Zero area singularities in general relativity and inverse mean curvature flow

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Abstract
First we restate the definition of a zero area singularity, recently introduced by H L Bray. We then consider several definitions of mass for these singularities. We use the inverse mean curvature flow to prove some new results about the mass of a singularity, the ADM mass of the manifold and the capacity of the singularity.

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1. Introduction
This paper consists of a study of some of the properties of zero area singularities, as recently introduced by Bray in [1] and developed by Bray and Jauregui in [2]. The motivating example of which is the spatial Schwarzschild metric with a negative mass parameter:

\[ g_{ij} = \left( 1 + \frac{m}{2r} \right)^4 \delta_{ij} \quad m < 0. \] (1)

In addition to being historically and physically important, the Schwarzschild solution is of particular mathematical interest since it is the case of equality of the Riemannian Penrose conjecture [3], and, in the case when \( m = 0 \), it is the case of equality of the Riemannian positive mass theorem [4]. Thus, this metric, and its generalizations, shows promise as objects of study. For a further development of ZAS, as well as an overview of some of the negative mass results in the field, see [2].

The main aims of this paper are, once we have defined the mass of a singularity, to extend the results of Huisken and Ilmanen in [5] to manifolds containing a single zero area singularity and a relationship between the capacity of the singularity and its mass.

2. Definitions

2.1. Asymptotically flat manifolds

We will use the following definition of asymptotic flatness.
Definition 2.1 [5]. A Riemannian 3-manifold \((M, g)\) is called asymptotically flat if it is the union of a compact set \(K\), and sets \(E_i\) each diffeomorphic to the complement of a compact set \(K_i\) in \(\mathbb{R}^3\), where the metric on each \(E_i\) satisfies
\[
|g_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, \quad |g_{ij,k}| \leq \frac{C}{|x|^2}
\] (2)
as \(|x| \to \infty\). Derivatives are taken in the flat metric \(\delta_{ij}\) on \(x \in \mathbb{R}^3\). Furthermore, the Ricci curvature must satisfy
\[
Rc \geq -\frac{Cg}{|x|^2}.
\] (3)
The set \(E_i\) is called an end of \(M\).

A manifold may have several ends, but our results will be relative to a single end. We will also be using the ADM mass of an asymptotically flat manifold and Hawking mass, capacity and minimizing hull of a surface:

Definition 2.2 [6]. The ADM mass of an end of an asymptotically flat manifold is
\[
m_{ADM} = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,i} - g_{ii,j}) n_j \, d\mu.
\] (4)

Definition 2.3. The Hawking mass of a surface \(\Sigma\) is given by
\[
m_H = \sqrt{|\Sigma|} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2\right).
\] (5)

Definition 2.4. Let \(\Sigma\) be a surface in an asymptotically flat manifold \(M\). Define the capacity of \(\Sigma\) by
\[
C(\Sigma) = \inf \left\{ \int_M \|\nabla \varphi\|^2 \, dV \left| \varphi(\Sigma) = 1, \varphi(\infty) = 0 \right. \right\}.
\] (6)

It is worth noting that if \(\Sigma\) and \(\Sigma'\) are two surfaces in \(M\) so that \(\Sigma\) divides \(M\) into two components, one containing infinity and the other containing \(\Sigma'\), then \(C(\Sigma') \leq C(\Sigma)\) since the infimum is over a larger set of functions.

Definition 2.5. Let \(\Sigma\) be a surface that is the boundary of an open set, \(E\), in a manifold \(M\). We call \(\Sigma\) a minimizing hull if
\[
|\partial E \cap K| \leq |\partial F \cap K|
\] (7)
for any \(F\) containing \(E\) where \(K\) is a compact set containing \(F \setminus E\).

2.2. Definition and mass of zero area singularities

The basic example of a zero area singularity is the negative Schwarzschild solution. This is the manifold \(\mathbb{R}^3 \setminus B_{-m/2}\) with the metric
\[
g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}
\] (8)
where \(m < 0\). This manifold fails the requirements of the positive mass theorem since it is not complete: geodesics reach the sphere at \(r = -m/2\) in finite distance. A straightforward calculation shows that the ADM mass of this manifold is given by \(m\). Furthermore, the far
field deflection of geodesics is the same as for a Newtonian mass of \(m\). These results are identical to the same results for a positive mass Schwarzschild solution.

Two important aspects of this example will be incorporated into the definition of a zero area singularity. One is that the point itself is not included. We still must describe the behavior of surfaces near the singularity. The manifold in that region should have surfaces whose areas converge to zero. In addition the capacity of these surfaces should go to zero. The second aspect is the presence of a background metric, in this case the flat metric. This background metric will provide a location where we can compute information about the singularity.

