Twisted $N = 4$ SUSY Algebra in Topological Models of Schwarz Type

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Abstract

We reinvestigate the twisted $N = 4$ supersymmetry present in Schwarz type topological field models. We show that Chern-Simons theory in three dimensions can be untwisted to a kind of sigma-model with reversed statistics only in the free case. By dimensional reduction we define then the two-dimensional BF-model. We establish an analog result concerning the untwisting. As a consequence of the definition through dimensional reduction we find new fermionic scalar symmetries that have been overlooked so far in the literature.

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\footnotesize

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1 Introduction

Topological field models have been a subject of high interest in the last years. One of the highlights is the presence of a particular kind of supersymmetry. The supercharges in the conventional case transform as spinors under the rotation group. In topological models however, they transform as scalar and vectors, giving rise to an algebra of the form

$$\{ s, \delta_\mu \} = \partial_\mu .$$  

(1)

This is usually called topological supersymmetry.

There are two different kinds of topological field theories. The first one is called Witten-type, and the prime example is topological Yang-Mills theory in four dimensions \[1\]. Here one starts with \(N = 2\) super Yang-Mills theory. The \(N = 2\) superalgebra has an \(SU(2)_{\text{left}} \otimes SU(2)_{\text{right}} \otimes SU(2)_{\text{I}}\) automorphism algebra. The rotation group \(SO(4)\) is given by the product of the first two \(SU(2)\) factors and \(SU(2)_{\text{I}}\) is an internal symmetry. The supercharges transform as \((\frac{1}{2}, 0, \frac{1}{2})\) and \((0, \frac{1}{2}, \frac{1}{2})\). In the topologically twisted theory one takes as a new rotation group the diagonal subgroup of \(SU(2)_{\text{left}} \otimes SU(2)_{\text{I}}\) times \(SU(2)_{\text{left}}\). The supercharges transform as \((0, 0), (3, 0)\) and \((2, 2)\) under this new rotation group. The singlet is a fermionic, nilpotent symmetry and constitutes a part of the BRS-operator of the theory. Together with the vector \[2\] it gives rise to an algebra of the form \(1\).

The other kind of topological field theory is called Schwarz-type. Here one starts with a Lagrangian that is an \(n\)-form, such that it can be integrated over a \(n\)-dimensional manifold without the need of a metric. Examples are Chern-Simons theories in \((2m + 1)\) dimensions and BF-models in arbitrary dimensions. The Lagrangian typically has a gauge symmetry. So upon quantization one is forced to introduce a gauge fixing \(\text{à la} \text{BRS}\). The action is written as an invariant part plus the gauge fixing part. The gauge fixing part depends on the metric and can be written as a BRS-variation. It turns out then, that the gauge fixing procedure gives rise to a fermionic vector symmetry that together with the BRS-operator obeys an algebra of the form \(1\). This vector supersymmetry has also played an important role in the algebraic study of field theories \[3\]. In particular it has given rise \(\text{e.g.}\) to a finiteness prove of Chern-Simons theory \[4\].

In Schwarz-type topological models the algebra \(1\) does not come through a twisted supersymmetric structure but rather expresses the triviality of translation as a consequence of the topological nature of the theory. Nevertheless it can sometimes be completed to an extended (twisted) supersymmetry. We will demonstrate this on the example of Chern-Simons theory. Using the fact that the gauge fixing term can not only be written as a BRS-variation but also as an anti-BRS-variation we derive a twisted \(N = 4\) algebra in gauge fixed Chern-Simons theory. With somewhat different methods this result has been established also in \[5\]. Then we investigate the possibility of defining Chern-Simons theory as a topological field theory of Witten-type\[6\]. This means that we look for a kind of sigma-model action whose topologically twisted version is gauge fixed Chern-Simons theory. We find that this is possible only in the case with vanishing interaction terms. The two-dimensional BF-model is then defined through dimensional reduction from the Chern-Simons action. In this way we are also able to establish a twisted \(N = 4\) supersymmetry for this model. As part of this we find two new fermionic, nilpotent (pseudo-)scalar symmetries, that have been overlooked so far in the literature. We also investigate the question of untwisting two-dimensional BF-model finding again that it is possible only in the non-interacting case.

\[4\] Often the action of Witten-type models can be written as a BRS-variation. This is not always so as the example of the twisted sigma-model in three dimension as defined in \[6\] shows.
2 The $N=4$ supersymmetry algebra in $D=3$

The $N=4$ supersymmetry algebra in Euclidean $D=3$ assuming \textit{a priori} the absence of central charges, is given by:

\[
\{Q_\alpha^{A}, Q_\beta^{B}\} = 2i\epsilon^{AB}\epsilon_{\mu\alpha\beta}\partial_\mu .
\]

(2)

