Gauge Theory of Massive Tensor Field

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Abstract

In order to construct a massive tensor theory with a smooth massless limit, we apply the Batalin-Fradkin algorithm to the ordinary massive tensor theory. By introducing an auxiliary vector field all second-class constraints are converted into first-class ones. We find a gauge-fixing condition which produces a massive tensor theory of desirable property.
1 Introduction

The purpose of the present paper is to construct a massive tensor field theory with a smooth massless limit. For a massive vector field it has been known that the Stueckelberg formalism has a smooth massless limit. For a massive tensor field Fronsdal and Heidenreich (FH) succeeded in constructing such a theory. Based on the Lagrangian formalism, they removed massless singularities from all two-point functions by introducing two kinds of auxiliary fields with spin-1 and 0. In the present paper we take a different way from FH for the same purpose. Our method based on the Hamiltonian formalism is more systematic than FH’s.

We apply the Batalin-Fradkin (BF) algorithm to the ordinary massive tensor theory. By introducing only a spin-1 auxiliary field (BF field), contrary to the case of FH, we convert the original second-class constrained system into a first-class one. To the resulting gauge-invariant system two kinds of gauge-fixings are imposed. One is unitary gauge-fixing that eliminates the BF field from the system to recover the original massive tensor theory. The other is massless-regular gauge-fixing that leads to a theory with a smooth massless limit. Based on the Hamiltonian formalism, however, our formulation is non-covariant from the beginning. We have not succeeded in getting final results in covariant form yet. In this regard we are below FH, while the number of auxiliary fields introduced here is smaller than in FH. To get some covariant expressions is to be left for future studies.

In §2, following Igarashi, we review the case of a massive vector field. This is to see how the Stueckelberg formalism follows from applying BF algorithm to a massive vector theory. Section 3 treats the usual canonical formalism of a massless tensor field. The canonical formalism of a massive tensor field is presented in §4. In §5 BF algorithm is applied to this system. By introducing an auxiliary vector field, the original second-class constrained system is converted into a first-class one. In §6 two kinds of gauge-fixings, unitary one and massless-regular one, are investigated. It is shown that the former recovers the original system, while the latter gives a system with a smooth massless limit. Section 7 is devoted to summary and discussion.

2 Massive vector field

A massive vector field is described by the Lagrangian

\[ L[A] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu, \]  

(1)

where

\[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \]  

(2)

This system has a primary constraint

\[ \varphi_1 \equiv \pi^0 \approx 0 \]  

(3a)

† In the present paper Greek indices run 0 − 3, while Latin indices 1 − 3. The metric is \( \eta^{\mu\nu} \equiv (-1, +1, +1, +1). \)
and a secondary constraint
\[ \varphi_2 \overset{d}{=} \partial_i \pi^i + m^2 A_0 \approx 0, \quad (3b) \]
where \( \pi^\mu \) are momenta conjugate to \( A_\mu \). The Poisson bracket between the constraints
\[ [\varphi_1(x), \varphi_2(x')] = -m^2 \delta^3(x - x') \quad (4) \]
shows that the constraints are of the second class when the mass \( m \) is finite. The Hamiltonian is calculated to be
\[ H = \frac{1}{2} \pi^i \dot{\pi}^i + \frac{1}{4} F_{ij} \dot{F}^{ij} + \frac{m^2}{2} (A_0^2 + A_i^2) + \partial_i A_i \varphi_1 - A_0 \varphi_2. \quad (5) \]
The time developments of the constraints are
\[ \dot{\varphi}_1 = [\varphi_1, H] = \varphi_2, \quad \dot{\varphi}_2 = [\varphi_2, H] = \partial_i^2 \varphi_1. \quad (6) \]

