TRACES ON THE SKEIN ALGEBRA OF THE TORUS

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Abstract. For a surface $F$, the Kauffman bracket skein module of $F \times [0,1]$, denoted $K(F)$, admits a natural multiplication which makes it an algebra. When specialized at a complex number $t$, nonzero and not a root of unity, we have $K_t(F)$, a vector space over $C$. In this paper, we will use the product-to-sum formula of Frohman and Gelca to show that the vector space $K_t(T^2)$ has five distinct traces. One trace, the Yang-Mills measure, is obtained by picking off the coefficient of the empty skein. The other four traces on $K_t(T^2)$ correspond to each of the four $\mathbb{Z}_2$ homology classes of the torus.

1. Introduction

Skein modules were introduced independently by Przytycki [9] and Turaev [11] and have been an active topic of research since their introduction. In particular, skein modules underlie quantum invariants [7, 6] and are connected to the representation theory of the fundamental group of the manifold [1, 10].

The skein module is spanned by the equivalence classes of framed links in the 3-manifold. The skein module of the cylinder over a surface has a multiplication that comes from laying one framed link on top of the other. With this multiplication, the skein module of the cylinder over a surface is an algebra.

In this paper, we will consider the skein algebra of the torus specialized at a complex number and describe the distinct traces on this vector space.

2. Preliminaries

Let $M$ be an orientable 3-manifold. A framed link in $M$ is the embedding of disjoint annuli into $M$. A framed link is depicted by drawing the core of each annulus. One typically uses the blackboard framing to produce the annulus from its core. We will use $M = F \times I$ for a surface $F$. In these cases, we will use the framing given by the surface to produce the annulus from its core.
Equivalence of framed links in $M$ is up to regular isotopy. That is, using only isotopy and Reidemeister’s II and III moves. A Reidemeister I move corresponds to a twist in the annulus and thus such a move does not preserve the equivalence class of a framed link.

Let $L(M)$ denote the equivalence class of framed links in $M$, including the empty link, $\phi$. Let $R = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials. Consider the free module $R L(M)$, with basis $L(M)$. Define $S(M)$ to be the smallest subspace of $R L(M)$ containing all expressions of the form $\times - t \bigcirc - t^{-1}$ and $\odot + t^2 + t^{-2}$, where the framed links in each expression are identical outside the region pictured in the diagrams. The Kauffman bracket skein module $K(M)$ is the quotient $R L(M)/S(M)$.

Because $K(M)$ is defined using local relations on framed links, two homeomorphic manifolds have isomorphic skein modules. Thus $K(M)$ is an invariant of the 3-manifold $M$.

Let $F$ be a compact, orientable surface and let $I = [0, 1]$ be the unit interval. $K(F \times I)$ has an algebra structure that comes from laying one link on top of the other. Given skein elements $\alpha, \beta \in K(F \times I)$, we can represent $\alpha, \beta$ with the links $L_\alpha, L_\beta \in F \times I$. Use isotopy to move $L_\alpha$ to $F \times (\frac{1}{2}, 1]$ and $L_\beta$ to $F \times [0, \frac{1}{2})$. The product $\alpha \ast \beta$ is the skein element represented by $L_\alpha \cup L_\beta$. A schematic of this product structure is shown in Figure 1.

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{product_structure.pdf}
\end{center}
\caption{The product structure on $K(F \times I)$}
\end{figure}

To simplify notation, and to emphasize that the algebra structure is determined by $F$ rather than by $F \times I$, we will use $K(F)$ to denote the skein module $K(F \times I)$ with the algebra structure described above. We will refer to $K(F)$ as the skein algebra of the surface $F$.

With the relations used to define $S(M)$, we can represent any skein element as the linear combination of simple diagrams, these are diagrams with no crossings and no trivial components. The skein elements induced by such diagrams form a basis for $K(F)$. This fact was observed by Bullock, Frohman, and Kania-Bartoszyńska in [2].

In this paper we will make a number of simplifying assumptions. Namely, $F$ is the standard 2-torus $T^2$ and $t$ is a complex number that is
nonzero and not a root of unity. Thus the polynomials in $R = \mathbb{Z}[t, t^{-1}]$ are evaluated at the complex number $t$ and the skein algebra $K(F)$ is \textit{specialized} at $t$ to form $K_t(F)$, a vector space over $\mathbb{C}$. Throughout this paper, we will divide by expressions of the form $(t^n - t^{-n})$. Choosing $t$ to be a complex number that is not a root of unity allows this type of division without requiring the use of rational functions.

