Construction of Codes for Wiretap Channel and Secret Key Agreement from Correlated Source Outputs by Using Sparse Matrices

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Abstract

The aim of this paper is to prove coding theorems for the wiretap channel and secret key agreement based on the notion of a hash property for an ensemble of functions. These theorems imply that codes using sparse matrices can achieve the optimal rate. Furthermore, fixed-rate universal coding theorems for a wiretap channel and a secret key agreement are also proved.

Index Terms

Shannon theory, hash property, linear codes, sparse matrix, maximum-likelihood decoding, minimum-divergence encoding, minimum-entropy decoding, secret key agreement from correlated source outputs, wiretap channel, universal codes

I. INTRODUCTION

The aim of this paper is to prove the coding theorems for the wiretap channel (Fig. 1) introduced in [23] and secret key agreement problem (Fig. 2) introduced in [12][1]. The proof of theorems is based on the notion of a hash property for an ensemble of functions introduced in [18][19]. This notion provides a sufficient condition for the achievability of coding theorems. Since an ensemble of sparse matrices has a hash property, we can construct codes by using sparse matrices where the rate of codes is close to the optimal rate. In the construction of codes, we use minimum-divergence encoding, maximum-likelihood decoding, and minimum-entropy decoding, where we can use the approximation methods introduced in [9][5] to realize these operations.

Wiretap channel coding using a sparse matrices is studied in [21] for binary erasure wiretap channels. On the other hand, our construction can be applied to any stationary memoryless channel. It should be noted here that the encoder design is based on the standard channel code presented in [14][18][19][13]. Furthermore, we prove the fixed-rate universal coding theorem for a wiretap channel, where our construction is reliable and secure for any channel under some conditions specified by the encoding rate. Universality is not considered in [23][21].
The secret key agreement from correlated source outputs using sparse matrices is studied in [15][16], where both non-universal and universal codes are considered. Our construction is the same as that proposed in [16]. It should be noted that the linearity of functions is not assumed in our proof of reliability and security while it is assumed in [16]. Furthermore, an expurgated ensemble of sparse matrices is not assumed in our proof while it is assumed in [16].

II. DEFINITIONS AND NOTATIONS

Throughout this paper, we use the following definitions and notations.

The cardinality of a set $\mathcal{U}$ is denoted by $|\mathcal{U}|$, $\mathcal{U}^c$ denotes the compliment of $\mathcal{U}$, and $\mathcal{U} \equiv \mathcal{U} \cap \mathcal{V}^c$.

Column vectors and sequences are denoted in boldface. Let $A u$ denote a value taken by a function $A : \mathcal{U}^n \rightarrow \mathcal{V}$ at $u \equiv (u_1, \ldots, u_n) \in \mathcal{U}^n$, where $\mathcal{U}^n$ is a domain of the function. It should be noted that $A$ may be nonlinear. When $A$ is a linear function expressed by an $l \times n$ matrix, we assume that $\mathcal{U}$ is a finite field and the range of functions is defined by $\mathcal{U} \equiv \mathcal{U}^l$. It should be noted that this assumption is not essential for general (nonlinear) functions because discussion is not changed if $l \log |\mathcal{U}|$ is replaced by $\log |\mathcal{U}|$. For a set $\mathcal{A}$ of functions, let $\text{Im} \mathcal{A}$ be defined as

$$\text{Im} \mathcal{A} \equiv \bigcup_{A \in \mathcal{A}} \{Au : u \in \mathcal{U}^n\}.$$

We define sets $C_A(c)$, $C_{AB}(c, m)$, and $C_{ABB}(c, m, w)$ as

$$C_A(c) \equiv \{u : Au = c\}$$

$$C_{AB}(c, m) \equiv \{u : Au = c, Bu = m\}$$

$$C_{ABB}(c, m, w) \equiv \{u : Au = c, Bu = m, \hat{B}u = w\}.$$
In the context of linear codes, \( C_A(c) \) is called a coset determined by \( c \).

Let \( p \) and \( p' \) be probability distributions and let \( q \) and \( q' \) be conditional probability distributions. Then entropy \( H(p) \), conditional entropy \( H(q|p) \), divergence \( D(p||p') \), and conditional divergence \( D(q||q'|p) \) are defined as

\[
H(p) \equiv \sum_u p(u) \log \frac{1}{p(u)}
\]

\[
H(q|p) \equiv \sum_{u,v} q(u|v)p(v) \log \frac{1}{q(u|v)}
\]

\[
D(p \parallel p') \equiv \sum_u p(u) \log \frac{p(u)}{p'(u)}
\]

\[
D(q \parallel q'|p) \equiv \sum_v p(v) \sum_u q(u|v) \log \frac{q(u|v)}{q'(u|v)},
\]

where we assume the base 2 of the logarithm.

Let \( \mu_U \) and \( \mu_V \) be the joint probability distribution of random variables \( U \) and \( V \). Let \( \mu_U \) and \( \mu_V \) be the respective marginal distributions and \( \mu_{U|V} \) be the conditional probability distribution. Then the entropy \( H(U) \), the conditional entropy \( H(U|V) \), and the mutual information \( I(U;V) \) of random variables are defined as

\[
H(U) \equiv H(\mu_U)
\]

\[
H(U|V) \equiv H(\mu_{U|V}|\mu_V)
\]

\[
I(U;V) \equiv H(U) - H(U|V).
\]

Let \( \nu_u \) and \( \nu_{u|v} \) be defined as

\[
\nu_u(u) \equiv \frac{|\{1 \leq i \leq n : u_i = u\}|}{n}
\]

\[
\nu_{u|v}(u|v) \equiv \frac{\nu_{uv}(u,v)}{\nu_v(v)}.
\]

We call \( \nu_u \) a type of \( u \in U^n \) and \( \nu_{u|v} \) a conditional type. Let \( U \equiv \nu_U \) be the type of a sequence and \( U|V \equiv \nu_{U|V} \) be the conditional type of a sequence given a sequence of type \( U \). Then a set of typical sequences \( T_U \) and a set of conditionally typical sequences \( T_{U|V}(v) \) are defined as

\[
T_U \equiv \{ u : \nu_u = \nu_U \}
\]

\[
T_{U|V}(v) \equiv \{ u : \nu_{u|v} = \nu_{U|V} \}.
\]

The empirical entropy, the empirical conditional entropy, and empirical mutual information are defined as

\[
H(u) \equiv H(\nu_u)
\]

\[
H(u|v) \equiv H(\nu_{u|v}|\nu_v)
\]

\[
I(u;v) \equiv H(u) - H(u|v).
\]

A set of typical sequences \( T_{U,\gamma} \) and a set of conditionally typical sequences \( T_{U|V,\gamma}(v) \) are defined as

\[
T_{U,\gamma} \equiv \{ u : D(\nu_u||\mu_U) < \gamma \}
\]

\[
T_{U|V,\gamma}(v) \equiv \{ u : D(\nu_{u|v}||\mu_{U|V}|\nu_v) < \gamma \}.
\]

We use several lemmas for the method of the types described in Appendix.
In the construction of codes, we use a minimum-divergence encoder
\[ \tilde{g}_{AB}(c, m, w) \equiv \arg\min_{x' \in C_{AB}(c, m, w)} D(\nu_{x'} \| \mu_X), \] (1)
a maximum-likelihood decoder
\[ g_{A}(c|y) \equiv \arg\max_{x' \in C_{A}(c)} \mu_X|Y(x'|y), \] (2)
and a minimum-entropy decoder
\[ \tilde{g}_{A}(c|y) \equiv \arg\min_{x' \in C_{A}(c)} H(x'|y). \] (3)
The minimum-divergence encoder assigns a message to a typical sequence as close as possible to the input distribution, where the typical sequence is in the coset determined by \( c \). The time complexity of encoding and decoding is exponential with respect to the block length by using the exhaustive search. It should be noted that the linear programming method introduced [9] and [5] can be applied to these encoder and decoders by assuming that \( X = Y = GF(2) \) and \( A, B, \) and \( \hat{B} \) are linear functions, where the linear programming method may not find the integral solution. Details are described in Section VIII. It should be noted here that we do not discuss the performance of the linear programming methods in this paper.

We define \( \chi(\cdot) \) as
\[ \chi(a = b) \equiv \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases} \]
\[ \chi(a \neq b) \equiv \begin{cases} 1, & \text{if } a \neq b \\ 0, & \text{if } a = b. \end{cases} \]

Finally, we use the following definitions in Appendix. For \( \gamma, \gamma' > 0 \), we define
\[ \lambda_U \equiv \frac{|U| \log [n + 1]}{n} \] (4)
\[ \zeta_U(\gamma) \equiv \gamma - \sqrt{2\gamma} \log \frac{\sqrt{2\gamma}}{|U|} \] (5)
\[ \zeta_U|V(\gamma'|\gamma) \equiv \gamma' - \sqrt{2\gamma'} \log \frac{\sqrt{2\gamma'}}{|U||V|} + \sqrt{2\gamma} \log |U| \] (6)
\[ \eta_U(\gamma) \equiv -\sqrt{2\gamma} \log \frac{\sqrt{2\gamma}}{|U|} + \frac{|U| \log [n + 1]}{n} \] (7)

It should be noted here that the product set \( U \times V \) is denoted by \( UV \) when it appears in the subscript of these functions. We define \( h(\theta) \) for \( 0 \leq \theta \leq 1 \) as
\[ h(\theta) \equiv -\theta \log \theta - [1 - \theta] \log (1 - \theta). \] (8)

We define \( |\cdot|^{+} \) as
\[ |\theta|^{+} \equiv \begin{cases} \theta, & \text{if } \theta > 0, \\ 0, & \text{if } \theta \leq 0. \end{cases} \] (9)
III. \((\alpha, \beta)\)-Hash Property

In the following, we review the notion of the hash property for an ensemble of functions, which is introduced in [18]. This provides a sufficient condition for coding theorems, where the linearity of functions is not assumed. We prove coding theorems based on this notion.

Definition 1 ([18]): Let \(\mathcal{A} \equiv \{\mathcal{A}_n\}_{n=1}^\infty\) be a sequence of sets such that \(\mathcal{A}_n\) is a set of functions \(A : \mathcal{U}^n \to \mathcal{U}_{\mathcal{A}_n}\) satisfying

\[
\lim_{n \to \infty} \log \frac{|\mathcal{U}_{\mathcal{A}_n}|}{n} = 0. \tag{H1}
\]

For a probability distribution \(p_{A,n}\) on \(\mathcal{A}_n\), we call a sequence \((\mathcal{A}, p_A) \equiv \{(\mathcal{A}_n, p_{A,n})\}_{n=1}^\infty\) an ensemble. Then, \((\mathcal{A}, p_A)\) has an \((\alpha_A, \beta_A)\)-hash property if there are two sequences \(\alpha_A \equiv \{\alpha_A(n)\}_{n=1}^\infty\) and \(\beta_A \equiv \{\beta_A(n)\}_{n=1}^\infty\) such that

\[
\lim_{n \to \infty} \alpha_A(n) = 1 \tag{H2}
\]

\[
\lim_{n \to \infty} \beta_A(n) = 0 \tag{H3}
\]

and

\[
\sum_{\mathcal{U} \in \mathcal{T}, \mathcal{U}' \in \mathcal{T}'} p_{A,n} \{A : Au = Au'\} \leq |\mathcal{T} \cap \mathcal{T}'| + \frac{|\mathcal{T}| |\mathcal{T}'| \alpha_A(n)}{|\text{Im} \mathcal{A}_n|} + \min\{|\mathcal{T}|, |\mathcal{T}'|\} \beta_A(n) \tag{H4}
\]

for any \(\mathcal{T}, \mathcal{T}' \subseteq \mathcal{U}^n\). Throughout this paper, we omit dependence of \(\mathcal{A}, p_A, \alpha_A\) and \(\beta_A\) on \(n\).

In the following, we present two examples of ensembles that have a hash property.

Example 1: In this example, we consider a universal class of hash functions introduced in [8]. A set \(\mathcal{A}\) of functions \(A : \mathcal{U}^n \to \mathcal{U}_{\mathcal{A}}\) is called a universal class of hash functions if

\[
|\{A : Au = Au'\}| \leq \frac{|\mathcal{A}|}{|\mathcal{U}_{\mathcal{A}}|}
\]

for any \(u \neq u'\). For example, the set of all functions on \(\mathcal{U}^n\) and the set of all linear functions \(A : \mathcal{U}^n \to \mathcal{U}_{\mathcal{A}}\) are classes of universal hash functions (see [8]). When \(\mathcal{A}\) is a universal class of hash functions and \(p_A\) is the uniform distribution on \(\mathcal{A}\), we have

\[
\sum_{\mathcal{U} \in \mathcal{T}, \mathcal{U}' \in \mathcal{T}'} p_A \{A : Au = Au'\} \leq |\mathcal{T} \cap \mathcal{T}'| + \frac{|\mathcal{T}| |\mathcal{T}'|}{|\text{Im} \mathcal{A}|}.
\]

This implies that \((\mathcal{A}, p_A)\) has a \((1, 0)\)-hash property, where \(1(n) \equiv 1\) and \(0(n) \equiv 0\) for every \(n\).

