DULL CUT OFF FOR CIRCULANTS

Abstract. Families of symmetric simple random walks on Cayley graphs of Abelian groups with a bound on the number of generators are shown to never have sharp cut off in the sense of [1], [3] or [5] for the convergence to the stationary distribution in the total variation norm. This is a situation of bounded degree and no expansion. Sharp cut off or the cut off phenomenon has been shown to occur in families such as random walks on a hypercube [1] in which the degree is unbounded as well as on a random regular graph where the degree is fixed, but there is expansion [4]. These examples agree with Peres’ conjecture in [3] relating sharp cut off, spectral gap and mixing time.

1. Introduction

A random walk on a finite Abelian group $G$ is called type $r$ if $|G| \geq \pi^r$ and the walk is given by applying one of the elements $\{\pm a_i, 0\}_{i \in [r]}$ each with equal probability or equivalently a simple random walk on the Cayley graph for the generating set $\{\pm a_i\}$ with the probability of staying at a vertex the same as that of following each edge.

Theorem 1. No family of walks all of the same type has sharp cut off.

See Section eight of [5] for more on the convergence rate of such walks.

If $A$ is an irreducible symmetric Markov matrix with unique stationary distribution $v_0$ (so that $Av_0 = v_0$ and $|v_0|_1 = 1$) and $x_0 = (1, 0, \ldots, 0)$ write

$$d_A(t) = |A^t x_0 - v_0|_1,$$

$$t_A(d) = \max \{t | d_A(t) \geq d\},$$

{$\lambda_k \subseteq (-1, 1]$ for the eigenvalues of $A$ with $\lambda_0 = 1$ and $|\lambda_m| = \max_{k \neq 0} |\lambda_k| < 1$.

Definition 2. A family $\{A_i\}$ of irreducible symmetric Markov matrices has sharp cut off if

$$\lim_{n \to \infty} \frac{t_{A_n}(\epsilon)}{t_{A_n}(1 - \epsilon)} = 1$$

for every $\epsilon \in (0, \frac{1}{2})$.

See Definition 3.3 in [5].

2. Proof of Theorem 1

The proof is given for the case in which every group is cyclic. Other Abelian groups work similarly.

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Proof. Consider a type \( r \) walk on \( \mathbb{Z}/n\mathbb{Z} \) with step directions \( \{\pm a_i\} \) and irreducible symmetric Markov transition matrix \( A \). Note the Fourier expansion:

\[
\lambda_k = \sum_{i=1}^{r} \frac{2}{2r+1} \cos \left( \frac{2\pi k a_i}{n} \right) + \frac{1}{2r+1}.
\]

**Lemma 3.**

\( \lambda_k^2 \leq d_A^2(t) \leq \sum_{k \neq 0} \lambda_k^2. \)

Proof. For the left inequality note that \( A = A^* \) is self adjoint and the stationary distribution is \( v_0 = \frac{1}{n}1 \) and write \( v_m \) for the eigenvector with \( Av_m = \lambda_m v_m \) and \( |v_m|_1 = 1. \) Since \( A \) is a transition matrix for a random walk on a Cayley graph for \( G \) it commutes with rotation (action by \( G \)). Thus \( v_m \) is an eigenvector for rotation and hence all entries of \( v_m \) have the same norm, which by the normalization is \( |v_m|_\infty = \frac{1}{n} \). Since \( d_A(t) = \max_{v} \langle A^t x_0, v \rangle \) this gives \( d_A(t) \geq |A^t x_0 - v_0, v_m| = |\lambda_m|^t. \)

For the right inequality if \( v_k \) is the eigenvector of \( A \) with eigenvalue \( \lambda_k \) and every entry having norm \( \frac{1}{n} \) then \( |\langle x_0, v_k \rangle| = \frac{1}{n} \) so that \( d_A^2(t) = |A^t x_0 - v_0|^2 \leq n |A^t x_0 - v_0|^2 = \sum_{k \neq 0} \lambda_k^2. \)

Replace trigonometric functions with exponentials using that if \( x \in [-\frac{3}{2}\pi, \frac{3}{2}\pi] \) then \( \cos(x) \leq e^{-\frac{3}{2} \pi x^2} \) and if \( x \in [-1, 1] \) then \( e^{-x^2} \leq \cos(x) \).

Commute sums with exponentials using that concavity of \( e^x \) and \( e^{-x^2} \) so that \( e^{-\frac{1}{2} a^2 - \frac{1}{2} b^2} \leq \frac{1}{2} e^{-a^2} + \frac{1}{2} e^{-b^2} \) for any \( a \) and \( b \) and if \( a, b \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \) then \( \frac{1}{2} e^{-a^2} + \frac{1}{2} e^{-b^2} \leq e^{-\frac{1}{2} (a^2 + b^2)}. \)

Eliminate the one norm by using that \( |\sigma|_1 \leq |\sigma|_2 \) and write \( \langle x \rangle \in (-\frac{1}{2}, \frac{1}{2}) \) for the smallest translate of \( x \) by an integer.

