OPE of the pseudoscalar gluonium correlator in massless QCD to three-loop order

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Abstract: In this paper analytical results are presented for higher order corrections to coefficient functions of the operator product expansion (OPE) for the correlator of two pseudoscalar gluonium operators $\hat{O}_1 = G_{\mu\nu} G^{\mu\nu}$. The Wilson coefficient in front of the scalar gluon condensate operator $O_1 = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu}$ is given at three-loop accuracy. The leading coefficient $C_0$ in front of the unity operator $O_0 = 1$ has been calculated up to three-loop order some time ago [1] but has been checked independently in this work. It is interesting to see that the coefficient $C_1$ in the pseudoscalar case is finite, whereas contact terms appear in $C_0$ in this case and in both coefficients $C_0$ and $C_1$ in the cases of the scalar gluonium correlator and the energy momentum tensor correlator [2]. For the corresponding Renormalization Group invariant Wilson coefficients which are also constructed the results are partially extended to four-loop accuracy. All results are given in the $\overline{\text{MS}}$-scheme at zero temperature.

Keywords: QCD, Quark-Gluon Plasma, Sum Rules
1 Motivation

Euclidian correlators of local operators are important objects in quantum field theory. Firstly, they have many important applications, e.g. in sum rules, where they are connected to physical quantities like spectral densities through dispersion relations. Secondly, they often have interesting properties in themselves, like their non-trivial renormalization, which are important for the understanding of quantum field theories. Such correlators are defined in momentum space as

$$i \int d^4x \, e^{iqx} T\{ [O](x)[O](0) \}$$

(1.1)

with a large Euclidian momentum $q$. Here and in the following the squared brackets indicate that the renormalized form of some operator $O$ is used. Usually, we are interested in the vacuum expectation value (VEV) of the correlator

$$\Pi(Q^2) = i \int d^4x \, e^{iqx} \langle 0| T\{ [O](x)[O](0)\}|0 \rangle \quad (Q^2 = -q^2)$$

(1.2)

which can be calculated in perturbation theory. But if we take $|0\rangle$ to be the physical vacuum state we also have to consider non-perturbative effects. Starting from the perturbative region of momentum space this is done by means of an operator product expansion (OPE). The idea is to expand the bilocal operator product (1.1) in a series of local operators with
Wilson coefficients depending on the large Euclidean momentum $q$ [3]:

$$\int d^4x \ e^{iqx} T\{ [O](x)[O](0) \} = \sum_i C_i^B(q)(Q^2)\frac{2 \dim(O) - \dim(O_i) - 4}{2} O_i^B$$  \hspace{1cm} (1.3)

$$= \sum_i C_i(q)(Q^2)\frac{2 \dim(O) - \dim(O_i) - 4}{2} [O_i]$$,  \hspace{1cm} (1.4)

where the index $B$ marks bare quantities and the factor $(Q^2)^{\frac{2 \dim(O) - \dim(O_i) - 4}{2}}$ constructed from the mass dimensions of the operators involved makes the Wilson coefficients $C_i(q)$ dimensionless.

In a sum rule approach to glueballs three operators are usually investigated as insertions on the lhs of (1.3) (see e.g. [5]):

$$O_1(x) = -\frac{1}{4} G^{a\mu\nu} G_{\mu\nu}^a(x) \quad \text{(scalar)},$$  \hspace{1cm} (1.5)

$$\hat{O}_1(x) = G^{a\mu\nu} \tilde{G}_{\mu\nu}^a(x) \quad \text{(pseudoscalar)},$$  \hspace{1cm} (1.6)

$$O_T^{\mu\nu}(x) = T^{\mu\nu}(x) \quad \text{(tensor)},$$  \hspace{1cm} (1.7)

where $G^{a\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ is the gluon field strength tensor,

$$\tilde{G}_{\mu\nu}^a = \varepsilon_{\mu\nu\rho\sigma} G^{a\rho\sigma}$$  \hspace{1cm} (1.8)

the dual gluon field strength tensor and $T^{\mu\nu}$ the energy-momentum tensor of QCD. Having discussed the correlators of $O_1$ and $O_T^{\mu\nu}$ in [2] the results for the correlator of (1.6)

$$X_t(q) := i \int d^4x \ e^{iqx} T\{ [\hat{O}_1](x)[\hat{O}_1](0) \}$$,  \hspace{1cm} (1.9)

whose VEV $\chi_t(q) := \langle 0|X_t(q)|0 \rangle$ is also known as the topological susceptibility of QCD$^2$ are presented here. This correlator has been connected to the mass of the $\eta'$-meson through the Witten-Veneziano formula [8–11]:

$$\frac{\alpha_s^2}{32 \pi^2} \chi_t(q) \bigg|_{q \to 0, \frac{n_f}{\Lambda} \to 0} = \frac{m_{\eta'}^2 F_{\pi}^2}{n_f} \frac{\pi}{n_f}$$ (leading order),  \hspace{1cm} (1.10)

where $F_{\pi} \approx 94 \text{ MeV}$ is the pion decay constant. An explicit sum rule calculation with an OPE at one-loop level using a Borel transformation has been done in [12]. In this work the value $m_{\eta'} \approx 1 \text{ GeV}$ is correctly estimated. A similar analysis at two-loop level but using only the leading coefficient $C_0$ has been done in [13].

$^1$Effectively this expansion separates the high energy physics, which is contained in the Wilson coefficients, from the low energy physics which is taken into account by the VEVs of the local operators, the so-called condensates [4]. These cannot be calculated in perturbation theory, but need to be derived from low energy theorems or be calculated on the lattice.

$^2$For a discussion of topological effects in QCD and the significance of the operator $\hat{O}_1$ and the correlator (1.9) in that respect see e.g. [6, 7].

$^3$It will be shown however in section 3.4 that the $\alpha_s$-expansion of the Wilson-coefficients, especially of $C_0$ converges rather badly at the low scales considered in these analyses. This should be taken into account in the treatment of pseudoscalar hadrons within the sum rule approach.
The correlator defined in (1.1) with renormalized operators is finite, i.e. all its matrix elements are finite, except for possible contact terms. These arise from the point where \( x \equiv 0 \) and manifest themselves as divergences \( \propto \delta(x) \) and derivatives of \( \delta(x) \) or in momentum space terms polynomial in \( q \). These local terms do not contribute to sum rules and can and should be subtracted with proper counterterms.

The leading term on the rhs of (1.3) is the coefficient in front of the unit operator \( 1 \) which is just the perturbative VEV of the correlator (1.1):

\[
(Q^2)^2 C_0(q) = \langle 0 | X(t) | q \rangle | \text{pert}. 
\]

(1.11)

The coefficient \( C_0 \) is known for the scalar case (1.5) at four-loop level [14] and for the pseudoscalar case (1.6) [1] and the energy-momentum tensor correlator [2] at three-loop level.\(^4\) The next important contribution in the OPE is the coefficient of the dimension four operator [\( O_1 \)] (1.5).\(^5\) The coefficient \( C_1 \) has been calculated at two-loop level for the scalar\(^6\) and tensor cases [2]. Here we present the coefficient \( C_1 \) for the pseudoscalar case at three-loop level which so far has only been known to one-loop accuracy [12, 19].

All physical matrix elements of \([O_1] = Z_G O_1^B\) are finite and so is the renormalized coefficient \( C_1 \):\(^7\)

\[
C_1 = \frac{1}{Z_G} C_1^B. 
\]

(1.12)

The renormalization constant

\[
Z_G = 1 + \alpha_s \frac{\partial}{\partial \alpha_s} \ln Z_{\alpha_s} = \left( 1 - \frac{\beta(\alpha_s)}{\varepsilon} \right)^{-1}
\]

(1.13)

has been derived in a simple way in [20] (see also an earlier work [21]). Here \( Z_{\alpha_s} \) is the renormalization constant\(^8\) for \( \alpha_s \) and the \( \beta \)-function is defined as

\[
\beta(\alpha_s) = \mu^2 \frac{d}{d \mu^2} \ln \alpha_s = - \sum_{i \geq 0} \beta_i \left( \frac{\alpha_s}{\pi} \right)^{i+1}. 
\]

(1.14)

The outline of this paper is as follows. In the next section the renormalization properties of \( \hat{O}_1 \) will be discussed. In section 3 the details of the calculation will be described (section

\(^4\)Two-loop results for \( C_0 \) in the scalar and pseudoscalar case [15] and in the tensor case in gluodynamics \((n_f = 0)\) [16] have been known for a long time.

