might not be constant.

We use importance sampled Monte Carlo integration to evaluate the Berry connection $A_{\tau_1}$ and $A_{\tau_2}$ numerically as

$$\tilde{A}_{\tau_1} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Im(\langle \psi_\tau(\tau) | \psi_\tau(\tau + \epsilon) \rangle)$$

$$\tilde{A}_{\tau_2} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Im(\langle \psi_\tau(\tau) | \psi_\tau(\tau + i\epsilon) \rangle).$$

In figure 3 we scan $\tilde{A}_{r}$ over a large part of the τ-plane. For illustrative purposes we show a relatively small system of $N_e = 8$ electrons in the figure. The features in this picture are also present in larger systems but will occur at different values of $\tau$. In the figure we can see that $\tilde{A}_{r} \approx -\frac{4N_0}{4\tau_2}$ in a large region around $\tau = \tau_2$ but that it deviates significantly from the expected value when $\tau_2 \to \infty$. For $N_e = 8$ particles this deviation begins at around $\tau_2 \approx 5$ and is almost independent of $\tau_2$.

We also see—figure 3(b)—that $\tilde{A}_{\tau_2} \approx 0$ everywhere except in an approximate circle of radius $\tau \approx 0.1$ centered at $\tau = \tau_r$. This non-zero value can be understood by appealing to modular covariance and noting that the this region is the modular image of the region $\tau_2 \geq 5$ for $\tilde{A}_{\tau_1}$ in figure 3(a).

Since the Berry connection is not a gauge invariant quantity, it can be difficult to see what part of the deviation in $\tilde{A}_{r}$ with respect to $A_r$ is due to an actual τ-dependence of the normalization, and what is due to a simple τ-dependence of the phase (at fixed $\phi$). For that purpose, we also compute the Berry curvature which is a gauge invariant quantity. Under the assumption of a constant $N_0$ the Berry curvature is
The curvature is closely related to the Hall viscosity [ASZ95, Rea09] which can be used as a probe to distinguish different topological phases at the same filling fraction. Hall viscosity is conjectured to be a probe containing the same information as the shift \( S \) [Rea09]. On a spherical geometry the Laughlin state is characterized by the shift \( \nu = -S_N \) in the relation

\[
\nu = -S_N e^\tau = -\frac{qN}{4\tau_2},
\]

between the number of flux quantum and the number of electrons. The shift appears as a consequence of the curvature of the sphere and the non-zero orbital spin \( \tilde{s} \) of the electrons. When the electrons move over the surface of the sphere a Berry phase is accumulated giving rise to an extra effective magnetic flux, \( S_{\text{sphere}} = 2\tilde{s} \) [Rea08, RR11].

On the torus there is no curvature, so here \( S_{\text{torus}} = 0 \) and \( N_0 = \nu^{-1}N_e \) precisely. However, as \( S_{\text{sphere}} \) is a topological characteristic of the Laughlin state, the same information should exist also on the torus, the question is how it manifests itself. Read [Rea09] showed that the topological information could be extracted by studying the Hall viscosity [ASZ95] \( \eta^H \) of the quantum fluid. Read conjectured that the viscosity in the thermodynamic limit (on any geometry, but especially on a torus) should be

\[
\eta^H = \frac{1}{4} \hbar \nu S_{\text{sphere}} = \frac{1}{2} \hbar \nu \tilde{s}, \tag{15}
\]

where here \( \tilde{n} \) is the electron number density. In a later paper Read & Rezayi [RR11] numerically showed this result to hold for the Laughlin state (amongst other things), when \( \tau \) was close to \( \tau = 1 \).
To analytically compute viscosity for a many body state is usually difficult, but it is simplified dramatically if the real space wave function can be written as

\[ \psi_{\tau\tau} = z f(z, \tau) \]

where \( f \) is holomorphic in \( \tau \). Under this assumption the viscosity can be computed to be

\[ \eta = \left( \frac{\pi}{\hbar c} \right) H e B N e^2 \]

This can be reformulated as the intensive quantity of effective average spin \( \bar{s} \) as

\[ \bar{s} = \frac{2P}{N_c} \]

The Laughlin wave function \( \psi_s \) in (7) is precisely of the form (16) where \( N(\tau) \) contains the factor \( \frac{\hbar c}{\tau^2} \), so \( \bar{s} = \frac{3}{2} \) for \( \psi_s \). In a similar manner the viscosity for the rest of the chiral Haldane–Halperin [Hal83a, Hal83b] hierarchy on the torus [FHS14] can also be computed. In doing so a viscosity is obtained that also agrees with Read’s conjecture (15).

The numerical work by Read & Rezayi [RR11] showed that \( s = \frac{3}{2} \) and supports that Read’s conjecture holds in the thermodynamic limit for the nearly square tori. This lends indirect evidence to the screening properties of the plasma analogy. However, in other numerical studies of the viscosity [FHS14, TH15, ZNS13] it is clear that \( \bar{s} \) has significant \( \tau \)-dependence, if the torus is asymmetric enough.

In figure 4(a) we plot the effective shift \( \bar{s} = \frac{2\tau_2 F_{\tau_2}}{N_c} \) in the \( \tau \)-plane for \( N_c = 8 \) particles. Again the same features as for \( A_{\tau_1} \) and \( A_{\tau_2} \) are present. However, one important difference is that in the TT-limit \( \bar{s} \) stabilizes at \( \frac{1}{2} \) instead of the expected \( \bar{s} = \frac{3}{2} \). Note that the features at large \( \tau_2 \) are modularly mapped to the ringlike structure at small \( |\tau| \).
In figure 4(b) we get further indication that the transition from \( \bar{s} = \frac{3}{2} \) to \( \bar{s} = \frac{1}{2} \) is a generic feature independent of system size. In the plot we see that the transition to \( \bar{s} = \frac{1}{2} \) happens for all systems sizes examined, but the point of transition scales as \( \tau_2 \propto \sqrt{N} \) (see inset).

This TT-limit value of \( \bar{s} = \frac{1}{2} \) can be understood if the state at \( \tau \to \infty \) is described not by a strongly correlated fluid, but rather by a single slater determinant, i.e. a Fock state \( \psi = \delta_{\bar{s},z} \) as in (5). We will expand on this observation further in section 5. For now we simply conclude that in a region around \( \tau \approx \frac{1}{2} \) the Berry curvature has the desired properties. The number of MC data points is here \( 10^6 \) for all system sizes.

Returning to the Berry connection, we can now also compute the Berry phases associated with the modular \( T \)-transform \( \tau \to \tau + 1 \) and \( S \)-transform \( \tau \to -\frac{1}{\tau} \). Starting with the \( T \)-transform, we find that the accumulated Berry phase for a straight path from \( \tau \) to \( \tau + 1 \) is

\[
\phi_T = \int_0^1 d\tau \langle \psi_L | \partial_\tau | \psi_L \rangle = A_\tau = -\frac{qN_0}{4\tau_2},
\]

where we assume that \( N_0 \) is constant.

The accumulated Berry phase \( \phi_T \) can also be approximated numerically by discretizing the path from \( \tau \) to \( \tau + 1 \) into \( n \) steps and computing the cumulative overlap along the path as

\[
\tilde{\phi} = \int dx \langle \psi(x) | \partial_\tau | \psi(x) \rangle \approx A \left\{ \prod_{j=1}^n \langle \psi_L(\tau_j) | \psi_L(\tau_{j+1}) \rangle \right\}.
\]

A comparison of the numerical value \( \tilde{\phi}_T \) and theoretical value \( \phi_T \) can be seen in figure 5 for \( 5 \leq N \leq 20 \) and \( 1 \leq \tau_2 \leq 20 \). The numeric data is represented by points and the theoretic expectation is shown by by dashed lines of the same colors as the points. One can clearly see that for small \( \tau_2 \), the numeric phase \( \tilde{\phi}_T \) agrees very well with (18). It is also possible to see that at larger \( \tau_2 \) the the accumulated phase \( \tilde{\phi}_T \) saturates at \( \tilde{\phi}_T \approx 0.5 \) instead of \( \phi_T \to 0 \) as expected. In section 5.3 we will show that this is actually constant is actually \( \tilde{\phi}_T = \frac{\pi}{6} \) in the limit \( \tau_2 \to \infty \). In the inset in the same figure, one sees that the \( \tau_2 \) at which \( \tilde{\phi}_T \) deviates from (18) grows monotonically with \( N \). This hints that in the thermodynamic limit the transition will be at \( \tau_2 \to \infty \).