Example 2: In this example, we consider a set of linear functions \(A : \mathcal{U}^n \to \mathcal{U}_{\mathcal{A}}\). It was discussed in the above example that the uniform distribution on the set of all linear functions has a \((1, 0)\)-hash property. In the following, we introduce the ensemble of \(q\)-ary sparse matrices proposed in [18]. Let \(\mathcal{U} \equiv \text{GF}(q)\) and \(l_A \equiv nR\) for given \(0 < R < 1\). We generate an \(l_A \times n\) matrix \(A\) with the following procedure, where at most \(\tau\) random nonzero elements are introduced in every row:

1) Start from an all-zero matrix.

2) For each \(i \in \{1, \ldots, n\}\), repeat the following procedure \(\tau\) times:

   a) Choose \((j, a) \in \{1, \ldots, l_A\} \times \text{GF}(q)\) uniformly at random.

   b) Add \(a\) to the \((j, i)\) component of \(A\).
Let \((A, p_A)\) be an ensemble corresponding to the above procedure, where \(\tau = O(\log l_A)\) is even. It is proved in [18, Theorem 2] that there is \((\alpha_A, \beta_A)\) such that \((A, p_A)\) has an \((\alpha_A, \beta_A)\)-hash property.

In the following, let \(A\) be a set of functions \(A : U^n \rightarrow U_A\) and assume that \(p_C\) is the uniform distribution on \(\text{Im} A\), and random variables \(A\) and \(C\) are mutually independent, that is,

\[
p_C(c) = \begin{cases} 
1/|\text{Im} A|, & \text{if } c \in \text{Im} A \\
0, & \text{if } c \notin U_A \setminus \text{Im} A 
\end{cases}
\]

for any \(A\) and \(c\). We have the following lemmas, where it is not necessary to assume the linearity of functions.

**Lemma 1** ([18, Lemma 1]): If \((A, p_A)\) satisfies (H4), then

\[
p_A(\{A : [G \setminus \{u\}] \cap C_A(Au) \neq \emptyset\}) \leq |G| \alpha_A |\text{Im} A| + \beta_A
\]

for all \(G \subset U^n\) and all \(u \in U^n\).

**Lemma 2** ([18, Lemma 2]): If \((A, p_A)\) satisfies (H4), then

\[
p_{AC}(\{(A, c) : T \cap C_A(c) = \emptyset\}) \leq \alpha_A - 1 + \frac{|\text{Im} A| (\beta_A + 1)}{|T|}
\]

for all \(T \neq \emptyset\).

Finally, we consider the independent joint ensemble \(p_{AB}\) of linear matrices. The following lemma asserts that it is sufficient to assume the hash property of \((A, p_A)\) and \((B, p_B)\) to satisfy the hash property of \((A \times B, p_{AB})\) when they are ensembles of linear matrices.

**Lemma 3** ([18, Lemma 7]): For two ensembles \((A, p_A)\) and \((B, p_B)\), of \(l_A \times n\) and \(l_B \times n\) linear matrices, respectively, let \(p_{AB}\) be the joint distribution defined as

\[
p_{AB}(A, B) \equiv p_A(A)p_B(B).
\]

Then \((A \times B, p_{AB})\) has an \((\alpha_{AB}, \beta_{AB})\)-hash property for the ensemble of functions \(A \oplus B : U^n \rightarrow U_{l_A+l_B}\) defined as

\[
A \oplus B(u) \equiv (Au, Bu),
\]

where

\[
\alpha_{AB}(n) = \alpha_A(n)\alpha_B(n) \\
\beta_{AB}(n) = \beta_A(n) + \beta_B(n).
\]

**IV. Wiretap Channel Coding**

In this section we consider the wiretap channel coding problem illustrated in Fig. 1, where no common message and perfect secrecy are assumed. A wiretap channel is characterized by the conditional probability distribution \(\mu_{YZ|X}\), where \(X, Y,\) and \(Z\) are random variables corresponding to the channel input of a sender, the channel output of a legitimate receiver and the channel output of an eavesdropper. Then the capacity\(^1\) of

\(^1\) It is stated in [20] that the auxiliary random variable can be eliminated by applying [10, Theorem 7] and [11, Theorem 3]. In fact, because of the authors misunderstanding about the result of [11, Theorem 3], the statement of [20] may not be true. They wish to thank Prof. Shamai (Shitz), Prof. Oohama, and Prof. Koga, for helpful discussions.
this channel is derived in [7, Eq. (11)] as

$$\text{Capacity} \equiv \max_{\hat{X}, X, \hat{X} \leftrightarrow X \leftrightarrow Y Z} \left[ I(\hat{X}; Y) - I(\hat{X}; Z) \right], \quad (10)$$

where the maximum is taken over all probability distribution $\mu_{\hat{X}X}$ and the joint distribution $\mu_{\hat{X}XYZ}$ is given by

$$\mu_{\hat{X}XYZ}(\hat{x}, x, y, z) \equiv \mu_{YZ|X}(y, z|x)\mu_{\hat{X}X}(\hat{x}, x). \quad (11)$$

If a channel between $X$ and $Y$ is more capable than a channel between $X$ and $Z$, that is,

$$I(X; Y) \geq I(X; Z)$$

is satisfied for every input $X$, then the capacity of this channel is simplified as

$$\text{Capacity} \equiv \max_X \left[ I(X; Y) - I(X; Z) \right], \quad (12)$$

where the maximum is taken over all random variables $X$ and the joint distribution of random variable $(X, Y, Z)$ is given by

$$\mu_{XYZ}(x, y, z) \equiv \mu_{YZ|X}(y, z|x)\mu_X(x). \quad (13)$$

This capacity formula is derived in [23] for a degraded broadcast channel, extended in [7] to the case where a channel between $X$ and $Y$ is more capable than a channel between $X$ and $Z$.

In the following, we assume that $\mu_X$ and $\mu_{YZ|X}$ are given, where it is not necessary to assume that a channel is degraded or a channel between $X$ and $Y$ is more capable than that between $X$ and $Z$. We fix functions

$$A : \mathcal{X}^n \rightarrow \mathcal{X}^{l_A}$$
$$B : \mathcal{X}^n \rightarrow \mathcal{X}^{l_B}$$
$$\hat{B} : \mathcal{X}^n \rightarrow \mathcal{X}^{l_{\hat{B}}}$$
and a vector $c \in \mathcal{X}^{l_A}$ available for an encoder, a decoder, and an eavesdropper, where
\[
\begin{align*}
  l_A &\equiv \frac{n[H(X|Y) + \varepsilon_A]}{\log |\mathcal{X}|} \\
  l_B &\equiv \frac{n[H(X|Z) - H(X|Y)]}{\log |\mathcal{X}|} \\
  l_{\hat{B}} &\equiv \frac{n[I(X;Z) - \varepsilon_{\hat{B}}]}{\log |\mathcal{X}|}.
\end{align*}
\]

We construct a stochastic encoder and assume that the encoder uses a random sequence $w \in \mathcal{X}^{l_{\hat{B}}}$, which is generated uniformly at random and independently of the channel and the message $m \in \mathcal{X}^{l_B}$. We define the encoder and the decoder
\[
\varphi : \mathcal{X}^{l_B} \times \mathcal{X}^{l_{\hat{B}}} \rightarrow \mathcal{X}^n
\]
\[
\varphi^{-1} : \mathcal{Y}^n \rightarrow \mathcal{X}^{l_B}
\]
as
\[
\varphi(m, w) \equiv g_{AB\hat{B}}(c, m, w)
\]
\[
\varphi^{-1}(y) \equiv Bg_A(c|y),
\]
where $g_{AB\hat{B}}(c, m, w)$ and $g_A(c|y)$ are defined by (1) and (2), respectively. It is noted that $g_{AB\hat{B}}$ is a deterministic map.

Let $M$ and $W$ be random variables corresponding to $m$ and $w$, respectively, where the probability distributions $p_M$ and $p_W$ are given by
\[
p_M(m) \equiv \begin{cases} 
\frac{1}{|\text{Im} B|} & \text{if } m \in \text{Im} B \\
0 & \text{if } m \notin \text{Im} B
\end{cases}
\]
(14)
\[
p_W(w) \equiv \begin{cases} 
\frac{1}{|\text{Im} \hat{B}|} & \text{if } w \in \text{Im} \hat{B} \\
0 & \text{if } w \notin \text{Im} \hat{B}
\end{cases}
\]
(15)
and the joint distribution $p_{MWYZ}$ of the messages, and the channel outputs is given by
\[
p_{MWYZ}(m, w, y, z) \equiv \mu_{YZ|X}(y, z|\varphi(m, w))p_M(m)p_W(w).
\]

The rate of this code is given by
\[
\text{Rate} = \frac{\log |\text{Im} \hat{B}|}{n} = I(X;Y) - I(X;Z) - \frac{\log |\mathcal{X}^{l_B}|}{n}
\]
which converges to $I(X;Y) - I(X;Z)$ as $n$ goes to infinity by assuming the condition (H1) for an ensemble $(B, p_B)$. The decoding error probability $\text{Error}_{Y|X}(A, B, \hat{B}, c)$ is given by
\[
\text{Error}_{Y|X}(A, B, \hat{B}, c) \equiv \sum_{m, w, y} \mu_{Y|X}(y|\varphi(m, w))p_M(m)p_W(w)\chi(\varphi^{-1}(y) \neq m).
\]
The information leakage \( \text{Leakage}_{Z|X}(A, B, \hat{B}, c) \) is given by

\[
\text{Leakage}_{Z|X}(A, B, \hat{B}, c) \equiv \frac{I(M; Z^n)}{n}.
\]

It should be noted that the vector \( c \) is considered to be part of a deterministic map, which is known by the eavesdropper.

We have the following theorem. It should be noted that alphabets \( X \) and \( Y \) is allowed to be non-binary, and the channel is allowed to be asymmetric, non-degraded.

**Theorem 1:** Let \( \mu_{Y|Z|X} \) be the conditional probability distribution of a stationary memoryless channel. For given \( l_A \) and \( l_B \), assume that ensembles \( (A, p_A) \), \( (A \times B, p_{AB}) \), and \( (A \times \hat{B}, p_{A\hat{B}}) \) have a hash property. Then for any \( \delta > 0 \) and all sufficiently large \( n \), there are \( \varepsilon_{\hat{B}} > \varepsilon_A > 0 \), functions (sparse matrices) \( A \in A \), \( B \in B \), \( \hat{B} \in \hat{B} \), and a vector \( c \in \text{Im}A \) such that

\[
\begin{align*}
\text{Rate} & > I(X; Y) - I(X; Z) - \delta \\
\text{Error}_{Y|X}(A, B, \hat{B}, c) & < \delta \\
\text{Leakage}_{Z|X}(A, B, \hat{B}, c) & < \delta.
\end{align*}
\]

By assuming that the channel between \( X \) and \( Y \) is more capable than that between \( X \) and \( Z \), \( \mu_X \) attains the secrecy capacity defined by (12), and \( \delta \to 0 \), the rate of the proposed code is close to the secrecy capacity.

For a general wiretap channel \( \mu_{Y|Z|X} \), let \( F : \hat{X} \to X \) be a channel (non-deterministic map) corresponding to a conditional probability distribution \( \mu_{X|\hat{X}} \) and assume that

\[
\mu_{\hat{X}|X}(\hat{x}, x) \equiv \mu_{X|\hat{X}}(x|\hat{x})
\]

achieves the maximum of the right hand side of (10). By using a proposed code for the channel \( \mu_{Y|Z|\hat{X}} \) defined as

\[
\mu_{Y|Z|\hat{X}}(y, z|\hat{x}) \equiv \sum_x \mu_{Y|Z|X}(y, z|x) \mu_{X|\hat{X}}(x|\hat{x})
\]

with the input distribution \( \mu_{\hat{X}} \), we construct a code for the channel \( \mu_{Y|Z|X} \) as

\[
\varphi(m, w) \equiv F(g_{A\hat{B}}(c, m, w))
\]

\[
\varphi^{-1}(y) \equiv Bg_A(c|y),
\]

where \( g_{A\hat{B}}(c, m, w) \) outputs the channel input \( \hat{x} \equiv (\hat{x}_1, \ldots, \hat{x}_n) \in \hat{X}^n \) of outer channel \( \mu_{Y|Z|\hat{X}} \), \( F \) is defined as

\[
F(\hat{x}) \equiv (F(\hat{x}_1), \ldots, F(\hat{x}_n)),
\]

and \( g_A(c|y) \) reproduces \( \hat{x} \) with small error probability. Then the rate of this code is close to the secrecy capacity of the channel \( \mu_{Y|Z|X} \) defined by (10).

