Scalar and tensor perturbations in vacuum inflation

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Abstract

Recently, it was proposed that a small true vacuum universe can inflate spontaneously, in principle. Furthermore, there should be matter creation in vacuum inflation due to quantum fluctuations, and the matter created will influence the inflation simultaneously. In this paper, scalar and tensor perturbations in this model are analyzed and confronted with recent observations. These perturbations are derived and expressed with Hubble flow-functions. By comparing our calculations with experimental results, we can determine all the parameters in this model. Finally, with the determined parameters, we compute the evolution of the matter density and show that the matter produced in inflation roughly fits the observations at present.

Keywords: vacuum inflation, perturbations, matter production

(Some figures may appear in colour only in the online journal)

1. Introduction

It has been widely believed that there was a fast expansion period of the early universe, called inflation, which successfully solved several severe problems in the Big Bang cosmology [1]. Although the concept of inflation is successful in the standard cosmology model, the origin and the power of the inflation are still unknown. It was thought that the Higgs field might drive the inflation, but there are some problems in the Higgs inflation model [2], such as the naturalness problem [3]. Recently, it was mathematically proved that the universe can be spontaneously created from nothing, in principle [4]. For a small true vacuum universe, its quantum potential can drive its inflation. When the universe becomes large enough, its accelerating expansion ends. This vacuum inflation model does not need a matter field to drive inflation,
which is different from slow-roll inflation [5]. However, vacuum inflation should be developed because there are some parameters to be determined. When the issue of matter creation during inflation is considered, the vacuum inflation derived by the quantum potential should be modified slightly. Furthermore, the quantum fluctuations of the early universe, viewed as seeds for the growth of the structure of the universe, should also be given [6].

This paper is organized as follows. Section 2, we first briefly review the vacuum inflation derived by the quantum potential and then investigate the modification of the inflation by the matter created in the inflation. In section 3, we calculate the scalar perturbations up to second-order perturbations of the Hamiltonian for the scalar field and gravity in ADM form, without considering the concrete form of the scalar fields. Then, we discuss the quantization of gravity waves and calculate the tensor perturbations. In section 4, by comparing our calculations with experimental observations, we determine all the necessary parameters in the model. In section 5, the matter density is calculated. With the previously determined parameters, we fit the value of matter density which is compatible with current observations. Finally, we give discussions and conclusions in section 6.

For simplicity, we use Planck units in this paper.

2. Vacuum inflation and matter correction

We first briefly review the vacuum inflation derived by the quantum potential and then show the modification of the vacuum inflation by the matter created in inflation.

2.1. Vacuum inflation

There is a long-standing debate over whether a vacuum can inflate spontaneously [7–9]. If the universe indeed came from a vacuum, the problem of creation in the Big Bang model can be avoided [10]. Recently, it was proved by the minisuperspace model that a small true vacuum bubble can inflate because of its quantum potential [4], which makes it possible to construct a theory in which the universe grows up spontaneously. For a true vacuum bubble, its action $S_{gr}$ can be written as

$$S_{gr} = \int \frac{1}{16\pi} \sqrt{-g} R \sqrt{g} d^4x,$$

(1)

where $\sqrt{-g}$ is the square root of the absolute value of the determinant of the metric, $\sqrt{g}$ is the curvature and $d^4x = d\tau d^3x$. $\tau$ is the conformal time defined as $\tau \equiv \int dt/\Delta t$, where $\Delta t$ is the scale factor. Because the vacuum bubble is homogeneous and isotropic, the metric can be expressed by one parameter, the scale factor $a$,

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & \frac{a^2}{1-kr^2} \delta_{ij} \end{pmatrix}.$$  

(2)

By substituting the metric above into equation (1), it is easy to obtain the Hamiltonian of this model $H = -i^2 \left( \frac{1}{2} \dot{a}^2 + \frac{1}{2} K a^2 \right)$, where $\dot{a} = -2i^2 a$ and $i^2 \equiv 8\pi/3$. Here, the primer denotes the derivative with respect to the conformal time. The parameter $K = 1, 0, -1$ represents closed, flat, and open bubbles, respectively. We adopt $K = 0$ as default for all calculations and only keep the $K$ explicitly in some formula for the purpose of comparing with classical case. In the minisuperspace model, the momentum operator is defined as
where $p$ is the factor of ordering and denotes the ambiguity of rules for quantization. Because there is no complete quantum theory for gravity, we treat $p$ as an uncertain parameter. In principle, the cosmological wavefunction satisfies the Wheeler–DeWitt equation, $\hbar \psi(a) = 0$. Decomposing the cosmological wavefunction into two real functions $\psi(a) = R(a) \exp(iS(a))$ and using equation (3), we have

$$4l^{-4}Q = -\frac{\partial R}{R} + \frac{p}{a} \frac{\partial R}{R},$$

which is called the quantum potential. Using the guidance relation [4]

$$\partial a^2 \mathcal{L} = -2l^{-2}a' = \partial S,$$

one can obtain

$$H^2 = -\frac{Q}{a^2} - \frac{K}{a^2},$$

where $H$ is the Hubble parameter. It is often defined as $d \ln a/dt$. The $\mathcal{L}$ in the guidance relation is the the Lagrangian of equation (1). The formula of the Hubble parameter above shows that the quantum potential $Q$ has the capacity to drive the vacuum bubble towards accelerating expansion.

