Asymptotically Anti-de Sitter spacetimes and scalar fields with a logarithmic branch

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Abstract: We consider a self-interacting scalar field whose mass saturates the Breitenlohner-Freedman bound, minimally coupled to Einstein gravity with a negative cosmological constant in $D \geq 3$ dimensions. It is shown that the asymptotic behavior of the metric has a slower fall-off than that of pure gravity with a localized distribution of matter, due to the back-reaction of the scalar field, which has a logarithmic branch decreasing as $r^{-\frac{D-1}{2}} \ln r$ for large radius $r$.

We find the asymptotic conditions on the fields which are invariant under the same symmetry group as pure gravity with negative cosmological constant (conformal group in $D - 1$ dimensions). The generators of the asymptotic symmetries are finite even when the logarithmic branch is considered but acquire, however, a contribution from the scalar field.

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1. Introduction

In the presence of matter fields, the asymptotic behavior of the metric can be different from that arising from pure gravity if the matter fields do not fall off sufficiently fast at infinity. This occurs, for instance, in the case of three-dimensional black holes with electric charge \[1, 2\], as well as in the presence of scalar fields \[3\]. The modified asymptotic form of the fields can, a priori, modify the original symmetry at infinity and its associated charges. However, as the example analyzed in \[3\] shows, it might be possible to relax the standard asymptotic conditions without losing the original symmetry, but modifying the charges in order to take into account the presence of the matter fields. This effect occurs in a dramatic way when the mass of the scalar field saturates the Breitenlohner-Freedman bound \[4, 5\], since in this case, a potentially dangerous logarithmic branch appears in the asymptotic form of the scalar field.

In this paper we deal with a self-interacting scalar field minimally coupled to \(D\)-dimensional Einstein gravity with a negative cosmological constant, whose B-F saturating mass is explicitly given by

\[
m^2_* = -\frac{(D - 1)^2}{4l^2},
\]

where \(l\) is the AdS radius\(^1\).

We prove that the back-reaction produced by the logarithmic branch in the scalar field requires an asymptotic behavior of the metric which differs from the standard asymptotic behavior found in \[1, 2\] by the addition of logarithmic terms. Nevertheless, this relaxed asymptotic metric still preserves the original asymptotic symmetry, which is \(SO(D - 1, 2)\) for \(D > 3\), and the conformal group in two spacetime dimensions for \(D = 3\) \(3\). Furthermore, the conserved charges acquire an extra contribution coming from the scalar field, and they are finite even when the logarithmic branch is switched on.

We consider the following action

\[
I[g, \phi] = \int d^D x \sqrt{-g} \left( \frac{R}{16\pi G} - \frac{1}{2}(\nabla \phi)^2 - V(\phi) \right),
\]

with a potential \(V(\phi)\) given by

\[
V(\phi) = \frac{\Lambda}{8\pi G} + \frac{m^2_*}{2} \phi^2 + \phi^3 U(\phi),
\]

where \(U(\phi)\) could be any smooth function\(^2\) around \(\phi = 0\). Here \(G\) is the gravitational constant, which is \(G = (8\pi)^{-1}\) in appropriate units, and the cosmological constant \(\Lambda\) is related to the AdS radius \(l\) through \(\Lambda = -l^{-2}(D - 1)(D - 2)/2\).

\(^1\)Scalar fields with masses lower than this value produce a perturbative instability of AdS spacetime, as shown in Ref. \[4\] in four dimensions, and generalized to other dimensions in \[5\]. The case previously considered in \[3\] does not saturate the Breitenlohner-Freedman bound, which is one of the reasons why we focus here on the case \((\square)\).