We may strengthen the picture that the normalization is constant for \( \tau \) near \( \tau \approx \frac{1}{2} \) by also considering the Berry phase accumulated under an \( S \)-transform. For the \( S \)-transform one has to be a bit careful, as there is no canonical path between \( \tau \) and \( -\frac{1}{\tau} \). Depending on the value of \( \tau \) some paths would be numerically more stable that others. We choose the a path that is self-dual, i.e. where \( \tau(s) = -\frac{1}{\tau(1-s)} \). One such path which is particularly nice is

\[
\tau(s) = \frac{1}{\tau} \left( \frac{1}{\tau} - 1 \right)^{-1} = r^{1-2s} e^{2\pi i (1-2s) + 2\pi i \sigma}, \tag{19}
\]

where \( r(0) = re^{i\theta} \). Using that \( A_\tau = -\frac{qN_0}{4\tau_2} \) and \( \frac{\partial \tau_2}{\partial \theta} = -2 \ln r \cdot \gamma(s) + (2\theta - \pi) \tau_2(s) \), the Berry phase becomes simply
Figure 5. The Berry phase $\phi_T$ accumulated by $\psi_\tau$ under a $T$ transform from $\tau$ to $\tau + 1$, as a function of $\tau_2$ for $\tau = 0$ and several $N_e$. Dashed lines show the expected values $\phi_T$ given by (18). Inset shows the same data as the difference $\phi_T - \phi_T'$. The path $\tau(s)$ is discretized in 100 steps with $7 \times 10^4$ MC points at each step. It is clear that $\phi_T$ agrees well with (18) when $\tau_2$ is not too large. However for large $\tau$ there is a clear deviation between $\phi_T$ and $\phi_T'$ which is not an effect of the discretization of the path. Note that as $\tau_2 \to \infty$ then $\phi_T \to \frac{7\pi}{6}$ instead of $\phi_T \to 0$.

$$
\phi_T = \int_0^1 \! ds \left( -2 \ln r \cdot \tau(s) + (2\theta - \pi)\tau(s) \right) A_{\tau_1} \\
= -\frac{qN_0}{4} \int_0^1 \! ds \left( -2 \ln r \cdot \frac{\tau(s)}{\tau_2(s)} + (2\theta - \pi) \right) \\
= -\frac{qN_0}{4} (\pi - 2\theta) 
$$

where we in the last step use that $\int_0^1 \! ds \tau \frac{d\tau}{\tau_2} = 0$. Again we assume that $N_0$ is constant.

In figure 6 we plot the phase $\phi_S$ accumulated by performing a $S$-transform along the self dual path (19) starting at $\tau = 0.4 + i\tau_2$. We see that this particular path gives a $\phi_S$ that agrees well with the expected $\phi_S$ as long as $\tau_2 \lesssim 0$ for all system sizes considered. Beyond this aspect it is possible to see clear deviations from (20). However, the deviation can be mostly attributed no Monte Carlo noise (upper left in inset). That the noise levels are bigger at larger $\tau_2$ is only natural as the length path through $\tau$-space will increase as $|\tau - i|$ grows.

Taking all the data into account (especially from $A_s$ and $\mathcal{F}_{\tau\tau}$) we conclude that the normalization $N_0$ is constant in a large region around $\tau \approx i$. We also conclude that the region of
constant $N_0$ increases with $N_e$. However we also see that if $\tau_2$ is sufficiently large, then $N_0$ develops a clear $\tau$-dependence and the ansatz (13) with $P = \frac{1}{4}$ is changed to $P = \frac{N_e}{4}$. This new power of $\tau_2$ is consistent with $\psi_L$ being described by only one Fock state.

To gain more understanding of the nature of the transition to this Fock state, we will in the next sections expand $\psi_s$ in a Fock basis and analytically send $\tau_2 \to \infty$.

4. Expanding Laughlin in a Fock basis

In this section we will rewrite the real space Laughlin wave function (7) in the single particle orbital basis of (5), i.e. the Fock basis. We do this to gain analytical control over the coefficients $a_k$ of the Laughlin state. Later, in section 5 we will use this analytic knowledge to make statements regarding the plasma analogy in the limit of a thin torus. In section 6 we will also outline how to perform the Fock expansion for the full chiral Haldane–Halperin hierarchy [Fre13–Hal83b] on the torus.

We begin the next section 4.1 by displaying and commenting on the main results. Readers who are interested in the derivation of these results may consult the supplementary material.
In the following part 4.2, we comment on the numerical efficiency of the Fock expansion and compare it to other methods of obtaining the same information.

4.1 Main results and formulas

In this part we present the main results and formulas. In what follows we will consider only the fermionic Laughlin wave function unless otherwise specified. The fermionic and bosonic Laughlin wave functions are so closely related that all results here carry over to the bosonic case with only minor modifications. Where there are differences these will be pointed out.

The Fock expansion of any state will in its most general form involve a sum over all partitions of the momentum numbers \( k_1, \ldots, k_N \) with the total momentum \( K_{\text{total}} = \sum_{i=1}^{N} k_i \mod N_e \). For periodic boundary conditions \( k_i \in \mathbb{Z} \) are integers and \( K_{\text{total}} = \frac{1}{2} q N_e (N_e - 1) + m N_e \mod q N_e \) for \( m \in \mathbb{Z} \). Since a Fock basis expansion could potentially contain all of these partitions it could become a very large sum. A convenient way to organize this sum is by changing to a variable offset from \( k_i \) which we call \( T_i \). The relation between the two is

\[
T_i = \phi_{k_i} + s \mod N_e,
\]

where the \( T_i \) are chosen such that \( \sum_{i=1}^{N} T_i = 0 \). The variable \( s \) is precisely the center of mass momentum label in (7). Because \( k_i \) is an integer, the index \( T_i \) will be an (half) integer when \( N_e \) is (even) odd, to match \( s \).

Using the auxiliary variable \( T_i \) the Laughlin wave function can be rewritten as

\[
\psi_s = \mathcal{N}(\tau) \sum_{T_i \in \mathbb{Z}_{N_e} + qN_e^{-1}} \mathcal{Z}(T) \prod_{i=1}^{N} \eta_{T_i+s}(z_i),
\]

which makes explicit the single particle orbital basis. Here, \( \eta_{k}(z) \) are the single particle orbitals from (1). Note that no explicit anti-symmetrization is needed as the coefficients \( \mathcal{Z}(T) \) are all fully antisymmetric by construction.

As the basis states \( \eta_k(z) \) are labeled modulo \( N_e \) the sum over \( T \) is finite. The half-integer offset to \( T \) is a remnant from the expansion of the Jastrow factor. Note that the CoM index \( s \) only enters the relation between \( k_i \) and \( T_i \) as a constant shift. This shift serves to relabel the basis states of the expansion. The same relabeling can be achieved by acting with the \( t_q \) translation operator as \( t_q \eta_{T_i} = \eta_{T_i + t_q} \) which shows again that all the \( q \) different Laughlin states are related by rigid magnetic translations.

The factor \( \mathcal{Z}(T) \) is anti-symmetric in the arguments \( T \) and is related to the Fock coefficient \( a_k \) in (6). As it stands (22) is not of the form of (6). However as \( \mathcal{Z}(T) \) is anti-symmetric, (22) can be recast on the form (6) with the identification \( a_k = \sqrt{N_e!} \mathcal{N}(\tau) \mathcal{Z}(T) \).

The Fock expansion coefficient \( \mathcal{Z}(T) \) is given by

\[
\mathcal{Z}(T) = \sum_{T_i \in \mathbb{Z} + q/2} \exp \left\{ i \pi \tau \frac{1}{q N_e} \sum_{i < j < k} (\tilde{T}_{ij} + \tilde{T}_{ik} + \tilde{T}_{jk}) \right\} \prod_{i < j} \left( Z^{(q)}_{T_{ij}} e^{\tau X_{T_{ij}}} \right),
\]

where the \( \tilde{T}_{ij} \) are for technical reasons (see the supplemental material) anti-symmetric half-integer numbers \( \tilde{T}_{ij} = -T_{ij} \). The sum over all the \( \tilde{T}_{ij} \) is constrained to fulfill
such that each particle has the desired momentum $T_i$.

To obtain the expression for $Z(T)$ one has to expand the $\vartheta$-functions in the Jastrow factors in terms of Fourier—or momentum—components $	ilde{T}_{ij}$. This makes it possible to extract the total momentum of a specific particle $z_i$ as

$$\sum_{j=1}^{N} \tilde{T}_{ij}$$

by multiplying together all the contributions from all the pairs of particles. Thus $\tilde{T}_{ij}$ appear naturally in $Z(T)$ as they enumerate all the ways of forming the total momentum $T_i$ from all the pairwise contributions $\tilde{T}_{ij}$.

In the above formula (23) the $\tilde{T}_{ij}$ are offset from integer by $\frac{q}{2}$ and subject to (24). This is a direct consequence of the fact that each $\vartheta$-function in the Jastrow factor is taken to the power of $q$. The higher power in the Jastrow factors is also noticeable in the factors $\tilde{T}_{ij}$. The $Z(T)$ can be thought of as structure factors for the $q$th power of a $\vartheta$-function. It enters the Fourier expansion of the $\vartheta$-function as

$$\sum_{T \in \mathbb{Z} + \frac{q}{2}} e^{i\pi \frac{1}{2} T^2} e^{i2\pi T \tau + bT} Z(T)^q$$

and was discussed briefly in [Fre13]. The sum over $T$ is infinite and offset from the integers by $aq$. Details of $Z(T)^q$ as well as explicit expressions for it are given in appendix.

The factor $e^{i\pi T \tau}$ in (23) also comes from the Jastrow factor. It is this piece that ensures that $Z(T)$ is anti-symmetric with respect to $T$. The anti-symmetry comes about since an interchange of $T_i$ and $T_j$ will only affect the factor $e^{i\pi T \tau}$. It does so by sending $e^{i\pi T} \rightarrow e^{-i\pi T}$. Since $T_i$ is a half-integer then $\tilde{T}_i - \tilde{T}_j \in 2\mathbb{Z} + q$ is an odd (even) integer and $e^{-i\pi \tilde{T}_i} = (-1)^{q} e^{i\pi \tilde{T}_j}$.

See figure 7(b) for and interpretation of the sum $\sum_{i<j<k} (\tilde{T}_{ij} + \tilde{T}_{jk} + \tilde{T}_{ki})^2$ as sum over triangles with corners in $(i,j),(j,k)$ and $(i,k)$.

In (23) we have for increased readability suppressed the common factors $N(\tau)$ and $\left(\frac{2N\pi^2}{\tau^2}\right)^{N_\tau}$ that come from the CFT normalization and normalization of the $\eta_k$ respectively.