The analytical formula of the wavefunction can be obtained by solving the Wheeler–DeWitt equation, $\psi(a) = iC_1a^{-p} / (1 - p) - C_2$, where $p \neq 1$ and $C_1$ and $C_2$ are arbitrary complex numbers. In order to get an inflation solution, we set $C_1$ and $C_2$ as real numbers. Because only the value of $|1 - p|$ is crucial for the inflation solution, we set $p < 1$ in the paper for convenience. Then, we have $S = \arctan[-C_1a^{-p}/C_2(1 - p)]$ [4]. With the guidance relation equation (5), when $|C_2(1 - p) a^{-p}|^2 \ll 1$, we can obtain the asymptotic form of the Hubble parameter

$$H = \frac{\dot{a}^2}{2} \frac{C_1}{C_2} a^{-2-p}.$$  

We can see how small $a$ should be when we determine $C_1$, $C_2$ and $p$. From the results of section 4, this equation is roughly right when $a < 50$. We can also get this conclusion from figure 1, where $H$ is almost constant till $a \sim 50$.

Now, it is easy to obtain the time-dependent evolution for the scale factor,

$$a(t) = \left\{ \begin{array}{ll} \frac{\dot{a}}{a} \frac{C_1}{C_2} (2 + p)(t - t_0) & p \neq -2, \\ e^{\frac{\dot{a}C_1}{C_2} (t - t_0)} & p = -2. \end{array} \right.$$  

When $p = -2$, we have an exponential inflation solution. When $-2 < p < -1$, we get power-law inflation.

In figure 1, we show the evolutions of the quantum potential and the Hubble parameter with the variable of the scale factor $a$ for a specific ordering factor $p = -2$. The Hubble parameter is nearly constant in the early stage of the universe, which is quite similar to slow-roll inflation. After a while, the Hubble parameter decreases rapidly to zero, which means that the inflation can exit spontaneously. The quantum potential is the power of the vacuum inflation and plays the role of the scalar field in the slow-roll inflation model [11]. We also find that different values of $p$ have different quantum potentials, which leads to different evolution of the universe.
2.2. Matter correction

There are quantum fluctuations such as the creation and annihilation of virtual particles even in the vacuum of space. Due to the accelerating expansion, the virtual particles may be separated before annihilation and then become real particles. Therefore, there is matter creation during vacuum inflation [12]. With the matter creation, the action of the early universe is modified by a scalar field,

\[ S_{\text{gr}} + S_{\text{SC}} = \int \left( \frac{1}{16\pi} R + \frac{1}{2} \phi_{,\alpha} \phi^{,\alpha} - V(\phi) \right) \sqrt{-g} d^4x, \] (8)

where \( V(\phi) \) is the potential of the scalar field. Here, we don’t need to restrict the form of the potential of the scalar field, which is different to the slow-roll inflation theory. By substituting equation (2) into equation (8), we can obtain the Hamiltonian of the system.

Figure 1. The quantum potential \( Q \) and Hubble parameter \( H \) versus the scale factor \( a \) with different \( p \). They are from equations (13) and (11) without matter correction. We use the parameter \( C_1/C_2 \) fixed in section 4 to get these exact results. When comparing it with asymptotic expression equation (6), we can easily find when equation (6) is approximately right. The late time of this plot means that the universe starts to exit the inflation. The quantum effect does decrease strongly, but is still much bigger than the influence of matter in that time.
\[ \mathcal{H} = -l^{-2} \left( \frac{1}{4} \dot{a}^2 P_a^2 + Ka^2 \right) + \left( \frac{1}{2} \frac{P_a^2}{a^2} + \frac{1}{2} a^4 \varphi_a^2 + a^4 V(\varphi) \right), \]  

(9)

where \( P_\varphi = a^2 \dot{\varphi} \) and \( P_a \) was defined in equation \( (3) \). The wave function of the universe still satisfies the Wheeler–DeWitt equation, \( \mathcal{H} \psi(a, \varphi) = 0 \), but it is difficult to obtain analytical solutions for this equation. For the vacuum inflation model, the inflation is driven by the quantum effects of the universe, particularly at the beginning of the early universe. Under this condition, the wavefunction of the universe can be approximately written as \( \psi(a, \varphi) = \psi_a(a) \psi_i(a, \varphi) \), where \( \psi_a(a) \) is the wavefunction for the space of the universe and \( \psi_i(a, \varphi) \) is the wavefunction of the scalar field. The quantity of \( P_a \psi_i(a, \varphi) \) is negligible compared with \( P_a \psi_a(a) \). In this way, equation \( (9) \) can be simplified as

\[
\begin{align*}
\left( \frac{1}{2} \frac{P_a^2}{a^2} + \frac{1}{2} a^4 \varphi_a^2 + a^4 V(\varphi) \right) \psi_i(a, \varphi) &= a^4 \rho(a) \psi_i(a, \varphi), \\
\left( -l^{-2} \left( \frac{1}{4} \dot{a}^2 P_a^2 + Ka^2 \right) + a^4 \rho(a) \right) \psi_a(a) &= 0,
\end{align*}
\]

