On some differential transformations of hypergeometric equations

M N Hounkonnou¹ and A Ronveaux¹,²

¹International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair), University of Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin
²Département de Mathématiques, Catholic University of Louvain, 2, Chemin du Cyclotron, B-1348, Louvain-la-Neuve, Belgium
E-mail: norbert.hounkonnou@cipma.uac.bj, andre.michel.ronveaux@gmail.com

Abstract. Many algebraic transformations of the hypergeometric equation $\sigma(x)z''(x) + \tau(x)z'(x) + lz(x) = 0$, where $\sigma, \tau, l$ are polynomial functions of degree 2 (at most), 1, 0, respectively, are well known. Some of them involve $x = x(t)$, a polynomial of degree $r$, in order to recover the Heun equation, extension of the hypergeometric equation by one more singularity. The case $r = 2$ was investigated by K. Kuiken (see 1979 SIAM J. Math. Anal. 10 (3) 655-657) and extended to $r = 3, 4, 5$ by R. S. Maier (see 2005 J. Differ. Equat. 213 171 - 203). The transformations engendered by the function $y(x) = A(x)z(x)$, also very popular in mathematics and physics, are used to get from the hypergeometric equation, for instance, the Schroedinger equation with appropriate potentials, as well as Heun and confluent Heun equations. This work addresses a generalization of Kimura’s approach proposed in 1971, based on differential transformations of the hypergeometric equations involving $y(x) = A(x)z(x) + B(x)z'(x)$. Appropriate choices of $A(x)$ and $B(x)$ permit to retrieve the Heun equations as well as equations for some exceptional polynomials. New relations are obtained for Laguerre and Hermite polynomials.

1. Introduction

In 1971, Kimura [1] investigated in detail all Fuchsian differential equations $F\{y(x)\} = 0$ reducible to hypergeometric equations $H\{z(x)\} = 0$ by a linear transformation $y(x) = P_0(x)z(x) + P_1(x)z'(x)$, with $P_0$ and $P_1$ rational functions of $x$. To compute $P_0(x)$ and $P_1(x)$, he assumed the same set of singularities and the same monodromy group for $H\{y(x)\} = 0$ and $F\{z(x)\} = 0$, giving information on the structures and properties of functions $P_0$ and $P_1$. The equation $F\{y\} = y'' + A_1(x)y' + A_2(x)y = 0, (y = y(x))$, being Fuchsian and written as a generalized Heun equation, contains regular singular points with many parameters, (not all free), in order to eliminate logarithmic situations and irreducibility (equation not factorizable).

Many theorems and properties are provided to compute $P_0(x)$ and $P_1(x)$ in particular situations, but without using the coupled differential equations satisfied by $P_0$ and $P_1$.

The aim of this work is to reverse this approach, making it possible to eliminate the Fuchsian constraints. We proceed as follows: inject an arbitrary linear transformation $y = A(x)z + B(x)z'$ into the hypergeometric equation and build a second order differential equation for $y$, (not always Fuchsian), depending on the two differentiable functions $A(x)$ and $B(x)$, polynomial or not.

Exceptional orthogonal polynomials (EOPs) like exceptional Jacobi, exceptional Laguerre,
and so on, (see [2] and references therein), can be retrieved from the same linear transformation
given a long time ago in the above mentioned seminal paper by Kimura.

Recall that the notion of the exceptional \((X_l)\) orthogonal polynomials was introduced in 2008
by Gomez-Ullate, Kamran and Milson [3] in the framework of Sturm Liouville theory. These
orthogonal polynomials are exceptional in the sense that they start at degree \(l(l \geq 1)\) instead of
degree 0 constant term, thus avoiding restrictions of Bochner’s theorem and they satisfy second-
order differential equations. These authors constructed the lowest examples, the \(X_1\) Laguerre
and \(X_1\) Jacobi polynomials explicitly. A quantum mechanical reformulation with shape-invariant
potentials was given by Quesne and collaborators [4, 5] within the framework of one-dimensional
quantum mechanics. The merit of quantum mechanical reformulation resides in the fact that
the orthogonality and completeness of the obtained eigenfunctions are guaranteed. Besides, the
well established solution mechanism of shape invariance combined with the Crum’s method [6],
or the so-called factorization method [7], or the susy quantum mechanics [8] is available.

