DIFFERENTIAL K-THEORY AND LOCALIZATION FORMULA FOR η-INVARIANTS

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Abstract. In this paper, we obtain a localization formula in differential K-theory for $S^1$-action. Then by combining an extension of Goette’s result on the comparison of two types of equivariant $η$-invariants, we establish a version of localization formula for equivariant $η$-invariants. An important step of our approach is to construct a pre-$λ$-ring structure in differential K-theory which is interesting in its own right.

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0. Introduction

The famous Atiyah-Singer index theorem [6] announced in 1963 computed the index of elliptic operators, which is defined analytically, in a topological way, more precisely, by using the characteristic classes. We can understand the index as a primitive spectral invariant of an elliptic operator, and the more refined spectral invariants such as the $η$-invariant of Atiyah-Patodi-Singer and the analytic torsion of Ray-Singer as the secondary spectral invariants.

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of an elliptic operator. In [5, Proposition 2.10], Atiyah and Segal established a localization formula for the equivariant index using topological K-theory which computes the equivariant index via the contribution of the fixed point sets of the group action. Thus it is natural to ask if the localization property holds for these secondary spectral invariants. Note that these secondary spectral invariants are not computable in a local way and not a topological invariant as the index.

Note that the holomorphic analytic torsion of Ray-Singer [46] (and its family version: Bismut-K"ohler torsion form [19]) is the analytic counterpart of the direct image in Arakelov geometry [50]. Bismut-Lebeau’s [16] embedding formula for the analytic torsion and Bismut’s family extension [11] are the essential analytic ingredient of the arithmetic Riemann-Roch-Grothendieck theorem [30, 28]. In the way to establish a version of the equivariant arithmetic Riemann-Roch theorem, in [33, Theorem 4.4], K"ohler-Roessler established a version of the fixed point formula of Lefschetz type in the equivariant arithmetic K-theory. At least in the arithmetic context, K"ohler and Roessler’s result gives some relation of the equivariant holomorphic torsions on the whole manifold and its contribution on the fixed point sets for a $n$-th roots of unity. In [34, Lemma 2.3], they discussed in detail this problem by combining their result and Bismut-Goette’s result [14] on the comparison of two different versions of equivariant holomorphic torsions, and made a conjecture for complex manifolds [34, Conjecture, p82]. For more applications of the equivariant arithmetic Riemann-Roch theorem, cf. [43] and later works.

Let’s consider the $\eta$-invariant now. Originally, the $\eta$-invariant for Dirac operators is the spectral invariant introduced by Atiyah-Patodi-Singer (APS) [3] as the boundary contribution in the course of establishing their famous APS index theorem for compact manifolds with boundary. Formally, it is equal to the number of positive eigenvalues of the Dirac operator minus the number of its negative eigenvalues. Cheeger-Simons [22] could give a formula in $\mathbb{R}/\mathbb{Q}$ for the $\eta$-invariant by using their differential characters, cf. also the recent work of Zhang [55]. We notice that the $\eta$-invariant (or its family version: Bismut-Cheeger $\eta$-form [12]) plays a similar role in differential K-theory as the holomorphic analytic torsion in Arakelov geometry. In particular, in Freed-Lott’s index theorem [26, Theorem 7.35] in differential K-theory, the embedding formula of Bismut-Zhang [18, Theorem 2.2] for the $\eta$-invariant plays an important role. Various extension of Bismut-Zhang’s embedding formula are recently established by B. Liu [37] and later work.

In this paper, we will establish a version of the localization formula in differential K-theory, even there is some similarity with [33, Theorem 4.4] in the final form, we need to apply totally different arguments. In certain sense for $S^1$-action, we get pointwise a relation of the equivariant $\eta$-invariant and its contribution on the fixed point set (its definition is an important part of the localization formula), modulo the values at $g \in S^1$ of rational functions with coefficients in $\mathbb{Z}$. By combining it with our recent result [39] which is an extension of Goette’s comparison formula for two kinds of equivariant $\eta$-invariants [31], we finally conclude our main result: except a finite set on the circle $S^1$, the difference of the equivariant $\eta$-invariant and its contribution on the fixed point set is the restriction of a rational function of $g \in S^1$ with coefficients in $\mathbb{Z}$. It seems that our result is the first geometric application of differential K-theory.
Let’s recall first the Atiyah-Segal localization formula for equivariant index. Let \( Y \) be an \( S^1 \)-equivariant compact Spin\(^c\) manifold, in particular, there exists an \( S^1 \)-equivariant complex line bundle \( L \) such that \( \omega_2(TY) = c_1(L) \mod (2) \), where \( \omega_2 \) is the second Stiefel-Whitney class and \( c_1 \) is the first Chern class \( \text{[33, Appendix D]} \). Let \( E \) be an \( S^1 \)-equivariant complex vector bundle over \( Y \). Let \( D^Y \otimes E \) be the spin\(^c\) Dirac operator on \( S(TY,L) \otimes E \), where \( S(TY,L) \) is the spinor associated with \( L \) (cf. \( \text{[1,17]} \)).

For any complex vector bundle \( F \) over a manifold \( X \), we use the notation

\[
\text{Sym}_1(F) = 1 + \sum_{k>0} \text{Sym}^k(F)t^k, \quad \lambda_1(F) = 1 + \sum_{k>0} \Lambda^k(F)t^k
\]

for the symmetric and exterior power of \( K(X)[[t]] \) respectively and denote by \( \text{Sym}(F) := \text{Sym}_1(F) \). Here 1 is understood as the trivial complex line bundle on \( X \) in \( K(X) \), the \( K \)-group of \( X \).

Let \( Y^{S^1} \) be the fixed point sets of the circle action on \( Y \), then each connected component \( Y^\alpha_{S^1} \), \( \alpha \in \mathfrak{B} \), of \( Y^{S^1} \), is a compact manifold. Let \( N_\alpha \) be the normal bundle of \( Y^\alpha_{S^1} \) in \( Y \) and we can choose a complex structure on \( N_\alpha \) through the circle action. For any \( \alpha \in \mathfrak{B} \), \( Y^\alpha_{S^1} \) also has an equivariant spin\(^c\) structure with associated equivariant line bundle \( L_\alpha = L|_{Y^\alpha_{S^1}} \otimes (\det N_\alpha)^{-1} \) (see \( \text{[17, 33]} \) for instance).

Let \( K^0_{S^1}(Y^\alpha_{S^1}) \) be the localization of the equivariant \( K \)-group \( K^0_{S^1}(Y^\alpha_{S^1}) \) at the prime ideal \( I(g) \), which consists of all characters of \( S^1 \) vanishing at \( g \).

Assume temporary that \( Y \) is even dimensional, then the spinor is naturally \( \mathbb{Z}_2 \)-graded: \( S(TY,L) = S^+(TY,L) \oplus S^-(TY,L) \). Let \( D^Y \otimes E \) be the restriction of \( D^Y \otimes E \) on \( C^\infty(Y;S^\pm(TY,L) \otimes E) \), the space of smooth sections of \( S^\pm(TY,L) \otimes E \) on \( Y \). Then \( \ker(D^Y \otimes E) \), the kernel of \( D^Y \otimes E \), are finite dimensional \( S^1 \)-complex vector spaces. Set

\[
\text{Ind}_g(D^Y \otimes E) = \text{Tr}|_{\ker(D^Y \otimes E)}[g] - \text{Tr}|_{\ker(D^Y \otimes E)}[g]
\]

be the equivariant index of \( D^Y \otimes E \) for \( g \in S^1 \). For \( x = (F^+ - F^-)/\chi \in K^0_{S^1}(Y^\alpha_{S^1}) \) with \( \chi \) a character of \( S^1 \) such that \( \chi(g) \neq 0 \), the equivariant index of \( D^Y \otimes E \) for \( g \in S^1 \) is defined by

\[
\text{Ind}_g(D^Y \otimes E) := \chi(g)^{-1} \left( \text{Ind}_g(D^Y \otimes F^+) - \text{Ind}_g(D^Y \otimes F^-) \right),
\]

which does not depend on the choices of \( F^+, F^- \in K^0_{S^1}(Y^\alpha_{S^1}) \) and \( \chi \). Here \( D^Y \otimes F^\pm \) is the spin\(^c\) Dirac operator on \( Y^\alpha_{S^1} \) defined in a similar way.

**Theorem 0.1.** \( \text{[3]} \) Lemma 2.7 and Proposition 2.8] If \( Y^{S^1} \) is the fixed point set of \( g \in S^1 \), then there exists an inverse \( \lambda_{-1}(N^\alpha_+)^{-1} \) in \( K^0_{S^1}(Y^\alpha_{S^1}) \). Moreover,

\[
\text{Ind}_g(D^Y \otimes E) = \sum_{\alpha} \text{Ind}_g \left( D^Y \otimes \lambda_{-1}(N^\alpha_+)^{-1} \otimes E \right)_{Y^{S^1}}.
\]

For simplicity, we fix a complex structure on \( N_\alpha \) such that the weights of the \( S^1 \)-action on \( N_\alpha \) are all positive. Then by \( \text{[11, (1.15)]} \), we can reformulate Theorem 0.1 as

\[
\text{Ind}_g(D^Y \otimes E) = \sum_{\alpha} \text{Ind}_g \left( D^Y \otimes \text{Sym}(N^{\alpha}_+) \otimes E \right)_{Y^{S^1}}
\]

as distributions on \( S^1 \).
Notice that \( \text{Ind}(D^Y \otimes E) = \ker(D^Y_+ \otimes E) - \ker(D^Y_- \otimes E) \in R(S^1) \), the representation ring of \( S^1 \), is a finite dimensional virtual representation of \( S^1 \) and for each \( \alpha \),

\[
(0.6) \quad \text{Ind} \left( D^{Y_{S^1}}_\alpha \otimes \text{Sym}(N^*_\alpha) \otimes E|_{Y^{S^1}} \right) \in R[S^1]
\]

is a formal representation of \( S^1 \), i.e., each weight \( k \) part of \( (0.6) \), denoted by

\[
\text{Ind} \left( D^{Y_{S^1}}_\alpha \otimes \text{Sym}(N^*_\alpha) \otimes E|_{Y^{S^1}} \right)_k
\]

is a finite dimensional virtual vector space. As a consequence of \( (0.5) \), for any \( |k| \gg 1 \),

\[
(0.7) \quad \sum_\alpha \text{Ind} \left( D^{Y_{S^1}}_\alpha \otimes \text{Sym}(N^*_\alpha) \otimes E|_{Y^{S^1}} \right)_k = 0.
\]

From now on, we assume that \( Y \) is odd dimensional. Let \( g^{TY} \) be an \( S^1 \)-invariant Riemannian metric on \( TY \), and \( \nabla^{TY} \) be the Levi-Civita connection on \((Y, g^{TY})\). Let \( h^L \) and \( h^E \) be \( S^1 \)-invariant metrics and \( \nabla^L \) and \( \nabla^E \) be \( S^1 \)-invariant Hermitian connections on \((L, h^L)\), \((E, h^E)\). Denote by

\[
(0.8) \quad (TY, g^{TY}, \nabla^{TY}), \quad L = (L, h^L, \nabla^L), \quad E = (E, h^E, \nabla^E).
\]

We call them equivariant geometric triples.

For \( g \in S^1 \), let \( \tilde{\eta}_g(TY, L, E) \) be the associated equivariant APS reduced \( \eta \)-invariant (cf. Definition 1.2).

In the rest of this paper, we always assume that \( Y^{S^1} \neq \emptyset \) except in Section 3.5.

Let \( L_\alpha, N^*_\alpha \) and \( \lambda_{-1}(N^*_\alpha) \) be the induced geometric triples on \( Y^{S^1}_\alpha \). In view of \( (0.5) \), it is natural to ask whether we can define \( \tilde{\eta}_g \left( TY^{S^1}_\alpha, L_\alpha, \text{Sym}(N^*_\alpha) \otimes E|_{Y^{S^1}} \right) \) as certain distribution on \( S^1 \) for each \( \alpha \) and how to compute the difference

\[
(0.9) \quad \tilde{\eta}_g(TY, L, E) - \sum_\alpha \tilde{\eta}_g \left( TY^{S^1}_\alpha, L_\alpha, \text{Sym}(N^*_\alpha) \otimes E|_{Y^{S^1}} \right)
\]

as certain distribution on \( S^1 \) by using geometric data on \( Y \).

In this paper, we give a realization of \( \lambda_{-1}(N^*_\alpha)^{-1} \) in the localization of equivariant differential K-theory, such that

\[
(0.10) \quad \sum_\alpha \tilde{\eta}_g \left( TY^{S^1}_\alpha, L_\alpha, \lambda_{-1}(N^*_\alpha)^{-1} \otimes E|_{Y^{S^1}} \right)
\]

is well-defined, and then we identify it to \( \tilde{\eta}_g(TY, L, E) \) up to a rational function on \( S^1 \) with coefficient in \( \mathbb{Z} \). The reminder challenging problem is to compute precisely this rational function on \( S^1 \) in a geometric way.

For \( g \in S^1 \), let \( \hat{K}^0_g(Y) \) be the \( g \)-equivariant differential K-group in Definition 2.12, which is the Grothendieck group of equivalent classes (see 2.83), which is the Grothendieck group of equivalent classes (see 2.83) for the equivalent relation) of cycles \((E, \phi)\), where \( E \) is an equivariant geometric triple and \( \phi \in \Omega^{odd}(Y^g, \mathbb{C})/d\Omega^{even}(Y^g, \mathbb{C}) \), the space of odd degree complex valued differential forms on the fixed point sets \( Y^g \) of \( g \), modulo exact forms. Let \( \hat{K}^0_g(Y)_{I(g)} \) be its localization at the prime ideal \( I(g) \).

With respect to the \( S^1 \)-action, we have the decomposition of complex vector bundles \( N_\alpha = \bigoplus_{v \geq 0} N_{\alpha, v} \) such that \( g \in S^1 \) acts on \( N_{\alpha, v} \) by multiplication by \( g^v \).

Using the pre-\( \lambda \)-ring structure of the differential K-theory constructed in Theorem 2.16, we obtain the differential K-theory version of the first part of Theorem 0.11.
**Theorem 0.2** (See Theorem [2.14]). There exists a finite subset $A \subset S^1$ (cf. (1.7)), such that for $g \in S^1 \setminus A$, \( [\lambda_{-1}(N^*_a), 0] \) is invertible in $\hat{K}^0_g(Y_{a}^{S^1})_{I(g)}$ and there exists $N_0 > 0$, which does not depend on $g \in S^1 \setminus A$, such that for any $N \in \mathbb{N}, N > N_0$, we have

\[
[\lambda_{-1}(N^*_a), 0]^{-1} = [\lambda_{-1}(N^*_a)_{N}, 0] \in \hat{K}^0_g(Y_{a}^{S^1})_{I(g)}.
\]

Here $\lambda_{-1}(N^*_a)_{N}$ is defined by cutting out up to degree $N > N_0$ from the expansion of $\lambda_{-1}(N^*_a)^{-1}$ by using the $\gamma$-filtration (see the precise definition in (2.61), (2.63), (2.64) and (2.32)).

Note that $\lambda_{-1}(N^*_a)_{N}$ is a sum of virtual vector bundles on $Y_{a}^{S^1}$ with coefficients in

\[
F(g)/ \prod_{x: N_x \neq 0} (g^v - 1)^{\text{rk} N_{a,v} + N}
\]

with $F(x) \in \mathbb{Z}[x]$, here $\mathbb{Z}[x]$ means the ring of polynomials on $x$ with coefficients in $\mathbb{Z}$ and \( \text{rk} N \) is the rank of the complex vector bundle $N$.

For $g \in S^1$, set

\[
Q_g := \{ P(g)/Q(g) \in \mathbb{C} : P, Q \in \mathbb{Z}[x] \} \subset \mathbb{C}.
\]

Let $\iota : Y^{S^1} \rightarrow Y$ be the natural embedding. Let $\check{\iota}^* : \hat{K}^0(Y)_{I(g)} \rightarrow \hat{K}^0_g(Y^{S^1})_{I(g)}$ be the induced homomorphism.

**Theorem 0.3.** For $g \in S^1$, the push-forward map $\check{f}_{Y*} : \hat{K}^0_g(Y)_{I(g)} \rightarrow \mathbb{C}/Q_g,$

\[
(E, \phi)/\chi \mapsto \chi(g)^{-1} \left( \int_{Y^g} \text{Td}_g(\nabla^{TY}, \nabla^L) \wedge \phi + \check{\eta}_g(TY, L, E) \right)
\]

is well-defined.

For any $g \in S^1 \setminus A$, $\check{\iota}^*$ is an isomorphism and the following diagram commutes

\[
\begin{array}{ccc}
\hat{K}^0(Y^{S^1})_{I(g)} & \xrightarrow{[\lambda_{-1}(N^*_a), 0]^{-1} \cup \check{\iota}^*} & \hat{K}^0_g(Y)_{I(g)} \\
\check{f}_{Y^{S^1}*} & \downarrow & \check{f}_{Y*} \\
\mathbb{C}/Q_g & \xleftarrow{\check{f}_{Y*}} & \mathbb{C}/Q_g \\
\end{array}
\]

(0.15)

here the cup product $\cup$ is defined in (2.84). In particular, from (0.14), for any $N \in \mathbb{N}$ and $N > N_0$ with $N_0$ in Theorem [0.2] we have

\[
\check{\eta}_g(TY, L, E) - \sum_{\alpha} \bar{\eta}_g(TY^a_{\alpha} \cup L, \lambda_{-1}(N^*_a)_{N} \otimes E|_{Y^{S^1}}) \in Q_g.
\]

Remark that $\mathbb{C}/Q_g$ here could be regarded as the localization of the $g$-equivariant differential $K^1$-group of a point. It could be computed from the analogue of (3.50) for the $g$-equivariant differential $K^1$-group. However, this topic exceeds the scope of this paper.

The final main result of our paper is as follows:
**Theorem 0.4.** Fixed $A \subset S^1$, $N_0 \in \mathbb{N}$ in Theorem 0.2, then for any $N \in \mathbb{N}$, $N > N_0$, for any equivariant geometric triple $E$ over $Y$, 

\[(0.17) \quad \bar{\eta}_g(\mathcal{T}Y, L, E) - \sum_{\alpha} \bar{\eta}_g\left(\mathcal{T}Y_{\alpha}^S, L_\alpha, \lambda_{-1}(N_\alpha)^{-1} \otimes E|_{Y_{\alpha}^S}\right)\]

is the restriction over $S^1 \setminus A$ of a rational function with coefficients in $\mathbb{Z}$ of $g \in S^1$, and it has no poles on $S^1 \setminus A$.

In the last part of this paper (see Section 3.5), we discuss the case when $Y^{S^1} = \emptyset$.

**Theorem 0.5** (See Theorem 3.13). If $Y^{S^1} = \emptyset$, $A = \{g \in S^1 : Yg \neq \emptyset\}$, then $\bar{\eta}_g(\mathcal{T}Y, L, E)$ as a function on $S^1 \setminus A$ is the restriction of a rational function on $S^1$ with coefficients in $\mathbb{Z}$, and it has no poles on $S^1 \setminus A$.

As the end of the introduction, we like to give a proof of Theorem 0.5 and a formal vanishing of (0.19) in a special case.

We suppose that there exists an oriented even dimensional $S^1$-equivariant Spin$^c$ Riemannian manifold $X$ with boundary $Y = \partial X$ and associated $S^1$-equivariant Hermitian line bundle $\mathcal{L} = (\mathcal{L}, h^\mathcal{L}, \nabla^\mathcal{L})$, and an $S^1$-equivariant Hermitian vector bundle $(\mathcal{E}, h^\mathcal{E})$ with $S^1$-invariant connection $\nabla^\mathcal{E}$ such that $(X, g^TX)$, $\mathcal{E}$ and $\mathcal{L}$ are of product structure near the boundary and their restriction on

\[(0.18) \quad S(TX, \mathcal{L}) = S^+(TX, \mathcal{L}) \oplus S^-(TX, \mathcal{L}), \quad S^+(TX, \mathcal{L})|_Y = S(TY, L).\]

Consider $\text{Ind}_{\text{APS}}(D^X \otimes \mathcal{E})$, the index of the Dirac operator $D^X \otimes \mathcal{E}$ with respect to the APS boundary condition gives in [3, 24]: For $g \in S^1 \setminus A_1$, with $A_1 = \{h \in S^1 : X^{S^1} \neq X^h\}$, we have

\[(0.19) \quad \text{Ind}_{\text{APS}, g}(D^X \otimes \mathcal{E}) = \int_{X^{S^1}} \text{Td}_g(TX, \mathcal{L}) \text{ch}_g(\mathcal{E}) - \bar{\eta}_g(\mathcal{T}Y, L, E).\]

Note that $\text{Ind}_{\text{APS}}(D^X \otimes \mathcal{E})$ is a finite dimensional virtual $S^1$-representation.

If $Y^{S^1} = \emptyset$, then in (0.19), $X^{S^1}$ is a manifold without boundary for $g \in S^1 \setminus A_1$. Thus (0.19) implies immediately that $\bar{\eta}_g(\mathcal{T}Y, L, E)$ on $S^1 \setminus A_1$ is the restriction of a rational function on $S^1$ with coefficients in $\mathbb{Z}$.