(10)

where \( \rho(a) \) can be treated as matter density, which we will explain using equation \( (11) \). By expressing \( \psi_a(a) \) as two real functions \( \psi_a(a) = R_a(a) \exp(i S_a(a)) \) and substituting equation \( (3) \) into equation \( (10) \), we can obtain

\[
\frac{1}{4} \dot{a}^2 (\partial_a S)^2 + Ka^2 + Q = \dot{a}^2 a^4 \rho(a),
\]

where \( Q \) is determined by equation \( (4) \). Using guidance relation equation \( (5) \) we find that the Hubble parameter takes the form

\[
H^2 = -\frac{Q}{a^2} - \frac{K}{a^2} + \frac{8\pi}{3} \rho.
\]

(11)

Comparing equation \( (11) \) with the classical Friedmann equation \( [13] \) when \( Q \to 0 \), we know that \( \rho(a) \) is exactly the matter density.

From the first equation of equations \( (10) \), we have \( \mathcal{H}_\varphi = a^4 \rho(a) \), where \( \mathcal{H}_\varphi \) is the \( \varphi \) part of the \( \mathcal{H} \). The derivative of the equation \( (11) \) with respect to the normal time can give

\[
2\dot{H} = \frac{3}{a^2} \frac{\dot{Q}}{a^2} - \frac{1}{a^2} \frac{dQ}{da} + K - \frac{8\pi}{3} \mathcal{S}_\varphi.
\]

(12)

where \( \mathcal{S}_\varphi \equiv P_\varphi^2 - \mathcal{H}_\varphi \). Equation \( (12) \) is the Einstein acceleration equation of this model. When \( Q \to 0 \), it goes back to the classical Einstein acceleration equation.

Because vacuum inflation is essentially driven by the quantum potential, the contribution of matter to the evolution of the early universe can be neglected. With equations \( (4) \) and \( (7) \), the quantum potential of the early universe can be written as

\[
Q = -\frac{1}{4} \frac{C_1^3 C_2^3 (p - 1)^2 a^{2p}}{(C_1^3 (p - 1)^2 a^{2p} + a^2 C_1^3)^2},
\]

(13)

where \( C_1 \) and \( C_2 \) are two parameters to be determined.

As discussed above, there are energy fluctuations as the virtual particles are created and annihilated. The virtual particles may become real particles probabilistically when tunneling through the cosmological horizon \( [14] \). If we consider equation \( (10) \) in de Sitter space, the radius \( r \) of the horizon is \( 1/H \), and the temperature of radiation from the horizon is \( H/(2\pi) \) \([12]\). Using the Stefan–Boltzmann law, the creation rate for the matter density is
\[ \dot{\rho} = 4j\pi r^2 \frac{3\sigma H^5}{16\pi^4}, \]  
\[ \dot{\rho} = 3\sigma H^5 \frac{5}{16\pi^4} - 4H\rho, \]  
where the dot denotes derivative with respect to the normal time \( t \), \( j \) is the energy flux from the horizon, and \( \sigma \) is the Stefan–Boltzmann constant. The particles created during inflation are relativistic due to the high temperature at this stage, so the mass-conservation equation gives \( \dot{\rho} = -4H\rho \). Considering both creation and dilution of matter simultaneously, we have

Combining equations (11) and (15), we can obtain the equation for the change of the matter density with the growth of the early universe. The evolution of the Hubble parameter can also be obtained, which is numerically shown in figure 2. When the universe is small, the Hubble parameter is modified slightly by the matter created during inflation, which is the previous assumption that the inflation is mainly driven by its quantum potential. When the early universe becomes large, the contribution of matter to inflation is relatively large, whereas the dynamics of the universe are classical because \( Q \to 0 \) for a large enough universe [13].

3. Cosmological perturbations

The measurement of CMBR shows that there were quantum fluctuations of gravity during inflation, which were the seeds of the galaxies of our universe. With the results of CMBR, we can look into the details of inflation and verify the validity of the inflation model. Although all kinds of fields can generate corresponding cosmology perturbations, we only consider scalar and tensor fluctuations during inflation, because they can be determined by the measurement of CMBR. In the following, we will first follow the standard cosmological perturbation theory but express the scalar and tensor perturbations with the Hubble parameter and its derivative. Then, we will show that it is right to use these classical conclusions in this model.

From the calculations, we see that the Hamiltonian of perturbations can be expressed by the Hubble parameter and its derivative. The perturbations are decided by how the universe
inflated rather than the concrete form of the scalar fields. In this model, quantum potential, together with field perturbations leads to cosmological perturbations. It is different from the standard model of inflation, where the scalar field and its perturbations lead to cosmological perturbations.