On this algebra acts an $SU(2)_{Spin} \otimes SU(2)_{I} \otimes SU(2)_{II}$. The supercharges transform as $(\frac{1}{2}, \frac{1}{2}, 1)$. Spinor indices are denoted by Greek letters ($\alpha, \beta, \gamma, \delta = 1, 2$), and the isospin indices for $SU(2)_{I}$ by capital Latin letters, the ones for $SU(2)_{II}$ by capital, barred Latin letters. The algebra can be obtained by dimensional reduction of the $N=2$ algebra in four dimensions. Let us describe now how the topological twist acts. We chose as the new rotation group the diagonal subgroup of $SU(2)_{Spin} \otimes SU(2)_{I}$. For practical purposes this is done by identifying the capital Latin indices with the Greek indices $A \rightarrow \alpha, \cdots$. So the indices for the new rotation group are also denoted by Greek letters. The Euclidean space indices are denoted by the Greek letters ($\mu, \nu, \rho, \sigma = 1, 2, 3$). When we decompose the generators of (2) into irreducible representations of the new rotation group with can define the operators $s^A$ and $\delta_\mu^A$:

\[
s^A = \left( \begin{array}{c} s \\ \bar{s} \end{array} \right) \equiv \frac{1}{2} \delta_\alpha^\beta Q_\beta^{A, \alpha} \quad (2 \text{ d.f. from } \text{Tr}(Q^A)) ;
\]

(3.a)

\[
\delta_\mu^A = \left( \begin{array}{c} \delta_\mu \\ \bar{\delta}_\mu \end{array} \right) \equiv -i \frac{2}{2} \sigma_{\mu\alpha} Q_\alpha^{A, \alpha} \quad (6 \text{ d.f. from } \text{Tr}(\sigma_{\mu}Q^A)) .
\]

(3.b)

The definitions given above (eqs. (3.a, 3.b)) together with the algebra (2) (considering the conventions presented in the Appendix), yield the anticommutation relations displayed below:

\[
\{s^A, s^B\} = 0 , \quad \{s^A, \delta_\mu^B\} = -\epsilon^{AB}\partial_\mu \quad \text{and} \quad \{\delta_\mu^A, \delta_\nu^B\} = \epsilon^{AB}\varepsilon_{\mu\nu\rho}\partial_\rho ,
\]

(4)

which give the following algebra:

\[
\{s, s\} = \{\bar{s}, s\} = \{s, \bar{s}\} = \{\bar{s}, \bar{s}\} = 0 ;
\]

\[
\{s, \delta_\mu\} = \partial_\mu , \quad \{\bar{s}, \bar{\delta}_\mu\} = -\partial_\mu ;
\]

\[
\{\delta_\mu, \delta_\nu\} = 0 , \quad \{\delta_\mu, \bar{\delta}_\nu\} = \varepsilon_{\mu\nu\rho}\partial_\rho ;
\]

\[
\{\delta_\mu, \bar{\delta}_\nu\} = -\varepsilon_{\mu\nu\rho}\partial_\rho , \quad \{\delta_\mu, \bar{\delta}_\nu\} = 0 .
\]

(5)

As it will be shown afterwards, the algebra (2) fulfilled by the operators $s, \bar{s}, \delta_\mu$ and $\bar{\delta}_\mu$, is exactly the same satisfied by the BRS and the anti-BRS (BRS) operators, together with their respective topological vector supersymmetries, in the case of the Chern-Simons model in a three-dimensional Euclidean space.

Now, it is possible to write the supercharges $Q_\alpha^{A, \beta}$ in terms of $s, \bar{s}, \delta_\mu$ and $\bar{\delta}_\mu$. By solving the equations (3.a) and (3.b), it is easily shown that the generators $Q_\alpha^{A, \beta}$ and $Q_\alpha^{A, -\beta}$ read

\[
Q_\alpha^{A, \beta} = \delta_\alpha^A s + i\sigma_{\mu\alpha}^{A} \bar{\delta}_\mu \quad \Rightarrow \quad \cancel{Q}^+ = 1s + i\sigma_{\mu}^{A} \bar{\delta}_\mu ,
\]

(6.a)

\[
Q_\alpha^{A, -\beta} = \delta_\alpha^A \bar{s} + i\sigma_{\mu\alpha}^{A} \delta_\mu \quad \Rightarrow \quad \cancel{Q}^- = 1\bar{s} + i\sigma_{\mu}^{A} \delta_\mu .
\]

(6.b)

\footnote{For conventions and notations adopted throughout the work see the Appendix.}
In a further Section it will be seen that, in the case of Chern-Simons, the supersymmetry charges $Q^+$ and $Q^-$ have the ghost numbers $+1$ of the BRS operator ($s$) and $-1$ of the BRS operator ($\bar{s}$). Therefore, it means that, in the case of Chern-Simons, these supersymmetry generators relates fields (in a non-linear way) with ghosts numbers differed by a unity. It should be stressed that the Chern-Simons model has actually a $N = 4$ supersymmetry, but in this case the “superpartners” are not related anymore by a half spin, in fact, they are related by a unity of ghost number.

