Dyson’s constants in the asymptotics of the determinants of Wiener-Hopf-Hankel operators with the sine kernel

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Abstract

Let $K^\pm_\alpha$ stand for the integral operators with the sine kernels $\frac{\sin(x-y) \pm \sin(x+y)}{\pi(x-y) \mp \pi(x+y)}$ acting on $L^2[0, \alpha]$. Dyson conjectured that the asymptotics of the Fredholm determinants of $I - K^\pm_\alpha$ are given by

$$\log \det(I - K^\pm_\alpha) = -\frac{\alpha^2}{4} + \frac{\alpha}{2} - \frac{\log \alpha}{8} + \frac{\log 2}{24} \pm \frac{\log 2}{4} + \frac{3}{2} \zeta'(-1) + o(1), \quad \alpha \to \infty.$$ 

In this paper we are going to give a proof of these two asymptotic formulas.

1 Introduction

In random matrix theory one is interested in the three Fredholm determinants

$$\det(I - K_\alpha), \quad \det(I - K^+_\alpha), \quad \det(I - K^-_\alpha),$$

where $K_\alpha$ is the integral operator on $L^2[0, \alpha]$ with the sine kernel

$$k(x, y) = \frac{\sin(x-y)}{\pi(x-y)}, \quad (1)$$

and $K^\pm_\alpha$ are the integral operators on $L^2[0, \alpha]$ with the Wiener-Hopf-Hankel sine kernels

$$k^\pm(x, y) = \frac{\sin(x-y)}{\pi(x-y)} \pm \frac{\sin(x+y)}{\pi(x+y)}, \quad (2)$$

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These determinants are related to the probabilities $E_\beta(n; \alpha)$ that in the bulk scaling limit of the three classical Gaussian ensembles of random matrices an interval of length $\alpha$ contains precisely $n$ eigenvalues. It is customary to associate the parameter $\beta = 2$ with the Gaussian Unitary Ensemble, $\beta = 1$ with the Gaussian Orthogonal Ensemble, and $\beta = 4$ with the Gaussian Symplectic Ensemble. The basic relationship between these probabilities and the Fredholm determinants is given by

\[ E_2(0; \alpha) = \det(I - K_\alpha), \quad E_1(0; \alpha) = \det(I - K_\alpha^+), \]

and

\[ E_4(0; \alpha) = \frac{1}{2} \left( \det(I - K_{2\alpha}^+ + \det(I - K_{2\alpha}^-) \right), \]

while expressions for $E_\beta(n; \alpha)$ with $n \geq 1$ also exist \[15, 4\].

A problem which has been open for a long time was the rigorous derivation of the asymptotics of these determinants as $\alpha \to \infty$. Dyson \[9\] was able to give a heuristic derivation and conjectured that

\[ \log \det(I - K_{2\alpha}) = -\frac{\alpha^2}{4} - \frac{\log \alpha}{2} + \frac{\log 2}{12} + 3\zeta'(-1) + o(1), \quad \alpha \to \infty, \quad (3) \]

where $\zeta$ stands for the Riemann zeta function. It is known \[15\] that

\[ \det(I - K_{2\alpha}^+) = \prod_{n=0}^{\infty} (1 - \lambda_{2n}(\alpha)), \quad \det(I - K_{2\alpha}^-) = \prod_{n=0}^{\infty} (1 - \lambda_{2n+1}(\alpha)), \quad (4) \]

where $\lambda_n(\alpha)$ are the decreasingly ordered eigenvalues of the operator $K_{2\alpha}$. Using \[3\] and a non-rigorous derivation of the asymptotics of the quotient

\[ \frac{\det(I - K_{2\alpha}^+)}{\det(I - K_{2\alpha}^-)} = \prod_{n=0}^{\infty} \frac{1 - \lambda_{2n}(\alpha)}{1 - \lambda_{2n+1}(\alpha)}, \quad (5) \]

which was given by des Cloiseaux and Mehta \[8\], Dyson obtained the asymptotics formulas

\[ \log \det(I - K_{2\alpha}^+) = -\frac{\alpha^2}{4} + \frac{\alpha}{2} - \frac{\log \alpha}{8} + \frac{\log 2}{24} \pm \frac{\log 2}{4} + \frac{3}{2} \zeta'(-1) + o(1), \quad \alpha \to \infty. \quad (6) \]

Recently the asymptotic formula \(3\) was proved independently by Krasovsky \[14\] and the author \[10\] using different methods. Yet another proof was given by Deift, Its, Krasovsky, and Zhou \[6\]. The proofs \[14, 6\] are based on the Riemann-Hilbert method, while the proof \[10\] is based on determinant identities and the asymptotics of Wiener-Hopf-Hankel determinants with certain Fisher-Hartwig symbols \[3\].

The goal of this paper is to give a proof of \(6\). In contrast to Dyson’s derivation we will not rely on \(3\) and \(5\). In fact, we will use methods similar to those of \[10\]. As a consequence
of (4), the asymptotic formulas (6) then imply the asymptotic formula (3). Hence the results
of the present paper give a fourth derivation of (3).

As was pointed out to the author by A. Its, another proof of (6), which is based on the
Riemann-Hilbert method, can very likely be accomplished. It would rely on (3) and (4) and
involve a (rigorous) derivation of the asymptotics of (5) based on observations made in [7,
p. 205/206].

Let us conclude this introduction with some remarks on what else is known about the
Fredholm determinants under consideration. It was shown by Jimbo, Miwa, Môri, and Sato
[13] (see also [17]) that the function

\[ \sigma(\alpha) = \alpha \frac{d}{d\alpha} \log \det(I - K_\alpha) \]

satisfies a Painlevé V equation. Widom [20, 21] was able to identify the highest term in
the asymptotics of \( \sigma(\alpha) \) as \( \alpha \to \infty \). Knowing these asymptotics one can derive a complete
asymptotic expansion for \( \sigma(\alpha) \). By integration it follows that the asymptotics of \( \det(I - K_{2\alpha}) \)
are given by

\[ \log \det(I - K_{2\alpha}) = -\frac{\alpha^2}{2} \log \alpha + \sum_{n=1}^{N} \frac{C_{2n}}{\alpha^{2n}} + O(\alpha^{2N+2}), \quad \alpha \to \infty, \quad (7) \]

with constants \( C_{2n} \) that can be computed recursively. However, the constant \( C \) cannot be
obtained in this way, and its rigorous identification was done - as mentioned above - only in
[14, 10, 6]. The asymptotic formula (7) was obtained in [7] as well; also, in the earlier work
by B. Suleimanov [16] a rigorous derivation of the leading term of the asymptotics of the
derivative of \( \sigma(\alpha) \) was obtained.

