Iterated Resolvent Function for the Ladder Bethe-Salpeter Equation

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Abstract

We develop the formal connection of the field theoretical Bethe-Salpeter equation including the ladder approximation with its representation on the light-front for a bosonic model. We use the light-front Green’s function for the N-particle system for two-particles plus N-2 intermediate bosons. We derive an infinite set of coupled hierarchy equations or iterated resolvents from which the Green’s function can be calculated. These equations allow a consistent truncation of the light-front Fock-space. We show explicitly the representation of the covariant two-body T-matrix and bound state vertex in the light-front.

1 Introduction

The Bethe-Salpeter (BS) equation provides a field-theoretical framework to study two-particle bound states\footnote{I}. In general, for practical applications its two-particle irreducible kernel is truncated in lowest order. Nowadays it is possible to solve it including an infinite
set of ladder and crossed ladder diagrams in a nonperturbative field theoretical calculation [2]. Recently the ladder BS in Minkowiski space has also been solved [3].

In this sense the discussion of three-dimensional reductions of the BS equation seems out of date [4]. However, much effort has been made in using the three-dimensional reduction using light-front coordinates. There is the hope that, by using light-front quantization, one could be able to understand several aspects of low energy QCD without loosing the kinematical boost invariance of the wave-function [5]. In the past, the concept of light-front wave-functions has also been applied in the context of nuclear physics to describe the deuteron and its properties [6, 7].

The representation of the ladder BS equation in the infinitum momentum frame limit was derived by Weinberg [8]. However, the result corresponds to the truncation of the intermediate light-front Fock-space to three-particles. In this approximation numerical results were obtained in different contexts, for bosonic models [9, 10] and fermionic models [5]. An explicit systematic expansion of the covariant BS equation in the light-front is still missing in the literature. It is our aim to fill this formal gap. At the same time it is also necessary to perform detailed numerical calculations comparing the covariant BS ladder approximation to its truncation in the light-front Fock-space.

It is well known that light-front perturbation theory is equivalent to the covariant perturbative expansion in field theory [11, 12]. Thus, in principle, it is possible to find the exact representation of the BS equation in the light-front. For this purpose we use a bosonic model for which the interaction Lagrangean is defined by

\[ \mathcal{L}_I = g_S \phi_1^\dagger \phi_1 \sigma + g_S \phi_2^\dagger \phi_2 \sigma, \]  

where the bosons \( \phi_1 \) and \( \phi_2 \) have equal masses, \( m \), the intermediate boson, \( \sigma \), has mass \( \mu \) and \( g_S \) is the coupling constant.

Beginning from Dirac’s idea [13] of representing the dynamics of the quantum system at light-front times \( x^+ = t + z \), we derive the two-body Green’s function from the covariant propagator that evolves the system from one light-front hyper-surface to another one. The light-front Green’s function is the probability amplitude for an initial state at \( x^+ = 0 \) do evolvey to a final state in the Fock-state at some \( x^+ \), where the evolution operator is defined by the light-front Hamiltonian [14]. The two-body Green’s function includes the propagation of intermediate states with any number of particles. When it is conveniently constrained, it results in the summation of the covariant ladder. The exact representation of the covariant off-shell two-body amplitude in terms of the light-front T-matrix is given here, as well as the light-front representation of the four-dimensional vertex of the bound state.

The two-body light-front Green’s function satisfies a hierarchy of coupled equations which includes virtual intermediate propagation for any number of particles. A consistent truncation can be performed, and in lowest order it is related to the Weinberg equation.
This work is divided as follows sections. In section $II$, we present the notation used through the work, discussing the one boson propagator in the light-front. In section $III$, we introduce the two-body light-front Green’s function in perturbation theory, and explicitly evaluated it up to $g_5^4$, including one boson exchange and the box-diagram. Then, in section $IV$, we generalize this discussion to any order in the ladder, and present the coupled hierarchy equations which, if solved, give the light-front two-body Green’s function. The hierarchy of equations does not include closed loops in the bosons $\Phi_1$ and $\Phi_2$. We relate the off-shell two-body scattering amplitude to the light-front T-matrix. In section $V$, we deduce the eigenvalue equation for the squared interacting mass operator of the bound state. We express the four dimensional vertex in terms of the light-front bound state wave-function. The approximation of the ladder BS equation in the light-front with up to four particles in lowest order in the kernel is given. Our conclusions are summarized in section $IV$.

2 Notation

The kinematics on the light-front are defined by the momentum canonically conjugate to the light-front coordinates, $x^- = t - z$ and $\vec{x}_\perp$. The momentum $k^+ = k_0 + k^3$ is canonically conjugate to $x^-$ and $\vec{k}_\perp$ to $\vec{x}_\perp$. The eigenfunctions of the momentum operators $k^+$ and $\vec{k}_\perp$ are defined by

$$< x|k > \equiv < x^-, \vec{x}_\perp|k^+, \vec{k}_\perp > = e^{-i(\frac{1}{2} k^+ x^- - \vec{k}_\perp \cdot \vec{x}_\perp)}.$$  \hspace{1cm} (2)

The basis states are eigenfunctions of the free $k^-_0$ operator

$$k^-_0 |k > = \frac{k^2 + m^2}{k^+} |k >.$$  \hspace{1cm} (3)