**V. Universal Wiretap Channel Coding**

In this section we consider the fixed-rate universal wiretap channel coding for any stationary memoryless channel \( \mu_{Y|Z|X} \), where an input distribution \( \mu_X \) is given and it is enough to know the upper bound of \( H(X|Y) \) and the lower bound of \( I(X; Z) \) before constructing the code. It should be noted here that we have to know the sizes of \( X \), \( Y \), and \( Z \) in advance.
For a given \( R_A, R_B > 0 \), let \( p_A \) and \( p_B \) be ensembles of functions

\[
A : X^n \to \mathcal{X}^{l_A} \\
B : X^n \to \mathcal{X}^{l_B} \\
\hat{B} : X^n \to \mathcal{X}^{l_B}
\]

satisfying

\[
R_A = \frac{\log |\text{Im}A|}{n} \\
R_B = \frac{\log |\text{Im}B|}{n} \\
R_{\hat{B}} = \frac{\log |\text{Im}\hat{B}|}{n},
\]

respectively. It should be noted that \( \text{Im}B \) represents the set of all messages, \( R_B \) represents the encoding rate of a confidential message.

We fix functions \( A, B, \hat{B} \) and a vector \( c \in \mathcal{X}^{l_A} \) available for an encoder, a decoder, and an eavesdropper. We construct a stochastic encoder and assume that the encoder uses a random sequence \( w \in \mathcal{X}^{l_{\hat{B}}} \), which is generated uniformly at random and independently of the channel and the message \( m \in \mathcal{X}^{l_B} \). We define the same encoder and decoder as defined in the last section except to replace and \( g_A \) by \( \tilde{g}_A \) defined by (3).

Let \( M \) and \( W \) be random variables corresponding to \( m \) and \( w \), respectively, where the probability distributions \( p_M \) and \( p_W \) are given by (14) and (15), respectively. The decoding error probability \( \text{Error}_{Y|X}(A, B, \hat{B}, c) \) and the information leakage \( \text{Leakage}_{Z|X}(A, B, \hat{B}, c) \) are given by (16) and (17), respectively.

We have the following theorem. It should be noted that alphabets \( \mathcal{X} \) and \( \mathcal{Y} \) is allowed to be non-binary, and the channel is allowed to be asymmetric.

**Theorem 2:** For \( R_A, R_B, \) and \( R_{\hat{B}} \), Assume that ensembles \((A, p_A), (A \times B, p_{AB}), \) and \((A \times B \times \hat{B}, p_{AB\hat{B}})\) have a hash property. Let \( \mu_X \) be the distribution of the channel input satisfying

\[
R_A + R_B + R_{\hat{B}} < H(X),
\]

where \( R_B \) represents the encoding rate of a confidential message. Then for any \( \delta > 0 \) and all sufficiently large \( n \), there are functions (sparse matrices) \( A \in \mathcal{A}, B \in \mathcal{B}, \hat{B} \in \hat{B} \), and a vector \( c \in \text{Im}A \) such that

\[
\text{Error}_{Y|X}(A, B, \hat{B}, c) < \delta \quad (22)
\]

\[
\text{Leakage}_{Z|X}(A, B, \hat{B}, c) < \delta \quad (23)
\]

for any stationary memoryless channel \( \mu_{YZ|X} \) satisfying

\[
R_A \geq H(X|Y) \quad (24)
\]

\[
R_{\hat{B}} \geq I(X; Z). \quad (25)
\]

**Remark 1:** It should be noted that (21), (24), and (25) imply

\[
0 < R_A < H(X|Z) \\
0 < R_B < I(X; Y) - I(X; Z).
\]
VI. SECRET KEY AGREEMENT FROM CORRELATED SOURCE OUTPUTS

In this section we construct codes for secret key agreement from the correlated source outputs \((X, Y, Z)\) introduced in [12] (see Fig. 2), where a sender, a receiver, and an eavesdropper have access to \(X, Y,\) and \(Z,\) respectively. The secret key capacity, which represents the optimal key generation rate, is given in [17] as

\[
\text{Capacity} = \sup_{n, t, (C^t, \hat{X}, \hat{Y})} \frac{1}{n} \left[ I(\hat{X}; \hat{Y}) - I(\hat{X}; Z, C^t) \right],
\]  

where the supremum is taken over all \(n, t,\) and protocols \((C^t, \hat{X}, \hat{Y})\) satisfying Markov conditions

\[
\begin{align*}
Y^n Z^n C^t_{i+1} \hat{X} \hat{Y} &\leftrightarrow X^n C^t_1 \leftrightarrow C_i, \text{ if } i \text{ is odd} \\
X^n Z^n C^t_{i+1} \hat{X} \hat{Y} &\leftrightarrow Y^n C^t_1 \leftrightarrow C_i, \text{ if } i \text{ is even} \\
Y^n \hat{Z} &\leftrightarrow X^n C^t_1 \leftrightarrow \hat{X} \\
X^n \hat{Z} &\leftrightarrow Y^n C^t_1 \leftrightarrow \hat{Y}
\end{align*}
\]

in which \(C^t\) represents the communication between the sender and the receiver via a public channel and finally the sender and the receiver generate \(\hat{X}\) and \(\hat{Y},\) respectively. It should be noted that \(\hat{X} \neq \hat{Y}\) is allowed with high probability. According to [3][4], there are three steps in a secret key agreement: advantage distillation, information reconciliation, and privacy amplification. This section deals with the combination of information reconciliation and privacy amplification studied in [1][4][15][16]. In the following, we assume that a fixed joint distribution \(\mu_{XYZ}\) satisfies

\[
I(X; Y) - I(X; Z) = H(X|Z) - H(X|Y) > 0
\]

and do not deal with advantage distillation. From (26), we can construct a protocol whose rate is close to the secret key capacity by combining an advantage distillation protocol \((C^t, \hat{X}, \hat{Y})\) with the following one-way secret key agreement protocol, where \(I(\hat{X}; \hat{Y}) - I(\hat{X}; Z, C^t)\) is close to the secret key capacity.

In the following, we focus on the one way secret key agreement protocol. When secret key agreement is allowed to be one-way from the sender to the receiver, the forward secret key capacity is given in [1] by

\[
\text{Capacity} = \max_{C, \hat{X}} \left[ I(\hat{X}; Y|C) - I(\hat{X}; Z|C) \right],
\]  

where the maximum is taken over all random variables \(C\) and \(\hat{X}\) that satisfy the Markov condition

\[
\hat{X} \leftrightarrow C \leftrightarrow X \leftrightarrow YZ.
\]

Since

\[
I(\hat{X}; Y|C) - I(\hat{X}; Z|C) = I(\hat{X}; Y, C) - I(\hat{X}; Z, C),
\]

then we can construct an optimal one-way secret key agreement protocol by applying the following protocol to the correlated source \((\hat{X}, (Y, C), (Z, C)),\) which achieves the maximum on the right hand side of (27).

The following construction is based on [16]. We fix functions

\[
A : \mathcal{X}^n \rightarrow \mathcal{X}^{t_A} \\
B : \mathcal{X}^n \rightarrow \mathcal{X}^{t_B}
\]
available for an encoder, a decoder, and an eavesdropper, where
\[
l_A \equiv n\frac{H(X|Y) + \varepsilon_A}{\log |X|}
\]
\[
l_B \equiv n\frac{H(X|Z) - H(X|Y)}{\log |X|}
\]
\[
= n\frac{I(X;Y) - I(X;Z)}{\log |X|}.
\]

Then a secret key agreement protocol is described below (see Fig. 4).

**Encoding:** Let \( x \in \mathcal{X}^n \) be a sender’s random sequence. The sender transmits \( c \) to a legitimate receiver via a public channel and generates a secret key by \( m \), where \( c \) and \( m \) are defined as
\[
c \equiv Ax
\]
\[
m \equiv Bx,
\]
respectively.

**Decoding:** Let \( y \in \mathcal{Y}^n \) be a receiver’s random sequence, and \( c \equiv Ax \) be a codeword received from the sender via a public channel. The receiver generates a secret key by \( Bg_A(c|y) \), where \( g_A \) is defined by (2).

Let \( C \) and \( M \) be random variables corresponding to \( c \) and \( m \) defined by (28) and (29), respectively. The key generation rate is given by
\[
\text{Rate} \equiv \frac{H(M)}{n}.
\]

The error probability of the secret key agreement is given by
\[
\text{Error}_{XY}(A, B) \equiv \mu_{XY} (\{(x, y) : Bg_A(Ax, y) \neq Bx\})\).
\]

The information leakage is given by
\[
\text{Leakage}_{XYZ}(A, B) \equiv \frac{I(M;Z^n,C)}{n}.
\]
We have the following theorem.

**Theorem 3:** For given $l_A$ and $l_B$, assume that ensembles $(\mathcal{A}, p_A)$ and $(\mathcal{A} \times \mathcal{B}, p_{AB})$ have a hash property. For all $\delta > 0$ and sufficiently large $n$, there are $\varepsilon_A > 0$ and functions (sparse matrices) $A \in A$ and $B \in B$ such that the above secret key agreement protocol satisfies

$$\text{Rate} > I(X;Y) - I(X;Z) - \delta$$  \hfill (33)

$$\text{Error}_{XY}(A, B) < \delta$$ \hfill (34)

$$\text{Leakage}_{XYZ}(A, B) < \delta.$$ \hfill (35)

By assuming that random variables $C$ and $\tilde{X}$ attain the forward secret key capacity given by (27) and the sender sends message $C$ via public channel before the protocol, the rate of the proposed secret key agreement protocol for correlated sources $(\tilde{X}, (Y, C), (Z, C))$ is close to the forward secret key capacity.

**VII. Universal Secret Key Agreement from Correlated Source Outputs**

In this section, we construct a fixed-rate universal secret key agreement scheme for any stationary memoryless sources $(X, Y, Z)$, where it is enough to know the upper bound of $H(X|Y)$ and the lower bound of $H(X|Z)$ before constructing the code. It should be noted here that we have to know the sizes of $\mathcal{X}, \mathcal{Y},$ and $\mathcal{Z}$ in advance.

For a given $R_A, R_B > 0$, let $p_A$ and $p_B$ be ensembles of functions

$$A : \mathcal{X}^n \to \mathcal{X}^{l_A}$$

$$B : \mathcal{X}^n \to \mathcal{X}^{l_B},$$

where

$$l_A \equiv \frac{nR_A}{\log |\mathcal{X}|}$$

$$l_B \equiv \frac{nR_B}{\log |\mathcal{X}|}.$$ 

We use the same secret key agreement protocol as that described in the last section except that we replace $g_A$ by $\tilde{g}_A$ defined by (3).

The key generation rate $\text{Rate}$, the error probability $\text{Error}_{XY}(A, B)$, and the information leakage $\text{Leakage}(A, B)$ are defined by (30), (31), and (32), respectively.

We have the following theorem.

**Theorem 4:** For given $R_A$ and $R_B$, assume that ensembles $(\mathcal{A}, p_A)$ and $(\mathcal{A} \times \mathcal{B}, p_{AB})$ have a hash property. For all $\delta > 0$ and sufficiently large $n$, there are functions (sparse matrices) $A \in A$ and $B \in B$ such that the above secret key agreement scheme satisfies

$$\text{Rate} > R_B - \delta$$ \hfill (36)

$$\text{Error}_{XY}(A, B) < \delta$$ \hfill (37)

$$\text{Leakage}_{XYZ}(A, B) < \delta.$$ \hfill (38)

for any stationary memoryless source $(X, Y, Z)$ satisfying

$$R_A > H(X|Y)$$ \hfill (39)
where $x$ April 12, 2010 DRAFT

and maximizations are the linear programming problems because the condition $A$ third condition. It should be noted that it is realized by the linear programming method because the conditions presented in [19]. We use the fact that the analysis of error probability in the proof of theorems is not changed if we replace the minimum-divergence encoder $\text{GF}(2)$ $B$ paper.

where $(\nu(t), \nu(1)) \equiv (1 - t/n, t/n)$. Then the function $g_{ABB}^\prime$ is realized by finding $x'$ that satisfies $Ax' = c$, $Bx' = m$, $\hat{B}x' = w$, and $\sum_{i=1}^{n} x'_i = t$ and declaring the encoding error if there is no such $x'$ that satisfies $Ax' = c$, $Bx' = m$, $\hat{B}x' = w$, and $\sum_{i=1}^{n} x'_i = t$, where we consider $x'$ as a real-valued vector in the third condition. It should be noted that it is realized by the linear programming method because the conditions $Ax' = c$, $Bx' = m$, $\hat{B}x' = w$ can be represented by linear inequalities by using the technique of [9].

