SU(n) ⊗ U(1)_c gauge models with spontaneous symmetry breaking

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(March 26, 2022)

Abstract

A possible generalization of the technique of the standard model to SU(n) ⊗ U(1) gauge models is proposed. A special Higgs mechanism and a new kind of Yukawa couplings in unitary gauge are introduced. These allow us to obtain a general method of deriving boson mass spectrum and coupling coefficients which will be used to find an exact solution of the Pisano-Pleitez three-generation SU(3) ⊗ U(1) model. A new anomaly-free one-generation model is briefly discussed.

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I. INTRODUCTION

The gauge models with spontaneous symmetry breaking must respect few basic rules to be renormalizable. The central point is to find a Higgs mechanism able to break the gauge symmetry up to a desired residual one and to produce a good boson mass spectrum. Moreover, the spinor mass terms could arise only from Yukawa interactions and certainly the model must be free of the axial anomaly \[1\]. In practice these rules follow to be applied in each particular case separately since we have not yet a general theory of the spontaneous breakdown of an arbitrary unitary gauge symmetry which satisfies all these requirements. However, if we consider the standard model (SM) and its recent $SU(3) \otimes U(1)$ and $SU(4) \otimes U(1)$ generalizations \[2–5\], we observe a some kind of regularity, namely, that the number of the vector Higgs multiplets may be equal to the dimension of the fundamental representation. This encourages us to try to find a generalization of the Higgs mechanisms of these models to any $SU(n) \otimes U(1)$ gauge model.

The problem is complicated since the whole structure of the Higgs sector depends on the possible Yukawa interactions and, therefore, on the concrete choice of the representations of the spinor multiplets. In order to avoid these difficulties we shall construct a special Higgs mechanism and a new kind of couplings among the spinors and the Higgs multiplets which will be referred as the minimal Higgs mechanism (mHm.). This may help us in the first stage of the model building when the structure and the parameterization of the model must be established. After that one can choose if the mHm. will be kept or will be replaced by an (equivalent or larger) traditional Higgs mechanism.

The mHm. of the $SU(n) \otimes U(1)$ gauge models we would like to propose will have at most $n$ Higgs multiplets, each one be of the fundamental representation of $SU(n)$, but having different chiral hypercharges. In addition, they will be seen as an orthonormal system with the "norm" defined by a scalar real field, which should play the same role as the neutral component of the Higgs doublet of the SM. Here we shall restrict ourselves to present only the
simplest version of this mHm. which will use $n$ such multiplets in order to break the gauge symmetry up to that of the $U(1)_{em}$ universal gauge. The Lagrangian density (Ld.) of mHm. will be constructed to have the potential energy of the simplest form but introducing a metric in its kinetic term to produce a non-degenerate boson mass spectrum. Furthermore, with the help of suitable direct products of Higgs multiplets and with scalar factors (depending on the "norm" of the vector multiplets), we shall define coupling terms which should lead to Yukawa couplings only in unitary gauge. These could produce adequate mass terms for any pair of spinor components we wish, giving thus more flexibility to the model. In this article our objective is to derive all the properties of this class of models with the hope to obtain a general method to solve models with high gauge symmetries.

To do this it is convenient to consider models with the spinor sector put in the (chiral) pure left (pl.) form, i.e. containing only left-handed chiral components. This does not restrict the generality since any model can be brought in this form replacing all the right-handed components by the charge conjugated left-handed ones, according to the method largely used in grand unification theory (GUT) [1]. For these models the group $U(1)$ will coincide with that of the chiral gauge, $U(1)_{c}$. Therefore in the following we shall speak about the pl. $SU(n) \otimes U(1)_{c}$ gauge models.

Starting, in Sec.II, with a short review of the $SU(n)$ representations, we shall define a suitable hybrid basis of the $su(n)$ algebra and we shall discuss the $SU(n) \otimes U(1)_{c}$ representations. In the next one we shall present the general structure of the spinor and gauge sectors of the pl. gauge models. The mHm. will be introduced in Sec.IV, where we shall study the breakdown of the gauge symmetry showing that the residual symmetry is determined by the hypercharges of the Higgs multiplets. In addition, we shall define the above mentioned Yukawa coupling in unitary gauge. The conclusion here is that our mHm. produces the same effects in unitary gauge as the Higgs mechanism of the SM. Based on these results, the properties of the gauge bosons will be discussed in Sec.V, giving special attention to the different possible parameterizations of the model. There the generalized Weinberg transformation will be defined and we shall obtain the masses of the non-hermitian gauge bosons,
the mass matrix of the hermitian ones and all the coupling coefficients of the charged and neutral currents. The next section is devoted to the structure of the spinor sector of the models with correct electric charge spectrum, which should be free of axial anomaly. This ends with the summary of the resulted method we shall use in Sec. VII to find an exact solution for the Pisano-Pleitez model \cite{2} and to discuss a possible new anomaly-free one-generation model. Finally, we shall briefly comment the role of the mHm and its possible extensions.

II. PRELIMINARIES

The general treatment of the $SU(n) \otimes U(1)_c$ gauge models in all their details requires some remarks concerning the unitary irreducible representations (irep.) of this gauge group and of its algebra. In this section we shall briefly present their construction as the direct product between an irep. of $SU(n)$ and an irep. of $U(1)_c$.

A. Conventions and notations

We shall respect the basic notations of the Lie groups and algebras representation theory \cite{6,7} with some particularities which will be pointed out in the next. Thus the summation convention will be systematically applied only for the pairs of indices in upper and lower positions. This rule will be extended even to the mathematical structures where these positions are perfectly equivalent (i.e. the Euclidean spaces and the orthogonal groups). Moreover, each family of indices we shall use will have its own fixed range, except the first Greek ones which will be held for the current needs.

Furthermore, we shall simplify the calculations by choosing a unique coupling constant, $g$, for both the groups involved here. This means that the ireps. of $U(1)_c$, which reduces to multiplications with phase factors $\exp(-igy\xi^0)$ of parameter $\xi^0$, will be defined by the
(real) character \( y \) instead of the chiral hypercharge, \( y_{ch} \). More precisely, \( y = g' y_{ch} / g \) where \( g' \) is the would be specific coupling constant of \( U(1)_e \).

The main pieces are the irreps. of \( SU(n) \). Their classification can be done by using the tensor method \([3]\). This starts with the fundamental irrep., \( n \), which defines the group \( SU(n) \) and its algebra, \( su(n) \). If \( \xi = \xi^+ \in su(n) \) then

\[
U = U(\xi) = e^{-ig\xi}
\]

are the unitary matrices of \( SU(n) \). These transform the components \( \psi_i \) of the \( n \)-dimensional (vector) multiplet, \( \psi \), like

\[
\psi_i \rightarrow (U\psi)_i = U_{i,j}\psi_j
\]

The transformation law of the complex conjugated components

\[
(\psi_i)^* \equiv \psi^j \rightarrow (U\psi)_i^* \equiv U_{i,j}^*\psi^j
\]

defines the complex conjugated of the fundamental irrep., \( n^* \), which has the matrices \( U_{i,j}^* = (U_{i,j})^* \). All the other irreps. of \( SU(n) \) correspond to the different classes of symmetry (given by the Young tableaus) of the tensors of the rank \((r, s)\) provided by the direct products \((\otimes n)^r \otimes (\otimes n^*)^s\). These tensors have \( r \) lower and \( s \) upper indices for which we shall reserve the notation, \( i, j, k, \cdots = 1, \cdots, n \). Their transformation laws are in accordance with the basic rules given by (2) and (3).

The irreps. of \( SU(n) \) will be denoted either symbolically by \( R \) or by indicating their dimension, \( n(R) \), in boldface (as we did already for the irreps. \( n \) and \( n^* \)). The notations \( R(U) \) and \( R(\xi) \) will stand for the transformation matrices and for the algebra elements of the irrep., \( R \). The complex conjugated of the irrep. \( R \) will be \( R^* \) of the matrices \( R^*(U) = R(U)^* \).

**B. The parameterization of the \( su(n) \) algebra**
The form of $\xi$ and implicitly that of $R(\xi)$ depends on the parameterization of the $su(n)$ algebra. The most natural way is to take the real parameters, $\xi^a$, and the hermitian generators, $T_a = T_a^+$, defined in Appendix A, for which we shall use the indices $a, b, c, \cdots = 1, \cdots, n^2 - 1$. In this parameterization $\xi = \xi^a T_a$. Another possibility is to express: $\xi = H^i_j \zeta^j_i$, by the real generators $H^i_j$ given by (93) and by the c-number parameters $\zeta^i_j = (\zeta^j_i)^*$. 

For the gauge models we intend to work out it is convenient to introduce a hybrid basis numbered by the indices $i, j, \cdots$ and by a new family of indices, $\hat{i}, \hat{j}, \cdots$, which range from 1 to $n - 1$. This basis will be defined as follows:

$$D_\hat{i} = T_{(\hat{i}+1)^2-1}, \quad E^i_j = \frac{1}{\sqrt{2}} H^i_j \quad i \neq j \quad (4)$$

The corresponding parameterization:

$$\xi = D_\hat{i} \xi^\hat{i} + \sum_{i \neq j} E^i_j \xi^j_i \quad (5)$$

contains $n - 1$ real parameters, $\xi^\hat{i}$, and $n(n - 1)/2$ c-number ones, $\xi^i_j = (\xi^j_i)^* = \sqrt{2} \zeta^i_j$, for $i \neq j$. This choice offers many advantages. The first is of a simple numeration of the diagonal generators, $D_\hat{i}$, (of the Cartan subalgebra). On the other hand, the parameters $\xi^i_j$ could be directly associated to the c-number gauge fields because of the factor $1/\sqrt{2}$ from (4) which gives their correct normalization. However, the most important is to have good trace orthogonality properties

$$Tr(D_\hat{i} D_\hat{j}) = \frac{1}{2} \delta_{\hat{i}\hat{j}} \quad , \quad Tr(D_\hat{i} E^i_j) = 0$$

$$Tr(E_j^i E^k_l) = \frac{1}{2} \delta^i_j \delta^k_l \quad (6)$$

In other respects the commutation relations of this basis can be calculated according to (4), (94) and (95).

