Inverse Dirichlet-to-Neumann problem for nodal curves

G. Henkin and V. Michel

Abstract. This paper proposes direct and inverse results for the Dirichlet and Dirichlet-to-Neumann problems for complex curves with nodal type singularities. As an application, it gives a method for reconstructing the conformal structure of a compact surface of $\mathbb{R}^3$ with constant scalar conductivity from electric current density measurements in a neighbourhood of one of its points.

Bibliography: 23 titles.

Keywords: conformal structure, Riemann surface, nodal curve, Green function, inverse Dirichlet-to-Neumann problem.

Contents

1. Introduction 1069
2. Direct problems for nodal curves 1073
   2.1. Nodal curves 1073
   2.2. Harmonic distributions 1073
   2.3. Green functions and Dirichlet problems 1076
3. Inverse problems for nodal curves 1077
   3.1. DN-data: hypotheses A and B 1078
   3.2. Proofs of results for the compact case 1080
   3.3. Uniqueness and reconstruction results for nodal curves 1081
4. Characterization of DN-data 1085
Bibliography 1088

1. Introduction

Let $Z$ be a compact or open bordered surface in $\mathbb{R}^3$ equipped with the complex structure induced by the standard Euclidean metric on $\mathbb{R}^3$; this approach to Riemann surfaces, which goes back to a result of Gauss about isothermal coordinates in 1822, is not restrictive since it has been proved, by Garsia [7] for the compact case and by Rüedy [19] for the bordered case, that any abstract Riemann surface is isomorphic to such a manifold. Let $\bar{\partial}$ be the Cauchy–Riemann operator on $Z$, $d^c = i(\bar{\partial} - \partial)$, and $d = \partial + \bar{\partial}$. If $Z$ has a constant scalar conductivity and if there is no time fluctuation and no current source or sink, then it follows from Maxwell’s
equations that the electric potential on an open set in $Z$ is a smooth function $U$ which satisfies the equation $dd^cU = 0$; the 1-form $d^cU = i(\overline{\partial}U - \partial U)$ can then be seen as modelling the physical current density arising from the potential $U$ (see, for example, [21]). An isolated finite charge induces a current whose density has a simple pole. When the current density $d^cU$ is theoretically allowed to have singularities on a discrete set, it is natural to limit them to simple charged poles. The fact that the charges should somehow compensate each other and arise from simple poles is mathematically natural because, according to Proposition 2, potentials with such singularities can be seen as harmonic distributions on complex nodal curves. The theorem below gives an electrostatic interpretation for a Dirichlet problem when a discrete set of finite charges is allowed.

**Theorem** (Riemann, 1851; Klein, 1882). Let $Z$ be a compact or bordered connected oriented smooth surface in $\mathbb{R}^3$ equipped with the conformal structure induced by the standard Euclidean metric on $\mathbb{R}^3$. Let $\overline{\partial}$ be its Cauchy–Riemann operator, $d^c = i(\overline{\partial} - \partial)$, and $d = \partial + \overline{\partial}$. Assume that $u$ is an electric potential on $bZ$ (this assumption is absent when $Z$ is compact) and that $Z$ has real electric charges $\pm c_j$ concentrated at points $a_j^\pm$, $1 \leq j \leq \nu$. Then there is a unique (up to an additive constant if $Z$ is compact) electric potential $U$ extending $u$ to $Z$ (when $Z$ is non-compact) such that $dd^cU = 0$ on $Z \setminus \{a_j^\pm; 1 \leq j \leq \nu\}$ and the residue $\text{Res}_{a_j^\pm}(d^cU) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{\text{dist}(\cdot, a_j^\pm) = \varepsilon} d^cU$ (for sufficiently small $\varepsilon > 0$) of $d^cU = i(\overline{\partial} - \partial)U$ at $a_j^\pm$ is $\pm c_j$, $1 \leq j \leq \nu$.

This problem was first considered by Gauss in 1840, and by Thomson (also known as Lord Kelvin) and Dirichlet in 1847. Riemann gave a mathematically incomplete proof in 1851. In 1882, Klein wrote an electrostatic interpretation which physicists have considered to be a quite convincing justification. Effective and correct constructions were given by Fredholm in 1900 and Hilbert in 1901. A good account on this story can be found in the book by De Saint-Gervais [5].

In 1962, Gelfand [8] formulated and obtained the first non-trivial result in the inverse problem of reconstructing the complex structure of a compact surface in $\mathbb{R}^3$ from the spectrum of its Laplacian. This problem was solved for most surfaces by Buser [4] in 1997. A related inverse question was enunciated by Wentworth in 2010: How does one recover the conformal structure of a compact Riemann surface from Dirichlet–Neumann type data for some subdomain? Theorem 2 below, which is a development of inverse results contained in [15], §2, is an inverse version for the compact case of the Riemann–Klein theorem and gives a constructive answer to Wentworth’s question.

Before we formulate Theorem 1, we set up some definitions and notation. Let $Z$ be a compact connected oriented smooth surface in $\mathbb{R}^3$ equipped with the conformal structure induced by the standard Euclidean metric on $\mathbb{R}^3$. We denote by $D_Z$ the set of pairs $(a, c)$ in $Z^6 \times \mathbb{R}^3$ such that $a = (a_0^\ell, a_2^\ell)_{0 \leq \ell \leq 2}$ is a family of 6 mutually distinct points and $c = (c_\ell)_{0 \leq \ell \leq 2} \in \mathbb{R}^3$. For $(a, c) \in D_Z$ let $U^{a, c}_{Z, \ell}$ denote a function which is harmonic on $Z \setminus \{a_0^\ell, a_2^\ell\}$ and such that $\partial U^{a, c}_{Z, \ell}$ has a simple pole at $a_1^\ell$ with residue $\pm c_\ell$; in fact, $U^{a, c}_{Z, \ell}$ is a standard Green bipolar function, and while it is determined only up to an additive constant, $\partial U^{a, c}_{Z, \ell}$ is unique. Hence,
as explained in §3, if the forms \( \partial U_{a,c}^{a,c} \) have no common zero, then we can define a map \( F_{Z}^{a,c} = (\partial U_{a,c}^{a,c} : \partial U_{Z,1}^{a,c} : \partial U_{Z,2}^{a,c}) \) from \( Z \setminus \{0 \leq \ell \leq 2\} \) to \( \mathbb{CP}_2 \) which has the affine representation 
\[
\begin{pmatrix}
\partial U_{a,c}^{a,c} \\
\partial U_{Z,1}^{a,c} \\
\partial U_{Z,2}^{a,c}
\end{pmatrix}
\] 
on the set \( \{ \partial U_{Z,0}^{a,c} \neq 0 \} \); the quotients here are well-defined meromorphic functions, because \( \dim Z = 1 \). Let \( S \) be an open subset of \( Z \). We denote by \( E_{Z}(S) \) the set of pairs \((a,c)\) in \( D_{Z} \) such that \( F_{Z}^{a,c} \) is well defined and injective outside some finite subset of \( Z \setminus S \). It is clear that \( E_{Z}(S) \) is an open subset of \( D_{Z} \). We can now give an inverse result for the compact case of the Riemann–Klein theorem.

**Theorem 1.** Let \( Z \) and \( Z' \) be compact connected oriented smooth surfaces in \( \mathbb{R}^3 \) equipped with the conformal structures induced by the standard Euclidean metric on \( \mathbb{R}^3 \). Assume that \( Z \cap Z' \) contains a surface \( S \) and let \( a = (a_{\ell}^0, a_{\ell}^+; 0 \leq \ell \leq 2) \) be a set of 6 mutually distinct points in \( S \). Assume that \((a,c) \in E_{Z}(S) \cap E_{Z'}(S) \) and \((U_{Z,S}^{a,c})_{0 \leq \ell \leq 2} = (U_{Z',S}^{a,c})_{0 \leq \ell \leq 2} \) for some \( c \in \mathbb{R}^3 \). Then \( Z \) and \( Z' \) are isomorphic. Moreover, \( Z \) can be explicitly reconstructed from the forms \((\partial U_{Z,S}^{a,c})_{0 \leq \ell \leq 2}\).

The practical interest of this result would be greatly improved if \( E_{Z} \cap (Z^6 \times \mathbb{R}^3) \) were dense in \( Z^6 \times \mathbb{R}^3 \). Although this seems very likely, it has yet to be proved. So we modify our point of view slightly in allowing small perturbations. Let us be precise. For \( n \in \mathbb{N}^* \) denote by \( D_{Z,n} \) the set of quadruples \((a,c,p,\kappa)\) in \( Z^6 \times \mathbb{C}^3 \times (\mathbb{Z}^n)^3 \times (\mathbb{C}^n)^3 \) such that \((a,c) \in D_{Z}, \ p = (p_{\ell})_{0 \leq \ell \leq 2} \), and \( \kappa = (\kappa_{\ell})_{0 \leq \ell \leq 2} \), where each \( p_{\ell} = (p_{\ell,j})_{1 \leq j \leq n} \) is a family of mutually distinct points of \( Z \setminus \{0 \leq \ell \leq 2\} \), and each point \( \kappa_{\ell} = (\kappa_{\ell,j})_{1 \leq j \leq n} \in \mathbb{C}^n \) satisfies the condition 
\[
\sum_{1 \leq j \leq n} \kappa_{\ell,j} = 0.
\]
For \((a,c,p,\kappa) \in E_{Z,n} \) let \( V_{Z,\ell}^{p,\kappa} \) be a function which is harmonic on \( Z \setminus \{p_{\ell,j}; 1 \leq j \leq n\} \) and such that \( \partial V_{Z,\ell}^{p,\kappa} \) has simple poles with residues \( \kappa_{\ell,j} \) at the points \( p_{\ell,j} \); \( V_{Z,\ell}^{p,\kappa} \) is unique up to an additive constant. Let \( U_{Z,\ell}^{a,c,p,\kappa} = U_{Z,\ell}^{a,c} + V_{Z,\ell}^{p,\kappa}, 0 \leq \ell \leq 2 \).

We denote by \( E_{Z,n}(S) \) the set of quadruples \((a,c,p,\kappa)\) in \( D_{Z,n} \) such that the map \( F_{Z}^{a,c,p,\kappa} = (\partial U_{Z,0}^{a,c,p,\kappa} : \partial U_{Z,1}^{a,c,p,\kappa} : \partial U_{Z,2}^{a,c,p,\kappa}) \) is well defined and injective outside some finite subset of \( Z \setminus S \). It is clear that \( E_{Z,n}(S) \) is open in \( D_{Z,n} \). According to Proposition 1, the elements of \( E_{Z,n}(S) \) can be said to be generic.

**Proposition 1.** Let \( Z \) be a compact connected oriented smooth surface in \( \mathbb{R}^3 \) equipped with the conformal structure induced by the standard Euclidean metric on \( \mathbb{R}^3 \), and let \( S \) be a subdomain of \( Z \). Consider a pair \((a,c)\) in \( D_{Z} \) with \( a \in S^6 \). Then for any neighbourhood \( W \) of \( 0 \) in \( \mathbb{C} \) there exist \( n \in \mathbb{N}^* \) and \((p,\kappa)\in (\mathbb{N}^*)^3 \times (\mathbb{W}^n)^3 \) such that \((a,c,p,\kappa) \in E_{Z,n}(S) \).