We have to be careful, since sometimes we are dealing with the topological manifold \(M\) and sometimes with the Riemannian manifold \(M \setminus \Pi\). So we should clarify what we mean by ‘convergence’ of surfaces. We, again, follow [2] and restrict a surface to mean a \(C^\infty\) closed embedded 2-manifold in the interior of \(M\) that is the boundary of a open region \(\Omega\). We will mostly be concerned with surfaces converging to \(\Pi\).

In this case, for a surface sufficiently close to \(\Pi\), we can consider coordinates \((x, s)\) on a tubular neighborhood of \(\Pi\), where \(x \in \Pi\) and \(s \in [0, \varepsilon)\). Then we will restrict ourselves further to ‘graphs’ over \(\Pi\). That is, surfaces that can be written as \((x, s(x))\). For such surfaces we define convergence as follows.

**Definition 2.6.** Let \(\{\Sigma_i\}\) be a sequence of surfaces that are graphs over \(\Pi\). Hence each \(\Sigma_i\) can be parametrized as \((x, s_i(x))\). We say that \(\{\Sigma_i\}\) converges to \(\Pi\) in \(C^2\) if the functions \(s_i : \Pi \to (0, \varepsilon)\) converge to \(0\) in \(C^2\).

With this in hand we make the following definition.

**Definition 2.7.** Let \(M^3\) be a smooth manifold with boundary, where the boundary is compact. Let \(\Pi\) be a compact connected component of the boundary of \(M\). Let the interior of \(M\) be a Riemannian manifold with smooth metric \(g\). Suppose that, for any smooth family of surfaces, \(\{\Sigma_i\}\), converging in \(C^2\) to \(\Pi\), the area of \(\Sigma_i\) with respect to \(g\) goes to zero as the surfaces converge to \(\Pi\). Then \(\Pi\) is a zero area singularity.

Bray and Jauregui in [2] provide several equivalent conditions to this definition. We will use ZAS for the singular and plural of zero area singularity. A particularly useful class of these singularities are regular zero area singularities.

**Definition 2.8.** Let \(M^3\) be a smooth manifold with boundary. Let the boundary of \(M\) consists of one compact component, \(\Pi\). Let \(\Pi\) be a ZAS. If there is a smooth metric \(\overline{g}\) on \(M\) and a smooth function \(\overline{\varphi}\) on \(M\) with nonzero differential on \(\Pi\) so that \(g = \overline{\varphi}^4 \overline{g}\), then we call \(\Pi\) a regular zero area singularity. We call the data \((M^3, \overline{g}, \overline{\varphi})\) a resolution of \(\Pi\).

Note that while \(\Pi\) is topologically a surface, and it is a surface in the Riemannian manifold \((M^3, \overline{g})\), the areas of surfaces near it in \((M^3 \setminus \Pi, g)\) approach zero, so we will sometimes speak of \(\Pi\) as being a point \(p\), when we are thinking in terms of the metric \(g\). Furthermore, note that the requirement that areas near \(\Pi\) go to zero under \(g\) tells us that \(\overline{\varphi} = 0\) on \(\Pi\). For a regular ZAS, \(g\) can be extended, as a symmetric two tensor, to \(\Pi\). In [2] they consider local and global resolutions of ZAS. However, since Geroch monotonicity under IMCF requires our surfaces to be connected, we only consider the case with a single ZAS. Thus, we have no need for a distinction between local and global resolutions.

We follow Bray in [1] and define the mass of a regular ZAS as follows.
Definition 2.9. Let \((M^3, \tilde{g}, \tilde{\varphi})\) be a resolution of a regular ZAS \(p = \Pi\). Let \(\varphi\) be the unit normal to \(\Pi\) in \(\tilde{g}\). Then the regular mass of \(p\) is defined to be

\[
m_{\text{reg}}(p) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Pi} \frac{\varphi}{\tilde{\varphi}} \frac{4}{3} \frac{\text{d}A}{4} \right)^{3/2}.
\]

We also define the mass of a ZAS that may not be regular.

Definition 2.10. Let \((M^3, g)\) be an asymptotically flat manifold, with a ZAS \(p\). Let \(\Sigma_i\) be a smooth family of surfaces converging to \(p\). Define \(h_i\) by

\[
\Delta h_i = 0 \quad \lim_{i \to \infty} h_i = 1 \quad h_i = 0 \quad \text{on} \quad \Sigma_i.
\]

Then the manifold \((M, h_i^4 g)\) has a ZAS at \(\Sigma_i = p\) which is resolved by \((M, g, h_i)\). Define the mass, \(m_{\text{ZAS}}(p)\), of \(p\) to be

\[
\sup \limsup_{\{\Sigma_i\} \to \infty} \frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma_i} \frac{\varphi(h_i)^{4/3} \text{d}A}{4} \right)^{3/2} = \sup \limsup_{i \to \infty} m_{\text{reg}}(p_i).
\]

Here the outer sup is over all possible smooth families of surfaces \(\{\Sigma_i\}\) which converge to \(p\).

