THE COHOMOLOGY OF
CERTAIN GROUPS

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January, 1992
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Introduction.

This dissertation comprises six chapters which are an improved version of the author’s PhD thesis (submitted in November 1990), together with a short chapter describing more recent work not directly related to the rest of the thesis. Most of the work contained in this dissertation consists of calculations in the integral and mod-$p$ cohomology rings of specific finite groups. A number of questions concerning group cohomology are resolved by means of these examples, as I shall explain below. I have tried to describe my results in sufficient detail that they may provide a tool for further calculations, which accounts for the length of the statements of some of the theorems. Of the seven chapters of this dissertation, all except the first and sixth involve, to some degree, calculations for specific groups.

Chapter one consists mainly of an explanation of a technique used throughout the calculational chapters. The technique was suggested by J. Huebschmann and P. H. Kropholler, but its subsequent development in this chapter, including its close connection with questions concerning Chern classes and their corestrictions, is my own work. The technique involves embedding a finite group in a compact Lie group whose identity component is a circle. The reason why I have chosen the ‘topologist’s notation’ for the cohomology of a group (as the cohomology of its classifying space) for this dissertation is to avoid confusion between the cases when I consider the cohomology of the circle as a topological space and as a Lie group. The remainder of chapter one consists of a generalisation whereby a finite group is embedded in a Lie group with identity component any torus, and a classification of Lie groups containing the circle as a subgroup of index $p^n$ for $n \leq 3$. The difficult cases of this classification are when $p = 2$ (so that the circle subgroup need not be central) and are not used in later chapters, but are included for completeness.

Chapter six of this dissertation is not very closely related to the other chapters, and might better be described as an appendix, except that I have chosen to reserve that title for the brief discussion of $p$-groups of small order (which is almost entirely unoriginal). Chapter six concerns an integer invariant of finite groups, due to Yagita [Ya], which is
related to free actions of the groups on products of spheres. The original invariant is defined in terms of images of the integral cohomology of the group, and the aim of the chapter is to investigate a similar definition using images of the Chern subring. Since the new invariant may be calculated directly from the complex representation ring of the group, it is not surprising that the invariant bears a relation to linear actions analogous to the relation of the original invariant to free actions.

Chapter two is the first of three chapters devoted to $p$-groups for odd primes $p$. The groups considered are the non-abelian groups expressible as central extensions of a cyclic group by the elementary abelian group of order $p^2$. There are two isomorphism classes of such group of given order. I determine, for each such group, its integral cohomology ring and the action of its outer automorphism group upon this ring. These calculations are referred to frequently throughout chapters three, four and five (often without comment). For the non-metacyclic groups I also make similar calculations for the mod-$p$ cohomology rings and determine the action thereon of the Steenrod algebra. Similar calculations could be made for the metacyclic groups, which may also be expressed as split extensions of a cyclic group by a group of order $p$, but I have not included these. Diethelm has shown that there are (for odd $p$) only two isomorphism types of ring that occur as mod $p$ cohomology rings of split metacyclic $p$-groups [Die], so one really ought to examine the actions of the automorphism group and the Steenrod algebra for all split metacyclics, rather than just those with a cyclic subgroup of index $p$. Neither the method used here, nor Diethelm’s method (which involves consideration of the spectral sequence for the ‘defining extension’) seems suited to this task. I feel that the best approach to this problem will involve the spectral sequence for these groups considered as extensions of their derived subgroups by their abelianisations. Some of the results of chapter two contain as special cases results of other authors; I verify Lewis’ calculations of the integral cohomology rings of non-abelian groups of order $p^3$, Thomas’ result that Chern classes generate the even degree cohomology of each of the groups considered in chapter two, and Thomas’ calculations of the integral
cohomology rings of the metacyclic $p$-groups with a cyclic subgroup of index $p$. Various preprints of Huynh-Mui dated from 1981 onwards concern the calculation of the mod-$p$ cohomology ring of the group of order $p^3$ and exponent $p$, but the methods used are different from those used here, and I believe that Huynh-Mui does not give a complete description of the relations between the generators that he uses.

Chapters three and four consist mainly of calculations in the integral cohomology of the $p$-groups of $p$-rank two and nilpotency class three. It is these groups (in the case $p \geq 5$) that are shown to provide counterexamples to the conjecture of C. B. Thomas, that for all $p$-groups of $p$-rank two Chern classes generate the even degree cohomology, and $p$-group counterexamples to the conjecture of M. F. Atiyah, that Chern classes generate the $E_\infty$-page of the Atiyah-Hirzebruch spectral sequence. These results have also been obtained by N. Yagita using Brown-Peterson cohomology. I also show that for $p = 3$ Chern classes do generate the even degree cohomology of these groups, and that for $p \geq 5$ corestrictions of Chern classes suffice. These results do not seem to be obtainable from the work of Yagita. In chapter four I calculate the cohomology rings of some of these groups for $p = 3$, and thus exhibit non-isomorphic groups of order $3^5$ with isomorphic integral cohomology rings. The best result of a similar nature for $p \geq 5$ in chapter four is the existence of non-isomorphic groups of order $p^6$ with isomorphic integral cohomology groups.

Chapter five consists of an extended application of the results of chapter two to the study of the Held group, one of the sporadic simple groups [He]. The $p$-torsion in the integral cohomology of the Held group is determined for all odd primes $p$, by calculating the image of the restriction map to a Sylow-$p$ subgroup. This provides enough information to determine the $E_2$-page of the Rector spectral sequence [Re]. I intend to examine this spectral sequence in the near future.

Chapter seven of this dissertation contains a description of a method due to Davis [Da] for constructing contractible simplicial complexes with groups acting with finite stabilisers. Among other things this method can yield important cohomological information concerning
Coxeter groups. This chapter does not contain any original results however.

I hope that the notation used in this dissertation is not unusual. My notation for cohomology is largely similar to that of [Br] and [Th3], except that I refer to $H^\ast(BG)$ instead of $H^\ast(G)$ for reasons explained above. Throughout chapter five much of the notation I use is an approximation to that of [Co]. I follow other authors in using Greek letters to stand for integral cohomology elements and Roman letters to stand for mod-$p$ elements. If $g, h$ are elements of a group $G$, then $h^g = g^{-1}hg$ and $[g, h] = g^{-1}gh$. If $H$ is a subgroup of $G$, then $c_g$ is the map from $H$ to $H^g$ given by $h \mapsto h^g$, and $c_g^\ast$ is the induced map in cohomology. I speak of extensions of normal subgroups by quotient groups, and say that these are central if the normal subgroup is contained in the centre of the big group.
1. The Circle Construction.

In this section we describe a method for examining the cohomology of a finite group by embedding it in a compact Lie group of dimension one. This method was suggested by P. Kropholler and J. Huebschmann, and has also been used by B. Moselle [Hu2], [Hu3], [Mo]. We derive general properties of the method, which demonstrate why it is particularly relevant to the study of the Chern subring. We also describe a generalisation of the original method, and classify the compact Lie groups of dimension one and \( p^n \) components, where \( n \leq 3 \).

Definition and Properties of \( \tilde{G} \).

Given a finite group \( G \) and a central cyclic subgroup \( C \), we fix an embedding of \( C \) into \( S^1 \), and define

\[
\tilde{G} = S^1 \times_C G.
\]

Then we have a commutative diagram:

\[
\begin{array}{ccc}
C & \rightarrow & G & \rightarrow & Q \\
\downarrow & & \downarrow & & \downarrow \\
S^1 & \rightarrow & \tilde{G} & \rightarrow & Q
\end{array}
\]

If \( M \) is a \( G \)-module on which \( C \) acts trivially, we may consider \( M \) as a \( \tilde{G} \)-module by letting \( S^1 \) act trivially, and the Lyndon-Hochschild-Serre spectral sequence for the second extension is often simpler than that for the first. To find \( H^*(BG; M) \), given \( H^*(B\tilde{G}; M) \), we use the Serre spectral sequence of the fibration

\[
S^1/C \cong \tilde{G}/G \rightarrow BG \rightarrow B\tilde{G}.
\]

(1)

This spectral sequence has \( E_2^{i,j} = 0 \) for \( j > 1 \), so the only possible non-zero differential is \( d_2 \). The above was first suggested to the author by P. Kropholler. A similar idea occurs in J. Huebschmann’s papers [Hu2], [Hu3].
We shall examine the cases in which $M$ is $\mathbb{Z}$ or $\mathbb{F}_p$, with the trivial $G$-action. If we let $R$ stand for either $\mathbb{Z}$ or $\mathbb{F}_p$, then

$$H^*(S^1; R) = \Lambda[\xi],$$

the exterior algebra over $R$ on one generator $\xi$ of degree 1, and

$$H^*(BS^1; R) = R[\tau],$$

the polynomial algebra over $R$ on one generator $\tau$ of degree 2. The $E_2^{*,*}$ page of our spectral sequence is given by

$$E_2^{*,*} \simeq H^*(\tilde{G}; R) \otimes \Lambda[\xi],$$

except that the product structure is changed by a sign. The effect of this sign change is to make $\xi$ anticommute with elements of $H^*(\tilde{G}; R)$. We shall call this ring the anticommutative ring generated by $H^*(\tilde{G}; R)$ and $\xi$. It is clear that

$$d_2(\xi) = \tau,$$

where $\tau$ is the inflation to $\tilde{G}$ of a generator $\tau$ for $H^2(\tilde{G}/G; R) \simeq H^2(BS^1; R)$. This follows by naturality of the spectral sequence with respect to maps of fibrations, in particular for the following commutative diagram.

$$\begin{array}{ccc}
S^1/C & \rightarrow & BG \\
\downarrow & & \downarrow \\
\tilde{G}/G & \rightarrow & B\{1\}
\end{array}$$

Here, all the vertical maps are induced by the quotient map

$$\tilde{G} \rightarrow \tilde{G}/G,$$

and the lower fibration is, up to homotopy, the path-loop fibration over $BS^1$. We may also think of $\tau$ as the first Chern class of $BG$ considered as a circle or $U(1)$ bundle over $B\tilde{G}$. 

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This follows because the first Chern class of the standard $S^1$ bundle over $BS^1$ generates $H^2(BS^1; R)$. With integer coefficients there is another interpretation for $\tau$. Recall that there is for any compact Lie group $K$ a natural isomorphism between $H^2(BK; \mathbb{Z})$ and $\text{Hom}(K, S^1)$. If we use this isomorphism to regard $\tau$ as a homomorphism from $\tilde{G}$ to $S^1$, its kernel is the subgroup $G$.

It appears that we may obtain another filtration of $H^*(BG; R)$ by examining the Eilenberg-Moore spectral sequence for the pullback square:

$$
\begin{array}{ccc}
BG & \rightarrow & B\tilde{G} \\
\downarrow & & \downarrow \\
\{\ast\} & \rightarrow & B\tilde{G}/G
\end{array}
$$

The $E_2^{\ast,\ast}$ page of this spectral sequence is

$$E_2^{\ast,\ast} \cong \text{Tor}^{R[\tau]}_{R[\tau]}(R, H^*(B\tilde{G}; R)).$$

To calculate $E_2^{\ast,\ast}$, we may construct a 2-stage free $R[\tau]$ resolution of $R$ by

$$0 \rightarrow R[\tau] \xleftarrow{\times \tau} R[\tau] \rightarrow R$$

and we see that

$$\text{Tor}^{i,\ast}_{R[\tau]}(R, H^*(B\tilde{G}; R)) \cong \begin{cases} H^*(B\tilde{G}; R)/\tau H^*(B\tilde{G}; R) & \text{for } i = 0 \\ \text{Ker} \times \tau : H^*(B\tilde{G}; R) \rightarrow H^*(B\tilde{G}; R) & \text{for } i = 1 \end{cases}$$

It is now easy to see that this spectral sequence collapses, and that its $E_2^{\ast,\ast}$ page gives the same filtration of $H^*(BG; R)$ as the Lyndon-Hochschild-Serre spectral sequence for the fibration (1). Of course, these spectral sequences are just alternative ways to view the Gysin sequence for the $S^1$ bundle $BG$. We now prove various results concerning $\tilde{G}$. We assume in the statements that $G$ is expressed as a central extension of $C$ by $Q$, and that $\tilde{G}$ is the corresponding extension of $S^1$ by $Q$.

**Lemma 1.1.** Given $G$ as above, consider the spectral sequence of the fibration (1) with coefficients in the trivial $G$-module $R$. 

a) For $R = \mathbb{Z}$, the spectral sequence does not collapse.

b) For $R = \mathbb{F}_p$, the spectral sequence collapses iff $C$ is not contained in $G^pG'$.

Proof. By $G^p$ we mean the subgroup of $G$ generated by $p$th powers of elements of $G$. If we let $\xi$ be a generator of $H^1(S^1; R)$, then as above, in each case $E_{2,\ast}^\ast$ is the anticommutative ring generated by $H^*(BG; R)$ and $\xi$, and the spectral sequence collapses if and only if $d_2(\xi) = 0$. $G$ is finite, so $H^1(BG; \mathbb{Z}) = 0$, so in case a) $d_2 : E^{0,1}_2 \to E^{2,0}_2$ must be injective.

$$\text{Hom}(G, \mathbb{F}_p) \cong H^1(BG; \mathbb{F}_p) \cong E^{0,1}_3 \oplus E^{1,0}_3,$$

and the map $E^{1,0}_2 \to E^{1,0}_3$ corresponds to restriction from $\tilde{G}$ to $G$. $E^{0,1}_3$ is either $C_p$ or 0, so the spectral sequence collapses iff there are homomorphisms from $G$ to $\mathbb{F}_p$ which cannot be extended to $\tilde{G}$. A homomorphism $f : G \to \mathbb{F}_p$ can be extended to $\tilde{G}$ iff $f(C) = 0$, and this will be true for all $f$ iff $CG'/G'$ is contained in $(G/G')^p$ iff $C$ is contained in $G^pG'$.

Lemma 1.2. For $n \geq 2$, the inclusion of $G$ in $\tilde{G}$ restricts to an isomorphism from $\Gamma_n(G)$ to $\Gamma_n(\tilde{G})$, and from $G^{(n-1)}$ to $\tilde{G}^{(n-1)}$.

Proof. Any element of $\tilde{G}$ can be expressed as a product $xg$, where $x$ is central, and $g$ is in the image of $G$. In any group, if $x$ and $x'$ are central, then for all $g, g'$ we have $[xg, x'g'] = [g, g']$.

Corollary 1.3. $G$ is nilpotent (resp. soluble) if and only if $\tilde{G}$ is, and if so they have the same nilpotency class (resp. soluble length).

Lemma 1.4. The following conditions on $G$ and $\tilde{G}$ are equivalent:

a) $S^1 \hookrightarrow \tilde{G} \twoheadrightarrow Q$ is split;

b) The extension class of $G$ in $H^2(BQ; C)$ is in the kernel of the Bockstein from $H^2(BQ; C)$ to $H^3(BQ; \mathbb{Z})$ associated to the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{Z} \twoheadrightarrow C$;
c) There is an extension $\overline{G}$ of $\mathbb{Z}$ by $Q$ and a map $\overline{G} \to G$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z} & \rightarrow & \overline{G} \\
\downarrow & & \downarrow \\
C & \rightarrow & G
\end{array}
\rightarrow
\begin{array}{ccc}
& & Q \\
& & \downarrow \\
& & S^1
\end{array}
\]

Proof. Using the classification of extensions up to equivalence, see for example [Th3], we see that a) holds if and only if the extension class of $G$ in $H^2(BQ;C)$ maps to zero under the map to $H^2(BQ;S^1)$ induced by the inclusion of $C$ in $S^1$. To verify that this is equivalent to b), we consider the following map of short exact sequences:

\[
\begin{array}{ccc}
\mathbb{Z} & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \rightarrow & \mathbb{R}
\end{array}
\rightarrow
\begin{array}{ccc}
& & C \\
& & \downarrow \\
& & S^1
\end{array}
\]

The Bockstein is natural for maps of short exact sequences, so we obtain the following commutative diagram:

\[
\begin{array}{ccc}
H^2(BQ;C) & \rightarrow & H^3(BQ;\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^2(BQ;S^1) & \rightarrow & H^3(BQ;\mathbb{Z}).
\end{array}
\]

The group $Q$ is finite, so the lower map is an isomorphism, and hence a) and b) are equivalent. The equivalence of b) and c) follows from the cohomology long exact sequence for the coefficient sequence $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow C$.

Lemma 1.5. Any complex representation of $G$ extends to one of $\tilde{G}$.

Proof. Given $\rho : G \to \text{Aut}(V)$, an irreducible representation of $G$, $\rho$ must restrict to $C$ as scalar multiplication, because an eigenspace for $C$ in $V$ would be a $G$ submodule. Hence $\rho$ may be extended to $S^1$, then to $\tilde{G}$ by defining:

\[
\rho(g) = \rho(x)\rho(h) \quad \text{where } g = xh \text{ for } x \in S^1 \text{ and } h \in i(G).
\]

The close connection between the representation ring of a group and the $K$ theory of its classifying space suggest the following:
LEMMA 1-6. $K^*(B\tilde{G})$ restricts onto $K^*(BG)$. Similarly, if $K(n)^*$ is the $n$th Morava $K$ theory, and $K(n)^{od}(BG)$ is trivial, then $K(n)^*(B\tilde{G})$ restricts onto $K(n)^*(BG)$.

Proof. The generalised Atiyah-Hirzebruch spectral sequence for the fibration

$$BG \to B\tilde{G} \to BS^1$$

has $E_2^{*,*}$ page $H^1(BS^1; \mathcal{H}^j(BG))$, converging to a filtration of $\mathcal{H}^{i+j}(B\tilde{G})$, and the fibre map is the restriction. If $\mathcal{H}$ is $K$ theory, or if $\mathcal{H}$ is Morava $K$ theory and $G$ satisfies the condition above, then $E_2^{i,j}$ is trivial if $i$ or $j$ is odd, so the spectral sequence collapses. The statement that this condition holds for all finite $G$ is known as Ravenel’s conjecture. ■

LEMMA 1-7. The subring $\text{Ch}(\tilde{G})$ of the integral cohomology ring of $\tilde{G}$ generated by Chern classes maps onto $\text{Ch}(G)$ under the restriction. Similarly $\overline{\text{Ch}}(\tilde{G})$ maps onto $\overline{\text{Ch}}(G)$, where $\overline{\text{Ch}}(H)$ is the subring of the integral cohomology generated by corestrictions of Chern classes.

Proof. The first statement follows immediately from lemma 1-5. For the second statement, we note that a subgroup of $\tilde{G}$ of finite index must contain $S^1$, so is of the form $\tilde{H}$ for some subgroup $H$ of $G$, so the image of $\overline{\text{Ch}}(\tilde{G})$ is contained in $\overline{\text{Ch}}(G)$. To show that the image is all of $\overline{\text{Ch}}(G)$, we note that the following diagram commutes:

$$
\begin{align*}
\text{Ch}(\tilde{H}) & \xrightarrow{\text{Cor}} H^{ev}(B\tilde{G}) \\
\text{Ch}(H) & \xrightarrow{\text{Cor}} H^{ev}(BG)
\end{align*}
$$

This is because $\tilde{H} \cap G = H$, and $\tilde{H}G = \tilde{G}$, so we only need one double coset in the restriction-corestriction formula (see [Br]). ■

LEMMA 1-8. $H^{n+1}(B\tilde{G}; \mathbb{Z})$ restricts onto $H^{n+1}(BG; \mathbb{Z})$ if and only if multiplication by $c(G)$, the first Chern class of $BG$ as an $S^1$ bundle over $B\tilde{G}$, is injective on $H^n(B\tilde{G}; \mathbb{Z})$.

Proof. Consider the spectral sequence for the fibration

$$\tilde{G}/G \to BG \to B\tilde{G}.$$
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The cokernel of the restriction map from $H^{n+1}(BG;\mathbb{Z})$ to $H^{n+1}(B\tilde{G};\mathbb{Z})$ is isomorphic to $E_{\infty}^{n,1}$, which is isomorphic to the kernel of multiplication by $c(G)$ in $H^n(B\tilde{G};\mathbb{Z})$.

**Theorem 1.9.** $H^{ev}(BG) = Ch(G)$ (resp. $H^{ev}(BG) = \text{Ch}(G)$) if and only if $H^{ev}(B\tilde{G}) = Ch(\tilde{G})$ (resp. $H^{ev}(B\tilde{G}) = \text{Ch}(\tilde{G})$) and multiplication by $c(G)$ is injective on $H^{od}(B\tilde{G})$.

**Proof.** By lemma 1.8 the condition on $c(G)$ is equivalent to $H^{ev}(B\tilde{G})$ maps onto $H^{ev}(BG)$ by the restriction. The two conditions on $\tilde{G}$ therefore imply that $Ch(\tilde{G})$ (resp. $\text{Ch}(\tilde{G})$) restricts onto $H^{ev}(BG)$, but by lemma 1.7 the image is $Ch(G)$ (resp. $\text{Ch}(G)$). Conversely, by lemma 1.7, the condition on $G$ implies that $H^{ev}(B\tilde{G})$ restricts onto $H^{ev}(BG)$.

If $H^{ev}(B\tilde{G})$ restricts onto $H^{ev}(BG) = Ch(G)$, we prove by induction that $Ch^{2n}(\tilde{G}) = H^{2n}(B\tilde{G};\mathbb{Z})$. For any compact Lie group $H$, $H^2(BH;\mathbb{Z}) = Ch^2(H)$, so without loss of generality we may assume that $n$ is at least 2. Lemma 1.7 tells us that $Ch^{2n}(\tilde{G})$ maps onto $Ch^{2n}(G)$, so $Ch^{2n}(\tilde{G})$ maps onto $E^{2n,0}_3$ in the spectral sequence for the $S^1$ bundle $BG$.

Therefore, given $x \in H^{2n}(B\tilde{G};\mathbb{Z})$, we may pick $y \in H^{2n-2}(B\tilde{G};\mathbb{Z})$ and $z \in Ch^{2n}(\tilde{G})$ such that $x = z + c(G)y$. Hence $x \in Ch^{2n}(\tilde{G})$. The proof for the case of $\text{Ch}(\tilde{G})$ is similar.

**A Generalisation of the Circle Construction.**

Let $G$ an extension of $A \cong (C_m)^n$ by $Q$ such that the ‘action map’ $\varphi : Q \to GL_n(\mathbb{Z}/m)$ lifts to a map $\overline{\varphi}$ from $Q$ to $GL_n(\mathbb{Z})$. We may use this map to define an action of $Q$ on the $n$ dimensional torus, $T^n$, which restricts to the original action on a subgroup of $T^n$ isomorphic to $A$, which we will identify with $A$. We may then form the semidirect product $T^n \rtimes G$, that is the manifold $T^n \times G$ with multiplication defined by

$$(x, g)(y, h) = (xy^{\overline{\varphi}(g^{-1})}, gh).$$

The set $B = \{(a^{-1}, a)|a \in A\}$ is a normal subgroup, and we define

$$\tilde{G} = T^n \rtimes_A G = T^n \rtimes G/B.$$

$G$ is a subgroup of $\tilde{G}$, so there is a fibration

$$T^n/A \cong \tilde{G}/G \longrightarrow BG \longrightarrow B\tilde{G},$$

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but the fundamental group of $\hat{B} \hat{G}$ (which is of course isomorphic to $Q$) will in general act non-trivially on the cohomology of the coset space $\hat{G}/G$. In fact the action of $\pi_1(BG)$ on $H^1(\hat{G}/G; \mathbb{Z})$ is exactly the action of $Q$ on $\mathbb{Z}^n$ given by $\varphi$. If we write $l(\_)$ for the soluble length, and $Q$ is soluble, then clearly the following inequalities hold.

$$l(Q) \leq l(G) \leq l(\hat{G}) \leq l(Q) + 1$$

There is however no analogue of lemma 1·2, and $\hat{G}$ is not in general nilpotent. $\hat{G}$ depends on the choice of lifting $\varphi$, which will not be unique. Even if we take $G$ to be a group of order 4 expressed as an extension of $C_2$ by $C_2$, we may obtain $\hat{G}$ isomorphic to either $S^1 \times C_2$ or the orthogonal group $O_2$ depending on our choice of $\varphi$ from $C_2$ to $\{\pm 1\}$.

There is an analogue of lemma 1·4, where we replace $C$ by $A$, and $\mathbb{Z}$ by $\mathbb{Z}^n$ considered as a $\mathbb{Z}Q$ module via $\varphi$, with exactly the same proof. Not all representations of $G$ will extend to $\hat{G}$, although in each particular case it seems easy to describe the ones that will. Lemma 1·6 breaks down because $G$ is not necessarily normal in $\hat{G}$. In spite of all these shortcomings I believe that $\hat{G}$ will prove useful in the study of various groups. For example, there is a family of 3-groups of 3-rank two, which may be presented as split extensions of $C_3 \oplus C_3$ by $C_3$, where the action is induced from a non-trivial action of $C_3$ on $\mathbb{Z}^2$. These groups are nilpotent of class $2t$, so are unsuited to study via central extensions.

**Classification of Circle groups on $p^n$ components.**

Here we classify Lie groups $G$ with maximal connected subgroup is $S^1$ and group of components of order dividing $p^3$. $\text{Aut}(S^1)$ has order two, so for $p$ odd $S^1$ is central in $G$. Central extensions of $S^1$ by $H$ are classified up to equivalence by $H^2(BH; S^1)$, which can be identified with $H^3(BH; \mathbb{Z})$ via the Bockstein for the sequence $\mathbb{Z} \to \mathbb{R} \to S^1$. In each of the cases we examine, either $H^3(BH; \mathbb{Z})$ is trivial, or the action of $\text{Aut}(H)$ is transitive on the non-zero elements, so there is at most one non-split extension. The groups $H$ for which $H^3(BH; \mathbb{Z})$ is non-trivial are $(C_p)^2$, $(C_p)^3$, $C_{p^2} \times C_p$, the dihedral group $D_8$ and $P_2$, the non-abelian group of exponent $p$ (see appendix). For odd $p$, section 2 contains sufficient
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information to check the third cohomology group of the groups $H$. The case when $p = 2$
may be calculated similarly, or see [CE] for $Q_8$ and [Ev2] for $D_8$. The split extensions
are just direct products $S^1 \times H$, and the non-split extensions are as described below. The
comments indicate why they are not isomorphic to the split extensions.

$H \cong (C_p)^2$. $H^3 \cong \mathbb{F}_p$. The non-split extension is not abelian, and may be obtained from
either of the non-abelian groups $G$ of order $p^3$ as $\tilde{G}$.

$H \cong C_p^2 \times C_p$. $H^3 \cong \mathbb{F}_p$. As above the non-split extension is not abelian, and may
be obtained from any of the non-abelian extensions of $C_p$ by $C_p^2 \times C_p$ using the circle
construction.

$H \cong (C_p)^3$. $H^3 \cong \mathbb{F}_p^3$. The non-split extension is the direct product of $C_p$ and the non-
split extension of $S^1$ by $(C_p)^2$. This follows from the fact that any element of $H^3(C_p^3)$ will
restrict trivially to some $C_p$ subgroup.

$H \cong P_2$. $H^3 \cong \mathbb{F}_p \oplus \mathbb{F}_p$. The non-split extension has nilpotency class three, and may
be obtained from any of the groups of order $p^4$ and nilpotency class three via the circle
construction.

$H \cong D_8$. $H^3 \cong \mathbb{F}_2$. The generator restricts non-trivially to either $C_2 \times C_2$ subgroup (this
may be verified readily using the spectral sequence for this extension), so the non-split
extension has a non-abelian subgroup of index two.

Our approach to the classification of the non-central extensions is slightly different. If
$G$ is a non-central extension of $S^1$ by a group of order $2^n$, let $K$ be the centraliser in $G$ of $S^1$.
Then $K$ has index 2, and the isomorphism type of $K$ as a group equipped with a map from
$C_2$ to $\text{Out}(K)$ is an invariant of the group. For fixed $K$ such a structure is determined by
an involution in $\text{Out}(K)$, and an isomorphism between structures specified by involutions
$t_1$ and $t_2$ is determined by an element $s$ of $\text{Out}(K)$ such that $st_1 = t_2s$. Thus the first
stage of our classification of non-central extensions of $S^1$ by a group of order $2^n$ is to take
each $K$, a central extension of $S^1$ by a group of order $2^{n-1}$, and find all conjugacy classes of involutions in $\text{Out}(K)$ which restrict non-trivially to $S^1$. We shall classify only groups with 2, 4, and 8 components, so we need consider only one case in which $K$ is non-abelian, that when $K$ is the non-split central extension of $S^1$ by $(C_2)^2$. First we shall consider the case when $K$ is abelian. In this case $\text{Out}(K) = \text{Aut}(K)$, and if we pick a representative for each of our conjugacy classes of involution, these define distinct $\mathbb{Z}C_2$-module structures on $K$ containing the submodule $\widehat{S^1}$, that is $S^1$ with the non-trivial $C_2$-action. For each such $M$ we calculate $H^2(BC_2; M)$ and the action of $\text{Aut}_{\mathbb{Z}C_2}(M)$ (which is isomorphic to the centraliser in $\text{Aut}(K)$ of the involution used to define $M$) upon it. Finally we decide whether or not elements in distinct $\text{Aut}(M)$ orbits really do give non-isomorphic groups. If $K$ is not abelian, we pick a representative for each conjugacy class of involution in $\text{Out}(K)$ restricting non-trivially to $S^1$, and use each of these to define a $\mathbb{Z}C_2$-module structure $M'$ on the centre $Z(K)$ of $K$. Either there are no extensions of $K$ by $C_2$ corresponding to this involution, or such extensions are classified up to equivalence by $H^2(BC_2; M')$ (see [Br]), depending on the vanishing of an obstruction in $H^3(BC_2; M')$. (In the only case we need the obstruction group is trivial.) Once again we are left to determine whether or not inequivalent extensions really give non-isomorphic groups. There are sufficiently few cases to examine when $K$ is non-abelian that ad hoc methods suffice. In the abelian case the following lemma, suggested to the author by R. E. Borcherds, is useful.

**Lemma 1.10.** If $H$ is a finite group with a fixed map from $C_p$ to $\text{Aut}(H)$, then a non-split extension of $H$ by $C_p$ and the split extension cannot be isomorphic.

**Proof.** If $G$ is an extension of $H$ by $C_p$, it splits if and only if there is an element of order $p$ in $G \setminus H$, so a non-split extension has the same number of elements of order $p$ as $H$, whereas a split extension has more.

This lemma also applies to the case when $H$ is a compact abelian Lie group of dimension one, because these contain only finitely many elements of order $p$. 

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The non-central extensions of $S^1$ by $H$ are described below. We consider $S^1$ as the complex numbers of unit modulus, with automorphism $z \mapsto \bar{z}$.

$H$ of order 2. In this case $M$ must be $S^1$ with the non-trivial $C_2$ action. $H^2(BC_2; M) \cong \mathbb{F}_2$, and we obtain two groups.

$$\hat{D} = \langle S^1, T \mid T^2 = 1, z^T = \bar{z} \rangle$$

$$\hat{Q} = \langle S^1, T \mid T^2 = -1, z^T = \bar{z} \rangle$$

$\hat{D}$ is isomorphic to $O_2$, and $\hat{Q}$ is isomorphic to the subgroup of $SU_2$ of elements of the forms

$$\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}.$$

$H$ of order 4. $\text{Aut}(S^1 \times C_2) \cong C_2 \times C_2$, so there are two $C_2$ structures on $S^1 \times C_2$ restricting non-trivially to $S^1$. Considering $C_2$ as a subgroup of $\mathbb{C}^\times$, the two actions are

$$M_1 \quad (z, a) \mapsto (\bar{z}, a)$$

$$M_2 \quad (z, a) \mapsto (\bar{z}a, a)$$

$H^2(BC_2; M_1) \cong (\mathbb{F}_2)^2$, but two elements are exchanged by $\text{Aut}(M_1)$, and we obtain three groups, with presentations

$$\langle S^1, A, T \mid A^2 = T^2 = 1, z^A = z, A^T = A, z^T = \bar{z} \rangle$$

$$\langle S^1, T \mid T^4 = 1, z^T = \bar{z} \rangle$$

$$\langle S^1, A, T \mid A^2 = 1, T^2 = -1, z^A = z, A^T = A, z^T = \bar{z} \rangle$$

$H^2(BC_2; M_2) \cong \mathbb{F}_2$, and so we obtain two groups.

$$\langle S^1, A, T \mid A^2 = T^2 = 1, z^A = z, A^T = -A, z^T = \bar{z} \rangle$$

$$\langle S^1, T \mid T^4 = -1, z^T = \bar{z} \rangle$$

Of these five groups, three have quotient isomorphic to $C_2 \times C_2$, and two $C_4$. 

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We do not give presentations for the twenty-one non-central extensions of $S^1$ by groups of order eight, but we list the possible ‘modules’ (we shall refer to a non-abelian $K$ equipped with a choice of involution in $\text{Out}(K)$ as a $C_2$-‘module’), and then tabulate the groups that arise for each of them. In the abelian cases lemma 1·10 together with the calculation of the isomorphism types of the finite quotients suffices to show that extensions whose classes are in distinct $\text{Aut}(M)$ orbits in $H^2(BC_2; M)$ do give rise to distinct groups.

$\text{Aut}(S^1 \times C_4)$ is a group of order 16 containing seven conjugacy classes of involution, four of which restrict non-trivially to $S^1$. Considering $C_4$ as contained in $\mathbb{C}^\times$, the following four modules are representatives for the distinct isomorphism types.

- $M_1 \ (z, a) \mapsto (\bar{z}, a)$
- $M_2 \ (z, a) \mapsto (\bar{z}a, a)$
- $M_3 \ (z, a) \mapsto (\bar{z}, a^{-1})$
- $M_4 \ (z, a) \mapsto (\bar{z}a^2, a^{-1})$

$\text{Aut}(S^1 \times C_2^2)$ is isomorphic to $C_2 \times S_4$, so has five conjugacy classes of involutions, three of which restrict non-trivially to $S^1$. The following are representatives of each class.

- $M_5 \ (z, a, b) \mapsto (\bar{z}, a, b)$
- $M_6 \ (z, a, b) \mapsto (\bar{z}, b, a)$
- $M_7 \ (z, a, b) \mapsto (\bar{z}ab, a, b)$

There is also the non-split central extension of $S^1$ by $C_2 \times C_2$, which we shall refer to as $\tilde{Q}$, since it could be obtained from the quaternion group by applying the circle construction. It may be presented as follows.

$$\tilde{Q} = \langle S^1, A, B | A^2 = B^2 = 1 \quad z^A = z^B = z \quad [A, B] = -1 \rangle$$

$\text{Inn}(\tilde{Q})$ is isomorphic to $(C_2)^2$, $\text{Out}(\tilde{Q})$ is isomorphic to $C_2 \times S_3$, and $\text{Aut}(\tilde{Q})$ is the split extension of $(C_2)^2$ by $C_2 \times S_3$ where $C_2$ acts trivially and $S_3$ acts as $GL_2(\mathbb{F}_2)$. It follows that any involution in $\text{Out}(\tilde{Q})$ will give rise to a split extension of $\tilde{Q}$ by $C_2$. There are two
conjugacy classes of involution in $\text{Out}(\widetilde{Q})$ which restrict non-trivially to $\text{Aut}(S^1)$, and we may choose as lifts to $\text{Aut}(\widetilde{Q})$ of representatives of them the following elements.

\[
t_1 : z \mapsto \overline{z} \quad A \mapsto A \quad B \mapsto B
\]

\[
t_2 : z \mapsto \overline{z} \quad A \mapsto B \quad B \mapsto A
\]

For each of these the $\mathbb{Z}C_2$-module structure of $Z(\widetilde{Q})$ is isomorphic to $\hat{S}^1$, the non-trivial $\mathbb{Z}C_2$-module structure on $S^1$. Since $H^2(BC_2; \hat{S}^1)$ has order 2 there are in each case two equivalence classes of extensions of $\widetilde{Q}$ by $C_2$. In each case these two groups can be shown to be distinct by comparing the orders of elements in the components of the group not contained in $\widetilde{Q}$.