Let $C_t(F) = [K_t(F), K_t(F)]$ be the vector space over $\mathbb{C}$ with basis consisting of the commutators on $K_t(F)$. A \textit{trace} on the algebra $K_t(F)$ is a linear functional $\varphi : K_t(F) \to \mathbb{C}$ satisfying $\varphi(\alpha \ast \beta) = \varphi(\beta \ast \alpha)$ for all $\alpha, \beta \in K_t(F)$. Since a trace is linear, this condition can also be written $\varphi(\alpha \ast \beta - \beta \ast \alpha) = 0$. Then a trace $\varphi$ on $K_t(F)$ has $C_t(F) \subset \ker(\varphi)$. So $\varphi$ descends to be a linear functional on the quotient $K_t(F)/C_t(F)$.

3. Examples of skein algebras

Before we explore the traces on $K_t(T^2)$ in detail, let’s look a few examples of skein algebras.

Let $F$ be the 2-dimensional disk $D^2$. Since every diagram in $D^2$ that has no crossings is trivial, the only simple diagram in $D^2$ is the empty skein $\phi$. Thus the skein algebra $K(D^2)$ is one-dimensional with basis $\{\phi\}$.

Let $F$ be the annulus $A = S^1 \times [0, 1]$. The simple diagrams of $A$ consist of the empty skein along with any number of parallel copies of the core of the annulus. Denote $n$ parallel copies of the core of the annulus by $z^n$ with $z^0 = \phi$. Then a basis for $K(A)$ is $\{z^0, z^1, z^2, \ldots\}$ and hence $K(A)$ is isomorphic to the algebra of polynomials in $z$ with coefficients from $R = \mathbb{Z}[t, t^{-1}]$.

Now let $F$ be the torus $T^2$. The collection of simple diagrams has a more intricate structure. We can have any $(p, q)$-curve with $p$ and $q$ relatively prime and also any number of parallel copies of a given $(p, q)$-curve. Thus a basis for $K(T^2)$ is $\{\phi, (p, q)^n \mid \gcd(p, q) = 1, n \in \mathbb{N}\}$, and we use the convention that $(p, q)^0 = \phi$.

Properties of the algebra $K(T^2)$ are explored by Frohman and Gelca in [4]. In particular, they give a basis for $K(T^2)$ that behaves nicely under multiplication. Let $(p, q)_T$ be the $(p, q)$-curve when $\gcd(p, q) = 1$ and let

$$(p, q)_T = T_{\gcd(p, q)} \left( \left( \frac{p}{\gcd(p, q)}, \frac{q}{\gcd(p, q)} \right) \right)$$

when $\gcd(p, q) \neq 1$. Here $T_n(x)$ is the $n$-th Chebyshev polynomial defined recursively by $T_0(x) = 2$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Using this basis for $K(T^2)$, we have the following \textit{product-to-sum} formula.
Theorem 1. (Frohman-Gelca)

\[(p, q)_T * (r, s)_T = t^{\|p\|_2}((p + r, q + s)_T + t^{-\|p\|_2}((p - r, q - s)_T)
\]

where \(\|p\|_2\) is the determinant.

4. Traces on \(K_t(T^2)\)

Let \(t\) be a fixed complex number, nonzero and not a root of unity. Let \(A = K_t(T^2)\) and let \(\delta : A \otimes A \rightarrow A\) be defined by \(\delta(a \otimes b) = ab - ba\). The image of \(\delta\) in \(A\) is the subalgebra of \(A\) generated by these commutators. Denote this subalgebra by \(C(A)\). To understand the nature of the traces on \(A\), we will focus our attention on the commutator quotient \(A/C(A)\) because the space of traces is dual to this quotient.

As a vector space \(A/C(A)\) is spanned by the cosets of all \((p, q)_T \in A\). To narrow this spanning set to a basis, we use the product-to-sum formula.

As cosets, \((x, y)_T + C(A) = (z, w)_T + C(A)\) if and only if \((x, y)_T - (z, w)_T \in C(A)\). \((x, y)_T - (z, w)_T \in C(A)\) if and only if \((x, y)_T - (z, w)_T\) is equal to some linear combination of commutators. The simplest case would be if

\[(x, y)_T - (z, w)_T = \lambda ((p, q)_T * (r, s)_T - (r, s)_T * (p, q)_T) \quad \lambda \in \mathbb{C}.
\]

We can use the product-to-sum formula to find the integers \(p, q, r, s\) when we are given \(x, y, z, w\).

