On the Conditional Bounds for Siegel Zeros

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Abstract In this paper, under a weakened version of Hardy–Littlewood Conjecture on the number of representations in Goldbach problem, we shall prove bounds for the Siegel zeros of real primitive Dirichlet characters for composite moduli.

Keywords Siegel zero, Goldbach problem

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1 Introduction

Let \( \chi \) be a Dirichlet character mod \( q \), \( s = \sigma + it \). It is well-known that there is an absolute constant \( c_1 > 0 \) such that the region

\[
1 - \frac{c_1}{\log(q(|t| + 2))} < \sigma
\]

contains no zero of Dirichlet \( L \)-function \( L(s, \chi) \) unless \( \chi \) is a real character, in which case \( L(s, \chi) \) has at most one real simple zero \( \beta \). If such a zero \( \beta \) exists, then we call it as exceptional zero and call the character \( \chi \) as exceptional character.

In 1935, Page [7] proved that there is an effective absolute constant \( c_2 > 0 \) such that for any real character \( \chi \) mod \( q \) (\( q \geq 3 \)), if \( \beta \) is a real zero of \( L(s, \chi) \), then

\[
\frac{c_2}{\sqrt{q} \log^2 q} \leq 1 - \beta.
\]

In the same year, Siegel [8] showed that for each \( \varepsilon > 0 \), there is a constant \( c_3(\varepsilon) > 0 \) such that for any real character \( \chi \) mod \( q \) (\( q \geq 3 \)), if \( \beta \) is a real zero of \( L(s, \chi) \), then

\[
\frac{c_3(\varepsilon)}{q^\varepsilon} \leq 1 - \beta,
\]

(1.1)
where \( c_3(\varepsilon) \) is an ineffective constant. The exceptional zero is also called Siegel zero. The study for Siegel zeros is an important topic in the number theory. Actually we need only to consider Siegel zeros for the real primitive characters.

Some people studied the connection between Siegel zeros and Goldbach problem. One could see [3] and [2].

Write

\[
R(n) = \sum_{\substack{p_1, p_2 \text{ prime} \, n = p_1 + p_2}} 1, \tag{1.2}
\]

where \( p_1, p_2 \) denote prime numbers. Hardy and Littlewood [4] conjectured that for the sufficiently large even integer \( n \), we have

\[
R(n) \sim \frac{n}{\varphi(n)} \prod_{p | n} \left( 1 - \frac{1}{(p-1)^2} \right) \cdot \frac{n}{\log^2 n},
\]

where \( \varphi(n) \) is the Euler totient function. There is a weakened version of Hardy–Littlewood Conjecture as follows.

**Conjecture 1.1** There is an absolute constant \( c_4 > 0 \) such that for the even integer \( n \geq 4 \), we have

\[
R(n) \geq \frac{c_4 n}{\log^2 n}.
\]

In 2016, under Conjecture 1.1, Fei [3] proved the following bounds of Siegel zeros which improve the bound (1.1).

**Theorem 1.2** Suppose that Conjecture 1.1 holds true. Let \( q \) be a prime number with \( q \equiv 3 \, (\text{mod} \, 4) \), \( \chi \) be the real primitive character \( \text{mod} \, q \) and \( \beta \) be the real zero of \( L(s, \chi) \). Then there is an absolute constant \( c_5 > 0 \) such that

\[
\frac{c_5}{\log^2 q} \leq 1 - \beta.
\]

Recently, in the first version of paper [1], Bhowmik and Halupczok generalized the result in Theorem 1.2 under the following Conjecture 1.3 which is weaker than Conjecture 1.1.

**Conjecture 1.3** Suppose that \( x \) is sufficiently large, \( q \leq x^2 \). There is an absolute constant \( c_6 > 0 \) such that for the even integers \( n(\frac{x}{2} < n \leq x, q | n) \), with at most \( \frac{x}{8q} \) exceptions, we have

\[
R(n) \geq \frac{c_6 n}{\log^2 n}.
\]

Actually they only proved the result for the situation of prime numbers, which is shown in the following Theorem 1.4. For the precise explanation, one could refer to the second version of our paper [5] on arXiv.