The results of the above calculations are contained in figure 1.1. Note that in each case there is a split extension, and the isomorphism type of the finite quotient of the split extension is listed first.

| Module | $|H^2(BC_2; M)|$ | Extensions | Quotients |
|--------|----------------|------------|-----------|
| $M_1$  | 4              | 3          | $C_2 \times C_4$, $C_2 \times C_4$, $C_8$ |
| $M_2$  | 2              | 2          | $C_2 \times C_4$, $C_8$ |
| $M_3$  | 4              | 3          | $D_8$, $D_8$, $Q_8$ |
| $M_4$  | 2              | 2          | $D_8$, $Q_8$ |
| $M_5$  | 8              | 3          | $(C_2)^3$, $(C_2)^3$, $C_2 \times C_4$ |
| $M_6$  | 2              | 2          | $D_8$, $D_8$ |
| $M_7$  | 2              | 2          | $(C_2)^3$, $C_2 \times C_4$ |
| $\hat{S}^1 = Z(\widetilde{Q})$ and $t_1$ | 2 | 2 | $(C_2)^3$, $(C_2)^3$ |
| $\hat{S}^1 = Z(\widetilde{Q})$ and $t_2$ | 2 | 2 | $D_8$, $D_8$ |

1.1. The non-central extensions of $S^1$ by groups of order eight.

We summarise our calculations in the following theorem.
Theorem 1.11. For $p$ an odd prime there are $8$ (resp. $3$, $1$) groups consisting of $p^3$ (resp. $p^2$, $p$) circles. There are $29$ (resp. $8$, $3$) groups consisting of $8$ (resp. $4$, $2$) circles, but in only $8$ (resp. $3$, $1$) of these is the $S^1$ subgroup central.

Remarks. It is desirable to have a stronger result than lemma 1.10, but informed opinion seems to be that for a fixed map from $Q$ to $\text{Aut}(N)$ it may be possible for the split extension and a non-split extension to be isomorphic groups. No examples seem to be known for $Q$ and $N$ both finite, but J. C. Rickard suggested the following.

Proposition 1.12. Let $N$ be $\prod_{n=1}^{\infty} C_4 \times \prod_{n=1}^{\infty} C_2$, and let $Q$ be $C_2$ acting trivially on $N$. Then

$$H^2(BQ; N) \cong \prod_{n=1}^{\infty} H^2(BC_2; C_4) \times \prod_{n=1}^{\infty} H^2(BC_2; C_2),$$

and any extension $G$ whose class restricts to zero in $\prod_{n=1}^{\infty} H^2(BC_2; C_4)$ is isomorphic to the split extension.

Proof. The split extension and all such $G$ are isomorphic to $N$. 

\[ \blacksquare \]
2. The Cohomology Rings of Various $p$-Groups.

In this section we shall calculate the integral and mod-$p$ cohomology rings of an infinite family of $p$-groups for odd $p$, which we shall call $P(n)$. The group $P(n)$ is defined for each $n \geq 3$, has order $p^n$, and may be presented as below.

$$\langle A, B, C \mid A^p = B^p = C^{p-n} = [A, C] = [B, C] = 1 \quad [A, B] = C^{p-n-3} \rangle$$

$P(3)$ is the non-abelian group of order $p^3$ and exponent $p$, which we also refer to as $P_2$, and $P(4)$ is the second group on Burnside’s list of groups of order $p^4$ (see appendix). The centre of $P(n)$ is the subgroup generated by $C$ of index $p^2$. Applying the circle construction described in section 1 to the whole centre we obtain the same group for all $n$, the unique non-abelian group consisting of $p^2$ circles, which we shall call $\tilde{P}$. The metacyclic groups with a cyclic subgroup of index $p$ which we shall refer to as $M(n)$, where $p^n$ is the order, also occur as subgroups of $\tilde{P}$, but their cohomology rings have been determined by other means [Th3], so we consider them only briefly.

The Calculation of $H^*(B\tilde{P}; \mathbb{Z})$

We now begin our calculation of $H^*(B\tilde{P}; \mathbb{Z})$ by examining the spectral sequence with integer coefficients for $\tilde{P}$ considered as an extension of $S^1$ by $C_p \oplus C_p$. The $E_2$ page is readily seen to be generated by elements $\alpha, \beta \in E_2^{2,0}, \gamma \in E_2^{3,0}$ and $\tau \in E_2^{0,2}$ subject only to the relations $p\alpha = p\beta = 0$, $p\gamma = 0$ and $\gamma^2 = 0$. Note that $\tau$ has infinite order. Since $E_2^{i,j}$ is trivial for $j$ odd, we see that all the even differentials must vanish. The behaviour of the differentials is summarised in the following lemma.

**Lemma 2.1.** In the above spectral sequence there are exactly two non-zero differentials, $d_3$ and $d_{2p-1}$. $d_3(\tau)$ is a non-zero multiple of $\gamma$, and $E_4$ is generated by the classes of the elements $\alpha, \beta, p\tau, \ldots, p\tau^{p-1}, \tau^p$ and $\tau^{p-1}\gamma$ (see figure 2.1). All of these generators are universal cycles except for $\tau^{p-1}\gamma$, which is mapped by $d_{2p-1}$ to a non-zero multiple of $\alpha^p\beta - \beta^p\alpha$. The $E_\infty$ page is generated by the elements $\alpha, \beta, p\tau, \ldots, p\tau^{p-1}, \tau^p$ subject only to the relations they satisfy as elements of $E_2$, and the relation $\alpha^p\beta = \beta^p\alpha$. 

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2.1. The $E_4$ page of the spectral sequence of lemma 2.1.

Proof. The derived subgroup of $\tilde{P}$ consists of the subgroup of its central $S^1$ of order $p$, so there can be no homomorphism from $\tilde{P}$ to $S^1$ that restricts to an isomorphism from the centre to $S^1$. It follows by considering the natural isomorphism $H^2(BG;\mathbb{Z}) \cong \text{Hom}(G,S^1)$ that the element $\tau$ cannot survive to $E_\infty$, so we must have $d_3(\tau)$ a non-zero multiple of $\gamma$. This determines $d_3$ completely. It may be checked that $E_4$ is isomorphic to the subring of $E_2$ generated by $\alpha, \beta, p\tau, \ldots, p\tau^{p-1}, \tau^p$ and $\tau^{p-1}\gamma$. All these elements must be universal cycles, with the possible exception of $\tau^{p-1}\gamma$, because the groups in which their images under $d_n$ lie are already trivial. (The $E_4$ page of the spectral sequence is depicted in figure 2.1, where $\chi_i = p\tau^i$, $\zeta = \tau^p$, and $\eta = \tau^{p-1}\gamma$.) The only remaining potentially
The Cohomology Rings of Various $p$-Groups

non-zero differential is $d_{2p-1}(\tau^{p-1}\gamma)$. To complete this proof it suffices to show that in the $E_\infty$ page the relation $\alpha^p \beta = \beta^p \alpha$ must hold.

Let $Q$ be the quotient of $\tilde{P}$ by its $S^1$ subgroup, and take generators $\alpha', \beta'$ for $H^2(BQ; \mathbb{Z})$ and $\gamma'$ for $H^3(BQ; \mathbb{Z})$. The statement that $\gamma$ does not survive to $E_\infty$ in the spectral sequence is equivalent to the statement that $\gamma'$ is mapped to zero by the inflation map from $Q$ to $\tilde{P}$. Now we calculate $\varphi(\gamma')$, where $\varphi$ is the integral cohomology operation $\delta_p P^1 \pi_*$, where $\pi_*$ is the map induced by the change of coefficients from $\mathbb{Z}$ to $\mathbb{F}_p$, $P^1$ is a reduced power, and $\delta_p$ is the Bockstein for the sequence $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{F}_p$. Taking $y, y' \in H^1(BQ; \mathbb{F}_p)$ such that $\delta_p(y) = \alpha'$, and $\delta_p(y') = \beta'$, we see that

$$
\varphi(\gamma') = \delta_p P^1 \pi_*(\gamma') = \delta_p P^1 (\beta_p(y)y' - \beta_p(y')y) = \delta_p (\beta_p(y)^p y' - \beta_p(y')^p y) = \alpha'^p \beta' - \beta'^p \alpha'.
$$

It follows that

$$
\alpha^p \beta - \beta^p \alpha = \text{Inf}(\alpha^p \beta' - \beta^p \alpha') = \text{Inf}(\varphi(\gamma')) = \varphi \text{Inf}(\gamma') = 0.
$$

We are now ready to state our theorem on $H^*(B\tilde{P})$.

**Theorem 2.2.** Let $p$ be an odd prime, and let $\tilde{P}$ be the group defined above. Then $H^*(B\tilde{P}; \mathbb{Z})$ is generated by elements $\alpha, \beta, \chi_1, \ldots, \chi_{p-1}, \zeta$, with

$$
deg(\alpha) = deg(\beta) = 2 \quad deg(\chi_i) = 2i \quad deg(\zeta) = 2p,
$$

subject to the following relations:

$$
p\alpha = p\beta = 0
$$

$$
\alpha^p \beta = \beta^p \alpha
$$

$$
\alpha \chi_i = \begin{cases} 
0 & \text{for } i < p - 1 \\
-\alpha^p & \text{for } i = p - 1 
\end{cases}
$$

$$
\beta \chi_i = \begin{cases} 
0 & \text{for } i < p - 1 \\
-\beta^p & \text{for } i = p - 1 
\end{cases}
$$

$$
\chi_i \chi_j = \begin{cases} 
p \chi_{i+j} & \text{for } i + j < p \\
p^2 \chi & \text{for } i + j = p \\
p \zeta \chi_{i+j-p} & \text{for } p < i + j < 2p - 2 \\
p \zeta \chi_{p-2} + \alpha^{2p-2} + \beta^{2p-2} - \alpha^{p-1} \beta^{p-1} & \text{for } i = j = p - 1 
\end{cases}
$$
Chern classes of representations of $\tilde{P}$ generate the whole ring. An automorphism of $\tilde{P}$ sends $\chi_i$ to $\chi_i$ (resp. $(-1)^i\chi_i$) and $\zeta$ to $\zeta$ (resp. $-\zeta$) if it fixes (resp. reverses) $S^1$. The effect of an automorphism on $\alpha$, $\beta$ may be determined from their definition. Considered as elements of $\text{Hom}(\tilde{P}, S^1)$, $\alpha$ has kernel $\langle S^1, B \rangle$ and sends $A$ to $e^{2\pi i/p}$, and $\beta$ has kernel $\langle S^1, A \rangle$ and sends $B$ to $e^{2\pi i/p}$. If we let $H$ be the subgroup generated by $B$ and elements of $S^1$ we may define

$$\chi_i = \begin{cases} \text{Cor}^\tilde{P}_{H}(\tau'^i) & \text{for } i < p - 1 \\ \text{Cor}^\tilde{P}_{H}(\tau'^{p-1}) - \alpha^{p-1} & \text{for } i = p - 1 \end{cases}$$

where $\tau'$ is any element of $H^2(BH; \mathbb{Z})$ restricting to $S^1$ as the generator $\tau$. Similarly, $\zeta = c_p(\rho)$, where $\rho$ is an irreducible representation of $\tilde{P}$ restricting to $S^1$ as $p$ copies of the representation $\xi$ with $c_1(\xi) = \tau$.

**Proof.** First we note that in the $E_\infty$ page of the above spectral sequence all the group extensions that we need to examine are extensions of finite groups by the infinite cyclic group, so are split. The elements $\alpha$ and $\beta$ defined in the statement above clearly yield generators for $E^{2,0}_\infty$, and the relations between them are exactly the relations that hold between the corresponding elements in the spectral sequence. Let $\beta'$ in $H^2(BH)$ be the restriction to $H$ of $\beta$, and take any choice of $\tau'$ as in the statement. We may show by considering $\beta'$ and $\tau'$ as homomorphisms from $H$ to $S^1$ that conjugation by $A^i$ induces the map on $H^2(BH)$ that fixes $\beta'$ and sends $\tau'$ to $\tau' - i\beta'$. Now applying the formula for $\text{Res}_K^G \text{Cor}^G_H$ (see for example [Br]) it follows that $\chi_i$ restricts to $S^1$ as $p\tau^i$, so yields a generator for $E^{0,2i}_\infty$.

Any irreducible representation of $\tilde{P}$ has degree 1 or $p$, because $\tilde{P}$ has an abelian subgroup of index $p$. Let $\rho$ be the representation of $\tilde{P}$ induced from a 1-dimensional representation of $H$ with first Chern class $\tau'$. $\rho$ restricts to $S^1$ as $p$ copies of the representation with first Chern class $\tau$, so its total Chern class restricts to $S^1$ as $(1 + \tau)^p$, and so $c_p(\rho)$ yields a generator for $E^{0,2p}_\infty$, and $c_i(\rho) = 1/p(p_i)\chi_i + P_i(\alpha, \beta)$ for some polynomial $P_i$. We shall show later that $P_i = 0$. 

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The restriction to $H$ of $\alpha$ is trivial, so by Frôbenius reciprocity

$$\alpha \text{Cor}_{\widetilde{P}}(\tau^i) = \text{Cor}_{\widetilde{P}}(\text{Res}_{\widetilde{P}}(\alpha)\tau^i) = 0,$$

and the expressions given for $\alpha \chi_i$ follow. By calculating $\alpha(\beta \chi_i) = \beta(\alpha \chi_i)$, we may deduce that $\beta \chi_i = 0$ for $i < p - 1$ and $\beta \chi_{p-1} = \lambda(\alpha^{p-1} \beta - \beta^p) - \alpha^{p-1} \beta$ for some scalar $\lambda$. To show that $\lambda = 1$ we use the restriction map to $H$, and the formula for corestriction followed by restriction.

$$\text{Res}_{\widetilde{P}}(\beta \chi_{p-1}) = \beta' \sum_{i=0}^{p-1} (\tau' + i\beta')^{p-1}$$

$$= \beta' \sum_{j=0}^{p-1} \tau'^{p-1-j} \beta'^j \sum_{i=0}^{p-1} i^j$$

Newton’s formula tells us that

$$\sum_{i=1}^{p-1} i^j \equiv \begin{cases} 0 \ (p) & \text{for } j \neq 0 \ (p - 1) \\ 1 \ (p) & \text{for } j \equiv 0 \ (p - 1) \end{cases}$$

so $\text{Res}_{\widetilde{P}}(\beta \chi_{p-1}) = -\beta'^p$, and the required relation follows.

We now know $\text{Res}_{\widetilde{S}^1}(\chi_i \chi_j)$, $\alpha \chi_i \chi_j$, and $\beta \chi_i \chi_j$, which together imply the relations given for $\chi_i \chi_j$. To complete the proof of the theorem we must determine the effect of automorphisms of $\widetilde{P}$ on the $\chi_i$. We know that an automorphism sends $c_i(\rho)$ to itself or $(-1)^i$ times itself depending whether or not it reverses the sense of $S^1$, so it will suffice to show that $\chi_i = 1/p(p) c_i(\rho)$. The character of $\rho$ is zero except on $S^1$, so if $\theta$ is a 1-dimensional representation of $\widetilde{P}$ restricting trivially to $S^1$, then $\rho \otimes \theta$ is isomorphic to $\rho$. If we apply the formula expressing $c.(\rho \otimes \theta)$ in terms of $c.(\rho)$ and $c.(\theta)$ (see [At]) we obtain

$$c_i(\rho) = c_i(\rho \otimes \theta) = \sum_{j=0}^{i} \binom{p - i + j}{j} c_1(\theta)^j c_{i-j}(\rho).$$

and hence inductively

$$c_i(\rho)c_1(\theta) = \begin{cases} 0 & \text{for } i < p - 1 \\ -c_1(\theta)^p & \text{for } i = p \end{cases}.$$
Since $\alpha$ and $\beta$ are possible values for $c_1(\theta)$ the required result follows. We may show inductively that $\chi_i$ is in the subring generated by Chern classes because $\chi_1$ is, and $\chi_1\chi_{i-1}$, $1/p\chi_i$ are coprime multiples of $\chi_i$.

We are now ready to state our theorem on the integral cohomology of $P(n)$.

**Theorem 2.3.** Let $p$ be an odd prime and let $P(n)$ be as defined above. Then $H^*(BP(n); \mathbb{Z})$ is generated by elements $\alpha, \beta, \mu, \nu, \chi^1, \ldots, \chi^{p-1}, \zeta$, with

\[
\begin{align*}
\deg(\alpha) &= \deg(\beta) = 2 & \deg(\mu) &= \deg(\nu) = 3 & \deg(\chi_i) &= 2i & \deg(\zeta) &= 2p \\
\end{align*}
\]

subject to the following relations:

\[
\begin{align*}
p\alpha &= p\beta = 0 & p\mu &= p\nu = 0 & p^{n-3}\chi_1 &= 0 & p^{n-2}\chi_i &= 0 & p^{n-1}\zeta &= 0 \\
\alpha\mu &= \beta\nu \\
\alpha^p\beta &= \beta^p\alpha & \alpha^p\mu &= \beta^p\nu \\
\alpha\chi_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\alpha^p & \text{for } i = p-1 \end{cases} \\
\beta\chi_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\beta^p & \text{for } i = p-1 \end{cases} \\
\mu\chi_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\beta^{p-1}\mu & \text{for } i = p-1 \end{cases} \\
\nu\chi_i &= \begin{cases} 0 & \text{for } i < p-1 \\ -\alpha^{p-1}\nu & \text{for } i = p-1 \end{cases} \\
\chi_i\chi_j &= \begin{cases} p\chi_{i+j} & \text{for } i+j < p \\ p^2\zeta & \text{for } i+j = p \\ p\chi_{p-2}^i + \alpha^{2p-2} + \beta^{2p-2} - \alpha^{p-1}\beta^{p-1} & \text{for } p < i+j < 2p-2 \\ \chi_3 & \text{for } n = 3, p = 3, \lambda = \pm1 \end{cases} \\
\mu\nu &= \begin{cases} 0 & \text{for } n > 3 \\ \lambda\chi_3 & \text{for } n = 3, p > 3, \lambda \in \mathbb{F}_p^\times \\ 3\lambda\zeta & \text{for } n = 3, p = 3, \lambda = \pm1 \end{cases}
\end{align*}
\]

Chern classes of representations of $P(n)$ generate $H^{ev}(BP(n); \mathbb{Z})$. Under an automorphism of $P(n)$ which restricts to the centre as $C \mapsto C^j$, $\chi_i$ is mapped to $j^i\chi_i$, and $\zeta$ is mapped to $j^p\zeta$. The effect of automorphisms on $\alpha$ and $\beta$ is determined by the natural isomorphism $H^2(BG; \mathbb{Z}) \cong \text{Hom}(G, \mathbb{R}/\mathbb{Z})$, under which

\[
\begin{align*}
\alpha : A &\mapsto 1/p & \beta : A &\mapsto 0 & \chi_1 : A &\mapsto 0 \\
B &\mapsto 0 & B &\mapsto 1/p & B &\mapsto 0 \\
C &\mapsto 0 & C &\mapsto 0 & C &\mapsto 1/p^{n-3}.
\end{align*}
\]
An automorphism of $P(n)$ which sends $\alpha$ to $n_1\alpha + n_2\beta$, $\beta$ to $n_3\alpha + n_4\beta$ and restricts to the centre as $C \mapsto C^j$ sends $\mu$ to $j(n_4\mu + n_3\nu)$ and $\nu$ to $j(n_2\mu + n_1\nu)$. If $\gamma'$ in $H^2(B\langle B, C \rangle; \mathbb{Z})$ is such that it maps to the following element of $\text{Hom}(\langle B, C \rangle, \mathbb{R}/\mathbb{Z})$
\[
\gamma' : B \mapsto 0
\]
\[
C \mapsto 1/p^{n-2},
\]
then $\chi_i$ is defined as follows:
\[
\chi_i = \begin{cases} 
\text{Cor}^{P(n)}_{\langle B, C \rangle}(\gamma''^i) & \text{for } i < p - 1 \\
\text{Cor}^{P(n)}_{\langle B, C \rangle}(\gamma'^{p-1}) - \alpha^{p-1} & \text{for } i = p - 1.
\end{cases}
\]

These are, up to scalar multiples, equal to $c_i(\rho)$, where $\rho$ is a $p$-dimensional irreducible representation of $P(n)$, whose restriction to $\langle C \rangle$ is a sum of $p$ copies of the representation $\theta$, with $c_1(\theta) = \text{Res}^{\langle B, C \rangle}_{\langle C \rangle}((\gamma'))$. In fact, $c_i(\rho) = 1/p^{(p^i)}\chi_i$. Also, we may define $\zeta = c_p(\rho)$.

**Proof.** We examine the spectral sequence for $B\langle P(n) \rangle$ as an $S^1$-bundle over $B\tilde{P}$. $E_2^{*0}$ is isomorphic to $H^*(B\langle P(n) \rangle; \mathbb{Z})$ and $E_2^{*,*}$ is freely generated by $E_2^{*0}$ and an element $\xi$ of infinite order in $E_2^{01}$. We know that $H^2(B\langle P(n) \rangle) \cong \text{Hom}(P(n), S^1) \cong C_{p^n-3} \oplus C_p \oplus C_p$, so $d_2(\xi)$ must be $\pm p^{n-3}\chi_1$. If we wanted to calculate the cohomology of the metacyclic groups $M(n)$ described above, the differential in this spectral sequence would send $\xi$ to $\pm p^{n-3}\chi_1 + \gamma$ for some non-zero $\gamma$ in $\langle \alpha, \beta \rangle$. It is now easy to see that $E_\infty$ is generated by the elements $\alpha$, $\beta$, $\mu = \beta\xi$, $\nu = \alpha\xi$, $\chi_1, \ldots, \chi_{p-1}$ and $\zeta$ subject to the relations they satisfy as elements of $E_2^{*,*}$ together with $p^{n-3}\chi_1 = 0$, $p^{n-2}\chi_i = 0$, and $p^{n-1}\zeta = 0$. For each $m$, the filtration of $H^m(B\langle P(n) \rangle)$ given by the $E_\infty$ page is trivial, so we may use the same symbols to denote elements of $H^m(B\langle P(n) \rangle)$, and the relations that hold in $E_\infty$ determine all the relations that hold in $H^m(B\langle P(n) \rangle)$ except for the product of the two odd dimensional generators.

We know that $p\mu\nu = 0$, and the relation $\alpha\mu = \beta\nu$ implies that $\alpha\mu\nu = \beta\mu\nu = 0$, and so $\mu\nu$ must be a multiple of $p^{n-3}\chi_3$ for $p \geq 5$ (resp. $3^{n-2}\zeta$ for $p = 3$). Note that these elements restrict to zero on all proper subgroups of $P(n)$. In the case of $P(3)$, Lewis [Lew]
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shows that $\mu\nu$ is not zero by considering the spectral sequence for $P(n)$ considered as an extension of a maximal subgroup by $C_p$. A similar method will work in general, but we offer an alternative proof that involves expressing $\mu$ and $\nu$ as Bocksteins of elements of $H^2(BP(n);\mathbb{F}_p)$. This proof is contained in lemma 2.4 and corollary 2.5.

The effect of automorphisms on $\chi_i$ and $\zeta$ is easily seen to be as claimed from their alternative definitions as Chern classes. To determine the effect of automorphisms on $\mu$ and $\nu$, we note that an automorphism of $P(n)$ restricting to the centre as $C \mapsto C^j$ extends to an endomorphism of $\tilde{P}$ which wraps the central circle $j$ times around itself, so induces a map of the above spectral sequence to itself sending $\xi$ to $j\xi$. This completes the proof of theorem 2.3 modulo lemma 2.4 and its corollary.

We now examine the spectral sequence with $\mathbb{F}_p$ coefficients for the central extension $C_{p^{n-2}} \to P(n) \to C_p \oplus C_p$. Take generators so that $H^*(BC_p \oplus C_p; \mathbb{F}_p) \cong \mathbb{F}_p[x, x'] \otimes \Lambda[y, y']$, where $\beta_p(y) = x, \beta_p(y') = x'$, and $H^*(BC_{p^{n-2}}; \mathbb{F}_p) \cong \mathbb{F}_p[t] \otimes \Lambda[u]$, where $\beta_p(u) = t$ for $n = 3$ (resp. $\beta_p(u) = 0$ for $n \geq 4$). Then the $E_2$ page is isomorphic to $\mathbb{F}_p[x, x', t] \otimes \Lambda[y, y', u]$, and the first two differentials are as described in the following lemma.

**Lemma 2.4.** With notation as above, identify the elements $x, x', y, y'$ in the spectral sequence with their images in $H^*(BP(n);\mathbb{F}_p)$ under the inflation map.

1) Let $n \geq 4$. Then $d_2$ is trivial, and $d_3(t)$ is a non-zero multiple of $xy' - x'y$. The elements $x, x', yy', u'y, u' y'$, form a basis for $H^2(BP(n))$, where $u'$ is any element of $H^1(BP(n))$ restricting to $C_{p^{n-2}}$ as $u$.

2) Let $n = 3$. Then $d_2(u)$ is a non-zero multiple of $yy'$, $d_2(t) = 0$, and $E_3$ is generated by $y, y', x, x', [uy], [uy']$ and $t$ subject to the relation $yy' = 0$ and those implied by the relations in $E_2$. In particular $[uy]y' = -[uy']y$ but this element is non-zero. As in the case $n \geq 4$, $d_3(t)$ is a non-zero multiple of $xy' - x'y$. Let $Y, Y'$ be elements of $H^2(BP(3))$ such that $x, x', Y, Y'$ form a basis for $H^2(BP(3))$, and let $X = \beta_p(Y), X' = \beta_p(Y')$. Then $yY', xy, xy', x'y', X, X'$, form a basis for $H^3(BP(3))$ and $xX, xX', x^2y, x^2y', xx'y', x^2y', YX'$, form a basis for $H^5(BP(3))$. The $E_4$ page of this spectral sequence is depicted in figure
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2.2, where boldface numerals indicate numbers of generators required in the case $p > 3$, and plain numerals the case $p = 3$.

|  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
|  |  | 1 1  | 1 1  | 0 2  | 2 2  |
|  | $[t^2y],[t^2y']$ | 3 4 | | 4 6 |
|  | $[uy][ty']$ | 1 1 | | 2 2 |
|  | $[ty],[ty']$ | $x[ty],x[ty']$, $x'[ty']$ | | 4 4 |
|  | $[uy],[uy']$ | $x[uy],x[uy']$, $x'[uy],x'[uy']$ | | 6 6 |
| 1 | $y,y'$ | $x,x'$ | $xy,xy'$, $x^2,xx'$, $x^2y,x^2y'$ | $x'y'$ | $x^2$, $xx'y',x^2y'$ |

2.2. The $E_4$ page of the spectral sequence of lemma 2.4.

Proof. 1). In this case $H^1$ has order $p^3$, so $u$ must survive. The element $xy' - x'y$ is the image under $\pi_*$ of a generator for $H^3(B(C_p \oplus C_p); \mathbb{Z})$, so must be killed by some differential. We have already shown that it cannot be killed by $d_2$, so the only possibility is that $t$ survives until $E_3$ and kills it. The rest of the statement follows easily.

2). In this case $H^1$ has order $p^2$, so $d_2(u)$ must be non-zero. It is true in general that if $G$ is a central extension of $C_p$ by $Q$, then in the corresponding spectral sequence with $\mathbb{F}_p$ coefficients $d_2 : E^{0,1}_2 \to E^{2,0}_2$ must kill the extension class. This follows by naturality, since one may regard the extension class as defining a homotopy class of maps from $BQ$ to $K(C_p,2)$ such that $BG$ is the $BC_p$-bundle induced by the path-loop fibration over $K(C_p,2)$. Since all subgroups of $P(3)$ of order $p^2$ are copies of $C_p \oplus C_p$, the extension class of $P(3)$ must restrict to zero on all cyclic subgroups, so must be a multiple of $yy'$. The transgression commutes with the Bockstein so $d_2(t) = 0$ and $d_3(t) = \beta_p d_2(u)$.

Given the values of these differentials it is routine to compute the $E_4$ page of the
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spectral sequence. If we write $E_r^n = \bigoplus_{i+j=n} E_r^{i,j}$, then $\{[uy], [uy'], x, x'\}$ forms a basis for $E_2^4 = E_\infty^2$, and $\{[ty], [ty'], [uy]y', xy, xy', x'y'\}$ forms a basis for $E_3^4 = E_\infty^3$. The spectral sequence operation $\beta$ introduced by Araki [Ar] and Vasquez [Va] maps $[uy]$ to $[ty]$ and $[uy']$ to $[ty']$, so if $Y$ and $Y'$ are chosen to yield the generators for $E_4^{1,1}$ their Bocksteins yield generators for $E_4^{1,2}$. A basis for $E_4^5$ is given by the eight elements of the statement, which we know to be universal cycles, and the elements $[t^2y], [t^2y']$. $E_4^4$ consists of universal cycles, and the universal coefficient theorem tells us that $H^5$ has order $p^8$, so $[t^2y]$ and $[t^2y']$ cannot be universal cycles.

The author has determined all the differentials in the above spectral sequence, using theorems 2.14 and 2.15. The non-trivial differentials are $d_2, d_3, d_4, d_2p−2,$ and $d_2p−1$. The $E_\infty$ page of this spectral sequence in the case $p = 7$ is depicted in figure 2.4 (on page 65).

**Corollary 2.5.** In $H^*(BP(n); \mathbb{Z})$ the product $\mu \nu$ is non-zero if and only if $n = 3$.

**Proof.** In the notation of lemma 2.4 it suffices to determine $\delta_p(u'y)\delta_p(u'y')$ in the case $n \geq 4$, and $\delta_p(Y)\delta_p(Y')$ in the case $n = 3$. In the case when $n = 3$,

$$\delta_p(Y)\delta_p(Y') = \delta_p(Y\beta_p(Y')) = \delta_p(YX').$$

The kernel of $\delta_p : H^5(BP(3); \mathbb{F}_p) \to H^6(BP(3); \mathbb{Z})$ is equal to $\pi_*(H^5(BP(3); \mathbb{Z}))$, which is generated by $xX$, $xX'$ and $x'X'$, so by lemma 2.4 $\delta_p(YX')$ is non-zero.

In the case when $n = 4$, $H^i(BP(4); \mathbb{Z})$ has exponent $p$ for $i = 2, 3$, so $\pi_*$ is injective from these groups, and $\text{Ker}\beta_p : H^2(BP(4)) \to H^3(BP(4))$ is equal to $\beta_p(H^1(BP(4)))$. $\beta_p(yy') = xy' - x'y = 0$, so we may choose the element $u'$ in lemma 2.4 so that $\beta_p(u') = \lambda yy'$ for some non-zero $\lambda$. Then we have

$$\delta_p(u'y)\delta_p(u'y') = \delta_p(u'y\beta_p(u'y')) = \delta_p(u'y(\lambda yy'y' - u'x')) = 0.$$

The case when $n \geq 5$ is similar but simpler, since $u'$ may be chosen so that $\delta_p(u') = p^{n-4}\chi_1$, which implies that $\beta_p(u') = 0$. ■
Remarks. The author can not think of a method to determine the non-zero constant \( \lambda \) in the equation for \( \mu \nu \) in the case \( \lambda = 3 \). If we replace the generators \( \alpha \) and \( \nu \) by \( \lambda^{-1} \alpha \) and \( \lambda^{-1} \nu \) respectively and leave the other generators unchanged, the new generating set will satisfy exactly the same relations as the old ones, except that the constant \( \lambda \) is replaced by 1.

Theorem 2.3 contains independent proofs of Thomas’ result that the even degree subring of \( H^*(BP(n); \mathbb{Z}) \) is generated by Chern classes [Th2], and Lewis’ calculation of \( H^*(BP(3); \mathbb{Z}) \) [Lew]. Our notation differs slightly from that of Lewis. We have renumbered the generators \( \chi_i \) (note that \( \chi_1 \) vanishes for \( n = 3 \)). Also our \( \chi_{p-1} \) and Lewis’ \( \chi_{p-2} \) are related by the formula

\[
\chi_{p-2}^{\text{Lewis}} = \chi_{p-1} + \alpha^{p-1} + \beta^{p-1}.
\]

Our result disagrees with that of AlZubaidy [Al2]. We now use a similar method to obtain the result of Lewis [Lew] and Thomas [Th1] concerning the metacyclic groups \( M(n) \), which may be presented as follows.

\[
M(n) = \langle A, B \mid A^p = B^{p^{n-1}} = 1 \quad [B, A] = B^{p^{n-2}} \rangle
\]

**Theorem 2.6.** Let \( M(n) \) be the metacyclic \( p \)-group defined above. Then \( H^*(BM(n); \mathbb{Z}) \) is generated by elements \( \alpha, \chi_1, \ldots, \chi_{p-1}, \zeta, \eta \), with

\[
\deg(\alpha) = 2 \quad \deg(\chi_i) = 2i \quad \deg(\zeta) = 2p \quad \deg(\eta) = 2p + 1
\]

subject to the following relations:

\[
p\alpha = 0 \quad p^{n-2} \chi_i = 0 \quad p^{n-1} \zeta = 0 \quad p\eta = 0
\]

\[
\alpha \chi_i = \begin{cases} 
0 & \text{for } i < p-1 \\
-\alpha^p & \text{for } i = p-1
\end{cases}
\]

\[
\eta \chi_i = \begin{cases} 
0 & \text{for } i < p-1 \\
-\alpha^{p-1} \eta & \text{for } i = p-1
\end{cases}
\]

\[
\chi_i \chi_j = \begin{cases} 
p\chi_i + j & \text{for } i + j < p \\
p^2 \zeta & \text{for } i + j = p \\
p \zeta \chi_i + j - p & \text{for } p < i + j < 2p - 2 \\
p \zeta \chi_{p-2} + \alpha^{2p-2} & \text{for } i = j = p - 1
\end{cases}
\]
The even-dimensional generators are multiples of Chern classes of irreducible representations of \( M(n) \). Considered as elements of \( \text{Hom}(G, \mathbb{R}/\mathbb{Z}) \), \( \alpha \) sends \( A \) to \( 1/p \) and \( B \) to \( 0 \), and \( \chi_1 \) sends \( A \) to \( 0 \) and \( B \) to \( 1/p^{n-2} \). Under an automorphism of \( M(n) \) which sends \( \alpha \) to \( n_1 \alpha + n_2 p^{n-3} \chi_1 \) and restricts to the centre as \( B^p \mapsto B^{3p} \), \( \chi_i \) is mapped to \( j^i \chi_i \), \( \zeta \) to \( j^p \zeta \), and \( \eta \) to \( jn_1 \eta \). Let \( H \) be the subgroup of \( M(n) \) generated by \( B \). If \( \beta' \) is the element of \( H^2(BH; \mathbb{Z}) \) mapping \( B \) to \( 1/p^{n-1} \), then we may define

\[
\chi_i = \begin{cases} 
\text{Cor}_{H}^{M(n)}(\beta'^i) & \text{for } i < p - 1, \\
\text{Cor}_{H}^{M(n)}(\beta'^{p-1}) - \alpha^{p-1} & \text{for } i = p - 1.
\end{cases}
\]

**Proof.** As in the case of \( P(n) \), we consider the spectral sequence for \( BM(n) \) as an \( S^1 \)-bundle over \( \widetilde{B}P \). The \( E_2 \) page is as in theorem 2.3, and we may take \( d_2(\xi) = \beta - p^{n-3} \chi_1 \). Since \( A \) is in the kernel of \( \beta \) this is consistent with the relation \( A^p = 1 \) in \( M(n) \). The \( E_\infty \) page is seen to be generated by the elements \( \alpha, \chi_i, \zeta \), and \( \xi \alpha(\alpha^{p-1} - \beta^{p-1}) \), subject to the relations they satisfy as elements of \( E_2 \) together with \( p^{n-2} \chi_i = 0 \), \( p^{n-1} \zeta = 0 \). These relations completely determine those that hold in \( H^*(BM(n); \mathbb{Z}) \). The action of \( \text{Aut}(M(n)) \) is determined as in theorem 2.3.  