\(^2\)This means that we consider any potential having a negative local maximum at \(\phi = \phi_0\), so that \(\Lambda = 8\pi G V(\phi_0)\), with \(V''(\phi_0) = m^2_*\).
In order to write down the asymptotic behavior of the fields, the metric is written as 
\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \]
where \( h_{\mu\nu} \) is the deviation from the AdS metric,
\[ ds^2 = -(1 + r^2/l^2)dt^2 + (1 + r^2/l^2)^{-1}dr^2 + r^2d\Omega_{D-2}^2. \] (1.4)

For matter-free gravity, the asymptotic behavior of the metric is given in [6, 7, 8]
\[ \begin{align*}
    h_{rr} &= O(r^{-D-1}), \\
    h_{rm} &= O(r^{-D}), \\
    h_{mn} &= O(r^{-D+3}).
\end{align*} \] (1.5)

Here the indices have been split as \( \mu = (r, m) \), where \( m \) includes the time coordinate \( t \) plus \( D-2 \) angles. It is easy to check that the asymptotic conditions (1.5) are invariant under \( SO(D-1, 2) \) for \( D > 3 \), and under the infinite-dimensional conformal group in two dimensions for \( D = 3 \). The asymptotic behavior of a generic asymptotic Killing vector field \( \xi^\mu \) is given by
\[ \begin{align*}
    \xi^r &= O(r), \\
    \xi^r_r &= O(1), \\
    \xi^m &= O(1), \\
    \xi^m_r &= O(r^{-3}).
\end{align*} \] (1.6)

The charges that generate the asymptotic symmetries involve only the metric and its derivatives, and are given by
\[ Q_0(\xi) = \frac{1}{2} \int d^{D-2}S_i \left\{ \bar{G}^{ijkl}(\xi^l g_{kl} - \xi^l_{\ |j} h_{kl}) + 2\xi^j \pi^i \right\}, \] (1.7)
where \( \bar{G}^{ijkl} = \frac{1}{2} g^{ij} (g^{kl} g^{ij} + g^{il} g^{jk} - 2g^{ij} g^{kl}) \), and the vertical bar denotes covariant differentiation with respect to the spatial AdS background. From (1.5) it follows that the momenta possess the following fall-off at infinity
\[ \begin{align*}
    \pi^{rr} &= O(r^{-1}), \\
    \pi^{rm} &= O(r^{-2}), \\
    \pi^{mn} &= O(r^{-5}),
\end{align*} \] (1.8)
and hence, the surface integral (1.7) is finite.

The Poisson brackets algebra of the charges yields the AdS group for \( D > 3 \) and two copies of the Virasoro algebra with a central charge given by
\[ c = \frac{3l}{2G} \] (1.9)
in three dimensions.

### 2. Switching on the scalar field

The asymptotic conditions (1.5) hold not just in the absence of matter but also for localized matter fields which fall off sufficiently fast at infinity, so as to give no contributions to the surface integrals defining the generators of the asymptotic symmetries. The scalar field would not contribute to the charges if it goes as \( \phi \sim r^{-(D-1)/2+\epsilon)} \) for large \( r \). However, when the
mass of the scalar field is given by (1.1), saturating the Breitenlohner-Freedman bound, the fall-off of the field is slower and generally produces a strong back-reaction that relaxes the asymptotic behavior of the metric.

In this case, the leading terms for $h_{\mu\nu}$ and $\phi$ as $r \to \infty$ read

$$\phi = \frac{1}{r^{(D-1)/2}} \left( a + b \ln \left( \frac{r}{r_0} \right) \right) + O \left( \frac{\ln \left( \frac{r}{r_0} \right)}{r^{(D+1)/2}} \right)$$

$$h_{rr} = -\frac{(D-1)b^2}{2(D-2)} \frac{\ln^2 \left( \frac{r}{r_0} \right)}{r^{(D+1)}}$$

$$+ \frac{1}{D-2} \frac{l^2 (b^2 - (D-1)ab) \ln \left( \frac{r}{r_0} \right)}{r^{(D+1)}} + O \left( \frac{1}{r^{(D+1)}} \right)$$

$$h_{mn} = O \left( \frac{1}{r^{(D-3)}} \right)$$

$$h_{mr} = O \left( \frac{1}{r^{(D-2)}} \right)$$

(2.1)

where $a = a(x^m)$, $b = b(x^m)$, and $r_0$ is an arbitrary constant.

