The modified cyclic multi-surface plastic model of sandy soil based on damping ratio

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Abstract. Traditional multi-surface plastic model always overestimates the accumulated plastic strain when it is applied into sandy soil under cyclic loading. Considering this overestimation can be reflected through equivalent damping ratio of the hysteretic loop and the damping ratio can also include the real anisotropic plastic hardening which is simply considered in kinematic hardening rule and isotropic hardening rule, series of strain-controlled cyclic simple shear tests were conducted to obtain the relationship between correction coefficient of hardening rule and initial state including initial effective, then a modified cyclic multi-surface plastic model was established by regarding the subsequent hysteretic loop as the new first loop with a new initial state which is the state at the start of the fourth quarter of the previous loop, thus, the plastic hardening formula can be upgraded in the process of loading.

1. Introduction

The behavior of granular material under different types of loading, especially in the nonlinear phase, has been modeled by many kinds of constitutive models, such as equivalent linear model, elastic-plastic model and Cambridge model. As for elastic-plastic model and Cambridge model, they are established on the condition of monotonic drained loading, then applied to cyclic loading, cause every cycle theoretically can be regarded as two parts of monotonic loading which are named as unloading and reloading respectively if the effect of loading rate is not taken into account. Actually, the most salient feature that these models need to accurately reflect is the relation between strain and stress after the yield condition has been reached so as to predict the deformation and evaluate safety factor in practical engineering, aiming to this, the plastic hardening rule in elastic-plastic model was postulated to simulate the variation of this relation and calculate the accumulated plastic strain which symbolizes energy dissipation.

In the last several decades, many kinds of elastic-plastic model were proposed and modified to try to fit the stress-strain line. A mathematical model capable of describing the soil behavior under any loading is present within the framework of critical state soil mechanism by Dafalias and Yannis[1]. The mathematical foundation of the general bounding surface constitutive formulation in plasticity was presented by Dafalias and Yannis[2], allowing the better understanding and improvement of existing bounding surface plasticity and hypo-plasticity models. Heidarzadeh and Oliaei[3] developed a generalized model by definition of a new unique plastic modulus for loading, unloading and reloading...
based on bounding surface plasticity, this model just needs 11 parameters to exhibit the sands behavior in all conditions. Shahram Shahrooi et al.[4] proposed an optimum model by analyzing two multi-surface and two non-linear kinematic hardening models of cyclic plasticity for fatigue life prediction of metals. Fang, JG et al.[5] showed that a greater hardening modulus would lead to an ascending branch of the stress versus strain curve; and the yield function may significantly affect the stress state and phase field damage. Chen, RP et al.[6] proposed the new hardening law related to accumulated deviatoric plastic strain for the inner surface to describe the cyclic shakedown and degradation. Leal, AN et al.[7] proposed microfabric-inspired rotational hardening rules for the plastic potential and bounding surfaces associated with the generalized bounding surface model for cohesive soils. Zhou, C et al.[8] used the anisotropic yield stresses associated with the parameters C (I“) and C (vp) to describe the effect of viscosity on the yield stress by following the approach of Perzyna's overstress visco-plasticity and Suklje's isochrona concept. Based on a new type of kinematic hardening and the theory of critical state soil mechanics, a two-surface model is herein developed for predicting the undrained behavior of saturated cohesive soils under cyclic loads by Li, Tao et al.[9]. Sukkarak et al.[10] modified the stress-dependent stiffness values control the shear hardening and the cap yield surfaces to exhibit different degrees of evolution with changes in the stress state.

Although these incremental plastic models were established through varying plastic hardening formula which can simulate the variation of loading surface, but these hardening rules can not accurately reflect the anisotropic change of loading surface which is considered by introducing kinematic hardening rule, meanwhile, yield surfaces applied into prevalent elastic-plastic model are incapable of modeling the intrinsic anisotropy of soil fully, these lead to the gap between stress-strain curves which are obtained through experiments and constitutive simulation so that the phenomenon of energy dissipation is not performed truly, another reason for this gap is about the method of fitting function of loading surface evolution, the function is always obtained by fitting scatter point, this lead to the difference between fitting result and reality.