3. Fundamental results

Before we continue we must verify that these definitions are consistent. First it must be verified that the regular mass of a regular ZAS is indeed intrinsic to the singularity, as shown in [1].

Lemma 3.1. The regular mass of a ZAS is independent of the resolution.

Proof. Let \((M^3, \tilde{g}, \tilde{\varphi})\) and \((M^3, \hat{g}, \hat{\varphi})\) be two resolutions of the same ZAS, \(p\). Then define \(\lambda\) by \(\tilde{\varphi} = \lambda \hat{\varphi}\). Thus, we note the following scalings:

\[
\tilde{g} = \lambda^4 \hat{g} \quad \tilde{\text{d}A} = \lambda^4 \hat{\text{d}A} \quad \tilde{\varphi} = \lambda^{-1} \hat{\varphi} \quad \tilde{\nu} = \lambda^{-2} \hat{\nu}.
\]

Now note that since \(\tilde{\varphi}, \hat{\varphi} = 0\) on \(\tilde{\Pi}, \hat{\Pi}\),

\[
\tilde{\nu}(\tilde{\varphi}) = \lambda^{-2} \hat{\nu}(\lambda^{-1} \hat{\varphi}) = \lambda^{-3} \hat{\nu}(\hat{\varphi}) + \lambda^{-4} \hat{\nu}(\lambda) \hat{\varphi}.
\]

The last term, \(\lambda^{-4} \hat{\nu}(\lambda) \hat{\varphi}\), needs discussion. Both \(\tilde{\varphi}\) and \(\hat{\varphi}\) are smooth functions with zero set \(\Pi\) and they both have nonzero differential on \(\Pi\). Hence, \(\lambda\) is smooth. Thus, since \(\tilde{\varphi}\) goes to zero on \(\Pi\), this last term is zero on \(\Pi\). Thus, the mass of \(p\) using the \((M^3, \tilde{g}, \tilde{\varphi})\) resolution is
given by

\[ m_{\text{reg}}(p) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} \nu(\bar{\psi})^{4/3} \, dA \right)^{3/2} \]

(14)

\[ = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} [\lambda^{-2} \nu(\lambda^{-1} \varphi)]^{4/3} \lambda^{4} dA \right)^{3/2} \]

(15)

\[ = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} [\lambda^{-3} \nu(\varphi)]^{4/3} \lambda^{4} dA \right)^{3/2} \]

(16)

\[ = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} \nu(\varphi)^{4/3} dA \right)^{3/2} . \]

(17)

□

Definition 2.10 seems to involve the entire manifold, as the definition of \( h_i \) takes place on the entire manifold. However that is not the case. The mass is actually local to the point \( p \).

**Lemma 3.2.** Let \((M^3, g)\) be a manifold with a ZAS \( p \). Let \( \bar{g} \) be a second metric on \( M \) that agrees with \( g \) in a neighborhood of \( p \). Then the mass of \( p \) in \((M^3, g)\) and \((M^3, \bar{g})\) is equal.

**Proof.** The goal is to show that for any selection of \( \{\Sigma_i\} \), the series \( m_{\text{reg}}(p_i) \) and \( \bar{m}_{\text{reg}}(p_i) \) obtained in the calculation of the mass of \( p \), with respect to \((M, g)\) and \((M, \bar{g})\), converge to the same value. Let \( S \) be a smooth, compact, connected surface separating \( p \) from infinity and contained in the region where \( g \) and \( \bar{g} \) agree. Fix \( i \) large enough so that \( \Sigma_i \) is inside of \( S \), and suppress the index \( i \) on all our functions. Then define the functions \( h, \bar{h} \) by

\[
\begin{align*}
    h = \bar{h} = 0 \quad &\text{on} \quad \Sigma_i \\
    \lim_{x \to \infty} h = \lim_{x \to \infty} \bar{h} = 1 \\
    \Delta h = \Delta \bar{h} = 0.
\end{align*}
\]

(18)

Here \( \Delta \) and \( \bar{\Delta} \) denote the Laplacian with respect to \( g \) and \( \bar{g} \) respectively.

Now inside \( S \), \( \Delta = \bar{\Delta} \) since \( g = \bar{g} \). Thus, there is only one notion of harmonic, and \( h \) and \( \bar{h} \) differ only by their boundary values on \( S \). Let \( \epsilon = 1 - \min_{\Sigma_i} \{ h, \bar{h} \} \). Consider the following two functions \( f^- \) and \( f^+ \) defined between \( S \) and \( \Sigma_i \):

\[
\begin{align*}
    f^- = f^+ &= 0 \quad &\text{on} \quad \Sigma_i \\
    \Delta f^- = \Delta f^+ &= 0 \\
    f^- &= 1 - \epsilon \quad &\text{on} \quad S \\
    f^+ &= 1 \quad &\text{on} \quad S.
\end{align*}
\]

(19)

The maximum principle gives us, inside \( S \),

\[
f^+ \geq h, \quad \bar{h} \geq f^-.
\]