Then, two sets of infinitely many shape invariant potentials, the deformed radial oscillator
potentials and the deformed trigonometric/hyperbolic Darboux-Pöschl-Teller (DPT) potentials,
and the corresponding \(X_l\) Laguerre and Jacobi polynomials \((l = 1, 2, \ldots \infty)\) were investigated
by Sasaki et al [9, 10, 11] in 2009.

This paper is organised as follows. In section 2, we provide a quick overview on
hypergeometric equations and Heun’s equations. Section 3 deals with our generalization
of Kimura’s approach, based on a differential transformation of hypergeometric equations. Known
special functions and orthogonal polynomials are retrieved from the presented general formalism.
Concluding remarks are presented in section 4.

2. Hypergeometric differential equation and Heun’s equations

The hypergeometric differential equation

\[
\frac{d^2 z}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x - 1} \right) \frac{dz}{dx} + \frac{\alpha \beta}{x(x - 1)} z = 0
\]

(1)
is characterized as a Fuchsian differential equation of the second order which has three
singularities at \(x = 0, 1, \infty\) corresponding to the exponents \((0, 1 - \gamma); (0, 1 - \delta)\) and \((\alpha, \beta)\)
respectively, satisfying the Fuchs relation

\[
\alpha + \beta - \gamma - \delta + 1 = 0.
\]

(2)

This equation has been studied in detail and extended in several directions by many
mathematicians. As one of those extensions we are led in a natural way to a second
order equation of Fuchsian type whose singularities and exponents are given by the following
Riemannian scheme

\[
\begin{array}{cccccc}
0 & 1 & a_1 & \cdots & a_k & \infty \\
0 & 0 & 0 & \cdots & 0 & \alpha \\
1 - \gamma & 1 - \delta & 1 - \epsilon_1 & \cdots & 1 - \epsilon_k & \beta,
\end{array}
\]

where the exponents are connected by Fuchs’s relation

\[
\alpha + \beta - \gamma - \delta - \epsilon_1 - \cdots - \epsilon_k + 1 = 0.
\]

The equation defined by these data contains just \(k\) accessory parameters, denoted by
\(\rho_1, \cdots, \rho_k\), and can be written in the form:
\[
d\frac{d^2 y}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \sum_{i=1}^{k} \frac{\epsilon_i}{x-a_i} \right) \frac{dy}{dx} + \frac{\alpha \beta x^k + \rho_1 x^{k-1} + \cdots + \rho_k y}{x(x-1) \prod_{i=1}^{k} (x-a_i)} = 0. \tag{3}
\]

In the case when \( k = 1 \), the equation is often called Heun’s equation that we write in the following form [12, 13]:

\[
d\frac{d^2 \omega}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{d\omega}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-a)} \omega = 0, \tag{4}
\]

with the Fuchs relation:

\[ \alpha + \beta + 1 = \gamma + \delta + \epsilon. \tag{5} \]

This equation has four regular singularities at \( 0, 1, a, \infty \) with corresponding exponents \( \{0, 1 - \gamma\}, \{0, 1 - \delta\}, \{0, 1 - \epsilon\}, \{\alpha, \beta\} \), respectively. All other homogeneous linear differential equations of the second order having four regular singularities in the extended complex plane \( \mathbb{C} \cup \{\infty\} \), can be transformed into (4). The parameters play different roles:

- \( a \) is the singularity parameter
- \( \alpha, \beta, \gamma, \delta, \epsilon \) are exponent parameters
- \( q \) is the accessory parameter.

The total number of free parameters is six.