In a similar way, it turns out that the functions

\[ \sigma_{\pm}(\alpha) = \alpha \frac{d}{d\alpha} \log \det(I - K_{\alpha}^\pm) \]

satisfy a Painlevé III equation [18, 19]. Moreover, the operators \( K_{\alpha}^\pm \) are related to special
cases of integral operators \( K_{\nu,\alpha} \) on \( L^2([0,\alpha]) \) with Bessel kernel,

\[ k_{\nu}(x, y) = \frac{J_{\nu}(\sqrt{x})\sqrt{y}J'_{\nu}(\sqrt{y}) - \sqrt{x}J'_{\nu}(\sqrt{x})J_{\nu}(\sqrt{y})}{2(x - y)}, \quad \nu > -1. \]

In fact, \( \det(I - K_{\alpha}^{\pm}) = \det(I - K_{\pm 1/2, \alpha^2}) \). In the Bessel case, functions defined similarly to
\( \sigma_{\pm}(x) \) satisfy also a Painlevé III equation. The determinants \( \det(I - K_{\nu,\alpha}) \) are the probabilities
that no eigenvalues lie in an interval of length \( \alpha \) for the Laguerre or Jacobi random
matrix ensembles in the hard edge scaling limit.
It is also interesting to observe that the following identity between \( \det(I - K^\pm) \) and \( \det(I - K^\alpha) \) exists (see, e.g., [17]):

\[
\log \det(I - K^\pm) = \frac{1}{2} \log \det(I - K_{2\alpha}) \mp \frac{1}{2} \int_0^\alpha \sqrt{-\frac{d^2}{dx^2} \log \det(I - K_{2x})} \, dx \tag{8}
\]

Using this formula it is possible to derive from (7) a complete asymptotic expansion for \( \log \det(I - K^\pm) \) at infinity with the exception of the constant, which remains undetermined due to the integration. Thus, once (7) had been proved, the only open problem was to identify the constant terms in (3) and (6).

Let us shortly outline how the paper is organized. In the following section we will fix the basic notation and make some additional comments about the idea of the proof. We will follow essentially the same lines as in [10]. In fact, the proof is even somewhat simpler since some technical results are not needed here (namely, Prop. 4.2 and Prop. 4.9 of [10]). The auxiliary results which are needed here are either the same as or analogous to those of [10]. In Section 3 we will prove a formula involving Hankel determinants and in Section 4 we will finally prove the asymptotic formula (6).

## 2 Basic notation and some remarks

We start with introducing some notation. We will denote the real line by \( \mathbb{R} \), the positive real half-axis by \( \mathbb{R}_+ \), and the complex unit circle by \( \mathbb{T} \). By \( L^p(M) \) we will denote the Lebesgue spaces \( (1 \leq p \leq \infty) \), where in our cases \( M \) is any of the above sets or a finite subinterval of \( \mathbb{R} \).

The \( n \times n \) Toeplitz and Hankel matrices are defined by

\[
T_n(a) = (a_{j-k})_{j,k=0}^{n-1}, \quad H_n(a) = (a_{j+k+1})_{j,k=0}^{n-1},
\]

where \( a \in L^1(\mathbb{T}) \) and

\[
a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta})e^{-ik\theta} \, d\theta, \quad k \in \mathbb{Z},
\]

are its Fourier coefficients. We will also need differently defined \( n \times n \) Hankel matrices

\[
H_n[b] = (b_{j+k+1})_{j,k=0}^{n-1},
\]

where the numbers \( b_k \) are the (scaled) moments of a function \( b \in L^1[-1, 1] \), i.e.,

\[
b_k = \frac{1}{\pi} \int_{-1}^1 b(x)(2x)^{k-1} \, dx, \quad k \geq 1.
\]
For $a \in L^{\infty}(\mathbb{T})$ the Toeplitz and Hankel operators are bounded linear operators acting on the Hardy space

$$H^2(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) : f_k = 0 \text{ for all } k < 0 \right\}$$

by

$$T(a) = PM(a)P|_{H^2(\mathbb{T})}, \quad H(a) = PM(a)JP|_{H^2(\mathbb{T})},$$

(11) where $P : \sum_{k=-\infty}^{\infty} f_k t^k \mapsto \sum_{k=0}^{\infty} f_k t^k$ stands for the Riesz projection, $J : f(t) \mapsto t^{-1} f(t^{-1})$ stands for a flip operator, and $M(a) : f(t) \mapsto a(t)f(t)$ stands for the multiplication operator. (These last three operators are acting on $L^2(\mathbb{T})$.) Finally, introduce the projections

$$P_n : \sum_{k\geq 0} f_k t^k \in H^2(\mathbb{T}) \mapsto \sum_{k=0}^{n-1} f_k t^k \in H^2(\mathbb{T}),$$

(12)

the image of which can be naturally identified with $\mathbb{C}^n$. Using this we can make the identifications $P_n T(a) P_n \cong T_n(a)$, $P_n H(a) P_n \cong H_n(a)$.

We will also need the notion of a trace class operator acting on a Hilbert space $H$. This is a compact operator $A$ such that the series of its singular $s_n(A)$ (i.e., the eigenvalues of $(A^*A)^{1/2}$ counted according to their algebraic multiplicities) converges. The class of all trace class operators can be made to a Banach space by introducing the norm

$$\|A\|_1 = \sum_{n \geq 1} s_n(A).$$

(13)

This class is also a two-sided ideal in the algebra of all bounded linear operators on $H$. The importance of trace class operators is that for such operators $A$, the operator trace “trace($A$)” and the operator determinant “det($I + A$)” can be defined as generalizations of matrix trace and matrix determinant. More detailed information on this subject can be found, e.g., in [12].