In order to convert the above second-class constrained system into a first-class one, we introduce a BF field \( \theta \) and its conjugate momentum \( \omega \), modifying the constraints and Hamiltonian as follows:
\[ \tilde{\varphi}_1 \overset{d}{=} \tilde{\pi}_0 \approx 0, \quad \tilde{\varphi}_2 \overset{d}{=} \partial_i \pi^i + m^2 \tilde{A}_0 \approx 0; \quad (7) \]
\[ \tilde{H} \overset{d}{=} \frac{1}{2} \tilde{\pi}^i \dot{\pi}^i + \frac{1}{4} \tilde{F}_{ij} \dot{F}^{ij} + \frac{m^2}{2} (\tilde{A}_0^2 + \tilde{A}_i^2) + \partial_i \tilde{A}_i \tilde{\varphi}_1 - \tilde{A}_0 \tilde{\varphi}_2, \quad (8) \]
where
\[ \tilde{A}_0 \overset{d}{=} A_0 + \theta, \quad \tilde{A}_i \overset{d}{=} A_i - \frac{1}{m^2} \partial_i \omega, \quad (9) \]
The resulting system is in fact of the first class:
\[ [\tilde{\varphi}_1, \tilde{\varphi}_2] = 0, \quad (10) \]
\[ [\tilde{\varphi}_\alpha, \tilde{H}] = 0. \quad (\alpha = 1, 2) \quad (11) \]

Now we come to the gauge-fixings. The first choice is the ‘unitary gauge’:
\[ \chi^1 \overset{d}{=} \theta \approx 0, \quad \chi^2 \overset{d}{=} \frac{1}{m^2} \omega \approx 0. \quad (12) \]
For this choice we get
\[ [\chi^\alpha(x), \tilde{\varphi}_\beta(x')] = -\delta^\alpha_\beta \delta^3(x - x'). \quad (13) \]
The path integral expression is given by
\[ Z = \int \mathcal{D} \pi^\mu \mathcal{D} A_\mu \mathcal{D} \omega \mathcal{D} \theta \delta(\chi^\alpha) \delta(\tilde{\varphi}_\alpha) \exp i \int d^4 x \left[ \pi^{\mu} \dot{A}_\mu + \omega \dot{\theta} - \tilde{H} \right]. \quad (14) \]
The variables $\pi^0, \theta$ and $\omega$ are integrated out, and the factor $\delta(\partial_i \pi^i + m^2 A_0)$ is exponentiated to give

\[
Z = \int D\pi^i D\dot{A}_i D\theta D\omega \delta(\partial_i \pi^i + m^2 A_0) \prod_t \text{Det} M \exp i \int d^4x \left[ \pi^i \dot{A}_i + \frac{1}{2} \pi^2 \theta - \frac{1}{4} F_{ij}^2 - \frac{m^2}{2} (A_i^2 + A_0^2) + \lambda (\partial_i \pi^i + m^2 A_0) \right].
\]

Integrating over the variables $\pi^i$ and $A_0$ and overwriting $A_0$ on $\lambda$, we obtain

\[
Z = \int D\dot{A}_\mu \exp i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \right].
\] (16)

This is nothing but the original system. That means the original system (1) can be regarded as a gauge-fixed version of the extended gauge system (7) and (8).

Another gauge-fixing we consider is

\[
\chi^1 \equiv A_0 \approx 0, \\
\chi^2 \equiv \partial^i A_i \approx 0.
\] (17)

In this case we have

\[
[\chi_\alpha(x), \tilde{\varphi}_\beta(x')] = \begin{pmatrix}
\delta^3(x-x') \\
-\partial_i \delta^3(x-x')
\end{pmatrix}.
\] (18)

The path integral expression is

\[
Z = \int D\pi^\mu D\dot{A}_\mu D\omega D\theta \delta(\chi_\alpha) \delta(\tilde{\varphi}_\alpha) \prod_t \text{Det} M \exp i \int d^4x \left[ \pi^\mu \dot{A}_\mu + \omega \dot{\theta} - \tilde{H} \right],
\] (19)

where

\[
M \equiv \partial_i \partial^i \delta^3(x-x').
\] (20)

The integrations over $\pi^0$ and $A_0$ and the exponentiation of the factor $\delta(\partial_i \pi^i + m^2 \theta)$ are easily carried out:

\[
Z = \int D\pi^i D\dot{A}_i D\omega D\theta D\lambda \delta(\partial^i A_i) \prod_t \text{Det} M \\
\times \exp i \int d^4x \left[ \pi^i \dot{A}_i + \omega \dot{\theta} - \frac{1}{2} \pi^2 \theta - \frac{1}{4} F_{ij}^2 - \frac{m^2}{2} (A_i - \frac{1}{m^2} \partial_i \omega)^2 \\
- \frac{m^2}{2} \theta^2 + \lambda (\partial_i \pi^i + m^2 \theta) \right].
\] (21)

