UNIVERSAL SPACES FOR ASYMPTOTIC DIMENSION

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Abstract. We construct a universal space for the class of proper metric spaces of bounded geometry and of given asymptotic dimension. As a consequence of this result, we establish coincidence of the asymptotic dimension with the asymptotic inductive dimensions.

1. Introduction

Asymptotic dimension asdim of a metric space was defined by Gromov for studying asymptotic invariants of discrete groups [14]. Then a successful application of the asymptotic dimension was found by G. Yu [19]. He proved the Higher Novikov Signature conjecture for finitely presented groups $\Gamma$ with a finite asymptotic dimension considered as metric spaces with the word metric. The word metric $d_S$ on a group $\Gamma$ depends on a generating set $S \subset \Gamma$. The distance $d_S(x, y)$ in the word metric is the minimal length of presentations of the word $x^{-1}y$ in the alphabet $S$. It turns out that the metric spaces $(\Gamma, d_S)$ and $(\Gamma, d_{S'})$ are coarsely equivalent (see Section 2 for the exact definition) for any two finite generating sets $S$ and $S'$, and hence $\text{asdim}(\Gamma, d_S) = \text{asdim}(\Gamma, d_{S'})$. Thus, in the case of a finitely generated group $\Gamma$, one can speak about its asymptotic dimension $\text{asdim}\, \Gamma$ without refereeing to a generating set. In view of Yu’s theorem, a finite-dimensionality results are of a particular interest. In [14] Gromov proved that all the hyperbolic groups have finite asymptotic dimension. In [11] the finite dimensionality was established for all Coxeter groups. In [2, 3] finite dimensionality theorems were proved for the fundamental group of graphs of groups. In particular, it was shown that the asymptotic finite dimensionality is preserved under the amalgamated product and the HNN-extension. In [3] an upper estimate for asdim of the fundamental group of a graph of groups is given which turns out to be exact in many cases. Strangely, it did not give an exact estimate in the case of the free product which is seemingly the simplest case of the graph of groups. Only recently the exact formula has been established in [4]

1991 Mathematics Subject Classification. 54F45.

Key words and phrases. Asymptotic dimension, universal space, asymptotic inductive dimension.

The paper was written during the second author’s visit to the University of Florida. He thanks the Department of Mathematics for hospitality.
with the use of the inductive dimension asInd introduced in [10]. The equality 
asdim = asInd is proven in the present paper by means of universal spaces.

In the classical dimension theory there are many embedding theorems of different kind. Here we mention three:

1. Nöbeling-Pontryagin Theorem: Every compactum $X$ with $\text{dim} X \leq n$ can be embedded in $\mathbb{R}^{2n+1}$.

2. Bowers Theorem [6]: Every compactum $X$ with $\text{dim} X \leq n$ can be embedded in the product of $n + 1$ dendrites.

3. Lefschetz-Menger Theorem [5]: For every $n$ there exists a universal Menger compactum $\mu^n$, i.e. every compactum $X$ with $\text{dim} X \leq n$ can be embedded in $\mu^n$.

We are interested in the asymptotic analogs of these theorems. The interest is partly motivated by a topological approach to Yu’s theorem taken in [9].

First, we note that in the large scale world there are direct analogs to (1)–(3) for $n = 0$. Here we give a description of a space $M^0$ analogous to the Cantor set $\mu^0$. We recall that the Cantor set is the set of all numbers $0 \leq x \leq 1$ which satisfy the following property:

\[ (*) \quad x \text{ can be written without use of 1 in the ternimal system (based on \{0, 1, 2\}). } \]

Then $M^0$ can be described as the set of all $x \in \mathbb{R}_+$ satisfying $(*)$. Every proper metric space $X$ with asdim $X = 0$ and of bounded geometry (see the definition below) can be embedded in the coarse sense in $M^0$ and hence in $\mathbb{R}$.

The direct analogy between classic and asymptotic dimension theories ends for $n > 0$. It is easy to show that the free group $F_2$ of two generators has dimension asdim $F_2 = 1$. It cannot be embedded into $\mathbb{R}^N$, since $F_2$ has an exponential volume growth. So there is no direct asymptotic analogy to (1). An asymptotic analogy of (2) was found in [9]: Every metric space of bounded geometry can be coarsely embedded in the product of $n + 1$ locally finite trees. Since the above product of trees can be embedded in $(2n + 2)$-dimensional nonpositively curved manifold, it gives some hope for analogy in the case of (1).

The main goal of this paper is an attempt to find a coarse version of the Lefschetz-Menger theorem, i.e. to construct universal spaces for the asymptotic dimension. We obtain a rather partial result (Theorem 3.10): For every $n$ there is a separable metric space $M_n$ with asdim $M_n = n$ which is universal for proper metric spaces $X$ with bounded geometry and with asdim $X \leq n$. Our universal space is neither proper nor of bounded geometry.

The construction of the space $M_n$ is analogous to those for the Menger and Nöbeling spaces. Besides the asymptotic strategy, the main difference is that we are building our fractals not in $\mathbb{R}^{2n+1}$ as in the classical case but in the product of trees. If one accomplish the local (classical) construction in the product of (finite) trees or even dendrites he will get the Menger space according to Bestvina’s criterion [5]. We construct our spaces $M_n$ out of product of locally finite trees. For locally finite trees this construction works even better but we have problems
Lipschitz; the 1-Lipschitz maps are called map is a map which is \((Lipschitz)\) bounded geometry can be embedded in the large scale sense.

turns out, however, (see Theorem 3.15) that there is no proper metric space of finite asymptotic dimension into which every asymptotically 1-dimensional space of bounded geometry into products of \(n+1\) nonpositively curved surfaces can be strengthened so that one can take the hyperbolic planes as these manifolds. It turns out, however, (see Theorem 3.15) that there is no proper metric space of finite asymptotic dimension into which every asymptotically 1-dimensional space of bounded geometry can be embedded in the large scale sense.

2. Preliminaries

The generic metric will be denoted by \(d\). If \(A\) is a subset of a metric space \(X\) and \(r \in \mathbb{R}\), then \(N_r(A)\) is defined as \(\{x \in X \mid d(x, A) < r\}\) if \(r > 0\) and \(A \setminus \{x \in X \mid d(x, A) < -r\}\) otherwise. The closed \(r\)-ball centered at \(x \in X\) will be denoted by \(B_r(x)\).

A map \(f: X \to Y\) of metric is called \((\lambda, s)\)-Lipschitz if \(d(f(x), f(y)) \leq \lambda d(x, y) + s\) for every \(x, y \in X\). The \((\lambda, 0)\)-Lipschitz maps are also called \(\lambda\)-Lipschitz; the 1-Lipschitz maps are called Lipschitz or short. An asymptotically Lipschitz map is a map which is \((\lambda, s)\)-Lipschitz for some \(\lambda > 0\), \(s > 0\).

For a cover \(\mathcal{U}\) of a metric space \(X\), we denote by \(L(\mathcal{U})\) its Lebesgue number, \(L(\mathcal{U}) = \inf \{\sup \{d(x, X \setminus U) \mid U \in \mathcal{U}\} \mid x \in X\}\), and for a family \(\mathcal{U}\) of subsets of a metric space we denote by \(mesh(\mathcal{U})\) the lowest upper bound of the diameters of the elements of \(\mathcal{U}\). A family \(\mathcal{A}\) of subsets of a metric space is called uniformly bounded if there exists a number \(C > 0\) such that \(diam(A) < C\) for every \(A \in \mathcal{A}\).

For \(r > 0\), a metric space \(X\) is called \(r\)-discrete if \(d(a, b) \geq r\) for every \(a, b \in X\), \(a \neq b\). A metric space \(X\) is called discrete if \(X\) is \(r\)-discrete for some \(r > 0\).

For \(r > 0\), the \(r\)-capacity of a subset \(Y\) of a metric space is the maximal cardinality \(K_r(Y)\) of an \(r\)-discrete subset of \(Y\). A metric space \(X\) is of bounded geometry if there exists a number \(r > 0\) and a function \(c: [0, \infty) \to [0, \infty)\) such that the \(r\)-capacity of every \(\varepsilon\)-ball \(B_{r}(x)\) does not exceed \(c(\varepsilon)\).

A map \(f: X \to Y\) between metric spaces is called uniformly cobounded if for every \(r > 0\) there exists \(C > 0\) such that for every \(y \in Y\) the diameter of the set \(B_{r}(y)\) does not exceed \(C\).

A space \(X\) is said to be a geodesic metric space if for every two points \(x, y \in X\) there is an isometric embedding \(j: [0, d(x, y)] \to X\) such that \(j(0) = x\) and \(j(d(x, y)) = y\).
A geodesic segment that connects points \( x, y \) in a geodesic metric space will be denoted by \([x, y]\).

A metric space is called uniformly arcwise connected if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every two points \( x, y \in X \) with \( d(x, y) < \varepsilon \) there exists a path in \( X \) connecting \( x \) and \( y \) and of diameter \( \leq \delta \).

The (finite) products of metric spaces are always endowed with the sup-metric.

Let \( C \) a decomposition of a proper metric space \( X \). Define the quotient pseudo-metric \( \rho \) on \( X \) by the following rule: \( \rho(x, y) \) is the greatest lower bound of the sums of the form \( \sum_{i=1}^{k-1} d(x_{2i}, x_{2i+1}) \), where \( x = x_1, y = x_{2k} \), and for every \( i = 1, \ldots, k \) there exists an element \( C_i \in C \) such that \( \{x_{2i-1}, x_{2i}\} \subset C_i \). Obviously, the identity map \( \text{id}: (X, d) \to (X, \rho) \) is short.

A map \( f: X \to Y \) between metric spaces is called coarse uniform if for every \( C > 0 \) there is \( K > 0 \) such that for every \( x, x' \in X \) with \( d(x, x') < C \) we have \( d(f(x), f(x')) < K \). A map \( f: X \to Y \) is called metric proper if the preimage \( f^{-1}(B) \) is bounded for every bounded set \( B \subset Y \). A map is coarse if it is both coarse uniform and metric proper. Two maps \( f, g: X \to Y \) between metric spaces are called bornotopic (see [17]) if there is \( C > 0 \) such that \( d(f(x), g(x)) < C \) for every \( x, y \in X \). Two metric space \( X \) and \( Y \) are coarse equivalent if there are coarse maps \( f: X \to Y, g: Y \to X \) such that the compositions \( fg \) and \( gf \) are bornotopic to the corresponding identity maps.

**Lemma 2.1.** [8] Let \( X \) be a geodesic metric space and \( f: X \to Y \) be a coarse uniform map. Then \( f \) is an asymptotically Lipschitz map.