For $a \in L^{\infty}(\mathbb{R})$, let $M_\mathbb{R}(a) : f(x) \mapsto a(x)f(x)$ stand for the multiplication operator acting on $L^2(\mathbb{R})$. The convolution operator $W_0(a)$ (or, “two-sided” Wiener-Hopf operator) is defined by

$$W_0(a) = \mathcal{F}M_\mathbb{R}(a)\mathcal{F}^{-1},$$

where $\mathcal{F}$ stands for the Fourier transform on $L^2(\mathbb{R})$. The continuous analogues of Toeplitz and Hankel operators are operators defined

$$W(a) = \Pi_+ W_0(a)\Pi_+|_{L^2(\mathbb{R}_+)}; \quad H_\mathbb{R}(a) = \Pi_+ W_0(a)\hat{J}\Pi_+|_{L^2(\mathbb{R}_+)},$$

(14) (15)

where $(\hat{J}f)(x) = f(-x)$, and $\Pi_+ = M_\mathbb{R}(\chi_{\mathbb{R}_+})$ is the projection operator on the positive real half axis. The operator $W(a)$ is usually called a Wiener-Hopf operator, and we will refer
to $H_R(a)$ as a Hankel operator, too. (The notation will avoid a possible confusion between $H_R(a)$ and $H(a).$) One can show that if $a \in L^1(\mathbb{R})$, then $W(a)$ and $H_R(a)$ are integral operators on $L^2(\mathbb{R})$ with the kernel $\hat{a}(x-y)$ and $\hat{a}(x+y)$, respectively, where

$$\hat{a}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} a(x) \, dx$$

stands for the Fourier transform of $a$. For $\alpha > 0$ we will define the projection operator

$$\Pi_\alpha : f(t) \in L^2(\mathbb{R}_+) \mapsto \chi_{[0,\alpha]}(x) f(x) \in L^2(\mathbb{R}_+).$$

The image of this operator can be identified with $L^2(0,\alpha]$.

With this notation the integral operators $K_\alpha$ and $K_\alpha^\pm$ can now be seen to be truncated Wiener-Hopf and Wiener-Hopf-Hankel operators,

$$K_\alpha = \Pi_\alpha W(\chi)\Pi_\alpha |_{L^2[0,\alpha]} , \quad K_\alpha^\pm = \Pi_\alpha (W(\chi) \pm H_R(\chi))\Pi_\alpha |_{L^2[0,\alpha]} ,$$

where $\chi$ stands for the characteristic function of the interval $[-1,1]$. Notice that $K_\alpha$ and $K_\alpha^\pm$ are trace class operators and that

$$\det(I - K_\alpha) = \det \left[ \Pi_\alpha W(1-\chi)\Pi_\alpha \right] , \quad (17)$$

$$\det(I - K_\alpha^\pm) = \det \left[ \Pi_\alpha \left(W(1-\chi) \pm H_R(1-\chi)\right)\Pi_\alpha \right] . \quad (18)$$

For determinants of Wiener-Hopf operators (and, more recently, also for determinants of Wiener-Hopf-Hankel operators) results describing the asymptotics as $\alpha \to \infty$ exist under the condition that the underlying symbol is sufficiently well behaved. These results are known as Achiezer-Kac formulas (if the symbol has no singularities) and as Fisher-Hartwig type formulas (if the symbol has a finite number of certain types of singularities). An overview about this topic can be found in [5]. In our case the symbol is the characteristic function $1-\chi$ vanishing on the interval $[-1,1]$, a state of affairs which is not covered by the just mentioned cases and which renders the situation completely non-trivial.

The main idea of the proof given in this paper is to relate the Fredholm determinants $\det(I - K_\alpha^\pm)$ to the determinants of different operators for which Fisher-Hartwig type formulas can be applied. Let us introduce the functions

$$\hat{u}_\beta(x) = \left( \frac{x - i}{x + i} \right)^\beta , \quad \hat{v}_\beta(x) = \left( \frac{x^2}{1 + x^2} \right)^\beta , \quad x \in \mathbb{R}, \ \beta \in \mathbb{C} ,$$

where these functions are supposed to be continuous on $\mathbb{R} \setminus \{0\}$ and to have their values approaching 1 as $x \to \pm \infty$. Then we are going to prove that

$$\det(I - K_\alpha^\pm) = \exp \left( -\frac{\alpha^2}{4} \mp \frac{\alpha}{2} \right) \det \left[ \Pi_\alpha (I \pm H_R(\hat{u}_{1/2}^{-1}))^{-1}\Pi_\alpha \right] . \quad (20)$$
It is now illuminating to point out that the determinants on the right hand side can be identified with determinants of truncated Wiener-Hopf-Hankel operators with Fisher-Hartwig symbols. In fact, it is proved in [3] that
\[
\det \left[ \Pi_\alpha(W(\hat{v}_\beta) + H_R(\hat{u}_\beta))\Pi_\alpha \right] = e^{-\alpha \beta} \det \left[ \Pi_\alpha(I + H_R(\hat{u}_{-\beta}))^{-1}\Pi_\alpha \right] \quad \text{if} \quad -\frac{1}{2} < \text{Re} \beta < \frac{3}{2},
\]
\[
\det \left[ \Pi_\alpha(W(\hat{v}_\beta) - H_R(\hat{u}_\beta))\Pi_\alpha \right] = e^{-\alpha \beta} \det \left[ \Pi_\alpha(I - H_R(\hat{u}_{-\beta}))^{-1}\Pi_\alpha \right] \quad \text{if} \quad -\frac{1}{2} < \text{Re} \beta < \frac{1}{2}.
\]
However, we will avoid making use of these formulas for two reasons. First of all, the determinants on the right hand side of (20) are those occurring primarily in the proof, and their asymptotics are computed also in [3]. Secondly, the left hand side of the last formula is, as it stands, not defined for \( \beta = -1/2 \). (It can be defined by analytic continuation in \( \beta \) because the right hand side makes sense for \(-3/2 < \text{Re} \beta < 1/2\).)

### 3 A Hankel determinant formula

In this section we are going to prove two formulas of the kind
\[
\det H_n[b] = G^n \det \left[ P_n(I + H(\psi))^{-1}P_n \right],
\]
where \( b \in L^1[-1, 1] \) is a (sufficiently smooth) continuous and nonvanishing function on \([-1, 1]\) multiplied in one case with the function \((1 + x)^{1/2}\) and in another case with \((1 - x)^{-1/2}\). The function \( \psi \) and the constant \( G \) depend on \( b \). A formula of the same type was proved in [10]. However, the conditions on the function \( b \) and the form of the function \( \psi \) were different.

Before we state the result we have to introduce more notation. Let \( \mathcal{W} \) stand for the Wiener algebra, i.e., the set of all functions in \( L^1(\mathbb{T}) \) whose Fourier series are absolutely convergent. Moreover, let
\[
\mathcal{W}_\pm = \{ a \in \mathcal{W} : a_n = 0 \text{ for all } \pm n < 0 \},
\]
be two Banach subalgebras of \( \mathcal{W} \), where \( a_n \) stand for the Fourier coefficients of \( a \). Notice that \( a \in \mathcal{W}_+ \) if and only if \( \tilde{a} \in \mathcal{W}_- \), where \( \tilde{a}(t) := a(t^{-1}) \), \( t \in \mathbb{T} \). Finally, we denote by \( GW \) and \( GW_\pm \) the group of invertible elements in the Banach algebras \( \mathcal{W} \) and \( \mathcal{W}_\pm \), respectively.