Figure 7. (a) A tabular view of relationship between $\tilde{T}_j$ and $T_j$ in equation (24) we well as the balance condition $\sum_T T_i = 0$. (b) How the signs of the terms in $\tilde{T}_j$ are added to form (23). In the figure it is assumed that $i < j < k$. 

$$\tilde{T}_j = \sum_{j=1}^{N} \tilde{T}_{ij}$$

(24) such that each particle has the desired momentum $T_i$.
4.2. Numerically evaluating the Fock coefficients

In the previous part we showed that the Laughlin state (7) could be expanded in a Fock basis with coefficients $\mathcal{Z}(\tilde{T})$ given by (23). Unfortunately the given expression is not particularly helpful when it comes to numerical evaluation. The culprit is the simultaneous infinite sums over all of the $\tilde{T}_{ij}$-s. To alleviate this problem a bit, $\mathcal{Z}(\tilde{T})$ can be rewritten such that the sums can be performed in a recursive manner. This will substantially reduce the scaling of the computation. The details of this manipulation can be found in the supplemental material.

A word of caution should be given. Although the optimized expression for $\mathcal{Z}(\tilde{T})$ can be put on a computer, the efficiency in evaluating it is still inferior to that of simply diagonalizing the Haldane pseudo-potential Hamiltonian. Using exact diagonalization the coefficients for around 12 particles can be extracted numerically, but in our current implementation of $\mathcal{Z}(\tilde{T})$, only 6 or so particles can be achieved.

Our method can as mentioned above not compete with exact diagonalization for the Laughlin state, but that is on the other hand not the main purpose of the expansion. An obvious advantage of our approach is the explicit access to the $\tau$-dependence of the Fock coefficients, something the exact diagonalization can not provide.

Also, the reader should also be aware that the pseudo-potential trick only exists for the Laughlin state. For states higher in the hierarchy there is no local Hamiltonian for which these state are the exact zero energy eigenstates. In these cases the results in this article is the only way we are aware of to analytically extract Fock coefficients.

There is also another setting in which analytical knowledge of these coefficients may be of use. This is when considering the non-chiral extensions of the chiral HH-hierarchy. For this extension the $K$-matrix [WZ91] is spit into two separate pieces as $K = \kappa - \bar{\kappa}$. This can be done even for the Laughlin case by splitting $K = q$ as $\kappa = q + p$ and $\bar{\kappa} = p$. On the torus, the CFT methods introduced in HSB+08 and [FHS14] can be generalized to handle general $K$-matrices in an analogous way to the procedure on the plane [HCJV07a, HCJV07b]. The wave function that is obtained [Fre13] contains factors $\vartheta(z) \vartheta(\bar{z}) \vartheta(z) \vartheta(\bar{z})$ such that it is not only in the LLL any more. This wave function can be projected on the LLL analytically using a basis of coherent states [Fre13]. In this projection, the coefficients $\mathcal{Z}(\tau)$ and $\tilde{\mathcal{Z}}(\bar{\tau})$ of the underlying Laughlin states appear naturally as ingredients. The details of the analytic projection mentioned in the above paragraph will be the subject of a future paper.

5. Asymptotic behavior in the TT-limit

In this section we will return to the main equations (22) and (23). We will analyze these expressions in the limit of $\tau \to \infty$ (or $L \to 0$ for constant area), which is what we call the thin torus (TT) limit. We are interested in studying the TT-limit to answer fundamental questions regarding the plasma analogy. Using techniques from conformal field theory [Rea09] a candidate for the normalization $\mathcal{N}(\tau)$ of the Laughlin state can be obtained—see equation (9). The assumption that $\mathcal{N}(\tau)$ captures the full $\tau$-dependence is related to the plasma analogy introduced by Laughlin [Lau83]. It states that the normalization of the Laughlin state is the Boltzmann weight for a single component plasma of charged particles with a logarithmic electrostatic interaction. As this plasma is know to be screening, the braiding properties of the quasi-particles can be deduced, assuming they are separated far enough. On the torus, the assumption of screening properties with respect to $\tau$ allows us to compute e.g. modular group
representations. In this section we will check if the screening holds also in the TT-limit. We will expand on this discussion in section 5.2.

This section has three parts. In the first section 5.1, we analytically take the TT-limit and extract the dominant Fock contributions. In this limit only a few of the Fock coefficients will remain nonzero, and their analytical expressions are drastically simplified. We also extract their relative scaling as a function of \( \tau \). This knowledge will then be used in sections 5.2 and 5.4 to make statements about the corrections to the CFT normalization and the Hall viscosity in the TT-limit.

Section 5.2 concerns the true normalization of (7) in the TT-limit. We will use the fact that there is only one particular configuration with non-zero Fock coefficient in the TT-limit—the root partition—to compute the full normalization when \( \tau \rightarrow \tau_{\infty} \). From this knowledge we compute the asymptotic correction to the normalization \( \mathcal{N}(\tau) \) proposed by the CFT construction in [Rea09]. The result shows that the plasma analogy does not hold in the TT-limit.

Under the assumption that \( \mathcal{N}(\tau) \) is the full normalization, i.e. that the unknown piece \( \mathcal{N}_0 \) contains no \( \tau \)-dependent contributions, the Hall viscosity \( \eta^{H} \) for \( \psi_L \) can be analytically extracted. In section 5.4 we will analyze how the viscosity changes as we approach the TT-limit. We will argue that since it is only the root configuration that is present in this limit, the viscosity is trivially \( \eta^{H} = \frac{1}{4} \hat{\eta} \) instead of \( \eta^{H} = \frac{1}{4} \hat{\eta} \) which is the value for the square torus. This is the same viscosity as that of a single Slater determinant state. To make the claim more robust, and also give some indications of why the viscosity is independent of system size in the regime of small \( L \), we compute the viscosity analytically for the root configuration and the sub-dominant Fock contribution in this limit.

### 5.1. Asymptotic scaling of the Laughlin state in TT-limit

Let us now take a look at the TT-limit for the Laughlin state. In this section we will show that the root configuration is the dominant configuration [BK08, SS81], and we will also give the relative scaling of all the two particle squeezed states.

We start from (23) and use the short hand notation \( A(N_i) \) for the Gaussian weight in \( Z(T) \):

\[
A(N_i) = \sum_{i<j<k} (\tilde{T}_{ij} + \tilde{T}_{jk} + \tilde{T}_{ki})^2. \tag{26}
\]

We are interested in the behavior deep in the TT-regime, when \( \tau \rightarrow \tau_{\infty} \). In this regime the Fock expansion will be exponentially dominated by the configuration of \( \tilde{T}_{ij} \) that can maximize the weight

\[
W(\tilde{T}_{ij}) = e^{\frac{1}{2\pi} A(N_i) \sum_{i<j} Z_{ij}^{(q)}}. \tag{27}
\]

For large \( \tau \) it is straightforward to show that \( \vartheta_{\beta}(z|\tau)^{H} \approx 1 \) for \( -\frac{\pi}{2} < \vartheta(z) < \frac{\pi}{2} \) and as a consequence \( \tilde{Z}_{ij}^{(q)} \rightarrow \delta_{ij} \) in this limit. That is, \( \tilde{Z}_{ij}^{(q)} \) has its largest value \( \tilde{Z}_{ij}^{(q)} \rightarrow 1 \) (and zero otherwise) when \( \tilde{T}_{ij} = \pm \frac{q}{2} \). We take as an ansatz that this configuration will also give the maximum for \( W(\tilde{T}_{ij}) \). Explicitly, we choose the ansatz

\[
\tilde{T}_{ij} = \frac{q}{2} (\theta_{ij} - \theta_{ij}), \tag{28}
\]

where \( \theta \) is the Heaviside function.
\[ \theta_{i,j} = \begin{cases} 1 & i < j \\ 0 & i \geq j \end{cases}. \]

In words: \( \tilde{T}_{ij} = \frac{q}{2} \) when \( i > j \) and \( \tilde{T}_{ij} = -\frac{q}{2} \) when \( i < j \). This particular ansatz is favorable since it produces the correct root partition. The electron with label \( i \) will \((s = 0)\) have momentum \( \tilde{T}_{ij} = \sum_{k<j} T_{kk} q_k \) such that \( \sum_{k=1}^{N_e} q_k N_k = 2 \). The value of \( A(N_e) \) for this configuration is

\[ A(N_e) = \sum_{i<j<k} \left( \frac{q}{2} + \frac{q}{2} - \frac{q}{2} \right)^2 = \sum_{i<j<k} \frac{q^2}{4} = \frac{q^2}{24}(N_e^3 - 3N_e^2 + 2N_e). \]

The sum \( P_{NN} = \sum_{i<j<k} 1 = \frac{1}{6}(N_e^3 - 3N_e^2 + 2N_e) \) is computed either as the third order polynomial that has roots \( P_0 = P_1 = P_2 = 0 \) and \( P_3 = 1 \) or by using Faulhaber’s formula \( \sum_{k=1}^{n} k^2 = \frac{1}{6}(2n^3 + 3n^2 + n) \).

The reference weight is thus

\[ W_{TT} = W\left(\frac{q}{2}(\theta_{i,j} - \theta_{j,i})\right) = e^{\pi\tau(qm - m^2)/2}. \] (29)

We must still ensure that the ansatz (28) yields a global minimum (up to permutations of the electrons) of (27), but here we only check that it is local\(^1\). To do so we investigate changes to \( A(N_e) \) as one of the \( \tilde{T}_{ij} \) is varied. Due to the permutation symmetry in \( A(N_e) \) we can without loss of generality choose \( i < j \). There is a subtlety here; as we have chosen the reference \( TT \)-configuration to be that of (28), the ordering of the electrons is implicit from 1 to \( N_e \). Another choice of the signs in (28) would lead to a corresponding permutation of the particles. Mathematically this shows up in (26) in such a way that \( \tilde{T}_{ij} \) is negative in some terms and positive in some others. We can interpret a change \( \tilde{T}_{ij} \) as electron \( i \) moving \( m \) steps to the left and electron \( j \) moves \( m \) steps to the right. Thus—because of the implicit ordering—\( (for \ i < j) \) for \( m > 0 \) is a squeeze and \( m < 0 \) is an anti-squeeze.