**Remarks.** Ths result confirms Thomas’ calculation of \( H^*(BM(n); \mathbb{Z}) \) [Th1] which generalised Lewis’ calculation for \( H^*(BM(3); \mathbb{Z}) \) [Lew], but note that our generator in degree \( 2p - 2 \) is not the usual one. Their method is simpler than the one used here (if one is not also interested in \( P(n) \)), but does not seem to yield the action of \( \text{Aut}(M(n)) \), which the author believes to be a new result. The method involves considering \( M(n) \) as an extension of \( C_{p^{n-1}} \) by \( C_p \). The \( C_{p^{n-1}} \) subgroups of \( M(n) \) are not characteristic however, so \( \text{Aut}(M(n)) \) does not act on this extension, or on the corresponding spectral sequence.
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**Calculation of $H^*(B\widehat{P}; \mathbb{F}_p)$.**

**The Spectral Sequence.** We proceed as for the corresponding calculations in integral cohomology. We start by examining the spectral sequence with mod-$p$ coefficients for the central extension:

$$S^1 \to \tilde{P} \to C_p^A \oplus C_p^B.$$ 

If we write

$$H^*(BC_p^A \oplus C_p^B; \mathbb{F}_p) = \mathbb{F}_p[x, x'] \otimes \Lambda[y, y'],$$

where $\beta(y) = x$ and $\beta(y') = x'$, and

$$H^*(BS^1; \mathbb{F}_p) = \mathbb{F}_p[t]$$

then we see that:

$$E_2^{i,j} = \mathbb{F}_p[x, x', t] \otimes \Lambda[y, y'].$$

We must have $d_3(t) = \lambda(xy' - x'y)$ for some non-zero $\lambda$, and so $E_4^{*,*}$ is generated by elements $y, y', x, x', [t^p], [tyy'], \ldots, [t^{p-1}yy']$, and $[t^{p-1}(xy' - x'y)]$, subject to the relations that they satisfy in $E_2^{*,*}$, together with the relations

$$t^i xy' = t^i x'y \text{ for } i \neq -1 \quad (p).$$

If we write $z$ for $[t^p]$, $c_i$ for $[t^{i-1}yy']$, and $d$ for $[t^{p-1}(xy' - x'y)]$, then $E_4^{*,*}$ is as shown in figure 2-3.

Using the universal coefficient theorem, and our calculation of $H^*(B\widehat{P}; \mathbb{Z})$, we find that

$$|H^{2i}(B\widehat{P}; \mathbb{F}_p)| = |H^{2i-1}(B\widehat{P}; \mathbb{F}_p)| = p^{i+2} \text{ for } i = 1, \ldots, p - 1,$$

and we deduce that $c_i$ is a universal cycle for $i \leq p - 1$, and hence $E_{2p-1}^{*,*} \cong E_4^{*,*}$.

The mod-$p$ reduction of $\zeta \in H^{2p}(B\widehat{P}; \mathbb{Z})$ restricts non-trivially to $S^1$, so $z$ is a universal cycle. Applying $P^1$, and $\beta P^1$ to $xy' - x'y$, we obtain:

$$0 = P^1(xy' - x'y) = x^py' - x'^py,$$

$$0 = \beta(x^py' - x'^py) = x^px' - x'^px.$$
2.3. The $E_4$ page of the spectral sequence discussed above.

We know that $z, z(\lambda y + \mu y')$ are universal cycles, so the only way the above relations can be introduced into $E_{\infty}^{\ast,\ast}$ is if

\[
d_{2p-1}(c_p) = \lambda(x^py' - x'^py)
\]

\[
d_{2p-1}(d) = \lambda'(x^px' - x'^px)
\]

for some non-zero scalars $\lambda, \lambda'$.

Now it is easy to see that $E_{2p}^{\ast,\ast} \cong E_{\infty}^{\ast,\ast}$, and is generated by elements $y, y', x, x', z, c_2, \ldots, c_{p-1}$, with bidegrees as above, subject to the relations:

\[
y^2 = y'^2 = 0 \quad xy' = x'y
\]

\[
x^py' = x'^py \quad x^px' = x'^px
\]
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$$c_i y = c_i y' = 0 \quad c_i x = c_i x' = 0 \quad \text{for all } i$$

$$c_i c_j = 0 \quad \text{for all } i, j.$$

**Multiplicative Relations.** We use the notation $\pi_*$ and $\delta_p$ introduced during our integral calculations for the maps

$$\pi_* : H^n(BG; \mathbb{Z}) \longrightarrow H^n(BG; \mathbb{F}_p)$$

$$\delta_p : H^n(BG; \mathbb{F}_p) \longrightarrow H^{n+1}(BG; \mathbb{Z}).$$

It is easy to see that we may choose the elements $\pi_*(\alpha), \pi_*(\beta), \pi_*(\chi_i),$ and $\pi_*(\zeta)$ respectively to yield the generators $x, x', c_i,$ and $z$ of $E^{*,*}_\infty.$ We define generators of $H^*(B\tilde{P}; \mathbb{F}_p)$ by:

$$x = \pi_*(\alpha)$$

$$x' = \pi_*(\beta)$$

$$c_i = \pi_*(\chi_i) \quad \text{for } i = 2, \ldots, p - 1$$

$$z = \pi_*(\zeta).$$

Then elements $y, y'$ are uniquely defined by the equations:

$$\beta(y) = x \quad \beta(y') = x'.$$

The relations between these generators follow from the spectral sequence and the above definitions, and we obtain the following theorem, which has also been obtained independently by Moselle [Mo].

**Theorem 2.7.** $H^*(B\tilde{P}; \mathbb{F}_p)$ is generated by elements $y, y', x, x', c_2, \ldots, c_{p-1}, z$ with

$$\deg(y) = \deg(y') = 1, \quad \deg(x) = \deg(x') = 2, \quad \deg(c_i) = 2i, \quad \deg(z) = 2p,$$

subject to the following relations:

$$xy' = x'y$$

$$x^p y' = x'^p y \quad x^p x' = x'^p x$$

$$\beta(y) = x \quad \beta(y') = x'.$$
c_i y = \begin{cases} 
0 & \text{for } i < p - 1 \\
-x^{p-1} y & \text{for } i = p - 1 
\end{cases}

\quad c_i y' = \begin{cases} 
0 & \text{for } i > p - 1 \\
-x^{p-1} y' & \text{for } i = p - 1 
\end{cases}

\quad c_i x = \begin{cases} 
0 & \text{for } i < p - 1 \\
-x^p & \text{for } i = p - 1 
\end{cases}

\quad c_i x' = \begin{cases} 
0 & \text{for } i > p - 1 \\
-x^p & \text{for } i = p - 1 
\end{cases}

\quad c_i c_j = \begin{cases} 
0 & \text{for } i+j < 2p - 2 \\
x^{2p-2} + x'^{2p-2} - \chi x^{p-1} x'^{p-1} & \text{for } i=j = p - 1
\end{cases}

An automorphism of $\tilde{P}$ sends $c_i$ to $c_i$ (resp. $(-1)^i c_i$) and $z$ to $z$ (resp. $-z$) if it fixes (resp. reverses) $S^1$. The effect of an automorphism on $y, y', x, x'$ may be determined by their definitions. Using the natural isomorphism $H^1(B\tilde{P}; \mathbb{F}_p) \cong \text{Hom}(\tilde{P}, \mathbb{F}_p)$, we define:

$y : A \mapsto 1 (p)$ \quad $y' : A \mapsto 0 (p)$

$B \mapsto 0 (p)$ \quad $B \mapsto 1 (p),$

and then define $x$ and $x'$ by the equations:

$x = \beta(y) \quad x' = \beta(y').$

We may define $c_i$ (resp. $z$) to be the image under $\pi_*$ of $\chi_i$ (resp. $\zeta$), as defined during the statement of theorem 3. $\pi_*(\chi_1)$ is $\lambda y y'$ for some non-zero $\lambda$.

**Proof.** We have done most of this already. We should check that the two definitions of $x$, as $\pi_*(\alpha)$ and $\beta(y)$, agree. It suffices for this to check that $\delta_p(y) = \alpha$, because $\beta = \pi_* \delta_p$. This is true by naturality of the Bockstein for maps of short exact sequences of coefficient modules, in particular for the map:

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{x^p} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} \times \mathbb{Z} & \xrightarrow{\times 1/p} & \mathbb{Q} \\
\downarrow & & \downarrow \\
\mathbb{Q} & \rightarrow & \mathbb{Q}/\mathbb{Z}.
\end{array}
\]

Also, we note that $\delta_p : H^{2i+1}(B\tilde{P}; \mathbb{F}_p) \rightarrow H^{2i+2}(B\tilde{P}; \mathbb{Z})$ is injective for all $i$, and $\delta_p(c_i y) = \chi_i x$, so the relations involving $c_i$ and $y, y'$ follow.

Much of the action of the Steenrod algebra $A_p$ on $H^*(B\tilde{P}; \mathbb{F}_p)$ is determined implicitly by theorem 2.7. For example $c_i$ is expressible as $\text{Cor}_H\tilde{P}(t^i) (-x^{p-1} \text{ when } i = p - 1)$, so $P^1(c_i)$ is easily calculated. An example of this calculation occurs in the proof of theorem 2.14. The only further piece of information required to specify the action of $A_p$ is $P^1(z)$, which does not follow immediately from theorem 2.7, but may be determined as follows.
PROPOSITION 2.8. With notation as in theorem 2.7,

\[ P^1(z) = zc_{p-1}. \]

Proof. Throughout this proof \( K \) shall stand for any of the \( p+1 \) subgroups of \( \tilde{P} \) of index \( p \) (each of which is isomorphic to \( S^1 \oplus C_p \)), and \( \text{Res} = \text{Res}_{\tilde{P}}^K \). Let \( H^*(B\tilde{K}; \mathbb{F}_p) = \mathbb{F}_p[t, \bar{x}] \otimes \Lambda[\bar{y}] \), where \( t \) restricts to \( S^1 \) as the mod-\( p \) reduction of the standard generator of \( H^2(BS^1; \mathbb{F}_p) \) (called \( \tau \) in the proof of theorem 2.2), and \( \beta(\bar{y}) = \bar{x} \). There are \( p-1 \) possible choices for \( \bar{y} \) (and hence for \( \bar{x} \)), and \( p \) choices for \( t \). If \( \rho \) is the representation of \( \tilde{P} \) mentioned in the statement of theorem 2.2, then

\[ \text{Res}(\pi_* c.(\rho)) = \prod_{i=0}^{p-1} (1 + t + i\bar{x}) = 1 - \bar{x}^{p-1} + t^p - \bar{x}^{p-1} t, \]

and this expression is independent of the choice of \( t \) and \( \bar{x} \). It follows that

\[ \text{Res}(c_i) = \begin{cases} 0 & \text{for } i < p-1 \\ -\bar{x}^{p-1} & \text{for } i = p-1 \end{cases} \]

\[ \text{Res}(z) = t^p - \bar{x}^{p-1} t. \]

It follows that \( \text{Res}(P^1(z)) = \bar{x}^{2p-2} t - \bar{x}^{p-1} t^p \). A typical element of \( H^{4p-2}(B\tilde{P}; \mathbb{F}_p) \) is of the form \( \lambda zc_{p-1} + zP(x, x') + Q(x, x') \). We shall show that \( zc_{p-1} \) is the only such element restricting correctly to each \( K \).

First, note that \( \text{Res}(Q(x, x')) = \lambda' \bar{x}^{2p-1} \), and that \( \lambda' = 0 \) for \( K = \langle S^1, AB^i \rangle \) (resp. \( K = \langle S^1, B \rangle \)) iff \( x' - ix \) (resp. \( x \)) divides \( Q \). Hence \( x^p x' - x'^p x \) must divide \( Q \), so \( Q \) is zero in \( H^*(B\tilde{P}) \). To complete the proof it suffices to show that for each \( \lambda'' \in \mathbb{F}_p \), no polynomial \( P(x, x') \) homogeneous of degree \( p-1 \) can restrict to all \( K \) as \( \lambda'' \bar{x}^{p-1} \). For every choice of \( K \) except \( \langle S^1, A \rangle \) we may choose \( \bar{x} \) to be \( \text{Res}(x') \). We have \( p \) such \( K \), and \( x \) will restrict to each of them as a distinct multiple of \( \bar{x} \). Hence \( P \) as above would have to satisfy \( P(X, 1) - \lambda'' = 0 \) for all \( X \in \mathbb{F}_p \), but \( P(X, 1) \) cannot have \( p \) roots. \( \blacksquare \)
Calculation of $H^*(BP(n); \mathbb{F}_p)$ for $n \geq 4$.

The Spectral Sequence. We follow the method used in the corresponding calculations for integral cohomology, and consider the spectral sequence of the fibration:

$$S^1/Z(P(n)) \longrightarrow BP(n) \longrightarrow \tilde{BP}(n).$$

This has

$$E_2^{i,j} \cong H^i(\tilde{BP}; H^j(S^1; \mathbb{F}_p)),$$

so if we set

$$H^*(S^1; \mathbb{F}_p) = \Lambda[u],$$

then $E_2^{*,*}$ is the anticommutative ring generated by $H^*(\tilde{BP}; \mathbb{F}_p)$ and $u$. We may apply lemma 1.1, and we deduce that the spectral sequence collapses iff $n > 3$. For $n > 3$ the relations in $H^*(BP(n); \mathbb{F}_p)$ follow easily from those in $H^*(\tilde{BP}; \mathbb{F}_p)$. In the case $n = 3$, $d_2(u) = \lambda(yy')$ for some non-zero $\lambda$, and so $E_\infty^{*,*}$ requires many generators which appear in $E_\infty^{*,1}$. There is a corresponding increase in the number of elements required to generate $H^*(BP(3); \mathbb{F}_p)$ and consequently in the number of relations we must calculate. We are forced to use new techniques, which shall be introduced in the next chapter.

Theorem 2.9. Let $n$ be at least 4. Then $H^*(BP(n); \mathbb{F}_p)$ is generated by elements $u, y, y', x, x', c_2, \ldots, c_{p-1}, z$ with

$$\deg(u) = \deg(y) = \deg(y') = 1, \quad \deg(x) = \deg(x') = 2, \quad \deg(c_i) = 2i, \quad \deg(z) = 2p,$$

subject to the following relations:

$$xy' = x'y$$
$$x^py' = x'^py \quad x^px = x'^px$$
$$\beta(y) = x \quad \beta(y') = x'$$
$$\beta(u) = \begin{cases} y'y & \text{for } n = 4 \\ 0 & \text{for } n > 4 \end{cases}$$
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\[
c_i y = \begin{cases} 
0 & \text{for } i < p - 1 \\
-x^{p-1} y & \text{for } i = p - 1 
\end{cases}
\]

\[
c_i y' = \begin{cases} 
0 & \text{for } i < p - 1 \\
-x'^{p-1} y' & \text{for } i = p - 1 
\end{cases}
\]

\[
c_i x = \begin{cases} 
0 & \text{for } i < p - 1 \\
-x^p & \text{for } i = p - 1 
\end{cases}
\]

\[
c_i x' = \begin{cases} 
0 & \text{for } i < p - 1 \\
-x'^p & \text{for } i = p - 1 
\end{cases}
\]

\[
c_i c_j = \begin{cases} 
0 & \text{for } i + j < 2p - 2 \\
x^{2p-2} + x'^{2p-2} - x^{p-1} x'^{p-1} & \text{for } i = j = p - 1 
\end{cases}
\]

Under an automorphism of \( P(n) \) which restricts to the centre as \( C \mapsto C^j \), \( c_i \) is mapped to \( j^i c_i \) and \( z \) is mapped to \( jz \). We define \( u, y, y' \) by regarding them as elements of \( \text{Hom}(P(n), \mathbb{F}_p) \) then

\[
u : A \mapsto 0 \quad y : A \mapsto 1 \quad y' : A \mapsto 0
\]

\[
B \mapsto 0 \quad B \mapsto 0 \quad B \mapsto 1.
\]

\[
C \mapsto 1 \quad C \mapsto 0 \quad C \mapsto 0.
\]

This determines the effect of automorphisms on \( u, y, y' \) and on \( x = \beta(y), \ x' = \beta(y') \). We may define

\[
c_i = \pi_*(\chi_i) \quad \text{and} \quad z = \pi_*(\zeta).
\]

Proof. Almost all of this follows from our work above. We may, of course choose \( u \) to be any element of \( H^1(BP(n); \mathbb{F}_p) \) not in the span of \( y, y' \). With our definition it follows that \( \delta_p(u) = p^{n-4} \chi_1 \), so we obtain

\[
\beta(u) = \begin{cases} 
\lambda yy' & \text{for some } \lambda \neq 0 \text{ for } n = 4 \\
0 & \text{for } n > 4.
\end{cases}
\]

It can be checked by explicit calculation with 1- and 2-cochains that in the case \( n = 4 \)

\[
\beta(u) = y'y = -yy',
\]

and hence also that for \( n = 4 \)

\[
\pi_*(\chi_1) = y'y.
\]

We leave the details to the interested reader.

Note that the action of \( \mathcal{A}_p \) on \( H^*(BP(n); \mathbb{F}_p) \) for \( n \geq 4 \) is determined completely by information contained in theorem 2·9 and proposition 2·8.
The Massey Product.

This section is not original, but it introduces techniques we shall use to construct various explicit elements in $H^*(BP_2; \mathbb{F}_p)$, and to find relations between them. Throughout this section $R$ will be a commutative ring on which $\pi_1(X)$ acts trivially, and $C^*(X; R)$ the singular cochain complex of $X$ with coefficients in $R$. Any cochains or cohomology classes represented by a single letter will be homogeneous, and we shall write, for example $(-1)^u$ for $(-1)^{\deg(u)}$. In particular, $(-1)^{uv}$ shall mean $(-1)^{\deg(u)\deg(v)}$, not $(-1)^{\deg(uv)}$.

**Definition.** Let $u, v, w$ be elements of $C^*(X; R)$, with

$$[uv] = 0 \quad [vw] = 0 \quad \text{in} \quad H^*(X; R).$$

Then choose $a \in C^{u+v-1}(X; R)$ and $b \in C^{v+w-1}(X; R)$ with

$$\delta a = uv \quad \delta b = vw,$$

and define the Massey product $\langle [u], [v], [w] \rangle$ by

$$\langle [u], [v], [w] \rangle = \left[(-1)^u ub - aw \right] \in H^{u+v+w-1}(X; R)/(uH^{u+v+w-1}(X; R) + wH^{u+v+w-1}(X; R)).$$

The Massey product is only well-defined modulo $uH^{u+v+w-1}(X; R) + wH^{u+v+w-1}(X; R)$ because of the freedom of choice of $a$ and $b$. It is easily seen to be linear in each of its arguments. Before stating some more properties enjoyed by the Massey product we state a relation due to Hirsch [Hi] between the cup-0 and cup-1 products. We recall that a cup-0 product is a product on cochains inducing the cup product on cohomology, for example the standard product defined by the Alexander-Whitney formula. A cup-1 product is a natural cochain transformation of degree $-1$

$$\sim_1: (C^*(\_))^{\otimes 2} \longrightarrow C^*(\_),$$

satisfying the following ‘coboundary formula’ for all (homogeneous) cochains $a$ and $b$:

$$\delta(a \sim_1 b) = -\delta a \sim_1 b - (-1)^{a} a \sim_1 \delta b + ab + (-1)^{ab} ba.$$
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**Theorem 2-10.** (Hirsch). With the standard choice of cup-0 product, there is a choice of cup-1 such that for all cochains $a, b, c$

$$(ab) \smile_1 c = (-1)^a a(b \smile_1 c) + (-1)^{bc}(a \smile_1 c)b.$$  

Note that Hirsch uses Steenrod’s original definition [St] of the cup-1 product, which satisfies a slightly different coboundary formula to the one stated above. This explains the difference between our statement of theorem 2-10 and Hirsch’s original statement. If we write $\smile_S$ for Steenrod’s definition of the cup-1 product, then we may define a choice of cup-1 in the modern sense by

$$a \smile_1 b = (-1)^{a+b-1}(a \smile_S b),$$

and this is a choice for cup-1 which satisfies the identity of theorem 2-10.

**Lemma 2-11.** The Massey product satisfies the following identities, which are valid whenever all the terms are defined, for any $u, v, w, x, y \in H^*(X; R)$:

$$(u, v, w)x + (-1)^u(u, v, w, x) \equiv 0 \mod uH^*x$$  

$$(1)$$

$$(1)^u(u, v, w, x, y) + (u, v, w, x, y) + (1)^v(u, v, w, x, y) \equiv 0 \mod uH^* + H^{u+v-1}wH^x+y^{-1} + yH^*$$  

$$(2)$$

$$(1)^{uw}(u, v, w) + (1)^uv(u, v, w) + (1)^v(u, v, w) \equiv 0 \mod uH^* + vH^* + wH^*$$  

$$(3)$$

$$(1)^{uw}(u, v, w) + (1)^uv(u, w, v) + (1)^v(u, w, v) \equiv 0 \mod uH^* + wH^*$$  

$$(4)$$

*Proof.* The verification of these is fairly straightforward. The proofs of (3) and (4) use theorem 2-10. For example, to prove (3), if we let $u, v, w$ also stand for representative cocycles, then pick $a, b$ and $c$ such that

$$\delta a = uv \quad \delta b = vw \quad \delta c = wu,$$
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then
\[
(-1)^{wu} \langle u, v, w \rangle + (-1)^{uv} \langle v, w, u \rangle + (-1)^{vw} \langle w, u, v \rangle
\]
contains the element
\[
(-1)^{wu+ub} - (-1)^{wu} aw + (-1)^{uv+v} vc - (-1)^{uv} bu + (-1)^{vw+w} wa - (-1)^{vw} cv
\]
\[
= - (-1)^{uv} (uv) \cup_1 w - (-1)^{vw} (vw) \cup_1 u - (-1)^{wu} (wu) \cup_1 v
\]
\[
= \delta((u \cup_1 v) \cup_1 w).
\]

Matrix Massey Products. This generalisation of the Massey product is due to May [Ma]. We consider now homogeneous ‘vectors’ of cocycles \((u_i), (v_{ij}), (w_j)\) of degrees \(u, v, w\) respectively, where \(1 \leq i \leq l\), and \(1 \leq j \leq m\), with the property that
\[
\sum_i [u_i v_{ij}] = 0 \quad \text{for all } j
\]
\[
\sum_j [v_{ij} w_j] = 0 \quad \text{for all } j.
\]
We choose \((a_j)\) and \((b_i)\) such that
\[
\delta a_j = \sum_i u_i v_{ij} \quad \delta b_i = \sum_j v_{ij} w_j,
\]
then define the matrix Massey product \(\langle [u_i], [v_{ij}], [w_j] \rangle\) by
\[
\langle [u_i], [v_{ij}], [w_j] \rangle = (-1)^u \sum_i u_i b_i - \sum_j a_j w_j
\]
\[
\in H^{u+v+w-1}(X; R)/(\sum_i u_i H^{v+w-1}(X; R) + \sum_j w_j H^{u+v-1}(X; R)).
\]
The obvious generalisations of the properties claimed in lemma 2.11 are valid for the matrix Massey product, with very similar proofs. For example suppose that we are given \((u_i), (v_{ij}), (w_{jk}), (x_k)\) for \(1 \leq i \leq l\), \(1 \leq j \leq m\), \(1 \leq k \leq n\) with
\[
\sum_i [u_i v_{ij}] = 0 \quad \text{for all } j
\]
\[
\sum_j [v_{ij} w_{jk}] = 0 \quad \text{for all } i, k
\]
\[
\sum_k [w_{jk} x_k] = 0 \quad \text{for all } j.
We wish to show that
\[
\sum_k \langle [u_i], [v_{ij}], [w_{jk}] \rangle [x_k] + (-1)^u \sum_i [u_i] \langle [v_{ij}], [w_{jk}], [x_k] \rangle \equiv 0 \mod \sum_{i,k} u_i H^* x_k.
\]
We choose \((a_j), (b_{ik}), (c_j)\) such that
\[
\delta a_j = \sum_i u_i v_{ij} \quad \delta b_{ik} = \sum_j v_{ij} w_{jk} \quad \delta c_j = \sum_k w_{jk} x_k,
\]
then
\[
\sum_k \langle [u_i], [v_{ij}], [w_{jk}] \rangle [x_k] - (-1)^u \sum_i [u_i] \langle [v_{ij}], [w_{jk}], [x_k] \rangle \text{ contains}
\]
\[
(-1)^u \sum_{i,k} u_i b_{i,k} x_k - \sum_{j,k} a_j w_{jk} x_k + (-1)^{u+v} \sum_{i,j} u_i v_{ij} c_j - (-1)^u \sum_{i,k} u_i b_{i,k} x_k = -\delta(\sum_j a_j c_j).
\]

We now give an example of a calculation using Massey products, which we shall refer to in our later work.

**Lemma 2.12.** Let \(p\) be a prime not equal to 2, and let \(y\) generate \(H^1(BC_p; \mathbb{F}_p)\). Then \(\langle y, y, y \rangle\) is a unique element of \(H^2(BC_p; \mathbb{F}_p)\), and
\[
\langle y, y, y \rangle = \begin{cases} 0 & \text{for } p > 3 \\ \beta(y) & \text{for } p = 3. \end{cases}
\]

**Proof.** We apply the ‘Jacobi identity’ (lemma 2.11, equation (3)) and obtain
\[
\langle y, y, y \rangle + \langle y, y, y \rangle + \langle y, y, y \rangle \equiv 0 \mod 0.
\]
Hence for \(p > 3\) we obtain \(\langle y, y, y \rangle = 0\). \(\beta(y)\) generates \(H^2(BC_p; \mathbb{F}_p)\), and \(\langle y, y, y \rangle\) is defined modulo 0, so for \(p = 3\) it only remains to find the constant \(\lambda\) in the equation
\[
\langle y, y, y \rangle = \lambda \beta(y).
\]
To find \(\lambda\) we resort to the definition in terms of cochains in the bar resolution.

For any \(G\), and any cocycle \(y \in C^1(BG; \mathbb{F}_p)\), the 1-cochain \(a\) defined by
\[
a([g]) = -\frac{1}{2} y([g])^2 \quad \text{satisfies} \quad \delta a([g|h]) = y([g])y([h]),
\]
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so for any such $y$, a choice for $(y, y, y)$ is

$$(y, y, y)([g|h]) = -y([g])y([h])(y([g]) + y([h])).$$

Now let $G = \langle A \rangle \cong C_3$, and define $y$ by

$$y : [A^r] \mapsto r \mod (3)$$

Now define $\bar{y} \in C^1(BG; \mathbb{Z})$ by

$$y : [A^r] \mapsto r \mod (3), \text{ where } 0 \leq r \leq 2.$$  

Then $b = \frac{1}{3}\delta \bar{y}$ is a cocycle representing $\beta(y)$, and satisfies

$$b([A^r|A^s]) = \begin{cases} 
0 & \text{for } 0 \leq r + s \leq 2 \\
1 & \text{for } 3 \leq r + s \leq 5,
\end{cases}$$

where we require $0 \leq r, s \leq 2$. Now it remains to solve for $\lambda$ in the equation over $\mathbb{F}_3$:

$$\lambda b([A^r|A^s]) + rs(r + s) = f(r) + f(s) - f(r + s),$$

where $f$ is some function from $\mathbb{F}_3$ to itself. We obtain $\lambda = 1$, as required.  

\[ \square \]
Calculation of $H^*(BP_2; F_p)$.

In this section we use the information given by the spectral sequence for the fibration

$$S^1 \longrightarrow BP_2 \longrightarrow \tilde{BP}$$

(1)

together with the properties of Massey products stated in the previous section to determine $H^*(BP_2; F_p)$. To show that a certain product in $H^5(BP_2; F_p)$ is non-zero we briefly consider the spectral sequence (discussed in lemma 2.4) for the central extension

$$\langle C \rangle \longrightarrow P_2 \longrightarrow C^A_p \oplus C^B_p.$$ 

We now examine the spectral sequence for the fibration (1). As before $E_2^{*,*}$ is the anticommutative ring generated by $H^*(BP; F_p)$ and $u$, a generator of $H^1(S^1; F_p)$. By naturality of the spectral sequence for changes of coefficients, we see that

$$d_2(u) = \lambda yy'$$

for some non-zero $\lambda$. Hence we see that

$$E_\infty^{*,0} \cong H^*(BP; F_p)/yy'H^*(BP; F_p)$$

$$E_\infty^{*,1} \cong \text{Ker} \times yy' : H^*(BP; F_p) \to H^*(BP; F_p),$$

and it follows that $E_\infty^{*,*}$ is generated by the elements $[uy], [uy'], [ux], [ux'], [uc_2], \ldots, [uc_{p-1}], y, y', x, x', c_2, \ldots, c_{p-1}, z$, subject to the relations implied by those that hold in $E_2^{*,*}$, together with the relation $yy' = 0$. $E_\infty^{*,0}$ is the subring of $H^*(BP; F_p)$ given by the image of the restriction from $\tilde{P}$. Throughout this section we shall use the same symbol for an element of $H^*(BP; F_p)$ and its image under the restriction. We shall write $H^*$ for $H^*(BP_2; F_p)$, unless there is danger of confusion.

We see immediately from the spectral sequence that

$$H^1H^1 = \{0\},$$
that $H^2$ has order $p^4$, and that we need to introduce two new generators in $H^2$ not in the image of the restriction. The Massey product of any three elements of $H^1$ is defined, and is a unique element of $H^2$. We define

$$Y = \langle y, y, y' \rangle$$

$$Y' = \langle y', y', y \rangle,$$

and claim that $x, x', Y$ and $Y'$ generate $H^2$. This is a consequence of the following lemma.

**Lemma 2.13.** Define $\overline{y}, d \in H^1(B\langle A, C \rangle; \mathbb{F}_p)$ and $\overline{y}', d' \in H^1(B\langle B, C \rangle; \mathbb{F}_p)$ by

$$\overline{y}: A^r C^s \mapsto r \quad \overline{y}' : B^r C^s \mapsto r$$

$$d : A^r C^s \mapsto s \quad d' : B^r C^s \mapsto s,$$

so that

$$\overline{y} = \text{Res}^{P_2}_{\langle A, C \rangle}(y), \quad \overline{y}' = \text{Res}^{P_2}_{\langle B, C \rangle}(y').$$

If $Y$ and $Y'$ are defined as above, then

$$\text{Res}^{P_2}_{\langle A, C \rangle}(Y) = \overline{y}d \quad \text{Res}^{P_2}_{\langle A, C \rangle}(Y') = 0$$

$$\text{Res}^{P_2}_{\langle B, C \rangle}(Y) = 0 \quad \text{Res}^{P_2}_{\langle B, C \rangle}(Y') = -\overline{y}'d'.$$

**Proof.** Recall from the proof of lemma 2.12 that the 1-cochain $a$ defined by

$$a([g]) = -\frac{1}{2}y([g])^2$$

satisfies

$$\delta a([g|h]) = y([g])y([h]).$$

Define a 1-cochain $b$ by

$$b([B^r A^s C^t]) = -t,$$

so that

$$\delta b([B^r A^s C^t|B^r' A^s' C^t']) = r's = y([B^r A^s C^t])y'([B^r' A^s' C^t']).$$

Now we may choose as cocycle representing $Y$ the following:

$$Y([g|h]) = y([g])(\frac{1}{2}y([g])y'([h]) - b([h])).$$
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$y$ restricts to $\langle B, C \rangle$ as the zero cocycle, so

$$\text{Res}_{P_2}^{P_2}(Y) = 0.$$  

Similarly, $y'$ restricts to $\langle A, C \rangle$ as the zero cocycle, and $b$ restricts to a cocycle representing $-d$. Hence

$$\text{Res}_{P_2}^{P_2}(Y) = \beta(y).$$

The results for $Y'$ are proven similarly. 

We know that

$$\text{Res}_{P_2}^{P_2}(x) = \beta(y) \quad \text{Res}_{P_2}^{P_2}(x') = 0 \quad \text{Res}_{P_2}^{P_2}(x) = 0 \quad \text{Res}_{P_2}^{P_2}(x') = \beta(y'),$$

so it follows that $x, x', Y, Y'$ are linearly independent.

We are now ready to state our theorems determining the ring structure of $H^*(BP_2; \mathbb{F}_p)$.

We consider separately the cases $p > 3$ in theorem 2·14, and $p = 3$ in theorem 2·15.

**Theorem 2·14.** Let $p$ be greater than 3. $H^*(BP_2; \mathbb{F}_p)$ is generated by elements $y, y', x, x'$, $Y, Y', X, X', d_4, \ldots, d_p, c_4, \ldots, c_{p-1}, z$, with

$$\deg(y) = \deg(y') = 1 \quad \deg(x) = \deg(x') = \deg(Y) = \deg(Y') = 2 \quad \deg(X) = \deg(X') = 3 \quad \deg(d_i) = 2i - 1 \quad \deg(c_i) = 2i \quad \deg(z) = 2p$$

$$\beta(y) = x \quad \beta(y') = x' \quad \beta(Y) = X \quad \beta(Y') = X'$$

$$\beta(c_i) = \begin{cases} c_i & \text{for } i < p \\ 0 & \text{for } i = p \end{cases}$$

subject to the following relations:

$$yy' = 0 \quad xy' = x'y \quad yY = y'Y' = 0 \quad yY' = y'Y$$
We define $y, y' \in H^1(BP_2; \mathbb{F}_p)$ by

$$y : A'B^sC^t \rightarrow r$$

$$y' : A'B^sC^t \rightarrow s,$$
Then define \( x, x' \) by the equations
\[
x = \beta(y) \quad x' = \beta(y').
\]
The equation \( yy' = 0 \) implies that we may define unique elements \( Y, Y' \) by
\[
Y = \langle y, y', y \rangle \quad Y' = \langle y', y', y \rangle
\]
and we also define
\[
X = \beta(Y) \quad X' = \beta(Y').
\]
The effect of automorphisms of \( P_2 \) on the generators \( y, y', x, x', Y, Y', X, X' \) is determined by the above definitions. An automorphism of \( P_2 \) which restricts to the centre as \( C \mapsto C^j \) sends
\[
d_i \mapsto j^i d_i
\]
\[
c_i \mapsto j^i c_i
\]
\[
z \mapsto jz.
\]
If we define \( d' \in H^1(B\langle B, C \rangle; \mathbb{F}_p) \) by
\[
d' : B^r C^s \mapsto s,
\]
then define \( c' \) by
\[
c' = \beta(d'),
\]
then we may define
\[
d_i = \begin{cases} 
\Cor_{B, C}^{P_2}(c'^{i-1} d') & \text{for } i < p - 1 \\
\Cor_{B, C}^{P_2}(c'^{p-2} d') - x^{p-2} y & \text{for } i = p - 1 \\
\Cor_{B, C}^{P_2}(c'^{p-1} d') + x^{p-2} X & \text{for } i = p.
\end{cases}
\]
We may define \( c_i \) in terms of either \( d_i \) or \( \chi_i \in H^{2i}(BP_2; \mathbb{Z}) \), using the equations
\[
c_i = \beta(d_i) = \pi_*(\chi_i),
\]
and we define $z$ by

$$z = \pi_*(\zeta).$$

We note that

$$\pi_*(\chi_2) = \lambda(xY' + x'Y)$$
$$\pi_*(\chi_3) = \lambda'XX'$$

for some non-zero $\lambda, \lambda'$.

**Proof.** We shall prove these assertions in the order in which they are stated. Throughout this proof we adopt the convention that $\text{Cor}(\_)$ should stand for $\text{Cor}^p_{(B,C)}(\_)$, and similarly $\text{Res}(\_)$ should stand for $\text{Res}^p_{(B,C)}(\_)$.