This result, as well as Proposition 4, will be proved in a separate paper because it involves methods and results of complex analysis which deserve attention in their own right. We can now give a generic version of Theorem 1.

**Theorem 2.** Let \( Z \) and \( Z' \) be compact connected oriented smooth surfaces in \( \mathbb{R}^3 \) equipped with the conformal structures induced by the standard Euclidean metric on \( \mathbb{R}^3 \). Assume that \( Z \cap Z' \) contains a surface \( S \) and let \( a = (a_{\ell}^0, a_{\ell}^+; 0 \leq \ell \leq 2) \) be a set of 6 mutually distinct points in \( S \). Let \( U_{Z,\ell}^{a,c,p,\kappa} = U_{Z,\ell}^{a,c} + V_{Z,\ell}^{p,\kappa} \). Assume that \((a,c,p,\kappa) \in E_{Z,n}(S) \cap E_{Z',n}(S) \) and \((U_{Z,\ell}^{a,c,p,\kappa})_{0 \leq \ell \leq 2} = (U_{Z',\ell}^{a,c,p,\kappa})_{0 \leq \ell \leq 2} \) for some
triplet \((c,p,\kappa) \in \mathbb{C}^3 \times (S^n)^3 \times (\mathbb{C}^n)^3\). Then \(Z\) and \(Z'\) are isomorphic. Moreover, \(Z\) can be explicitly reconstructed from the forms \(\partial U_{Z,\ell}^{c,p,\kappa}\) \(0 \leq \ell \leq 2\).

Remarks. 1. As a consequence, the genus of \(Z\) is fully determined by the given \(S\) and \(U_{Z,\ell}^{c,p,\kappa}\) considered here, but a formula has yet to be found.

2. The results of Theorem 2 remain true for compact nodal curves if the potentials are associated with generic admissible families for \(Z\) and \(Z'\) (see §§2 and 3 for definitions).

3. It could be interesting to compare Theorem 2 with Gonchar’s results [9] on characterizing meromorphic extensions of germs of holomorphic functions to a fixed domain in the complex plane in terms of Padé approximations.

Our next theorem is an inverse version for the bordered case of the Riemann–Klein result. Without electric charges, it is contained in [15], Theorem 1. The precise definition of Dirichlet–Neumann data and how they are linked to the Neumann operator is given in the third section. In short, such data consist of a smooth oriented real curve \(\gamma\) which is the boundary of an open complex curve \(Z\), a triplet \(u = (u_0, u_1, u_2)\) of smooth real functions defined on \(\gamma\), and a triple \(\theta u = (\theta u_0, \theta u_1, \theta u_2)\) of smooth \((1,0)\)-forms, each \(\theta u_{\ell}\) being the boundary value of a form \(\partial \tilde{u}_{\ell}\), where \(\tilde{u}_{\ell}\) is the harmonic extension of \(u_{\ell}\) to \(Z \setminus \{a_j^\pm; 1 \leq j \leq \nu\}\) such that \(\partial \tilde{u}_{\ell}\) has residue \(\pm c_j\) at \(a_j^\pm\), \(1 \leq j \leq \nu\). An important requirement for \((\gamma,u,\theta u)\) to be Dirichlet–Neumann data is that the map \((\partial \tilde{u}_0^\pm : \partial \tilde{u}_1^\pm : \partial \tilde{u}_2^\pm)\) is well defined and injective outside a finite subset of \(Z\) and embeds \(\gamma\) in \(\mathbb{CP}_2\). However, as will be seen later, this is a generic condition (see Proposition 4). For simplicity, and because the charges constitute information to be recovered, we have chosen to let them be independent of \(\ell\).

**Theorem 3.** Let \(Z\) and \(Z'\) be bordered oriented smooth surfaces in \(\mathbb{R}^3\) equipped with the conformal structures induced by the standard Euclidean metric on \(\mathbb{R}^3\). Let \(\nu\) (respectively, \(\nu'\)) be the number of pairs of mutually distinct points \(a_j^\pm\) (respectively, \(a_j'^\pm\)) fixed in \(Z\) (respectively, \(Z'\)). Assign to each pair \(a_j^\pm\) (respectively, \(a_j'^\pm\)) non-zero complex “electric” charges \(\pm c_j\) (respectively, \(\pm c_j'\)) satisfying the generic conditions \(|c_j^\pm| \neq |c_k^\pm|\) (respectively, \(|c_j'^\pm| \neq |c_k'^\pm|\)), \(1 \leq j < k \leq \nu\) (respectively, \(1 \leq j < k \leq \nu'\)).

Assume that \((\gamma,u,\theta u)\) is Dirichlet–Neumann data for \(Z\) with each pair of points \(a_j^\pm\) carrying the charges \(c_j^\pm\), as well as Dirichlet–Neumann data for \(Z'\) with each pair of points \(a_j'^\pm\) carrying the charges \(c_j'^\pm\).

Then \(\nu = \nu'\), \((c_j)_{1 \leq j \leq \nu} = (c_j')_{1 \leq j \leq \nu'}\), and there is an isomorphism \(\varphi: Z \rightarrow Z'\) of Riemann surfaces such that \(\varphi|_\gamma = \text{Id} _\gamma\) and \(\varphi(a_j^\pm) = a_j'^\pm, 1 \leq j \leq \nu\). Moreover, \(Z\), \(\{a_j^\pm; 1 \leq j \leq \nu\}\), and \((c_j)_{1 \leq j \leq \nu}\) can be explicitly reconstructed from \((\gamma,u,\theta u)\).

The proof of Theorem 2 is given in §3.2. Because the pairs \(\{a_j^-, a_j^+\}\) can be seen as the singularities of a nodal curve, Theorem 3 is a consequence of Theorems 4 and 5, which deal with (generalized) nodal curves. Its proof is given at the end of §3 where our main theorems about inverse problems are stated. Section 2 is devoted to the definition of nodal surfaces and harmonic distributions, and to Dirichlet problems. A characterization of Dirichlet–Neumann data for a nodal curve is given in §4.
2. Direct problems for nodal curves

2.1. Nodal curves. In this paper, an open bordered Riemann surface is the interior of a one-dimensional compact complex manifold with boundary all of whose connected components have a non-trivial real one-dimensional smooth boundary. An open bordered nodal curve $X$ is the quotient of an open bordered Riemann surface $Z$ by an equivalence relation identifying a finite number of interior points; $Z$ is said to lie over $X$. We define compact nodal curves similarly, but for that case we require the connectedness of the compact Riemann surface.

The points of the singular set $\text{Sing} X$ of a nodal curve $X$ are called nodes. As a consequence, the intersections of the irreducible components of $X$ with neighbourhoods of nodes are germs of Riemann surfaces. If $a$ is any point of $X$, then we define the branches of $X$ at $a$ to be a family $(X_{a,j})_{1 \leq j \leq \nu(a)}$ of connected Riemann surfaces intersecting only at $a$ and whose union is a relatively compact neighbourhood of $a$ in $X$. Thus, the nodes of $X$ are the points $a$ of $X$ where $\nu(a) \geq 2$. If $\nu = 2$ at each node of $X$, a restriction not relevant for our theorems, then $X$ is a nodal curve as often defined in the literature. That is why we usually omit the word generalized when describing the nodal curves we consider here. If $X$ is compact and has a trivial automorphism group, and $\nu \equiv 2$, then $X$ is said to be stable (see, for instance, [11]).

Below we use $X_{a,j}^*$ to denote $X_{a,j} \setminus \{a\}$. The boundary of $X$ is denoted by $bX$; by definition $\overline{X} = X \cup bX$ is a manifold with boundary outside the set $\text{Sing} X$; its regular part $\text{Reg} \overline{X}$ is defined to be $bX \cup \text{Reg} X$, where $\text{Reg} X = X \setminus \text{Sing} X$. Note that if $X$ is an open bordered nodal curve and $\overline{X} \xrightarrow{\pi} \overline{X}$ ($\hat{X} \xrightarrow{\pi} X$ for short, but with a slight abuse of notation) is one of its normalizations, then $\hat{X}$ is an open bordered Riemann surface, and the set $\pi^{-1}(\text{Sing} X)$ is finite.

Two open bordered nodal curves $X$ and $X'$ are said to be isomorphic if there exists a bijective map $\varphi : \overline{X} \rightarrow \overline{X'}$ which is an isomorphism of Riemann surfaces from $\text{Reg} X$ onto $\text{Reg} X'$, a diffeomorphism of manifolds with boundary between certain open neighbourhoods of the real curves $bX$ and $bX'$ in $\overline{X}$ and $\overline{X'}$, and such that for each node $a$ of $X$ the (germs of the) branches of $X'$ at $\varphi(a)$ are the images under $\varphi$ of the (germs of the) branches of $X$ at $a$. In particular, such a map $\varphi$ has to be a homeomorphism.

A weaker notion of equivalence between nodal curves arises naturally in this paper. If the above map $\varphi$ has the first two properties but only sends the set (of germs) of branches of $X$ bijectively to the set (of germs) of branches of $X'$, then we say that $X$ and $X'$ are roughly isomorphic. Assuming that $X$ (respectively, $X'$) is the quotient of an open bordered Riemann surface $Z$ (respectively, $Z'$) and that $Z \xrightarrow{\pi} X$ (respectively, $Z' \xrightarrow{\pi} X'$) is the natural projection, another way of stating this is to ask for an isomorphism of open bordered Riemann surfaces from $\overline{Z}$ onto $\overline{Z'}$ which sends $\pi^{-1}(\text{Sing} X)$ onto $\pi'^{-1}(\text{Sing} X')$.

In order to settle appropriate Dirichlet problems, in the next subsection we define what is meant by a harmonic distribution on a nodal curve.

2.2. Harmonic distributions. If a nodal curve $X$ is an analytic subset of some open set in an affine space, then one can define smooth functions as the restrictions to $X$ of smooth functions on the ambient space. In the simple case when $X$ is the
union of two lines, it turns out that a function is smooth on \( X \) if and only if it is smooth on each branch of \( X \) and is continuous at the singular point of \( X \). Bearing in mind that every open bordered nodal curve can be embedded in a complex affine space (see [23]), we take this model as a guideline for a general definition. Translating the algebraic definitions of [18], [10], and [20] into analytic terms would have given the same result.

Let \( X \) be a (generalized) nodal curve, \( W \) an open set in \( X \), and \( r \in [0, +\infty] \). A function \( u \) on \( W \) is said to be of class \( C^r \) if it is continuous and if \( u|_B \in C^r(B) \) for any branch \( B \) of \( X \) contained in \( W \); the space of such functions is denoted by \( C^r(W) \) or \( C^r_{0,0}(W) \).

If \( p, q \in \{0, 1\} \) and \( p + q > 0 \), then a \((p, q)\)-form \( \omega \) in \( C^r_{p,q}(W \cap \text{Reg} \ X) \) is said to be of class \( C^r \) on \( W \) if for any branch \( B \) of \( X \) contained in \( W \) the form \( \omega|_B \) extends as an element of \( C^r_{p,q}(B) \). The space of such forms is denoted by \( C^r_{p,q}(W) \). Note that the question of the continuity of a form at nodes is relevant only when it is a function, since the tangent spaces to branches of \( X \) at the same node may be different.