Assume now $Y^{S^1} \neq \emptyset$. We denote by $\text{Ind}_{\text{APS}}(D^X \otimes \mathcal{E}, n)$ the multiplicity of weight $n$ part of $S^1$-representation in $\text{Ind}_{\text{APS}}(D^X \otimes \mathcal{E})$. Then

\[(0.20) \quad \text{Ind}_{\text{APS}, g}(D^X \otimes \mathcal{E}) = \sum_n \text{Ind}_{\text{APS}}(D^X \otimes \mathcal{E}, n) \cdot g^n, \quad \text{for any } g \in S^1.\]

Introduce the notation (cf. (3.11))

\[(0.21) \quad R(q) = q^{-\frac{1}{2}} \sum_{v \in \mathbb{R}} \text{rk} N_v + \frac{1}{2} \bigotimes_{v > 0} \text{Sym}_{q^{-v}}(N_v) \otimes \left( \sum_{v > 0} E_v q^v \right) = \sum_k R_k q^k \in K(X)[[g, q^{-1}]].\]

Now in view of (0.3), as $X^{S^1}$ is a manifold with boundary $Y^{S^1} = \partial X^{S^1}$, we apply the usual APS-index theorem [3] for the operator $D^{X^{S^1}} \otimes R_k$ and then obtain

\[(0.22) \quad \text{Ind}_{\text{APS}}(D^{X^{S^1}} \otimes R_k) = \int_{X^{S^1}} \text{Td}(TX^{S^1}, \mathcal{L}) \text{ch}(R_k) - \sum_{\alpha} \bar{\eta}(TY_{\alpha}^{S^1}, L_\alpha, R_k),\]
where $\mathcal{L}'$ is the associated line bundle over $X^{S^1}$ defined as in (1.33). By the general analytic localization technique in local index theory by Bismut-Lebeau [16], we can expect that the arguments in [41] §1.2 extend to the APS-index case, i.e., we can expect that

$$\text{Ind}_{\text{APS}}(D^X \otimes \mathcal{E}, k) = \text{Ind}_{\text{APS}}(D^{X^{S^1}} \otimes R_k), \quad \text{for any } k \in \mathbb{Z}. \quad (0.23)$$

Now formally as for manifolds without boundary, we have

$$\sum_{k \in \mathbb{Z}} \text{Td}(TX^{S^1}, \mathcal{L}') \cdot \text{ch}(R_k) \cdot g^k = \text{Td}_g(TX, \mathcal{L}) \cdot \text{ch}_g(\mathcal{E}). \quad (0.24)$$

Thus at least formally, for $g \in S^1 \backslash A_1$,

$$\bar{\eta}_g(TY, L, E) = \sum_{k \in \mathbb{Z}} \sum_{\alpha} \bar{\eta}(TY^{S^1}, L_\alpha, R_k) \cdot g^k. \quad (0.25)$$

Thus we show formally that (0.19) is zero. Of course, from the above argument for the case $Y^{S^1} = \emptyset$, we also understand that in general, (0.19) will not be zero. At least this discussion indicates the possibilities to compute (0.19) in geometric way.

However, the authors do not know a theorem stating that for $(TY, L, E)$ as above, there exist $k > 0$ and $S^1$-equivariant $X, L, \mathcal{E}$ such that $\partial X$ consists of $k$ properly oriented copies of $Y$ and on the restriction of $\partial X$, we get $TY, L, E$. Another difficulty is that how to make a proper sense for the right-hand side of (0.25). This explains that our intrinsic formulation of Theorem 0.4 does not rely on the existence of such an $X$, also that the usefulness of the $\gamma$-filtration we introduced in differential K-theory.

Finally, it is natural to ask whether there is the similar localization formula (0.17) for the real analytic torsion [45, 17]. However, unlike the holomorphic torsion and the $\eta$-invariant, a suitable $K$-theory where the real analytic torsion is the analytic ingredient of a Riemann-Roch type theorem in this theory (cf. [15]) is still lacking.

The main result of this paper is announced in [38].

This paper is organized as follows. In Section 1 we introduce the main object of our paper: the equivariant $\eta$-invariant, and we review some of its analytic properties, which we will use in this paper, such as the variation formula, the embedding formula and the comparison of equivariant $\eta$-invariants. In Section 2 we prove that the differential K-ring is a pre-$\lambda$-ring and construct the inverse of $[\lambda_1(N^*), 0]$ in $R^0_g(Y^{S^1}_\alpha)$ explicitly. In Section 3 we prove Theorem 0.3 and 0.4 and study the case for $Y^{S^1} = \emptyset$. We compute also in detail the equivariant $\eta$-invariant in the case $Y = S^1$.

**Notation:** For any vector space $V$, $B \in \text{End}(V)$, we denote by $\text{Tr}[B]$ the trace of $B$ on $V$.

We denote by dim the real or complex dimension of a vector space, if it’s clear in the context, otherwise, we add the subscript $\mathbb{R}$ or $\mathbb{C}$. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\Omega^\bullet(X, \mathbb{K})$ the space of smooth $\mathbb{K}$-valued differential forms on a manifold $X$, and its subspaces of even/odd degree forms by $\Omega^{\text{even/odd}}(X, \mathbb{K})$. Let $d$ be the exterior differential, then the image of $d$ is the space of exact forms, $\text{Im} d$.

Let $R(S^1)$ be the representation ring of the circle group $S^1$. For any finite dimensional virtual $S^1$-representation $V = M - M' \in R(S^1)$ and $h \in S^1$, the character $\chi_V(h) = \text{Tr}|_M[h] - \text{Tr}|_{M'}[h] \in \mathbb{Z}[h, h^{-1}]$, a polynomial on $h$ and $h^{-1}$. Conversely, for any $f \in \mathbb{Z}[h, h^{-1}]$,
there exists a finite dimensional virtual \( S^1 \)-representation \( V_f \in R(S^1) \) such that \( f = \chi_{V_f} \) on \( S^1 \). So in this paper, we will not distinguish the finite dimensional virtual \( S^1 \)-representation and \( f \in \mathbb{Z}[h, h^{-1}] \) as an element of \( R(S^1) \).

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1. **Equivariant \( \eta \)-invariants**

In this section, we review some of the facts on the equivariant \( \eta \)-invariants. In Section 1.1, we study the equivariant decomposition of \( TY \). In Section 1.2, we define the equivariant \( \eta \)-invariant. In Section 1.3, we recall the variation formula. In Section 1.4, we explain the embedding formula. In Section 1.5, we compare the equivariant \( \eta \)-invariant with the equivariant infinitesimal \( \eta \)-invariant.

1.1. **Circle action.** Let \( Y \) be a smooth compact manifold with a smooth circle action. For \( g \in S^1 \), set

\[
Y^g = \{ y \in Y : gy = y \},
\]

\[
Y^{S^1} = \{ y \in Y : hy = y \text{ for any } h \in S^1 \}.
\]

Then \( Y^g \) is the fixed point set of \( g \)-action on \( Y \) and \( Y^{S^1} \) is the fixed point set of the circle action on \( Y \) with connected components \( \{Y^{S^1}_\alpha\}_{\alpha \in \mathcal{B}} \). Since \( Y \) is compact, the index set \( \mathcal{B} \) is a finite set. Certainly, for any \( g \in S^1 \), \( Y^{S^1} \subset Y^g \).

Unless otherwise stated, we assume that \( Y^{S^1} \neq \emptyset \).

If \( g \in S^1 \) is a generator of \( S^1 \), that is, \( g = e^{2\pi it} \) with \( t \in \mathbb{R} \) irrational, then \( Y^{S^1} \) is the fixed point set \( Y^g \) of \( g \). We have the decomposition of real vector bundles over \( Y^{S^1}_\alpha \)

\[
TY|_{Y^{S^1}_\alpha} = TY^{S^1}_\alpha \oplus \bigoplus_{v \neq 0} N^R_{\alpha,v},
\]

where \( N^R_{\alpha,v} \) is the underlying real vector bundle of a complex vector bundle \( N_{\alpha,v} \) over \( Y^{S^1}_\alpha \) such that \( g \) acts on \( N_{\alpha,v} \) by multiplication by \( g^v \). Let \( N_\alpha \) be the normal bundle of \( Y^{S^1}_\alpha \) in \( Y \). Then (1.2) induces the canonical identification \( N_\alpha = \bigoplus_{v \neq 0} N^R_{\alpha,v} \). We will regard \( N_\alpha \) as a complex vector bundle. As complex vector bundles, complex conjugation provides a \( \mathbb{C} \)-anti-linear isomorphism between \( N_{\alpha,v} \) and \( \overline{N}_{\alpha,-v} \). Since we can choose either \( N_{\alpha,v} \) or \( \overline{N}_{\alpha,v} \) as the complex vector bundle for \( N^R_{\alpha,v} \), in what follows, we may and we will assume that

\[
TY|_{Y^{S^1}_\alpha} = TY^{S^1}_\alpha \oplus \bigoplus_{v > 0} N^R_{\alpha,v}, \quad N_\alpha = \bigoplus_{v > 0} N_{\alpha,v}.
\]

Since the dimension of \( TY \) is finite, there are only finitely many \( v \) such that \( \text{rk} N_{\alpha,v} \neq 0 \) for each \( \alpha \). Here \( \text{rk} N \) is the rank of the complex vector bundle \( N \). Set

\[
q = \max \{ v : \text{there exists } \alpha \in \mathcal{B} \text{ such that } \text{rk} N_{\alpha,v} \neq 0 \}.
\]
Let $\mathcal{F}_q$ be the set of Farey sequence of order $q$:
\begin{equation}
\mathcal{F}_q = \{0 < k/n \leq 1 : k, n \text{ coprime}, n \leq 2q\}.
\end{equation}

Obviously, $\mathcal{F}_q$ is a finite set. Set
\begin{equation}
A = \{g \in S^1 : Y^S \neq Y^g\}.
\end{equation}

**Proposition 1.1.** For any $g \in S^1 \setminus A$, we have $Y^g = Y^{S^1}$. The set $A \subset S^1$ is finite and
\begin{equation}
A \subset \{e^{2\pi it} : t \in \mathcal{F}_q\}.
\end{equation}

Moreover, we have the decomposition
\begin{equation}
TY|_{Y^S} = TY^{S^1} \oplus \bigoplus_{v=1}^q N_{\alpha,v}^R,
\end{equation}
such that $g \in S^1$ acts on $N_{\alpha,v}$ by multiplication by $g^v$.

**Proof.** From (1.6), for any $g \in S^1 \setminus A$, we have $Y^g = Y^{S^1}$.

To establish (1.7), we only need to prove that if $g \in S^1 \setminus \{e^{2\pi it} : t \in \mathcal{F}_q\}$ is not a generator of $S^1$, then $Y^g = Y^{S^1}$. In this case, there exists a rational number $t = k/n \in (0, 1]$, $k, n$ coprime, $n > 2q$, such that $g = e^{2\pi it}$. Thus $g$ generates the cyclic subgroup $\mathbb{Z}_n \subset S^1$ of order $n$.

Let $Y(n)$ be the fixed point set of the induced $\mathbb{Z}_n$-action on $Y$. Let $N(n)$ be the normal bundle of $Y(n)$ in $Y$. Similarly as in (1.4), as real vector bundles, we have the decomposition
\begin{equation}
TY|_{Y(n)} = TY(n) \oplus N(n),
\end{equation}
\begin{equation}
N(n) = \bigoplus_{1 \leq v < n/2} N(n)_v \oplus N(n)_{n/2}.
\end{equation}

Here for $1 \leq v < n/2$, $N(n)_v$ is a real vector bundle over $Y(n)$ which has a complex structure such that $g$ acts by multiplication by $g^v$ on it and $N(n)_{n/2}$ is a real vector bundle over $Y(n)$ with $g$ acting by multiplication by $-1$ on it.

Let $N_{Y^S}/Y(n)$ be the normal bundle of $Y^S$ in $Y(n)$. Then by (1.3), as real vector bundles,
\begin{equation}
N_{Y^S}/Y(n) = \bigoplus_{v > 0, v \in \mathbb{Z}} N_{\alpha,v}^R.
\end{equation}

By (1.3) and (1.9), the restriction of $N(n)_v$ ($1 \leq v \leq n/2$) to $Y^S$ is given by the direct sum of real vector bundles
\begin{equation}
N(n)_v|_{Y^S} = \bigoplus_{v' > 0, v' \equiv v \mod n} N_{\alpha,v'}^R \bigoplus \bigoplus_{v' > 0, v' \equiv -v \mod n} N_{\alpha,v'}^R.
\end{equation}

From (1.10) and (1.11), we have the following identifications of real vector bundles over $Y^S$,
\begin{equation}
N(n)_{n/2}|_{Y^S} = \bigoplus_{v > 0, v \equiv n/2 \mod n} N_{\alpha,v}^R,
\end{equation}
\begin{equation}
TY(n)|_{Y^S} = TY^S \oplus \bigoplus_{v \neq 0, v \in \mathbb{Z}} N_{\alpha,v}^R.
\end{equation}

Since $n > 2q$, by (1.4), (1.11) and (1.12), we have
\begin{equation}
TY(n)|_{Y^S} = TY^S, \quad N(n)_v|_{Y^S} = N_{\alpha,v}^R, \quad N(n)_{n/2} = \{0\}.
\end{equation}
So we have $Y(n) = Y^{S^1}$ and $[\mathbb{13}]$ is also the $\mathbb{Z}_n$-equivariant decomposition of $TY$ over $Y(n)$.

The proof of Proposition $[\mathbb{13}]$ is completed. \hfill $\square$

1.2. Equivariant $\eta$-invariants. In the rest of this section, let $Y$ be an odd dimensional compact oriented manifold with a circle action. Certainly the circle action preserves the orientation of $Y$.

Let $g^{TY}$ be an $S^1$-invariant metric on $TY$.

Assume that $Y$ has an $S^1$-equivariant spin$^c$ structure, i.e., the $S^1$-action on $Y$ lifts naturally on the associated Spin$^c$ principal bundle, in particular, there exists an $S^1$-equivariant complex line bundle $L$ such that $\omega_2(TY) = c_1(L) \mod 2$, where $\omega_2$ is the second Stiefel-Whitney class and $c_1$ is the first Chern class $[35, \text{Appendix D}]$. Let $\mathcal{S}(TY, L)$ be the fundamental complex spinor bundle for $L$ which locally may be written as

\begin{equation}
\mathcal{S}(TY, L) = \mathcal{S}_0(TY) \otimes L^{1/2},
\end{equation}

where $\mathcal{S}_0(TY)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on $TY$ and $L^{1/2}$ is the (possibly non-existent) square root of $L$. It is naturally an $S^1$-equivariant complex vector bundle.

Let $E$ be an $S^1$-equivariant complex vector bundle over $Y$. Then $S^1$ acts on $\mathcal{C}^\infty(Y, \mathcal{S}(TY, L) \otimes E)$ by

\begin{equation}
(gs)(x) = g.s(g^{-1}x), \quad \text{for } g \in S^1.
\end{equation}

Let $h^L$ and $h^E$ be $S^1$-invariant Hermitian metrics on $L$ and $E$ respectively. Let $h^{\mathcal{S}^c}$ be the $S^1$-invariant Hermitian metric on $\mathcal{S}(TY, L)$ induced by $g^{TY}$ and $h^L$.

Let $\nabla^{TY}$ be the Levi-Civita connection on $(TY, g^{TY})$. Let $\nabla^L$ and $\nabla^E$ be $S^1$-invariant Hermitian connections on $(L, h^L)$ and $(E, h^E)$ respectively. Let $\nabla^{\mathcal{S}^c}$ be the connection on $\mathcal{S}(TY, L)$ induced by $\nabla^{TY}$ and $\nabla^L$ $[35, \text{Appendix D}]$. Let $\nabla^{\mathcal{S}^c} \otimes E$ be the connection on $\mathcal{S}(TY, L) \otimes E$ induced by $\nabla^{\mathcal{S}^c}$ and $\nabla^E$,

\begin{equation}
\nabla^{\mathcal{S}^c} \otimes E = \nabla^{\mathcal{S}^c} \otimes 1 + 1 \otimes \nabla^E.
\end{equation}

Let $\{e_j\}$ be a locally oriented orthonormal frame of $(TY, g^{TY})$. We denote by $c(\cdot)$ the Clifford action of $TY$ on $\mathcal{S}(TY, L)$. Let $D^Y \otimes E$ be the spin$^c$ Dirac operator on $Y$ defined by

\begin{equation}
D^Y \otimes E = \sum_j c(e_j) \nabla^{\mathcal{S}^c} \otimes E : \mathcal{C}^\infty(Y, \mathcal{S}(TY, L) \otimes E) \to \mathcal{C}^\infty(Y, \mathcal{S}(TY, L) \otimes E).
\end{equation}

Then $D^Y \otimes E$ is an $S^1$-equivariant first order self-adjoint elliptic differential operator on $Y$ and its kernel $\text{Ker}(D^Y \otimes E)$ is an $S^1$-complex vector space.

Let $\exp(-u(D^Y \otimes E)^2)$, $u > 0$, be the heat semi-group of $(D^Y \otimes E)^2$.

We denote by $TY, L, E$ the equivariant geometric data

\begin{equation}
TY = (TY, g^{TY}, \nabla^{TY}), \quad L = (L, h^L, \nabla^{L}), \quad E = (E, h^E, \nabla^E).
\end{equation}

We also call $TY, L, E$ equivariant geometric triples over $Y$. 
Definition 1.2. For \( g \in S^1 \), the equivariant \( \eta \)-invariant associated with \( TY, L, E \) is defined by
\[
\tilde{\eta}_g(TY, L, E) = \int_0^{+\infty} \text{Tr} \left[ g(D^Y \otimes E) \exp(-u(D^Y \otimes E)^2) \right] \frac{du}{2\sqrt{\pi u}} + \frac{1}{2} \text{Tr} |\text{Ker}(D^Y \otimes E)| [g] \in \mathbb{C}.
\]

Note that the convergence of the integral in (1.19) is nontrivial (see e.g., [13, Theorem 2.6], [24], [53 Theorem 2.1]).

1.3. Variation formula. Since \( g^{TY} \) is \( S^1 \)-invariant, for any \( g \in S^1 \), the fixed point set \( Y^g \) is an odd dimensional totally geodesic submanifold of \( Y \). Since the \( S^1 \)-action preserves the spin\(^c\) structure, we see that \( Y^g \) is canonically oriented (cf. [8, Proposition 6.14], [40, Lemma 4.1]).

As in (1.9), for \( g = e^{2\pi it} \in S^1 \), we have the decomposition of real vector bundles over \( Y^g \),
\[
(TY|_{Y^g} = Ty^g \oplus \bigoplus_{c>0} N_c,
\]
where \( N_c \) is a real vector bundle over \( Y^g \) which has a complex structure such that \( g \) acts by multiplication by \( g^c \) if \( g^c \neq -1 \) or an even dimensional oriented real vector bundle on which \( g \) acts by multiplication by \( -1 \) if \( g^c = -1 \).

Let \( N_{Y^g/Y} \) be the normal bundle of \( Y^g \) in \( Y \). Then \( N_{Y^g/Y} = \oplus_{c>0} N_c \). Since \( g^{TY} \) is \( S^1 \)-invariant, the decomposition (1.20) is orthogonal and the restriction of \( \nabla^{TY} \) on \( Y^g \) is split under the decomposition. Let \( g^{TY^g}, g^N \) and \( g^{N_c} \) be the metrics induced by \( g^{TY} \) on \( Y^g \), \( N \) and \( N_c \). Let \( \nabla^{TY^g}, \nabla^N \) and \( \nabla^{N_c} \) be the corresponding induced connections on \( Y^g \), \( N \) and \( N_c \), with curvatures \( R^{TY^g}, R^N \) and \( R^{N_c} \). Then under the decomposition (1.20),
\[
g^{TY} = g^{TY^g} \oplus g^N, \quad g^N = \bigoplus_v g^{N_v}, \quad \nabla^{TY}|_{Y^g} = \nabla^{TY^g} \oplus \bigoplus_v \nabla^{N_v}.
\]

Similarly as (1.20), we have the orthogonal decomposition of complex vector bundles with connections on \( Y^g \)
\[
E|_{Y^g} = \bigoplus_v E_v, \quad \nabla^E|_{Y^g} = \bigoplus_v \nabla^{E_v},
\]
here \( g \in S^1 \) acts as multiplication by \( g^v \) on \( E_v \) and the connection \( \nabla^{E_v} \) on \( E_v \) is induced by \( \nabla^E \). Let \( R^E, R^{E_v} \) be the curvatures of \( \nabla^E, \nabla^{E_v} \).

Definition 1.3. For \( g = e^{2\pi it} \in S^1 \), set
\[
\hat{A}(TY^g, \nabla^{TY^g}) := \text{det}^{1/2} \left( \frac{i}{4\pi} R^{TY^g} \text{ sinh} \left( \frac{i}{4\pi} R^{TY^g} \right) \right),
\]
\[
\hat{A}_g(N, \nabla^N) := \prod_{c>0} \left( i^{\dim N_c} \text{det}^{1/2} |N_c| \left( 1 - g \cdot \exp \left( \frac{i}{2\pi} R^{N_c} \right) \right) \right)^{-1},
\]
\[
\hat{A}_g(TY, \nabla^TY) := \hat{A}(TY^g, \nabla^{TY^g}) \cdot \hat{A}_g(N, \nabla^N) \in \Omega^*(Y^g, \mathbb{C}),
\]
\[
\text{ch}_g(E) := \sum_v \text{Tr} \left[ \exp \left( \frac{i}{2\pi} R^{E_v} + 2i\pi vt \right) \right] \in \Omega^*(Y^g, \mathbb{C}).
\]
The sign convention in $\hat{A}_g(N, \nabla^N)$ is that the degree 0 part in $\prod_{\nu>0}$ is given by $\left(\frac{\nu\pi i}{\nu-1}\right)^{\frac{1}{2}\dim N^\nu}$.

The forms in (1.23) are closed forms on $Y^g$ and their cohomology class does not depend on the $S^1$-invariant metrics $g^{TY}, h^E$ and connection $\nabla^E$. We denote by $\hat{A}(TY^g), \hat{A}_g(TY)$, $\text{ch}_g(E)$ their cohomology classes, two of which appear in the equivariant index theorem [8, Chapter 6].

Comparing with (1.23), if $g$ acts on $L|_{Y^g}$ by multiplication by $g^l$, we write

$$\text{ch}_g(L^{1/2}) := \exp \left(\frac{i}{4\pi} \hat{R}^L_{|Y^g} + i\pi l t\right) \in \Omega^*(Y^g, \mathbb{C}).$$

We denote by

$$\text{Td}_g(\nabla^{TY}, \nabla^L) := \hat{A}_g(TY, \nabla^{TY}) \text{ch}_g(L^{1/2}).$$

Let $\pi : Y \times \mathbb{R} \to Y$ be the natural projection. Then the $S^1$-action lifts naturally on $Y \times \mathbb{R}$, by acting only on the factor $Y$. Let $TY_j = (TY, g^j_TY, \nabla^j_TY), L_j = (L, h^j_L, \nabla^j_L)$ and $E_j = (E, h^j_E, \nabla^j_E)$ for $j = 0, 1$. Let $g^{*TY}, h^{*L}$ and $h^{*E}$ be $S^1$-invariant metrics on $\pi^*TY$, $\pi^*L$ and $\pi^*E$ over $Y \times \mathbb{R}$ such that for $j = 0, 1$,

$$g^{*TY}|_{Y \times \{j\}} = g^j_TY, \quad h^{*L}|_{Y \times \{j\}} = h^j_L, \quad h^{*E}|_{Y \times \{j\}} = h^j_E.$$

Let $\nabla^{*TY}, \nabla^{*L}$ and $\nabla^{*E}$ be $S^1$-invariant Hermitian connections on $(\pi^*TY, g^{*TY}), (\pi^*L, h^{*L})$ and $(\pi^*E, h^{*E})$ such that for $j = 0, 1$,

$$\nabla^{*TY}|_{Y \times \{j\}} = \nabla^j_TY, \quad \nabla^{*L}|_{Y \times \{j\}} = \nabla^j_L, \quad \nabla^{*E}|_{Y \times \{j\}} = \nabla^j_E.$$

Let $\pi^*E = (\pi^*E, h^{*E}, \nabla^{*E})$ be the associated geometric triple on $Y \times \mathbb{R}$.