\(^5\)In the case of massive fermion flavours \( f \) we would also have contributions proportional to the dimension two operator \( O^f = m_f^2 1 \) and the dimension four operator \( O^f_1 = \bar{\psi}_f \psi_f \). In the case of temperature \( T \neq 0 \) Lorentz variant operators like \( T_a^b \sim e + p \) with the energy density \( e \) and the pressure \( p \) have to be considered as well. At \( T = 0 \), however, only Lorentz and gauge invariant scalar operators contribute to the the VEV in (1.2) which is the quantity that we are ultimately interested in. For a discussion of the correlator \( X(t) \) at finite temperature up to \( O(\alpha_s) \) see [17].

\(^6\)The one-loop result for the scalar case was first derived in [18].

\(^7\)In the massless case \( O_1 \) only mixes with unphysical operators whose matrix elements with physical external states vanish. The renormalization of \( O_1 \) including these unphysical contributions as well as the mixing with \( O^f_1 \) in the massive case can be found in [20].

\(^8\)Often in the literature \( Z_{\alpha_s} \) is used instead of \( Z_G \) and \( \alpha_s G^{\mu \nu} G_{\mu \nu} \) instead of \( O_1 \). This renormalization is only valid up to first order in \( \alpha_s \) as the renormalization constants \( Z_G \) and \( Z_{\alpha_s} \) coincide to this accuracy. In higher orders, however, \( Z_G \) and \( Z_{\alpha_s} \) differ.
3.1) and the results for the OPE of (1.9) will be presented (section 3.2). After that Renormalization Group invariant (RGI) operators and Wilson coefficients will be constructed (section 3.3) followed by a numerical evaluation of the main results (section 3.4). Finally, some conclusions and acknowledgements will be given.

2 Renormalization of \( \tilde{O}_1 \) and its correlator

The operator \( \tilde{O}_1 \) forms a closed set under renormalization with the pseudoscalar fermionic operator

\[
\partial_\mu J_5^\mu := \varepsilon^{\mu \nu_1 \mu_1 \mu_3} \partial_\mu \sum_f \bar{\Psi}_f \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \Psi,
\]

(2.1)

which can be written as

\[
\partial_\mu J_5^\mu = \partial_\mu \sum_f \bar{\Psi}_f \gamma^\mu \gamma_5 \Psi
\]

(2.2)

in the Larin scheme for \( \gamma_5 \) [22].

The \( \varepsilon \)-tensors appearing in (1.6) and (2.1) are then drawn out of the R-operation performed in dimensional regularization. In the correlators which have to be calculated there are always two \( \varepsilon \)-tensors involved which can be contracted and expressed through metric tensors:

\[
\varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \varepsilon_{\nu_1 \nu_2 \nu_3 \nu_4} = -g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} g^{\mu_3 \nu_3} g^{\mu_4 \nu_4},
\]

(2.3)

where \([ \ldots ]\) means complete antisymmetrization. These operators are renormalized like [22]

\[
[\partial_\mu J_5^\mu] = Z_s^g Z_{M_5} \partial_\mu J_5^{B \mu} = Z_s^g \partial_\mu J_5^{B \mu},
\]

(2.4)

\[
[\tilde{O}_1] = Z_{GG} \tilde{O}_1^B + Z_G J \partial_\mu J_5^{B \mu},
\]

(2.5)

where \( Z_{M_5} \) is an \( \overline{\text{MS}} \) renormalization constant, \( Z_s^g \) a finite renormalization constant fixed by the requirement that the one-loop character of the axial anomaly relation

\[
[\partial_\mu J_5^\mu] = \frac{\alpha_s}{4\pi} \eta_f T_F (\tilde{O}_1) + \text{CT}
\]

(2.6)

is valid in dimensional regularization.\(^9\) CT stands for contact terms of \( \partial_\mu J_5^\mu \) with fermion fields. In the gluon sector these can be neglected. \( Z_{GG} \) is an \( \overline{\text{MS}} \) renormalization constant again and \( Z_{GJ} \) starts at \( \mathcal{O}(\alpha_s) \). In [22] \( Z_{M_5} \) and \( Z_s^g \) are given up to \( \mathcal{O}(\alpha_s^3) \) and \( \mathcal{O}(\alpha_s^2) \) respectively. Furthermore it is shown that \( Z_{GG} = Z_n \) \( Z_n \) being the renormalization constant for \( \alpha_n \). The constant \( Z_{GJ} \) is only given at one-loop level in the literature [1, 22] but for the Wilson coefficient \( C_1 \) at three-loop level it is needed to two-loop accuracy. In section 3.3 we will also need the corresponding three-loop anomalous dimension. The simplest way to determine \( Z_{GJ} \) is by constructing the matrix elements of \( \tilde{O}_1 \) and \( \partial_\mu J_5^{B \mu} \) with two external fermions (see Fig. 1) using a projector

\[
P(q) := q^{\mu_1} \gamma^{\mu_2} \gamma_5 \gamma^{\mu_4} \varepsilon_{\mu_1 \mu_2 \mu_3 \mu_4}
\]

(2.7)
Figure 1. Diagrams for the calculation of $Z_{GJ}$

on the external fermion line. From this we get

$$Z_{GJ} = \frac{\alpha_s}{4\pi\varepsilon} 12 C_F + \frac{\alpha_s^2}{(4\pi)^2\varepsilon^2} \left\{ 142 C_A C_F - \frac{42 C_F^2 - \frac{8}{3} C_F n_f T_F}{3} \right\}$$

$$+ \frac{\alpha_s^2}{(4\pi)^2\varepsilon^2} \left\{ 16 C_F n_f T_F - 44 C_A C_F \right\}$$

$$+ \frac{\alpha_s^3}{(4\pi)^3\varepsilon^3} \left\{ \frac{484}{3} C_A^2 C_F - \frac{352}{3} n_f C_A T_F C_F + \frac{64}{3} n_f^2 T_F^2 C_F \right\}$$

$$+ \frac{\alpha_s^3}{(4\pi)^3\varepsilon^3} \left\{ \frac{550}{3} C_A C_F^2 - \frac{2378}{9} C_A^2 C_F - \frac{32}{3} n_f T_F C_F^2 \right\}$$

$$+ \frac{1136}{9} n_f C_A T_F C_F - \frac{32}{9} n_f^2 T_F^2 C_F$$

$$+ \frac{\alpha_s^3}{(4\pi)^3\varepsilon^3} \left\{ 178 C_F^3 - \frac{2947}{9} C_A C_F^2 + \frac{1607}{9} C_A^2 C_F - \frac{1096}{9} n_f T_F C_F^2 \right\}$$

$$+ \frac{328}{9} n_f C_A T_F C_F - \frac{208}{9} n_f^2 T_F^2 C_F + 192 \zeta_3 n_f T_F C_F^2 - 192 \zeta_3 n_f C_A T_F C_F \right\}. \quad (2.8)$$

An interesting additional application of this result is to check the connection between the anomalous dimensions of the operator set $\{\tilde{O}_1, \partial_{\nu}J_5^{\nu}\}$. In [22] the following relations have been motivated:

$$\gamma_{GG} = \frac{\beta(\alpha_s)}{\alpha_s^2}, \quad (2.9)$$

$$\gamma_{GJ} = \left( \frac{\alpha_s}{4\pi} n_f T_F \right)^{-1} \gamma_j^*, \quad (2.10)$$

with

$$\gamma_{ij} = \left( \frac{\mu^2}{d\mu^2} Z_{ik} \right) (Z^{-1})_{kj}, \quad Z = \begin{pmatrix} Z_{G\tilde{G}} & Z_{GJ} \\ Z_{GJ} & Z_j \end{pmatrix}. \quad (2.11)$$

The first relation (2.9) has been explicitly checked to three-loop level in [22] the second one (2.10) only to one-loop accuracy. Now we can check this equation with $\gamma_{G,J}$ at two-loop

