ON THE STRUCTURE OF FINITELY GENERATED MODULES OVER
QUOTIENTS OF COHEN-MACAULAY LOCAL RINGS

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Abstract. Let \((R, \mathfrak{m})\) be a homomorphic image of a Cohen-Macaulay local ring and \(M\) a finitely generated \(R\)-module. We use the splitting of local cohomology to shed a new light on the structure of non-Cohen-Macaulay modules. Namely, we show that every finitely generated \(R\)-module \(M\) is associated by a sequence of invariant modules. This modules sequence expresses the deviation of \(M\) with the Cohen-Macaulay property. Our result generalizes the unmixed theorem of Cohen-Macaulayness for any finitely generated \(R\)-module. As an application we construct a new extended degree in sense of Vasconcelos.

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1. Introduction

Throughout this paper, let \((R, \mathfrak{m})\) be a Noetherian local ring and \(M\) a finitely generated \(R\)-module of dimension \(d\). Let \(x_1, \ldots, x_d\) be a system of parameters of \(M\).

Standard setting. We always assume that \(R\) is a homomorphic image of a Cohen-Macaulay local ring.

Cohen-Macaulay rings and modules are the central objects of commutative algebra. The unmixed theorem says that \(M\) is Cohen-Macaulay if and only if for every \(i < d\) all associated prime ideals of \(M/(x_1, \ldots, x_i)M\) have the same height \(i\) (or dimension \(d - i\)), that is, \(M/(x_1, \ldots, x_i)M\) is an unmixed module for all \(i < d\) and for every system of parameters \(x_1, \ldots, x_d\). Suppose \(\cap_{p \in \text{Ass}M} N(p) = 0\) is a reduced primary decomposition of the zero submodule of \(M\), then the unmixed component of \(M\) is defined by

\[
UM(0) = \bigcap_{p \in \text{Ass}M, \dim R/p = d} N(p).
\]

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Then $U_M(0)$ is just the largest submodule of $M$ of dimension strictly less than $d$. The following is the unmixed component version of unmixed theorem.

**The unmixed theorem.** A finitely generated $R$-module $M$ is Cohen-Macaulay if and only if for some (and hence for all) system of parameters $x_1, \ldots, x_d$ of $M$ all unmixed components

$$U_M(0), U_{M/x_1M}(0), \ldots, U_{M/(x_1, \ldots, x_{d-1})M}(0)$$

are vanished.

The unmixed theorem can be expressed in another form as follows. A finitely generated $R$-module $M$ is Cohen-Macaulay if and only if every system of parameters $x_1, \ldots, x_d$ of $M$ is an $M$-regular sequence. Recall that $x_1, \ldots, x_d$ is an $M$-regular sequence if for all $i \leq d$ all relations

$$x_1a_1 + \cdots + x_ia_i = 0$$

are trivial, that is, $a_i \in (x_1, \ldots, x_{i-1})M$ for all $i \leq d$. In general we have $a_i \in (x_1, \ldots, x_{i-1})M : x_i$, so $x_1, \ldots, x_d$ is an $M$-regular sequence if the sub-quotient module

$$\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M} = 0$$

for all $i = 1, \ldots, d$. Since

$$((x_1, \ldots, x_{i-1})M : x_i)/(x_1, \ldots, x_{i-1})M = (0 : x_i)_{M/(x_1, \ldots, x_{i-1})M}$$

is a submodule of $M/(x_1, \ldots, x_{i-1})M$ of dimension less than or equal to $d-i = \dim M/(x_1, \ldots, x_{i-1})M - 1$, we have

$$((x_1, \ldots, x_{i-1})M : x_i)/(x_1, \ldots, x_{i-1})M \subseteq U_{M/(x_1, \ldots, x_{i-1})M}(0)$$

for all $i < d$. Set

$$\mathfrak{b}(M) = \bigcap_{i \leq d} \text{Ann}(x_1, \ldots, x_{i-1})M : x_i/(x_1, \ldots, x_{i-1})M,$$

where $x = x_1, \ldots, x_d$ runs over all systems of parameters of $M$. It is clear that the ideal $\mathfrak{b}(M)$ kills all non-trivial relations of systems of parameters of $M$.

The Cohen-Macaulayness of $M$ can be characterized by local cohomology: $M$ is Cohen-Macaulay if and only if the local cohomology $H^i_{\mathfrak{m}}(M) = 0$ for all $i < d = \dim M$. Thus if $M$ is not Cohen-Macaulay, then $H^i_{\mathfrak{m}}(M) \neq 0$ for some $i < d$. Notice that $H^i_{\mathfrak{m}}(M)$ is always Artinian but it is rarely Noetherian. So $H^i_{\mathfrak{m}}(M)$ may not be annihilated by $\mathfrak{m}$-primary ideals. The ideals $\mathfrak{a}_i(M) = \text{Ann}H^i_{\mathfrak{m}}(M)$, $i = 0, \ldots, d$, play important role in many areas in commutative algebra such as the homological conjectures, the tight closure theory, etc. Set $\mathfrak{a}(M) = \mathfrak{a}_0(M) \cdots \mathfrak{a}_{d-1}(M)$.

Schenzel proved the following inclusions [30, Satz 2.4.5]

$$\mathfrak{a}(M) \subseteq \mathfrak{b}(M) \subseteq \mathfrak{a}_0(M) \cap \cdots \cap \mathfrak{a}_{d-1}(M).$$

Notice that our ring is always a homomorphic image of a Cohen-Macaulay local ring. This condition gives us a critical fact that $\dim M/\mathfrak{a}(M) < \dim M$ for all finitely generated $R$-modules. Therefore we can choose a parameter element $x$ contained in $\mathfrak{a}(M)$ (and hence in $\mathfrak{b}(M)$). Furthermore, we have a special system of parameters satisfying that

$$x_d \in \mathfrak{a}(M), x_{d-1} \in \mathfrak{a}(M/x_dM), \ldots, x_1 \in \mathfrak{a}(M/(x_2, \ldots, x_d)M).$$

Such a system of parameters is called a $p$-standard system of parameters [6]. The $p$-standard systems of parameters play a key ingredient in Kawasaki’s proof for the Macaulayfication problem [17]. By [9, Theorem 1.2] $R$ is a homomorphism of a Cohen-Macaulay local ring if and only if every
finitely generated $R$-module admits a $p$-standard system of parameters.

In this paper, we will use a kind of $p$-standard system of parameters to study the splitting of local cohomology modules. As mentioned above we know that $0 : x \subseteq U_M(0)$ for every parameter element $x$ of $M$. Moreover, if $x \in b(M)$ then we have $0 : x = U_M(0)$, so we get the following short exact sequence

$$0 \to M/U_M(0) \xrightarrow{\pi} M \to M/xM \to 0.$$  
Furthermore if $x \in b(M)^2$ then the above short exact sequence deduces the short exact sequence of local cohomology for any ideal $I$ (see Lemma 3.1)

$$0 \to H^i_I(M) \to H^i_I(M/xM) \to H^{i+1}_I(M/U_M(0)) \to 0$$

for all $i < d - \dim R/I - 1$. Using [11] we can study the splitting of these local cohomology exact sequences. Namely, the following is the first main result of this paper.

**Theorem 1.1.** Let $I$ be an ideal of $R$ and $x$ a parameter element of $M$ contained in $b(M)^3$. Then for all $i < d - \dim R/I - 1$ we have

$$H^i_I(M/xM) \cong H^i_I(M) \oplus H^{i+1}_I(M/U_M(0)).$$

In the case $I = m$ we have the following consequence.

**Corollary 1.2.** Let $x$ be a parameter element of $M$ contained in $b(M)^3$. Then

$$H^{i-1}_m(M/xM) \cong H^{i-1}_m(M) \oplus H^i_m(M/U_M(0))$$

for all $i < d - 1$, and

$$0 : H^{d-1}_m(M/xM) b(M) \cong H^{d-1}_m(M) \oplus 0 : H^d_m(M) b(M).$$

These splitting results lead a new kind of system of parameters $x_1, \ldots, x_d$ satisfying that

$$x_d \in b(M)^3, x_{d-1} \in b(M/x_M)^3, \ldots, x_1 \in b(M/(x_2, \ldots, x_d)M)^3.$$  
We call such a system of parameters a $C$-system of parameters of $M$. Similar to $p$-standard system of parameters, every finitely generated $R$-module admits $C$-systems of parameters if and only if $R$ is a quotient of a Cohen-Macaulay local ring. It should be noted that the right hand sides of the above isomorphisms do not depend of the choice of $C$-parameter element $x \in b(M)^3$. Thus the local cohomology modules $H^i_I(M/xM)$, $i < d - \dim R/I - 1$, are invariants (up to an isomorphism).

As consequences, we can expect several invariant properties of quotient modules $M/(x_1, \ldots, x_d)M$ regarding $C$-systems of parameters. For example, by using the fact $U_M(0) = H^0_{b(M)}(M)$, as the second main result of this paper, we generalize the unmixed theorem for any finitely generated $R$-module.

**Theorem 1.3.** Let $M$ be a finitely generated $R$-module of dimension $d$ and $\underline{x} = x_1, \ldots, x_d$ a $C$-system of parameters of $M$. Then the unmixed component $U_{M/(x_{i+1}, \ldots, x_d)M}(0)$ is independent of the choice of $\underline{x}$ for all $1 \leq i \leq d$ (up to an isomorphism).

The above theorem assigns to any finitely generated $R$-module $M$ of dimension $d$ a sequence of modules $U_0(M), \ldots, U_{d-1}(M)$, which satisfies that $U_i(M) \cong U_{M/(x_{i+2}, \ldots, x_d)M}(0)$ for every $C$-system of parameters $x_1, \ldots, x_d$ of $M$. Notice that $M$ is Cohen-Macaulay if and only if $U_i(M) = 0$ for all $i = 0, \ldots, d - 1$ by the unmixed theorem. This modules sequence gives information about the distance between $M$ and the Cohen-Macaulayness. We call $U_0(M), \ldots, U_{d-1}(M)$ the *Cohen-Macaulay deviated sequence* of $M$. The name of Cohen-Macaulay deviated sequence comes from the notion...
of \textit{Cohen-Macaulay deviation} of Vasconcelos in his theory of extended degrees.

Let $I$ be an $m$-primary ideal. We denote by $\deg(I, M)$ the ordinary multiplicity of $M$ with respect to $I$, and call the \textit{degree} of $M$ with respect to $I$. The degree, $\deg(I, M)$, is a basic invariant that measures the complexity of $M$ with respect to $I$. Vasconcelos et al. \cite{13, 34, 35} introduced the notion of \textit{extended degree} in order to capture the size of a module along with some of the complexity of its structure. It is a numerical function on the category of finitely generated modules over local or graded rings which generalizes the ordinary degree. Let $\mathcal{M}(R)$ be the category of finitely generated $R$-modules. An \textit{extended degree} on $\mathcal{M}(R)$ with respect to $I$ is a numerical function

$$\text{Deg}(I, \bullet) : \mathcal{M}(R) \to \mathbb{R}$$

satisfying the following conditions

(i) $\text{Deg}(I, M) = \text{Deg}(I, \overline{M}) + \ell(H^0_m(M))$, where $\overline{M} = M/H^0_m(M)$.
(ii) (Bertini’s rule) $\text{Deg}(I, M) \geq \text{Deg}(I, M/xM)$ for every generic element $x \in I \setminus mI$ of $M$.
(iii) If $M$ is Cohen-Macaulay then $\text{Deg}(I, M) = \deg(I, M)$.

The difference $\text{Deg}(I, M) - \deg(I, M)$ is called the \textit{Cohen-Macaulay deviation} of $M$ with respect to $I$. The prototype of an extended degree is the \textit{homological degree}, $h\text{deg}(I, M)$, was introduced and studied by Vasconcelos in \cite{34} (see Definition 5.4). Until nowadays, the homological degree is the unique extended degree that we can describe in an explicit formula. Using the Cohen-Macaulay deviated sequence we introduce a new degree of $M$, which we call the \textit{unmixed degree} of $M$ with respect to $I$, and denote by $\text{udeg}(I, M)$. We define

$$\text{udeg}(I, M) = \deg(I, M) + \sum_{i=0}^{d-1} \delta_{i, \dim U_i(M)} \deg(I, U_i(M)),$$

where $\delta_{i, \dim U_i(M)}$ is Kronecker’s symbol. The unmixed degree is a natural generalization of the ordinary degree as well as the \textit{arithmetic degree} (for the definition of arithmetic degree, $\text{adeg}(I, M)$, we refer to Definition 5.1). We prove the last main result of this paper as follows.

\textbf{Theorem 1.4.} The \textit{unmixed degree} $\text{udeg}(I, \bullet)$ is an extended degree on the category of finitely generated $R$-modules $\mathcal{M}(R)$.

Let us talk about the structure of this paper. In the next section we collect useful results about the annihilator of local cohomology, the unmixed component and some special systems of parameters. We also mention the method of \cite{11} to study the splitting of local cohomology.

