Large $N$ planar or bare vertex approximation and critical behavior of a SU($N$) invariant four-fermion model in 2+1 dimensions

Manuel Reenders

Department of Polymer Chemistry and Materials Science Center,
University of Groningen, Nijenborgh 4,
9747 AG Groningen, The Netherlands
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Abstract

A four-fermion model in 2+1 dimensions describing $N$ Dirac fermions interacting via SU($N$) invariant $N^2 - 1$ four-fermion interactions is solved in the leading order of the $1/N$ expansion. The $1/N$ expansion corresponds to ’t Hooft's topological $1/N$ expansion in which planar Feynman diagrams prevail. For the symmetric phase of this model, it is argued that the planar expansion corresponds to the ladder approximation. A truncated set of Schwinger-Dyson equations for the fermion propagator and composite boson propagator representing the relevant planar diagrams is solved analytically. The critical four-fermion coupling and various critical exponents are determined as functions of $N$. The universality class of this model turns out to be quite distinct from the Gross-Neveu model in the large $N$ limit.
I. INTRODUCTION

Already in the early seventies K.G. Wilson [1] argued that four-fermion theories in dimensions $2 < d < 4$ are nontrivially renormalizable at least in leading order of the $1/N$ expansion, with $N$ the number of fermion flavors. The model studied by Wilson is nowadays referred to as the $d$ dimensional generalization of Gross-Neveu model [4]. In the Gross-Neveu model with $N$ fermion flavors there is one composite or auxiliary field $\sigma \propto -\sum_{\alpha=1}^{N} \bar{\psi}_\alpha \psi_\alpha$ describing the scalar bound states; the number of fermions is taken to be much larger than the number of light bound states. In the $1/N$ expansion for such models, Feynman diagrams with fermion loops dominate over other types of diagrams.

The Gross-Neveu model in 2+1 dimensions is well-known to exhibit dynamically symmetry breaking whenever the four-fermion coupling exceeds some critical value. About a decade ago renewed interest in the Gross-Neveu model led to a number of papers establishing the renormalizability of the Gross-Neveu model and related models in the $1/N$ expansion due to the presence of a finite ultraviolet stable fixed point and conformal invariance at this critical point [3, 4, 5, 6, 7]. Probably the most extensive study of the Gross-Neveu model, was performed by Hands, Kocić, and Kogut [8]. In their paper analytical results in next-to-leading order in $1/N$ were verified in numerical lattice simulations. The 1+1 dimensional four-fermion models cannot exhibit dynamical symmetry breaking, due to the Coleman-Mermin-Wagner theorem, whereas 3+1 dimensional models usually suffer from the triviality problem [4, 5].

In this paper a SU($N$) $\times$ U(1) invariant four-fermion model, closely related to the Gross-Neveu model, is studied in 2+1 dimensions. In this model the four-fermionic potential contains $N^2 - 1$ terms, i.e., the four-fermion interactions are in the adjoint representation of the SU($N$) symmetry. Thus the model presented here describes the interaction of $N$ Dirac fermions with $N^2 - 1$ scalar composite states. Consequently the large $N$ treatment will be quite different from the Gross-Neveu model. Instead of the large $N$ expansion à la Wilson, 't Hooft's topological $1/N$ expansion [11] will be adopted. We assume that the $1/N$ expansion of 't Hooft is particularly useful for theories with a global U($N$) symmetry containing fields with two U($N$) or SU($N$) indices, e.g., see Refs. [3, 11, 12, 13]. In such an expansion planar diagrams dominate over diagrams with topologies other than planar.

The leading large $N$ critical behavior of the proposed model is studied, with the main focus on the determination of the scaling behavior of the fermion wave function and of the $N^2 - 1$ scalar or $\sigma$ boson propagators. Once the anomalous dimensions of the fermion propagator and the $\sigma$ boson propagators are determined, the gap equation for the order parameter will be studied. The dynamical breaking of the SU($N$) symmetry to U(1) gives rise to the Goldstone realization with $N^2 - 2$ Nambu-Goldstone bosons and one massive scalar in the broken phase.

The main motivation for studying this particular three dimensional four-fermionic model is its possible relationship with low temperature behavior of the 2+1 dimensional Hubbard-Heisenberg and $t-J$ models near the antiferromagnetic wave vector [13, 14, 15, 16].

The setup of the paper is the following. The model Lagrangian is introduced in the next section. We present a truncation scheme for the Schwinger-Dyson (SD) equations which generates all planar and thus leading $1/N$ diagrams in Sec. III. This closed set of SD equations is subsequently solved analytically in Sec. IV. In Sec. V the dynamical symmetry breaking is analyzed and the critical exponents, hyperscaling, and universality are discussed. Next-to-leading order $1/N$ corrections are discussed in Sec. VI. The conclusions are presented in
In the Appendix, the applicability of the topological $1/N$ expansion is motivated for this 2+1 dimensional model.

II. THE MODEL

We will consider the following 2+1 dimensional four-fermion model given by the Lagrangian

$$\mathcal{L} = \bar{\psi}_\alpha \hat{i} \partial_\alpha \psi_\alpha + \frac{G}{2} \sum_{A=1}^{N^2-1} (\bar{\psi} A \tau^A_\alpha \psi_\beta)^2,$$

where the index $\alpha = 1, \ldots, N$ labels the $N$ fermion flavors. The fermions are described by four-component Dirac spinors. The Lagrangian Eq. (1) is globally invariant under SU($N$) $\times$ U(1), see also Ref. [12]. By the way, in Refs. [9, 10] a U($N$) $\times$ U($N$) invariant version of Eq. (1) is described in 3 + 1 dimensions. The Hermitian generators, $\tau^A$, of the SU($N$) symmetry satisfy

$$\text{Tr}[\tau^A] = 0, \quad \text{Tr}[\tau^A \tau^B] = \delta^{AB}, \quad \sum_{A=1}^{N^2-1} \tau^A_{\alpha\beta} \tau^A_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\gamma\beta} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta}.$$

After a Hubbard-Stratonovich transformation, the Lagrangian (1) reads

$$\mathcal{L} = \bar{\psi}_\alpha \hat{i} \partial_\alpha \psi_\alpha - \sum_A \bar{\psi} A \tau^A_{\alpha\beta} \psi_\beta \sigma_A - \frac{1}{2G} \sum_A \sigma^2_A,$$

with the real auxiliary bosonic fields $\sigma^A = -G \bar{\psi}^A \tau^A \psi$ conveniently describing the composite fermion–antifermionic degrees of freedom. Another equivalent representation with $\sigma_{ij} = \sum_A \sigma^A \tau^A_{ij}$ can be adopted [12], see also the Appendix. Along the lines of Ref. [10], $N^2 - 1$ $\sigma$ boson propagators $\Delta^A_{\sigma}(p)$ and Yukawa vertices $\Gamma^A_{\sigma}(p + q, p)$ (“fully amputated”) are introduced. The bare Yukawa vertices are $\gamma^A_{\sigma} = \tau^A_{\sigma}$. Since in the symmetric phase the $\Delta^A_{\sigma}$ propagators are identical, the index $A$ is often discarded.