Combining these gives for every \( k \) that

\[
\lambda_k \leq e^{-\frac{8}{(2r+1)^2} \frac{1}{(\frac{k a_i}{n})^2}}.
\]

Since \( n \geq \pi^r \) and \( \text{Vol}[\mathbb{R}^n/(\mathbb{Z}^n + \frac{a_i}{n})] = \frac{1}{n} \) there is some \( k \) with every \( |(\frac{k a_i}{n})| \leq \frac{1}{2n} \) so that

\[
|\lambda_m| \geq |\lambda_k| \geq e^{-\frac{8}{(2r+1)^2} \frac{1}{(\frac{k a_i}{n})^2}}.
\]

**Lemma 4.** For every \( r \) there is \( \kappa_r \) so that if \( \Lambda \) is a full rank lattice in \( \mathbb{R}^r \) and \( \mu = \min_{0 \neq \sigma \in \Lambda} \{|\sigma|_2\} \) then

\[
\sum_{\sigma \in \Lambda} e^{-|\sigma|_2^2} \leq 1 + \frac{\kappa_r}{e^{a \mu^2} - 1}
\]

or

\[
\sum_{\sigma \in \Lambda} e^{-|\sigma|_2^2} \leq 1 + \frac{\kappa_r}{(e^{a \mu^2} - 1)^r}.
\]

Proof.

**Definition 5.** A full rank lattice \( \Lambda \subseteq \mathbb{R}^r \) is tight if \( \mu = \min_{0 \neq \sigma \in \Lambda} \{|\sigma|_2\} = 1 \) and there is a basis for \( \Lambda \) so that \( \{|\sigma| \} = \{1\} \) is maximal among lattices generated by rescalings of this basis with shortest nonzero element of length one.

Note that the tight lattices form a compact subset of all full rank lattices in \( \mathbb{R}^n \).
Definition 6. An acute cone in a lattice $\Lambda \subseteq \mathbb{R}^n$ is a free submonoid generating $\Lambda$ as a group and with all inner products of nonzero elements positive.

Note that acuteness of a cone is an open condition in the space of lattices and every lattice has a finite decomposition into acute cones. Write

$$
\kappa_r = \max_{\Lambda \subseteq \mathbb{R}^n, \Lambda = \cup_i C_i} \min_{\Lambda} |I|
$$

where $\Lambda$ is a full rank lattice and $\{C_i\}_{i \in I}$ is a decomposition into interiors of acute cones. By compactness $\kappa_r$ is finite. Note that for any rank $r$ lattice $\Lambda$ with $\mu = \min_{\sigma \in \Lambda} \{ |\sigma|^2 \} = 1$ there is a tight lattice $\Lambda'$ with $\sum_{\sigma \in \Lambda} e^{-a|\sigma|^2} \leq \sum_{\sigma' \in \Lambda'} e^{-a|\sigma'|^2}$ obtained by rescaling some generators.

If $\Omega$ is the interior of an acute cone of dimension $d > 0$ in a tight lattice then

$$
\sum_{\omega \in \Omega} e^{-a|\omega|^2} \leq \sum_{\omega \in \Omega} e^{-a|\omega|^2} \leq \sum_{\omega \in \mathbb{Z}^d} e^{-a|\omega|^2} = \left( e^a - 1 \right)^{-d} \leq \max\{ \left( e^a - 1 \right)^{-1}, \left( e^a - 1 \right)^{-r} \}.
$$

Thus using the second inequality from Lemma 3

$$
d_A^2(t) \leq \sum_{\sigma \in \mathbb{Z}^d} e^{-\frac{16a}{(2r+1)^2} |\sigma|^2} - 1 \leq \max \left\{ \frac{\kappa_r}{e^{\frac{16a}{(2r+1)^2} |\sigma|_2^2} - 1}, \frac{\kappa_r}{e^{\frac{16a}{(2r+1)^2} |\sigma|_2^2} - 1} \right\}
$$

$$
\leq \max \left\{ \frac{\kappa_r}{\lambda_{m}(2r+1)^2} - 1, \frac{\kappa_r}{\lambda_{m}(2r+1)^2} - 1 \right\}.
$$

Using the first inequality from Lemma 3 at $d = \epsilon$, the above inequality at $d = 1 - \epsilon$ and taking $\epsilon < e^{-2\pi^2(2r+1)\kappa_r}$ gives

$$
\frac{t_A(\epsilon)}{t_A(1 - \epsilon)} \geq \frac{-\ln \epsilon}{2\pi^2(2r+1)\kappa_r} > 1.
$$

Note that the same bound as above gives $t_A\left(\frac{1}{2}\right) \leq \frac{\kappa_r e^{2\pi^2(2r+1)\kappa_r}}{-\ln |\lambda|_m}$ so that $(1 - |\lambda|_m^d t_A\left(\frac{1}{2}\right))$ is uniformly bounded and this class of examples agrees with Peres’ conjecture in [3].

References

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