A normality criterion corresponding to the defect relations

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Abstract

Let \( F \) be a family of meromorphic functions on a domain \( D \). We present a quite general sufficient condition for \( F \) to be a normal family. This criterion contains many known results as special cases. The overall idea is that certain comparatively weak conditions on \( F \) locally lead to somewhat stronger conditions, which in turn lead to even stronger conditions on the limit function \( g \) in the famous Zalcman Lemma. Ultimately, the defect relations for \( g \) force normality of \( F \).

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1. Introduction and main result

A family \( F \) of meromorphic functions on a domain \( D \subseteq \mathbb{C} \) is normal if every sequence \( \{ f_n \} \subseteq F \) contains a subsequence which converges spherically uniformly on compact subsets of \( D \).

We refer to [10] for all necessary background information.

The starting point for this paper was the following recent result.

Theorem 1.1. [16, Theorem 1.6] Let \( F \) be a family of meromorphic functions defined in a domain \( D \), \( M \) be a positive number and \( S = \{ \alpha, \beta \} \), where \( \alpha, \beta \) are distinct elements of \( \mathbb{C} \cup \{ \infty \} \). Further suppose that

(i) each pair of functions \( f, g \in F \) share the set \( S \) in \( D \);

(ii) there exists a \( \gamma \in \mathbb{C} \setminus \{ \alpha, \beta \} \) such that for each \( f \in F \), \( |f'(z)| \leq M \) whenever \( f(z) = \gamma \) in \( D \);

(iii) each \( f \in F \) has no simple \( \beta \)-points in \( D \).

Then \( F \) is normal in \( D \).
Condition (iii) is a bit of a nuisance because it destroys the symmetry between \( \alpha \) and \( \beta \). And indeed, condition (iii) can be completely omitted. This can be seen as follows.

To prove normality at a point \( z_0 \) we can restrict \( F \) to a small neighbourhood of \( z_0 \). Given a sequence of functions \( f_n \) from \( F \), we can make this neighbourhood so small that the sequence \( f_n \) contains a subsequence \( f_{n_k} \) such that in this small neighbourhood all \( f_{n_k} \) share \( \alpha \) and omit \( \beta \). Or they all share \( \beta \) and omit \( \alpha \), in which case we simply switch the names. See Step 1 in the proof of Theorem 1.2 given in Section 3 for more details and a slightly more general version. So in final instance, Theorem 1.1 itself, applied locally, proves the version without condition (iii).

Besides Theorem 1.1 the improved version also improves [13, Theorem 1]. Actually, both of them, and several other published results, can also be obtained as special cases of our quite general main result, which we will formulate as Theorem 1.2.

In addition to the local argument that we just outlined, the proof of Theorem 1.2 also makes heavy use of the impact that certain conditions satisfied by all \( f \in F \) will have on the limit function \( g \) in Zalcman’s Lemma. These steps follow the ideas from the proofs of [3, Theorem 2] and [8, Theorem 1.3]. Then in the end everything will boil down to the defect relations applied to \( g \).

**Theorem 1.2. (Main Theorem)** Let \( F \) be a family of meromorphic functions on a domain \( D \). Let there be pairwise disjoint, finite sets \( A \subset \mathbb{C} \cup \{ \infty \} \), \( B \subset \mathbb{C}^* \) and \( C \subset \mathbb{C} \cup \{ \infty \} \) of cardinalities \( n, s \) and \( r \), and with the following properties:

(a) All \( f \in F \) share the set \( A \), i.e., for every \( z_1 \in D \) we have \( f(z_1) \in A \) for all \( f \in F \) or \( f(z_1) \notin A \) for all \( f \in F \).

(b) For every \( b_i \in B \), every \( f \in F \) and every \( z_1 \in D \) we have \( f(z_1) = b_i \Rightarrow f'(z_1) = b_i \).

(c) If \( \infty \in C \), say \( c_1 = \infty \), then on \( D \) every pole of every \( f \in F \) has at least multiplicity \( m_1 \geq 2 \). For every finite \( c_j \in C \) there exists an integer \( m_j \geq 2 \) and a number \( M_j \) such that for every \( f \in F \) and every \( z_1 \in D \) we have \( f(z_1) = c_j \Rightarrow |f^{(k)}(z_1)| \leq M_j \) for \( k = 1, \ldots, m_j - 1 \).