If \( K \) is a compact subset of \( X \) and \( p, q \in \{0, 1\} \), then the space \( C_{p,q}^\infty(K) \) of smooth \((p, q)\)-forms supported in \( K \) is equipped with the topology induced by the seminorms \( \sup_{B \cap K} \| D^{(m)} \omega \|_B \); where \( m \) is any positive integer, \( B \) is any branch of \( X \) intersecting \( K \), and \( D \) is the total differential acting on the coefficients. The space \( D_{p,q}(W) \) of smooth \((p, q)\)-forms compactly supported in \( W \) is equipped with the topology of the inductive limit of the spaces \( C_{p,q}^\infty(K) \), where \( K \) is any compact subset of \( W \). The space \( D'_{p,q}(W) \) of currents on \( W \) of bidegree \((p, q)\) is the topological dual of \( D_{p,q}(W) \); the elements of \( D'_{1,1}(W) \) are the distributions on \( W \).

The exterior differential \( d \) of smooth forms is well defined along branches of \( X \), as are the differentials \( \partial \) and \( \bar{\partial} \). These operators extend to currents by duality.

A distribution \( u \in D'_{1,1}(W) \) is said to be (weakly) harmonic if the current \( i\partial\bar{\partial}u \) vanishes, that is, \( \langle i\partial\bar{\partial}u, \phi \rangle = 0 \) for all \( \phi \in C_{\infty}^\infty(W) \); this is equivalent to \( \partial u \) being (weakly) holomorphic in the sense of Rosenlicht [18]. Such distributions are ordinary harmonic functions near regular points. While a harmonic function on \( \text{Reg} \ X \) may have complicated singularities at a node, the following proposition shows that harmonic distributions have in the worse case logarithmic singularities which in a certain sense compensate each other.

**Proposition 2** (characterization of harmonic distributions). Let \( X \) be a nodal curve. Assume that \( u \) is a harmonic distribution on a neighbourhood of some point \( a \) in \( X \). Let \((X_{a,j})_{1 \leq j \leq \nu(a)}\) be the branches of \( X \) at \( a \), and assume that they are small enough for there to exist on each \( X_{a,j} \) a holomorphic coordinate \( z_j \) centered at \( a \). Then there exists a family \((c_{a,j})_{1 \leq j \leq \nu(a)}\) of complex numbers such that \( u|_{X^*_a,j} = -2c_{a,j} \log |z_j| \) extends to an ordinary harmonic function in a neighbourhood of \( a \) on \( X_{a,j} \) and \( \sum_{1 \leq j \leq \nu(a)} c_{a,j} = 0 \). In particular, \( \partial u \) is a meromorphic \((1, 0)\)-form whose singularities are simple poles at nodes of \( X \) and have residues \( c_{a,j} \) on \( X_{a,j} \) for \( a \in \text{Sing} \ X \). Conversely, \( u \) is a harmonic distribution if \( \partial u \) and \( (c_{a,j}) \) satisfy the indicated conditions.
Remark. The condition that the singularities of \( \partial u \) are only simple poles with vanishing sums of residues at each node is a criterion for the (weak) holomorphicity of \( \partial u \). This is in correspondence with the definition of dualizing sheaves given by Grothendieck in [10] and Hartshorne in [12], which represents an algebraic approach to (weakly) holomorphic forms.

Proof. For each \( k \) we assume that \( X_{a,k} \) is small enough so that \( z_k \) maps \( X_{a,k} \) bijectively onto \( \mathbb{D} = D(0,1) \), and we fix some function \( \xi_k \in C_c^\infty(X_{a,k}) \) such that \( \xi_k = 1 \) in a neighbourhood of \( a \) in \( X_{a,k} \). Consider some \( j \) in \( \{1, \ldots, \nu(a)\} \). The extension operator which associates \( \chi \in D_{1,1}(X_{a,j}) \) with the form \( E_j \chi \) satisfying \( (E_j \chi)_{|X_{a,k}} = 0 \) if \( k \neq j \) and \( (E_j \chi)_{|X_{a,j}} = \chi \) is a continuous operator from \( D_{1,1}(X_{a,j}) \) to \( D_{1,1}(W) \), where \( W = \bigcup_{1 \leq j \leq \nu(a)} X_{a,j} \). Hence, \( u_j = u \circ E_j \) is a distribution on \( X_{a,j} \). Since \( u|_{X_{a,j}}^\ast \) is an ordinary harmonic function, \( v_j = u_j \circ z_j^{-1} \) is an ordinary harmonic function on \( \mathbb{D}^* = \mathbb{D} \setminus \{0\} \) which extends to \( \mathbb{D} \) as a distribution.

This is possible only if the holomorphic function \( \frac{\partial v_j}{\partial z} \) does not have an essential singularity at 0. Indeed, consider \( \varepsilon \in \]0, 1[\), \( \xi \in C_c^\infty \left[\frac{1}{4}, \frac{3}{4}\right] \), \( p \in \mathbb{Z} \cap ]-\infty, -2[ \),

\[
\chi_{\varepsilon} = \chi \left( \frac{|z|}{\varepsilon} \right) \left( \frac{z}{|z|} \right)^{-p} i dz \wedge d\overline{z},
\]

and \( \chi_{\varepsilon,j} = (z_j)^* \chi_{\varepsilon} \). Then \( E_j \chi_{\varepsilon,j} = 0 \) on \( \bigcup_{k \neq j} X_{a,k} \) and on \( \{|z_j| \leq \frac{\varepsilon}{4}\} \). Hence if the Laurent series of \( \frac{\partial v_j}{\partial z} \) has the form \( (\sum c_{j,n} z^n) \), then

\[
\langle v_j, \chi_{\varepsilon} \rangle = \langle u_j, \chi_{\varepsilon,j} \rangle = \langle u, E_j \chi_{\varepsilon,j} \rangle = \int_{\{\varepsilon/4 \leq |z_j| \leq 3\varepsilon/4\}} u \chi_{\varepsilon} = \sum_{n \in \mathbb{Z}} \int_{\varepsilon/4}^{3\varepsilon/4} \int_0^{2\pi} c_{j,n} \chi \left( \frac{r}{\varepsilon} \right) e^{i\pi(n-p)} r^{n+1} dr d\chi
\]

\[
= 2\pi c_{j,p} \int_{\varepsilon/4}^{3\varepsilon/4} \chi \left( \frac{r}{\varepsilon} \right) r^{p+1} dr = I_p c_{j,p} \varepsilon^{p+2},
\]

where \( I_p = 2\pi \int_{1/4}^{3/4} \chi(s)s^{p+1} ds \). On the other hand, the fact that \( v_j \) is a distribution on \( \mathbb{D} \) implies that there exists a pair \( (C_j, n_j) \in \mathbb{R}_{+} \times \mathbb{N} \) such that

\[
|\langle u_j, \theta \rangle| \leq C_j \sup_{0 \leq m \leq n_j} \|D^m \theta\|_{\infty}
\]

for any \( \theta \in C_c^\infty \left( \frac{3}{4} \mathbb{D} \right) \). Since \( \|\chi_{\varepsilon}\|_{n_j} \leq \frac{\text{const}}{\varepsilon^{n_j}} \), the last equality above implies that \( c_{j,p} = 0 \) if \( p < -d_j = -n_j - 2 \). Thus, if \( \delta_0 \) denotes the Dirac measure, then

\[
\frac{\partial^2 v_j}{\partial z \partial \overline{z}} = \sum_{1 \leq n \leq d_j} \tilde{c}_{j,-n} \frac{\partial^{n-1} \delta_0}{\partial z^{n-1}},
\]

where \( \tilde{c}_{j,-n} = \pi \left( \frac{-1}{(n-1)!} \right) c_{j,-n} \) since \( \frac{\partial}{\partial z} \frac{1}{z} = \pi \delta_0 \).

Assume now that \( \chi \) is any smooth function compactly supported in \( \bigcup_k \{\xi_k = 1\} \). It follows from the definition that the \((1,1)\)-form \( i \partial \overline{\partial} \chi \) can be written as the sum of the smooth forms \( E_j(i \partial \overline{\partial} \chi_j) \), where \( \chi_j = \chi|_{X_{a,j}} \). Hence, setting \( \theta_j = \chi_j \circ z_j^{-1} \),

\[
\frac{\partial^2 v_j}{\partial z \partial \overline{z}} = \sum_{1 \leq n \leq d_j} \tilde{c}_{j,-n} \frac{\partial^{n-1} \delta_0}{\partial z^{n-1}},
\]

where \( \tilde{c}_{j,-n} = \pi \left( \frac{-1}{(n-1)!} \right) c_{j,-n} \) since \( \frac{\partial}{\partial z} \frac{1}{z} = \pi \delta_0 \).
we get that

\[
0 = \langle i\partial\bar{\partial}u, \chi \rangle = \langle u, i\partial\bar{\partial}\chi \rangle = \sum_{1 \leq j \leq \nu(a)} \langle u, E_j(i\partial\bar{\partial}x_j) \rangle
\]

\[
= \sum_{1 \leq j \leq \nu(a)} \langle u_j, i\partial\bar{\partial}x_j \rangle = \sum_{1 \leq j \leq \nu(a)} \langle i\partial\bar{\partial}v_j, \theta_j \rangle
\]

\[
= \chi(0) \sum_{1 \leq j \leq \nu(a)} \tilde{c}_{j,-1} + \sum_{1 \leq j \leq \nu(a)} \sum_{2 \leq n \leq d_j} \tilde{c}_{j,-n} \frac{\partial^{n-1}\theta_j}{\partial z^{n-1}}(0).
\]

Since \( \left( \frac{\partial^m \theta_j}{\partial z^m}(0) \right)_{m \geq 1} \) can be any sequence of complex numbers, the above equality implies that \( c_{j,-n} = 0 \) for \( n \leq 2 \) and \( \sum_{1 \leq j \leq \nu(a)} c_{j,-1} = 0 \). The converse statement of the proposition is obvious. \( \square \)

2.3. Green functions and Dirichlet problems. Because our proofs use principal Green functions for smooth curves and because inverse problems require constructive methods, we take the opportunity in this paper to recall how these functions can be built with constructive tools. In 1828 Green conjectured the existence of such functions for domains in \( \mathbb{R}^3 \). We recall that a Green function for an open bordered Riemann surface \( Z \) is a symmetric function \( g \) defined on \( \overline{Z} \times \overline{Z} \) outside its diagonal such that for any \( z \in Z \) the function \( g_z = g(\cdot, z) \) is harmonic on \( Z \setminus \{z\} \), smooth on \( \overline{Z} \setminus \{z\} \), and has the singularity \((2\pi)^{-1} \log \text{dist}(\cdot, z)\) at \( z \), the distance being computed in any Hermitian metric on \( Z \). It is said to be principal if \( g_z|_{\partial Z} = 0 \) for any \( z \in Z \). The existence of Green functions for smooth bordered Riemann surfaces follows from classical works of Fredholm and Hilbert. In [16] an explicit construction was derived from Cauchy type formulae even for singular Riemann surfaces. The theorem below recalls how to obtain a principal Green function from an ordinary one.