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In Pauli-Villars regularization for example this relation is automatically fulfilled. In $d \neq 4$ dimensions, however, the operators $\partial_{\nu}J_5^{\nu}$ and $\tilde{O}_1$ become linearly independent.
level and $\gamma^3_j$ at three-loop level and it turns out to hold there as well. Using (2.8) and the renormalization constants $Z^3_j$ and $Z_a$ [22, 23] the following anomalous dimension is derived.\(^{10}\)

\[
\gamma_{GJ} = -12C_F \left( \frac{\alpha_s}{4\pi} \right) + \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ -\frac{284}{3} C_A C_F + 36C_F^2 + \frac{16}{3} C_F n_f T_F \right\} + \left( \frac{\alpha_s}{4\pi} \right)^3 \left\{ -\frac{1607}{3} C_A^2 C_F + 461 C_A C_F^2 + 576 C_A C_F n_f T_F \right\} - \frac{328}{3} C_A C_F n_f T_F
\]

(2.12)

\[
-126C_F^3 - 576C_F^2 n_f T_F \gamma^3 - 428C_F n_f T_F^2 + \frac{208}{3} C_F n_f T_F^2 \}
\]

Now we can write the correlator $X_i(q)$ as

\[
i \int d^4 x e^{i q x} T\{ [\hat{O}_i(x)][\hat{O}_i(0)] \}
\]

\[
= i \int d^4 x e^{i q x} T\left\{ Z^2_G G \hat{O}_i(x) \hat{O}_i(0) + 2Z_G G_{GJ} \hat{O}_i(x) \partial_\mu J^\mu_5(0) + Z^2_G \partial_\mu J^\mu_5(x) \partial_\nu J^\nu_5(0) \right\}.
\]

(2.13)

In [2] it has been discovered that there are contact terms at two-loop level in the coefficient $C_1$ for the correlator of $O_1$. The coefficient $C_0$ also has contact terms for the correlator of two operators $O_1$ or two operators $T^{\mu\nu}$. For the operator $\hat{O}_1$ we can make an important restriction on possible contact terms due to the fact that it can be exactly expressed as the divergence of the Chern-Simons current:

\[
\hat{O}_1 = \partial_\mu K^\mu
\]

(2.14)

with

\[
K^\mu = \varepsilon^{\mu\nu\rho\sigma} \left\{ 4G^a \partial_\rho G^a_\sigma + \frac{4}{3} g_s f^{abc} G^{a\rho} G^{b\sigma} \right\}.
\]

(2.15)

From this follows for (2.13)

\[
i \int d^4 x e^{i q x} T\{ [\hat{O}_1(x)][\hat{O}_1(0)] \}
\]

\[
= q_\mu q_\nu i \int d^4 x e^{i q x} T\left\{ Z^2_G K^\mu(x) K^B(x) (0) + 2Z_G G_{GJ} K^\mu(x) J^B_5(x) J^B_5(x) + Z^2_G J^\mu_5(x) J^B_5(x) J^B_5(x) \right\}
\]

\[
\rightarrow q_\mu q_\nu \left\{ q^2 C_0^{\mu\nu}(q^2) + \frac{1}{q^2} C_1^{\mu\nu}(q^2) + \ldots \right\} \text{ for } q^2 \rightarrow -\infty \text{ (OPE)}
\]

(2.16)

with dimensionless coefficients $C_0^{\mu\nu}(q^2)$ and $C_1^{\mu\nu}(q^2)$. Because of the non-local factor $\frac{1}{q^2}$ the coefficient $C_1^{\mu\nu}(q^2)$ cannot contain any contact terms. This makes the Wilson coefficient $C_1(q^2) = \frac{q_\mu q_\nu}{q^2} C_1^{\mu\nu}(q^2)$ for the correlator (2.13) finite and unambiguous due to the absence of contact terms.
Figure 2. Diagrams for the calculation of the coefficient $C_0(Q^2)$

3 Calculation and results

3.1 Details of the calculation

The leading coefficient $C_0$ is just the perturbative VEV of the correlator eq. (2.13)

$$\langle 0 | X_t(q) | 0 \rangle_{\text{pert}} =$$

$$Z_a^2 \left( \begin{array}{c}
\tilde{O}_1^B \\
\partial^\nu J_5^B
\end{array} \right) + 2Z_aZ_GJ \left( \begin{array}{c}
\tilde{O}_1^B \\
\partial^\nu J_5^B
\end{array} \right) + Z_G^2 \left( \begin{array}{c}
\tilde{O}_1^B \\
\partial^\nu J_5^B
\end{array} \right)$$

which has been computed up to order $\alpha_s^2$ (three loops). In Figure (2) some sample Feynman diagrams contributing to this calculation are shown. The operators $\tilde{O}_1^B$ and $\partial^\mu J_5^B$ play the roles of external currents. The Feynman diagrams have been produced with the program QGRAF [24]. As all diagrams in this problem are propagator-like the relevant integrals can be computed with the FORM package MINCER [25–27]. For the colour part of the diagrams the FORM package COLOR [28] has been used.

In order to compute the coefficient $C_1(Q^2)$ the method of projectors [29, 30] has been applied, which allows to express coefficient functions for any OPE of two operators in terms of massless propagator type diagrams only. The method is based the fact that in dimensional regularization every massless tadpole-like Feynman integral is set to zero. We apply a projector to both sides of (1.3) which sets every operator on the rhs to zero.

$^{10}$ $\gamma_{GG}$ and $\gamma_J$ can be found in [22, 23] at three-loop level. All renormalization constants and anomalous dimensions are available at \url{http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp13/ttp13-003/}
except for $O_1^B$:
\[
P\{X_i(q)\} = \sum_i (Q^2)^{\frac{4-d(Q)}{2}} C_i^{B,(\nu)}(Q^2) P\{O_i^B\},
\]  
with $P\{O_1^B\} = 1$ and $P\{O_{i\neq 1}^B\} = 0$. This is done in the same way as described in [2] leading to
\[
C_{1,B}(Q^2) = Z_{GG}^2 C_{1,B}^{(\tilde{O}_1^B,\tilde{O}_1^B)}(Q^2) + 2Z_{GG}Z_G C_{1,B}^{(\tilde{O}_1^B,\partial_\nu J_5^B)}(Q^2) + Z_G^2 C_{1,B}^{(\partial_\nu J_5^B,\partial_\nu J_5^B)}(Q^2),
\]
with
\[
C_{1,B}^{(O_2^B, O_1^B)}(Q^2) = \frac{\delta^{ab}}{n_g} \frac{g^{\mu_1 \mu_2}}{(D-1)D} \frac{1}{\partial k_1 \partial k_2} \left[ O^B_{\alpha} \right] \left[ O^B_{\beta} \right],
\]  
where the blue circle represents the sum of all (bare) Feynman diagrams which become 1PI after formal gluing of the two external lines representing the operators on the lhs of the OPE.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{diagram3}
\caption{Diagrams for the calculation of $C_{1,B}^{(\tilde{O}_1^B,\tilde{O}_1^B)}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{diagram4}
\caption{Diagrams for the calculation of $C_{1,B}^{(\tilde{O}_1^B,\partial_\nu J_5^B)}$.}
\end{figure}

Table (1) shows the number of diagrams generated for the different contributions to $C_0$ and $C_1$. Sample diagrams for the calculation of the bare coefficients $C_{1,B}^{(\tilde{O}_1^B,\tilde{O}_1^B)}$, $C_{1,B}^{(\tilde{O}_1^B,\partial_\nu J_5^B)}$ and $C_{1,B}^{(\partial_\nu J_5^B,\partial_\nu J_5^B)}$ are shown in Figures (3), (4) and (5) respectively.
All results are given in the $\overline{\text{MS}}$ scheme with $\alpha_s = \frac{g_s^2}{\pi}$, $\alpha_s = \frac{g_s^2}{4\pi}$, and the abbreviation $\mu_q = \ln \left( \frac{\mu^2}{Q^2} \right)$ where $\mu$ is the $\overline{\text{MS}}$ renormalization scale. They can be retrieved from http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp13/ttp13-003/

The gauge group factors are defined in the usual way: $C_F$ and $C_A$ are the quadratic Casimir operators of the quark and the adjoint representation of the corresponding Lie algebra, $d_R$ is the dimension of the quark representation, $n_g$ is the number of gluons (dimension of the adjoint representation), $T_F$ is defined so that $T_F \delta^{ab} = \text{Tr} \left( T^a T^b \right)$ is the trace of two group generators of the quark representation.$^{11}$ For QCD (colour gauge group $SU(3)$) we have $C_F = \frac{4}{3}, \ C_A = 3, \ T_F = 1/2$ and $d_R = 3$. By $n_f$ we denote the number of active quark flavours.