If

\[
n + s + \sum_{j=1}^{r} \left( 1 - \frac{1}{m_j} \right) > 2,
\]

then \( F \) is normal on \( D \).

The proof will be given in Section 3.

In practice the conditions will often come in a less technical form. If for example on \( D \) all \( f \in F \) omit the value \( \omega \in \mathbb{C}^* \), the we could take \( \omega \) into any one of the three
sets $A$, $B$ or $C$ (with $m = \infty$), but of course only into one because of the pairwise disjointness.

A value $\omega$ such that all $\omega$-points of all $f \in \mathcal{F}$ have multiplicity at least $m$ clearly belongs into the set $C$ with $m_i = m$. In fact, the condition for $C$ is weaker.

Note however that if for all $f \in \mathcal{F}$ we have $f \in \{d_1, \ldots, d_t\} \Rightarrow f' \in \{d_1, \ldots, d_t\}$ with $t > 1$, then the values $d_1, \ldots, d_t$ are only booked as belonging to the set $C$ (with $m = 2$), even if all $d_i$ are nonzero. See Section 4 for more on this.

Also note that from a condition $f = a \Leftrightarrow f' = a$ we only use one direction. For more on this, see Section 4 as well.

**Corollary 1.3.** [14, Theorem 2] Let $a_1, a_2, a_3$ be three distinct complex numbers. If for each $f \in \mathcal{F}$ we have $f = a_i \Rightarrow f' = a_i$ ($i = 1, 2, 3$) in $D$, then $\mathcal{F}$ is normal.

**Proof.** Obviously at least two of the $a_i$ are nonzero. So we have $|B| = 2$ and $|C| = 1$ with $m_1 = 2$. \hfill \Box

Other corollaries, which we don’t want to all spell out in detail, include our improved version of Theorem 1.1, [3, Theorem 2], [8, Theorem 1.3], [10, Theorem 4.1.4, p.105], [11, Theorem 1.3], [13, Theorems 1 and 2], and [17, Theorem 5].

Most of them are generalizations of Montel’s Fundamental Normality Criterion (see for example [10, p.74]), which says that a family whose functions all omit the same three values must be normal. They cover many of the possibilities with $|A| + |B| + |C| = 3$. In Section 2 we discuss the cases that do not imply normality. We present one more constellation that we couldn’t find in the literature.

**Corollary 1.4.** Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D$ with the following properties:

(a) For any $f \in \mathcal{F}$ and any $z_1 \in D$ we have $f'(z_1) = 1$ whenever $f(z_1) = 1$.

(b) All poles of all $f \in \mathcal{F}$ have multiplicity at least 3.

(c) The set $\bigcup_{f \in \mathcal{F}} f'(f^{-1}\{0\})$ is bounded.

Then $\mathcal{F}$ is normal on $D$.

**Proof.** In Theorem 1.2 this gives $B = \{1\}$ and $C = \{\infty, 0\}$ with $m_1 = 3$ and $m_2 = 2$. \hfill \Box
2. (Counter-)Examples

In this section we will construct several examples that show that the conditions in Theorem 1.2 are sharp. What we mean by that is that for numbers \( n, s, r \) and \( m_1, \ldots, m_r \) such that \( n + s + \sum_{k=1}^{r}(1 - \frac{1}{m_k}) \leq 2 \) one can construct a non-normal family satisfying the other conditions of Theorem 1.2 for these numbers.

We need some preparation. The spherical derivative of a meromorphic function \( f \) is
\[
f^\#(z) := \frac{|f'(z)|}{1 + |f(z)|^2}.
\]

**Theorem 2.1.** [2, Lemma 1] Let \( f \) be a meromorphic function on \( \mathbb{C} \). If \( f \) has bounded spherical derivative on \( \mathbb{C} \), \( f \) is of order at most 2. If, in addition, \( f \) is entire, then the order of \( f \) is at most 1.

**Corollary 2.2.** Let \( f \) be a meromorphic function on \( \mathbb{C} \). If the order of \( f \) is bigger than 2, the family
\[
\mathcal{F} = \{ f(z + \omega) : \omega \in \mathbb{C} \}
\]
is not normal at any point \( z_0 \in \mathbb{C} \).