Section 3 is devoted the splitting of local cohomology Theorem \textbf{1.1} and Corollary \textbf{1.2} (see Theorem \textbf{3.4} and Corollary \textbf{3.5}). Then we introduce the notion of $C$-system of parameters, that plays a key role in this paper.

Theorem \textbf{1.3} is proved in Section 4. We also prove the invariance of local cohomology of quotient modules regarding $C$-systems of parameters (cf. Theorem \textbf{4.1}). As an application of the Cohen-Macaulay deviated sequence $U_0(M), \ldots, U_{d-1}(M)$, we compute the length function $\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M)$ when $x_1, \ldots, x_d$ is a $C$-system of parameters (cf. Proposition \textbf{4.8}). Other applications for sequentially Cohen-Macaulay modules and the Serre condition $(S_2)$ are also given.

The unmixed degree will be introduced in Section 5. Theorem \textbf{1.4} follows from Proposition \textbf{5.7}, Theorems \textbf{5.11} and \textbf{5.16}. The most difficulty is to prove the Bertini rule of unmixed degree. For that we show that for certain \textit{superficial element} $x$ of $M$ with respect to $I$ we have $\text{udeg}(I, M/xM) \leq \text{udeg}(I, M)$. We also compare the unmixed degree with the ordinary degree, the arithmetic degree and the homological degree.
2. Preliminaries

We start with the notion of annihilator of local cohomology which will be used frequently in this paper.

**Notation 2.1.** Let \((R, \mathfrak{m})\) be a Noetherian local ring and \(M\) a finitely generated \(R\)-module of dimension \(d > 0\).

(i) For all \(i < d\) we set \(a_i(M) = \text{Ann} H^i_{\mathfrak{m}}(M)\), and set \(a(M) = a_0(M) \ldots a_{d-1}(M)\).

(ii) Put \(b(M) = \bigcap_{i=1}^d \text{Ann}(0 : x_i)_{M/(x_1, \ldots, x_{i-1})M}\) where \(x = x_1, \ldots, x_d\) runs over all systems of parameters of \(M\).

**Remark 2.2.** (i) Schenzel [30] Satz 2.4.5] proved that

\[
a(M) \subseteq b(M) \subseteq a_0(M) \cap \cdots \cap a_{d-1}(M).
\]

(ii) If \(R\) is a homomorphic image of a Cohen-Macaulay local ring, then \(\dim R/a_i(M) \leq i\) for all \(i < d\) [30 Theorem 1.2]. Furthermore, \(\dim R/a_i(M) = i\) if and only if there exists \(p \in \text{Ass} M\) such that \(\dim R/p = i\) (see [3] Theorem 8.1.1)).

(iii) If \(R\) is a homomorphic image of a Cohen-Macaulay local ring, then Faltings’ annihilator theorem claims that \(p \in \text{supp}(M)\) and \(p \notin V(a(M))\) if and only if \(M_p\) is Cohen-Macaulay and \(\dim M_p + \dim R/p = d\) (see [2] 9.6.6], [4]).

(iv) The condition that \(R\) is a homomorphic image of a Cohen-Macaulay local ring cannot be removed in (ii) and (iii) by Nagata’s example [21 Example 2, pp. 203–205].

Since we always assume that \((R, \mathfrak{m})\) is a homomorphic image of a Cohen-Macaulay local ring, Remark 2.2(ii) ensures that \(\dim R/a(M) < d\). Therefore we can choose a parameter element \(x \in a(M)\). Following [6] such a parameter element is called \(p\)-standard.

**Definition 2.3.** A system of parameters \(x_1, \ldots, x_d\) of \(M\) is called \(p\)-standard if \(x_d \in a(M)\) and \(x_i \in a(M/(x_{i+1}, \ldots, x_d)M)\) for all \(i = d-1, \ldots, 1\).

We recall a property of \(p\)-standard system of parameters which will be used in the sequel. Let \(\underline{x} = x_1, \ldots, x_d\) be a system of parameters of \(M\). Let \(\underline{n} = (n_1, \ldots, n_d)\) be a \(d\)-tuple of positive integers and \(\underline{x}^{\underline{n}} = x_1^{n_1}, \ldots, x_d^{n_d}\). We consider the difference

\[
I_{M, \underline{x}}(\underline{n}) = e(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}M)
\]

as function in \(\underline{n}\), where \(e(\underline{x}; M)\) is the Serre multiplicity of \(M\) with respect to the sequence \(\underline{x}\). Although \(I_{M, \underline{x}}(\underline{n})\) may be not a polynomial for \(n_1, \ldots, n_d\) large enough, it is bounded above by polynomials. Moreover, the first author in [4] proved that the least degree of all polynomials in \(\underline{n}\) bounding above \(I_{M, \underline{x}}(\underline{n})\) is independent of the choice of \(\underline{x}\), and it is denoted by \(p(M)\). The invariant \(p(M)\) is called the polynomial type of \(M\). If \((R, \mathfrak{m})\) is a homomorphic image of a Cohen-Macaulay local ring, then \(p(M) = \dim R/a(M)\) (see [5]). In addition, if \(\underline{x} = x_1, \ldots, x_d\) is \(p\)-standard then we have the following.

**Proposition 2.4** ([6], Theorem 2.6(ii)). Let \(x_1, \ldots, x_d\) be a \(p\)-standard system of parameters of \(M\). Then for all \(n_1, \ldots, n_d > 0\) we have

\[
I_{M, \underline{x}}(\underline{n}) = \sum_{i=0}^{p(M)} n_1 \ldots n_i e_i,
\]

where \(e_i = e(x_1, \ldots, x_i; 0 : M/(x_{i+2}, \ldots, x_d)M x_{i+1})\) and \(e_0 = e(0 : M/(x_2, \ldots, x_d)M x_1)\).

Recently, Cuong and the first author introduced the notion of \(dd\)-sequence which is a special case of the notion of \(d\)-sequences of Huneke.
Definition 2.5 ([16][15]). A sequence of elements \( \underline{x} = x_1, \ldots, x_s \) is called a \( d \)-sequence of \( M \) if \((x_1, \ldots, x_{i-1})M : x_j = (x_1, \ldots, x_{i-1})M : x_{i-1}x_j \) for all \( i \leq j \leq s \). A sequence \( \underline{x} = x_1, \ldots, x_s \) is called a strong \( d \)-sequence if \( \underline{x^n} = x_1^{n_1} \cdots x_s^{n_s} \) is a \( d \)-sequence for all \( n = (n_1, \ldots, n_s) \in \mathbb{N}^s \).

For important properties of \( d \)-sequence we refer to [16][33].

Definition 2.6 ([7]). A sequence of elements \( \underline{x} = x_1, \ldots, x_s \) is call a \( dd \)-sequence of \( M \) if \( \underline{x} \) is a strong \( d \)-sequence of \( M \) and the following conditions are satisfied

(i) \( s = 1 \) or,
(ii) \( s > 1 \) and \( \underline{x'} = x_1, \ldots, x_{s-1} \) is a \( dd \)-sequence of \( M/x_s^n \) for all \( n \geq 1 \).

The following is a characterization of \( dd \)-sequence in terms of \( I_{M, \underline{x}}(n) \) ([7] Theorem 1.2]).

Proposition 2.7. A system of parameters \( \underline{x} = x_1, \ldots, x_d \) of \( M \) is a \( dd \)-sequence if and only if for all \( n_1, \ldots, n_d > 0 \) we have

\[
I_{M, \underline{x}}(n) = \sum_{i=0}^{p(M)} n_1 \cdots n_1 e_i,
\]

where \( e_i = e(x_1, \ldots, x_i; 0 : M/(x_{i+1}, \ldots, x_d)M x_{i+1}) \) and \( e_0 = \ell(0 : M/(x_2, \ldots, x_d)M x_1) \).

Remark 2.8. (i) By Propositions 2.6 and 2.7 if a system of parameter \( x_1, \ldots, x_d \) of \( M \) is \( p \)-standard, then it is a \( dd \)-sequence. Conversely, if \( x_1, \ldots, x_d \) is a \( dd \)-sequence then \( x_1^{n_1}, \ldots, x_d^{n_d} \) with \( n_i \geq 1 \) \( i = 1, \ldots, d \) is \( p \)-standard (see [7] Section 3).

(ii) An \( R \)-module \( M \) admits a \( p \)-standard (or \( dd \)-sequence) system of parameters if and only if \( R/\text{Ann}M \) is a homomorphic image of a Cohen-Macaulay local ring [9] Theorem 1.2].

We next recall the notion of unmixed component of \( M \) and its relations with the ideal \( b(M) \).

Definition 2.9. The largest submodule of \( M \) of dimension less than \( d \) is called the unmixed component of \( M \), and denoted by \( U_M(0) \).

Remark 2.10. (i) If \( \cap_{p \in \text{Ass}M} N(p) = 0 \) is a reduced primary decomposition of the zero submodule of \( M \), then \( U_M(0) = \cap_{\dim R/p = d} N(p) \), where \( \text{Ass}M = \{ p \in \text{Ass}M \mid \dim R/p = d \} \).

(ii) Since \( \dim U_M(0) < d \), there exists a parameter element \( x \) of \( M \) contained in \( \text{Ann}U_M(0) \). Therefore \( U_M(0) \subseteq 0 : x. \) But \( x \) is a parameter element, so \( \dim 0 : x < d \). Hence \( U_M(0) = 0 : x. \) Following the definition of \( b(M) \) we have \( b(M) \subseteq \text{Ann}U_M(0). \) Thus if \( x \in b(M) \) is a parameter element of \( M \) then \( U_M(0) = 0 : x. \) We also have \( U_M(0) \cong H^0_{b(M)}(M) \).

(iii) By (ii) we have \( \cap_{b} \text{Ann}(0 : M x) = \text{Ann}U_M(0) \), where \( x \) runs over all parameter elements of \( M. \) Therefore

\[
b(M) = \bigcap_{x_{i=1}^d} \text{Ann}(0 : x_i)M/(x_{i+1}, \ldots, x_d)M x_{i+1}
\]

\[
= \bigcap_{x_{i=1}^d} \text{Ann}U_M/(x_{i+1}, \ldots, x_d)M(0),
\]

where \( \underline{x} = x_1, \ldots, x_d \) runs over all systems of parameters of \( M. \)

Problem of the splitting of local cohomology is started in [11]. For convenience we recall some results of [11] (with slight generalizations). Suppose we are given an integer \( t \), an ideal \( a \) of \( R \) and a submodule \( U \) of \( M. \) Set \( M = M/U. \) We say that an element \( x \in a \) satisfies the condition (**) if \( 0 :_M x = U \) and the short exact sequence

\[
0 \longrightarrow \overline{M} \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0
\]
induces short exact sequences
\[ 0 \rightarrow H^t_a(M) \rightarrow H^t_a(M/xM) \rightarrow H^{t+1}_a(M/x^tM) \rightarrow 0 \]
for all \( i < t - 1 \). When this is the case, we consider the above exact sequence as an extension of \( H^{t+1}_a(M) \) by \( H^t_a(M) \), therefore as an element of \( \text{Ext}^1_R(H^{t+1}_a(M), H^t_a(M)) \) (see [19, Chapter 3]). We denote this element by \( F^t_x \). Especially, if \( H^t_a(M) \cong H^t_a(M) \), then we have the short exact sequence
\[ 0 \rightarrow H^t_a(M) \rightarrow H^t_a(M/xM) \rightarrow 0 : H^t_a(M) x \rightarrow 0. \]

Let \( b \) be an ideal such that \( x \in b \). We denote by \( F^{t-1}_x \) the element of \( \text{Ext}^1_R(0 : H^t_a(M) b, 0 : H^{t-1}_a(M) b) \) which represented by the following short exact sequence
\[ 0 \rightarrow 0 : H^{t-1}_a(M) b \rightarrow 0 : H^{t-1}_a(M/xM) b \rightarrow 0 : H^t_a(M) b \rightarrow 0 \]
provided the exact sequence is determined by applying the \( \text{Hom}(R/b, \bullet) \) functor. It should be noted here that an extension of \( R \)-module \( A \) by an \( R \)-module \( C \) is split if it is the zero-element of \( \text{Ext}^1_R(C, A) \). The two next theorems can be proven by the same method as used in [11, Theorem 2.2]

**Theorem 2.11.** Let \( t \) be a positive integer and \( U \) a submodule of \( M \). Let \( M = M/U \). Suppose \( x \) and \( y \) are elements satisfying the condition (\( x \)) and \( 0 : M (x + y) = U \). Then

(i) \( x + y \) also satisfies the condition (\( x \)) and \( E^i_{x+y} = E^i_x + E^i_y \) for all \( i < t - 1 \).

(ii) If \( H^t_a(M) \cong H^t_a(M) \) and \( F^{t-1}_x, F^{t-1}_y \) are determined, then \( F^{t-1}_{x+y} \) is determined and \( F^{t-1}_{x+y} = F^{t-1}_x + F^{t-1}_y \).