Integrating out $\pi^i$ and $\theta$ and writing $A_0$ and $\omega$ over $\lambda$ and $\frac{1}{m^2} \omega$ respectively, we obtain

\[
Z = \int D\dot{A}_\mu D\omega D\delta(\partial^i A_i) \prod_t \text{Det} M \exp i \int d^4x L[A, \omega],
\] (22)
where
\[ L[A, \omega] \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} (A_\mu - \partial_\mu \omega)^2. \] (23)

Note that the Lagrangian \( L[A, \omega] \) is invariant under the gauge transformation with an arbitrary function \( \varepsilon(x) \):
\[ \delta A_\mu = \partial_\mu \varepsilon, \]
\[ \delta \omega = \varepsilon. \] (24)

Considering this fact, the factor
\[ \delta(\partial^i A_i) \prod_i \text{Det} M \] (25)
can be replaced by the covariant expression
\[ \delta(\partial^\mu A_\mu - f) \text{Det} N, \] (26)
where \( N \) is defined by
\[ N \equiv \partial_\mu \partial^\mu \delta^4(x - x'), \] (27)
and \( f \) is an arbitrary function of \( x \). By using the Nakanishi-Lautrup (NL) field \( B \) and the Faddeev-Popov (FP) ghosts \( (c, \bar{c}) \), we can exponentiate the factor (26) to get the final form of the path integral
\[ Z = \int \mathcal{D}A_\mu \mathcal{D}\omega \mathcal{D}B \mathcal{D}c \mathcal{D}\bar{c} \]
\[ \times \exp i \int d^4x \left[ L[A, \omega] + B(\partial^\mu A_\mu + \frac{\alpha}{2} B) + ic\partial_\mu \partial^\mu c \right], \] (28)
where \( \alpha \) is an arbitrary constant, gauge parameter. The transformation to the Stueckelberg formalism has been completed. The BF field \( \omega \) is now identified with the Stueckelberg field. The expression (28) has a smooth massless limit: when the mass \( m \) tends to zero, the field \( \omega \) becomes redundant and the expression (28) goes to the usual one for the Abelian gauge theory
\[ Z = \int \mathcal{D}A_\mu \mathcal{D}B \mathcal{D}c \mathcal{D}\bar{c} \exp i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + B(\partial^\mu A_\mu + \frac{\alpha}{2} B) + ic\partial_\mu \partial^\mu c \right]. \] (29)

In this respect we call the gauge choice (17) 'massless-regular gauge'.

3 Massless tensor field

The Lagrangian is
\[ L[h, m = 0] = -\frac{1}{2}(\partial_\lambda h_{\mu\nu}\partial^{\lambda} h^{\mu\nu} - \partial_\lambda h^\mu_{\mu} \partial^{\lambda} h^\nu_{\nu}) + \partial_\lambda h_{\mu\nu} \partial^{\nu} h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h^{\lambda}. \] (30)

This system has two primary constraints
\[ \varphi^0 \equiv \pi^0 + \partial^m h_m \approx 0, \]
\[ \varphi^m \equiv \pi^m - \partial^m h_0 - \partial^m h^n_n \approx 0, \] (31a)
and two secondary constraints
\begin{align}
\varphi^0_1 & \equiv \partial_m \partial^m h_n - \partial^m \partial^m h_{nn} \approx 0, \\
\varphi^m_1 & \equiv \partial_n \pi_{mn} + \partial^m \partial^n h_n \approx 0,
\end{align}
(31b)
where \( \pi^0, \pi^m \) and \( \pi^{mn} \) are momenta conjugate to \( h^d \equiv h^0_0, h^m_m \) and \( h^{mn} \). The Poisson brackets between these constraints \( \varphi^A \equiv (\varphi^0, \varphi^m, \varphi^0_1, \varphi^m_1) \) are
\[ [\varphi^A(x), \varphi^B(x')] = 0. \]
(32)
The Hamiltonian is
\[ H = H_0(m = 0) + \lambda_A \varphi^A, \]
(33)
where \( H_0(m = 0) \) is defined by
\begin{align*}
H_0(m = 0) & \equiv \frac{1}{2} \pi^{mn} \pi_{mn} - \frac{1}{4} \pi^m \pi_n + 2 \pi^{mn} \partial_m h_n - \frac{1}{2} \pi^m \partial^n h_n \\
& \quad + 2 \partial_m h_n \partial^n h^m - \frac{3}{4} \partial^m h_m \partial^n h_n + \partial_m h_0 \partial^m h^n - \partial_m h_0 \partial_n h^{mn} \\
& \quad + \frac{1}{2} \left( \partial_l h_{mn} \partial^l h^{mn} - \partial_l h^m_0 \partial^0 h^n - \partial_l h^{mn} \partial^n h^m + \partial_m h^{mn} \partial_n h^l \right),
\end{align*}
(34)
and \( \lambda_A \) are arbitrary coefficients. The time developments of the constraints are
\begin{align}
\dot{\varphi}^0 & = [\varphi^0, H] = \varphi^0_1, \\
\dot{\varphi}^m & = [\varphi^m, H] = 2 \varphi^m_1, \\
\dot{\varphi}^0_1 & = [\varphi^0_1, H] = - \partial_n \varphi^m_1, \\
\dot{\varphi}^m_1 & = [\varphi^m_1, H] = 0. 
\end{align}
(35)
Equations (32) and (35) show that this system is a first-class constrained system.