3.1. Scalar perturbations

Scalar perturbations are gravity perturbations caused by fluctuations of scalar fields. We need the four quantities $B$, $\psi$, $E$ and $\phi$ to describe the most general form of scalar metric perturbations

$$g_{ij} = \begin{pmatrix} \mathcal{N}^2 - \mathcal{N}_i \mathcal{N}^i & -\mathcal{N}_i \\ -\mathcal{N}_i & -\gamma_{ij} \end{pmatrix},$$

(16)

where $\mathcal{N}_i = a^2 \partial_i B$, $\gamma_{ij} = a^2 (1 - 2\psi)\delta_{ij} + 2a^2 \partial_i \partial_j E$ and $\mathcal{N} = a (1 + \phi - \frac{1}{2} \phi^2 + \frac{1}{2} \partial_i \partial_i B)$. Furthermore, we use the quantity $\phi_0 + \delta \phi$ to describe perturbations of the scalar field.

The scalar perturbations can be tested by the intrinsic curvature perturbation of the comoving hypersurface $R$, which is a gauge invariance quantity,

$$R \equiv -\partial_{\tau} \phi \delta \phi,$$

(17)

where $\mathcal{H} \equiv a'/a$. Using the background equations to cancel the background action, the first-order perturbations and the background scalar field $\phi_0$, we can simplify the action in equation (8) to the second-order

$$\delta_2 S = \frac{1}{6\ell^2} \int \left\{ a^2 \left[ -6(\psi')^2 - 12\mathcal{H}\phi\psi' - 2(\mathcal{H'} + 2\mathcal{H}^2) \phi^2 - 2\psi' (2\phi, \psi) + 3\ell^2 ((\delta \phi')^2 - \delta \phi, \delta \phi, \partial_i \partial_j \delta \phi) \right] + 2(\mathcal{B} - \mathcal{E}) \right\} \, d^4 x.$$

(18)

Furthermore, we choose a simple gauge as $\delta \phi = 0$ and $E = 0$ to eliminate $B$ and $\phi$. A straightforward calculation gives the action that only contains one field,

$$\delta_2 S = \frac{1}{6\ell^2} \int \left[ -2a^2 \frac{H}{\mathcal{H}^2} [ (\psi')^2 - \psi, \psi ] \right] \, d^4 x,$$

(19)

where we use both the normal time and the conformal time to simplify our calculations. $\mathcal{H}$ is $dH/dt$ and $\psi'$ is $d\psi/d\tau$. For convenience, we introduce two variables $z = \sqrt{-a^2 \mathcal{H}^2} = a \mathcal{H}/H$ and $u = -zR/(2\sqrt{\pi})$ in our calculations. Varying equation (19) and using Fourier transformation in the space variables, we can obtain

$$\left( \frac{d^2}{d\tau^2} + k^2 \right) u_k = 0,$$

(20)

where $u_k$ is the image function of $u$. For easy comparison with experimental results, we use the Hubble flow-functions (HFFs) [16] to simplify our calculations,

$$\epsilon_1 = -\frac{d \ln H}{d \ln a}, \quad \epsilon_{i+1} = \frac{d \ln \epsilon_i}{d \ln a},$$

(21)
where \(i \geq 1\). Expressing \(z\) with HFFs, we obtain \(z = a \sqrt{\epsilon_1}\) and

\[
\frac{1}{z} \frac{d^2z}{d\tau^2} = a^2 H^2 \left( \frac{\epsilon_1^2}{4} + \frac{3\epsilon_2}{2} - \frac{\epsilon_1 \epsilon_2}{2} + \frac{\epsilon_1^2}{2} - \epsilon_1 + 2 \right).
\]

(22)

Integrating the conformal time \(\tau = \int 1/a \, dt = \int 1/(a^2 H) \, da\) by parts repeatedly gives

\[
\tau = -\sum_{i=0}^{\infty} f_i \epsilon_i, \quad f_i = f_i \left( \frac{d \ln f_i}{d \ln a} + \epsilon_i \right),
\]

(23)

where \(f_0 \equiv 1\). Combining equations (22) and (23), we obtain

\[
\frac{1}{z} \frac{d^2z}{d\tau^2} = \left( \sum_{i=0}^{\infty} f_i \right)^2 \left( \frac{\epsilon_1^2}{4} + \frac{3\epsilon_2}{2} - \frac{\epsilon_1 \epsilon_2}{2} + \frac{\epsilon_1^2}{2} - \epsilon_1 + 2 \right) \frac{\tau^2}{\tau^2}.
\]