In this paper, considering the energy dissipation can implicitly reflect the anisotropy mentioned above, although the concrete affecting way has not been figured out, the damping ratio formulation of sample under cyclic loading is derived from the existing multi-surface plastic model firstly, then the strain controlled cyclic simple shear tests were conducted to compare the experimental energy dissipation with the result of the plastic model under condition of certain strain amplitude to get the relationship between the difference of damping ratio and initial state including initial effective mean stress and initial void ratio. According to the relationship, the modification factor and corresponding parameter can be obtained and introduced into incremental constitutive relationship. Finally, the modified cyclic multi-surface plastic model was proved to be capable of reproducing the curve of strain-stress under cycle loading more accurately than precedent version.

2. Multi-surface plastic model
The multi-surface plasticity reserves the concept of yield surface and loading surface as showed in figure 1, loading surface evolution is established by introducing isotropic hardening rule which can not reflect strength anisotropy, but the kinematic hardening rule can to some degree. Von-mise yield principle is applied to this model, the basic expression is:

\[ f_0 = \sqrt{\frac{2}{3} J_2} - k_0 = 0 \]  

(1)

Where \( J_2 \) is the second deviatoric stress invariant, \( k_0 \) is the stress strength on the octahedral surface. Actually this expression has assumed the isotropy of shear strength at original state. After taking hardening rule into account, the loading surface can be obtained as follows:

\[ f_l = \sqrt{\frac{2}{3} (s_{ij} - a_{ij}) : (s_{ij} - a_{ij})} - k_l = 0 \]  

(2)

Where \( s_{ij} \) is deviatoric stress tensor, \( a_{ij} \) is back stress of kinematic hardening rule which symbolizes the isotropic change of loading surface in the process of loading, and the general form of parameter of
kinematic hardening rule is always a scale value which coarsely defines the effect of state parameter on back stress in different direction, $k_i$ is a scale value from isotropic hardening rule which is related to state parameter.

3. The damping ratio of multi-surface plastic model

The damping ratio measures the energy dissipation during cyclic loading, which is caused by the plastic strain, this ratio can be calculated through the area in figure 2.

![Figure 1. Scheme of yield surface and loading surface in multi-surface plastic model](image1)

![Figure 2. Hysteretic curve for damping calculation](image2)

The equations can be derived as follows:

In terms of Masing second method, the hysteretic loop can be expressed in form of full quantity constitutive which changes with the skeleton curve during loading, so it is easier to calculate the area of triangle $OAD$ and circle $ABB’CC’A’$, symbolizing the total work and energy dissipation respectively, than incremental constitutive of multi-surface model, but the general expression of damping ratio can still be derived as follows:

The equation of loading surface in form of general stress state is

$$f(\sigma_{ij}, \alpha_{ij}, k) = 0 \quad (3)$$

According to the consistency rule:

$$\frac{\partial f}{\partial \sigma_{ij}} : d\sigma_{ij} + \frac{\partial f}{\partial a_{ij}} : da_{ij} + \frac{\partial f}{\partial k} : dk = 0 \quad (4)$$

Take the relation between back stress and state parameter which is selected as accumulative plastic strain and the relation between isotropic hardening rule and state parameter into account, the equation (4) can be changed into equation (5) as follows:

$$\frac{\partial f}{\partial \sigma_{ij}} : d\sigma_{ij} + \frac{\partial f}{\partial a_{ij}} : \frac{da_{ij}}{d\varepsilon^p_{ij}} : d\varepsilon^p_{ij} + \frac{\partial f}{\partial k} : \frac{dk}{d\varepsilon^p} : d\varepsilon^p = 0 \quad (5)$$

According to the flow rule, the plastic potential function is assumed as

$$g(\sigma_{ij}, \alpha_{ij}, k) \quad (6)$$

Where $d\varepsilon^p$ is norm of tensor of plastic strain increment, $d\varepsilon^p_{ij}$ is tensor of plastic strain increment.