Making the substitution \( \omega(z) = z^{-\gamma/2}(z-1)^{-\delta/2}(z-a)^{-\epsilon/2} W(z) \), (4) can be transformed into the normal form:

\[
d\frac{d^2 W}{dz^2} = \left( \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-a} + \frac{D}{z^2} + \frac{E}{(z-1)^2} + \frac{F}{(z-a)^2} \right) W, \tag{6}
\]

\[ A + B + C = 0, \]

\[ A = -\frac{\gamma \delta}{2} - \frac{\gamma \epsilon}{2a} + \frac{q}{a}, \quad B = \frac{\gamma \delta}{2} - \frac{\delta \epsilon}{2(a-1)} - \frac{q - \alpha \beta}{a-1}, \]

\[ C = \frac{\gamma \epsilon}{2a} + \frac{\delta \epsilon}{2(a-1)} - \frac{a \alpha \beta - q}{a(a-1)}, \quad D = \frac{1}{2}\gamma \left( \frac{1}{2}\gamma - 1 \right), \]

\[ E = \frac{1}{2} \delta \left( \frac{1}{2} \delta - 1 \right), \quad F = \frac{1}{2} \epsilon \left( \frac{1}{2} \epsilon - 1 \right). \]

Confluent forms of Heun’s differential equation (4) arise when two or more of the regular singularities merge to form an irregular singularity. This is analogous to the derivation of the confluent hypergeometric equation. There are four standard forms, as follows (see [14] and references therein):
(i) Confluent Heun Equation

\[
\frac{d^2y(x)}{dx^2} + \left( \alpha + \frac{\beta + 1}{x} + \gamma + \frac{1}{x - 1} \right) \frac{dy(x)}{dx} + \frac{(2\delta + \alpha(\beta + \gamma + 2))x + 2\eta + \beta + (\gamma - \alpha)(\beta + 1)}{2x(x - 1)} y(x) = 0.
\]  

(7)

It has regular singularities at \( x = 0 \) and 1, and an irregular singularity of rank 1 at \( x = \infty \). Mathieu functions, spheroidal wave functions and Coulomb spheroidal functions are special cases of solutions of the confluent Heun equation.

(ii) Doubly-confluent Heun equation

\[
x^2 \frac{d^2y(x)}{dx^2} + \alpha \left( x + \frac{1}{x} \right) \left( x \frac{dy(x)}{dx} \right) + \left( \beta_1 + \frac{1}{2} \right) \alpha x + \left( \frac{\alpha^2}{2} - \gamma \right) + \left( \beta_1 - \frac{1}{2} \right) \frac{\alpha}{x} y(x) = 0
\]

(8)

has irregular singularities at \( x = 0 \) and \( \infty \), each of rank 1.

(iii) Biconfluent Heun Equation

\[
\frac{d^2y(x)}{dx^2} + \left( \frac{\alpha + 1}{x} - \beta - 2x \right) \frac{dy(x)}{dx} + \left( \gamma - \alpha - 2 - \frac{\delta + (\alpha + 1)\beta}{2x} \right) y(x) = 0
\]

(9)

has a regular singularity at \( x = 0 \), and an irregular singularity at \( x = \infty \) of rank 2.

(iv) Triconfluent Heun Equation

\[
\frac{d^2y(x)}{dx^2} - \left( \gamma + 3x^2 \right) \frac{dy(x)}{dx} + [\alpha + (\beta - 3)x] y(x) = 0
\]

(10)

has one singularity, an irregular singularity of rank 3 at \( x = \infty \).

3. Differential transformation of hypergeometric equations

In this section, we consider the following hypergeometric equation:

\[
\sigma(x)z''(x) + \tau(x)z'(x) + \lambda z(x) = 0,
\]

(11)

and transformation:

\[
y(x) = A(x)z(x) + B(x)z'(x),
\]

(12)

where \( \sigma \equiv \sigma(x) \) is a polynomial of degree less or equal to 2, \( \tau \equiv \tau(x) \) a polynomial of degree exactly equal to 1, \( \lambda \) is a constant; \( A = A(x) \) and \( B = B(x) \) are polynomials of degrees \( r \) and \( s \), respectively. Then, we build a general linear second order ordinary differential equation \( L_2[y(x)] = 0 \) satisfied by the function \( y \), polynomial or not, and investigate a set of situations. For that, we proceed as follows:

- First derive (12) with respect to \( x \), using (11), and get:

\[
\sigma y' = \bar{A}z + \bar{B}z'
\]

(13)

where \( \bar{A} = \sigma A' - B\lambda, \bar{B} = \sigma A + \sigma B' - \tau B \).
• Derive (13) with respect to $x$ and obtain
\[ \sigma (\sigma y')' = \bar{A} z + \bar{B} z', \tag{14} \]
where $\bar{A} = \sigma \bar{A}' - \bar{B} \lambda$, $\bar{B} = \sigma \bar{A} + \sigma \bar{B}' - \tau \bar{B}$.