The dynamical symmetry breaking SU($N$) $\rightarrow$ U(1) is described by the order parameter $\langle \sigma_3 \rangle \simeq -G \bar{\psi} \tau^3 \psi$, which is a particular choice for the symmetry breaking term. In principle we have the freedom to choose from $N^2 - 1$ order parameters $\bar{\psi}^A \tau^A \psi$. In order to “force” the symmetry breaking in the $\tau^3$ direction a bare mass term $-h \bar{\psi} \tau^3 \psi$ is added to the Lagrangian in Eq. (3). In the end the limit $h \rightarrow 0$ can be taken.

In the fermion propagator the symmetry breaking is given by the mass function $M$ which is introduced as follows

$$S_{\alpha\beta}(p) = \frac{A(p) \hat{p} \delta_{\alpha\beta} + \sqrt{N} M \tau^3_{\alpha\beta}}{A^2(p) p^2 - M^2}.$$

The function $A(p)$ is referred to as the fermion wave function. The fermion dynamical mass $m_{\text{dyn}}$ is defined as the mass pole in the fermion propagator; it is related to $M$ via

$$A^2(p) m_{\text{dyn}}^2 - M^2.$$
FIG. 1: The planar approximation; the dashed lines are the real $\sigma$ boson propagators, the black lines are fermion propagators. The blobs correspond to the fully dressed propagators. The dark blob in the fermion propagator corresponds to a wave function part being proportional to $\hat{p}$, whereas the white blob with a cross corresponds to a mass insertion proportional to the $\tau^3$ generator.

III. LARGE $N$ PLANAR AND BARE VERTEX APPROXIMATION

The present model describes $N$ species of fermions interacting via $N^2 - 1$ four-fermionic interactions in the so-called adjoint representation of the SU($N$) flavor symmetry. Therefore, when $N$ is large, we assume the applicability of 't Hooft's topological $1/N$ expansion [11], see also Ref. [10]. Moreover, since the U($N$) symmetry is global, we assume that the corresponding topological $1/N$ expansion is independent of the space-time dimension $d$. The applicability of 't Hooft's $1/N$ expansion to the present model is motivated in the Appendix. The topological $1/N$ expansion states that the Feynman diagrams can be classified in terms of surfaces with a specific topology. The so-called planar Feynman diagrams, with the fermions at the boundary, describe the leading contributions to the Green functions. Diagrams with other than planar topological structures are multiplied by factors of at least $1/N$, and in the limit of large $N$, their contribution can be neglected with respect to planar graphs. For instance diagrams with one handle on their representing surface are suppressed by a factor $1/N^2$ relatively to their planar counterparts, whereas diagrams with a hole are suppressed by a factor $1/N$.

In this paper we are mainly interested in the SD equations for the $\sigma$ boson propagator $\Delta_\sigma$ and the fermion wave function $A$. In the Appendix, it is shown that the leading large $N$ behavior for $\Delta_\sigma$ is given by those diagrams which do not contain Yukawa vertex ($\Gamma_{\sigma}$) corrections, see the first two truncated SD equations in Fig. 1. In other words, only self energy corrections ($\Sigma$) and $\sigma$ boson vacuum polarization corrections ($\Pi_{\sigma}$) should be considered. One can straightforwardly verify that the bare vertex approximation (see Fig. 2), $\Gamma^A_{\sigma}(p + q, p) \approx \tau^A \mathbf{1}$, generates only planar and thus leading large $N$ graphs for $\Delta_\sigma$ and $A$. Here we mention that the propagators of the real scalar fields $\sigma^A$ are represented by dashed lines, whereas the propagators of the Hermitian fields $\sigma_{\alpha\beta}$ of the Appendix are represented by double lines.

For the symmetry breaking fermion mass $M$ only the tad-pole graph contributes in the leading large $N$ limit, see Fig. 1. At a first glance this might seem inconsistent, since parts
of the fermion propagator are now described by different truncations. However, the self-
consistency of the truncated set of SDEs given in Fig. 1 can be verified using the SU($N$)
Ward identities, e.g., see Ref. [10]. These Ward identities relate the $N^2 - 1$ Yukawa vertices
$\Gamma^A_{\sigma}(p, p)$ at zero \( \sigma \) boson momentum transfer to the fermion dynamical mass \( M(p) \) in the
following way:

$$\Gamma^A_{\sigma}(p, p) = \tau^A \frac{M(p)}{\langle \sigma_3 \rangle}, \hspace{1cm} A = 1, \ldots N^2, \hspace{0.5cm} A \neq 3, \hspace{1cm} (5)$$

$$\Gamma^3_{\sigma}(p, p) = \tau^3 \frac{\partial M(p)}{\partial \langle \sigma_3 \rangle}, \hspace{1cm} (6)$$

when the symmetry breaking is in the \( \tau^3 \) direction. Thus if we apply the bare vertex
approximation $\Gamma^A_{\sigma} = \tau^A$ (Fig. 2), we get the equation $M = \langle \sigma_3 \rangle$. This equation is indeed
represented by the third SD equation in Fig. 1.

Summarizing; the leading large \( N \) behavior can be described consistently in terms of the
truncated set of SDEs given in Fig. 1.

IV. SCALING BEHAVIOR AT CRITICALITY

Having introduced the leading large \( N \) planar or bare vertex approximation for this model,
we proceed by studying the truncated set given in Fig. 1 in the symmetric phase first. These
equations are in Euclidean formulation

$$\Delta^{-1}_{\sigma}(p) = -\frac{1}{G} + \Pi_{\sigma}(p), \hspace{1cm} (7)$$

$$\Pi_{\sigma}(p) = \int_E \frac{d^3k}{(2\pi)^3} \frac{4(k^2 + k \cdot p)}{(k + p)^2 k^2 A(k + p) A(k)}, \hspace{1cm} (8)$$

and

$$A(p) = 1 - N \Sigma_A(p), \hspace{1cm} (9)$$

$$\Sigma_A(p) = \int_E \frac{d^3k}{(2\pi)^3} \frac{p \cdot k}{p^2 k^2 A(k)} \Delta_{\sigma}(k - p). \hspace{1cm} (10)$$

For the dimensionless coupling constant \( g \equiv 2GA/\pi^2 \) there exists an ultraviolet stable fixed
point \( g_c \) above which the SU($N$) is dynamically broken. Exactly at the critical coupling
\( g = g_c \) the infrared mass scale vanishes and the above propagators will be given in terms of
pure power-laws. Therefore, we first consider the two coupled SD equations at the critical point \((g = g_c)\), where we can safely assume a power-law form for the fermion wave function

\[ A(p) = (p^2/\Lambda^2)^{-\zeta/2}, \]  

(11)

where \(\Lambda\) is the ultraviolet cutoff, and where both the critical coupling \(g_c\) and the exponent \(\zeta\) are to be determined from the above SD equations and will depend on \(N\) via Eq. (9). Using the above ansatz for the fermion wave function, we subsequently solve the SD for the \(\sigma\) boson propagator. For the model to be nontrivially renormalizable, the \(\sigma\) boson propagator should also exhibit a pure power-law form at the critical coupling;

\[ \Delta_\sigma(p) \propto \left( \frac{\Lambda}{p} \right)^{2-\eta}, \]  

(12)

where \(\eta\) is the anomalous dimension of the \(\sigma\) boson propagator. Therefore, in what follows, we solve the SD equations (7)–(10) assuming the power-law forms (11) and (12) in order to obtain the exponents \(\zeta\) and \(\eta\) and the critical coupling \(g_c\) as function of \(N\).