A function \( a \in \mathcal{W} \) is said to admit a canonical Wiener-Hopf factorization in \( \mathcal{W} \) if it can be written in the form
\[
a(t) = a_-(t)a_+(t), \quad t \in \mathbb{T},
\]
where \( a_\pm \in GW_\pm \). It is easy to see that \( a \in \mathcal{W} \) admits a canonical Wiener-Hopf factorization in \( \mathcal{W} \) if and only if \( a \in GW \) and if the winding number of \( a \) is zero (see, e.g., [5]). Moreover,
this condition is equivalent to the existence of a logarithm \( \log a \) which belongs to \( \mathcal{W} \). If this is fulfilled, then one can unambiguously define the geometric mean of \( a \) by

\[
G[a] := \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log a(e^{i\theta}) \, d\theta \right). \tag{23}
\]

The following result (which is not yet what we ultimately want) is cited from [10, Thm. 4.5]. The invertibility statement is taken from [10, Prop. 4.3]. Recall that a function \( a \) on \( \mathbb{T} \) is called even if \( \tilde{a} = a \), where \( \tilde{a}(t) := a(t^{-1}) \).

**Theorem 3.1** Let \( a \in G\mathcal{W} \) be an even function which possesses a canonical Wiener-Hopf factorization \( a(t) = a_{-}(t)a_{+}(t) \). Define \( \psi(t) = \tilde{a}_{+}(t)a_{+}^{-1}(t) \), and let \( b \in L^{1}(\mathbb{T}) \) be

\[
b(\cos \theta) = a(e^{i\theta}) \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}}. \tag{24}
\]

Then \( I + H(\psi) \) is invertible on \( H^{2}(\mathbb{T}) \) and

\[
\det H_{n}[b] = G[a]^{n} \det \left[ P_{n}(I + H(\psi))^{-1}P_{n} \right]. \tag{25}
\]

In order to be able to state the desired result we introduce (for \( \tau \in \mathbb{T} \) and \( \beta \in \mathbb{C} \)) the functions

\[
u_{\beta,\tau}(e^{i\theta}) = \exp(i\beta(\theta - \theta_{0} - \pi)), \quad 0 < \theta - \theta_{0} < 2\pi, \quad \tau = e^{i\theta_{0}}. \tag{26}
\]

These functions are continuous on \( \mathbb{T} \setminus \{\tau\} \) and have a jump discontinuity at \( t = \tau \) whose size is determined by \( \beta \).

The promised formulas are now given in the following theorem. Notice that the difference between Theorem 3.1 and Theorem 3.2 (as well as to Thm. 4.6 of [10]) is in the conditions on the underlying functions.

**Theorem 3.2** Let \( c \in G\mathcal{W} \) be an even function which possesses a canonical Wiener-Hopf factorization \( c(t) = c_{-}(t)c_{+}(t) \). Define \( b^{+}, b^{-} \in L^{1}[-1, 1] \) and \( \psi^{+}, \psi^{-} \in L^{\infty}(\mathbb{T}) \) by

\[
b^{+}(\cos \theta) = c(e^{i\theta}) \sqrt{2 + 2\cos \theta}, \quad \psi^{+}(e^{i\theta}) = \tilde{c}_{+}(e^{i\theta})c_{+}^{-1}(e^{i\theta})u_{-1/2,1}(e^{i\theta}),
\]

\[
b^{-}(\cos \theta) = \frac{c(e^{i\theta})}{\sqrt{2 - 2\cos \theta}}, \quad \psi^{-}(e^{i\theta}) = \tilde{c}_{+}(e^{i\theta})c_{+}^{-1}(e^{i\theta})u_{1/2,-1}(e^{i\theta}).
\]

Then the operators \( I + H(\psi^{\pm}) \) are invertible on \( H^{2}(\mathbb{T}) \) and

\[
\det H_{n}[b^{\pm}] = G[c]^{n} \det \left[ P_{n}(I + H(\psi^{\pm}))^{-1}P_{n} \right]. \tag{27}
\]
For the proof of this theorem we will apply some auxiliary results, which are stated in [10] in connection with Thm. 4.6 and which we are not going to restate here. However, we will recall the following notation, which is used here and later on. For $r \in [0, 1)$ and $\tau \in \mathbb{T}$ let $G_{r, \tau}$ be the following operator acting on $L^\infty(\mathbb{T})$:

$$G_{r, \tau} : a(t) \mapsto b(t) = a \left( \frac{t + r}{1 + rt} \right) \quad (28)$$

Proof of Theorem 3.2. The first problem is to verify the invertibility of $I + H(\psi^\pm)$. In the special case $c_+ \equiv 1$, i.e., for $I + H(u_{-1/2, 1})$ and $I + H(u_{1/2, -1})$, this was done in [3, Thm. 3.6]. (Notice that $I + H(u_{1/2, -1})$ is similar to $I - H(u_{1/2, 1})$.) The proof in the case where $c_+ \not\equiv 1$ can be done in the same way as in [3 Sec. 3.2] or [10, Prop. 4.2]. We refrain from copying the proof with the little modifications necessary and make instead only the following remarks.

The proof in [3] consists of two parts. First one determines the essential spectrum of the Hankel operators. Since $\psi^\pm$ have their discontinuities at the same places as $u_{\pm 1/2, \pm 1}$ and the one-sided limits there are also the same, the essential spectrum of $H(\psi^\pm)$ is the same as that of $H(u_{\pm 1/2, \pm 1})$. The second step is to determine the kernel of the operators $I + H(u_{\pm 1/2, \pm 1})$. (Passing to the adjoints, gives similarly information about the cokernel.) The crucial point in [3] is to write, e.g., $u_{\beta, 1} = \xi_{-\beta} \eta_{\beta}$ with $\xi_{-\beta} = (1 - t^{-1})^{-\beta}$, $\eta_{\beta}(t) = (1 - t)^{\beta}$, $t \in \mathbb{T}$. What one uses about these functions are the facts that $\xi_{-\beta}(t) = 1/\eta_{\beta}(t^{-1})$, that they and their inverses belong to certain Hardy spaces. In our case, one has to write, e.g., $\psi^+ = (\tilde{c}_+ \xi_{1/2}) \cdot (c_{-1} \eta_{-1/2})$. The factors in this product have the just mentioned properties, too. Hence the proof works in the same way.

