MEASURE TRANSFER AND $S$-ADIC DEVELOPMENTS FOR SUBSHIFTS

NICOLAS BÉDARIDE, ARNAUD HILION, AND MARTIN LUSTIG

Abstract. Based on previous work of the authors, to any $S$-adic development of a subshift $X$ a "directive sequence" of commutative diagrams is associated, which consists at every level $n \geq 0$ of the measure cone and the letter frequency cone of the level subshift $X_n$ associated canonically to the given $S$-adic development.

The issuing rich picture enables one to deduce results about $X$ with unexpected directness. For instance, we exhibit a large class of minimal subshifts with entropy zero that all have infinitely many ergodic probability measures.

As a side result we also exhibit, for any integer $d \geq 2$, an $S$-adic development of a minimal, aperiodic, uniquely ergodic subshift $X$, where all level alphabets $\mathcal{A}_n$ have cardinality $d$, while none of the $d-2$ bottom level morphisms is recognizable in its level subshift $X_n \subset \mathcal{A}_n^\mathbb{Z}$.

1. Introduction

A subshift over a finite alphabet $\mathcal{A}$ is a non-empty, closed and shift-invariant subset $X \subset \mathcal{A}^\mathbb{Z}$. A very efficient tool to investigate such a subshift $X$ is given by an $S$-adic development of $X$: the latter is obtained by a directive sequence $\overline{\sigma}$ of monoid morphisms $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ for all integers $n \geq 0$, where each $\mathcal{A}_n$ is again a finite alphabet, and $\mathcal{A}_n^*$ denotes the free monoid over $\mathcal{A}_n$. The morphisms $\sigma_n$ here are all assumed to be non-erasing, i.e. none of the letters of $\mathcal{A}_{n+1}$ is mapped to the empty word. The directive sequence $\overline{\sigma}$ generates the given subshift $X$ if for some identification $\mathcal{A} = \mathcal{A}_0$ any finite factor $x_k \ldots x_\ell$ of any bi-infinite word $x = \ldots x_{-1}x_0x_1\ldots \in X$ is also a factor of some $\sigma_0 \circ \ldots \circ \sigma_{n-1}(a_i)$ with $a_i \in \mathcal{A}_n$, and conversely: any such $x$ belongs to $X$. One usually also assumes that $\overline{\sigma}$ is everywhere growing, which means that $\liminf(\min\{|\sigma_0 \circ \ldots \circ \sigma_{n-1}(a_i)| \mid a_i \in \mathcal{A}_n\}) = \infty$.

It is well known that any subshift $X \subset \mathcal{A}^\mathbb{Z}$ is generated by some everywhere growing directive sequence $\overline{\sigma}$.

A directive sequence $\overline{\sigma}$ as above determines at every level $n \geq 0$ a level subshift $X_n \subset \mathcal{A}_n^\mathbb{Z}$, which is the subshift generated by the truncated sequence $\overline{\sigma}^\downarrow_n$, obtained from $\overline{\sigma}$ through forgetting all levels $k < n$ and the corresponding level morphisms. It is a straightforward observation that every level morphism $\sigma_n$ induces a map $X_{n+1} \to X_n$ which is surjective on shift-orbits.

More generally, any non-erasing morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ between free monoids over finite alphabets $\mathcal{A}$ and $\mathcal{B}$ respectively, defines for any subshift $X \subset \mathcal{A}^\mathbb{Z}$ an image subshift $\sigma(X)$, and it is natural to ask which properties of $X$ are inherited (under suitable hypotheses) by the image subshift $\sigma(X)$. In our cousin paper [5] we have formally introduced and studied, for any such morphism $\sigma$, a measure transfer map $\sigma_\mathcal{M}^\mathcal{M} : \mathcal{M}(X) \to \mathcal{M}(\sigma(X))$, where $\mathcal{M}(X)$ denotes the measure cone on $X$, i.e. the set of all shift-invariant Borel measures on the subshift $X$. The map $\sigma_\mathcal{M}^\mathcal{M}$ is the restriction/co-restriction of a map $\sigma_\mathcal{M} : \mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z})$ which is linear, functorial and commutes with the support map on subshifts (see section 3.1).
We thus obtain canonically, for any everywhere growing directive sequence \( \overline{\sigma} = (\sigma_n: A^*_{n+1} \rightarrow A^*_{n+1})_{n \geq 0} \) as above, an induced sequence \( \mathcal{M}(\overline{\sigma}) = (\sigma_n^\mathcal{M}: \mathcal{M}(X_{n+1}) \rightarrow \mathcal{M}(X_n))_{n \geq 0} \) of linear maps \( \sigma_n^\mathcal{M} := \sigma_{X_{n+1}} \) on the measure cones \( \mathcal{M}(X_{n+1}) \).

Furthermore, any invariant measure \( \mu \) on a subshift \( X \subseteq A^\mathbb{Z} \) defines canonically a letter frequency vector \( \vec{v}(\mu) \) in the non-negative cone \( \mathbb{R}^A_{\geq 0} \) of the vector space \( \mathbb{R}^A \), where for each letter \( a_i \in A \) the coordinate of \( \vec{v}(\mu) \) is given by the measure \( \mu([a_i]) \) of the cylinder \([a_i]\). The latter consists of all biinfinite words \( x \in A^\mathbb{Z} \) as above for which the letter with index 1 satisfies \( x_1 = a_i \). The cone of all such letter frequency vectors is denoted by \( C(X) \subseteq \mathbb{R}^A_{\geq 0} \); it gives rise to a canonical linear evaluation map \( \zeta_X: \mathcal{M}(X) \rightarrow C(X) \) which by definition is surjective.

It has been shown in [5] that the linear map \( \mathbb{R}^A \rightarrow \mathbb{R}^B \), defined by the incidence matrix \( M(\sigma) \) of any non-erasing free monoid morphism \( \sigma: A^* \rightarrow B^* \), commutes via the evaluation maps \( \zeta_{A^\mathbb{Z}} \) and \( \zeta_{B^\mathbb{Z}} \) with the measure transfer map \( \sigma^\mathcal{M} \). We thus obtain, for any directive sequence \( \overline{\sigma} \) as above, a rather useful commutative diagram:

\[
\begin{array}{ccccccc}
\ldots & \xrightarrow{\sigma_n^\mathcal{M}^{-1}} & \mathcal{M}(X_{n+1}) & \xrightarrow{\sigma_n^\mathcal{M}} & \mathcal{M}(X_n) & \xrightarrow{\sigma_n^\mathcal{M}^{-1}} & \ldots & \xrightarrow{\sigma_1^\mathcal{M}} & \mathcal{M}(X_1) & \xrightarrow{\sigma_1^\mathcal{M}} & \mathcal{M}(X) \\
\downarrow \zeta_{X_{n+1}} & & \downarrow \zeta_{X_n} & & \downarrow \zeta_{X_1} & & \downarrow \zeta_X \\
\ldots & \xrightarrow{M(\sigma_n+1)} & C(X_{n+1}) & \xrightarrow{M(\sigma_n)} & C(X_n) & \xrightarrow{M(\sigma_{n+1})} & \ldots & \xrightarrow{M(\sigma_2)} & C(X_1) & \xrightarrow{M(\sigma_1)} & C(X)
\end{array}
\]

A measure tower \( \overline{\mu} = (\mu_n)_{n \geq 0} \) on a directive sequence \( \overline{\sigma} \) as above, defined by postulating \( \mu_n \in \mathcal{M}(X_n) \) and \( \sigma_n^\mathcal{M}(\mu_{n+1}) = \mu_n \), defines a tower of letter frequency vectors \( \vec{v}(\overline{\mu}) = (\vec{v}(\mu_n))_{n \geq 0} \) which satisfy \( M(\sigma_{n+1}) \cdot \vec{v}(\mu_{n+1}) = \vec{v}(\mu_n) \). This last equality had been used in [3] as defining equality for what was called there a vector tower over the directive sequence \( \overline{\sigma} \). A linear evaluation map \( \mathbf{m}: \mathcal{V}(\overline{\sigma}) \rightarrow \mathcal{M}(X) \), from the set \( \mathcal{V}(\overline{\sigma}) \) of all such vector towers, to the measure cone \( \mathcal{M}(X) \) of the subshift \( X \) generated by \( \overline{\sigma} \), has been established in [3], and the map \( \mathbf{m} \) is shown in [3] to be always surjective, as long as \( \overline{\sigma} \) is everywhere growing (but no other hypotheses are needed). We obtain:

**Proposition 1.1.** For any everywhere growing directive sequence \( \overline{\sigma} \) there is a canonical linear bijection between the cone \( \mathcal{V}(\overline{\sigma}) \) of vector towers and the cone \( \mathcal{M}(\overline{\sigma}) \) of measure towers on \( \overline{\sigma} \), given by the letter frequency map

\[
\overline{\nu} = (\nu_n)_{n \geq 0} \mapsto \overline{\nu} = (\vec{v}(\mu_n))_{n \geq 0} ,
\]

with \( \vec{v}_n = \vec{v}(\mu_n) = (\mu_n([a_k]))_{a_k \in A} \) for all levels \( n \geq 0 \).

Based on the main result of our previous paper [3] (quoted below as Theorem 2.7) we derive from this set-up the following consequence (see Proposition 4.4):

**Theorem 1.2.** For any non-erasing monoid morphism \( \sigma: A^* \rightarrow B^* \) and any subshift \( X \subseteq A^\mathbb{Z} \) the induced measure transfer map \( \sigma^\mathcal{M} \) maps the measure cone \( \mathcal{M}(X) \) of \( X \) surjectively to the measure cone \( \mathcal{M}(\sigma(X)) \) of the image subshift \( \sigma(X) \):

\[
\sigma^\mathcal{M}(\mathcal{M}(X)) = \mathcal{M}(\sigma(X))
\]

This general surjectivity result for the measure transfer map \( \sigma^\mathcal{M} \) is mirrored in the special case where \( \sigma \) is recognizable in \( X \) (see Definition 3.5) by the following fact, proved below in Corollary 3.9:

**Proposition 1.3.** If a non-erasing morphism \( \sigma: A^* \rightarrow B^* \) is recognizable in a subshift \( X \subseteq A^\mathbb{Z} \), then the measure transfer map \( \sigma_X^\mathcal{M}: \mathcal{M}(X) \rightarrow \mathcal{M}(\sigma(X)) \) is injective.
We apply this injectivity result to any directive sequence $\sigma = (\sigma_n)_{n \geq 0}$, where each level map $\sigma_n$ is assumed to be recognizable in the corresponding level subshift $X_n$. Such totally recognizable directive sequences (or slight variations of it) have recently received a lot of attention (see for instance [1], [9], [13], [17]), and they are shown to play a central role in the S-adic approach to symbolic dynamics. We obtain (see Theorem 5.7):

**Theorem 1.4.** For any totally recognizable everywhere growing directive sequence $\sigma$, with generated subshift $X = X_{\sigma}$, the linear surjective map of cones

$$m : V(\sigma) \to M(X)$$

is a bijection.

We combine this result with a construction from our earlier paper [4], where for any integer $d \geq 2$ a subshift $X$ with $d$ distinct invariant ergodic probability measures has been shown to exist, while $X$ is defined by an everywhere growing directive sequence with level alphabets $A_n$ that all have cardinality $\text{card}(A_n) = d$. This construction is used in section 7 below to define a large “diagonal” family $\mathcal{X}$ of directive sequences $\sigma$ and to give a quick proof (see Theorem 7.4) that they all generate subshifts $X_{\sigma}$ which have a remarkable property, exhibited first by a quite different and more elaborate method for very particular subshifts in a recent paper by Cyr-Kra (see [11]):

**Corollary 1.5.** For any directive sequence $\sigma \in \mathcal{X}$ the subshift $X_{\sigma}$ is minimal, has topological entropy $h_{X_{\sigma}} = 0$ and admits infinitely many distinct ergodic probability measures in $M(X_{\sigma})$.

The directive sequences considered in the last corollary are all totally recognizable, and they are “large”, in that their alphabet rank, i.e. the limit inferior of the cardinality of the level alphabets, is infinite. For finite alphabet rank, on the other hand, the condition “totally recognizable” can be replaced by a distinctly weaker condition: in this case, the linear map defined by the incidence matrix $M(\sigma_n)$ is for any sufficiently high level $n \geq 0$ a fortiori (from the surjectivity result in Theorem 1.2) injective on the subspace spanned by the cone $M(X_n)$. In the special – but rather frequent – case that this injectivity property of the $M(\sigma_n)$ is also true for all low levels, the bijectivity of the map $m$ as in Theorem 1.4 above is a direct consequence of our set-up. We thus obtain (see Corollary 6.3):

**Corollary 1.6.** Let $X \subseteq \mathcal{A}^\mathbb{Z}$ be a subshift generated by an everywhere growing directive sequence $\sigma = (\sigma_n)_{n \geq 0}$ of finite alphabet rank. Assume that for any $n \geq 0$ the incidence matrix $M(\sigma_n)$ is invertible over $\mathbb{R}$. Then any invariant measure $\mu$ on the subshift $X$ is determined by the letter frequency vector associated to $\mu$, i.e. by the values $\mu([a_k])$ for all $a_k \in \mathcal{A}$.

This generalizes a result of [6], obtained under additional hypotheses by very different methods.

A slightly more general situation than considered in Theorem 1.4, which deserves some particular interest, occurs if the given directive sequence is only eventually recognizable, i.e. only for sufficiently high levels one assumes that the level morphisms are recognizable in the corresponding level subshift. In section 8 we investigate non-recognizable morphisms, and in particular we show in Corollary 8.5 the following result, which is somewhat surprising, in view of the claims in [13] and [17] (see Remark 8.7).

**Proposition 1.7.** For any integer $n_0 \geq 0$ there exists an everywhere growing directive sequence $\sigma = (\sigma_n)_{n \geq 0}$ with the following properties:

1. For any $n \geq n_0$ the level alphabets satisfy $A_n = A_{n_0}$ and the level morphisms are stationary: $\sigma_n = \sigma_{n_0}$. Furthermore, each level morphism $\sigma_n$ is recognizable in the level subshift $X_{n+1}$.
2. For any level $n$ with $0 \leq n \leq n_0 - 1$ we have $\text{card}(A_n) = n + 2 = \text{card}(A_{n+1}) - 1$, and none of the level morphisms $\sigma_n$ is recognizable in the level subshift $X_{n+1}$.
3. All level subshifts $X_n$ are minimal, uniquely ergodic, and aperiodic.
(In fact, each level subshift $X_n$ is actually an interval exchange subshift, obtained from the stable lamination of a pseudo-Anosov homeomorphism on a suitably punctured surface.)

Acknowledgements: The authors would like to thank Fabien Durand and Samuel Petite for encouraging remarks and interesting comments.

2. Terminology, notation, conventions and some quotes

In this section we first recall some standard terminology from symbolic dynamics (see subsection 2.1), then summarize the notation introduced in [5] and some of its results (see subsection 3.1), and in subsection 2.2 we recall some classical $S$-adic terminology and quote the main result from [3], which plays a key role later in this paper.

2.1. Standard terminology from symbolic dynamics.

Throughout this paper we denote by $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$ non-empty finite sets, called alphabets, and by $\mathcal{A}^*, \mathcal{B}^*$ or $\mathcal{C}^*$ the free monoid over those alphabets. Every element $w \in \mathcal{A}^*$ is a word in the letters $a_1, a_2, \ldots, a_d$ of $\mathcal{A}$, i.e.

$$w = x_1x_2\ldots x_n \quad \text{with} \quad x_i \in \{a_1, a_2, \ldots, a_d\} = \mathcal{A}$$

for any $i = 1, \ldots, n$, and the empty word is denoted by $\varepsilon$. Here $n$ is the length of $w$, denoted by $|w|$, and one sets $|\varepsilon| = 0$. We immediately verify the formula $|w| = \sum_{a_j \in \mathcal{A}} |w|_{a_j}$, where $|w|_{a_j}$ denotes the number of occurrences of the letter $a_j$ in $w$. More generally, for any second word $u \in \mathcal{A}^*$ we denote by $|w|_u$ the number of (possibly overlapping) occurrences of $u$ as subword $x_k \ldots x_{\ell}$ (also called a factor) of $w$.

Any monoid morphism $\sigma: \mathcal{A}^* \to \mathcal{B}^*$ is determined by the family of letter images $\sigma(a_i) \in \mathcal{B}^*$ for all $a_i \in \mathcal{A}$, and this family can be chosen freely. Such a morphism $\sigma$ is non-erasing if $|\sigma(a_i)| \geq 1$ for all $a_i \in \mathcal{A}$. All morphisms considered in this paper will be non-erasing. Note that any composition of non-erasing morphisms is non-erasing.