Inserting Eq. (11) in Eq. (8) we obtain the integral

\[ \Pi_\sigma(p) = \frac{2\Lambda^{-2\zeta}}{\pi^2} \int_0^\Lambda dk \int d\Omega \frac{4\pi}{k^{-\zeta}(k^2 + p^2 + 2k \cdot p)^{1-\zeta/2}} \left[ (k^2 + \zeta k^2 + \zeta k p - p^2)(k + p)^\zeta \right. 
\]  

(13)

The angular integral is straightforward

\[ \Pi_\sigma(p) = \left( \frac{2\Lambda^{-2\zeta}}{\pi^2} \right) \left( \frac{p}{\Lambda} \right)^{1+2\zeta} \left[ J_1(\zeta, x) + J_2(\zeta, x) + J_3(\zeta, x) \right], \]  

(15)

with \(x = p/\Lambda\) and where

\[ J_1(\zeta, x) = \frac{(1 + \zeta)}{2\zeta(2 + \zeta)} \left[ \int_0^1 dt t^{\zeta+1} f_1(t) + \int_x^1 dt t^{-2\zeta-3} f_1(t) \right], \]  

(16)

\[ J_2(\zeta, x) = \frac{1}{2(2 + \zeta)} \left[ \int_0^1 dt t^{\zeta} f_2(t) + \int_x^1 dt t^{-2\zeta-2} f_2(t) \right], \]  

(17)

\[ J_3(\zeta, x) = -\frac{1}{2\zeta(2 + \zeta)} \left[ \int_0^1 dt t^{\zeta-1} f_1(t) + \int_x^1 dt t^{-2\zeta-1} f_1(t) \right], \]  

(18)

with

\[ f_1(t) \equiv (1 + t)^\zeta - (1 - t)^\zeta, \quad f_2(t) \equiv (1 + t)^\zeta + (1 - t)^\zeta. \]  

(19)
The integrals can be performed straightforwardly using the Table of Integrals [17] and the \( J \) function can be expressed in terms of the \( \Gamma \) function and the hypergeometric function \( _2F_1 = F \). In this way, the function \( J_1 \) can be expressed as (with \( x = p/\Lambda \))

\[
J_1(\zeta, x) = \frac{(1 + \zeta)}{2\zeta(2 + \zeta)} \left\{ \frac{x^{-2\zeta - 2}}{(2 + 2\zeta)} [F(-2 - 2\zeta, -\zeta; -1 - 2\zeta; -x) - F(-2 - 2\zeta, -\zeta; -1 - 2\zeta; x)] + \frac{1}{(2 + \zeta)} F(2 + \zeta, -\zeta; 3 + \zeta; -1) - \frac{1}{(2 + 2\zeta)} F(-2 - 2\zeta, -\zeta; -1 - 2\zeta; -1) \right. \\
- \left. \frac{\Gamma(2 + \zeta)\Gamma(1 + \zeta)}{\Gamma(3 + 2\zeta)} - \frac{\Gamma(-2 - 2\zeta)\Gamma(1 + \zeta)}{\Gamma(-1 - \zeta)} \right\}.
\]

Subsequently \( J_2(\zeta, x) \) and \( J_3(\zeta, x) \) are computed in a similar manner, giving

\[
J_2(\zeta, x) = \frac{1}{2(2 + \zeta)} \left\{ \frac{x^{-2\zeta - 1}}{(1 + 2\zeta)} [F(-1 - 2\zeta, -\zeta; -2\zeta; -x) + F(-1 - 2\zeta, -\zeta; -2\zeta; x)] + \frac{1}{(1 + \zeta)} F(1 + \zeta, -\zeta; 2 + \zeta; -1) - \frac{1}{(1 + 2\zeta)} F(-1 - 2\zeta, -\zeta; -2\zeta; -1) \right. \\
+ \left. \frac{\Gamma(1 + \zeta)\Gamma(1 + \zeta)}{\Gamma(2 + 2\zeta)} + \frac{\Gamma(-1 - 2\zeta)\Gamma(1 + \zeta)}{\Gamma(-\zeta)} \right\},
\]

and

\[
J_3(\zeta, x) = -\frac{1}{2\zeta(2 + \zeta)} \left\{ \frac{x^{-2\zeta}}{2\zeta} [F(-2\zeta, -\zeta; 1 - 2\zeta; -x) - F(-2\zeta, -\zeta; 1 - 2\zeta; x)] + \frac{1}{\zeta} F(\zeta, -\zeta; 1 + \zeta; -1) - \frac{1}{2\zeta} F(-2\zeta, -\zeta; 1 - 2\zeta; -1) \right. \\
- \left. \frac{\Gamma(\zeta)\Gamma(1 + \zeta)}{\Gamma(1 + 2\zeta)} - \frac{\Gamma(-2\zeta)\Gamma(1 + \zeta)}{\Gamma(-\zeta)} \right\}.
\]

The condition for the existence of these integrals is that \( \zeta > 0 \). Assuming that \( 0 < \zeta < 1/2 \), we can determine the asymptotic behavior or infrared limit of the above \( J \) functions, by taking the \( x \to 0 \) limit (\( p \ll \Lambda \)). We write

\[
J_i(\zeta, x) \simeq A_i(\zeta)x^{-1-2\zeta} - B_i(\zeta) + C_i(\zeta)x^{1-2\zeta} + \ldots, \quad i = 1, 2, 3,
\]

or

\[
J(\zeta, x) \simeq A(\zeta)x^{-1-2\zeta} - B(\zeta) + C(\zeta)x^{1-2\zeta} + \ldots,
\]

with \( A(\zeta) = \Sigma_{i=1}^{3} A_i(\zeta) \), \( B = \Sigma_{i=1}^{3} B_i \), \( C = \Sigma_{i=1}^{3} C_i \), and where

\[
A_1 = \frac{1}{2(2 + \zeta)} + \frac{1}{2(2 + \zeta)(1 + 2\zeta)}, \quad A_2 = \frac{1}{(2 + \zeta)(1 + 2\zeta)}, \quad A_3 = 0.
\]
and
\[
\mathcal{B}_1(\zeta) = -\frac{(1+\zeta)}{2(2+\zeta)} \left\{ \frac{1}{(2+\zeta)} F(2+\zeta,-\zeta;3+\zeta;-1) \\
- \frac{1}{(2+2\zeta)} F(-2-2\zeta,-\zeta;-1-2\zeta;-1) - \frac{\Gamma(2+\zeta)\Gamma(1+\zeta)}{\Gamma(3+2\zeta)} \\
- \frac{\Gamma(-2-2\zeta)\Gamma(1+\zeta)}{\Gamma(-1-\zeta)} \right\},
\]
\[
\mathcal{B}_2(\zeta) = -\frac{1}{2(2+\zeta)} \left\{ \frac{1}{(1+\zeta)} F(1+\zeta,-\zeta;2+\zeta;-1) - \frac{1}{(1+2\zeta)} F(-1-2\zeta,-\zeta;-2\zeta;-1) \\
+ \frac{\Gamma(1+\zeta)\Gamma(1+\zeta)}{\Gamma(2+2\zeta)} + \frac{\Gamma(-1-2\zeta)\Gamma(1+\zeta)}{\Gamma(-\zeta)} \right\},
\]
\[
\mathcal{B}_3(\zeta) = \frac{1}{2(2+\zeta)} \left\{ \frac{1}{(1+\zeta)} F(1+\zeta,-\zeta;1+\zeta;-1) - \frac{1}{2\zeta} F(-2\zeta,-\zeta;1-2\zeta;-1) \\
- \frac{\Gamma(\zeta)\Gamma(1+\zeta)}{\Gamma(1+2\zeta)} - \frac{\Gamma(-2\zeta)\Gamma(1+\zeta)}{\Gamma(1-\zeta)} \right\}.
\]