By submitting equation (6) into equation (5), the expression of plastic hardening modulus can be derived as follows:

$$H_m = -\left(\frac{\partial g}{\partial \sigma_{ij}} : \frac{da_{ij}}{d\varepsilon^p_{ij}} + \frac{\partial g}{\partial \sigma_{ij}} : \frac{dk}{d\varepsilon^p} \right) \quad (7)$$

It need to be noted that the result of $\frac{da_{ij}}{d\varepsilon^p_{ij}}$ is assumed as scale value in Prager hardening rule, but it...
actually should be a fourth rank tensor, whose form is like tensor of modulus, to model the real variation of loading surface in different direction of body of soil.

So the incremental constitutive can be expressed as follows:

$$d \varepsilon_{ij} = C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$  \hspace{1cm} (8)

In every cycle of stress and strain, $ABB'CC'A'$, there are two initial yield surface, the phase of $AB'$ during unloading and the phase of $CC'$ during reloading, and every process of unloading and reloading actually can be regarded as monotonic loading with new initial condition, there are only elastic deformation happening in these two phases. The expressions of these two yield surfaces are showed as follows:

$$f_{AB'}(\sigma_{ij}, \alpha_{ij}^{AB'}, k_{AB'}) = 0$$  \hspace{1cm} (9)

$$f_{CC'}(\sigma_{ij}, \alpha_{ij}^{CC'}, k_{CC'}) = 0$$  \hspace{1cm} (10)

Where $\alpha_{ij}^{AB'}, k_{AB'}, \alpha_{ij}^{CC'}, k_{CC'}$ indicate the location and the shape of yield surface in stress space., the physical value of these four value can be calculated by computer iteration.

Then the corresponding loading surface in phase of $B'C$ and $C'A'$ are expressed as follows:

$$f_{B'C}(\sigma_{ij}, \alpha_{ij}^{AB'} + \alpha_{ij}, k_{AB'} + k) = 0$$  \hspace{1cm} (11)

$$f_{C'A'}(\sigma_{ij}, \alpha_{ij}^{CC'} + \alpha_{ij}, k_{CC'} + k) = 0$$  \hspace{1cm} (12)

The stress states of points $A,B'C,C'C'A'$ are noted as $\sigma_{ij}^A, \sigma_{ij}^{B'}, \sigma_{ij}^C, \sigma_{ij}^{C'}, \sigma_{ij}^{A'}$ respectively so that the plastic work or plastic energy in one cycle can be calculated as follows:

Because the $d\varepsilon_{ij}^p$ can be obtained through

$$d\varepsilon_{ij}^p = \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$  \hspace{1cm} (13)

For phase of $B'C$, the plastic energy is:

$$W_{B'C}^P = \int_{\sigma_{ij}^p}^{\sigma_{ij}^{B'}} \sigma_{ij} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$  \hspace{1cm} (14)

For phase of $C'A'$, the plastic energy is:

$$W_{C'A'}^P = \int_{\sigma_{ij}^p}^{\sigma_{ij}^{C'}} \sigma_{ij} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$  \hspace{1cm} (15)

The total work in one cycle can also be integrated as follows:

$$W_{tot} = \int_{\sigma_{ij}^p}^{\sigma_{ij}^{C'}} \sigma_{ij} \left( C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \right) + \int_{\sigma_{ij}^p}^{\sigma_{ij}^{B'}} \sigma_{ij} \left( C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \right)$$  \hspace{1cm} (16)