$yy' = 0$ follows from the spectral sequence, as does $xy' = x'y$. For the other relations in $H^3$, we note that

$$yY = y\langle y, y, y' \rangle$$
$$= \langle y, y, y \rangle y' \mod yH^1y' = \{0\}$$

and similarly for $y'Y'$. Also

$$Yy' = \langle y, y, y' \rangle y'$$
$$\equiv y\langle y, y', y' \rangle \mod yH^1y' = \{0\}$$
$$\equiv y\langle y', y', y \rangle \mod yyH^1 + yy'H^1 = \{0\}$$

$$= yY'.$$

The spectral sequence implies that $yY'$ is non-zero.

Before moving on to $H^4$ we note that

$$\langle y, y, Y \rangle \equiv 0 \mod H^2y + H^2y',$$

which follows from

$$-\langle y, y, Y \rangle + \langle y, \langle y, y, y \rangle, y' \rangle - \langle \langle y, y, y \rangle, y, y' \rangle \equiv 0 \mod H^2y + H^2y'.$$
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Now

$$Y^2 = Y\langle y, y, y' \rangle$$

$$\equiv -\langle Y, y, y' \rangle y' \mod YH^1y' = \{0\}$$

$$\equiv -\langle y, Y, y' \rangle y' \mod YH^1y' + yH^2y' = \{0\}$$

$$\equiv 0 \mod H^2yy' + H^2y'^2 = \{0\}$$

Similarly, $Y'^2 = 0$ and $(Y + Y')^2 = 0$, because an automorphism of $P_2$ sending $y$ to $y - y'$ and fixing $y'$ sends $Y$ to $Y + Y'$, and we deduce that $YY' = 0$.

The relations involving $yX$ and $y'X'$ follow by applying the Bockstein to the relations $yY = 0$ and $y'Y' = 0$ respectively. For the remaining relations in $H^4$ we introduce some matrix Massey products. We consider

$$\langle (x, x'), \left( \begin{array}{c} y' \\ -y \end{array} \right), y \rangle,$$

which is defined modulo $H^2H^1$.

$$\langle (x, x'), \left( \begin{array}{c} y' \\ -y \end{array} \right), y \rangle y \equiv -(x, x') \left( \begin{array}{c} \langle y', y, y \rangle \\ -\langle y, y, y \rangle \end{array} \right) \mod xH^1y + x'H^1y = \{0\}$$

$$= -xY,$$

and

$$\langle (x, x'), \left( \begin{array}{c} y' \\ -y \end{array} \right), y \rangle y' \equiv -(x, x') \left( \begin{array}{c} \langle y', y, y' \rangle \\ -\langle y, y, y' \rangle \end{array} \right) \mod xH^1y' + x'H^1y' = \{0\}$$

but

$$\langle y', y, y' \rangle \equiv -2Y' \mod \{0\}$$

so

$$\langle (x, x'), \left( \begin{array}{c} y' \\ -y \end{array} \right), y \rangle y' = 2xY' + x'Y.$$
hence
\[ X \in \langle (x, x'), \left( \frac{y'}{y} \right), y \rangle \quad X' \in \langle (x', x), \left( \frac{y}{y'} \right), y' \rangle \]
and the remaining relations in \( H^4 \) follow. Our results allow us to deduce that \( X, X', yY', xy, xy', x'y' \) form a basis for \( H^3 \). In \( H^5 \) the relations
\[ XY = 0 \quad X'Y' = 0 \quad XY' = -X'Y \quad xX' = -x'X \]
follow by applying the Bockstein to the relations
\[ Y^2 = 0 \quad Y'^2 = 0 \quad YY' = 0 \quad X'y = 2x'Y + xY' \]
respectively. For the relations stated in \( H^6 \), we note that
\[ xyY' = x'yY' = 0, \]
and then apply the Bockstein to these relations, noting also that
\[ \beta(yY') = 2xY' + 2x'Y. \]

We see now that in the \( E_{\infty}^{\ast, \ast} \) page of the spectral sequence \( Y \) yields \( \lambda[u_y] \) and \( Y' \) yields \( -\lambda[u_y'] \) for some non-zero \( \lambda \), and deduce that
\[ xy' + x'Y \equiv 0 \pmod{\langle x^2, xx', x'^2, c_2 \rangle}, \]
then the relations in \( H^6 \) imply that
\[ xy' + x'Y = \lambda'c_2 \]
for some \( \lambda' \). \( yY' \) is not in the kernel of the Bockstein, so \( \lambda' \) must be non-zero, and we see that products of elements in \( H^2 \) generate \( H^4 \).

It follows from lemma 2·4 that \( H^5 \) is generated by \( x^2y, x^2y', xx'y', x^2y', xX, xX', x'X' \) and \( XY' \). Now \( XY' \notin \text{Im}(\pi_\ast) \), and since \( H^6(BP_2; \mathbb{Z}) \) has exponent \( p \), we deduce that
\[ XX' = -\beta(XY') \neq 0. \]
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$XX'x = 0,$

but it is apparent from the spectral sequence that in $H^6,$

$$\text{Ker}(\times x) = \langle c_3 \rangle,$$

so

$$XX' = \lambda c_3 \quad \lambda \neq 0$$

and hence $H^6$ is generated by $XX'$ and products of elements of $H^2.$

We already have the relations

$$x^py' = x'^py \quad x^px' = x'^px,$$

and we obtain

$$x^pY' + x'^pY = 0$$

by applying $P^1$ to the relation

$$xY' + x'Y = \lambda c_2,$$

but firstly we must find $\text{Res}(z).$ We recall that

$$z = \pi_*(\zeta) = \pi_*(c_p(\rho)),$$

and that $\rho$ restricts to $\langle B, C \rangle$ as the sum of $p$ representations, whose first Chern classes map under $\pi_*$ to $c', c' + x', \ldots, c' + (p - 1)x'.$ Hence we see that

$$\text{Res}(z) = \prod_{i=0}^{p-1} (c' + ix') = c'^p - x'^{p-1}c'.$$

Now we may calculate $P^1(c_2)$ as follows:

$$P^1(c_2) = P^1(\text{Cor}(c'^2))$$

$$= \text{Cor}(2c'^{p+1})$$

$$= 2\text{Cor}(\text{Res}(z)c' + \text{Res}(x'^{p-1})c'^2)$$

$$= 2z y y' + 2x'^{p-1}c_2$$

$$= 0$$

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The last of these equations follows because
\[ x'c_2 = \pi_*(\beta\chi_2) = 0. \]

Now
\[ 0 = P^1(xY' + x'Y) \]
\[ = x^pY' + xY'p + x^pY + x'Yp \]
\[ = x^pY' + x^pY. \]

Now we apply the Bockstein to obtain the relation
\[ x^pX' + x^pX = 0. \]

We may verify that in degrees greater than 6 all products of the generators \(y, y', x, x', Y, Y', X, X'\) may be expressed in the form
\[ f_1 + f_2Y + f_3Y' \quad \text{for even total degree} \]
\[ f_1y + f_2y' + f_3X + f_4X' \quad \text{for odd total degree} \]
where \(f_i\) is a polynomial in \(x\) and \(x'\). With the exception of \(xY' + x'Y\), such expressions satisfy ‘the same’ relations as elements of \(H^*\) as they do as elements of \(E^*\). Elements that are expressible as above form a subspace of \(H^n\) of codimension 1 for \(7 \leq n \leq 2p\), so we introduce the elements \(c_i\) for \(i > 3\) and \(z\) to our generating set. \(\delta_p\) sends elements of the above form and odd total degree to polynomials in \(\alpha\) and \(\beta\), so any element mapping under \(\delta_p\) to \(\chi_i\) (resp. \(p\zeta\)) will suffice to complete a basis for \(H^{2i-1}\) (resp. \(H^{2p-1}\)). Hence we may add \(d_i\) defined as above to our generating set.

Before verifying the relations involving \(c_i\) and the low dimensional generators, we consider the automorphism \(\theta\) of \(P_2\), given by
\[ \theta : A \mapsto B \]
\[ B \mapsto A, \]
which has the effect of exchanging the ‘primed’ and ‘unprimed’ low dimensional generators of \(H^*\). We know the effect of this automorphism on \(\chi_i\), and we deduce that it sends
\[ \theta^* : c_i \mapsto (-1)^ic_i. \]
We already know the relations between \( c_i \) and \( y, y', x, x' \) because they are exactly the relations that they satisfy in \( H^*(\tilde{B}P; \mathbb{F}_p) \). For \( c_iY \), note that

\[
\text{Cor}(c^iY) = \text{Cor}(c^i\text{Res}(Y)) = \text{Cor}(0) = 0,
\]

so

\[
c_iY = \begin{cases} 0 & \text{for } i < p - 1, \\ -x^{p-1}Y & \text{for } i = p - 1. \end{cases}
\]

For the results concerning \( c_iX \) we apply the Bockstein to the above results. For the corresponding results concerning the ‘primed’ generators, we apply the automorphism \( \theta \) described above. The results concerning \( c_ic_j \) follow from the relations in \( H^*(\tilde{B}P; \mathbb{F}_p) \), or from those that hold in \( H^*(BP_2; \mathbb{Z}) \).

Now we examine the relations between \( d_i \) and the low dimensional generators. We do not yet know the effect of the automorphism \( \theta \) on \( d_i \), so we must examine both the ‘primed’ and ‘unprimed’ relations.

\[
\text{Cor}(c^{i-1}d')y = \text{Cor}(c^{i-1}d'\text{Res}(y)) = 0,
\]

so

\[
d_iy = \begin{cases} 0 & \text{for } i < p - 1, \\ x^{p-2}Xy = -x^{p-1}Y & \text{for } i = p - 1. \end{cases}
\]

\[
\text{Cor}(c^{i-1}d')y' = \text{Cor}(c^{i-1}d'\overline{y'}),
\]

where \( \overline{y'} \) is as defined during the statement of lemma 2.13, but

\[
\text{Cor}(c^{i-1}d'\overline{y'}) = -\text{Cor}(c^{i-1})Y' = \begin{cases} 0 & \text{for } i < p, \\ -(c_{p-1} + x^{p-1})Y' = (x^{p-1} - x^{p-1})Y' & \text{for } i = p. \end{cases}
\]

Hence

\[
d_iy' = \begin{cases} 0 & \text{for } i < p, \\ x^{p-1}Y' & \text{for } i = p. \end{cases}
\]
Similarly, 
\[
\text{Cor}(c^{i-1}d')Y = 0
\]
\[
\text{Cor}(c^{i-1}d')Y' = \text{Cor}(c^{i-1}d'\overline{y}'d') = 0,
\]
so
\[
d_iY = 0 \quad \text{for all } i
\]
\[
d_iY' = 0 \quad \text{for all } i.
\]
For the relations involving \(d_i\) and \(x, x', X, X'\), we apply the Bockstein to the above relations and substitute for terms involving \(c_i\). For example,
\[
0 = \beta(d_{p-1}y) = c_{p-1}y - d_{p-1}x,
\]
so
\[
d_{p-1}x = c_{p-1}y = -x^{p-1}y.
\]
We now use the formula for ResCor:
\[
\text{ResCor} = \sum_{j=0}^{p-1} c_{A^j}^*
\]
where \(c_{A^j}^*\) is the map of \(H^*(B\langle B, C \rangle; \mathbb{F}_p)\) induced by conjugation by \(A^j\). It is easy to verify that
\[
c_{A^j}^* : d' \mapsto d' + j\overline{y}'
\]
\[
c' \mapsto c' + j\overline{x}'.
\]
Then
\[
\text{ResCor}(c^i) = \sum_{j=0}^{p-1} (c' + j\overline{x})^i.
\]
Similarly,
\[
\text{ResCor}(c^{i-1}d') = \sum_{j=0}^{p-1} (c' + j\overline{x}')^{i-1}(d' + j\overline{y}).
\]
Hence
\[
\text{Cor}(c^{i-1}d')\text{Cor}(c^{j-1}d') = \text{Cor}(c^{i-1}d' \sum_{k=0}^{p-1} (c' + k\overline{x}')^{i-1}(d' + k\overline{y}'))
\]
\[
= \text{Cor}(c^{i-1}d' \sum_{k=0}^{p-1} k\overline{y}' (c' + k\overline{x})^{j-1})
\]
\[
= \text{Cor}(c^{i-1}d' \sum_{l=0}^{j-1} \binom{j-1}{l} \overline{y}' c^{j-1-l} \overline{x}' \sum_{k=0}^{p-1} k^{l+1})
\]
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but

$$\sum_{k=1}^{p-1} k^{l+1} = \begin{cases} -1 & \text{if } p - 1 \text{ divides } l + 1, \\ 0 & \text{otherwise} \end{cases}$$

We may assume that $i > j$, so the only non-zero case we need consider is $j = p - 1$ and $l = p - 2$.

$$\text{Cor}(c^{p-1}d')\text{Cor}(c'^{p-2}d') = -\text{Cor}(c^{p-1}d'y'x^{p-2})$$
$$= \text{Cor}(c^{p-1})Y'x^{p-2}$$
$$= (c_{p-1} + x^{p-1})Y'x^{p-2}$$
$$= x^{p-1}x'^{p-2}Y' - x'^{2p-3}Y',$$

so

$$d_pd_{p-1} = (\text{Cor}(c^{p-1}d') + x^{p-2}X)(\text{Cor}(c'^{p-2}d') - x^{p-2}y)$$
$$= x^{p-1}x'^{p-2}Y' + x^{2p-3}Y - x'^{2p-3}Y',$$

and we obtain the relations between the $d_i$.

Similarly

$$\text{Cor}(c'^{i-1}d')\text{Cor}(c'^j) = \text{Cor}(c'^{i-1}d')\sum_{l=0}^{j} \binom{j}{l} c'^{j-l}x^{p-1}x'^{l}\sum_{k=0}^{p-1} k^{l},$$

so

$$\text{Cor}(c'^{i-1}d')\text{Cor}(c'^j) = \begin{cases} 0 & \text{for } j < p - 1 \\ -\text{Cor}(c'^{i-1}d'x^{p-1}) & \text{for } j = p - 1 \end{cases}$$
$$= \begin{cases} 0 & \text{for } i < p - 1 \text{ or } j < p - 1 \\ -(d_{p-1} + x^{p-2}y)x'^{p-1} & \text{for } i = j = p - 1 \\ (d_{p-1} + x^{p-2}X)x'^{p-1} & \text{for } i = p \text{ and } j = p - 1 \end{cases}$$

Hence we obtain the relations for $d_i c_j$ as claimed.

All that remains to be calculated is the effect of automorphisms of $P_2$ on $d_i$. The effect of automorphisms on $c_i$ and $z$ follow immediately from the corresponding results for integral cohomology. Let $\varphi$ be an automorphism of $P_2$ that restricts to the centre as

$$\varphi : C \mapsto C^j.$$ 

We know that

$$\delta_p\varphi^*(d_i) = \varphi^*\delta_p(d_i) = j^i\delta_p(d_i),$$

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so

$$\varphi^*(d_i) \equiv j^i d_i \pmod{\ker(\delta_p)}.$$ 

Also $d_i$ was defined so that it corresponds, in the $E_{\infty}^{*,*}$ page of the spectral sequence, to $\lambda[uc_{i-1}]$ for some non-zero $\lambda$. If $j \neq \pm 1$ then $\varphi$ does not extend to an automorphism of $\tilde{P}$, but it can be extended to an endomorphism $\overline{\varphi}$ of $\tilde{P}$, defined on $S^1$ by

$$\overline{\varphi}: v \mapsto jv \quad \text{for all } v \in S^1 \cong \mathbb{R}/\mathbb{Z}.$$ 

It may be verified that $\overline{\varphi}^*(c_i) = j^i c_i$, by first verifying that $\overline{\varphi}^*(\chi_i) = j^i \chi_i$. Note that $\overline{\varphi}^*$ is an automorphism of $H^*(BP; F_p)$ although for $j \neq \pm 1$ it is not an automorphism of $H^*(BP; \mathbb{Z})$. The map induced by $\overline{\varphi}$,

$$\overline{\varphi}: \tilde{P}/P_2 \mapsto \tilde{P}/P_2 \cong S^1$$

has degree $j$, so $u$ is sent to $ju$ under the map of $H^*(S^1; F_p)$ induced by $\overline{\varphi}$.

Hence the element $[uc_{i-1}]$ is sent to $j^i[uc_{i-1}]$ by the map of the spectral sequence induced by $\overline{\varphi}$, so it follows that

$$\varphi^*(d_i) \equiv j^i d_i \pmod{\text{im}(\text{Res}_{\tilde{P}/P_2})}.$$ 

$$\text{Res}_{\tilde{P}/P_2}(H^{2i-1}(BP; F_p)) \cap \ker(\delta_p) = \{0\},$$

so we see that

$$\varphi^*(d_i) = j^i d_i.$$ 

\begin{theorem}
Let $p = 3$. Then $H^*(BP_2; F_3)$ is generated by elements $y, y', x, x', Y, Y', X, X', z$, with

$$\deg(y) = \deg(y') = 1 \quad \deg(x) = \deg(x') = \deg(Y) = \deg(Y') = 2$$

$$\deg(X) = \deg(X') = 3 \quad \deg(z) = 6$$

\end{theorem}
subject to the following relations:

\[ \begin{align*}
yy' &= 0 \quad xy' = x'y \\
Y &= y'Y' = xy' \quad Y' = y'Y \\
XY &= x'X \quad X'Y' = xx \\
X'Y' &= -X'Y \quad xX' = -x'X \\
XX' &= 0 \quad xx' + x'y = -xx' \quad x(x'y' + x'y) = -x'x^2 \\
x'Y' &= x'y - x'Y' \quad x^3y' - x^3y = 0 \quad x^3x' - x^3x = 0 \\
x^3Y' + x^3Y = -x^2x' \quad x^3X' + x^3X = 0
\end{align*} \]

We define \( y, y' \in H^1(BP_2; \mathbb{F}_3) \) by

\[ y : A^r B^s C^t \mapsto r \]
\[ y' : A^r B^s C^t \mapsto s, \]

then define \( x, x' \) by the equations

\[ x = \beta(y) \quad x' = \beta(y'), \]

The equation \( yy' = 0 \) implies that we may define unique elements in \( H^2(BP_2; \mathbb{F}_3) \) by forming the Massey product of any three elements of \( H^1(BP_2; \mathbb{F}_3) \), and we define \( Y, Y' \) by

\[ Y = \langle y, y', y \rangle \quad Y' = \langle y', y', y \rangle \]

and we also define

\[ X = \beta(Y) \quad X' = \beta(Y'), \]
The effect of automorphisms of \( P_2 \) on the generators \( y, y', x, x', Y, Y', X, X' \) is determined by the above definitions. An automorphism of \( P_2 \) which restricts to the centre as \( C \mapsto C^j \) sends
\[
z \mapsto jz,
\]
and we may define \( z \) to be \( \pi_*(\zeta) \). We also note that
\[
\pi_*(\chi_2) = -xY' - x'Y - x^2 - x'^2.
\]

Proof. Many of the relations may be proven exactly as in the case \( p > 3 \). The relations involving \( Y^2, Y'^2 \), and \( YY' \) must be proven differently, and we exhibit a basis for \( H^4 \) before attempting to prove them.

\( yy' = 0 \) and \( xy' = x'y \) follow from the spectral sequence. Now
\[
yY = y\langle y, y, y' \rangle
\]
\[
\equiv \langle y, y, y \rangle y' \mod yH^1 y' = \{0\}
\]
\[
= xy',
\]
and similarly for \( y'Y' \). The relation \( yY' = y'Y \) follows exactly as in the proof of theorem 2·14. For the relations involving \( yX \) and \( y'X' \) we apply the Bockstein to the relations \( yY = xy' \) and \( y'Y' = xy' \). As in the proof of theorem 2·14, any
\[
Z \in \langle (x, x'), \left( \begin{array}{c} y' \\ -y \end{array} \right), y \rangle
\]
satisfies
\[
 Zy = -xY + xx' \quad Zy' = x'Y - xY'.
\]
Similarly, any
\[
Z' \in \langle (x', x), \left( \begin{array}{c} y \\ -y' \end{array} \right), y' \rangle
\]
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satisfies

$$Z'y' = -x'Y' + xx' \quad Z'y = xY' - x'Y.$$  

We deduce that $X, X'$ satisfy the relations claimed in $H^4$, and that $X, X', xy, xy', x'y', y'y'$ form a basis for $H^3$. Using the relation for $X'y$ we see that

$$\beta(Y'y) = -xY' - x'Y.$$  

The relations

$$xY'y = x'^2y \quad x'Y'y = x^2'y'$$

follow easily from the relations we have already proven, now we apply the Bockstein to them, and obtain

$$x(xy' + x'Y) = -xx'^2 \quad x'(xy' + x'Y) = -x'x^2.$$  

It follows that the relation in $H^4$ yielding the relation in $E^{3,1}_\infty$ of the spectral sequence,

$$x[uy'] = x'[uy]$$

must be

$$0 = xY' + x'Y + c_2 + x^2 + x'^2.$$  

We deduce that $x^2, xx', x'^2, xy, xy', x'Y$, and $x'Y'$ form a basis for $H^4$. Now we shall return to the other relations we wish to prove in $H^4$.

Consider the map

$$(\text{Res}^{P_2}_{A,C}, \text{Res}^{P_2}_{B,C}) : H^4(BP_2; F_3) \rightarrow H^4(B\langle A, C \rangle; F_3) \times H^4(B\langle B, C \rangle; F_3).$$

Lemma 2.13 tells us that $YY'$ is in the kernel of this map, and that $xx', xy'$ and $x'Y$ form a basis for this kernel. Hence

$$YY' = \lambda xx' + \lambda' xy' + \lambda'' x'Y,$$  

(*)
for some \( \lambda, \lambda', \lambda'' \). The automorphism \( \theta \) of \( P_2 \) which exchanges the ‘primed’ and ‘unprimed’ generators fixes \( YY' \), so
\[
\lambda'' = \lambda'.
\]
Now we multiply equation (*) by \( y \), and obtain
\[
xx'y = \lambda xx'y - \lambda' x'^2 y.
\]
Hence we see that \( YY' = xx' \). For the remaining relations in \( H^4 \), we consider the effect of an automorphism \( \theta' \) of \( P_2 \) having the following effect on \( H^1 \):
\[
\theta'^*: y \mapsto y + y' \\
y' \mapsto y'.
\]
We verify that
\[
\theta'^*: x \mapsto x + x' \\
x' \mapsto x' \\
Y \mapsto Y + x' - Y' \\
Y' \mapsto Y' + x',
\]
and apply \( \theta'^* \) to the equation \( YY' = xx' \), obtaining
\[
YY' + x'^2 - Y'^2 + x'Y = xx' + x'^2,
\]
and hence
\[
Y'^2 = x'Y.
\]
We obtain the relation for \( Y^2 \) similarly.

We apply the Bockstein to the relations
\[
Y^2 = xY' \\
Y'^2 = x'Y \\
YY' = xx' \\
0 = xY' + x'Y + c_2 + x^2 + x'^2
\]
respectively to obtain the relations
\[ XY = -xX' \]
\[ X'Y' = -x'X \]
\[ XY' = -X'Y \]
\[ xX' = -x'X. \]

Lemma 2.4 implies that \( XY' \) is linearly independent of elements of the form
\[ f_1y + f_2y' + f_3X + f_4X', \]
where \( f_i \) is a polynomial in \( x \) and \( x' \), and hence as in theorem 2.14 we see that no new
generator is needed in \( H^5 \). \( XX' = \beta(YX') \) is in the image of \( \pi_* \), which, in \( H^6 \), is spanned
by \( z, x^3, x^2x', xx'^2, \) and \( x'^3 \). It is easily checked that
\[ XX'x = XX'x' = 0, \]
but
\[ \text{Ker}(x) \cap \text{Ker}(x') \cap \langle z, x^3, x^2x', xx'^2, x'^3 \rangle = \{0\}, \]
and so
\[ XX' = 0. \]

We already have the relations
\[ x^3y' = x'^3y \quad x^3x' = x'^3x, \]
and as in the case \( p > 3 \), we prove the relation
\[ x^3Y' + x'^3Y = -x^2x'^2 \]
by applying \( P^1 \) to the relation
\[ 0 = x'Y + xY' + c_2 + x^2 + x'^2. \]
Noting that $\text{Res}^{P_2}_{(B,C)}(z) = c^3 - x'^2 c'$, we have
\[
P^1(c_2 + x^2 + x'^2) = P^1(\text{Cor}^{P_2}_{(B,C)}(c^2) + x'^2)
= -\text{Cor}^{P_2}_{(B,C)}(c'^4) - x'^4
= -\text{Cor}^{P_2}_{(B,C)}(\text{Res}^{P_2}_{(B,C)}(z)c' + x'^2 c'^2) - x'^4
= -zyy' - x'^2 c_2 + x'^2 x^2 - x'^4
= x'^2 x^2
\]
and
\[
P^1(xY' + x'Y) = x'^3 Y' + x'^3 Y + xY'^3 + x'y'^3
= x'^3 Y' + x'^3 Y - x^2 x'^2,
\]
so
\[
x'^3 Y' + x'^3 Y = -x^2 x'^2.
\]
We apply the Bockstein to this relation to obtain
\[
x'^3 X' + x'^3 X = 0.
\]

It may be checked that all products of degree at least 6 of elements of degree at most three may be expressed in the form
\[
f_1 + f_2 Y + f_3 Y' \quad \text{for even total degree}
f_1 y + f_2 y' + f_3 X + f_4 X' \quad \text{for odd total degree}
\]
where $f_i$ is a polynomial in $x$ and $x'$. The relations we have given between such elements are sufficient to imply the relations that hold between the corresponding elements of the spectral sequence, hence our presentation of the ring $H^*(BP_2; \mathbb{F}_3)$ is complete. The effect of automorphisms on $z$ follows from its definition as $\pi_*(\zeta)$, and we already have the required expression for $\pi_*(\chi_2) = c_2$.

**Remarks.** Using the results of theorems 2·14 and 2·15, the author has determined the differentials in the spectral sequence with $\mathbb{F}_p$ coefficients for $P_2$ expressed as a central extension of $C_p$ by $C_p$. The $E_\infty$-page in the case $p = 7$ is depicted in figure 2·4.
2.4. The $E_\infty$-page for the extension $C_7 \to P_2 \to C_7 \oplus C_7$ with $\mathbb{F}_7$ coefficients.
It is unreasonable to expect that by rechoosing generators we could make the statement of theorem 2.15 look more like the statement of theorem 2.14. For example, for $p > 3$, $H^2(BP_2; \mathbb{F}_p)$ may be expressed as the direct sum of two $\text{Aut}(P_2)$-invariant subspaces:

$$H^2(BP_2; \mathbb{F}_p) = \langle x, x' \rangle \oplus \langle Y, Y' \rangle.$$ 

For $p = 3$ the subspace $\langle x, x' \rangle$ has no $\text{Aut}(P_2)$-invariant complement.

We recall that central extensions of $C_p$ by $G$ are classified up to equivalence of extensions by $H^2(BG; \mathbb{F}_p)$ (see [Br] or [Th3] for details), where $E$ is equivalent to $E'$ if there is a homomorphism $\theta$ making the following diagram commute:

$$
\begin{array}{ccc}
C_p & \rightarrow & E & \rightarrow & G \\
\downarrow\text{Id} & & \downarrow\theta & & \downarrow\text{Id} \\
\tilde{C}_p & \rightarrow & \tilde{E}' & \rightarrow & \tilde{G}
\end{array}
$$

It is easily seen that a two generator group $G$ of exponent $p$ and order $p^4$ must be a central extension of $C_p$ by $P_2$. Conversely, such an extension has exponent $p$ iff its extension class restricts to zero on all cyclic subgroups, and in this case $G$ will be a two generator group iff its extension class is non-zero. For $p > 3$, the elements $\lambda Y + \mu Y'$ restrict to zero on all cyclic subgroups, whereas for $p = 3$ there is no such non-zero element. We thus have verified a result due to Burnside [Bu], that there are two generator groups of exponent $p$ and order $p^4$ only when $p > 3$.

The action of the Steenrod algebra $\mathcal{A}_p$ on the 1- and 2-dimensional generators of $H^*(BP_2; \mathbb{F}_p)$, and on the $c_i$ and $d_i$ (which are expressed in terms of corestrictions from an abelian subgroup) is apparent. The generator $z$ is the restriction of the generator of $H^{2p}(B\tilde{P}; \mathbb{F}_p)$ of the same name, so $P^1(z)$ is determined by lemma 2.8. The following proposition completes the description of the action of $\mathcal{A}_p$ on $H^*(BP_2; \mathbb{F}_p)$.

**Proposition 2.16.** With notation as in theorems 2.14 and 2.15,

$$P^1(X) = x^{p-1}X + zy$$

$$P^1(X') = x'^{p-1}X' - zy'.$$
Proof. The spectral sequence operation \(BP^1\) defined by Araki and Vasquez ([Ar], [Va]) on the \(E_\infty\) page of the spectral sequence for \(BP_2\) as an \(S^1\)-bundle over \(B\tilde{P}\) sends \(ux\) to \(ux^p\), so we deduce that \(P^1(X) \equiv x^{p-1}X\) modulo the image of the restriction from \(\tilde{P}\). Let \(K\) be a subgroup of \(P_2\) of index \(p\), and let \(H^*(BK; \mathbb{F}_p) = \mathbb{F}_p[\bar{x}, c] \otimes \Lambda[\bar{y}, d]\), where \(\beta(d) = c\), \(\beta(\bar{y}) = \bar{x}\), and \(d\), considered as a morphism from \(K\) to \(\mathbb{F}_p\) sends \(C\) to 1. Then if we let \(\text{Res} = \text{Res}_{K}^{P_2}\) we have \(\text{Res}(X) = \lambda(\bar{x}d - c\bar{y})\), since \(X\) is in the image of the Bockstein and the image of the Bockstein in \(H^3(BK)\) is generated by \(\bar{x}d - c\bar{y}\). We obtain

\[
\text{Res}(P^1(X) - x^{p-1}X) = -\lambda(c^p - \bar{x}^{p-1}c)\bar{y}.
\]

If \(P\) is an expression of degree \(2p + 1\) involving only \(y, y', x, \text{ and } x'\), then \(\text{Res}(P)\) is a multiple of \(\bar{x}^p\bar{y}\), and if \(Q\) is in the span of \(zy\) and \(zy'\), then \(\text{Res}(Q)\) is a multiple of \((c^p - \bar{x}^{p-1}c)\bar{y}\). We know that \(P^1(X) - x^{p-1}X = P + Q\) for some such choices of \(P\) and \(Q\), and we deduce that for all \(K\), \(\text{Res}(P) = 0\). Thus \(\text{Res}(\beta P) = 0\), and hence \(\beta P\) is a multiple of \(x^p x' - x'^p x\), so is zero in \(H^*(BP_2; \mathbb{F}_p)\). The Bockstein is injective on the subspace of \(H^{2p+1}\) generated by \(x, x', y, \text{ and } y'\), so we deduce that \(P = 0\). From lemma 2.13 we can determine \(\lambda\) in the case \(K = \langle A, C \rangle\) and \(\bar{y} = \text{Res}(y)\) (resp. \(K = \langle B, C \rangle\) and \(\bar{y} = \text{Res}(y')\)), and conclude that \(Q = -zy\). The result for \(P^1(X')\) follows from this result or may be deduced similarly.

The above information tells us what happens in the Atiyah-Hirzebruch spectral sequence for \(P_2\).

Corollary 2.17. In the Atiyah-Hirzebruch spectral sequence for \(P_2\) \((E^1_2 \cong H^i(BP_2; \mathbb{Z})\) converging to a filtration of the representation ring of \(P_2\)) the only non-zero differential is \(d_{2p-1}\), which sends \(\mu\) to a multiple of \(\zeta \beta\), and \(\nu\) to a multiple of \(\zeta \alpha\).

Proof. In this spectral sequence the first potentially non-zero differential is \(d_{2p-1}\), which is \(\delta_p P^1 \pi_*\).
3. The Size of the Chern Subring and its Closure.

We recall that a group $G$ is said to have $p$-rank $n$ if $n$ is maximal such that $G$ has a subgroup isomorphic to $(C_p)^n$. In [Bl], Blackburn classified, for odd primes $p$, the $p$-groups having no normal subgroup isomorphic to $C_p \times C_p \times C_p$. It can be shown, independently of Blackburn’s result, that all such groups satisfy the apparently stronger condition of having $p$-rank at most two, see for example [Go]. The classification is as follows:

**Theorem 3.1.** (Blackburn) Let $p$ be an odd prime. Then the $p$-groups of $p$-rank two are the following groups:

1. The (non-cyclic) metacyclic $p$-groups.
2. $P(n)$, where $n \geq 3$ as defined in the introduction to section 2.
3. $B(n, \varepsilon)$, where $n \geq 4$, there are two groups for each $n$, depending whether $\varepsilon$ is 1 or a quadratic non-residue modulo $p$. $B(n, \varepsilon)$ has order $p^n$, and may be presented as:

\[
\langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [B, C] = 1 \quad [A, C^{-1}] = B \quad [B, A] = C^{\varepsilon p^{n-3}} \rangle
\]

For $p > 3$ the groups $B(4, \varepsilon)$ are the seventh and eighth groups on Burnside’s list of groups of order $p^4$, and for $p = 3$, $B(4, -1)$ is the tenth group on Burnside’s list ([Bu] or the appendix to this dissertation), and $B(4, 1)$ is the eighth group.

4. In the case $p = 3$, every 3-group of maximal nilpotency class except $C_3$, $C_9$ and the wreath product of $C_3$ with itself (the sixth group on Burnside’s list of groups of order 81 [Bu]).

Let us say that a group $G$ has property $C$ if $\text{Ch}(G) = H^{\text{ev}}(BG)$ (recall that $\text{Ch}(G)$ is the subring generated by Chern classes of representations of $G$). Various of the $p$-groups of $p$-rank two have been shown to have property $C$, for example in section 2 we verified Thomas’ result that $P(n)$ has property $C$ [Th2]. Thomas also verified that the split metacyclics have property $C$ and conjectured that all $p$-groups of $p$-rank two would have property $C$ [Th1], [Th2]. He also gave the example $A_4$ to show that the conjecture could not
be extended to arbitrary groups of $p$-rank at most two for all primes. AlZubaidy claimed to have verified the conjecture for $p \geq 5$, see [Al1], [Al2], but some of his proofs are flawed. Recently Tezuka-Yagita have shown that all metacyclic $p$-groups have property $C$ [TY2], and Huebschmann has shown independently [Hu1] that all finite metacyclic groups have property $C$. Moselle has suggested that property $\overline{C}$, that of satisfying $\overline{\text{Ch}}(G) = H^{ev}(BG)$ might be more natural, and points out that $G$ has property $\overline{C}$ if and only if each of its Sylow subgroups does [Mo].

We shall show that for $p \geq 5$ the groups $B(n, \varepsilon)$ do not have property $C$, but that for $p = 3$ the groups $B(n, \varepsilon)$ do have property $C$. We also show that for all odd $p$ the groups $B(n, \varepsilon)$ have property $\overline{C}$. Yagita has shown independently that for $p \geq 5$ the groups $B(n, \varepsilon)$ do not have property $C$ [Ya2].

In [At] Atiyah showed that for any finite group $G$, $K^0(BG)$ is the completion of the representation ring of $G$ with respect to a certain topology. The filtration of $K^0(BG)$ given by the $E_\infty$ page of the Atiyah-Hirzebruch spectral sequence ($H^i(BG; K^j(\ast)) \Rightarrow K^{i+j}(BG)$) gives rise to a filtration on the representation ring of $G$. He conjectured that this filtration coincided with another filtration defined algebraically, and remarked that this conjecture is equivalent to the conjecture that $\text{Ch}(G)$ maps onto the $E_\infty$ page of the AHSS. Many counterexamples have been found of composite order [Wei]. We show that for $p \geq 5$ the groups $B(n, \varepsilon)$ are counterexamples. These seem to be the first counterexamples of prime power order.