Let $s$ be the coordinate of $\mathbb{R}$ in $Y \times \mathbb{R}$. If $\alpha = ds \wedge \alpha_0 + \alpha_1$ with $\alpha_0, \alpha_1 \in \Lambda^*(T^*Y)$, we denote by

$$\{\alpha\}^{ds} := \alpha_0.$$

For $g \in S^1$, the equivariant Chern-Simons classes $\tilde{\text{ch}}_g(E_0, E_1), \tilde{Td}_g(\nabla^0_TY, \nabla^0_L, \nabla^1_TY, \nabla^1_L) \in \Omega^{odd}(Y^g, \mathbb{C})/\text{Im } d$ are defined by

$$\tilde{\text{ch}}_g(E_0, E_1) = \int_0^1 \{\text{ch}_g(\pi^*E)\}^{ds} ds \in \Omega^{odd}(Y^g, \mathbb{C})/\text{Im } d,$$

$$\tilde{Td}_g(\nabla^0_TY, \nabla^0_L, \nabla^1_TY, \nabla^1_L) = \int_0^1 \{\text{Td}_g(\nabla^{*TY}, \nabla^{*L})\}^{ds} ds \in \Omega^{odd}(Y^g, \mathbb{C})/\text{Im } d.$$

Moreover, we have

$$d \tilde{\text{ch}}_g(E_0, E_1) = \text{ch}_g(E_1) - \text{ch}_g(E_0),$$

$$d \tilde{Td}_g(\nabla^0_TY, \nabla^0_L, \nabla^1_TY, \nabla^1_L) = \text{Td}_g(\nabla^1_TY, \nabla^1_L) - \text{Td}_g(\nabla^0_TY, \nabla^0_L).$$

Note that the Chern-Simons classes depend only on $\nabla^j_TY, \nabla^j_L$ and $\nabla^j_E$ for $j = 0, 1$, i.e., they do not depend on the choice of the path of connections neither of metrics (see [12, Theorem B.5.4]).

Let $\tilde{\eta}_0(TY_0, L_0, E_0)$ and $\tilde{\eta}_0(TY_1, L_1, E_1)$ be the equivariant reduced $\eta$-invariants associated with $(TY_0, L_0, E_0)$ and $(TY_1, L_1, E_1)$ respectively. The following variation formula is proved
in [36, Proposition 2.14], which extends the usual well-known non-equivariant variation formula for \( \eta \)-invariant (cf. [4, p95] or [13, Theorem 2.11]).

**Theorem 1.4.** There exists \( V \in R(S^1) \) such that for any \( g \in S^1 \),

\[
\bar{c} \neq c
\]

\[
\bar{c}(u) = \sqrt{2} u^* \wedge, \quad \bar{c}(\bar{u}) = -\sqrt{2} i \pi \quad \text{for any } u \in \mathbb{N},
\]

Here \( \wedge, i \) are the exterior and interior product on forms.

Set

\[
\Lambda \eta = L|_{Y^S} \otimes (\det_\mathbb{N})^{-1}.
\]

Then \( TY^S \) has an equivariant spin\(^c\) structure since \( \omega_2(TY^S) = c_1(L_\eta) \mod 2 \) (cf. [LMZ, (1.47)]). Let \( S(TY^S, L_\eta) \) be the fundamental spinor bundle for \( (TY^S, L_\eta) \). As in (1.14), locally, we have

\[
S(TY^S, L_\eta) = S_0(TY^S) \otimes L^{1/2}|_{Y^S} \otimes (\det_\mathbb{N})^{-1/2},
\]

\[
\Lambda(\mathbb{N}) = S_0(\mathbb{N}) \otimes (\det_\mathbb{N})^{1/2}.
\]

Therefore, we have

\[
S(TY, L)|_{Y^S} = S(TY^S, L_\eta) \otimes \Lambda^*(\mathbb{N}).
\]

Note that the equivariant geometric triple \( \mathbb{N} = (\mathbb{N}, h^N, \nabla^N) \) induces equivariant geometric triples \( \Lambda^{even}(\mathbb{N}) \), \( \Lambda^{odd}(\mathbb{N}) \) and \( \det_\mathbb{N} \). Denote by

\[
\Lambda(-1)(\mathbb{N}) = \Lambda^{even}(\mathbb{N}) - \Lambda^{odd}(\mathbb{N}).
\]

We set

\[
\Lambda(-1)(\mathbb{N}) = \Lambda^{even}(\mathbb{N}) - \Lambda^{odd}(\mathbb{N}).
\]

Let \( L_\eta \) be the equivariant geometric triple induced from (1.33).

From [10] (6.26)], we have

\[
\chi_g \left( \Lambda(-1)(\mathbb{N}) \right) = \tilde{A}_g(N_\alpha, \nabla^{N_\alpha})^{-1} \cdot \chi_g \left( (\det_\mathbb{N})^{1/2} \right).
\]

From [12,23] and (1.33), on \( Y^S \), we have

\[
\text{Td}_g(\nabla^{TY}, \nabla^L) \chi_g \left( \Lambda(-1)(\mathbb{N}) \right) = \tilde{A}(TY^S, \nabla^{TY^S}) \chi_g \left( L^{1/2}_\alpha \right) = \text{Td}_g \left( \nabla^{TY^S}, \nabla^L \right).
\]
Let \((\mu, h^\mu)\) be an \(S^1\)-equivariant Hermitian vector bundle over \(Y^{S^1}\) with an \(S^1\)-invariant Hermitian connection \(\nabla^\mu\). Let \(\iota : Y^{S^1} \to Y\) be the natural embedding. In the followings, we describe Atiyah-Hirzebruch’s geometric construction of the direct image \(\iota_*\mu\) of \(\mu\) for the embedding in K-theory [2, 18]. It will be clear from its construction that it is compatible with the group action.

For any \(\varepsilon > 0\), set \(N_{\alpha,\varepsilon} := \{Z \in N_\alpha : |Z| < \varepsilon\}\). Then there exists \(\varepsilon_0 > 0\) such that the exponential map \((y, Z) \in N_{\alpha,\varepsilon} \to \exp_y(Z)\) is a diffeomorphism between \(N_{\alpha,2\varepsilon_0}\) and an open \(S^1\)-equivariant tubular neighbourhood of \(Y^{S^1}_\alpha\) in \(Y\) for any \(\alpha\). Without confusion we will also regard \(N_{\alpha,2\varepsilon_0}\) as this neighbourhood of \(Y^{S^1}_\alpha\) in \(Y\) via this identification. We choose \(\varepsilon_0 > 0\) small enough such that for any \(\alpha \neq \beta \in \mathfrak{F}, N_{\alpha,2\varepsilon_0} \cap N_{\beta,2\varepsilon_0} = \emptyset\).

Let \(\pi_\alpha : N_\alpha \to Y^{S^1}_\alpha\) denote the projection of the normal bundle \(N_\alpha\) over \(Y^{S^1}_\alpha\). For \(Z \in N_{\alpha}\), let \(\bar{c}(Z) \in \text{End}(\Lambda^*(N^*_\alpha))\) be the adjoint action of the canonical Clifford action \(c(Z)\) on \(\Lambda^*(N^*_\alpha)\) in (1.32). Let \(\pi_\alpha^*(\Lambda^*(N^*_\alpha))\) be the pull back bundle of \(\Lambda^*(N^*_\alpha)\) over \(N_\alpha\). For any \(Z \in N_{\alpha}\) with \(Z \neq 0\), let \(\bar{c}(Z) : \pi_\alpha^*(\Lambda^{\text{even/odd}}(N^*_\alpha))|_Z \to \pi_\alpha^*(\Lambda^{\text{odd/even}}(N^*_\alpha))|_Z\) denote the corresponding pull back isomorphism at \(Z\).

As \(S^1\) acts trivially on \(Y^{S^1}_\alpha\), we can just apply [33, Chapter I, Corollary 9.9] for each weight part to see that (cf. also [19, Proposition 2.4]) there exists an \(S^1\)-equivariant vector bundle \(F_\alpha\) over \(Y^{S^1}_\alpha\) such that \((\Lambda^{\text{odd}}(N^*_\alpha) \otimes \mu_\alpha) \oplus F_\alpha\) is an \(S^1\)-equivariant trivial complex vector bundle over \(Y^{S^1}_\alpha\). Then

\[
\bar{c}(Z) \oplus \pi_\alpha^*\text{Id}_{F_\alpha} : \pi_\alpha^*(\Lambda^{\text{even}}(N^*_\alpha) \otimes \mu_\alpha \oplus F_\alpha)|_Z \to \pi_\alpha^*(\Lambda^{\text{odd}}(N^*_\alpha) \otimes \mu_\alpha \oplus F_\alpha)|_Z
\]

induces an \(S^1\)-equivariant isomorphism between two equivariant trivial vector bundles over \(N_{\alpha,2\varepsilon_0}\) \(\setminus Y^{S^1}_\alpha\).

By adding the trivial bundles with corresponding \(S^1\)-action, we could assume that for

\[
\text{rk}(\Lambda^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha \oplus F_\alpha) = \text{rk}(\Lambda^{\text{even/odd}}(N^*_\beta) \otimes \mu_\beta \oplus F_\beta) \text{ for any } \alpha \neq \beta \in \mathfrak{F}.
\]

Clearly, \(\{\pi_\alpha^*(\Lambda^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha \oplus F_\alpha)|_{\partial N_{\alpha,2\varepsilon_0}}\}_{\alpha \in \mathfrak{F}}\) extend smoothly to two equivariant trivial complex vector bundles over \(Y \setminus \cup_{\alpha \in \mathfrak{F}} N_{\alpha,2\varepsilon_0}\). Take equivariant geometric triple \(F_\alpha = (F_\alpha, h^{F_\alpha}, \nabla^{F_\alpha})\). By a partition of unity argument, we get an equivariant \(\mathbb{Z}_2\)-graded Hermitian vector bundle \((\xi, h^\xi)\) such that

\[
\xi_\pm|_{N_{\alpha,\varepsilon_0}} = \pi_\alpha^*(\Lambda^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha \oplus F_\alpha)|_{N_{\alpha,\varepsilon_0}},
\]

\[
h^\xi|_{N_{\alpha,\varepsilon_0}} = \pi_\alpha^*(h^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha \oplus h^{F_\alpha})|_{N_{\alpha,\varepsilon_0}},
\]

where \(h^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha\) is the \(S^1\)-invariant Hermitian metric on \(\Lambda^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha\) induced by \(g^{N_\alpha}\) and \(h^{\mu_\alpha}\). Again by a partition of unity argument, we get an \(S^1\)-invariant \(\mathbb{Z}_2\)-graded Hermitian connection \(\nabla^\xi = \nabla^{\xi_+} \oplus \nabla^{\xi_-}\) on \(\xi = \xi_+ \oplus \xi_-\) over \(Y\) such that

\[
\nabla^{\xi_\pm}|_{N_{\alpha,\varepsilon_0}} = \pi_\alpha^*(\nabla^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha \oplus \nabla^{F_\alpha})|_{N_{\alpha,\varepsilon_0}},
\]

where \(\nabla^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha\) is the Hermitian connection on \(\Lambda^{\text{even/odd}}(N^*_\alpha) \otimes \mu_\alpha\) induced by \(\nabla^{N_\alpha}\) and \(\nabla^{\mu_\alpha}\). Then the direct image of \(\mu\) by \(\iota\) is given by

\[
\iota_\ast \mu = \xi_+ - \xi_-.
\]
We denote now
\[ \iota_* \mu = \xi_+ - \xi_- \] with \( \xi_{\pm} = (\xi_{\pm, h^{\xi_{\pm}}, \nabla^{\xi_{\pm}}}) \) over \( Y \).

The following theorem is proved in [37, Corollary 2.10] by applying for \( G = S^1, g \in S^1 \setminus A \), which is an equivariant extension of [18, Theorem 2.2] (cf. also [23, Theorem 4.1] or [25, Theorem 2.1]).

**Theorem 1.5.** There exists \( V \in R(S^1) \), such that for any \( g \in S^1 \setminus A \),
\[ \eta_g(T_Y, L, \xi_+) - \eta_g(T_Y, L, \xi-) = \sum_{\alpha} \eta_g(T^S_1, L_{\alpha, \mu}) + \chi_V(g). \] (1.46)

**Remark 1.6.** Note that in the general setting of [37, Theorem 2.7] for the embedding \( i : Y \to X \), there are two additional terms: the Chern-Simons class and the equivariant Bismut-Zhang current. In our case, since \( (Y^g)^g = Y^g \), this term is again zero.

### 1.5. Comparison of equivariant \( \eta \)-invariants.

In this subsection, we review the comparison formula for equivariant \( \eta \)-invariants in [39] which is an extension of the result of [31] and the analogue of the comparison formulas for the holomorphic torsions [14] and for the de Rham torsions [15].

For \( K \in \text{Lie}(S^1) \), let \( K^Y(x) = \frac{d}{dt}|_{t=0} e^{tK} \cdot x \) be the induced vector field on \( Y \) and \( L_K \) be its Lie derivative, by \( L_K s = \frac{d}{dt}|_{t=0} (e^{-tK} s) \) for \( s \in \mathcal{C}^\infty(Y, E) \) (cf. (1.15)). The associated moments are defined by [8, Definition 7.5],
\[ m^E(K) := \nabla^E_{K^Y} - L_K \in \mathcal{C}^\infty(Y, \text{End}(E)), \]
\[ m^TY(K) := \nabla^{TY}_{K^Y} - L_K = \nabla^{TY} K^Y \in \mathcal{C}^\infty(Y, \text{End}(TY)). \] (1.47)

Here the last equation holds since the Levi-Civita connection \( \nabla^{TY} \) is torsion free.

Let \( R^E_K \) and \( R^{TY}_K \) be the equivariant curvatures of \( E \) and \( TY \) defined in [8, §7.1]:
\[ R^E_K = R^E - 2i\pi m^E(K), \quad R^{TY}_K = R^{TY} - 2i\pi m^{TY}(K). \] (1.48)

Observe that \( m^{TY}(K)|_{Y^g} \) commutes with the circle action for any \( g \in S^1 \). Then it preserves the decomposition (1.20). Let \( m^{TY}_{\sigma}(K) \) and \( m^{N^g}_{\sigma}(K) \) be the restrictions of \( m^{TY}(K)|_{Y^g} \) to \( TY^g \) and \( N^g \), respectively. Similarly, \( m^E_{\sigma}(K) \) preserves the decomposition (1.22). Let \( m^E_{\sigma}(K) \) be the restrictions of \( m^E(K)|_{Y^g} \) to \( E^g \). We define the corresponding equivariant curvatures \( R^{TY}_{\sigma}, R^{N^g}_{\sigma} \) and \( R^E_{\sigma} \) as in (1.48). The following definition is an analogue of Definition 1.3 and (1.24).
**Definition 1.7.** For $K \in \text{Lie}(S^1)$, $|K|$ small enough, set

$$
\tilde{\mathcal{A}}_{g,K}(TY, \nabla^{TY}) := \det^{1/2} \left( \frac{\pi R_K^{TY \gamma}}{\sinh \left( \frac{\pi R_K^{TY \gamma}}{4} \right)} \right) \cdot \prod_{\nu > 0} \left( \frac{i^{2 \dim \mathbb{R}^\nu} \det^{1/2} \left( 1 - g \cdot \exp \left( \frac{i}{2\pi} R_K^{TY \gamma} \right) \right) }{2\pi} \right)^{-1},
$$

(1.49)

$$
\text{ch}_{g,K}(E) := \sum_v \text{Tr} \left[ \exp \left( \frac{i}{2\pi} R_K^{E \gamma} |_{YY} + 2i\pi vt \right) \right],
$$

$$
\text{ch}_{g,K}(L^{1/2}) := \exp \left( \frac{i}{4\pi} R_K^{L \gamma} |_{YY} + i\pi \ell t \right).
$$

For $K \in \text{Lie}(S^1)$, set

$$
d_K = d - 2i\pi i_{KY}.
$$

(1.50)

Then by [32, Theorem 7.7], $\tilde{\mathcal{A}}_{g,K}(TY, \nabla^{TY})$, $\text{ch}_{g,K}(E)$ and $\text{ch}_{g,K}(L^{1/2})$ are $d_K$-closed.

As in (1.25), we denote by

$$
\text{Td}_{g,K}(\nabla^{TY}, \nabla^L) := \tilde{\mathcal{A}}_{g,K}(TY, \nabla^{TY}) \text{ch}_{g,K}(L^{1/2}).
$$

(1.51)

For $K \in \text{Lie}(S^1)$, let $\vartheta_K \in T^*Y$ be the 1-form which is dual to $K^Y$ by the metric $g^{TY}$, i.e.,

$$
\vartheta_K(X) = \langle K^Y, X \rangle \text{ for } X \in T_Y.
$$

(1.52)

For $g \in S^1$, $K \neq 0$, set

$$
\mathcal{M}_{g,K}(TY, L, E) = -\int_0^\infty \left\{ \int_{\mathbb{R}} \frac{\vartheta_K}{2i\pi} \exp \left( \frac{vd_K \vartheta_K}{2i\pi} \right) \text{Td}_{g,K}(\nabla^{TY}, \nabla^L) \text{ch}_{g,K}(E) \right\} dv.
$$

(1.53)

By [32, Proposition 2.2], $\mathcal{M}_{g,K}(TY, L, E)$ is well-defined for $|K|$ small enough. Moreover, for $K_0 \in \text{Lie}(S^1)$, $t \in \mathbb{R}$ and $|t|$ small enough, $\mathcal{M}_{g,tK_0}(TY, L, E)$ is smooth at $t \neq 0$ for $g \in S^1 \setminus A$ and there exist $c_j(K_0) \in \mathbb{C}$ such that as $t \to 0$, we have

$$
\mathcal{M}_{g,tK_0}(TY, L, E) = \sum_{j=1}^{(\dim \mathbb{R}^\gamma + 1)/2} c_j(K_0) t^{-j} + \mathcal{O}(t^0).
$$

(1.54)

**Definition 1.8.** [39] For $g \in S^1$, $K \in \text{Lie}(S^1)$ and $|K|$ small enough, the equivariant infinitesimal $\eta$-invariant is defined by

$$
\bar{\eta}_{g,K}(TY, L, E) = \int_0^{+\infty} \frac{1}{2\sqrt{\pi t}} \text{Tr} \left[ g \left( D^Y \otimes E - \frac{c(K^Y)}{4t} \right) \right] \cdot \exp \left( -t \left( D^Y \otimes E + \frac{c(K^Y)}{4t} \right)^2 - L_{K^Y} \right) \right] dt + \frac{1}{2} \text{Tr} \left|_{\text{Ker}(D^Y \otimes E)} \right| [ge^K] \in \mathbb{C}.
$$

(1.55)

The following two theorems are the special cases of the main results in [39] when the equivariant $\eta$-forms are just equivariant $\eta$-invariants and the compact Lie group is $S^1$.

**Theorem 1.9.** Fix $K_0 \in \text{Lie}(S^1)$. There exists $\beta > 0$ such that for $t \in \mathbb{R}$ and $|t| < \beta$, the equivariant infinitesimal $\eta$-invariant $\bar{\eta}_{g,tK_0}(TY, L, E)$ is well-defined and depends analytically on $t$. Furthermore, as a function of $t$ near 0, $t^{(\dim \mathbb{R}^\gamma + 1)/2} \mathcal{M}_{g,tK_0}(TY, L, E)$ is real analytic.
Theorem 1.10. For any \( g \in S^1 \), \( K \in \text{Lie}(S^1) \), \( K \neq 0 \) and \(|K|\) small enough,

\[
\tilde{\eta}_{g,K}(TY, L, E) = \tilde{\eta}_{ge^K}(TY, L, E) + \mathcal{M}_{g,K}(TY, L, E).
\tag{1.56}
\]

Remark that since \( \tilde{\eta}_{g,tK_0}(TY, L, E) \) is analytic on \( t \), when \( t \to 0 \), the singular terms of \( \tilde{\eta}_{g,tK_0}(TY, L, E) \) is the same as that of \( -\mathcal{M}_{g,tK_0}(TY, L, E) \) in \((1.54)\). Thus from Theorem 1.10, we know \( \tilde{\eta}_{g}(TY, L, E) \) as a function of \( g \in S^1 \), is smooth on \( S^1 \setminus A \), and near \( g \in A \), it has the same singularity as \( -\mathcal{M}_{g,tK_0}(TY, L, E) \).

2. Differential K-theory

In this section, we prove that the differential K-ring is a pre-\( \lambda \)-ring and use this fact to construct the inverse of \( \lambda_{-1}(N^*) \) explicitly in differential K-theory level. In Section 2.1 we define the pre-\( \lambda \)-ring structure and study some examples. In Section 2.2 we construct the pre-\( \lambda \)-ring structure in differential K-theory. In Section 2.3 we obtain the locally nilpotent property of the \( \gamma \)-filtration in differential K-theory. In Section 2.4 we define the \( g \)-equivariant differential K-theory and construct the inverse of \( \lambda_{-1}(N^*) \) explicitly in differential K-theory level.

2.1. Pre-\( \lambda \)-ring.

Definition 2.1. [9, Definition 2.1] For a commutative ring \( R \) with identity, a pre-\( \lambda \)-ring structure is defined by a countable set of maps \( \lambda^n : R \to R \) with \( n \in \mathbb{N} \) such that for all \( x, y \in R \),

a) \( \lambda^0(x) = 1 \);

b) \( \lambda^1(x) = x \);

c) \( \lambda^n(x + y) = \sum_{j=0}^{n} \lambda^j(x)\lambda^{n-j}(y) \).

If \( R \) has a pre-\( \lambda \)-ring structure, we call it a pre-\( \lambda \)-ring.

Remark that in [7, §1], the pre-\( \lambda \)-ring here is called the \( \lambda \)-ring. If \( t \) is an indeterminate, for \( x \in R \), we define

\[
\lambda_t(x) = \sum_{n \geq 0} \lambda^n(x)t^n.
\tag{2.1}
\]

Then the relations a), c) show that \( \lambda_t \) is a homomorphism from the additive group of \( R \) into the multiplicative group \( 1 + R[[t]]^+ \), of formal power series in \( t \) with constant term 1, i.e.,

\[
\lambda_t(x + y) = \lambda_t(x)\lambda_t(y) \quad \text{for any } x, y \in R.
\tag{2.2}
\]

Now we study some pre-\( \lambda \)-rings which we will use later.