**Proof.** By Marty’s Criterion [10, p.75], a family \( \mathcal{F} \) of meromorphic functions on a domain \( D \) is normal if and only if for every compact subset \( K \subseteq D \) there exists a constant \( C(K) \) such that \( f^\#(z) \leq C(K) \) for all \( z \in K \) and all \( f \in \mathcal{F} \). With this, the corollary is an immediate consequence of Theorem 2.1. \( \square \)

One of the many uses of normal families is the following. Given a function \( f \) that is meromorphic in the complex plane, and certain sharing conditions for \( f \) and \( f' \), one wants to show \( f = f' \) or that \( f \) has a certain form. In some cases a good strategy is not to start immediately with arguments from Nevanlinna Theory, but to first look at the family \( \{ f(z + \omega) : \omega \in \mathbb{C} \} \). If this family turns out to be normal, this implies that the order of the function \( f \) is at most 2, which might be quite helpful for the successive arguments (for example from Nevanlinna Theory). This is for example the approach in [8].

Here we will go exactly in the opposite direction. We find a function \( h \) of order bigger than 2 that has some desired properties like omitting a value, or sharing a value with its derivative, or having a totally ramified value. Because of the order of \( h \) the family \( \{ h(z + \omega) : \omega \in \mathbb{C} \} \) cannot be normal. But it inherits the desired conditions. This proves that these conditions are not sufficient for normality.

Constructing suitable functions of order bigger than 2 is actually quite easy, using the following result.

**Theorem 2.3.** [5, Corollary 1.2] Let \( F \) be a meromorphic function which is not of
order zero and let $g$ be an entire function which is not a polynomial. Then $F(g(z))$ is of infinite order.

**Example 2.4.** Consider the function $h(z) = \sin(e^z)$.

It has infinite order by Theorem 2.3 as $\sin(z)$ has order one. So by Corollary 2.2 the family

$$
F = \{ h(z + \omega) : \omega \in \mathbb{C} \}
$$

cannot be normal. Obviously, every $f \in F$ omits the value $\infty$. Moreover, every 1-point and every $-1$-point of every $f \in F$ has multiplicity 2. So in the sense of Theorem 1.2 we have $A = \{\infty\}$, $B = \emptyset$ and $C = \{1,-1\}$ with $m_1 = m_2 = 2$.

Applying a linear transformation to get all three interesting values into $\mathbb{C}^*$, e.g. considering

$$
h(z) = 2 - \frac{1}{\sin(e^z)},
$$

this can also be interpreted as an example for $A = \emptyset$, $B = \{2\}$, $C = \{1,3\}$ with $m_1 = m_2 = 2$, or alternatively as an example for $A = B = \emptyset$, $|C| = 3$ with $m_1 = m_3 = 2$ and $m_2 \geq 2$.

We can also apply a linear transformation to place the totally ramified values at $\infty$ and 0. For example,

$$
h(z) = \frac{\sin(e^z) - 1}{\sin(e^z) + 1}
$$
is of infinite order, has only multiple zeroes and multiple poles, and omits the value 1. So the non-normal family $\{h(z + \omega) : \omega \in \mathbb{C} \}$ answers the question asked in [12, Remark 5]. Even if the function $\psi$ in [12, Theorem 2] is constant 1, the condition (2) in that theorem is best possible in the sense that it cannot be weakened to “all poles of $f$ are multiple”.

**Example 2.5.** This example is included to show that in Theorem 1.2 the condition $f = 0 \Rightarrow f' = 0$ is really weaker than $f = b_1 \Rightarrow f' = b_1$ for a nonzero complex number $b_1$. Let $F$ be a family of analytic functions. If for all $f \in F$ we have $f = 1 \Rightarrow f' = 1$, and all $c$-points of $f$ are multiple, then Theorem 1.2 implies that $F$ is normal. On the other hand, the analytic family $F = \{ h(z + \omega) : \omega \in \mathbb{C} \}$ with

$$
h(z) = \frac{c}{2}(\sin(e^z) + 1),
$$

where $c$ is a fixed nonzero complex number, is not normal. But for every $f \in F$ we have $f = 0 \Rightarrow f' = 0$, and every $c$-point of $f$ is multiple.

**Example 2.6.** The function $h(z) = \exp(z^3)$ has order 3. So by Corollary 2.2
the family \( \mathcal{F} = \{ h(z + \omega) : \omega \in \mathbb{C} \} \) is not normal. In the terminology of Theorem 1.2 this is an example with \( A = \{ \infty, 0 \} \), \( B = C = \emptyset \) or with \( A = \{ \infty \} \), \( B = \emptyset \), \( C = \{ 0 \} \) and \( m_1 \) any integer we want.