We start by considering the simplest case of changing only \( \tilde{T}_{12} \). The effect on \( W(\tilde{T}_{ij}) \) comes from two factors. The first is the change in \( A \) that is \( \Delta A = (N_e - 2)m(m-q) \) and the second is from \( Z_{TT}^{(q)} \). This factor becomes

\[ \tilde{Z}_{TT}^{(q)} \approx \frac{q}{2} \left| \frac{m}{q} \right| e^{\pi\tau\left|\frac{m^2 - m^2}{q}\right|}. \] (30)

for \( \frac{q}{2} \leq m \leq \frac{q}{2} \).

By looking just at \( \Delta A \) we see that \( \Delta A < 0 \) when \( 0 < m < q \), which translates into the particles being squeezed. This must be a pathological result as it would render the \( TT \)-state unstable to clustering of electrons. However, when we also take \( Z_{TT}^{(q)} \) into account we get

\[ W(\tilde{T}_{ij}) \approx \left( \frac{q}{|m|} \right) e^{\pi\tau\left|\frac{m^2 - m^2}{q}\right|} \times W_{TT}. \] (31)

The combined difference \( \Delta W \) is

\[ \Delta W = -2m^2 + 2|m|q \times \begin{cases} 1 & m > 0 \\ N_e - 1 & m < 0 \end{cases}. \] (32)

\(^1\) A global proof has not been attempted.
which for $|m| \leq \frac{q}{2}$ is always a positive. Note the somewhat counterintuitive behavior that $…01000001010…$ is more penalized than $…01000100010…$ in the TT-limit. The former would have an excitation energy of $\Delta E = 2\Delta_3 + \Delta_4 + \cdots$ and the latter $\Delta E = \Delta_2 + 2\Delta_3 + \cdots$ which is larger. The terms $\Delta_k$ are greater than zero and can be thought of as pseudo-potentials, although their origin is slightly different. For simple potentials like the Coulomb potential $\Delta > \Delta > \Delta > \cdots > 0$. See e.g. equation (19) in [BK08] for details. Of course, using perturbation theory, the first correction would not be given by the energy, but by the amplitudes of the hopping terms. In this case the hopping term for squeezing $V_{3, -1}$ is larger than anti-squeezing $V_{3, 1}$. Thus, the relative scaling of the squeezed $m = +1$ state is $q e^{\pi \tau \Delta_W q^{N_0 - q - 1}}$ whereas it is $q e^{\pi \tau \Delta_W q^{N_0 - q - 1}}$ for the anti-squeezed state.

In figure 8 we see the relative scaling of the root partition in comparison to the sub-dominant Fock-states for $N_e = 6$ particles. We see that the two leading sub-dominant states are a squeezed $(0011000100101001010)$ and a doubly squeezed $(00110001000100110010)$ state. The order and scaling of these is well captured by the leading order expansion (32). In the figure it is possible also possible to identify a triply squeezed state $(0011000110001100100)$ and this is also well described by (32) taken three times.

Observe that as the area is fixed to $L^2 \tau = 2\pi N q$ we can rewrite the scaling difference as $e^{\pi \tau \Delta_W} = e^{\pi \tau \Delta_W} e^{-\pi \tau \frac{L}{2} \Delta_W}$ where the scale depends only on $L$, and $\tau$ only changes the relative phase. This shows that the relative scaling in the TT-limit is independent of system size and depends only on $L$ instead of $\tau$. This is actually to be expected since in the thin torus limit the only relevant length scale is the short circumference $L$.

We can now also consider the more generic case where particle $i$ and particle $i + \Delta$ are squeezed $m$ steps. The change in $A$ this time is
\[ \Delta A = (N_e - 2)m^2 - mq(N_e - 2\Delta), \]  

(33)  

where \( \Delta = j - i \). Taking into account the change from \( \tilde{T}_i^\alpha \) gives again (31) but this time with  

\[ \Delta W = -2m^2 + 2|m|q \times \begin{cases} \Delta & m > 0 \\ N_e - \Delta & m < 0 \end{cases} \]  

(34)  

Putting \( \Delta = 1 \) yields the result in (32). Just as with the special case \( \tilde{T}_i^\alpha \), \( \Delta W > 0 \) for small values of \( m \). Note also that the formula for \( m > 0 \) can be mapped on the case \( m < 0 \) by letting \( m \to -m \) and \( \Delta \to N_e - \Delta \). This symmetry is natural since an inwards squeeze of \( m \) steps a distance \( \Delta \) away can alternatively be seen as outward squeeze of two particles \( N_e - \Delta \) steps away.  

Looking again at figure 8 we see that we can identify the one particle squeezed states \( (00\ldots001010100010010010\ldots) \) corresponding to \( (\Delta = 2, m = 1) \). We can also see \( (00\ldots001010100010001100\ldots) \) which is a combination of \( (\Delta = 2, m = 1) \) and \( (\Delta = 1, m = 1) \) acting on different pairs. The scaling of both of these are well described by (34).  

5.2. \( N_0 \) in the TT-limit  

In this part we will investigate the possibility of TT-limit corrections to the normalization of the Laughlin state proposed by the CFT construction in [Rea09]. If TT-corrections turn out to be present in the limit \( N_e \to \infty \), it will have immediate consequences for the validity of the plasma analogy. The plasma analogy states—in the form relevant here—that the free energy of a single component plasma in two dimensions on a torus does not depend on the geometry of the torus, provided the area is held constant. The original plasma analogy formulated by Laughlin states that the free energy does not depend on the positions of test charges in the plasma, provided they are sufficiently far apart [Lau83]. In both cases the validity of the analogy relies on the property that the plasma should be screening.  

At the end of section 2.2 we discussed the behavior of the true normalization \( N_0 \) for a small system of \( N_e = 6 \) electrons. We found by consulting figure 2 that as long as the torus is reasonably thick in both directions then \( N_0 \) is also constant. For \( \tau \approx 1 \), variations of \( N_0 \) appeared only in the fourth decimal (see figure 2(b)). However, looking at more asymmetric tori \( (\tau_2 > 10) \) we noted that \( N_0 \) deviates from being constant and develops clear \( \tau \)-dependence. This we interpret as one of the torus handles being so small that the plasma no longer can screen in that direction. As a consequence the plasma stops to screen and the normalization obtained from CFT is incorrect.  

We will now develop an approximate expression for the \( \tau \)-dependence of \( N_0 \) in the TT-limit. This is enabled since the expression for \( \psi_L \) simplifies in this limit. Due to the differences in scaling, only one of the Fock states will survive and \( \psi_L \) will be just a product state \( \psi_{TT}(z) = \tilde{\theta}_{TT}(z) \).  

(35)  

This product state is precisely that of the root partition. Also, since \( \psi_L = N_0^{-1} \psi_L \) the limit can be formulated as  

\[ \psi_L \to N_0\psi_L \sim \mathcal{N}(\tau) \mathcal{Z}(\mathcal{T}) \sqrt{N_e!} \tilde{\theta}_{TT}(z), \]  

(36)  

where both \( \mathcal{N}(\tau) \) and \( \mathcal{Z}(\mathcal{T}) \) are known analytically. Comparing (35) and (36) leads to the the form of \( N_0 \) in the TT-limit \( (\tau_2 \to \infty) \) as  

\[ N_0 \to N_{TT} = \mathcal{N}(\tau) \mathcal{Z}(\mathcal{T}) \sqrt{N_e!}. \]  

(37)
Note that for \( q = 1 \), \( N_0 = N_{TT} \) for all \( \tau \) as in this case there exist only one Fock state, the filled Landau level.

The expressions both for \( N(\tau) \) and \( Z(\tau) \) do also simplify when \( \tau \to \infty \) as all sub-dominant terms will vanish. For instance, the Dedekind eta function \( \eta(\tau) \) that appears in \( N(\tau) \) can be expressed as 
\[
\eta(\tau) = e^{\pi \tau^2} \prod_{n=1}^{\infty} (1 - e^{2\pi \tau n}),
\]
so the asymptotic expansion is simply \( \eta(\tau) \to e^{\pi \tau^2} \). Inserting this into (9) gives asymptotic behavior of \( N(\tau) \) as
\[
N(\tau) \approx \frac{\vartheta N}{e^{\pi \tau^2} [\vartheta N(N_0 - 3) + 2]}.
\]

Similarly for \( Z(\tau) \) the dominant \( \tau \)-dependent factor of (23) is given by (29) and so
\[
Z(\tau) \to \left( \frac{2N_0 \pi^3}{\tau_2} \right)^{\frac{N}{2}} e^{\pi \tau^2} \frac{N^{N_0 - 1}}{\tau^2}.
\]
in the TT-limit. Inserting the above simplifications into (37) gives
\[
N_{TT} = \sqrt{N_0} \left( 2qN_0 \pi^3 \right) \frac{N_0}{\tau_2^2} e^{\pi \tau^2} \frac{N_0 - 1}{\tau_2}. \tag{39}
\]

We should at this points ask ourselves how well \( N_{TT} \) actually approximates \( N_0 \), as it is obtained from a rather crude analysis. It turns out that \( N_{TT} \) does a good job, provided \( \tau \) is large enough. In figure 9(a) is displayed a comparison of \( N_0 \) and \( N_{TT} \) from \( \tau_2 = 1 \) and deep in the TT-regime. The solid lines display \( N_0 \) and the dotted lines \( N_{TT} \). Two things should be noted in this figure:

Firstly, from figure 2 it was already clear that while \( N_0 \) is constant over a large region of \( \tau \)-space, it does depend on \( \tau \) for thin tori. In figure 9, we now see that when this happens, e.g. at \( \tau_2 \approx 6 \) for \( N_0 = 7 \), \( N_{TT} \) matches \( N_0 \) very well for all values of \( N_0 \) plotted. We thus conclude that \( N_{TT} \) manages to capture the full \( \tau \)-dependence of \( N_0 \) where this exists. We should however note that in the region where \( N_0 \) actually is constant, then the approximation \( N_{TT} \) breaks down as it is not constant there.