**Theorem 1.4** Suppose that Conjecture 1.3 holds true. Let \( q \) be a sufficiently large prime number, \( \chi \) be the real primitive character \( \text{mod} \, q \) with \( \chi(-1) = -1 \) and \( \beta \) be the Siegel zero of \( L(s, \chi) \). Then there is an effective absolute constant \( c_7 > 0 \) such that

\[
\frac{c_7}{\log^2 q} \leq 1 - \beta.
\]

In this paper, we shall assume the following Conjecture 1.5, which is a little stronger than Conjecture 1.3 but is still a weakened version of Hardy–Littlewood Conjecture.
Conjecture 1.5  Suppose that \( x \) is sufficiently large, \( q \leq \frac{x}{2} \). There is an absolute constant \( c_8 > 0 \) such that for the even integers \( n \left( \frac{x}{2} < n \leq x, q \mid n \right) \), we have

\[
R(n) \geq \frac{c_8 n}{\varphi(n)} \cdot \frac{n}{\log^2 n}.
\]

Under the assumption of Conjecture 1.5, we shall prove the following Theorem 1.6.

**Theorem 1.6**  Suppose that Conjecture 1.5 holds true. Let \( q \) be a sufficiently large composite number, \( \chi \) be the real primitive character \( \mod q \) with \( \chi(-1) = -1 \) and \( \beta \) be the Siegel zero of \( L(s, \chi) \). Then there is an effective absolute constant \( c_9 > 0 \) such that

\[
\frac{c_9}{\log^2 q} \leq 1 - \beta.
\]

Throughout this paper, we assume that \( c_i \) is a positive constant. Let \( p, p_i \) denote the prime numbers, \( \varphi(n) \) denote the Euler totient function, \( \mu(n) \) denote the Möbius function.

## 2  Some Lemmas

**Lemma 2.1**  We have

\[
\frac{n}{\varphi(n)} \geq \frac{\mu^2(d)}{d} \geq \frac{6}{\pi^2} \cdot \frac{n}{\varphi(n)}.
\]

*Proof*  Firstly, by the expression of the Euler totient function, we have

\[
\frac{n}{\varphi(n)} = \frac{1}{\prod_{p \mid n} (1 - \frac{1}{p})} = \frac{\prod_{p \mid n} (1 + \frac{1}{p})}{\prod_{p \mid n} (1 - \frac{1}{p})} \geq \prod_{p \mid n} \left( 1 + \frac{1}{p} \right) = \sum_{d \mid n} \frac{\mu^2(d)}{d}.
\]

Secondly, in a similar way to the above, we get

\[
\sum_{d \mid n} \frac{\mu^2(d)}{d} = \prod_{p \mid n} \left( 1 + \frac{1}{p} \right) = \prod_{p \mid n} \left( 1 - \frac{1}{p^2} \right)
\]

\[
\geq \prod_{p} \left( 1 - \frac{1}{p^2} \right) \cdot \frac{n}{\varphi(n)}
\]

\[
= \frac{6}{\pi^2} \cdot \frac{n}{\varphi(n)},
\]

where the fact

\[
\prod_{p} \left( 1 - \frac{1}{p^2} \right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
\]

is used. \( \square \)

**Lemma 2.2**  Let \( \chi \) be the real primitive character \( \mod q \) \( (q \geq 3) \), \( \beta \) be the Siegel zero of \( L(s, \chi) \). If \( \gcd(a, q) = 1 \), then there is an effective absolute constant \( c_{10} > 0 \) such that

\[
\sum_{p \leq x \atop p \equiv a \pmod{q}} 1 = \frac{\text{li}(x)}{\varphi(q)} - \frac{\chi(a)}{\varphi(q)} \text{li}(x^\beta) + O(x \exp(-c_{10} \sqrt{\log x}))
\]

where

\[
\text{li}(x) = \int_2^x \frac{du}{\log u}.
\]
One could see [6, p. 381, Corollary 11.20].