In order to make the constraints second-class, we impose the gauge-fixing conditions \( \chi_A \equiv (\chi_0, \chi_m, \chi_{10}, \chi_{1m}) \):
\begin{align}
\chi_0 & \equiv h_0 \approx 0, \\
\chi_m & \equiv h_m \approx 0, \\
\chi_{10} & \equiv \pi^m \approx 0, \\
\chi_{1m} & \equiv \partial^n h_{mn} - \frac{1}{2} \partial_n h^n \approx 0.
\end{align}
(36)
The path integral expression is
\[ Z = \int D\varphi^0 D\varphi^m D\varphi^{mn} D\varphi^0_1 D\varphi^m_1 D\varphi^{0_1} D\varphi^{m_1} \prod_t \text{Det} M \\
\times \exp i \int d^4 x \left[ \pi^0 \dot{h}_0 + \pi^m \dot{h}_m + \pi^{mn} \dot{h}^{mn} - H \right], \]
(37)
where
\[ M \equiv \delta^\beta_\alpha \partial_m \partial^m \delta^3(x - x'). \] 
(38)
The integrations over $h_0, h_m, \pi^0$ and $\pi^m$ are easily carried out. The factors $\delta(\partial_m \partial^m h_n^\alpha - \partial^m \partial^\alpha h_{mn})$ and $\delta(\partial_\alpha \pi^{mn})$ are exponentiated by

$$\delta(\partial_m \partial^m h_n^\alpha - \partial^m \partial^\alpha h_{mn}) = \int \mathcal{D}h_0 \exp i \int d^4x \left[ -\partial_m h_0 \partial^m h_n^\alpha + \partial_m h_0 \partial_\alpha h^{mn} \right],$$

$$\delta(\partial_\alpha \pi^{mn}) = \int \mathcal{D}h_m \exp i \int d^4x \left[ -\pi^{mn} (\partial_m h_n + \partial_n h_m) \right].$$

We further integrate over $\pi^{mn}$ to get

$$Z = \int \mathcal{D}h_0 \mathcal{D}h_m \mathcal{D}h_{mn} \delta(\partial^m h_m - \frac{1}{2} \partial_m h_0) \prod_t \text{Det} M$$

$$\times \exp i \int d^4x \left[ L[h, m = 0] + \frac{4}{3} (\partial^m h_m - \frac{1}{2} \partial_m h_0)^2 \right].$$

Considering the fact that the Lagrangian $L[h, m = 0]$ is invariant (up to total divergence) under the gauge transformation with four arbitrary functions $\varepsilon_\mu(x)$

$$\delta h_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu,$$

we can give various expressions to the generating functional $Z$. For example, for ‘Coulomb-like gauge’ we have

$$Z = \int \mathcal{D}h_0 \mathcal{D}h_m \mathcal{D}h_{mn} \delta(\partial^m h_m - \frac{1}{2} \partial_m h_0 - f_0) \delta(\partial^m h_{mn} - \frac{1}{2} \partial_m h_n^\alpha) \prod_t \text{Det} M$$

$$\times \exp i \int d^4x L[h, m = 0]$$

$$= \int \mathcal{D}h_0 \mathcal{D}h_m \mathcal{D}h_{mn} \delta(\partial^m h_m - \frac{1}{2} \partial_m h_0) \prod_t \text{Det} M$$

$$\times \exp i \int d^4x \left[ L[h, m = 0] + \frac{1}{2\alpha} (\partial^m h_m - \frac{1}{2} \partial_m h_0)^2 \right],$$