(20)

Furthermore, since all four functions are zero on \( \Sigma_i \),

\[
v(f^+) \geq v(h), \quad v(\bar{h}) \geq v(f^-).
\]

(21)

Here \( v \) is the normal derivative on \( \Sigma_i \). Now define \( \mathcal{F}(\varphi) \) by the formula

\[
\mathcal{F}(\varphi) = \int_{\Sigma} v(\varphi)^{4/3} \, dA.
\]

(22)

Then the ordering of the derivatives gives the ordering

\[
\mathcal{F}(f^+) \geq \mathcal{F}(h), \quad \mathcal{F}(\bar{h}) \geq \mathcal{F}(f^-).
\]

(23)
However, since \( f^- = (1 - \epsilon)f^+ \),
\[
\nu(f^-) = (1 - \epsilon)\nu(f^+); \tag{24}
\]
hence,
\[
\mathcal{F}(f^-) = (1 - \epsilon)^{4/3}\mathcal{F}(f^+). \tag{25}
\]
Now, without loss of generality assume that the limit of the capacities of \( \{\Sigma_i\} \) is zero, as the mass would be \(-\infty\) otherwise.

Thus, as \( i \to \infty \), \( \Sigma_i \) has capacity going to zero (see subsection 4.2 for more discussion of capacity of ZAS.) Hence, \( \epsilon_i \) goes to zero, and so \( \mathcal{F}(f_i^-)/\mathcal{F}(f_i^+) \) goes to 1. Thus, equation (23) forces \( \mathcal{F}(h_i) \) and \( \mathcal{F}({\tilde h}_i) \) to equality. This forces the masses of \( p_i \) in the two metrics to equality as well. \( \square \)

**Corollary 3.3.** In definition 2.10 we may replace the condition that \( h_i \) be 1 at infinity with the condition that \( h_i \) be 1 on a fixed surface outside \( \Sigma_i \) for \( i \) sufficiently large.

4. **Zero area singularity results**

4.1. **Zero area singularities and IMCF**

First recall that (weak) IMCF finds a (weak) solution to the equation
\[
\text{div}_M \left( \frac{\nabla u}{|\nabla u|} \right) = |\nabla u|. \tag{26}
\]
Wherever \( u \) is smooth with \( \nabla u \neq 0 \), the level sets of \( u \) form a flow of surfaces where the flow speed is given by \( 1/H \).

The main features we will be using of this are the following two facts.

**Theorem 4.1.** Assume \( M \) is asymptotically flat, let \( (N_t)_{t \geq 0} \) be the surfaces obtained from a weak solution to IMCF in \( M \). Then
\[
\lim_{t \to \infty} m_H(N_t) \leq m_{\text{ADM}}(M). \tag{27}
\]

**Theorem 4.2** (Geroch mononicity, 6.1 from [5]). Let \( \tilde M \) be an asymptotically flat region of a manifold with \( R > 0 \) exterior to a surface \( \partial\tilde M \). For each connected component \( N \) of \( \partial\tilde M \), there exists a flow of compact \( C^{1,\alpha} \) surfaces \( (N_t)_{t \geq 0} \) such that \( N_0 = N \), \( m_H(N_t) \) is monotone nondecreasing function for all \( t \) and for sufficiently large \( t \), \( N_t \) satisfies the IMCF.

There is an assumption here that our starting surface is a minimizing hull. In order to apply IMCF to ZAS we will extend Geroch monotonicity down to \( t = 0 \) in the case where our initial surface has negative Hawking mass.

**Lemma 4.3.** Let \( \Sigma \) be a surface in an asymptotically flat 3-manifold. Let \( \Sigma' \) be the boundary of the minimizing hull of \( \Sigma \). Let \( \Sigma \) or \( \Sigma' \) have negative Hawking mass. Then
\[
m_H(\Sigma) \leq m_H(\Sigma'). \tag{28}
\]

**Proof.** If \( \Sigma' \) has nonnegative Hawking mass, then \( m_H(\Sigma') \geq 0 \geq m_H(\Sigma) \) and we are done. Thus, we can assume that \( \Sigma' \) has negative Hawking mass. Since \( \Sigma' \) has negative Hawking mass, it must intersect \( \Sigma \) on a set of positive measure. Otherwise, \( \Sigma' \) would be a minimal surface, with Hawking mass \( \sqrt{\frac{1}{16\pi}} > 0 \). We define the following sets:
\[
\Sigma_0 = \Sigma' \cap \Sigma \quad \Sigma_+ = \Sigma' \setminus \Sigma_0 \quad \Sigma_- = \Sigma \setminus \Sigma_0. \tag{29}
\]
Recalling that $|\Sigma_+| \leq |\Sigma_-|$ by the minimization property, and that $H = 0$ on $\Sigma_+$, we observe the following:

$$0 > m_H(\Sigma') = \sqrt{|\Sigma_0| + |\Sigma_+|} \left( 16\pi - \int_{\Sigma_0} H^2 \right) \quad (30)$$

$$\geq \sqrt{|\Sigma_0| + |\Sigma_-|} \left( 16\pi - \int_{\Sigma_0} H^2 \right) \quad (31)$$

$$\geq \sqrt{|\Sigma_0| + |\Sigma_-|} \left( 16\pi - \int_{\Sigma_0} H^2 - \int_{\Sigma_-} H^2 \right) \quad (32)$$

$$= m_H(\Sigma). \quad \square$$

With this lemma and Geroch Monotonicity we can prove the following lemma.