The sum

\[ c_q(n) = \sum_{\substack{a=1 \\ (a, q) = 1}}^{q} e\left(\frac{an}{q}\right) \]

is called as Ramanujan sum.

**Lemma 2.3** For the Ramanujan sum, we have

\[ c_q(n) = \frac{\mu\left(\frac{q}{(q, n)}\right)}{\phi\left(\frac{q}{(q, n)}\right)} \cdot \phi(q). \]

One could see [6, p. 110, Theorem 4.1].

For the character \( \chi \mod q \), we write

\[ \tau(\chi, k) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ak}{q}\right) \]

and

\[ \tau(\chi) = \tau(\chi, 1). \]

**Lemma 2.4** If \( \chi \) is a primitive character \( \mod q \), then we have that

\[ \tau(\chi, k) = \overline{\chi}(k) \tau(\chi) \]

and

\[ |\tau(\chi)| = \sqrt{q}. \]

One could see [6, p. 287, Theorems 9.5 and 9.7].

**Lemma 2.5** If \( \chi \) is the real primitive character \( \mod q \), then we have

\[ \tau^2(\chi) = \chi(-1)q. \]

**Proof** It is easy to see

\[ \overline{\tau}(\chi) = \sum_{a=1}^{q} \chi(a) e\left(-\frac{a}{q}\right) = \chi(-1) \sum_{b=1}^{q} \chi(b) e\left(\frac{b}{q}\right) = \chi(-1) \tau(\chi). \]

Then by Lemma 2.4, we have

\[ \tau^2(\chi) = \chi(-1)|\tau(\chi)|^2 = \chi(-1)q. \]

\[ \square \]

3 Proof of Theorem 1.6

Following the research route in [3], we shall consider the lower bound and upper bound of the sum

\[ S = \sum_{k=1}^{q} \left( \sum_{2 < p \leq x} e\left(\frac{kp}{q}\right) \right)^2. \]  

3.1

Firstly we take

\[ x = \exp\left(\frac{36}{c_{10}} \log^2 q\right). \]  

3.2
where \(c_{10}\) is the constant defined in Lemma 2.2. Since \(q\) is sufficiently large, so is \(x\). It is easy to see 

\[
q = \exp \left( \frac{c_{10}}{6} \sqrt{\log x} \right) < \frac{x}{4}.
\]

We have

\[
S = \sum_{k=1}^{q} \sum_{2 < p_1, p_2 \leq x} e \left( \frac{k(p_1 + p_2)}{q} \right) = \sum_{n \leq 2x} \sum_{k=1}^{q} e \left( \frac{kn}{q} \right) \sum_{2 < p_1, p_2 \leq x} 1
\]

\[
= \sum_{n \leq 2x} \sum_{q|n\ n \neq 1} 1 \geq q \sum_{n \leq 2x} \sum_{p_1, p_2 \leq x} \sum_{n=p_1+p_2} 1 = q \sum_{n \neq 1} R(n).
\]

Under the assumption of Conjecture 1.5, for the even integers \(n(\frac{x}{2} < n \leq x, q|n)\), we have

\[
R(n) \geq c_{8n} \varphi(n) \cdot \frac{n}{\log^2 n}.
\]

Hence, by Lemma 2.1,

\[
S \gg \frac{qx}{\log^2 x} \sum_{\frac{x}{2} < n \leq x} \frac{n}{\varphi(n)} \geq \frac{qx}{\log^2 x} \sum_{\frac{x}{2} < n \leq x} \frac{n}{\varphi(n)} \geq \frac{qx}{\log^2 x} \sum_{d|q} \frac{\mu^2(d)}{d} \sum_{\frac{x}{2} < n \leq x} 1 \gg \frac{qx}{\log^2 x} \sum_{d|q} \frac{\mu^2(d)}{d} \sum_{\frac{x}{2} < n \leq x} 1 \gg \frac{qx}{\log^2 x} \cdot \frac{x^2}{\log^2 x}.
\]