The states $|k >$ form an orthonormal and complete basis:

$$\int \frac{dk^+ d^2 k_\perp}{2(2\pi)^3} < x'|k > |k| x > = \delta(x'^+ - x^+) \delta(\vec{x}'_\perp - \vec{x}_\perp).$$  \hspace{1cm} (4)

The free one-body Green’s function for particle propagation is defined by the operator

$$G^{(1p)}_0(k^-) = \frac{\theta(k^+)}{k^- - k^-_0 + i\varepsilon};$$  \hspace{1cm} (5)

and, for antiparticle propagation, is given by

$$G^{(1a)}_0(k^-) = \frac{\theta(-k^+)}{k^- + k^-_0 - i\varepsilon}.$$  \hspace{1cm} (6)

The function defined by Eq.(5) is the Green’s function of the operator equation

$$(k^- - k^-_0) \left( G^{(1p)}_0(k^-) + G^{(1a)}_0(k^-) \right) = 1.$$  \hspace{1cm} (7)
The Feynman propagator is defined by
\[
S^{(1)}(k^\mu) = \frac{i}{k^+} G_0^{(1)p} - \frac{i}{|k^+|} G_0^{(1)a} = \frac{i}{k^+(k^- - \frac{k^2 + m^2 - i\epsilon}{k^+})},
\]
where the phase-space factor $1/|k^+|$ has been introduced.

The Green’s function of Eq.(5) is the Fourier transform of the single-boson propagator for forward direction in the light-front time while, for antiparticle states, the propagation is backward in the light-front time. The space-time propagator is given by:
\[
\tilde{S}^{(1)}(x^+) = \int \frac{dk^-}{2\pi} \frac{i e^{-\frac{i}{2} k^- x^+}}{k^+(k^- - \frac{k^2 + m^2 - i\epsilon}{k^+})} = 1
\]
\[
e^\frac{ik^\mu}{k^+} x^\mu \theta(k^+) \theta(x^+) + 1
\]
\[
e^\frac{ik^\mu}{k^+} x^\mu \theta(-k^+) \theta(-x^+).
\]

3 Two-body Green’s Function

3.1 Free Green’s Function

The two-body Green’s functions can be derived from the covariant propagator for two particles propagating at equal light-front times. Without losing generality, we are going to restrict our calculation to total momentum $K^+$ positive and the corresponding forward light-front time propagation. In this case the propagator from $x^+ = 0$ to $x^+ > 0$ is given by:
\[
\tilde{S}^{(2)}(x^+; x_1^+, x_2^+) = \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{ie^{-ik^\mu_1(x_1^\mu - x_1^\mu)} e^{-ik^\mu_2(x_2^\mu - x_2^\mu)}}{k_1^2 - m^2 + i\epsilon \ k_2^2 - m^2 + i\epsilon}.
\]

At equal light-front times $x_1^+ = x_2^+ = 0$ and $x_1^+ = x_2^+ = x^+$, the propagator is written as:
\[
\tilde{S}^{(2)}(x^+) = \tilde{S}^{(1)}(x^+) \tilde{S}^{(1)}(x^+),
\]
where the one-body propagators, $\tilde{S}^{(1)}_i$, corresponding to the light-front propagators of particles $i = 1$ or 2, are defined by Eq.(9). We have explicitly:
\[
\tilde{S}^{(2)}(x^+) = -\int \frac{dk^-_1}{(2\pi)} \frac{dk^-_2}{(2\pi)} \frac{e^{-\frac{i}{2} k^-_1 x^+}}{k^+_1 \left(k^-_1 - \frac{k^2_1 + m^2 - i\epsilon}{k^+_1}\right)} \frac{e^{-\frac{i}{2} k^-_2 x^+}}{k^+_2 \left(k^-_2 - \frac{k^2_2 + m^2 - i\epsilon}{k^+_2}\right)}.
\]

The Fourier transform to the total light-front energy $(K^-)$ is given by
\[
S^{(2)}(K^-) := \frac{1}{2} \int dx^+ e^{\frac{i}{2} K^- x^+} \tilde{S}^{(2)}(x^+),
\]
which result in

\[ S^{(2)}(K^-) = -\frac{1}{(2\pi)} \int \frac{dk_1^-}{k_1^+ k_2^+} \left( \frac{1}{k_1^- - k_1^+ + m^2 - i\epsilon/k_1^-} \right) \left( \frac{1}{k_1^- - k_1^+ - k_2^+ + m^2 - i\epsilon/k_1^-} \right), \]  

(14)

where \( K^- = k_1^- + k_2^- \).

We perform the analytical integration in the \( k_1^- \) momentum by evaluating the residue at the pole \( k_1^- = (k_1^2 + m^2 - i\epsilon)/k_1^+ \). It implies that only \( k_1^+ \) in the interval \( 0 < k_1^+ < K^+ \) gives a nonvanishing contribution to the integration. The result is

\[ S^{(2)}(K^-) = \frac{\theta(k_1^+)}{k_1^+} \frac{\theta(K^+ - k_1^+)}{(K^- - K_0^{(2)-} + i\epsilon)}, \]  

(15)

where

\[ K_0^{(2)-} = \frac{k_1^2 + m^2}{k_1^+} + \frac{k_2^2 + m^2}{K^+ - k_1^+}, \]  

(16)

with \( K_0^{(2)-} \) being the light-front Hamiltonian of the free two-particle system. For \( x^+ < 0 \), \( S^{(2)}(x^+) = 0 \) due to our choice of \( K^+ > 0 \). Observe that \( S^{(2)}(K^-) \) is written in Eq.(15) in operator form with respect to \( k^+ \) and \( \vec{k}_{\perp} \).