Next, we construct the maximum-likelihood decoder $g_A$ defined by (2). The following construction is equivalent to [9]. The function $g_A$ is realized by

$$g_A(c|y) \equiv \begin{cases} 
\arg\min_{x':Ax'=c} \sum_{i=1}^{n} x'_i, & \text{if } 0 \leq \mu_{X|Y}(1|0), \mu_{X|Y}(1|1) \leq 1/2 \\
\arg\max_{x':Ax'=c} \sum_{i=1}^{n} [-1]^{y_i} x'_i, & \text{if } 0 \leq \mu_{X|Y}(0|0), \mu_{X|Y}(0|1) \leq 1/2 \\
\arg\min_{x':Ax'=c} \sum_{i=1}^{n} [-1]^{y_i} x'_i, & \text{if } 0 \leq \mu_{X|Y}(0|0), \mu_{X|Y}(0|1) \leq 1/2 \\
\arg\max_{x':Ax'=c} \sum_{i=1}^{n} x'_i, & \text{if } 0 \leq \mu_{X|Y}(1|0), \mu_{X|Y}(1|1) \leq 1/2 
\end{cases}$$

where $x'$ and $y$ are considered as real-valued vectors in $\sum_{i=1}^{n} x'_i$ and $\sum_{i=1}^{n} [-1]^{y_i} x'_i$. The above minimizations and maximizations are the linear programming problems because the condition $Ax' = c$ can be represented by linear inequalities by using the technique of [9].
Finally, we construct the minimum-entropy decoder \( \tilde{g}_A \) defined by (3). The following construction is presented in [19], which is based on the idea presented in [5]. The function \( \tilde{g}_A \) can be realized as

\[
x_{t,\text{min}} \equiv \arg \min_{x'} \sum_{i=1}^{n} y_i x'_i \\
x_{t,\text{max}} \equiv \arg \max_{x'} \sum_{i=1}^{n} y_i x'_i
\]

where \( x' \) and \( y \) are considered as real-valued vectors in \( \sum_{i=1}^{n} x'_i \) and \( \sum_{i=1}^{n} y_i x'_i \). The derivation of (44) is presented in [19, Appendix A]. We can use the linear programming method to obtain \( x_{t,\text{min}} \) and \( x_{t,\text{max}} \) because the constraint \( Ax' = c \) can be represented by linear inequalities by using the technique introduced in [9]. It should be noted that \( g_A \) can be replaced by

\[
g'_A(c|y) \equiv \arg \min_{x' \in \bigcup_{n=0}^{\infty} \{x_{t,\text{min}},x_{t,\text{max}}\}} H(x'|y),
\]

where \( x' \) and \( y \) are considered as real-valued vectors in \( \sum_{i=1}^{n} x'_i \) and \( \sum_{i=1}^{n} y_i x'_i \). The derivation of (44) is presented in [19, Appendix A]. We can use the linear programming method to obtain \( x_{t,\text{min}} \) and \( x_{t,\text{max}} \) because the constraint \( Ax' = c \) can be represented by linear inequalities by using the technique introduced in [9]. It should be noted that \( g_A \) can be replaced by

\[
g'_A(c|y) \equiv \arg \min_{x' \in C_A(c) \cap T} H(x'|y)
\]

by assuming that \( U \) defined by (76) or \( t \) defined by (41) is shared by the encoder and the decoder, where \( x_{t,\text{min}} \) and \( x_{t,\text{max}} \) are defined by (42) and (43), respectively.

**IX. PROOF OF THEOREMS**

**A. Proof of Theorem 1**

We use the following lemma which is proved in Appendix.

**Lemma 4:** Let \( g_{AB}(c, m|z) \) be defined as

\[
g_{AB}(c, m|z) \equiv \max_{x' \in C_{AB}(c, m)} \mu_{X|Z}(x'|z).
\]

Then, for all \( \delta' > 0 \), all sufficiently small \( \gamma > 0 \), and all sufficiently large \( n \), there are functions (sparse matrices) \( A \in A, B \in B, \tilde{B} \in \tilde{B} \), and a vector \( c \in \text{Im}A \) such that

\[
p_{MWY|z}(m, w, y, z) : \left\{ \begin{array}{l} g_{AB}(c, m, w) / \notin T_{X, \gamma} \\
or z / \notin T_{Z|X, \gamma}(g_{AB}(c, m, w)) \\
or g_{AB}(c, m, w) / \neq g_A(c|y) \\
or g_{AB}(c, m, w) / \neq g_A(c|y) \end{array} \right\} \leq \delta'. \tag{46}
\]

Now we prove Theorem 1. The equality (18) has already been shown. Since \( g_{AB}(c, m, w) = g_A(c|y) \) implies

\[
\varphi^{-1}(y) = Bg_A(c|y) = Bg_{AB}(c, m, w) = m
\]

for all \( c \) and \( w \), the inequality (19) comes immediately from Lemma 4 by letting \( \delta' < \delta \).
In the following we prove (20). From Lemma 4 and Fano’s inequality, we have

$$H(g_{ABB}(c, M, W)|Z^n, M) \leq h(\delta') + n\delta' \log |\mathcal{X}|$$

(47)

for all $\delta > 0$ and all sufficiently large $n$, where $h$ is defined by (8).

Let $\tilde{x} \equiv g_{ABB}(c, m, w)$, $\tilde{X}^n \equiv g_{ABB}(c, M, W)$ and $\tilde{X}$ be defined as

$$\tilde{X} \equiv \{g_{ABB}(c, m, w) : m \in \mathcal{X}^n, w \in \mathcal{X}^l\}.$$  

Then the probability distribution $P_{\tilde{X}Z}$ is given by

$$P_{\tilde{X}Z}(\tilde{x}, z) = \sum_{m, w} \mu_{Z|X}(z|x)P_M(m)P_W(w),$$

(48)

where the summation equals zero when $\tilde{x} \notin \tilde{X}$ and the second equality comes from the fact that if $\tilde{x} \in \tilde{X}$ then there is a unique pair $(m, w)$ such that $\tilde{x} = g_{ABB}(c, m, w)$. From Lemma 15, we have

$$\mu_{Z|X}(z|x) \leq 2^{-n[H(Z|X) - \zeta_{Z|X}(\gamma|\gamma)]}$$

(49)

for $x \in \mathcal{T}_{X, \gamma}$ and $z \in \mathcal{T}_{Z|X, \gamma}(x)$. Then the joint entropy $H(\tilde{X}^n, Z^n)$ is given by

$$H(\tilde{X}^n, Z^n) \geq \sum_{\tilde{x} \in \mathcal{T}_{X, \gamma}} \sum_{\tilde{z} \in \mathcal{T}_{Z|X, \gamma}(\tilde{x})} P_{\tilde{X}Z}(\tilde{x}, \tilde{z}) \log \frac{1}{P_{\tilde{X}Z}(\tilde{x}, \tilde{z})}$$

$$\geq \sum_{\tilde{x} \in \mathcal{T}_{X, \gamma}} \sum_{\tilde{z} \in \mathcal{T}_{Z|X, \gamma}(\tilde{x})} P_{\tilde{X}Z}(\tilde{x}, \tilde{z}) \left[n[H(Z|X) - \zeta_{Z|X}(\gamma|\gamma)] + \log |\text{Im}B||\text{Im}\tilde{B}|\right]$$

$$\geq n[1 - \delta'] \left[H(Z|X) + \frac{1}{n} \log |\text{Im}B||\text{Im}\tilde{B}| - \zeta_{Z|X}(\gamma|\gamma)\right]$$

$$\geq n[H(Z|X) + I(X; Y)] - \log \frac{|\mathcal{X}|^{|l_x| + |l_g|}}{|\text{Im}B||\text{Im}\tilde{B}|} - n[\delta' \log |\mathcal{X}|] + \zeta_{Z|X}(\gamma|\gamma) + \varepsilon_{\tilde{B}}$$

(50)

for sufficiently large $n$, where the second inequality comes from (48) and (49), and the third inequality comes from Lemma 4. Then we have

$$I(M; Z^n) = H(M) + H(Z^n) - H(Z^n, M)$$

$$= H(M) + H(Z^n) - H(Z^n, g_{ABB}(c, M, W)) + H(g_{ABB}(c, M, W)|Z^n, M)$$

$$= H(M) + H(Z^n) - H(Z^n, g_{ABB}(c, M, W)) + H(g_{ABB}(c, M, W)|Z^n, M)$$

$$\leq H(M) + H(Z^n) - H(Z^n, g_{ABB}(c, M, W)) + h(\delta') + n\delta' \log |\mathcal{X}|$$

$$\leq n[I(X; Y) - I(X; Z)] + H(Z^n) - n[H(Z|X) + I(X; Y)] + \log \frac{|\mathcal{X}|^{|l_x| + |l_g|}}{|\text{Im}B||\text{Im}\tilde{B}|}$$

$$+ n[\delta' \log |\mathcal{X}|] + \zeta_{Z|X}(\gamma|\gamma) + \varepsilon_{\tilde{B}} + h(\delta') + n\delta' \log |\mathcal{X}|$$

$$< n\delta$$

(51)

for sufficiently large $n$, where the third inequality comes from the fact that $Bg(c, M, W) = M$, the first inequality comes from (47), the second inequality comes from (50), and we choose suitable $\varepsilon_{\tilde{B}}, \gamma, \delta' > 0$ to satisfy the last inequality. From this inequality we have (20).
B. Proof of Theorem 2

We use the following lemmas, which are proved in Appendix.

**Lemma 5:** If \( I(X; Z) \leq R \), then for all \( \epsilon > 0 \) there is a random variable \( \tilde{Z} \) taking values in \( \tilde{Z} \equiv X \times Z \) and a function \( f \) such that

\[
I(X; \tilde{Z}) = R + \epsilon
\]

\[
Z = f(\tilde{Z}).
\]

**Lemma 6:** Let \( \tilde{g}_{AB}(c, m|\tilde{z}) \) be defined as

\[
\tilde{g}_{AB}(c, m|\tilde{z}) \equiv \arg \min_{x' \in C_{AB}(c, m)} H(x'|\tilde{z}).
\]

Then, for all \( \delta' > 0 \), all sufficiently small \( \gamma > 0 \), and sufficiently large \( n \), there are functions (sparse matrices) \( A \in A, B \in B, \tilde{B} \in \tilde{B} \), and a vector \( c \in \text{Im} A \) such that

\[
p_{M,W,Y,Z} \left( \begin{array}{c}
\tilde{g}_{ABB}(c, m, w) \notin T_{X,\gamma} \\
\text{or} \tilde{z} \notin T_{Z|X,\gamma}(\tilde{g}_{ABB}(c, m, w)) \\
\text{or} \tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_{A}(c|y) \\
\text{or} \tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_{AB}(c, m|\tilde{z})
\end{array} \right) \leq \delta'
\]

for any \( \mu_{Y|Z,X} \) satisfying

\[
R_A + R_B + R_{\tilde{B}} < H(X)
\]

\[
R_A > H(X|Y)
\]

\[
R_A + R_B > H(X|\tilde{Z}).
\]

Now we prove Theorem 2. The inequality (22) is shown similarly to the proof of (19).

In the following we prove (23). From Lemma 5, there is \( \tilde{Z} \in \tilde{Z} \) such that

\[
I(X; \tilde{Z}) = R_B + \epsilon,
\]

where \( \epsilon > 0 \) is specified later. From Lemma 6 and Fano’s inequality, we have

\[
H(\tilde{g}_{ABB}(c, M, W)|\tilde{Z}^n, M) \leq h(\delta') + n\delta' \log |X|
\]

for all \( \delta' > 0 \) and sufficiently large \( n \), where \( h \) is defined by (8). Similarly to the proof of (50), we have

\[
H(\tilde{Z}^n: \tilde{g}_{ABB}(c, M, W)) \geq n(1 - \delta') \left[ H(\tilde{Z}|X) + \frac{1}{n} \log |\text{Im}B||\text{Im}\tilde{B}| - \zeta_{Z|X}(\gamma|\gamma) \right]
\]

\[
\geq nH(\tilde{Z}|X) + \log |\text{Im}B||\text{Im}\tilde{B}| - n \left[ \delta' \log |X||\tilde{Z}| + \zeta_{Z|X}(\gamma|\gamma) \right]
\]

\[
\geq n[H(\tilde{Z}|X) + R_B + R_{\tilde{B}}] - n \left[ \delta' \log |X||\tilde{Z}| + \zeta_{Z|X}(\gamma|\gamma) \right].
\]

where the second inequality comes from the fact that \( R_B + R_{\tilde{B}} < H(X) \leq \log |X| \). Then we have

\[
I(M; Z^n) = I(M; f(\tilde{Z}_1), \ldots, f(\tilde{Z}_n))
\]

\[
\leq I(M; \tilde{Z}^n)
\]

\[
= H(M) + H(\tilde{Z}^n) - H(\tilde{Z}^n, M, \tilde{g}_{ABB}(c, M, W)) + H(\tilde{g}_{ABB}(c, M, W)|\tilde{Z}^n, M)
\]
\[ \leq H(M) + H(\tilde{Z}^n) - H(\tilde{Z}^n, \tilde{g}_{ABB}(c, M, W)) + h(\delta') + n\delta' \log |\mathcal{X}| \]
\[ \leq nR_B + H(\tilde{Z}^n) - n[H(\tilde{Z}|X) + R_B + R_{\mathcal{G}}] + n \left[ \delta' \log |\mathcal{X}| |\tilde{Z}| + \zeta_{Z|X}(\gamma|\gamma) \right] + h(\delta') + n\delta' \log |\mathcal{X}| \]
\[ \leq n[I(X; \tilde{Z}) - R_{\mathcal{G}}] + n \left[ \delta' \log |\mathcal{X}| |\tilde{Z}| + \zeta_{Z|X}(\gamma|\gamma) \right] + h(\delta') + n\delta' \log |\mathcal{X}| \]
\[ \leq n \left[ \epsilon + \delta' \log |\mathcal{X}| |\tilde{Z}| + \zeta_{Z|X}(\gamma|\gamma) \right] + h(\delta') \]
\[ < n\delta \]  

where the second inequality comes from (57) and \( M = Bg_{ABB}(c, M, W) \), the third inequality comes from (58), the fifth inequality comes from (56), and we choose a suitable \( \gamma > 0 \), a suitable \( \epsilon > 0 \), and a suitable \( \delta' > 0 \) to satisfy the last inequality. From this inequality, we have (23).