Therefore there is an absolute constant \(c_{11} (0 < c_{11} < \frac{1}{4})\) such that

\[
S \geq c_{11} \cdot \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^2 x}. \tag{3.3}
\]

On the other hand, the application of Lemmas 2.2, 2.3 and 2.4 produces that

\[
\sum_{2 < p \leq x} e \left( \frac{kp}{q} \right) = \sum_{2 < p \leq x} e \left( \frac{kp}{q} \right) + \sum_{2 < p \leq x} e \left( \frac{kp}{q} \right)
\]

\[
= \sum_{p \leq x} e \left( \frac{kp}{q} \right) + O(\log q)
\]

\[
= \sum_{a=1}^{q} e \left( \frac{ka}{q} \right) \sum_{p \leq x} 1 + O(\log q)
\]

\[
= \sum_{a=1}^{q} e \left( \frac{ka}{q} \right) \cdot \frac{\text{li}(x)}{\varphi(q)} - \sum_{a=1}^{q} \chi(a) e \left( \frac{ka}{q} \right) \cdot \frac{\text{li}(x^a)}{\varphi(q)}
\]
\[ + O(qx \exp(-c_{10} \sqrt{\log x})) \]
\[ = c_q(k) \cdot \frac{\text{li}(x)}{\varphi(q)} - \tau(\chi, k) \cdot \frac{\text{li}(x^\beta)}{\varphi(q)} \]
\[ + O(qx \exp(-c_{10} \sqrt{\log x})) \]
\[ = \frac{\mu(\frac{q}{(q, k)})}{\varphi(\frac{q}{(q, k)})} \cdot \text{li}(x) - \chi(k) \tau(\chi) \cdot \frac{\text{li}(x^\beta)}{\varphi(q)} \]
\[ + O(qx \exp(-c_{10} \sqrt{\log x})). \]

Hence,
\[
\sum_{k=1}^{q} \left( \sum_{2 < p \leq x} e \left( \frac{kp}{q} \right) \right)^2
= \sum_{k=1}^{q} \mu^2(\frac{q}{(q, k)}) \cdot \varphi^2(\frac{q}{(q, k)}) \cdot \text{li}^2(x) + \sum_{k=1}^{q} \chi^2(k) \tau^2(\chi) \cdot \frac{\text{li}^2(x^\beta)}{\varphi^2(q)}
- 2 \sum_{k=1}^{q} \mu(\frac{q}{(q, k)}) \cdot \text{li}(x) \cdot \chi(k) \tau(\chi) \cdot \frac{\text{li}(x^\beta)}{\varphi(q)}
+ O \left( \sum_{k=1}^{q} q^2 x^2 \exp(-c_{10} \sqrt{\log x}) \right). \]

Note that
\[ \text{li}(x) = \frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right). \]