Every monoid morphism $\sigma: \mathcal{A}^* \to \mathcal{B}^*$ induces canonically a linear map $\mathbb{R}_{\geq 0}^\mathcal{A} \to \mathbb{R}_{\geq 0}^\mathcal{B}$, given by the incidence matrix

$$M(\sigma) = (|\sigma(a_j)|_{b_i})_{b_i \in \mathcal{B}, a_j \in \mathcal{A}}. \quad (2.1)$$

To any alphabet $\mathcal{A}$ there is also associated the full shift $\mathcal{A}^\mathbb{Z}$; its elements are written as biinfinite words

$$x = \ldots x_{i-1}x_ix_{i+1} \ldots$$

with $x_i \in \mathcal{A}$ for any index $i \in \mathbb{Z}$. The set $\mathcal{A}^\mathbb{Z}$ is naturally equipped with the product topology (with respect to the discrete topology on $\mathcal{A}$), and $\mathcal{A}^\mathbb{Z}$ is a Cantor set unless $\text{card}(\mathcal{A}) = 1$. Furthermore, the space $\mathcal{A}^\mathbb{Z}$ comes naturally with a shift-operator $T$, defined for any $x$ as in (2.2) by $T(x) = \ldots y_{i-1}y_iy_{i+1} \ldots$ with $y_i = x_{i+1}$ for any $i \in \mathbb{Z}$. The shift-operator acts as homeomorphism on the space $\mathcal{A}^\mathbb{Z}$; for convenience it will always be denoted by the symbol $T$, independently of the choice of the given alphabet $\mathcal{A}$.

For any integers $k \leq l$ we denote by $x_{[k,l]}$ the subword (again also called factor) $x_k \ldots x_{\ell}$ of the biinfinite word $x$ as in (2.2). We also consider the one-sided infinite positive half-word $x_{(1,\infty)} = x_1x_2 \ldots$ of $x$.

To any word $w \in \mathcal{A}^*$ there is associated the cylinder $[w] \subseteq \mathcal{A}^\mathbb{Z}$, which consists of all words $x \in \mathcal{A}^\mathbb{Z}$ which satisfy $x_{[1,|w|]} = w$. If $w$ is the empty word, then $[w] = \mathcal{A}^\mathbb{Z}$. The set of all cylinders $[w]$ together with their shift translates $T^m([w])$ for any $m \in \mathbb{Z}$ constitute a basis for the above specified topology of the space $\mathcal{A}^\mathbb{Z}$.
A non-empty subset \( X \subseteq \mathbb{A}^{\mathbb{Z}} \) is a subshift if \( X \) is closed and if \( T(X) = X \). A subshift \( X \) is minimal if none of its subsets is a subshift except \( X \) itself. This is equivalent to the statement that for any \( x \in X \) the shift-orbit \( O(x) = \{ T^m(x) \mid m \in \mathbb{Z} \} \) is dense in \( X \). A minimal subshift \( X \) is either uncountably infinite or else it is finite: in this case \( X \) consists of the single shift-orbit \( X = O(w^{\pm \infty}) \) of some periodic word \( w^{\pm \infty} = \ldots www \ldots \), which is well defined for any non-empty \( w \in \mathbb{A}^* \) by the convention \( w^{\pm \infty}_{[1,\infty]} = www \ldots \). It follows that any infinite minimal subshift is in particular aperiodic, which means that \( X \) doesn’t contain any periodic word \( w^{\pm \infty} \).

Any subshift \( X \subseteq \mathbb{A}^{\mathbb{Z}} \) defines a language \( \mathcal{L}(X) \) which consists of all words \( w \in \mathbb{A}^* \) that occur as factor in some \( x \in X \). Conversely, every infinite subset \( \mathcal{L} \subseteq \mathbb{A}^* \) generates a subshift \( X(\mathcal{L}) \subseteq \mathbb{A}^{\mathbb{Z}} \), defined by the property that any word from \( \mathcal{L}(X) \) must occur as factor in some \( w' \in \mathcal{L} \).

For any subshift \( X \subseteq \mathbb{A}^{\mathbb{Z}} \) and any \( n \in \mathbb{N} \) one denotes by \( p_X(n) \) the number of words in \( \mathcal{L}(X) \) of length \( n \). The following limit is well defined and is known as topological entropy \( h_X \) of the subshift \( X \):

\[
(2.3) \quad h_X = \lim_{n \to \infty} \frac{\log p(n)}{n}
\]

Any non-erasing monoid morphism \( \sigma : \mathbb{A}^* \to \mathbb{B}^* \) defines canonically a map

\[
(2.4) \quad \sigma^Z : \mathbb{A}^Z \to \mathbb{B}^Z
\]

where for any \( x \in \mathbb{A}^{\mathbb{Z}} \) the image \( y = \sigma^Z(x) \in \mathbb{B}^{\mathbb{Z}} \) is defined by extending \( \sigma \) first to the positive half-word \( x_{[1,\infty]} \) to define \( y_{[1,\infty]} \), and subsequently extending \( \sigma \) to all of \( x \).

For almost all subshifts \( X \subseteq \mathbb{A}^{\mathbb{Z}} \) the image set \( \sigma^Z(X) \) will not be shift-invariant and hence not be a subshift. However, there is a canonical image subshift \( \sigma(X) \) of \( X \), which admits several naturally equivalent definitions:

**Remark 2.1.** The following three definitions of the image subshift \( Y := \sigma(X) \) are equivalent, for any non-erasing monoid morphism \( \sigma : \mathbb{A}^* \to \mathbb{B}^* \):

1. \( Y \) is the intersection of all subshifts that contain the set \( \sigma^Z(X) \).
2. \( Y \) is the union of all shift-orbits \( O(\sigma(x)) \), for any \( x \in X \). [Note here (see Lemma 2.4 of [5]) that this union is always closed, a fact that a priori can not be taken for granted.]
3. \( Y \) is the subshift generated by the language \( \sigma(\mathcal{L}(X)) \). Thus \( Y \) consists of all biinfinite words \( y \in \mathbb{B}^{\mathbb{Z}} \) with the property that every factor of \( y \) is also a factor of some word in \( \sigma(\mathcal{L}(X)) \).

We observe directly the following consequence:

**Lemma 2.2.** Let \( \sigma : \mathbb{A}^* \to \mathbb{B}^* \) be a non-erasing monoid morphism, and let \( X \subseteq \mathbb{A}^{\mathbb{Z}} \) be any subshift. If \( \mathcal{L} \subseteq \mathbb{A}^* \) is a language that generates \( X \), then \( \sigma(\mathcal{L}) \) generates \( \sigma(X) \). \( \Box \)

An invariant measure on \( \mathbb{A}^{\mathbb{Z}} \) is a finite Borel measure \( \mu \) on \( \mathbb{A}^{\mathbb{Z}} \) which is invariant under the homeomorphism \( T \) (= the shift operator). The set of all such invariant measures is denoted by \( \mathcal{M}(\mathbb{A}^{\mathbb{Z}}) \). For any subshift \( X \subseteq \mathbb{A}^{\mathbb{Z}} \) we denote by \( \mathcal{M}(X) \subseteq \mathcal{M}(\mathbb{A}^{\mathbb{Z}}) \) the set of those invariant measures \( \mu \) for which their support satisfies \( \text{Supp}(\mu) \subseteq X \). For notational convenience we identify any such \( \mu \) with its restriction to \( X \).

Any invariant measure \( \mu \in \mathcal{M}(\mathbb{A}^{\mathbb{Z}}) \) defines a function

\[
\mathbb{A}^* \to \mathbb{R}_{\geq 0} , \ w \mapsto \mu([w])
\]

which for convenience is also denoted by \( \mu \), yielding \( \mu(w) = \mu([w]) \) for any \( w \in \mathbb{A}^* \). This function is a weight function in that it satisfies the Kirchhoff equalities

\[
(2.5) \quad \mu(w) = \sum_{a_i \in A} \mu(a_i w) = \sum_{a_i \in A} \mu(wa_i)
\]

5
for any \( w \in \mathcal{A}^* \). Conversely, it is well known that any weight function \( \mu : \mathcal{A}^* \to \mathbb{R}_{\geq 0} \) defines an invariant measure \( \mu \in \mathcal{M}(\mathcal{A}^Z) \) which satisfies \( \mu([w]) = \mu(w) \). The set \( \mathcal{M}(\mathcal{A}^Z) \) can hence be understood as subset of the infinite dimensional non-negative cone \( \mathbb{R}^{\mathcal{A}^*}_{\geq 0} = \{ \sum_{w \in \mathcal{A}^*} x_w \vec{e}_w \mid x_w \geq 0 \} \), from which it inherits the product topology; the latter coincides with the more generally known weak* -topology on the measure cone \( \mathcal{M}(\mathcal{A}^Z) \).

A measure \( \mu \in \mathcal{M}(\mathcal{A}^Z) \) is a probability measure if its total mass satisfies \( \mu(\mathcal{A}^Z) = 1 \). A measure \( \mu \in \mathcal{M}(\mathcal{A}^Z) \) is ergodic if \( \mu \) can not be written as linear combination with positive coefficients of two distinct probability measures. For any subshift \( X \subseteq \mathcal{A}^Z \) the number \( e(X) \) of ergodic probability measures in \( \mathcal{M}(X) \) can be finite or infinite; it is equal to the dimension of the linear convex cone \( \mathcal{M}(X) \subseteq \mathbb{R}^{\mathcal{A}^*}_{\geq 0} \). For any subshift \( X \subseteq \mathcal{A}^Z \) we have \( e(X) \geq 1 \); if \( e(X) = 1 \) the subshift \( X \) is called uniquely ergodic.

The support \( \text{Supp}(\mu) \) of any \( \mu \in \mathcal{A}^Z \) is always a subshift \( X \subseteq \mathcal{A}^Z \); if \( \mu \) is ergodic, then \( X = \text{Supp}(\mu) \) is a minimal subshift. The converse conclusion doesn’t hold (see section 7 below).

Any non-trivial word \( w \in \mathcal{A}^* \) defines a characteristic measure \( \mu_w \in \mathcal{M}(\mathcal{A}^Z) \): if \( w \) is not a proper power, then \( \mu_w \) is given by

\[
(2.6) \quad \mu_w(B) := \text{card}(B \cap \mathcal{O}(w^{\pm \infty}))
\]

for any measurable set \( B \subseteq \mathcal{A}^Z \). If on the other hand \( w = u^m \) for some \( u \in \mathcal{A}^* \) and some integer \( m \geq 2 \), where \( u \) is assumed not to be a proper power, then one has

\[
(2.7) \quad \mu_w := m \cdot \mu_u
\]

In either case, it follows that \( \frac{1}{|w|} \mu_w \) is a probability measure. The set of weighted characteristic measures \( \lambda \mu_w \) (for any \( \lambda > 0 \)) is known to be dense in \( \mathcal{M}(\mathcal{A}^Z) \). The support of any characteristic measure is given by

\[
(2.8) \quad \text{Supp}(\mu_w) = \mathcal{O}(w^{\pm \infty}).
\]

To any alphabet \( \mathcal{A} \) one associates canonically the non-negative alphabet cone \( \mathbb{R}^{\mathcal{A}}_{\geq 0} = \{ \sum_{a_k \in \mathcal{A}} x_k \vec{e}_{a_k} \mid x_k \geq 0 \} \). For any invariant measure \( \mu \) on \( \mathcal{A}^Z \) the evaluation on the letter cylinders \( [a_k] \) for all \( a_k \in \mathcal{A} \) defines a letter frequency vector

\[
(2.9) \quad \vec{v}(\mu) := \sum_{a_k \in \mathcal{A}} \mu([a_k]) \vec{e}_{a_k},
\]

so that one has a canonical \( \mathbb{R}_{\geq 0} \)-linear map of cones, denoted by

\[
(2.10) \quad \zeta_{\mathcal{A}} : \mathcal{M}(\mathcal{A}^Z) \to \mathbb{R}^{\mathcal{A}}_{\geq 0}, \mu \mapsto \vec{v}(\mu).
\]

For any subshift \( X \subseteq \mathcal{A}^Z \) the restriction of this map to \( \mathcal{M}(X) \) will be denoted by \( \zeta_X \). The image of this map is a cone, denoted by

\[
(2.11) \quad C(X) := \zeta_X(\mathcal{M}(X)) \subseteq \mathbb{R}^{\mathcal{A}}_{\geq 0},
\]

and called the letter frequency cone of the subshift \( X \). For simplicity we will below, for any subshift \( X \subseteq \mathcal{A}^Z \) and any morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \), use the symbol \( M(\sigma) \) to denote all three linear maps defined by the incidence matrix of the morphism \( \sigma \).

More details about these basic facts and some references can be found in section 2 of [5].
2. Measures on subshifts via vector towers on directive sequences.

In order to state Theorem 2.7 below, which is the main purpose of this subsection, we first recall some standard notation that is also used later.

A **directive sequence** $\overline{\sigma} = (\sigma_n)_{n \geq 0}$ consists of level morphisms

\[(2.12) \quad \sigma_n : A^*_n \to A^*_n\]

for any level $n \geq 0$, where each $A_n$ is a finite non-empty set, called the **level $n$ alphabet**. We sometimes use the less formal but more suggestive notation

$\overline{\sigma} = \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \ldots$

to denote a directive sequence.

For any integers $m > n \geq 0$ we define the **telescoped level morphism**

$\sigma_{[n,m]} := \sigma_n \circ \sigma_{n+1} \circ \ldots \circ \sigma_{m-1}$

as well as the level $n$ **truncated** directive sequence

\[(2.13) \quad \overline{\sigma} |_n = (\sigma_k)_{k \geq n}.
\]

Any directive sequence $\overline{\sigma}$ as in (2.12) above generates a subshift $X = X_{\overline{\sigma}}$ over the base alphabet $A_0$, defined by the convention that $x \in A^*_0$ belongs to $X$ if and only if for any finite factor $w$ of $x$ there exists some level $n \geq 1$ and some letter $a_j \in A_n$ such that $w$ is also a factor of $\sigma_{[0,n-1]}(a_j)$.

For any level $n \geq 0$ a directive sequence $\overline{\sigma}$ as above defines an **intermediate level subshift** $X_n \subseteq A_n$ which is generated by the truncated sequence $\overline{\sigma} |_n$:

\[(2.14) \quad X_n := X_{\overline{\sigma} |_n}.
\]

The subshift $X_n$ is the image subshift of the analogously defined level $n + 1$ intermediate subshift $X_{n+1}$ under the morphism $\sigma_n$, i.e.:\n
\[(2.15) \quad X_n = \sigma_n(X_{n+1}) \quad \text{for any level} \quad n \geq 0\]

In this paper we will almost exclusively consider directive sequences which are **everywhere growing**, by which we mean that the sequence of **minimal level letter image lengths**

\[(2.16) \quad \beta_-(n) := \min \{|\sigma_{[0,n-1]}(a_j)| \mid a_j \in A_n\}\]

tends to $\infty$ for $n \to \infty$. We have (see for instance Proposition 5.10 of [3]):

**Fact 2.3.** For every subshift $X \subseteq A^*$ there exists an everywhere growing directive sequence $\overline{\sigma}$ that generates $X$. More precisely, using the notation from (2.12), one has

\[(2.17) \quad A_0 = A \quad \text{and} \quad X_{\overline{\sigma}} = X.\]

**Remark 2.4.** If a directive sequence $\overline{\sigma} = (\sigma_n)_{n \geq 0}$ is everywhere growing, it could well be that some of the telescoped level maps $\sigma_{[n,m]} : A^*_m \to A^*_n$ maps a generator $a_i \in A_{m+1}$ to the empty word in $A^*_n$. In this case, since the equivalences from Remark 2.1 could fail to hold (see Example 5.1 of [2]), it is advisable to admit as “generated subshift” $X_{\overline{\sigma}}$ only those $x \in A^*_0$ which can be lifted to any level of $\overline{\sigma}$.

With this alteration a simple direct argument shows that quotienting out any such “eventually erased” letter yields new alphabets $A'_n \subseteq A_n$ and new level maps $\sigma'_n : A'^*_n \to A'^*_n$ such that the issuing directive sequence $\overline{\sigma}' = (\sigma'_n)_{n \geq 0}$ is everywhere growing and generates the same subshift as the original sequence $\overline{\sigma}$. In addition, any of the morphisms $\sigma'_n$ or $\sigma'_{[n,m]}$ is non-erasing.

We can hence (and will from now on quietly) assume that any everywhere growing directive sequence consists of non-erasing level maps only.
Remark 2.5. (1) A directive sequence $\sigma$ that generates a subshift $X$ is also called an $S$-adic development (or $S$-adic expansion) of $X$, where $S$ stands sometimes for an (often assumed to be finite) set of substitutions which contains all level morphisms. This concept and in particular the terminology “$S$-adic” has been introduced by S. Ferenczi in [19]. In this context one often assumes that the sequence $\sigma$ has finite alphabet rank. By this we mean that there is a uniform upper bound to the cardinality of any level alphabet, so that we can identify all level alphabets with a single finite alphabet $A$.