We will need the following simple result whose proof mimicks that of Proposition 2 from [10].

**Lemma 2.2.** For every proper metric space \( X \) and every \( C > 0 \) there is a \( C \)-discrete subset \( Y \) of \( X \) such that \( d(x, Y) \leq C \) for every \( x \in X \).

A map \( f: X \to Y \) of metric spaces is called a large scale embedding (see [17]) if there exist increasing functions \( \varphi_1, \varphi_2: [0, \infty) \to [0, \infty) \) with \( \lim_{t \to \infty} \varphi_1(t) = \lim_{t \to \infty} \varphi_2(t) = \infty \) such that \( \varphi_1(d(x, y)) \leq d(f(x), f(y)) \leq \varphi_2(d(x, y)) \) for every \( x, y \in X \). It is easy to see that every metric space is coarsely equivalent to its image under a coarse embedding.

2.1. **Asymptotic dimension.** The notion of asymptotic dimension is introduced by Gromov [14].

**Definition 2.3.** The asymptotic dimension of a metric space \( X \) does not exceed \( n \) (written as \( \text{asdim} \ X \leq n \)) if for every \( D > 0 \) there exists a uniformly bounded cover \( \mathcal{U} \) of \( X \) such that \( \mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n \), where all \( \mathcal{U}^i \) are \( D \)-disjoint.
A piecewise-euclidean $n$-dimensional complex of mesh $D$ is a complex which is a union of isometric copies of the standard $n$-dimensional symplex of mesh $D$ in $\mathbb{R}^{n+1}$.

**Lemma 2.4.** For a proper metric space $X$ the following two conditions are equivalent:

1) $\operatorname{asdim} X \leq n$;

2) for any $D > 0$ there is a uniformly cobounded short proper map of $X$ into a piecewise-euclidean complex of mesh $D$ and dimension $n$.

**Proof.** See [14]. □

A family of metric spaces $X_\alpha$ satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ uniformly (see [2]) if for arbitrary large $D > 0$ there exists $R > 0$ and $R$-bounded $D$-disjoint families $U_\alpha^0, \ldots, U_\alpha^n$ such that $\bigcup_{i=0}^n U_\alpha^i$ is a cover of $X_\alpha$.

**Theorem 2.5.** [2] Assume that $X = \bigcup_\alpha X_\alpha$ and $\operatorname{asdim} X_\alpha$ uniformly. Suppose that for any $R$ there is $Y_R \subset X$ with $\operatorname{asdim} Y_R \leq n$ and such that the family $\{X_\alpha \setminus Y_R\}$ is $R$-disjoint. Then $\operatorname{asdim} X \leq n$.

3. **Asymptotic embeddings into product of trees**

### 3.1. Trees

A geodesic metric space $T$ is called an $\mathbb{R}$-tree (a real tree) if (1) every two points in $T$ are connected by a unique geodesic segment and (2) if $[x, y] \cap [y, z] = \{y\}$ then $[x, y] \cup [y, z] = [x, z]$ for all $x, y, z \in T$.

Recall that in this definition $[a, b]$ stands for a geodesic segment connecting $a$ and $b$. We will also use self-explaining notations $[a, b)$ and $(a, b]$ for (half)open geodesic segments.

Every tree (connected connected acyclic graph) is an $\mathbb{R}$-tree. We assume that the graphs are endowed with a geodesic metric whose restriction to every edge is isometric to the standard unit segment.

A (half)open segment in $T$ is *free* if it is an open subset of $T$.

The *mesh* of $T$ is the greatest lower bound of the diameters of the maximal (with respect to the inclusion) free (half)open segments in $T$.

**Definition 3.1.** An $\mathbb{R}$-tree $T$ is said to be *regular* if there exists a sequence of $\mathbb{R}$-trees $T = T_0 \supset T_1 \supset T_2 \supset \ldots$ with the following properties:

1) $\cap_{i=0}^{\infty} T_i = \emptyset$;

2) for every $i$, the set $T_i \setminus T_{i+1}$ is a disjoint union of a uniformly bounded family of maximal free in $T_i$ half-open segments;

3) $D_0 < D_1 < D_2 < \ldots$ and $\lim_{i \to \infty} D_i = \infty$, where $D_i$ denotes the mesh of the $\mathbb{R}$-tree $T_i$. 
If in the above definition $D_i = 2^i$, we say that $T$ is binary regular.

Suppose that $T^{(0)}, \ldots, T^{(n)}$ is a sequence of regular $\mathbb{R}$-trees. For every $j$, a sequence of $\mathbb{R}$-trees $T^{(j)} = T_0^{(j)} \supset T_1^{(j)} \supset T_2^{(j)} \supset \ldots$ is given such that properties 1)–3) from Definition 3.1 are satisfied with $T_i = T_i^{(j)}$. For every $i$ and $j$ denote by $r_i^{(j)}: T^{(j)} \to T_i^{(j)}$ the retraction that maps every component $C$ of the set $T^{(j)} \setminus T_i^{(j)}$ onto its boundary $\partial C$ (which is a singleton). We will refer to these retractions as the canonical retractions. For $k \leq i$ we have $r_i^{(j)} = r_k^{(j)} r_i^{(j)}$. Let

$$M_j = \left\{ (x_0, \ldots, x_n) \in \prod_{i=0}^n T^{(i)} \mid \text{there exists } i \text{ such that } r_i(x_i) \in \partial(T_{j-1}^{(i)} \setminus T_j^{(i)}) \right\}$$

and $M = M(T^{(0)}, \ldots, T^{(n)}) = \bigcap_{j=1}^\infty M_j$.

Denote by $\mathcal{M}_n$ the class of spaces of the form $M(T^{(0)}, \ldots, T^{(n)})$ for some sequence $T^{(0)}, \ldots, T^{(n)}$ of regular $\mathbb{R}$-trees.

Suppose that $T$, $S$ are regular $\mathbb{R}$-trees given with filtrations $(T_j)_{j=0}^\infty$, $(S_j)_{j=0}^\infty$ satisfying the conditions from Definition 3.1. We denote by $r_j: T \to T_j$, $g_j: S \to S_j$ the canonical retractions.

**Definition 3.2.** A map $f: T \to S$ is said to be regular if the following conditions are satisfied:

1. $f(T_j) \subset S_j$ for every $j$;
2. $f(r_j(\partial(T_{j-1} \setminus T_j))) \subset g_j(\partial(S_{j-1} \setminus S_j))$ for every $j$.

The construction $M(T^{(0)}, \ldots, T^{(n)})$ is functorial in the following sense. Suppose that $T^i$, $S^i$ are regular $\mathbb{R}$-trees given with filtrations $(T_j^i)_{j=0}^\infty$, $(S_j^i)_{j=0}^\infty$, $i \in \{0, 1, \ldots, n\}$ satisfying the conditions from Definition 3.1. We denote by $r_j^i: T^i \to T_j^i$, $g_j^i: S^i \to S_j^i$ the canonical retractions. Suppose that $f_i: T^i \to S^i$ are regular maps.
Lemma 3.3. Under these conditions
\[ (f_0 \times \cdots \times f_n)(M(T^{(0)}, \ldots, T^{(n)})) \subset M(S^{(0)}, \ldots, S^{(n)}). \]

Proof. Obvious. \hfill \Box

Proposition 3.4. For every \( M \in \mathcal{M}_n \) we have \( \text{asdim} M \leq n \).

Proof. We apply Lemma 2.4. Note that for every \( j \) the map
\[ r_j = \prod_{i=0}^{n} r_{j}^{(i)} : \prod_{i=0}^{n} T^{(i)} \to \prod_{i=0}^{n} T^{(i)} \]
maps \( M \) onto an \( n \)-dimensional piecewise-euclidean polyhedron of mesh \( D_j \).

Since \( r_j \) is uniformly cobounded, we are done. \hfill \Box

Theorem 3.5. For every proper metric space \( X \) of \( \text{asdim} X \leq n \) there exist locally finite binary regular trees \( T^0, \ldots, T^n \) such that \( X \) is large scale embeddable into \( M(T^0, \ldots, T^n) \).

The proof is a modification of the corresponding result from [8].

Proposition 3.6. Let \( X \) be a proper metric space with base point \( x_0 \) and \( \text{asdim} X \leq n \). Then there exists a sequence \( (\mathcal{U}_k)_{k=1}^{\infty} \) of open covers of \( X \) such that every \( \mathcal{U}_k \) splits into the union \( \mathcal{U}_k = \mathcal{U}_k^0 \cup \cdots \cup \mathcal{U}_k^n \) of \( d_k \)-disjoint families and the following conditions are satisfied:

1) \( L(\mathcal{U}_k) > d_k \) and for every \( i \) and every \( U \in \mathcal{U}_k^i \) we have \( N_{-d_k}(U) \neq \emptyset \);
2) \( d_{k+1} = 2^{k+2}m_k \), where \( m_k \) is the mesh of \( \mathcal{U}_k \);
3) for every \( m \in \mathbb{N} \) and for every \( i \) there is \( k \in \mathbb{N} \) and an element \( U \in \mathcal{U}_k^i \) such that \( N_m(x_0) \subset U \);
4) for every \( k, l, k < l \) and every \( U \in \mathcal{U}_k^i, V \in \mathcal{U}_l^i \), if \( U \not\subset V \), then \( d(U, V) \geq d_k/2 \).

Proof. We start with repeating the construction from the proof of Proposition 1 from [9]. We proceed by induction. Let \( \hat{\mathcal{U}}_0 \) be an open cover of \( X \) which is \( \hat{d}_0 \)-discrete with \( \hat{d}_0 > 2 \). We enumerate the partition \( \hat{\mathcal{U}}_0 = \hat{\mathcal{U}}_0^0 \cup \cdots \cup \hat{\mathcal{U}}_0^n \) in such a way that \( d(x_0, X \setminus U) > \hat{d}_0 \) for some \( U \in \hat{\mathcal{U}}_0^0 \).

Assume that families \( \hat{\mathcal{U}}_k \) are constructed for all \( k \leq l \). Define \( d_{l+1} = 2^{l+2}m_l \) and consider a uniformly bounded cover \( \tilde{\mathcal{U}}_{l+1} \) with \( L(\tilde{\mathcal{U}}_{l+1}) > 2d_{l+1} \) and such that there is a splitting \( \tilde{\mathcal{U}}_{l+1} = \tilde{\mathcal{U}}_{l+1}^0 \cup \cdots \cup \tilde{\mathcal{U}}_{l+1}^n \), where \( \tilde{\mathcal{U}}_{l+1}^i \) are \( d_{l+1} \)-disjoint and \( N_{-2d_{l+1}}(U) \neq \emptyset \) for all \( U \in \tilde{\mathcal{U}}_{l+1} \). The families \( \tilde{\mathcal{U}}_{l+1}^0, \ldots, \tilde{\mathcal{U}}_{l+1}^n \) are enumerated in such a way that \( d(x_0, X \setminus U) > 2d_{l+1} \) for some \( U \in \tilde{\mathcal{U}}_{l+1}^0 \), where \( i = l + 1 \mod n + 1 \). For every \( U \in \tilde{\mathcal{U}}_{l+1} \) let
\[ \tilde{U} = U \setminus \{ N_i(V) \mid V \in \mathcal{U}_k^i, k \leq l, V \not\subset U \} \]
and \( \mathcal{U}_{l+1}^i = \{ \tilde{U} \mid U \in \mathcal{U}_{l+1}^i \} \).