(24)

where \(F(\epsilon_1, \epsilon_2, \ldots)\) represents the multinomial of \(\epsilon_1, \epsilon_2, \ldots\), and does not contain a constant term. In this model, HFFS and \(F(\epsilon_1, \epsilon_2, \ldots)\) is constant during the early stage of inflation. We will show that in section 4. With equation (24) and constant \(F(\epsilon_1, \epsilon_2, \ldots)\), we can solve equation (20) analytically. In the limit \(k/(aH) \to \infty\), \(u_k\) has a normalized solution as \(u_k(\tau) \to -e^{-i\kappa \tau}/\sqrt{2k}\). Thus, the solution of equation (20) is

\[
u
\]

\[
\frac{\sqrt{\pi}}{2} e^{i\pi \nu + \nu i + 1} \sqrt{-\tau} H_\nu(-k\tau),
\]

where \(H_\nu\) is the Hankel function of the first kind with \(\nu \equiv \frac{1}{2} + F(\epsilon_1, \epsilon_2, \ldots)\). We can then calculate the vacuum fluctuations of the curvature perturbation \(\mathcal{R}\). The power spectrum \(P_\mathcal{R}(k)\) is defined in terms of the expectation value of inflation state \(\langle \mathcal{R}_k \mathcal{R}_l^* \rangle \equiv 2\pi^2 P_\mathcal{R} \delta^3(k-l)/k^3\), where \(\mathcal{R}_k\) is defined as

\[
\mathcal{R} = \int \frac{d^3k}{(2\pi)^3/2} \mathcal{R}_k(\tau) e^{ik\cdot x}.
\]

Considering the quantization of \(u_k\), we have \(\langle \mathcal{R}_k \mathcal{R}_l^* \rangle = 4\pi \int |u_k|^2 |u_l|^2 \delta^3(k-l)/z^2\).

Finally, we can obtain

\[
P_{\mathcal{R}}^{1/2}(k) = \frac{1}{\sqrt{\pi}} 2^{\nu-3/2} \Gamma(\nu) \Gamma(3/2) \sum_{i=0}^{\infty} f_i \left( \frac{H}{|\kappa|} \right)_{k=aH}.
\]

(25)

The scalar perturbations do not depend on special scalar fields. All scalar fields in the fundamental theory contribute a part of the scalar perturbations \(P_\mathcal{R}(k)\). In the standard model, we have scalar fields \(\Phi\) with complex doublets. Therefore, there are two scalar field contributions to the scalar perturbations [17], and the total power spectrum \(P_{\mathcal{R}}(k) = 2P_{\mathcal{R}}(k)\). We show how curvature perturbations change along with the scale factor in figure 3. The perturbations are close to constant in the early stage of inflation.

### 3.2. Gravitational waves

Gravitational waves are gravitational perturbations that are produced by the vacuum fluctuations of gravity. The linear tensor perturbations can be written as \(g_{\mu\nu} = a^2(\tau) \left[ h_{\mu\nu} + h_{\mu\nu} \right]\). In the transverse traceless gauge as \(h_{00} = h_{0i} = \partial_i h_{ij} = h_{ii} = 0\), there are two independent states [18]. The action of the perturbations can be expressed as
\[
S_g = \frac{1}{64\pi} \int a^2(\tau) \partial_\mu h^\mu \partial_\mu h^\mu d^4x.
\] (26)

It is convenient to calculate the perturbations with rescaled variables
\[\nu(\tau) = \frac{3}{2} \frac{a(\tau)}{2\pi} \lambda(\tau),\]
where \(h(x)\) and \(\lambda = +, \times\) represent two independent states. By varying equation (26) and using Fourier transformation in the space variables, we can obtain
\[
\frac{d^2\nu}{d\tau^2} + \left( k^2 - \frac{1}{a} \frac{d^2a}{d\tau^2} \right) \nu = 0,
\] (27)
where \(\nu\) is the image function of \(v\). Using of equation (21), we can simplify the following expression
\[
\frac{1}{a} \frac{d^2a}{d\tau^2} = 2a^2H^2 \left( 1 - \frac{1}{2} \epsilon_1 \right).
\] (28)

Substituting equation (23) into the equation above, we can obtain
\[
\frac{1}{a} \frac{d^2a}{d\tau^2} = \left( \sum_{i=0}^{\infty} f_i(\epsilon) \right)^2 \left( 1 - \frac{1}{2} \epsilon_1 \right) \frac{\tau^2}{\epsilon_1} = \left( \left( \frac{\epsilon_2}{2} + G(\epsilon_1, \epsilon_2, ...) \right)^2 - \frac{1}{2} \right) \frac{\tau^2}{\epsilon_1},
\] (29)
where \(G(\epsilon_1, \epsilon_2, ...)\) represents the multinomial of \(\epsilon_1, \epsilon_2, \ldots\) and does not contain a constant term. Repeating the same procedure as the case of scalar perturbations, we find that \(v(\tau)\) has the same solutions as that of \(u(\tau)\) except that \(\nu\) is replaced by \(\mu\), and \(\mu\) is defined as \(\mu = \frac{3}{2} + G(\epsilon_1, \epsilon_2, ...)\).

