PRESENTATIONS OF DEHN QUANDLES

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Abstract. The paper gives two approaches to write explicit presentations for the class of Dehn quandles using presentations of their underlying groups. The first approach gives finite presentations for Dehn quandles of a class of Garside groups and Gaussian groups. The second approach is for general Dehn quandles when the centralisers of generators of their underlying groups are known. Several examples including Dehn quandles of spherical Artin groups, surface groups and mapping class groups of orientable surfaces are given to illustrate the results.

1. Introduction

A quandle is an algebraic system with a binary operation that encodes the three Reidemeister moves of planar diagrams of links in the 3-space. Besides being fundamental to knot theory as observed first in [22, 23, 28], these objects appear in the study of mapping class groups [40], set-theoretic solutions of the quantum Yang-Baxter equation and Yetter-Drinfeld Modules [12], Riemannian symmetric spaces [26] and Hopf algebras [2], to name a few. Many classical topological, combinatorial and geometric knot invariants such as the knot group [22, 28], the knot coloring, the Conway polynomial, the Alexander polynomial [22] and the volume of the complement in the 3-sphere of a hyperbolic knot [21] can be retrieved from the knot quandle. Though quandles give strong invariants of knots, the isomorphism problem for them is hard. This has motivated search for newer properties, constructions and invariants of quandles themselves.

Understanding of presentations of quandles is a fundamental problem and determining a presentation is challenging in general. Even for the simplest quandles arising from groups, such as conjugation quandles of infinite abelian groups, the number of generators and relations turn out to be infinite. In this paper, we give two approaches to write explicit presentations for a fairly large class of quandles called Dehn quandles introduced recently in [11]. We believe that understanding of presentations has the potential to lead to a combinatorial quandle theory.

The notion of a Dehn quandle of a group is motivated by two classes of examples. The first one being the free quandle on a set $S$, which is simply the union of conjugacy classes of elements of $S$ in the free group on $S$, equipped with the quandle operation of conjugation. The second class of examples is given by surfaces. Let $\mathcal{M}_g$ be the mapping class group of a closed orientable surface $S_g$ of genus $g \geq 1$ and $D_g^{ns}$ the set of isotopy classes of non-separating simple closed curves on $S_g$. It is known that $\mathcal{M}_g$ is generated by Dehn twists along finitely many simple closed curves from $D_g^{ns}$ [13, Theorem 4.1]. The binary operation

$$\alpha \ast \beta = T_\beta(\alpha),$$

where $\alpha, \beta \in D_g^{ns}$ and $T_\beta$ is the Dehn twist along $\beta$, turns $D_g^{ns}$ into a quandle called the Dehn quandle of the surface $S_g$. It turns out that $D_g^{ns}$ can be seen as a subquandle of the conjugation quandle $\text{Conj}(\mathcal{M}_g)$ of the mapping class group $\mathcal{M}_g$ of $S_g$, by identifying the isotopy class of a simple

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closed curve with the isotopy class of the corresponding Dehn twist. These quandles originally appeared in the work of Zablow [40, 41]. He derived a homology theory based on Dehn quandles of surfaces [42] and showed that isomorphism classes of Lefschetz fibrations over a disk correspond to quandle homology classes in dimension two. Further, in [6], the Dehn quandle structure of the torus has been extended to a quandle structure on the set of its measured geodesic foliations and the quandle homology of this extended quandle has been explored. On the other hand, [25, 38, 39] considered a quandle structure on the set of isotopy classes of simple closed arcs on an orientable surface with at least two punctures, and called it the quandle of cords. In the case of a disk with \( n \) punctures, this is simply the Dehn quandle of the braid group \( B_n \) with respect to its standard Artin generators. A presentation for the quandle of cords of the real plane and the 2-sphere has been given in [25]. Beyond these special cases, we have not seen explicit presentations of Dehn quandles of surfaces in the literature. In fact, other than the well-known procedure of writing presentations of link quandles of tame links via their planar diagrams, general results on presentations of quandles do not seems to be known. However, an analogue of Tietze’s theorem relating two finite presentations of a quandle is known due to Fenn and Rourke [14, Theorem 4.2].

Dehn quandles of groups with respect to their subsets include many well-known constructions of quandles from groups including conjugation quandles, free quandles, Coxeter quandles [11], Dehn quandles of closed orientable surfaces, quandles of cords of orientable surfaces, knot quandles of prime knots, core quandles of groups and generalized Alexander quandles of groups with respect to fixed-point free automorphisms, to name a few. See [11] for details.

The paper is organized as follows. We give two approaches to write explicit presentations of Dehn quandles using presentations of their underlying groups. In Section 2, using Garside theory, we give finite presentations of Dehn quandles of groups of fractions of Garside monoids and Gaussian monoids (Theorem 2.20 and Theorem 2.28). We give examples of presentations for Dehn quandles using this approach including those of spherical Artin quandles. In Section 3 we prove a general result giving a presentation of the Dehn quandle of a group with respect to a generating set when the centraliser of each generator is known (Theorem 3.1). As a consequence, it follows that is \( G = \langle S \mid R \rangle \) is a finitely presented group such that the centraliser of each generator from \( S \) is finitely generated, then the Dehn quandle \( D(S^G) \) is finitely presented (Corollary 3.2). Although Theorem 3.1 is general, finding generating sets for centralisers of elements in interesting classes of groups like Garside groups and Artin groups is usually challenging. We give presentations for Dehn quandles of surface groups, braid groups and mapping class groups of orientable surfaces to illustrate the result. Examples of braid groups show that presentations of Garside quandles given by Theorem 3.1 usually have larger number of relations than the ones given by Theorem 2.20.

2. Presentations of Gaussian Quandles and Garside Quandles

We refer the reader to [22, 23, 28] for basic facts on quandles. Throughout the paper, we consider only right distributive quandles and follow conventions used in [11]. If \((X, \ast)\) is such a quandle with a generating set \( S \), then by [37, Lemma 4.4.8], any element of \( X \) can be written in a left-associated product of the form

\[
((\cdots ((a_0 \ast^{e_1} a_1) \ast^{e_2} a_2) \ast^{e_3} \cdots) \ast^{e_{n-1}} a_{n-1}) \ast^{e_n} a_n
\]

for \( a_i \in S \) and \( e_i \in \{1, -1\} \). For simplicity of notation, we write such an element as

\[
a_0 \ast^{e_1} a_1 \ast^{e_2} \cdots \ast^{e_n} a_n,
\]

Let \( G \) be a group, \( S \) a non-empty subset of \( G \) and \( S^G \) the set of all conjugates of elements of \( S \) in \( G \). The Dehn quandle \( D(S^G) \) of \( G \) with respect to \( S \) is the set \( S^G \) equipped with the binary
operation of conjugation, that is,
\[ x * y = yxy^{-1} \]
for \( x, y \in D(S^G) \). We refer to our recent work [11] for examples and basic results on Dehn quandles.

Understanding of presentations of quandles is fundamental to the development of quandle theory. In this section, we give presentations of Dehn quandles of certain Gaussian groups and Garside groups, which we shall refer as Gaussian quandles and Garside quandles, respectively. The notion of a Gaussian group and a Garside group was first introduced by Dehornoy and Paris [10] and developed further in the works of Dehornoy [9] and Picantin [31, 32, 33]. The definition of a Garside group which many authors use first appeared in [9]. Many equivalent definitions of a Garside group can be seen in the literature. For example, see [4, 8, 15, 18]. Note that a Garside group is referred to Artin groups of spherical type. Dehornoy and Paris [10] and Exel and Henry [8] treated braid groups, and by the work of Brieskorn and Saito [5] generalizing the work of Garside quandles.

A right Gaussian monoid (denoted by \( R = \{x \in M \} \)) is the unique right least common multiple (right l.c.m.) of \( x \) and \( y \) if \( x \) is a left divisor of \( y \), or \( y \) is a right multiple of \( x \), if there exists an element \( z \) satisfying \( y = xz \). Right divisors and left multiples are defined analogously. We denote by \( x \leq_L y \) if \( x \) is a left divisor of \( y \), and \( x \leq_R y \) if \( x \) is a right divisor of \( y \). In an atomic monoid \( M \), the left and right divisibility relations (i.e. the relations \( \leq_L \) and \( \leq_R \)) are respectively left and right invariant partial orders on \( M \) (see [10, Proposition 2.3]).

A right Gaussian monoid is a monoid \( M \) satisfying the following properties:

- \( M \) is atomic;
- \( M \) is left and right cancellative;
- \( (M, \leq_L) \) is a lattice (i.e. the left g.c.d. and the right l.c.m. exist and they are unique for every \( x, y \in M \), see the definitions below).

A left Gaussian monoid is defined similarly. A Gaussian monoid is both left as well as right Gaussian monoid. Note that the preceding definition of a right Gaussian monoid is different from that of [10]. We have placed an extra assumption of right cancellativity to insure that such a monoid embeds in its group of fractions (see [7, Theorem 1.23] for Ore’s criterion).

For elements \( x \) and \( y \) in a right Gaussian monoid, the left greatest common divisor (left g.c.d.) of \( x \) and \( y \) is a common left divisor \( z \) of \( x \) and \( y \) such that every common left divisor of \( x \) and \( y \) is a left divisor of \( z \). The right least common multiple (right l.c.m.) of \( x \) and \( y \) is a common right multiple \( z \) of \( x \) and \( y \) such that every common right multiple of \( x \) and \( y \) is a right multiple of \( z \). Similarly, in a left Gaussian monoid, we can define the right g.c.d. and the left l.c.m. of elements.

For elements \( x \) and \( y \) in a right Gaussian monoid, denote the left g.c.d. and the right l.c.m. of \( x \) and \( y \) respectively by \( x \wedge y \) and \( x \vee y \). The right residue of \( x \) in \( y \) (denoted by \( x \setminus y \)) is the unique element \( z \) satisfying \( x \vee y = xz \). Thus, we have
\[ x \vee y = x(x \setminus y) = y(y \setminus x) = y \setminus x. \]
A Garside element of a monoid $M$ is an element $\Delta \in M$ such that the left divisors of $\Delta$ coincide with the right divisors of $\Delta$, they are finite in number, and they generate $M$. We say that a Gaussian monoid $M$ is a Garside monoid if it contains a Garside element. By [10, Proposition 2.2], the set of atoms in $M$ with the right divisors of $\Delta$, they are finite in number, and they generate a Gaussian monoid $M$. In [10, 31, 32, 33], a Garside monoid is referred as a small Gaussian monoid or a thin Gaussian monoid. One can look at [9, Proposition 2.1] for a necessary and sufficient condition on a monoid to be a Garside monoid. Note that the notion of a Garside monoid in [10] is slightly restricted in the sense that the left l.c.m. and the right l.c.m. of atoms in a Garside monoid should coincide and it is a Garside element in the sense defined above.

The group of fractions of a monoid $M$ is the group which has the same presentation as that of the monoid $M$. In other words, if a monoid $M = (S \mid R)$ is the quotient of the free monoid on $S$ modulo the relations in $R$, then its group of fractions is the quotient of the free group on $S$ modulo the relations in $R$.

A Gaussian group (respectively, a Garside group) is the group of fractions of a Gaussian monoid (respectively, of a Garside monoid). Similarly, we can also define a right Gaussian group and a left Gaussian group.

An Ore monoid is one that embeds in its group of fractions. Ore’s criterion says that if a monoid $M$ is left and right cancellative, and if any two elements of $M$ have a common right multiple, then $M$ embeds in its group of fractions [7, Theorem 1.23]. Note that a Gaussian monoid (respectively, a Garside monoid) satisfies Ore’s conditions, and thus embeds in the corresponding Gaussian group (respectively, in the corresponding Garside group). The same is true for right Gaussian monoids as well as left Gaussian monoids.

Let $M$ be a right Gaussian monoid, $A$ the set of atoms in $M$ and $G$ the group of fractions of $M$. Then the Dehn quandle $D(A^G)$ will be referred as a right Gaussian quandle. The terms left Gaussian quandles, Gaussian quandles and Garside quandles are defined similarly. For example, in case of a spherical Artin group, the Artin quandle as defined in [11, Section 3] is a Garside quandle. Here, note that, by [5] and [10, Example 1], a spherical Artin group is a Garside group.

Let $S$ be any finite set. We set the following notations:

- $S^{-1}$ - a set which is in one to one correspondence with $S$ (for each $s \in S$, the corresponding element in $S^{-1}$ is denoted by $s^{-1}$ which denotes the inverse of $s$ in the free group on $S$).
- $S^*$ - the set of words on $S$ together with the empty word (i.e. it is nothing but the free monoid on $S$ and the empty word corresponds the identity element of the free monoid).

Note that, by the notation itself, the set $(S \cup S^{-1})^*$ is nothing but the free monoid on $S \cup S^{-1}$. The empty word represents the identity element in $S^*$ (or, in $(S \cup S^{-1})^*$) and we denote it by $\epsilon$ or by 1, as per convenience of notation. For a word $x = s_1^{\delta_1} s_2^{\delta_2} \cdots s_n^{\delta_n}$ in $(S \cup S^{-1})^*$, we denote the word $s_n^{-\delta_n} s_{n-1}^{-\delta_{n-1}} \cdots s_1^{-\delta_1}$ by $x^{-1}$, where $s_i \in S$ and $\delta_i \in \{1, -1\}$. Note that, for any $x \in (S \cup S^{-1})^*$, the elements $xx^{-1}$ and $x^{-1}x$ are different elements of $(S \cup S^{-1})^*$, and they differ from the empty word. A word $x \in (S \cup S^{-1})^*$ is called positive if $x \in S^*$. For a positive word $x \in S^*$, the length of $x$ (denoted by $\ell(x)$) is the number of letters in $x$ (i.e. $\ell(x) = n$ if $x = s_1 s_2 \cdots s_n$, where $s_i \in S$). Note that the length of a positive word is zero if and only if it is the empty word. A word $w \in (S \cup S^{-1})^*$ is called symmetric if it is of the form $x^{-1} s^\delta x$ for some $x \in (S \cup S^{-1})^*$, $s \in S$ and $\delta \in \{-1, 1\}$. The word $w$ is called positive symmetric if $\delta = 1$ and it is called negative symmetric if $\delta = -1$. We denote the set of positive symmetric words on $S \cup S^{-1}$ by $S((S \cup S^{-1})^*)^+$, i.e.,

$$S((S \cup S^{-1})^*)^+ = \{x^{-1} s x \mid s \in S \text{ and } x \in (S \cup S^{-1})^* \}.$$
Let $M$ be a monoid generated by a finite set $S$ and $G$ the group of fractions of $M$. For words $x$ and $y$ in $(S \cup S^{-1})^*$, we write $x =_G y$ if $x$ and $y$ represent the same element in $G$. Similarly, we write $x =_M y$ if $x$ and $y$ are equivalent in $M$ if and only if they are equivalent in $G$. In particular, this is true for right Gaussian monoids, left Gaussian monoids and Garside monoids, since such monoids are Ore monoids.

For a finite set $S$, a complement on $S$ is a map $f : S \times S \to S^*$ such that $f(s,t)$ is the empty word if and only if $s = t$. A complement $f$ on $S$ is said to be homogeneous if $\ell(f(s,t)) = \ell(f(t,s))$ for every $(s,t) \in S \times S$. A presentation of a monoid (or, of a group) is called a complemented presentation if it is of the form

$$\langle S \mid sf(s,t) = tf(t,s) \text{ for } s,t \in S \rangle$$

or

$$\langle S \mid g(s,t)t = g(t,s)s \text{ for } s,t \in S \rangle,$$

where $f$ and $g$ are complements on $S$. A complemented presentation is called homogeneous if the associated complement is homogeneous. In general, a finite presentation $\langle S \mid R \rangle$ of a monoid is homogeneous if $\ell(x) = \ell(y)$ for every relation $x = y$ in $R$, where $x, y \in S^*$. A monoid is homogeneous if it has a finite homogeneous presentation. For example, Artin monoids (equipped with Artin presentations) are homogeneous.

Let $M$ be a right Gaussian monoid (or, a Garside monoid) and $S$ a finite generating set for $M$. A right l.c.m. selector on $S$ in $M$ is a complement $f$ on $S$ such that $f(s,t)$ represents the element $s/t$ in $M$ for every $(s,t) \in S \times S$. By [10, Theorem 4.1], $M$ has the complemented presentation

$$\langle S \mid sf(s,t) = tf(t,s) \text{ for } s,t \in S \rangle,$$

where $f$ is the right l.c.m. selector on $S$ in $M$. The corresponding right Gaussian group (respectively, the corresponding Garside group) is the quotient of the free monoid $(S \cup S^{-1})^*$ by relations $ss^{-1} = 1, s^{-1}s = 1$ and $sf(s,t) = tf(t,s)$ for $s,t \in S$.

Let $M$ be a left Gaussian monoid (or, a Garside monoid) and $S$ a finite generating set for $M$. A left l.c.m. selector on $S$ in $M$ is a complement $g$ on $S$ such that $g(s,t)$ represents the element $s/t$ in $M$ for every $(s,t) \in S \times S$. The monoid $M$ has the complemented presentation

$$\langle S \mid g(s,t)t = g(t,s)s \text{ for } s,t \in S \rangle,$$

where $g$ is the left l.c.m. selector on $S$ in $M$. The corresponding left Gaussian group (respectively, the corresponding Garside group) has the same complemented presentation as given above.