It is proved in [9] that the sequence \( (\hat{\mathcal{U}}_k) \) satisfies the following properties:
1) $L(\hat{U}_k) > d_k$ and for every $i$ and every $U \in \hat{U}_k^i$ we have $N_{-d_k}(U) \neq \emptyset$;
2) $d_{k+1} = 2^k m_k$, where $m_k$ is the mesh of $\hat{U}_k$;
3) for every $m \in \mathbb{N}$ and for every $i$ there is $k \in \mathbb{N}$ and an element $U \in \hat{U}_k^i$ such that $N_m(x_0) \subseteq U$;
4) for every $U \in \hat{U}_k^i$, $V \in \hat{U}_i^q$, $U \not\subseteq V$ we have $d(U,V) \geq 4$.

We are going to modify the sequence $(\hat{U}_k)_{k=1}^\infty$. Passing, if necessary, to a subsequence of the sequence $(\hat{U}_k)_{k=1}^\infty$ we may assume that the sequence $(\hat{U}_k)_{k=1}^\infty$ itself satisfies conditions 1), 3), 4) and the following condition:

2*) $d_{k+1} \geq 2^{4k+6} m_k$.

Define, by induction, numbers $\tilde{d}_k$ and families $\tilde{U}_k^i$. Let $\tilde{d}_0 = d_0$ and $\tilde{U}_0 = U_0$ and suppose $\tilde{d}_j$ and $\tilde{U}_j$ have been defined for every $j < k$. For every $U \in \hat{U}_k^i$ apply induction by $p$ to define the sets $\tilde{U}(p)$. Let $\tilde{U}(0) = \hat{U}$ and suppose sets $\tilde{U}(p)$ have been defined for all $p < q$, for some $q \leq k$. Let

$$\tilde{U}(q) = \tilde{U}(q-1) \cup \bigcup \{ V \in \tilde{U}_q^i \mid d(\tilde{U}(q-1),V) < \tilde{d}_q/2 \}.$$  

Then, by the definition, $\tilde{U} = \tilde{U}(k-1)$ and $\tilde{U}_k^i = \{ \tilde{U} \mid U \in \hat{U}_k^i \}$.

We are going to verify conditions 1)–4) (with $U_k$, $d_k$, and $m_k$ replaced by $\tilde{U}_k$, $\tilde{d}_k$, and $\tilde{m}_k$ respectively). Conditions 1) and 3) are obvious.

Denote by $\tilde{m}_k$ the mesh of $\tilde{U}_k$ and let

$$(3.1) \quad \tilde{d}_k = d_k - 2^{k+2} \tilde{m}_{k-1}.$$  

Note that for every $\tilde{U}, \tilde{V} \in \tilde{U}_k^i$, $\tilde{U} \not\subseteq \tilde{V}$ we have

$$d(\tilde{U}, \tilde{V}) = d(\tilde{U}(k-1), \tilde{V}(k-1)) \geq d(\tilde{U}(k-2), \tilde{V}(k-2)) - \tilde{d}_{k-1} - 2\tilde{m}_{k-1}$$
$$\geq d(\tilde{U}(k-2), \tilde{V}(k-2)) - 3\tilde{m}_{k-1}$$
$$\geq d(\tilde{U}(k-3), \tilde{V}(k-3)) - 3\tilde{m}_{k-1} - 3\tilde{m}_{k-2}$$
$$\cdots$$
$$\geq d(U,V) - 3 \sum_{i=0}^{k-1} \tilde{m}_i \geq d_k - 3km_{k-1} \geq \tilde{d}_k,$$

and therefore the family $\tilde{U}_k^i$ is $\tilde{d}_k$-disjoint.

Condition 2). Let $\tilde{U} \in \tilde{U}_k$, then $\tilde{U} = \tilde{U}(k-1)$ and

$$\diam \tilde{U} \leq \diam \tilde{U}(k-2) + 2\tilde{m}_{k-1} + \tilde{d}_{k-1} \leq \diam \tilde{U}(k-2) + 3\tilde{m}_{k-1}$$
$$\cdots$$
$$\leq \diam \tilde{U}(0) + 3 \sum_{j=0}^{k-1} \tilde{m}_j \leq m_k + 3k\tilde{m}_{k-1} \leq m_k + 2^{k+2} \tilde{m}_{k-1}.$$
(here we used an obvious equality \( m_j \geq d_j \), for every \( j \)), whence
\[
(3.2) \quad \tilde{m}_k \leq m_k + 2^{k+2} \tilde{m}_{k-1}.
\]
Then
\[
\tilde{m}_k \leq m_k \sum_{j=0}^{k-1} 2^{j(k+2)} \leq m_k 2^{2k+3}
\]
and using condition 2\(^*\) we obtain
\[
(3.3) \quad d_k = d_k - 2^{k+2} \tilde{m}_{k-1} \geq d_k - 2^{2k+3} m_{k-1} \\
\geq 2^{4k+6} \tilde{m}_{k-1} - 2^{2k+3} m_{k-1} \geq 2^{4k+5} m_{k-1} \\
\geq 2^{2k+2} \tilde{m}_{k-1}.
\]

Let us verify condition 4\(^\circ\)). Suppose the contrary. Then there exist \( \tilde{U} \in \tilde{U}_k^i \), \( \tilde{V} \in \tilde{U}_k^i \), \( \tilde{U} \not\subset \tilde{V} \), and \( k \leq l \) such that \( d(\tilde{U}, \tilde{V}) < d_k / 2 \). We also suppose that \( l \) is the minimal number with that property. Since \( d(\tilde{U}, \tilde{V}) < d_k / 2 \), there exists \( x \in \tilde{V} \) such that \( d(x, \tilde{U}) < d_k / 2 \). Let \( j \geq 0 \) be the minimal integer with the property \( x \in \tilde{V}(j) \). It follows from the definition of \( \tilde{U}_k^i \) that \( j \leq l - 1 \). Since \( x \in \tilde{V}(j) \setminus \tilde{V}(j - 1) \), it follows from the definition of \( \tilde{V}(j) \) that there exists \( \tilde{W} \in \tilde{U}_k^i \) such that \( x \in \tilde{W} \). Since \( j < l \), this contradicts to the choice of \( l \).

\[ \square \]

For every element \( U \in U_i \) denote by \( \psi(U) \) the minimal (with respect to inclusion) \( V \in U_i \) such that \( U \) is a proper subset of \( \psi(U) \).

**Proposition 3.7.** Let \( X \) be a proper metric space and \( (U_k) \) a sequence of open covers of \( X \) satisfying properties 1\(^\circ\)–4\(^\circ\)) from Proposition 3.6. Let \( U \in U_k \) and \( \varrho \) be the quotient metric on \( U \) with respect to the decomposition of \( U \) into singletons and the elements of the family \( V = \psi^{-1}(U) \). For every \( V \in \mathcal{V} \cap U_k^i \), \( W \in \mathcal{V} \cap U_k^i \), where \( j \leq l \), we have \( \varrho(V, W) \geq 2^{2j} \).

**Proof.** The assertion is obvious for \( j = 0 \). Let \( V \in \mathcal{V} \cap U_k^j \), \( W \in \mathcal{V} \cap U_k^i \), where \( 1 \leq j \leq l \). Then \( \varrho(V, W) \) is the greatest lower bound of the sums of the form \( \sum_{i=1}^{p-1} d(x_{2i}, x_{2i+1}) \), where \( x \in V \), \( x_{2p} \in W \), and for every \( i = 1, \ldots, p \) there exists an element \( C_i \in \psi^{-1}(U) \) such that \( \{x_{2i-1}, x_{2i}\} \subset C_i \). Without loss of generality, we may assume that in the sums above \( C_i \in \psi^{-1}(U) \setminus \mathcal{V}_j \) for all \( j < p \). Then \( (p-2)m_{j-1} + (p-1)d_{j-1} \geq d_j \) and therefore, by condition 2\(^\circ\) of Proposition 3.6,
\[
p - 1 \geq \frac{d_j - d_{j-1}}{m_{j-1} + d_{j-1}} \geq \frac{d_j}{4m_{j-1}} \geq 2^{2j}
\]
and \( \sum_{i=1}^{p-1} d(x_{2i}, x_{2i+1}) \geq (p-1)d_0 \geq 2^{2j} \). Hence \( \varrho(V, W) \geq 2^{2j} \). \[ \square \]

**Proposition 3.8.** Let \( X \), \( U \), and \( \varrho \) be as in Proposition 3.7. There exists \( x \in U \) with \( \varrho(x, X \setminus U) \geq 2^{2k} \).
Proof. By condition 1) of Proposition 3.6, there exists \( x \in U \) with \( d(x, \partial U) \geq d_k \).

Arguing like in the proof of Proposition 3.7 we conclude that \( \varrho(x, X \setminus U) \geq 2^{2^k} \).

Fix a sequence \( (\mathcal{U}_k) \) of open covers of \( X \) satisfying properties 1)–4) from Proposition 3.6. Given \( U \in \mathcal{U}_k \) define a Lipschitz map \( \varphi_U : \bar{U} \rightarrow I_U = [0, 2^{2^k}] \) by the following procedure. Let \( \varrho \) denote the pseudometric generated by the decomposition of \( U \) into singletons and the elements of the family \( \psi^{-1}(U) \) and let \( \mathcal{V}_j = (\{ V \in \mathcal{U}_l \mid j \leq l < k \}) \cap \psi^{-1}(U) \).

We are going to define by induction with respect to \( j \) maps \( \vartheta_j : \partial U \cup \bigcup \mathcal{V}_j \rightarrow I_U \). Define the map \( \vartheta_j : \partial U \cup \bigcup \mathcal{V}_{j-1} \rightarrow I_U \) by the formula \( \vartheta_j(x) = 2^{-k} \varrho(x, X \setminus U) \).