3 The operators $Q^+$ and $Q^-$ in topological models

In this Section, we consider the inverse problem by supposing a topological model (Chern-Simons, as a particular case) which the BRS, the $\bar{s}$ and their respectives topological symmetries satisfy the algebra given by eq.(5).

3.1 The inverse problem

Assuming the algebra (5) and defining the $Q^+$ and $Q^-$ operators as in eqs.(6.a–6.b), it can be easily checked that their anti-commutation relations read

$$\{Q^+, Q^+\} = 0 \text{ and } \{Q^-, Q^-\} = 0 ;$$

$$\{Q^+, Q^-\} = \left[ i(\mathbb{I} \wedge \sigma_\mu)\partial_\mu + \frac{1}{2}(\sigma_\mu \wedge \sigma_\nu)\varepsilon_{\mu\nu\rho}\partial_\rho \right] = -2ie^{\gamma\delta}\sigma_{\mu\alpha}\partial_\mu ;$$

$$\{Q^-, Q^+\} = \left[ -i(\mathbb{I} \wedge \sigma_\mu)\partial_\mu + \frac{1}{2}(\sigma_\mu \wedge \sigma_\nu)\varepsilon_{\mu\nu\rho}\partial_\rho \right] = 2ie^{\gamma\delta}\sigma_{\mu\alpha}\partial_\mu ,$$

(7.a)

(7.b)

(7.c)

where the $N = 4$ supersymmetry charges $Q^+$ and $Q^-$ are nilpotent operators with ghost numbers $+1$ and $-1$, respectively. Therefore, a two new nilpotent operators appear, in a natural way, as a by-product of the topological nature of a model and its $N = 4$ supersymmetry.

3.2 The Chern-Simons model

Here, our goal, is to show that the Chern-Simons model has a $N = 4$ supersymmetry as a consequence of being a topological model, as well as, to find its untwisted version with the help of the nilpotent operators (6.a–6.b).

The Chern-Simons action is given by

$$\Sigma_{CS} = \frac{k}{4\pi} \text{Tr} \int d^3x \varepsilon_{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) ,$$

(8)

where the parameter $k$ is quantized $[1]$ and $A_\mu$ is a Lie algebra valued gauge field, as well as all the fields we shall introduce later on. Denoting such a generic field by $\phi$, we define

$$\phi(x) \equiv \phi^a(x)\tau_a ,$$

(9)

where the matrices $\tau$ are the generators of the group and obey

$$[\tau_a, \tau_b] = f_{abc}\tau_c \text{ and } \text{Tr}(\tau_a\tau_b) = \frac{1}{2}\delta_{ab} .$$

(10)

The gauge transformations read

$$\delta A_\mu = -(\partial_\mu \omega + [A_\mu, \omega]) \equiv -D_\mu \omega ,$$

(11)
with $\omega \equiv \omega^a \tau_a$. The gauge group is chosen to be simple and compact. These transformations change the integrand of the action (8) into a total derivative, leaving thus the action invariant if there are no boundary contributions and if the topology is trivial [1].

The gauge fixing is of the Landau-type. It is implemented by adding to the gauge invariant action (8) the term

$$\Sigma_{gf} = \text{Tr} \int d^3x \ (b \partial_\mu A_\mu + \bar{c} \partial_\mu D_\mu c) ,$$

where the Lie algebra valued fields $c$, $\bar{c}$ and $B$, are the ghost, the antighost and a Lagrange multiplier, respectively.

The gauge fixing action (12) can also be written either as a BRS-variation or as a BRS-variation:

$$\Sigma_{gf} = \begin{cases} s \text{ Tr} \int d^3x \ (\bar{c} \partial_\mu A_\mu) \\ -\bar{s} \text{ Tr} \int d^3x \ (c \partial_\mu A_\mu) \end{cases} .$$

Therefore, the action $\Sigma_{inv} = \Sigma_{CS} + \Sigma_{gf}$ is invariant under the following BRS and BRS transformations:

$$\begin{align*}
sA_\mu &= -D_\mu c , & s\bar{A}_\mu &= -D_\mu \bar{c} , \\
sc &= c^2 , & s\bar{c} &= \bar{c}^2 , \\
s\bar{c} &= b , & s\bar{c} &= \{\bar{c}, c\} - b , \\
sb &= 0 , & \bar{s}b &= [\bar{c}, b] .
\end{align*}$$

Since the action $\Sigma_{inv}^{(3)}$ depends on the metric only through the gauge fixing part (13), which is a BRS or a BRS-variation, the energy-momentum tensor will be a BRS or a BRS-variation as well:

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta \Sigma_{gf}}{\delta g_{\mu\nu}} \bigg|_{g_{\mu\nu} = \delta_{\mu\nu}} = \begin{cases} sA_{\mu\nu} \\
\bar{s}A_{\mu\nu} \end{cases} .$$

Beyond the original topological vector supersymmetry ($\delta_\mu$) associated to the BRS symmetry [7, 4], from eq.(13), we can infer about the existence of another topological vector supersymmetry associated to the BRS symmetry ($\bar{\delta}_\mu$), since, due to the gauge fixing (13), the energy-momentum tensor can also be written as a BRS-variation. By using the same approach to the usual case[3], it can be found that the action $\Sigma_{inv}^{(3)}$ is invariant under the following topological vector supersymmetries:

$$\begin{align*}
\delta_\mu A_\nu &= \frac{2\pi}{k} \epsilon_{\mu\nu\rho} \partial_\rho \bar{c} , & \bar{\delta}_\mu A_\nu &= \frac{2\pi}{k} \epsilon_{\mu\nu\rho} \partial_\rho c , \\
\delta_\mu c &= -A_\mu , & \bar{\delta}_\mu \bar{c} &= A_\mu , \\
\delta_\mu b &= \partial_\mu \bar{c} , & \bar{\delta}_\mu b &= D_\mu c , \\
\delta_\mu \bar{c} &= 0 , & \bar{\delta}_\mu c &= 0 .
\end{align*}$$

It should be stressed that the BRS ($s$) and the BRS ($\bar{s}$), eq.(14), together with their respective topological vector supersymmetries ($\delta_\mu$) and ($\bar{\delta}_\mu$), eq.(16), form the algebra given by eq.(5), which closes up to contact-terms.

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6For details concerning the topological vector supersymmetry see [3, 8, 9] and references therein.
3.3 Untwisting Chern-Simons

Given the fact that the Chern-Simons model with Landau-type gauge fixing possesses a twisted $N = 4$ supersymmetry it is tempting to speculate if it can be defined as a topological model of Witten-type. This would imply that there exists a kind of sigma-model with $N = 4$ supersymmetry, whose twisted version would be the gauge-fixed Chern-Simons action. Actually it would be a sigma-model of reversed statistics, since the scalar fields should be related to the ghosts of the Chern-Simons theory. On the other hand the spinor fields would be bosons since they should, after twisting, give the gauge field $A_\mu$. Such a spinor field can be defined using the bosonic fields of the gauge-fixed Chern-Simons model as follows:

$$\chi^A_\alpha = \frac{1}{2} \left( \delta^A_\alpha \sqrt{\frac{2\pi}{k}} b - i\sigma^A_\mu \sqrt{\frac{2\pi}{k}} A_\mu \right),$$

$$b = \sqrt{\frac{k}{2\pi}} \text{Tr}(\chi) \quad \text{and} \quad A_\mu = i\sqrt{\frac{2\pi}{k}} \text{Tr}(\sigma^\mu_\chi). \quad (17)$$

According to our index conventions it is made explicit that we take $\chi$ to transform as $(\frac{1}{2}, \frac{1}{2})$ under the $SU(2)_{\text{Spin}} \otimes SU(2)_I$ whereas $b$ and $A_\mu$ transform as a singlet and a triplet under the diagonal subgroup. Of course this is precisely how the topological twist works in three dimensions. Let us study now a kinetic term making the $SU(2)_{\text{Spin}} \otimes SU(2)_I \otimes SU(2)_{II}$ structure of the $N = 4$ algebra explicit. Demanding invariance under this product group fixes the kinetic terms to be

$$S_{\text{kin}} = \text{Tr} \int d^3 x \left( i\epsilon^A_{\mu\nu\rho} \chi^A_\alpha \sigma^\mu_\mu \chi^B_\beta + \frac{1}{2} \epsilon^{AB} c^A c^B \right), \quad (18)$$

where we defined $c^A = (\bar{c}, c)$. Using (17) one sees that this indeed reproduces the terms containing derivatives of the Chern-Simons action. In particular we also obtain the gauge fixing term $b\partial_\mu A_\mu$! Non-Abelian Chern-Simons theory also contains interaction terms. The question is now if one can also obtain these Chern-Simons interactions from the twisted version of some interactions between the spinors $\chi^A_\alpha$. From the form of the interaction

$$S_{\text{int}} = \text{Tr} \int d^3 x \left[ \frac{k}{6\pi} \varepsilon_{\mu\nu}\partial_\mu A_\mu A_\nu + \bar{c} \partial_\mu ([A_\mu, c]) \right] \quad (19)$$