(2) If the set $S$ consists of a single endomorphism $\sigma : A^* \to A^*$, then the $S$-adic subshift $X$, which is generated by the stationary directive sequence $\sigma = (\sigma_n)_{n \geq 0}$ with $\sigma_n = \sigma$ for all $n \geq 0$, is called substitutive. It is important to note that we require here the substitution $\sigma$ (or rather: the above stationary directive sequence $\sigma$) to be everywhere growing. The term “substitution” itself is often used synonymous to “endomorphism of a free monoid”, but sometimes (varying) additional conditions are imposed (see for instance [16]).

(3) A very convenient criterion to ensure the condition “everywhere growing” is given as follows:

Let $\sigma = (\sigma_n)_{n \geq 0}$ be a directive sequence, and assume that for every level $n \geq 0$ there exists a level $m > n$ such that the telescoped incidence matrix $M(\sigma_{[n,m]})$ is positive (i.e. it has all coefficients $> 0$).

One verifies easily that any directive sequence $\sigma$ which satisfies the criterion (2) is indeed everywhere growing.

(4) The criterion (2) has another important consequence, namely that the subshift $X$ generated by $\sigma$ is minimal. For this conclusion we cite Theorem 5.3 of [7], proved originally in [14].

In [3] to any directive sequence $\sigma$ as in (2.12) there has been associated the set $\mathcal{V}(\sigma)$ of vector towers $\vec{v} = (\vec{v}_n)_{n \geq 0}$ over $\sigma$. Such a vector tower$^1$ consists of non-negative vectors

$$\vec{v}_n = \sum_{a_j \in A} \vec{v}_n(a_j) e_{a_j} \in \mathbb{R}_{\geq 0}^{|A|}$$

that are subject to the compatibility condition

$$\vec{v}_n = M(\sigma_n) \cdot \vec{v}_{n+1}$$

for all $n \geq 0$. It has been shown (see [3], Remark 9.5) that for any word $w \in A_0^*$ and any such vector tower $\vec{v}$ the sequence of sums

$$\sum_{a_j \in A_n} \vec{v}_n(a_j) |\sigma_{[0,n]}(a_j)|_w$$

is bounded above and increasing, as long as $\sigma$ is everywhere growing (but no other condition is needed). This gives:

Remark 2.6. (1) The value

$$\mu_{\vec{v}}(w) := \lim_{n \to \infty} \sum_{a_j \in A_n} \vec{v}_n(a_j) |\sigma_{[0,n]}(a_j)|_w$$

is well defined for any $w \in A_0^*$. It is shown in [3], Propositions 7.4 and 9.4, that the issuing function $\mu_{\vec{v}} : A_0^* \to \mathbb{R}_{\geq 0}$ satisfies the Kirchhoff equalities (2.5), so that we can summarize:

(2) Any vector tower $\vec{v}$ on an everywhere growing subshift $X$ generated by $\sigma$, denoted by $\mu_{\vec{v}} \in \mathcal{M}(X)$.

$^1$The terminological specification $\sigma$-compatible vector tower used in [4] has been dropped here, as all “vector towers” occurring in the present paper satisfy the compatibility condition (2.19) for any $n \geq 0$. 

8
In terms of \( S \)-adic language, the main result of \([3]\) translates directly into the following (see also section 3 of \([4]\)):

**Theorem 2.7** ([3]). Let \( \sigma = (\sigma_n)_{n \geq 0} \) be an everywhere growing directive sequence which generates the subshift \( X := X_\sigma \). Then the map

\[
m_\sigma : \mathcal{V}(\sigma) \to \mathcal{M}(X), \quad \bar{v} \mapsto \mu^{\bar{v}}
\]

is linear and surjective.

For any of the level alphabets \( A_n \) of a directive sequence \( \sigma \) as above we consider the projection map \(^2\) of the set of vector towers to the corresponding non-negative alphabet cone:

\[
pr_n : \mathcal{V}(\sigma) \to \mathbb{R}_{\geq 0}^A, \quad \bar{v} = (\bar{v}_n)_{n \geq 0} \mapsto \bar{v}_n
\]

On the base level \( n = 0 \) this projection splits over the evaluation map \( \zeta_{A_0} \) from (2.9) via the map \( m_\sigma \) from the last theorem. More precisely, this gives (see \([3]\), Proposition 10.2 (1) and (2)):

**Proposition 2.8.** For any subshift \( X \subseteq \mathcal{A}^\mathbb{Z} \), generated by an everywhere growing directive sequence \( \sigma \) as in (2.12) and (2.17), one has:

1. The map \( \zeta_X : \mathcal{M}(X) \to \mathbb{R}_{\geq 0}^A, \mu \mapsto (\mu([a_k])_{a_k \in A} \) satisfies:

\[
pr_0 = \zeta_X \circ m_\sigma
\]

2. In particular, for the letter frequency cone \( \mathcal{C}(X) = \text{im}(\zeta_X) \) (see (2.10)) this gives:

\[
\mathcal{C}(X) = \zeta_X(m_\sigma(\mathcal{V}(\sigma)))
\]

3. Alternatively, the letter frequency cone is obtained as nested intersection as follows:

\[
\mathcal{C}(X) := \bigcap_{n \geq 1} M(\sigma_{[0,n]})(\mathbb{R}_{\geq 0}^A)
\]

4. In particular, \( \dim \mathcal{C}(X) \) is a lower bound to the number \( e(X) \) of distinct ergodic probability measures on \( X \). \( \square \)

The following statement is the translation of Remark 9.2 (3) of \([3]\) into the terminology used here.

**Lemma 2.9.** For any vector tower \( \bar{v} = (\bar{v}_n)_{n \geq 0} \) over an everywhere growing directive sequence \( \sigma \) as in (2.12) one has

\[
\lim_{n \to \infty} \sum_{a_j \in A_n} \bar{v}_n(a_j) = 0,
\]

where the coefficient \( \bar{v}_n(a_j) \in \mathbb{R}_{\geq 0} \) is defined in equality (2.18). \( \square \)

3. The Measure Transfer and Its Injectivity for Recognizable Morphisms

In this section we will first recall the definition of the measure transfer map and quote some basic properties derived in \([5]\) (see subsection 3.1 below), then recall the definition and some related properties of recognizable morphisms (see subsection 3.2 below), and in subsection 3.3 we will derive the injectivity result from the title of this section.

---

\(^2\) The map \( pr_n \) was denoted in \([3]\) and \([4]\) by \( m_n \), but we decided to reserve this notation here for the more telling maps introduced below in section 5.
3.1. The measure transfer and some results from [5]. For any non-erasing monoid morphism \( \sigma : A^* \to B^* \) we define the subdivision alphabet \( A_\sigma = \{ a_i(k) \mid a_i \in A \text{ and } 1 \leq k \leq |\sigma(a_i)| \} \). The morphism \( \sigma \) now defines a subdivision morphism \( \pi_\sigma : A^* \to A^*_\sigma \) and a letter-to-letter morphism \( \alpha_\sigma : A^*_\sigma \to B^* \), given for any \( a_i \in A \) and any \( a_i(k) \in A_\sigma \) by:

\[
\pi_\sigma(a_i) = a_i(1) a_i(2) \ldots a_i(|\sigma(a_i)|) \quad \text{and} \quad \alpha_\sigma(a_i(k)) = [\sigma(a_i)]_k
\]

Here by \([\sigma(a_i)]_k\) we mean the \( k \)-th letter of the word \( \sigma(a_i) \in B^* \). We obtain directly:

**Fact 3.1.** For any non-erasing monoid morphism \( \sigma : A^* \to B^* \) one has:

\[
\sigma = \alpha_\sigma \circ \pi_\sigma
\]

For any word \( w \in A^*_\sigma \) we denote by \( \hat{w} \in A^* \) the shortest word such that \( \pi_\sigma(\hat{w}) \) contains \( w \) as factor. If such \( \hat{w} \) exists, it is unique; otherwise we treat \( \hat{w} \) as formal symbol and we set

\[
(3.1) \quad \mu(\hat{w}) = 0
\]

for any \( \mu \in \mathcal{M}(A^\mathbb{Z}) \).

For any measure \( \mu \in \mathcal{M}(A^\mathbb{Z}) \) a measure \( \mu^{\pi_\sigma} \in \mathcal{M}(A^*_\sigma) \) is defined in section 3.1 of [5] by setting \( \mu^{\pi_\sigma}([w]) = \mu([\hat{w}]) \), where \([\hat{w}]\) is the cylinder associated to the word \( \hat{w} \) (see subsection 2.1). On the other hand, for any measure \( \mu' \in \mathcal{M}(A^*_\sigma) \) the classical push-forward measure \( (\alpha_\sigma)_\#(\mu') \) is an invariant measure on \( B^\mathbb{Z} \), since \( \alpha_\sigma \) is letter-to-letter. We thus obtain (see [5], section 3):

**Theorem 3.2.** Let \( \sigma : A^* \to B^* \) be a non-erasing morphism of free monoids.

1. For any invariant measure \( \mu \) on \( A^\mathbb{Z} \) an invariant measure \( \mu^\sigma \) on \( B^\mathbb{Z} \) is given by

\[
\mu^\sigma = (\alpha_\sigma)_\#(\mu^{\pi_\sigma});
\]

2. For any word \( w' \in B^\mathbb{Z} \) the “transferred measure” \( \mu^\sigma \) takes on the cylinder \([w']\) the value

\[
\mu^\sigma([w']) = \sum_{w_1 \in \alpha_\sigma^{-1}(w')} \mu([\hat{w}_1]).
\]

3. The issuing measure transfer map

\[
\sigma^\mathcal{M} : \mathcal{M}(A^\mathbb{Z}) \to \mathcal{M}(B^\mathbb{Z}), \ \mu \mapsto \mu^\sigma
\]

induced by the morphism \( \sigma \) has the following properties:

3a) The map \( \sigma^\mathcal{M} \) is linear (over \( \mathbb{R} \)) and continuous (with respect to the weak*-topology).

3b) The map \( \sigma^\mathcal{M} \) is functorial.

3c) If \( X \) is the support of \( \mu \), then \( \sigma(X) \) is the support of \( \mu^\sigma \). Hence \( \sigma^\mathcal{M} \) induces in particular on any subshift \( X \subseteq A^\mathbb{Z} \) a restriction/co-restriction map

\[
\sigma^\mathcal{M}_X : \mathcal{M}(X) \to \mathcal{M}(\sigma(X)).
\]

\[ \square \]

We also list the following more technical properties derived in [5]:

**Proposition 3.3.** Let \( \sigma : A^* \to B^* \) be a non-erasing free monoid morphism, and let \( \sigma^\mathcal{M} \) be the induced transfer map on the measure cones. Let \( \mu \in \mathcal{M}(A^\mathbb{Z}) \) be an invariant measure on the full shift \( A^\mathbb{Z} \), and denote as before by \( \mu^\sigma = \sigma^\mathcal{M}(\mu) \) the transferred measure on \( B^\mathbb{Z} \). Then one has:

(a) The total mass of the transferred measure \( \mu^\sigma \) is given by the formula

\[
\mu^\sigma(B^\mathbb{Z}) = \sum_{a_i \in A} \sum_{b_j \in B} |\sigma(a_k)|_{b_j} \cdot \mu(a_k).
\]

In particular, if \( \mu \) is a probability measure, then in general \( \mu^\sigma \) will not be probability.
(b) For any generator \( b_j \in B \) we have:
\[
\mu^\sigma([b_j]) = \sum_{a_k \in A} |\sigma(a_k)| b_j \cdot \mu(a_k)
\]
In particular, for the letter frequency vectors from (2.8) we obtain:
\[
(3.2) \quad \bar{v}(\mu^\sigma) = M(\sigma) \cdot \bar{v}(\mu)
\]
In other words (see Proposition 4.5 of [5]), the measure transfer map \( \sigma^M \) commutes via the evaluation maps \( \zeta_A \) and \( \zeta_B \) from (2.9) with the linear map induced by \( \sigma \) on the non-negative cone \( \mathbb{R}_{\geq 0}^A \):
\[
\zeta_B \circ \sigma^M = M(\sigma) \circ \zeta_A
\]
(c) For any \( w \in A^* \) the cylinder measures satisfy:
\[
\mu^\sigma([\sigma(w)]) \geq \mu([w])
\]
(d) For any word \( w \in A^* \) the characteristic measure \( \mu_w \) satisfies:
\[
\sigma^M(\mu_w) = \mu_{\sigma(w)}
\]
\[\square\]
It remains to quote a useful evaluation technique for the transferred measure, derived in section 4 of [5] from what is stated above as part (2) of Theorem 3.2. For this purpose we define for any non-erasing morphism \( \sigma : A^* \rightarrow B^* \) and any \( w \in A^*, u \in B^* \) the number \( |\sigma(w)|_u \) of essential occurrences of \( u \) in \( \sigma(w) \), by which we mean that the first letter of \( u \) occurs in the \( \sigma \)-image of first letter of \( w \), and the last letter of \( u \) occurs in the \( \sigma \)-image of last letter of \( w \). By \( \langle \sigma \rangle \) we denote the smallest length of any of the letter images \( \sigma(a_i) \).

Proposition 3.4. ([5], Proposition 4.2) Let \( \sigma : A^* \rightarrow B^* \) be any non-erasing monoid morphism, and let \( \mu \in \mathcal{M}(A^Z) \). Then for any \( w' \in B^* \) with \( |w'| \geq 2 \) the transferred measure \( \mu^\sigma = \sigma^M(\mu) \) takes on the cylinder \([w']\) the value
\[
\mu^\sigma([w']) = \sum_{\{w_j \in A^* \mid w_j \subseteq [w']_\sigma + 2\}} |\sigma(w_j)|_{w'} \cdot \mu([w_j])
\]
\[\square\]

3.2. Recognizable morphisms and some related properties.

The following notion has become more and more central to symbolic dynamics (see for instance [9], [13], [15] or [16]):

Definition 3.5. Let \( \sigma : A^* \rightarrow B^* \) be a non-erasing morphism, and let \( X \subseteq A^Z \) be a subshift over \( A \). Then \( \sigma \) is said to be recognizable in \( X \) if the following conclusion is true:

Consider biinfinite words \( x, x' \in X \subseteq A^Z \) and \( y \in B^Z \) which satisfy:

\((*)\) \( y = T^k(\sigma^Z(x)) \) and \( y = T^\ell(\sigma^Z(x')) \) for some integers \( k, \ell \) which satisfy \( 0 \leq k \leq |\sigma(x_1)| - 1 \) and \( 0 \leq \ell \leq |\sigma(x'_1)| - 1 \), where \( x_1 \) and \( x'_1 \) are the first letters of the positive half-words \( x_{[1, \infty)} = x_1 x_2 \ldots \) of \( x \) and \( x'_{[1, \infty)} = x'_1 x'_2 \ldots \) of \( x' \) respectively.

Then one has \( x = x' \) and \( k = \ell \).

As we will see in the next subsection, recognizability in a subshift is much related to the following:

Definition 3.6 ([5], Section 5). For any non-erasing monoid morphism \( \sigma : A^* \rightarrow B^* \) and any subshift \( X \subseteq A^Z \) we define the following two properties:

(1) \( \sigma \) is shift-orbit injective in \( X \): Any \( x \) and \( y \) in \( X \) have images \( \sigma(x) \) and \( \sigma(y) \) in the same shift-orbit if and only \( x \) and \( y \) lie in a common shift-orbit.
(2) $\sigma$ is shift-period preserving in $X$: For any periodic biinfinite word $w^{\pm \infty} = \ldots w w \ldots \in X$ the word $w$ can be written as proper power if and only if $\sigma(w)$ can be written as proper power.

Here $w \in A^* \setminus \{\varepsilon\}$ is a proper power\(^3\) if $w = u^m$ for some $u \in A^*$ and some integer $m \geq 2$.

The following useful property is a direct consequence of the previous definition (see Lemma 5.2 of [5]).