Let \( \vartheta_1(x) = 2^{-k-1}[2^{-k+1} \vartheta_1(x)] \). The map \( \vartheta_1 \) is a 2\(^{-k+1}\)-Lipschitz map into \( I_U \) defined on the closed subset \( \partial U \cup \bigcup \mathcal{V}_{k-1} \) of \( \bar{U} \).

Suppose the maps \( \vartheta_j, \vartheta_j \) are defined for all \( j \), \( j_0 < j < k \) and the maps \( \vartheta_j \) are 2\(^{-j}\)-Lipschitz. By the theorem on extension of Lipschitz maps (see [8]) there exists a 2\(^{-j_0+1}\)-Lipschitz extension of \( \vartheta_{j_0-1} \) onto the set \( \partial U \cup \bigcup \mathcal{V}_{j_0} \). Let \( \vartheta_{j_0}(x) = 2^{j_0}[2^{-j_0} \vartheta_{j_0}(x)] \).

As a final result, we obtain a 1-Lipschitz map

\[
\vartheta_0 : \partial U \cup \bigcup \mathcal{V}_0 = \partial U \cup \bigcup \psi^{-1}(U) \rightarrow I_U.
\]

Denote by \( \varphi_U \) its 1-Lipschitz extension onto \( \bar{U} \).

We will follow [9] and construct a locally finite tree \( T^i \) by the following procedure. Assign to every \( U \in \mathcal{U}_k \), \( k = 0, 1, \ldots, \) a line segment \( I_U = [0, 2^k] \). The tree \( T^i \) will be the quotient space of the disjoint union \( \sqcup \{ I_U \mid U \in \cup_{k=0}^\infty \mathcal{U}_k \} \) with respect to the following equivalence relation \( \sim \): suppose \( U \in \mathcal{U}_k \), \( V \in \mathcal{U}_l \), where \( k < l \) and \( U \in \psi^{-1}(V) \). Then \( 0 \in I_U \) is identified with \( f_V(\partial U) \in I_V \).

We argue like in the proof of Theorem 3 from [9] to show that the quotient space \( \sqcup \{ I_U \mid U \in \cup_{k=0}^\infty \mathcal{U}_k \} / \sim \) is a tree. We are going to show that \( T^i \) is locally finite. Assume the contrary and let \( y \) be a vertex of infinite order in \( T^i \). Then there exists an infinite family \( \mathcal{V} \subset \mathcal{U} \) such that \( y = q(0_U) \) for every \( U \in \mathcal{V} \). Since the set \( \psi^{-1}(U) \) is finite for every \( U \in \mathcal{U} \), we conclude that there exists an increasing sequence \((k(j)))_{j=1}^\infty \) of positive integers and sets \( U_{k(j)} \in \mathcal{U}_{k(j)}^i \) such that \( U_{k(j)} \in \psi^{-1}U_{k(j+1)} \) and \( f_{U_{k(j+1)}}(U_{k(j)}) = 0 \). By conditions 3) and 4), there exists \( j > 1 \) such that \( x_0 \in U_{k(j)} \) and \( U_{k(1)} \subset U_{k(j)} \). Then, by condition 3), \( U_{k(j+1)} \supset N_{m_{k(j)}+2^k(j)+2d_k}(x_0) \) and, since \( \text{diam}(U_{k(j)}) \leq m_{k(j)} \), by condition 3), \( \text{d}(U_{k(j)}, X \setminus U_{k(j+1)}) \geq 2^k(j)+2d_k \) and, therefore, by Proposition 3.8, \( \varrho(U_{k(j)}, X \setminus U_{k(j+1)}) \geq 2^{k(j)} \). Hence \( f_{U_{k(j+1)}}(U_{k(j)}) \geq 1 \neq 0 \) and we obtain a contradiction.

Let \( q : \sqcup \{ I_U \mid U \in \cup_{k=0}^\infty \mathcal{U}_k \} \rightarrow T^i \) be the quotient map. Define the map \( f = f^i : X \rightarrow T^i \) as follows. Let \( x \in X \). By condition 3) of Proposition 3.6,
there exists minimal $k$ such that $x \in U$ for some $U \in \mathcal{U}_i^k$. Put $f^i(x) = q f_U(x)$. It can be easily seen that $f^i$ is well-defined and is Lipschitz.

The diagonal map $f = (f^i)_{i=0}^n: X \to \prod_{i=0}^n T^i$ is Lipschitz. Show that $f$ is a large-scale embedding. Assuming the contrary we can find sequences $(x_i)$ and $(y_i)$ in $X$ such that $d(x_i, y_i) \to \infty$ while $d(f(x_i), f(y_i)) < C$ for some $C > 0$. Let $k$ be a positive integer such that $2^k > C$. There exists an integer $l$ such that $d(x, y) > m_k$. There exists $i \in \{0, \ldots, n\}$ such that $x_l \in U$ and $d(x, X \setminus U) > d_k$ for some $U \in \mathcal{U}_i^k$. Then $f_U(x) = 2^k$, By Proposition 3.8. Since $d(x, y) > m_k$, we see that $y \notin U$ and $f_U(y) = 0$. Thus, $d(f(x), f(y)) \geq 2^k > C$, which gives a contradiction.

Note that every $T^i$ is a regular tree. Indeed, for every $j$ denote by $T^i_j$ the subspace $q(\bigcup \{ I_U \mid U \in \bigcup_{k=j}^{\infty} \mathcal{U}_k^i \})$. Let us verify conditions 1)--3) of Definition 3.1 (with $T$ and $T^j_i$ replaced by $T$ and $T^j_i$ respectively). Condition 1) is obvious. The set $T^i_{j+1} \setminus T^i_j$ is a disjoint union of isometric copies of the half-open segment $(0, 2^{i+1}]$ and is therefore uniformly bounded. To verify condition 3) note that the mesh of the tree $T^i_j$ is $2^i$, by the construction.

The trees $T^0, \ldots, T^n$ together with filtrations $T^0 \supset T^1 \supset T^2 \supset \ldots$ determine the subspace $M(T^0, \ldots, T^n)$ of $\prod_{i=1}^n T^i$. We are going to show that $f(X) \subset M(T^0, \ldots, T^n)$. Suppose $x \in X$. For every $k = 0, 1, \ldots$ there is $i(k) \in \{0, 1, \ldots, n\}$ such that $x \in U$ for some $U \in \mathcal{U}_k^{i(k)}$. Then $r^{i(k)}_{k+1}(f(x)) \in \partial T^{i(k)}_{k+1} \setminus T^{i(k)}_k$, i. e. $f(x) \in M_{k+1}$. Therefore, $f(x) \in M_{k+1}$ for every $k$, i. e. $f(x) \in M(T^0, \ldots, T^n)$.

3.2. Universal space. A class of metric spaces $\mathcal{C}$ is said to be universal for a class of metric spaces $\mathcal{D}$ if for every $D \in \mathcal{D}$ there is a large scale embedding of $C$ into some $C \in \mathcal{C}$.

Theorem 3.5 can be reformulated as follows.

**Theorem 3.9.** The class $\mathcal{M}_n$ is universal for the class of proper metric spaces of asymptotic dimension $\leq n$.

A metric space $X$ is said to be universal for a class of metric spaces $\mathcal{D}$ if for every $D \in \mathcal{D}$ there is a large scale embedding of $D$ into $X$.

Construct an $\mathbb{R}$-tree $T$ by the following procedure. Define inductively a sequence $T_i$ of $\mathbb{R}$-trees and retractions $r_i: T_i \to T_0$. Let $T_0 = \mathbb{R}_+$. Denote by $T_i$ an $\mathbb{R}$-tree obtained by attaching to every integer point in $\mathbb{R}_+$ a countable set of isometric copies of the unit segment $[0, 1]$. Denote by $r_1$ a retraction of $T_1$ onto $T_0$ that sends every component $C$ which is constant on every component of the complement $T_1 \setminus T_0$. Suppose that $\mathbb{R}$-trees $T_i$ and retractions $r_i$ are defined for all $i < j$. Denote by $S$ the subtree $r_1^{-1}([0, 2^j])$ with base point $2^j$. The $\mathbb{R}$-tree $T_j$ is obtained from $T_{j-1}$ by attaching to every point of the form $k2^j$, $k \in \mathbb{N}$, of $\mathbb{R}_+ \subset T_{j-1}$ a countable family of isometric copies of $S$ by the base point.
Denote by $r_j$ a retraction of $T_j$ onto $T_0$ that sends every component $C$ which is constant on every component of the complement $T_j \setminus T_0$.

Let $T = \bigcup_{i=1}^{\infty} T_i$. It is easy to see that $T$ is a regular $\mathbb{R}$-tree that contains every locally finite binary regular tree so that the inclusion preserves the filtration. By Lemma 3.3, we conclude that for every sequence $T^0, T^1, \ldots, T^n$ of locally finite binary regular trees we have

$$M(T^0, T^1, \ldots, T^n) \subset M_n = M(S^0, S^1, \ldots, S^n),$$

where $S^i$ is isomorphic to $T$ for every $i = 0, 1, \ldots, n$.

We therefore obtain the following

**Theorem 3.10.** There exists a separable metric space $M_n$ with $asdim M_n = n$ universal for the class of proper metric spaces $X$ with $asdim X \leq n$.

For the spaces of asymptotic dimension zero the result can be improved.

**Theorem 3.11.** There exists a proper metric space $M_0$ of bounded geometry with $asdim M_0 = 0$ universal for the class of proper metric spaces $X$ of bounded geometry with $asdim X \leq 0$.

**Proof.** Let $M_0$ denote the set of integers whose ternary expansion consists only from 0s and 1s. Obviously, $asdim M_0 = 0$ and $M_0$ is of bounded geometry. For every $k \in \mathbb{N}$ the set $M_0$ can be represented as the union of $3^k$-discrete family of the intervals of length $3^k$ in the set of natural numbers.

Let $X$ be a proper metric space of bounded geometry with $asdim X \leq 0$. Let $x_0 \in X$ be a base point. There exists a sequence $(U_k)$ of uniformly bounded covers of $X$ with the following properties:

1. $U_k$ refines $U_{k+1}$;
2. $U_k$ is $k$-discrete;
3. $\bigcup\{U \in \bigcup k U_k \mid x_0 \in U\} = X$.

For $U \in U_k$, denote by $\psi(U)$ the unique $V \in U_{k+1}$ that contains $U$.

Since $X$ is of bounded geometry, for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that $|\psi(U)| \leq C_k$ for every $U \in U_{k+1}$.

Using the mentioned decomposition of $M_0$ into the union of intervals, we can easily construct a sequence of covers $(V_k)$ satisfying the properties:

4. $V_k$ refines $V_{k+1}$;
5. $V_k$ is $k$-discrete;
6. $|\psi^{-1}(V)| > C_k$ for every $V \in V_{k+1}$ (as above, for every $W \in V_k$ by $\psi(V)$ we denote the unique element of $V_{k+1}$ that contains $V$).