The spectrum of gravitational waves \(P_g(k)\) is defined as \(\langle h_{k,\lambda} h_{l,\lambda}^* \rangle \equiv 2\pi^2 P_g \delta^3(k - 1)/k^3\) [5]. Using the relation between \(v(\tau)\) and \(h_\lambda(x)\), we can obtain
\[
\langle v_{k,\lambda} v_{l,\lambda}^* \rangle \equiv \frac{1}{32\pi} a(\tau)^2 2\pi^2 P_g \delta^3(k - 1)/k^3.
\] (30)

With the quantization of \(v_{k,\lambda}\), we have that \(\langle v_{k,\lambda} v_{l,\lambda}^* \rangle = |v_{k,\lambda}|^2 \delta^3(k - 1) \delta_{\lambda,\lambda'}\). Furthermore, we can obtain the spectrum of the gravitational waves as

![Figure 3](image-url)
\[ P^{1/2}_g(k) = \frac{2}{\sqrt{\pi}} 2^{\mu - 1/2} \Gamma(\mu) \left( \sum_{i=0}^{\infty} f_i \right)^{-\mu+1/2} \left| k = aH \right. \]

(31)

In figure 3, we show the change of the power spectrum of gravitational waves with the scale factor. The power spectrum is also close to a constant in the early stage of inflation. It decreases in the same way as the Hubble parameter does when the scale factor becomes large. The perturbations are nearly constant when they leave the horizon, all the calculation and experimental observations are based on \( k = aH \) \cite{18}.

3.3. Perturbations and quantum potential

The calculation of sections 3.1 and 3.2 are totally classical. This model is based on quantum theory. It seems unreasonable to use equation (19) in this model. We need to discuss how to get a reasonable perturbation theory in this model. The derivation satisfies the following principle. First, all the variables of the Hamiltonian are operator. They may not communicate with each other. Second, the operator must satisfy the classical equations of motion. Third, when the operator acts on the wavefunction which is not the eigenstate of this operator, we need add the corresponding quantum potential in and replace the operator with its classical definition. For example, we take

\[ \hat{\tilde{H}}^2 \psi_a(a) = \left( \hat{H}^2 + \frac{Q}{a^2} \right) \psi_a(a), \]

\[ \hat{\tilde{H}} \psi_a(a) = \left( \hat{H} + \frac{3Q}{2a^2} - \frac{1}{2a^4} \frac{dQ}{da} \right) \psi_a(a). \]

(32)

With equation (32), we can easily obtain equation (11) from equation (10).

First, we discuss the gravitational waves. The wave function of the universe satisfies

\[ \left( \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\varphi}) + \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\phi}, \hat{\psi}, \hat{\beta}, \hat{\delta}) \right) \psi(a, \varphi, h) = 0, \]

(33)

where \( \delta \mathcal{L}(\hat{a}, \hat{\varphi}) \) is defined by equation (9) and \( \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}) \) is the Hamiltonian of equation (26). We can separate the wave function similar to that of the section 2.2 and obtain \( \psi(a, \varphi, h) = \psi_b(a, \varphi)\psi_b(a, \varphi, h), \) with that we can simplify equation (33) to

\[ \delta \mathcal{L}(\hat{a}, \hat{\varphi}) \psi_b(a, \varphi, h) = a^4 \rho_b(a) \psi_b(a, \varphi, h), \]

\[ \left( \delta \mathcal{L}(\hat{a}, \hat{\varphi}) + a^4 \rho_b(a) \right) \psi_b(a, \varphi) = 0. \]

(34)

Comparing it with equation (10), we can know that \( \rho_b(a) \) is the energy of gravitational waves and \( \psi_b(a, \varphi) \) is the background wave function of an inflation universe which contains the correction of gravitational waves. \( \psi_b(a, \varphi) \) represents gravitational waves at the scale factor \( a. \) As the energy of perturbations is negligible, the \( \psi_b(a, \varphi) \) approximates to the \( \psi(a, \varphi) \) of the section 2.2. The \( \psi_b(a, \varphi, h) \) goes back to the calculation of section 3.2.

Second, we discuss the scalar perturbations. The Hamiltonian is

\[ \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\varphi}) + \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\phi}, \hat{\psi}, \hat{\beta}, \hat{\delta}), \]

(35)

where \( \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\phi}, \hat{\psi}, \hat{\beta}, \hat{\delta}) \) is the total Hamiltonian of scalar perturbations. Under the second principle, We can use the Hamiltonian of equation (19) which means the wave function satisfies

\[ \left( \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\varphi}) + \delta \mathcal{L}(\hat{a}, \hat{\tilde{H}}, \hat{\psi}) \right) \psi(a, \varphi, \psi) = 0, \]

(36)
where $H_R$ is the Hamiltonian of equation (19). We can separate the wave function similar to that of the section 2.2 and obtain $\psi(a, \varphi) = \psi_b(a, \varphi)\psi_R(a, \varphi, \psi)$. With that we can simplify equation (36) to

$$
S_R(a, H, \dot{\psi}) \psi_R(a, \varphi, \psi) = a^2 \rho_R(a) \psi_R(a, \varphi, \psi),
$$

$$
\left( S(a, H, \dot{\varphi}) + a^2 \rho_R(a) \right) \psi_b(a, \varphi) = 0.
$$

(37)

compared with equation (10), the $\rho_R(a)$ is the energy of scalar perturbations and $\psi_b(a, \varphi)$ is the background wave function of an inflation universe which contains the correction of scalar perturbations. $\psi_R(a, \varphi, \psi)$ represents scalar perturbations at the scale factor $a$. Since the Hubble parameter has been determined for the given $a$ in this model, the $H$ and $H$ in $S_R$ will not bring any vagueness. As the energy of perturbations is negligible, the $\psi_b(a, \varphi)$ here approximates to the $\psi(a, \varphi)$ of the section 2.2. The $\psi_R(a, \varphi, \psi)$ goes back to the calculation of section 3.1.