Moving the two omitted values to \( a \) and \( b \) by considering

\[
h(z) = \frac{ae^{z^3} + b}{e^{z^3} + 1}
\]

also furnishes examples with \( A = B = \emptyset \), \( C = \{ a, b \} \), and \( m_1 \) and \( m_2 \) as desired, and so on.

Alternatively, we could do everything in this example using \( \exp(\exp(z)) \) instead of \( \exp(z^3) \).

Example 2.7. Fix two distinct nonzero complex numbers \( a, b \) and let \( \wp(z) \) be the Weierstrass function with differential equation

\[
(\wp'(z))^2 = 4\wp(z)(\wp(z) - a)(\wp(z) - b).
\]

Since \( \wp \) has order one,

\[
h(z) = \wp(e^z)
\]

has infinite order by Theorem 2.3. So by Corollary 2.2 the meromorphic family \( \mathcal{F} = \{ h(z + \omega) : \omega \in \mathbb{C} \} \) cannot be normal. If \( f \in \mathcal{F} \), then all poles, zeroes, \( a \)-points and \( b \)-points of \( f \) have multiplicity 2. This is an example with \( A = B = \emptyset \), \( |C| = 4 \) and \( m_1 = \cdots = m_4 = 2 \). A similar but more complicated example is given in [7].

If \( b = -a \), we can also take

\[
h(z) = (\wp(e^z))^2.
\]

Then for every \( f \in \mathcal{F} = \{ h(z + \omega) : \omega \in \mathbb{C} \} \) all poles and zeroes have multiplicity 4, and all \( a^2 \)-points have multiplicity 2. This is an example with \( A = B = \emptyset \), \( |C| = 3 \) and \( m_1 = m_2 = 4 \), \( m_3 = 2 \).

Example 2.8. Consider the Weierstrass equation

\[
(\wp'(z))^2 = 4(\wp(z))^3 + c
\]

with \( c \neq 0 \). Then \( \wp'(z) \) is an elliptic function. All its poles are triple. Differentiating the differential equation we see that its \( \sqrt[3]{c} \)-points and its \(-\sqrt[3]{c} \)-points also all have multiplicity 3. Taking \( h(z) = \wp'(e^z) \) we get a non-normal family with \( A = B = \emptyset \), \( |C| = 3 \) and \( m_1 = m_2 = m_3 = 3 \).

Moreover, \( (\wp'(z))^2 \) has poles of order 6, zeroes of order 2 and \( c \)-points of multiplicity 3. So

\[
h(z) = (\wp'(e^z))^2
\]

gives a non-normal family with \( A = B = \emptyset \), \( |C| = 3 \), \( m_1 = 6 \), \( m_2 = 2 \), \( m_3 = 3 \).
3. Proof of the Main Theorem

Like practically every paper on normal families from the last decade, our paper too makes heavy use of the very powerful Zalcman Lemma.

**Theorem 3.1. (Zalcman’s Lemma) [15]** Let $\mathcal{F}$ be a family of meromorphic (analytic) functions on the unit disk. If $\mathcal{F}$ is not normal at 0, then there exist

(i) a number $0 < r < 1$,

(ii) points $z_n, |z_n| < r$,

(iii) functions $f_n \in \mathcal{F}$,

(iv) positive numbers $\rho_n \to 0$,

such that

$$f_n(z_n + \rho_n \xi) =: g_n(\xi) \to g(\xi)$$

spherically uniformly (uniformly) on compact subsets of $\mathbb{C}$, where $g$ is a nonconstant meromorphic (entire) function.

The second tool that we crucially need is the following.

**Theorem 3.2. (defect relations, slightly modified)** Let $f$ be a non-constant meromorphic function on the complex plane. For each totally ramified value $c_j$ of $f$ let $m_j \geq 2$ be an integer such that each $c_j$-point of $f$ has multiplicity at least $m_j$.

(a) If $f$ is a transcendental meromorphic function, let $U \subset \mathbb{C} \cup \{\infty\}$ be the set of all values that $f$ takes at only finitely many points.

(b) If $f$ is a non-constant rational function, let $U \subset \mathbb{C} \cup \{\infty\}$ be the set of all values that $f$ (as a function on the complex plane) omits.

(c) Alternatively, if $f$ is a non-constant rational function and there is a value $a_1 \in \mathbb{C} \cup \{\infty\}$ such that there is exactly one $z_1 \in \mathbb{C}$ with $f(z_1) = a_1$, we can also take for $U$ the union of $\{a_1\}$ and all values that $f$ (as a function on the complex plane) omits.