For $0 < \zeta < 1/2$, only the leading scaling terms or power-laws $x^{-1-2\zeta}$ and $x^0$ of Eq. (24) are relevant for our purposes, therefore the functions $C_i$ are not displayed. Then the vacuum polarization is
\[
\Pi_{\sigma}(p) \simeq \frac{2\Lambda}{\pi^2} \left[ \frac{1}{1+2\zeta} - \mathcal{B}(\zeta) \left( \frac{p}{\Lambda} \right)^{1+2\zeta} \right],
\]
where $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3$. The critical coupling $g_c$ is given by the condition $\Delta_{\sigma}^{-1}(0) = 0$ in Eq. (7), thus
\[
\frac{1}{g_c} = A_1 + A_2 + A_3 = \frac{1}{1+2\zeta}.
\]

With the expression (29), the boson propagator (Eq. (7)) takes the pure power-law form (12) at the critical coupling $g = g_c = 1 + 2\zeta$:
\[
\Delta_{\sigma}(p) \simeq -\frac{\pi^2}{2\Lambda\mathcal{B}(\zeta)} \left( \frac{\Lambda}{p} \right)^{1+2\zeta},
\]
consequently, the anomalous dimension $\eta = 1 - 2\zeta$.

It can be shown that $\mathcal{B}_3(\zeta \to 0) \approx \pi^2/8$, and that $\mathcal{B}_1$ and $\mathcal{B}_2$ approach zero when $\zeta \to 0$, thus $\mathcal{B}(0) = \pi^2/8$. The case $\zeta = 0$ (i.e. $A(p) = 1$) in the above equation simply corresponds to the one-loop or leading $1/N$ correction, $\Pi_{\sigma}(p) \simeq (2\Lambda/\pi^2)(1 - \mathcal{B}(0)p/\Lambda)$, see Ref. [6].

At $\zeta = 1/2$ it can be shown that $\mathcal{B}$ diverges, but so that
\[
\mathcal{B}(\zeta \to 1/2) = C(\zeta \to 1/2), \quad \mathcal{B}(\zeta \to 1/2) \approx \frac{3}{8(1-2\zeta)},
\]
and $\mathcal{J}(\zeta, x)$ reduces to
\[
\mathcal{J}(\zeta, x) \simeq \frac{1}{2x^2} + \frac{3}{8} \ln x
\]

and
at $\zeta = 1/2$. Consequently, at $\zeta = 1/2$, the vacuum polarization $\Pi_\sigma$ of Eq. (8) reads

$$\Pi_\sigma(p) \simeq \frac{2\Lambda}{\pi^2} \left[ \frac{1}{2} + \frac{3p^2}{8\Lambda^2} \ln \left( \frac{p}{\Lambda} \right) \right].$$  \hspace{1cm} (34)$$

This means there is no pure power-law form at $\zeta = 1/2$ and consequently the model becomes non-renormalizable in this case. This situation resembles the situation in 3 + 1 dimensional four-fermion models such as the gauged Nambu–Jona-Lasinio model, when the gauge coupling approaches zero, e.g., see Ref. [18].

With the asymptotic form for the boson propagator, Eq. (31) and the Ansatz (11) for the wave function $A$, we can compute the Euclidean self-energy contribution Eq. (10) and verify the self-consistency of Eq. (11). Thus

$$\Sigma_A(p) = -\frac{\Lambda^\zeta}{4B(\zeta)} \int_0^\Lambda dk \int \frac{d\Omega}{4\pi} \frac{k \cdot p}{p^2 k^{-\zeta} (k^2 + p^2 - 2k \cdot p)^{1/2+\zeta}}. \hspace{1cm} (35)$$

Again the angular integral can be performed straightforwardly;

$$\Sigma_A(p) = -\frac{\Lambda^\zeta}{4B(\zeta)2(3-2\zeta)(1-2\zeta)} \int_0^\Lambda dk \frac{k^{\zeta-1}}{p^3} \left\{ k^2 - (1 - 2\zeta)kp + p^2 \left( (k^2 - (1 - 2\zeta)kp + p^2) (k + p)^{1-2\zeta} - \left[ k^2 + (1 - 2\zeta)kp + p^2 \right] |k - p|^{1-2\zeta} \right) \right\}. \hspace{1cm} (36)$$

The integral is expressed as follows:

$$\Sigma_A(p) = -\frac{\mathcal{F}(\zeta, p/\Lambda)}{8B(\zeta)(3 - 2\zeta)(1 - 2\zeta)} \left( \frac{p}{\Lambda} \right)^{-\zeta}, \hspace{1cm} \text{ (37)}$$

where

$$\mathcal{F}(\zeta, p/\Lambda) \equiv \int_0^\frac{p}{k} \frac{dk}{k} \frac{k^{\zeta-1}}{p^{\zeta-1}} \left\{ \left[ k^2 - (1 - 2\zeta)kp + p^2 \right] \left( 1 + \frac{k}{p} \right)^{1-2\zeta} - \left[ k^2 + (1 - 2\zeta)kp + p^2 \right] \left( 1 - \frac{k}{p} \right)^{1-2\zeta} \right\} + \int_\frac{p}{k}^\Lambda \frac{dk}{k} \frac{k^{-\zeta+3}}{p^{-\zeta+3}} \left\{ \left[ 1 - (1 - 2\zeta)p + p^2 \right] \left( 1 + \frac{p}{k} \right)^{1-2\zeta} - \left[ 1 + (1 - 2\zeta)p - p^2 \right] \left( 1 - \frac{p}{k} \right)^{1-2\zeta} \right\}. \hspace{1cm} (38)$$