**Theorem.** Let \( Z \) be an open bordered Riemann surface and \( g \) a Green function for \( Z \). Assume that \( \overline{Z} \) lies on some complex curve \( \overline{Z} \), for instance, its double. Consider the operator \( T: C^\infty(\gamma) \to C^\infty(\overline{Z} \setminus \gamma) \) defined by \( T v: z \mapsto 2i \int v \overline{\partial g_z} \). If \( v \in C^\infty(\gamma) \), then let \( T^+ v \) denote the restriction of \( T v \) to \( Z^+ = Z \) and \( T^- v \) the restriction of \( T v \) to \( Z^- = \overline{Z} \setminus Z \). Then the following hold.

1. For any \( v \in C^\infty(\gamma) \) the function \( T^\pm v \) is harmonic on \( Z^\pm \) and extends continuously to \( \gamma \).

2. (Sohotsky (1873) for \( Z \subset \mathbb{C} \)) \( v = (T^+ v)|_{\gamma} - (T^- v)|_{\gamma} \) for any \( v \in C^\infty(\gamma) \).

3. (Fredholm (1900) for \( Z \subset \mathbb{C} \)) For any \( u \in C^\infty(\gamma) \) the unique harmonic extension \( E u \) of \( u \) to \( Z \) is the solution \( u \) of the integral equation \( u = v + (T^- v)|_{\gamma} \).

4. The principal Green function for \( Z \) is the function \( G \) defined by \( G(z, \zeta) = g(z, \zeta) - (E g_z)|_{\gamma}(\zeta) \) for all \( (z, \zeta) \in \mathbb{Z}_2 \) with \( z \neq \zeta \).

We define an admissible family for an open bordered nodal curve \( X \) to be a family \( (c_{a,j})_{a \in \text{Sing} X, 1 \leq j \leq \nu(a)} \) of complex numbers such that \( \sum_{1 \leq j \leq \nu(a)} c_{a,j} = 0 \) for each node \( a \) of \( X \). The following proposition generalizes the initial formulation of Riemann and Klein given in the Introduction. It is a consequence of the above classical result and Proposition 2.


Proposition 3 (solution of the nodal Dirichlet problem). Let $X$ be an open bordered or compact nodal curve, let $c = (c_{a,j})_{a \in \text{Sing } X, 1 \leq j \leq \nu(a)}$ be an admissible family, and for each sufficiently small branch $X_{a,j}$ at a node $a$ in $X$ fix some holomorphic coordinate $z_j$ on $X_{a,j}$ centered at $a$. If $X$ is non-compact, then also fix a $u \in C^0(bX)$. Then there exists a unique (up to an additive constant if $X$ is compact) harmonic distribution $\tilde{u}^c$ on $X$ such that $|\tilde{u}^c|_{bX} = u$ (if $X$ is non-compact) and $|\tilde{u}^c|_{X_{a,j}} - 2c_{a,j} \log |z_j|$ extends as an ordinary harmonic function to a neighbourhood of $a$ on $X_{a,j}$. Equivalently, $\tilde{u}^c$ is the harmonic distribution $U$ extending $u$ to $X$ such that $\partial U$ is a meromorphic $(1,0)$-form whose singularities are simple poles at nodes of $X$ with residue $c_{a,j}$ along $X_{a,j}$ for $a \in \text{Sing } X$.

Proof. Assume that $X$ is non-compact. Then let $\hat{g}$ be a Green function for a normalization $\hat{X} \xrightarrow{\pi} X$ of $X$ such that $\hat{g}_\zeta \overset{\text{def}}{=} \hat{g}(\zeta, \cdot ) = 0$ on $b\hat{X}$ for any point $\zeta \in \hat{X}$. Since $\bar{X}$ is a smooth manifold with boundary near $bX$, $b\hat{X} \xrightarrow{\pi} bX$ is a diffeomorphism and $v = \pi^*u$ is a well-defined continuous function on $b\hat{X}$. Let $V$ be the distribution defined on $\hat{X}$ by

$$V = \tilde{v} + \sum_{a \in \text{Sing } X} \sum_{1 \leq j \leq \nu(a)} c_{a,j} \hat{g}_{a,j},$$

where $\tilde{v}$ is the harmonic extension of $v$ to $\hat{X}$ and where $\{a_1, \ldots, a_{\nu(a)}\} = \pi^{-1}(a)$ and $X_{a,j} = \pi(W_j)$ for each $a \in \text{Sing } X$, $W_j$ being some neighbourhood of $a_j$ in $\hat{X}$. Then $\tilde{u}^c = \pi_* V$ is a distribution on $X$ which is an ordinary harmonic function on $\text{Reg } \hat{X}$ that extends $u$. The same kind of calculations as in Proposition 2 show that $\tilde{u}^c$ is a harmonic distribution on $X$. Since $\bar{X}$ has a smooth boundary, $\tilde{u}^c$ has the same regularity as $u$ in a neighbourhood of $bX$ in $\bar{X}$.

To prove uniqueness, we have to show that if $U$ is a harmonic distribution on $X$ which vanishes on $bX$ and has an extension to any branch of $X$ as an ordinary harmonic function, then $U = 0$. Let us consider such a distribution $U$. Then $V = \pi^* U$ is a well-defined function; if $a_j \in \pi^{-1}(a)$ is in the closure of $\pi^{-1}(X_{a,j}^*)$, then $V(a_j)$ is the value at $a$ of the harmonic extension of $U|_{X_{a,j}^*}$. Of course, $V$ is harmonic on $\hat{X}$, continuous up to the boundary, and vanishes on the boundary. Hence, $V = 0$, and therefore $U = 0$.

When $X$ is compact, the classical construction techniques for bipolar Green functions can be adapted to obtain on a normalization of $X$ multipolar Green functions which, due to the properties of admissible families, can be seen as harmonic distributions on $X$. □

Proposition 3 shows in particular that any continuous function on the boundary of $X$ has many (weakly) harmonic distribution extensions to $X$ if no data are specified at nodes. This non-uniqueness phenomenon also occurs for principal Green functions on nodal surfaces.

3. Inverse problems for nodal curves

The inverse Dirichlet-to-Neumann problem (IDN problem for short) for a given (smooth) Riemann surface $X$ with smooth boundary $\gamma$ is to reconstruct $X$ from
the data for $\gamma$, $T_\gamma X$, and its Dirichlet-to-Neumann operator, that is, the operator associating with a smooth function on $\gamma$ the restriction to $\gamma$ of the normal derivative of the harmonic extension of this function to $X$. This topic was initiated by Belishev and Kurylev [3] in a non-stationary setting. For the stationary case, uniqueness results based on a full knowledge of the DN-operator are obtained in [17] and [2]. The constructive reconstruction method given in [15] is extended here to nodal Riemann surfaces, compact or non-compact.

3.1. DN-data: hypotheses A and B. Let $X$ be an open bordered nodal curve. Since $\overline{X}$ has smooth boundary, we can select two vector fields $\tau$ and $\nu$ along $bX$ such that $\tau$ is a smooth generating section of the tangent bundle $T(bX)$ of $bX$ and for each $x$ in $bX$ the vectors $(\nu_x, \tau_x)$ form a positively oriented orthonormal basis in $T_x \overline{X}$. Then the Dirichlet-to-Neumann operator for $X$ with some admissible family $c = (c_{a,j})_{a \in \text{Sing} X, 1 \leq j \leq \nu(a)}$ is the operator $N_{X,c}$ defined for any $u \in C^1(bX)$ by

$$N_{X,c} u = \frac{\partial \tilde{u}^c}{\partial \nu} \bigg|_{bX},$$

where $\tilde{u}^c$ is the extension of $u$ to $X$ as a harmonic distribution such that $\partial \tilde{u}^c$ has residue $c_{a,j}$ at $a$ for all $a \in \text{Sing} X$ and $1 \leq j \leq \nu(a)$.

Since an admissible family does not ‘feel’ the complex structure of $X$ but only reflects the existence of nodes, a natural inverse Dirichlet-to-Neumann problem is to find a process reconstructing $X$ from the data for $bX$ and the action of some operator $N_{X,c}$ on some function $u \in C^1(bX)$, where $c$ belongs to an a priori unknown set of admissible families.

It is very natural to ask whether admissible families can be determined from boundary data if one considers the physical origin of the problem as discussed in the Introduction: such families correspond to the charges located at the nodes.

With these inverse reconstruction problems there arises the question of the uniqueness of an open bordered nodal curve having given boundary data. Thus, let $\gamma$ be a smooth compact oriented real curve with no components reducing to a point. Let $\tau$ be a smooth generating section of $T\gamma$ and $\nu$ another vector field along $\gamma$ such that the bundle $\mathcal{T}$ generated by $(\nu_x, \tau_x)_{x \in \gamma}$ has rank 2; $\gamma$ is assumed to be oriented by $\tau$ and $\mathcal{T}$ by $(\nu, \tau)$. Consider an operator $N$ from $C^1(\gamma)$ to the space of currents on $\gamma$ of degree 0 and order 1 (that is, functionals on $C^1$-smooth 1-forms on $\gamma$). As in [15], we use a setting which emphasizes the complex analysis involved. With $N$ come two other operators $L$ and $\theta$ defined on a $u \in C^1(\gamma)$ by

$$Lu = \frac{1}{2}(Nu - iTu) \quad \text{and} \quad \theta u = (Lu)(\nu^* + i\tau^*),$$

where $T$ is the tangential derivation along $\tau$ and $(\nu_x^*, \tau_x^*)$ is the dual basis of $(\nu_x, \tau_x)$ at $x \in \gamma$.

Note that if $\gamma$ is actually the smooth boundary of a bordered nodal curve $X$ and $(\nu_x, \tau_x)$ is a positively oriented orthonormal basis in $T_x \overline{X}$, then the equality $Nu = N_{X,c} u$ is equivalent to the identity $(\partial \tilde{u}^c)_{|\gamma} = \theta u$.

In the smooth case we know from [15] that knowledge of $\gamma$ and of the action of $N$ on only three generic (in the sense detailed below) continuous functions is sufficient to reconstruct such a Riemann surface when it exists. Since admissible families
(c_{a,j}) are not tied to the complex structure of X, it is natural to let the given boundary data correspond to different admissible families and hence to different operators N. Thus, we are led to consider the following.

**A.** We consider three operators \( N_0, N_1, \) and \( N_2 \) from \( C^1(\gamma) \) to the space of currents on \( \gamma \) of degree 0 and order 1, their corresponding operators \( \theta_\ell = (\nu^* + i\tau^*)L_\ell \), where \( L_\ell = (N_\ell - iT)/2 \), and three real-valued functions \( u_0, u_1, u_2 \in C^\infty(\gamma) \) governed only by the hypothesis that

\[
f = (f_1, f_2) = \left( \frac{L_\ell u_\ell}{L_0 u_0} \right)_{\ell=1,2} = \left( \frac{\theta_\ell u_\ell}{\theta_0 u_0} \right)_{\ell=1,2}
\]

(3.2)

is an embedding of \( \gamma \) in \( \mathbb{C}^2 \) regarded as the complement of the line \( \{w_0 = 0\} \) in the complex projective plane \( \mathbb{CP}_2 \) with homogeneous coordinates \( (w_0 : w_1 : w_2) \). This is in some way generic, since Proposition 4 below shows in particular that if it happens that \( \gamma \) is the smooth boundary of a complex curve and \( N \) is a Dirichlet-to-Neumann operator, then the set of \( (u_\ell)_{0 \leq \ell \leq 2} \in C^\infty(\gamma)^3 \) such that the above map \( f \) is an embedding is a dense open subset of \( C^\infty(\gamma)^3 \).