In any case, let $R = \{c_1, \ldots, c_r\}$ be the set of totally ramified values of $f$ that are not in $U$. Then

$$|U| + \sum_{j=1}^{r} (1 - \frac{1}{m_j}) \leq 2.$$

**Proof.** With $U$ as in part (a) or (b) this is a special case of the defect relations [4, §5.2].
For the proof when $U$ is as in (c), we first recall that if $\phi: Y \to X$ is a covering of degree $d$ of compact Riemann surfaces, then by the Hurwitz formula we have

$$2g(Y) - 2 = d(2g(X) - 2) + \sum_{\omega \in X} (d - \#(\phi^{-1}(\omega)))$$

where $g$ denotes the genus. Since a rational function $f$ of degree $d \geq 1$ is a covering from the Riemann sphere to the Riemann sphere, its total ramification is

$$\sum_{\omega \in C \cup \{\infty\}} (d - \#(f^{-1}(\omega))) = 2d - 2.$$

Also, a rational function $f$, considered as a function on the complex plane, can omit at most one value, namely $f(\infty)$. In that case $f(\infty)$ must be taken with multiplicity $d$. Moreover, $a_1$ must then also be taken with multiplicity $d$. So together they use up all ramification. Hence we have $|U| = 2$ and $R$ is empty.

If $f$ omits no values, then $f^{-1}(a_1)$ can consist of one or two points, the second one being $\infty$. In either case the value $a_1$ uses up at least $d-2$ from the total ramification. What is left suffices for at most two totally ramified values (with $m_1 = m_2 = 2$) or at most one totally ramified value (possibly with bigger $m$).

The function $f(z) = \frac{(z-1)^2}{z^2+1}$ takes each of the values 0, 1 and 2 exactly once (on $\mathbb{C}$). This shows that for rational functions we cannot take $U$ to be the set of all values that are taken at most once.

**Proof of Theorem 1.2.** As normality is a local property, we can prove it at each point individually. So fix $z_0 \in D$.

**Step 1:** We pick a function $f_0$ from $\mathcal{F}$.

Let us first examine the case $f_0(z_0) \in A$. If $f_0$ is constant, then, because of the sharing of $A$, all $f \in \mathcal{F}$ are constants from $A$, and thus $\mathcal{F}$ is normal. If $f_0$ is not constant, there is a small disk around $z_0$ such that on the punctured disk $f_0$ omits all values from $A$. By condition (i) the same must hold for all $f \in \mathcal{F}$. We can replace $D$ by this small disk and $\mathcal{F}$ by its restrictions to this small disk.

Now let $f_n$ be a sequence of functions from (our new) $\mathcal{F}$. Our task is to show that it contains a subsequence that converges spherically uniformly. Since $f_n(z_0) \in A$ and $A$ is finite, there exists (at least) one element of $A$ (after renaming we can assume that it is $a_1$) with $f_n(z_0) = a_1$ for infinitely many $n$. We replace the sequence $f_n$ with the subsequence for these $n$. Our task still is to show that this subsequence contains a subsubsequence that converges spherically uniformly. For that we can now assume that all $f \in \mathcal{F}$ share the value $a_1$ and omit the values $a_2, \ldots, a_n$. The conditions related to the sets $B$ and $C$ remain intact.

If $f_0(z_0) \notin A$, we can make the disk around $z_0$ so small that $f_0$, and hence every $f \in \mathcal{F}$ completely omits all values from $A$. So in that case we can assume without
loss of generality that all \( f \in \mathcal{F} \) omit the set \( A \) and the conditions (ii) and (iii) still hold.

**Step 2:** Now we assume that the family \( \mathcal{F} \) with the stronger conditions from Step 1 is not normal at \( z_0 \). In a few more steps we bring this to a contradiction.

To that end we invoke Zalcman’s Lemma. Without loss of generality we can assume that \( z_0 = 0 \) and that \( D \) is the unit disk.

We begin by showing that the limit function \( g(\xi) \) from Zalcman’s Lemma also omits the values \( a_2, \ldots, a_n \in A \) and takes the value \( a_1 \) at most once.

For the omitted values this follows easily from Hurwitz’s Theorem [10, Corollary 3.8.2, p.98] because \( g \) is not constant.