First we state a result (presumably well known) concerning abelian $p$-groups and property $C$, and show that $p$-groups with an elementary abelian maximal subgroup of rank at least 3 do not have property $\overline{C}$.

**Proposition 3-2.** Let $A$ be an abelian $p$-group. Then the following are equivalent:

1) $A$ has $p$-rank at most two.

2) $\text{Ch}(A) = H^{ev}(BA)$.

3) $\overline{\text{Ch}}(A) = H^{ev}(BA)$. 

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Proof. An abelian group has only 1-dimensional irreducible representations, so \( \text{Ch}(A) \) and \( \overline{\text{Ch}}(A) \) are generated in degree two. For any finite group \( G \),

\[
\text{Ch}^2(G) = H^2(BG; \mathbb{Z}) \cong \text{Hom}(G, U(1)),
\]

so 2) and 3) are equivalent to each other and to the statement that \( H^{\text{ev}}(BA) \) is generated in degree two. If \( A \) is cyclic then \( H^*(BA; \mathbb{Z}) \) is a polynomial algebra on a generator of degree two, and if \( A \) is a product of two cyclic groups then the Künneth theorem implies that \( H^{\text{ev}}(BA) \) is a polynomial algebra on two generators of degree two. For the general case, let

\[
A \cong C_{p^{n_1}} \oplus C_{p^{n_2}} \oplus \cdots \oplus C_{p^{n_m}}, \quad \text{where} \quad n_1 \leq n_2 \leq \cdots \leq n_m.
\]

Then

\[
H^*(BA; \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_m] \otimes A[y_1, \ldots, y_m],
\]

where the \( n_i \)-th higher Bockstein maps \( y_i \) to \( x_i \). The image of \( \overline{\text{Ch}}(A) \) under reduction mod-\( p \) is the subring generated by the \( x_i \), and the image of \( H^*(BA; \mathbb{Z}) \) is the universal cycles in the Bockstein spectral sequence. (Recall that the Bockstein spectral sequence has \( E^2_i \cong H^i(BA; \mathbb{F}_p) \) and converges to \( \mathbb{F}_p \) concentrated in degree zero, with differentials the higher Bocksteins and \( B^*_i \) the image under reduction mod-\( p \) of elements of order dividing \( p^{i-1} \).) The first non-zero differential in this spectral sequence sends \( y_1y_2y_3 \) to \( x_1y_2y_3 + \varepsilon x_2y_1y_3 + \varepsilon' x_3y_1y_2 \), where \( \varepsilon, \varepsilon' \) are either 0 or 1, so this element is in the image of \( H^{\text{ev}}(BA) \), but not in the image of \( \overline{\text{Ch}}(A) \).

**Proposition 3.3.** If \( G \) is a \( p \)-group with a \( (C_p)^n \) subgroup of index \( p \) for some \( n \geq 3 \), then \( \overline{\text{Ch}}(G) \neq H^{\text{ev}}(BG) \).

**Proof.** The subgroup is maximal, so is normal. Call it \( N \), and let \( Q \) be the quotient \( G/N \cong C_p \). By proposition 3.2 we may assume that \( Q \) acts non-trivially on \( N \). We may identify \( H^1 = H^1(BN; \mathbb{F}_p) \) with \( \text{Hom}(N, \mathbb{F}_p) \) and then we see that

\[
H^*(BN; \mathbb{F}_p) \cong S(\beta H^1) \otimes E(H^1),
\]

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where \( S(\beta H^1) \) and \( E(H^1) \) are respectively the symmetric and exterior algebras on the vector space \( H^1 \). Now \( H^2 \) splits as a direct sum of \( \beta H^1 \) and \( E^2(H^1) \), \( H^3 \) splits as a direct sum of \( \beta H^1 \otimes H^1 \) and \( E^3(H^1) \), and the action of \( Q \) on \( H^1 \) respects these splittings. Let \( y \) be an element of \( H^1 \) with an orbit of length \( p \) so that its images under \( Q \) are \( y + iy' \), for some \( y' \) fixed by \( Q \), and let \( \alpha = \delta_p(y) \) and \( \alpha' = \delta_p(y') \). Also let \( w \) be an element of \( E^3(H^1) \) fixed by \( Q \), and let \( \xi = \delta_p(w) \). Now writing \( N_G^G \) for Evens multiplicative transfer map from \( N \) to \( G \) [Ev1], we have

\[
\text{Res}_G^G N_G^G (\alpha + \xi) = p^{-1} \prod_{i=0}^{p-1} (\alpha + i\alpha' + \xi) = \alpha_p - \alpha'^p - \alpha - \alpha'^{-1} \xi.
\]

Since \( \pi_*(\alpha'^{-1} \xi) \) does not lie in \( S(\beta H^1) \) it follows that \( \alpha'^{-1} \xi \) is not in \( \overline{\text{Ch}}(N) \), and hence the component of \( N_G^G (\alpha + \xi) \) in degree \( 2p + 2 \) cannot be in \( \overline{\text{Ch}}(G) \).

**The Groups** \( B(n, \varepsilon) \). Throughout this section we shall consider \( B(n, \varepsilon) \) as being presented, as in the introduction, by elements \( A, B, \) and \( C \) subject to the above relations. We note that the centre of \( B(n, \varepsilon) \) is cyclic generated by \( C^p \), and that if we apply the circle construction to the whole centre we obtain a group \( \tilde{B} \) which is independent of both \( n \) and \( \varepsilon \). This is because there is only one circle group on \( p^3 \) components of nilpotency class three. The subgroup generated by \( A, B, \) and \( C^p \) is isomorphic to \( P(n - 1) \), with centre also generated by \( C^p \), and so we have a commutative diagram of extensions.

\[
\begin{array}{ccc}
P(n - 1) & \rightarrow & B(n, \varepsilon) \rightarrow C_p \\
\downarrow & & \downarrow \quad \downarrow id \\
\tilde{P} & \rightarrow & \tilde{B} \rightarrow C_p
\end{array}
\]

We note also that the subgroup generated by \( B \) and \( C \) is abelian and isomorphic to \( C_{p^n - 1} \oplus C_p \). The corresponding subgroup of \( \tilde{B} \) (that is the one generated by \( B, C \) and \( S^1 \)) is also abelian and is isomorphic to \( S^1 \oplus C_p \oplus C_p \). The action induced on \( H^*(BP(n - 1); \mathbb{Z}) \) by conjugation by \( C \) fixes \( \chi_i, \zeta, \alpha \) and \( \nu \) and sends \( \beta \) to \( \beta + \alpha \), and \( \mu \) to \( \mu + \nu \). The action induced on \( H^*(B\tilde{P}; \mathbb{Z}) \) is similar.
Lemma 3.4. The fixed point subring of $H^*(B\tilde{P};\mathbb{Z})$ (resp. $H^{ev}(BG;\mathbb{Z})$) is generated by the elements $\alpha, \chi_i, \zeta, \text{ and } \beta^n(\beta^p - \alpha^{p-1}\beta)$ for $n \geq 0$.

Proof. The element $\beta^p - \alpha^{p-1}\beta$ is the product of all conjugates of $\beta$ so must be fixed. Also $\alpha(\beta^p - \alpha^{p-1}\beta) = 0$ and $\beta^n$ is sent to itself modulo multiples of $\alpha$, so $\beta^n(\beta^p - \alpha^{p-1}\beta)$ is fixed. $\xi P(\alpha, \beta)$ is fixed if and only if $P(\alpha, \beta)$ is, so we only need to check that there are no more fixed points in the subring generated by $\alpha$ and $\beta$. We may take as basis for the degree $n$ piece of this the elements $\alpha^n, \alpha^{n-1}\beta, \ldots \beta^n$ if $n \leq p$ and $\alpha^n, \alpha^{n-1}\beta, \ldots \alpha^{n-p+1}\beta^{p-1}, \beta^n$ if $n \geq p$. In either case we are left to show that there are no fixed points of the form $\sum_{i=1}^m \lambda_i \alpha^{n-i}\beta^i$, where $m \leq p - 1$. Equating coefficients of $\alpha^{n-m+1}\beta^{m-1}$ in

$$\sum_{i=1}^m \lambda_i \alpha^{n-i}\beta^i = \sum_{i=1}^m \lambda_i \alpha^{n-i}(\beta + \alpha)^i$$

we obtain

$$\lambda_{m-1} = \lambda_{m-1} + m\lambda_m$$

so $\lambda_m = 0$, and inductively each $\lambda_i$ is zero.

Lemma 3.5. The image of $\text{Ch}(\tilde{B})$ under restriction to $\tilde{P}$ (resp. the image of $\text{Ch}(B(n, \varepsilon))$ under restriction to $P(n-1)$) is generated by $\chi_i, \zeta, \alpha, \text{ and } \beta^p - \alpha^{p-1}\beta$.

Proof. By lemma 1.7 it suffices to prove the assertion for the Lie groups. A direct proof for the finite groups would be very similar. $\tilde{B}$ has an abelian subgroup of index $p$, so has only 1- and $p$-dimensional irreducible representations. By considering the natural isomorphisms

$$\text{Ch}^2(G) = H^2(BG;\mathbb{Z}) \cong \text{Hom}(G, S^1)$$

we see that $\chi_1$ and $\alpha$ span the image of $\text{Ch}^2(\tilde{B})$. For any $p$-dimensional representation $\rho$ of $\tilde{P}$ the representation $\text{Ind}_{\tilde{B}}^P(\rho)$ will split into $p$ $p$-dimensional representations of $\tilde{P}$, any of which will restrict to $\rho$. Hence $\chi_i$ and $\zeta$ are in the image. If $\theta$ is the 1-dimensional representation of $\tilde{P}$ with $c_1(\theta) = \beta$, then

$$c_p(\text{Ind}_{\tilde{B}}^P(\theta)) = \prod_{i=0}^{p-1} (\beta + i\alpha) = \beta^p - \alpha^{p-1}\beta.$$
We do not need any more generators because the image must be generated in degrees at most $2p$. In these degrees we already have all of the fixed points by lemma 3.4. ■

Let $H$ be the subgroup of $\tilde{B}$ generated by $S^1$ and $B$, and define elements $\tau', \beta'$ of $H^2(BH; \mathbb{Z}) \cong \text{Hom}(H, S^1)$ by the following formulae.

$$
\begin{align*}
\tau'|_{S^1} &= 1_{S^1} & \beta'|_{S^1} &= 0 \\
\tau'(B) &= 0 & \beta'(B) &= \exp(2\pi i/p)
\end{align*}
$$

Let $M$ be the abelian maximal subgroup of $\tilde{B}$, that is the subgroup generated by $S^1$, $B$, and $C$. The restriction map from $H^*(BM; \mathbb{Z})$ to $H^*(BH; \mathbb{Z})$ is clearly onto.

**Lemma 3.6.** If $\varphi \in H^{2(p+n)}(BM; \mathbb{Z})$ restricts to $H$ as $-\tau^{p-1}\beta^{n+1}$, then $\text{Cor}_{B}(\varphi)$ restricts to $\tilde{P}$ as $\beta^n(\beta^p - \alpha^{p-1}\beta)$.

**Proof.** $M\tilde{P} = \tilde{B}$, and $M \cap \tilde{P} = H$, so the restriction-corestriction formula shows us that

$$
\text{Res}_{\tilde{P}}^B \text{Cor}_{M}(\varphi) = \text{Cor}_{H}^B \text{Res}_{H}^M(\varphi)
$$

$$
= -\text{Cor}_{H}^B (\tau^{p-1}\beta^{n+1})
$$

$$
= -\text{Cor}_{H}^B (\tau^{p-1})\beta^{n+1}
$$

$$
= -(\chi_{p-1} + \alpha^{p-1})\beta^{n+1}
$$

$$
= \beta^n(\beta^p - \alpha^{p-1}\beta). 
$$

**Corollary 3.7.** For $p \geq 5$, $H^{\text{ev}}(BB(n, \varepsilon))$ is not generated by Chern classes.

**Proof.** $\beta^{p+1} - \alpha^{p-1}\beta^2$ is in the image of the restriction from $B(n, \varepsilon)$ to $P(n-1)$ by lemma 3.6. For $p \geq 5$ it may be verified that it is not expressible in terms of the generators for the image of $\text{Ch}(B(n, \varepsilon))$ given in lemma 3.5. For $p = 3$,

$$
\beta^4 - \alpha^2\beta^2 = \chi^2 - 3\zeta\chi - \alpha^4.
$$

Yagita has also proved corollary 3.7 [Ya2] using a method involving his calculation of the Brown-Peterson cohomology of $BP_2$ [TY].
Corollary 3.8. For \( p \geq 5 \) and the groups \( B(n, \varepsilon) \), the Chern subring does not map surjectively to the \( E_\infty \) page of the Atiyah Hirzebruch spectral sequence.

Proof. Let \( K \) be the abelian maximal subgroup of \( B(n, \varepsilon) \). Then by lemma 3.5 and lemma 1.7 there is a \( \varphi \in \text{Ch}^{2p+2}(K) \) satisfying

\[
\text{Res}_{P(n-1)}^{B(n,\varepsilon)} \text{Cor}_K^{B(n,\varepsilon)}(\varphi) = \beta^{p+1} - \alpha^{p-1} \beta^2.
\]

From the naturality properties of the AHSS it follows that \( \text{Cor}(\varphi) \) is a universal cycle. We show that it is not the sum of a Chern class and a universal boundary by considering the restriction to the subgroup generated by \( B \).

\[
H^*(B\langle B \rangle; \mathbb{Z}) = \mathbb{Z}[\overline{\beta}]/(p\overline{\beta}),
\]

where \( \overline{\beta} \) is the restriction from \( P(n-1) \) of \( \beta \). The AHSS for \( \langle B \rangle \) collapses because \( E_{2}^{i,j} \) is trivial if \( i \) or \( j \) is odd. Hence the universal boundaries \( B_\infty(B(n, \varepsilon)) \) restrict trivially to \( \langle B \rangle \). By lemma 3.5 the image of \( \text{Ch}(B(n, \varepsilon)) \) under this restriction is generated by \( \overline{\beta}^p \) and \( \overline{\beta}^{p-1} \). So for \( p \geq 5 \),

\[
\text{Res}_{\langle B \rangle}^{B(n,\varepsilon)} \text{Cor}_K^{B(n,\varepsilon)}(\varphi) = \overline{\beta}^{p+1} \notin \text{Res}_{\langle B \rangle}^{B(n,\varepsilon)}(B_\infty(B(n, \varepsilon)) + \text{Ch}(B(n, \varepsilon))).
\]

Corollary 3.9. \( H^*(B\widetilde{B}) \) restricts onto the fixed point subring of \( H^*(B\widetilde{P}) \).

Proof. Immediate from lemma 3.4 and lemma 3.6.

Corollary 3.10. \( H^{ev}(B\widetilde{B}) \) is generated by corestrictions of Chern classes, and by Chern classes for \( p = 3 \).

Proof. Consider the spectral sequence with integer coefficients for the extension

\[
\widetilde{P} \rightarrow \widetilde{B} \rightarrow C_p.
\]

For any extension with quotient \( C_p \) the \( E_2 \) page of the corresponding spectral sequence is generated by \( E_2^{0,*}, E_2^{1,*} \) and \( E_2^{2,0} \cong \mathbb{F}_p \). The inflation of a generator for \( H^2(BC_p) \) is a
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Chern class, and yields a generator for $E_2^{2,0}$. Also corollary 3.9 implies that $E_2^{0,*}$ consists of universal cycles, and lemma 3.6 implies that corestriction of Chern classes (resp. Chern classes in the case when $p = 3$) yield generators for the even degree subring of $E_2^{*,*}$. ■

We shall now work towards theorem 3.13, which will tell us that $H^{\text{ev}}(BB(n, \varepsilon))$ is generated by corestrictions of Chern classes, and will involve the application of theorem 1.9. Corollary 3.10 has verified one of the hypotheses required for this, so it suffices now to check that multiplication by the extension class $c(B(n, \varepsilon))$ in $H^2(\tilde{B})$ is injective on $H^{\text{od}}(B\tilde{B})$. This is proven for all $n$ as a corollary of lemma 3.12, but we first present a simpler proof in the case when $n \geq 5$. The reader who objects to redundancy should turn directly to the end of the proof of lemma 3.11.

Since $\tilde{P}$ has index $p$ in $\tilde{B}$ and trivial odd degree cohomology it follows by the usual corestriction-restriction argument that $H^{\text{od}}(B\tilde{B})$ has exponent $p$. It is easily checked that $c(B(n, \varepsilon))$ is of the form $p^{n-4}$ times an element of infinite order plus a non-zero element of the image of the inflation from $\tilde{B}/\tilde{P} \cong C_p$. Thus it will follow that $H^{\text{ev}}(BB(n, \varepsilon))$ is generated by corestrictions of Chern classes (resp. Chern classes for $p = 3$) for $n \geq 5$ provided that multiplication by a generator for $\text{Inf}(H^2(\tilde{B}/\tilde{P}; \mathbb{Z}))$ is injective on $H^{\text{od}}(B\tilde{B})$. This is a case of the following lemma.

**Lemma 3.11.** Consider a fibration

$$F \longrightarrow E \xrightarrow{\pi} BC_p,$$

such that $H^*(F)$ is concentrated in even degrees and $H^*(E)$ maps onto $H^*(F)^{C_p}$, and let $\gamma$ be a generator for $H^2(BC_p)$. Then multiplication by $\pi^*(\gamma)$ is injective on $H^{\text{od}}(E)$.

**Proof.** Consider the spectral sequence for the fibration. The element $\pi^*(\gamma)$ yields a generator for $E_2^{0,2}$ which we shall call $\gamma$, and all elements in even total degree are universal cycles. It suffices to prove that multiplication by $\gamma$ is an isomorphism from $E_2^{2i+1, j}$ to $E_2^{2i+3, j}$ for all $i$ and $j$. In fact we shall show inductively that multiplication by the corresponding element of $E_2^{2,0}$ (which we shall also call $\gamma$) is an isomorphism from $E_2^{i,j}$ to $E_2^{i+2, j}$.
provided that either $i$ is odd or $i \geq 2n$. This is true for $n = 1$. Now $d_{2n}$ is trivial because it maps between even and odd values of $j$. It is possible for $d_{2n+1}$ to be non-trivial, but only from odd total degree to even total degree. Since by induction the odd degree part of $E_{2n+1}^{*,*}$ is generated by $E_{2n+1}^{1,*}$, $d_{2n+1}$ is completely determined by its values on $E_{2n+1}^{1,*}$. We obtain the following commutative diagram, where the horizontal maps are isomorphisms by induction.

\[ \begin{array}{ccccccc}
E_{2n+1}^{1,j} & \xrightarrow{\times \gamma} & E_{2n+1}^{3,j} & \xrightarrow{\times \gamma} & E_{2n+1}^{5,j} & \rightarrow & \ldots \\
E_{2n+1}^{2n+2,j-2n-1} & \xrightarrow{\times \gamma} & E_{2n+1}^{2n+4,j-2n-1} & \xrightarrow{\times \gamma} & E_{2n+1}^{2n+6,j-2n-1} & \rightarrow & \ldots
\end{array} \]

Hence multiplication by $\gamma$ induces isomorphisms on the kernels and cokernels, and the inductive step is proven.

We shall now deduce that $H_{\text{ev}}^*(BG) = \overline{\text{Ch}}(G)$ (resp. $\text{Ch}(G)$ for $p = 3$) for any $p$-group $G$ of $p$-rank two and nilpotency class three using the following lemma.

**Lemma 3.12.** As above, let $M$ be the abelian subgroup of $\tilde{B}$ of index $p$. Then the restriction from $H_{\text{od}}^*(B\tilde{B})$ to $H_{\text{od}}^*(BM)$ is injective.

**Proof.** First let us fix a presentation for $\tilde{B}$.

\[ \tilde{B} = \langle S^1, A, B, C \mid S^1 \text{ central}, \quad A^p = B^p = C^p = 1 \]
\[ [A, B] = C, \quad [B, C] = 1, \quad [A, C] = e^{2\pi i/p} \]

Note that $\tilde{B}/S^1$ is generated by the images of $A$ and $B$, which we shall call $A$ and $B$ too, so that our notation coincides with that of section 2. Note that the subgroup $M$ is expressible as an extension of $S^1$ by $\langle B, C \rangle$. We shall show that the map of spectral sequences induced by the following map of extensions is injective on the $E_\infty$ page.

\[ \begin{array}{ccc}
S^1 & \rightarrow & M \\
\downarrow \text{Id} & & \downarrow i \\
S^1 & \rightarrow & \tilde{B}
\end{array} \]
\[ \begin{array}{ccc}
M & \rightarrow & \langle B, C \rangle \\
\downarrow i & & \downarrow \bar{i} \\
\tilde{B} & \rightarrow & P_2
\end{array} \]

Let $E^{*,*}_s$ stand for the spectral sequence for $\tilde{B}$, and let $\overline{E}^{*,*}_s$ for the spectral sequence for $M$. Notice that $\overline{E}_s^{*,*}$ collapses. If we use the notation of section 2 for elements of $H^*(BP_2; \mathbb{Z})$
and let $H^*(BS^1; \mathbb{Z}) \cong \mathbb{Z}[\tau]$, then we see that

$$E_2^{*,*} \cong \mathbb{Z}[\tau] \otimes H^*(BP_2; \mathbb{Z}).$$

If we let $\rho$ be a 1-dimensional representation of $M$ such that $\text{Res}^{M}_{S^1}(c_1(\rho)) = \tau$, then

$$\text{Res}^{\tilde{B}}_{S^1}(c.(\text{Ind}^{\tilde{B}}_{M}(\rho))) = (1 + \tau)^p.$$ 

Hence $p\tau^i$ and $\tau^p$ are universal cycles, the $E_\infty$ page is periodic vertically with period $2p$, and $E_\infty^{*,*} \cong E_{2p}^{*,*}$. Since $\tilde{B}' \cap S^1$ has order $p$ there can be no homomorphism from $\tilde{B}$ to $S^1$ that restricts to $S^1$ as an isomorphism, so $d_3(\tau)$ must be non-zero. The kernel of the map from $E_2^{3,0}$ to $E_2^{3,0}$ is generated by $\nu$, and $d_3(\tau)$ must lie in this kernel (because $E_*^{*,*}$ collapses), so without loss of generality $d_3(\tau) = \nu$.

It may be verified that $E_5^{*,*}$ is generated by $\alpha, \beta, \chi_2, \chi_4, \ldots, \chi_{p-1}, \zeta, \mu, \tau^i\beta\mu, \tau^i\chi_2, \tau^i\chi_4, \ldots, \tau^i\chi_{p-2}, \tau^i(\chi_{p-1} + \alpha^{p-1}), p\tau^i, \tau^p$, and $\tau^{p-1}\nu$, where $1 \leq i \leq p - 1$, subject to various relations. For example, we have that $[\tau^i\beta\mu]\alpha = 0$ unless $i$ is congruent to $-1$ modulo $p$. Figure 3-1 depicts this $E_5$ page in the case when $p \geq 7$.

We now claim that $\tau^{p-1}\nu$ survives until the $E_{2p-1}$ page, and that $d_{2p-1}(\tau^{p-1}\nu)$ is a non-zero multiple of $\zeta\alpha$ (which is equal to $\delta_pP^1\pi_*(\nu)$—see proposition 2-16 and corollary 2-17). To show this, we first consider the spectral sequence with $\mathbb{Z}_{(p)}$ coefficients for the path-loop fibration over $K(\mathbb{Z}, 3)$. Let $i_3$ be the (image in $\mathbb{Z}_{(p)}$ coefficients of the) fundamental class in $H^3(K(\mathbb{Z}, 3))$, and $i_2$ that in $H^2(\Omega K(\mathbb{Z}, 3))$, so that $d_3(i_2) = i_3$. It is readily verified that $i_2^{p-1}i_3$ survives until the $E_{2p-1}$ page, and that $d_{2p-1}(i_2^{p-1}i_3)$ is a unit multiple of $\delta_pP^1\pi_*(i_3)$. The claim concerning $\tau^{p-1}\nu$ now follows by naturality, because $\nu$ may be thought of as a map from $BP_2$ to $K(\mathbb{Z}, 3)$ classifying the $K(\mathbb{Z}, 2)$ bundle $\text{B}^{-1}$. 

Even on $E_5$ pages the map from $E_5^{\text{od},2i}$ to $E_5^{\text{od},2i}$ is injective except when $i$ is congruent to $-1$ modulo $p$, and its kernel is the submodule generated by $\tau^{p-1}\nu$. Thus it will suffice to show that no differential lower than $d_{2p-1}$ can hit anything in the ideal of $E_5^{*,0}$ generated by $\zeta\alpha$. This follows because multiplication by $\alpha$ is injective on this ideal but trivial on $E_5^{\text{od},2i}$ for $1 \leq i < p - 1$. 

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| $p\tau^3$ | $-$ | $-$ | $-$ | $\tau^3\chi_2$ | $\tau^3\beta\mu$ | $-$ | $\tau^3\beta^2\mu$ | $\tau^3\chi_4$ |
|-----------|-----|-----|-----|-----------------|-----------------|-----|-----------------|----------------|
| $-$       | $-$ | $-$ | $-$ | $-$             | $-$             | $-$ | $-$             | $-$             |
| $p\tau^2$ | $-$ | $-$ | $-$ | $\tau^2\chi_2$ | $\tau^2\beta\mu$ | $-$ | $\tau^2\beta^2\mu$ | $\tau^2\chi_4$ |
| $-$       | $-$ | $-$ | $-$ | $-$             | $-$             | $-$ | $-$             | $-$             |
| $p\tau$   | $-$ | $-$ | $-$ | $\tau\chi_2$  | $\tau\beta\mu$  | $-$ | $\tau\beta^2\mu$  | $\tau\chi_4$  |
| $-$       | $-$ | $-$ | $-$ | $-$             | $-$             | $-$ | $-$             | $-$             |
| $1$       | $-$ | $\alpha,\beta$ | $\mu$ | $\alpha^2,\alpha\beta,\beta^2,\chi_2$ | $\beta\mu$ | $\alpha^3,\alpha^2\beta,\alpha\beta^2,\beta^3,\beta^2\mu$ | $\alpha^4,\alpha^3\beta,\alpha^2\beta^2,\alpha\beta^3,\beta^4,\chi_4$ | $-$ |

3.1. The $E_5$-page of the spectral sequence of lemma 3.12 in the case $p \geq 7$.

**Theorem 3.13.** If $G$ is any finite normal subgroup of $\tilde{B}$ of $p$-rank two with quotient isomorphic to $S^1$, then $H^{ev}(BG) = \overline{Ch}(G)$, and if $p = 3$ then $H^{ev}(BG) = Ch(G)$. Examples of such $G$ include the $p$-groups $B(n, \varepsilon)$ and for $p = 3$ the seventh, eighth and tenth groups on Burnside’s list of groups of order 81 ([Bu] or the appendix).

**Proof.** Let $c(G)$ be the extension class of $G$ in $H^2(B\tilde{B})$, that is the element of $H^2(B\tilde{B})$ corresponding to the homomorphism from $\tilde{B}$ to $S^1$ with kernel $G$. Since $H = G \cap M$ is abelian of $p$-rank at most two, $Ch(H) = H^{ev}(BH)$, so by theorem 1.9 multiplication by $c(H)$ is injective on $H^{od}(BM)$. But now $c(H) = \text{Res}_{\tilde{B}}^B(c(G))$, and so by lemma 3.12 multiplication by $c(G)$ is injective on $H^{od}(B\tilde{B})$. The result now follows from theorem 1.9, since corollary 3.10 tells us that $H^{ev}(B\tilde{B})$ is generated by corestrictions of Chern classes.
and by Chern classes for \( p = 3 \). The groups listed in the statement do have \( p \)-rank two, and they are normal subgroups of \( \tilde{B} \), because they have nilpotency class three, and a central cyclic subgroup with quotient group \( P_2 \), so yield \( \tilde{B} \) when the circle construction is applied.

\[ \square \]

**Corollary 3.14.** For \( p \geq 5 \), if \( G \) is a \( p \)-group of \( p \)-rank two, then \( H^{ev}(BG) \) is generated by corestrictions of Chern classes.

**Proof.** This is immediate from a combination of Blackburn’s classification [Bl] (or see theorem 3.1), Huebschmann’s results on the metacyclic groups [Hu1], Thomas’ result for the groups \( P(n) \) [Th2] (or theorem 2.3), and theorem 3.13.  

\[ \blacksquare \]
4. Further Calculations at the Prime Three.

In this section we shall make a more detailed study of the integral cohomology of the group \( \tilde{B} \) and the \( p \)-groups \( B(n, \varepsilon) \), particularly when \( p = 3 \). In this case the proof of lemma 3.12 describes most of the behaviour of a spectral sequence converging to a filtration of \( H^*(\tilde{B}; \mathbb{Z}) \). Using this spectral sequence we determine the cohomology of \( \tilde{B} \), and then calculate the cohomology of \( B(n, \varepsilon) \) using the spectral sequence for \( BB(n, \varepsilon) \) as an \( S^1 \)-bundle over \( B\tilde{B} \). The main result of the section is that (for \( p = 3 \)) the integral cohomology ring of \( B(n, 1) \) is isomorphic to that of \( B(n, -1) \), so that there are two groups of order \( 3^5 \) with isomorphic integral cohomology rings. The author believes that these are the first examples of such \( p \)-groups, although non-isomorphic groups of composite order with isomorphic cohomology rings have been known for some time, and Larson has even exhibited metacyclic examples [La]. For \( p > 3 \) it seems far more difficult to determine the cohomology of \( \tilde{B} \). Nevertheless, and in spite of the title of this section, we present a proof that the cohomology groups (ignoring the ring structure) of \( B(n, \varepsilon) \) are independent of \( \varepsilon \) for all odd \( p \), without actually determining these groups. One response to the main result of this section has been “Are you sure that the groups are different?”, so before starting our calculations we show that as \( \varepsilon \) varies in \( \mathbb{F}_p^\times \) there really are two non-isomorphic groups of the form \( B(n, \varepsilon) \).

**Lemma 4.1.** For any fixed odd prime \( p \) and \( n \geq 5 \), define for each \( \varepsilon \in \mathbb{F}_p \) a group \( G(\varepsilon) \) of order \( p^n \) by the presentation

\[
G(\varepsilon) = \langle A, B, C, D \mid A^p = B^p = C^p = 1, \quad C \text{ central, } B = [A, D], \quad C = [A, B], \quad [B, D] = 1, \quad D^{p^{n-3}} = C^\varepsilon \rangle.
\]

Then there are three isomorphism classes of such \( G \), depending as \( \varepsilon \) is 0, a quadratic residue, or a quadratic non-residue mod-\( p \).

**Proof.** Note that the relations given suffice to express any element of \( G \) in the form \( A^i B^j C^k D^l \). Note that the subgroup of \( G \) generated by \( A \) and \( B \) is normal, and isomorphic
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to $P_2$, with quotient cyclic of order $p^{n-3}$. First we claim that if $g \in \langle A, B \rangle$, then the order of $gD^j$ is at least the order of $D^j$. First note that

$$(gD^j)^p = \left( \prod_{k=0}^{p-1} g^{D^j_k} \right) D^{jp},$$

and that the expression on the right is not 1 unless $D^{jp}$ is. Now since $D^p$ commutes with elements of $\langle A, B \rangle$, we see that

$$(gD^j)^{p^2} = D^{jp^2}. \quad (*)$$

It follows that if $\varepsilon \neq 0$, then $\langle A, B \rangle$ is the subgroup of $G$ of elements of order $p$, and we deduce that $G(0)$ is not isomorphic to $G(\varepsilon)$ for $\varepsilon \neq 0$ because it contains more elements of order $p$. From now on, let us consider only the case $\varepsilon \neq 0$.

The elements $A$ and $D$ generate $G$, and any automorphism of $G$ must the form

$$D \mapsto A^i B^j C^k D^l \quad \text{where } l \neq 0 \mod p$$

$$A \mapsto A^r B^s C^t \quad \text{where } 1 \leq r \leq p - 1.$$

If we define $D'$ to be the image under such an automorphism of $D$, $A'$ to be the image of $A$, $B' = [A', D']$ and $C' = [A', B']$, then the primed elements define a presentation of the form $G(\varepsilon')$. To complete the proof it suffices to show that $\varepsilon' r^2 = \varepsilon$. From equation $(*)$ it follows that $D'^{p^{n-3}} = D^{lp^{n-3}}$. Now

$$B' = [A^r B^s C^t, A^i B^j C^k D^l]$$

$$\equiv [A^r, D^l] \mod \langle C \rangle$$

$$\equiv B^{rl} \mod \langle C \rangle,$$

$$C' = [A^r B^s C^t, B^{rl} C^2] = C'^{2l},$$

from which it is apparent that $D'^{p^{n-3}} = C'^{\varepsilon/r^2}$.

We shall now begin our calculation of $H^*(\tilde{B}^{\tilde{B}}; \mathbb{Z})$ in the case when $p = 3$ by considering the spectral sequence for $\tilde{B}$ expressed as an extension of $S^1$ by $P_2$. This spectral sequence was studied for arbitrary $p$ during the proof of lemma 3.12, where $d_3$ and part of $d_{2p-1}$ were calculated. In the case when $p = 3$ not much more work is required to complete the spectral sequence. For the rest of this chapter we shall consider only the prime three unless otherwise stated.

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Lemma 4.2. Consider the spectral sequence with integral coefficients for the extension

$$S^1 \rightarrow \tilde{B} \rightarrow P_2.$$ 

Use notation as in the proof of lemma 3.12, so that

$$E_2^{**} \cong \mathbb{Z}[	au] \otimes H^*(BP_2; \mathbb{Z}).$$

Then $d_3(\tau) = \nu$, and $E_5^{**}$ is generated by $\alpha, \beta, \chi_2, \zeta, \mu, 3\tau, \tau(\chi_2 + \alpha^2), \tau \beta \mu, 3\tau^2, \tau^2 \nu, \tau^2(\chi_2 + \alpha^2), \tau^2 \beta \mu$, and $\tau^3$. All of these are universal cycles except for $\tau^2 \nu$ and $\tau^2 \beta \nu$, which satisfy

$$d_5(\tau^2 \nu) = \zeta \alpha \quad d_5(\tau^2 \beta \mu) = \zeta(\beta^2 + \alpha^2 + \chi_2).$$

There are no other non-zero differentials, so $E_\infty^{**}$ is generated by $\alpha, \beta, \chi_2, \zeta, \mu, 3\tau, \tau(\chi_2 + \alpha^2), \tau \beta \mu, 3\tau^2, \tau^2(\chi_2 + \alpha^2), \tau^3$ and $\tau^2(\beta^2 - \alpha^2) \mu$.