### 3.2 Results

As we have seen from (2.13) contact terms in $C_0$ are possible and it turns out that they appear starting from one loop. Because of these contact terms an unambiguous result for $C_0$ can only be given up to local (that is $q$-independent) contributions. To avoid the

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$^{11}$For an SU($N$) gauge group these are $d_R = N$, $C_A = 2T_FN$ and $C_F = T_F \left( N - \frac{1}{N} \right)$. 

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Table 1. Number of diagrams needed for $C_0$ and $C_1$
ambiguity the $Q^2$-derivative is presented:

\[
Q^2 \frac{d}{dQ^2} C_0 = \frac{n_f}{\pi^2} \left[ -1 + a_s \left( -\frac{97}{12} C_A + \frac{7}{3} n_f T_F \right) + a_t l_{\mu q} \left( -\frac{11}{6} C_A + \frac{2}{3} n_f T_F \right) + a_s^2 \left( -\frac{51959}{864} C_A^2 + \frac{107}{12} n_f T_F C_F + \frac{3793}{108} n_f C_A T_F - \frac{251}{94} n_f T_F^2 \right)
\right.
\]
\[
+ \frac{55}{8} \zeta_3 C_A^3 - 3 \zeta_3 n_f T_F C_F + \frac{1}{2} \zeta_3 n_f C_A T_F \right]
\]
\[
+ a_t^2 l_{\mu q} \left( -\frac{1135}{48} C_A + 2 n_f T_F C_F + \frac{46}{3} n_f C_A T_F - \frac{7}{3} n_f T_F^2 \right) \]
\[
+ a_t^2 l_{\mu q} \left( \frac{121}{48} C_A + \frac{11}{6} n_f C_A T_F - \frac{1}{3} n_f T_F^2 \right) \right].
\]

This result has been derived before [1] which serves as a nice check for the setup. As discussed above the coefficient $C_1$ is unambiguous and is therefore given in full:

\[
C_1 = 64 \left\{ 1 + a_s \left( \frac{157}{36} C_A - \frac{5}{9} n_f T_F + \frac{11}{12} l_{\mu q} C_A - \frac{1}{3} l_{\mu q} n_f T_F \right) + a_s^2 \left( \frac{25945}{1296} C_A^2 - \frac{11}{2} n_f T_F C_F - \frac{4355}{648} n_f C_A T_F + \frac{25}{81} n_f T_F^2 + \frac{1727}{216} l_{\mu q} C_A^2 \right)
\right.
\]
\[
- \frac{3}{2} l_{\mu q} n_f T_F C_F - \frac{106}{27} l_{\mu q} n_f C_A T_F + \frac{10}{27} l_{\mu q} n_f T_F^2 + \frac{121}{144} l_{\mu q} C_A^2 - \frac{11}{18} l_{\mu q} n_f C_A T_F \right.
\]
\[
+ \frac{1}{9} l_{\mu q} n_f T_F^2 - \frac{33}{8} \zeta_3 C_A^3 + 3 \zeta_3 n_f T_F C_F - \frac{3}{2} \zeta_3 n_f C_A T_F \right)
\]
\[
+ a_t^3 \left( \frac{19360399}{186624} C_A^3 + \frac{461}{144} n_f T_F C_F^2 - \frac{614501}{10368} n_f C_A T_F C_F \right.
\]
\[
- \frac{1857805}{31104} n_f C_A^2 T_F + \frac{2592}{2592} n_f T_F C_F^2 + \frac{126415}{15552} n_f T_F C_F^2 - \frac{125}{729} n_f T_F^2 \right.
\]
\[
+ \frac{594247}{10368} l_{\mu q} C_A^3 + \frac{35}{32} l_{\mu q} n_f T_F C_F^2 - \frac{1623}{64} l_{\mu q} n_f C_A T_F C_F - \frac{68935}{1728} l_{\mu q} n_f C_A T_F C_F \right.
\]
\[
+ \frac{275}{5040} l_{\mu q} n_f T_F C_F^2 - \frac{2795}{96} l_{\mu q} n_f C_A T_F C_F + \frac{8}{27} l_{\mu q} n_f T_F^2 - \frac{1331}{1728} l_{\mu q} C_A^2 C_F + \frac{24}{24} l_{\mu q} n_f T_F^2 - \frac{1}{12} l_{\mu q} C_A^3 \right.
\]
\[
- \frac{5}{27} l_{\mu q} n_f T_F^3 + \frac{55}{8} \zeta_3 C_A^3 - \frac{15}{3} l_{\mu q} n_f T_F C_F^2 + \frac{15}{2} \zeta_3 n_f C_A T_F C_F + \frac{5}{2} \zeta_3 n_f C_A^2 T_F \right.
\]
\[
- \frac{6893}{144} \zeta_3 C_A^3 + \frac{145}{12} \zeta_3 n_f T_F C_F^2 + \frac{1291}{48} \zeta_3 n_f C_A T_F C_F - \frac{349}{144} \zeta_3 n_f C_A T_F \right.
\]
\[
- \frac{13}{2} \zeta_3 l_{\mu q} n_f T_F^2 C_F + \frac{121}{36} \zeta_3 l_{\mu q} C_A T_F - \frac{33}{4} \zeta_3 l_{\mu q} n_f C_A T_F \right.
\]
\[
- \frac{3}{2} \zeta_3 l_{\mu q} n_f T_F^2 C_F + \frac{3}{2} \zeta_3 l_{\mu q} n_f C_A T_F \right) \right].
\]

The cancellation of all divergences is a strong check for this result. Another important check is the independence of the gauge parameter $\xi$ as all calculations have been done for an arbitrary $R_\xi$ gauge. The leading term of (3.6) is in agreement with [12] and the part
\( \propto a_s l_{\mu q} \) has been derived in \[19\] if we set the colour factors to their QCD values. For QCD colour factors we get

\[
C_1 = 64 \left\{ 1 + a_s \left( \frac{157}{12} - \frac{5}{18} n_f + \frac{11}{4} l_{\mu q} - \frac{1}{6} l_{\mu q} n_f \right) \\
+ a_s^2 \left( \frac{25945}{144} - \frac{5939}{432} n_f + \frac{25}{324} n_f^2 + \frac{1727}{24} l_{\mu q} \\
- \frac{62}{9} l_{\mu q} n_f + \frac{5}{54} l_{\mu q} n_f^2 + \frac{121}{16} l_{\mu q}^2 \\
- \frac{11}{12} l_{\mu q}^2 n_f + \frac{1}{36} l_{\mu q}^2 n_f^2 - \frac{297}{8} \zeta_3 - \frac{1}{4} \zeta_3 n_f \right) \\
+ a_s^3 \left( \frac{19360399}{6912} - \frac{7972411}{20736} n_f + \frac{611093}{62208} n_f^2 - \frac{125}{5832} n_f^3 \right) \\
+ \frac{594247}{384} l_{\mu q} n_f - \frac{264113}{1152} l_{\mu q} n_f^2 + \frac{1981}{1152} l_{\mu q} n_f^3 \\
+ \frac{9779}{32} l_{\mu q} n_f^4 + \frac{9485}{192} l_{\mu q} n_f^5 + \frac{288}{288} l_{\mu q}^2 n_f^2 - \frac{5}{216} l_{\mu q}^2 n_f^3 \right\} (3.7)
\]

A nice consistency check for these results is to perform an OPE of the correlator

\[
i \int d^4 x e^{iqx} T\{[\partial_\mu J_\mu^5](x)[\partial_\mu J_\mu^5](0)\} = (Q^2)^2 C_0^{JJ} + C_1^{JJ}[O_1] + \ldots
\]

and then see that (2.6) is fulfilled (except for possible contact terms):

\[
Q^2 \frac{d}{dQ^2} C_0^{JJ} = \left( \frac{a_s}{4\pi n_f T_F} \right)^2 Q^2 \frac{d}{dQ^2} C_0,
\]

\[
C_1^{JJ} = \left( \frac{3 a_s}{4\pi n_f T_F} \right)^2 C_1.
\]

Indeed we find

\[
Q^2 \frac{d}{dQ^2} C_0^{JJ} = \frac{n_g}{\pi^2} \left[ -\frac{a_s^2}{16} n_f^2 T_F^2 \right]
\]

and

\[
C_1^{JJ} = 4a_s^2 n_f^2 T_F \left\{ 1 + a_s \left( \frac{157}{36} C_A - \frac{5}{9} n_f T_F + \frac{11}{12} l_{\mu q} C_A - \frac{1}{3} l_{\mu q} n_f T_F \right) \right\} (3.12)
\]

satisfying (3.9) and (3.10) up to the calculated accuracy of \(O(a_s^2)\) and \(O(a_s^3)\) respectively.