C. Proof of Theorem 3

We use the following lemma which is proved in Appendix.

**Lemma 7:** Let \( g_{AB}(c, m|z) \) be defined as
\[
g_{AB}(c, m|z) = \arg \max_{x' \in A(c, m)} \mu_{X|Z}(x'|z).
\]

Then, for any \( \delta' > 0 \), and all sufficiently large \( n \), there are functions (sparse matrices) \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that
\[
p_{XYZ} \left( \{ (x, y, z) : g_A(Ax|y) \neq x \ \text{or} \ g_{AB}(Ax, Bx|z) \neq x \} \right) \leq \delta'.
\]

Now we prove Theorem 3.

First, we prove (34). Since \( g_A(Ax|y) = x \) implies \( Bg_A(Ax|y) = Bx \), then the inequality (34) comes immediately from Lemma 7 by letting \( \delta' < \delta \).

Next, we prove (35). From Lemma 7 and Fano’s inequality, we have
\[
H(X^n|Z^n, C, M) \leq h(\delta') + n\delta' \log |X|
\]
for all \( \delta > 0 \) and all sufficiently large \( n \), where \( h \) is defined by (8). This implies that
\[
H(Z^n, C, M) \geq H(X^n, Z^n, C, M) - h(\delta') - n\delta' \log |X|
\]
\[
= H(X^n, Z^n) - h(\delta') - n\delta' \log |X|
\]
\[ (61) \]
for all \( \delta > 0 \) and all sufficiently large \( n \), where the equality comes from the definitions (28) and (29) of \( C \) and \( M \). Then we have
\[
I(M; Z^n, C) = H(Z^n, C) + H(M) - H(Z^n, C, M)
\]
\[
\leq H(Z^n) + H(C) + H(M) - H(Z^n, C, M)
\]
\[
\leq H(Z^n) + H(C) + H(M) - H(X^n, Z^n) + h(\delta') + n\delta' \log |X|
\]
\[
\leq H(Z^n) + n[H(X|Y) + \epsilon_{A}] + n[H(X|Z) - H(X|Y)] - H(X^n, Z^n) + h(\delta') + n\delta' \log |X|
\]
\[
= n\epsilon_{A} + h(\delta') + n\delta' \log |X|
\]
\[ < n\delta,
\]
\[ (62) \]
where the second inequality comes from (61), the third inequality comes from the definitions (28) and (29) of \( C \) and \( M \), and we choose a suitable \( \varepsilon_\mathcal{A} > 0 \) and a suitable \( \delta' > 0 \) to satisfy the last inequality. From this inequality we have (35).

Finally, we prove (33). We have

\[
H(M) = H(M) + H(Z^n, C) \geq H(Z^n, C, M) - H(C)
\]

\[
\geq H(X^n, Z^n) - h(\delta') - n\delta' \log |X| - H(Z^n) - n[H(X|Y) + \varepsilon_\mathcal{A}]
\]

\[
= n[I(X; Z) - I(X; Y)] - n\varepsilon_\mathcal{A} - h(\delta') - n\delta' \log |X|
\]

\[
\geq n[I(X; Z) - I(X; Y)] - n\delta,
\]

where the second inequality comes from (61), and we choose a suitable \( \varepsilon_\mathcal{A} > 0 \) and a suitable \( \delta' > 0 \) to satisfy the last inequality. From this inequality we have (33).

D. Proof of Theorem 4

We use the following lemmas which are proved in Appendix.

Lemma 8: If \( H(X|Z) \geq R \), then for all \( \varepsilon > 0 \) there is a random variable \( \tilde{Z} \) taking values in \( \tilde{Z} \equiv \mathcal{X} \times \mathcal{Z} \) and a function \( f \) such that

\[
H(X|\tilde{Z}) = R - \varepsilon
\]

\[
Z = f(\tilde{Z}).
\]

Lemma 9: Let \( \tilde{g}_{AB}(c, m|\tilde{z}) \) be defined as

\[
\tilde{g}_{AB}(c, m|\tilde{z}) \equiv \arg \min_{x' \in C_{AB}(c, m)} H(x'|\tilde{z}).
\]

Then, for any \( \delta' > 0 \), and all sufficiently large \( n \), there are functions (sparse matrices) \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that

\[
p_{XY\tilde{Z}} \left( \left\{ (x, y, \tilde{z}) : g_A(Ax|y) \neq x \right. \left. \text{ or } g_{AB}(Ax, Bx|\tilde{z}) \neq x \right\} \right) \leq \delta'
\]

for any \( \mu_{XY\tilde{Z}} \) satisfying

\[
R_A > H(X|Y)
\]

\[
R_A + R_B > H(X|\tilde{Z}).
\]

Now we prove Theorem 4. In the following, we prove (38) and (36). The proof of (37) is similar to that of (64).

First, from Lemma 8 and (66), there is \( \tilde{Z} \in \tilde{Z} \) such that

\[
H(X|\tilde{Z}) = R_A + R_B - \varepsilon,
\]
where $\varepsilon > 0$ is specified later. From Lemma 9 and Fano’s inequality, we have

$$H(X^n|\tilde{Z}^n, C, M) \leq h(\delta') + n\delta' \log |X|$$

for all $\delta > 0$ and all sufficiently large $n$, where $h$ is defined by (8). This implies that

$$H(\tilde{Z}^n, C, M) \geq H(X^n, \tilde{Z}^n) - h(\delta') - n\delta' \log |X|. \quad (68)$$

Next, we prove (38). We have

$$I(M; Z^n, C) = I(M; f(\tilde{Z}_1), \ldots, f(\tilde{Z}_n), C) \leq I(M; \tilde{Z}^n, C) \leq H(\tilde{Z}^n) + H(C) + H(M) - H(\tilde{Z}^n, C, M) \leq H(\tilde{Z}^n) + H(C) + H(M) - H(X^n, \tilde{Z}^n) + h(\delta') + n\delta' \log |X| \leq H(\tilde{Z}^n) + nR_A + nR_B - H(X^n, \tilde{Z}^n) + h(\delta') + n\delta' \log |X| = n\varepsilon + h(\delta') + n\delta' \log |X| < n\delta, \quad (69)$$

where the third inequality comes from (68), the fourth inequality comes from the definitions (28) and (29) of $C$ and $M$, the second equality comes from (67) and we choose a suitable $\varepsilon > 0$ and a suitable $\delta' > 0$ to satisfy the last inequality. From this inequality, we have (38).

Finally, we prove (36). We have

$$H(M) \geq H(\tilde{Z}^n, C, M) - H(\tilde{Z}^n) - H(C) \geq H(X^n, \tilde{Z}^n) - h(\delta') - n\delta' \log |X| - H(\tilde{Z}^n) - nR_A = nR_B - n\varepsilon - h(\delta') - n\delta' \log |X|, \geq nR_B - n\delta, \quad (70)$$

where the second inequality comes from (68), the equality comes from (67), and we choose a suitable $\varepsilon > 0$ and a suitable $\delta' > 0$ to satisfy the last inequality. From this inequality, we have (36).

X. CONCLUSION

The constructions of codes for the wiretap channel and secret key agreement from correlated source outputs were presented. The optimality, reliability, and security of the codes were proved and the universal reliability and security were also proved. The proof of the theorems is based on the notion of a hash property for an ensemble of functions. Since an ensemble of sparse matrices has a hash property, we can construct codes by using sparse matrices and practical encoding and decoding methods are expected to be effective. We believe that our construction can be applied to a quantum channel to realize a quantum cryptography. However, it should be noted that the security criteria should be revised to the quantum version.
A. Proof of Lemmas

Before the proof of Lemmas 4 and 6, we prepare the following lemmas.

**Lemma 10 ([18, Lemma 8]):** For any \( A \) and \( u \in U^n \),
\[
p_C \left( \{ c : Au = c \} \right) = \sum_c p_C(c) \chi(Au = c) = \frac{1}{|\text{Im}A|}
\]
and for any \( u \in U^n \)
\[
E_{AC} [\chi(Au = c)] = \sum_{A, c} p_{AC}(A, c) \chi(Au = c) = \frac{1}{|\text{Im}A|}.
\]

**Lemma 11 ([18, Lemma 3]):** If \((A, p_A)\) satisfies (H4), then
\[
p_{AC} \left( \left\{ (A, c) : G \cap C_A(c) \neq \emptyset \right\} \right) \leq \frac{|G| \alpha_A |\text{Im}A|^2}{2} + \frac{\beta_A |\text{Im}A|}{2}
\]
for all \( G \subset U^n \) and all \( u / \notin G \).

**Lemma 12:** Assume that \( \varepsilon_B > \varepsilon_A > 0 \). For \( \beta_A \) satisfying \( \lim_{n \to \infty} \beta_A(n) = 0 \) and any \( \gamma > 0 \), there is a sequence \( \kappa \equiv \{ \kappa(n) \}_{n=1}^{\infty} \) and \( T \subset T_U \subset T_{X, \gamma} \) such that
\[
\lim_{n \to \infty} \kappa(n) = \infty \quad (71)
\]
\[
\lim_{n \to \infty} \kappa(n) \beta_A(n) = 0 \quad (72)
\]
\[
\lim_{n \to \infty} \frac{\log \kappa(n)}{n} = 0 \quad (73)
\]
and
\[
\gamma \geq D(\nu_U \| \mu_X) \quad (74)
\]
\[
\kappa \leq \frac{|T|}{|\text{Im}A||\text{Im}B||\text{Im}B|} \leq 2\kappa \quad (75)
\]
for all sufficiently large \( n \), where \( U \) is defined as
\[
U \equiv \arg \min_{U'} D(\nu_{U'} \| \mu_X). \quad (76)
\]

In the following, \( \kappa \) denotes \( \kappa(n) \).

**Proof:** Let
\[
\kappa(n) \equiv \begin{cases} 
 n^\xi & \text{if } \beta_A(n) = o\left(n^{-\xi}\right), \xi > 0 \\
 \frac{1}{\sqrt{\beta_A(n)}} & \text{otherwise}
\end{cases} \quad (77)
\]
for every \( n \). It is clear that \( \kappa \) satisfies (71) and (72). It is also clear that \( \kappa \) satisfies (73) when \( \beta_A(n) = o\left(n^{-\xi}\right) \), \( \xi > 0 \). If \( \beta_A(n) \) is not \( o\left(n^{-\xi}\right) \), there is \( \kappa' > 0 \) such that \( \beta_A(n)n^{\xi} > \kappa' \) and
\[
\frac{\log \kappa(n)}{n} = \frac{\log \frac{1}{\sqrt{\beta_A(n)}}}{2n} \leq \frac{\log \frac{\xi^\xi}{2n}}{2n} = \frac{\xi \log n - \log \kappa'}{2n} \quad (78)
\]
for all sufficiently large \( n \). This implies that \( \kappa \) satisfies (73). The inequality (74) comes from Lemma 21. From Lemma 21 and \( \varepsilon_B > \varepsilon_A > 0 \), we have

\[
R_A + R_B + R_B^\hat{B} + \frac{\log \kappa}{n} \leq H(U) - \lambda_X \leq H(X)
\]  

(79)

for all sufficiently large \( n \). Then we have

\[
|T_{X,\gamma}| \geq |T_U| \\
\geq 2^n[H(U) - \lambda_X], \\
\geq \kappa 2^n[R_A + R_B + R_B^\hat{B}] \\
= \kappa |\text{Im}A||\text{Im}B||\text{Im}\hat{B}|
\]

(80)

for all sufficiently large \( n \), where the first inequality comes from (74). This implies that there is \( T \subset T_U \subset T_{X,\gamma} \) such that

\[
\kappa \leq \frac{|T|}{|\text{Im}A||\text{Im}B||\text{Im}\hat{B}|} \leq 2\kappa
\]

(81)

for all sufficiently large \( n \).

**Remark 3:** It should be noted that we can let \( \xi \) be arbitrarily large in (77) when \( \beta_A(n) \) vanishes exponentially fast. This parameter \( \xi \) affects the upper bound of (46) and (52).

**B. Proof of Lemma 4**

In the following, we assume that \( \varepsilon_A, \varepsilon_B, \) and \( \gamma > 0 \) satisfy

\[
\varepsilon_B > \varepsilon_A > \max \{\zeta_{X,Y}(2\gamma)|2\gamma|, \zeta_{X,Z}(2\gamma)|2\gamma|\}
\]

(82)

Let \( \kappa \equiv \{\kappa(n)\}_{n=1}^\infty \) be a sequence satisfying (71)–(73), and \( U \) be defined by (76). Then (74) is satisfied for all \( \gamma > 0 \) and all sufficiently large \( n \). From Lemma 12, there is \( T \subset T_U \subset T_{X,\gamma} \) satisfying (75).