For every $U \in U_k$ and $V \in V_k$ denote by $g_{UV}$ arbitrary constant map from $U$ to $V$. Suppose that for every $i \leq k$ and every $U \in U_i$ and $V \in V_i$ a map $g_{UV}: U \to V$ is defined. Given $U \in U_{k+1}$, $V \in V_{k+1}$, consider arbitrary injective map $\alpha: \psi^{-1}(U) \to \psi^{-1}(V)$. Define $g_{UV}: U \to V$ as follows: $g_{UV}|W = g_{W\alpha(W)}$ for every $W \in \psi^{-1}(U)$. 

Now we are ready to define a map $f: X \to M_0$. For every $k \in \mathbb{N}$ denote by $U_k$ the unique element of $\mathcal{U}_k$ that contains $x_0$. By induction, define a sequence of maps $f_k: U_k \to M_0$ with the following properties:

1. $f_{k+1}|U_k = f|k$, for every $k$;
2. $f(U_k)$ is contained in an element of $\mathcal{U}_{k+1}$.

Let $f_1 = g_{U_1V}: U_1 \to V \subset M_0$, for some $V \in \mathcal{V}_1$. Suppose that $f_i$ are defined for every $i \leq k$. Let $V' \in \mathcal{V}_k$ be such that $f_k(U_k) \subset V'$. Denote by $\alpha: \psi^{-1}(\psi(U_k)) \to \psi^{-1}(\psi(V'))$ an embedding such that $\alpha(U_k) = V'$. Define $f_{k+1}$ by the conditions $f_{k+1}|U_k = f_k$ and $f_{k+1}|W = g_{W\alpha(W)}$ for every $W \in \psi^{-1}(\psi(U_k)) \setminus \{U_k\}$.

By condition (1'), the sequence of maps $(f_k)$ uniquely determines a map $f: X \to M_0$. Using properties (1)--(6) it is easy to see that $f$ is a coarse embedding. \hfill $\square$

3.3. Nonexistence results. There is no counterpart of Theorem 3.11 in higher dimensions.

**Theorem 3.12.** There is no proper metric space universal for the class of proper metric spaces $Y$ with $\operatorname{asdim} Y \leq 1$.

**Proof.** Suppose the contrary and let $X$ be a proper metric space universal for the class of proper metric spaces $Y$ with $\operatorname{asdim} Y \leq 1$. Let $x_0 \in X$ be a base point. Denote by $\alpha(r)$ the 1-capacity of the ball $N_r(x_0)$, i.e. the number $K_1(N_r(x_0))$. Note that for every $x \in X$ we have

$$K_1(N_r(x)) \leq K_1(N_{r+d(x,x_0)}(x_0)) \leq \alpha(r + d(x,x_0)).$$

Let $T$ be a locally finite tree with a base point $t_0$ and the index function $\varphi(r)$ which we specify later. Suppose that $f: T \to X$ is a coarse embedding. Since $T$ is a geodesic space, the map $f$ is proper and $(\lambda,s)$-Lipschitz for some $\lambda > 0$, $s \geq 0$. There exists $a > 0$ such that $d(x,y) \geq a$ implies $d(f(x),f(y)) \geq 1$. There exist $\lambda',s' > 0$ such that $f(N_r(t_0)) \subset N_{\lambda' r + s'}(x_0)$. Therefore

$$K_\alpha(N_r(t_0)) \leq K_1(N_{\lambda' r + s'}(x_0)) = \alpha(\lambda' r + s').$$

Now suppose that $\varphi: \mathbb{N} \to \mathbb{N}$ satisfies the following property: for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $\alpha(nr+n) < \varphi(r)$ for all $r \geq m$. Define $T$ as follows: Let $T_0 = \mathbb{R}_+$. For every $m \in \mathbb{N}$ denote by $T_m$ a tree obtained by attaching to every integer point $n \in \mathbb{R}_+$, $n \geq m$, $\varphi(n)$ copies of the segment $[0,m]$ by its endpoints. Let $T = \cup\{T_n \mid n \in \mathbb{N} \cup \{0\}\}$.

Now $K_\alpha(N_r(t_0)) \leq \alpha(\lambda' r + s') \leq \alpha(nr+n)$, where $n \in \mathbb{N}$, $n \geq \max\{\lambda', s'\}$. Let $m \in \mathbb{N}$, $m \geq a$. Then for every $r \geq 2m + 1$ we obtain $K_\alpha(N_r(t_0)) \geq K_m(N_r(t_0)) \geq \varphi(\lfloor r \rfloor)$, a contradiction with the choice of $\varphi$. \hfill $\square$
Show that there is no universal proper metric space of given asymptotic dimension in the class of proper metric spaces of bounded geometry.

We fix natural $n \geq 1$.

For natural $k$, denote by $X(1,k)$ the set
\[
\{(x_0, \ldots, x_n) \in [0,k]^{n+1} \mid x_i \notin \mathbb{Z} \text{ for at most one } i = 0, \ldots, n\}.
\]
For natural $m$, let $X(m,k) = \{mx \mid x \in X(1,k)\}$.

Denote by $\mathcal{V}$ the set of covers $\mathcal{V}$ of $X(1,k)$ such that $\mathcal{V} = \bigcup_{i=0}^{n} \mathcal{V}_i$, where $\mathcal{V}_i$ are 3-discrete, $i = 0, 1, \ldots, n$. For $\mathcal{V} \in \mathcal{V}$, denote by $\mu_{\mathcal{V}}(k)$ the maximal 1-capacity of $V \in \mathcal{V}$ and let $\mu_k = \min\{\mu_{\mathcal{V}}(k) \mid \mathcal{V} \in \mathcal{V}\}$.

Given $\lambda > 0$, we say that a subset $A$ of a metric space $X$ is a $\lambda$-component of $X$ if $A$ is a maximal (with respect to inclusion) subset of $B$ with respect to the property that every two points of $A$ can be connected by a $\lambda'$-chain, for some $\lambda' < \lambda$.

**Lemma 3.13.** $\mu_k \to \infty$ as $k \to \infty$.

**Proof.** Assume the opposite, i.e. there exists a constant $S$ such that $\mu_k \leq S$, for all $k$. Given $\mathcal{V} \in \mathcal{V}$ with $\mathcal{V}^\infty_{i=0} = \bigcup \mathcal{V}_i$, where $\mathcal{V}_i$ are 3-discrete, $i = 0, \ldots, n$, for every $V \in \mathcal{V}$, denote by $\hat{V}$ the family of $<3$-connected components of $V$.

Let $\hat{\mathcal{V}}_i = \cup \{\hat{V} \mid V \in \mathcal{V}_i\}, i = 0, \ldots, n$, and $\hat{V} = \cup_{i=0}^{n} \hat{\mathcal{V}}_i$. From our assumption it easily follows that there exists a constant $C > 0$ such that for every $W \in \hat{V}$ we have $\text{diam}(W) \leq C$.

Now, for every $W \in \hat{V}$ define $\hat{W}$ as the 1-neighborhood of $W$ in $[0,k]^{n+1}$. The families $\{\hat{W} \mid W \in \hat{\mathcal{V}}_i\}, i = 0, 1, \ldots, n$, are 1-discrete, of mesh $\leq C$, and together form a cover of $[0,k]^{n+1}$. Then the families $\{(1/k)\hat{W} \mid W \in \hat{\mathcal{V}}_i\}, i = 0, 1, \ldots, n$, are discrete, of mesh $\leq C/k$, and together form a cover of $[0,1]^{n+1}$. Because of arbitrariness of $k$, we obtain a contradiction with Ostrand’s characterization of the covering dimension [16].

Passing to the images of spaces under the homothety with coefficient $m$ and centered at the origin, we derive from Lemma 3.13 the following statement.

**Proposition 3.14.** For every $m, s \in \mathbb{N}$ there exists $r(m, s) \in \mathbb{N}$ such that for any $r \geq r(m, s)$ the following holds: given a cover $\mathcal{V}$ of $X(m,r)$ that splits into union of $n + 1$ $3m$-disjoint families, there is $V \in \mathcal{V}$ with $K_m(V) \geq s$.

**Theorem 3.15.** There is no proper metric space $Y$ of bounded geometry, asdim $Y = n$, with the following property: for every proper metric space $X$ of bounded geometry, asdim $X = n$, there exists a large scale embedding of $X$ into $Y$.

**Proof.** Suppose the opposite and let $Y$ be such a space. There exists a sequence of covers $(\mathcal{U}_k)$ of $Y$ with the following properties:

1) $\mathcal{U}_k$ is uniformly bounded;
2) $\mathcal{U}_k = \bigcup_{i=0}^{n} \mathcal{U}_i^k$, where $\mathcal{U}_i^k$ are $d_k$-discrete with $d_k \geq k^2$.

Since $Y$ is of bounded geometry, $c_k = \max\{K_1(U) \mid U \in \mathcal{U}_k\} < \infty$.

For every $m \in \mathbb{N}$, choose $R_m$ so that for any $r \geq r(m, R_m)$ the following holds: given a cover $\mathcal{V}$ of $X(m, r)$ that splits into union of two $3m$-disjoint families, there is $V \in \mathcal{V}$ with $K_m(V) \geq \sum_{i=1}^{m} c_m + 1$.

Define a space $X$ as follows: attach to $\mathbb{R}_+$ at every point $m \in \mathbb{N} \subset \mathbb{R}_+$ a copy of space $X(m, R_m)$, with the geodesic metric on the adjunction space. Note that $\text{asdim} X = 1$.

Suppose that $f: X \to Y$ is a large scale embedding, i.e. there exist monotone, increasing to infinity functions $\varphi_1, \varphi_2: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi_1(d(x, y)) \leq d(f(x), f(y)) \leq \varphi_2(d(x, y))$$

for every $x, y \in X$.

Since $X$ is a geodesic space, by Lemma 2.1 one may choose $\varphi_2$ linear, $\varphi_2(t) = at + b$, $a > 0$. There exists $m_0 \in \mathbb{N}$ such that $d(f(x), f(y)) \geq 1$ as $d(x, y) \geq m_0$.

For any $m \geq m_0$, find minimal $k(m) \in \mathbb{N}$ such that the preimage $f^{-1}(\mathcal{A})$ of any $d_k(m)$-discrete family is $3m$-discrete. We have for $x \in A$, $y \in B$, $A, B \in \mathcal{A}$, $A \neq B$:

$$d_k(m) \leq d(f(x), f(y)) \leq \varphi_2(d(x, y)) = ad(x, y) + b$$

whence $d(x, y) \geq (d_k(m) - b)/a$ and if $(d_k(m) - b)/a \geq 3m$, we are done. Therefore, the minimal $k(m)$ is found from the inequality $d_k(m) \geq 3ma + b$.