The free two-body Green’s function is given by

\[ G_0^{(2)}(K^-) = \frac{\theta(k_1^+)}{k_1^+} \frac{\theta(K^+ - k_1^+)}{(K^- - K_0^{(2)-} + i\epsilon)}, \]  

(17)

The difference between the free two-body Green’s function and \( S^{(2)}(K^-) \), Eq.(15), is the phase-space factor for particles 1 and 2.

The generalization for the \( N \) particle system is

\[ S^{(N)}(K^-) = \prod_{j=1}^N \theta(k_j^+) \frac{\theta(K^+ - k_j^+)}{k_j^+} \frac{1}{(K^- - K_0^{(N)-} + i\epsilon)}, \]  

(18)

where \( K_0^{(N)-} \) is the free light-front Hamiltonian of the \( N \)-particle system, which is given by

\[ K_0^{(N)-} = \sum_{j=1}^N \frac{k_{j\perp}^2 + m_j^2}{k_j^+}, \]  

(19)

with \( k_j^+ > 0 \) and \( K^+ = \sum_{j=1}^N k_j^+ \). The many-body free Green’s function is given by

\[ G_0^{(N)}(K^-) = \prod_{j=1}^N \theta(k_j^+) \frac{\theta(K^+ - k_j^+)}{k_j^+} \frac{1}{(K^- - K_0^{(N)-} + i\epsilon)}, \]  

(20)

In the following subsection, we will include the exchange of a boson between the two-particles, and calculate the correction to the light-front propagator.
3.2 Green’s Function $O(g_5^2)$

The perturbative correction to the two-body propagator which comes from the exchange of one intermediate virtual boson, is given by

$$
\Delta \tilde{S}_{g_5^{(2)}}^{(2)}(x^+) = \left(\frac{ig_5}{2}\right)^2 \int d\bar{x}_1^+ d\bar{x}_2^+ \tilde{S}^{(1)}_{\nu}(x^+ - \bar{x}_1^+) \tilde{S}^{(1)}_{\gamma}(x^+ - \bar{x}_2^+) S^{(1)}_{\sigma}(\bar{x}_1^+ - \bar{x}_2^+) S^{(1)}_{\gamma}(\bar{x}_1^+) \tilde{S}^{(1)}_{\nu}(\bar{x}_2^+) .
$$

(21)

The intermediate boson, $\sigma$, propagates between the time interval $\bar{x}_1^+ - \bar{x}_2^+$. The labels 1 and 2 and 1’ and 2’ in the particle propagators indicate initial and final states, respectively.

Performing the Fourier transform from $x^+$ to $K^-$, for the total kinematical momentum $K^+$, which we choose positive, and for $K^\perp$, we find

$$
\Delta S^{(2)}_{g_5^{(2)}}(K^-) = \frac{i(g_5)^2}{(2\pi)^2} \int \frac{dk_{1'}^- dk_1^-}{k_{1'}^-(K^+ - k_{1'}^+) k_1^-(K^+ - k_1^+)} \frac{1}{(k_{1'}^- - k_1^- - (k_{1'} - k_1)^2 + m^2 - i\varepsilon) k_{1'}^- - k_1^-} \frac{1}{(K^- - k_1^- - (K - k_1)^2 + m^2 - i\varepsilon) K^+ - k_1^-} \times

\frac{1}{(k_2^- - k_{1'}^- + m^2 - i\varepsilon) k_2^- - k_{1'}^-} \frac{1}{(K^- - k_2^- - (K - k_2)^2 + m^2 - i\varepsilon) K^+ - k_2^-}
$$

(22)

The double integration in $k^-$ is performed analytically in Eq.(22). To simplify the notation, we define $q \equiv k_{1'}$ and $k \equiv k_1$. The integration is nonzero for $K^+ > k^+ > 0$ and $K^+ > q^+ > 0$. Two possibilities also appear for the forward propagation of $\sigma$. For $k^+ > q^+$ it is created by particle 1 and for $k^+ < q^+$ it is annihilated by particle 1:

$$
\Delta S^{(2)}_{g_5^{(2)}}(K^-) = (g_5)^2 \frac{\theta(q^+)\theta(K^+ - q^+)}{q^+ (K^+ - q^+)} \frac{i}{K^- - K_0^{(2)-} + i\varepsilon} \times \frac{\theta(k^+ - q^+)}{(k^+ - q^+)} \frac{i}{K^- - K_0^{l(3)-} + i\varepsilon} + [k \leftrightarrow q] \times \frac{\theta(k^+)}{k^+ (K^+ - k^+)} \frac{i}{K^- - K_0^{(2)-} + i\varepsilon},
$$

(23)

where the light-front energies of the intermediate state propagation are given by Eq.(16) for the initial and final two particle intermediate states and by $K_0^{l(3)-}$ which is given by Eq.(19).
We have

\begin{align*}
K_0^{(2)-} & = \frac{q^2 + m^2}{q^+} + \frac{(K - q)^2_+ + m^2}{(K^+ - q^+)} , \\
K_0^{(3)-} & = \frac{q^2 + m^2}{q^+} + \frac{(K - k)^2_+ + m^2}{(K^+ - k^+)} + \frac{k^2_\sigma + \mu^2}{k^+} , \\
K_0^{(2)-} & = \frac{k^2 + m^2}{k^+} + \frac{(K - k)^2_+ + m^2}{(K^+ - k^+)} ,
\end{align*}

where \( k^+ = k^+ - q^+ \) and \( \vec{k}_\perp = \vec{k}_\perp - \vec{q}_\perp \).