• Rewrite the coefficients $\bar{A}$ and $\bar{B}$ of (14) in terms of $\sigma$, $\tau$, $\lambda$, $A$, $B$ and their derivatives as follows:
\[
\bar{A} = A'' \sigma^2 + \sigma A' \sigma' - 2 \lambda \sigma B' - \lambda A \sigma + \lambda B \tau \tag{15} \\
\bar{B} = 2A' \sigma^2 - \lambda \sigma B + \sigma A \sigma' + \sigma^2 B'' + \sigma B' \sigma' - 2 \sigma B' \tau - \sigma B \tau' - \tau A \sigma + B \tau^2. \tag{16} 
\]

• Then, using (13) and (14) we form the following determinantal equation giving the ordinary differential equation satisfied by $y$:
\[
\mathcal{L}_2[y(x)] = \left| \begin{array}{ccc}
A & B \\
\sigma y' & \bar{A} & \bar{B} \\
\sigma (\sigma y')' & \bar{A} & \bar{B}
\end{array} \right| = 0, \tag{17} 
\]
or, equivalently,
\[
\hat{\mathcal{L}}_2[y(x)] = \left| \begin{array}{ccc}
y & A & B \\
\sigma y' & \hat{A} & \hat{B} \\
\sigma^2 y'' & \hat{A} & \hat{B}
\end{array} \right| = 0 \tag{18} 
\]
with
\[
\hat{A} = \sigma \bar{A}' - \bar{B} \lambda - \sigma' \bar{A} \tag{19} \\
\hat{B} = \sigma \bar{A} + \sigma \bar{B}' - \bar{B} (\tau + \sigma'). \tag{20} 
\]

The following observations are in order:

(i) The coefficient of $y''$ is $\sigma^2 (A\bar{B} - B\bar{A})$ which introduces new singular points depending on the degrees $r$ and $s$ of the polynomials $A$ and $B$. In general the obtained ODE is no more Fuchsian.

(ii) This general formulation does not take into account the possible logarithmic solutions and apparent singularities for $\mathcal{L}_2[y(x)] = 0$ which strongly restrict the choice of the polynomials $A$ and $B$. Such a freedom allows to cover a larger class of equations with $r + s + 2$ parameters such as Heun, generalized Heun equations and their various confluencies and to propose some solutions.

(iii) An obvious way to remain in the Fuchsian’s class is to choose $B(x) = \sigma(x)$ or more generally $B(x) = S(x) \sigma(x)$. Such a simplified choice has been already considered in the work by Kimura [1] and in recent literature on exceptional polynomials (see [2] and references therein).

Let us investigate some relevant particular cases in the sequel.
3.1. Basic choice: $A(x)$ is arbitrary and $B(x) = \sigma(x)$

For arbitrary $A(x)$ and $B(x) = \sigma(x)$, the equation (17) can be transformed into the following second order differential equation:

$$\mathcal{L}_2[y(x)] = \sigma P y'' + Q y' + R y = 0$$

(21)

where

$$P \equiv P(x) = A^2 + A(\sigma' - \tau) + \sigma(\lambda - A')$$

(22)

$$Q \equiv Q(x) = A''\sigma^2 + \sigma(A\tau' + \tau\lambda - \lambda\sigma' - A\sigma'' - 2AA')$$

(23)

$$R \equiv R(x) = \sigma[A''(\tau - A - \sigma') + A'(2A' + \sigma'' - 3\lambda - \tau') + \lambda(\lambda + \tau' - \sigma'') + \sigma'\[2\lambda A + \lambda\sigma' - \lambda\tau - \tau A'] + (A - \tau)(\lambda A - A'\tau).$$