**Theorem 2.12.** Let \( t \) be a positive integer and \( U \) a submodule of \( M \). Let \( M = M/U \). Suppose \( x \) and \( y \) are elements such that \( x \) satisfies the condition (\( x \)) and \( 0 : M xy = U \). Then

(i) \( xy \) satisfies the condition (\( x \)) and \( E^i_{xy} = E^i_x y \) for all \( i < t - 1 \). Suppose that \( H^t_a(M) \cong H^t_a(M) \) and \( F^{t-1}_x \) is determined. Then \( F^{t-1}_{xy} \) is determined and \( F^{t-1}_{xy} = y F^{t-1}_x \).

(ii) Suppose that \( H^t_a(M) \cong H^t_a(M) \) and \( y H^t_a(M) = 0 \) for all \( i < t - 1 \). Moreover, \( F^{t-1}_{xy} \) is determined and \( F^{t-1}_{xy} = 0 \).

The following is a prime avoidance theorem for a product of ideals.

**Lemma 2.13** ([11] Lemma 3.1). Let \((R, m)\) be a Noetherian local ring, \( a, b \) ideals and \( p_1, \ldots, p_n \) prime ideals such that \( ab \nsubseteq p_j \) for all \( j \leq n \). Let \( x \in ab \) with \( x \notin p_j \) for all \( j \leq n \). There are elements \( a_1, \ldots, a_r \in a \) and \( b_1, \ldots, b_r \in b \) such that \( x = a_1b_1 + \cdots + a_rb_r \), and that \( a_ib_j \notin p_j \) and \( a_1b_1 + \cdots + a_rb_r \notin p_j \) for all \( i \leq r \) and all \( j \leq n \).

**Corollary 2.14.** Let \((R, m)\) be a Noetherian local ring, \( M \) a finitely generated \( R \)-module of dimension \( d > 0 \), \( a \) and \( b \) two ideals such that \( \dim R/ab < d \). Let \( x \in ab \) be a parameter element of \( M \).

There exist parameter elements \( a_1, \ldots, a_r \in a \) and \( b_1, \ldots, b_r \in b \) of \( M \) such that \( x = a_1b_1 + \cdots + a_rb_r \), and that \( a_1b_1 + \cdots + a_rb_r \) is a parameter element for all \( i \leq r \).

**Proof.** Note that an element \( x \) is a parameter element of \( M \) if and only if \( x \notin p \) for all \( p \in \text{Ass}M \). The assertion now follows from Lemma 2.13.

3. **The splitting of local cohomology**

In this section we prove splitting theorems for local cohomology in local rings. These results lead a new kind of systems of parameters. We need the following key ingredient about the annihilator of local cohomology supported at an arbitrary ideal that is of independent interest.
Proposition 3.1. Let $M$ be a finitely generated $R$-module of dimension $d$ and $I$ an ideal of $R$. We have $b(M)H_i^j(M) = 0$ for all $i < d - \dim R/I$.

To prove the above result we use the following isomorphism of Nagel and Schenzel (see [24, Proposition 3.4]). Recall that a sequence $x_1, \ldots, x_t$ of elements contained in $I$ is an $I$-filter regular sequence of $M$ if

$$\text{Supp} ((x_1, \ldots, x_i)M : x_i)/(x_1, \ldots, x_i-1)M \subseteq V(I)$$

for all $i = 1, \ldots, t$, where $V(I)$ denotes the set of prime ideals containing $I$. This condition is equivalent to that $x_i \notin p$ for all $p \in \text{Ass}_R M/(x_1, \ldots, x_i-1)M \setminus V(I)$ for all $i = 1, \ldots, t$. Moreover we can choose an $I$-filter regular sequence on $M$ of any length by the prime avoidance theorem.

Lemma 3.2 (Nagel-Schenzel’s isomorphism). Let $I$ be an ideal of $R$ and $x_1, \ldots, x_t$ an $I$-filter regular sequence of $M$. For each $j \leq t$ we have

$$H^j_I(M) \cong \begin{cases} H^j_{(x_1, \ldots, x_i)}(M) & \text{with } j < t, \\ H^{j-t}(H^1_{(x_1, \ldots, x_i)}(M)) & \text{with } j \geq t. \end{cases}$$

Proof of Proposition 3.1. Set $t = d - \dim R/I$. Suppose $t < d$, by the prime avoidance theorem we can choose an element $x_1 \in I$ such that $x_1 \notin p$ for all $p \in \text{Ass}_R M \cup (\text{Ass} \setminus V(I))$. Thus $x_1$ is a parameter element of $M$ that is also an $I$-filter regular element. We continue this progress to obtain a part of a system of parameters $x_1, \ldots, x_t$ of $M$ that is also an $I$-filter regular on $M$. By Lemma 3.2 for $i < t$, we have

$$H^j_I(M) \cong H^0_I(H^j_{(x_1, \ldots, x_i)}(M))$$

$$\cong H^0_I(\lim_{\rightarrow} M/(x_1^n, \ldots, x_i^n)M)$$

$$\cong \lim_{\rightarrow} H^0_I(M/(x_1^n, \ldots, x_i^n)M)$$

$$\cong \lim_{\rightarrow} \frac{(x_1^n, \ldots, x_i^n)M : I^\infty}{(x_1^n, \ldots, x_i^n)M}$$

$$\cong \lim_{\rightarrow} \frac{(x_1^n, \ldots, x_i^n)M : x_{i+1}^\infty}{(x_1^n, \ldots, x_i^n)M} = 0$$

for all $n \geq 1$ by the definition of $b(M)$. Hence $b(M)H^j_I(M) = 0$ for all $i < d - \dim R/I$. The proof is complete.

Lemma 3.3. Let $I$ be an ideal of $R$ and $x, y \in b(M)$ parameter elements of $M$. Let $U_M(0)$ be the unmixed component of $M$. Put $\overline{M} = M/U_M(0)$ and $t = d - \dim R/I$. Then for all $i < t - 1$ we have the following short exact sequence

$$0 \rightarrow H^i_I(M) \rightarrow H^i_I(M/xyM) \rightarrow H^{i+1}_I(\overline{M}) \rightarrow 0.$$ 

Furthermore, if $H^j_I(M) \cong H^j_I(\overline{M})$ then we have the short exact sequence

$$0 \rightarrow H^{i-1}_I(M) \rightarrow H^{i-1}_I(M/xyM) \rightarrow 0 : H^j_I(M) xy \rightarrow 0.$$
Proof. By Remark 2.10 (ii) we have $U_M(0) = 0 :_M x = 0 :_M xy$. Therefore the following diagram commutes

$$
\begin{array}{ccc}
0 & \xrightarrow{\cdot} & \overline{M} \\
 & \downarrow{id} & \downarrow{y} \\
0 & \xrightarrow{\cdot} & \overline{M}
\end{array}
\xrightarrow{\cdot} \begin{array}{ccc}
0 & \xrightarrow{x} & M \\
 & \downarrow{id} & \downarrow{y} \\
0 & \xrightarrow{x} & M/xyM
\end{array} \xrightarrow{\cdot} 0.
$$

Applying the functor $H^i_I(\bullet)$ to the above diagram we obtain the following commutative diagram for all $i < t - 1$

$$
\cdots \xrightarrow{} H^i_I(M) \xrightarrow{\psi^i} H^i_I(M/xyM) \xrightarrow{\cdot} \cdots
$$

where $\psi^i$ and $\varphi^i$ are derived from homomorphisms $\overline{M} \xrightarrow{\cdot} M$ and $\overline{M} \xrightarrow{\cdot x} M$, respectively. By Proposition 3.1, $yH^i_I(M) = 0$ for all $i \leq t - 1$, so $\varphi^i = 0$ for all $i \leq t - 1$. Thus we have the short exact sequences

$$0 \rightarrow H^i_I(M) \rightarrow H^i_I(M/xyM) \rightarrow H^{i+1}_I(M) \rightarrow 0$$

for all $i < t - 1$. Thus we have the exact sequence

$$0 \rightarrow H^{t-1}_I(M) \rightarrow H^{t-1}_I(M/xyM) \rightarrow H^{t}_I(M/xyM) \rightarrow H^{t}_I(M/\overline{M}) \rightarrow 0.$$

Moreover, if $H^1_I(M) \cong H^1_I(M)$ then we have the following short exact sequence

$$0 \rightarrow H^{t-1}_I(M) \rightarrow H^{t-1}_I(M/xyM) \rightarrow 0 :_{H^1_I(M)} xy \rightarrow 0.$$

The proof is complete. □

Let $xy$ be a parameter element of $M$ such that $x, y \in b(M)$. Lemma 3.3 says that $xy$ satisfies the condition ($\sharp$) mentioned in Section 2 with $t = d - \dim R/I$ and $U = U_M(0)$. Let $x \in b(M)^2$ be a parameter element of $M$, for all $i < t - 1$, we denote by $E^i_x$ the element in $\Ext(H^{i+1}_I(M), H^1_I(M))$ represented by the following short exact sequence provided it is determined

$$0 \rightarrow H^i_I(M) \rightarrow H^i_I(M/xyM) \rightarrow H^{i+1}_I(M) \rightarrow 0.$$

In the case $i = t - 1$ and assume that $H^1_I(M) \cong H^1_I(M)$, we have the short exact sequence

$$0 \rightarrow H^{t-1}_I(M) \rightarrow H^{t-1}_I(M/xyM) \rightarrow 0 :_{H^1_I(M)} x \rightarrow 0.$$

Suppose we obtain the following short exact sequence by applying the $\Hom(R/b(M), \bullet)$ to above short exact sequence

$$0 \rightarrow H^{t-1}_I(M) \rightarrow 0 :_{H^{t-1}_I(M/xyM)} b(M) \rightarrow 0 :_{H^1_I(M)} b(M) \rightarrow 0.$$

Then we denote by $F^{t-1}_x$ the element of $\Ext(0 :_{H^1_I(M)} b(M), H^{t-1}_I(M))$ represented by the above short exact sequence. The main result of this section as follows.

**Theorem 3.4.** Let $M$ be a finitely generated $R$-module of dimension $d$, $I$ an ideal of $R$ and $x$ a parameter element of $M$. Let $U_M(0)$ be the unmixed component of $M$ and set $\overline{M} = M/U_M(0)$. Let $t = d - \dim R/I$. Then

(i) If $x \in b(M)^2$ then $E^i_x$ is determined for all $i < t - 1$.

(ii) If $x \in b(M)^3$ then $E^i_x = 0$ for all $i < t - 1$. Moreover, if $H^1_I(M) \cong H^1_I(M)$ then $F^{t-1}_x = 0$. 


Proof. (i) Notice that \( b(M) \not\subset p \) for all \( p \in \text{Assh} M \). By Corollary 2.13 there exist parameter elements \( a_1, \ldots, a_r, b_1, \ldots, b_r \in b(M) \) of \( M \) such that \( x = a_1b_1 + \cdots + a_r b_r \), and \( a_1b_1 + \cdots + a_j b_j \) are parameter elements for all \( j \leq r \). By Lemma 3.3 \( E_{a_k b_k}^i \) is determined for all \( i < t - 1 \) and for all \( 1 \leq k \leq r \). By Theorem 2.11 we have

\[
E_x^i = E_{a_1 b_1}^i + \cdots + E_{a_r b_r}^i
\]

is determined for all \( i < t - 1 \).

(ii) Similarly, we choose parameter elements \( a_1, \ldots, a_r \in b(M)^2 \) and \( b_1, \ldots, b_r \in b(M) \) of \( M \) such that \( x = a_1b_1 + \cdots + a_r b_r \), and \( a_1b_1 + \cdots + a_j b_j \) are parameter elements for all \( j \leq r \). By Theorem 2.12 (ii) we have \( E_{a_k b_k}^i = 0 \) for all \( i < t - 1 \) and for all \( 1 \leq k \leq r \). So \( E_x^2 = 0 \) for all \( i < t - 1 \). For the last assertion, by the same method, it is sufficient to show that \( F_{ab}^{t-1} = 0 \) for all parameter elements \( a \in b(M)^2 \) and \( b \in b(M) \) provided \( H_1^f(M) \cong H_1^f(M) \). Indeed, since \( E_a^i \) and \( E_{ab}^i \) are determined for all \( i < t - 1 \), the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & M[a] & \longrightarrow & M & \longrightarrow & M/abM & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M & \longrightarrow & M/abM & \longrightarrow & 0.
\end{array}
\]

duces the following diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_1^f(M) & \longrightarrow & H_1^f(M/aM) & \longrightarrow & 0 : H_1^f(M) a & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1^f(M) & \longrightarrow & H_1^f(M/abM) & \longrightarrow & 0 : H_1^f(M) ab & \longrightarrow & 0,
\end{array}
\]

where \( \alpha : 0 : H_1^f(M) a \rightarrow 0 : H_1^f(M) ab \) is injective. By Proposition 3.4 \( bH_1^f(M) = 0 \), so \( \beta \circ i = 0 \).