One can look at [10, sections 3 and 4], for necessary and sufficient conditions on a monoid with complemented presentations to be a Gaussian monoid (or, a right Gaussian monoid, or a left Gaussian monoid). Also, see [8, Criterion 5.9], [9, propositions 5.4, 5.12 and 6.14] and [10, Corollary 3.11 and Theorem 4.2], for necessary and sufficient conditions on a monoid with complemented presentations to be a Garside monoid.

### 2.2. Presentations of right Gaussian quandles and Garside quandles of certain types.

For this subsection, we set the following notations:

- $M$ - a right Gaussian monoid;
- $S$ - a finite generating set for $M$;
- $A$ - the set of atoms in $M$;
- $N$ - a Garside monoid;
- $T$ - a finite generating set for $N$;
- $B$ - the set of atoms in $N$;
- $\Delta$ - a Garside element in $N$;
• \((M, S), (M, A), (N, T), (N, B), (N, \Delta)\) and \((N, T, \Delta)\) - pairs and a triple of objects with the meaning above.

For a pair \((M, S)\), we assume throughout this subsection that elements in \(S\) are pairwise distinct in \(M\). The same is assumed in case of a pair \((N, T)\).

For any pair \((N, \Delta)\), by [9] lemmas 2.2 and 2.3, the map

\[
x \mapsto (x \setminus \Delta) \Delta
\]

is a permutation of the set of divisors of \(\Delta\) and it extends to an automorphism of \(\phi\) extends to an automorphism \(\phi\) of the Garside group \(H\) corresponding to \(N\).

**Remark 2.1.** By [9] Lemma 2.2, we have \(x \Delta = \Delta \phi(x)\) for all \(x \in H\). Since \(\phi(N) = N\) and \(\phi(B) = B\), we have \(N \Delta = \Delta N\) and \(B \Delta = \Delta B\).

Consider the following conditions on a pair \((M, S)\):

(i) \((s \setminus t)s \in S\) whenever \((s, t) \in S \times S\) and \(s \leq_L t\).

(ii) \(s \setminus t \leq_L s \vee t\) for all \((s, t) \in S \times S\).

(iii) \((s \setminus t)(s \vee t) \in S\) for all \((s, t) \in S \times S\).

(iv) \(M\) has a finite homogeneous presentation \(\langle S \mid R\rangle\).

(v) For each \(s \in S\) and each \(x \in M\), there exists \(y = y(s, x)\) in \(M\) such that \(xy \leq_L sxy\).

Recall that the set of atoms in \(M\) is contained in every generating subset of \(M\). Thus, if the set of atoms does not satisfy condition (ii), then there does not exist any generating set for \(M\) satisfying condition (ii).

**Remark 2.2.** Note that, in an atomic monoid, an atom \(x\) is a divisor of an atom \(y\) if and only if \(x = y\). Thus, a pair \((M, A)\) of a right Gaussian monoid \(M\) and the set \(A\) of atoms in \(M\) satisfies condition (ii) trivially.

**Lemma 2.3.** If a pair \((M, S)\) satisfies conditions (ii) and (iii), then it satisfies conditions (i) and (iv). Moreover, in this case \(S\) must be the set of atoms in \(M\).

**Proof.** Since \((M, S)\) satisfies condition (iii), we have \(\ell(x) = \ell(y)\) for any words \(x, y \in S^\ast\) with \(x =_M y\) and \(\ell(z) = \ell(x) + \ell(y)\) for any words \(x, y, z \in S^\ast\) with \(z =_M xy\). Also, it is easy to see that \(S\) must be the set of atoms in \(M\). By Remark 2.2, \((M, S)\) satisfies condition (ii). Since \(S\) satisfies condition (iii), we have

\[
(s \setminus t)(s \setminus t)(s \vee t) = s \vee t = s(s \setminus t)
\]

for all \(s, t \in S\). Let \(s\) and \(t\) be any elements in \(S\). Suppose \(x\) and \(y\) be positive words representing \(s \setminus t\) and \((s \setminus t)(s \vee t)\), respectively. Then \(xy =_M sx\) and hence \(\ell(x) + \ell(y) = \ell(s) + \ell(x)\). This implies that \(\ell(y) = \ell(s) = 1\). Let \(y = s_1s_2\cdots s_n\) for some \(s_i \in S\). Since \(\ell(y) = 1\), we must have \(n = 1\). Thus, \(y \in S\) and consequently \((s \setminus t)(s \vee t) \in S\). Hence, \(S\) satisfies condition (iii). \(\square\)

We define the following terms:

- A pair \((M, S)\) is of
  - type \(R_1\) if it satisfies conditions (i), (ii), (iii) and (iv).
  - type \(R_2\) if it satisfies conditions (i), (ii) and (iii), and if there exist a triple \((N, T, \Delta)\) with \(T \Delta = \Delta T\) and an epimorphism \(\pi : (N, T) \to (M, S)\).

- A pair \((N, T)\) is of type \(R_3\) if it satisfies conditions (i), (iii) and (iv), and there exists a Garside element \(\Delta \in N\) such that \(T \Delta = \Delta T\).

- A pair \((M, A)\) is of
  - type \(R_4\) if it satisfies conditions (i), (iii) and (iv).
Then, by Proposition 2.4, the monoid of atoms in positive integers $\mathbb{N}$ we say a Garside monoid $\mathcal{D}$ is a Garside monoid.

Let $\nu \in \mathbb{N}$.

Example 2.5. \([10, \text{Example 5.2}]\)

For $i = 4, 5, 6, 7$, we say a right Gaussian monoid $M$ is of type $\mathcal{R}_i$ if the pair $(M, A)$ is of type $\mathcal{R}_i$, where $A$ is the set of atoms in $M$. In this case, we also say the right Gaussian group $G$ corresponding to $M$ and the right Gaussian quandle $\mathcal{D}(A^G)$ are of type $\mathcal{R}_i$. Similarly, for $i = 8, 9$, we say a Garside monoid $N$ is of type $\mathcal{R}_i$ if the pair $(N, B)$ is of type $\mathcal{R}_i$, where $B$ is the set of atoms in $N$. In this case, we also say the Garside group $H$ corresponding to $N$ and the Garside quandle $\mathcal{D}(B^H)$ are of type $\mathcal{R}_i$.

Many classes of Garside monoids are of type $\mathcal{R}_8$ and $\mathcal{R}_9$. For example, braid monoids, and more generally, Artin monoids of spherical type are Garside monoids of type $\mathcal{R}_9$. Note that, by [5, 17] and [10, Example 1], such monoids are Garside monoids. The next two propositions give a machinery to produce families of Garside monoids of types $\mathcal{R}_8$ and $\mathcal{R}_9$.

**Proposition 2.4.** \([10, \text{Proposition 5.2}]\) Consider a finite set $S = \{x_1, x_2, \ldots, x_n\}$, $n$ positive words $u_1, u_2, \ldots, u_n$ in $S^*$, and a permutation $\delta$ of $\{1, 2, \ldots, n\}$. We assume that:

1. There exists a map $\nu : S \rightarrow \mathbb{Z}_{\geq 0}$ which when extended to $S^*$ by setting $\nu(\epsilon) = 0$ and $\nu(uv) = \nu(u) + \nu(v)$, satisfies
   \[
   \nu(x_1 u_1 x_{\delta(1)}) = \nu(x_2 u_2 x_{\delta(2)}) = \cdots = \nu(x_n u_n x_{\delta(n)}).
   \]
2. For each index $k$, there exists an index $j$ satisfying
   \[x_k u_k = u_j x_{\delta(j)}.
   \]

Let $M$ be the monoid defined by the presentation
\[
\langle x_1, x_2, \ldots, x_n \mid x_1 u_1 x_{\delta(1)} = x_2 u_2 x_{\delta(2)} = \cdots = x_n u_n x_{\delta(n)} \rangle.
\]
Then $M$ is a Garside monoid.

In Proposition 2.4, one can check that the map $k \mapsto j$ is a permutation of $\{1, 2, \ldots, n\}$. It follows from the proof of the proposition that $S$ is the set of atoms in $M$. If the words $x_i u_i x_{\delta(i)}$ and $x_j u_j x_{\delta(j)}$ both represent a right l.c.m. of $x_i$ and $x_j$ for all $i$ and $j$, then the set $S$ satisfies conditions \([\mathbb{I}]\) and \([\mathbb{III}]\). Consequently, $M$ is a Garside monoid of type $\mathcal{R}_8$.

**Example 2.5.** \([10, \text{Example 4}]\) Consider a set $S = \{x_1, x_2, \ldots, x_n\}$ and integers $p_1, p_2, \ldots, p_n$ strictly greater than 1. Take $\delta$ to be the identity permutation and $u_i = x_i^{p_i-2}$ for each $i$. Choose $n$ positive integers $k_1, \ldots, k_n$ satisfying $k_1 p_1 = \cdots = k_n p_n$ and set $\nu(x_{i_1} x_{i_2} \cdots x_{i_n}) = k_{i_1} + \cdots + k_{i_n}$. Then, by Proposition 2.4, the monoid
\[
\langle x_1, x_2, \ldots, x_n \mid x_1^{p_1} = x_2^{p_2} = \cdots = x_n^{p_n} \rangle
\]
is a Garside monoid. Moreover, it is of type $\mathcal{R}_8$. Note that monoids with presentations $\langle x, y \mid x^p = y^q \rangle$ have torus knot groups as their groups of fractions.
Example 2.6. [10] Example 5] For $p$ letters $x_1, x_2, \ldots, x_p$ and a positive integer $m \geq 2$, let $\prod(x_1, x_2, \ldots, x_p; m)$ denotes the word
\[
x_1 x_2 \cdots x_p x_1 x_2 \cdots .
\]

Now, let $S = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 2$. By Proposition 2.4 the monoid
\[
\langle x_1, x_2, \ldots, x_n \mid \prod(x_1, x_2, \ldots, x_n; m) = \prod(x_2, x_3, \ldots, x_n, x_1; m) = \cdots = \prod(x_n, x_1, x_2, \ldots, x_{n-1}; m)\rangle
\]
is a Garside monoid. Further, it is of type $R_9$.

As a special case of this example, the monoid
\[
\langle x_1, x_2, \ldots, x_n \mid x_1 x_2 \cdots x_n = x_2 x_3 \cdots x_n x_1 = \cdots = x_n x_1 \cdots x_{n-1}\rangle
\]
has the group of fractions as the fundamental group of the complement of $n$ lines through the origin in $\mathbb{C}^2$ [36]. And, the monoid
\[
\langle x_1, x_2, \ldots, x_n \mid x_1 x_2 = x_2 x_3 = \cdots = x_n x_1\rangle
\]
has the Artin group of type $I_2(n)$ as its group of fractions.

Proposition 2.7. [10] Proposition 5.3] Consider Garside monoids $M_1, M_2, \ldots, M_n$ and positive integers $p_1, p_2, \ldots, p_n$ for $n \geq 2$. Let $\Delta_i$ denote the minimal Garside element of $M_i$. For each $i$, we assume that:

1. There is a map $\nu_i : M_i \to \mathbb{Z}_{\geq 0}$ satisfying $\nu_i(a) > 0$ for all $a \in M_i$ with $a \neq 1$ and $\nu_i(ab) = \nu_i(a) + \nu_i(b)$ for all $a, b \in M_i$.
2. If $M_i$ has only one atom, that is, if $M_i$ is isomorphic to $\mathbb{Z}^+$, then $p_i \geq 2$.

Let $M$ be the quotient of the free product $M_1 \ast M_2 \ast \cdots \ast M_n$ of monoids modulo the congruence $\equiv$ generated by $\Delta_i^{p_i} = \Delta_j^{p_j}$ for all $i, j$, that is,
\[
M = (M_1 \ast M_2 \ast \cdots \ast M_n)/\equiv.
\]
Then $M$ is a Garside monoid.

It is easy to see that if the Garside monoids $M_1, M_2, \ldots, M_n$ in Proposition 2.7 are of type $R_8$, and if $S_i$ is the set of atoms in $M_i$ satisfying conditions (i) and (ii), then the monoid $M = (M_1 \ast M_2 \ast \cdots \ast M_n)/\equiv$ is a Garside monoid of type $R_8$ with the set of atoms $S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n$ satisfying conditions (i) and (ii). It follows from the proof of the proposition that $S$ is, in fact, the set of atoms in $M$.

Example 2.8. [10] Example 6] Mixing presentations in examples 2.5 and 2.6 gives new examples. For example,
\[
\langle x_1, x_2, y_1, y_2, y_3 \mid x_1^2 = x_2^5 = y_1 y_2 y_3 y_1 = y_2 y_3 y_1 y_2 = y_3 y_1 y_2 y_3\rangle
\]
is a Garside monoid by Proposition 2.7. Further, it is of type $R_8$.

Example 2.9. [10] Example 7] Applying Proposition 2.7 to Artin monoids of type $B_3$ and $A_3$ shows that the monoid
\[
\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid x_1 x_2 x_1 x_2 = x_2 x_1 x_2 x_1, \quad x_1 x_3 = x_3 x_1, \quad x_2 x_3 x_2 = x_3 x_2 x_3, \quad y_1 y_2 y_1 = y_2 y_1 y_2, \quad y_1 y_3 = y_3 y_1, \quad y_2 y_3 y_2 = y_3 y_2 y_3, \quad (x_1 x_2 x_3)^6 = (y_1 y_2 y_3 y_1 y_2 y_3)^3\rangle
\]
is a Garside monoid of type $R_9$. 
Proof. For a right Gaussian monoid $R$ one can choose Garside monoids of type $R$. Proposition 2.10. Pairs of types $R_2$ through $R_9$ are of type $R_1$.

There are many examples of Garside monoids in the literature, for example, \cite{4,10,15,17,18,31}. One can choose Garside monoids of type $R_8$ and/or of type $R_9$ from these examples. Then using Proposition 2.11 one can produce more Garside monoids of type $R_8$ and/or of type $R_9$. In \cite{31}, the cross product of monoids has been defined, and it has been proved that the cross product of Garside monoids is a Garside monoid. This allows us to construct more Garside monoids of type $R_8$ and/or of type $R_9$ once we have some families of such monoids.

Proposition 2.10. Pairs of types $R_2$ through $R_9$ are of type $R_1$.

Proof. For a right Gaussian monoid $M$, by \cite{10} Proposition 2.2, the set of atoms in $M$ is a finite generating set for $M$. With this in mind, together with Remark 2.1, Remark 2.2 and Lemma 2.3, we can conclude that:

- Pairs of types $R_4$ and $R_5$ are of type $R_1$.
- Pairs of types $R_6$ and $R_7$ are of type $R_2$.
- Pairs of types $R_8$ and $R_9$ are of type $R_3$.

Since every Garside monoid is of course a right Gaussian monoid, a pair $(M, S)$ of a right Gaussian monoid with $M = N$ and a finite generating set $S = T$ for $M$. Thus, by taking $M = N, S = T$, $\pi = Id_N$ (where $Id_N$ denotes the identity map of $N$), we conclude that a pair of type $R_3$ is of type $R_2$. Now, it is enough to prove that a pair of type $R_2$ is of type $R_1$.

Let $(M, S)$ be a pair of type $R_2$. Then, there exist a triple $(N, T, \Delta)$ with $T \Delta = \Delta T$ and an epimorphism $\pi : (N, T) \rightarrow (M, S)$. In order to prove that the pair $(M, S)$ is of type $R_1$, we only have to show that it satisfies condition (\forall). Let $s \in S$ and $x \in M$ be any elements. Since $\pi$ is surjective and it maps $T$ onto $S$, there exist $s' \in T$ and $x' \in N$ such that $\pi(s') = s$ and $\pi(x') = x$. Since divisors of $\Delta$ generate $N$, we can write $x' = d_1 d_2 \cdots d_n$ for some divisors $d_1, d_2, \ldots, d_n$ of $\Delta$. Here, note that $d_i$’s are not necessarily pairwise distinct. Recall that the map $\phi$ given by $x \mapsto (x \Delta) \backslash \Delta$ is a permutation of the set of divisors of $\Delta$. It is easy to see that $d \Delta = \Delta \phi(d)$ for a divisor $d$ of $\Delta$.