it is clear that one will need the product of three $\chi^A_\alpha$’s to produce the term with three gauge fields. Let us concentrate now on the $SU(2)_{\text{Spin}}$-group, under which $\chi$ transforms as spin $\frac{1}{2}$ representation. The product of three of these fields transforms as $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$ which decomposes as $\frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$. Notice that there is no scalar part in this decomposition. This means that we can not form a term invariant under rotations with three spinors. This would be needed however to write down an interaction term corresponding to the Chern-Simons interaction. A similar consideration applies to the $SU(2)_I$ symmetry. This argument shows that it is impossible to untwist Chern-Simons theory. Several remarks are in order now. The possibility of untwisting Chern-Simons based on the twisted $N = 4$ algebra has been considered before in [11]. There

\[\text{We think of the fields corresponding to differential forms as being complexified. This is necessary since in three dimensions with Euclidean-signature we do not have Majorana spinors. As explained in [10] going from the untwisted to the twisted form of the Lagrangian also includes a restriction to real fields. Similarly, after defining the (untwisted) spinor fields we could go to Minkowski-signature and declare the spinors to be Majorana. Here we take the twist as a rather formal manipulation and ignore these subtleties in the following.}\]
the authors started however with a three dimensional $N = 4$ sigma-model with conventional statistics. The twist had to be supplemented by a change of statistics of the fields and the substitutions of an $\epsilon_{AB}$ instead of an $\delta_{AB}$ in the kinetic term of the fermions. As was shown here it is much more natural to start from the beginning with a sigma-model of reversed statistics since one automatically deals then with the “correct” kinetic term for the spin $1/2$ fields. Furthermore, in [11] the non-Abelian interaction terms have not been addressed. On the other hand we could give a fairly general group-theoretical argument showing that it is in fact impossible to obtain non-Abelian Chern-Simons theory by twisting some kind of sigma-model action. More generally, the study of sigma-models with reversed statistics might be an interesting area for future research. Of course the actual sigma-model should then not be taken seriously as a starting point for quantization. What we have in mind is that the topologically twisted versions of such sigma-models could give interesting analogs of Chern-Simons theory in three dimensions.

4 The dimensional reduction to $D = 2$

In this Section we perform a dimensional reduction à la Scherk of the algebra presented in Section 2 as well as the Chern-Simons model and its symmetries displayed in Section 3. In the latter case, such a dimensional reduction gives rise to the BF-model in $D = 2$ and its respectives symmetries. A new hidden symmetry of the BF-model arises as a direct consequence of the dimensional reduction of the topological vector supersymmetry generator.

4.1 Dimensional reducing the algebra

By dimensional reducing the twisted algebra (2) giving

$$\{Q^{A}_{\alpha}, Q^{B}_{\beta}\} = 2i\gamma^{\delta} \epsilon^{AB} \sigma_{m\alpha\beta} \partial_{m} ,$$

(20)

(m = 1, 2), the operators (3.a–3.b) turn out to be

$$s^{A} = \left( \begin{array}{c} s \\ \bar{s} \end{array} \right) \equiv \frac{1}{2} \delta^{\alpha}_{\beta} Q^{A}_{\beta} \alpha \quad (2 \text{ d.f. from } \text{Tr}(Q^{A})) ;$$

(21.a)

$$\delta^{\hat{A}}_{m} = \left( \begin{array}{c} \delta_{m} \\ \bar{\delta}_{m} \end{array} \right) \equiv -\frac{i}{2} \sigma_{ma} Q^{A}_{\beta} \alpha \quad (4 \text{ d.f. from } \text{Tr}(Q^{A})) ;$$

(21.b)

$$\zeta^{\hat{A}} = \left( \begin{array}{c} \bar{\zeta} \\ \zeta \end{array} \right) \equiv -\frac{i}{2} \sigma_{3a} Q^{A}_{\beta} \alpha \quad (2 \text{ d.f. from } \text{Tr}(Q^{A})) .$$

(21.c)

The operators (21.a–21.c) together with the supersymmetry algebra (20) yield the following anticommutation relations :

$$\{s^{A}, s^{B}\} = 0 , \quad \{s^{A}, \zeta^{B}\} = 0 \quad \text{and} \quad \{s^{A}, \delta^{B}_{m}\} = -\epsilon^{AB} \partial_{m} ;$$

(22)

that lead to the algebra displayed below

$$\{s, s\} = \{\bar{s}, \bar{s}\} = \{s, \bar{s}\} = \{\bar{s}, s\} = 0 ;$$

$$\{s, \zeta\} = \{s, \bar{\zeta}\} = \{\bar{s}, \zeta\} = \{\bar{s}, \bar{\zeta}\} = 0 ;$$