The function $\mathcal{F}$ is written as

$$\mathcal{F}(\zeta, x) = \int_0^1 dt t^{\zeta-1} \kappa(t) + \int_x^1 dt t^{\zeta-4} \kappa(t), \hspace{1cm} (39)$$

where

$$\kappa(t) \equiv \left[ t^2 - (1 - 2\zeta)t + 1 \right] (1 + t)^{1-2\zeta} - \left[ t^2 + (1 - 2\zeta)t + 1 \right] (1 - t)^{1-2\zeta}. \hspace{1cm} (40)$$
Performing the integral, and determining the leading small \( x = p / \Lambda \) behavior of the function \( \mathcal{F} \), we find for \( \mathcal{F} \)

\[
\mathcal{F} = \frac{\Gamma(\zeta) \Gamma(2 - 2\zeta)(1 + \zeta)(2\zeta - 3)}{\Gamma(2 - \zeta)(3 - \zeta)} + \frac{1}{\zeta} \left( \mathcal{F}(\zeta, -1 + 2\zeta; 1 + \zeta; -1) \right)
\]

\[
- \left( \frac{1 - 2\zeta}{1 + \zeta} \right) \mathcal{F}(1 + \zeta, -1 + 2\zeta; 2 + \zeta; -1) + \frac{1}{2 + \zeta} \mathcal{F}(2 + \zeta, -1 + 2\zeta; 3 + \zeta; -1)
\]

\[
+ \left( \frac{\Gamma(\zeta - 3) \Gamma(2 - 2\zeta)(2 - \zeta)(2\zeta - 3)}{\Gamma(-1 - \zeta)} \right) - \frac{1}{1 - \zeta} \mathcal{F}(-1 + \zeta, -1 + 2\zeta; \zeta; -1)
\]

\[
+ \frac{1 - 2\zeta}{2 - \zeta} \mathcal{F}(-2 + \zeta, -1 + 2\zeta; -1 + \zeta; -1) - \frac{1}{3 - \zeta} \mathcal{F}(-3 + \zeta, -1 + 2\zeta; -2 + \zeta; -1)
\]

\[
+ \frac{x^\zeta (2 - 2\zeta)(2\zeta^3 - 5\zeta^2 + 2)}{(\zeta - 2)} + \mathcal{O}\left( x^{\zeta + 2} \right).
\]  (41)

Thus for small \( x = p / \Lambda \), the \( \mathcal{F} \) can be written as

\[
\mathcal{F}(\zeta, x) \simeq \mathcal{F}(\zeta) + \mathcal{O}\left( x^{\zeta} \right),
\]  (42)

where we write \( \mathcal{F}(\zeta) = \mathcal{F}(\zeta, 0) \). For values of \( \zeta \) with \( 0 < \zeta < 1/2 \), the self energy has the scaling form

\[
\Sigma_A(p) \simeq -\lambda(\zeta) \left( \frac{p}{\Lambda} \right)^{-\zeta},
\]  (43)

where

\[
\lambda(\zeta) \equiv \frac{\mathcal{F}(\zeta)}{8B(\zeta)(3 - 2\zeta)(1 - 2\zeta)}.
\]  (44)

Using the scaling form Eq. (43) and the Ansatz (11) in Eq. (9), we get the self-consistency relation

\[
\left( \frac{p}{\Lambda} \right)^{-\zeta} = 1 - N\Sigma_A(p) = N\lambda(\zeta) \left( \frac{p}{\Lambda} \right)^{-\zeta} + \mathcal{O}(1).
\]  (45)

This gives the eigenvalue equation (for \( 0 < \zeta < 1/2 \))

\[
\lambda(\zeta) = 1/N.
\]  (46)

This equation fixes the dependence of the anomalous dimension \( \zeta \) on \( N \).

In the limit \( \zeta \to 0 \), the function \( \mathcal{F} \) given by Eq. (41) reduces to

\[
\mathcal{F}(0, x) \approx -2 \ln x.
\]  (47)

At \( \zeta = 0 \), using \( B(0) = \pi^2/8 \) and Eq. (17), we find that \( \Sigma_A(p) \approx 2/(3\pi^2) \ln(p/\Lambda) \) which is consistent with the one-loop or leading \( 1/N \) correction to \( \Sigma_A \) in the Gross-Neveu model [3].

At \( \zeta = 1/2 \) the function \( \mathcal{F}(\zeta) \) has a first order root and we can extract the residue by taking
FIG. 3: Exact solution of Eq. (46) compared with $\zeta = 1/2 - 5/(4\pi N)$; the exponent $\zeta$ is plotted versus $1/N$.

Thus $F(\zeta) \approx 12\pi(1-2\zeta)/5$ for $\zeta \uparrow 1/2$. Near $\zeta = 1/2$, we obtain by using Eq. (32), Eq. (44), and Eq. (48) that

$$\lambda(\zeta) \approx \frac{2\pi(1-2\zeta)}{5} = \frac{1}{N} \implies \zeta \approx \frac{1}{2} - \frac{5}{4\pi N}.$$

The exact solution Eq. (46) and the above approximate solution are depicted in Fig. 3. Clearly Fig. 3 shows that if $N \to \infty$ then $\zeta \to 1/2$. This behavior is a result of the fact that the functions $1/B(\zeta)$ as defined in Sec. IV and the function $F(\zeta)$ defined in Eq. (42) have a first order root at $\zeta = 1/2$. There, the eigenvalue equation is of the form given by Eq. (49).

Finally we mention that the leading scaling behavior is independent of the choice of momenta flowing in the self energy diagram $\Sigma_A$ given by Eq. (10); an alternative choice, e.g., $k + p$ for the fermion propagator and $k$ for $\Delta_{\sigma}$ leads to the same scaling form (43) and eigenvalue equation (46).
In the previous section the scaling laws for $A(p)$ and $\Delta_\sigma(p)$ at the critical coupling $g = g_c$ were obtained. This yielded the anomalous dimension $\eta = 1 - 2\zeta$ with $\zeta$ given by Eq. (46). In order to obtain the other critical exponents the set of SD equations should be studied either in the subcritical ($g \uparrow g_c$) or supercritical ($g \downarrow g_c$) regime. In these regimes the analysis of the SD equations depicted in Fig. 1 is more complicated, due to the fact that a nonzero infrared mass scale (the inverse correlation length) enters the problem. For instance in the subcritical regime the inverse correlation length or $m_\sigma$ is given by the complex pole $p^2 \sim m^2_\sigma \exp(-i\theta)$ in $\Delta_\sigma(p)$ [20]. The critical exponent $\nu$ can then be derived from the scaling law $m_\sigma \sim (g_c - g)^\nu$. If we consider Eqs. (7) and (29) in the subcritical regime, it follows that $\nu = 1/(1 + 2\zeta)$. Consequently the anomalous dimension $\gamma = 1$ as can be determined from the limit $\Delta_\sigma(p \to 0) \sim 1/|g_c - g| \sim \chi$ with $\chi$ the usual susceptibility [8, 10].