---

\(^{12}\)In \[19\], however, the leading term differs from this result and the one derived in \[12\] by a minus sign and the non-logarithmic term of \(O(a_s)\) is also missing there.
3.3 RGI operators and Wilson coefficients

Note that the coefficients (3.5) and (3.6) are not Renormalization Group invariant (RGI). In this section we take RGI versions of all operators and construct RGI Wilson coefficients. For an operator that is renormalized multiplicatively like $\partial_\mu J_5^\mu$ in (2.5) constructing a finite and RGI operator is straightforward (see e.g. [31]). Because of

$$\mu^2 \frac{d}{d\mu^2} [\partial_\mu J_5^\mu] = \gamma^s_J(a_s(\mu)) [\partial_\mu J_5^\mu]$$  \hspace{1cm} (3.13)

we can define

$$[\partial_\mu J_5^\mu]_{\text{RGI}} := \exp \left\{ - \int \frac{a_s(a)}{a \beta(a)} da \right\} [\partial_\mu J_5^\mu] =: E_2(a_s)$$ \hspace{1cm} (3.14)

which fulfills $\mu^2 \frac{d}{d\mu^2} [\partial_\mu J_5^\mu]_{\text{RGI}} = 0$. A remarkable feature of the operator (3.14) is its renormalization scheme independence [32]. If we start with a different renormalized operator $[\partial_\mu J_5^\mu]' := Z(a_s)[\partial_\mu J_5^\mu]$ (3.15)

we get

$$\gamma^s_J(a_s) = \gamma^s_J(a_s) + \mu^2 \frac{d}{d\mu^2} \ln(Z(a_s))$$ \hspace{1cm} (3.16)

which leads to

$$E_2'(a_s) = \frac{E_2(a_s)}{Z(a_s)}$$ \hspace{1cm} (3.17)

and therefore to the same RGI operator

$$[\partial_\mu J_5^\mu]_{\text{RGI}} = E_2(a_s)[\partial_\mu J_5^\mu]' = E_2(a_s)[\partial_\mu J_5^\mu].$$ \hspace{1cm} (3.18)

If we apply the same procedure to the non-diagonal operator $\hat{O}_1$ we get an RG variant operator

$$[\hat{O}_1]_{\text{RGV}} := \exp \left\{ - \int \frac{\gamma^s_{G\hat{G}}(a)}{a \beta(a)} da \right\} [\hat{O}_1],$$ \hspace{1cm} (3.19)

where $E_1(a_s) = a_s$ because of (2.9). Taking the derivative wrt the renormalization scale we find

$$\mu^2 \frac{d}{d\mu^2} [\hat{O}_1]_{\text{RGV}} = E_1(a_s) \gamma_{GJ}(a_s)[\partial_\mu J_5^\mu] = \frac{E_1(a_s)}{E_2(a_s)} \gamma_{GJ}(a_s)[\partial_\mu J_5^\mu]_{\text{RGI}}$$ \hspace{1cm} (3.20)

which leads to the definition of the RGI operator

$$[\hat{O}_1]_{\text{RGI}} := [\hat{O}_1]_{\text{RGV}} - \int \frac{E_1(a)}{E_2(a)} \gamma_{GJ}(a) \frac{da}{a \beta(a)} [\partial_\mu J_5^\mu]_{\text{RGI}} \hspace{1cm} (3.21)$$

$$= a_s \left\{ Z_{G\hat{G}}(a_s) \hat{O}_1^B + \left( Z_{GJ}(a_s) - E_2(a_s) \bar{Z}(a_s) Z^s_J(a_s) \right) \partial_\mu J_5^B \right\} $$

- 12 -
fulfilling \( \mu^2 \frac{d}{d\mu^2} [\tilde{O}_1]^{\text{RGI}} = 0 \). In similar way as for (3.14) it can be shown that (3.21) is invariant under transformations [\( \tilde{O}_1 \rightarrow [\tilde{O}_1]’ = Z_1(a_s)[\tilde{O}_1] \)]. Even if we allow for redefinitions of the kind [\( \tilde{O}_1 \rightarrow [\tilde{O}_1]’ = Z_1(a_s)[\tilde{O}_1] + Z_2(a_s)[\partial_\mu J^\mu_5] \)] the RGI operator derived with this method is the same:

\[
[\tilde{O}_1]^{\text{RGV}’} = [\tilde{O}_1]^{\text{RGV}} + \frac{E_1(a_s)Z_2(a_s)}{E_2(a_s)Z_1(a_s)}[\partial_\mu J^\mu_5]^{\text{RGI}}.
\] (3.22)

\[
\Rightarrow \mu^2 \frac{d}{d\mu^2} [\tilde{O}_1]^{\text{RGV}’} = \left[ \frac{E_1(a_s)}{E_2(a_s)} \gamma_{GJ}(a_s) + \mu^2 \frac{d}{d\mu^2} \left( \frac{E_1(a_s)Z_2(a_s)}{E_2(a_s)Z_1(a_s)} \right) \right] [\partial_\mu J^\mu_5]^{\text{RGI}}.
\] (3.23)

\[
\Rightarrow [\tilde{O}_1]^{\text{RGI}’} = [\tilde{O}_1]^{\text{RGV}’} - \left[ \int \left( \frac{E_1(a)}{E_2(a)} \gamma_{GJ}(a) \frac{da}{a} \beta'(a) \right) \frac{E_1(a_s)Z_2(a_s)}{E_2(a_s)Z_1(a_s)} \right] [\partial_\mu J^\mu_5]^{\text{RGI}}
\]

\[
= [\tilde{O}_1]^{\text{RGV}} - a_s \tilde{Z}(a_s)[\partial_\mu J^\mu_5]^{\text{RGI}} = [\tilde{O}_1]^{\text{RGI}}.
\] (3.24)

The leading RGI Wilson coefficient

\[
C_0^{\text{RGI}}(q) = \frac{1}{(Q^2)^2} \left. \langle 0 | X_t^{\text{RGI}}(q) | 0 \rangle \right|_{\text{pert}}
\] (3.25)

in an OPE of the RGI correlator

\[
X_t^{\text{RGI}}(q) := i \int d^4x e^{iqx} T \{ [\tilde{O}_1]^{\text{RG1}}(x) | \tilde{O}_1]^{\text{RG1}}(0) \}
\] (3.26)

can now be calculated from the same three bare correlators as \( C_0 \) and the result for its \( Q^2 \)-derivative is

\[
Q^2 \frac{d}{dQ^2} C_0^{\text{RGI}} = \frac{a_s^2(Q^2)n_s}{\pi^2} \left[ -1 + a_s(Q^2) \left( -\frac{97}{12} C_A + \frac{7}{3} n_f T_F \right) \right.
\]

\[
+ \frac{a_s(Q^2)}{11C_A - 4n_f T_F} \frac{182n_f T_F C_F}{18n_f T_F C_F}
\]

\[
+ \frac{a_s^2(Q^2)}{11C_A - 4n_f T_F} \left( -\frac{51959}{864} C_A^2 + \frac{107}{12} n_f T_F C_F + \frac{3793}{108} n_f C_A T_F \right)
\]

\[
- \frac{251}{54} n_f T_F^2 + \frac{55}{8} \zeta_3 C_A^2 - 3 \zeta_3 n_f T_F C_F + \frac{1}{2} \zeta_3 n_f C_A T_F
\]

\[
+ \frac{a_s^2(Q^2)}{11C_A - 4n_f T_F} \left( \frac{291}{2} n_f C_A T_F C_F - 42 T_F^2 C_F \right)
\]

\[
+ \frac{a_s^2(Q^2)}{11C_A - 4n_f T_F} \left( \frac{291}{2} n_f C_A T_F C_F + \frac{475}{4} n_f C_A^2 T_F C_F \right)
\]

\[
- 108 n_f T_F^2 C_F - 37 n_f^2 C_A T_F^2 C_F + 4 n_f^3 T_F^2 C_F \right) \]

where the logarithmic pieces have been resummed into \( a_s(\mu^2 = Q^2) \) for brevity. These terms can easily be recovered from the RG equations (see (3.33) below). They have been calculated explicitly however using the above definitions in order to be able to use the RGI condition \( \mu^2 \frac{d}{d\mu^2} \left( Q^2 \frac{d}{dQ^2} C_0^{\text{RGI}} \right) = 0 \) as a consistency check.
As explained in [2] a finite and RGI version of $O_1$ can be defined as
\[
O_1^{\text{RGI}} := \hat{\beta}(a_s) [O_1], \quad \hat{\beta}(a_s) := -\frac{\beta(a_s)}{\beta_0} = a_s \left( 1 + \sum_{i \geq 1} \frac{\beta_i}{\beta_0 a_s^i} \right). 
\]  