Thus we have a homomorphism \( \epsilon : 0 : H_1^f(M) a \rightarrow H_1^f(M/abM) \) which makes the following diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_1^f(M) & \longrightarrow & H_1^f(M/aM) & \longrightarrow & 0 : H_1^f(M) a & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1^f(M) & \longrightarrow & H_1^f(M/abM) & \longrightarrow & 0 : H_1^f(M) ab & \longrightarrow & 0,
\end{array}
\]

By applying the \( \text{Hom}_R(R/b(M), \bullet) \) to the above diagram we have the following diagram

\[
\begin{array}{cccccc}
0 : H_1^f(M) b(M) & \longrightarrow & 0 : H_1^f(M/abM) b(M) & \longrightarrow & 0 : H_1^f(M) b(M),
\end{array}
\]

where the row is an exact sequence and the vertical map is an identification. Since \( \pi \circ \epsilon = \text{id} \), the homomorphism \( \pi \) is split. Thus \( F_{ab}^{t-1} = 0 \). The proof is complete. \( \square \)

In the case \( I = m \), the following is a generalization of [11, Corollary 4.1] and [25, Proposition 3.4].
Corollary 3.5. Let \( x \in b(M)^3 \) be a parameter element of \( M \). Let \( U_M(0) \) be the unmixed component of \( M \) and set \( \overline{M} = M/U_M(0) \). Then
\[
H^i_m(M/xM) \cong H^i_m(M) \oplus H^{i+1}_m(\overline{M})
\]
for all \( i < d - 1 \), and
\[
0 : H^{d-1}_m(M/xM) b(M) \cong H^{d-1}_m(\overline{M}) \oplus 0 : H^{d}_m(M) b(M).
\]

By the above splitting theorems it is natural to consider the following system of parameters.

**Definition 3.6** ([20], Definition 2.15). A parameter element \( x \in b(M)^3 \) is called a \( C \)-parameter element of \( M \). A system of parameters \( x_1, \ldots, x_d \) is called a \( C \)-system of parameters of \( M \) if \( x_d \in b(M)^3 \) and \( x_i \in b(M/(x_{i+1}, \ldots, x_d)M)^3 \) for all \( i = d - 1, \ldots, 1 \). A sequence of elements \( x_i, \ldots, x_d \) is called a part of a \( C \)-system of parameters if we can expand it to a \( C \)-system of parameters \( x_1, \ldots, x_d \).

It is evident that \( C \)-systems of parameters are closely related with \( p \)-standard systems of parameters. Lemmas below will be very useful in the sequel.

**Lemma 3.7.** Let \( x \) be a parameter element of \( M \). Then \( b(M) \subseteq b(M/xM) \).

**Proof.** It follows from the definition of \( b(M) \). \( \square \)

**Lemma 3.8.** Let \( x_1, \ldots, x_d \) be a \( C \)-system of parameters of \( M \). Then, for all \( j \leq d \) we have \( x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d \) is a \( C \)-system of parameters \( M/x_jM \).

**Proof.** The case \( j = d \) is clear. For \( j \neq d \) by Lemma 3.7 we have \( b(M) \subseteq b(M/x_jM) \). Therefore \( x_d \) is a \( C \)-parameter element of \( M/x_jM \). Notice that \( x_1, \ldots, x_{d-1} \) is a \( C \)-system of parameters of \( M/x_dM \). The claim follows from the induction on \( d \). \( \square \)

4. The Cohen-Macaulay deviated sequences

In this section we use the splitting theorem 3.4 to shed a new light on the structure of non-Cohen-Macaulay modules. Let \( M \) be a finitely generated \( R \)-module of dimension \( d \). The unmixed characterization of Cohen-Macaulay modules says that \( M \) is Cohen-Macaulay if and only if for some (and hence for all) system of parameters \( x_1, \ldots, x_d \) we have \( U_{M/(x_{i+1}, \ldots, x_d)M}(0) = 0 \) for all \( 1 \leq i \leq d \). If \( M \) is a generalized Cohen-Macaulay module and \( m^{n_0} H^i_m(M) = 0 \) for all \( i < d \) and for some positive integer \( n_0 \), then by [11, Corollary 4.2] we have
\[
U_{M/(x_{i+1}, \ldots, x_d)M}(0) = H^0_m(M/(x_{i+1}, \ldots, x_d)M) \cong \bigoplus_{j=0}^{d-i} H^j_m(M)^{d-i-j}.
\]

for any system of parameters \( x_1, \ldots, x_d \in m^{2n_0} \). Thus \( U_{M/(x_{i+1}, \ldots, x_d)M}(0) \) is independent of the choice of system of parameters \( x_1, \ldots, x_d \) contained in \( m^{2n_0} \) for all \( 1 \leq i \leq d \) (up to an isomorphism). The main aim this section is to generalize this fact for any finitely generated \( R \)-module. Concretely, we will show that for all \( 1 \leq i \leq d \) the modules \( U_{M/(x_{i+1}, \ldots, x_d)M}(0) \) is independent (up to an isomorphism) of the choice of a \( C \)-system of parameters \( x_1, \ldots, x_d \). We start with the following result about the invariance of local cohomology of quotient modules regarding \( C \)-systems of parameters.

**Theorem 4.1.** Let \( \underline{x} = x_1, \ldots, x_d \) be a \( C \)-system of parameters of \( M \). Then the local cohomology module \( H^j_m(M/(x_{i+1}, \ldots, x_d)M) \) is independent of the choice of \( \underline{x} \) for all \( j < i < d \) (up to an isomorphism).
Proof. We set $M_i = M/(x_{i+1}, \ldots, x_d)M$ for all $i < d$. We consider another $C$-system of parameters $y = y_1, \ldots, y_d$ of $M$, and put $M'_i = M/(y_{i+1}, \ldots, y_d)M$ for all $i < d$. We proceed by induction on $d$ that $H^j_{m}(M_i) \cong H^j_{m}(M'_i)$ for all $j < i < d$. The assertion is trivial if $d = 1$. For $d > 1$ and $i = d - 1$ since $x_d$ and $y_d$ are $C$-parameter elements, Corollary 4.5 implies that

$$H^i_{m}(M_{d-1}) \cong H^i_{m}(M) \oplus H^{i+1}_{m}(M/U_M(0)) \cong H^i_{m}(M'_{d-1})$$

for all $j < d - 1$. Suppose $i < d - 1$. Since $\dim R/b(M_{i+1}) < \dim M_{i+1} = i + 1$ and $\dim R/b(M'_{i+1}) < \dim M'_{i+1} = i + 1$ we can choose a $C$-parameter element $z$ of both $M_{i+1}$ and $M'_{i+1}$. By the inductive hypothesis we have

$$H^i_{m}(M_i) = H^i_{m}(M_{i+1}/x_{i+1}M_{i+1}) \cong H^i_{m}(M/(z, x_{i+2}, \ldots, x_d)M),$$

and

$$H^i_{m}(M'_i) = H^i_{m}(M'_{i+1}/y_{i+1}M'_{i+1}) \cong H^i_{m}(M/(z, y_{i+2}, \ldots, y_d)M)$$

for all $j < i$. Notice that $z, x_{i+2}, \ldots, x_d$ and $z, y_{i+2}, \ldots, y_d$ are parts of $C$-systems of parameters of $M$. By Lemma 4.3 we have $x_{i+2}, \ldots, x_d$ and $y_{i+2}, \ldots, y_d$ are parts of $C$-systems of parameters of $M/zM$. Applying the inductive hypothesis for $M/zM$ we have

$$H^i_{m}(M/(z, x_{i+2}, \ldots, x_d)M) \cong H^i_{m}(M/(z, y_{i+2}, \ldots, y_d)M)$$

for all $j < i$. The assertion follows from the isomorphisms (1), (2) and (3).

Corollary 4.2. Let $x = x_1, \ldots, x_d$ be a $C$-system of parameters of $M$. Then the ideals $a(M/(x_{i+1}, \ldots, x_d)M)$ and $\sqrt{a(M/(x_{i+1}, \ldots, x_d)M)} = \sqrt{b(M/(x_{i+1}, \ldots, x_d)M)}$ are independent of the choice of $x$ for all $i < d$.

We need the following result.

Lemma 4.3. Let $x$ be a $C$-parameter element of $M$. Then $U_{M/xM}(0)$ is independent of the choice of $x$ (up to an isomorphism).

Proof. By Corollary 4.2 we have the ideal

$$b' = \sqrt{a(M/xM)} = \sqrt{b(M/xM)}$$

is independent of the choice of $C$-parameter element $x$. By Remark 2.10 (ii) we have $U_{M/xM}(0) \cong H^0_{b'}(M/xM)$. Since $\dim R/b' \leq \dim M/xM - 1 = d - 2$, Theorem 3.4 (ii) implies that

$$H^0_{b'}(M/xM) \cong H^0_{b'}(M) \oplus H^1_{b'}(M/U_M(0)),$$

and the right hand side does not depend on $x$. Thus the unmixed component $U_{M/xM}(0)$ is independent of the choice of $C$-parameter element $x$ (up to an isomorphism). \hfill $\Box$

Using Lemma 4.3 and by the same method as used in the proof of Theorem 4.1 we obtain the main result of this section as follows.

Theorem 4.4. Let $M$ be a finitely generated $R$-module of dimension $d$ and $x = x_1, \ldots, x_d$ a $C$-system of parameters of $M$. Then for all $1 \leq i \leq d$, the unmixed component $U_{M/(x_{i+1}, \ldots, x_d)M}(0)$ is independent of the choice of $x$ (up to an isomorphism).

Definition 4.5. For all $0 \leq i \leq d - 1$ we denote by $U_i(M)$ the module satisfying that $U_i(M) \cong U_{M/(x_{i+2}, \ldots, x_d)M}(0)$ for all $C$-systems of parameters $x_1, \ldots, x_d$ of $M$. Notice that $\dim U_i(M) \leq i$ for all $0 \leq i \leq d - 1$, and $U_{d-1}(M) \cong U_M(0)$. We call the modules sequence $U_0(M), \ldots, U_{d-1}(M)$ the Cohen-Macaulay deviated sequence of $M$. Notice that the Cohen-Macaulay deviated sequence of $M$ vanishes if and only if $M$ is Cohen-Macaulay.
We next use the Cohen-Macaulay deviated sequence to prove some properties of $C$-systems of parameters.

**Corollary 4.6.** Let $\underline{x} = x_1, \ldots, x_d$, $i > 1$, be a part of a $C$-system of parameters of $M$. Then $b(M/(x_1, \ldots, x_d)M) = b(M/(x_{d-i}^i, \ldots, x_d^i)M)$ for all $n_j \geq 1$ and all $i \leq j \leq d$.

**Proof.** For $i = d$, notice that $y = y_1, \ldots, y_{d-1}$ is a system of parameters of $M/x_dM$ if and only if it is also a system of parameters of $M/x_d^{n_d}M$ for all $n_d \geq 1$. By Lemma 3.7 we have $x_d$ and hence $x_d^{n_d}$ are contained in $b(M/(y_1, \ldots, y_{d-1})M)$ for all $1 \leq j \leq d - 1$. So Theorem 4.4 claims that

$$
U_{M/(y_1, \ldots, y_{d-1}, x_d)}(0) \cong U_{M/(y_1, \ldots, y_{d-1}, x_d^{n_d})}(0)
$$

for all $1 \leq j \leq d - 1$. By Remark 2.10 (iii) we have

$$
b(M/x_dM) = \bigcap_{y_{d-j} = 1}^{d-j} \text{Ann} U_{M/(y_1, \ldots, y_{d-1}, x_d)}(0)
$$

$$
= \bigcap_{y_{d-j} = 1}^{d-j} \text{Ann} U_{M/(y_1, \ldots, y_{d-1}, x_d^{n_d})}(0)
$$

$$
= b(M/x_d^{n_d}M),
$$

where $y = y_1, \ldots, y_{d-1}$ runs over all systems of parameters of $M/x_dM$.

We now proceed by induction on $d$. The case $d = 2$ follows from the above fact since $i = 2$. Suppose $d \geq 3$ and $i < d$. Applying the inductive hypothesis for $M/(x_{i+1}, \ldots, x_d)M$ we have

$$
b(M/(x_1, x_{i+1}, \ldots, x_d)M) = b(M/(x_1^{n_1}, x_{i+1}, \ldots, x_d)M)
$$

for all $n_i \geq 1$. By Lemma 4.8 we have $x_{i+1}, \ldots, x_d$ is a part of a $C$-system of parameters of $M/x_1^{n_1}M$. By using the inductive hypothesis for $M/x_1^{n_1}M$ we obtain

$$
b(M/(x_1^{n_1}, x_{i+1}, \ldots, x_d)M) = b(M/(x_1^{n_1}, x_{i+1}, \ldots, x_d^{n_d})M)
$$

for all $n_{i+1}, \ldots, n_d \geq 1$. The proof is complete. \qed

**Corollary 4.7.** Let $\underline{x} = x_1, \ldots, x_d$ be a $C$-system of parameters of $M$. Then for all $d$-tuples of positive integers $\underline{n} = (n_1, \ldots, n_d)$ we have $x_1^{n_1}, \ldots, x_d^{n_d}$ is also a $C$-system of parameters.