The matrix elements of the interaction Hamiltonian that creates or destroys a quantum of the intermediate boson are given by

\begin{align*}
< qk_\sigma | V | k > & = 2\delta(q + k_\sigma - k) \frac{g}{\sqrt{q^+ k^+_\sigma k^+}} \theta(k^+_\sigma) , \\
< q | V | k_\sigma k > & = 2\delta(k + k_\sigma - q) \frac{g}{\sqrt{q^+ k^+_\sigma k^+}} \theta(k^+_\sigma) .
\end{align*}

The perturbative correction of the propagator can be written as a perturbative correction to the two-particle Green’s function. Using Eqs. (17), (18) and (25), we have:

\begin{align*}
\Delta G_{g^2_S}^{(2)-(2)} (K^-) = G_{0}^{(2)-(2)} (K^-) V G_{0}^{(3)-(3)} (K^-) V G_{0}^{(2)-(2)} (K^-) .
\end{align*}

The correction \( \Delta G_{g^2_S}^{(2)-(2)} (K^-) \) also contains the self energies diagrams for the bosons \( \Phi_1 \) and \( \Phi_2 \). Imposing the restriction that only ladder diagrams are allowed in Eq.(26), we recover Eq.(23).

The nonpertubative Green’s function up to order \( g_S^2 \) also contains the covariant propagation in the light-front time up to one boson exchange, which is written as:

\begin{align*}
G_{g^2_S}^{(2)-(2)} (K^-) = G_0^{(2)-(2)} (K^-) + G_0^{(2)-(2)} (K^-) K_I^{(3)-(3)} G_{g^2_S}^{(2)-(2)} (K^-) ,
\end{align*}

where the interaction is

\begin{align*}
K_I^{(3)-(3)} = V G_0^{(3)-(3)} (K^-) V .
\end{align*}

The above Green’s function up to order \( g_S^2 \) reduces to the covariant propagator for the light-front time. Restricting the kernel to the ladder aproximation,

\begin{align*}
K_{I,l}^{(3)-(3)} = \left[ V G_0^{(3)-(3)} (K^-) V \right]_{\text{ladder}} ,
\end{align*}

the bound state solutions of Eq.(27) satisfy the Weinberg equation [8], which has already been solved numerically [9]. Without the ladder restriction, the bound state solution of Eq.(27) has been obtained numerically in Ref.[10].
3.3 Green’s Function $\mathcal{O}(g_s^4)$

The perturbative correction to the Feynman two-body propagator of order $g_s^4$ in the ladder approximation (box diagram) is given by:

$$
\Delta S^{(2)}_{g_s^4}(x^+) = \left(\frac{ig_s}{2}\right)^4 \int d\tau_1^+ d\tau_2^+ d\tau_1^- d\tau_2^- S^{(1)}_{\tau_1^-}(x^+ - \tau_1^+) \tilde{S}^{(1)}_{\tau_2^-}(x^+ - \tau_2^+) \times
$$

$$
\tilde{S}^{(1)}_{\tau_1^-}(\tau_1^+ - \tau_1^-) \tilde{S}^{(1)}_{\tau_2^-}(\tau_2^+ - \tau_2^-) \tilde{S}^{(1)}_\tau(\tau_1^+ - \tau_2^+),
$$

(30)

Writing the Fourier transform of the perturbative correction to the propagator for $K^+ > 0$, we have:

$$
\Delta S^{(2)}_{g_s^4}(K^-) = \left(\frac{ig_s}{2}\right)^4 \int \frac{dk^- dp^- dq^-}{(2\pi)^3} \frac{dk^+ dp^+ dq^+}{(2\pi)^3} \times
$$

$$
\frac{1}{q^- - \frac{q^2 + m^2 - ie}{q^+}} \frac{1}{K^- - q^- - \frac{(K-q)^2 + m^2 - ie}{K^- - q^+}} \frac{1}{q^- - p^- - \frac{(q-p)^2 + m^2 - ie}{q^- - p^+}} \times
$$

$$
\frac{1}{p^- - \frac{p^2 + m^2 - ie}{p^+}} \frac{1}{K^- - p^- - \frac{(K-p)^2 + m^2 - ie}{K^- - p^+}} \frac{1}{k^- - \frac{k^2 + m^2 - ie}{k^+}}
$$

(31)

where we have defined $q \equiv k_{\nu}$, $p \equiv k_{\tau}$ and $k \equiv k_1$ to simplify the notation.