In the subcritical regime the $\sigma$ boson propagator will be of the renormalizable form:

$$\Delta_\sigma(p) \approx -\frac{\pi^2}{2\Lambda B(\zeta)} \left(\frac{\Lambda}{p}\right)^{1+2\zeta} \frac{1}{1 + (m^2_\sigma/p^2)^{(1+2\zeta)/2}},$$

(50)

where $m_\sigma$ is the generic notation for inverse correlation length ($m_\sigma \propto m_{dyn} \sim 1/\xi$). If this expression (50) is inserted in Eq. (10) a careful analysis will show that the fermion wave function $A$ behaves as follows in the subcritical regime,

$$A(p) \approx \left(\frac{\max(p, c_- m_\sigma)}{\Lambda}\right)^{-\zeta},$$

(51)

where $c_-$ is a nonuniversal constant. This constant $c_-$ should be determined from the nonlinear SD equations (7)–(10), but for the scaling arguments its precise value is irrelevant. In the supercritical regime the wave function will be of the same form as Eq. (51), but with a different constant of proportionality, e.g., $c_+$ instead of $c_-$. In rest of this section the supercritical regime ($g > g_c$) is considered and we study the dynamical symmetry breaking $SU(N) \to U(1)$. As was explained in Sec. III, in the leading large $N$ approximation only the tadpole graph is included in the gap equation for $M$. This gap equation with bare mass $h$ reads (in Euclidean formulation)

$$M = h + 4G \int_E \frac{d^3p}{(2\pi)^3} \frac{M}{A^2(p)p^2 + M^2}.$$  

(52)

For the wave function $A$ the solution which is valid in the broken phase should be substituted. However, since the scaling laws for the order parameter and dynamical mass are mainly determined by the ultraviolet behavior $m_\sigma \ll p \leq \Lambda$ of the theory, we can safely use Eq. (11). Since the mass $M$ in Eq. (52) is independent of momentum, the above integral can be performed. Using Eq. (11) for the fermion wave function $A$, we obtain

$$1 \approx \frac{h}{M} + g \left(\frac{\Lambda^2}{3M^2}\right)^2 \frac{1}{2} \left(1, \frac{3}{2-2\zeta}; 1 + \frac{3}{2-2\zeta}; \frac{\Lambda^2}{M^2}\right).$$

(53)

The argument of the hypergeometric function approaches $-\infty$ as the critical curve is approached ($g \to g_c, M \to 0$) at $h = 0$. Therefore, close to criticality, the gap equation can be
approximated as follows:

\[-\left(\frac{g - g_c}{g_c}\right) = \frac{h}{M} + \frac{g}{3} \Gamma (1 + b) \Gamma (1 - b) \left(\frac{M}{\Lambda}\right)^{2b-2},\]  

(54)

where \(b = \frac{3}{2(2 - 2\zeta)}\) and \(g_c = 1 + 2\zeta\).

Note that within our approximation \(M = \langle \sigma_3 \rangle\). Therefore, treating \(M\) as the order parameter, the gap equation (54) is the quantum mechanical equivalent of the equation of state known in statistical mechanics and in analogy various critical exponents can be derived from it (see Refs. \cite{8, 19}). For instance, the critical exponent \(\delta\) follows from the scaling law \(h \sim M^\delta\) at \(g = g_c\):

\[h \propto M^{2b-2+1} \implies \delta = 2b - 1 = \frac{2 + \zeta}{1 - \zeta} \]  

(55)

and the critical exponent \(\beta\) follows from \(M \sim (g - g_c)\beta\) at \(h = 0\):

\[g - g_c \sim M^{2b-2} \implies \beta = \frac{1}{2b - 2} = \frac{1 - \zeta}{1 + 2\zeta}. \]  

(56)

In a similar manner the critical exponent \(\gamma\) can be computed from \(\chi = \partial M/\partial h\big|_{h=0} \sim (g - g_c)^{-\gamma}\):

\[\frac{\partial M}{\partial h} \bigg|_{h=0} \sim M^{2-2b} \sim (g - g_c)^{-1} \implies \gamma = 1. \]  

(57)

The other critical exponents can be derived from the effective potential and the scaling form for the composite boson propagator Eq. (31). Summarizing, the critical coupling \(g_c\) and the critical exponents are expressed in terms of the exponent \(\zeta\) of the fermion wave function \(A\) which is determined by Eq. (46):

\[g_c = 1 + 2\zeta, \quad \beta = (1 - \zeta)/(1 + 2\zeta), \quad \delta = (2 + \zeta)/(1 - \zeta), \quad \nu = 1/(1 + 2\zeta), \quad \gamma = 1, \quad \eta = 1 - 2\zeta. \]  

(58)

These exponents satisfy the three dimensional hyperscaling equations \(\gamma = \beta(\delta - 1), \gamma = \nu(2 - \eta), \) and \(3\nu = 2\beta + \gamma\). The critical exponent \(\nu\) follows from the scaling law for the dynamical mass \(m_{\text{dyn}}\) (inverse correlation length) describing the physical fermion mass in the broken phase close to criticality; \(m_{\text{dyn}} \sim M^{1/(1-\zeta)}\) (see Sec. \[\text{II}\]).

In the limit \(N \to \infty\), keeping only terms to order \(1/N\) by using the expression (49) for \(\zeta\), the critical exponents are:

\[\beta = \frac{1}{4} + \frac{15}{16\pi N}, \quad \delta = 5 - \frac{15}{\pi N}, \quad \nu = \frac{1}{2} + \frac{5}{8\pi N}, \quad \gamma = 1, \quad \eta = \frac{5}{2\pi N}. \]  

(59)

These exponents clearly differ from the next-to-leading large \(N\) exponents of the Gross-Neveu model \[\cite{3}\]:

\[\beta = 1, \quad \delta = 2 + \frac{8}{\pi^2 N}, \quad \nu = 1 + \frac{8}{3\pi^2 N}, \quad \gamma = 1 + \frac{8}{N\pi^2}, \quad \eta = 1 - \frac{16}{3\pi^2 N}. \]  

(60)

Note, for instance, the fractions of \(1/(\pi N)\) appearing in the exponents (59), whereas the Gross-Neveu exponents contain fractions of \(1/(\pi^2 N)\).
From Eq. (11) and (31), it follows that the anomalous dimension \( \eta_\psi \) of the fermion field is \( \eta_\psi = -\zeta/2 \approx -1/4 + 5/(8\pi N) \) and that the anomalous dimension \( \eta_\sigma \) of the composite boson field is \( \eta_\sigma = \zeta \approx 1/2 - 5/(4\pi N) \). Since the \( \beta \)-function of \( g \) vanishes at critical coupling \( g = g_c \) (i.e., \( \beta(g_c) = 0 \)), the Yukawa vertex should satisfy the following Callan-Symanzik equation at \( g = g_c \):

\[
\left( \eta_\sigma + 2\eta_\psi + \Lambda \frac{\partial}{\partial \Lambda} \right) \Gamma^A(k+p,k) \approx 0.
\]  

(61)

Clearly, the bare Yukawa vertex approximation \( \Gamma^A(p+q,p) = \tau^A \) is consistent with this scaling constraint, since the scaling dimension of the bare vertex is zero in agreement with the equality \( \eta_\sigma = -2\eta_\psi \).

VI. DISCUSSION

In Eq. (59), we have summarized the critical exponents obtained within our bare vertex approximation to order \( 1/N \). Together with the anomalous dimensions \( \zeta = 1/2 - 5/(4\pi N) \) and \( \eta = 1 - 2\zeta \) for respectively the fermion and \( \sigma \) boson propagators for the model presented, this is the main result of this paper. However, the truncation scheme depicted in Fig. 1 describes merely the leading behavior in the \( 1/N \) expansion. Interestingly, the solutions of the truncated system depicted in Fig. 1 satisfy the hyperscaling equations for arbitrary but finite \( N \). However, we cannot say with certainty that the values obtained in Eq. (59) are the correct critical exponents to order \( 1/N \), without a thorough investigation of the \( \mathcal{O}(1/N) \) diagrams for the Yukawa vertex. Examples of such \( 1/N \) Yukawa vertex corrections are depicted in Fig. 4. The factors of \( N \) depicted in this figure represent index-loops which can be straightforwardly determined, using ’t Hoofts double line representation (as explained in the Appendix).