C. The irreducible representations of $SU(n) \otimes U(1)_c$
Let us denote either by \( \rho = (R_\rho, y_\rho) \) or by \( \rho = (n_\rho, y_\rho) \), with \( n_\rho = n(R_\rho) \), the irep. of \( SU(n) \otimes U(1)_c \) defined as the direct product of the irep. \( R_\rho \) of \( SU(n) \) and the irep. of the character \( y_\rho \) of \( U(1)_c \). This will have the matrices:

\[
U^\rho(\xi^0, \xi) = U^0(\xi)e^{-ig\rho \xi^0} = e^{-ig(\xi^0 + y_\rho \xi)}
\]  

(7)

where \( \xi^\rho = R_\rho(\xi) \). The transformation law of a \( n_\rho \)-dimensional multiplet, \( \psi^\rho \), of this irep. can be written in the matrix form as

\[
\psi^\rho \rightarrow U^\rho(\xi^0, \xi)\psi^\rho
\]  

(8)

understanding that its tensor components (in the direct product basis of the representation space) transform like:

\[
(\psi^\rho)_{i_1 \cdots i_r}^{j_1 \cdots j_s} \rightarrow e^{-igy_\rho \xi^0}U_{i_1}^i \cdots U_{i_r}^i \cdot \cdot \cdot U_{j_1}^j \cdots U_{j_s}^j (\psi^\rho)_{i_1 \cdots i_r}^{j_1 \cdots j_s}
\]  

(9)

We note that for \( y_\rho = 0 \) the irep. \( \rho \) coincides with \( R_\rho \) and therefore the notations \((R, 0)\) and \( R \) are equivalent.

The generator of the irep. \( \rho \) which corresponds to the \( SU(n) \) generator \( T \) will be \( T^\rho = R_\rho(T) \). In the next we shall use the generators defined by (4) and the hermitian ones. Their matrix elements in the direct product basis can be calculated according to the tensor method. For example, if \( D \) is a diagonal \( SU(n) \) generator, of matrix elements \( D_{i_1}^j = d_i \delta_i^j \), then \( D^\rho \) will be also diagonal and we have:

\[
(D^\rho \psi^\rho)_{i_1 \cdots i_r}^{j_1 \cdots j_s} = \left( \sum_{\alpha=1}^r d_{i_\alpha} - \sum_{\beta=1}^s d_{j_\beta} \right) (\psi^\rho)_{i_1 \cdots i_r}^{j_1 \cdots j_s}
\]  

(10)

It is clear that the \( U(1)_c \) generator of the irep. \( \rho \) is \( T^\rho_0 = y_\rho \cdot 1_\rho \) where \( 1_\rho \) is the unit matrix of the irep. \( R_\rho \) (to be omitted when no confusion danger is present).

The complex conjugated of the irep. \( \rho \) will be \( \rho^* = (R^*_\rho, -y_\rho) \). This transforms the tensor components of \((\psi^\rho)^*\) which are related to those of \( \psi^\rho \) according to the raising an lowering rule of indices defined by (3). The hermitian \( SU(n) \) generators of the irep. \( \rho^* \) are

\[
T^*_{a} = -(T^\rho_a)^T
\]  

(11)
while those defined by (4) become

\[ D^\rho \hat{\sigma}_i = -D^\rho \hat{\sigma}_i, \quad E^\rho \hat{\sigma}_i \hat{\tau}_j = -E^\rho \hat{\tau}_j \]

if the parameterization remains unchanged. The complex conjugation changes also the sign of the \( U(1)_c \) generator.

III. THE PURE LEFT GAUGE MODELS

The Ld. of any gauge model with spontaneous symmetry breaking has three terms

\[ \mathcal{L} = \mathcal{L}_S + \mathcal{L}_G + \mathcal{L}_H \]

They correspond to the spinor (\( \mathcal{L}_S \)), gauge (\( \mathcal{L}_G \)) and Higgs (\( \mathcal{L}_H \)) sectors \([1]\). Let us start with an arbitrary spinor sector.

A. The pure left form of the spinor sector

In general any spinor sector contains a given number of left and right-handed chiral components. After second quantization all the spinor fields will be fermions. Thus it is natural to suppose from the beginning that all their components anticommute among themselves (as Grassmann classical variables or as quantum fermion fields). Thereby we have the possibility to replace each right-handed component by a charge conjugated left-handed one, as it is shown in Appendix B. BY doing this we shall obtain a pl. model containing only the left-handed components grouped in the multiplet \( L = (L_1, L_2, \ldots, L_N)^T \) the Dirac adjoint of which is \( \bar{L} = (\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_N) \). (The superscript \( T \) stands for matrix transposition.) The components of these two multiplets represent a set of \( 2N \) independent Grassmann formal variables (each one being in fact a 2-component spinor). Thus the spinor sector of any
model can be put in the pl. form. The number $N$ of the left-handed components will give the dimension of the model.

The most general form of the Ld. of the spinor sector of a pl. model can be written in the compact matrix notation as:

$$\mathcal{L}_S = \frac{i}{2} \bar{L} \gamma^\mu \partial_\mu L - \frac{1}{2} \mathcal{L} \chi L^c + \mathcal{L}^c \chi^+ L$$

(14)

where $\bar{\mathcal{L}} = \gamma^\mu \partial_\mu L - \gamma^\mu \partial_\mu \mathcal{L}$ and $L^c = (L_1^c, L_2^c, \cdots, L_N^c)^T$ is the charge conjugated of the multiplet $L$. The field $\chi$ has been introduced to give rise to the spinor masses after the breakdown of the gauge symmetry and therefore this may depend on the Higgs field components. Moreover, according to (107), $\chi$ results to be a $N \times N$ symmetric matrix of scalar fields which could couple all the spinor components among themselves, being able to generate simultaneously the Dirac and Majorana mass terms (like those of (108) and (109) respectively), in the limits of the possible Yukawa interactions.

We observe that generally in (14) quadratic terms in the anticommuting spinor variables could arise. Consequently, the correct Euler-Lagrange equations will be obtained by using Grassmann derivatives.

B. The global symmetries

One of the advantages of the pl. form of the spinor sector is to point out the maximal global symmetry of the model, which is partially hidden when the left and right-handed multiplets are treated separately. This will be defined as the global symmetry of the kinetic part (i.e. the first term) of (14). It is clear that the maximal symmetry is given by the group $SU(N) \otimes U(1)_c$ since the kinetic term remains invariant if the multiplets $L$ and $\mathcal{L}$ transform according to the ireps. $(N, y)$ and $(N^*, -y)$ of this group, for any $y$. Moreover, if $\chi$ would behave as a symmetric tensor of the rep. $(N(N+1)/2, 2y)$ of the same group then the whole Ld. of the spinor sector will have this symmetry. Hence, as long as we do not make
supplementary restrictive hypotheses relative to the covariance properties of the field \( \chi \), the maximal global symmetry of the spinor sector is determined only by its dimension, \( N \). Its knowledge is important for the choice of the gauge group since this can be any subgroup of \( SU(N) \otimes U(1)_c \).

Let us choose the group \( SU(n) \otimes U(1)_c \), with \( n < N \), to be the gauge group of our model. Then \( L \) will be of a reducible representation defined as the direct sum of a given set of ireps., \( \rho \). Therefore, we have,

\[
L = \sum_{\rho} \oplus L^\rho \tag{15}
\]

where each multiplet, \( L^\rho \), transforms according to its own irep., \( \rho \), defined by (I). The corresponding charge conjugated multiplets, \( (L^\rho)^c \), will transform according to the complex conjugated ireps., \( \rho^* \). Furthermore, we see that the matrix \( \chi \) contains all the blocks, \( \chi^{\rho \rho'} \), which couple the pairs of the multiplets \( L^\rho \) and \( (L^\rho')^c \). Then the Ld. (14) can be written as:

\[
\mathcal{L}_{S_0} = \frac{i}{2} \sum_{\rho} \bar{L}^{\rho^*} \rho L^\rho - \frac{1}{2} \sum_{\rho \rho'} \left( \overline{L^\rho} \chi^{\rho \rho'} (L^{\rho'})^c + h.c. \right) \tag{16}
\]

This remains invariant under the global \( SU(n) \otimes U(1)_c \) transformations if the blocks \( \chi^{\rho \rho'} \) will transform like

\[
\chi^{\rho \rho'} \rightarrow U^\rho (\xi^0, \xi) \chi^{\rho \rho'} \left( U^{\rho'} (\xi^0, \xi) \right)^T \tag{17}
\]

according to the representations \( (R_{\rho} \otimes R_{\rho'}, y_{\rho} + y_{\rho'}) \) which generally are reducible.

We note that there some discrete symmetries could be added. These can be chosen from the list of the discrete groups of the coset space \( SU(N)/SU(n) \).

**C. Gauging \( SU(n) \otimes U(1)_c \)**

Now we shall gauge the \( SU(n) \otimes U(1)_c \) group and, consequently, we shall introduce the gauge fields: \( A^0_\mu = (A^0_\mu)^* \) and \( A_\mu = A^+_\mu \in su(n) \) which can be expressed in the basis (III) as:
\[ A_\mu = D_\mu A_\mu + \sum_{i \neq j} E_{ij} A_{i\mu} \]  

(18)

This depends on \( n - 1 \) real fields, \( A_{i\mu} \), and \( n(n - 1)/2 \) c-number ones which satisfy: \( A_{i\mu} = (A_{j\mu})^*, i \neq j \). The next step is to replace the ordinary derivatives of the Ld (16) by the covariant ones:

\[ D_\mu L^\rho = \partial_\mu L^\rho - ig(R_\rho(A_\mu) + y_\rho A_\mu^0)L^\rho \]  

(19)

which will give the interaction terms of the whole Ld. of the spinor sector. In the basis (4) this is:

\[ \mathcal{L}_S = \mathcal{L}_{S_0} + g \sum_\rho \bar{L} \left( D_\rho A_{i\mu} \right)^* \sum_{i \neq j} E_{ij \rho} E_{ij \mu} + y_\rho A_\mu^0 \right) \gamma^\mu L^\rho \]  

(20)

The gauge invariance of this Ld. requires the gauge fields to transform like:

\[ A_\mu \rightarrow U A_\mu U^+ - i g (\partial_\mu U) U^+ \]  

\[ A_\mu^0 \rightarrow A_\mu^0 + \partial_\mu \xi^0 \]  

(21)

where \( U = U(\xi(x)) \) is given by (1). The field strength tensors:

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] \]  

\[ F_{\mu\nu}^0 = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0 \]  