The correction to the propagator is found by analytical integration in the light-front energies in Eq (31). To separate the intermediate four particle propagation, that occurs for $k^+$, $p^+$ and $q^+$ such that $0 < k^+ < p^+ < q^+ < K^+$, the following factorization is necessary

$$
\frac{1}{K^- - p^- - \frac{(K-p)^2 + m^2 - ie}{K^- - p^+}} \times \frac{1}{p^- - k^- - \frac{(k-p)^2 + m^2 - ie}{p^- - k^+}} \times
$$

$$
\frac{1}{K^- - k^- - \frac{(K-k)^2 + m^2 - ie}{K^- - k^+}}
$$

(32)

After the Cauchy integration the result for the correction to the two body propagator,
with the condition that \(0 < k^+ < p^+ < q^+ < K^+\) \(\left(\Delta_a S^{(2)}_{g_S}\right)\), is given by

\[
\Delta_a S^{(2)}_{g_S}(K^-) = (i g_S) \frac{\theta(k^+)\theta(K^- - k^+)}{k^+(K^- - k^+)} \frac{i}{k^+ - \frac{k^2 + m^2 - i\varepsilon}{K^- - \frac{k^2 + m^2 - i\varepsilon}{k^+ + k^-}} - \frac{(k-k)^2 + m^2 - i\varepsilon}{K^- - k^-}\frac{\theta(p^+)\theta(q^+ - p^+)\theta(p^+ - k^+)}{(q^+ - p^+) (p^+ - k^+) (K^- - p^+) p^+} \frac{\theta(q^+)\theta(k^+ - q^+)}{q^+ (K^- - q^+)} \frac{i}{q^+ - \frac{q^2 + m^2 - i\varepsilon}{K^- - \frac{q^2 + m^2 - i\varepsilon}{q^+ + q^-}} - \frac{(q-q)^2 + m^2 - i\varepsilon}{K^- - q^-}} \times \left[ F'(K^-) + F''(K^-) \right] \times \frac{i}{K^- - \frac{k^2 + m^2}{k^+}} - \frac{(K-k)^2 + m^2}{p^+ - k^+} \times \frac{i}{K^- - \frac{p^2 + m^2}{p^+}} - \frac{(K-p)^2 + m^2}{p^+ - p^-} \times \frac{i}{K^- - \frac{q^2 + m^2}{q^+}} - \frac{(q-p)^2 + m^2}{q^+ - p^+} \times \frac{i}{K^- - \frac{m^2}{k^+}} - \frac{(K-q)^2 + m^2}{K^- - q^-} - \frac{(q-p)^2 + m^2}{q^- - p^-} - \frac{(p-k)^2 + m^2}{p^- - k^+} \times \frac{i}{K^- - \frac{m^2}{k^+}} - \frac{(K-q)^2 + m^2}{K^- - q^-} - \frac{(q-p)^2 + m^2}{q^- - p^-} - \frac{(p-k)^2 + m^2}{p^- - k^+}. \tag{33}
\]

with

\[
F'(K^-) = \frac{i}{K^- - \frac{k^2 + m^2}{k^+}} - \frac{(K-k)^2 + m^2}{p^+ - k^+} \times \frac{i}{K^- - \frac{p^2 + m^2}{p^+}} - \frac{(K-p)^2 + m^2}{p^+ - p^-} \times \frac{i}{K^- - \frac{q^2 + m^2}{q^+}} - \frac{(q-p)^2 + m^2}{q^+ - p^+} \times \frac{i}{K^- - \frac{m^2}{k^+}} - \frac{(K-q)^2 + m^2}{K^- - q^-} - \frac{(q-p)^2 + m^2}{q^- - p^-} - \frac{(p-k)^2 + m^2}{p^- - k^+}. \tag{34}
\]

\[
F''(K^-) = \frac{i}{K^- - \frac{k^2 + m^2}{k^+}} - \frac{(K-k)^2 + m^2}{p^+ - k^+} \times \frac{i}{K^- - \frac{p^2 + m^2}{p^+}} - \frac{(K-p)^2 + m^2}{p^+ - p^-} \times \frac{i}{K^- - \frac{q^2 + m^2}{q^+}} - \frac{(q-p)^2 + m^2}{q^+ - p^+} \times \frac{i}{K^- - \frac{m^2}{k^+}} - \frac{(K-q)^2 + m^2}{K^- - q^-} - \frac{(q-p)^2 + m^2}{q^- - p^-} - \frac{(p-k)^2 + m^2}{p^- - k^+}. \tag{35}
\]

The part of the propagator given by Eq.\((33)\) contains the virtual light-front propagation of intermediate states with up to four particles. The function \(F'\) contains only intermediate states up to three particles, while \(F''\) has one intermediate state propagation with four particles which can be recognized as the last term of Eq.\((33)\). The other possibility that includes up to four particles in the intermediate state propagation is given by \(0 < q^+ < p^+ < k^+ < K^+\). To obtain this contribution, we make the transformation \(q \leftrightarrow k\) in Eq.\((33)\).

The correction to the propagator \(\left(\Delta_a S^{(2)}_{g_S}\right)\) for \(0 < p^+ < k^+ < q^+ < K^+\) contains only
up to three-particle intermediate states only and is given by

\[
\Delta b^{(2)}_{g^{S}}(K^-) = (igS)^4 \frac{\theta(k^+)}{k^+(K-k)} \frac{\theta(k^+ - k^+)}{K^- - \frac{k^2 + m^2 - i\varepsilon}{k^+}} \frac{i}{K^- - \frac{k^2 + m^2 - i\varepsilon}{k^+}} \times \\
\theta(q^+ - k^+) \theta(k^+ - p^+) \\
\frac{(q^+ - p^+)(k^+ - p^+)(K^+ - p^+)}{K^- - \frac{p^2 + m^2}{p^+}} \\
\times i \\
\frac{i}{K^- - \frac{p^2 + m^2}{p^+}} \\
\theta(q^+) \theta(k^+ - q^+) \\
q^+(K^+ - q^+) \\
K^- - \frac{q^2 + m^2 - i\varepsilon}{q^+} - \frac{(K-q)^2 + m^2}{K^- - q^+} \\
\times \frac{(K-q)^2 + m^2}{K^- - q^+} .
\]