For the set of even degree real closed forms \( Z^{\text{even}}(Y, \mathbb{R}) \), we define the Adams operation \( \Psi^k : Z^{\text{even}}(Y, \mathbb{R}) \to Z^{\text{even}}(Y, \mathbb{R}) \) for \( k \in \mathbb{N} \) by

\[
\Psi^k(x) = k^lx \quad \text{for } x \in Z^2(Y, \mathbb{R}).
\tag{2.3}
\]

For \( x \in Z^{\text{even}}(Y, \mathbb{R}) \), we define

\[
\lambda_t(x) = \sum_{n \geq 0} \lambda^n(x)t^n := \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Psi^k(x)t^k}{k} \right).
\tag{2.4}
\]
From the Taylor expansion of the exponential function, we have $\lambda^0(x) = 1$ and $\lambda^1(x) = x$. Since

$$\lambda_t(x + y) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Psi^k(x + y)t^k}{k}\right)$$

$$= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Psi^k(x)t^k}{k} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Psi^k(y)t^k}{k}\right) = \lambda_t(x)\lambda_t(y),$$

we have

$$\lambda^n(x + y) = \sum_{j=0}^{n} \lambda^j(x)\lambda^{n-j}(y).$$

Thus (2.4) gives a pre-$\lambda$-ring structure on $Z^{\text{even}}(Y, \mathbb{R})$.

Consider the vector space

$$\Gamma(Y) := Z^{\text{even}}(Y, \mathbb{R}) \oplus \left(\Omega^{\text{odd}}(Y, \mathbb{R}) / \text{Im} d\right).$$

We define a pairing on $\Gamma(Y)$ by the formula

$$(\omega_1, \phi_1) \ast (\omega_2, \phi_2) := (\omega_1 \wedge \omega_2, \omega_1 \wedge \phi_2 + \phi_1 \wedge \omega_2 - d\phi_1 \wedge \phi_2).$$

Easy to verify that this pairing is commutative and associative. Since $\ast$ is clearly bilinear and $(1,0)$ is a unit, the pairing $\ast$ defines a graded associative, commutative and unitary $\mathbb{R}$-algebra structure on $\Gamma(Y)$ (see [29, Theorem 7.3.2]). We define the Adams operation $\Psi^k : \Gamma(Y) \to \Gamma(Y)$ for $k \in \mathbb{N}$ by

$$\Psi^k(\alpha, \beta) = (k^l\alpha, k^l\beta) \quad \text{for} \quad (\alpha, \beta) \in Z^{2l}(Y, \mathbb{R}) \oplus \left(\Omega^{2l-1}(Y, \mathbb{R}) / \text{Im} d\right).$$

By using the pairing $\ast$ to replace the multiplicity in (2.4), similarly as $Z^{\text{even}}(Y, \mathbb{R})$, we could obtain a pre-$\lambda$-ring structure on $\Gamma(Y)$.

Set

$$p : \Gamma(Y) \to Z^{\text{even}}(Y, \mathbb{R}), \quad (\omega, \varphi) \to \omega$$
$$j : Z^{\text{even}}(Y, \mathbb{R}) \to \Gamma(Y), \quad \omega \to (\omega, 0).$$

Then $p$ is the projection from $\Gamma(Y)$ to its component $Z^{\text{even}}(Y, \mathbb{R})$ and $j$ is the natural injection. By (2.8), $p, j$ are homomorphism of pre-$\lambda$ rings, in particular,

$$\lambda^k(\omega, 0) = (\lambda^k\omega, 0).$$

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. A polynomial $\varphi : \mathfrak{g} \to \mathbb{C}$ is called a $G$-invariant polynomial if

$$\varphi(\text{Ad}(g^{-1})A) = \varphi(A), \quad \text{for any } g \in G, \ A \in \mathfrak{g}. $$

The set of all $G$-invariant polynomials is denoted by $\mathbb{C}[\mathfrak{g}]^G$. Let $U(r)$ be the unitary group with Lie algebra $\mathfrak{u}(r)$. For $A \in \mathfrak{u}(r)$, the characteristic polynomial of $-A$ is

$$\det(\lambda I + A) = \lambda^r + c_1(A)\lambda^{r-1} + \cdots + c_r(A).$$

So $c_j \in \mathbb{C}[\mathfrak{u}(r)]^{U(r)}$ for $1 \leq j \leq r$. It is well-known that $\mathbb{C}[\mathfrak{u}(r)]^{U(r)}$ is generated by $c_1, \ldots, c_r$:

$$\mathbb{C}[\mathfrak{u}(r)]^{U(r)} = \mathbb{C}[c_1, \ldots, c_r].$$
Let $T^r = \{(e^{it_1}, \cdots, e^{it_r}) : t_1, \cdots, t_r \in \mathbb{R}\}$ be a maximal torus of $U(r)$ with Lie algebra $t^r$. Then
\begin{equation}
\mathbb{C}[t^r]^{T^r} = \mathbb{C}[u_1, \cdots, u_r],
\end{equation}
where $u_j(x) = x_j$, for any $x = \sum_{j=1}^{r} x_j \frac{\partial}{\partial x_j}|_{t=0} \in T_e(T^r) = t^r$. Let
\begin{equation}
\theta : T^r \rightarrow U(r), \quad (e^{it_1}, \cdots, e^{it_r}) \mapsto \text{diag}(e^{it_1}, \cdots, e^{it_r})
\end{equation}
be the natural injection, here diag($\cdots$) is the diagonal matrix. It is well-known that
\begin{equation}
\theta^* : \mathbb{C}[u(r)]^{U(r)} \rightarrow \mathbb{C}[t^r]^{T^r}
\end{equation}
is an injective homomorphism and
\begin{equation}
\theta^*(c_j) = \sigma_j(u_1, \cdots, u_r), \quad \theta^*(\mathbb{C}[u(r)]^{U(r)}) = \mathbb{C}[\sigma_1, \cdots, \sigma_r],
\end{equation}
where $\sigma_j$ is the $j$-th elementary symmetric polynomial. We define the Adams operations for any $k \in \mathbb{N}$
\begin{equation}
\Psi^k : \mathbb{C}[u(r)]^{U(r)} \rightarrow \mathbb{C}[u(r)]^{U(r)}, \quad \Psi^k : \mathbb{C}[t^r]^{T^r} \rightarrow \mathbb{C}[t^r]^{T^r}
\end{equation}
by
\begin{equation}
\Psi^k(c_j) = k^j c_j, \quad \Psi^k(u_j) = k u_j.
\end{equation}
By constructing $\lambda^n$ as in (2.4), $\mathbb{C}[u(r)]^{U(r)}$ and $\mathbb{C}[t^r]^{T^r}$ are equipped now pre-$\lambda$-ring structures.

Let $E$ be a complex vector bundle over $Y$. Let $h^E$ be a Hermitian metric on $E$. Let $\nabla^E$ be a Hermitian connection on $(E, h^E)$. We also denote by $\underline{E} = (E, h^E, \nabla^E)$ the geometric triple for this non-equivariant setting. For $\varphi \in \mathbb{C}[u(r)]^{U(r)}$, we define the characteristic form $\varphi(\underline{E})$ by
\begin{equation}
\varphi(\underline{E}) = \psi_Y \varphi(-R^E) \in \Omega^{\text{even}}(Y, \mathbb{C}),
\end{equation}
where $\psi_Y : \Omega^{\text{even}}(Y, \mathbb{C}) \rightarrow \Omega^{\text{even}}(Y, \mathbb{C})$ is defined by
\begin{equation}
\psi_Y \omega = (2i\pi)^{-j} \omega \quad \text{for } \omega \in \Omega^{2j}(Y, \mathbb{C}).
\end{equation}
Thus the triple $\underline{E}$ induces a homomorphism of rings
\begin{equation}
f_{\underline{E}} : \mathbb{C}[u(r)]^{U(r)} \rightarrow \mathbb{C}[\varphi(\underline{E})] = Z^{\text{even}}(Y, \mathbb{C}), \quad \varphi \mapsto \varphi(\underline{E}).
\end{equation}

Let $\pi^* \underline{E} = (\pi^* E, h^{\pi^* E}, \nabla^{\pi^* E})$ be the triple defined in Section 1.2 without the group action. As in (1.29), the Chern-Simons class $\overline{\varphi}(\underline{E}_0, \underline{E}_1) \in \Omega^{\text{odd}}(Y, \mathbb{C})/\text{Im} \, d$ is defined by (cf. [42, Definition B.5.3])
\begin{equation}
\overline{\varphi}(\underline{E}_0, \underline{E}_1) := \int_0^1 \{ \varphi(\pi^* \underline{E}) \}^d ds \in \Omega^{\text{odd}}(Y, \mathbb{C})/\text{Im} \, d.
\end{equation}
Then by [42, Theorem B.5.4],
\begin{equation}
d \overline{\varphi}(\underline{E}_0, \underline{E}_1) = \varphi(\underline{E}_1) - \varphi(\underline{E}_0),
\end{equation}
and the Chern-Simons class depends only on $\nabla^{E_0}$ and $\nabla^{E_1}$, i.e., it does not depend on the path of connections neither of metrics.
Lemma 2.2. Let $E_j = (E, h_j^F, \nabla_j^F)$ for $j = 0, 1, 2$. Let $\varphi, \varphi' \in \mathbb{C}[u(r)]^{U(r)}$. Then we have

(a) $\tilde{\varphi}(E_0, E_2) = \tilde{\varphi}(E_0, E_1) + \varphi(E_0, E_1)$;
(b) $\varphi + \varphi'(E_0, E_1) = \tilde{\varphi}(E_0, E_1) + \varphi'(E_0, E_1)$;
(c) $\varphi \varphi'(E_0, E_1) = \tilde{\varphi}(E_0, E_1) \varphi'(E_1) + \varphi(E_0) \tilde{\varphi}'(E_0, E_1)$;
(d) $(\varphi(E_1), \tilde{\varphi}(E_0, E_1)) \ast (\varphi'(E_1), \tilde{\varphi}'(E_0, E_1)) = (\varphi \varphi'(E_1), \tilde{\varphi} \varphi'(E_0, E_1))$.

Proof. By (2.24), (a) and (b) are obvious and (c) could be calculated directly. We only need to prove (d).

From (2.8), (2.25) and (a)-(c), we have

\begin{equation}
(2.26) \quad (\varphi(E_1), \tilde{\varphi}(E_0, E_1)) \ast (\varphi'(E_1), \tilde{\varphi}'(E_0, E_1))
\end{equation}

\begin{equation}
= \left( \varphi(E_1) \varphi'(E_1), \varphi(E_1) \tilde{\varphi}'(E_0, E_1) + \tilde{\varphi}(E_0, E_1) \varphi'(E_1) - (\varphi(E_1) - \varphi(E_0)) \tilde{\varphi}'(E_0, E_1) \right)
\end{equation}

\begin{equation}
= (\varphi \varphi'(E_1), \tilde{\varphi} \varphi'(E_0, E_1)).
\end{equation}

The proof of Lemma 2.2 is completed.

Similarly as in (2.23), the triple $\pi^*E$ induces a map

\begin{equation}
\tilde{f}_E : \mathbb{C}[u(r)]^{U(r)} \to \Gamma(Y), \quad \varphi \mapsto (\varphi(E_1), \tilde{\varphi}(E_0, E_1)).
\end{equation}

It is a ring homomorphism by Lemma 2.2.

Lemma 2.3. The ring homomorphisms $p$, $\theta^*$, $f_E$ and $\tilde{f}_E$ in the following diagram are all homomorphisms of pre-$\lambda$-rings.

\[
\begin{array}{ccc}
\Gamma(Y) & \xleftarrow{\tilde{f}_E} & \mathbb{C}[u(r)]^{U(r)} \xrightarrow{\theta^*} \mathbb{C}[t^r]^{T^r} \\
p & & \downarrow{f_E} \\
Z^{even}(Y, \mathbb{R}) & \xleftarrow{p} & \mathbb{C}[u(r)]^{U(r)}
\end{array}
\]

Moreover, we have

\begin{equation}
(2.28) \quad f_E = p \circ \tilde{f}_E.
\end{equation}

Proof. From (2.3), (2.9), (2.10), (2.11), (2.17), (2.20), (2.23) and (2.27), we see that all homomorphisms here commute with the corresponding Adams operations. So by (2.4), they are all homomorphisms of pre-$\lambda$-rings.

The relationship (2.28) follows directly from (2.23) and (2.27).

The proof of our lemma is completed.

By (2.13), $(\theta^*)^{-1}(H) \in \mathbb{C}[u(r)]^{U(r)}$ is well-defined for any homogeneous symmetric polynomial $H$ on $u_1, \cdots, u_r$. We define the Chern character to be the formal power series

\begin{equation}
(2.29) \quad \text{ch} = \sum_{k=0}^{\infty} (\theta^*)^{-1} \left( \frac{1}{k!} \sum_{j=1}^{r} u_j^k \right).
\end{equation}
It is easy to see that

\[
(2.30) \quad f_E(ch) := \sum_{k=0}^{\infty} f_E \circ (\theta^*)^{-1} \left( \frac{1}{k!} \sum_{j=1}^{r} u_j^k \right).
\]

It is the same as the canonical Chern character \( ch(E) \) in Definition 1.3 for \( g = 1 \).

Since the manifold is finite dimensional, when we consider the characteristic forms, in the image of \( f_E \), the right hand side of (2.30) is a finite sum. In this paper, we only care about the characteristic forms. We could calculate \( \theta^* \) and \( \lambda^i \) on \( ch \) formally and obtain the rigorous equality of the characteristic forms after taking the map \( f_E \).

From this point of view, we write

\[
(2.31) \quad ch = (\theta^*)^{-1} \left( \sum_{j=1}^{r} \exp(u_j) \right).
\]

From Lemma 2.3, (2.4), (2.20) and (2.31), we have

\[
(2.32) \quad \theta^* \circ \lambda_t(ch) = \lambda_t \left( \sum_{j=1}^{r} \exp(u_j) \right) = \exp \left( \sum_{k=1}^{\infty} \left( -1 \right)^{k-1} t^k \Psi_k \left( \sum_{j=1}^{r} \exp(u_j) \right) \right)
\]

\[
= \exp \left( \sum_{j=1}^{r} \sum_{k=1}^{\infty} \left( -1 \right)^{k-1} t^k \exp(ku_j) \right) = \exp \left( \sum_{j=1}^{r} \log(1 + t \exp(u_j)) \right)
\]

\[
= \prod_{j=1}^{r} \left( 1 + t \exp(u_j) \right).
\]

From (2.32), we get the following equality, which is firstly proved in [29, Lemma 7.3.3],

\[
(2.33) \quad \lambda^k(ch)(E) = ch(\Lambda^k(E)) \in Z^{even}(Y, \mathbb{R}).
\]

**Lemma 2.4.** The following identity holds,

\[
(2.34) \quad \lambda^k \left( ch(E_\lambda), \tilde{ch}(E_0, E_\lambda) \right) = \left( ch(\Lambda^k(E_\lambda)), \tilde{ch}(\Lambda^k(E_0), \Lambda^k(E_\lambda)) \right) \in \Gamma(Y).
\]

**Proof.** From (2.24) and (2.33), modulo exact forms, we have

\[
(2.35) \quad \tilde{\lambda^k}(ch)(E_0, E_\lambda) = \int_{0}^{1} \left\{ \lambda^k(ch(\pi^*E)) \right\} ds ds
\]

\[
= \int_{0}^{1} \left\{ ch(\Lambda^k(\pi^*E)) \right\} ds ds = \tilde{ch}(\Lambda^k(E_0), \Lambda^k(E_\lambda)).
\]

So from Lemma 2.3, (2.27), (2.33) and (2.35), we get

\[
(2.36) \quad \lambda^k \left( ch(E_\lambda), \tilde{ch}(E_0, E_\lambda) \right) = \lambda^k(\tilde{f}_E(ch)) = \tilde{f}_E(\lambda^k(ch))
\]

\[
= (\lambda^k(ch(E_\lambda), \tilde{\lambda^k}(ch)(E_0, E_\lambda)) = \left( ch(\Lambda^k(E_\lambda)), \tilde{ch}(\Lambda^k(E_0), \Lambda^k(E_\lambda)) \right).
\]

The proof of Lemma 2.4 is completed. \( \square \)
2.2. Pre-λ-ring structure in differential K-theory. In this subsection, we introduce a pre-λ-ring structure for differential K-ring. It can be understood as the differential K-theory version of the pre-λ-ring structure for arithmetic K-theory in [29, Theorem 7.34].

**Definition 2.5.** [26, Definition 2.16] A cycle for differential K-theory is the pair \((E, \phi)\) where \(\phi \in \Omega^{\text{odd}}(Y, \mathbb{R})/\text{Im } d\). Two cycles \((E_1, \phi_1)\) and \((E_2, \phi_2)\) are equivalent if there exists a geometric triple \(E_3\) and a vector bundle isomorphism

\[
\Phi : E_1 \oplus E_3 \rightarrow E_2 \oplus E_3
\]

such that

\[
\tilde{\text{ch}}(E_1 \oplus E_3, \Phi^*(E_2 \oplus E_3)) = \phi_2 - \phi_1.
\]

We define the sum in the obvious way by

\[
(E, \phi) + (F, \psi) = (E \oplus F, \phi + \psi).
\]

We define the differential K-group \(\hat{K}^0(Y)\) as the Grothendieck group of equivalent classes of cycles. It is an abelian group. We define the ring structure of \(\hat{K}^0(Y)\) by

\[
[E, \phi] \cup [F, \psi] = [E \otimes F, [(\text{ch}(E), \phi) * (\text{ch}(F), \psi)]_\text{odd}],
\]

where \([:)_\text{odd}\) is the component of \(\Gamma(Y)\) in \(\Omega^{\text{odd}}(Y, \mathbb{R})/\text{Im } d\).

It is easy to check that this cup product is well-defined, commutative and associative. The element

\[
1 := [\mathbb{C}, 0]
\]

is a unit. Here \(\mathbb{C}\) denotes by the trivial line bundle over \(Y\) with the trivial metric and connection.

From (1.30), (2.37) and (2.38), if \([E, 0] = [F, 0] \in \hat{K}^0(Y)\), we have

\[
\text{ch}(E) = \text{ch}(F) \in \Omega^*(Y, \mathbb{R}).
\]

**Theorem 2.6.** There exists a pre-λ-ring structure on \(\hat{K}^0(Y)\).

**Proof.** Observe that \((\text{ch}(E), \phi) \in \Gamma(Y)\). Since \(\Gamma(Y)\) is a pre-λ-ring, we define

\[
\lambda^k(E, \phi) := (\Lambda^k(E), [\lambda^k(\text{ch}(E), \phi)]_\text{odd}).
\]

It is clear that

\[
\lambda^0(E, \phi) = 1, \quad \lambda^1(E, \phi) = (E, \phi).
\]
By (2.33), (2.40) and (2.43), for cycles \((E, \phi)\) and \((E, \psi)\), we have

\[
\lambda^k((E, \phi) + (E, \psi)) = \lambda^k(E \oplus E, \phi + \psi)
\]

\[
= (\Lambda^k(E \oplus E), [\lambda^k(ch(E \oplus E), \phi + \psi)]_{odd})
\]

\[
= \left( \sum_{j=0}^{k} \Lambda^j(E) \oplus \Lambda^{k-j}(E), \left[ \sum_{i=0}^{k} \lambda^i(ch(E), \phi) \ast \lambda^{k-i}(ch(E), \psi) \right]_{odd} \right)
\]

\[
= \sum_{j=0}^{k} \left( \Lambda^j(E) \oplus \Lambda^{k-j}(E), \left[ \left( \operatorname{ch}(\Lambda^j(E)), [\lambda^j(ch(E), \phi)]_{odd} \right) \ast \left( \Lambda^{k-j}(E), [\lambda^{k-j}(ch(E), \psi)]_{odd} \right) \right. \right)
\]

\[
= \sum_{j=0}^{k} \left( \Lambda^j(E), [\lambda^j(ch(E), \phi)]_{odd} \right) \cup \left( \Lambda^{k-j}(E), [\lambda^{k-j}(ch(E), \psi)]_{odd} \right)
\]

\[
= \sum_{j=0}^{k} \lambda^j(E, \phi) \cup \lambda^{k-j}(E, \psi),
\]

here the second equality is from (2.43), the third equality is from Definition 2.1 c) and the pre-\(\lambda\)-ring structure of \(\Gamma(Y)\), the fourth equality is from the definition

\[
\lambda^j(ch(E), \phi) = \left( \lambda^j(ch(E), \phi), [\lambda^j(ch(E), \phi)]_{odd} \right)
\]

and (2.33), the last two equalities are from (2.40) and (2.43). So we only need to prove that \(\lambda^k\) is well-defined on \(\tilde{K}^0(Y)\).