6
\[
\{s, \delta_m\} = \partial_m \ , \ \{s, \bar{\delta}_m\} = 0 \ ; \\
\{s, \delta_m\} = 0 \ , \ \{s, \bar{\delta}_m\} = -\partial_m \ ; \\
\{\zeta, \zeta\} = \{\bar{\zeta}, \bar{\zeta}\} = \{\zeta, \bar{\zeta}\} = 0 \ ; \\
\{\zeta, \delta_m\} = 0 \ , \ \{\zeta, \bar{\delta}_m\} = \varepsilon_{mn} \partial_n \ ; \\
\{\bar{\zeta}, \delta_m\} = -\varepsilon_{mn} \partial_n \ , \ \{\bar{\zeta}, \bar{\delta}_m\} = 0 \ ; \\
\{\delta_m, \delta_n\} = \{\delta_m, \bar{\delta}_n\} = \{\bar{\delta}_m, \bar{\delta}_n\} = 0 \ . \\
\]

Now, in two dimensions, the supercharges \(Q^+\) and \(Q^-\) read
\[
Q^+ = i \lambda s + i \sigma_3 \bar{\zeta} + i \sigma_m \bar{\delta}_m \quad \text{and} \quad Q^- = i \bar{\lambda} s + i \sigma_3 \zeta + i \sigma_m \delta_m ,
\]
where it can be checked with the help of the algebra (23) that \(Q^+\) and \(Q^-\) satisfy the \(N = 4\) supersymmetry algebra (24).

### 4.2 The BF-model in \(D = 2\) and its new symmetries

The dimensional reduction à la Scherk of the Chern-Simons action (8) to \(D = 2\) is achieved by assuming that
\[
A_\mu(x) \xrightarrow{\text{DR}} (A_m(\hat{x}), \phi(\hat{x})) \quad \text{where} \quad \begin{cases} x_\mu, \quad \mu = 1, 2, 3 \\ \hat{x}_m, \quad m = 1, 2 \end{cases}
\]
with \(\phi\) being a real scalar and \(\varepsilon_{mn3} \equiv \varepsilon_{mn}\). Since we are adopting the trivial dimensional reduction, the derivative, \(\partial_3\), of all fields vanishes, \(\partial_3 \mathcal{F} = 0\). Therefore, the following action stems
\[
\Sigma_{\text{BF}} = \frac{k}{4\pi} \text{Tr} \int d^2 \hat{x} \varepsilon_{mn} F_{mn} \phi ,
\]
with \(F_{mn} = \partial_m A_n + [A_m, A_n]\).

Now, let us perform the dimensional reduction of the gauge fixing (12), by assuming a curved three-dimensional manifold described by the metric tensor \(g_{\mu\nu}(x)\) which determinant is denoted by \(g\), it follows that
\[
g_{\mu\nu}(x) \xrightarrow{\text{DR}} (g_{mn}(\hat{x}), v_m(\hat{x}), \varphi(\hat{x})) \ ,
\]
\[
g = \hat{g}(\varphi - v_m v_m) ,
\]
where \(\hat{g}\) is the determinant of the two-dimensional metric \(g_{mn}\). The reduced gauge fixing in a curved two-dimensional space read
\[
\Sigma_{\text{gf}} = \begin{cases} s \text{ Tr} \int d^2 \hat{x} \sqrt{\hat{g}(\varphi - v_k v_k)} \ g_{mn} \hat{c} (v_m \partial_n \phi + \partial_m A_n) \\ -\bar{s} \text{ Tr} \int d^2 \hat{x} \sqrt{\hat{g}(\varphi - v_k v_k)} \ g_{mn} \ c (v_m \partial_n \phi + \partial_m A_n) \end{cases}
\]
(28)

Thus, from the energy-momentum tensor (13)
\[
T_{\mu\nu} \xrightarrow{\text{DR}} (T_{mn}, T_m, T) \quad \text{where} \quad \begin{cases} T_{mn} = \frac{2}{\sqrt{\hat{g}}} \frac{\partial \Sigma_{\text{gf}}}{\partial g_{mn}} \bigg|_{v_m = 0} = s \Lambda_{mn} = \bar{s} \bar{\Lambda}_{mn} \\ T_m = \frac{2}{\sqrt{\hat{g}}} \frac{\partial \Sigma_{\text{gf}}}{\partial v_m} \bigg|_{v_m = 0} = s \Lambda_m = \bar{s} \bar{\Lambda}_m \end{cases}
\]
(29)

\[
T = \frac{2}{\sqrt{\hat{g}}} \frac{\partial \Sigma_{\text{gf}}}{\partial \varphi} \bigg|_{\varphi = 0} = 0 \ .
\]
such that

\[
\Lambda_{mn} = -\mathrm{Tr} \, \bar{c} [\delta_{mn} \partial_l A_l - (\partial_m A_n + \partial_n A_m)] ,
\]
(30.a)

\[
\bar{\Lambda}_{mn} = \mathrm{Tr} \, c [\delta_{mn} \partial_l A_l - (\partial_m A_n + \partial_n A_m)] ,
\]
(30.b)

\[
\Lambda_m = 2 \, \mathrm{Tr} \, \bar{c} \partial_m \phi ,
\]
(30.c)

\[
\bar{\Lambda}_m = -2 \, \mathrm{Tr} \, c \partial_m \phi .
\]
(30.d)