**Proof.** The assertion follows immediately from Corollary 4.6 and the definition of $C$-system of parameters. \qed

**An application to $dd$-sequences.** We use the Cohen-Macaulay deviated sequence to compute the function $I_{M, \underline{x}}(\underline{n})$.

**Proposition 4.8.** Let $\underline{x} = x_1, \ldots, x_d$ be a $C$-system of parameters of $M$. Let $U_i(M)$, $0 \leq i \leq d-1$, be the Cohen-Macaulay deviated sequence of $M$. Then the difference

$$
I_{M, \underline{x}}(\underline{n}) = \ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) - n_1 \ldots n_d c(x_1, \ldots, x_d; M)
$$

is a polynomial in $\underline{n} = n_1, \ldots, n_d$. More precisely

$$
I_{M, \underline{x}}(\underline{n}) = \sum_{i=0}^{p(M)} n_1 \ldots n_i c(x_1, \ldots, x_i; U_i(M))
$$

for all $n_i \geq 1$, where $p(M)$ is the polynomial type of $M$. In particular, $\underline{x} = x_1, \ldots, x_d$ is a $dd$-sequence system of parameters.
Proof. For all $d$-tuples of positive integers $\mathbf{n} = (n_1, \ldots, n_d)$ by Corollary 4.7 we have $x_1^{n_1}, \ldots, x_d^{n_d}$ is a $C$-system of parameters. By Theorem 4.4 and Remark 2.10 (ii) we have
\[(x_{i+1}^{n_{i+1}}, \ldots, x_d^{n_d})M : M x_{i+1}^{n_{i+1}}/(x_{i+1}^{n_{i+1}}, \ldots, x_d^{n_d})M \cong U_i(M)\]
for all $0 \leq i \leq d - 1$. By the Auslander-Buchsbaum formula (cf. [1, Corollary 4.3]) we have
\[I_{M, \mathbf{x}}(\mathbf{n}) = \sum_{i=0}^{d-1} e(x_1^{n_1}, \ldots, x_i^{n_i}; (x_{i+1}^{n_{i+1}}, \ldots, x_d^{n_d})M : M x_{i+1}^{n_{i+1}}/(x_{i+1}^{n_{i+1}}, \ldots, x_d^{n_d})M)\]
\[= \sum_{i=0}^{d-1} e(x_1^{n_1}, \ldots, x_i^{n_i}; U_i(M))\]
\[= \sum_{i=0}^{d-1} n_1 \ldots n_i e(x_1, \ldots, x_i; U_i(M))\]
is a polynomial in $n_1, \ldots, n_d$. By Remark 2.10 (iii) we have $\text{Ann} U_i(M) \supseteq b(M)$ for all $i \leq d - 1$. Thus $\dim U_i \leq p(M)$ for all $i \leq d - 1$ since $\dim R/b(M) = \dim R/a(M) = p(M)$. Therefore $e(x_1, \ldots, x_i; U_i(M)) = 0$ for all $p(M) < i \leq d - 1$. Hence
\[I_{M, \mathbf{x}}(\mathbf{n}) = \sum_{i=0}^{p(M)} n_1 \ldots n_i e(x_1, \ldots, x_i; U_i(M)).\]
The last assertion follows from Proposition 2.7. The proof is complete.

The following is in some sense a generalization of Proposition 2.7 (see also [12, Theorem 3.7]).

**Corollary 4.9.** Let $\mathbf{x} = x_1, \ldots, x_d$ be a dd-sequence system of parameters of $M$. Let $U_i(M)$, $0 \leq i \leq d - 1$, be the Cohen-Macaulay deviated sequence of $M$. Then the difference
\[I_{M, \mathbf{x}}(\mathbf{n}) = \sum_{i=0}^{p(M)} n_1 \ldots n_i e(x_1, \ldots, x_i; U_i(M))\]
for all $n_i \geq 1$, where $p(M)$ is the polynomial type of $M$.

**Proof.** Notice that if $\mathbf{x} = x_1, \ldots, x_d$ is a dd-sequence system of parameters of $M$, then $\mathbf{x}^k = x_1^k, \ldots, x_d^k$ is a $C$-system of parameters for some $k \geq 1$ (see Remark 2.8). So we have
\[I_{M, \mathbf{x}^k}(\mathbf{n}) = \sum_{i=0}^{p(M)} k^i n_1 \ldots n_i e(x_1, \ldots, x_i; U_i(M))\]
for all $n_i \geq 1$. By Proposition 2.7 we have
\[I_{M, \mathbf{x}}(kn_1, \ldots, kn_d) = \sum_{i=0}^{p(M)} k^i n_1 \ldots n_i e(x_1, \ldots, x_i; 0 : M/(x_{i+2}, \ldots, x_d)M x_{i+1})\]
for all $n_i \geq 1$. However it is clear that $I_{M, \mathbf{x}}(\mathbf{n}) = I_{M, \mathbf{x}}(kn_1, \ldots, kn_d)$. By the above equality we have
\[e(x_1, \ldots, x_i; U_i(M)) = e(x_1, \ldots, x_i; 0 : M/(x_{i+2}, \ldots, x_d)M x_{i+1})\]
for all $i \leq p(M)$. Therefore
\[I_{M, \mathbf{x}}(\mathbf{n}) = \sum_{i=0}^{p(M)} n_1 \ldots n_i e(x_1, \ldots, x_i; U_i(M))\]
for all $n_i \geq 1$ by Proposition 4.4 again. The proof is complete. \hfill \Box

**Sequentially Cohen-Macaulay modules.** We give an application of the Cohen-Macaulay deviate sequence to characterize sequentially Cohen-Macaulay modules. This notion firstly introduced by Stanley in the graded rings [29], and for modules over local rings by Schenzel in [31], and by Nhan and the first author in [10].

**Remark 4.10** [8]. (i) The filtration of submodules $D : D_0 \subset D_1 \subset \cdots \subset D_t = M$ of $M$ is called the dimension filtration if $D_i = U_{D_{i+1}}(0)$ for all $i \leq t - 1$.
(ii) We call $M$ is a sequentially Cohen-Macaulay module if $D_{t+1}/D_t$ is Cohen-Macaulay for all $i \leq t - 1$.
(iii) A system of parameters $\underline{x} = x_1, \ldots, x_d$ of $M$ is called good if $D_i \cap (x_i, x_{i+1}, \ldots, x_d)M = 0$ for $i = 0, 1, \ldots, t - 1$, where $d_i = \dim D_i$ for all $i \leq t$. Notice that every $dd$-sequence system of parameters is good.

**Remark 4.11.** Let $M$ be a finitely generated $R$-module of dimension $d$ with the dimension filtration

$$D : D_0 \subset D_1 \subset \cdots \subset D_t = M,$$

with $d_i = \dim D_i$ for all $i \leq t$. Let $\underline{x} = x_1, \ldots, x_d$ be a $C$-system of parameters of $M$. For each $i < t$ and $d_i \leq j \leq d - 1$ we have

$$D_i \cap (x_{i+1}, \ldots, x_d)M = 0.$$

Therefore we can identify $D_i$ with a submodule of $M/(x_{i+1}, \ldots, x_d)M$. Moreover, since $\dim D_i = d_i < j + 1 = \dim M/(x_{i+1}, \ldots, x_d)M$, $D_i$ is isomorphism to a submodule of $U_j(M)$ for all $d_i \leq j \leq d - 1$. So without any confusion we write $D_i \subseteq U_j(M)$ for all $d_i \leq j \leq d - 1$.

The following is a characterization of sequentially Cohen-Macaulay modules.

**Proposition 4.12.** Let $M$ be a finitely generated $R$-module of dimension $d$ with the dimension filtration

$$D : D_0 \subset D_1 \subset \cdots \subset D_t = M,$$

with $d_i = \dim D_i$ for all $i \leq t$. Let $U_i(M)$, $0 \leq i \leq d - 1$, be the Cohen-Macaulay deviated sequence of $M$. The following statements are equivalent

(i) $M$ is a sequentially Cohen-Macaulay module.
(ii) $D_i = U_j(M)$ for all $i < t$ and for all $d_i \leq j < d_{i+1}$.

**Proof.** (i) $\Rightarrow$ (ii) Let $\underline{x} = x_1, \ldots, x_d$ be a $C$-system of parameters of $M$. By Proposition 4.8 it is a $dd$-sequence. By [7] Lemma 6.4, $M/(x_{i+1}, \ldots, x_d)M$ is a sequentially Cohen-Macaulay module with the dimension filtration

$$D_0 \cong \frac{D_0 + (x_{i+1}, \ldots, x_d)M}{(x_{i+1}, \ldots, x_d)M} \subset \cdots \subset D_i \cong \frac{D_i + (x_{i+1}, \ldots, x_d)M}{(x_{i+1}, \ldots, x_d)M} \subset M/(x_{i+1}, \ldots, x_d)M$$

for all $i < t$ and for all $d_i \leq j < d_{i+1}$. Thus $D_i = U_j(M)$ for all $i < t$ and for all $d_i \leq j < d_{i+1}$.

(ii) $\Rightarrow$ (i) Let $\underline{x} = x_1, \ldots, x_d$ be a $C$-system of parameters of $M$. By Proposition 4.8 we have

$$I_{M_{\underline{x}}}(n) = \sum_{j=0}^{d-1} n_1 \cdots n_j e(x_1, \ldots, x_j; U_j(M)).$$
for all \( n_1, ..., n_d \geq 1 \). Since \( D_i = U_j(M) \) for all \( i < t \) and for all \( d_i \leq j < d_{i+1} \) we have 
\[ e(x_1, \ldots, x_j; U_j(M)) = 0 \]
for all \( i < t \) and for all \( d_i < j < d_{i+1} \). Therefore 
\[ I_{M,x}(n) = \sum_{i=0}^{t-1} n_1 \cdots n_d e(x_1, \ldots, x_d; D_i) \]
for all \( n_1, ..., n_d \geq 1 \). Hence \( M \) is a sequentially Cohen-Macaulay module by \[8\], Theorem 4.2. The proof is complete. \( \square \)

**Relation with the Serre condition \((S_2)\).** For each \( R \)-module \( M \) we have a set of invariant modules \( U_i(M), 0 \leq i \leq d - 1 \), as Theorem 4.1. Therefore we have a special set of prime ideals, \( \bigcup_{i=0}^{d-1} \text{Ass} \ U_i(M) \), attached to \( M \). If \( p \in \text{Ass} M \) and \( \dim R/p < d \), then \( p \in \text{Ass} U_M(0) = \text{Ass} U_{d-1}(M) \). In the following we consider the relation between \( \text{Ass} U_{d-2}(M) \) and the Serre condition \((S_2)\).

**Definition 4.13.** For all \( n \geq 1 \), we say that \( M \) satisfies the Serre condition \((S_n)\) at the prime ideal \( p \in \text{Supp}(M) \) if
\[ \text{depth} M_p \geq \min \{ \dim M_p, n \} \]
Moreover, \( M \) has property \((S_n)\) if it satisfies the Serre condition \((S_n)\) at all \( p \in \text{Supp}(M) \).

It is obvious that \( R \) satisfies the condition \((S_1)\) if and only if \( \text{Ass} R = \text{minAss} R \). Furthermore, if \( R \) satisfies the condition \((S_2)\) and \( R \) is cartenary (this condition is always true if \( R \) is a homomorphic image of a Cohen-Macaulay ring), then \( \text{Ass} R = \text{Assh} R \) (see \[8\], Corollary 2.24). Conversely, Goto and Nakamura \[14\] Lemma 3.2 proved that if \( \text{Ass} R \subseteq \text{Assh} R \cup \{ m \} \), then the set \( F(R) = \{ p \in \text{Spec}(R) | \dim R_p > 1 = \text{depth} R_p, p \neq m \} \) is finite, i.e. \( R \) does not satisfy the Serre condition \((S_2)\) at only finitely many prime ideals. The set \( F(R) \) can be described as follows.

**Proposition 4.14.** Suppose that \( \text{Ass} M \subseteq \text{Assh} M \cup \{ m \} \). Set 
\[ F(M) = \{ p \in \text{Supp}(M) | \dim M_p > 1 = \text{depth} M_p, p \neq m \}. \]
Then 
\[ F(M) = \text{Ass} U_{d-2}(M) \setminus \{ m \}. \]

**Proof.** Let \( x \) be a \( C \)-parameter element of \( M \). For all \( p \in \text{Ass} U_{d-2}(M) \setminus \{ m \} \) we have \( p \in \text{Ass} M/xM \) and \( \dim R/p \leq d - 2 \). Hence \( \dim M_p > 1 = \text{depth} M_p \), So \( \text{Ass} U_{d-2}(M) \setminus \{ m \} \subseteq F(M) \).