Since $\phi^{n-i}(d_i)$ is a divisor (in particular, a left divisor) of $\Delta$, we have $\phi^{n-i}(d_i) \phi^{n-i}(d_i) \backslash \Delta = \Delta$. Let us set $e_i = \phi^{n-i}(d_i) \backslash \Delta$ for $i = 1, 2, \ldots, n$. Then, for each $i = 1, 2, \ldots, n$, we have

\[
\begin{align*}
d_i \Delta^{n-i} e_i &= \Delta^n \phi^{n-i}(d_i) e_i \\
&= \Delta^{n-i} \phi^{n-i}(d_i) (\phi^{n-i}(d_i) \backslash \Delta) \quad \text{(since $d_i \Delta = \Delta \phi(d_i)$)} \\
&= \Delta^{n-i+1} \quad \text{(since $e_i = \phi^{n-i}(d_i) \backslash \Delta$)} \\
&= \Delta \phi^{n-i}(d_i) (\phi^{n-i}(d_i) \backslash \Delta) = \Delta.
\end{align*}
\]

Let $y' = e_n e_{n-1} \cdots e_1$. Then, using $\Delta^{n-i+1} = d_i \Delta^{n-i} e_i$ once for each $i = 1, 2, \ldots, n$, we get

\[
\begin{align*}
\Delta^n &= d_1 \Delta^{n-1} e_1 \\
&= d_1 d_2 \Delta^{n-2} e_2 e_1 \\
&= d_1 d_2 \cdots d_n \Delta^0 e_n e_{n-1} \cdots e_1 \\
&= x' y'.
\end{align*}
\]

We see that

\[
\begin{align*}
s' x' y' &= s' \Delta^n \\
&= \Delta^n t' \quad \text{for some $t' \in T$} \\
&= x' y' t' \quad \text{(since $T \Delta = \Delta T$)} \\
&= x' y'.
\end{align*}
\]
Let \( y = \pi(y') \) and \( t = \pi(t') \). Then
\[
sxy = \pi(s'x'y') \quad \text{(since } \pi \text{ is a morphism of monoids)}
\]
\[
= \pi(x'y't') \quad \text{(since } s'x'y' = x'y't')
\]
\[
= xyt,
\]
and hence \( xy \leq_L sxy \). Thus, the pair \((M, S)\) satisfies condition (v), and hence is of type \( \mathcal{R}_1 \). \( \square \)

Figure 1 summarises relations among the pairs \( \mathcal{R}_i \), where an edge \( \mathcal{R}_i \rightarrow \mathcal{R}_j \) means that a pair of type \( \mathcal{R}_i \) is of type \( \mathcal{R}_j \).

![Figure 1](image)

**Figure 1.** A diagram for pairs of types \( \mathcal{R}_i \)

Suppose that \((M, S)\) is a pair that satisfies condition (i). Then, we have a map
\[
\alpha : \{(s, t) \in S \times S \mid s \leq_L t\} \rightarrow S
\]
defined by
\[
(2.2.1) \quad (s, t) \mapsto (s \setminus t)s.
\]
Note that \( \alpha(s, s) = s \) for all \( s \in S \).

**Remark 2.11.** Let \( s, t \in S \) be such that \( s \leq_L t \). Then \( t = s(s \setminus t) \) and \( \alpha(s, t) = (s \setminus t)s \). Suppose that \( f \) is a right l.c.m. selector on \( S \) in \( M \). Since \( f(s, t) \) represents the element \( s \setminus t \), we have \( t = M sf(s, t) \) and \( \alpha(s, t) = M f(s, t)s \). Thus, \( s^{-1}ts = G f(s, t)s = G \alpha(s, t) \), where \( G \) is the right Gaussian group corresponding to \( M \).

Suppose that \((M, S)\) is a pair that satisfies conditions (ii) and (iii). Then, we have a map
\[
\beta : S \times S \rightarrow S
\]
defined by
\[
(2.2.2) \quad (s, t) \mapsto (s \setminus t)(s \lor t).
\]
Note that \( \beta(s, s) = s \) for all \( s \in S \).

**Remark 2.12.** Let \( s \) and \( t \) be elements in \( S \). Since \((M, S)\) satisfies condition (iii), we have \( s \lor t = (s \setminus t)((s \setminus t)(s \lor t)) \). In other words, \( s(s \setminus t) = (s \setminus t)((s \setminus t)(s \lor t)) \). Suppose that \( f \) is a right l.c.m. selector on \( S \) in \( M \). Since \( f(s, t) \) represents the element \( s \setminus t \) and \( \beta(s, t) = (s \setminus t)(s \lor t) \), we have \( sf(s, t) = M f(s, t)\beta(s, t) \). Thus, \( f(s, t)^{-1}sf(s, t) = G \beta(s, t) \), where \( G \) is the right Gaussian group corresponding to \( M \).
Suppose that \((M, S)\) is a pair that satisfies conditions [1], [2] and [3]. Let \(\alpha\) and \(\beta\) be the maps defined by [2.2.1] and [2.2.2], respectively. Let \(G\) be the right Gaussian group corresponding to \(M\) and \(f\) a right l.c.m. selector on \(S\) in \(M\). Then, the right \(f\)-equivalence on \(S^{(S \cup S^{-1})^*}\) is defined to be the equivalence relation generated by the following transformations:

\[
(R_\alpha) \quad x^{-1}s^{-1}t(sx) \leftrightarrow x^{-1}\alpha(s,t)x
\]

where \(s, t \in S\) with \(s \leq_L t\) and \(x \in (S \cup S^{-1})^*\).

\[
(R_{f, \beta}) \quad x^{-1}f(s,t)^{-1}sf(s,t)x \leftrightarrow x^{-1}\beta(s,t)x
\]

where \(s, t \in S\) and \(x \in (S \cup S^{-1})^*\).

\[
(R_G) \quad x^{-1}sx \leftrightarrow y^{-1}sy
\]

where \(s \in S\) and \(x, y \in (S \cup S^{-1})^*\) with \(x =_G y\).

For \(x, y \in S^{(S \cup S^{-1})^*}\), we denote \(x \simeq_R^f y\) if \(x\) and \(y\) are right \(f\)-equivalent.

**Lemma 2.13.** \(\text{Let } x, y \in S^{(S \cup S^{-1})^*}\) and \(z \in (S \cup S^{-1})^*\). If \(x \simeq_R^f y\), then \(z^{-1}xz \simeq_R^f z^{-1}yz\).

**Proof.** Let \(x, y \in S^{(S \cup S^{-1})^*}\) and \(z \in (S \cup S^{-1})^*\) such that \(x \simeq_R^f y\). It is clear that if \(x\) and \(y\) differ by a single transformation \(R_\alpha\) or \(R_{f, \beta}\) or \(R_G\), then the transformation itself, \(z^{-1}xz \simeq_R^f z^{-1}yz\). Since the right \(f\)-equivalence on \(S^{(S \cup S^{-1})^*}\) is generated by transformations \(R_\alpha\), \(R_{f, \beta}\) and \(R_G\), we have \(z^{-1}xz \simeq_R^f z^{-1}yz\) in the general case as well. \(\Box\)

The next result is motivated by [24, Theorem 1.4].

**Theorem 2.14.** Suppose that \((M, S)\) is a pair of type \(R_1\). Let \(G\) be the right Gaussian group corresponding to \(M\) and \(f\) a right l.c.m. selector on \(S\) in \(M\). Then, two words in \(S^{(S \cup S^{-1})^*}\) are right \(f\)-equivalent if and only if they represent the same element in \(G\).

We require some lemmas to prove the preceding theorem.

**Lemma 2.15.** \(\text{Let } (M, S)\) be a pair that satisfies conditions [1], [2] and [3]. Let \(G\) be the right Gaussian group corresponding to \(M\) and \(f\) a right l.c.m. selector on \(S\) in \(M\). Let \(s, t \in S\) and \(x \in S^*\) be such that \(x^{-1}sx =_G t\). Then \(x^{-1}sx \simeq_R^f t\).

**Proof.** Since \(M\) is atomic, by [10, Proposition 2.1], there exists a map \(\nu : M \to \mathbb{Z}_{\geq 0}\) such that \(\nu(a) + \nu(b) \leq \nu(ab)\) for all \(a, b \in M\), and \(\nu(a) = 0\) if and only if \(a = 1\). For \(w \in S^*\), let \(\bar{w}\) denote the element in \(M\) represented by \(w\). We prove the lemma by induction on \(\nu(\bar{x})\). Suppose that \(x^{-1}sx =_G t\) and \(\nu(\bar{x}) = 0\). Then \(\bar{x} = 1\), and it follows from properties of \(\nu\) that \(x\) is the empty word. This implies that \(s =_M t\), i.e. \(s\) and \(t\) are the same elements of \(S\). Thus \(x^{-1}sx \simeq_R^f t\). Suppose that the lemma is true when \(\nu(\bar{x}) \leq n\). Let \(x \in S^*\) be such that \(\nu(\bar{x}) = n + 1\) and \(x^{-1}sx =_G t\). Let \(x = uy\) for \(u \in S\) and \(y \in S^*\). Then \(x^{-1}sx =_G y^{-1}u^{-1}suy\), and hence \(y^{-1}u^{-1}suy =_G t\). Let us consider the following cases:

1. \(u \leq_L s\): By transformation \(R_\alpha\), we have

\[
y^{-1}u^{-1}suy \simeq_R^f y^{-1}\alpha(u,s)y,
\]

where \(\alpha\) is the map defined by [2.2.1]. Since \(x = uy\) and \(u \neq 1\), it follows from properties of \(\nu\) that \(\nu(\bar{y}) < \nu(\bar{x})\). In other words, \(\nu(\bar{y}) \leq n\). Using Remark 2.11 we get

\[
y^{-1}\alpha(u,s)y =_G y^{-1}u^{-1}suy =_G t.
\]
By induction hypothesis, we have

$$y^{-1}a(u, s)y \succeq^f_R t,$$

and hence

$$x^{-1}sx \succeq^f_R y^{-1}u^{-1}suy \succeq^f_R y^{-1}a(u, s)y \succeq^f_R t.$$

(2) \(u \not\leq_L s\): Since \(y^{-1}u^{-1}suy = G t\) and \(M\) embeds in \(G\), we have \(suy = M uy\). Thus, the words \(suy\) and \(uy\) both represent the right l.c.m. multiple \(\bar{suy} = \bar{uy}\) in \(M\). Since \(sf(s, u)\) represents the right l.c.m. of \(s\) and \(u\), we have \(suy = M sf(s, u)z\) for some \(z \in S^*\). By cancellation on the left, we get \(uy = M f(s, u)z\). Thus

$$y^{-1}u^{-1}suy \succeq^f_R z^{-1}f(s, u)^{-1}sf(s, u)z \quad \text{(by transformation } R_G\text{)}$$

$$\succeq^f_R z^{-1}\beta(s, u)z \quad \text{(by transformation } R_{f, \beta}\text{)},$$

where \(\beta\) is the map defined by (2.2.2). Since \(u(u')s = s(s'\backslash u)\) and \(u \not\leq_L s\), we have \(s'\backslash u \neq 1\). In other words, \(f(s, u)\) does not represent the identity element of \(M\). Since \(x = M uy = M f(s, u)z\), it follows from properties of \(\nu\) that \(\nu(\bar{z}) < \nu(\bar{x})\). Thus, we have \(\nu(\bar{z}) \leq n\). Using Remark 2.12, we get

$$z^{-1}\beta(s, u)z = _G y^{-1}u^{-1}suy = _G t.$$

By induction hypothesis, we have

$$z^{-1}\beta(s, u)z \succeq^f_R t.$$

Hence, we obtain

$$x^{-1}sx \succeq^f_R y^{-1}u^{-1}suy \succeq^f_R z^{-1}\beta(s, u)z \succeq^f_R t,$$

and the proof is complete. \(\Box\)

**Lemma 2.16.** Let \((M, S)\) be a pair that satisfies conditions (1), (13) and (17). Let \(s \in S\) and \(x, y \in S^*\) be such that \(sx =_M xy\). Then, there exists \(t \in S\) such that \(y =_M t\).

**Proof.** By [10 Proposition 2.1], we have a map \(\nu : M \to \mathbb{Z}_{\geq 0}\) such that \(\nu(a) + \nu(b) \leq \nu(ab)\) for all \(a, b \in M\), and \(\nu(a) = 0\) if and only if \(a = 1\). For \(w \in S^*\), let \(\bar{w}\) denote the element in \(M\) represented by \(w\). We prove the lemma by induction on \(\nu(\bar{w})\). Suppose \(sx =_M xy\) and \(\nu(\bar{x}) = 0\). Then \(\bar{x} = 1\), and it follows from the properties of \(\nu\) that \(x\) is the empty word. This implies that \(y =_M s\), and the lemma holds. Assume that the lemma is true when \(\nu(\bar{w}) \leq n\). Suppose \(s \in S\) and \(x, y \in S^*\) be such that \(\nu(\bar{w}) = n + 1\) and \(sx =_M xy\). Let \(x = uz\) for \(u \in S\) and \(z \in S^*\). Then \(suz =_M uz\). We now consider two cases below:

1. \(u \leq_L s\): Since \(x = uz\) and \(u \neq 1\), it follows from properties of \(\nu\) that \(\nu(\bar{z}) < \nu(\bar{w})\). In other words, \(\nu(\bar{z}) \leq n\). Let \(\alpha\) be the map defined by (2.2.1). Then, Remark 2.11 gives \(su =_M u\alpha(u, s)\). Since \(suz =_M uz\), we have \(u\alpha(u, s)z =_M uz\). Cancellation on the left gives \(\alpha(u, s)z =_M zy\). By induction hypothesis, there exists \(t \in S\) such that \(y =_M t\).

2. \(u \not\leq_L s\): Since \(suz =_M uz\), the words \(suz\) and \(uz\) both represent the common right multiple \(\bar{suz} = \bar{uz}\) of \(s\) and \(u\) in \(M\). Let \(f\) be a right l.c.m. selector on \(S\) in \(M\). Since \(sf(s, u)\) represents the right l.c.m. of \(s\) and \(u\), we have \(suz =_M sf(s, u)w\) for some \(w \in S^*\). Cancellation on the left gives \(uz =_M f(s, u)w\). Since \(u \not\leq_L s\), we have \(s'\backslash u \neq 1\). Thus, \(f(s, u)\) does not represent the identity element of \(M\). Since \(x =_M uz =_M f(s, u)w\), it follows from properties of \(\nu\) that
\[ \nu(\bar{w}) < \nu(\bar{x}), \] and hence \( \nu(\bar{w}) \leq n. \) Let \( \beta \) be the map defined by (2.2.2). Then

\[
f(s, u)\beta(s, u)w =_M sf(s, u)w \quad \text{(by Remark 2.12)}
\]

\[
=_M suz \quad \text{(since } f(s, u)w =_M uz) \]

\[
=_M uzy \quad \text{(since } suz =_M uzy) \]

\[
=_M f(s, u)wy \quad \text{(since } uz =_M f(s, u)w). \]

Left cancellation gives \( \beta(s, u)w =_M wy. \) Thus, by induction hypothesis, there exists \( t \in S \) such that \( y =_M t, \) and the proof is complete. \( \square \)

**Lemma 2.17.** Let \((M, S)\) be a pair of type \( \mathcal{R}_1. \) Then, for each \( s \in S \) and each \( x \in S^* \), there exist \( t = t(s, x) \) in \( S \) and \( y = y(s, x) \) in \( S^* \) such that \( sxy =_M xyt. \)

**Proof.** Let \( s \in S \) and \( x \in S^*. \) For \( w \in S^* \), let \( \bar{w} \) denote the element in \( M \) represented by \( w. \) By condition (3), there exist \( y = y(s, x) \) and \( z = z(s, x) \) in \( S^* \) such that \( \bar{sz} = \bar{xz}. \) Thus, we have \( sxy =_M xyz. \) By Lemma 2.16, there exists \( t \in S \) such that \( z =_M t. \) Hence, we have \( sxy =_M xyt. \) \( \square \)

**Lemma 2.18.** Let \((M, S)\) be a pair of right Gaussian monoid \( M \) and a finite generating set \( S \) for \( M, \) and \( G \) the group of frictions of \( M. \) Then, for each element \( x \in (S \cup S^{-1})^*, \) there exist \( y, z \in S^* \) such that \( x =_G yz^{-1}. \)

**Proof.** Let \( f \) be a right l.c.m. selector on \( S \) in \( M. \) Then, by [10] Theorem 4.2], \( (S, f) \) satisfies condition \( \text{III}_R, \) and hence \( R^f_R(x) \) exists for any \( x \in (S \cup S^{-1})^*. \) In other words, for every \( x \in (S \cup S^{-1})^*, \) there exist \( y, z \in S^* \) such that \( x =_G yz^{-1}. \) See the definitions just after Proposition 3.2 in [10] for definitions of \( R^f_R(x) \) and condition \( \text{III}_R. \) \( \square \)

**Lemma 2.19.** Let \((M, S)\) be a pair of type \( \mathcal{R}_1, G \) the right Gaussian group corresponding to \( M \) and \( f \) a right l.c.m. selector on \( S \) in \( M. \) Let \( s, t \in S \) and \( x \in (S \cup S^{-1})^* \) be such that \( x^{-1}sx =_G t. \) Then \( x^{-1}sx \simeq_R f t. \)

**Proof.** Let \( s, t \in S \) and \( x \in (S \cup S^{-1})^* \) be such that \( x^{-1}sx =_G t. \) By Lemma 2.18, there exist \( y, z \in S^* \) such that \( x =_G yz^{-1}. \) Further, by Lemma 2.17, there exist \( u \in S \) and \( w \in S^* \) such that \( syw =_M gwu. \) Thus \( w^{-1}y^{-1}syw =_G u, \) and hence by Lemma 2.15

\[
w^{-1}y^{-1}syw \simeq_R f u. \]

Since \( x^{-1}sx =_G t \) and \( x =_G yz^{-1}, \) we have \( y^{-1}sy =_G z^{-1}tz. \) Thus \( w^{-1}z^{-1}tzw =_G w^{-1}y^{-1}syw =_G u. \) Again, by Lemma 2.15 we have

\[
w^{-1}z^{-1}tzw \simeq_R f u, \]

and consequently

\[
w^{-1}y^{-1}syw \simeq_R f w^{-1}z^{-1}tzw. \]

Now, using transformation \( R_G \) and Lemma 2.13 alternatively, we get

\[
x^{-1}sx \simeq_R f zww^{-1}y^{-1}syww^{-1}z^{-1} \quad \text{(since } x =_G yww^{-1}z^{-1}) \]

\[
\simeq_R f zww^{-1}z^{-1}tzww^{-1}z^{-1} \quad \text{(since } w^{-1}y^{-1}syw \simeq_R f w^{-1}z^{-1}tzw) \]

\[
\simeq_R f t \quad \text{(since } zw^{-1}z^{-1} =_G \epsilon, \text{ where } \epsilon \text{ is the empty word).} \]

This proves the lemma. \( \square \)
Proof of Theorem 2.21. It follows from remarks 2.11 and 2.12 that if two words in \( S(S \cup S^{-1})^* \) are right \( f \)-equivalent, then they represent the same element in \( G \). For the converse part, let \( x, y \in (S \cup S^{-1})^* \) and \( s, t \in S \) be such that \( x^{-1}sx = y^{-1}t y \). Then \( yx^{-1}sx y^{-1} = G t \). By Lemma 2.19 we have \( yx^{-1}sx y^{-1} \simeq_R t \). Thus, by Lemma 2.13 together with transformation \( R_G \), we get \( y^{-1}ty \simeq_R y^{-1}yx^{-1}xy y^{-1}y \simeq_R x^{-1}sx \).