(24)

Note that by using appropriate choices of the coefficients in the polynomial $A$, we can reduce the above equation into hypergeometric equations and generate known or unknown contiguous relations between the solutions $y(x)$. For instance, setting:

$$\sigma(x) = B(x) = 1 - x^2, \quad \tau(x) = \beta - \alpha - (\alpha + \beta + 2)x \quad \text{and} \quad \lambda = (n - 1)(n + \alpha + \beta), \quad n \in \mathbb{N},$$

(25)

then, the equation (21), after simplification, coincides with the hypergeometric equation satisfied by the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$:

$$\mathcal{L}_2[y(x)] = (1 - x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + (n - 1)(n + \alpha + \beta)y(x) = 0$$

(26)

giving $y(x) = P_{n-1}^{(\alpha,\beta)}(x)$, polynomial of degree $n - 1$, as solution. One can then use the well known contiguous relation [15]:

$$(2n + \alpha + \beta)(1 - x^2) \frac{d}{dx} P_n^{(\alpha,\beta)}(x) - n[\alpha - \beta - (2n + \alpha + \beta)x]P_n^{(\alpha,\beta)}(x)$$

$$= 2(n + \alpha)(n + \beta)P_{n-1}^{(\alpha,\beta)}(x)$$

(27)

to find the polynomial coefficient $A(x)$ of $P_n^{(\alpha,\beta)}(x)$. Of course if $\lambda$ is arbitrary, non polynomial solutions like the second kind solutions of (11) can be exploited to generate similar contiguous relations for the $z(x)$ as for the polynomial solutions $y(x)$.

3.2. Second choice: $A$ and $B$ are constant

Let us consider now the trivial case where $A$ and $B$ are constants. Then, the equation (17) can be reduced to a simpler second order differential equation as follows:

$$\mathcal{L}_2[y(x)] = \sigma P y'' + Q y' + R y = 0$$

(28)

where

$$P \equiv P(x) = A(\sigma - \tau B) + \lambda B^2$$

(29)

$$Q \equiv Q(x) = \sigma(\tau A^2 + AB\tau') + (\lambda B^2\tau + \lambda\sigma' B^2 - AB\tau^2 - \sigma' AB\tau)$$

(30)

$$R \equiv R(x) = \lambda A^2\sigma + (\lambda^2 B^2 - \lambda AB\sigma' + \lambda B^2\tau' - \lambda B A\tau).$$

(31)

This equation is also not hypergeometric in general, but suitable choices of the quantities $\sigma, \tau, \lambda$ and $A, B$ can also permit to deduce relevant contiguous relations. For instance, we retrieve the following particular equations:
3.3.1. General Heun’s equation

- Laguerre equation
  (i) If \( \sigma(x) = x, \tau(x) = 1 + \alpha - x \) and \( \lambda = n \), the equation (28) yields the hypergeometric equation respected by Laguerre polynomials \( L_n^{(\alpha)}(x) \).
  (ii) The choice \( A = 1 = -B \) gives again a Laguerre polynomial for \( y(x) \) confirming the well known relation [15]
  \[
  L_n^{(\alpha)}(x) - [L_n^{(\alpha)}(x)]' = [L_{n+1}^{(\alpha)}(x)]' = L_n^{(\alpha+1)}(x). \tag{32}
  \]

- Hermite equation
  If \( \sigma = 1 \) and \( \tau = -2x \), the equation (28) takes the form:
  \[
  L_2[y(x)] = y''(A^2 + B^2\lambda + 2ABx) - y'[2AB + 2x(A^2 + B^2\lambda) + 4x^2AB] + y\lambda[B^2(\lambda - 2) + A(A + 2xB)] = 0 \tag{33}
  \]
  which can be further simplified to give:
  (i) For the particular case of \( B = 0 \) with arbitrary \( A \), the well known Hermite equation:
  \[
  L_2[y(x)] = y'' - 2xy' + \lambda y = 0. \tag{34}
  \]
  (ii) In the opposite case, when \( A = 0 \) and \( B \) is arbitrary, the equation (33) is transformed into a modified Hermite equation
  \[
  L_2[y(x)] = y'' - 2xy' + \tilde{\lambda}y = 0 \text{ with } \tilde{\lambda} = \lambda - 2. \tag{35}
  \]

3.3. Link with the Heun equations and exceptional Jacobi polynomials

Besides, Heun general equation, its confluent forms as well as exceptional Jacobi polynomials can also be retrieved from the general approach developed here.