Conversely, let \( p \in F(M) \). Since \( \text{depth} M_p = 1 \), for every parameter element \( z \in p \) we have \( p \in \text{Ass} M/zM \). Therefore \( p \in \text{Ass} M/(xz)M \). Notice that \( xz \) is a \( C \)-parameter element of \( M \) and \( \dim R/p \leq d - 2 \), so \( p \in \text{Ass} U_{M/(xz)M}(0) \cong \text{Ass} U_{d-2}(M) \). The proof is complete. \( \square \)

**Remark 4.15.** Let \( M \) be a finitely generated \( R \)-module.

(i) Suppose that \( \text{Ass} M \subseteq \text{Assh} M \cup \{ m \} \) and \( F(M) \) as the previous proposition. Let \( x \) be a parameter element of \( M \) such that \( x \notin p \) for all \( p \in F(M) \). Then \( M \) satisfies the Serre condition \((S_2)\) at all prime ideals \( p \in \text{supp} M \) containing \( x \) and \( p \neq m \). So \( M/xM \) satisfies the Serre condition \((S_1)\) at all \( p \in \text{Supp}(M/xM) \) and \( p \neq m \). Hence 
\[ \text{Ass} (M/xM) \subseteq \text{minAss}(M/xM) \cup \{ m \} = \text{Assh}(M/xM) \cup \{ m \} \]
(ii) Let \( \overline{M} = M/U_M(0) \). Let \( x \in b(M)^3 \cap b(\overline{M})^3 \) be a parameter element of \( M \) and hence of \( \overline{M} \). Put \( b' = b(M/xM), b'' = b(M/\overline{M}) \) and \( b = b' \cap b'' \). We have \( \dim R/b \leq d - 2 \). By Remark 2.10 we have 
\[ U_{d-2}(M) \cong H^0_{b'}(M/xM) \subseteq H^0_b(M/xM) \]. However \( \dim H^0_b(M/xM) < d - 1 \),
so \( U_{d-2}(M) \cong H_0^0(M/xM) \). Similarly, we have \( U_{d-2}(\overline{M}) \cong H_0^0(\overline{M}/x\overline{M}) \). By the proof of Lemma 4.13 we have
\[
U_{d-2}(M) \cong H_0^0(M) \oplus H_1^1(\overline{M}/x\overline{M})
\]
and
\[
U_{d-2}(\overline{M}) \cong H_0^0(\overline{M}) \oplus H_1^1(\overline{M}/x\overline{M}) = H_1^1(\overline{M}/x\overline{M}).
\]
Therefore \( U_{d-2}(\overline{M}) \) is isomorphism to a direct summand of \( U_{d-2}(M) \).

The following plays an important role in the next section.

**Proposition 4.16.** Let \( M \) be a finitely generated \( R \)-module of dimension \( d \geq 2 \). Let \( x \) be a parameter element of \( M \) such that \( x \notin \mathfrak{p} \) for all \( \mathfrak{p} \in (\text{Ass } U_M(0) \cup \text{Ass } U_{d-2}(M)) \setminus \{ \mathfrak{m} \} \). Then we have the following short exact sequence
\[
0 \to U_M(0)/xU_M(0) \to U_{M/xM}(0) \to H_m^0(\overline{M}/x\overline{M}) \to 0,
\]
where \( \overline{M} = M/U_M(0) \).

**Proof.** Since \( U_M(0) \cap xM = x(U_M(0) :_M x) = xuM(0) \), we have the following short exact sequence
\[
0 \to U_M(0)/xU_M(0) \xrightarrow{\varphi} M/xM \to \overline{M}/x\overline{M} \to 0.
\]
If \( \dim U_M(0) = 0 \) then \( \dim U_M(0)/xU_M(0) < d - 1 \). If \( \dim U_M(0) > 0 \) then \( x \) is a parameter element of both \( M \) and \( U_M(0) \) so \( \dim U_M(0)/xU_M(0) = \dim U_M(0) - 1 < d - 1 \). Notice that \( \text{Im}(\varphi) = (U_M(0) + xM)/xM \). Thus we always have \( (U_M(0) + xM)/xM \) is a submodule of \( M/xM \) of dimension less than \( d - 1 \). Hence \( \text{Im}(\varphi) = (U_M(0) + xM)/xM \subseteq U_{M/xM}(0) \). So we have the short exact sequence
\[
0 \to U_M(0)/xU_M(0) \to U_{M/xM}(0) \to U_{\overline{M}/x\overline{M}}(0) \to 0.
\]
On the other hand \( x \notin \mathfrak{p} \) for all \( \mathfrak{p} \in \text{Ass } U_{d-2}(M) \setminus \{ \mathfrak{m} \} \). So \( x \notin \mathfrak{p} \) for all \( \mathfrak{p} \in \text{Ass } U_{d-2}(\overline{M}) \setminus \{ \mathfrak{m} \} \) by Remark 4.15 (ii). By Remark 4.15 (i) we have
\[
\text{Ass } (\overline{M}/x\overline{M}) \subseteq \text{Assh}(\overline{M}/x\overline{M}) \cup \{ \mathfrak{m} \}.
\]
Therefore \( U_{\overline{M}/x\overline{M}}(0) = H_m^0(\overline{M}/x\overline{M}) \). Thus we obtain the short exact sequence
\[
0 \to U_M(0)/xU_M(0) \to U_{M/xM}(0) \to H_m^0(\overline{M}/x\overline{M}) \to 0.
\]
The proof is complete. \( \square \)

**5. The unmixed degree**

In this section let \( I \) be an \( \mathfrak{m} \)-primary ideal and \( M \) a finitely generated \( R \)-module of dimension \( d > 0 \). Let \( U_i(M), 0 \leq i \leq d - 1 \), be the Cohen-Macaulay deviated sequence of \( M \). The purpose of this section is to construct a new degree for \( M \) in terms of \( U_i(M) \). Firstly, recalling that the length function \( \ell(M/I^nM) \) becomes a polynomial of degree \( d \) when \( n \gg 0 \) and
\[
\ell(M/I^{n+1}M) = \sum_{i=0}^d (-1)^i e_i(I, M) \binom{n+d-i}{d-i}.
\]
The coefficients \( e_i(I, M), i = 0, \ldots, d \) are called the Hilbert coefficients of \( M \) with respect to \( I \). Especially, the leading coefficient \( e_0(I, M) \) is called the *Hilbert-Samuel multiplicity* of \( M \) with respect to \( I \). If \( I = \mathfrak{m} \), the multiplicity is written by \( e_0(M) \) for simply. In the present paper we denote by \( \deg(I, M) \) (resp. \( \deg(M) \)) the multiplicity \( e_0(I, M) \) (resp. \( e_0(M) \)) and call the degree of \( M \) with respect to \( I \) (resp. the degree of \( M \)). The following associativity formula for degree
says that \( \deg(I, M) \) depends only on the associated prime ideals of the highest dimension (see [3, Corollary 4.7.8])

\[
\deg(I, M) = \sum_{p \in \text{Ass} M} \ell_{R_p}(M_p) \deg(I, R/p).
\]

Notice that if \( p \in \text{minAss} M \), then \( M_p \) has finite length and \( M_p = H^0_{R_p}(M_p) \). So we have

\[
\deg(I, M) = \sum_{p \in \text{Ass} M} \ell_{R_p}(H^0_{R_p}(M_p)) \deg(I, R/p).
\]

We next recall some other degrees of \( M \) related to \( \deg(I, M) \) (see [35]).

**Definition 5.1.** The **arithmetric degree** of \( M \) with respect to \( I \), denoted by \( \text{adeg}(I, M) \), is the integer

\[
\text{adeg}(I, M) = \sum_{p \in \text{Ass} M} \ell_{R_p}(H^0_{R_p}(M_p)) \deg(I, R/p).
\]

**Remark 5.2.**

(i) Let \( D : D_0 \subseteq D_1 \subseteq \cdots \subseteq D_t = M \) be the dimension filtration of \( M \) we have

\[
\text{adeg}(I, M) = \sum_{i=0}^{t} \deg(I, D_i).
\]

So \( \text{adeg}(I, M) \geq \deg(I, M) \) and the equation occurs if and only if \( U_M(0) = 0 \).

(ii) Moreover, if \((R, \mathfrak{m})\) is a homomorphic image of a Gorenstein local ring \((S, \mathfrak{n})\) of dimension \( n \).

Then \( \text{adeg}(I, M) \) can be determined without the knowledge of the primary decomposition as follows

\[
\text{adeg}(I, M) = \sum_i \deg(I, \text{Ext}^i_S(M, S)).
\]

Vasconcelos et al. [13, 34, 35] introduced the notion of **extended degree of graded modules** in order to capture the size of a module along with some of the complexity of its structure. The prototype of an extended degree is the **homological degree** was introduced and studied by Vasconselos in [34] (see also [35]). The extended degree for local rings was considered by Rossi, Trung and Valla in [28]. This notion is associated by an \( m \)-primary ideal \( I \) in [18].

**Definition 5.3.** Let \( \mathcal{M}(R) \) be the category of finitely generated \( R \)-modules. An **extended degree** on \( \mathcal{M}(R) \) with respect to \( I \) is a numerical function

\[
\text{Deg}(I, \bullet) : \mathcal{M}(R) \to \mathbb{R}
\]

satisfying the following conditions

(i) \( \text{Deg}(I, M) = \text{Deg}(I, \overline{M}) + \ell(H^0_{m}(M)), \) where \( \overline{M} = M/H^0_m(M) \);

(ii) (Bertini’s rule) \( \text{Deg}(I, M) \geq \text{Deg}(I, M/xM) \) for every generic element \( x \in I \setminus \mathfrak{m}I \) of \( M \);

(iii) If \( M \) is Cohen-Macaulay then \( \text{Deg}(I, M) = \deg(I, M) \).

The homological degree is a typical extended degree that is defined as follows.

**Definition 5.4 ([34]).** Suppose that \((R, \mathfrak{m})\) be a homomorphic image of a Gorenstein local ring \((S, \mathfrak{n})\) of dimension \( n \), and \( M \) a finitely generated \( R \)-module of dimension \( d \). Then the **homological degree**, \( h\text{deg}(I, M) \), of \( M \) with respect to \( I \) is defined by the following recursive formula

\[
\text{hdeg}(I, M) = \deg(I, M) + \sum_{i=n-d+1}^{n} \binom{d-1}{i-n+d-1} \text{hdeg}(I, \text{Ext}^i_S(M, S)).
\]

**Remark 5.5.**

(i) The Definition 5.4 is recursive on dimension since \( \dim \text{Ext}^i_S(M, S) \leq n-i < d \) for all \( \text{i} = n-d+1, \ldots, n \).
(ii) $\text{hdeg}(I, \bullet)$ is an extended degree on $\mathcal{M}(R)$, and $\text{hdeg}(I, M) = \deg(I, M)$ if and only if $M$ is Cohen-Macaulay.

(iii) If $M$ is a generalized Cohen-Macaulay module, then $\ell(\text{Ext}_S^{n-i}(M, S)) = \ell(H^i_m(M))$ for all $i = 0, \ldots, d - 1$ by the local duality theorem. We have

$$\text{hdeg}(I, M) = \deg(I, M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_m(M)).$$

(iv) ([35, Proposition 3.5]) If $\dim M = \dim S = 2$ then

$$\text{hdeg}(I, M) = \text{udeg}(I, M) + \ell(\text{Ext}_S^2(\text{Ext}_S^1(M, S), S)).$$

Until nowadays, the homological degree is the uniquely explicit extended degree. The purpose of this section is to introduce another extended degree on $\mathcal{M}(R)$ in terms of the Cohen-Macaulay deviated sequence $U_i(M)$, $i = 0, \ldots, d - 1$. Notice that $\dim U_i(M) \leq i$ for all $0 \leq i \leq d - 1$.

**Definition 5.6.** Let $M$ be a finitely generated $R$-module of dimension $d$ and $U_i(M)$, $0 \leq i \leq d - 1$, be a generalized Cohen-Macaulay deviated sequence of $M$. We define the *unmixed degree* of $M$ with respect to $I$, $\text{udeg}(I, M)$, as follows

$$\text{udeg}(I, M) = \deg(I, M) + \sum_{i=0}^{d-1} \delta_{i, \dim U_i(M)} \deg(I, U_i(M)),$$

where $\delta_{i, \dim U_i(M)}$ is the Kronecker symbol.

It is worth noting that in the above definition and Proposition 5.8 we consider the subsequence of modules of the Cohen-Macaulay deviated sequence consisting $U_i(M)$ with $\dim U_i(M) = i$. We call this subsequence the *reduced Cohen-Macaulay deviated sequence* of $M$. In the rest of this paper, we shall prove that the unmixed degree is an extended degree. The first condition of Definition 5.3 follows from the following.