(36)

For the momenta satisfying \(0 < q^+ < k^+ < p^+ < K^+\), the correction to the propagator can be obtained from Eq. (36) by performing the transformation \(q \leftrightarrow K - q\) and \(k \leftrightarrow K - k\). From Eqs. (33) and (36), the following result is obtained

\[
\Delta S^{(2)}_{g^{S}}(K^-) = \Delta a^{(2)}_{g^{S}}(K^-) + \Delta a^{(2)}_{g^{S}}(K^-) [q \leftrightarrow k] + \\
\Delta b^{(2)}_{g^{S}}(K^-) + \Delta b^{(2)}_{g^{S}}(K^-) [q \leftrightarrow K - q, \ k \leftrightarrow K - k] .
\]

(37)

Finally, the correction in order \(g^{S}\) to the Green’s function can be found identifying the free Green’s functions for two, three and four body propagation in Eqs. (33) and (36). In its general form,

\[
\Delta C^{(2)}_{g^{S}}(K^-) = G^{(2)}_{0}(K^-) V G^{(3)}_{0}(K^-) V G^{(2)}_{0}(K^-) V G^{(3)}_{0}(K^-) V G^{(2)}_{0}(K^-) + \\
G^{(2)}_{0}(K^-) V G^{(3)}_{0}(K^-) V G^{(4)}_{0}(K^-) V G^{(3)}_{0}(K^-) V G^{(2)}_{0}(K^-) ,
\]

(38)

it contains the self energies corrections for the bosons \(\Phi_{1}\) and \(\Phi_{2}\), vertex corrections and the crossed box diagram. Imposing the restriction that only ladder diagrams are allowed in Eq. (36), we recover Eq. (31).

The nonperturbative Green’s function up to order \(g^{S}\) contains the covariant propagation of the system in the ladder approximation and is written as:

\[
G^{(2)}_{g^{S}}(K^-) = G^{(2)}_{0}(K^-) + G^{(2)}_{0}(K^-) \left(K^{(3)}_{l} + K^{(4)}_{l} \right) G^{(2)}_{g^{S}}(K^-) ,
\]

(39)

where

\[
K^{(4)}_{l} = V G^{(3)}_{0}(K^-) V G^{(4)}_{0}(K^-) V G^{(3)}_{0}(K^-) V .
\]

(40)

The Green’s function, (39), up to order \(g^{S}\) yields the covariant propagator for light-front times. Separating the terms in the kernel corresponding to the covariant ladder,

\[
K^{(4)}_{l, l} = \left[V G^{(3)}_{0}(K^-) V G^{(4)}_{0}(K^-) V G^{(3)}_{0}(K^-) V \right]_{\text{ladder}} ,
\]

(41)
together with the contribution of the one boson exchange diagram, Eq. (29), we obtain the nonperturbative Green’s function equation which, up to order $g_s^4$, corresponds to the covariant ladder propagator for light-front times, including one boson exchange and the box diagram. It is given by

$$G^{(2)}_{g_s^4,l}(K^-) = G^{(2)}_{0}(K^-) + G^{(2)}_{0}(K^-) \left(K^{(3)}_{l,l} + K^{(4)}_{l,l}\right) G^{(2)}_{g_s^4,l}(K^-),$$

in which the bound state solutions of Eq. (42) have up to four particles in lowest order in the intermediate state. We have solved it numerically to study the contribution of this Fock component in the nonperturbative regime in which the bound state appears [15].

### 3.4 Green’s Function $O(g_s^{2n})$

The perturbative correction to the light-front propagator of order $g_s^{2n}$ for the covariant ladder can be obtained by generalizing our previous results as

$$\Delta S^{(2)}_{g_s^n}(K^-) = i^{3n+2}(ig_s)^{2n} \int \frac{dk^-}{k^+(K^- - k^-)} \frac{1}{k^- - k^+_1 + m^2 - i\epsilon} \frac{1}{K^- - k^-_1 - (K - k^-_1)^2 + m^2 - i\epsilon} \times$$

$$\prod_{i=2}^n \frac{dk^-}{k^+_i (k^- - k^-_i)(k^+_i - k^-_{i-1})} \frac{1}{k^-_i - k^-_{i-1} - (k^-_i - k^-_{i-1})^2 + m^2 - i\epsilon} \frac{1}{k^+_i - k^-_{i-1} - (K - k^-_1)^2 + m^2 - i\epsilon}.$$

In Eq. (43), light-front intermediate states of up to $n + 2$ particles appear.