Secondly, the maximum of \( N_{TT} \) (at \( \tau_2 = \frac{3N_0}{\pi} \)) lies in a region where \( N_0 \) is still approximately constant. Under the assumption that this will be the case also for larger system sizes, we may refine the approximation of \( N_{TT} \) in the region \( 1 < \tau_2 < \frac{3N_0}{\pi} \). We simply assume that if \( \tau_2 < \frac{3N_0}{\pi} \) then no \( \tau \)-dependent correction is needed and if \( \tau_2 \geq \frac{3N_0}{\pi} \) then the correction should be according to (39). The constant correction for \( \tau_2 < \frac{3N_0}{\pi} \) is naturally set to the value of \( N_{TT} \) at \( \tau_2 = \frac{3N_0}{\pi} \). The revised approximation is thus

\[
N_{TT}' = \begin{cases} 
N_{TT}(\tau) & 1 < \tau_2 < \frac{3N_0}{\pi} \\
N_{TT}(\tau_2) & \tau_2 \geq \frac{3N_0}{\pi}
\end{cases}
\]

\[
= \sqrt{N_0} \left( 2qN_0 \pi^3 \right) \frac{N_0}{\tau_2^2} e^{\pi \tau_2^2} \frac{N_0 - 1}{\tau_2} \begin{cases} 
\left( \frac{3N_0}{\pi^2} \right)^{\frac{N_0 - 1}{4}} & 1 < \tau_2 < \frac{3N_0}{\pi} \\
\frac{N_0 - 1}{\tau_2} e^{\pi \tau_2^2} \frac{N_0 - 1}{\tau_2} & \tau_2 \geq \frac{3N_0}{\pi}
\end{cases} \tag{40}
\]
We define $Q = \frac{N_0}{N_{TT}}$ as the quotient between the exact normalization and the revised approximation $N'_{TT}$. In figure 10(a) we show that the revised normalization $N'_{TT}$ is correct up to factors of order unity for all $\tau$. To the best of our knowledge this is the first estimate of the normalization of the Laughlin state spanning both the thick torus and thin torus regimes, although we acknowledge that a calculation for the thick cylinder has been performed in [TGV09]. We also note, in the same figure, that $Q$ is close to 1, but often slightly below, for all system sizes considered, e.g. $1 \lesssim Q \gtrsim 0.8$ for $N_e = 7$.

Further, as we argue that $L$ and not $\tau$ is the natural variable to describe the physics, we may re-plot $Q$ as a function of $L$. This is a reasonable action since $\tau = \frac{3N_e}{\pi}$ is equivalent to $L = \pi \sqrt{\frac{q}{2\pi}} \approx 4.44 \sqrt{\frac{q}{2\pi}}$ which is independent of $N_e$. This means that when $L < \pi \sqrt{\frac{q}{2\pi}}$ then $N_0$ has $\tau$-dependence irrespective of the system size. This is another example showing that the physics in the TT-limit is dominated by the short torus length, and insensitive to the overall size of the system.
Figure 10(b) plots $Q$ as a function of $L$, where we take the $N_i$th root to cancel the difference in scaling at different $N_i$. As should be apparent, all the curves $Q_{N_i}$ have collapsed approximately on a single curve $f(L)$. Here, $f(L)$ is an unknown function that is independent of $N_i$. By graphical inspection the form of $f(L)$ seems to converge for system sizes as small as $N_i = 5$. Also the value of $f(L)$ seems to stabilize around $\approx 0.973(3)$.

To obtain the numerical values for $N_0$ we used exact diagonalization of the Haldane pseudo-potential to access the properly normalized $\psi_L$. This method will only work for system sizes where we actually can perform the exact diagonalization and generate real space samples. To see that $Q_{TT} = f(L)$ beyond this regime we resort to estimating the norm by Monte-Carlo (MC) and importance sampling. We then generate samples according to the probability distribution $p(z)$ that we choose to be the absolute square of the $\nu = 1$ wave function. The MC estimate of the norm is then

$$N_{MC} \approx (2\pi q N_i)^{-1} \sum_{j=1}^{N_{MC}} \left| \frac{\psi(z_j)}{p(z_j)} \right|^2,$$

where $Z = \sum_{j=1}^{N_{MC}} \frac{1}{p(z_j)}$ and $N_{MC}$ is the number of MC samples. The factor $(2\pi q N_i)^{-1}$ is the volume integrated over, as importance sampling will only compute the average value of $|\psi|$. In figure 11 we compare $N_{MC}$ to $N_0$ and $N_{TT}^\prime$, for $\tau = \ell$ (square torus) and up to $N_e = 20$ electrons. Exact numeric values for $N_0$ exist for up to $N_e = 11$ but beyond that it is hard to evaluate in real space the ground state wave function $\psi_L$. In the figure 11 we see that $N_{MC}$ and $N_0$ agree (as they should) to within MC errors for all system sizes where we can compute $N_0$ exactly. In the main figure we see that $Q(N_0, L) \approx f(N_i)$ seems to hold.
beyond where we have exact data for \( N_0 \). We note that \( N_{MC} \) has a tendency to overestimate \( N_0 \), which is why the lower values (e.g. \( N_e = 19 \)) have a much larger error than the higher ones (e.g. \( N_e = 18 \)). We conclude that using Monte Carlo strengthens the picture that \( \psi_0 \approx Q_{NL} f_L \), and that \( L_f \approx 80.973 \) is constant. Another way of stating the same is that \( N_T \) at \( \tau \approx \frac{1}{e} \) is correct up to a factor 0.973\( N_e \), and sub-leading multiplicative corrections must be polynomial in \( N_e \).

5.3. Modular transformation phases in the TT-limit

We are now in a position to answer what happened with the Berry phase \( \phi_T \) in the thermodynamic limit \( \tau \to \infty \) in figure 5. In the TT-limit we know that \( \psi_L \) is just a Slater determinant, so we know that it is on the form used to obtain equations (18) and (20), but his time with \( P = \frac{N}{4} \). However we also know that \( \psi_s \) needs to be normalized by the factor \( N_{TT} \) in (39) which contains a factor of \( e^{i \pi q/2} \). As a result, the the normalized \( \psi_L = \frac{\psi_L}{P_{TT}} \) actually has the form \( r^2 e^{i \pi q/2} \psi \). The Berry phase picked up by this state is therefore

\[
\phi_T^{(TT)} = -\frac{N_e}{4\gamma_2} - \pi q - \frac{1}{12}.
\]

(41)

Now, using the relation \( L_2^2 = 2\pi q N \), we can rewrite the two curves (18) and (41) as
Thus plotting $\tilde{\phi}_T$ in units of $L$ instead of $\tau$ all the data for different $N_e$ should now collapse onto the same curve. Indeed, looking at Figure 12 this is precisely what we see. Also, the crossover from $\phi_T^{(CFT)}$ to $\phi_T^{(TT)}$ happens again around $L = \pi \sqrt{\frac{2}{3q}}$.

5.4. Hall viscosity in the TT-limit

In this section we will investigate the consequence for the Hall viscosity \[\text{[ASZ95]}\] when there is $\tau$-dependence in $N_0$. From studying $\tilde{\phi}_T$ in figure 12 we know that the proper parameter that determines $\tilde{\phi}_T$ is $L$ instead of $\tau$. It would thus seem likely that $L$ would also determine the behavior of the viscosity $\bar{s}$.

An example of this is shown in figure 13, where the data from figure 4 is replotted as a function of $0 < L < \sqrt{2\pi q N_0}$. The different curves extend to different $L$, since the torus area increases with system size. Two things should be noted here. Firstly, all the curves of $\bar{s}$ for different $N_e$ are approximately identical when plotted against $L$. This again lends support to the interpretation that it is $L$ that is important for the physics. Secondly, as the thin torus is approached—$L \to 0$—the value of $\bar{s}$ drops to $\bar{s} = \frac{1}{2}$ from the square torus value of $\bar{s} = \frac{3}{2}$. This indicates that there are extra Berry phase contributions from the normalization $N_0$ and is direct evidence that the plasma is not screening in the TT-limit. This last observation leads naturally to the need for systematic corrections to $N_0$, as a function of $\tau$. 

![Figure 12. $\tilde{\phi}_T$ plotted as function of $L$, such that all data collapses on the curves $-\frac{L^2}{8\pi}$ (dotted) and $-\frac{L^2}{24\pi} - \frac{q}{6}$ (dashed). The vertical (dotted) line marks $L = \pi \sqrt{\frac{2}{3q}}$.](image-url)
Using the results from section 5.1 and equation (40) we can approximately compute the viscosity of the Laughlin state in the thin torus region and in the square torus region separately. We first consider 
\[ \psi = N_{TT} \left( \sum_{i,j} \right) \] 
where \( N_{TT} \) is constant for \( L > \pi \sqrt{\frac{3}{2}} q \), giving \( \bar{s} = \frac{\tau}{2} \) as above. However for \( L < \pi \sqrt{\frac{3}{2}} q \) there is an extra contribution \( \frac{N}{2} \left( \tau^{-1} \right) \). Thus, according to (17) the viscosity will not be \( \bar{s} = \frac{\tau}{2} \) as expected from the plasma analogy. Instead we see that the viscosity is \( \bar{s} = \frac{\tau}{2} \frac{1}{2} \) for all values of \( \tau \). This is also an example that shows that to compute the Hall viscosity analytically it is essential to have control over the non-holomorphic \( \tau \)-dependence of the wave function. None of the holomorphic factors will affect the value of the viscosity, which allows for the possibility that \( N_0 \) can have holomorphic dependence on \( \tau \) without affecting the viscosity of the state.