Indeed, for an asymptotically AdS spacetime, the leading terms in the metric determine the fall-off of $\phi$ through its field equation, and in turn, a back-reaction with logarithmic terms in $h_{rr}$ is obtained by solving the constraints.

When the logarithmic branch of the scalar field is switched on ($b \neq 0$), these relaxed asymptotic conditions still preserve the original asymptotic symmetry, provided

$$a = -\frac{2}{(D-1)} b \ln \left( \frac{b}{b_0} \right) ,$$

where $b_0$ is a constant. Not that for $a = 0$, $b$ must be a constant. For $b = 0$, the asymptotic symmetry does not impose restrictions on $a$.

Using the Regge-Teitelboim approach [9] the contributions of gravity and the scalar field to the conserved charges, $Q_G(\xi)$ and $Q_\phi(\xi)$ are given by

$$\delta Q_G(\xi) = \frac{1}{2} \int d^{D-2}S_l \left[ G^{ijkl} (\xi^l \delta g_{ij,k} - \xi^l \delta g_{ij}) \right.$$

$$+ \frac{1}{2} \int d^{D-2}S_l (2\xi^k \delta \pi_{kl} + (2\xi^k \pi^{jl} - \xi^l \pi^{jk}) \delta g_{jk}) \right]$$

$$\delta Q_\phi(\xi) = - \int d^{D-2}S_l \left( \xi^l g^{1/2} g^{ij} \partial_j \phi \delta \phi + \xi^l \pi_{\phi} \delta \phi \right) .$$

Making use of the the relaxed asymptotic conditions, the momenta at infinity are found to be

$$\pi^{rr} = O(r^{-1}) , \quad \pi^{rm} = O(r^{-2}) , \quad \pi^{mn} = O(r^{-5} \ln^2(r)) ,$$

$$\pi_\phi = O(r^{(D-7)/2} \ln(r)) ,$$

(2.3)

(2.4)
and hence Eqs. (2.2), and (2.3) acquire the form

\[ \delta Q_G(\xi) = \delta Q_0(\xi)|_{\phi=0} + \frac{1}{2} \int d\Omega^{D-2} \frac{\xi^t}{l^2} \left\{ -\frac{D-1}{2} \delta b^2 \ln^2(r/r_0) \right. \]
\[ \left. + \delta(b^2 - (D-1)ab) \ln(r/r_0) \right\} \]
\[ (2.5) \]

\[ \delta Q_\phi(\xi) = -\frac{1}{2} \int d\Omega^{D-2} \frac{\xi^t}{l^2} \left\{ (2b - (D-1)a)\delta a \right. \]
\[ \left. - \frac{D-1}{2} \delta b^2 \ln^2(r/r_0) + \delta(b^2 - (D-1)ab) \ln(r/r_0) \right\} . \]
\[ (2.6) \]

Asymptotically, \( \xi^t \sim O(1) \) and therefore \( \delta Q_G \), and \( \delta Q_\phi \), both pick up logarithmic divergences, but these divergent pieces exactly cancel out. Thus, the total variation is well defined and the total charge \( Q = Q_G + Q_\phi \) can be integrated, obtaining

\[ Q(\xi) = Q_0(\xi) + \int d\Omega^{D-2} g^{1/2} r \frac{l^2}{l^2} \left\{ \frac{D-1}{8} \phi^2 + \frac{r^2}{2(D-1)} (\partial_\nu \phi)^2 \right\} , \]
\[ (2.7) \]

with \( Q_0(\xi) \) given by (1.7). Consequently, the conserved charges acquire an extra contribution coming from the scalar field, and they are finite even when the logarithmic branch is switched on. Note that in the case \( b = 0 \), the asymptotic behavior of the metric reduces to the standard one (1.5), and the original asymptotic symmetry is preserved, but nevertheless, the charges (2.7) still give a non-trivial contribution coming from the scalar field. One should expect that a similar expression for \( Q(\xi) \) in Eq. (2.7) could also be found using covariant methods as in [10].