Let \( x \) an input of the channel, and \( y \) and \( z \) be the channel outputs of the receiver and the eavesdropper, respectively. Let \( m \) be a message and \( w \) be a random sequence. We define

\[
\text{• } g_{ABB}(c, m, w) \in T \subset T_{X,\gamma}
\]

(W1)

\[
\text{• } y \in T_{Y|X,\gamma}(g_{ABB}(c, m, w))
\]

(W2)

\[
\text{• } z \in T_{Z|X,\gamma}(g_{ABB}(c, m, w))
\]

(W3)

\[
\text{• } g_A(c|y) = g_{ABB}(c, m, w)
\]

(W4)

\[
\text{• } g_{AB}(c, m|z) = g_{ABB}(c, m, w).
\]

(W5)
Then the left hand side of (46) is upper bounded by

\[
p_{MWYZ} \left( \begin{array}{l}
g_{ABB}(c, m, w) \notin T_{X, \gamma} \\
or z \notin T_{Z|X, \gamma}(g_{ABB}(c, m, w)) \\
or g_{ABB}(c, m, w) \neq g_A(c|y) \\
or g_{ABB}(c, m, w) \neq g_{AB}(c, m|z)
\end{array} \right) \leq p_{MWYZ}(S_1^c) + p_{MWYZ}(S_1 \cap S_2^c) + p_{MWYZ}(S_1 \cap S_3^c) + p_{MWYZ}(S_1 \cap S_2 \cap S_3^c) + p_{MWYZ}(S_1 \cap S_3 \cap S_3^c),
\]

where

\[
S_i \equiv \{(m, w, y, z) : (Wi)\}.
\]

First, we evaluate \(E_{ABB|C}[p_{MWYZ}(S_1^c)]\). From Lemma 2 and (81), we have

\[
E_{ABB|C}[p_{MWYZ}(S_1^c)] = p_{ABBCM} \left( \begin{array}{l}
(A, B, \hat{B}, c, m, w) : g_{ABB}(c, m, w) \notin \mathcal{T}
\end{array} \right)
\leq p_{ABBCM} \left( \begin{array}{l}
(A, B, \hat{B}, c, m, w) : \mathcal{T} \cap C_{ABB}(c, m, w) = \emptyset
\end{array} \right)
\leq \alpha_{ABB} - 1 + \frac{|\text{Im}A||\text{Im}B||\text{Im}\hat{B}|}{|\mathcal{T}|} \left( \beta_{ABB} + 1 \right)
\leq \alpha_{ABB} - 1 + \frac{\beta_{ABB} + 1}{\kappa}
\leq \frac{\delta'}{5}
\]

for all \(\delta' > 0\) and sufficiently large \(n\), where the last inequality comes from (71) and the properties (H2) and (H3) of an ensemble \((\mathcal{A} \times \mathcal{B} \times \hat{\mathcal{B}}, p_{ABB})\).

Next, we evaluate \(E_{ABB|C}[p_{MWYZ}(S_1 \cap S_2^c)]\) and \(E_{ABB|C}[p_{MWYZ}(S_1 \cap S_3^c)]\). From Lemma 16, we have

\[
E_{ABB|C}[p_{MWYZ}(S_1 \cap S_2^c)] \leq \frac{\delta'}{5}
\]

(85)

\[
E_{ABB|C}[p_{MWYZ}(S_1 \cap S_3^c)] \leq \frac{\delta'}{5}
\]

(86)

for all \(\delta' > 0\) and sufficiently large \(n\).

Next, we evaluate \(E_{ABB|C}[p_{MWYZ}(S_1 \cap S_2 \cap S_3^c)]\) and \(E_{ABB|C}[p_{MWYZ}(S_1 \cap S_3 \cap S_3^c)]\). In the following, we assume that

- \(x \in \mathcal{T} \subset \mathcal{T}_{X, \gamma}\)
- \(y \in \mathcal{T}_{Y|X, \gamma}(x)\)
- \(g_A(c|y) \neq x\).

From Lemma 14, we have \((x, y) \in \mathcal{T}_{XY, 2\gamma}\) and \(x \in \mathcal{T}_{X|Y, 2\gamma}(y)\). Then there is \(x' \in \mathcal{C}_A(c)\) such that \(x' \neq x\) and

\[
\mu_{X|Y}(x'|y) \geq \mu_{X|Y}(x|y) 
\geq 2^{-n[H(X|Y) + \xi_{X|Y}(2\gamma/2\gamma)]},
\]

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where the second inequality comes from Lemma 16. This implies that \([\mathcal{G}(y) \setminus \{x\}] \cap C_A(c) = \emptyset\), where
\[
\mathcal{G}(y) \equiv \left\{ x' : \mu_{X'|Y}(x'|y) \geq 2^{-[H(X|Y) + \zeta_X|Y(2\gamma)]} \right\}.
\]

Then we have
\[
E_{ABBC}[p_{MWYZ}(\mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_C)]
\leq E_{ABBCMW}\left[ \sum_{x \in T} \chi(g_{ABB}(C, M, W) = x) \sum_{y \in T_{Y|X,y}(x)} \mu_{Y|X}(y|x) \chi(g_A(C|y) \neq x) \right]
\leq E_{ABBCMW}\left[ \sum_{x \in T} \sum_{y \in T_{Y|X,y}(x)} \chi(Ax = C) \chi(Bx = M) \chi(\hat{B}x = W) \sum_{y \in T_{Y|X,y}(x)} \mu_{Y|X}(y|x) \chi(g_A(C|y) \neq x) \right]
= \sum_{x \in T} \sum_{y \in T_{Y|X,y}(x)} \mu_{Y|X}(y|x) E_{AC}\left[ \chi(g_A(C|y) \neq x) \chi(Ax = C) \chi(Bx = W) \right]
= \frac{1}{|\text{Im}B||\text{Im}B|} \sum_{x \in T} \sum_{y \in T_{Y|X,y}(x)} \mu_{Y|X}(y|x) E_{AC}\left[ \chi(g_A(C|y) \neq x) \chi(Ax = C) \right]
\leq \frac{1}{|\text{Im}B||\text{Im}B|} \sum_{x \in T} \sum_{y \in T_{Y|X,y}(x)} \mu_{Y|X}(y|x) p_{AC}\left( \left\{ (A, c) : [\mathcal{G}(y) \setminus \{x\}] \cap C_A(c) = \emptyset \right\} \right)
\leq \frac{1}{|\text{Im}B||\text{Im}B|} \sum_{x \in T} \sum_{y \in T_{Y|X,y}(x)} \mu_{Y|X}(y|x) \left[ \frac{2^{[n(H(X|Y) + \zeta_X|Y(2\gamma)]}\alpha_A}{|\text{Im}A|^2} + \frac{\beta_A}{|\text{Im}A|} \right]
\leq \frac{2|\text{Im}|[A 2^{-n|\xi_A - \zeta_X|Y(2\gamma)|}\alpha_A}{|\text{Im}A||\text{Im}B|} + \frac{2\kappa\beta_A}{|\text{Im}A|}
\leq \frac{\delta'}{5}
\] (87)
for all \(\delta' > 0\) and sufficiently large \(n\), where the second inequality comes from Lemma 10, the fourth inequality comes from Lemma 11 and the fact that
\[
|\mathcal{G}(y)| \leq 2^{n[H(X|Y) + \zeta_X|Y(2\gamma)]},
\]
the sixth inequality comes from (81), and the last inequality comes from (72), (82) and the properties (H1)–(H3) of an ensemble \((\mathcal{A}, \mathbf{p}_A)\). Similarly, we have
\[
E_{ABBC}[p_{MWYZ}(\mathcal{S}_1 \cap \mathcal{S}_3 \cap \mathcal{S}_C)] \leq \frac{2|\text{Im}|[A + \sigma 2^{-n|\xi_A - \zeta_X|Y(2\gamma)}]\alpha_{AB}}{|\text{Im}A||\text{Im}B|} + 2\kappa\beta_{AB}
\leq \frac{\delta'}{5}
\] (88)
for all \(\delta' > 0\) and sufficiently large \(n\).

Finally, from (83)–(88), we have the fact that for all \(\delta' > 0\) and sufficiently large \(n\) there are \(A \in \mathcal{A}, B \in \mathcal{B},\)
\[ \hat{B} \in \hat{B}, \text{ and } c \in \text{Im} A \text{ such that} \]

\[ p_{MWYZ} \left( \begin{array}{l}
g_{AB\hat{B}}(c, m, w) \notin T_{X, \gamma} \\
or \ z \notin T_{Z|X, \gamma}(g_{AB\hat{B}}(c, m, w)) \\
or \ g_{AB\hat{B}}(c, m, w) \neq g_A(c|y) \\
or \ g_{AB\hat{B}}(c, m, w) \neq g_{AB}(c, m|z) \\
\end{array} \right) \leq \delta'. \]

\[ \blacksquare \]

C. Proof of Lemma 5

The following proof is based on [16, Lemma 1]. If there is a random variable \( X' \) taking values in \( \mathcal{X} \) such that

\[ H(X|X', Z) = R' \]  

(89)

for given \((X, Z)\) and \(0 \leq R' \leq H(X|Z)\) the lemma is proved by letting

\[ R' = H(X) - R - \varepsilon \leq H(X|Z) \]

\[ \tilde{Z} = (X', Z) \]

\[ f(\tilde{z}) \equiv z \text{ for } \tilde{z} = (x', \tilde{z}) \]

because

\[ I(X; \tilde{Z}) = H(X) - H(X|\tilde{Z}) \]

\[ = H(X) - R' \]

\[ = R + \varepsilon. \]

(90)

The following proves the existence of \( X' \) satisfying (89). It is clear that \( 0 \leq H(X|X', Z) \leq H(X|Z) \) for any \((X, X', Z), H(X|X', Z) = H(X|Z) \) when \( X' \) is independent of \((X, Z)\), and \( H(X|X', Z) = 0 \) when \( X' = X \). Since \( H(X|X', Z) \) is a continuous function of the conditional distribution \( p_{X'|XZ} \), we have the existence of \( p_{X'|XZ} \) satisfying \( H(X|X', Z) = R' \) from the intermediate value theorem, where \( p_{X'|XZ} \) is given by

\[ p_{X'XZ}(x, x', v) = \mu_{XZ}(x, z)p_{X'|XZ}(x'|x, z) \]

for \((x, x', z) \in \mathcal{X} \times \mathcal{X} \times \mathcal{Z} \).

\[ \blacksquare \]

D. Proof of Lemma 6

Let \( \kappa \equiv \{ \kappa(n) \}_{n=1}^{\infty} \) be a sequence satisfying (71)–(73). Let \( U \) be defined by (76). Then (74) is satisfied for all \( \gamma > 0 \) and all sufficiently large \( n \). From Lemma 12, there is \( \mathcal{T} \subset \mathcal{T}_U \subset \mathcal{T}_{X, \gamma} \) satisfying (75).

We define

\[ \bullet \ g_{AB}(c, m, w) \in \mathcal{T} \subset \mathcal{T}_U \subset \mathcal{T}_{X, \gamma} \]  

(UW1)

\[ \bullet \ \tilde{z} \notin T_{Z|X, \gamma}(g_{AB\hat{B}}(c, m, w)) \]  

(UW2)

\[ \bullet \ g_A(c|y) = g_{AB\hat{B}}(c, m, w) \]  

(UW3)
where we assume that $n$ is large enough to satisfy $\mathcal{T}_{Z|X,\gamma}(x) \neq \emptyset$ for all $x \in \mathcal{T}_{X,\gamma}$. Then the left hand side of (22) is upper bounded by

\[
p_{MWYZ} \left\{ \begin{array}{l}
g_{AB}(c, m, w) \notin \mathcal{T}_{X,\gamma} \\
(\mathbf{m}, w, y, \bar{z}) : \\
\text{or } \bar{z} \notin \mathcal{T}_{Z|X,\gamma}(\tilde{g}_{ABB}(c, m, w)) \\
\text{or } \tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_{A}(c|y) \\
\text{or } \tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_{AB}(c, m|\bar{z})
\end{array} \right\} \leq p_{MWYZ}(S_1^c) + p_{MWYZ}(S_1 \cap S_2^c) + p_{MWYZ}(S_1 \cap S_2^c) + p_{MWYZ}(S_1 \cap S_3^c),
\]

where

\[
S_i = \{ (\mathbf{m}, w, y, \bar{z}) : (UW_i) \}.
\]

First, we evaluate $E_{ABBC}[p_{MWYZ}(S_1^c)]$. Similarly to the proof of (84), we have

\[
E_{ABBC}[p_{MWYZ}(S_1^c)] \leq \alpha_{ABBC} - 1 + \frac{\beta_{ABBC} + 1}{\kappa}. \tag{92}
\]

Next, we evaluate $E_{ABBC}[p_{MWYZ}(S_1 \cap S_2^c)]$. From Lemma 16, we have

\[
E_{ABBC}[p_{MWYZ}(S_1 \cap S_2^c)] \leq 2^{-n[\gamma - \lambda_{xy}]} \tag{93}
\]