Proof. The claims concerning $d_3$ and $E_5^{**}$ were proved in lemma 3.12. Of the generators required for $E_5^{**}$, all are universal cycles with the possible exception of $\tau^2 \nu$, $\tau^2(\chi_2 + \alpha^2)$ and $\tau^2 \beta \mu$. It was shown in lemma 3.12 that $d_5(\tau^2 \nu) = \zeta \alpha$. For the remaining cases we again (as in lemma 3.12) compare the spectral sequence $E_\infty^{**}$ with the spectral sequence $\overline{E}_\infty^{**}$ for $M$ (the abelian maximal subgroup of $\tilde{B}$) using the map induced by the following map of extensions.

$$
\begin{array}{ccc}
S^1 & \rightarrow & M \\
\downarrow \text{Id} & & \downarrow i \\
S^1 & \rightarrow & \tilde{B} \\
\end{array}
\rightarrow
\begin{array}{ccc}
& \rightarrow & \langle B, C \rangle \\
& \downarrow i & \downarrow i \\
& \rightarrow & P_2
\end{array}
$$

Define generators $\beta', \gamma'$ for $H^2(B\langle B, C \rangle; \mathbb{Z}) \cong \text{Hom}(\langle B, C \rangle, S^1)$ by

$$
\beta' : B \mapsto \omega \quad \gamma' : B \mapsto 1
$$

$$
C \mapsto 1 \quad C \mapsto \omega,
$$

where $\omega = \exp(2\pi i/3)$, so that $\beta' = \text{Res}(\beta)$. Also define a generator $\mu$ for $H^3(B\langle B, C \rangle; \mathbb{Z})$ by $\mu' = \text{Res}(\mu)$. Now we know that $\overline{E}_\infty^{**}$ collapses and that $\overline{E}_2^{**} \cong \mathbb{Z}[\tau', \beta', \gamma'] \otimes \Lambda[\mu']$.
To show that $d_5(\tau^2(\chi_2 + \alpha^2)) = 0$, we note that it lies in $E^{9,0}_5$, which is generated by $\beta^3 \mu$ and $\zeta \mu$, so maps injectively to $\bar{E}^{0,0}_5$ (recall that $\text{Res}(\zeta) = \gamma'^3 - \beta^2 \gamma'$). The result follows because $\bar{E}^{*,*}_n$ collapses. Similarly, we know that $d_5(\tau^2 \beta \mu)$ must be in the kernel of $\text{Res}^{*,*}_n$. The equation $(\tau^2 \nu) \beta^2 = (\tau^2 \beta \mu) \alpha$ implies that
\[
\alpha d_5(\tau^2 \beta \mu) = \zeta \alpha \beta^2,
\]
so $d_5(\tau^2 \beta \mu) = \zeta \beta^2 + Q$ for some $Q$ in the kernel of multiplication by $\alpha$ as a map from $E^{10,0}_5$ to $E^{12,0}_5$. This kernel is spanned by $\zeta(\alpha^2 + \chi_2)$ and $\beta^5 - \alpha^2 \beta^3$, which map under $\text{Res}^{*,*}_n$ to $(\beta^2 \gamma' - \gamma'^3) \beta^2$ and $\beta^5$ respectively. It follows that $d_5(\tau^2 \beta \mu)$ is as claimed. The $E\infty$-page of this spectral sequence is depicted in figure 4.1 (on page 87), but with elements named by the elements of $H^*(\tilde{B}\tilde{B}; \mathbb{Z})$ that yield them, rather than as in this statement.

**Corollary 4.3.** In the spectral sequence of lemma 4·2, the subring of $E^{*,*}_\infty$ generated by $\tau^3$ and $\zeta$ is of the form $\mathbb{Z}[x, y]/(3y)$, where $x = \tau^3$ and $y = \zeta$. As a module for this subring $E^{*,*}_\infty$ is generated by $1, 3\tau, 3\tau^2, \alpha^{i+1}, \beta^{i+1}, \alpha \beta^{i+1}, \alpha^2 \beta^{i+1}, \chi_2, \beta^i \mu, \tau \beta^{i+1} \mu, \tau(\alpha^2 + \chi_2) \beta^i, \tau^2(\beta^2 - \alpha^2) \beta^i \mu$ and $\tau^2(\alpha^2 + \chi_2) \beta^i$, where $i \geq 0$. Each of these module generators has order three, except that $1, 3\tau$ and $3\tau^2$ have infinite order. Multiplication by $\tau^3$ is injective on $E^{*,*}_\infty$, and the kernel of multiplication by $\zeta$ is generated (as a $\mathbb{Z}[\tau^3, \zeta]$-module) by $3\tau, 3\tau^2, \alpha^{i+1}, \alpha \beta^{i+1}, \alpha^2 \beta^{i+1}$ and $\beta^2 + \chi_2$.

**Proof.** Almost immediate from lemma 4·2.

An immediate consequence of lemma 4·2 is that $H^*(\tilde{B}\tilde{B}; \mathbb{Z})$ may be generated by nine elements of even degree and three of odd degree, since $E^{*,*}_\infty$ can be. It has, however, already been shown in corollary 3·10 that six elements suffice to generate $H^{ev}(\tilde{B}\tilde{B})$. We shall now define six elements of even degree and one of odd degree which (as we shall demonstrate in lemma 4·7) generate $H^*(\tilde{B}\tilde{B}; \mathbb{Z})$. To determine the relations that they satisfy we shall have to calculate their restrictions to $M$, which are listed as lemma 4·6. Information given by restriction to proper subgroups does not quite suffice to show that the elements generate $H^*(\tilde{B}\tilde{B}; \mathbb{Z})$, essentially because there are elements of degree four that restrict trivially to
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every proper subgroup. We are forced, therefore, to prove separately in lemma 4.5 that the elements of even degree defined below generate $H^\text{ev}(B\tilde{B})$, using the same method as in corollary 3.10.

**Definition 4.4.** Define $\tau', \beta', \gamma' \in H^2(BM; \mathbb{Z}) \cong \text{Hom}(M, S^1)$ by

\[
\begin{align*}
\tau' & : zB^iC^j \mapsto z \\
\beta' & : zB^iC^j \mapsto \omega^i \\
\gamma' & : zB^iC^j \mapsto \omega^j,
\end{align*}
\]

where $\omega = \exp(2\pi i/3)$. Now define elements $\alpha, \beta, \gamma, \delta_1, \delta_2, \delta_3, \mu$ of $H^*(B\tilde{B}; \mathbb{Z})$ as follows. Let $\alpha, \beta$ and $\mu$ be the images under inflation from $\tilde{B}/S^1 = P_2$ of the elements with the same names. Let $\rho$ be a 3-dimensional irreducible representation of $\tilde{B}$, whose restriction to $M$ contains the 1-dimensional representation with first Chern class $\tau'$, and define

\[
\delta_3 = c_3(\rho).
\]

Define also

\[
\begin{align*}
\delta_1 & = \text{Cor}_{BM}(\tau') \\
\delta_2 & = \text{Cor}_{BM}(\tau'^2) \\
\gamma & = \text{Cor}_{BM}(\tau'^2(\beta' - \gamma')) = \delta_2 \beta - \text{Cor}_{BM}(\tau'^2 \gamma').
\end{align*}
\]

**Lemma 4.5.** The elements $\alpha, \beta, \gamma, \delta_1, \delta_2$ and $\delta_3$ defined in 4.4 generate $H^\text{ev}(B\tilde{B})$.

**Proof.** If we consider $\beta$ as a homomorphism from $\tilde{B}$ to $S^1$, it has image $C_3$ and kernel $\tilde{P}$, with $C_3$ acting on $\tilde{P}$ via outer automorphisms. This is, modulo choice of generators, the extension considered in corollary 3.10. Recall that in corollary 3.10 it was shown (for $p = 3$) that Chern classes generate $H^\text{ev}(B\tilde{B})$. The method involved checking that their restrictions to $\tilde{P}$ generate $H^*(B\tilde{P}; \mathbb{Z})^{C_3}$. By a similar argument we may show that the elements of the statement generate $H^\text{ev}(B\tilde{B})$ by showing that the restrictions of $\alpha, \gamma,$
δ₁, δ₂ and δ₃ to P̃ (that is, to Ker(β)) generate $H^*(B\tilde{P};\mathbb{Z})^{C₂}$, which was described in lemma 3·4. Some caution is required since the notation used in lemma 3·4 for elements of $H^*(B\tilde{P};\mathbb{Z})$ clashes with our current notation (which was designed to harmonise with our notation for elements of $H^*(B\tilde{B}/S^1;\mathbb{Z})$ rather than with the cohomology of any subgroup).

We claim that the restrictions are as described below, where the elements of $H^*(B\tilde{B};\mathbb{Z})$ are as defined in 4·4, and their images are named as in lemma 3·4.

$$
\text{Res}(\alpha) = \alpha \quad \text{Res}(\delta_1) = \chi_1 \quad \text{Res}(\delta_2) = \chi_2 + \alpha^2$$

$$
\text{Res}(\gamma) = \beta^3 - \alpha^2 \beta \quad \text{Res}(\delta_3) = \zeta.
$$

For the elements of degree two this may be checked by considering them as homomorphisms from $\tilde{B}$ to $S^1$. For the elements defined as corestrictions from $M$ to $\tilde{B}$ we use the formula

$$
\text{Res}_{\tilde{P}}(\mu) \text{Cor}_{M \cap \tilde{P}} = \text{Cor}_{M \cap \tilde{P}}\text{Res}_{M \cap \tilde{P}}(\tau^2 \gamma'),
$$

which holds because $\tilde{P} M = \tilde{B}$. For example,

$$
\text{Res}_{\tilde{P}}(\delta_2 \beta - \text{Cor}_{M}(\tau^2 \gamma')) = 0 - \text{Cor}_{M \cap \tilde{P}}(\tau^2 \gamma')
$$

$$
= -\text{Cor}_{M \cap \tilde{P}}(\tau^2 \beta)
$$

$$
= -(\chi_2 + \alpha^2) \beta
$$

$$
= \beta^3 - \alpha^2 \beta.
$$

For δ₃, we note that the representation ρ restricts to P̃ as the (unique) irreducible representation of degree three that restricts to S¹ as three copies of the identity, so $c_3(\rho)$ must restrict to $\zeta$. It is clear now from lemma 3·4 that the restrictions of $\alpha$, $\gamma$, $\delta_1$, $\delta_2$ and $\delta_3$ generate $H^*(B\tilde{P};\mathbb{Z})^{C₂}$, and the result follows.

**Lemma 4·6.** The restrictions to the subgroup $M$ of $\tilde{B}$ of the elements defined in 4·4 are as follows, where the equation for Res($\mu$) is used to define $\mu'$, a generator for $H^3(BM;\mathbb{Z}) \cong \mathbb{F}_3$.

$$
\text{Res}(\alpha) = 0 \quad \text{Res}(\beta) = \beta' \quad \text{Res}(\mu) = \mu' \quad \text{Res}(\delta_1) = 3\tau' + \beta' \quad \text{Res}(\gamma) = \gamma'^3 - \beta'^2 \gamma'
$$

$$
\text{Res}(\delta_2) = 3\tau'^2 - \tau' \beta' + \beta'^2 - \gamma' \beta' - \gamma'^2 \quad \text{Res}(\delta_3) = \tau'^3 + \tau'^2 \beta' - \tau' \gamma' \beta' - \tau' \gamma'^2
$$
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Proof. The claims for $\alpha$, $\beta$ and $\mu$ are trivial. For $\delta_1$, $\delta_2$ and $\gamma$, we use the restriction-corestriction formula, which in this case tells us that $\text{Res}_M \text{Cor}_M$ is equal to the sum of conjugation by the distinct powers of $A$. By considering them as homomorphisms from $\tilde{B}$ to $S^1$, it is easily verified that conjugation by $A$ fixes $\beta$ and sends $\tau'$ to $\tau' - \gamma'$ and $\gamma'$ to $\gamma' - \beta'$. For example,

$$\text{Res}_M(\delta_1) = \sum_{i=0}^{2} c_{A^i}^*(\tau') = \tau' + (\tau' - \gamma') + (\tau' + \gamma' + \beta') = 3\tau' + \beta'.$$

For $\delta_3$, which we recall was defined as $c_3(\rho)$, we note that since the restriction to $M$ of $\rho$ contains a summand with Chern class $\tau'$ and is invariant under conjugation by $A$ it must also contain summands with Chern class $\tau' - \gamma'$ and $\tau' + \gamma' + \beta'$. Thus

$$\text{Res}(\delta_3) = \tau'(\tau' - \gamma')(\tau' + \gamma' + \beta').$$

Lemma 4.7. The seven elements defined in 4.4 generate $H^*(B\tilde{B};\mathbb{Z})$. Their relation to the generators for the $E_\infty$-page of the spectral sequence of lemma 4.2 is as follows.

1. The generators $\alpha$, $\beta$ and $\mu$ yield the spectral sequence elements of the same name.
2. The generator $\delta_1$ (resp. $\delta_2$, $\delta_3$) yields the generator $3\tau$ (resp. $3\tau^2$, $\tau^3$) for $E^{0,2}_\infty$ (resp. for $E^{0,4}_\infty$, $E^{0,6}_\infty$).
3. The subgroup of $H^4(B\tilde{B};\mathbb{Z})$ generated by $\alpha^2$, $\alpha \beta$, $\beta^2$ and $\delta_1 \beta$ is mapped bijectively to $E^{4,0}_\infty$ (which is generated by $\alpha^2$, $\alpha \beta$, $\beta^2$ and $\chi_2$).
4. The subgroup of $H^6(B\tilde{B};\mathbb{Z})$ generated by $\alpha^3$, $\alpha^2 \beta$, $\alpha \beta^2$, $\beta^3$ and $\gamma$ is mapped bijectively to $E^{6,0}_\infty$ (which is generated by $\alpha^3$, $\alpha^2 \beta$, $\alpha \beta^2$, $\beta^3$ and $\zeta$).
5. $\delta_2 \beta$ (resp. $\delta_2 \mu$) yields the generator $\tau(\alpha^2 + \chi_2)$ for $E^{4,2}_\infty$ (resp. $-\tau \beta \mu$ for $E^{5,2}_\infty$).
6. $\delta_2^2 \delta_1$ (resp. $\delta_2^2 \mu$) yields the generator $-\tau^2(\alpha^2 + \chi_2)$ for $E^{4,4}_\infty$ (resp. $\tau^2(\beta^2 - \alpha^2)\mu$ for $E^{7,4}_\infty$).

The $E_\infty$-page of this spectral sequence is depicted in figure 4.1.

Proof. To show that the elements generate $H^*(B\tilde{B};\mathbb{Z})$ it will suffice to show that they yield a set of generators for the $E_\infty$-page of the spectral sequence of lemma 4.2. Thus it suffices
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to prove the numbered assertions made above. Many of these assertions follow easily from lemma 4.6. For example the restriction to $M$ of $\delta_2$ is $3\tau^2$ modulo terms of lower degree in $\tau'$, the restriction from $E^{0,4}_\infty$ to $E^{0,4}_\infty$ is injective, and hence $\delta_2$ must yield the generator $3\tau^2$. Similarly $\gamma$ yields an element of $E^{6,0}_\infty$ which restricts to $E^{6,0}_\infty$ as $\gamma^3 - \beta^2\gamma'$, while the image of the subgroup of $E^{6,0}_\infty$ generated by $\alpha^3$, $\alpha^2\beta$, $\alpha\beta^2$ and $\beta^3$ in $E^{6,0}_\infty$ is generated by $\beta^3$, so $\gamma$ is independent of this subgroup, and so the five elements of item 4 above yield generators for $E^{6,0}_\infty$. All the assertions except that of item 3 can be proved similarly, and we will consider them proved.

\[
\begin{array}{cccccc}
\delta_3 & \quad & \quad & \quad & \quad & \quad \\
\delta_2 & \quad & \quad & \quad & \quad & \quad \\
\delta_1 & \quad & \quad & \quad & \quad & \quad \\
1 & \quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

4.1. The $E_{\infty}$-page of the spectral sequence of lemmata 4.2 and 4.7.

It is clear that $\delta_1\beta$ yields an element of $E^{4,0}_{\infty}$. To prove that it does not lie in the subgroup of $E^{4,0}_{\infty}$ generated by $\alpha^2$, $\alpha\beta$ and $\beta^2$ is more difficult. From the spectral sequence it can be seen that $H^4(B\tilde{B}; \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus C_3^4$. The products of the elements of definition 4.4 that have degree four are $\delta_2$, $\delta_1^4$, $\delta_1\beta$, $\delta_1\alpha$, $\alpha^2$, $\alpha\beta$ and $\beta^2$. In the group

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generated by these elements the subgroup of elements of order at most three is generated by $\delta_1^2 - 3\delta_2$, $\delta_1\beta$, $\delta_1\alpha$, $\alpha^2$, $\alpha\beta$ and $\beta^2$. We now determine some relations between these elements. By Fröbenius reciprocity

$$\delta_1\alpha = \text{Cor}(\tau')\alpha = \text{Cor}(\tau'\text{Res}(\alpha)) = 0.$$  

Similarly,

$$\delta_1^2 = \text{Cor}(\tau')\text{Cor}(\tau') = \text{Cor}(\tau'\text{ResCor}(\tau'))$$

$$= \text{Cor}(3\tau'^2 + \tau'\beta')$$

$$= 3\text{Cor}(\tau'^2) + \text{Cor}(\tau')\beta'$$

$$= 3\delta_2 + \delta_1\beta.$$  

It follows that $\delta_1\alpha$ and $\delta_1^2 - 3\delta_2$ are dependent on $\delta_1\beta$, $\alpha^2$, $\alpha\beta$, $\beta^2$. Lemma 4-5 tells us that $\alpha$, $\beta$, $\gamma$, $\delta_1$, $\delta_2$ and $\delta_3$ generate $H^{ev}(B\tilde{B})$, so it must be that $\delta_1\beta$ is not contained in the subgroup generated by $\alpha^2$, $\alpha\beta$ and $\beta^2$. □

Theorem 4-8. $H^*(B\tilde{B};\mathbb{Z})$ is generated by the elements of definition 4-4 subject to the following relations.

$$3\alpha = 3\beta = 0 \quad 3\mu = 0 \quad 3\gamma = 0$$

$$\delta_1^2 = \delta_1\beta + 3\delta_2 \quad \gamma\delta_1 = \gamma\beta \quad \alpha\mu = 0$$

$$\delta_1\mu = \beta\mu \quad \delta_1\alpha = 0 \quad \gamma\alpha = 0$$

$$\delta_1\delta_2 = \delta_2\beta + 9\delta_3 \quad \delta_1\beta^2 = \beta^3 - \alpha^2\beta$$

$$\delta_2\alpha = 0 \quad \alpha^3\beta = \alpha\beta^3$$

$$\delta_2^3 - 27\delta_3^2 = -\delta_3(\beta^3 - \alpha^2\beta) + \delta_2^2\beta^2 + \delta_2\beta^4 - \gamma^2 - \beta^6 + \alpha^2\beta^4$$

Proof. If we assume that these relations hold, it follows easily from corollary 4-3 and lemma 4-7 that they are all the relations that we require. (Note in particular that the relations given imply that $3(\delta_2^2 - 3\delta_3\delta_1) = 0$.) Thus it remains to demonstrate that these relations hold.

That the elements $\alpha$, $\beta$, $\gamma$ and $\mu$ have order three is immediate from their definitions. The relations claimed in degree 4 were checked during the proof of lemma 4-7. The relations
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$\alpha \mu = 0$ and $\delta_1 \mu = \beta \mu$ hold because the restrictions of these equations to $M$ are valid, and the restriction is injective from $E_{\infty}^{5,0}$ to $\overline{E}_{\infty}^{5,0}$. The relations $\delta_2 \alpha = 0$, $\gamma \alpha = 0$, and $\gamma \delta_1 = \gamma \beta$ follow readily by a Fröbenius reciprocity argument. The relation for $\delta_1 \delta_2$ may also be deduced by using Fröbenius reciprocity as follows.

$$\delta_2 \delta_1 = \text{Cor}(\tau'^2) \delta_1 = \text{Cor}(\tau'^2 (3\tau' + \beta'))$$
$$= \text{Cor}(3\tau'^3) + \text{Cor}(\tau'^2) \beta$$
$$= \text{Cor}(\text{Res}(3\delta_3)) + \delta_2 \beta$$
$$= 9\delta_3 + \delta_2 \beta.$$ 

The relation $\alpha^3 \beta = \alpha \beta^3$ holds because the corresponding relation holds in the spectral sequence. We know (because $\delta_2 \beta$ yields an element of $E_{\infty}^{4,0}$) that $\delta_1 \beta^2$ is in the span of $\alpha^3$, $\alpha^2 \beta$, $\alpha \beta^2$, $\beta^3$, $\gamma$, that its restriction to $M$ is $\beta^3$, and that it is annihilated by $\alpha$. Since the kernel of the restriction from $E_{\infty}^{6,0}$ is spanned by $\alpha^3$, $\alpha^2 \beta$ and $\alpha \beta^2$ whereas the kernel of multiplication by $\alpha$ is spanned by $\gamma$ and $\beta^3 - \alpha^2 \beta$, these facts suffice to determine $\delta_1 \beta^2$.

It is clear from corollary 4.3 and lemma 4.7 that all we are left to do is obtain an expression for $\delta_2^3$. It may be checked that the relation given is correct when restricted to $M$, and that when it is multiplied by $\alpha$ the valid equation $0 = 0$ is obtained. The result follows, since it may be checked that in degree 12 the kernel of restriction to $M$ is generated by $\delta_3 \alpha^3$, $\delta_3 \alpha^2 \beta$, $\delta_3 \alpha \beta^2$, $\alpha^6$, $\alpha^2 \beta^4$ and $\alpha \beta^5$, and that multiplication by $\alpha$ is injective on this subgroup.

\textbf{Lemma 4.9.} For any odd prime $p$, define $\delta', \beta \in H^2(B\tilde{B}; \mathbb{Z}) \cong \text{Hom}(\tilde{B}, S^1)$, where $\tilde{B}$ is presented as in the proof of lemma 3.12, by

$$\delta' : A^i B^k C^l z \mapsto z^p$$
$$\beta : A^i B^k C^l z \mapsto \exp(2\pi ik/p).$$

(In the case when $p = 3$, $\beta$ is the element $\beta$ defined in 4.4 and $\delta'$ is $\delta_1 - \beta$.) then for any $\varepsilon$, the subgroup of $\tilde{B}$ with extension class $p^{n-4} \delta' - \varepsilon \beta$ is isomorphic to $B(n, \varepsilon)$. 

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The assertions relating $\delta'$ and $\beta$ to $\delta_1$ and $\beta$ in the case $p = 3$ are easily proved. If we write $\varphi = p^{n-4} \delta' - \varepsilon \beta$, then as a homomorphism from $\widetilde{B}$ to $S^1$, $\varphi$ has the following form.

$$\varphi : A^j B^k C^l z \mapsto z^{p^{n-3} \omega^{-\varepsilon j}},$$

where $\omega = \exp(2\pi i / p)$. The kernel of $\varphi$ (which we recall is one of the equivalent interpretations of the group with characteristic class $\varphi$) is the subgroup generated by $A$, $C$, and $B' = B\eta^\varepsilon$, where $\eta = \exp(2\pi i / p^{n-2})$. It is easily checked that the map from this group to $B(n, \varepsilon)$ (as presented in the statement of theorem 3.1) sending $A$ to $A$, $C$ to $B^{-1}$ and $B'$ to $C$ is an isomorphism.

Before calculating $H^*(BB(n, \varepsilon); \mathbb{Z})$ we require one more lemma which will determine a generator for $H^5(BB(n, \varepsilon); \mathbb{Z})$.

**Lemma 4.10.** Let $A$ be the abelian maximal subgroup of $B(n, \varepsilon)$, so that $A \cong C_2 \oplus C_3$ and $A = B(n, \varepsilon) \cap M$. Also take generators $\tau$, $\gamma$ and $\mu$ for $H^*(BA; \mathbb{Z})$, where $\tau = \text{Res}_A^M(\tau')$, $\gamma = \text{Res}_A^M(\gamma')$, and $\mu = \text{Res}_A^M(\mu')$. Note that $\tau$ and $\mu$ have order 3 while $\gamma$ has order $3^{n-2}$. Then there is an element of $H^5(BB(n, \varepsilon); \mathbb{Z})$ that restricts to $A$ as $\gamma \mu$, and this element is in the image of the corestriction from a $P(n-1)$ subgroup to $B(n, \varepsilon)$.

**Proof.** First we claim that if $H$ is the subgroup of $A$ isomorphic to $C_{2^{n-3}} \oplus C_3$, and $\mu$ is a generator for $H^3(BH; \mathbb{Z})$, then $\text{Cor}_H^A(\mu) = \nu$. To show this, note that if $y$ is a generator for $H^1(BC_{3^{n-3}}; \mathbb{F}_3)$ (which is isomorphic to $\text{Hom}(C_{3^{n-3}}, C_3)$), and we consider $C_{3^{n-3}}$ as a subgroup of $C_{2^{n-2}}$, then the classical description of the transfer (as a map from $G/G'$ to $H/H'$ where $H \subset G$) allows us to check that $\text{Cor}(y)$ is a generator for $H^1(BC_{3^{n-2}}; \mathbb{F}_3)$. Now if we define $y \in H^1(BH; \mathbb{F}_3)$ to be the inflation of a generator for $H^1(BH/(\mathbb{Z}C_3); \mathbb{F}_3)$, and $y'$ the inflation of a generator for $H^1(BH/(C_{2^{n-2}} \oplus C_3); \mathbb{F}_3)$ then Fröbenius reciprocity implies that $\text{Cor}_H^A(yy') = \text{Cor}_H^A(y)y'$ is an element of $H^2(BA; \mathbb{F}_3)$ not in the image of $H^2(BA; \mathbb{Z})$. The claim now follows since corestriction commutes with the Bockstein.

Now note that $H = A \cap P(n-1)$, and that from the above paragraph we see that we
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may take generators \( \tilde{\tau}, \tilde{\gamma} \) and \( \tilde{\mu} \) for \( H^*(BH; \mathbb{Z}) \), where \( \tilde{\tau} = \text{Res}(\tau) \) and \( \tilde{\gamma} = \text{Res}(\gamma) \) while \( \overline{\tau} = \text{Cor}(\tilde{\mu}) \). If we use the notation of theorem 2·3 and lemma 3·4 for elements of the group \( H^*(BP(n-1); \mathbb{Z}) \), then after a suitable choice of generators the action of \( B(n, \varepsilon)/P(n-1) \cong C_3 \) on \( H^*(BP(n-1); \mathbb{Z}) \) may be taken to fix \( \alpha, \chi_1 \) and \( \nu \), and send \( \beta \) to \( \beta + \alpha \) and \( \mu \) to \( \mu + \nu \). The kernel of the restriction from \( P(n-1) \) to \( H \) in degree two is generated by \( \alpha \) (this may be checked by considering \( H^2(BP(n-1); \mathbb{Z}) \cong \text{Hom}(P(n-1), S^1) \)), which implies that the kernel of the restriction from \( P(n-1) \) to \( H \) in degree three is generated by \( \nu \). Thus we see that

\[
\text{Res}_{A} B^{(n, \varepsilon)} \text{Cor}_{P(n-1)} (\beta \mu) = \text{Cor}^A \text{Res}_H P(n-1) (\beta \mu)
\]

\[
= \pm \text{Cor}^A (\tilde{\gamma} \tilde{\mu})
\]

\[
= \pm \tilde{\gamma} \tilde{\mu}
\]

Theorem 4·11. For \( n \geq 5 \) (and \( p = 3 \)), \( H^*(BB(n, \varepsilon); \mathbb{Z}) \) is generated by elements \( \alpha, \delta_1, \mu, \delta_2, \nu, \delta_3 \) and \( \gamma \) of degrees 2, 2, 3, 4, 5, 6 and 6 respectively. The generators other than \( \nu \) may be taken to be the restrictions from \( \tilde{B} \) of the generators for \( H^*(B\tilde{B}; \mathbb{Z}) \) of the same name. They are subject to the following relations.

\[
\begin{align*}
3\alpha &= 0 & 3 \gamma &= 0 & 3 \mu &= 0 & 3 \nu &= 0 \\
3n^{-3} \delta_1 &= 0 & 3^{n-2} \delta_2 &= 0 & 3^{n-1} \delta_3 &= 0 \\
3\delta_2 &= \delta_1^2 (1 - 3^{n-4} \varepsilon) & \delta_1 \alpha &= 0 & \alpha \mu &= 0 \\
9\delta_3 &= \delta_1 \delta_2 (1 - 3^{n-4} \varepsilon) & \delta_1 \mu &= 0 & \alpha \delta_2 &= 0 \\
\delta_2^3 &= 27 \delta_3^2 - \gamma^2 & \gamma \delta_1 &= 0 & \gamma \alpha &= 0 \\
\delta_1 \nu &= 0 & \mu \nu &= 0 & \delta_2 \nu &= -\gamma \mu & \gamma \nu &= \delta_2^3 \mu
\end{align*}
\]

Proof. We consider the spectral sequence for \( BB(n, \varepsilon) \) as an \( S^1 \)-bundle over \( B\tilde{B} \). It follows from lemma 4·9 that the differential in this spectral sequence sends \( \xi \) (a generator for \( E^0_{2,1} \)) to \( \varepsilon \beta - 3^{-4} \delta_1 \) (because \( n \geq 5 \) we have that \( 3^{n-4} \delta_1 = 3^{n-4} \delta' \)). The relations
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given between the generators that are in the image of the restriction from \( \tilde{B} \) are the relations of theorem 4·8 after the substitution of \( \beta \) by \( 3^{n-4} \varepsilon \delta_1 \). In the spectral sequence \( E^\ast_{\infty,1} \) corresponds to the kernel of multiplication by \( \varepsilon \beta - 3^{n-4} \delta_1 \), which is the ideal of \( H^\ast(B\tilde{B};\mathbb{Z}) \) generated by \( \delta_1 \beta - \beta^2 + \alpha^2 \). This ideal is a free module of rank one for \( \mathbb{F}_3[\delta_3,\alpha] \), and the product of \( \delta_1 \beta - \beta^2 + \alpha^2 \) with any of \( \delta_1, \delta_2, \gamma \) or \( \mu \) is zero. It follows that we only need one new generator \( \nu \) in degree five, and that the only extra relations that we need are expressions for \( \delta_1 \nu, \delta_2 \nu, \gamma \nu \) and \( \mu \nu \) in terms of the generators in the image of restriction from \( \tilde{B} \). We may take \( \nu \) as in lemma 4·10, so that its restriction to \( A \) (the subgroup of \( B(n,\varepsilon) \) of lemma 4·10) is \( \gamma \bar{\mu} \). The restrictions to \( A \) of the other generators may be determined easily from lemma 4·6 (which describes \( \text{Res}^B_{\tilde{B}}(M) \)), if we note that \( \text{Res}^M_A(\tau') = \tau, \text{Res}^M_A(\mu') = \mu, \text{Res}^M_A(\gamma') = \gamma \) and \( \text{Res}^M_A(\beta') = 3^{n-4} \varepsilon \tau \). Since \( \text{Res}^B_M \) is injective on \( H^\text{od}(B\tilde{B}) \) (see lemma 3·12) it follows that \( \text{Res}^B_{A(n,\varepsilon)} \) is injective on the image of \( H^\text{od}(B\tilde{B}) \). To check the relations given for \( \delta_1 \nu, \delta_2 \nu \) and \( \gamma \nu \) it therefore suffices to check that they yield valid relations when restricted to \( A \). The relation \( \mu \nu = 0 \) follows by Fröbenius reciprocity, because \( \nu \) may be taken to be a corestriction from \( P(n-1) \), and the product of any element of \( H^5(BP(n-1);\mathbb{Z}) \) and any element of \( H^3(BP(n-1);\mathbb{Z}) \) is zero.

\[ \begin{align*}
3\delta_2^2 &= \delta_2 \delta_1^2 (1 - 3^{n-4} \varepsilon) = 9\delta_3 \delta_1 
\end{align*} \]

imply that \( \delta_2^2 \) has order \( 3^{n-4} \).
For $p \geq 5$ it is much harder to calculate $H^*(B\tilde{B}; \mathbb{Z})$. Nevertheless we are able to prove the following theorem comparing the additive structure of $H^*(BB(n, \varepsilon); \mathbb{Z})$ for arbitrary $p$.

**Theorem 4.13.** For all $m \geq 0$, for all odd $p$, and all $n \geq 6$, the group $H^m(BB(n, \varepsilon); \mathbb{Z})$ is independent of $\varepsilon$.

**Proof.** First we consider the groups $H^m(B\tilde{B}; \mathbb{Z})$. The $E_2$-page of the spectral sequence for $\tilde{B}$ as an extension of $S^1$ by $P_2$ has the property that $E^{0,2j}_2$ is isomorphic to $\mathbb{Z}$, whereas $E^{i,j}_2$ is a finite $p$-group for $(i, j)$ not of the form $(0, 2j')$. It follows that $E^{*,*}_\infty$ also has this property, and hence that $H^{2m}(B\tilde{B}; \mathbb{Z})$ is an extension of a finite $p$-group by $\mathbb{Z}$ (which must of course be split) and $H^{2m+1}(B\tilde{B}; \mathbb{Z})$ is a finite $p$-group. We know that $\tilde{B}$ has a subgroup isomorphic to $\tilde{P}$ of index $p$, and that the torsion in $H^*(\tilde{B}\tilde{P}; \mathbb{Z})$ in degree $2m$ (resp. in degree $2m+1$) has exponent $p$ (resp. is trivial). Since $\text{Cor}_{\tilde{P}}^{\tilde{B}} \text{Res}_{\tilde{P}}^{\tilde{B}}$ is multiplication by $p$ it follows that in $H^{2m}(B\tilde{B}; \mathbb{Z})$ the torsion has exponent at most $p^2$, and that $H^{2m+1}(B\tilde{B}; \mathbb{Z})$ has exponent at most $p$ (the odd degree case could also be deduced from lemma 3.12).

Now fix for each $m$ an isomorphism

$$H^{2m}(B\tilde{B}; \mathbb{Z}) \cong T^{2m} \oplus \mathbb{Z},$$

where $T^{2m}$ is the torsion subgroup of $H^{2m}(B\tilde{B}; \mathbb{Z})$.

Now we consider the spectral sequence for $BB(n, \varepsilon)$ as an $S^1$-bundle over $B\tilde{B}$. With notation as in lemma 4.9, we see that the differential $d_2$ from $E^{*,1}_2$ to $E^{*,0}_2$ corresponds to multiplication by $p^{n-4} \delta' - \varepsilon \beta$ from $H^*(B\tilde{B}; \mathbb{Z})$ to itself. During theorem 3.13 it was shown that $H^{ev}(B\tilde{B})$ maps on to $H^{ev}(BB(n, \varepsilon))$, or equivalently that $E^{\text{odd},1}_\infty$ is trivial. If we let $\varphi$ stand for the map

$$\varphi : T^{2m} \oplus \mathbb{Z} \longrightarrow T^{2m+2} \oplus \mathbb{Z}$$

corresponding to multiplication by $p^{n-4} \delta' - \varepsilon \beta$ from $H^{2m}(B\tilde{B}; \mathbb{Z})$ to $H^{2m+2}(B\tilde{B}; \mathbb{Z})$, then

$$E^{2m+2,0}_\infty \cong (T^{2m+2} \oplus \mathbb{Z})/\text{Im}(\varphi) \quad E^{2m,1}_\infty \cong \text{Ker}(\varphi)$$

$$E^{2m+3,0}_\infty \cong H^{2m+3}(B\tilde{B}; \mathbb{Z})/(p^{n-4} \delta' - \varepsilon \beta)H^{2m+1}(B\tilde{B}; \mathbb{Z}).$$

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Since $H^{\text{od}}(B\tilde{B})$ has exponent $p$ and $n > 5$, multiplication by $p^{n-4} \delta'$ annihilates the group $H^{2m+1}(B\tilde{B}; \mathbb{Z})$. Also $-\varepsilon$ is a unit modulo $p$, so

$$E^{2m+3,0}_\infty \cong H^{2m+3}(B\tilde{B}; \mathbb{Z})/\beta H^{2m+1}(B\tilde{B}; \mathbb{Z}).$$

Writing elements of $H^2(B\tilde{B}; \mathbb{Z})$ as ‘row vectors’ via the isomorphism with $T^m \oplus \mathbb{Z}$, it may be checked that the map $\varphi$ has matrix

$$\varphi = \begin{pmatrix} -\varepsilon \beta & 0 \\ p^{n-4} \delta' & p^l \end{pmatrix} : T^m \oplus \mathbb{Z} \rightarrow T^{m+2} \oplus \mathbb{Z},$$

where $l$ is either $n-4$, $n-3$, or $n-2$, but depends only on $m$, not on $\varepsilon$. In any case, if $p^{n-4}$ annihilates $T^m$, which certainly happens for $n \geq 6$, then $\text{Ker}(\varphi)$ and $T^{m+2} \oplus \mathbb{Z}/\text{Im}(\varphi)$ are independent of $\varepsilon$.