If \( f(0) = a_1 \) for all \( f \in \mathcal{F} \), then obviously \( g_n(\xi) \) takes the value \( a_i \) only at \( \xi = \frac{1}{\rho_n}z_n \). So, since \( g \) is not constant, by Hurwitz’s Theorem it can take the value \( a_1 \) at a point \( \xi_1 \) only if every neighbourhood of \( \xi_1 \) contains \( \frac{1}{\rho_n}z_n \) for all big enough \( n \). Thus \( \lim_{n \to \infty} \frac{1}{\rho_n}z_n \), if it exists, is the only possible candidate for \( \xi_1 \).

**Step 3:** Next we show that \( g \) takes each value \( c_j \in \mathbb{C} \) with multiplicity at least \( m_j \). For fixed \( n \) we differentiate \( g_n(\xi) \) with respect to \( \xi \) and get

\[
g_n^{(k)}(\xi) = \rho_n^k f_n^{(k)}(z_n + \rho_n \xi).
\]

Now suppose that \( g(\xi_0) = c_j \in \mathbb{C} \). Then there is a small neighbourhood of \( \xi_0 \) on which \( g \) is holomorphic. Because of the locally uniform convergence, for big enough \( n \) the functions \( g_n \) must be also be holomorphic in this neighbourhood. Since \( g \) is nonconstant, by Hurwitz’s theorem there exist \( \xi_n, \xi_n \to \xi_0 \), such that for sufficiently large \( n \) we have \( c_j = g(\xi_0) = g_n(\xi_n) \). Hence our assumptions imply \( |g_n^{(k)}(\xi_n)| \leq \rho_n^k M_j \) for \( k = 1, 2, \ldots, m_j - 1 \) and \( n \) sufficiently large. Since \( g_n^{(k)}(\xi) \) converges locally uniformly to \( g^{(k)}(\xi) \), we obtain

\[
g^{(k)}(\xi_0) = \lim_{n \to \infty} g_n^{(k)}(\xi_n) = 0
\]

for \( k = 1, 2, \ldots, m_j - 1 \).

If \( c_j = \infty \), we use that \( \frac{1}{g_n(\xi)} \to \frac{1}{g(\xi)} \) locally uniformly (compare [10, Theorem 3.1.3, p.72]) and that all zeroes of \( \frac{1}{g_n(\xi)} \) have multiplicity at least \( m_j \).

**Step 4:** Now we show that \( g \) also omits all values from the set \( B \). Let \( b \in \mathbb{B} \). To start with, by exactly the same argument as in Step 3 we see that \( g \) has no simple \( b \)-points.

So suppose \( g(\xi_0) = b \in \mathbb{B} \). Let \( m(\geq 2) \) be the multiplicity of this \( b \)-point. Then \( g^{(m)}(\xi_0) \neq 0 \). We can find a small disk around \( \xi_0 \) such that none of \( g(\xi), g'(\xi), \ldots, g^{(m)}(\xi) \) vanishes at any point of the punctured disk. Now we fix an even smaller disk \( \tilde{D} \) around \( \xi_0 \). Since \( g \) is not constant, by Rouché’s Theorem, for big enough \( n \) the function \( g_n \) must have \( m \) \( b \)-points \( \xi_{n,1}, \ldots, \xi_{n,m} \) (counted with multiplicities) in \( \tilde{D} \). Actually,

\[
g_n^{(m)}(\xi_{n,j}) = \rho_n f_n^{(m)}(z_n + \rho_n \xi_{n,j}) = \rho_n b \neq 0;
\]
so these $b$-points are all simple. Moreover, $\lim_{n \to \infty} g'_n(\xi_{n,j}) = \lim_{n \to \infty} \rho_n b = 0$. Since $g'_n(\xi) - \rho_n b$ has $m$ zeroes on $\tilde{D}$, again by Rouché’s Theorem $g'$ must have $m$ zeroes on $\tilde{D}$ (counted with multiplicity). But the only zero of $g'$ on $\tilde{D}$ is $\xi_0$ with multiplicity $m - 1$. This contradiction disproves the assumption that $g$ has a $b$-point.

Step 5: In Steps 2 to 4 we have shown that in the sense of Theorem 3.2 for the non-constant function $g$ we have $U = A \cup B$ and $R = C$ with the same $m_j$. This yields the final contradiction (between the condition in Theorem 1.2 and the conclusion of Theorem 3.2).

□

4. Limitations of the criterion

In [9] Pang and Zalcman proved, among other results, the following strong statement.