Finally, the damping ratio in one cycle can be presented:

$$D = \frac{W_{B'C}^P + W_{C'A'}^P}{W_{tot}} = \frac{\int_{\sigma_{ij}^p}^{\sigma_{ij}^{B'}} \sigma_{ij} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \int_{\sigma_{ij}^p}^{\sigma_{ij}^{C'}} \sigma_{ij} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\int_{\sigma_{ij}^p}^{\sigma_{ij}^{C'}} \sigma_{ij} \left( C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \right) + \int_{\sigma_{ij}^p}^{\sigma_{ij}^{B'}} \sigma_{ij} \left( C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \right)$$  \hspace{1cm} (17)

By the way, the stress state of points $A,B'$ satisfies the yield surface of equation (9), the stress state of points $C,C'$ satisfy the yield surface of equation (10).

Considering the need of correction below, the equation (17) was transformed into equation (18)

$$\frac{1-D}{D} = \frac{\int_{\sigma_{ij}^p}^{\sigma_{ij}^{C'}} \sigma_{ij} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \int_{\sigma_{ij}^p}^{\sigma_{ij}^{B'}} \sigma_{ij} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}}{\int_{\sigma_{ij}^p}^{\sigma_{ij}^{C'}} \sigma_{ij} \left( C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \right) + \int_{\sigma_{ij}^p}^{\sigma_{ij}^{B'}} \sigma_{ij} \left( C_{ijkl}^{-1} d\sigma_{ij} + \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \right)}$$  \hspace{1cm} (18)
4. The modified multi-surface plastic model

In order to compare the difference of the damping ratio between experimental modelling and multi-surface plastic constitutive, series of cyclic simple shear test were conducted under five different kinds initial effective axial stress, 20kPa, 50kPa, 100kPa, 200kPa, 300kPa respectively, with different axial strain amplitude. In this paper, for every sample, the damping ratio of first hysteresis loop was calculated, because the subsequent loop can be considered as sample with new initial effective axial stress and new particle arrangement. The variation of damping ratio with strain amplitude is showed in figure 3, it is indicated that the damping ratio is close to 0, and no obvious increase within medium strain comparing with the situation of large strain. Temporarily, the result of multi-surface plastic modeling was obtained to compare with experimental result, as for the stress amplitude during modeling, it can be adjusted to be as close as possible to the stress value corresponding to the strain amplitude in experiment, this can guarantee the accuracy of difference of damping ratio between constitutive and experimental result.

![Figure 3. The variation of damping ratio of sand sample under different effective axial stress](image)

The physical difference of damping ratio for initial effective axial stress mentioned above is respectively showed in figure 4(a), (b),(c),(d),(e) the multi-surface plastic constitutive always overestimates the equivalent damping ratio especially when the strain is beyond 0.1%, and the difference increase with the increase of strain amplitude, considering the effect of initial axial stress to the initial fabric of sample and evolution of arrangement of soil particle, the overestimation of damping ration become more intense for axial stress of 200kPa and 300kPa when the strain amplitude reaches 10%, this is because that high particle density lead to larger size of yield surface in tensile and compression direction, larger plastic hardening modulus and slower evolution of loading surface in practice.

![Images](image)
The difference of $\frac{D-1}{D}$ is showed more apparently in figure 5(a), it is noticed there is a similar relationship between strain amplitude and difference. The difference was derived by initial effective axial stress to exclude the influence of it, the result is showed in figure 5(b), the concrete formula of the relationship can be fitted as equation (19), the value of coefficient in equation (19a) is larger than others because of the unstable of sample under low axial stress, the variation of coefficient in equation (19b),(19c),(19d),(19e) can be related to the initial particle structure which is always symbolized by void ratio $e_0$, considering the pre-consolidation process is isotropic which is expressed in normal consolidation line in $e-p$ space, the relationship between value of three parameters in equation(19) and initial void $e_0$ can be fitted respectively as equation (20), these relationships are related to the over-consolidation ratio and species of soil which will be investigated in the future.

