Critical Momenta of Lattice Chiral Fermions

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Abstract

We determine the critical momenta for chiral fermions in the domain wall model recently suggested by Kaplan. For a wide range of domain wall masses $m$ and Wilson couplings $r$ we explicitly exhibit the regions in momentum space where the fermions are chiral. We compare the critical momenta for the infinitely large system with those obtained on a finite lattice.
1 Introduction

In a recent paper [1] a new method for simulating chiral fermions on a lattice was suggested. The proposal uses the fact that by introducing a domain wall in an odd dimensional – and therefore vectorlike– theory one finds zero modes bound to this domain wall [2, 3, 4]. Regarding the domain wall as a lower (even) dimensional world it was demonstrated in [1] that for the infinite lattice these zero modes are chiral fermions and that the resulting theory on the domain wall exhibits the desired anomaly structure.

The same features have been shown to survive on a finite lattice where boundary conditions require one to consider a wall/anti-wall pair. For low energies a chiral fermion is bound to one of the domain walls with its chiral partner of opposite chirality living on the other domain wall. In addition, it has been demonstrated that in the presence of weak external gauge fields the divergence of the gauge current satisfies the continuum anomaly equation [5]. This surprising result has been explained as being a consequence of the fact that on the lattice the divergence of the Goldstone-Wilczek current [3, 4] has the same form as in the continuum [6].

One of the most important properties of the domain wall method is that the fermions are only chiral in the low energy limit. For the special choice of the Wilson coupling \( r = 1 \) it was shown in [1] that there exists a critical value of the momentum \( p_c \) for which the fermions cease to be chiral. However, it is expected that also for other choices of the Wilson coupling such critical momenta exist. This was confirmed on the finite lattice where the existence of the critical momentum could be demonstrated for an \( r = 1.8 \) [5].

Here we want to perform a systematic study of the values of the critical momenta for which one will find chiral modes. We compare the analytical results obtained on the infinite lattice with the spectrum of the finite lattice Hamiltonian. Aside from the theoretical interest of this question, our results are of practical importance for future numerical simulations of this system.

2 Zeromodes on the Infinite Lattice

To be specific we will first discuss the zeromodes for the case of a 3-dimensional model, though our results will be generalized to arbitrary dimensions at the end. We start with the Dirac-Wilson operator on an infinite lattice with lattice spacing \( a = 1 \)

\[
K_{3D} = \sum_{\mu=1}^{3} \sigma_\mu \partial_\mu + m\theta(s) + \frac{r}{2} \sum_{\mu=1}^{3} \Delta_\mu
\]  

(1)

where \( \partial \) denotes the lattice derivative \( \partial_\mu = \frac{1}{2} [\delta_{z,z+\mu} - \delta_{z,z-\mu}] \), \( \Delta \) the lattice Laplacian \( \Delta_\mu = [\delta_{z,z+\mu} + \delta_{z,z-\mu} - 2\delta_{z,z}] \), the \( \sigma_\mu \) are the usual Pauli matrices and \( r \) the Wilson coupling. We will denote by \( s \) the extra dimension along which the mass defect appears, while \( x, t \) are the
2-dimensional coordinates. The domain wall is taken to be a step function \( \theta \),

\[
\theta(s) = \begin{cases} 
-1 & s < 0 \\
0 & s = 0 \\
+1 & s > 0 
\end{cases}
\]  

where the height of the domain wall is given by the mass parameter \( m \) which we will choose to be positive throughout this paper.

We are looking for solutions which are plane waves in the \((x,t)\)-plane

\[
\Psi^\pm = e^{i(p_t t + p_x x)} \Phi(s) u^\pm
\]  

where \( u^\pm \) are the eigenspinors of \( \sigma_3 \)

\[
\sigma_3 u^\pm = \pm u^\pm.
\]

With this ansatz the Dirac operator becomes

\[
K_{3D} = \sum_{i=1}^{2} i\sigma_i \sin(p_i) + \sigma_3 \theta(s) + m\theta(s) + r(\sum_{i=1}^{2} \cos(p_i) - 1) + \frac{r}{2} \Delta_s.
\]  

Our final goal is to diagonalize the 3 dimensional Dirac operator in such a way that it reduces to the 2 dimensional Dirac operator for free massless fermions,

\[
K_{3D} \Psi = K_{2D} \Psi
\]

where \( K_{2D} \) acting on \( \Psi \) is given by

\[
K_{2D} = \sum_{\mu=1}^{2} \sigma_\mu \partial_\mu = i(\sigma_1 \sin(p_t) + \sigma_2 \sin(p_x)).
\]

Hence the equation to solve is

\[
\left[ \sigma_3 \partial_s + m\theta(s) - rF + \frac{r}{2} \Delta_s \right] \Phi u^\pm = 0
\]  

where

\[
F = \sum_{i=t,x} (1 - \cos(p_i)).
\]

Following [1] we choose an exponential ansatz for \( \Phi \) away from the domain wall

\[
\Phi(s + 1) = z\Phi(s).
\]

Inserting this into (3) one finds four solutions

\[
z = \frac{r - m_{eff} \pm \sqrt{m_{eff}(m_{eff} - 2r) + 1}}{r \pm 1}
\]  

2
where $m_{\text{eff}} = m\theta(s) - rF$, the $\pm$ in the nominator stand for the two roots and the $\pm$ in the denominator stand for the chirality. Note that in the limit $r = 1$ eq.(11) can be reduced to the corresponding expressions in [3].