Formally we can write the light-front propagator up to order $g_s^n$ as a sum given by:

$$S^{(2)}_{g_s^n}(K^-) = S^{(2)}(K^-) + \sum_{m=1}^n \Delta S^{(2)}_{g_s^n}(K^-).$$

### 4 Hierarchy equations

The light-front Green’s function for the two-body system obtained from the solution of the covariant BS equation that contains all two-body irreducible diagrams, with the exception of those including closed loops of bosons $\Phi_1$ and $\Phi_2$ and part of the cross-ladder diagrams,
is given by:

\[
\begin{align*}
G^{(2)}(K^-) &= G_0^{(2)}(K^-) + G_0^{(2)}(K^-) VG^{(3)}(K^-) VG^{(2)}(K^-), \\
G^{(3)}(K^-) &= G_0^{(3)}(K^-) + G_0^{(3)}(K^-) VG^{(4)}(K^-) VG^{(3)}(K^-), \\
G^{(4)}(K^-) &= G_0^{(4)}(K^-) + G_0^{(4)}(K^-) VG^{(5)}(K^-) VG^{(4)}(K^-), \\
&\cdots \\
G^{(N)}(K^-) &= G_0^{(N)}(K^-) + G_0^{(N)}(K^-) VG^{(N+1)}(K^-) VG^{(N)}(K^-), \\
&\cdots
\end{align*}
\]  

(45)

The hierarchy of equations (45) corresponds to a truncation in the light-front Fock space in which only two bosons states with the two particles \(\Phi_1\) and \(\Phi_2\) are allowed in the intermediate state, with no any restriction on the number of bosons \(\sigma\), which thus excludes the complete representation of the crossed ladder diagrams. To obtain the two-body propagator for light-front times in the covariant ladder approximation, the kernel of the hierarchy equations must be restricted.

A systematic expansion by the consistent truncation of the light-front Fock space up to \(N\) particles in the intermediate states (boson 1, boson 2 and \(N-2\) \(\sigma\)'s) in the set of Eqs. (45), amounts to substitution

\[
G^{(N)}(K^-) \rightarrow G_0^{(N)}(K^-),
\]  

(46)

and subsequent solution of the coupled-hierarchy equations.

By restricting to up to four-particles in the intermediate state propagation, we obtain the following nonperturbative equation for the Green’s function:

\[
\begin{align*}
G^{(2)}(K^-) &= G_0^{(2)}(K^-) + G_0^{(2)}(K^-) VG^{(3)}(K^-) VG^{(2)}(K^-), \\
G^{(3)}(K^-) &= G_0^{(3)}(K^-) + G_0^{(3)}(K^-) VG^{(4)}(K^-) VG^{(3)}(K^-). \\
\end{align*}
\]  

(47) (48)

The kernel of Eq. (47) still contains an infinite sum of light-front diagrams, that are obtained solving by Eq. (48). To obtain the ladder approximation up to order \(g_4^\lambda\), Eq. (42), only the free and first order terms are kept in Eq. (48), with the restriction of only one and two boson covariant exchanges.

The bound-state solution of Eq. (45) is covariant in the sense that only the initial and final states are defined for specific light-front times, and thus the position of the bound state pole is frame independent. The residue of \(G^{(2)}(K^-)\) at the bound state energy \(K^- = K_E^-\) is related to the two-body wave-function at given light-front time.

The two-body T-Matrix is written in terms of the light-front two-body Green’s function, Eq. (45), as

\[
T^{(2)}(K^-) = VG^{(3)}(K^-) V + VG^{(3)}(K^-) VG^{(2)}(K^-) VG^{(3)}(K^-) V .
\]  

(49)
The T-matrix also satisfies the inhomogeneous integral equation:

\[
T^{(2)}(K^-) = VG^{(3)}(K^-)V + VG^{(3)}(K^-)V G_0^{(2)}(K^-) T^{(2)}(K^-) .
\]  

(50)

Note that the T-matrix equation above is exact in providing the non-perturbative two-body propagator given by the Green’s function \( G^{(2)}(K^-) \). The interaction that appears in Eq. (50) has an infinite number of terms, as one sees in the second equation of the hierarchy Eqs. (45). By truncating \( G^{(3)}(K^-) \) it is possible to define several approximate equations. In particular the choice of \( G_0^{(3)}(K^-) \) to approximate \( G^{(3)}(K^-) \), gives the bound state equation found by Weinberg [8].

The covariant on-shell scattering amplitude, \( t^{(2)}(K^-) \), is obtained from the T-matrix, by taking into account the phase space factors:

\[
< q, K - q | t^{(2)}(K^-) | k, K - k > = \frac{< q, K - q | T^{(2)}(K^-) | k, K - k >}{[q^+(K^+ - q^+)k^+(K^+ - k^+)]^{\frac{1}{2}}} .
\]  

(51)

The full-off-shell covariant two-body amplitude, is formally obtained from the light-front Green’s function, Eq. (45), by substituting the values of the on-shell initial and final “-” momentum by their off-shell values

\[
< q, K - q | t^{(2)}_{(off)}(K^-) | k, K - k > = \frac{\left[ VG^{(3)}(K^-)V + VG^{(3)}(K^-)V G^{(2)}(K^-)V G^{(3)}(K^-)V \right]}{[q^+(K^+ - q^+)k^+(K^+ - k^+)]^{\frac{1}{2}}} ;
\]  

(52)

such that the initial off-energy-shell momenta are \( k^- \) and \( K^- - k^- \) and the final off-energy-shell ones are \( q^- \) and \( K^- - q^- \).

