Cubic Transmuted Gompertz Distribution: As a Life Time Distribution

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Authors’ contributions

This work was carried out in collaboration among all authors. All authors contributed immensely to the development of the article in all stages of the article formation. All authors read and approved the final manuscript.

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Abstract

In this work, we proposed and studied the Cubic Transmuted Gompertz (CTG) distribution using the Cubic Transmuted family of distributions which was introduced by Rahman et al. [8] and based on cubic transmutation map. We studied the statistical properties of the new distribution which includes: $r^{th}$ moment, moment generating function order statistics, mean, variance, Renyl entropy. The CTG distribution was fitted to a real data set to demonstrate its flexibility and tractability in modelling real life data.

Keywords: Renyl entropy; order statistics; cubic transmutation map; $r^{th}$ moment.

1 Introduction

Several attempts have been made to develop a new generalised families of distribution. The common properties of this distribution is that they are flexible, tractable and more adaptable to any kind of real life data. Eugene et al. [1] developed family of beta-generated family of distribution. Cordeiro and Castro [2]

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proposed the Kumaraswamy-G family of distribution. The Weibull-G family of probability distribution was developed by Bourguignon et al. [3]. Nadarajah et al. [4] developed the generalised G family of distribution. The Topp-Leone-G family of distribution was introduced by Rezaie et al [5]. Granzotto et al. [6] Aslam Mohammad et al. [7], Rahman et al. [8] studied the Generalized Cubic Transmuted G-family of distribution using different approaches that found it root from The Generalised transmuted family of distribution by Shaw et al. [9]. This work generalize the Gompertz distribution using the generalized Cubic Transmuted-G distribution to impute flexibility and tractability into the Gompertz distribution.

1.1 Aim and objectives of the research

The purpose of this work is to generalize the Gompertz distribution to obtain a new distribution named Cubic Transmuted Gompertz (CTG) which can be used to model bi-modal data with the aim of achieving the following objectives:

- To derive the probability density function and the cumulative density function of the new distribution.
- To determine some structural properties of the proposed distribution which includes: Moment, moment generating function, Renyi entropy and order statistics.
- To determine the flexibility and tractability of the new distribution with application to a real data set.

2 General Transmuted Family of Distributions

A general transmuted family of distributions that can be used to generate new families. Let $X$ be a random variable with cdf $F(x)$, then a general transmuted family; called $p$-transmuted family; is defined by

$$F(x) = G(x) + [1 - G(x)]^p \sum_{i=1}^{k} \lambda_i [G(x)]^i$$

(1)

With $\lambda_i \in [-1, 1]$ for $i = 1, 2, \ldots, k$ and $-p \leq \sum_{i=1}^{p} \lambda_i \leq 1$. The general transmuted family reduces to the base distribution for $\lambda_i = 0$, for $i = 1, 2, 3, \ldots, p$. The density function corresponding to (1) is

$$f(x) = g(x) \left\{1 - \sum_{i=1}^{p} \lambda_i [G(x)]^i + [1 - G(x)] \sum_{i=1}^{p} i \lambda_i G^i(x)\right\}$$

(2)

The density function of the quadratic transmuted quadratic family of distributions which was obtained by letting $p = 1$ as defined by Shaw et al. [9], is given as

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x)$$

(3)

Where $\lambda \in (1, -1)$ is the transmutation parameter. The quadratic transmuted family of distributions given in (3) has a wider area of applications to any baseline $G(x)$. The quadratic transmuted distribution can be regarded as a “defective” model because of its inability to address the problem of bi-modality which is sometimes present data.

To address the problem of bi-modality encountered in real life data, the cubic transmuted family of distributions was obtained by setting $k = 2$ in (2), which was demonstrated in the work of Aslam Mohammad et al. [7], and is given as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x)$$

(4)

Where $\lambda_1 \in [-1, 1], \lambda_2 \in [-1, 1]$ are the transmutation parameters and obey the condition.
\[-2 \leq \lambda_1 + \lambda_2 \leq 1.\]

The pdf corresponding to the equation (4) is defined as

\[
f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) - 3\lambda_2G^2(x)]
\]

The work of Granzotto et al. [6], stipulate that the cdf in equation (4) can also be obtained by defining the variables in the following way. Let \(K_1, K_2\) and \(K_3\) be independent and identically random variables with common distribution function \(f(t)\). Then, consider \(K_{1:3} = \min(K_1, K_2, K_3), K_{2:3} = \max(K_1, K_2), K_{3:3} = \max(K_1, K, K_3)\),

(i) \(Y \sim K_{1:3}\) with probability \(r_1\),
(ii) \(Y \sim K_{2:3}\) with probability \(r_2\),
(iii) \(Y \sim K_{3:3}\) with probability \(r_3\).