Suppressing factors of \( 1/N \) appear in the following way. From section IV, using Eqs. (31) and (32), we find that the \( \sigma \) boson propagator at the critical coupling has the approximate form

\[
\Delta_\sigma(p) \approx -\frac{4\pi^2(1-2\zeta)}{3\Lambda} \left( \frac{\Lambda}{p} \right)^{1+2\zeta}.
\]

(62)

The eigenvalue equation (19) identifies a factor of \( 1/N \) with \( 1 - 2\zeta \) \( (1 - 2\zeta \propto 1/N) \). Hence, each internal full \( \sigma \) boson line in a Feynman diagram carries a suppressing factor \( 1/N \). This
illustrates that the diagrams in Fig. 4 are indeed of order $1/N$. Now the factor $1 - 2\zeta$ or $1/N$ is related to the Yukawa coupling $g_Y$ mentioned in the Appendix. Since a four-fermion scattering amplitude, exchanging a virtual $\sigma$ boson would correspond to a factor $g_Y^2$, we can associate the square of the Yukawa coupling with $1/N$, $g_Y^2 \sim 1/N$, so that the factors $g_Y^2 N$ are indeed of order one, in retrospect. Using the above observation that each internal $\sigma$ boson propagator corresponds to a factor $1/N$ and each index-loop to a factor $N$, it is straightforward to verify that the vertex corrections depicted in Fig. 4 are order $1/N$. Moreover, diagrams containing fermion loops with an odd number of out-going $\sigma$ lines are zero in the subcritical regime, due to restrictions imposed by the symmetry of the model.

If $N = \infty$, we should get the exponents within the accuracy of the approximation. It turned out, however, that at $N = \infty$, $\zeta = 1/2$ and no pure power-law expression exists for the $\sigma$ boson propagator (see Eq. (34)). This logarithmic deviation from the power-law behavior causes the violation of hyperscaling and consequently spoils the renormalizability of the model. Such a violation could be a sign that the $1/N$ expansion is at the most an asymptotic expansion, since continuation to negative values of $1/N < 0$ are physically meaningless. But at the moment, this is a mere hypothesis.

Nevertheless, when we take finite but large $N$ then the model does have power-law renormalizability and respects hyperscaling. The main question remains whether the exponents in Eq. (59) are correct to order $1/N$. In order to settle this question adequately one needs to investigate the $1/N$ corrections to the Yukawa vertex. We can think of three scenarios. 1. The next-to-leading $1/N$ corrections as depicted in Fig. 4 are finite and will contribute only to the “conformal invariant” or scale invariant part of the Yukawa vertex and do not alter its scaling dimension. This so-called scale invariant part of the Yukawa vertex depends only on the ratios of the in- and out-going momenta. 2. Each of the diagrams depicted in Fig. 4 is logarithmically divergent and the infinite sum of such diagrams give rise to a new scaling dimension of order $1/N$ for the Yukawa vertex. 3. The $1/N$ expansion breaks down due to the appearance of kinematic factors such as $1/(1 - 2\zeta)$ terms, which give rise to additional factors of $N$ and render the whole idea of topological $1/N$ expansion meaningless. In case that scenario 1 would turn out to be correct, then the set of critical exponents (59) will be correct to order $1/N$. Scenario 2 leads to an anomalous dimension for the Yukawa vertex of order $1/N$ and the expressions for the critical exponents and anomalous dimension are only valid in the leading order and up to logarithmic accuracy. In case that scenario 3 unfolds, our analytical approach has failed, since then we lost track of the perturbative parameter $1/N$ and no standard techniques, except perhaps lattice simulations, are adequate to tackle the critical behavior of the presented model. By the way, each of three scenarios can be reconciled with the full conformal symmetry at the critical point of four-fermion models as advocated in Ref. [7].

VII. SUMMARY AND CONCLUSION

In this paper we have studied various aspects of dynamical symmetry breaking $SU(N) \rightarrow U(1)$ in a specific $SU(N)$ four-fermion model in the large $N$ limit in 2+1 dimensions, by making use of ’t Hooft’s topological $1/N$ expansion. We advocated the applicability of ’t Hooft’s topological $1/N$ expansion to the presented model. Within the adopted $1/N$ expansion, the Yukawa vertex was approximated by its bare form. The main result of this paper is the determination of the exponent $\zeta$ or anomalous dimension of the fermion wave function $A(p)$ as a function of $N$ in the planar approximation at the critical point $g = g_c$. 

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The functional relationship between $\zeta$ and $N$ given in Eq. (46) and Fig. 3. The approximate relationship was found to be $\zeta \simeq 1/2 - 5/(4\pi N)$. Subsequently the critical coupling $g_c$ and various critical exponents are given in terms of the exponent $\zeta$. In the large $N$ limit, the critical exponents for this SU($N$) invariant four-fermion model differ considerably from the critical exponents of the Gross-Neveu model. The presence of $N^2 - 1$ light composite bosons turned out to drastically changes the scaling behavior of the fermion propagator in the vicinity of the critical point, giving rise to a new three dimensional universality class, see Eq. (58). Similar to the 2+1 dimensional Gross-Neveu model, the presented model is nonperturbatively renormalizable for large but finite $N$. The obtained anomalous dimensions of the fermion and $\sigma$ boson propagators are self-consistent with the bare Yukawa vertex approximation.

For the future, it could be interesting to extend the presented approach to arbitrary space-time dimension $d$ in order to investigate the perseverance of triviality at the upper critical dimension $d = 3 + 1$.

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APPENDIX: THE TOPOLOGICAL 1/$N$ EXPANSION

In this appendix, we illustrate the applicability of ’t Hoofts topological $1/N$ expansion to our model Lagrangian (1). As is explicitly mentioned in Ref. [11], the topological expansion is also valid for models, with a global U($N$) symmetry, having scalar fields with two U($N$) indices (i.e., the adjoint representation). Therefore, for large $N$, we generalize the SU($N$) symmetry to U($N$). Subsequently, we perform a Hubbard-Stratonovich transformation, after which the Lagrangian (1) is expressed in terms of Hermitian scalar fields $\sigma_{\alpha\beta} = \sigma^*_{\beta\alpha}$,

$$\mathcal{L} = \bar{\psi}_\alpha i \hat{\partial} \psi_\alpha - \bar{\psi}_\alpha \psi_\beta \sigma_{\alpha\beta} - \frac{1}{2G} \sigma_{\alpha\beta} \sigma_{\beta\alpha},$$

(A.1)

with the flavor or “color” indices $\alpha, \beta$ running from 1 to $N$. Contrary to the QCD and U($N$) gauge theories, “physical” observables are not restricted to be “color” blind. In other words, external fields or sources are allowed to carry U($N$) flavor indices. The Hermitian scalar field are linearly related to the $N^2$ real scalar fields $\sigma^A$ (for U($N$)) of Eq. (3),

$$\sigma_{\alpha\beta} = \sum_{A=0}^{N^2-1} \sigma^A r^A_{\alpha\beta}, \quad \sigma^A = \tau^A_{\alpha\beta} \sigma_{\beta\alpha},$$

(A.2)

with the U(1) generator $r^0_{\alpha\beta} = \delta_{\alpha\beta}/\sqrt{N}$.