The following main result is inspired by [25, Theorem 3.1] for the quandle of cords of the plane.

Theorem 2.20. Suppose that \((M, S)\) is a pair of type \( R_1 \). Let \( \alpha \) and \( \beta \) be maps defined by (2.2.1) and (2.2.2), respectively. Let \( G \) be the right Gaussian group corresponding to \( M \) and \( f \) a right l.c.m. selector on \( S \) in \( M \). For \((s, t) \in S \times S \) with \( s \neq t \), let \( f_1(s, t), f_2(s, t), \ldots, f_{n_{st}}(s, t) \) be elements in \( S \) such that \( f(s, t) = f_1(s, t)f_2(s, t) \cdots f_{n_{st}}(s, t) \). Then, the Dehn quandle \( D(S^G) \) has a presentation with \( S \) as the set of generators and defining relations as follows:

1. \( \alpha(s, t) * s = t \) if \( s \leq_L t \),
2. \( \beta(s, t) * f_1(s, t) * f_2(s, t) * \cdots * f_{n_{st}}(s, t) = s \),
3. \( u * f_1(s, t) * f_2(s, t) * \cdots * f_{n_{st}}(s, t) = u * f_1(t, s) * f_2(t, s) * \cdots * f_{n_{st}}(t, s) * t \)

for \( s, t, u \in S \) with \( s \neq t \).

Proving Theorem 2.20 requires some preparation in the form of following lemmas.

Lemma 2.21. Let \((M, S)\) be a pair that satisfies conditions 10, 11 and 12. Let \( \alpha \) and \( \beta \) be maps defined by (2.2.1) and (2.2.2), respectively. Let \( f \) be a right l.c.m. selector on \( S \) in \( M \). Then, the right \( f \)-equivalence on \( S(S \cup S^{-1})^* \) is same as the equivalence relation generated by the following transformations:

\[ (T_\alpha) \quad x s \alpha(s, t) x^{-1} \leftrightarrow x t x^{-1}, \]

where \( s, t \in S \) with \( s \leq_L t \) and \( x \in (S \cup S^{-1})^* \).

\[ (T_{f, \beta}) \quad x f(s, t) \beta(s, t) f(s, t) x^{-1} \leftrightarrow x s x^{-1}, \]

where \( s, t \in S \) and \( x \in (S \cup S^{-1})^* \).

\[ (T_0) \quad x s x^{-1} \leftrightarrow y s y^{-1}, \]

where \( s \in S \) and \( x, y \in (S \cup S^{-1})^* \) are words differing by a single use of a relation of the form \( t t^{-1} = \epsilon \) or \( t^{-1} t = \epsilon \) for \( t \in S \). Here \( \epsilon \) denotes the empty word.

\[ (T_f) \quad x s x^{-1} \leftrightarrow y s y^{-1}, \]

where \( s \in S \) and \( x, y \in (S \cup S^{-1})^* \) are words differing by a single use of a relation of the form \( v f(u, v) = v f(v, u) \) for \( u, v \in S \).

Proof. For \( x, y \in S(S \cup S^{-1})^* \), we write \( x \simeq_f y \) if \( x \) and \( y \) are equivalent under the equivalence relation generated by \( T_\alpha, T_{f, \beta}, T_0 \) and \( T_f \). By [19, Theorem 4.1], \( M \) has a complemented presentation

\[ \langle S \mid sf(s, t) = tf(s, t) \text{ for } s, t \in S \rangle. \]

Let \( G \) be the right Gaussian group corresponding to \( M \). Then, \( G \) is the quotient of the free monoid \( (S \cup S^{-1})^* \) by relations \( s s^{-1} = 1, s^{-1} s = 1 \) and \( s f(s, t) = t f(s, t) \) for \( s, t \in S \). To prove the lemma, we need to show that each single transformation \( R_\alpha, R_{f, \beta} \) and \( R_G \) can be obtained from a combination of transformations \( T_\alpha, T_{f, \beta}, T_0 \) and \( T_f \), and vice-versa.
Let $s \in S$ and $x, y \in (S \cup S^{-1})^*$. If $x =_G y$, then there is a sequence $x = x_1, x_2, \ldots, x_n = y$ of words such that $x_i$ and $x_{i+1}$ differ by a single use of a relation of the form $uu^{-1} = \varepsilon$ or $uf(u, v) = vf(v, u)$ for $u, v \in S$. In this case, by virtue of transformations $T_0$ and $T_f$, we have $x_i sx_i^{-1} \simeq F x_{i+1} sx_{i+1}^{-1}$ for all $i$. Consequently, $sx^{-1} \simeq F ysy^{-1}$. This suggests that, for the equivalence relation $\simeq F$, transformations $T_0$ and $T_f$ can be replaced by the following transformation:

$$(T_G) \quad xsx^{-1} \longleftrightarrow ysy^{-1},$$

where $s \in S$ and $x, y \in (S \cup S^{-1})^*$ with $x =_G y$. Note that if $w \in (S \cup S^{-1})^*$, then $w^{-1} \in (S \cup S^{-1})^*$ and $(w^{-1})^{-1} = w$. Further, $x^{-1} =_G y^{-1}$ if and only if $x =_G y$. Thus, transformations $T_G$ and $R_G$ are the same. It is now sufficient to prove the following statements:

1. If $x, y \in S(S \cup S^{-1})^*$ are words differing by a single use of $R_\alpha$ or $R_{f_\beta}$, then $x \simeq F y$.
2. If $x, y \in S(S \cup S^{-1})^*$ are words differing by a single use of $T_\alpha$ or $T_{f_\beta}$, then $x \simeq F y$.

With analogy to Lemma 2.13, we can say that if $y, z \in S(S \cup S^{-1})^*$ and $x \in (S \cup S^{-1})^*$ such that $y \simeq F z$, then $xyx^{-1} \simeq F xzx^{-1}$. Since $x^{-1} \in (S \cup S^{-1})^*$ and $(x^{-1})^{-1} = x$, we also have $x^{-1}xy \simeq F x^{-1}zx$ whenever $y \simeq F z$. Thus, this together with the following statements will prove statement (1).

1. $s^{-1} \alpha(s, t) \simeq F t$ for $s, t \in S$ with $s \leq L t$.
2. $f(s, t)^{-1} \beta(s, t) \simeq F s$ for $s, t \in S$.

Thus, to prove statement (1), it is enough to prove (2.2.3) and (2.2.4). Using transformations $T_\alpha$ and $T_{f_\beta}$, we have the following:

1. $s \alpha(s, t) s^{-1} \simeq F t$ for $s, t \in S$ with $s \leq L t$.
2. $f(s, t)^{-1} \beta(s, t) f(s, t)^{-1} \simeq F s$ for $s, t \in S$.

Let $x \in (S \cup S^{-1})^*$ and $y, z \in S(S \cup S^{-1})^*$ be such that $xyx^{-1} \simeq F z$. Then

$$x^{-1}zx \simeq F x^{-1}xyx^{-1}$$

(by transformation $T_G$).

Thus, $x^{-1}zx \simeq F y$ whenever $xyx^{-1} \simeq F z$. This together with (2.2.5) and (2.2.6) implies (2.2.3) and (2.2.4), respectively. This proves the statement (1). Statement (2) can be proved similarly.

The following lemma is a direct consequence of quandle axioms. See, for example, [37, Lemma 4.4.8].

**Lemma 2.22.** Let $Q$ be quandle. Then, any left associated product

$$u_0 \ast_{\delta_1} u_1 \ast_{\delta_2} \cdots \ast_{\delta_m} u_m \ast_{\eta_1} v_1 \ast_{\eta_2} v_2 \ast_{\eta_3} \cdots \ast_{\eta_n} v_n$$

can be written as

$$(u_0 \ast_{\eta_1} v_1 \ast_{\eta_2} v_2 \ast_{\eta_3} \cdots \ast_{\eta_n} v_n) \ast_{\delta_1} (u_1 \ast_{\eta_1} v_1 \ast_{\eta_2} v_2 \ast_{\eta_3} \cdots \ast_{\eta_n} v_n) \ast_{\delta_2} \cdots \ast_{\delta_m} (u_m \ast_{\eta_1} v_1 \ast_{\eta_2} v_2 \ast_{\eta_3} \cdots \ast_{\eta_n} v_n)$$

where $u_i, v_j \in Q$ and $\delta_i, \eta_j \in \{-1, 1\}$. 
Proof of Theorem 2.20. Since $S$ generates $G$, it follows from [11, Proposition 3.2] that $S$ also generates the Dehn quandle $\mathcal{D}(S^G)$. Note that relations (1), (2) and (3) (in the statement of the theorem) can be written respectively as follows:

\begin{align*}
(2.2.7) \quad s\alpha(s,t)s^{-1} &= t \quad \text{if } s \leq L t, \\
(2.2.8) \quad f(s,t)\beta(s,t)f(s,t)^{-1} &= s, \\
(2.2.9) \quad sf(s,t)uf(s,t)^{-1}s^{-1} &= tf(t,s)uf(t,s)^{-1}t^{-1}
\end{align*}

for $s, t, u \in S$ with $s \neq t$. By remarks 2.11 and 2.12, relations (2.2.7) and (2.2.8) are in fact relations in $\mathcal{D}(S^G)$, and hence so are relations (1) and (2). Also, since $sf(s,t) = G tf(t,s)$ for any $(s,t) \in S \times S$. (2.2.9) is a relation in $\mathcal{D}(S^G)$, and hence so is a relation as in (3). By Theorem 2.14 together with Lemma 2.21, it is sufficient to prove that relations in $\mathcal{D}(S^G)$ corresponding to transformations $T_a, T_{f,\beta}, T_0$ and $T_f$ can be obtained by relations (1), (2) and (3). Note that a relation corresponding to $T_0$ is a trivial relation. Also, a relation corresponding to $T_a$ or $T_{f,\beta}$ is trivial if $s = t$, and a relation corresponding to $T_f$ is trivial if $u = v$. The remaining relations can be written as follows:

\begin{align*}
(2.2.10) \quad \alpha(s,t) &= s \delta 1 u_1 \delta 2 u_2 \delta 3 \ldots \delta m u_m = t \delta 1 u_1 \delta 2 u_2 \delta 3 \ldots \delta m u_m \quad \text{if } s \leq L t, \\
(2.2.11) \quad \beta(s,t) &= f_{ns}(s,t) f_{ns-1}(s,t) \ldots f_1(s,t) \delta 1 u_1 \delta 2 u_2 \delta 3 \ldots \delta m u_m \\
&= s \delta 1 u_1 \delta 2 u_2 \delta 3 \ldots \delta m u_m, \\
(2.2.12) \quad u_0 \delta 1 u_1 \delta 2 \ldots \delta m u_m &= f_{ns}(s,t) f_{ns-1}(s,t) \ldots f_1(s,t) s \epsilon_1 v_1 \epsilon_2 v_2 \epsilon_3 \ldots \epsilon_n v_n \\
&= u_0 \delta 1 u_1 \delta 2 \ldots \delta m u_m f_{ns-1}(t,s) f_{ns-1}(t,s) \ldots f_1(t,s) t \epsilon_1 v_1 \epsilon_2 v_2 \epsilon_3 \ldots \epsilon_n v_n
\end{align*}

for $s, t, u, v_i \in S$ with $s \neq t$ and $\delta_i, \epsilon_i \in \{-1, 1\}$. Here, note that (2.2.10), (2.2.11) and (2.2.12) are non-trivial relations corresponding to transformations $T_a$, $T_{f,\beta}$, $T_0$ and $T_f$, respectively. Relations (2.2.10) and (2.2.11) can be obtained by relations (1) and (2), respectively. By right cancellation, (2.2.12) takes the form

\begin{align*}
(2.2.13) \quad u_0 \delta 1 u_1 \delta 2 \ldots \delta m u_m &= f_{ns}(s,t) f_{ns-1}(s,t) \ldots f_1(s,t) s \\
&= u_0 \delta 1 u_1 \delta 2 \ldots \delta m u_m f_{ns-1}(t,s) f_{ns-1}(t,s) \ldots f_1(t,s) t.
\end{align*}

It follows from Lemma 2.22 that (2.2.13) can be written as

\begin{align*}
(2.2.14) \quad v_0 \delta 1 v_1 \delta 2 \ldots \delta m v_m &= w_0 \delta 1 w_1 \delta 2 \ldots \delta m w_m,
\end{align*}

where

\begin{align*}
v_i &= u_i f_{ns}(s,t) f_{ns-1}(s,t) \ldots f_1(s,t) s \\
w_i &= u_i f_{ns}(t,s) f_{ns-1}(t,s) \ldots f_1(t,s) t.
\end{align*}

It follows from relations as in (3) that $v_i = w_i$ for each $i$. Thus (2.2.14) can be obtained by relations as in (3), and hence so are (2.2.13) and (2.2.12). This completes the proof of the theorem.

By Proposition 2.10, theorems 2.14 and 2.20 are also true for pairs of types $\mathcal{R}_2$ through $\mathcal{R}_9$. For $i \in \{4, 5, 6, 7\}$, let $(M, A)$ be a pair of type $\mathcal{R}_i$. Then, for $a, b \in A$, $a \leq L b$ if and only if $a = b$ (see Remark 2.22). Thus, we have the following corollary of Theorem 2.20.

**Corollary 2.23.** For $i \in \{4, 5, 6, 7\}$, let $M$ be a right Gaussian monoid of type $\mathcal{R}_i$. Let $A$ be the set of atoms in $M$ and $\beta : A \times A \to A$ the map defined by $(a,b) \mapsto (a \land b) \lor (a \lor b)$. Let $G$ be the right Gaussian group corresponding to $M$ and $f$ a right l.c.m. selector on $A$ in $M$. For $(a,b) \in A \times A$ with $a \neq b$, let $f_1(a,b), f_2(a,b), \ldots, f_{n_{ab}}(a,b)$ be elements in $A$ such that $f(a,b) = f_{n_{ab}}(a,b)$. Then, for $i \in \{4, 5, 6, 7\}$, let $\mathcal{D}(i)$ be the Dehn quandle of $\mathcal{R}_i$. Then, $\mathcal{D}(i)$ is a right Gaussian monoid of type $\mathcal{R}_i$. Moreover, let $\mathcal{D}(i, a, b)$ be the Dehn quandle of $\mathcal{R}_i$ corresponding to $(a,b)$. Then, $\mathcal{D}(i, a, b)$ is a right Gaussian monoid of type $\mathcal{R}_i$.
Let $f_1(a,b)f_2(a,b)\cdots f_{n_{ab}}(a,b)$. Then, the right Gaussian quandle $D(A^G)$ has a presentation with $A$ as its set of generators and defining relations as follows:

1. $\beta(a,b) * f_{n_{ab}}(a,b) * f_{n_{ab}-1}(a,b) * \cdots * f_1(a,b) = a$,
2. $c * f_{n_{ab}}(a,b) * f_{n_{ab}-1}(a,b) * \cdots * f_1(a,b) * a = c * f_{n_{ba}}(b,a) * f_{n_{ba}-1}(b,a) * \cdots * f_1(b,a) * b$

for $a,b,c \in A$ with $a \neq b$.

For $i \in \{8,9\}$, let $(N,B)$ be a pair of type $R_i$. Then, for $a,b \in B$, $a \leq_L b$ if and only if $a = b$. Thus, we have the following corollary.