Then we have
\[
\sum_{k=1}^{q} \frac{\mu^2(\frac{q}{(q, k)})}{\varphi^2(\frac{q}{(q, k)})} \cdot \text{li}^2(x) = \sum_{d|q} \left( \sum_{l=1}^{\frac{q}{d}} 1 \right) \frac{\mu^2(\frac{q}{d})}{\varphi^2(\frac{q}{d})} \cdot \text{li}^2(x)
= \sum_{d|q} \left( \sum_{l=1}^{\frac{q}{d}} 1 \right) \frac{\mu^2(\frac{q}{d})}{\varphi^2(\frac{q}{d})} \cdot \text{li}^2(x)
= \sum_{d|q} \frac{\mu^2(\frac{q}{d})}{\varphi(\frac{q}{d})} \cdot \text{li}^2(x)
= \sum_{r|q} \mu^2(r) \cdot \text{li}^2(x)
= \prod_{p|q} \left( 1 + \frac{1}{\varphi(p)} \right) \left( \frac{x^2}{\log^2 x} + O \left( \frac{x^2}{\log^3 x} \right) \right)
= \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^2 x} + O \left( \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^3 x} \right),
\]
and
\[
\sum_{k=1}^{q} \chi^2(k) \tau^2(\chi) \cdot \frac{\text{li}^2(x^\beta)}{\varphi^2(q)} = \frac{x(-1)^q}{\varphi(q)} \cdot \text{li}^2(x^\beta)
\]
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\[ \chi(-1)q \left\{ \frac{x^3}{\beta \log x} + O\left( \frac{x^3}{\log^2 x} \right) \right\}^2 = \chi(-1)q \left( \frac{x^{2\beta}}{\beta^2 \log^2 x} + O\left( \frac{q \cdot x^2}{\varphi(q) \cdot \log^3 x} \right) \right). \]

We also have

\[ -2 \sum_{k=1}^{q} \frac{\mu\left( \frac{q}{(q,k)} \right)}{\varphi\left( \frac{q}{(q,k)} \right)} \cdot \ln(x) \cdot \chi(k) \tau(\chi) \cdot \frac{\ln(x^\beta)}{\varphi(q)} = -\frac{2}{\varphi(q)} \cdot \ln(x) \ln(x^\beta) \tau(\chi) \sum_{k=1}^{q} \frac{\mu(q)}{\varphi(q)} \cdot \chi(k) = 0. \]

Combining the above estimates, we obtain

\[ S = \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^2 x} + \frac{\chi(-1)q}{\varphi(q)} \cdot \frac{x^{2\beta}}{\beta^2 \log^2 x} + O\left( \frac{q \cdot x^2}{\varphi(q) \cdot \log^3 x} \right) + O(q^3 x^2 \exp(-c_{10} \sqrt{\log x})) = \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^2 x} - \frac{q}{\varphi(q)} \cdot \frac{x^{2\beta}}{\beta^2 \log^2 x} + O\left( \frac{q \cdot x^2}{\varphi(q) \cdot \log^3 x} \right). \]  

(3.4)

Comparing (3.3) with (3.4), we get

\[ c_{11} \cdot \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^2 x} \leq \frac{q}{\varphi(q)} \cdot \frac{x^2}{\log^2 x} - \frac{q}{\varphi(q)} \cdot \frac{x^{2\beta}}{\beta^2 \log^2 x} + O\left( \frac{q \cdot x^2}{\varphi(q) \cdot \log^3 x} \right), \]

which yields

\[ \frac{x^{2\beta-2}}{\beta^2} \leq (1 - c_{11}) + \frac{c_{11}}{2}. \]

Therefore we have

\[ x^{2\beta-2} \leq 1 - \frac{c_{11}}{2}, \]

which yields

\[ 1 - \beta \geq \frac{-\log(1 - \frac{c_{11}}{2})}{2 \log x} - \frac{c_{11}^2}{72 \log^2 q} = \frac{c_9}{\log^2 q}, \]

where \( c_9 > 0 \) is an effective absolute constant.

So far the proof of Theorem 1.6 is complete.

**Remark 3.1** Theorem 1.6 in this paper was published in the first version of our paper [5] on arXiv. Some days after the publication of the first version of [5], G. Bhowmik and K. Halupczok published the second version of [1] on arXiv, in which they assumed Conjecture 1.3 and used our new discussion to get a weaker result than that in Theorem 1.6. But they did not mention our paper so that we had to make some explanation. One could refer to the second version of [5].

**Remark 3.2** In order to prove Theorems 1.4 and 1.6, one needs only to suppose that Conjectures 1.3 and 1.5 hold true for \( q \leq \exp(C \sqrt{\log x}) \), where \( C > 0 \) is some large constant.

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