The action \( \Sigma_{\text{inv}}^{(2)} = \Sigma_{\text{BF}} + \Sigma_{\text{gl}} \) is invariant under the following BRS and \text{BRS} transformations:

\[
\begin{align*}
\delta_m A_n &= 0 , \\
\bar{\delta}_m A_n &= 0 , \\
\delta_m \phi &= -\frac{2\pi}{k} \varepsilon_{mn} \partial_n \bar{c} , \\
\bar{\delta}_m \phi &= -\frac{2\pi}{k} \varepsilon_{mn} \partial_n c , \\
\delta_m c &= -A_m , \\
\bar{\delta}_m c &= A_m , \\
\delta_m b &= \partial_m \bar{c} , \\
\bar{\delta}_m b &= D_m c , \\
\delta_m \bar{c} &= 0 , \\
\bar{\delta}_m c &= 0 .
\end{align*}
\]
(32)

It should be stressed that beyond the usual topological symmetries (32) the action \( \Sigma_{\text{inv}}^{(2)} \) is also invariant under a new hidden “gauge” symmetry, a “topological pseudo-scalar supersymmetry”, obtained from (30.c–30.d)

\[
\begin{align*}
\zeta A_m &= \frac{2\pi}{k} \varepsilon_{mn} \partial_n \bar{c} , \\
\bar{\zeta} A_m &= \frac{2\pi}{k} \varepsilon_{mn} \partial_n c , \\
\zeta \phi &= 0 , \\
\bar{\zeta} \phi &= 0 , \\
\zeta c &= -\phi , \\
\bar{\zeta} c &= \phi , \\
\zeta b &= 0 , \\
\bar{\zeta} b &= [\bar{c}, c] , \\
\zeta \bar{c} &= 0 , \\
\bar{\zeta} c &= 0 .
\end{align*}
\]
(33)

All the symmetries displayed above in (31), (32) and (33), satisfy the algebra (23) up to contact terms. These symmetries of the two-dimensional BF-model \( \Sigma_{\text{inv}}^{(2)} \) (including the new one (33)) could be gotten simply by dimensional reducing the symmetries of the original Chern-Simons model \( \Sigma_{\text{inv}}^{(3)} \) (given by eqs. (14) and (16)).

### 4.3 Untwisting the BF-model

In order to investigate the possibility of untwisting the two-dimensional BF-model we find it convenient to introduce complex coordinates

\[
\begin{pmatrix}
z \\
\bar{z}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & i \\
1 & -i
\end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} .
\]
(34)
The gauge fixed BF-model can then be written as
\[
\Sigma_{\text{inv}}^{(2)} = \text{Tr} \int d^2 z \left( \chi \bar{\partial} A_z + \bar{\chi} \partial A_\bar{z} + \frac{2\pi}{k} (\bar{\chi} - \chi)[A_z, A_\bar{z}] + \bar{c} \partial \bar{c} + \frac{1}{2} \{c, \partial \bar{c}\} A_\bar{z} + \frac{1}{2} \{c, \bar{\partial} \bar{c}\} A_z \right),
\] (35)
where we introduced \( \chi = \frac{b}{2} + i \frac{k}{4\pi} \phi \). In two dimensions the rotation group is Abelian. The advantage of working with (35) is that all the fields have definite spin eigenvalues. The topological twist consists now of just adding the spin eigenvalues and the eigenvalues of the fields under the Cartan generators of the \( SU(2)_I \otimes SU(2)_{II} \). The question if one can untwist the BF-model boils down then to the question of which spin eigenvalues one can assign to the fields. Let us concentrate again for a moment on the kinetic term alone. Since \( (\partial, \bar{\partial}) \) have spin \((1, -1)\) and the Lagrangian should of course have spin zero we see that we can assign spin eigenvalues to \( \chi \) and \( A_z \) such that
\[
\text{spin}(\chi) + \text{spin}(A_z) = 1.
\] (36)
We also demand \( \text{spin}(\chi) = -\text{spin}(\bar{\chi}) \) and \( \text{spin}(A_\bar{z}) = -\text{spin}(A_z) \). From the interaction terms we get the equations
\[
\text{spin}(\chi) + \text{spin}(A_z) + \text{spin}(A_\bar{z}) = \text{spin}(\bar{\chi}) + \text{spin}(A_z) + \text{spin}(A_\bar{z}) = 0.
\] (37)
These equations together with analogous ones including \( (c, \bar{c}) \) tell us that the only possible spin assignments are the ones which we have in the original BF-model namely \( \text{spin}(\chi, A_z, c, \bar{c}) = (0, 1, 0, 0) \). If the interaction terms vanish however we can in particular assign spin \( \frac{1}{2} \) to \((A_z, \chi)\). The action is then the one for a linear sigma-model with reversed statistics in two dimensions. Analogous to what we had in the case of the three-dimensional Chern-Simons theory we find again that we can untwist only the non-interacting theory.