We have to impose the condition that the solutions are normalizable to obtain sensible wavefunctions. This means that $|z| > 1$ for $s < 0$ and $|z| < 1$ for $s > 0$. The boundaries of the regions where chiral solutions exist are obtained by setting $|z| = 1$. Explicit matching of the normalizable solutions for positive and negative $s$ at $s = 0$ enables us to determine the regions with chiral fermions. We find that existence and chirality of the solutions is independent of the sign of $r$ and that a negative $m$ leads to opposite chiralities. Depending on the values of $m/r$ we get $m = rF$ and $m = r(F + 2)$ as the boundaries for the critical momenta where $F$ is defined as above in eq.(9).

The results are summarized in fig.1. We show the Brillouin zone $-\pi \leq p_t \leq \pi$, $-\pi \leq p_x \leq \pi$ for different ratios of $m/r$. The white areas indicate the region in momentum space where chiral fermions exist. Starting with $m/r = 0$ we find no chiral fermions. For increasing $m/r > 0$ the region where chiral modes exist grows from a small circle around $\vec{p} = (0, 0)$ until it hits the boundary of the Brillouin zone for $m/r = 2$. The boundary of the circle, i.e. the upper critical momenta, is given by $m = rF$. Increasing $m/r$ further opens up the two “doubler” modes at $\vec{p} = (0, \pi)$ and $\vec{p} = (\pi, 0)$ which have flipped chirality, while the original mode at $\vec{p} = (0, 0)$ disappears. Here the boundaries of the white regions are given by $m = rF$ for the lower and $m = r(F + 2)$ for the upper critical momenta. For $m/r > 4$ the two “doublers” disappear and we get a zero mode at $\vec{p} = (\pi, \pi)$ with the same chirality as the mode at $\vec{p} = (0, 0)$. The boundary for the lower critical momenta is given by $m = r(F + 2)$. This mode is finally also lost as $m/r$ is increased to $m/r \geq 6$.

We want to remark that this spectrum stems from $\Psi^+$ solutions only, and that there are no $\Psi^-$ solutions for positive $m$. The change of the chirality is the usual reinterpretation of the chirality at different corners of the Brillouin zone.

The generalization to arbitrary dimensions $d = 2n + 1$ consists merely in replacing the function $F$ in eq.(9) by

$$F = \sum_{i=1}^{d-1} \left(1 - \cos(p_i)\right).$$

(12)

We find in $d$ dimensions that for $2k < |m/r| < 2k + 2$ ($0 \leq k < d - 1$), the number of chiral zero modes $N_{zm}$ bound to the $d - 1$ dimensional domain wall is given by

$$N_{zm} = \binom{d - 1}{k}$$

(13)

and their chirality is $(-1)^k$.

The regions of chiral fermions as found here correspond exactly to the values of $m/r$ where the lattice Chern-Simons current induced by Wilson fermions changes its value [6] giving contributions to the Goldstone-Wilczek current. The calculation of the Chern-Simons current in [6] uses the observation that the fermion propagator in momentum space can be
interpreted as a map from the torus to the sphere \((T^d \to S^d)\). The winding number of this map is closely related to the zeromode spectrum and changes only at particular values of \(m/r\) which agree exactly with our results for the values of \(m/r\) where the number of zeromodes changes.

## 3 Finite Size Effects

Any numerical work on this system will necessarily involve finite lattices, and so we now compare the critical momenta obtained on the infinite system with the ones of a finite lattice. On the finite lattice we have to choose some boundary conditions which generate a second anti-domain wall. The mass term is therefore modified to be \(m\theta_L(s)\) with

\[
\theta_L(s) = \begin{cases} 
-1 & 2 \leq s \leq \frac{L_s}{2} \\
+1 & \frac{L_s}{2} + 2 \leq s \leq L_s \\
0 & s = 1, \frac{L_s}{2} + 1 
\end{cases}
\]  

(14)

We searched for the zeromodes on the finite lattice by solving the Hamiltonian problem numerically. If we again assume plane waves in the \(x\)-direction the Hamiltonian is given by

\[
H = -\sigma_1 \left[ i\sigma_2 \sin(p_x) + \sigma_3 \partial_s + m\theta_L(s) + r(\cos(p_x) - 1) + \frac{r}{2} \Delta_s \right].
\]  