where $f_\mu(\mu = 0 - 3)$ are arbitrary functions of $x$, and $\alpha$ is an arbitrary constant, gauge parameter. The expression (41) is a special case of (44). The covariant expressions are also obtained as follows:

$$Z = \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h^\nu - f_\mu) \text{Det} N \exp i \int d^4x L[h, m = 0]$$

$$= \int \mathcal{D}h_{\mu\nu} \mathcal{D}B_\mu \mathcal{D}c_\mu \mathcal{D}\bar{c}^\mu$$

$$\times \exp i \int d^4x \left[ L[h, m = 0] + B^\mu (\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h^\nu) + \frac{\alpha}{2} B^\mu B_\mu + i\bar{c}^\mu \partial_\nu \partial^\nu c_\mu \right],$$

where $N$ is defined by

$$N \equiv \delta^\beta_\alpha \partial_\mu \partial^\mu \delta(x - x'),$$

$$\text{and NL field } B_\mu \text{ and FP ghosts } (c_\mu, \bar{c}^\nu) \text{ have been introduced.}$$
4 Massive tensor field

A massive tensor field is described by the Lagrangian

\[ L[h] = L[h, m = 0] - \frac{m^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^\mu_{\mu} h^\nu_{\nu}). \tag{48} \]

This system has the same primary constraints as (31a)

\[
\phi^0 \equiv \pi^0 + \partial^m h_m \approx 0,
\phi^m \equiv \pi^m - \partial^m h_0 - \partial^m h^n_n \approx 0,
\tag{49a}
\]

but the different secondary constraints from (31b)

\[
\phi^0_1 \equiv \partial_m \partial^m h^n_n - \partial^m \partial^m h_m - m^2 h^n_m \approx 0,
\phi^m_1 \equiv \partial_n \pi^{mn} + \partial^m \partial^n h_n + m^2 h^m_m \approx 0,
\phi^0_2 \equiv \pi^m_m + \partial^m h_m \approx 0.
\tag{49b}
\]

The Poisson brackets between these constraints are calculated as

\[
[\varphi^m(x), \varphi_1^n(x')] = -m^2 \eta^{mn} \delta^3(x - x'),

[\varphi^m(x), \varphi_2^n(x')] = -2\partial^m \delta(x - x'),

[\varphi^0_1(x), \varphi_1^n(x')] = m^2 \partial^m \delta(x - x'),

[\varphi^0_2(x), \varphi_2^n(x')] = (2\partial^m \partial^m - 3m^2) \delta(x - x'),

\text{The others} = 0.
\tag{50}
\]

The Hamiltonian is

\[ H = H_0 + \lambda_0 \varphi^0 + \partial^m h^{mn} \varphi^m + (h_0 - h^m_m) \varphi^0_1, \tag{51} \]

where \( H_0 \) is defined by

\[ H_0 = H_0(m = 0) + \frac{m^2}{2} (h_{mn} h^{mn} - h^m_m h^n_n - 2h_m h^m + 2h_0 h^m_m), \tag{52} \]

and \( \lambda_0 \) is an arbitrary coefficient. The Poisson brackets between the constraints and Hamiltonian are

\[
[\varphi^0, H] = 0,
[\varphi^m, H] = 2\varphi_1^m,
[\varphi_1^0, H] = -\partial^m \varphi_1^m + \frac{m^2}{2} \varphi_2^0,
[\varphi_1^m, H] = \partial^m \varphi_1^0 + \frac{1}{2} \partial_n \partial^m \varphi^m + \frac{1}{2} \partial^m \partial_n \varphi^n,
[\varphi_2^0, H] = \partial_m \varphi^m + 4\varphi^0_1.
\tag{53}
\]

Equations (50) and (53) show that \( \varphi^0_1 \) is a first-class constraint and the other constraints are of the second class.
In order to make $\varphi^0$ of the second class too, we impose the gauge-fixing condition

$$\chi_0 \overset{d}{=} h_0 \approx 0.$$  \hspace{1cm} (54)

For the total set of constraints $\Phi^M \overset{d}{=} (\varphi^A, \chi_0)$ with $\varphi^A \overset{d}{=} (\varphi^0, \varphi^m, \varphi_1^0, \varphi_1^m, \varphi_2^0)$, we have

$$\text{Det} \left[ \Phi^M(x), \Phi^N(x') \right] \propto 1.$$  \hspace{1cm} (55)