Before defining DN-data, we must explain how a triple \( \omega = (\omega_0, \omega_1, \omega_2) \) of smooth \((1,0)\)-forms not vanishing simultaneously induces a map from \( X \) to \( \mathbb{CP}_2 \); such triples exist because a normalization of \( X \) has some. We define a map from \( X \) to \( \mathbb{CP}_2 \), denoted by \([\omega]\) or \( (\omega_0 : \omega_1 : \omega_2) \), by the formulae \([\omega] = \left( 1 : \frac{\omega_1}{\omega_0} : \frac{\omega_2}{\omega_0} \right)_{\omega_0 \neq 0} \),

\[
[\omega] = \left( \frac{\omega_0}{\omega_1} : 1 : \frac{\omega_2}{\omega_1} \right)_{\omega_1 \neq 0}, \text{ and } [\omega] = \left( \frac{\omega_0}{\omega_2} : \frac{\omega_1}{\omega_2} : 1 \right)_{\omega_2 \neq 0};
\]

here each quotient is a well-defined (along any branch) meromorphic function, since \( \dim X = 1 \). In [15], with a slight abuse of language, we identified \([\omega] \) with \([\omega]_{\{\omega_0 \neq 0\}} \), with the affine coordinates \( \left( \frac{\omega_1}{\omega_0}, \frac{\omega_2}{\omega_0} \right) \) in the affine hyperplane \( \{w_0 \neq 0\} \).

Assume now that \( u = (u_\ell)_{0 \leq \ell \leq 2} \in C^\infty(\gamma)^3 \) and let \( \theta u = (\theta_\ell u_\ell)_{0 \leq \ell \leq 2} \). We call \((\gamma, u, \theta u)\) DN-data for an open bordered nodal curve \( X \) if \((\gamma, u, \theta u)\) satisfies A and the following conditions hold.

**B1.** \( X \) has smooth boundary \( \gamma \).

**B2.** For each \( \ell \in \{0, 1, 2\} \) the equality \( \theta_\ell u_\ell = (\partial \tilde{u}_\ell^{c\ell}) \big|_\gamma \) holds for some admissible family \( c_\ell \).

**B3.** The forms \( \partial \tilde{u}_\ell^{c\ell} \) have no common zeros, and the map

\[
F = (\partial \tilde{u}_0^{c_0} : \partial \tilde{u}_1^{c_1} : \partial \tilde{u}_2^{c_2})
\]

extends to \( X \) the map \((1 : f_1 : f_2)\) defined by (3.2), in the sense that for every \( x_0 \in \gamma \) the limit

\[
\lim_{x \to x_0, x \in X} \left( \frac{\partial \tilde{u}_1^{c_1}}{\partial \tilde{u}_0^{c_0}}(x), \frac{\partial \tilde{u}_2^{c_2}}{\partial \tilde{u}_0^{c_0}}(x) \right)
\]

exists and equals \((f_1(x_0), f_2(x_0))\); this last condition holds automatically if \( \gamma \) and \( f \) are real-analytic.

**B4.** There is a finite subset \( A \) of \( X \) such that \( F \) is an embedding of \( Z \setminus A \) in \( \mathbb{CP}_2 \).
If we wish to emphasize the admissible families \( c_\ell \), then we say that \( (\gamma, u, \theta u) \) and the \( c_\ell \) are associated. If each node \( a_n \), \( 1 \leq n \leq N \), of \( X \) is obtained by identifying the points in a subset \( (a_{n,j})_{1 \leq j \leq \nu_n} \) on the Riemann surface \( Z \) and if the family of charges or residues corresponding to \( a_n \) and \( u_\ell \) is \( (c_{\ell,n,j})_{1 \leq j \leq \nu_n} \) \( (0 \leq \ell \leq 2) \), then we also say that \( (\gamma, u, \theta u) \) is DN-data for \( Z \) and for the \( a_{n,j} \) with charges \( c_{\ell,n,j} \) (or \( c_{n,j} \) if the \( c_{\ell,n,j} \) do not depend on \( \ell \)).

The condition B4 may seem restrictive but it is an open and dense condition in the following sense.

**Proposition 4.** Assume that \( \gamma \) is the boundary of an open nodal Riemann surface \( X \) and \( c = (c_\ell)_{0 \leq \ell \leq 2} \) is a triple of admissible families. Then the set \( E_{X,c} \) of \( u \) in \( C^\infty(\gamma)^3 \) such that \( (\gamma, u, \theta u) \) is DN-data for \( X \) and \( c \) is a dense open subset of \( C^\infty(\gamma)^3 \).

As stated in the Introduction, this result, together with Proposition 1, will be proved in a separate paper because the proofs involve methods and results of complex analysis which deserve attention in their own right.

When \( X \) is smooth, there are no nodes and the only triple of admissible families is the empty one. Dropping any reference to admissible families in the above definition gives a concept of DN-data in the smooth case. Because of B4, the DN-data thus defined here are more specific than those considered in [15]. Actually, in the exceptional case when \( (\gamma, u, \theta u) \) satisfies only B1–B3, it is possible that, contrary to the assertion in the proof of Lemma 7 in [15], the direct image under \( F \) of the integration current on \( X \) may not always be an integration current over a subvariety of \( \mathbb{CP}_2 \setminus f(\gamma) \). The reason is that in the general case \( F_*[X] \) might not even be a locally flat current as defined in [6]. However, all statements of [15] are true with the above refined definition of DN-data.

It follows from the definitions that if \( X \) is an open bordered nodal curve obtained by identifying some points in an open bordered Riemann surface \( Z \) and \( \pi: Z \to X \) is the natural projection, then the direct image under \( \pi \) of any harmonic function on \( Z \) continuous up to the boundary of \( \bar{Z} \) is a harmonic distribution on \( X \) solving a Dirichlet problem with a zero admissible family. Hence, the data of its differential along \( bX \) fail to provide any information about the nodal curve except about its normalization. This motivates the following definition. We say that a finite family \( (w_s)_{s \in \Sigma} \) of complex numbers is generic for a partition \( \{\Sigma_1, \ldots, \Sigma_N\} \) of the index set \( \Sigma \) if the following hold:

(i) \( \sum_{s \in \Sigma_j} w_s = 0 \) for any \( j \);

(ii) for any family of sets \( (T_j)_{1 \leq j \leq N} \) such that \( T_j \subseteq \Sigma_j \) for all \( j \) and \( \bigcup_{1 \leq j \leq N} T_j \neq \emptyset \) we have \( \sum_{1 \leq j \leq N} \sum_{t \in T_j} w_t \neq 0 \).

An admissible family \( c = (c_{a,j})_{(a,j) \in \Sigma} \) \( (\Sigma = \bigcup_{a \in \text{Sing} X} \Sigma_a, \Sigma_a = \{a\} \times \{1, \ldots, \nu(a)\}) \) on an open bordered nodal curve \( X \) is said to be generic for \( X \) if it is generic for the partition \( \{\Sigma_a; a \in \text{Sing} X\} \), that is, if the only way to achieve the equalities \( \sum_{a \in \text{Sing} X} \sum_{j \in J_a} c_{a,j} = 0 \) for all \( a \in \text{Sing} X \) with \( J_a \subseteq \{1, \ldots, \nu(a)\} \) is to have either \( J_a = \{1, \ldots, \nu(a)\} \) or \( J_a = \emptyset \) for all \( a \).

### 3.2. Proofs of results for the compact case.

**Proof of Theorem 1.** We assume without loss of generality that \( S \) is smoothly bordered and \( \gamma = bS \) is then equipped with the orientation induced by \( Z \setminus S \). Let
Theorem 4 (Uniqueness for IDN problems). Assume that \( X \) and \( X' \) are bordered nodal curves with DN-data \((\gamma, u, \theta u)\) on them associated with admissible families \(c_0, c_1, c_2\) for \( X \) and \( c'_0, c'_1, c'_2\) for \( X'\). Then the following hold.

1. \( X \) and \( X' \) are obtained by identifying certain finite sets of points in the same open bordered Riemann surface.

2. If at least one of the admissible families associated with \((\gamma, u, \theta u)\) has no zero coefficients, then \( X \cup \gamma \) and \( X' \cup \gamma \) are roughly isomorphic via a map which is the identity on \( \gamma \).

3. If at least one of the \( c_\ell \) and one of the \( c'_\ell \) are generic for \( X \) and \( X' \), respectively, then there is an isomorphism of bordered nodal curves between \( X \cup \gamma \) and \( X' \cup \gamma \) whose restriction to \( \gamma \) is the identity.

Remarks. 1. If \( E \subset \gamma \) and \( h^1(E \cap c) > 0 \) for each connected component \( c \) of the real curve \( \gamma \), then meromorphic functions are uniquely determined by their values on \( E \) and the conclusions of Theorem 4 hold even when \( N_{X',c'}u_\ell = N_{X,c}u_\ell \) only on \( E \), and the meromorphic functions \((\partial \tilde{u}^{c'})/(\partial \tilde{u}^{c}_0)\) and \((\partial \tilde{u}'^{c'})/(\partial \tilde{u}'^{c_0})\) are continuous near \( \gamma \).

2. The assertion 2 of the theorem shows that if two bordered nodal curves \( X \) and \( X' \) share the same DN-data \((\gamma, u, \theta u)\), then there is only a finite indeterminacy between \( X \) and \( X' \).

3. If one of the families \( c_\ell \) is generic for \( X \), then there is a holomorphic surjective map from \( X \) onto \( X' \), that is, a continuous map holomorphic along any branch of

\[
u = (U^{a,c}_{Z,\ell}|_{\gamma})_{0 \leq \ell \leq 2} \quad \text{and} \quad \theta u = (\partial_Z U^{a,c}_{Z,\ell}|_{\gamma})_{0 \leq \ell \leq 2} \quad \text{(respectively, } u' = (U^{a,c}_{Z',\ell}|_{\gamma})_{0 \leq \ell \leq 2} \quad \text{and} \quad \theta' u' = (\partial_{Z'} U^{a,c}_{Z',\ell}|_{\gamma})_{0 \leq \ell \leq 2})\text{, where } \partial_Z \text{ and } \partial_{Z'} \text{ are the Cauchy–Riemann operators on } Z \text{ and } Z'.\]
X which sends a node of X to a node of X'. Hence, X' may be considered as a quotient of X and, equivalently, X as a partial normalization of X'.