Now the scalar fields $\sigma_{\alpha\beta}$ are analogous to the gluon gauge fields of Ref. [11], carrying two U($N$) indices, and consequently the topological $1/N$ expansion can be applied. Moreover, due to the absence cubic and quartic self interactions for the auxiliary scalar fields, the “flavor” Feynman rules are considerably simpler than the corresponding rules for the U($N$) gauge field theories. These flavor Feynman rules for the U($N$) symmetric phase are depicted in Fig. 5, using the double line representation of ’t Hooft, see also Ref. [13]. Each index loop in a Feynman diagram gives rise to factor $N$. In Fig. 5, the propagators are fully
FIG. 5: The double index Feynman rules for the symmetric phase, with $\Delta^{-1}_\sigma(p) = -1/G + \Pi_\sigma(p)$ and $S^{-1}(p) = \hat{p} - \Sigma(p)$, where $\Sigma$ is fermion self-energy.

dressed. This is necessarily so, because of the fact that in a critical theory, with relevant four-fermion interactions close to or at a critical point, the vacuum polarization $\Pi_\sigma$ is of a comparable magnitude as the bare mass term $1/G$. The fermion self-energy $\Sigma$ is of the order of the canonical or free form $\hat{p}$ or larger (this will be shown in Sec. IV). At the critical point the bare mass term $1/G$ of the $\sigma$ boson propagator $\Delta_\sigma(p)$ is canceled by the momentum independent $\sigma$ boson vacuum polarization $(\Pi_\sigma)$ corrections and the remaining momentum dependent terms give rise to the anomalous scaling law for the scalar propagator. For the fermion propagator a similar argument holds, see also Sec. IV.

For $d = 4$ the leading $1/N$ expansion for $U(N)$ gauge theories consists of complicated planar networks of gluon exchanges. This is mainly due to the local nature of the symmetry, which introduces cubic and quartic gluon self-couplings and gauge-fixing interactions, whose particular structure depends on the space-time dimension. For $d = 2$, ’t Hooft showed that the set of planar diagrams can be reduced to self-energy and ladder diagrams. As a consequence, the SDEs form a closed set and the $d = 2$ gauge theory is solvable in the leading $1/N$ expansion.

Contrary to the local $U(N)$ symmetries, the topological $1/N$ expansion is not space-time dependent for the global $U(N)$ symmetries of our interest, merely because of the absence of gauge fixing constraints or Faddeev-Popov ghosts. The space-time dimension $d$ does not play a role in the selection rules for the planarity and thus the large $N$ behavior of Feynman diagrams for global $U(N)$. For internal $\sigma$ boson exchanges is does not matter whether we use the real representation or the Hermitian representation for the composite fields to leading order in $1/N$, since

$$
\langle \sigma_{\alpha\beta}(p)\sigma_{\gamma\delta}(-p) \rangle_C = \sum_A \sum_B \tau^A_{\alpha\beta}\tau^B_{\gamma\delta}\langle \sigma^A(p)\sigma^B(-p) \rangle_C = \sum_A \tau^A_{\alpha\beta}\tau^A_{\gamma\delta}i\Delta^A_\sigma(p)
$$

$$
= i\Delta_\sigma(p)\delta_{\alpha\delta}\delta_{\gamma\beta} + \frac{1}{N} \left[i\Delta^0_\sigma(p) - i\Delta_\sigma(p)\right] \delta_{\alpha\beta}\delta_{\gamma\delta},
$$

where the $N^2 - 1$ propagators $\Delta^A_\sigma$ are degenerate, $\Delta^A_\sigma = \Delta_\sigma$, and $\Delta^0_\sigma$ is the propagator of the U(1) field $\sigma^0$. In general $\Delta^0_\sigma \neq \Delta_\sigma$, thus from Eq. (A.3) it follows that the double line representation of the $\sigma$ boson propagator is violated at the most by a $1/N$ suppressed term.

The planar expansion for the present 2+1 dimensional model is equivalent to the planar expansion for the $d = 2$ U(N) gauge theory or ’t Hooft model, since both models have a
single interaction vertex, see Fig. 6. This planarity can be verified order by order in the diagrammatic expansion of full $\sigma$ propagators $\Delta_\sigma$. We assume that with each internal full $\sigma$ boson exchange in a Feynman diagram an “effective” Yukawa coupling, $g_Y$ is associated. We assume that combination $g_Y^2 N$ is fixed and of the order of one [10], so that all diagrams with powers of $g_Y^2 N$ are of the same order and need to be summed nonperturbatively. The assumption that $g_Y^2 N$ is fixed for $N \to \infty$ can only be validated in retrospect, since $g_Y$ is not a free parameter like the gauge coupling $g$ in QCD; $g_Y$ is self-consistently determined from the planar SDEs and therefore depends on $N$ and $G$. For the $\sigma$ boson vacuum polarization $\Pi_\sigma$, each planar diagram with $n$ internal exchanges of $\sigma$ bosons is associated with a factor $N^n g_Y^{2n}$. For the fermion self-energy $\Sigma$, each planar diagram with $n$ $\sigma$ exchanges also corresponds to a factor $N^n g_Y^{2n}$. Examples of vacuum diagrams with different topology (planar, hole, handle) are depicted in Fig. 7.

For the fermion self-energy, the planar expansion is generated by the rainbow approximation. Moreover for the present model, a certain class of planar diagrams can be neglected. This class comprises all planar diagrams, which can be identified as vertex corrections. This can be understood in the following way. As depicted in Fig. 6, a full Yukawa vertex can either connect the left upper in-going line in $\Pi_\sigma$ with a right upper outgoing line, or connect the
upper left line with the lower left line. Whereas the bare vertex corresponds to the former, the ladder vertex corrections correspond to the latter. Since the generator $\tau^0$ is the only generator having a nonzero trace, it is straightforward to check that diagrams connecting upper left lines with lower left lines give only nonzero contributions to the propagator $\Delta^0_\sigma$ of the $U(1)$ field, thereby causing the inequality $\Delta^0_\sigma \neq \Delta_\sigma$. For instance, in the real representation, the vacuum polarization $\Pi^A_\sigma$ of the propagator $\Delta^A_\sigma$ gets a two-loop vertex correction which is proportional to $\sum_B \text{Tr} \tau^A \tau^B \tau^A \tau^B$. This trace vanishes, except for $A = 0$. Using Eq. (A.3) in the fermion self energy, it follows that the contribution of $\Delta^0_\sigma$ is suppressed by a factor $1/N^2$ with respect to the degenerate $SU(N)$ propagator $\Delta_\sigma$. Therefore, we assume that all relevant planar diagrams, i.e., diagrams contributing to $\Delta_\sigma$, are given by the bare vertex approximation depicted in Fig. 2. A similar planar diagram approach for the $SU(N)$ extrapolation of the Hubbard model was adopted by Foerster [13].

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