The proof of (27) will be carried out by an approximation argument and with the help of Theorem 3.1. For $r \in [0, 1)$ consider the even functions

$$a_r^\pm(t) = c(t) \left( (1 \mp rt)(1 \mp rt^{-1}) \right)^{\pm 1/2}, \quad t \in \mathbb{T}.$$  

Clearly, $a_r^\pm \in GW$. The functions $b_r^\pm$ defined in terms of $a_r^\pm$ by formula (24) evaluate to

$$b_r^\pm(x) = b^\pm(x) \left( 1 \mp 2x \mp 2r x \right)^{\mp 1/2} \left( 1 + r^2 \mp 2r x \right)^{\pm 1/2} \sqrt{1 + x \over 1 - x}, \quad x \in (-1, 1).$$

Then the functions $b_r^\pm$ converge to $b^\pm$ in the norm of $L^1[-1, 1]$. Hence, if we fix $n$,

$$\det H_n[b^\pm] = \lim_{r \to 1} \det H_n[b_r^\pm].$$

It is now easily seen that the canonical Wiener-Hopf factorization of $a_r^\pm$ is given by $a_r^\pm(t) = a_{r,+}^\pm(t) a_{r,-}^\pm(t)$ with the factors

$$a_{r,+}^\pm(t) = c_+(t)(1 \mp rt)^{\pm 1/2}, \quad a_{r,-}^\pm(t) = c_-(t)(1 \mp rt^{-1})^{\pm 1/2}.$$
Notice also that $G[a_r] = G[c]$. If we define

$$\psi_r^\pm(t) := \tilde{a}_{r,+}^\pm(t)(a_{r,+}^\pm(t))^{-1} = \tilde{c}_+^1(t) c_+^{-1}(t) \left( \frac{1 + rt}{1 + rt^{-1}} \right)^{\pm 1/2},$$

we can apply Theorem 3.1 and conclude that

$$\det H_n[b_r^\pm] = G[c]^n \det \left[ P_n(I + H(\psi_r^\pm))^{-1} P_n \right].$$

It follows that

$$\det H_n[b_r^\pm] = G[c]^n \lim_{r \to 1} \det \left[ P_n(I + H(\psi_r^\pm))^{-1} P_n \right].$$

Next define

$$f_r^\pm(t) := \left( \frac{1 + rt}{1 + rt^{-1}} \right)^{\pm 1/2}$$

and observe that $f_r^\pm \to u_{1/2, \pm 1}$ in measure as $r \to 1$. Hence also $\psi_r^\pm \to \psi^\pm$ in measure. Because the sequence $\psi_r^\pm$ is bounded in the $L^\infty$-norm it follows that $H(\psi_r^\pm)$ converges strongly to $H(\psi^\pm)$ on $H^2(\mathbb{T})$ (see, e.g., Lemma 4.7 of [10]). In order to obtain that

$$(I + H(\psi_r^\pm))^{-1} \to (I + H(\psi^\pm))^{-1}$$

strongly on $H^2(\mathbb{T})$, it is necessary and sufficient that the following stability condition,

$$\sup_{r \in [r_0, 1)} \| (I + H(\psi_r^\pm))^{-1} \| < \infty,$$

is satisfied (see, e.g., Lemma 4.8 of [10]). Here $r_0$ is some number in $[0, 1)$.

Stability criteria for such a type of operator sequences were established in [11] (see Sections 4.1, 4.2, and 5.2 therein), and we are going to apply the corresponding results. First of all, there exist certain mappings $\Phi_0$ and $\Phi_\tau$, $\tau \in \mathbb{T}$, which are defined by

$$\Phi_0[\psi_r] := \mu^- \lim_{r \to 1} \psi_r, \quad \Phi_0[\psi_r] := \mu^- \lim_{r \to 1} G_{r, \tau} \psi_r.$$

Here $\mu^-$-lim stands for the limit in measure. It is now easy to see that these mappings evaluate as follows,

$$\Phi_0[f_r^\pm] = u_{1/2, \pm 1}, \quad \Phi_\tau[f_r^\pm] = u_{1/2, \pm 1}(\tau),$$

if $\tau \neq \pm 1$, and

$$\Phi_{\pm 1}[f_r^\pm] = \mu^- \lim_{r \to 1} G_{r, \pm 1} f_r^\pm = \mu^- \lim_{r \to 1} \left( \frac{1 + rt}{1 + rt^{-1}} \right)^{\pm 1/2} = u_{1/2, -1}$$
if \( \tau = \pm 1 \). Because of \( \psi_r^{\pm} = \bar{c}_+ c_-^{-1} f_r^{\pm} \) it follows immediately that

\[
\Phi_0[\psi_r^\pm] = \bar{c}_+ c_-^{-1} u_{\mp 1/2, \pm 1},
\]

\[
\Phi_{\pm 1}[\psi_r^\pm] = u_{\pm 1/2, -1},
\]

\[
\Phi_r[\psi_r^\pm] = \text{constant function}, \quad \tau \in \mathbb{T} \setminus \{\pm 1\}.
\]

The stability criterion in [11] (Thm. 4.2 and Thm. 4.3) says that \( I + H(\psi_r^\pm) \) is stable if and only if each of the following operators is invertible:

(i) \( \Psi_0[I + H(\psi_r^\pm)] = I + H(\Phi_0[\psi_r^\pm]) = I + H(\psi^\pm) \),

(ii) \( \Psi_{\pm 1}[I + H(\psi_r^\pm)] = I \pm H(\Phi_{\pm 1}[\psi_r^\pm]) = I \pm H(u_{\pm 1/2, -1}) \),

(iii) \( \Psi_\mp[I + H(\psi_r^\pm)] = I \mp H(\Phi_\mp[\psi_r^\pm]) = I \),

(iv) \( \Psi_\tau[I + H(\psi_r^\pm)] =
\]

\[
\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
\]

\[
+ 
\begin{pmatrix}
P & 0 \\
0 & Q
\end{pmatrix}
\]

\[
\begin{pmatrix}
M(\Phi_r[\psi_r^\pm]) & 0 \\
0 & M(\Phi_r[\psi_r^\pm])
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & I \\
0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
P & 0 \\
0 & Q
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
\]

(\( \tau \in \mathbb{T}, \text{Im}(\tau) > 0 \))

The invertibility is obvious for (iii) and (iv). As to (i) and (ii) the invertibility has been stated at the beginning of the proof. Notice that \( I \pm H(U_{\pm 1/2, -1}) \) is similar to \( I \mp H(u_{\pm 1/2, -1}) \).