**Proposition 5.7.** Let $N$ be a submodule of finite length of $M$. Then

$$\text{udeg}(I, M) = \text{udeg}(I, M/N) + \ell(N).$$

**Proof.** Let $x_1, \ldots, x_d$ be a $C$-system of parameters of both $M$ and $M/N$. By Proposition 4.8 $x_1, \ldots, x_d$ is a $dd$-sequence of $M$. So $H^0_m(M) \cap (x_1, \ldots, x_d)M = 0$. For all $0 \leq j \leq d - 1$, we have the short exact sequence

$$0 \to N \to M/(x_{j+2}, \ldots, x_d)M \to M/(N + (x_{j+2}, \ldots, x_d)M) \to 0.$$

Therefore $U_j(M/N) \cong U_j(M)/N$ for all $0 \leq j \leq d - 1$. Thus

$$\delta_{j, \dim U_j(M/N)} \deg(I, U_j(M/N)) = \delta_{j, \dim U_j(M)} \deg(I, U_j(M))$$

for all $1 \leq j \leq d - 1$ and

$$\delta_{0, \dim U_0(M/N)} \deg(I, U_0(M/N)) = \delta_{0, \dim U_0(M)} \deg(I, U_0(M)) - \ell(N).$$

The claim is now obvious. \hfill $\square$

The next result shows that $\text{udeg}(M)$ agrees with $\text{hdeg}(M)$ for generalized Cohen-Macaulay modules.

**Proposition 5.8.** Let $M$ be a generalized Cohen-Macaulay $R$-module of dimension $d$. Then

$$\text{udeg}(I, M) = \deg(I, M) + \sum_{j=0}^{d-1} \binom{d-1}{j} \ell(H^j_m(M)).$$
Proof. Let \( x_1, \ldots, x_d \) be a \( C \)-system of parameters of \( M \). By Corollary 5.5 (see also [11, Corollary 4.2]) we have

\[
U_i(M) \cong H_m^0(M/(x_{i+2}, \ldots, x_d)M) \cong \bigoplus_{j=0}^{d-i-1} H_m^j(M)_{i-j-1}
\]

for all \( 0 \leq i \leq d - 1 \). So \( \dim U_i(M) = 0 \) for all \( i \leq d - 1 \). Therefore \( \delta_{i, \dim U_i(M)} \deg(I, U_i(M)) = 0 \) for all \( 1 \leq i \leq d - 1 \) and

\[
\delta_{0, \dim U_0(M)} \deg(I, U_0(M)) = \sum_{j=0}^{d-1} \binom{d-1}{j} \ell(H_m^j(M)).
\]

The proof is complete. \( \square \)

We next compute the unmixed degree when \( \dim M \) is small.

**Proposition 5.9.** The following statements hold true.

(i) If \( d = 1 \) then \( \udeg(I, M) = \adeg(I, M) \).

(ii) If \( d = 2 \) then \( \udeg(I, M) = \adeg(I, M) + \ell(H_m^1(M/U_M(0))) \).

*Proof.* (i) It is clear.

(ii) We consider the following two cases.

The case \( \dim U_M(0) = 0 \), we have \( M \) is a generalized Cohen-Macaulay modules. Therefore by Proposition 5.8 we have

\[
\udeg(I, M) = \deg(I, M) + \ell(H_m^0(M)) = \adeg(I, M) + \ell(H_m^1(M/U_M(0))).
\]

The case \( \dim U_M(0) = 1 \). Consider the dimension filtration \( H_m^0(M) \subset U_M(0) \subset M \) of \( M \). By Remark 5.2 (i) we have

\[
\adeg(I, M) = \deg(I, M) + \deg(I, U_M(0)) + \ell(H_m^0(M)).
\]

On the other hand \( U_1(M) \cong U_M(0) \) so \( \delta_{1, \dim U_1(M)} \deg(I, U_1(M)) = \deg(I, U_M(0)) \). Let \( x_2 \) be a \( C \)-parameter element of \( M \). By Corollary 5.5 we have

\[
U_0(M) \cong H_m^0(M/x_2M) \cong H_m^0(M) \oplus H_m^1(M/U_M(0)).
\]

Thus \( \delta_{0, \dim U_0(M)} \deg(I, U_0(M)) = \ell(H_m^0(M)) + \ell(H_m^1(M/U_M(0))) \). Therefore we also have

\[
\udeg(I, M) = \adeg(I, M) + \ell(H_m^1(M/U_M(0))).
\]

The proof is complete. \( \square \)

**Corollary 5.10.** Suppose \((R, \mathfrak{m})\) is a homomorphic image of a Gorenstein local ring and \( \dim M = 2 \). Then \( \udeg(I, M) = \hdeg(I, M) \).

*Proof.* Without loss of generality we may assume that \((R, \mathfrak{m})\) is a Gorenstein local ring of dimension two. If \( U_M(0) = H_m^0(M) \) we have \( M \) is generalized Cohen-Macaulay, the claim follows from Proposition 5.8 and Remark 5.5 (iii). Suppose \( \dim U_M(0) = 1 \), by Proposition 5.9 and Remark 5.5 (iv) we need only to show that

\[
\ell(H_m^1(M/U_M(0))) = \ell(\Ext_R^1(M, R)).
\]

Since \( \Ass M/U_M(0) = \{ \mathfrak{p} \mid \mathfrak{p} \in \Ass M, \dim R/\mathfrak{p} = 2 \} \) we have \( \Ext_R^1(M/U_M(0), R) \) is a module of finite length, and \( \ell(\Ext_R^1(M/U_M(0), R)) = \ell(H_m^1(M/U_M(0))) \) by local duality theorem. By local
Thus adeg(I, M) follows from (ii), so it is enough to prove (ii). If M be the dimension filtration of

Conversely, suppose adeg(I, M) ≤ udeg(I, M). We have

By [32, Lemma 1.9] (v) we have Ext^1_R(U_M(0), R) = 0. Therefore

The proof is complete. □

In the following we prove the third condition of Definition 5.3. Moreover we also give a characterization of sequentially Cohen-Macaulay modules in terms of unmixed degrees.

**Theorem 5.11.** Let M be a finitely generated R-module of dimension d. We have

\[ \deg(I, M) \leq \text{adeg}(I, M) \leq \text{udeg}(I, M). \]

**Furthermore**

(i) \( \deg(I, M) = \text{udeg}(I, M) \) if and only if M is a Cohen-Macaulay module.

(ii) \( \text{adeg}(I, M) = \text{udeg}(I, M) \) if and only if M is a sequentially Cohen-Macaulay module.

**Proof.** The first inequality is clear. Let

\[ \mathcal{D} : D_0 \subset D_1 \subset \cdots \subset D_t = M \]

be the dimension filtration of M with \( d_i = \dim D_i \) for all \( i \leq t \). Recalling that

\[ \text{adeg}(I, M) = \deg(I, M) + \sum_{i=0}^{t-1} \deg(I, D_i). \]

For all \( i < t \) by Remark 4.11 we have \( D_i \subseteq U_{d_i}(M) \). So \( \dim U_{d_i}(M) = d_i \) and then

\[ \deg(I, D_i) \leq \deg(I, U_{d_i}(M)) = \delta_{d_i, \dim U_{d_i}(M)} \deg(I, U_{d_i}(M)). \]

Thus \( \text{adeg}(I, M) \leq \text{udeg}(I, M) \).

We have (i) follows from (ii), so it is enough to prove (ii). If M is sequentially Cohen-Macaulay, then by Proposition 4.12 we have \( \text{adeg}(I, M) = \text{udeg}(I, M) \).

Conversely, suppose \( \text{adeg}(I, M) = \text{udeg}(I, M) \). We have

\[ \deg(I, D_i) = \deg(I, U_{d_i}(M)) \quad (\ast) \]

for all \( i < t \), and

\[ \delta_{j, \dim U_j(M)} \deg(I, U_j(M)) = 0 \quad (\ast\ast) \]

for all \( i < t \) and \( d_i < j < d_{i+1} \). Let \( \underline{x} = x_1, \ldots, x_d \) be a C-system of parameters of M. By (\ast\ast) and the associative formula we have

\[ e(x_1, \ldots, x_{d_i}; D_{i}) = \deg((\underline{x}), D_i) = \deg((\underline{x}), U_{d_i}(M)) = e(x_1, \ldots, x_{d_i}; U_{d_i}(M)) \]

for all \( i < t \). By (\ast\ast) we have \( \dim U_j(M) < j \) for all for all \( d_i < j < d_{i+1} \) and \( i < t \), so

\[ e(x_1, \ldots, x_j; U_j(M)) = 0 \]
for all \( d_i < j < d_{i+1} \) and \( i < t \). By Proposition 4.8 we have

\[
I_{M,x}(n) = \sum_{j=0}^{d-1} n_1 \ldots n_j e(x_1, \ldots, x_j; U_j(M)).
\]

for all \( n_1, \ldots, n_d \geq 1 \). Thus we have

\[
I_{M,x}(n) = \sum_{i=0}^{t-1} n_1 \ldots n_{d_i} e(x_1, \ldots, x_{d_i}; \ell_i).
\]

for all \( n_1, \ldots, n_d \geq 1 \). Hence \( M \) is a sequentially Cohen-Macaulay module by [8, Theorem 4.2].

The proof is complete. \( \square \)

In order to prove the Bertini rule of Definition 5.3 we will show that the unmixed degree has good behavior by passing to the quotient modules regarding certain superficial elements.

**Definition 5.12.** An element \( x \in I \setminus mI \) is called a superficial element of \( M \) with respect to \( I \) if there exists a positive integer \( c \) such that

\[
(I^{n+1}M : x) \cap I^cM = I^nM
\]

for all \( n \geq c \).

**Remark 5.13.**

(i) Let \( G_I(R) = \oplus_{n \geq 0} I^n / I^{n+1} \) be the associated graded ring of \( R \) with respect to \( I \) and \( G_I(M) = \oplus_{n \geq 0} I^nM / I^{n+1}M \) the graded \( G_I(R) \)-module. Set \( (G_I(R))_+ = \oplus_{n \geq 1} I^n / I^{n+1} \). Then \( x \) is a superficial element of \( M \) with respect to \( I \) if and only if the initial \( x^* \) of \( x \) in \( G_I(R) \) is a \( (G_I(R))_+ \)-filter regular element of \( G_I(M) \) i.e. \( \ell(0 : G_I(M) x^*) < \infty \) (notice that in our context \( I \) is \( m \)-primary). Moreover, if \( x \) is a superficial element, then it is an I-filter regular element of \( M \).

(ii) A superficial element of \( M \) with respect to \( I \) always exist if the residue field \( R/m \) is infinite, a hypothesis which never cause us any problem because we can replace \( R \) by the local ring \( R[X]_m R[X] \), where \( X \) is an indeterminate. In the sequel we assume that the residue field is infinite.

(iii) (cf. [21, 22.6]) Let \( x \) be a superficial element of \( M \) with respect to \( I \). For \( n \gg 0 \) we have

\[
I^{n+1}M :_{M} x = 0 :_{M} x + I^nM
\]

so

\[
\ell(M/(I^{n+1} + (x))M) = \ell(M/I^{n+1}M) - \ell(M/I^nM) + \ell(0 :_M x)
\]

for all \( n \gg 0 \).

(iv) Let \( x \) be a superficial element of \( M \) with respect to \( I \). By (iii) we have \( \deg(I, M/xM) = \deg(I, M) \) if \( d \geq 2 \), and \( \ell(M/xM) = \deg(I, M/xM) = \deg(I, M) + \ell(0 :_M x) \) if \( d = 1 \).

We need some lemmas before proving the Bertini rule of unmixed degrees.

**Lemma 5.14.** Let \( M \) be a finitely generated \( R \)-module of dimension \( d \geq 2 \). Let \( x \) be a parameter element of \( M \) such that \( x \) is a superficial element of \( U_M(0) \) with respect to \( I \) and \( x \notin \mathfrak{p} \) for all \( \mathfrak{p} \in \text{Ass} U_{d-2}(M) \setminus \{m\} \). Then

\[
\delta_{d-2, \dim U_M/\mathfrak{m}M(0)} \deg(I, U_M/xM(0)) = \delta_{d-1, \dim U_M(0)} \deg(I, U_M(0))
\]

if \( d \geq 3 \), and

\[
\delta_{0, \dim U_M/\mathfrak{m}M(0)} \deg(I, U_M/xM(0)) = \delta_{1, \dim U_M(0)} \deg(I, U_M(0)) + \ell(0 : H_{R}^0(M) x) + \ell(0 : H_{R}^0(M/U_M(0)) x)
\]

if \( d = 2 \).
Hence by Remark 5.13 (iv) we have \( \deg(I, U/M) = \delta_{d-1, \dim(U/M)} \deg(I, U/M(0)) \).