Next, we evaluate $E_{ABBC}[p_{MWYZ}(S_1 \cap S_3^c)]$. Let

\[
\mathcal{G}(y) \equiv \{ x' : H(x'|y) \leq H(U|V) \}
\]

and assume that $(x, y) \in \mathcal{T}_{UV}$. Then we have

\[
E_{AC}[\chi(Ax = C) \chi(\tilde{g}_A(c|y) \neq x)] = p_{AC}\left\{ \begin{array}{l}
Ax = c \\
(A, c) : \exists x' \neq x \text{ s.t. } H(x'|y) \leq H(x|y) \text{ and } Ax' = c
\end{array} \right\} = p_{A}\left\{ \begin{array}{l}
\exists x' \neq x \text{ s.t. } H(x'|y) \leq H(x|y) \text{ and } Ax' = Ax
\end{array} \right\} p_{C}(\{ c : Ax = c \}) = \frac{1}{|\text{Im.}A|} p_{A}\left\{ \begin{array}{l}
\exists x' \neq x \text{ s.t. } H(x'|y) \leq H(U|V) \text{ and } Ax' = Ax
\end{array} \right\} \leq \frac{1}{|\text{Im.}A|} \max \left\{ \sum_{x' \in \mathcal{G}(y) \setminus\{x\}} p_{A}(\{ A : Ax = Ax' \}) , 1 \right\} \leq \frac{1}{|\text{Im.}A|} \max \left\{ 2^{n[H(U|V) + \lambda_{xy}]} \alpha_{A} + \beta_{A}, 1 \right\} = \frac{1}{|\text{Im.}A|} \max \left\{ 2^{-n[R_{A} - H(U|V) - \lambda_{xy}]} \alpha_{A} + \beta_{A}, 1 \right\} \leq \frac{1}{|\text{Im.}A|} \left[ \max \{ \alpha_{A}, 1 \} 2^{-n[H(U|V) + \lambda_{xy}]} + \beta_{A} \right], \tag{94}
\]

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where $|\cdot|^+$ is defined by (9), the third equality comes from Lemma 10, and the second inequality comes from Lemma 20 and the property (H4) of $(A, p_A)$. Let

$$F_{Y|X}(R) \equiv \min_{V \mid U} \left[D(\nu_{V|U} \mid \mu_{Y|X} \mid \nu_U) + |R - H(U|V)|^+\right],$$

where $V \mid U$ denotes the conditional type given type $U$. Then we have

$$E_{\text{ABC}} \left[p_{MWY\tilde{Z}}(S_1 \cap S_6)\right] \leq E_{\text{ABC}} \left[\sum_{x \in T} \sum_{y \in T_U} \mu_{Y|X}(y|x) \chi(g_{AB}(c, m, w) = x) \chi(\tilde{g}_A(c|y) \neq x)\right]$$

$$= E_{\text{ABC}} \left[\sum_{x \in T} \sum_{y \in T_U} \mu_{Y|X}(y|x) \chi(\tilde{g}_{AB}(C, M) = x) \chi(\tilde{g}_A(C|y) \neq x)\right]$$

$$\leq E_{\text{ABC}} \left[\sum_{x \in T} \sum_{y \in T_U} \mu_{Y|X}(y|x) \chi(Ax = C) \chi(Bx = M) \chi(\tilde{B}x = W) \chi(g_A(C|y) \neq x)\right]$$

$$= \sum_{x \in T} \sum_{y \in T_U} \mu_{Y|X}(y|x) E_{AC} \left[\chi(Ax = C) \chi(g_A(C|y) \neq x)\right] E_{\text{B\tilde{B}MW}} \left[\chi(Bx = M) \chi(\tilde{B}x = W)\right]$$

$$\leq \frac{1}{|\text{Im}A||\text{Im}B||\text{Im}C|} \sum_{x \in T} \sum_{y \in T_U} \sum_{\tilde{y} \in T_{\tilde{U}|\tilde{x}}} \mu_{Y|X}(y|x) \left[\max \{\alpha_A, 1\} 2^{-\min\{R_A - H(U|V)|^+ - \lambda_{xy}\} + \beta_A\right]$$

$$= \frac{1}{|\text{Im}A||\text{Im}B||\text{Im}C|} \sum_{x \in T} \sum_{y \in T_U} \sum_{\tilde{y} \in T_{\tilde{U}|\tilde{x}}} \mu_{Y|X}(y|x) \left[\max \{\alpha_A, 1\} 2^{-\min\{D(\nu_{V|U} \mid \mu_{Y|X} \mid \nu_U) + |R_A - H(U|V)|^+ - \lambda_{xy}\} + \beta_A\right]$$

$$\leq \frac{|T|}{|\text{Im}A||\text{Im}B||\text{Im}C|} \left[\max \{\alpha_A, 1\} 2^{-\min\{F_{Y|X}(R_A) - 2\lambda_{xy}\} + \beta_A\right]$$

$$\leq 2\kappa \left[\max \{\alpha_A, 1\} 2^{-\min\{F_{Y|X}(R_A) - 2\lambda_{xy}\} + \beta_A\right],$$

(95)

where the third inequality comes from Lemma 10 and (94), the fourth inequality comes from Lemmas 13 and 19, the fifth inequality comes from the definition of $F_{Y|X}$ and Lemma 18, and the last inequality comes from (81). Similarly, we have

$$E_{\text{ABC}} \left[p_{MWY\tilde{Z}}(S_1 \cap S_6)\right] \leq 2\kappa \left[\max \{\alpha_{AB}, 1\} 2^{-\min\{F_{Y|X}(R_A + R_B) - 2\lambda_{xy}\} + \beta_{AB}\right],$$

(96)

where

$$F_{Z|X}(R) \equiv \min_{V \mid U} \left[D(\nu_{V|U} \mid \mu_{Z|X} \mid \nu_U) + |R - H(U|V')|^+\right].$$
From (91)–(93), (95) and (96), we have
\[
E_{ABB} \left[ \begin{pmatrix}
\tilde{g}_{AB}(c, m, w) \notin T_{X, \gamma} \\
\tilde{z} \notin T_{Z|X, \gamma}(\tilde{g}_{ABB}(c, m, w)) \\
\tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_A(c|y) \\
\tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_{AB}(c, m|\tilde{z})
\end{pmatrix} : (m, w, y, \tilde{z}) \right]
\leq \alpha_{ABB} - 1 + \frac{\beta_{ABB} + 1}{\kappa} + 2^{-n[\gamma - \lambda_{xZ}]} + 2\kappa \left[ \max \{\alpha_A, 1\} 2^{-n[\inf F_{Z|X}(R_A - 2\lambda_{xy}) + \beta_A]}ight]
+ 2\kappa \left[ \max \{\alpha_{AB}, 1\} 2^{-n[\inf F_{Z|X}(R_A + R_B - 2\lambda_{xy}) + \beta_{AB}]} \right],
\tag{97}
\]

where the infimum is taken over all \(\mu_{Y\tilde{Z}|X} \) satisfying (53)–(55). This implies that there are \(A \in \mathcal{A}, B \in \mathcal{B}, \tilde{B} \in \tilde{B}, \) and \(c \in \text{Im}A\) such that
\[
P_{MWYZ} \left[ \begin{pmatrix}
\tilde{g}_{AB}(c, m, w) \notin T_{X, \gamma} \\
\tilde{z} \notin T_{Z|X, \gamma}(\tilde{g}_{ABB}(c, m, w)) \\
\tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_A(c|y) \\
\tilde{g}_{ABB}(c, m, w) \neq \tilde{g}_{AB}(c, m|\tilde{z})
\end{pmatrix} : (m, w, y, \tilde{z}) \right]
\leq \alpha_{ABB} - 1 + \frac{\beta_{ABB} + 1}{\kappa} + 2^{-n[\gamma - \lambda_{xZ}]} + 2\kappa \left[ \max \{\alpha_A, 1\} 2^{-n[\inf F_{Z|X}(R_A + R_B - 2\lambda_{xy}) + \beta_{AB}]} \right].
\]

Since
\[
\inf_{H(Y|X) < R_A} F_{Y|X}(R_A) > 0
\]
\[
\inf_{H(Z|X) < R_A + R_B} F_{Z|X}(R_A + R_B) > 0,
\]
then the right hand side of (97) goes to zero as \(n \to \infty\) by assuming (71)–(73) and the properties (H2) and (H3) of ensembles \((\mathcal{A}, p_A), (\mathcal{A} \times \mathcal{B}, p_{AB})\) and \((\mathcal{A} \times \mathcal{B} \times \tilde{B}, p_{ABB})\).

**E. Proof of Lemma 7**

In the following, we assume that \(\varepsilon_A > 0\) and \(\gamma > 0\) satisfy
\[
\varepsilon_A > \max \{\zeta_X(Y|\gamma), \zeta_X(Z|\gamma)\}. \tag{98}
\]

Let \(x, y, \) and \(z\) be outputs of correlated sources. We define
\[
\bullet (x, y, z) \in T_{XYZ, \gamma} \quad \text{(SKA1)}
\]
\[
\bullet g_A(Ax|y) = x \quad \text{(SKA2)}
\]
\[
\bullet g_{AB}(Ax, Bx|z) = x. \quad \text{(SKA3)}
\]

Then the left hand side of (34) is upper bounded by
\[
p_{XYZ} \left[ \begin{pmatrix}
g_{AB}(Ax|y) \neq x \\
g_{AB}(Ax, Bx|z) \neq x
\end{pmatrix} : (x, y, z) \right] \leq p_{XYZ}(S_1^c) + p_{XYZ}(S_1 \cap S_2^c) + p_{XYZ}(S_1 \cap S_3^c),
\tag{99}
\]

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where

\[ S_i \equiv \{(x, y, z) : (SKA_i)\} \].

First, we evaluate \( E_{AB} [p_{XYZ}(S_i^2)] \). From (81), we have

\[
E_{AB} [p_{XYZ}(S_i^2)] \leq 2^{-n[\gamma - \lambda_{XYZ}]}
\leq \frac{\delta'}{3}
\]

(100)

for all \( \delta' > 0 \) and sufficiently large \( n \).

Next, we evaluate \( E_{AB} [p_{XYZ}(S_i \cap S_j^2)] \) and \( E_{AB} [p_{XYZ}(S_i \cap S_j^3)] \). From Lemma 14, we have \((x, y) \in T_{XYZ, \gamma}\) and \( x \in T_{X|Y, \gamma}(y) \). Then there is \( x' \in C_A(Ax) \) such that \( x' \neq x \) and

\[
\mu_{X|Y}(x'|y) \geq \mu_{X|Y}(x|y)
\geq 2^{-n[H(X|Y) + \zeta_{X|Y}(\gamma \gamma)]}
\]

(101)

where the second inequality comes from Lemma 16. This implies that \([\mathcal{G}(y) \setminus \{x\}] \cap C_A(Ax) \neq \emptyset\), where

\[
\mathcal{G}(y) \equiv \left\{ x' : \mu_{X|Y}(x'|y) \geq 2^{-n[H(X|Y) + \zeta_{X|Y}(\gamma \gamma)]} \right\}.
\]

Then we have

\[
E_{AB} [\mu_{XYZ}(S_i \cap S_j^2)] \leq \sum_{(x, y, z) \in T_{XYZ, \gamma}} \mu_{XYZ}(x, y, z) p_A (\{A : [\mathcal{G}(y) \setminus \{x\}] \cap C_A(Ax) \neq \emptyset\})
\leq \sum_{(x, y, z) \in T_{XYZ, \gamma}} \mu_{XYZ}(x, y, z) \left\lfloor \frac{[\mathcal{G}(y)\alpha_A + \beta_A]}{\text{Im}A} \right\rfloor
\leq \sum_{(x, y, z) \in T_{XYZ, \gamma}} \mu_{XYZ}(x, y, z) \left\lfloor \frac{2n[H(X|Y) + \zeta_{X|Y}(\gamma \gamma)]\alpha_A}{\text{Im}A} + \beta_A \right\rfloor
\leq \left\lfloor \frac{|X|^lA\alpha_A 2n[H(X|Y) + \zeta_{X|Y}(\gamma \gamma)]}{\text{Im}A} \right\rfloor + \beta_A
\leq \left\lfloor \frac{|X|^lA\alpha_A 2^{-n[\epsilon_A - \zeta_{X|Y}(\gamma \gamma)]}}{\text{Im}A} \right\rfloor + \beta_A
\leq \frac{\delta'}{3}
\]

(102)

for all \( \delta' > 0 \) and sufficiently large \( n \) by taking an appropriate \( \gamma > 0 \), where the second inequality comes from Lemma 1 and the third inequality comes from the fact that

\[
|\mathcal{G}(y)| \leq 2^n[H(X|Y) + \zeta_{X|Y}(\gamma \gamma)],
\]

the fifth inequality comes from the definition of \( l_A \), and the last inequality comes from (98) and the properties (H1)–(H3) of an ensemble \((A, p_A)\). Similarly, we have

\[
E_{AB} [p_{XYZ}(S_i \cap S_j^3)] \leq \left\lfloor \frac{|X|^{l_A + l_{AB}}\alpha_A 2^{-n[\epsilon_A - \zeta_{X|Z}(\gamma \gamma)]}}{\text{Im}A|\text{Im}B|} \right\rfloor + \beta_{AB}
\leq \frac{\delta'}{3}
\]

(103)

for all \( \delta' > 0 \) and sufficiently large \( n \).
Finally, from (99)–(103), we have the fact that for all \( \delta' > 0 \) and sufficiently large \( n \) there are \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that

\[
p_{X_{\mathcal{A}} Y Z} \left( \left\{ \left( x, y, z \right) : g_A(Ax|y) \neq x \right. \left. \text{or } g_{AB}(Ax, Bx|z) \neq x \right\} \right) \leq \delta'
\]

for all \( \delta' > 0 \) and sufficiently large \( n \). \[\blacksquare\]

**Remark 4:** It should be noted that the property (H2) of ensembles \((\mathcal{A}, p_A)\) and \((\mathcal{A} \times \mathcal{B}, p_{AB})\) can be replaced by

\[
\lim_{n \to \infty} \frac{\log \alpha_A(n)}{n} = 1
\]

\[
\lim_{n \to \infty} \frac{\log \alpha_{AB}(n)}{n} = 1,
\]

respectively. In particular, there are expurgated ensembles \((\mathcal{A}, p_A)\) and \((\mathcal{B}, p_B)\) of sparse matrices that have an \((\alpha_A, 0)\)-hash property, where the condition (H2) for \( \alpha_A \) and \( \alpha_{AB} \) is replaced by the above respective conditions (see [2]).