Where \(\sum_{i=1}^{3} r_i = 1\) and \(0 \leq r_i \leq 1, i = 1, 2, 3\). The cdf \(F_y(t)\) can be written as

\[
F_y(t) = r_1Pr(\min(K_1, K_2, K_3) \leq t) + r_2Pr(K_{2:3} \leq t) + r_3Pr(\max(K_1, K_2, K_3) \leq t).
\]

On substituting the respective probabilities and the simplifying to obtain

\[
F_y(t) = 3r_1f(t) + 2r_1(\tau_2 - \tau_1)f^2(t) + (1 - \tau_1)f^3(t)
\]

The above expression transforms to the same as in equation (4), if we consider \(3r_1 = \lambda_1\) and \(3(\tau_2 - \tau_1) = (\lambda_2 - \lambda_1)\).

### 3 Cubic Transmuted Gompertz (CTG) Distribution

The Gompertz distribution was proposed by Gompertz [10] in 1825. It is useful in modelling survival times, actuarial and human mortality data. It found application in several areas such as gerontology, biology, marketing etc. Several extensions of Gompertz distribution has been proposed and studied in literature.

Roozegar et al. [11] studied the McDonald Gompertz distribution, The Generalised Gompertz (GG) distribution was studied by El-Gohary et al. [12], Jafari et al. [13] studied the structure and properties of Beta Gompertz distribution.

The cdf of CTG was obtained as follows:

The cdf of Gompertz distribution is given by

\[
G(x) = 1 - e^{-\frac{x}{\beta}(e^{\beta x} - 1)}, \quad x \in [0, \infty)
\]

And the associated probability density function (pdf) is given by

\[
g(x) = \alpha e^{\beta x - \frac{\alpha}{\beta}}(e^{\beta x} - 1), \quad x \in [0, \infty)
\]

The Transmuted Gompertz (TG) distribution has been proposed and studied by Abdul-Moniem et al. [7]. Using (7) in (3), the cdf of TG distribution was obtained as

\[
F(x) = \left[1 - e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)}\right] \left[1 + \lambda e^{-\frac{\alpha}{\beta}(e^{\beta x} - 1)}\right]
\]

Where \(\alpha, \beta > 0, \lambda \in (-1, 1)\). We propose an extension of TG distribution called the Cubic Transmuted Gompertz (CTG) distribution by using (7) in (4). It cdf is given by
\[ F(x) = \left[ 1 - e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right] \left[ 1 + (\lambda_1 + \lambda_2)e^{-\frac{2\alpha}{\beta} e^{\beta x - 1}} + \lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right] \] (10)

Where \( \alpha, \beta > 0, \lambda_1 \in (-1,1), \lambda_2 \in (-1,1) \) and \(-2 \leq \lambda_1 + \lambda_2 \leq 1\). If we let \( \lambda_2 = 0 \), the distribution function given in (10), reverse back to (9) and when \( \lambda_1 = \lambda_2 = 0 \), then we obtain its baseline distribution function. The CTG Distribution can be used to analyze various forms of data even those that exhibits bi-modal failure rate.

Drawn below is the graph of the cdf of CTG distributions for various values of the parameters. Where \( a = \alpha, b = \beta, b1 = \lambda_1 \) and \( b2 = \lambda_2 \)

**Proposition 1:** Suppose a continuous random variable \( X \) follows a Gompertz distribution with parameters \( \alpha, \beta \), then the pdf of CTG distribution is given as

\[
 f(x) = \alpha e^{\beta x} e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2)e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right]
\] (11)

Since, \( f(x) = \frac{dF(x)}{dx} \)

where \( x \in [0, \infty) \), \( \lambda_1 \in [-1,1] \) and \( \lambda_2 \in (-1,1) \) are the transmutation parameters which holds the condition \(-2 \leq \lambda_1 + \lambda_2 \leq 1\)

**Proposition 2:** The limit of CTG distribution as \( x \to 0 \) is \( \alpha [1 + \lambda_1] \) and \( x \to \infty \) is 0

Fig. 2 represents the graph of the pdf of CTG distributions for various values of the parameters. Where \( a = \alpha, b = \beta, b1 = \lambda_1 \) and \( b2 = \lambda_2 \).