(22)

are covariant and consequently we can define the invariant Ld. of the gauge sector as

\[ \mathcal{L}_G = -\frac{1}{2} Tr (F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} F_{\mu\nu}^0 F^{0\mu\nu} \]  

(23)

All these relations can be written in the basis (4) starting with the form of the matrix \( F_{\mu\nu} \) in this basis:

\[ F_{\mu\nu} = D_i F_{i\mu\nu} + \sum_{i \neq j} E_{ij \rho} F_{ij \mu\nu} \]  

(24)

where, according to (4), we have

\[ F_{\mu\nu}^i = 2 Tr(D_i F_{\mu\nu}) \quad , \quad F_{ij \mu\nu} = (F_{ij \mu\nu})^* = 2 Tr(E_{ij \mu\nu}) \]  

(25)
Furthermore, the Ld. \((23)\) will get the form:

\[
\mathcal{L}_G = -\frac{1}{4} \left( \sum_i F_{\mu\nu}^i F_{\mu\nu}^i + F_{\mu\nu}^0 F_{\mu\nu}^0 \right) - \frac{1}{2} \sum_i \sum_{j<i} (F_{\mu\nu}^j)^* F_{\mu\nu}^{ji}
\]

Its kinetic part, \(\mathcal{L}_{G|\mu=0}\), will contain only the kinetic parts of the fields, \(F_{\mu\nu}^0\) and \(F_{\mu\nu}^{\mu\nu}|\mu=0\). This will be combined with the mass term resulted from the breakdown of the gauge symmetry, in order to obtain the free Ld. of the gauge sector.

**IV. THE MINIMAL HIGGS MECHANISM**

Our aim is to introduce the mHm. as a generalization the Higgs mechanism of the SM by using a minimal number of field variables and a simple Ld. which should take in unitary gauge the same familiar form as that of SM. Moreover, this mHm. must break the \(SU(n) \otimes U(1)_c\) gauge symmetry up to that of the \(U(1)_{em}\) producing only one nonvanishing vacuum expectation value (vev.). This means that only one scalar real field, \(\phi\), should survive the gauge fixing. On the other hand, it is clear that in these conditions the blocks of \(\chi\), which generally could be tensors of any rank, does not coincide with the Higgs fields even though they may have nonvanishing vevs.. The solution is to suppose that the \(\chi\)-blocks are direct products of Higgs multiplets, in accordance to their gauge covariance, but having suitable scalar factors which should lead to the Yukawa couplings only in unitary gauge. However, these factors may depend only on the scalar \(\phi\).

**A. The Higgs sector**

Let us take \(n^2\) \(c\)-number Higgs variables, \(\phi_j^{(i)}\), organized into \(n\) multiplets, \(\phi^{(i)}\), each one transforming according its own irep., \((n, y^{(i)})\). These represent a set of \(2n^2\) real field variables from which at most \(n^2\) could be removed by fixing the gauge. Thus the danger is to remain
with some Goldstone bosons after the breakdown of the gauge symmetry. To avoid this, we shall reduce the number of field variables by introducing a priori the following constraints:

$\phi^{(i)} + \phi^{(j)} = \phi^2 \delta_{ij}$

where $\phi$ is a gauge invariant real field variable. These are $n^2$ real equations and, therefore, the fields $\phi^{(i)}$ will have only $n^2$ independent real components. We note that the indices included in parentheses are not $SU(n)$ vectorial ones even though they still range from 1 to $n$. Thus, in fact, we have introduced an orthonormal basis in the representation space of the irrep. $n$ of $SU(n)$ the vectors of which transform differently under the $U(1)_c$ group. Their $U(1)_c$ characters, $y^{(i)}$, will be considered as arbitrary parameters. They can be grouped into the matrix:

$Y = diag \left( y^{(1)}, y^{(2)}, \ldots, y^{(n)} \right)$

In order to obtain a Higgs Ld. with very simple interaction terms but able to produce a non trivial boson mass spectrum, it needs to introduce free parameters in its kinetic term. These will be: $\eta_0 \in [0, 1)$ and the nonvanishing elements $\eta^{(i)}$ of the matrix $\eta = diag \left( \eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)} \right)$ with the property,

$Tr(\eta^2) = 1 - \eta_0^2$

Now we can use $(\eta_0^2, \eta^2)$ as the metric of the kinetic part of the Ld. of the Higgs sector which will be

$\mathcal{L}_H = \frac{1}{2} \eta_0^2 \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} \sum_i (\eta^{(i)})^2 \left( \mathcal{D}^\mu \phi^{(i)} \right)^+ \left( \mathcal{D}_\mu \phi^{(i)} \right) - V(\phi)$

where

$\mathcal{D}^\mu \phi^{(i)} = \partial^\mu \phi^{(i)} - ig \left( A^\mu + y^{(i)} A^0_\mu \right) \phi^{(i)}$

and

$V(\phi) = -\frac{1}{2} \mu^2 \sum_i (\phi^{(i)})^+ \phi^{(i)} + \frac{1}{4} \lambda_1 \left( \sum_i (\phi^{(i)})^+ \phi^{(i)} \right)^2 + \frac{1}{4} \lambda_2 \sum_{i,j} (\phi^{(i)})^+ \phi^{(j)} (\phi^{(j)})^+ \phi^{(i)}$
is the potential which has been chosen to have the simplest algebraic form (allowed by its
gauge invariance).

Furthermore, we shall look for the absolute minimum of $V(\phi)$. To this end, we can
apply two methods. The first one is to give up (27) and to require the vanishing of all the
derivatives of $V(\phi)$ with respect to $\phi^{(i)}_j$. This leads to the equations:

$$\delta_{ij} \left( -\mu^2 + \lambda_1 \sum_k \phi^{(k)}_j \phi^{(k)}_j + \phi^{(i)}_j \phi^{(j)}_j \right) \lambda_2 \phi^{(i)}_j + \phi^{(j)}_j = 0$$

which have as solutions only the set of orthonormal multiplets (27) with $\phi$ given by:

$$-\mu^2 + (n\lambda_1 + \lambda_2)\phi^2 = 0$$

Another possibility is to express (32) only on $\phi$ with the help of (27) and to take the
minimum in this unique variable. The result is the same, namely (34). Thus it results that
the constraints (27) are compatible with the (absolute) minimum of $V(\phi)$. This will define
the vacuum state in which $\phi$ will have a nonvanishing expectation value, $< \phi >$. Then,
$\phi = < \phi > + \sigma$ where $\sigma$ is the physical Higgs field (with zero vev.). From (34) we obtain in
the zero-th order of the perturbations

$$< \phi > = | \mu | / \sqrt{n\lambda_1 + \lambda_2}$$

Moreover, because of (27), we are sure that here exists a gauge in which

$$\hat{\phi}^{(i)}_k = \delta_{ik} \phi = \delta_{ik} (< \phi > + \sigma)$$

This will be the unitary gauge. To go now to an arbitrary gauge one needs to perform a
("boost") transformation, $\hat{U} = \hat{U}(\phi^{(i)})$, so that $\phi^{(i)} = \hat{U}\hat{\phi}^{(i)}$. Therefore, the components of
$\phi^{(i)}$ in an arbitrary gauge can be written as:

$$\phi^{(i)}_j = \hat{U}_j^{\cdot i} (< \phi > + \sigma)$$

The Ld. of the Higgs sector in unitary gauge can be calculated by using (29)-(32), and
(36). We find that this is
\[ \mathcal{L}_H = \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma - \frac{1}{2} m^2_\sigma \sigma^2 - \frac{m^2_\sigma}{2} < \phi > \sigma^3 - \frac{m^2_\sigma}{8} < \phi >^2 \sigma^4 + \]
\[ + \frac{g^2}{2} ( < \phi > + \sigma)^2 \text{Tr}[(A_\mu + YA^0_\mu)\eta^2 (A^\mu + YA^{0\mu})] \] (38)

where \( m_\sigma = \sqrt{2n} | \mu | \) is the mass of the field \( \sigma \). We see that it has the same form like in the SM (or in the Weinberg-Salam model \cite{8,9}), except the last term, which contains here the both kinds of the free parameters introduced above (\( y^{(i)} \) and \( \eta^{(i)} \)). This will produce the boson mass spectrum which will be analyzed in more detail in the next section.

Particularly, for \( n = 2 \), the pair of the Higgs doublets of the SM, \( \phi_{SM} \) and \( \tilde{\phi}_{SM} \) \cite{9}, can be recognized to be in our notation:

\[ \frac{1}{\sqrt{2}} \phi^{(2)} = \phi_{SM} = \begin{bmatrix} \phi_+ \\ \phi_0 \end{bmatrix} ; \quad \frac{1}{\sqrt{2}} \tilde{\phi}^{(1)} = \tilde{\phi}_{SM} = \begin{bmatrix} \phi_0^* \\ -\phi_+^* \end{bmatrix} \] (39)

Moreover, it can be proved that in this case the Ld. (38) with \( \eta_0 = 0 \) and the homogeneous metric,

\[ \eta^{(1)} = \eta^{(2)} = 1/\sqrt{2} \] (40)

coincides with that of the SM. Thus, the mHm. appears as a natural generalization of the Higgs mechanism of the SM.

**B. The residual symmetry**

From the apparently large collection of Higgs field variables, we remain only with the physical scalar \( \sigma \). However, the other ones (i.e. the Goldstone bosons) do not disappear without trace, since they provide us with the characters \( y^{(i)} \) which will be involved in all the sectors of the model.

These will determine even the existence of the residual symmetry. This is given by the little group which leaves invariant the form (36) of the Higgs fields in unitary gauge. Therefore, its transformations must satisfy
for all $i = 1 \cdots n$. This requires $\xi$ to be from the Cartan subalgebra, i.e. $\xi = D_i \xi^i$, and to have:

$$D_i \xi^i + Y \xi^0 = 0$$

(42)

The nontrivial solutions of this homogeneous system will span the parameter space of the little group. These could arise only if $Y$ will be also of the Cartan subalgebra so that:

$$Tr(Y) = 0$$

(43)

Then the system (42) will admit as a unique nontrivial solution the one-dimensional subspace of the equations $\xi^i + 2\xi^0 Tr(D_i Y) = 0$ and nothing else. This will be just the subspace of the parameter $\xi^{em}$ of the little group $U(1)_{em}$. In this case $\hat{U}$ will be a transformation of the coset space $SU(n) \otimes U(1)_{c}/U(1)_{em}$, (depending on $n^2 - 1$ real parameters). Moreover, the condition (43) will reduce to $n - 1$ the number of the independent characters, $y^{(i)}$. Their concrete values will determine the direction of $\xi^{em}$ in the $n$-dimensional parameter subspace $\{\xi^0, \xi^i\}$.