The algebra of the charges (2.7) is identical to the standard one, namely AdS for \( D > 3 \) [6, 7], and two copies of the Virasoro algebra with the same central extension for \( D = 3 \) [8]. This can be readily obtained following Ref [11], where it is shown that the bracket of two charges provides a realization of the asymptotic symmetry algebra with a possible central extension. The central charge can be determined by the variation of the charges in the vacuum, represented here by AdS spacetime with \( \phi = 0 \).

3. Discussion

The presence of matter fields with nontrivial asymptotic behavior has generically two effects: It gives rise to a back reaction that modifies the asymptotic form of the geometry, and it generates additional contributions to the charges that depend explicitly on the matter fields at infinity which are not already present in the gravitational part. These two effects were observed in 2+1 dimensions in Ref. [3], and it is also seen here in the presence of a scalar field with a logarithmic branch. Furthermore, even when the logarithmic branch is switched off (\( b = 0 \) in Eq. (2.6)), the scalar field still gives a contribution to the charge, even though the asymptotic form of the metric is unchanged.

As shown here, the presence of the logarithmic branch in the scalar field is consistent with asymptotically AdS symmetry (both kinematically in the sense that the boundary conditions...
are preserved, and dynamically in the sense that the associated charges are finite), provided one takes into account the back reaction in the metric. In other words, it would be inconsistent to treat the scalar field as a probe in this case. It should also be stressed that it is only the sum of the gravitational contribution and of the scalar field contribution which is conserved and which defines a meaningful AdS charge. Each term separately may vary as one makes asymptotic AdS time translations.

We note that the AdS charges of metric-scalar field configurations with a logarithmic branch can also be computed through the method of holographic renormalization, as explicitly performed in [12, 13] for $D = 5$. Those papers contain a detailed discussion of the AdS/CFT correspondence in this context. The counterterms needed to render the action finite explicitly depend on the scalar field. This was also shown for the $D = 3$ case of [3] in the article [14].

Another question that deserves further study is the positivity of the AdS energy in the wider context considered here (positivity has been proved under standard, supersymmetric boundary conditions in [15, 16] following the asymptotic flat space derivation of [17, 18]). A related issue is whether the slower fall-off of the metric and the logarithmic tail of the scalar field are compatible with supersymmetry. In that respect, the spin 1/2-partner of the scalar field should play a crucial role and is expected to contribute to the supercharge. This question is under current investigation.

Scalar fields are involved in a recent controversy concerning cosmic censorship [19]. In particular, in Ref. [20], the cosmic censorship conjecture is challenged through the evolution of a scalar field in five dimensions, whose logarithmic branch is switched on. One may consider the following set of initial conditions without introducing a cut-off: $\phi = \phi_0$, for $r < R_0$, and $\phi = b_0 r^{-2} \ln r$, for $r > R_0$, so that $\phi_0 = b_0 R_0^{-2} \ln R_0$. The mass of this configuration is obtained through Eq. (2.7), with $\xi = \partial_t$, and reads

$$M = \frac{\pi^2 b_0^2}{4 l^2} (1 + 4 \ln^2 R_0) ,$$

(3.1)

which is manifestly finite and positive. It is argued in [20], that a naked singularity will develop at $r = R_s \sim \phi_0^{2/3} R_0$, for $\phi_0 << 1$. However, the mass as given by (3.1) has an associated Schwarzschild radius $r_+$ satisfying $r_+ >> R_s$, so that it encloses the singularity. This may help resolve the controversy.

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