Fig. 2 shows that the CTG distribution can be used to model real life data that are bi-modal in nature.

### 3.1 Reliability analysis

The reliability function is given as \( R(t) = 1 - F(t) \); the reliability function of CTG distribution is given by
where \( x \in [0, \infty) \), \( \lambda_1 \in [-1,1] \) and \( \lambda_2 \in (-1,1] \) are the transmutation parameters which holds the condition \(-2 \leq \lambda_1 + \lambda_2 \leq 1\)

Fig. 2. The graph of the pdf of CTG distribution

Fig. 3 represents the graph of the pdf of CTG distributions for various values of the parameters. Where \( a = \alpha, b = \beta, b_1 = \lambda_1 \) and \( b_2 = \lambda_2 \)

Fig. 3. Graph of the survival function of CTG distribution

And the Hazard function \( (j(x)) \) is given by

\[
j(x) = \frac{ae^\beta x e^{-\frac{a}{\beta}e^{\beta x - 1}}} \left[ 1 \right. - \left( 1 - \frac{a}{\beta}e^{\beta x - 1} \right) \left[ 1 + \left( \lambda_1 + \lambda_2 \right)e^{-\frac{a}{\beta}e^{\beta x - 1}} + \lambda_2 e^{-\frac{a}{\beta}e^{\beta x - 1}} \right] \right] - 2(\lambda_1 + 2\lambda_2) e^{-\frac{a}{\beta}e^{\beta x - 1}} - 3\lambda_2 e^{-\frac{a}{\beta}e^{\beta x - 1}} \right] \right] \right]
\]

(13)

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The (13) above is the CTG model and when $\lambda_1 = \lambda_2 = 0$, it reverses back to Gompertz model given by $j(x) = ae^{\beta x}$.

### 3.2 $r^{th}$ moment of CTG distribution

The $r^{th}$ moment of a distribution can be obtained using the relation

$$E(X^r) = \mu'_r = \int_{-\infty}^{\infty} x^r f(x; \alpha, \beta, \lambda_1, \lambda_2) dx$$  \hspace{1cm} \text{(14)}$$

Inserting equation (11) in (14), we have

$$\mu'_r = \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{a}{\beta}(e^{bx}-1)} \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2)e^{-\frac{a}{\beta}(e^{bx}-1)} - 3\lambda_2 e^{-\frac{2a}{\beta}(e^{bx}-1)} \right] dx$$

This can further be reduced to

$$\mu'_r = (1 - \lambda_1 - \lambda_2) \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{a}{\beta}(e^{bx}-1)} dx + 2(\lambda_1 + 2\lambda_2) \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{2a}{\beta}(e^{bx}-1)} dx$$

$$-3\lambda_2 \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{a}{\beta}(e^{bx}-1)} dx$$  \hspace{1cm} \text{(15)}$$

We let,

$$I_1 = (1 - \lambda_1 - \lambda_2) \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{a}{\beta}(e^{bx}-1)} dx$$  \hspace{1cm} \text{(16)}$$

$$I_2 = 2(\lambda_1 + 2\lambda_2) \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{2a}{\beta}(e^{bx}-1)} dx$$  \hspace{1cm} \text{(17)}$$

$$I_3 = 3\lambda_2 \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{a}{\beta}(e^{bx}-1)} dx$$  \hspace{1cm} \text{(18)}$$

Then,

$$\mu'_r = I_1 + I_2 - I_3$$  \hspace{1cm} \text{(19)}$$

From (16) we have

$$I_1 = (1 - \lambda_1 - \lambda_2) \int_{-\infty}^{\infty} x^r a e^{bx} e^{-\frac{a}{\beta}(e^{bx}-1)} dx$$

Let $y = e^{bx}, \ln y = \beta x, x = \frac{\ln y}{\beta}, dx = \frac{dy}{y\beta}$ and substitute it in the above expression, we have
Integrating (20) by part, we have

\[
I_1 = (1 - \lambda_1 - \lambda_2)e^{\frac{a}{\beta}} \frac{1}{\beta^{r+1}} \int_{-\infty}^{\infty} (\ln y)^r e^{-\frac{a}{\beta}y} \, dy
\]  

(20)

Where

\[
E_s^k(y) = \frac{1}{k!} \int_{-\infty}^{\infty} (\ln x)^k x^{-s} e^{-\gamma x} \, dx
\]  

(22)

Is the generalised integro-exponential.