**Proof.** Let \( Z \xrightarrow{\pi} X \) be a normalization of X and \( g = f \circ \pi \), where \( f \) is defined by (3.2). Then \( \hat{\gamma} = \pi^{-1}(\gamma) \) is a smooth oriented real curve and bounds \( \overline{Z} \) smoothly. Due to hypothesis A, \( \delta = f(\gamma) = g(\hat{\gamma}) \) is a smooth compact oriented real curve in \( \mathbb{C}P_2 \) with no components reducing to a point. It follows from B2 that \( G = F \circ \pi \) is a meromorphic extension of \( g \) to \( Z \). To prove the assertion 1, we now follow the proof of Theorem 1 in [15], for which the key point is that the embedding \( g \) of the real curve \( \hat{\gamma} \) into \( \mathbb{C}P_2 \) extends meromorphically to the complex curve Z as the map \( G \). The gap in the proof arising under the conditions of Lemma 7 in [15] is now avoided thanks to the refined but still generic definition of DN-data.

Since \( (\gamma, u, \theta u) \) satisfies B4 and X is nodal, there is a finite set \( A \) in \( Z \) such that \( G \) is an isomorphism of \( Z \setminus A \) to \( G(X) \setminus G(A) \). Thus, \( Y = G(X) \setminus \delta \) has to be a complex curve in \( \mathbb{C}P_2 \) satisfying \( d[Y] = [\delta] \). It contains no compact complex curves because X has none, and it has finite mass by a theorem of Wirtinger (see [13]).

Note that \( A \) may intersect \( G^{-1}(\delta) \) but that each point of \( Z \) has a neighbourhood \( V \) in \( Z \) such that \( F: V \rightarrow F(V) \) is a diffeomorphism between manifolds with smooth boundaries. Hence, the conclusions of Lemma 7 in [15] are valid.

Let \( Z_o = Z \setminus G^{-1}(\delta) \), \( B = \text{Sing}Y \), and \( \tilde{A} = G^{-1}(B) \); we note that \( S = \pi^{-1}(\text{Sing} X) \subset \tilde{A} \). The proof of Lemma 9 in [15] gives us that \( G: Z_o \setminus \tilde{A} \rightarrow Y \setminus B = \text{Reg} Y \) is an isomorphism of complex manifolds, \( G: \overline{Z} \rightarrow \overline{Y} \) is a proper map, and \( G: \overline{Z} \setminus A \rightarrow \overline{Y} \setminus B = \text{Reg} \overline{Y} \) is an isomorphism of manifolds with smooth boundaries. The same construction can be carried out for \( X' \). Using the prime symbol to denote all objects connected with \( X' \), we can repeat the end of the proof of Theorem 1 of [16] to get that \( \overline{Z} \) and \( \overline{Z'} \) are isomorphic via a map \( \Phi: X \rightarrow X' \) which is the identity on \( \gamma \). Thus, the first assertion of the theorem is proved.

The \((1,0)\)-forms \( U_1 = \pi^* \partial \overline{u}_1^c \) and \( U'_1 = \Phi^* \pi'^* \partial \overline{u}_{1'}^c \) are meromorphic on \( Z \) and they are equal on \( \gamma = bZ \). Hence they are equal on \( Z \). In particular, they have the same poles and the same residues there. If \( c_1 \) or \( c'_1 \) has no zero coefficients, then \( \Phi^{-1}(S') = S \). Thus, the second assertion of the theorem is proved.

Let \( \Sigma = \bigcup_{a \in \text{Sing} X} \{ a \} \times \{ 1, \ldots, \nu(a) \} \) and \( \Sigma' = \bigcup_{a' \in \text{Sing} X'} \{ a' \} \times \{ 1, \ldots, \nu(a') \} \). For each \( a \) in \( \text{Sing} X \) (respectively, each \( a' \) in \( \text{Sing} X' \)), we fix an enumeration \( (X_{a,j})_{1 \leq j \leq \nu(a)} \) (respectively, \( (X'_{a',j'})_{1 \leq j' \leq \nu(a')} \)) of the branches of X at a (respectively, of \( X' \) at \( a' \)). Let \( (a, j) \) be a pair in \( \Sigma \) and \( W_{a,j} = \pi^{-1}(X_{a,j}) \). Then the restriction \( \pi_j = \pi|_{W_{a,j}} \) is bijective, so we can define \( s_j = \pi_j^{-1}(a) \in S \), \( s_j' = \Phi(s_j) \), \( b(a,j) = \pi'(s') = \pi'(\Phi_j(\pi_j^{-1}(a))) \), and \( \sigma(a,j) = (b(a,j), k(a,j)) \), where \( k(a,j) \) is the integer such that \( \pi'(\Phi(W_{a,j})) \) and \( X'_{b(a,j),k(a,j)} \) have the same germ at \( b(a,j) \). The map \( \sigma: \Sigma \rightarrow \Sigma' \) is bijective by construction; we set \( \sigma^{-1} = (b', k') \).

The map \( \varphi = \pi' \circ \Phi \circ \pi_r^{-1} \left( \pi_r \overset{\text{def}}{=} \pi|_{\text{Reg} X} \right) \) extends as a many-valued map (still denoted by \( \varphi \)) continuously along each branch of X if \( \varphi|_{X^*_a,j} \) is extended at \( a \) by the value \( b(a,j) \). Likewise, we denote by \( \psi \) the many-valued extension of \( \pi \circ \Phi^{-1} \circ \pi'^{-1} \left( \pi' \overset{\text{def}}{=} \pi'|_{\text{Reg} X'} \right) \) such that for any \( (a', j') \in \Sigma' \) the restriction \( \psi|_{X'_{a',j'}} \) is continuous and, hence, takes the value \( b'(a', j') \) at \( (a', j') \). To see that
X and X′ are actually isomorphic, we check that if a is a given node of X, then 
\[ b(a, 1) = \cdots = b(a, \nu(a)) \] def = a′ and \( \nu(a′) = \nu(a) \).

Assume now that \( c_1 \) is generic for X. Let \( a′ \) be a node of X′. The family 
\( (X_{a′, j′})_{1 \leq j′ \leq \nu(a′)} \) of its branches can be decomposed into disjoint subfamilies 
\( (X_{a′, j′})_{j′ \in J_m^a} \), 1 \( \leq m \leq \mu \), such that for each m and each \( j′ \in J_m^a \) the set 
\( \{a_m\} \cup \varphi^{-1}(X_{a′, j′}) \) is some branch \( X_{a_m, \ell_m(j′)} \) of X at some node a_m of X. Since 
\( \widetilde{w}_1 e_1 \) is a harmonic distribution solving a Dirichlet problem associated with the 
admissible family \( (c_1′) \), we know that 
\[
0 = \sum_{1 \leq j \leq \nu(a′)} c_1′_{a, j} = \sum_{1 \leq j \leq \nu(a′)} c_{1, \sigma^{-1}(a′, j′)} = \sum_{1 \leq m \leq \mu} \sum_{j′ \in J_m^a} c_{1, a_m, \ell_m(j′)}.
\]

Because \( (c_1) \) is generic for X, this implies that \( \ell_m \) is a bijection from \( J_m^a \) onto 
\{1, \ldots, \nu(a_m)\}. Hence if a is one of the points a_m, then any branch X_{a, j} of X at a 
is sent by \( \varphi \) to a branch of X′ at a′. Since any node of X is sent by \( \varphi \) to a node of 
X′, this proves that \( \varphi \) is actually continuous. If also one of the \( (c_1′) \) is generic 
for X′, then the same conclusions apply for \( \psi \). □

Now that a reasonable uniqueness has been proved for the nodal IDN-problem, 
the question of reconstructing solutions from the boundary data arises. The second 
assertion of Theorem 5 below shows how to first recover \( F(X) \) and \( \partial u_\ell \) from \( \theta u_\ell \) and 
the intersection of \( F(X) \) with the lines \( \Delta_\xi = \{z_2 := \frac{w_2}{w_0} = \xi\} \), \( \xi \in \mathbb{C} \). Once this 
is done, the third assertion gives a process for recovering a normalization of \( F(X) \). 
The fourth assertion enables one to reconstruct X itself if the admissible family is 
generic.

**Theorem 5.** Let X be an open bordered nodal curve with DN-data \((\gamma, u, \theta u)\) associated 
with an admissible family with no zero coefficients. Consider any normalization 
\( \hat{X} \xrightarrow{\pi} X \) of X. Then the following assertions hold.

1. The map \( \pi^* f \) with f defined by (3.2) has a meromorphic extension \( \hat{F} \) to \( \hat{X} \), 
and there are discrete sets \( \hat{T} \) and \( \hat{S} \) in \( \hat{X} \) and \( Y = F(X) \setminus f(\gamma) \), respectively, such 
that \( F: \hat{X} \setminus \hat{T} \to Y \setminus S \) is one-to-one.

2. Almost all \( \xi_\ast \in \mathbb{C} \) have a neighbourhood \( W_{\xi_\ast} \) such that \( Y_{\xi_\ast} = Y \cap \Delta_\xi = \bigcup_{1 \leq j \leq p} \{(h_j(\xi), \xi)\} \) for all \( \xi \in W_{\xi_\ast} \), where \( h_1, \ldots, h_p \) are p mutually distinct holomorphic functions on \( W_{\xi_\ast} \) whose symmetric functions \( S_{h, m} = \sum_{1 \leq j \leq p} h_j^m \) can be 
recovered using the Cauchy type integral formulae
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f^m}{f^m - \xi} df = S_{h, m}(\xi) + P_m(\xi), \quad m \in \mathbb{N}, \quad (E_m, \xi)
\]
with \( P_m \) a polynomial of degree at most \( m \) under the generic condition \((0 : 1 : 0) \notin Y \).
More precisely, the system \( E_\xi = (E_{m, \xi}, 0 \leq m \leq M - 1) \) of relations enables explicit calculation 
of suitable \( h_j(\xi_\ast) \) and \( P_m \) if \( N > M \geq 2p + 1 \) and \( \xi_0, \ldots, \xi_N \) are mutually 
distinct points.

3. Consider \( U_\ell = \pi^*_\ast \widetilde{u}_\ell e_\ell \), 0 \( \leq \ell \leq 2 \). Then \( \partial U_\ell \) are meromorphic \((1, 0)\)-forms, 
and for almost all \( \xi_\ast \in \mathbb{C} \) there is a neighbourhood \( W_{\xi_\ast} \) such that \( S \cap \bigcup_{\xi \in W_{\xi_\ast}} Y_\xi = \emptyset \).
and $\partial\widehat{U}_\ell$ can be reconstructed in the set $\widehat{F}^{-1}(\bigcup_{\xi\in W_\xi} Y_\xi)$ from the well-defined meromorphic quotient $(\partial\widehat{U}_\ell)/(\partial\widehat{F}_2)$ in view of the Cauchy type formulæ

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f_j^m}{f_2 - \xi} \theta_{u\ell} = \sum_{1 \leq j \leq p} h_j(\xi)^m \frac{\partial\widehat{U}_\ell}{\partial\widehat{F}_2} (\widehat{F}^{-1}(h_j(\xi), \xi)) + Q_m(\xi), \quad (T_m, \xi)$$

where $m$ is any integer and $Q$ is a polynomial of degree at most $m$.