3.3.1. General Heun’s equation

- With the choice \( A = ax + b \) and \( \sigma = x^2 - x \), the degrees of \( P, Q, R \) in equation (21) are 2, 3 and 2, respectively. Then, the HEUN equation
  \[
  \frac{d^2y}{dx^2} \left[ \gamma + \frac{\delta}{x - 1} + \frac{\epsilon}{x - \mu} \right] \frac{dy}{dx} + \frac{\alpha\beta x + \rho}{x(x - 1)(x - \mu)} y = 0, \tag{36}
  \]
  is recovered when the coefficients \( a, b \) and \( \tau \) are chosen so that \( P(x) \) reduces to \( x - \mu \), and \( Q(x) = (x - \mu)Q_1(x) \) and \( R(x) = (x - \mu)R_1(x) \), where \( Q_1(x) \) and \( R_1(x) \) are polynomials.

- Other appropriate choices also produce general Heun and confluent Heun equations [13] with explicit solutions \( y(x) = A(x)z(x) + B(x)z'(x) \). Kimura gives [1] solutions in two relevant cases:
  (i) for \( \epsilon = -1 \) with \( A(x) \) of degree 1 and \( B(x) = x(x - 1) \),
  (ii) for \( \epsilon = -2 \) with \( A(x) \) of degree 2 and \( B(x) \) of degree 3.

The Kimura method is very nice, but complicated and restrictive, the confluent equations being excluded. Its intrinsic complexity resides in the step \( \epsilon = -1 \) to \( -2 \) increasing the degree of polynomials \( A(x) \) and \( B(x) \). In our approach, the problem is entirely algebraic, although not excluding, of course, also some difficulties.
3.3.2. Confluent Heun equation

- Setting $\sigma = 2x$, $A = B = 1$, the relations (29)-(31) give:

$$\tau = x; \quad P(x) = x - 1; \quad Q(x) = x^2 - x - 2 \quad \text{and} \quad R(x) = -x + 2$$

providing $\lambda = -1$ and the confluent Heun equation (7) with the coefficients

$$(\alpha, \beta, \gamma, \delta, \eta) = \left(\frac{1}{2}, 0, -\frac{1}{2}, -\frac{9}{4}\right).$$

3.3.3. Biconfluent Heun equation

- Taking $\tau = x$, $A = B = 2$, equations (29)-(31) yield:

$$\sigma = -\frac{1}{2}; \quad \lambda = \frac{1}{2}; \quad P(x) = -4x; \quad Q(x) = -4x^2 - 2 \quad \text{and} \quad R(x) = -2x + 2$$

leading to the biconfluent version of Heun equation (7) with the parameters

$$(\alpha, \beta, \delta, \gamma) = (-2, 0, -2, -1).$$

3.3.4. Exceptional $\chi_1$-Jacobi polynomials

Finally, let us mention that the exceptional $\chi_1$–Jacobi polynomials investigated in [2] can be also easily retrieved from our method. Indeed, set, for $g, h \notin \{-1/2, -3/2, -5/2, \ldots\}$:

$$\zeta(\eta) = \frac{g - h}{2} \eta + \frac{g + h + 1}{2}, \quad \tilde{\zeta}(\eta) = \frac{g - h}{2} \eta + \frac{g + h + 3}{2}$$

and consider the polynomials

$$A(\eta) = \frac{1}{k + h + \frac{1}{2}} \left( h + \frac{1}{2} \right) \tilde{\zeta}(\eta)$$

and

$$B(\eta) = \frac{1}{k + h + \frac{1}{2}} \left( 1 + \eta \right) \zeta(\eta)$$

of degrees 1 and 2, respectively. Let $z(\eta)$ be the Jacobi polynomial parametrized as [2]:

$$P_k(\eta) = \frac{(g + \frac{1}{2})_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j (k + g + h + 2)_j}{j! (g + \frac{1}{2})_j} \left( \frac{1 - \eta}{2} \right)^j.$$  \hspace{1cm} (40)