If \((E_1, \phi_1) \sim (E_2, \phi_2)\), there exist \(E_3\) and isomorphism \(\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3\) such that

\[
\tilde{\chi}(E_1 \oplus E_3, \Phi^* (E_2 \oplus E_3)) = \phi_2 - \phi_1.
\]

From (2.34) and (2.43), we have

\[
\lambda^k(E_2 \oplus E_3, \tilde{\chi}(E_1 \oplus E_3, \Phi^* (E_2 \oplus E_3)))
\]

\[
= (\Lambda^k(E_2 \oplus E_3), [\lambda^k(ch(E_2 \oplus E_3), \tilde{\chi}(E_1 \oplus E_3, \Phi^* (E_2 \oplus E_3)))]_{odd})
\]

\[
= (\Lambda^k(E_2 \oplus E_3), \tilde{\chi}(\Lambda^k(E_1 \oplus E_3, \Phi^* \Lambda^k(E_2 \oplus E_3))).
\]

By Def. 2.5 (2.11) and (2.43), we get

\[
\lambda^k(E_1 \oplus E_3, 0) = (\Lambda^k(E_1 \oplus E_3), [\lambda^k(ch(E_1 \oplus E_3), 0)]_{odd})
\]

\[
= (\Lambda^k(E_1 \oplus E_3), 0) = (\Lambda^k(E_2 \oplus E_3), \tilde{\chi}(\Lambda^k(E_1 \oplus E_3), \Phi^* \Lambda^k(E_2 \oplus E_3))).
\]

From (2.48) and (2.49), we get

\[
\lambda^k(E_1 \oplus E_3, 0) \sim \lambda^k(E_2 \oplus E_3, \tilde{\chi}(E_1 \oplus E_3, \Phi^* (E_2 \oplus E_3))).
\]

Since

\[
\lambda_t(E_j \oplus E_3, \phi_j) = \lambda_t((E_j \oplus E_3, 0) + (0, \phi_j)) = \lambda_t(E_j \oplus E_3, 0) \cup \lambda_t(0, \phi_j)
\]
for $j = 1, 2$, by (2.47) and (2.50), we have

\[(2.52)\quad \lambda^k(E_1 \oplus E_2, \phi_1) = \sum_{n=0}^{k} \lambda^n(E_1 \oplus E_2, 0) \cup \lambda^{k-n}(0, \phi_1) \sim \sum_{n=0}^{k} \lambda^n(E_2 + E_2, \phi_2 - \phi_1) \cup \lambda^{k-n}(0, \phi_1) = \lambda^k(E_2 \oplus E_2, \phi_2).\]

By Def. 2.1 c), for any $k \geq 1$, $j = 1, 2$, we have

\[(2.53)\quad \lambda^k(E_1 \oplus E_2, \phi_j) = \sum_{n=0}^{k} \lambda^n(E_1, \phi_j) \cup \lambda^{k-n}(E_2, 0).\]

Note that $\lambda^0(E_1, \phi_1) = (E_1, \phi_1) \sim (E_2, \phi_2) = \lambda^1(E_2, \phi_2)$. We assume that $\lambda^n(E_1, \phi_1) \sim \lambda^n(E_2, \phi_2)$ holds for all $1 \leq n \leq k - 1$. Then

\[(2.54)\quad \sum_{n=0}^{k-1} \lambda^n(E_1, \phi_1) \cup \lambda^{k-n}(E_2, 0) \sim \sum_{n=0}^{k-1} \lambda^n(E_2, \phi_2) \cup \lambda^{k-n}(E_3, 0).\]

From (2.52), (2.53) and (2.54), since $\lambda^0(x) = 1$, $\lambda^k(E_1, \phi_1) \sim \lambda^k(E_2, \phi_2)$. So by induction, for any $k \geq 1$, we have $\lambda^k(E_1, \phi_1) \sim \lambda^k(E_2, \phi_2)$. The proof of Theorem 2.7 is completed. □

**Remark 2.7.** In [9] Chapter V, the $\lambda$-ring is well studied, which needs two additional conditions in Definition 2.1. It is well-known that the topological $K$-group $K^0(Y)$ is a $\lambda$-ring (cf. [7] Theorem 1.5). In [18], the author proves that the arithmetic $K$-group is also a $\lambda$-ring. It is natural to ask whether the differential $K$-group $\hat{K}^0(Y)$ is also a $\lambda$-ring. However, for our application here, the pre-$\lambda$-ring structure for differential $K$-theory is enough.

### 2.3. $\gamma$-filtration

**Definition 2.8.** Let $R$ be any pre-$\lambda$-ring with an augmentation homomorphism $rk : R \to \mathbb{Z}$. The $\gamma$-operations are defined by

\[(2.55)\quad \gamma_t(x) = \sum_{j \geq 0} \gamma^j(x)t^j := \lambda_{\frac{t}{1-t}}(x).\]

By Definition 2.1, we have

\[(2.56)\quad \gamma^0 = 1, \quad \gamma^1(x) = x, \quad \gamma_t(x + y) = \gamma_t(x)\gamma_t(y), \quad \text{for any } x, y \in R.\]

**Definition 2.9.** Set $F^n R := R$ for $n \leq 0$ and $F^1 R$ the kernel of $rk : R \to \mathbb{Z}$. Let $F^n R$ be the additive subgroup generated by $\gamma^{r_1}(x_1) \cdots \gamma^{r_k}(x_k)$, where $x_1, \cdots, x_k \in F^1 R$ and $\sum_{j=1}^{k} r_j \geq n$. The filtration

\[(2.57)\quad F^1 R \supseteq F^2 R \supseteq F^3 R \supseteq \cdots\]

is called the $\gamma$-filtration of $R$. The $\gamma$-filtration is said to be **locally nilpotent at** $x \in F^1 R$, if there exists $N(x) \in \mathbb{N}$, such that $\gamma^{r_1}(x) \cdots \gamma^{r_k}(x) = 0$ for any $\sum_{j=1}^{k} r_j > N(x)$.

It is well-known that the classical $\gamma$-filtration of $K^0(Y)$ is locally nilpotent for any $x \in F^1 K^0(Y)$ [1] Proposition 3.1.5]. Since $\hat{K}^0(Y)$ is a pre-$\lambda$-ring, the augmentation homomorphism $rk(E, \phi) := rk E$ defines a $\gamma$-filtration of $\hat{K}^0(Y)$. Recall that $rk E$ is the rank of the complex vector bundle $E$. 

For geometric triple $E$, by (2.41), (2.43), we have
\begin{equation}
\lambda_i(E) = \sum_{j \geq 0} \Lambda^j(E) t^j.
\end{equation}

So $\gamma^i(E)$, defined as in (2.55), is a finite dimensional virtual Hermitian vector bundle with induced metric and connection. We also denote it by $\overline{\gamma^i(E)}$. Let $rk E$ be the $rk E$-dimensional trivial complex vector bundle with trivial metric and connection. The $n$-ized by cycles $(E - rk E, \phi)$ for $\phi \in \Omega^{odd}(Y, \mathbb{R}) / Im d$. By (2.58),
\begin{equation}
\lambda_i(\mathbb{C}) = 1 + t, \quad \gamma_i(\mathbb{C}) = 1 + \frac{t}{1 - t} \cdot 1 = \frac{1}{1 - t}.
\end{equation}

From (2.55), (2.56) and (2.59), letting $r = rk E$, we have
\begin{equation}
\gamma_i(E - rk E) = \lambda_i(E) \gamma_i(\mathbb{C})^{-r} = \lambda_i(E) t^i (1 - t)^{-r} = \sum_{i=0}^{r} \Lambda^i(E) t^i (1 - t)^{-r - i}
\end{equation}
\begin{equation}
= \sum_{i=0}^{r} \sum_{j=0}^{r-i} (-1)^j \left( \begin{array}{c} r - i \\ j \end{array} \right) \Lambda^i(E) t^{i+j} = \sum_{k=0}^{r} \left( \sum_{i=0}^{k} (-1)^{k-i} \left( \begin{array}{c} r - i \\ k - i \end{array} \right) \Lambda^i(E) \right) t^k.
\end{equation}

So
\begin{equation}
\gamma^k(E - rk E) = \begin{cases} 
\sum_{i=0}^{k} (-1)^{k-i} \left( \begin{array}{c} r - i \\ k - i \end{array} \right) \Lambda^i(E), & \text{if } 0 \leq k \leq r; \\
0, & \text{if } k > r.
\end{cases}
\end{equation}

Since $\lambda_i(x) = \gamma_i(1+t)(x)$, we have
\begin{equation}
\lambda_i(E) = \lambda_i(E - rk E) \cdot \lambda_i(rk E) = \lambda_i(E - rk E) \cdot (1 + t)^r
\end{equation}
\begin{equation}
= (1 + t)^r \left( 1 + \sum_{i=1}^{r} \gamma^i(E - rk E) t^i (1 + t)^{-i} \right).
\end{equation}

Formally, we have
\begin{equation}
\lambda_i(E)^{-1} = (1 + t)^{-r} \left( 1 + \sum_{i=1}^{r} \gamma^i(E - rk E) t^i (1 + t)^{-i} \right)^{-1}
\end{equation}
\begin{equation}
= (1 + t)^{-r} \left( 1 + \sum_{j=1}^{\infty} (-1)^j \left( \sum_{i=1}^{r} \gamma^i(E - rk E) t^i (1 + t)^{-i} \right)^j \right)
\end{equation}
\begin{equation}
= (1 + t)^{-r} \left( 1 + \sum_{k=1}^{\infty} t^k (1 + t)^{-k} \left( \sum_{(n_1, \ldots, n_r) \in \mathbb{N}_r,} (-1)^{\sum_{i=1}^{r} n_i} \frac{(\sum_{i=1}^{r} n_i)!}{\prod_{i=1}^{r} n_i!} \prod_{i=1}^{r} \left( \gamma^i(E - rk E) \right)^{n_i} \right) \right).
\end{equation}

To simplify the notations, by using (2.61), in (2.63), we denote by
\begin{equation}
\lambda_i(E)^{-1} = (1 + t)^{-r} \left( 1 + \sum_{k=1}^{\infty} t^k (1 + t)^{-k} \left( P_{k,+(E)} - P_{k,-(E)} \right) \right).
\end{equation}
Remark that \( P_{k,\pm}(E) \) here are finite dimensional Hermitian vector bundles with induced metrics and connections.

**Theorem 2.10.** The \( \gamma \)-filtration of \( \hat{K}^0(Y) \) is locally nilpotent at \([E - \text{rk} E, 0]\). Explicitly, for any \((n_1, \cdots, n_r) \in \mathbb{N}^r\) such that

\[
\sum_{i=1}^{r} i \cdot n_i > r^2((m+1)(2m+1)r^2 - 1),
\]

with \( r = \text{rk} E, m = \dim Y \), we have

\[
\prod_{i=1}^{r} (\gamma^i([E - \text{rk} E, 0]))^{n_i} = \left[ \prod_{i=1}^{r} (\gamma^i(E - \text{rk} E))^{n_i}, 0 \right] = 0 \in \hat{K}^0(Y).
\]

**Proof.** From the classical property of the \( \gamma \)-operation in K-theory [1, Proposition 3.1.5], we know that

\[
\prod_{i=1}^{r} (\gamma^i(E - \text{rk} E))^{n_i} = 0 \in K^0(Y)
\]

for any \((n_1, \cdots, n_r) \in \mathbb{N}^r\) such that \( \sum_{i=1}^{r} i \cdot n_i > r^2 \).

Let \( \text{Gr}(r, \mathbb{C}^p) \) be the Grassmannian which is a space parameterizing all complex linear subspaces of \( \mathbb{C}^p \) of given dimension \( r \). Let \( H \) be the tautological bundle over \( \text{Gr}(r, \mathbb{C}^p) \) with the canonical metric \( h^H \). By a theorem of Narasimhan and Ramanan [44, Theorem 1], for

\[
p = (m+1)(2m+1)r^3,
\]

there exists a map \( f : Y \to \text{Gr}(r, \mathbb{C}^p) \) and a universal Hermitian connection \( \nabla^H \) on \((H, h^H)\) such that \( f^*H = E \). Let \( f^* : \hat{K}^0(\text{Gr}(r, \mathbb{C}^p)) \to \hat{K}^0(Y) \) be the induced map. Let \( \text{rk} H \) be the \( r \)-dimensional trivial vector bundle over \( \text{Gr}(r, \mathbb{C}^p) \) with trivial metric and connection. From (2.43) and (2.55), we have

\[
\prod_{i=1}^{r} (\gamma^i([H - \text{rk} H, 0]))^{n_i} = \left[ \prod_{i=1}^{r} (\gamma^i(H - \text{rk} H))^{n_i}, 0 \right] = f^* \left\{ \prod_{i=1}^{r} (\gamma^i(H - \text{rk} H))^{n_i}, 0 \right\} \in \hat{K}^0(Y).
\]

By (2.67), if \( \sum_{i=1}^{r} i \cdot n_i > r^2 \), we have

\[
\prod_{i=1}^{r} (\gamma^i(H - \text{rk} H))^{n_i} = 0 \in K^0(\text{Gr}(r, \mathbb{C}^p)).
\]

From Definition 2.5 and (2.70), there exists \( \alpha \in \Omega^{\text{odd}}(\text{Gr}(r, \mathbb{C}^p), \mathbb{R}) \), such that if \( \sum_{i=1}^{r} i \cdot n_i > r^2 \),

\[
\left[ \prod_{i=1}^{r} (\gamma^i(H - \text{rk} H))^{n_i}, 0 \right] = [0, \alpha]
\]
and

\[
(2.72) \quad -d\alpha = \text{ch} \left( \prod_{i=1}^{r} (\gamma^i (H - \text{rk} H))^{n_i} \right) = \prod_{i=1}^{r} (\text{ch} (\gamma^i (H - \text{rk} H)))^{n_i} \\
\in \Omega_{\text{even}}(\text{Gr}(r, \mathbb{C}^p), \mathbb{R}).
\]

From (2.23), (2.33) and (2.60), we have

\[
(2.73) \quad \text{ch}(\gamma^i (H - \text{rk} H)) = \text{ch}(\lambda_{r/(1-t)}(H))(1-t)^r = \sum_{i=0}^{r} \text{ch}(\lambda^i(H))t^i(1-t)^{r-i} \\
= \sum_{i=0}^{r} \lambda^i(\text{ch}(H))t^i(1-t)^{r-i} = \sum_{i=0}^{r} f_H(\lambda^i(\text{ch}))t^i(1-t)^{r-i},
\]

where \(f_H\) is the map (2.23) with respect to \(H\). So from (2.32), we have

\[
(2.74) \quad \sum_{i=0}^{r} \theta^* \lambda^i(\text{ch})t^i(1-t)^{r-i} = \sum_{i=0}^{r} \sigma_i(e^{u_1}, \ldots, e^{u_r})t^i(1-t)^{r-i} \\
= \prod_{j=1}^{r} \left( (1-t) + te^{u_j} \right) = \prod_{j=1}^{r} \left( 1 + t(e^{u_j} - 1) \right) = \sum_{i=0}^{r} \sigma_i(e^{u_1} - 1, \ldots, e^{u_r} - 1)t^i.
\]

Since \(\theta^*\) is injective and \(\sigma_i(e^{ku_1} - 1, \ldots, e^{ku_r} - 1)\) is symmetric with respect to \(u_i\), from (2.73) and (2.74), we have

\[
(2.75) \quad \text{ch}(\gamma^i (H - \text{rk} H)) = f_H \circ (\theta^*)^{-1}(\sigma_i(e^{u_1} - 1, \ldots, e^{u_r} - 1)) \in \Omega_{\text{even}}(Y, \mathbb{R}).
\]

From Lemma 2.3, we see that the Adams operation \(\Psi^k\) commutes with \(f_H \circ (\theta^*)^{-1}\). Since

\[
(2.76) \quad \Psi^k(\sigma_i(e^{u_1} - 1, \ldots, e^{u_r} - 1)) = \sigma_i(e^{ku_1} - 1, \ldots, e^{ku_r} - 1)
\]

is a power series with respect to \(k\) such that the coefficients of \(1, k, \ldots, k^{i-1}\) vanish, so is \(\Psi^k(\text{ch}(\gamma^i (H - \text{rk} H)))\). Since \(\Psi^k\beta = k^i\beta\) for \(\beta \in Z^2(\text{Gr}(r, \mathbb{C}^p), \mathbb{R})\), we have

\[
(2.77) \quad \text{ch}(\gamma^i (H - \text{rk} H)) \in \Omega_{\geq 2i}(\text{Gr}(r, \mathbb{C}^p), \mathbb{R}).
\]

Since \(\dim \mathbb{R} \text{Gr}(r, \mathbb{C}^p) = 2r(p-r) = 2r^2((m+1)(2m+1)r^2 - 1)\), for any \((n_1, \ldots, n_r) \in \mathbb{N}^r\) such that \(\sum_j i \cdot n_i > r^2((m+1)(2m+1)r^2 - 1)\), we have

\[
(2.78) \quad \text{ch} \left( \prod_{i=1}^{r} (\gamma^i (H - \text{rk} H))^{n_i} \right) = 0 \in \Omega_{\text{even}}^r(\text{Gr}(r, \mathbb{C}^p), \mathbb{R}).
\]

By (2.72) and (2.78), we have \(d\alpha = 0\).

Since the cohomology of the Grassmannian vanishes for the odd degree, the differential form \(\alpha\) is exact. From (2.71), \(\prod_{i=1}^{r} (\gamma^i (H - \text{rk} H))^{n_i}, 0 \rightleftharpoons \hat{K}^0(\text{Gr}(r, \mathbb{C}^p))\). By (2.69), for any \((n_1, \ldots, n_r) \in \mathbb{N}^r\) such that \(\sum_j i \cdot n_i > r^2((m+1)(2m+1)r^2 - 1)\), we have

\[
(2.79) \quad \left[ \prod_{i=1}^{r} (\gamma^i (E - \text{rk} E))^{n_i}, 0 \right] = 0 \in \hat{K}^0(Y).
\]

The proof of Theorem 2.10 is completed. \(\Box\)

From Theorem 2.10, (2.63) and (2.64), we have the following corollary.
Corollary 2.11. For \( r = \text{rk} \, E, \ m = \dim Y, \) set
\begin{equation}
N_{r,m} = r^2((m+1)(2m+1)r^2-1).
\end{equation}
If \( k > N_{r,m}, \) we have
\begin{equation}
[P_k^+(E), 0] = [P_k^-(E), 0] \in \hat{K}_0^g(Y).
\end{equation}

2.4. The \( g \)-equivariant differential K-theory. In the followings, we adapt the notations in Section 1.

The following definition is an extension of Definition 2.5.

Definition 2.12. For \( g \in S^1 \), a cycle for the \( g \)-equivariant differential K-theory is the pair \((E, \phi)\) where \( E = (E, h^E, \nabla^E) \) is an \( S^1 \)-equivariant geometric triple on \( Y \) and \( \phi \in \Omega^{\text{odd}}(Y^g, \mathbb{C})/\text{Im} \, d \). Two cycles \((E_1, \phi_1)\) and \((E_2, \phi_2)\) are equivalent if there exists an \( S^1 \)-equivariant geometric triple \( E_3 = (E_3, h^{E_3}, \nabla^{E_3}) \) and an \( S^1 \)-equivariant vector bundle isomorphism
\begin{equation}
\Phi: E_1 \oplus E_3 \to E_2 \oplus E_3
\end{equation}
such that
\begin{equation}
\tilde{\text{ch}}_g(E_1 \oplus E_3, \Phi^*(E_2 \oplus E_3)) = \phi_2 - \phi_1.
\end{equation}
We define the sum in the same way as in (2.39). We define the \( g \)-equivariant differential K-group \( \hat{K}_g^0(Y) \), the Grothendieck group of equivalent classes of cycles. It is an abelian group. We define the ring structure of \( \hat{K}_g^0(Y) \) by
\begin{equation}
[E, \phi] \cup [E, \psi] = [E \oplus E, \text{ch}_g(E) \wedge \psi + \phi \wedge \text{ch}_g(E) - d\phi \wedge \psi].
\end{equation}
As in Definition 2.5, this cup product is well defined, commutative and associative. The element \( 1 := [C, 0] \) is a unit. The circle action on the total space \( Y \times \mathbb{C} \) is defined by \( g(y, c) = (\gamma y, c) \) for any \((y, c) \in Y \times \mathbb{C}\). The trivial metric and connection are obviously \( S^1 \)-invariant.

Remark 2.13. Certainly, we can replace \( S^1 \) by any compact Lie group in Definition 2.12.

As in (2.42), if \([E, 0] = [E, 0] \in \hat{K}_g^0(Y)\), we have
\begin{equation}
\text{ch}_g(E) = \chi_g(E) \in \Omega^*(Y^g, \mathbb{C}).
\end{equation}
Note that \( g \in S^1 \) defines a prime ideal \( I(g) \) in \( R(S^1) \), the representation ring of \( S^1 \), namely all characters of \( S^1 \) which vanish at \( g \). For any \( R(S^1) \)-module \( \mathcal{M} \), we denote by \( \mathcal{M}_{I(g)} \) the module obtained from \( \mathcal{M} \) by localizing at this prime ideal. An element of \( R(S^1)_{I(g)} \) is a “fraction” \( u/v \) with \( u, v \in R(S^1) \) and \( \chi_v(g) \neq 0 \), but two fractions \( u/v \) and \( u'/v' \) represent the same element of \( R(S^1)_{I(g)} \) if there exists \( t \in R(S^1) \) with \( \chi_t(g) \neq 0 \) and \( tuv = tuv' \in R(S^1) \). Elements of \( \mathcal{M}_{I(g)} \) are ”fractions” \( u/v \) (\( u \in \mathcal{M}, v \in R(S^1), \chi_v(g) \neq 0 \)) with a similar equivalence relation. Thus \( \mathcal{M}_{I(g)} \) is a module over the local ring \( R(S^1)_{I(g)} \). Since we do not distinguish the finite dimensional virtual representations and the characters of elements in \( R(S^1) \), we usually write the element of \( \mathcal{M}_{I(g)} \) by
\begin{equation}
u/\chi \quad \text{with} \ u \in \mathcal{M}, \chi \in \mathbb{Z}[h, h^{-1}] \quad \text{for} \ h \in S^1, \chi(h) \neq 0.
\end{equation}
For a finite dimensional $S^1$-representation $M$, we take the $S^1$-action on $Y \times M$ by $g(y, u) = (gy, gu)$. Thus $Y \times M \to Y$ is an equivariant vector bundle over $Y$. We denote this equivariant vector bundle by $E_M$. From this construction, the trivial metric $h^M$ and the trivial connection $\nabla^M$ on $E_M$ are naturally $S^1$-invariant. Let $\mathcal{M} = (E_M, h^M, \nabla^M)$. Note that $E \to E_M \otimes E$ makes $K^0_{S^1}(Y)$ a $R(S^1)$-module. From (2.93), since $\chi_g(M) = \chi_M(g)$, constant on $Y^g$, $(E, \phi) \mapsto (\mathcal{M} \otimes E, \chi_M(g) \cdot \phi)$ makes $\hat{K}^0_g(Y)$ a $R(S^1)$-module.

In the reminder of this section, we will use the geometric triple with or without the group action together. In order to distinguish the notations, we will denote by $\mathcal{E}$ the corresponding geometric triple forgetting the group action.