5.4.1 Approximating the TT-limit viscosity. In section 5.1 we computed the relative scaling of the root partition and the one-pair squeezed states. In this section we will now compute the viscosity for the Laughlin state in the TT-limit under the approximation that only the root configuration and the leading one-pair squeezed states contribute. We will obtain a viscosity that is not dependent on \( \tau \), where \( L \) instead of \( \tau \) is the relevant length scale. As the approximation is quite crude, we will not be able to reproduce the full features of \( \bar{s}_{\text{eff}} \) with this analysis, but we will see good agreement when \( L \ll 5 \). Due to the difference in scaling between the root partition and the sub-dominant contributions the viscosity will reduce to \( \bar{s}_{\text{eff}} = \frac{1}{2} \) in the TT-limit.

We thus consider the root partition of the \( q = 3 \) Laughlin state \( |TT\rangle = \ldots 001001001 \ldots \) as well as the sub-dominant configuration \( |\bar{i}, i+1\rangle = \ldots 00100010001001 \ldots \). The number of nearest neighbor one-pair squeezed states are \( N_{\tau} \). From (32) we get the relative scaling between these terms as \( a(\tau) = 3e^{\tau\pi} \). The normalized asymptotic state can thus be written as

\[
|\tau\rangle = N(\tau) \left( |TT\rangle + a(\tau) \sum_{i=1}^{N} |i, i+1\rangle \right), \tag{42}
\]

where \( N^{-2}(\tau) = 1 + N_{\tau} e^{-\tau\pi} \) is the normalization. To compute the viscosity we use
which is an (for $\bar{s}$ instead of $\eta$) adapted version of equation (2.99) in [RR11]. When applied to $|\tau\rangle$ and identifying $\alpha_0 = N(\tau)$ and $\alpha_i = N(\tau)a(\tau)$ we obtain

$$\bar{s}(\tau) = \frac{1}{2} - \frac{2r_2^2}{N\eta} \sum_k (\partial_{\tau_1}\bar{a}_k \cdot \partial_{\tau_1}\alpha_k - \partial_{\tau_2}\bar{a}_k \cdot \partial_{\tau_2}\alpha_k),$$

which is an (for $\bar{s}$ instead of $\eta$) adapted version of equation (2.99) in [RR11]. When applied to $|\tau\rangle$ and identifying $\alpha_0 = N(\tau)$ and $\alpha_i = N(\tau)a(\tau)$ we obtain

$$\bar{s}(\tau) = \frac{1}{2} - \frac{2r_2^2}{N\eta} \left[ (\partial_{\tau_1}\bar{N} \cdot \partial_{\tau_1}\bar{N} - \partial_{\tau_2}\bar{N} \cdot \partial_{\tau_2}\bar{N}) \right.$$  
$$\left. + \sum_{i=1}^N (\partial_{\tau_1}(Na^*) \cdot \partial_{\tau_1}(Na) - \partial_{\tau_2}(Na^*) \cdot \partial_{\tau_2}(Na)) \right]$$  

$$= \frac{1}{2} - i2r_2^2(\partial_{\tau_1}(Na^*)\partial_{\tau_1}(Na) - \partial_{\tau_2}(Na^*)\partial_{\tau_2}(Na))$$

$$= \frac{1}{2} + \frac{64e^{8\pi^2}}{N^2} \frac{\pi^2}{2} \frac{1}{L^2} + \frac{1}{2} + \frac{2304\pi^4 e^{16\pi^2}}{L^2} + 9N\eta$$

$$= \frac{1}{2} \left( 1 + \frac{16\pi^2}{L^2} + 9N\eta \right)^2 \frac{1}{L^2} + 9N\eta.$$ 

Figure 13. (Data points) The viscosity (measured as effective average spin $\bar{s}$) of the Laughlin state as function of $L$ for different $N_e$. Note that the curves are almost identical for all $N_e$, even for as small systems as $N_e = 3$, hinting that the physics is dictated by the shorter length scale $L$. For $L \lesssim 10\ell$ the viscosity has stabilized at $\bar{s} = 1/2$, which is the expected thermodynamic value. (Solid lines) Approximation of $\bar{s}$ using (44) for different values of $N_e$ as a function of $L$. Note that this function has the right features of rising from $\bar{s} = 1/2$ along the same curve almost independent of $N_e$. For $L \gtrsim 5$ the analytic and numerical results in the plot are in good agreement. For $L \gtrsim 5$ the curves start to deviate as the approximation breaks down.
which apart from a factor of $N_e$ in the denominator only depends on $L$. The contribution from the first term ($a_0$) in (42) disappears completely as $\partial_\tau N = 0$. Note that $\bar{s}(\tau)$ does not depend on $\eta$ and only on $L$. This is an encouraging result as it tells us that the viscosity is insensitive to skewing of the torus geometry. Again a natural result if we expect the physics to be dictated by the shorter length scale $L$.

In figure 13 we plot (44) as a function of $L$ and $N_e$. We observe that $\bar{s}(\tau)$ rises from $\bar{s} = \frac{1}{2}$ almost independently of $N_e$. If we set (artificially) $N_e = 0$ we get the idealized upper curve $\bar{s}(\tau) = \frac{1}{2} + \frac{2304 \pi^4}{L^4} e^{-\frac{L^2}{\bar{e}^2}}$ which starts to deviate from $\bar{s} = \frac{1}{2}$ at about $L \gtrsim 4$. We can with this construction show explicitly that in the deep TT-limit we will approach $\bar{s} = \frac{1}{2}$ as $\frac{\bar{e}^2}{L^2} \to 0$ when $L \to 0$.

Now, if we compare to the exact viscosity in the same figure we see that there is good quantitative agreement between (44) and the viscosity of the full Laughlin state when $L \lesssim 5$. For $L$ larger than this value the analytic approximation for the viscosity breaks down. We stress that we should not expect perfect agreement for all $L$ with this simple model. The reasons are several. Firstly, when $\tau_2 \to 1$ there are going to be increasingly more states that are relevant, which we ignore here. Secondly, the relative scaling in (34) is only the leading expansion of relative scaling, and will also break down when $\tau_2 \to 1$.

We summarize this section by stating that the viscosity in the TT-limit is not $\bar{s} = \frac{q}{2}$ as expected from the Read conjecture. However, the $L$-dependence of $\bar{s}$ is—up to small corrections—indeed independent of system size. Numerical studies of e.g. $\nu = \frac{2}{5}$ suggests that this holds true for, at least some, more complicated filling fractions [Mor15].

6. Fock expansion of the hierarchy states

In this section we will generalize the results obtained in section 4 for the chiral Haldane–Halperin hierarchy states recently constructed on the torus using CFT techniques [FHS14, HSB + 08]. The Laughlin wave function fits in to a much larger framework of trial wave functions, namely the Haldane–Halperin hierarchy of incompressible states [Hal83a, Hal83b]. These states have been realized using correlation functions in CFT on both plane [HCJV07a, HCJV07b], sphere [Kvo13] and torus [FHS14, HSB + 08]. In order to make this section reasonably self-contained we review the basic structure of the torus Haldane–Halperin hierarchy [Hal83a, Hal83b] wave functions. We here summarize the construction of the Haldane–Halperin hierarchy on the torus and refer to [FHS14] and references therein for more details.

The topological information regarding the quantum Hall fluids are encoded in the Wen–Zee $K$-matrix [WZ91]. The matrix is symmetric with integer entries and has the same dimensionality $n$ as the level of the hierarchy state. The $K$-matrix formalism describes $n$ different layers with $N_e$ particles in layer $\alpha$. The distribution of particles in the different layers are determined by the requirement that the quantum Hall droplet should be homogeneous. Under this requirement, the group sizes are given by $N_e \sum_\beta K_{\alpha \beta}^{-1} = N_e$. The filling fraction is readily extracted as $\nu = \sum_\alpha \beta K_{\alpha \beta}^{-1}$.

For the Jain series [Jai07] at $\nu = \frac{n}{m_0 + 1}$, the $K$-matrix is $n$-dimensional and has a particularly simple form $K_{\alpha \beta} = m + \delta_{\alpha \beta}$. The Jain series is a subset of the chiral hierarchy, which will be considering in this work. As examples, the $K$-matrices for the Laughlin $\nu = \frac{1}{q}$ states and the $\nu = \frac{2}{5}$ Jain state are given by $K = q$ and $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

In this article we are exclusively working in the basis where $t = (1, \ldots, 1)$.
Electronic wave function for the HH-hierarchy are constructed using correlation functions in CFT. The correlation functions consist of insertions of \( n \) different types of electronic operators, one for each layer, and a neutralizing background charge operator. Many-particle wave functions with well defined single particle boundary conditions are identified with the conformal blocks of primary operators in the CFT. At filling fraction \( \nu = \frac{p}{q} \), the many-particle wave functions can be written as

\[
\psi(z) = \mathcal{N}(\tau) e^{i \pi N_\ell \sum \beta_i \left( \frac{z_i}{L} \right) } \prod_{i < j} (z_i - z_j)^{K_{ij}} \mathcal{F}(z, \tau). \tag{45}
\]

Here \( K_{ij} \) should be interpreted as \( K_{\alpha \beta} \) for the groups \( \alpha, \beta \) that \( i, j \) belong to. Just as with the Laughlin state, the label \( s \) enumerates the \( q \) different degenerate states that has to exist when the denominator of \( \nu = \frac{p}{q} \) is \( q \). Also, in analogy to the Laughlin case, these states can be transformed into each other by rigid translations of all the particles. The CFT construction also gives a proposal for the normalization factor \( \mathcal{N}(\tau) \).