With these derivation, we know it is right to use the conclusions of sections 3.1 and 3.2. The quantum potential will have impact on the background wave function. It can give the relation between the scale factor and the Hubble parameter, which will determine the perturbations.

4. Experimental constraints of vacuum inflation

There are two parameters in the vacuum inflation model that should be determined: $C_1/C_2$ and $p$. Only when all parameters are determined, can this model be falsified by experimental observations. With the progress of measuring CMBR, this model can be fixed by observational results.

There are some parameters that constrain inflation in the current observations: the tensor-to-scalar ratio $r \equiv P_T/P_R < 0.11(95\% \text{CL})$, the spectral index of curvature perturbations $n_s = 1 + dP_R/d\ln k = 0.968 \pm 0.006$ and its scale dependence $dn_s/d\ln k = -0.003 \pm 0.007$ [19]. From equations (25) and (31), we express these parameters up to the second-order of HFFs:

$$
n_s - 1 = -2\epsilon_1 - \epsilon_2 - 2\epsilon_1^2 - (2C + 3)\epsilon_1\epsilon_2 - C\epsilon_2\epsilon_3,
$$

(38)

where $C = -2 + \ln 2 + \gamma$. Because the perturbations are mainly produced in the early stage of inflation, we calculate the fluctuations when the scale factor is small. At that time, the influence of matter is negligible. In this case, equation (6) gives

$$
\epsilon_1 = p + 2 \quad \epsilon_i = 0 \quad |i| > 1.
$$

(39)

Then, equation (38) can be represented by the ordering factor $p$ as $r = 8(p + 2)(p + 3)$, $n_s - 1 = -2(p + 2)(p + 3)$ and $dn_s/d\ln k = 0$.

The measurement precision of $n_s$ is more dependable among the constraints of inflation, so we choose it to determine $p$, and get $p + 2 = 0.016 \pm 0.003$. If we choose

$$
p + 2 = 0.013,
$$

(40)

the limitation on $r$ can also be fixed. The restriction on $dn_s/d\ln k$ is satisfied regardless of the value of $p$. The $r$ and $n_s - 1$ are constant in the early stage of inflation, which is similar to slow-roll inflation.
To constrain the parameter $C_1/C_2$, we need to specify the starting point problem of scale factor. In this model, we can not give the starting point of $a$. Luckily enough, with equation (11), we can roughly determinate the ending point of the inflation by comparing the quantum potential with the contribution of matter. Once the ending point of $a$ is determined, we can use it to fix the starting point of $a$ with the general requirement of $60\, e$-folds of expansion [5]. Generally, the observational perturbations are mainly crossed the Hubble radius in the range from 50 to $60\, e$-foldings before the end of inflation [5].

With these general considerations, from equations (25) and (39), we can get

$$P'_R = \frac{(1 - (C + 1)(p + 2))^2}{8(p + 2)(p + 3)} \frac{16}{\pi} H^2. \tag{41}$$

The observations give that $P'_R \approx 2 \times 10^{-9}$ [19]. Before we constrain the parameter $C_1/C_2$ with this observation result, we still have an unsolved problem. We can not numerically compute the ending point of the inflation without parameter $C_1/C_2$. Since $p + 2$ is a very small number, $P'_R$ is a slowly varying function of $a$. The estimation of the end point of inflation is pretty accurate. Using $a = 1$ in equation (6) and equation (41), we can get $C_1/C_2 \approx 1.55 \times 10^{-6}$. By numerical calculation, we can roughly get the end point of inflation around $a \approx 10^4$. This means the observational perturbations are mainly crossed the Hubble radius in the range from $a = 10^{-22}$ to $a = 2 \times 10^{-18}$. With equations (6) and (41) and this range, we can rederive $C_1/C_2$ more accurately, which ranges from $C_1/C_2 = 8.0 \times 10^{-7}$ to $C_1/C_2 = 9.2 \times 10^{-7}$. The range is really narrow and we set

$$C_1/C_2 = 8.6 \times 10^{-7} \tag{42}$$

for simplification. With this value of $C_1/C_2$, we can recompute the ending point of inflation. It changes within a magnitude of 10, which proves that the estimation for $C_1/C_2$ is pretty good. Thus, all the parameters in the vacuum inflation model have been determined.

### 5. Comparison and prediction

With the fixed parameters $C_1, C_2$ and $p$, we will first calculate the matter generation. Then, we can test the model by comparing the calculated matter density with current experimental observations.