Hence, it is obvious that the mHm. with $n$ vector multiplets is able to break the $SU(n) \otimes U(1)_{c}$ gauge symmetry up to the $U(1)_{em}$ residual one, when the condition (43) is accomplished. If it will be applied to the $SU(n) \otimes U(1)_{c}$ subgroup a $S(n') \otimes SU(n) \otimes U(1)_{c}$ gauge group then the residual symmetry group will be $SU(n') \otimes U(1)_{em}$. This means that this mHm. could be used for the actual generalizations of the SM. We note that the mHm. with a reduced number of vector multiplets will lead to larger residual symmetries. Thus if we shall use only $n - k$ orthogonal multiplets in a $SU(n) \otimes U(1)_{c}$ gauge model then a $SU(n-k)$ symmetry will be broken and the residual symmetry will be given by $SU(k) \otimes U(1)$ (where $SU(k)$ is the maximal subgroup of the coset space $SU(n)/SU(n-k)$).

C. The Yukawa couplings in unitary gauge
As we have mentioned, to be in accordance with the gauge covariance, we must define each block of $\chi$ as a direct product of some Higgs multiplets. In addition we shall introduce factors of the form $\phi^{-p}$ to obtain Yukawa couplings in unitary gauge.

Let us take the block $\chi^{\rho\rho'}$ which transforms like $\{17\}$ according to the irep.

$(R_\rho \otimes R_{\rho'}, y_\rho + y_{\rho'})$. If the ireps. $\rho$ and $\rho'$ are the ranks $(r, s)$ and $(r', s')$ respectively, then we find the components of $\chi^{\rho\rho'}$ to be tensors of the rank $(r + r', s + s')$. These will be defined as follows:

$$
(\chi^{\rho\rho'})_{\substack{j_1 \cdots j_s j'_1 \cdots j'_{s'} \\
i_1 \cdots i_r i'_1 \cdots i'_{r'}}} = \phi^{-p} \sum_{k,l} G^{l_1,l_2,\cdots,l_{s'+r'}}_{k_1,k_2,\cdots,k_r} \times \phi^{(k_1)} \times \cdots \times \phi^{(k_{p+r'})} \times \left( \phi^{(l_1)} \right) \times \cdots \left( \phi^{(l_{q'+p'})} \right) 
$$

(44)

where $G^{l_1 \cdots l_r}_{k_1 \cdots k_r}$ are coupling constants. This form corresponds to the reducible representation $R_\rho \otimes R_{\rho'}$ of $SU(n)$ but to fix the value of its character to $y_\rho + y_{\rho'}$ it requires a supplementary selection rule. This is: the coupling constants of (44) can have non-zero arbitrary values only for those combinations of indices for which we have:

$$
y^{(k_1)} + \cdots + y^{(k_{p+r'})} - y^{(l_1)} + \cdots + y^{(l_{q'+p'})} = y_\rho + y_{\rho'}
$$

(45)

When this condition is not satisfied the coupling constants must vanish. Thus the components of $\chi$ are well defined according to the gauge invariance of $L_s$.

In (44) we have introduced the scalar factor, $\phi^{-p}$, to control the formal dimensions of the coupling terms [10,11]. It is known that the model will be renormalizable only if each block $\chi^{\rho\rho'}$ will be of the dimension $d(\chi^{\rho\rho'}) \leq 1$. These can be easily pointed out in unitary gauge where (44) becomes:

$$
\left( \chi^{\rho\rho'} \right)_{\substack{j_1 \cdots j_s j'_1 \cdots j'_{s'} \\
i_1 \cdots i_r i'_1 \cdots i'_{r'}}} = <\phi >^{d(\chi^{\rho\rho'})} G^{j_1 \cdots j_s j'_1 \cdots j'_{s'}}_{i_1 \cdots i_r i'_1 \cdots i'_{r'}}
$$

(46)

with the obvious identification

$$
d(\chi^{\rho\rho'}) = r + r' + s + s' - p
$$

(47)

Thus the power $p$ of the scalar factor will be the parameter giving the desired formal dimension of the coupling terms. This will be fixed to $p = r + r' + s + s' - 1$ in order to obtain
the Yukawa couplings in unitary gauge (when \(d(\chi^{\rho\rho'}) = 1\)). In the other possible case (of \(d(\chi^{\rho\rho'}) = 0\)) the field \(\phi\) is completely decoupled and the mass terms appear as put by hand.

Here it is important to note that the factor \(\phi^{-p}\) can not produce singularities as long as \(p\) is at most equal to the number of the fields \(\phi^{(i)}\) of (44). The argument is that \(\phi^{(i)}/\phi\) results from (37) to be just a matrix element of the "boost" \(\hat{U}\) which must be regular. On the other hand, we observe that, despite the unusual definition of the \(\chi\)-blocks, their form in unitary gauge leads to the familiar couplings among \(\sigma\) and spinors like in the SM. Thus we can conclude that the \(m_{\text{H}m}\) produces the same effects in unitary gauge as the Higgs mechanism of the SM. In addition it has the advantage to be able to couple any pair of spinor multiplets in the mass terms, with the unique restriction given by (45).

V. THE PHYSICAL GAUGE BOSONS AND THEIR COUPLING COEFFICIENTS

Generally, the \(c\)-number gauge fields can be directly associated to the non-hermitian gauge bosons but the real gauge fields are linear combinations of the hermitian physical fields. These arise from some global transformations which leave invariant the kinetic part of the \(L_d\) (26) and diagonalize the mass matrix generated by the last term of (38). In the following we shall explicitly obtain these transformation which will help us to calculate the coupling coefficients of the charged and neutral currents i.e., the electric and the neutral charges.

A. The separation of the electromagnetic potential

The first step is to extract the electromagnetic potential, \(A_{\mu}^{em}\), associated to the \(U(1)^{em}\) parameter, \(\xi^{em}\). In the previous section we have seen that \(\xi^{em}\) belongs to a direction, of the parameter subspace \(\{\xi^0, \xi^i\}\), depending on the values of the characters (28) which satisfy the
condition (43). We shall try to separate $A^\mu_{em}$ from the other real gauge fields by changing the basis of $\{\xi^0, \xi^i\}$ in order to bring $\xi^{em}$ along the new zero-direction.

This can be done since the kinetic part of the first term of the Ld (26) is invariant under the $SO(n)$ global transformations of the subspace $\{\xi^0, \xi^i\}$ of the form:

$$
\xi^0 = V_{0}^0 \xi^0 + V_{0}^i \xi^i \\
\xi^i = V_{i}^0 \xi^0 + V_{i}^j \xi^j
$$

(48)

which change like the group parameters, the gauge fields as well as the kinetic parts of the field strength tensors. We shall choose a special $SO(n)$ transformation defined as a rotation of angle $\theta$ around the axis of versor $\nu$ orthogonal to the $\xi^0$ direction (i.e. $\nu_0 = 0$, $\nu_i = \nu_i \nu^i = 1$) and we shall require $\xi^{0}$ to coincide with $\xi^{em}$. This transformation has the matrix elements:

$$
V_{0}^0 = \cos \theta \\
V_{i}^0 = -V_{i}^0 = \nu_i \sin \theta \\
V_{i}^j = \delta^i_j - \nu^i \nu_j (1 - \cos \theta)
$$

(49)

Now we see that $\xi^{em}$ will be the unique solution of (42) only if we have:

$$
Y = -D_i \nu^i \tan \theta \equiv -(D \cdot \nu) \tan \theta
$$

(50)

With the help of (5), we can calculate the values of $\theta$ and $\nu_i$ for a given matrix $Y$ which satisfy the condition (43). Thus the transformation (49) leading the gauge fields $A^0_{\mu}$ and $A^i_{\mu}$ into the new ones, $A^{em}_{\mu}$ and $A^{\hat{i}}_{\mu}$, is completely determined.

On the other hand, (50) can be interpreted as the change of the $n - 1$ arbitrary parameters, $y^{(i)}$, to the new ones, $\theta$ and $\nu_i$ (with $n - 2$ independent components). In the next we shall use this new parameterization which will be more efficient.

B. The massive gauge bosons
Let us take the mass term of the gauge fields given by the last term of the Ld. (38). This is:

\[
g^2 \frac{\langle \phi \rangle}{2} Tr \left[ \left( A_\mu + Y A^0_\mu \right) \eta^2 \left( A^\mu + Y A^{0\mu} \right) \right]
\]

(51)

where \( \eta^2 \) satisfies the condition (29). It can be expressed in terms of the fields \( A^e_\mu \) and \( A^\hat{i}_\mu \) and of the parameters \( \theta \) and \( \nu_i \) instead of \( y^{(i)} \). Thus by (18), (19) and (50), after few manipulations, we can put it in the form:

\[
\frac{1}{2} (M^2)_{ij} A^i_\mu A^j_\mu + \sum_i \sum_{j<i} \left( M^i_j \right)^2 (A^i_\mu)^* A^j_\mu
\]

(52)

Here \( M^2 \) is a non-diagonal matrix with the elements given by

\[
(M^2)_{ij} = \langle \phi \rangle^2 Tr(B_i B_j)
\]

(53)

where

\[
B_i = g \left( D_i + \nu_i (D \cdot \nu) \frac{1 - \cos \theta}{\cos \theta} \right) \eta
\]

(54)

while

\[
M^i_j = \frac{1}{2} g \langle \phi \rangle \left( (\eta^{(i)})^2 + (\eta^{(j)})^2 \right)^{1/2}
\]

(55)

are just the masses of the c-number gauge fields, \( A^\hat{i}_\mu \).