We can thus conclude that the sequence \( I + H(\psi_r^\pm) \) is stable and the strong convergence (29) follows. Hence the matrices \( P_n(I + H(\psi_r^\pm))^{-1} P_n \) converge to \( P_n(I + H(\psi^\pm))^{-1} P_n \) as \( r \to 1 \). This implies that their determinants also converge and proves the assertion. \( \square \)

4 Proof of the asymptotic formula

In order to prove the asymptotic formula (3), we are going to discretize the underlying Wiener-Hopf-plus-Hankel operators \( I - K_\alpha^\pm \). This will give us Toeplitz-plus-Hankel operators. Let \( \chi_\alpha \) denote the characteristic function of the subarc \( \{e^{i\theta} : \alpha < \theta < 2\pi - \alpha\} \) of \( \mathbb{T} \).

Proposition 4.1 For each \( \alpha > 0 \) we have

\[
\det(I - K_\alpha^\pm) = \lim_{n \to \infty} \det \left[ T_n(\chi_\alpha^\pm) \pm H_n(\chi_\alpha^\pm) \right].
\] (30)
Proof. The operator $K^\pm_\alpha$ is the integral operator on $L^2[0,\alpha]$ with the kernel $K(x - y) \pm K(x + y)$, where $K(x) = \frac{\sin x}{\pi x}$. Consider the $n \times n$ matrices

$$
A^\pm_n = \left[ \frac{\alpha}{n} K \left( \frac{(j - k)}{n} \right) \pm \frac{\alpha}{n} K \left( \frac{(j + k - 1)}{n} \right) \right]_{j,k=0}^{n-1},
$$

$$
B^\pm_n = \left[ \frac{\alpha}{n} \int_0^1 \int_0^1 \left\{ K \left( \frac{(j - k + \xi - \eta)}{n} \right) \pm K \left( \frac{(j + k + \xi + \eta)}{n} \right) \right\} d\xi d\eta \right]_{j,k=0}^{n-1}.
$$

The entries of $A^\pm_n - B^\pm_n$ can be estimated uniformly by $O(n^{-2})$ using the mean value theorem. Hence the Hilbert-Schmidt norm of $A^\pm_n - B^\pm_n$ is $O(n^{-1})$, and the trace norm is $O(1/\sqrt{n})$.

The rest of the proof can be completed in the same way as in [10, Prop. 5.1] by showing that $\det(I - K^\pm_\alpha) = \det(I_n - B^\pm_n)$ and $\det(I - A^\pm_n) = \det \left[ T_n(\chi_{\alpha,n}) \pm H_n(\chi_{\alpha,n}) \right]$. \hfill \Box

After discretizing, the next goal is to reduce the Toeplitz-plus-Hankel determinants to Hankel determinants. For this purpose we use an exact identity which is stated in the following result cited from [2, Thm. 2.3].

**Proposition 4.2** Let $a \in L^1(\mathbb{T})$ be an even function, and let $b \in L^1[-1,1]$ be given by

$$
b(\cos \theta) = a(e^{i\theta}) \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}}.
$$

Then $\det \left[ T_n(a) + H_n(a) \right] = \det H_n[b]$. 

Notice that the assumption $b \in L^1[-1,1]$ implies that $a \in L^1(\mathbb{T})$. Applying the previous result yields the following.

**Proposition 4.3** For each $\alpha > 0$ and $n \in \mathbb{N}$ we have

$$
\det \left[ T_n(\chi_{\alpha,n}) \pm H_n(\chi_{\alpha,n}) \right] = (\mu_{\alpha,n})^{\pm n/2} \left( \frac{\varrho_{\alpha,n} + 1}{2} \right)^{n^2} \det H_n[b^\pm_{\alpha,n}],
$$

where

$$
b^+_{\alpha,n}(x) = \sqrt{\frac{2 + 2x}{1 + \mu^2_{\alpha,n} - 2\mu_{\alpha,n}x}}, \quad b^-_{\alpha,n}(x) = \sqrt{\frac{1 + \mu^2_{\alpha,n} + 2\mu_{\alpha,n}x}{2 - 2x}},
$$

and $\varrho_{\alpha,n}$ and $\mu_{\alpha,n}$ are numbers (unambiguously) defined by

$$
\varrho_{\alpha,n} = \cos \left( \frac{\alpha}{n} \right), \quad \frac{1 + \mu^2_{\alpha,n}}{2\mu_{\alpha,n}} = \frac{3 - \varrho_{\alpha,n}}{1 + \varrho_{\alpha,n}}, \quad 0 < \mu_{\alpha,n} < 1.
$$
Proof. In the plus-case, we apply Proposition 4.2 with
\[ a(e^{i\theta}) = \chi_{\alpha}^\omega(e^{i\theta}), \quad b(x) = \hat{b}_{\alpha,n}^+(x) := \chi_{[-1,\alpha_n]}(x) \sqrt{\frac{1+x}{1-x}}. \]
In the minus-case, we apply this proposition with
\[ a(e^{i\theta}) = \chi_{\alpha}^\omega(-e^{i\theta}), \quad b(x) = \hat{b}_{\alpha,n}^-(x) := \chi_{[-\alpha_n,1]}(x) \sqrt{\frac{1+x}{1-x}}. \]
Hence we obtain (by using the general formula \( \det(T_n(f) + H_n(f)) = \det(T_n(\hat{f}) - H_n(\hat{f})) \)) with \( \hat{f}(t) = f(-t) \) in the minus-case)
\[ \det \left[ T_n(\chi_{\alpha}^\omega) \pm H_n(\chi_{\alpha}^\omega) \right] = \det H_n[\hat{b}_{\alpha,n}^\pm]. \]
The entries of \( H_n[\hat{b}_{\alpha,n}^\pm] \) are the moments \( [\hat{b}_{\alpha,n}^\pm]_{1+j+k}, 0 \leq j, k \leq n - 1 \). Computing them yields
\[
[\hat{b}_{\alpha,n}^+]_k = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} (2x)^{k-1} dx \\
= \frac{1}{\pi} \left( \frac{\alpha_n+1}{2} \right)^k \int_{-1}^{1} \sqrt{\frac{1+y}{1+y \frac{1+\alpha_n}{\alpha_n}}} \left( 2y - \frac{1-\alpha_n}{1+\alpha_n} \right)^{k-1} dy \\
= \frac{\sqrt{\mu_{\alpha,n}}}{\pi} \left( \frac{\alpha_n+1}{2} \right)^k \int_{-1}^{1} b_{\alpha,n}^+(y) \left( 2y - \frac{1-\alpha_n}{1+\alpha_n} \right)^{k-1} dy
\]
and
\[
[\hat{b}_{\alpha,n}^-]_k = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} (2x)^{k-1} dx \\
= \frac{1}{\pi} \left( \frac{\alpha_n+1}{2} \right)^k \int_{-1}^{1} \sqrt{\frac{1+y}{1+y \frac{1+\alpha_n}{\alpha_n}}} \left( 2y + \frac{1-\alpha_n}{1+\alpha_n} \right)^{k-1} dy \\
= \frac{1}{\pi \sqrt{\mu_{\alpha,n}}} \left( \frac{\alpha_n+1}{2} \right)^k \int_{-1}^{1} b_{\alpha,n}^-(y) \left( 2y + \frac{1-\alpha_n}{1+\alpha_n} \right)^{k-1} dy.
\]
Hence
\[
H_n[\hat{b}_{\alpha,n}^\pm] = \left[ (\mu_{\alpha,n})^{1/2} \left( \frac{\alpha_n+1}{2} \right)^{j+k+1} \frac{1}{\pi} \int_{-1}^{1} b_{\alpha,n}^\pm(y) (2y \mp 2\alpha_n)^{j+k} dy \right]_{j,k=0}^{n-1}.
\]
with certain $\tau_{\alpha,n}$. One can pull out certain diagonal matrices from the left and the right, which give the terms $(\mu_{\alpha,n})^{\pm n/2}((1 + g_{\alpha,n})/2)^{n^2}$ after taking the determinant. The remaining matrix can be written as