We see that there exists $m_1 \in \mathbb{N}$ such that $d_m \geq (3m - b)/a$ for every $m \geq m_1$.

Now consider $m > \max\{m_0, m_1\}$. The cover $\mathcal{V}_m$ splits into the union of $d_m$-disjoint families $\mathcal{V}_m^i$, $i = 0, 1, \ldots, n$.

The cover $f^{-1}(\mathcal{V}_m)$ of $X(m, R_m)$ splits into the union of $3m$-disjoint families $f^{-1}(\mathcal{V}_m^i)$, $i = 0, \ldots, n$. By Proposition 3.14, there exists an element $V \in \mathcal{V}_m$ such that $K_m(f^{-1}(V)) \geq \sum_{i=1}^{m} c_i + 1 > c_m$ as $m > m_0$, we see that $K_1(V) > c_m$, a contradiction.

\[\square\]

4. Higson property

A metric space $X$ with $\text{asdim} X \leq n$ is said to satisfy the Higson property if there exists $C > 0$ such that for every $D > 0$ there exists a cover $\mathcal{U}$ of $X$ with $\text{mesh}(\mathcal{U}) < CD$ and such that $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$, where $\mathcal{U}^0, \ldots, \mathcal{U}^n$ are $D$-disjoint. Equivalently, $X$ has the Higson property if there exists a sequence $(\mathcal{U}_k)$ of uniformly bounded covers of $X$ and $C > 0$ such that $\text{mesh}(\mathcal{U}_k) \to \infty$ and every ball in $X$ of radius $\leq C\text{mesh}(\mathcal{U}_k)$ intersects at most $n + 1$ element of $(\mathcal{U}_k)$. The latter is a large scale analog of the Nagata dimension $\text{N-dim}$ defined by Assouad [1] as follows: for a metric space $X$ we have $\text{N-dim} X \leq n$ if there is a constant $C$ such that, for each $r > 0$, there is an open cover $\mathcal{U}(r)$ of $X$ by
sets of diameter \( \leq Cr \) such that each open ball of radius \( r \) meets at most \( n + 1 \) members of \( U(r) \).

**Proposition 4.1.** Every proper metric space of finite asymptotic dimension is coarsely equivalent to a proper metric space that satisfies the Higson property.

**Proof.** By Theorem 3.5, there is a large scale embedding of \( X \) into a space of the form \( M(T^0, \ldots, T^n) \), for a sequence of regular locally finite trees \( T^0, \ldots, T^n \) together with filtrations \( T_0^0 \supset T_1^0 \supset T_2^0 \supset \ldots \). Since the Higson property is preserved by subspaces, it is sufficient to show that the space \( M(T^0, \ldots, T^n) \) satisfies it.

Note that the set \( K_j = r_j(M(T^0, \ldots, T^n) \) is a cubic piecewise euclidean \( n \)-dimensional complex with \( \text{mesh}(K_j) = D_j = 2^j \). Passing to the second barycentric subdivision of \( K_j \) we are able to produce \( D_k/3 \)-discrete families \( A \) with \( \text{mesh}(A) \leq 2D_k/3 \leq D_k \), \( l \in \{0, \ldots, n\} \). Let \( U_k = \{ r^{-1}_k(A) \mid A \in A \} \), then

\[
\text{mesh}(U_k) \leq 2(D_k + D_{k-1} + \cdots + D_0) \leq 4D_k.
\]

Since the map \( r_k \) is short, the family \( U_k \) is also \( D_k/3 \)-discrete. \( \square \)

A metric space \( X \) satisfies the \( n \)-dimensional Nagata property (briefly denoted by \( (P_n) \)) if for every \( r > 0 \), every \( x \in X \), and every \( y_1, \ldots, y_{n+2} \) such that \( d(y_i, N_r(x)) < 2r \) for every \( i = 1, \ldots, n + 2 \), there exist \( i, j \in \{1, \ldots, n + 2\} \), \( i \neq j \), such that \( d(y_i, y_j) < 2r \) (see [15]).

**Theorem 4.2.** For a proper metric space \( X \) the following are equivalent:

1. \( \text{asdim } X \leq n \);
2. \( X \) is large scale equivalent to a metric space with property \( (P_n) \).

**Proof.** \((1) \Rightarrow (2)\) By Proposition 4.1, \( X \) is coarsely equivalent to a proper metric space with the Higson property. Moreover, as we can see from the proof of Proposition 4.1, without loss of generality, we may assume that there exists a sequence \( (U_k) \) of uniformly bounded open covers of \( X \) and \( C > 0 \) such that \( \text{mesh}(U_k) = 2^k \) and \( U_k \) splits into the union of \( n+1 \) families that are \( C2^k \)-discrete. Equivalently, every ball of radius \( C2^{k-1} \) intersects at most \( n + 1 \) element of the cover \( U_k \). By Lemma 2.2, there exists a 1-discrete subset \( X' \) of \( X \) which is coarsely equivalent to \( X \). For every \( r \in (0, \infty) \), put

\[
U'(r) = \begin{cases} 
\{ \{x\} \mid x \in X' \} & \text{whenever } 0 < r < 1, \\
\{ U \cap X' \mid U \in U[1] \} & \text{whenever } r \geq 1,
\end{cases}
\]

and \( C' = C/4 \). Then every ball in \( X' \) of radius \( \leq C'r \) intersects at most \( n + 1 \) element of the cover \( U'(r) \), \( r \in (0, \infty) \). The latter means that \( \text{N-dim } X' \leq n \) and by

It is proved in [1][Proposition 2.2] that then there exists a metric \( \delta \) on \( X' \) such that \( \delta \) satisfies \( (P_n) \) and \( \delta^p \) is Lipschitz equivalent to the original metric,
\[ A\delta(x, y)^p \leq d(x, y) \leq B\delta(x, y)^p, \quad x, y \in X'. \]

(2) $\implies$ (1) The proof is completely parallel to that in the case of dimension $\dim(X)$ (see [15]). Let $X$ be a proper metric space whose metric, $d$, satisfies property $(P_n)$. Given $D > 0$, find a maximal (with respect to inclusion) set $Y \subset X$ which is $2D$-discrete. Then $\mathcal{U} = \{ N_{2D}(y) \mid y \in Y \}$ is a uniformly bounded cover of $X$.

Show that for every $x \in X$ the set $N_D(x)$ intersects at most $n + 1$ element of $\mathcal{U}$. Indeed, suppose the contrary; then there $y_1, \ldots, y_{n+2} \in Y$ such that $N_D(x) \cap N_{2D}(y_i) \neq \emptyset$ for every $i = 1, \ldots, n + 2$. Property $(P_n)$ implies that there exist $i, j \in \{1, \ldots, n + 2\}, i \neq j$, such that $d(y_i, y_j) < 2D$. This is a contradiction with the choice of $Y$. Because of arbitrariness of $D$, $\mathrm{asdim} X \leq n$.

5. Coincidence of asymptotic dimensions

5.1. Higson compactification and Higson corona. Let $\varphi : X \to \mathbb{R}$ be a function defined on a metric space $X$. For every $x \in X$ and every $r > 0$ let $V_r(x) = \sup\{|\varphi(y) - \varphi(x)| \mid y \in N_r(x)\}$. A function $\varphi$ is called slowly oscillating whenever for every $r > 0$ we have $V_r(x) \to 0$ as $x \to \infty$ (the latter means that for every $\varepsilon > 0$ there exists a compact subspace $K \subset X$ such that $|V_r(x)| < \varepsilon$ for all $x \in X \setminus K$). Let $\bar{X}$ be the compactification of $X$ that corresponds to the family of all continuous bounded slowly oscillation functions. The Higson corona of $X$ is the remainder $\nu X = \bar{X} \setminus X$ of this compactification.

It is known that the Higson corona is a functor from the category of proper metric space and coarse maps into the category of compact Hausdorff spaces. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$.

For any subset $A$ of $X$ we denote by $A'$ its trace on $\nu X$, i.e. the intersection of the closure of $A$ in $\bar{X}$ with $\nu X$. Obviously, the set $A'$ coincides with the Higson corona $\nu A$.

5.2. Asymptotic inductive dimensions. The notion of asymptotic inductive dimension $\mathrm{asInd}$ is introduced in [10].

Recall (see, e.g. [12]) that a closed subset $C$ of a topological space $X$ is a separator between disjoint subsets $A, B \subset X$ if $X \setminus C = U \cup V$, where $U, V$ are open subsets in $X$, $U \cap V = \emptyset, A \subset U, V \subset B$.

Let $X$ be a proper metric space. A subset $W \subset X$ is called an asymptotic neighborhood of a subset $A \subset X$ if $\lim_{r \to \infty} d(X \setminus N_r(x_0), X \setminus W) = \infty$. Two sets $A, B$ in a metric space are asymptotically disjoint if $\lim_{r \to \infty} d(A \setminus N_r(x_0), B \setminus N_r(x_0)) = \infty$. In other words, two sets are asymptotically disjoint if the traces $A', B'$ on $\nu X$ are disjoint.
A subset $C$ of a metric space $X$ is an *asymptotic separator* between asymptotically disjoint subsets $A_1, A_2 \subset X$ if the trace $C'$ is a separator in $\nu X$ between $A_1'$ and $A_2'$.

By the definition, $\text{asInd} X = -1$ if and only if $X$ is bounded. Suppose we have defined the class of all proper metric spaces $Y$ with $\text{asInd} Y \leq n - 1$. Then $\text{asInd} X \leq n$ if and only if for every asymptotically disjoint subsets $A_1, A_2 \subset X$ there exists an asymptotic separator $C$ between $A_1$ and $A_2$ with $\text{asInd} C \leq n - 1$. The dimension function $\text{asInd}$ is called the *asymptotic inductive dimension*.

We can similarly define the small inductive asymptotic dimension $\text{asInd}$. By the definition, $\text{asInd} X = -1$ if and only if $X$ is bounded. Suppose we have defined the class of all proper metric spaces $Y$ with $\text{asInd} Y \leq n - 1$. Then $\text{asInd} X \leq n$ if and only if for every $A \subset X$ and every $x \in \nu X \setminus A'$ there exists a subset $C'$ of $X$ such that $C'$ is a separator between $A'$ and $\{x\}$ in $\nu X$ and $\text{asInd} C \leq n - 1$.