\[
F = 216.74r^2 - 55.73r + 2.78 \quad (19a) \\
F = 15.94r^2 - 8.26r + 0.82 \quad (19b) \\
F = 10.39r^2 - 7.56r + 0.88 \quad (19c) \\
F = 0.82r^2 - 0.83r + 0.17 \quad (19d) \\
F = 19.48r^2 - 14.48r + 1.78 \quad (19e)
\]

\[
c_1 = 0.039e_0^2 - 6.58e_0 + 308.73 \quad (20a) \\
c_2 = -0.008e_0^2 + 1.44e_0 - 75.27 \quad (20b) \\
c_3 = 0.0002e_0^2 - 0.047e_0 + 3.34 \quad (20c)
\]

Where $F$ is the difference of damping ratio between experiment and constitutive model, $r$ means the strain amplitude in every circle of unloading and reloading, $c_1, c_2, c_3$ are three parameters in equation (19), $e_0$ is the initial void ratio of every hysteretic loop.

So the difference can be expressed in equation (21) as follows:

\[
F = (0.039e_0^2 - 6.58e_0 + 308.73)r^2 + (-0.008e_0^2 + 1.44e_0 - 75.27)r + (0.0002e_0^2 - 0.047e_0 + 3.34) \quad (21)
\]

In every circle of unloading-reloading, the strain amplitude is decided by the previous loading which can be regarded as initial loading for soil with new state after precedent cyclic loading, this means the initial $e_0$ should be updated at the end of third quarter of every loop. According to the constitutive in equation (8), every strain amplitude can be calculated as follows:

\[
\varepsilon_{ij}^{ap} = \int d\varepsilon_{ij} = \int_{\sigma_{ij}}^{\sigma_{ij}^{ap}} \left( C_{ijkl}^{-1} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \right) : d\sigma_{ij} \quad (22)
\]

Where $\varepsilon_{ij}^{ap}$ is the strain amplitude corresponding to the next hysteretic loop, $\sigma_{ij}^o$ and $\sigma_{ij}^{ap}$ are the stress state at the beginning and end of third quarter of current loop respectively.

To substitute the equation (22) into equation (21), the norm of tensor $\varepsilon_{ij}^{ap}$ was calculated as follows
Then the correction coefficient can be obtained through equation (24)

\[ m = 1 + p' \ast F \]

Where \( p' \) is the initial mean effective principal stress, \( m \) is the correction coefficient.

By introducing the correction coefficient into equation (18) the modification to plastic hardening rule can be induced as follows:

The plastic work in equation (18) is modified:

\[ \frac{\partial f}{\partial \sigma_{ij}} \cdot d\sigma_{ij} \]

Because of equation (7), the correction coefficient \( m \) can be introduced into isotropic and kinematic hardening rule, \( \frac{da_{ij}}{de_{ij}}/m \), \( \frac{de_{ij}}{de_{ij}}/m \), respectively. After integration, the new loading surface was obtained as follows:

\[ f_{lm} = \sqrt{\frac{2}{3}(s_{ij} - (1 + p' \ast F) \ast a_{ij}) \ast (s_{ij} - (1 + p' \ast F) \ast a_{ij}) - (1 + p' \ast F) \ast k_l} = 0 \]  \[ (25) \]

It is emphasized again that correction coefficient \( m \) is updated, this means the initial void ratio \( e_0 \) and the mean effective principal stress are updated for every new hysteretic loop, their value can be calculated at the end of third quarter of previous loop by using the Rowe’s stress-dilatancy theory[11] which is expressed in equation (26)

\[ \frac{de_{p}}{de_{q}} = M_{pt} - \frac{q}{p} \]  \[ (26) \]

Where \( de_{p} \) and \( de_{q} \) are increments of plastic volumetric strain and plastic general shear strain., \( M_{pt} \) is the transformation line in \( p - q \) space.