To complete the proof, it will suffice to show that $H^{2m+3}(BB(n, \varepsilon); \mathbb{Z})$ is isomorphic to $E^{2m+2,1}_\infty \oplus E^{2m+3,0}_\infty$ for any choice of $\varepsilon$. The group $E^{2m+3,0}_\infty$ is a subgroup of $H^{2m+3}(BB(n, \varepsilon); \mathbb{Z})$, and it follows from lemma 3.12 that this subgroup maps injectively to $H^{2m+3}(BA; \mathbb{Z})$, where $A$ is the abelian maximal subgroup of $B(n, \varepsilon)$. Since $A$ is isomorphic to $C_{p^{n-2}} \oplus C_p$, the exponent of $H^{2m+3}(BA; \mathbb{Z})$ is $p$, and hence if $\xi$ is any element of $H^{2m+3}(BB(n, \varepsilon); \mathbb{Z})$ such that $p\xi$ is an element of $E^{2m+3,0}_\infty$, then $p\xi = 0$. ■
5. The Integral Cohomology of the Held Group.

In this section we shall examine integral cohomology of some finite groups by considering the restriction maps to the Sylow subgroups, and applying the results of section 2. Groups that we shall consider include extensions of \((C_p)^2\) by various subgroups of \(GL_2(p)\), and the Held group, a sporadic simple group of order \(2^{10} . 3^3 . 5^2 . 7^3 . 17\) [He], [Co]. For the Held group we obtain explicitly the \(p\)-torsion subring of the integral cohomology for all odd primes \(p\). C. B. Thomas has suggested [Th4] that the Atiyah-Hirzebruch-Rector spectral sequence [Re] might be a useful tool in studying the modular representation rings of groups. Since the \(E_2\) page of this spectral sequence for the \(p\)-modular ring consists of the cohomology of the group with coefficients of exponent prime to \(p\) our results for the Held group may have consequences concerning its 2-modular representation ring.

Methods.

If \(M\) is any \(p\)-local module for a finite group \(G\) (that is a module for which multiplication by any prime other than \(p\) is an isomorphism) then the restriction map from the cohomology of \(G\) with coefficients in \(M\) to that of its Sylow \(p\)-subgroup \(G_p\) is split injective, with splitting map a multiple of the corestriction. There is a characterisation of the image of this restriction due to Cartan-Eilenberg [CE] as the ‘stable elements’ of \(H^*(BG_p; M)^{N(G_p)}\), that is the elements \(\xi\) satisfying

\[
\text{Res}^{gHg^{-1}}_{H \cap gHg^{-1}} c_g^*(\xi) = \text{Res}^H_{H \cap gHg^{-1}} (\xi).
\]

Recent work of P. Webb [We] gives, for each of a wide variety of \(G\)-simplicial complexes, an alternative characterisation of the image of \(H^*(BG; M)\) in \(H^*(BG_p; M)\) as the kernel of a map from the direct sum of the cohomology of the vertex stabilisers to that of the edge stabilisers. If \(G\) never reverses an edge of the complex the maps involved are composed solely of restriction maps, so this characterisation can be simpler than the usual one. All the proofs given in this section use the stable element characterisation, but others were used in some of the author’s earlier proofs of these results. The author is indebted to
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J. C. Rickard for many discussions concerning Webb’s work.

If $G_p$ (the Sylow subgroup of $G$) is abelian and we consider only trivial coefficients then a theorem of Swan [Sw] tells us that the stable elements are exactly the fixed points of $H^*(BG_p)$ under the action of its normaliser. (In [Sw] this theorem is stated for arbitrary coefficients, but this is incorrect. The author is indebted to D. J. Green for pointing this out to him and providing a counterexample.) The only cases of this theorem that we shall need are implied by the following lemma.

**Lemma 5.1.** The corestriction from $H^i(BC_p;\mathbb{Z})$ to $H^i(BC_p \oplus C_p;\mathbb{Z})$ is trivial for $i > 0$.

*Proof.* $H^i(BC_p \oplus C_p;\mathbb{Z})$ has exponent $p$ for $i > 0$, and the restriction from $C_p \oplus C_p$ to $C_p$ is onto. ■

**Corollary 5.2.** If a Sylow $p$-subgroup $G_p$ of $G$ is elementary abelian of rank two then $H^*(BG;\mathbb{Z}) \cong H^*(BG_p;\mathbb{Z})^{N(G_p)}$. If $G_p$ is isomorphic to $P_2$ then the only non-trivial conditions for an element of $H^*(BG_p;\mathbb{Z})^{N(G_p)}$ to be stable arise from intersections $G_p \cap G_p^g$ of order $p^2$. ■

We require the following lemma which is due to Dickson [Di].

**Lemma 5.3.** Let $GL_2(p)$ act in the usual way on a vector space spanned by $x$ and $x'$. Then $\mathbb{F}_p[x, x']^{SL_2(p)}$ is a polynomial algebra on generators $a$ and $b$ where

$$a = x^p x' - x'^p x \quad b = x^{p(p-1)} + x^{(p-1)^2} x'^{p-1} + \ldots + x'^{(p-1)},$$

and $\mathbb{F}_p[x, x']^{GL_2(p)}$ is a polynomial algebra on $a^{p-1}$ and $b$.

*Proof.* Consider the polynomials as functions from $\mathbb{F}_p^2$ to $\mathbb{F}_p$. The action of $SL_2(p)$ on $\mathbb{F}_p^2$ is transitive on non-zero elements, so if $P$ is fixed it must represent either the trivial function, in which case it must divide by each of $x'$, $x + ix'$, and hence by $a$, or we may assume that it represents the function sending every non-zero point of $\mathbb{F}_p^2$ to 1. We shall describe such a polynomial as being identically one. We claim now that if $P$ is identically
one then the degree of $P$ is divisible by $p(p-1)$. First, setting $x=0$ and letting $x'$ vary we see that $P$ is of the form $xQ + x'^n$, and that $p-1$ divides $n$. Now

$$\sum \lambda_ix^ix'^{n-i} = P(x,x') = P(x,x+x') = \sum \lambda_ix^i(x+x')^{n-i},$$

where $\lambda_0 = 1$. Balancing coefficients of $xx'^{n-1}$ gives $\lambda_1 = \lambda_1 + n\lambda_0$, so $p$ must divide $n$.

$$b = (\prod_{i=0}^{p-1} (x + ix'))^{p-1} + x'^p(p-1) \quad (\ast)$$

Equation $(\ast)$ makes apparent the fact that $b$ is identically one, because the first term of the right hand side is one if $x'=0$ and zero otherwise. It is also apparent from $(\ast)$ that $b$ is fixed by $x' \mapsto x'$, $x \mapsto x + x'$, and there is a similar expression with $x$ and $x'$ interchanged that illustrates the fact that $b$ is fixed by $x' \mapsto x' + x$, $x \mapsto x$. These two elements generate $SL_2(p)$, so $b$ is invariant as claimed. Now given $Q$ an arbitrary homogeneous polynomial fixed by $SL_2(p)$, either $Q$ is identically zero and we may divide $Q$ by $a$, or $Q$ is identically one and of degree divisible by $p(p-1)$, in which case for some $n$, $Q - b^n$ is homogeneous of degree the same as $Q$ and identically zero and fixed by $SL_2(p)$.

The group $GL_2(p)$ is generated by $SL_2(p)$ and the map that fixes $x$ and sends $x'$ to $\lambda x'$, for $\lambda$ a generator of $F_p^\times$. Hence $F_p[x,x']^{GL_2(p)}$ is the fixed point subring of $F_p[a,b]$ under the map which fixes $b$ and sends $a$ to $\lambda a$.  

**Corollary 5.4.** If $H^\ast(BC_p \oplus C_p;\mathbb{Z}) = \mathbb{Z}[x,x'] \otimes \Lambda[w]$ and $GL_2(p)$ acts in the usual way on $C_p \oplus C_p$, then

$$H^\ast(BC_p \oplus C_p;\mathbb{Z})^{SL_2(p)} = \mathbb{Z}[a,b] \otimes \Lambda[w] \quad \text{and}$$

$$H^\ast(BC_p \oplus C_p;\mathbb{Z})^{GL_2(p)} = \mathbb{Z}[a^{p-1},b] \otimes \Lambda[a^{p-2}w].$$

**Proof.** If $\lambda \in GL_2(p)$, the action of $\lambda$ sends $w$ to $\det(\lambda)w$.  

Let $G$ be an extension of $(C_p)^2$ by $SL_2(p)$ or any subgroup of $GL_2(p)$ containing $SL_2(p)$. The metacyclic group $P_1$ has a $(C_p)^2$ subgroup, but this contains a characteristic
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subgroup of order \( p \) (the subgroup of \( p \)-th powers). It follows that the Sylow subgroups of \( G \) are isomorphic to \( P_2 \), and so \( G \) must be the split extension. The groups \((C_p)^2: SL_2(p)\) and \((C_p)^2: GL_2(p)\) seem to arise frequently in cohomology calculations, so we determine the \( p \)-part of their cohomology rings below.

**Theorem 5.5.** Let \( H \cong C_p \oplus C_p \), and let \( Q \) be a subgroup of \( GL_2(p) \) containing \( SL_2(p) \) and acting in the standard way on \( H \). Also let \( C \) be a Sylow subgroup of \( Q \), so \( C \cong C_p \), and \( H:C \cong P_2 \) is a Sylow subgroup of \( H:Q \). Then the image of \( H^*(B\cdot H:Q;\mathbb{Z}) \) in \( H^*(B\cdot H:C;\mathbb{Z}) \) is the subring

\[
\{ \xi \in H^*(B\cdot H:C;\mathbb{Z})^{N(H:C)} \mid \text{Res}_{H:C}^H(\xi) \in H^*(B\cdot H;\mathbb{Z})^Q \}.
\]

Note also that \( N_{H:Q}(H:C) = H:N_Q(C) \), that \( N_Q(C) \) is isomorphic to \( C_{p-1} \oplus \text{det}(Q) \), and \( N_Q(C)/C_Q(C) \) is isomorphic to \( C_{(p-1)/2} \).

**Proof.** The remarks at the end of the statement are standard group theory, and were included to indicate how the main result can be applied. By considering the restriction maps from \( H:Q \) to \( H:C \) and \( H \) it may be seen that the image is no bigger than claimed. Conversely, corollary 5.4 tells us that given an element \( \xi \) of \( H^*(B\cdot H:C;\mathbb{Z}) \) fixed by the action of the normaliser we need only check that for each \( g \notin N(H:C) \),

\[
\text{Cor}^H\cdot C_{g^\ast}^{g^{-1}}(\xi) = |C|\xi.
\]

Since \( H \) is normal

\[
\text{Cor}^H\cdot C_{g^\ast}^{g^{-1}}(\xi) = \text{Cor}^H\cdot C_{g^\ast}^{g^{-1}}(\xi),
\]

and this will be equal to \( |C|\xi \) if \( \text{Res}(\xi) \) is fixed by \( Q \).

**Corollary 5.6.** If we take \( P_2 \leq (C_p)^2: SL_2(p) \) such that the subgroup \( \langle B, C \rangle \) is the normal subgroup, then the image of \( H^*(B(C_p)^2: SL_2(p);\mathbb{Z}) \) in \( H^*(BP_2;\mathbb{Z}) \) has module generators

\[
\zeta^i \alpha^j \beta^k \mu^\varepsilon \quad \text{for } j > 0, \varepsilon = 0, 1 \text{ and } i + 2j \equiv k \quad (p - 1),
\]

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\[
\zeta^{(p-1)i-j} \chi_j \quad \text{for } j < p-1, \quad \zeta^{i(p-1)}(\chi_{p-1} + \beta^{p-1}),
\]

\[
\zeta^i \alpha^j \nu \quad \text{where } i + 2j \equiv -3 \pmod{p-1}, \quad \text{and } (\zeta \beta)^i (\zeta^{p-1} + \beta^{p(p-1)})^j \mu^\varepsilon.
\]

Proof. First we determine the fixed points of \(H^*(BP_2)\) under the action of its normaliser, which is \(P_2: C_{p-1}\). This group contains a chain of normal subgroups \(\langle C \rangle, \langle B, C \rangle, P_2\). Let \(\lambda\) be a generator for \(F^\times_p\). The action by conjugation of a generator of \(C_{p-1}\) on \(\langle B, C \rangle\) must be faithful, preserve \(\langle C \rangle\), and have determinant one, so without loss of generality may be taken to be \(C \mapsto \lambda C, B \mapsto \lambda^{p-1} B\). Also this element acts on the quotient group \(P_2/\langle C \rangle = \langle A, B \rangle\). The action on \(\langle C \rangle\) is the determinant of the action on \(\langle A, B \rangle\), so the element must send \(A\) to \(A \lambda^2\). The induced action on \(H^*(BP_2)\) is

\[
\alpha \mapsto \lambda^2 \alpha \quad \mu \mapsto \mu \quad \zeta \mapsto \lambda^p \zeta
\]

\[
\beta \mapsto \lambda^{p-1} \beta \quad \nu \mapsto \lambda^3 \nu \quad \chi_i \mapsto \lambda^i \chi_i
\]

Since each monomial is an eigenvector, it is easy to identify the fixed subspace, which has the following module generators.

\[
\zeta^i \alpha^j \beta^k \mu^\varepsilon \quad \text{for } j > 0, \varepsilon = 0, 1 \quad \text{and } i + 2j \equiv k \pmod{p-1}, \quad \zeta^{(p-1)i-j} \chi_j, \quad \zeta^i \alpha^j \nu \quad \text{for } i + 2j + 3 \equiv 0 \pmod{p-1}.
\]

The kernel of the restriction to \(\langle B, C \rangle\) is the ideal generated by \(\alpha, \chi_2, \ldots, \chi_{p-2}, \chi_{p-1} + \beta^{p-1}, p\zeta, \) and \(\nu\), and this ideal together with the subring generated by \(\zeta, \beta, \) and \(\mu\) spans \(H^*(BP_2)\) as a module. Identifying elements of this subring with their images under restriction to \(\langle B, C \rangle\), it follows from corollary 5·4 that the fixed points of \(H^*(B(B, C))\) under the action of \(N(\langle B, C \rangle)\) are generated by \(\zeta \beta, \zeta^{p-1} + \beta^{p(p-1)}, \) and \(\mu\). The result follows. ■

The Held Group. Now we shall consider the cohomology of the Held group \(He\), a sporadic simple group of order \(2^{10}.3^3.5^2.7^3.17\), \([He]\). The Sylow 3- and 7-subgroups are isomorphic to \(P_2\), so the results of section 2 will be relevant. By the end of this section we shall have determined \(H^*(BH\ell; \mathbb{Z}[\frac{1}{2}])\) by considering separately each of the other primes that
divide $|He|$. We list our results for the primes in order of increasing difficulty. In the sequel we shall assume any results concerning $He$ and other groups that are stated explicitly in the Atlas [Co], but we shall prove everything else that we require in a series of lemmata, one for each $p$. Much of the content of these is in Held’s original paper [He].

The prime 17. If $G$ has a cyclic Sylow subgroup $C$, it follows from Swan’s theorem (or directly from the stable element criterion if $C$ has order $p$) that $H^*(BG;\mathbb{Z})_p$ is a polynomial algebra on a generator of degree $2|N_G(C) : C_G(C)|$. More recently Thomas has shown that in this case the generator may be taken to be a Chern class of a representation (see [Th3], together with the observation that $c.(-\rho)c.(\rho) = 1$ so the Chern subring is generated by classes of actual representations). In $He$ there are two classes of elements of order 17, so $|N_{He}(C_{17}) : C_{He}(C_{17})|$ has order 8, and $H^*(BHe;\mathbb{Z})_{17}$ is a polynomial algebra on a generator of degree 16.

The prime 5.

Lemma 5.7. The group $He$ has a unique conjugacy class of elements of order 5, and its Sylow 5-subgroup is $5A^2$, that is it is isomorphic to $C_5 \oplus C_5$. The centraliser of $5A^2$ is $5A^2$, and $N(5A^2) \cong (C_5)^2 : C_4A_4$. The group $\text{Aut}((C_5)^2)$ contains a unique class of $C_4A_4$ subgroups.

Proof. The assertions concerning $He$ are stated explicitly in the Atlas, except for the claim that $5A^2$ is its own centraliser. Note that $C(5A)$ has order 300, and of course $C(5A^2) \subset C(5A)$. $He$ has only two classes of maximal subgroups of order divisible by 25, $N(5A^2)$, and another isomorphic to $S_4(4) : C_2$. The only maximal subgroups of $S_4(4) : C_2$ with order divisible by 25 are isomorphic to $(A_5 \times A_5) : (C_2 \times C_2)$. In this group the subgroup $C_5 \times A_5$ has order 300 and centralises an element of order 5, so must be $C(5A^2)$, but $C_5 \times C_5$ is its own centraliser in $C_5 \times A_5$.

It remains to check the assertions concerning $GL_2(5)$. A Sylow 2-subgroup of $GL_2(5)$, for example the group of diagonal and ‘anti-diagonal’ matrices, is non-abelian of order
32. It follows that any abelian subgroup of $GL_2(5)$ of order 16 must contain the centre. $PGL_2(5) \cong S_5$, so the result follows since $S_5$ has a unique class of $A_4$ subgroups.

**Theorem 5.8.** $H^*(BHe; \mathbb{Z})_5$ is generated by elements $\alpha$, $\beta$, $\gamma$ and $\chi$, where $\alpha$ has degree 16, $\beta$ and $\gamma$ have degree 24 and $\chi$ has degree 15, subject only to the relations that all generators have order 5 and $\gamma^2 = 3(\beta^2 + \gamma^3)$. The cohomology operation $\delta_p P^1 \pi_*$ sends $\chi$ to a multiple of $\gamma$.

**Proof.** It follows from corollary 5.2 that $H^*(BHe; \mathbb{Z})_5$ is isomorphic to the fixed points of $H^*(B5A^2; \mathbb{Z})$ under the action of $N(5A^2)$. By lemma 5.7 we may take as generators for $N(5A^2)/5A^2$ considered as a subgroup of $GL_2(p)$ the matrices

$$M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}.$$ 

Note that these define a composition series for this group, in the sense that the subgroup generated by the first $i$ of them is normal in the subgroup generated by the first $i + 1$ of them. Let $H^*(B5A^2; \mathbb{Z})$ be generated by elements $\delta$ and $\delta'$ of degree 2 and $\varepsilon$ of degree 3, such that the above matrices describe the action of $C_4 A_4$ on the basis $\delta, \delta'$ for $H^2$. Thus $M_1$ sends $\delta$ to $2\delta$, $\delta'$ to $3\delta'$, and fixes $\varepsilon$. The fixed point subring under the action of $M_1$ is easily seen to be generated by $\delta^4, \delta'^4, \delta \delta'$, and $\varepsilon$. Since $M_2$ normalises $M_1$, $M_2$ acts on this fixed point subring, and in fact it fixes $\delta^4$ and $\delta'^4$ and sends $\delta \delta'$ to $-\delta \delta'$ and $\varepsilon$ to $-\varepsilon$. The fixed point subring is generated by $\delta^4, \delta'^4, \delta^2 \delta'^2$ and $\delta \delta' \varepsilon$. The action of $M_3$ fixes $\delta^2 \delta'^2$, exchanges $\delta^4$ and $\delta'^4$ and sends $\delta \delta' \varepsilon$ to minus itself, so has fixed point subring generated by $\delta^4 + \delta'^4$, $\delta^2 \delta'^2$, and $(\delta^5 \delta' - \delta'^5 \delta) \varepsilon$. This is a polynomial algebra on two generators of degree 8 tensored with an exterior algebra on one generator of degree 15.

The action of $M_4$ clearly fixes the exterior generator, but has no fixed point in degree 8. To complete the proof it suffices to show that whenever $C_3$ acts non-trivially on $\mathbb{F}_5[x, x']$, then the fixed point subring is generated by elements $a, b, c$, of degrees two, three and three respectively (where $x$ is of degree one), subject only to $c^2 = 3(b^2 + a^3)$. For this, extend the action to one on $\mathbb{F}_{25}[x, x']$, where it may be diagonalised. Note that if $y$ is an eigenvector
in degree one with eigenvalue $\omega$, then $\bar{y}$ (defined by conjugating the coefficients of $y$ as an expression in $x$ and $x'$) is an eigenvector of value $\bar{\omega}$. The fixed point subring over $\mathbb{F}_{25}$ is generated by $y\bar{y}$, $y^3$ and $\bar{y}^3$. An element of this ring is in $\mathbb{F}_5[x, x']$ iff it is invariant under conjugation. Elements invariant under conjugation are the sum of a symmetric polynomial over $\mathbb{F}_5$ in $y$ and $\bar{y}$, and $\sqrt{3}$ times an antisymmetric polynomial over $\mathbb{F}_5$ in $y$ and $\bar{y}$. The symmetric polynomials in $y$ and $\bar{y}$ expressible in terms of $y\bar{y}$, $y^3$ and $\bar{y}^3$ form a polynomial ring on $y\bar{y}$ and $y^3 + \bar{y}^3$, while the antisymmetric polynomials form a free module for the symmetric ones with basis $y^3 - \bar{y}^3$. Our three generators are $a = y\bar{y}$, $b = y^3 + \bar{y}^3$ and $c = \sqrt{3}(y^3 - \bar{y}^3)$.

\[\square\]

The author wishes to thank Dr. D. J. Benson who found an error in an earlier statement of theorem 5·8.

**The prime 3.**

Lemma 5·9. Let $P$ be a Sylow 3-subgroup of $He$. Then $P \cong P_2$, and $He$ contains two (self-inverse) conjugacy classes of elements of order 3, 3A and 3B. $N(P)$ is isomorphic to $P: D_8$ and $D_8$ acts faithfully. The centre of $P$ is 3A. There are two classes of $C_3 \oplus C_3$ subgroups of $He$, $3A^2$ which is contained in a unique Sylow 3-subgroup, and $3A3B$, which has

\[N(3A3B) \cong C_3 \cdot (S_3 \times S_4) \quad C(3A3B) \cong C_3 \cdot (C_3 \times V_4)\]

There is a unique class of $D_8$ subgroups of $GL_2(3)$.

**Proof.** The order of $C(3B)$ is not divisible by 27, so $Z(P)$ must be 3A. One of the maximal subgroups of $He$ is $N(3A) \cong C_3 \cdot S_7$, and we shall work largely within this group. For example, $N(P) \leq N(3A)$, and we see that

\[N(P) = C_3 \cdot N_{S_7}(C_3 \times C_3) \cong P: D_8.\]

If we write $3X$ for the conjugacy class in $He$ represented by the 3-cycles in $S_7$, and $3Y$ for that represented by products of two disjoint 3-cycles, then

\[C(3A, 3X) \cong C_3 \cdot (C_3 \times V_4) \quad C(3A, 3Y) \cong C_3 \times C_3\]
We deduce that there are two classes of $C_3 \oplus C_3$ subgroup in $He$, one of which must be $3A^2$, and the other $3A3B$ containing a unique $3A$ subgroup. Now

$$|N(3A, 3B) : C(3A, 3B)| \leq 12 \quad \text{and} \quad |N(3A^2) : C(3A^2)| \leq 48,$$

but $N(3A, 3Y) \cap N(3A) \cong C_3 \cdot (S_3 \times S_3).C_2$, which already has order 24 times that of $C(3A, 3Y)$. Therefore $3Y = 3A$, and $N(3A^2)$ is either $C_3 \cdot (S_3 \times S_3).C_2$ or a group containing this with index two. In either case $N(3A^2)$ contains a unique Sylow subgroup. Also $3X = 3B$, and $N(3A, 3B)$ is as claimed.

Theorem 5·10. $H^*(BHe; \mathbb{Z})_3$ is isomorphic to the subring of $H^*(BP_2; \mathbb{Z})$ generated by $\chi_2$, $\alpha^2 + \beta^2$, $\alpha^2 \beta^2$, $\zeta(\alpha \nu - \beta \mu)$, and $\zeta^2$.

Proof. Identifying $\text{Aut}(P_2)/P_2$ with $GL_2(p)$, it follows from lemma 5·9 that we may take $N(P_2)$ as being generated by

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

(Of course we only need $M_1$ and $M_3$). The action of $M_1$ fixes $\beta$, $\nu$, and $\chi_2$, and multiplies the other generators by $-1$. The action of $M_2$ fixes $\alpha$, $\mu$, and $\chi_2$ and multiplies the other generators by $-1$. It follows that the fixed point subring under $M_1$ and $M_2$ is spanned by the elements

$$\zeta^{2i}, \quad \zeta^{2i} \chi_2, \quad \zeta^{2i+\varepsilon} \alpha^{2j+\varepsilon} \nu,$$

$$\zeta^i \alpha^j \beta^k \nu^\varepsilon \quad \text{where} \quad \varepsilon + j \equiv i \equiv k \pmod{2}, \quad \text{and} \quad j = 0 \text{ or } k + \varepsilon \leq 3.$$

The action of $M_3$ on the generators is given by

$$\alpha \mapsto -\beta \quad \mu \mapsto \nu \quad \zeta \mapsto -\zeta$$

$$\beta \mapsto \alpha \quad \nu \mapsto -\mu \quad \chi_2 \mapsto \chi_2$$

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and a simple calculation shows that the fixed points under the action of \(D_8\) are spanned by

\[
\zeta^{2i} \chi_2, \quad \zeta^{2i+1}(\alpha^{2j+1} \nu - \beta^{2j+1} \mu),
\]

\[
\zeta^{2i}(\alpha^{2j} + \beta^{2j}), \quad \zeta^{2i} \alpha^{2j} \beta^2,
\]

and further that this is the subring generated by the elements of the statement. To complete the proof we must show that all these elements are stable. Without loss of generality we may take \(\langle A, C \rangle\) and \(\langle B, C \rangle\) to be the 3A3B subgroups of \(P_2\), and by corollary 5·2 it suffices to check that for \(H = \langle B, C \rangle\) and \(\xi\) any of the generators of the statement then

\[
\text{Cor}_{P_2} \xi^* \text{Res}_{H}^P_2 (\xi) = 3 \xi = \text{Cor}_{H}^P \xi \text{Res}_{H}^P_2 (\xi) \quad \text{for all } g \in N_{He}(H).
\]

If we take \(H^*(BH; \mathbb{Z}) \cong \mathbb{Z}[\beta', \gamma] \otimes \Lambda[\delta]\), then the action of an element of \(N_{He}(H)\) sends \(\beta'\) to \(\lambda \beta'\), \(\gamma\) to \(\lambda' \gamma + \lambda'' \beta'\) and \(\delta\) to \(\lambda \lambda' \delta\), where \(\lambda, \lambda' \in \mathbb{F}_3^\times\) and \(\lambda'' \in \mathbb{F}_3\). The restriction from \(P_2\) to \(H\) is given by

\[
\begin{align*}
\alpha & \mapsto 0 & \mu & \mapsto \delta & \zeta & \mapsto \gamma^3 - \beta^2 \gamma \\
\beta & \mapsto \beta' & \nu & \mapsto 0 & \chi_2 & \mapsto -\beta^2
\end{align*}
\]

It can be checked that the images of the generators of the statement are fixed by \(N_{He}(H)\), so the generators satisfy the stability condition. 

\[\blacksquare\]

**The prime 7.**

**Lemma 5·11.** Let \(P\) be a Sylow 7-subgroup of \(He\). then \(P \cong P_2\), and \(He\) contains five conjugacy classes of element of order seven, \(7A, 7B\) which are inverse to each other, \(7C\) which is self-inverse, and \(7D, 7E\) which are inverse to each other. The central elements are \(7C\), and in fact \(N(7C) = N(P) \cong P : (S_3 \times C_3)\), with \(S_3 \times C_3\) acting faithfully on \(P\).

There is a unique class of such subgroups of \(GL_2(7)\). There are three conjugacy classes of \(C_7 \times C_7\) subgroups of \(He\), with normalisers

\[
N(7C, 7AB) \cong P : (C_3 \times C_3) \quad N(7C^2) \cong (C_7)^2 : SL_2(7) \quad N(7C, 7DE) \cong P : (C_2 \times C_3).
\]
$7C^2$ is its own centraliser. Some other subgroups are

$$(C_7)^2 : C_6 \cong N(7DE) \leq N(P) \quad N(7AB) \cong C_7 : C_3 \times L_2(7),$$

$$(C_7)^2 : (C_3 \times C_3) \cong N(7C, 7AB) \cap N(7AB) \leq N(P).$$

**Proof.** The normaliser of a Sylow 3-subgroup of $GL_2(7)$ is $C_6 \wr C_2$, which contains a unique conjugacy class of $C_3 \wr C_2 \cong S_3 \times C_3$ subgroup. The Atlas tells us that $N(7C^2)$ is as claimed above, but $L_2(7)$ is not a subgroup of $GL_2(7)$, so the only possibility is that $7C^2$ is self-centralising. The Atlas tells us that $N(7C)$ is $P : (S_3 \times C_3)$ as claimed, and since no element of $S_3 \times C_3$ can centralise a $7C^2$ subgroup of $P$ we see that $S_3 \times C_3$ acts faithfully on $P$. We already know that such an action is essentially unique, and so we fix generators for this group considered as acting on $C_7^A \oplus C_7^B = P_2/\langle C \rangle$ as below.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \quad \text{where } \lambda^3 = \lambda'^3 = 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$}

Direct computation shows that there are four conjugacy classes of $C_7$ subgroup in $P$ under the action of $N(P)$, which are tabulated in figure 5·1 together with their centralisers in $N(P)$. For each of them except $\langle C \rangle$ the centraliser has index three in the normaliser.

| $C_7$ subgroup | Centraliser in $N(P)$ |
|----------------|----------------------|
| $\langle C \rangle$ | $P.C_3$ |
| $\langle AC^i \rangle \sim \langle BC^i \rangle$ | $(C_7)^2 : C_3$ |
| $\langle AB^{2i}C^j \rangle$ | $(C_7)^2 : C_2$ |
| $\langle AB^{-2i}C^j \rangle$ | $(C_7)^2$ |

5·1. Table of $C_7$ subgroups of $N(P)$.

He has only two classes of maximal subgroup whose order divides by $7^3$, $N(7C^2)$ and $N(7C)$. We see that $\langle A, C \rangle$ cannot be $7C^2$ because its normaliser has order divisible by 9. Also its normaliser must be contained in $N(7C)$, so is $P.(C_3 \times C_3)$. $\langle A \rangle$ cannot be $7C$ by the above, but neither can it be $7DE$ since the order of $C(7DE)$ is coprime to 3, hence
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\[ \langle A \rangle \text{ must be } 7AB. \] There are \( 7C^2 \) subgroups, but \( \langle C, AB^2 \rangle \) cannot be \( 7C^2 \) because it has elements centralised by an involution. It follows that \( \langle AB^2 \rangle \) must be \( 7DE \) and \( \langle AB^{-2} \rangle \) is \( 7C \). Comparing orders tells us that \( N(7DE) \) is contained in \( N(P) \) and is as claimed. The normaliser of \( 7AB \) is a maximal subgroup listed in the Atlas. The intersection of \( N(7C, 7AB) \) and \( N(7AB) \) has index seven in \( N(7C, 7AB) \), because this group permutes its \( 7AB \) subgroups transitively, so is as claimed.

**Theorem 5·12.** With notation as in lemma 5·11, \( H^*(BN(P); \mathbb{Z})_7 \) is mapped via restriction to the subring of \( H^*(BP; \mathbb{Z}) \) generated by

\[
\begin{align*}
\alpha^3 + \beta^3, & \quad \alpha^3 \beta^3, & \quad \chi_6, & \quad \alpha^5 \beta \mu - \alpha^2 \beta^4 \mu, & \quad \zeta \alpha \mu & \quad \zeta \chi_5, & \quad \zeta (\alpha^5 \beta^2 - \alpha^2 \beta^5), & \quad \zeta^2 \alpha \beta, \\
\zeta^2 (\alpha^2 \nu - \beta^2 \mu), & \quad \zeta^2 \chi_4, & \quad \zeta^3 \chi_3, & \quad \zeta^3 (\alpha^3 - \beta^3), & \quad \zeta^4 \chi_2, & \quad \zeta^5 (\alpha^2 \nu + \beta^2 \mu), & \quad \zeta^6.
\end{align*}
\]

**Proof.** As in theorem 5·10 we take a basis for \( H^*(BP; \mathbb{Z}) \otimes \mathbb{F}_7 \) consisting of the monomials

\[ \zeta^i, \quad \zeta^i \chi_j, \quad \zeta^i \alpha^j \nu, \quad \zeta^i \alpha^j \beta^k \mu^\varepsilon \quad \text{where } j = 0 \text{ or } k + \varepsilon \leq 6. \]

All monomials are eigenvectors for the action of the \( C_3 \times C_3 \) subgroup of \( N(P)/P \), and it may be checked that the fixed monomials are

\[ \zeta^i, \quad \zeta^i \chi_j, \quad \zeta^i \alpha^j \nu, \quad \zeta^i \alpha^j \beta^k \mu^\varepsilon \quad \text{where } i + j + \varepsilon \equiv 0 \text{ and } i + k \equiv \varepsilon \mod 3. \]

The action of the involution represented by the matrix in the statement of lemma 5·11 on the original generators is

\[
\begin{align*}
\alpha & \mapsto \beta \\
\mu & \mapsto -\nu \\
\zeta & \mapsto -\zeta
\end{align*}
\]

\[
\begin{align*}
\beta & \mapsto \alpha \\
\nu & \mapsto -\mu \\
\chi_i & \mapsto (-1)^i \chi_i
\end{align*}
\]

and the fixed points under the whole of \( N(P)/P \) are seen to be spanned by

\[
\begin{align*}
\zeta^{6i}, & \quad \zeta^{6i-j} \chi_j, & \quad \zeta^{6i+2}(\alpha^{6j+2} \nu - \beta^{6j+2} \mu), & \quad \zeta^{6i}(\alpha^{3j} + \beta^{3j}),
\end{align*}
\]

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\[ \zeta^{6i-1}(\alpha^{6j-1}\nu + \beta^{6j-1}\mu), \quad \zeta^{i}\alpha^j\beta^k\mu^\varepsilon \quad \text{where } i \equiv 2j - \varepsilon \equiv j + k \quad (6), \]

\[ \zeta^{i}(\alpha^j\beta^k\mu^\varepsilon + \alpha^{i-3}\beta^{k+3}\mu^\varepsilon) \quad \text{where } i \equiv 2j - \varepsilon \equiv j + k + 3 \quad (6), \]

\[ \zeta^{i}(\alpha^j\beta^k\mu^\varepsilon - \alpha^{i-3}\beta^{k+3}\mu^\varepsilon) \quad \text{where } i \equiv 2j - \varepsilon + 3 \equiv j + k \quad (6). \]

More calculation shows that the fifteen elements of the statement generate this subring.

In a sense this almost completes the calculation of \( H^*(BHe;\mathbb{Z})_7 \), which may be described as the subring of stable elements of \( H^*(BN(P);\mathbb{Z})_7 \). We know from corollary 5·2 that only conjugates \( P^g \) of \( P \) such that \( P \cap P^g \) has order 49 can give any extra conditions, and by lemma 5·11 the only such intersections are of type \( 7C^2 \). It follows as in theorem 5·5 that \( \xi \in H^*(BN(P);\mathbb{Z})_7 \) is in the image of \( H^*(BHe;\mathbb{Z})_7 \) iff

\[ \text{Cor}_{(AB^{-1},C)}^P \varphi^{*}\text{Res}_{(AB^{-1},C)}^P(\xi) = p\xi, \quad (*) \]

where \( \varphi \) is some fixed element of \( \text{Aut}(\langle A, C \rangle) \) of order 7 not fixing \( \langle C \rangle \). (It is not obvious, but true, that all such \( \varphi \) will give the same condition.)