$$\left[ \frac{1}{\pi} \int_{-1}^{1} b_{\alpha,n}^{\pm}(y)(2y \mp 2\tau_{\alpha,n})^j(2y \mp 2\tau_{\alpha,n})^k \, dy \right]_{j,k=0}^{n-1}.$$ 

After expanding $(2y \mp 2\tau_{\alpha,n})^j$ and $(2y \mp 2\tau_{\alpha,n})^k$ into two binomial series it is easily seen that the previous matrix is the matrix $H_n[b_{\alpha,n}^{\pm}]$ multiplied from the left and right with triangular matrices having ones on the diagonal. This implies the desired assertion. □

In the following result and also later on we use the functions

$$\psi_{\alpha,n}(t) := \left( \mp \frac{t \mp \mu_{\alpha,n}}{1 \pm \mu_{\alpha,n} t} \right)^{\mp 1/2} \quad (35)$$

with the sequence $\mu_{\alpha,n}$ defined by (34).

**Proposition 4.4** For each $\alpha > 0$ we have

$$\lim_{n \to \infty} \det \left[ T_n(\chi_{\frac{\alpha}{n}}) \pm T_n(\chi_{\frac{\alpha}{n}}) \right] = \exp \left( -\frac{\alpha^2}{8} \pm \frac{\alpha}{2} \right) \lim_{n \to \infty} \det \left[ P_n(I + H(\psi_{\alpha,n}^{\pm}))^{-1} P_n \right]. \quad (36)$$

Proof. The asymptotics (as $n \to \infty$) of the numbers appearing in (32) of Proposition 4.3 are given by

$$\frac{1 + g_{\alpha,n}}{2} = 1 - \frac{\alpha^2}{4n^2} + O(n^{-4}), \quad \mu_{\alpha,n} = 1 - \frac{\alpha}{n} + O(n^{-2}).$$

Hence using this proposition it follows that

$$\lim_{n \to \infty} \det \left[ T_n(\chi_{\frac{\alpha}{n}}) \pm H_n(\chi_{\frac{\alpha}{n}}) \right] = \exp \left( -\frac{\alpha^2}{8} \pm \frac{\alpha}{2} \right) \lim_{n \to \infty} \det H_n[b_{\alpha,n}^{\pm}].$$

Next we introduce

$$c(e^{i\theta}) = ((1 \mp \mu_{\alpha,n} t)(1 \mp \mu_{\alpha,n} t^{-1}))^{\mp 1/2},$$

and we are going to employ Theorem 3.2. It can be verified easily that $G[c] = 1$ and that $c(t) = c_{-}(t)c_{+}(t)$ is a canonical Wiener-Hopf factorization of $c$ where $c_{+}(t) = (1 \mp \mu_{\alpha,n} t)^{\mp 1/2}$ and $c_{-}(t) = (1 \mp \mu_{\alpha,n} t^{-1})^{\mp 1/2}$. Moreover,

$$\tilde{c}_{+}(t)c_{+}^{-1}(t) = \left( \frac{1 \mp \mu_{\alpha,n} t}{1 \pm \mu_{\alpha,n} t^{-1}} \right)^{\mp 1/2}.$$
The functions $b^\pm$ and $\psi^\pm$ defined in Theorem 3.2 now evaluate to

$$
\begin{align*}
   b^\pm(x) &= (1 + \mu_{\alpha,n}^2 + 2\mu_{\alpha,n} x)^{\pm 1/2}(2 \pm 2x)^{\pm 1/2} = b_{\alpha,n}^\pm(x), \\
   \psi^\pm(t) &= \left(\frac{1 \mp \mu_{\alpha,n} t}{1 + \mu_{\alpha,n} t^{-1}}\right)^{\pm 1/2}(\mp t)^{\mp 1/2} = \psi_{\alpha,n}^\pm(t).
\end{align*}
$$

Combining all this we obtain from Theorem 3.2 that

$$
\det H_n[b_{\alpha,n}^\pm] = \det \left[ P_n(I + H(\psi_{\alpha,n}^\pm))^{-1}P_n \right],
$$

which concludes the proof.

The next step is to identify the limit on the right hand side of (36). For this purpose we resort to an auxiliary result, which was stated in [10] (with a slight change of notation). In order to make the reference correct, we allow (for the time being) $\mu_{\alpha,n} \in [0,1)$ to be an arbitrary sequence and define the functions

$$
h_{\alpha}(t) = \exp\left(-\alpha \frac{1-t}{1+t}\right), \quad h_{\alpha,n}(t) = \left(\frac{t + \mu_{\alpha,n}}{1 + \mu_{\alpha,n}t}\right)^n. \quad (37)
$$

Moreover, we also consider the functions $\psi_{\alpha,n}^\pm$ as being defined by (35) with this arbitrary sequence.