**F. Proof of Lemma 8**

It has already been proved in the proof of Lemma 5 that there is a random variable \( X' \) taking values in \( \mathcal{X} \) such that

\[
H(X|X', Z) = R'
\]

for given \((X, Z)\) and \( 0 \leq R' \leq H(X|Z) \). The lemma is proved by letting

\[
R' \equiv R - \varepsilon
\]

\[
\tilde{Z} \equiv (X', Z)
\]

\[
f(\tilde{z}) \equiv z \text{ for } \tilde{z} = (x', z).
\]

\[\blacksquare\]

**G. Proof of Lemma 9**

Let \( x, y, \tilde{z} \) be outputs of the correlated sources. We define

- \( \tilde{g}_A(Ax|y) = x \) \hspace{1cm} (USKA1)
- \( \tilde{g}_{AB}(Ax, Ax|\tilde{z}) = x \) \hspace{1cm} (USKA2)

Then the left hand side of (64) is upper bounded by

\[
p_{X_{\mathcal{A}} Y Z} \left( \left\{ \left( x, y, \tilde{z} \right) : g_{AB}(Ax|y) \neq x \right. \left. \text{or } g_{AB}(Ax, Bx|\tilde{z}) \neq x \right\} \right) \leq p_{X_{\mathcal{A}} Y Z}(S_1^c) + p_{X_{\mathcal{A}} Y Z}(cS_2^c),
\]

where

\[
S_i \equiv \{ (x, y, \tilde{z}) : \text{(USKA}i) \}.
\]
In the following, we evaluate $E_{AB} \left[ p_{XY \hat{Z}}(S^*_i) \right]$ and $E_{AB} \left[ p_{XY \hat{Z}}(\mathcal{S}^*_i) \right]$. Let $UV$ be the type of sequence $(x, y) \in \mathcal{X}^n \times \mathcal{Y}^n$ and $V|U$ be the conditional type given type $U$. In the following, we assume that $(x, y) \in \mathcal{T}_{UV}$. If $\tilde{g}_A(Ax|y) \neq x$, then there is $x' \in C_A(Ax)$ such that $x' \neq x$ and

$$H(x'|y) \leq H(x|y) \leq H(U|V).$$

This implies that $[\mathcal{G}(y) \setminus \{x\}] \cap C_A(Ax) \neq \emptyset$, where

$$\mathcal{G}(y) \equiv \{x' : H(x'|y) \leq H(U|V)\}.$$

Then we have

$$E_{AB} \left[ \chi(\tilde{g}_A(Ax|y) \neq x) \right] \leq p_A \left( [A : [\mathcal{G}(y) \setminus \{x\}] \cap C_A(Ax) \neq \emptyset] \right)$$

$$\leq \max \left\{ \frac{[\mathcal{G}(y)] \alpha_A}{|\text{Im } A|} + \beta_A, 1 \right\}$$

$$\leq \max \left\{ \frac{2^{n[H(U|V) + \lambda_{xy}]} \alpha_A}{|\text{Im } A|} + \beta_A, 1 \right\}$$

$$\leq \max \left\{ \frac{|X|^+ \alpha_A}{|\text{Im } A|}, 1 \right\} 2^{-n[R_A - H(U|V)]^+ - \lambda_{xy}} + \beta_A, \quad (105)$$

where $| \cdot |^+$ is defined by (9), the second inequality comes from Lemma 1, and the third inequality comes from Lemma 20. Let

$$F_{XY}(R) \equiv \min_{UV} \left[ D(\nu_{xy}\|\mu_{XY}) + |R - H(U|V)|^+ \right].$$

Then we have

$$E_{AB} \left[ \mu_{XY \hat{Z}}(S^*_i) \right] = \sum_{UV} \sum_{(x, y) \in \mathcal{T}_{UV}} \mu_{XY}(x, y) E_{AB} \left[ \chi(\tilde{g}_A(Ax|y) \neq x) \right]$$

$$\leq \sum_{UV} \sum_{(x, y) \in \mathcal{T}_{UV}} \mu_{XY}(x, y) \left[ \max \left\{ \frac{|X|^+ \alpha_A}{|\text{Im } A|}, 1 \right\} 2^{-n[R_A - H(U|V)]^+ - \lambda_{xy}} + \beta_A \right]$$

$$\leq \max \left\{ \frac{|X|^+ \alpha_A}{|\text{Im } A|}, 1 \right\} \sum_{UV} 2^{-n[D(\nu_{xy}\|\mu_{XY}) + |R_A - H(U|V)|^+ - \lambda_{xy}]} + \beta_A$$

$$\leq \max \left\{ \frac{|X|^+ \alpha_A}{|\text{Im } A|}, 1 \right\} 2^{-n[F_{XY}(R_A) - 2\lambda_{xy}]} + \beta_A, \quad (106)$$

where the first inequality comes from (105), the second inequality comes from Lemmas 13 and 19, and the last inequality comes from Lemma 18 and the definition of $F_{XY}$. Similarly, we have

$$E_{AB} \left[ p_{XY \hat{Z}}(\mathcal{S}^*_i) \right] \leq \max \left\{ \frac{|X|^i \alpha + p_{AB}}{|\text{Im } A||\text{Im } B|}, 1 \right\} 2^{-n[F_{X\hat{Z}}(R_A + R_B) - 2\lambda_{x\hat{z}}]} + \beta_{AB}, \quad (107)$$

where

$$F_{X\hat{Z}}(R) \equiv \min_{UV} \left[ D(\nu_{x\hat{z}}\|\mu_{X\hat{Z}}) + |R - H(U|V')|^+ \right].$$

Finally, from (104), (106), and (107), we have

$$E_{AB} \left[ p_{XY \hat{Z}} \left( \left\{ (x, y, \hat{z}) : g_{AB}(Ax|y) \neq x \right\} \right) \right]$$

$$\leq \max \left\{ \frac{|X|^+ \alpha_A}{|\text{Im } A|}, 1 \right\} 2^{-n[F_{XY}(R_A) - 2\lambda_{xy}]} + \max \left\{ \frac{|X|^i \alpha + p_{AB}}{|\text{Im } A||\text{Im } B|}, 1 \right\} 2^{-n[F_{X\hat{Z}}(R_A + R_B) - 2\lambda_{x\hat{z}}]}$$

$$+ \beta_A + \beta_{AB},$$
where the infimum is taken over all \( \mu_{XYZ} \) satisfying (65) and (66). This implies that there are \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that

\[
p_{XYZ} \left( \left\{ (x, y, z) : g_{AB}(Ax|y) \neq x \right\} \right)
\leq \max \left\{ \frac{|X|^\lambda \alpha_A}{|\Im A|}, 1 \right\} 2^{-n \inf F_{XYZ}(R_A - 2\lambda x)} + \max \left\{ \frac{|X|^\lambda \alpha_{AB}}{|\Im A||\Im B|}, 1 \right\} 2^{-n \inf F_{XZ}(R_A + R_B - 2\lambda x)} + \beta_A + \beta_{AB}.
\]

(108)

Since

\[
\inf_{\mathcal{H}(X|Y) \leq R_A} F_{XY}(R_A) > 0
\]

\[
\inf_{\mathcal{H}(X|Z) \leq R_A + R_B} F_{XZ}(R_A + R_B) > 0,
\]

then the right hand side of (108) goes to zero as \( n \to \infty \) by assuming the properties (H1)–(H3) of \( (\mathcal{A}, p_A) \) and \( (\mathcal{A} \times \mathcal{B}, p_{AB}) \).

\( \blacksquare \)

### H. Method of Types

We use the following lemmas for a set of typical sequences.

**Lemma 13** ([6, Lemma 2.6][18, Lemma 21]):

\[
\frac{1}{n} \log \frac{1}{\mu_{UV}(u,v)} = H(\nu_{uv}) + D(\nu_{uv}||\mu_{UV})
\]

\[
\frac{1}{n} \log \frac{1}{\mu_{U|V}(u|v)} = H(\nu_{uv}) + D(\nu_{uv}||\mu_{U|V})
\]

**Lemma 14** ([22, Theorem 2.5][18, Lemma 22]): If \( uv \in \mathcal{T}_{V,\gamma} \) and \( uv \in \mathcal{T}_{U|V,\gamma}(v) \), then \( (u,v) \in \mathcal{T}_{UV,\gamma+\gamma'} \).

If \( (u,v) \in \mathcal{T}_{UV,\gamma} \), then \( u \in \mathcal{T}_{U,\gamma} \) and \( u \in \mathcal{T}_{U|V,\gamma}(v) \).

**Lemma 15** ([22, Theorem 2.7][18, Lemma 24]): Let \( 0 < \gamma \leq 1/8 \). Then,

\[
\left| \frac{1}{n} \log \frac{1}{\mu_U(u)} - H(U) \right| \leq \zeta_U(\gamma)
\]

for all \( u \in \mathcal{T}_{U,\gamma} \), and

\[
\left| \frac{1}{n} \log \frac{1}{\mu_{U|V}(u|v)} - H(U|V) \right| \leq \zeta_{U|V}(\gamma')\gamma
\]

for \( v \in \mathcal{T}_{V,\gamma} \) and \( u \in \mathcal{T}_{U|V,\gamma}(v) \), where \( \zeta_U(\gamma) \) and \( \zeta_{U|V}(\gamma')\gamma \) are defined in (5) and (6), respectively.

**Lemma 16** ([22, Theorem 2.8][18, Lemma 25]): For any \( \gamma > 0 \) and \( v \in \mathcal{V}^n \),

\[
\mu_U([T_{U,\gamma}]^c|\gamma) \leq 2^{-n[\gamma - \lambda_U]}
\]

\[
\mu_{U|V}([T_{U|V,\gamma}(v)]^c|v) \leq 2^{-n[\gamma - \lambda_{UV}]}
\]

where \( \lambda_U \) and \( \lambda_{UV} \) are defined in (4).

**Lemma 17** ([22, Theorem 2.9][18, Lemma 26]): For any \( \gamma > 0 \),

\[
\left| \frac{1}{n} \log |\mathcal{T}_{U,\gamma}| - H(U) \right| \leq \eta_U(\gamma),
\]

where \( \eta_U(\gamma) \) is defined in (7).
Lemma 18 ([6, Lemma 2.2]): The number of different types of sequences in $U^n$ is fewer than $[n + 1]|U|$. The number of conditional types of sequences in $U^n \times V^n$ is fewer than $[n + 1]|U||V|$.

Lemma 19 ([6, Lemma 2.3]): For a type $U$ of a sequence in $X^n$,
$$2^{n[H(U) - \lambda_X]} \leq |T_U| \leq 2^{nH(U)},$$
where $\lambda_X$ is defined in (4).

Lemma 20 ([19, Lemma 7],[16, Lemma 2]): For $y \in T_V$,
$$|\{x' : H(x') \leq H(U)\}| \leq 2^{n[H(U)+\lambda_X]},$$
$$\left|\{x' : H(x'|y) \leq H(U|V)\}\right| \leq 2^{n[H(U|V)+\lambda_{XY}]},$$
where $\lambda_X$ and $\lambda_{XY}$ are defined in (4).

Lemma 21 ([19, Lemma 7]): For any probability distribution $\mu_X$ on $X$,
$$\min_U d(\nu_U, \mu_X) \leq \frac{|X|}{n},$$
$$\min_U D(\nu_U||\mu_X) \leq \frac{|X|}{n \min_{x, \mu_X(x) > 0} \mu_X(x)}$$
$$\min_U |H(X) - H(U)| \leq \frac{2|X|}{n \min_{x, \mu_X(x) > 0} \mu_X(x)},$$
where minimum is taken over all types $U$ of the sequence in $X^n$.

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