**Corollary 2.24.** For $i \in \{8,9\}$, let $N$ be a Garside monoid of type $R_i$. Let $B$ be the set of atoms in $N$ and $\beta : B \times B \to B$ the map defined by $(a,b) \mapsto (a\setminus b)\setminus (a \lor b)$. Let $H$ be the Garside group corresponding to $N$ and $f$ a right l.c.m. selector on $B$ in $N$. For $(a,b) \in B \times B$ with $a \neq b$, let $f_1(a,b), f_2(a,b), \ldots, f_{n_{ab}}(a,b)$ be elements in $B$ such that $f(a,b) = f_1(a,b)f_2(a,b)\cdots f_{n_{ab}}(a,b)$. Then, the Garside quandle $D(B^H)$ has a presentation with $B$ as its set of generators and defining relations as:

1. $\beta(a,b) * f_{n_{ab}}(a,b) * f_{n_{ab}-1}(a,b) * \cdots * f_1(a,b) = a$,
2. $c * f_{n_{ab}}(a,b) * f_{n_{ab}-1}(a,b) * \cdots * f_1(a,b) * a = c * f_{n_{ba}}(b,a) * f_{n_{ba}-1}(b,a) * \cdots * f_1(b,a) * b$

for $a,b,c \in A$ with $a \neq b$.

### 2.3. Presentations of left Gaussian quandles and Garside quandles of certain types.

This subsection is a left analogue of the preceding section and we present it without details. We set the following notations:

- $M$ - a left Gaussian monoid;
- $S$ - a finite generating set for $M$;
- $A$ - the set of atoms in $M$;
- $N$ - a Garside monoid;
- $T$ - a finite generating set for $N$;
- $B$ - the set of atoms in $N$;
- $\Delta$ - a Garside element in $N$;
- $(M, S), (M, A), (N, T), (N, B), (N, \Delta)$ and $(N, T, \Delta)$ - pairs and a triple of objects with the meaning above.

For a pair $(M, S)$, we assume throughout this subsection that elements in $S$ are pairwise distinct in $M$. The same is assumed in case of a pair $(N, T)$. For elements $x$ and $y$ in a left Gaussian monoid, denote the right g.c.d. and the left l.c.m. of $x$ and $y$ respectively by $x\land y$ and $x\lor y$. The *left residue* of $x$ in $y$ (denoted by $y/x$) is the unique element $z$ satisfying $x\lor y = zx$. Thus, we have $x\lor y = (y/x)x = (x/y)y$.

Consider the following conditions on a pair $(M, S)$:

1. $t(s/t) \in S$ whenever $(s, t) \in S \times S$ and $t \leq_R s$.
2. $s/t \leq_R s\lor t$ for all $(s, t) \in S \times S$.
3. $(s\lor t)/(s/t) \in S$ for all $(s, t) \in S \times S$.
4. $M$ has a finite homogeneous presentation $\langle S \mid R \rangle$.
5. For each $s \in S$ and each $x \in M$, there exists $y = y(s, x)$ in $M$ such that $yx \leq_R yxs$.

Define the following terms:

- A pair $(M, S)$ is of
  - type $L_1$ if it satisfies conditions (vi), (vii), (viii) and (ix).
- type \( \mathcal{L}_2 \) if it satisfies conditions (vi), (vii), and (viii), and if there exist a triple \((N, T, \Delta)\) with \(T\Delta = \Delta T\) and an epimorphism \(\pi : (N, T) \rightarrow (M, S)\).
- A pair \((N, T)\) is of type \( \mathcal{L}_3 \) if it satisfies conditions (vi), (vii), and (viii), and there exists a Garside element \(\Delta \in N\) such that \(T\Delta = \Delta T\).
- A pair \((M, A)\) is of
  - type \( \mathcal{L}_4 \) if it satisfies conditions (vi), (vii), and (ix).
  - type \( \mathcal{L}_5 \) if it satisfies conditions (viii), (ix), and (x).
  - type \( \mathcal{L}_6 \) if it satisfies conditions (vii) and (ix), and if there exists a triple \((N, T, \Delta)\) with \(T\Delta = \Delta T\) and an epimorphism \(\pi : (N, T) \rightarrow (M, A)\).
- A pair \((N, B)\) is of
  - type \( \mathcal{L}_7 \) if it satisfies conditions (vi), (vii) and (viii), and if there exists a triple \((N, T, \Delta)\) with \(T\Delta = \Delta T\) and an epimorphism \(\pi : (N, T) \rightarrow (M, A)\).

For \(i = 4, 5, 6, 7\), we say a left Gaussian monoid \(M\) is of type \( \mathcal{L}_i \) if the pair \((M, A)\) is of type \( \mathcal{L}_i \).

In this case, we also say that the left Gaussian group \(G\) corresponding to \(M\) and the left Gaussian quandle \(\mathcal{D}(A^G)\) are of type \( \mathcal{L}_1 \). Similarly, for \(i = 8, 9\), we say a Garside monoid \(N\) is of type \( \mathcal{L}_i \) if the pair \((N, B)\) is of type \( \mathcal{L}_i \). In this case, we also say that the Garside group \(H\) corresponding to \(N\) and the Garside quandle \(\mathcal{D}(B^H)\) are of type \( \mathcal{L}_i \).

The next proposition is analogous to Proposition 2.10.

**Proposition 2.25.** Pairs of types \( \mathcal{L}_2 \) through \( \mathcal{L}_9 \) are of type \( \mathcal{L}_1 \).

Let \((M, S)\) be a pair that satisfies condition (vi). Then, we have a map
\[
\alpha : \{(s, t) \in S \times S \mid t \leq_R s\} \rightarrow S
\]
defined by
\[
(2.3.1) \quad (s, t) \mapsto t(s/t).
\]
Note that \(\alpha(s, s) = s\) for all \(s \in S\).

**Remark 2.26.** Let \(s, t \in S\) be such that \(t \leq_R s\). Then \(s = (s/t)t\) and \(\alpha(s, t) = t(s/t)\). Suppose \(f\) be a left l.c.m. selector on \(S\) in \(M\). Since \(f(s, t)\) represents the element \(s/t\), we have \(s =_M f(s, t)t\) and \(\alpha(s, t) =_M tf(s, t)\). Thus \(tst^{-1} =_G tf(s, t) =_G \alpha(s, t)\), where \(G\) is the left Gaussian group corresponding to \(M\).

Let \((M, S)\) be a pair that satisfies conditions (vii) and (viii). Then, we have a map \(\beta : S \times S \rightarrow S\) defined by
\[
(2.3.2) \quad (s, t) \mapsto (s\tilde{v}t)/(s/t).
\]
Note that \(\beta(s, s) = s\) for all \(s \in S\).

**Remark 2.27.** Let \(s\) and \(t\) be elements in \(S\). Then \(s\tilde{v}t = ((s\tilde{v}t)/(s/t))(s/t)\). In other words, \((s/t)t = ((s\tilde{v}t)/(s/t))(s/t)\). Suppose \(f\) be a left l.c.m. selector on \(S\) in \(M\). Since \(f(s, t)\) represents the element \(s/t\) and \(\beta(s, t) = (s\tilde{v}t)/(s/t)\), we have \(f(s, t)t =_M \beta(s, t)f(s, t)\). Thus \(f(s, t)f(s, t)^{-1} =_G \beta(s, t)\), where \(G\) is the left Gaussian group corresponding to \(M\).

The next theorem is analogous to Theorem 2.20.

**Theorem 2.28.** Suppose \((M, S)\) be a pair of type \( \mathcal{L}_1 \). Let \(\alpha\) and \(\beta\) be maps defined by (2.3.1) and (2.3.2), respectively. Suppose \(G\) be the left Gaussian group corresponding to \(M\) and \(f\) a left l.c.m.
selector on $S$ in $M$. For $(s,t) \in S \times S$ with $s \neq t$, let $f_1(s,t), f_2(s,t), \ldots, f_{n_{st}}(s,t)$ be elements in $S$ such that $f(s,t) = f_1(s,t)f_2(s,t) \cdots f_{n_{st}}(s,t)$. Then, the Dehn quandle $\mathcal{D}(S^G)$ has a presentation with $S$ as its set of generators and defining relations as:

1. $s \ast t = \alpha(s,t)$ if $t \leq_R s$,
2. $t \ast f_{st}(s,t) \ast f_{n_{st}-1}(s,t) \ast \cdots \ast f_1(s,t) = \beta(s,t)$,
3. $u \ast t \ast f_{st}(s,t) \ast f_{n_{st}-1}(s,t) \ast \cdots \ast f_1(s,t) = u \ast s \ast f_{ts}(t,s) \ast f_{n_{ts}-1}(t,s) \ast \cdots \ast f_1(t,s)$

for $s,t,u \in S$ with $s \neq t$.

In view of Proposition 2.25, Theorem 2.28 is also true for pairs of types $\mathcal{L}_2$ through $\mathcal{L}_9$. Further, we can deduce corollaries of Theorem 2.28 analogous to corollaries 2.23 and 2.24.

### 2.4. Examples of presentations of Garside quandles

Theorem 2.20 and Corollary 2.24 can be used to write presentations of certain Garside quandles.

**Example 2.29.** Let $A$ be an Artin group with Artin presentation

$$\langle S \mid (s)_{mst} = (ts)_{mts} \text{ for } s,t \in S \text{ with } s \neq t \rangle,$$

where $S$ is a finite set, $(uv)_k$ denotes the word $uvuvu \cdots$ of length $k$, $mst = mts \geq 2$ for all $s \neq t$ and $mst = \infty$ if there is no relation. Let $M$ be the monoid with the same presentation as above, i.e. $M$ is the quotient of the free monoid $S^*$ by relations $(st)_{mst} = (ts)_{mts}$ for all $s \neq t$. We refer to such a monoid as an *Artin monoid* and call it of *spherical type* if the corresponding Artin group is of spherical type. Further, we call the Dehn quandle $\mathcal{D}(S^A)$ as an *Artin quandle* and say that it is of *spherical type* if the Artin group $A$ is of spherical type. Note that, in these definitions, $S$ is an Artin generating set for $A$.

For the further discussion, we assume that $M$ is a spherical Artin monoid. It follows from [5] (see also [10, Example 1]) that $M$ is a Garside monoid. Note that, for $s \neq t$, the words $(st)_{mst}$ and $(ts)_{mts}$ both represent the right l.c.m. of $s$ and $t$. It is easy to see that the pair $(M,S)$ satisfies conditions [11] and [14]. By Lemma 2.3 the set $S$ is the set of atoms in $M$. Thus, $M$ is a Garside monoid of type $\mathcal{R}_9$, and hence the Artin quandle $\mathcal{D}(S^A)$ is a Garside quandle of type $\mathcal{R}_9$. Again by Lemma 2.3 the pair $(M,S)$ satisfies condition [11], and thus we have the map $\beta : S \times S \to S$ defined by $(s,t) \mapsto (s,t)(s \vee t)$. Let $f : S \times S \to S^*$ be the map defined by $f(s,t) = \epsilon$ if $s = t$ and $f(s,t) = (ts)_{mu - 1}$ if $s \neq t$, where $\epsilon$ is the empty word. For $s \neq t$, since $s(ts)_{mu - 1} = (st)_{mst}$ represents $s(s^{-1}) = s \vee t$, the word $(ts)_{mu - 1}$ must represent $s^{-1}$. Thus, $f$ is a right l.c.m. selector on $S$ in $M$. For $s \neq t$ and $i = 0,1,\ldots,mst$, let $f_i(s,t) = s$ if $i$ is even and $f_i(s,t) = t$ if $i$ is odd. Then, for any $s \neq t$ and any $i = 1,2,\ldots,mst$, we have $f_i(s,t) = f_{i-1}(t,s)$ and $f(s,t) = f_1(s,t)f_2(s,t) \cdots f_{mst - 1}(s,t)$. Let $s \neq t$ be elements in $S$. Then

$$f(s,t)f_{mst}(s,t) = (ts)_{mst - 1}f_{mst}(s,t)$$

$$= (ts)_{mst - 1}f_{mst - 1}(t,s)$$

$$= tf_{1}(t,s)f_{2}(t,s) \cdots f_{mst - 1}(t,s)$$

$$= tf(t,s)$$

$$= t(ts)_{mst - 1}$$

$$= (ts)_{mst}.$$
represents \((s\setminus t) \cup (s \cup t)\). Thus, we have \(\beta(s, t) = f_{mst}(s, t)\). By Corollary 2.24, the Artin quandle \(D(S^A)\) has a presentation with \(S\) as its set of generators and defining relations as follows:

\[
\begin{align*}
(2.4.2) & \quad f_{mst}(s, t) * f_{mst-1}(s, t) * \cdots * f_1(s, t) = f_0(s, t) \quad \text{and} \\
(2.4.3) & \quad u * f_{mst-1}(s, t) * f_{mst-2}(s, t) * \cdots * f_1(s, t) = f_0(s, t) \\
& \quad = u * f_{mst-1}(s, t) * f_{mst-2}(t, s) * \cdots * f_1(t, s) * f_0(t, s)
\end{align*}
\]

for \(s, t, u \in S\) with \(s \neq t\) (note that \(s = f_0(s, t)\) and \(t = f_0(t, s)\)). We shall show that \((2.4.3)\) can be obtained by \((2.4.2)\). Let \(s, t, u \in S\) with \(s \neq t\). Then, by Corollary 2.24, it can be seen that, for each \(f\), \(\beta(x, y) = \beta(x, y)\) is redundant. Hence, if \(A\) is spherical, then the Artin quandle \(D(S^A)\) has a presentation

\[
D(S^A) = \left\langle S \mid (s * t)_{mst} = s \quad \text{if} \ mst \ \text{is even, and} \right.

\[
(t * s)_{mst} = s \quad \text{if} \ mst \ \text{is odd for all} \ s \neq t \right\},
\]

where \((u * v)_k\) denotes the left associated product \(u * v * u * v * \cdots\) of length \(k\).

**Example 2.30.** Let \(M\) be the Garside monoid as in Example 2.25 and \(G\) its group of fractions. The set \(S = \{x_1, x_2, \ldots, x_n\}\) is the set of atoms in \(M\) (see the proof of Proposition 2.3, i.e. 10 Proposition 5.2). It can be seen that, for each \(i \neq j\), the word \(x_i^p\) represents \(x_i \cup x_j\). Let \(f(x_i, x_i) = 0\) for \(i = 1, 2, \ldots, n\) and \(f(x_i, x_j) = x_i^{p_{i,j}}\) for \(i \neq j\), where \(\epsilon\) is the empty word. Then, \(f\) is a right l.c.m. selector on \(S\) in \(M\). Let \(\beta(x_i, x_j) = x_i\) for any \(i\) and \(j\). Then \(x_i f(x_i, x_j) = f(x_i, x_j) \beta(x_i, x_j)\) which, of course, represents \(x_i \cup x_j\). In other words, \(\beta(x_i, x_j) = (x_i \setminus x_j) \setminus (x_i \cup x_j)\). Then, by Corollary 2.24, the Garside quandle \(D(S^G)\) has a presentation

\[
D(S^G) = \langle x_1, x_2, \ldots, x_n \mid x_i \beta^p \quad x_j = x_i \quad \text{for all} \ i \neq j \rangle.
\]

**Example 2.31.** Let \(M\) be the Garside monoid as in Example 2.6 and \(G\) its group of fractions. The set \(S = \{x_1, x_2, \ldots, x_n\}\) is the set of atoms in \(M\). For an integer \(i\), let \(x_i = x_j\) for \(1 \leq j \leq n\) such that \(i \equiv j \pmod{n}\). One can see that, for each \(i \neq j\), the word \(x_i x_{i+1} \cdots x_{i+m-1}\) represents \(x_i \cup x_j\). Let \(f(x_i, x_i) = 0\) for \(i = 1, 2, \ldots, n\) and \(f(x_i, x_j) = x_i^{p_{i,j}} \cdots x_{i+m-2} \cdots x_{j-1}\) for \(i \neq j\), where \(\epsilon\) is the empty word. Then, \(f\) is a right l.c.m. selector on \(S\) in \(M\). Let \(\beta(x_i, x_i) = x_i\) for \(i = 1, 2, \ldots, n\) and \(\beta(x_i, x_j) = x_i x_{i+m}\) for \(i \neq j\). Then \(x_i f(x_i, x_j) = f(x_i, x_j) \beta(x_i, x_j)\), i.e. \(\beta(x_i, x_j) = (x_i \setminus x_j) \setminus (x_i \cup x_j)\). Then, by Corollary 2.24, the Garside quandle \(D(S^G)\) is generated by \(S\) and has defining relations

\[
x_i x_{j+m} = x_{i+1} \cdots x_i x_{j+m-1} \cdots x_{j+1} = x_i x_{j+m-1} x_{j+m-2} \cdots x_{j+1}
\]

for all \(i, j, m = 1, 2, \ldots, n\) with \(j \neq k\). Note that relations of second type are redundant. For, we have

\[
x_i x_{j+m-1} x_{j+m-2} \cdots x_{j+1} = (x_i x_{j+m-1} x_{j+m-2} \cdots x_{j+1}) x_{j+m} x_{j+m-1} \cdots x_{j+1}
\]
for all $i, j = 1, 2, \ldots, n$, where the first equality follows from relations of first type and the second follows from Lemma 2.22. Iterating the process yields relations of second type, and hence the final presentation is

$$D(S^G) = \langle x_1, x_2, \ldots, x_n \mid x_{i+m} * x_{i+m-1} * x_{i+m-2} * \cdots * x_{i+1} = x_i \text{ for } i = 1, 2, \ldots, n \rangle.$$  

**Remark 2.32.** It follows from [35, Proposition 7.1] that the link quandle $Q(T(n, m))$ of the torus link $T(n, m)$ has a presentation

$$Q(T(n, m)) = \langle x_1, x_2, \ldots, x_n \mid x_{m+i} * x_m * x_{m-1} * \cdots * x_1 = x_i \text{ for } i = 1, 2, \ldots, n \rangle,$$

where $x_l = x_k$ for $l \in \mathbb{Z}$ and $1 \leq k \leq n$ such that $l \equiv k \pmod{n}$. Using induction on $j$ and relations $x_{m+i} * x_m * x_{m-1} * \cdots * x_1 = x_i$, we can obtain relations $x_{m+i} * x_{m+j} * x_{m+j-1} * \cdots * x_{j+1} = x_i$ for $i, j = 1, 2, \ldots, n$. Further, it suffices to consider only $x_{m+i} * x_{m+i-1} * x_{m+i-2} * \cdots * x_{i+1} = x_i$ among latter relations. Thus, $Q(T(n, m))$ is isomorphic to the Garside quandle in Example 2.31.