**Lemma 4.4.** Let $(M, g)$ be an asymptotically flat manifold with ADM mass $m$, nonnegative scalar curvature and a single regular ZAS $p$. Let $\{\Sigma_i\}$ be a smooth family of surfaces converging to $p$, which eventually have negative Hawking mass. Then for sufficiently large $i, m_H(\Sigma_i) \leq m$.

**Proof.** Since for large enough $i$, $\Sigma_i$ has non-positive Hawking mass, we can apply lemma 4.3 to show that $\Sigma'_i$ must have larger Hawking mass. From this surface, we start inverse mean curvature flow. Geroch monotonicity tells us that the Hawking masses of the surfaces $N_t$ defined by IMCF starting with $\Sigma'_i$ only increase. Theorem 7.4 in [5] tells us that the increasing limit of the Hawking masses of these surfaces is less than the ADM mass. Thus, the Hawking mass of the starting surface was also less than the ADM mass. $\square$

Now we relate the limit of the Hawking masses to the regular mass.

**Lemma 4.5.** Let $(M, g)$ be an asymptotically flat manifold with nonnegative scalar curvature and a single regular ZAS $p$. Then there is a smooth family of surfaces $\{\Sigma_i\}$ converging to $p$ such that

$$\lim_{i \to \infty} m_H(\Sigma_i) = -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} \psi(\phi)^{4/3} dA \right)^{3/2} = m_{ZAS}(p). \quad (33)$$

**Proof.** The Hawking mass of a surface $\Sigma_i$ is given by

$$m_H(\Sigma_i) = \sqrt{|\Sigma_i|_{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_i} H^2 dA \right). \quad (34)$$

Since the areas of the surfaces are converging to zero, we have

$$\lim_{i \to \infty} m_H(\Sigma_i) = -\lim_{i \to \infty} \sqrt{|\Sigma_i|_{16\pi}} \int_{\Sigma_i} H^2 dA. \quad (35)$$

By the Hölder inequality this is bounded as follows:

$$-\sqrt{|\Sigma_i|_{16\pi}} \int_{\Sigma_i} H^2 dA \leq -\frac{1}{(16\pi)^{3/2}} \left( \int_{\Sigma_i} H^{4/3} dA \right)^{3/2}. \quad (36)$$

Switching to the resolution space, we use the formula

$$H = \psi^{-2}H + 4\psi^{-3} \psi(\phi). \quad (37)$$
Putting this into the previous equation we get
\[
\int_{\Sigma} H^{4/3} \ dA = \int_{\Sigma} (\varphi^{-2} \bar{H} + 4\varphi^{-3} \nabla(\varphi))^{4/3} \ dA \quad \text{(38)}
\]
\[
= \int_{\Sigma} (\varphi \bar{H} + 4\nabla(\varphi))^{4/3} \ dA. \quad \text{(39)}
\]

Since \( \varphi \) is zero on \( \Sigma \) and \( \bar{H} \) is bounded, the first term goes to zero. The second term converges since the family of surfaces \( \{\Sigma_i\} \) are converging smoothly:
\[
\lim_{i \to \infty} \int_{\Sigma_i} (\varphi \bar{H} + 4\nabla(\varphi))^{4/3} \ dA = \frac{4}{3} \int_{\Sigma} \nabla(\varphi)^{4/3} \ dA. \quad \text{(40)}
\]

Combining all of these equations we have
\[
\lim_{i \to \infty} m_H(\Sigma_i) \leq -\frac{1}{4} \left( \frac{1}{\pi} \int_{\Sigma} \nabla(\varphi)^{4/3} \ dA \right)^{3/2} = m_{\text{ZAS}}(p). \quad \text{(41)}
\]

To see when this estimate is sharp, we look at inequality (36) since that is the only inequality is our estimate. In the limit, this inequality is an equality exactly when the ratio of the maximum and minimum values of \( H \) approaches 1. We choose a resolution such that \( \nabla(\varphi) = 1 \) on the boundary. We also choose a family of surfaces \( \Sigma_i \) given by level sets of \( \varphi \). Then if we look at the ratio
\[
\lim_{\varphi \to 0} \frac{H_{\min}}{H_{\max}} = \lim_{\varphi \to 0} \frac{\varphi \bar{H}_{\min} + 4\nabla(\varphi)}{\varphi \bar{H}_{\max} + 4\nabla(\varphi)}, \quad \text{(42)}
\]
and remember that \( \bar{H} \) is bounded, we see that the \( \nabla(\varphi) \) terms dominate, and as \( \varphi \to 0 \), this ratio approaches 1. Thus, with this resolution and this family of surfaces, inequality (41) will turn to an equality.