4. Let $y$ be a singular point of $Y$ and $C$ a representative of an irreducible component of the germ of $Y$ at $y$ whose boundary is a smooth real curve in $\text{Reg}Y$. Then $C$ is the image under $F$ of a branch of $X$ if and only if $\int_{\partial C} \partial F_* \bar{u}_\ell^{c\ell} \neq 0$. In this case $\frac{1}{2\pi i} \int_{\partial C} \partial F_* \bar{u}_\ell^{c\ell} = c_{a,j}$, where $a$ is a node of $X$ and $j$ is the index such that $X_{a,j} = F^{-1}(C)$ is one of the branches of $X$ at $a$. Equivalently, $C$ is the image under $F$ of a branch of $X$ if and only if $\int_C (\partial F_* \bar{u}_\ell^{c\ell}) = +\infty$.

Remarks. 1. Let $p_2$ be the second natural projection of $\mathbb{C}^2$ onto $\mathbb{C}$ and let $\gamma_2$ denote the real curve $p_2(f(\gamma))$. Then a point $\xi_*$ in the first assertion can be any element of $\mathbb{C} \setminus \gamma_2$ outside a discrete set $\Delta$. More precisely, let $\Gamma$ be a connected component of $\mathbb{C} \setminus \gamma_2$ and let $H$ be a holomorphic function on $\mathbb{C} \times \Gamma$ such that $Y \cap p_2^{-1}(\Gamma) = \{H = 0\}$ and $p_2^{-1}(\Gamma) \cap H = H(\cdot, z_2)$ is a unitary polynomial for every $z_2 \in \Gamma$. Then $\xi_*$ can be any element of the complement of the discrete set where the discriminant $\text{disc} H_{z_2}$ vanishes. If $\xi_*$ is chosen in this way, then $W_{\xi_*}$ can be taken to be $\{\text{disc} H_{z_2} \neq 0\}$. Note that the equations $(E_{m, \xi})$ allow one to recover $H$.

2. The first three assertions of the theorem above are equivalent to Theorem 2 in [15]. The assertion 4 is a device for detecting which singularities of $Y$ lie in $\pi^{-1}(\text{Sing} X)$ and which have appeared because $F$ is not necessarily an embedding. Note that the characterizations given for $C$ have an invariant interpretation. Indeed, if $C = F(B)$, where $B$ is a branch of $X$ at $a$, then $\int_C (\partial F_* \bar{u}_\ell^{c\ell}) = \int_B (\partial \bar{u}_\ell^{c\ell})$ and $\int_{\partial C} \partial F_* \bar{u}_\ell^{c\ell} = \int_{\partial B} \partial \bar{u}_\ell^{c\ell}$.

3. If an admissible family has a zero coefficient, then the above method recovers branches corresponding to non-zero coefficients but cannot determine the others. In particular, the method does not recognize nodes corresponding to a zero admissible family.

4. The fourth assertion of the theorem allows one to reconstruct in $Z = \widehat{X}$ the set $S = \pi^{-1}(\text{Sing} X)$ and the map $\mu: S \rightarrow \mathbb{N}$ such that $X_{\pi(s), \mu(s)} = \pi(W_s)$ for some neighbourhood $W_s$ of $s$ in $S$ and

$$c_{\ell, \pi(s), \mu(s)} = \frac{1}{2\pi i} \int_{\partial F(W_s)} \partial F_* \bar{u}_\ell^{c\ell} \overset{\text{def}}{=} \kappa_{\ell, s}.$$
the condition that if \( T_1, \ldots, T_\ell \) is another partition of \( S \) such that \( \sum_{s \in T_m} \kappa_{\ell, s} = 0 \) for any \( m \), then each \( T_m \) is the union of some of the sets \( S_1, \ldots, S_k \). If no admissible family is known to be generic, then \( X \) is roughly isomorphic to the nodal curve \( X_S \) determined by \( Z \) and \( S_1, \ldots, S_k \) and is obtained by supplementary identifications in the set of nodes of \( X_S \).

**Proof.** Denote by \( F \) the meromorphic extension of \( f \) to \( X \) and let \( Y = F(X) \setminus \delta \), where \( \delta = f(\gamma) \). Let \( \hat{X} \xrightarrow{\pi} X \) be a normalization of \( X \). Then \( \hat{X} \xrightarrow{\tilde{F} = F \circ \pi} Y \) is a normalization of \( Y \), and \( \tilde{\gamma} = \pi^{-1}(\gamma) \) is a real curve diffeomorphic to \( \gamma \). The functions \( \tilde{u}_\ell = \pi^* u_\ell \) are well-defined functions on \( \tilde{\gamma} \) and they extend as harmonic functions \( \tilde{U}_\ell \) on \( \hat{X} \setminus \pi^{-1}(\text{Sing} \ X) \) and as distributions on \( \hat{X} \) with logarithmic singularities at points of \( \pi^{-1}(\text{Sing} \ X) \). However, the forms \( \partial \tilde{u}_\ell \) extend to \( \hat{X} \) as meromorphic \((1,0)\)-forms with simple poles. Hence, \( \tilde{f} = \pi^* f \) extends to \( \hat{X} \) as a meromorphic function which is of course \( \tilde{F} \). This is sufficient to apply Theorem 2 of [15], whose statements are reproduced here by assertions 1–3. Note that the points of \( \tilde{F}^{-1}(\text{Sing} \ Y) \) which are not in \( \pi^{-1}(\text{Sing} \ X) \) occur only because \( F \) is not necessarily an embedding.

Since admissible families are assumed not to have zero coefficients, the harmonic distributions \( \tilde{u}_\ell \) have logarithmic singularities along each branch at each node of \( X \) while they are ordinary harmonic functions at regular points of \( X \). This implies that the functions \( \tilde{U}_\ell = \pi^* \tilde{u}_\ell \) have singularities at each point over a node of \( X \) and are bounded near points of \( \tilde{F}^{-1}(\text{Sing} \ Y) \setminus \pi^{-1}(\text{Sing} \ X) \). Since the assertion 3 explains how to recover the quotient \( (\partial \tilde{U}_\ell)/(\partial F_2) \) and hence the form \( \Theta_\ell = \partial \tilde{U}_\ell \) from boundary data by explicit formulae, the same applies to the set \( \pi^{-1}(\text{Sing} \ X) \).

Now let \( y \in Y \) be a singularity of \( Y \) and \( C \) an irreducible component of \( Y \) at \( y \) whose boundary is a smooth real curve in \( \text{Reg} \ Y \). Assume that \( C \) is the image under \( F \) of a branch \( X_{a,j} \) of \( X \), where \( a \) is some node of \( X \). Then \( C \) is smooth, and since for a holomorphic coordinate \( z_j \) on \( X_{a,j} \) centered at \( a \) the function \( \tilde{u}_\ell \) centered at \( a \) extends to an ordinary harmonic function on \( X_{a,j} \), it follows that \( \partial F_2 \tilde{u}_\ell \) is a meromorphic \((1,0)\)-form whose only singularity is at \( y \), where it has a simple pole with residue \( c_{a,j} \). If \( C \) contains no points of \( F(\text{Sing} \ X) \), then \( \tilde{u}_\ell \) is an ordinary harmonic function on \( F^{-1}(C) \) and \( \int_{\partial C} \partial F_2 \tilde{u}_\ell \) has to be zero. The equivalent characterization by the infiniteness of the Dirichlet integral is straightforward. \( \square \)

**Proof of Theorem 3.** The first part of the theorem is a particular case of Theorem 4 applied to the nodal curves \( \mathcal{X} \) and \( \mathcal{X}' \) obtained by identifying in \( X \) (respectively, \( X' \)) the points \( a_j^+ \) and \( a_j^- \) (respectively, \( a_j^+ \) and \( a_j^- \)), \( 1 \leq j \leq \nu \). The reconstruction part is a particular case of Theorem 5 applied to \( \mathcal{X} \) and \( \mathcal{X}' \). \( \square \)

### 4. Characterization of DN-data

We give here some criteria characterizing DN-data. The theorem we present here is a development of Theorem 3 of [15] for the nodal case. Theorems 3b and 4 in [15] on characterization of boundary data and the characterization results in [22] can also be adapted to the nodal case, but in this paper we omit the cumbersome formulations they require.
For simplicity, open surfaces were assumed in the preceding sections to have smooth boundaries even though they may be singular in their interior. However, for our characterization method we have to consider almost smooth boundaries, as in [15]. An open bordered nodal curve $X$ is said to have an almost smooth boundary $\gamma$ if $h^2(X \cup \gamma) < \infty$, $\gamma$ is a smooth oriented real curve without components reducing to a point, and for some open neighbourhood $W$ of $\gamma$ in $X \cup \gamma$ the set $W \setminus \gamma$ is an open Riemann surface such that the set $W_{\text{sing}}$ of points in $\gamma$ where $W$ does not have a smooth boundary satisfies the condition $h^1(W_{\text{sing}}) = 0$.

If $X$ is an open bordered nodal curve with an almost smooth boundary $\gamma$, then the arguments above can be adapted to obtain analogues of the classical results (see, for example, [1]): namely, for any admissible family $c$ a real-valued function $u$ of class $C^1$ on $\gamma$ has a unique extension $\tilde{u}^c$ to $X$ as a harmonic distribution such that $\int_W i \partial \tilde{u}^c \wedge \overline{\partial \tilde{u}}^c < +\infty$ for some open neighbourhood $W$ of $\gamma$ in $X$ and such that $\text{Res}_a (\partial u|_{X_{a,j}}) = c_{a,j}$ for each branch $X_{a,j}$ at a node $a$ of $X$. Moreover, $N_{X,c}u$ still makes sense as the element of the dual space of $C^1(\gamma)$ which equals $\partial \tilde{u}^c / \partial \nu$ on $\gamma \setminus X_{\text{sing}}$ (see [15], Proposition 12).

The first condition for $(\gamma, u, \theta u)$ to be DN-data is for $\gamma$ to border a complex curve whose tangent bundle along $\gamma$ is the real two-dimensional bundle given with the data and to which $f$ extends meromorphically. Parts (a) and (b) of the theorem below give necessary and sufficient conditions for this to occur. This fact does not really depend on whether or not $u$ and $\theta u$ are restrictions of first derivatives of a harmonic distribution. Part (c) gives a necessary and sufficient condition for $u_\ell$ and $\theta u_\ell$ to be boundary values coming from a harmonic distribution with logarithmic singularities. For simplicity, the real curve $\gamma$ is assumed in parts (b) and (c) to be connected.

**Theorem 6.** Assume that the hypothesis $\text{A}$ is valid and consider

$$G : \mathbb{C}^2 \ni (\xi_0, \xi_1) \mapsto \frac{1}{2\pi i} \int_{\gamma} f_1 d(\xi_0 + \xi_1 f_1 + f_2) \xi_0 + \xi_1 f_1 + f_2.$$ (4.1)

(a) If an open bordered nodal curve $X$ has DN-data $(\gamma, u, \theta u)$, then almost all points $\xi_\ast$ of $\mathbb{C}^2$ have a neighbourhood on which one can find a family (possibly empty) $(h_1, \ldots, h_p)$ of mutually distinct holomorphic functions satisfying

$$0 = \frac{\partial^2}{\partial \xi_0^2} \left( G - \sum_{1 \leq j \leq p} h_j \right)$$ (4.2)

and the Riemann–Burgers equations

$$h_j \frac{\partial h_j}{\partial \xi_0} = \frac{\partial h_j}{\partial \xi_1}, \quad 1 \leq j \leq p.$$ (4.3)

(b) Conversely, assume that $\gamma$ is connected and the conclusion of (a) holds in a connected neighbourhood $W_{\xi_\ast}$ of a point $(\xi_{0\ast}, \xi_{1\ast})$.