As it was expected, \( A^e_\mu \) does not appear in the mass term and, consequently, it remains massless. The other real gauge fields \( A^\hat{i}_\mu \) have the non-diagonal mass matrix (53). This can be brought in diagonal form with the help of a new \( SO(n-1) \) transformation:

\[
A^\hat{i}_\mu = \omega^{\hat{i}j}_\mu Z^j_\mu
\]

(56)

which will lead to the neutral gauge bosons \( Z^j_\mu \) with well-defined masses. The special form of the matrix (53) allows us to find a general method to derive the transformation \( \omega \) and the mass spectrum of the Z-bosons but this is too complicated to be presented here. Anyway it is obvious that \( \omega \) will depend on \( \theta \), \( \nu_i \) and \( \eta^{(i)} \).
Now we can appreciate the role of the parameters $\eta^{(i)}$ which determine the structure of the mass spectrum of the gauge bosons. In the simplest case of the homogeneous $\eta^2$ metric (when $\eta^{(i)} \sim 1/\sqrt{n}$) this spectrum results to be deeply degenerated. Indeed, then the matrix (53) will get the form $(M_2)^{(i)j} \sim (\delta^{(i)j} + \nu_i \nu_j \tan^2 \theta)/2n$, and, consequently, it will have only two distinct eigenvalues: $\sim 1/2n \cos^2 \theta$ (for the eigenvector along the $\nu$ direction) and $\sim 1/2n$ (for all the other orthogonal eigenvectors). Thereby we shall obtain one $Z$-boson of the mass $\sim 1/\sqrt{2n} \cos \theta$ while the masses of all the other gauge bosons including the $c$-number ones will be $\sim 1/\sqrt{2n}$. This reproduces the well known result (i.e. $m_W/m_Z = \cos \theta_W$) of the SM which has a Higgs mechanism equivalent to the $mHm$. with the metric (40) and $\eta_0 = 0$. However, for the gauge models with $n > 2$, the two values mass spectrum corresponding to the homogeneous metric it is less likely to be satisfactory. Thus the conclusion is that, in general, the parameters $\eta^{(i)}$ must be chosen according to (29) in a manner that permits us to differentiate the masses (55) among themselves.

C. The generalized Weinberg transformation and the coupling coefficients

The whole transformation which brings the original gauge fields, $A^0_{\mu}$ and $A^i_{\mu}$ into the physical ones, $A^e_{\mu}$ and $Z^i_{\mu}$, will be called the generalized Weinberg transformation (gWt.). According to (48), (49) and (56) this is:

$$A^0_{\mu} = A^e_{\mu} \cos \theta - \nu_i \omega^{\hat{i}j}_{\mu} Z^j_{\mu} \sin \theta$$
$$A^k_{\mu} = \nu^k A^e_{\mu} \sin \theta + \left( \delta^k_i - \nu^k \nu_i (1 - \cos \theta) \right) \omega^{\hat{i}j}_{\mu} Z^j_{\mu}$$

(57)

Let us now turn to the spinor sector looking for the coupling coefficients of the spinor-gauge field interactions. It is convenient to take the electric and neutral charges in units of the elementary electric charge, $e_0$, by using $g_0 = g/e_0$ instead of $g$. First we shall introduce (57) in the interaction term of the Ld. (21). Thus, we find that the spinor multiplet $L^\rho$ (of the irep. $\rho$) has the following electric charge matrix:
Q^\rho = g_0 [(D^\rho \cdot \nu) \sin \theta + y_\rho \cos \theta] \quad (58)

and the \( n-1 \) neutral charge matrices:

\[ Q^\rho(Z^i) = g_0 [D^\rho_k - \nu_k (D^\rho \cdot \nu)(1 - \cos \theta) - y_\rho \nu_k \sin \theta] \omega^{k \cdot i} \quad (59) \]

corresponding to the \( n-1 \) gauge fields, \( Z^i_\mu \). All the other gauge fields, \( A^i_{\mu} \), will have the same coupling constant, \( g/\sqrt{2} \). From (58) and (59) we can calculate the coupling coefficients corresponding to the fundamental irep., \( (n,0) \). These are given by the electric charge matrix,

\[ Q \equiv diag(q_1, q_2, \ldots, q_n) = g_0 (D \cdot \nu) \sin \theta \quad (60) \]

and by the neutral charge matrices,

\[ Q(Z^i) \equiv diag(q^{(i)}_1, q^{(i)}_2, \ldots, q^{(i)}_n) = g_0 [D^k_k - \nu_k (D \cdot \nu)(1 - \cos \theta)] \omega^{k \cdot i} \quad (61) \]

All these matrices are traceless since they depend only on the generators \( D^i \).

Now we can change again the parameterization of the model going from the parameters, \( (g, \theta, \nu_i) \), to the new ones, \( (\theta, q_i) \). To remain in the spirit of the SM, we shall keep the angle \( \theta \) as the main parameter of the model, while \( g_0 \) and \( \nu_i \) will be expressed by the electric charges \( q_i \) of the fundamental multiplet (which represent a set of \( n-1 \) independent parameters since \( TrQ = \sum_i q_i = 0 \)). This can be done by using the formulae:

\[ g_0 \nu_i \sin \theta = 2Tr(D_i Q) \quad , \quad g_0^2 \sin^2 \theta = 2Tr(Q^2) \quad (62) \]

and the new form of the matrix (28):

\[ Y = -Q \tan \theta / \sqrt{2Tr(Q^2)} \quad (63) \]

resulted from (60), (6) and (50). These will help us to calculate the matrix (53), the transformation \( \omega \) and the gWt. (57) in this last parameterization.

Based on these results we can find out the charges of each tensor component of the multiplets \( L^\rho \). First we shall express the neutral charges (61) of the fundamental irep. only
on $\theta$ and $q_i$ and then, with the help of the general rule given by (10) and by using (58), (59) and (62), we find the electric charges of the component $(L^\rho)^{j_1\cdots j_s}_{i_1\cdots i_r}$

$$(Q^\rho)^{j_1\cdots j_s}_{i_1\cdots i_r} = \sum_{\alpha=1}^{r} q_{i_\alpha} - \sum_{\beta=1}^{s} q_{j_\beta} + y_\rho \sqrt{2Tr(Q^2)} \cot \theta$$

(64)

and the corresponding neutral charges

$$(Q^\rho(Z^{\dot{i}}))^{j_1\cdots j_s}_{i_1\cdots i_r} = \sum_{\alpha=1}^{r} q_{i_\alpha} - \sum_{\beta=1}^{s} q_{j_\beta} - 2y_\rho Tr(D_\dot{j}Q)\omega_{\dot{i}}^{\dot{j}}$$

(65)

The electric charge of the gauge boson $A^j_{\mu}$ is $q_i - q_j$. Moreover, we specify that charge conjugation simultaneously changes the signs of the electric and neutral charges of the spinor components since $(L^\rho)^c$ is of the irep. $\rho^*$ which has the generators $[12]$.

Now, the last term of (64) will allow us to define different kind of chiral hypercharges (and of the second coupling constants, $g_i^i$) we wish to use in concrete models. For our future developments it is convenient to define

$$\hat{y} = y \sqrt{2Tr(Q^2)} \cot \theta$$

(66)

where $y$ can be $y_\rho$ or $y^{(i)}$. Thus, in the following we could use the new ireps. notation, $(R, \hat{y})$, and the matrix $\hat{Y} = -Q$ instead of $Y$. We note that these definitions would be not consistent before to introduce the angle $\theta$ (of the transformation [49] which separates $A^m_{\mu}$)

VI. THE CONSTRUCTION OF THE RENORMALIZABLE MODELS

Our previous results indicate that the models with $mH_m$ have good boson sectors and, therefore, they will be renormalizable when an adequate spinor sector will be added. The structure of this sector is determined by the choice of its ireps. and by the values of one of the equivalent set of parameters we have introduced above: $(g, y^{(i)}) \sim (g, \theta, \nu_i) \sim (\theta, q_i)$. Finally, it must have a charge spectrum in accordance to the particle-antiparticle principle and be free of axial anomaly. Therefore, in order to complete our method, we need to briefly review the specific formulation of these requirements for the pl. models.
A. The particle-antiparticle principle

The physical meaning of the model will strongly depend on the form of the mass terms of the spinor $\psi_d$. which may give rise to the masses of the Dirac fields. These are represented here by pairs of left-handed components with equal and opposite signs electric charges, i.e. pairs of Dirac partners as that of the the $\psi_{d}$. In our approach, the desired good Dirac mass terms will appear because of the condition (108) which, according to (13) and (54), restricts each mass term to be neutral, coupling only Dirac partners which will gain the same mass. When there are many pairs with the same electric charges, some mixings could also occur.

Hence, a good spinor mass spectrum could be obtained only if each charged left-handed component will have a partner of the opposite sign of charge. If this should not happen then the physical content of the model could be compromised by the appearance of several charged massless components. For this reason, it is necessary to introduce the following supplementary requirement: the electric charge spectrum of the spinor sector must be symmetric with respect to zero. This could be considered as the specific form of the particle-antiparticle principle for pl. models. Only when this is accomplished, we could introduce adequate mass terms to produce mass for all the Dirac fields.

However, it is difficult to find a general mathematical formulation of this principle since this would be dependent by the concrete values of the parameters. Therefore, in applications, we are determined to verify step by step while building the model if the structure of the charge spectrum remains correct and if all the needed mass terms have been introduced. To be efficient, it is recommended to start the construction of the model directly with the parameterization $(\theta, q_i)$.
B. The cancellation of the axial anomaly

In the case of pl. models, where we have no right-handed multiplets, the conditions of anomaly cancellation have the same simple form as in GUT (or in the current algebra) \[12\]. In terms of the hermitian generators this amounts to have:

\[
\sum_{\rho} Tr(T^\rho_\alpha \{ T^\rho_\beta, T^\rho_\gamma \}) = 0
\]

(67)
for all the values \(\alpha, \beta, \gamma = 0, 1, \cdots, n^2 - 1\) when the sum is tacked over all the irreps. \(\rho\) of the left-handed multiplets.