With these results we can prove the following theorem.

**Theorem 4.6.** Let \((M, g)\) be an asymptotically flat manifold with nonnegative scalar curvature and a single regular ZAS \( p \). Then the ADM mass of \( M \) is at least the mass of \( p \).

**Proof.** First consider the case when \( p \) can be enclosed by a surface, \( \Sigma \), with nonnegative Hawking mass. The minimizing hull of a surface with nonnegative Hawking mass has nonnegative Hawking mass. Thus, we can run IMCF from \( \Sigma' \), and the AMD mass of \( M \) is at least \( m_H(\Sigma') \geq 0 \). However, the regular mass of \( p \) is always nonpositive so in this case we are done.

Now assume that \( p \) cannot be enclosed by a surface with nonnegative Hawking mass. By lemma 4.4 we know that the ADM mass is greater than the Hawking masses of any sequence of surface converging to \( p \) which have negative Hawking mass. By lemma 4.5 we know that there is a family of surfaces converging to \( p \) which have the mass of \( p \) as the limit of their Hawking mass; hence, the ADM mass is greater than their Hawking masses which limit to the regular mass.

This can be extended to a general ZAS. However, first we need to consider the effect of multiplication by a harmonic conformal factor on the ADM mass of a manifold.

**Lemma 4.7.** Let \((M^3, g)\) be an asymptotically flat manifold. Let \( \varphi \) be a harmonic function with respect to \( g \) with asymptotic expansion
\[
\varphi = 1 + \frac{C}{|x|_g^2} + O \left( \frac{1}{|x|_g^4} \right) \quad \text{(43)}
\]
Then if the ADM mass of \((M^3, g)\) is \( m \), the ADM mass of \((M^3, \varphi^4 g)\) is \( m + 2C \).
Proof. This is a direct calculation. We write $g^\phi = \phi^4 g$, and calculate, only keeping the terms of lowest order in $|x|^{-1}$ since we are taking limits as $|x| \to \infty$:

$$m_\phi = \lim_{|x| \to \infty} \frac{1}{16\pi} \int_{S^i} (g^\phi_{ij,i} - g^\phi_{ii,j})nj \, dA$$

\hspace*{1cm} (44)

$$= \lim_{|x| \to \infty} \phi^4 \frac{1}{16\pi} \int_{S^i} (g_{ij,i} - g_{ii,j})nj \, dA + \lim_{|x| \to \infty} \frac{\phi^3}{4\pi} \int_{S^i} (\delta_{ij}\phi_i - \delta_{ii}\phi_j)nj \, dA$$

\hspace*{1cm} (45)

$$= \lim_{|x| \to \infty} \phi^4 m - \lim_{|x| \to \infty} \phi^3 \lim_{|x| \to \infty} \frac{1}{4\pi} \int_{S^i} (\phi_j - 3\phi_j)nj \, dA$$

\hspace*{1cm} (46)

$$= m - \lim_{|x| \to \infty} \frac{1}{2\pi} \int_{S^i} \phi_jnj \, dA$$

\hspace*{1cm} (47)

$$= m - \lim_{|x| \to \infty} \frac{1}{2\pi} \int_{S^i} \langle \nabla \phi, v \rangle \, dA$$

\hspace*{1cm} (48)

$$= m + 2C.$$ \hspace*{1cm} (49)

Using this we can now extend theorem 4.6 to a general ZAS.

**Theorem 4.8.** Let $(M, g)$ be an asymptotically flat manifold with nonnegative scalar curvature and a single zero area singularity $p$. Then $m$, the ADM mass of $M$, is at least the mass of $p$.

**Proof.** If the capacity of $p$ is nonzero, then the statement is trivial. Thus, we assume the capacity of $p$ is zero. Using the terminology of definition 2.10, theorem 4.6 tells us that the ADM mass of $(M, h^4_i g)$ is at least the mass of the regular singularity at $\Sigma_i = p_i$. Each $h_i$ is defined by the equations

$$\Delta h_i = 0$$

$$\lim_{r \to \infty} h_i = 1$$

$$h_i = 0 \text{ on } \Sigma_i.$$ \hspace*{1cm} (50)

Thus, it has asymptotic expansion

$$h_i = 1 - \frac{C_i}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right),$$ \hspace*{1cm} (51)

where $4\pi C_i$ is the capacity of $\Sigma_i$. Thus, the ADM mass, $m_i$, of $(M, h^4_i g)$ is given by $m - 2C_i$.