**Lemma 3.7.** Let $\sigma_1 : A^* \to B^*$ and $\sigma_2 : B^* \to C^*$ be two non-erasing morphisms, and consider a subshift $X \subseteq A^\mathbb{Z}$ as well as its image subshift $Y = \sigma_1(X) \subseteq B^\mathbb{Z}$. Then we have:

1. The composed morphism $\sigma_2 \circ \sigma_1 : A^* \to C^*$ is shift-orbit injective in $X$ if and only if $\sigma_1$ is shift-orbit injective in $X$ and $\sigma_2$ is shift-orbit injective in $Y$.
2. The composed morphism $\sigma_2 \circ \sigma_1 : A^* \to C^*$ is shift-period preserving in $X$ if and only if $\sigma_1$ is shift-period preserving in $X$ and $\sigma_2$ is shift-period preserving in $Y$.

\[ \square \]

### 3.3. Injectivity of the Measure Transfer for Recognizable Morphisms.

Let $\sigma : A^* \to B^*$ be a non-erasing morphism of free monoids, and let $\pi_\sigma : A^* \to A_\sigma^*$ and $\alpha_\sigma : A_\sigma^* \to B^*$ be the canonical subdivision morphism and the induced letter-to-letter morphism associated to $\sigma$ which satisfy $\sigma = \alpha_\sigma \circ \pi_\sigma$ (see Fact 3.1). For any subshift $X \subseteq A^\mathbb{Z}$ we consider the image subshift $\pi_\sigma(X) \subseteq A_\sigma^\mathbb{Z}$ and the induced restriction/co-restriction

\[ \alpha_\sigma^X : \pi_\sigma(X) \to \sigma(X) \]

of the map $\alpha_\sigma^Z : A_\sigma^Z \to B^Z$ to $\pi_\sigma(X)$ and $\sigma(X)$ respectively.

**Proposition 3.8.** For any non-erasing morphism $\sigma : A^* \to B^*$ and any subshift $X \subseteq A^\mathbb{Z}$ the following statements are equivalent:

1. $\sigma$ is recognizable in $X$.
2. $\alpha_\sigma^X$ is an isomorphism of subshifts.
3. $\alpha_\sigma$ is shift-orbit injective and shift-period preserving in $\pi_\sigma(X)$.
4. $\sigma$ is shift-orbit injective and shift-period preserving in $X$.

**Proof.** We first note that by definition $\alpha_\sigma^X$ is continuous and surjective, so that claim (2) is equivalent to stating that $\alpha_\sigma^X$ is injective.

Next we observe that claim (1) is equivalent to stating that $\alpha_\sigma^X$ is recognizable in $\pi_\sigma(X)$. This is a direct consequence of the product decomposition $\sigma = \alpha_\sigma \circ \pi_\sigma$ from Fact 3.1 and of Lemma 3.5 of [9], since every subdivision morphism $\pi_\sigma$ is recognizable in the full shift, as follows directly from the definition of $\pi_\sigma$.

In order to show the equivalence (1) $\iff$ (2) we apply Definition 3.5 to the morphism $\alpha_\sigma$ and the subshift $\pi_\sigma(X)$: we observe that, since $|\alpha_\sigma(x)| = 1$ for any letter $x \in A_\sigma$, in the hypothesis (*) of Definition 3.5 the integers $k$ and $\ell$ are necessarily equal to 0. But in this case the conclusion $x = x'$ stated there amounts precisely to assuring that the map $\alpha_\sigma^Z$ is injective on $\pi_\sigma(X)$, or in other words, that $\alpha_\sigma^X$ is injective.

The equivalence (2) $\iff$ (3) is immediate, since any subshift-isomorphism preserves orbits and shift-periods, while conversely, any shift-orbit injective letter-to-letter morphism could only fail to be injective if on some periodic orbit the shift-period is not preserved.

Finally, the equivalence (3) $\iff$ (4) is a direct consequence of Lemma 3.7, since every subdivision morphism $\pi_\sigma$ is shift-orbit injective and shift-period preserving in the full shift (see Lemma 5.3 of [5]).

\[ \square \]

---

\(^3\)Elements in $A^*$ which are not a proper power are sometimes called “primitive”. However, since $A^*$ is canonically embedded into the free group $F(A)$, where the notion of “primitive elements” is classical, but has a different meaning, we believe it is better not to use this terminology for a different purpose.
Note that the equivalence of the statements (1) and (2) from Proposition 3.8 has already been observed in [16], Proposition 2.4.24. Indeed, Fabien Durand has suggested to us to use this equivalence in order to derive the following corollary. In the mean time we have obtained a result which is actually a bit stronger: it turns out (see Theorem 5.5 of [5]) that the hypothesis “shift-orbit injective” suffices to obtain the same conclusion as stated in Corollary 3.9 below, but the proof is much less direct.

**Corollary 3.9.** For any non-erasing morphism \( \sigma : A^* \to B^* \) and any subshift \( X \subseteq A^\mathbb{Z} \) the measure transfer map \( \sigma_X^M : \mu \to \mu^\sigma \) is injective if \( \sigma \) is recognizable in \( X \).

**Proof.** We decompose \( \sigma = \alpha_\sigma \circ \pi_\sigma \) as in Fact 3.1, so that from the functoriality of the measure transfer (see property (3b) of Theorem 3.2) we have \( \sigma_X^M = (\alpha_\sigma^X)^M \circ \pi_\sigma^M \). The injectivity of \( \pi_\sigma^M \) is immediate from the definition of a subdivision morphism (see Lemma 5.4 of [5]), and the injectivity of \( (\alpha_\sigma^X)^M \) is a direct consequence of Proposition 3.8 (2). \( \square \)

**Remark 3.10.** Consider any non-erasing morphism \( \sigma : A^* \to B^* \) and any subshift \( X \subseteq A^\mathbb{Z} \) with image subshift \( Y = \sigma(X) \subseteq B^\mathbb{Z} \).

1. Assume that the subshift \( Y \) contains a periodic word \( w_{\pm \infty} \) for some \( w \in B^* \setminus \{\varepsilon\} \), and that the morphism \( \sigma \) is shift–orbit injective. Then, in order for \( \sigma \) to be shift-period preserving in \( X \), a necessary condition is that at least one of the letters \( a_i \in A \) satisfies \(|\sigma(a_i)| \leq |w|\).

As a consequence, unless a given subshift \( Y \) is aperiodic, in any everywhere growing \( S \)-adic development of \( Y \) there will always be infinitely many level morphisms which are not recognizable in their corresponding level subshift.

2. This has sparked the following weakening of the notion of “recognizability” which has become recently very popular (see for instance [2]).

The morphism \( \sigma \) is said to be **recognizable for aperiodic points in \( X \)** if the conclusion in Definition 3.5 holds under the strengthened assumption that \( y \) is not a periodic word.

3. From the above proof of Proposition 3.8 we observe that the property “shift-orbit injective in \( X \)” implies the property “recognizable for aperiodic points in \( X \)”.

Indeed, since the subdivision morphism \( \pi_\sigma \) is always shift–orbit injective and shift-periodic preserving (and thus recognizable) in the full shift, the property “\( \sigma \) is recognizable for aperiodic points in \( X \)” is equivalent to “\( \alpha_\sigma^X \) is recognizable for aperiodic points in \( \pi_\sigma(X) \)”.

This in turn is equivalent to stating that every non-periodic word in \( \sigma(X) \) has precisely one preimage under the letter-to-letter map \( \alpha_\sigma \). But since we assume that \( \sigma \) and hence \( \alpha_\sigma^X \) is shift–orbit injective, two distinct such preimages must lie in the same shift–orbit, which implies that their image in \( \sigma(X) \) must be periodic.

### 4. The Measure Transfer via Vector Towers

In this section we will consider a subshift \( X \) given by means of a directive sequence, an invariant measures \( \mu \) on \( X \) given by means of a vector tower on this directive sequence, and a morphism \( \tau : X \to Y = \tau(X) \) which we use to build a new directive sequence for \( Y \) by simply adding \( \tau \) at the bottom to the given sequence. Then the given vector tower is naturally transferred to a new vector tower on the new directive sequence, and, as do all vector towers, it defines an invariant measure \( \mu' \) on the subshift \( Y \) generated by this new sequence. The main goal of this section is to show that the new measure \( \mu' \) is precisely the image of given measure \( \mu \) under the transfer map \( \tau^M \) induced by the morphism \( \tau \) (see Theorem 3.2 (1)).

For convenience we summarize the running hypotheses for this section as follows:

**Assumption 4.1.** Let \( \tau : A^* \to B^* \) be a non-erasing morphism of free monoids over finite alphabets \( A \) and \( B \), and let \( \bar{\sigma} = (\sigma_n)_{n \geq 0} \) be an everywhere growing directive sequence with base level alphabet \( A_0 = A \). Let \( X := X_{\bar{\sigma}} \subseteq A^\mathbb{Z} \) be the subshift generated by \( \bar{\sigma} \), and denote by \( Y := \tau(X) \) the image subshift of \( X \) given by the morphism \( \tau \) (see Remark 2.1).
Definition-Remark 4.2. Let \( \tau \) and \( \sigma = (\sigma_n)_{n \geq 0} \) as well as the subshifts \( X \) and \( Y \) be as in Assumption 4.1.

(1) We define a second “prolonged” directive sequence \( \tau^\tau = (\sigma'_n)_{n \geq 0} \) by setting \( \sigma'_n := \sigma_{n-1} \) for any level \( n \geq 1 \) and \( \sigma'_0 := \tau \), and observe from Lemma 2.2 that the subshift \( X_{\tau^\tau} \) generated by \( \tau^\tau \) agrees precisely with the \( \tau \)-image subshift \( Y = \tau(X) = B^\mathbb{Z} \).

(2) Consider now any vector tower \( \bar{\omega} = (\bar{\omega}_n)_{n \geq 0} \) over \( \tau \), and let \( \mu = m_\tau(\bar{\omega}) \) be the invariant measure on \( X \) associated to \( \bar{\omega} \) via Theorem 2.7. We obtain a prolonged vector tower \( \bar{\omega}^\tau = (\bar{\omega}'_n)_{n \geq 0} \) over \( \tau^\tau \) by setting \( \bar{\omega}'_n = \bar{\omega}_{n-1} \) for any level \( n \geq 1 \), and by setting \( \bar{\omega}'_0 := M(\tau) \cdot \bar{\omega}_0 \). Let \( \mu' \) be the associated measure on \( Y \), i.e.

\[
\mu' = m_{\tau^\tau}(\bar{\omega}^\tau).
\]

We can now link up the measure transfer map defined and studied in [5] with the technology of vector towers from our previous papers [3],[4]. The following will be the basis for all results presented in this paper:

Proposition 4.3. Let \( \tau, \sigma \) and \( X \) be as in Assumption 4.1, and let \( \bar{\omega} = (\bar{\omega}_n)_{n \geq 0} \) be a vector tower over \( \tau \), with associated invariant measure \( \mu = m_\tau(\bar{\omega}) \) on \( X \). Let \( \sigma^\tau = (\sigma'_n)_{n \geq 0} \), \( \bar{\omega}^\tau = (\bar{\omega}'_n)_{n \geq 0} \) and \( \mu' = m_{\tau^\tau}(\bar{\omega}^\tau) \) be as in Definition-Remark 4.2.

Then the measure transfer map \( \tau^M : M(A^\mathbb{Z}) \to M(B^\mathbb{Z}) \) induced by the morphism \( \tau \) satisfies:

\[
\mu' = \tau^M(\mu) \quad \mu' = \mu^\tau
\]

Proof. In this proof we will freely use the terminology from [5] as recalled in section 3.1.

For any word \( w' \in B^\mathbb{Z} \) we consider in the subdivision monoid \( A^* \) the subset \( W(w') \) of preimages \( w_i \) of \( w' \) under the induced letter-to-letter morphism \( \alpha_\tau : A^*_\rightarrow B^* \). For each \( w_i \in W(w') \) consider (as in the paragraph subsequent to Fact 3.1) the word \( \hat{w}_i \in A^* \) defined by the conditions that (a) its canonically subdivided image \( \pi_r(\hat{w}_i) \) contains \( w_i \), and that (b) the word \( \hat{w}_i \) is shortest among all words in \( A^* \) which satisfy (a). Recall from (3.1) that either \( \hat{w}_i \) exists and is unique, or else we formally set \( \mu(\hat{w}_i) = 0 \) for any \( \mu \in M(A^\mathbb{Z}) \). From Theorem 3.2 (2) we know that \( \mu^\tau(w') = \sum_{w_i \in W(w')} \mu(\hat{w}_i) \) (with \( \mu^\tau = \tau^M(\mu) \) as before).

For the purpose of using formula (2.20) we consider now the value of the approximating sum on its right hand side of this formula for any (large) level \( n-1 \), for each of the words \( \hat{w}_i \) and the given vector tower \( \bar{\omega} \), i.e. the term (see (2.18) for the notation)

\[
\sum_{a \in A_{n-1}} \tilde{\nu}_{n-1}(a) |\sigma_{[0,n-1]}(a)| \hat{w}_i.
\]

We sum up the results of (2.2) over all \( w_i \in W(w') \) to get

\[
\sum_{w_i \in W(w')} \sum_{a \in A_{n-1}} \tilde{\nu}_{n-1}(a) |\sigma_{[0,n-1]}(a)| \hat{w}_i
\]

and compare the obtained sum to the limit on the right hand side of (2.20), when this formula is applied to \( w' \) and to the vector tower \( \bar{\omega}^\tau \) over the prolonged directive sequence \( \tau^\tau \). The \( n \)-th term of this limit gives the sum

\[
\sum_{a \in A_{n-1}} \tilde{\nu}'_n(a) |\sigma'_{[0,n]}(a)| w'.
\]

When comparing the two sums (4.3) and (4.4) we keep in mind that according to the set-up from Definition-Remark 4.2 for any \( n \geq 1 \) we have \( \tilde{\nu}'_n(a) = \tilde{\nu}_{n-1}(a) \) for any \( a \in A_{n-1} \), as well as \( \sigma'_{[0,n]} = \tau \circ \sigma_{[0,n-1]} \).

14
We now notice that each occurrence of any of the \(\hat{w}_i\) in any of the image words \(\sigma_{[0,n-1]}(a)\) with \(a \in \mathcal{A}_{n-1}\) defines precisely an occurrence of \(w_i\) in \(\pi_r(\sigma_{[0,n-1]}(a))\), and thus an occurrence of \(w'\) in \(\alpha_r(\pi_r(\sigma_{[0,n-1]}(a))) = \tau(\sigma_{[0,n-1]}(a)) = \sigma'_{[0,n]}(a)\). Furthermore, two distinct occurrences of \(\hat{w}_i\) in some \(\sigma_{[0,n-1]}(a)\) define distinct occurrences of \(w_i\) in \(\pi_r(\sigma_{[0,n-1]}(a))\), and thus distinct occurrences of \(w'\) in \(\sigma'_{[0,n]}(a)\). The same is true for occurrences of distinct \(\hat{w}_i\) in \(\sigma_{[0,n-1]}(a)\). It follows (using the above recalled equality \(\bar{v}'_n = \bar{v}_{n-1}\)) that

\[
\sum_{w_i \in W_0(w')} \sum_{a \in \mathcal{A}_{n-1}} \bar{v}_{n-1}(a) |\sigma_{[0,n-1]}(a)|_{\hat{w}_i} \leq \sum_{a \in \mathcal{A}_{n-1}} \bar{v}'_n(a) |\sigma'_{[0,n]}(a)|_{w'}
\]

But the opposite inequality is also true, up to a constant \(K_n\) which only depends on \(\overline{\tau}\) and not on \(\overline{\tau}\):

\[
\sum_{a \in \mathcal{A}_{n-1}} \bar{v}'_n(a) |\sigma'_{[0,n]}(a)|_{w'} \leq \sum_{w_i \in W_0(w')} \sum_{a \in \mathcal{A}_{n-1}} \bar{v}_{n-1}(a) |\sigma_{[0,n-1]}(a)|_{\hat{w}_i} + K_n
\]

Indeed, any occurrence of \(w'\) in \(\sigma'_{[0,n]}(a)\) defines in a unique manner an occurrence of some \(w_i\) in \(\pi_r(\sigma_{[0,n-1]}(a))\). The latter defines (again uniquely) an occurrence of \(\hat{w}_i\) in \(\sigma_{[0,n-1]}(a)\), unless the corresponding occurrence of \(w_i\) in \(\pi_r(\sigma_{[0,n-1]}(a))\) takes place in a suffix or prefix of length bounded by the maximum \(m(w') \geq 0\) of all \(|\hat{w}_i|\). We hence deduce:

\[
K_n \leq 2m(w') \sum_{a \in \mathcal{A}_{n-1}} \bar{v}_{n-1}(a)
\]

It follows now from Lemma 2.9 that the right hand side of the last inequality tends to 0 for \(n \to \infty\), so that we obtain from (4.5) and (4.6) through the above definitions \(\mu = \mathfrak{m}_{\overline{\tau}}(\overline{\tau}) = \mu^\overline{\tau}\) and \(\mu' = \mathfrak{m}_{\overline{\tau}}(\overline{\tau}) = \mu^\overline{\tau}\) the desired result

\[
\mu^\overline{\tau}(w') = \sum_{w_i \in W_0(w')} \left( \lim_{n \to \infty} \sum_{a \in \mathcal{A}_{n-1}} \bar{v}_{n-1}(a) |\sigma_{[0,n-1]}(a)|_{\hat{w}_i} \right)
\]

\[
= \lim_{n \to \infty} \sum_{a \in \mathcal{A}_{n-1}} \bar{v}'_n(a) |\sigma'_{[0,n]}(a)|_{w'} = \mu'(w')
\]

for any \(w' \in \mathcal{B}^\ast\). \(\square\)

As a first application of the above shown “basic” Proposition 4.3 we derive:

**Proposition 4.4.** For any non-erasing morphism \(\sigma : \mathcal{A}^\ast \to \mathcal{B}^\ast\) and any subshift \(X \subseteq \mathcal{A}^\mathbb{Z}\) with image subshift \(\sigma(X)\) the induced measure transfer map

\[
\sigma^\mathcal{M} : \mathcal{M}(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}(\mathcal{B}^\mathbb{Z}), \mu \mapsto \mu^\sigma
\]

maps the measure cone \(\mathcal{M}(\mathcal{A}^\mathbb{Z})\) surjectively to the measure cone \(\mathcal{M}(\sigma(X))\).