Also for \(I_2\) using the substitution, (17) we transform to

\[
I_2 = 2(\lambda_1 + 2\lambda_2) \frac{1}{\beta^{r+1}} \int_{-\infty}^{\infty} (\ln y)^r e^{-\frac{2a}{\beta}y} e^{\frac{2a}{\beta}y} \, dy
\]  

(23)

Also integrating (23) using integration by part, we have,

\[
I_2 = 2(\lambda_1 + 2\lambda_2) \frac{1}{\beta^{r+1}} e^{\frac{2a}{\beta}E_1^{r-1}} \left( \frac{2\alpha}{\beta} \right)
\]  

(24)

Considering \(I_3\) using the same substitution, (18) we transform to

\[
I_3 = 3\lambda_2 \frac{1}{\beta^{r+1}} \int_{-\infty}^{\infty} (\ln y)^r e^{-\frac{3a}{\beta}y} e^{\frac{3a}{\beta}y} \, dy
\]  

(25)

Integrating (25) by part will transform to

\[
I_3 = 3\lambda_2 \frac{1}{\beta^{r+1}} e^{\frac{3a}{\beta}E_1^{r-1}} \left( \frac{3\alpha}{\beta} \right)
\]  

(26)

Finally, by combining (21), (24) and (26) we obtained an expression for the moment generating function of CTG distribution given by

\[
E(X^r) = \alpha e^{\frac{a}{\beta}} \frac{1}{\beta^{r+1}} \left[ (1 - \lambda_1 - \lambda_2)E_1^{r-1} \left( \frac{\alpha}{\beta} \right) + 2(\lambda_1 + 2\lambda_2)e^{\frac{2a}{\beta}E_1^{r-1}} \left( \frac{2\alpha}{\beta} \right) - 3\lambda_2 e^{\frac{3a}{\beta}E_1^{r-1}} \left( \frac{3\alpha}{\beta} \right) \right]
\]  

(27)

From (27), we can obtain the mean, variance, kurtoses, skewness of the CTG distribution.

### 3.3 Moment Generating Function (mgf) of CTG distribution

The mgf of the a random variable \(x\) is

\[
M_x(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(T^r)
\]  

(28)
Order statistics are among the most fundamental tools in non-parametric statistics and inference. The pdf of the order statistic for a random sample \( x_1, x_2, \ldots, x_n \) from the CTG distribution is given by

\[
g_{i:n}(x) = \binom{n}{i} g(x) \sum_{i=1}^{n-i} (-1)^j \binom{n-i}{j} G^{i+j-1}(x)
\]

3.4 Renyi entropy

The Renyi entropy of a random variable \( X \) represents a measure of uncertainty. The measure has been shown to be effective in comparing the tails and shapes of various standard distributions, Song [14]. A large value of entropy indicates the greater level of uncertainty in the data. The Renyi, A. [15] introduced the Renyi entropy defined as

\[
H_\phi(T) = \frac{1}{1 - \phi} \log \int_{-\infty}^{\infty} g(x)^\phi dx , \quad \phi > 0 \text{ and } \phi \neq 1
\]

The Renyi entropy tends to Shannon entropy as \( \phi \to 1 \)

Putting (11) in (30), we have

\[
H_\phi(T) = \frac{1}{1 - \phi} \log \int_{-\infty}^{\infty} a e^{\beta x} e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2) e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right]^\phi dx
\]

By letting \( Z_1 = \left\{ a e^{\beta x} e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right\}^\phi \) and \( Z_2 = \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2) e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right]^\phi \)

This implies that

\[
H_\phi(T) = \frac{1}{1 - \phi} \log \left[ \int_{-\infty}^{\infty} Z_1 dx \right] \left[ \int_{-\infty}^{\infty} Z_2 dx \right]
\]

Finally, the Renyi entropy of CTG is given by

\[
H_\phi(T) = \frac{1}{1 - \phi} \log \left[ \frac{\beta^\phi}{\alpha^\phi} \Gamma(\phi + 1) \int_{-\infty}^{\infty} \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2) e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x - 1}} \right]^\phi dx \right]
\]

The expression above can further be solved numerically.