The resulting functional integral expression is, therefore,

$$Z = \int D\pi^0 D\pi^m D\pi^{mn} Dh_0 Dh_m Dh_{mn} \delta(\varphi^A) \delta(\chi_0) \times \exp i \int d^4 x \left[ \pi^0 \dot{h}_0 + \pi^m \dot{h}_m + \pi^{mn} \dot{h}_{mn} - H_0 \right].$$  \hspace{1cm} (56)

5 Batalin-Fradkin extension

In this section, following BF, we convert the second-class constraints among (49) into first-class ones. This is done by introducing a spin-1 auxiliary field (BF field) $\theta_\mu$ and its conjugate momentum $\omega^\mu$ to modify the constraints $\varphi^A$ (49) and the Hamiltonian $H$ (51).

The modification of the constraints is easily carried out as follows:

$$\tilde{\varphi}^0 \overset{d}{=} \varphi^0 \approx 0,$$

$$\tilde{\varphi}^m \overset{d}{=} \varphi^m - \omega^m \approx 0,$$

$$\tilde{\varphi}_1^0 \overset{d}{=} \varphi_1^0 + \partial_m \omega^m + m^2 \theta_0 \approx 0,$$

$$\tilde{\varphi}_1^m \overset{d}{=} \varphi_1^m + m^2 \theta^m \approx 0,$$

$$\tilde{\varphi}_2^0 \overset{d}{=} \varphi_2^0 - 2 \partial^m \theta_m + 3 \omega^0 \approx 0.$$  \hspace{1cm} (57)

In fact all Poisson brackets between them vanish:

$$\left[ \tilde{\varphi}^A(x), \varphi^B(x') \right] = 0.$$  \hspace{1cm} (58)

The next step is to construct a modified Hamiltonian $\tilde{H}$

$$\tilde{H} = H + \Delta H(h, \pi; \theta, \omega)$$  \hspace{1cm} (59)

that has vanishing Poisson brackets with all the constraints (57) and satisfies the boundary condition

$$\Delta H(h, \pi; 0, 0) = 0.$$  \hspace{1cm} (60)

The conditions $\left[ \tilde{\varphi}^A, \tilde{H} \right] = 0$ give the following equations:

$$- \frac{\partial \Delta H}{\partial h_0} + \partial^m \frac{\partial \Delta H}{\partial \pi^m} = 0,$$  \hspace{1cm} (61)

$$2 \varphi_1^m \frac{\partial \Delta H}{\partial h_m} - \partial^m \frac{\partial \Delta H}{\partial \pi^0} - \partial^m \frac{\partial \Delta H}{\partial \pi^k} \eta^{kl} + \frac{\partial \Delta H}{\partial \theta_m} = 0,$$  \hspace{1cm} (62)
These equations can be solved successively: Eqs. (62), (63), (64) and (65) determine in what form $\Delta H$ contains the variables $\theta_m, \omega^0, \omega^m$ and $\theta_0$ respectively; Eq. (61) just tells that the variable $h_0$ in $\Delta H$ is to be combined with the variable $\pi^m$ to the form $\pi^m - \partial^m h_0$. After some lengthy but straightforward calculations we can find the solution satisfying all the conditions (60) and (61) – (65):

\[
\Delta H = \varphi^m \left[ \frac{1}{3} \left( 1 + \frac{2}{m^2} \partial_m \partial^m \right) \partial_m \theta_0 - \frac{1}{2m^2} \partial^m \left( \partial_n \omega_m + \partial_m \omega_n \right) \right] \\
+ \varphi_0^m \left[ \frac{1}{3} \left( 4 + \frac{2}{m^2} \partial_m \partial^m \right) \theta_0 + \frac{1}{m^2} \partial_m \omega^m \right] \\
+ \varphi_1^m \left( -2 \theta_m + \frac{1}{m^2} \partial_m \omega^0 \right) + \varphi_0 \left( -\frac{1}{2} \omega^0 \right) \\
- \frac{1}{3} \partial_m \theta_0 \partial^m \theta_0 + \frac{2}{3} m^2 \theta_0^2 + \theta_0 \partial_m \omega_m - \frac{1}{8m^2} \left( \partial_m \omega_n - \partial_n \omega_m \right)^2 \\
- m^2 \partial_m \partial^m - \frac{3}{4} \omega^0_0. \tag{66}
\]