(15)

We computed the eigenvalues and eigenfunctions of the Hamiltonian eq.(15) numerically\(^1\). Choosing the Hamiltonian instead of the Dirac operator reduces the numerical effort substantially as we only have to search for a single critical momentum instead of a pair. To find the critical momenta we studied the ratio

\[
R = \frac{\bar{\Psi}\Psi}{\bar{\Psi}\sigma_1\Psi}
\]  

(16)

which is a normalized measure for whether the fermions are chiral or not. It is zero if the fermions are chiral and \(R > 0\) for non-chiral modes (see fig.2b in [5]). To determine whether we still have chiral fermions we defined a threshold value for \(R\). If \(R < 0.01\) we regarded the fermions to be chiral.

We show the critical momenta as functions of \(r\) at three different values of \(m\) in Fig.2. As discussed above we should find two boundary curves for the critical momenta. Note that because we now use the Hamiltonian, the function \(F\) is only one dimensional \(F = 1 - \cos(p_x)\). For \(m/r < 2\) the chiral fermions appear for momenta bounded by \(m = rF\). This is the solid

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\(^1\)In principle it is, of course, possible to perform a similar analysis as in the infinite system. However, on the finite lattice one also has to take into account solutions which are non-normalizable in the infinite system. In addition, on the finite lattice the modes are not exactly chiral. A mode bound to one of the walls always gets an exponentially suppressed contribution from the mode on the other wall. These features render analytic calculations on the finite lattice quite intractable.
line in Fig.2. For $2 < m/r < 4$ the momenta for which chiral fermions appear are given by $m = rF$ for the lower and $m = r(F + 2)$ for the upper critical momenta. We plot the curve for the upper critical momenta as a dashed line in Fig.2.

We compare the results from the infinite system with our finite lattice calculations. The crosses correspond to a system size of $L = 100$ and the open circles to a size of $L = 20$. We find that the $L = 100$ lattice is practically indistinguishable from the infinite system. For $L = 20$, a lattice size realistic for simulations, a small shift occurs. Fixing $m$ and $r$ we find for $m/r < 2$ a smaller value and for $2 < m/r < 4$ a larger value of the critical momentum. Note, that for $m = 0.4$ the circles belong only to the solid curve. We did not find the momenta which correspond to the dashed line as our resolution in the numerical computation was not fine enough.

In conclusion, we find the differences between $L = \infty$ and $L = 20$ to be small. This means that the lattice has not to be too large to reproduce the basic features of the model at $L = \infty$ which makes the domain wall model feasible for numerical investigations. Therefore we have shown that the domain wall model can be used for numerical simulations on realistic lattices. We also give the values of the domain wall mass $m$ and the Wilson coupling $r$ with which numerical simulations should eventually be performed.

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References

[1] D.B. Kaplan, *A Method for Simulating Chiral Fermions on the Lattice*, UCSD preprint, UCSD/PTH 92-16, to appear in Phys.Lett.B.

[2] R. Jackiw and C. Rebbi, Phys.Rev.D 13 (1976) 3398.

[3] J. Goldstone and F. Wilczek, Phys.Rev.Lett. 47 (1981) 986.

[4] C.G. Callan, Jr. and J.A. Harvey, Nucl.Phys. B250 (1985) 427.

[5] K. Jansen, *Chiral Fermions and Anomalies on a Finite Lattice*, UCSD preprint, UCSD/PTH 92-18, to appear in Phys.Lett.B.

[6] M.F.L. Golterman, K. Jansen and D. Kaplan, *Chern-Simons Currents and Chiral Fermions on the Lattice*, UCSD preprint, UCSD/PTH 92-28.
Figure Caption

**Fig.1** We plot the regions within the Brillouin zone $-\pi < p_t < +\pi$, $-\pi < p_x < +\pi$ where chiral fermions exist (white areas) as a function of $m/r$ with $m$ the domain wall mass and $r$ the Wilson coupling. For the different cases $0 < m/r < 2$, $2 < m/r < 4$ and $4 < m/r < 6$ we have one chiral fermion with positive chirality, two chiral fermions with negative chirality and again one chiral fermion with positive chirality, respectively. For $m/r < 0$ and $m/r > 6$ there exist no chiral fermions.

**Fig.2** We plot the two lines which give the upper and lower critical momenta on the infinite system, $m = rF$ (solid lines) and $m = r(F+2)$ dashed lines, where $F = 1 - \cos(p_x)$. We show these lines at three different values of the domain wall mass $m$ as a function of the Wilson coupling $r$. We compare the curves from the infinite system with results from finite lattice calculations with lattice sizes $L = 100$ (crosses) and $L = 20$ (open circles). For $m = 0.4$ our resolution in the numerical computations was not fine enough to find the critical momenta corresponding to the dashed line. Note that though a shift in the critical momenta on the finite lattice is visible we find the same structure as for the infinite system.