**Proposition 5.1.** *For every metric space $X$ we have $\text{asInd} X \leq \text{asInd} X$.***

*Proof.* Induction by $\text{asInd} X$. By the definition, the properties $\text{asInd} X = -1$ and $\text{asInd} X = -1$ are equivalent. Suppose $\text{asInd} X \leq n$ and $A \subset X, x \in \nu X \setminus A'$. There exists a continuous function $\varphi: X \to [-1, 1]$ such that $\varphi|A = -1, \varphi(x) = 1$. The sets $A$ and $B = (\varphi|X)^{-1}([0, 1]) = \varphi^{-1}([0, 1]) \cap X$ are asymptotically disjoint subsets of $X$ ad therefor there exists an asymptotic separator $C$ between $A$ and $B$ with $\text{asInd} C \leq n - 1$. Then obviously, $C'$ is a separator between $A'$ and $x$ in $\nu X$ and, by inductive hypothesis, $\text{asInd} C \leq n - 1$. Therefore $\text{asInd} X \leq n$. □

The following is a counterpart of Theorem 1 in [10].

**Proposition 5.2.** *Let $X$ be a proper metric space. Then $\text{asInd} X \geq \text{ind} \nu X$.***

*Proof.* Induction by $\text{asInd} X$. Obviously, $\text{asInd} X = -1$ if and only if $\text{ind} \nu X = -1$. Suppose that $\text{asInd} X \leq n$ and show that $\text{ind} \nu X \leq n$. Let $A$ be a closed subset of $\nu X$ and $x \in \nu X \setminus A$. There exists a continuous function $\varphi: X \to [-1, 1]$ such that $\varphi|A = -1, \varphi(x) = 1$. Let $B = \varphi^{-1}([-1, 0])$. Then $x \notin B'$ and, since $\text{asInd} X \leq n$, there exists a subset $C \subset X$ with $\text{asInd} C \leq n - 1$ and such that $C'$ is a separator between $B'$ and $x$ in $\nu X$. By the inductive hypothesis,

$$\text{ind} C' = \text{ind} \nu C \leq \text{asInd} C \leq n - 1.$$ 

Since $C$ is a separator between $B'$ and $x$ in $\nu X$, it is also a separator between $A$ and $x$.

Therefore, for every closed subset $A$ of $\nu X$ and $x \in \nu X \setminus A$ we are able to find a separator $K$ with $\text{ind} K \leq n - 1$. This means that $\text{ind} \nu X \leq n$. □

It is proved in [10] (see Proposition 1 therein) that $\text{asInd} X = \text{asInd} Y$ for coarsely equivalent spaces $X$ and $Y$. 

Proposition 5.3. Let \( X \) and \( Y \) be coarsely equivalent metric spaces. Then \( \text{asind} \ X = \text{asind} \ Y \).

Proof. Analogous to that of Proposition 1 in [10]. \( \square \)

Lemma 5.4. Suppose that \( A \cup B \) are asymptotically disjoint subsets of \( X \subset Y \) and \( C \) is an asymptotic separator in \( Y \) between \( A \) and \( B \) with \( \text{asdim} \ C \leq m \). Then there exists an asymptotic separator \( \tilde{C} \) between \( A \) and \( B \) in \( X \) with \( \text{asdim} \ C \leq m \).

Proof. Define subsets \( D_k \) of \( X \) by induction. Let \( D_1 = N_1(C) \cap X \) and \( D_{k+1} = (N_{k+1}(C) \cap X) \setminus N_k(D_k) \). By Lemma 2.2, there exists a \((k+1)\)-net \( \tilde{D}_{k+1} \) in \( D_{k+1} \) which is \( k \)-discrete. Let \( \tilde{C} = \bigcup_{k=1}^\infty \tilde{D}_k \).

We first show that \( \text{asdim} \ \tilde{C} \leq m \). Remark that for every \( k > 0 \) the complement to the subset \( \bigcup \{ \tilde{D}_j \mid j \leq k \} \) is the union of a \( k \)-disjoint family \( \{ \tilde{D}_j \mid j > k \} \).

To apply Theorem 2.5, we have to verify that \( \text{asdim} \ \tilde{D}_k \leq m \) uniformly. If \( k > 0 \), then for some \( R > 0 \) there exists an \( R \)-bounded cover \( U \) of the set \( \bigcup \{ \tilde{D}_j \mid j \leq k \} \) that splits into the union \( U^0 \cup \cdots \cup U^m \) of \( k \)-discrete families. Let \( V^0 = U^0 \cup \{ \{ x \} \mid x \in \bigcup \{ \tilde{D}_j \mid j > k \} \} \), then the family \( V^0 \cup U^1 \cup \cdots \cup U^m \) is a \( \min\{ R, k \} \)-uniformly bounded family that splits into the union of \( m+1 \) \( k \)-disjoint families.

Show that for every integer \( k > 0 \) there exists \( R = R(k) > 0 \) such that \( N_k(C) \cap X \subset N_{R(k)}(\tilde{C}) \). Indeed, if \( x \in N_k(C) \cap X \), then either \( x \in D_k \subset N_{2k}(\tilde{D}_k) \) or \( x \in N_k(C) \setminus D_k \subset N_{k-1}(D_{k-1}) \subset N_{k-1}(N_{k-1}(\tilde{D}_{k-1})) \subset N_{2k-2}(\tilde{D}_{k-1}) \subset N_{2k-2}(\tilde{C}) \) and we can choose \( R(k) = 2k \).

Now suppose that \( C \) is an asymptotic separator between \( A \) and \( B \) in \( Y \). Show that \( \tilde{C} \) is an asymptotic separator between \( A \) and \( B \) in \( X \). It is sufficient to show that \( \tilde{C}' \supset C' \cap \nu X \). Suppose that \( a \in C' \cap \nu X \) and \( U \) is a closed neighborhood of \( a \) in \( Y \). Then there exist sequences \( (x_i) \) in \( X \cap U \) and \( (c_l) \) in \( C \cap U \) and \( k \in \mathbb{N} \) such that \( d(x_i, c_l) \leq k \) for every \( i \).

Case 1). For infinite number of \( i \) (passing to a subsequence we then assume that for all \( i \) \( x_i \in D_k \). Then there exist \( \tilde{x}_i \in \tilde{D}_k \) with \( d(x_i, \tilde{x}_i) \leq k \).

Case 2). For infinite number of \( i \) (passing to a subsequence we then assume that for all \( i \) \( x_i \in N_{k-1}(D_{k-1}) \). Then there exist \( y_i \in D_{k-1} \) such that \( d(x_i, y_i) \leq k - 1 \) and there exist \( \tilde{x}_i \in \tilde{D}_{k-1} \) such that \( d(y_i, \tilde{x}_i) \leq k - 1 \).

In both cases, \( \tilde{x}_i \in \tilde{C} \) for every \( i \) and \( d(x_i, \tilde{x}_i) \leq \max\{ k, 2k - 1 \} = 2k \) and therefore \( \tilde{x}_i \in U \cap X \) for all but finitely many \( i \). This means that the sets \( \tilde{C} \) and \( U \) are not asymptotically disjoint and therefore \( C' \cap U \neq \emptyset \). Because of arbitrariness of \( U \) we conclude that \( a \in \tilde{C}' \). \( \square \)

Theorem 5.5. For every space \( X \in \mathcal{M}_n \) we have \( \text{asInd} \ X \leq n \).
We need some auxiliary results. Let \( T^0, \ldots, T^n \) be a sequence of regular locally finite trees.

**Proposition 5.6.** For every disjoint asymptotically disjoint subsets \( A, B \) in \( M(T^0, \ldots, T^n) \) there exists a separator \( C \) between them such that \( \text{asdim}(C) \leq n - 1 \) and the sets \( A, B, C \) are asymptotically disjoint.

**Proof.** Let
\[
U = \{ x \in M(T^0, \ldots, T^n) \mid d(x, A) \leq (1/2)d(x, B) \},
\]
\[
V = \{ x \in M(T^0, \ldots, T^n) \mid d(x, B) \leq (1/2)d(x, A) \},
\]
then \( U, V \) are closed, disjoint, asymptotically disjoint asymptotic neighborhoods of the sets \( A, B \) respectively. For every \( j \) let \( R_j \) be a number such that
\[
d(r_j(U \setminus N_{R_j}(x_0)), r_j(V \setminus N_{R_j}(x_0))) \geq 2^{j+2}
\]
(such a number exists because \( U, V \) are asymptotically disjoint and the retraction \( r_j \) is \( 2^{j+1} \)-close to the identity map. We will assume that \( R_{j+1} \geq 2R_j \). Put
\[
U_j = r_j\left(U \cap (N_{R_{j+1}}(x_0) \setminus N_{R_j}(x_0))\right).
\]

Denote by \( \hat{U}_j \) the union of all \( n \)-dimensional cubic simplices in the subcomplex \( r_j(M(T^0, \ldots, T^n)) \) that intersect \( U_j \). Let \( S_j = r^{-1}(\partial(N_{1/2}(\hat{U}_j))) \) and \( S = \cup_{i=0}^{\infty} S_j \).

We are going to show that \( S \) contains a separator between \( U \) and \( V \) and \( \text{asdim}(S) \leq n - 1 \).

Put \( S' = \cup \{ S_{2k} \mid k \in \mathbb{N} \} \), \( S'' = \cup \{ S_{2k-1} \mid k \in \mathbb{N} \} \). By finite sum theorem for asymptotic dimension (see, e.g., [2]), it is sufficient to prove that \( \text{asdim}(S' \leq n - 1 \), \( \text{asdim}(S'' \leq n - 1 \) and, because of complete analogy, we only prove that \( \text{asdim}(S' \leq n - 1 \).

Prove that the family \( \{ S_{2k} \} \) has asymptotic dimension \( \leq n - 1 \) uniformly. Let \( D > 1 \). There exists natural \( j \) such that every \( (n - 1) \)-dimensional cubic complex of mesh \( 2^j \) can be covered by a \( 2^j \)-bounded family of its subsets that splits into the union of \( n \) families which are \( 2D \)-disjoint.