Recall that $Y^S = \{Y^S_a\}_{a \in \mathbb{N}}$ is the fixed point set of the circle action and $N_a$ is the normal bundle of $Y^S_a$ in $Y$. We consider $N_a$ as a complex vector bundle. By [49, Proposition 2.2],

\[(2.87) \quad K^0_{S^1}(Y^S_a) \simeq R(S^1) \otimes K^0(Y^S_a).
\]

By Proposition 1.1 for $g \in S^1 \setminus A$, in the sense of (2.87), we write

\[(2.88) \quad \lambda_{-g^{-v}}(N^*_a) = \bigotimes_{v=1}^q \lambda_{-g^{-v}}(N^*_a) = \bigotimes_{v=1}^q \left( 1 + \sum_{k=1}^{\text{rk} N_a} (-g^{-v})^k \cdot \Lambda^k (N^*_a) \right).
\]

Remark that from (1.4)-(1.7), for $g \in S^1 \setminus A$, $g^v - 1 \neq 0$ if $\text{rk} N_a \neq 0$. Set

\[(2.89) \quad r_{a,v} = \text{rk} N_{a,v}, \quad m_a = \dim Y^S_a.
\]

By (2.63) and (2.64), formally,

\[(2.90) \quad \lambda_{-g^{-v}}(N^*_a) = (1 - g^{-v})^{-r_{a,v}} \left( 1 + \sum_{k=1}^\infty (-g^{-v})^k (1 - g^{-v})^{-k} \left( P_{k,+}(N^*_a) - P_{k,-}(N^*_a) \right) \right)
\]

\[\quad = \left( g^{r_{a,v}} (g^v - 1)^{r_{a,v}} \right) \left( 1 + \sum_{k=1}^\infty (-1)^k (g^v - 1)^k \left( P_{k,+}(N^*_a) - P_{k,-}(N^*_a) \right) \right).
\]

Furthermore, by Corollary 2.11 we know that for any $k \geq N_a$,

\[(2.91) \quad \left[ P_{k,+}(N^*_a) - P_{k,-}(N^*_a), 0 \right] = 0 \in \hat{K}^0(Y^S_a).
\]

We define

\[(2.92) \quad \lambda_{-g^{-v}}_{N_a} := \left( g^{r_{a,v}} (g^v - 1)^{r_{a,v}} \right)^{N_a} \left( 1 + \sum_{k=1}^{N_a} (-1)^k (g^v - 1)^k \left( P_{k,+}(N^*_a) - P_{k,-}(N^*_a) \right) \right),
\]

\[\quad \lambda_{-1}(N^*_a)_{N_a} := \bigotimes_{v: r_{a,v} \neq 0} \lambda_{-g^{-v}}(N^*_a)^{-1}_{N_a}.
\]

By (2.80) and (2.91), we see that for any $N_a', N_a' > \sup_v N_r_{a,v}, m_a$,

\[(2.93) \quad \left[ \lambda_{-1}(N^*_a)_{N_a'}, 0 \right] = \left[ \lambda_{-1}(N^*_a)_{N_a}, 0 \right] \in \hat{K}^0_g(Y^S_a)_{I(g)}.
\]
Then from (2.88)-(2.93), for any $N > \sup_v \mathcal{N}_{r_0,v,m_0}$, we have
\begin{equation}
\left[ \lambda^{-1}_1(N^*_\alpha), 0 \right] \cup \left[ \lambda^{-1}_1(N^*_\alpha)^{-1}, 0 \right] = 1 \in \hat{K}_g^0(Y^{S^1})_{I(g)}.
\end{equation}

Summarizing, we obtain that
\begin{equation}
\left[ \lambda^{-1}_1(N^*_\alpha), 0 \right] \cup \left[ \lambda^{-1}_1(N^*_\alpha)^{-1}, 0 \right] = 1 \in \hat{K}_g^0(Y^{S^1})_{I(g)}.
\end{equation}

**Theorem 2.14.** For $g \in S^1 \setminus A$, $\left[ \lambda^{-1}_1(N^*_\alpha), 0 \right]$ is invertible in $\hat{K}_g^0(Y^{S^1})_{I(g)}$ and for any $N > \sup_v \mathcal{N}_{r_0,v,m_0}$ in (2.80), we have
\begin{equation}
\left[ \lambda^{-1}_1(N^*_\alpha), 0 \right] \cup \left[ \lambda^{-1}_1(N^*_\alpha)^{-1}, 0 \right] = 1 \in \hat{K}_g^0(Y^{S^1})_{I(g)}.
\end{equation}

Remark that the lower bound $\sup_v \mathcal{N}_{r_0,v,m_0}$ does not depend on $g \in S^1 \setminus A$.

From (2.92) and (2.94), for any $k \geq \mathcal{N}_{r_0,v,m_0}$,
\begin{equation}
\text{ch} \left( P_{k,+} \left( 'N^*_{\alpha,v} \right) \right) = \text{ch} \left( P_{k,-} \left( 'N^*_{\alpha,v} \right) \right) \in \Omega^*(Y^{S^1}, \mathbb{R}).
\end{equation}

From (2.92), we define
\begin{equation}
\text{ch}_g \left( \lambda^{-1}_1(N^*_\alpha) \right)
= \prod_{v: r_0, v \neq 0} \frac{g^{v,r_0,v}}{(g^v - 1)^{r_0,v}} \left( 1 + \sum_{k=1}^N (-1)^k \left( \text{ch} \left( P_{k,+} \left( 'N^*_{\alpha,v} \right) \right) - \text{ch} \left( P_{k,-} \left( 'N^*_{\alpha,v} \right) \right) \right) \right).
\end{equation}

The following corollary follows directly from (2.85), (2.94) and (2.96).

**Corollary 2.15.** For any $N > \sup_v \mathcal{N}_{r_0,v,m_0}$, $g \in S^1 \setminus A$,
\begin{equation}
\text{ch}_g \left( \lambda^{-1}_1(N^*_\alpha) \right) \cdot \text{ch}_g \left( \lambda^{-1}_1(N^*_\alpha)^{-1} \right) = 1 \in \Omega^*(Y^{S^1}, \mathbb{C}).
\end{equation}

3. **Localization Formula for Equivariant $\eta$-Invariants**

In this section, we establish the main result of this paper, Theorem 0.3 and 0.4. In Section 3.1, we define a push-forward map in $g$-equivariant differential K-theory. In Section 3.2, we state our main result Theorem 3.3. In Section 3.3, we prove Theorem 3.3 and leave the proof of Theorem 3.5 to the next subsection. In Section 3.4, we prove Theorem 3.5. In Section 3.5, we study the case when $Y^{S^1} = \emptyset$.

3.1. **Push-forward map in $g$-equivariant differential K-theory**. Note that for $g \in S^1$, $\mathbb{Q}_g \subset \mathbb{C}$ was defined in (0.13) and $\text{ch}_g(R(S^1)_{I(g)}) = \mathbb{Q}_g$.

**Definition and Theorem 3.1.** Assume that $Y$ has an $S^1$-equivariant spin$^c$ structure. For equivariant geometric triple $(E, \phi) \in \Omega^{\text{odd}}(Y^g, \mathbb{C})/\text{Im} d$, $\chi \in R(S^1)$ such that $\chi(g) \neq 0$, the map
\begin{equation}
\widehat{f}_{Y*}((E, \phi)/\chi) := \chi(g)^{-1} \left( \int_{Y^g} Td_g(\nabla^Y, \nabla^L) \wedge \phi + \bar{\eta}_g(TY, \xi, E) \right)
\end{equation}
defines a push-forward map $\widehat{f}_{Y*} : \hat{K}_g^0(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$. 
Proof. For an equivariant vector bundle isomorphism \( \Phi : E \to E \), by Definition \[3.2\] we have
\[
(3.2) \quad \bar{\eta}_g(TY, L, \Phi^*E) = \bar{\eta}_g(TY, L, E).
\]
From Definition \[3.2\] for any finite dimensional \( S^1 \)-representation \( M \) and triples \( E, E_1, E_2 \), we have
\[
(3.3) \quad \bar{\eta}_g(TY, L, M \otimes E) = \chi_M(g) \cdot \bar{\eta}_g(TY, L, E),
\]
\[
\bar{\eta}_g(TY, L, E_1 \oplus E_2) = \bar{\eta}_g(TY, L, E_1) + \bar{\eta}_g(TY, L, E_2).
\]
If \( (E_1, \phi_1) \sim (E_2, \phi_2) \) in \( \hat{K}_g^0(Y) \), there exists \( E_3 \) such that \( \Phi : E_1 \oplus E_3 \to E_2 \oplus E_3 \) is an equivariant vector bundle isomorphism. From the variation formula \((3.1), (3.2) \) and \((3.3) \), there exists \( \alpha_g \in \mathbb{Z}[g, g^{-1}] := \{ f(g) \in \mathbb{C} : f \in \mathbb{Z}[x, x^{-1}] \} \) such that
\[
(3.4) \quad \bar{\eta}_g(TY, L, E_2) - \bar{\eta}_g(TY, L, E_1) = \bar{\eta}_g(TY, L, E_2 \oplus E_3) - \bar{\eta}_g(TY, L, E_1 \oplus E_3) - \bar{\eta}_g(TY, L, \Phi^*(E_2 + E_3)) + \alpha_g.
\]
From \((3.1) \) and \((3.3) \), for cycles \((E_1, \phi_1)/\chi_1\) and \((E_2, \phi_2)/\chi_2\) of \( \hat{K}_g^0(Y)_{I(g)} \), we have
\[
(3.5) \quad \widehat{f}_Y^*(((E_1, \phi_1)/\chi_1 + (E_2, \phi_2)/\chi_2) = \widehat{f}_Y^*((E_1, \phi_1)/\chi_1) + \widehat{f}_Y^*((E_2, \phi_2)/\chi_2).
\]
If \([E_2 - E_1, \phi]/\chi = 0 \in \hat{K}_g^0(Y)_{I(g)} \), there exists finite dimensional \( S^1 \)-representation \( M \) such that \([M \otimes (E_2 - E_1), \chi_M(g) \phi] = 0 \in \hat{K}_g^0(Y) \). Thus from Definition \[2.12\] there exist \( E_3 \) and an equivariant vector bundle isomorphism \( \Phi : (M \otimes E_1) \oplus E_3 \to (M \otimes E_2) \oplus E_3 \) such that
\[
(3.6) \quad \phi = -\chi_M(g)^{-1} \tilde{c}_g((M \otimes E_1) \oplus E_3, \Phi^*((M \otimes E_2) \oplus E_3)).
\]
From \((3.1), (3.2) \) and \((3.6) \), there exists \( \alpha_g \in \mathbb{Z}[g, g^{-1}] \) such that
\[
(3.7) \quad \widehat{f}_Y^*((E_2 - E_1, \phi)/\chi) = \chi(g)^{-1} \left[ \bar{\eta}_g(TY, L, E_2) - \bar{\eta}_g(TY, L, E_1) + \int_{Y_g} Td_g(\nabla^{TY}, \nabla^L) \phi \right]
= \chi(g)^{-1} \chi_M(g)^{-1} \left[ \bar{\eta}_g(TY, L, M \otimes E_2) - \bar{\eta}_g(TY, L, M \otimes E_1) \right] + \int_{Y_g} Td_g(\nabla^{TY}, \nabla^L) \tilde{c}_g((M \otimes E_1) \oplus E_3, \Phi^*((M \otimes E_2) \oplus E_3))
\]
\[
= \chi(g)^{-1} \chi_M(g)^{-1} \cdot \alpha_g \in \mathbb{Q}_g.
\]
The proof of Theorem \[3.1\] is completed.

3.2. Main result: Theorem \[1.4\] From \([2.92]\), there exist equivariant geometric triples \( \mu_{\alpha_N^+, \alpha} \) and \( \mu_{\alpha_N^-, \alpha} \) such that
\[
(3.8) \quad \lambda_1(N_{\alpha}^+) = \prod_{\alpha \neq 0, \, \alpha \notin 0} \frac{1}{(g - 1)^{r_{\alpha, \alpha} = N} \cdot \mu_{\alpha_N^+, \alpha} - \mu_{\alpha_N^-, \alpha}}.
\]
In (3.9), we identify \( f(g) \cdot E \) with \( M_f \otimes E \) for triple \( E, f \in \mathbb{Z}[x] \) and virtual \( S^1 \)-representation \( M_f \) associated with \( f \). For \( g \in S^1 \setminus A \), we define

\[
(3.9) \quad \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \lambda_1(N^*_{\alpha})^1 \otimes E|_{Y^S_{\alpha}} \right) = \prod_{v : N_{\alpha,v} \neq 0} (g^v - 1)^{-r_{\alpha,v}} \cdot N_{\alpha,v}^{-N} \left( \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \mu_{\alpha,N,+} \otimes E|_{Y^S_{\alpha}} \right) - \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \mu_{\alpha,N,-} \otimes E|_{Y^S_{\alpha}} \right) \right).
\]

**Remark 3.2.** Note that from (2.92) and (3.8),

\[
(3.10) \quad \mu_{\alpha,N,\pm} = \bigoplus_{k \geq 0} \xi_{\alpha,k,\pm} \in K_{S^1}(Y^S_{\alpha})
\]

and \( S^1 \) acts fiberwisely on \( \xi_{\alpha,k} \) by weight \( k \). Let \( S^1 \) acts on \( L \) by sending \( g \in S^1 \) to \( g^l_{\alpha} \) \((l_{\alpha} \in \mathbb{Z})\) on \( Y^S_{\alpha} \). Then by [11, p139],

\[
(3.11) \quad \sum_v \text{rk} N_{\alpha,v} + l_{\alpha} = 0 \mod 2.
\]

Now by (1.22), (1.31), (3.10) and (3.11), for \( g \in S^1 \), we have

\[
(3.12) \quad \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \mu_{\alpha,N,+} \otimes E|_{Y^S_{\alpha}} \right) - \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \mu_{\alpha,N,-} \otimes E|_{Y^S_{\alpha}} \right) = g^{-\frac{1}{2} \sum_v \text{rk} N_{\alpha,v} + \frac{1}{2} l_{\alpha}} \sum_{k \geq 0, v} g^{k+v} \left( \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \xi_{\alpha,k,+} \otimes E_\nu \right) - \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \xi_{\alpha,k,-} \otimes E_\nu \right) \right).
\]

Now we state our main result of this paper, which is a precise formulation of Theorem 1.4.

**Theorem 3.3.** There exists \( N_0 > 0 \) such that for any \( N, N' \in \mathbb{N} \) and \( N' > N > N_0 \),

\[
(3.13) \quad P_{N,N'}(g) := \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \lambda_1(N^*_{\alpha})^1 \otimes E|_{Y^S_{\alpha}} \right) - \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \lambda_1(N^*_{\alpha})^{-1} \otimes E|_{Y^S_{\alpha}} \right)
\]

and

\[
(3.14) \quad Q_N(g) := \tilde{\eta}_g (TY, L, E) - \sum_\alpha \tilde{\eta}_g \left( TY^S_{\alpha}, L_\alpha, \lambda_1(N^*_{\alpha})^{-1} \otimes E|_{Y^S_{\alpha}} \right)
\]

as functions on \( S^1 \setminus A \), are restrictions of rational functions on \( S^1 \) with coefficients in \( \mathbb{Z} \) which does not have poles on \( S^1 \setminus A \).

### 3.3. A proof of Theorem 3.3

**Lemma 3.4.** There exists \( N_0 > 0 \) such that for any \( N, N' \in \mathbb{N} \) and \( N' > N > N_0 \), \( g \in S^1 \setminus A \), we have \( P_{N,N'}(g) \in \mathbb{Q}_g \).

**Proof.** From (2.93), for any \( N, N' > \sup_v N_{\alpha,v,m_{\alpha}} \), \( g \in S^1 \setminus A \),

\[
(3.15) \quad \left[ \lambda_1(N^*_{\alpha})^1 \otimes E|_{Y^S_{\alpha}}, 0 \right] = \left[ \lambda_1(N^*_{\alpha})^{-1} \otimes E|_{Y^S_{\alpha}}, 0 \right] \in \hat{K}_g^0(Y^S_{\alpha})_{l(g)}.
\]

Thus our lemma follows directly from Definition and Theorem 3.1. \( \square \)
(3.16) \[ \widehat{K}_g^0(Y^{S^1})_{I(g)} = \bigoplus_{\alpha \in \mathbb{N}} \widehat{K}_g^0(Y^S_{\alpha})_{I(g)}. \]

From Theorem 2.14, for \( g \in S^1 \setminus A \), \( \left[ \lambda_1(N_{\alpha}^a), 0 \right] \) is invertible in \( \widehat{K}_g^0(Y^{S^1})_{I(g)} \). We denote by

(3.17) \[ \left[ \lambda_1(N^a), 0 \right]^{-1} = \bigoplus_{\alpha} \left[ \lambda_1(N_{\alpha}^a), 0 \right]^{-1} \in \widehat{K}_g^0(Y^{S^1})_{I(g)}. \]

Let \( \iota : Y^{S^1} \to Y \) be the embedding. Let \( \hat{i}^* : \widehat{K}_g^0(Y)_{I(g)} \to \widehat{K}_g^0(Y^{S^1})_{I(g)} \) be the induced homomorphism.

Let \( \widehat{f}_{Y^{S^1}} \) be the push-forward map \( \widehat{f}_{Y^{S^1}} : \widehat{K}_g^0(Y^{S^1})_{I(g)} \to \mathbb{C}/\mathbb{Q}_g \) defined in Definition 3.1. Explicitly, for any \( [E, \phi]/\chi \in \widehat{K}_g^0(Y^{S^1})_{I(g)}, \)

(3.18) \[ \widehat{f}_{Y^{S^1}}([E, \phi]/\chi) := \chi(g)^{-1} \sum_{\alpha} \left( \int_{Y^S_{\alpha}} T_d g(\nabla^{Y_{\alpha}^S}, \nabla L_{\alpha}) \wedge \phi \right. \]
\[ \left. + \bar{\eta}_g \left( T_{Y^S_{\alpha}}^L L_{\alpha}, E \right) \right) \mod Q_g. \]

We restate now Theorem 0.3 as the following result which will be proved in Section 3.4

**Theorem 3.5.** For \( g \in S^1 \setminus A \), \( \iota^* \) is an isomorphism and the following diagram commutes

(3.19) \[ \widehat{K}_g^0(Y^{S^1})_{I(g)} \xleftarrow{\iota^* \cup \iota^* \cup \iota^*} \widehat{K}_g^0(Y)_{I(g)} \]

\[ \widehat{f}_{Y^{S^1}} \quad \widehat{f}_{Y^{S^1}} \]

**Corollary 3.6.** There exists \( N_0 > 0 \) such that for any \( N > N_0, g \in S^1 \setminus A, Q_N(g) \in \mathbb{Q}_g \).

**Proof.** By Definition 3.1, we have

(3.20) \[ \widehat{f}_{Y,*}([E, 0]) = \bar{\eta}_g(T_{Y^S_{\alpha}} L_{\alpha}, E) \mod Q_g. \]

By Theorems 2.14, 3.1, 3.17 and 3.18, we get for \( N \) large enough, for any \( g \in S^1 \setminus A, \)

(3.21) \[ \widehat{f}_{Y^{S^1}} \left( \left[ \lambda_1(N^a), 0 \right]^{-1} \cup \iota^* ([E, 0]) \right) = \widehat{f}_{Y^{S^1}} \left( \bigoplus_{\alpha} \left[ \lambda_1(N_{\alpha}^a), 0 \right]^{-1} \cup \iota^* ([E, 0]) \right) \]
\[ = \sum_{\alpha} \bar{\eta}_g \left( T_{Y_{\alpha}^S}^L L_{\alpha}, \lambda_1(N_{\alpha}^a), 0 \right) \mod Q_g. \]

Thus by Theorem 3.5, for \( N \) large enough and \( g \in S^1 \setminus A, \) we have \( Q_N(g) \in \mathbb{Q}_g \).

The proof of Corollary 3.6 is completed.

**Lemma 3.7.** Let \( K \in \text{Lie}(S^1) \). For \( g \in S^1 \setminus A \), there exists \( \beta > 0 \), such that for any \( t \in \mathbb{R}, \)
\[ |t| \leq \beta, P_N^\beta(g e^{iK}) \text{ and } Q_N(g e^{iK}) \] are real analytic on \( t \).
Proof. Recall that \( r_{\alpha,v} = \text{rk} N_{\alpha,v} \). By (3.18), we have
\[
(3.22) \quad F_{\alpha,N}(g) \cdot \lambda_{N}^{-1} = \mu_{\alpha,N,+} - \mu_{\alpha,N,-} \text{ with } F_{\alpha,N}(g) = \prod_{v: r_{\alpha,v} \neq 0} (g^v - 1)^{r_{\alpha,v} + N}.
\]

Recall that \( \vartheta_K \in T^*Y \) is the 1-form which is dual to \( K^N \) by the metric \( g_T^N \). It vanishes on \( Y_{S^1} \). Since for any \( g \in S^1 \setminus A, Y^g = Y_{S^1} \), from (1.53), for any \( \alpha \) and \( g \in S^1 \setminus A, \)
\[
(3.23) \quad \mathcal{M}_{g,K}(TY, L, E) = \mathcal{M}_{g,K}(TY_{S^1}^N, L_\alpha, (\mu_{\alpha,N,+} - \mu_{\alpha,N,-}) \otimes E|_{Y_{S^1}^N}) = 0.
\]

Set
\[
(3.24) \quad F_N(g) = \prod_{v: r_{\alpha,v} \neq 0} (g^v - 1)^{r_{\alpha,v} + N}.
\]

From Theorem 1.10, (3.8), (3.13), (3.14) and (3.23), for \( g \in S^1 \setminus A, N' > N > N_0 \), we have
\[
(3.25) \quad F_N'(ge^{iK}) \cdot P_{N,N'}(ge^{iK})
\]
\[
\quad = \sum_{\alpha} \eta_{g,tK}(TY_{S^1}^N, L_\alpha, (\mu_{\alpha,N,+} - \mu_{\alpha,N,-}) \otimes E|_{Y_{S^1}^N}) \cdot \frac{F_N'(ge^{iK})}{F_{N,N}(ge^{iK})} - \sum_{\alpha} \bar{\eta}_{g,tK}(TY_{S^1}^N, L_\alpha, (\mu_{\alpha,N,+} - \mu_{\alpha,N,-}) \otimes E|_{Y_{S^1}^N}) \cdot \frac{F_N'(ge^{iK})}{F_{N,N}(ge^{iK})},
\]
and
\[
(3.26) \quad F_N'(ge^{iK}) \cdot Q_N'(ge^{iK}) = F_N'(ge^{iK})\bar{\eta}_{g,tK}(TY, L, E)
\]
\[
\quad - \sum_{\alpha} \bar{\eta}_{g,tK}(TY_{S^1}^N, L_\alpha, (\mu_{\alpha,N,+} - \mu_{\alpha,N,-}) \otimes E|_{Y_{S^1}^N}) \cdot \frac{F_N'(ge^{iK})}{F_{N,N}(ge^{iK})}.
\]

Recall that for \( g \in S^1 \setminus A, g^v - 1 \neq 0 \) if \( r_{\alpha,v} \neq 0 \). So there exists \( \beta > 0 \) such that for \( |t| \leq \beta, \)
\( F_N'(ge^{iK})^{-1} \) is real analytic on \( t \). Thus by Theorem 1.9 for \( g \in S^1 \setminus A \), there exists \( \beta > 0 \)
such that for \( |t| \leq \beta, P_{N,N'}(ge^{iK}) \) and \( Q_N'(ge^{iK}) \) are real analytic on \( t \).

The proof of Lemma 3.7 is completed. \( \square \)

**Proposition 3.8.** There exists \( N_0 > 0 \) such that for any \( N' > N > N_0 \), the functions \( P_{N,N'} \) and \( Q_N \) are restrictions on \( S^1 \setminus A \) of rational functions on \( S^1 \) with coefficients in \( \mathbb{Z} \).