The description of \( H^*(BHe;\mathbb{Z})_7 \) given by condition \((*)\) is not very explicit, since the condition is difficult to check. For example, this description is not much use for calculating the Poincaré series of \( H^*(BHe;\mathbb{Z})_7 \otimes \mathbb{F}_7 \). The following lemma gives a more useful description of \( H^*(BHe;\mathbb{Z})_7 \), which will allow us to completely determine \( H^*(BHe;\mathbb{Z})_7 \) in theorem 5·14.

Lemma 5·13. Let \( K \) be the subgroup \( \langle AB^{-1}, C \rangle \) of \( P \), let \( \text{Res} = \text{Res}_K^P \), and define \( \zeta', \varepsilon, \delta \), elements of \( H^*(BK;\mathbb{Z}) \) by

\[ \zeta' = \text{Res}(\zeta) \quad \varepsilon = \text{Res}(\alpha) = -\text{Res}(\beta) \quad \delta = \text{Res}(\mu) = -\text{Res}(\nu). \]

Now let \( S \) be the subring of \( H^*(BK;\mathbb{Z}) \) generated by \( \zeta', \varepsilon, \zeta'^6 + \varepsilon^4 \) and \( \delta \). Then \( H^*(BHe;\mathbb{Z})_7 \) is isomorphic to \( (\text{Res})^{-1}(S) \cap H^*(BN(P);\mathbb{Z})_7 \).
Proof. From corollary 5.2 it follows that an element $\xi$ of $H^*(BN(P);\mathbb{Z})_7$ is in the image of the restriction from $H^*(BHe;\mathbb{Z})_7$ iff

$$c_g^*\text{Res}_g^{P}\left(\text{Res}_g^{P}\left(\xi\right)\right) = \text{Res}_g^{P}\left(\text{Res}_g^{P}\left(\xi\right)\right)$$

for all $g$ such that $P^g \cap P$ has order $p^2$. This can only happen if $P^g \cap P$ is of type $7C^2$ (see lemma 5.11). Since $N(P)$ acts transitively on such subgroups we may restrict to the case when $P^g \cap P$ is the subgroup $K = \langle AB^{-1}, C \rangle$ and $g$ normalises $K$. Lemma 5.11 tells us that $N(7C^2) \equiv (C_7)^2; SL_2(7)$, and the ring $S$ described in the statement is (by corollary 5.4) the fixed point subring of $H^*(BK;\mathbb{Z})$ under the action of $SL_2(7)$.

THEOREM 5.14. With notation as in lemma 5.11, $H^*(BHe;\mathbb{Z})_7$ is mapped via restriction to the subring of $H^*(BP;\mathbb{Z})$ generated by the following elements.

$$\alpha^3 + \beta^3, \chi_6 - \alpha^3\beta^3, \zeta\alpha\mu, \zeta\chi_5, \zeta^2\alpha\beta, \zeta^2(\alpha^2\nu - \beta^2\mu),$$

$$\zeta^2\chi_4, \zeta^3\chi_3, \zeta^3(\alpha^3 - \beta^3), \chi_2^4, \zeta^5(\alpha^2\nu + \beta^2\mu), \zeta^6 - \alpha^39\beta^3.$$ 

Proof. The twelve elements of the statement with $\alpha^3\beta^3$, $(\alpha^5\beta - \alpha^2\beta^4)\mu$, and $\zeta(\alpha^5\beta^2 - \alpha^2\beta^5)$ generate the subring of $H^*(BP)$ described in theorem 5.12. Also the images of the twelve elements under restriction from $P$ to the subgroup $K$ lie in the subring $S$ (see lemma 5.13), so these elements are contained in $H^*(BHe;\mathbb{Z})_7$. It remains to show that no other generators are required. It may be verified that the image of the subring of this statement under restriction to $K$ is the subring $S'$ of $S$ generated by $\zeta^4 + \varepsilon^{42}$, $\zeta^2\varepsilon^2$, $\zeta^3\varepsilon^3$, $\zeta'\varepsilon^2\delta$ and $\zeta'^2\varepsilon^2\delta$. The elements $\alpha^3\beta^3$, $(\alpha^5\beta - \alpha^2\beta^4)\mu$, and $\zeta(\alpha^5\beta^2 - \alpha^2\beta^5)$ restrict to $-\varepsilon^6$, $-2\varepsilon^6\delta$, and $2\zeta'\varepsilon^7$ respectively. We claim now that the intersection of $S$ and $\text{Res}_K^P(H^*(BN(P));\mathbb{Z}))$ is the subring $S'$ of $S$ generated by the elements of the statement. Equivalently we claim that the intersection of $S$ and the ring generated by $S', \varepsilon^6, \varepsilon^6\delta$ and $\zeta'\varepsilon^7$ is $S'$. In even degrees $S \setminus S'$ is the set of elements of $S$ divisible exactly once by $\varepsilon$, while elements of even degree in the ring generated by $S', \varepsilon^6$ and $\zeta'\varepsilon^7$ will be divisible by
\( \varepsilon \) either at least twice or not at all. A similar argument works in odd degrees, since the odd degree part of \( S \setminus S' \) is the set of elements not divisible by \( \varepsilon \).

Now we have shown that the subring of \( H^*(BP; \mathbb{Z}) \) generated by the elements of the statement has the same image under restriction to \( K \) as \( H^*(BHe; \mathbb{Z})_7 \), so we may assume that any remaining elements of \( H^*(BHe; \mathbb{Z})_7 \) are in the kernel of \( \text{Res}_K^P \). It will suffice to show that the kernel of \( \text{Res}_K^P \) as a map from \( H^*(BN(P); \mathbb{Z})_7 \) is contained in the subring generated by the elements of the statement. We claim that this kernel is the ideal of \( H^*(BN(P); \mathbb{Z})_7 \) generated by \( \alpha^3 + \beta^3 \), \( \zeta^5(\alpha^2\nu + \beta^2\mu) \), \( \zeta^3\chi_2 \), \( \zeta^3\chi_3 \), \( \zeta^2\chi_4 \), \( \zeta\chi_5 \) and \( \chi_6 - \alpha^3\beta^3 \). We easily reduce to the case of elements of the kernel of the form \( \zeta^iP(\alpha, \beta) \) or \( \zeta^i(P(\alpha, \beta) + \lambda\alpha^r\nu) \). Notice that if \( n \geq 7 \) and

\[
\zeta^j \left( \sum_{i=0}^{6} \lambda_i \alpha^{n-i} \beta^i + \lambda_7 \beta^n \right) \in H^*(BN(P); \mathbb{Z})_7,
\]

then

\[
\zeta^{6k+j} \left( \sum_{i=0}^{6} \lambda_i \alpha^{6m+n-i} \beta^i + \lambda_7 \beta^{6m+n} \right) \in H^*(BN(P); \mathbb{Z})_7,
\]

so the remaining cases reduce to \( \zeta^iP(\alpha, \beta) \) and \( \zeta^i(P(\alpha, \beta) + \lambda\alpha^r\nu) \) where the degree of \( P \) is at most 13 and \( i \leq 5 \). It can be checked by lengthy calculation that all such elements are in the ideal of \( H^*(BN(P); \mathbb{Z})_7 \) generated by \( \alpha^3 + \beta^3 \) and \( \zeta^5(\alpha^2\nu + \beta^2\mu) \). To check that the kernel is contained in the subring of the statement it suffices to find an expression for each product of one of the generators of the kernel (as an ideal) and one of \( \alpha^3\beta^3 \), \( (\alpha^5\beta - \alpha^2 \beta^4)\mu \) and \( \zeta(\alpha^5\beta^2 - \alpha^2 \beta^5) \) in terms of the twelve elements of the statement. This too is a routine calculation.

\( \blacksquare \)
6. Yagita’s Invariant.

The following short section concerns an invariant related to free actions of groups on products of spheres. In [Ya] Yagita defined an integer invariant $p^G$ for a prime $p$ and a finite group $G$ which must divide $n$ if $G$ acts freely on a product $(S^{n-1})^k$ with trivial action on $H^*((S^{n-1})^k; Z)$.

**The Invariant.** Let $u$ be a generator for $H^2(BC_p; Z)$. To each inclusion $i$ of $C_p$ in $G$ Yagita associates an integer $m(i)$, which is the largest $m$ such that $i^*(H^*(BG; Z))$ is contained in the subring of $H^2(BC_p; Z)$ generated by $u^m$. Then $p^G$ is defined to be twice the lowest common multiple of all such $m(i)$.

We recall that given a CW complex $X$ with a free cellular action of $G$ (such a space is referred to as a free $G$ complex), there is a Cartan-Leray spectral sequence with $E_2$ page $H^i(BG; H^j(X))$ converging to a filtration of $H^{i+j}(X/G)$. Now let $X$ be a finite free $G$ complex, with cohomology ring isomorphic to an exterior algebra on $k$ generators with trivial $G$ action. An inclusion $i : C_p \hookrightarrow G$ induces a map of Cartan-Leray spectral sequences, where the map on $E_2$ pages is essentially the restriction, and the map on $E_\infty$ pages is induced by the projection from $X/C_p$ to $X/G$. By considering this map of spectral sequences Yagita shows that $m(i)$ must divide $n$.

While exploring the possibility of defining a similar invariant in terms of the images of $\text{Ch}(G)$, the author discovered two errors in [Ya], and wrote the following letter to Yagita. At the time of writing there seemed to be no connection between the new invariant and actions on spheres, but the author has recently found a connection which is explained in lemma 6.1.

The preprint referred to in the letter contained an incorrect proof of corollary 3.7, and a determination of the order of $H^n(BG(a, 1); Z)$ which disagrees with Yagita’s theorem 2.4. We explain the method of calculation after the letter.

*Dear Prof. Yagita,*

I am writing to you to report some work that I have done which was inspired by your paper ‘On the dimension of spheres whose product admits a free action by a non-abelian
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group’, Quart. J. Math. Oxford 36 (1985) 117–127. I have also found two mistakes in this paper. I find that theorem 2.4 is incorrect for odd primes \( p \) for the groups \( G_2(a, 1) \), but that your other results that rely on this theorem are correct. Secondly, your proof of lemma 1.7 is incorrect; in this case I have been unable to find a proof or a counterexample.

Before explaining my criticisms, I shall define my adaptation, \( pc(G) \), of your invariant. My invariant is easier to calculate than yours, and gives an upper bound for yours, but does not seem to bear any direct relationship to actions on spheres. I shall consider only odd primes \( p \).

You define, for each \( i : \mathbb{Z}/p \rightarrow G \), \( m(i) \) to be the greatest integer \( m \) such that \( \iota^*(R) \subset \mathbb{Z}[u^m]/(pu^m) \), where \( R \) is \( H^*(BG; \mathbb{Z}) \). I define \( n(i) \) similarly, except that I take \( R \) to be the subring of \( H^*(BG; \mathbb{Z}) \) generated by Chern classes of representations of \( G \). I then define

\[
pc(G) = 2\text{l.c.m.}\{n(i) \mid i : \mathbb{Z}/p \rightarrow G\}.
\]

(Incidentally, your stated definition of \( p\circ(G) \) is \( \text{l.c.m.}\{2m(i)\} \), which gives \( p\circ(G) = 1 \) if \( p \) does not divide the order of \( G \), and gives easy counterexamples to lemma 1.7. I shall use the definition \( 2\text{l.c.m.}\{m(i)\} \).

Various properties of \( pc(G) \), such as 1 to 4 below, may be proven using the proofs you use for \( p\circ(G) \):

1. If \( G \) is abelian, then \( pc(G) = 2 \).
2. If \( N \hookrightarrow G \twoheadrightarrow Q \), then \( pc(G) \mid \text{l.c.m.}\{pc(Q), pc_N(G)\} \).
3. If \( H < G \), then \( pc(H) \mid pc_H(G) \).
4. If \( H < G \), then \( pc_H(G) \mid pc(G) \).
5. \( p\circ(G) \mid pc(G) \), so my invariant gives an upper bound for yours. A lower bound would of course be more useful! It follows from 1 and 2 that
6. \( pc(G) = pc_{G'}(G) \), where \( G' \) is the derived subgroup of \( G \). Similarly, it can be shown that
6. \( pc(G) = pc_G(G) \).

If \( G \) is a minimal non-abelian \( p \)-group or an extra-special \( p \)-group, then \( G' \) is isomorphic to \( \mathbb{Z}/p \), so \( 6' \) could be used to simplify your calculations for these groups.

Using 6 it may be shown that

7. If \( G \) is a minimal non-abelian \( p \)-group, then \( pc(G) = 2p \), and hence

8. If \( G \) is a \( p \)-group, then \( pc(G) = 2 \) if and only if \( G \) is abelian.

The regular representation of \( G \) restricts to a \( \mathbb{Z}/p \) subgroup as \( \left| G : \mathbb{Z}/p \right| \) copies of the regular representation of \( \mathbb{Z}/p \), so its total Chern class restricts to \( \mathbb{Z}/p \) as

\[
\prod_{i=0}^{p-1} (1 - iu)\right|G:\mathbb{Z}/p| = (1 - u^{p-1})\right|G:\mathbb{Z}/p|,
\]

and so

9. \( pc(G) \mid 2(p - 1)p^{n-1} \), where \( p^n \) is the \( p \)-part of \( |G| \).

Using also representations of \( G \) induced up from irreducible representations of \( \mathbb{Z}/p \) I can show that \( pc(G) \mid 2|G| \), but the calculation is longer.

I first noticed the flaw in lemma 1·7 while trying to prove the analogous result for \( pc(G) \). Let \( N \triangleleft G \), and let \( x \in H^*(BN) \). Then letting \( \zeta_{\ast}^t \) be the map \( H^*(BN) \longrightarrow H^*(BN) \) induced by \( n \mapsto n^t \), we have:

\[
\text{Res}_{N\ast}^G \zeta_{N\ast}^G(x) = \prod_{t \in G/N} \zeta_{\ast}^t(x)
\]

(compare this with the formula for \( \text{Res}_{N\ast}^G \text{Cor}_{N\ast}^G \)), so the equation in your proof of lemma 1·7 is incorrect in general. I have been unable either to prove lemma 1·7 or to find a counterexample. However, the analogous statement for \( pc(G) \) is false:

Let the cyclic group of order \( p^n - 1 \) act on \( (\mathbb{Z}/p)^n \) so as to permute the non-zero elements transitively (this can be done – it is the action by multiplication of the multiplicative group of the field of order \( p^n \) on its additive group). Now let \( N = (\mathbb{Z}/p)^n \), \( Q = \mathbb{Z}/p^{n-1} \), and \( G \) the (unique) extension of \( N \) by \( Q \) corresponding to the above action. \( G \) has \( p^n - 1 \).
1-dimensional representations, which restrict trivially to any subgroup of order \( p \), and one \((p^n - 1)\)-dimensional representation, whose total Chern class restricts to any \( \mathbb{Z}/p \) subgroup as \((1 - u^{p-1})p^{n-1} = 1 - u^{(p-1)p^{n-1}}\). Hence \( \text{pc}(G) = 2(p - 1)p^{n-1} \), which does not divide \((p^n - 1)\text{pc}(N)\).

To calculate \( \text{pc}(G) \), I followed a suggestion of the late Prof. J. F. Adams: Firstly extend the action of \( Q \) on \( H^*(BN; \mathbb{Z}) = \mathbb{Z}/p[x_1, \ldots, x_n] \) to an action of \( Q \) on \( \mathbb{F}_p[x_1, \ldots, x_n] \), calculate the fixed subring under this action, then calculate the subring of this fixed by the Galois group of \( \mathbb{F}_p^n \) over \( \mathbb{Z}/p \). Using this method it may be shown that such groups \( G \) do not give counterexamples to \( \text{pc}(G) | (|Q|, \text{pc}(N)) \).

Your theorem 2·4, in the case \( G_1 \) (the metacyclic groups) is a special case of a result of C. T. C. Wall from ‘Resolutions for extensions of groups’, Proc. Camb. Phil. Soc. 57 (1961) 251–5. The case \( G_2(1,1) \) is studied in G. Lewis ‘The integral cohomology rings of groups of order \( p^3 \)’ Trans. Amer. Math. Soc. 132 (1968) 501–29. In this spectral sequence \( d_2 \) is not trivial; for example \( d_2 : E_2^{1,4} \rightarrow E_2^{3,3} \) is not trivial. In my preprint, which I have enclosed, I give sufficient information (during the proof of theorem 3·6) to calculate the order of \( H^n(BG_2(a,1); \mathbb{Z}) \) for \( a > 1 \) (I use the name \( M(a + 2) \) for your \( G_2(a,1) \)). I find that the order of \( H^n(BG_2(a,1); \mathbb{Z}) \) is 0, \( p^{a+1} \), \( p^2 \), \( p^{a+3} \) for \( n = 1, 2, 3, 4 \) respectively. For \( i + j \leq 4 \), I calculate that the additive structure of \( E_2 \) for the spectral sequence you consider is

| \( \mathbb{Z}/p^a \oplus \mathbb{Z}/p \) | \( \mathbb{Z}/p \) | \( \mathbb{Z}/p \) | \( \mathbb{Z}/p \) |
| \( \mathbb{Z}/p \) | \( \mathbb{Z}/p \) |
| \( \mathbb{Z}/p^a \) | \( \mathbb{Z}/p \) | \( \mathbb{Z}/p \) |
| \( \mathbb{Z} \) | \( - \) | \( - \) |
| \( \mathbb{Z}/p \) | \( - \) |
| \( \mathbb{Z}/p \) |

I believe, therefore, that in the spectral sequence for \( \langle A, C \rangle \Rightarrow G_2(a,1) \Rightarrow \mathbb{Z}/p \), where \( a > 1 \), there is a non-trivial \( d_n : E_n^{i,A-i} \rightarrow E_n^{i+n,5-i-n} \) for some \( i \) and \( n \). I think that the
values of $i$ and $n$ are 1 and 2 respectively.

I have no plans to publish this work, but I shall include it in my Ph. D. thesis.

**Remarks.** Since receiving the above letter, Yagita has retracted the non-metacyclic cases of his theorem 2·4, and has sent the author a proof of lemma 1·7 under the strong condition that the action of $G$ on $H^*(BN)$ is nilpotent. The groups $G(a,1)$ considered by Yagita may be presented as follows.

$$\langle A, B, C \mid A^p = B^p = C^p = [A, C] = [B, C] = 1, [A, B] = C \rangle$$

Yagita's theorem 2·4 claimed that the spectral sequence with integer coefficients for $G(a,1)$ expressed as an extension of $\langle A, C \rangle$ by $C_p$ collapses. We may consider $G(a,1)$ as a central extension of $\langle A^p \rangle$ by $P_2$. Since $G(a,1)$ is nilpotent of class two the corresponding circle group is isomorphic to $S^1 \times P_2$, which has cohomology given by the Künneth theorem.

$$H^*(BS^1 \times P_2; \mathbb{Z}) \cong \mathbb{Z}[\tau] \otimes H^*(BP_2; \mathbb{Z})$$

The class $\tau p^{n-1} + \beta$ in $H^2(BS^1 \times P_2; \mathbb{Z})$ is the first Chern class of a bundle over $B(S^1 \times P_2)$ with total space $BG(a,1)$. We could of course choose any other non-zero element of $H^2(BP_2; \mathbb{Z})$ instead of $\beta$. It is easy now to calculate the order of $H^n(BG(a,1); \mathbb{Z})$ using the spectral sequence for this $S^1$ bundle.

A linear action of $G$ on $S^{2n-1}$ is an action induced by an $n$ dimensional complex representation of $G$. Linear actions on products of spheres are products of linear actions. Linear actions induce the trivial action on the cohomology of the spheres because $GL_n(\mathbb{C})$ is connected. An element $g$ of $G$ will act fixed point freely in the linear action given by a product of representations $\rho_1 \times \ldots \times \rho_k$ if and only if there is a $j$ such that the restriction of $\rho_j$ to $\langle g \rangle$ does not contain the trivial representation of $\langle g \rangle$. We obtain the following lemma.

**Lemma 6·1.** If $G$ has a linear action on a product of $2n-1$ dimensional spheres such that every element of $G$ of order $p$ acts fixed point freely, then $pc(G)$ divides $2n$.  

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Proof. It suffices to check that for each $i : C_p \to G$ then $n(i)$ (as defined in the above letter) divides $n$. Let the action be specified by maps $\rho_1, \ldots, \rho_k$ from $G$ to $\text{GL}_n(\mathbb{C})$, and let $C$ be the image of $C_p$ in $G$. If $C$ acts fixed point freely, then there is a $\rho = \rho_j$ such that $\text{Res}_C^G(\rho)$ does not contain the trivial representation. Then

$$c.(\text{Res}_C^G(\rho)) = \prod_{i=1}^{n} (1 + \lambda_i u)$$

where $u$ is a generator for $H^2(BC; \mathbb{Z})$ and $\lambda_i$ is non-zero. But then $c_n(\text{Res}_C^G(\rho))$ is a non-zero multiple of $u^n$. 

\[ \blacksquare \]
7. The Davis Complex.

The following chapter, which is largely expository although we hope to obtain new results in this area eventually, concerns a construction due to M. Davis [Da], which assigns to a Coxeter group $G$ a simplicial complex $D(G)$, which is contractible and has a simplicial $G$-action with finite stabilisers. We give a simplified account of a special case of Davis’ construction, and an elementary account of an example due to Bestvina [Be] of a group whose cohomological dimension over $\mathbb{F}_2$ is greater than its cohomological dimension over $\mathbb{Q}$. None of these results are original, although our proofs differ from those of [Da] and [Be]. Finally we suggest an application to the construction of a 4-manifold with interesting homological properties.

**Definition 7.1.** A group $G$ is said to be a graph product if it is generated by a collection $G_1, \ldots, G_l$ of finite subgroups, such that for each $i \neq j$ the subgroup generated by $G_i$ and $G_j$ is either the free product $G_i \ast G_j$ or the direct product $G_i \times G_j$. In the special case when each $G_i$ is a group of order two, $G$ is a right angled Coxeter group. A subgroup of $G$ generated by a subset of the $G_i$ shall be called a special subgroup. Note that the trivial subgroup is defined to be special (generated by the empty set of $G_i$'s).

**Definition 7.2.** If $G$ is a graph product, then its Davis complex $D(G)$ is the simplicial complex associated to the poset whose elements are the left cosets of the finite special subgroups of $G$, ordered by inclusion.

Write $G(S)$ for the special subgroup of $G$ generated by the set of $G_i$ for $i$ an element of $S$. We may represent any $n$-simplex of $D(G)$ in the form $(g, S_0, S_1, \ldots, S_n)$, where $g$ is an element of $G$ and $S_0, \ldots, S_n$ is a strictly increasing chain of subsets of $\{1, \ldots, l\}$ such that $G(S_n)$ is finite. Another such symbol $(h, T_0, \ldots, T_n)$ represents the same simplex if and only if $T_i = S_i$ for all $i$ and $gG(S_0) = hG(S_0)$. The stabiliser of the simplex $(g, S_0, \ldots, S_n)$ is the subgroup $gG(S_0)g^{-1}$ of $G$, which is finite.

We shall not prove that the Davis complex is contractible, but we shall prove the
Proposition 7.3 (Davis). The Davis complex of a graph product is acyclic.

Proof. If for all pairs $i, j$ in $\{1, \ldots, l\}$ the subgroup of $G$ generated by $G_i$ and $G_j$ is $G_i \times G_j$, then $G$ is itself the direct product of all of the $G_i$’s, and hence finite, and so $D(G)$ is a cone with vertex the unique left coset of $G$ itself. If not then without loss of generality we may assume that $G_1$ and $G_l$ generate $G_1 \ast G_l$. Define subgroups $H$, $K$ and $L$ of $G$ by

$$H = G(\{1, \ldots, l-1\}), \quad K = G(\{2, \ldots, l\}), \quad L = G(\{2, \ldots, l-1\}).$$

Then $H \cap K = L$, and $G = H \ast_L K$. Consider now the subcomplex $X$ of $D(G)$ with simplices those $(g, S_0, \ldots, S_n)$ such that $l \notin S_n$. By fixing a transversal $T$ to $H$ and expressing $g = th$ for some $t \in T$ and $h \in H$, we see that $X$ is isomorphic to a disjoint union of copies of $D(H)$, indexed by $T$. Note however that the action of $H$ on the $D(H)$ corresponding to $t$ is twisted by conjugation by $t$. Since inductively $D(H)$ may be assumed to be acyclic we see that $X$ has no homology except in degree zero, and that $H_0(X)$ is isomorphic as a $\mathbb{Z}G$-module to $\mathbb{Z}G/H$ (the permutation module with basis the left cosets of $H$). Similarly, if we define $Y$ and $Z$ by

$$Y = \{(g, S_0, \ldots, S_n)|1 \notin S_n\}, \quad Z = \{(g, S_0, \ldots, S_n)|1, l \notin S_n\},$$

then these subcomplexes have only degree zero homology, and as $\mathbb{Z}G$-modules

$$H_0(Y) \cong \mathbb{Z}G/K, \quad H_0(Z) \cong \mathbb{Z}G/L.$$

Any simplex $(g, S_0, \ldots, S_n)$ of $D(G)$ must be contained in either $X$ or $Y$ since $S_n$ cannot contain both 1 and $l$, and the intersection of $X$ and $Y$ is $Z$. The Mayer-Vietoris sequence for $D(G)$ implies that $H_i(D(G)) = 0$ for $i > 1$, and gives

$$0 \rightarrow H_1(D(G)) \rightarrow \mathbb{Z}G/L \xrightarrow{\eta} \mathbb{Z}G/H \oplus \mathbb{Z}G/K \rightarrow H_0(D(G)) \rightarrow 0.$$

The map $\eta$ is often studied in the proof of the Mayer-Vietoris theorem for group cohomology (see [Br]), and it is known that $\eta$ is injective and has cokernel $\mathbb{Z}$. ■
Corollary 7.4. If $H$ is a torsion-free subgroup of a graph product $G$ (for example the kernel of the map from $G$ to the direct product of the $G_i$’s), then the chain complex for $D(G)$ is an $H$-free resolution for $\mathbb{Z}$. The chain complex for $D(G)$ with rational coefficients is a $\mathbb{Q}G$-projective resolution for $\mathbb{Q}$.

Proof. The chain complex for $D(G)$ is a resolution for the trivial module by Proposition 7.3. The modules that occur in this resolution are permutation modules with basis $G/K$ for various finite subgroups $K$. If $H$ is torsion-free, then it cannot meet such a $K$, so acts freely. The module $RG/K$ is projective provided that the order of $K$ is invertible in $R$. ■

Before stating any further theorems it is useful to introduce another simplicial complex $K(G)$ associated to the graph product $G$. $K(G)$ has as simplices the subsets $S$ of $\{1,\ldots,l\}$ such that $G(S)$ is finite. Equivalently, $K(G)$ has vertices $\{1,\ldots,l\}$, edges the pairs $i$ and $j$ such that $G_i$ and $G_j$ commute, and is the full simplicial complex on its 1-skeleton. (Recall that a simplicial complex is said to be full if whenever its 1-skeleton contains a complete graph on $n$ vertices then the complex contains an $(n-1)$-simplex whose boundary is that graph.) The barycentric subdivision of any simplicial complex is full. Given a (finite) full simplicial complex $K$, and a finite group $G_v$ for each vertex $v$ of $K$, we may define a graph product $G$ with $K(G) = K$, by taking the free product of all the $G_v$’s and adding the relations that $G_v$ and $G_w$ commute if $(v,w)$ is an edge of $K$.

Proposition 7.5. The Davis complex $D(G)$ may be obtained by taking a free $G$-orbit of copies of the cone on the barycentric subdivision of $K(G)$, and identifying together parts of their boundaries as described below.

Proof. By the boundary of a cone, we mean the simplices not containing the cone point. Simplices of $K(G)$ correspond to non-empty subsets $S$ of $\{1,\ldots,l\}$ such that $G(S)$ is finite, so simplices of its barycentric subdivision correspond to ascending chains $(S_0,\ldots,S_n)$ of non-empty subsets such that $G(S_n)$ is finite. We may add a cone point to this by al-
lowing our chains to contain the empty set. Now we see that simplices of $G \times CK(G)'$ are representable uniquely by symbols $(g, S_0, \ldots, S_n)$, where $S_0, \ldots, S_n$ is a chain of subsets with $G(S_n)$ finite. Putting the equivalence relation on these symbols described after Definition 7·2 is equivalent to identifying parts of the boundaries of the cones. ■

From now onwards, we shall only consider the case when $G$ is a right-angled Coxeter group.

**Theorem 7·6 (Bestvina).** Let $X$ be the topological space obtained by attaching a disc $D^2$ to a circle $S^1$ using the map $z \mapsto z^n$ from $S^1$ to $S^1$. Let $G$ be any right-angled Coxeter group such that $K(G) \cong X$, and let $H$ be a torsion-free subgroup of $G$ of finite index. Then $H$ has cohomological dimension three, but $H^3(H; M)$ has exponent dividing $n$ for all modules $M$. Also $H^3(H; \mathbb{Z}H)$ is cyclic of order $n$.

**Proof.** By Corollary 7·4, the Davis complex provides a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ of length three. Using Proposition 7·3 we shall study this resolution. The rest of this proof is very similar to a ‘bare hands’ proof that $H^2(X)$ is cyclic of order $n$. Firstly, orient $C(K(G)')$ (the cone on the barycentric subdivision of $K(G)$) by coherently orienting the cone on a disc having the same 2-simplices as $K(G)'$, and then identifying the sides of the cone. The third stage $(C_3)$ of the resolution consists of a direct sum of copies of $\mathbb{Z}G$, one for each 2-simplex of $K(G)'$. The second stage $(C_2)$ of the resolution consists of a direct sum of three sorts of pieces. Firstly, copies of $\mathbb{Z}G$ corresponding to interior 2-simplices of $C(K(G)')$ containing the cone point. The map from $C_3$ to each of these components will be non-zero on exactly two of $C_3$’s summands, and will be the identity on one and minus the identity on the other. Secondly, copies of $\mathbb{Z}G$ corresponding to 2-simplices of $C(K(G)')$ containing the cone point and an edge of the disc. The map from $C_3$ to each of these components will be the identity map on $n$ of $C_3$’s summands, and zero on the rest. Finally, for each 2-simplex $\sigma$, a copy of $\mathbb{Z}G/L$, where $L$ is the minimal vertex of $\sigma$ (necessarily a subgroup of order two). The map from $C_3$ to this component will be
projection on one of $C_3$’s summands, and zero on the rest.

Now apply $\text{Hom}_{ZH}(-, M)$ to this resolution, and consider the ‘adjoint map’ from $C^2$ to $C^3$. $C^3$ consists of a direct sum of copies of $N = \text{Hom}_{ZH}(ZG, M)$, and $C^2$ consists of a direct sum of three types of module, the first and second types being isomorphic to $N$. The map from a summand of the first type to $C^3$ is of the form

$$m \mapsto (0, \ldots, 0, m, 0, \ldots, 0, -m, 0, \ldots, 0),$$

and since the interior of the disc is connected the images of the first type summands suffice to identify all the copies of $N$ that comprise $C^3$. The map from a second type summand to this quotient $N$ is multiplication by $n$, and we see that $H^3(H; M)$ has exponent dividing $n$.

Now specialise to the case when $M = ZH$, so that $N \cong ZG$. Already by quotienting $C^3$ by the image of first and second type summands of $C^2$ we have shown that $H^3$ is a quotient of $RG$, where $R = Z/(n)$. A summand of $C^3$ of the third type has the form $ZG/(g_i)$, where the $g_i$ are the Coxeter generators for $G$, and the effect of quotienting by these is that

$$H^3(H; ZH) \cong RG/(\sum_i (1 + g_i)RG).$$

Since $G$ is presented by the $g_i$ subject only to relations involving even powers of each $g_i$ we see that the subgroup $G'$ of the group of units of $RG$ generated by $-g_1, \ldots, -g_l$ is isomorphic to $G$ and $RG' = RG$. Now $H^3$ is just $RG'$ modulo its augmentation ideal, or $R$.

Next we shall look for conditions for $D(G)$ to be a combinatorial manifold.

**Lemma 7.7.** The link of a simplex of $D(G)$ is isomorphic to a suspension of the link of a simplex of $K(G)'$, except that the 0-simplex $(g, \emptyset)$ has link isomorphic to $K(G)'$.

**Proof.** The link of the simplex $(g, S_0, S_1, \ldots, S_n)$ is the join of various pieces, which we shall examine separately. Firstly there is the complex corresponding to the poset of special
cosets strictly contained in $gG(S_0)$. This is the barycentric subdivision of the Coxeter complex of $G(S_0)$, that is the subdivision of the $m$ dimensional analogue of the octahedron (where $S_0$ contains $m+1$ elements), so is homeomorphic to $S^m$ except when $S_0 = \emptyset$, when it is empty. Secondly, there is the complex corresponding to the poset of special cosets lying strictly between $gG(S_i)$ and $gG(S_{i+1})$. If $S_{i+1} \setminus S_i$ contains $m$ elements then this complex is the barycentric subdivision of the surface of an $(m-1)$-simplex (or empty if $m = 1$)—this is because faces of an $(m-1)$-simplex correspond to non-empty subsets of its vertices. Thirdly, there is the complex corresponding to the poset of special cosets strictly containing $gG(S_n)$. This is isomorphic to the link in $K(G)'$ of any $|S_n|-1$-simplex with $S_n$ as its minimal vertex, except when $S_n = \emptyset$, in which case it is the whole of $K(G)'$. 

Corollary 7.8. $D(G)$ is a combinatorial $(n+1)$-manifold if and only if $K(G)$ is homeomorphic to $S^n$.

Proof. Keep track of the dimensions of the spheres that arise in the proof of the preceding lemma.

I. M. Chiswell has used the Davis complex to provide a formula for the Euler characteristic of a graph product [Ch], which we state here only in the case when $G$ is a right-angled Coxeter group.

Theorem 7.9 (Chiswell). If $G$ is a right-angled Coxeter group, then

$$\chi(G) = 1 - \frac{1}{2} \sum_{i \geq 0} \frac{n_i}{2^i},$$

where $n_i$ is the number of $i$-simplices in $K(G)$.

P. H. Kropholler has suggested combining Corollary 7.8 and Theorem 7.9 to try to exhibit a closed aspherical 4-manifold of negative Euler characteristic as follows. Given any full triangulation of the three sphere, form the corresponding right-angled Coxeter group $G$, and let $H$ be a torsion-free subgroup of finite index. $H$ acts freely and cocompactly on $D(G)$, so $D(G)/H$ is a closed manifold. The strong version of Davis’ theorem (namely
that $D(G)$ is contractible) implies that $D(G)/H$ is aspherical, and its Euler characteristic is $|G : H|$ times that of $G$. Unfortunately this seems not to work. For example:

**Proposition 7.10.** If $K$ is any triangulation of $S^3$ having $m_i$ $i$-simplices, and $G$ is the Coxeter group corresponding to the barycentric subdivision of $K$, then $\chi(G)$ is positive.

**Proof.** Find $n_i$ (the number of $i$-simplices of $K'$ in terms of the $m_j$, and substitute in to the formula of Theorem 7.9. It is helpful to note that the $n_i$ (and the $m_i$) satisfy the following relations:

$$n_0 - n_1 + n_2 - n_3 = 0 \quad 2n_3 = n_2.$$  

(The first of these comes from $\chi(S^3) = 0$, and the second from the fact that each 2-simplex in a 3-manifold bounds exactly two 3-simplices.

In the case of triangulations of $S^3$ arising from convex polytopes, the theory of toric varieties may be brought to bear on this question. G. K. Sankaran has pointed out that given a triangulation of $S^3$, the signature of the cup product as a quadratic form on $H^4$ of the corresponding toric variety is a positive multiple of the expression occuring in Theorem 7.9.
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