To separate the invariant factors of these equations (which depend only on the choice of the irreps.), we shall use the form of the \(U(1)_c\) generator, \(T^\rho_0 = y^\rho_1\), and the well known properties of the hermitian \(SU(n)\) generators of the irreps. \(R_\rho\), which generalize (\[99\]) and (\[100\]) according to the Wigner-Eckart theorem. In our notations these are:

\[
Tr(T^\rho_a T^\rho_b) = \frac{1}{2} \alpha(R_\rho) \delta_{ab}
\]

\[
Tr(T^\rho_a \{ T^\rho_b, T^\rho_c \}) = \frac{1}{2} \beta(R_\rho) d_{abc}
\]

(68)
where \(d_{abc}\) is the symmetric tensor of (\[100\]). The reduced matrix elements, \(\alpha(R)\) and \(\beta(R)\), have specific values for each irrep. \(R\). Few examples are given in the Table \[1\]. Furthermore for the complex conjugate irreps. we have

\[
\alpha(R^*) = \alpha(R) , \quad \beta(R^*) = -\beta(R)
\]

(69)

Now, from (\[88\]) and the evident properties, \(Tr(T^\rho_a) = 0\) and \(Tr(1_\rho) = n(R_\rho)\), we find that (\[77\]) are equivalent to the following conditions:

\[
\sum_{\rho} \beta(R_\rho) = 0
\]

\[
\sum_{\rho} \alpha(R_\rho) \hat{y}_\rho = 0
\]

\[
\sum_{\rho} n(R_\rho) \hat{y}^3_\rho = 0
\]

(70)
which will assure the cancellation of the axial anomaly. Particularly, for \( n = 2 \) the symmetric tensor \( d_{abc} \) vanishes and, consequently, in the case of the \( SU(2) \otimes U(1)_c \) gauge models the first of the equations (70) do not occur.

On the other hand, we have seen that the spinor charge spectrum must be symmetric with respect to zero. Consequently, we can add to (70) the minimal necessary condition to accomplish that,

\[
\sum_\rho Tr(Q^\rho) \sim \sum_\rho n(R_\rho)\hat{y}_\rho = 0
\]  

(71)

obtaining, thus, the complete set of restrictions which should select the irreps. of the spinor sector. We note that (70) gives, in addition, \( \sum_\rho Tr((Q^\rho)^3) = 0 \), but this will not suffice to determine the symmetry of the charge spectrum and, therefore, adequate values of the parameters \( q_i \) are also needed.

C. The summary of the method

Now we have all the elements of the the method we intend to propose for the construction of high unitary symmetry gauge model. This starts with the choice of the irreps. of the spinor sector according to (70) and (71). It follows to introduce suitable parameters \( q_i \) in order to define the desired (symmetric) charge spectrum. Then, from (63) and (62) the characters \( y^{(i)} \) as well as their corresponding \( \hat{y}^{(i)} \) can be obtained. Thus the irreps. of the Higgs multiplets will be completely determined only by the electric charges of the fundamental irrep.. Moreover, the versor \( \nu \) of the transformation (49) will also result from (62). As we have mentioned, the angle \( \theta \) of this transformation remains the main free parameter of the model. Furthermore, we shall take arbitrary \((\eta_0, \eta)\) which will represent a set of \( n \) independent parameters because of (55). These will be involved in the structure of the whole gauge boson mass spectrum, giving the masses (55) and the neutral boson mass matrix (53). This matrix will be diagonalized by the transformation \( \omega \) which may be calculated in each
particular case separately. The next step is to derive the \( gW_t \), which will point out the physical neutral gauge bosons. Their coupling coefficients will result from \( (65) \) and \( (55) \).

The last operation is to introduce the constants \( G \), for all the \( \chi \)-blocks one needs. Finally, the resulted LD. in unitary gauge may be written in the standard Dirac form, (by using the formulae of the Appendix B).

We note that the parameters \((\eta_0, \eta)\) could be stable under renormalization since they appear initially only in the kinetic term of the Higgs LD.. Therefore, they could be kept arbitrary when we shall analyse the properties of the model. In these conditions even the effects of the radiative corrections would be better pointed out. For this reason we believe that the physical interpretation, including the fit of these parameters, may be done only after we have studied the behavior of the model.

The Weinberg-Salam model in the pl. form is the simplest example which can be solved easily by using this method, starting directly with the parameterization \((\theta_W, q_i)\). It is of dimension \( N = 15 \) containing the well-known left-handed doublets, \((\nu_L, e_L)^T, 3 \times (u_L, d_L)^T\), while the right-handed singlets will be replaced by their charge conjugated: \((e_R)^c, 3 \times (u_R)^c\) and \(3 \times (d_R)^c\). Denoting by \( T_i^W \) the \( SU(2) \) generators, we find that the electric charges of the fundamental multiplet, \((2, 0)\), are given by the matrix \( Q = T_3^W \) and, consequently, \( \hat{Y} = -Q = -T_3^W \). Furthermore, from \( (62) \) we obtain \( g \sin \theta_W = \epsilon_0 \). Now it is clear that the values of \( \tilde{y}_\rho \) giving the desired electric charges of the spinor multiplets must be (in the above order): \(-1/2, 1/6\) for the doublets and \(-2/3, 1/3\) for the singlets. On the other hand, from the form of \( \hat{Y} \) it results that the Higgs doublets, \( \phi^{(1)} \) and \( \phi^{(2)} \), transform like \((2, -\frac{1}{2})\) and \((2, \frac{1}{2})\) respectively. As we have mentioned, these doublets can be related to the standard doublet of the Weinberg-Salam model according to \( (39) \). If, in addition, we shall take the metric \( (11) \) then our mHM. will coincide with its known Higgs mechanism, producing the same boson mass spectrum. Moreover, by using \( (55) \) and \( (56) \) we can find the well-known values of the neutral charges expressed in terms of \( \epsilon_0 \) and \( \theta_W \) \( [9] \). Thus, in our approach, we can obtain directly the final results of the model.

This example shows that the \( SU(n) \otimes U(1)_c \) pl. models with mHM. we have proposed
are a natural generalization of the SM. For this reason we shall use the above developed
technique to solve the models with $n = 3$ which would contain the SM as a submodel.

**VII. $SU(3) \otimes U(1)_C$ MODELS**

In the following we shall study the $SU(3) \otimes U(1)$ electroweak pl. models with mHm.. In
fact we are interested only by the models which contain the lepton triplet $(\nu_{eL}, e_L, (e_R)^c)^T$
since these have a virtue: their gauge group combines the gauge group of the SM to that of
the old Pauli-Pursey-Gursey (PPG) symmetry [13]. This is nothing else than the $SU(2) \otimes$
$U(1)_c$ group of the *maximal* global symmetry of the Dirac Ld. (108). In addition, if one
gauges the PPG group one finds that the doubly charged boson of the actual $SU(3) \otimes U(1)$
models is due just to this gauge. This is important since the PPG gauge submodel is
anomaly-free and, consequently, its phenomenology could be separately treated in the future
investigations related to the electromagnetic implications of the doubly charged boson. On
the other hand, we shall see that a basis of the $su(3)$ algebra defined so that its first three
generators be those of the PPG $SU(2)$ will offer certain technical advantages.

**A. Three-generations $SU(3) \otimes U(1)_c$ model**

Let us consider the pl. model which should have: (I) the spinor sector of the Pisano-
Pleitez model [2] put in pl. form and (II) a mHm. with arbitrary $(\eta_0, \eta)$, which should satisfy
(29), and Yukawa couplings in unitary gauge. We shall try to solve this model supposing
that, in addition: (III) its unique coupling constant, $g$, coincides to the first one of the SM
and (IV) at least one $Z$-boson should satisfy the condition of the SM, $m_Z = m_W / \cos \theta_W$.

This model is an $SU(3)_{col} \otimes SU(3) \otimes U(1)_c$ gauge model but here we shall consider only
the electroweak interactions gauged by $SU(3) \otimes U(1)_c$. Therefore, the color indices will be
omitted, restricting ourselves to indicate only the triplication of the quark multiplets. For
this gauge group we shall use the hermitian $SU(3)$ generators $T_a = \lambda_a / 2$ ($a = 1, \cdots, 8$) represented by the Gell-Mann matrices in the basis where

$$\lambda_3 = diag(0, 1, -1), \quad \lambda_8 = diag(-2, 1, 1) / \sqrt{3}$$

(72)

Hence, the first three generators, $T_{\hat{a}} = \lambda_{\hat{a}} / 2$, $\hat{a} = 1, 2, 3$, are just those of the PPG $SU(2)$ group while the $SU(2)$ group of the SM will be generated by: $T^W_1 = \lambda_6 / 2$, $T^W_2 = \lambda_7 / 2$ and $T^W_3 = -(\sqrt{3} \lambda_8 + \lambda_3) / 4$. This is an unusual basis but it is indicated since here $D_1 = T_3$ will be proportional to $Q$. The other diagonal generator in our previous notation is $D_2 = T_8$.

It follows to introduce the left-handed spinor multiplets in the pl. form. Thus, all the triplets will be those of Ref. [2] (up to an unitary transformation) while the singlets will be replaced by their charge conjugated and, therefore, all their coupling coefficients will appear with opposite signs. The resulted spinor sector will have the lepton triplets,

$$L_l = \begin{pmatrix} \nu_l \\ l \\ l^c \end{pmatrix} \sim (3, 0), \quad l = e, \mu, \tau$$

(73)

the quark triplets ($\times 3$),

$$\begin{pmatrix} u \\ d \\ j_1 \end{pmatrix} \sim (3, 2/3), \quad \begin{pmatrix} s \\ -c \\ j_2 \end{pmatrix}, \quad \begin{pmatrix} b \\ -t \\ j_3 \end{pmatrix} \sim (3^*, -1/3)$$

(74)

and the quark singlets ($\times 3$)

$$(d_R)^c, (s_R)^c, (b_R)^c \sim (1, 1/3) \quad (u_R)^c, (c_R)^c, (t_R)^c \sim (1, -2/3) \quad (j_{1R})^c \sim (1, -5/3) \quad (j_{2R})^c, (j_{3R})^c \sim (1, 4/3)$$

(75)

The electric charge matrix of the fundamental irep., $(3, 0)$, is $Q = -2T_3$ while those of the ireps. $\rho = (n_\rho, \hat{y}_\rho)$ of the above lists are given by

$$Q^\rho = -2T_3^\rho + \hat{y}_\rho$$

(76)
Furthermore, the condition (III) gives: \( g \sin \theta_W = c_0 \). Then, according to (62), we obtain

\[
\sin \theta = 2 \sin \theta_W
\]

(77)

which fixes the value of \( \theta \). Hereby it results that \( \sin^2 \theta_W \leq 1/4 \) \([4]\). It remains to find the irreps. of the three Higgs triplets of our \( m_{Hm} \), \( \phi^{(1)} \), \( \phi^{(2)} \) and \( \phi^{(3)} \). These are defined by \( \hat{Y} = -Q = 2T_3 \) to be, \((3, 0)\), \((3, 1)\) and \((3, -1)\) respectively, like in Ref. \([2]\).