---

$h^d$ is the $d$-dimensional Hausdorff measure.
In this case if \( (\partial^2 G/\partial \xi_0^2)|_{W_{\ell, \gamma}} \neq 0 \), then there is an open Riemann surface \( Z \) with almost smooth boundary \( \gamma \) to which \( f \) extends meromorphically.\(^2\)

If, however, \( (\partial^2 G/\partial \xi_0^2)|_{W_{\ell, \gamma}} = 0 \), then the same conclusion holds for \( \gamma \) or for \(-\gamma\), where \(-\gamma\) denotes the same curve but with the opposite orientation. More precisely, if \( (\partial^2 G/\partial \xi_0^2)|_{W_{\ell, \gamma}} = 0 \) and the expected conclusion holds for \(-\gamma\), then \( f(-\gamma) \) is the boundary (in the sense of currents) of a possibly singular complex curve \( Y \) in \( \mathbb{C}^2 \setminus f(\gamma) \), and the expected conclusion holds for \( \gamma \) if and only if \( Y \) is an algebraic curve.

(c) Assume that \((\bar{Z}, \gamma)\) is a Riemann surface with an almost smooth boundary, let \( \mathcal{D} \) be some smooth domain in the double of \( Z \) containing \( \bar{Z} \), and let \( g \) be a Green function for \( \mathcal{D} \).

Then \((\gamma, u, \theta u)\) is actually DN-data for the open bordered nodal curve \( X \) obtained from \( Z \) by identifying the points within each family \((a, j)_{1 \leq j \leq \nu_n}, 1 \leq n \leq N \), if and only if there exists a family \((c_{\ell, n, j})_{0 \leq \ell \leq 2}\) of non-zero complex numbers such that

\[
\sum_{1 \leq j \leq \nu_n} c_{\ell, n, j} = 0 \quad \text{for any } (\ell, n) \in \{0, 1, 2\} \times \{1, \ldots, N\} \quad \text{with the property that for any } z \in \mathcal{D} \setminus \bar{Z} \text{ and any } \ell \in \{0, 1, 2\}
\]

\[
\frac{2}{i} \int_{\gamma} u_\ell(\zeta) \partial_\xi g(\zeta, z) + g(\zeta, z) \theta u_\ell(\zeta) = 2\pi \sum_{1 \leq n \leq N} \sum_{1 \leq j \leq \nu_n} c_{\ell, n, j} g(a_{n, j}, z). \tag{4.4}
\]

Remarks. 1. (b) is actually the second part of Theorem 3a in [15], but with more details about the case \( (\partial^2 G/\partial \xi_0^2)|_{W_{\ell, \gamma}} = 0 \). The algebraic criterion for the curve is effective, since \( Y \) can be explicitly reconstructed from the Cauchy type formulae given in Theorem 5 or in Theorem 2 of [15].

2. The points \( a_{n, j} \) and the coefficients \( c_{\ell, n, j} \) are unique if they exist, because the right-hand side of (4.4) extends to \( Z \) as a distribution \( T \) such that

\[
2i \partial \bar{T} = 2\pi \sum_{1 \leq n \leq N} \sum_{1 \leq j \leq \nu_n} c_{\ell, n, j} \delta a_{n, j} \, dV,
\]

where \( dV \) is some volume form on \( Z \).

Proof. Part (a) follows from Theorem 3a in [15], since we can apply this theorem to \( Z \), where \( Z \xrightarrow{\pi} X \) is a normalization. In fact, we only need to know that \( \pi^* f \) embeds \( \pi^* \gamma \) in \( \mathbb{CP}_2 \) and extends meromorphically to \( Z \). The case \( (\partial^2 G/\partial \xi_0^2)|_{W_{\ell, \gamma}} \neq 0 \) of part (b) is the same as Theorem 3a, B in [15]. Assume that \( (\partial^2 G/\partial \xi_0^2)|_{W_{\ell, \gamma}} = 0 \).

As in the proof of Theorem 3a, B in [15], \( \pm f(\gamma) \) is then the boundary of a complex curve in some two-dimensional affine subspace of \( \mathbb{CP}_2 \), and therefore the expected conclusion holds for \( \pm \gamma \). If both \(-f(\gamma)\) and \( f(\gamma)\) are the boundaries in \( \mathbb{CP}_2 \setminus f(\gamma) \) of complex curves \( Y^- \) and \( Y^+ \), then \( Y^- \cup f(\gamma) \cup Y^+ \) is a complex algebraic curve.

Assume now that (4.4) is satisfied. Let \( \Omega_\ell^+ \) denote the form \( u_\ell \partial g(\cdot, z) + g(\cdot, z) \theta u_\ell \) for \( z \) in \( \mathcal{D} \setminus \gamma \), and define \( U_\ell^+ \) (respectively, \( U_\ell^- \)) to be the restriction to \( \mathcal{D}^+ = Z \) (respectively, \( \mathcal{D}^- = \mathcal{D} \setminus \bar{Z} \)) of the function \( \mathcal{D} \setminus \gamma \ni z \mapsto \frac{2}{i} \int_{\gamma} \Omega_\ell^z \). Then we know from

\(^2\)Using Example 10.5 in [14], one can construct smooth DN-data for which the solution of the IDN-problem is a manifold whose boundary is only almost smooth.
the Plemelj–Sohotsky formula in Lemma 15 of [15] that $U_±^\pm$ is a harmonic function on $\mathcal{D}^\pm$ which extends continuously to $\overline{\mathcal{D}}^\pm$, $u_\ell = U_\ell^+ - U_\ell^-$, and $\theta u_\ell = \overline{\partial}U_\ell^+ - \partial U_\ell^-$ almost everywhere on $\gamma$. Hence, $u_\ell$ is the boundary value of the distribution $T_\ell$ on $Z$ defined by $T_\ell = U_\ell^+ + 2\pi \sum_{1 \leq n \leq N} \sum_{1 \leq j \leq \nu_n} c_{\ell,n,j} g(a_{n,j}, z)$, and by construction $\partial T_\ell = \theta u_\ell$ on $\gamma$.

If $X$ is the nodal curve obtained as in the statement of the theorem, then the equalities $\sum_{1 \leq j \leq \nu_n} c_{\ell,n,j} = 0$ imply that $T_\ell$ can be considered as a harmonic distribution on $X$. As a consequence, $(\gamma, u, \theta u)$ is DN-data for $X$. The converse follows directly from the definitions. □

Bibliography

[1] L.V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Math. Ser., vol. 26, Princeton Univ. Press, Princeton, NJ 1960, xi+382 pp.

[2] M.I. Belishev, “The Calderon problem for two-dimensional manifolds by the BC-method”, *SIAM J. Math. Anal.* 35:1 (2003), 172–182 (electronic).

[3] M.I. Belishev and Y.V. Kurylev, “To the reconstruction of a Riemannian manifold via its spectral data (BC-method)”, *Comm. Partial Differential Equations* 17:5-6 (1992), 767–804.

[4] P. Buser, “Inverse spectral geometry on Riemann surfaces”, *Progress in inverse spectral geometry*, Trends Math., Birkhäuser, Basel 1997, pp. 133–173.

[5] H.P. de Saint-Gervais, *Uniformisation des surfaces de Riemann, Retour sur un théorème centenaire*, ENS Éditions, Lyon 2010, 544 pp.

[6] H. Federer, *Geometric measure theory*, Grundlehren Math. Wiss., vol. 153, Springer-Verlag, New York 1969, xiv+676 pp.

[7] A.M. Garsia, “An imbedding of closed Riemann surfaces in Euclidean space”, *Comment. Math. Helv.* 35 (1961), 93–110.

[8] I.M. Gel’fand, “Automorphic functions and the theory of representations”, *Proc. Internat. Congress Math.* (Stockholm 1962), Inst. Mittag-Leffler, Djursholm 1963, pp. 74–85.

[9] A.A. Гончар, “Полюсы строк таблицы Паде и мероморфное продолжение функций”, *Матем. сб.* 115(157):4(8) (1981), 590–613; English transl., A.A. Gonchar, “Poles of rows of the Padé table and meromorphic continuation of functions”, *Math. USSR-Sb.* 43:4 (1982), 527–546.

[10] A. Grothendieck, “The cohomology theory of abstract algebraic varieties”, *Proc. Internat. Congress Math.* (Edinburgh 1958), Cambridge Univ. Press, New York 1960, pp. 103–118.

[11] J. Harris and I. Morrison, *Moduli of curves*, Grad. Texts in Math., vol. 187, Springer-Verlag, New York 1998, xiv+366 pp.

[12] R. Hartshorne, “Residues and duality”, *Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64*, with an appendix by P. Deligne, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin–New York 1966, vii+423 pp.

[13] R. Harvey, “Holomorphic chains and their boundaries”, *Several complex variables* (Williams Coll., Williamstown, MA 1975), Proc. Sympos. Pure Math., vol. 30, part 1, Amer. Math. Soc., Providence, RI 1977, pp. 309–382.

[14] F.R. Harvey and H.B. Lawson, Jr., “On boundaries of complex analytic varieties. I”, *Ann. of Math.* (2) 102:2 (1975), 233–290.
[15] G. Henkin and V. Michel, “On the explicit reconstruction of a Riemann surface from its Dirichlet–Neumann operator”, *Geom. Funct. Anal.* **17**:1 (2007), 116–155.

[16] G. Henkin and V. Michel, *Problème de Plateau complexe feuilleté. Phénomènes de Hartogs–Severi et Bochner pour des feuilletages CR singuliers*, 2011, arXiv: 1109.4300.

[17] M. Lassas and G. Uhlmann, “On determining a Riemannian manifold from the Dirichlet-to-Neumann map”, *Ann. Sci. École Norm. Sup.* (4) **34**:5 (2001), 771–787.

[18] M. Rosenlicht, “Generalized Jacobian varieties”, *Ann. of Math.* (2) **59** (1954), 505–530.

[19] R. A. Rüedy, “Embeddings of open Riemann surfaces”, *Comment. Math. Helv.* **46** (1971), 214–225.

[20] J.-P. Serre, *Groupes algébriques et corps de classes*, Publications de l’Institut de Mathématique de l’Université de Nancago, vol. VII, Hermann, Paris 1959, 202 pp.

[21] J. Sylvester, “An anisotropic inverse boundary value problem”, *Comm. Pure Appl. Math.* **43**:2 (1990), 201–232.

[22] R. A. Walker, “Extended shockwave decomposability related to boundaries of holomorphic 1-chains within $\mathbb{C}P^2$”, *Indiana Univ. Math. J.* **57**:3 (2008), 1133–1172.

[23] K.-W. Wiegmann, “Einbettungen komplexer Räume in Zahlenräume”, *Invent. Math.* **1** (1966), 229–242.

G. Henkin
Université Pierre et Marie Curie, Paris, France; Central Economics and Mathematics Institute of the Russian Academy of Sciences, Moscow, Russia
E-mail: henkin@math.jussieu.fr

V. Michel
Université Pierre et Marie Curie, Paris, France
E-mail: michel@math.jussieu.fr

Received 31/OCT/12