If \( \dim(U_M(0)) = d - 1 \) we have \( \dim(U/M) = d - 2 > 0 \). So \( \deg(I, U/M(0)) = \deg(I, U/M(0)/xU_M(0)) \). By Remark 5.13 (iv) we have \( \deg(I, U_M(0)) = \deg(I, U_M(0)/xU_M(0)) \). Thus we also have

\[
\delta_{d-2, \dim(U/M(0))} \deg(I, U/M(0)) = \delta_{d-1, \dim(U/M(0))} \deg(I, U/M(0)).
\]

The case \( d = 2 \), we have \( U/M(0) \) has finite length. Therefore

\[
\delta_{0, \dim(U/M(0))} \deg(I, U/M(0)) = \ell(U/M(0)) = \ell(U_M(0)/xU_M(0)) + \ell(H^0_{M/M(0)}).
\]

If \( \dim(U_M(0)) = 1 \), by Remark 5.13 (iv) we have

\[
\ell(U_M(0)/xU_M(0)) = \deg(I, U_M(0)) + \ell(0 : U_M(0), x) = \delta_{1, \dim(U_M(0))} \deg(I, U_M(0)) + \ell(0 : H^0_{M(0), x}).
\]

If \( \dim(U_M(0)) = 0 \) then we have \( U/M = H^0_{M(M)} \) and hence \( \delta_{1, \dim(U_M(0))} \deg(I, U_M(0)) = 0 \). Moreover one can check that \( \ell(H^0_{M(0)/xH^0_{M(0)}}) = \ell(0 : H^0_{M(0), x}) \). Thus we always have

\[
\ell(U_M(0)/xU_M(0)) = \delta_{1, \dim(U_M(0))} \deg(I, U_M(0)) + \ell(0 : H^0_{M(0), x}).
\]

On the other hand the short exact sequence

\[
0 \to M/xM \to M/xM \to 0
\]

induces the exact sequence of local cohomology modules

\[
0 \to H^0_{M/M(0)} \to H^1_{M(0)} \to H^1_{M(0)} \to 0.
\]

Therefore \( \ell(H^0_{M/M(0)}) = \ell(0 : H^0_{M(0), x}) \). Hence

\[
\delta_{0, \dim(U/M(0))} \deg(I, U/M(0)) = \delta_{1, \dim(U_M(0))} \deg(I, U_M(0)) + \ell(0 : H^0_{M(0), x}) + \ell(0 : H^0_{M(0)/U_M(0), x}).
\]

The proof is complete.

We need one more technical lemma.

**Lemma 5.15.** Let \( M \) be a finitely generated \( R \)-module of dimension \( d \geq 2 \). Let \( x \) be a parameter element of \( M \) such that \( x \notin p \) for all \( p \in \text{Ass}(U_M(0)) \setminus \{m\} \). Then we can choose a \( C \)-parameter element \( x_d \) of \( M \) such that \( x \) is a parameter element of \( M/x_d M \).

**Proof.** If \( \dim(U_M(0)) < d - 1 \) then \( \dim(R/b(M)) \leq d - 2 \) by Remark 2.2 (ii). Therefore we can choose a \( C \)-parameter element \( x_d \) such that \( x \) and \( x_d \) are a part of a system of parameters of \( M \) by the prime avoidance theorem. Hence \( x \) is a parameter element of \( M/x_d M \).

We now assume that \( \dim(U_M(0)) = d - 1 \). Set \( M = M/Um(0) \). The short exact sequence

\[
0 \to U_M(0) \to M \to M \to 0.
\]

induces the exact sequence of local cohomology modules

\[
\cdots \to H^i_{M(0)} \to H^i_{M(0)} \to H^i_{M(0)} \to \cdots.
\]

Hence \( \alpha_i(M) = \text{Ann} H^i_{M(0)} \supseteq \text{Ann} U_M(0) \alpha_i(M) \) for all \( i \geq 0 \). So

\[
\sqrt{b(M)} = \sqrt{a(M)} \supseteq \sqrt{\text{Ann} U_M(0) a(M)} = \sqrt{\text{Ann} U_M(0) b(M)}.
\]
We claim that $b(M) \not\subseteq q$ for all $q \in \text{Ass} M/xM$. Indeed, by Remark 2.2(ii) we have $\dim R/b(M) \leq d - 2$. Therefore $b(M) \not\subseteq q$. Suppose $\text{Ann}_U M(0) \subseteq q$. Then $q \in \text{Ass} U_M(0)$ since $\dim U_M(0) = \dim R/q = d - 1$. It contrasts to our assumption that $x \notin p$ for all $p \in \text{Ass} U_M(0) \setminus \{m\}$. So $\text{Ann}_U M(0) \not\subseteq q$, and hence $b(M) \not\subseteq q$ for all $q \in \text{Ass} M/xM$. Thus there exists $x_d \in b(M)^3$ such that $x_d$ is a parameter element of $M/xM$ by the prime avoidance theorem. Such an element $x_d$ satisfies the requirements. The proof is complete. 

We are now ready to prove that the unmixed degrees satisfy the Bertini rule of extended degrees.

**Theorem 5.16.** Let $M$ be a finitely generated $R$-module of dimension $d$. Let $x$ be a superficial element of $M$ and of all $U_i(M)$, $1 \leq i \leq d - 1$, with respect to $I$. Then

$$\text{udeg}(I, M) \geq \text{udeg}(I, M/xM).$$

**Proof.** Notice that since $x$ is a superficial element of $U_i(M)$, $1 \leq i \leq d - 1$, with respect to $I$ we have $x \notin p$ for all $p \in \text{Ass} U_i(M) \setminus \{m\}$, $1 \leq i \leq d - 1$ by Remark 5.1(i). The case $d = 1$ is clear since $\text{udeg}(I, M) = \deg(I, M) + \ell(H^0_0(M))$ and $\text{udeg}(I, M/xM) = \ell(M/xM) = \deg(I, M) + \ell(0:_{M} x)$. Suppose $d \geq 2$, by Lemma 5.13 we can choose a part of a $C$-system of parameters $x_2, \ldots, x_d$ of $M$ such that $x, x_2, \ldots, x_d$ is also a system of parameters of $M$. By Lemma 3.7 we have $x_2, \ldots, x_d$ is a $C$-system of parameters of $M/xM$. Therefore, we have

$$\text{udeg}(I, M) = \deg(I, M) + \sum_{i=0}^{d-1} \delta_i \dim U_i(M) \deg(I, U_i(M))$$

$$= \deg(I, M) + \sum_{j=2}^{d+1} \delta_{j-2} \dim U_{j-2}(x_j, \ldots, x_d)M(0) \deg(I, U_{j-2}(x_j, \ldots, x_d)M(0)),$$

and

$$\text{udeg}(I, M/xM) = \deg(I, M/xM) + \sum_{i=0}^{d-2} \delta_i \dim U_i(M/xM) \deg(I, U_i(M/xM))$$

$$= \deg(I, M/xM) + \sum_{j=3}^{d+1} \delta_{j-3} \dim U_{j-3}(x_j, x_j, \ldots, x_d)M(0) \deg(I, U_{j-3}(x_j, x_j, \ldots, x_d)M(0)).$$

Since $x$ is a superficial element of $M$ with respect to $I$ we have $\deg(I, M/xM) = \deg(I, M)$. For $j > 3$ we have $\dim M/(x_j, \ldots, x_d)M = j - 1 \geq 3$. By Lemma 5.13 we obtain

$$\delta_{j-2} \dim U_{j-2}(x_j, \ldots, x_d)M(0) \deg(I, U_{j-2}(x_j, \ldots, x_d)M(0)) = \delta_{j-3} \dim U_{j-3}(x_j, x_j, \ldots, x_d)M(0) \deg(I, U_{j-3}(x_j, x_j, \ldots, x_d)M(0))$$

for all $3 < j \leq d + 1$. For $j = 3$, set $M' = M/(x_3, \ldots, x_d)M$ we have $\dim M' = 2$. By Lemma 5.13 we have

$$\delta_{0} \dim U_{2}(x_2)M(0) \deg(I, U_{2}(x_2)M(0)) = \delta_{1} \dim U_{1}(0) \deg(I, U_{1}(0)) + \ell(0 : H^{1}_{m}(M')) + \ell(0 : H^{1}_{m}(M'/U_{M'}(0)) \cdot x).$$

By Corollary 3.5 we have

$$U_0(M') = H^0_{m}(M'/x_2M') \cong H^0_{m}(M') \oplus H^{1}_{m}(M'/U_{M'}(0)),$$

So

$$\delta_{0} \dim U_{0}(M') \deg(I, U_{0}(M')) = \ell(H^0_{m}(M')) + \ell(H^1_{m}(M'/U_{M'}(0)))$$

$$\geq \ell(0 : H^{1}_{m}(M')) + \ell(0 : H^{1}_{m}(M'/U_{M'}(0)) \cdot x).$$
Therefore
\[
\delta_{0, \dim U_{M'/(x)}(0)} \deg(I, U_{M'/(x)}(0)) \leq \delta_{1, \dim U_{M'}(0)} \deg(I, U_{M'}(0)) + \delta_{0, \dim U_0(M')} \deg(I, U_0(M')).
\]
More precisely, we have
\[
\delta_{0, \dim U_{M/(x,x_3,\ldots,x_d)M}(0)} \deg(I, U_{M/(x,x_3,\ldots,x_d)M}(0)) \leq \sum_{j=2}^{3} \delta_{j-2, \dim U_{M/(x_j,\ldots,x_d)M}(0)} \deg(I, U_{M/(x_j,\ldots,x_d)M}(0)).
\]
In conclusion, \(\udeg(I, M) \geq \udeg(I, M/xM)\). The proof is complete. \(\square\)

**Remark 5.17.** By the prime avoidance theorem we always can choose \(x\) satisfying the condition of Theorem 5.16. Furthermore, according to the above proof we have \(\udeg(I, M/xM) = \udeg(I, M)\) provided \(x\) annihilates \(H_0^0(M')\) and \(H_1^0(M'/UM'(0))\), where \(M' = M/(x_3, \ldots, x_d)M\). This is the case if \(xU_0(M) = 0\) by Corollary 3.5.

By Proposition 5.7, Theorems 5.11 and 5.16, we have the main result of this section.

**Theorem 5.18.** For every \(m\)-primary ideal \(I\), the unmixed degree \(\udeg(I, \bullet)\) is an extended degree on the category of finitely generated \(R\)-modules \(\mathcal{M}(R)\).

We next compare the unmixed degree and the homological degree for sequentially Cohen-Macaulay modules.

**Remark 5.19.** Suppose \((R, \mathfrak{m})\) be a homomorphic image of a Gorenstein local ring \(S\) of dimension \(n\), and \(M\) a sequentially Cohen-Macaulay \(R\)-module. It is easy to see that \(\Ext^n_S(M, S)\) is either a Cohen-Macaulay module or zero module for all \(i\). By Theorem 5.11, we have
\[
\udeg(I, M) = \adeq(I, M) = \deg(I, M) + \sum_{i=0}^{d-1} \deg(\Ext^n_S(M, S))
\]
for the last equation see [22, Theorem 3.11]. Furthermore by [22, Theorem 3.5] we have
\[
\hdeg(I, M) = \deg(I, M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \deg(\Ext^n_S(M, S)).
\]
Therefore \(\udeg(I, M) \leq \hdeg(I, M)\). The equation occurs if and only if \(\Ext^n_S(M, S) = 0\) for all \(1 \leq i \leq d - 2\). In this case the dimension filtration of \(M\) is either \(H_0^0(M) \subseteq M\) or \(H_0^0(M) \subseteq U_M(0) \subseteq M\) with \(\dim U_M(0) = d - 1\).

We close this paper with some examples and an open question.

**Example 5.20.** Let \(R = k[[X_1, \ldots, X_4]]/(X_1^2, X_1X_2, X_1X_3)\) where \(k\) is a field and \(X_i, 1 \leq i \leq 4\), are indeterminates. We denote by \(x_i\) the image of \(X_i\) in \(R\). We have \(R\) is a sequentially Cohen-Macaulay ring of dimension 3 with the dimension filtration \(\mathcal{D} : 0 \subseteq (x_1) \subseteq R\). We have
\[
\deg(R) = 1 < \adeq(R) = \udeg(R) = 2 < \hdeg(R) = 3.
\]

**Example 5.21.** Let \(R = k[[X_1, \ldots, X_7]]/(X_1, X_2, X_3) \cap (X_4, X_5, X_6)\) where \(k\) is a field and \(X_i, 1 \leq i \leq 7\), are indeterminates. It is easy to see that \(\deg(R) = \adeq(R) = 2\). Moreover, we can compute that \(\hdeg(R) = 5\) and \(\udeg(R) = 4\).

**Question 1.** Is it true that \(\udeg(I, M) \leq \hdeg(I, M)\) for all finitely generated \(R\)-modules \(M\) and all \(m\)-primary ideals \(I\)?
Acknowledgement. Most results of this paper are contained in the second author thesis [26] under the supervisor of the first author that was defended in 2013 at Hanoi Institute of Mathematics. After that many applications of this work have been found about the attached primes of local cohomology [24], the Hilbert polynomial and its coefficients [12] and the big Cohen-Macaulay algebra in positive characteristic [27]. The readers are encouraged to these consequences.

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