**Proof.** We consider any everywhere growing directive sequence \(\overline{\tau}\) which generates \(X\); from Fact 2.3 we know that such \(\overline{\tau}\) exists for any subshift \(X\). By prolonging \(\overline{\tau}\) through the morphism \(\sigma\) as explained above in Definition-Remark 4.2 we obtain any everywhere growing directive sequence \(\overline{\tau}' := \overline{\tau}^\sigma\) which generates \(\sigma(X)\). We then apply Theorem 2.7 to obtain for any measure \(\mu' \in \mathcal{M}(\sigma(X))\) a vector tower \(\overline{\nu}'\) on \(\overline{\tau}'\) with \(\mathfrak{m}_{\overline{\tau}}(\overline{\tau}') = \mu\). Truncating now the last term of \(\overline{\nu}'\) gives a vector tower \(\overline{\nu}\) on \(\overline{\tau}\), which by Remark 2.6 (2) defines a measure \(\mu := \mathfrak{m}_{\overline{\tau}}(\overline{\nu})\) on \(X\). We can now apply Proposition 4.3 to obtain \(\mu' = \mu^\sigma \left[= \sigma^\mathcal{M}(\mu)\right]\). \(\square\)

**Remark 4.5.** We’d like to note that as a result of the material presented in this section we have now derived an alternative way how to understand the transferred measure \(\mu^\sigma = \sigma^\mathcal{M}(\mu) \in \mathcal{M}(\mathcal{B}^\mathbb{Z})\), for any non-erasing morphism \(\sigma : \mathcal{A}^\ast \to \mathcal{B}^\ast\) and any invariant measure \(\mu \in \mathcal{M}(\mathcal{A}^\mathbb{Z})\).
It turns out that in many circumstances the use of vector towers as presented here is more convenient when dealing with $\mu^\sigma$ in practice, compared to the definition as studied in sections 3 and 4 of [5], and also compared to the approximation method via weighted characteristic measures indicated in Remark 3.9 of [5].

5. Measure towers and vector towers

Throughout this section we will assume that

$$\overrightarrow{\sigma} = (\sigma_n : A_{n+1}^0 \to A_n^0)_{n \geq 0}$$

is an everywhere growing directive sequence which generates a subshift $X = X_0 \subseteq A_0^\mathbb{Z}$ (and where all level maps $\sigma_n$ are non-erasing, see Remark 2.4). As in (2.14) we denote for any level $k \geq 0$ by $X_k \subseteq A_k^\mathbb{Z}$ the intermediate subshift of level $k$, which is generated by the truncated sequence $\overrightarrow{\sigma} |_k = (\sigma_n : A_{n+1}^0 \to A_n^0)_{n \geq k}$ from (2.13).

**Definition 5.1.** A measure tower on $\overrightarrow{\sigma}$, denoted by $\overrightarrow{\mu} = (\mu_n)_{n \geq 0}$, is given by a sequence of measures $\mu_n \in \mathcal{M}(A_n^\mathbb{Z})$ which satisfy:

$$\mu_n = \sigma_n^\mathcal{M}(\mu_{n+1})$$

The set of measure towers on $\overrightarrow{\sigma}$ will be denoted by $\mathcal{M}(\overrightarrow{\sigma})$.

We will now construct a particular type of measure towers on a given directive sequence $\overrightarrow{\sigma}$ as above, starting from a vector tower $\overrightarrow{v} = (\overrightarrow{v}_n)_{n \geq 0}$ on $\overrightarrow{\sigma}$. We first observe that for any intermediate level $k \geq 0$ we obtain from $\overrightarrow{\sigma}$ via the truncated directive sequence $\overrightarrow{\sigma} |_k$ a “truncated evaluation map” $m_k := m_{\sigma_k} : \mathcal{V}(\overrightarrow{\sigma} |_k) \to \mathcal{M}(X_k)$. From the vector tower $\overrightarrow{v}$ we obtain similarly a “truncated” vector tower $\overrightarrow{v} |_k = (\overrightarrow{v}_n)_{n \geq k}$ on $\overrightarrow{\sigma} |_k$, which defines the corresponding shift-invariant “level $k$ measure”

$$\mu_k := m_k(\overrightarrow{v} |_k)$$

on the subshift $X_k \subseteq A_k^\mathbb{Z}$. From Proposition 4.3 we obtain directly that $\sigma_k^\mathcal{M}(\mu_{k+1}) = \mu_k$ for all $k \geq 0$, so that we have:

**Lemma 5.2.** For any vector tower $\overrightarrow{v}$ on an everywhere growing directive sequence $\overrightarrow{\sigma}$ the family of level $k$ measures $\mu_k$ as in (5.1), for all $k \geq 0$, defines a measure tower $\overrightarrow{m}(\overrightarrow{v}) := (\mu_k)_{k \geq 0}$ on $\overrightarrow{\sigma}$. □

**Definition-Remark 5.3.** Conversely, every measure tower $\overrightarrow{\mu} = (\mu_n)_{n \geq 0}$ on a directive sequence $\overrightarrow{\sigma}$ as above determines a vector tower $\overrightarrow{\zeta}(\overrightarrow{\mu}) = (\overrightarrow{\zeta}_n(\mu_n))_{n \geq 0}$ on $\overrightarrow{\sigma}$, given by the letter frequency vectors $\overrightarrow{\zeta}_n := \overrightarrow{v}(\mu_n) = \zeta_{X_n}(\mu_n)$ from (2.8) and (2.9). The fact that $\overrightarrow{\zeta}(\overrightarrow{\mu})$ is indeed a vector tower, i.e. that compatibility conditions (2.19) are satisfied, is a direct application of Proposition 4.5 of [5], stated above as equality (3.2).

The above set-up of measure towers and vector towers over a given directive sequence is very natural, and indeed, it turns out that the two are essentially equivalent. More precisely, we obtain:

**Proposition 5.4.** For any everywhere growing directive sequence $\overrightarrow{\sigma}$ there is a canonical $\mathbb{R}_{\geq 0}$-linear bijection

$$\overrightarrow{\zeta} : \mathcal{M}(\overrightarrow{\sigma}) \to \mathcal{V}(\overrightarrow{\sigma})$$

between the cone of measure towers on one hand and the cone of vector towers on the other, given by the map

$$\overrightarrow{\mu} \mapsto \overrightarrow{\zeta}(\overrightarrow{\mu})$$

and its inverse

$$\overrightarrow{v} \mapsto \overrightarrow{m}(\overrightarrow{v}).$$
Proof. The fact, that the composition $\bar{\zeta} \circ \bar{m}$ gives the identity on $\mathcal{V}(\bar{\sigma})$, follows directly from Proposition 2.8 (1), when applied to all truncated sequences $\sigma_k$ with $k \geq 0$. We obtain in particular that the map $\bar{m}$ is injective.

On the other hand, we can apply Theorem 2.7 to each of the truncated sequences $\sigma_k$ to obtain the surjectivity of the map $m_k : \mathcal{V}(\bar{\sigma}_k) \to \mathcal{M}(X_k)$ for any level $k \geq 0$. It follows then directly from the definition set up in Lemma 5.2 above that the map $\bar{m} : \mathcal{V}(\bar{\sigma}) \to \mathcal{M}(\bar{\sigma})$ must be surjective.

Hence $\bar{m}$ is a bijective map, which implies that $\bar{\zeta}$ must also be bijective, and that $\bar{m} \circ \bar{\zeta}$ is the identity on $\mathcal{M}(\bar{\sigma})$.

The linearity of the maps $\bar{\zeta}$ and $\bar{m}$ is a direct consequence of the linearity (see section 2.2) of the maps $m_k$ and $\zeta_X$ used in the above definitions of the measure or vector towers $\bar{m}(\bar{v}) = (m_k(\sigma_k))_{k \geq 0}$ and $\bar{\zeta}(\bar{\mu}) = (\zeta_X(\mu_n))_{n \geq 0}$ respectively. \hfill $\square$

Although a bit similar in notation, the two cones $\mathcal{M}(\bar{\sigma})$ and $\mathcal{M}(X_\sigma)$ should not be confused. Indeed, without further assumptions on the given set-up, the structure of the cone $\mathcal{M}(\bar{\sigma})$ of measure towers will not only depend on the given subshift $X = X_\sigma$ but can vary quite a bit depending on the choice of the $S$-adic development $\sigma$ of $X$. More precisely, we have:

Remark 5.5. For any everywhere growing directive sequence $\sigma$ which generates a subshift $X = X_\sigma$ the composition

$$(5.2) \quad m_\sigma \circ \bar{\zeta} : \mathcal{M}(\bar{\sigma}) \to \mathcal{M}(X)$$

is linear and surjective since $\bar{\zeta}$ is linear and bijective by Proposition 5.4, and $m_\sigma$ is linear and surjective by Theorem 2.7. However, in general the map $m_\sigma \circ \bar{\zeta}$ will be far from being injective.

We thus consider the following strengthening on the hypotheses of the given directive sequence, which has been considered already by several other authors in a related context (compare Definition 4.1 of [9] or subsection 3.3 of [13]):

Definition 5.6. A directive sequence (or an $S$-adic development) $\sigma = (\sigma_n)_{n \geq 0}$ is called totally recognizable if every level map $\sigma_n$ is recognizable in the corresponding subshift $X_{n+1}$ (see Definition 3.5 and Proposition 3.6). If all but finitely many of the level maps $\sigma_n$ are recognizable in $X_{n+1}$, we call $\sigma$ eventually recognizable.

Theorem 5.7. For any everywhere growing totally recognizable $S$-adic development $\sigma$ of a subshift $X$ and its associated cone $\mathcal{V}(\bar{\sigma})$ of vector towers the canonical linear map

$$m_\sigma : \mathcal{V}(\bar{\sigma}) \to \mathcal{M}(X)$$

is a bijection.

In particular, for any level $n \geq 0$ the map $\sigma_n^M : \mathcal{M}(X_n) \to \mathcal{M}(X)$ is a linear bijection of cones. Similarly, the same conclusion follows for the map $m_\sigma \circ \bar{\zeta}$ from (5.2).

Proof. From the assumption that $\bar{\sigma}$ is totally recognizable it follows (using statement (3d) of Theorem 3.2) that the induced linear map

$$(\sigma_n)^M_{X_{n+1}} : \mathcal{M}(X_{n+1}) \to \mathcal{M}(X_n)$$

is bijective for any level $n \geq 0$. It follows that the composed map $m_\sigma \circ \bar{\zeta} : \mathcal{M}(\bar{\sigma}) \to \mathcal{V}(\bar{\sigma}) \to \mathcal{M}(X)$ from (5.2) is bijective. Since we know from Proposition 5.4 that the map $\bar{\zeta}$ is a bijection, we deduce that $m_\sigma$ must be bijective. \hfill $\square$
6. Directive sequences with “small” intermediate letter frequency cones

In this section we will give a first application of the machinery set up in the previous two sections. But before doing so we want to summarize, for the convenience of the reader, the various ingredients that the rich picture issuing from this set-up offers, and to list some basic facts in order to avoid potential misunderstandings. As an illustration, we give at the end of this section a detailed example, where all the data listed now can be seen in practice.

We use the same terminology as previously, i.e. \( X \in A^\mathbb{Z} \) is a subshift over the finite alphabet \( A = A_0 \), and \( \sigma = (\sigma_n : A_{n+1}^* \rightarrow A_n^*)_{n \geq 0} \) is an everywhere growing directive sequence which generates \( X \). We are particularly interested in the intermediate letter frequency cones \( C_n = C_n(\sigma) \subseteq \mathbb{R}_{\geq 0}^A \) and in particular in their dimension

\[
c_n := \dim C_n \leq \text{card}(A_n) .
\]

The cone \( C_n \) is given as the image of the set \( \mathcal{V}(\sigma) \) of vector towers under the level \( n \) projection map \( pr_n \), which amounts to stating that \( C_n \) is the intersection of the nested images of the non-negative alphabet cones of level \( m \geq n \) under the telescoped level maps, i.e.

\[
C_n = \bigcap \{ \mathbb{R}_{\geq 0}^A \supseteq \ldots \supseteq M(\sigma_{[n,m]})(\mathbb{R}_{\geq 0}^{A_{m+1}}) \supseteq \ldots \}
\]

In particular one always has

\[
(6.1) \quad C_n = M(\sigma_n)(C_{n+1})
\]

and thus

\[
c_n \leq c_{n+1}
\]

for all \( n \geq 0 \).

An alternative interpretation of the intermediate level frequency cones is given through the level subshifts \( X_n \) (defined by the truncated directive sequences \( \sigma\uparrow_n \)) and their measure cones \( \mathcal{M}(X_n) \), which, when evaluated via the associated letter frequency vectors, results into

\[
C_n = \zeta_{X_n}(\mathcal{M}(X_n)),
\]

where \( \zeta_{X_n} : \mathcal{M}(X_n) \rightarrow \mathbb{R}_{\geq 0}^A \) is given for \( A_n = \{a_{n,1}, \ldots , a_{n,d(n)}\} \) by \( \mu \mapsto ([\mu(a_{n,1})], \ldots , [\mu(a_{n,d(n)})]) \) for any \( \mu \in \mathcal{M}(X_n) \).

Our main focus here is to explain how this set-up and in particular the value of the \( c_n \) can be used in order to find out information about the number \( e(X) \in \mathbb{N} \cup \{ \infty \} \) of invariant ergodic probability measures on \( X \).

**Remark 6.1.** Under the above stated conditions the following conclusions are immediate:

1. It is quite possible that \( e(X) > c_n \) for some “low” level \( n \geq 0 \), even if \( \sigma \) is totally recognizable.
2. The converse inequality, \( e(X) < c_n \), is also possible, but in this case the directive sequence \( \sigma \) is not totally recognizable. More precisely, in this case the telescoped morphism \( \sigma_{[0,n]} \) is not recognizable.
3. In any case, we always have

\[
e(X) \leq \lim c_n \leq \lim \text{inf}(\text{card} A_n),
\]

but in general both inequalities may well be strict.
4. However, if \( \sigma \) is totally recognizable, then we have

\[
e(X) = \lim c_n
\]

In particular, we recover the well known upper bound \( e(X) \leq \lim \text{inf}(\text{card} A_n) \), as well as the lower bounds \( c_n \leq e(X) \) for all \( n \geq 0 \).
From Remark 6.1 (3) we observe directly that for any directive sequence $\sigma$ with finite alphabet rank (i.e. $\lim \inf (\text{card } A_n) < \infty$) there is a critical level $n_0 \geq 0$ such that one has

$$c_n = c_{n_0} \text{ for all } n \geq n_0 \text{ and } c_n < c_{n_0} \text{ for all } n < n_0.$$  \hspace{1cm} (6.2)

More generally, any everywhere growing directive sequence $\sigma$ (possibly with infinite alphabet rank) which possesses such a critical level has been termed in [3] thinning, and in the particular case where the critical level agrees with the base level $n_0 = 0$, the sequence $\sigma$ has been called thin. Of course, any thinning sequence can be made thin by simply truncating it at its critical level (or any level higher up); furthermore, we can telescope all levels below the critical level into a single “thinning” morphism. Subshifts that are “thin” in that they are generated by a thin (and in particular everywhere growing) directive sequence have the following useful property:

**Proposition 6.2** ([3]). Let $X \subseteq A^\mathbb{Z}$ be a subshift generated by a thin directive sequence $\sigma$. Then the letter frequency map $\zeta_X : M(X) \to \mathbb{R}_{\geq 0}^A$ co-restricts to a linear bijection

$$M(X) \to C(X), \, \mu \mapsto (\mu(a_k))_{a_k \in A}.$$  

In particular, any two invariant measures $\mu_1$ and $\mu_2$ on $X$ are equal if and only if one has $\mu_1([a_k]) = \mu_2([a_k])$ for the finitely many cylinders $[a_k]$ given by all letters $a_k \in A$.