**Claim.** \( r_{j_0}^{-1}(\partial(N_{1/2}(\hat{U}_{j_0})) \subset N_{3/4}(r_{j_0}^{-1}(\hat{U}_{j_0})^{(n-1)} \).

To prove Claim, suppose that \( r_{j_0}(x) \in \partial(N_{1/2}(\hat{U}_{j_0})) \). Then there is a cubic \( n \)-dimensional symplex \( K \) in \( \hat{U}_{j_0} \) such that \( r_{j_0}(x) \in \partial(N_{1/2}(K)) \). The symplex \( K \) is of the form \( K = \prod_{i=0}^{n} K_i \), where \( K_i \) is a 1-dimensional symplex in \( T_{j_0}^{i} \) and there is \( l \in \{ 0, \ldots, n \} \) such that \( K_l \) is a singleton, \( K_l = \{ a_l \} \). Then \( \partial N_{1/2}(K) = (\cup_{k=0}^{n} R_k) \cap M_{j_0}^{i} \), where
\[
R_k = \left( \prod_{p \in \{ 0, \ldots, n \} \setminus \{ k \} N_{p/2}(K_p) \right) \times \partial N_{1/2}(K_k).
\]
Then \( r_{j'}^{-1}(\partial N_{1/2}(K_k)) = \partial N_{1/2}(K_k) \) and we obtain

\[
r_{j'}^{-1}(R_k) \cap M_j = \left( \prod_{p \in \{0, \ldots, n\} \setminus \{k\}} (r_{j'}^{-1}(\partial N_{1/2}(K_p))) \right) \times \partial N_{1/2}(K_k) \cap M_j.
\]

For every \( x = (x_0, \ldots, x_n) \in r_{j'}^{-1}(R_k) \cap M_j \) there exists \( p \in \{0, \ldots, n\} \setminus \{k\} \) such that \( x_p \in \partial(T_{j-1}^p \setminus T_j^p) \). Then \( d(x, (x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)) = 1/2 \) for some \( y_k \in \partial K_k \) (the boundary is considered in \( T^k \); note that then \( y_k \in r_{j'}^{k}(\partial(T_{j-1}^k \setminus T_j^k)) \) and \((x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \in N_{3/4}(r_{j'}^{k}(\hat{U_j}))(n-1) \). Claim is proved.

Denote by \( V \) a cover of the cubic symplex \( M_j \) that splits into the union \( V = \bigcup_{V \in \Sigma_k} = \{ N_k(V) \cap r_{j'}(\hat{U}_2k) \mid V \in \Sigma_k \} \) of uniformly \( 2^j \)-bounded covers which are \( 2D \)-discrete. Now for every \( k \) and every \( i \in \{0, \ldots, n\} \) consider the family \( \Sigma_k = \{ N_k(V) \cap r_{j'}(\hat{U}_2k) \mid V \in \Sigma_k \} \) of the set \( r_{j'}^{k}(\hat{U}_2k) \). The cover \( V_k = \bigcup_{\Sigma_k} \) is uniformly \( (2^j + 3) \)-bounded and splits into the union of \( D \)-discrete families \( \Sigma_k = \{ N_k(V) \cap r_{j'}(\hat{U}_2k) \mid V \in \Sigma_k \} \) and \( D \)-discrete and, because \( S_{2k} = r_{j'}^{-1}(\hat{U}_2k) \), we see that \( \bigcup_{i=0}^n \Sigma_k \) covers \( S_{2k} \).

Applying Theorem 2.5 we conclude that \( \operatorname{asdim} S \leq n - 1 \). Finally, note that \( S \) contains a separator between \( A \) and \( B \), namely, the set \( \partial N_{1/2}(\bigcup_{j=1}^\infty \hat{U}_j) \).

**Proposition 5.7.** Let \( C \) be a separator between asymptotically disjoint subsets \( A, B \) in a proper uniformly arcwise connected space \( X \) and the sets \( C \) and \( A \cup B \) are asymptotically disjoint. Then \( C \) is an asymptotic separator between \( A \) and \( B \) in \( X \).

**Proof.** Denote by \( U \) the set of points that can be connected to \( A \) by an arc in the set \( X \setminus C \). Show that the sets \( (U) \cap X \) and \( B \) are asymptotically disjoint. Indeed, assuming the opposite we obtain sequences \( (a_i) \) and \( (b_i) \) in \( U \) and \( B \) respectively such \( d(a_i, x_0) \to \infty \), \( d(b_i, x_0) \to \infty \) (here \( x_0 \) is an arbitrary base poin in \( X \)) and that there exists \( R > 0 \) for which \( d(a_i, b_i) < R, i \in \mathbb{N} \). Since \( X \) is uniformly arcwise connected, there exists \( S > 0 \) such that, for every \( i \), the points \( a_i \) and \( b_i \) can be connected by an arc of diameter \( \leq S \). There exists natural \( i_0 \) such that for every \( i \geq i_0 \) we have \( d(b_i, C) \geq R + S \). Then obviously there exists an arc in \( X \setminus C \) that connects \( a_i \) and \( b_i, i \geq i_0 \). Therefore, \( b_i \in U \) and we obtain a contradiction.

Thus, \( B' \subset (X \setminus U)' \). In order to prove that \( C \) is an asymptotic separator between \( A \) and \( B \) it remains to prove that \( \hat{U} \cap B \cap \nu X \subset C' \). Indeed, suppose the opposite, i.e. that there exists a point \( y \in \hat{U} \cap B \cap \nu X \setminus C' \). Then there exist disjoint neighborhoods \( V \) of \( y \) and \( W \) of \( C \) in \( X \) and obviously \( U \cap V \cap B \neq \emptyset \). Therefore, there exist sequences \( (a_i) \) in \( U \cap V \) and \( (b_i) \) in \( B \) such that \( \lim_{i \to \infty} d(a_i, x_0) = \infty \), \( \lim_{i \to \infty} d(b_i, x_0) = \infty \), and the sequence \( (d(a_i, b_i)) \) is
bounded. Arguing similarly as above we conclude that then there exists a sequence \((c_i)\) in \(C\) for which the sequence \((d(a_i, c_i))\) is bounded. This contradicts to our choice of \(V\).

\[ \square \]

**Lemma 5.8.** The space \(M(T^0, \ldots, T^n)\) is uniformly arcwise connected.

**Proof.** Note that for every \(l\) and every \(z \in r_l(M(T^0, \ldots, T^n))\) there exists a path of diameter \(\leq 2^{l+1}\) in \(M(T^0, \ldots, T^n)\) connecting \(z\) with \(r_l(z)\).

Let \(x = (x_0, \ldots, x_n), y = (y_0, \ldots, y_n) \in M(T^0, \ldots, T^n)\). Let \(j\) be the minimal natural number such that \(d(x, y) < 2^j\). Then \(r_j(x) = r_j(y)\). Let \(J\) be a path connecting \(x\) and \(y\) that consists of paths of diameter \(\leq 2^{l+1}\) connecting \(r_l(x)\) with \(r_l(y)\), \(1 \leq l \leq j\) (recall that \(r_0\) is the identity map). Then

\[
\text{diam}(J) \leq \sum_{l=1}^{j} d(r_{l-1}(x), r_l(x)) + \sum_{l=1}^{j} d(r_{l-1}(y), r_l(y)) \\
\leq 2 \sum_{l=1}^{j} 2^{l+1} \leq 2^{j+3} \leq 16d(x, y).
\]

\[ \square \]

Now we are going to prove Theorem 5.5. It is proved in [10] that \(\text{asInd } X \geq \text{Ind } \nu X\). This result together with the classical theorem on the comparison of \(\text{Ind}\) and \(\text{dim}\) implies the inequality \(\text{asInd } X \geq \text{dim } \nu X\). Suppose \(\text{asdim } X < \infty\).

Then \(\text{asdim } X = \text{dim } \nu X\) (see [7]), and we obtain \(\text{asInd } X \geq \text{asdim } X\) for every \(X\) with \(\text{asdim } X < \infty\).

To prove the opposite inequality we apply induction on \(\text{asdim } X\). It is known (see [10]) that the properties \(\text{asdim } X = 0\) and \(\text{asInd } X = 0\) are equivalent. Assume that the inequality \(\text{asInd } X \leq \text{asdim } X\) is proved for every \(X\) with \(\text{asdim } X \leq n - 1\).

Suppose that \(\text{asdim } X \leq n\). Then by Theorem 3.5 there is a large scale embedding of \(X\) into the space \(M(T^0, \ldots, T^n)\), for some regular locally finite trees \(T^0, \ldots, T^n\). Since the image of any space under a large scale embedding is coarse equivalent to this space and the dimension \(\text{asInd}\) is a coarse invariant (see [10, Proposition 1]), we may assume that \(X\) is a subspace of \(M(T^0, \ldots, T^n)\). Let \(A, B\) be asymptotically disjoint subsets of \(X\). Without loss of generality we assume \(A, B\) to be closed and disjoint. By Proposition 5.6, there exists a separator \(C\) between \(A\) and \(B\) with \(\text{asdim } C \leq n - 1\). Since, by Lemma 5.8, the space \(M(T^0, \ldots, T^n)\) is uniformly arcwise connected, it follows from Proposition 5.7 that \(C\) is also an asymptotic separator between \(A\) and \(B\) in \(M(T^0, \ldots, T^n)\).
By Lemma 5.4, there exists an asymptotic separator $\tilde{C}$ between $A$ and $B$ in $X$ with $\text{asdim} \tilde{C} \leq n - 1$. Applying the induction assumption we see that $\text{asInd} \tilde{C} \leq n - 1$.

Since for every asymptotically disjoint subsets in $X$ there exists an asymptotic separator between them whose dimension $\text{asInd}$ does not exceed $n - 1$, we conclude that $\text{asInd} X \leq n$.

The equality $\text{asInd} X = \text{asdim} X$ is therefore proven for all $X$ with $\text{asdim} X < \infty$.

One can similarly prove the following result.

**Theorem 5.9.** For all proper metric spaces $X$ with $\text{asdim} X < \infty$ we have $\text{asind} X = \text{asdim} X$.

We finally obtain that the dimensions $\text{asdim}$, $\text{asInd}$, and $\text{asind}$ coincide in the class of proper metric spaces that are finite dimensional with respect to $\text{asdim}$. This fact can be regarded as a counterpart in the asymptotic topology of the classical result on coincidence of the covering dimension, the small inductive dimension, and the large inductive dimension (see [12]).

6. **Open problems**

The coincidence of the dimension functions $\text{asdim}$ and $\text{asInd}$ is proved under the assumption of finiteness of $\text{asdim}$. This leads to the following natural question.

**Question 6.1.** Is there a proper metric space $X$ with $\text{asdim} X = \infty$ and $\text{asInd} X < \infty$?

A similar question can be formulated for $\text{ind}$.

A closed subset $C$ of a topological space $X$ is a cut between disjoint subsets $A, B \subset X$ if every continuum (compact connected space) $T \subset X$ that intersect both $A$ and $B$ also intersects $C$.

The notion of the large inductive dimension can be based on the notion of cut instead of separator. It turns out (see [13]) that the obtained dimension (it was defined by Brouwer who called it Dimensiongrad) coincides with the classical large inductive dimension in the class of separable metrizable spaces).

**Question 6.2.** Does the large asymptotic inductive dimension defined on the base of asymptotic cut coincides with $\text{asInd}$?

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