(3.28)

The RGI Wilson coefficient
\[
C_{1}^{\text{RGI}}(Q^2) = \frac{a_s^2}{\hat{\beta}(a_s)^2} \left\{ Z_{GG}^2 \, C_{1,B}^{(O_1^B,O_1^B)}(Q^2) 
+ (2Z_{GJ} Z_{GJ} - 2E_2 Z_{GJ} Z_{J^r}) C_{1,B}^{(O_1^B,\partial_v J_r^B)}(Q^2) 
+ (Z_{GJ}^2 - 2E_2 Z_{GJ} Z_{J^r} Z_{J^r}) C_{1,B}^{(\partial_v J_{r}^B,\partial_v J_{r}^B)}(Q^2) \right\} 
\]  

(3.29)

which satisfies
\[
C_{1}^{\text{RGI}}[O_1]^{\text{RGI}} = C_1[O_1] 
\]  

(3.30)
in the OPE of (3.26). The result (again with logarithms resummed into $a_s(\mu^2 = Q^2)$) is
\[
C_{1}^{\text{RGI}} = 64a_s(Q^2) \left\{ 1 + a_s(Q^2) \left( \frac{157}{36} C_A - \frac{5}{9} n_f T_F \right) 
+ \left( \frac{11 C_A - 4n_f T_F}{2} \right) \left( -\frac{17}{2} C_A^2 - 15n_f T_F C_F + 5n_f C_A T_F \right) 
+ a_s^2(Q^2) \left( \frac{25945}{1296} - \frac{11}{2} n_f T_F C_F - \frac{4355}{648} n_f C_A T_F + \frac{25}{81} n_f^2 T_F^2 
- \frac{33}{8} \zeta_3 n_f C_A^2 + 3\zeta_3 n_f T_F C_F - \frac{3}{2} \zeta_3 n_f C_A T_F \right) 
+ \frac{a_s^2}{(11 C_A - 4n_f T_F)^2} \left( -\frac{2669}{72} C_A^3 - \frac{785}{12} n_f C_A T_F C_F + \frac{955}{36} n_f C_A^2 T_F \right) 
+ \frac{25}{3} n_f T_F^2 C_F - \frac{25}{9} n_f C_A^2 T_F \right) + \frac{a_s^2}{(11 C_A - 4n_f T_F)^2} \left( -\frac{10619}{288} C_A^4 
- \frac{561}{8} n_f C_A T_F C_F + \frac{1451}{48} n_f C_A^2 T_F C_F + \frac{3013}{48} n_f C_A^3 T_F + \frac{129}{2} n_f T_F^2 C_F 
- \frac{301}{6} n_f C_A T_F^2 C_F - \frac{211}{8} n_f^2 C_A^2 T_F - \frac{1}{3} n_f^3 C_A T_F + \frac{79}{18} n_f^3 C_A^2 T_F \right) 
+ a_s^3(Q^2) \left( \frac{19360399}{186624} C_A^3 + \frac{461}{144} n_f T_F C_F - \frac{614501}{10368} n_f C_A T_F C_F 
- \frac{1857805}{31104} n_f C_A^2 T_F + \frac{28981}{2592} n_f T_F^2 C_F + \frac{126415}{15552} n_f C_A T_F^2 + \frac{125}{729} n_f^3 C_A^3 T_F 
+ \frac{55}{8} \zeta_3 n_f C_A^3 - 15\zeta_3 n_f C_A T_F C_F + \frac{15}{2} \zeta_3 n_f C_A^2 T_F 
- \frac{6893}{144} \zeta_3 C_A^4 + \frac{145}{12} \zeta_3 n_f T_F C_F + \frac{1291}{48} \zeta_3 n_f C_A T_F C_F - \frac{349}{144} \zeta_3 n_f C_A^2 T_F 
- \frac{13}{2} \zeta_3 n_f T_F^2 C_F + \frac{121}{36} \zeta_3 n_f C_A^2 T_F \right) + \frac{a_s^3(Q^2)}{(11 C_A - 4n_f T_F)^2} \left( -\frac{441065}{2592} C_A^4 
- \frac{109529}{432} n_f C_A^2 T_F + \frac{1415}{9} n_f C_A^3 T_F + \frac{165}{2} n_f T_F^2 C_F + \frac{15835}{216} n_f^2 C_A T_F^2 C_F 
+ \frac{7825}{216} n_f^2 C_A^2 T_F - \frac{125}{27} n_f^3 C_A T_F + \frac{125}{81} n_f^3 C_A^2 T_F + \frac{561}{16} \zeta_3 C_A^4 + \frac{291}{8} \zeta_3 n_f C_A^2 T_F C_F \right) \right\} 
\]  

(3.31)
\[-\frac{63}{8} \zeta_3 n_f C_A^3 T_F - 45 \zeta_3 n_f^2 T_F^2 C_A^2 + \frac{75}{2} \zeta_3 n_f^2 C_A T_F^2 C_F - \frac{15}{2} \zeta_3 n_f^2 C_A^2 T_F^2 + \frac{a^2(Q^2)}{(11 C_A - 4 n_f T_F)^2} \left( -\frac{1667183}{10368} C_A^5 + \frac{29359}{96} n_f C_A^2 T_F C_F^2 + \frac{227807}{1728} n_f C_A^3 T_F C_F + \frac{1525313}{5184} n_f C_A^3 T_F^2 + \frac{727}{3} n_f^2 C_A^2 T_F^2 C_F - \frac{33923}{144} n_f^2 C_A^2 T_F^2 C_F - \frac{129511}{864} n_f^2 C_A^3 T_F^2 - \frac{215}{6} n_f^3 T_F^3 C_F^2 + \frac{317}{12} n_f^3 C_A T_F^3 C_F + \frac{10949}{324} n_f^3 C_A^2 T_F^3 C_F + \frac{5}{27} n_f^4 T_F^4 C_F - \frac{395}{162} n_f^4 C_A T_F^4 \right) + \frac{a^2(Q^2)}{11 C_A - 4 n_f T_F} \left( \frac{7623}{16} n_f C_A^2 T_F C_F^3 + \frac{22121}{32} n_f C_A^3 T_F C_F^2 + \frac{31207}{32} n_f C_A^4 T_F C_F \right) - \frac{2079}{4} n_f^2 C_A T_F^2 C_F^3 + \frac{13533}{8} n_f^2 C_A T_F^2 C_F^3 - \frac{29647}{12} n_f^2 C_A^3 T_F^2 C_F - 45 n_f^3 T_F^3 C_F^3 - \frac{1911}{2} n_f^3 C_A T_F^3 C_F^2 + 1443 n_f^3 C_A^2 T_F^3 C_F + 178 n_f^4 T_F^4 C_F - 384 n_f^4 C_A T_F^4 C_F + \frac{104}{3} n_f^5 T_F^5 C_F - 2178 \zeta_3 n_f^3 C_A^2 T_F^2 C_F^2 + 2178 \zeta_3 n_f^3 C_A^3 T_F^3 C_F + 1584 \zeta_3 n_f^3 C_A^3 T_F^3 C_F - 1584 \zeta_3 n_f^3 C_A^3 T_F^3 C_F - 288 \zeta_3 n_f^4 T_F^4 C_F^2 + 288 \zeta_3 n_f^4 C_A T_F^4 C_F \right) \}

Again an explicit calculation including all logarithmic pieces for an arbitrary scale \( \mu \) confirms that indeed \( \mu^2 \frac{d}{d \mu^2} C_1^{\text{RGI}} = 0 \) which is a welcome consistency check.