In our notations the gauge fields are \( A_\mu^0 \) and \( A_\mu \in su(3) \) given by

\[
A_\mu = \frac{1}{2} \begin{pmatrix}
-2A_\mu^8/\sqrt{3} & \sqrt{2}W_\mu & \sqrt{2}V_\mu \\
\sqrt{2}W_\mu^* & A^3_\mu + A_\mu^8/\sqrt{3} & \sqrt{2}U_\mu \\
\sqrt{2}V_\mu^* & \sqrt{2}U_\mu^* & -A^3_\mu + A_\mu^8/\sqrt{3}
\end{pmatrix}
\]

(78)

where \( W \) is the Weinberg charged boson while \( U \) and \( V \) are the new charged bosons due to this gauge symmetry. Now one can see that the doubly charged boson, \( U \), corresponds to the non-diagonal generators of the PPG \( SU(2) \) group. It is interesting to note that it couples lepton Cooper-like currents of the form \( g(\tilde{T} \gamma^\mu (1 - \gamma^5) t^c) / 2\sqrt{2} \).

**B. The solution of the model**

The masses of the charged bosons are given by (54), having the same form as those of Ref. \([2]\). In order to find the physical neutral bosons one needs to calculate the transformation \( \omega \), which should diagonalize the matrix (53), and the gWeT. (57). To this end, it is convenient to start with the following parameterization

\[
\eta^2 = (1 - \eta_0^2) \text{diag}(\frac{1}{2}a - b, \frac{1}{2}a + b, 1 - a)
\]

(79)

Then, the masses of the charged bosons will be

\[
m_W^2 = m^2a, \quad m_V^2 = m^2(1 - \frac{1}{2}a - b), \quad m_U^2 = m^2(1 - \frac{1}{2}a + b)
\]

(80)

where
\[ m^2 = \frac{1}{4} g^2 < \phi >^2 (1 - \eta_0^2) = \frac{1}{2} (m_U^2 + m_V^2 + m_W^2) \]  \hspace{1cm} (81)

and the matrix (53) will get the form
\[
M^2 = m^2 \begin{pmatrix}
\frac{1}{\cos^2 \theta} (1 - \frac{1}{2} a + b) & -\frac{1}{\sqrt{3} \cos \theta} (1 - \frac{3}{2} a - b) \\
-\frac{1}{\sqrt{3} \cos \theta} (1 - \frac{3}{2} a - b) & \frac{1}{3} + \frac{1}{2} a - b
\end{pmatrix} \hspace{1cm} (82)
\]

We observe that this matrix has a singularity for \( \sin^2 \theta_W = \frac{1}{4} \) and, consequently, \( \sin^2 \theta_W \) must remain less than \( \frac{1}{4} \). The transformation \( \omega \) reduces here to a simple rotation of the angle \( \theta' \) (with \( \omega_1 = \omega_2'' = \cos \theta' \) and \( \omega_1 = -\omega_2 = \sin \theta' \)). Furthermore, we find that the condition (IV) is satisfied if and only if
\[ b = -\frac{3}{2} a \tan^2 \theta_W \]  \hspace{1cm} (83)

This will lead to the following exact boson mass spectrum
\[
m_W^2 = m_Z^2 \cos^2 \theta_W = m^2 a \\
m_Z^I^2 = \frac{m^2}{1 - 4 \sin^2 \theta_W} \left( \frac{4}{3} \cos^2 \theta_W - a \left( 1 - \left( 1 - 4 \sin^2 \theta_W \right) \tan^2 \theta_W \right) \right) \\
m_V^2 = m^2 \left( 1 - \frac{a}{2} \frac{1 - 4 \sin^2 \theta_W}{\cos^2 \theta_W} \right) \\
m_U^2 = m^2 \left( 1 - \frac{a}{2} \frac{1 + 2 \sin^2 \theta_W}{\cos^2 \theta_W} \right) \hspace{1cm} (84)
\]

depending on the arbitrary parameter \( a \in (0, 1] \). Therefore, \( Z^2 = Z \) is the Weinberg neutral boson while \( Z^1 = Z' \) is the new one of this model.

Now we have the surprise to find the angle \( \theta' \) does not depend on \( a \), when (83) is accomplished. Its value is given by
\[ \tan 2 \theta' = -\sqrt{3} \frac{\sqrt{1 - 4 \sin^2 \theta_W}}{1 + 2 \sin^2 \theta_W} \]  \hspace{1cm} (85)

Consequently, the gWt. will be also independent on \( a \). Its versor, calculated from (62), is \( \nu = (-1, 0) \) and, therefore, it can be written as,
\[
A_\mu^0 = A_\mu^{em} \cos \theta + (Z'_\mu \cos \theta' - Z_\mu \sin \theta') \sin \theta \\
A_\mu^3 = -A_\mu^{em} \sin \theta + (Z'_\mu \cos \theta' - Z_\mu \sin \theta') \cos \theta \\
A_\mu^8 = Z'_\mu \sin \theta' + Z_\mu \cos \theta' \hspace{1cm} (86)
\]
where $\theta$ and $\theta'$ are given by (76) and (85) respectively. The gWt. allows us to calculate the coupling coefficients of the fundamental multiplet, $L$. Its neutral charges corresponding to the Weinberg boson $Z$ are just those predicted by the SM while the matrix of those corresponding to the other neutral boson, $Z'$, is

$$Q(Z') = -\frac{1}{\sqrt{3}\sin 2\theta_W} \sqrt{1 - 4\sin^2 \theta_W} diag(1, 1, -2)$$

(87)

The neutral charge matrices of the others multiplets, $L^\rho$, can be derived from (65) and (66).

These are

$$Q^\rho(Z) = -\frac{1}{\sin 2\theta_W} \left( (1 - 4\sin^2 \theta_W) T^\rho_3 + \sqrt{3} T^\rho_8 + 2\hat{y}_\rho \sin \theta_W \right)$$

(88)

$$Q^\rho(Z') = \frac{\sqrt{1 - 4\sin^2 \theta_W}}{\sin 2\theta_W} \left( \sqrt{3} T^\rho_3 - T^\rho_8 - 2\sqrt{3} \hat{y}_\rho \frac{\sin^2 \theta_W}{1 - 4\sin^2 \theta_W} \right)$$

The present exact results are similar to those of Ref. [4] obtained in the tree approximation. In fact only the last term of $Q^\rho(Z')$ is different (notice that $\hat{y}_\rho$ coincides to the $X$-hypercharge used therein). Moreover, we must specify that in our approach the coupling coefficients of the vector and axial neutral currents of a fermion $f$ are

$$Q(Z, f)_V = \frac{1}{2}(Q(Z, f) - Q(Z, f^c)), \quad Q(Z, f)_A = -\frac{1}{2}(Q(Z, f) + Q(Z, f^c))$$

(89)

because of the pl. form of the spinor sector.

Under mHm. the quark masses will be generated by the usual Yukawa interactions involving only the fields $\phi^{(i)}$ like in Ref. [2] but to produce the lepton masses we need to use our Yukawa couplings in unitary gauge. Therefore, we shall introduce the 2-rank symmetric tensors, $\chi_l = \phi^{-1} G_l (\phi^{(2)} \otimes \phi^{(3)} + \phi^{(3)} \otimes \phi^{(2)})$, and the coupling terms, $\bar{L}_l \chi_l L_l^c + h.c$, for all the leptons ($l = e, \mu, \tau$). These will give rise to the lepton masses while the neutrinos remain massless. If we wish massive neutrinos we could use the tensors $\chi'_l = \phi^{-1} G_l (\phi^{(1)} \otimes \phi^{(1)})$.

Hence, the model is completely solved without any kind of approximations. It depends on the parameter $a$ which will determine the structure of the boson mass spectrum. We observe that for $a \simeq 0$ the new gauge bosons will be very massive as in Refs. [2,4] while for $a \simeq 1$
their masses become smaller than $m_W$. Our solution offers the possibility to investigate both of these cases. On the other hand, the exact values of the neutral charges we have derived bring nothing new concerning the suppression of the flavour-changing neutral currents of this model [2,14].

C. A possible new one-generation $SU(3) \otimes U(1)_c$ model

Our $m_{Hm.}$ eliminates the usual restrictions due to the Yukawa interactions and gives us the hope to find a new anomaly-free one-generation model in which the suppression of the flavor-changing neutral currents is more natural [15]. However, this can not be done without introducing new particles. There are many possibilities but here we shall restrict ourselves to present only the most strange of them. We shall start with the observation that the anomaly is canceled when one uses the following ireps.: $(3, 0)$, $3 \times (3, \hat{y})$, $3 \times (3, -\hat{y})$ and $(6^*, 0)$. Indeed, according to Table I and (69), we have $\beta(6^*) = -7$ and, consequently, the conditions (70) and (71) are fulfilled for any $\hat{y}$. Therefore, the first irep. may be of $L_e$, defined by (73), while the others would involve the quarks $u$, $d$ and only one exotic quark, namely that of the electric charge $5/3$ (denoted now by $j$), as well as the corresponding anti-quarks. The choice $\hat{y} = \frac{2}{3}$ will give the desired electric charges to the quark triplets

$$3 \times \begin{vmatrix} u \\ d \\ j \end{vmatrix} \sim (3, 2/3), \quad 3 \times \begin{vmatrix} u^c \\ j^c \\ d^c \end{vmatrix} \sim (3, -2/3)$$

(90)

In addition, we find that the irep. $(6^*, 0)$ will contain two massive leptons, $x$ and $y$, of the electric charges $q(x) = -2$ and $q(y) = -1$, and their corresponding neutrinos, $\nu_x$ and $\nu_y$. This sextet is:

$$\begin{vmatrix} \nu_y & y^c & y \\ y^c & x^c & \nu_x \\ y & \nu_x & x \end{vmatrix} \sim (6^*, 0)$$

(91)
In this model the gauge and the Higgs sectors will be identical to those of the previous one. Therefore, all the results obtained above remain valid. However, differences will appear in the Yukawa sector since here all the fermion masses may be produced only by tensors. The masses of the quarks and of the usual leptons can be obtained with the help of the 2-th rank tensors like $\chi$ and $\chi'$ defined above but the masses of the new leptons will be produced by 4-th rank tensors. To give rise to the mass of $x$ we need a tensor of the form $$\phi^{-3}G((\phi^{(2)})^* \otimes (\phi^{(2)})^* \otimes (\phi^{(3)})^* \otimes (\phi^{(3)})^* + (2 \leftrightarrow 3))$$ while for $y$ we can use the tensor resulted from the symmetric direct product of $$((\phi^{(1)})^* \otimes (\phi^{(2)})^* + (1 \leftrightarrow 2))$$ with $$((\phi^{(1)})^* \otimes (\phi^{(3)})^* + (1 \leftrightarrow 3)).$$ What is important to observe here is that our method allows us to give independent mass values to each spinor component, piece by piece. Thus we could take the exotic quark $j$ and the leptons $x$ and $y$ to be very massive.