Now we know that $m_i \geq m_{\text{reg}}(p_i)$. Taking lim sup of both sides gives us

$$\limsup_{i \to \infty} m_i \geq \limsup_{i \to \infty} m_{\text{reg}}(p_i).$$ \hspace*{1cm} (52)

Since $C_i$ is going to zero, the left-hand side is simply $m$, and so has no dependence on which $\{\Sigma_i\}$ we chose in our mass calculation. Thus, we get

$$m \geq \sup_{\{\Sigma_i\}} \limsup_{i \to \infty} m_{\text{reg}}(p_i) = m_{\text{ZAS}}(p).$$ \hspace*{1cm} (53)

as desired. \hspace*{1cm} $\Box$
4.2. Capacity and ZAS

The capacity of a surface provides a measure of its size as seen from infinity. We extend the definition of the capacity of surface to the capacity of a zero area singularity. We then show that if a ZAS has non-zero capacity, the Hawking mass of any family of surfaces converging to it must go to negative infinity. We now define the capacity of a singular point. The natural definition is the one we want.

**Definition 4.9.** Let \( p \) be singular point in an asymptotically flat manifold \( M \). Chose a sequence of surfaces \( \Sigma_i \) of decreasing diameter enclosing \( p \). Then define the capacity of \( p \) by the limit

\[
C(p) = \lim_{i \to \infty} C(\Sigma_i)
\]

Before using this definition we have to show that it is well defined.

**Lemma 4.10.** Let \( \Sigma_i \) and \( \tilde{\Sigma}_i \) be two sequences of surfaces approaching the point \( p \). If \( \lim_{i \to \infty} C(\Sigma_i) = K \), \( \lim_{i \to \infty} C(\tilde{\Sigma}_i) = K \). Hence \( C(p) \) is well defined.

**Proof.** Since \( \Sigma_i \) are going to \( p \), for any given \( \tilde{\Sigma}_i \), we can choose \( i_0 \) such that for all \( i > i_0 \), \( \Sigma_i \) is contained within \( \tilde{\Sigma}_i \). Thus, if \( \varphi \) is a capacity test function for \( \tilde{\Sigma}_i \), i.e. \( \varphi(\tilde{\Sigma}_i) = 1 \) and \( \varphi \to 0 \) at infinity, then \( \varphi \) is also a capacity test function for \( \Sigma_i \). Since \( C(\Sigma_i) \) is taking the infimum over a larger set of test functions than \( C(\tilde{\Sigma}_i) \), \( C(\Sigma_i) \leq C(\tilde{\Sigma}_i) \). Thus, if we create a new sequence of surfaces \( \Sigma_i \), alternately choosing from \( \Sigma_i \) and \( \tilde{\Sigma}_i \), such that each surface contains the next, we get a nonincreasing sequence of capacities. Thus, if either original sequence of surfaces has a limit of capacity, then this new sequence must as well, and it must be the same. Hence, \( \lim_{i \to \infty} C(\Sigma_i) = \lim_{i \to \infty} C(\tilde{\Sigma}_i) \).

Now we look at the relationship between capacity and the Hawking mass of a surface. We will use techniques similar to those used in [7].

**Theorem 4.11.** Let \( M \) be an asymptotically flat 3-manifold with nonnegative scalar curvature, and ZAS \( p \). Let \( \Sigma_i \) be a family of surfaces converging in \( C^2 \) to \( p \). Assume each \( \Sigma_i \) is a minimizing hull. Assume the areas of \( \Sigma_i \) are going to zero. Then if the Hawking mass of the surfaces is bounded below, the capacities of surfaces converging to \( p \) must go to zero.

**Proof.** To use Geroch monotonicity, we need to know that our IMCF surfaces stay connected. In the weak formulation of IMCF, the level sets \( \Sigma_t \) always bound a region in \( M \). Thus, if \( \Sigma_i \) is not connected, one of its components \( \Sigma_i^* \) must not bound a region. That is, \( \Sigma_i^* \) is not homotopic to a point in \( M \). Since \( M \) is smooth, it must have finite topology on any bounded set. Thus, we know that near \( p \), there is a minimum size for a surface that does not bound a region. Call this size \( A_{\min} \). Thus, if we have any surface that does not bound a region, it must have area greater than \( A_{\min} \). The area of our surfaces grows exponentially. Thus, if we restrict ourselves to starting IMCF with a surface with area \( A_{\min} / e \), and only run the flow for time 1, we will stay connected. At first glance it seems we may need to worry about the jumps in weak IMCF; however, Geroch monotonicity does not depend on smoothness of the flow, and neither does the area growth formula. Thus, even with jumps, the area of our surfaces will remain below \( A_{\min} \).