The full results for the RGI coefficients at a general scale \( \mu \) are available at http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp13/ttp13-003/.

These full results can now be used to obtain the logarithmic pieces of \( Q^2 \frac{d}{d Q^2} C_0^{\text{RGI}} \) and \( C_1^{\text{RGI}} \) at four-loop level. If a generic RGI quantity has the structure

\[
Q^{\text{RGI}} = a_s(\mu) A_1 + a_\mu^2(\mu)(A_2 + l_{\mu} B_2) + a_\mu^3(\mu)(A_3 + l_{\mu} B_3 + l_{\mu}^2 C_3) + a_\mu^4(\mu)(A_4 + l_{\mu} B_4 + l_{\mu}^2 C_4 + l_{\mu}^3 D_4) + a_\mu^5(\mu)(A_5 + l_{\mu} B_5 + l_{\mu}^2 C_5 + l_{\mu}^3 D_5 + l_{\mu}^4 E_5) + \mathcal{O}(a_6^6)
\]

(3.32)

with scale independent coefficients \( (A_i, B_i, \ldots) \) the requirement \( \mu^2 \frac{d}{d \mu^2} Q^{\text{RGI}} \perp 0 \) leads to the conditions

\[
B_2 = A_1 \beta_0, \quad C_3 = B_2 \beta_0, \quad B_3 = A_1 \beta_1 + 2 A_2 \beta_3, \quad D_4 = C_3 \beta_0, \quad C_4 = \frac{1}{2} (3 B_3 \beta_0 + 2 B_2 \beta_1), \quad B_4 = A_1 \beta_2 + 2 A_2 \beta_1 + 3 A_3 \beta_0
\]

(3.33)

which in the cases of \( Q^2 \frac{d}{d Q^2} C_0^{\text{RGI}} \) and \( C_1^{\text{RGI}} \) can be used as checks for the result with an arbitrary scale \( \mu \) or to reconstruct the logarithmic pieces from the result for \( \mu^2 = Q^2 \). In \( \mathcal{O}(a_6^6) \) we find

\[
E_5 = D_4 \beta_0, \quad D_5 = \frac{1}{3} (4 C_4 \beta_0 + 3 C_3 \beta_1), \quad C_5 = \frac{1}{2} (2 B_2 \beta_2 + 3 B_3 \beta_1 + 4 B_4 \beta_0),
\]

(3.34)
The results read

\[
B_5 = A_1\beta_3 + 2A_2\beta_2 + 3A_3\beta_1 + 4A_4\beta_0.
\]

Using the four-loop \(\beta\)-function\(^\text{13}\) of QCD [39, 40] the following four-loop contributions (for QCD colour factors) are derived:

\[
Q^2 \frac{d}{dQ^2} C_{0}^{\text{RGI}, \text{4loop}} = a_s^5 n_g \left( \left( \frac{n_f^4}{1296} - \frac{11n_f^3}{216} + \frac{121n_f^2}{96} - \frac{1331n_f}{96} + \frac{14641}{256} \right) t_\mu^\nu + \left( \frac{5n_f^4}{972} - \frac{1595n_f^3}{2592} + \frac{4355n_f^2}{216} - \frac{293975n_f}{1152} + \frac{424105}{384} \right) t_\mu^\nu + \left( \frac{25n_f^4}{1944} - \frac{n_f^3}{24} - \frac{6937n_f^3}{2304} - \frac{77n_f^2}{16} + \frac{1812625n_f^2}{13824} \right) t_\mu^\nu + \left( \frac{580n_f^4}{9} + \frac{13206877n_f^4}{5184} + \frac{38583n_f^3}{4} + \frac{2695n_f^3}{3} \right) \right) l_\mu + \text{const.}
\]

\[
C_1^{\text{RGI}, \text{4loop}} = 64 a_s^5 \left( \left( \frac{n_f^4}{1296} - \frac{11n_f^3}{216} + \frac{121n_f^2}{96} - \frac{1331n_f}{96} + \frac{14641}{256} \right) t_\mu^\nu + \left( \frac{5n_f^4}{972} - \frac{1595n_f^3}{2592} + \frac{4355n_f^2}{216} - \frac{293975n_f}{1152} + \frac{424105}{384} \right) t_\mu^\nu + \left( \frac{25n_f^4}{1944} - \frac{n_f^3}{24} - \frac{6937n_f^3}{2304} - \frac{77n_f^2}{16} + \frac{1812625n_f^2}{13824} \right) t_\mu^\nu + \left( \frac{580n_f^4}{9} + \frac{13206877n_f^4}{5184} + \frac{38583n_f^3}{4} + \frac{2695n_f^3}{3} \right) \right) l_\mu + \text{const.}
\]

For completeness we also give the RGI Wilson coefficients for the correlator

\[
i \int d^4x e^{i\eta x} [\{\partial_\mu J_5^\mu\}^{\text{RGI}}(x)[\partial_\mu J_5^\mu\}^{\text{RGI}}(0)] = (Q^2)^2 C_0^{JJ, \text{RGI}} + C_1^{JJ, \text{RGI}} [O_1]^{\text{RGI}} + \ldots \quad (3.37)
\]

The results read

\[
Q^2 \frac{d}{dQ^2} C_{0}^{JJ, \text{RGI}} = a_s^5 \left( \left( \frac{n_f^4}{1296} - \frac{11n_f^3}{216} + \frac{121n_f^2}{96} - \frac{1331n_f}{96} + \frac{14641}{256} \right) t_\mu^\nu + \left( \frac{5n_f^4}{972} - \frac{1595n_f^3}{2592} + \frac{4355n_f^2}{216} - \frac{293975n_f}{1152} + \frac{424105}{384} \right) t_\mu^\nu + \left( \frac{25n_f^4}{1944} - \frac{n_f^3}{24} - \frac{6937n_f^3}{2304} - \frac{77n_f^2}{16} + \frac{1812625n_f^2}{13824} \right) t_\mu^\nu + \left( \frac{580n_f^4}{9} + \frac{13206877n_f^4}{5184} + \frac{38583n_f^3}{4} + \frac{2695n_f^3}{3} \right) \right) l_\mu + \text{const.}
\]

\(^\text{13}\)The one-loop, two-loop and three-loop results are known from [23, 33–38].
and

\[ C_{1}^{\text{J,RGI}} = 4a_{s}n_{f}^{2}T_{F}^{2} \left\{ 1 + a_{s} \left( \frac{157}{36}C_{A} - \frac{5}{9}n_{f}T_{F} + \frac{11}{12}l_{\mu}C_{A} - \frac{1}{3}l_{\mu}n_{f}T_{F} \right) \right. \\
+ \left. \frac{a_{s}}{(11C_{A} - 4n_{f}T_{F})} \left( -\frac{17}{2}C_{A}^{2} - 15n_{f}C_{F} + 5n_{f}C_{A}T_{F} \right) \right\} . \] (3.39)

The four-loop extension of these results with QCD colour factors are given by

\[ Q^{2} \frac{d}{dQ^{2}} C_{0}^{\text{J,RGI, 4loop}} = \frac{a_{s}^{3}n_{f}}{\pi^{2}} \left[ l_{\mu}n_{f}^{2}\frac{(-33 + 2n_{f})}{384} + \text{const.} \right] \] (3.40)

and

\[ C_{1}^{\text{J,RGI, 4loop}} = 4a_{s}^{3} \left\{ \frac{1}{864}n_{f}^{2}(14166 - 1533n_{f} + 20n_{f}^{2}) \right. \\
+ \left. l_{\mu}^{2}\frac{121n_{f}^{2}}{64} - \frac{11n_{f}^{3}}{48} + \frac{n_{f}^{4}}{144} + \text{const.} \right\} . \] (3.41)