Then the transformation (12) gives the $\chi_1$–Jacobi polynomials $y(\eta) \equiv \tilde{P}_k(\eta)$ satisfying the following differential equation:

$$(1 - \eta^2)y''(\eta) + \left( h - g - (g + h + 3)\eta - 2\frac{(1 - \eta^2)\zeta'(\eta)}{\zeta(\eta)} \right) y'(\eta) + \left( -2(h + \frac{1}{2})(1 - \eta)^2 \frac{\tilde{\zeta}'(\eta)}{\tilde{\zeta}(\eta)} + k(k + g + h + 2) + g - h \right) y(\eta) = 0.$$  \hspace{1cm} (41)

Here also, transforming $\eta$ into a new variable and assigning adequate relations between parameters lead to specific Heun’s differential equations as shown, for instance, with the particular variable change $\eta = 1 - 2x$. For more details about such relations between Heun’s equations and differential equation describing the exceptional Jacobi polynomials, see [2] and references therein.
4. Concluding remarks
This work addressed a generalization of Kimura’s approach based on differential transformation of hypergeometric equations. This transformation involving \( y(x) = A(x)z(x) + B(x)z'(x) \), where \( z(x) \) is a solution of a hypergeometric equation, led to a general second order differential equation for \( y \) that encompasses various ODEs of mathematical physics as particular cases such as Heun equations and equations for exceptional polynomials. In particular, the equations (21) and (28), defined with the relations (22)-(24) and (29)-(31), permitted us to generate new as well as known relations for Laguerre and Hermite polynomials.

Acknowledgments
The authors are grateful to anonymous referees for their useful comments and careful reading of the manuscript. MNH thanks Professor Tudor Ratiu, Chair of Geometric Analysis, and his collaborators for their hospitality during his stay as Visiting Professor at the Centre Interfacultaire Bernoulli (CIB) at the Ecole Polytechnique Fédérale de Lausanne (EPFL) where this paper was completed. This work is partially supported by the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) through the Office of External Activities (OEA) - Prj-15. The ICMPA is in partnership with the Daniel Iagolnitzer Foundation (DIF), France.

References
[1] Kimura T 1971 Funkcialaj Ekvacioj 13 213-232
[2] Takemura K 2012 J. Phys. A: Math. Theor. 45 085211
[3] Gómez-Ullate D, Kamran N and Milson R 2008 (arXiv:0805.3376 [math-ph]); Gómez-Ullate D, Kamran N and Milson R 2009 J. Math. Anal. Appl. 359 352–367 (arXiv:0807.3939 [math-ph])
[4] Quesne C 2008 J. Phys. A 41 392001
[5] Bagchi B, Quesne C and Roychoudhury R 2009 Pramana J. Phys 73 337-347
[6] Crum M M 1955 Quart. J. Math. Oxford Ser.(2) 6 121-127
[7] Infeld L and Hull T E 1951 Rev. Mod. Phys. 23 21-68
[8] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267-385
[9] Odake S, Sasaki R 2009 Phys. Lett B. 679
[10] Ho C-L, Odake S and Sasaki R 2011 SIGMA 7 107 (arXiv:0912.5447 [math-ph])
[11] Sasaki R, Tsujimoto S and Zhedanov A 2010 J. Phys. A 43 315204
[12] Sleeman B D and Kuznetsov V B 2010 Heun Functions in NIST Handbook of Mathematical Functions, Chapter 31(Cambridge: Cambridge University Press)
[13] Ronveaux A 1995 Heun’s differential equations (Oxford: Oxford University Press) (ed.)
[14] Hounkonnou M N and Ronveaux A 2009 Appl. Math. Comput. 209 421-424
[15] Rainville E O 2002 Special functions (New York: Mcmillan)