This statement can be derived directly from Proposition 10.2 (1) and Corollary 10.4 of [3]. For convenience of the reader we give here a proof in the terminology introduced above.

**Proof of Proposition 6.2.** For any two measure $\mu, \mu' \in M(X)$ there exist, by Theorem 2.7, vector towers $\overline{v} = (\overline{v}_n)_{n \geq 0}$ and $\overline{v}' = (\overline{v}'_n)_{n \geq 0}$ on $\overline{v}$ with $m_\sigma(\overline{v}) = \mu$ and $m_\sigma(\overline{v}') = \mu'$. Thus $\mu \neq \mu'$ implies $\overline{v} \neq \overline{v}'$ and hence $\overline{v}_n \neq \overline{v}'_n$ for some $n \geq 0$. But then we deduce from (6.1) and the hypothesis that $\dim C(X_n) = c_n = c_0 = \dim C(X)$ that $\overline{v}_0 = M(\sigma_{[0,n]})(\overline{v}_n) \neq M(\sigma_{[0,n]})(\overline{v}'_n) = \overline{v}'_0$. From Proposition 2.8 (1) we know that $\overline{v}_0 = pr_0(\overline{v}) = \zeta_X(\mu)$ and $\overline{v}'_0 = pr_0(\overline{v}') = \zeta_X(\mu')$, which shows that the map $\zeta_X$ is injective. For the linearity of $\zeta_X$ and the equality $\zeta_X(M(X)) = C(X)$ see (2.9) and (2.10). \hfill $\square$

Directive sequences of finite alphabet rank occur naturally in many important contexts in the symbolic dynamics literature (e.g. substitutive subshifts, IETs, etc). Furthermore, the extra invertibility condition from the following proposition is rather frequently satisfied.

**Corollary 6.3.** Let $X \subseteq A^\mathbb{Z}$ be a subshift generated by an everywhere growing directive sequence $\sigma = (\sigma_n)_{n \geq 0}$ of finite alphabet rank. Assume that for every $n \geq 0$ the incidence matrix $M(\sigma_n)$ is invertible over $\mathbb{R}$. Then any invariant measure $\mu$ on the subshift $X$ is determined by the evaluation of $\mu$ on the letter cylinders, i.e. by the values $\mu([a_k])$ for all $a_k \in A$.

**Proof.** From (6.1) and the hypothesis that $M(\sigma_n)$ is invertible it follows directly that $c_{n+1} = c_n$ for all $n \geq 0$, so that the directive sequence $\sigma$ is thin. Hence the hypotheses of Proposition 6.2 are satisfied, which gives directly the claimed statement. \hfill $\square$

Note that the conclusion of Corollary 6.3 has recently been proved by Berthé et al. under somewhat more restrictive hypotheses (see Corollary 4.2 of [6]); in particular it is required there that every $M(\sigma_n)$ has determinant equal to 1 or to $-1$, and that $X$ is minimal.

**Remark 6.4.** (1) If in Proposition 6.2 the hypothesis “thin” is replaced by “thinning”, with critical level $n_0 \geq 1$, then the conclusion, that any two distinct measures $\mu \neq \mu' \in M(X)$ can be distinguished by the evaluation on the letter cylinders $[a_k]$ for all $a_k \in A$, may in some cases still hold, despite the fact that from the definition of the critical level we have

$$\dim C_0 = c_0 < c_{n_0} = \dim C_{n_0} = \dim M(X_{n_0}).$$
Here the last equality follows from Proposition 6.2, applied to the directive sequence truncated at the critical level $n_0$. The reason, why the above strict inequality doesn’t contradict the presumed equality $c_0 = \dim C_0 = \dim \mathcal{M}(X)$, is that the measure transfer map $\sigma^M_{[0,n_0)} : \mathcal{M}(X_{n_0}) \to \mathcal{M}(X)$ may well not be injective, in the case that the telescoped level map $\sigma_{(0,n_0)}$ is not recognizable in the level subshift $X_{n_0}$.

However, if $\overline{\sigma}$ is totally recognizable, or if at least $\sigma_{(0,n_0)}$ is recognizable in $X_{n_0}$, and if furthermore $\overline{\sigma}$ is thinning but not thin, then the conclusion of Proposition 6.2 necessarily fails: this case is treated in Example 6.5 below.

(2) In view of the fact that the measure transfer map $\sigma^M$ induced by a non-recognizable monoid morphisms $\sigma$ is in general far from being injective, it seems noteworthy that in Proposition 6.2 and Corollary 6.3 no recognizability condition on the level maps $\sigma_n$ is imposed. One should recall in this context that in [9], Theorem 5.2, it has been proved that directive sequences of bounded alphabet rank, with aperiodic level subshifts, are “eventually recognizable”, i.e. all level maps above some “other critical level” must be recognizable in their level subshift. But this “other critical level” may well be a lot bigger than the above critical level $n_0$, and indeed we give in Corollary 8.5 (2) examples of thin directive sequences where this “other critical level” can be chosen to be arbitrarily high up, while none of the level morphisms below it is recognizable in its corresponding level subshift (which is aperiodic for any level).

We now present the promised “detailed example with all above data made visible”:

**Example 6.5.** The subshift $X$ in this example consists of two periodic words and is hence all by itself not so interesting. We chose it in order to give a transparent presentation of a simple subshift via some not so obvious everywhere growing directive sequence, which we now describe in detail. We first describe the level $n = 1$, then pass to the base level $n = 0$, and finally built the higher levels $n \geq 2$ on top of the two lowest levels. We also include for each level $n$ a description of the measure cone $\mathcal{M}(X_n)$ and of the associated letter frequency cone $C_n$.

Set $A_1 = \{a, b\}$, and let $X_1 \subseteq A_1^{\mathbb{Z}}$ be the union of the two periodic subshifts $O(w^\pm \infty)$ and $O(w'^\pm \infty)$, defined by the words $w = a^2b$ and $w' = b^2a$. We consider the two characteristic measures $\mu := \mu_0$ and $\mu' := \mu_{w'}$, and observe that $\mathcal{M}(X_1)$ consists of all non-negative linear combinations of these two measures. The letter frequency map $\zeta_{X_1} : \mathcal{M}(X_1) \to \mathcal{C}(X_1) \subseteq \mathbb{R}_{\geq 0}$ is injective, in that $\zeta_{X_1}(\mu) = 2\vec{e}_a + \vec{e}_b$ and $\zeta_{X_1}(\mu') = \vec{e}_a + 2\vec{e}_b$. This results into $c_1 = \dim(C_1) = 2$.

For $A_0 = \{c, d\}$ consider now the “Thue-Morse” morphism $\sigma_0 : A_1^{\mathbb{Z}} \to A_0^{\mathbb{Z}}$, $a \rightarrow cd$, $b \rightarrow dc$, and recall (see Proposition 3.3 (d)) that $\sigma_0^M(\mu) = \mu_{\sigma_0(w)}$ and $\sigma_0^M(\mu') = \mu_{\sigma_0(w')}$, with $\sigma_0(w) = cdcdc$ and $\sigma_0(w') = dcdcd$. Since $cdcdc$ and $dcdcd$ can not be obtained from each other by a cyclic permutation, we have $O((cdcdc)^\pm \infty) \neq O((dcdcd)^\pm \infty)$, so that from (2.7) it follows that $\text{Supp}(\mu_{cdcdc}) \neq \text{Supp}(\mu_{dcdcd})$. We thus deduce for the image subshift $X_0 = \sigma_0(X_1)$ that the measure cone $\mathcal{M}(X_0)$, which is spanned by $\mu_{cdcdc}$ and $\mu_{dcdcd}$, is of dimension 2.

On the other hand, using Proposition 3.4 (or more directly, equality (2.6)) we readily compute $\mu_{cdcdc}([cd]) = \mu_{dcdcd}([dc]) = 2$ as well as $\mu_{cdcdc}([cd]) = \mu_{dcdcd}([dc]) = 2$. It follows that the frequency map $\zeta = \zeta_{X_0}$ is not injective, and that $C_0$ has dimension $c_0 = 1$.

We now define the higher up levels of the directive sequence by setting $A_n = \{x, y\}$ for any $n \geq 2$, and by defining all level morphisms $\sigma_n : A_{n+1} \to A_n$ for $n \geq 2$ be equal to the substitution defined by $x \mapsto x^z$, $y \mapsto y^z$. It follows that for $n \geq 2$ all level subshifts $X_n$ consist of the two biinfinite periodic words $x^\pm \infty$ and $y^\pm \infty$. Moreover, we easily see that the incidence matrix of $\sigma_n$ is equal to 2 times the 2-by-2 identity matrix, $M(\sigma_n) = 2 \cdot I_2$, so that we have $\mathcal{M}(X_n) = C_n = \mathbb{R}_{\geq 0}^{\{x,y\}}$.

It remains now to define $\sigma_1 : A_2 \to A_1$ via $x \mapsto w$, $y \mapsto w'$, which ensures $\sigma_1(X_2) = X_1$, in order to obtain a directive sequence $\overline{\sigma} = (\sigma_n)_{n \geq 0}$ over alphabets that all have cardinality 2. We
have shown above that the critical level of this directive sequence is \( n_0 = 1 \), while the evaluation on the cylinders \([\sigma_0(a)] = [d] \) and \([\sigma_0(b)] = [dc] \) does not suffice to distinguish the two measures \( \mu_{edecd} \neq \mu_{dced} \) that span \( M(X_0) \).

7. Minimal subshifts with zero entropy and infinitely many ergodic probability measures

A subshift \( X \), which is “small” in that it has topological entropy \( h_X = 0 \) (see (2.3)), and simultaneously “large” in that the number \( e(X) \) of ergodic probability measures carried by \( X \) is infinite, is a bit of a contradiction in itself (if one restricts to non-atomic measures). However, such subshifts are known to exist, but they are not easy to come by. One of the first such subshift known to us is the Pascal-adic subshift, treated in [22]; more recent such examples (with additional strong properties, in particular minimality) are exhibited in [11]. Not surprisingly, there is always a certain amount of labor involved in order to simultaneously achieve the above two opposite properties.

In this section we will present an alternative way to construct minimal subshifts \( X \) which satisfy both, \( h_X = 0 \) and \( e(X) = \infty \). The main purpose of this section is to underline how directly such examples can be exhibited by means of the technology established in the previous sections.

We first recall two known results. The first appears as Theorem 4.3 in [7] and is attributed there to Thierry Monteil; alternatively it can be found in [8] as Lemma 6.7.1 of Chapter 6, written by Fabien Durand, who told us that the result can actually be traced back to the paper [10] by Boyle-Handelman.

**Proposition 7.1.** Let \( X \) be a subshift which is generated by a directive sequence \( \overline{\sigma} = (\sigma_n)_{n \geq 0} \) with level alphabets \( A_n \). Then, for the minimal level letter image length \( \beta_-(n) \) from (2.16), one has:

\[
h_X \leq \inf_{n \geq 0} \frac{\log(\text{card}(A_n))}{\beta_-(n)}
\]

**Proposition 7.2** (Section 4.1 of [4]). For any integer \( d \geq 2 \) let \( X \) be a subshift which is generated by a directive sequence \( \overline{\sigma} = (\sigma_{n,d})_{n \geq 0} \) with level alphabets that are all of uniform cardinality \( d \) (and are thus identified with \( A_d = \{a_1, \ldots, a_d\} \)). Assume that for any level \( n \geq 0 \) the incidence matrix of the level map \( \sigma_{n,d} \) is given by

\[
M(\sigma_{n,d}) = M_{\ell(n),d} := \ell(n)I_d + 1_{d \times d},
\]

where \( I_d \) is the identity matrix of size \( d \times d \), \( 1_{d \times d} \) is the \( d \times d \)-matrix with all entries equal to 1, and \( \ell(n) \) is a positive integer depending on \( n \).

Then \( X \) is minimal, and for any sufficiently fast growing sequence \( (\ell(n))_{n \in \mathbb{N}} \) the subshift \( X \) admits precisely \( d \) distinct invariant ergodic probability measures.

The use of Proposition 7.1 will be an ingredient in our proof below. Proposition 7.2, on the other hand, will not be formally used in the sequel, but it may pay anyway for the reader to look it up: We use below the very same basic idea as in this earlier result, but do not carry out all calculations as had been done in section 4 of [4] (where in particular precise lower bounds for the integers \( \ell(n) \) are computed which guarantee the “sufficiently fast growing” in the above statement).

For the proof below we first need to define for any integer \( d \geq 2 \) and alphabet \( A_{(d)} = \{a_1, \ldots, a_d\} \) the morphism \( \tau_d : A_{(d+1)}^* \rightarrow A_d^* \), given by \( a_i \mapsto a_i^2 \) for any \( a_i \) with \( 1 \leq i \leq d \) and \( a_{d+1} \mapsto a_1a_2 \ldots a_d \).

**Remark 7.3.** (1) For the morphism \( \tau_d \) as given above it is easy to see that any biinfinite word \( y \in A_{(d)}^\mathbb{Z} \setminus \{a_1^{\pm \mathbb{Z}}, a_2^{\pm \mathbb{Z}}, \ldots, a_d^{\pm \mathbb{Z}}\} \) can be “desubstituted” in at most one way (compare Remark 6.2 (2) of [5]) to give a biinfinite word \( x \in A_{(d+1)}^\mathbb{Z} \) with \( \tau_d(O(x)) = O(y) \). Since for any \( i = 1, \ldots, n \)
the periodic word $a_i^{±\infty}$ is the only element $x \in \mathcal{A}_{(d+1)}^\mathbb{Z}$ with $\tau_d(\mathcal{O}(x)) = \mathcal{O}(a_i^{±\infty})$, it follows that $\tau_d$ is recognizable in every subshift which doesn’t contain any of the periodic words $a_i^{±\infty}$.

(2) Again by elementary desubstitution arguments one verifies quickly that any morphism $\sigma_{n,d}$ with incidence matrix given by equality (7.1), with $\ell(n) \geq 2$, is recognizable in the full shift $\mathcal{A}_{(d)}^\mathbb{Z}$.

[Indeed, it suffices to check in any biinfinite word $y \in \sigma_{n,d}(\mathcal{A}_{(d)}^\mathbb{Z})$ for a factor $w \in \mathcal{A}_{(d)}^*$ which is “distinguished” in that some letter $a_i \in \mathcal{A}_{(d)}$ occurs precisely 3 times in $w$, while all other letters $a_j \in \mathcal{A}_{(d)}$ occur at most twice. Such a distinguished word $w$ occurs in $\sigma_{n,d}(a_i)$, and any such occurrence is contained in the image of some word from $\mathcal{A}_{(d)}^*$ of length at most 3. In either case one verifies quickly that the middle occurrence of $a_i$ in $w$ must belong to $\sigma_{n,d}(a_i)$. For this middle occurrence $y_s$ in the factor $w = y_r \ldots y_t$ of $y = y_{n-1}y_{n+1} \ldots$ one considers the factors $w^+ = y_{s-1} \ldots y_t$ and $w^- = y_{r-1} \ldots y_t$ of $y$, with $y_s = y_{s+1} = \ldots = y_{r-1} = a_i$ and $y_r \neq a_i$, and similarly $y_s = y_{s-1} = \ldots = y_{r+1} = a_i$ and $y_r \neq a_i$. From the fact that $\sigma_{n,d}(a_i)$ contains each letter $a_j \neq a_i$ precisely once one deduces directly that the words $w^+$ and $w^-$ determine which occurrence of $a_i$ in $\sigma_{n,d}(a_i)$ is given by the letter $y_s$. It follows that, starting from $y_s$, the biinfinite word $y$ can be desubstituted in precisely one way.]

(3) From the conditions on the incidence matrix $M(\sigma_n)$ in (7.1) it follows directly that every word in $\sigma_{n,d}(\mathcal{A}_{(d)}^\mathbb{Z})$ must contain each of the letters of $\mathcal{A}_{(d)}$. Hence we observe that $\sigma_{n,d}(\mathcal{A}_{(d)}^\mathbb{Z})$ can not contain any of the periodic words $a_i^{±\infty}$.

(4) As a consequence of the above observations (1) - (3) we deduce for the alternating directive sequence

$$\sigma = \sigma_2 \circ \sigma_3 \circ \sigma_3 \circ \ldots,$$

where we set $\sigma_d := \sigma_{d,d}$, that each level map is recognizable in its corresponding level subshift, so that the sequence $\sigma$ is fully recognizable.

**Theorem 7.4.** For any integer $d \geq 2$ let $\mathcal{A}_{(d)} = \{a_1, \ldots, a_d\}$ and let $\sigma_d : \mathcal{A}_{(d)}^* \rightarrow \mathcal{A}_{(d)}^*$ be a morphism with incidence matrix $M(\sigma_d) = M_{\ell(d),d}$ from (7.1), for some integer $\ell(d) \geq 2$ depending on $d$. Let $X$ be the subshift generated by the alternating directive sequence $\sigma$ given in (7.2).