### 3.4 Numerics

We now consider the two cases \( n_{f} = 0 \) (pure gluodynamics) and \( n_{f} = 3 \) which are most important for applications. Furthermore we set \( Q^{2} = \mu^{2} \), i.e. \( l_{\mu} = 0 \). The numerical results for \( C_{1} \) and \( C_{1}^{\text{RGI}} \) are then

\[ C_{1}(Q^{2} = \mu^{2}, n_{f} = 0) = 64\{1 + 13.0833a_{s} + 135.547a_{s}^{2} + 1439.88a_{s}^{3}\}, \] (3.42)

\[ C_{1}(Q^{2} = \mu^{2}, n_{f} = 3) = 64\{1 + 12.25a_{s} + 94.0971a_{s}^{2} + 646.69a_{s}^{3}\}, \] (3.43)

\[ C_{1}^{\text{RGI}}(Q^{2} = \mu^{2}, n_{f} = 0) = 64a_{s}\{1 + 10.7652a_{s} + 102.475a_{s}^{2} + 1089.78a_{s}^{3}\}, \] (3.44)

\[ C_{1}^{\text{RGI}}(Q^{2} = \mu^{2}, n_{f} = 3) = 64a_{s}\{1 + 9.13889a_{s} + 55.9532a_{s}^{2} + 361.615a_{s}^{3}\}. \] (3.45)

In order to estimate the numerical significance of the higher order corrections we evaluate \( C_{1} \) at \( \mu = M_{Z} \), \( \mu = 3.5 \) GeV and \( \mu = 2 \) GeV with

\[ \alpha_{s}(n_{f}=5)(M_{Z}) \approx 0.118 , \ \alpha_{s}(n_{f}=3)(3.5 \text{GeV}) \approx 0.24 \text{ and } \alpha_{s}(n_{f}=3)(2 \text{GeV}) \approx 0.30 \] [41] (3.46)

for the cases \( n_{f} = 5 \) and \( n_{f} = 3 \) respectively.

\[ C_{1}(Q^{2} = \mu^{2} = M_{Z}^{2}, n_{f} = 5) = 64 \left( 0.0116 + 0.0949 + 0.4393 + 1 \right), \] (3.47)

\[ C_{1}(Q^{2} = \mu^{2} = (3.5 \text{ GeV})^{2}, n_{f} = 3) = 64 \left( 0.2883 + 0.5492 + 0.9358 + 1 \right), \] (3.48)

\[ C_{1}(Q^{2} = \mu^{2} = (2 \text{ GeV})^{2}, n_{f} = 3) = 64 \left( 0.5631 + 0.8581 + 1.1698 + 1 \right). \] (3.49)

At the scale \( \mu^{2} = M_{Z}^{2} \) the two and three-loop contributions are about 9% and 1% wrt tree-level, whereas at a scale \( \mu^{2} = (2 \text{ GeV})^{2} \) these contributions become so large that perturbation theory stops to work (as is expected). From this evaluation we can assume
that in the case of $Q^2 = \mu^2$ the Wilson coefficient to this accuracy in perturbation theory is a valid approximation down to a scale of about $\mu^2 = (3.5 \text{ GeV})^2$.

It is interesting to compare this with the numerics for the Adler function of the coefficient $C_0$, i.e. the purely perturbative part of the pseudoscalar gluonium correlator:

$$
\left[ Q^2 \frac{d}{dQ^2} C_0 \right] (Q^2 = \mu^2, n_f = 0) = -\frac{n_g}{\pi^2} \{ 1 + 24.25a_s + 466.862a_s^2 \}, \quad (3.50)
$$

$$
\left[ Q^2 \frac{d}{dQ^2} C_0 \right] (Q^2 = \mu^2, n_f = 3) = -\frac{n_g}{\pi^2} \{ 1 + 20.75a_s + 305.953a_s^2 \}, \quad (3.51)
$$

$$
\left[ Q^2 \frac{d}{dQ^2} C_0^{\text{RGI}} \right] (Q^2 = \mu^2, n_f = 0) = -\frac{a_s^2 n_g}{\pi^2} \{ 1 + 24.25a_s + 466.862a_s^2 \}, \quad (3.52)
$$

$$
\left[ Q^2 \frac{d}{dQ^2} C_0^{\text{RGI}} \right] (Q^2 = \mu^2, n_f = 3) = -\frac{a_s^2 n_g}{\pi^2} \{ 1 + 19.4167a_s + 277.194a_s^2 \}. \quad (3.53)
$$

Evaluated at the same scales as $C_1$ we find:

$$
\left[ Q^2 \frac{d}{dQ^2} C_0 \right] (Q^2 = \mu^2 = M_{\gamma}^2, n_f = 5) = -\frac{n_g}{\pi^2} \left( \frac{0.2967 + 0.6917 + 1}{3 \text{ loop}} \right), \quad (3.54)
$$

$$
\left[ Q^2 \frac{d}{dQ^2} C_0 \right] (Q^2 = \mu^2 = (3.5 \text{ GeV})^2, n_f = 3) = -\frac{n_g}{\pi^2} \left( \frac{1.7856 + 1.5852 + 1}{3 \text{ loop}} \right), \quad (3.55)
$$

$$
\left[ Q^2 \frac{d}{dQ^2} C_0 \right] (Q^2 = \mu^2 = (2 \text{ GeV})^2, n_f = 3) = -\frac{n_g}{\pi^2} \left( \frac{2.7900 + 1.9815 + 1}{3 \text{ loop}} \right). \quad (3.56)
$$

We see that the purely perturbative part of the OPE is much less convergent than $C_1$. In fact it stops to converge already at a scale of about $\mu^2 = (20 \text{ GeV})^2$ which corresponds to

$$
\alpha_s^{(n_f=4)}(20\text{GeV}) \approx 0.15 \quad [41] \quad (3.57)
$$

and hence

$$
\left[ Q^2 \frac{d}{dQ^2} C_0 \right] (Q^2 = \mu^2 = (20\text{GeV})^2, n_f = 4) = -\frac{n_g}{\pi^2} \left( \frac{0.5858 + 0.9350 + 1}{3 \text{ loop}} \right). \quad (3.58)
$$

This behaviour should be taken into account in any application that approaches low energies, e.g. in sum rules. We note however that in sum rules, e.g. in $[12, 18]$, a Borel transformation is used on the $1/\tau^2$-series of the OPE. The Borel operator

$$
\hat{\mathcal{L}}_M = \lim_{\substack{n \to \infty, \; Q^2 \to \infty \; \frac{Q^2}{n} = M^2}} \frac{(Q^2)^n}{(n-1)!} \left( -\frac{d}{dQ^2} \right)^n, \quad (3.59)
$$

strongly enhances the convergence of the OPE (i.e. the expansion in $1/\tau^2$) and the scale $\mu$ is then usually set to the finite Borel mass $M$. Nevertheless, the numerical evaluation presented here suggests that the convergence of the $\alpha_s$-expansion is a problem in sum rules using the OPE of the pseudoscalar gluonium correlator at low scales.
4 Discussion and Conclusions

I have presented higher order corrections for the coefficient function $C_1$ of the OPE of the correlator $X_t$ of two pseudoscalar gluonium operators. This result extends the previously known accuracy by two loops. It is also worth of notice that no contact terms can appear in this coefficient due to the relation between the operator $\tilde{O}_1$ and the Chern-Simons current, a fact that has been explicitly checked and verified up to $O(\alpha_s^3)$ by this calculation. The OPE of the correlator of two operators $\partial_\mu J_5^\mu$ which mixes with $\tilde{O}_1$ under renormalization has been performed as well and the corresponding coefficients $C_{0}^{JJ}$ and $C_{1}^{JJ}$ have been given at three-loop level. In addition the construction of RGI operators and Wilson coefficients has been discussed, the coefficients $C_{0}^{\text{RGI}}$, $C_{1}^{\text{RGI}}$, $C_{0}^{JJ,\text{RGI}}$ and $C_{1}^{JJ,\text{RGI}}$ have been presented and their logarithmic part has been derived at four-loop level from the principle of scale invariance. Finally, the numerical evaluation of $C_0$ and $C_1$ shows large coefficients in the $\alpha_s$-expansion causing a breakdown of the applicability of perturbation theory already at $Q^2 = \mu^2 = (20 \text{ GeV})^2$ for $\frac{d}{dQ^2} C_0$ and at $Q^2 = \mu^2 = (3.5 \text{ GeV})^2$ for $C_1$.

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All calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers using the thread-based [42] version of FORM [25]. The Feynman diagrams have been drawn with the Latex package Axodraw [43].

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