3.5 Order statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. The pdf \( f_{i:n}(x) \) of the \( i \)th order statistic for a random sample \( x_1, x_2, \ldots, x_n \) from the CTG distribution is given by

\[
g_{i:n}(x) = \binom{n}{i} g(x) \sum_{i=1}^{n-i} (-1)^j \binom{n-i}{j} G^{i+j-1}(x)
\]
Putting (11) in (35) give an expression for the $i$th order statistics of CTG distribution as

$$g_{i:n}(x) = \frac{n!}{(n-i)!} \left\{ a e^{\beta x} e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2)e^{-\frac{\alpha}{\beta} e^{\beta x-1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \right] \right\}$$

$$+ \sum_{j=1}^{n-i} (-1)^j \binom{n-i}{j} \left\{ 1 - e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \right\} \left[ 1 + (\lambda_1 + \lambda_2)e^{-\frac{2\alpha}{\beta} e^{\beta x-1}} + \lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \right]^{j-1}$$

(36)

From (36) the first order statistics can be obtained by taking $i = 1$ and the largest order can also be obtained by taking $i = n$.

### 4 Estimation of the Parameters

The likelihood function of CTG distribution is given by

$$L(x; \alpha, \beta, \lambda_1, \lambda_2) = \prod_{i=1}^{n} \left\{ a e^{\beta x} e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2)e^{-\frac{\alpha}{\beta} e^{\beta x-1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x-1}} \right] \right\}$$

(37)

The log likelihood ($l$) function is

$$l = n \ln(a) + \beta \sum_{i=1}^{n} x_i - \frac{\alpha}{\beta} \sum_{i=1}^{n} (e^{\beta x_i} - 1) + \sum_{i=1}^{n} \ln \left[ 1 - \lambda_1 - \lambda_2 + 2(\lambda_1 + 2\lambda_2)e^{-\frac{\alpha}{\beta} e^{\beta x_i-1}} - 3\lambda_2 e^{-\frac{\alpha}{\beta} e^{\beta x_i-1}} \right]$$

(38)

Differentiating ($l$) with respect to parameters $\alpha, \beta, \lambda_1$ and $\lambda_2$ and equating the non-linear equation to zero and solving the resulting equation gives the parameter estimates. The solution may not be obtained as a closed form expression, rather the parameter estimates can be obtained numerically by using software which may include, OX program, R, MAPLE etc.

### 5 Application

In this section, we provide an application to real data set to prove the tractability and flexibility of CTG model. The measure of goodness-of-fit statistics for this model are compared with other competing models and the MLEs of the model parameters were obtained. We compare the fits of Cubic transmuted Gompertz with the Weibull power series (W-Ps) distribution Tahir et al. [16], Kumaraswamy Pareto (Kw-P) distribution, Bourguignon et al. [17], New modified Weibull (N-MW) distribution Almalki and Yuan [18], Transmuted Weibull (T-W) distribution Aryal and Tsokos [19] and Complementary Bur III Poison (C-BIII-P) distribution Hassan et al. [20].

The data set consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang (2003). The starting point of the iterative processes for the cancer patient’s data set is $(1:0.089; 10:0.1; 0:1)$. The data point is given as: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63 and 22.69.

Table 1 present the Exploratory Data Analysis (EDA) for the cancer remission time data set.
Table 1. EDA of the remission time of the bladder cancer patients

| min | Q₁ | median | mean | Q₃ | max | kurtosis | skew. | var. | range |
|-----|----|--------|------|----|-----|----------|-------|------|-------|
| 0.08| 3.348 | 6.395 | 9.366 | 11.840 | 79.05 | 16.154 | 3.326 | 110.425 | 78.85 |

The Total Time on Test (TTT) plot for the remission time of cancer bladder patients’ data is displayed in Fig. 4, which clearly indicates that the data exhibits (Bathtub) shape failure rate.