**VIII. COMMENTS**

We have seen that the "geometrization" of the Higgs mechanism which leads to the mHm offers the possibility to derive a method which should help us to easily analyse the models with high gauge symmetries. This has the advantage to give general solutions to several technical problems related to the model behavior (the breakdown of the gauge symmetry, the definition of the gWt. and of the coupling coefficients, parameterizations, etc.). In its simplest version, presented here, the mHm. has a metric introduced by hand which is suitable for the study of the structure of the gauge boson mass spectrum (as we have seen in the previous section). However, one could reproach that these parameters have not a dynamical origin. This is not an impediment since the presented mHm. can be replaced any time even by a new extended mHm. without a such a metric or by an usual Higgs mechanism able to supply the effects of this metric.

The extended mHm. can be defined by giving up the metric $\eta^2$ and by introducing the new gauge invariant field variables, $\sigma^{(i)}$, which should allow us to modify (27) so that
Now, with an adequate Ld., each field $\sigma^{(i)}$ will gain its own vev. and thus the system of vevs., $<\sigma^{(i)}>$, would play the role of the eliminated metric.

The second option can be realized by taking each $\eta^{(i)}\phi^{(i)}$ as an usual Higgs multiplet (free of constraints) and by introducing the suitable Higgs field variables corresponding to all the tensors involved in the Yukawa couplings in unitary gauge. Of course, new adequate terms in the Higgs Ld. may be also added. After this operation we shall obtain a Higgs mechanism with new mass sources (i.e. the new tensor Higgs multiplets) which could modify the results obtained with the help of mHm.. Therefore, if one desires to conserve these results one may arrange the contribution of the new sources to be very small.

Finally we note that all these mechanisms will give only one mass scale. However, one can imagine extended mHm. producing two mass scales. These may have, in addition, an $n \times n$ matrix, $H$, of scalar fields, like in GUT. To remain in the spirit of our mHms. we could suppose that the fields $\phi^{(i)}$ are just the eigenvectors of this matrix corresponding to the field eigenvalues, $h^{(i)}(x)$. When a such a mechanism with $n - k$ vector multiplets will be introduced in a $SU(n) \otimes U(1)_c$ gauge model it could produce a good $k/(n - k)$ splitting and two mass scales.

**IX. THE SU(N) GENERATORS**

Let us consider the group $SU(n)$ and its algebra, $su(n)$, in the fundamental irep. The real generators, $H^i_j$, defined by

$$ (H^i_j)_{k}^l = \delta^i_k \delta^l_j - \frac{1}{n} \delta^i_j \delta^k_l $$

are $n^2 - 1$ linear independent traceless matrices because of the condition $\sum_i H^i_i = 0$. Their commutation relations are

$$ [H^i_j, H^k_l] = \delta^i_k H^j_l - \delta^j_k H^i_l $$

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Starting with these matrices the hermitian generators $T_a = T_a^+$, $a = 1, 2, \cdots, (n^2 - 1)$ can be constructed \cite{16}. The diagonal ones are of the form:

$$T_{l^2-1} = \frac{1}{\sqrt{2l(l-1)}} \left( \sum_{i=1}^{l-1} H_i^l - (l-1)H_l^1 \right)$$

(95)

for all $l = 2, 3, \cdots, n$. The non-diagonal generators

$$T_{(k^2-1)+2j-1} = \frac{1}{2} \left( H_{k+1}^j + H_j^{k+1} \right)$$

(96)

$$T_{(k^2-1)+2j} = -\frac{i}{2} \left( H_{k+1}^j - H_j^{k+1} \right)$$

are numbered by $j = 1, \cdots, k$ for each $k = 1, 2, \cdots, n-1$. They satisfy the canonical commutation relations:

$$[T_a, T_b] = ic_{ab}^c T_c$$

(97)

where $a, b, c = 1, 2, \cdots, (n^2 - 1)$. The structure constants, $c_{ab}^c = c_{abc}$, are real and completely antisymmetric. Furthermore we have:

$$\{T_a, T_b\} = \frac{1}{n} \delta_{ab} + d_{abc} T_c$$

(98)

and the trace properties:

$$Tr(T_a) = 0$$

$$Tr(T_a T_b) = \frac{1}{2} \delta_{ab}$$

(99)

which give

$$Tr(T_a \{T_b, T_c\}) = \frac{1}{2} d_{abc}$$

(100)

Thereby $d_{abc}$ results to be completely symmetric.

The hermitian generators transform according to the adjoint irep. (Adj) of $SU(n)$:

$$U(\xi)T_a U^+(\xi) = Adj(\xi)_a^b T_b$$

(101)

defined as follows:

$$Adj(\xi) = e^{-ig_{adj}(\xi)}$$

(102)

where $adj(\xi) = \xi^a adj(T_a)$ is of the adjoint irep. of $su(n)$ for which $adj(T_a)_b^c = ic_{abc}$. The corresponding transformation law of the real generators \cite{93} defines the adjoint irep. in the tensor basis (for the multiplets with real components $\psi_i^j$ satisfying $\psi_i^i = 0$).
X. CHIRALITY AND CHARGE CONJUGATION

The Dirac Lagrangian

\[ \mathcal{L}_D = \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \]  \hspace{1cm} (103)

which depends on the Dirac field \( \psi \) and on its Dirac adjoint, \( \bar{\psi} = \psi^+ \gamma^0 \). It is well known that by using the left and the right-handed chiral projections of \( \psi \) (i.e. \( \psi_L = (1 - \gamma^5)\psi/2 \)) and \( \psi_R = (1 + \gamma^5)\psi/2 \) this Ld. can be written as [1, 10]:

\[ \mathcal{L}_D = \frac{i}{2} (\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R \gamma^\mu \partial_\mu \psi_R) - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \]  \hspace{1cm} (104)

Furthermore, we have an important property namely that the charge conjugation, \( \psi \rightarrow \psi^c = C\bar{\psi}^T \), changes the chirality. Indeed, bearing in mind that the matrix \( C = i\gamma^2\gamma^0 \) satisfies \( C = \overline{C} = -C^+ = -C^T = -C^{-1} \) and \( C^{-1}\gamma^\mu C = -\gamma^\mu \), we can verify the following relations:

\[ (\psi_L)^c = \frac{1 + \gamma^5}{2} \psi^c = (\psi_R^c)_R \hspace{0.5cm} (\psi_R)^c = \frac{1 - \gamma^5}{2} \psi^c = (\psi_L^c)_L \]  \hspace{1cm} (105)

Now if we consider that the spinor components anticommute we find that two arbitrary Dirac fields, \( \psi_1 \) and \( \psi_2 \) fulfill:

\[ \bar{\psi}_1 \gamma^\mu \psi_2 = -\bar{\psi}_2 \gamma^\mu \psi_1^c \hspace{0.5cm} \bar{\psi}_1 \psi_2 = \bar{\psi}_2^c \psi_1 \]  \hspace{1cm} (106)

from which we have:

\[ \bar{\psi}_{1R} \gamma^\mu \psi_{2R} = (\bar{\psi}_2^c)_L \bar{\psi}(\psi_1^c)_L \hspace{0.5cm} \bar{\psi}_1 L \psi_{2R} = (\bar{\psi}_2^c)_L (\psi_1^c)_R \]  \hspace{1cm} (107)

These are the basic relations which allow us to bring any spinor sector in the pl. form.

Particularly for only one Dirac field, \( \psi \), we can do that if we put \( L_1 = \psi_L \) and \( L_2 = (\psi_R^c)_c \) so that \( \psi = L_1 + (L_2)^c \) and the Ld. (104) becomes:

\[ \mathcal{L}_D = \frac{i}{2} (\bar{\mathcal{L}}_1 \gamma^\mu \partial_\mu \mathcal{L}_1 + \bar{\mathcal{L}}_2 \gamma^\mu \partial_\mu \mathcal{L}_2) - m(\bar{\mathcal{L}}_1 \mathcal{L}_2 + \bar{\mathcal{L}}_2 \mathcal{L}_1) \]  \hspace{1cm} (108)
In this approach the usual Dirac equations will be: \( i \partial \mathcal{L}_1 = mL_2 \) and \( i \partial \mathcal{L}_2 = mL_1^c \).
The special case is of the (neutral) Majorana field, \( \psi_M = \psi_M^c \). This can be written as
\( \psi_M = L + L^c \) depending on only one left-handed component, \( L \). Then, the corresponding
\( \mathcal{L}_D \) has the form:
\[
\mathcal{L}_D = \frac{i}{2}(\mathcal{L}^\leftrightarrow \partial \mathcal{L}) - \frac{m}{2}(\mathcal{L}L^c + \mathcal{L}^c L)
\]
and the field equation (calculated by using Grassmann derivatives) is: \( i \partial \mathcal{L} = mL^c \). For
\( m = 0 \), (109) gives the familiar \( \mathcal{L}_D \) of the neutrino field.
TABLE I. Reduced matrix elements for the second-rank tensor irreps.

| tensor | scalar | vector | antisymmetric tensor | symmetric tensor | Adjoint |
|--------|--------|--------|----------------------|------------------|---------|
| R:     | $\psi$ | $\psi_i$ | $\psi_{ij} = -\psi_{ji}$ | $\psi_{ij} = \psi_{ij}$ | $\psi^i_j$ |
| $n(R)$ | 1      | $n$    | $n(n-1)/2$           | $n(n+1)/2$      | $n^2 - 1$ |
| $\alpha(R)$ | 0      | 1      | $n - 2$             | $n + 2$         | $2n$    |
| $\beta(R)$      | 0      | 1      | $n - 4$             | $n + 4$         | 0       |
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