**Proof.** We only prove this property for \( Q_N \) here. The proof for \( P_{N,N'} \) is almost the same.

By Corollary 3.6 for \( g = e^{2\pi it} \in S^1 \setminus A, \) if \( N > N_0, Q_N(g) \in \mathbb{Q} \). We could write
\[
(3.27) \quad Q_N(g) = \frac{\sum_{k=0}^{N(g)} a_k(g) g^k}{\sum_{k=0}^{M(g)} b_k(g) g^k},
\]
where \( a_k(g), b_k(g) \in \mathbb{Z}, \) \( N(g), M(g) \in \mathbb{N}, \) and the polynomials \( \sum_{k=0}^{N(g)} a_k(g) g^k \) and \( \sum_{k=0}^{M(g)} b_k(g) g^k \)
are relatively prime.

Let \( T_{M,N} = \{ g \in S^1 \setminus A: M(g) \leq M, N(g) \leq N \} \). Then
\[
(3.28) \quad \bigcup_{M,N=1}^{\infty} T_{M,N} = S^1 \setminus A.
\]

Fix \( g_0 \in S^1 \setminus A \). Let \( U \) be a connected open neighbourhood of \( g_0 \) in \( S^1 \setminus A \) such that \( Q_N \) is real analytic on \( U \) by Lemma 3.7. We have \( \bigcup_{M,N=1}^{\infty} (T_{M,N} \cap U) = U \). Since \( U \) is an uncountable
set, there exist \(M_0, N_0 \in \mathbb{N}\) such that \(T_{M_0, N_0} \cap U\) is an uncountable set. We define the map \(\Phi : T_{M_0, N_0} \cap U \to \mathbb{Z}^{M_0+N_0+2}\) such that

\[
\Phi(g) = (a_0(g), \ldots, a_{N_0}(g), b_0(g), \ldots, b_{M_0}(g)).
\]

Since \(\mathbb{Z}^{M_0+N_0+2}\) is a countable set, there exists \(x = (a_0, \ldots, a_{N_0}, b_0, \ldots, b_{M_0}) \in \text{Im}(\Phi)\) such that \(\Phi^{-1}(x)\) is an uncountable set. Set \(h(g) = \sum_{k=0}^{N_0} a_k g^k \sum_{k=0}^{M_0} b_k g^k\) on \(U\). Then there is an open subset \(U' \subset U\) such that \(h\) is real analytic on \(U'\) and \(Q_N = h\) on an uncountable subset of \(U'\). Moreover, since \(h\) is a meromorphic function near \(U\), \(Q_N\) can be extended as a holomorphic function near \(U\) and \(U\) is connected, we have \(h \equiv Q_N\) on \(U\). So for any \(g_0 \in S^1\setminus A\), there is an open neighborhood \(U\) of \(g_0\) in \(S^1\setminus A\) such that \(Q_N\) is a rational function on \(U\) with coefficients in \(\mathbb{Z}\). It means that \(Q_N\) is a rational function on each connected component of \(S^1\setminus A\) with coefficients in \(\mathbb{Z}\).

For \(g \in A\), by Theorem 1.10, for small \(t, t \neq 0\), similarly as in (3.26), we have

\[
Q_Nge^{tK} = \tilde{\eta}_{g,tK}(TY, L, E) - \mathcal{M}_{g,tK}(TY, L, E) - F_N(ge^{tK})^{-1} \sum_{\alpha} \tilde{\eta}_{g,tK}(TY^{S^1}_{\alpha}, L_{\alpha}, (\mu_{\alpha,N} + \mu_{\alpha,N}^{-1}) \otimes E_{S^1_{\alpha}}) \cdot \frac{F_N(ge^{tK})}{F_{\alpha,N}(ge^{tK})}.
\]

From Theorem 1.9, (3.24), and (3.30), \(Q_N(ge^{tK})\) is a meromorphic function on \(t\) near 0. But we know for \(t > 0\) small

\[
Q_N(ge^{tK}) = \frac{P_+(ge^{tK})}{Q_+(ge^{tK})}
\]

is a rational function on \(ge^{tK}\). As \(\frac{P_+(ge^{tK})}{Q_+(ge^{tK})}\) is a meromorphic function on \(t\) near 0, this implies for \(t\) near 0, (3.31) holds. In particular, (3.31) holds for \(t < 0\) small.

The proof of Proposition 3.8 is completed. \(\square\)

By Lemma 3.7 and Proposition 3.8, the proof of Theorem 3.3 is completed.

3.4. A proof of Theorem 3.5 We explain first the \(S^1\)-equivariant \(K^1\)-theory on the fixed point set \(Y^{S^1}\) of \(S^1\)-action on \(Y\), and the odd Chern character for an element in \(K^1\)-group.

Let \(K_{S^1}^1(Y^{S^1})\) be the \(S^1\)-equivariant \(K^1\)-group of \(Y^{S^1}\) (see [19] Definitions 2.7 and 2.8)). By Definition, it is characterized by the following exact sequence

\[
0 \to K_{S^1}^1(Y^{S^1}) \to K_{S^1}^0(Y^{S^1} \times S^1) \overset{\iota^*}{\to} K_{S^1}^0(Y^{S^1}) \to 0,
\]

where \(\widehat{S^1}\) is a copy of \(S^1\) and \(i : Y^{S^1} \to Y^{S^1} \times S^1\) is an \(S^1\)-equivariant embedding with trivial \(S^1\)-action on \(\widehat{S^1}\). Note that \(Y^{S^1} \times S^1 = Y^{S^1} \times \mathbb{R}/\mathbb{Z}\). We assume that \(i(Y^{S^1}) = Y^{S^1} \times \{\frac{1}{2}\}\).

By (3.32), the element of \(K_{S^1}^1(Y^{S^1})\) could be represented as an element \(0 \neq x \in K_{S^1}^0(Y^{S^1} \times \widehat{S^1})\) such that \(i^*x = 0 \in K_{S^1}^0(Y^{S^1})\). We write \(x = W - U\), where \(W\) and \(U\) are equivariant complex vector bundles over \(Y^{S^1} \times \widehat{S^1}\). Then we have the decomposition of complex vector bundles

\[
W = \bigoplus_v W_v, \quad U = \bigoplus_v U_v,
\]

where \(W_v\) and \(U_v\) are the complex vector bundles over \(Y^{S^1} \times \widehat{S^1}\) on which \(g \in S^1\) acts by multiplication by \(g^v\). Since \(K_{S^1}^0(Y^{S^1} \times \widehat{S^1}) \simeq R(S^1) \otimes K^0(Y^{S^1} \times \widehat{S^1})\), we may and we will
assume that $U_v$ is a trivial vector bundle for any $v$. Since $i^* x = W|_{Y_s^1 \times \{1/2 \}} - U|_{Y_s^1 \times \{1/2 \}} = 0 \in K^{0}_{S_1}(Y^S_1)$, we may and we will assume that $W_v|_{Y_s^1 \times \{1/2 \}}$ is a trivial vector bundle over $Y^S_1 \times \{1/2 \}$ for any $v$. Since $[0, 1]$ is contractible, there exist finite dimensional $S^1$-representations $M_v$, on which $g \in S^1$ acts by $g^v$, and morphisms $F_v$ on $M_v$ such that

$$W_v = Y^S_1 \times [0, 1] \times M_v / \sim_{F_v},$$

where $\sim_{F_v}$ is the glue: $(y, 1, m) \sim_{F_v} (y, 0, F_v(y)m)$ for $y \in Y^S_1$, $m \in M_v$.

We denote by $F := \oplus_v F_v$ the morphism of the $S^1$-representation $M := \oplus_v M_v$. Then it induces an equivariant vector bundle isomorphism $F : E^S_1 \rightarrow E^S_M$, where

$$E^S_M := Y^S_1 \times M,$$

is an equivariant vector bundle over $Y^S_1$. Recall that for finite dimensional $S^1$-representation $M$, the $S^1$-action on $Y \times M$ defined by $g(y, m) = (gy, gm)$ makes an $S^1$-equivariant vector bundle over $Y$

$$E_M := Y \times M.$$

It is obvious that $E^S_M = E_M|_{Y^S_1}$.

Let $\nabla$ be an $S^1$-invariant connection on $E^S_M$. Then $F^* \nabla \cdot = F^{-1} \nabla \cdot (F \cdot)$ is also an $S^1$-invariant connection on $E^S_M$. From (3.33) and (3.34),

$$\nabla^W = dt \land \frac{\partial}{\partial t} + (1 - t)^{1/2} \nabla + t F^* \nabla = dt \land \frac{\partial}{\partial t} + \nabla + t F^{-1} \nabla F$$

is a well-defined $S^1$-invariant connection on $W$ over $Y^S_1 \times \widehat{S}^1$.

Recall that the equivariant Chern character form $\text{ch}_g(E)$ and the equivariant Chern-Simons class $\widetilde{\text{ch}}_g(E_0, E_1)$ defined in (1.23) and (1.29) are independent of the metrics. We often denote the equivariant Chern character form by $\text{ch}_g(E, \nabla^E)$ and the equivariant Chern-Simons class by $\widetilde{\text{ch}}_g(E, \nabla^{E_0}, \nabla^{E_1})$.

Let $\nabla^U$ be the trivial connection on $U$. It is naturally $S^1$-invariant. For $g \in S^1 \setminus A$, the odd equivariant Chern character for $x \in K^1_{S_1}(Y^S_1)$ as above, is defined by

$$\text{ch}_g(x) := \left[\int_{S^1} \left( \text{ch}_g(W, \nabla^W) - \text{ch}_g(U, \nabla^U) \right) \right] \in H^{**}(Y^S_1, \mathbb{C}) \subset \Omega^{**}(Y^S_1, \mathbb{C})/\text{Im} \, d.$$

As $[\text{ch}_g(W, \nabla^W)] \in H^{**}(Y^S_1 \times \widehat{S}^1, \mathbb{C})$ does not depend on the choice of $\nabla$, thus $\text{ch}_g(x)$ also does not depend on $\nabla$. From (1.23), (1.28), (3.37) and (3.38), we have

$$\text{ch}_g(x) = \left[\int_{[0, 1]} \left\{ \text{ch}_g \left( [0, 1] \times E^S_M, dt \land \frac{\partial}{\partial t} + (1 - t)^{1/2} \nabla + t F^* \nabla \right) \right\} dt \right]$$

$$= \widetilde{\text{ch}}_g(E^S_M, \nabla, F^* \nabla) \in \Omega^{**}(Y^S_1, \mathbb{C})/\text{Im} \, d.$$

If we choose $\nabla$ as the trivial connection $d$ on $E^S_M$, by (3.37), the curvature $R^W$ of $\nabla^W$ is given by

$$R^W = (\nabla^W)^2 = dt \land (F^{-1} dF) \land t (1 - t) (F^{-1} dF)^2.$$
From \([1,23]\), \([1,26]\), \([3,39]\) and \((3.40)\), we calculate that
\[
(3.41) \quad \text{ch}_g(x) = \sum_{n \geq 0} \frac{1}{(2\pi)^{n+1}(2n+1)!} \left[ \text{Tr} \left( g(F^{-1}dF)^{2n+1} \right) \right].
\]
This is just the equivariant version of the odd Chern character in \([27]\) and \([54, (1.50)]\).

From \((3.32)\), \(K^1_{S^1}(Y^{S^1})\) is a \(R(S^1)\)-module. Moreover, \(\phi \mapsto \chi_M(g) \cdot \phi\) makes \(\Omega^{\text{odd}}(Y^{S^1}, \mathbb{C})/\text{Im} \, d\) a \(R(S^1)\)-module. The following Proposition is the \(g\)-equivariant extension of the corresponding results in \([20, \text{Proposition 2.20}], [26, (2.21)]\) and \([21, \text{Proposition 2.24}]\).

**Proposition 3.9.** The following sequence is exact as \(R(S^1)\)-modules for \(g \in S^1 \setminus A\):
\[
(3.42) \quad K^1_{S^1}(Y^{S^1}) \xrightarrow{\text{ch}_g} \Omega^{\text{odd}}(Y^{S^1}, \mathbb{C})/\text{Im} \, d \xrightarrow{a} \tilde{K}^0_g(Y) \xrightarrow{\tau} K^0_{S^1}(Y) \rightarrow 0,
\]
where
\[
(3.43) \quad a(\phi) = [0, -\phi], \quad \tau([E, \phi]) = [E].
\]

**Proof.** From Definition \(2.12\) it is obvious that \(\tau\) is surjective and \(\tau \circ a = 0\).

If \(x \in \text{Ker} \, \tau\), from Definition \(2.12\) it is easy to see that \(x \in \text{Im} \, a\).

Now we prove \(a \circ \text{ch}_g = 0\). For \(x \in K^1_{S^1}(Y^{S^1})\), we could construct equivariant vector bundles \(E^S_M\) over \(Y^{S^1}\) and \(E_M\) over \(Y\) as in \((3.35)\) and \((3.36)\). Let \(h\) be the metric on \(E_M\) induced by an \(S^1\)-invariant metric on \(M\) via \((3.36)\) and \(\nabla\) be the trivial connection on \((E_M, h)\). By \((3.39)\), since \(E^S_M = E_M|_{Y^{S^1}}\), we have
\[
(3.44) \quad a(\text{ch}_g(x)) = [0, -\text{ch}_g(E^S_M, \nabla|_{Y^{S^1}}, F^*\nabla|_{Y^{S^1}})] = [0, -\text{ch}_g(E_M, \nabla, F^*\nabla)]
\]
\[
= [(E_M, h, \nabla), 0] - [(E_M, h, \nabla), \text{ch}_g((E_M, h, \nabla), (E_M, F^*h, F^*\nabla))].
\]
By Definition \(2.12\) under the equivariant vector bundle isomorphism \(F\), \(((E_M, h, \nabla), 0)\) and \(((E_M, h, \nabla), \text{ch}_g((E_M, h, \nabla), (E_M, F^*h, F^*\nabla)))\) are equivalent. That is, \(a(\text{ch}_g(x)) = 0 \in \tilde{K}^0_g(Y)\).

At last, we prove \(\text{Ker} \, a = \text{Im} \, \text{ch}_g\). For \(\phi' \in \text{Ker} \, a\), i.e., \([0, -\phi'] = 0 \in \tilde{K}^0_g(Y)\). By Definition \(2.12\) there exists an equivariant geometric triple \(E' = (E', h', \nabla^{E'})\) and an equivariant vector bundle isomorphism
\[
(3.45) \quad \Phi' : E' \to E' \quad \text{such that} \quad \phi' = \text{ch}_g(E', \Phi'^*E').
\]
By \([49, \text{Proposition 2.4}]\), there exists an \(S^1\)-vector bundle \(E\) on \(Y\) such that \(E \oplus E' = E_M\) where \(M\) is a finite dimensional \(S^1\)-representation. Set
\[
(3.46) \quad \Phi : E \oplus E' \to E \oplus E', \quad (u, v) \mapsto (u, \Phi'(v)).
\]
Let \(h^E\) be an \(S^1\)-invariant Hermitian metric on \(E\) and \(\nabla^E\) be an \(S^1\)-invariant Hermitian connection on \((E, h^E)\). Then \(\text{ch}_g(E \oplus E', \Phi^* (E \oplus E')) = \text{ch}_g(E', \Phi'^*E')\). As in \((3.37)\),
\[
(3.47) \quad \nabla^W = dt \wedge \frac{\partial}{\partial t} + (1 - t)(\nabla^E \oplus \nabla^{E'})|_{Y^{S^1}} + t\Phi^*(\nabla^E \oplus \nabla^{E'})|_{Y^{S^1}}
\]
is an $S^1$-invariant connection on $W = Y^{S^1} \times [0, 1] \times M/\sim_\phi$. Therefore, by \eqref{1.29}, \eqref{3.37} and \eqref{3.39}, modulo exact forms, we have
\begin{equation}
(3.48) \quad \tilde{\chi}_g (E \oplus E', \Phi^* (E \oplus E')) = \tilde{\chi}_g \left( E \oplus E', (\nabla E \oplus \nabla E')|_{Y^{S^1}}, \Phi^* (\nabla E \oplus \nabla E')|_{Y^{S^1}} \right) = \int_{S^1} \chi_g (W, \nabla^W).
\end{equation}

From \eqref{3.39}, \eqref{3.46} and \eqref{3.48}, $\Phi|_{Y^{S^1}}$ defines an element $x \in K^1_{S^1} (Y^{S^1})$ and $\phi' = \chi_g (x)$.

It is obvious that the $R(S^1)$-action commutes with $\chi_g$, $a$ and $\tau$.

The proof of Proposition 3.9 is completed.

The following localization theorem, which is the differential K-theory version of the classical localization theorem in topological K-theory \cite[Theorem 1.1]{5} is inspired by \cite[Theorem 5.5]{51} and \cite[Theorem 3.27]{21}.

**Theorem 3.10** (Localization Theorem). Let $\iota : Y^{S^1} \to Y$ be the natural inclusion. For $g \in S^1 \setminus A$, we have a $R(S^1)_{I(g)}$-module isomorphism
\begin{equation}
(3.49) \quad \iota^* : \widehat{K}^0_g (Y)_{I(g)} \xrightarrow{\sim} \widehat{K}^0_{g} (Y^{S^1})_{I(g)}.
\end{equation}

**Proof.** Since the localization produces an exact functor, from Proposition 3.9, we get an exact sequence of $R(S^1)_{I(g)}$-modules
\begin{equation}
(3.50) \quad K^1_{S^1} (Y^{S^1})_{I(g)} \xrightarrow{\text{ch}_g} \left( \Omega^\text{odd} (Y^{S^1}, \mathbb{C})/\text{Im } d \right)_{I(g)} \xrightarrow{a} \widehat{K}^0_g (Y)_{I(g)} \xrightarrow{\tau} K^0_{S^1} (Y)_{I(g)} \longrightarrow 0.
\end{equation}

Replacing $Y$ by $Y^{S^1}$, since $(Y^{S^1})^{S^1} = Y^{S^1}$, we get an exact sequence of $R(S^1)_{I(g)}$-modules
\begin{equation}
(3.51) \quad K^1_{S^1} (Y^{S^1})_{I(g)} \xrightarrow{\text{ch}_g} \left( \Omega^\text{odd} (Y^{S^1}, \mathbb{C})/\text{Im } d \right)_{I(g)} \xrightarrow{a} \widehat{K}^0_g (Y^{S^1})_{I(g)} \xrightarrow{\tau} K^0_{S^1} (Y^{S^1})_{I(g)} \longrightarrow 0.
\end{equation}

Furthermore, we have the commutative diagram
\begin{equation}
(3.52) \quad \begin{array}{ccc}
K^1_{S^1} (Y^{S^1})_{I(g)} & \xrightarrow{\text{ch}_g} & \left( \Omega^\text{odd} (Y^{S^1}, \mathbb{C})/\text{Im } d \right)_{I(g)} \\
\Pi \downarrow & & \Pi \\
K^1_{S^1} (Y^{S^1})_{I(g)} & \xrightarrow{\text{ch}_g} & \left( \Omega^\text{odd} (Y^{S^1}, \mathbb{C})/\text{Im } d \right)_{I(g)}
\end{array} \xrightarrow{a} \begin{array}{ccc}
\widehat{K}^0_g (Y)_{I(g)} & \xrightarrow{\tau} & K^0_{S^1} (Y)_{I(g)} \\
\tau |_K & & \tau |_K \\
\widehat{K}^0_g (Y^{S^1})_{I(g)} & \xrightarrow{\tau} & K^0_{S^1} (Y^{S^1})_{I(g)}
\end{array} \longrightarrow 0.
\end{equation}

Here $\tau |_K : K^0_g (Y)_{I(g)} \to K^0_g (Y^{S^1})_{I(g)}$ is the $R(S^1)_{I(g)}$-module map induced by $\tau$. From the localization theorem in topological K-theory \cite[Theorem 1.1]{5}, $\tau |_K$ is an isomorphism. By the five lemma, we know that $\tau^*$ is an isomorphism.

The proof of Theorem 3.10 is completed.

In the following definition, we adapt the notations in Section 1.4.