Note that for two particles in the same group, the anti-symmetry properties of (45) are dictated by the Jastrow factor \( \prod \left( z_{ij} \right)^{\vartheta_{\alpha \beta}} \) alone and is not influenced by the CoM piece \( \mathcal{F}(z, \tau) \). This means that each group is fully anti-symmetric within the group. The same is not true for particles in different groups.

The chiral hierarchy has the property that all the edge modes travel in the same direction. This manifests itself in that all the eigenvalues of \( K \) are positive. This enables us to parametrize \( K \) using charge vectors \( \alpha_q \). These are real \( n \)-dimensional vectors such that \( \ell \cdot \alpha_q = 0 \). The vectors \( \alpha_q \) span an infinite \( n \)-dimensional charge lattice \( \Gamma \). The charge lattice \( \Gamma \) is important for the construction of the CoM function. The CoM function is given by

\[
\mathcal{F}(z, \tau) = \sum_{q \in \Gamma} e^{i \pi \ell \cdot (q + h) + h \cdot (z + t)} e^{i \pi \ell \cdot (q + h - t)}, \tag{46}
\]

where \( Z = \frac{1}{L} \sum_{q \in \Gamma} q \cdot z \). The parameters \( h \) and \( t \) carry the many-body momentum index. They are chosen such that for periodic boundary conditions \( e^{i \pi \ell \cdot q} = e^{i \pi \ell \cdot h} = e^{i \pi \ell \cdot (q + h - t)} \). In the last equation appears the total charge \( Q = \sum q_i \) of the combined charge vectors. To ensure that all the particles are at the same flux the homogeneity condition

\[
Q \cdot q_j = \sum_{j=1}^{N} K_{ij} = N_{q_j} \tag{47}
\]

has to apply. This can also be formulated in terms of \( K_{\alpha \beta} \) as the homogeneity condition that dictates the group sizes; \( \sum_{\beta} K_{\alpha \beta} N_{\beta} = N_{q_s} \). To obtain the physical wave function from the CFT conformal blocks, equation (45) has to be anti-symmetrized and external translation operators have to be added as

\[
\tilde{\psi}_s = \mathcal{A} \left\{ \prod_{s=1}^{n} \mathbb{W}_{(\alpha)}^{(n, -1)} \psi_s \right\}. \tag{48}
\]

where the \( \mathcal{A} \) stands for anti-symmetrization. The operator \( \mathbb{W}_{(\alpha)} \) is the torus counterpart of the external derivatives on the plane \( \mathbb{W}_{(\alpha)}^{(\text{plane})} = \prod_{\alpha_{i, j}} \partial_{\alpha_i} \). The anti-symmetrization in the above equation is there to ensure that all particles are indistinguishable. For the Laughlin state in (7) the \( \mathcal{A} \) is redundant as that state is by construction an anti-symmetric single-component wave function.
function. Note also that \( \mathcal{A} \) acts primarily between groups as all the particles within the same group are by construction anti-symmetrized. For a detailed discussion about the construction of \( \mathcal{D}_\alpha \), \( \psi_s \) and \( \widetilde{\psi}_s \) we refer to [FHS14] and references therein.

6.1. Main results and formulas

The Fock expansion for the chiral hierarchy in (45) can be obtained by straightforward generalization of the method used to analyze the Laughlin state. As such, also the form and interpretation of the different terms are very similar. Thus, the more general (45) can be written as

\[
\psi_s = \mathcal{N}(\tau) \sum_{\{\tau \in \mathbb{Z}^+ \}} \mathcal{Z}(\tilde{T}_\tau) \sum_{\{m_\alpha \}} \zeta_k(z_i),
\]

(49)

The structure is again analogous to that of (22). The main difference is that there is an extra summation over the \( n \) indexes \( \alpha \), \( m \). These indexes \( \alpha \) come from the CoM function (46) and are here brought to the form where the they only appear in the definition of the momentum \( k_i \)

\[
k_i = T_i + N_0 \sum_{\alpha} m_\alpha \delta_{i,N_0} + s.
\]

(50)

As (49) is written, the sums over \( \alpha \) can immediately be performed to produce

\[
\sum_{\{m_\alpha \}} \zeta_k(z_i) = \left( \frac{2\pi \pi^2}{\tau} \right)^2 \prod_{\alpha} \eta_{i_{\alpha}}(z_i).
\]

However we write (49) with this summation unperformed as it only affects \( n \) of the \( N_e \) factors in \( \prod_{i=1}^{N_e} \zeta_k(z_i) \).

This time, \( \mathcal{Z}(\tilde{T}_\tau) \) is not invariant under \( T_i \rightarrow T_i + N_0 \) which means that we can not write (49) on the from \( \sum_{\{\tilde{T}_\tau \}} \tilde{Z}(\tilde{T}_\tau) \prod_{i=1}^{N_e} \eta_{i}(z_i) \) with a simple expression for \( \tilde{Z}(\tilde{T}_\tau) \). This is not to say that it cannot be done, only that \( \tilde{Z}(\tilde{T}_\tau) \) will not have the simple structure that \( \mathcal{Z}(\tilde{T}_\tau) \) in (23) has. The \( T_i \) still obey the same balance condition \( \sum_{i} T_i = 0 \) as earlier. The total momentum is just as above \( K_{\text{total}} = \sum_i k_i = N_s \times \text{mod} \ N_0 \).

The generalized weight is now

\[
\mathcal{Z}(\tilde{T}_\tau) = \exp \left\{ i \pi \tau \left( \sum_{i<j} \frac{T_{ij}^2}{K_{ij}} \right) \right\} \prod_{i<j} \left( e^{i \pi T_i K_{ij}} \right)^{2},
\]

(51)

which has the same type of structure as (23). Again the \( \tilde{T}_{ij} \) are anti-symmetric just as for the Laughlin state. Similarly the factor \( e^{i \pi T_{ij}} \) is responsible of for the anti-symmetry under the exchange of two \( T_{ij} \) from the same group.

Note that, as the different groups of electrons now are distinguishable, \( \tilde{Z}(\tilde{T}_\tau) \) is not proportional to the Fock coefficients, which was the case for the Laughlin state. In fact the states in the hierarchy needs to go through (48) to become physical electronic wave functions for a single component state. As a result, the coefficients given by \( \tilde{Z}(\tilde{T}_\tau) \) would not even be the Fock coefficients, but rather the components that enter into them.

6.2. \( T \bar{T} \)-limit scaling for HH-states

In section 5 the asymptotic scaling of the Fock coefficients where extracted for the Laughlin state. A similar analysis should be possible also for the general hierarchy states using (51).
Just as with the Laughlin state the dominant state should be the TT-state \cite{BK08,TT83}. This is the state where the electrons are maximally separated along the long dimension of the torus. The analysis is made more complicated by the fact that $Z(T)$ is not invariant under $T_i \rightarrow T_i + N$, which gives more than one weight that corresponds to the root partition.

We can however, without performing that full analysis, shed some light on an observation made in an earlier paper. In \cite{FHS14} it was noted that in comparison to the Coulomb ground state at $\nu = \frac{2}{5}$, some of the translation operators that appear in (48) gives better overlaps than others. For instance, in the TT-limit, the translation operator $D_{1,0}$ gives the best overlap with coulomb, whereas $D_{0,1}$ yields an overlap that is almost zero. See figure 3 of that paper. The understanding we had was that $D_{1,0}$ constituted a smaller translation and therefore resembled a derivative more than $D_{0,1}$ did. From a technical point of view we can now also say that since the TT-limit is dominated by the single slater determinant of the root partition the effect of $D_{1,0}$ and $D_{0,1}$ are drastically different. All Fock-states are eigenstates of the operator $D_{1,0}$, and so it will only give rise to a phase factor. The $D_{0,1}$ operator on the other hand will shift all the electrons in one group relative to all the others, which changes the configuration. As an example, for $\nu = \frac{2}{5}$ the TT-configuration is ... 010100101001010 ... When acted on by $D_{1,0}$ it will just pick up a phase and becomes $e^{i2\pi \sum \phi e_{i}^{j}} 010100101001010 ...$ Under the action of $D_{0,1}$ however, it becomes ... 001100011000110 ... which is a configuration with almost no overlap with the Coulomb ground state.

7. Summary

In this paper we have rewritten the Laughlin state in the momentum Fock-basis. The techniques developed here can also be applied to the full chiral Haldane–Halperin hierarchy wave functions on the torus, although the results are not as easily interpreted as for the Laughlin wave function. We have further re-expressed the Fock-coefficients of the Laughlin state in a recursive form for faster numerical evaluation.

We used the analytic expressions for the Fock coefficients to study the Laughlin state deep in the TT-limit. We did this by analyzing the asymptotic $\tau$-dependence of the Fock coefficients. Using this asymptotic expansion we have analytically computed viscosity in the TT-limit as well as the leading $\tau$-dependence on the correction to the normalization. We found a correction to the proposed CFT-normalization that is accurate up to a factor $f^{N}(L)$.