**Example 2.33.** Let $M$ be the Garside monoid as in Example 2.8 and $G$ be its group of fractions. The set $S = \{x_1, x_2, y_1, y_2, y_3\}$ is the set of atoms in $M$. Let $p_1 = 2$ and $p_2 = 5$. For an integer $i$, let $y_i = y_j$ for $1 \leq j \leq 3$ such that $i \equiv j \pmod{3}$. Using Corollary 2.24 and after reducing the relations, we get a presentation

$$D(S^G) = \left\langle x_1, x_2, y_1, y_2, y_3 \mid y_{k+4} * y_{k+3} * y_{k+2} * y_{k+1} = y_k, \ x_i * y_i = y_i \text{ for } i = 1, 2, 3 \right\rangle.$$  

**Example 2.34.** Let $M$ be the Garside monoid as in Example 2.9 and $G$ its group of fractions. The set $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ is the set of atoms in $M$. Let $\Delta_1$ and $\Delta_2$ be minimal Garside elements in underling Artin monoids $M_1$ and $M_2$ of type $B_3$ and $A_3$, respectively. Then, the words $(x_1 x_2 x_3)^3$, $(x_2 x_3 x_1)^3$ and $(x_3 x_1 x_2)^3$ all represent $\Delta_1$, and the words $(y_1 y_2 y_3 y_1 y_2 y_1)$, $(y_2 y_1 y_3 y_2 y_1)$ and $(y_3 y_1 y_2 y_3 y_2 y_1)$ all represent $\Delta_2$ (see [4] examples 3 and 4 for the minimal Garside elements in Artin monoids of spherical type). Note that $M = (M_1 * M_2) / \equiv$, where $\equiv$ is the equivalence relation on $M_1 * M_2$ generated by $\Delta_1^2 = \Delta_2^2$. One can see that both $\Delta_1^2$ and $\Delta_2^2$ represent the first $\Delta_i$ and the second $\Delta_i$ for $i = 1, 2, 3$. For an integer $i$, let $x_i = x_j$ and $y_i = y_j$ for $1 \leq j \leq 3$ such that $i \equiv j \pmod{3}$. Define $f(x_i, y_j) = x_{i+1} x_{i+2} \cdots x_{i+17}, f(y_k, x_j) = y_{k+17} y_{k+16} \cdots y_{k+1} x_j$ and $f(y_3, x_j) = y_1 y_2 \cdots x_{i+17}$ for $i, j = 1, 2, 3$. Then $f$ can be extended to a right l.c.m. selector on $S$ in $M$. For $i = 1, 2, 3$, let $\psi_i$ be the permutation of the set of divisors of $\Delta_i$ given by $x \mapsto x \setminus \Delta_i$, and $\phi_i$ be the automorphism of $M_i$ defined by $\phi_i(x) = \psi_i^2(x)$ for a divisor $x$ of $\Delta_i$ (see [3] lemmas 2.2 and 2.3). One can verify that $x \psi_i(x) = \Delta_1 = \psi_1(x) \phi_i(x)$ and $x \Delta_1 = \Delta_1 \phi_i(x)$ for a divisor $x$ of $\Delta_i$. Note that $\phi_i$ maps atoms to atoms. Let $z_1 = y_3, z_2 = y_2$ and $z_3 = y_1$. It is easy to see that $\phi_1(x_i) = x_i$ and $\phi_2(y_i) = z_i$ for $i = 1, 2, 3$. Thus

$$x_i \psi_1(x_i) \Delta_1 = \Delta_1^2 = \psi_1(x_i) \Delta_1 x_i$$

and

$$y_i \psi_2(y_i) \Delta_2^2 = \Delta_2^3 = \psi_2(y_i) \Delta_2^2 z_i$$

for $i = 1, 2, 3$. Define $\beta(x_i, y_j) = x_i$ and $\beta(y_j, x_i) = z_j$ for $i, j = 1, 2, 3$. Then $\beta(x_i, y_j) = (\psi_1(x_i) \Delta_1) \Delta_1^2 = (x_i \setminus y_j) \setminus (x_i \setminus y_j)$ and $\beta(y_j, x_i) = (\psi_2(y_j) \Delta_2^2) \Delta_2^3 = (y_j \setminus x_i) \setminus (y_j \setminus x_i)$ for all $i, j = 1, 2, 3$. Note that $\beta$ can be extended to the map $S \times S \to S$ defined as in (2.22). Using
Corollary 2.24 and after reducing the relations, we see that \( \mathcal{D}(S^G) \) has a presentation
\[
\left\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid x_1 \star x_2 \star x_1 = x_1 \star x_3 = x_1, \quad x_2 \star x_1 \star x_2 \star x_1 = x_3 \star x_2 \star x_3 = x_2, \\
x_3 \star x_1 = x_2 \star x_3 \star x_2 = x_3, \quad y_2 \star y_1 \star y_2 = y_1 \star y_3 = y_1, \quad y_1 \star y_2 \star y_1 = y_3 \star y_2 \star y_3 = y_2, \\
y_3 \star y_1 = y_2 \star y_3 \star y_2 = y_3, \quad x_{i+18} \star x_{i+17} \star \cdots \star x_{i+1} = x_i, \quad y_1 \star y_2 \star \cdots \star y_{12} = y_1, \\
y_2 \star y_3 \star \cdots \star y_{19} = y_2, \quad y_12 \star y_{11} \star \cdots \star y_1 = y_3, \quad x_i \star y_2 \star y_3 \star \cdots \star y_{19} = x_i, \\
y_1 \star x_{18} \star x_{17} \star \cdots \star x_1 = y_1 \star y_2 \star \cdots \star y_7, \quad y_2 \star x_{18} \star x_{17} \star \cdots \star x_1 = y_2, \\
y_3 \star x_{18} \star x_{17} \star \cdots \star x_1 = y_1 \text{ for } i = 1, 2, 3 \right\rangle.
\]

3. Presentations of Dehn quandles of groups

In this section, we prove a general result giving presentations of Dehn quandles of groups when the centraliser of each generator is known. Although the result is general, determining generating sets for centralisers of elements in interesting classes of groups like Garside groups and Artin groups is usually challenging. See, for example, [11, 15, 16, 18, 20, 29, 33] for related works. Further, presentations obtained for Garside quandles using Theorem 3.1 usually have larger number of relations than the one given by Theorem 2.20. We shall see many examples later in this section.

3.1. Presentations of Dehn quandles. The following theorem gives a presentation of the Dehn quandle of a group \( G \) with respect to a generating set \( S \) when a generating set for the centraliser of each element in \( S \) is known.

**Theorem 3.1.** Let \( G \) be a group with a presentation \( \langle S \mid R \rangle \). For \( s \in S \), let \( A_s \) be a generating set for the centraliser \( C_G(s) \) of \( s \) in \( G \). Let \( T = \{ (s, t) \in S \times S \mid s \text{ and } t \text{ are conjugate in } G \} \), and \( f : T \rightarrow G \) a map such that
- \( f(t, s)sf(t, s)^{-1} = t \),
- \( f(t, s) = f(s, t)^{-1} \),
- \( f(u, t)f(t, s) = f(u, s) \)

for all \( (s, t), (t, u) \in T \). Then, the Dehn quandle \( \mathcal{D}(S^G) \) has a presentation with the generating set \( S \) and defining relations as follows:

1. For each \( s \in S \) and each relation \( s = s_{r_1} \delta_1 \delta_2 \cdots \delta_k \) in \( R \), where \( s_{r_i} \in S \) and \( \delta_i \in \{-1, 1\} \), we have
   \[ s \star \delta_k \star s_{r_k-1}^{-1} \star \delta_{k-1}^{-1} \star \cdots \star \delta_1^{-1} \star s_{r_1} = s. \]

2. For each \( s \in S \) and each \( w = s_{\epsilon_1}^{l_1} s_{\epsilon_2}^{l_2} \cdots s_{\epsilon_j}^{l_j} \) in \( A_s \), where \( s_{\epsilon_i} \in S \) and \( \epsilon_i \in \{-1, 1\} \), we have
   \[ s \star \epsilon_i \star s_{\epsilon_i-1} \star s_{\epsilon_i-2} \star \cdots \star s_{\epsilon_1}^{-1} = s. \]

3. For each \( (s, t) \in T \) such that \( f(t, s) = f_1(t, s)^{\mu_1} f_2(t, s)^{\mu_2} \cdots f_n(t, s)^{\mu_n} \), where \( f_i(t, s) \in S \) and \( \mu_i \in \{-1, 1\} \), we have
   \[ s \star \mu_n f_n(t, s) \star \mu_{n-1}^{-1} \star f_{n-1}(t, s) \star \mu_{n-2}^{-1} \star \cdots \star \mu_1^{-1} f_1(t, s) = t. \]

**Proof.** Let \( \psi : S \rightarrow S \) be a bijection of \( S \) onto another set \( S \), where, for brevity, we denote \( \psi(s) \) by \( S \). Let \( (Q, \star) \) be a quandle that has a presentation with the set of generators \( S \) and defining relations as in (1), (2) and (3) written in terms of elements of \( S \). We write elements of \( Q \) in bold to differentiate them from elements of \( \mathcal{D}(S^G) \). We claim that \( \mathcal{D}(S^G) \cong Q \).

Let \( \phi : S \rightarrow S \) given by \( \phi(s) = s \) be the inverse of \( \psi \). The map \( \phi \) induces a quandle homomorphism \( \phi : Q \rightarrow \mathcal{D}(S^G) \), which is clearly surjective. It suffices to show that \( \phi \) is injective.
We first claim that if \( x = s_{x_1}^{\eta_1} s_{x_2}^{\eta_2} \cdots s_{x_n}^{\eta_n} \) and \( y = s_{y_1}^{\theta_1} s_{y_2}^{\theta_2} \cdots s_{y_m}^{\theta_m} \), where \( s_{x_i}, s_{y_j} \in S \) and \( \eta_i, \theta_j \in \{1, -1\} \), represent the same element of \( G \), then
\[
s \star s_{x_n}^{\eta_n} \star s_{x_{n-1}}^{\eta_{n-1}} \star \cdots \star s_{x_1}^{\eta_1} = s \star s_{y_m}^{\theta_m} \star s_{y_{m-1}}^{\theta_{m-1}} \star \cdots \star s_{y_1}^{\theta_1}
\]
for all \( s \in S \).

Since \( x = y \) in \( G \), there is a sequence of elements \( x = g_0, g_1, g_2, \ldots, g_k = y \) in \( G \) such that \( g_i \) and \( g_{i+1} \) differ by a single relation from \( R \cup R' \), where \( R' = \{s^{-1} s, ss^{-1} \mid s \in S \} \) is the set of trivial relations. Thus, it is enough to consider the case when \( x \) and \( y \) differ by a single relation i.e. when \( x = s_{x_1}^{\eta_1} s_{x_2}^{\eta_2} \cdots s_{x_n}^{\eta_n} \) and \( y = s_{x_1}^{\eta_1} s_{x_2}^{\eta_2} \cdots s_{x_n}^{\eta_n} s_{x_{i+1}}^{\eta_{i+1}} \cdots s_{x_k}^{\eta_k} \) for some \( r \in R \cup R' \). If \( r \in R' \), then the claim holds by the second quandle axiom (i.e. bijectivity of right multiplication). Suppose that \( r = s_{r_1}^{\delta_{r_1}} s_{r_2}^{\delta_{r_2}} \cdots s_{r_k}^{\delta_{r_k}} \) is an element of \( R \). Then, Lemma \[2.22\] gives
\[
\begin{align*}
&= (s_{x_1}^{s_{x_1}^{\delta_{r_1}} r_{r_1}} s_{x_1}^{s_{x_2}^{\delta_{r_2}} r_{r_2}} \cdots s_{x_n}^{s_{x_n}^{\delta_{r_n}} r_{r_n}}) s_{x_1}^{s_{x_1}^{\delta_{r_1}} r_{r_1}} \cdots s_{x_n}^{s_{x_n}^{\delta_{r_n}} r_{r_n}}.
\end{align*}
\]
This proves the claim.

Now, to prove injectivity of \( \phi_i \), take two elements \( s \star \alpha_p s_{a_p} \star \alpha_{p-1} s_{a_{p-1}} \star \cdots \star \alpha_1 s_{a_1} \) and \( t \star \beta_q s_{b_q} \star \beta_{q-1} s_{b_{q-1}} \star \cdots \star \beta_1 s_{b_1} \) in \( Q \) such that
\[
(3.1.1) \quad s \star \alpha_p s_{a_p} \star \alpha_{p-1} s_{a_{p-1}} \star \cdots \star \alpha_1 s_{a_1} = t \star \beta_q s_{b_q} \star \beta_{q-1} s_{b_{q-1}} \star \cdots \star \beta_1 s_{b_1}
\]
in \( \mathcal{D}(S(G)) \). Rewriting \( (3.1.1) \) gives
\[
(3.1.2) \quad s \star \alpha_p s_{a_p} \star \alpha_{p-1} s_{a_{p-1}} \star \cdots \star \alpha_1 s_{a_1} - \beta_1 s_{b_1} - \beta_2 s_{b_2} - \beta_3 \cdots - \beta_q s_{b_q} = t.
\]
Since \( s \) and \( t \) are conjugate in \( G \), we have \( (s, t) \in T \). Thus, we have
\[
(3.1.3) \quad s \star \mu_n f_n(t, s) \star \mu_{n-1} f_{n-1}(t, s) \star \mu_{n-2} \cdots \star \mu_1 f_1(t, s) = t,
\]
where \( f(t, s) \in G \) such that \( f(t, s) = f_1(t, s)^{\mu_1} f_2(t, s)^{\mu_2} \cdots f_n(t, s)^{\mu_n} \) for \( f_i(t, s) \in S \) and \( \mu_i \in \{1, -1\} \). Using \( (3.1.2) \) and \( (3.1.3) \), we have
\[
(3.1.4) \quad s = s \star \alpha_p s_{a_p} \star \alpha_{p-1} s_{a_{p-1}} \star \cdots \star \alpha_1 s_{a_1} \star \beta_1 s_{b_1} \star \beta_2 s_{b_2} \star \cdots \star \beta_q s_{b_q} \star -\mu_1 f_1(t, s) \star -\mu_2 f_2(t, s) \star -\mu_3 \cdots -\mu_n f_n(t, s).
\]
Writing \( (3.1.4) \) in terms of conjugation in \( G \) implies that the element
\[
-f_n(t, s)^{-\mu_n} f_{n-1}(t, s)^{-\mu_{n-1}} \cdots f_1(t, s)^{-\mu_1} -\beta_1 s_{b_1} - \beta_2 s_{b_2} - \cdots - \beta_q s_{b_q} \star \alpha_p s_{a_p} \star \alpha_{p-1} s_{a_{p-1}} \star \cdots \star \alpha_1 s_{a_1}
\]
commutes with \( s \), and hence equals to an element, say, \( w \) of \( C_G(s) \). Without loss of generality, we can assume that \( w \in A_v \). If not, then \( w \) will be a product of elements of \( A_v \) and the argument will be similar. Let \( w = s_{w_1}^{\delta_{w_1}} s_{w_2}^{\delta_{w_2}} \cdots s_{w_l}^{\delta_{w_l}} \) written in terms of generators \( S \) of \( G \). Then the equality
\[
\begin{align*}
\delta_{w_1} s_{w_1}^{\delta_{w_2}} s_{w_2}^{\delta_{w_3}} \cdots s_{w_l}^{\delta_{w_l}} &= f_n(t, s)^{-\mu_n} f_{n-1}(t, s)^{-\mu_{n-1}} \cdots f_1(t, s)^{-\mu_1} -\beta_1 s_{b_1} - \beta_2 s_{b_2} - \cdots - \beta_q s_{b_q} \star \alpha_p s_{a_p} \star \alpha_{p-1} s_{a_{p-1}} \star \cdots \star \alpha_1 s_{a_1}
\end{align*}
\]
of elements in \( G \) can be rewritten as
\[
(3.1.5) \quad s_{w_1}^{\delta_{w_2}} s_{w_2}^{\delta_{w_3}} \cdots s_{w_l}^{\delta_{w_l}} f_1(t, s)^{\mu_1} f_2(t, s)^{\mu_2} \cdots f_n(t, s)^{\mu_n} s_{w_1}^{\delta_{w_2}} s_{w_2}^{\delta_{w_3}} \cdots s_{w_l}^{\delta_{w_l}} = s_{a_1}^{\alpha_p} s_{a_2}^{\alpha_p} \cdots s_{a_p}^{\alpha_p}.
\]
Our earlier proved claim gives
\[ s^{\alpha_p} s_{\alpha_p}^{\alpha_p-1} s_{\alpha_p-1}^{\alpha_p-2} \cdots s_{\alpha_1} \]
\[ = s^{\partial_1} s_{w_1}^{\partial_1-1} s_{w_1-1}^{\partial_1-2} \cdots s_{w_1}^{\partial_1} f_n(t,s) s_{n-1}^{\mu_{n-1}} f_{n-1}(t,s) \cdots s_{n}^{\mu_1} f_1(t,s) \]
\[ \star s_{b_q}^{\beta_q} s_{b_q-1}^{\beta_q-2} \cdots s_{b_1} \]
\[ = s^{\mu_n} f_n(t,s) s_{n-1}^{\mu_{n-1}} f_{n-1}(t,s) \cdots f_1(t,s) s_{b_q}^{\beta_q} s_{b_q-1}^{\beta_q-2} \cdots s_{b_1} , \]
by relation of type (2)
\[ = t^{\beta_q} s_{b_q}^{\beta_q-1} s_{b_q-1}^{\beta_q-2} \cdots s_{b_1} , \] by relation of type (3).