If the exponent sequence $(\ell(n))_{n \in \mathbb{N}}$ is sufficiently fast growing, then the subshift $X$ is minimal, has entropy $h_X = 0$ and admits infinitely many distinct invariant ergodic probability measures.

(We denote by $\mathcal{X}$ the class of all subshifts $X \subseteq \mathcal{A}_{(d)}^\mathbb{Z}$ which satisfy all of the above conditions.)

**Proof.** For each integer $d \geq 2$ we identify the finite alphabet $\mathcal{A}_{(2)} = \{a_1, a_2, \ldots, a_d\}$ with the corresponding subset of an infinite alphabet, via $\mathcal{A}_{(2)} \subseteq \mathcal{A}_{(3)} \subseteq \ldots \subseteq \mathcal{A}_{(x)} = \{a_1, a_2, \ldots\}$. For the issuing infinite non-negative cone $\mathbb{R}_{\geq 0}^{\mathcal{A}_{(x)}}$ we abbreviate for notational convenience the base unit vectors to $\bar{e}_i := e_{a_i}$.

For any level $n = 2d - 2$ or $n = 2d - 1$ we consider the subcone $\mathcal{C}_n := \mathbb{R}_{\geq 0}^{\mathcal{A}_{(d)}} \subseteq \mathbb{R}_{\geq 0}^{\mathcal{A}_{(x)}}$, and in particular the “center vector” $\bar{e}_n = \sum \bar{e}_i$ of $\mathcal{C}_n$. We observe that both families, the morphisms $\sigma_d$ as well as the morphisms $\tau_d$, induce maps $M(\sigma_d) : \mathcal{C}_n \rightarrow \mathcal{C}_n$ and $M(\tau_d) : \mathcal{C}_n \rightarrow \mathcal{C}_n$ respectively which each maps the center vector $\bar{e}_n$ (for $\sigma_n$) or $\bar{e}_{n+1}$ (for $\tau_n$) to a scalar multiple of the center vector $\bar{e}_n$. Furthermore, any unit vector $\bar{e}_i$ with $1 \leq i \leq d$ is mapped by both, $M(\sigma_d)$ and $M(\tau_d)$, to a non-negative linear combination $\lambda_1 \bar{e}_i + \lambda_2 \bar{e}_d$. Note here that (again for both, $\sigma_d$ and $\tau_d$)

$$\frac{\lambda_2}{\lambda_1}$$

by choosing $\ell(d)$ sufficiently large.

We now fix some level $n_0 = 2d - 2 \geq 0$, and for any index $i$ with $1 \leq i \leq d$ we look for a vector tower $\bar{v}_i = (\bar{v}_n^i)_{n \geq n_0}$ on the truncated directed sequence $\sigma_{n_0}^\dagger = \sigma_{n_0} \circ \tau_n \circ \sigma_{n_0+1} \circ \tau_{n_0+1} \circ \ldots$ with
the property that $\Psi_i$ has for any level $n \geq n_0$ a level vector $\vec{v}_n^i = \lambda_1 n \vec{e}_i + \lambda_2 n \vec{c}_n$, with coefficients (7.4)

$$\lambda_{1, n} > 0 \quad \text{and} \quad \lambda_{2, n} > 0$$

(which must both tend to 0 for $n \to \infty$). From (7.3) we deduce that a sufficiently large choice of the exponents $\ell(d)$ effects indeed that there exist families of such coefficients where both of the inequalities in (7.4) are satisfied, while the compatibility condition (2.19) is maintained, for any $n \geq n_0$. It follows that on the lowest level $n = n_0$ (and thus similarly also on all levels $n \geq n_0$) the level vectors $\vec{v}_0^1, \vec{v}_0^2, \ldots, \vec{v}_d$ are linearly independent.

For the level subshift $X_{n_0} \subseteq \mathcal{A}^d_{\ell(d)}$, generated by the truncated sequence $\Psi_{\ell(n_0)}$, the truncated evaluation map $m_{n_0} := m_{\Psi_{\ell(n_0)}} : \mathcal{V}(\Psi_{\ell(n_0)}) \to \mathcal{M}(X_{n_0})$ from (5.1) defines $d$ invariant measures $\mu_1, \ldots, \mu_d$ on the level subshift $X_{n_0}$ as images of the $d$ vector towers $\vec{v}_1, \ldots, \vec{v}_d$ respectively:

$$\mu_i = m_{n_0}(\Psi_i)$$

It follows from Proposition 2.8 (1) that the subcone

$$\mathcal{M}_{n_0} := \mathbb{R}_{\geq 0} \langle \mu_1, \ldots, \mu_d \rangle \subseteq \mathcal{M}(X_{n_0})$$

spanned by the $\mu_i$ has dimension $d$. Since we verified in Remark 7.3 (4) above that each of the maps $\sigma_j$ and $\tau_i$ is recognizable in its corresponding level subshift, it follows from Theorem 3.2 (3d) that $\mathcal{M}_{n_0}$ is mapped by $\sigma_2^M \circ \tau_2^M \circ \ldots \circ \sigma_{n_0-1}^M \circ \tau_{n_0-1}^M$ to a subcone of $\mathcal{M}(X)$ that also has dimension $d$.

We have thus proved that $\mathcal{M}(X)$ contains subcones of arbitrary large dimension, and hence must be infinite dimensional, i.e. $e(X) = \infty$. The desired equality $h_X = 0$ is immediate from Proposition 7.1 for large $\ell(d)$, and the minimality of $X$ follows directly from the positivity of the matrices $M(\sigma_d)$, see Remark 2.5 (4). \hfill $\Box$

8. Non-recognizable directive sequences

The purpose of this section is to show how non-recognizable morphisms appear naturally in a well known context (IETs and pseudo-Anosov surface homeomorphisms), and how this phenomenon can be exploited to construct interesting directive sequences that are not totally recognizable or even not eventually recognizable.

Our construction will be presented in 4 steps, organized below as follows: In subsection 8.1 we present our basic quotient construction in geometric language. In subsection 8.2 we show how the canonical “inverse quotient construction” is obtained in a natural geometric context, to define a non-recognizable monoid morphism. In subsection 8.3 the results from the previous subsections are properly “pasted together” to give a directive sequence where every level morphism is non-recognizable (and in addition it is a particularly nice letter-to-letter factor map). Finally, in subsection 8.4 we modify this sequence slightly to obtain the desired everywhere growing but not (eventually) recognizable directive sequences. Note that all intermediate level subshifts which occur in our constructions turn out to be minimal; they are furthermore both, substitutive and IET.

Before starting the detailed description, we will highlight its essential features in a special case, in a language that may be more easily accessible to those of us who are less familiar with Thurston’s work on surface homeomorphisms:

Remark 8.1. (1) Let us consider the tiling of the real plane $\mathbb{R}^2$ by squares of side length 1 that have their vertices on the points with integer coordinates. We now pick a slope $s$, say $0 < s < 1$, and we foliate the plane by lines that have slope $s$. By choosing the slope $s$ to be irrational, we make sure that on any line of the foliation there is at most one vertex of our square tiling. To every line $\ell$ that avoids any such vertex one can associate canonically a biinfinite word $w(\ell)$ in the letters $h$ and $v$, which records the sequence of intersections of $\ell$ with a horizontal (“$h$”) or vertical (“$v$”)
line of our square grid. In order to fix an indexing of the letters of \( w(\ell) \) we pick a distinguished “base square” \( Q \) and require that \( \ell \) passes through the interior of \( Q \). We quickly observe that the orbits in our family of lines \( \ell \), with respect to the canonical \( \mathbb{Z} \oplus \mathbb{Z} \)-action on \( \mathbb{R}^2 \), are in 1-1 relation with the shift-orbits of the resulting set of words \( w(\ell) \). Indeed, for this 1-1 relation it suffices to consider the positive half-words of any \( w(\ell) \), so that it extends naturally to the lines \( \ell \) that pass over any of the vertices.

We consider now more closely any of the “troublesome” lines \( \ell_P \) that cross over a vertex \( P \) of the square grid. To \( \ell_P \) we associate two words \( w_{\text{above}}(\ell_P) \) and \( w_{\text{below}}(\ell_P) \) in \( \{ h, v \}^\mathbb{Z} \), which are read off from \( \ell_P \) after isotopying it slightly in the neighborhood of \( P \) so that it passes either above or below \( P \). From the above observed 1-1 relation between the \( \mathbb{Z} \oplus \mathbb{Z} \)-orbits of the lines \( \ell \) and the shift-orbits of the corresponding words \( w(\ell) \) we deduce that the words \( w_{\text{above}}(\ell_P) \) and \( w_{\text{below}}(\ell_P) \) do not belong to the same shift-orbit.

The set \( X_s \subseteq \{ h, v \}^\mathbb{Z} \) of all biinfinite words \( w(\ell) \), including the above defined \( w_{\text{above}}(\ell_P) \) and \( w_{\text{below}}(\ell_P) \), for any line \( \ell \) that passes through our distinguished base square \( Q \), is a subshift - indeed, a well known Sturmian subshift.

(2) We now proceed by subdividing the top and bottom side of each square into segments of equal length through introducing a new vertex at the midpoint of any horizontal segment of the square grid. Any transition of a line \( \ell \) through the left half of the subdivided horizontal square side will now be recorded by the letter \( h_{\text{left}} \), and any transition through the right half by \( h_{\text{right}} \), to give a new biinfinite word \( w'(\ell) \in \{ h_{\text{left}}, h_{\text{right}}, v \}^\mathbb{Z} \). The morphism \( \sigma : \{ h_{\text{left}}, h_{\text{right}}, v \}^\mathbb{Z} \rightarrow \{ h, v \}^\mathbb{Z} \) defined by \( h_{\text{left}} \mapsto h, h_{\text{right}} \mapsto h \) and \( v \mapsto v \) maps any \( w'(\ell) \) to \( w(\ell) \), and it will be 1-1, except for the new “troublesome” lines \( \ell_R \) that pass through any of the new vertices \( R \) in the middle of our original horizontal square grid intervals. For such lines we have as before 2 words \( w'_{\text{above}}(\ell_R) \) and \( w'_{\text{below}}(\ell_R) \), and both have the same image word \( w(\ell_R) \). Since \( w'_{\text{above}}(\ell_R) \) and \( w'_{\text{below}}(\ell_R) \) belong as above to distinct shift-orbits, the morphism \( \sigma \) is not shift-orbit injective, and hence not a recognizable (see Proposition 3.6 (1)).

Clearly, this process can be iterated arbitrarily often, and every time the obtained morphism is shift-orbit injective except for two particular shift-orbits, which have the same image orbit.

(3) The above set-up of lines in a square grid of \( \mathbb{R}^2 \) admits a particularly convincing translation into an IET setting, since for any of the squares we can use the left hand and the bottom sides together as “bottom intervals”, and the top side together with the right hand side as “top intervals”, and the line segments of our foliation that are contained in the chosen square give canonically a classical IET system. If the chosen square agrees with the above picked base square \( Q \), then the interval coding associated traditionally to the IET defines a subshift that agrees precisely with the one given by the set of biinfinite words \( w(\ell) \) (or similarly for \( w'(\ell) \)), which have been read off above from the intersections of the lines \( \ell \) with the given square grid.

After this “appetizer” we now give a detailed description of our construction in the subsequent 4 subsections. We assume a minimal familiarity with the basic terminology of Thurston’s work on surfaces, such as “pseudo-Anosov homeomorphism”, “stable lamination” or “invariant train track”.

8.1. The basic geometric quotient construction.

We will start by describing our basic geometric construction, using a pseudo-Anosov homeomorphism \( h \) of a compact orientable surface \( \Sigma \), and its expanding invariant lamination \( \Lambda^s \), which consists of uncountably many biinfinite geodesics (called “leaves”) with respect to a fixed hyperbolic structure on \( \Sigma \). [The family \( \Lambda^s \) was called “the stable lamination” by Thurston, as he was looking at its behavior when lifted to the universal covering of \( \Sigma \), identified with the hyperbolic plane \( \mathbb{H}^2 \), in the neighborhood of a \( \partial \tilde{h} \)-fixed point on \( \partial \mathbb{H}^2 \) (where \( \tilde{h} \) is a lift of \( h \) to \( \mathbb{H}^2 \) and \( \partial \tilde{h} \) is the canonical extension of \( \tilde{h} \) to \( \partial \mathbb{H}^2 \)).]
It is a standard procedure to translate such laminations (for instance by using an $h$-invariant train track neighborhood of $\Lambda^s$) into a classical interval exchange setting, which in turn (assuming that $\Lambda^s$ is orientable and $\Sigma$ has at least one boundary component) allows a direct translation of $\Lambda^s$ into a subshift $X \subseteq \mathcal{A}^Z$, where $\mathcal{A}$ is given by the intervals in the IET. Since both of these translations are well known (see for instance [12], [20], [21]), we will restrict ourselves here only to a description of the geometry of $h$ and $\Lambda^s$.

For our purposes it is convenient to impose the following extra conditions:

(H1) Assume that $\Sigma$ has $r \geq 2$ boundary components, which are all fixed by $h$.

(H2) Each complementary component of $\Lambda^s$ contains precisely one boundary component.

(Note that this assumption effects that there is a natural identification of $\pi_1 \Sigma$ with the free group $F(\mathcal{A})$.)

(H3) Each complementary component has at least 2 cusps, and each cusp is fixed by $h$.

We now pick a particular complementary component $\Sigma_i \subseteq \Sigma$ of $\Lambda^s$, and assume that $\Sigma_i$ has precisely two cusps, and thus also precisely two boundary leaves $\ell_1$ and $\ell_2$, which (do all boundary leaves of complementary components) will then both belong to $\Lambda^s$. We now pass to a quotient surface $\Sigma'$ by “filling in” the boundary component of $\Sigma$ that is contained in $\Sigma_i$, through identifying all points of the boundary curve in $\Sigma_i$ into a single point $P$ of $\Sigma'$. Then $h$ induces a pseudo-Anosov homeomorphism $h' : \Sigma' \to \Sigma'$ with stable lamination $\Lambda'^s$, and there is a canonical quotient map $q : \Lambda^s \to \Lambda'^s$ that commutes with $h$ and $h'$ respectively. The map $q$ is 1-1 everywhere, except at points on the leaves $\ell_1$ and $\ell_2$, which are identified by $q$ to a single leaf $\ell' \in \Lambda'^s$. The leaf $\ell'$ is fixed and expanded by $h'$, and the sole $h'$-fixed point on $\ell'$ is precisely the above point $P$. This can be seen for example by the canonical passage from the stable lamination $\Lambda^s$ to the associated stable foliation $\mathcal{F}^s$ for $h$.

Remark 8.2. (1) There is a remarkable feature here in that both, $\Lambda^s$ and $\Lambda'^s$ are minimal laminations (i.e. each leaf is dense), while the map $q$ induces on the leaf spaces of $\Lambda^s$ and $\Lambda'^s$ a map that is surjective, but not injective.

(2) This is translated (via the associated IETs as indicated above) into a subshift $X \subseteq \mathcal{A}^Z$ that is mapped by a morphism $\sigma : \mathcal{A}^s \to \mathcal{A}^s$ to a subshift $\sigma(X) = X' \subseteq \mathcal{A}^Z$, (for $\mathcal{A}^s \subseteq F(\mathcal{A}) = \pi_1 \Sigma'$, in complete analogy to $\mathcal{A}$ and $\Sigma$ in the above set-up). Here both, $X$ and $X'$, are minimal, while the map induced by $\sigma$ on $X$ is not shift-orbit injective, so that $\sigma$ is not recognizable in $X$.

(3) More precisely, since there is a natural 1-1 correspondence between the shift-orbits of $X$ and the leaves of $\Lambda^s$ (and similarly for $X'$ and $\Lambda'^s$), we observe that $\sigma$ maps precisely two shift-orbits of $X$ to a common image shift-orbit of $X'$, while everywhere else the induced map on shift-orbits is 1-1.