Lemma 1. Pick a complex number \(t\) that is nonzero and not a root of unity and let \(A = K_t(T^2)\) with basis \((p, q)_T\) as described above. Then \((x, y)_T + C(A) = (z, w)_T + C(A)\) if \(x + z\) is even, \(y + w\) is even, and \(\|\frac{x + z}{2}\| \neq 0\).

Proof. Suppose we have integers \(x, y, z, w\) such that \(x + z\) is even, \(y + w\) is even and \(\|\frac{x + z}{2}\| \neq 0\). Let \(p = \frac{x + z}{2}, q = \frac{y + w}{2}, r = \frac{x - z}{2},\) and \(s = \frac{y - w}{2}\). Since \(\|\frac{x + z}{2}\| \neq 0\), elementary matrix operations lead to \(\|\frac{p}{r, s}\| \neq 0\). Let \(\alpha = |\frac{p}{r, s}| \neq 0\). Then

\[
(p, q)_T * (r, s)_T = t^\alpha (x, y)_T + t^{-\alpha} (z, w)_T
\]
\[
(r, s)_T * (p, q)_T = t^{-\alpha} (x, y)_T + t^\alpha (-z, -w)_T
\]

Since orientation doesn’t matter in \(A\), \((-z, -w)_T = (z, w)_T\) and therefore

\[
(p, q)_T * (r, s)_T - (r, s)_T * (p, q)_T = (t^\alpha - t^{-\alpha})((x, y)_T - (z, w)_T).
\]
Since $\alpha \neq 0$ and $t$ is not a root of unity, we can divide by $(t^\alpha - t^{-\alpha})$ to get

$$(x, y)_T - (z, w)_T = \frac{1}{t^\alpha - t^{-\alpha}} \left( (p, q)_T \ast (r, s)_T - (r, s)_T \ast (p, q)_T \right) \in C(A).$$

Thus $(x, y)_T + C(A) = (z, w)_T + C(A)$ as cosets in $A/C(A)$.

In other words, if $x$ and $z$ have the same parity, $y$ and $w$ have the same parity, and $(x, y)$ and $(z, w)$ are linearly independent, then $(x, y)_T$ is equivalent to $(z, w)_T$ in $A/C(A)$. This fact allows us to reduce our basis for $A/C(A)$ somewhat, and it suggests that the parity of $(p, q)$ will determine the class of $(p, q)_T$ in $A/C(A)$.

**Theorem 2.** Pick a fixed complex number $t$, nonzero and not a root of unity. Let $A = K_4(T^2)$ and let $C(A)$ be the subalgebra of $A$ generated by commutators. Then $A/C(A)$ is a five dimensional vector space over $\mathbb{C}$.

**Proof.** Motivated by Lemma 1, we define a map

$$\varphi: A \to \mathbb{C} \{\phi, ee, eo, oe, oo\}$$

by

$$(p, q)_T \mapsto \left\{ \begin{array}{cl} \phi & \text{if } p = 0, \ q = 0 \\ ee & \text{if } p \text{ even, } q \text{ even} \\ eo & \text{if } p \text{ even, } q \text{ odd} \\ oe & \text{if } p \text{ odd, } q \text{ even} \\ oo & \text{if } p \text{ odd, } q \text{ odd} \end{array} \right.$$

then extend linearly.

We need to show that $\ker(\varphi) = C(A)$. Recall that $C(A)$ is the vector space generated by $(p, q)_T \ast (r, s)_T - (r, s)_T \ast (p, q)_T$ for $(p, q)_T, (r, s)_T \in A$.

Choose $(p, q)_T, (r, s)_T \in A$. To show that $C(A) \subset \ker(\varphi)$, it suffices to show that $c = (p, q)_T \ast (r, s)_T - (r, s)_T \ast (p, q)_T \in \ker(\varphi)$.

$$c = (p, q)_T \ast (r, s)_T - (r, s)_T \ast (p, q)_T$$

$$= \left( t^{|\frac{r}{2}|} - t^{-|\frac{r}{2}|} \right) \left( (p + r, q + s)_T - (p - r, q - s)_T \right)$$

If $(p + r, q + s)_T = (0, 0)_T$, then $(p, q) = -(r, s)$. Hence $(p, q)_T = (r, s)_T$. Thus $c = 0 \in \ker(\varphi)$. If $(p - r, q - s)_T = (0, 0)_T$, then $(p, q) = (r, s)$. Hence $(p, q)_T = (r, s)_T$. Thus $c = 0 \in \ker(\varphi)$.