This completes the proof of the theorem.

\[ \square \]

**Corollary 3.2.** If \( G = \langle S \mid R \rangle \) is a finitely presented group such that the centraliser of each generator from \( S \) is finitely generated, then the Dehn quandle \( D(S^G) \) is finitely presented.

It follows from [16, Theorem 12] that the centraliser of an element in a Garside group is finitely generated. Thus, the corollary holds for Garside groups, which includes spherical Artin groups.

**Remark 3.3.** One can reduce the number of relations in Theorem 3.1 as follows:

1. Relations of type (2) need to be checked only for those generators in \( S \) that represent distinct conjugacy classes in \( G \).

2. We write \( S = \bigsqcup X_i \) as a disjoint union of sets where \( X_i \) consists of elements that are conjugate to each other. We equip each \( X_i \) with a partial order such that the corresponding poset graph is a tree. Then, for each \( i \), relations of type (3) need to be checked only for elements in \( X_i \times X_i \) that are adjacent in the corresponding poset graph.

### 3.2. Examples of presentations of Dehn quandles.

As applications of Theorem 3.1, we give several examples of presentations of Dehn quandles.

**Example 3.4.** Let \( \text{Core} (\mathbb{Z}_n) \) be the core quandle of the cyclic group of order \( n \) (also called the dihedral quandle of order \( n \)). Then, the quandle operation in \( \text{Core} (\mathbb{Z}_n) \) is given by \( x \star y = 2y - x \mod n \). It is shown in [11, Proposition 3.1] that the core quandle of any group is a Dehn quandle of some group. Let \( D_n = \langle s_1, s_2 \mid s_1^n = 1, s_2 = 1, (s_1 s_2)^n = 1 \rangle \) be the Coxeter presentation of the dihedral group of order \( 2n \), and set \( S = \{ s_1, s_2 \} \). It follows from a direct computation (see also [3, Lemma 6.3]) that \( \text{Core} (\mathbb{Z}_n) \cong D(S^{D_n}) \).

First consider the case when \( n \) is odd. In this case, \( s_1 \) and \( s_2 \) are conjugate, and \( C_{D_n}(s_i) = \langle s_i \rangle \) for all \( i \). Set \( s_1 < s_2 \) and \( f(s_2, s_1) = (s_1 s_2)^{n+1} \). Let \( (s_1 * s_2)_n \) denote the left associated product \( s_1 * s_2 * s_1 * s_2 \cdots \) of length \( n \). Then Theorem 3.1 gives the following set of relations:

\[
(3.2.1) \quad s_i * s_j * s_j = s_i \quad \text{for } i, j = 1, 2, \\
(3.2.2) \quad s_i * s_2 * s_1 * s_2 * s_1 * s_2 * s_1 = s_i \quad \text{for } i = 1, 2, \\
(3.2.3) \quad (s_1 * s_2)_n = s_2.
\]

Note that (3.2.1) for \( i = j \) are trivial relations. Since \( n \) is odd, using (3.2.1), the relation (3.2.3) can be turned into the relation \( (s_2 * s_1)_n = s_1 \). Now, using (3.2.3) and \( (s_2 * s_1)_n = s_1 \), we can recover (3.2.2). Thus, the final presentation is

\[
\text{Core} (\mathbb{Z}_n) \cong \langle s_1, s_2 \mid s_2 * s_1 * s_1 = s_2, \quad s_1 * s_2 * s_2 = s_1, \quad (s_1 * s_2)_n = s_2 \rangle.
\]
Suppose now that $n$ is even. In this case, $s_1$ and $s_2$ lie in different conjugacy classes. Also, $C_{D_n}(s_i) = \langle s_i, (s_1 s_2)^2 \rangle$. Again, Theorem 3.11 gives the following set of relations:

(3.2.4) $s_i s_j s_i = s_i$ for $i, j = 1, 2$,

(3.2.5) $s_i s_j s_k s_i = s_i$ for $i = 1, 2$,

(3.2.6) $s_1 s_2 s_1 s_2 s_1 \cdots s_2 s_1 = s_1$,

(3.2.7) $s_2 s_1 s_2 s_1 s_2 \cdots s_2 s_1 = s_2$.

Again we ignore the trivial relations. Further, using (3.2.6) and (3.2.7), we can recover relations in (3.2.5). Hence, the final presentation is

$\text{Core}(\mathbb{Z}_n) \cong \langle s_1, s_2 \mid s_2 s_1 s_2 s_1 = s_2, s_1 s_2 s_2 s_1 = s_1, (s_1 s_2)_n = s_1, (s_2 s_1)_n = s_2 \rangle$.

**Remark 3.5.** If $n$ is even, it follows from the presentation of $Q(T(n, m))$ as given by Remark 2.32 and the presentation of $\text{Core}(\mathbb{Z}_n)$ that the map $x_i \to s_j$ for $i \equiv j (\text{mod} \ 2)$ defines a surjective quandle homomorphism from $Q(T(n, m))$ onto $\text{Core}(\mathbb{Z}_n)$. Thus, for $n$ even, the torus link $T(n, m)$ is always $m$-colorable. This observation seems to be well-known.

**Example 3.6.** The fundamental group of a closed orientable surface $S_g$ of genus $g \geq 2$ has a presentation

$$\pi_1(S_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$ 

Setting $S = \{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$, we note that no two elements of $S$ are conjugate to each other. Further, it is known that $C_{\pi_1(S_g)}(a_i) = \langle a_i \rangle$ and $C_{\pi_1(S_g)}(b_i) = \langle b_i \rangle$ [13] Section 1.1.3. Thus, by Theorem 3.11 we get

$$\mathcal{D}(\pi_1(S_g)) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid a_i b_g a_g b_g^{-1} b_g b^{-1} a_g \cdots b_1 a_1 b_1^{-1} a_1 \rangle.$$

We note that the presentation of the enveloping group of $\mathcal{D}(\pi_1(S_g))$ given in [11] Theorem 3.18 can be recovered using (3.2.8) and [37] Theorem 5.1.7.

**Example 3.7.** For $n \geq 3$, recall the Artin presentation of the braid group

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2, \ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i \rangle.$$ 

Let us set $S = \{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and the total ordering $\sigma_1 < \sigma_2 < \cdots < \sigma_{n-1}$ on $S$. Note that all the elements of $S$ are conjugate to each other. Choose $f(\sigma_{i+1}, \sigma_i) = \sigma_i \sigma_{i+1}$ for each $1 \leq i \leq n - 2$. It follows from [20] Theorem 4 that $C_{B_n}(\sigma_1)$ is generated by $X \cup Y$, where

$$X = \{\sigma_1, \sigma_3, \sigma_4, \ldots, \sigma_{n-1}, \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \}$$

and

$$Y = \{\sigma_1^2, (\sigma_2 \sigma_1)(\sigma_3 \sigma_2) \cdots (\sigma_r \sigma_1 \sigma_r) (\sigma_r \sigma_{r+1}) \cdots (\sigma_2 \sigma_3)(\sigma_1 \sigma_2) \mid 1 \leq r \leq n - 2 \}.$$ 

It follows from the discussion after [20] Theorem 4 that generators from $Y$ can be written in terms of generators from $X$. In fact, setting $x_r = (\sigma_2 \sigma_1)(\sigma_3 \sigma_2) \cdots (\sigma_r \sigma_1 \sigma_r) (\sigma_r \sigma_{r+1}) \cdots (\sigma_2 \sigma_3)(\sigma_1 \sigma_2)$, one can show that $x_r = x_{r-1} \sigma_{r+1} \sigma_r \cdots (\sigma_r \sigma_{r+1} \sigma_r) \cdots (\sigma_3 \sigma_1 \sigma_3 \sigma_4 \sigma_2) \cdots (\sigma_1 \sigma_2 \sigma_1 \sigma_2) \sigma_r^{-1} \sigma_{r+1}^{-1} \cdots \sigma_r^{-1} \sigma_{r+1}^{-1}$ for each $r \geq 2$. Thus, we obtain

$$C_{B_n}(\sigma_1) = \langle \sigma_1, \sigma_3, \sigma_4, \ldots, \sigma_{n-1}, \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \rangle.$$
First we consider \( n = 3 \) to clarify the general idea. We have \( f(\sigma_2, \sigma_1) = \sigma_1 \sigma_2 \) and \( C_{B_3}(\sigma_1) = \langle \sigma_1, \sigma_2 \sigma_1 \sigma_2 \rangle \). Thus, by Theorem 3.1, the defining relations in \( \mathcal{D}(S_{B_3}) \) are given by

\[
\begin{align*}
(3.2.10) \quad \sigma_i \ast \sigma_1 \ast \sigma_2 \ast \sigma_1 \ast^{-1} \sigma_2 \ast^{-1} \sigma_1 \ast^{-1} \sigma_2 &= \sigma_i \quad \text{for } i = 1, 2, \\
(3.2.11) \quad \sigma_1 \ast \sigma_2 \ast \sigma_1 \ast \sigma_2 &= \sigma_1, \\
(3.2.12) \quad \sigma_1 \ast \sigma_2 \ast \sigma_1 &= \sigma_2.
\end{align*}
\]

By using (3.2.12) in (3.2.11), we get

\[
(3.2.13) \quad \sigma_2 \ast \sigma_1 \ast \sigma_2 = \sigma_1.
\]

It follows from relations (3.2.12) and (3.2.13) that (3.2.10) is redundant. Thus, we obtain

\[
\mathcal{D}(S_{B_3}) = \langle \sigma_1, \sigma_2 \mid \sigma_1 \ast \sigma_2 \ast \sigma_1 = \sigma_2, \ \sigma_2 \ast \sigma_1 \ast \sigma_2 = \sigma_1 \rangle.
\]

Now, assume that \( n \geq 4 \). In this case, defining relations for \( \mathcal{D}(S_{B_n}) \) are as follows:

\[
\begin{align*}
(3.2.14) \quad \sigma_k \ast \sigma_i \ast \sigma_{i+1} \ast \sigma_i &= \sigma_k \quad \text{for } 1 \leq k \leq n - 1 \text{ and } 1 \leq i \leq n - 2, \\
&\ast^{-1} \sigma_{i+1} \ast^{-1} \sigma_i \ast^{-1} \sigma_{i+1} \\
(3.2.15) \quad \sigma_k \ast \sigma_i \ast \sigma_j \ast \sigma_i \ast^{-1} \sigma_j &= \sigma_k \quad \text{for } 1 \leq k \leq n - 1 \text{ and } 3 \leq i + 2 \leq j \leq n - 1, \\
(3.2.16) \quad \sigma_1 \ast \sigma_j &= \sigma_1 \quad \text{for } 3 \leq j \leq n - 1, \\
(3.2.17) \quad \sigma_1 \ast \sigma_2 \ast \sigma_1 \ast \sigma_2 &= \sigma_1, \\
(3.2.18) \quad \sigma_i \ast \sigma_{i+1} \ast \sigma_i &= \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 2.
\end{align*}
\]

Using relations in (3.2.18), we can recover relations in (3.2.14), and the relation (3.2.17) can be rewritten as \( \sigma_3 \ast \sigma_1 \ast \sigma_2 = \sigma_1 \). Taking \( k = i - 1 \) and using induction on \( n \), we can rewrite relations in (3.2.15) as \( \sigma_i \ast \sigma_j = \sigma_i \) for \( 3 \leq i + 2 \leq j \leq n - 1 \). The remaining relations in (3.2.15) can be recovered using these relations. Thus, the presentation \( \mathcal{D}(S_{B_n}) \) is

\[
(3.2.19) \quad \mathcal{D}(S_{B_n}) = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \ast \sigma_{i+1} \ast \sigma_i = \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 2, \right.
\]

\[
\left. \sigma_2 \ast \sigma_1 \ast \sigma_2 = \sigma_1, \ \sigma_i \ast \sigma_j = \sigma_i \quad \text{for } 3 \leq i + 2 \leq j \leq n - 1. \right\rangle
\]

**Remark 3.8.** Using induction on \( n \), one can show that the presentation for \( \mathcal{D}(S_{B_n}) \) given by (2.4.1) can also be reduced to the one given in (3.2.19).

**Example 3.9.** Recall the presentation of an Artin group from (2.4.1). We say that \( A \) is a right angled Artin group if each \( m_{ij} \) is either 2 or \( \infty \). Setting \( S = \{x_1, x_2, \ldots, x_n\} \) to be the Artin generating set of \( A \), we see that no two elements from \( S \) are conjugate to each other. Further, \( C_A(x_i) = \langle x_j \mid m_{ij} = 2 \rangle \). Thus, by Theorem 3.1, \( \mathcal{D}(A) \) has a presentation with generating set \( S \) and defining relations as

\[
\begin{align*}
(3.2.20) \quad x_k \ast x_i \ast x_j \ast^{-1} x_i \ast^{-1} x_j &= x_k \quad \text{for } 1 \leq k \leq n \text{ and } m_{ij} = 2, \\
(3.2.21) \quad x_i \ast x_j &= x_i \quad \text{for } m_{ij} = 2.
\end{align*}
\]

Clearly, relations in (3.2.20) can be recovered from relations in (3.2.21). Thus, the presentation is

\[
(3.2.22) \quad \mathcal{D}(A) = \langle x_1, x_2, \ldots, x_n \mid x_i \ast x_j = x_i \quad \text{whenever } m_{ij} = 2 \rangle.
\]

In view of examples (2.29) and (3.9) we propose the following.
Hence, the isotopy class of the right hand Dehn twist quandle of the braid group becomes an isomorphism of quandles since further, [25, 39] considered the quandle structure on the set of isotopy classes of simple closed arcs [13, Corollary 4.15]. Thus, if \( S \) and \( \leq 2 \), and called it half twist about the punctures joined by any arc representing \( y \). This defines an injective map
\[
\tau : D_{g,p}^{ns} \hookrightarrow M_{g,p}
\]
by setting
\[
\tau(x) = \begin{cases} 
T_x & \text{if } x \text{ is the isotopy class of a non-separating simple closed curve}, \\
H_x & \text{if } x \text{ is the isotopy class of a non-separating simple closed arc}.
\end{cases}
\]
For \( x, y \in D_{g,p}^{ns} \), defining \( x * y = \tau(y)(x) \) turns \( D_{g,p}^{ns} \) into a quandle (see [11] for details), called the Dehn quandle of the surface \( S_{g,p} \). Note that the set of all Dehn twists along non-separating simple closed curves forms one conjugacy class in \( M_{g,p} \), whereas the set of all half twists along simple closed arcs forms another conjugacy class. For each \( g, p \geq 0 \), the group \( M_{g,p} \) is generated by finitely many Dehn twists about non-separating simple closed curves and half twists about simple closed arcs [13 Corollary 4.15]. Thus, if \( S \) is such a generating set for \( M_{g,p} \), then \( \tau : D_{g,p}^{ns} \to D(S_{g,p}) \) becomes an isomorphism of quandles since
\[
\tau(x * y) = \tau(\tau(y)(x)) = \tau(y)\tau(x)\tau(y)^{-1} = \tau(x) \cdot \tau(y).
\]
Hence, \( D_{g,p}^{ns} \) has the structure of the Dehn quandle of the group \( M_{g,p} \) with respect to \( S \).