Fig. 4. Total test on time plot

Table 2. MLEs (Standard error in parenthesis) to the remission times of a cancer patients

| Model | Estimates |
|-------|-----------|
| CTG \((α, β, λ₁, λ₂)\) | (0.2097 \pm 0.0526) | -0.0320 | -0.7333 | -0.4943 |
| \(W - P\) \((α, β, β, α)\) | (9.7237 \pm 5.2316) | 0.08284 | 12.5913 | 84.7241 |
| Kw-P \((α, β, α, β)\) | (27.2081 \pm 8.1380) | 15.5401 | 0.31896 | 0.00055 |
| N-MW \((α, β, α, Y, λ)\) | (0.0763 \pm 0.1674) | 0.0176 | 1.0476 | 1.0479 | 1E-10 |
| \(TW\) \((α, β)\) | (9.5607 \pm 0.8529) | 1.0478 | 1E-10 | (0.00023) |
| \(AW\) \((α, β, α)\) | (1E-10 \pm 1.144E - 10) | 0.7570 | 0.0939 | 1.0478 |
| EW \((α, β, λ)\) | (0.2055 \pm 1.144E - 10) | 1E-10 | 0.1068 | (0.0094) |
| C-BIII-P \((α, β, λ)\) | (1.2853 \pm 0.0832) | 1.5895 | 4.5730 |

5.1 Model selection criteria

In each case the parameters were estimated using the maximum likelihood approach and also model selection was carried out using the log-likelihood function evaluated at the MLEs \(\hat{\theta}\). Akaike information Criterion (AIC), Cramer-von Mises (\(W^*\)), Probability value (P-Value) and Kolmogorov Smirnov (K-S) Statistics to compare the fitted models. The statistics \(W^*\) are well defined by Chen and Balakrishnan [22]. The smaller the values of AIC, \(W^*\) and K-S and the larger the P-Value statistics the better the model fits to the data. The statistics are given by
\[ AIC = -2\hat{\ell} + 2p, \quad W^* = \left( \frac{0.5}{n} + 1 \right) \left( \sum_{i=1}^{n} \left( \mathcal{K}_i - \frac{2i - 1}{n} \right)^2 + \frac{1}{12n} \right), \quad K - S = \max \left\{ \frac{i}{n} - \mathcal{K}_i, \mathcal{K}_i - \frac{i + 1}{n} \right\} \]

Where \( \hat{\ell} \) denotes the maximized log-likelihood function, \( p \) is the number of parameters, \( n \) is the sample size. \( \mathcal{K}_i = \text{cdf} \left( Y_{(i)} \right) \) and \( Y_{(i)} \)'s are the ordered observations. The estimates of the parameters and the standard error are listed in Tables 2 and 3 gives the statistics as \(-2\hat{\ell}, AIC, W^*, K - S\) and \( P - values \).

Table 2 shows the CTG distribution fits the data better than the other competitive models.

| CTG       | 820.80 | 828.80 | 0.0326 | 0.0563 | 0.8128 |
|-----------|--------|--------|--------|--------|--------|
| W-P       | 827.836| 835.837| 0.1373 | 0.0703 | 0.5525 |
| Kw-P      | 833.739| 841.739| 0.1497 | 0.0632 | 0.6864 |
| N-MW      | 828.174| 838.174| 0.1314 | 0.069991 | 0.5575 |
| TW        | 828.174| 834.174| 0.1314 | 0.070017 | 0.5570 |
| AW        | 828.174| 836.174| 0.1314 | 0.069993 | 0.5574 |
| E-W       | 828.684| 834.684| 0.1193 | 0.084640 | 0.3183 |
| C-BIII-P  | 832.928| 838.928| 0.1356 | 0.49197 | 2.2E-16 |

The MLEs and their corresponding standard errors (in parenthesis) of the model parameters and the values of \(-2\hat{\ell}, AIC, CAIC, HQIC, BIC, W^* and A^*\) statistics for the data set considered. It is clear from Table 3 that the Cubic transmuted Gompertz distribution produce a best fits to cancer data set.

### 6 Conclusion

We proposed and developed a cubic transmuted Gompertz distribution which generalises the conventional Gompertz distribution by adding extra two shape parameters. We provide some mathematical properties of the new distribution which includes moment, moment generating function, entropy and order statistics. By means of a real data set we were able to show that the Cubic Gompertz distribution provide a better fits than other class of models considered and this further proof its tractability and flexibility in modeling real life data even those that a bi-modal.

### Competing Interests

Authors have declared that no competing interests exist.

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