**Definition 3.11.** The direct image map
\begin{equation}
(3.53) \quad \hat{i}^*: \widehat{K}^0_g (Y^{S^1})_{I(g)} \to \widehat{K}^0_{g} (Y)_{I(g)}
\end{equation}
is defined by
\begin{equation}
(3.54) \quad \hat{i}^*([\mu, \phi]/\chi) = [\xi_+, \chi_g (\Lambda^\text{even} (N^*)) \wedge \phi]/\chi - [\xi_-, \chi_g (\Lambda^\text{odd} (N^*)) \wedge \phi]/\chi.
\end{equation}
Theorem 3.12. The direct image map $\hat{i}_*$ is a well-defined isomorphism and

$$(3.55) \quad \hat{i}^* \circ \hat{i}_* = \left[\lambda_{-1}(N^*), 0\right] \cup: \hat{K}_g^0(Y^{S_1})_{I(g)} \xrightarrow{\sim} \hat{K}_g^0(Y^{S_1})_{I(g)}.$$

Proof. From the construction of $\xi_{\pm}$ in (1.42) and (1.43), we have

$$\begin{align*}
\hat{i}^* \left\{ \left[\xi_+, \hat{\chi}_g(\Lambda^{\text{even}}(N^*)) \wedge \phi\right]/\chi - \left[\xi_-, \hat{\chi}_g(\Lambda^{\text{odd}}(N^*)) \wedge \phi\right]/\chi \right\} &= \left( \left(\Lambda^{\text{even}}(N^*) \otimes \mu_\wedge \right) + \mathcal{F}, \hat{\chi}_g(\Lambda^{\text{even}}(N^*)) \wedge \phi \right)/\chi \\
&\quad - \left( \left(\Lambda^{\text{odd}}(N^*) \otimes \mu_\wedge \right) + \mathcal{F}, \hat{\chi}_g(\Lambda^{\text{odd}}(N^*)) \wedge \phi \right)/\chi \\
&= \left( \left(\Lambda^{\text{even}}(N^*) \otimes \mu_\wedge \right) + \mathcal{F}, \hat{\chi}_g(\Lambda^{\text{even}}(N^*)) \wedge \phi \right)/\chi - \left( \left(\Lambda^{\text{odd}}(N^*) \otimes \mu_\wedge \right) + \mathcal{F}, \hat{\chi}_g(\Lambda^{\text{odd}}(N^*)) \wedge \phi \right)/\chi.
\end{align*}$$

Since $\lambda_{-1}(N^*) = \Lambda^{\text{even}}(N^*) - \Lambda^{\text{odd}}(N^*)$, by Definition 2.12 and (3.56), we have

$$(3.57) \quad \hat{i}^* \left\{ \left[\lambda_0 \Lambda^{\text{even}}(N^*) \wedge \lambda_0 \phi\right]/\chi - \left[\lambda_0 \Lambda^{\text{odd}}(N^*) \wedge \lambda_0 \phi\right]/\chi \right\} = \hat{i}^* \left\{ \left[\lambda_{-1}(N^*) \wedge \lambda_0 \phi\right]/\chi - \left[\lambda_{-1}(N^*) \wedge \lambda_0 \phi\right]/\chi \right\}.$$  

Therefore, on the cycles, we have

$$(3.58) \quad \hat{i}^* \circ \hat{i}_* = \left[\lambda_{-1}(N^*), 0\right] \cup.$$

By Theorem 3.10, $\hat{i}^* : \hat{K}_g^0(Y)_{I(g)} \xrightarrow{\sim} \hat{K}_g^0(Y^{S_1})_{I(g)}$ is invertible. From Theorem 2.14

$$(3.59) \quad \hat{i}^* \circ \lambda_{-1}(N^*) = \lambda_{-1}(N^*) \cup: \hat{K}_g^0(Y^{S_1})_{I(g)} \xrightarrow{\sim} \hat{K}_g^0(Y)_{I(g)}$$

is a well-defined isomorphism. Equations (3.58) and (3.59) imply that $\hat{i}_*$ in (3.54) is a well-defined isomorphism and (3.55) holds. The proof of Theorem 3.12 is completed. \hfill \square

Proof of Theorem 3.5. By Theorem 1.5 (3.59), (3.11), (3.18) and (3.54), for any $[\lambda, \phi]/\chi \in \hat{K}_g^0(Y^{S_1})_{I(g)}$, we have

$$(3.60) \quad \hat{f}_{Y^*} \circ \hat{i}_*([\lambda, \phi]/\chi) = \hat{f}_{Y^*}\left\{ \left[\xi_+, \hat{\chi}_g(\Lambda^{\text{even}}(N^*)) \wedge \phi\right]/\chi - \left[\xi_-, \hat{\chi}_g(\Lambda^{\text{odd}}(N^*)) \wedge \phi\right]/\chi \right\} = \chi(g)^{-1} \left\{ \int_{Y^{S_1}} \text{Td}_g(\nabla^{TY}, \nabla^L) \hat{\chi}_g(\Lambda_{-1}(N^*)) \wedge \phi + \bar{\gamma}_g(TY, L, \xi_+) - \bar{\gamma}_g(TY, L, \xi_-) \right\} = \chi(g)^{-1} \sum_a \left\{ \int_{Y^{S_1}} \text{Td}_a(\nabla^{TY_a}, \nabla^L_a) \wedge \phi + \bar{\gamma}_a(TY_a^{S_1}, L_a, \mu) \right\} \mod \mathbb{Q}_g = \hat{f}_{Y^{S_1}}([\lambda, \phi]/\chi).$$

It means that

$$(3.61) \quad \hat{f}_{Y^*} \circ \hat{i}_* = \hat{f}_{Y^{S_1}} : \hat{K}_g^0(Y^{S_1})_{I(g)} \to \mathbb{C}/\mathbb{Q}_g.$$

From (3.55) and (3.61), we have

$$(3.62) \quad \hat{f}_{Y^*} = \hat{f}_{Y^{S_1}} \circ [\lambda_{-1}(N^*), 0]^{-1} \cup \hat{i}^* : \hat{K}_g^0(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g.$$

The proof of Theorem 3.5 is completed. \hfill \square
The case when $Y^S^1 = \emptyset$. In the reminder of this paper, we discuss the case when $Y^S^1 = \emptyset$.

As $Y^S^1 = \emptyset$, it is easy to see that $A = \{g \in S^1 : Y^g \neq \emptyset\}$ is a finite set. From the variation formula (1.31), for $g \in S^1 \setminus A$, up to $\mathbb{Q}_g$, $\bar{\eta}_g(TY, L, E)$ does not depend on the geometric data $g^TY$, $(h^L, \nabla^L)$, $(h^E, \nabla^E)$. Similarly as in Definition 3.1, the map $f_* : K^0_S(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$ defined by

$$f_*([E]/\chi) = \chi(g)^{-1} \bar{\eta}_g(TY, L, E) \mod \mathbb{Q}_g$$

is well-defined. By [3, Proposition 1.5], $K^0_S(Y)_{I(g)} = 0$ for $g \in S^1 \setminus A$. So $\bar{\eta}_g(TY, L, E) \in \mathbb{Q}_g$. Since Theorem 1.4 and 1.10 still hold for $Y^S^1 = \emptyset$, following the same process as in Lemma 3.7 and Proposition 3.8, we obtain that

**Theorem 3.13.** If $Y^S^1 = \emptyset$, $A = \{g \in S^1 : Y^g \neq \emptyset\}$, then $\bar{\eta}_g(TY, L, E)$ as a function on $S^1 \setminus A$ is the restriction of a rational function on $S^1$ with coefficients in $\mathbb{Z}$, and it has no poles on $S^1 \setminus A$.

**Example 3.14.** For $k \in \mathbb{N}^*$, we consider the circle action on $Y = \widehat{S}^1$ with

$$g.e^{i\theta} = e^{2\pi i k \phi + i \theta}, \quad \text{for } g = e^{2\pi i \phi} \in S^1.$$ 

Here $\widehat{S}^1$ is a copy of $S^1$. For $x = e^{i\theta} \in \widehat{S}^1$, if $g.x = x$, we have $k\phi \in \mathbb{Z}$, which means that $g^k = 1$. So $Y^S^1 = \emptyset$ and $A = \{g \in S^1 : g^k = 1\}$.

We identify $[0, 2\pi)$ with $\widehat{S}^1$ by sending $\theta$ to $e^{i\theta}$. Then the canonical metric on $\widehat{S}^1$ is defined by $|\frac{\partial}{\partial \theta}| = 1$, the spinor of $\widehat{S}^1$ is $S(\widehat{S}^1) = \mathbb{C}$, and the Clifford action is defined by $c\left(\frac{\partial}{\partial \theta}\right) = -i \in \text{End}(S(\widehat{S}^1))$. Thus the untwisted Dirac operator on $\widehat{S}^1$ is

$$D = -i \frac{\partial}{\partial \theta}. $$

From (3.64) and (3.65), we see that the circle action commutes with $D$. From (3.65), the eigenvalues of $D$ are $\lambda_n = n$, $n \in \mathbb{Z}$, with eigenspaces $E_{\lambda_n} = \mathbb{C}\{e^{in\theta}\}$. For $g = e^{2\pi i \phi} \in S^1$, $s \in \mathbb{C}$ and $\text{Re}(s) > 1$, we see that

$$\eta_g(s) := \sum_{n=1}^{+\infty} \frac{\text{Tr}[E_{\lambda_n}[g]}{n^s} - \sum_{n=1}^{+\infty} \frac{\text{Tr}[E_{\lambda_{-n}[g]}]}{n^s} = \sum_{n=1}^{+\infty} \frac{e^{2\pi i n k \phi}}{n^s} - \sum_{n=1}^{+\infty} \frac{e^{-2\pi i n k \phi}}{n^s}$$

is well-defined.

For $x, y \in \mathbb{R}$, $s \in \mathbb{C}$, let $S_1(x, y, s)$ be the Kronecker zeta function [52, p53],

$$S_1(x, y, s) = \sum_{n \in \mathbb{Z}}' (x + n)(x + n) - 2s e^{-2\pi iy},$$

where $\sum'_{n \in \mathbb{Z}}$ is a sum over $n \in \mathbb{Z}$, $n \neq -x$. The series in (3.67) converges absolutely for $\text{Re}(s) > 1$, and defines a holomorphic function of $s$. Moreover, $s \mapsto S_1(x, y, s)$ has a holomorphic continuation to $\mathbb{C}$ [52, p57]. By (3.66) and (3.67), we have

$$\eta_g(s) = -S_1\left(0, k\phi, \frac{s + 1}{2}\right).$$
Thus \( \eta_g(s) \) has a holomorphic continuation to \( \mathbb{C} \). Also by [52, p57], we have the functional equation for \( S_1(x, y, s) \),

\[
\Gamma(s) S_1(x, y, s) = -i \pi^{2s-3/2} e^{2\pi i xy} \Gamma \left( \frac{3}{2} - s \right) S_1 \left( y, -x, \frac{3}{2} - s \right).
\]

From (3.68) and (3.69), we have

\[
\eta_g(0) = \frac{i}{\pi} S_1(k \phi, 0, 1).
\]

By [52, p57], for \( x \notin \mathbb{Z} \), we have

\[
S_1(x, 0, 1) = \pi \cot(\pi x).
\]

If \( g \in S^1 \setminus A \), \( k \phi \notin \mathbb{Z} \). Thus by (3.64), (3.70) and (3.71), for \( g \in S^1 \setminus A \), we have

\[
\eta_g(\hat{S}^1) = \eta_g(0) = i \cot(\pi k \phi) = -\frac{g^{k/2} + g^{-k/2}}{g^{k/2} - g^{-k/2}} = \frac{2}{1 - g^k} - 1.
\]

Since \( \ker(D) \) is the space of the complex valued constant functions on \( \hat{S}^1 \), we have \( \text{Tr} |_{\ker(D)}[g] = 1 \). Thus from (3.72), for \( g \in S^1 \setminus A \), the reduced eta invariant

\[
\bar{\eta}_g(\hat{S}^1) = \frac{\eta_g(\hat{S}^1) + \text{Tr} |_{\ker(D)}[g]}{2} = \frac{1}{1 - g^k}.
\]

It is a rational function on \( S^1 \) with coefficients in \( \mathbb{Z} \) and poles in \( A \).

For \( g \in A \), then from (3.66), we get \( \eta_g(s) = 0 \), for any \( s \in \mathbb{C} \). Since \( \text{Tr} |_{\ker(D)}[g] = 1 \),

\[
\bar{\eta}_g(\hat{S}^1) = \bar{\eta}_1(\hat{S}^1) = \bar{\eta}(\hat{S}^1) = \frac{1}{2}, \quad \text{for } g \in A.
\]

Remark that this reduced eta invariant could also be computed from the equivariant APS index theorem (0.19). Let \( B \) be the unit disc in \( \mathbb{R}^2 \) and \( \partial B = \hat{S}^1 \). In polar coordinate, the circle action on \( B \) is defined by

\[
g(\rho e^{i \theta}) = r e^{2 \pi i k \phi + i \theta}, \quad \text{for } g = e^{2 \pi i \phi}, \quad k \in \mathbb{N}^*.
\]

It induces the circle action on \( \partial B = \hat{S}^1 \) in (3.61). If \( g \in S^1 \setminus A \), then \( g^k \neq 1 \) and the only fixed point set of \( g \) on \( B \) is \( B^g = \{0\} \). Let \( N \) be the normal bundle of \( \{0\} \) in \( B \). Then the \( g \)-action on \( N \) is as multiplication by \( g^k \).

We take the metric on \( B \) such that it is of product near the boundary, induces the canonical metric on \( \hat{S}^1 \) and commutes with the circle action. We denote by \( D^B_{+, \geq 0} \) the Fredholm operator with respect to the Dirac operator \( D^+_{S^1} \) on \( B \) and the APS boundary condition. The equivariant APS index is defined by

\[
\text{Ind}_{APS, g}(D^B) = \text{Tr} |_{\ker(D^B_{+, \geq 0})}[g] - \text{Tr} |_{\coker(D^B_{+, \geq 0})}[g].
\]

From the equivariant APS index theorem, we have for \( g \in S^1 \setminus A \),

\[
\bar{\eta}_g(\hat{S}^1) = \int_{B^g} \frac{1}{\det(1 - g^k[N])} - \text{Ind}_{APS, g}(D^B_{+}) = \frac{1}{1 - g^k} - \text{Ind}_{APS, g}(D^B_{+}).
\]

Note that the equivariant APS index is invariant when the metric varies near the boundary, without changing the metrics on the boundary. We only need to compute \( \text{Ind}_{APS, g}(D^B_{+}) \) using
the canonical metric on $B$. In this case, with the coordinate $z = x + iy$ in $\mathbb{R}^2 \simeq \mathbb{C}$, the spinor of $B$ is $\mathcal{S}(B) = \mathbb{C} \oplus \mathbb{C}(\partial z/\sqrt{2}) \simeq \mathbb{C} \oplus \mathbb{C}$ and the Clifford action is given by

$$c \left( \frac{\partial}{\partial x} \right) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c \left( \frac{\partial}{\partial y} \right) := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.78}$$

Thus $D^B = c(\frac{\partial}{\partial x}) \frac{\partial}{\partial x} + c(\frac{\partial}{\partial y}) \frac{\partial}{\partial y}$ has the form

$$D^B = \begin{pmatrix} 0 & D^B_x \\ D^B_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} + \frac{-\partial}{\partial z} \frac{i}{\sqrt{2}} \\ \frac{\partial}{\partial y} -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \tag{3.79}$$

In polar coordinates,

$$D^B_+ = e^{i\theta} \left( \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right). \tag{3.80}$$

Note that $-\frac{\partial}{\partial r}$ here is the inward normal vector. Let $P_{\geq 0}$ be the orthogonal projection onto the direct sum of the eigenspaces associated with the nonnegative eigenvalues of $A := -\frac{i}{r} \frac{\partial}{\partial \theta} |_{\partial B} = D$. Recall that the eigenvalues of $A$ are $\lambda_n = n, n \in \mathbb{Z}$, with eigenspaces $\mathbb{C}\{e^{in\theta}\}$. Thus in complex coordinates, for $f \in \mathcal{C}^\infty(B, \mathbb{C})$, the APS boundary condition

$$P_{\geq 0}f|_{\partial B} = 0 \iff f|_{\partial B} = \sum_{n < 0, n \in \mathbb{Z}} a_n z^n. \tag{3.81}$$

If $D^B_+ f = 0, f$ is holomorphic on $B$. Thus

$$\operatorname{Ker}(D^B_{+, \geq 0}) = \{ f \in \mathcal{C}^\infty(B, \mathbb{C}) : D^B_+ f = 0, P_{\geq 0}(f|_{\partial B}) = 0 \} = 0. \tag{3.82}$$

By (3.80), the adjoint of $D^B_{+, \geq 0}$ is $D^B_{-, < 0}$, i.e., $D^B$ with boundary condition

$$P_{< 0}(e^{-i\theta} f)|_{\partial B} = 0, \text{ with } P_{\geq 0} + P_{< 0} = \text{Id on } \mathcal{C}^\infty(\partial B, \mathbb{C}). \tag{3.83}$$

By (3.79) and (3.83), if $D^B_{-, < 0} f = 0$, then $\tilde{f}$ is holomorphic on $B$ and $f|_{\partial B} = \sum_{n > 0, n \in \mathbb{Z}} a_n z^n$. Thus $\operatorname{Coker}(D^B_{+, \geq 0}) = \operatorname{Ker}(D^B_{-, < 0}) = 0$. From (3.78), (3.77) and (3.52), we have $\bar{g}(S^1) = (1 - g^k)^{-1}$ for $g \in S^1 \setminus A$, which is the same as (5.73).

References

[1] M. F. Atiyah, K-theory, Lecture notes by D. W. Anderson, W. A. Benjamin, Inc., New York-Amsterdam, 1967.
[2] M. F. Atiyah and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds, Bull. Amer. Math. Soc. 65 (1959), 276–281.
[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry. I, Math. Proc. Cambridge Philos. Soc. 77 (1975), 43–69.
[4] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry. III, Math. Proc. Cambridge Philos. Soc. 79 (1976), 71–99.
[5] M. F. Atiyah and G. B. Segal, The index of elliptic operators. II, Ann. of Math. (2) 87 (1968), 531–545.
[6] M. F. Atiyah and I. M. Singer, The index of elliptic operators on compact manifolds, Bull. Amer. Math. Soc. 69 (1963), 422–433.
[7] M. F. Atiyah and D. O. Tall, Group representations, λ-rings and the J-homomorphism, Topology 8 (1969), 253–297.
[8] N. Berline, E. Getzler, and M. Vergne, Heat kernels and Dirac operators, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original.
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[9] P. Berthelot, A. Grothendieck, and L. Illusie, *Théorie des Intersections et Théorème de Riemann-Roch (SGA6)*, Lecture Notes in Mathematics., vol. 225, Springer-Verlag, Berlin, 1971.

[10] J.-M. Bismut, *Equivariant immersions and Quillen metrics*, J. Differential Geom. 41 (1995), no. 1, 53–157.

[11] J.-M. Bismut, *Holomorphic families of immersions and higher analytic torsion forms*, Astérisque (1997), no. 244, viii+275.

[12] J.-M. Bismut and J. Cheeger, $\eta$-invariants and their adiabatic limits, J. Amer. Math. Soc. 2 (1989), no. 1, 33–70.

[13] J.-M. Bismut and D. Freed, *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem*, Comm. Math. Phys. 107 (1986), no. 1, 103–163.

[14] J.-M. Bismut and S. Goette, *Holomorphic equivariant analytic torsions*, Geom. Funct. Anal. 10 (2000), no. 6, 1289–1422.

[15] J.-M. Bismut and S. Goette, *Equivariant de Rham torsions*, Ann. of Math. (2) 159 (2004), no. 1, 53–216.

[16] J.-M. Bismut and D. Freed, *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem*, Comm. Math. Phys. 107 (1986), no. 1, 103–163.

[17] J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp.

[18] J.-M. Bismut and W. Zhang, *Real embeddings and $\eta$-invariants*, Math. Ann. 295 (1993), no. 4, 661–684.

[19] J.-M. Bismut and K. Köhler, *Higher analytic torsion forms for direct images and anomaly formulas*, J. Algebraic Geom. 1 (1992), no. 4, 647–684.

[20] U. Bunke and T. Schick, *Smooth $K$-theory*, Astérisque (2009), no. 328, 45–135 (2010).

[21] U. Bunke and T. Schick, *Differential orbifold $K$-theory*, J. Noncommut. Geom. 7 (2013), no. 4, 1027–1104.

[22] J. Cheeger and J. Simons, *Differential characters and geometric invariants*. volume 1167 of Lecture Notes in Math., pages 50–80. Springer, Berlin, 1985.

[23] X. Dai and W. Zhang, *Real embeddings and the Atiyah-Patodi-Singer index theorem for Dirac operators*, Asian J. Math. 4 (2000), no. 4, 775–794.

[24] H. Donnelly, *Eta invariants for $G$-spaces*, Indiana Univ. Math. J. 27 (1978), no. 6, 889–918.

[25] H. Feng, G. Xu, and W. Zhang, *Real embeddings, $\eta$-invariant and Chern-Simons current*, Pure Appl. Math. Q. 5 (2009), no. 3, 1113–1137.

[26] D. S. Freed and J. Lott, *An index theorem in differential $K$-theory*, Geom. Topol. 14 (2010), no. 2, 903–966.

[27] E. Getzler, *The odd Chern character in cyclic homology and spectral flow*, Topology 32 (1993), no. 3, 489–507.

[28] H. Gillet, D. Roessler, and C. Soulé, *An arithmetic Riemann-Roch theorem in higher degrees*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 6, 2169–2189.

[29] H. Gillet and C. Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metric. II*, Ann. of Math. (2) 131 (1990), no. 1, 205–238.

[30] H. Gillet and C. Soulé, *An arithmetic Riemann-Roch theorem*, Invent. Math. 110 (1992), no. 3, 473–543.

[31] H. Gillet and C. Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metric. II*, Ann. of Math. (2) 131 (1990), no. 1, 205–238.

[32] S. Goette, *Equivariant $\eta$-invariants and $\eta$-forms*, J. Reine Angew. Math. 526 (2000), 181–236.

[33] S. Goette and D. Roessler, *A fixed point formula of Lefschetz type in Arakelov geometry. I. Statement and proof*, Invent. Math. 145 (2001), no. 2, 333–396.

[34] S. Goette and D. Roessler, *A fixed point formula of Lefschetz type in Arakelov geometry. II. A residue formula*, Ann. Inst. Fourier (Grenoble) 52 (2002), no. 1, 81–103.

[35] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.

[36] B. Liu, *Equivariant eta forms and equivariant differential $K$-theory*, arXiv: 1610.02311 (2016).

[37] B. Liu, *Real embedding and equivariant eta forms*, arXiv:1706.07121 (2017). Math. Z. to appear.

[38] B. Liu and X. Ma, *Differential $K$-theory and localization formula for $\eta$-invariants*, C. R. Math. Acad. Sci. Paris, to appear.

[39] B. Liu and X. Ma, *Comparison of two equivariant eta forms*, arXiv: 1808.0...., to appear.
[40] K. Liu, X. Ma, and W. Zhang, Spin$^c$ manifolds and rigidity theorems in $K$-theory, Asian J. Math. 4 (2000), no. 4, 933–959.

[41] K. Liu, X. Ma, and W. Zhang, Rigidity and vanishing theorems in $K$-theory, Comm. Anal. Geom. 11 (2003), no. 1, 121–180.

[42] X. Ma and G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007.

[43] V. Maillot and D. Roessler. On the periods of motives with complex multiplication and a conjecture of Gross-Deligne. Ann. of Math. (2), 160 (2004), 727–754.

[44] M. S. Narasimhan and S. Ramanan, Existence of universal connections, Amer. J. Math. 83 (1961), 563–572.

[45] D. B. Ray and I. M. Singer, $R$-torsion and the Laplacian on Riemannian manifolds, Advances in Math. 7 (1971), 145–210.

[46] D. B. Ray and I. M. Singer, Analytic torsion for complex manifolds, Ann. of Math. (2) 98 (1973), 154–177.

[47] D. Roessler, An Adams-Riemann-Roch theorem in Arakelov geometry, Duke Math. J. 96 (1999), no. 1, 61–126.

[48] D. Roessler, Lambda-structure on Grothendieck groups of Hermitian vector bundles, Israel J. Math. 122 (2001), 279–304.

[49] G. Segal, Equivariant $K$-theory, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 129–151.

[50] C. Soulé, Lectures on Arakelov geometry, Cambridge Studies in Advanced Mathematics, vol. 33, Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[51] S. Tang, Concentration theorem and relative fixed point formula of Lefschetz type in Arakelov geometry, J. Reine Angew. Math. 665 (2012), 207–235.

[52] A. Weil, Elliptic functions according to Eisenstein and Kronecker, Erg. Math. Grenzg. 88. Berlin-Heidelberg-New-York: Springer 1976.

[53] W. Zhang, A note on equivariant eta invariants, Proc. Amer. Math. Soc. 108 (1990), no. 4, 1121–1129.

[54] W. Zhang, Lectures on Chern-Weil theory and Witten deformations, Nankai Tracts in Mathematics, vol. 4, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[55] W. Zhang, η-invariant and Chern-Simons current. Chinese Ann. Math. Ser. B 26 (2005), no. 1, 45–56.

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