The construction of Dehn quandle of \( S_{g,p} \) for \( p = 0 \) first appeared in the work of Zablow [10, 11]. Further, [25, 39] considered the quandle structure on the set of isotopy classes of simple closed arcs in \( S_{g,p} \) for \( p \geq 2 \), and called it quandle of cords. In general, the quandle of cords is a subquandle of \( D_{g,p}^{ns} \). In the case of a disk with \( n \) marked points, this quandle can be identified with the Dehn quandle of the braid group \( B_n \) with respect to its standard set of generators, that is, half twists along the cords. The reader may refer to [11 Section 5] for generalities on Dehn quandles of surfaces, and [13] for basic facts about mapping class groups.

**Example 3.11.** Recall from [13 Section 5.1.3] that a presentation of the mapping class group \( \mathcal{M}_1 \) of the closed orientable surface of genus one (namely, the torus) is given by
\[
\mathcal{M}_1 = \langle T_a, T_b \mid T_aT_bT_a = T_bT_aT_b, \quad (T_aT_b)^6 = 1 \rangle,
\]
where the curves \( a, b \) are as shown in Figure 2(A).

Let us set \( S = \{T_a, T_b\} \) and \( T_b < T_a \). Note that \( T_a \) and \( T_b \) are conjugates of each other and we can take \( f(T_b, T_a) = T_aT_b \). Note that \( \Psi : \mathcal{M}_1 \to \text{SL}(2, \mathbb{Z}) \) given by
\[
\Psi(T_a) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Psi(T_b) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\]
is an isomorphism of groups. An elementary check gives $C_{\text{SL}(2,\mathbb{Z})}(\Psi(T_a)) = \langle \Psi(T_a), -\text{Id} \rangle$ and $-\text{Id} = (\Psi(T_b)\Psi(T_b))^3$. Thus, we have $C_{\mathcal{M}_1}(T_a) = \langle T_a, T_b T_a T_b T_b \rangle$. By Theorem 3.1 a presentation of $\mathcal{D}_1$ has $S$ as its generating set and defining relations as follows:

\begin{align*}
(3.2.23) & \quad T_a * T_a * T_b * T_a * T_b * T_a * T_b = T_a, \\
(3.2.24) & \quad T_b * T_a * T_b * T_a * T_b * T_a = T_a, \\
(3.2.25) & \quad T_a * T_b * T_a * T_b * T_a * T_b * T_a = T_b, \\
(3.2.26) & \quad T_b * T_a * T_b * T_a * T_b * T_a * T_b = T_b, \\
(3.2.27) & \quad T_a * T_b * T_a * T_b = T_a, \\
(3.2.28) & \quad T_a * T_b * T_a = T_b.
\end{align*}

Using the relation (3.2.28), we can write the relation (3.2.27) as $T_b * T_a * T_b = T_a$. Now, using this new relation together with (3.2.28), one can recover relations (3.2.23) through (3.2.26). Thus, we obtain the presentation

\begin{equation}
\mathcal{D}_1^{ns} \cong \langle T_a, T_b \mid T_a * T_b * T_a = T_b, \quad T_b * T_a * T_b = T_a \rangle.
\end{equation}

**Remark 3.12.** Notice that, the right hand side of (3.2.29) is the presentation of the knot quandle of the trefoil. This recovers the main result of [30] Theorem 3.1].

**Example 3.13.** Next, we give a presentation of the Dehn quandle $\mathcal{D}_2^{ns}$ of the closed orientable surface $S_2$ of genus two. To simplify the notation, we set $\alpha_1 := T_{a_1}, \alpha_2 := T_{b_1}, \alpha_3 := T_{c_1}, \alpha_4 := T_{b_2}$, and $\alpha_5 := T_{a_2}$, where the curves are as in Figure 2(B). Then, by [27] Section 4], a presentation of the mapping class group $\mathcal{M}_2$ of $S_2$ is

\begin{equation}
\mathcal{M}_2 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid [\alpha_i, \alpha_j] = 1 \text{ for } |i-j| \geq 2, \quad \alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1} \text{ for } 1 \leq i \leq 4, \\
(3.2.30) & \quad (\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5)^6 = 1, \quad (\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1)^2 = 1, \\
& \quad [\alpha_1, (\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1)] = 1 \rangle.
\end{equation}

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and consider the total order $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ on $S$. Note that all Dehn twists along non-separating simple closed curves are conjugate in $\mathcal{M}_2$. We choose $f$ such that $f(\alpha_2, \alpha_1) = \alpha_1 \alpha_2$, $f(\alpha_3, \alpha_2) = \alpha_2 \alpha_3$, $f(\alpha_4, \alpha_3) = \alpha_3 \alpha_4$, and $f(\alpha_5, \alpha_4) = \alpha_4 \alpha_5$. Using [19] Theorem 1] we get a generating set for the mapping class group of a genus one surface with two boundary components. Then embedding this surface into a closed surface of genus two and using [34] Remark 1.19] gives

\begin{equation}
C_{\mathcal{M}_2}(\alpha_1) = \langle \iota, \alpha_1, \alpha_3, \alpha_4, \alpha_5 \rangle.
\end{equation}
where \( \iota \) is the hyperelliptic involution. Note that \( \iota = \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1 = \alpha_5 \alpha_4 \alpha_3 \alpha_2 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \) (see [27, Section 4]). Thus, by Theorem 3.1 \( D_2^{ns} \) is generated by \( S \) and has defining relations as follows. Relations of type (1) are

\[
\begin{align*}
(3.2.31) & \quad \alpha_i * \alpha_j * \alpha_{j+1} * \alpha_j *^{-1} \alpha_{j+1} *^{-1} \alpha_j *^{-1} \alpha_{j+1} = \alpha_i \quad \text{for } 1 \leq i \leq 5 \\
& \quad \text{and } 1 \leq j \leq 4, \\
(3.2.32) & \quad \alpha_i * \alpha_1 * \alpha_j *^{-1} \alpha_1 *^{-1} \alpha_j = \alpha_i \quad \text{for } 1 \leq i \leq 5 \\
& \quad \text{and } j = 3, 4, 5, \\
(3.2.33) & \quad \alpha_i * \alpha_2 * \alpha_j *^{-1} \alpha_2 *^{-1} \alpha_j = \alpha_i \quad \text{for } 1 \leq i \leq 5, \\
& \quad \text{and } j = 4, 5, \\
(3.2.34) & \quad \alpha_i * \alpha_3 * \alpha_5 *^{-1} \alpha_3 *^{-1} \alpha_5 = \alpha_i \quad \text{for } 1 \leq i \leq 5, \\
(3.2.35) & \quad \alpha_i * \alpha_5 * \alpha_4 * \alpha_3 * \alpha_2 * \alpha_1 * \cdots * \alpha_5 * \alpha_4 * \alpha_3 * \alpha_2 * \alpha_1 = \alpha_i \quad \text{for } 1 \leq i \leq 5, \\
& \quad 6 \text{ times} \\
(3.2.36) & \quad \alpha_i * \alpha_1 * \alpha_2 * \alpha_3 * \alpha_4 * \alpha_5 * \alpha_4 * \alpha_3 * \alpha_2 * \alpha_1 = \alpha_i \quad \text{for } 1 \leq i \leq 5, \\
& \quad \ast \alpha_1 * \ast \alpha_2 * \ast \alpha_3 * \ast \alpha_4 * \ast \alpha_5 * \ast \alpha_4 * \ast \alpha_3 * \ast \alpha_2 * \ast \alpha_1 = \alpha_i \quad \text{for } 1 \leq i \leq 5, \\
& \quad \ast^{-1} \alpha_1 * \ast^{-1} \alpha_2 * \ast^{-1} \alpha_3 * \ast^{-1} \alpha_4 * \ast^{-1} \alpha_5 * \ast^{-1} \alpha_4 * \ast^{-1} \alpha_3 * \ast^{-1} \alpha_2 * \ast^{-1} \alpha_1 = \alpha_i \quad \text{for } 1 \leq i \leq 5.
\end{align*}
\]

Relations of type (2) are

\[
\begin{align*}
(3.2.38) & \quad \alpha_1 * \alpha_5 * \alpha_4 * \alpha_3 * \alpha_2 * \alpha_1 = \alpha_1, \\
& \quad \alpha_1 * \alpha_2 * \alpha_3 * \alpha_4 * \alpha_5 = \alpha_1, \\
& \quad \alpha_1 * \alpha_1 = \alpha_1 \quad \text{for } i = 3, 4, 5,
\end{align*}
\]

and relations of type (3) are

\[
\begin{align*}
(3.2.40) & \quad \alpha_i * \alpha_{i+1} * \alpha_i = \alpha_{i+1} \quad \text{for } 1 \leq i \leq 4.
\end{align*}
\]

Note that relations (3.2.31) can be recovered using relations (3.2.40). Along similar lines, we can recover relations (3.2.32) using relations (3.2.39). We can rewrite the relation (3.2.35) as \( \alpha_2 * \alpha_3 * \alpha_2 = \alpha_1 \) using relations (3.2.39). Similarly, we can rewrite relations (3.2.33) and (3.2.34) as \( \alpha_2 * \alpha_4 = \alpha_2, \alpha_2 * \alpha_5 = \alpha_2 \) and \( \alpha_3 * \alpha_5 = \alpha_3 \), respectively, using relations (3.2.39) and (3.2.40). One can see that relations (3.2.35) can be recovered using relations (3.2.39), (3.2.39), and (3.2.40) and relations \( \alpha_2 * \alpha_4 = \alpha_2, \alpha_2 * \alpha_5 = \alpha_2 \) and \( \alpha_3 * \alpha_5 = \alpha_3 \). For \( i = 2 \), the relation (3.2.37) can be rewritten as \( \alpha_1 * \alpha_2 * \alpha_3 * \alpha_4 * \alpha_5 * \alpha_4 * \alpha_3 * \alpha_2 = \alpha_1 \), which further gives relations (3.2.36). Thus, \( D_2^{ns} \) has a presentation

\[
D_2^{ns} = \left\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \begin{array}{l}
\alpha_1 * \alpha_3 = \alpha_1, \\
\alpha_1 * \alpha_4 = \alpha_1, \\
\alpha_1 * \alpha_5 = \alpha_1,
\end{array}
\begin{array}{l}
\alpha_2 * \alpha_4 = \alpha_2, \\
\alpha_2 * \alpha_5 = \alpha_2, \\
\alpha_3 * \alpha_5 = \alpha_3,
\end{array}
\begin{array}{l}
\alpha_1 * \alpha_2 * \alpha_1 = \alpha_2, \\
\alpha_2 * \alpha_3 * \alpha_2 = \alpha_3, \\
\alpha_3 * \alpha_4 * \alpha_3 = \alpha_4,
\end{array}
\begin{array}{l}
\alpha_4 * \alpha_5 * \alpha_4 = \alpha_5, \\
\alpha_1 * \alpha_2 * \alpha_3 * \alpha_4 * \alpha_5 * \alpha_4 * \alpha_3 * \alpha_2 = \alpha_1 \end{array}\right\rangle.
\]

3.3. An alternate approach to presentation of \( D_2^{ns} \). We conclude by giving an alternate proof of the presentation of \( D_2^{ns} \) by identifying it with \( D_2^{ns}_{0,6} \). It follows from [13, Section 5.1.3] that a
presentation of the mapping class group of $S_{0,6}$ is

$$
\mathcal{M}_{0,6} = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i, \quad (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5)^6 = 1, \quad \sigma_1 \sigma_2 \sigma_3 \sigma_5 \sigma_4 \sigma_5 \sigma_2 \sigma_1 = 1 \rangle.
$$

Recall that, the enveloping group $\text{Env}(X)$ of a quandle $(X,\ast)$ is defined as

$$
\text{Env}(X) = \langle e_x, \ x \in X \mid e_{x+y} = e_y e_x e_y^{-1}, \ x, y \in X \rangle.
$$

The natural map $\eta : X \to \text{Env}(X)$ given by $\eta(x) = e_x$ is a homomorphism of quandles when we view $\text{Env}(X)$ with the conjugation quandle structure. A presentation of the Dehn quandle $D_{0,6}^{ns}$ of $S_{0,6}$ is given in [25, Theorem 3.2] as

$$
D_{0,6}^{ns} = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \mid \sigma_i \ast \sigma_j = \sigma_i \text{ for } |i - j| \geq 2, \quad \sigma_i \ast \sigma_j \ast \sigma_i = \sigma_j \text{ for } |i - j| = 1, \quad \sigma_1 \ast \sigma_2 \ast \sigma_3 \ast \sigma_4 \ast \sigma_5 = \sigma_5 \ast \sigma_4 \ast \sigma_3 \ast \sigma_2 \ast \sigma_1 \rangle.
$$

Further, a presentation of the enveloping group of $D_{0,6}^{ns}$ follows from [25, Theorem 3.7] (also from [37, Theorem 5.1.7]) and given as

$$
\text{Env}(D_{0,6}^{ns}) = \langle e_{\sigma_1}, e_{\sigma_2}, e_{\sigma_3}, e_{\sigma_4}, e_{\sigma_5}, e_{\sigma_1} e_{\sigma_2} e_{\sigma_3} e_{\sigma_4} e_{\sigma_5} e_{\sigma_2} e_{\sigma_1} \mid e_{\sigma_i} e_{\sigma_{i+1}} e_{\sigma_i} = e_{\sigma_{i+1}} e_{\sigma_i} e_{\sigma_{i+1}} \text{ for all } i, \quad [e_{\sigma_1}, e_{\sigma_2} e_{\sigma_3} e_{\sigma_4} e_{\sigma_5} e_{\sigma_2} e_{\sigma_1} e_{\sigma_2} e_{\sigma_1}] = 1 \text{ for all } j \rangle.
$$

Note that $D_{0,6}^{ns}$ is referred as the quandle of cords of $S_{0,6}$ in [25]. It is easy to see that commutativity of $e_{\sigma_1} e_{\sigma_2} e_{\sigma_3} e_{\sigma_4} e_{\sigma_5} e_{\sigma_2} e_{\sigma_1}$ with $e_{\sigma_j}$ for $2 \leq j \leq 5$ follows from braid relations, and hence the last relation in (3.3.2) is needed only for $j = 1$. The following theorem together with (3.3.1) gives a presentation of $D_2^{ns}$.

**Proposition 3.14.** $D_2^{ns} \cong D_{0,6}^{ns}$.

**Proof.** The map $\eta : D_{0,6}^{ns} \to \text{Env}(D_{0,6}^{ns})$, given by $\eta(x) = e_x$, induces a quandle homomorphism $\eta' : D_{0,6}^{ns} \to \mathcal{D}(\eta(S)^{\text{Env}(D_{0,6}^{ns})})$, where $S$ is the generating set of $D_{0,6}^{ns}$ as in (3.3.1). By [11, Section 3.2], the map

$$
\Phi : \text{Env}(D_{0,6}^{ns}) \to \mathcal{M}_{0,6}
$$

given by $\Phi(e_x) = x$ is a surjective group homomorphism, and hence induces a surjective quandle homomorphism $\Phi' : \mathcal{D}(\eta(S)^{\text{Env}(D_{0,6}^{ns})}) \to D_{0,6}^{ns}$. Since $\eta'\Phi'$ and $\Phi'\eta'$ are both identity maps, $\Phi'$ is a quandle isomorphism. Now, recall the presentation of $\mathcal{M}_2$ from (3.2.30). It follows that we can factorise $\Phi = \Phi_2\Phi_1$, where $\Phi_1 : \text{Env}(D_{0,6}^{ns}) \to \mathcal{M}_2$ and $\Phi_2 : \mathcal{M}_2 \to \mathcal{M}_{0,6}$ are surjective group homomorphisms. These maps induce surjective quandle homomorphisms $\Phi_1' : \mathcal{D}(\eta(S)^{\text{Env}(D_{0,6}^{ns})}) \to D_2^{ns}$ and $\Phi_2' : D_2^{ns} \to D_{0,6}^{ns}$ such that $\Phi' = \Phi_2'\Phi_1'$. Since $\Phi'$ is an isomorphism, it follows that both $\Phi_2'$, $\Phi_1'$ are isomorphisms, and hence $D_2^{ns} \cong D_{0,6}^{ns}$. \qed

**Remark 3.15.** Note that, the additional braid and far commutativity induced relations in (3.3.1) can be recovered using relations from (3.2.41). Further, the long relation $\alpha_1 \ast \alpha_2 \ast \alpha_3 \ast \alpha_4 \ast \alpha_5 \ast \alpha_4 \ast \alpha_3 \ast \alpha_2 = \alpha_1$ in (3.2.41) is equivalent to the long relation $\sigma_1 \ast \sigma_2 \ast \sigma_3 \ast \sigma_4 \ast \sigma_5 = \sigma_5 \ast \sigma_4 \ast \sigma_3 \ast \sigma_2 \ast \sigma_1$ in (3.3.1).
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