8.2. The “inverse” geometric quotient construction.

After having presented our basic geometric quotient construction, we will now describe the precise converse procedure: For this purpose we assume in this subsection that $\sigma_0, h_0, \Lambda^s_0, A_0$ and $X_0$ are as $\Sigma, h, \Lambda^s, \mathcal{A}$ and $X$ in subsection 8.1 above, and that in particular the conditions (H1) - (H3) are satisfied, except that in (H1) we lower the assumption on the number $r$ of boundary components of $\Sigma_0$ to $r \geq 1$. We now select any non-boundary leaf $\ell_0$ of $\Lambda^s_0$ which is fixed by $h_0$:

\begin{equation}
\label{eq:8.1}
h_0(\ell_0) = \ell_0
\end{equation}

Since $\Lambda^s_0$ is expanded by $h_0$, it follows that there is precisely one fixed point $P = h_0(P) \in \ell$. We derive the surface $\Sigma_1$ from $\Sigma_0$ by puncturing a hole in $\Sigma_0$ at the point $P$, and observe from (8.1) that $h_0$ induces a homeomorphism $h_1 : \Sigma_1 \to \Sigma_1$. Again from considering the stable foliation $\mathcal{F}^s_{\Lambda^s_0}$ associated to $\Lambda^s_0$, we obtain the stable lamination $\Lambda^s_1 \subseteq \Sigma_1$ for $h_1$ from $\Lambda^s_0$ by doubling the leaf $\ell_0$ into two leaves $\ell_0$ and $\ell_0$, which are boundary leaves of a new complementary component $\Sigma_1' \subseteq \Sigma_1$
that has no further boundary leaf. The component \( \Sigma_1 \) contains a new boundary component of \( \Sigma_1 \) that runs around the puncture where formerly the point \( P \in \Sigma_0 \) was located.

From this construction we obtain a quotient map \( q_0 : \Lambda_1^s \to \Lambda_0^s \) that satisfies

\[
(8.2) \quad h_0 \circ q_0 = q_0 \circ h_1,
\]

and \( q_0 \) is 1-1 everywhere except on the leaves \( \hat{\ell}_0 \) and \( \hat{\ell}_0' \), which are identified by \( q_0 \) to the single leaf \( \ell_0 \in \Lambda_0^s \). We thus observe that the “quotient procedure” from \( \Sigma_1, h_1 \) and \( \Lambda_1^s \) to \( \Sigma_0, h_0 \) and \( \Lambda_0^s \) is precisely the same as described in subsection 8.1 when passing from \( \Sigma, h \) and \( \Lambda^s \) to \( \Sigma', h' \) and \( \Lambda'^s \).

**Remark 8.3.** In the passage from \( \Lambda_0^s \) to \( \Lambda_1^s \), when translated into the IET language as in Remark 8.2, we observe that the IET for \( \Lambda_1^s \) derives from the IET for \( \Lambda_0^s \) by subdividing one of the intervals (namely the one onto which we choose to isotope \( P \) along the leaf \( \ell_0 \)). Hence the alphabet \( \mathcal{A}_1 \) for \( \Lambda_1^s \) derives from \( \mathcal{A}_0 \) by doubling one of its letters, namely the one corresponding to the subdivided interval.

For the minimal subshift \( X_1 \subseteq \mathcal{A}_1^\mathbb{Z} \) associated to \( \Lambda_1 \) and the morphism \( \sigma_0 : \mathcal{A}_1^s \to \mathcal{A}_0^s \) determined by the map \( q_0 \), which maps \( X_1 \) to \( X_0 \) and is non-recognizable in \( X_1 \), it follows that \( \sigma_0 \) is letter-to-letter, so that \( X_0 \) is actually a factor of \( X_1 \).

**8.3. Iteration of the inverse quotient construction.**

We now look for a leaf \( \ell_1 \in \Lambda_1^s \) with \( h_1(\ell_1) = \ell_1 \). As shown in the previous subsection, this is the only ingredient needed in order to repeat the above procedure to obtain a surface \( \Sigma_2 \), a pseudo-Anosov homeomorphism \( h_2 : \Sigma_2 \to \Sigma_2 \) with stable lamination \( \Lambda_2^s \), a map \( q_1 : \Lambda_2^s \to \Lambda_1^s \) and a morphism \( \sigma_1 : \mathcal{A}_2^s \to \mathcal{A}_1^s \) that is non-recognizable on the minimal subshift \( X_2 \) which satisfies \( \sigma_1(X_2) = X_1 \).

Hence, in order to be able to repeat this procedure infinitely often, with the purpose to get for any \( n \geq 0 \) a morphism \( \sigma_n : \mathcal{A}_{n+1}^s \to \mathcal{A}_n^s \) that is non-recognizable on a minimal subshift \( X_{n+1} \) with \( \sigma_n(X_{n+1}) = X_n \), we just need for any \( \Lambda_n^s \) a leaf \( \ell_n \in \Lambda_n^s \) with \( h_n(\ell_n) = \ell_n \). However, up to replacing \( h_n \) by a power \( h_n^{t(n)} \) for some suitable integer \( t(n) \geq 1 \), this is no problem: It is well known that any pseudo-Anosov map \( h \) has infinitely many \( h \)-periodic leaves in its stable lamination. We obtain the following result, which is however only an intermediate step in our construction: In particular, the subshifts \( X_n \) are not the intermediate level subshifts of the given directive sequence \( \mathcal{S} \).

**Proposition 8.4.** There exists a directive sequence \( \mathcal{S} = (\sigma_n : \mathcal{A}_{n+1}^s \to \mathcal{A}_n^s)_{n \geq 0} \) and subshifts \( X_n \subseteq \mathcal{A}_n^\mathbb{Z} \), such that for any \( n \geq 0 \) the following hold:

1. \( \sigma_n(X_{n+1}) = X_n \), and \( \sigma_n \) is not recognizable in \( X_{n+1} \).
2. \( \text{card}(\mathcal{A}_{n+1}) = \text{card}(\mathcal{A}_n) + 1 \)
3. \( \sigma_n \) is letter-to-letter. In particular, \( \sigma_n \) commutes with the shift operator, and \( X_n \) is a factor of \( X_{n+1} \).
4. \( X_n \) is minimal, aperiodic and uniquely ergodic.
5. \( X_n \) is substitutive (see Remark 2.5 (2)) for some primitive substitution \( \tau_n : \mathcal{A}_n^s \to \mathcal{A}_n^s \).
6. \( \tau_n^{t(n)} \circ \sigma_n = \sigma_n \circ \tau_{n+1} \) for some integer \( t(n) \geq 1 \).

**Proof.** Properties (1), (2) and (3) have been derived in the construction described above. The substitution \( \tau_n \) from (5) is the translation of the homeomorphism \( h_n \) into the monoid setting through the canonical embedding \( \mathcal{A}_n^s \subseteq F(\mathcal{A}_n) = \pi_1 \Sigma_n \). The primitivity of \( \tau_n \) is a direct consequence of the assumption “pseudo-Anosov” for \( h \) and thus for all \( h_n \). Property (4) is a direct consequence of (5), and (6) is the translation into the monoid setting of the commutativity relation \( h_n^{t(n)} \circ \sigma_n = q_n \circ h_{n+1} \), which is a consequence of the equality (8.2) together with the above replacement of \( h_n \) by \( h_n^{t(n)} \). \( \square \)
8.4. Everywhere growing directive sequences that are not (eventually) recognizable.

The sequence \( \sigma \) from Proposition 8.4 is not everywhere growing; in fact, for any integers \( m > n \geq 0 \) the telescoped level map \( \sigma_{[m,n]} \) is letter-to-letter. However, by choosing suitable “diagonal” or “eventually horizontal” paths through the infinite commutative diagram built from the above morphisms \( \sigma_n \) (“vertical”) and \( \tau_n \) (“horizontal”) we will derive below everywhere growing directive sequences with interesting properties.

Using the terminology from Proposition 8.4, we first define for each \( n \geq 0 \) the morphism

\[
\sigma'_n := \tau'_n \circ \sigma_n \quad (= \sigma_n \circ \tau'_n)
\]

where we set \( t'(n) := s(n) t(n) \) for some suitably chosen integer \( s(n) \geq 1 \) which ensures that the incidence matrix \( M(\tau'_n) \) is positive. Such \( s(n) \) exists because of property (5) of Proposition 8.4, and since \( M(\sigma_n) \) has no zero-columns, it follows furthermore that

\[
(8.3) \quad \text{the incidence matrix } M(\sigma'_n) \text{ is positive, for any index } n \geq 0.
\]

We now define a directive sequence \( \sigma' = (\sigma'_n : A^*_n \rightarrow A^*_n)_{n \geq 0} \) with intermediate level subshifts called \( X'_n \). Since \( \tau_n(X_n) = X_n \) and \( \sigma_n(X_{n+1}) = X_n \), we have

\[
(8.4) \quad \sigma'_n(X_{n+1}) = X_n
\]

for any \( n \geq 0 \), so that from the minimality of \( X_n \) we can deduce \( X_n \subseteq X'_n \). In particular, we obtain from statement (1) of Proposition 8.4 together with Remark 3.7 that \( \sigma'_n \) is not recognizable in \( X_{n+1} \), and thus neither in \( X'_{n+1} \). From (8.3) we obtain directly (see Remark 2.5 (3)) that the sequence \( \sigma' \) is everywhere growing.

Furthermore, we define for any integer \( k \geq 0 \) a directive sequence \( \sigma_k = (\tau'_n)_{n \geq 0} \) through setting \( \tau'_{n+k} := \tau_k \) for all \( n \geq k \) and \( \tau'_n = \sigma'_n \) if \( 0 \leq n \leq k-1 \). We also specify the starting surface \( \Sigma_0 \) to be a punctured torus, so that one has \( |A_0| = 2 \), and \( X_0 \) is Sturmian. It follows that for any level \( n \geq k \) the intermediate level \( n \) subshift of \( \sigma_k \) is equal to the substitutive subshift \( X_k \) defined by the substitution \( \tau_k \) from statement (5) of Proposition 8.4, so that for every \( 0 \leq n \leq k-1 \) we deduce from (8.4) that the level \( n \) subshift is equal to \( X_n \). The primitivity of \( \tau_k \) implies in particular that the directive sequence \( \sigma_k \) is everywhere growing. Recall also that (as is true for all stationary sequences, see [1] and the references given there) the truncated stationary sequence \( \sigma_k = (\tau'_n)_{n \geq k} \) is totally recognizable.

We obtain hence as immediate consequence of Proposition 8.4 the following result; we observe that its parts (2) and (3) give directly the statements that have been rephrased in the Introduction and stated there as Proposition 1.7:

**Corollary 8.5.** (1) The directive sequence \( \sigma' \) is everywhere growing and satisfies the properties (1), (2), (4), (5) and (6) from Proposition 8.4, with \( \sigma_n \) replaced by \( \sigma'_n \).

(2) For any integer \( k \geq 0 \) there exists a directive sequence \( \sigma_k \), with level alphabets \( A_n \) of size \( \text{card}(A_n) = k+2 \) for any level \( n \geq k \), and \( \text{card}(A_n) = n+2 \) if \( n \leq k \).

The sequence \( \sigma_k \) is everywhere growing and eventually recognizable: each of the first \( k \) level morphisms on the bottom of \( \sigma_k \) is not recognizable in its corresponding level subshift, while all level morphisms of level \( n \geq k \) are recognizable in their corresponding level subshift. Indeed, the sequence \( \sigma_k \) is stationary above level \( k \).

(3) All intermediate level subshifts of the above directive sequences \( \sigma_k \) are minimal, uniquely ergodic and aperiodic. In particular, the properties “recognizable”, “shift-orbit injective” (see Definition 3.6) and “recognizable for aperiodic points” (see Remark 3.10 (2)) are equivalent, for each level morphism in its corresponding intermediate level subshift.

**Remark 8.6.** It turns out that property (3) of Corollary 8.5 is also true for the directive sequence \( \sigma' \). Indeed, from property (6) of Proposition 8.4 and the well known North-South dynamics induced
by any pseudo-Anosov homeomorphism of $\Sigma$ on the projectivized space of all measured laminations (= the boundary of Teichmüller space for $\Sigma$) one can deduce that the inclusion $X_n \subseteq X'_n$ derived after (8.4) is actually an equality. However, laying out the details of these arguments would go beyond our self-imposed limits on the amount of Nielsen-Thurston theory imported into this section.

**Remark 8.7.** Given any eventually recognizable everywhere growing directive sequence $\sigma = (\sigma_n)_{n \geq 0}$ of finite alphabet rank, one may ask whether there is an upper bound to the number level morphisms $\sigma_n$ which are not recognizable in their corresponding intermediate level subshift. This question has sparked some interest, see [9] and [13]. It seems, however, that the examples given in part (2) of Corollary 8.5 above contradict the bound claimed in Theorem 3.7 of [13]. This could also effect the upper bound given in Corollary 1.5 of [17] on the number of successive factor maps, for a large class of subshifts.

In this context we also want to point to Example 7.5 of the very recent paper [2], where a family of directive sequences is presented that has the same properties as exhibited in Corollary 8.5 (2) above for the sequences $\sigma_k$. The examples from [2] are easier to describe, but fail to have the extra properties listed in part (3) of Corollary 8.5.

Another construction of a similar kind (but closer our Corollary 8.5 above) has been communicated to us by Bastián Espinoza [18] in the final stages of the revision of this paper.

### References

[1] M.-P. Béal, D. Perrin and A. Restivo, *Recognizability of morphisms*. arXiv:2110.10267
[2] M.-P. Béal, D. Perrin, A. Restivo and W. Steiner, *Recognizability in S-adic shifts*. arXiv:2302.06258
[3] N. Bédaride, A. Hilion and M. Lustig, *Graph towers, laminations and their invariant measures*. J. London Math. Soc. (2) 101 (2020), 1112–1172
[4] N. Bédaride, A. Hilion and M. Lustig, *Tower power for S-adics*. Math. Z. 297 (2021), 1853–1875
[5] N. Bédaride, A. Hilion and M. Lustig, *The measure transfer for subshifts induced by a morphism of free monoids*. arXiv:2211.11234
[6] V. Berthé, P. Cecchi Bernales, F. Durand, J. Leroy, D. Perrin and S. Petite, *On the dimension group of unimodular S-adic subshifts*. Monatsh. Math. 194 (2021), 687–717
[7] V Berthé and V. Delecroix, *Beyond substitutive dynamical systems: S-adic expansions*. RIMS Kôkyûroku Bessatsu B46 (2014), 81–123
[8] V. Berthé and M. Rigo (eds.), *Combinatorics, automata and number theory*. Encyclopedia Math. Appl. 135. Cambridge Univ. Press, Cambridge, 2010.
[9] V. Berthé, W. Steiner, J. Thuswaldner and R. Yassawi, *Recognizability for sequences of morphisms*. Ergodic Theory Dynam. Systems 39 (2019), 2896–2931
[10] M. Boyle and D. Handelman, *Entropy versus orbit equivalence for minimal homeomorphisms*. Pacific J. Math. 164 (1994), 1–13
[11] V. Cyr and B. Kra, *Realizing ergodic properties in zero entropy subshifts*. Isr. J. Math. 240 (2020), 119–148
[12] V. Delecroix, *Interval exchange transformations*. Lecture notes from Salta (Argentina), November 2016
[13] S. Donoso, F. Durand, A. Maass and S. Petite, *Interplay between finite topological rank minimal Cantor systems, S-adic subshifts and their complexity*. Trans. Am. Math. Soc. 374 (2021), 3453–3489
[14] F. Durand, *Linearly recurrent subshifts have a finite number of non-periodic subshift factors*. Ergodic Theory Dynam. Systems 20 (2000), 1061–1078. Corrigendum and addendum, Ergodic Theory Dynam. Systems 23 (2003), 663–669.
[15] F. Durand, J. Leroy and G. Richomme, *Do the properties of an S-adic representation determine factor complexity?* J. Integer Sequences 16 (2013), Article 13-2-6
[16] F. Durand and D. Perrin, *Dimension Groups and Dynamical Systems, Bratteli diagrams and Cantor systems*. Camb. Stud. Adv. Math. 196. Cambridge University Press, Cambridge 2022.
[17] B. Espinoza, *On symbolic factors of S-adic subshifts of finite topological rank*. To appear in Ergodic Theory Dynam. Systems. arXiv:2012.00715
[18] B. Espinoza, *Worst-case number of non-recognizable levels for finite alphabet rank S-adic sequences*. https://sites.google.com/ug.uchile.cl/espinoza/miscellaneous
[19] S. Ferenczi, *Rank and symbolic complexity*. Ergodic Theory Dynam. Systems, 16 (1996), 663–682.
[20] V. H. Gadre, *Dynamics of non-classical interval exchanges*. Ergodic Theory Dynam. Systems 32 (2012), 1930–1971
[21] M. Kapovich, *Laminations, Foliations, and Trees*. In: “Hyperbolic Manifolds and Discrete Groups.” Modern Birkhäuser Classics. Birkhäuser Boston 2009

[22] X. Méla and K. Petersen. *Dynamical properties of the Pascal adic transformation*. Ergodic Theory Dynam. Systems 25 (2005), 227–256

Aix Marseille Université, CNRS, I2M UMR 7373, 13453 Marseille, France

*Email address*: Nicolas.Bedaride@univ-amu.fr

Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, UPS F-31062 Toulouse Cedex 9, France

*Email address*: arnaud.hilion@math.univ-toulouse.fr

Aix Marseille Université, CNRS, I2M UMR 7373, 13453 Marseille, France

*Email address*: Martin.Lustig@univ-amu.fr