If we introduce new variables

\[
\tilde{h}_0 \equiv h_0, \\
\tilde{h}_m \equiv h_m + \theta_m - \frac{1}{m^2} \partial_m \omega^0, \\
\tilde{h}_{mn} \equiv h_{mn} - \frac{1}{2m^2} \left( \partial_m \omega_n + \partial_n \omega_m \right) - \frac{1}{3} \left( \eta_{mn} + \frac{2}{m^2} \partial_m \partial_n \right) \theta_0, \\
\tilde{\pi}^0 \equiv \pi^0 - \partial^m \theta_m + \frac{1}{m^2} \partial_m \partial^m \omega^0, \\
\tilde{\pi}^m \equiv \pi^m - \left( \eta^m + \frac{1}{m^2} \partial^m \partial^0 \right) \omega_n - \frac{1}{3} \left( 3 + \frac{2}{m^2} \partial_m \partial^0 \right) \partial^m \theta_0, \\
\tilde{\pi}^{mn} \equiv \pi^{mn} - \eta^{mn} \partial^0 \theta_1 + \left( \eta^{mn} + \frac{1}{m^2} \partial^m \partial^n \right) \omega^0, \tag{67}
\]

the modified constraints $\tilde{\varphi}^A$ and Hamiltonian $\tilde{H}$ have more transparent expressions. These are what we have through the simple replacement of the variables $(h, \pi)$ by $(\tilde{h}, \tilde{\pi})$ in Eqs. (49) and (51):

\[
\tilde{\varphi}^A = \varphi^A \left[ h, \pi \to \tilde{h}, \tilde{\pi} \right] \approx 0, \tag{68}
\]

\[
\tilde{H} = H \left[ h, \pi \to \tilde{h}, \tilde{\pi} \right]. \tag{69}
\]

We thus constructed a first-class constrained system for a massive tensor field.
6 Gauge-fixings

6.1 Unitary gauge

To eliminate the BF field, we impose the following gauge-fixing conditions $\chi_A \overset{d}{=} (\chi_0, \chi_m, \chi_{10}, \chi_{1m}, \chi_{20})$:

$$
\begin{align*}
\chi_0 & \overset{d}{=} h_0 \approx 0, \\
\chi_m & \overset{d}{=} \theta_m \approx 0, \\
\chi_{10} & \overset{d}{=} \omega^0 \approx 0, \\
\chi_{1m} & \overset{d}{=} \omega^m \approx 0, \\
\chi_{20} & \overset{d}{=} \theta_0 \approx 0.
\end{align*}
$$

(70)

Then the path integral

$$
Z = \int D\pi^0 D\pi^m D\omega^0 D\omega^m D\theta_0 D\theta_m \delta(\tilde{\varphi}^A) \delta(\chi_A)
$$

$$
\times \exp i \int d^4 x \left[ \pi^0 \dot{h}_0 + \pi^m \dot{h}_m + \pi^{mn} \dot{h}_{mn} + \omega^0 \dot{\theta}_0 + \omega^m \dot{\theta}_m - \tilde{H} \right]
$$

(71)

immediately reduces to (56). This shows that for a massive tensor field the original system (48) is a gauge-fixed version of the extended system (68) and (69).

6.2 Massless-regular gauge

As the second gauge-fixing we impose

$$
\begin{align*}
\chi_0 & \overset{d}{=} h_0 \approx 0, \\
\chi_m & \overset{d}{=} h_m \approx 0, \\
\chi_{10} & \overset{d}{=} \pi^m + 3\omega^0 \approx 0, \\
\chi_{1m} & \overset{d}{=} \partial^h h_{mn} - \frac{1}{2} \partial_m h_n^m \approx 0, \\
\chi_{20} & \overset{d}{=} \theta_0 \approx 0.
\end{align*}
$$

(72)

In this case the path integral is given by

$$
Z = \int D\pi^0 D\pi^m D\omega^0 D\omega^m D\theta_0 D\theta_m \delta(\tilde{\varphi}^A) \delta(\chi_A) \prod_t \text{Det} M
$$

$$
\times \exp i \int d^4 x \left[ \pi^0 \dot{h}_0 + \pi^m \dot{h}_m + \pi^{mn} \dot{h}_{mn} + \omega^0 \dot{\theta}_0 + \omega^m \dot{\theta}_m - \tilde{H} \right],
$$