Theorem 4.1. [9, Theorem 2] Let $\mathcal{F}$ be a family of meromorphic functions on the unit disk $D$, and let $a$ and $b$ be distinct complex numbers. If $f$ and $f'$ share $a$ and $b$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal on $D$.

This is a result that our approach cannot achieve. Let’s even assume that $a$ and $b$ are both nonzero. Then the impact of the sharing $f = a \Leftrightarrow f' = a$ on the limit function $g$ in our proof is that $g$ omits $a$, the same as if we only had $f = a \Rightarrow f' = a$. And similarly for $b$. But the conditions $f = a \Rightarrow f' = a$ and $f = b \Rightarrow f' = b$ only imply normality if $\mathcal{F}$ is an analytic family. That’s the content of [8, Theorem 1.3]. For a counter-example with meromorphic $f$ see Example 2.6.

We also point out that if every $f \in \mathcal{F}$ shares the two-element-set $S = \{a, b\}$ with its derivative $f'$, this does not suffice to imply normality, not even when the family is analytic. The following counter-example shows up in several papers, e.g. [3, Example 1] and [1, Example 6].

Example 4.2. Consider the analytic family $\mathcal{F} = \{f_n(z) : n = 2, 3, 4, \ldots\}$ on the unit disk where

$$f_n(z) = \frac{n + 1}{2n} e^{nz} + \frac{n - 1}{2n} e^{-nz}.$$ 

One easily checks that $n^2(f_n^2 - 1) = (f'_n)^2 - 1$. So $f_n$ and $f'_n$ share the set $S = \{1, -1\}$. But $\mathcal{F}$ is not normal at 0.

See also [1, Examples 7 and 8] for a non-normal meromorphic family in which $f$ and $f'$ share the set $\{0, b\}$ resp. the set $\{-1, 3\}$. Actually, [1] contains a much deeper study of families whose functions share a two-element-set with their derivative.

Example 4.3. Now we consider the family $\mathcal{H}$ of all $h(z) = \frac{1}{f(z)}$ with $f(z)$ from
the family \( \mathcal{F} \) in Example 4.2. Then \( \mathcal{H} \) is also not normal on the unit disk. Since 
\[ h'(z) = \frac{f'(z)}{(f(z))^2}, \]
we see that the family \( \mathcal{H} \) still satisfies the partial sharing condition 
\[ h(z) \in \{1, -1\} \Rightarrow h'(z) \in \{1, -1\}. \]

Also, all zeroes are multiple, for the trivial reason that there are no zeroes.

This example shows that Theorem 3 in [3] is incorrect.

The proofs of Theorem 1 and Theorem 3 in [3] are similar, and the mistake seems to be that on lines 6 and 7 of page 1476 the value \( a_l \) in \( g_l'(\xi_{l(j)}) = \rho_n a_l \) depends on \( j \), and therefore on line 13 one cannot conclude that \( g^n_l(\xi) - \rho_n a_l \) has \( k \) zeroes.

So the status of Theorem 1 in [3], for which we neither know an alternative proof nor a counter-example, is unclear at the moment. That theorem claims that if \( S = \{a_1, a_2, a_3\} \) with three nonzero complex numbers \( a_i \), and if \( f \in S \Rightarrow f' \in S \) for all \( f \in \mathcal{F} \), then \( \mathcal{F} \) is normal. So apart from the condition \( a_i \neq 0 \) it would be a common generalization of Corollary 1.3 and the following result.

**Theorem 4.4.** [6, Theorem 1] Let \( \mathcal{F} \) be a family of meromorphic functions on the unit disk \( D \), and let \( a_1, a_2, a_3 \) be three distinct complex numbers. If for every \( f \in \mathcal{F}, f \) and \( f' \) share the set \( S = \{a_1, a_2, a_3\} \), then \( \mathcal{F} \) is normal on \( D \).

The reason why Theorem 1.2 cannot yield this result is that from the information \( f \in S \Leftrightarrow f' \in S \) it only sees \( f \in S \Rightarrow f' \in S \); and for that to force normality it would need that \( S \) is a bounded set with at least 5 elements.

The problematic point in the proof of Theorems 1 and 3 in [3] is exactly the attempt to show that for a finite set \( S \) of nonzero \( a_i \) the condition \( f \in S \Rightarrow f' \in S \) also forces the limit function \( g \) to omit \( a_i \).

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