If neither $(p + r, q + s)_T$ nor $(p - r, q - s)_T$ is $(0, 0)_T$, then since $p + r$ and $p - r$ have the same parity and $q + s$ and $q - s$ have the same parity, we have $\varphi((p + r, q + s)_T) = \varphi((p - r, q - s)_T)$ which implies that $c \in \ker(\varphi)$. Hence $C(A) \subset \ker(\varphi)$. 
Now we show that \( \ker(\varphi) \subset C(A) \). Take \( k \in \ker(\varphi) \). So \( k \in A \) and \( \varphi(k) = 0 \). Then

\[
k = \sum_{\text{finite}} \lambda_{(p,q)}(p,q)_T
\]

(1) \[
= \lambda_{(0,0)}(0,0)_T + \sum_{ee} \lambda_{(p,q)}(p,q)_T + \cdots + \sum_{oo} \lambda_{(p,q)}(p,q)_T
\]

Here we are breaking the sum into five parts, according to the parity of \( (p,q) \). In each of these five parts, the coefficients must sum to zero since \( k \in \ker(\varphi) \). As a model for the other cases, we work the case where

\[
k = \sum_{ee} \lambda_{(p,q)}(p,q)_T
\]

and so

\[
\sum_{ee} \lambda_{(p,q)} = 0.
\]

Now,

\[
\sum_{ee} \lambda_{(p,q)}(p,q)_T
\]

is a finite sum and \( (p,q)_T \) are all of even-even parity, and \( (p,q) \neq (0,0) \). Choose integers \( r \) and \( s \) such that \( (r,s)_T \) is of even-even parity and \( (r,s) \) is linearly independent to each of the \( (p,q) \) in the sum for \( k \). That is, \( \frac{s}{r} \) is a rational slope that is different from the finite number of rational slopes \( \frac{q}{p} \). Now using Lemma \[\square\] each \( (p,q)_T = (r,s)_T \) in the quotient \( A/C(A) \). Hence

\[
k = \sum_{ee} \lambda_{(p,q)}(p,q)_T
\]

\[
= \left( \sum_{ee} \lambda_{(p,q)}(r,s)_T \right) + \text{commutators}
\]

\[
= \left( \sum_{ee} \lambda_{(p,q)} \right) (r,s)_T + \text{commutators}
\]

\[
= \text{commutators, since } \sum_{ee} \lambda_{(p,q)} = 0.
\]

Thus \( k \in C(A) \).

We could repeat this process for each of the \( eo, oe, \) and \( oo \) sums given in Equation \[\square\] Thus for a general \( k \in \ker(\varphi) \), we have \( k \in C(A) \). Hence \( \ker(\varphi) \subset C(A) \). Now \( \ker(\varphi) = C(A) \) and we have

\[
A/C(A) \cong \mathbb{C} \{ \phi, ee, eo, oe, oo \}.
\]
Thus $A/C(A)$ is a five dimensional vector space over $\mathbb{C}$.

Recall that a trace is a linear functional defined on $A$ that is zero on $C(A)$. The space of traces is dual to the quotient $A/C(A)$. Thus Theorem 2 implies that there are five traces on $A = K_1(T^2)$. There is a trace for each $\mathbb{Z}_2$ homology class of $T^2$, with one more trace for the empty skein. Each trace picks off the coefficients of the basis elements in its corresponding class. We could denote these traces by $\varphi_\phi$, $\varphi_{ee}$, $\varphi_{eo}$, $\varphi_{oe}$, $\varphi_{oo}$. The trace $\varphi_\phi$ is what Bullock, Frohman, and Kania-Bartoszyńska call the Yang-Mills measure in [3].

5. FURTHER INVESTIGATION

The Yang-Mills measure, $\varphi_\phi$, is defined on the skein algebra of any closed surface $F$. We have shown that there are four additional traces when $F = T^2$. Frohman and Kania-Bartoszyńska have connected $\varphi_\phi$ to the $SU(2)$-characters of $\pi_1(F)$ in [5] and gave a state-sum formula for computing $\varphi_\phi$.

It is natural to ask if the other four traces, $\varphi_{ee}$, $\varphi_{eo}$, $\varphi_{oe}$, and $\varphi_{oo}$, could be used in a similar way to study the skein algebra of the torus or to study the skein modules of manifolds with a torus boundary.

The commutator quotient $A/C(A)$ can also be seen as the zeroth Hochschild homology of the skein algebra of the torus. Understanding the structure of the commutator quotient may have an impact on the Hochschild homology of genus one Heegaard splittings as explored by the author in [8].

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