Now recall that capacity of a surface is defined by

\[
C(\Sigma) = \inf \left\{ \int_M \| \nabla \varphi \|^2 \, dV \mid \varphi(\Sigma) = 1, \varphi(\infty) = 0 \right\}.
\]

Here, the integral is only over the portion of \( M \) outside of \( \Sigma \). Call this integral, \( \mathcal{E}(\varphi) \), the energy of \( \varphi \). Thus, for any \( \varphi \) with \( \varphi(\infty) = 0 \) and \( \varphi(\Sigma) = 1 \), we have \( \mathcal{E}(\varphi) \geq C(\Sigma) \). So we will find an estimate that relates the Hawking mass and the energy of a test function \( \varphi \).
Choose a starting surface $\Sigma$ with sufficiently small starting area. Let $f$ be the level set function of the associated weak IMCF starting with the surface $\Sigma$. Call the resulting level sets $\Sigma_t$. Now if we use a test function of the form $\varphi = u(f)$, then the energy of $\varphi$ is given by

$$E(\varphi) = \int_M \|\nabla f\|^2 (u')^2 \, dV. \tag{55}$$

Since $f$ is given by IMCF, we know that $\|\nabla f\| = H$ where $H$ is the mean curvature of the level sets. Next we use the co-area formula with the foliation $\Sigma_t$ and our integral becomes

$$E(\varphi) = \int_0^\infty (u'(t))^2 \int_{\Sigma_t} |H| \, dA_t \, dt. \tag{56}$$

Here the co-area gradient term cancels one of the $|H| = \|\nabla f\|$ terms. Now we will bound the interior integral of curvature. We know that IMCF causes the Hawking mass to be nondecreasing in $t$. We first rewrite the definition of the Hawking mass $m_H(\Sigma^i_t) = m(t)$ as

$$\int H^2 \, dA_t = 16\pi \left(1 - m(t) \sqrt{\frac{16\pi}{A(t)}}\right). \tag{57}$$

Here $A(t)$ is the area of $\Sigma_t$. Since the Hawking mass is nondecreasing under IMCF, we have

$$\int H^2 \, dA_t \leq 16\pi \left(1 - m(0) \sqrt{\frac{16\pi}{A(t)}}\right). \tag{58}$$

Thus, we can use Cauchy–Schwartz to get

$$\int |H| \, dA_t \leq \sqrt{A(t)} \sqrt{16\pi \left(1 - m(0) \sqrt{\frac{16\pi}{A(t)}}\right)}, \tag{59}$$

We can rewrite this as

$$\int |H| \, dA_t \leq \sqrt{\alpha A(t) + \beta \sqrt{A(t)}}. \tag{60}$$

Furthermore, since $A(t)$ grows exponentially in $t$, we can write this as

$$\int |H| \, dA_t \leq \sqrt{\alpha e^t + \beta e^{t/2}} = v(t), \tag{61}$$

where $A_0$ has been absorbed into $\alpha$ and $\beta$. Thus, our energy formula has become

$$E(\varphi) \leq \int_0^\infty (u'(t))^2 v(t) \, dt \tag{62}$$

with

$$v(t) = \sqrt{\alpha e^t + \beta e^{t/2}} \tag{63}$$

where $\alpha = 16\pi A_0$, $\beta = (16\pi)^{3/2} A_0^{1/2} |m_0|$ and $A_0$ is $A(\Sigma_0)$. This means we can pick our test function $u(t)$ to be as simple as

$$u(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & t \geq 1. \end{cases} \tag{64}$$

Then our integral becomes

$$E(\varphi) \leq \int_0^1 v(t) \, dt \tag{65}$$

$$\leq 2\sqrt{\alpha} + 2\sqrt{\beta}. \tag{66}$$
Since $m_{11}(\Sigma) \leq \sqrt{\frac{|\Sigma|}{16\pi}}$, $m_0$ is bounded above. By assumption $m_0$ is bounded below, so $\alpha$ and $\beta$ are bounded by multiples of $A_0$ and $\sqrt{A_0}$ respectively. Thus, $E(\varphi)$ goes to zero if $A_0 \to 0$ and $m_0$ is bounded. Hence, $C(p)$ must be zero since it is the infimum over a positive set with elements approaching zero.

**Theorem 4.12** (Capacity theorem). Let $M$ be an asymptotically flat 3-manifold with nonnegative scalar curvature, and ZAS $p$, such that there exists a family of surfaces, $\Sigma_i$, converging in $C^2$ to $p$. Then if the capacity of $p$ is nonzero, the Hawking masses of the surfaces $\Sigma_i$ must go to $-\infty$.

**Proof.** Any such family of surfaces will generate a family, $\{\Sigma'_i\}$, of minimizing hulls that will also converge to $p$. By theorem 4.11, the masses of $\{\Sigma'_i\}$ must go to $-\infty$. Thus, the masses of $\{\Sigma'_i\}$ must go to $-\infty$. Thus, for sufficiently large $i$, the masses of the minimizing hulls are all negative. From then on lemma 4.3 applies, and the masses of $\Sigma_i$ must be less than the masses of $\Sigma'_i$. Hence, they also converge to $-\infty$.

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