5. Performance

To verify the modified multi-surface plastic model, cyclic undrained torsional shear tests was conducted on Toyoura[12] sand, the initial mean effective stress \( p' \) is 100 kPa, the frequencies of test sample is 0.5, the initial relative density \( D_r \) is 70%, simultaneously, the original multi-surface plastic model and the modified model were used to prove the modification effect.

![Figure 5. Difference of Damping ratio](image-url)
As the comparison showed in figure 6, for the original model in figure (b), the speed of degradation of mean effective stress is higher than that of experiment figure 6(a), and it reached the unstable state where the shear dilatancy take place earlier than result of experiment, all these are due to overestimation of plastic strain presented in curve of stress-strain. Apparently, this difference was alleviated in the modification model in figure 6(c) comparing with the original model.

6. summary
This paper proposes a modified cyclic multi-surface plastic model based on the damping ratio during cyclic loading. Comparing with the traditional multi-surface plastic model, the plastic hardening rule is adjusted for every new hysteretic loop which is considered as the first unloading-reloading loop with upgraded initial loading and initial state, this adjustment prevents the prediction of accumulated plastic strain from overestimating. The comparison result proved the utility of the modification.

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References
[1] Dafalias, Y. F. (1979) A Model for Soil Behavior under Monotonic and Cyclic Loading Conditions. Trans.int.conf.on Structural Mechanics in Reactor Technology.
[2] Dafalias, and F. Yannis. (1986) Bounding Surface Plasticity. I: Mathematical Foundation and
[3] Heidarzadeh, Heisam, and M. Oliaei. (2017) Development of a generalized model using a new plastic modulus based on bounding surface plasticity. Acta Geotechnica.

[4] Shahrooi, Shahram, I. H. Metselaar, and Z. Huda. (2010) Evaluation of cyclic plasticity models of multi-surface and non-linear hardening by an energy-based fatigue criterion. Journal of Mechanical Science and Technology 24.6:1255-1260.

[5] Fang, JG, Wu, CQ, Li, J (Li, Jun), Liu, Q, Wu, C, Sun, GY, Li, Q. (2019) Phase field fracture in elasto-plastic solids: Variational formulation for multi-surface plasticity and effects of plastic yield surfaces and hardening. International Journal of Mechanical Science 3.12:382-396.

[6] Chen, RP, Zhu, S, Hong, PY, Cheng, W, Cui, YJ. (2019) A two-surface plasticity model for cyclic behavior of saturated clay. Acta Geotechnica 14.2: 279-293.

[7] Nieto Leal, Andrés, V. N. Kaliakin, and M. Mashayekhi. (2017) Improved rotational hardening rule for cohesive soils and definition of inherent anisotropy. International Journal for Numerical and Analytical Methods in Geomechanics.

[8] Zhou, C, Leroueil, S, Fafard, M, Yin, JH. (2018) A kinematic hardening and elastic visco-plastic model of saturated cohesive anisotropic soils. KSCE Journal of Civil Engineering 22.2: 532-543.

[9] Li, Tao, and H. Meissner. (2002) Two-Surface Plasticity Model for Cyclic Undrained Behavior of Clays. Journal of Geotechnical and Geoenvironmental Engineering 128.7:613-626.

[10] Sukkarak, et al. (2016) A modified elasto-plastic model with double yield surfaces and considering particle breakage for the settlement analysis of high rockfill dams. KSCE Journal of Civil Engineering 21.3:1-12.

[11] Rowe, P. W.. (1962) The Stress-Dilatancy Relation for Static Equilibrium of an Assembly of Particles in Contact. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 269.1339:500-527.

[12] Zhang, Jian Min, Y. Shamoto, and K. Tokimatsu. (1997) Moving Critical and Phase-Transformation Stress State Lines of Saturated Sand during Undrained Cyclic Shear. SOILS AND FOUNDATIONS 37.2:51-59.