Our analytical approximation for the viscosity show qualitative but not quantitative agreement with the numerically obtained viscosity. We conclude that the parameter that dictates the viscosity in the TT-limit is the short length scale $L$ and not $\tau$. Thus, any state sufficiently deep into the TT-limit will always have the ‘trivial’ viscosity of $s = \frac{1}{2}$. Or findings suggests that the plasma is not screening in the TT-limit. This is supported by the need to correct the normalization by a $\tau$ dependent factor when $L \lesssim 4.4 \ell_{p}$. This results thus warns that Hall viscosity can not be used as a probe to detect topological information on this asymmetric geometry. We still believe, however, that the plasma analogy holds in the thermodynamic limit, both for the Laughlin state and for the rest of the hierarchy states, and we have given some numerical evidence that this is the case in this paper. We further believe that you can study the viscosity as a function of $L$ to determine whether a topological phase has stabilized or not. By thermodynamic limit, we here mean the limit where both the torus axes are much larger than the magnetic length.

One of the advantages of studying the TT-limit is that it can be exactly solved while still being adiabatically connected to the bulk state at $\tau = 1$ \cite{BK06, BK08}. The gap to excitations...
remains open when tuning $\tau$ between $\tau = i$ and $\tau = i\infty$ and the charges of the quasi-particles are the same. What we have seen however in this paper is that some other properties does change, such as viscosity, but there are other quantities that also get altered. One such is the exclusion statistic [Ha91] for the quasi-particles which changes [KK11] at about the same $L$ as the viscosity. This serves as a reminder that the TT-physics is in many cases the same as in the bulk, but that there are notable exceptions that one needs to be weary of.

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Appendix. A recursive formula for $Z_k^{(N)}$

In this appendix we list some of the properties of the $\vartheta$-function structure factor $Z_k^{(N)}$. This object appears in the exponentiation of the generalized $\vartheta$-function as

$$\exp\left[\frac{1}{N} \sum_{\ell \in \mathbb{Z}} e^{i\pi \vartheta(\tau) \tau} \right] Z_k^{(N)}.$$  

(A.1)

We define $Z_k^{(N)}$ as

$$Z_k^{(N)} = \sum_{\{t_i\} = -\infty}^{\infty} e^{i\pi \vartheta(\tau) \sum_{i=1}^N t_i},$$

where $\sum_i t_i = 0$ and $\sum_i t_i = k$ and $t_i = \tilde{t} + \frac{k}{N}$. This enables us to rewrite it as

$$Z_k^{(N)} = \sum_{\{t_i\} = -\infty}^{\infty} e^{i\pi \vartheta(\tau) \left(k - \frac{k}{N}\right)}$$

(A.2)

where $\sum_i t_i = k$. By letting $t_i \rightarrow -t_i$ we easily see that $Z_k^{(N)} = Z_k^{(N)}$. As one of the $k_i$ is linearly dependent on the others, we may without loss of generality choose it to be $t_N = k - \sum_{i=1}^{N-1} t_i$, such that $\frac{k}{N} - t_N = \frac{k}{N} - k + \sum_{i=1}^{N-1} t_i = \sum_{i=1}^{N-1} t_i - \frac{k}{N}(N-1)$. We may now rewrite $Z_k^{(N)}$ as depending on $N-1$ linearly independent variables $\{t_i\}^{N-1}$ as

$$Z_k^{(N)} = \sum_{\{t_i\} = -\infty}^{\infty} e^{i\pi \vartheta(\tau) \left(k - \frac{k}{N}\right)} e^{i\pi \left(T_{N-1} - \frac{k}{N}(N-1)\right)},$$

where we introduced the dummy variable $T_{N-1} = \sum_{i=1}^{N-1} t_i$. We note that $Z_k^{(N)} = Z_k^{(N)}$ as the shift $k \rightarrow k + N$ is reverted by the re-summation $t_i \rightarrow t_i + 1$, i.e. $T_{N-1} \rightarrow T_{N-1} + 1 - N$. We now note that we may extract a reduced form of (A.2) by identifying $k$ with $T_{N-1}$ and $N$ with $N-1$. We then get

$$Z_k^{(N)} = \sum_{t_{N-1} = -\infty}^{\infty} e^{i\pi \vartheta(\tau) N-1 \left(T_{N-1} - \frac{k}{N}(N-1)\right)} Z_{T_{N-1}}^{(N-1)}.$$
Since $T_{N-1} \rightarrow T_{N-1} + N - 1$ and $k \rightarrow k + N$ cancel each other we can rewrite the recursion using $\vartheta$-functions as

$$Z_{k}^{(N)} = \sum_{t \in \mathbb{Z}_{N-1}} Z_{t}^{(N-1)} \vartheta \left[ \frac{t}{N-1} - \frac{k}{N} \right] (0|N(N-1)\tau)$$

(A.3)

with the root function $Z_{1}^{(1)} = 1$.

References

[ASW84] Arovas D, Schrieffer J R and Wilczek F 1984 Fractional statistics and the quantum hall effect Phys. Rev. Lett. 53 722

[ASZ95] Avron J E, Seiler R and Zograf P G 1995 Viscosity of quantum hall fluids Phys. Rev. Lett. 75 697–700

[BBr12] Bernevig B A, Bonderson P and Regnault N 2012 Screening behavior and scaling exponents from quantum hall wavefunctions (arXiv e-prints)

[BH08] Andrei Bernevig B and Haldane F D M 2008 Model fractional quantum hall states and jack polynomials Phys. Rev. Lett. 100 246802

[BK06] Bergholtz E J and Karlhede A 2006 ‘One-dimensional’ theory of the quantum hall system J. Stat. Mech. L04001

[BK08] Bergholtz E J and Karlhede A 2008 Quantum hall system in taq-thouless limit Phys. Rev. B 77 155308

[Cv13] Cincio L and Vidal G 2013 Characterizing topological order by studying the ground states on an infinite cylinder Phys. Rev. Lett. 110 067208

[FHS14] Fremling M, Hansson T H and Suorsa J 2014 Hall viscosity of hierarchical quantum hall states Phys. Rev. B 89 125303

[Fre13] Fremling M 2013 Coherent state wave functions on a torus with a constant magnetic field J. Phys. A: Math. Theor. 46 275302

[Hal83a] Haldane F D M 1983 Fractional quantization of the hall effect: a hierarchy of incompressible quantum fluid states Phys. Rev. Lett. 51 605

[Hal83b] Halperin B I 1983 Theory of the quantized hall conductance Helv. Phys. Acta 56 75

[Hal85] Haldane F D M 1985 Many-particle translational symmetries of two-dimensional electrons at rational landau-level filling Phys. Rev. Lett. 55 2095

[Hal91] Haldane F D M 1991 ‘Fractional statistics’ in arbitrary dimensions: a generalization of the pauli principle Phys. Rev. B 31 2529

[HCJV07a] Hansson T H, Chang C-C, Jain J K and Viefers S 2007 Composite-fermion wave functions as correlators in conformal field theory Phys. Rev. B 76 075347

[HCJV07b] Hansson T H, Chang C-C, Jain J K and Viefers S 2007 Conformal field theory of composite fermions Phys. Rev. Lett. 98 076801

[HR85] Haldane F D M and Rezayi E H 1985 Periodic laughlin-jastrow wave functions for the fractional quantized hall effect Phys. Rev. B 31 2529

[HSB+08] Hermans M, Suorsa J, Bergholtz E J, Hansson T H and Karlhede A 2008 Quantum hall wave functions on the torus Phys. Rev. B 77 125321

[Jai07] Jain J K 2007 Composite Fermions (Cambridge: Cambridge University Press)

[KK11] Kardell M and Karlhede A 2011 Exclusion statistics for quantum hall states in the taq-thouless limit J. Stat. Mech. P02037

[KL99] Kjønsberg H and Leinaas J M 1999 Charge and statistics of quantum hall quasi-particles—a numerical study of mean values and fluctuations Nucl. Phys. B 559 705

[KM99] Kjønsberg H and Myrheim J 1999 Numerical study of charge and statistics of laughlin quasiparticles Int. J. Mod. Phys. A 14 537–57

[Kvo13] Kvorning T 2013 Quantum hall hierarchy in a spherical geometry Phys. Rev. B 87 195131

[Lau85] Laughlin R B 1983 Anomalous quantum hall effect: an incompressible quantum fluid with fractionally charged excitations Phys. Rev. Lett. 50 1395

[LLM00] Lapointe L, Lascoux A and Morse J 2000 Determinantal expression and recursion for jack polynomials Electron. J. Comb. 7 N1
[Mor15] Moran N 2015 private communication
[Rea08] Read N 2008 Quasiparticle spin from adiabatic transport in quantum hall trial wavefunctions (arXiv e-prints)
[Rea09] Read N 2009 Non-abelian adiabatic statistics and hall viscosity in quantum hall states and $p_x + ip_y$ paired superfluids Phys. Rev. B 79 045308
[RR11] Read N and Rezayi E H 2011 Hall viscosity, orbital spin, and geometry: Paired superfluids and quantum hall systems Phys. Rev. B 84 085316
[SS81] Su W P and Schrieffer J R 1981 Fractionally charged excitations in charge-density-wave systems with commensurability 3 Phys. Rev. Lett. 46 738–41
[TT83] Tao R and Thouless D J 1983 Fractional quantization of hall conductance Phys. Rev. B 28 1142–4
[WZ91] Wen X G and Zee A 1991 Topological structures, universality classes, and statistics screening in the anyon superfluid Phys. Rev. B 44 274–84
[ZMP13] Zaletel M P, Mong R S K and Pollmann F 2013 Topological characterization of fractional quantum hall ground states from microscopic hamiltonians Phys. Rev. Lett. 110 236801
[ZNS13] Zhou Z, Nussinov Z and Seidel A 2013 Heat equation approach to geometric changes of the torus laughlin state Phys. Rev. B 87 115103