The evolution of the early universe is roughly determined by equations (11) and (15). Before we do any calculation, we must clarify the evolution process of universe in this model. As all the parameter have been determined, we can divide the evolution process into several stages. At first, the cosmos is beginning to inflate due to quantum potential. The scale factor grew very fast from about $10^{-22}$ to about 50, which is obvious in figure 1. In this stage, matter is produced with little influence on the inflation. With the growth of the scale factor, the quantum potential decreases rapidly. It means that the universe is about to exit the inflation. This process lasts from about 50 to the magnitude of $10^4$, which can be seen from figures 1, 2 and 4. After the end of inflation, the radiation produced by inflation takes the role of the quantum potential and drives the expansion of the cosmos slowly. Combining equations (15) and (11) and considering that $H$ is very small in this stage, we have the relation

$$H \propto \frac{1}{a^2}. \tag{43}$$

The quantum potential is negligible at this stage, which is shown in figure 4. After the radiation-dominated stage, the matter produced during inflation replaces the radiation and continues to drive the expansion of the cosmos.
Because the Hubble parameter grew rapidly during the initial stage of inflation and slowly after that stage, we divide the calculations into two parts to obtain more accurate results. First, we plot the function $H(a)$ to find when the quantum potential becomes negligible. Figure 4 shows that the quantum potential and matter production no longer play a part when $a$ is very large. The relation between $H$ and $a$ exactly fits the picture of the radiation-dominated stage. In this case, we can ignore the effects of the quantum potential.

When the quantum potential dominates the inflation, we use

$$t_f - t_i = \int_{a_i}^{a_f} \frac{1}{aH} \, da$$

(44)

to evaluate the time $t_f$, we can use $a = 2 \times 10^4$ from figure 4. We use the time when $a = 1$ as $t_i$. The reason why we use $t_i$ to calculate $t_f$ is that we have to calculate this period with sufficiently long step length. The scale factor $a = 1$ does not mark the start of the universe as we clarified before. If we choose $a = 10^{-22}$ at the start of time, $t_i$ will be a very short time. The universe is still inflating at $t_i$ according to figure 1. The Hubble parameter can be described by equation (6) at this time. Therefore, we have

$$t_i = \int_{10^{-22}}^{1} \frac{2}{P^2 C_1} a^{-p-3} \, da \approx 1.0 \times 10^7 t_P,$$

(45)

where $t_P$ is the Planck time. Using equations (11), (13) and (15), we can obtain the evolution of the Hubble parameter along with the scale factor numerically. With the numeric solution of the Hubble parameter we can integrate equation (44) numerically. We calculate $t_f - t_i \approx 3.40 \times 10^{16} t_P$, which is much bigger than $t_i$. The matter density can be calculated as $\rho_f = 1.58 \times 10^{-35} m_p \cdot t_P^{-3}$. We now address the classic case. With the Friedmann equation $H^2 = 8\pi \rho/3$ and the mass-conservation equation $\dot{\rho} = -3(w + 1)H \rho$ we can obtain the evolution of matter density

$$\rho \propto t^{-2},$$

(46)

where $w$ represents the ratio of the pressure to the density of matter, which we treat as constant because it changes very slowly. Before we use equation (46), we must be careful about the meaning of $t$. $t = 0$ does not mark the time when $a = 10^{-22}$. We must calculate asymptotic
axis where $\rho \to \infty$ from equation (46) and figure 4. We can get $t_u \approx -0.95 \times 10^{16} t_P$ by comparing $\rho$ and $t$ of points after $a = 2 \times 10^3$. The right relation can be get by replacing $t$ with $t - t_a$ in equation (46). The age of the universe is estimated as $t_{\text{now}} \approx 4.3 \times 10^{17} \text{s}$ [20]. Using revised equation (46), we estimate

$$\rho_{\text{now}} = \rho f \frac{(t_f - t_a)}{t_{\text{now}}}^2 \approx 2.5 \times 10^{-27} \text{kg} \cdot \text{m}^{-3},$$

which is roughly in agreement with experimental observation, which gives $\rho \approx 1.8 \times 10^{-27} \text{kg} \cdot \text{m}^{-3}$ [21].

The model can also provide some new predictions for further test. The HFFs of this model are different from other models. The high-order terms of the HFFs are zero as shown in the equation (39). Independent of the value of $p$, we have two strict relations

$$r/(1 - n_s) = 4, \quad n_s/d \ln k = 0$$

(48)

These can be used to falsify this model in future, when more accurate observations are available.

6. Discussion and conclusion

The vacuum inflation model derived by the quantum potential is interesting because it can avoid the problem of creation [10]. In contrast to other inflation models, the early universe is a vacuum bubble that grows rapidly due to its quantum potential. Matter can be created during the inflation stage, and inflation ends when the early universe becomes large enough. In this paper, we have considered the scalar perturbations and gravitational waves for the vacuum inflation derived by the quantum potential with HFFs directly. We find that matter contributes little to inflation, but becomes significant after the inflation. All the parameters in the vacuum inflation model are determined by comparison with observations. The matter density of the current universe is calculated with the vacuum inflation model, and the result roughly fits the observations, which indicates that the vacuum inflation model may be the correct answer.

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