**Proposition 4.5** Let $\alpha > 0$ be fixed, and consider (35) and (37). Assume that

$$
\mu_{\alpha,n} = 1 - \frac{\alpha}{n} + O(n^{-2}) \quad \text{as } n \to \infty. \quad (38)
$$

Then the following is true:

(i) The operators $H(\psi_{\alpha,n}^\pm)$ are unitarily equivalent to the operators $\pm H(u_{\mp 1/2,1}).$

(ii) The operator

$$
P_n(I + H(\psi_{\alpha,n}^\pm))^{-1}P_n - P_n
$$

is unitarily equivalent to operators

$$
A_n = H(h_{\alpha,n})(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_{\alpha,n}) - H(h_{\alpha,n})^2,
$$

which are trace class operators and converge as $n \to \infty$ in the trace norm to

$$
A = H(h_{\alpha})(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_{\alpha}) - H(h_{\alpha})^2.
$$

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Proof. These results are proved in [10, Prop. 4.12] with a change of the condition on the sequence \( \mu_{\alpha,n} \). This change is consistent with the different notation for \( h_\alpha \). In fact, one has only to replace \( \alpha \) by \( 2\alpha \). Moreover, instead of the functions \( \psi_\alpha^\pm \), functions \( \psi_\alpha^\pm - 1 \) occur, which do not change the Hankel operators. The fact that the operators \( I \pm H(u_{\mp 1/2,1}) \) are invertible has already been stated in the proof of Theorem 3.2 (see also [10, Prop. 4.1] or [3, Thm. 3.6]).

In the following proposition we identify the limit of the determinant appearing in the right hand side of (36). We return to the specific definition of \( \mu_{\alpha,n} \) given in (34) and to the definitions (35) and (37) in terms of this sequence.

Proposition 4.6 For each \( \alpha > 0 \) we have

\[
\lim_{n \to \infty} \det \left[ P_n(I + H(\psi_{\alpha,n}^\pm))^{-1}P_n \right] = \det \left[ H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha) \right]. \tag{39}
\]

Proof. Proceeding as in [10, Prop. 5.5] we notice that \( H(h_\alpha)^2 \) is a projection operator. (The slight change in notation, \( \alpha \mapsto 2\alpha \), does not affect the statements made here). Hence, in the same way it is established that

\[
\det \left[ H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha) \right] = \det \left[ I + H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha) - H(h_\alpha)^2 \right].
\]

where the determinant on the left hand side is of that an operator acting on the image of \( H(h_\alpha)^2 \), while the right hand side corresponds to an operator acting on \( L^2(\mathbb{R}) \).

Similarly, the determinant on the left hand side of (39) can be written as

\[
\det \left[ P_n(I + H(\psi_{\alpha,n}^\pm))^{-1}P_n \right] = \det \left[ I + P_n(I + H(\psi_{\alpha,n}^\pm))^{-1}P_n - P_n \right]
\]

\[
= \det \left[ I + H(h_{\alpha,n})(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_{\alpha,n}) - H(h_{\alpha,n})^2 \right].
\]

As stated in the proof of Proposition 4.4, the sequence \( \mu_{\alpha,n} \) has the asymptotics (38). By applying the previous proposition the desired assertion follows.

We are now finally able to identify the determinants \( \det(I - K_\alpha^\pm) \). Recall in this connection the definition (19) of the functions \( \hat{u}_\beta \).

Theorem 4.7 For each \( \alpha > 0 \) we have

\[
\det(I - K_\alpha^\pm) = \exp \left( -\frac{\alpha^2}{4} \mp \frac{\alpha}{2} \right) \det \left[ \Pi_\alpha(I \pm H_R(\hat{u}_{\mp 1/2}))^{-1}\Pi_\alpha \right]. \tag{40}
\]

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Proof. From Propositions 4.1, 4.4 and 4.6 it follows that

\[ \det(I - K^\pm_\alpha) = \exp\left( -\frac{\alpha^2}{4} \mp \frac{\alpha}{2} \right) \det\left[ H(h_\alpha)(I \pm H(u_{\mp 1/2,1}))^{-1}H(h_\alpha) \right]. \]

As noted in the proof of [10, Thm. 5.6], there exists a unitary transform \( S \) from \( H^2(\mathbb{T}) \) onto \( L^2(\mathbb{R}_+) \) such that \( H_\mathbb{R}(a) = SH(b)S^* \) with \( a(x) = b(\frac{1+ix}{1-ix}) \). In the specific case we obtain

\[ H_\mathbb{R}(\hat{u}_{\pm 1/2}) = SH(u_{\pm 1/2,1})S^*, \quad H_\mathbb{R}(e^{ix\alpha}) = SH(h_\alpha)S^*. \]

This together with the remark that \( H(e^{ix\alpha})^2 = \Pi_\alpha \) implies (40).

Finally we are using result of [3] in order to establish the promised asymptotic formula. Recall that \( \zeta \) stands for the Riemann zeta function.

**Theorem 4.8** The following asymptotic formula holds as \( \alpha \to \infty \):

\[
\log \det(I - K^\pm_\alpha) = -\frac{\alpha^2}{4} \mp \frac{\alpha}{2} - \log \frac{\alpha}{8} + \frac{\log 2}{24} \mp \frac{\log 2}{4} + \frac{3}{2} \zeta'(-1) + o(1) \quad (41)
\]

Proof. In Sect. 3.6 of [3] it has been proved that

\[
\det\left[ \Pi_\alpha(I \pm H_\mathbb{R}(\hat{u}_{\mp 1/2}))^{-1}\Pi_\alpha \right] \sim \alpha^{-1/8} \pi^{1/4} 2^{1/4} G(1/2), \quad \alpha \to \infty, \quad (42)
\]

where \( G(z) \) stands for the Barnes \( G \)-function [11]. Notice that \( G(3/2) = G(1/2)\Gamma(1/2) \), \( \Gamma(1/2) = \pi^{1/2} \), and \( G(1) = 1 \). This together with the previous theorem implies that

\[
\log \det(I - K^\pm_\alpha) = -\frac{\alpha^2}{2} \mp \frac{\alpha}{2} - \log \frac{\alpha}{8} + \frac{\log \pi}{4} \mp \frac{\log 2}{4} + \log G(1/2) + o(1). \quad (43)
\]

Finally observe that

\[
\log G(1/2) = -\frac{\log \pi}{4} + \frac{3}{2} \zeta'(-1) + \frac{\log 2}{24},
\]

which follows from a formula for \( G(1/2) \) in terms of Glaisher’s constant \( A = \exp(-\zeta'(-1) + 1/12) \) given in [11, page 290]. \( \square \)

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