(73)

where $M$ is what is defined by Eq. (38). Carry out the integrations over $h_0, h_m, \pi^0, \pi^m$ and $\theta_0$. Do the following exponentiations

$$
\delta \left( \partial_m \partial^m h_n^m - \partial^m \partial^m h_{mn} - m^2 h_m^m + \partial_m \omega^m \right)
$$

$$
= \int Dh_0 \exp i \int d^4 x \left( -\partial_m h_0 \partial^m h_n^m + \partial^m h_0 \partial^m h_{mn} - m^2 h_0 h_m^m + h_0 \partial_m \omega^m \right),
$$

(74)
\[ \delta \left( \partial_n \pi^{mn} + m^2 \theta^m \right) = \int \mathcal{D} h \exp i \int d^4 x \left[ -\pi^{mn} (\partial_m h_n + \partial_n h_m) + 2m^2 h_m \theta^m \right], \tag{75} \]

\[ \delta (-2\partial^m \theta_m) = \int \mathcal{D} \lambda_0 \exp i \int d^4 x \left( 2\partial_m \lambda_0 \theta^m \right). \tag{76} \]

And then integrate over \( \omega^0, \pi^{mn} \) and \( \theta_m \). Write \( \theta_0 \) and \( \theta_m \) over \( -\frac{1}{m^2} \lambda_0 \) and \( \frac{1}{2m^2} \omega_m \) respectively. The final result we arrive at is

\[ Z = \int \mathcal{D} h_0 \mathcal{D} h_m \mathcal{D} \theta_0 \mathcal{D} \theta_m \delta \left( \partial^m h_m - \frac{1}{2} \dot{h}_m^m \right) \delta \left( \partial^m h_m - \frac{1}{2} \partial_m h_n^n \right) \prod_t \text{Det} M \times \exp i \int d^4 x \left[ L[h, m = 0] + R \right], \tag{77} \]

where

\[ R \equiv \frac{d}{2} \left( h_{mn} - \partial_m \theta_n - \partial_n \theta_m \right)^2 \left( h_m^m - 2 \partial^m \theta_m \right)^2 - 2 h_0 \left( h_m^m - 2 \partial^m \theta_m \right) - 2 \left( h_m - \dot{\theta}_m - \partial_m \theta_0 \right)^2. \tag{78} \]

When the mass \( m \) goes to zero, the fields \( \theta_0 \) and \( \theta_m \) become redundant and the path integral (77) tends to the expression obtained by setting \( f_\mu = 0 \) in Eq. (43). It has turned out that the BF-extended theory of massive tensor field equipped with the gauge-fixing conditions (72) does smoothly reduce to the massless tensor theory of Coulomb-like gauge.

The expression (77) is not covariant. For the purpose of having a covariant expression, it may be useful to rewrite Eq. (77) as

\[ Z = \int \mathcal{D} h_\mu \mathcal{D} \theta_\mu \delta \left( \partial^\mu h_{0\mu} - \frac{1}{2} \partial_\mu h^\mu \right) \delta \left( \partial^\mu h_{mn} - \frac{1}{2} \partial_m h_n^n \right) \prod_t \text{Det} M \times \exp i \int d^4 x \left[ L[h, m = 0] \right. \left. - \frac{m^2}{2} \left( (h_{\mu\nu} - \partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2 - \left( h_\mu^\mu - 2 \partial^\mu \theta_\mu \right)^2 \right) \right. \]

\[ \left. - 2m^2 \partial_0 \theta_0 \left( h_m^m - 2 \partial^m \theta_m \right) \right] \tag{79} \]

In this expression the non-covariant factors are underlined, all of which turn out to be related to gauge-fixings. To find a covariant expression by setting some suitable gauge conditions is left to be solved.

### 7 Summary and discussion

We have applied BF algorithm to a massive tensor field theory. By introducing an auxiliary vector field we have been able to convert the original second-class constrained system into a first-class one. Two gauge-fixings have been discussed: one is what recovers the original massive theory; the other is the one that has a smooth massless limit.

Our final expressions are still non-covariant. To find some suitable gauge-fixing conditions that lead to covariant expressions remains to be solved. Furthermore,
our formalism has been restricted to the linear free theory. To construct a complete nonlinear theory such as smoothly reduces to general relativity in the massless limit is another big problem.

Investigation in this direction seems to be useful for solving the infrared problem in quantum gravity.

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