5 Light-Front Bound state Equation

The homogeneous equation for the light-front two-body bound state wave-function is obtained the solution of

\[
| \Psi_B > = G_0^{(2)}(K^-_B)V G^{(3)}(K^-_B)V | \Psi_B > ,
\]  

(53)

with the kernel defined by the hierarchy Eqs. (45). It can also be written as an eigenvalue equation for the squared mass operator:

\[
\left[ \left( M_0^{(2)} \right)^2 + K^+ V G^{(3)}(K^-_B)V \right] | \Psi_B > = (M_2)^2 | \Psi_B > ,
\]  

(54)

where \( (M_2)^2 = K^+ K^-_B - K^2_\perp \) and \( M_0^{(2)} = K^+ K_0^{(2)-} - K^2_\perp \).
The vertex function for the bound state wave-function is defined as
\[ \Gamma_{\text{LF}}(\bar{q}_\perp, q^+) = \langle q, K - q | \left( G_0^{(2)}(K_B^-) \right)^{-1} | \Psi_B \rangle. \] (55)

The light-front wave-function at \( x^+ = 0 \) is obtained from the residue of the covariant two-body propagator at the bound state pole:
\[ G_0^{(2)}(K_B^-) \frac{\Gamma_{\text{LF}}(\bar{q}_\perp, q^+)}{\sqrt{q^+(K^+ - q^+)}} = \frac{1}{\sqrt{q^+(K^+ - q^+)}} < q, P - q | \Psi_B > \]
\[ = \int dq^- \frac{F(q^+)}{(q^2 - m^2 + i\varepsilon)((k - q)^2 - m^2 + i\varepsilon)}. \] (56)

Approaching the bound-state pole of the off-shell T-matrix, Eq. (52), we relate the residue at this pole to the full four-dimensional vertex \( F(q^+) \), which is the solution of the field-theoretical BS equation. In terms of the light-front wave-function, the four-dimensional vertex is given by
\[ F(q^+) = \frac{1}{\sqrt{q^+(K^+ - q^+)}} < q, K - q |_{(\text{off})} [VG^{(3)}(K_B^-)V] | \Psi_B >. \]

The operator \( |_{(\text{off})} [VG^{(3)}(K_B^-)V] \) has off-shell energy values for \( q_1^- = q^- \) and \( q_2^- = K_B^- - q^- \) in the positions where the corresponding on-shell values appear in the many-body Green’s function.

We redefine the light-front vertex function by including the phase-space factor, in order to simplify the formula for the bound state equation: \( F_{\text{LF}} = \sqrt{q^+(K^+ - q^+)}/\Gamma_{\text{LF}}. \)

In the actual numerical calculations to quantify the effects of the higher Fock-components, we have used the bound state solution of Eq. (42) \[15\]. The Green’s function obtained from this equation, up to order \( g_5^4 \), reproduces the covariant two-body propagator between two light-front hypersurfaces. In this approximation, the vertex function satisfies the following integral equation,
\[ F_{\text{LF}}(\bar{q}_\perp, y) = \frac{1}{(2\pi)^3} \int \frac{d^2k_\perp dx}{2x(1-x)} \frac{K^{(3)-}_{I,l}(\bar{q}_\perp, y; \vec{k}_\perp, x) + K^{(4)-}_{I,l}(\bar{q}_\perp, y; \vec{k}_\perp, x)}{M^2_T - M^2_0} F_{\text{FL}}(\vec{k}_\perp, x), \] (57)
where the momentum fractions are \( y = q^+/K^+ \) and \( x = k^+/K^+ \), with \( 0 < y < 1 \).

The part of the kernel which contains only the propagation of virtual three particle states forward in the light-front time is obtained from Eq. (29) as,
\[ \frac{g_5^2}{(x - y)} \left( \frac{\theta(x - y)}{M^2_2 - \frac{q^2 + m^2}{y} - \frac{k^2 + m^2}{1-x} - \frac{q_\perp - k_\perp}{x-y}^2 + m^2} \right) + \left[ x \leftrightarrow y, \vec{k}_\perp \leftrightarrow \vec{q}_\perp \right]. \] (58)
where the momentum fractions are \( y = q^+/K^+ \) and \( x = k^+/K^+ \). The denominator in Eq. (58), in the non-relativistic limit gives origin to the Yukawa potential of range \( \mu^{-1} \).

The contribution to the kernel from the virtual four-body propagation is obtained from Eq. (41) as,

\[
K^{(4)}_{I,J}(q_\perp, y; k_\perp, x) = \frac{g_s^4}{(2\pi)^3} \int \frac{d^2p_\perp dz}{2z(1-z)(z-x)(y-z)} \frac{\theta(z-y)\theta(x-z)}{M_2^2 - q_\perp^2 + m^2 - \frac{p_\perp^2 + m^2}{1-z} - \frac{(q_\perp - p_\perp)^2 + \mu^2}{z-y}} \times \frac{1}{M_2^2 - \frac{p_\perp^2 + m^2}{1-x} - \frac{k_\perp^2 + m^2}{z-y} - \frac{(q_\perp - k_\perp)^2 + \mu^2}{x-z}} + \left[ x \leftrightarrow y, k_\perp \leftrightarrow q_\perp \right].
\] (59)

6 Conclusion

We have developed a general framework for constructing the light-front two-body Green’s function and have discussed the formal representation of the covariant BS equation in the light-front. We have show how to obtain the covariant off-shell T-matrix and vertex of the bound state from the light-front quantities. Although our discussion has been performed in the context of a bosonic Lagrangean, it can be extended to general cases \[16\]. We have found a hierarchy of coupled equations which gives the exact two-body propagator in several cases, including the ladder approximation. Truncation of the hierarchy can be performed consistently by approximating the \( N \)-body Green’s function with the free operator, which implies that the \( N \)-body light-front intermediate state is accounted for in lowest order of the two-body propagation.

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