5 Conclusions

We reinvestigated the twisted \( N = 4 \) supersymmetry present in the three-dimensional Chern-Simons theory and the two-dimensional BF-model with Landau-type gauge fixing. We could show that it is impossible to untwist the non-Abelian theories. For the Abelian case we also showed that some puzzles concerning the untwisting procedure can be avoided by choosing an approach through sigma-models with reversed statistics in comparison to the G-twist of [11]. For the two-dimensional BF-model we established new fermionic (pseudo-)scalar symmetries \( (\zeta, \bar{\zeta}) \) who together with the BRS, anti-BRS and the respective fermionic vector supersymmetries \( (\delta_\mu, \bar{\delta}_\mu) \) give rise to the twisted \( N = 4 \) algebra. We expect these symmetries to be useful in the investigation of possible counterterms as \( e.g. \) in [12]. Finally one could imagine also to apply the methods presented here to higher-dimensional analogs of BF- and Chern-Simons models as recently investigated in [13].

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A Notations and conventions for the Euclidean $D=3$

Throughout this work the metric is assumed to be Euclidean, $\delta_{\mu\nu} = \text{diag}(+, +, +)$, $\mu, \nu = (1, 2, 3)$. The Dirac $2\times2$ $\gamma$-matrices are in fact the Pauli $\sigma$-matrices that satisfy the Clifford algebra

$$\{\sigma_\mu, \sigma_\nu\} = 2\delta_{\mu\nu} \mathbb{I} \implies \{\sigma_\mu, \sigma_\nu\}^\beta_\alpha = 2\delta_{\mu\nu}\delta^\beta_\alpha \ ,$$

are as follows

$$\sigma_\mu \equiv \sigma_{\mu\alpha} = \sigma_1, \sigma_2, \sigma_3 \ ,$$

where the Pauli matrices read

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Some useful relations involving the $\sigma$-matrices, their traces (Tr) and the $\varepsilon$-tensor used in the calculations are given by:

$$\sigma_\mu\sigma_\nu = \delta_{\mu\nu} \mathbb{I} + i\varepsilon_{\mu\rho\gamma\rho} \implies \sigma_{\mu\alpha}\sigma_{\nu\beta} = \delta_{\mu\nu}\delta^\gamma_\alpha + i\varepsilon_{\mu\rho\sigma\rho}\sigma^\gamma_\sigma \ ;$$

$$\text{Tr}(\sigma_\mu\sigma_\nu) = 2\delta_{\mu\nu} \implies \sigma_{\mu\alpha}\sigma_{\nu\beta} = 2\delta_{\mu\nu} \ ;$$

$$\varepsilon_{\mu\nu\kappa\varepsilon_{\mu\rho\sigma}} = \delta_{\sigma\nu}\delta_{\rho\kappa} - \delta_{\rho\nu}\delta_{\sigma\kappa} \ .$$

The spinor indices are raised and lowered according to the rules

$$X^\alpha = \epsilon^{\alpha\beta}X_\beta \quad \text{and} \quad X_\alpha = X^\beta\epsilon_{\beta\alpha} \ ,$$

with the $\epsilon_{\alpha\beta}$ tensor defined by

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ ,$$

where the following relation comes

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\delta} = \delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma \ .$$

Other important identities follow:

$$A_{[\alpha}^{[\beta}B_{\gamma]}^{\delta]} = -\frac{1}{2}\epsilon^{\beta\delta}A_{(\alpha}^\epsilon B_{\gamma)\epsilon} \ ;$$

$$(\mathbb{I} \wedge \sigma_\mu)^{\beta\delta}_{\alpha\gamma} \equiv \delta^{[\beta}_{[\alpha}\sigma_{\mu]^{\delta]} = -\frac{1}{2}\epsilon^{\beta\delta}\delta^{\epsilon}_{(\alpha\epsilon\sigma_{\mu})\epsilon} = -\epsilon^{\beta\delta}\sigma_{\mu\alpha\gamma} \ ;$$

$$(\sigma_\mu \wedge \sigma_\nu)^{\beta\delta}_{\alpha\gamma} \equiv \sigma_{[\mu}^{[\beta}\sigma_{\nu\gamma]}^{\delta]} = -\frac{1}{2}\epsilon^{\beta\delta}\sigma_{\mu(\alpha\epsilon\sigma_{\nu})\epsilon} = -i\epsilon^{\beta\delta}\varepsilon_{\mu\rho\sigma\rho\sigma_{\nu}} \ .$$

It should be noticed that the symmetrizations ( ) and antisymmetrizations [ ] are just related to the spinor indices $(\alpha, \beta, \gamma, \delta)$ without a factor $\frac{1}{2}$. 

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